ANALYSIS OF AN ITERATIVE SCHEME OF FRACTIONAL STEPS TYPE ASSOCIATED TO THE NONLINEAR PHASE-FIELD EQUATION WITH NON-HOMOGENEOUS DYNAMIC BOUNDARY CONDITIONS

ALAIN MIRANVILLE
Université de Poitiers, Laboratoire de Mathématiques et Applications
UMR CNRS 7348, SP2MI, 86962 Chasseneuil Futuroscope Cedex, France

COSTICĂ MOROŞANU
University “Al. I. Cuza” of Iași, 700506 Iași, Romania

Abstract. The paper concerns with the existence, uniqueness, regularity and the approximation of solutions to the nonlinear phase-field (Allen-Cahn) equation, endowed with non-homogeneous dynamic boundary conditions (depending both on time and space variables). It extends the already studied types of boundary conditions, which makes the problem to be more able to describe many important phenomena of two-phase systems, in particular, the interactions with the walls in confined systems. The convergence and error estimate results for an iterative scheme of fractional steps type, associated to the nonlinear parabolic equation, are also established. The advantage of such method consists in simplifying the numerical computation. On the basis of this approach, a conceptual numerical algorithm is formulated in the end.

1. Introduction. Let us consider the following nonlinear parabolic boundary value problem with respect to the unknown function $\varphi$:

\[
\begin{cases}
\alpha \xi \frac{\partial}{\partial t} \varphi - \xi \Delta \varphi = \frac{1}{2\xi}(\varphi - \varphi^3) + g(t,x) & \text{in } Q \\
\xi \frac{\partial}{\partial \nu} \varphi + \alpha \xi \frac{\partial}{\partial t} \varphi - \Delta_\Gamma \varphi + c_0 \varphi = w(t,x) & \text{on } \Sigma \\
\varphi(0,x) = \varphi_0(x) & \text{on } \Omega,
\end{cases}
\]

where:

- $\Omega$ denotes some bounded domain in $\mathbb{R}^n$ ($1 < n \leq 3$) with a $C^2$ boundary $\partial \Omega = \Gamma$. We set $Q = (0,T] \times \Omega$, $\Sigma = (0,T] \times \partial \Omega$, where $T > 0$ stands for some final time. Of course, $t \in [0,T]$ while $x$ varies in $\Omega$;
- $\varphi(t,x)$ is the phase function (the order parameter), used to distinguish between the states (phases) of material which occupies the region $\Omega$ at every time $t \in [0,T]$;
- $\alpha, \xi, c_0$ are physical parameters representing: the relaxation time, the measure of the interface thickness and a positive constant, respectively;

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• \( \Delta \Gamma \) is the Laplace-Beltrami operator;
• \( g(t, x) \in L^p(Q) \) is a given function (can be interpreted as distributed control), where \( p \) satisfies
  \[
  p \geq \frac{n + 2}{2};
  \]
• \( w(t, x) \in W^{1-\frac{2}{p}, \frac{2}{p} - 1}_p(\Sigma) \) is a given function (can be interpreted as boundary control);
• \( \varphi_0 \in W^{2, -2}_{\infty}(\Omega) \) verifying \( \xi \frac{\partial}{\partial \nu} \varphi_0 - \Delta \Gamma \varphi_0 + c_0 \varphi_0 = w(0, x) \) on \( \Gamma \).

The equation (1) was introduced initially by Allen-Cahn (see [1]) to describe the motion of anti-phase boundaries in crystalline solids. Recently, the Allen-Cahn equation has been widely applied to many complex moving interface problems, like: the mixture of two incompressible fluids, the nucleation of solids, vesicle membranes, etc. Also, the nonlinear parabolic equation (1) occurs in the Caginalp’s phase-field transition system (see [6]) describing the transition between the solid and liquid phases in the solidification process of a material occupying a region \( \Omega \).

The main novelty of this paper refer to the presence of non-homogeneous dynamic boundary conditions in equation (1), untreated until now (to our knowledge) in the mathematical literature, which makes the present nonlinear parabolic problem to be more able describing many important phenomena of two-phase systems: superheating, supercooling, the effects of surface tension, separating zones etc. in particular, the interactions with the walls in confined systems. Consequently, a wide variety of industrial applications are covered. Endowed with homogeneous dynamic boundary conditions and singular potentials, problem (1) was treated in [7], [10]-[12] and [19]. For detailed discussions on the phase-field transition system we refer to [5]-[13], [15]-[20], [23], [30]-[33].

At the moment \( t \) the material is considered to be liquid if \( \varphi \) is close to \( 1 + \delta_1 \), while it is considered to be solid if \( \varphi \) is close to \( -1 - \delta_1 \), with \( \delta_1 \) a positive number. Let us consider the interface as a continuous region, more vast (in which the liquid can coexist with the solid), of finite thickness, in which the change of phase occurs continuously. In this spirit we define the separating region (the interface at the moment \( t \)) as being the set:

\[
\Omega_t = \{ x \in \Omega; |\varphi(t, x)| \leq 1 + \delta_1 \}.
\]

Here we will use the standard notations for Sobolev spaces, namely, given positive integer \( k \) and \( 1 \leq p \leq \infty \), we denote by \( W^{2k}_p(Q) \) the usual Sobolev space on \( Q \):

\[
W^{2k}_p(Q) = \left\{ y \in L^p(Q) : \frac{\partial^r y}{\partial x^s}, \frac{\partial^s y}{\partial x^r} \in L^p(Q), \text{ for } 2r + s \leq k \right\},
\]

i.e., the space of functions whose \( t \)-derivatives and \( x \)-derivatives up to the order \( k \) and \( 2k \), respectively, belong to \( L^p(Q) \). Also, we will use the Sobolev spaces \( W^l_p(\Omega) \), \( W^l_p(\Sigma) \) with non-integer \( l \) for the initial and boundary conditions, respectively (see [24], p. 70 and p. 81).

In the following we will denote by \( C \) several positive constants, with the remark that the extra dependencies will be set out on occurrence. In addition, every product is understood in the \( L^2 \)-space, except when otherwise specified. In particular, the norm and the scalar product in \( L^2(\Omega) \) are denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), respectively, while the corresponding symbols in \( L^2(\Gamma) \) will be marked by a subscript \( \Gamma \).
The outline of the paper is as follows: In Section 2 we prove the existence, regularity, stability and uniqueness of the solution to the nonlinear problem (1) in the presence of non-homogeneous dynamic boundary conditions, while, in Section 3 we are concerned with the convergence and error estimate of the approximating scheme of fractional steps type associated to the phase-field nonlinear (Allen-Cahn) equation. The paper concludes with the formulation of a conceptual numerical algorithm.

2. On the existence, regularity, stability and uniqueness of solutions to the phase field nonlinear equation with non-homogeneous dynamic boundary conditions. In order to study the nonlinear problem (1), we will appeal to the strategy used in [26]. In this sense, we will consider a further variable \( \psi = \varphi \), \( \psi(0, x) = \varphi_0 \) on \( \Gamma \) and we will treat the dynamic boundary conditions (1) as a parabolic equation for \( \psi \) on the boundary, i.e.

\[
\begin{align*}
\varphi &= \psi & \text{on } \Sigma \\
\xi \frac{\partial}{\partial t}\varphi + \alpha \xi \frac{\partial}{\partial t} \psi - \Delta \Gamma \psi + c_0 \psi &= w(t, x) & \text{on } \Sigma \\
\psi(0, x) &= \psi_0(x) & \text{on } \Gamma,
\end{align*}
\]

\( \psi_0 \in W^{2-\frac{2}{p}}(\Gamma) \). For the remaining data in (1), we keep the meanings already formulated above. Consequently, the nonlinear parabolic boundary value problem (1) can be rewritten suitably in the following form

\[
\begin{align*}
\alpha \xi \frac{\partial}{\partial t} \varphi - \xi \Delta \varphi &= \frac{1}{2\xi}(\varphi - \varphi^3) + g(t, x) & \text{in } Q \\
\varphi &= \psi & \text{on } \Sigma \\
\xi \frac{\partial}{\partial t} \varphi + \alpha \xi \frac{\partial}{\partial t} \psi - \Delta \Gamma \psi + c_0 \psi &= w(t, x) & \text{on } \Sigma \\
\varphi(0, x) &= \varphi_0(x) & \text{on } \Omega \\
\psi(0, x) &= \psi_0(x) & \text{on } \Gamma.
\end{align*}
\]

We first analyze the linearized version of problem (4), that is

\[
\begin{align*}
\alpha \xi \frac{\partial}{\partial t} \varphi - \xi \Delta \varphi &= \hat{f}(t, x) & \text{in } Q \\
\varphi &= \psi & \text{on } \Sigma \\
\varphi(0, x) &= \varphi_0(x) & \text{on } \Omega \\
\xi \frac{\partial}{\partial t} \varphi + \alpha \xi \frac{\partial}{\partial t} \psi - \Delta \Gamma \psi + c_0 \psi + h_3 &= 0 & \text{on } \Sigma \\
\psi(0, x) &= \psi_0(x) & \text{on } \Gamma,
\end{align*}
\]

which are useful in the proof of the main result of this Section. We have

Lemma 2.1. If \( \hat{f}(t, x) \in L^p(Q) \), \( h_3 \in L^p(\Sigma) \), \( \varphi_0 \in W^{2-\frac{2}{p}}(\Omega) \), \( \psi_0 \in W^{2-\frac{2}{p}}(\Gamma) \), then problem (5) possesses a unique solution \((\varphi, \psi) \in W^{1,2}_p(Q) \times W^{1,2}_p(\Sigma)\) such that

\[
\|\varphi\|_{W^{1,2}_p(Q)} + \|\psi\|_{W^{1,2}_p(\Sigma)} \leq C \left[ \|\varphi_0\|_{W^{2-\frac{2}{p}}(\Omega)} + \|\psi_0\|_{W^{2-\frac{2}{p}}(\Gamma)} + \|\hat{f}\|_{L^p(Q)} + \|h_3\|_{L^p(\Sigma)} \right],
\]

where \( C \) depends on \( |\Omega|, T, n, p, \alpha, \xi, c_0 \), but is independent of \( \varphi, \psi, \hat{f} \) and \( h_3 \).
Proof. Applying Lemma 2.3 in [26] with \( h_3 = -w \) and making use of the embeddings
\( W^{2-\frac{2}{p}}_{\infty}(\Omega) \subset W^{2-\frac{2}{p}}_{p}(\Gamma) \) and \( W^{1-\frac{2}{p}}_{p}(\Omega) \subset L^p(\Sigma) \) (see (2)), we can easily conclude
that the results set out by Lemma 2.1 are true.

The main result of this Section establishes the dependence of the solution \((\varphi, \psi)\)
in the nonlinear parabolic problem (4) on the terms \( g(t, x) \) and \( w(t, x) \). We have

**Theorem 2.2.** There exists a unique solution \((\varphi, \psi) \in W^{1,2}_p(\Omega) \times W^{1,2}_p(\Sigma)\) to (4)
and \((\varphi, \psi)\) satisfies

\[
\|\varphi\|_{W^{1,2}_p(\Omega)} + \|\psi\|_{W^{1,2}_p(\Sigma)} \leq C \left[ 1 + \|\varphi_0\|_{W^{2-\frac{2}{p}}_{\infty}(\Omega)}^{3-\frac{2}{p}} + \|\psi_0\|_{W^{2-\frac{2}{p}}_{\infty}(\Gamma)}^{3-\frac{2}{p}} + \|g\|_{L^p(\Omega)} + \|w\|_{W^{1-\frac{2}{p}}_{p}(\Omega)} \right],
\]

where the constant \( C \) depends on \(|\Omega|\), \( T \), \( n \), \( \alpha \), \( \varepsilon \) and \( p_0 \) but is independent of \( \varphi \),
\( \psi \), \( g \) and \( w \).

If \((\varphi_1, \psi_1), (\varphi_2, \psi_2)\) are two solutions to (4), corresponding to \((\varphi_0^1, \psi_0^1), (\varphi_0^2, \psi_0^2)\)
\( \in W^{2-\frac{2}{p}}_{\infty}(\Omega) \times W^{2-\frac{2}{p}}_{\infty}(\Gamma) \) and \((g^1, w^1), (g^2, w^2)\), respectively, such that for some \( M_1 \in (0, \infty) \) \( \) we have

\[
\|\varphi_1\|_{W^{1,2}_p(\Omega)} \leq M_1, \quad \|\varphi_2\|_{W^{1,2}_p(\Omega)} \leq M_1,
\]

then

\[
\|\varphi_1 - \varphi_2\|_{W^{1,2}_p(\Omega)} + \|\psi_1 - \psi_2\|_{W^{1,2}_p(\Sigma)} \leq C \left[ \|\varphi_0^1 - \varphi_0^2\|_{W^{2-\frac{2}{p}}_{\infty}(\Omega)} + \|\psi_0^1 - \psi_0^2\|_{W^{2-\frac{2}{p}}_{\infty}(\Gamma)} \right. \\
+ \|g^1 - g^2\|_{L^p(\Omega)} + \|w^1 - w^2\|_{W^{1-\frac{2}{p}}_{p}(\Omega)} \right],
\]

where the constant \( C \) depends on \(|\Omega|\), \( T \), \( M_1 \), \( n \), \( \alpha \), \( \varepsilon \), \( c_0 \) and \( b_0 \), but is independent of \((\varphi_1, \psi_1), (\varphi_2, \psi_2), (g^1, g^2), (w^1, w^2)\).}

The results in Theorem 2.2, with homogeneous Neumann boundary conditions
\( \frac{\partial}{\partial \nu} \varphi = 0 \) but with a general nonlinearity (which include the classical regular potential
\( \frac{1}{2\varepsilon}(\varphi - \varphi^3) \)), was proved in [32].

2.1. **Proof of Theorem 2.2.** Here we will prove the existence, uniqueness and
regularity of the solution to problem (4), considering as nonlinear term the classical
regular potential \( f(\varphi) = \frac{1}{2\varepsilon}(\varphi - \varphi^3) \) which verifies for \( n \leq 3 \) and \( r = 3 \) the general
assumptions \( H_1 \) and \( H_3 \) formulated in [32], namely:

\( H_1 : (\varphi - \varphi^3)|\varphi|^{3p-4} \varphi \leq 1 + |\varphi|^{3p-1} - |\varphi|^{3p}; \)

\( H_3 : \) There are a function \( \bar{F} : \mathbb{R}^2 \rightarrow \mathbb{R} \) and a constant \( b_0 > 0 \) verifying relations:

\[
((\varphi_1 - \varphi^3_2) - (\varphi_2 - \varphi^3_2))^2 \leq \bar{F}(\varphi_1, \varphi_2)(\varphi_1 - \varphi_2)^2,
\]

\( \bar{F}(\varphi_1, \varphi_2) \leq b_0(1 + |\varphi_1|^4 + |\varphi_2|^4), \forall \varphi_1, \varphi_2 \in \mathbb{R}. \)

Basic tools in treating the problem (4) are the Leray-Schauder degree theory
[14], the \( L^p \)-theory of linear and quasi-linear parabolic equations [24], as well as the
Lions and Peetre embedding Theorem [25], p. 24, which ensures the existence of a
continuous embedding $W^{1,2}_p(Q) \subset L^\mu(Q)$, where the number $\mu$ is defined as follows

$$
\mu = \begin{cases} 
\text{any positive number } \geq 3p & \text{if } \frac{1}{p} - \frac{2}{n+2} \leq 0, \\
\left( \frac{1}{p} - \frac{2}{n+2} \right)^{-1} & \text{if } \frac{1}{p} - \frac{2}{n+2} > 0.
\end{cases}
$$

In order to use the Leray-Schauder degree theory, as we have mentioned above, we will choose as suitable Banach space $B = L^{3p}(Q) \times L^p(\Sigma)$, endowed with the norm

$$
\|(v_1, v_2)\|_B = \|v_1\|_{L^{3p}(Q)} + \|v_2\|_{L^p(\Sigma)} \quad \forall v = (v_1, v_2) \in B.
$$

Let us define the nonlinear operator $T : B \times [0, 1] \to B$ as

$$
T(v, \lambda) = ((\varphi, \psi) = ((\varphi(v, \lambda), \psi(v, \lambda))) \quad \forall v \in B, \ \forall \lambda \in [0, 1],
$$

where $(\varphi, \psi)$ is the solution to the linear problem

$$
\begin{align*}
\alpha & \frac{\partial}{\partial t} \varphi - \xi \Delta \varphi = \lambda \left[ \frac{1}{2\xi} (v_1 - v_1^3) + g(t, x) \right] & & \text{in } Q \\
\varphi & = \psi & & \text{on } \Sigma \\
\varphi(0, x) & = \lambda \varphi_0(x) & & \text{in } \Omega \\
\xi & \frac{\partial}{\partial t} \varphi + \alpha \xi \frac{\partial}{\partial t} \psi - \Delta \Gamma \psi + c_0 \psi = \lambda w(t, x) & & \text{on } \Sigma \\
\psi(0, x) & = \lambda \psi_0(x) & & \text{on } \Gamma.
\end{align*}
$$

The nonlinear operator $T$ defined by (12), enjoys the following properties:

**A. $T$ is well-defined (problem (13) has a solution).** We remember that $g \in L^p(Q)$ and $w(t, x) \in W^{1-{\frac{2}{p}}-{\frac{2}{3}}}(\Sigma)$ are given functions. It follows from the right hand of (13) that $\forall v_1 \in L^{3p}(Q)$, we deduce that $\frac{1}{2\xi} (v_1 - v_1^3) \in L^p(Q)$. Then $\frac{1}{2\xi} (v_1 - v_1^3) + g(t, x) \in L^p(Q)$, $h_3 = -\lambda w \in W^{1-{\frac{2}{p}}-{\frac{2}{3}}}(\Sigma) \subset L^p(\Sigma)$ (see (2)), the solution $(\varphi, \psi)$ to problem (13) exists and is unique. Furthermore, $\forall v \in B$ and $\forall \lambda \in [0, 1],

$$
(\varphi, \psi) = ((\varphi(v, \lambda), \psi(v, \lambda)) \in W^{1,2}_p(Q) \times W^{1,2}_p(\Sigma).
$$

Since $\mu = \frac{p (n + 2)}{n - 2 - 2p} \geq 3p$ if $\frac{1}{p} - \frac{2}{n+2} > 0$, we can take $\mu > 3p$ in all cases required by (10). Consequently, we have the continuous inclusions (see [14], p. 24)

$$
\begin{align}
W^{1,2}_p(Q) & \subset L^\mu(Q) \subset L^{3p}(Q) \\
W^{1,2}_p(\Sigma) & \subset L^p(\Sigma)
\end{align}
$$

which means that $T(v, \lambda) = ((\varphi, \psi)) \in B$ for all $v \in B$ and $\forall \lambda \in [0, 1]$.

**B. $T$ is continuous and compact.** Let $v_n = (v^n_1, v^n_2) \to v = (v_1, v_2)$ in $B$ and $\lambda_n \to \lambda$ in $[0, 1]$. Denote $(\varphi^n_\lambda, \psi^n_\lambda) = T(v_n, \lambda_n)$, $(\varphi^\lambda_n, \psi^\lambda_n) = T(v_n, \lambda)$ and

$$
\int_0^1 \int_\Omega (\alpha \varphi - \xi \Delta \varphi) \varphi^n_\lambda \psi^n_\lambda \, dx \, dt + \frac{1}{p} \int_0^1 \int_\Omega (\varphi^n_\lambda - \varphi^n_\lambda^3) \varphi^n_\lambda \psi^n_\lambda \, dx \, dt + \int_0^1 \int_\Omega g(t, x) \varphi^n_\lambda \psi^n_\lambda \, dx \, dt = \int_0^1 \int_\Omega h(t, x) \varphi^n_\lambda \psi^n_\lambda \, dx \, dt
$$

$$
\int_0^1 \int_\Omega \left( (\alpha \varphi - \xi \Delta \varphi) \varphi - \xi \Delta \varphi \right) \varphi_\lambda \psi_\lambda \, dx \, dt + \frac{1}{p} \int_0^1 \int_\Omega (\varphi_\lambda - \varphi_\lambda^3) \varphi_\lambda \psi_\lambda \, dx \, dt + \int_0^1 \int_\Omega g(t, x) \varphi_\lambda \psi_\lambda \, dx \, dt = \int_0^1 \int_\Omega h(t, x) \varphi_\lambda \psi_\lambda \, dx \, dt
$$

$$
\int_0^1 \int_\Omega (\alpha \varphi - \xi \Delta \varphi) \varphi_\lambda \psi_\lambda \, dx \, dt + \frac{1}{p} \int_0^1 \int_\Omega (\varphi_\lambda - \varphi_\lambda^3) \varphi_\lambda \psi_\lambda \, dx \, dt + \int_0^1 \int_\Omega g(t, x) \varphi_\lambda \psi_\lambda \, dx \, dt = \int_0^1 \int_\Omega h(t, x) \varphi_\lambda \psi_\lambda \, dx \, dt
$$
(\varphi^{\lambda}, \psi^{\lambda}) = T(v, \lambda). \text{ From (12) and (13) we obtain}
\begin{align*}
\frac{\alpha \xi}{\partial t} \bar{\varphi} - \xi \Delta \bar{\varphi} &= (\lambda_n - \lambda) \left[ \frac{1}{2\xi} \left[ (v_1^n)^{(3)} + g(t, x) \right) \right] \quad \text{in } Q \\
\bar{\varphi} &= \bar{\psi} \quad \text{on } \Sigma \\
\bar{\varphi}(0, x) &= (\lambda_n - \lambda) \varphi_0(x) \quad \text{in } \Omega \quad (16) \\
\xi \frac{\partial}{\partial \nu} \bar{\varphi} + \alpha \xi \frac{\partial}{\partial t} \bar{\psi} - \Delta \bar{\psi} + c_0 \bar{\psi} &= (\lambda_n - \lambda)w(t, x) \quad \text{on } \Sigma \\
\bar{\psi}(0, x) &= (\lambda_n - \lambda) \psi_0(x) \quad \text{on } \Gamma,
\end{align*}
where \(\bar{\varphi} = \varphi^{\lambda_n} - \varphi^{\lambda} \) and \(\bar{\psi} = \psi^{\lambda_n} - \psi^{\lambda} \). Using Lemma 2.1 to the linear problem (16), we have
\begin{align*}
\|\varphi^{\lambda_n} - \varphi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\psi^{\lambda_n} - \psi^{\lambda}\|_{W^{1,2}_p(\Sigma)} &
\leq C|\lambda_n - \lambda| \left[ \|\varphi_0\|_{L^{\infty}_p(\Omega)} + \|\psi_0\|_{L^{\infty}_p(\Gamma)} + \|v^n_1 - (v_1^n)^{(3)}\|_{L^p(\Omega)} + \|w\|_{L^p(\Sigma)} \right].
\end{align*}
Having \(v_1^n\) bounded in \(L^p(\Omega)\), we see that \(v^n_1 - (v_1^n)^{(3)}\) is bounded in \(L^p(\Omega)\) (see [14]). Thus, making use of the convergence \(\lambda_n \to \lambda\), from the above inequality we get
\begin{align*}
\|\varphi^{\lambda_n} - \varphi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\psi^{\lambda_n} - \psi^{\lambda}\|_{W^{1,2}_p(\Sigma)} &\to 0 \text{ as } n \to \infty. \quad (17)
\end{align*}
From (12) and (13) we also obtain
\begin{align*}
\begin{cases}
\alpha \xi \frac{\partial}{\partial t} (\varphi^{\lambda_n} - \varphi^{\lambda}) - \xi \Delta (\varphi^{\lambda_n} - \varphi^{\lambda}) = \frac{1}{2\xi} \left[ (v_1^n - v_1) + (v_1^n - (v_1^n)^{(3)}) \right] \quad \text{in } Q \\
\varphi^{\lambda_n} - \varphi^{\lambda} = \psi^{\lambda_n} - \psi^{\lambda} \quad \text{on } \Sigma \\
(\varphi^{\lambda_n} - \varphi^{\lambda})(0, x) = 0 \quad \text{in } \Omega \quad (18) \\
\xi \frac{\partial}{\partial \nu} (\varphi^{\lambda_n} - \varphi^{\lambda}) + \alpha \xi \frac{\partial}{\partial t} (\psi^{\lambda_n} - \psi^{\lambda}) - \Delta (\psi^{\lambda_n} - \psi^{\lambda}) + c_0 (\psi^{\lambda_n} - \psi^{\lambda}) = 0 \quad \text{on } \Sigma \\
(\psi^{\lambda_n} - \psi^{\lambda})(0, x) = 0 \quad \text{on } \Gamma.
\end{cases}
\end{align*}
Lemma 2.1 applied to linear problem (18) gives the estimate
\begin{align*}
\|\varphi^{\lambda_n} - \varphi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\psi^{\lambda_n} - \psi^{\lambda}\|_{W^{1,2}_p(\Sigma)} &\leq CA \left[ \|(v_1^n - v_1) + (v_1^n - (v_1^n)^{(3)})\|_{L^p(\Omega)} \right],
\end{align*}
for a constant \(C > 0\). Then the convergence \(v^n_1 \to v_1\) in \(L^p(\Omega)\) and the continuity of the Nemitskij operator (see [14]) allow to conclude that
\begin{align*}
\|\varphi^{\lambda_n} - \varphi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\psi^{\lambda_n} - \psi^{\lambda}\|_{W^{1,2}_p(\Sigma)} &\to 0 \quad \text{as } n \to \infty. \quad (19)
\end{align*}
Making use of the continuous embedding (15) and relations (17), (19), we derive the continuity of the nonlinear operator \(T\) defined in (12). Furthermore, \(T\) is compact. Indeed, since \(\mu > 3p\) (see (10) and (15)), the inclusion \(W^{1,2}_p(Q) \hookrightarrow L^{3p}(Q)\) is compact (see [25], p. 21). Moreover, writing \(T\) as the composition
\begin{align*}
B \times [0, 1] &\to W^{1,2}_p(Q) \times W^{1,2}_p(\Sigma) \hookrightarrow L^{3p}(Q) \times L^p(\Sigma) = B,
\end{align*}
the compactness of \(T\) immediately follows.
We will continue with the proof of Theorem 2.2.
The regularity of the solution. Now, we will establish the existence of a number \( \delta > 0 \) such that (see (11) and (12))

\[
(\varphi, \psi, \lambda) \in B \times [0, 1] \quad \text{with} \quad (\varphi, \psi) = T(\varphi, \psi, \lambda) \implies \| (\varphi, \psi) \|_B < \delta. \tag{20}
\]

The equality \((\varphi, \psi) = T(\varphi, \psi, \lambda)\) is equivalent to

\[
\begin{align*}
\alpha \xi \frac{\partial}{\partial t} \varphi - \xi \Delta \varphi &= \lambda \left[ \frac{1}{2\xi} (\varphi - \varphi^3) + g(t, x) \right] \quad \text{in} \ Q \\
\varphi &= \psi \quad \text{on} \ \Sigma \\
\varphi(0, x) &= \lambda \varphi_0(x) \quad \text{in} \ \Omega \\
\xi \frac{\partial}{\partial t} \varphi + \alpha \xi \frac{\partial}{\partial t} \psi - \psi \varphi \psi + c_0 \psi = \lambda w(t, x) \quad \text{on} \ \Sigma \\
\psi(0, x) &= \lambda \psi_0(x) \quad \text{on} \ \Gamma.
\end{align*}
\tag{21}
\]

Multiplying the first equation in (21) by \(|\varphi|^{3p-4}\varphi\), integrating over \(Q_t := (0, t) \times \Omega\), \(t \in (0, T]\) and using Green’s Theorem and (21)2, we get

\[
\frac{\alpha \xi}{3p-2} \left[ \int_{\Omega} |\varphi(t, x)|^{3p-2} \, dx + \int_{\Gamma} |\psi(t, x)|^{3p-2} \, d\gamma \right] + \int_{Q_t} |\nabla \varphi|^2 |\varphi|^{3p-4} \, dx + \int_{Q_t} |\nabla \psi|^2 |\psi|^{3p-4} \, dx + c_0 \xi \int_{\Sigma_t} |\psi|^{3p-2} \, d\gamma \\
+ \frac{\lambda}{3p-2} \left[ \int_{\Omega} |\varphi_0(x)|^{3p-2} \, dx + \int_{\Gamma} |\psi_0(x)|^{3p-2} \, d\gamma \right] + \lambda \int_{Q_t} g|\varphi|^{3p-4} \varphi \, dx + \lambda \int_{Q_t} w|\varphi|^{3p-4} \varphi \, d\gamma. \tag{22}
\]

By \(H_1\) and Young’s inequality, from (22) we obtain

\[
\frac{\alpha \xi}{3p-2} \left[ \int_{\Omega} |\varphi(t, x)|^{3p-2} \, dx + \int_{\Gamma} |\psi(t, x)|^{3p-2} \, d\gamma \right] + (p-1) \left[ \xi \int_{Q_t} |\nabla \varphi|^2 |\varphi|^{3p-4} \, dx + \int_{Q_t} |\nabla \psi|^2 |\psi|^{3p-4} \, dx \right] + c_0 \xi \int_{\Sigma_t} |\psi|^{3p-2} \, d\gamma \\
+ \frac{\lambda}{3p-2} \left[ \int_{\Omega} |\varphi_0(x)|^{3p-2} \, dx + \int_{\Gamma} |\psi_0(x)|^{3p-2} \, d\gamma \right] + \lambda \int_{Q_t} g|\varphi|^{3p-4} \varphi \, dx + \lambda \int_{Q_t} w|\varphi|^{3p-4} \varphi \, d\gamma \\
\leq \lambda \frac{\alpha \xi}{3p-2} \left[ \int_{\Omega} |\varphi_0(x)|^{3p-2} \, dx + \int_{\Gamma} |\psi_0(x)|^{3p-2} \, d\gamma \right] + \lambda \left[ \frac{1}{3p} |\Omega| + \frac{1}{3p} \varepsilon^{-3p} \frac{1}{2\xi} |\Omega| T + \frac{1}{p} |\varepsilon^{-p} g\|_{L^p(\Omega)} + \frac{1}{p} |\varepsilon^{-p} w\|_{L^p(\Sigma)} \right] \\
+ \lambda \left[ \frac{3p-1}{2\xi} \frac{1}{3p} - \frac{\lambda}{p} \right] \int_{Q_t} |\varphi|^{3p} \, dx. \tag{23}
\]

Taking \(\varepsilon\) small enough, inequality (23) yields

\[
\lambda \| |\varphi|^{3p} \|_{L^p(Q)} \leq C_1 \left[ 1 + \| \varphi_0 \|_{L^{3p-2}(\Omega)}^{3p-2} + \| \psi_0 \|_{L^{3p-2}(\Gamma)}^{3p-2} + \| g \|_{L^p(\Omega)} + \| w \|_{L^p(\Sigma)} \right], \tag{24}
\]
for a constant $C_1 = C(\Omega, T, n, p, \alpha, \xi) > 0$.

Owing to (24) and making use of the embedding $L^{3p-2}(\Sigma) \subset L^p(\Sigma)$ (see (2)), we deduce from (23) that
\[
\lambda^p \|\psi\|_{L^p(\Sigma)}^p \leq \lambda \|\psi\|_{L^{3p-2}(\Sigma)}^{3p-2},
\]
\[
\leq C_1 \left[ 1 + \|\varphi_0\|_{L^{3p-2}(\Omega)}^{3p-2} + \|\psi_0\|_{L^{3p-2}(\Gamma)}^{3p-2} + \|g\|_{L^p(Q)}^p + \|w\|_{L^p(\Sigma)}^p \right],
\]
(25)
where $C_1 = C(\Omega, T, n, p, \alpha, \xi, c_0) > 0$ denotes a new positive constant.

Applying Lemma 2.1 to problem (21), we get
\[
\|\varphi\|_{W_{-1}^p(Q)} + \|\psi\|_{W_{-1}^p(\Sigma)} \leq C_1 \left[ \|\varphi_0\|_{W_{-2}^p(\Omega)}^{3p-2} + \|\psi_0\|_{W_{-2}^p(\Gamma)}^{3p-2} + \lambda \|\varphi - \varphi^3\|_{L^p(Q)} + \lambda \|\psi\|_{L^p(\Sigma)} + \lambda \|w\|_{L^p(\Sigma)} \right],
\]
(26)
Taking into account Lemma 1.1 in [24] and relation (24), we deduce that
\[
\lambda \|\varphi - \varphi^3\|_{L^p(Q)} \leq \lambda C^p \left( 1 + |\varphi|^3 \right)^p \int_Q dt dx \leq 2^{p-1} C^p \left[ |\Omega| T + \lambda \|\varphi\|_{L^p(Q)}^p \right]
\]
\[
\leq C_1 \left[ 1 + \|\varphi_0\|_{L^{3p-2}(\Omega)}^{3p-2} + \|\psi_0\|_{L^{3p-2}(\Gamma)}^{3p-2} + \|g\|_{L^p(Q)}^p + \|w\|_{L^p(\Sigma)}^p \right],
\]
(27)
i.e.
\[
\lambda \|\varphi - \varphi^3\|_{L^p(Q)} \leq C_1 \left[ 1 + \|\varphi_0\|_{L^{3p-2}(\Omega)}^{3p-2} + \|\psi_0\|_{L^{3p-2}(\Gamma)}^{3p-2} + \|g\|_{L^p(Q)} + \|w\|_{L^p(\Sigma)} \right].
\]

Substituting the above inequality in (26), we get
\[
\|\varphi\|_{W_{-1}^p(Q)} + \|\psi\|_{W_{-1}^p(\Sigma)} \leq C_1 \left[ 1 + \|\varphi_0\|_{W_{-2}^p(\Omega)}^{3p-2} + \|\psi_0\|_{W_{-2}^p(\Gamma)}^{3p-2} + \|\varphi_0\|_{L^{3p-2}(\Omega)}^{3p-2} + \|\psi_0\|_{L^{3p-2}(\Gamma)}^{3p-2} + \|g\|_{L^p(Q)} + \|w\|_{L^p(\Sigma)} \right].
\]

The continuous embedding in (15) ensures that
\[
\|\varphi\|_{L^p(Q)} + \|\psi\|_{L^p(\Sigma)} \leq C \left[ \|\varphi\|_{W_{-1}^p(Q)} + \|\psi\|_{W_{-1}^p(\Sigma)} \right],
\]

which means that the claim in (20) holds true.

Denoting
\[
B_\delta := \left\{ (\varphi, \psi) \in B : \|{(\varphi, \psi)}\|_B < \delta \right\},
\]
(28)
relation (20) ensures that
\[
T(\varphi, \psi, \lambda) \neq (\varphi, \psi) \quad \forall (\varphi, \psi) \in \partial B_\delta, \quad \forall \lambda \in [0, 1],
\]
provided that $\delta > 0$ is sufficiently large. Furthermore, we conclude that problem (4) has a solution $(\varphi, \psi) \in W_{1,2}^p(Q) \times W_{1,2}^p(\Sigma)$ (for more details, see [4], p. 21).

Finally, making use of the embeddings $W_{-2}^{2,2}(\Omega) \subset W_{-1}^{2,2}(\Sigma) \subset L^{3p-2}(\Omega)$ and $W_{-1}^{2,2,2}(\Sigma) \subset L^p(\Sigma)$ (see (2)), the inequality (27) permit us to conclude that the estimate (7) is valid.
The uniqueness of the solution. Next, we will establish the stability result \((9)\) and, as a consequence, the uniqueness of the solution to problem \((4)\). By hypothesis, \((\varphi_1, \psi_1), (\varphi_2, \psi_2) \in W^{1,2}_p(\Omega) \times W^{2,2}_p(\Sigma)\) solve problem \((4)\) corresponding to \(g^1, w^1, \varphi_0^1\) and \(g^2, w^2, \varphi_0^2\), respectively. Thus \(\varphi_1 - \varphi_2 \in W^{1,2}_p(\Omega), \psi_1 - \psi_2 \in W^{1,2}_p(\Sigma)\) and

\[
\begin{align*}
\alpha \xi \frac{\partial}{\partial t} (\varphi_1 - \varphi_2) &- \xi \Delta (\varphi_1 - \varphi_2) \\
&= \frac{1}{2\xi} \left[ (\varphi_1 - \varphi_2) - (\varphi_1^2 - \varphi_2^2) \right] + (g^1 - g^2) \quad \text{in } Q \\
\varphi_1 - \varphi_2 &= \psi_1 - \psi_2 \quad \text{on } \Sigma \\
(\varphi_1 - \varphi_2)(0, x) &= \varphi_0^1 - \varphi_0^2 \quad \text{in } \Omega \\
\xi \frac{\partial}{\partial n} (\varphi_1 - \varphi_2) + \alpha \xi \frac{\partial}{\partial t} (\psi_1 - \psi_2) &- \Delta_G (\psi_1 - \psi_2) + c_0 (\psi_1 - \psi_2) = w^1 - w^2 \quad \text{on } \Sigma \\
(\psi_1 - \psi_2)(0, x) &= \psi_0^1 - \psi_0^2 \quad \text{on } \Gamma.
\end{align*}
\]

Multiplying \((2.27)_1\) by \(|\varphi_1 - \varphi_2|^{p-2}(\varphi_1 - \varphi_2)\), integrating over \(Q_t, t \in (0, T]\), and using Green’s formula as well as the Cauchy-Schwarz inequality, we obtain

\[
\begin{align*}
\frac{\alpha \xi}{p} &\int_{\Omega} |\varphi_1 - \varphi_2|^p \, dx + \int_{\Gamma} |\psi_1 - \psi_2|^p \, d\gamma \\
&+ (p - 1) \left[ \xi \int_{Q_t} |\nabla (\varphi_1 - \varphi_2)|^2 |\varphi_1 - \varphi_2|^{p-2} \, dsdx \\
&+ \int_{\Sigma_t} |\nabla_G (\psi_1 - \psi_2)|^2 |\psi_1 - \psi_2|^{p-2} \, dsd\gamma \right] + c_0 \xi \int_{\Sigma_t} |\psi_1 - \psi_2|^p \, dsd\gamma \\
&\leq \frac{\alpha \xi}{p} \left[ \int_{\Omega} |\varphi_0^1 - \varphi_0^2|^p \, dx + \int_{\Gamma} |\psi_0^1 - \psi_0^2|^p \, d\gamma \right] \\
&+ \frac{p - 1}{p} \frac{1}{2^{p-1}} \int_{Q_t} |\varphi_1 - \varphi_2|^p \, dsdx + \frac{2p}{p} \|g^1 - g^2\|_{L^p(\Omega)} + \frac{2p}{p} \|w^1 - w^2\|_{L^p(\Sigma)} \\
&+ \frac{1}{2\xi} \int_{Q_t} \left[ (\varphi_1 - \varphi_2) - (\varphi_1^2 - \varphi_2^2) \right] |\varphi_1 - \varphi_2|^{p-2}(\varphi_1 - \varphi_2) \, dsdx, \forall t \in (0, T].
\end{align*}
\]

Thanks to the inequality

\[
[(\varphi_1 - \varphi_0^1) - (\varphi_2 - \varphi_0^2)](\varphi_1 - \varphi_2) \leq (\varphi_1 - \varphi_2)^2, \forall \varphi_1, \varphi_2 \in IR,
\]

and making uses of Gronwall’s inequality, it results from above that

\[
\begin{align*}
&\|\varphi_1 - \varphi_2\|_{L^p(\Omega)} + \|\psi_1 - \psi_2\|_{L^p(\Sigma)} \\
&\leq C(T, p, \alpha, \xi) \left[ \|\varphi_0^1 - \varphi_0^2\|_{L^p(\Omega)} + \|\psi_0^1 - \psi_0^2\|_{L^p(\Gamma)} \\
&\quad + \|g^1 - g^2\|_{L^p(\Omega)} + \|w^1 - w^2\|_{L^p(\Sigma)} \right],
\end{align*}
\]
Due to relation (15), we have $\frac{1}{M_1^2}[(\varphi_1 - \varphi_2) - (\varphi_3^1 - \varphi_3^2)] \in L^p(\Omega)$. Thus we may apply Lemma 2.1 to problem (29) which gives the estimate
\[
\|\varphi_1 - \varphi_2\|_{W^{1,2}_p(\Omega)} + \|\psi_1 - \psi_2\|_{W^{1,2}_p(\Sigma)} 
\leq C(\Omega, T, n, p, \alpha, \xi) \left[\|\varphi_1^0 - \varphi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Omega)} + \|\psi_1^0 - \psi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Gamma)} + \|(\varphi_1 - \varphi_2) - (\varphi_3^1 - \varphi_3^2)\|_{L^p(\Omega)} + \|g_1 - g_2\|_{L^p(\Omega)} + \|w_1 - w_2\|_{L^p(\Sigma)}\right].
\]
(32)

Further we will focus our attention on the term $\|(\varphi_1 - \varphi_2) - (\varphi_3^1 - \varphi_3^2)\|_{L^p(\Omega)}$. Owing to (2) and following the work [32], the next sequence of embeddings holds
\[
W^{1,2}_p(\Omega) \subset L^m(\Omega) \subset L^{3p}(Q) \subset L^p(\Omega) \subset L^2(\Omega).
\]
(33)

From H2, Hölder’s inequality and relations (8), (33), we get
\[
\|(\varphi_1 - \varphi_2) - (\varphi_3^1 - \varphi_3^2)\|_{L^p(\Omega)} \leq C(\Omega, T, p, b_0) \left[1 + 2M_1^2\right]\|\varphi_1 - \varphi_2\|_{L^m(\Omega)},
\]
(34)

(for more details, see [4], inequality (38), or [32], inequality (2.21)).

Thanks to (34), the estimate (32) leads to
\[
\|\varphi_1 - \varphi_2\|_{W^{1,2}_p(\Omega)} + \|\psi_1 - \psi_2\|_{W^{1,2}_p(\Sigma)} 
\leq C(\Omega, T, n, p, \alpha, \xi, c_0) \left[\|\varphi_1^0 - \varphi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Omega)} + \|\psi_1^0 - \psi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Gamma)} + \|\varphi_1 - \varphi_2\|_{L^m(\Omega)} + \|(g_1^1 - g_2^1)\|_{L^p(\Omega)} + \|w_1 - w_2\|_{L^p(\Sigma)}\right]
\]
(35)

\[
\leq C(\Omega, T, n, p, \alpha, \xi, c_0, b_0, M_1) \left[\|\varphi_1^0 - \varphi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Omega)} + \|\psi_1^0 - \psi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Gamma)} + \|g_1 - g_2\|_{L^p(\Omega)} + \|w_1 - w_2\|_{L^p(\Sigma)}\right]
\]
(36)

By the embedding in (33), standard interpolation inequalities yield that $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that (see [25], pp. 58)
\[
\|v_1\|_{L^m(\Omega)} \leq \varepsilon\|v_1\|_{W^{1,2}_p(\Omega)} + C(\varepsilon)\|v_1\|_{L^p(\Omega)}, \ \forall v_1 \in W^{1,2}_p(\Omega).
\]
(37)

Using (31) and (36), from (35), we derive that
\[
\left(1 - \varepsilon C(\Omega, T, n, p, \alpha, \xi, c_0, b_0, M_1)\right)\|\varphi_1 - \varphi_2\|_{W^{1,2}_p(\Omega)} + \|\psi_1 - \psi_2\|_{W^{1,2}_p(\Sigma)} 
\leq C(\Omega, T, n, p, \alpha, \xi, c_0, b_0, M_1) \left[\|\varphi_1^0 - \varphi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Omega)} + \|\psi_1^0 - \psi_2^0\|_{W^{2,\frac{2}{p}}_\infty(\Gamma)} + \|g_1 - g_2\|_{L^p(\Omega)} + \|w_1 - w_2\|_{L^p(\Sigma)}\right]
\]
(38)

\[
+ C(\varepsilon)C(T, p, \alpha, \xi) \left(\|\varphi_1^0 - \varphi_2^0\|_{L^p(\Omega)} + \|\psi_1^0 - \psi_2^0\|_{L^p(\Gamma)} + \|g_1 - g_2\|_{L^p(\Omega)} + \|w_1 - w_2\|_{L^p(\Sigma)}\right).
\]
(39)

For $\varepsilon > 0$ with $1 - \varepsilon C(\Omega, T, n, p, \alpha, \xi, c_0, b_0, M_1) > 0$, and thanks to the embedding $W^{1,2}_p(\Omega) \subset L^p(\Sigma)$, the inequality (37) leads to the estimate (9), which finishes the proof of Theorem 2.2.

As a consequence, the uniqueness of solution to problem (4) is valid.

**Corollary 1.** For the same initial conditions and under hypotheses H1, H3, the problem (4) possesses a unique solution $(\varphi, \psi) \in W^{1,2}_p(\Omega) \times W^{1,2}_p(\Sigma)$. 
Proof. Let \( g^1 = g^2 = g \) and \( w^1 = w^2 = w \) in the Theorem 2.2. Then (9) shows that the conclusion of the corollary is true. \( \square \)

Remark 1. In order to approximate the unique solution in (1) with homogeneous Neumann boundary conditions \( \frac{\partial}{\partial \nu} \varphi = 0 \), a scheme of fractional steps type has been introduced and analyzed (convergence and error estimates) in [32].

The results established by Theorem 2.2 highlight the solutions dependence of physical parameters, very useful in the error analysis and numerical simulations.

3. Approximating scheme. Convergence and error estimate. The aim of this Section is to use the fractional steps method in order to approximate the solution of nonlinear boundary value problem (4) (in fact, the solution of problem (1)), whose uniqueness is guaranteed by Corollary 1. Such a method consists in associating to problem (4) for every \( \varepsilon > 0 \) the following approximating scheme (see also [2],[3],[28],[29],[31]):

\[
\begin{cases}
\alpha \xi \frac{\partial}{\partial t} \varphi^\varepsilon - \xi \Delta \varphi^\varepsilon = \frac{1}{2\xi} \varphi^\varepsilon + g(t, x) & \text{in } Q^\varepsilon \times [i\varepsilon, (i+1)\varepsilon] \times \Omega \\
\xi \frac{\partial}{\partial t} \varphi^\varepsilon + \alpha \xi \frac{\partial}{\partial t} \psi^\varepsilon - \Delta \Gamma \psi^\varepsilon + c_0 \psi^\varepsilon = w(t, x) & \text{on } \Sigma^\varepsilon \times [i\varepsilon, (i+1)\varepsilon] \times \Gamma \\
\varphi^\varepsilon(i\varepsilon, x) = z(\varepsilon, \varphi_{-}^\varepsilon(i\varepsilon, x)) & \text{on } \Omega \\
\psi^\varepsilon(i\varepsilon, x) = \varphi^\varepsilon(i\varepsilon, x) & \text{on } \Gamma,
\end{cases}
\]  

(38)

where \( z(\varepsilon, \varphi_{-}^\varepsilon(i\varepsilon, x)) \) is the solution of Cauchy problem:

\[
\begin{cases}
z'(s) + \frac{1}{2\xi} z^3(s) = 0 & s \in [0, \varepsilon] \\
z(0) = \varphi_{-}^\varepsilon(i\varepsilon, x) & \text{on } \Omega \\
\varphi_{-}^\varepsilon(0, x) = \varphi_0(x) & \text{on } \Omega \\
\varphi_{-}^\varepsilon(0, x) = \psi_0(x) & \text{on } \Gamma,
\end{cases}
\]  

(39)

for \( i = 0, 1, \cdots, M_\varepsilon - 1 \), with \( M_\varepsilon = \left[ \frac{T}{\varepsilon} \right] \), \( Q_{M_\varepsilon-1} = [(M_\varepsilon - 1)\varepsilon, T] \times \Omega \), \( \Sigma_{M_\varepsilon-1} = [(M_\varepsilon - 1)\varepsilon, T] \times \Gamma \) and \( \varphi_{-}^\varepsilon \) stands for the left-hand limit of \( \varphi^\varepsilon \).

We point out that the sequence of approximating problems (38)-(39) supplies a decoupling method for the original problem (4) into a linear parabolic boundary value problem (38) and a nonlinear evolution equation (39). Accordingly, the advantage of this approach consists in simplifying the numerical computation of the process of approximation for the solution of nonlinear problem (1).

The main question is the convergence of the sequence \( (\varphi^\varepsilon, \psi^\varepsilon) \) of solutions to the approximate problems (38)-(39) to the unique solution \( (\varphi, \psi) \) of problem (4) as \( \varepsilon \to 0 \). We will treat the convergence of this numerical scheme on the basis of compactness (in particular Helly-Foias theorem).

For later use, we set:

\[ W_Q = L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q) \quad \text{and} \quad W_\Sigma = L^2([0, T]; H^1(\Gamma)) \cap L^\infty(\Sigma). \]

Definition 3.1. By weak solution of the nonlinear problem (4) we mean a pair of functions \( (\varphi, \psi) \in W_Q \times W_\Sigma, \varphi = \psi \text{ on } \Sigma \), which satisfies (4) in the following sense:

\[
\alpha \xi \int_Q \left( \frac{\partial}{\partial t} \varphi, \varphi_1 \right) dt dx + \xi \int_Q \nabla \varphi \nabla \varphi_1 dt dx
\]
satisfies (38)-(39) in the following sense:

\[
\begin{align*}
&+ \alpha \xi \int_{\Sigma} \left( \frac{\partial}{\partial t} \psi, \phi_2 \right) dt \, d\gamma + \int_{\Sigma} \nabla \psi \nabla \phi_2 \, dt \, d\gamma + c_0 \int_{\Sigma} \psi \phi_2 \, dt \, d\gamma \\
&= \frac{1}{2\xi} \int_{Q} (\varphi - \varphi^3) \psi_1 \, dt \, dx + \int_{\Sigma} g \psi_1 \, dt \, dx + \int_{\Sigma} w \phi_2 \, dt \, d\gamma \\
&\quad \forall (\phi_1, \phi_2) \in L^2([0, T]; H^1(\Omega)) \times L^2([0, T]; H^1(\Gamma)),
\end{align*}
\]

with \( \phi_1 = \phi_2 \) on \( \Sigma \) and \( \varphi(0, x) = \varphi_0(x) \) in \( \Omega \).

**Definition 3.2.** By weak solution of the linear problem (38)-(39) we mean a pair of functions \((\varphi^\varepsilon, \psi^\varepsilon) \in W^{1,2}_{Q,T} \times W^{1,2}_{\Sigma,T}, \varphi_i^\varepsilon = \psi_i^\varepsilon \) on \( \Sigma_i \), \( i = 0, 1, \ldots, M_\varepsilon - 1 \), which satisfies (38)-(39) in the following sense:

\[
\begin{align*}
&\quad \alpha \xi \int_{Q} \left( \frac{\partial}{\partial t} \varphi^\varepsilon, \zeta_1 \right) dt \, dx + \xi \int_{\Sigma} \nabla \varphi^\varepsilon \nabla \zeta_1 \, dt \, dx \\
&+ \alpha \xi \int_{\Sigma} \left( \frac{\partial}{\partial t} \psi^\varepsilon, \zeta_2 \right) dt \, dx + \int_{\Sigma} \nabla \psi^\varepsilon \nabla \zeta_2 \, dt \, d\gamma + c_0 \int_{\Sigma} \psi^\varepsilon \zeta_2 \, dt \, d\gamma \\
&= \frac{1}{2\xi} \int_{Q} \varphi^\varepsilon \zeta_1 \, dt \, dx + \int_{\Sigma} g \zeta_1 \, dt \, dx + \int_{\Sigma} w \zeta_2 \, dt \, d\gamma \\
&\quad \forall (\zeta_1, \zeta_2) \in L^2([0, T]; H^1(\Omega)) \times L^2([0, T]; H^1(\Gamma)),
\end{align*}
\]

together with \( \varphi^\varepsilon_0(0, x) = \varphi_0(x) \) in \( \Omega \) and \( \varphi^\varepsilon_0(0, x) = \psi_0(x) \) in \( \Gamma \).

The symbols \( \int \) and \( \int \) above denote the duality between \( L^2([0, T]; H^1(\Omega)) \) and \( L^2([0, T]; H^1(\Omega)^\prime) \) as well as \( L^2([0, T]; H^1(\Gamma)) \) and \( L^2([0, T]; H^1(\Gamma)^\prime) \), respectively.

### 3.1. Convergence of the approximating scheme

In this section, we will prove the convergence of the iterative scheme (38)-(39) of fractional steps type to the nonlinear parabolic boundary value problem (4). We have

**Theorem 3.3.** Assume that \( \varphi_0(x) \in W^{2-\frac{2}{p}}_{\infty}(\Omega) \), satisfying \( \xi \frac{\partial}{\partial t} \varphi_0 - \Delta \varphi_0 + c_0 \varphi_0 = w(0, x) \) on \( \Gamma \) and \( w(t, x) \in W^{1-\frac{2}{p}, 2-\frac{2}{p}}(\Sigma) \). Let \((\varphi^\varepsilon, \psi^\varepsilon) \) be the solution of the approximating scheme (38)-(39). Then for \( \varepsilon \to 0 \), one has

\[
(\varphi^\varepsilon, \psi^\varepsilon) \to (\varphi^*, \psi^*) \quad \text{strongly in} \quad L^2(\Omega) \times L^2(\Gamma) \quad \text{for any} \quad t \in (0, T],
\]

where \((\varphi^*, \psi^*) \in L^2([0, T]; H^1(\Omega)) \times L^2([0, T]; H^1(\Gamma)) \) is the weak solution to the nonlinear phase transition equation (4).

The following lemmas, which targets the Cauchy problem (39) and which are very useful in the proof of the main result of this Section (Theorem 3.3) were established for the first time in the work [28]. For reader convenience we fully reproduce their proofs.

**Lemma 3.4.** If \( \varphi^\varepsilon(i\varepsilon, x) \in L^\infty(\Omega), \ i = 0, 1, \ldots, M_\varepsilon - 1 \), then \( \varphi^\varepsilon(i\varepsilon, x) \in L^\infty(\Omega) \) and

\[
\| \varphi^\varepsilon(i\varepsilon, x) \|^2_{L^2(\Omega)} \leq \| \varphi^\varepsilon(i\varepsilon, x) \|^2_{L^2(\Omega)},
\]
We deduce that problem (38) has the solution $(\psi_L)$. The foregoing equality leads us to the estimate (45).

Now, integrating the above inequality on $(0, \varepsilon)$, we obtain the estimate (46). The following estimate holds
\[ \|\nabla \varphi^e(i\varepsilon, x)\|_{L^2(\Omega)} \leq \|\nabla \varphi^{-}_e(i\varepsilon, x)\|_{L^2(\Omega)}. \] (45)

Proof. Let us set $\theta(t, x) = \nabla z(t, x)$. Thus (39) becomes
\[ \theta'(s, x) + \frac{3}{2\xi} \theta^2(s, x) = 0, \quad s \in [0, \varepsilon] \]
\[ \theta(0, x) = \nabla \varphi^e(i\varepsilon, x), \]
whose solution is
\[ \theta(\varepsilon, x) = e^{\int_0^\varepsilon - \frac{3}{2\xi} \theta^2(t, x) dt} \theta(0, x). \]
The foregoing equality leads us to the estimate (45).

Lemma 3.6. The following estimate holds
\[ \|z(\varepsilon, x) - \varphi^{-}_e(i\varepsilon, x)\|_{L^2(\Omega)} \leq \varepsilon L \] (46)
where $L > 0$ is a constant depending on $\Omega$, $\|\varphi^{-}_e\|_{L^\infty(\Omega)}$ and $\xi$.

Proof. From (39), using the inequality $(a^3 - b^3)(a - b) \geq 0$ for all $a, b \in \mathbb{R}$, we get
\[ \frac{1}{2} \frac{d}{dt} |z(t, x) - z(0, x)|^2 \leq -\frac{1}{2\xi} \frac{3}{2} (t, x) (z(t, x) - z(0, x)) \leq -\frac{1}{2\xi} \varphi^3(0, x) (z(t, x) - z(0, x)). \]
Now, integrating the above inequality on $(0, \varepsilon)$, we obtain the estimate
\[ |z(\varepsilon, x) - z(0, x)| \leq \frac{\varepsilon}{2\xi} |z^3(0, x)| = \frac{\varepsilon}{2\xi} |\varphi^e(i\varepsilon, x)|^3, \]
which, taking into account (38)$_3$, permit us to conclude the estimate (46).

Proof of Theorem 3.3. Consider firstly $i = 0$. Corresponding, from Lemma 3.4 we derive that the solution of the Cauchy problem (39) $z(\varepsilon, x)$ belongs to $L^\infty(\Omega)$. Since $W^{2-\frac{3}{2}}(\Omega) \subset W^{1,4}_\infty(\Omega)$ then $z(\varepsilon, x) \in W^{1,4}_\infty(\Omega)$. Using Theorem 2.2 to the problem (38) we ensure the existence of a solution $(\varphi^e, \psi^e) \in W^{1,2}_p(\Omega^e) \cap L^\infty(Q^e) \times W^{1,2}_p(\Sigma^e) \cap L^\infty(\Sigma^e)$. By induction, $\varphi^e(i\varepsilon, x) \in L^\infty(\Omega), i = 1, 2, ..., M_\varepsilon - 1$. Accordingly, we deduce that problem (38) has the solution $(\varphi^e, \psi^e) \in W^{1,2}_p(Q^e_i) \cap L^\infty(Q^e_i) \times W^{1,2}_p(\Sigma^e_i) \cap L^\infty(\Sigma^e_i)$, for all $i \in \{0, 1, ..., M_\varepsilon - 1\}$.

Next we will give some a priori estimates in $Q^e_i, i = 0, 1, ..., M_\varepsilon - 1$. *AN ITERATIVE SCHEME OF FRACTIONAL STEPS TYPE 549*
Multiplying firstly (38) by $\varphi_\varepsilon^t$, using Green’s formula and integrating by parts, we obtain
\[
\alpha \xi \int_\Omega |\varphi_\varepsilon|^2 dx + \alpha \xi \int_\Gamma |\psi_\varepsilon|^2 d\gamma \\
+ \frac{\xi}{2} \frac{d}{dt} \int_\Omega |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Gamma |\nabla \varphi_\varepsilon|^2 d\gamma + \frac{c_0}{2} \frac{d}{dt} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma = \frac{1}{4 \xi} \frac{d}{dt} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma + \int_\Gamma w \varphi_\varepsilon^t d\gamma + \int_\Omega g \varphi_\varepsilon^t dx.
\]
(47)

We now focus on the right terms $\int_\Gamma w \varphi_\varepsilon^t d\gamma$ and $\int_\Omega g \varphi_\varepsilon^t dx$ in (47). Using Hölder’s inequality, we have
\[
\int_\Gamma w \varphi_\varepsilon^t d\gamma \leq \frac{\alpha \xi}{2} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma + \frac{1}{2 \alpha \xi} \int_\Gamma |w|^2 d\gamma,
\]
(48)
\[
\int_\Omega g \varphi_\varepsilon^t dx \leq \frac{\alpha \xi}{2} \int_\Omega |\varphi_\varepsilon|^2 dx + \frac{1}{2 \alpha \xi} \int_\Omega |g|^2 dx.
\]
(49)

Substituting (48) and (49) in (47), we derive
\[
\frac{\alpha \xi}{2} \int_\Omega |\varphi_\varepsilon|^2 dx + \frac{\alpha \xi}{2} \int_\Gamma |\psi_\varepsilon|^2 d\gamma \\
+ \frac{\xi}{2} \frac{d}{dt} \int_\Omega |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Gamma |\nabla \varphi_\varepsilon|^2 d\gamma + \frac{c_0}{2} \frac{d}{dt} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma \\
\leq \frac{1}{4 \xi} \frac{d}{dt} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma + \frac{1}{2 \alpha \xi} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma + \frac{1}{2 \alpha \xi} \int_\Omega |g|^2 dx.
\]
(50)

Multiplying now (3.1) by $\frac{1}{\alpha \xi} \varphi_\varepsilon^t$, integrating over $\Omega$ and using Green’s formula, we get
\[
\frac{1}{2 \xi} \frac{d}{dt} \int_\Omega |\varphi_\varepsilon|^2 dx + \frac{1}{2 \xi} \frac{d}{dt} \int_\Gamma |\psi_\varepsilon|^2 d\gamma \\
+ \frac{1}{\alpha \xi} \int_\Omega |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{\alpha \xi} \int_\Gamma |\nabla \varphi_\varepsilon|^2 d\gamma + \frac{c_0}{\alpha \xi^2} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma = \frac{1}{2 \alpha \xi^2} \int_\Omega |\varphi_\varepsilon|^2 dx + \frac{1}{\alpha \xi^2} \int_\Gamma w \varphi_\varepsilon^t d\gamma + \frac{1}{\alpha \xi^2} \int_\Gamma g \varphi_\varepsilon^t d\gamma.
\]
(51)

Using again Hölder’s inequality, i.e.
\[
\frac{1}{\alpha \xi^2} \int_\Gamma w \varphi_\varepsilon^t d\gamma \leq \frac{c_0}{\alpha \xi^2} \int_\Gamma |\varphi_\varepsilon|^2 d\gamma + \frac{1}{4 c_0 \alpha \xi^2} \int_\Gamma |w|^2 d\gamma,
\]
\[
\frac{1}{\alpha \xi^2} \int_\Omega g \varphi_\varepsilon^t dx \leq \frac{1}{2 \alpha \xi^2} \int_\Omega |\varphi_\varepsilon|^2 dx + \frac{1}{2 \alpha \xi^2} \int_\Omega |g|^2 dx.
\]
from (51) we obtain
\[
\frac{1}{2\xi} \frac{d}{dt} \int_{\Omega} |\varphi^\varepsilon|^2 dx + \frac{1}{2\xi} \frac{d}{dt} \int_{\Gamma} |\psi^\varepsilon|^2 d\gamma \\
+ \frac{1}{\alpha\xi} \int_{\Omega} |\nabla \varphi^\varepsilon|^2 dx + \frac{1}{\alpha\xi} \int_{\Gamma} |\nabla \psi^\varepsilon|^2 d\gamma \\
\leq C(\alpha, \xi, c_0) \left[ \int_{\Omega} |\varphi^\varepsilon|^2 dx + \int_{\Gamma} |\psi^\varepsilon|^2 d\gamma + \int_{\Omega} |w|^2 d\gamma + \int_{\Omega} |g|^2 dx \right].
\] (52)

Adding (50) and (52), we derive
\[
\frac{\partial}{\partial t} \left[ \frac{1}{4\xi} \int_{\Omega} |\varphi^\varepsilon|^2 dx + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \int_{\Gamma} |\psi^\varepsilon|^2 d\gamma + \frac{\xi}{2} \int_{\Omega} |\nabla \varphi^\varepsilon|^2 dx + \frac{1}{2} \int_{\Gamma} |\nabla \psi^\varepsilon|^2 d\gamma \right] \\
+ \frac{\alpha\xi}{2} \int_{\Omega} |\nabla \varphi^\varepsilon|^2 dx + \frac{\alpha\xi}{2} \int_{\Gamma} |\psi^\varepsilon|^2 d\gamma + \frac{1}{\alpha\xi} \int_{\Omega} |\nabla \varphi^\varepsilon|^2 dx + \frac{1}{\alpha\xi} \int_{\Gamma} |\nabla \psi^\varepsilon|^2 d\gamma \\
\leq C(\alpha, \xi, c_0) \left[ \int_{\Omega} |\varphi^\varepsilon|^2 dx + \int_{\Gamma} |\psi^\varepsilon|^2 d\gamma + \int_{\Omega} |w|^2 d\gamma + \int_{\Omega} |g|^2 dx \right].
\]

Integrating the above inequality over \((0, \varepsilon)\) leads to
\[
\frac{1}{4\xi} \|\varphi^\varepsilon(\varepsilon, x)\|_{L^2(\Omega)}^2 + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \|\psi^\varepsilon(\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \frac{\xi}{2} \|\nabla \varphi^\varepsilon(\varepsilon, x)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \psi^\varepsilon(\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \int_0^\varepsilon \left[ \frac{\alpha\xi}{2} \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\alpha\xi}{2} \|\psi^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{\alpha\xi} \|\nabla \varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\alpha\xi} \|\nabla \psi^\varepsilon\|_{L^2(\Gamma)}^2 \right] ds \\
\leq \frac{1}{4\xi} \|\varphi_0\|_{L^2(\Omega)}^2 + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \|\psi_0\|_{L^2(\Gamma)}^2 + \frac{\xi}{2} \|\nabla \varphi_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \psi_0\|_{L^2(\Gamma)}^2 \\
+ C(\alpha, \xi, c_0) \left\{ \int_0^\varepsilon \left[ \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 + \|\psi^\varepsilon\|_{L^2(\Gamma)}^2 \right] ds + \|w\|_{L^2(\Sigma_\delta)}^2 + \|g\|_{L^2(\Sigma_\delta)}^2 \right\}. \tag{53}
\]

Similarly, for \(i = 1, 2, \ldots, M_\varepsilon - 2\) (i.e. on \(Q^\varepsilon_i\) and \(\Sigma^\varepsilon_i\), \(i = 1, M_\varepsilon - 2\)), we have
\[
\frac{1}{4\xi} \|\varphi^\varepsilon((i+1)\varepsilon, x)\|_{L^2(\Omega)}^2 + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \|\psi^\varepsilon((i+1)\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \frac{\xi}{2} \|\nabla \varphi^\varepsilon((i+1)\varepsilon, x)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \psi^\varepsilon((i+1)\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \int_{\varepsilon}^{(i+1)\varepsilon} \left[ \frac{\alpha\xi}{2} \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\alpha\xi}{2} \|\psi^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{\alpha\xi} \|\nabla \varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\alpha\xi} \|\nabla \psi^\varepsilon\|_{L^2(\Gamma)}^2 \right] ds \\
\leq \frac{1}{4\xi} \|\varphi^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)}^2 + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \|\psi^\varepsilon(i\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \frac{\xi}{2} \|\nabla \varphi^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \psi^\varepsilon(i\varepsilon, x)\|_{L^2(\Gamma)}^2 \\
+ \int_{\varepsilon}^{(i+1)\varepsilon} \left[ \frac{\alpha\xi}{2} \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\alpha\xi}{2} \|\psi^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{\alpha\xi} \|\nabla \varphi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\alpha\xi} \|\nabla \psi^\varepsilon\|_{L^2(\Gamma)}^2 \right] ds \\
\leq C(\alpha, \xi, c_0) \left\{ \int_{\varepsilon}^{(i+1)\varepsilon} \left[ \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 + \|\psi^\varepsilon\|_{L^2(\Gamma)}^2 \right] ds + \|w\|_{L^2(\Sigma_\delta)}^2 + \|g\|_{L^2(\Sigma_\delta)}^2 \right\}. \tag{54}
\]
\[ + C(\alpha, \xi, c_0) \left\{ \int_{\mathcal{M}^{-1}} \left[ \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \| \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds + \| w \|^2_{L^2(\Sigma^e)} + \| g \|^2_{L^2(Q^e)} \right\}. \] (54)

while on \( Q^e_{M^{-1}} \) and \( \Sigma^e_{M^{-1}} \), relation (53) gets
\[ \frac{1}{4\xi} \| \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \| \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + \frac{\xi}{2} \| \nabla \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + \int_{\mathcal{M}^{-1}} \left[ \frac{\alpha \xi}{2} \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \frac{\alpha \xi}{2} \| \psi^\varepsilon \|^2_{L^2(\Gamma)} + \frac{1}{\alpha \xi} \| \nabla \varphi^\varepsilon \|^2_{L^2(\Omega)} + \frac{1}{\alpha \xi} \| \nabla \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds \]
\[ \leq \frac{1}{4\xi} \| \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \| \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + \frac{\xi}{2} \| \nabla \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + C(\alpha, \xi, c_0) \left\{ \int_{\mathcal{M}^{-1}} \left[ \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \| \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds + \| w \|^2_{L^2(\Sigma^e)} + \| g \|^2_{L^2(Q^e)} \right\}. \] (55)

Adding (53)-(55) and making use of the inequalities (43) and (45), we obtain
\[ \frac{1}{4\xi} \| \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \| \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + \frac{\xi}{2} \| \nabla \varphi^\varepsilon (T, x) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \psi^\varepsilon (T, x) \|^2_{L^2(\Gamma)} \]
\[ + \int_0^T \left[ \frac{\alpha \xi}{2} \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \frac{\alpha \xi}{2} \| \psi^\varepsilon \|^2_{L^2(\Gamma)} + \frac{1}{\alpha \xi} \| \nabla \varphi^\varepsilon \|^2_{L^2(\Omega)} + \frac{1}{\alpha \xi} \| \nabla \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds \]
\[ \leq \frac{1}{4\xi} \| \varphi_0 \|^2_{L^2(\Omega)} + \left( \frac{c_0}{2} + \frac{1}{2\xi} \right) \| \psi_0 \|^2_{L^2(\Gamma)} + \frac{\xi}{2} \| \nabla \varphi_0 \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \psi_0 \|^2_{L^2(\Gamma)} \]
\[ + C(\alpha, \xi, c_0) \left\{ \int_0^T \left[ \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \| \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds + \| w \|^2_{L^2(\Sigma^e)} + \| g \|^2_{L^2(Q^e)} \right\}. \]

Continuing by applying Gronwall inequality, we finally deduce
\[ \int_0^T \left[ \| \varphi^\varepsilon \|^2_{L^2(\Omega)} + \| \psi^\varepsilon \|^2_{L^2(\Gamma)} + \| \nabla \varphi^\varepsilon \|^2_{L^2(\Omega)} + \| \nabla \psi^\varepsilon \|^2_{L^2(\Gamma)} \right] ds \leq C, \] (56)
where \( C \) does not depend to \( \varepsilon \) and \( M_\varepsilon \).

Thanks to estimate (46) established in Lemma 3.6 and using (38), (38), we derive
\[ \sum_{i=0}^{M_\varepsilon-1} \| \varphi^\varepsilon (i\varepsilon, x) - \varphi^\varepsilon (i\varepsilon, x) \|_{L^2(\Omega)} \leq TL = C_1, \] (57)
\[ \sum_{i=0}^{M_\varepsilon-1} \| \psi^\varepsilon (i\varepsilon, x) - \psi^\varepsilon (i\varepsilon, x) \|_{L^2(\Gamma)} \leq C_2, \] (58)
where $C_1$ and $C_2$ do not depend on $M_\varepsilon$ and $\varepsilon$. Adding (56)-(58), we deduce

$$\frac{T}{0} V_1 \varphi^\varepsilon + \frac{T}{0} V_2 \psi^\varepsilon + \int_0^T \left[ \|\varphi^\varepsilon\|_{L_x^2(\Omega)}^2 + \|\psi^\varepsilon\|_{L_x^2(\Omega)}^2 + \|\nabla \varphi^\varepsilon\|_{L_x^2(\Gamma)}^2 + \|\nabla \psi^\varepsilon\|_{L_x^2(\Gamma)}^2 \right] ds \leq C,$$

where the positive constant $C$ do not depend on $M_\varepsilon$ and $\varepsilon$, while $V_1^T \varphi^\varepsilon$ and $V_2^T \psi^\varepsilon$, stand for the variation of $\varphi^\varepsilon : [0, T] \to L^2(\Omega)$ and $\psi^\varepsilon : [0, T] \to L^2(\Gamma)$, respectively.

Since the injection of $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact and the set $\{\varphi^\varepsilon(t)\}$ is bounded in $L^2(\Omega)$ for every $t \in [0, T]$, we conclude that there exists a bounded variation $\varphi^*(t) \in BV([0, T]; H^{-1}(\Omega))$ and a subsequence $\varphi^\varepsilon(t)$ (see Helly-Foiaş theorem) such that

$$\varphi^\varepsilon(t) \to \varphi^*(t) \text{ strongly in } H^{-1}(\Omega) \text{ for every } t \in [0, T].$$

A similar reasoning carried out for the unknown $\psi$ allows us to conclude on the convergence

$$\psi^\varepsilon(t) \to \psi^*(t) \text{ strongly in } H^{-1}(\Gamma) \text{ for every } t \in [0, T].$$

Further, we deduce from (59) that

$$\begin{cases}
\varphi^\varepsilon \to \varphi^* \text{ weakly in } L^2(0, T; H^1(\Omega)) \\
\psi^\varepsilon \to \psi^* \text{ weakly in } L^2(0, T; H^1(\Gamma)).
\end{cases}$$

By the embedding $H^1(\Omega) \subset L^3(\Omega) \subset H^{-1}(\Omega)$ and $H^1(\Gamma) \subset L^2(\Gamma) \subset H^{-1}(\Gamma)$, standard interpolation inequalities (see [25], pp. 58) yield that $\forall \kappa > 0$, there exists some constant $C(\kappa)$ such that

$$\begin{cases}
\|\varphi^\varepsilon(t) - \varphi^*(t)\|_{L^2(\Omega)} \leq \kappa \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^1(\Omega)} + C(\kappa) \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^{-1}(\Omega)}, \\
\|\psi^\varepsilon(t) - \psi^*(t)\|_{L^2(\Gamma)} \leq \kappa \|\psi^\varepsilon(t) - \psi^*(t)\|_{H^1(\Gamma)} + C(\kappa) \|\psi^\varepsilon(t) - \psi^*(t)\|_{H^{-1}(\Gamma)},
\end{cases}$$

$\forall \varepsilon > 0$ and $\forall t \in [0, T]$.

Our assertion (42) holds true from (60)-(63). This achieves the proof of Theorem 3.3.

Corollary 2. Let $\varphi_0 \in W^{2, \frac{2}{p}}_{\infty}(\Omega)$ satisfying $\xi \frac{\partial}{\partial n} \varphi_0(x) - \Delta_\Gamma \varphi_0 + c_0 \varphi_0(x) = w(0, x)$ on $\Gamma$ and $w(t, x) \in W_{p}^{1, \frac{2}{p} - 2, \frac{2}{p}}(\Sigma)$. Then $\varphi^* \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$ is a weak solution of nonlinear equation (1).

3.2. Error analysis of the approximating scheme. Now we search the error of the approximating scheme (38)-(39) with respect to the terms $g(t, x)$ and $w(t, x)$.

By Theorem 2.2 we have that, for each $g \in L^p(Q)$ and $w \in W_{p}^{1, \frac{2}{p} - 2, \frac{2}{p}}(\Sigma)$, the problem (4) has a unique solution $(\varphi, \psi) \in W_{p}^{1, 2}(Q) \times W_{p}^{1, 2}(\Sigma)$, and the estimate below holds (see (7))

$$\|\varphi\|_{W_{p}^{1, 2}(Q)} + \|\psi\|_{W_{p}^{1, 2}(\Sigma)} \leq C \left[ 1 + \|\varphi_0\|_{W_{\infty}^{2, \frac{2}{p}}(\Omega)}^{3 - \frac{2}{p}} + \|\psi_0\|_{W_{\infty}^{2, \frac{2}{p}}(\Gamma)}^{3 - \frac{2}{p}} + \|g\|_{L^p(Q)} + \|w\|_{W_{p}^{1, \frac{2}{p} - 2, \frac{2}{p}}(\Sigma)} \right],$$

with a fixed $\psi_0 \in W_{\infty}^{2, \frac{2}{p}}(\Gamma)$ and $\varphi_0 \in W_{\infty}^{2, \frac{2}{p}}(\Omega)$ verifying $\xi \frac{\partial}{\partial n} \varphi_0 - \Delta_\Gamma \varphi_0 + c_0 \varphi_0 = w(0, x)$. We have
Theorem 3.7. Let \( g \in L^p(Q) \) and \( w \in W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega) \). Let \( g_k \in L^p(Q) \) and \( w_k \in W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega) \) be two sequences such that \( g_k \to g \) in \( L^p(Q) \) and \( w_k \to w \) in \( W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega) \) as \( k \to \infty \). Denote by \( (\varphi_m,\psi_m) \subset W^{1,2}_p(\Omega) \times W^{1,2}_p(\Sigma) \) and \( (\varphi_{m,k},\psi_{m,k}) \subset W^{1,2}_p(\Omega) \times W^{1,2}_p(\Sigma) \) the approximating sequences given by (38), (39), for \( (g,w) \) and \( (g_k,w_k) \), respectively, with a fixed \( \varphi_0 \in W^{2-\frac{2}{p}}\Omega(\Omega) \), (see Theorem 3.3).

Then the following error estimate holds

\[
\limsup_{m \to \infty} \left[ \| \varphi_{m,k} - \varphi \|_{L^p(Q)} + \| \psi_{m,k} - \psi \|_{L^p(\Sigma)} \right] \leq C \left[ \| g_k - g \|_{L^p(Q)} + \| w_k - w \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega)} \right]
\]

for all \( k \geq 1 \), where \( C \) is a positive constant depending on \( |\Omega|, T, n, p, \alpha, \xi, c_0, \| \varphi_0 \|_{W^{2-\frac{2}{p}}\Omega(\Omega)} \), \( \| \psi_0 \|_{L^p(\Sigma)} \) and \( \| w_k \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Sigma)} \).

In particular, there exists a subsequence of \( (\varphi_{m,k},\psi_{m,k}) \), denoted by \( (\varphi_{m_k},\psi_{m_k}) \), such that \( (\varphi_{m_k},\psi_{m_k}) \to (\varphi,\psi) \) in \( L^p(Q) \times L^p(\Sigma) \) and a.e. in \( Q \times \Sigma \) as \( k \to \infty \).

Proof. Thanks to (64) we can assume that

\[
\| \varphi_k \|_{W^{1,2}_p(Q)} + \| \psi_k \|_{W^{1,2}_p(\Sigma)} \leq C \left[ 1 + \| \varphi_0 \|_{W^{2-\frac{2}{p}}\Omega(\Omega)}^{3-\frac{2}{p}} + \| \psi_0 \|_{W^{2-\frac{2}{p}}\Gamma(\Gamma)}^{3-\frac{2}{p}} + \| g_k \|_{L^p(Q)} + \| w_k \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega)} \right]
\]

\[
\leq C \left[ 1 + \| \varphi_0 \|_{W^{2-\frac{2}{p}}\Omega(\Omega)}^{3-\frac{2}{p}} + \| \psi_0 \|_{W^{2-\frac{2}{p}}\Gamma(\Gamma)}^{3-\frac{2}{p}} + \| g \|_{L^p(\Sigma)} + \| w \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Sigma)} \right],
\]

where \( C \) is a positive constant. The foregoing estimate provides us the constant \( M_1 \) in (8). This ensures that we can apply the estimate (9) in Theorem 2.2 with the same initial conditions \( \varphi_0 \in W^{2-\frac{2}{p}}\Omega(\Omega), \psi_0 \in W^{2-\frac{2}{p}}\Gamma(\Gamma) \), to obtain

\[
\| \varphi_k - \varphi \|_{W^{1,2}_p(Q)} + \| \psi_k - \psi \|_{W^{1,2}_p(\Sigma)} \leq C_1 \left[ \| g_k - g \|_{L^p(Q)} + \| w_k - w \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega)} \right], \quad \forall k \geq 1,
\]

where \( C_1 \) is a positive constant depending on \( |\Omega|, T, M_1, n, p, \alpha, \xi, c_0, b_0, \| \varphi_0 \|_{W^{2-\frac{2}{p}}\Omega(\Omega)}, \| \psi_0 \|_{W^{2-\frac{2}{p}}\Gamma(\Gamma)}, \| g \|_{L^p(\Sigma)}, \| w \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Sigma)} \).

For \( k \geq 1 \), Theorem 3.3 gives the convergence

\[
(\varphi_{m,k}(t,\cdot),\psi_{m,k}(t,\cdot) \to (\varphi_k(t,\cdot),\psi_k(t,\cdot)) \quad \text{in} \quad L^2(\Omega) \times L^2(\Gamma),
\]

uniformly for \( t \in [0,T] \), as \( m \to \infty \). In particular, for each \( k \geq 1 \) we deduce that

\[
(\varphi_{m,k},\psi_{m,k}) \to (\varphi_k,\psi_k) \quad \text{in} \quad L^2(\Omega) \times L^2(\Sigma), \quad \text{as} \quad m \to \infty.
\]

Using (66) and due to the sequence of embeddings in (33), we get the estimate

\[
\| \varphi_{m,k} - \varphi \|_{L^2(\Omega)} + \| \psi_{m,k} - \psi \|_{L^2(\Sigma)} \leq \| \varphi_{m,k} - \varphi_k \|_{L^2(\Omega)} + \| \psi_{m,k} - \psi_k \|_{L^2(\Sigma)} + \| \varphi_k - \varphi \|_{L^2(\Omega)} + \| \psi_k - \psi \|_{L^2(\Sigma)}
\]

\[
\leq C_1 \left[ \| g_k - g \|_{L^p(Q)} + \| w_k - w \|_{W^{1-\frac{1}{p},2-\frac{1}{p}}_p(\Omega)} \right], \quad \forall m, k \geq 1.
\]
Taking into account (67) we may pass in the foregoing estimate to the superior limit as \( m \to \infty \) to conclude that (65) is valid. The last statement in Theorem 3.7 follows directly from the estimate (65).

The general scheme of the conceptual algorithm to compute the approximate solution of nonlinear parabolic problem (1) by means of a fractional steps method is obtained by the following sequence (\( i \) denotes the time level):

\begin{verbatim}
Begin ALGFRAC Allen-Cahn NonHomDBC
\end{verbatim}

\begin{verbatim}
i := 0 \rightarrow \varphi_0 \text{ from the initial conditions (39)};
For \( i := 0 \) to \( M \varepsilon - 1 \) do
    Compute \( z(\varepsilon, \cdot) \) from (39);
    \( \varphi^\varepsilon(i\varepsilon, \cdot) := z(\varepsilon, \cdot) \);
    \( \psi^\varepsilon(i\varepsilon, \cdot) := \varphi^\varepsilon(i\varepsilon, \cdot) \);
    Compute \( (\varphi^\varepsilon((i+1)\varepsilon, \cdot), \psi^\varepsilon((i+1)\varepsilon, \cdot)) \) solving the linear system (38);
End-for;

End.
\end{verbatim}

The Cauchy problem (39) can be solved directly, using the method of separation of variables, which gives us (see [2], [3], [28], [29]):

\[ z^2(\varepsilon) = \left( \frac{\varphi^\varepsilon(i\varepsilon, x)}{1 + \frac{i}{\xi}(\varphi^\varepsilon(i\varepsilon, x))^2} \right)^2, \quad i = 0, 1, \cdots, M \varepsilon - 1. \]

Numerical implementation of the above conceptual algorithm, as well as various simulations regarding the physical phenomena described by nonlinear parabolic problem (1) (in particular, the separating region), represent a matter for further investigation.

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E-mail address: Alain.Miranville@math.univ-poitiers.fr
E-mail address: costica.morosanu@uaic.ro