Composite Fermions and the Fermion-Chern-Simons Theory

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ABSTRACT. The concept of composite fermions, and the related Fermion-Chern-Simons theory, have been powerful tools for understanding quantum Hall systems with a partially full lowest Landau level. We shall review some of the successes of the Fermion-Chern-Simons theory, as well as some limitations and outstanding issues.

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1 Introduction

In the 23 years since Klaus von Klitzing discovered the integer quantized Hall effect, experiments on quantum Hall systems have produced a myriad of surprising results. To understand these, we have needed to introduce a number of new theoretical tools and concepts. The composite fermion picture, and the related Fermion-Chern-Simons theory have been among the most successful of these tools, particularly for understanding electrons in the lowest Landau level, when it is partially full. The general approach has been useful for describing both incompressible fractional quantized Hall states and compressible unquantized quantum Hall states.

The composite fermion picture was introduced by Jain in 1989, in the form of trial wavefunctions for the groundstates and quasiparticle excitations in the principal quantized Hall states, such as 1/3, 2/5, 3/7, 4/9, states which have filling fractions of the form $\nu = p/(2p + 1)$, where $p$ is an integer. The composite fermion picture correctly predicted the most prominent observed fractional quantized Hall states, and Jain showed that the trial wavefunctions had excellent overlap with the exact wavefunctions for small numbers of particles on a sphere, for the appropriate filling fractions, with, e.g. Coulomb interactions between the electrons.

The Fermion-Chern-Simons (FCS) theory was first applied to Jain’s fractional quantized Hall states by Lopez and Fradkin. It was used by Moore and Read, and Greiter, Wen and Wilczek, to gain insight into possible wavefunctions for observed even-denominator quantized Hall states at $\nu = 5/2$ in a single layer system, and $\nu = 1/2$ in certain bilayer systems. The approach was
applied to unquantized fractions, such as \( \nu = 1/2 \) in single layer systems, by Halperin, Lee and Read [11], and by Kalmeyer and Zhang [12]. The FCS theory was particularly useful for understanding dynamic properties of the quantum Hall systems, such as linear response functions and transport properties, as well as the dispersion of collective excitations. Similar mathematical methods had been used earlier, before the first applications to quantum Hall systems, by Laughlin and others, in order to explore properties of systems of “anyons”, advanced as a model for high-temperature superconductivity. [13, 14, 15, 16, 17, 18] A Boson-Chern-Simons approach had also been used to develop an analogy between quantized Hall states and superfluidity, and to describe the dynamics of quantized Hall states [19].

In this article, I shall briefly review the FCS theory and some of its key predictions. I will then say a few word about what is a composite fermion, contrasting the bare FCS particles with the low-energy quasiparticles that emerge from the theory. I shall also discuss some attempts to formulate a Fermi-liquid like description directly in terms of the low-energy quasiparticles, at compressible fractions such as \( \nu = 1/2 \). I will not say much about the trial wavefunction approach here, as this will be the subject of Jainendra Jain’s presentation [20].

2 Fermion-Chern-Simons theory.

The FCS approach begins with an exact unitary transformation of the electron problem to a system of “bare composite fermions”, which interact with each other via a fictitious gauge field \( a(r) \), known as the Chern-Simons gauge field, as well as through the usual Coulomb potential. [11] [21] [22] As a next step, one makes a mean-field approximation on the transformed Hamiltonian. Finally, one tries to calculate corrections to the mean field results using perturbation theory methods, such as the Random Phase Approximation, Feynman diagram expansions, etc.

The nature of the Chern-Simons gauge field is that there are an even integer number \( m \) of flux quanta of the Chern-Simons magnetic field, \( b \equiv \nabla \times a \), attached to each fermion. (By choosing \( m \) to be an even integer, one preserves the antisymmetry of the wavefunction under interchange of two particles.) For the purpose of this presentation, I will confine myself to the case \( m = 2 \). I will also assume that the electron spins are completely aligned by the Zeeman field.

If we make a mean field approximation, (more precisely, a Hartree approximation) to the transformed FCS Hamiltonian, we replace the true Chern-Simons magnetic field \( b \), which depends on the position of every particle, by its average value \( < b > \). For a system with a uniform electron density \( n_e \), in units where the flux quantum is \( 2\pi \), one finds that \( < b > = 2\pi mn_e \). In the Hartree approximation, one also replaces the Coulomb interaction of the electrons, with each other and with the positive background, by its average value, which is just a constant for a uniform system. Thus the mean field version of the FCS problem describes a collection of non-interacting fermions in a uniform effective magnetic field

\[
B_{\text{eff}} = B - 2m\pi n_e .
\]

Recalling that the Landau-level filling factor for electrons is defined by \( \nu = 2\pi n_e / B \), we may define an effective filling factor \( p \) for the composite fermions by \( p = 2\pi n_e / B_{\text{eff}} \). It follows that \( p^{-1} = \nu^{-1} - m \). For \( m = 2 \), we then find that \( \nu = p / (2p + 1) \). If \( p \) is an integer, the mean-field
theory for the composite fermions is an integer quantized Hall state with an energy gap, so we may hope that the interacting electron state will also have an energy gap and be incompressible. Of course, the corresponding values of $\nu$ are precisely the Jain fractions, which are the most prominent fractional quantized Hall states.\cite{1}

Let us now consider the case where the electron filling factor is $\nu = 1/2$, so that there are precisely two quanta of magnetic flux for each electron in the system. The external magnetic field is then precisely canceled by the mean Chern-Simons field $< b >$, so that the effective magnetic field seen by the composite fermions is $B_{\text{eff}} = 0$. The mean-field solution in this case is then a filled Fermi sea, with a Fermi wavevector $k_F = (4\pi n_e)^{1/2}$, appropriate for a system of spin-aligned fermions in two dimensions. This suggests that the groundstate for the interacting electron system should have no energy gap, and therefore should be compressible. In the presence of scattering due to impurities, we would expect there to be a finite, non-zero value of the electrical resistivity, $\rho_{xx}$. (We ignore here interaction-corrections to the resistivity, which are predicted to diverge logarithmically at very low temperatures, but are negligible in practice.) Furthermore, if the theory is correct, we should be able to understand properties of the electron system near $\nu = 1/2$ by using perturbation theory, starting from the Hartree ground state, and putting in the Coulomb interaction and the fluctuating Chern-Simons gauge field as perturbations. There is reason to hope that this approach will work, even though the mean field solution is not protected by an energy gap, because the filled Fermi sea has a very low density of states for multiple particle-hole excitations with low total energy. (We contrast this with the original electron problem, before the unitary transformation, where the partially full Landau level has an infinitely degenerate ground state before interactions are turned on.) On the other hand, there is no small parameter in the perturbation theory, as the coupling to the fluctuating gauge field is not small. Hence, quantitative predictions are only possible when one can argue that a certain result is correct to all orders in perturbation theory.

For dynamic calculations, the Hartree approximation is not by itself satisfactory. However, reasonable results are obtained if one employs the Random Phase Approximation, or a Time-Dependent Hartree Approximation. Here the fermions respond to a self consistent field which includes, in addition to any external driving potentials, the self-consistent Chern-Simons magnetic field $\delta B$ produced by fluctuations in the particle density $< \delta n >$, and a Chern-Simons electric field, given by

$$
e = -2\pi m \frac{\hat{z} \times j}{r},$$

where $j$ is the induced current density.

The FCS approach has several strengths and weaknesses, compared to the composite fermion trial wavefunction approach. A great strength of the FCS approach is that it can address, analytically, the low energy behavior at compressible filling fractions, such as $\nu = 1/2$. It can also address the asymptotic behavior of quantized Hall systems close to $\nu = 1/2$; for example, it makes non-trivial predictions for the behavior of the energy gaps and for the low-energy collective modes in fractional quantized Hall states of the form $\nu = p/(2p+1)$, in the limit $p \to \infty$.\cite{11, 23, 24} However, the FCS approach is not very good for calculating the absolute value of energy gaps or of the effective mass for quasiparticles at compressible filling factors. These are properties which depend on an accurate description of short distance behavior, that are not readily obtained using the FCS perturbation expansion. As an example, with the FCS approach, it is not apparent that the energy gaps at fractional quantized Hall states are proportional to the strength of the electron-electron interaction, and must vanish if the interaction goes to zero while the electron mass is held fixed. In the trial
wavefunction approach, this property is guaranteed by projection onto the lowest Landau level, so that the kinetic energy is automatically a constant, and the differences in energy levels arise only from the interactions. Of course, accurate evaluations of energies using trial wavefunctions still require complicated numerical calculations and extrapolations to infinite systems.

3 Predictions at $\nu = 1/2$.

Let us summarize some of the interesting predictions of the FCS theory for the case of a compressible fraction, such as $\nu = 1/2$, at zero temperature, in the absence of disorder. The first prediction is that the system is in fact compressible, i.e., that the energy cost of a small long-wavelength fluctuation in the density is quadratic in the density fluctuation, and independent of the wavelength, if the singular potential energy due to the long-range Coulomb interaction is subtracted. (Strictly speaking, the limit should be taken in which the size of the density fluctuation goes to zero while the wavelength is fixed, and then the wavelength goes to infinity.)

Dynamically, one finds that in the absence of disorder, the longitudinal electrical conductivity vanishes linearly with the wave vector $q$ in the limit $q \to 0$, with a coefficient that can be calculated analytically. Specifically, at $\nu = 1/2$, for $q \parallel \hat{x}$, one finds

$$\sigma_{xx}(q) = \frac{e^2 q}{8\pi \hbar k_F}. \quad (3)$$

This implies that fluctuations in the electron density relax very slowly at long wavelengths. In the case of long-range Coulomb interactions, the relaxation rate $\gamma(q)$ is proportional to $q^2$, and is given (in cgs units) by

$$\gamma(q) = \frac{q^2 e^2}{4\hbar k_F \epsilon}, \quad (4)$$

where $\epsilon$ is the dielectric constant of the background semiconductor. In the case of short-range interactions, one finds even slower relaxation, $\gamma(q) \propto q^3$ for $q \to 0$.

The longitudinal conductivity $\sigma_{xx}(q)$ is of direct experimental interest, as it is reflected in the attenuation and velocity shift in a surface acoustic wave (SAW) experiment. The wavevector $q$ is the wavevector of the SAW, and we use the zero-frequency limit of $\sigma_{xx}(q)$, because we are assuming that the sound velocity is smaller than the effective Fermi velocity of the composite fermions. This gives an explanation for the anomaly in SAW propagation at $\nu = 1/2$ that was first observed by Willett and coworkers in 1990. For an SAW of sufficiently high frequency, such that the wavelength is shorter than the mean-free-path of quasiparticles due to disorder scattering, the conductivity is enhanced at $\nu = 1/2$, proportional to $q$, as predicted by FCS theory for a clean system. By contrast, at lower frequencies, where the SAW wavelength is long compared to the mean-free-path, one sees the ordinary dc conductivity, which shows no anomaly at $\nu = 1/2$. 

4
4 Slightly away from $\nu = 1/2$.

At $\nu = 1/2$, the composite fermions see zero effective magnetic field, and travel in straight lines until they are scattered by an impurity. Slightly away from $\nu = 1/2$, they see a small $B_{\text{eff}}$, given by the difference between $B$ and the field corresponding to $\nu = 1/2$. Then the composite fermions will move in circular orbits, with radius:

$$R^*_c = \frac{\hbar k_F}{e|B_{\text{eff}}|}.$$  \hspace{1cm} (5)

The orbit diameter $2R^*_c$ has been measured in geometric resonance experiments with density modulations created by superimposed periodic structures, as well as in SAW experiments, and in magnetic focusing experiments. The results agree well with the theoretical prediction, with measured values of $2R^*_c$ which are as large as $\approx 1\mu$m. This length is of order 100 times larger than the actual cyclotron radius for the electrons in the lowest Landau level.

In Figure 1, we show results of Willett et al., comparing measurements of the shift in SAW velocity near $\nu = 1/2$, at a frequency of 8.5 GHz, with predictions of the FCS theory of Halperin, Lee and Read. The theoretical curve is broadened by 1.5% (FWHM) to account for sample inhomogeneities, and there is an adjustment to the overall conductivity scale of the theory. The theory is essentially an RPA calculation of $\sigma_{xx}(q)$ as a function of magnetic field. The results are quite different from what one would have obtained if one had ignored the self-consistent Chern-Simons electric field. The minima in the sound velocity occur when $2R^*_c$ is equal to $5/4$ times the wavelength of the sound wave, or when the deviation of $B$ from the value at $\nu = 1/2$ is equal to $3.83\hbar cqk_F/e$. Thus, the fact that the minima coincide in the theoretical and experimental curves is a confirmation that the Fermi momentum of the composite fermions is the same as that for spin-aligned electrons in zero magnetic field.

Experiments which combine SAW propagation and a static density modulation induced by a superimposed gate array show additional peculiar features, which have been explained, at least in part, using FCS theory.

The peculiar wavevector-dependent conductivity has implications for experiments in which one tunnels an electron into the center or the edge of a compressible quantum Hall system, and also for drag experiments, where one measures the transresistance in a system of two separated layers, each near $\nu = 1/2$. The FCS theory has been applied, with some success, to such experiments.

5 Beyond RPA.

Other interesting predictions of the FCS theory go beyond the mean field approximation and the RPA. For example, it was found by Halperin, Lee and Read that at $\nu = 1/2$ the effective mass of the quasiparticles should diverge as the energy approaches the Fermi energy. For unscreened Coulomb interactions this divergence is only logarithmic, and is not a large effect in the range accessible to experiments, but it poses some interesting questions of principle. The divergence in the effective mass does not affect the long-wavelength linear response functions, as it is cancelled.
by interaction effects in the Fermi liquid theory. Thus the results of Eqs. (3) and (4), which do not involve the quasiparticle effective mass, are expected to remain correct beyond the RPA approximation. The divergent effective mass should be reflected, however, in the energy gaps for the fractional quantized Hall states at \( \nu = p/(2p + 1) \), in the limit \( p \to \infty \). For the case of unscreened Coulomb interactions, the prediction is \[ \Delta_p \sim \frac{\pi e^2}{2e l_0 (2p + 1)[\ln(2p + 1) + C']}, \] (6)

where \( l_0 \) is the magnetic length, and \( C' \) is a constant that depends on the short-range part of the electron-electron interaction and cannot be calculated within the FCS theory. However, it is believed that the coefficient of the logarithmic term in the denominator of Eq. (6) is exact, even though there is no small parameter in the FCS perturbation theory.

In Figure 2, we show results from a recent paper by Morf, d’Ambrumenil and Das Sarma, who have performed calculations of the energy gap for finite systems of up to 18 electrons on a sphere, and have carefully extrapolated to the limit of an infinite system, at filling fractions \( \nu = 1/3, 2/5, 3/7, 4/7 \), corresponding to \( p = 1, 2, 3, 4 \). They find that with a proper choice of \( C' \), the gaps fit well to Eq. (6), and that the fit is better than would be obtained without the logarithmic correction. However, it is not clear \textit{a priori} that the asymptotic formula (6) should apply for such small values of \( p \).

Unfortunately, it is not possible to draw firm conclusions about the intrinsic energy gaps from existing experiments. In order to compare experiments with theoretical predictions, it has been customary to subtract a large constant \( \Gamma \) from the experimental gaps, in order to account for effects of impurities. This constant varies from sample to sample, but has been generally taken to be independent of the \( p \), and thus of precise filling fraction. Recently, Morf and d’Ambrumenil have proposed an alternative correction, with \( \Gamma \) varying inversely with \( p \), which seems to give better agreement between theory and experiment. However, there is still no satisfactory basic understanding of the effects of impurities on energy gaps extracted from transport measurements when the corrections to the gaps are large.

### 6 Bare composite fermions and low energy quasiparticles.

There are some important distinctions between the bare composite fermions which enter the FCS theory, and the low energy quasiparticles which emerge from the theory. The bare fermions have charge \( e \) at all filling fractions. The low-energy quasiparticles have a reduced charge \( e^* \), which in the case of the quantized Hall fractions \( \nu = p/(2p + 1) \) is given by

\[ e^* = \frac{e}{2p + 1}. \] (7)

At \( \nu = 1/2 \), or the limit \( p \to \infty \) this becomes \( e^* = 0 \), but the quasiparticles have an electric dipole moment, \( \mathbf{d} \), related to the quasiparticle momentum \( \mathbf{k} \), by

\[ \mathbf{d} = e l_0^2 \hat{z} \times \mathbf{k}. \] (8)
It is also known that the quasiparticles at fractional quantized Hall states are not properly fermions, but rather are anyons, with a statistical angle given by \( \theta = \pi(2p - 1)/(2p + 1) \). At \( \nu = 1/2 \), however, the statistical angle becomes \( \theta = \pi \), corresponding to fermions.

At filling fraction \( \nu = p/(2p + 1) \), the effective field seen by a composite fermion is related to the external magnetic field by \( B_{\text{eff}} = B/(2p + 1) \). Thus we see that the cyclotron radius for a quasiparticle of charge \( e^* \) given by (7) in the external field \( B \) is the same as the effective cyclotron radius given by (5), provided one uses the same value of the Fermi momentum \( k_F \).

As the low-energy quasiparticles at \( \nu = 1/2 \) are indeed fermions, it should be possible to construct something like a Landau Fermi liquid theory description directly in terms of low-energy neutral, dipolar quasiparticles, as first proposed by Shankar and Murthy. Various methods for constructing such a description have been explored by several groups of researchers. When done properly, these descriptions all lead to the same predictions as the FCS theory for low energy observable properties, such as the response functions for electrons. However, the Fermi liquid itself has some rather peculiar properties, as emphasized by Stern et al.

One peculiar property of the Fermi liquid at \( \nu = 1/2 \) is that energy is unchanged if a constant \( \mathbf{K} \) is added simultaneously to the momentum of every quasiparticle (i.e., the Fermi surface is displaced by \( \mathbf{K} \)). Within Landau Fermi liquid theory this is equivalent to assuming a Landau interaction parameter \( F_1 = -1 \) for the \( l = 1 \) angular momentum channel. With an appropriate normalization for \( F_1 \), the energy cost of displacing the Fermi surface by \( \mathbf{K} \) is given by

\[
\delta E = \frac{n_e K^2}{2m^*}(1 + F_1),
\]

where \( m^* \) is the quasiparticle effective mass.

A second peculiarity of the Fermi liquid description is that the local electron density \( \rho_e(\mathbf{r}) \) is related to the momentum density of quasiparticles \( \mathbf{g}(\mathbf{r}) \) by

\[
\rho_e = \frac{e^2}{4\pi}\nabla \times \mathbf{g}.
\]

This relation has a simple interpretation. Comparing it with Eq. (8), we see that the electron charge density is given by the divergence of the polarization density of the dipolar quasiparticles. The density of quasiparticles is also constrained to be equal to a constant times \( \nabla \times \mathbf{g} \). The equations of motion for the quasiparticles are necessarily consistent with this constraint.

Another way of describing the independence of the energy under a constant momentum displacement is that the neutral quasiparticles have a strong momentum-dependent interaction, which cancels their bare kinetic energy in this situation. There is no Chern-Simons vector potential in this Fermi liquid description, as it has already been eliminated in constructing the neutral quasiparticles.

Murthy and Shankar have calculated many properties of quantum Hall systems using a Hamiltonian approach for describing composite fermions, which begins with the FCS theory, and proceeds through several additional mathematical transformations and additional approximations. The objects described by their Hamiltonian, at \( \nu = 1/2 \), are the neutral quasiparticles and high energy...
oscillator modes, describing inter-Landau-level excitations. Although the correct behavior at low energies is not immediately apparent in the Murthy-Shankar approach, their method does allow them to obtain relatively simple analytic estimates of quantities such as the energy gaps at fractional quantized Hall states, or the energy cost for partial spin polarization of the electrons at \( \nu = 1/2 \), which cannot be obtained from the original FCS theory. These estimates generally agree at the level of \( \pm 10\% \) with results from numerical calculations using trial wavefunctions, which are much more difficult to carry out. The reader is referred to a detailed review article by Murthy and Shankar, which describes their approach.\[45\]

7 Conclusion

As indicated above, the composite fermion picture and the FCS theory have been particularly successful in understanding properties of single layer quantum Hall systems with electrons in the lowest Landau level. For electrons above the second Landau level (i.e., for \( \nu > 4 \)), the composite fermion picture has not, so far, proven useful. It appears that the system is better described by the ordinary Hartree-Fock theory, where the partially filled Landau level forms inhomogeneous structures such as charge-density waves or crystals.\[49, 50, 51\] For electrons in the second Landau level the system is more delicate. It appears\[52, 53\] that the quantized Hall states observed at \( \nu = 5/2 \) and \( \nu = 7/2 \) are well-described by the Moore-Read Pfaffian state,\[5\] which can be understood in the context of FCS theory as a state where a gap is opened at the Fermi surface of composite fermions due to a \( p \)-wave BCS pairing.\[54\] However, numerical calculations on finite systems indicate that these states are close to an instability, and small changes in the electron-electron interaction at short distances can drive the system into an anisotropic charge-density-wave state similar to those in higher Landau levels.\[52, 53\] Experimentally, it appears that application of a magnetic field parallel to the electron layer can drive this transition.\[55\] Away from \( \nu = 5/2 \) and 7/2, at other filling fractions in the second Landau level, alternations have been observed in a perpendicular magnetic field, between fractional quantized Hall states consistent with the composite fermion picture, and insulating states suggestive of a straightforward Hartree-Fock state.\[56\]

The FCS theory has been applied, with some success to bilayer systems in the lowest Landau level, but a proper discussion of bilayer systems is beyond the scope of this article.\[57, 58\] Other topics omitted from this article, where FCS theory has been applied, include transitions between states with different spin polarizations,\[45\] effects of disorder on transport properties in compressible states,\[59\] and transitions between different fractional quantized Hall plateaus in the presence of disorder.\[12, 59\]

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FIGURE CAPTIONS

Figure 1. Experimental results and theoretical prediction of the shift in surface acoustic wave velocity, as a function of magnetic field, in a GaAs sample with a two-dimensional electron gas below the surface, near filling fraction $\nu = 1/2$. The theoretical curve is the fermion-Chern-Simons prediction of Ref. [11], broadened to account for sample inhomogeneity, with an adjustment to the overall scale of the conductivity. The SAW frequency is 8.5 GHz. From Ref. [28].

Figure 2. Energy gaps at quantized Hall fractions of the form $\nu = p/(2p + 1)$. Dots are obtained exact calculations on systems of up to 18 electrons on a sphere, extrapolated to infinite system size. The dot-dashed curve is a fit to the form expected for non-interacting composite fermions with a constant effective mass; the solid curve includes the logarithmic correction predicted by FCS theory. Each curve has one adjustable constant, chosen to fit the data point at $\nu = 1/3$. From Ref. [38].
Figure 1. B. I. Halperin
Figure 2. B. I. Halperin