Abstract

We consider the problem of synthesizing structured controllers for complex systems over directed graphs. In particular, we consider three tasks, i.e., $P_1$: structured stabilization, $P_2$: structured optimal control, and $P_3$: communication topology design. These problems are known to be non-convex, and previous attempts have addressed them separately using different approaches, e.g., by characterizing special systems that allow convex formulations, or by employing certain techniques to obtain convex relaxations. In this paper, we provide a simple and unified convex strategy to deal with these three problems systematically by assuming that the closed-loop system admits a block-diagonal Lyapunov function. Also, two sufficient conditions are introduced to guarantee the block-diagonal stabilization. The block-diagonal assumption naturally guarantees a property of sparsity invariance, indicating that the structured constraint on controllers can be exactly transformed to constraints on Lyapunov variables. Using this assumption, a linear matrix inequality (LMI) approximation is used to solve $P_1$; an LMI approximation and a gradient projection method are proposed to address $P_2$; and an iterative algorithm is introduced to solve $P_3$. Two illustrative examples are used to demonstrate the effectiveness of the proposed solutions.

Key words: Structured controller, block-diagonal Lyapunov function, convex design, block-diagonal stability

1 Introduction

1.1 Motivation

The analysis and synthesis problem for large-scale complex systems has attracted considerable attention in recent years [1–4]. These problems arise in a wide range of engineering applications, such as the smart grid [5], biological networks [6], and automated highways [7]. One central and challenging issue in the case of complex systems is to develop analytical and computational methods for the tractable design of structured controllers, where the information that is available for feedback has structured constraints according to the communication connections in the systems.

It is well-known that solutions to linear feedback control synthesis, such as LQR, $H_2$, and $H_{\infty}$, are available [8]. However, the resulting controllers are usually dense and typically implemented in a centralized fashion. This usually requires information from every subsystem in the complex system to form the control input, which imposes a prohibitive communication burden in large-scale complex networks. Alternatively, many efforts have focused on the design of structured controllers for complex systems, explicitly taking the structured information constraints into account. Three major issues have been discussed in the literature: 1) structured stabilization [9–11]; 2) structured optimal control [12–15]; and 3) communication topology design [16–18]. One common challenge is to deal with the sparse structure of the feedback arising naturally from the need of implementing control policies in a distributed fashion. In fact, the general problem of designing linear feedback gains with structured constraints is NP-hard [19]. Numerous attempts have been made to provide analytical and computational approaches for structured controller synthesis by either restricting the problems to special classes of systems [1,12], using convex approximations [14,18], or employing non-convex optimization techniques [15,20].

In this paper we consider three issues of designing structured controllers, i.e., 1) structured stabilization, 2) structured optimal control, and 3) communication topology design. We show that under a simple assumption that the closed-loop system admits a block-diagonal Lyapunov function, we can obtain a unified convex approach to deal with the three tasks. Here, we first concisely describe the problems addressed in this paper.

Problem Statement: Consider a complex system of heterogeneous subsystems over graphs with a vertex set $\mathcal{V} = \{1, \ldots, N\}$: each vertex in $\mathcal{V}$ represents a subsystem and a
The problems considered in this paper are: performance measure. 

The overall state-space model is then given by 

\[ \dot{x}(t) = Ax(t) + B_1d(t) + B_2u(t), \] 

where \( x(t) = [x_1(t)^T, \ldots, x_N(t)^T]^T \), and the vectors \( u(t) \) and \( d(t) \) are defined similarly. We look for linear static state feedback controllers 

\[ u_i(t) = -K_{ii}x_i(t) - \sum_{j \in N_i^c} K_{ij}x_j(t), \] 

where \( N_i^c \) denotes the neighbours of subsystem \( i \) in the communication graph \( G_c \), i.e., those vertices that send their state information to vertex \( i \). The compact form of the overall controller is 

\[ u(t) = -Kx(t), \quad K \in \mathcal{K}, \] 

where \( \mathcal{K} \) denotes the block-sparsity pattern determined by the communication topology (see the precise definition in Section II). The closed-loop system is 

\[ \dot{x}(t) = (A - B_2K)x(t) + B_1d, \quad K \in \mathcal{K}. \] 

The problems considered in this paper are:

\[ \mathcal{P}_1 : \text{Structured Stabilization} \] 

Find \( K \in \mathcal{K} \), such that \( A - B_2K \) is asymptotically stable. \( (P_1) \)

\[ \mathcal{P}_2 : \text{Structured Optimal Control} \] 

which is to search for a structured controller that minimizes a certain quadratic performance measure. 

\[ \min_K \|G_{dz}(K)\|^2 \] 

s.t. \( K \) stabilizing 

\( K \in \mathcal{K}, \) \( (P_2) \)

where \( G_{dz}(K) \) denotes the transfer function of the closed-loop system from disturbance \( d \) to a certain performance output \( z \), and \( \| \cdot \| \) is typically the \( H_2 \) or \( H_{\infty} \) norm. In this paper, we consider \( H_2 \) performance, and the performance output is chosen as \( z = Cx + Du \).
decentralized controller [24]. Anderson and Clements derived an algebraic characterization for the existence of fixed modes [27]. Recently, the emphasis has shifted to the development of computational methods according to Lyapunov theory via linear matrix inequalities (LMIs). For example, Zecèvić and Šiljak proposed an LMI approximation for the design of robust control laws subject to sparsity constraints [10], and this technique was extended to the design of static output-feedback controllers in [11]. In addition, a sufficient condition was established using LMIs for the output feedback negative imaginary synthesis with structural constraints [28].

Stability is in general only one basic expected feature of a closed-loop system. The next stage is to address certain performance specifications, which are typically expressed in terms of input/output properties such as $H_2$ or $H_{\infty}$ performances (see $P_2$ as a basic formulation). In general, this problem is known to be NP-hard when imposing structured constraints [19]. Previous approaches can be categorized into three classes: 1) finding exact solutions for special classes of systems [1, 12, 13, 29]; 2) seeking tractable approaches via convex approximations [14, 18, 30]; and 3) obtaining suboptimal solutions using non-convex optimizations [15, 20, 31]. In the first case, for a class of systems with the property of quadratic invariance (QI), it is possible to find optimal structured controllers in the frequency domain via a Youla parameterization [1]. In principle, a system with QI means the convexity of structured constraints can still be preserved in the Youla parameterization, which has received considerable attention recently [29]. In another special class of systems modeled by partially ordered sets [12], Shah and Parrilo derived explicit state-space solutions by solving a number of uncoupled Ricatti equations. Other classes of convex structured control problems include partially nested systems [32] and positive systems [13, 33]. In the second case, the strategy is to derive a convex relaxation of the original problem, and obtain an approximate solution. For instance, structured optimal control can be formulated as a rank-constrained LMI in both the continuous-time [18] and discrete-time setting [30]: a convex relaxation is obtained by dropping the rank constraint. Here, the difficulty is in finding a low-rank solution [18, 30]. In addition, a new convex control objective is formulated using singular values of the linear mapping from disturbances to states for both $H_2$ or $H_{\infty}$ norms [14]. Finally, the third approach is to search for structured controllers by solving the original non-convex problem directly. For instance, generalised gradients and bundling techniques were used to solve $H_{\infty}$ synthesis problems under structural constraints in [31]. Apkarian et al. employed nonsmooth optimization techniques to compute mixed $H_2/H_{\infty}$ controllers locally [15], which may have a predefined structure. Lin et al. proposed an augmented Lagrangian method to numerically solve the optimal control problem with sparsity constraints [20].

Recently, it has been demonstrated that the design of information structure of a control law is as important as the design of the control law itself [2, 16, 34]. Langbort and Gupta pointed out that adding communication links may harm the system performance if there exist costs in the communication [17]. On the other hand, it was also shown that for certain fixed communication topologies, there exist closed-loop performance limitations which are independent of the choice of feedback gains [35, 36]. Ideas from control theory and optimization have been combined to design the communication topology, aiming to strike a balance between the closed-loop performance and controller structures. This problem is typically formulated as a regularized optimal control problem by augmenting quadratic performance measures with sparsity-promoting functions (see $P_3$) [2]. In [16], the sparsity structure of controllers was identified using an alternating direction method of multipliers (ADMM), where the number of communication links was penalized. This technique was later applied to identify the optimal control structure for wide-area control of power networks [5]. More recently, Matni and Chandrasekaran proposed a computationally tractable approach for the co-design of structured controllers and communication architectures [34].

Although the structural constraint poses a great challenge for the analysis and synthesis of large-scale systems, it has the potential to bring certain benefits from the perspective of numerical computations after appropriate relaxations. In fact, the speed and accuracy of numerically computing a controller can actually be improved if the structure of the resulting relaxation problem is taken advantage of [37, 38]. Also, there exists some recent work that aims to exploit the inherent structural properties of large-scale systems to make the computation scalable [39] or locally within subsystems [38, 40], which is beyond the scope of our current work.

1.3 Contribution

The aforementioned diverse body of research provides powerful tools for the synthesis of structured controllers. However, most of them addressed the three tasks, i.e., 1) structured stabilization, 2) structured optimal control, and 3) communication topology design, separately using different approaches. The major contribution of this paper is to provide a simple and unified convex strategy to deal with the three tasks $P_1$–$P_3$ systematically. This is achieved using a simple assumption that the closed-loop system admits a block-diagonal Lyapunov function. In fact, this property naturally guarantees sparsity invariance, indicating that the structured constraints on feedback gains are exactly transformed to constraints on Lyapunov variables. This idea convexifies the constraint on the controller design (i.e., $K \in \mathcal{K}$) when using Lyapunov theory to synthesize controllers.

Specifically, an LMI approximation is used to solve $P_1$, and we propose an LMI approximation and a gradient projection method to address $P_2$. Then, an iterative algorithm is introduced to solve $P_3$ based on the reweighted $l_1$ scheme [41]. While diagonal stability has recently been exploited in the analysis and synthesis of positive systems [13, 33], the application of block-diagonal stability on structured controller synthesis is discussed in our paper. A block-diagonal Lyapunov function can be viewed as an intermediate step between a diagonal Lyapunov function and a full one. The advantage of using block-diagonal Lyapunov functions is clearly demonstrated by the solutions to $P_1$–$P_3$. Also, we
note that the idea of applying block-diagonal Lyapunov functions was previously used to design localized controllers in [42] and address consensus problems in [43]. In contrast to these results, we focus on the convex strategy for controller synthesis given arbitrary structures (see $\mathcal{P}_1$-$\mathcal{P}_3$), and employ a gradient-projection and a reweighted $l_1$ scheme to address the problems. Besides, we introduce two sufficient conditions under which there always exists a structured controller such that the closed-loop system admits a block-diagonal Lyapunov function. Moreover, the sufficient conditions can be tested using local information based on each subsystem.

The rest of this paper is organized as follows. In Section 2, we introduce the notion of sparsity invariance and review block-diagonal stability. Section 3 presents two sufficient conditions under which the closed-loop system can be block-diagonally stable. The unified treatments on the convex design of structured controllers $\mathcal{P}_1$-$\mathcal{P}_3$ are presented in Section 4. In Section 5, we use two examples to demonstrate our methods, before concluding the paper in Section 6.

Notation: $K_{ij}$ denotes a sub-matrix (block) of matrix $K$ (its dimension should be clear from the context). We use $0$ to denote a block with all entries being zero. The cardinality $\text{card}(K)$ of a block matrix $K$ represents the number of its non-zero blocks. $\text{diag}(X_1,\ldots,X_n)$ represents a block-diagonal matrix with matrices $X_i, i=1,\ldots,n$, on its diagonal. Given a symmetric matrix $X \in \mathbb{S}^n$, $X \succ (\succeq) 0$ means that the matrix is positive (semi-) definite. We use $X^T$ to denote the transpose of $X$. The trace of a matrix $X$ is denoted by $\text{Tr}(X)$. The notation $\|X\|_F$ denotes the Frobenius norm of $X$. $\nabla J(K)$ denotes the gradient of a differentiable function $J(K)$ with respect to $K$.

2 Sparsity Invariance and Block-diagonal Stability

In this paper, we use two directed graphs, i.e., $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$ and $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$, to model a complex interconnected system (see Fig. 1). Precisely, a graph is represented by a set of vertices $\mathcal{V} = \{1, 2, \ldots, N\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote $(i,j) \in \mathcal{E}$ if and only if there is a directed edge from vertex $i$ to vertex $j$. The out-degree of vertex $i$ is the number of edges coming out of the corresponding vertex.

The communication graph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$ imposes a constraint on the admissible controllers. If $(j,i) \notin \mathcal{E}^c$, then there is no communication link from vertex $j$ to vertex $i$. Thus, subsystem $j$ cannot use the information of subsystem $i$ to compute its control input. This leads to the structured constraint $\mathcal{K}$, defined by

$$\mathcal{K} = \{ K \in \mathbb{R}^{\hat{m} \times \hat{n}} \mid K_{ij} = 0, \text{ if } (j,i) \notin \mathcal{E}^c \},$$

(6)

where $\hat{m} = \sum_{i=1}^N m_i, \hat{n} = \sum_{i=1}^N n_i$. Throughout the paper, a subscript denotes the blocks corresponding to the subsystems, e.g., $K_{ij} \in \mathbb{R}^{m_i \times n_j}$. In $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$, it is assumed that $(i,i) \in \mathcal{E}^c, \forall i \in \mathcal{V}$, which means subsystem $i$ can always use its own state information for feedback. Take the system in Fig. 1 as an example. Its structured constraint $\mathcal{K}$ has a block-sparsity pattern as

$$\mathcal{K} = \begin{bmatrix}
* & * & 0 \\
0 & 0 & * \\
0 & 0 & 0
\end{bmatrix},$$

where each block $*$ or $0$ has compatible dimensions with the corresponding subsystems. It is easy to see that $\mathcal{K}$ is linear and convex.

In this section, we first introduce a simple sufficient condition that preserves the sparsity pattern $\mathcal{K}$ when performing a change of variables to synthesize the controller, and then present a discussion on block-diagonal stability.

2.1 Sparsity invariance

Many problems in control can be formulated as LMIs using Lyapunov theory, which are subsequently solved by well-established convex optimization algorithms [44]. For synthesis problems, one standard technique to obtain LMI formulations is to introduce a change of variables:

$$Z = KX,$$

(7)

where $K$ is the static feedback gain, $P = X^{-1}$ usually serves as a Lyapunov function $V(x) = x^T(t)PX(t)$ for the closed-loop system, and $Z$ is a new variable.

However, the change of variables (7) is in general not suitable when imposing structured constraints on $K$. Although the constraint on the feedback gains $K \in \mathcal{K}$ (see $\mathcal{P}_1$-$\mathcal{P}_3$) is linear and thus convex, the corresponding constraint on $Z$ and $X$ (i.e., $ZX^{-1} \in \mathcal{K}$) is nonlinear and nonconvex. If we restrict the structure of $X$, and assume the matrix $X$ is block-diagonal with the block sizes compatible to those of the subsystems, i.e., $X = \text{diag}(X_1,\ldots,X_N)$, the following property of sparsity invariance is naturally guaranteed (see Fig. 2 for an illustration):

$$ZX^{-1} \in \mathcal{K} \iff Z \in \mathcal{K}.$$  

(8)

In other words, (8) means the sparsity pattern of feedback gains is exactly transformed to the pattern of Lyapunov variables. This also indicates that the convexity of structured constraints is preserved in the Lyapunov parameterization under the block-diagonal assumption in the time
domain. Note that the block-diagonal assumption on $X$ is a sufficient condition for (8) given an arbitrary sparse pattern $K$.

In this paper, we apply the block-diagonal assumption to solve the structured controller synthesis $P_1\cdot P_3$. Since this assumption indicates the closed-loop system admits a block-diagonal Lyapunov function, one related concept is the so-called diagonal stability or block-diagonal stability, which is an active area of research [45]. For the sake of completeness, we discuss this topic in the next section.

**Remark 1** The block-diagonal assumption also means the closed-loop system admits a fully decentralized Lyapunov function:

$$V(x) = \sum_{i=1}^{N} x_i^T(t)P_ix_i(t),$$

which is a composite form of the property of each subsystem $V_i(x_i) = x_i^T(t)P_ix_i(t)$. In fact, the idea of decentralized Lyapunov functions has been exploited in some recent studies on large-scale systems; see e.g., [37–40, 42, 43, 46].

### 2.2 Block-diagonal Stability

It is well-known that a linear system $\dot{x}(t) = Ax(t)$ is asymptotically stable if and only if there exists a $P > 0$ satisfying the Lyapunov LMI

$$A^TP + PA \prec 0.$$  \hspace{1cm} (10)

In general, the solution $P$ is a dense matrix, which defines a Lyapunov function of the form $V(x) = x^T(t)Px(t)$ [44]. Recently, it has been shown that the elements of $P$ are usually spatially localized or decaying quickly away from a banded pattern given a banded matrix $A$ [47]. Also, some efforts have been devoted to search for a diagonal or block-diagonal solution, leading to the notion of diagonal stability or block-diagonal stability [45].

**Definition 1** A linear system is called block-diagonally stable if there exists a block-diagonal $P$ satisfying (10), where the block sizes of $P$ are compatible with those of the subsystems. Further, if the subsystem size is scalar, then $P$ is diagonal and the system is called diagonally stable.

#### 2.2.1 Characterization

There exist necessary and sufficient conditions for diagonal stability of systems of dimension three or four [45]. For higher dimensional systems, conditions have been established under certain assumptions on the dynamical systems, two of which are: 1) positive systems, where the system matrix $A$ is Metzler [33]; 2) hierarchical systems with a directed acyclic graph topology, where there exists a permutation matrix $H$ such that $H^T AH$ is a lower triangular matrix [12, 38]. The first type of system adds nonnegative constraints on certain entries, while the second one introduces sparsity requirements on the connections of subsystems.

Another broader class of systems is based on the notion of $H$ matrices, and we refer the interested reader to [48] for details. In addition, necessary and sufficient conditions were derived for diagonal stability of special classes of interconnected systems that are widespread in biological networks, such as the secant criterion in cyclic structures [49] and its generalization on cactus graphs [50]. Moreover, the conditions for block-diagonal stability were discussed in [51–53]. For instance, a generalization of the result for Metzler matrices was presented in [52]. In [53], the problem of block-diagonal stability was reformulated as a common Lyapunov function problem and then addressed using the Dubovikii-Milyutin theorem.

#### 2.2.2 Application

One major benefit of (block-) diagonal stability is reflected on the improvement of numerical computations. An obvious aspect is the reduction of decision variables when checking the Lyapunov LMI (10). Also, many analysis and synthesis problems for positive systems can be solved using linear programming (LP) instead of semidefinite programming (SDP), which offers a scalable approach to large-scale systems [33]. Another important observation is that the Lyapunov LMI (10) preserves the (block-) sparsity pattern of the original system when $P$ is (block-) diagonal. This fact motivates some recent work to develop scalable approaches using decomposition techniques [37, 39, 40].

In our current work, we focus on the property of sparsity invariance (8) for block-diagonal stability. We show that if the closed-loop system can be synthesized to be block-diagonally stable, then all of the three tasks $P_1\cdot P_3$ can be solved using convex optimization methods.

### 3 Sufficient Conditions on Block-diagonal Stabilization

In this section, we discuss two sufficient conditions under which the closed-loop system admits a block-diagonal Lyapunov function.

#### 3.1 Scalar subsystems

The block-diagonal assumption may seem restrictive. However, it can be easily fulfilled for a complex system with scalar subsystems. We have the following result.

**Theorem 1** Given a linear system (1) with any plant graph $\mathcal{G}^p = (V, \mathcal{E}^p)$ and communication graph $\mathcal{G}^c = (V, \mathcal{E}^c)$, if the subsystem size is scalar and $(A_{ii}, B_{2i})$ is controllable for all $i \in V$, then there exists a structured controller $K \in \mathcal{K}$ such that the closed-loop system is diagonally stable.

**Proof.** Since $(A_{ii}, B_{2i})$ is controllable and $B_{2i}$ is scalar, the scalars $B_{2i}$ are non-zero. According to [48], for any matrix $A$ there exists an $\alpha > 0$ such that $A - \alpha I$ is diagonally stable. Therefore, we can simply set

$$K_{ii} = \frac{\alpha}{B_{2i}}; K_{ij} = 0, i \neq j,$$  \hspace{1cm} (11)

which clearly satisfies $K \in \mathcal{K}$ for any communication graph.
Then, the resulting closed-loop system $A - B_2K$ is diagonally stable.

The sufficient condition in Theorem 1 can be viewed as local controllability for each subsystem. We note that the value of $\alpha$ is closely related to the plant graph $G^p$ and the strength of dynamical interactions $A_{ij}$, which is highlighted in Remark 2.

### 3.2 General subsystems

Here, we present a generalization of Theorem 1 to block subsystems. First, we need the following lemma.

**Lemma 1** Given two matrices $X, Y$ of appropriate dimensions, then we have

$$X^TWX + Y^TW^{-1}Y \succeq X^TY + Y^TX$$  \hspace{1cm} (12)

for any $W \succ 0$ of appropriate dimension.

*Proof.* It is easy to see that

$$X^TWX + Y^TW^{-1}Y - (X^TY + Y^TX) = (WX - Y)^TW^{-1}(WX - Y) \succeq 0.$$  \hspace{1cm} (12)

This means (12) holds.

**Theorem 2** Given a linear system (1) with any plant graph $G^p = (V, E^p)$ and communication graph $G^c = (V, E^c)$, there exists a structured controller $K \in \mathbb{K}$ such that the closed-loop system is block-diagonally stable, if the following condition

$$\hat{A}_{ii}^TP_i + P_i\hat{A}_{ii} + P_i \left( \sum_{j \in N_p^i} A_{ij}W_{ij}^{-1}A_{ji}^T \right)P_i + \sum_{j \in N_p^i} W_{ji} \prec 0,$$  \hspace{1cm} (13)

holds for some $W_{ij} \succ 0$, $P_i > 0$, where $\hat{A}_{ii} = A_{ii} - B_{2i}K_{ii}$.

*Proof.* Consider a block-diagonal controller

$$K = \text{diag}(K_{11}, \ldots, K_{NN}),$$

which naturally satisfies $K \in \mathbb{K}$ for any communication graph. By defining $\hat{A}_{ii} = A_{ii} - B_{2i}K_{ii}$, and ignoring the disturbance, the closed-loop dynamics for each subsystem become

$$\dot{x}_i(t) = \hat{A}_{ii}x_i(t) + \sum_{j \in N_p^i} A_{ij}x_j(t), \quad \forall i \in V. \hspace{1cm} (14)$$

Next, we consider a block-diagonal Lyapunov function

$$V(x) = \sum_{i=1}^{N} x_i^T(t)P_ix_i(t).$$

The derivative of $V(x)$ along the closed-loop trajectory (14) is

$$\dot{V}(x) = \sum_{i=1}^{N} (\dot{x}_i^T(t)P_ix_i(t) + x_i^TP_i\dot{x}_i(t)).$$

For the coupling term in (15), according to Lemma 1, we have

$$\sum_{j \in N_p^i} (\sum_{j \in N_p^i} A_{ij}W_{ij}^{-1}A_{ji}^TP_jx_j(t)) + x_j^TP_iW_{ij}x_j(t),$$

where $W_{ij} \succ 0$. Substituting (16) into (15), we know

$$\dot{V}(x) \leq \sum_{i=1}^{N} \left( \sum_{j \in N_p^i} (A_{ij}^TP_jA_{ji})x_j(t) + x_j^TP_iW_{ij}x_j(t) \right).$$

If the condition (13) holds for some $W_{ij} \succ 0$, $P_i > 0$, then, $\dot{V}(x)$ is negative definite. It means we have found a block-diagonal Lyapunov function for the closed-loop system. This completes the proof.

We have two propositions that provide local tests to verify the sufficient condition (13) in Theorem 2.

**Proposition 1** The condition (13) is true if we can assign the poles of $(A_{ii} - B_{2i}K_{ii}) + (A_{ii} - B_{2i}K_{ii})^T$ anywhere by choosing appropriate $K_{ii}$.

*Proof.* There exists a local controller $K_{ii}$, such that

$$\lambda_{\text{max}}(\hat{A}_{ii}^T + \hat{A}_{ii}) < -\lambda_{\text{max}}\left( \sum_{j \in N_p^i} A_{ij}A_{ji}^T \right) - \sigma,$$  \hspace{1cm} (17)

where $\sigma_i$ is the out-degree of node $i$ in the plant graph $G^p$, and $\lambda_{\text{max}}(\cdot)$ denotes the maximum eigenvalue of a symmet-
ric matrix. Then, we have

\[ \dot{\hat{X}}_i + \hat{A}_{ii} + \left( \sum_{j \in \mathbb{N}_+^p} A_{ij}A_{ij}^T \right) + \sigma_i I < 0. \]

It means that \( P_i = I, W_{ij} = I \) satisfies (13).

**Remark 2** When the subsystem size is scalar, the condition in Proposition 1 is equivalent to the controllability of \((A_{ii}, B_{ii})\), which coincides with the result in Theorem 1. As shown in (17), the value of \( \alpha \) in (11) has close relationship to the plant graph \((i.e., \sigma_j)\) and the strength of dynamical interactions \((i.e., \sum_{j \in \mathbb{N}_+^p} A_{ij}A_{ij}^T)\). For general block subsystems, the condition in Proposition 1 is stronger than the controllability of \((A_{ii}, B_{ii})\).

**Proposition 2** Given the following isolated systems

\[
\begin{aligned}
\dot{x}_i &= A_{ii}x_i + B_{2i}u_i + \hat{B}_i w_i, \\
\dot{z}_i &= \sqrt{\sigma_i} x_i, \quad i \in \mathcal{V},
\end{aligned}
\]

where \( \hat{B}_i \) satisfies \( \hat{B}_i \hat{B}_i^T = \sum_{j \in \mathbb{N}_+^p} A_{ij}A_{ij}^T \), \( \sigma_i \) is the out-degree of node \( i \) in the plant graph \( \mathcal{G}^p \), \( w_i \) and \( z_i \) denote virtual signals. If there exists a local controller \( K_{ii} \), such that the closed-loop isolated system satisfies

\[
\int_0^\infty z_i(t)^T z_i(t)dt < \int_0^\infty w_i(t)^T w_i(t)dt,
\]

then condition (13) holds.

**Proof.** According to [44, Section 2.7], the closed-loop isolated system satisfies (19), if and only if there exists a positive-definite \( P_i > 0 \) such that

\[
\dot{\hat{X}}_i^T P_i + P_i \dot{\hat{X}}_i + P_i \hat{B}_i \hat{B}_i^T P_i + \sigma_i I < 0. \tag{20}
\]

This condition implies (13), since \( \hat{B}_i \hat{B}_i^T = \sum_{j \in \mathbb{N}_+^p} A_{ij}A_{ij}^T \), and \( W_{ij} = I \).

It is easy to see that the conditions in Propositions 1 and 2 can be tested locally within each subsystem using limited model information. Also, the non-expansive condition in Proposition 2 has an intuitive understanding: the signals are attenuated when propagated among different nodes in the closed-loop interconnected system.

**Remark 3** The sufficient condition in Proposition 2 is essentially an \( \mathcal{H}_\infty \) synthesis problem. It aims to find a local controller \( K_{ii} \) such that the closed-loop isolated system satisfies

\[
\|\sqrt{\sigma_i} [sI - (A_{ii} + B_{2i}K_{ii})]^{-1} \hat{B}_i]\|_{\mathcal{H}_\infty} < 1. \tag{21}
\]

According to [8], we can show that (21) is equivalent to the condition that the following LMI have a solution \( X_i > 0 \)

\[
\begin{bmatrix}
A_{ii}X_i + B_{2i}Z_i + (A_{ii}X_i + B_{2i}Z_i)^T \hat{B}_i & X_i \\
\hat{B}_i^T & -I & 0 \\
X_i & 0 & -\frac{1}{\sigma_i}I
\end{bmatrix} < 0,
\]

where \( K_{ii} = Z_iX_i^{-1} \) satisfies (21). Note that the test (22) is completely decoupled, and only requires local model information based on each subsystem.

**Remark 4** In (13), by setting \( W_{ij} = \gamma_{ij} I \), we can also obtain the following sufficient condition

\[
\dot{A}_{ii}^T P_i + P_i \dot{A}_{ii} + P_i \left( \sum_{j \in \mathbb{N}_+^p} \frac{1}{\gamma_{ij}} A_{ij}A_{ij}^T \right) P_i + \sum_{i \in \mathbb{N}_+^p} \gamma_{ij} I < 0, \tag{23}
\]

where \( \gamma_{ij} \) is a positive real number. Compared to (20), the inequality (23) offers some tunable parameters \( \gamma_{ij} \) to reduce the conservatism. As indicated in [54], one way to a priori choose \( \gamma_{ij} \) is based on the information of the gains of the interconnections, e.g., \( \gamma_{ii} = \sigma(A_{ii}) \), where \( \sigma(\cdot) \) denotes the maximal singular value of a matrix. Note that there are some other tests for block-diagonal stability based on the notion of scaled diagonal dominance [54].

### 4 Convex Design of Structured Controllers

This section presents our strategies to solve problems \( \mathcal{P}_1 - \mathcal{P}_3 \) using Lyapunov theory. The key idea is to use the property of sparsity invariance (8) to preserve the convexity of \( K \in \mathcal{K} \) for the Lyapunov variables. In this way, we implicitly assume that the closed-loop system can be designed to be block-diagonally stable.

#### 4.1 Structured Stabilization

According to the Lyapunov LMI (10), it is easy to see that problem \( \mathcal{P}_1 \) is equivalent to

\[
\begin{bmatrix}
(A - B_2K)^T P + P(A - B_2K) & < 0 \\
P > 0, K \in \mathcal{K}
\end{bmatrix}.
\]

The first inequality in (24) does not linearly depend on \( P \) and \( K \). A standard change of variables \( P = X^{-1} \), and \( Z = KX \) leads to

\[
\begin{bmatrix}
X A^T + AX - Z^T B_2^T - B_2Z & < 0 \\
X > 0, ZX^{-1} & \in \mathcal{K}
\end{bmatrix}.
\]

A new nonlinear constraint arises in (25) due to the structured constraint determined by the communication graph. Here, we assume \( X \) is block-diagonal, then we have sparsity invariance (8). Following this sequence of steps, we obtain a convex approximation of the structured stabilization \( \mathcal{P}_1 \)

\[
\begin{bmatrix}
X A^T + AX - Z^T B_2^T - B_2Z & < 0 \\
X > 0, X \text{ is block-diagonal} & \in \mathcal{K}
\end{bmatrix}.
\]

\[\text{(26)}\]
This is an LMI in terms of the Lyapunov variables \( Z \) and \( X \). If a solution \( (Z, X) \) is found, the resulting controller \( K = ZX^{-1} \) naturally satisfies the original structured constraint \( K \).

4.2 Structured Optimal Control

In this section, we apply the idea above to the structured optimal control problem \( P_2 \). Specifically, we consider \( H_2 \) performance, and the performance output is chosen as \( z = Cx + Du \), where \( C = \begin{bmatrix} Q^{1/2} & 0 \end{bmatrix}^T \) and \( D = \begin{bmatrix} 0 & R^{1/2} \end{bmatrix}^T \).

It is known that the \( H_2 \) norm can be calculated using an LMI [44].

Lemma 2 Consider a stable linear system \( \dot{x} = Ax + Bd, z = Cx. \) The \( H_2 \) norm of the transfer function from \( d \) to \( z \) can be computed by

\[
\|G_d\|^2 = \inf_{X > 0} \{\text{Tr}(CXC^T) \mid AX + XA^T + B^TB \leq 0\}.
\]

Next, we introduce two solutions to solve \( P_2 \), i.e., LMI approximation and gradient projection method. We note that the first method relies on the LMI in Lemma 2, which follows the similar idea of the solution to \( P_1 \); in contrast, the gradient projection method uses a slightly different idea, where an iterative algorithm is designed without using any LMIs.

4.2.1 LMI Approximation

According to Lemma 2, the structured \( H_2 \) synthesis \( P_2 \) can be equivalently reformulated as

\[
\begin{align*}
\min_{X,K} & \quad \text{Tr}((Q + K^TRK)X) \\
\text{s.t.} & \quad (A - B_2K)X + X(A - B_2K)^T + B_1B_1^T \leq 0, \\
& \quad X > 0, K \in K.
\end{align*}
\]

In (27), we have used the property of the trace operator \( \text{Tr}(MN) = \text{Tr}(NM), \forall M, N \) with compatible dimensions. Note that the stability of the closed-loop system is implied by the LMI constraint and positive definiteness of \( X \) in (27). Similarly, introducing a change of variables \( (T) \), and using the Schur complement, (27) is equivalent to

\[
\begin{align*}
\min_{X,Y,Z} & \quad \text{Tr}(QX) + \text{Tr}(RY) \\
\text{s.t.} & \quad (AX - B_2Z) + (AX - B_2Z)^T + B_1B_1^T \leq 0, \\
& \quad \begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0, \quad Z \in K, \\
& \quad X > 0, ZX^{-1} \in K.
\end{align*}
\]

Again, we use the property of sparsity invariance by assuming \( X \) is block-diagonal. Then, an LMI approximation to

Algorithm 1 Gradient projection method to solve \( P_2 \)

**Input:** System dynamics \( A, B_1, B_2 \); performance matrices \( Q, R \); and structured constraint \( K \).

**Output:** Structured optimal controller \( K \in K \).

Obtain an initial point \( K_0 \in K \) by solving LMI (26);

for \( i = 0, 1, \ldots, i_{\text{max}} \) do

1) Calculate \( X_i \) and \( P_i \) by solving Lyapunov equations (31) and (32) with \( K = K_i \);

2) Compute the gradient \( \nabla J(K_i) \) using (30) with \( X = X_i, P = P_i, K = K_i \);

3) Update \( K_{i+1} = K_i - s_i \cdot \hat{K}_i \), where \( \hat{K}_i \) is the projection of \( \nabla J(K_i) \) onto \( K \), and \( s_i \) is determined by the Armijo rule;

\[
\text{if } \|\hat{K}_i\|_F \leq \epsilon \text{ then} \break \end{\text{if}}
\]

end if

Armijo rule [55]: for the step-size \( s_i \);

Initialization \( s_i = 1 \);

repeat \( s_i = \beta s_i \), until

\[
J(K_i) - J(K_i - s_i \cdot \hat{K}_i) \geq \alpha \langle \nabla J(K_i), s_i \cdot \hat{K}_i \rangle
\]

where \( \alpha, \beta \in (0, 1) \).

If a solution \((X, Y, Z)\) is found, then the resulting controller \( K = ZX^{-1} \) has the sparsity pattern \( K \). It is not difficult to see that the above strategy is also applicable to \( P_2 \) with \( H_\infty \) norm.

4.2.2 Gradient Projection Method

The above LMI approximation offers a convex way to solve problem \( P_2 \) at the cost of a certain performance loss in general. Motivated by the augmented Lagrangian approach in [20], we can derive a numerical approach based on gradient projection to solve \( P_2 \). Here, our idea is to view \( P_2 \) as a special constrained nonlinear optimization problem, and apply the general gradient projection method to get a locally optimal solution [55]:

(1) Given an initial stabilizing feedback controller: \( K_0 \in K \);

(2) Update the gain in the direction of the projected gradient: \( K_{i+1} = K_i - s_i \cdot \hat{K}_i \).

Here, \( s_i \) is an appropriate step length, which can be chosen using the Armijo rule [55], and \( \hat{K}_i \) is a projected gradient of the cost function onto the convex set \( K \).

To obtain an initial stabilizing point \( K_0 \), we use the block-diagonal assumption and then solve the structured sta-
bilization $\mathcal{P}_1$ via LMI (26). As for the updating direction, using the standard techniques in [56], the gradient of $J(K) = \|G_{d2}(K)\|^2$ with respect to any stabilizing point can be expressed as

$$\nabla J(K) = 2(RK - B_2^TP)X,$$  \hspace{1cm} (30)

where $P$ and $X$ are observability and controllability grammans of the closed-loop system

$$(A - B_2K)X + X(A - B_2K)^T = -B_1B_1^T,$$  \hspace{1cm} (31)

$$(A - B_2K)^TP + P(A - B_2K) = -(Q + K^TRK).$$  \hspace{1cm} (32)

The solvability of $P$ and $X$ in (31) and (32) is implied by the stability of $A - B_2K$. Note that the observability and controllability grammans $P$ and $X$ are in general dense, which means the block-diagonal assumption is relaxed during each updating process for our gradient-projection method. Also, in the absence of structured constraints, by letting $\nabla J(K) = 2(RK - B_2^TP)L = 0$, we obtain the optimal unstructured feedback gain $K_c = R^{-1}B_2^TP$, where $P$ is the positive definite solution of the algebraic Riccati equation $A^TP + PA + Q - PB_2R^{-1}B_2P = 0$.

The projection of $\nabla J(K)$ onto the convex set $\mathcal{K}$ is straightforward and computationally cheap: we only need to set the elements of $\nabla J(K)$ outside $\mathcal{K}$ to zero. Then, we are ready to obtain a gradient projection method to solve $\mathcal{P}_2$, which is summarized in Algorithm 1.

It is easy to see that the feedback controller $K_1$ in each step of Algorithm 1 satisfies the structured constraint $\mathcal{K}$ and stabilizes the whole system as well. In contrast, the augmented Lagrangian approach in [20] only generates a stable structured feedback controller at the end of the algorithm. The Armijo rule in Algorithm 1 is essentially a method of successive step-size reduction, which facilitates the guarantee of theoretical convergence to a locally optimal solution [55]. The reduction factors $\alpha$ and $\beta$ are fixed, and the convergence to a locally optimal solution can be achieved under any $\alpha, \beta \in (0, 1)$. It is recommended in [55, Section 1.2] that $\alpha$ is usually chosen close to zero, e.g., $\alpha \in (10^{-5}, 10^{-1})$, and $\beta$ is usually chosen between $\frac{1}{10}$ and $\frac{1}{2}$.

**Remark 5** The calculation of the gradient using (30) is only valid for stabilizing $K$. Thus, it is important to use a stabilizing initial structured gain to iterate. We notice that the idea of gradient descent was applied to solve structured optimal control in [57], where simulated trajectories were used to calculate the gradient direction. However, the authors did not discuss how to obtain an initial stabilizing feedback gain. In this paper, we offer a solution by solving the non-trivial structured stabilization $\mathcal{P}_1$ using a block-diagonal assumption. Note that the iterative steps in Algorithm 1 do not need the block-diagonal assumption.

### 4.3 Communication Topology Design

For problems $\mathcal{P}_1$ and $\mathcal{P}_2$, the structured constraint $\mathcal{K}$ determined by the communication topology is a priori fixed and specified, which may impose certain limits on the achievable performance [2, 35, 36]. When the communication topology is flexible and not fixed, it is desirable to design a communication topology that strikes a trade-off between closed-loop performance and controller structure (see $\mathcal{P}_3$). In this section, we discuss the strategy to solve the problem of communication topology design $\mathcal{P}_3$. Similar to the proposed solutions to $\mathcal{P}_1$ and $\mathcal{P}_2$, one key step of our solution is the use of sparsity invariance under the block-diagonal assumption.

In the formulation of $\mathcal{P}_3$, $\|G_{d2}(K)\|^2$ denotes the square of the $\mathcal{H}_2$ norm of the closed-loop system, and $g(K)$ represents a sparsity-promoting function. The non-negative parameter $\gamma$ encodes the emphasis on the sparsity of $K$. As $\gamma$ increases, the resulting controller $K$ becomes sparser. If the value of $\gamma$ is equal to zero, $\mathcal{P}_3$ is reduced to a standard centralized $\mathcal{H}_2$ design. Typically, $g(K)$ is a cardinality function to count the non-zero blocks of $K$ [2],

$$g(K) = \sum_{i,j} \text{card}(\|K_{ij}\|_F),$$

which in turn penalizes the number of communication links in the system.

In principle, we can use the same strategy for $\mathcal{P}_2$ to deal with the term $\|G_{d2}(K)\|^2$ in $\mathcal{P}_3$. This directly leads to an equivalent formulation of $\mathcal{P}_3$, as shown in (33).

$$\min_{X,Y,Z} \text{Tr}(QX) + \text{Tr}(RY) + \gamma g(ZX^{-1})$$

subject to:

$$AX - B_2Z + (AX - B_2Z)^T + B_1B_1^T \preceq 0,$$

$$\begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,$$

$$X > 0.$$  \hspace{1cm} (33)

Similar to the cases of $\mathcal{P}_1$ and $\mathcal{P}_2$, we assume $X$ is block-diagonal, and use the property of sparsity invariance. Then, it is easy to check that the following equality holds.

$$g(K) = g(ZX^{-1}) = g(Z).$$  \hspace{1cm} (34)

This means the sparsity promotions of $K$ and $Z$ are equivalent under the block-diagonal assumption. Following this sequence of steps, we obtain an approximation of $\mathcal{P}_3$.

$$\min_{X,Y,Z} \text{Tr}(QX) + \text{Tr}(RY) + \gamma g(Z)$$

subject to:

$$AX - B_2Z + (AX - B_2Z)^T + B_1B_1^T \preceq 0,$$

$$\begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,$$

$$X > 0, X \text{ is block-diagonal.}$$  \hspace{1cm} (35)

Now, the difficulty of solving $\mathcal{P}_3$ is reduced to how to deal with the cardinality functions, since (35) typically requires an intractable combinatorial search. As suggested in [41], a weighted $l_1$ norm is a useful convex proxy for promoting sparsity

$$g_1(Z) = \sum_{i,j} (\omega_{ij} \|Z_{ij}\|_F),$$  \hspace{1cm} (38)
Algorithm 2 Communication topology design $\mathcal{P}_3$

**Input:** System dynamics $A, B_1, B_2$; performance matrices $Q, R$; and a regularization parameter $\gamma$.

**Output:** Sparsity pattern $\mathcal{K}$, and structured optimal controller $K \in \mathcal{K}$.

Initialize the weights $\omega_{ij}^0 = 1$;

for $k = 0, 1, \ldots, k_{\text{max}}$ do

1) Solve the following weighted $l_1$ optimization problem to obtain $X^k, Z^k$

$$\min_{X,Y,Z} \text{Tr}(QX) + \text{Tr}(RY) + \gamma \cdot \sum_{i,j} (\omega_{ij}^k \|Z_{ij}\|_F)$$

s.t. $$(AX - B_2Z) + (AX - B_2Z)^T + B_1B_1^T \preceq 0,$$
$$\begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,$$
$$X > 0, \text{ } X \text{ is block-diagonal.}$$

(36)

2) Update the weights

$$\omega_{ij}^{k+1} = \frac{1}{\|Z_{ij}^k\|_F + \tau},$$

(37)

where $\tau$ is a small positive number.

if Converged: $\forall |\omega_{ij}^{k+1} - \omega_{ij}^k| \leq \epsilon$ then

break;

end if

end for

Polishing step: Apply the LMI approximation or gradient projection method to find a structured optimal controller $K \in \mathcal{K}$.

where $\omega_{ij}$ are non-negative weights. This strategy has been applied by some recent work on topology design [2, 16]. In addition, if we choose $\omega_{ij}$ to be inversely proportional to the Frobenius norm of $Z_{ij}$, i.e.,

$$\omega_{ij} = \begin{cases} 1/\|Z_{ij}\|_F, & \text{if } Z_{ij} \neq 0, \\ \delta, & \text{otherwise} \end{cases},$$

(39)

where $\delta$ is a non-zero constant, then the weighted $l_1$ norm coincides with the cardinality function of $Z$

$$\sum_{i,j} (\omega_{ij} \|Z_{ij}\|_F) = \sum_{i,j} \text{card}(\|Z_{ij}\|_F).$$

One difficulty is that the weights depend on the unknown blocks $Z_{ij}$. This motivates the notion of reweighted $l_1$ norm, proposed in [41], which solves a sequence of weighted $l_1$ optimization problems.

Combining the ideas of block-diagonal assumption and reweighted $l_1$ norm, we are ready to propose Algorithm 2 to solve $\mathcal{P}_3$.

**Remark 6** Problem (36) in each iteration is a convex approximation to $\mathcal{P}_3$, which is readily solved using existing SDP solvers, e.g., SeDuMi [58]. The reweighted scheme is implemented in (37). Here, the small positive number $\tau$ ensures that the weights are well-defined when $Z_{ij}^k$ is a zero block. Also, it is possible to iteratively decrease the value of $\tau$. Note that the communication topology (or $\mathcal{K}$) is encoded in $Z$. The final step in Algorithm 2 is to compute the optimal controller by fixing the sparsity pattern $K \in \mathcal{K}$ that is identified by the reweighted $l_1$ scheme.

5 Numerical Illustrations

In this section, we present two illustrative examples to demonstrate the effectiveness of the proposed solutions. In particular, we consider a mesh network of unstable nodes to demonstrate $\mathcal{P}_1$ and $\mathcal{P}_2$, and an example of circular formation to illustrate $\mathcal{P}_3$. We ran the simulations in the MATLAB environment, and used YALMIP [59] and SeDuMi [58] to solve the LMIs.

5.1 $\mathcal{P}_1$ and $\mathcal{P}_2$: A Mesh Network of Unstable Nodes

Consider an $n \times n$ mesh network of nodes distributed in a square region. Motivated by [16], each node is assumed to be an unstable second-order system coupled with other nodes through a decaying function of the Euclidean distance $\alpha(i, j)$ between them,

$$\dot{x}_i = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x_i + \sum_{j \in \mathcal{N}_i} \epsilon^{-\alpha(i,j)} x_j + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (d_i + u_i).$$

(40)

We assume that both the plant and communication graphs have the structure shown in Fig. 3, which means the dynamics of each node are only coupled with its nearest neighbours and the nodes can only use the information of their nearest neighbours for feedback. Fig. 4 demonstrates the sparsity pattern of the structured controller of the system when $n = 10$. The control objective is to move each subsystem to a desired position in the pre-specified square region. It is assumed that each subsystem knows its desired position.

In the simulations, the state and control weights were chosen as $Q = I$ and $R = I$. We used the proposed structured stabilization (26), LMI approximation (29), and gradient projection method (see Algorithm 1) to synthesize the...
structured controllers. For comparison, the centralized $H_2$ controller was computed as well. For the mesh network with dimensions $n = 2, 4, 6, 8, 10$, the results of $H_2$ performance of the closed-loop system are summarized in Table 1. It is shown that the method of structured stabilization (26) can stabilize the unstable system using neighbouring information, but with bad $H_2$ performance. This performance can be improved using the LMI approximation (29) for structured optimal control. In addition, the gradient projection method (i.e., Algorithm 1) can generally find a locally optimal solution, which is very close to the unstructured centralized optimal controller (note that the centralized controller requires all-to-all communication). This means additional communication links only offer limited performance improvement. The important communication links can be identified using the proposed communication design method, as shown in the next section.

5.2 $P_3$: Circular Formation of Unstable Nodes

Our next example is a formation of unstable nodes in a circle (see Fig. 5). The dynamics of each node is given by (40). The control objective is to keep constant distances between neighbouring nodes along the pre-specified circle.

![Figure 4. Sparsity pattern of the structured controller of the system in Fig. 3.](image)

![Figure 5. Circular formation of unstable nodes, where the dynamics of each node is given by (40) and the plant graph is assumed to be complete.](image)

![Figure 6. Design of communication topology $G^c = (\mathcal{V}, \mathcal{E}^c)$ with different regularization parameter $\gamma$ for the circular formation.](image)

![Figure 7. Tradeoff between performance loss and communication density for the circular formation. The arrow point shows that the $H_2$ performance of the nearest communication is only around 15\% worse than the performance of the all-to-all communication.](image)
systems. Similar communication patterns were identified using the ADMM method in [16].

6 Conclusion

This paper addressed the three tasks \( \mathcal{P}_1 - \mathcal{P}_3 \) of structured controller synthesis. We have shown that the block-diagonal assumption naturally guarantees the property of sparsity invariance, resulting in convexity preservation in the Lyapunov parameterization. This idea offers a simple and unified method to deal with the structural constraints \( \mathcal{K} \) in the design of structured controllers. Two sufficient conditions were introduced under which the closed-loop system admits a block-diagonal Lyapunov function. Note that the conditions can be tested locally within each subsystems. Using the block-diagonal assumption, we have proposed 1) the LMI approximation (26) to solve \( \mathcal{P}_1 \), 2) another LMI (29) and the gradient projection method (Algorithm 1) to address \( \mathcal{P}_2 \), and 3) the iterative Algorithm 2 to solve \( \mathcal{P}_3 \). Future work will address the conservatism of using block-diagonal Lyapunov functions. In addition, another interesting direction is to develop efficient decomposition methods to exploit sparsity in the systems.

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