Double Copy of Form Factors and Higgs Amplitudes:  
An Example of Turning Spurious Poles in Yang-Mills into Physical Poles in Gravity

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We extend the double copy picture of scattering amplitudes to a class of matrix elements (so-called form factors) that involve local gauge invariant operators. Both the Bern, Carrasco and Johansson (BCJ) and the Kawai, Lewellen and Tye (KLT) formalisms are considered and novel properties are observed. One remarkable feature is that through the double-copy construction, certain spurious poles hidden in the gauge form factors become physical propagators in gravity. This mechanism also reveals new hidden relations for form factors which can be understood as a generalization of the BCJ relations. The same double-copy prescription applies as well to tree-level QCD amplitudes involving a color-singlet Higgs particle. The double copy of form factors suggests a possible new class of observables in gravity and string theory.

INTRODUCTION

Despite the very different nature, gauge and gravity theories are known to be intimately related. The celebrated AdS/CFT correspondence shows that a gravity theory in the AdS space can be equivalent to a gauge theory living on the AdS boundary. Moreover, the perturbative amplitudes in gauge and gravity theories are also closely linked via the double copy as “gravity = (gauge theory)$^n$, realized in various formalisms including the Kawai, Lewellen and Tye (KLT) relations, the Bern, Carrasco and Johansson (BCJ) double copy stemming from the color-kinematics (CK) duality, and the Cachazo, He and Yuan (CHY) formula. An excellent review about the double copy can be found in.

Apart from scattering amplitudes, which involve on-shell asymptotic states only, gauge invariant local operators also play important roles in gauge theories and it is natural to ask: does a consistent double copy picture exist for physical quantities involving local operators? Nevertheless, the answer is not obvious at all, since for example, local operators in gravity would break the diffeomorphism invariance.

In this paper, we make a concrete step towards addressing this question, by realizing both BCJ and KLT double copy for the form factors. Form factors are defined as matrix elements between a gauge invariant operator $O$ and $n$ on-shell states (see for a recent introduction and review),

$$\mathcal{F}_{\mathcal{O},n} = \int d^D x e^{-iq \cdot x} (12...n|\mathcal{O}(x)|0),$$

where $q = \sum_{i=1}^n p_i$ is the off-shell momentum associated with the operator. We find that in realizing the double copy, the inclusion of gauge invariant local operators indeed leads to intriguing new features.

One novel feature is that special spurious poles appear in the construction of CK-dual numerators in gauge-theory form factors, and after double copy they become new physical propagators in the gravity quantities, i.e.

$$\text{spurious} \xrightarrow{\text{double-copy}} \text{physical propagators}.$$ 

Besides, the factorization on the new propagators in gravity implies that the gauge-theory form factors satisfy hidden relations when evaluated on the spurious poles, which can be schematically shown as

$$\vec{v} \cdot \overline{\mathcal{F}}_n |_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m},$$

and may be understood as a generalization of BCJ relations for form factors.

Below we explain these properties in detail with examples of tree-level form factors with $\mathcal{O} = \text{tr}(\phi^2)$ in the scalar-Yang-Mills theory. Similar discussions also apply to form factors of other operators, such as $\bar{\psi} \psi$ in QCD, of which the form factors are equivalent to a class of Higgs plus quarks and gluons amplitudes. This provides for the first time a double copy for amplitudes involving a color singlet particle. We will discuss more on this in the last section.

INVITATION: A THREE-POINT EXAMPLE

Most new features of the form-factor double copy can be illustrated by considering a simple example: the three-point tree-level form factor $F_{\text{tr}(\phi^2),3}(1^\phi, 2^\phi, 3^\phi)$. In this example, there are two cubic Feynman diagrams $\Gamma_{a,b}$ as given in Figure and the full-color form factor can be written as

$$F_3(1^\phi, 2^\phi, 3^\phi) = \frac{C_a N_a(\varepsilon_3, \{p_i\})}{s_{13}} + \frac{C_b N_b(\varepsilon_3, \{p_i\})}{s_{23}},$$

where $s_{13} = s_{23} = s$.
where $C_{a,b}$ are color factors, and $N_{a,b}$ are kinematic numerators depending on momenta $p_i$, $i = 1, 2, 3$ and the gluon polarization vector $\varepsilon_3$. We emphasize that the operator only couples to the scalar lines and $C_a = C_b = f^{123}$ since the color factor of the operator is a $\delta$-function in color space.

To obtain the double copy, one can apply the standard operation by squaring the kinematic numerators and propose the following quantity in gravity:

$$G_3(1^\phi, 2^\phi, 3^b) = \frac{N_a^2(\varepsilon_3, \{p_1\})}{s_{13}} + \frac{N_b^2(\varepsilon_3, \{p_1\})}{s_{23}}. \tag{4}$$

If $G_3$ is a well-defined quantity in gravity, it must be invariant under the diffeomorphism transformation, which acts on the graviton polarization tensor as $\varepsilon_3^{\mu} \rightarrow \varepsilon_3^{\mu} + p_3^{(\mu} \varepsilon_3^{\nu)}$. Here $\xi$ is a reference vector satisfying $\xi \cdot p_3 = 0$ and $\varepsilon_3^{\mu} = \varepsilon_3^{(\mu} \varepsilon_3^{\nu)}$ with the brackets indicating the symmetric-traceless part. However, if naively plugging in the numerators from Feynman rules for $N_{a,b}$, one easily finds that the diffeomorphism invariance is broken.

The key condition to restore the invariance is the color-kinematics duality, as in the amplitude cases \cite{3, 4}. Given the aforementioned color relation $C_a = C_b$, one can require the numerators to satisfy a parallel relation and get the solution

$$N_a = N_b = \frac{s_{13}s_{23}}{s_{13} + s_{23}} F_3(1^\phi, 3^g, 2^\phi), \tag{5}$$

where $F_3$ is the color-ordered three-point form factor. Note that the CK-dual numerators are uniquely determined and manifestly gauge invariant. With this solution, the $G_3$ proposed in \cite{6} is given as

$$G_3 = \frac{s_{13}s_{23}}{s_{13} + s_{23}} (F_3(1^\phi, 3^g, 2^\phi))^2, \tag{6}$$

which is indeed diffeomorphism invariant. Also, to get \cite{9} it is enough to require only one copy of the numerators in \cite{4} to satisfy \cite{5}, similar to the amplitude case \cite{14}.

Having diffeomorphism invariance of $G_3$ is not without any price though. In particular, the numerators \cite{5} contain a spurious pole $s_{13} + s_{23}$. In the gauge form factor, this pole is cancelled by adding two terms in \cite{3}; however, after double copy it becomes a real pole in \cite{6}. Does this pole have a physical meaning in gravity?

Remarkably, the pole $s_{13} + s_{23} = -(s_{12} - q^2)$ looks like a massive Feynman propagator, and the residue of $G_3$ on this “spurious” pole can be nicely organized as

$$\text{Res}_{s_{12} = q^2} [G_3] = (F_2(1^\phi, 2^\phi))^2 \times (A_3(q_2^S, 3^g, -q^S))^2, \tag{7}$$

where $F_2(1^\phi, 2^\phi) = 1$ is the minimal form factor, and

$$A_3(q_2^S, 3^g, -q^S) = \varepsilon_3 \cdot q, \quad \text{with} \quad q_2 = p_1 + p_2, \tag{8}$$

is the three-point planar amplitude of a gluon and one pair of massive scalar particle $S$ with mass $m^2 = q^2 = q_2^2$, see e.g. \cite{13}. In this way, \cite{9} can be interpreted as a factorization formula

$$\text{Res}_{s_{12} = q^2} [G_3] = G_2(1^\phi, 2^\phi) \cdot M_3(q_2^S, -q^S, 3^b), \tag{9}$$

where $G_2 = (F_2)^2$ is the double copy of the minimal form factor, and $M_3(q_2^S, -q^S, 3^b) = (A_3(q_2^S, 3^g, -q^S))^2$ is the three-point amplitude of a graviton coupled to two massive scalars (such amplitudes appear extensively in the gravitational wave studies via double copy, see e.g. \cite{13}).

The factorization property \cite{9} implies that the spurious pole $s_{13} + s_{23}$ in gauge theory should be understood as a physical pole in gravity! Indeed, it represents the factorization of the Feynman diagram $\Gamma_3$ in Figure 2. This new diagram naturally appears since graviton couples to everything including the “operator” leg. As further consistency checks, \cite{9} also gives correct factorization on the $s_{13}$ and $s_{23}$ pole, which are associated with diagrams $\Gamma_a$ and $\Gamma_b$ respectively. Finally, $G_3$ matches the expression from the Feynman diagrams of Figure 2 in gravity.

An additional nice property to emphasize is that by taking the “square-root” of the double copy factorization \cite{7}, one can get an intriguing relation for the gauge-theory form factor:

$$s_{13} F_3(1^\phi, 3^g, 2^\phi) \big|_{s_{12} = q^2} = F_2(1^\phi, 2^\phi) \cdot A_3(q_2^S, 3^g, -q^S). \tag{10}$$

We can summarize the main features in the three-point example: (1) diffeomorphism invariance requires that the numerators satisfy complete CK-dual relations; (2) such numerators contain spurious poles which can be interpreted as physical poles in gravity, on which the double-copy result factorizes as a product of double copies of
the lower-point form factor and amplitude; (3) the gravity factorization implies an interesting relation for the gauge form factor. While the three-point example seems too simple, we show below that the same features also hold for non-trivial higher-point cases.

THE GENERAL KLT DOUBLE COPY

To generalize the above discussion to higher points, we develop a systematic way to obtain the CK-dual numerators and the KLT-type double copy. We consider the four-point form factor $F_4(1^{\phi}, 2^{\phi}, 3^{\sigma}, 4^{\sigma})$ as an explicit example, and the generalization to $n$-point is straightforward.

As in the three-point case, we first express the four-point form factor in terms of eight cubic diagrams shown in Figure 3.

\[ F_4(1^{\phi}, 2^{\phi}, 3^{\sigma}, 4^{\sigma}) = \sum_{\Gamma} C_\Gamma N_{\Gamma} D(\Gamma), \]  \hspace{1cm} (11)

where $D(\Gamma)$ is the product of propagators of $\Gamma$. Since the color factor of the operator is a $\delta$-function, the eight $C_\Gamma$ takes only three different values, and we can classify the diagrams into three groups accordingly. For instance, the first three graphs $\Gamma_1, i = 1, 2, 3$ share the same color factor $C_s = f^{\alpha_i \alpha_j} f^{\beta_0 \alpha_{12}}$; and $\Gamma_{t1}$ and $\Gamma_{u1}$ have color factors $C_t$ and $C_u$, respectively. The three color factors satisfy the Jacobi relation $C_s = C_t + C_u$ like the four-point amplitude.

For the purpose of double copy, we consider the CK duality and require the numerators to satisfy

\[ N_{s1} = N_{s2} = N_{s3} = N_s, \quad N_{t1} = N_{t2} = N_t, \]
\[ N_{u1} = N_{u2} = N_{u3} = N_u, \quad N_s = N_t + N_u, \]  \hspace{1cm} (12)

which lead to an expression of the form factor as

\[ F_4 = \frac{C_s N_s}{P_s} + \frac{C_t N_t}{P_t} + \frac{C_u N_u}{P_u}, \]  \hspace{1cm} (13)

with $P_s^{-1} = \sum_{i=1}^3 (D(\Gamma_{s_i}))^{-1}$, and $P_{t,u}$ are defined likewise. Alternatively, since $C_s, C_u$ form a color basis (the DDM color decomposition [17]), $F_4$ can be expanded using color-ordered form factors $F_4$ as

\[ F_4 = C_s F_4(1, 3, 4, 2) + C_u F_4(1, 4, 3, 2), \]  \hspace{1cm} (14)

which can be easily generalized to the $n$-point case as

\[ F_n = \sum_{\sigma \in S_{n-2}} C_{n,\{\sigma\}} F_n(1^{\phi}, \sigma \{3^{\sigma}, \ldots, n^{\sigma}\}, 2^{\phi}), \]  \hspace{1cm} (15)

with $C_{n,\{\sigma\}}$ the $n$-point DDM basis.

Matching [14] with [13], one finds the following relation:

\[ F_n^4 = \Theta_n^F \cdot \bar{N}_4, \quad F_n^4 = \left( F_n(1, 3, 4, 2) \right), \quad \bar{N}_4 = \begin{pmatrix} N_s \\ N_u \\ N_t \end{pmatrix}, \]  \hspace{1cm} (16)

where $\Theta_n^F$ is a matrix of propagators as

\[ \Theta_n^F = \begin{pmatrix} \frac{1}{P_s} + \frac{1}{P_t} & \frac{1}{P_u} \\ -\frac{1}{P_t} & \frac{1}{P_u} \\ -\frac{1}{P_u} & \frac{1}{P_u} \end{pmatrix}. \]  \hspace{1cm} (17)

One can check that $\Theta_n^F$ has full rank [18] and thus by simply inverting [17] one obtains the CK-dual numerators as

\[ N_4[\alpha] = \sum_{\beta \in S_2} S_n^F[\alpha | \beta] F_4[\beta], \quad S_n^F \equiv (\Theta_n^F)^{-1}, \]  \hspace{1cm} (18)

where $\alpha, \beta$ label the vector/matrix components in [16]. We point out that the CK-dual numerators contain spurious poles (such as [22]) which are introduced by $S_n^F$, and we will discuss more on this shortly.

Given the CK-dual numerators [13], one can perform the double copy to get the gravitational quantity as [19]

\[ \mathcal{G}_n = \sum_{\alpha, \beta \in S_{n-2}} \mathcal{F}_n[\alpha] S_n^F[\alpha | \beta] \mathcal{F}_n[\beta], \]  \hspace{1cm} (19)

which is manifestly diffeomorphism invariant. Such a bilinear form is very similar to the KLT form for amplitudes, and $S_n^F$ serves as a (four-point) KLT kernel.

Clearly, [19] can be easily generalized to higher points such that the double copy in the KLT form is

\[ \mathcal{G}_n = \sum_{\alpha, \beta \in S_{n-2}} \mathcal{F}_n[\alpha] S_n^F[\alpha | \beta] \mathcal{F}_n[\beta], \]  \hspace{1cm} (20)

where $S_n^F \equiv (\Theta_n^F)^{-1}$ is the $n$-point KLT kernel determined by the propagator matrix $\Theta_n^F$ and $\mathcal{F}_n[\alpha]$ are selected as in the aforementioned DDM basis [17].

We have verified that the double copy construction is consistent with physical requirements by considering all factorization channels. The factorization on “physical” poles (as in the gauge form factors) is similar to the known amplitude cases, and the main concern here is about the new “spurious”-type poles. Below we show that these poles become physical propagators on which $\mathcal{G}_n$ has nice factorization properties, as in the three-point case.
\( \Theta^F \)-Matrix and New Factorization

The “spurious” poles in the CK-dual numerators originate from the propagator matrices \( \Theta^F \), which have surprisingly nice properties. For the above four-point example, one finds the determinant takes a very simple form as

\[
\det(\Theta^F) = \frac{(q^2 - s_{12})(q^2 - s_{123})(q^2 - s_{1234})}{s_{134}s_{1234}s_{234}}. \tag{21}
\]

The highly non-trivial fact is that the numerators of \( \det(\Theta^F) \) is a product of terms such as \( s_{123} - q^2 \). Similar structures extend to higher-point cases, where the matrix \( \Theta^F \) can be rather complicated but its determinant remains a simple form like \( \Theta^F \).

The zeros of \( \det(\Theta^F) \) provide the information of poles of the KLT kernel \( S^F \). The crucial and striking point is that \( S^F \) has only simple poles \( \{s_{12} - q^2, s_{123} - q^2, s_{124} - q^2\} \) like massive Feynman propagators \( \llbracket \). For higher \( n \)-point cases, we have explicitly checked that \( S^F \) also have only simple poles like \( s_{(123\ldots n-2)} - q^2 \) up to highly non-trivial seven points. We stress that these beautiful structures rely on the fact that the local operator couples only to the matter fields and should not be inserted on the gluon lines.

The kernel \( S^F \) gives directly the pole structure of \( N_4 \), for example, \( N_4 \) in \( \llbracket \) can be given as

\[
N_4 = \frac{4}{(s_{12} - q^2)(s_{124} - q^2)} (s_{12} - q^2)
+ (1, 3 \leftrightarrow 2, 4), \tag{22}
\]

where \( f^\mu_\nu \equiv \varepsilon^\mu_\nu p^\nu \varepsilon^\nu_\mu \). Importantly, there are only spurious-type simple poles.

It is necessary to check such new poles after double copy are well-defined physical poles in \( G_n \). For \( G_4 \), the factorizations on the three new poles are presented in Figure 4 for example, on the pole \( s_{123} - q^2 \) it can be written as

\[
\text{Res}[G_4]_{s_{123}=q^2} = G_3(1, 2, 3): M_3(q_3^S, -q^S, 4^b), \tag{23}
\]

where \( q_3 = p_1 + p_2 + p_3 \). As in the three-point case, these factorization relations show that new Feynman diagrams (with massive propagators) contribute.

For the generic \( n \)-point case, we have

\[
\text{Res}[G_n]_{q_n^2=q^2} = G_n(1, \ldots, m): M_n(m^S, -q^S, m + 1, \ldots, n), \tag{24}
\]

where \( q_n = \sum_{i=1}^m p_i \), \( m' = n + 2 - m \) and \( M_m \) is a \( m' \)-point amplitude of gravitons couple to a pair of massive scalars. We have also checked this up to seven points.

Hidden Relations in Gauge Theory

The factorization relations \( \llbracket \) imply new relations for gauge form factors, which can be understood as generalizations of BCJ relations. Consider again the four-point example and \( \llbracket \) implies the relation

\[
(\vec{v}_4 \cdot \vec{F}_4)_{s_{123}=q^2} = F_3(1^g, 3^g, 2^g) \cdot A_3(q_3^S, 4^g, -q^S), \tag{25}
\]

where the (row) vector \( \vec{v}_4 \) and (column) vector \( \vec{F}_4 \) are

\[
\vec{v}_4 = (\tau_{42}, \tau_{42} + \tau_{43}) \quad \vec{F}_4 = \left( F_4(1, 3, 4, 2), F_4(1, 3, 4, 2) \right), \tag{26}
\]

with \( \tau_{ij} = 2p_i \cdot p_j \). One may notice that this is reminiscent of the BCJ relation for four-point amplitudes \( \llbracket \):

\[
\tau_{42} A_4(1, 3, 4, 2) + \tau_{42} + \tau_{43} A_4(1, 4, 3, 2) = 0. \tag{27}
\]

Here the RHS of \( \llbracket \) is not zero; instead, it offers a relation involving form factors with different number of external legs and scalar-Yang-Mills amplitudes.

Similar relations exist for higher-point cases taking the schematic form as \( \llbracket \). Such a general \( n \)-point relation is

\[
\sum_{i=3}^n \tau_{n,(2+i+\ldots+(n-1))} F_n(1, 3, \ldots, i - 1, n, i, \ldots, n - 1, 2)_{q_{n-1}=q^2} = F_{n-1}(1, 3, \ldots, n - 1, 2) A_3(q_{n-1}, n, -q), \tag{28}
\]

where \( q_{n-1} = \sum_{i=1}^{n-1} p_i \) and \( \tau_{n,(j+\ldots+k)} = 2p_n \cdot (p_j + \ldots + p_k) \).

A proof for this relation for MHV form factors, as well as complete relations for four and five-point form factors are given in the appendix.

The bridge between the gauge theory relation \( \llbracket \) and the gravity factorization \( \llbracket \) is a decomposition relation of \( S^F \), in which \( \vec{v}_4 \) also plays a central role. Concretely, for the 2 \times 2 matrix \( S^F \), we have

\[
\text{Res}[S^F]_{s_{123}=q^2} = \vec{v}_4^T \cdot (S^F \otimes S^A) \cdot \vec{v}_4, \tag{29}
\]

where \( S^F \) is \( s_{1323}^3 + s_{1323}^3 \) in \( \llbracket \) and \( S^A \) is the KLT kernel for three-point amplitudes, which equals to 1. For the generic \( n \)-point case, one has the similar decomposition as

\[
\text{Res}[S^F]_{q_{n-1}=q^2} = V_n^T \cdot (S^F \otimes S^A) \cdot V_n, \tag{30}
\]

where \( m' = n + 2 - m \) and \( V_n \) is a matrix as a collection of BCJ-like vectors similar to \( \vec{v}_4 \).

The decomposition for \( S^F \) provides a connection between gauge \( \llbracket \) and gravity \( \llbracket \) factorization relations.
in the sense that
\[ \text{Res} \left[ g_n \right]_{q^2 = q^2} = \mathcal{F}_n \cdot \text{Res} \left[ S_n \right] \cdot \mathcal{F}_n \mid_{q^2 = q^2} \] (31)
\[ = \left( \mathcal{F}_m \mathcal{M}_m \right) \cdot \left( S_n^F \otimes S_m^A \right) \cdot \left( \mathcal{F}_m \mathcal{M}_m \right) \\
= \left( \mathcal{F}_m \cdot S_n^F \cdot \mathcal{F}_m \right) (\mathcal{M}_m \cdot S_m^A \cdot \mathcal{M}_m) = \mathcal{G}_m \mathcal{M}_m. \]
We explain these relations in more detail in the appendix.

GENERALIZATION AND OUTLOOK

The structure of propagator matrix \( \Theta^F \) is of fundamental importance in the above discussions. The double-copy and factorization properties are expected to be true for form factors, of different operators and of different theories, with the same \( \Theta^F \)-matrix. One application of the above double copy is to the tree-level Higgs amplitudes \( A(q^1, 1 \psi, 2 \psi, 3 \psi, \ldots, n \psi) \) in the standard model, which is equivalent to the form factor of a bilinear quark operator \( \bar{q} q \) in the Yukawa coupling vertex with \( q^2 = m^2_H \). The corresponding double-copy quantities are the amplitudes of Higgs plus two photons and arbitrary number of gravitons (with an interaction vertex \( H F_{\mu \nu} F^{\mu \nu} \) in the theory), and explicit checks have been performed up to five points.

Some remarks are in order here. First, the Higgs particle considered here is a color singlet in QCD amplitudes, which is very different in nature from other considerations such as in [21, 22]. Besides, the spacetime dimensions, the mass and color-representation of the matter fields in the operator will not affect the validity of the previous discussions. Also, other propagator matrices, including those corresponding to \( \text{tr}(\phi^3) \) form factors, also share desirable properties so that the double copy is possible. These will be discussed in detail elsewhere [23].

Another important question is whether the above tree-level picture can be generalized to quantum loop level. This is certainly expected from the unitarity method [24–29]. We find that with the new tree-level prescription, the double-copy picture can be extended to loop form factors, which requires modifications of the previous loop constructions via CK-duality [27, 31]. We leave details of the loop generalization to another paper [32]. Here we highlight that one intriguing feature is that the massive propagators are not allowed in the internal loops after double-copy. This seems to indicate that the double copy of the loop form factors involves a semi-classical non-local operator in gravity, which can radiate gravitons (as in the tree-level discussion) but receive no loop correction for the operator itself. It would be interesting to explore such non-local observables in gravity and string theory.

There are many open problems associated with the observations made in this paper and here we mention a few. (i) It is important to understand and give general proofs of the properties of the \( \Theta^F \)-matrix, such as the simplicity of its determinant. Our finding suggests that a more general class of KLT matrices containing spurious poles can be physically meaningful, and it would be interesting to explore this using the KLT bootstrap method [33]. (ii) The propagator matrix \( \Theta^F \) can be regarded as the form factors in bi-adjoint scalar (BAS) theory [34, 35], where many Feynman diagrams have the same propagator structures and trivial numerators [36]. Denoting the BAS form factors as \( m^F_n \equiv \Theta^F_n \), one has by definition a double-copy relation as
\[
m^F_n[\alpha|\beta] = \sum_{\gamma,\delta \in S_{n-2}} m^F_n[\alpha|\gamma] S^F_n[\gamma|\delta] m^F_n[\delta|\beta], \] (32)
and by replacing one or both \( m^F_n \) on the RHS to be \( \mathcal{F}_n \), one obtains the KLT form for \( \mathcal{F}_n \) and \( g_n \) respectively.

Furthermore, the double copy relations in this paper are valid in \( D \)-dimension. These evidences imply that there should be a CHY formalism [7, 8], as a generalization of the four-dimensional connected description of form factors [38, 39]. (iii) As in the original KLT prescription [4], it would be very interesting to have a string theory generalization for the KLT formula (29) and also find a Z-function [10, 41] that encodes the \( \alpha' \)-expansion for form factors. (iv) Finally, to have a better understanding and a proof for the BCJ-like relations [2] of form factors, using field theory [42] or string theory methods [43, 44], is very important. In particular, a string theory prescription for the spurious pole factorizations should be fascinating.

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As in the three-point example, here only one copy of the trivalent diagrams in Figure 3.

We also remark that none of the propagators (physical poles) such as $s_{12}$ appears in the denominators of matrix elements of $S^c$, so that the master numerators also have no such poles. This is similar to the amplitude case where the master numerators can be select to be local, which further explains the physical pole factorization of the double copy.

In the BAS form factor, in order to preserve the propagator matrix, two external on-shell legs are distinguished from others in the sense that only they have interaction with the operator leg. This structure can be understood as replacing the gluons with bi-adjoint scalars in the above gauge-theory form factors.

We also remark that none of the propagators (physical poles) such as $s_{12}$ appears in the denominators of matrix elements of $S^c$, so that the master numerators also have no such poles. This is similar to the amplitude case where the master numerators can be select to be local, which further explains the physical pole factorization of the double copy.
Supplemental material

I. New factorization relations for four- and five-point form factors

In this appendix, we give complete generalized BCJ vectors for four- and five-point form factors and the related identities. We use the notation \( q_m = \sum_{i=1}^{m} p_i \), \( \tau_{ij} = 2p_i \cdot p_j \) and \( \tau_{n,(j+...+k)} = 2p_n \cdot (p_j + ... + p_k) \).

The four-point case

The four-point case has two types of factorization relations, corresponding to (A) two-particle \( s_{12} = q^2 \) and (B) three-particle \( s_{123} = q^2 \) spurious poles (the result for the \( s_{124} = q^2 \) pole can be obtained by a trivial relabeling), respectively. Introducing the form factor basis \( \vec{F}_4 \) as a (column) vector

\[
\vec{F}_4 = \{ F_4(1,3,4,2), F_4(1,4,3,2) \},
\]

we have

\[
\begin{align*}
(v_B^4 \cdot \vec{F}_4)_{s_{12}=q^2} &= F_2(1^\phi, 2^\phi) A_4(q_1^S, 3^g, 4^g, -q^S), \\
(v_B^4 \cdot \vec{F}_4)_{s_{123}=q^2} &= F_3(1^\phi, 3^g, 2^g) A_3(q_3^S, 4^g, -q^S),
\end{align*}
\]

where

\[
\begin{align*}
v_A^4 &= \left\{ \frac{\tau_{31} \tau_{42}}{\tau_{3q_2}}, \frac{\tau_{32} \tau_{41}}{\tau_{3q_2}} \right\}, \\
v_B^4 &= \left\{ \tau_{42}, \tau_{42} + \tau_{43} \right\},
\end{align*}
\]

with \( \tau_{ij} = 2p_i \cdot p_j \). The vectors \( v \) also appear in the decomposition of the KLT kernel as

\[
\begin{align*}
\text{Res } [S_4^T]_{s_{12}=q^2} &= (v_A^4)^T (S_2^T \otimes S_4^T) v_A^4 |_{s_{12}=q^2}, \\
\text{Res } [S_4^T]_{s_{123}=q^2} &= (v_B^4)^T (S_4^T \otimes S_4^T) v_B^4 |_{s_{123}=q^2}.
\end{align*}
\]

Besides, we also give explicit form of the four-point propagator matrix \( \Theta_4^F \) and the KLT kernel \( S_4^F \):

\[
\begin{align*}
\Theta_4^F &= \left( \begin{array}{cccc}
\frac{1}{s_{13}s_{24}} + \frac{1}{s_{14}s_{32}} & -\frac{1}{s_{14}s_{23}} & \frac{1}{s_{12}s_{34}} & -\frac{1}{s_{13}s_{24}} \\
-\frac{1}{s_{14}s_{23}} & \frac{1}{s_{13}s_{24}} & \frac{1}{s_{13}s_{24}} & -\frac{1}{s_{14}s_{23}} \\
\frac{1}{s_{12}s_{34}} & \frac{1}{s_{13}s_{24}} & \frac{1}{s_{13}s_{24}} & -\frac{1}{s_{12}s_{34}} \\
-\frac{1}{s_{13}s_{24}} & -\frac{1}{s_{14}s_{23}} & -\frac{1}{s_{12}s_{34}} & \frac{1}{s_{13}s_{24}} 
\end{array} \right), \\
S_4^F &= \left( \begin{array}{cccc}
\frac{1}{s_{13}s_{24}(s_{13}q_{23} + s_{14}q_{12} + s_{12}q_{24})} - \Delta_t & \Delta_t & \Delta_t & \Delta_t \\
\frac{1}{s_{12}s_{34}(s_{12}q_{24} + s_{13}q_{12} + s_{14}q_{23})} - \Delta_t & \frac{1}{s_{12}s_{34}(s_{12}q_{24} + s_{13}q_{12} + s_{14}q_{23})} - \Delta_t & \frac{1}{s_{12}s_{34}(s_{12}q_{24} + s_{13}q_{12} + s_{14}q_{23})} - \Delta_t & \frac{1}{s_{12}s_{34}(s_{12}q_{24} + s_{13}q_{12} + s_{14}q_{23})} - \Delta_t 
\end{array} \right).
\end{align*}
\]

where \( \Delta_t = \frac{s_{13}s_{14}s_{23}s_{24}}{s_{12} - q^2} \left( \frac{1}{s_{12} - q^2} + \frac{1}{s_{12} - q^2} \right) \).

The five-point case

The five-point case has three types of generalized BCJ vectors, corresponding to the (A) \( s_{12} = q^2 \), (B) \( s_{123} = q^2 \) and (C) \( s_{1234} = q^2 \) poles respectively. We introduce the form factor basis \( \vec{F}_5 \) as the following vector

\[
\vec{F}_5 = \{ F_5(1,3,4,5,2), F_5(1,3,5,4,2), F_5(1,4,3,5,2), F_5(1,4,5,3,2), F_5(1,5,3,4,2), F_5(1,5,4,3,2) \}.
\]

(A) For the factorization on the \( s_{12} = q^2 \) pole, one has

\[
(v_A^5 \cdot \vec{F}_5) |_{s_{12}=q^2} = F_2(1^\phi, 2^\phi) A_5(q_2^S, 3^g, \sigma\{4^g, 5^g\}, -q^S),
\]

where \( \sigma \in \{1, \sigma_2 \} = S_2 \) and the two corresponding vectors are

\[
\begin{align*}
v_{A,1}^5 &= \left( \frac{\tau_{31} \tau_{52}}{\tau_{3q_2}}, \frac{\tau_{41}(1+\tau_{42})}{\tau_{3q_2}} \right), \\
v_{A,2}^5 &= \left( \frac{\tau_{31} \tau_{52} \tau_{41}(1+\tau_{42})}{\tau_{3q_2}}, \frac{\tau_{41}(1+\tau_{42})}{\tau_{3q_2}} \right),
\end{align*}
\]

(41)
One can combine the two vectors to form a $2 \times 6$ matrix:

$$
V_5^A = \begin{pmatrix}
\vec{v}_5^{A,1} \\
\vec{v}_5^{A,2}
\end{pmatrix},
$$

and it appears in the decomposition of the KLT kernel as

$$
\text{Res} \left[ S_5^F \right]_{s_{12} = q^2} = (V_5^A)^T \cdot (S_2^F \otimes S_5^A) \cdot V_5^A \bigg|_{s_{12} = q^2}.
$$

(B) For the factorization on the $s_{123} = q^2$ pole, one has

$$
\langle \vec{v}_5^B \cdot \vec{F}_5 \rangle|_{s_{123} = q^2} = \mathcal{F}_3(1^φ, 3^φ, 2^φ) \mathcal{A}_4(q_3^S, 4^φ, 5^φ, -q^S),
$$

where the generalized BCJ vectors are

$$
\vec{v}_5^B = \left\{ \frac{\tau_{4,1}(1+3)T_{52}}{\tau_{4q_3}}, \frac{\tau_{42T_5(1+3)}}{\tau_{4q_3}}, \frac{\tau_{41T_5(2+3)}}{\tau_{4q_3}}, \frac{\tau_{42T_51}}{\tau_{4q_3}}, \frac{\tau_{41T_5(2+3)}}{\tau_{4q_3}}, \frac{\tau_{42T_51}}{\tau_{4q_3}} \right\},
$$

and it satisfies

$$
\text{Res} \left[ S_5^F \right]_{s_{123} = q^2} = (\vec{v}_5^B)^T \cdot (S_2^F \otimes S_5^A) \cdot \vec{v}_5^B \bigg|_{s_{123} = q^2}.
$$

Similar results for the $s_{124} = q^2$ or $s_{125} = q^2$ pole can be obtained by a trivial relabeling.

(C) For the factorization on the $s_{1234} = q^2$ pole, one has

$$
\langle \vec{v}_5^{C,1} \cdot \vec{F}_5 \rangle|_{s_{1234} = q^2} = \mathcal{F}_4(1^φ, \sigma \{3^φ, 4^φ\}, 2^φ) \mathcal{A}_3(q_4^S, 3^φ, -q^S),
$$

where $\sigma \in \{1, 2\} = S_2$ and the two corresponding vectors are

$$
\vec{v}_5^{C,1} = \{ \tau_{52}, \tau_5(2+4), 0, 0, \tau_5(2+3+4), 0 \},
$$

$$
\vec{v}_5^{C,2} = \{ 0, 0, \tau_{52}, \tau_5(2+3), 0, \tau_5(2+3+4) \}.
$$

One can combine the two vectors to form a $2 \times 6$ matrix:

$$
V_5^C = \begin{pmatrix}
\vec{v}_5^{C,1} \\
\vec{v}_5^{C,2}
\end{pmatrix},
$$

and it appears in the factorization of the KLT kernel as

$$
\text{Res} \left[ S_5^F \right]_{s_{1234} = q^2} = (V_5^C)^T \cdot (S_2^F \otimes S_5^A) \cdot V_5^C \bigg|_{s_{1234} = q^2}.
$$

We will not give the explicit form of $\Theta_5^F$ (and $S_5^F$), but only give its determinant which has the following nice structure as promised in the main text:

$$
\det(\Theta_5^F) = \frac{[q^2 - s_{12}] \prod_{1 \leq i < j \leq 5} (q^2 - s_{12j})^2 \prod_{i=3}^{5} (q^2 - s_{12i})}{[s_{1345} s_{2345} \prod_{i \neq (12)} s_{i j}]^2 s_{345} \prod_{3 \leq i < j \leq 5} (s_{1ij} s_{2ij})}. 
$$

We also point out that the power of the propagators have clear physical meanings and will be explained in [28].

II. A proof for the general factorization relation [28] for MHV form factors

We consider the generalized BCJ relation [28] in the main text, which are reproduced here:

$$
\mathcal{F}_{n-1}(1, 3, \ldots, n-1, 2) \mathcal{A}_3(q_{n-1}, n, -q)
$$

$$
= \left[ \tau_{n2} \mathcal{F}_n(1, 3, \ldots, n, 2) + \sum_{t=3}^{n-1} \tau_{n,(2+i-\ldots+(n-1))} \mathcal{F}_n(1, 3, \ldots, i-1, n, i, \ldots, n-1, 2) \right] q_{n-1}^2 = q^2,
$$

for MHV form factors.
where \( q_{n-1} = \sum_{i=1}^{n-1} p_i \). We will give a recursive proof of this relation for the four-dimensional MHV form factors of \( \text{tr}(\phi^2) \), expressed as

\[
\mathcal{F}_n(1^\phi, \sigma \{3^+, \ldots, n^+\}, 2^\phi) = \frac{\langle 12 \rangle^2}{\langle 1(3) \rangle \cdots \langle \sigma(n)2 \rangle / \langle 21 \rangle}.
\]  

(55)

To prove (28), we perform a standard BCFW shift, that is the \( \langle 21 \rangle \)-shift: \( |2\rangle = |2\rangle - |1\rangle, \langle 1\rangle = |1\rangle + |2\rangle \). We first focus on the LHS, in which only \( \mathcal{F}_{n-1}(1, 3, \ldots, n-1, 2) \) is affected by the shift. We define

\[
E_L(z) = \frac{1}{z} \mathcal{F}_{n-1}(\hat{1}, 3, \ldots, n-1, \hat{2}; z) A_3(q_{n-1}, n, -q),
\]

so that \( \text{Res}[E_L(z); z=0] \) gives the LHS of (28). Apart from \( z = 0 \), \( E_L \) have only one other pole on the complex plane \( z_p = (2(n-1))/(1(n-1)) \), on which the MHV form factors factorize as (here \( P = p_2 + p_{n-1} \))

\[
\text{Res}[E_L(z); z=z_p] = -\mathcal{F}_{n-2}(\hat{1}, 3, \ldots, \hat{P}; z_p) \frac{1}{s_{2(n-1)}} A_3(\hat{2}, n-1, -\hat{P}; z_p) A_3(q_{n-1}, n, -q).
\]

(57)

For the RHS of (28), we can define \( E_R \) similarly as

\[
E_R(z) = \frac{1}{z} \left[ \tau_{2} \mathcal{F}_n(\hat{1}, 3, \ldots, n, \hat{2}; z) + \sum_{i=3}^{n-1} \tau_{n,(2+i+\ldots+(n-1))} \mathcal{F}_n(\hat{1}, 3, \ldots, i-1, n, i, \ldots, n-1, \hat{2}; z) \right]|_{q_{n-1}^2 = q^2}.
\]

(58)

First we point out that the condition \( q_{n-1}^2 = q^2 \) will not be spoiled by the shift since \( \hat{p}_1 + \hat{p}_2 = p_1 + p_2 \). Next we examine the possible poles of \( E_R(z) \). For the form factors in the sum, only \( z_p \) pole appears. For the first form factor \( \mathcal{F}_{n}(1, 3, \ldots, n, \hat{2}; z) \), naively one may expect a pole \( (2n)/(1n) \), however, it is canceled by the \( \tau_{n,2} = (2n)(n2) - z(1n)(n2) \) factor. Moreover, we need to be careful about the pole at infinity: although the MHV form factors themselves do not contribute to the pole at infinity, the \( \tau \) factors do. One can compute the corresponding residue for each term in \( E_R(z) \) as

\[
\text{Res}[\tau_{n,(2+\ldots)} \mathcal{F}_n(\hat{1}, \sigma \{3, \ldots, n\}, \hat{2})]_{z=\infty} = \frac{\langle 1n \rangle [n2]/(21)}{\langle 1(3) \rangle \cdots \langle \sigma(n)1 \rangle}.
\]

(59)

where \( \sigma \) can be arbitrary permutations, and it turns out that the sum of the residues actually vanishes:

\[
\text{Res}[E_R(z)]_{z=\infty} = \sum_{i=3}^{n} \frac{\langle 1n \rangle [n2]/(21)}{\langle 13 \rangle \cdots \langle (i-1)n \rangle \langle ni \rangle \cdots \langle (n-1)1 \rangle} = 0,
\]

(60)

which is equivalent to an \((n-1)\)-point U(1) decoupling relation. Therefore, we find that \( E_R \) have also only the \( z_p \) pole (apart from the \( z = 0 \) one). And the residue is

\[
\text{Res}[E_R(z)]_{z=z_p} = -\sum_{i=3}^{n-1} \tau_{n,(\hat{p}+i+\ldots+(n-2))} \mathcal{F}_{n-1}(\hat{1}, 3, \ldots, i-1, n, i, \ldots, \hat{P}; z_p) \frac{1}{s_{2(n-1)}} A_3(\hat{2}, n-1, -\hat{P}; z_p)|_{q_{n-1}^2 = q^2}.
\]

(61)

Comparing (57) and (61), one can see that \( \text{Res}[E_L(z)]_{z=z_p} = \text{Res}[E_R(z)]_{z=z_p} \) by using a \((n-1)\)-point relation (54), and the residue theorem guarantees that \( \text{Res}[E_L(z)]_{z=0} = \text{Res}[E_R(z)]_{z=0} \), so that (54) is valid for the \( n \)-point case.

III. Further detail about \( n \)-point double copy and hidden relations

Here we discuss generic \( n \)-point form factor double copy and hidden relations, as well as explain the notation of (51).

We start from the generalized factorization relations (2) for the generic \( n \)-point case which reads in a more precise form as

\[
\sum_{\alpha \in S_{n-2}} \tilde{v}(\tilde{\kappa}, \tilde{\rho}) \mathcal{F}_n[\alpha] |_{\vec{q}^2 = q^2} = \mathcal{F}_m[\tilde{\kappa}] \mathcal{A}_{m'}[\tilde{\rho}] \equiv (\mathcal{F}\mathcal{A})_{(m, m')}[\tilde{\kappa}, \tilde{\rho}],
\]

(62)
with \( q_m = \sum_{i=1}^{m} p_i \) and \( m' = n - m + 2 \). Here \( F_m[\vec{r}] \) and \( A_{m'}[\vec{\rho}] \) are the color-ordered \( m \)-point form factors and \( m' \)-point amplitudes defined explicitly as

\[
F_m[\vec{r}] = F_m(1, \vec{r} \{3, \ldots, m \}, 2), \quad \vec{r} \in S_{m-2};
A_{m'}[\vec{\rho}] = A_{m'}(q_m, m + 1, \vec{\rho} \{m + 2, \ldots, n \} , -q), \quad \vec{\rho} \in S_{m'-3}.
\]

(63)

For the form factors, we use the DDM basis, and for amplitudes, we use the BCJ basis with \( q_m, q \) and \( (m + 1) \) (adjacent to \( q_m \)) fixed. From (62), we see that the generalized BCJ vectors \( \vec{v}(\vec{r}, \vec{\rho}) \) depend on the ordering \( (\vec{r}, \vec{\rho}) \) of the basis amplitudes and form factors, and each of them is assigned a generalized BCJ relation.

To relate (62) to the gravitational factorization property as in (31), we note that the same vectors \( \vec{v}(\vec{r}, \vec{\rho})[\alpha] \) also induce a decomposition of the KLT matrix \( S^n_r \):

\[
\text{Res} [S^n_r]_{\vec{q}^2} = \sum_{\vec{r}, \vec{\rho} \in S_{m-2}} \vec{v}(\vec{r}, \vec{\rho})[\alpha_1 \alpha_2] (S^n_r[F_m][\vec{r}][\vec{\rho}]) (S^n_r[A_{m'}][\vec{\rho}]) \vec{v}(\vec{r}, \vec{\rho})[\alpha_2].
\]

(64)

This equation can be regarded as a matrix product, as shown in the schematic equation (30) in the main text, and the collection of \( \vec{v}(\vec{r}, \vec{\rho})[\alpha] \) form a \((m - 2)! (m' - 3)! \times (n - 2)!\) matrix denoted as \( V \) in (30). Explicit examples can be found in the previous five-point examples such as (42) and (51).

A nice consequence of (62) and (51) is the factorization formula of the gravity theory (24):

\[
\text{Res} [G_n]_{\vec{q}^2} = \sum_{\alpha_1, \alpha_2} F_n[\alpha_1] \text{Res} [S^n_r]_{\vec{q}^2} (S^n_r[F_m][\vec{r}][\vec{\rho}]) (S^n_r[A_{m'}][\vec{\rho}]) = G_{m} M_{m'},
\]

(65)

which is a detailed explanation to (31) in the paper.

We make a remark that the derivation (35) shows that once we acknowledge the validity of two properties: (1) the generalized BCJ relation (62), which is practically easier to check in gauge theories, and (2) the decomposition of the KLT kernel (51), which is irrelevant to the specific theory or operator, there is no need to worry about the factorization on the new \( q_m^2 = q^2 \) poles exposed by double copy. This is a crucial step to confirm that \( G_n[\rho] \) is indeed a physically meaningful quantity in gravity.

Other physical requirements are easier to address, such as the manifest diffeomorphism invariance and factorization properties on the physical poles, i.e. those poles appearing already in gauge form factors. To argue the latter point, it is more convenient to look at an alternative form of the KLT double copy

\[
G_n = \sum_{\alpha, \beta \in S_{n-2}} N_n[\alpha] \Theta^{\alpha}[\alpha/\beta] N_n[\beta].
\]

(66)

Since \( \Theta^{\alpha} \) can be understood as the form factor in the bi-adjoint scalar theory, it also has a factorization on, say the \( P^2_m = s(1 i_1 \ldots i_m) = 0 \) pole, so that \( \text{Res}[\Theta^{\alpha}](P^2_m) = U \cdot (\Theta^{\alpha} \otimes \Theta^{\alpha}) \cdot U \). Here \( U \) is the orthogonal complement of BCJ vectors for amplitudes. The numerators can also factorize schematically \( U \cdot \vec{N}_n[\rho] = N_{m+2} N_{n-m} \). Thus, starting from (65) and performing some refinements similar to (65), we have

\[
\text{Res}[G_n]_{P^2_m} = \left( N_{m+2} \Theta^{\alpha} \Theta_{m+2} \right) \times \left( N_{n-m} \Theta^{\alpha} \Theta_{n-m} \right) = \hat{M}_{m+2} \hat{G}_{n-m}.
\]

(67)

here \( \hat{M} \) refers to gravity amplitudes with all massless scalar \( \phi \) and gravitons.