Integrable dispersive chains and energy dependent Schrödinger operator

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Abstract
In this paper we consider integrable dispersive chains associated with the so-called ‘energy dependent’ Schrödinger operator. In a general case multi-component reductions of these dispersive chains are new integrable systems, which are characterized by two arbitrary natural numbers. Also we show that integrable three-dimensional linearly degenerate quasilinear equations of a second order possess infinitely many differential constraints. Corresponding dispersive reductions are integrable systems associated with the ‘energy dependent’ Schrödinger operator.

Keywords: dispersive chain, differential constraint, integrable reduction, quasilinear equation, Hamiltonian structure, conservation law, commuting flow

1. Introduction

The remarkable Korteweg–de Vries equation is associated with the linear Schrödinger equation

\[ \psi_{xx} = (\lambda + u) \psi. \]

This paper is devoted to the description of integrable systems associated with the more general linear equation \( \psi_{xx} = u \psi \), where a dependence \( u(x, t, \lambda) \) with respect to the spectral parameter \( \lambda \) can be much more complicated (for instance, rational). According to [14] we call such a linear equation \( \psi_{xx} = u(x, t, \lambda) \psi \) the energy dependent Schrödinger equation.

* on the occasion of the 77th birthday of Professor Alexey Borisovich Shabat

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The function $\psi(x, t, \lambda)$ satisfies the pair of linear equations in partial derivatives
\[ \psi_{xt} = u \psi, \quad \psi_t = a \psi_x - \frac{1}{2} a_x \psi. \] (1)

Then the compatibility condition $(\psi_{xt})_{x} = (\psi_{t})_{xx}$ yields the relationship
\[ u_t = \left( -\frac{1}{2} a_x^3 + 2 u a_x + u_x \right) a \] (2)

between functions $u(x, t, \lambda)$ and $a(x, t, \lambda)$.

If we choose the linear dependences $u(x, t, \lambda) = \lambda + u'(x, t)$ and $a(x, t, \lambda) = \lambda + a_t(x, t)$, we obtain nothing but the famous Korteweg–de Vries equation
\[ uu_x + uu_{xx} = -\frac{1}{4} u^4 + \frac{3}{2} u^2 u_x^2. \] (3)

where $a_t = -\frac{1}{2} u$. An accurate derivation leads to $a_t = -\frac{1}{2} u + \xi$, where $\xi$ is an arbitrary constant. However, corresponding equation $u_{t, x} = \frac{1}{2} u_{x, x, x} - \frac{1}{2} u_t u_{x, x} + \xi u_t$, reduces to the form (3) under the transformation $t \to t, x + \xi t \to x$.

If we choose the quadratic dependence $u(x, t, \lambda) = \lambda^2 + \lambda u'(x, t) + u^2(x, t)$ and again the linear dependence $a(x, t, \lambda) = \lambda + a_t(x, t)$, we obtain nothing but the well-known Kaup–Boussinesq system
\[ u_t = -u u_x + \left( -\frac{1}{2} u^2 + \xi \right) u_x^2 + u_x^2, \quad u_x^2 = \frac{1}{4} u_{x, x, x} - u u_x^2 + \left( -\frac{1}{2} u^2 + \xi \right) u_x^2, \]
where $a_t = -\frac{1}{2} u + \xi$. The arbitrary constant $\xi$ can be removed by a linear change of independent variables ($t \to t, x + \xi t \to x$) exactly as in the previous case. Thus finally the Kaup–Boussinesq system takes the form
\[ u_t = -u u_x + \frac{3}{2} u^2 u_x^2, \quad u_x^2 = \frac{1}{4} u_{x, x, x} - u u_x^2 - \frac{1}{2} u'u_x^2. \] (4)

Multi-component polynomial (with respect to the spectral parameter $\lambda$) generalization
\[ u(x, t, \lambda) = \lambda^M + \lambda^{M-1} u'(x, t) + ... + u^M(x, t) \]
was investigated in several papers [6, 7]. Multi-component rational (with respect to the spectral parameter $\lambda$) generalization ($\epsilon_k$ are arbitrary parameters)
\[ u(x, t, \lambda) = \frac{\lambda^M u^M(x, t) + \lambda^{M-1} u^{M-1}(x, t) + ... + u^M(x, t)}{\epsilon_M \lambda^M + \epsilon_{M-1} \lambda^{M-1} + ... + \epsilon_0} \] (5)

was studied in [7] and in [15]. The authors considered two main subclasses selected by the conditions: $\epsilon_M = 0$ and $u^0 = 1$ (the so-called ‘generalized KdV–type systems’); $\epsilon_M = 0$ but $u^M = 1$ (the so-called ‘generalized Harry Dym–type systems’).

This paper is devoted to an open question: description of multi-component integrable systems associated with different dependencies $u(x, t, \lambda)$ with respect to the spectral parameter $\lambda$. In this paper we construct infinitely many multi-component dispersive reductions of the integrable dispersive chains introduced in [19]. Thus we found infinitely many new multi-component integrable systems, which can be written in a compact form (28), (29) and (41). Existence of dispersive reductions for linearly degenerate integrable hydrodynamic chains was discovered in [4]. The hierarchy of these hydrodynamic chains is associated with five integrable three-dimensional quasilinear equations of a second order (see, for instance, [11]).

First such a three-dimensional quasilinear equation of a second order can be written in a

3 Here we follow SJ Alber, see [2].
hydrodynamic form \((44)\). This equation possesses \(N\) component hydrodynamic reductions parameterized by \(N\) arbitrary functions of a single variable (see [16]). In this paper we show that integrable three-dimensional quasilinear system of a first order \((44)\) also possesses \(N\) component dispersive reductions, which are integrable dispersive systems \((28), (47)\) characterized by two arbitrary natural numbers.

The structure of the paper is as follows. In section 2 we consider general properties of integrable dispersive chains (associated with the energy dependent Schrödinger equation) such as higher commuting flows, conservation laws and local Hamiltonian structures. In section 3 we construct new dispersive integrable systems. In section 4 we separately present an exceptional case with a dispersive chain but written in non-evolution form. In section 5 we investigate three-dimensional linearly degenerate quasilinear equations of a second order, and find explicit differential constraints which allow us to reduce these quasilinear equations to the dispersive integrable systems discussed in previous sections. In section 6 we briefly derive a dispersionless limit of integrable dispersive chains including their multi-component dispersive reductions. In section 7 we formulate a program for further research.

2. Dispersive integrable chains

According to [19], instead of the linear dependence\(^4\) \(u(x, t, \lambda) = \lambda + u^1(x)\) (see (3)) and quadratic dependence \(u(x, t, \lambda) = \lambda^2 + \lambda u^1(x, t) + u^2(x, t)\) (see (4)) below we consider \((M = 1, 2,...)\)

\[
u(x, t, \lambda) = \lambda \nu \left(1 + \frac{u^1(x, t)}{\lambda} + \frac{u^2(x, t)}{\lambda^2} + \frac{u^3(x, t)}{\lambda^3} + \ldots \right),
\]

where \(\nu^k\) are infinitely many unknown functions\(^5\).

The substitution (6) and the linear dependence \(a^{(1)} = \lambda + a_1(x, t)\) into (2) yields \(M\)th integrable dispersive chain\(^6\)

\[
u^k = u^k + 1 - \frac{1}{2} u^1 u^k - u^1 u^k + \frac{1}{4} \delta_M^{(1)} u^1, \quad k = 1, 2, ..., \]

where \(\delta_M^{(1)}\) is the Kronecker delta and

\[
a_1 = -\frac{1}{2} u^1.
\]

2.1. Higher commuting flows

Higher commuting flows of the Korteweg–de Vries hierarchy are determined by the linear spectral system

\[
\psi_{\nu} = \left(\lambda + u^1\right)\psi, \quad \psi_{\nu^1} = a^{(1)} \psi_{\nu} - \frac{1}{2} a^{(1)} \psi.
\]

\(^4\) The bold script chosen for vector \(t\) means that we consider a family of commuting integrable systems where vector components \(t^j\) play a role of the so-called extra group parameters.

\(^5\) This Laurent expansion was suggested in [19]; see other detail in [1].

\(^6\) Earlier in [19] these integrable dispersive chains were written just in a compact symbolic form.
where
\[ a^{(s)} = \lambda^s + \sum_{m=1}^{s} a_m \lambda^{s-m}, \]  
\hspace{1cm} \text{(9)}

and functions \(a_m\) and \(u^1\) depend on the ‘space’ variable \(x\) and infinitely many extra ‘time’ variables \(t^k\) (obviously, \(i \equiv t^i\)). Substitution (6) and (9) into (2) leads to higher commuting flows (here we also define \(a_0 = 1\))
\[ u^k_i = \sum_{s=0}^{\infty} \left( u^{k+s} \partial_s + \partial_s u^{k+s} - \frac{1}{2} \delta^s_{m-k} \partial_i^s \right) a_{s-m}, \hspace{1cm} s = 1, 2, ..., \]  
\hspace{1cm} \text{(10)}

where all differential polynomials \(a_m\) can be found iteratively from the linear system\(^7\) (here we also define \(u^0 = 1\) and \(u^{-m} = 0\) for all \(m = 1, 2, ...\))
\[ \sum_{m=0}^{s} \left( u^{m-k} \partial_s + \partial_s u^{m-k} - \frac{1}{2} \delta^m_{s-k} \partial_i^s \right) a_{s-m} = 0, \hspace{1cm} k = 0, 1, ..., s - 1. \]  
\hspace{1cm} \text{(11)}

For instance,
\[ a_1 = -\frac{1}{2} u^1, \hspace{1cm} a_2 = -\frac{1}{2} u^2 + \frac{3}{8} (u^1)^2 - \frac{1}{8} \delta^1_2 u^1_{xx}, \]
\[ a_3 = -\frac{1}{2} u^3 + \frac{3}{4} u^1 u^2 - \frac{5}{16} (u^1)^3 + \frac{1}{32} \delta^1_3 \left( 10 u^1 u^1_{xx} + 5 (u^1)^2 - u^1_{xxxx} - 4 u^2_{xx} \right) - \frac{1}{8} \delta^1_2 \delta^1_2 u^1_{xx}, \]

Thus all higher commuting flows are written in an evolution form. For instance, the first commuting flow to (7) is (here we identify \(y \equiv t^2\))
\[ u^k_i = u^k_x + \frac{1}{2} u^1 u^k_x + 1 \left( -\frac{1}{2} u^2 + \frac{3}{8} (u^1)^2 - \frac{1}{8} \delta^1_2 u^1_{xx} \right) u^k_x - u^1 u^1_x \]  
\[ + u^1 \left( -u^2_x + \frac{3}{4} u^1 u^1_x - \frac{1}{4} \delta^1_1 u^1_{xxx} \right) + \frac{1}{4} \delta^k_{s=1} u^1_{xxx} + \frac{1}{4} \delta^k_{s=1} \left( u^2_{xxx} - \frac{3}{4} (u^1)^2_{xxx} + \frac{1}{2} \delta^1_2 u^1_{xxxx} \right). \]  
\hspace{1cm} \text{(12)}

A generating function of higher commuting flows is determined by the choice (instead of (9))
\[ a = \frac{a(x, t, \zeta)}{\lambda - \zeta}, \]  
\hspace{1cm} \text{(13)}

where \(\zeta\) is an arbitrary parameter. Indeed, substitution of the asymptotic expansion (\(\zeta \to \infty\))
\[ a(x, t, \zeta) = -\zeta \left( \frac{1}{\zeta} \frac{a_1(x, t)}{\zeta} + \frac{a_2(x, t)}{\zeta^2} + \frac{a_3(x, t)}{\zeta^3} + ... \right) \]  
\hspace{1cm} \text{(14)}

into (13) leads to
\[ a = 1 + \frac{\lambda + a_1}{\zeta} + \frac{\lambda^2 + a_1 \lambda + a_2}{\zeta^2} + \frac{\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3}{\zeta^3} + ..., \]

where all polynomial expressions with respect to the spectral parameter \(\lambda\) are precisely (9).

The substitution (6) and (13) into (2) (reminder: \(u^0 = 1\)) implies

\(^7\) Linear system (11) can be obtained formally from (10), if to put \(u^0 = 1\) and all other unknown functions \(u^{-m} = 0\) for \(m = 1, 2, ...\).
where we introduced the special ‘time’ variable \( \tau(\zeta) \) instead of \( t \) in (2) to emphasize that we deal with a generating function of commuting flows. Taking into account that \( u_{1(\zeta)} = 2a \), and iteratively expressing all others \( u_{r(\zeta)} \) via higher derivatives of \( u^m \) and \( a \), we come to the more explicit form
\[
\sum_{m=0}^{k} \zeta^m \left( u^{k-m} \partial_x + \partial_x u^{k-m} - \frac{1}{2} \delta_M^{k-m} \partial_x^3 \right) a, \quad k = 0, 1, \ldots,
\]
which again yields (10), if substituting (14) together with another formal expansion \( (\zeta \to \infty) \)
\[
\partial_{\zeta(\zeta)} = \partial_x + \frac{1}{\zeta} \partial_x + \frac{1}{\zeta^2} \partial_x + \frac{1}{\zeta^3} \partial_x + \ldots
\]
Moreover, now we can identify \( x \equiv t^\rho \).

2.2. Conservation laws

The substitution
\[
\psi = \exp \left( \int rdx \right)
\]
into (1) yields
\[
r_x + r^2 = u, \quad r_\lambda = \left( ar - \frac{1}{2} a_x \right) x.
\]
(15)

Thus \( r(x, t, \lambda) \) is a generating function of conservation law densities (with respect to the spectral parameter \( \lambda \)), while the second equation of (15) is a generating function of conservation laws.

The differential substitution \( (\epsilon \text{ is an arbitrary constant}) \)
\[
r = \frac{q_x}{2\psi} + \frac{\epsilon}{2\psi}
\]
into the first equation of (15) yields
\[
2\psi \psi_x - q_x^2 + \epsilon^2 = 4\psi \psi_x^2,
\]
(17)
which is nothing but the first integral of a well-known equation (in this case, the Korteweg–de Vries equation)
\[
q_{xx} = 4\psi \psi_x + 2u_x \psi.
\]
(18)

Indeed, the function \( \psi = \psi + \psi^* \) satisfies two linear equations
\[
q_{xx} = 4\psi \psi_x + 2u_x \psi, \quad \psi_i = a\psi_x - a_x \psi,
\]
(19)
where \( \psi \) and \( \psi^* \) are two linearly independent solutions (see (1)), which (obviously functionally dependent according to the Wronskian relationship \( \psi \psi^* - \psi^* \psi = \text{const} \neq 0 \) ) are usually determined by their asymptotic behavior \( (\lambda \to \infty): \psi \to \exp (\lambda^{M/2} x), \psi^* \to \exp (-\lambda^{M/2} x). \)
The second equation can be written in the conservative form

\[
\left( \frac{1}{\varphi} \right)_j = \left( \frac{a}{\varphi} \right)_j.
\]

(20)

Thus the function \( p = 1/\varphi \) is a generating function of conservation law densities, and the above equation is a generating function of conservation laws. Taking into account (16), one can see that the second equation in (15) is equivalent to the above generating function of conservation laws (up to a total derivative).

**Theorem:** Dispersive chains (7) also have an alternative generation function of conservation laws

\[
\left( u' \varphi \right)_j = \left( \left[ 2u + (\lambda + a_j)u' \right] \varphi - \frac{1}{2} \varphi_{xx} \right)_j.
\]

(21)

where \( u' \equiv u'(\lambda) \), i.e.

\[
u = \lambda^M + \sum_{k=1}^{\infty} \lambda^{M-k}u', \quad u' = M\lambda^{M-1} + \sum_{k=1}^{\infty} (M-k)\lambda^{M-k-1}u'.
\]

**Proof:** We seek a generating function of conservation law densities in the form \( b\varphi \), where \( b \) is a linear expression with respect field variables \( u' \), whose coefficients depend on the spectral parameter \( \lambda \) only. Differentiating this product \( b\varphi \) with respect to the independent variable \( \tau \) and taking into account that \( u\varphi = (2u\varphi - \frac{1}{2}\varphi_{xx}) \) (see the first equation in (19)), we expect that the flux of this generating function will be proportional to the function \( \varphi \) and its second derivative \( \varphi_{xx} \). A straightforward computation yields that \( b = u' \), the flux is \([2u + (\lambda + a_j)u']\varphi - \frac{1}{2}\varphi_{xx} \) and

\[
\left( u' \varphi \right)_j = \left( \left[ 2u + (\lambda + a_j)u' \right] \varphi - \frac{1}{2} \varphi_{xx} \right)_j,
\]

where the function \( a_j \) is defined by (8). The theorem is proved.

Taking into account that in (6) we seek an asymptotic expansion of (18) in the form \( \varphi \), any solution of linear equations is determined up to an arbitrary factor; thus without loss of generality and for the sake of simplicity we choose the normalization to unity at infinity with respect to the spectral parameter \( \lambda \)

\[
\varphi(x, \tau, \lambda) = 1 + \frac{\varphi_1(x, \tau, \lambda)}{\lambda} + \frac{\varphi_2(x, \tau, \lambda)}{\lambda^2} + \frac{\varphi_3(x, \tau, \lambda)}{\lambda^3} + \cdots, \quad \lambda \to \infty.
\]

(22)

Then we obtain (here we define \( \varphi_0 = 1 \) and remind that \( u^0 = 1 \))

\[
\frac{1}{2} \sum_{k=0}^{\infty} \frac{\varphi_{k+1,xx}}{\lambda^{2k+2}} = \sum_{k=0}^{\infty} \frac{1}{\lambda^{2k+2}} \sum_{n=1}^{k+1} (2\frac{\varphi_{k+1-m} u_{n,xx}}{\lambda^{k+1-m}} + \varphi_{k+1-m} u_{n,xx}).
\]

Selecting factors of each degree of the spectral parameter \( \lambda \), one can see that expressions for coefficients \( \varphi_k \) coincide with expressions for coefficients \( a_k \) in (11), i.e. (22) becomes

\[
\varphi(x, \tau, \lambda) = 1 + \frac{a_1(x, \tau, \lambda)}{\lambda} + \frac{a_2(x, \tau, \lambda)}{\lambda^2} + \frac{a_3(x, \tau, \lambda)}{\lambda^3} + \cdots
\]

(23)

Thus substituting this asymptotic expansion into generating function (21), infinitely many conservation laws can be presented explicitly.

\[\text{8 This equation can be obtained directly from (2) by differentiation with respect to the spectral parameter } \lambda.\]
2.3. Local Hamiltonian structures

Construction of local multi-Hamiltonian structures for polynomial and rational cases (5) was presented in [3, 6, 7].

A hierarchy of integrable dispersive chains (7) possesses infinitely many local Hamiltonian structures, the first two of which are $(s = 1, 2, ...)$

\[
\begin{align*}
    u_k^j &= \sum_{m=1}^{s+1} \left( u^{k+m-1} \partial_x + \partial_x u^{k+m-1} - \frac{1}{2} \delta_M^{k+m-1} \partial_x^3 \right) \frac{\delta H_{j+1}}{\delta u^m}, \quad k = 1, 2, ...; \\
    u_0^j &= -2 \partial_x \left( \frac{\partial H_{j+2}}{\partial u^1} \right), \quad u_1^j = \sum_{m=2}^{s+1} \left( u^{k+m-2} \partial_x + \partial_x u^{k+m-2} - \frac{1}{2} \delta_M^{k+m-2} \partial_x^3 \right) \frac{\delta H_{j+2}}{\delta u^m}, \quad k > 1.
\end{align*}
\]

Remark: The first local Hamiltonian structure follows from (10), where we utilized the observation

\[
a_m = \frac{\delta H_{j+1}}{\delta u^m}, \quad m = 0, 1, ...; \quad s = 1, 2, ...
\]

This is an alternative approach for construction of local polynomial conservation laws (cf (15), (20), (21)). In such a case all Hamiltonians can be found from the above variation derivatives, for instance

\[
\begin{align*}
    H_1 &= \int u_i dx, \quad H_2 = \int \left( u_i^2 - \frac{1}{4} \left( u_i^2 \right)^2 \right) dx, \\
    H_3 &= \int \left( u_i^3 - \frac{1}{2} u_i u_i^2 + \frac{1}{8} (u_i^3)^3 + \frac{1}{16} \delta_i^1 \left( u_i^1 \right)^2 \right) dx, \\
    H_4 &= \int \left( u_i^4 - \frac{1}{2} u_i u_i^3 + \frac{1}{4} (u_i^2)^2 + \frac{3}{8} (u_i^3)^2 u_i^2 - \frac{5}{64} (u_i^4)^4 \right) \\
    &\quad + \frac{1}{32} \delta_i^1 \left( -5 u_i^1 \right)^2 - \frac{1}{2} \left( u_i^1 \right)^2 + 4 u_i^1 u_i^2 \right) + \frac{1}{16} \delta_i^2 \left( u_i^1 \right)^2 dx, ...
\end{align*}
\]

All other higher local Hamiltonian structures can be constructed utilizing the relationship (24), i.e.

\[
\frac{\delta H_i}{\delta u^m} = \frac{\delta H_{j+1}}{\delta u^{m+1}}, \quad m, s, k = 1, 2, ...
\]

For instance, the third local Hamiltonian structure is $(s = 1, 2, ...)$

\[
\begin{align*}
    u_0^1 &= -2 \partial_x \left( \frac{\partial H_{j+3}}{\partial u^2} \right), \quad u_1^2 = -2 \partial_x \left( \frac{\partial H_{j+2}}{\partial u^1} \right) - \left( u_i^2 \partial_x + \partial_x u_i^2 - \frac{1}{2} \delta_i^1 \partial_x^3 \right) \frac{\delta H_{j+3}}{\delta u^2}, \\
    u_k^j &= \sum_{m=1}^{s+1} \left( u^{k+m-1} \partial_x + \partial_x u^{k+m-1} - \frac{1}{2} \delta_M^{k+m-1} \partial_x^3 \right) \frac{\delta H_{j+3}}{\delta u^m}, \quad k > 2.
\end{align*}
\]

3. Multi-component reductions

The theory of multi-component semi-Hamiltonian hydrodynamic reductions of integrable hydrodynamic chains was built in [13] and applied in several papers [9, 16] (see also [5, 18]). A corresponding theory of multi-component integrable dispersive reductions of integrable dispersive chains does not exist at this moment. Nevertheless infinitely many multi-component integrable dispersive systems extracted from (7) are presented in this section.
3.1. Elementary reductions

Obviously for any natural number $N \geq M$ the reduction $u^{N+1} = 0$ of $M$th dispersive chain (7) leads to $N$ component integrable dispersive systems:

1. $N = M = 1$, the Korteweg–de Vries equation (3);

2. $N = 2$, $M = 1$, the Ito system

$$u_t^1 = u_x^2 - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}, \quad u_t^2 = -\frac{1}{2} u^1 u_x - u^1 u_x,$$

(25)

3. $N > 2$, $M = 1$

$$u_t^1 = u_x^2 - \frac{3}{2} u^1 u_x + \frac{1}{4} u_{xxx},$$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^{k} - u^1 u_x, \quad k = 2, ..., N - 1,$$

$$u_t^N = -\frac{1}{2} u^1 u_x^{N-1} - u^1 u_x.$$

4. $N = M > 1$, (if $N = M = 2$; this is the Kaup–Boussinesq equation, see (4))

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^{k} - u^1 u_x, \quad k = 1, 2, ..., N - 1,$$

$$u_t^N = -\frac{1}{2} u^1 u_x^{N-1} - u^1 u_x + \frac{1}{4} u_{xxx}.$$  

5. $N = M + 1$, $M > 1$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^{k} - u^1 u_x, \quad k = 1, 2, ..., N - 2,$$

$$u_t^{N-1} = u_x^N - \frac{1}{2} u^1 u_x^{N-1} - u^1 u_x + \frac{1}{4} u_{xxx},$$

$$u_t^N = -\frac{1}{2} u^1 u_x^{N-1} - u^1 u_x.$$  

6. $N > M + 1$, $M > 1$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^{k} - u^1 u_x, \quad k = 1, ..., M,$$

$$u_t^M = u_x^{M+1} - \frac{1}{2} u^1 u_x^{M} - u^1 u_x + \frac{1}{4} u_{xxx},$$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^{k} - u^1 u_x, \quad k = M + 1, ..., N - 1,$$

$$u_t^N = -\frac{1}{2} u^1 u_x^{N-1} - u^1 u_x.$$
3.2. Rational constraints with movable singularities

Now we consider more complicated $N$ component reductions ($M = 1, 2, ...$)

$$u(x, t, \lambda) = \frac{\lambda^{M+K} + \lambda^{M+K-1}v_{M+K-1}(x, t) + \ldots + \lambda v_1(x, t) + v_0(x, t)}{\lambda^K + \lambda^{K-1}w_{K-1}(x, t) + \ldots + \lambda w_1(x, t) + w_0(x, t)}, \quad K = 0, 1, \ldots$$  \hspace{1cm} (26)

Suppose for simplicity that all roots of these two polynomials are pairwise distinct. Then the substitution

$$u(x, t, \lambda) = \frac{\prod_{m=1}^{M+K}(\lambda - s^n(x, t))}{\prod_{k=1}^{K}(\lambda - r^k(x, t))}$$  \hspace{1cm} (27)

into (2) together with the linear dependence $a^{(i)} = \lambda + a_i(x, t)$ yields new multi-component integrable dispersive systems

$$r^k_i = (r^i + a_i)r^k, \quad s^n_i = (s^n + a_i)s^n_i + \frac{1}{2} \prod_{k=1}^{K}(s^n - s^n) a_{1,\text{sat}},$$  \hspace{1cm} (28)

where

$$a_i = \frac{1}{2} \left( \sum_{n=1}^{M+K} s^n - \sum_{k=1}^{K} r^k \right).$$  \hspace{1cm} (29)

Remark: In the particular case $r^i = \text{const}$, the ansatz (26) ($e_k$ are symmetric functions with respect to $r^k$ according to the Viète theorem)

$$u(x, t, \lambda) = \frac{\lambda^{M+K} + \lambda^{M+K-1}v_{M+K-1}(x, t) + \ldots + \lambda v_1(x, t) + v_0(x, t)}{\lambda^K + \epsilon_{K-1}\lambda^{K-1} + \ldots + \epsilon_1\lambda + \epsilon_0}$$

was investigated in [7]. Then $2K + M$ component system (28) reduces to the form

$$s^n_i = (s^n + a_i)s^n_i + \frac{1}{2} \prod_{k=1}^{K}(s^n - s^n) a_{1,\text{sat}}, \quad i = 1, 2, \ldots, M + K,$$

where $r^i = \text{const}$. Thus all integrable systems considered in [7] also can be written in the above symmetric form. For instance, the Kaup–Boussinesq system (4) becomes

$$s^1 = \frac{1}{2} (3s^1 + s^2) s^1 + \frac{(s^1 + s^2)_{\text{sat}}}{4(s^1 - s^2)};$$

$$s^2 = \frac{1}{2} (s^1 + 3s^2) s^2 + \frac{(s^1 + s^2)_{\text{sat}}}{4(s^1 - s^2)};$$  \hspace{1cm} (30)

and the Ito system (25) takes the form

$$s^1 = \frac{1}{2} (3s^1 + s^2) s^1 + \frac{s^1(s^1 + s^2)_{\text{sat}}}{4(s^1 - s^2)}, \quad s^2 = \frac{1}{2} (s^1 + 3s^2) s^2 + \frac{s^2(s^1 + s^2)_{\text{sat}}}{4(s^1 - s^2)}.$$
### 3.3. Negative flows

Substitution of the expansion \((\zeta \to 0)\)

\[
a(x, t, \zeta) = -a_{-1}(x, t) - \zeta a_{-2}(x, t) - \zeta^2 a_{-3}(x, t) + \ldots
\]

into (13) yields (cf (14))

\[
a = -\frac{a_{-1}}{\lambda} = \left(\frac{a_{-2}}{\lambda} + \frac{a_{-1}}{\lambda^2}\right)\zeta - \left(\frac{a_{-3}}{\lambda} + \frac{a_{-2}}{\lambda^2} + \frac{a_{-1}}{\lambda^3}\right)\zeta^2 - \ldots
\]

Substitution of this expansion and the rational ansatz (27) into (2) implies infinitely many lower (negative) flows of the integrable hierarchies, whose first nontrivial (positive) members are (28). For instance, the choice

\[
a = -\frac{a_{-1}}{\lambda}
\]

(31)
determines non-evolution system

\[
r^k_{r^{-1}} = \frac{a_{-1}}{a_{-1}} r^k_{-1}, \quad s^i_{r^{-1}} = \frac{a_{-1}}{s^i} s^i_{-1} = \frac{1}{2s^i} \frac{\prod_{i=1}^{r} (s^i - r^i)}{\prod_{s \neq i}^{M} (s^i - s^m)} a_{-1,ss},
\]

(32)

where

\[
a_{1,r^{-1}} = a_{-1,r^{-1}}.
\]

(33)

Taking into account (29), one can derive an ordinary differential equation \((\xi\tilde{\xi}\text{is an integration constant})\) on the function \(a_{-1}\), i.e. if any \(r^i \neq 0\), then

\[
\frac{(-1)^u}{2} \left(\frac{a_{-1,ss}}{a_{-1}} - \frac{1}{2} \frac{a_{-1,ss}}{a_{-1}}\right) + \frac{\xi}{a_{-1}} = \frac{\prod_{i=1}^{K} r^k_{-1} s^i}{\prod_{s \neq i}^{M} r^i}.
\]

(34)

If, for instance, \(r^i = 0\), then non-evolution system (32) becomes the evolution system \((k = 2, 3, \ldots, K; i = 1, 2, \ldots, M + K)\), where \((\xi\tilde{\xi}\text{ is an integration constant})\)

\[
a_{-1} = \tilde{\xi} \left(\frac{\prod_{i=1}^{K} r^k_{-1} s^i}{\prod_{s \neq i}^{M} s^m}\right)^{1/2}.
\]

(35)

### 4. The exceptional case

The concept of \(Mth\) dispersive integrable chain can be extended to the case \(M = 0\) (see (6)). Indeed, the compatibility condition \((\psi_\alpha) = (\psi_\beta)\) of the linear system

\[
\psi_\beta = \left(1 + \frac{u^i}{\lambda} + \frac{u^j}{\lambda^2} + \ldots\right)\psi_\beta, \quad \psi_\alpha = (\lambda + a_1)\psi_\alpha - \frac{1}{2} a_{1,ss} \psi
\]

yields the zeroth dispersive integrable chain

\[
u^k = u_s^{k+1} + a_1 u^k + 2u^i a_{1,ss}, \quad k = 1, 2, \ldots,
\]

(36)

where \((\xi\text{ is an arbitrary constant})\)

\[
\frac{1}{2} a_{1,ss} - 2a_1 = u^i + \xi.
\]

(37)
Since the function \( a_1 \) cannot be expressed via the function \( u_1 \) and a finite number of its derivatives, this zeroth dispersive chain is not an evolution system (in comparison with dispersive chains (7) with \( M > 1 \)).

For this particular case we omit consideration of conservation laws, commuting flows and Hamiltonian structures, because such an investigation can be made exactly as in previous sections. Here we will just mention the most important non-evolution reductions:

1. \( N = 1 \), the Camassa–Holm equation

\[
\frac{\partial u_1^1}{\partial t} = u_1 u_1^2 + 2 u_1^2 a_{1,\xi}, \quad \frac{1}{2} a_{1,\xi\xi} - 2 a_1 = u_1^1 + \xi; \tag{38}
\]

2. \( N > 1 \), the multi-component generalization of the Camassa–Holm equation

\[
\frac{\partial u_1^k}{\partial t} = u_1^{k+1} + a_{1,\xi} u_1^k + 2 u_1^2 a_{1,\xi}, \quad k = 1, 2, ..., N - 1, \tag{39}
\]

\[
\frac{\partial u_1^N}{\partial t} = a_{1,\xi} u_1^N + 2 u_1^2 a_{1,\xi}, \quad \frac{1}{2} a_{1,\xi\xi} - 2 a_1 = u_1^1 + \xi.
\]

3. Suppose for simplicity that all roots of two polynomials in (26) are pairwise distinct. Then the substitution

\[
u(x, t, \lambda) = \prod_{m=1}^{K} \left( \lambda - s^m(x, t) \right)
\]

into (2) yields multi-component non-evolution systems (cf (28))

\[
\frac{\partial r_1^k}{\partial t} = \left( r_1^k + a_1 \right) r_1^k, \quad s_1^k = \left( s_1^k + a_1 \right) s_1^k + \frac{1}{2} \prod_{m=1}^{K} \left( s_1^k - r_1^k \right) a_{1,\xi\xi},
\]

where \( a_1 \) is determined by \( (\xi \) is an arbitrary constant, see (37))

\[
2a_1 - \frac{1}{2} a_{1,\xi\xi} = \sum_{m=1}^{K} s^m - \sum_{k=1}^{K} r^k - \xi. \tag{41}
\]

In this case lower commuting flows are determined by (31) and (40). Then (2) leads to (cf (32))

\[
\frac{\partial r_1^k}{\partial t} = - \frac{a_{1,\xi\xi} r_1^k}{2 r^k}, \quad s_1^k = - \frac{a_{1,\xi\xi}}{s_1^k} s_1^k + \frac{1}{2} \prod_{m=1}^{K} \left( s_1^k - r_1^k \right) a_{1,\xi\xi}, \tag{42}
\]

where the function \( a_{1,\xi\xi} \) satisfies (33). Taking into account (41), one can derive an ordinary differential equation \( (\xi \) is an integration constant) on the function \( a_{1,\xi} \), i.e. if any \( r^k \neq 0 \), then (cf (34))

\[
\frac{1}{2} (-1)^{\nu} \left( a_{1,\xi\xi} + 1, a_{1,\xi} = \frac{1}{2} a_{1,\xi}^2 \right) + \frac{\xi}{2} a_{1,\xi}^2 = \prod_{m=1}^{K} \frac{s^m}{r^k} \prod_{l=1}^{K} r^l.
\]

If, for instance, \( r^1 = 0 \), then non-evolution system (42) becomes the evolution system

\[
\frac{\partial r_1^k}{\partial t} = - \frac{a_{1,\xi\xi} r_1^k}{2 r^k}, \quad s_1^k = - \frac{a_{1,\xi\xi}}{s_1^k} s_1^k + \frac{1}{2} \prod_{m=1}^{K} \left( s_1^k - r_1^k \right) a_{1,\xi\xi}
\]
(k = 2, 3, ..., K; i = 1, 2, ..., K), where $\xi_i$ is an integration constant, cf (35)

$$a_{i-1} = \xi_i \left( \prod_{k=2}^{K} p^k \prod_{m=1}^{k} \xi_m \right)^{1/2}.$$

### 5. Three-dimensional linearly degenerate qausilinear equations

Hierarchies of integrable dispersive chains (7) and (10) have generating functions of conservation laws

$$p_{k,t} = (a_{i}^{(k)} p)_{i},$$

where $p = 1/\varphi$ (see (20)). The compatibility conditions $(p_{i,t}) = (p_{k,t})_{i}$ must be fulfilled because corresponding dispersive integrable chains (and their multi-component reductions) commute to each other. For instance, consistency of the two first such equations (reminder: $x = t^0, t = t^1, y = t^2$)

$$p_1 = \left( (\lambda + a_1) p \right), \quad p_2 = \left( (\lambda^2 + a_1 \lambda + a_2) p \right),$$

leads to the three-dimensional quasilinear system

$$a_{1,t} = a_{2,x}, \quad a_{1} a_{2,x} + a_{1,y} = a_{2} a_{1,x} + a_{2,y},$$

whose integrability by the method of hydrodynamic reductions was investigated in [16]. This three-dimensional quasilinear system also belongs to the class of such integrable systems which are called linearly degenerate. This means that any of these systems admits at least one $N$ component two-dimensional hydrodynamic reduction ($N$ must be arbitrary) which is linearly degenerate (see detail in [8, 17]). Moreover in such a case all hydrodynamic reductions can be completely described (see, for instance, [16]). They are parameterized by $N$ arbitrary functions of a single variable.

In this section we formulate the following:

**Statement**: Three-dimensional quasilinear system (44) possesses infinitely many differential constraints $a_{i} (u, u, ..., a_{2} (u, u, ..., a_{3} (u, u, ...)$, where field variables $u_i$ are solutions of dispersive integrable systems determined by linear spectral problem (1) and described in sections 3 and 4.

Already in two previous papers [4] a concept of differential constraints was introduced for the hydrodynamic chain

$$a_{k,t} = a_{k+1,t} + a_{k} a_{k,x} - a_{k} a_{k,t}, \quad k = 1, 2, ...,$$

which can be obtained from the second equation in (1), where one should substitute the linear ansatz $a^{(1)} = \lambda + a_1$ instead of $a$ and formal expansion (23) instead of $\varphi$. L Martinez Alonso and AB Shabat also considered such remarkable examples like the Korteweg–de Vries equation and the nonlinear Schrödinger equation.

The first commuting flow to (45)

$$a_{k,t} = a_{k+1,t} + a_{k} a_{k+1,x} - a_{k+1} a_{k,x} + a_{k} a_{k+1} + a_{k} a_{k+1}, \quad k = 1, 2,...$$

can be obtained from the second equation in (1), where one should substitute the quadratic ansatz $a^{(2)} = \lambda^2 + a_1 \lambda + a_2$ instead of $a$ and formal expansion (23) instead of $\varphi$. Taking the first two equations $a_{1,t} = a_{2,x}, a_{2,t} = a_{1,x} + a_{1} a_{2,x} - a_{2} a_{1,x}$ from (45), the first equation
$a_{i,s} = a_{j,s}$ from (46), and excluding $a_{i,s}$, one can obtain three-dimensional quasilinear system (44).

Remark: Hydrodynamic chain (45), under differential invertible polynomial substitutions $a_i(u, u, ...)$ from (11), transforms into dispersive chains (7), where $M$ is any natural number. This means that any $M$th dispersive chain is equivalent to another $K$th dispersive chain by appropriate invertible differential substitutions (also including the exceptional case $M = 0$). However, corresponding differential reductions are not equivalent to each other.

Substitution of the quadratic dependence $a^{22} = \lambda^2 + \lambda a_1 + a_2$ into (2) together with ansatz (27) yields first commuting flow to (28)

\[
\frac{r_i^k}{r_i^k} = \left( a_2 + a_1 r^i + \left( r^i \right)^2 \right) r_i^k,
\]

\[
s_j^i = \left( a_2 + a_1 s^j + \left( s^j \right)^2 \right) s_j^i + \frac{1}{2} \prod_{k=i}^{K} \left( s^k - s^n \right) \left( s^j a_{j,xxx} + a_{z,xxx} \right),
\]

where $a_1$ is determined by (29) and

\[
a_2 = \frac{1}{4} \sum_{m=1}^{K} \left( s^m \right)^2 - \frac{1}{4} \sum_{i=1}^{K} \left( s^i \right)^2 + \frac{1}{2} a_1^2 + \frac{1}{4} \delta_{M}^1 a_{l,xxx},
\]

Thus in this section we selected dispersive constraints (29), (48) of three-dimensional quasilinear system (44), where $M + K$ functions $s^i(x, t, y)$ and $K$ functions $r^i(x, t, y)$ are solutions of $2K + M$ component commuting dispersive integrable systems (28) and (47).

Now we present the first four important examples of finite component differential reductions of three-dimensional quasilinear system (44).

1. The Korteweg–de Vries equation ($N = M = 1$). In this case commuting dispersive chains (7) and (12) reduce to (see (3))

\[
u_1^i = \left( \frac{1}{4} \nu_1^i - \frac{3}{4} (u^i)^2 \right),
\]

and its first commuting flow

\[
u_1^i = \left( \frac{5}{8} (u^i)^3 - \frac{5}{16} (u_1^i)^2 - \frac{5}{8} u_1^i u_1^i + \frac{1}{16} u_1^i u_1^i \right).
\]

Substitution of (see (11))

\[
a_1 = -\frac{1}{2} u_1^i, \quad a_2 = \frac{3}{8} (u^i)^2 - \frac{1}{8} u_1^i
\]

into three-dimensional quasilinear system (44) leads to an identity.

2. The Ito system ($N = 2, M = 1$). In this case commuting dispersive chains (7) and (12) reduce to (see (25))

\[
u_1^i = \nu_1^i - \frac{3}{2} u^i u_1^i + \frac{1}{4} u_1^i u_1^i, \quad u_2^i = -\frac{1}{2} u_1^i u_1^i - u^i u_1^i,
\]

and its first commuting flow

\[
u_1^i = -\frac{3}{2} u^i u_1^2 + \left( -\frac{3}{2} u^2 + \frac{15}{8} (u^i)^2 \right) u_1^i
\]
\[
- \frac{1}{4} u' u_x^2 + \frac{1}{4} u_x^3 - \frac{3}{16} [(u')^2]_{xx} + \frac{1}{16} u_{xxxxx},
\]

\[
u_{x}^2 = \left( -\frac{3}{2} u^2 + \frac{3}{8} (u')^2 - \frac{1}{8} u_x^4 \right) u_x^2 + \frac{3}{2} u_x u_x u_x - \frac{1}{4} u_x u_x u_x^2.
\]

Substitution of (see (11))

\[
a_1 = -\frac{1}{2} u', \quad a_2 = -\frac{1}{2} u^2 + \frac{3}{8} (u')^2 - \frac{1}{8} u_x^4,
\]

into three-dimensional quasilinear system (44) leads to an identity.

3. The Kaup–Boussinesq equation \((N = M = 2)\). In this case commuting dispersive chains (7) and (12) reduce to (see (4))

\[
u_{x}^1 = u_x^2 - \frac{3}{2} u_x^2 u_x^1, \quad u_x^2 = \frac{1}{4} u_x^3 - u_x u_x^1 - \frac{1}{2} u_x u_x^1,
\]

and its first commuting flow

\[
u_{x}^1 = -\frac{3}{2} u_x u_x^2 - \frac{3}{2} u_x u_x^1 + \frac{15}{8} (u')^2 u_x^1 + \frac{1}{4} u_x^4,
\]

\[
u_{x}^2 = -\frac{3}{2} u_x u_x^2 - \frac{3}{8} (u')^2 u_x^2 + \frac{3}{2} u_x u_x^1 + \frac{1}{4} u_x^2 - \frac{3}{16} [(u')^2]_{xx}.
\]

Substitution of (see (11))

\[
a_1 = -\frac{1}{2} u', \quad a_2 = -\frac{1}{2} u^2 + \frac{3}{8} (u')^2
\]

into three-dimensional quasilinear system (44) leads to an identity.

4. The Camassa–Holm equation \((N = 1, M = 0)\). Substitution expressions \((\xi_1, \xi_2)\) are arbitrary parameters)

\[
a_1 = -\frac{1}{2} \left( 1 - \frac{1}{4} a_x^2 \right) \left( u^1 + \frac{1}{4}, \xi_1 \right),
\]

\[
a_2 = \left( 1 - \frac{1}{4} a_x^2 \right) \left( \frac{1}{4} a_x a_x a_x + \frac{3}{8} a_x^2 + \xi_1 a_1 + \xi_2 \right)
\]

into three-dimensional quasilinear system (44) leads to an identity, if the function \(u^1(x, t, y)\) is a solution of the pair of nonlocal equations (see (38))

\[
u_{x}^1 = a_1 u_x^1 + 2 a_x a_x^1, \quad u_x^1 = 2 a_x a_x^1 + a_2 u_x^1.
\]

5.1. Negative flows

Now we consider another asymptotic expansion of the function (cf (13), (14))

\[
a(x, t, \zeta) = -a_{-1} - \zeta a_{-2} - \zeta^2 a_{-3} - \ldots, \quad \zeta \to 0
\]

(49)

The consistency of two generating functions of conservation laws (cf (43))

\[
p_{x} = -\frac{1}{2} (a_{-1} p_{x}), \quad p_{x} = \left[ \left( \frac{1}{2} + a_{1} \right) p_{x} \right]_{x}
\]
implies the second three-dimensional quasilinear system (cf (44))
\[ a_{1,1} = a_{-1,1}, \quad a_{-1,1} = a_{1,1} - a_{1,1}; \]  
(50)
consistency of two generating functions of conservation laws
\[ p_{r-1} = -\left( \frac{a_{-1}}{\lambda} p \right)_t, \quad p_{r-2} = -\left[ \left( \frac{a_{-1}}{\lambda} + \frac{a_{-1}}{\lambda^2} \right) p \right]_t \]  
(51)
leads to the third three-dimensional quasilinear system
\[ a_{-1,1} = a_{-2,1} - a_{1,1}, \quad a_{-1,1} = a_{-2,1} - a_{1,1}. \]  
(52)

**Remark**: Taking into account that \( p = 1/\phi \), substitution of formal expansion (23) instead of \( \phi \) into (51) yields two commuting hydrodynamic chains\(^9\) to (45), (46)
\[ a_{k+1,1} = a_{k,1} - a_{k,1}, \quad k = 0, 1, 2, \ldots \]  
(53)
\[ a_{k+2,2} = a_{k+1,3} - a_{k,3} a_{k+1,3} + a_{k,3} a_{k,3} - a_{k,3} a_{k,3}, \quad k = -1, 0, 1, \ldots \]  
(54)
Moreover, all of the above hydrodynamic chains can be extended on negative values of the index \( k \) (see, for instance, [16] and [5]). Thus, taking the first ‘negative’ equation of hydrodynamic chain (45)
\[ a_{1,1} = a_{1,1} - a_{1,1} \]

(together with the first equation of hydrodynamic chain (53) \( a_{1,1} = a_{-1,1} \), one can obtain three-dimensional quasi-linear system (50); taking the first two ‘negative’ equations of hydrodynamic chain (53)
\[ a_{1,2} = a_{1,2} - a_{1,2}, \quad a_{2,1} = a_{2,1} - a_{2,1}, \]  
(53)
\[ a_{-1,1} = a_{-2,1} - a_{1,1} a_{1,1} + a_{1,1} a_{1,1} - a_{1,1} a_{1,1}, \]  
(54)
and eliminating the common block \( a_{-2,1} - a_{1,1} a_{1,1} \), one can obtain three-dimensional quasi-linear system (52).

Instead of asymptotic expansion (49), we can consider an asymptotic expansion at any fixed point \( \zeta = \lambda_0 \). If we choose \( \lambda_0 = 1 \), then instead of (31), we obtain (see (13))
\[ a = b(x)(\lambda - 1)^{-1}, \quad b(x) = a(x, t, 1); \]  
if we keep \( \lambda_0 \) as an arbitrary constant, then we have another choice \( a = c(x)(\lambda - \lambda_0)^{-1} \), where \( c(x) = a(x, t, \lambda_0) \). Thus, we must introduce two extra independent variables \( y^{-1}, z^{-1} \) such that two extra generating functions of conservation laws
\[ p_{r-1} = -\left( \frac{b_{-1}}{\lambda - 1} p \right)_t, \quad p_{r-1} = -\left( \frac{c_{-1}}{\lambda - \lambda_0} p \right)_t \]
can be determined as a part of an integrable hierarchy, which contains hydrodynamic chain (45) together with all its commuting flows. This means that all functions \( a_k \) (see, for instance, (45), (46), (49)), all functions \( u^k \) (see, for instance, (7)) and these three extra functions \( b_{-1}, c_{-1} \) and \( p \) depend on ‘time’ variables \( t \) and \( y^{-1}, z^{-1} \) simultaneously. Consistency of two generating functions of conservation laws
\[ p_{r-1} = -\left( \frac{1}{\lambda - 1} a_{-1} p \right)_t, \quad p_{r-1} = -\left( \frac{b_{-1}}{\lambda - 1} p \right)_t \]
\[ p_{r-1} = -\left( \frac{1}{\lambda - 1} a_{-1} p \right)_t, \quad p_{r-1} = -\left( \frac{b_{-1}}{\lambda - 1} p \right)_t \]
\[ p_{r-1} = -\left( \frac{1}{\lambda - 1} a_{-1} p \right)_t, \quad p_{r-1} = -\left( \frac{b_{-1}}{\lambda - 1} p \right)_t \]

\(^9\) Let us remind that \( a_0 = 1 \).
yields the fourth three-dimensional quasilinear system
\[ a_{-1,t} = b_{-1,t}, \quad a_{-1,y} = b_{-1}a_{-1,t} - a_{-1}b_{-1,t}, \]
(55)

**Remark:** Introducing a potential function \( W \) such that
\[ a_{-1,t} = W_t, \quad a_{-1,y} = W_y, \quad b_{-1,t} = W_{t,t}, \quad b_{-1,y} = W_{t,y}, \quad \] systems (44), (50), (52), (55) can be written as four three-dimensional quasilinear equations of a second order (reminder: \( x = t^0, \ y = t^1, \ y = t^2 \)), respectively
\[ W_tW_{t,t} + W_yW_{t,y} = W_tW_y + W_{t,t} - W_{t,y}, \quad W_{t,t} = W_{t,y} - W_{t,y}, \quad W_{t,t} = W_{t,y} - W_{t,y}. \]
(56)

Now we consider three generating functions of conservation laws
\[ p_{t,1} = -\frac{1}{\lambda}(a_{-1}p)_{t,1}, \quad p_{y,1} = -\frac{1}{\lambda - 1}(b_{-1}p)_{t,1}, \quad p_{z,1} = -\frac{1}{\lambda - \lambda_0}(c_{-1}p)_{t,1}, \]

where \( a_{-1} = W_t, \ b_{-1} = W_{t,t} \) and \( c_{-1} = W_{t,y} \). One can introduce a potential function \( S \) such that
\[ dS = pdx - \frac{1}{\lambda}a_{-1}pdt - \frac{b_{-1}}{\lambda - 1}pdz - \frac{c_{-1}}{\lambda - \lambda_0}pdz. \]

Then the compatibility conditions \((S_{-1})_{t,1} = (S_{-1})_{t,1}\) and \((S_{-1})_{y,1} = (S_{-1})_{t,1}\) yield
\[ q_{t,1} = \left( \frac{\lambda}{\lambda - 1} \right) b_{-1} q_{t,1}, \quad q_{y,1} = \left( \frac{\lambda}{\lambda - \lambda_0} \right) a_{-1} q_{t,1}, \]

where \( q = a_{-1}p/\lambda \). The compatibility condition \((S_{-1})_{t,1} = (S_{-1})_{t,1}\) is equivalent to the compatibility condition \((q_{t,1})_{t,1} = (q_{t,1})_{t,1}\), which implies the fifth three-dimensional quasilinear system
\[ (\lambda_0 - 1)W_tW_{t,y} + W_{t,y}W_{t,y} - \lambda_0 W_{t,y}W_{t,y} = 0, \]
(57)

Recently a complete classification of linearly degenerate three-dimensional quasilinear equations of a second order was presented in [11]. The list of these equations\(^{10}\) coincides with the five equations (56), (57).

Since all of these five equations belong to the same hierarchy (the function \( W \) is common), the computation of differential reductions of the last four three-dimensional quasilinear equations of a second order reduces to the problem of computation of higher (positive) and lower (negative) commuting flows of integrable dispersive systems (28). For instance, the second three-dimensional quasilinear system (50) possesses infinitely many differential constraints \( a_{-1}(u, u, ..., u), a_{-1}(u, u, ..., u) \) (see expressions (29), (34), (35)), where field variables \( u^i(t, t, t) \) are solutions of dispersive systems (28) and (32). Moreover all lower differential constraints \( a_{-2}(u, u, ..., u) \) can be found directly from (17) by substitution of the expansion (cf (23))
\[ \varphi(x, t, \lambda) = a_{-1} + \lambda a_{-2} + \lambda^2 a_{-3} + ..., \ \lambda \to 0, \]

Together with (27). Indeed, the ordinary differential equation on the function \( a_{-1} \) coincides with (34). The next coefficient \( a_{-2} \) is determined by the ordinary differential equation\(^{11}\)

\[^{10}\text{All of these three-dimensional quasilinear equations of a second order can be found, for instance, in [16] and [5]. Some other references are in [11].} \]
\[^{11}\text{Constants } \zeta_0, \zeta_i \text{ are first coefficients of the expansion } e^x = \zeta_0 + \lambda \zeta_1 + ... \text{ (see (17)). Obviously, the coefficient } \xi \text{ coincides with } \zeta_0. \]
\[ (-1)^M \pi^{\frac{1}{14}} \prod_{k=1}^{M+K} s^{\pi} \left( \sum_{p=1}^{K} \frac{1}{r^p} - \sum_{k=1}^{M+K} \frac{1}{s^k} \right) = \frac{a_{-2, \pi}}{2a_{-1}} - \frac{a_{-1, \pi}}{2a_{-1}} \]

\[ \left( a_{-2, \pi} - \frac{a_{-1, \pi}}{2a_{-1}} \right) a_{-2} + \frac{\xi}{4a_{-1}} \]

where the function \( a_{-1} \) as already determined by (34). Thus the third three-dimensional quasilinear system (52) possesses differential reductions (32) and

\[ r^k = - \left( \frac{a_{-2}}{r^k} + \frac{a_{-1}}{(r^k)^2} \right) r_s^k \]

\[ s^k = - \left( \frac{a_{-2}}{s^k} + \frac{a_{-1}}{(s^k)^2} \right) s_s^k - \frac{1}{2(s^k)^2} \prod_{k=1}^{M} \left( s^k - r^k \right) \]

where differential constraints \( a_{-1}(u, u, ...), a_{-2}(u, u, ...) \) are determined by (34) and (58). More lengthy computations lead to similar but more complicated expressions for the fourth three-dimensional quasilinear system (55) and for the fifth three-dimensional quasilinear equation of a second order (57). We omit corresponding derivations here.

6. The dispersionless limit

Hydrodynamic chain (45) also can be written in the form (see formula (53) in [16])

\[ u^k = u^k - \frac{1}{2} u^k u_s^k - u^k u_s^k. \]

Indeed, the substitution \( q = q^{-1/2} \) into the second equation in (19) yields

\[ q_t = \left( \lambda - \frac{1}{2} u_t \right) q_t = - q u_s^k, \]

where \( a_t = - u_t / 2 \) and (taking into account (23))

\[ q = 1 + \frac{u_t}{\lambda} + \frac{u_t}{\lambda^2} + \frac{u_t}{\lambda^3} + ... \]

Thus hydrodynamic chain (59) is a dispersionless limit of \( M \)th dispersive chain (7) for any \( M > 0 \) (also this statement is valid for \( M = 0 \), see section 4). So we come to an interesting observation: hydrodynamic chain (45) under infinitely many triangular invertible transformations (11) can be written as dispersive chain (7) (by infinitely many different ways depending on the natural number \( M \)), whose dispersionless limit (59) is equivalent to original hydrodynamic chain (45) up to invertible triangular point transformations. Hydrodynamic chain (59) and its higher commuting flows\(^\text{12}\) have infinitely many local Hamiltonian structures. They can be easily obtained by dispersionless limit from local Hamiltonian structures of \( M \)th dispersive chain (see subsection 2.3). The first three of them are (\( s = 1, 2, ... \))

\(^\text{12}\) These commuting flows have a dispersionless limit of (10). Here coefficients (see (24)) \( a_t = \frac{h_m}{a^m}, m = 0, 1, s = 1, 2, ... \), where Hamiltonian densities \( h_t(u) \) are determined below.
All Hamiltonian densities \( h_k(u) \) can be found from (43) by substitution \( p = 1/\varphi \), where expansion of \( \varphi \) is determined by (23). For instance first conservation law densities are (corresponding conservation laws follow from (43) and (59); see other detail in [16])

\[
        h_1 = u^1, \quad h_2 = u^2 - \frac{1}{4}(u^1)^2, \quad h_3 = u^3 - \frac{1}{2}u'u^2 + \frac{1}{8}(u^1)^3, \quad \ldots \\
        h_4 = u^4 - \frac{1}{2}u'^2 - \frac{1}{4}(u^1)^2 + \frac{3}{8}(u^2)^2 - \frac{5}{64}(u^1)^4, \ldots
\]

A dispersionless limit of dispersive reductions associated with ansatz (5) was investigated in [12]. In this section we consider a dispersionless limit of dispersive reductions (28), i.e. the dispersionless systems

\[
    r^k_i = (r^k + a_i)r^k_i, \quad s^i_j = (s^i + a_i)s^i_j,
\]

where

\[
    a_i = \frac{1}{2} \left( \sum_{m=1}^{M} s^m - \sum_{i=1}^{N} r^k \right).
\]

Thus the field variables \( r^k, s^m \) are Riemann invariants (cf formula (6) in [12]). These coordinates \( r^k, s^m \) are most convenient for comparison of distinct integrable systems. For instance, Kaup–Boussinesq system (4), Ito system (25) and two component generalization of the Camassa–Holm equation (39) have the same dispersionless limit (see the end of subsection 3.2)

\[
    s^1_1 = \frac{1}{2}(3s^1 + s^2)s^1_1, \quad s^2_1 = \frac{1}{2}(s^1 + 3s^2)s^2_1.
\]

However, these three integrable dispersive systems are not connected by any differential substitutions. They are different, but have the same dispersionless limit.

In paper [16] we proved that hydrodynamic chain (45) possesses \( N \) component hydrodynamic reductions

\[
    r^k_i = (r^k + a_i)r^k_i, \quad \ldots
\]

where \( f_1(r^k) \) are arbitrary functions

\[
    a_i = \sum_{m=1}^{N} f_k r^m.
\]

Thus, in this paper we proved that hydrodynamic type system (60) has a dispersive integrable extension (28) if
a_k = \frac{1}{2} \sum_{n=1}^{N} e_k^n r^n,\

where \( e_k = \pm 1 \), while in [12] all constants \( e_k = 1 \).

**Remark**\(^{13} \): Kaup–Boussinesq system (4) can be written in the conservative form

\[
u_1 = \left( w - \frac{1}{2} (u')^2 \right)_x, \quad \nu_2 = \left( \frac{1}{4} u_1^2 - w' \right)_x. \tag{61}
\]

where

\[
w = u^2 - \frac{1}{4} (u')^2.
\]

Under the invertible substitution

\[
w = \rho \pm \frac{1}{2} u_1^2, \tag{61}
\]

becomes the Kaup–Broer system

\[
u_1 = \left( \rho - \frac{1}{2} (u')^2 \pm \frac{1}{2} u_1^2 \right)_x, \quad \rho_1 = - \left( u' \rho \pm \frac{1}{2} \rho_1 \right)_x. \tag{62}
\]

Under the invertible substitution

\[
v = u_1^2 + \frac{1}{2} \rho_1, \tag{62}
\]

reduces to the Hasimoto form of the nonlinear Schrödinger equation

\[
u_1 = \left( \rho - \frac{1}{2} u_1^2 \pm \frac{1}{2} \rho_1 \right)_x. \tag{63}
\]

Indeed, under the Madelung substitution

\[
\psi = \sqrt{\rho} e^{i\phi}, \quad \psi^* = \sqrt{\rho} e^{-i\phi},
\]

(63) transforms into the nonlinear Schrödinger equation

\[
\psi_t + \frac{1}{2} \psi_{xx} - \psi \psi^* \psi = 0, \quad \psi_t^* - \frac{1}{2} \psi_{xx}^* + \psi^* \psi \psi^* = 0. \tag{64}
\]

Thus nonlinear Schrödinger equation (64) is reducible to the form (30), where

\[
\rho = - \frac{1}{4} (s_1 - s_2)^2 \pm \frac{1}{2} (s_1 + s_2), \quad v = - s_1 - s_2 \pm \frac{1}{2} (\ln \rho),
\]

Also nonlinear Schrödinger equation (64) is well known in the complex form

\[
i \psi_t - \frac{1}{2} \psi_{xx} - \psi \psi^* \psi = 0, \quad i \psi_t^* + \frac{1}{2} \psi_{xx}^* + \psi^* \psi \psi^* = 0, \tag{65}
\]

which follows from (64) by the complex change of independent variables \( x \to ix, t \to it \). In such a case (65) can be written in the form (cf (30))

\(^{13} \) This derivation of the relationship between the Kaup–Boussinesq system (4), the Kaup–Broer system (62) and the nonlinear Schrödinger equation (63) belongs to Solomon J Alber.
Thus, all of the above systems possess the same dispersionless limit. The field variables \( s^1, s^2 \) are natural coordinates for the nonlinear Schrödinger equation. For investigation of shock waves (\( \epsilon \to 0 \)) the nonlinear Schrödinger equation can be written in the form (by rescaling \( \partial_x \to \epsilon \partial_x, \partial_t \to \epsilon \partial_t \))

\[
s^1_i = \frac{1}{2} \left( 3s^i + s^2 \right) x_s^i - \frac{e^2}{4} \left( s^i + s^2 \right), \quad s^2_i = \frac{1}{2} \left( s^i + 3s^2 \right) x_s^i + \frac{e^2}{4} \left( s^i + s^2 \right).
\]

Such a consideration obviously is valid just if the difference \( s^i - s^2 \) is not small.

7. Conclusion

In this paper we found new multi-component integrable dispersive systems (28) associated with the energy dependent Schrödinger equation. We believe that the same approach based on construction of dispersive chains can be utilized for all other linear spectral problems. Moreover we hope that each linear spectral problem can be associated with some three-dimensional linearly degenerate quasilinear system of a first order. Thus we would like to establish a link between three-dimensional linearly degenerate quasilinear systems of a first order and two-dimensional dispersive integrable systems.

A more general alternative approach for extraction of integrable dispersive systems from (2) is based on the extension of a method of hydrodynamic reductions (see detail in [10]) to integrable dispersive deformations. In this case functions \( a_1 \) and \( a_2 \) (see (44)) depend on \( N \) Riemann invariants \( r^k \) and can be written in the quasipolynomial form

\[
a_1 = a_{10}(r) + \epsilon b_{1k}(r) r^k + e^2 \left( c_{10m}(r) r^x_k r^m_x + c_{1k}(r) r^k_x \right) + ..., \quad a_2 = a_{20}(r) + \epsilon b_{2k}(r) r^k + e^2 \left( c_{20m}(r) r^x_k r^m_x + c_{2k}(r) r^k_x \right) + ...,
\]

where Riemann invariants \( r^k(\chi, t, y) \) satisfy two commuting systems

\[
r_{i}^{k} = \lambda_{i}^{k}(r) r_{x k}^{i} + e \left( \lambda_{i}^{k}(r) r_{x k}^{i} + \lambda_{i}^{k}(r) r_{x k}^{i} \right) + ... \quad (66)
\]

\[
r_{j}^{k} = \mu_{j}^{k}(r) r_{x k}^{j} + e \left( \mu_{j}^{k}(r) r_{x k}^{j} + \mu_{j}^{k}(r) r_{x k}^{j} \right) + ... \quad (67)
\]

Functions \( a_{10}(r), b_{1k}(r), \lambda_{i}(r), ..., \) are not yet determined. However, substitution of these expressions into (44) leads to (see detail in [16])

\[
a_{10}(r) = \sum_{n=1}^{N} \Phi_n(r^m), \quad a_{20}(r) = \frac{1}{2} a_{10}^2 + \sum_{m=1}^{N} \int f_m(r^m) \Phi_n(r^m) dr^m, \quad \lambda_{i}(r) = f_{i} (r) + a_{10} f_i (r) + a_{20}.
\]

All other coefficients \( c_{10m}(r), c_{1k}(r), \lambda_{i}^{k}(r), \lambda_{j}^{k}(r), ..., \) can be found iteratively for each degree of the parameter \( \epsilon \). Integrability of systems (66), (67) follows from (2). Corresponding
determine a common function $u(x, t, y, \lambda)$, which one can look for in the form

$$u = u_0(r, \lambda) + cu_{1k}(r, \lambda) r^k + \epsilon^2 \left[ a_{2k}(r, \lambda) r^k + u_{2mn}(r, \lambda) r^k r^m \right] + \ldots$$

This investigation should be made in a separate paper.

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