Bateman method for two-body scattering without partial-wave decomposition

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We explore the use of Bateman method for solving the two-variable version of the Lippmann-Schwinger equation for two-body scattering without invoking the partial-wave decomposition. In our adaptation of the Bateman method to the Lippmann-Schwinger equation, the momentum-space kernel of the potential is interpolated by a separable expansion constructed from its sections on a multi-variate grid. We describe a suitable scheme for constructing a multi-variate Cartesian grid that allows for the treatment of the singularity of the free propagator with due care. The method is tested in the nucleon-nucleon scattering employing a model two-nucleon potential. Our results demonstrate that the Bateman method can produce quite accurate solutions with relatively small number of grid points.

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I. INTRODUCTION

The traditional strategy in the treatments of quantum scattering problems have been the elimination of angular variables via expansions over angular-momentum states. Certain drawbacks of this strategy have been noted in recent years, especially for high-energy collisions and within the context of few-body problems. As a result, computational methods that avoid the traditional decomposition of wave functions and scattering amplitudes into partial waves have been explored recently by a number of groups [1-13]. Various direct multi-variable solution techniques for two-body Lippmann–Schwinger (LS) equation have been investigated. Most studies employed the Nystrom method (i.e., discretization of the integral equation via a suitable multi-variate quadrature) [14]. Although the Nystrom method can produce very accurate results, the matrix dimensions in the multi-variable Nystrom approach can grow very fast to computationally prohibitive levels. Multi-variable methods that lead to a reduction in the matrix sizes are therefore of considerable interest.

Methods based on basis-set expansions, such as Galerkin [3,5], collocation [13], and Schwinger variational methods [13], have been investigated with various choices of multi-variable bases. These expansion methods give rise to matrix elements involving multi-variable integrals that are typically evaluated by quadrature. Assuming that the quadrature rule used in Nystrom method is also employed to construct the matrix elements of, say, the Galerkin method,
the final linear equation system of the Galerkin method can be viewed as a contraction of the linear system of equations that the Nystrom approach gives rise to. In effect, the basis-set expansion methods represent replacement of the (large) equation system of the Nystrom method by a smaller set of approximate equations, by demanding that a residual vanishes on a chosen test space [13]. In numerical linear algebra, such contractions/projections form the basis for many (iterative) methods to solve large systems of equations.

In this article we consider the Bateman method as a tool for the solution of the multi-variable integral equations of the two-body scattering problem. This method was originally proposed by Bateman [15] to solve single-variable integral equations by interpolating the bi-variate kernel by its sections on an interpolation grid. Analysis of its convergence and error bound is given in Refs. [16] and [17].

The Bateman method had drawn some interest in 1970's in the context of few-body scattering calculations[18-23]. However, almost all discussions and applications have been restricted to partial-wave (single-variable) LS equations. As far as the present author is aware, the only Bateman application that avoids partial-wave expansion is by Gianini and Lim [22] who applied the Bateman method to the two-variable LS equation for a Yukawa potential. The interpolation grid in their implementation had been severely limited, and as a result the full potential of the method does not seem to have been realized.

Although the Bateman method could be used directly on the three-dimensional LS equation, the present paper concentrates on its application to the reduced (two-variable) version of the LS equation in which the azimuthal angle dependence is integrated out. We use a cartesian bi-variate grid that pays due care for a symmetric treatment of the singularity of the free propagator in the kernel of the LS equation. The results of our calculations for the Malfliet-Tjon potential [24] demonstrate that the promise of the Bateman method as a simple and viable tool to solve multi-variable scattering equations is in fact borne out. It appears that the choice for the interpolation grid is instrumental in obtaining rather accurate results with relatively few interpolation points.

Another observation that does not appear to have been noticed before is the intimate relation between Nystrom and Bateman methods: A Nystrom calculation (with a given quadrature grid) can also be viewed as a Bateman calculation in which the quadrature grid employed in the Nystrom method serves as both interpolation and quadrature grids. This equivalence is borne out in our numerical calculations as well, even though our implementations of Nystrom and Bateman methods are organized differently. When the quadrature grid used in Nystrom calculation is employed as both interpolation and quadrature grid in the Bateman calculation, the results of two calculations are indistinguishable within at least 8 digits after the decimal point.

Plan of this article is as follows: In Sect. II, we fix notation and give a derivation of the reduced two-variable LS equation from the full three-dimensional LS equation. Sect. III defines and elaborates the Bateman method for the two-
variable LS equation. In Sect. IV, the treatment of the singular integrals, the selection of the interpolation and quadrature grids, and the connection between Nystrom and Bateman approaches is presented. In Sec. V, the results of calculations for a model two-nucleon potential are discussed and compared to benchmark results of Nystrom calculations. In Sec. VI, we summarize our conclusions.

II. LIPPMANN-SCHWINGER EQUATION

The Lippmann-Schwinger (LS) equation for two-body scattering in operator form reads

\[ T(z) = V + V G_0(z) T(z) , \]

where \( T \) is the transition operator, \( V \) the two-body potential, \( G_0 = (z-H_0)^{-1} \), with \( H_0 \) being the free hamiltonian and \( z \) the (complex) energy of the two-body system. Working in the center-of-mass frame, the eigenstates of \( H_0 \) are the relative momentum states \( |q> \), viz., \( H_0 |q> = (\frac{q^2}{2\mu}) |q> \). For on-shell scattering, \( z = E + i0 \), with \( E = q_0^2/2\mu \), where \( \mu \) is the reduced mass. The momentum-space matrix elements \( T(q,q_0,z) \) satisfy

\[ T(q,q_0,z) = V(q,q_0) + \int dq' \frac{V(q,q') T(q',q_0,z)}{z - q'^2/2\mu} \]

The \( z \)-dependence of the \( T \)-matrix elements \( T(q,q_0;z) \) will be suppressed, unless there is a need to explicitly show the energy dependence. The momentum-space representation \( V(q,q') \) of the potential \( V \) is given as

\[ V(q,q') = <q|V|q'> = \int d\mathbf{r} <q|\mathbf{r}> V(\mathbf{r}) <\mathbf{r}|q'> , \]

with \( <q|\mathbf{q}> = e^{i\mathbf{r}\cdot\mathbf{q}}/(2\pi)^{3/2} \). For central potentials, \( V(q,q') \) and \( T(q,q') \) depend only on \( q, q' \) and \( x_{qq'} \). Here, \( x_{qq'} \) denotes the cosine of the angle between vectors \( q \) and \( q' \). We denote the polar and azimuthal angles of the momentum vectors \( q \) by \( \theta \) and \( \phi \), respectively. We then have \( x_{qq'} = \hat{q} \cdot \hat{q}' = \cos \theta_{qq'} = xx' + ss' \cos (\phi - \phi') \), where \( x = \cos \theta \) and \( s = \sqrt{1-x^2} \). To emphasize this functional dependence on \( x_{qq'} \), we will occasionally use the notation \( T(q,q',x_{qq'}) \) to stand for \( T(q,q') \).

For central potentials, the azimuthal-angle dependence in Eq. (2) can be integrated out to obtain a two-dimensional integral equation [1]. Towards this end, we introduce the averaged momentum states \( |qx> \) via

\[ |qx> = (2\pi)^{-1/2} \int_0^{2\pi} d\phi |q> \]

with \( |q> \) as the classical wavefunction of a two-body system. For \( qx \), we have

\[ |qx> = (2\pi)^{-1/2} \int_0^{2\pi} d\phi |q\theta\phi> . \]
For a two-body operator $A$, we introduce reduced matrix elements by
\[
A(q, x; q', x') = \langle qx | A | q' x' \rangle = \int_0^{2\pi} d\phi \, A(q, q', x_q x'_q). \tag{5}
\]

For a rotationally invariant operator $A$, the above integral is independent of the variable $\phi'$. Integrating Eq. (2) over $\phi$, we obtain the two-variable LS equation for the reduced $T$-matrix:
\[
T(q, x; q_0, x_0) = V(q, x; q_0, x_0) + 2\mu \int_0^{\infty} q'^2 dq' \int_{-1}^{1} dx' \frac{V(q, x; q', x') T(q', x'; q_0, x_0)}{q'^2 - q^2 + i0}. \tag{6}
\]

If we take the initial momentum vector $q_0$ along the z-axis, the half-off-shell $T$-matrix element for a general final momentum vector $q_f$ is then given by
\[
\langle q_f | T | q_0 z \rangle = T(q_f, q_0, x_f) = (2\pi)^{-1} T(q_f, x_f : q_0, 1).
\]

Direct numerical solution of this two-variable Lippmann-Schwinger (LS) equation without invoking the partial wave expansion can be performed nowadays in commonly available computational platforms. The approach used most frequently is the Nystrom method [14] in which the integrals over $q'$ and $x'$ are approximated by a suitable two-variable quadrature and then $x$ and $q$ variables are collocated at the quadrature points. This gives rise to a system of linear equations. Although the number of linear equations for the two-variable case is manageable and does not require special computing environment, going beyond two variables makes the matrix size quickly become computationally prohibitive. Therefore, in the contexts of three and four body problems, alternatives to Nystrom method would be welcome. Bateman method described in the next section is such an alternative.

### III. BATEMAN METHOD

Bateman method is based on a special kind of interpolation of $V(q, x; q', x')$ on a finite set of grid points (nodes) in the $q - x$ domain. Suppose two sets of nodes have been prescribed: $N$ distinct points $\{q_1, q_2, ..., q_N\}$ for $q$ in the interval $[0, \infty)$ and $M$ distinct points $\{x_1, x_2, ..., x_M\}$ for $x$ in the interval $[-1, +1]$. The Cartesian grid generated by the Cartesian product of these two sets is the set
\[
\mathcal{N} = \{q_1, q_2, ..., q_N\} \times \{x_1, x_2, ..., x_M\} = \{(q_n, x_m) : 1 \leq n \leq N, 1 \leq m \leq M\}. \tag{7}
\]
The set \( \mathcal{N} \) will be referred to as the interpolation grid. The Bateman interpolate \( V^B(q, x; q', x') \) of \( V(q, x; q', x') \) is defined as

\[
V^B(q, x; q', x') = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{n'=1}^{N} \sum_{m'=1}^{M} V(q, x; q_n, x_m) A_{nm,n'm'} V(q_{n'}, x_{m'}; q', x'),
\]

where the matrix \( A \) is defined via

\[
(A^{-1})_{nm,n'm'} = V(q_n, x_m; q_{n'}, x_{m'}).
\]  

(9)

The (exact) transition operator \( T^B \) for the separable potential \( V^B \) is then given as

\[
T^B(q, x; q', x') = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{n'=1}^{N} \sum_{m'=1}^{M} V(q, x; q_n, x_m) D_{nm,n'm'} V(q_{n'}, x_{m'}; q', x'),
\]

where the matrix \( D \) is defined via

\[
(D^{-1})_{nm,n'm'} = \langle q_n x_m | V - V G_0 V | q_{n'} x_{m'} \rangle,
\]

\[
= V(q_n, x_m; q_{n'}, x_{m'}) - 2\mu \int_0^\infty q'^2 dq' \int_{-1}^1 dx' \frac{V(q_n, x_m; q', x') V(q', x'; q_{n'}, x_{m'})}{q_0^2 - q'^2 + i0}.
\]  

(10)

This result corresponds to another instance of the weighted-residual approach [13] for the solution of the LS equation. It also follows from Schwinger variational method [23,25,26] if the wave function is expanded in the set of reduced momentum states \( \{ |q_n x_m \rangle, n = 1, ..., N, m = 1, ..., M \} \). On the other hand, \( V^B \) and \( T^B \) can as well be viewed, in the terminology of Ref. [27], as inner-projection approximations to the operators \( V \) and \( V(V - V G_0 V)^{-1}V \) [26], respectively.

IV. COMPUTATIONAL IMPLEMENTATION

A. Singular Integrals

To computationally implement the Bateman method, we need to introduce a quadrature rule on the \( q - x \) computational domain for the evaluation of the matrix elements \( \langle q_n x_m | V G_0 V | q_{n'} x_{m'} \rangle \). We opt for a tensor-product quadrature scheme. Suppose \( \{ q_\alpha, \alpha = 1, ..., N_q \} \) denote a suitable set of quadrature points for the \( q \)-variable, with corresponding weights \( \{ w_\alpha, \alpha = 1, ..., N_q \} \). Similarly, let \( \{ x_\beta, \beta = 1, ..., N_x \} \) denote a set of quadrature points for the \( x \)-variable, with corresponding weights \( \{ \rho_\beta, \beta = 1, ..., N_x \} \). (Quadrature points are indexed by Greek letters \( \alpha \) and \( \beta \), while indices \( n \) and \( m \) are reserved for interpolation points.) A quadrature rule of order \( N_q N_x \) is thus provided by the set \( \{ (q_\alpha, x_\beta) \} \)
of quadrature points (referred to as the quadrature grid) and the set \( \{(w_\alpha, \rho_\beta)\} \) of quadrature weights.

The evaluation of the matrix elements \( <q_n x_m| V G_0 V | q_n' x_{m'}> \) are carried out using essentially the same subtraction procedure described in detail in Ref. [13]. The singular integral is separated into its real and imaginary parts as

\[
<q_n x_m| V G_0 V | q_n' x_{m'}> = 2 \mu A_{nm,n'm'} - i\pi \mu q_0 B_{nm,n'm'}(q_0),
\]

where

\[
A_{nm,n'm'} = \mathcal{P} \int_{0}^{q_{\text{max}}} dq \frac{q^2 B_{nm,n'm'}(q)}{q_0^2 - q^2},
\]

\[
B_{nm,n'm'}(q) = \int_{-1}^{1} dx <q_n x_m| V| qx> <qx| V| q_n' x_{m'}> ,
\]

where \( \mathcal{P} \) stands for principle-value integral. By adding and subtracting a singular integral that can be evaluated analytically, singular term \( A_{nm,n'm'} \) is rearranged as a sum of non-singular and singular terms:

\[
A_{nm,n'm'} = A_{nm,n'm'}^{(ns)} + A_{nm,n'm'}^{(s)},
\]

where

\[
A_{nm,n'm'}^{(ns)} = \int_{0}^{q_{\text{max}}} dq \frac{q^2 B_{nm,n'm'}(q)}{q_0^2 - q^2},
\]

\[
A_{nm,n'm'}^{(s)} = B_{nm,n'm'}(q_0) \int_{0}^{q_{\text{max}}} dq \frac{q_0^2}{q_0^2 - q^2} = B_{nm,n'm'}(q_0) \frac{q_0}{2} \ln \frac{q_{\text{max}} + q_0}{q_{\text{max}} - q_0}.
\]

As the integrals involved in \( B_{nm,n'm'}(q) \) and \( A_{nm,n'm'}^{(ns)} \) are non-singular, they are amenable to approximation by quadrature with the result

\[
B_{nm,n'm'}(q) \approx \Sigma_{\beta=1}^{N_q} \rho_\beta <q_n x_m| V| q x_\beta> <q x_\beta| V| q_n' x_{m'}> ,
\]

\[
A_{nm,n'm'} \approx \Sigma_{\alpha=1}^{N_q} w_\alpha q_0^2 \frac{B_{nm,n'm'}(q_\alpha)}{q_0^2 - q_\alpha^2} + C_{\text{sing}} q_0^2 B_{nm,n'm'}(q_0),
\]

where

\[
C_{\text{sing}} = \frac{1}{2q_0} \ln \frac{q_{\text{max}} + q_0}{q_{\text{max}} - q_0} - \sum_{\alpha=1}^{N_q} \frac{w_\alpha}{q_0^2 - q_\alpha^2}.
\]

Note that as long as the quadrature grid treats the singularity symmetrically and is sufficiently dense in its vicinity, the correction term involving \( C_{\text{sing}} \) comes out to be very small and can even be omitted without causing a significant loss of accuracy.

**B. Interpolation Grid and Quadrature**
To construct the Cartesian interpolation grid, we need to specify two uni-variate grids, one in $q$ and one in $x$. To define the $q$-grid $\{q_n\}$, the full $q$-domain is divided into two intervals: $[0, 2q_0]$ and $[2q_0, \infty)$. This scheme is adopted to treat the singularity of the LS kernel at $q' = q_0$ in Eqs. (2) or (6) as symmetrically as possible, and to have a $q$-grid that is denser in the vicinity of $q_0$. To this end, the first interval $[0, 2q_0]$, is subdivided into $I_1$ (equal) subintervals (finite elements).

The second interval $[2q_0, \infty)$, however, is first mapped to $[-1, +1]$ via the transformation

$$u = \frac{q - 2q_0 - f}{q - 2q_0 + f}, \quad \text{or} \quad q = 2q_0 + f \frac{1 + u}{1 - u}, \quad (13)$$

where $f$ is a scale factor. The $q$-variable is cut off at some large but finite value $q_{\text{max}}$ by adopting an upper limit $u_{\text{max}}$ ($< 1$) to the variable $u$. In the calculations reported in this paper, we used $u_{\text{max}} = 0.99$, which corresponds to $q_{\text{max}}$ values of several thousand. This variable transformation is instrumental for a discretization of the semi-infinite interval $[0, \infty)$ with relatively few finite elements. The interval $[-1, u_{\text{max}}]$ is divided into $I_2$ equal finite elements. Note that this gives rise to a non-uniform partitioning for the $q$-variable in the interval $[2q_0, q_{\text{max}}]$. The total number of finite elements covering the computational interval $[0, q_{\text{max}}]$ is $I = I_1 + I_2$. The choice $I_2 = 3I_1$ (hence $I = 4I_1$) was found adequate after some experimentation. The $q$-grid $\{q_1, q_2, ..., q_N\}$ consists of the end points and mid-points of this finite-element partitioning. Note that $N = 2I_1 + 1$, $q_1 = 0$, and $q_N = q_{\text{max}}$.

The specification of the $x$-grid proceeds similarly to that of the $q$-grid. The interval $[-1, 1]$ is partitioned into $J$ equal subintervals (finite-elements). Collecting and ordering the endpoints and midpoints of the finite elements together, we define the set of grid points $\{x_1, x_2, ..., x_M\}$, where $M = 2J + 1$, $x_0 = -1$ and $x_M = +1$.

To construct a composite Gauss-Legendre quadrature rule for $q$, each finite-element $[q_{2i-1}, q_{2i+1}]$, $i = 1, 2, ..., I_1$, in the interval $[0, 2q_0]$, is mapped to $[-1, 1]$ via $s = (2q - q_{2i-1} - q_{2i+1})/(q_{2i+1} - q_{2i-1})$. For finite elements in $[2q_0, q_{\text{max}}]$, we map the finite elements $[u_{2i-1}, u_{2i+1}]$, for $i = 1, ..., I_2$, into $[-1, +1]$ via the map $s = (2u - u_{2i-1} - u_{2i+1})/(u_{2i+1} - u_{2i-1})$. We choose a set of $n_q$ Gauss-Legendre quadrature points in the local variable $s$, and then transform them back to $q$-variable. The Gauss-Legendre quadrature points for all elements are then combined and ordered to form a composite quadrature rule with the set of quadrature points $\{q_\alpha, \alpha = 1, 2, ..., N_q\}$, where $N_q = I_1n_q$. The quadrature weights are similarly collected in the set $\{w_\alpha, \alpha = 1, 2, ..., N_q\}$. In the calculations reported in the next section, values of $n_q$ ranged from 4 to 32, depending on the fineness of the finite-element partitioning. To ensure results stable within 6-7 digits after the decimal point, the total number $N_q$ of quadrature points were typically in the order of 160-200, although 3-4 digit accuracy could be achieved with, say, $N_q = 64$. 

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For the $x$-variable, each finite element $[x_{2j-1}, x_{2j+1}]$, is mapped to $[-1, +1]$ via the map $s = (2x - x_{2i-1} - x_{2i+1})/(x_{2i+1} - x_{2i-1})$ for $i = I_1, ..., I_n$. We choose $n_x$ Gauss-Legendre quadrature points in $s$ and transform back to the $x$-variable. The quadrature points and their weights over individual finite elements are collected in the sets $\{x_\beta, \beta = 1, 2, ..., N_x\}$, and $\{\rho_\beta, \beta = 1, 2, ..., N_x\}$, where $N_x = J n_x$. In our calculations, typically $N_x = 80$ was sufficient to obtain results stable within 6 digits.

C. Connection between Nystrom and Bateman Methods

The reference results against which the results of Bateman method will be tested are obtained by solving the two-variable integral equation via the Nystrom method. Details of the computational implementation of the Nystrom method is given in [13]. Nystrom result for T-matrix elements on the quadrature grid can be recast as

$$T^N(q_\alpha, x_\beta; q_{\alpha'}, x_{\beta'}) = \sum_{\alpha''=1}^{N_q} \sum_{\beta''=1}^{M_x} V(q_\alpha, x_\beta; q_{\alpha''}, x_{\beta''}) D_{\alpha\beta', \alpha'' \beta''} V(q_{\alpha''}, x_{\beta''}; q_{\alpha'}, x_{\beta'}) , \quad (14)$$

where the matrix $D$ is given as

$$(D^{-1})_{\alpha\beta', \alpha'' \beta''} = < q_\alpha, x_\beta | V - V G_0 V | q_{\alpha''}, x_{\beta''} > . \quad (15)$$

In Eq. (15), the multivariable integrals implicit in $< q_\alpha, x_\beta | V G_0 V | q_{\alpha'}, x_{\beta'} >$ are, of course, evaluated using the quadrature grid $(q_\alpha, x_\beta)$ with weights $(w_\alpha, \rho_\beta)$.

Eq. (14) represents a somewhat unusual depiction of the Nystrom method. This form manifestly shows the connection between Nystrom and Bateman methods. Eq. (14) is in fact the Bateman expression when quadrature grid also doubles as the interpolation grid. Therefore, if a given grid is employed as both interpolation and quadrature grids in the Bateman method, then the ensuing Bateman calculation becomes equivalent to the Nystrom calculation (using the same quadrature rule). Stated differently, a Nystrom calculation can indeed be viewed as a Bateman calculation as well. This formal equivalence is borne out in computations. Although computational organizations of our Nystrom and Bateman codes are quite different, their results agree to at least 8 digits when interpolation and quadrature grids are chosen to coincide, giving confidence that computer codes are performing satisfactorily.

V. RESULTS

We have tested the Bateman method for the Malfliet-Tjon III (MT-III) model [24] for the two-nucleon potential:

$$V(r) = V_R e^{-\mu_R r} - V_A e^{-\mu_A r}$$
whose momentum-space representation is given as

\[
V(q, q') = \frac{1}{2\pi^2} \left( \frac{V_R}{(q - q')^2 + \mu_R^2} - \frac{V_A}{(q - q')^2 + \mu_A^2} \right).
\]

For this potential the azimuthal integration in Eq. (4) can be carried out analytically to give

\[
V(q, x; q', x') = \frac{V_R/\pi}{\sqrt{(q^2 + q'^2 - 2qq'xx' + \mu_R^2)^2 - 4q^2q'^2(1 - x^2)(1 - x'^2)}} - \frac{V_A/\pi}{\sqrt{(q^2 + q'^2 - 2qq'xx' + \mu_A^2)^2 - 4q^2q'^2(1 - x^2)(1 - x'^2)}}.
\]

The parameters for MT-III potential are taken from Ref. [5]: \(V_A = 626.8932\) MeV fm, \(V_R = 1438.723\) MeV fm, \(\mu_A = 1.55\) fm and \(\mu_R = 3.11\) fm\(^{-1}\). For the two-nucleon calculations, we set nucleon mass and \(\hbar\) to unity and take fm as the unit of length. The nucleon mass adopted yields the conversion factor \(1\) fm\(^{-2}\) = 41.47 MeV.

For general potentials, \(V(q, x; q', x')\) may not be available analytically. Its numerical generation by applying a suitable quadrature to the integral over the azimuthal angle \(\phi\) is quite feasible. In fact, we have tested this aspect on the present potential. Using a composite 64-point Gauss-Legendre rule for the \(\phi\)-integral, the results were indistinguishable within 7-8 digits from those of the analytical reduced potential.

Tables I-IV show the convergence pattern of the Bateman results as the number of grid points in \(q\) and \(x\) variables are increased. Two collision energies considered are \(E = 150\) MeV and \(E = 400\) MeV. Shown are the on-shell T-matrix elements \(T(q_0, q_0, x) (\equiv T(q_0, x; q_0, 1)/2\pi)\), in units of MeV - fm\(^3\), for three values of \(x\). Also shown is the s-wave component of the on-shell T-matrix, obtained by numerically averaging \(T(q_0, q_0, x)\over x\) over \(x\). Reference values in these tables were obtained using \(N_q = 200\) and \(N_x = 80\) in the Nystrom method. These are stable to within at least the number of digits shown against further increases in the computational parameters like \(N_q\), \(N_x\), \(q_{max}\) and against the variations in the distribution pattern of quadrature points in the \(q-x\) plane.

An examination of these tables show that results accurate to 3-4 digits can be obtained with relatively few grid points. However, going beyond this level of accuracy may require much finer interpolation grids. Especially, the peak at the forward direction requires many grid points to reach 6 digit accuracy. For directions other than forward, convergence is rather rapid. As a remedy for the slow convergence around the forward direction, a non-uniform \(x\)-grid with more points around \(x = 1\) might be considered.

Whenever the number of interpolation points are comparable to the number of quadrature points, the Bateman and Nystrom methods will yield similar levels of accuracy. In fact, as mentioned earlier, when a quadrature grid is also used
as interpolation grid, Bateman method degenerates into the Nystrom method. Conversely, a Nystrom calculation is at the same time a Bateman calculation. In a sense, Nystrom is a restricted type of Bateman method in which interpolation and quadrature grids are the same. That Bateman approach distinguishes between interpolation and quadrature grids is a strength of the Bateman approach over the Nystrom approach. For a given (crude) interpolation grid, the integral involved in the matrix element \( \langle q_n x_m | V G_0 V | q_{n'} x_{m'} \rangle \) can be calculated with a finer quadrature grid, without affecting the the order of the matrix \( D \). In contrast, in Nystrom method, the quadrature grid used to discretize the integral will have to be used as collocation points in order to obtain a consistent set of equations.

VI. DISCUSSION and CONCLUSIONS

We have shown that the multi-variate Bateman interpolation of \( V(q, x; q', x') \) on a grid provide a simple and viable computational scheme to solve the LS equation without invoking angular momentum decomposition. In terms of computational complexity, Bateman approach is quite similar to the Nystrom method. For a given quadrature grid, both methods involve essentially the same sampling of the potential and the free propagator, but in fact, when interpolation and quadrature grids are taken to coincide, Bateman and Nystrom methods become equivalent.

The use of two grids in the Bateman approach, one for interpolation/collocation and one for quadrature evaluation of matrix elements, gives it an additional flexibility. The matrix dimension is determined by the interpolation grid. However, calculation of the integral in Eq. (10) for the construction of the matrix \( D^{-1} \) can be carried out with a higher-order quadrature rule without affecting the matrix size. From this point of view, Nystrom approach is a restricted type of Bateman method with the same grid used for the purposes of interpolation and quadrature both. Thus, Bateman can be expected to be more effective for small interpolation grids than the Nystrom method (with a quadrature grid of similar size).

We find that Bateman method can yield 3-4 digit accuracy with relatively small numbers of interpolation points. However, as the size of the interpolation grid is increased to achieve higher level of accuracy, the advantage associated with the reduction in matrix dimension may disappear to some extent. With large sets of interpolation and quadrature points, both Nystrom and Bateman methods are expected to perform satisfactorily.

In three- and four-particle contexts, the two-particle \( T \)-matrix \( \langle q | T(E) | q' \rangle \) is needed for many different two-particle energies \( E \) and off-shell momenta \( q \) and \( q' \). Bateman method could provide an effective way of generating arbitrary off-shell \( T \)-matrix elements needed in direct momentum-vector approaches to solve three-particle Faddeev equations without employing partial-wave decomposition [28,29].
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TABLE I. Convergence of the Bateman method with respect to the number points in the $q$-grid. Shown are the on-shell T-matrix elements $T(q_0, q_0, x)$ for $q_0 = 1.901860 \text{ fm}^{-1}$ or $E = 150 \text{ MeV}$. Parameters $N$ and $M$ denote the number of points in the $q$- and $x$-grids, respectively.

| N  | M  | $s$-wave | $x = +1.0$ | $x = 0.0$ | $x = -1.0$ |
|----|----|----------|-----------|-----------|-----------|
|    |    |          | Real part of $T(q_0, q_0, x)$ |          |           |
| 9  | 41 | 0.102424 | -6.012713 | 0.478064  | 0.232340  |
| 17 | 41 | 0.103792 | -6.084988 | 0.490515  | 0.234057  |
| 25 | 41 | 0.103967 | -6.092244 | 0.491699  | 0.233998  |
| 33 | 41 | 0.103982 | -6.092730 | 0.491771  | 0.233959  |
| 41 | 41 | 0.103982 | -6.092758 | 0.491778  | 0.233969  |
| 51 | 41 | 0.103986 | -6.092771 | 0.491771  | 0.233962  |
| 49 | 41 | 0.103984 | -6.092766 | 0.491774  | 0.233962  |
| 65 | 41 | 0.103984 | -6.092765 | 0.491774  | 0.233962  |
| 81 | 41 | 0.103983 | -6.092763 | 0.491776  | 0.233967  |
| 51 | 41 | 0.103986 | -6.092774 | 0.491771  | 0.233959  |
|    |    | 0.103988 | -6.092782 | 0.491768  | 0.233958  |
|    |    |          | Imaginary part of $T(q_0, q_0, x)$ |          |           |
| 9  | 41 | 0.0207717 | -1.837919 | 0.272400  | 0.329956  |
| 17 | 41 | 0.0213556 | -0.1927475 | 0.284770  | 0.362303  |
| 25 | 41 | 0.0214309 | -1.936595 | 0.286015  | 0.365442  |
| 33 | 41 | 0.0214373 | -1.937290 | 0.286094  | 0.365635  |
| 41 | 41 | 0.0214371 | -1.937226 | 0.286101  | 0.365647  |
| 51 | 41 | 0.0214389 | -1.937237 | 0.286098  | 0.365647  |
| 49 | 41 | 0.0214383 | -1.937230 | 0.286099  | 0.365643  |
| 65 | 41 | 0.0214381 | -1.937230 | 0.286100  | 0.365643  |
| 81 | 41 | 0.0214376 | -1.937231 | 0.286100  | 0.365647  |
| 51 | 41 | 0.0214391 | -1.937238 | 0.286098  | 0.365646  |
|    |    | 0.0214399 | -1.937247 | 0.286097  | 0.365649  |
TABLE II. Convergence of the Bateman method with respect to number of points in the $x$-grid. Shown are the on-shell T-matrix elements $T(q_0, q_0, x)$ for $q_0 = 1.901860 \text{ fm}^{-1}$ or $E = 150$ MeV. Parameters $N$ and $M$ denote the number of points in the $q$- and $x$-grids, respectively.

| $N$ | $M$ | $s$-wave | $x=+1.0$ | $x=0.0$ | $x=-1.0$ |
|-----|-----|----------|----------|----------|----------|
|     |     | Real part of $T(q_0, q_0, x)$ |          |          |          |
| 41  | 21  | 0.103941  | -6.092561| 0.491835 | 0.234068 |
| 31  |     | 0.103979  | -6.092735| 0.491782 | 0.233966 |
| 41  |     | 0.103982  | -6.092758| 0.491778 | 0.233969 |
| 51  |     | 0.103986  | -6.092771| 0.491771 | 0.233962 |
| Nystrom |  | 0.103988  | -6.092782| 0.491768 | 0.233958 |
|     |     | Imaginary part of $T(q_0, q_0, x)$ |          |          |          |
| 41  | 21  | 0.0214173 | -1.937046| 0.286109 | 0.365694 |
| 31  |     | 0.0214358 | -1.937200| 0.286102 | 0.365633 |
| 41  |     | 0.0214371 | -1.937226| 0.286101 | 0.365647 |
| 51  |     | 0.0214389 | -1.937237| 0.286098 | 0.365647 |
| Nystrom |  | 0.0214399 | -1.937247| 0.286097 | 0.365649 |
TABLE III. Convergence of the Bateman method with respect to the number of points in the $q$-grid. Shown are the on-shell T-matrix elements $T(q_0, q_0, x)$ for $q_0 = 3.105725 \, \text{fm}^{-1}$ or $E = 400$ MeV. Parameters $N$ and $M$ denote the number of points in the $q$- and $x$-grids, respectively.

| $N$ | $M$ | $s$-wave | $x=+1.0$ | $x=0.0$ | $x=-1.0$ |
|-----|-----|----------|----------|---------|---------|
|     |     | Real part of $T(q_0, q_0, x)$ |          |         |         |
| 9   | 41  | -0.092720 | -6.051275 | 0.472848 | 0.283085 |
| 17  | 41  | -0.080018 | -6.122230 | 0.453371 | 0.253529 |
| 25  | 41  | -0.078365 | -6.154789 | 0.454285 | 0.249604 |
| 33  | 41  | -0.078171 | -6.162070 | 0.454812 | 0.249213 |
| 41  | 41  | -0.078140 | -6.163427 | 0.454910 | 0.249137 |
| 51  | 41  | -0.078138 | -6.163477 | 0.454910 | 0.249148 |
| 49  | 41  | -0.078134 | -6.163683 | 0.454928 | 0.249129 |
| 65  | 41  | -0.078135 | -6.163743 | 0.454934 | 0.249141 |
| 81  | 41  | -0.078135 | -6.163750 | 0.454934 | 0.249145 |
| 51  | 41  | -0.078137 | -6.163762 | 0.454939 | 0.249155 |
| Nystrom | -0.078129 | -6.163808 | 0.454930 | 0.249139 |
|     |     | Imaginary part of $T(q_0, q_0, x)$ |          |         |         |
| 9   | 41  | 0.0293396 | -1.270232 | 0.079004 | -0.0885716 |
| 17  | 41  | 0.0212913 | -1.288721 | 0.102911 | -0.0768172 |
| 25  | 41  | 0.0203595 | -1.306253 | 0.109280 | -0.0773454 |
| 33  | 41  | 0.0202510 | -1.310558 | 0.110482 | -0.0775932 |
| 41  | 41  | 0.0202334 | -1.311349 | 0.110694 | -0.0776496 |
| 51  | 41  | 0.0202332 | -1.311435 | 0.110704 | -0.0776371 |
| 49  | 41  | 0.0202306 | -1.311405 | 0.110731 | -0.0776569 |
| 65  | 41  | 0.0202305 | -1.311531 | 0.110737 | -0.0776534 |
| 81  | 41  | 0.0202303 | -1.311535 | 0.110738 | -0.0776498 |
| 51  | 41  | 0.0202312 | -1.311613 | 0.110745 | -0.0776478 |
| Nystrom | 0.0202292 | -1.311641 | 0.110753 | -0.0776420 |
TABLE IV. Convergence of the Bateman method with respect to the number of points in the \( x \)-grid. Shown are the on-shell T-matrix elements \( T(q_0, q_0, x) \) for \( q_0 = 3.105725 \text{ fm}^{-1} \) or \( E = 400 \text{ MeV} \). Parameters \( N \) and \( M \) denote the number of points in the \( q \)- and \( x \)-grids, respectively.

| \( N \) | \( M \) | s-wave | \( x = +1.0 \) | \( x = 0.0 \) | \( x = -1.0 \) |
|---|---|---|---|---|---|
| \( \text{Real part of } T(q_0, q_0, x) \) |
| 41 | 21 | -0.0781765 | -6.164592 | 0.454996 | 0.249352 |
| 31 | -0.0781459 | -6.163433 | 0.454913 | 0.249137 |
| 41 | -0.0781400 | -6.163427 | 0.454910 | 0.249137 |
| 51 | -0.0781380 | -6.163477 | 0.454910 | 0.249148 |
| \( \text{Nystrom} \) | -0.0781299 | -6.163808 | 0.454930 | 0.249139 |
| \( \text{Imaginary part of } T(q_0, q_0, x) \) |
| 41 | 21 | 0.0201645 | -1.309207 | 0.110317 | -0.0778120 |
| 31 | -0.0202295 | -1.311040 | 0.110645 | -0.0776740 |
| 41 | 0.0202334 | -1.311349 | 0.110694 | -0.0776496 |
| 51 | 0.0202332 | -1.311435 | 0.110704 | -0.0776371 |
| \( \text{Nystrom} \) | 0.0202292 | -1.311641 | 0.110753 | -0.0776420 |