FINK TYPE CONJECTURE ON AFFINE-PERIODIC SOLUTIONS AND LEVINSON’S CONJECTURE TO NEWTONIAN SYSTEMS

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Abstract. This paper concerns the existence of affine-periodic solutions for differential systems (including functional differential equations) and Newtonian systems with friction. This is a kind of pattern solutions in time-space, which may be periodic, anti-periodic, subharmonic or quasi periodic corresponding to rotation motions. Fink type conjecture is verified and Lyapunov’s methods are given. These results are applied to study gradient systems and Newtonian (including Rayleigh or Lienard) systems. Levinson’s conjecture to Newtonian systems is proved.

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1. Introduction. In the study of the qualitative theory of differential equations, one of the center topics is to prove the existence of periodic type solutions, and the following questions is hence paid high attention: Does a system with certain periodicity have correspondingly the same periodic solutions, quasi periodic solutions or almost periodic solutions if it possesses suitable stability? Without a doubt, the study of the questions is important in applications.

Through long term research, some profound results have been obtained, for example:

1) For a periodic system or an almost periodic system, if there exists a globally uniformly asymptotically stable bounded solution, then it admits a periodic solution or almost periodic solution, see the books [22, 3]. Here the key is to ask the existence of bounded solutions. Unfortunately, asymptotic stability does not imply necessarily the existence of bounded solutions. For example, consider the equation

\[
x' = \begin{cases} 
-\frac{(t-x)^2}{2(x^2+t^2)+1} + 1, & x > t, \\
1, & x = t, \\
\frac{(t-x)^2}{2(x^2+t^2)+1} + 1, & x < t.
\end{cases}
\]

The solution \(x = t\) is globally asymptotically stable, but unbounded. In particular, the equation also satisfies the Lipschitz condition.

2) Fink [9] proved that for a periodic system, the exponential uniform asymptotic stability implies the existence of bounded solutions, thus it follows the existence of periodic solutions, and he conjectured that for an almost periodic system the exponential uniform asymptotic stability implies the existence of almost periodic solutions, which was proved by Lin [15] when the system satisfies the Lipschitz condition and is uniformly asymptotically stable. For some periodic solution theory, see [5, 4, 11, 7, 2] and references therein.

However, in physical models, a stronger nonlinear system may not satisfy the Lipschitz conditions, and only satisfy the local Lipschitz one. Naturally consider the following problem.

Problem 1. Assume initial value problems are unique, does the uniformly asymptotic stability imply the existence of periodic solutions, or affine periodic solutions?

The concept of affine-periodic solutions was introduced and investigated in the literature [24, 13, 16, 18, 6, 19, 21, 20]. A \((Q, T)\) affine-periodic solution \(x(t)\) of a differential equation is the following:

\[x(t + T) = Qx(t) \quad \forall t \in \mathbb{R}^1,\]

where \(Q\) is a nonsingular matrix in the finite dimension case and an invertible operator in the infinite dimension case. Obviously, this kind of solutions is, in turn, periodic, subharmonic, or quasi-periodic when \(Q\) is, in turn, an identity matrix \(I\), a power identity matrix, i.e., \(Q^k = I\) for some integer \(k \neq 0\), or an orthogonal matrix except for the previous cases, i.e., \(Q \in O(n)\). All these correspond to affine-periodic solutions, describing rotation motions in body from mechanics. When \(Q \in GL(n) \setminus O(n)\), an affine-periodic solution might be spiral like \((e^{it} \cos \omega t, e^{it} \sin \omega t)\). In a word, affine-periodic solutions express a kind of patterns in space-time.
In this paper, we will touch the problem mentioned above. We will prove that a system has periodic solutions or affine-periodic solutions under weaker stability conditions, for instance, equi-asymptotic stability.

Throughout the paper, we let \( Q \in O(n) \) unless otherwise stated. Consider the following three kinds of equations

\[
\begin{align*}
\dot{x} &= f(t, x), \\
\dot{x} &= F(t, x_t), \\
\ddot{x} + A(t, x) \dot{x} + \nabla V(x) &= e(t), \quad \nabla = \frac{\partial}{\partial x},
\end{align*}
\]

where \( x_t(s) = x(t + s), \quad s \in [-r, 0], \quad r > 0 \) is a constant, \( f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n, \)
\( F : \mathbb{R}^1 \times C \to \mathbb{R}^n, \)
\( A : \mathbb{R}^1 \times \mathbb{R}^m \to \mathbb{R}^{m \times m}, \)
\( V : \mathbb{R}^m \to \mathbb{R}^1, \)
\( e : \mathbb{R}^1 \to \mathbb{R}^m \)

satisfy some regularity conditions. \( C = C([-r, 0], \mathbb{R}^n) \) with the norm \( \|\phi\| = \max_{[-r,0]} |\phi(s)|. \)

Obviously, system (1) is an ordinary differential equation, system (2) is a functional differential equation, and system (3) is a Newtonian equation of motion with friction \( A(t, x) \dot{x} \) when \( A \geq 0. \)

We make the following assumptions.

H1) \( f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and satisfies

\[
f(t + T, x) = Qf(t, Q^{-1}x)
\]

for all \((t, x)\). Moreover, (1) admits the uniqueness of solutions with respect to initial values.

H2) \( F : \mathbb{R}^1 \times C \to \mathbb{R}^n \) is completely continuous and satisfies

\[
F(t + T, \varphi) = QF(t, Q^{-1}\varphi)
\]

for all \((t, \varphi)\). Moreover, (2) admits the uniqueness of solutions with respect to initial values.

H3) \( A : \mathbb{R}^1 \times \mathbb{R}^m \to \mathbb{R}^{m \times m}, \)
\( V : \mathbb{R}^m \to \mathbb{R}^1, \)
\( e : \mathbb{R}^1 \to \mathbb{R}^m \)

are continuous, and satisfy the local Lipschitz condition in \( x \), \( V \) is \( C^1 \). Moreover,

\[
\begin{align*}
A(t + T, x)y &= QA(t, Q^{-1}x)Q^{-1}y, \\
\nabla V(x) &= Q\nabla V(Q^{-1}x), \\
e(t + T) &= Qe(t).
\end{align*}
\]

Thus systems (1), (2) or (3) are said to be a \((Q, T)\) affine-periodic ordinary differential equation, a \((Q, T)\) affine-periodic functional differential equation, or a \((Q, T)\) affine-periodic Newtonian equation, respectively.

As previously mentioned, when \( Q^k \neq I \) for any positive integer \( k \), an affine-periodic solution is just a quasi periodic one. It is very difficult to investigate the existence of quasi periodic solutions due to “small divisor”. In nonresonant situations, see the celebrated KAM theory [10, 1, 17]; and in resonant cases, see [8, 14]. The present paper provides a topological way and the Lyapunov’s method to study quasi periodic solutions for some differential systems with certain time-space structures.

It has been a fundamental principle in physics that a Newtonian system with friction must have a periodic solution. Mathematically in 1940’s, Levinson [12] conjectured that the principle is true, which brought important dissipativeness theory. For some literature, see [3, 4, 5, 11, 22, 23] and references therein. However, answers to Newtonian systems with friction (3) seem not complete. Hence, we pose the following:
Problem 2. Does a Newtonian system with friction admit a \((Q,T)\) affine periodic solution?

The paper is organized as follows. In Section 2, we first consider ordinary differential equations and prove Fink type conjecture for affine periodic solutions without the Lipschitz condition, which concludes that the equi-asymptotic stability of an ordinary differential equation implies the existence of affine periodic solutions. As a result, we give corresponding Lyapunov's methods, in which the assumptions on the directional derivative are quite weak, and the negative definiteness is removed. Next in Section 3, we give an analog for functional differential equations (2). All the arguments are based on the modular degree theory due to Zabreiko and Krasnosel'skii [23]. Finally, we apply our results to gradient systems, harmonic oscillators, and Newtonian (including Rayleigh or Liénard) systems with \(m\) degree of freedom, and obtain some criteria of affine-periodicity in the presence of friction. We verify essentially Levinson's conjecture for Newtonian systems with friction.

2. Ordinary differential equation. To establish the results, we first recall some basic definitions. Denote by \(x(t,t_0,x_0)\) the solution of (1) with the initial condition \(x(t_0) = x_0\).

Definition 2.1. System (1) is called equi-stable if for any \(\epsilon > 0\) there exists \(\delta > 0\) such that as \(|x_0 - y_0| < \delta\),

\[|x(t,0,y_0) - x(t,0,x_0)| < \epsilon\]

for all \(t \geq 0\); System (1) is called uniformly equi-stable if for any \(\epsilon > 0\) there exists \(\delta > 0\) such that as \(|x_0 - y_0| < \delta\),

\[|x(t,t_0,y_0) - x(t,t_0,x_0)| < \epsilon\]

for all \(t_0 \geq 0\) and \(t \geq t_0\).

Definition 2.2. System (1) is called equi-asymptotically stable if it is equi-stable, and there exists \(\delta_0 > 0\) such that for any \(\epsilon > 0\) there exists \(L = L(\epsilon) > 0\), as \(|x_0 - y_0| < \delta_0\),

\[|x(t,0,y_0) - x(t,0,x_0)| < \epsilon \quad \forall t \geq L;\]

System (1) is called uniformly equi-asymptotically stable if it is uniformly equi-stable, and there exists \(\delta_0 > 0\) such that for any \(\epsilon > 0\) there exists \(L = L(\epsilon) > 0\), as \(t_0 \geq 0\) and \(|x_0 - y_0| < \delta_0\),

\[|x(t,t_0,y_0) - x(t,t_0,x_0)| < \epsilon \quad \forall t \geq t_0 + L.\]

The first main result of this paper is the following.

Theorem 2.3. Assume besides H1) system (1) is asymptotically equi-stable. Then system (1) has a unique asymptotically stable \((Q,T)\) affine-periodic solution.

Remark 1. The article [15] discussed the Fink conjecture. It gives an equivalent relation between the uniformly asymptotic stability and the existence of Lyapunov functions. The Lyapunov functions constructed are closely related to the global Lipschitz condition.

Remark 2. The proof of the theorem is simple and intuitive, which is described below: We build walls in the east and west, respectively. Assume the east ground is higher than the west. The bricks on both sides are the same length and width, but the bricks in the west are thicker than the east. If we put the bricks flat and
use the same number to build walls, then when bricks increase, the west wall will ultimately exceed the height of the east at the same level.

To prove the theorem, we need the following.

**Theorem 2.4.** (the modular degree theorem [23]) Let $X$ be a Banach space, $\Omega \subset X$ be a bounded open set, and $P : \bar{\Omega} \to X$ be a completely continuous map. Let $N$ be a prime number. If

$$0 \notin (id - P)(\partial \Omega), \quad j = N, N + 1,$$

then

$$\text{deg}(id - P, \Omega, 0) = \text{deg}(id - P^N, \Omega, 0) (\text{modular}(N)).$$

We now give the proof of Theorem 2.3.

**Proof.** For simplicity, we prove the case with $n = 1$. Let $x(t, 0, 0)$ be a solution of (1) with the initial condition $x(t_0) = 0$ regarded as the ground. Then it exists on $\mathbb{R}^1$ by equi-stability and Peano’s theorem. According to the definition of equi-asymptotic stability, it follows that for $\varepsilon_0 = \frac{\delta_0}{32}$, there exists a prime number $N$ such that for any $x_0, y_0 \in \mathbb{R}^1$, as $|x_0 - y_0| < \delta_0$,

$$|x(t, y_0) - x(t, x_0)| < \varepsilon_0 \quad \forall t \geq NT,$$

where $x(t, x_0) = x(t, 0, x_0)$ for simplicity. Let the thickness of each brick of west and east walls be $\frac{\delta_0}{8}$ and $\frac{\delta_0}{32}$, respectively. To make the west wall higher than the east at the same level, we choose the brick number $K$ so that

$$K \cdot \frac{\delta_0}{8} > \max\{|x(NT, 0)|, |x((N + 1)T, 0)|\} + K \cdot \frac{\delta_0}{32} = \eta_0.$$  \hspace{1cm} (4)

Note that

$$|Q^{-j}x(jT, x_0)| = |x(jT, x_0)| \quad \forall j \in \mathbb{N}^1.$$ \hspace{1cm} (5)

Let

$$D = \{x_0 \in \mathbb{R}^1 : |x_0| < K \cdot \frac{\delta_0}{8}\},$$

and define a Poincaré map $P$ by

$$P(x_0) = Q^{-1}x(T, x_0).$$

By induction, we have

$$P^j(x_0) = P \circ P^{j-1}(x_0) = Q^{-j}x(jT, x_0) \quad \forall j \geq 1.$$  

It follows from (4) and (5) that

$$P^j(x_0) \in D, \quad j = N, N + 1.$$  

Thus by the modular degree theorem, we have

$$\text{deg}(id - P, D, 0) = \text{deg}(id - P^N, D, 0) (\text{modular}(N)).$$

In addition,

$$\text{deg}(id - P^N, D, 0) = 1.$$  

Hence,

$$\text{deg}(id - P, D, 0) \neq 0.$$  

Therefore, $P$ has a fixed point $x_*$ in $D$, i.e.,

$$x(T, x_*) = Qx_*.$$
Again by the uniqueness of solutions to the initial value problems, we have
\[ x(t + T, x_*) \equiv Qx(t, x_*), \]
which shows that \( x(t, x_*) \) is a \((Q, T)\) affine-periodic solution of equation (1).

As is well known, the Lyapunov method is a powerful tool to study the stability of differential equations. In the following, we will combine Theorem 2.3 with Lyapunov’s method to study the existence of \((Q, T)\) affine-periodic solutions for equation (1).

**Definition 2.5.** A function \( a : \mathbb{R}^n \rightarrow \mathbb{R}_{+}^1 \) is called positive definite if it is continuous and satisfies
\[ a(0) = 0, \quad a(x) > 0 \quad \forall x \neq 0. \]

**Definition 2.6.** A function \( a : \mathbb{R}_{+}^1 \rightarrow \mathbb{R}_{+}^1 \) is called a wedge function if it is positive definite, strictly increasing and with \( a(\infty) = \infty \).

We have the following result.

**Theorem 2.7.** Assume H1). Consider the adjoint system of (1)
\[ \dot{x} = f(t, x), \quad \dot{y} = f(t, y). \] (6)
Let \( l : \mathbb{R}_{+}^1 \rightarrow \mathbb{R}_{+}^1 \) be continuous and satisfy:
1) There exists \( L \geq 0 \), such that \( \int_{0}^{L} l(s) ds > 0 \ \forall t \geq L; \)
2) There exists a sequence \( t_k \) monotonically increasing to \( \infty \), such that
\[ \lim_{k \rightarrow \infty} \int_{t_k-1}^{t_k} l(s) ds \geq \sigma > 0. \]
Assume
3) Equation (1) has a solution \( x_0(t) \) defined on \( \mathbb{R}_{+}^1; \)
4) There exists a \( C^1 \) function \( V(t,x,y) : \mathbb{R}_{+}^1 \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{+}^1 \), such that the directional derivative along (6)
\[ \dot{V}(6)(t,x,y) \leq -l(t)V(t,x,y), \]
and
\[ a(|x - y|) \leq V(t,x,y), \quad V(t,0,0) = 0, \]
where \( a : \mathbb{R}_{+}^1 \rightarrow \mathbb{R}_{+}^1 \) is a wedge function.

Then system (1) is equi-asymptotically stable, and hence it admits a unique asymptotically stable \((Q, T)\) affine-periodic solution.

**Remark 3.** a) In the above theorem, it does not require the function \( l(t) \) to be positive. In some classical results, the function \( l(t) \) is assumed to be positive, see the book [22].

b) In particular, no periodicity in \( t \) is added on the Lyapunov function \( V(t,x,y) \). However, this periodicity conditions is all needed among previous results to our knowledge.

Let’s begin the proof of Theorem 2.7.
Proof. Firstly we prove that all solutions of (1) are defined on \( \mathbb{R}_{+}^{1} \), and they are equi-stable. Let \( x(t) \) be any solution of (1) starting from \( t = 0 \), and let \( y(t) = x_{0}(t) \). Then from 4), we have
\[
a(||x(t) - y(t)||) \leq V(t, x(t), y(t)) \leq V(t_{0}, x_{0}(t_{0}), y(t_{0})) \exp\left\{-\int_{t_{0}}^{t} l(s)ds\right\} \forall t \geq t_{0}.
\]
In particular, by 1), we know that
\[
a(||x(t) - y(t)||) \leq V(t, x(t), y(t)) \leq V(0, x(0), y(0)) \exp\left\{-\int_{0}^{t} l(s)ds\right\} \forall t \geq 0,
\]
and assumption 4) is strengthened accordingly
\[
\text{Then from 4), we have}
\]
\[
where \( b \) is defined in (7). Then system (1) is uniformly equi-asymptotically stable. Hence it has a unique affine-periodic solution.

Remark 4. If assumption 1) is changed into
\[
\text{1') For any } t_{0} \geq 0, \text{ there exists } L = L(t_{0}) \geq 0, \text{ such that } \int_{0}^{t} l(s)ds \geq 0 \forall t \geq t_{0} + L; \text{ and assumption 4) is strengthened accordingly}
\]
\[
a(||x - y||) \leq V(t, x, y) \leq b(||x - y||), \quad (7)
\]
where \( b: \mathbb{R}_{+}^{1} \rightarrow \mathbb{R}_{+}^{1} \) is a wedge function, then the conclusion of Theorem 2.7 holds. This is the following theorem.

Theorem 2.8. Under the conditions of Theorem 2.7 except for 1), assume 1') and (7). Then system (1) is uniformly equi-asymptotically stable. Hence it has a unique uniformly asymptotically stable \((Q, T)\) affine-periodic solution.
Corollary 1. Suppose that there exists a solution of system (1) defined on $\mathbb{R}_+^1$, and

$$\langle x - y, f(t, x) - f(t, y) \rangle \leq -l(t)|x - y|^2,$$

where $l(t)$ is the same as in Theorem 2.7. Then system (1) is equi-asymptotically stable, and hence it has a unique asymptotically stable $(Q, T)$ affine-periodic solution.

Proof. Put a Lyapunov function

$$V(t, x, y) = \frac{1}{2}|x - y|^2 \quad \forall x, y \in \mathbb{R}^n.$$ 

Thus,

$$\dot{V}(t, x, y) = \langle x - y, f(t, x) - f(t, y) \rangle \leq -2l(t)V(t, x, y).$$

Now the conditions of Theorem 2.7 are all satisfied. Therefore, there exists a unique asymptotically stable $(Q, T)$ affine-periodic solution of system (1). The proof is complete.

Remark 5. If suppose that $f(t, x)$ satisfies the Lipschitz condition, then every solution of system (1) is defined on $\mathbb{R}_+^1$.

3. A version to functional differential equations. In this section, we will discuss the existence of $(Q, T)$ affine-periodic solutions and the asymptotic stability for system (2). That is, for functional differential equations with finite delay, we hope to establish a result corresponding to ordinary differential equations. The main difficulty is that the phase space of system (2) is $C = C([-r, 0], \mathbb{R}^n)$ but not $\mathbb{R}^n$, therefore Poincaré map $Q^{-1}x_T(\cdot, \varphi), \varphi \in C$ loses the compactness when period $T < 2r$, where $x(t, \varphi) = x(t, 0, \varphi)$. Here we denote by $x(t, t_0, \varphi)$ a solution of (2) with the initial condition $x_{t_0} = \varphi$.

Definition 3.1. System (2) is called equi-stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for arbitrary two solutions $x(t, 0, \varphi), x(t, 0, \psi)$,

$$|x(t, 0, \varphi) - x(t, 0, \psi)| < \varepsilon \quad \forall t \geq 0,$$

whenever $\|\varphi - \psi\| < \delta$; System (7) is called uniformly equi-stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for arbitrary two solutions $x(t, t_0, \varphi), x(t, t_0, \psi)$,

$$|x(t, t_0, \varphi) - x(t, t_0, \psi)| < \varepsilon \quad \forall t \geq t_0,$$

whenever $\|\varphi - \psi\| < \delta$.

Definition 3.2. System (2) is called equi-asymptotically stable if it is equi-stable, and there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$, for arbitrary two solutions $x(t, 0, \varphi), x(t, 0, \psi)$,

$$|x(t, 0, \varphi) - x(t, 0, \psi)| < \varepsilon \quad \forall t \geq L,$$

whenever $\|\varphi - \psi\| < \delta_0$; System (2) is called uniformly equi-asymptotically stable if it is uniformly equi-stable, and there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$, there exists $L = L(\varepsilon) > 0$, for any $t_0 \geq 0$ and arbitrary two solutions $x(t, t_0, \varphi), x(t, t_0, \psi)$,

$$|x(t, t_0, \varphi) - x(t, t_0, \psi)| < \varepsilon \quad \forall t \geq t_0 + L,$$

whenever $\|\varphi - \psi\| < \delta_0$.

We now state the main result of this section.
Theorem 3.3. Assume system (2) is equi-asymptotically stable, then system (2) has a unique asymptotically stable \((Q,T)\) affine-periodic solution.

Proof. Similar to the proof of ordinary differential equations, but we need to consider the wall thickness \(r\). For simplicity, we also prove the case \(n = 1\). Take \(\varepsilon_0 = \frac{1}{16}\delta_0\). Then according to the equi-asymptotic stability, there exists a positive number \(L\) such that for arbitrary two solutions \(x(t, \varphi)\) and \(x(t, \psi)\) of (2),

\[
|x(t, \varphi) - x(t, \psi)| < \eta = \frac{1}{16}\delta_0 \quad \forall t \geq L,
\]
whenever \(\|\varphi - \psi\| < \delta_0\).

Take a prime number \(N\), such that \(NT - r > L\).

Put a positive integer \(K\), such that

\[
\frac{1}{4}\delta_0 K > \frac{1}{16}\delta_0 K + \max_{[NT-r,(N+1)T]} \|x(t,0,0)\| \equiv B.
\]

As shown in the proof of Theorem 2.3, we build a west wall at \(t = 0\), and two east walls at \(t = NT, (N+1)T\).

West wall brick : length \times width \times height = \(r \times \eta_1 \times \eta_1\),

Two east walls brick : length \times width \times height = \(r \times \eta_2 \times \eta_2\),

where \(\eta_1 = \frac{1}{4}\delta_0, \eta_2 = \frac{1}{16}\delta_0\). Respectively using \(K\) block bricks, it follows from (9) that the west wall is higher than the two east. Set

\[
D = \{ \varphi \in C([[-r,0], R^1]) : \|\varphi(s)\| < \frac{1}{4}\delta_0 K \}.
\]

By equi-asymptotic stability, we have

\[
|x(t, \varphi)| < B \quad \forall \varphi \in D, \forall t \in [NT - r, (N+1)T].
\]

Indeed, for every \(\varphi \in D\), there exists \(\varphi_i (i = 1, 2, \cdots, K - 1)\), such that

\[
\|\varphi_1 - 0\| < \frac{1}{4}\delta_0,
\]

\[
\|\varphi_{i-1} - \varphi_i\| < \frac{1}{4}\delta_0, \quad i = 2, \cdots, K - 1,
\]

\[
\|\varphi_{K-1} - \varphi_i\| < \frac{1}{4}\delta_0.
\]

Then by (8), (9) and the construct of \(D\), we have (10).

Since system (2) is equi-stable, and the completely continuous map \(F(t, \varphi)\) maps the bounded set of \(C\) into the one of \(R^n\), there exist positive numbers \(h\) and \(H\), such that

\[
|x(t, \varphi)| \leq h \quad \forall \varphi \in D, \forall t \in [-r, (N+1)T],
\]

\[
|F(t, \varphi)| \leq H \quad \forall \varphi \in D, \forall t \in [0, (N+1)T].
\]

Let

\[
\bar{D} = \{ \varphi \in C : \|\varphi\| \leq h, |\varphi(s_1) - \varphi(s_2)| \leq H|s_1 - s_2| \quad \forall s_1, s_2 \in [-r, 0] \}.
\]
Then \( \bar{D} \) is a compact convex subset in \( C \). Consequently, from functional analysis, there exists a continuous retract \( \alpha : C \to \bar{D} \), i.e. \( \alpha \) is continuous on \( C \) with the restriction map

\[
\alpha|_{\bar{D}} = \text{id}.
\]

Define a modified Poincaré map \( P : C \to C \) by

\[
P(\varphi) = (Q^{-1}x)_T(\cdot, \alpha \circ \varphi).
\]

Based on Arzela-Ascoli theorem, it is easy to see that \( P : C \to C \) is a completely continuous map.

By the definition of \( \alpha \), \( \bar{D} \) and (10)-(12), we obtain

\[
|\varphi(t, \alpha \circ \varphi)| \leq h, \quad t \in [-r, (N + 1)T];
|\varphi(t, \alpha \circ \varphi)| < B, \quad t \in [NT - r, (N + 1)T] \quad \forall \varphi \in D.
\]

(13)

Note that for any \( \varphi \in \bar{D} \), it follows from (13) that

\[
P^2(\varphi) = P \circ P(\varphi)
= P \circ ((Q^{-1}x)_T(\cdot, \alpha \circ \varphi))
= (Q^{-1}x)_T(\cdot, \alpha \circ (Q^{-1}x)_T(\cdot, \alpha \circ \varphi))
= (Q^{-1}x)_T(\cdot, (Q^{-1}x)_T(\cdot, \alpha \circ \varphi))
= (Q^{-2}x)_T(\cdot, \alpha \circ \varphi).
\]

Generally, we can prove by induction

\[
P^i(\varphi) = (Q^{-i}x)_T(\cdot, \alpha \circ \varphi) \quad \forall \varphi \in \bar{D}, i \geq 1.
\]

By (13), for any fixed point \( \varphi_* \in \bar{D} \) of \( P^N \), we have,

\[
P(\varphi_*) = P^{N+1}(\varphi_*) \in D.
\]

Thus, from the modular degree theorem, we have

\[
\text{deg}(\text{id} - P, D, 0) = \text{deg}(\text{id} - P^N, D, 0) \quad \text{(modular}(N)).
\]

On the other hand,

\[
\text{deg}(\text{id} - P^N, D, 0) = 1.
\]

Hence,

\[
\text{deg}(\text{id} - P, D, 0) \neq 0.
\]

Therefore, \( P \) has a fixed point \( \varphi_0 \) in \( D \), i.e.,

\[
x_T(\cdot, \alpha_0 \circ \varphi_0) = Q\varphi_0.
\]

Combining (13) implies

\[
\alpha \circ \varphi_0 = \varphi_0.
\]

Again by the uniqueness of solutions to initial value problems, we have

\[
x(t + T, \varphi_0) \equiv Qx(t, \varphi_0) \quad \forall t,
\]

which shows that \( x(t, \varphi_0) \) is a \((Q, T)\) affine-periodic solution of system (2).

Finally, we have to show that \( x(t, \varphi_0) \) is globally equi-asymptotically stable, which implies the uniqueness of the \((Q, T)\) affine-periodic solutions. We now claim: for any \( \varphi \in C \),

\[
\lim_{t \to \infty} |x(t, \varphi) - x(t, \varphi_0)| = 0.
\]
It suffices to prove that for any $\epsilon > 0$, there exists $L = L(\epsilon) > 0$, such that

$$|x(t, \varphi) - x(t, \varphi_0)| < \epsilon \quad \forall t \geq L.$$ 

Put a positive integer $K$, such that

$$\|\varphi - \varphi_0\| < \frac{1}{4\delta_0}K. \quad (14)$$

Since system (2) is equi-asymptotically stable, there exists $L_1 > 0$, such that for arbitrary two solutions $x(t, \psi_1)$ and $x(t, \psi_2)$ of (2),

$$|x(t, \psi_1) - x(t, \psi_2)| < \frac{\epsilon}{K} \quad \forall t \geq L_1,$$ 

whenever $\|\psi_1 - \psi_2\| \leq \frac{1}{4\delta_0}$.

According to (14), there exists $\{\varphi_i\}_{i=1}^{K-1}$, such that

$$\|\varphi_0 - \varphi_1\| < \frac{1}{4\delta_0}, \quad \|\varphi_{i-1} - \varphi_i\| < \frac{1}{4\delta_0}, \quad i = 1, 2, \ldots, K - 1, \quad \|\varphi_{K-1} - \varphi\| < \frac{1}{4\delta_0}.$$ 

Further by (15), we have

$$|x(t, \varphi) - x(t, \varphi_0)|$$

$$\leq |x(t, \varphi_1) - x(t, \varphi_0)| + |x(t, \varphi_2) - x(t, \varphi_1)| + \cdots + |x(t, \varphi) - x(t, \varphi_{K-1})|$$

$$< \frac{\epsilon}{K} + \cdots + \frac{\epsilon}{K}$$

$$= \epsilon \quad \forall t > L_1 \equiv L,$$

as claimed. This completes the proof. \qed

Combining Theorem 3.3 with Lyapunov’s methods, we have the following.

**Theorem 3.4.** Assume H2) holds. Consider the adjoint system

$$\dot{x} = F(t, x), \quad \dot{y} = F(t, y). \quad (16)$$

Assume that there exists a $C^1$ Lyapunov functional $V : \mathbb{R}_+^1 \times C \times C \rightarrow \mathbb{R}_+^1$ such that

1) on $\mathbb{R}_+^1 \times C \times C$,

$$a(|\varphi(0) - \psi(0)|) \leq V(t, \varphi, \psi),$$

where $a : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1$ is a wedge;

2) the directional derivative of $V(t, \varphi, \psi)$ along (16)

$$\dot{V}(16)(t, \varphi, \psi) \leq -l(t)V(t, \varphi, \psi)$$

for all $t \geq 0$, and $\varphi, \psi \in C$, where $l(t)$ is the same as in Theorem 2.7.

Then system (2) is equi-asymptotically stable, and as a consequence, it admits a unique asymptotically stable $(Q, T)$ affine-periodic solution.

**Proof.** This can follow the proof of Theorem 2.7, and hence we omit the details. \qed

4. **Gradient and Newtonian systems.** In this section, we give some discrete applications.
4.1. **Gradient systems.** Consider the gradient system
\[ \dot{x} = -\nabla U(t, x), \quad (17) \]
where \( U : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1 \) is continuous and \( C^2 \) with respect to \( x \). We have the following.

**Theorem 4.1.** Assume \( U(t, x) \) satisfies:
1) For all \((t, x)\),
\[ \nabla U(t + T, x) = Q \nabla U(t, Q^{-1}x); \]
2) For all \( t \geq 0 \) and \( x \in \mathbb{R}^n \),
\[ -\frac{\partial^2 U}{\partial x^2} \leq -l(t)I, \]

where \( l : \mathbb{R}_+^1 \to \mathbb{R}^1 \) is the same as in Theorem 2.7.

Then system (17) admits a unique asymptotically stable \((Q, T)\) affine-periodic solution.

**Remark 6.** As an example, let us put
\[ U(t, x) = \frac{1}{2} a(t)|x|^2 + \frac{1}{4} b(t)|x|^4 + \left( \begin{array}{c} e_1(t) \\ \vdots \\ e_k(t) \\ \sin \omega_1 t \\ \cos \omega_1 t \\ \vdots \\ \sin \omega_m t \\ \cos \omega_m t \end{array} \right) \top x, \]

where \( a, b, e_i(i = 1, \cdots, m) : \mathbb{R}^1 \to \mathbb{R}^1 \) are continuous, \( T \)-periodic and satisfy
\[ \int_0^T a(t)dt = \sigma > 0, \quad b(t) \geq 0. \]

Set
\[ Q = \text{diag} \left( I_k \left( \begin{array}{cc} \cos \omega_1 T & \sin \omega_1 T \\ -\sin \omega_1 T & \cos \omega_1 T \end{array} \right) \cdots \left( \begin{array}{cc} \cos \omega_m T & \sin \omega_m T \\ -\sin \omega_m T & \cos \omega_m T \end{array} \right) \right). \]

It is easy to check
\[ \nabla U(t + T, x) = Q \nabla U(t, Q^{-1}x). \]

We choose
\[ V(t, x, y) = \frac{1}{2} |x - y|^2. \]

Then the conditions of Theorem 2.7 are all satisfied.

The proof of Theorem 4.1 is as follows:

**Proof.** Let
\[ V(x, y) = \frac{1}{2} |x - y|^2. \]
Then along any two solutions \( x(t) \) and \( y(t) \) of (17),
\[
\dot{V}_{(17)}(x,y) = (x - y)^\top (\dot{x} - \dot{y}) \\
= -(x - y)^\top (\nabla U(t,x) - \nabla U(t,y)) \\
= -(x - y)^\top \frac{\partial^2 U}{\partial x^2} (t,y + \theta(x - y))(x - y) \\
\leq -l(t)|x - y|^2 \\
= -2l(t)V(t,y).
\]

Thus the conclusion follows from Theorem 2.7.

\[
\square
\]

4.2. **Newtonian systems.** Finally let us consider Newtonian systems with friction of the form
\[
\ddot{x} + C(t,x,\dot{x})\dot{x} + \nabla V(x) = e(t),
\]
where \( C : \mathbb{R}^1 \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m \times m} \), \( V : \mathbb{R}^m \to \mathbb{R}^1 \), \( e : \mathbb{R}^1 \to \mathbb{R}^m \) are continuous, \( C^1 \) with respect to \((x,\dot{x})\), and satisfy the following \((Q,T)\) affine-periodicity:

\[
C(t+T,x,y) = QC(t,\dot{Q}^{-1}x,\dot{Q}^{-1}y)Q^{-1}y, \\
\nabla V(x) = Q\nabla V(Q^{-1}x), \\
e(t+T) = Qe(t),
\]
where \( Q \in O(m) \) is given.

We have the following.

**Theorem 4.2.** Assume besides continuity, smoothness, and \((Q,T)\) affine-periodicity that

i) \( C^\top (t,x,y) = C(t,x,y) \),

ii) there exist \( \alpha, \eta, \gamma, l > 0 \) such that

\[
C(t,x,y) \geq \alpha I; \\
x^\top \nabla V(x) \geq \eta|x|^2 \gamma x^\top Cx \quad \forall (t,x,y) \text{ with } |x| \geq l.
\]

Then system (18) admits a \((Q,T)\) affine-periodic solution.

**Remark 7.** We make some comments to condition ii) in Theorem 4.2.

a) When \( C(t,x,y) \) is bounded, particularly \( C = C(t) \), the assumption

\[
x^\top \nabla V(x) \geq \gamma x^\top C(t,x,y)x \quad \forall |x| \geq l
\]

is certainly true.

b) When \( Q = I \) corresponding to the periodic solution case and \( m = 1 \), \( V \) and \( C \) may have the following forms:

\[
V(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2p} x^{2p}, \\
C(t,x,y) = C_0(t) + C_1(t)x + \cdots + C_{2q}(t)x^{2q},
\]

where \( p \geq q \geq 0 \) are integers, \( a_{2p} > 0 \), \( C_{2q}(t) \geq \alpha \), or \( C_0(t) \geq \alpha > 0 \) if \( q = 0 \).

In this case, the system (18) includes quite wide Lienard’s equations and Duffing’s equations.

This is the following.
Corollary 2. Under the assumptions in b), the system
\[ \ddot{x} + (C_0(t) + C_1(t)x + \cdots + C_{2q}(t)x^{2q})\dot{x} + b_0 x + b_1 x + \cdots + b_{2p-1}x^{2p-1} = e(t) = e(t + T), \]
where \( C_i : \mathbb{R}^1 \to \mathbb{R}^1 \) (\( i = 0, 1, \ldots, 2q \)) are continuous and \( T \)-periodic, \( C_{2q}(t) \geq \alpha > 0, b_{2p-1} > 0 \), admits a \( T \)-periodic solution.

c) We do not require that the external force \( e(t) \) is sufficiently small with \(|e(t)| \ll 1\).

We start to prove Theorem 4.2.

Proof. Let
\[ U(x, y) = \lambda|y|^2 + |x + y|^2 + (2\lambda + 2)V(x), \]
where \( \lambda \gg 1 \). Then along (18) for \((x, y) = (x, \dot{x})\),
\[ \dot{U}(18) = 2\lambda \dot{x}^\top(C\dot{x} - \nabla V + e) + 2(\dot{x} + x)^\top(C\dot{x} - \nabla V + e + \dot{x}) \]
\[ + (2\lambda + 2)x^\top\nabla V = - (2\lambda + 2)x^\top\nabla V + 2\lambda x^\top e + 2\lambda \dot{x}^\top e + 2\dot{x}^\top x + 2\dot{x}^\top Cx \]
\[ - 2x^\top \nabla V + 2x^\top e = - (\lambda + 2)x^\top C\dot{x} - x^\top \nabla V \]
\[ - \lambda(||\sqrt{C}\dot{x}||^2 - 2(\sqrt{C}\dot{x})^\top(\lambda\sqrt{C})^{-1}\dot{x} - 2(\sqrt{C}\dot{x})^\top(\lambda\sqrt{C})^{-1}x \]
\[ - 2(\sqrt{C}\dot{x})^\top \frac{\sqrt{C}}{\lambda} x - 2(\sqrt{C}\dot{x})^\top(\lambda\sqrt{C})^{-1}e + 2(\sqrt{C}\dot{x})^\top(\lambda\sqrt{C})^{-1}e \]
\[ + (|\eta||x|^2 - x^\top \nabla V) - \eta(|x|^2 - 2x^\top \frac{e}{\eta}) \]
\[ = - (\lambda + 2)x^\top C\dot{x} - x^\top \nabla V + (|\eta||x|^2 - x^\top \nabla V) \]
\[ - \lambda|\sqrt{C}\dot{x}| - ((\lambda\sqrt{C})^{-1}x + (\lambda\sqrt{C})^{-1}x + (\sqrt{C})^{-1}e \]
\[ + (\lambda(\sqrt{C})^{-1}e)^2 - \eta|x|^2 - \frac{e}{\eta}^2 + \lambda(\lambda\sqrt{C})^{-1}x + (\lambda\sqrt{C})^{-1}x + \frac{\sqrt{C}}{\lambda} x \]
\[ + (\sqrt{C})^{-1}e + (\lambda\sqrt{C})^{-1}e^2 + \eta(\frac{e}{\eta})^2 \]
\[ \leq - (\lambda + 2)x^\top C\dot{x} - x^\top \nabla V + (|\eta||x|^2 - x^\top \nabla V) + \lambda(\lambda\sqrt{C})^{-1}x \]
\[ + (\lambda\sqrt{C})^{-1}x + (\sqrt{C})^{-1}e + (\lambda\sqrt{C})^{-1}e^2 + \eta\frac{e}{\eta}^2 \]
\[ \leq - (\lambda + 2)x^\top C\dot{x} - x^\top \nabla V + (|\eta||x|^2 - x^\top \nabla V) \]
\[ + \frac{K}{\lambda}(|\dot{x}|^2 + |x|^2 + |e|^2) + \lambda K|e|^2 + \frac{1}{\lambda} x^\top Cx + \eta \frac{e}{\eta}^2, \]
for some \( K > 0 \). Hence, for \( \lambda \) large and \( \mu(>0) \) small, and some \( L > 0 \),
\[ \dot{U} \leq -\mu(|x|^2 + |\dot{x}|^2) + L. \] (19)

Obviously, there exists \( \sigma > 0 \) such that
\[ U(x, y) \geq \sigma(|x|^2 + |y|^2). \] (20)
It follows from (19), (20) that along any solution \( z(t, z_0) = (x(t, x_0, y_0), y(t, x_0, y_0)) \) of (18) with the initial value \( z(0) = (x(0), y(0)) = (x_0, y_0) \) for \( y(t, x_0, y_0) = x(t, x_0, y_0) \),
\[
\sigma |z(t, z_0)|^2 \leq U(z(t, z_0)) \leq U(z_0)e^{\int_0^t (-\mu |z(s, z_0)|^2 + L)ds},
\]
which implies that for any \( z_0 = (x_0, y_0) \), the solution \( z(t, z_0) \) exists on \( \mathbb{R}_+^1 \).

Take \( M = \max\{U(x, y) : |x|^2 + |y|^2 \leq \frac{L+1}{\mu}\} \),
\[
N_1 \in \mathbb{N} \quad \text{with} \quad Me^{-N_1T} < \frac{\sigma L}{\mu}.
\]
Set
\[
B = \left( \frac{M}{\sigma}e^{LN_1T} \right)^{\frac{1}{2}}.
\]
We claim that for any \( R > 0 \) with \( R^2 \geq \frac{L+1}{\mu} \), there exists a positive integer \( N(R) \) such that
\[
|z(t, z_0)| \leq B \quad \forall t \geq N(R)T, \tag{21}
\]
whenever \( |z_0| \leq R \). Indeed, set
\[
M_R = \max\{U(x, y) : |x|^2 + |y|^2 \leq R\}.
\]
Take a prime \( N(R) \) such that
\[
M_Re^{-N(R)T} < \frac{\sigma L}{\mu}. \tag{22}
\]
Then it follows from (19), (20), (22) that (21) holds.

Set \( \Omega = \{z_0 \in \mathbb{R}^{2m} : |z_0| < B + 1\} \).
Define a Poincaré map \( P \) by
\[
P(z_0) = \text{diag}(Q^{-1}, Q^{-1})z(T, z_0).
\]
Then \( P \) is well defined on \( \mathbb{R}^{2m} \). Note that
\[
P^j(z_0) = \left( \begin{array}{cc} Q & 0 \\ 0 & Q \end{array} \right)^{-j}z(jT, z_0) \quad \forall j \geq 1.
\]
Since \( Q \in O(m) \), by (21), we have
\[
|P^j(z_0)| = |\left( \begin{array}{cc} Q & 0 \\ 0 & Q \end{array} \right)^{-j}z(jT, z_0)|
= |z(jT, z_0)|
\leq B \quad \forall j = N(B + 1), N(B + 1) + 1,
\]
for any \( z_0 \in \partial \Omega \). By virtue of Theorem 2.4, \( P \) has a fixed point \( z_* = (x_*, y_*) \) in \( \Omega \).
Then
\[
z(t + T, z_*) = \text{diag}(Q, Q)z(t, z_*) \quad \forall t.
\]
Consequently, \( x(t, x_*, y_*) \) is a \( (Q, T) \) affine-periodic solution of (18). The proof is complete.

As an application, we have the following.
Corollary 3. Consider the Newtonian (including Rayleigh or Lienard) system with friction
\[ \ddot{x} + \frac{d}{dt} \nabla F(x) + \nabla V(x) = e(t). \]  
\[ (23) \]
Assume
i) \( F : \mathbb{R}^m \to \mathbb{R}^m \) is \( C^2 \), \( V : \mathbb{R}^m \to \mathbb{R} \) is \( C^1 \), \( e : \mathbb{R} \to \mathbb{R}^m \) is continuous.
ii) For given \( Q \in O(m) \),
\[ \frac{\partial^2 F(x)}{\partial x^2} = \frac{\partial^2 F(x)}{\partial x^2}, \]
\[ \frac{\partial^2 F(Q^{-1}x)}{\partial x^2} = Q \frac{\partial^2 F(Q^{-1}x)}{\partial x^2}, \]
\[ \nabla V(x) = Q \nabla V(Q^{-1}x), \]
\[ e(t + T) = Qe(t). \]
iii) There exist positive constants \( \alpha, \gamma, \sigma, l > 0 \) such that
\[ \frac{\partial^2 F(x)}{\partial x^2} \geq \alpha I, \quad x^2 \nabla V \geq \gamma |x|^2, \quad \sigma x^2 \frac{\partial^2 F(x)}{\partial x^2} \geq \alpha I, \forall x \text{ with } |x| \geq l. \]
Then system (23) has a \((Q,T)\) affine-periodic solution.

Remark 8. We can set
\[ F(x) = |x|^{2p}, \quad V(x) = |x|^{2q}, \quad e(t + T) = Qe(t), \]
where \( p > q > 0 \) are integers, and \( Q \in O(m) \).

We sketch the proof of Corollary 3.

Proof. Let
\[ U(x,y) = \lambda |y|^2 + |x + y|^2 + 2(\lambda + 1)V(x), \]
where \( \lambda \gg 1 \). The rest arguments are the same as in Theorem 4.2, and we omit the details. \( \square \)

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