ABELIAN CLASS GROUPS OF REDUCTIVE GROUP SCHEMES

CRISTIAN D. GONZÁLEZ-AVILÉS

Abstract. We introduce the abelian class group $C_{ab}(G)$ of a reductive group scheme $G$ over a ring $A$ of arithmetical interest and study some of its basic properties. For example, we show that if the fraction field of $A$ is a global field without real primes, then there exists a surjection $C(G) \twoheadrightarrow C_{ab}(G)$, where $C(G)$ is the class set of $G$.

1. INTRODUCTION

Let $K$ a global field, i.e., $K$ is either a number field or a function field in one variable over a finite field $k$. Let $S$ be a nonempty open subscheme of the spectrum of the ring of integers of $K$ (in the number field case) or a nonempty open affine subscheme of the unique smooth, projective and irreducible curve over $k$ whose function field is $K$ (in the function field case). For any nonempty open subscheme $U$ of $S$, let $U_0$ denote the set of closed points of $U$. For each $v \in S_0$, let $\mathcal{O}_v$ denote the completion of the local ring of $S$ at $v$ and let $K_v$ denote the fraction field of $\mathcal{O}_v$. Let

$$A_S(U) = \prod_{v \in S_0 \setminus U_0} K_v \times \prod_{v \in U_0} \mathcal{O}_v$$

be the ring of $U$-integral adeles of $S$. The rings $A_S(U)$ form an inductive system when the sets $U$ are ordered by reverse inclusion (i.e., $U \leq U'$ if, and only if, $U \supset U'$) and the ring of adeles of $S$ is, by definition,

$$A_S = \varprojlim U A_S(U).$$

Now let $G$ be an affine group scheme of finite type over $S$ with smooth generic fiber. Then the class set of $G$,

$$C(G) = G(\mathbb{A}_S(S)) \setminus G(\mathbb{A}_S)/G(K),$$

encodes important arithmetic information about $G$. Unfortunately, the above set is notoriously difficult to compute and, in general, carries no additional structure, which makes the tools of Algebra rather useless in its study. It is perhaps for this reason that, at least in specific cases, researchers have

2000 Mathematics Subject Classification. Primary 20G35; Secondary 20G30, 11E72.

Key words and phrases. Reductive group schemes, abelian cohomology, flasque resolutions, class sets, class groups.

The author is partially supported by Fondecyt grant 1080025.
sought to study certain abelian groups which in some sense “approximate” $C(G)$. A well-known instance of this approach can be found in the theory of quadratic forms, where the genus class set of a quadratic form $q$ (or, equivalently, the class set of a natural integral model of its orthogonal group) is studied via an “abelian group approximation”, namely the spinor genus class group of $q$. In this paper we present the beginnings of a general development of this idea. More precisely, we introduce the abelian class group of a reductive $S$-group scheme $G$ and study some of its main properties (see below for statements). Thus the theory developed here applies to (connected) reductive algebraic groups over $K$ which extend to a reductive group scheme over $S$, i.e., which have good reduction over $S$. We work under this restriction because the general case, where $G$ is allowed to have “bad” (non-reductive) fibers, requires a long series of preparations which will be the subject of separate papers. In this sense, therefore, the present paper is less general than the work of C.Demarche [Dem2], who considered arbitrary flat integral models of finite type of a (connected) reductive algebraic group $G_K$ over a number field $K$. See [Dem2], §4. However, in a different sense to be explained below, the present paper is more general than [op.cit.]. Indeed, the arguments of [op.cit.] are valid only under the hypothesis (H) that the simply-connected central cover $\tilde{G}_K$ of $G_K$ satisfies the strong approximation property with respect to the set $\Sigma$ of primes of $K$ which do not correspond to a point of $S_0$. Under hypothesis (H), the class sets of the models considered in [Dem2] are naturally equipped with the structure of (finite) abelian groups, and Demarche is able to use the Brauer-Manin pairing to obtain a duality theorem for these groups [Dem2], Theorem 4.1. We stress here the evident fact that, since the Brauer group of a scheme is abelian, the Brauer-Manin pairing can only “detect” abelian groups associated to $G$. In this regard, we also note that (H) implies as well that the defect of strong approximation for $G_K$ relative to $\Sigma$ is naturally equipped with the structure of an abelian group, and [Dem2], Theorem 3.14, can be interpreted as a duality statement for this group. In this paper we dispense with the hypothesis (H) in the study of class sets of reductive group schemes over $S$ and, in particular, abandon the compact-noncompact-type dichotomy which is familiar from the discussion of class sets contained in [PR], §§8.2 and 8.3. We are thus able to handle both cases simultaneously (again, under a “good reduction” hypothesis on $G_K$). Note, further, that this paper also covers the function field case.

We now state the main results of the paper.

Let $G$ be a reductive group scheme over $S$, let $G^{\text{der}}$ be the derived group of $G$ and set $G^{\text{tor}} = G/G^{\text{der}}$. Further, let $\tilde{G}$ be the simply-connected central cover of $G^{\text{der}}$ and let $\mu$ for the fundamental group of $G^{\text{der}}$, i.e., the kernel of $\tilde{G} \to G^{\text{der}}$. Now let $S_{\text{fl}}$ be the small fppf site over $S$ and let $H^i_{\text{ab}}(S_{\text{fl}}, G)$ be

1 Including the development of a homotopy theory for smooth (abelian) sheaves over a discrete valuation ring and a theory of Néron models of (2-term) complexes of tori.
the abelian cohomology groups of $G$ introduced in [GA3]. Restriction to the
generic fiber of $G$ yields a map $H^1_{\text{ab}}(S_\text{fl}, G) \to H^1_{\text{ab}}(K_\text{fl}, G)$, and the abelian
class group of $G$ is by definition

$$C_{\text{ab}}(G) = \ker\left[H^1_{\text{ab}}(S_\text{fl}, G) \to H^1_{\text{ab}}(K_\text{fl}, G)\right].$$

When $K$ has no real primes and $\tilde{G}_K$ satisfies hypothesis (H), so that $C(G)$
has a natural abelian group structure as mentioned above, then there exists a
canonical isomorphism $C(G) \simeq C_{\text{ab}}(G)$ (see Remark 3.12(b)). Then results
from Section 3, which essentially follow from the main theorem of [GA3],
yield the following statement.

**Theorem 1.1.** (=Theorem 3.11) Assume that $K$ has no real primes. Then
there exists a natural right action of $H^0_{\text{ab}}(S_\text{fl}, G)$ on $H^1(S_{\text{ét}}, \tilde{G})$ and a canoni-
cal exact sequence of pointed sets

$$1 \to H^1(S_{\text{ét}}, \tilde{G})/H^0_{\text{ab}}(S_\text{fl}, G) \to C(G) \to C_{\text{ab}}(G) \to 1,$$

where the first nontrivial map is injective.

In Section 3 we also study some basic properties of $C_{\text{ab}}(G)$. In par-
cular, using the flasque resolutions of $G$ constructed in [GA4], we obtain the
following results.

**Theorem 1.2.** (=Theorem 3.13) Let $G$ be a reductive group scheme over $S$
and let $1 \to F \to H \to G \to 1$ be a flasque resolution of $G$. Set $R = H^\text{tor}$. Then
the given resolution induces an exact sequence of finitely generated
abelian groups

$$\mu(S) \hookrightarrow F(S) \to R(S) \to H^0_{\text{ab}}(S_\text{fl}, G) \to H^1(S_{\text{ét}}, F) \to C(R) \to C_{\text{ab}}(G) \to 1.$$

**Theorem 1.3.** (=Corollary 3.14) Let $L/K$ be a finite Galois extension and
let $S' \to S$ be the normalization of $S$ in $L$. Let $G$ be a reductive group
scheme over $S$ and let $1 \to F \to H \to G \to 1$ be a flasque resolution of $G$.
Then the given resolution defines a corestriction homomorphism

$$\text{cores}_{S'/S}: C_{\text{ab}}(S', G) \to C_{\text{ab}}(S, G),$$

where $C_{\text{ab}}(S', G)$ (respectively, $C_{\text{ab}}(S, G)$) is the abelian class group of $G \times_S S'$
(respectively, $G$).

The homomorphism of the theorem is independent, up to isomorphism, of
the chosen flasque resolution of $G$ and is functorial in $S' \to S$. See Remark
3.15. When $K$ has no real primes and $\tilde{G}_K$ satisfies hypothesis (H), so that
$C(G) \simeq C_{\text{ab}}(G)$ as noted above, the preceding theorem shows that $C(G)$ is
endowed with natural corestriction maps. When $K$ is an arbitrary number
field and $\tilde{G}_K$ satisfies hypothesis (H), C.Demarche obtained in [Dem2], The-
orem 4.6, a similar corestriction homomorphism which presumably coincides

\footnote{When $K$ is a number field with real primes, the relation between $C(G)$ and $C_{\text{ab}}(G)$
is more complicated. See Remark 3.12(c).}
with the above one when $K$ is totally imaginary. The existence of such corestriction maps (under hypothesis (H) and for any $K$) was first established in [Th], Theorem 14, p.36.

In Sections 4 and 5 we use results of C.Demarche [Dem1] and M.Borovoi and J.van Hamel [BvH] to obtain the following result. Set $G_K = G \times_S \text{Spec} K$.

**Theorem 1.4.** (Theorem 5.5) Assume that $G_K = G \times_S \text{Spec} K$ admits a smooth $K$-compactification. There exists a perfect pairing of finite groups

$$C_{ab}(G) \times \text{Br}_{a,\text{nr}}(G_K)/\text{Br}(G_K) \rightarrow \mathbb{Q}/\mathbb{Z},$$

where $\text{Br}_{a,\text{nr}}(G_K)$ and $\text{Br}(G_K)$ are the subgroups of the algebraic Brauer group of $G_K$ given by (5.4) and (5.1), respectively.

When $K$ is a totally imaginary number field and $\tilde{G}_K$ satisfies hypothesis (H), the pairing of the theorem should be closely related to that obtained by Demarche in [Dem2], Theorem 4.14. We hope to clarify this issue in a future publication.

**Acknowledgements**

I thank Bas Edixhoven, Philippe Gille, Niko Naumann and Adrian Vasiu for helpful comments.

2. **Preliminaries**

Let $K$ and $S$ be as in the Introduction. We will write $S_{\text{fl}}$ (respectively, $S_{\text{ét}}, S_{\text{Nis}}$) for the small fppf (respectively, étale, Nisnevich) site over $S$. If $\tau = \text{fl}$ or $\text{ét}$, $G$ is an $S$-group scheme and $i = 0$ or 1, $H^i(S_\tau, G)$ will denote the $i$-th cohomology set of the sheaf on $S_\tau$ represented by $G$. If $G$ is commutative, these cohomology sets are in fact abelian groups and are defined for every $i \geq 0$. If $S' \rightarrow S$ is a morphism of schemes, we will write $H^i(S_\text{fl}', G)$ for $H^i(S_\text{fl}', G \times_S S')$. When $G$ is smooth, the canonical map $H^i(S_{\text{ét}}, G) \rightarrow H^i(S_{\text{fl}}, G)$ is bijective (see [Mi1], Remark III.4.8(a), p.123). In this case, the preceding sets will be identified. If $K$ is a field, we will write $\overline{K}$ for a fixed separable algebraic closure of $K$, $\Gamma$ for $\text{Gal}(\overline{K}/K)$ and $H^i(K, G)$ for the Galois cohomology set (or group) $H^i(\Gamma, G(\overline{K}))$. If $S = \text{Spec} K$, $H^i(S_{\text{ét}}, G)$ and $H^i(K, G)$ will be identified.

An $S$-group scheme $G$ is called reductive (respectively, semisimple) if it is affine and smooth over $S$ and its geometric fibers are connected reductive (respectively, semisimple) algebraic groups. If $G$ is a reductive $S$-group scheme, $G^* = \text{Hom}_S(G, G_m)$ is the twisted-constant $S$-group scheme of characters of $G$. The derived group of $G$ (see [SGA3], XXII, Theorem 6.2.1(iv)), will be denoted by $G^{\text{der}}$. It is a normal semisimple subgroup scheme of $G$ and the quotient

$$G^{\text{tor}} := G/G^{\text{der}}$$

This is certainly the case if $K$ is a number field, by Hironaka’s theorem.
is an $S$-torus (in [SGA3], XXII, 6.2, $G^{tor}$ is denoted by corad$(G)$ and called the coradical of $G$). Note that, since $G^{der*} = 0$, the exact sequence of reductive $S$-group schemes

$$1 \to G^{der} \to G \to G^{tor} \to 1$$

induces the equality $G^* = G^{tor*}$.

A semisimple $S$-group scheme $G$ is called simply-connected if it admits no nontrivial central cover, i.e., any central $S$-isogeny $G' \to G$ from a semisimple $S$-group scheme $G'$ to $G$ is an isomorphism. If $G$ is any semisimple $S$-group scheme, then there exists a simply-connected $S$-group scheme $	ilde{G}$ and a central isogeny $\varphi: \tilde{G} \to G$. The pair $(\tilde{G}, \varphi)$ is unique up to unique isomorphism and its formation commutes with arbitrary extensions of the base. It is called the simply-connected central cover of $G$. The fundamental group of $G$ is by definition the kernel of $\varphi$ and will be denoted by $\mu$ (or by $\mu_G$, if necessary). It is a finite $S$-group scheme of multiplicative type. Now, if $G$ is any reductive $S$-group scheme, then there exists a simply-connected $S$-group scheme $\tilde{G}$ and a central isogeny $\tilde{\phi}: \tilde{G} \to G$. The pair $(\tilde{G}, \tilde{\phi})$ is unique up to unique isomorphism and its formation commutes with arbitrary extensions of the base. It is called the simply-connected central cover of $G$. The fundamental group of $G$ is by definition the kernel of $\tilde{\phi}$ and will be denoted by $\mu$ (or by $\mu_G$). There exists a canonical central extension

$$1 \to \mu \to \tilde{G} \to G^{der} \to 1.$$ 

We will write $\partial: \tilde{G} \to G$ for the composition $\tilde{G} \to G^{der} \to G$. Clearly, there exists a canonical exact sequence

$$1 \to \mu \to \tilde{G} \xrightarrow{\partial} G \to G^{tor} \to 1.$$ 

Now there exists a canonical “conjugation” action of $G$ on $\tilde{G}$ such that $\tilde{G} \xrightarrow{\partial} G$, regarded as a two-term complex with $\tilde{G}$ and $G$ placed in degrees $-1$ and $0$, respectively, is (a left) quasi-abelian crossed module on $S_{fl}$, in the sense of [GA3], Definition 3.2. See [Br], Example 1.9, p.28, and [GA3], Example 2.2(iii). Thus $\partial$ induces a homomorphism $\partial Z: Z(\tilde{G}) \to Z(G)$ and the embedding of crossed modules

$$(Z(\tilde{G}) \xrightarrow{\partial Z} Z(G)) \hookrightarrow (\tilde{G} \xrightarrow{\partial} G)$$

is a quasi-isomorphism (see [GA3], Proposition 3.4). In particular, $\partial Z$ induces an exact sequence of $S$-groups of multiplicative type

$$1 \to \mu \to Z(\tilde{G}) \xrightarrow{\partial Z} Z(G) \to G^{tor} \to 1.$$ 

Let $i \geq -1$ be an integer. The $i$-th abelian (flat) cohomology group of $G$ is by definition the hypercohomology group

$$H^i_{ab}(S_{fl}, G) = \mathbb{H}^i(S_{fl}, Z(\tilde{G}) \xrightarrow{\partial Z} Z(G)),$$

where $Z(G)$ is placed in degree 0. We will also need the dual abelian cohomology groups of $G$. By definition, these are the groups

$$H^i_{ab}(S_{et}, G^*) = \mathbb{H}^i(S_{et}, Z(G)^* \xrightarrow{\partial Z} Z(\tilde{G})^*),$$

where $Z(G)^*$ is placed in degree 0.
where $Z(G)^*$ is placed in degree $-1$. Note that, as the Cartier dual of an $S$-group of multiplicative type is étale over $S$, the preceding groups coincide with the flat hypercohomology groups $\mathbb{H}^i(S_{\text{fl}}, Z(G)^*) \to Z(\tilde{G})^*$). If $S = \text{Spec } K$, where $K$ is a field, we will write $H^i_{\text{ab}}(K, G^*)$ for $H^i_{\text{ab}}(\text{Spec } S, G^*)$.

Clearly,

$$H^i_{\text{ab}}(K, G^*) = \mathbb{H}^i(\Gamma, Z(G(\overline{K}))^* \to Z(\tilde{G}(\overline{K}))^*)$$

(Galois hypercohomology). Now, by (2.3), there exist exact sequences

(2.7)

$$\ldots \to H^{i-1}(S_{\text{ét}}, G^{\text{tor}}) \to H^{i+1}(S_{\text{fl}}, \mu) \to H^i_{\text{ab}}(S_{\text{fl}}, G) \to H^i(S_{\text{ét}}, G^{\text{tor}}) \to \ldots .$$

and

(2.8)

$$\ldots \to H^{i-1}(S_{\text{ét}}, \mu^*) \to H^{i+1}(S_{\text{ét}}, G^{\text{tor}*}) \to H^i_{\text{ab}}(S_{\text{ét}}, G^*) \to H^i(S_{\text{ét}}, \mu^*) \to \ldots .$$

**Examples 2.1.**

(a) If $G$ is semisimple, i.e., $G^{\text{tor}} = 0$, then $H^i_{\text{ab}}(S_{\text{fl}}, G) = H^{i+1}(S_{\text{fl}}, \mu)$ and $H^i_{\text{ab}}(S_{\text{ét}}, G^*) = H^i(S_{\text{ét}}, G^*)$.

(b) If $G$ has trivial fundamental group, i.e., $\mu = 0$, then $H^i_{\text{ab}}(S_{\text{fl}}, G) = H^i(S_{\text{ét}}, G^{\text{tor}})$ and $H^i_{\text{ab}}(S_{\text{ét}}, G^*) = H^{i+1}(S_{\text{ét}}, G^{\text{tor}*})$.

By [GA3], there exist canonical abelianization maps

(2.9)

$$ab^i = ab^i_{G/S} : H^i(S_{\text{fl}}, G) \to H^i_{\text{ab}}(S_{\text{fl}}, G)$$

so that the following holds.

**Theorem 2.2.** Let $G$ be a reductive group scheme over $S$.

(i) There exists an exact sequence of pointed sets

$$1 \to \mu(S) \to \tilde{G}(S) \to G(S) \xrightarrow{\text{ab}^0} H^0_{\text{ab}}(S_{\text{fl}}, G) \xrightarrow{\text{ab}^1} H^1_{\text{ab}}(S_{\text{fl}}, \tilde{G}) \xrightarrow{\delta_0} H^1(S_{\text{fl}}, \tilde{G}) \xrightarrow{\delta_1} H^2(S_{\text{fl}}, \tilde{G}) \to 1.$$ 

(ii) The group $H^0_{\text{ab}}(S_{\text{fl}}, G)$ acts on the right on the set $H^1(S_{\text{fl}}, \tilde{G})$ compatibly with the map $\delta_0$ and the preceding exact sequence induces an exact sequence of pointed sets

$$1 \to H^1(S_{\text{fl}}, \tilde{G}) / H^0_{\text{ab}}(S_{\text{fl}}, G) \xrightarrow{\tilde{\delta}^1} H^1(S_{\text{fl}}, G) \xrightarrow{\text{ab}^1} H^1_{\text{ab}}(S_{\text{fl}}, G) \to 1,$$

where the map $\tilde{\delta}^1$, induced by $\delta^1$, is injective.

**Proof.** By [GA3], Example 5.4(iii), $S$ is a scheme of Douai type, i.e., every class of the Giraud cohomology set $H^2(S_{\text{fl}}, \tilde{G})$ is neutral. Thus the theorem follows from [GA3], Theorem 5.1, Theorem 5.5 and Proposition 3.14(b). The action mentioned in (ii) is defined in [op.cit.], Remark 3.9(b).

**Remark 2.3.** The exact sequence in part (ii) of the theorem is compatible with inverse images, i.e., if $S' \to S$ is a morphism of schemes of Douai type,
then the following diagram commutes

\[
\begin{array}{ccccccc}
1 & \rightarrow & H^1(S_{\text{ét}}, \tilde{G})/H^0_{\text{ab}}(S_{\text{fl}}, G) & \rightarrow & H^1(S_{\text{ét}}, G) & \rightarrow & H^1_{\text{ab}}(S_{\text{fl}}, G) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & H^1(S'_{\text{ét}}, \tilde{G})/H^0_{\text{ab}}(S'_{\text{fl}}, G) & \rightarrow & H^1(S'_{\text{ét}}, G) & \rightarrow & H^1_{\text{ab}}(S'_{\text{fl}}, G) & \rightarrow & 1
\end{array}
\]

This follows from [GA3], Remark 4.3, and the fact that the action of \( H^0_{\text{ab}}(S_{\text{fl}}, G) \) on \( H^1(S_{\text{ét}}, \tilde{G}) \) is compatible with inverse images (see [op.cit.], Remark 3.9(b)).

We will write \( S_0 \) for the set of closed points of \( S \) and \( \Sigma \) for the set of primes of \( K \) which do not correspond to a point of \( S_0 \). Thus \( \Sigma \) is nonempty and contains all archimedean primes of \( K \) in the number field case. For any prime \( v \) of \( K \), \( K_v \) will denote the completion of \( K \) at \( v \). If \( v \in S_0 \), we will write \( \mathcal{O}_v \) for the ring of integers of \( K_v \) and \( k(v) \) for the corresponding residue field. Note that, since \( k(v) \) is finite, every \( k(v) \)-torus is cohomologically trivial by [Se], Propositions 5(iii) and 6(b), pp. II-7-8. If \( v \) is a real prime of \( K \), \( i \) is any integer and \( C \) is a cohomologically bounded complex of abelian sheaves on \( (\text{Spec} \ K_v)_{\text{fl}} \), \( \mathbb{H}^i(K_{v, \text{fl}}, C) \) will denote the modified (Tate) \( i \)-th hypercomology group of \( C \) defined in [HS], p.103. In particular, the groups \( H^i_{\text{ab}}(K_{v, \text{fl}}, G) = \mathbb{H}^i(K_v, Z(\tilde{G}) \rightarrow Z(G)) \) coincide with the groups denoted \( H^i_{\text{ab}}(K_v, G) \) in [Bor].

3. The abelian class group

Let \( K \) and \( S \) be as in the Introduction. Let \( G \) be a (connected) reductive algebraic group over \( K \) and let \( T \) be a \( K \)-torus. Set

\[
\begin{align*}
\mathbb{H}^1(K, G) &= \text{Ker} \left[ H^1(K, G) \rightarrow \prod_{v} H^1(K_v, G) \right], \\
\mathbb{H}^2(K, T) &= \text{Ker} \left[ H^2(K, T) \rightarrow \prod_{v} H^2(K_v, T) \right]
\end{align*}
\]

and

\[
(3.1) \quad \mathbb{H}^1_{\text{ab}}(K, G) = \text{Ker} \left[ H^1_{\text{ab}}(K_{\text{fl}}, G) \rightarrow \prod_{v} H^1_{\text{ab}}(K_v, G, T) \right].
\]

Proposition 3.1. Let \( G \) be a (connected) reductive algebraic group over \( K \). Let \( 1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1 \) be a flasque resolution of \( G \). Then the given resolution defines an isomorphism

\[
\mathbb{H}^1_{\text{ab}}(K, G) \simeq \mathbb{H}^2(K, F).
\]
Proof. Since $\mathbb{III}^2(L, \mathbb{G}_m) = 0$ for every finite separable extension $L$ of $K$, $\mathbb{III}^2(K, R) = 0$. The result is now immediate from [GA4], Proposition 4.5.

□

Remark 3.2. By [GA3], Corollary 5.10, there exists a bijection $\mathbb{III}^1(K, G) \cong \mathbb{III}^1_{ab}(K, G)$. Thus the proposition yields a bijection $\mathbb{III}^1(K, G) \cong \mathbb{III}^2(K, F)$ which extends to an arbitrary global field $K$ the bijection of [CT], Theorem 9.4(ii). Now, if $G$ is semisimple, then [GA4], (3.4) (with $G^\text{tor} = 0$), yields an isomorphism $\mathbb{III}^2(K, \mu) \cong \mathbb{III}^2(K, F)$. Thus there exists a bijection $\mathbb{III}^1(K, G) \cong \mathbb{III}^2(K, \mu)$ which extends to the function field case the bijection of [San], Corollary 4.4.

Now let

\begin{equation}
\mathbb{III}^1_S(K, G) = \text{Ker} \left[ H^1(K, G) \to \prod_{v \in S_0} H^1(K_v, G) \right]
\end{equation}

and

\begin{equation}
\mathbb{III}^1_{ab, S}(K, G) = \text{Ker} \left[ H^1_{ab}(K_{\text{fl}}, G) \to \prod_{v \in S_0} H^1_{ab}(K_{v, \text{fl}}, G) \right].
\end{equation}

The latter group is torsion since, by (2.7), $H^1_{ab}(K_{\text{fl}}, G)$ is a torsion group.

**Lemma 3.3.** The abelianization map $\text{ab}_{G/K}^1 : H^1(K, G) \to H^1_{ab}(K_{\text{fl}}, G)$ induces a surjection

$\mathbb{III}^1_S(K, G) \twoheadrightarrow \mathbb{III}^1_{ab, S}(K, G)$.

**Proof.** This follows from the commutative diagram

\[
\begin{array}{ccc}
H^1(K, G) & \xrightarrow{\text{ab}_{G/K}} & H^1_{ab}(K_{\text{fl}}, G) \\
\downarrow & & \downarrow \\
\prod_{v \in S_0} H^1(K_v, G) & \xrightarrow{\sim} & \prod_{v \in S_0} H^1_{ab}(K_{v, \text{fl}}, G),
\end{array}
\]

whose top map is surjective by [GA3], Theorem 5.5(i), and bottom map is bijective by [op.cit.], Theorem 5.8(i) and Remark 5.9(a).

□

Now let $G$ be any affine $S$-group scheme of finite type with smooth generic fiber. Let $\mathbb{A}_S$ (respectively, $\mathbb{A}_S(S)$) denote the ring of adeles (respectively, integral adeles) of $S$ and let

$C(G) = G(\mathbb{A}_S(S)) \setminus G(\mathbb{A}_S)/G(K)$

be the class set of $G$. Since $G$ is generically smooth, Ye.Nisnevich has shown that there exists a canonical bijection $C(G) \cong H^1(S_{\text{Nis}}, u_*G)$, where $u : S_{\text{et}} \to S_{\text{Nis}}$ is the canonical morphism of sites. See [GA2], Theorem 3.5.
Theorem 3.4. (Ye.Nisnevich) Let $G$ be an affine group scheme of finite type over $S$ with smooth generic fiber. Assume that $H^1(O_{v, \text{ét}}, G)$ is trivial for every $v \in S_0$. Then there exists an exact sequence of pointed sets

$$1 \to C(G) \xrightarrow{\varphi} H^1(S_\text{ét}, G) \to \Pi^1_S(K, G) \to 1,$$

where the map $\varphi$ is injective and $\Pi^1_S(K, G)$ is the set (3.2).

Proof. This follows from [Gi], Proposition V.3.1.3, p.323, using the bijection $C(G) \simeq H^1(S_{\text{Nis}}, u_* G)$ recalled above and [Nis], Proposition 1.37, p.282. □

Remark 3.5. The fibers of the map $\lambda: H^1(S_\text{ét}, G) \to \Pi^1_S(K, G)$ appearing above may be computed as follows (see [Gi], Corollary V.3.1.4, p.324): let $p \in H^1(S_\text{ét}, G)$, choose a $G$-torsor $P$ representing $p$ and let $P^G$ be the twist of $G$ by $P$ (see [Gi], Proposition III.2.3.7, p.146). Let $\theta_p: H^1(S_\text{ét}, G) \xrightarrow{\sim} H^1(S_\text{ét}, P^G)$ be the bijection defined in [op.cit.], Remark III.2.6.3, p.154. Then $\theta_p^{-1} \circ P \circ \varphi$ induces a bijection $C(P^G) \xrightarrow{\sim} \{p' \in H^1(S_\text{ét}, G): \lambda(p') = \lambda(p)\}$.

Corollary 3.6. Let $G$ be a reductive group scheme over $S$. Then there exists an exact sequence of pointed sets

$$1 \to C(G) \to H^1(S_\text{ét}, G) \to H^1(K, G) \to 1.$$

Proof. By the theorem, we only need to check that $H^1(O_{v, \text{ét}}, G)$ is trivial for every $v \in S_0$. By [SGA3], XXIV, Proposition 8.1, there exists a canonical bijection $H^1(O_{v, \text{ét}}, G) \simeq H^1(k(v), G)$. Now, since $G \times_S \text{Spec } k(v)$ is a connected algebraic group over the finite field $k(v)$, the set $H^1(k(v), G)$ is trivial by Lang’s theorem [La], Theorem 2, p.557. □

Corollary 3.7. Let $G$ be a semisimple and simply-connected group scheme over $S$.

(i) There exists an exact sequence of pointed sets

$$1 \to C(G) \to H^1(S_\text{ét}, G) \to H^1(K, G) \to 1.$$

(ii) The canonical localization map $H^1(K, G) \to \prod_{v \text{real}} H^1(K_v, G)$ is bijective.

Proof. Since $H^1(K_v, G)$ is trivial for every nonarchimedean prime $v$ of $K$ (see [PR], Theorem 6.4, and [BT], Theorem 4.7(ii)), we have $\Pi^1_S(K, G) = H^1(K, G)$. Thus assertion (i) is immediate from the previous corollary. Assertion (ii) is well-known (see [PR], Theorem 6.6, p.286, and [Har], Theorem A, p.125). □
Henceforth, $G$ will denote a reductive group scheme over $S$. By [GA3], Remark 4.3, there exists a commutative diagram

$$
\begin{array}{ccc}
H^1(S_{\acute{e}t}, G) & \longrightarrow & \Sha^1_S(K, G) \\
\downarrow \ab_{G/S} & & \downarrow \\
H^1_{ab}(S_{\acute{e}t}, G) & \longrightarrow & H^1_{ab}(K_{\acute{e}t}, G),
\end{array}
$$

where the top arrow is surjective by Corollary 3.6, the left-hand vertical map is surjective by [GA3], Proposition 5.5(i), and the right-hand vertical map (which is the restriction of $\ab^1_{G/K}$ to $\Sha_S^1(K, G)$) maps $\Sha^1_S(K, G)$ onto $\Sha^1_{ab,S}(K, G)$ by Lemma 3.3. We conclude that the bottom map in the above diagram induces a surjection $H^1_{ab}(S_{\acute{e}t}, G) \twoheadrightarrow \Sha^1_{ab,S}(K, G)$.

**Definition 3.8.** Let $G$ be a reductive group scheme over $S$. The **abelian class group of $G$** is the group

$$
C_{ab}(G) = \ker [H^1_{ab}(S_{\acute{e}t}, G) \rightarrow \Sha^1_{ab,S}(K, G)].
$$

Thus, there exists an exact sequence

$$
1 \rightarrow C_{ab}(G) \rightarrow H^1_{ab}(S_{\acute{e}t}, G) \rightarrow \Sha^1_{ab,S}(K, G) \rightarrow 1. \tag{3.4}
$$

**Examples 3.9.**

(a) If $G$ is semisimple with fundamental group $\mu$ then, by Example 2.1(a), $H^1_{ab}(S_{\acute{e}t}, G) = H^2(S_{\acute{e}t}, \mu)$ and $\Sha^1_{ab,S}(K, G) = \Sha^2_S(K, \mu)$, where

$$
\Sha^2_S(K, \mu) = \ker \left[ H^2(K_{\acute{e}t}, \mu) \rightarrow \prod_{v \in S_0} H^2(K_v, \mu) \right].
$$

Thus there exists an exact sequence of abelian groups

$$
1 \rightarrow C_{ab}(G) \rightarrow H^2(S_{\acute{e}t}, \mu) \rightarrow \Sha^2_S(K, \mu) \rightarrow 1.
$$

Note that $C_{ab}(G)$ is annihilated by the exponent of $\mu$. When $n \geq 2$ is an integer and $\mu = \mu_{n,S}$ is the group scheme of $n$-th roots of unity on $S$, $C_{ab}(G)$ can be computed explicitly. Indeed, taking cohomology of the exact sequence of fppf sheaves $1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \twoheadrightarrow \mathbb{G}_{m} \rightarrow 1$ over $S$ and over $K$, we obtain an exact commutative diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & \Pic(S)/n \\
\downarrow & & \downarrow \\
1 & \longrightarrow & H^2(S_{\acute{e}t}, \mu_n) \longrightarrow \Br(S)/n \longrightarrow 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & H^2(K_{\acute{e}t}, \mu_n) \longrightarrow \Br(K)/n \longrightarrow 1.
\end{array}
$$

The right-hand vertical map in the above diagram is injective by [ADT], proof of Proposition II.2.1, p.164, line -10, whence there exists a canonical isomorphism

$$
C_{ab}(G) = \ker [H^2(S_{\acute{e}t}, \mu_n) \rightarrow H^2(K_{\acute{e}t}, \mu_n)] \simeq \Pic(S)/n.
$$
(b) If $G$ has trivial fundamental group, then $H^1_{ab}(S_{\text{fl}}, G) = H^1(S_{\text{ét}}, G_{\text{tor}})$ and $\text{III}^1_{ab, S}(K, G) = \text{III}^1_S(K, G_{\text{tor}})$ by Example 2.1(b). Thus, by Corollary 3.6, $C_{ab}(G) = C(G_{\text{tor}})$. The latter group is the Néron-Raynaud $\Sigma$-class group of the $K$-torus $G_{\text{tor}}^r$ introduced in [GA2], where $\Sigma$ is the set of primes of $K$ which do not correspond to a point of $S$. To check this, we only need to show that the $S$-torus $G_{\text{tor}}^r$ is the identity component of the Néron-Raynaud model $\mathcal{N}$ of $G_{K_{\text{tor}}}$ over $S$. There exists a unique $S$-morphism $i: G_{\text{tor}}^r \to \mathcal{N}$ which extends the identity morphism on $G_{K_{\text{tor}}}$ (see [BLR], §10.1, p.289). By [SGA3], VI_B, Lemma 3.10.1, we only need to check that $i$ is an open immersion. Let $S' \to S$ be a connected finite étale cover of $S$ with function field $L$ such that $G_{\text{tor}}^r \times_S S' \simeq G_{m, S'}^r$. Then $G_{\text{tor}}^r \times_S S' \to \mathcal{N} \times_S S'$ is the embedding of $G_{m, S'}^r$ into $\mathcal{N} \times_S S'$, which is the Néron-Raynaud model of $G_K \times_{\text{Spec} K} \text{Spec} L \simeq G_{m, L}^r$ over $S'$ (see [BLR], §7.2, Theorem 1(iii), p.176). Thus $i \times_S S': G_{\text{tor}}^r \times_S S' \to \mathcal{N} \times_S S'$ is an open immersion and hence so is $i$.

Consider the exact commutative diagrams

\[
\begin{array}{ccccccccc}
1 & \to & C(G) & \to & H^1(S_{\text{ét}}, G) & \to & \text{III}^1_S(K, G) & \to & 1 \\
& & \downarrow & \ & \downarrow & \ & \downarrow & \ & \downarrow \\
& & \prod_{v \in \Sigma} H^1(K_v, G) & \to & \prod_{v \in \Sigma} H^1(K_v, G) & \to & & & \\
\end{array}
\]

and

\[
\begin{array}{ccccccccc}
1 & \to & C_{ab}(G) & \to & H^1_{ab}(S_{\text{fl}}, G) & \to & \text{III}^1_{ab, S}(K, G) & \to & 1 \\
& & \downarrow & \ & \downarrow & \ & \downarrow & \ & \downarrow \\
& & \prod_{v \in \Sigma} H^1_{ab}(K_v, G) & \to & \prod_{v \in \Sigma} H^1_{ab}(K_v, G) & \to & & & \\
\end{array}
\]

whose top rows are given by Corollary 3.6 and (3.4), respectively. The middle vertical maps in the above diagrams are induced by the compositions $\text{Spec} K_v \to \text{Spec} K \to S$ for $v \in \Sigma$. Then (3.5) and (3.6) induce exact sequences of pointed sets

\[
\begin{array}{ccccccc}
1 & \to & C(G) & \to & D^1(S, G) & \to & \text{III}^1(K, G) & \to & 1 \\
\end{array}
\]

and

\[
\begin{array}{ccccccc}
1 & \to & C_{ab}(G) & \to & D^1_{ab}(S, G) & \to & \text{III}^1_{ab}(K, G) & \to & 1, \\
\end{array}
\]

\[I\] thank B.Edixhoven for sending me this proof.
where
\[ D^1(S, G) = \text{Ker} \left[ H^1(S_{\text{et}}, G) \to \prod_{v \in \Sigma} H^1(K_v, G) \right] \]
and
\[ D^1_{ab}(S, G) = \text{Ker} \left[ H^1_{ab}(S_{\text{fl}}, G) \to \prod_{v \in \Sigma} H^1_{ab}(K_{v, \text{fl}}, G) \right]. \]

Clearly, the restriction of \( ab^1_{G/S} : H^1(S_{\text{et}}, G) \to H^1_{ab}(S_{\text{fl}}, G) \) to \( D^1(S, G) \) defines a map \( D^1(S, G) \to D^1_{ab}(S, G) \). Further, there exists an exact commutative diagram

\[
\begin{array}{cccccccc}
1 & \longrightarrow & C(G) & \longrightarrow & D^1(S, G) & \longrightarrow & \Theta^1(K, G) & \longrightarrow & 1 \\
1 & \longrightarrow & C_{ab}(G) & \longrightarrow & D^1_{ab}(S, G) & \longrightarrow & \Theta^1_{ab}(K, G) & \longrightarrow & 1,
\end{array}
\]

where the right-hand vertical map is induced by \( ab^1_{G/K} \). That the latter map is bijective is [GA3], Corollary 5.10. We may now define a map
\[(3.10) \quad C(G) \to C_{ab}(G) \]

to be that induced by the composition \( C(G) \to D^1(S, G) \to D^1_{ab}(S, G) \). Note that \((3.10)\) is surjective if, and only if, \( C_{ab}(G) \subset ab^1_{G/S}(D^1(S, G)) \).

Now let \( C(\tilde{G})' \) denote the kernel of the composition
\[(3.11) \quad H^1(S_{\text{et}}, \tilde{G}) \xrightarrow{\tilde{\lambda}} H^1(K, \tilde{G}) \xrightarrow{\pi} H^1(K, \tilde{G})/H^0_{ab}(K_{\text{fl}}, G), \]
where \( \tilde{\lambda} \) is induced by Spec \( K \to S \) and \( \pi \) is the canonical projection. If \( \delta_0 : H^0_{ab}(K_{\text{fl}}, G) \to H^1(K, \tilde{G}) \) is the map appearing in Theorem 2.2(i), then \( \text{Ker} \pi = \text{Im} \delta_0 \). On the other hand, by Corollary 3.7(i), \( \text{Ker} \tilde{\lambda} \) is in bijection with \( C(\tilde{G}) \). Thus, by the surjectivity of \( \tilde{\lambda} \), the pair of maps \((3.11)\) induces an exact sequence of pointed sets
\[ 1 \to C(\tilde{G}) \to C(\tilde{G})' \to \text{Im} \delta_0 \to 1, \]
where the first nontrivial map is injective. Note that, by Corollary 3.7(ii), \( \text{Im} \delta_0 \) is in bijection with a subset of \( \prod_{v \in \text{real}} H^1(K_v, G) \).

**Proposition 3.10.** There exists an exact sequence of pointed sets
\[ 1 \to \mu(S) \to \tilde{G}(S) \to G(S) \to H^0_{ab}(S_{\text{fl}}, G) \to C(\tilde{G})' \to C(G) \to C_{ab}(G), \]
where \( C(\tilde{G})' \) is the kernel of the composition \((3.11)\).
Proof. This follows from the bijection $C(G) \simeq \text{Ker } [H^1(S_{\text{et}}, G) \to H^1(K, G)]$ of Theorem 3.4 and the exact commutative diagram

\[
\begin{array}{ccccccc}
\ldots & H^0_{ab}(S_{\text{fl}}, G) & \longrightarrow & H^1(S_{\text{et}}, \tilde{G}) & \longrightarrow & H^1(S_{\text{et}}, G) & \longrightarrow & H^1_{ab}(S_{\text{fl}}, G) \\
& & & \uparrow \pi \circ \lambda & & & & \\
& 1 & \longrightarrow & H^1(K, \tilde{G})/H^0_{ab}(K_{\text{fl}}, G) & \longrightarrow & H^1(K, G) & \longrightarrow & H^1_{ab}(K_{\text{fl}}, G),
\end{array}
\]

whose top and bottom rows are given by Theorem 2.2(i) and (ii), respectively. For the commutativity of the above diagram, see Remark 2.3. □

Theorem 3.11. Assume that $K$ has no real primes. Then there exists an exact sequence of pointed sets

\[
1 \rightarrow H^1(S_{\text{et}}, \tilde{G})/H^0_{ab}(S_{\text{fl}}, G) \rightarrow C(G) \rightarrow C_{ab}(G) \rightarrow 1,
\]

where the first nontrivial map is injective.

Proof. By [GA3], Theorem 5.8(i) and Remark 5.9(a), the abelianization map $ab^1_{G/K} : H^1(K, G) \to H^1_{ab}(K_{\text{fl}}, G)$ is bijective. The theorem is now immediate from the exact commutative diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & H^1(S_{\text{et}}, \tilde{G})/H^0_{ab}(S_{\text{fl}}, G) & \longrightarrow & H^1(S_{\text{et}}, G) & \longrightarrow & H^1_{ab}(S_{\text{fl}}, G) \\
& & & \downarrow & & & \downarrow \\
& 1 & \longrightarrow & H^1(K, G) & \sim & H^1_{ab}(K_{\text{fl}}, G),
\end{array}
\]

whose top row is given by Theorem 2.2(ii). □

Remarks 3.12.

(a) The maps appearing in the exact sequences of Proposition 3.10 and Theorem 3.11 are induced by the corresponding maps appearing in the exact sequence of Theorem 2.2(i), all of which are explicitly described in [GA3].

(b) Assume that $K$ has no real primes and that $\tilde{G}_K$ has the strong approximation property with respect to $\Sigma$ (see [PR], §7.1). Then the set $H^1(S_{\text{et}}, \tilde{G}) \simeq C(\tilde{G})$ (see Corollary 3.7) is trivial and $C(G)$ is known to have a natural structure of abelian group (see, e.g., [Th], Proposition 13, p.32). This group, which was denoted $G_{\text{cl}}(G)$ in [PR], §8, has been studied in [PR], §8.2, [Th], §§4.4 4.5, and [Dem2], §4. Now the theorem and a twisting argument (see [GA3], Corollary 3.15) show that there exists an isomorphism of abelian groups $G_{\text{cl}}(G) \simeq C_{ab}(G)$. In particular, by Corollary 3.14 below, $G_{\text{cl}}(G)$ is endowed with natural corestriction homomorphisms. For the case of number fields with real primes, see [Dem2], Theorem 4.6.

(c) In general, the map $C(G) \rightarrow C_{ab}(G)$ is not surjective. More precisely, let $c$ be a class in $C_{ab}(G) \subset H^1_{ab}(S_{\text{fl}}, G)$ and consider the
exact commutative diagram
\[
\begin{array}{ccccccc}
H^1(S_{\text{ét}}, \tilde{G}) & \xrightarrow{\partial(1)} & H^1(S_{\text{ét}}, G) & \xrightarrow{\lambda} & H^1(K, \tilde{G}) & \xrightarrow{\partial^1_K} & H^1(K, G) & \xrightarrow{\partial^1_K} & H^1(K, G) & \xrightarrow{\lambda} & H^1(K, G) & \xrightarrow{\lambda} & H^1(K, G) & \xrightarrow{\lambda} & 1, \\
\end{array}
\]
whose left-hand vertical map is surjective by Corollary 3.7(i). Let \( c' \) be a preimage of \( c \) in \( H^1(S_{\text{ét}}, G) \). Since \( c \) maps to zero in \( H^1(K, G) \), \( \lambda(c') = \partial^1_K(x') \) for some \( x' \in H^1(K, \tilde{G}) \). Let \( p' \in H^1(S_{\text{ét}}, \tilde{G}) \) be such that \( \lambda(p') = x' \) and let \( p = \partial^1(p') \). Choose a \( \tilde{G} \)-torsor \( P' \) representing \( p' \) and let \( P = P' \wedge \tilde{G} \) be a \( G \)-torsor representing \( p \). Since \( \lambda(c') = \lambda(p) \), Remark 3.5 shows that \( c' = (\theta_\varphi^{-1} \circ \varphi)(c'') \) for some class \( c'' \in C(PG) \). Now, by [GA3], Lemma 3.13, we have
\[
\left(P_{ab/G/S}^{\varphi}(c')\right) = \left(P_{ab/G/S}^{\varphi}(\theta_\varphi^{-1} \circ \varphi)(c'')\right) = ab_{G/S}(c') = c.
\]
Thus, the following holds. Let \( \mathcal{S} \) be a complete set of representatives for the classes in \( H^1(K, G) \cong \prod_{v \text{ real}} H^1(K_v, \tilde{G}) \). For each \( \tilde{G} \)-torsor \( x \in \mathcal{S} \), choose an extension of \( x \) to a \( \tilde{G} \)-torsor \( X \) and let \( ^x\!G \) denote the \((X \wedge \tilde{G})\)-twist of \( G \). Then there exists a surjection \( \prod_{x \in \mathcal{S}} C(^x\!G) \rightarrow C_{ab}(G) \). Consequently, since each set \( C(^x\!G) \) is in bijection with \( C(G) \), we conclude that there exists a surjection
\[
\prod_{x \in \mathcal{S}} C(G) \rightarrow C_{ab}(G).
\]
In particular, \( |C_{ab}(G)| \) is a lower bound for \( \prod_{v \text{ real}} \#H^1(K_v, \tilde{G}) \cdot \#C(G) \).

(d) As mentioned in Remark 2.3, the right action of \( H^0_{ab}(S_{\text{fl}}, G) \) on \( H^1(S_{\text{ét}}, \tilde{G}) \) is compatible with inverse images. Thus, via the bijection \( C(\tilde{G}) \cong \text{Ker} [H^1(S_{\text{ét}}, \tilde{G}) \rightarrow H^1(K, \tilde{G})] \) of Theorem 3.4, it induces a right action of the abelian group [GA3] on the set \( C(\tilde{G}) \). Using [GA3], Proposition 3.14(a), it can be shown that the stabilizer in \( C^0_{ab}(G) \) of a class \( p \in C(\tilde{G}) \) represented by a \( \tilde{G} \)-torsor \( P \) is
\[
Q_{ab/G/S}^0 \left(QG(S) \cap Q\vartheta(Q\tilde{G}(K))\right) \subset C^0_{ab}(G),
\]
where \( Q = P \wedge \tilde{G} \), \( Q\vartheta \) is the \( Q \)-twist of \( \vartheta : \tilde{G} \rightarrow G \) and the intersection takes place inside \( QG(K) \). In the interesting particular case where \( \mu = \mu_{n,S} \) is the group scheme of \( n \)-th roots of unity on \( S \),

\[\text{[GA3]}
\]
Proposition 4.2. Thus, it remains only to check the exactness of the sequence 
the sequence of the theorem is exact up to the term

\[ C_{ab}(G) \rightarrow \text{Pic}(S)^n. \]

In the remainder of this Section we establish some basic properties of
\(C_{ab}(G)\) using flasque resolutions of \(G\). Recall that a flasque resolution of
\(G\) is an exact sequence \(1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1\), where
\(F\) is a flasque \(S\)-torus, \(R = H^{tor}\) is a quasi-trivial \(S\)-torus and \(H^{der}\) is a semisimple and
simply-connected \(S\)-group scheme. See \[GA4\] for more details.

For any torus \(T\) over \(S\), set

\[ D^2(S, T) = \text{Ker} \left[ H^2(S_{\text{ét}}, T) \rightarrow \prod_{v \in \Sigma} H^2(K_v, T) \right], \]

where the map involved is induced by the compositions \(\text{Spec } K_v \rightarrow \text{Spec } K \rightarrow S\) for \(v \in \Sigma\). If \(v \in S_0\), the preceding composition coincides with the com-
position \(\text{Spec } K_v \rightarrow \text{Spec } O_v \rightarrow S\). Since \(H^2(\mathcal{O}_{v, \text{ét}}, T) = H^2(k(v), T) = 0\) by \[Mi1\], III.3.11(a), p.116 (recall that \(k(v)\)-tori are cohomologically triv-
ial), we conclude that the map \(H^2(S_{\text{ét}}, T) \rightarrow H^2(K, T)\) induces a map
\(D^2(S, T) \rightarrow \Pi^2(K, T)\).

**Theorem 3.13.** Let \(G\) be a reductive group scheme over \(S\) and let \(1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1\) be a flasque resolution of \(G\). Set \(R = H^{tor}\). Then the
given resolution induces an exact sequence of abelian groups

\[ \mu(S) \hookrightarrow F(S) \twoheadrightarrow R(S) \rightarrow H^0_{ab}(\text{fl}, G) \rightarrow H^1(S_{\text{ét}}, F) \rightarrow C(R) \rightarrow C_{ab}(G) \rightarrow 1. \]

**Proof.** Since \(R\) is quasi-trivial, Theorem 3.4 and Hilbert’s Theorem 90 show that
\(C(R) = H^1(S_{\text{ét}}, R)\). Thus, since \(H^{-1}_{ab}(\text{fl}, G) = \mu(S)\) by \[GA3\], (2.1), the sequence of the theorem is exact up to the term \(H^1(S_{\text{ét}}, F)\) by \[GA4\],
Proposition 4.2. Thus, it remains only to check the exactness of the sequence
\(H^1(S_{\text{ét}}, F) \rightarrow C(R) \rightarrow C_{ab}(G) \rightarrow 1\). By \[GA4\], Propositions 4.2 and 4.5,
there exists an exact commutative diagram

\[
\begin{array}{cccccc}
\cdots & \rightarrow & C(R) & \rightarrow & H^1_{ab}(\text{fl}, G) & \rightarrow & H^2(S_{\text{ét}}, F) & \rightarrow & H^2(S_{\text{ét}}, R) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \prod_{v \in \Sigma} H^1_{ab}(K_v, G) & \rightarrow & \prod_{v \in \Sigma} H^2(K_v, F) & \rightarrow & \prod_{v \in \Sigma} H^2(K_v, R). & & \\
\end{array}
\]

The right-hand vertical map in the above diagram is injective by \[ADT\],
Proposition II.2.1, p.163. Thus the preceding diagram yields an exact se-
quence

\[ H^1(S_{\text{ét}}, F) \rightarrow C(R) \rightarrow D^1_{ab}(S, G) \rightarrow D^2(S, F) \rightarrow 1. \]

Now, since \(F\) is flasque, the map \(H^2(S_{\text{ét}}, F) \rightarrow H^2(K, F)\) is injective (see
\[CTS2\], Theorem 2.2(ii), p.161), whence \(D^2(S, F) \rightarrow \Pi^2(K, F)\) is injective.
as well. The theorem now follows from the diagram

\[
\begin{array}{cccccc}
H^1(S_{\text{ét}}, F) & \longrightarrow & C(R) & \longrightarrow & D^1_{ab}(S, G) & \longrightarrow & D^2(S, F) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Xi^1_{ab}(K, G) & \xrightarrow{\sim} & \Xi^2(K, F),
\end{array}
\]

where the bottom map is an isomorphism by Proposition 3.1.

**Corollary 3.14.** Let \( L/K \) be a finite Galois extension and let \( S' \to S \) be the normalization of \( S \) in \( L \). Let \( G \) be a reductive group scheme over \( S \) and let \( 1 \to F \to H \to G \to 1 \) be a flasque resolution of \( G \). Then the given resolution defines a corestriction homomorphism

\[
\text{cores}_{S'/S} : C_{ab}(S', G) \to C_{ab}(S, G),
\]

where \( C_{ab}(S', G) \) (respectively, \( C_{ab}(S, G) \)) is the abelian class group of \( G \times_S S' \) (respectively, \( G \)).

**Proof.** Let \( R = H^1 \text{tor} \) and recall that \( C(S, R) = H^1(S_{\text{ét}}, R) \) (and similarly for \( R \times_S S' \)). There exists a canonical corestriction homomorphism

\[
\text{cores}_{S'/S} : H^1(S'_{\text{ét}}, F) \to H^1(S_{\text{ét}}, F),
\]

namely the composite

\[
H^i(S'_{\text{ét}}, F) \xrightarrow{\sim} H^i(S_{\text{ét}}, R_{S'/S}(F \times_S S')) \to H^i(S_{\text{ét}}, F),
\]

where the second map is induced by the trace morphism \( R_{S'/S}(F \times_S S') \to F \) of \([\text{SGA}4]\), XVII, 6.3.13.2. See \([\text{CTS}2]\), (0.4.1), p.154. Similarly, there exists a corestriction homomorphism \( \text{cores}_{S'/S} : C(S', R) \to C(S, R) \). Now, by \([\text{CTS}2]\), Proposition 1.4, p.158, \( 1 \to F \times_S S' \to H \times_S S' \to G \times_S S' \to 1 \) is a flasque resolution of \( G \times_S S' \), and the theorem yields an exact commutative diagram

\[
\begin{array}{cccccc}
H^1(S'_{\text{ét}}, F) & \longrightarrow & C(S', R) & \longrightarrow & C_{ab}(S', G) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(S_{\text{ét}}, F) & \longrightarrow & C(S, R) & \longrightarrow & C_{ab}(S, G) & \longrightarrow & 1.
\end{array}
\]

This establishes the existence of the right-hand vertical map in the above diagram, which is the assertion of the corollary. □

**Remark 3.15.** The map of the corollary is independent, up to isomorphism, of the chosen flasque resolution of \( G \), i.e., if \( 1 \to F_1 \to H_1 \to G \to 1 \) is another flasque resolution of \( G \) and \( \text{cores}_{S'/S,1} : C_{ab}(S', G) \to C_{ab}(S, G) \) is the corestriction map that it defines, then \([\text{GA}4]\), Proposition 3.7(iii), shows that there exist automorphisms \( \sigma' \) of \( C_{ab}(S', G) \), \( \sigma \) of \( C_{ab}(S, G) \) and
a commutative diagram

\[
\begin{array}{ccc}
C_{ab}(S', G) & \xrightarrow{\sigma'} & C_{ab}(S', G) \\
\downarrow \text{cores}_{S'/S} & & \downarrow \text{cores}_{S'/S, 1} \\
C_{ab}(S, G) & \xrightarrow{\sigma} & C_{ab}(S, G)
\end{array}
\]

Further, cores_{S'/S} is functorial in S' \to S. This follows from the functoriality of the corestriction homomorphisms in the case of tori mentioned above.

Recall that a flasque S-torus F is called invertible if it is a direct factor of a quasi-trivial S-torus.

**Corollary 3.16.** Assume that G admits an invertible resolution, i.e., there exists a flasque resolution 1 \to F \to R \to T \to 1 with F invertible. Then the given resolution induces an exact sequence of abelian groups

\[1 \to \mu(S) \to F(S) \to R(S) \to H^0_{ab}(S_{fl}, G) \to C(F) \to C(R) \to C_{ab}(G) \to 1,\]

where \(R = H^{tor}\).

**Proof.** Since F is invertible, \(H^1(K, F) = 0\) by Hilbert’s Theorem 90. Thus \(H^1(S_{\acute{e}t}, F) = C(F)\) by Theorem 3.4 and the corollary is now immediate from the theorem. □

**Remark 3.17.** The corollary extends the “good reduction” case of [GA2], Theorem 6.1, from K-tori to arbitrary connected reductive K-groups. Indeed, let T be a K-torus with multiplicative reduction over S which admits an invertible resolution 1 \to F \to R \to T \to 1, i.e., F is invertible and R is quasi-trivial (this is the case, for example, if T is split by a metacyclic extension of K, by [CTS1], Proposition 2, p.184). Let j: Spec K \to S be the canonical morphism. Since \(R^1j_*F = 0\) for the smooth topology on S (see [GA2], Lemma 4.8(b)), the given resolution induces an exact sequence 1 \to \mathcal{T} \to R \to T \to 1 of Néron-Raynaud models over S. The latter sequence induces, in turn, an exact sequence 1 \to \mathcal{T}^0 \to R^0 \to T^0 \to 1, where \(\mathcal{T}^0\) (respectively, \(\mathcal{T}^0, \mathcal{F}^0\)) denotes the identity component of \(\mathcal{T}\) (respectively, \(\mathcal{R}, \mathcal{F}\)). Further, \(\mathcal{I}^{-1}(\mathcal{F}^0)/\mathcal{F}^0 \simeq \text{Ker}[\Phi(\mathcal{F}) \to \Phi(\mathcal{R})]\), where \(\Phi(\mathcal{F}) = \mathcal{F}/\mathcal{F}^0\) and \(\Phi(\mathcal{R}) = \mathcal{R}/\mathcal{R}^0\). See [B], Theorems 2.2.4 and 2.3.1, pp.49-50. On the other hand, by [B], Theorem 2.3.4, p.52, \(\text{Ker}[\Phi(\mathcal{F}) \to \Phi(\mathcal{R})] \subset \Phi(\mathcal{F})_{\text{tors}}\), which is zero by [GA2], Lemma 4.8(c). Thus the given invertible resolution of T induces an exact sequence of connected Néron-Raynaud models

\[1 \to \mathcal{T}^0 \to \mathcal{R}^0 \to \mathcal{T}^0 \to 1.\]

Since T has multiplicative reduction over S, \(\mathcal{T}^0\) is an S-torus and the preceding sequence is an invertible resolution of \(\mathcal{T}^0\). Thus the corollary yields

---

6This is the case if both \(G^{tor}\) and \(Z(\tilde{G})\) are split by a metacyclic Galois cover of S. See [GA1], Remark 3.3.
an exact sequence
\[ 1 \to \mathcal{F}^\otimes(S) \to \mathcal{A}^\otimes(S) \to \mathcal{F}(S) \to C(\mathcal{F}^\otimes) \to C(\mathcal{A}^\otimes) \to C(\mathcal{F}) \to 1. \]

The groups $C(\mathcal{F}^\otimes), C(\mathcal{A}^\otimes)$ and $C(\mathcal{F})$ are the Néron-Raynaud class group of $T, R$ and $F$ over $S$, respectively (see Example 3.9(b)), and the last exact sequence coincides with the exact sequence of [GA2], Theorem 6.1, for the $K$-torus $T$.

4. The duality theorem

Let $K$ and $S$ be as in the Introduction. If $T$ is an $S$-torus such that $\Pi_3^4(K,T) = 0$, then $C(T) = H^1(S_{\text{et}}, T)$ (see Theorem 3.4) is known to satisfy a duality theorem. Namely, there exists a perfect pairing of finite groups
\[ (4.1) \quad C(T) \times H^2_c(S_{\text{et}}, T^*) \to \mathbb{Q}/\mathbb{Z} \]
induced by the natural pairing $T \times T^* \to \mathbb{G}_{m,S}$, where $H^i_c$ denotes cohomology with compact support. See [ADT], Theorem II.4.6(a), p.191. The purpose of this Section is to extend the above duality theorem to an arbitrary reductive group scheme $G$ over $S$, with $C_{\text{ab}}(G)$ replacing $C(T)$. See Theorem 4.12. In particular, Corollary 4.14 below extends (4.1) to an arbitrary $S$-torus $T$ (see Remark 4.15).

For any prime number $\ell$, abelian group $B$ and positive integer $m$, we will write $B_{\ell^m}$ for the $\ell^m$-torsion subgroup of $B$ and $B/\ell^m B$. Set $B(\ell) = \bigcup_{m \geq 1} B_{\ell^m}$, $B(\ell) = \lim_{\leftarrow m} B/\ell^m$ and $T_\ell B = \lim_{\leftarrow m} B_{\ell^m}$. Further, set $B_{\ell, \text{div}} = \bigcap_{m} \ell^m B$ and $B/\ell - \text{div} = B/B_{\ell, \text{div}}$.

If $B$ is an abelian topological group and $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology, let $B^D = \text{Hom}_{\text{conts}}.(B, \mathbb{Q}/\mathbb{Z})$ be the Pontryagin dual of $B$. It is endowed with the compact-open topology. If $B$ is discrete and torsion (respectively, profinite), then $B^D$ is profinite (respectively, discrete and torsion). A continuous pairing of topological abelian groups $A \times B \to \mathbb{Q}/\mathbb{Z}$ is called non-degenerate on the right (respectively, left) if the induced homomorphism $B \to A^D$ (respectively, $A \to B^D$) is injective. It is called non-degenerate if it is non-degenerate both on the right and on the left. The pairing is said to be perfect if the homomorphisms $B \to A^D$ and $A \to B^D$ are (topological) isomorphisms. It is not difficult to see that, for any prime $p$, a perfect pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ induces pairings $A(p) \times (B(p)-\text{div}) \to \mathbb{Q}/\mathbb{Z}$ and $(A(p)-\text{div}) \times B(p) \to \mathbb{Q}/\mathbb{Z}$, which are non-degenerate on the left and on the right, respectively. Further, two pairings $(-,-), (-,-)': A \times B \to \mathbb{Q}/\mathbb{Z}$ are said to be isomorphic if there exist automorphisms $\alpha$ of $A$ and $\beta$ of $B$ such that $(a, b)' = (\alpha(a), \beta(b))$ for all $a \in A$ and $b \in B$.

By [HJ], beginning of §3, and [GA1], beginning of §5, for any cohomologically bounded complex $C$ of abelian sheaves on $S_\text{et}$ there exist hypercohomology groups with compact support $\mathbb{H}^i_c(S_{\text{et}}, C)$ which fit into an exact
sequence
\[ \cdots \to \mathbb{H}^i_c(S_{\text{fl}}, C) \to \mathbb{H}^i(S_{\text{fl}}, C) \to \prod_{v \in \Sigma} \mathbb{H}^i(K_{v, \text{fl}}, C) \to \mathbb{H}^{i+1}_c(S_{\text{fl}}, C) \to \cdots, \]

where \( \Sigma \) is the set of primes of \( K \) that do not correspond to a point of \( S \).

Set
\[ \mathbb{D}^i(S, C) = \text{Ker} \left[ \mathbb{H}^i(S_{\text{fl}}, C) \to \prod_{v \in \Sigma} \mathbb{H}^i(K_{v, \text{fl}}, C) \right] \]
\[ = \text{Im} \left[ \mathbb{H}^i_c(S_{\text{fl}}, C) \to \mathbb{H}^i(S_{\text{fl}}, C) \right] \]

and
\[ \mathbb{H}^i(K, C) = \text{Ker} \left[ \mathbb{H}^i(K_{\text{fl}}, C) \to \prod_{\text{all } v} \mathbb{H}^i(K_{v, \text{fl}}, C) \right]. \]

Now let \( T_1 \) and \( T_2 \) be arbitrary \( S \)-tori and let \( C = (T_1 \to T_2) \), where \( T_1 \) and \( T_2 \) are placed in degrees \(-1\) and \(0\), respectively. Let \( C^* = (T_2^* \to T_1^*) \)
be the dual complex of twisted-constant \( S \)-groups, where \( T_2^* \) and \( T_1^* \) are placed in degrees \(-1\) and \(0\), respectively. Then the canonical morphism
\[ C \otimes^L C^* \to \mathbb{G}_m[1] \]
defined in [Dem1], beginning of §2, induces a pairing
\[ \langle -, - \rangle : \mathbb{H}^i(S_{\text{fl}}, C) \times \mathbb{H}^{2-i}_{\text{et}}(S_{\text{et}}, C^*) \to \mathbb{Q}/\mathbb{Z} \]
(cf. [HS], p.108). The next result extends [Dem1], Corollary 4.7, to the function field case.

**Proposition 4.1.** The pairing \( \langle -, - \rangle \) induces a perfect pairing of finite groups
\[ \mathbb{D}^1(S, C) \times \mathbb{D}^1(S, C^*) \to \mathbb{Q}/\mathbb{Z}. \]

**Proof.** The pairing is defined as follows: if \( a \in \mathbb{D}^1(S, C) \subset \mathbb{H}^1(S_{\text{fl}}, C) \) and \( a' \in \mathbb{D}^1(S, C^*) \subset \mathbb{H}^1(S_{\text{et}}, C^*) \) is the image of \( b' \in \mathbb{H}^1_{\text{et}}(S_{\text{et}}, C^*) \), then \( \{a, a'\} = \langle a, b' \rangle \), where \( \langle -, - \rangle \) is the pairing \( \langle 4.4 \rangle \) for \( i = 1 \). We begin by proving the finiteness statement. As noted in [Dem1], proof of Corollary 4.7, the finiteness of \( \mathbb{D}^1(S, C) \) follows from (a) the finiteness of \( \Sigma \), (b) [ADT], Corollary I.2.4, p.29, and Theorem II.4.6(a), p.191, and (c) the finiteness of \( D^2(S, T) \), where \( T \) is an \( S \)-torus. To prove the latter, assume first that \( T \) is flasque. Then \( D^2(S, T) \) injects into \( \mathbb{H}^2(K, T) \) (see the proof of Theorem 3.13), which is finite by [Oes], Theorem 2.7(a), p.52. In the general case, let \( 1 \to T \to F \to P \to 1 \) be a flasque resolution of \( T \), where \( F \) is flasque and

---

7 Recall that, if \( v \) is a real prime, \( \mathbb{H}^i(K_{v, \text{fl}}, C) \) denotes the \( i \)-th modified (Tate) hypercohomology group of \( C \times_S \text{Spec } K_v \).

8 Recall that, since the Cartier dual of an \( S \)-group scheme of multiplicative type is étale, the groups \( \mathbb{H}^{2-i}_{\text{et}}(S_{\text{et}}, C^*) \) and \( \mathbb{H}^{2-i}_c(S_{\text{fl}}, C^*) \) are canonically isomorphic.
$P$ is quasi-trivial (see [CTS], (1.3.2), p.158). Then there exists an exact commutative diagram

$$
\begin{array}{ccl}
H^1(S_{\text{ét}}, P) & \longrightarrow & H^2(S_{\text{ét}}, T) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Sigma} H^2(K_v, T)
\end{array}
\quad
\begin{array}{ccl}
\longrightarrow & \longrightarrow & \prod_{v \in \Sigma} H^2(K_v, F)
\end{array}
$$

which yields an exact sequence $H^1(S_{\text{ét}}, P) \to D^2(S, T) \to D^2(S, F)$. Since $H^1(S_{\text{ét}}, P)$ and $D^2(S, F)$ are both finite, $D^2(S, T)$ is finite as well. As regards the finiteness of $D^1(S, C^*)$, it again follows from [ADT], Corollary I.2.4, p.29, the finiteness of $\Sigma$ and the finiteness of both $D^2(S, T^*)$ (see [HS], proof of Proposition 3.7, p.111), and $H^1(S, T^*)$ (see [op.cit.], proof of Lemma 3.2(3), p.108). Now, it was shown in [Dem1], Corollary 4.7, that the pairing $D^1(S, C)_{(\ell)} \times D^1(S, C^*)_{(\ell)} \to Q/Z$, induced by the pairing of the statement, is perfect for any prime $\ell \neq p$, where $p = \text{char } K$ in the function field case. The following lemmas will show that it is perfect for $\ell = p$ as well.

**Lemma 4.2.** For every $i \in Z$ and every $m \geq 1$, there exists a perfect pairing

$$
\mathbb{H}^i(S_{\text{fl}}, C \otimes L Z/p^m) \times \mathbb{H}^{1-i}_c(S_{\text{ét}}, C^* \otimes L Z/p^m) \to Q/Z,
$$

where the left-hand group is discrete and torsion and the right-hand group is profinite.

**Proof.** The proof is similar to the proof of [Dem1], Proposition 4.2, using [ADT], Theorem III.8.2, p.290, in place of [op.cit.], Corollary II.3.3(b), p.177. □

Now define\(^9\)

$$
\mathbb{H}^i(S_{\text{fl}}, C \otimes L Q_p/Z_p) = \lim_{m \to \infty} \mathbb{H}^i(S_{\text{fl}}, C \otimes L Z/p^m)
$$

and

$$
\mathbb{H}^i(S_{\text{fl}}, C \otimes L Z_p) = \lim_{m \to \infty} \mathbb{H}^i(S_{\text{fl}}, C \otimes L Z/p^m).
$$

Similar definitions apply with $\mathbb{H}^i_c$ in place of $\mathbb{H}^i$. Then the previous lemma yields perfect pairings

$$
\mathbb{H}^i(S_{\text{fl}}, C \otimes L Q_p/Z_p) \times \mathbb{H}^{1-i}_c(S_{\text{ét}}, C^* \otimes L Z_p) \to Q/Z
$$

and

$$
\mathbb{H}^i(S_{\text{fl}}, C \otimes L Z_p) \times \mathbb{H}^{1-i}_c(S_{\text{ét}}, C^* \otimes L Q_p/Z_p) \to Q/Z.
$$

\(^9\)Note that the maps $Z/p^m \to Z/p^{m+1}$ (respectively, $Z/p^{m+1} \to Z/p^m$) are induced by multiplication by $p$ on $Z$ (respectively, the identity map of $Z$).
Thus we obtain an injection nondegenerate on the right, interchange in the above argument, i.e., the pairing of the lemma is nondegenerate on the left. To see that it is using Lemma 4.2 above in place of \cite{op.cit.}, Proposition 4.2.

On the other hand, there exists an exact sequence

\[ 1 \rightarrow \mathbb{H}^i(S, C)(p) / p - \text{div} \rightarrow \mathbb{H}^i(S, C) \rightarrow T_p \mathbb{H}^{i+1}(S, C). \]

See \cite{Dem1}, p.16. Therefore \( \mathbb{H}^i(S, C)(p) / p - \text{div} \rightarrow \mathbb{H}^i(S, C) \) (p) see \cite{op.cit.}, Lemma 2.1 and an exact sequence

\[ 1 \rightarrow \mathbb{H}^i(S, C)(p) / p - \text{div} \rightarrow \mathbb{H}^i(S, C) \rightarrow T_p \mathbb{H}^{i+1}(S, C). \]

See \cite{Dem1}, p.16. Therefore \( \mathbb{H}^i(S, C)(p) = \mathbb{H}^i(S, C \otimes \mathbb{L} \mathbb{Z}_p) / p - \text{div} \) and the left-hand nondegeneracy of \( (4.6) \) yields an injection

\[ \mathbb{H}^i(S, C)(p) / p - \text{div} \rightarrow (\mathbb{H}^i_c(S, C^*) \otimes \mathbb{L} \mathbb{Q}_p / \mathbb{Z}_p) / p - \text{div} \]

On the other hand, there exists an exact sequence

\[ \mathbb{H}^i_c(S, C^*) \otimes \mathbb{Q}_p / \mathbb{Z}_p \rightarrow \mathbb{H}^i_c(S, C^* \otimes \mathbb{L} \mathbb{Q}_p / \mathbb{Z}_p) \rightarrow \mathbb{H}^{i-1}_c(S, C^*)(p) \]

which identifies \( \mathbb{H}^{i-1}(S, C^*) / p - \text{div} \) and \( \mathbb{H}^{i-1}(S, C^* \otimes \mathbb{L} \mathbb{Q}_p / \mathbb{Z}_p) / p - \text{div} \). Thus we obtain an injection

\[ \mathbb{H}^i(S, C)(p) / p - \text{div} \rightarrow (\mathbb{H}^{i-1}_c(S, C^*) / p - \text{div} \]

i.e., the pairing of the lemma is nondegenerate on the left. To see that it is nondegenerate on the right, interchange in the above argument \( C \) and \( C^* \), \( i \) and \( 2 - i \), \( \mathbb{H} \) and \( \mathbb{H}_c \) and use the right-hand nondegeneracy of \( (4.5) \).

**Lemma 4.4.** Assume that \( a \in \mathbb{D}^1(S, C) \cap p^m \mathbb{H}^1(S, C) \) is orthogonal to \( \mathbb{D}^1(S, C^*) \) under the pairing of Proposition 4.1. Then \( a \in p^m \mathbb{D}^1(S, C) \).

**Proof.** The proof is formally the same as the proof of \cite{Dem1}, Lemma 4.5, using Lemma 4.2 above in place of \cite{op.cit.}, Proposition 4.2.

We can now complete the proof of Proposition 4.1. By Lemmas 4.3 and 4.4, the pairing of Proposition 4.1 induces a nondegenerate (and therefore perfect) pairing of finite groups \( \mathbb{D}^1(S, C)(p) \times \mathbb{D}^1(S, C^*)(p) \rightarrow \mathbb{Q}/\mathbb{Z} \). See \cite{Dem1}, proof of Corollary 4.6, p.19.

Let \( G \) be a reductive group scheme over \( S \). Then there exist dual abelian cohomology groups with compact support

\[ H^i_{ab,c}(S_{\text{ét}}, G^*) = \mathbb{H}^i_c(S_{\text{ét}}, Z(G)^* \rightarrow Z(\widehat{G})^*) \]
which fit into an exact sequence
\[\ldots \rightarrow H^i_{ab,c}(S_{\text{ét}}, G^*) \rightarrow H^i_{ab}(S_{\text{ét}}, G^*) \rightarrow \prod_{v \in \Sigma} H^i_{ab}(K_v, G^*) \rightarrow H^{i+1}_{ab,c}(S_{\text{ét}}, G^*) \rightarrow \ldots \]

These groups have the following property (cf. [ADT], Proposition III.0.4(c), p.221, and Remark III.0.6, p.223). Let \( V \) be a nonempty open subscheme of \( S \). Then there exists an exact sequence
\[\ldots \rightarrow H^i_{ab,c}(V_{\text{ét}}, G^*) \rightarrow H^i_{ab,c}(S_{\text{ét}}, G^*) \rightarrow \prod_{v \in S \setminus V} H^i_{ab}(O_{v,\text{ét}}, G^*) \rightarrow \ldots \]

Further, by applying an analog of [ADT], Proposition III.0.4(b), p.220, to the short exact sequence of complexes
\[1 \rightarrow (G^{\text{tor}} \rightarrow 1) \rightarrow (Z(G)^* \rightarrow Z(\tilde{G})^*) \rightarrow (Z(G^{\text{det}})^* \rightarrow Z(\tilde{G})^*) \rightarrow 1\]
and using the quasi-isomorphism \( (Z(G^{\text{det}})^* \rightarrow Z(\tilde{G})^*) \simeq (1 \rightarrow \mu^*) \), we conclude that the groups \( H^i_{ab,c}(S_{\text{ét}}, G^*) \) fit into an exact sequence
\[\ldots \rightarrow H^{i-1}_c(S_{\text{ét}}, \mu^*) \rightarrow H^{i+1}_c(S_{\text{ét}}, G^{\text{tor}}) \rightarrow H^i_{ab,c}(S_{\text{ét}}, G^*) \rightarrow H^i_c(S_{\text{ét}}, \mu^*) \rightarrow \ldots \]

**Examples 4.5.**

(a) If \( G \) is semisimple, i.e., \( G^{\text{tor}} = 0 \), then \( H^1_{ab,c}(S_{\text{ét}}, G^*) = H^i_c(S_{\text{ét}}, \mu^*) \).

(b) If \( G \) has trivial fundamental group, i.e., \( \mu = 0 \), then \( H^i_{ab,c}(S_{\text{ét}}, G^*) = H^{i+1}_c(S_{\text{ét}}, G^{\text{tor}}) \).

Now let \( 1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1 \) be a flasque resolution of \( G \) and let \( C = (F \rightarrow R) \), where \( R = H^{\text{tor}} \). By [GA4], Proposition 4.2, the given resolution induces isomorphisms \( H^i_{ab}(S_{\text{fl}}, G) \simeq \mathbb{H}^i(S_{\text{fl}}, C) \), \( H^i_{ab}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i(S_{\text{ét}}, C^*) \) and \( H^i_{ab,c}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i_c(S_{\text{ét}}, C^*) \). In particular, via the above isomorphisms, the pairing (4.4) defines a pairing
\[H^1_{ab}(S_{\text{fl}}, G) \times H^1_{ab,c}(S_{\text{ét}}, G^*) \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

Clearly, a different choice of flasque resolution of \( G \) yields another such pairing which is isomorphic to (4.10), i.e., (4.11) is independent up to isomorphism of the chosen flasque resolution of \( G \).

Similarly, for every \( v \in \Sigma \), there exist isomorphisms \( H^i_{ab}(K_v, G \simeq \mathbb{H}^i(K_v, C) \) and \( H^i_{ab}(K_v, G^*) \simeq \mathbb{H}^i(K_v, C^*) \). Thus, if \( D^1_{ab}(S, G) \) is the group (3.9) and
\[D^1_{ab}(S, G^*) = \text{Ker} \left[ H^1_{ab}(S_{\text{ét}}, G^*) \rightarrow \prod_{v \in \Sigma} H^1_{ab}(K_v, G^*) \right] \]
\[= \text{Im} \left[ H^1_{ab,c}(S_{\text{ét}}, G^*) \rightarrow H^1_{ab}(S_{\text{ét}}, G^*) \right],\]
then there exist isomorphisms $D_{ab}^1(S, G) \simeq D^1(S, C)$ and $D_{ab}^1(S, G^*) \simeq D^1(S, C^*)$. Consequently, the following is an immediate corollary of Proposition 4.1.

**Proposition 4.6.** The pairing (4.10) induces a perfect pairing of finite groups

$$D_{ab}^1(S, G) \times D_{ab}^1(S, G^*) \rightarrow \mathbb{Q}/\mathbb{Z},$$

where $D_{ab}^1(S, G)$ and $D_{ab}^1(S, G^*)$ are the groups (3.9) and (4.11), respectively. □

**Examples 4.7.**

(a) If $G$ is semisimple, then the proposition yields a perfect pairing of finite groups

$$D_2(S, \mu) \times D_1(S, \mu^*) \rightarrow \mathbb{Q}/\mathbb{Z}.$$ See Example 2.1(a). Compare [ADT], Corollary II.3.4, p.178, and [GA1], Lemma 4.7.

(b) If $G$ has trivial fundamental group, then the proposition yields a perfect pairing of finite groups

$$D_1(S, G_{tor}) \times D_2(S, G_{tor}^*) \rightarrow \mathbb{Q}/\mathbb{Z}.$$ See Example 2.1(b). Compare [ADT], Corollary II.4.7, p.192, and [GA1], Theorem 5.7 (for $M = (1 \rightarrow G_{tor})$).

**Lemma 4.8.** Let $Y$ be an $S$-group scheme which is étale-locally isomorphic to $\mathbb{Z}^r$ for some $r \geq 1$. Then the canonical map $H^2(S_{\text{ét}}, Y) \rightarrow H^2(K, Y)$ is injective.

**Proof.** Let $K_S$ be the maximal subfield of the separable closure of $K$ which is unramified at all primes of $K$ which correspond to a (closed) point of $S$. Further, let $S'$ be the normalization of $S$ in $K_S$ and write $I_S = \text{Gal}(K_S/K)$. Then $Y \times_S S'$ is constant and the proof of [ADT], Lemma II.2.10, p.172, shows that $H^2(S'_{\text{ét}}, Y)$ is a quotient of $H^1(S'_{\text{ét}}(Y \times \mathbb{Q}))/Y$. Since the latter group is zero by [op.cit.], proof of Proposition II.2.9, p.171, so is $H^2(S'_{\text{ét}}, Y)$. Now, since $H^1(S'_{\text{ét}}, Y)$ is also zero [loc.cit.], the exact sequence of terms of low degree belonging to the Hochschild-Serre spectral sequence $H^p(I_S, H^q(S'_{\text{ét}}, Y)) \Rightarrow H^{p+q}(S_{\text{ét}}, Y)$ yields an isomorphism $H^2(I_S, Y) \simeq H^2(S_{\text{ét}}, Y)$. On the other hand, the canonical map $H^2(I_S, Y) \rightarrow H^2(K, Y)$ is injective (see [HS], p.112, lines 10-15), and the lemma follows. □

Now define

$$\Pi_{ab}^1(K, G^*) = \text{Ker} \left[ H_{ab}^1(K, G^*) \rightarrow \prod_v H_{ab}^1(K_v, G^*) \right].$$

**Proposition 4.9.** The pairing of Proposition 4.6 induces a perfect pairing of finite groups

$$\Pi_{ab}^1(K, G) \times \Pi_{ab}^1(K, G^*) \rightarrow \mathbb{Q}/\mathbb{Z},$$
where $\Pi^1_{ab}(K, G)$ and $\Pi^1_{ab}(K, G^*)$ are the groups (3.1) and (4.12), respectively.

**Proof.** We will show that there exist a nonempty open subset $W$ of $S$ and canonical isomorphisms $D^1_{ab}(W, G) \simeq \Pi^1_{ab}(K, G)$ and $D^1_{ab}(W, G^*) \simeq \Pi^1_{ab}(K, G^*)$. The proposition will then follow immediately from Proposition 4.6. Let $1 \to F \to H \to G \to 1$ be a flasque resolution of $G$ and set $R = H^{\text{tor}}$. Then, for any nonempty open subset $U'$ of $S$, the given resolution induces isomorphisms $D^1_{ab}(U, G) \simeq D^1(U, C)$ and $D^1_{ab}(U, G^*) \simeq D^1(U, C^*)$, where $C = (F \to R)$. Similarly, there exist isomorphisms $\Pi^1_{ab}(K, G) \simeq \Pi^1(K, C)$ and $\Pi^1_{ab}(K, G^*) \simeq \Pi^1(K, C^*)$, where $\Pi^1(K, C)$ (respectively, $\Pi^1(K, C^*)$) is the group (4.3). Thus, it suffices to find a set $W$ as above and isomorphisms $D^1(V, C) \simeq \Pi^1(K, C)$ and $D^1(V, C^*) \simeq \Pi^1(K, C^*)$. There exists a nonempty open subset $U$ of $S$ such that $H^1(V_{\text{ét}}, R) = 0$ for every open subset $V$ of $U$ (indeed, $R$ is a finite product of tori of the form $R_{S'/S}(G_\mathfrak{m}, S')$ where each $S'$ is finite and étale over $S$ and contains a nonempty open subscheme with trivial Picard group). It follows that the canonical map $H^1(V_{\text{ét}}, C) \to H^2(V_{\text{ét}}, F)$ is injective (cf. [GA4], Proposition 4.2). Thus there exists an injection $D^1(V, C) \hookrightarrow D^2(V, F)$. Further, the canonical map $D^2(V, F) \to \Pi^1(K, F)$ is injective (see the proof of Theorem 3.12) and $\Pi^1(K, C) \to \Pi^1(K, F)$ is an isomorphism (cf. Proposition 3.1). Now the commutative diagram

$$
\begin{array}{ccc}
D^1(V, C) & \xrightarrow{\simeq} & D^2(V, F) \\
\downarrow & & \downarrow \\
\Pi^1(K, C) & \xrightarrow{\simeq} & \Pi^1(K, F)
\end{array}
$$

shows that the canonical map $D^1(V, C) \to \Pi^1(K, C)$ is injective. This map is shown to be surjective in [Dem1], proof of Theorem 5.7, p.22, lines 11-12. Thus, for every $V \subset U$, the canonical map $D^1(V, C) \to \Pi^1(K, C)$ is, in fact, an isomorphism. On the other hand, Lemma 4.8 shows that the map $H^2(S_{\text{ét}}, R^*) \to H^2(K, R^*)$ is injective and [GA1], proof of Lemma 6.2, shows that the canonical map $H^1(S_{\text{ét}}, F^*) \to H^1(K, F^*)$ is an isomorphism. In particular, $H^1(S_{\text{ét}}, R^*) \simeq H^1(K, R^*) = 0$. Thus there exists an exact commutative diagram

$$
\begin{array}{cccc}
1 & \to & H^1(S_{\text{ét}}, F^*) & \xrightarrow{\simeq} & H^1(S_{\text{ét}}, C^*) & \to & H^2(S_{\text{ét}}, R^*) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & H^1(K, F^*) & \xrightarrow{\simeq} & H^1(K, C^*) & \to & H^2(K, R^*)
\end{array}
$$

(cf. [GA4], Proposition 4.2), which shows that there exists an injection $H^1(S_{\text{ét}}, C^*) \hookrightarrow H^1(K, C^*)$. In particular, $D^1(V, C^*)$ may be identified with a subgroup of $H^1(K, C^*)$. Now the same arguments used in [Dem1], proofs of Lemma 5.6 and Theorem 5.7, show that there exist a nonempty open subset
$U^*$ of $S$ and isomorphisms $\mathbb{D}^1(V, C^*) \cong \Pi^1(K, C^*)$ for every nonempty open subset $V$ of $U^*$. Further, these isomorphisms are compatible with respect to inclusions $V' \subset V \subset U^*$, in the sense that the diagram

\begin{equation}
\begin{array}{ccc}
\mathbb{D}^1(V', C^*) & \rightarrow & \Pi^1(K, C^*) \\
\downarrow & & \downarrow \\
\mathbb{D}^1(V, C^*) & \rightarrow & \Pi^1(K, C^*)
\end{array}
\end{equation}

(whose vertical map is induced by the canonical map $H_1^c(V', C) \rightarrow H_1^c(V, C)$) commutes. We conclude that, if $W = U \cap U^*$, then there exist isomorphisms $\mathbb{D}^1(V, C^*) \cong \Pi^1(K, C^*)$ for every nonempty open subset $V$ of $W$, as desired. \hfill \Box

Examples 4.10.

(a) If $G$ is semisimple, the pairing of the proposition is a pairing

$$\Pi^2(K, \mu) \times \Pi^1(K, \mu^*) \rightarrow \mathbb{Q}/\mathbb{Z}$$

which is isomorphic to the natural one, i.e., that induced by the pairing $\mu \times \mu^* \rightarrow \mathbb{G}_{m,S}$ (the so-called Poitou-Tate pairing of [ADT], Theorem I.4.10(a), p.57, and [GA1], Theorem 1.1). See Example 2.1(a).

(b) If $G$ has trivial fundamental group, then the pairing of the proposition is isomorphic to the natural pairing

$$\Pi^1(K, G^\text{tor}) \times \Pi^2(K, G^\text{tor}^*) \rightarrow \mathbb{Q}/\mathbb{Z}$$

for the $K$-torus $G^\text{tor} \times \text{Spec } K$. See Example 2.1(b).

As seen in the proof of Proposition 4.9, the canonical map $H_1^c(S_{\text{et}}, C^*) \rightarrow H_1^c(K, C^*)$ is injective. Thus, by [GA1], Proposition 4.2, the canonical map $H^1_{ab}(S_{\text{et}}, G^*) \rightarrow H^1_{ab}(K, G^*)$ is injective as well. Its image

\begin{equation}
H^1_{ab, \text{nr}}(K, G^*) = \text{Im}[H^1_{ab}(S_{\text{et}}, G^*) \rightarrow H^1_{ab}(K, G^*)]
\end{equation}

is called the subgroup of (S-)unramified classes of $H^1_{ab}(K, G^*)$. The following statement, which will be used in the next Section, is immediate from (4.13).

**Proposition 4.11.** The canonical map $H^1_{ab}(S_{\text{et}}, G^*) \rightarrow H^1_{ab}(K, G^*)$ induces an isomorphism

$$D^1_{ab}(S, G^*) \cong H^1_{ab, \text{nr}}(K, G^*) \cap \text{Ker} \left[ H^1_{ab}(K, G^*) \rightarrow \prod_{v \in \Sigma} H^1_{ab}(K_v, G^*) \right].$$

We now recall the exact sequence (3.8):

$$1 \rightarrow C_{ab}(G) \rightarrow D^1_{ab}(S, G) \rightarrow \Pi^1_{ab}(K, G) \rightarrow 1.$$  

By Propositions 4.6 and 4.9, the dual of the preceding exact sequence of finite abelian groups is an exact sequence

$$1 \rightarrow \Pi^1_{ab}(K, G^*) \rightarrow D^1_{ab}(S, G^*) \rightarrow C_{ab}(G)^D \rightarrow 1.$$
Thus the following holds.

**Theorem 4.12.** The pairings of Propositions 4.6 and 4.9 induce a perfect pairing of finite groups

$$C_{ab}(G) \times D^1_{ab}(S, G^*)/\Pi^1_{ab}(K, G^*) \to \mathbb{Q}/\mathbb{Z}.$$  

In other words, the exact annihilator of $C_{ab}(G) \subset D^1_{ab}(S, G)$ under the pairing of Proposition 4.6 is the group $\Pi^1_{ab}(K, G^*) \subset D^1_{ab}(S, G^*)$. □

By Examples 4.7 and 4.10, the following statements are immediate consequences of the theorem.

**Corollary 4.13.** Let $G$ be a semisimple $S$-group scheme with fundamental group $\mu$. Then there exists a perfect pairing of finite groups

$$C_{ab}(G) \times D^1(S, \mu^*)/\Pi^1(K, \mu^*) \to \mathbb{Q}/\mathbb{Z}. \quad □$$

**Corollary 4.14.** Let $T$ be an $S$-torus. Then there exists a perfect pairing of finite groups

$$C(T) \times D^2(S, T^*)/\Pi^2(K, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad □$$

**Remark 4.15.** If $T$ is an $S$-torus such that $\Pi^1_{S}(K, T) = 0$, then $\Pi^1(K, T) \subset \Pi^1_{S}(K, T)$ is zero as well, whence $\Pi^2(K, T^*) \simeq \Pi^1(K, T)^D = 0$. On the other hand, $H^1(S_{\text{ét}}, T) \to \prod_{v \in \Sigma} H^1(K_v, T)$ is the zero map (see diagram (4.5)), whence its dual $\prod_{v \in \Sigma} H^1(K_v, T^*) \to H^2_c(S_{\text{ét}}, T^*)$ is also zero. Thus the map $H^2_c(S_{\text{ét}}, T^*) \to H^2(S_{\text{ét}}, T^*)$ is injective and consequently $D^2(S, T^*) \simeq H^2_c(S_{\text{ét}}, T^*)$. Therefore the corollary yields a perfect pairing $C(T) \times H^2_c(S_{\text{ét}}, T^*) \to \mathbb{Q}/\mathbb{Z}$ which is isomorphic to the pairing (4.1).

Let

$$\delta_S: H^1_{ab, c}(S_{\text{ét}}, G^*) \to \prod_{v \in S_0} H^1_{ab}(O_{v, \text{ét}}, G^*)$$

be the canonical map (see (4.13)). Then we have the following alternative description of $D^1_{ab}(S, G^*)/\Pi^1_{ab}(K, G^*) \simeq C_{ab}(G)^D$.

**Proposition 4.16.** There exists a canonical isomorphism of finite groups

$$D^1_{ab}(S, G^*)/\Pi^1_{ab}(K, G^*) \simeq \text{Im}\delta_S,$$

where $\delta_S$ is the map (4.15).

**Proof.** Recall the set $W$ introduced in the proof of Proposition 4.9. For every nonempty open subset $V$ of $W$, there exist isomorphisms $D^1_{ab}(V, G^*) \xrightarrow{\sim} \Pi^1_{ab}(K, G^*)$ which are compatible with respect to inclusions $V' \subset V \subset W$, i.e., the following diagram commutes

$$\begin{array}{ccc}
D^1_{ab}(V', G^*) & \xrightarrow{\sim} & \Pi^1_{ab}(K, G^*) \\
\downarrow & & \\
D^1_{ab}(V, G^*) & \xrightarrow{\sim} & \Pi^1_{ab}(K, G^*)
\end{array}$$
Here the vertical arrow is induced by $\mathbb{H}^1_{ab,c}(V, G^*) \to \mathbb{H}^1_{ab,c}(V, G^*)$ (see (4.13)). On the other hand, for each $V$ the composition $D^1_{ab}(V, G^*) \to \mathbb{H}^1_{ab,c}(V, G^*)$ is induced by the canonical map $\mathbb{H}^1_{ab,c}(V, G^*) \to \mathbb{H}^1_{ab,c}(S, G^*)$, and these maps fit into a commutative diagram

(4.17) \[
\begin{array}{ccc}
D^1_{ab}(V, G^*) & \to & D^1_{ab}(S, G^*) \\
\downarrow & & \downarrow \\
D^1_{ab}(V, G^*)
\end{array}
\]

We will now compute the cokernel of the map $D^1_{ab}(V, G^*) \to D^1_{ab}(S, G^*)$ for any nonempty open subset $V$ of $W$. Since $D^1_{ab}(S, G^*) = \text{Im} [\mathbb{H}^1_{ab,c}(S, G^*) \to \mathbb{H}^1_{ab,c}(S, G^*)]$ by (4.11), (4.17) induces an isomorphism

(4.18) \[
\text{Coker} \left[ \prod_{v \in \Sigma} H^0_{ab}(K_v, G^*) \xrightarrow{\beta_S} H^1_{ab,c}(S, G^*) \right] \simeq D^1_{ab}(S, G^*),
\]

and similarly over $V$. Now consider the pairs of maps

(4.19) \[
\prod_{v \notin V} H^0_{ab}(K_v, G^*) \xrightarrow{\alpha_{V,S}} \prod_{v \in \Sigma} H^0_{ab}(K_v, G^*) \xrightarrow{\beta_S} H^1_{ab,c}(S, G^*)
\]

and

(4.20) \[
\prod_{v \notin V} H^0_{ab}(K_v, G^*) \xrightarrow{\beta_V} H^1_{ab,c}(V_\text{et}, G^*) \xrightarrow{\gamma_{V,S}} H^1_{ab,c}(S, G^*)
\]

which satisfy $\beta_S \circ \alpha_{V,S} = \gamma_{V,S} \circ \beta_V$. Then (4.19) and (4.18) show that \[
\text{Coker} (\gamma_{V,S} \circ \beta_V) = \text{Coker} (\beta_S \circ \alpha_{V,S}) = \text{Coker} \beta_S \simeq D^1_{ab}(S, G^*).
\]

Now (4.18) over $V$, the kernel-cokernel exact sequence of the pair of maps (4.20) (see [ADT], Proposition I.0.24, p.16), and the injectivity of $D^1_{ab}(V, G^*) \to D^1_{ab}(S, G^*)$ yield an exact sequence

(4.21) \[
1 \to D^1_{ab}(V, G^*) \to D^1_{ab}(S, G^*) \to \text{Coker} \gamma_{V,S} \to 1.
\]

Finally, we partially order the family of nonempty open subsets $V$ of $S$ by setting $V \leq V' \iff V' \subseteq V$. Then, by the commutativity of (4.17), (4.21) is an exact sequence of inverse systems of finite abelian groups. By (4.16),

(4.22) \[
\varprojlim_{V \subseteq S} D^1_{ab}(V, G^*) \simeq \mathbb{H}^1_{ab,c}(K, G^*).
\]

On the other hand, by (4.13), there exists a canonical isomorphism

\[
\text{Coker} \gamma_{V,S} \simeq \text{Im} \left[ H^1_{ab,c}(S_\text{et}, G^*) \xrightarrow{\delta_{V,S}} \prod_{v \in S \setminus V} H^1_{ab,c}(O_{v,\text{et}}, G^*) \right],
\]
where the map $\delta_{V,S}$ fits into a commutative diagram

$$
\begin{array}{ccc}
H^1_{ab,c}(S_{\et}, G^*) & \xrightarrow{\delta} & \prod_{v \in S_0} H^1_{ab}(\mathcal{O}_{v,\et}, G^*) \\
\downarrow^{\delta_{V,S}} & & \downarrow \\
\prod_{v \in S \setminus V} H^1_{ab}(\mathcal{O}_{v,\et}, G^*).
\end{array}
$$

Consequently

$$
\lim_{\leftarrow V \subset S} \coker \gamma_{V,S} \simeq \lim_{\leftarrow V \subset S} \delta_{V,S} = \im \delta_S.
$$

Thus, by (4.22) and (4.23), the inverse limit of (4.21) is an exact sequence\(^{10}\)

$$
1 \to \prod^1_{ab}(K, G^*) \to D^1_{ab}(S, G^*) \to \im \delta_S \to 1,
$$

which completes the proof. \(\square\)

The preceding proposition has interesting consequences, as we will now see.

Recall the pairing (4.10):

$$
H^1_{ab}(S_{fl}, G) \times H^1_{c}(S_{\et}, G^*) \to \mathbb{Q}/\mathbb{Z}.
$$

When $G$ is semisimple with fundamental group $\mu$, the above pairing is isomorphic to the canonical pairing

$$
H^2(S_{fl}, \mu) \times H^1_{c}(S_{\et}, \mu^*) \to \mathbb{Q}/\mathbb{Z}
$$

of [ADT], III, Corollary 3.2, p.253, and Theorem 8.2, p.290 (see Examples 2.1(a) and 4.5(a)). The latter is a perfect pairing between the discrete torsion group $H^2(S_{fl}, \mu)$ and the profinite group $H^1_{c}(S_{\et}, \mu^*)$. On the other hand, by Examples 2.1(b) and 4.5(b), when $G = T$ is a torus, (4.10) is isomorphic to the pairing

$$
H^1(S_{\et}, T) \times H^2_{c}(S_{\et}, T^*) \to \mathbb{Q}/\mathbb{Z}
$$

of [ADT], Theorem II.4.6(a), p.191, which is also perfect (both groups are finite by [ADT], Theorem II.4.6(a), p.191). Thus (4.10) is perfect in two important particular cases\(^{11}\). Now, when (4.10) is perfect, the dual of (3.4) is an exact sequence

$$
1 \to \prod^1_{ab,S}(K, G)^D \to H^1_{ab,c}(S_{\et}, G^*) \to C_{ab}(G)^D \to 1.
$$

\(^{10}\)Recall that the inverse limit functor is exact on the category of finite abelian groups by [Jen], Proposition 2.3, p.14.

\(^{11}\)However, (4.10) is not perfect in general. More precisely, its perfectness does not follow from that of the two particular cases just mentioned (via the standard five-lemma argument) because the Pontryagin dual of $H^1_{c}(S_{\et}, T^*)$ is not $H^0(S_{\et}, T)$, but rather its completion relative to the topology of subgroups of finite index. See [ADT], Theorem II.4.6(a), p.191.
On the other hand, by Theorem 4.12 and Proposition 4.16, there exists a canonical isomorphism \( C_{ab}(G)^D \simeq \text{Im} \delta_S \). Set

\[(4.25) \quad \text{III}_{ab,c}^1(S, G^*) = \text{Ker} \left[ H^1_{ab,c}(S_{\text{ét}}, G^*) \xrightarrow{\delta_S} \prod_{v \in S_0} H^1_{ab}(O_v, G^*) \right]. \]

Then (4.24) and the isomorphism \( C_{ab}(G)^D \simeq \text{Im} \delta_S \) yield the following statement.

**Proposition 4.17.** Assume that the pairing (4.10) is perfect. Then it induces a perfect pairing

\[ \text{III}_{ab,S}^1(K, G) \times \text{III}_{ab,c}^1(S, G^*) \to \mathbb{Q}/\mathbb{Z}, \]

where the left-hand group is the discrete torsion group (3.3) and the right-hand group is the profinite group (4.25).

When \( G \) is semisimple with fundamental group \( \mu \), Example 3.9(a) shows that \( \text{III}_{ab,S}^1(K, G) \) is isomorphic to

\[(4.26) \quad \text{III}_S^2(K, \mu) = \text{Ker} \left[ H^2(K_{\text{fl}}, \mu) \to \prod_{v \in S_0} H^2(K_v, \mu) \right]. \]

On the other hand, since the map

\[ H^1_{ab}(O_v, G^*) = H^1(O_v, \mu^*) \to H^1(K_v, \mu^*) \]

is injective for every \( v \in S_0 \) by [ADT], III, Lemma 1.1(a), p.237, and p.280, we conclude that \( \text{III}_{ab,c}^1(S, G^*) \) is isomorphic to

\[(4.27) \quad \text{III}_c^1(S, \mu^*) = \text{Ker} \left[ H^1_c(S_{\text{ét}}, \mu^*) \to \prod_{v \in S_0} H^1(K_v, \mu^*) \right]. \]

Thus the following is an immediate consequence of the proposition.

**Corollary 4.18.** Let \( G \) be a semisimple \( S \)-group scheme with fundamental group \( \mu \). Then the natural pairing \( \mu \times \mu^* \to \mathbb{G}_{m,S} \) induces a perfect pairing

\[ \text{III}_S^2(K, \mu) \times \text{III}_c^1(S, \mu^*) \to \mathbb{Q}/\mathbb{Z}, \]

where the left-hand group is the discrete torsion group (4.26) and the right-hand group is the profinite group (4.27).

When \( G = T \) is an \( S \)-torus, we have \( \text{III}_{ab,S}^1(K, G) = \text{III}_S^1(K, T) \) by Example 3.9(b). Note that, since both \( \text{III}_S^1(K, T) \) and \( \prod_{v \in \Sigma} H^1(K_v, T) \) are finite, \( \text{III}_S^1(K, T) \) is finite as well. On the other hand, by [Dem1], Lemma 3.2, p.111, for every \( v \in S_0 \) the canonical map \( H^1_{ab}(O_v, G^*) = H^2(O_v, T^*) \to \)

\[ \]
$H^2(K_v, T^*)$ is injective. Thus, by Examples 2.1(b) and 4.5(b), we conclude that $\ker_1 \gamma(S, T^*)$ equals

$$\ker_1 \gamma(S, T^*) = \ker \left[ H^2(S_{\text{et}}, T^*) \to \prod_{v \in S_0} H^2(K_v, T^*) \right].$$

We have already noted that $H^2(S_{\text{et}}, T^*)$ is finite, so $\ker_2 \gamma(S, T^*)$ is finite as well. Thus the following statement is also an immediate consequence of Proposition 4.17.

**Corollary 4.19.** The natural pairing $T \times T^* \to \mathbb{G}_{m,S}$ induces a perfect pairing of finite groups

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$

**Remark 4.20.** The dualities of the above two corollaries should not be confused with those contained in [ADT], Theorem I.4.20(a), p.65. For example, the groups denoted $\ker_1 \gamma(K, M)$ in [op.cit.], p.56, are not the same as the groups so denoted here when $M = \mu$ or $\mu^*$.  

## 5. Brauer groups and class groups

Let $K$ and $S$ be as in the Introduction and let $G$ be a reductive group scheme over $S$. In this Section we use results from [BvH] to relate the dual of $C_{ab}(G)$ to the algebraic Brauer group of $G_K = G \times_S \text{Spec } K$.

If $X$ is any smooth and geometrically integral $K$-variety, we will write $\overline{X}$ for $X \times_{\text{Spec } K} \text{Spec } \overline{K}$.

There exists a canonical complex of abelian groups $\text{Br } K \to \text{Br } G_K \to \text{Br } G_K$ and we define

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$

Further, set

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$

For any smooth and geometrically integral $K$-variety $X$, let $\text{UPic}(\overline{X})$ be the complex of $\Gamma$-modules defined in [BvH], §2.1. We note that many of the proofs in [BvH] are in fact independent of the characteristic of $K$, provided attention is restricted to reductive groups in [op.cit.], §§3 and 5, and certain references to the literature in [loc.cit.] are replaced by references to [San]. Those which are not are either irrelevant to the matters discussed in this Section, or else there exist published alternative characteristic-free proofs of these results (see [BvH], Remarks 2.14 and 4.15).

There exist a canonical divisor map $\overline{\text{Pic}}(\overline{X})^* / \overline{K}^* \to \text{Div}(\overline{X})$ and a canonical quasi-isomorphism

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$

Further, set

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$

For any smooth and geometrically integral $K$-variety $X$, let $\text{UPic}(\overline{X})$ be the complex of $\Gamma$-modules defined in [BvH], §2.1. We note that many of the proofs in [BvH] are in fact independent of the characteristic of $K$, provided attention is restricted to reductive groups in [op.cit.], §§3 and 5, and certain references to the literature in [loc.cit.] are replaced by references to [San]. Those which are not are either irrelevant to the matters discussed in this Section, or else there exist published alternative characteristic-free proofs of these results (see [BvH], Remarks 2.14 and 4.15).

There exist a canonical divisor map $\overline{\text{Pic}}(\overline{X})^* / \overline{K}^* \to \text{Div}(\overline{X})$ and a canonical quasi-isomorphism

$$\ker_1 \gamma(K, T) \times \ker_2 \gamma(S, T^*) \to \mathbb{Q}/\mathbb{Z}. \quad \square$$
where $K[X]^*/K^*$ is placed in degree $-1$ and $\text{Div}(X)$ in degree 0 (see [BvH], Corollary 2.5 and Remark 2.6). Now let $1 \to F \to H \to G \to 1$ be a flasque resolution of $G$. Then its generic fiber $1 \to F_K \to H_K \to G_K \to 1$ is a flasque resolution of $G_K$ (see [CTS2], Proposition 1.4, p.158), and the following holds.

**Proposition 5.1.** Assume that $G_K$ admits a smooth $K$-compactification. Then the flasque resolution $1 \to F_K \to H_K \to G_K \to 1$ defines a quasi-isomorphism of complexes of $\Gamma$-modules

$$\text{UPic}(G_K)[1] \simeq (R_K^* \to F_K^*),$$

where $R_K = H_K^{\text{tor}}$.

**Proof.** By (5.2), $\text{UPic}(G_K)[1]$ is quasi-isomorphic to $(K[G]^*/K^* \to \text{Div}(G_K))$. Now let $X$ be a smooth $K$-compactification of $G_K$. Then, by [CT], Proposition B.2(iii) and Remark B.2.1(2), pp.130-131, there exist quasi-isomorphisms

$$(K[G]^*/K^* \to \text{Div}(G_K)) \simeq (\text{Div}_{X \setminus G_K}(X) \to \text{Pic}(X)) \simeq (R_K^* \to F_K^*).$$

This completes the proof. □

**Corollary 5.2.** Under the hypotheses of the above proposition, there exists an isomorphism $\text{Br}_nG_K \simeq H^1_{\text{ab}}(K, G^*)$.

**Proof.** There exists an isomorphism $\text{Br}_nG_K \simeq H^1(K, \text{UPic}(G_K)[1])$ by [BvH], Corollary 2.20(ii). On the other hand, the proposition and [GA4], Proposition 4.2, yield isomorphisms

$$H^1(K, \text{UPic}(G_K)[1]) \simeq H^1(K, R^* \to F^*) \simeq H^1_{\text{ab}}(K, G^*),$$

whence the result follows. □

Similarly, for each prime $v$ of $K$, there exist isomorphisms $\text{Br}_nG_{K_v} \simeq H^1_{\text{ab}}(K_v, G^*)$ which are compatible with the isomorphism of the preceding corollary. Thus the chosen flasque resolution of $G$ induces an isomorphism

$$(5.3) \quad \text{B}(G_K) \simeq H^1_{\text{ab}}(K, G^*).$$

Therefore the following statement is immediate from Proposition 4.9.

**Proposition 5.3.** Assume that $G_K$ admits a smooth $K$-compactification. Then there exists a perfect pairing of finite groups

$$H^1_{\text{ab}}(K, G) \times \text{B}(G_K) \to \mathbb{Q}/\mathbb{Z},$$

where $H^1_{\text{ab}}(K, G)$ and $\text{B}(G_K)$ are the groups (3.1) and (5.1), respectively. □

**Remark 5.4.** We stress the fact that the definition of the above pairing depends on the choice of a flasque resolution of $G$ (note, however, that a different choice of flasque resolution leads to an isomorphic pairing). Now, the proposition and [GA3], Corollary 5.10, establish the existence of a pairing $H^1(K, G) \times \text{B}(G_K) \to \mathbb{Q}/\mathbb{Z}$ which induces a bijection $H^1(K, G) \simeq \text{B}(G_K)^D$ over any global field $K$. The existence of such a bijection in the
number field case was established in [San], Theorem 8.5, where the underlying pairing is the Brauer-Manin pairing. This raises the question of clarifying the relationship of the pairing of the proposition with the Brauer-Manin pairing, which we hope to address in a future publication.

Now recall the subgroup $H^1_{ab, nr}(K, G^*)$ of $H^1_{ab}(K, G^*)$ given by (4.14) and let $\text{Br}_{a, nr}(G_K)$ denote the subgroup of $\text{Br}_a G_K$ which corresponds to $H^1_{ab, nr}(K, G^*)$ under the isomorphism of Corollary 5.2. Set

\[(5.4) \quad \text{Br}^S_{a, nr}(G_K) = \text{Br}_{a, nr}(G_K) \cap \ker \left[ \text{Br}_a G_K \to \prod_{v \in \Sigma} \text{Br}_a G_{K_v} \right] \subset \text{Br}_a G_K.\]

Then, by Corollary 5.2 and Proposition 4.11, there exist isomorphisms

\[\text{Br}^S_{a, nr}(G_K) \cong H^1_{ab, nr}(K, G^*) \cap \ker \left[ H^1_{ab}(K, G^*) \to \prod_{v \in \Sigma} H^1_{ab}(K_v, G^*) \right] \cong D^1_{ab}(S, G^*).\]

Thus, using (5.3), the following statement is an immediate consequence of Theorem 4.12.

**Theorem 5.5.** Assume that $G_K$ admits a smooth $K$-compactification. Then there exists a perfect pairing of finite groups

\[C_{ab}(G) \times \text{Br}^S_{a, nr}(G_K)/B(G_K) \to \mathbb{Q}/\mathbb{Z},\]

where the groups $\text{Br}^S_{a, nr}(G_K)$ and $B(G_K)$ are given by (5.4) and (5.1), respectively.

\[\Box\]

**References**

[Bor] Borovoi, M.: *Abelian Galois cohomology of reductive groups*. Mem. Amer. Math. Soc. **132** (1998), no. 626.

[BvH] Borovoi, M. and van Hamel, J.: *Extended Picard complexes and linear algebraic groups*. J. reine angew. Math. **627** (2009), 53-82.

[BLR] Bosch, S., Lütkebohmert, W. and Raynaud, M. *Néron Models*. Springer Verlag, Berlin 1989.

[B] Brahm, B.: *Néron-Modelle algebraischer Tori*. Schriftenreihe des Mathematischen Instituts der Universität Münster. 3. Serie 31. Münster: Univ. Münster, Mathematisches Institut; Münster: Univ. Münster, Fachbereich Mathematik und Informatik (Dissertation). x, 134 pp. (2003).

[Br] Breen, L. *On the classification of 2-gerbes and 2-stacks*. Asterisque no. **225** (1994).

[BT] Bruhat, F. and Tits, J.: *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34**, no. 3 (1987), 671-698.

[CTS1] Colliot-Thélène, J.-L. and Sansuc, J.-J.: *R-équivalence sur les tores*. Ann. sci. Éc. Norm. Sup., 4e série, vol. **10** (1977), 175-229.

[CTS2] Colliot-Thélène, J.-L. and Sansuc, J.-J.: *Principal homogeneous spaces under flasque tori: applications*. J. Algebra **106** (1987), 148-205.

[CT] Colliot-Thélène, J.-L.: *Résolutions flasques des groupes linéaires connexes*. J. reine angew. Math. **618** (2008), 77-133.
ABELIAN CLASS GROUPS OF REDUCTIVE GROUP SCHEMES

[SGA3] Demazure, M. and Grothendieck, A. (Eds.): Schémas en groupes. Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Lecture Notes in Math. 151-153, Springer, Berlin-Heidelberg-New York, 1972.

[Dem1] Demarche, C.: Théorèmes de dualité pour les complexes de tores. arXiv:0906.3453v1 [math.NT].

[Dem2] Demarche, C.: Le défaut d’approximation forte dans les groupes linéaires connexes. Proc. Lond. Math. Soc. (3) 102 (2011), no. 3, 563-597.

[Gi] Giraud, J. Cohomologie non abélienne. Die Grundlehren der mathematischen Wissenschaften, vol. 179. Springer-Verlag, Berlin-New York, 1971.

[GA1] González-Avilés, C.D.: Arithmetic duality theorems for 1-motives over function fields. J. reine angew. Math. 632 (2009), 203-231.

[GA2] González-Avilés, C.D.: On Néron-Raynaud class groups of tori and the Capitulation Problem. J. reine angew. Math. 648 (2010), 149-182.

[GA3] González-Avilés, C.D.: Quasi-abelian crossed modules and nonabelian cohomology. arXiv:1110.4542v2 [math.NT].

[GA4] González-Avilés, C.D.: Flasque resolutions of reductive group schemes. arXiv:1112.6020v1 [math.NT].

[EGA] Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique. Étude locale des schémas et des morphismes de schémas, Quatrième partie (EGA IV). Publ. Math. de l'IHÉS 20 (1964), 5259.

[SGA4] Grothendieck, A. and Verdier, J. (Eds.): Théorie de Topos et Cohomologie Étale des Schémas. Séminaire de Géométrie Algébrique du Bois Marie 1963-64 (SGA 4III). Lecture Notes in Math. 305, Springer, Berlin-Heidelberg-New York, 1972.

[HS] Harari, D. and Szamuely, T.: Arithmetic duality theorems for 1-motives. J. reine angew. Math. 578, pp. 93-128 (2005), and Errata: available from http://www.renyi.hu/~szamuely.

[Har] Harder, G.: Über die Galoiskohomologie halbeinfacher algebraischer Gruppen, III. J. Reine Angew. Math. 274-275 (1975), 125-138.

[Jen] Jensen, C.U.: Les Foncteurs Dérivés de lim et leurs Applications en Théorie des Modules. Lect. Notes in Math. vol. 254. Springer-Verlag, Heidelberg 1972.

[La] Lang, S.: Algebraic groups over finite fields. Amer. J. Math. 78 (1975), no.3, 555-563.

[Mil] Milne, J.S.: Étale Cohomology. Princeton University Press, Princeton, 1980.

[ADT] Milne, J.S.: Arithmetic Duality Theorems. Second Ed. (electronic version), 2006.

[Nis] Nisnevich, Ye. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, pp. 241–342. Kluwer Acad. Publ., Dordrecht, 1989.

[Oes] Oesterlé, J.: Nombres de Tamagawa et groupes unipotentes en caractère p. In: Invent. Math. 78 (1984), 13-88.

[PR] Platovon, V. and Rapinchuk, A.: Algebraic Groups and Number Theory. Academic Press Inc., 1994.

[San] Sansuc, J.: Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. J. reine angew. Math. 327 (1981), 12-80.

[Se] Serre, J.-P.: Cohomologie Galoisienne. Lect. Notes in Math. 5, Springer-Verlag, New York 1986.

[Th] Thaŭg, N.Q.: Corestriction Principle for non-abelian cohomology of reductive group schemes over Dedekind rings of integers of local and global fields. preprint (2008).

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LA SERENA, CHILE
E-mail address: cgonzalez@userena.cl