Isoperimetric and Sobolev inequalities on hypersurfaces in sub-Riemannian Carnot groups

Francescopaolo Montefalcone *

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Abstract

The geometric setting of this paper is that of sub-manifolds immersed in a sub-Riemannian $k$-step Carnot group $G$ of homogeneous dimension $Q$. Our main result is an isoperimetric inequality for the case of a compact hypersurface $S$ of class $C^2$ with (or without) boundary $\partial S$; see Theorem 4.1. We stress that $S$ and $\partial S$ are endowed with the homogeneous measures $\sigma_{n-1}^H$, $\sigma_{n-2}^H$, which (up to bounded densities) turn out to be equivalent, respectively, to the $(Q-1)$-dimensional and $(Q-2)$-dimensional spherical intrinsic Hausdorff measures. This extends a classical inequality, involving the mean curvature of the hypersurface, proven by Michael and Simon [47] and Allard [1], independently. We also deduce some related Sobolev-type inequalities; see Section 5.

Key words and phrases: Carnot groups; Sub-Riemannian Geometry; Hypersurfaces; Isoperimetric Inequality; Sobolev Inequalities; Blow-up; Coarea Formula.

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1 Introduction

In the last decades considerable efforts have been made to extend to the general setting of metric spaces the methods of Analysis and Geometric Measure Theory. This philosophy, in a sense already contained in Federer’s treatise [25], has been pursued, among other authors, by Ambrosio [2], Ambrosio and Kirchheim [3], Capogna, Danielli and Garofalo [9], Cheeger [13], Cheeger and Kleiner [14], David and Semmes [23], De Giorgi [24], Gromov [34], Franchi, Gallot and Wheeden [28], Franchi and Lanconelli [29], Franchi, Serapioni and Serra Cassano [30, 31], Garofalo and Nhieu [32], Heinonen and Koskela [35], Koranyi and Riemann [40], Pansu [56, 57], but the list is far from being complete.

In this respect, sub-Riemannian or Carnot-Carathéodory geometries have become a subject of great interest also because of their connections with many different areas of Mathematics and Physics, such as PDE’s, Calculus of Variations, Control Theory, Mechanics and Theoretical Computer Science. For references, comments and other perspectives, we refer the reader to Montgomery’s book [55] and the surveys by Gromov, [31], and Vershik and Gershkovich, [70]. We also mention, specifically for sub-Riemannian geometry, [66] and [58]. More recently, the so-called Visual Geometry has also received new impulses from this field; see [62], [18] and references therein.

The setting of the sub-Riemannian geometry is that of a smooth manifold \(N\), endowed with a smooth non-integrable distribution \(H \subset TN\) of \(h\)-planes, or horizontal subbundle (\(h \leq \dim N\)), where a metric \(g_{\mu}\) is defined. The manifold \(N\) is said to be a Carnot-Carathéodory space or CC-space when one introduces the so-called CC-metric \(d_{CC}\); see Definition 2.2. With respect to such a metric, the only paths on \(N\) which have finite length are tangent to \(H\) and therefore called horizontal. Roughly speaking, in connecting two points we are only allowed to follow horizontal paths joining them.

A \(k\)-step Carnot group \((G, \bullet)\) is an \(n\)-dimensional, connected, simply connected, nilpotent and stratified Lie group (with respect to the group multiplication \(\bullet\)) whose Lie algebra \(g \cong \mathbb{R}^n\) satisfies the following:

\[
g = H_1 \oplus \ldots \oplus H_k, \quad [H_i, H_{i-1}] = H_i, \quad \forall \ i = 2, \ldots, k, \quad H_{k+1} = \{0\}.
\]

Let \(X_H := \{X_1, \ldots, X_k\}\) be a frame of left-invariant vector fields for the horizontal bundle \(H\). This horizontal frame can be completed to a global left-invariant frame \(X := \{X_1, \ldots, X_n\}\) for \(TG\). In fact, the standard basis \(\{e_i : i = 1, \ldots, n\}\) of \(\mathbb{R}^n\) can be relabeled to be graded or adapted to the stratification. Every Carnot group \(G\) is endowed with a one-parameter group of positive dilations (adapted to the grading of \(g\)) making it a homogeneous group of homogeneous dimension \(Q := \sum_{i=1}^{k} i h_i\) (where \(h_i = \dim H_i\)), in the sense of Stein’s definition; see [63]. The number \(Q\) coincides with the Hausdorff dimension of \((G, d_{CC})\) as a metric space with respect to the CC-distance. Carnot groups are of special interest for many reasons and, in particular, because they constitute a wide class of examples of sub-Riemannian geometries. Note that, by a well-know result due to Mitchell [49] (see also [55]), the Gromov-Hausdorff tangent cone at any regular point of a sub-Riemannian manifold turns out to be a suitable Carnot group. This fact motivates the interest towards Carnot groups which play for sub-Riemannian geometries an analogous role to that of Euclidean spaces in Riemannian geometry. The initial development of Analysis in this setting was motivated by some works published in the first eighties. Among others, we cite the paper by Fefferman and Phong [27] about the so-called “sub-elliptic estimates” and that of Franchi and Lanconelli [29], where a Hölder regularity theorem was proven for a class of degenerate elliptic operators in divergence form. Meanwhile, the beginning of Geometric Measure Theory was perhaps an intrinsic isoperimetric inequality proven by Pansu in his thesis [56], for the Heisenberg group \(\mathbb{H}^1\). For further results about isoperimetric inequalities on Lie groups and Carnot-Carathéodory spaces, see also [69, 34, 58, 32, 9, 28, 35]. For results on these topics, and for more detailed bibliographic references, we refer the reader to [2], [9], [30, 31], [22], [33, 32], [11, 32], [51, 52, 38]. We also quote [10], [16, 33, 79, 61], for some results about minimal and constant mean-curvature hypersurfaces immersed in Heisenberg groups.
In this paper we are concerned with immersed hypersurfaces in Carnot groups, endowed with the so-called $H$-perimeter measure $\sigma_R^{n-1}$; see Definition 2.1. We first study some technical tools. In particular, we extend to hypersurfaces with non-empty characteristic sets, the 1st-variation of $\sigma_R^{n-1}$, proved in [50, 52] for the non-characteristic case; see Section 3.2. We then discuss a blow-up theorem, which also holds for characteristic points and a horizontal Coarea Formula for smooth functions on hypersurfaces; see Section 3.3 and Section 3.1. In Section 4 these results will be used to investigate the validity in this context of an isoperimetric inequality, proved by Michael and Simon in [47] for a general setting including Riemannian geometries and, independently, by Allard in [11] for varifolds; see below for a more precise statement. In Section 5, we shall deduce some related Sobolev inequalities, following a classical pattern by Federer-Fleming [26] and Mazja [46]. Some similar results in this direction have been obtained by Danielli, Garofalo and Nhieu in [22], where a monotonicity estimate for the $H$-perimeter has been proven for graphical strips in the Heisenberg group $\mathbb{H}^1$.

Now we would like to make a short comment about the Isoperimetric Inequality for compact hypersurfaces immersed in the Euclidean space $\mathbb{R}^n$.

**Theorem 1.1** (Euclidean Isoperimetric Inequality for $S \subset \mathbb{R}^n$). Let $S \subset \mathbb{R}^n (n > 2)$ be a compact hypersurface of class $C^2$ with -or without- piecewise $C^1$ boundary. Then

$$\left( \sigma_R^{n-1}(S) \right)^{\frac{n-2}{n-1}} \leq C_{\text{Isop}} \left( \int_S |H_R| \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S) \right)$$

where $C_{\text{Isop}} > 0$ is a dimensional constant.

In the above statement, $H_R$ is the mean curvature and $\sigma_R^{n-1}$ and $\sigma_R^{n-2}$ denote, respectively, the Riemannian measures on $S$ and $\partial S$. The first step in the proof is a linear isoperimetric inequality. More precisely, one has

$$\sigma_R^{n-1}(S) \leq r \left( \int_S |H_R| \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S) \right),$$

where $r$ is the radius of a Euclidean ball $B(x, r)$ containing $S$. Starting from this linear inequality and using Coarea Formula, one gets the so-called monotonicity inequality, i.e.

$$- \frac{d}{dt} \sigma_R^{n-1}(S_t) \leq \frac{1}{t^{n-1}} \left( \int_{S_t} |H_R| \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S \cap B(x, t)) \right)$$

for every $x \in \text{Int } S$, for $L^1$-a.e. $t > 0$, where $S_t = S \cap B(x, t)$. Note that every interior point of a $C^2$ hypersurface $S$ is a density-point.\footnote{We say that $x \in S$ is a density-point, if $\lim_{t \to 0^+} \sigma_R^{n-1}(S_t) = \omega_{n-1}$, where $\omega_{n-1}$ denotes the measure of the unit ball in $\mathbb{R}^{n-1}$.}

By applying the monotonicity inequality along with a contradiction argument, one obtains a calculus lemma which, together with a standard Vitali-type covering theorem, allows to achieve the proof. We also remark that the monotonicity inequality is equivalent to an asymptotic exponential estimate, i.e.

$$\sigma_R^{n-1}(S_t) \geq \omega_{n-1} t^{n-1} e^{-H_0 t}$$

for $t \to 0^+$, where $x \in \text{Int } S$ and $H_0$ is any positive constant such that $|H_R| \leq H_0$. In case of minimal hypersurfaces (i.e. $H_R = 0$), this implies that

$$\sigma_R^{n-1}(S_t) \geq \omega_{n-1} t^{n-1}$$

as long as $t \to 0^+$.

We now give a quick overview of the paper.
Section 2 introduces Carnot groups, immersed hypersurfaces and submanifolds. In particular, we describe some geometric structures and basic facts about stratified Lie groups, Riemannian and sub-Riemannian geometries, intrinsic measures and connections.

If \( S \subset \mathbb{G} \) is a hypersurface of class \( C^1 \), then \( x \in S \) is a characteristic point if \( H_x \subset T_x S \). If \( S \) is non-characteristic, the unit \( H \)-normal along \( S \) is given by \( \nu_\mu := \frac{P_\mu \nu}{|P_\mu \nu|} \), where \( \nu \) is the Riemannian unit normal of \( S \) and \( P_\mu : T \mathbb{G} \to H \) is the orthogonal projection operator onto \( H \). By using the contraction operator \( \mathcal{J} \) on differential forms, we can define a \((Q - 1)\)-homogeneous measure \( \sigma^{n-1}_H \in \wedge^{n-1}(T^*S) \), called \( H \)-perimeter, by setting

\[
\sigma^{n-1}_H \llcorner S := (\nu_\mu \mathcal{J} \sigma^n_\mu)|_S,
\]

where \( \sigma^n_\mu := \wedge_{i=1}^n \omega_i \in \wedge^n(T^*\mathbb{G}) \) denotes the Riemannian (left-invariant) volume form on \( \mathbb{G} \) (obtained by wedging together the elements of the “dual”basis \( \omega = \{\omega_1, \ldots, \omega_n\} \) of \( T^*\mathbb{G} \), where \( \omega_i = X_i^* \), for every \( i = 1, \ldots, n \). Note that the characteristic set \( C_S \) of \( S \) can be seen as the set of all points at which the horizontal projection of the unit normal vanishes, i.e. \( C_S = \{x \in S : |P_\mu \nu| = 0\} \).

Analogously, we can define a \((Q - 2)\)-homogeneous measure \( \sigma^{n-2}_H \) on any \((n-2)\)-dimensional smooth submanifold \( N \) of \( \mathbb{G} \). To this aim, let \( \nu_\mu = \nu^1_\mu \wedge \nu^2_\mu \in \wedge^2(H) \) be a unit horizontal normal 2-vector along \( N \); see Definition 2.21. Then, we define a \((Q - 2)\)-homogeneous measure \( \sigma^{n-2}_H \in \wedge^{n-2}(T^*S) \), by setting

\[
\sigma^{n-2}_H \llcorner S := (\nu_\mu \mathcal{J} \sigma^n_\mu)|_S.
\]

The measures \( \sigma^{n-1}_H \) and \( \sigma^{n-2}_H \) turn out to be equivalent (up to bounded densities called metric factors), respectively, to the \((Q - 1)\)-dimensional and \((Q - 2)\)-dimensional spherical Hausdorff measures \( S^{Q-1}_\mathbb{G} \) and \( S^{Q-2}_\mathbb{G} \) associated with a homogeneous distance \( \rho \) on \( \mathbb{G} \); see Section 3.3.

Section 3 contains the technical preliminaries. In Section 3.1 we state a smooth Coarea Formula for the HS-gradient. More precisely, let \( S \subset \mathbb{G} \) be a compact hypersurface of class \( C^2 \) and let \( \varphi \in C^1(S) \). Then

\[
\int_S \psi(x) \left| \text{grad}_{HS} \varphi(x) \right| \sigma^{n-1}_H(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \psi(y) \sigma^{n-2}_H(y)
\]

for every \( \psi \in L^1(S; \sigma^{n-1}_H) \).

In Section 3.2 we state a horizontal divergence theorem for hypersurfaces with boundary endowed with the homogeneous measures \( \sigma^{n-1}_H \) and \( \sigma^{n-2}_H \); see Theorem 3.3. More importantly, we discuss 1st variation formula of \( \sigma^{n-1}_H \); see Theorem 3.9. These results, proved in \( [50] [52] \) for non-characteristic hypersurfaces, are generalized to the case of non-empty characteristic sets. Roughly speaking, we shall show that the “infinitesimal” 1st variation of \( \sigma^{n-1}_H \) is given by

\[
L_W \sigma^{n-1}_H = \left( -\mathcal{H}_W \langle W, \nu \rangle + \text{div}_{TS} \left( W^\top |P_\mu \nu| - \langle W, \nu \rangle \nu_\mu^\top \right) \right) \sigma^{n-1}_H,
\]

where \( L_W \sigma^{n-1}_H \) is the Lie derivative of \( \sigma^{n-1}_H \) with respect to the initial velocity \( W \) of the variation. We stress that \( \mathcal{H}_W = -\text{div}_W \nu_\mu \) is the so-called horizontal mean curvature of \( S \). Moreover, the symbols \( W^\perp \), \( W^\top \) denote the normal and tangential components of \( W \), respectively. If \( H_\mu \) is \( L^1_{\text{loc}}(S; \sigma^{n-1}_H) \), then the function \( L_W \sigma^{n-1}_H \) turns out to be integrable on \( S \) and the integral of \( L_W \sigma^{n-1}_H \) on \( S \) gives the 1st variation of \( \sigma^{n-1}_H \). Note that the third term in the previous formula depends on the normal component of \( W \). We stress that this term was omitted in \( [52] \). Using

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Recall that \( \mathcal{J} : \Lambda^k(T^*\mathbb{G}) \to \Lambda^{k-1}(T^*\mathbb{G}) \) is defined, for \( X \in T \mathbb{G} \) and \( \alpha \in \Lambda^k(T^*\mathbb{G}) \), by

\[
(X \mathcal{J} \alpha)(Y_1, ..., Y_{k-1}) := \alpha(X, Y_1, ..., Y_{k-1})
\]

This operator extends, in a simple way, to \( p \)-vectors; see \( [37] [25] \).
a generalized divergence-type formula, the divergence term can be integrated on the boundary. It is worth observing that a central point of this paper is to make an appropriate choice of the variation vector field in the 1st variation formula \( (11) \); see Theorem 3.9.

In Section 3.3 we state a blow-up theorem for the horizontal perimeter \( \sigma_{H}^{n-1} \). In other words, we study the density of \( \sigma_{H}^{n-1} \) at \( x \in S \), or the limit

\[
\lim_{r \to 0^+} \frac{\sigma_{H}^{n-1}(S \cap B_{\rho}(x,r))}{r^{Q-1}},
\]

where \( B_{\rho}(x,r) \) is a homogeneous \( \rho \)-ball of center \( x \in S \) and radius \( r \). We first discuss the blow-up procedure at non-characteristic points of a \( C^1 \) hypersurface \( S \); see, for instance, \([30, 31]\), \([5, 41, 42]\). Then, under more regularity assumptions on \( S \), we tract the characteristic case; see Theorem 3.12. A similar result was proven in \([43]\) for 2-step groups.

Section 4 is devoted to our main result, that is an isoperimetric inequality for compact hypersurfaces with (or without) boundary, depending on the horizontal mean curvature \( H \). This extends to Carnot groups an inequality proved by Michael and Simon \([47]\) and Allard \([1]\), independently.

**Theorem 1.2** (Isoperimetric Inequality on hypersurfaces). Let \( S \subset \mathbb{G} \) be a compact hypersurface of class \( C^2 \) with boundary \( \partial S \) (piecewise) \( C^1 \) and assume that the horizontal mean curvature \( \mathcal{H}_H \) of \( S \) is integrable, i.e. \( \mathcal{H}_H \in L(1)(S;\sigma_{H}^{n-1}) \). Then, there exists a constant \( C_{Isop} > 0 \), only dependent on \( \mathbb{G} \) and on \( \rho \), such that

\[
(\sigma_{H}^{n-1}(S))^{\frac{Q-2}{Q-1}} \leq C_{Isop} \left( \int_S |\mathcal{H}_H| \sigma_{H}^{n-1} + \sigma_{H}^{n-2}(\partial S) \right).
\]

The proof of this result is heavily inspired from the classical one, for which we refer the reader to the book by Burago and Zalgaller \([8]\). A similar strategy was useful in proving isoperimetric and Sobolev inequalities in abstract metric setting such as weighted Riemannian manifolds and graphs; see \([17]\).

The starting point is a linear isoperimetric inequality (see Proposition 4.8) that is used to obtain a monotonicity formula for the \( H \)-perimeter; see Theorem 4.13. Exactly as in the Euclidean/Riemannian case, the monotonicity inequality is an ordinary differential inequality, concerning the first derivative of the density-quotient

\[
\frac{\sigma_{H}^{n-1}(S \cap B_{\rho}(x,t))}{t^{Q-1}},
\]

where \( x \in \text{Int} (S \setminus C_{S}) \); see Section 4.1.

In Section 4.2 we prove some key estimates dependent on standard density theorems. Section 4.3 is devoted to the proof of the Isoperimetric Inequality.

In Section 4.4 we discuss some straightforward applications of the monotonicity formula. More precisely, let \( S \subset \mathbb{G} \) be a hypersurface of class \( C^2 \), let \( x \in \text{Int}(S \setminus C_{S}) \) and, without loss of generality, assume that, near \( x \), the horizontal mean curvature \( \mathcal{H}_H \) is bounded by a positive constant \( \mathcal{H}_H^0 \). Then, we will show that

\[
\sigma_{H}^{n-1}(S_t) \geq \kappa_{\rho}(\nu_{\rho}(x)) t^{Q-1} e^{-t \mathcal{H}_H^0},
\]

as long as \( t \wedge 0^+ \), where \( \kappa_{\rho}(\nu_{\rho}(x)) \) denotes the so-called metric factor at \( x \); see Corollary 4.27.

We also consider the case where \( x \in C_{S} \); see Corollary 4.29 and Corollary 4.30.

Finally, in Section 5 we discuss some Sobolev-type inequalities which can easily be deduced by using the Isoperimetric Inequality, following a classical argument by Federer-Fleming \([26]\) and Mazja \([46]\). The main result is the following:
**Theorem 1.3** (Sobolev-type inequality on hypersurfaces). Let $G$ be a $k$-step Carnot group endowed with a homogeneous metric $\varrho$ as in Definition 2.4. Let $S \subset G$ be a hypersurface of class $C^2$ without boundary. Let $\mathcal{H}_H$ be the horizontal mean curvature of $S$ and assume that $\mathcal{H}_H \in L^1_{loc}(S;\sigma_H^{n-1})$. Then

$$
\left( \int_S |\psi|^{\frac{n-1}{2}} \sigma_H^{n-1} \right)^{\frac{2}{n-1}} \leq C_{iso} \int_S (|\psi| |\mathcal{H}_H| + |\text{grad}_{Hs}\psi|) \sigma_H^{n-1}
$$

for every $\psi \in C_0^1(S)$, where $C_{iso}$ is the constant appearing in Theorem 1.1.

Note that $C_{iso}$ is the same constant appearing in Theorem 1.2.

## 2 Carnot groups, submanifolds and measures

### 2.1 Sub-Riemannian Geometry of Carnot groups

In this section we introduce the definitions and the main features concerning the sub-Riemannian geometry of Carnot groups. References for this large subject can be found, for instance, in [9], [32], [34], [41], [49], [55], [56, 57, 58], [66]. Let $N$ be a $C^\infty$-smooth connected $n$-dimensional manifold and let $H \subset TN$ be an $h$-dimensional smooth subbundle of $TN$. For any $x \in N$, let $T_x^k$ denote the vector subspace of $T_xN$ spanned by a local basis of smooth vector fields $X_1(x), \ldots, X_k(x)$ for $H$ around $x$, together with all commutators of these vector fields of order $\leq k$. The subbundle $H$ is called generic if, for all $x \in N$, $\dim T_x^k$ is independent of the point $x$ and horizontal if $T_x^k = TN$, for some $k \in \mathbb{N}$. The pair $(N, H)$ is a $k$-step CC-space if is generic and horizontal and if $k = \inf\{r : T_x^r = TN\}$. In this case

$$
0 = T^0 \subset H = T^1 \subset T^2 \subset \ldots \subset T^k = TN
$$

is a strictly increasing filtration of subbundles of constant dimensions $n_i := \dim T^i, i = 1, \ldots, k$. Setting $(H_i)_x := T_x^i \setminus T_x^{i-1}$, then $\text{gr}(T_xN) = \oplus_{i=1}^k(h_i)_x$ is the associated graded Lie algebra at $x \in N$, with respect to the Lie product $[\cdot, \cdot]$. We set $h_i := \dim H_i = n_i - n_{i-1}$ ($n_0 = h_0 = 0$) and, for simplicity, $h := h_1 = \dim H$. The $k$-vector $\overline{h} = (h, h_2, \ldots, h_k)$ is the growth vector of $H$.

**Definition 2.1** (Graded frame). We say that $X = \{X_1, \ldots, X_n\}$ is a graded frame for $N$ if $\{X_{ij} : n_{j-1} < i_j \leq n_j\}$ is a basis for $H_{j,x}$ for all $x \in N$ and $j = 1, \ldots, k$.

**Definition 2.2.** A sub-Riemannian metric $g_H = \langle \cdot, \cdot \rangle_H$ on $N$ is a symmetric positive bilinear form on $H$. If $(N, H)$ is a CC-space, the CC-distance $d_{CC}(x, y)$ between $x, y \in N$ is defined by

$$
d_{CC}(x, y) := \inf \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_H} dt,
$$

where the infimum is taken over all piecewise-smooth horizontal paths $\gamma$ joining $x$ to $y$.

In fact, Chow’s Theorem implies that $d_{CC}$ is metric on $N$ and that any two points can be joined with at least one horizontal curve. The topology induced on $N$ by the CC-metric is equivalent to the standard manifold topology; see [34], [55].

The above setting is the starting point of sub-Riemannian geometry. A large class of these geometries is represented by Carnot groups which, for many reasons, play in sub-Riemannian geometry an analogous role to that of Euclidean spaces in Riemannian geometry. For the geometry of Lie groups we refer the reader to Helgason’s book [37] and Milnor’s paper [48], while for sub-Riemannian geometry, to Gromov, [34], Pansu, [50, 58], and Montgomery, [55].
A \textit{k-step Carnot group} \((G, \bullet)\) is an \(n\)-dimensional, connected, simply connected, nilpotent and stratified Lie group (with respect to the multiplication \(\bullet\)). Let 0 be the identity on \(G\). The Lie algebra \(g \cong T_0G\) of \(G\) is an \(n\)-dimensional vector space such that:

\[ g = H_1 \oplus \ldots \oplus H_k, \quad [H_1, H_{i-1}] = H_i \quad \forall \ i = 2, \ldots, k, \quad H_{k+1} = \{0\}. \]

The smooth subbundle \(H_1\) of the tangent bundle \(TG\) is called \textit{horizontal} and henceforth denoted by \(H\). Let \(V := H_2 \oplus \ldots \oplus H_k\) be the \textit{vertical subbundle} of \(TG\). We set \(h_i = \dim H_i, n_i := h_1 + \ldots + h_i\) for every \(i = 1, \ldots, k\) \((h_1 = h, n_k = n)\). We assume that \(H\) is generated by a frame \(X_h := \{X_1, \ldots, X_h\}\) of left-invariant vector fields. This frame can always be completed to a global, graded, left-invariant frame \(X := \{X_i : i = 1, \ldots, n\}\) for the tangent bundle \(TG\), in a way that \(H_l = \operatorname{span}_{\mathbb{R}}\{X_i : n_l-1 < i \leq n_l\}\) for \(l = 1, \ldots, k\). In fact, the standard basis \(\{e_i : i = 1, \ldots, n\}\) of \(\mathbb{R}^n \cong g\) can be relabeled to be \textit{adapted to the stratification}. Each left-invariant vector field belonging to \(X\) is given by \(X_i(x) = L_{x^*} e_i(i = 1, \ldots, n)\), where \(L_{x^*}\) is the differential of the left-translation by \(x\).

**Notation 2.3.** We set \(I_h := \{1, \ldots, h\}, I_{h-1} := \{n_1 + 1, \ldots, n_2\}, \ldots, I_v := \{h + 1, \ldots, n\}\). Unless otherwise specified, Latin letters \(i, j, k, \ldots\) are used for indices belonging to \(I_h\) and Greek letters \(\alpha, \beta, \gamma, \ldots\) for indices belonging to \(I_v\). The function \(\operatorname{ord} : \{1, \ldots, n\} \rightarrow \{1, \ldots, k\}\) is defined by \(\operatorname{ord}(a) := i\), whenever \(n_i-1 < a \leq n_i, i = 1, \ldots, k\).

We use exponential coordinates of 1st kind so that \(G\) will be identified with its Lie algebra \(g\), via the (Lie group) exponential map \(\exp : g \rightarrow G\); see [85]. The Baker-Campbell-Hausdorff formula gives the group law \(\bullet\) of the group \(G\), starting from a corresponding operation on the Lie algebra \(g\). In fact, one has

\[ \exp(X) \bullet \exp(Y) = \exp(X \star Y) \quad (X, Y \in g), \]

where \(\star : g \times g \rightarrow g\) is the \textit{Baker-Campbell-Hausdorff product} defined by

\[ X \star Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \text{brackets of length } \geq 3. \quad (1) \]

In exponential coordinates, the group law \(\bullet\) on \(G\) is polynomial and explicitly computable; see [19]. Note that \(0 = \exp(0, \ldots, 0)\) and the inverse of each point \(x = \exp(x_1, \ldots, x_n) \in G\) is given by \(x^{-1} = \exp(-x_1, \ldots, -x_n)\).

When \(H\) is endowed with a metric \(g_H = \langle \cdot, \cdot \rangle_H\), we say that \(G\) has a \textit{sub-Riemannian structure}. It is always possible to define a left-invariant Riemannian metric \(g = \langle \cdot, \cdot \rangle\) on \(G\) such that \(X\) is orthonormal and \(g_{|H} = g_H\). Note that, if we fix a Euclidean metric on \(g = T_0G\) such that \(\{e_i : i = 1, \ldots, n\}\) is an orthonormal basis, this metric extends to the whole tangent bundle, by left-translations.

Since Chow’s Theorem holds true for Carnot groups, the \textit{Carnot-Carathéodory distance} \(d_{CC}\) associated with \(g_H\) can be defined and the pair \((G, d_{CC})\) turns out to be a complete metric space where every couple of points can be joined by at least one \(d_{CC}\)-geodesic.

Carnot groups are \textit{homogeneous groups}, in the sense that they admit a 1-parameter group of automorphisms \(\delta_t : G \rightarrow G\) \((t \geq 0)\) defined by

\[ \delta_t x := \exp \left( \sum_{j,i} t^j x_{ij} e_{ij} \right), \]

for every \(x = \exp \left( \sum_{j,i} x_{ij} e_{ij} \right) \in G\). We stress that the \textit{homogeneous dimension} of \(G\) is the integer \(Q := \sum_{i=1}^k i h_i\), coinciding with the \textit{Hausdorff dimension} of \((G, d_{CC})\) as a metric space; see [49, 55, 81].
Definition 2.4. A continuous distance $\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_+$ is called homogeneous if, and only if, the following hold:

(i) $\rho(x, y) = \rho(z \cdot x, z \cdot y)$ for every $x, y, z \in \mathbb{G}$;
(ii) $\rho(\delta_t x, \delta_t y) = t \rho(x, y)$ for all $t \geq 0$.

The CC-distance $d_{CC}$ is an example of homogeneous distance. Another example can be found in [31]. Moreover, any Carnot group admits a smooth, subadditive, homogeneous norm, that is, there exists a function $\| \cdot \|_\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_+ \cup \{0\}$ such that:

(i) $\|x \cdot y\|_\rho \leq \|x\|_\rho + \|y\|_\rho$;
(ii) $\|\delta_t x\|_\rho = t \|x\|_\rho$ (t $\geq 0$);
(iii) $\|x\|_\rho = 0 \iff x = 0$;
(iv) $\|x\|_\rho = \|x^{-1}\|_\rho$;
(v) $\|\cdot\|_\rho$ is continuous and smooth on $\mathbb{G} \setminus \{0\}$; see [36].

Example 2.5. A smooth homogeneous norm $\rho$ on $\mathbb{G} \setminus \{0\}$, can be defined by setting

$$\|x\|_\rho := \left(\|x\|^{\lambda} + C_2|x_{i_2}|^{\lambda/2} + C_3|x_{i_3}|^{\lambda/3} + \ldots + C_k|x_{i_k}|^{\lambda/k}\right)^{1/\lambda},$$  \hfill (2)

where $\lambda$ is any positive number evenly divisible by $i = 1, \ldots, k$ and $|x_{i_j}|$ denotes the Euclidean norm of the projection of $x$ onto the $i$-th layer $H_i$ of $\mathfrak{g}$.

Henceforth, we shall assume the following:

Definition 2.6. Let $\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_+$ be a homogeneous distance such that:

(i) $\rho$ is (piecewise) $C^1$;
(ii) $|\text{grad}_x \rho| \leq 1$ at each regular point of $\rho$;
(iii) $|x_{i_j}| \leq \rho(x)$ for every $x \in \mathbb{G}$, where $\rho(x) = \rho(0, x) = \|x\|_\rho$. Furthermore, we shall assume that there exist constants $c_i \in \mathbb{R}_+$ such that $|x_{i_j}| \leq c_i \rho(x)$ for every $i = 2, \ldots, k$.

Example 2.7. Let us consider the case of Heisenberg groups $\mathbb{H}^n$; see Example 2.12. It can be shown that the CC-distance $d_{CC}$ satisfies all the previous assumptions. Another important example is the so-called Koranyi norm, defined by

$$\|y\|_\rho := \rho(y) = \sqrt[4]{|y_{i_1}|^4 + 16t^2} \quad (y = \exp(y_{i_1}, t) \in \mathbb{H}^n),$$

is homogeneous and $C^\infty$-smooth outside $0 \in \mathbb{H}^n$ and satisfies (ii) and (iii) of Definition 2.6.

Since we have fixed a Riemannian metric on $T\mathbb{G}$, we may define the left-invariant co-frame $\omega := \{\omega_i : i = 1, \ldots, n\}$ dual to $X$. In fact, the left-invariant 1-forms $\omega_i$ are uniquely determined by the condition:

$$\omega_i(X_j) = \langle X_i, X_j \rangle = \delta_i^j \quad \text{for every } i, j = 1, \ldots, n,$$

where $\delta_i^j$ denotes “Kronecker delta”. Recall that the structural constants of the Lie algebra $\mathfrak{g}$ associated with the left invariant frame $X$ are defined by

$$C^p_{ijr} := \langle [X_i, X_j], X_r \rangle \quad \text{for every } i, j, r = 1, \ldots, n.$$  \hfill (3)

They satisfy the following:\footnote{That is, $L^*_p \omega_l = \omega_l$ for every $p \in \mathbb{G}$.}
of the Levi-Civita connection \( \nabla \) properties easily follow from the very definition of divergence of \( g \) with the metric \( g \).

Definition 2.8 (Matrices of structural constants). We set

(i) \( C^\alpha_{ij} := [C^\alpha_{ij}]_{i,j \in I_H} \in M_{h \times h}(\mathbb{R}) \quad \forall \alpha \in I_{h} \);

(ii) \( C^\alpha := [C^\alpha_{ij}]_{i,j = 1, \ldots, n} \in M_{n \times n}(\mathbb{R}) \quad \forall \alpha \in I_{v} \).

The linear operators associated with these matrices are denoted in the same way.

Definition 2.9. Let \( \nabla \) be the (unique) left-invariant Levi-Civita connection on \( \mathbb{G} \) associated with the metric \( g \). If \( X, Y \in \mathfrak{X}(H) := \mathbb{C}\infty(\mathbb{G}, H) \), we set \( \nabla^H_X Y := \mathcal{P}_H(\nabla_X Y) \).

Remark 2.10. The operation \( \nabla^H \) is called horizontal \( H \)-connection; see [52] and references therein. By using the properties of the structural constants of the Levi-Civita connection, one can show that \( \nabla^H \) is flat, i.e. \( \nabla^H_X X = 0 \) for every \( i, j \in I_H \). \( \nabla^H \) turns out to be compatible with the sub-Riemannian metric \( g_H \); i.e. \( X \langle Y, Z \rangle = \langle \nabla^H_X Y, Z \rangle + \langle Y, \nabla^H_X Z \rangle \) for all \( X, Y, Z \in \mathfrak{X}(H) \). Moreover, \( \nabla^H \) is torsion-free, i.e. \( \nabla^H_X Y − \nabla^H_Y X − \mathcal{P}_H[X, Y] = 0 \) for all \( X, Y \in \mathfrak{X}(H) \). All these properties easily follow from the very definition of \( \nabla^H \) together with the corresponding properties of the Levi-Civita connection \( \nabla \) on \( \mathbb{G} \). Finally, we recall a fundamental property of \( \nabla \), i.e.

\[
\nabla_X X_j = \frac{1}{2} \sum_{r=1}^{n} \left( C^q_{ri} - C^q_{jr} + C^q_{rij} \right) X_r \quad \forall i, j = 1, \ldots, n.
\]

Definition 2.11. If \( \psi \in \mathbb{C}\infty(\mathbb{G}) \) we define the horizontal gradient of \( \psi \) as the unique horizontal vector field \( \text{grad}_H \psi \) such that \( \langle \text{grad}_H \psi, X \rangle = d\psi(X) = X\psi \) for all \( X \in \mathfrak{X}(H) \). The horizontal divergence of \( X \in \mathfrak{X}(H) \), \( \text{div}_H X \), is defined, at each point \( x \in \mathbb{G} \), by

\[
\text{div}_H X(x) := \text{Trace}(Y \rightarrow \nabla^H_Y X)(x) \quad (Y \in H_x).
\]

Example 2.12 (Heisenberg group \( \mathbb{H}^n \)). Let \( h_n := T_0 \mathbb{H}^n = \mathbb{R}^{2n+1} \) denote the Lie algebra of \( \mathbb{H}^n \). The only non-trivial algebraic rules are given by \( [e_i, e_{i+1}] = e_{2n+1} \) for every \( i = 2k + 1 \) where \( k = 0, \ldots, n - 1 \). We have \( h_n = H \oplus \mathbb{R} e_{2n+1} \), where \( H = \text{span}_\mathbb{R} \{ e_i : i = 1, \ldots, 2n \} \) and the second layer turns out to be the 1-dimensional center of \( h_n \). The Baker-Campbell-Hausdorff formula determines the group law \( \bullet \). For every \( x = \exp \left( \sum_{i=1}^{2n+1} x_i X_i \right), \ y = \exp \left( \sum_{i=1}^{2n+1} y_i X_i \right) \in \mathbb{H}^n \) it turns out that

\[
x \bullet y = \exp \left( x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{1}{2} \sum_{k=1}^{n} (x_{2k-1}y_{2k} - x_{2k}y_{2k-1}) \right).
\]

The matrix of structural constants is given by

\[
C^i_{2n+1} := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdot \\
-1 & 0 & 0 & 0 & \cdot \\
0 & 0 & 0 & 1 & \cdot \\
0 & 0 & -1 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\]
2.2 Hypersurfaces, homogeneous measures and geometric structures

Hereafter, \( \mathcal{H}^m \) and \( \mathcal{S}^m \) will denote the Hausdorff measure and the spherical Hausdorff measure, respectively, associated with a homogeneous distance \( \rho \) on \( \mathbb{G} \).

The Riemannian left-invariant volume form on \( \mathbb{G} \) is given by \( \sigma^n := \Lambda_{i=1}^n \omega_i \in \Lambda^n(T^* \mathbb{G}) \). This measure is the Haar measure of \( \mathbb{G} \) and equals (the push-forward of) the \( n \)-dimensional Lebesgue measure \( \mathcal{L}^n \) on \( \mathbb{G} \cong \mathbb{R}^n \).

**Definition 2.13.** Let \( S \subset \mathbb{G} \) be a hypersurface of class \( C^1 \). We say that \( x \in S \) is a characteristic point if \( \dim H_x = \dim(H_x \cap T_x S) \) or, equivalently, if \( H_x \subset T_x S \). The characteristic set \( C_S \) of \( S \) is the set of all characteristic points, i.e. \( C_S := \{ x \in S : \dim H_x = \dim(H_x \cap T_x S) \} \).

Note that a hypersurface \( S \subset \mathbb{G} \) oriented by the outward-pointing normal vector \( \nu \) turns out to be non-characteristic if, and only if, \( H_x \subset T_x S \) is transversal to \( S \). We have to remark that the \((Q-1)\)-dimensional CC-Hausdorff measure of \( C_S \) vanishes, i.e. \( \mathcal{H}^{Q-1}_{CC}(C_S) = 0 \); see [41]. The \((n-1)\)-dimensional Riemannian measure along \( S \) can be defined by setting

\[
\sigma^{n-1}_H S := (\nu \cup \sigma^n_H)|_S,
\]

where \( \cup \) denotes the “contraction” operator on differential forms; see footnote 2. Just as in [50, 52] (see also [10, 38, 51]), since we are studying “smooth” hypersurfaces, instead of the variational definition à la De Giorgi (see, for instance, [30, 31, 32, 50] and bibliographies [50, 52]), we define an \((n-1)\)-differential form which, by integration on smooth boundaries, yields the usual \( H \)-perimeter measure.

**Definition 2.14 (\( \sigma^{n-1}_H \)-measure).** Let \( S \subset \mathbb{G} \) be a \( C^1 \) non-characteristic hypersurface and \( \nu \) the outward-pointing unit normal vector. We call unit \( H \)-normal along \( S \) the normalized projection of \( \nu \) onto \( H \), i.e. \( \nu_H := \frac{\nu}{|\nu_H|} \). The \( H \)-perimeter along \( S \) is the homogeneous measure associated with the \((n-1)\)-differential form \( \sigma^{n-1}_H \in \Lambda^{n-1}(T^* S) \) given by \( \sigma^{n-1}_H S := (\nu_H \cup \sigma^n_H)|_S \).

If we allow \( S \) to have characteristic points, one extends \( \sigma^{n-1}_H \) by setting \( \sigma^{n-1}_H C_S = 0 \). It turns out that

\[
\sigma^{n-1}_H S = |\mathcal{P}_H \nu| \sigma^{n-1}_H S
\]

and that \( C_S = \{ x \in S : |\mathcal{P}_H \nu| = 0 \} \). We also remark that

\[
\sigma^{n-1}_H (S \cap B) = k_{\rho}(\nu_H) S^{Q-1}_\rho (S \cap B),
\]

for all \( B \in \text{Bor}(\mathbb{G}) \), where the bounded density-function \( k_{\rho}(\nu_H) \), called metric factor, only depends on \( \nu_H \) and on the (fixed) homogeneous metric \( \rho \) on \( \mathbb{G} \); see [41]; see also Section 3.3.

---

4We recall that:

(i) \( \mathcal{H}_\rho(S) = \lim_{\delta \to 0^+} \mathcal{H}^m_{\rho,\delta}(S) \), where

\[
\mathcal{H}^m_{\rho,\delta}(S) = \inf \left\{ \sum_i (\text{diam}_{\rho}(C_i))^m : S \subset \bigcup_i C_i; \text{diam}_{\rho}(C_i) < \delta \right\}
\]

and the infimum is taken with respect to any non-empty family of closed subsets \( \{C_i\} \subset \mathbb{G} \);

(ii) \( \mathcal{S}_\rho(S) = \lim_{\delta \to 0^+} \mathcal{S}^m_{\rho,\delta}(S) \), where

\[
\mathcal{S}^m_{\rho,\delta}(S) = \inf \left\{ \sum_i (\text{diam}_{\rho}(B_i))^m : S \subset \bigcup_i B_i; \text{diam}_{\rho}(B_i) < \delta \right\}
\]

and the infimum is taken with respect to closed \( \rho \)-balls \( B_i \).
Definition 2.15. Setting $H_x S := H_x \cap T_x S$ for every $x \in S \setminus C_S$, yields $H_x = H_x \oplus \text{span}_R \{ \nu_h (x) \}$ and this uniquely defines the subbundles $H S$ and $\nu_h S$, called, respectively, horizontal tangent bundle and horizontal normal bundle. We have $H = HS \oplus \nu_h S$ and $\dim H_x S = \dim H - 1 = 2n - 1$ at each non-characteristic point $x \in S \setminus C_S$.

The horizontal tangent bundle $H S$ turns out to be well-defined even at the characteristic set $C_S$. More precisely, in this case $H_x S = H_x$ for every $x \in C_S$. In particular, one has $\dim H S |_{C_S} = \dim H = h$.

For the sake of simplicity, unless otherwise mentioned, we assume that $S \subset G$ is a $C^2$ non-characteristic hypersurface. We first remark that, if $\nabla^{TS}$ is the connection induced on $T S$ from the Levi-Civita connection $\nabla$ on $T G$,
\footnote{\(\nabla^{TS}\) is the Levi-Civita connection on $S$; see [11].}
then $\nabla^{TS}$ induces a “partial connection” $\nabla^{HS}$ on $H S \subset T S$, that is defined by
\[ \nabla^{HS}_X Y := \mathcal{P}_{HS} (\nabla^{TS}_X Y) \quad \forall \, X, Y \in \mathfrak{x}^1 (HS) := C^1 (S, HS). \]

Note that the orthogonal decomposition $H = HS \oplus \nu_h S$ enable us to define $\nabla^{HS}$ in analogy with the definition of “connection on submanifolds”;
\footnote{The map $\mathcal{P}_{HS} : TS \rightarrow HS$ denotes the orthogonal projection onto $HS$.}
see [34], [58], [31], [41, 45]. Furthermore, if the horizontal tangent bundle $H S$ is generic and horizontal, then the couple $(S, HS)$ turns out to be a $k$-step $CC$-space; see Section 2.1.

\[ \nabla^{HS}_X Y = \nabla^{H}_X Y - \langle \nabla^{H}_X Y, \nu_h \rangle \nu_h \quad \forall \, X, Y \in \mathfrak{x}^1 (HS). \]

Definition 2.16. Let $S \subset G$ be a $C^2$ non-characteristic hypersurface and $\nu$ the outward-pointing unit normal vector. The $HS$-gradient $\text{grad}_H \psi$ of $\psi \in C^1 (S)$ is the unique horizontal tangent vector field such that $\langle \text{grad}_H \psi, X \rangle = \text{div}_H (\psi) = \psi$ for all $X \in \mathfrak{x}^1 (HS)$. We denote by $\text{div}_HS$ the divergence operator on $HS$, i.e. if $X \in \mathfrak{x}^1 (HS)$ and $x \in S$, then
\[ \text{div}_H S (x) := \text{Trace} (Y \rightarrow \nabla^{HS}_X Y) (x) \quad (Y \in H_x S). \]

The horizontal 2nd fundamental form of $S$ is the continuous map given by $B_H (X, Y) := \langle \nabla^{H}_X Y, \nu_h \rangle$ for every $X, Y \in \mathfrak{x}^1 (HS)$. The horizontal mean curvature $H_H$ is the trace of the linear operator $B_H$, i.e. $H_H := \text{Tr} B_H = - \text{div}_H \nu_h$. We set
\[
\begin{align*}
(i) \quad & \omega_\alpha := \frac{\langle X_\alpha, \nu \rangle}{\| \nu_h \|} \quad (\nu_\alpha := \langle X_\alpha, \nu \rangle) \quad \forall \, \alpha \in I_V; \\
(ii) \quad & \omega := \frac{\partial \nu}{\| \nu_h \|} = \sum_{\alpha \in I_V} \omega_\alpha X_\alpha; \\
(iii) \quad & C_H := \sum_{\alpha \in I_H} \omega_\alpha C_H^{\alpha}. 
\end{align*}
\]

Note that $\frac{\partial \nu}{\| \nu_h \|} = \nu_h + \omega$. The horizontal 2nd fundamental form $B_H (X, Y)$ is a (continuous) bilinear form of $X$ and $Y$. However, in general, $B_H$ is not symmetric and so it can be written as a sum of two matrices, one symmetric and the other skew-symmetric, i.e. $B_H = S_H + A_H$. It turns out that $A_H = \frac{1}{2} C_H |_{HS}$; see [32].

Remark 2.17 (Induced stratification on $TS$; see [34]). The stratification of $G$ induces a natural decomposition of the tangent space of any smooth submanifold of $G$. Let us analyze the case of a hypersurface $S \subset G$. To this aim, at each point $x \in S$, we intersect $T_x S \subset T_x G$ with $T_x^i G = \bigoplus_{j=1}^i (H_j)_x$. Setting $T^i S := T S \cap T^i G$, $n_i' := \dim T^i S$, $H^i S := T^i S \setminus T^{i-1} S$ and $HS = H_1 S$, yields $TS := \bigoplus_{i=1}^k H_i S$ and $\sum_{i=1}^k n_i' = n - 1$. Henceforth, we shall set $V S := \bigoplus_{i=2}^k H_i S$. It turns out that the Hausdorff dimension of any smooth hypersurface $S$ is $Q - 1 = \sum_{i=1}^k n_i'$; see [33], [58], [31], [41] [45]. Furthermore, if the horizontal tangent bundle $HS$ is generic and horizontal, then the couple $(S, HS)$ turns out to be a $k$-step $CC$-space; see Section 2.1.
Example 2.18. Let $S \subset \mathbb{H}^n$ be a smooth hypersurface. If $n = 1$, the horizontal tangent bundle $HS$ turns out to be 1-dimensional at each non-characteristic point. But if $n > 1$, $HS$ turns out to be generic and horizontal along any non-characteristic domain $U \subset S$.

Definition 2.19. Let $N \subset G$ be a $(n - 2)$-dimensional submanifold of class $C^1$. At each point $x \in N$, the horizontal tangent space is given by $H_xN := H_x \cap T_xN$. We say that $N$ is $H$-regular, or non-characteristic, at $x \in N$, if there exist two linearly independent vectors $\nu^1_\mu, \nu^2_\mu \in H_x$ which are transversal to $N$ at $x$. Without loss of generality, we assume that $\nu^1_\mu, \nu^2_\mu$ are orthonormal and such that $|\nu^1_\mu \wedge \nu^2_\mu| = 1$. When the condition holds true for every $x \in N$, we say that $N$ is $H$-regular or non-characteristic. In this case, we define in the obvious way the associated vector bundles $HN(\subset TN)$ and $\nu_\mu N$, called, respectively, horizontal tangent bundle and horizontal normal bundle. Note that $H := HN \oplus \nu_\mu N$, where $\nu_\mu N \cong \text{span}_R(\nu^1_\mu \wedge \nu^2_\mu)$.

Definition 2.20 (see [41]). Let $N \subset G$ be a $(n - 2)$-dimensional submanifold of class $C^1$. The characteristic set $C_N$ is the set of all characteristic points of $N$. Equivalently, one has $C_N := \{x \in N : \dim H_x - \dim(H_x \cap T_xN) \leq 1\}$.

Let $N \subset G$ be a submanifold of class $C^1$; then the $(Q - 2)$-dimensional Hausdorff measure (associated with a homogeneous metric $g$ on $G$) of $C_N$ is 0, i.e. $\mathcal{H}^{Q-2}(C_N) = 0$; see [41].

Definition 2.21 ($\sigma^{n-2}_\mu$-measure). Let $N \subset G$ be a $(n - 2)$-dimensional $H$-regular submanifold of class $C^1$; let $\nu^1_\mu, \nu^2_\mu \in \nu_\mu N$ be as in Definition 2.19 and set $\nu_\mu := \nu^1_\mu \wedge \nu^2_\mu$. In particular, we are assuming that $\nu_\mu$ is a unit horizontal normal 2-vector field along $N$. Then, we define a homogeneous measure $\sigma^{n-2}_\mu$ on $N$ by setting $\sigma^{n-2}_\mu L_{|S} := (\nu_\mu \wedge \sigma^1_\mu)|_{S}$.

Equivalently, the measure $\sigma^{n-2}_\mu$ is defined to be the contraction of the top-dimensional volume form $\sigma^1_\mu$ by the unit horizontal normal 2-vector $\nu_\mu := \nu^1_\mu \wedge \nu^2_\mu$ which spans $\nu_\mu N$. The measure $\sigma^{n-2}_\mu$ can be represented in terms of the $(n - 2)$-dimensional Riemannian measure $\sigma^{n-2}_\mu$. More precisely, let $\nu N$ denote the Riemannian normal bundle of $N$ and let $\nu_1, \nu_2 \in \nu N$ be the unit normal 2-vector field orienting $N$. By standard Linear Algebra, we get that

$$\nu_\mu = \frac{\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2}{|\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2|}.$$

Moreover, it turns out that

$$\sigma^{n-2}_\mu = |\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2| \sigma^{n-2}_\mu.$$

Note that if $C_N \neq \emptyset$, then $C_N = \{x \in N : |\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2| = 0\}$ and $\sigma^{n-2}_\mu$ can be extended up to $C_N$ just by setting $\sigma^{n-2}_\mu L_{|C_N} = 0$. By construction, $\sigma^{n-2}_\mu$ is $(Q - 2)$-homogeneous with respect to Carnot dilations $\{\delta_t\}_{t>0}$, i.e. $\delta^*_t \sigma^{n-2}_\mu = t^{Q-2} \sigma^{n-2}_\mu$. Finally, it can be shown that $\sigma^{n-2}_\mu$ is equivalent, up to a bounded density-function called metric-factor, to the $(Q - 2)$-dimensional Hausdorff measure associated with a homogeneous distance $g$ on $G$; see [45].

We also recall a recent result (see [7]) about the size of horizontal tangencies to non-involutive distributions.

Theorem 2.22 (Generalized Derridj’s Theorem; see [7]). Let $G$ be a $k$-step Carnot group.

(i) If $S \subset G$ is a hypersurface of class $C^2$, then the Euclidean-Hausdorff dimension of the characteristic set $C_S$ of $S$ satisfies the inequality $\text{dim}_{\text{Eu-Hau}}(C_S) \leq n - 2$.

(ii) Let $V = H^1 \subset T^*G$ be such that $\dim V \geq 2$. If $N \subset G$ is a $(n - 2)$-dimensional submanifold of class $C^2$, then the Euclidean-Hausdorff dimension of the characteristic set $C_N$ of $N$ satisfies the inequality $\text{dim}_{\text{Eu-Hau}}(C_N) \leq n - 3$.

---

For the most general definition of $\mathcal{J}$, see [22], Ch.1. It is unique, up to the sign.
If $N \subset \mathbb{G}$ is a $(n - 2)$-dimensional submanifold, we have some general blow-up theorems by Magnani and Vittone [45] and Magnani [43].

**Theorem 2.23** (Blow-up for $(n - 2)$-dimensional submanifolds; see [45]). Let $N \subset \mathbb{G}$ be a $(n - 2)$-dimensional submanifold of class $C^{1,1}$ and let $x \in N$ be non-characteristic. Then

$$
\delta_{r}(x^{-1} \bullet N) \cap B_{r}(0,1) \to T^{2}(\nu_{h}(x)) \cap B_{r}(0,1)
$$

as long as $r \to 0^{+}$, where $T^{2}(\nu_{h}(x))$ denotes the $(n - 2)$-dimensional subgroup of $\mathbb{G}$ given by $T^{2}(\nu_{h}(x)) := \{y \in \mathbb{G} : y = \exp(Y), Y \wedge \nu_{h}(x) = 0\}$ and $\nu_{h} = \nu_{h}^{1} \wedge \nu_{h}^{2}$ is a unit horizontal normal 2-vector along $N$. If $\nu = \nu_{1} \wedge \nu_{2}$ is a unit normal 2-vector field orienting $N$, then

$$
\lim_{r \to 0^{+}} \frac{\sigma_{h}^{n-2}(N \cap B_{r}(x, r))}{r^{Q-2}} = \frac{\kappa(\nu_{h}(x))}{|P_{\nu} \nu(x)|},
$$

where $\kappa(\nu_{h}(x)) := \sigma_{h}^{n-2}(B_{r}(0,1) \cap T^{2}(\nu_{h}(x)))$ is a strictly positive and bounded density-function, called metric factor and $P_{\nu}$ denotes the orthogonal projection on horizontal 2-vectors. Moreover, if $\mathcal{H}_{Q}^{2}(C_{N}) = 0$, then

$$
\sigma_{h}^{n-2}(N) = \int_{N} \kappa(\nu_{h}(x)) dS_{Q}^{2}.
$$

It is worth observing that the convergence in (3) is understood with respect to the Hausdorff distance of sets.

## 3 Preliminary tools

### 3.1 Coarea Formula for the HS-gradient

**Theorem 3.1.** Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^{2}$ and let $\varphi \in C^{1}(S)$. Then

$$
\int_{S} \psi(x) |\text{grad}_{\text{HS}} \varphi(x)| \sigma_{h}^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \psi(y) \sigma_{h}^{n-2}(y)
$$

for every $\psi \in L^{1}(S, \sigma_{h}^{n-1})$.

**Proof.** This result can be deduced by the Riemannian Coarea Formula. Indeed, we have

$$
\int_{S} \phi(x) |\text{grad}_{\text{HS}} \varphi(x)| \sigma_{h}^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \sigma_{h}^{n-2}(y)
$$

for every $\phi \in L^{1}(S, \sigma_{h}^{n-1})$; see [8, 25]. Choosing

$$
\phi = \psi \left| \frac{\text{grad}_{\text{HS}} \varphi}{|\text{grad}_{\text{HS}} \varphi|} \right| \frac{\text{P}_{\nu}}{|\text{P}_{\nu}|},
$$

for some $\psi \in L^{1}(S, \sigma_{h}^{n-1})$, yields

$$
\int_{S} \phi |\text{grad}_{\text{HS}} \varphi| \sigma_{h}^{n-1} = \int_{S} \psi \left| \frac{\text{grad}_{\text{HS}} \varphi}{|\text{grad}_{\text{HS}} \varphi|} \right| |\text{grad}_{\text{HS}} \varphi| |\text{P}_{\nu}| \sigma_{h}^{n-1} = \int_{S} \psi |\text{grad}_{\text{HS}} \varphi| \sigma_{h}^{n-1}.
$$
Since \( \eta = \frac{\text{grad}_S \varphi}{|\text{grad}_S \varphi|} \) along \( \varphi^{-1}[s] \), it follows that \( |\mathcal{P}_{HS} \eta| = \frac{|\text{grad}_{HS} \varphi|}{|\text{grad}_S \varphi|} \). Therefore

\[
\int_R ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \sigma_R^{n-2} = \int_R ds \int_{\varphi^{-1}[s] \cap S} \psi \frac{|\text{grad}_{HS} \varphi|}{|\text{grad}_S \varphi|} |\mathcal{P}_{H \nu}| \sigma_R^{n-2} = \int_R ds \int_{\varphi^{-1}[s] \cap S} \psi \left|\mathcal{P}_{HS} \eta\right| |\mathcal{P}_{H \nu}| \sigma_R^{n-2} = \sigma_R^{n-2} \psi \sigma_R^{n-2}.
\]

\[\square\]

### 3.2 Horizontal divergence theorem and first variation of \( \sigma_R^{n-1} \)

Let \( S \subset \mathbb{G} \) be a \( \mathbb{C}^2 \) hypersurface and let \( U \subset S \) be a relatively compact open set with (piecewise) \( \mathbb{C}^1 \) boundary. If \( X \in \mathfrak{X}^1(TS) := \mathbb{C}^1(S, TS) \), using the very definition of \( \sigma_R^{n-1} \) together with the “infinitesimal” Riemannian Divergence Formula (see [54]), yields

\[
d(X \bigodot \sigma_R^{n-1})|_U = d(|\mathcal{P}_{H \nu}| X \bigodot \sigma_R^{n-1}) = \text{div}_{TS} (|\mathcal{P}_{H \nu}| X) \sigma_R^{n-1} = \left( \text{div}_{TS} X + \left\langle X, \frac{\text{grad}_S |\mathcal{P}_{H \nu}|}{|\mathcal{P}_{H \nu}|} \right\rangle \right) \sigma_R^{n-1} L U,
\]

where \( \text{grad}_{TS} \) and \( \text{div}_{TS} \) denote the tangential gradient and the tangential divergence. By using formula (5) and Stokes’ formula, one obtains a integration by parts formula for the \( H \)-perimeter. However, in the sub-Riemannian setting, we can do something more intrinsic. At this regard, we now discuss a horizontal integration by parts formula; see [50], [52] or [21].

**Remark 3.2** (The measure \( \sigma_R^{n-2} \) along \( \partial S \)). Let \( S \subset \mathbb{G} \) be a hypersurface of class \( \mathbb{C}^2 \) with piecewise \( \mathbb{C}^1 \) boundary \( \partial S \). Let \( \eta \in TS \) be the outward-pointing unit normal vector along \( \partial S \) and denote by \( \sigma_R^{n-2} \) the Riemannian measure on \( \partial S \), given by \( \sigma_R^{n-2} L \partial S = (\eta \bigodot \sigma_R^{n-1})|_{\partial S} \). We recall that \( (X \bigodot \sigma_R^{n-1})|_{\partial S} = \left\langle X, \eta \right\rangle |\mathcal{P}_{H \nu}| \sigma_R^{n-2} L \partial S \) for every \( X \in \mathfrak{X}^1(TS) \). The characteristic set \( C_{\partial S} \) of \( \partial S \) turns out to be given by \( C_{\partial S} = \{ p \in \partial S : |\mathcal{P}_{H \nu}| |\mathcal{P}_{H \eta} \left| = 0 \right. \} \). Furthermore, by applying Definition [2.21] one has

\[
\sigma_R^{n-2} L \partial S = \left( \frac{\mathcal{P}_{HS} \eta}{|\mathcal{P}_{HS} \eta|} \bigodot \sigma_R^{n-1} \right)|_{\partial S},
\]

or, equivalently \( \sigma_R^{n-2} L \partial S = |\mathcal{P}_{H \nu}| |\mathcal{P}_{H \eta} \sigma_R^{n-2} L \partial S \). The unit horizontal normal along \( \partial S \) is given by \( \eta_{HS} := \frac{\mathcal{P}_{HS} \eta}{|\mathcal{P}_{HS} \eta|} \). Note that \( (X \bigodot \sigma_R^{n-1})|_{\partial S} = \left\langle X, \eta_{HS} \right\rangle \sigma_R^{n-2} L \partial S \) for every \( X \in \mathfrak{X}^1(HS) \).

**Theorem 3.3** (Horizontal divergence theorem). Let \( \mathbb{G} \) be a k-step Carnot group. Let \( S \subset \mathbb{G} \) be an immersed hypersurface of class \( \mathbb{C}^2 \) and let \( U \subset S \setminus C_S \) be a non-characteristic relatively compact open set. We assume that \( \partial U \) is a (piecewise) \( \mathbb{C}^1 \)-smooth \( (n-2) \)-dimensional manifold oriented by the outward-pointing unit normal \( \eta \). Then, we have

\[
\int_U \left( \text{div}_{HS} X + (C_H \nu_H, X) \right) \sigma_R^{n-1} = -\int_U \mathcal{H}_U \left\langle X, \nu_H \right\rangle \sigma_R^{n-1} + \int_{\partial U} \left\langle X, \eta_{HS} \right\rangle \sigma_R^{n-2}
\]

for every \( X \in \mathfrak{X}^1(H) \).

The proof can be found in [52]. We remark that this formula extends to the characteristic case; see [54]. Actually, this formula is a particular case of a more general integral formula, the so-called first variation formula for the \( H \)-perimeter.
Definition 3.4. Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathcal{C}^2$. Let $\vartheta : S \to \mathbb{G}$ be the inclusion of $S$ in $\mathbb{G}$ and let $\vartheta : \mathbb{I} \to \mathbb{G}$ be the $\mathcal{C}^2$-smooth map. We say that $\vartheta$ is a variation of $\vartheta$ if, and only if:

(i) every $\vartheta_t := \vartheta(t, \cdot) : S \to \mathbb{G}$ is an immersion;

(ii) $\vartheta_0 = \vartheta$.

We say that $\vartheta$ keeps the boundary $\partial S$ fixed if:

(iii) $\vartheta_t|_{\partial S} = \vartheta|_{\partial S}$ for every $t \in ] - \epsilon, \epsilon [$. The variation vector of $\vartheta$ is defined by $X := \frac{\partial \vartheta}{\partial t}|_{t=0}$ and we also set $\tilde{X} = \frac{\partial \vartheta}{\partial t}$.

Notation 3.5. Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathcal{C}^2$. Let $X \in \mathfrak{X}(\mathbb{G})$ and let $\nu$ be the outward-pointing unit normal vector along $S$. Hereafter, we shall denote by $X^\perp$ and $X^\top$ the standard decomposition of $X$ into its normal and tangential components, i.e. $X = (X, \nu) \nu$ and $X^\top = X - X^\perp$.

By definition, the 1st variation formula of $\sigma^n_{\nu^{-1}}$ along $S$ is given by

\[ I_S(\sigma^n_{\nu^{-1}}) := \frac{d}{dt} \left( \int_S \vartheta_t^* (\sigma^n_{\nu^{-1}}) \right) \Big|_{t=0}, \]  

(6)

where $\vartheta_t^*$ denotes the pull-back by $\vartheta_t$ and $(\sigma^n_{\nu^{-1}})_t$ denotes the $H$-perimeter along $S_t := \vartheta_t(S)$.

A natural question arises: is it possible to bring the time-derivatives inside the integral sign? Clearly, if we assume that $\mathfrak{F}$ is non-characteristic, then the answer is affirmative. In the general case, we can argue as follows. We first note that

\[ \int_S \vartheta_t^* (\sigma^n_{\nu^{-1}})_t = \int_S |\mathcal{P}_{\nu_t} \nu'_t| \mathcal{J}ac \vartheta_t \sigma^n_{\nu^{-1}}, \]

where $\mathcal{J}ac \vartheta_t$ denotes the usual Jacobian of the map $\vartheta_t$; see [33], Ch. 2, § 8, pp. 46-48. Indeed, by definition, we have $(\sigma^n_{\nu^{-1}})_t = |\mathcal{P}_{\nu_t} \nu'_t| (\sigma^n_{\nu^{-1}})_t$ and hence the previous formula follows from the well-known Area formula of Federer; see [25] or [63]. Let us set $f := ] - \epsilon, \epsilon [ \times S \to \mathbb{R}$,

\[ f(t, x) := |\mathcal{P}_{\nu_t} \nu'_t(x)| \mathcal{J}ac \vartheta_t(x). \]  

(7)

In this case, we also set $C_S := \{ x \in S : |\mathcal{P}_{\nu_t} \nu'_t(x)| = 0 \}$. With this notation, our original question can be solved by applying to $f$ the Theorem of Differentiation under the integral; see, for instance, [39], Corollary 1.2.2, p.124. More precisely, let us compute

\[ \frac{df}{dt} = \frac{d |\mathcal{P}_{\nu_t} \nu'_t|}{dt} \mathcal{J}ac \vartheta_t + |\mathcal{P}_{\nu_t} \nu'_t| \frac{d \mathcal{J}ac \vartheta_t}{dt} \]  

(8)

\[ = \left( \mathcal{X}, \text{grad} |\mathcal{P}_{\nu_t} \nu'_t| \right) \mathcal{J}ac \vartheta_t + |\mathcal{P}_{\nu_t} \nu'_t| \frac{d \mathcal{J}ac \vartheta_t}{dt} \]

\[ = \left( \left( \mathcal{X}^\perp, \text{grad} |\mathcal{P}_{\nu_t} \nu'_t| \right) + \left( \mathcal{X}^\top, \text{grad} |\mathcal{P}_{\nu_t} \nu'_t| \right) \right) \mathcal{J}ac \vartheta_t \]

\[ = \left( \left( \mathcal{X}^\perp, \text{grad} |\mathcal{P}_{\nu_t} \nu'_t| \right) + \text{div}_{TS_t} \mathcal{X} \right) \mathcal{J}ac \vartheta_t, \]

where we have used the very definition of tangential divergence and the well-known calculation of $\frac{d \mathcal{J}ac \vartheta_t}{dt}$, which can be found in Chavel’s book [12]; see Ch.2, p.34. Now since $|\mathcal{P}_{\nu_t} \nu'_t|$ is a Lipschitz continuous function, it follows that $\frac{df}{dt}$ is bounded on $S \setminus C_S$ and so lies to $I^1_{loc}(S; \sigma^n_{\nu^{-1}})$. This shows that: \textit{we can pass the time-derivative through the integral sign.}

At this point the 1st variation formula follows from the calculation of the Lie derivative of $\sigma^n_{\nu^{-1}}$ with respect to the initial velocity $X$ of the flow $\vartheta_t$. 
Remark 3.6. Let $M$ be a smooth manifold, let $\omega \in \Lambda^k(T^*M)$ be a differential $k$-form on $M$ and let $X \in \mathfrak{X}(TM)$ be a differentiable vector field on $M$, with associated flow $\phi_t : M \rightarrow M$. We recall that the Lie derivative of $\omega$ with respect to $X$, is defined by $\mathcal{L}_X \omega := \frac{d}{dt}\phi_t^* \omega|_{t=0}$, where $\phi_t^* \omega$ denotes the pull-back of $\omega$ by $\phi_t$. In other words, the Lie derivative of $\omega$ under the flow generated by $X$ can be seen as the “infinitesimal 1st variation” of $\omega$ with respect to $X$. Then, Cartan’s identity says that

$$\mathcal{L}_X \omega = (X \lrcorner \ d\omega) + d(X \lrcorner \ \omega).$$

This formula is a very useful tool in proving variational formulas. For the case of Riemannian volume forms, we refer the reader to Spivak’s book [64]; see Ch. 9, pp. 411-426 and 513-535.

The Lie derivative of the differential $(n-1)$-form $\sigma^{n-1}_H$ with respect to $X$ can be calculated elementarily as follows. We have

$$X \lrcorner \ d\sigma^{n-1}_H = X \lrcorner \ d(\nu_H \lrcorner \ \sigma^n_H) = X \lrcorner \ (\text{div} \ \nu_H \sigma^n_H) = \langle X, \nu \rangle \text{div} \ \nu_H \sigma^{n-1}_R.$$

Note that $\text{div} \ \nu_H = \text{div}_H \nu_H = -\mathcal{H}_H$. More precisely, we have

$$\text{div} \ \nu_H = \sum_{i=1}^n \langle \nabla_{X_i} \nu_H, X_i \rangle = \sum_{i=1}^n \sum_{j=1}^h X_i(\nu_{H_{i,j}}) = \text{div}_H \nu_H = -\mathcal{H}_H.$$

The second term in Cartan’s identity can be computed using the following:

**Lemma 3.7.** If $X \in \mathfrak{X}^1(T\mathbb{G})$, then $(X \lrcorner \ \sigma^{n-1}_H)|_S = ((X^\top | \mathcal{P}_H \nu - \langle X, \nu \rangle \nu^\top_H) \lrcorner \ \sigma^{n-1}_R)|_S$. Moreover, at each non-characteristic point of $S$, we have

$$d(X \lrcorner \ \sigma^{n-1}_H)|_S = d_{TS} \left( X^\top | \mathcal{P}_H \nu - \langle X, \nu \rangle \nu^\top_H \right) \sigma^{n-1}_R |_S.$$

**Proof.** We have

$$d(X \lrcorner \ \sigma^{n-1}_H)|_S = (X \lrcorner \nu_H \lrcorner \ \sigma^n_H)|_S$$

$$= d \left( (X^\top + X^\perp) \lrcorner \nu_H^\top \lrcorner \ \sigma^n_H \right)|_S$$

$$= \left( X^\top \lrcorner \nu_H^\top \lrcorner \ \sigma^n_H \right)|_S + d \left( \nu_H^\top \lrcorner X^\perp \lrcorner \sigma^n_H \right)|_S$$

$$= \left( X^\top \lrcorner \sigma^{n-1}_H \right)|_S + d \left( \nu_H^\top \langle X, \nu \rangle \sigma^{n-1}_R \right)|_S$$

$$= d_{TS} \left( X^\top | \mathcal{P}_H \nu - \langle X, \nu \rangle \nu^\top_H \right) \sigma^{n-1}_R |_S.$$

\[\square\]

Remark 3.8. The previous calculation corrects a mistake in [22], where the normal component of the vector field $X$ was omitted and this caused the loss of some divergence-type terms in the variational formulas proved there.

Thus, we can conclude that

$$\mathcal{L}_X \sigma^{n-1}_H = \left( -\mathcal{H}_H \langle X, \nu \rangle + d_{TS} \left( X^\top | \mathcal{P}_H \nu - \langle X, \nu \rangle \nu^\top_H \right) \right) \sigma^{n-1}_R,$$

at each non-characteristic point of $S$. Furthermore, if $\mathcal{H}_H \in L^1_{\text{loc}}(S; \sigma^{n-1}_R)$, we can integrate this formula over all of $S$. Indeed, in this case, all terms in the formula above turn out to be in $L^1(S; \sigma^{n-1}_R)$; see also [54].
Theorem 3.9 (1st variation of $\sigma_{n-1}^H$). Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$ with -or without- boundary $\partial S$ and let $\vartheta : - \epsilon, \epsilon ] \times S \to \mathbb{G}$ be a $C^2$-smooth variation of $S$. Let $X = \frac{d\vartheta}{dt} |_{t=0}$ be the variation vector field and denote by $X^\perp$ and $X^\top$ the normal and tangential components of $X$ along $S$, respectively. If $\mathcal{H}_H \in L^1(S; \sigma_{n-1}^H)$, then

$$I_S(X, \sigma_{n-1}^H) = \int_S \left( -\mathcal{H}_H \langle X^\perp, \nu \rangle + \text{div}_S \left( X^\top |\mathcal{P}_H \nu | - \langle X^\perp, \nu \rangle \nu^\top \right) \right) \sigma_{n-1}^H \quad (10)$$

$$\quad = \int_S -\mathcal{H}_H \langle X^\perp, \nu \rangle |\mathcal{P}_H \nu | \sigma_{n-1}^H + \int_{\partial S} \left( \left( X^\top - \frac{\langle X^\perp, \nu \rangle \nu^\top}{|\mathcal{P}_H \nu |} \right) \cdot \frac{\eta}{|\mathcal{P}_{\mathcal{H}_S} \eta|} \right) |\mathcal{P}_H \nu | |\mathcal{P}_{\mathcal{H}_S} \eta| \sigma_{n-2}^H. \quad (11)$$

The second equality follows by applying the generalized Stokes’ formula stated in the next:

Proposition 3.10. Let $M$ be an oriented $k$-dimensional manifold of class $C^2$ with boundary $\partial M$. Then $\int_M d\alpha = \int_{\partial M} \alpha$ for every compactly supported $(k-1)$-form $\alpha$ such that $\alpha \in L^\infty(M)$, $d\alpha \in L^1(M)$ -or $d\alpha \in L^\infty(M)$- and $i_M^* \alpha \in L^\infty(\partial M)$, where $i_M : \partial M \to M$ is the natural inclusion.

Remark 3.11. This result can be deduced by applying a standard procedure\cite{3} from a divergence-type theorem proved by Anzellotti; see, more precisely, Theorem 1.9 in \cite{4}. More recent and more general results can be found in the paper by Chen, Torres and Ziemer \cite{15}. See also \cite{67}, formula (G.38), Appendix G.

3.3 Blow-up of the horizontal perimeter $\sigma_{n-1}^H$ up to $C_S$

Let $S \subset \mathbb{G}$ be a hypersurface. In this section we shall study the following density-limit:

$$\lim_{r \to 0^+} \frac{\sigma_{n-1}^H(S \cap B_\varrho(x, r))}{r^{Q-1}}, \quad (12)$$

where $B_\varrho(x, r)$ is the $\varrho$-ball of center $x \in \text{Int} S$ and radius $r$. Note that the point $x$ is not necessarily non-characteristic. For a similar analysis, we refer the reader to \cite{41, 45, 44} and to \cite{43}, for the characteristic case in the setting of 2-step Carnot groups; see also \cite{6, 60, 30, 31}.

Theorem 3.12. Let $\mathbb{G}$ be a $k$-step Carnot group.

Case (i) Let $S$ be a hypersurface of class $C^1$ and let $x \in \text{Int}(S \setminus C_S)$; then

$$\sigma_{n-1}^H(S \cap B_\varrho(x, r)) \sim \kappa_\varrho(\nu_H(x)) r^{Q-1} \quad \text{for} \quad r \to 0^+, \quad (13)$$

where the density-function $\kappa_\varrho(\nu_H(x))$ is called metric factor. It turns out that

$$\kappa_\varrho(\nu_H(x)) = \sigma_{n-1}^H (\mathcal{I}(\nu_H(x)) \cap B_\varrho(x, 1)), \quad \text{where} \quad \mathcal{I}(\nu_H(x)) \text{ denotes the vertical hyperplane}^{10} \text{ through } x \text{ and orthogonal to } \nu_H(x).$$

Case (ii) Let $x \in \text{Int}(S \cap C_S)$ and let $\varrho \in I_v$ be such that, locally around $x$, $S$ can be represented as an $X_{\alpha_i}$-graph of class $C^i$, where $i = \text{ord}(\alpha) \in \{2, \ldots, k\}$. In this case, we have

$$S \cap B_\varrho(x, r) \subset \exp \left\{ (\zeta_1, \ldots, \zeta_{\alpha-1}, \psi(\zeta), \zeta_{\alpha+1}, \ldots, \zeta_n) : \zeta := (\zeta_1, \ldots, \zeta_{\alpha-1}, 0, \zeta_{\alpha+1}, \ldots, \zeta_n) \in e_\alpha^1 \right\},$$

\[9\text{See, for instance, Federer} \cite{25}, \text{paragraph 3.2.46, p. 280; see also} \cite{60}, \text{Remark 5.3.2, p. 197.}\]

\[10\text{Note that } \mathcal{I}(\nu_H(x)) \text{ corresponds to an ideal of the Lie algebra } g. \text{ We also remark that the } H \text{-perimeter on a vertical hyperplane equals the Euclidean-Hausdorff measure } \mathcal{H}_{Ew}^{n-1} \text{ on the hyperplane.}\]
it follows that

We shall set:

for any \( x \in \mathbb{G} \) and \( \psi(0) = 0 \). If

\[
\frac{\partial^{(i)} \psi}{\partial \zeta_{j_1} \ldots \partial \zeta_{j_i}}(0) = 0 \quad \text{whenever} \quad \text{ord}(j_1) + \ldots + \text{ord}(j_i) < i,
\]

then

\[
\sigma_{\mu}^{n-1}(S \cap B_{\varrho}(x, r)) \sim \kappa_{\varrho}(C_S(x)) \, r^{Q-1} \quad \text{as long as} \quad r \to 0^+,
\]

where the constant \( \kappa_{\varrho}(C_S(x)) \) can be computed by integrating the measure \( \sigma_{\mu}^{n-1} \) along a polynomial hypersurface, which is the graph of the Taylor’s expansion up to \( i = \text{ord}(\alpha) \) of \( \psi \) at \( \zeta = 0 \in e_{\alpha}^\perp \). More precisely, one has

\[
\kappa_{\varrho}(C_S(x)) = \sigma_{\mu}^{n-1}(S_\infty \cap B_{\varrho}(x, 1)),
\]

where \( S_\infty = \{(\zeta_1, \ldots, \zeta_{\alpha-1}, \tilde{\psi}^o(\zeta), \zeta_{\alpha+1}, \ldots, \zeta_{\alpha}) : \zeta \in e_{\alpha}^\perp \} \) and

\[
\tilde{\psi}^o(\zeta) = \sum_{\text{ord}(j_1) = i} \frac{\partial \psi}{\partial \zeta_{j_1}}(0) \zeta_{j_1} + \ldots + \sum_{\text{ord}(j_1) = i} \frac{\partial^{(i)} \psi}{\partial \zeta_{j_1} \ldots \partial \zeta_{j_i}}(0) \zeta_{j_1} \ldots \zeta_{j_i}.
\]

Finally, if (14) does not hold, then \( S_\infty = \emptyset \) and \( \kappa_{\varrho}(C_S(x)) = 0 \).

**Remark 3.13** (Order of \( x \in S \)). The rescaled hypersurfaces \( \delta_1 S \) locally converge to a limit-set \( S_\infty \), i.e.

\[
\delta_1 S \to S_\infty \quad \text{for} \quad r \to 0^+,
\]

where the convergence is understood with respect the Hausdorff convergence of sets; see \([45, 43]\). At every \( x \in \text{Int}(S \setminus C_S) \) the limit-set \( S_\infty \) is the vertical hyperplane \( \tilde{I}(\nu_S(x)) \). Otherwise, \( S_\infty \) is a polynomial hypersurface. Assume that \( S \) is smooth enough near its characteristic set \( C_S \), say of class \( C^k \). Then, there exists a minimum \( i = \text{ord}(\alpha) \) such that (14) holds. The number \( \text{ord}(x) = Q - i \) is called the order of the characteristic point \( x \in C_S \).

**Proof of Theorem 3.12**. We preliminarily note that the limit (12) can be computed, without loss of generality, at \( 0 \in \mathbb{G} \), just by left-translating \( S \). We have

\[
\sigma_{\mu}^{n-1}(S \cap B_{\varrho}(x, r)) = \sigma_{\mu}^{n-1}(x^{-1} \bullet (S \cap B_{\varrho}(x, r))) = \sigma_{\mu}^{n-1}((x^{-1} \bullet S) \cap B_{\varrho}(0, r))
\]

for any \( x \in \text{Int} S \), where the second equality follows from the additivity of the group law \( \bullet \).

**Notation 3.14. We shall set:**

(i) \( S_r(x) := S \cap B_{\varrho}(x, r) \);

(ii) \( \tilde{S} := x^{-1} \bullet S \);

(iii) \( \tilde{S}_r := x^{-1} \bullet S_r(x) = \tilde{S} \cap B_{\varrho}(0, r) \).

By using the homogeneity of \( \varrho \) and the invariance of \( \sigma_{\mu}^{n-1} \) under positive Carnot dilations\(^{11}\), it follows that

\[
\sigma_{\mu}^{n-1}(\tilde{S}_r) = r^{Q-1} \sigma_{\mu}^{n-1}(S_{1/r} \cap B_{\varrho}(0, 1))
\]

\(^{11}\)This means that \( \delta_t \sigma_{\mu}^{n-1} = t^{Q-1} \sigma_{\mu}^{n-1}, t \in \mathbb{R}_+ ; \) see Section 2.4.
for all \( r \geq 0 \). Therefore
\[
\sigma^{n-1}_{\mu} (\hat{\delta}_t) = \sigma^{n-1}_{\mu} \left( \delta_{1/r} \hat{S} \cap B_g(0,1) \right),
\]
and it remains to compute
\[
\lim_{r \to 0^+} \sigma^{n-1}_{\mu} \left( \delta_{1/r} \hat{S} \cap B_g(0,1) \right).
\] (16)

We begin by studying the non-characteristic case; see also [43, 44].

Case (i). Blow-up for non-characteristic points. Let \( S \subset \mathbb{G} \) be a hypersurface of class \( C^1 \) and let \( x \in \text{Int} S \) be non-characteristic. Locally around \( x \), the hypersurface \( S \) is oriented by unit \( H \)-normal \( \nu_{\mu}(x) \), i.e. \( \nu_{\mu}(x) \) is transversal to \( S \) at \( x \). Thus, at least locally around \( x \), we may think of \( S \) as a \( C^1 \)-graph with respect to the horizontal direction \( \nu_{\mu}(x) \). Moreover, we can find an orthonormal change of coordinates on \( g \cong T_0 \mathbb{G} \) such that
\[
e_1 = X_1(0) = (L_{x^{-1}})_* \nu_{\mu}(x).
\]

With no loss of generality, by the Implicit Function Theorem we can write \( \tilde{S}_r = x^{-1} \bullet S_r(x) \), for some (small enough) \( r > 0 \), as the exponential image in \( \mathbb{G} \) of a \( C^1 \)-graph. So let
\[
\Psi = \{ (\psi(\xi), \xi) : \xi \in \mathbb{R}^{n-1} \} \subset g,
\]
where \( \psi : e_1^+ \cong \mathbb{R}^{n-1} \to \mathbb{R} \) is a \( C^1 \)-function satisfying:

(i) \( \psi(0) = 0 \);

(ii) \( \partial \psi / \partial \xi_j(0) = 0 \) for every \( j = 2, ..., h \) (\( = \dim H \)),

for \( \xi \in e_1^+ \cong \mathbb{R}^{n-1} \). Therefore \( \tilde{S}_r = \exp_r \Psi \cap B_g(0,r) \), for all (small enough) \( r > 0 \). This remark can be used to compute the limit (16). So let us fix a positive \( r_0 \) satisfying the previous assumptions and let \( 0 \leq r \leq r_0 \). Then
\[
\delta_{1/r} \hat{S} \cap B_g(0,1) = \exp \left( \delta_{1/r} \Psi \right) \cap B_g(0,1),
\] (17)
where \( \{ \delta_t \}_{t \geq 0} \) are the induced dilations on \( g \), i.e. \( \delta_t = \exp \circ \delta_t \) for every \( t \geq 0 \). Henceforth, we will consider the restriction of \( \delta_t \) to the hyperplane \( e_1^+ \cong \mathbb{R}^{n-1} \). So, with a slight abuse of notation, instead of \( (\delta_t)_{|e_1^+}(\xi) \) we shall write \( \delta_t \xi \). Moreover, we shall assume \( \mathbb{R}^{n-1} = \mathbb{R}^{h-1} \oplus \mathbb{R}^{n-h} \). Note that the induced dilations \( \{ \delta_t \}_{t \geq 0} \) make \( e_1^+ \cong \mathbb{R}^{n-1} \) a graded vector space, whose grading respects that of \( g \). We have
\[
\delta_{1/r} \Psi = \delta_{1/r} \{ (\psi(\xi), \xi) : \xi \in \mathbb{R}^{n-1} \} = \left\{ \left( \frac{\psi(\xi)}{r}, \delta_{1/r} \xi \right) : \xi \in \mathbb{R}^{n-1} \right\}.
\]

By using the change of variables \( \zeta := \delta_{1/r} \xi \), we get that
\[
\delta_{1/r} \Psi = \left\{ \left( \frac{\psi(\zeta)}{r}, \zeta \right) : \zeta \in \mathbb{R}^{n-1} \right\}.
\]

By hypothesis \( \psi \in C^1(U_0) \), where \( U_0 \) is a suitable open neighborhood of \( 0 \in \mathbb{R}^{n-1} \). Using a Taylor’s expansion of \( \psi \) at \( 0 \in \mathbb{R}^{n-1} \) and the assumptions (i) and (ii), yields
\[
\psi(\xi) = \psi(0) + \langle \text{grad}_{\mathbb{R}^{n-1}} \psi(0), \xi \rangle_{\mathbb{R}^{n-1}} + o(\| \xi \|_{\mathbb{R}^{n-1}}) = \langle \text{grad}_{\mathbb{R}^{n-h}} \psi(0), \xi_{\mathbb{R}^{n-h}} \rangle_{\mathbb{R}^{n-h}} + o(\| \xi \|_{\mathbb{R}^{n-1}}),
\]
where \( \text{grad}_{\mathbb{R}^{n-1}} \psi(0) \) is a unit normal vector along \( S \).

We say that \( X \) is transversal to \( S \) at \( x \), in symbols \( X \not\subset T_x S \), if \( \langle X, \nu \rangle \neq 0 \) at \( x \), where \( \nu \) is a unit normal vector along \( S \).
as long as $\xi \to 0 \in \mathbb{R}^{n-1}$. Note that $\hat{\delta}_r \xi \to 0 \in \mathbb{R}^{n-1}$ for $r \to 0^+$. By the previous change of variables, we get that

$$
\psi(\hat{\delta}_r \xi) = \left\langle \text{grad}_{\mathbb{R}^{n-h}} \psi(0), \hat{\delta}_r (\xi_{\mathbb{R}^{n-h}}) \right\rangle_{\mathbb{R}^{n-h}} + o(r)
$$

for $r \to 0^+$. Since $\left\langle \text{grad}_{\mathbb{R}^{n-h}} \psi(0), \hat{\delta}_r (\xi_{\mathbb{R}^{n-h}}) \right\rangle_{\mathbb{R}^{n-h}} = o(r)$ for $r \to 0^+$, we easily get that the limit-set (obtained by blowing-up $\hat{S}$ at the non-characteristic point 0) is given by

$$
\Psi_\infty = \lim_{r \to 0^+} \hat{\delta}_{1/r} \Psi = \exp(e_1^+) = \mathcal{I}(X_1(0)),
$$

where $\mathcal{I}(X_1(0))$ denotes the vertical hyperplane through the identity 0 $\in \mathbb{G}$ and orthogonal to $X_1(0)$. We have shown that (16) can be computed by means of (17) and (18). More precisely

$$
\lim_{r \to 0^+} \sigma^{n-1}_h \left( \hat{\delta}_{1/r} \hat{S} \cap B_{\mathbb{g}}(0,1) \right) = \sigma^{n-1}_h (\mathcal{I}(X_1(0)) \cap B_{\mathbb{g}}(0,1))
$$

By remembering the previous change of variables, it follows that $S_\infty = \mathcal{I}(\nu_h(x))$ and that

$$
\kappa_\mathcal{H}(\nu_h(x)) = \lim_{r \to 0^+} \frac{\sigma^{n-1}_h (S \cap B_{\mathbb{g}}(x,r))}{r^{q-1}} = \sigma^{n-1}_h (\mathcal{I}(\nu_h(x)) \cap B_{\mathbb{g}}(x,1))
$$

which was to be proven.

Case (ii). Blow-up at the characteristic set. We are now assuming that $S \subset \mathbb{G}$ is a hypersurface of class $C^i$ for some $i \geq 2$ and that $x \in \text{Int}(S \cap C_S)$. Near $x$ the hypersurface $S$ is oriented by some vertical vector. Hence, at least locally around $x$, we may think of $S$ as the exponential image of a $C^i$-graph with respect to some vertical direction $X_\alpha$ transversal to $S$ at $x$. Note that $X_\alpha$ is a vertical left-invariant vector field of the fixed left-invariant frame $\mathbb{X} = \{X_1, \ldots, X_n\}$ and $\alpha \in I_V = \{h+1, \ldots, n\}$ denotes a “vertical” index; see Notation 2.3. Furthermore, we are assuming that $\text{ord}(\alpha) := i$, for some $i = 2, \ldots, k$. As in the non-characteristic case, for the sake of simplicity, we left-translate $S$ in such a way that $x$ coincides with 0 $\in \mathbb{G}$. To this end, it is sufficient to replace $S$ by $\tilde{S} = x^{-1} \cdot S$. At the level of the Lie algebra $\mathfrak{g}$, let us consider the hyperplane $e_\alpha^+$ through the origin 0 $\in \mathfrak{g}$ $\cong \mathbb{R}^n$ and orthogonal to $e_\alpha = X_\alpha(0)$. Note that $e_\alpha^+$ is the natural “parameter space” of a $e_\alpha$-graph. By the classical Implicit Function Theorem, we may write $\tilde{S}_r = x^{-1} \cdot S_r(x)$ as the exponential image in $\mathbb{G}$ of a $C^i$-graph. We have

$$
\Psi = \left\{ \left( \xi_1, \ldots, \xi_{\alpha - 1}, \psi(\xi), \xi_{\alpha + 1}, \ldots, \xi_n \right)_{\alpha-th\ place} : \xi := (\xi_1, \ldots, \xi_{\alpha - 1}, 0, \xi_{\alpha + 1}, \ldots, \xi_n) \in e_\alpha^+ \cong \mathbb{R}^{n-1} \right\}
$$

where $\psi : e_\alpha^+ \cong \mathbb{R}^{n-1} \to \mathbb{R}$ is a $C^i$-smooth function satisfying:

(j) $\psi(0) = 0$;

(jj) $\partial \psi / \partial \xi_j(0) = 0$ for every $j = 1, \ldots, h (= \dim H)$.

Thus we get that $\tilde{S}_r = \exp \Psi \cap B_{\mathbb{g}}(0,r)$, for every (small enough) $r > 0$. Hence, we can use the above remarks to compute (16) and as in the non-characteristic case, we use (17). We have

$$
\tilde{\delta}_{1/r} \Psi = \tilde{\delta}_{1/r} \left\{ (\xi_1, \ldots, \xi_{\alpha - 1}, \psi(\xi), \xi_{\alpha + 1}, \ldots, \xi_n) : \xi \in e_\alpha^+ \right\}
$$

= \left\{ \left( \frac{\xi_1}{r}, \ldots, \frac{\xi_{\alpha - 1}}{r^{\text{ord}(\alpha - 1)}}, \frac{\psi(\xi)}{r}, \frac{\xi_{\alpha + 1}}{r^{\text{ord}(\alpha + 1)}}, \ldots, \frac{\xi_n}{r^k} \right) : \xi \in e_\alpha^+ \right\}.
$$
Setting
\[ \zeta := \hat{\delta}_{1/r} \xi = \left( \frac{\xi_1}{r}, \ldots, \frac{\xi_{\alpha-1}}{r^{\text{ord}(\alpha-1)}}, 0, \frac{\xi_{\alpha+1}}{r^{\text{ord}(\alpha+1)}}, \ldots, \frac{\xi_n}{r^k} \right), \]
where \( \zeta = (\zeta_1, \ldots, \zeta_{\alpha-1}, 0, \zeta_{\alpha+1}, \ldots, \zeta_n) \in e^+_{\alpha} \), yields
\[ \hat{\delta}_{1/r} \Psi = \left\{ \left( \zeta_1, \ldots, \zeta_{\alpha-1}, \frac{\psi(\hat{\delta}_r \zeta)}{r^i}, \zeta_{\alpha+1}, \ldots, \zeta_n \right) : \zeta \in e^+_{\alpha} \right\}. \]

By hypothesis \( \psi \in C^1(U_0) \), where \( U_0 \) is an open neighborhood of \( 0 \in e^+_{\alpha} \cong \mathbb{R}^{n-1} \). Furthermore, one has \( \delta_r \zeta \to 0 \) as long as \( r \to 0^+ \). So we have to study the following limit
\[
\hat{\psi}(\zeta) := \lim_{r \to 0^+} \frac{\psi(\hat{\delta}_r \zeta)}{r^i}, \tag{19}
\]
whenever exists. The first remark is that, when this limit equals \( +\infty \), we have
\[
\lim_{r \to 0^+} \frac{\sigma_{n-1}^{r^{-1}}(\delta_r)}{r^{Q-1}} = \lim_{r \to 0^+} \sigma_{n-1}^{r^{-1}} \left( \exp \left( \hat{\delta}_{1/r} \Psi \right) \cap B_\rho(0,1) \right) = 0,
\]
because \( \exp \left( \hat{\delta}_{1/r} \Psi \right) \cap B_\rho(0,1) \to \emptyset \) as long as \( r \to 0^+ \).

At this point, making use of a Taylor’s expansion of \( \psi \) together with (j) and (jj), yields
\[
\psi(\hat{\delta}_r \zeta) = \psi(0) + \sum_{j_1} r^{\text{ord}(j_1)} \frac{\partial \psi}{\partial \zeta_{j_1}}(0) \zeta_{j_1} + \sum_{j_1, j_2} r^{\text{ord}(j_1)+\text{ord}(j_2)} \frac{\partial^2 \psi}{\partial \zeta_{j_1} \partial \zeta_{j_2}}(0) \zeta_{j_1} \zeta_{j_2}
\]
\[
+ \ldots + \sum_{j_1, \ldots, j_i} r^{\text{ord}(j_1)+\ldots+\text{ord}(j_i)} \frac{\partial^l \psi}{\partial \zeta_{j_1} \ldots \zeta_{j_i}}(0) \zeta_{j_1} \ldots \zeta_{j_i} + o(r^i)
\]
\[
= \sum_{j_1} r^{\text{ord}(j_1)} \frac{\partial \psi}{\partial \zeta_{j_1}}(0) \zeta_{j_1} + \sum_{j_1, j_2} r^{\text{ord}(j_1)+\text{ord}(j_2)} \frac{\partial^2 \psi}{\partial \zeta_{j_1} \partial \zeta_{j_2}}(0) \zeta_{j_1} \zeta_{j_2}
\]
\[
+ \ldots + \sum_{j_1, \ldots, j_i} r^{\text{ord}(j_1)+\ldots+\text{ord}(j_i)} \frac{\partial^l \psi}{\partial \zeta_{j_1} \ldots \zeta_{j_i}}(0) \zeta_{j_1} \ldots \zeta_{j_i} + o(r^i)
\]
as \( r \to 0^+ \). Therefore
\[
\frac{\psi(\hat{\delta}_r \zeta)}{r^i} = \sum_{j_1} r^{\text{ord}(j_1)-i} \frac{\partial \psi}{\partial \zeta_{j_1}}(0) \zeta_{j_1} + \sum_{j_1, j_2} r^{\text{ord}(j_1)+\text{ord}(j_2)-i} \frac{\partial^2 \psi}{\partial \zeta_{j_1} \partial \zeta_{j_2}}(0) \zeta_{j_1} \zeta_{j_2}
\]
\[
+ \ldots + \sum_{j_1, \ldots, j_i} r^{\text{ord}(j_1)+\ldots+\text{ord}(j_i)-i} \frac{\partial^l \psi}{\partial \zeta_{j_1} \ldots \zeta_{j_i}}(0) \zeta_{j_1} \ldots \zeta_{j_i} + o(1)
\]
as \( r \to 0^+ \). By applying the hypothesis
\[
\frac{\partial^l \psi}{\partial \zeta_{j_1} \ldots \zeta_{j_i}}(0) = 0 \quad \text{whenever} \quad \text{ord}(j_1) + \ldots + \text{ord}(j_i) < i,
\]
it follows that (19) exists. Setting
\[
\Psi_{\infty} = \lim_{r \to 0^+} \hat{\delta}_{1/r} \Psi = \left\{ \left( \zeta_1, \ldots, \zeta_{\alpha-1}, \hat{\psi}(\zeta), \zeta_{\alpha+1}, \ldots, \zeta_n \right) : \zeta \in e^+_{\alpha} \right\},
\]
where $\tilde{\psi}$ is the polynomial function of homogeneous order $i = \text{ord}(\alpha)$ given by

$$\tilde{\psi}(\zeta) = \sum_{\text{ord}(j_1) = i} \frac{\partial \psi}{\partial \zeta_{j_1}}(0) \zeta_{j_1} + \ldots + \sum_{\text{ord}(j_1) + \ldots + \text{ord}(j_l) = i} \frac{\partial^l \psi}{\partial \zeta_{j_1} \ldots \partial \zeta_{j_l}}(0) \zeta_{j_1} \ldots \zeta_{j_l},$$

yields $S_\infty = x \cdot \Psi_\infty$ and the thesis follows.

**Remark 3.15.** The metric factor is not constant, in general. It turns out to be constant, for instance, by assuming that $\alpha$ is symmetric on all layers; see (45). Anyway, it is uniformly bounded by two positive constants $K_1$ and $K_2$. This can be seen by using the "ball-box metric" and a homogeneity argument. Let $S$ be as in Theorem 3.12. Case (ii). Let $B_\delta(x,1)$ the unit $\delta$-ball centered at $x \in \text{Int}(S \setminus C_S)$ and let $r_1, r_2 \geq 0$ be such that $0 < r_1 \leq 1 \leq r_2$. In particular, $\text{Box}(x, r_1) \subseteq B_\delta(x,1) \subseteq \text{Box}(x, r_2)$. Recall that

$$k_\delta(\nu_\delta(x)) = \sigma_{H}^{n-1}(\mathcal{I}(\nu_\delta(x)) \cap B_\delta(x,1)) = \mathcal{H}_{E\nu}^{n-1}(\mathcal{I}(\nu_\delta(x)) \cap B_\delta(x,1)),$$

where $\mathcal{I}(\nu_\delta(x))$ denotes the vertical hyperplane orthogonal to $\nu_\delta(x)$. By homogeneity, one has $\delta_1 \text{Box}(0,1/2) = \text{Box}(0,t/2)$ for every $t \geq 0$ and by an elementary computation we get that

$$(2r_1)^{Q-1} \leq k_\delta(\nu_\delta(x)) \leq \sqrt{n-1}(2r_2)^{Q-1}.$$

Set $K_1 := (2r_1)^{Q-1}, \quad K_2 := \sqrt{n-1}(2r_2)^{Q-1}$. Therefore, one can always choose two positive constants $K_1, K_2$, independent of $S$, such that

$$K_1 \leq k_\delta(\nu_\delta(x)) \leq K_2 \quad \forall \ x \in \text{Int}(S \setminus C_S).$$

### 4 Isoperimetric Inequality on hypersurfaces

**Theorem 4.1** (Main Result: Isoperimetric Inequality). Let $S \subseteq \mathbb{G}$ be a compact hypersurface of class $C^2$ with boundary $\partial S$ (piecewise) $C^1$ and assume that the horizontal mean curvature $\mathcal{H}_{\nu}$ of $S$ is integrable, i.e. $\mathcal{H}_{\nu} \in L^1(S;\sigma_{\nu}^{-1})$. Then, there exists a constant $C_{\text{Isop}} > 0$, only dependent on $\mathbb{G}$ and on $\nu$, such that

$$\left(\frac{\sigma_{H}^{n-1}(S)}{\sigma_{\nu}^{n-1}(\partial S)}\right)^{\frac{Q-2}{Q-1}} \leq C_{\text{Isop}} \int_S |\mathcal{H}_{\nu}| \sigma_{\nu}^{n-1} + \sigma_{H}^{n-2}(\partial S).$$

---

By definition one has $\text{Box}(x,r) = x \cdot \text{Box}(0,r)$ for every $x \in \mathbb{G}$, where

$$\text{Box}(0,r) = \left\{ y = \exp \left( \sum_{i=1}^k y_n \right) \in \mathbb{G} : \|y_n\|_\infty \leq r \right\}.$$

We stress that $y_n = \sum_{j_i \in I_i} y_{j_i} c_{j_i}$ and that $\|y_n\|_\infty$ denotes the sup-norm on the $i$-th layer of $\mathbb{G}$; see (52), (55).

The unit box $\text{Box}(x,1/2)$ is the left-translated at $x$ of $\text{Box}(0,1/2)$ and so, by left-invariance of $\sigma_{\nu}^{n-1}$, the computation can be done at $0 \in \mathbb{G}$. Since $\text{Box}(0,1/2)$ is the unit hypercube of $\mathbb{R}^n \cong \mathbb{G}$, it remains to estimate the $\sigma_{\nu}^{n-1}$-measure of the intersection of $\text{Box}(0,1/2)$ with a generic vertical hyperplane through the origin $0 \in \mathbb{R}^n$. If $\mathcal{I}(X)$ is the vertical hyperplane through $0 \in \mathbb{R}^n$ and orthogonal to $X \in H$, we get that

$$1 \leq \mathcal{H}_{E\nu}^{n-1}(\text{Box}(0,1/2) \cap \mathcal{I}(X)) \leq \sqrt{n-1},$$

where $\sqrt{n-1}$ is the diameter of any face of the unit hypercube of $\mathbb{R}^n$. Therefore

$$(\delta_{2r_1} \text{Box}(0,1/2) \cap \mathcal{I}(X)) \subseteq (B_\delta(0,1) \cap \mathcal{I}(X)) \subseteq (\delta_{2r_2} \text{Box}(0,1/2) \cap \mathcal{I}(X))$$

and so

$$(2r_1)^{Q-1} \leq (2r_2)^{Q-1} \mathcal{H}_{E\nu}^{n-1}(\text{Box}(0,1/2) \cap \mathcal{I}(X)) \leq \mathcal{H}_{E\nu}^{n-1}(B_\delta(0,1) \cap \mathcal{I}(X)) \leq k_\delta(x) \leq (2r_2)^{Q-1} \mathcal{H}_{E\nu}^{n-1}(\text{Box}(0,1/2) \cap \mathcal{I}(X)).$$
The next sections are devoted to prove this theorem. Finally, in Section 5 we shall discuss some related Sobolev-type inequalities.

Nevertheless, we would like to state an immediate corollary of this theorem, which can be meaningful only in few cases. Among them the Heisenberg group $\mathbb{H}^1$ is perhaps the more interesting one; see, for instance, Theorem 2.22.

**Corollary 4.2.** Let $G$ be a $k$-step Carnot group. Under the assumptions of Theorem 4.1, if $\partial S$ is horizontal, then $S$ cannot be $H$-minimal.

Note that if $\partial S$ is horizontal this means that $\partial S = C_{\partial S}$.

**Proof.** If $\partial S$ is horizontal, then $\sigma_{\nu}^{n-2}(\partial S) = 0$. Furthermore, if $H_{\nu} = 0$ the right-hand side of (21) vanishes.

### 4.1 Linear isoperimetric inequality and monotonicity formula

Let $S \subset G$ be a compact hypersurface of class $C^2$ with boundary $\partial S$. Let $\nu$ denote the outward-pointing unit normal vector along $S$ and $\varpi = \frac{P_v \nu}{|P_v \nu|}$. Furthermore, we shall set

$$\varpi_{u_i} := P_{u_i} \varpi = \sum_{\alpha \in I_{H_i}} \varpi_\alpha X_\alpha$$

for $i = 2, \ldots, k$. Note that $\frac{\nu}{|P_v \nu|} = \nu + \sum_{i=2}^k \varpi_{u_i}$.

**Notation 4.3.** Let $\eta$ be the outward-pointing unit normal vector along $\partial S$. Note that, at each point $x \in \partial S$, $\eta(x) \in T_x S$. In the sequel, we shall set

(i) $\chi := \frac{P_v s \varpi}{|P_v s \varpi|}$;

(ii) $\chi_{u_i} := P_{u_i} \chi \quad \forall \ i = 2, \ldots, k$;

see Remark 2.17.

We have $\chi = \sum_{i=2}^k \chi_{u_i}$ and $\frac{\eta}{|P_v s \varpi|} = \eta_{u_S} + \chi$; see also Remark 3.2.

**Definition 4.4.** Fix a point $x \in G$ and consider the “Carnot homothety” centered at $x$, i.e. $\delta^x(t, y) := x \cdot \delta_t(x^{-1} \cdot y)$. The variation vector of $\delta^x_t(y) := \delta^x(t, y)$ at $t = 1$ is given by

$$Z_x := \left. \frac{\partial \delta^x_t}{\partial t} \right|_{t=1} = \delta_t(x^{-1} \cdot y)$$

Let us apply the 1st variation of $\sigma_{\nu}^{n-1}$, with a special choice of the variation vector. More precisely, fix a point $x \in G$ and consider the Carnot homothety $\delta^x_t(y) := x \cdot \delta_t(x^{-1} \cdot y)$ centered at $x$.

**Remark 4.5.** Without loss of generality, by using group translations, we can choose $x = 0 \in G$. In this case, we have

$$\delta^0(t, y) = \delta_t y = \exp \left( ty_h, t^2 y_{h_2}, t^3 y_{h_3}, \ldots, t^i y_{h_i}, \ldots, t^k y_{h_k} \right) \quad \forall \ t \in \mathbb{R},$$

where $y_{u_i} = \sum_{j_i \in I_{H_i}} y_{j_i} e_{j_i}$ and $\exp$ is the Carnot exponential mapping; see Section 2.1. Thus the variation vector related to $\delta^0_t(y) := \delta^0(t, y)$, at $t = 1$, is simply given by

$$Z_0 := \left. \frac{\partial \delta^0_t}{\partial t} \right|_{t=1} = \left. \frac{\partial \delta_t}{\partial t} \right|_{t=1} = y_h + 2y_{h_2} + \ldots + ky_{h_k}.$$
By invariance of $\sigma^{n-1}_H$ under Carnot dilations, one gets

$$
\frac{d}{dt} \delta^i \sigma^{n-1}_H \bigg|_{t=1} = (Q - 1) \sigma^{n-1}_H(S).
$$

Furthermore, by using the 1st variation formula, it follows that

$$(Q - 1) \sigma^{n-1}_H(S) = - \int_S \mathcal{H}_H \left( \langle Z_x, \frac{\nu}{|\mathcal{P}_H \nu|} \rangle \sigma^{n-1}_H + \int_{\partial S} \left( \langle Z_x^+, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \mathcal{H}_H \left( \frac{\nu}{|\mathcal{P}_H \nu|} \right) \right) \right) \sigma^{n-2}_H(S).
$$

**Lemma 4.6.** The following holds

$$
\frac{1}{q_x} \left| \left\langle Z_x, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \right| \leq \left( 1 + \sum_{i=2}^k c_i q_x^{i-1} |\omega_{h_i}| \right).
$$

Furthermore, we have $\left( 1 + \sum_{i=2}^k c_i q_x^{i-1} |\omega_{h_i}| \right) \leq 1 + O \left( \frac{q_x}{|\mathcal{P}_H \nu|} \right)$ as long as $q_x \to 0^+$.

**Proof.** Without loss of generality, by left-invariance, let $x = 0 \in \mathbb{G}$. Note that

$$
\left\langle Z_0, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle = \langle Z_0, \nu_H + \omega \rangle = \langle y_H, \nu_H \rangle + \sum_{i=2}^k \langle y_{h_i}, \omega_{h_i} \rangle.
$$

By Cauchy-Schwarz inequality, we immediately get that

$$
\left| \left\langle Z_0, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \right| \leq |y_H| + \sum_{i=2}^k |y_{h_i}| |\omega_{h_i}|.
$$

According with Definition 2.6 let $c_i \in \mathbb{R}_+$ be constants such that $|y_{h_i}| \leq c_i q^i(y)$ for $i = 2, \ldots, k$. Using the last inequality yields

$$
\left| \left\langle Z_0, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \right| \leq q \left( 1 + \sum_{i=2}^k c_i q^{i-1} |\omega_{h_i}| \right) \leq q \left( 1 + O \left( \frac{q}{|\mathcal{P}_H \nu|} \right) \right)
$$

as long as $q_x \to 0^+$. \hfill \Box

**Definition 4.7.** Let $G$ be a $k$-step Carnot group and $S \subset \mathbb{G}$ be a hypersurface of class $C^2$ with (piecewise) $C^1$ boundary $\partial S$. Moreover, let $S_r := S \cap B_r(x, r)$, where $B_r(x, r)$ is the open $q$-ball centered at $x \in \mathbb{G}$ and of radius $r > 0$. We shall set

$$
\mathcal{A}(r) := \int_{S_r} \left| \mathcal{H}_H \right| \left( 1 + \sum_{i=2}^k c_i q_x^{i-1} |\omega_{h_i}| \right) \sigma^{n-1}_H,
$$

$$
\mathcal{B}_0(r) := \int_{\partial S_r} \frac{1}{q_x} \left| \left\langle Z_x^+, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \mathcal{H}_H \left( \frac{\nu}{|\mathcal{P}_H \nu|} \right) \right| \sigma^{n-2}_H,
$$

$$
\mathcal{B}_1(r) := \int_{\partial B_r(x, r) \cap S} \frac{1}{q_x} \left| \left\langle Z_x^+, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \mathcal{H}_H \left( \frac{\nu}{|\mathcal{P}_H \nu|} \right) \right| \sigma^{n-2}_H,
$$

$$
\mathcal{B}_2(r) := \int_{S \cap B_r(x, r) \cap S} \frac{1}{q_x} \left| \left\langle Z_x^+, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \mathcal{H}_H \left( \frac{\nu}{|\mathcal{P}_H \nu|} \right) \right| \sigma^{n-2}_H,
$$

where $q_x(y) := q(x, y)$ for $y \in S$, i.e. $q_x$ denotes the $q$-distance from a fixed point $x \in \mathbb{G}$.
We clearly have the following:

**Proposition 4.8** (Linear Inequality). Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$ with (piecewise) $C^1$ boundary $\partial S$. Let $r$ be the radius of a $g$-ball centered at $x \in \mathbb{G}$. Then

$$(Q - 1) \sigma_{n-1}^n(S_r) \leq r (A(r) + B_0(r)).$$

Proof. Immediate.

**Remark 4.9.** In the sequel we will need the following property: there exists $r_S > 0$ such that

$$\int_{S_{r+h} \setminus S_r} \frac{1}{\varrho_x} \left< \left( Z^\top_x - \frac{\langle Z^\top_x, \nu \rangle}{[P_H \nu] \nu_H} \right), \text{grad}_S \varrho_x \right> \sigma_{n-1}^n(S_{r+h} \setminus S_r) \tag{23}$$

for $\sigma_{n-1}^n$-a.e. $x \in \text{Int} S$, for $L^1$-a.e. $r, h > 0$ such that $r + h \leq r_S$. In the classical setting the previous inequality easily follows from a key-property of the Euclidean metric $d_{Eu}$, that is the Ikonal equation $|\text{grad}_{Eu}d_{Eu}| = 1$. In fact, $Z_x(y) = y - x$ and, since $\nu_H$ coincides with $\nu$, one has $\nu_H^\top = 0$. Thus setting $\varrho_x(y) := d_{Eu}(x, y) = |x - y|$ yields

$$\left| \langle Z_x(y), \text{grad}_S \varrho_x(y) \rangle \over \varrho_x(y) \right| = 1 - \left< \frac{y - x}{|y - x|}, n_0 \right>^2 \leq 1,$$

where $n_0$ denotes the Euclidean unit normal of $S$. In particular, we may take $r_S = +\infty$. A stronger version of (23) is a natural assumption in the Riemannian setting. At this regard we refer the reader to a paper by Chung, Grigor’yan and Yau where this hypothesis is the starting point of a general theory about isoperimetric inequalities on weighted Riemannian manifolds and graphs; see [17].

It is worth observing that\(^{16}\)

$$\frac{1}{\varrho_x} \langle Z_x, \text{grad} \varrho_x \rangle = 1 \tag{24}$$

for every $x \in \mathbb{G}$. The last identity can be used to rewrite (23). We have

$$\left| \langle Z^\top_x(y) - \frac{\langle Z^\top_x(y), \nu \rangle}{[P_H \nu] \nu_H} \rangle, \text{grad}_S \varrho_x(y) \right| \over \varrho_x(y) = 1 - \left< \text{grad}_H \varrho_x(y), \nu_H(y) \right> \langle Z_x(y), \nu(y) \rangle \over \varrho_x(y).$$

Hence (23) can be formulated as follows:

\(\heartsuit\) There exists $r_S > 0$ such that:

$$\int_{S_{r+h} \setminus S_r} \left| 1 - \frac{\left< \text{grad}_H \varrho_x(y), \nu_H(y) \right> \left( Z_x(y), (\nu_H(y) + \varpi(y)) \right)}{\varrho_x(y)} \right| \sigma_{n-1}^n(y) \leq \sigma_{n-1}^n(S_{r+h} \setminus S_r) \tag{25}$$

for $\sigma_{n-1}^n$-a.e. $x \in \text{Int} S$, for $L^1$-a.e. $r, h > 0$ such that $r + h \leq r_S$.

\(^{16}\)We stress that (24) holds true for every (smooth enough) homogeneous distance on any Carnot group $\mathbb{G}$.

**Lemma 4.10.** Let $\mathbb{G}$ be a k-step Carnot group and let $\varrho : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}_+$ be any $C^1$-smooth homogeneous norm. Then $\frac{1}{\varrho_x} \langle Z_x, \text{grad} \varrho_x \rangle = 1$ for every $x \in \mathbb{G}$.

Proof. By homogeneity and left-invariance of $\varrho$. More precisely, we have $t \varrho(z) = \varrho(tz)$ for all $t > 0$ and for every $z \in \mathbb{G}$. Setting $z := x^{-1} \cdot y$, we get that $t \varrho(x, y) = t \varrho(z) = \varrho(tz) = \varrho(x \cdot t \delta x \cdot t \delta z)$ for every $x, y \in \mathbb{G}$ and for all $t > 0$. Hence $\varrho(x, y) = \varrho(x, y) \varrho(x \cdot t \delta x \cdot t \delta z)_{t=1} = \langle \text{grad}_x(y), Z_x(y) \rangle$, and the claim follows. \(\square\}
Lemma 4.11 (Key result). Let \( x \in \text{Int}(S \setminus C_S) \). Set
\[
\mu(S_t) := \int_{S_t} \left| 1 - \frac{\langle \text{grad}_H \varrho_x(y), \nu_H(y) \rangle \langle Z_x(y), (\nu_H(y) + \varpi(y)) \rangle}{\varrho_x(y)} \right| \sigma_H^{n-1}(y)
\]
for \( t > 0 \). Then
\[
\lim_{t \to 0^+} \frac{\mu(S_t)}{\sigma_H^{n-1}(S_t)} = 1.
\]

Remark 4.12. By using standard results about differentiation of measures (see, for instance, Theorem 2.9.7 in [25]), it follows from Lemma 4.11 that \( \mu(S_t) = \sigma_H^{n-1}(S_t) \) for \( \sigma_H^{n-1} \)-a.e. \( x \in S \), for all \( t > 0 \). Thus we get that \( \mu(S_{t+h} \setminus S_t) = \sigma_H^{n-1}(S_{t+h} \setminus S_t) \) for \( \sigma_H^{n-1} \)-a.e. \( x \in S \) and for every \( t, h \geq 0 \). In particular, we may choose \( r_S = +\infty \).

Proof of Lemma 4.11 Let \( x \in \text{Int}(S \setminus C_S) \) and note that \( S_t = \partial t^x \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \) for all \( t > 0 \). So we have
\[
\frac{\mu(S_t)}{\sigma_H^{n-1}(S_t)} = \frac{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \left| 1 - \frac{\langle \text{grad}_H \varrho_x(y), \nu_H(y) \rangle \langle Z_x(y), (\nu_H(y) + \varpi(y)) \rangle}{\varrho_x(y)} \right| \sigma_H^{n-1}(y)}{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \sigma_H^{n-1}(y)}
\]
\[
= \frac{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \left| 1 - \frac{\langle \text{grad}_H \varrho_x(\delta^t(z)), \nu_H(\delta^t(z)) \rangle \langle Z_x(\delta^t(z)), (\nu_H(\delta^t(z)) + \varpi(\delta^t(z))) \rangle}{\varrho_x(\delta^t(z))} \right| \sigma_H^{n-1}(z)}{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \sigma_H^{n-1}(z)} + O(t),
\]
as long as \( t \to 0^+ \), where we have used the “Big-O” notation. It is worth observing that the horizontal gradient of \( \varrho_x \) turns out to be homogeneous of degree 0, and hence independent of \( t \). Note also that \( [Z_x(z)]_H = \mathcal{P}_H(Z_x)(z) = z_H - x_H \). Therefore
\[
\lim_{t \to 0^+} \frac{\mu(S_t)}{\sigma_H^{n-1}(S_t)} = \frac{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \left| 1 - \frac{\langle \text{grad}_H \varrho_x(\delta^t(z)), \nu_H(\delta^t(z)) \rangle \langle [Z_x(z)]_H, (\nu_H(\delta^t(z)) + \varpi(\delta^t(z))) \rangle}{\varrho_x(\delta^t(z))} \right| \sigma_H^{n-1}(z)}{\int_{\partial t^x} \left( \delta^t \mathcal{H} \cap B_0(x, 1) \right) \sigma_H^{n-1}(S_{\infty} \cap B_0(x, 1))}
\]
\[
=: L.
\]
Recall that, by Theorem 3.12, we have \( S_{\infty} = \mathcal{I}(\nu_H(x)) \). This implies that \( \langle (z_H - x_H), \nu_H(x) \rangle = 0 \) whenever \( z \in \mathcal{I}(\nu_H(x)) \) and hence \( L = 1 \), as wished.

At this point, starting from Proposition 4.8, we may prove a monotonicity formula for the \( H \)-perimeter \( \sigma_H^{n-1} \). Henceforth, we shall set \( S_t := S \cap B_0(x, t) \), for \( t > 0 \).
**Theorem 4.13** (Monotonicity of $\sigma_{\mu}^{n-1}$). Let $S \subset \mathbb{R}^n$ be a $C^2$ compact hypersurface. Then for every $x \in \mathrm{Int}(S \setminus C_S)$ the following ordinary differential inequality holds

$$-rac{d}{dt} \frac{\sigma_{\mu}^{n-1}(S_t)}{t^{q-1}} \leq \frac{\mathcal{A}(t) + \mathcal{B}_2(t)}{t^{q-1}} \tag{26}$$

for $\mathcal{L}^1$-a.e. $t > 0$.

**Proof.** By applying Sard’s Theorem we get that $S_t$ is a $C^2$ manifold with boundary for $\mathcal{L}^1$-a.e. $t > 0$. From the inequality in Proposition 4.8 we have

$$(Q - 1)\sigma_{\mu}^{n-1}(S_t) \leq t(\mathcal{A}(t) + \mathcal{B}_0(t))$$

for $\mathcal{L}^1$-a.e. $t > 0$, where $t$ is the radius of a $\varrho$-ball centered at $x \in \mathrm{Int} S$. Since

$$\partial S_t = \{\partial B_\varrho(x,t) \cap S\} \cup \{\partial S \cap B_\varrho(x,t)\},$$

we get that

$$(Q - 1)\sigma_{\mu}^{n-1}(S_t) \leq t(\mathcal{A}(t) + \mathcal{B}_1(t) + \mathcal{B}_2(t)).$$

We estimate $\mathcal{B}_1(t)$ by using (23) and Coarea Formula. For every $t, h > 0$ one has

$$\int_{t}^{t+h} \mathcal{B}_1(s) \, ds = \int_{t}^{t+h} \int_{\partial B_\varrho(x,s) \cap S} \frac{1}{t} \left| \left( Z_{x,s} - \frac{\langle Z_{x,s}^+, \nu \rangle}{|\nu|} \nu_{\mu}^{s,t} \right) \right| |\nu| |\sigma_{\mu}^{n-2}$$

$$= \int_{S_{t+h} \setminus S_t} \frac{1}{t} \left| \left( Z_{x,s} - \frac{\langle Z_{x,s}^+, \nu \rangle}{|\nu|} \nu_{\mu}^{s,t} \right) \right| |\nu| |\sigma_{\mu}^{n-1}$$

where we have used the following facts:

- $\eta = \frac{\text{grad}_{\partial S_t} \varrho_x}{|\text{grad}_{\partial S_t} \varrho_x|}$ and $\eta_{\mu s} = \frac{\text{grad}_{\partial S_t} \varrho_x}{|\text{grad}_{\partial S_t} \varrho_x|}$ along $\partial B_\varrho(x,s) \cap S$ for $\mathcal{L}^1$-a.e. $s \in ]t, t + h[$;

- Coarea formula (4) together with Lemma 4.11 and Remark 4.12

Therefore

$$\int_{t}^{t+h} \mathcal{B}_1(s) \, ds = \frac{\sigma_{\mu}^{n-1}(S_{t+h} \setminus S_t)}{h}$$

as long as $h \to 0^+$ and hence $\mathcal{B}_1(t) = \frac{d}{dt} \sigma_{\mu}^{n-1}(S_t)$ for $\mathcal{L}^1$-a.e. $t > 0$. So we get that

$$(Q - 1)\sigma_{\mu}^{n-1}(S_t) \leq t(\mathcal{A}(t) + \mathcal{B}_2(t) + \frac{d}{dt} \sigma_{\mu}^{n-1}(S_t))$$

which is equivalent to (26).

\[\square\]

### 4.2 Further estimates

In this section we study the integrals $\mathcal{A}(t)$ and $\mathcal{B}_2(t)$ appearing in the right-hand side of the monotonicity formula (26).

**Estimate of $\mathcal{A}(t)$**.
Lemma 4.14. Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^k$, let $x \in \text{Int } S$ and let $S_t = S \cap B_\varrho(x,t)$ for some $t > 0$. Then there exists a constant $b_\varrho > 0$, only dependent on $\varrho$ and $\mathbb{G}$, such that

$$\lim_{t \to 0^+} \frac{\int_{S_t} |\varpi u_i| \sigma_{\mu}^{n-1}}{t^{Q-i}} \leq h_i b_\varrho \quad \text{for every } i = 2, \ldots, k$$

(27)

where $h_i = \dim H_i$.

Proof. For any $\alpha = h + 1, \ldots, n$, we have $(X_{\varrho} \mathcal{J} \sigma_{\mu}^n) \big|_S = (\langle X_\alpha, \nu \rangle \sigma_{\mu}^{n-1}) \big|_S = (\omega_\alpha) \big|_S$, where $*$ denotes the Hodge star operator on $T^*\mathbb{G}$; see \cite{37}. Moreover $\delta_t^*(\omega_\alpha) = t^{Q-\ord(\alpha)}(\omega_\alpha)$ for every $t > 0$. So we get that

$$\int_{S_t} |\varpi u_i| \sigma_{\mu}^{n-1} = \int_{S_t} |\varpi u_i| \sigma_{\mu}^{n-1} \leq \sum_{\ord(\alpha) = i} \int_{S_t} |X_\alpha \mathcal{J} \sigma_{\mu}^n| = \sum_{\ord(\alpha) = i} t^{Q-i} \int_{\delta_{1/t}^x S \cap B_\varrho(x,1)} |(\omega_\alpha) \circ \delta_t^*|,$$

Since

$$\int_{\delta_{1/t}^x S \cap B_\varrho(x,1)} |(\omega_\alpha) \circ \delta_t^*| \leq \sigma_{\mu}^{n-1} \left(\delta_{1/t}^x S \cap B_\varrho(x,1)\right),$$

by using Theorem 3.12 we may pass to the limit as $t \to 0^+$ the right-hand side. More precisely, if $x \in \text{Int}(S \setminus C_S)$ the rescaled hypersurfaces $\delta_{1/t}^x S$ converge to the vertical hyperplane $\mathcal{I}(\nu_\mu(x))$ as $t \to 0^+$. Otherwise, $x \in \text{Int}(S \cap C_S)$ and we can assume that $\ord(x) = Q - i$, for some $i = 2, \ldots, k$. We also recall that the limit-set $S_\infty$ is a polynomial hypersurface of homogeneous order $i$ passing through $x$; see Remark 3.13. So let us set $b_1 := \sup_{x \in H_i, \|X\|_1 = 1} \sigma_{\mu}^{n-1}(\mathcal{I}(X) \cap B_\varrho(0,1))$, where $\mathcal{I}(X)$ denotes the vertical hyperplane through $0 \in \mathbb{G}$ and orthogonal to $X$. In order to study the characteristic case, let $b_2 := \sup_{\Psi \in \mathcal{P}_0^k} \sigma_{\mu}^{n-1}(\Psi \cap B_\varrho(0,1))$, where $\mathcal{P}_0^k$ denotes the class of all graphs of polynomial functions of homogeneous order $\leq k$, passing through $0 \in \mathbb{G}$. Using the left-invariance of $\sigma_{\mu}^{n-1}$ and setting

$$b_\varrho := \max\{b_1, b_2\},$$

(28)

yields $\lim_{t \to 0^+} \sigma_{\mu}^{n-1} \left(\delta_{1/t}^x S \cap B_\varrho(x,1)\right) \leq b_\varrho$. Therefore

$$\lim_{t \to 0^+} \frac{\int_{S_t} |\varpi u_i| \sigma_{\mu}^{n-1}}{t^{Q-i}} \leq \lim_{t \to 0^+} h_i \sigma_{\mu}^{n-1} \left(\delta_{1/t}^x S \cap B_\varrho(x,1)\right) \leq h_i b_\varrho,$$

which achieves the proof of (27).

\hfill \Box

Remark 4.15. If $S$ is just of class $\mathbf{C}^2$, then (27) holds for every $x \in \text{Int}(S \setminus C_S)$. Moreover, if $x \in C_S$ has order $Q - i$ for some $i = 2, \ldots, k$, then the same claim holds if $S$ is of class $\mathbf{C}^i$.

Let $S \subset \mathbb{G}$ be of class $\mathbf{C}^2$, let $x \in \text{Int}(S \setminus C_S)$ and $S_t = S \cap B_\varrho(x,t)$. Moreover, let $A(t)$ be as in Definition 4.7. By applying Theorem 2.22 we get that $\dim_{\text{Eu-Hau}}(C_S) \leq n - 2$. In particular $\sigma_{\mu}^{n-1}$-a.e. interior point of $S$ is non-characteristic.

Lemma 4.16. Under the previous assumptions, one has

$$A(t) \leq \int_{S_t} |\mathcal{H}_\mu| \sigma_{\mu}^{n-1}.$$

(29)
Proof. First, note that $\varrho_t(y) = \varrho(x, y) \to 0^+$ as $t \to 0^+$. So we have

$$A(t) = \int_{S_t} |\mathcal{H}_t| \left(1 + \sum_{i=2}^k i c_i \varrho_x^{i-1} |\nu_{H_i}| \right) \sigma_{H_i}^n \leq \int_{S_t} |\mathcal{H}_t| \left(1 + \frac{2c_2 \varrho_x (1 + o(1))}{|\mathcal{P}_{H_i} \nu|} \right) \sigma_{H_i}^n$$

as long as $t \to 0^+$. Note that $\frac{1}{|\mathcal{P}_{H_i} \nu|}$ is continuous near $x \in \text{Int}(S \setminus C_S)$. Since $\mathcal{H}_t$ turns out to be continuous near every non-characteristic point, by using standard differentiation results in Measure Theory (see Theorem 2.9.7 in [25]), we get that

$$\lim_{t \to 0^+} \int_{S_t} |\mathcal{H}_t| \left(\frac{2c_2 \varrho_x (1 + o(1))}{|\mathcal{P}_{H_i} \nu|} \right) \sigma_{H_i}^n = 0$$

and (29) follows. □

Actually, a similar result holds true even if $x \in \text{Int}(S \cap C_S)$, at least whenever $\mathcal{H}_t$ is bounded and $S$ is smooth enough near $C_S$. Below we shall make use of Theorem 3.12 (Case (ii)).

**Lemma 4.17.** Let $S$ be a hypersurface of class $C^k$ and assume that $\mathcal{H}_t$ is bounded on $S$. Let $x \in \text{Int}(S \cap C_S)$ be an interior characteristic point such that $\text{ord}(x) = Q - i$, for some $i = 2, \ldots, k$. This means that there exists $\alpha = h + 1, \ldots, n$, $\text{ord}(\alpha) = i$, such that $S$ can be represented, locally around $x$, as a $X_\alpha$-graph for which (14) holds. Then there exists a constant $d_\varrho > 0$, only dependent on $\varrho$ and $S$, such that $A(t) \leq \|\mathcal{H}_t\|_{L^\infty(S)} (\kappa_\varrho(C_S(x)) + d_\varrho) t^{Q-1}$ as long as $t \to 0^+$. In particular, we have

$$A(t) \leq \|\mathcal{H}_t\|_{L^\infty(S)} (\kappa_\varrho(C_S(x)) + d_\varrho) S_\varrho^{Q-1}(S_t)$$

for all $t > 0$, where $S_\varrho^{Q-1}$ denotes the spherical Hausdorff measure computed with respect to the homogeneous distance $\varrho$.

Proof. Using Lemma 4.14 we obtain

$$\sum_{i=2}^k \int_{S_t} i c_i \varrho_x^{i-1} |\nu_{H_i}| \sigma_{H_i}^n \leq \sum_{i=2}^k i c_i h_i b_\varrho$$

as $t \to 0^+$, where $b_\varrho$ is the constant defined by (28). Finally, the thesis follows by setting $d_\varrho := \sum_{i=2}^k i c_i h_i b_\varrho$ and by using a well-known density estimate; see Theorem 2.10.17 in [25]. □

The previous result will be applied in Section 4.3.

**Estimate of $B_2(t)$**.

**Lemma 4.18.** Let $x \in \text{Int}(S \setminus C_S)$. Then, there exists $C_{\dim} = C(\varrho, \mathbb{G})$ such that

$$B_2(t) \leq \sigma_{H_i}^{n-2} (\partial S \cap B_\varrho(x, t)) + C_{\dim} t \sigma_{H_i}^{n-2} (\partial S \cap B_\varrho(x, t))$$

for every $t \leq 1$.

Proof. Fix $x \in \text{Int}(S \setminus C_S)$ and let $f : \partial S \setminus C_\varrho S \to \mathbb{R}_+$ be defined as

$$f(y) := \frac{1}{\varrho_x(y)} \left| \langle Z^\perp_x(y) - \langle Z^\perp_x(y), \nu(y) \rangle \nu^\perp_x(y), \frac{\eta(y)}{|\mathcal{P}_{H_i} \eta(y)|} \rangle \right|$$.
Then

\[ f \equiv \frac{1}{\varrho_x} \left| \left( \frac{Z_x}{\mathcal{P}_x \nu} - \frac{\nu}{\mathcal{P}_x \nu} \right)^\top, (\eta_{HS} + \chi) \right| \]

\[ = \frac{1}{\varrho_x} \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} - \frac{\nu}{\mathcal{P}_x \nu} \right)^\top, (\eta_{HS} + \chi) \]

\[ = \frac{1}{\varrho_x} \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} \right)^\top, (\eta_{HS} + \chi) \]

\[ = \frac{1}{\varrho_x} \left( \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} \right)^\top, (\eta_{HS} + \chi) \right) \]

\[ = \frac{1}{\varrho_x} \left( \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} \right)^\top, \chi \right) \]

\[ \leq 1 + \frac{\left( \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} \right)^\top, \chi \right)}{\varrho_x}. \]

Note that \( \mathcal{H} = O(\varrho_x^2) \) as \( \varrho_x \to 0^+ \). More precisely, by using (iii) of Definition 2.4 we easily get that there exists \( C = C(\varrho, \mathcal{G}) \) such that \( \mathcal{H} \leq C \varrho_x \) for every \( y \) such that \( \varrho_x \leq 1 \). In fact, we see that

\[ \left\langle \left( \frac{\mathcal{H}_x}{\mathcal{P}_x \nu} \right)^\top, \chi \right\rangle \leq \frac{2 |\mathcal{H}|}{\varrho_x |\mathcal{P}_x \nu| |\mathcal{P}_x \eta|} \leq \frac{2 \sum_{i=2}^{k} c_i \varrho_x^{i-1}}{|\mathcal{P}_x \nu| |\mathcal{P}_x \eta|} \leq \frac{2 \varrho_x \sum_{i=2}^{k} c_i \varrho_x^{i-2}}{|\mathcal{P}_x \nu| |\mathcal{P}_x \eta|}. \]

Set

\[ C = C_{\text{dim}} := 2 \sum_{i=2}^{k} c_i \]

Hence

\[ f(y) \leq 1 + C \left( \frac{\varrho_x(y)}{|\mathcal{P}_x \nu(y)| |\mathcal{P}_x \eta(y)|} \right) \]

for every \( y \in \partial S \cap B_\varrho(x, 1). \)

### 4.3 Proof of the Isoperimetric Inequality

By applying the results of Section 4.2 together with Theorem 4.13 we get the following version of the monotonicity inequality:

**Corollary 4.19.** Let \( S \subset \mathcal{G} \) be a hypersurface of class \( \mathcal{C}^2 \) with (piecewise) \( \mathcal{C}^1 \) boundary \( \partial S \). Then, for every \( x \in \text{Int}(S \setminus C_S) \) we have

\[ - \frac{d}{dt} \sigma_{\mu}^{n-1}(S_t) \leq \frac{1}{t^{q-1}} \left( \int_{S_t} |\mathcal{H}_\mu| \sigma_{\mu}^{n-1} + \sigma_{\mu}^{n-2}(\partial S \cap B_\varrho(x, t)) + C t \sigma_{\mu}^{n-2}(\partial S \cap B_\varrho(x, t)) \right) \]

for \( L^1 \)-a.e. \( t \in [0, 1] \), where \( C = C_{\text{dim}} \) is a dimensional constant only dependent on \( \varrho \) and \( \mathcal{G} \).

**Proof.** The proof follows by applying Theorem 4.13, Lemma 4.16 and Lemma 4.18. \( \square \)
Notation 4.20. Let \( x \in \text{Int}(S \setminus C_S) \). Henceforth, we shall set
\[
D(t) := \int_{S_t} |\mathcal{H}_n| \sigma_{n-1}^n + \sigma_{n-2}^n(\partial S \cap B_g(x,t)) + C't \sigma_{n-2}^n(\partial S \cap B_g(x,t)) \quad \forall \ t \in [0,1].
\]

Lemma 4.21. Let \( S \subset \mathbb{R}^n \) be a hypersurface of class \( C^2 \) with (piecewise) \( C^1 \) boundary \( \partial S \). Let \( x \in \text{Int}(S \setminus C_S) \) and let
\[
r_0(x) := \min \left\{ 1, \frac{1}{2} \left( \frac{\sigma_{n-1}^n(S)}{k_{n}(\nu_n(x))} \right)^{1/Q-1} \right\},
\]
(32)
For every \( \lambda \geq 2 \) there exists \( r \in ]0, r_0(x) [ \) such that
\[
\sigma_{n-1}^n(S_{\lambda r}) \leq \lambda^{Q-1} r_0(x) D(r).
\]

Due to Remark 3.15, the number \( r_0(x) > 0 \) can be globally estimated from above and below.

Proof of Lemma 4.21. Fix \( r \in ]0, r_0(x) [ \) and note that \( \sigma_{n-1}^n(S_t) \) is a monotone non-decreasing function of \( t \) on \( [r, r_0(x)] \). We start from the identity
\[
\sigma_{n-1}^n(S_t)/t^{Q-1} = \left( \sigma_{n-1}^n(S_t) - \sigma_{n-1}^n(S_{r_0(x)}) \right) /t^{Q-1} + \sigma_{n-1}^n(S_{r_0(x)}) /t^{Q-1}.
\]
The first addend is an increasing function of \( t \), while the second one is an absolutely continuous function of \( t \). Therefore, by integrating the differential inequality (32), we get that
\[
\frac{\sigma_{n-1}^n(S_t)}{t^{Q-1}} \leq \frac{\sigma_{n-1}^n(S_{r_0(x)})}{(r_0(x))^{Q-1}} + \int_r^{r_0(x)} D(t) t^{-(Q-1)} dt.
\]

Therefore
\[
\beta := \sup_{r \in ]0, r_0(x)[} \frac{\sigma_{n-1}^n(S_t)}{r^{Q-1}} \leq \frac{\sigma_{n-1}^n(S_{r_0(x)})}{(r_0(x))^{Q-1}} + \int_0^{r_0(x)} D(t) t^{-(Q-1)} dt.
\]

Now we argue by contradiction. If the lemma is false, it follows that for every \( r \in ]0, r_0(x)[ \)
\[
\sigma_{n-1}^n(S_{\lambda r}) > \lambda^{Q-1} r_0(x) D(t).
\]
From the last inequality we infer that
\[
\int_0^{r_0(x)} D(t) t^{-(Q-1)} dt \leq \frac{1}{\lambda^{Q-1} r_0(x)} \int_0^{r_0(x)} \sigma_{n-1}^n(S_{\lambda t}) t^{-(Q-1)} dt
\]
\[
= \frac{1}{\lambda r_0(x)} \int_0^{\lambda r_0(x)} \sigma_{n-1}^n(S_s) s^{-(Q-1)} ds
\]
\[
= \frac{1}{\lambda r_0(x)} \int_0^{r_0(x)} \sigma_{n-1}^n(S_s) s^{-(Q-1)} ds + \frac{1}{\lambda r_0(x)} \int_{r_0(x)}^{\lambda r_0(x)} \sigma_{n-1}^n(S_s) s^{-(Q-1)} ds
\]
\[
\leq \beta \frac{\lambda - 1}{\lambda} \frac{\sigma_{n-1}^n(S)}{(r_0(x))^{Q-1}}.
\]

Therefore, using (33) yields
\[
\beta \leq \frac{\sigma_{n-1}^n(S_{r_0(x)})}{(r_0(x))^{Q-1}} + \beta \frac{\lambda - 1}{\lambda} \frac{\sigma_{n-1}^n(S)}{(r_0(x))^{Q-1}}.
\]
and so
\[
\frac{\lambda - 1}{\lambda} \beta \leq \frac{2\lambda - 1}{\lambda} \left( \frac{\sigma_h^{n-1}(S)}{(r_0(x))^{Q-1}} \right) \leq \frac{2\lambda - 1}{\lambda} \left( \frac{k_\nu(\nu_\rho(x))}{2^{Q-1}} \right).
\]

By its own definition, one has
\[
k_\nu(\nu_\rho(x)) = \lim_{r \searrow 0^+} \frac{\sigma_h^{n-1}(S_r)}{r^{Q-1}} \leq \beta.
\]

Furthermore, since \(Q - 1 \geq 3\), we get that
\[
\lambda - 1 \leq \frac{2\lambda - 1}{8},
\]
or equivalently \(\lambda \leq \frac{7}{6}\), which contradicts the hypothesis \(\lambda \geq 2\).

The next covering lemma is well-known and can be found in [8]; see also [25].

**Lemma 4.22** (Vitali’s Covering Lemma). Let \((X, \rho)\) be a compact metric space and let \(A \subseteq X\). Moreover, let \(C\) be a covering of \(A\) by closed \(\varrho\)-balls with centers in \(A\). We also assume that each point \(x\) of \(A\) is the center of at least one closed \(\varrho\)-ball belonging to \(C\) and that the radii of the balls of the covering \(C\) are uniformly bounded by some positive constant. Then, for every \(\lambda > 2\) there exists a no more than countable subset \(C_\lambda \subseteq C\) of pairwise non-intersecting closed balls \(\overline{B}_\varrho(x_k, r_k), k \in \mathbb{N}\), such that
\[
A \subseteq \bigcup_{k \in \mathbb{N}} B_\varrho(x_k, \lambda r_k).
\]

**Notation 4.23.** Henceforth, we shall set
\[
r_0(S) := \sup_{x \in \text{Int}(S \setminus C_S)} r_0(x).
\]

We are now in a position to prove our main result.

**Proof of Theorem 4.14**. First we shall apply Lemma 4.21. To this aim, let \(\lambda > 2\) be fixed and, for every \(x \in \text{Int}(S \setminus C_S)\), let \(r(x) \in ]0, r_0(S)[\) be such that
\[
\sigma_h^{n-1}(S_r(x)) \leq \lambda^{Q-1} r_0(S) \mathcal{D}(r(x)). \tag{34}
\]

Let \(C = \{\overline{B}_\varrho(x, r(x)) : x \in \text{Int}(S \setminus C_S)\}\) be a covering of \(S\). By Lemma 4.22, there exists a non more than countable subset \(C_\lambda \subseteq C\) of pairwise non-intersecting closed balls \(\overline{B}_\varrho(x_k, r_k), k \in \mathbb{N}\), such that
\[
S \setminus C_S \subset \bigcup_{k \in \mathbb{N}} B_\varrho(x_k, \lambda r_k),
\]
where we have set \(r_k := r(x_k)\). We therefore get
\[
\sigma_h^{n-1}(S) \leq \sum_{k \in \mathbb{N}} \sigma_h^{n-1}(S \cap B_\varrho(x_k, \lambda r_k)) \leq \lambda^{Q-1} r_0(S) \sum_{k \in \mathbb{N}} \mathcal{D}(r_k) \quad \text{(by (34))}
\]
\[
= \lambda^{Q-1} r_0(S) \sum_{k \in \mathbb{N}} \left( \int_{S_{r_k}} |\mathcal{H}_h| \sigma_h^{n-1} + \sigma_h^{n-2}(\partial S \cap B_\varrho(x_k, r_k)) + C r_k \sigma_h^{n-2}(\partial S \cap B_\varrho(x, r_k)) \right)
\]
\[
\leq \lambda^{Q-1} r_0(S) \left[ \left( \int_S |\mathcal{H}_h| \sigma_h^{n-1} + \sigma_h^{n-2}(\partial S) \right) + \sum_{k \in \mathbb{N}} C r_k \sigma_h^{n-2}(\partial S \cap B_\varrho(x, r_k)) \right].
\]

\[17\] Indeed, the first non-abelian Carnot group is the Heisenberg group \(\mathbb{H}^1\) for which \(Q = 4\).
Hence
\[
\sigma_n^{-1}(S) \leq \lambda^{Q-1} r_0(S) \left( \int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) + \varepsilon C_{\dim} \sigma_n^{-2}(\partial S) \right),
\]
where we have set
\[
\varepsilon := \sup_{k \in \mathbb{N}} r_k.
\]
Note that the constant $C_{\dim}$ has been defined by (30). By letting $\lambda \searrow 2$, we get that
\[
\sigma_n^{-1}(S) \leq 2^{Q-1} r_0(S) \left( \int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) + \varepsilon C_{\dim} \sigma_n^{-2}(\partial S) \right).
\]
Since
\[
2^{Q-1} r_0(S) \leq 2^{Q-1} \sup_{x \in \text{Int}(S \setminus C_S)} 2 \left( \frac{\sigma_n^{-1}(S)}{K_0(\nu_{\sigma}(x))} \right)^{\frac{1}{Q-1}} = 2Q \sup_{x \in \text{Int}(S \setminus C_S)} \frac{(\sigma_n^{-1}(S))^{\frac{1}{Q-1}}}{K_0^{\frac{1}{Q-1}}},
\]
using (20) yields
\[
2^{Q-1} r_0(S) \leq 2Q \frac{(\sigma_n^{-1}(S))^{\frac{1}{Q-1}}}{K_0^{\frac{1}{Q-1}}};
\]
see Remark 3.15. Therefore
\[
(\sigma_n^{-1}(S))^{\frac{Q-2}{Q-1}} \leq \frac{2Q}{K_0^{\frac{1}{Q-1}}} \left( \int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) + \varepsilon C_{\dim} \sigma_n^{-2}(\partial S) \right). \tag{35}
\]
Set $C_1 := \frac{2Q}{K_0^{\frac{1}{Q-1}}}$.

Claim 4.24. If $\sigma_n^{-2}(\partial S) > 0$ and $\int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) > 0$, there exists $\varepsilon > 0$ such that
\[
(\sigma_n^{-1}(S))^{\frac{Q-2}{Q-1}} \leq 2C_1 \left( \int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) \right).
\]

Proof. It is sufficient to choose $\varepsilon \leq \frac{\int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S)}{C_{\dim} \sigma_n^{-2}(\partial S)}$. \hfill \qed

Obviously, if $\sigma_n^{-2}(\partial S) = 0$ there is nothing to prove. So it remains to consider the case $\sigma_n^{-2}(\partial S) > 0$ and $\int_S |H_u| \sigma_n^{-1} + \sigma_n^{-2}(\partial S) = 0$. In such a case, from (35) we get that
\[
(\sigma_n^{-1}(S))^{\frac{Q-2}{Q-1}} \leq C_1 \varepsilon C_{\dim} \sigma_n^{-2}(\partial S). \tag{36}
\]
Replace $S$ by $S_t := \delta_t S$ for $t > 0$, where $\delta_t$ is the Carnot homothety with center at $0 \in \mathbb{G}$. Now using the $(Q-1)$-homogeneity of $\sigma_n^{-1}$ yields
\[
(\sigma_n^{-1}(S))^{\frac{Q-2}{Q-1}} \leq C_1 \varepsilon C_{\dim} \sigma_n^{-2}(\partial S_t). \tag{36}
\]
Remark 4.25 (Homogeneity). Let $k \geq 2$ be the step of $\mathbb{G}$. Then, the pull-back $\delta^*_t \sigma^{n-2}_R$ splits into different homogeneous components of (homogeneous) degree $Q-i$ for some $i = 2, 3, ..., 2k$. Hence, there exist differential $(n-1)$-forms of class $C^1$, say $(\sigma^{n-2}_R)_{Q-2}, (\sigma^{n-2}_R)_{Q-3}, ..., (\sigma^{n-2}_R)_{Q-i}, ...$, such that

$$
\delta^*_t \sigma^{n-2}_R = t^{Q-2} (\sigma^{n-2}_R)_{Q-2} + \ldots + t^{Q-i} (\sigma^{n-2}_R)_{Q-i} + \ldots + t^{Q-2k} (\sigma^{n-2}_R)_{Q-2k} \quad \forall \ t > 0.
$$

Therefore, we have

$$
\frac{\sigma^{n-2}_R(\partial S_t)}{t^{Q-2}} = \left( (\sigma^{n-2}_R)_{Q-2} + \frac{\sigma^{n-2}_R(\partial S)}{t^{Q-2}} (1 + o(1)) \right) \| \partial S \| = (\sigma^{n-2}_R)_{Q-2}(\partial S) + o(1)
$$
as long as $t \to +\infty$. Hence $\lim_{t \to +\infty} \frac{\sigma^{n-2}_R(\partial S_t)}{t^{Q-2}} = (\sigma^{n-2}_R)_{Q-2}(\partial S)$. Furthermore, it is worth observing that the differential $(n-2)$-form $(\sigma^{n-2}_R)_{Q-2}$ can be estimated in terms of $\sigma^{n-2}_R$. More precisely, it is elementary to see that

$$
(\sigma^{n-2}_R)_{Q-2} = \mathcal{P}_R (\nu \wedge \eta) \mathcal{J} \sigma^R.
$$

Indeed, the homogeneous degree of any simple 2-vector $X_{i_1} \wedge X_{i_2} \in \Lambda^2(T \mathbb{G})$ is the sum of degrees of $X_{i_1}$ and $X_{i_2}$ $(r, s = 1, \ldots, k)$. By definition, we have

$$
\sigma^{n-2}_R = \sum_{r,s=1}^k \sum_{i_r \in I_{H^R}, i_s \in I_{H^\nu}} \nu_{i_r} \eta_{i_s} \ast X_{i_r} \wedge X_{i_s},
$$

where $\ast$ denotes the Hodge star operator on $T^* \mathbb{G}$; see [37]. This is a sum of $(n-2)$-forms of homogeneous degree $\geq Q-2$, with equality if and only if $r = s = 1$. This proves (37). Finally

$$
\| (\sigma^{n-2}_R)_{Q-2} \| = \| \mathcal{P}_R (\nu \wedge \eta) \mathcal{J} \sigma^R \| \leq \| \mathcal{P}_R (\nu \wedge \eta) \sigma^{n-2}_R \| \leq \| \sigma^{n-2}_R \|,
$$

where $\| \cdot \|$ denotes the usual norm of a differential form; see Paragraph 1.8 of [27].

By applying Remark 4.25 we easily get that the left-hand side of (36), i.e. $\frac{\varepsilon C_{dim} \sigma^{n-2}_R(\partial S_t)}{t^{Q-2}}$, can be made arbitrarily small (even with respect to the measure $\sigma^{n-2}_R$) by varying the dilation parameter $t > 0$ and, possibly, the upper bound $\varepsilon \in [0, 1]$. Consequently, we have shown the following result:

Claim 4.26. If $\int_S \mathcal{H}_R | \sigma^{n-1}_R + \sigma^{n-2}_R(\partial S) = 0$, then $\sigma^{n-1}_R(S) = 0$.

For our purposes, the proof of the Isoperimetric Inequality [21] can be achieved by setting $C_{I sop} := 2C_1$. 

\[ \square \]

4.4 An application of the monotonicity formula: asymptotic behavior of $\sigma^{n-1}_R$

The monotonicity formula (26) (see Theorem 4.13) can be formulated as follows:

$$
\frac{d}{dt} \left( \frac{\sigma^{n-1}_R(S_t)}{t^{Q-1}} \right) \exp \left( \int_0^t A(s) + B_2(s) \frac{ds}{\sigma^{n-1}_R(S_s)} \right) \geq 0
$$

(38)

for $L^1$-a.e. $t \in [0, r]$, and for every $x \in \text{Int}(S \setminus C_S)$. For the sake of simplicity, let $\partial S = \emptyset$ (and hence $B_2(s) = 0$). By Theorem 3.12, Case (i), we may pass to the limit as $t \searrow 0^+$ in the previous inequality; see Section 3.3. Hence

$$
\sigma^{n-1}_R(S_t) \geq \kappa_\varepsilon(\nu_R(x)) t^{Q-1} \exp \left( - \int_0^t \frac{A(s)}{\sigma^{n-1}_R(S_s)} ds \right),
$$

(39)

for every $x \in \text{Int}(S \setminus C_S)$.
Corollary 4.27. Let \( G \) be a \( k \)-step Carnot group and let \( S \subseteq G \) be a hypersurface of class \( C^2 \) without boundary. Assume that \( |\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty \). Then, for every \( x \in \text{Int}(S \setminus C_S) \), one has
\[
\sigma_H^{n-1}(S_t) \geq \kappa_e(\nu_H(x)) t^{Q-1} e^{-t \mathcal{H}_H^0}
\]
as long as \( t \to 0^+ \).

Proof. We just have to bound \( \int_0^t \frac{A(s)}{\sigma_H^{n-1}(S_s)} \, ds \) from above. Using Lemma 4.16 yields
\[
\int_0^t \frac{A(s)}{\sigma_H^{n-1}(S_s)} \, ds \leq \mathcal{H}_H^0 (1 + o(1))
\]
as long as \( t \to 0^+ \) and (40) follows from (39).

Warning 4.28. If \( S \) is smooth enough near its characteristic set \( C_S \) and \( \mathcal{H}_H \) is globally bounded, the previous asymptotic estimate can be generalized by applying the results of Section 4.2. In the following corollaries, however, we need to assume that condition (25) holds at the point where the monotonicity inequality has to be proved.

Corollary 4.29. Let \( G \) be a \( k \)-step Carnot group. Let \( S \subseteq G \) be a hypersurface without boundary and such that \( |\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty \). Let \( x \in \text{Int}(S \setminus C_S) \) and assume that \( \text{ord}(x) = Q-i \), for some \( i = 2, \ldots, k \). With no loss of generality, we suppose that there exists \( \alpha \in \{h+1, \ldots, n\} \), \( \text{ord}(\alpha) = i \), such that \( S \) can be represented, locally around \( x \), as \( X_{\alpha} \)-graph of a \( C^1 \) function satisfying (14). We assume that condition (25) holds at the point \( x \). Then
\[
\sigma_H^{n-1}(S_t) \geq \kappa_e(\nu_H(x)) t^{Q-1} e^{-t \mathcal{H}_H^0 (\kappa_e(C_S(x)) + d_e)}
\]
as long as \( t \to 0^+ \).

We recall that the function \( \kappa_e(C_S(x)) \) has been defined in Theorem 3.12 see Case (ii). Notice that \( d_e = \sum_{i=2}^k i c_i h_i b_e \), where \( b_e \) is the constant defined by (28).

Proof. By arguing as above, we may pass to the limit in (38) as \( t \searrow 0^+ \) and we get that
\[
\sigma_H^{n-1}(S_t) \geq \kappa_e(\nu_H(x)) t^{Q-1} \exp \left( - \int_0^t \frac{A(s)}{\sigma_H^{n-1}(S_s)} \, ds \right).
\]
By using Lemma 4.17 we get that
\[
\int_0^t \frac{A(s)}{\sigma_H^{n-1}(S_s)} \, ds \leq \|\mathcal{H}_H\|_{L^\infty(S)} (\kappa_e(C_S(x)) + d_e) \, t \leq \mathcal{H}_H^0 (\kappa_e(C_S(x)) + d_e) \, t
\]
as long as \( t \to 0^+ \). This achieves the proof.

In particular, in the case of the Heisenberg group \( \mathbb{H}^n \), the following holds:

Corollary 4.30. Let \((\mathbb{H}^n, g)\) be the Heisenberg group endowed with the Koranyi distance; see Example 2.7. Let \( S \subseteq \mathbb{H}^n \) be a hypersurface of class \( C^2 \) without boundary and assume that \( |\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty \). Furthermore, let \( x \in \text{Int}(S \setminus C_S) \) be such that condition (25) holds. Then
\[
\sigma_H^{2n}(S_t) \geq \kappa_e(\nu_H(x)) t^{Q-1} e^{-t \mathcal{H}_H^0 (\kappa_e(C_S(x)) + b_e)}
\]
as long as \( t \to 0^+ \).
The density-function $\kappa_\psi(C_S(x))$ has been defined in Theorem 3.12; see Case (ii). Moreover, $b_\psi$ is the constant defined by (28).

**Proof.** By arguing as for the non-characteristic case, we may pass to the limit in (38) as $t \downarrow 0^+$. As above, we have

$$\sigma_H^{2n}(S_t) \geq \kappa_\psi(C_S(x)) t^{2n-1}\exp \left(- \int_0^t \frac{A(s)}{\sigma_H^{2n}(S_s)} ds \right),$$

as $t \downarrow 0^+$, for every $x \in S \cap C_S$. By applying Lemma 4.14

$$\frac{A(s)}{\sigma_H^{2n}(S_s)} \leq \mathcal{H}_n^0 (\kappa_\psi(C_S(x)) + 2 c_2 b_\psi) = \mathcal{H}_n^0 (\kappa_\psi(C_S(x)) + b_\psi),$$

for every (small enough) $s > 0$, since in this case $c_2 = \frac{1}{2}$.

**Example 4.31.** Let $(\mathbb{H}^n, g)$, where $g$ is the Koranyi distance and $Q = 2n + 2$. Let

$$S = \{ \exp(x_t, t) \in \mathbb{H}^n : t = 0 \}.$$ 

We have $C_S = 0 \in \mathbb{H}^n$ and $\nu_t = -\frac{1}{2} C^{2n+1}_n x_t$. Furthermore

$$-\mathcal{H}_n = \text{div}_n \nu_t = \frac{1}{2} \text{div}_{\mathbb{R}^{2n}}(-x_2, x_1, \ldots, -x_{2n}, x_{2n-1}) = 0.$$ 

Note that $\kappa_\psi(C_S) = \frac{O_{2n}}{4n}$, where $O_{2n-1}$ is the surface measure of the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Thus (42) says that $\sigma_H^{2n}(S_t) \geq \frac{O_{2n}}{4n} t^{2n-1}$. This inequality can also be proved by using the formula $\sigma_H^{2n} = \frac{|x|}{2} d\mathcal{L}^{2n}$ and then by introducing spherical coordinates on $\mathbb{R}^{2n} \simeq H$.

## 5 Sobolev-type inequalities on hypersurfaces

The isoperimetric inequality (21) turns out to be equivalent to a Sobolev-type inequality. The proof is analogous to that of the equivalence between the (Euclidean) Isoperimetric Inequality and the Sobolev one; see [8].

**Theorem 5.1.** Let $G$ be a $k$-step Carnot group endowed with a homogeneous metric $g$ as in Definition 2.6. Let $S \subset G$ be a hypersurface of class $C^2$ without boundary. Let $\mathcal{H}_n$ be the horizontal mean curvature of $S$ and assume that $\mathcal{H}_n \in L^1_{\text{loc}}(S; \sigma_n^{-1})$. Then

$$\left( \int_S |\psi|^{\frac{Q-1}{Q}} \sigma_n^{-1} \right)^{\frac{Q-2}{Q}} \leq C_{\text{isop}} \int_S (|\psi| |\mathcal{H}_n| + |\text{grad}_{\nu} \psi|) \sigma_n^{-1}$$

(43)

for every $\psi \in C^1_0(S)$, where $C_{\text{isop}}$ is the constant appearing in Theorem 4.1.

**Proof.** The proof follows a classical argument; see [25], [46]. Since $|\text{grad}_{\nu} \psi| \leq |\text{grad}_{\nu} \psi|$, without loss of generality we assume $\psi \geq 0$. Set $S_t := \{ x \in S : \psi(x) > t \}$. Since $\psi$ has compact support, the set $S_t$ is a bounded open subset of $S$ and, by applying Sard’s Lemma, we see that its boundary $\partial S_t$ is $C^1$ for $\mathcal{L}^1$-a.e. $t > 0$. Furthermore, $S_t = \emptyset$ for each (large enough) $t > 0$. The main tools are Cavalieri’s principle[18] and Coarea Formula; see Theorem 4.1. We start by the identity

$$\int_S |\psi|^{\frac{Q-1}{Q}} \sigma_n^{-1} = \frac{Q-1}{Q} \int_0^\infty t^{\frac{1}{Q-2}} \sigma_n^{-1}(S_t) dt$$

(44)

---

[18] The following lemma, also known as Cavalieri’s principle, is a simple consequence of Fubini’s Theorem:

**Lemma 5.2.** Let $X$ be an abstract space, $\mu$ a measure on $X$, $\alpha > 0$, $\varphi \geq 0$ and $A_t = \{ x \in X : \varphi > t \}$. Then

$$\int_0^\infty t^{\alpha-1} \mu(A_t) dt = \frac{1}{\alpha} \int_{A_0} \varphi^\alpha d\mu.$$
which follows from Lemma 5.2 with $\alpha = \frac{Q-1}{Q-2}$. We also recall that, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive decreasing function and $\alpha \geq 1$, then

$$\alpha \int_0^{+\infty} t^{\alpha-1} \varphi(t) dt \leq \left( \int_0^{+\infty} \varphi(t) dt \right)^\alpha.$$ 

Using (44) and the last inequality yields

$$\int_S \psi \sigma^{n-1}_H = \frac{Q-1}{Q-2} \int_0^{+\infty} t^{\frac{Q-2}{Q-1}} \sigma^{n-1}_H(S_t) dt$$

$$\leq \left( \int_0^{+\infty} (\sigma^{n-1}_H(S_t))^{\frac{Q-2}{Q-1}} dt \right)^\frac{Q-1}{Q-2}$$

$$\leq \left( \int_{S_t} C_{I,sop} \left( \int_{H(H_0)} \sigma^{n-2}_H(\partial S_t) \right) dt \right)^\frac{Q-1}{Q-2} \text{ (by (2.1) with } S = S_t)$$

$$= \left( C_{I,sop} \int_S (|\psi| |H_0| + |\text{grad}_H \psi|) \sigma^{n-1}_H \right)^\frac{Q-1}{Q-2},$$

where we have used Cavalieri’s principle and Coarea Formula.

**Notation 5.3.** For any $p > 0$, set $\frac{1}{p'} = \frac{1}{p} - \frac{1}{Q-1}$. Furthermore, we denote by $p'$ the Hölder conjugate of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. In the sequel, any $L^p$ norm will be understood with respect to the measure $\sigma^{n-1}_H$.

Henceforth, we shall assume that $H_0$ is globally bounded on $S$ and set $H_0 := \|H_0\|_{L^\infty(S)}$.

**Corollary 5.4.** Under the previous assumptions, one has

$$\|\psi\|_{L^p(S)} \leq C_{I,sop} \left( H_0^0 \|\psi\|_{L^p(S)} + c_{p'} \|\text{grad}_H \psi\|_{L^p(S)} \right)$$

for every $\psi \in C_0^1(S)$, where $c_{p'} := p' \frac{Q-2}{Q-1}$. Thus, there exists $C_p := C_{p'}(H_0^0, q, G)$ such that

$$\|\psi\|_{L^p(S)} \leq C_p \left( \|\psi\|_{L^p(S)} + \|\text{grad}_H \psi\|_{L^p(S)} \right)$$

for every $\psi \in C_0^1(S)$.

**Proof.** Let us apply (43) with $\psi$ replaced by $\psi |\psi|^{t-1}$, for some $t > 0$. It follows that

$$\left( \int_S |\psi|^{t} \sigma^{n-1}_H \right)^{\frac{Q-2}{Q-1}} \leq C_{I,sop} \int_S \left( H_0^0 |\psi|^{t} + t|\psi|^{t-1} |\text{grad}_H \psi| \right) \sigma^{n-1}_H.$$  \hfill (45)

If we put $(t-1)p' = p^*$, one gets $p^* = t \frac{Q-2}{Q-1}$. Using Hölder inequality yields

$$\left( \int_S |\psi|^{p^*} \sigma^{n-1}_H \right)^{\frac{Q-2}{Q-1}} \leq C_{I,sop} \left( \int_S |\psi|^{p^*} \sigma^{n-1}_H \right)^{\frac{Q-2}{Q-1}} \left( H_0^0 \|\psi\|_{L^p(S)} + t \|\text{grad}_H \psi\|_{L^p(S)} \right).$$

**Corollary 5.5.** Under the previous assumptions, let $p \in [1, Q-1]$. For all $q \in [p, p^*]$ one has

$$\|\psi\|_{L^q(S)} \leq (1 + H_0^0 C_{I,sop}) \|\psi\|_{L^p(S)} + c_p C_{I,sop} \|\text{grad}_H \psi\|_{L^p(S)}$$

for every $\psi \in C_0^1(S)$. In particular, there exists $C_q = C_q(H_0^0, q, G)$ such that

$$\|\psi\|_{L^q(S)} \leq C_q \left( \|\psi\|_{L^p(S)} + \|\text{grad}_H \psi\|_{L^p(S)} \right)$$

for every $\psi \in C_0^1(S)$.
Proof. For any given $q \in [p, p^*]$ there exists $\alpha \in [0, 1]$ such that $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$. Hence
\[
\|\psi\|_{L^q(S)} \leq \|\psi\|_{L^p(S)}^{\alpha} \|\psi\|_{L^{p^*}(S)}^{1-\alpha} \leq \|\psi\|_{L^p(S)} + \|\psi\|_{L^{p^*}(S)},
\]
where we have used interpolation inequality and Young’s inequality. The thesis follows from Corollary 5.4.

Corollary 5.6 (Limit case: $p = Q - 1$). Under the previous assumptions, let $p = Q - 1$. For every $q \in [Q - 1, +\infty[$ there exists $C_q = C_q(\mathcal{H}_H^0, \varrho, \mathcal{G})$ such that
\[
\|\psi\|_{L^q(S)} \leq C_q \left( \|\psi\|_{L^p(S)} + \|\operatorname{grad}_{H^S} \psi\|_{L^p(S)} \right)
\]
for every $\psi \in C_0^1(S)$.

Proof. By using (15) we easily get that there exists $C_1 = C_1(\mathcal{H}_H^0, t, \varrho, \mathcal{G}) > 0$ such that
\[
\left( \int_S |\psi|^t \frac{\varrho - 1}{\varrho - t} \sigma_H^{-1} \right)^{\frac{\varrho - 2}{\varrho - t}} \leq C_1 \int_S (|\psi|^t + |\psi|^{t-1} |\operatorname{grad}_{H^S} \psi|) \sigma_H^{-1}
\]
for every $\psi \in C_0^1(S)$. From now on we assume that $t \geq 1$. Using Hölder inequality with $p = Q - 1$, yields
\[
\|\psi\|_{L^t(\frac{\varrho - 1}{\varrho - 2} S)} \leq C_1 \left( \|\psi\|_{L^t(S)} + \|\psi\|_{L^{(\varrho - 1)(Q - 1)}(S)} \right)
\]
for every $\psi \in C_0^1(S)$ and $t \geq 1$. By means of Young’s inequality, we get that there exists another constant $C_2 = C_2(\mathcal{H}_H^0, t, \varrho, \mathcal{G})$ such that
\[
\|\psi\|_{L^t(\frac{\varrho - 1}{\varrho - 2} S)} \leq C_2 \left( \|\psi\|_{L^t(S)} + \|\psi\|_{L^{(\varrho - 1)(Q - 1)}(S)} + \|\operatorname{grad}_{H^S} \psi\|_{L^{Q - 1}(S)} \right).
\]
By setting $t = Q - 1$ in the last inequality we get that
\[
\|\psi\|_{L^{(Q - 1)^2}(S)} \leq C_2 \left( \|\psi\|_{L^{Q - 1}(S)} + \|\operatorname{grad}_{H^S} \psi\|_{L^{Q - 1}(S)} \right).
\]
By reiterating this procedure for $t = Q, Q + 1, \ldots$ one can show that for all $q \geq Q - 1$ there exists $C_q = C_q(\mathcal{H}_H^0, \varrho, \mathcal{G})$ such that
\[
\|\psi\|_{L^q(S)} \leq C_q \left( \|\psi\|_{L^{Q - 1}(S)} + \|\operatorname{grad}_{H^S} \psi\|_{L^{Q - 1}(S)} \right)
\]
for every $\psi \in C_0^1(S)$.

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Francescopaolo Montefalcone:  
Dipartimento di Matematica Pura ed Applicata  
Università degli Studi di Padova  
Address: Via Trieste, 63, 35121 Padova (Italy)  
E-mail: montefal@math.unipd.it