Research Article

Alternative Legendre Polynomials Method for Nonlinear Fractional Integro-Differential Equations with Weakly Singular Kernel

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Abstract

In this paper, we present a numerical scheme for finding numerical solution of a class of weakly singular nonlinear fractional integro-differential equations. This method exploits the alternative Legendre polynomials. An operational matrix, based on the alternative Legendre polynomials, is derived to be approximated the singular kernels of this class of the equations. The operational matrices of integration and product together with the derived operational matrix are utilized to transform nonlinear fractional integro-differential equations to the nonlinear system of algebraic equations. Furthermore, the proposed method has also been analyzed for convergence, particularly in context of error analysis. Moreover, results of essential numerical applications have also been documented in a graphical as well as tabular form to elaborate the effectiveness and accuracy of the proposed method.

1. Introduction

In this study, the operational alternative Legendre method is introduced and employed to solve a class of nonlinear fractional integro-differential equations with weakly singular kernel:

\[
D_\alpha^\alpha y(t) = F(y(t)) + \int_0^t (t-s)^{-\beta} G(y(s)) ds + f(t), \quad \alpha > 0, \ 0 \leq \beta < 1, \ t \in [0,1],
\]

\[
y^{(i)}(0) = y_0^{(i)}, \quad i = 0, 1, 2, \ldots, [\alpha] - 1,
\]

where \( F: C([0,1]) \to \mathbb{R} \) and \( G: C([0,1]) \to \mathbb{R} \) (sufficiently smooth operators) are considered to be nonlinear. \( D_\alpha^\alpha \) are the Caputo fractional derivative operators, \([\alpha]\) is the ceiling function of \( \alpha \), \( f(t) \) is a continuous function, and \( y(t) \) is an unknown function to be determined.

These equations come from the mathematical modeling of various phenomena, such as radiative equilibrium, heat conduction problems, elasticity, and fracture mechanics (see [1–5]). Since the numerical solution of the nonlinear fractional equations is almost a new subject and because of having a singular kernel, there are many schemes for solving this kind of equations. Heydari et al. have utilized the Chebyshev wavelet method to solve systems of the linear and nonlinear singular fractional Volterra integro-differential equations (see [6]). Mohammadi has applied the block pulse functions for the linear and nonlinear singular fractional
integro-differential equations (see [7]). Zhao et al. have proposed the piecewise polynomial collocation method for solving the fractional integro-differential equations with weakly singular kernels (see [8]). In [9, 10], the operational Tau method has been used to solve this kind of equations. A spectral method based on the Chebyshev polynomials of the second kind has been applied in [11]. Yi et al. have dealt with the CAS wavelets and Legendre wavelets method for solving the linear and nonlinear fractional weakly singular integro-differential equations (see [12, 13]), and so on (see [14–16]).

In this paper, application of the alternative Legendre polynomials is extended to solve the nonlinear weakly singular fractional order integro-differential equations. For this purpose, the fractional operational matrices of integration and product are derived. Also, an operational matrix is derived to approximate the integral part with the singular kernel in equation (1). The matrices and approximations result are substituted into the given equation to convert it into the nonlinear system of algebraic equations. The nonlinear systems can be solved by the well-known Newton iteration method. Also, error analysis and convergence analysis of the proposed method are presented.

2. Definitions of Fractional Derivatives and Integrals

**Definition 1.** Riemann–Liouville fractional integral of order \( \alpha \), \((\alpha \geq 0)\) (see [17]):

\[
I^\alpha_0 y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} y(r) \, dr, \quad t > 0,
\]

(3)

**Definition 2.** Caputo’s fractional derivative of order \( \alpha \), \((\alpha \geq 0)\) (see [17]):

\[
D^\alpha_t y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{y^{(n)}(r)}{(t-r)^{\alpha+n-1}} \, dr, \quad 0 \leq r-1 < \alpha < r.
\]

(4)

Specially, the operator \( D^\alpha_t \) satisfies the following properties (\( c \) is a constant) (see [18]):

\[
D^\alpha_0 c = 0,
\]

\[
D^\alpha_d \beta = \begin{cases} 
0, & \beta \in N_0, \beta < |\alpha|, \\
\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \beta^{\beta-\alpha}, & \beta \in N_0, \beta \geq |\alpha|; \beta \notin N_0, \beta > |\alpha|, \\
D^\alpha t_0 y(t) = y(t), \\
I^\alpha_0 D^\alpha_t y(t) = y(t) - \sum_{k=0}^{[\alpha]-1} \frac{y^{(k)}(0^+)}{k!} t^k, & t > 0.
\end{cases}
\]

(5)

3. Alternative Legendre Polynomials and Their Operational Matrices

The set \( P_n = [P_{nk}(t)]_{k=0}^n \) of alternative Legendre polynomials of degree \( n \) in \([0, 1]\) (see [19]) is given by

\[
P_{nk}(t) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \binom{n+k+j+1}{n-k} t^{k+j}, \quad k = 0, 1, \ldots, n.
\]

(6)

They are orthogonal over the interval \([0, 1]\) with respect to the weight function \( \omega(t) = 1 \) and satisfy the property as follows:

\[
\int_0^1 P_{nl}(t)P_{ml}(t) \, dt = \begin{cases} 
1 & k = l, k, l = 0, 1, \ldots, n, \\
0, & k \neq l.
\end{cases}
\]

(7)

Equation (6) may be rewritten as Rodrigues’s type:

\[
P_{nk}(t) = \frac{1}{(n-k)!} \frac{d^{n-k}}{dt^{n-k}} (t^{n+k+1} (1-t)^{-k}), \quad k = 0, 1, \ldots, n.
\]

(8)

Integrating equation (8) from 0 to 1, we have

\[
\int_0^1 P_{nk}(t) \, dt = \frac{1}{n+k+1}, \quad k = 0, 1, \ldots, n.
\]

(9)

Suppose the alternative Legendre polynomials vector as

\[
\Phi(t) = [P_{n0}(t), P_{n1}(t), \ldots, P_{nn}(t)]^T.
\]

(10)

Equation (10) can be rewritten as the form

\[
\Phi(t) = Q \cdot T(t),
\]

(11)

where

\[
T(t) = [1, t, t^2, \ldots, t^n]^T,
\]

(12)

and \( Q \) is the upper triangular matrix defined by

\[
Q = [q_{kj}]_{k,j=0}^n, \quad q_{kj} = \begin{cases} 
0, & 0 \leq j < k, \\
(-1)^j \binom{n-k}{j} \binom{n+j+1}{n-k}, & k \leq j \leq n.
\end{cases}
\]

(13)

A square integrable function \( y(t) \in L^2(0, 1) \) can be expressed in terms of the alternative Legendre polynomials basis as follows:

\[
y(t) = \sum_{k=0}^{\infty} c_k P_{nk}(t).
\]

(14)

In practice, only the first \((n+1)\) term of alternative Legendre polynomials is considered. Hence, one has
Theorem 1. Let \( \Phi(t) \) be the alternative Legendre polynomials vector obtained by equation (10), then we have the following:

\[
y(t) = \sum_{k=0}^{n} c_k P_{nk}(t) = C^T \Phi(t),
\]

where \( C = [c_0, c_1, \ldots, c_n]^T \), \( c_k = \langle y, P_{nk} \rangle / \langle P_{nk}, P_{nk} \rangle = (2k+1) \langle y, P_{nk} \rangle, k = 0, 1, \ldots, n \) are called alternative Legendre polynomials coefficients, and \( \langle y, y \rangle = \int_0^1 y^2(t)dt \).

In implementing the operations on the alternative Legendre basis, we frequently encounter the integration and the product of the vector \( \Phi(t) \) and also it is necessary to evaluate the integration, the product of the vector \( \Phi(t) \). For this purpose, some operational matrices will be derived. To pursue, we need the following lemmas. The proofs of these lemmas are quite easy by using the definition.

Lemma 1. Let \( P_{nk}(t) = \sum_{r=0}^{n} p_{kr}^{(k)} t^r \) and \( P_{nj}(t) = \sum_{r=0}^{n} p_{jr}^{(j)} t^r \) be the \( k^{th} \) and \( j^{th} \) alternative Legendre polynomials, respectively. Then, the product of \( P_{nk}(t) \) and \( P_{nj}(t) \) can be written as

\[
P_{nk}(t)P_{nj}(t) = \sum_{r=0}^{2n} q_r^{(k,j)} t^r,
\]

where \( q_r^{(k,j)} = \left\{ \begin{array}{ll} \sum_{l=0}^{r} c_l^{(k)} c_{r-l}^{(j)} & r \leq n \allowbreak \vspace{1ex} \sum_{l=n-r}^{n} c_l^{(k)} c_{r-l}^{(j)} & r > n \end{array} \right. \).

Lemma 2. Let \( r \) be a positive integer, then we have

\[
I^r(\Phi(t)) = J \cdot \Phi(t),
\]

The alternative Legendre polynomials operational matrix of fractional integration is shown by the following theorem.

Theorem 1. Let \( \Phi(t) \) be the alternative Legendre polynomials vector obtained by equation (10), then we have the following:

\[
P_{kr} = (2r+1) \sum_{j=0}^{n-k} \frac{(-1)^j}{\Gamma(\alpha+k+j+1)} \binom{n-k}{j} \binom{n+k+j+1}{n-k} \sum_{l=0}^{n-r} \frac{(-1)^l}{\Gamma(\alpha+k+l+1)} \binom{n-r}{l} \binom{n+r+l+1}{n-r},
\]

where \( \tilde{C} = [\tilde{c}_k]_{k=0}^{n} \) is the alternative Legendre polynomials operational matrix of the product in which

\[
\tilde{c}_k = (2k+1) \sum_{j=0}^{n} c_j \xi_{ijk}, \tag{22}
\]

\[
\xi_{ijk} = \int_0^1 P_{ni}(t)P_{nj}(t)P_{nk}(t)dt.
\]
The values of \( \xi_{ijk} \) are calculated by using Lemma 3.

**Proof (see [20]).**

**Theorem 3.** Suppose that \( y(t) \in C([0,1]), 0 < \beta < 1, \text{ and } y(t) = C^T \Phi(t) \), then

\[
\int_0^t \frac{y(s)}{(t-s)^\beta} ds = C^T B^{(\beta)} \Phi(t),
\]

where \( B^{(\beta)} = [b^{(\beta)}_{jk}]_{k=0}^n \) is a \((n+1) \times (n+1)\) matrix and its elements are as follows:

\[
b^{(\beta)}_{jk} = (2r+1) \sum_{j=0}^{n-k} (-1)^j \frac{\Gamma(k+j+2-\beta)}{\Gamma(k+j+\beta+2)} \frac{\Gamma(n-k+j+1)}{\Gamma(n-k)} \frac{n-r}{(n-r+1)} \frac{n+r+l+1}{l}.
\]

**Proof.** According to the definition of the vector \( \Phi(t) \), one has

\[
\int_0^t \frac{y(s)}{(t-s)^\beta} ds = C^T \int_0^t \frac{\Phi(s)}{(t-s)^\beta} ds,
\]

\[
= C^T \left[ \int_0^t \frac{P_{00}(s)}{(t-s)^\beta} ds, \int_0^t \frac{P_{01}(s)}{(t-s)^\beta} ds, \ldots, \int_0^t \frac{P_{nn}(s)}{(t-s)^\beta} ds \right]^T.
\]

Using equation (6) leads to

\[
\int_0^t \frac{P_{nk}(s)}{(t-s)^\beta} ds = \sum_{j=0}^{n-k} (-1)^j \frac{n-k}{j} \frac{n+k+j+1}{n-k} \frac{n-r}{(n-r+1)} \frac{n+r+l+1}{l} \int_0^t \frac{k^{j-\beta+1}}{(t-s)^\beta} ds,
\]

where, by Lemma 2, one can obtain

\[
k^{j-\beta+1} = \sum_{r=0}^{n} b_{kj} P_{nr}(t),
\]

where

\[
b_{kj} = (2r+1) \int_0^1 k^{j-\beta+1} P_{sr}(t) dt = (2r+1) \sum_{l=0}^{n-r} (-1)^l \frac{n-r}{l} \frac{n+r+l+1}{n-r} \frac{n+r+l+1}{k+l+r+j+2-\beta}.
\]
Therefore,

\[\int_0^t P_m(s) \frac{d^\alpha s}{(t-s)^\beta} ds = \sum_{r=0}^n (2r+1) \left[ \sum_{j=0}^{n-k} \frac{(-1)^j}{\Gamma(k+j+2-\beta)} \binom{n-k}{j} \binom{n+k+j+1}{n-k} \Gamma(k+j+1) \Gamma(1-\beta) \right] \sum_{l=0}^{n-r} \frac{(-1)^l}{\Gamma(k+l+r+j+2-\beta)} \binom{n-r}{l} P_{nr}(t),\]

= \sum_{r=0}^n b_{r1} P_{nr}(t).

(29)

So, equation (25) can be written as follows:

\[\int_0^t \Phi(s) \frac{d^\alpha s}{(t-s)^\beta} ds = \left[ \int_0^t \frac{P_{n0}(s)}{(t-s)^\beta} ds, \int_0^t \frac{P_{n1}(s)}{(t-s)^\beta} ds, \ldots, \int_0^t \frac{P_{nm}(s)}{(t-s)^\beta} ds \right]^T,\]

\[= \begin{bmatrix} b_{00} & b_{01} & \ldots & b_{0m} \\ b_{10} & b_{11} & \ldots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} P_{n0}(t) \\ P_{n1}(t) \\ \vdots \\ P_{nm}(t) \end{bmatrix}.\]

(30)

This proof is completed.

4. Solution of Nonlinear Fractional Integro-Differential Equations with Weakly Singular Kernel

To explain the implementation process of the proposed method, the following case of equation (1) is considered:

\[D_x^\alpha y(t) = F(y^{m_1}(t)) + \int_0^t (t-s)^{\beta_1} y^{m_2}(s) ds + f(t), \quad \alpha > 0, 0 \leq \beta < 1, t \in [0, 1],\]

where \(m_1\) and \(m_2\) are the positive integers.

By applying the introduced matrices and the basis vector in the previous section, the terms of the equations under investigation may be approximated in terms of the alternative Legendre polynomials as follows.

\[D_x^\alpha y(t) \approx C_x^\alpha \Phi(t).\]

(32)

By applying the fractional integral operator of order \(\alpha\) to equation (28), an approximation of unknown function \(y(t)\) results as follows:

\[y(t) \approx C_x^\alpha \Phi(t) + \sum_{k=0}^{[\alpha]} \frac{\alpha^k}{k!} y^{(k)}(0) \frac{t^k}{k!} = C_x^\alpha \Phi(t) + X^\alpha \Phi(t) = Y^\alpha \Phi(t),\]

(33)
\[ \int_{0}^{t} (t - s)^{-\beta} \gamma_{m}^\beta (s) ds = Y^T C_{m-1} Y^T C_{m-1}^{-1} \int_{0}^{t} (t - s)^{-\beta} \Phi (s) ds = Y^T C_{m-1} B^\beta \Phi (t). \]  

(36)

After substituting approximations equations (32)–(36) into equation (31), the main equation is converted into the following nonlinear algebraic equation:

\[ C^T \Phi (t) = F (Y^T C_{m-1} \Phi (t)) + Y^T C_{m-1} B^\beta \Phi (t) + f (t). \]  

(37)

Equation (37) is collocated at the \( n + 1 \) Gauss–Chebyshev nodes (see [21]). By solving the nonlinear systems of the generated algebraic equations, the unknown vector \( C^T \) can be determined. The resultant nonlinear systems can be solved by the Newton iteration method.

5. Error Analysis and Convergence

5.1. Error Estimate. Assume that \( C^{\infty} ([0, 1]) \) is the space of functions and \( y : [0, 1] \rightarrow R \) are the continuous derivatives,

\[ y^{(i)} (t) = \frac{d^i}{dt^i} y (t), \quad t \in [0, 1], \]  

(38)

for all \( i \) such that \( 0 \leq i \leq n + 1 \), then we have the following theorem.

**Theorem 4.** Suppose that \( y (t) \in C^{n+1} ([0, 1]) \) and \( y_n (t) = C^T \Phi (t) \) are \( y (t) \) expansion in terms of alternative Legendre polynomials. Then, for \( t \in [0, 1] \),

\[ \| y (t) - y_n (t) \|_2 \leq \frac{C}{(n + 1)! 2^{2n+1}}, \]  

(39)

where \( C \) is a constant such that \( \| y^{(n+1)} (t) \| \leq C. \)  

(40)

**Proof.** Assuming that \( q_n (t) \) are the interpolating polynomials to \( y (t) \) at points \( t_i \) \( (t_i \) are the roots of the shifted Chebyshev polynomials of degree \( n + 1 \)\), one has

\[ y (t) - q_n (t) = \left[ y^{(n+1)} (t) \right] \frac{\eta !}{(n + 1)!} \sum_{i=0}^{n} (t - t_i), \quad \eta \in [0, 1]. \]  

(41)

\[ g (t) = \sum_{i=0}^{\lfloor a/1 \rfloor} y_0^\beta (t - t_i)^{\alpha-1} F (y (t)) + \frac{1}{\Gamma (\alpha)} \int_{0}^{t} (t - \tau)^{\alpha-1} f (\tau) d\tau. \]  

(46)

\[ k_1 (t, \tau) = \frac{(1 - \beta)}{\Gamma (\alpha - \beta + 1)} (t - \tau)^{\alpha-1}. \]  

(47)

**Proof.** Applying the Riemann–Liouville integral operator \( I^\alpha_0 \) to both sides of equation (1), we have

\[ y (t) = g (t) + \frac{1}{\Gamma (\alpha)} \int_{0}^{t} (t - \tau)^{\alpha-1} \left( \int_{0}^{\tau} (\tau - s)^{-\beta} G (y (s)) ds \right) d\tau, \]  

(48)
where \( g(t) \) is defined by equation (46).

Using the Dirichlet’s formula to the last part of the right-hand side of equation (48), we get

\[
\frac{1}{\Gamma(a)} \int_0^t \int_0^\tau (t-\tau)^{a-1} (\tau-s)^{-\beta} G(y(s)) ds \, dr = \frac{1}{\Gamma(a)} \int_0^t \int_\tau^t (t-\tau)^{a-1} (\tau-s)^{-\beta} G(y(s)) ds \, dr, 
\]

\[
= \frac{1}{\Gamma(a)} \int_0^t \int_\tau^t (t-s)^{a-1} (s-\tau)^{-\beta} G(y(\tau)) ds \, dr. 
\]  (49)

Let \( w = (s-\tau/t-\tau) \), then equation (49) will be

\[
\frac{1}{\Gamma(a)} \int_0^t \int_\tau^t (t-s)^{a-1} (s-\tau)^{-\beta} G(y(\tau)) ds \, dr = \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-\beta} \left( \int_0^1 (1-w)^{a-1} w^{-\beta} dw \right) G(y(\tau)) ds \, dr, 
\]

\[
= \frac{B(a, 1-\beta)}{\Gamma(a)} \int_0^t (t-\tau)^{a-\beta} G(y(\tau)) ds \, dr 
\]

\[
= \frac{\Gamma(a) \Gamma(1-\beta)}{\Gamma(a-\beta+1)} \int_0^t (t-\tau)^{a-\beta} G(y(\tau)) ds \, dr. 
\]  (50)

where \( B \) is the beta function.

This proof is completed.

Let \( g_n(t) \) be the approximation of \( g(t) \) and \( f_n(t) \) be the alternative Legendre approximation of \( f(t) \) and suppose the nonlinear terms \( F(u) \) and \( G(u) \) are satisfied in Lipschitz condition such that

\[
\|F(u) - F(v)\| \leq L_1 \|u - v\|, 
\]  (51)

\[
\|G(u) - G(v)\| \leq L_2 \|u - v\|. 
\]  (52)

\[
\|y(t) - y_n(t)\| = \left\| g(t) - g_n(t) + \int_0^t k_1(t, \tau) G(y(\tau)) ds \, dr - \int_0^t k_1(t, \tau) G(y_n(\tau)) ds \, dr \right\| 
\]

\[
\leq \left\| g(t) - g_n(t) \right\| + \left\| \int_0^t k_1(t, \tau) [G(y(\tau)) - G(y_n(\tau))] ds \, dr \right\| 
\]

\[
= I_1(t) + I_2(t), 
\]  (53)

where \( I_1(t) = \| g(t) - g_n(t) \| \) and \( I_2(t) = \| \int_0^t k_1(t, \tau) [G(y(\tau)) - G(y_n(\tau))] ds \, dr \| \).

**Theorem 6.** The solution of the nonlinear fractional order integro-differential equation (45) by using alternative Legendre polynomials approximations is convergent when \( n \to +\infty \).

**Proof.**
\[ I_1 (t) = \| g(t) - g_n(t) \| \]
\[ = \frac{1}{\Gamma (\alpha)} \left\| \int_0^t (t - r)^{\alpha - 1} [F(y(r)) - F(y_n(r))] \, dr + \int_0^t (t - r)^{\alpha - 1} [f(r) - f_n(r)] \, dr \right\| , \]
\[ \leq \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \| F(y(r)) - F(y_n(r)) \| \, dr + \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \| f(r) - f_n(r) \| \, dr \]
\[ \leq \frac{L_1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \| y(r) - y_n(r) \| \, dr + \frac{1}{\Gamma (\alpha)} \int_0^t (t - r)^{\alpha - 1} \| f(r) - f_n(r) \| \, dr \]
\[ \leq \frac{L_1 C}{\Gamma (\alpha) (n + 1)! 2^{2m+1}} \int_0^t (t - r)^{\alpha - 1} \, dr + \frac{C}{\Gamma (\alpha) (n + 1)! 2^{2m+1}} \int_0^t (t - r)^{\alpha - 1} \, dr \]
\[ \leq \frac{(L_1 + 1) C}{\Gamma (\alpha + 1) (n + 1)! 2^{2m+1}}. \]}

Combining Theorem 4 and equation (52) together, one has
\[ I_2 (t) = \left\| \int_0^t k_1 (t, \tau) [G(y(\tau)) - G(y_n(\tau))] \, d\tau \right\|, \]
\[ \leq \int_0^t \| k_1 (t, \tau) \| \cdot \| G(y(\tau)) - G(y_n(\tau)) \| \, d\tau \]
\[ \leq L_2 \int_0^t \| k_1 (t, \tau) \| \cdot \| y(\tau) - y_n(\tau) \| \, d\tau \]
\[ \leq \frac{L_2 C}{(n + 1)! 2^{2m+1}} \int_0^t \| k_1 (t, \tau) \| \, d\tau \]
\[ \leq \frac{L_2 C (1 - \beta)}{(n + 1)! 2^{2m+1} \Gamma (\alpha - \beta + 1)} \int_0^t (t - \tau)^{\alpha - \beta} \, d\tau \]
\[ \leq \frac{L_2 C (1 - \beta)}{\Gamma (\alpha - \beta + 2) (n + 1)! 2^{2m+1}}. \]

Combining equations (53)–(55), we can get the relationship between the convergence order and \( \alpha, \beta \) as follows:
\[ \| y(t) - y_n(t) \| \leq \left( \frac{(L_1 + 1) C}{\Gamma (\alpha + 1)} + \frac{L_2 C (1 - \beta)}{\Gamma (\alpha - \beta + 2)} \right) \frac{(n + 1)! 2^{2m+1}}{(n + 1)! 2^{2m+1}}. \]}

6. Numerical Examples

In this section, some numerical examples are presented to illustrate the proposed alternative Legendre polynomials method. In order to demonstrate the error of the method, the notation is shown as
\[ e_n = \max_{0 \leq t \leq n} | y(t_i) - y_n(t_i) |, \]
where \( y(t) \) and \( y_n(t) \) are the exact solution and the numerical solution, respectively, \( t_i \in [0, 1] \). All algorithms are performed by Mathematica 10.0.
Example 1. Consider the following nonlinear fractional order integro-differential equation with weakly singular kernel (see [23]):

\[ D_t^{\alpha} y(t) = g(t) + p(t) y(t) + \int_0^t (t-s)^{-\frac{1}{2}} y^2(s) ds, \quad t \in [0, 1], \]  

where

\[ g(t) = \frac{3\Gamma(1/2)}{4\Gamma(11/6)} t^{5/6} - t^{5/2} - \frac{32}{35} t^{7/2}, \quad p(t) = t, \]  

with initial value \( y(0) = 0 \).

The exact solution of this equation is \( y(t) = t^{3/2} \). The absolute errors between the numerical solutions and the exact solution with different values of \( n \) are displayed in Table 1 and compared with the results obtained by MHFs (modification of hat functions) (see [23]). From Table 1, it can be seen that the absolute errors become smaller and smaller with \( n \) increasing. Satisfactory results would be acquired by a small number of alternative Legendre polynomials. Figure 1 shows the comparison of \( e_n \) obtained by our method and MHFs for different \( n \) on logarithmic scale. From Figure 1, we find that \( e_n \) are smaller than that, and the accuracy of our method is higher.

Example 2. Consider the nonlinear fractional integro-differential equation with weakly singular kernel as follows (see [23]):

\[ g(t) = \frac{\Gamma(4)}{\Gamma(16/5)} t^{11/5} + \frac{\Gamma(7/3)}{\Gamma(23/15)} t^{8/15} - t^6 - 2t^{13/3} - t^{8/3} - \sqrt{\pi} \left( \frac{\Gamma(7)}{\Gamma(15/2)} t^{13/2} + \frac{\Gamma(11/3)}{\Gamma(25/6)} t^{19/6} + \frac{2\Gamma(16/3)}{\Gamma(35/6)} t^{29/6} \right), \]  

with initial value \( y(0) = 0 \).

The exact solution of this equation is \( y(t) = t^3 + t^{4/3} \). The figures of the numerical solutions and the exact solution for the problem with different \( n \) are shown in Figures 4–7. From Figures 4–7, we can see that the approximate solutions are getting close to the exact solutions with \( n \) increasing.

Example 3. Consider this equation:

\[ D_t^{\alpha} y(t) = g(t) + y^2(t) + \int_0^t (t-s)^{-1/2} y^2(s) ds, \quad t \in [0, 1], \]  

where

\[ g(t) = 3t^2 \left( \frac{\sqrt{\pi} \Gamma(7/15)}{\Gamma(15/2)} \right) t^{23/2}, \]  

with \( y(0) = 0 \).

The exact solution of the nonlinear equation for \( \alpha = 1 \) is \( y(t) = t^3 \). The numerical solutions and comparisons are given in Table 2 and Figure 2. The absolute errors for \( \alpha = 1 \) are listed in Table 2. It can be seen that the values of the absolute errors decay as \( n \) increases from 5 to 8. The comparisons of \( e_n \) obtained by our method and MHFs for different \( n \) on logarithmic scale are plotted in Figure 2. This figure shows good coincidence between the numerical and exact solutions.

According to Figure 2, it can be deduced that the results obtained by the alternative Legendre polynomials method are more precise than those obtained by MHFs. Figure 3 displays the numerical results for \( \alpha = 0.7, 0.8, 0.9, 1 \) and the exact result for \( \alpha = 1 \). It shows that the numerical solutions converge to the exact solution as \( \alpha \rightarrow 1 \).

Example 4. Consider the following nonlinear fractional integro-differential equation with weakly singular kernel:

\[ D_t^{\alpha} y(t) = g(t) + \int_0^t (t-s)^{-1/2} y^2(s) ds, \quad t \in [0, 1], \]  

where

\[ g(t) = m t^{m-1} - \sin(t) \cdot t^m - \cos(t) \cdot t^{2m} - \left( \sqrt{\pi} \Gamma(2m+1)/(\Gamma(2m+1.5)) \right) t^{2m+0.5}, \]  

such that \( y(0) = 0 \).

The exact solution is \( y(t) = t^m \) if \( \alpha = 1 \), and \( m \) is a positive integer. The graphs of \( e_n \) for different \( m \) on logarithmic scale are depicted in Figures 8–10. From Figures 8–10, one can find that for large \( n \), the error norm becomes smaller and their logarithm behaves approximately as a linear function of \( n \), which means that the error norm decreases linearly.

Example 5. As the last example, consider the nonlinear fractional integro-differential equation with weakly singular kernel as follows:

\[ D_t^{\alpha} y(t) = g(t) + \sin(t) \cdot y(t) + \cos(t) \cdot y^2(t) + \int_0^t (t-s)^{-1/2} y^2(s) ds, \quad t \in [0, 1], \]  

where

\[ g(t) = m t^{m-1} - \sin(t^m) - \left( \sqrt{\pi} \Gamma(2m+1)/(\Gamma(2m+1.5)) \right) t^{2m+0.5}, \]  

such that \( y(0) = 0 \).

The exact solution of this equation is also \( y(t) = t^m \) for \( \alpha = 1 \). The approximate solutions are plotted in Figure 11 for the values of \( \alpha = 0.7, 0.8, 0.9, 1 \) and \( m = 4, n = 6 \). Figure 11 shows that the exact and approximate solutions are in good agreement when \( \alpha = 1 \). It also illustrates that the numerical solutions converge to the exact solution as \( \alpha \rightarrow 1 \).
Table 1: The absolute errors with different $n$ for Example 1.

| $t$  | $n = 4$       | $n = 5$       | $n = 6$       | $n = 7$       | $n = 8$ (see [23]) |
|------|---------------|---------------|---------------|---------------|-------------------|
| 0    | 0.00000       | 0.00000       | 0.00000       | 0.00000       | 0.00000           |
| 0.125| 2.82184e-03   | 1.28015e-03   | 6.59465e-04   | 3.98172e-04   | 1.59756e-03       |
| 0.25 | 1.27453e-03   | 5.94191e-04   | 4.86154e-04   | 4.02646e-04   | 5.69812e-04       |
| 0.375| 1.10784e-03   | 8.65911e-04   | 6.26588e-04   | 3.89888e-04   | 6.53198e-04       |
| 0.5  | 1.00480e-03   | 1.09916e-03   | 5.90305e-04   | 3.92234e-04   | 8.56779e-04       |
| 0.625| 2.46826e-03   | 1.12048e-03   | 7.02201e-04   | 5.1706e-04    | 8.81666e-04       |
| 0.75 | 2.88819e-03   | 1.45003e-03   | 1.07896e-03   | 7.13425e-04   | 1.42594e-03       |
| 0.875| 3.92612e-03   | 2.48974e-03   | 1.50493e-03   | 1.05219e-03   | 1.81991e-03       |
| 1    | 7.69399e-03   | 3.50573e-03   | 2.68452e-03   | 1.64642e-03   | 3.30803e-03       |

Figure 1: $e_n$ on logarithmic scale for different methods (Example 1).

Table 2: The absolute errors with different $n$ for Example 2 ($\alpha = 1$).

| $t$  | $n = 5$       | $n = 6$       | $n = 7$       | $n = 8$       | $n = 16$ (see [23]) |
|------|---------------|---------------|---------------|---------------|---------------------|
| 0    | 0.00000       | 0.00000       | 0.00000       | 0.00000       | 0.00000             |
| 0.125| 5.24999e-04   | 2.26218e-05   | 8.97476e-07   | 9.3225e-08    | 4.75615e-09         |
| 0.25 | 4.42941e-04   | 1.70919e-05   | 6.88273e-07   | 8.62125e-08   | 1.46318e-07         |
| 0.375| 3.55136e-04   | 1.66330e-05   | 7.81163e-07   | 9.59676e-08   | 7.47778e-07         |
| 0.5  | 3.81245e-04   | 1.94631e-05   | 8.00897e-07   | 8.92258e-08   | 2.25373e-06         |
| 0.625| 4.57360e-04   | 1.95593e-05   | 7.51114e-07   | 9.51004e-08   | 5.19464e-06         |
| 0.75 | 4.84045e-04   | 1.86419e-05   | 8.55455e-07   | 1.03849e-07   | 1.01464e-05         |
| 0.875| 4.77051e-04   | 2.25186e-05   | 9.59522e-07   | 1.08441e-07   | 1.76618e-05         |
| 1    | 7.12625e-04   | 2.23668e-05   | 1.22798e-06   | 1.22564e-07   | 2.81379e-05         |

Figure 2: $e_n$ on logarithmic scale for different methods (Example 2).
Figure 3: Numerical solutions (different $\alpha$) and exact solution ($\alpha = 1$) with $n = 6$ for Example 2.

Figure 4: Comparison of the numerical solutions with exact solution for $n = 6$.

Figure 5: Comparison of the numerical solutions with exact solution for $n = 8$.

Figure 6: Comparison of the numerical solutions with exact solution for $n = 10$.

Figure 7: Comparison of the numerical solutions with exact solution for $n = 12$.

Figure 8: $e_n$ on logarithmic scale for $m = 3$ (Example 4).
7. Conclusion

The alternative Legendre polynomials operational matrix method has been developed for solving the weakly singular nonlinear integro-differential equations with fractional derivatives. The method exploits the operational matrices of the fractional integration and the product of alternative Legendre polynomials as the primary underlying tool. A given nonlinear equation is converted into a nonlinear system of algebraic equations. The method has also been analyzed for error and convergence. A new operational matrix is obtained to approximate the integral parts with the singular kernels. To illustrate the reliability and efficiency of the suggested approach, the alternative Legendre operational matrices and the relevant approximations are employed to solve several examples. In the examples in which the order of the fractional derivatives is known, the numerical results show good agreement between the numerical and exact solutions. When the order of the fractional derivative is uncertain, the numerical solutions for the various values of the approach to the exact solutions as $\alpha \to 1$. The results obtained by our method are also compared with those obtained by some existing methods. From the tables and figures, it can be concluded that the results obtained by the suggested method are more precise than those. The results confirm the ability of the alternative Legendre polynomials method to solve the nonlinear fractional integro-differential equations with weakly singular kernels.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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