On the best Ulam constant of the linear differential operator with constant coefficients

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Received: date / Accepted: date

Abstract The linear differential operator with constant coefficients

\[ D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in C^n(\mathbb{R}, X) \]

acting in a Banach space \( X \) is Ulam stable if and only if its characteristic equation has no roots on the imaginary axis. We prove that if the characteristic equation of \( D \) has distinct roots \( r_k \) satisfying \( \Re r_k > 0, 1 \leq k \leq n \), then the best Ulam constant of \( D \) is

\[ K_D = \frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^n (-1)^k V_k e^{-r_k x} \right| dx, \]

where \( V = V(r_1, r_2, \ldots, r_n) \) and \( V_k = V(r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n), 1 \leq k \leq n \), are Vandermonde determinants.

Keywords Linear differential operator, Ulam stability, Best constant, Banach space

Mathematics Subject Classification (2010) 34D20, 39B82

1 Introduction

In this paper, we denote by \( K \) the field of real numbers \( \mathbb{R} \) or the field of complex numbers \( \mathbb{C} \). Let \( M \) and \( N \) be two linear spaces over the field \( K \).

Definition 1 A function \( \rho_M : M \to [0, \infty] \) is called a gauge on \( M \) if the following properties hold:

i) \( \rho_M(x) = 0 \) if and only if \( x = 0 \);
ii) $\rho_M(\lambda x) = |\lambda|\rho_M(x)$ for all $x \in M$, $\lambda \in \mathbb{K}$, $\lambda \neq 0$.

Throughout this paper we denote by $(X, \|\cdot\|)$ a Banach space over the field $\mathbb{C}$ and by $\mathcal{C}^n(\mathbb{R}, X)$ the linear space of all $n$ times differentiable functions with continuous $n$-th derivatives, defined on $\mathbb{R}$ with values in $X$. $\mathcal{C}^0(\mathbb{R}, X)$ will be denoted as usual by $\mathcal{C}(\mathbb{R}, X)$. For $f \in \mathcal{C}^n(\mathbb{R}, X)$ define

$$
\|f\|_\infty = \sup\{\|f(t)\| : t \in \mathbb{R}\}.
$$

(1)

Then $\|f\|_\infty$ is a gauge on $\mathcal{C}^n(\mathbb{R}, X)$. We suppose that $\mathcal{C}^n(\mathbb{R}, X)$ and $\mathcal{C}(\mathbb{R}, X)$ are endowed with the same gauge $\|\cdot\|_\infty$.

Let $\rho_M, \rho_N$ be two gauges on the linear spaces $M$ and $N$, respectively and let $L : M \to N$ be a linear operator.

We denote by $\ker L = \{x \in M | Lx = 0\}$ and $R(L) = \{Lx | x \in M\}$ the kernel and the range of the operator $L$, respectively.

**Definition 2** We say that the operator $L$ is Ulam stable if there exists $K \geq 0$ such that for every $\varepsilon > 0$ with $\rho_N(Lx) \leq \varepsilon$ there exists $z \in \ker L$ with the property $\rho_M(x - z) \leq K\varepsilon$.

The Ulam stability of the operator $L$ is equivalent with the stability of the associated equation $Lx = y$, $y \in R(L)$. An element $x \in M$ satisfying $\rho_N(Lx) \leq \varepsilon$ for some positive $\varepsilon$ is called an approximate solution of the equation $Lx = y$, $y \in R(L)$. Consequently, Definition 2 can be reformulated as follows: The operator $L$ is Ulam stable if for every approximate solution of $Lx = y$, $y \in R(L)$ there exists an exact solution of the equation near it. The problem of Ulam stability can be traced back to 1940 and is due to Ulam [23]. Ulam formulated this problem during a conference at Madison University, Wisconsin, for the equation of the homomorphisms of a metric group. The first answer to Ulam’s question was given by D.H. Hyers for the Cauchy functional equation in Banach spaces in [8]. In fact, a problem of this type was formulated in the famous book by Polya and Szegö for the Cauchy functional equation on the set of integers; see [19]. Since than, this research area received a lot of attention and was extended to the context of operators, functional, differential or difference equations. For a broad overview on the topic we refer the reader to [30].

The number $K$ from Definition 2 is called an Ulam constant of $L$. In what follows the infimum of all Ulam constants of $L$ is denoted by $K_L$. Generally, the infimum of all Ulam constants of the operator $L$ is not an Ulam constant of $L$ (see [7, 15]) but if it is, it will be called the best Ulam constant of $L$, or simply the Ulam constant of the operator $L$. Finding the best Ulam constant of an equation or operator is a challenging problem because it offers the best measure of the error between the approximate and the exact solution. For linear and bounded operators acting on normed spaces in [7, 22] is given a characterization of their Ulam stability as well as a representation of their best Ulam constant. Using this result D. Popa and I. Raşa obtained the best Ulam constant for Bernstein, Kantorovich and Stancu operators; see [14, 16–18]. For more information on Ulam stability with respect to gauges and on the best Ulam constant of linear operators we refer the reader to [3, 4].
To the best of our knowledge the first result on Ulam stability of differential equations was obtained by M. Obłoza [13]. Thereafter, the topic was deeply investigated by T. Miura, S. Miyajima, S.E. Takahasi in [11, 12, 21]; and S. M. Jung in [10], who gave results for various differential equations and partial differential equations. For further details on Ulam stability we refer the reader to [3, 9, 23].

Let \( a_1, \ldots, a_n \in \mathbb{C} \) and consider the linear differential operator \( D : \mathcal{C}^n(\mathbb{R}, X) \to \mathcal{C}(\mathbb{R}, X) \) defined by

\[
D(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X).
\]  

(2)

Denote by \( P(z) = z^n + a_1 z^{n-1} + \cdots + a_n \) the characteristic polynomial of the operator \( D \) and let \( r_1, \ldots, r_n \) be the complex roots of the characteristic equation \( P(z) = 0 \).

The problem of finding the best Ulam constant was first posed by Th. Rassias in [20]. Since then, various papers on this topic appeared, but there are only few results on the best Ulam constant of differential equations and differential operators. In the sequel we will provide a short overview of some important results concerning the Ulam stability and best Ulam constant of the differential operator \( D \). In [12] it is proved that the operator \( D \) is Ulam stable with the Ulam constant \( \frac{1}{\prod_{k=1}^n |\text{Re} r_k|} \) if and only if its characteristic equation has no roots on the imaginary axis. In [14] D. Popa and I. Raşa obtained sharp estimates for the Ulam constant of the first order linear differential operator and the higher order linear differential operator with constant coefficients. The best Ulam constant of the first order linear differential operator with constant coefficients is obtained in [11]. Later, A.R. Baias and D. Popa obtained the best Ulam constant for the second order linear differential operator with constant coefficients [2]. Recent results on Ulam stability for linear differential equations with periodic coefficients and on the best constant for Hill’s differential equation were obtained by R. Fukutaka and M. Onitsuka in [5, 6]. Important steps in finding the best Ulam constant were made also for higher order difference equations with constant coefficients. For details we refer the reader to [1] and the references therein.

The aim of this paper is to determine the best Ulam constant for the \( n \) order linear differential operator with constant coefficients acting in Banach spaces, for the case of distinct roots of the characteristic equation. Through this result we improve and complement some extant results in the field.

2 Main results

Let \( a_1, \ldots, a_n \in \mathbb{C} \) and consider the linear differential operator \( D : \mathcal{C}^n(\mathbb{R}, X) \to \mathcal{C}(\mathbb{R}, X) \) defined by

\[
D(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X).
\]  

(3)
If \( r_1, r_2, \ldots, r_n \) are distinct roots of the characteristic equation \( P(z) = 0 \), then the general solution of the homogeneous equation \( D(y) = 0 \) is given by

\[
y_H(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x},
\]

where \( C_1, \ldots, C_n \in X \) are arbitrary constants. Consequently

\[
\ker D = \left\{ \sum_{k=1}^{n} C_k e^{r_k x} | C_1, C_2, \ldots, C_n \in X \right\}.
\]

The operator \( D \) is surjective, so according to the variation of constants method, for every \( f \in C(\mathbb{R}, X) \) there exists a particular solution of the equation \( D(y) = f \) of the form

\[
y_P(x) = \sum_{k=1}^{n} C_k(x) e^{r_k x}, \quad x \in \mathbb{R},
\]

where \( C_1, \ldots, C_n \) are functions of class \( C^1(\mathbb{R}, X) \) which satisfy

\[
\begin{pmatrix}
e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\
r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\
\vdots & \vdots & \ddots & \vdots \\
r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x}
\end{pmatrix}
\begin{pmatrix}
C'_1(x) \\
C'_2(x) \\
\vdots \\
C'_n(x)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix}, \quad x \in \mathbb{R}.
\]

In what follows, we denote for simplicity the Vandermonde determinants by \( V := V(r_1, r_2, \ldots, r_n) \) and \( V_k := V(r_1, r_2, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n) \), \( 1 \leq k \leq n \). Consequently, we obtain

\[
C'_k(x) = (-1)^{n+k} \frac{V_k}{V} e^{-r_k x} f(x), \quad k = 1, \ldots, n.
\]

Hence, a particular solution of the equation \( D(y) = f \) is given by

\[
y_P(x) = \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} dt, \quad x \in \mathbb{R}.
\]

The main result concerning the Ulam stability of the operator \( D \) for the case of distinct roots of the characteristic equation is given in the next theorem.

**Theorem 1** Suppose that \( r_k, 1 \leq k \leq n \), are distinct roots of the characteristic equation with \( \text{Re} r_k \neq 0 \) and let \( \varepsilon > 0 \). Then for every \( y \in C^n(\mathbb{R}, X) \) satisfying

\[
\|D(y)\|_{\infty} \leq \varepsilon
\]

there exists a unique \( y_H \in \ker D \) such that

\[
\|y - y_H\|_{\infty} \leq K \varepsilon
\]
where

\[
K = \begin{cases}
\frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^n (-1)^k V_k e^{-r_k x} \right| \, dx, & \text{if } \Re r_k > 0; \\
\frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^n (-1)^k V_k e^{r_k x} \right| \, dx, & \text{if } \Re r_k < 0; \\
\frac{1}{|V|} \int_0^\infty \left( \left| \sum_{k=1}^p (-1)^k V_k e^{-r_k x} \right| + \left| \sum_{k=p+1}^n (-1)^k V_k e^{r_k x} \right| \right) \, dx, & \text{if } \Re r_k > 0; \\
\frac{1}{|V|} \int_0^\infty \left( \left| \sum_{k=p}^n (-1)^k V_k e^{-r_k x} \right| + \left| \sum_{k=p+1}^n (-1)^k V_k e^{r_k x} \right| \right) \, dx, & \text{if } \Re r_k < 0.
\end{cases}
\]

(10)

**Proof** Existence. Suppose that \( y \in C^n(\mathbb{R}, X) \) satisfies (10) and let \( D(y) = f \). Then \( \|f\|_\infty \leq \varepsilon \) and

\[
y(x) = \sum_{k=1}^n C_k e^{r_k x} + \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} \, dt, \quad x \in \mathbb{R},
\]

for some \( C_k \in X, \ 1 \leq k \leq n \).

i) Let first \( \Re r_k > 0, \ 1 \leq k \leq n \). Define \( y_H \in \text{Ker}D \) by the relation

\[
y_H(x) = \sum_{k=1}^n \widetilde{C}_k e^{r_k x}, \quad x \in \mathbb{R}, \quad \widetilde{C}_k \in X,
\]

where

\[
\widetilde{C}_k = C_k + (-1)^{n+k} \frac{V_k}{V} \int_0^\infty f(t) e^{-r_k t} \, dt, \quad 1 \leq k \leq n.
\]

Since \( \|f(t) e^{-r_k t}\| \leq \varepsilon |e^{-r_k t}| = \varepsilon e^{-t \Re r_k}, \ t \geq 0 \), and \( \int_0^\infty e^{-t \Re r_k} \, dt \) is convergent it follows that \( \int_0^\infty f(t) e^{-r_k t} \, dt \) is absolutely convergent, so the constants \( \widetilde{C}_k, 1 \leq k \leq n \) are well defined. Then:

\[
y(x) - y_H(x) = \sum_{k=1}^n C_k e^{r_k x} + \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} \, dt - \sum_{k=1}^n \widetilde{C}_k e^{r_k x}
\]

\[
= \sum_{k=1}^n C_k e^{r_k x} + \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} \, dt
\]

\[
- \sum_{k=1}^n C_k e^{r_k x} - \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x} \int_0^\infty f(t) e^{-r_k t} \, dt
\]

\[
= \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x} \int_x^\infty f(t) e^{-r_k t} \, dt
\]

\[
= \frac{1}{V} \sum_{k=1}^n (-1)^{n+k} V_k \int_x^\infty f(t) e^{r_k (x-t)} \, dt, \quad x \in \mathbb{R}.
\]
Now, letting \( t - x = u \) in the above integral we obtain
\[
y(x) - y_H(x) = -\frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k \int_{0}^{\infty} f(u + x)e^{-ru} du,
\]
\[
= \frac{(-1)^{n+1}}{V} \int_{0}^{\infty} \left( \sum_{k=1}^{n} (-1)^k V_k e^{-ru} \right) f(u + x) du, \quad x \in \mathbb{R}.
\]

Hence
\[
\|y(x) - y_H(x)\| \leq \int_{0}^{\infty} \frac{1}{|V|} \sum_{k=1}^{n} (-1)^k V_k e^{-ru} \cdot \|f(u + x)\| du, \quad x \in \mathbb{R}
\]
\[
\leq \frac{\varepsilon}{|V|} \int_{0}^{\infty} \sum_{k=1}^{n} (-1)^k V_k e^{-ru} du, \quad x \in \mathbb{R},
\]
therefore
\[
\|y - y_0\|_\infty \leq K\varepsilon.
\]

ii) Let \( \text{Re} r_k < 0, 1 \leq k \leq n \). The proof follows analogously, defining
\[
y_H(x) = \sum_{k=1}^{n} \tilde{C}_k e^{rx}, x \in \mathbb{R}, \quad \tilde{C}_k \in X,
\]
with
\[
\tilde{C}_k = C_k - (-1)^{n+k} \frac{V_k}{V} \int_{-\infty}^{0} f(t)e^{-ru} dt \quad 1 \leq k \leq n.
\]

Then
\[
y(x) - y_H(x) = \sum_{k=1}^{n} C_k e^{rx} + \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{0}^{x} f(t)e^{-ru} dt
\]
\[
- \sum_{k=1}^{n} C_k e^{rx} + \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{-\infty}^{0} f(t)e^{-ru} dt
\]
\[
= \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{-\infty}^{0} f(t)e^{-ru} dt,
\]
\[
= \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k \int_{-\infty}^{x} f(t)e^{r(x-t)} dt,
\]
\[
= \frac{(-1)^n}{V} \int_{0}^{\infty} \left( \sum_{k=1}^{n} (-1)^k V_k e^{ru} \right) f(x - u) du, \quad x \in \mathbb{R},
\]
where \( u = x - t \). Hence
\[
\|y(x) - y_H(x)\| \leq \frac{1}{|V|} \int_{0}^{\infty} \left| \sum_{k=1}^{n} (-1)^k V_k e^{ru} \right| \cdot \|f(x - u)\| du = K\varepsilon, x \in \mathbb{R},
\]
which entails
\[
\|y - y_H\|_\infty \leq K\varepsilon.
\]
iii) Let \( \text{Re} r_k > 0, 1 \leq k \leq p \), and \( \text{Re} r_k < 0, p + 1 \leq k \leq n \). Define \( y_H \) by the relation

\[
y_H(x) = \sum_{k=1}^{n} \tilde{C}_k e^{r_k x}, x \in \mathbb{R}, \quad \tilde{C}_k \in X,
\]

with

\[
\tilde{C}_k = C_k + (-1)^{n+k} \frac{V_k}{V} \int_{0}^{\infty} f(t)e^{-r_k t} dt, \quad 1 \leq k \leq p,
\]

\[
\tilde{C}_k = C_k - (-1)^{n+k} \frac{V_k}{V} \int_{-\infty}^{0} f(t)e^{-r_k t} dt, \quad p + 1 \leq k \leq n.
\]

Then

\[
y(x) - y_H(x) = \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_{0}^{\infty} f(t)e^{-r_k t} dt
\]

\[
- \frac{1}{V} \sum_{k=1}^{p} (-1)^{n+k} V_k e^{r_k x} \int_{0}^{\infty} f(t)e^{-r_k t} dt + \frac{1}{V} \sum_{k=p+1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_{0}^{\infty} f(t)e^{-r_k t} dt,
\]

\[
= - \frac{1}{V} \sum_{k=1}^{p} (-1)^{n+k} V_k e^{r_k x} \int_{0}^{\infty} f(t)e^{-r_k t} dt + \frac{1}{V} \sum_{k=p+1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_{-\infty}^{0} f(t)e^{-r_k t} dt,
\]

\[
= - \frac{1}{V} \sum_{k=1}^{p} (-1)^{n+k} V_k \int_{x}^{\infty} f(t)e^{r_k (x-t)} dt + \frac{1}{V} \sum_{k=p+1}^{n} (-1)^{n+k} V_k \int_{-\infty}^{0} f(t)e^{r_k (x-t)} dt.
\]

Letting \( x - t = -u \), respectively \( x - t = u \) in the previous integrals it follows

\[
y(x) - y_H(x) = - \frac{1}{V} \sum_{k=1}^{p} (-1)^{n+k} V_k \int_{0}^{\infty} f(x + u)e^{-r_k u} du
\]

\[
+ \frac{1}{V} \sum_{k=p+1}^{n} (-1)^{n+k} V_k \int_{0}^{\infty} f(x - u)e^{r_k u} du, \quad x \in \mathbb{R}
\]

and

\[
\|y(x) - y_H(x)\| \leq \int_{0}^{\infty} \left( \left| \int_{0}^{\infty} \sum_{k=1}^{p} (-1)^{n+k} V_k e^{-r_k u} \left\| f(x + u) \right\| du \right| + \int_{0}^{\infty} \left| \sum_{k=p+1}^{n} (-1)^{n+k} V_k e^{r_k u} \left\| f(x - u) \right\| du \right| \right) du
\]

\[
\leq \frac{\varepsilon}{|V|} \int_{0}^{\infty} \left( \sum_{k=1}^{p} (-1)^{n+k} V_k e^{-r_k u} \left\| f(x + u) \right\| + \sum_{k=p+1}^{n} (-1)^{n+k} V_k e^{r_k u} \left\| f(x - u) \right\| \right) du, \quad x \in \mathbb{R}.
\]

Therefore we get

\[
\|y - y_0\|_{\infty} \leq K \varepsilon.
\]
The existence is proved.

**Uniqueness.** Suppose that for some \( y \in C^n(\mathbb{R}, X) \) satisfying (8) there exist \( y_1, y_2 \in \ker D \) such that

\[
\|y - y_j\|_\infty \leq K\varepsilon, \quad j = 1, 2.
\]

Then

\[
\|y_1 - y_2\|_\infty \leq \|y_1 - y\|_\infty + \|y - y_2\|_\infty \leq 2K\varepsilon.
\]

But \( y_1 - y_2 \in \ker D \), hence there exist \( C_k \in X, 1 \leq k \leq n \) such that

\[
y_1(x) - y_2(x) = \sum_{k=1}^{n} C_k e^{r_k x}, \quad x \in \mathbb{R}.
\] (11)

If \((C_1, C_2, \ldots, C_n) \neq (0, 0, \ldots, 0)\), then

\[
\|y_1 - y_2\|_\infty = \sup_{x \in \mathbb{R}} \|y_1(x) - y_2(x)\| = +\infty,
\]

contradiction with the boundedness of \( y_1 - y_2 \). We conclude that \( C_k = 0, 1 \leq k \leq n \), therefore \( y_1 = y_2 \). The theorem is proved.

**Theorem 2** If \( r_k \) are distinct roots of the characteristic equation with \( \text{Re} r_k \neq 0, 1 \leq k \leq n \), then the best Ulam constant of \( D \) is given by

\[
K_D = \begin{cases} 
\frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^{n} (-1)^k V_k e^{-r_k x} \right| \, dx, & \text{if } \text{Re} r_k > 0; \\
\frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^{n} (-1)^k V_k e^{r_k x} \right| \, dx, & \text{if } \text{Re} r_k < 0; \\
\frac{1}{|V|} \int_0^\infty \left( \sum_{k=1}^{p} (-1)^k V_k e^{-r_k x} \right) + \left| \sum_{k=p+1}^{n} (-1)^k V_k e^{r_k x} \right| \, dx, & \text{if } \text{Re} r_k > 0; \\
\text{Re} r_k < 0.
\end{cases}
\] (12)

**Proof** Suppose that \( D \) admits an Ulam constant \( K < K_D \).

i) First, let \( \text{Re} r_k > 0, 1 \leq k \leq n \). Then

\[
K_D = \frac{1}{|V|} \int_0^\infty \left| \sum_{k=1}^{n} (-1)^k V_k e^{-r_k x} \right| \, dx.
\]

Let \( h(x) = \sum_{k=1}^{n} (-1)^k V_k e^{-r_k x}, \ x \in \mathbb{R} \). Take \( s \in X, \|s\| = 1 \), and \( \theta > 0 \) arbitrary chosen, and consider \( f : \mathbb{R} \to X \) given by

\[
f(x) = \frac{h(x)}{|h(x)| + \theta e^{-s}}, \quad x \in \mathbb{R}.
\]

Obviously, the function \( f \) is continuous on \( \mathbb{R} \) and \( \|f(x)\| \leq 1 \) for all \( x \in \mathbb{R} \).
Let $\tilde{y}$ be the solution of $D(y) = f$, given by

$$
\tilde{y}(x) = \sum_{k=1}^{n} C_k e^{rx} + \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{0}^{x} f(t) e^{-r_k t} dt
$$

(13)

with the constants

$$
C_k = -(-1)^{n+1} \frac{V_k}{V} \int_{0}^{\infty} f(t) e^{-r_k t} dt, \quad 1 \leq k \leq n.
$$

The improper integrals in the definition of $C_k$ $1 \leq k \leq n$ are obviously absolutely convergent since $\|f(x)\| \leq 1$, $x \in \mathbb{R}$, and $\text{Re} r_k > 0$, $1 \leq k \leq n$. Then

$$
\tilde{y}(x) = \frac{1}{V} \sum_{k=1}^{n} \left( (-1)^{n+k} V_k \int_{0}^{\infty} f(t) e^{-r_k t} dt \right) e^{rx}
$$

$$
+ \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{0}^{x} f(t) e^{-r_k t} dt
$$

$$
= \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{rx} \int_{x}^{\infty} f(t) e^{-r_k t} dt
$$

$$
= \frac{(-1)^{n+1}}{V} \sum_{k=1}^{n} (-1)^{k} V_k \int_{x}^{\infty} f(t) e^{r_k (x-t)} dt.
$$

Using the substitution $x-t = -u$, $\tilde{y}(x)$ becomes

$$
\tilde{y}(x) = \frac{(-1)^{n+1}}{V} \sum_{k=1}^{n} (-1)^{k} V_k \int_{0}^{\infty} f(x+u) e^{-r_k u} du, \quad x \in \mathbb{R}. \quad (14)
$$

Since $f$ is bounded and $\text{Re} r_k > 0$, $1 \leq k \leq n$, it follows that $\tilde{y}(x)$ is bounded on $\mathbb{R}$. Furthermore $\|D(\tilde{y})\|_{\infty} \leq 1$ and the Ulam stability of $D$ for $\varepsilon = 1$ with the constant $K$, leads to the existence of $y_H \in \ker D$, given by

$$
y_H(x) = \sum_{k=1}^{n} C_k e^{rx}, \quad x \in \mathbb{R},
$$

$C_k \in X$, $1 \leq k \leq n$, with the property

$$
\|\tilde{y} - y_H\|_{\infty} \leq K. \quad (15)
$$

If $(C_1, C_2, \ldots, C_n) \neq (0, 0, \ldots, 0)$ we get, in view of the boundedness of $\tilde{y}$

$$
\lim_{x \to \infty} \|\tilde{y}(x) - y_H(x)\| = +\infty,
$$

(16)

contradiction with the existence of $K$ satisfying (15). Therefore $C_1 = C_2 = \cdots = C_n = 0$, and the relation (15) becomes

$$
\|\tilde{y}(x)\| \leq K, \quad \text{for all } x \in \mathbb{R}. \quad (17)
$$
Now let \( x = 0 \) in (17). We get, in view of (14),
\[
\frac{1}{|V|} \left\| \int_0^\infty \sum_{k=1}^n (-1)^{k} V_k e^{-r_k u} f(u) du \right\| \leq K,
\]
or equivalently
\[
\frac{1}{|V|} \left\| \int_0^\infty h(u) f(u) du \right\| = \frac{1}{|V|} \int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du \leq K, \quad \forall \theta > 0. \tag{18}
\]
Let \( I(\theta) = \int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du \) and \( I_0 = \int_0^\infty |h(u)| du \). We show that
\[
\lim_{\theta \to 0} I(\theta) = I_0.
\]
Indeed,
\[
|I(\theta) - I_0| \leq \int_0^\infty \left| \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} - |h(u)| \right| du
\leq \int_0^\infty \frac{|h(u)| e^{-u}}{|h(u)| + \theta e^{-u}} du
\leq \int_0^\infty e^{-u} du = \theta, \quad \theta > 0,
\]
consequently \( \lim_{\theta \to 0} I(\theta) = I_0 \). Letting \( \theta \to 0 \) in (18) we get \( K_D \leq K \), a contradiction to the supposition \( K < K_D \).

ii) The case \( \text{Re} r_k < 0 \), \( 1 \leq k \leq n \), follows analogously.

Let \( h(x) = \sum_{k=1}^n (-1)^k V_k e^{r_k x} \), \( x \in \mathbb{R} \) and \( f \) be given by
\[
f(x) = \frac{h(-x)}{|h(-x)| + \theta e^x},
\]
for \( s \in X, \|s\| = 1 \), \( x \in \mathbb{R} \) and \( \theta > 0 \) arbitrary chosen. Obviously, the function \( f \) is continuous on \( \mathbb{R} \) and \( \|f(x)\| \leq 1 \) for all \( x \in \mathbb{R} \).

Let \( \tilde{y} \) be the solution of \( D(y) = f \), given by
\[
\tilde{y}(x) = \sum_{k=1}^n C_k e^{r_k x} + \sum_{k=1}^n (-1)^{n+k} V_k \int_0^x f(t) e^{-r_k t} dt \tag{19}
\]
with the constants
\[
C_k = (-1)^{n+k} \frac{V_k}{V} \int_{-\infty}^0 f(t) e^{-r_k t} dt, \quad 1 \leq k \leq n.
\]
Using a similar reasoning as in the previous case, we obtain
\[
\tilde{y}(x) = \frac{(-1)^n}{V} \int_0^\infty \left( \sum_{k=1}^n (-1)^k V_k e^{r_k u} \right) f(x-u) du, \quad x \in \mathbb{R}.
\]
Since $f$ is bounded and $\Re r_k < 0$ for $1 \leq k \leq n$, it follows that $\hat{y}(x)$ is bounded on $\mathbb{R}$. Furthermore $\|D(\hat{y})\|_\infty \leq 1$ and the Ulam stability of $D$ for $\varepsilon = 1$ with the constant $K$ leads to the existence of $y_H \in \ker D$, given by

$$y_H(x) = \sum_{k=1}^{n} C_k e^{r_k x}, \quad x \in \mathbb{R},$$

$\tilde{C}_k \in \mathbb{X}$, $1 \leq k \leq n$, such that

$$\|\hat{y} - y_H\|_\infty \leq K. \tag{20}$$

If $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n) \neq (0, 0, \ldots, 0)$ it follows that $\hat{y} - y_H$ is unbounded, a contradiction to the existence of $\tilde{K}$ satisfying (21). Therefore $\tilde{C}_1 = \tilde{C}_2 = \cdots = \tilde{C}_n = 0$, and the relation (20) becomes

$$\|\hat{y}(x)\| \leq K, \quad \forall x \in \mathbb{R}. \tag{21}$$

Now let $x = 0$ in (21). We get

$$\frac{1}{|V|} \left| \int_{0}^{\infty} \left( \sum_{k=1}^{n} (-1)^k V_k e^{r_k u} \right) f(-u) du \right| \leq K,$$

or equivalently

$$\frac{1}{|V|} \left| \int_{0}^{\infty} h(u) f(-u) du \right| = \frac{1}{|V|} \int_{0}^{\infty} \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du \leq K, \quad \forall \theta > 0. \tag{22}$$

Let $I(\theta) = \int_{0}^{\infty} \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du$ and $I_0 = \int_{0}^{\infty} |h(u)| du$. The arguments used in the proof of the previous case lead to $\lim_{\theta \to 0} I(\theta) = I_0$.

Letting $\theta \to 0$ in (22) we get $K_D \leq K$, a contradiction to the supposition $K < K_D$.

iii) Consider $\Re r_k > 0$, $1 \leq k \leq p$ and $\Re r_k < 0$, $p + 1 \leq k \leq n$. Let

$$h_1(x) = \sum_{k=1}^{p} (-1)^k V_k e^{r_k x}, \quad h_2(x) = \sum_{k=p+1}^{n} (-1)^k V_k e^{-r_k x}, \quad x \in \mathbb{R}.$$ 

Take an arbitrary $\theta > 0$, $s \in \mathbb{X}$, $\|s\| = 1$ and define

$$f(x) = \begin{cases} \frac{-h_2(x)}{|h_2(x)| + \theta e^{-x}}, & \text{if } x \in (-\theta, 0] \\ \frac{h_1(x)}{|h_1(x)| + \theta e^{x}}, & \text{if } x \in [\theta, +\infty) \\ \varphi(x), & \text{if } x \in (-\theta, \theta), \end{cases} \tag{23}$$

where $\varphi : (-\theta, \theta) \to \mathbb{X}$ is an affine function chosen such that $f$ is continuous on $\mathbb{R}$. Remark that $\|f\|_\infty \leq 1$.

Let $\breve{y}$ be the solution of $D(\breve{y}) = f$, given by
\[ \tilde{y}(x) = \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t)e^{-r_k t} dt \]  \hspace{1cm} (24)

with the constants

\[ C_k = -(-1)^n \frac{V_k}{V} \int_0^\infty f(t)e^{-r_k t} dt, \quad 1 \leq k \leq p. \]

\[ C_k = (-1)^{n+k} \frac{V_k}{V} \int_{-\infty}^0 f(t)e^{-r_k t} dt, \quad p + 1 \leq k \leq n. \]

Consequently

\[ \tilde{y}(x) = \left( -1 \right)^n V \int_0^\infty \left( \sum_{k=p+1}^{n} (-1)^k V_k e^{r_k u} \right) f(x-u) - \left( \sum_{k=1}^{p} (-1)^k V_k e^{-r_k u} \right) f(x+u) \right) du. \]

Since \( f \) is bounded, taking account of the sign of \( \text{Re} \, r_k, 1 \leq k \leq n \), it follows that \( \tilde{y}(x) \) is bounded. The relation \( \|D(y)\|_\infty = \|f\|_\infty < 1 \) and the stability of \( D \) for \( \varepsilon = 1 \) with the Ulam constant \( K \), leads to the existence of an exact solution \( y_H \in \ker D \) given by

\[ y_H(x) = \sum_{k=1}^{n} \tilde{C}_k e^{r_k x}, \quad x \in \mathbb{R}, \]

such that

\[ \|\tilde{y} - y_H\|_\infty \leq K. \]  \hspace{1cm} (25)

For \( (\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n) \neq (0,0,\ldots,0) \), the solution \( y_H \) is unbounded, therefore the relation (25) is true only for \( y_H(x) = 0 \), \( x \in \mathbb{R} \). Consequently relation (25) becomes

\[ \|\tilde{y}(x)\| \leq K, \quad x \in \mathbb{R}. \]  \hspace{1cm} (26)

For \( x = 0 \) we get \( \|\tilde{y}(0)\| \leq K \). But

\[
\tilde{y}(0) = \left( -1 \right)^n V \int_0^\infty \left( h_1(u)f(-u) - h_2(u)f(u) \right) du \\
= \left( -1 \right)^n V \left\{ \int_0^\infty h_1(u)f(-u)du - \int_0^\infty h_2(u)f(u)du \right\} \\
+ \left( -1 \right)^n V \int_0^\theta \left( h_1(u)f(-u) - h_2(u)f(u) \right) du \\
= \left( -1 \right)^n V \left\{ \int_0^\infty \frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} du + \int_0^\infty \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} du \right\} \\
+ \left( -1 \right)^n V \int_0^\theta \left( h_1(u)f(-u) - h_2(u)f(u) \right) du.
\]
Analogously to the previous cases it can be proved that if \( \theta \to 0 \) then
\[
\int_{\theta}^{\infty} \frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} du \to \int_{0}^{\infty} |h_1(u)| du
\]
and
\[
\int_{\theta}^{\infty} \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} du \to \int_{0}^{\infty} |h_2(u)| du,
\]
in view of the relation
\[
\|f(u)\| = \|\varphi(u)\| \leq 1, \quad u \in [-\theta, \theta].
\]
Hence, letting now \( \theta \to 0 \) in (26) we get \( K_D < K \), a contradiction.

**Theorem 3** If \( r_k, 1 \leq k \leq n, \) are real and distinct roots of the characteristic equation and \( a_n \neq 0 \), then the best Ulam constant of the operator \( D \) is
\[
K_D = \frac{1}{\prod_{k=1}^{n} r_k} = \frac{1}{|a_n|}.
\]

**Proof** Suppose that \( D \) admits an Ulam constant \( K < K_D \). Let \( \varepsilon > 0 \) and
\[
\tilde{y}(x) = \frac{\varepsilon}{a_n}, \quad x \in \mathbb{R}.
\]
Then \( \|D(\tilde{y})\|_\infty = \varepsilon \) and since \( D \) is Ulam stable with the constant \( K \) it follows that there exists \( y_H \in \ker D \) such that
\[
\|\tilde{y} - y_H\|_\infty \leq K \varepsilon.
\]
Clearly if \( y_H \) is not identically \( 0 \in X \), then it is unbounded so relation (28) cannot hold. Therefore \( y_H(x) = 0 \) for all \( x \in \mathbb{R} \) and relation (28) becomes \( \|\tilde{y}\|_\infty \leq K \varepsilon \), or \( K_D \leq K \), a contradiction.

The previous results lead to the following identity.

**Proposition 1** If \( r_k, 1 \leq k \leq n, \) are real distinct, nonzero numbers then
\[
\frac{1}{|r_1 r_2 \cdots r_n|} = K_D,
\]
where \( K_D \) is given by (12).

**Proof** For real and distinct roots \( r_k, 1 \leq k \leq n, \) of the characteristic equation, the best Ulam constant is given on one hand by relation (12), Theorem 2 and on the other hand by relation (27) in Theorem 3.
Next, we get as well an explicit representation of the best Ulam constant for the case of complex and distinct roots of the characteristic equation having the same imaginary part.

**Theorem 4** If the characteristic equation of \( D \) admits outside of the imaginary axis distinct roots having the same imaginary part, then the best Ulam constant of \( D \) is given by

\[
K_D = \frac{1}{\prod_{k=1}^{n} |\text{Re} r_k|}.
\] (30)

**Proof** Suppose that \( r_k = \rho_k + i\alpha, \rho_k \in \mathbb{R} \setminus \{0\}, 1 \leq k \leq n, \alpha \in \mathbb{R} \). Then the best Ulam constant of \( D \) is given by (12) with \( r_k = \rho_k, 1 \leq k \leq n, \) and \( V = V(\rho_1, \rho_2, \ldots, \rho_n), V_k = (\rho_1, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_n) \). Now taking account of Theorem 2 it follows

\[
K_D = \frac{1}{\prod_{k=1}^{n} |\rho_k|}.
\]

Theorem 2 is an extension of the result given in [2] for distinct roots of the characteristic equation. Indeed, the particular case \( n = 2 \) corresponds to the second order linear differential operator.

\[
D(y) = y'' + a_1 y' + a_2 y, \quad a_1, a_2 \in \mathbb{C},
\] (31)

and the best Ulam constant in this case is

\[
K_D = \begin{cases} 
\frac{1}{|a_2|}, & \text{if } a_1^2 - 4a_2 \geq 0, \\
\frac{1}{a_2} \coth \frac{|a_1|}{\sqrt{4a_2 - a_1^2}}, & \text{if } a_1^2 - 4a_2 < 0.
\end{cases}
\] (33)

**Proof** Let \( \delta = a_1^2 - 4a_2 \).

i) If \( \delta \geq 0 \) then \( r_1, r_2 \in \mathbb{R} \) and in view of [2, Theorem 3]

\[
K_D = \frac{1}{|r_1 r_2|} = \frac{1}{|a_2|}.
\]
ii) If $\delta < 0$ then $r_{1,2} = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

Suppose first $\alpha > 0$. Then

$$K_D = \frac{1}{2|\beta|} \int_0^\infty e^{-\alpha x} |e^{-i\beta x} - e^{i\beta x}| dx$$

$$= \frac{1}{2|\beta|} \int_0^\infty e^{-\alpha x} | - 2i \sin \beta x| dx = \frac{1}{|\beta|} \int_0^\infty e^{-\alpha x} |\sin(|\beta| x)| dx.$$ 

Now, letting $|\beta| x = t$ in the above integral we obtain, taking account of

$$\int_0^\infty e^{-px} |\sin x| dx = \frac{1}{1 + p^2} \coth \frac{p\pi}{2}, \quad p > 0,$$

$$K_D = \frac{1}{\beta^2} \int_0^\infty e^{-\frac{a_1|t|}{\sqrt{4a_2 - a_1^2}}} |\sin t| dt = \frac{1}{\alpha^2 + \beta^2} \coth \frac{\alpha \pi}{2|\beta|} \pi = \frac{1}{a_2} \coth \frac{|a_1|\pi}{\sqrt{4a_2 - a_1^2}}.$$ 

Analogously for $\alpha < 0$.

It will be interesting to obtain a closed form (if possible) for the best Ulam constant of the $n$ order differential operator also for the case of multiple roots of the characteristic equation. This problem might be quite challenging and we intend to leave it, for the moment, as an open problem.

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