Generalizing the $\mathcal{N} = 2$ supersymmetric RG flow solution of IIB supergravity

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We explicitly construct the supersymmetry transformations for the $\mathcal{N} = 2$ supersymmetric RG flow solution of chiral IIB supergravity. We show that the metric, dilaton/axion, five-index tensor and half of the three index tensor are determined algebraically in terms of the Killing spinor of the unbroken supersymmetry. The algebraic nature of the solution allows us to generalize this construction to a new class of $\mathcal{N} = 2$ supersymmetric solutions of IIB supergravity. Each solution in this class is algebraically determined by supersymmetry and is parametrized by a single function of two variables that satisfies a non-linear equation akin to the Laplace equation on the space transverse to the brane.

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1. Introduction

There continues to be a strong interest in finding supersymmetric backgrounds with non-trivial $RR$-fluxes. Indeed, one of the most interesting forms of this problem is to start with a geometric background with a higher level of supersymmetry and then use the fluxes to break some, or all of this supersymmetry. This has been a theme of much recent research, but here we will focus on its application in holographic field theory, and most particularly in the four-dimensional AdS/CFT correspondence. The problem is then one of finding supersymmetric solutions of IIB supergravity in which there are non-trivial fluxes that vanish suitably at infinity. To be more specific, this paper will examine the problem of holographic flows of $\mathcal{N} = 4$, $SU(N)$ Yang-Mills theory in which the supersymmetry is softly broken to $\mathcal{N} = 2$ (eight supersymmetries).

Such flows are well-understood in field theory, and the Wilsonian effective action has been computed in [1,2]. They thus provide a valuable test of the holographic correspondence. These theories have a Coulomb branch parametrized by the invariants of the scalar field, $\Phi$, that lies in the $\mathcal{N} = 2$ vector multiplet:

$$ u_m \equiv \text{Tr} \left( \Phi^m \right), \quad m = 1, 2, \ldots N, \quad (1.1) $$

and in the large-$N$ limit there is thus an infinite series of such invariants. In terms of branes, this Coulomb branch involves the $D3$ branes spreading out in the two directions, $(v_1, v_2)$, that correspond to (complex) vevs of $\Phi$. The $u_m$ are then the moments of the brane distribution. For large $N$, a general point on the Coulomb branch therefore corresponds to an arbitrary function, $\rho(v_1, v_2)$, that defines the density of the $D3$-brane distribution in the two-dimensions in which the spreading occurs. If the supersymmetry were not softly broken then a solution with maximal supersymmetry could be obtained via the usual “harmonic rule,” with the harmonic function sourced by the distribution, $\rho(v_1, v_2)$. The corresponding solution with soft supersymmetry breaking to $\mathcal{N} = 2$ is unknown for general $\rho(v_1, v_2)$, and indeed this has been an open problem for several years.

The solution for one particular, relatively uniform distribution of branes was obtained in [3], and the precise brane distribution was computed in [4,5]. This solution has found, and continues to find (see, for example, [6,7]) interesting applications, and it would be rather useful to find $\mathcal{N} = 2$ supersymmetric holographic flows to a broader section of the Coulomb branch. The essential problem is the apparent complexity of the solution of [3]. In this paper we re-examine the solution of [3] and explicitly compute the Killing spinors.
We find that while the geometry is complicated, the structure of the Killing spinors is remarkably simple: The Killing spinors are determined algebraically in terms of the metric coefficients. Conversely, we find that much of the geometry and the fluxes can be inferred from these Killing spinors. This enables us to make a more general Ansatz that will capture rotationally symmetric distributions of $D3$ branes, that is, density functions, $\rho(v_1, v_2)$, that only depend upon $v \equiv \sqrt{v_1^2 + v_2^2}$.

Our approach to finding the softly broken flows is thus parallel to that of [8]: We make a very general Ansatz for the metric and for the fluxes, and then impose projectors that determine the Killing spinor spaces algebraically in terms of the metric coefficients. There is an invaluable constraint on the Killing spinors: Namely, if $\epsilon^a$ and $\epsilon^b$ are two commuting (non-grassmann) Killing spinors, then

$$K_{(ab)}^\mu \equiv \epsilon^a \gamma^\mu \epsilon^b + \epsilon^b \gamma^\mu \epsilon^a,$$

must be a Killing vector of the underlying background metric. We will refer to this linked property of the Killing vectors and spinors as the “Killing Structure.” This structure provides strong constraints on the projectors that define the Killing spinors, and also enables us to completely fix the normalizations of the Killing spinors. Thus the simple structure of the flow solutions lies in the definition of the Killing spinors. We then use the supersymmetry transformations to reconstruct everything else. These transformations can be used to fix the fluxes and metric algebraically in terms of the functions (and their derivatives) that define the Killing spinors. Finally, the Bianchi identities provide the necessary differential equations. Indeed, we find that metric and all the fluxes are can be algebraically determined in terms of a single function, $c$, and its derivatives. The function, $c$, must itself satisfy a second order, partial differential equation. The only difficulty is that this differential equation is non-linear, but like the corresponding $M$-theory result [8], this differential equation has a rather straightforward perturbation theory that shows that it does indeed admit solutions with a functional degree of freedom corresponding to a general, rotationally symmetric, two-dimensional brane distribution, $\rho(v)$. Thus, while the governing differential equation is non-linear, we have found a natural generalization of the “harmonic rule” to softly broken supersymmetry.

In the next section of we summarize the key properties of the solution of [3], and in Section 3 we compute the Killing spinors of the supersymmetry explicitly. In particular, we show how the space of Killing spinors can be defined by two projectors, $\Pi^{(j)}$, $j = 1, 2,$
that are algebraically related to the metric coefficients. Each of these projectors reduces the dimension of the spinor space by a factor of two, and together they define the space of the eight supersymmetries. One of the projectors is naturally interpreted in terms of a helicity projector on the moduli space, \((v_1, v_2)\), of the \(D3\)-branes, while the other projector is a “dielectric deformation” of the usual projector \(\left(\frac{1}{2}(1 - i\gamma^{1234})\right)\) associated with a stack of \(D3\)-branes. The Killing structure helps fix this deformation, and determines the normalization of the Killing spinors. In Section 4 we use these observations to obtain a more general Ansatz for \(\mathcal{N} = 2\) flows with a rotationally symmetric distribution of branes, and we then solve this Ansatz and show that the entire solution is generated by the single function, \(c\), that satisfies a particular second order, non-linear PDE. We then find some simple solutions of this PDE and use them to generate new backgrounds with eight supersymmetries. Section 5 contains some final remarks.

2. The \(\mathcal{N} = 2\) supersymmetric RG flow solution

In this section we summarize the details of the solution of the chiral IIB supergravity obtained in [3]. This solution corresponds to an \(\mathcal{N} = 2\) supersymmetric RG flow of the \(\mathcal{N} = 4\) super-Yang-Mills, and was obtained by lifting to ten dimensions a solution of \(\mathcal{N} = 8\) gauged supergravity in five dimensions. In such a construction it is natural to use coordinates in ten dimensions in which both the flow and the lift are manifest, even though, as will become clear later, the ten-dimensional geometry may be somewhat obscured. For the moment we will follow the convention in [3,9] so that a comparison with the result there is more straightforward. In particular, \(x^\mu, \mu = 0, \ldots, 3\) are coordinates along the brane, \(r\) is the coordinate along the flow, while \(\theta, \alpha^i\) and \(\phi\) are coordinates on the deformed \(S^5\), where \(\alpha^i, i = 1, 2, 3\), are the Euler angles of the unbroken \(SU(2)\). We will denote the ten-dimensional coordinates collectively by \(x^M\), where \(M\) runs from 1 to 10. Following [3,9], our metrics will be “mostly minus.”

The solution involves two functions, \(c(r)\) and \(\rho(r)\), that are related to the mass and the Coulomb branch deformations, \(\chi(r)\) and \(\alpha(r)\), of the \(\mathcal{N} = 4\) super-Yang-Mills theory along the RG flow:

\[
c = \cosh(2\chi), \quad \rho = \exp(\alpha).
\]

They satisfy a system of first-order differential (flow) equations:

\[
\frac{dc}{dr} = \rho^4 (1 - c^2), \quad \frac{d\rho}{dr} = \frac{1}{3} \left( \frac{1}{\rho} - c\rho^5 \right).
\]

\[\tag{2.1}\]

\[\tag{2.2}\]
whose general solution is given by [3,10]:

\[ \rho^6 = c + (c^2 - 1) \left[ \gamma + \frac{1}{2} \log \left( \frac{c-1}{c+1} \right) \right], \quad (2.3) \]

where \( \gamma \) is a constant of integration that parametrizes different flows.

In the solution of the IIB supergravity, all the bosonic fields, the metric, \( g_{MN} \), the dilaton/axion, \( \tau \), the five-index tensor, \( F_{(5)} \), and the three index tensor, \( G_{(3)} \), are non-vanishing. The metric is diagonal:

\[ ds^2 = \Omega^2(dx_\mu dx^\mu) - (V_1^2 dr^2 + V_2^2 d\theta^2 + V_3^2 (\sigma^1)^2 + V_4^2 ((\sigma^2)^2 + (\sigma^3)^2) + V_5^2 d\phi^2), \quad (2.4) \]

with the functions \( \Omega(r, \theta) \) and \( V_a(r, \theta), a = 1, \ldots, 5 \), given by:

\[ \Omega(r, \theta) = \frac{c^{1/8} \rho^{3/2} X_1^{1/8} X_2^{1/8}}{(c^2 - 1)^{1/2}}, \quad (2.5) \]

and

\[ V_1(r, \theta) = \frac{c^{1/8} X_1^{1/8} X_2^{1/8}}{\rho^{1/2}}, \]
\[ V_2(r, \theta) = \frac{X_1^{1/8} X_2^{1/8}}{c^{3/8} \rho^{3/2}}, \]
\[ V_3(r, \theta) = \frac{\rho^{3/2} X_1^{1/8}}{c^{3/8} X_2^{3/8}} \cos \theta, \]
\[ V_4(r, \theta) = \frac{c^{1/8} \rho^{3/2} X_2^{1/8}}{X_1^{3/8}} \cos \theta, \]
\[ V_5(r, \theta) = \frac{c^{1/8} X_1^{1/8}}{\rho^{3/2} X_2^{3/8}} \sin \theta, \quad (2.6) \]

where

\[ X_1(r, \theta) = \cos^2 \theta + c \rho^6 \sin^2 \theta, \quad X_2(r, \theta) = c \cos^2 \theta + \rho^6 \sin^2 \theta. \quad (2.7) \]

The \( SU(2) \) Maurer-Cartan forms \( \sigma^i, i = 1, 2, 3 \), are normalized by \( d\sigma^1 = 2 \sigma^2 \wedge \sigma^3 \). (Note that this normalization differs by a factor of two from that used in [3].) The metric has the Poincaré invariance along the brane directions, \( x^\mu \), and is also manifestly \( SU(2) \times U(1)^2 \) invariant, where the first \( U(1) \) rotates \( \sigma^2 \) and \( \sigma^3 \), while the second \( U(1) \) is a phase rotation in the angle \( \phi \).

We alert the reader that the form of the metric (2.4) differs slightly from the one in [3] in that we have combined all the warp factors in the metric along the brane into a single function, \( \Omega(r, \theta) \). The orthonormal frames, \( e^M, M = 1, \ldots, 10 \), are the same as in [3].
The dilaton/axion fields \((\Phi, C_{(0)})\) form a complex scalar, which is related to the supergravity field, \(B\), in the \(SU(1, 1)\) basis by
\[
\tau \equiv C_{(0)} + i e^{-\Phi} = i \left( \frac{1 - B}{1 + B} \right).
\] (2.8)

The latter is explicitly given by
\[
B(r, \theta, \phi) = \left( \frac{b^{1/4} - b^{-1/4}}{b^{1/4} + b^{-1/4}} \right) e^{2i\phi}, \quad b(r, \theta) = \frac{c X_1}{X_2}.
\] (2.9)

We also recall that the scalar one-form, \(P\), and the connection form, \(Q\), are defined by:
\[
P = \frac{dB}{1 - BB^*}, \quad Q = \frac{1}{2i} \frac{B dB^* - B^* dB}{1 - BB^*}.
\] (2.10)

The RR four-form potential, \(C_{(4)}\), and the corresponding five-index field strength, \(F_{(5)}\), are:
\[
C_{(4)} = w dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad F_{(5)} = dC_{(4)} + *dC_{(4)},
\] (2.11)

where the function \(w(r, \theta)\) can be written in the form \([3]\):
\[
w(r, \theta) = \frac{\Omega^4}{4} \frac{X_1^{1/2}}{c X_2^{1/2}}.
\] (2.12)

Finally, the three-index tensor field, \(G_{(3)}\), is:
\[
G_{(3)} = (1 - BB^*)^{-1/2} (dA_{(2)} - B dA_{(2)}^*).
\] (2.13)

with the two-form potential, \(A_{(2)}\), given by:
\[
A_{(2)} = e^{i\phi} \left( a_1 \, d\theta \wedge \sigma^1 + a_2 \, \sigma^2 \wedge \sigma^3 + a_3 \, \sigma^1 \wedge d\phi \right),
\] (2.14)

where
\[
a_1(r, \theta) = -i c \frac{1}{c} (c^2 - 1)^{1/2} \cos \theta,
\]
\[
a_2(r, \theta) = i \frac{b^6}{X_1} (c^2 - 1)^{1/2} \cos^2 \theta \sin \theta,
\] (2.15)
\[
a_3(r, \theta) = -\frac{1}{X_2} (c^2 - 1)^{1/2} \cos^2 \theta \sin \theta.
\]

We note that \(G_{(3)}\) has only six non-vanishing components when expanded in the orthonormal basis, namely
\[
G_{5710}, \quad G_{6710}, \quad G_{8910}, \quad G_{567}, \quad G_{589}, \quad G_{689},
\] (2.16)

where, for \(\phi = 0\), the first three are real while the remaining three are purely imaginary.

\[\text{We correct here the misprint in [3], brought to our attention by the authors of [4] and [5].}\]
3. Supersymmetry

The supersymmetry variations for the gravitino, $\psi_M$, and the spin-$\frac{1}{2}$ field, $\lambda$, in IIB supergravity read [9]:

$$\delta \psi_M = D_M \epsilon + \frac{i}{480} F_{PQRST} \gamma^{PQRST} \gamma_M \epsilon + \frac{1}{96} \left( \gamma_M^{PQR} - 9 \delta_M^P \gamma^{QR} \right) G_{PQR} \epsilon^*, \quad (3.1)$$

and

$$\delta \lambda = i P_M \gamma^M \epsilon^* - \frac{i}{24} G_{MNP} \gamma^{MNP} \epsilon, \quad (3.2)$$

where $\epsilon$ is a complex chiral spinor satisfying

$$\gamma^{11} \epsilon = -\epsilon. \quad (3.3)$$

The conditions for unbroken supersymmetry are $\delta \psi_M = \delta \lambda = 0$, which gives a combination of algebraic and first order differential equations for the Killing spinor, $\epsilon$. We write these equations schematically as

$$\partial_M \epsilon = \Delta_M \epsilon \quad \text{and} \quad \Delta_{1/2} \epsilon = 0. \quad (3.4)$$

Here $\partial_M$ denotes the partial derivative $\partial/\partial x^M$, except for the $SU(2)$ directions, where we take $\partial_M$, $M = 7, 8, 9$, to be the $SU(2)$ invariant (Killing) vector fields dual to the Maurer-Cartan forms, $\sigma^i$, $i = 1, 2, 3$, respectively. The operators $\Delta_M$ and $\Delta_{1/2}$ are purely algebraic and depend on all background fields. In particular, the dependence on the metric and the dilaton/axion in (3.1) arises from the connection terms in the covariant derivative:

$$D_M \epsilon = \partial_M \epsilon + \frac{1}{4} \omega_{MPQ} \gamma^{PQ} \epsilon - \frac{i}{2} Q_M \epsilon. \quad (3.5)$$

The operators $\Delta_M$ and $\Delta_{1/2}$ will in general involve the operation of complex conjugation, which will be denoted by $'*$'. In practice it is often convenient to pass to a real realization of operators by decomposing spinors into the real and imaginary parts.

It has been emphasised recently (see, for example, [11–13]) that a lot of information about a supersymmetric background can be recovered from the Killing spinors by constructing canonical vector fields and differential forms associated with them – the so-called G-structures. Here, we will concentrate on vector fields. That is, given two Killing spinors, $\epsilon^a$ and $\epsilon^b$, consider:

$$K^M_{ab} = \bar{\epsilon}^a \gamma^M \epsilon^b, \quad (3.6)$$

2 We use the same notation and $\gamma$-matrix conventions as in [3]. Also, see appendix A.
which, as expected, is a Killing vector of the metric. More generally, one can consider the differential forms:

\[ \Omega^{(ab)}_{M_1 M_2 \ldots M_p} \equiv \bar{e}^r \gamma_{M_1 M_2 \ldots M_p} e^b. \]  

(3.7)

Such forms can be used to partially determine the potentials for the antisymmetric tensor fields [8].

At this point the standard procedure would be to examine integrability conditions for the equations, however, since we are interested in an explicit form of the Killing spinors, we will proceed directly with the solution by first recalling the standard calculation at the maximally supersymmetric point (see, for example, [14,15]). This will set a proper stage for the discussion of the general case.

3.1. Supersymmetry at the \( \mathcal{N} = 8 \) point

The maximally supersymmetric point is the \( \text{AdS}_5 \times S^5 \) solution of the IIB supergravity, which in the present set-up is recovered by taking the limit:

\[ c(r) \to 1, \quad \rho(r) \to 1, \quad \Omega(r, \theta) \to e^r. \]  

(3.8)

The only background fields are the metric and the five-index tensor given by:

\[ ds^2 = e^{2r} (dx_\mu dx^\mu) - dr^2 - (d\theta^2 + \cos^2 \theta ((\sigma^1)^2 + \ldots + (\sigma^3)^2) + \sin^2 \theta d\phi^2), \]  

\[ F_{(5)} = e^{4r} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dr + \sin \theta \cos^3 \theta d\theta \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge d\phi. \]  

As in the standard calculation we write \( \Delta_M \) as products of projectors and invertible operators using (3.3) along the way, but no other conditions. We obtain:

\[ \Delta_1 = -\frac{1}{2} e^r \gamma^1 \gamma^5 (1 + i \gamma^1 \ldots \gamma^4), \]

\[ \Delta_j = \frac{1}{2} e^r \gamma^j \gamma^5 (1 + i \gamma^1 \ldots \gamma^4), \quad j = 2, 3, 4 \]  

(3.11)

and

\[ \Delta_5 = -\frac{i}{2} \gamma^1 \ldots \gamma^4, \]

\[ \Delta_6 = \frac{i}{2} \gamma^1 \ldots \gamma^4 \gamma^5 \gamma^6. \]  

(3.12)

For the \( SU(2) \) directions we have:

\[ \Delta_i = \frac{1}{2} \gamma^j \gamma^k (1 + i \cos \theta \gamma^6 \gamma^{10} + \sin \theta \gamma^1 \ldots \gamma^4 \gamma^5 \gamma^{10}) \quad i, j, k = 7, 8, 9, \]  

(3.14)
and, finally,
\[ \Delta_{10} = \frac{i}{2} (i \cos \theta \gamma^6 \gamma^{10} + \sin \theta \gamma^1 \ldots \gamma^4 \gamma^5 \gamma^{10}) . \]  

(3.15)

While this point has sixteen supersymmetries (in the presence of the brane), we are interested in isolating the eight supersymmetries that will remain unbroken under the flow solution described in the previous section. These eight supersymmetries transform as a doublet under the residual $SU(2) \mathcal{R}$-symmetry. Thus, we are interested in determining supersymmetries that are invariant along the brane, i.e. $\epsilon$ is independent of $x^\mu$, and transform non-trivially under $SU(2)$. Using (3.11) and (3.14) these two conditions are equivalent to two algebraic equations:

\[ \Pi^{(1)}_0 \epsilon = \epsilon, \quad \Pi^{(2)}(\theta) \epsilon = \epsilon, \]  

(3.16)

where

\[ \Pi^{(1)}_0 = \frac{1}{2} (1 - i \gamma^1 \gamma^2 \gamma^3 \gamma^4). \]  

(3.17)

and

\[ \Pi^{(2)}(\theta) = \frac{1}{2} \left(1 + i (\sin \theta \gamma^5 + \cos \theta \gamma^6) \gamma^{10}\right). \]  

(3.18)

These are mutually commuting projectors and thus define a space of eight (real) supersymmetry parameters. Note that $\Pi^{(1)}_0$ is the standard projector parallel to the $D3$-brane.

Using

\[ \cos \theta \gamma^6 + \sin \theta \gamma^5 = \mathcal{O}(\theta)^{-1} \]  

(3.19)

where

\[ \mathcal{O}(\theta) = \cos \theta \frac{1}{2} - \sin \theta \frac{1}{2} \gamma^5 \gamma^6, \]  

(3.20)

we also have

\[ \Pi^{(2)}(\theta) = \mathcal{O}(\theta) \Pi^{(2)}_0 \mathcal{O}(\theta)^{-1}, \quad \Pi^{(2)}_0 = \frac{1}{2} (1 + i \gamma^6 \gamma^{10}). \]  

(3.21)

Given that $\mathcal{O}(\theta)$ satisfies the equation

\[ \partial_\theta \mathcal{O}(\theta) = -\frac{1}{2} \gamma^5 \gamma^6 \mathcal{O}(\theta). \]  

(3.22)

it is now straightforward to integrate all the equations with the result

\[ \epsilon = e^{r/2} e^{i \phi/2} \mathcal{O}(\theta) \epsilon_0, \]  

(3.23)
where $\epsilon_0$ depends only on the $SU(2)$ directions and satisfies:

$$\Pi_0 \epsilon_0 = \epsilon_0, \quad \Pi_0 \equiv \Pi_0^{(1)} \Pi_0^{(2)}. \tag{3.24}$$

If one combines (3.14) with (3.18) one sees that the action of $SU(2)$ is generated by the products of gamma matrices $\gamma^{67}$, $\gamma^{68}$ and $\gamma^{78}$. However, because of the projection matrices, there are several ways to express this action on $\epsilon_0$. Indeed, the $SU(2)$ action can be generated by the matrices:

$$t_1 = \gamma^8 \gamma^9, \quad t_2 = -\gamma^5 \gamma^8, \quad t_3 = -\gamma^5 \gamma^9, \tag{3.25}$$

which can be related to the more canonical set using the identities:

$$\gamma^7 \gamma^9 \Pi_0 = \gamma^5 \gamma^8 \Pi_0, \quad \gamma^7 \gamma^8 \Pi_0 = -\gamma^5 \gamma^9 \Pi_0. \tag{3.26}$$

We will find the generators (3.25) more convenient in the $\mathcal{N} = 2$ flow solution. The dependence of the Killing spinors on the Euler angles, $\alpha^i$, of the $SU(2)$ can thus be obtained by exponentiating these matrices in much the same manner as was done in [16,8].

Finally, consider the vector field:

$$K = \bar{\epsilon} \gamma^M \epsilon \partial_M = e^r \bar{\epsilon}_0 \mathcal{O}(\theta) \gamma^M \mathcal{O}(\theta) \epsilon_0 \partial_M. \tag{3.27}$$

Using (3.16), we find that for $M \geq 5$ we have:

$$K_M \propto \bar{\epsilon}_0 \gamma^M \epsilon_0$$

$$= \bar{\epsilon}_0 (1 - \Pi_0^{(1)}) \gamma^M \Pi_0^{(1)} \epsilon_0$$

$$= 0. \tag{3.28}$$

Thus, the non-vanishing (frame) components are:

$$(K^M) = e^r k^M, \quad M = 1, \ldots, 4, \tag{3.29}$$

where $k^M$ are constants as the dependence on $\theta$, $\phi$ and the Euler angles drops out in (3.27). We thus see explicitly that the Killing spinors yield only those Killing vectors which correspond to the “trivial” symmetry of the metric (3.9), namely translations along the brane. However, this result is quite non-trivial if we reverse the order of reasoning and observe that the $r$-dependence of the Killing spinors is completely fixed once we require that $K$ yields some non-zero Killing vectors of the metric. In fact, this observation will be crucial for the calculation of the Killing spinors in the next section.
3.2. Supersymmetry along the $\mathcal{N}=2$ flow

We now return to the $\mathcal{N}=2$ background in Section 2. Our general strategy will be exactly as above, and will be parallel to corresponding approach in M-theory [8]. First we consider those equations in (3.4) that are purely algebraic and construct the projector onto the space of solutions. Then we use the relation with the Killing vectors to integrate the remaining differential equations explicitly.

The resulting Killing spinors must continuously reduce to the ones we have already found at the $\mathcal{N}=8$ point. Because of the Poincaré symmetry along the brane, and because of the residual $\mathcal{R}$-symmetry we know that: (i) the Killing spinors, $\epsilon$, are constant along the brane, (ii) the dependence on rotational coordinates, $x^7, x^8, x^9$, can be obtained by exponentiating suitably defined $SU(2)$ generators, and (iii) the $\phi$ dependence of the Killing spinors appears through the simple phase as in (3.23). On a more mechanistic level, one can verify a posteriori that this simplyfying Ansatz yields all unbroken supersymmetries. One can also easily check that this $\phi$-dependence, given the phases in (2.9), (2.10), (2.14) and (2.13), is in fact the only that is consistent with the appearance of $\epsilon$ and $\epsilon^*$ in the spin-$\frac{1}{2}$ variation, (3.2).

Our Ansatz yields six algebraic equations with four of them manifestly equivalent:

$$\gamma^1 \Delta_1 \epsilon = \ldots = \gamma^4 \Delta_4 \epsilon = 0.$$  

(3.30)

Guided by the same strategy as in the construction of the lift in [3], we now seek a linear combination of $\Delta_1, \Delta_{10}$ and $\Delta_{1/2}$ in which the dependence on the three-index tensor field cancels out. We find that such a combination is indeed possible and that the result is quite simple, namely:

$$2(\gamma^1 \Delta_1 + \gamma^{10}(\Delta_{10} - \frac{i}{2})\epsilon - (\Delta_{1/2}\epsilon)^*) = (\cos \theta \gamma^6 + \sqrt{c} \rho^3 \sin \theta \gamma^5 + i \sqrt{X_1} \gamma^{10}) \epsilon,$$  

(3.31)

where the left-hand side is required to vanish. Here we have used the background values for the fields, but it is easy to see that only the metric and the dilaton/axion contribute to the right hand side. We can now rewrite the resulting equation in terms of a projector. Indeed the vanishing of the right hand side of (3.31) is equivalent to:

$$\Pi^{(2)}(\alpha) \epsilon = \epsilon,$$  

(3.32)

where the projector $\Pi^{(2)}(\alpha)$ is the same as in (3.18) with the angle $\alpha(r, \theta)$ determined by

$$\tan \alpha = \sqrt{c} \rho^3 \tan \theta,$$  

(3.33)
or, equivalently,
\[
\cos \alpha = \frac{\cos \theta}{X_1^{1/2}}, \quad \sin \alpha = \frac{\sqrt{c} \rho^3 \sin \theta}{X_1^{1/2}}. \tag{3.34}
\]

Now, rather than chasing other linear combinations to obtain further projectors, we proceed quite directly. We can simplify the spin-$\frac{1}{2}$ variation by restricting it to the subspace of spinors satisfying (3.32). After some algebra it is then possible to write down the general form of the solution and construct the corresponding projector:
\[
\frac{1}{2} \left( 1 - i \gamma^1 \gamma^2 \gamma^3 \gamma^4 (p_1(r, \theta) + p_2(r, \theta) \gamma^7 \gamma^{10} \ast) \right), \tag{3.35}
\]
where
\[
p_1(r, \theta) = \frac{X_1^{1/2}}{c^{1/2} X_2^{1/2}}, \quad p_2(r, \theta) = \frac{(c^2 - 1)^{1/2} \cos \theta}{c^{1/2} X_2^{1/2}}. \tag{3.36}
\]
Introduce an operator
\[
O^*(\beta) = \cos \frac{\beta}{2} + \sin \frac{\beta}{2} \gamma^7 \gamma^{10} \ast. \tag{3.37}
\]
Then (3.35) can be simply written as:
\[
\Pi^{(1)}(\beta) = O^*(\beta) \Pi^{(1)}_0 O^*(\beta)^{-1}, \tag{3.38}
\]
where $\Pi^{(1)}_0$ is defined in (3.17) and the deformation angle $\beta(r, \theta)$ is determined by:
\[
\cos \beta = \frac{X_1^{1/2}}{c^{1/2} X_2^{1/2}}, \quad \sin \beta = -\frac{(c^2 - 1)^{1/2} \cos \theta}{c^{1/2} X_2^{1/2}}. \tag{3.39}
\]
It is now easy to verify that all algebraic equations have been solved, and thus $\epsilon$ must satisfy:
\[
\Pi(\alpha, \beta) \epsilon = \epsilon, \quad \Pi(\alpha, \beta) \equiv \Pi^{(1)}(\beta) \Pi^{(2)}(\alpha). \tag{3.40}
\]
It is interesting that in fact (3.40) follows from the spin-$\frac{1}{2}$ variation alone. We have verified this by brute force algebra after we have obtained the projection condition using the simplifications outlined above. It would be useful to have a more direct proof of this observation.

The next step is to integrate explicitly the remaining first order equations. We start with the $SU(2)$ directions and first verify that:
\[
\Delta_i \Pi(\alpha, \beta) = \Pi(\alpha, \beta) \Delta_i \Pi(\alpha, \beta), \tag{3.41}
\]
which shows, as expected, that the solutions to the algebraic equations form a representation of $SU(2)$. More specifically, we find the identities:

$$\Delta_i \Pi(\alpha, \beta) = \mathcal{O}(\alpha) t_i \mathcal{O}(\alpha)^{-1} \Pi(\alpha, \beta) = \Pi(\alpha, \beta) \mathcal{O}(\alpha) t_i \mathcal{O}(\alpha)^{-1}, \quad (3.42)$$

where the generators $t_i$ are given by (3.25). This provides an explicit action of $SU(2)$ on the solutions of (3.40). This can then be exponentiated to yield the dependence of the solution upon the Euler angles. Equivalently, this equation shows how to map the $SU(2)$ dependence at the maximally supersymmetric point onto the $SU(2)$ dependence anywhere along the flow.

Consider spinors of the form $\Pi(\alpha, \beta) \mathcal{O}(\alpha) \epsilon_0$, where $\epsilon_0$ satisfies (3.24) and transforms under $SU(2)$ with the generators $t_i$. By construction, such spinors solve (3.4), except for the two equations along the $r$ and $\theta$ directions. Since we expect to find eight (real) independent components of the Killing spinors corresponding to the unbroken $\mathcal{N} = 2$ supersymmetry, and since the dimension of the range of the projector $\Pi(\alpha, \beta)$ is equal to eight, it remains only to fix the overall normalization of the solution. Rather than trying to integrate the remaining equations explicitly, we take a shortcut and require that using (3.6) the solution gives rise to at least one Killing vector along the brane. This leads to an prescription:

$$\epsilon = \frac{\Omega^{1/2} e^{i\phi/2}}{\cos(\beta/2)} \Pi(\alpha, \beta) \mathcal{O}(\alpha) \epsilon_0 = \frac{\Omega^{1/2} e^{i\phi/2}}{\cos(\beta/2)} \mathcal{O}(\alpha) \mathcal{O}^*(\beta) \Pi_0^{(1)} \mathcal{O}^*(\beta)^{-1} \epsilon_0, \quad (3.43)$$

with $\epsilon_0$ satisfying

$$\Pi_0^{(1)} \Pi_0^{(2)} \epsilon_0 = \epsilon_0. \quad (3.44)$$

It is then quite straightforward to check that (3.43) is indeed the general solution to the Killing spinor equations (3.4) for this IIB supergravity background.

3.3. Comments: New coordinates and the “Killing structure”

We have now shown explicitly that our solution is indeed $\mathcal{N} = 2$ supersymmetric. By comparing (3.43) with (3.23) we see that the deformation of the Killing spinor as we go away from the maximally supersymmetric point is encoded in the operator $\mathcal{O}^*(\beta)$, which rotates the projector $\Pi_0^{(1)}$ into $\Pi^{(1)}(\beta)$. The other rotation, $\mathcal{O}(\alpha)$, is merely an artifact of a particular choice of orthonormal frames for the metric and clearly can be removed by
rotating \( e^5 \) and \( e^6 \), the frames along the \( r \) and \( \theta \) coordinates, to a new set of frames, \( e^u \) and \( e^v \), given by:
\[
\begin{pmatrix}
  e^u \\
  e^v 
\end{pmatrix} = \begin{pmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha 
\end{pmatrix} \begin{pmatrix}
  e^5 \\
  e^6 
\end{pmatrix}.
\] (3.45)

Obviously, one would like to know whether there is a corresponding change of variables, \((r, \theta) \rightarrow (u, v)\), which induces this rotation.

A similar issue appears in [8], where it was shown that such a change of variables does indeed exist. Using the parallels between the solution in [8] and the present one, we find that the new coordinates are given by:
\[
u(r, \theta) = \frac{\rho^3 \cos \theta}{(c^2 - 1)^{1/2}}, \quad v(r, \theta) = \frac{\sin \theta}{(c^2 - 1)^{1/2}}.
\] (3.46)

In the next section we will show that the existence of these coordinates is a direct consequence of the supersymmetry of the background.

In terms of those new coordinates the metric, and other background fields, have a relatively simple structure. In particular, the metric becomes
\[
ds^2 = \Omega^2 (dx_\mu dx^\mu) - \frac{\Omega^{-2}}{\cos \beta} \left( du^2 + \frac{1}{c} dv^2 + u^2 ((\sigma^2)^2 + (\sigma^3)^2) \right) - \Omega^{-2} \cos \beta \left( u^2 (\sigma^1)^2 + cv^2 (d\phi)^2 \right),
\] (3.47)

where \( \Omega(u, v), c(u, v) \) and \( \beta(u, v) \) are now functions of \( u \) and \( v \). We will label the orthonormal frames for this metric as before with the correspondence \( e^u \leftrightarrow e^5 \) and \( e^v \leftrightarrow e^6 \). Also note that in these new coordinates:
\[
\Omega^4 = \frac{u^2 \cos \beta}{\sin^2 \beta}.
\] (3.48)

Finally, we find that (3.6) gives rise to five Killing vectors, which, with respect to the orthonormal frame above, have the components:
\[
(K^M) = \Omega (k_1, k_2, k_3, k_4, 0, 0, k_7 \sin \beta, 0, 0, 0),
\] (3.49)

where \( k_i \) are constants. From (3.47) and (3.48) one can see that the coordinate components, \( K^\mu \), are all constants. Thus, as we move away from the \( \mathcal{N} = 8 \) point there is one additional Killing vector, which corresponds to the \( U(1) \) symmetry that rotates \( \sigma^2 \) and \( \sigma^3 \). The fact that this new Killing vector is not forbidden by (3.28) is a direct consequence of the deformation of \( \Pi^{(1)}_0 \) in (3.17) to \( \Pi^{(1)} \) in (3.35) via the rotation (3.38).

It is now an interesting problem to find to what extent the metric or the entire background is determined by the form of the Killing spinors (3.43). We will study this more general problem in the next section.
4. Generalized $\mathcal{N} = 2$ backgrounds

In this section we discuss in more detail the constraints on the background imposed by the requirement of $\mathcal{N} = 2$ supersymmetry (i.e. eight supersymmetries). More specifically, we want to determine the most general $\mathcal{N} = 2$ supersymmetric solution of the chiral IIB supergravity arising from the following Ansatz:

(i) The metric is diagonal of the form (2.4), where the $\Omega(r, \theta)$ and $V_a(r, \theta)$, $a = 1, \ldots, 5$ are arbitrary functions.

(ii) The dilaton/axion is given by (2.8) and (2.9) with an arbitrary function $b(r, \theta)$.

(iii) The five-index tensor has only four non-vanishing components: $F_{12345}$, $F_{12346}$, and the ones related by the condition of self-duality. These components are to be functions of $r$ and $\theta$ alone.

(iv) The non-vanishing components of the three-index tensor are, at most, those listed in (2.16). Their dependence on $\phi$ is only through the overall phase, $\exp(i\phi)$, and they satisfy the same reality conditions at $\phi = 0$ as those in (2.10).

(v) The Killing spinors of the unbroken supersymmetry have the form (3.43), where $\alpha(r, \theta)$ and $\beta(r, \theta)$ are arbitrary functions.

(vi) The Killing spinors transform under $SU(2)$ with the same action as in (3.42).

(vii) The Killing vectors that are generated by bilinears of Killing spinors include the $U(1)$ rotation as in (3.49).

4.1. Solving the Ansatz

We will solve our problem in two steps. First we insert the general Ansatz for the background fields and the Killing spinors into the supersymmetry equations (3.4) and show that all unknown functions, $\Omega(r, \theta)$, $V_a(r, \theta)$, $b(r, \theta, \phi)$, $\beta(r, \theta)$ and the components of the three-index and five-index tensors are completely determined in terms of a single function, $c(r, \theta)$. The dependence on the angle, $\alpha(r, \theta)$, can be removed by a suitable change of coordinates. The Bianchi identities for the tensor fields turn out to be equivalent to a second order partial differential equation for $c$. In the second step we verify that all field equations are satisfied.

We begin by performing a rotation by the angle $\alpha$ from $(e^5, e^6)$ to $(e^u, e^v)$ as in (3.43). Then

$$E_u = \frac{\cos \alpha}{V_1} \frac{\partial}{\partial r} - \frac{\sin \alpha}{V_2} \frac{\partial}{\partial \theta}, \quad E_v = \frac{\sin \alpha}{V_1} \frac{\partial}{\partial r} + \frac{\cos \alpha}{V_2} \frac{\partial}{\partial \theta},$$

(4.1)
are vector fields dual to $e^u$ and $e^v$, respectively. We denote the corresponding tensor indices in the new frame by $u$ and $v$.

We can now rewrite the supersymmetry equations (3.4) in the new basis and in terms of derivatives $E_u$ and $E_v$. In Appendix C we have compiled a list of independent equations and discuss briefly how they arise.

We note that the assumption (vii) about the existence of the $U(1)$ Killing vector of the form (3.49) is equivalent to the following condition on the functions $V_3$ and $V_4$:

$$\frac{V_3}{V_4} = \cos(\beta).$$  (4.2)

Then, using (C.3), we obtain

$$V_4 = \frac{\sin(2\beta)}{\cos(\beta) G_{89\ 10} - i G_{567}}.$$  (4.3)

Guided by the form of the metric (3.47), we define a function

$$u(r, \theta) = \Omega V_4 (\cos \beta)^{1/2}.$$  (4.4)

Using (C.7), (C.8), (C.10) and then (4.3), one can check that

$$E_u u = \Omega (\cos \beta)^{1/2}, \quad E_v u = 0.$$  (4.5)

Similarly, we define

$$v(r, \theta) = \frac{\Omega V_5}{c^{1/2} (\cos \beta)^{1/2}},$$  (4.6)

where

$$c(r, \theta) = \frac{1}{\cos \beta} \left( \frac{1 + B}{1 - B} \right)_{\phi=0}.$$  (4.7)

Using (C.7), (C.8) and (C.11), we find

$$E_u v = 0, \quad E_v v = \Omega c^{1/2} (\cos \beta)^{1/2}.$$  (4.8)

This shows that upon the change of coordinates from $(r, \theta)$ to $(u, v)$ given in (4.4) and (4.6), the forms $e^u$ and $e^v$ are proportional to $du$ and $dv$:

$$e^u = \frac{\Omega^{-1}}{(\cos \beta)^{1/2}} du, \quad e^v = \frac{\Omega^{-1}}{c^{1/2} (\cos \beta)^{1/2}} dv,$$  (4.9)

and thus the metric has a diagonal form in the new coordinates as well.
Thus, just as in the special solution in Section 3, the dependence on the angle, $\alpha$, can be removed by a change of coordinates, and the existence of the new coordinates, $u$ and $v$, which essentially is the content of the equations (4.3) and (4.3), is a consequence of supersymmetry. From now on we will work in the new coordinates and define $V_u$ and $V_v$ as the coefficients of $du$ and $dv$ in (4.9), respectively. It also follows from (4.2)-(4.9) that in the new coordinates the metric has precisely the form (3.47), where $\Omega(u, v)$, $\beta(u, v)$ and $c(u, v)$ are, at this point, arbitrary functions, which we will now restrict further by solving the equations (a)-(f) in Appendix C.

First, we evaluate

$$E_u c = \Omega (\cos \beta)^{1/2} \partial_u c, \quad (4.10)$$

and

$$E_v c = \Omega c^{1/2} (\cos \beta)^{1/2} \partial_v c. \quad (4.11)$$

Using (4.7), (C.4)-(C.6) and (C.7), we find

$$E_u c = -ic \tan(\beta) G_{567}, \quad (4.12)$$

and

$$E_v c = \frac{\Omega}{v c^{1/2} (\cos \beta)^{3/2}} \left(1 - c^2 \cos^2 \beta\right), \quad (4.13)$$

where, in the last equation, all dependence on the the antisymmetric tensor field cancelled out and we used (4.6) to eliminate $V_5$. Combining (4.11) and (4.13), we can now determine $\beta$ in terms of $c$:

$$\beta(u, v) = -\arctan \left(c^2 + v c \partial_v c - 1\right)^{1/2}. \quad (4.14)$$

Next we use (4.3), (4.4), (4.12) and (C.7) to express $G_{567}$, $G_{u89}$, $G_{v89}$ and $G_{8910}$ in terms of $\Omega$ and $c$. Then we substitute the result in (C.8) and solve for $\partial_u \Omega$ and $\partial_v \Omega$ in terms of $u$, $v$, $c(u, v)$ and its partial derivatives. We verify that the system is integrable and obtain:

$$\Omega(u, v) = \frac{u^{1/2} c^{1/8} (c + v \partial_v c)^{1/8}}{(c^2 + v c \partial_v c - 1)^{1/4}}, \quad (4.15)$$

together with

$$B(u, v, \phi) = \frac{c - c^{1/2} (c + v \partial_v c)^{1/2}}{c + c^{1/2} (c + v \partial_v c)^{1/2}} e^{2i\phi}, \quad (4.16)$$

which follows from (4.7) and (4.14). This shows that the metric and the dilaton/axion are given in terms of a single function $c(u, v)$. We also verify that all equations (a)-(f) in Appendix C are now satisfied for an arbitrary $c(u, v)$.
Further, using (C.12), (C.2) and the relations obtained above, we can express all the components of the antisymmetric tensors field strengths, \( G_{(3)} \) and \( F_{(5)} \), in terms of \( c(u,v) \) and its derivatives. In order to compute the corresponding potentials, \( C_{(4)} \) and \( A_{(2)} \), we first check whether Bianchi identities [9]

\[
dF_{(5)} + \frac{i}{8} G_{(3)} \wedge G_{(3)}^* = 0,
\]

(4.17)

and

\[
d \left[ (1 - BB^*)^{-1/2} (G_{(3)} + BG_{(3)}^*) \right] = 0,
\]

(4.18)

are satisfied.

The result of the calculation can be succinctly expressed in terms of

\[
\mathcal{L}(c) = \frac{\partial}{\partial u} \left( \frac{v^3}{u} \frac{\partial c}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{v^3}{u} c \frac{\partial c}{\partial v} \right).
\]

(4.19)

The equations resulting from the two Bianchi identities read

\[
vc \partial_u \mathcal{L}(c) + (v \partial_v c - c) \mathcal{L}(c) = 0,
\]

(4.20)

and

\[
\mathcal{Z}(c) \mathcal{L}(c) = 0, \quad \mathcal{Z}(c) (v \partial_u \mathcal{L}(c) - 2\mathcal{L}(c)) = 0,
\]

(4.21)

respectively, where

\[
\mathcal{Z}(c) = 2 c^{1/2} (c + v \partial_v c) + (2c + c \partial_v c)(c + v \partial_v c)^{1/2}.
\]

(4.22)

An obvious solution to those equations is obtained by setting

\[
\mathcal{L}(c) = 0,
\]

(4.23)

and so we will henceforth impose this on \( c(u,v) \).

The explicit form of \( C_{(4)} \) is now simply obtained by setting,

\[
w(u,v) = \frac{1}{4} \Omega^4 \cos \beta,
\]

(4.24)

where \( w(u,v) \) is defined in (2.11) and (2.12). Finally, to calculate the two-form potential, \( A_{(2)} \), we consider an Ansatz of the form:

\[
A_{(2)} = e^{i\phi} (a_1 dv \wedge \sigma^1 + a_2 \sigma^2 \wedge \sigma^3 + a_3 \sigma^1 \wedge d\phi),
\]

(4.25)
with arbitrary functions $a_i(u,v)$, $i = 1, 2, 3$. We find that, provided (4.23) is satisfied, $A_{(2)}$ is the desired potential if we set

$$
egin{align*}
a_1(u,v) &= \frac{i}{c}, \\
a_2(u,v) &= i \frac{v}{v \partial_v c + c}, \\
a_3(u,v) &= -uv \partial_u c + 2vc.
\end{align*}
$$

We have thus shown that all fields in our Ansatz are determined by the supersymmetry and Bianchi identities. We have also verified that all field equations are now satisfied without any further conditions on $c(u,v)$.

4.2. Summary

Here we pull together the key elements of the foregoing derivation to show how a complete solution can be obtained trivially once one finds a solution of the “master equation,” defined by (4.23) and (4.19).

The metric functions, $V_u$, $V_v$ and $V_j$, $j = 3, 4, 5$ can be read off from (4.9), (4.4), (4.6) and (4.2). The resulting metric can then be written in the form:

$$
ds^2 = \Omega^2 (dx_\mu dx^\mu) - \Omega^{-2} \left[ H_1 (du^2 + u^2 ((\sigma^2)^2 + (\sigma^3)^2)) + H_1^{-1} u^2 (\sigma^1)^2 \\
+ H_2 dv^2 + H_2^{-1} v^2 d\phi^2 \right],
$$

where

$$
H_1(u,v) \equiv \frac{1}{\cos \beta}, \quad H_2(u,v) \equiv \frac{1}{c \cos \beta}.
$$

More importantly, from (4.28) and (4.13), we have:

$$
H_1 H_2^{-1} = c, \quad H_1 H_2 = \partial_v (v c),
$$

which shows that the $H_1$ and $H_2$ can be trivially generated from $c$. The expression, (4.13), for $\Omega$ may be written algebraically in terms of $H_1$:

$$
\Omega = \frac{u^{1/2}}{(H_1 - H_1^{-1})^{1/4}}.
$$

Similarly, the expression for the dilaton and axion may be re-written in terms of $H_2$:

$$
B = \frac{(1 - H_2)}{(1 + H_2)} e^{2i\phi} \Leftrightarrow \tau = -\frac{(H_2 \sin \phi + i \cos \phi)}{(H_2 \cos \phi - i \sin \phi)}.
$$
The expression, (4.24), for the function that governs the 4-form potential may be re-written as:

\[ w(u, v) = \frac{1}{4} \Omega^4 \cos \beta = \frac{1}{4} \frac{u^2}{(H_1^2 - 1)} , \]  

(4.32)

while the expression for the 2-form potential is already given in terms of \( c \) by (4.26). Note that:

\[ a_1 = i H_2 H_1^{-1} , \quad a_2 = i v H_1^{-1} H_2^{-1} , \quad a_3 = -u^2 \partial_u \left( \frac{v}{u^2} c \right) \Rightarrow \partial_v a_3 = -u^2 \partial_u \left( \frac{1}{u^2} H_1 H_2 \right) . \]  

(4.33)

The structure of this solution is manifestly very similar to that of [8].

4.3. Some examples

We have obtained a general class of \( \mathcal{N} = 2 \) solutions which are determined by a single function \( c(u, v) \) satisfying (4.23). While the equation is non-linear, one can easily find some explicit solutions.

**Example 1.** The original solution in Section 2 does indeed fall into this class of solutions. We can use (3.46) to obtain:

\[ \rho^6 = \frac{u^2 (c^2 - 1)}{1 - v^2 (c^2 - 1)} . \]  

(4.34)

Then we substitute the result in (2.3) and differentiate with respect to \( u \) and \( v \) to eliminate the integration constant, \( \gamma \). This yields a first-order system of equations for \( c(u, v) \):

\[ \begin{align*}
    \frac{\partial c}{\partial u} &= \frac{u (c^2 - 1)^2 [v^2 (c^2 - 1) - 1]}{(1 + v^2)^2 + u^2 v^2 c - 2 v^2 (1 + v^2) c^2 - 2 u^2 v^2 c^3 + v^4 c^4 + u^2 v^2 c^5} , \\
    \frac{\partial c}{\partial v} &= -\frac{u^2 v (c^2 - 1)}{(1 + v^2)^2 + u^2 v^2 c - 2 v^2 (1 + v^2) c^2 - 2 u^2 v^2 c^3 + v^4 c^4 + u^2 v^2 c^5} ,
\end{align*} \]  

(4.35)

from which (4.23) follows directly.

**Example 2.** A simpler example can be generated by looking for solutions in which the two terms in (4.19) vanish separately. This yields a three-parameter family of solutions with

\[ c = \mu (1 + b u^2) \left( 1 - \frac{a}{v^2} \right)^{1/2} , \]  

(4.36)

for some constants \( a, b \) and \( \mu \). One then finds

\[ H_1 = \mu (1 + b u^2) , \quad H_2 = \frac{v}{(v^2 - a)^{1/2}} . \]  

(4.37)
This is analogous to the solution found in [8]. For non-zero $a$ and $b$ the background is highly non-trivial. Indeed, if $b \neq 0$, the six-dimensional metric in the square brackets of (4.27) is asymptotic (at large distances) to an $S^1$ of constant radius non-trivially fibered over a flat $\mathbb{R}^5$. To have this six-metric asymptote to a flat $\mathbb{R}^6$ one must $b = 0$ and $\mu = 1$, however this leads to singular expression for $\Omega$. One can arrive at a non-singular result by taking $b = 0$, $\mu = 1 + \eta$, and then taking the limit $\eta \to 0$ while scaling $u,v$ and $x^\mu$ by suitable powers of $\eta$. The end result is a metric of the form:

$$
 ds^2 = H_0^{-\frac{1}{2}} (dx_\mu dx^\mu) - H_0^\frac{1}{2} ds_6^2,
$$

(4.38) where

$$
 H_0 \equiv \Omega^{-4} = \frac{m}{u^2},
$$

(4.39) for some constant, $m$. For $a = 0$, the metric, $ds_6^2$, is the flat Euclidean metric on $\mathbb{R}^6$, and the solution is the standard “harmonic” form for D3-branes spread over the two-dimensional $v$-plane. For $a \neq 0$ the metric, $ds_6^2$, is not flat, but has a curvature singularity at $v^2 = a$. The dilaton/axion background is also non-zero, and as pointed out in [17], the corresponding eight-metric in $F$-theory must be hyper-Kähler.

5. Final comments

We have managed to push beyond the “harmonic rule” for brane configurations and obtain a solution with softly-broken supersymmetry. In particular we have shown how to construct a IIB supergravity solution that is the holographic dual of $\mathcal{N} = 4$ Yang-Mills theory, softly broken to $\mathcal{N} = 2$ Yang-Mills, at an arbitrary, $SO(2)$-invariant point on the Coulomb branch of the $\mathcal{N} = 2$ theory. More generally, we have found a natural method for constructing supersymmetric solutions in the presence of multiple independent $RR$ fluxes. This method is based upon defining the Killing spinors in terms of projectors that are algebraic in the metric Ansatz. The overall Killing structure can then be used to constrain the projectors and fix the spinor normalization.

The only complication in our procedure here is the fact that the crucial underlying function, $c$, must satisfy a non-linear differential equation. However, for the metric to be asymptotically $AdS_5 \times S^5$, one must have $c \to 1$ at infinity. Moreover, one can then make a perturbative expansions, $c = 1 + \sum_n c_n$. One then finds that $c_1$ must satisfy the linearized form of (4.23):

$$
 \frac{\partial}{\partial u} \left( \frac{v^3}{u} \frac{\partial c_1}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{v^3}{u} \frac{\partial c_1}{\partial v} \right) = 0.
$$

(5.1)
Each of the functions, $c_n$, then satisfies the same linear equation, but with a source that is quadratic in the $c_j$, $j < n$, and their derivatives. This perturbation series is essentially “seeded” by, $c_1$, a homogeneous solution to a second order, linear, elliptic equation in two variables. The function, $c_1$, is, in turn, defined by some distribution of sources, $\rho(v)$, that is assumed to lie at $u = 0$. Whether or not this perturbation series is a practical solution remains to be seen, however it does show that we have, in principle, solved the problem of $\mathcal{N} = 2$ supersymmetric flows with a rotationally symmetric distribution of $D3$-branes in the $(v_1, v_2)$ directions.

We suspect that the techniques employed here could be used to solve a variety of flow problems, and, more generally, find supersymmetric backgrounds that involve several independent fluxes. Work on this is continuing.

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Appendix A. Some $\gamma$-matrix conventions and identities

We use the mostly minus convention and label the frames and $\gamma$-matrices from 1 to 10, with 1 being the time-like direction. Otherwise our conventions are the same as in [3]. In particular, we use a pure imaginary representation of $\gamma$-matrices,

$$ (\gamma^M)^* = -\gamma^M, \quad M = 1, \ldots, 10. \quad \text{(A.1)} $$

which then have the following symmetry properties:

$$ (\gamma^1)^T = -\gamma^1, \quad (\gamma^M)^T = \gamma^M, \quad M > 1. \quad \text{(A.2)} $$

The matrix,

$$ \gamma^{11} = \gamma^1 \gamma^2 \ldots \gamma^{10}, \quad \text{(A.3)} $$

is real and symmetric. The Dirac conjugate of a spinor $\eta$ is $\bar{\eta} = \eta^\dagger \gamma^1$. The charge conjugation matrix is $C = \gamma^1$. Then

$$ C \gamma^M_1 \ldots \gamma^M_p = \begin{cases} \text{symmetric for } p = 1, 2, 5, 6, 9, 10. \\ \text{antisymmetric for } p = 0, 3, 4, 7, 8. \end{cases} \quad \text{(A.4)} $$

The spinor, $\epsilon_0$, which parametrizes the unbroken supersymmetries, is defined by

$$ \gamma^{11} \epsilon_0 = -\epsilon_0, \quad \gamma^1 \gamma^2 \gamma^3 \gamma^4 \epsilon_0 = i \epsilon_0, \quad \gamma^6 \gamma^{10} \epsilon_0 = -i \epsilon_0. \quad \text{(A.5)} $$

Using hermiticity properties of $\gamma$ matrices we then find

$$ \epsilon_0^\dagger \gamma^{11} = -\epsilon_0^\dagger, \quad \epsilon_0^\dagger \gamma^1 \gamma^2 \gamma^3 \gamma^4 = i \epsilon_0^\dagger, \quad \epsilon_0^\dagger \gamma^6 \gamma^{10} = -i \epsilon_0^\dagger, \quad \text{(A.6)} $$

and similar relations for $\epsilon_0^T$. In the notation of Section 3 we can rewrite (A.6) as identities for the conjugate spinor $\bar{\epsilon}_0 = \epsilon_0^\dagger \gamma^1$,

$$ \bar{\epsilon}_0 (1 - \Pi_0^{(1)}) = \bar{\epsilon}_0, \quad \bar{\epsilon}_0 \Pi_0^{(2)} = \bar{\epsilon}_0. \quad \text{(A.7)} $$

In particular the first identity implies that $\bar{\epsilon}_0 \epsilon_0 = 0$.

Since $(\gamma^{11})^2 = 1$, we find that for any product of odd number of $\gamma$-matrices

$$ \epsilon_0^T \gamma^{M_1} \ldots \gamma^{M_{2n+1}} \epsilon_0 = 0, \quad \text{and} \quad \epsilon_0^\dagger \gamma^{M_1} \ldots \gamma^{M_{2n+1}} \epsilon_0, \quad M_i = 1, \ldots, 10. \quad \text{(A.8)} $$

Similarly, using $(\gamma^1 \gamma^2 \gamma^3 \gamma^4)^2 = -1$ and $(\gamma^6 \gamma^{10})^2 = -1$, together with (A.3) and (A.6), we obtain additional vanishing relations, such as

$$ \epsilon_0^T \gamma^{M_1} \ldots \gamma^{M_{2n}} \epsilon_0 = 0, \quad M_i = 5, \ldots, 10. \quad \text{(A.9)} $$

Further vanishing relations follow from symmetry combined with the statistics of $\epsilon_0$. 

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Appendix B. Killing vectors

Here we outline a proof of the general result used frequently throughout the text:

Let $\epsilon$ and $\eta$ be two commuting Killing spinors satisfying (3.1) and (3.2). Then

$$(K^M) = \bar{\eta} \gamma^M \epsilon + \bar{\epsilon} \gamma^M \eta,$$

is a Killing vector of the metric.

The proof uses symmetry properties of $\gamma$-matrices and is essentially identical with the one in [13] for eleven-dimensional supergravity. We show directly that the Killing vector equation

$$D_M K_N + D_N K_M = 0,$$

is satisfied. First rewrite (3.1) in the form:

$$D_M \epsilon = -\frac{i}{480} F_{(5)} \gamma_M \epsilon - \frac{1}{96} (\gamma_M \gamma^{PQR} - 12 \delta_M^P \gamma^{QR}) G_{PQR} \epsilon^*, \tag{B.2}$$

where $F_{(5)} = F_{PQRST} \gamma^{PQRST}$. Conjugating this equation we obtain

$$D_M \bar{\epsilon} = \frac{i}{480} \bar{\tau} \gamma_M F_{(5)} - \frac{1}{96} \bar{\epsilon} \bar{\epsilon}^* (\gamma^{PQR} \gamma_M \pm 12 \delta_M^P \gamma^{QR}) G^*_{PQR}, \tag{B.3}$$

where

$$D_M \bar{\epsilon} = \partial_M \bar{\epsilon} - \frac{1}{4} \bar{\epsilon} \gamma^{PQ} \omega_{MPQ} + \frac{i}{2} \bar{\epsilon} Q_M. \tag{B.4}$$

Now, the left hand side of (B.1) becomes

$$(D_M \bar{\eta}) \gamma_N \epsilon + \bar{\eta} \gamma_N (D_M \epsilon) + (M \leftrightarrow N) + (\eta \leftrightarrow \epsilon). \tag{B.5}$$

Substituting (B.2) and (B.3) in (B.5), we find that, upon symmetrization over $M$ and $N$, the terms with $F_{(5)}$ vanish, while in the terms with $G_{(3)}$ only products of three $\gamma$-matrices remain. Those terms are of the form

$$\eta^T C \gamma^{PQR} \epsilon g_{MN} G_{PQR} \quad \text{and} \quad \eta^T C \gamma^{PQ}_{(M} \epsilon G_{N)PQ}, \tag{B.6}$$

and the complex conjugates. Using (A.4), we conclude that they vanish after symmetrization in $\epsilon$ and $\eta$. 

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Appendix C. The supersymmetry equations

In this appendix we list some of the equations resulting from the substitution of the Ansatz in Section 4 in (3.4) and indicate the variations which have given rise to each of them. One should note that most of the equations differ from their original form resulting from (3.4) as we systematically eliminate the $G_{u710}$ and $G_{v710}$ components of the three-index tensor field and the $F_{1234u}$ and $F_{1234v}$ components of the five-index tensor field using the relations

\[ G_{u710} = -i G_{567} + \cos(\beta) (G_{8910} - i G_{v89}) , \]
\[ G_{v710} = i \cos(\beta) G_{u89} , \]

and

\[ F_{1234u} = \frac{i}{4} \csc(\beta) G_{567} + \frac{i}{4} \cot(\beta) G_{v89} , \]
\[ F_{1234v} = -\frac{1}{4} \cot(\beta) G_{u89} , \]

which follow from $\delta \lambda = 0$ and $\delta \psi = 0$, respectively. Throughout this appendix, after evaluating derivatives, we set $\phi = 0$.

There is one further algebraic equation from $\delta \psi_8 = 0$:

\[ \frac{V_3}{V_4^2} = -\frac{i}{2} \csc(\beta) G_{567} - \frac{1}{2} \cot(\beta) G_{8910} . \]

The remaining equations involve derivatives of the functions in the Ansatz.

(a) Form $\delta \lambda = 0$:

\[ P_u = -\frac{1}{4} \sin(\beta) (G_{8910} - i G_{v89}) , \]
\[ P_v - iP_{10} = -\frac{i}{4} \sin(\beta) G_{u89} , \]

where, we recall,

\[ P_u = \frac{E_u B}{1 - B^2} , \quad P_v = \frac{E_v B}{1 - B^2} , \]

and

\[ P_{10} = \frac{2i}{V_5} \frac{B}{1 - B^2} , \quad Q_{10} = -\frac{2}{V_5} \frac{B^2}{1 - B^2} . \]

(b) From $\delta \psi_u$ and $\delta \psi_v$:

\[ E_u \beta = -i G_{567} + \frac{1}{2} \cos(\beta) (G_{8910} - i G_{v89}) , \]
\[ E_v \beta = \frac{i}{2} \cos(\beta) G_{u89} , \]
\[ E_u \log(\Omega) = \frac{i}{4} \cot(\beta) G_{567} \]
\[ + \frac{i}{16} (3 + \cos(2\beta)) \csc(\beta) G_{89} + \frac{1}{8} \sin(\beta) G_{8910}, \quad (C.8) \]
\[ E_v \log(\Omega) = -\frac{i}{16} (3 + \cos(2\beta)) \csc(\beta) G_{u89}, \]

(c) From \( \delta\psi_7 = 0 \):
\[ E_u \log V_3 = -\frac{3i}{4} \cot(\beta) G_{567} + \frac{1}{16} (5 + 3 \cos(2\beta)) \csc(\beta) G_{8910} \]
\[ -\frac{i}{16} (1 + 3 \cos(2\beta)) \csc(\beta) G_{v89}, \quad (C.9) \]
\[ E_v \log V_3 = \frac{i}{16} (1 + 3 \cos(2\beta)) \csc(\beta) G_{u89}, \]

(d) From \( \delta\psi_{8,9} = 0 \):
\[ E_u \log V_4 = \frac{2}{V_4} + \frac{i}{4} \cot(\beta) G_{567} - \frac{1}{16} (7 + \cos(2\beta)) \csc(\beta) G_{8910} \]
\[ -\frac{i}{16} (5 - \cos(2\beta)) \csc(\beta) G_{v89}, \quad (C.10) \]
\[ E_v \log V_4 = \frac{i}{16} (5 - \cos(2\beta)) \csc(\beta) G_{u89} \]

(e) From \( \delta\psi_{10} = 0 \):
\[ E_u \log V_5 = \frac{E_u B}{1 - B^2} - 2E_u (\Omega^{1/2}) \]
\[ E_v \log V_5 = \frac{E_v B}{1 - B^2} + \frac{1}{V_5} \left( \frac{1 + B}{1 - B} \right) + \frac{i}{16} (3 + \cos(2\beta)) \csc(\beta) G_{u89} \quad (C.11) \]

After passing to the new new coordinates \( u \) and \( v \), we obtain equations for \( V_u \) and \( V_v \).

(f) From \( \delta\psi_\mathit{u} = 0 \) and \( \delta\psi_\mathit{v} = 0 \):
\[ E_u \log V_v = -\frac{i}{4} \cot(\beta) G_{567} - \frac{i}{16} (5 - \cos(2\beta)) \csc(\beta) G_{v89} \]
\[ + \frac{1}{8} \sin(\beta) G_{8910}, \quad (C.12) \]
\[ E_v \log V_u = \frac{i}{16} (5 - \cos(2\beta)) \csc(\beta) G_{u89}. \]
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