ON THE HEEGAARD GENUS OF CONTACT 3-MANIFOLDS

BURAK OZBAGCI

ABSTRACT. It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. We show that Heegaard genus of contact 3-manifolds is not necessarily additive under contact connected sum. We also prove some basic properties of the contact genus (a.k.a. open book genus [8]) of 3-manifolds, and compute this invariant for some 3-manifolds.

1. Introduction

We assume that all 3-manifolds are closed, connected and oriented and all contact structures are co-oriented and positive throughout this paper. Let $Y$ denote a 3-manifold. Given an open book $(B, \pi)$ on $Y$, where $B$ denotes the binding and $\pi$ denotes the fibration of $Y - B$ over $S^1$. It follows that $(\pi^{-1}([0, 1/2]) \cup B)$ and $(\pi^{-1}([1/2, 1]) \cup B)$ are handlebodies which induce a Heegaard splitting of $Y$, where we view $S^1$ as the interval $[0, 1]$ whose endpoints are identified with each other. In this sense an open book is can be viewed as a special Heegaard splitting. Note that a stabilization of an open book at hand corresponds to a stabilization of the induced Heegaard splitting.

We define the Heegaard genus $Hg(Y, \xi)$ of a contact 3-manifold $(Y, \xi)$ as the minimal genus of a Heegaard surface in any Heegaard splitting of $Y$ induced from an open book supporting $\xi$. Equivalently, $Hg(Y, \xi) = 1 + sn(\xi) = \min \{1 - \chi(\Sigma) \mid \Sigma \text{ is a page of an open book supporting } \xi\}$, where $sn(\xi)$ denotes the support norm of $\xi$ (cf. [4]) and $\chi(\Sigma)$ denotes the Euler characteristic of $\Sigma$. This is certainly a generalization of the Heegaard genus adapted to contact 3-manifolds. It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. Here we show that Heegaard genus is sub-additive but not necessarily additive under contact connected sum of contact 3-manifolds.

Moreover we define the contact genus $cg(Y)$ of a 3-manifold $Y$ as the minimal Heegaard genus over all contact structures, i.e., $cg(Y) = \min\{Hg(Y, \xi) \mid \xi \text{ is a contact structure on } Y\}$ which, by Giroux’s correspondence [5], is the minimal genus of a Heegaard surface in any Heegaard splitting of $Y$ induced from an open book. In other words, the contact genus of a 3-manifold is a topological invariant obtained by taking the minimum of the sum $2g + r - 1$ over all open books, where $g$ and $r$ denote the genus of the page and the
number of binding components of the open book, respectively. We show that contact genus is sub-additive (and conjecture that it is additive) under connected sum of 3-manifolds.

We would like to point out that the contact invariant was first studied by Rubinstein who named it the open book genus of \( Y \) (cf. \[8\]). We prefer to call it the contact genus to emphasize its connection with contact topology. It is clear by definition that for any contact structure \( \xi \) on \( Y \) we have

\[
Hg(Y) \leq cg(Y) \leq Hg(Y, \xi),
\]

where \( Hg(Y) \) denotes the Heegaard genus of \( Y \). In \[8\], it was shown that “most” 3-manifolds of Heegaard genus 2 have contact genus \( > 2 \), which implies the existence of 3-manifolds where the first inequality above is strict. In particular, it follows that not every Heegaard splitting of a 3-manifold comes from an open book.

Here we show that “most” 3-manifolds of Heegaard genus 1 have contact genus \( > 1 \). Namely we show that a lens space which is not diffeomorphic to an oriented circle bundle over \( S^2 \) have contact genus \( \geq 2 \). On the other hand, the contact genus of any oriented circle bundle over \( S^2 \) is equal its Heegaard genus. We also show that there are many small Seifert fibered 3-manifolds (which are not lens spaces) which have this property. Examples of such 3-manifolds were constructed in \[8\], but our examples are much simpler. We refer the reader to \[3\] and \[7\] for more on open books and contact structures.

2. Heegaard genus and contact connected sum

Given any two contact 3-manifolds \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\). By removing a Darboux ball from each of these contact 3-manifolds and gluing them along their convex boundaries by an orientation reversing map carrying respective characteristic foliations onto each other we get a well defined contact structure \( \xi_1 \# \xi_2 \) on the connected sum \( Y_1 \# Y_2 \). The contact 3-manifold \((Y_1 \# Y_2, \xi_1 \# \xi_2)\) is called the contact connected sum of \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\). It is well-known that Heegaard genus is additive under connected sum of smooth 3-manifolds, which follows from Haken’s Lemma. Here we show that

**Theorem 1.** The Heegaard genus is sub-additive but not necessarily additive under connected sum of contact 3-manifolds.

**Proof.** Let \( OB_i \) denote the open book realizing \( Hg(Y_i, \xi_i) \), for \( i = 1, 2 \). Then the contact structure \( \xi_1 \# \xi_2 \) on \( Y_1 \# Y_2 \) is supported by the open book \( OB \) obtained by plumbing the pages of the open books \( OB_1 \) and \( OB_2 \) by Torisu \[9\]. Denote a page of the open book \( OB_i \) by \( \Sigma_i \). It follows that

\[
-\chi(\Sigma) = -\chi(\Sigma_1) - \chi(\Sigma_2) + 1,
\]

where \( \Sigma \) denotes the page of the open book \( OB \). Thus we have

\[
Hg(Y_1 \# Y_2, \xi_1 \# \xi_2) \leq Hg(Y_1, \xi_1) + Hg(Y_2, \xi_2),
\]

which implies that \( Hg \) is sub-additive under contact connected sum.
Next we show that $Hg$ is not necessarily additive under contact connected sum. Let $\xi_d$ denote the overtwisted contact structure in $S^3$ whose $d_3$ invariant (cf. [6]) is equal to the half integer $d$. The following result was obtained in [1]: If $(Y, \xi)$ is a contact structure with $c_1(\xi)$ torsion, then

$$d_3(Y, \xi \# \xi_d) = d_3(Y, \xi) + d_3(S^3, \xi_d) + 1/2.$$ 

Now suppose that $Y$ is an integral homology sphere. It follows that $c_1(\xi) = 0$ for every contact structure $\xi$ on $Y$, and $Y$ carries a unique spin$^c$ structure. Thus for an arbitrary contact structure $\xi$ on $Y$ we have

$$d_3(Y, \xi \# \xi - \frac{1}{2}) = d_3(Y, \xi) + d_3(S^3, \xi - \frac{1}{2}) + \frac{1}{2} = d_3(Y, \xi),$$

which implies that the connected sum $\xi \# \xi - \frac{1}{2}$ is homotopic to $\xi$ as oriented plane fields (cf. [6]). In fact, $\xi \# \xi - \frac{1}{2}$ is isotopic to $\xi$ by the classification of overtwisted contact structures due to Eliashberg [2]. As a consequence we have

$$Hg(Y, \xi \# \xi - \frac{1}{2}) = Hg(Y, \xi).$$

On the other hand, in ([4], Lemma 5.5), it was proved that $Hg(S^3, \xi - \frac{1}{2}) = 2$. Note that an open book realizing $Hg(S^3, \xi - \frac{1}{2})$ can be described by taking a pair of pants as a page and $t_1t_2^2t_3^{-3}$ as the monodromy, where $t_i$ denotes a right-handed Dehn twist along a boundary component. Consequently we have

$$Hg(Y \# S^3, \xi \# \xi - \frac{1}{2}) < Hg(Y, \xi) + Hg(S^3, \xi - \frac{1}{2}).$$

□

3. Contact genus of three dimensional manifolds

Here we provide some basic properties of the contact genus of 3-manifolds, and compute this invariant for some 3-manifolds.

**Proposition 2.** Let $Y$ denote a 3-manifold. Then we have

(a) $cg(Y) \geq 0$ (=$0$ if and only if $Y \cong S^3$),

(b) $cg(Y) = 1$ if and only if $Y$ is an oriented circle bundle over $S^2$ (which is not diffeomorphic to $S^3$).

**Proof.** For a 3-manifold $Y$, $cg(Y)$ is obtained by taking the minimum of the sum $2g + r - 1$ over all open books, where $g$ and $r$ denote the genus of the page and the number of binding components of an open book, respectively. Hence we have $0 \leq cg(Y)$ for an arbitrary 3-manifold $Y$, since $g \geq 0$ and $r \geq 1$. It is clear that the absolute minimum of the expression $2g + r - 1$ is realized when $g = 0$ and $r = 1$ and the open book with disk pages and trivial monodromy supports the unique tight contact structure on $S^3$, which proves (a).
To prove (b), we note that \( cg(Y) = 1 \) is realized if and only if \( g = 0 \) and \( r = 2 \). Any self-diffeomorphism of an annulus is given by \( t^m_c \), for some \( m \in \mathbb{Z} \), where \( c \) is the core of the annulus, and \( t_c \) denotes a right-handed Dehn twist along \( c \). If \( m \geq 0 \), this open book supports the unique tight contact structure on the lens space \( L(m, -1) \) which is an oriented circle bundle over \( S^2 \) with Euler number \( m \). Otherwise (i.e., when \( m < 0 \)) the induced contact structure is the overtwisted contact structure on \( L(-m, 1) \) which is an oriented circle bundle over \( S^2 \) with Euler number \( m \). Combining, we showed that \( cg(Y) = 1 \) if and only if \( Y \) is an oriented circle bundle over \( S^2 \), which is not diffeomorphic to \( S^3 \).

\[ \square \]

Note that oriented circle bundles over \( S^2 \) are very special lens spaces and therefore we immediately conclude from Proposition 2 that

**Corollary 3.** Most 3-manifolds of Heegaard genus 1 have contact genus > 1.

For example, \( cg(L(5, 3)) = 2 \), since \( L(5, 3) \) is not a circle bundle over \( S^2 \) and it carries a (tight) contact structure which is supported by a planar open book with three binding components.

**Lemma 4.** We have \( cg(Y_{p,q,r}) \leq 2 \), where \( Y_{p,q,r} \) denotes the 3-manifold depicted in Figure 1 with \( p, q, r \in \mathbb{Z} \). Moreover if \( |p| > 1, |q| > 1 \) and \( |r| > 1 \) then \( cg(Y_{p,q,r}) = 2 \).

**Proof.** It follows from [4] that \( Y_{p,q,r} \) has a planar open book with at most three binding components, which indeed proves that \( cg(Y_{p,q,r}) \leq 2 \). Moreover, under the assumption that \( |p| > 1, |q| > 1 \), and \( |r| > 1 \), the 3-manifold \( Y_{p,q,r} \) is not diffeomorphic to any lens space and hence \( cg(Y_{p,q,r}) = 2 \) by Proposition 2.

![Figure 1. Integral surgery diagram for the small Seifert fibered 3-manifold \( Y_{p,q,r} \)](image)

When we drop the assumption on \( p, q \) and \( r \) in Lemma 4, we observe that \( Y_{p,q,r} \) is diffeomorphic to either \( S^3 \), \( S^1 \times S^2 \), a lens space, or certain connected sums of these for some values of the integers \( p, q \) and \( r \).
Remark 5. Note that Lemma 4 exhibits examples of $3$-manifolds $Y = Y_{p,q,r}$ for which $Hg(Y) = cg(Y) = 2$, although most $3$-manifolds of Heegaard genus 2 have contact genus $> 2$ as was shown by Rubinstein [8].

Lemma 6. We have $cg(\#_k S^1 \times S^2) = k$, for $k \geq 1$.

Proof. Since $Hg(\#_k S^1 \times S^2) = k$, we know that $cg(\#_k S^1 \times S^2) \geq k$. Hence to show that $cg(\#_k S^1 \times S^2) = k$, we just need to realize this lower bound by a Heegaard splitting of $\#_k S^1 \times S^2$ induced from an open book. We use the fact that the unique tight contact structure on $\#_k S^1 \times S^2$ is supported by an planar open book with $k+1$ binding components, whose monodromy is the identity map.

The proof of the following result is similar to the proof of Theorem 1.

Proposition 7. Let $Y_i$ denote a $3$-manifold, for $i = 1, 2$. Then we have $cg(Y_1 \# Y_2) \leq cg(Y_1) + cg(Y_2)$.

Conjecture 8. Contact genus is additive under connected sum of $3$-manifolds.

Note that if $Hg(Y_i) = cg(Y_i)$ for $i = 1, 2$, then $cg(Y_1 \# Y_2) = cg(Y_1) + cg(Y_2)$.

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REFERENCES

[1] F. Ding, H. Geiges, and A. Stipsicz, Surgery diagrams for contact 3-manifolds, Turkish J. Math. 28 (2004), 41–74.
[2] Y. Eliashberg, Contact 3-manifolds twenty years since J. Martinet’s work, Ann. Inst. Fourier 42 (1992), 165–192.
[3] J. B. Etnyre, Lectures on open book decompositions and contact structures, Lecture notes from the Clay Mathematics Institute Summer School on Floer Homology, Gauge Theory, and Low Dimensional Topology at the Alfréd Rényi Institute.
[4] J. B. Etnyre and B. Ozbagci, Invariants of contact structures from open books, Trans. Amer. Math. Soc., 360(6):3133–3151, 2008.
[5] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, Proceedings of the International Congress of Mathematicians (Beijing 2002), Vol. II, 405–414.
[6] R. Gompf, Handlebody construction of Stein surfaces, Ann. of Math. (2), 148 (1998) no. 2, 619–693.
[7] B. Ozbagci and A. Stipsicz, Surgery on contact 3-manifolds and Stein surfaces, Bolyai Soc. Math. Stud., Vol. 13, Springer, 2004.
[8] H. Rubinstein, *Comparing open book and Heegaard decompositions of 3-manifolds*, Turkish J. Math. 27 2003, 189–196.
[9] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices 2000, no. 9, 441–454.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 17 GAUSS WAY, BERKELEY, CA, 94720-5070
E-mail address: bozbagci@msri.org