Universal vortex formation in rotating traps with bosons and fermions

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When a system consisting of many interacting particles is set rotating, it may form vortices. This is familiar to us from everyday life: you can observe vortices while stirring your coffee or watching a hurricane. In the world of quantum mechanics, famous examples of vortices are superconducting films and rotating bosonic 4He or fermionic 3He liquids. Vortices are also observed in rotating Bose-Einstein condensates in atomic traps and are predicted to exist for paired fermionic atoms. Here we show that the rotation of trapped particles with a repulsive interaction leads to a similar vortex formation, regardless of whether the particles are bosons or (unpaired) fermions. The exact, quantum mechanical many-particle wave function provides evidence that in fact, the mechanism of this vortex formation is the same for boson and fermion systems.

Let us now consider a number of identical particles with repulsive interparticle interactions confined in a harmonic trap under rotation. These particles could be electrons, positive or negative ions, or neutral atoms in boson or fermion condensates. Though simple to describe, this quantum mechanical many-body problem is extremely complex and in general not solvable exactly. Consequently, in rotating systems the formation of vortices and their mutual interaction is usually described using a mean field approximation. In superconductors this is the Ginzburg-Landau method. For Bose-Einstein condensates, one often applies the Gross-Pitaevskii equation. For Bose-Einstein condensates, one often applies the Gross-Pitaevskii equation. In this way, Butts and Rokhsar found successive transitions between stable patterns of singly-quantised vortices, as the angular momentum was increased. A single vortex appears when the angular momentum \( L \) is equal to the number of particles \( N \), two vortices appear at \( L \sim 1.75N \) and three vortices at \( L \sim 2.1N \) (see Refs. [13,14,15,16]). For quantum dots in strong magnetic fields, the occurrence of vortices was very recently discussed by Saarikoski et al.

Based on the rigorous solution of the many-particle Hamiltonian, we show that striking similarities between the boson and fermion systems exist: the vortex formation is indeed universal for both kinds of particles, and the many-particle configurations generating these vortices are the same. For a small number of particles, the many-body Hamiltonian operator can be diagonalised numerically. We use a single particle basis of Gaussian functions to span the Hilbert space. These Gaussians are eigenstates of the trap for radial quantum number \( n = 0 \) and different single-particle angular momenta. These states dominate for large total angular momenta \( L \). We only consider one spin state, i.e. bosons with zero spin or spin-polarised fermions. Numerical feasibility limits calculations to small particle numbers \( N \). However, the advantages of our approach as compared to mean field methods are that our solutions (i) are exact (up to numerical accuracy), (ii) maintain the circular symmetry and thus have a good angular momentum, and (iii) allow the direct, quantitative comparison between boson and fermion states and thus serve to uncover the origin of the vortices in small systems.

The many-particle energies for rotating clouds of bosons or fermions are compared in Figs. 1 (bosons) and 2 (fermions). The low-lying states are shown as a function of total angular momentum \( L \). Following the tradition in nuclear physics, the line connecting the lowest states at fixed \( L \) is called the “yrast” line. (The word “yrast” originates from Swedish language and means “the most dizzy”. It is marked with a red line in Figs. 1 and 2. For bosons the two spectra appear almost identical, although one of them is calculated with a short-range contact interaction and the other with a long-range Coulomb interaction.)

The comparison between the lowest energy states of the spectra for bosons and Coulomb-interacting fermions, as displayed in Figs. 1 and 2, reveals striking similarities: The yrast line has the same kinks and vortices can be found in both systems appearing at similar angular momenta, as we explain below.

When studying the appearance of vortices in the boson or fermion densities, we should remember that in contrast to mean field methods, for an exact calculation with good angular momentum the particle density has circular symmetry and thus does not display the internal structure directly. To find the vortices we therefore would need to study pair-correlation functions. In the fermion case, however, this can be problematic due to the disturbance of the exchange hole. Alternatively, as done in the insets of Figs. 1 and 2, we break the circular symmetry with a small perturbation of the form \( V_l(r, \varphi) = V_0 \cos(\ell \varphi) \) which has \( \ell \) minima around the center. The perturbation can only couple states which differ in angular momentum by \( \pm \ell \). Since the lowest energy state for each angular momentum is also the most important in the perturbation expansion, we can estimate the effect of the perturbation by the mixture of three yrast states \( \Psi(L) = \Psi_0(L) + \eta[\Psi_0(L-\ell) + \Psi_0(L+\ell)] \), where \( L \) is the angular momentum and \( \eta \) is the mixing parameter, which is of the order 0.1 or smaller. If, for instance, the state in question has a two-vortex structure, then this will appear in \( \Psi(L) \) as two distinct minima for \( \ell = 2 \) already at very small mixing ratio \( \eta \). The insets to Figs. 1 and 2 are obtained in this way. In the boson case, for \( N=8 \) the perturbative densities show a single vortex for \( L = 13 \).
has a non-zero angular momentum $L$ state by a symmetric polynomial. A vortex can be formed by multiplying the boson ground state by a symmetric polynomial. For bosons with short-range repulsive interaction, a single vortex still exists at $L/N = 1$ for a single vortex and $L/N = 1.75$ for two vortices.

This universality in the vortex formation can be understood by looking more in detail at the many-particle states of the rotating system. In the case of non-interacting bosons the many-particle ground state is

\[ \Psi_B = e^{-\sum_k |z_k|^2}, \]

where the coordinates in two dimensions are expressed by complex numbers $z_j = x_j + iy_j$. It turns out that in the case of spin-polarised Fermions the corresponding “condensate” is the so-called maximum density droplet (MDD) \(21\),

\[ \Psi_F = \prod_{j<k} (z_j - z_k) e^{-\sum_i |z_i|^2}, \]

which is a Slater determinant of the consecutive single particle states, filled from $m = 0$ to $m = N - 1$ and thus has a non-zero angular momentum $L_F = N(N - 1)/2$. The state $\Psi_F$ corresponds to the Laughlin wave function for the integer quantum Hall effect \(25\). For bosons with short-range repulsive interaction, a single vortex can be formed by multiplying the boson ground state by a symmetric polynomial \(15,22\),

\[ P_{1V} = \prod_k (z_k - z_0), \]

where $z_0$ is the center of mass. If one multiplies the MDD with the same polynomial \(23\), this gives a good approximation for the exact single vortex state for charged fermions. By noticing that for a system with many particles, the center of mass can be put at the origin, $z_0 = 0$, we can make an Ansatz for the state with $n$ fixed vortices forming a ring around the origin:

\[ \Psi_{nV} = \prod_{j=1}^{N} (z_{j1} - ae^{i\alpha_1}) \times \cdots \times \prod_{j=n}^{N} (z_{jn} - ae^{i\alpha_n}) \Psi_{B,F} \]

\[ = \prod_{j} (z_j - a^n) \Psi_{B,F}, \]

where $\Psi_{B,F}$ is either the boson condensate or the fermion MDD, and the vortex centers are localised on a ring of radius $a$ ($\alpha_k = k \cdot 2\pi \alpha \). This state does not have a good angular momentum, but such a state can be projected out by collecting only terms corresponding to a specific power of the constant $a$,

\[ \Psi_{nV} = a^{n(N-K)} \mathcal{S} \left( \prod_{k} z_k^n \right) \Psi_{B,F}, \]

where $\mathcal{S}$ means symmetrisation. Note that with $n = 1$, $K = N$ and $a = 0$ this state describes a single vortex fixed at the origin. Figure 3 shows schematically the single particle occupation of these “vortex generating states”
for bosons and fermions. In both cases the \( n \) vortices are generated by exciting \( K \) single particles by \( n \) units of angular momentum. The quantum states of the numerical

\[
\text{FERMIONS } L=42 \\
\text{BOSONS } L=14
\]

exact solutions shows that the dominating configurations in cases where we see 1, 2, or 3 vortices (independent of the number of particles) are indeed those shown in Fig. 3, for both bosons and fermions.

In the exact diagonalisation, by multiplying the exact boson wave function with the wave function \( \Psi \), we can determine the overlap between the fermion and boson states. It turns out to be even larger than the weights of the most important configurations. For example, the overlap between the two-vortex states shown in Figs. 1 and 2 is 57\% while the weight of the most important configuration (as shown in Fig. 3) in the boson case is only 15\% and in the fermion case 47\%.

The vortices are born by the rotational motion and consequently carry angular momentum. In the single particle picture the angular momentum is associated with the phase of the complex wave function: Going around the angular momentum axis the phase changes by \( 2\pi \). Similarly, the phase changes by \( 2\pi \) in going around a vortex center. In the many-particle picture the phase of the wave function depends on the coordinates of all the particles. In this case, the phase change around the vortex cores can be visualised by fixing the coordinates of \( N-1 \) particles and plotting the phase as a function of the last coordinate. This is done in Fig. 4 for the vortex generating configuration for \( N=8 \). The state \( \Psi_F \) has maximum amplitude, when the electrons are located so that one electron is in the center and seven electrons form a ring around it. To study the phase, we fix six of the electrons on the ring and one at a slightly off-center position in the middle. The resulting phase is shown in Fig. 4a. One can clearly see that each electron carries a vortex with it, as known from the theory of the integer quantum Hall effect. When the wave function is multiplied with the polynomial generating the vortices, Eq. (4) with \( n=2 \), two additional vortices appear (Fig. 4b). When the fermion state \( \Psi_F \) is replaced with the corresponding boson state \( \Psi_B \) only the two additional vortices are seen (Fig. 4c).

Finally, we will return to the fermion spectrum shown in Fig. 2. The maximum amplitude of the MDD, i.e. the fermion “condensate”, corresponds to the equilibrium particle positions of a classical system with logarithmic repulsive interactions. In small systems a rigid rotation of this localised state gives some of the high angular momentum states. For example, in the case of eight particles there are two classical configurations: a single ring of eight particles, which we label by \( (0,7) \), and a ring of seven particles with one particle at the center, \( (1,7) \). The former allows rigid rotation at angular momenta \( 28+8=36 \), \( 28+16=44 \), etc., while the latter at \( 28+7=35 \), \( 28+14=42 \), etc. (as marked by arrows in Fig. 2). In the fermion systems, in some cases both localisation and vortex structure coincide. This is for example the case at \( L=42 \) for eight fermions, which in the pair correlation function shows two vortices as well as very weakly localised particles arranged in a \( (1,7) \)-configuration.

To summarise, we have shown with exact solutions of the many-particle systems that the vortex formation in rotating traps of bosons and fermions have universal features:

(i) They appear at certain angular momenta determined
by the number of particles and number of vortices, (ii) the many-particle excitations generating the vortices are the same for bosons and fermions, and (iii) the vortex formation does not depend on the shape of the repulsive interaction between the particles.

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