Group-like Structures in Quantum Lie Algebras 
and the Process of Quantization

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Abstract.

For a certain class of Lie bialgebras \((A, A^*)\) the corresponding quantum universal enveloping algebras \(U_q(A)\) are proved to be equivalent to quantum groups \(\text{Fun}_q(F^*)\), \(F^*\) being the factor group for the dual group \(G^*\). This property can be used to simplify the process of quantization. The described class is wide enough to contain all the standard quantizations of infinite series. The properties of \(F^*\) are explicitly demonstrated for the standard deformations \(U_q(sl(n))\). It is shown that for different \(A^*\) (remaining in the described class of Lie bialgebras) the same algorithm leads to nonstandard quantizations.

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Introduction
The problem of equivalence between the categories of quantum algebras $U_q(A)$ and quantum groups $\text{Fun}_q(G)$ was first mentioned by V.G.Drinfeld [1]. The quantum duality principle formulated by M.A.Semenov-Tian-Shansky [2] reveals the background of this equivalence. Consider the quantum deformation of Lie bialgebra $(A, A^*)$, i.e. the pair of Hopf algebras $(U_q(A), U_q(A^*))$. For the universal covering group $G^*$ (with Lie algebra $A^*$) there exists the quantum group $\text{Fun}_q(G^*)$ dual to $U_q(A^*)$. Certain reservations must be taken into consideration in noncompact case [3]. So the duality in the bialgebra we have started from leads to the conclusion that $U_q(A)$ can be treated as $\text{Fun}_q(G^*)$.

\[ U_q(A) \quad \leftrightarrow \quad U_q(A^*) \]

\[ \text{Fun}_q(G^*) \quad \text{Fun}_q(G) \]

In this report the scheme is proposed for the explicit realization of $U_q(A)$ not only as $\text{Fun}_q(G^*)$ but (when certain conditions are fulfilled) also as $\text{Fun}_q(G^*/N^*)$. The group $G^*/N^* \equiv F^*$ is proved to be the deformation of the additive group of the vector space spanned by the generators of $A$. Different $G^*$ and $N^*$ can be constructed for fixed $A$. When the space $Z^2(A, A \otimes A)$ is difficult to describe explicitly general considerations on the structure of $A^*$ are become important. In [4] it was shown that simply connected solvable groups are the natural candidates for $F^*$. In particular these groups are flat and often tolerate the noncommutative coordinates. The possibility to use $F^*$ instead of $G^*$ simplifies considerably the process of quantization.

In section 1 the principle scheme is exposed. The properties of the appropriate groups $F^*$ for a given $A$ and the way to use them in the process of quantization $U(A) \Rightarrow U_q(A)$ are described. The mechanism proposed in [3] when applied to quantum algebras $U_q(A)$, provides the groups isomorphic to $F^*$ (section 2). Both procedures are illustrated for the standard quantum deformations $U_q(sl(n))$ and different limits in the family of quantum algebras are considered (section 3). In section 4 the applications of the proposed scheme are discussed and supplied by examples.

1.
Consider the Lie bialgebra $(A, A^*)$. Let $U(A)$ be the universal covering algebra of $A$ with generators $\{y_j : j = 1, \ldots, m\}$ and the relations $\mu(\{y_j\}) = 0$. Define the subspace $V(\{y_j^j\})$ of the vector space $V_{A^*}$ as the linear span of $\{y_j^j\}$. Write down the direct sum decomposition
\( V_{A^*} \approx V(\{y^j\}) \oplus V' \). When the restriction \( A^*_{V'} \equiv J^* \) is an ideal of \( A^* \), the factor algebra \( F^* \equiv A^*/J^* \) can be defined on \( V(\{y^j\}) \). Let \( F^* \) be the universal covering group with Lie algebra \( F^* \). We shall be interested in the case where \( F^* \) is flat, i.e. its space is Euclidean,

\[
\dim F^* = \dim V(\{y^j\}) = m.
\]

In particular the solvable simply connected groups \( F^* \) will be considered with the canonical decomposition

\[
F^* \approx H \triangleright (H^1 \triangleright (H^2 \triangleright \cdots) \cdots)
\]

(1)

In this case we shall suppose that the coordinates \( \{v_j\} \) are correlated with the decomposition (1). For the product \( (v' \ast v'') \) of two elements \( v', v'' \in F^* \) the coordinates can always be written in the power series form

\[
(v' \ast v'')_j = \sum_l \Gamma'_l(\{v'_l\}) \cdot \Gamma''_l(\{v''_l\}) = v'_j + v''_j + \ldots,
\]

(2)

Here \( \Gamma' \) and \( \Gamma'' \) are the monomes depending on the coordinates of the first and the second factors respectively. For these monomes two characteristics will be important: the power \( p_\Gamma \) and the subalphabet \( \{v_{j_1}, \ldots, v_{j_s}\}_\Gamma \) – the subset of coordinates \( \{v_j; j = 1, \ldots, m\} \) encountered in \( \Gamma \).

We shall also consider algebra \( L \) of dimension \( m \) as the factor algebra of the free associative unital algebra modulo the relations leading to the commutativity of the first \( r \) basic elements. The corresponding commutative subalgebra will be denoted by \( L^c \).

Suppose that the following requirements are fulfilled.

\begin{itemize}
  \item[u1.] \( A^*_{V'} \equiv J^* \) is an ideal in \( A^* \).
  \item[u2.] The group \( F^* \) is flat.
  \item[u3.] \( F^* \) preserves the group structure when the coordinates \( v_j \) belong to the algebra \( L \) (so that the first \( r \) coordinates remain commutative). The group \( F^* \) with such noncommutative coordinates will be denoted by \( Q \).
  \item[u4.] For any \( \Gamma \) with \( p_\Gamma > 1 \) all the elements of the subalphabet \( \{v_{j_1}, \ldots, v_{j_s}\}_\Gamma \) commute, i.e. \( v_{j_{l=1,\ldots,s}} \in L^c \).
  \item[u5.] For any indice \( j_l \) appearing in the subalphabet \( \{v_{j_1}, \ldots, v_{j_s}\}_\Gamma \) for \( p_\Gamma > 1 \), all the monomes \( \Gamma_{(j_l)} \) commute.
\end{itemize}

Let us construct the algebra of functions on \( Q \). For \( \mathcal{A} \equiv \text{Fun}Q \) we choose the following generators:

\[
Y_j = \begin{cases} 
H_k & | \ H_k(v) = v_k \in L^c \quad k = 1, \ldots, r; \\
X_l & | \ X_l(v) = v_l \in L \setminus L^c \quad l = r + 1, \ldots, m, 
\end{cases}
\]

(3)
and the trivial unit function $1_{\mathcal{A}}$. On the generators the Hoph structure is defined by the relations

$$\begin{align*}
(Y_i \cdot_{\mathcal{A}} Y_j)(v) &= Y_i(v) \cdot_{\mathcal{L}} Y_j(v), \\
\Delta Y_j(v) &= Y_j(v)^{*_{\otimes n}} \equiv \sum \Gamma'_{(j)} t(\{Y_i\}) \otimes \Gamma''_{(j)} t(\{Y_i\}), \\
SY_j(v) &= Y_j(-1), \\
\varepsilon Y_j &= 0,
\end{align*}$$

(4), (5), (6), (7)

Operations (4), (5), and (7) are extended to $\mathcal{A}$ by homomorphisms and the antipode – by antihomomorphisms.

**Proposition 1.** $\mathcal{A}$ is a Hopf algebra.

**Proof.** The associativity is trivially induced by $\mathcal{L}$. The properties of the counit (7) with respect to the coproduct (5) are equivalent to the statement that the zero vector is the unit in $Q$ (see (2)). Checking the coassociativity on the generators $Y_j$ one comes to the expressions that in general differ (due to noncommutativity of the coordinates in $Q$) from the relations describing the associativity of the multiplication (2). Only thanks to condition $u5$ the coassociativity is restored. Similarly, the property $(\cdot)(id \otimes S)\Delta = (\cdot)(S \otimes id)\Delta = \eta \circ \varepsilon$ is valid only when the condition $u4$ is fulfilled.

The relations $\mu(\{Y_j\}) = 0$ may disagree with the newly defined Hopf structure (4-7). Write down the deformed relations $\mu_{\xi}(\{Y_j\}) = 0$. The consistency conditions lead to the deformation equations. Every solution of these equations defines the Hopf algebra. Among them quantum deformations of $U(A)$ are easily identified. To do this remember that the topological space of $\mathcal{F}^*$ is flat. Thus $Q$ can always be included in such a family of groups $Q_\lambda$ with the abelian vector additive group $Q_0 = \lim_{\lambda \to 0} Q_\lambda$. All the constructions above are valid for the whole ensemble $\{Q_\lambda\}$ and one can find the family $\{\mathcal{A}_\lambda\}$ and the sets of the deformed relations $\{\mu_{\xi}(\lambda) = 0\}$.

Take only those $\mu_{\xi}(\lambda)$ that a) has the common limit $\lim_{\lambda \to 0} \mu_{\xi}(\lambda) = \mu$ b) do not violate the Poincare-Birkhoff-Witt theorem. Every algebra $\mathcal{A}$ modulo the relations $\mu_{\xi}(\lambda)(\{Y\}) = 0$ is the quantum universal enveloping algebra $U_\lambda(A)$.

Algebras $U_\lambda(A)$ inherit the following properties:

$h1$. $H_k H_l = H_l H_k$.

$h2$. Every monome $\Gamma$ in $\Delta Y_j$ (3) of power $p > 1$ has the subalphabet of commuting basic elements.

$h3$. The subalphabet of every $\Gamma$ with $p > 1$ contains $Y_j$ with only such $j$'s that all $\Gamma'_{(j)}$ and $\Gamma''_{(j)}$ commute.

$h4$. The system of equations

$$(\cdot)(id \otimes S)\Delta(Y_i) = (\cdot)(S \otimes id)\Delta(Y_i) = \eta \circ \varepsilon(Y_i)$$

3
can be solved to find a unique expression for $S(Y_i)$ using only the multiplication property $h1$.

The last feature is due to the fact that $Q$ is a group and this group is flat.

It is evident that in general the obtained class of Hopf algebras $\{U_\lambda(A)\}$ may not cover the set $\{Fun_q(G^*)\}$ of all possible quantizations of the group $G^*$. Nevertheless this class is wide enough. In section 3 it will be demonstrated that it contains all the standard deformations of classical Lie algebras. Moreover being defined only by the factor group $G^*/N^*$ it may contain algebras that cannot be presented in the form of $Fun_q(G^*)$.

Note that in the exposed algorithm we do not appeal neither to semisimplicity of the initial algebra $A$ nor to the properties of its main field. The method can be applied to an arbitrary Lie bialgebra $(A,A^*)$ defined over an arbitrary field. The latter is especially important for physical applications.

2.

Starting with the Hopf algebra of the type $U_\lambda(A)$ one can reconstruct the groups $Q$ and $F^*$ and establish the correspondence between classes of $U_\lambda(A)$ and the sets of isomorphic flat vector groups $F^*(A^*)$.

Let $U_\lambda(A)$ be the Hopf algebra with generators $\{1, H_k, X_l\}$, relations $\mu_\lambda(Y) = 0$ and the properties $h1 - h5$. Then there exists the Hopf algebra $A_\lambda$ with the same costructure and antipode as in $U_\lambda(A)$, whose multiplication is almost free [4]. Consider the set of vector space morphisms $\text{Mor}(V(\{Y_j\}), L)$. The subset

$$\text{Mor}^h \equiv \{\phi \in \text{Mor}(V(\{Y_j\}), L) \mid \phi(H_k) \in L^c\}$$

is in turn a vector space. Each element $\phi \in \text{Mor}^h$ is defined by the coordinates $\{\phi(Y_j) \equiv \phi_j\}$. Consider the multiplication

$$(\phi' \ast \phi'')_j = ((\cdot_L)(\phi'_t \Delta \phi''_t)\Delta)_j$$

(9)

where "$\cdot$" denotes the homomorphic extension. It can be verified that the space $\text{Mor}^h$ with the multiplication (9) is a vector Lie group [4], where the inverse element and the unit have the form

$$\phi^{-1} = \phi_t \circ S,$$
$$\phi(0) = \eta_L \circ \epsilon_A.$$  

(10)

(11)

Suppose now that the Hopf algebra $U_\lambda(A)$ has been constructed as in Section 1 starting with the group $F^*$. Insert the coproduct (5) into the multiplication law (9). The initial form
of the group product will be obtained, \( \phi_j \) playing the role of coordinates of \( F^* \). The conclusion is as follows.

**Proposition 2.** The flat groups \( F^* \equiv G^*(A^*)/N^*(A^*) \) with the properties \( u1 - u5 \) are in one-to-one correspondence with the classes of quantum deformations \( U_\lambda(A) \) with the fixed form of coproduct \( (2) \), antipode \( (3) \) and counit \( (7) \).

3.

Let us illustrate the above constructions for the well known case of standard quantum deformations \( U_q(sl(n)) \). Write the defining relations in the following form \( (3) \).

\[
\begin{align*}
[H_i, H_j] &= 0; \\
[H_i, E_{\pm(j,j+1)}^\pm] &= \pm \alpha_{ij}(H_i) E_{\pm(j,j+1)}; \\
[E_\pm(i,i+1), E_{\mp(j,j+1)}^\mp]_q &= \delta_{ij} \frac{e^{-hH_i} - 1}{e^{-hH_i} - 1}; \\
\Delta H_i &= H_i \otimes 1 + 1 \otimes H_i; \\
\Delta E_{\pm(i,i+1)} &= E_{\pm(i,i+1)} \otimes 1 + e^{-hH_i/2} \otimes E_{\pm(i,i+1)}; \\
S(H_i) &= -H_i; \\
S(E_{\pm(i,i+1)}) &= -e^{hH_i/2} E_{\pm(i,i+1)} \\
\varepsilon(E_{\pm(i,i+1)}) &= \varepsilon(H_i) = 0; \\
(ad(E_{\pm(i,i+1)}))^{1-\alpha_{ij}} E_{\pm(j,j+1)}^\pm &= 0; \text{ for } |i - j| = 1; \\
i, j &= 1, \ldots, n - 1;
\end{align*}
\]

Here \( \alpha \) is the Cartan matrix for \( sl(n) \). The adjoint operator has the usual form

\[
ad = (L \otimes R) \circ (\text{id} \otimes S) \circ \Delta.
\]

The Cartan-Weil basis is specified recursively,

\[
E_{\pm(i,j)} \equiv \text{ad}(E_{\pm(i,i+1)}^\pm) E_{\pm(i+1,j)^\pm}, \quad i < j.
\]

Let us redefine it,

\[
X_{\pm(i,j)} \equiv E_{\pm(i,j)} e^{-hH_{j,n-1}/2};
\]

with

\[
H_{j,n-1} \equiv H_j + \cdots + H_{n-1},
\]

so that the coproduct for all the basic elements \( X_{\pm(i,j)} \) acquire the compact form:

\[
\Delta X_{\pm(i,j)} = X_{\pm(i,j)} \otimes e^{-hH_{j,n-1}/2} + e^{-hH_{i,n-1}/2} \otimes X_{\pm(i,j)} + (1 - e^{\pm h}) \sum_{k=i+1}^{j-1} (X_{\pm(i,k)} \otimes X_{\pm(k,j)}).
\]
It is significant that the conditions $h_1 - h_5$ are fulfilled here not only for the generic elements of the Hopf algebra but for the whole Cartan-Weil basis of it. This means that the group

$$(\text{Mor}_{sl(n)}^h, \ast)_V(sl(n)) = Q_{sl(n)}. \quad (20)$$

exists on the space of Mor($V_{sl(n)}$, $L$):

$$\text{Mor}_{sl(n)}^h \equiv \{ \phi \in \text{Mor}(V_{sl(n)}, L) \mid \phi(H_i) \in L^c \}. $$

The elements $\phi \in Q_{sl(n)}$ are defined by $n^2 - 1$ coordinates $\phi = (b_i, l_{\pm(i,j)})$. The coordinates $b_i$ commute. The group multiplication and the inverse element take the following form

$$\phi' \ast \phi'' = (b'_i + b''_i, l_{\pm(i,j)}') e^{-(h/2)l_{j,n-1}} + e^{-(h/2)l'_{j,n-1}} l''_{\pm(i,j)} + (1 - e^{\pm h}) \sum_{k=1}^{n-1} l_{\pm(i,k)} l_{\pm(k,j)}), \quad (21)$$

$$\phi^{-1} = (-b_i, \sum_{i > k_1 > \ldots > k_m > j} (-1)^{m+1}(1 - e^{\pm h})^m \cdot \exp[(h/2)b_{i,n-1}] l_{\pm(i,k_1)} \exp[(h/2)b_{k_1,n-1}] l_{\pm(k_1,k_2)} \cdot \ldots \cdot \exp[(h/2)b_{k_m,n-1}] l_{\pm(k_m,j)}), \quad (22)$$

$$l \in L, \quad b \in L^c, \quad b_{i;j} \equiv b_i + \ldots + b_{j-1}.$$ 

Putting the commuting coordinates instead of $\phi_i$ one obtains the solvable flat Lie group, that is just the dual group $(SL(n))^* \equiv P^*$.

$$P^* \supset (P^*)_1 \supset (P^*)_2 \supset \ldots \supset (P^*)_s \supset (P^*)^{s+1} = e,$$

$$\sup(1 + 2^{s-1}) \leq n.$$ 

Its Lie algebra $(sl(n))^*$ has the following commutation relations

$$[H^i, H^j] = 0;$$

$$[X^i_{\pm}, X^p_{\pm,q}] = 0;$$

$$[H^k, X^i_{\pm}] = \begin{cases} 0, & k < i; \quad k \geq j; \\ -(h/2)X^i_{\pm,j}, & i \leq k \leq j - 1; \end{cases} \quad (23)$$

$$[X^i_{\pm,j}, X^p_{\pm,q}] = (1 - e^{\pm h})(X^i_{\pm,q} \delta_{jp} - X^p_{\pm,j} \delta_{iq}).$$

Note that the algebra $(sl(n))^*$ has the ideal $J^*$ generated by $X^p_{\pm,q}$ for $q \geq p + 2$. The factor algebra $(sl(n))^*/J^* \equiv F^*(sl(n))$ is defined on the space spanned by the duals of the Chevalley generators $y_j$ of $sl(n)$:

$$V_{(sl(n))^*} \approx V_{J^*} \oplus V(\{y^j\}).$$
Its structure is very simple

\[
\begin{align*}
[H^i, H^j] &= 0; \\
[H^i, X^{k,k+1}_\pm] &= -(\hbar/2)X^{k,k+1}_\pm \delta_{ik}; \\
[X^{i,i+1}_\pm, X^{j,j+1}_\pm] &= 0.
\end{align*}
\] (24)

The corresponding group \(\mathcal{F}^*\) is flat and coincides with the factor group \(Q_{sl(n)}/(Q_{sl(n)})^2\), where the coordinates are considered commutative. So \(\mathcal{F}^*\) obviously tolerates the noncommutative coordinates. \(Q_{\mathcal{F}} \equiv Q_{sl(n)}/(Q_{sl(n)})^2\) plays here the role of the group \(Q\) introduced in Section 1. For the elements \(v', v'' \in Q_{\mathcal{F}}^{(h)}\) the product can be easily written in terms of coordinates,

\[
v' \ast v'' = (b'_i + b''_i, l'_\pm(i,i+1)e^{-(\hbar/2)h'_{i+1,n-1}} + e^{-(\hbar/2)h''_{i,n-1}} l''_\pm(i,i+1)).
\]

\(\mathcal{F}^*\) and \(Q_{\mathcal{F}}\) are in fact one-parameter families of groups and the limit \(\lim_{\hbar \to 0} \mathcal{F}^* \equiv \mathcal{F}^*_{(0)}\) is obviously an abelian vector additive group.

Now it is clear that the standard quantization \(U_q(sl(n))\) could have been obtained by the method described in Section 1. The starting point is the Lie bialgebra \((sl(n), (sl(n))^*)\) defined by relations (12), (16), (18) and (23). After the estimation of the groups \(\mathcal{F}^*\) and \(Q_{\mathcal{F}}\) the Hopf algebra \(A\) was to be defined with costructure (13 - 15). The relations (12) and (16) must have been deformed and the deformation parameter correlated with \(\hbar\). The deformation function

\[\Phi(E_{+(i,i+1)} A E_{+(j,j+1)}) = \delta_{ij} \left(\frac{e^{-\hbar H'_i}}{e^{-\hbar H'_i}} - 1\right) - H_i\]

(zero elsewhere)

is just the solution leading to the standard quantization \(U_q(sl(n))\).

We have demonstrated explicitely that \(U_q(sl(n))\) is a quantum group

\[U_q(sl(n)) \approx \text{Fun}_q(\mathcal{F}^*_{sl(n)}).
\]

It is useful to check the possible limiting procedures for such Hopf algebras.

\[
\begin{array}{cccc}
\text{Fun}(Q(A^*)) & \equiv & A & \text{factorization} & U_h(A) & \text{alg. contr.} & \text{Fun}(\mathcal{F}^*(A^*)) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fun}(Q_{ab}) & \equiv & U(A) & \text{alg. contr.} & U(A) = \text{Fun}(\mathcal{F}^*_{ab})
\end{array}
\]
The algebraic contraction
\[ \lim_{h \to 0} hY_j = \tilde{Y}_j \] (25)
leads to the abelian Hopf algebra while the structure of the coproduct \[19\] remains stable. This illustrates the quantum duality in our case:
\[ U_q(sl(n)) \approx \text{Fun}_q((Sl(n))^*). \]

The group contractions (the vertical arrows) forsee the redefinition of the group structure on the vector space, quantum algebras are treated as quantum groups. The doubling of limits manifests the quantum duality principle. It must be noticed that both contractions can induce the changes in the complementary parts of Hopf structure. For example the classical limit can be considered as induced by a group contraction, but the result is the contraction of multiplication law as well as the comultiplication one.

### 4.

In the previous Section the standard Lie bialgebras for \( sl(n) \) where in the game. Here we consider the 3-dimentional Lie algebra \( B^* \) defined by the relations
\[
[H^*, X^-] = -\lambda H^*; \quad [X^-, X^+] = \lambda X^+; \\
[H^*, X^-] = 0.
\]

It is easy to check that \((sl(2), B^*)\) is also a Lie bialgebra. Generators \( \{H^*, X^\pm\} \) are dual to the ordinary Chevalley basic elements \((12)\) of \( sl(2) \). In this case \( F^* = B^* \). For the corresponding simply connected group \( F^* \) the multiplication can be written in terms of the flat coordinates \( \{b, l_\pm\} \):
\[
v' \ast v'' = (b'e^{-\lambda l''} + e^{\lambda l'} b', l'_+ e^{-\lambda l''} + e^{\lambda l'} l'_+, l'_- + l''_+)
\]
The group structure of \( F^* \) survives when the coordinates \( \{b, l_\pm\} \) are placed in the free associative algebra \( L \). The group \( Q \) is thus obtained. The algebra \( \text{Fun}(Q) \equiv A \) is a free assiciative algebra supplied by the operations
\[
\begin{align*}
\Delta H &= H \otimes e^{-\lambda X_-} + e^{\lambda X_-} \otimes H; \\
\Delta X_+ &= X_+ \otimes e^{-\lambda X_-} + e^{\lambda X_-} \otimes X_+; \\
\Delta X_- &= X_- \otimes 1 + 1 \otimes X_-; \\
S(H) &= -e^{-\lambda X_-} H e^{\lambda X_-}; \\
S(X_+) &= -e^{-\lambda X_-} X_+ e^{\lambda X_-}; \\
S(X_-) &= -X_-; \\
\varepsilon(H) &= \varepsilon(X_\pm) = 0.
\end{align*}
\] (26)
The deformed commutators
\[
[H, X_{\pm}] = \pm 2X_{\pm} + \Phi_{\pm};
\]
\[
[X_{+}, X_{-}] = H + \Phi;
\]
correlated with operations (26) lead to the deformation equations. Their solutions give the following multiplication law:
\[
[H, X_{+}] = \frac{\lambda}{\text{sh} \lambda}(2(\text{ch} \lambda X_{-})X_{+} - \lambda(\text{sh} \lambda X_{-}));
\]
\[
[H, X_{-}] = -2\frac{\text{sh} \lambda X_{-}}{\text{sh} \lambda};
\]
\[
[X_{+}, X_{-}] = H;
\]
Together with (26) these relations define the quantum algebra \( U_q(sl(2)) \). The obtained quantization appears to be equivalent to the so called nonstandard quantization of \( sl(2) \) \[6\]. This can be proved using simple but cumbersome transformations.

**Concluding Remarks**

The quantum duality principle is a powerful tool in studying quantum algebras. To simplify the applications of the exposed method one can use the ”universal \( \mathcal{F}^* \)-group” anzatz as it was done in \[7\]. There it was supposed that the trivial solvable groups (the semidirect products of two abelian ones) can play the role of \( \mathcal{F}^* \) for a large class of triangular Lie algebras.

The right-hand side of the duality diagramm (see page 1) is also of great importance. In particular it was used in studying the exponential mappings for quantum groups \[8\], \[9\].

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