STRING\textsuperscript{c} STRUCTURES AND MODULAR INVARIANTS

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**Abstract.** In this paper, we study some algebraic topology aspects of String\textsuperscript{c} structures, more precisely, from the aspect of Whitehead tower and the aspect of the loop group of Spin\textsuperscript{c}(n). We also extend the generalized Witten genus constructed for the first time in [7] to String\textsuperscript{c} structures of various levels and apply them to study actions of compact non-abelian simply connected Lie groups.

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**Introduction**

Let $V$ be a real rank $n$ oriented vector bundle over a manifold $M$. Let $F_{SO(n)}$ be the oriented orthonormal frame bundle of $V$ over $M$. $V$ is called Spin if $F_{SO(n)}$ has an equivariant lift with respect to the double covering $\rho : \text{Spin}(n) \rightarrow \text{SO}(n)$. More precisely, a spin structure is a pair $(P, f_P)$ with $\pi_P : P \rightarrow M$ being a principal Spin$(n)$-bundle over $M$ and $f_P : P \rightarrow F_{SO(n)}$ being an equivariant 2-fold covering map such that $P$ is called the bundle of Spin frames of $V$.

A String structure is a higher version of Spin structure, which is related to quantum anomaly in physics [21]. One mathematical way to look at String structure is from the perspective of Whitehead tower. Let $V$ a vector bundle with the spin structure $(P, f_P)$. Let
$g: M \to B\text{Spin}(n)$ be the classifying map of $P$. $V$ is called String, if there is a lift,

$$
\begin{array}{ccc}
B\text{String}(n) & \xrightarrow{\tau} & B\text{Spin}(n) \\
M & \xrightarrow{g} & B\text{Spin}(n).
\end{array}
$$

The obstruction to the lift is $\frac{1}{2} p_1(V)$, and if it vanishes then the distinct String structures lifting the prescribed spin structure on $M$ are one to one correspondence to the elements in $H^3(M; \mathbb{Z})$. Another way to look at String structure is from the perspective of free loop space, namely by looking at lift of the structure group of the looped frame bundle from the loop group to its universal central extension [37]. Under this point of view, the obstruction to the existence of String structure is the transgression of $\frac{1}{2} p_1(V)$, and if it vanishes then the distinct String structures lifting the prescribed Spin structure on $M$ are one to one correspondence to the elements of $H^3(LM; \mathbb{Z})$. These two approaches to view String structures are equivalent when $M$ is 2-connected.

More geometrically, Stolz and Teichner give the profound link of the String structure on $M$ to the fusive Spin structure on $LM$ [44]. This has been developed by Waldorf and Kottke-Melrose [22, 46, 47]. In [4], Bunke studies the Pfaffian line bundle of a certain family of real Dirac operators and shows that String structures give rise to trivialisations of that Pfaffian line bundle. See also the study of String structures from the differential and the twisted point of view [41, 42].

Let $M$ be a $4m$ dimensional closed oriented smooth manifold. Let $\{ \pm 2\pi \sqrt{-1} z_j, 1 \leq j \leq 2m \}$ denote the formal Chern roots of $T\tilde{C}M$, the complexification of the tangent vector bundle $TM$ of $M$. Then the famous Witten genus of $M$ can be written as (cf. [27])

$$
W(M) = \left\{ \prod_{j=1}^{2m} z_j^{\tau'(0, \tau)} \right\} [M] \in \mathbb{Q}[[q]],
$$

with $\tau \in \mathbb{H}$, the upper half-plane, and $q = e^{\sqrt{-1} \tau}$. The Witten genus was first introduced in [49] and can be viewed as the loop space analogue of the $A$-genus. It can be expressed as a $q$-deformed $\tilde{A}$-genus as

$$
W(M) = \left\langle \tilde{A}(T\tilde{C}M), \text{ch} (\Theta(T\tilde{C}M)), [M] \right\rangle,
$$

where

$$
\Theta(T\tilde{C}M) = \sum_{n=1}^{\infty} S_{2n}(T\tilde{C}M), \quad \text{with} \quad T\tilde{C}M = T\tilde{C}M - \mathbb{C}^{4m},
$$

is the Witten bundle defined in [49]. When the manifold $M$ is Spin, according to the Atiyah-Singer index theorem [2], the Witten genus can be expressed analytically as index of twisted Dirac operators, $W(M) = \text{ind}(D \otimes \Theta(T\tilde{C}M)) \in \mathbb{Z}[[q]]$, where $D$ is the Atiyah-Singer-Dirac operator on $M$ (cf. [17]). Moreover, if $M$ is String, i.e. $\frac{1}{2} p_1(TM) = 0$, or even weaker, if $M$ is spin and the first rational Pontryagin class of $M$ vanishes, then $W(M)$ is a modular form of weight $2k$ over $SL(2, \mathbb{Z})$ with integral Fourier expansion ([50]). The homotopy theoretical refinements of the Witten genus on String manifolds leads to the theory of $tmf$ (topological modular form) developed by Hopkins and Miller [19]. The String condition is the orientability condition for this generalized cohomology theory.

As one of the important applications, the Witten genus can be used as obstruction to continuous symmetry on manifolds. In [26], Liu discovered a profound vanishing theorem for the Witten genus under the condition that $p_1(M)_{\mathbb{Z}} = n \cdot \pi^* u^2$, where $p_1(M)_{\mathbb{Z}}$ is the
the obstruction to the lift is 

Spin
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groups at one time. For instance, Nikolaus-Sachse-Wockel in larger Spin groups, we can define a Z

bundle involved to understand the choice of this obstruction. Indeed, if this obstruction line bundle plays a role in several senses. First, There is a geometric way with the line bundle involved to understand the choice of this obstruction. Indeed, if this obstruction

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is the generator and n is an integer, which, as observed by Dessai, when the S1-action is induced from an S3-action, is equivalent to that the manifold is String, i.e. the free loop space is spin [37]. Liu’s vanishing theorem has been generalized in [28–31, 33] for family case, in [32] for foliation case and recently in [13] for proper actions of non compact Lie groups on non compact manifolds.

Spin
 structure is the complex analogue of Spin structure. It is known that there exists a Spin structure on the vector bundle V over M if and only if its second Stiefel-Whitney class w2(V) = 0. However, if w2(V) is only assumed to be trivial after Bockstein, or equivalently, w2(V) is the mod 2 reduction of the first Chern class c1(ξ) of some complex line bundle ξ over M, then the product of the frame bundle and the circle bundle of ξFSO(n) × S(ξ) rather than FSO(n) admits an equivariant double covering P Spin (V) with the structural group Spin(n). By definition, this specifies a Spin-structure on V associated to ξ, and we may often refer to the Spin bundle V as the pair (V, ξ). An excellent introduction to the structural and index theoretical aspects of Spin

structures can be found in Appendix D of the famous book of Lawson-Michelsohn [24].

In this paper, we study String
 structures, which can be viewed as higher versions of Spin
 structures. We study String
 structures from the perspectives of algebraic topology, including their definitions and explanations, the explicit constructions of String
 groups as well as the obstruction and classification of String
 structures.

As in the real case, String
 structures can be understood from the perspective of Whitehead tower. One of the significant differences for this complex situation is that there are infinitely many essentially distinct String
-structures. Indeed, by embedding Spin
(n) groups in larger Spin groups, we can define a Z-indexed family of topological groups String
ξ(n) as particular group extensions of Spin
(n) groups by suitable group models of K(Z, 2) (Section 2). Indeed, Any group model of String will induce group models of all the String
 groups at one time. For instance, Nikolaus-Sachse-Wockel [39] constructed an infinite-dimensional Lie group model for String. Also in the famous paper [43] of Stolz-Teichner, they showed a model of String in terms of a group extension by a projective unitary group PU(A) as a model of K(Z, 2). In particular, for our String
 groups we have the extension of topological groups

\{1\} \to PU(A) \to String\xi(n) \xrightarrow{\delta} Spin\xi(n) \to \{1\}.

We then have the classifying spaces BString\xi(n) to define String
-structures. We call a Spin
 bundle (V, ξ) or simply V a String
 of level 2k + 1 for some k ∈ Z, if there is a lift

BString\xi(n)

-M \xrightarrow{g'} BS\text{Spin}(n),

where g’ is the classifying map of P Spin (V). Due to our constructions of the String
 groups, the obstruction to the lift is

\frac{p_1(V) - (2k + 1)c(ξ)^2}{2}.

Another point of the significant differences for the complex situation is that the complex line bundle plays a role in several senses. First, There is a geometric way with the line bundle involved to understand the choice of this obstruction. Indeed, if this obstruction
class vanishes then the stable spin bundle $V \oplus \xi \oplus (-2k-1)$ is String. Under this point of view, the String$^c$ and String structures are connected. Furthermore, it can be showed that the distinct String$^c$ structures on $(V, \xi)$ are in one-to-one correspondence with the elements in the image of

$$\rho^*: H^3(M; \mathbb{Z}) \rightarrow H^3(S(\xi); \mathbb{Z}),$$

where $\rho$ is the bundle projection for the sphere bundle $S(\xi)$ of $\xi_\mathbb{R}$. In other words, the complex line bundle of Spin$^c$ affects the structural classification of String$^c$. With mild restrictions, $\rho^*$ is surjective or injective and then the String$^c$-structures are classified by the third cohomology $H^3(S(\xi); \mathbb{Z})$ or $H^3(M; \mathbb{Z})$. These discussions on the structural theory of String$^c$, in a strong sense comparing to the approach in the sequel, are carried out explicitly in Section 2.

The String$^c$-structures can be also understood from the perspective of free loop spaces as discussed in Section 3. Indeed, if $V$ is String$^c$ of level $2k + 1$ then the structural group $LSpin^c(n)$ of the loop principal bundle $LP_{Spin^c}(V)$ over $LM$ can be lifted to $LSpin^c(n)$, the universal central extension of $LSpin^c(n)$ by $U(1)$. As in the case of String structures, the latter description via loop spaces in general is weaker than the one via classifying spaces $BSpin^c(n)$, though in good cases they are equivalent. On the contrary, the distinct String$^c$ structures on $(V, \xi)$ via the loop spaces approach are in one-to-one correspondence with the elements in the image of

$$(L\rho)^*: H^2(LM; \mathbb{Z}) \rightarrow H^2(LS(\xi); \mathbb{Z}).$$

In good situation, $(L\rho)^*$ can be surjective and then the weak String$^c$-structures are classified by the second cohomology $H^2(LS(\xi); \mathbb{Z})$. In particular, we see the role of the complex bundle in the loop spaces approach as well.

Furthermore, despite the comparison of the two notions of String$^c$ structures themselves, the distinct String$^c$-structures in the non-loop world can be transgressed to their weak counterparts in the loop world via the transgression diagram (see Section 4 for details)

$$\xymatrix{ H^3(M) \ar[r]^\rho \ar[d]^\nu & H^3(S(\xi)) \ar[d]^\nu \\
H^2(LM) \ar[r]^{(L\rho)^*} & H^2(LS(\xi)).}$$

Nevertheless, there are possibly strictly weaker String$^c$-structures than the strong ones. For instance, it is quite likely that is the case if $H^3(M; \mathbb{Z})$ is trivial.

Similar to the Witten genus for String manifolds, we also construct generalized Witten genus $W_{2k+1;\vec{a},\vec{b}}^c(M)$ for String$^c$ manifolds of level $2k + 1$ indexed by two integral vectors $\vec{a}, \vec{b}$ satisfying (5.1), (5.2). Such kind of invariants were constructed in [7] for the first time. In this paper, we enrich them for String$^c$ manifolds of various levels. We remark that $W_{2k+1;\vec{a},\vec{b}}^c(M)$ are more flexible due to the freedom of the double vector-valued indices. As application, we obtain Liu’s type vanishing theorem for $W_{2k+1;\vec{a},\vec{b}}^c(M)$ as follows.

In the following, we always assume $G$ is a compact non-abelian simply connected Lie group. If $M$ is level $2k + 1$ String$^c$, then rationally $p_1(M) - (2k + 1)c_1(\xi)^2 = 0$. Suppose $G$ acts smoothly on $M$ and lifts to $\xi$. Since $G$ is simply connected, for $G$-equivariant characteristic classes, we must have

$$p_1(M)_G - (2k + 1)c_1(\xi)^2_G = n \cdot \pi^* q,$$

for details).
where \( n \in \mathbb{Z} \), \( \pi : M \times_G EG \to BG \) is the projection of the Borel fibre bundle, and \( q \in H^4(BG) \) is the canonical generator corresponding to the generator \( u^2 \in H^4(BS^1) \) (see Section 5 for details). We call that the \( G \)-action is positive on the level \( 2k + 1 \) Stringc manifold \( M \) if \( n > 0 \).

**Theorem 1** (Theorem 5.2). Let \( M \) be a compact level \( 2k + 1 \) Stringc manifold with \( 2k + 1 > 0 \). If \( M \) admits a positive effective action of \( G \), then
\[
W_{2k+1,\xi}(M) = 0.
\]

Suppose \( (M, J) \) is a closed stable almost complex manifold. Then \( M \) is Spinc and the determinant line bundle \( \xi \) of the Spinc structure has \( c_1 \) mod 2 congruent to \( w_2(TM) \). If \( G \) acts smoothly on \( M \) and preserves the stable almost complex structure \( J \), then the action of \( G \) can be lifted to \( \xi \). Applying the above vanishing theorem, we obtain

**Theorem 2** (Theorem 5.3). Let \( (M, J) \) be a closed stable almost complex manifold, which is level \( 2k + 1 \) Stringc, i.e. \( p_1(TM) = (2k + 1)c_1^2 \) and suppose \( 2k + 1 > 0 \). If \( M \) admits a positive effective action of \( G \) preserving \( J \), then
\[
W_{2k+1,\xi}(M) = 0.
\]

In the following, we will give examples to illustrate and apply the above theorem.

**Example 3.** On \( \mathbb{CP}^6 \), consider the stable almost complex structure \( J \) such that
\[
T \mathbb{CP}^6 \oplus \mathbb{R}^2 \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1),
\]
where \( \mathcal{O}(1) \) is the canonical line bundle and \( \mathcal{O}(-1) \) is the dual. It is clear that the determinant line bundle is \( \mathcal{O}(1) \) and \( c_1(\mathbb{CP}^6, J) = x \). As \( p_1(\mathbb{CP}^6) - 7c_1(\mathbb{CP}^6, J)^2 = 0 \), we see that it is level \( 7 \) Stringc.

The linear action of \( SU(7) \) on \( \mathbb{CP}^6 \) obviously preserves \( J \). Since \( SU(7) \) is simply connected, we have that for \( SU(7) \)-equivariant characteristic classes,
\[
p_1(\mathbb{CP}^6)_{SU(7)} - 7c_1(\mathcal{O}(1))_{SU(7)}^2 = n \cdot \pi^* q,
\]
where \( q \in H^4(BSU(7)) \) is the generator corresponding to the generator \( u^2 \in H^4(BS^1) \). We claim that \( n = 1 \) and in particular the \( SU(7) \)-linear action is positive.

Actually consider the embedding of the circle group \( S^1 \) into \( SU(7) \) defined by
\[
f : S^1 \to SU(7), \quad \lambda \to \begin{pmatrix} \lambda^{a_0} & & & \\ & \ddots & & \\ & & \lambda^{a_6} & \end{pmatrix},
\]
such that \( a_0, a_1, \cdots, a_6 \in \mathbb{Z} \) are distinct and \( \sum_{i=0}^6 a_i = 0 \). Then \( S^1 \) acts on \( \mathbb{CP}^6 \) through the linear action of \( SU(7) \) in the following manner
\[
\lambda [z_0, z_1, \cdots, z_{2l-1}] = [\lambda^{a_0} z_0, \lambda^{a_1} z_1, \cdots, \lambda^{a_6} z_6].
\]
Since \( a_i \)'s are distinct from each other, we see that this action has 7 fixed points
\[
[1, 0, \cdots, 0], [0, 1, 0, \cdots, 0], \cdots, [0, 0, \cdots, 1].
\]
The tangent space of the first fixed point has weights \( a_1 - a_0, a_2 - a_0, \cdots, a_6 - a_0 \) and \( \mathcal{O}_1 \) restricted at the first fixed point has weight \( a_0 \). Then when restricted at the first fixed point, we have
\[
p_1(\mathbb{CP}^6)_{S^1} - 7c_1(\mathcal{O}(1))_{S^1}^2 = [(a_1 - a_0)^2 + (a_2 - a_0)^2 + \cdots + (a_6 - a_0)^2 - 7a_0^2]u^2
\]
\[
= \left( \sum_{i=0}^6 a_i^2 \right) \pi^* u^2.
\]
Indeed, similar computations imply that when restricted to any fixed point we always have

\[(3) \quad p_1(\mathbb{C}P^6)_{S^1} - 7c_1(\mathcal{O}(1))_{S^1}^2 = \left( \sum_{i=0}^{6} a_i^2 \right) \pi^* u^2.\]

However it is also clear that for \(Bf : BS^1 \to BSU(7)\), we have \((Bf)^*(q) = (\sum_{i=0}^{6} a_i^2) u^2\) and then by the naturality of equivariant characteristic classes and (2)

\[(4) \quad p_1(\mathbb{C}P^6)_{S^1} - 7c_1(\mathcal{O}(1))_{S^1}^2 = \left( \sum_{i=0}^{6} a_i^2 \right) \cdot n \cdot \pi^* u^2.\]

Hence from (3) and (4), we see that \(n = 1\). As \(n > 0\), Theorem 2 asserts that \(W^{c_1}_{7,6}\) \((\mathbb{C}P^6)\) must vanish. Indeed, this vanishing can be shown through directly computing by using the residue theorem as in [7]. We left the details to interested readers.

In [18] W.-C. Hsiang and W.-Y. Hsiang used the quaternionic projective spaces of Eells-Kuiper [11, 12] to construct a family of homotopy complex projective spaces \(H\mathbb{C}P^6(p)\) indexed by an integer \(p\) such that \((13p + 1)(6p + 1)p \equiv 0 \mod 31\). They can be distinguished by their first Pontryagin classes. When \(p = 0\), it is just the standard \(\mathbb{C}P^6\). In Lemma 5.4, we show that each \(H\mathbb{C}P^6(p)\) admits a stable almost complex structure \(J\) with the first Chern class \(c_1 = (2l + 1)x\) for any \(l \in \mathbb{Z}\). Here \(x \in H^2(\mathbb{C}P^6(p); \mathbb{Z})\) is a generator of the second cohomology. See Section 5 for details. Applying the above theorem, we can see that, unlike the standard \(\mathbb{C}P^6\), the following holds

**Corollary 4** (Corollary 5.5). Let \(J\) be the stable almost complex structure on \(H\mathbb{C}P^6(p)\) with \(p > 0\) and \(c_1(J) = \pm x\). Then \(H\mathbb{C}P^6(p)\) does not admit a positive effective action of \(G\) preserving \(J\).

**Remark 5.** The Petrie conjecture [40] concerns non-trivial \(S^1\) actions on homotopy projective spaces. It claims that if \(S^1\) acts smoothly and non-trivially on the homotopy projective space \(X^{2n}\), then the total Pontryagin class \(p(X^{2n})\) of \(X\) must agree with that of \(\mathbb{C}P^6\). The conjecture was proved particularly for \(X^{2n}\) with \(n \leq 4\). Furthermore, Hatorri [15] proved the conjecture when \(X^{2n}\) admits an \(S^1\) invariant stable almost complex structure with \(c_1 = (n + 1)x\). He also showed that when \(c_1 = k_0x\) with \(|k_0| > n + 1\), \(X^{2n}\) admits no \(S^1\) action preserving \(J\). Hence the first non-spin case which is not covered by the known results related to Petrie conjecture is the one when \(n = 6\) and \(c_1 = \pm x\).

The paper is organized as follows. In Section 1, we first introduce the basic and necessary algebraic topology around \(\text{Spin}^c\) groups including cohomology of related spaces in low dimensions, then compute the free suspension (transgression) of \(B\text{Spin}^c\) which is the key to link the strong and weak String\(^c\) structures together. In Section 2 and Section 3, we establish the basis of String\(^c\)-theory in the strong and weak sense respectively, including their definitions, the construction of String\(^c\) groups, the geometric explanations and their structural theories. We then discuss their relations in Section 4. In Section 5, we construct generalized Witten genus \(W^c_{2k+1,6}(M)\) for String\(^c\) manifolds of level \(2k + 1\) and prove Liu’s type vanishing theorem for them. Then we apply it to stable almost complex manifolds and the concrete example of homotopy \(\mathbb{C}P^6\). We also add four appendixes for reference. Appendix A, B, and C are devoted to various homotopy techniques used in this paper including fibration diagram trick, cohomology suspension and transgression, and the Blakers-Massey type theorems. These materials, though some of which may be not included in standard textbooks of algebraic topology, are well known to homotopy theorists.
We add them here mainly for the readers and experts in other fields, especially for geometers and mathematical physicists. The final section, Appendix D, provides necessary number theoretical preliminaries for defining and computing the generalized Witten genus in Section 5.

Conventions:

- We always use $\simeq$ to denote homotopy equivalence;
- In many places, $H^*(X)$ is used to denote the singular cohomology $H^*(X; \mathbb{Z})$ unless we want emphasis the coefficient group $\mathbb{Z}$.

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1. Some aspects of algebraic topology around $\text{Spin}^c$

Throughout this section, we may use same notation for both a map and its homotopy class, and add subscripts to cohomology classes to indicate their degrees unless otherwise stated. For our purpose, we only need cohomology of spaces under consideration up to dimension 4, and the cohomology $H^i(-; \mathbb{Z})$ here should be understood as reduced cohomology with one $\mathbb{Z}$-summand omitted in $H^0$.

1.1. $\text{Spin}^c(n)$ and $B\text{Spin}^c(n)$. By definition, the topological group $\text{Spin}^c(n)$ is given by

\[ \text{Spin}^c(n) = (\text{Spin}(n) \times S^1)/\{\pm 1\}, \]

where $\text{Spin}(n) \cap S^1 = \{\pm 1\}$. Alternatively, it is the central extension of $\text{SO}(n)$ by the circle group $S^1$

\[ \{1\} \to S^1 \xrightarrow{i} \text{Spin}^c(n) \xrightarrow{\rho} \text{SO}(n) \to \{1\}. \]

From (1.1) we have a principal bundle

\[ \text{Spin}(n) \xrightarrow{i} \text{Spin}^c(n) \xrightarrow{\rho} S^1, \]

where $i(x) = [x, 1]$ and $\pi([x, z]) = z^2$ for any $(x, z) \in \text{Spin}(n) \times S^1$. It is then easy to see that $\pi_1(\text{Spin}^c(n)) \cong \mathbb{Z}$, the generator $s_1$ of which serves as a right homotopy inverse of $\pi$. Hence the composition map

\[ \text{Spin}(n) \times S^1 \xrightarrow{i \times 1} \text{Spin}^c(n) \times \text{Spin}^c(n) \xrightarrow{\mu} \text{Spin}^c(n) \]

is a weak equivalence, that is, induces isomorphisms of homotopy groups, where $\mu$ is the group multiplication of $\text{Spin}^c(n)$. Then by Whitehead theorem it follows that (1.3) splits as spaces

\[ \text{Spin}^c(n) \simeq \text{Spin}(n) \times S^1. \]
Since $H^{\leq 4}(Spin(n); \mathbb{Z}) \cong \mathbb{Z}\{u_3\}$ with the degree $|\mu_3| = 3$, we have
\begin{equation}
H^{\leq 4}(Spin(n)) \cong \mathbb{Z}\{s_1\} \oplus \mathbb{Z}\{\mu_3\} \oplus \mathbb{Z}\{s_1\mu_3\},
\end{equation}
where $xy$ denotes the cup product of $x$ and $y$.

For classifying spaces, it is well known that with the help of Serre spectral sequence, the cohomological transgression (see Appendix B) $\tau$ connects the cohomology of $BSpin(n)$ with that of $Spin(n)$. In particular,
\[ \tau : H^3(BSpin(n)) \to H^4(BSpin(n)) \]
is an isomorphism such that $\tau(\mu_3) = q_4$ is a typical generator of $H^4(BSpin(n))$. Similarly, from (1.5) it is easy to show that
\begin{equation}
H^{\leq 4}(BSpin(n)) \cong \mathbb{Z}\{c_2\} \oplus \mathbb{Z}\{q_4\} \oplus \mathbb{Z}\{c_2^2\},
\end{equation}
such that $\tau(s_1) = c_2$.

1.2. $LSpin^c(n)$ and $BLSpin^c(n)$. For any pointed space $X$, we have the canonical fibration
\begin{equation}
\Omega X \xrightarrow{s} LX \xrightarrow{p} X,
\end{equation}
where $LX = \text{map}(S^1, X)$ is the free loop space of $X$, and $p(\lambda) = \lambda(1)$. It is clear that there is a cross section $s : X \to LX$ defined by constant loops such that $p \circ s = \text{id}_X$. It follows that whenever $X$ is an $H$-space, we have
\begin{equation}
LX \cong \Omega X \times X
\end{equation}
as spaces, while $LX$ inherits an $H$-structure naturally from that of $X$ by point-wise multiplications. When $X = G$ is a topological group, $LG$ is the so-called loop group, and
\begin{equation}
LG \cong \Omega G \times G.
\end{equation}
Moreover, if $G(X)$ is commutative (homotopically commutative), then $LG(LX)$ splits as groups ($H$-spaces) in (1.9) ((1.8)).

The classifying space of $LG$ satisfies
\begin{equation}
BLSpin \cong LBG,
\end{equation}
and we have a fibration
\begin{equation}
G \to BLG \xrightarrow{p} BG,
\end{equation}
which is fibrewise homotopy equivalent to the Borel fibration
\begin{equation}
G \to G \times_G EG \to BG
\end{equation}
induced by the adjoint action of $G$.

We are now interested in $LSpin^c(n)$. First by applying the free loop functor to (1.3) we obtain the fibration
\begin{equation}
LSpin(n) \to LSpin^c(n) \to LS^1,
\end{equation}
where $LS^1 \cong \Omega S^1 \times S^1 \cong \mathbb{Z} \times S^1$ as groups. Since there is a one-one correspondence between the components of $\Omega Spin^c(n)$ and of $\Omega S^1 (\pi_0(\Omega Spin^c(n)) \cong \mathbb{Z} \cong \pi_0(\Omega S^1))$, we see that
\begin{equation}
\Omega Spin(n) \simeq \Omega S^1 Spin^c(n),
\end{equation}
where $\Omega S^1 Spin^c(n)$ denotes the $k$-th component of $\Omega Spin^c(n)$ indexed by $k \in \mathbb{Z} \cong \pi_0(\Omega Spin^c(n))$. It should be noticed that $\Omega_0 Spin^c(n)$ is a normal subgroup of $\Omega Spin^c(n)$, and the splitting
\((1.14)\) is an \(A_\infty\)-equivalence in this case (that is, a group isomorphism up to homotopy). Hence the \(k\)-th component of \(LSpin^c(n)\)

\[
L_k Spin^c(n) \simeq \Omega_k Spin^c(n) \times Spin^c(n) \\
\simeq \Omega Spin(n) \times Spin^c(n) \\
\simeq \Omega Spin(n) \times Spin(n) \times S^1,
\]

and

\[
H^*(L_k Spin^c(n)) \cong H^*(LSpin(n)) \otimes H^*(S^1).
\]

In particular,

\[
H^{\leq 4}(L_k Spin^c(n)) \cong \mathbb{Z}[s_1, x_2, \mu_3],
\]

where \(\mathbb{Z}[\leq m][-]\) denotes graded truncated polynomial ring consisting of elements of degree not greater than \(m\), the generator \(x_2 \in H^2(\Omega Spin(n))\) satisfies \(\tau(x_2) = \mu_3\).

\(L_0 Spin^c(n)\) is also a normal subgroup of \(LSpin^c(n)\), and we have the group extension

\[
\{1\} \to L_0 Spin^c(n) \to LSpin^c(n) \to \mathbb{Z} \to \{1\}.
\]

Then we see that \(BL_0 Spin^c(n)\) is the universal covering of \(BLSpin^c(n)\)

\[
\mathbb{Z} \to BL_0 Spin^c(n) \to BLSpin^c(n).
\]

Moreover, from \((1.13)\) we have

\[
LSpin(n) \to L_0 Spin^c(n) \to S^1,
\]

which implies that \(BLSpin(n)\) is the 2-connected cover of \(BL_0 Spin^c(n)\), and then of \(BLSpin^c(n)\)

\[
S^1 \to BLSpin(n) \to BL_0 Spin^c(n).
\]

In conclusion, we have the first two stages of the Whitehead tower of \(BLSpin^c(n)\)

\[
\cdots \to BLSpin(n) \to BL_0 Spin^c(n) \to BLSpin^c(n).
\]

Now the cohomology of \(BL_0 Spin^c(n)\) can be computed via the Serre spectral sequence of \((1.19)\), while the cohomology of \(BLSpin(n)\) and \(BLSpin^c(n)\) can be calculated via that of the loop space fibration \((1.11)\). Here we need to use the fact that \(p^*: H^*(BG) \to H^*(BLG)\) is an injection due to the existence of cross section of \((1.11)\), which allows us to handle the \(E_2\)-terms in low degrees easily. We summarise the results in the next subsection.

### 1.3. Cohomology in low dimensions.

Table 1 summarises the cohomology of dimensions up to 4 for groups and their classifying spaces around \(Spin^c\) based on the discussion in the last two subsections.

In the table, we indicate the generators of each group by abuse of notations, which indeed show their connections through computations and ring structures, and any two generators corresponding to each other via some map are denoted by same letter. For later use, let us also recall that we have nontrivial transgressions

\[
\tau(s_1) = c_2, \quad \tau(x_2) = \mu_3, \quad \tau(\mu_3) = q_4.
\]

Notice that in Table 1, we do not consider \(SO(n)\) and its relatives. Indeed, there are relations among the generators of classifying spaces. Recall that

\[
H^*(BSO(n); \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots],
\]

where \(p_i\) is the \(i\)-th universal Pontryagin class. Then by abuse of notations we have relations \((1.10)\)

\[
2q_4 = p_1 \in H^*(BSpin(n); \mathbb{Z}), \quad 2q_4 + c_2^2 = p_1 \in H^*(BSpin^c(n); \mathbb{Z}).
\]
1.4. Evaluation map and free suspension. Let $X$ be a pointed space. We define the free evaluation map

\[ ev : S^1 \times LX \to X \]

by $ev((t, \lambda)) = \lambda(1)$. The free suspension

\[ \nu : H^{n+1}(X) \to H^n(LX) \]

is then determined by the formula $ev^*(x) = 1 \otimes p^*(x) + s_1 \otimes \nu(x)$ for any $x \in H^{n+1}(X)$. It is then easy to check that the free suspension satisfies the following properties (the map $i$ and $p$ are defined in (1.7); see Section 3 of [23] or Section 2 of [20]):

1. $i^* \circ \nu = \sigma^* : H^{n+1}(X) \to H^n(\Omega X)$;
2. $\nu(xy) = \nu(x)p^*(y) + (-1)^{|x|}\nu(y)p^*(x)$, for any $x$ and $y \in H^{n+1}(X)$,

where $\sigma^*$ is the classic cohomology suspension (for details see Appendix B). The Property (2) means that $\nu$ is a module derivation under $p^*$ (but since $p^*$ is always injective, we may omit it and simply write $\nu(x)y$ for $\nu(x)p^*(y)$, etc). It is helpful to mention that the transgression $\tau$ is a partial inverse of $\sigma^*$, and in good cases they are isomorphisms (again refer to Appendix B). In particular, for the transgressions in (1.21) we have

\[ \sigma^*(c_2) = s_1, \quad \sigma^*(\mu_3) = x_2, \quad \sigma^*(q_4) = \mu_3. \]

Let us now study the free suspension for $X = BSpin^c(n)$. We then form a commutative diagram of evaluation maps

\[ S^1 \times LB Spin(n) \xrightarrow{ev} B Spin(n) \]

(1.27)
which implies the diagram

\[
\begin{array}{c}
H^4(BSpin(n)) \xrightarrow{\nu} H^3(BLSpin(n)) \\
| \\
H^4(BSpin^c(n)) \xrightarrow{\nu} H^3(BLSpin^c(n)) \\
| \\
H^4(BS^1) \xrightarrow{\nu} H^3(BLS^1)
\end{array}
\]

(1.28)

commutes. The morphisms \(\nu\) for \(BSpin(n)\) and \(BS^1\) in Diagram 1.28 are easy. Indeed, since

\[i^* \circ \nu(c_2) = \sigma^*(c_2) = s_1,\]

and \(i^* : H^1(LBS^1) \to H^1(\Omega BS^1)\) is an isomorphism, we see that

\[\nu(c_2) = s_1.\]

Similarly, since

\[i^* \circ \nu(q_4) = \sigma^*(q_4) = \mu_3,\]

and \(i^* : H^3(LBSpin(n)) \to H^3(\Omega BSpin(n))\) is an isomorphism, we see that

\[\nu(q_4) = \mu_3.\]

**Lemma 1.1.** \(\nu : H^4(BSpin^c(n); \mathbb{Z}) \to H^3(BLSpin^c(n); \mathbb{Z})\) satisfies

\[\nu(q_4) = \mu_3 - s_1 c_2, \quad \nu(c_2^2) = 2s_1 c_2,\]

while the \(i\)-th component of the cohomology suspension \(\sigma_i^* : H^1(BLSpin^c(n); \mathbb{Z}) \to H^2(LSpin^c(n); \mathbb{Z})\) satisfies

\[\sigma_i^*(\mu_3) = x_2, \quad \sigma_i^*(s_1 c_2) = 0\]

for each \(i \in \mathbb{Z} \cong \pi_1(LSpin^c(n))\).

**Proof.** The computations of the value of \(\sigma_i^*\) are easy and will be omitted here. For the free suspension, based on the previous calculations we have

\[(1.29) \quad \nu(c_2^2) = 2s_1 c_2, \quad \nu(q_4) = \mu_3 + \lambda s_1 c_2,\]

for some \(\lambda \in \mathbb{Z}\) by Property (2) of \(\nu\) and the commutativity of Diagram 1.28. In order to get the exact value of \(\lambda\), we consider the homotopy commutative diagram of fibrations

\[
\begin{array}{c}
* \xrightarrow{\ast} BSpin(n) \xrightarrow{B_{ln}} BSpin(n) \\
| \\
K(\mathbb{Z}, 2) \xrightarrow{B_{ln}} BSpin^c(n) \xrightarrow{B_{p}} BSO(n) \\
| \\
K(\mathbb{Z}, 2) \xrightarrow{B_{p}} K(\mathbb{Z}, 2) \xrightarrow{2} K(\mathbb{Z}/2, 2),
\end{array}
\]

(1.30)
where 2 is a square map of $H$-spaces. By applying the functor $L$ to Diagram 1.30, we can form a commutative diagram

$$
\begin{array}{ccc}
H^4(BS^1) & \xrightarrow{(Bl)^*} & H^4(BSpin^c(n)) \\
\downarrow v & & \downarrow v \\
H^3(BLS^1) & \xrightarrow{(Bl)^*} & H^3(BLSpin^c(n)) \\
\downarrow v & & \downarrow v \\
H^3(BLS^1) & \xrightarrow{(Bl)^*} & H^3(BLSpin^c(n)) \\
\end{array}
$$

(1.31)

where $(Bl)^* = 4$. Now we need to calculate the two sides of the following equality:

$$
(Bl)^* v(q_4) = (v \circ (Bt)^*)(q_4).
$$

For the left hand side of (1.32), (1.29) implies that

$$
(Bl)^* v(q_4) = (Bl)^*(\mu_3) + \lambda (Bl)^*(s_1c_2).
$$

Recall that $H^3(SO(n)) \cong \mathbb{Z}\{e_3\}$ such that $\tau(e_3) = p_1$, and $H^3(Spin^c(n)) \cong \mathbb{Z}\{\mu_3\}$. Then since $\tau(\mu_3) = q_4$ and $2q_4 = p^*(p_1) \in H^4(BSpin^c(n)) \xrightarrow{\tau^*} H^4(BSO(n))$, by the naturality of $\tau$ we see that $p^*(e_3) = 2\mu_3$ and $2(Bl)^*(\mu_3) = (Bl)^*(Bl)^*(\mu_3) = 0$. It follows that $(Bl)^*(\mu_3) = 0$. Also, $(Bl)^*(s_1c_2) = (Bl)^*(BLS\pi)^*(s_1c_2) = 4s_1c_2$. Hence

$$
(Bl)^* v(q_4) = 4\lambda s_1c_2.
$$

(1.33)

For the right hand side of (1.32), we know that $(Bp)^*(p_1) = 2q_4 + c_2^2$ by (1.23), and it follows that

$$
0 = (Bl)^*(Bp)^*(p_1) = (Bl)^*(2q_4 + c_2^2) = 2(Bl)^*(q_4) + 4c_2^2.
$$

Hence, $(Bl)^*(q_4) = -2c_2^2$ and by (1.29)

$$
(v \circ (Bt)^*)(q_4) = -2\nu(c_2^2) = -4s_1c_2.
$$

(1.34)

Combining (1.32), (1.33) and (1.34) together, we see that $\lambda = -1$. This proves the lemma for the value of $\nu$.

\[\square\]

2. Strong String$^c$-structures

Let $V$ be an $n$-dimensional oriented vector bundle over a closed oriented smooth manifold $M$. $V$ is said to have a Spin$^c$-structure if and only if its second Stiefel-Whitney class $\omega_2(V)$ is in the image of the mod 2 reduction homomorphism

$$
\rho_2 : H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2).
$$

To specify such a structure is then equivalent to choose a particular class $c \in H^2(M; \mathbb{Z})$ such that $\rho_2(c) = \omega_2(V)$, which determines and is determined by a complex line bundle $\xi$ with its associated circle bundle

$$
S^1 \to S(\xi) \to M.
$$

(2.1)

Let $F_{SO}(V) \to M$ be the principal orthonormal frame bundle of $V$ with fibre $SO(n)$. Then there exists a principal Spin$^c(n)$-bundle

$$
Spin^c(n) \xrightarrow{i} P_{Spin^c}(V) \xrightarrow{\pi} M,
$$

(2.2)
defined as the fibrewise double cover of $P_{SO}(V) \times S(\xi)$ with classifying map $g : M \to BSpin^c(n)$.

**Definition 2.1.** Let $V$ be an $n$-dimensional real $Spin^c$-vector bundle over $M$ associated with a complex line bundle $\xi$. For any $k \in \mathbb{Z}$, $V$ is said to have a **level $2k + 1$ (strong)** $String^c$-structure if the characteristic class

$$\frac{p_1(V) - (2k + 1)c^2}{2} = 0,$$

where $p_1(V)$ is the first Pontryagin class of $V$ and $c = c_1(\xi)$ is the first Chern class of $\xi$.

In particular, a manifold $M$ admits a **$String^c$-structure of level $2k + 1$** if its tangent bundle $TM$ has a level $2k + 1$ $String^c$-structure.

Let us look at the universal case and define $BString^c_k(n)$ to be the homotopy fibre of the map

$$\frac{p_1 - (2k + 1)c^2}{2} : BSpin^c(n) \to K(\mathbb{Z}, 4).$$

At this moment, this is just a space with specified notation. We want to construct $BString^c_k(n)$ explicitly as the classifying space of the $(2k + 1)$-level strong $String^c$-structure, and then the bundle $V$ is a strong $String^c$-bundle of level $2k + 1$ if the classifying map of the associated frame $Spin^c(n)$-bundle of $V$ can be lifted to a map to $BString^c_k$ served as the classifying map of the desired $(2k + 1)$-level $String^c$-structure

$$BString^c_k(n) \leftarrow M \overrightarrow{\pi} BSpin^c(n).$$

For this purpose, we need to show that $BString^c_k(n)$ is really a classifying space of some topological group $String^c_k(n)$ with suitable group model, which justifies our choice of notation as well.

Let us firstly consider the case when $k < 0$. The first step is to embed the group $Spin^c(n)$ into a larger spin group $Spin(n - 4k - 2)$ through the pullback of groups

$$\begin{array}{ccc}
Spin^c(n) & \overset{\lambda_{2k+1}}{\longrightarrow} & Spin(n - 4k - 2) \\
\rho \downarrow & & \downarrow \rho \\
SO(n) \times S^1 & \overset{id_{SO(n)} \times \Delta_{-2k-1}}{\longrightarrow} & SO(n) \times S^1 \times \cdots \times S^1 \overset{\chi_{-2k-1}}{\longrightarrow} SO(n - 4k - 2),
\end{array}$$

where $\rho([x, z]) = (\rho(x), z^2)$ ($\rho$ is the standard projection map; see (1.2)), $\Delta_{-2k-1}$ is the diagonal map, and

$$\chi_{-2k-1}(A, z_1, \ldots, z_{-2k-1}) = \text{diag}(A, z_1, \ldots, z_{-2k-1})$$

is the standard embedding mapping any $(n \times n)$-matrix $A$ and $(2 \times 2)$-matrix $z_i$ to be block diagonal matrix. Then we may use the group embedding of Diagram (2.5) to define the
group $\text{String}^c_k(n)$ as the pullback

$$
\begin{array}{ccc}
\text{String}^c_k(n) & \xrightarrow{j_k} & \text{String}(n-4k-2) \\
\downarrow & & \downarrow j \\
\text{Spin}^c(n) & \xrightarrow{\lambda_{2k+1}} & \text{Spin}(n-4k-2),
\end{array}
$$

(2.6)

where $j : \text{String}(n-4k-2) \to \text{Spin}(n-4k-2)$ can be chosen as any group extension by group model of $K(\mathbb{Z},2)$.

In order to get similar definitions of $\text{String}^c_k(n)$ for $k \geq 0$, we need to modified our embeddings in Diagram (2.5). Recall that the stable special orthogonal group $SO = SO(\infty) = \lim_n SO(n)$ is an infinity loop space, and in particular there is a group homomorphism

$$
v : SO \to SO,
$$

which is the homotopy inverse of the identity map. We then can embed the group $\text{Spin}^c(n)$ in the stable group $\text{Spin}$ twisted by $v$ through the pullback diagram

$$
\begin{array}{ccc}
\text{Spin}^c(n) & \xrightarrow{\lambda_{2k+1}} & \text{Spin} \\
\downarrow \rho & & \downarrow \rho \\
SO(n) \times S^1 & \xrightarrow{\text{id}_{SO(n)} \times \chi} & SO(n) \times SO \xrightarrow{\text{id}_{SO(n)} \times v} SO(n) \times SO \xrightarrow{j} SO,
\end{array}
$$

(2.7)

where $j$ is the standard embedding and $\chi$ is defined as the composition

$$
S^1 \xrightarrow{\lambda_{2k+1}} \cdots \xrightarrow{\lambda_{2k+1}} SO(2k+1) \hookrightarrow SO.
$$

However, since $\text{Spin}^c(n)$ is compact, the map $\lambda_{2k+1}$ indeed maps it into some finite stage of $\text{Spin}$ as subgroup, that is, for sufficient large $m$

$$
\lambda_{2k+1} : \text{Spin}^c(n) \hookrightarrow \text{Spin}(m+n+4k+2).
$$

Hence, as in Diagram (2.6) we may define $\text{String}^c_k(n)$ as the pullback of $\lambda_{2k+1}$ along $j : \text{String}(m+n+4k+2) \to \text{Spin}(m+n+4k+2)$ when $k \geq 0$, which is clearly independent of the choice of $m$.

So far we have defined $\text{String}^c_k(n)$ for any $k \in \mathbb{Z}$, the group structure of which can be understood through that of $\text{String}$ group. In particular, the group models of $\text{String}$ will induce group models of $\text{String}^c$. Indeed, there is topological group model of $\text{String}$ by Stolz and Teichner [43] in terms of group extension by a projective unitary group $PU(A)$ as a model of $K(\mathbb{Z},2)$. On the other hand, Nikolaus, Sachse and Wockel [39] constructed an infinite-dimensional Lie group model for $\text{String}$. In either case, we obtain a real topological or smooth group $\text{String}^c_k(n)$.

Let us check that our constructions are the right choices for the determinant obstructions of $\text{String}^c_k(n)$ structures. Applying the classifying functor $B$ to Diagram (2.5), there is particularly an $SO(n-4k-2)$-bundle over $BSO(n) \times BS^1$ with first Pontryagin class $p_1 - (2k+1)c_2^2$ presented by the bottom composition. Since by (1.23)

$$
(Bp)^\ast (p_1 - (2k+1)c_2^2) = 2q_4 - 2kc_2^2,
$$

and also $p_1 = 2q_4$ in $H^4(\text{Spin}(n+4k+2))$, we see that

$$
(\lambda_{2k+1})^\ast (q_4) = q_4 - kc_2^2.
$$

(2.8)
Applying the classifying functor $B$ to Diagram (2.6), by (2.8) we have the commutative diagram

$$\begin{array}{c}
BSpin^c(n) \xrightarrow{B\lambda_{2k+1}} BSpin(n-4k-2) \\
\downarrow \quad \downarrow \\
K(\mathbb{Z}, 4),
\end{array}$$

(2.9)

which justifies the definition of $BString^c(n)$ by (2.3) for $k < 0$. The case when $k \geq 0$ can be treated similarly with the fact that $(Bv)^*(p_1) = -p_1$ and then $p_1$ will be pulled back to $p_1 - (2k + 1)c_2^k \in H^4(SO(n) \times S^3)$ along the bottom composition in Diagram (2.7).

The process of constructing these groups also suggests geometric explanations for the String$^c$-structures. For our Spin$^c$-bundle $V$ over $M$, let us consider the real $(n-4k-2)$-bundle $V \oplus \mathbb{R}^{\otimes (-2k-1)}$ when $k < 0$. Then it is easy to calculate its second Stiefel-Whitney class

$$\omega_2(V \oplus \mathbb{R}^{\otimes (-2k-1)}) = \omega_2(M) - (2k + 1)c(\xi) \mod 2 = 0.$$  

In particular, the principal frame bundle $P_{SO}(V \oplus \mathbb{R}^{\otimes (-2k-1)})$ has a fibrewise two-sheeted covering denoted by $P^k_{Spin}(V, \xi)$, which is a Spin bundle

$$Spin(n-4k-2) \xrightarrow{i} P^k_{Spin}(V, \xi) \xrightarrow{\pi} M.$$  

(2.10)

Then by expecting Diagram (2.5) and our above calculations, there is a bundle embedding

$$\begin{array}{c}
Spin^c(n) \xrightarrow{\lambda_{2k+1}} P_{Spin}^c(V) \\
\downarrow \quad \downarrow \\
Spin(n-4k-2) \xrightarrow{i} P^k_{Spin}(V, \xi) \xrightarrow{\pi} M \xrightarrow{g} BSpin^c(n) \\
\downarrow \quad \downarrow \quad \downarrow \\
Spin(n-4k-2) \xrightarrow{i} P^k_{Spin}(V, \xi) \xrightarrow{\pi} M \xrightarrow{h} BSpin(n-4k-2).
\end{array}$$

(2.11)

When $k \geq 0$, we may consider the stable vector bundle $V \oplus \mathbb{R}^{\otimes (-2k-1)}$, and go through the argument above with Diagram (2.7) to embed the bundle $P_{Spin^c}(V, \xi)$ into $P^k_{Spin}(V, \xi)$ similarly. Note that in this case the bundle $P^k_{Spin}(V, \xi)$ is of dimension $(m + n + 4k + 2)$.

**Theorem 2.2.** Let $V$ be an $n$-dimensional Spin$^c$-vector bundle over $M$ associated with a complex line bundle $\xi$. $V$ admits a strong String$^c$-structure if and only if the stable spin bundle associated to $V \oplus \mathbb{R}^{\otimes (-2k-1)}$ admits a String structure for some $k \in \mathbb{Z}$.

Furthermore, if the obstruction class $p_1(V) - (2k + 1)c_2^k = 0$, then the $(2k + 1)$-level String$^c$-structures on $V$ are in one-one correspondence with the elements in the image of the morphism

$$\rho^* : H^3(M) \rightarrow H^3(S(\xi)),$$

where $\rho : S(\xi) \rightarrow M$ is the circle bundle of $\xi$. 

**Proof.** We may consider the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & B\text{String}_k(n) \\
\downarrow{(1)} & & \downarrow{B\lambda_{2k+1}} \\
BS\text{pin}^c(n) & \xrightarrow{b_j} & B\text{Spin},
\end{array}
\]

(2.12)

where the square is a homotopy pullback by (2.6) or (2.7). Then by the universal property of homotopy pullback, the existence of a lifting at (1) in the diagram is equivalent to the existence of a lifting at (2). For the bundle \( V \oplus \xi \oplus (-2k-1) \), it is easy to show that its first Pontryagin class

\[
p_1(V \oplus \xi \oplus (-2k-1)) = p_1(V) - (2k+1)c(\xi)^2
\]

(notice that \( c_1(\xi \oplus \xi) = 0 \) and \( p_1(\xi \oplus (-2k-1)) = - (2k+1)c(\xi)^2 \)). Then by definition, the spin bundle associated to \( V \oplus \xi \oplus (-2k-1) \) admits a (strong) String structure if and only if

\[
p_1(V) - (2k+1)c(\xi)^2 = 0.
\]

This proves the first claim of the theorem.

For the second claim of the theorem, we first proved that the different String\(^c\)-structures on \( V \) are classified by the image of

\[
\pi^*: H^3(M) \to H^3(\text{P}_{\text{Spin}^c}(V)).
\]

By Diagram (2.11) we can construct a commutative diagram

\[
\begin{array}{cccccc}
H^3(M) & \xrightarrow{\pi^*} & H^3(\text{P}_{\text{Spin}^c}(V)) & \xrightarrow{\iota^*} & H^3(\text{Spin}^c(n)) & \xrightarrow{\bar{\delta}} & H^4(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3(M) & \xrightarrow{\pi^*} & \text{Im} \Theta^* & \xrightarrow{\iota^*} & H^3(\text{Spin}^c(n)) & \xrightarrow{\bar{\delta}} & H^4(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3(M) & \xrightarrow{\pi^*} & H^3(\text{P}_{\text{Spin}^c}(V, \xi)) & \xrightarrow{\iota^*} & H^3(\text{Spin}) & \xrightarrow{\bar{\delta}} & H^4(M),
\end{array}
\]

(2.13)

where the third row is exact by the dual Blakers-Massey theorem (Theorem C.3), and \( \bar{\delta} \) is defined by \( \bar{\delta}(\mu_5) = q_1(V) - kc(\xi)^2 \). Here

\[
q_1(V) = g^*(q_4): H^4(M) \leftarrow H^4(\text{B}_{\text{Spin}^c}(n))
\]

is the characteristic class defined by universal class \( q_4 \). It is easy to see that the second row of the diagram is exact (this gives a second proof for the first claim). Notice that \( \text{Ker} \Theta^* \subseteq H^3(M) \) and the first morphism \( \pi^* \) in the second row has \( \text{Ker} \Theta^* \) as its kernel. Hence the distinct String\(^c\) structures on \( V \) are classified by

\[
\text{Ker}^* \cong \text{Im} \pi^* \cong H^3(M)/\text{Ker} \Theta^*.
\]
On the other hand, there is a bundle morphism

\[
\begin{align*}
\text{Spin}(n) & \longrightarrow P_{\text{Spin}}(V) \xrightarrow{\pi} S(\xi) \xrightarrow{\tilde{\delta}} B\text{Spin}(n) \\
\text{Spin}^c(n) & \longrightarrow P_{\text{Spin}}(V) \xrightarrow{\pi} M \xrightarrow{\rho} B\text{Spin}^c(n),
\end{align*}
\]

where the existence of lifting \( \tilde{\delta} \) is due to the vanishing of the second Stiefel-Whitney class of \( S(\xi) \). This diagram then induces a commutative diagram of cohomology groups

\[
\begin{align*}
0 = H^2(\text{Spin}(n)) & \longrightarrow H^3(S(\xi)) \longrightarrow H^3(P_{\text{Spin}}(V)) \\
H^3(M) & \longrightarrow H^3(P_{\text{Spin}}(V)),
\end{align*}
\]

where the first row is exact again by Theorem C.3. Hence \( \text{Im} \rho^* \cong \text{Im} \rho^* \) and the proof of the theorem is completed.

There are some cases when \( \rho^* \) are surjective or injective.

**Corollary 2.3.** Let \((V, \xi)\) as in Theorem 2.2. Then

1. if that the cup product by \( c(\xi) \)
   \[
   \cup c(\xi) : H^2(M; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z})
   \]
   is injective, then \( \rho^* \) is surjective. In particular, the strong String\(^c\) structures of level \( 2k + 1 \) on \( V \) are in one-one correspondence with elements of \( H^3(S(\xi)) \);
2. if the fundamental group \( \pi_1(M) \) is a torsion group (e.g., when \( M \) is simply connected), then \( \rho^* \) is injective. In particular, the strong String\(^c\) structures of level \( 2k + 1 \) on \( V \) are in one-one correspondence with elements of \( H^3(M) \).

**Proof.** Let us look at the Gysin sequence of the line bundle \( \xi \)

\[
\cdots \longrightarrow H^1(M) \xrightarrow{\cup c(\xi)} H^3(M) \xrightarrow{\rho^*} H^3(S(\xi)) \xrightarrow{\delta} H^2(M) \xrightarrow{\cup c(\xi)} H^4(M) \longrightarrow \cdots.
\]

For Case (1), in the exact sequence the second cup product \( \cup c(\xi) \) is injective, which implies that \( \delta \) is trivial. Hence \( \rho^* \) is surjective, and by Theorem 2.2 the String\(^c\) structures on \( V \) are classified by \( \text{Im} \rho^* = H^3(S(\xi)) \).

For Case (2), the condition on the fundamental group of \( M \) is equivalent to that \( H^1(M) = 0 \). Then from the Gysin sequence above we see that \( \rho^* \) is injective, and again by Theorem 2.2 the String\(^c\) structures on \( V \) are classified by \( \text{Im} \rho^* \cong H^3(M) \).

### 3. Weak String\(^c\)-structures

Motivated by the philosophy that String structures can be studied in terms of spin structures on loop spaces, we may define String\(^c\)-structures in terms of Spin\(^c\)-structures on loop spaces in a reasonable way, which in general is weaker than the notion of String\(^c\) defined in Section 2.

Let \((V, \xi)\) be the Spin\(^c\)-bundle defined in Section 2. By applying free loop functor to (2.2), we get a principal fibre bundle

\[
LS\text{pin}^c(n) \xrightarrow{L_i} LP_{\text{Spin}}(V) \xrightarrow{L_{\xi}} LM
\]
classified by $Lg : LM \to BLSpin^c(n)$. In particular, we may define the $LSpin^c$ characteristic classes of $M$ as the pullbacks of the elements of $H^*(BLSpin^c(n))$ in $H^*(LM)$ through $Lg$. In the low degrees, let us denote that $s = s(L_\xi), c = c(\xi), \mu_1(V) = \mu_1(V, \xi), q_1(V) = q_1(V, \xi)$ and $p_1(V)$ as the $LSpin^c$-classes of $LV$ corresponding to the universal classes $s_1, c_2, \mu_3, q_4$ and $p_1$ respectively. We then notice that $c(\xi)$ and $p_1(V)$ correspond to usual Euler class of $\xi$ and the first Pontryagin class of $V$ respectively via the projection $p$ in the loop fibration (1.7), which justifies our notations.

Throughout the remaining part of this section, let us assume $x_2 \in H^2(LSpin^c(n))$ is always chosen from the 0-th component $H^2(LSpin^c(n))$ unless otherwise stated.

**Definition 3.1.** Let $V$ be an $n$-dimensional Spin$^c$-vector bundle over a manifold $M$ associated with a complex line bundle $\xi$. $V$ is said to have a level $2k + 1$ weak String$^c$-structure if the obstruction class

$$\delta_k(x_2) = \mu_1(V) - (2k + 1)c$$

vanishes, where $\delta_k$ is the composition

$$H^2(LSpin^c(n); \mathbb{Z}) \xrightarrow{s_k} H^3(BLSpin^c(n); \mathbb{Z}) \xrightarrow{(Lg)^*} H^3(LM; \mathbb{Z}),$$

and $s_k$ is a section of cohomology suspension $\sigma^* : H^3(BLSpin^c(n); \mathbb{Z}) \to H^2(LSpin^c(n); \mathbb{Z})$ defined by $s_k(x_2) = \mu_3 - (2k + 1)s_1c_2$ for each integer $k \in \mathbb{Z}$.

This definition of String$^c$-structures also has geometric explanations. By using the group and bundle embeddings (e.g., Diagram (2.5), Diagram (2.11)) constructed in Section 2, we want to construct a commutative diagram (when $k < 0$)

$$H^2(LSpin^c(n)) \xrightarrow{(Lg)^*} H^3(LSpin^c(n)) \xrightarrow{(Lg)^*} H^3(LM)$$

(3.3)

$$H^2(LSpin(n - 4k - 2)) \xrightarrow{(L\lambda_{2k+1})^*} H^3(BLSpin(n - 4k - 2)) \xrightarrow{(Lh)^*} H^3(LM).$$

For this purpose, firstly apply free loop functor to Diagram (2.5), and denote $\phi = \chi_{2k+1} \circ (id_{SO(n)} \times \Delta_{-2k-1})$. Recall that (2.8)

$$(B\phi)^*(p_1) = p_1 - (2k + 1)c_2^2, \quad (B\lambda_{2k+1})^*(q_4) = q_4 - kc_2^2.$$  

Then by the naturality of the free suspension $\nu$ and Lemma 1.1, the homomorphism

$$(B\lambda_{2k+1})^* : H^3(BLSpin(n + 4k + 2)) \to H^3(BLSpin^c(n))$$

satisfies

$$\delta_k(x_2) = x_2.$$  

(3.4)

Similarly, by applying cohomology suspensions for the both sides of (3.4), we obtain

$$(L\lambda_{2k+1})^*(x_2) = x_2.$$  

(3.5)

Combining (3.4), (3.5) and the fact $\tau(x_2) = \mu_3$ for the transgression homomorphism, we see that the left square of Diagram (3.3) commutes. The right square of Diagram (3.3) is natural by applying loop functor $L$ to Diagram (2.11).

We have showed the commutativity of Diagram (3.3) when $k < 0$, while the case when $k \geq 0$ can be done similarly. From the diagram, we notice that the composition of the morphisms in the first row is the defined String$^c$-obstruction $\delta_k$, while the composition $\delta$ of those in the second row is the obstruction to the existence of String structure on the bundle $\oplus (L_\xi \oplus (-2k))$ from the point of view of loop spaces. Indeed, by expecting
Serre spectral sequence of the spin bundle (2.10) after looping (or simply applying the dual Blakers-Massey Theorem), the second row of Diagram 3.3 can be fitted into an exact sequence

\[(\delta \circ \lambda) : H^3(\Sigma \Sigma \Sigma \Sigma LM) \rightarrow H^3(\Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma \Sigma 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On the other hand, by considering the bundle morphism in Diagram (2.14) after looping, we obtain the commutative diagram of cohomology groups

$$0 = H^1(LSpin(n)) \xrightarrow{\lambda_0} H^2(LS(\xi)) \xrightarrow{(L\pi)^*} H^2(LP_{Spin}(V))$$

(3.8)

where the first row is exact again by Theorem C.3. Then $\text{Im}(L\pi)^* \cong \text{Im}(L\rho)^*$ and the proof of the theorem is completed.

Remark 3.3. We can also interpret the weak obstructions of String$^c$-structures from the perspective of classifying spaces by simply applying loop functor to Diagram (2.12).

Corollary 3.4. Let $(V, \xi)$ as in Theorem 3.2. Suppose that $M$ is simply connected, and $c(\xi)$ is a generator element of $H^2(M)$, then $(L\rho)^*: H^2(LM) \to H^2(LS(\xi))$ is surjective. In particular, the strong String$^c$-structures of level $2k + 1$ on $V$ are in one-one correspondence with elements of $H^2(LS(\xi))$.

Proof. We need to analyse the homotopy commutative diagram of fibrations

$$\begin{array}{cccc}
\Omega S(\xi) & \xrightarrow{\Omega\rho} & \Omega M & \xrightarrow{\Omega c} & S^1 \\
\downarrow & & \downarrow & & \\
LS(\xi) & \xrightarrow{L\rho} & LM & \xrightarrow{Lc} & S^1 \times K(\mathbb{Z}, 2) \\
\downarrow & & \downarrow & & \\
S(\xi) & \xrightarrow{\rho} & M & \xrightarrow{c} & K(\mathbb{Z}, 2)
\end{array}$$

(3.9)

using the Serre spectral sequences. First, from the Serre spectral sequence (or Gysin sequence) of the fibration in the third row of Diagram 3.9 there is a short exact sequence

$$0 \to H^2(K(\mathbb{Z}, 2)) \xrightarrow{\cdot c} H^2(M) \xrightarrow{\rho^*} H^2(S(\xi)) \to 0.$$  

(3.10)

On the other hand, since $M$ is simply connected, $H^2(M) \cong \text{Hom}(H_2(M), \mathbb{Z}) \cong \text{Hom}(\pi_2(M), \mathbb{Z})$ is torsion free and $(\Omega c)_*: \pi_1(\Omega M) \to \pi_1(S^1)$ is surjective. Then the fibration in the top row of Diagram (3.9) splits

$$\Omega M \cong S^1 \times \Omega S(\xi),$$

which particularly implies that $(\Omega c)^*: H^2(\Omega M) \to H^2(\Omega S(\xi))$ is surjective. Now since $c(\xi)$ is a generator element of $H^2(M)$ by assumption, $S(\xi)$ is simply connected. We then can consider Serre spectral sequences of the fibrations in the first two columns of Diagram (3.9). By the naturality of Serre spectral sequences and the fact that loop projection induces monomorphism on cohomology, we have the induced morphism of short exact sequences

$$\begin{array}{cccc}
0 & \xrightarrow{\rho^*} & H^2(M) & \xrightarrow{(L\rho)^*} H^2(LM) & \xrightarrow{(\Omega \rho)^*} H^2(\Omega M) & \to 0 \\
0 & \xrightarrow{c^*} & H^2(S(\xi)) & \xrightarrow{(Lc)^*} H^2(LS(\xi)) & \xrightarrow{(\Omega c)^*} H^2(\Omega S(\xi)) & \to 0.
\end{array}$$

(3.11)
Since we have shown that $\rho^*$ and $(\Omega \rho)^*$ are surjective, we see the middle morphism $(L \rho)^*$ in the diagram is also surjective by the (sharp) five lemma. Hence the lemma follows.

4. Relation Between Strong and Weak String$^c$ Structures

The relations between strong String$^c$ and weak String$^c$-structures are characterized by the following theorems:

Theorem 4.1. Let $V$ be an $n$-dimensional Spin$^c$-vector bundle over $M$ associated with a complex line bundle $\xi$. If $V$ is strong String$^c$ of level $2k + 1$, then $V$ is level $2k + 1$ weak String$^c$. The converse is also true, if the image of the cohomology of the classifying map

$$g^*: H^4(B\operatorname{Spin}^c(n); \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z})$$

is a subgroup of the dual of the Hurewicz image $h: \pi_4(M) \rightarrow H_4(M; \mathbb{Z})$, and the rational Hurewicz morphism

$$h \otimes \mathbb{Q}: \pi_3(LM) \otimes \mathbb{Q} \rightarrow H_3(LM; \mathbb{Q})$$

is injective.

Proof. We use the free suspension $\nu$ to prove the theorem. By Lemma 1.1, for the universal case

$$\nu\left(\frac{p_1 - (2k + 1)c_2}{2}\right) = \nu(q_4 - kc_2) = \mu_3 - (2k + 1)s_1c_2 = s_4(x_2).$$

By the naturality of $\nu$, we see that the obstructions to the weak and strong String$^c$-structures are connected via the equality

$$\nu\left(\frac{p_1(V) - (2k + 1)c^2}{2}\right) = \mu_1(V) - (2k + 1)sc.$$  \hspace{1cm} (4.1)

Hence the first claim of the theorem follows immediately. For the converse part of the theorem, we use the similar strategy used in the proof of Theorem 3.1 in [37] for the String case. The idea is to describe the free suspension $\nu$ geometrically at least for the elements in the Hurewicz image. Choose any $f \in \pi_4(M)$. $S^4$ can be covered by loops which meet only at one point (say the base point), and the parameter space for this set of loops is its equator $S^3$. By this view we obtain a class $g \in \pi_3(LM)$; indeed, this operation is equivalent to take the adjoint of $f$ to get $g \in \pi_3(\Omega M)$ and we notice that in general $\pi_3(LM) \cong \pi_3(M) \oplus \pi_3(\Omega M)$. In either way, this operation is the free suspension after taking the composition of Hurewicz map and the dual map, that is, we have the commutative diagram

$$\begin{array}{ccc}
\pi_3(LM) & \xrightarrow{h} & H_3(LM; \mathbb{Z}) \\
\downarrow \text{i} & & \downarrow \nu \\
\pi_4(M) & \xrightarrow{h} & H_4(M; \mathbb{Z}) \\
\end{array}$$

Now by assumption, the obstruction class $\mu_1(M) - (2k + 1)c_2^2 \in H^4(M)$ is from an element $f \in \pi_4(M)$. If $f$ is a torsion, then the dual of $h(f)$ will be 0 (recall here dual is defined by the natural pairing $H^4(M; \mathbb{Z}) \times H_4(M; \mathbb{Z}) \rightarrow \mathbb{Z}$). Otherwise $f$ is torsion free. Then by the above argument, we obtain an element $g = i(f) \in \pi_3(LM)$ such that $h(g)$ is non-zero by assumption. Take the dual of $h(g)$, we obtain the free suspension $\nu\left(\frac{\mu_1(M) - (2k + 1)c^2}{2}\right)$ which is non-zero. This is a contradiction, and then $\frac{\mu_1(M) - (2k + 1)c^2}{2} = 0$. The converse statement is proved.
Theorem 4.2. Let \((V, \xi)\) as in Theorem 4.1. Suppose \((V, \xi)\) is (strong) String\(^c\) of level \(2k + 1\). Then the distinct strong String\(^c\) -structures lifting the original Spin\(^c\) -structure on \(V\) transgress to the weak String\(^c\) -structures via the transgression \(\nu\)

\[
\begin{array}{ccc}
H^3(M) & \xrightarrow{\rho^*} & H^3(S(\xi)) \\
\downarrow\nu & & \downarrow\nu \\
H^2(LM) & \xrightarrow{(L\rho)^*} & H^2(LS(\xi)).
\end{array}
\]  

(4.3)

Proof. This follows immediately from the naturality of the involved constructions. \(\square\)

Corollary 4.3. Let \((V, \xi)\) as in Theorem 4.2. Suppose \(M\) is simply connected, and the Euler class \(c(\xi)\) is a generator element of \(H^2(M)\). Then the distinct String\(^c\) -structures on \(V\) transgress to the weak String\(^c\) -structures via the composition of the free suspension and the pullback

\[
(L\rho)^* \circ \nu = \nu \circ \rho^* : H^3(M) \to H^2(LS(\xi)).
\]

Proof. The corollary follows immediately from Theorem 4.2, Corollary 2.3 and Corollary 3.4. \(\square\)

5. MODULAR INVARIANTS AND GROUP ACTIONS ON STRING\(^c\) MANIFOLDS

In this section, for even dimensional level \((2k + 1)\) String\(^c\) manifolds with \(2k + 1 > 0\), we construct Witten type genus, which are modular invariants taking values in \(\mathbb{Z}[t]\) and prove Liu’s type vanishing theorem for them. They extend the generalized Witten genus for level 1 and level 3 String\(^c\) manifolds constructed in [6, 7]. As applications, we obtain vanishing theorems about positive effective actions of a compact non-abelian simply connected Lie group on almost complex manifolds.

Let \(M\) be Spin\(^c\) manifold, who is level \(2k + 1\) String\(^c\) with \(2k + 1 > 0\). Let \(\vec{a} = (a_1, a_2, \ldots, a_r)\in \mathbb{Z}^r\), \(\vec{b} = (b_1, b_2, \ldots, b_s)\in \mathbb{Z}^s\) be two vectors of integers such that \(\sum_{j=1}^r a_j + \sum_{j=1}^s b_j\) is even. If \(M\) is 4\(m\) dimensional, we require that

\[
3|\vec{a}|^2 + |\vec{b}|^2 = 2k - 2;
\]
and if \(M\) is 4\(m + 2\) dimensional, we require that

\[
3|\vec{a}|^2 + |\vec{b}|^2 = 2k.
\]

Let \(\xi\) be the determinat line bundle of the Spin\(^c\) structure. Let \(h^\xi\) be a Hermitian metric on \(\xi\) and \(\nabla^\xi\) be a Hermitian connection. Let \(\nabla^\xi\) and \(\nabla^\xi\) be the induced Euclidean metric and connection on \(\xi_R\). Construct

\[
\Theta_{\vec{a}, \vec{b}}(T_C M, \xi_R \otimes \mathbb{C})
:= \left( \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n}}(T_C M) \right) \otimes \bigotimes_{j=1}^{r} \left( \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n}}(\xi_R^{\otimes a_j} \otimes \mathbb{C}) \right) \otimes \bigotimes_{j=1}^{s} \left( \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n}}(\xi_R^{\otimes b_j} \otimes \mathbb{C}) \right).
\]

Then \(\nabla^T_M\) and \(\nabla^\xi\) induce connections \(\nabla^{\Theta_{\vec{a}, \vec{b}}(T_C M, \xi_R \otimes \mathbb{C})}\) on \(\Theta_{\vec{a}, \vec{b}}(T_C M, \xi_R \otimes \mathbb{C})\). Let \(c\) be the first Chern form of \((\xi, \nabla^\xi)\).
If \( \dim M = 4m \), define the type \((2k + 1; \vec{a}, \vec{b})\) Witten forms

\[
\mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M) := \tilde{A}(TM, \nabla^{TM}) e^{\sum_{j=1}^{r} \frac{a_j c}{2} \prod_{j=1}^{s} \sinh \left( \frac{b_j c}{2} \right)} \cdot \text{ch} \left( \Theta_{\vec{a}, \vec{b}}(T_{C^*} M, \xi_R \otimes \mathbb{C}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n+1}}(\xi_R \otimes \mathbb{C}) \bigotimes_{n=1}^{\infty} \Lambda_{-q^{2n+1}}(\xi_R \otimes \mathbb{C}) \right).
\]

If \( \dim M = 4m + 2 \), define the type \((2k + 1; \vec{a}, \vec{b})\) Witten forms

\[
\mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M) := \tilde{A}(TM, \nabla^{TM}) e^{\sum_{j=1}^{r} \frac{a_j c}{2} \prod_{j=1}^{s} \sinh \left( \frac{b_j c}{2} \right)} \cdot \text{ch} \left( \Theta_{\vec{a}, \vec{b}}(T_{C^*} M, \xi_R \otimes \mathbb{C}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n+1}}(\xi_R \otimes \mathbb{C}) \bigotimes_{n=1}^{\infty} \Lambda_{-q^{2n+1}}(\xi_R \otimes \mathbb{C}) \right).
\]

We can express the generalized Witten forms by using the Chern-root algorithm. Let \( \{ \pm 2\pi \sqrt{-1} z_j \} \) be the formal Chern roots for \( (T_{C^*} M, \nabla^{T_{C^*} M}) \) and set \( u = -\sqrt{-1} c \). In terms of the theta-functions (the details about which are discussed in Appendix D), we get through direct computations that (c.f. [6, 7, 26, 27])

\[
\mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m}) = \left( \prod_{j=1}^{2m} z_j \right) \frac{1}{\theta(z_j, \tau)} \prod_{j=1}^{r} \frac{\theta_1(u, \tau) \theta_2(u, \tau) \theta_3(u, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \prod_{j=1}^{r} \frac{\theta_2(a_j u, \tau) \theta_2(a_j u, \tau) \theta_3(a_j u, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \prod_{j=1}^{r} \sqrt{-1} \theta(b_j u, \tau) \cdot \prod_{j=1}^{s} \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)
\]

and

\[
\mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m+2}) = \left( \prod_{j=1}^{2m} z_j \right) \frac{1}{\theta(z_j, \tau)} \prod_{j=1}^{r} \frac{\theta_1(a_j u, \tau) \theta_2(a_j u, \tau) \theta_3(a_j u, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \prod_{j=1}^{r} \sqrt{-1} \theta(u, \tau) \cdot \prod_{j=1}^{s} \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)
\]

Define the type \((2k + 1; \vec{a}, \vec{b})\) Witten genus by

\[
W^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m}) := \int_{M^{4m}} \mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m})
\]

and

\[
W^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m+2}) := \int_{M^{4m+2}} \mathcal{W}^{c,\ell}_{2k+1; \vec{a}, \vec{b}}(M^{4m+2})
\]

Note that

\[
\prod_{j=1}^{r} \cosh \left( \frac{a_j c}{2} \right) \prod_{j=1}^{s} \sinh \left( \frac{b_j c}{2} \right) = \frac{1}{2^{r+s}} e^{\sum_{j=1}^{r} (e^{a_j c} + 1) \prod_{j=1}^{s} \frac{1}{2^{0} (e^{b_j c} - 1)} - \sum_{j=1}^{s} (e^{b_j c} - 1)}.
\]
However since $\sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j$ is even, $e^{\frac{-\sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j}{k}} \prod_{j=1}^{k} (e^{a_j + 1} - 1) \prod_{j=1}^{k} (e^{b_j} - 1)$ is Chern character of some vector bundle. Hence by the Atiyah-Singer index theorem, $2^{r+s} W_{2k+1, d, e}^c (M^{4m})$ and $2^{r+s} W_{2k+1, d, e}^c (M^{4m+2})$ are analytic, i.e they are indices $q$-series of twisted Spin$^c$ Dirac operators. We therefore see that $W_{2k+1, d, e}^c (M^{4m}) \in \mathbb{Z}[\frac{1}{2}]$ and $W_{2k+1, d, e}^c (M^{4m+2}) \in \mathbb{Z}[\frac{1}{2}]$.

By the same method in [25], using the conditions (5.1) or (5.2) when performing the transformation laws of theta functions, we have

**Theorem 5.1.** If $\dim M = 4m$, then $W_{2k+1, d, e}^c (M^{4m}) \in \mathbb{Z}[\frac{1}{2}]$ is a modular form of weight $2m$ over $SL(2, \mathbb{Z})$; if $\dim M = 4m+2$, then $W_{2k+1, d, e}^c (M^{4m+2}) \in \mathbb{Z}[\frac{1}{2}]$ is a modular form of weight $2m$ over $SL(2, \mathbb{Z})$.

For the generalized Witten genus $W_{2k+1, d, e}^c (M)$, we have the following Liu’s type vanishing theorem.

**Theorem 5.2.** Let $M$ be a compact level $2k+1$ String$^c$ manifold with $2k+1 > 0$. If $M$ admits a positive effective positive action of a simply connected compact non-abelian Lie group that can be lifted to the determinant line bundle of the Spin$^c$ structure, then $W_{2k+1, d, e}^c (M) = 0$.

**Proof.** Let $G$ be the simply connected compact non-abelian Lie group. It has been shown in [36] that $G$ contains $SU(2)$ or $SO(3)$ as subgroup. Since there exists the standard 2-sheet covering $p : SU(2) \rightarrow SO(3)$, in either case we see that there exists a $SU(2)$-action on $M$ factoring through $G$. Then choose any subgroup $S^1 \hookrightarrow SU(2)$. Since $G$ acts effectively on $M$ and can be lifted to the Spin$^c$ determinant line bundle $\xi$, we have that the induced $S^1$-action is non-trivial and can be lifted to the bundle $TM \otimes _{\xi} G$. In particular, through the induced composition map of classifying spaces

$$BS^1 \rightarrow BSU(2) \rightarrow BG,$$

there exists the canonical generator $q \in H^4(BG)$ restricted to the generator $a^2 \in H^4(BS^1)$.

We now can apply the similar argument of Dessai [9] for the bundle $TM \otimes _{\xi} G \otimes (2k+1)$. If the $S^1$-action has no fixed points, the generalized Witten genus vanishes by the Atiyah-Bott-Segal-Singer-Lefschetz fixed point formula ([1, 2]). Otherwise suppose that there are some fixed points. Let $EG$ be the universal $G$-principal bundle over the classifying space $BG$ of any topological group $G$. By applying the dual Blakers-Massey theorem (Theorem C.3 in Appendix C) to the Borel fibre bundle

$$M \xrightarrow{i} M \times G \xrightarrow{\pi} EG \xrightarrow{p} BG$$

(with the fact that $BG$ is 3-connected), we see that there exists a commutative diagram

$$\begin{array}{cccccc}
0 & \xrightarrow{0} & H^4(BG) & \xrightarrow{\pi^*} & H^4(M \times G EG) & \xrightarrow{i^*} & H^4(M) & \xrightarrow{0} \\
& & \| & & \| & & \| \\
& & \| & & \| & & \| \\
& & H^4(BS^1) & \xrightarrow{\pi^*} & H^4(M \times G ES^1) & \xrightarrow{i^*} & H^4(M) \\
\end{array}$$

such that the first row is exact, and maps to the second row by restricting the action to $S^1$. On the other hand, since the level $2k+1$ String$^c$ condition tells us that $p_1(TM \otimes _{\xi} G \otimes (2k+1)) = 0$, we have

$$p_1(TM)_G - (2k+1)c_1(\xi)_G^2 = p_1(TM \otimes _{\xi} (2k+1))_G = n \cdot \pi^* q$$
for some \( n > 0 \) by positive assumption (1). Hence by the above commutative diagram we see that the restriction of the equivariant Pontryagin class
\[
p_1(TM)_{S^1} - (2k+1)c_1(\xi)^2_{S^1} = p_1(TM \oplus \xi^{(2k+1)}|_{S^1}) = n \cdot \pi^* u^2.
\]
The theorem then follows by the proof of Liu’s vanishing theorem [26] for nonzero anomaly about Witten genus.

Suppose \((M,J)\) is a closed stable almost complex manifold. Then \(M\) is Spin\(^c\) and the determinant line bundle \(\xi\) of the Spin\(^c\) structure has \(c_1 \mod 2\) congruent to \(w_2(TM)\). If \(G\) acts smoothly on \(M\) and preserves the stable almost complex structure \(J\), then the action of \(G\) can be lifted to \(\xi\). Applying Theorem 5.2, we obtain the following result:

**Theorem 5.3.** Let \((M,J)\) be a closed stable almost complex manifold, which is level \(2k+1\) String\(^c\), i.e. \(p_1(TM) = (2k+1)c_1^2\) and suppose \(2k+1 > 0\). If \(M\) admits a positive effective action of a simply connected compact non-abelian Lie group that preserves the stable almost complex structure \(J\), then \(W_{c^{2k+1}}(M) = 0\).

As an application of the above theorem, let us consider homotopy \(\mathbb{C}P^6\) with stable almost complex structures. A complex homotopy projective space \(X = X^{2n}\), by definition, is a closed differential manifold homotopically equivalent to the standard complex projective space \(\mathbb{C}P^n\). For the non-abelian Lie groups actions on it, we are interested in \(X^{12}\) with stable almost complex structures. Indeed, in [18] W.-C. Hsiang and W.-Y. Hsiang used the quaternionic projective spaces of Eells-Kuiper [11, 12] to construct a family of homotopy complex projective spaces \(H \mathbb{C}P^6(p)\) indexed by an integer \(k\) such that
\[
(5.11) \quad (13p + 1)(6p + 1)q \equiv 0 \mod 31.
\]
Moreover they can be distinguished by their first Pontryagin classes
\[
(5.12) \quad p_1(H \mathbb{C}P^6(p)) = (672p + 7)x^2.
\]

**Lemma 5.4.** Each \(H \mathbb{C}P^6(p)\) admits a stable almost complex structure \(J\) with the first Chern class \(c_1 = (2l+1)x\) for any \(l \in \mathbb{Z}\). Here \(x \in H^2(H \mathbb{C}P^6(p); \mathbb{Z})\) is a generator of the second cohomology.

**Proof.** It is known that an orientable differentiable manifold \(M\) admits a stable almost complex structure if and only if the classifying map \(f\) of its stable tangent bundle can be lifted to \(BU\)

\[
\begin{align*}
&\xymatrix{ BU \ar[dd]^r & \cr M \ar[r]^f & BSO. }
\end{align*}
\]

For our \(H \mathbb{C}P^6(p)\), the obstructions of this lifting problem are certain cohomology elements
\[
\sigma_{i+1} \in H^{i+1}(H \mathbb{C}P^6(p); \pi_i(SO/U)).
\]

It is clear that \(\sigma_{2i+1} (0 \leq i \leq 5)\) is trivial by degree reason. Further, Bott [3] has completely determined the homotopy groups of \(SO/U\). In particular,
\[
\pi_{2i+1}(SO/U) = 0 \text{ for } 0 \leq i \leq 5 \text{ and } i \neq 3, \quad \pi_7(SO/U) \cong \mathbb{Z}/2.
\]

Hence the only possibly nontrivial obstruction is \(\sigma_8 \in H^8(H \mathbb{C}P^6(p); \mathbb{Z}/2) \cong \mathbb{Z}/2\). However, Massey [34] has showed that this obstruction has indeterminacy at least
\[
S_0^2 H^8(H \mathbb{C}P^6(p); \mathbb{Z}/2) \cong H^8(H \mathbb{C}P^6(p); \mathbb{Z}/2).
\]
This implies that $\theta_8$ also vanishes and there is no obstruction for the lifting. The lemma then follows from Lemma 2.1 of [16] which states that we can modify the lifting to make sure that $c_1$ of the associated stable complex structure to be any integral lifting of the second Stiefel-Whitney class of the manifold. □

**Corollary 5.5.** Let $J$ be a stable almost complex structure on $H\mathcal{C}P^6(p)$ with $p > 0$ and satisfying (5.11). If $c_1(J) = \pm x$, then $H\mathcal{C}P^6(p)$ does not admit a positive effective action of a simply connected compact non-abelian Lie group preserving $J$.

**Proof.** Since
\begin{equation}
(5.13) \quad p_1(H\mathcal{C}P^6(p)) = (672p + 7)x^2 = (672p + 7)c_1^2.
\end{equation}
Choose $\bar{b} = (b_1, b_2, \cdots, b_6) = (1, 1, 1, 3, 3)$. Consider the equation (see (5.1)),
\begin{equation}
(5.14) \quad 3||a||^2 + 1^2 + 1^2 + 1^2 + 3^2 + 3^2 = 672k + 4,
\end{equation}
or
\begin{equation}
(5.15) \quad ||a||^2 = 224p - 6.
\end{equation}
As $p > 0$, by the classical Jacobi theorem on the expression of a positive integer as sum of squares, this equation always have positive integer solution.

By (5.6), we have
\begin{equation}
(5.16) \quad W_{672p+7,a,b}^{c_1}(H\mathcal{C}P^6(p)) = \int_{H\mathcal{C}P^6(p)} \left( \prod_{j=1}^{6} z_j \right) \frac{\theta_1(u, \tau) \theta_2(u, \tau) \theta_3(u, \tau) \prod_{j=1}^{6} \theta_1(a_ju, \tau) \theta_2(a_ju, \tau) \theta_3(a_ju, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau) \prod_{j=1}^{6} \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \cdot \left( \frac{\sqrt{-1} \theta(u, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \right)^4 \left( \frac{\sqrt{-1} \theta(3u, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \right)^2.
\end{equation}
Note that $\theta(v, \tau)$ is an odd function of $v$ starting from $2\pi \sqrt{-1} v$ in the expansion while $\theta_i(v, \tau)$ are all even functions of $v$ for $i = 1, 2, 3$, we have
\begin{equation}
(5.17) \quad W_{672p+7,a,b}^{c_1}(H\mathcal{C}P^6(p)) = \int_{H\mathcal{C}P^6(p)} 9e^6 = 9 \neq 0.
\end{equation}
So by Theorem 5.3, we see that $H\mathcal{C}P^6(p)$ admits no effective action of a compact non-abelian Lie group preserving $J$. □

Similar calculations will show the following corollary for the case when $c_1 = \pm 5x$:

**Corollary 5.6.** Let $J$ be the stable almost complex structure on $H\mathcal{C}P^6(25h + 19)$ with $h \geq 0$ and
\begin{equation}
(2h + 19)(h + 1)h \equiv 0 \mod 31.
\end{equation}
If $c_1 = \pm 5x$, $H\mathcal{C}P^6(25h + 19)$ admits no effective positive action of a simply connected compact non-abelian Lie group preserving $J$. □
APPENDIX A. BASICS ON HOMOTOPY FIBRE SEQUENCES

For any pointed map \( f : X \rightarrow Y \), there is a canonical way to turn it into a fibration with a homotopy fibre \( F \)
\[ F \xrightarrow{f} X \xrightarrow{i} Y. \]

Continue the process for the leftmost maps, we then obtain the so-called Puppe sequence of \( f \) (e.g., See Chapter 2 of [45])
\[ \cdots \xrightarrow{\Omega_i} \Omega F \xrightarrow{\Omega i} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i} F \xrightarrow{f} X \xrightarrow{i} Y, \]

of which any three consecutive terms give a homotopy fibration. The following lemma is used frequently in this paper without further reference:

Lemma A.1 (Lemma 2.1 of [8]). A homotopy commutative diagram
\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & D \\
\end{array} \]

can be embedded in a homotopy commutative diagram
\[ \begin{array}{ccc}
Q & \xrightarrow{f} & J \\
\downarrow & & \downarrow \\
F & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & C \\
\end{array} \xrightarrow{f} \begin{array}{ccc}
B \\
\downarrow & & \downarrow \\
D \\
\end{array} \]

in which the rows and columns are fibration sequences up to homotopy.

APPENDIX B. COHOMOLOGY SUSPENSION AND TRANSGRESSION

In cohomology theory there are two classical kinds of suspensions (e.g., see Section 1.3 of [14]): Mayer-Vietoris suspension
\[ \Delta^* : \tilde{H}^n(X) \rightarrow H^{n+1}(\Sigma X), \]

and cohomology suspension
\[ \sigma^* : H^{n+1}(X) \rightarrow H^n(\Omega X). \]

The MV-suspension \( \Delta^* \) is also known as part of the axioms of general reduced cohomology theories and is always an isomorphism. The cohomology suspension is then not in general, and can be defined as
\[ \sigma^* : H^{n+1}(X) \xrightarrow{p} H^{n+1}(PX, \Omega X) \xrightarrow{\delta} H^n(\Omega X), \]

where \( p : (PX, \Omega X) \rightarrow (X, *) \) is the canonical path fibration, \( \delta \) is the connecting homomorphism in the long exact sequence of the cohomology of the pair \((PX, \Omega X)\).

There are other two useful alternative descriptions. Firstly we may identify cohomology groups with groups of homotopy classes of maps into Eilenberg-Maclane spaces via Brown representation theorem
\[ \tilde{H}^n(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)]. \]
Then the MV-suspension is just to take the adjoint map and the cohomology suspension is to take the loop functor
\[(B.5) \quad \Omega : [X, K(\mathbb{Z}, n + 1)] \rightarrow [\Omega X, K(\mathbb{Z}, n)].\]

We may also define the cohomology suspension via the evaluation map
\[(B.6) \quad \text{ev} : S^1 \times \Omega X \rightarrow X \]
defined by \(\text{ev}(t, \omega) = \omega(1)\). In this case, \(\sigma^*\) is a slant-product by the fundamental class \([S^1]\) of \(S^1\)
\[(B.7) \quad \text{ev}^*(x) = s_1 \otimes \sigma^*(x).\]

Both MV-suspension and cohomology suspension are natural and have a useful connection, that is,
\[(B.8) \quad \Delta \circ \sigma^* = \bar{\text{ev}}^* : H^{n+1}(X) \rightarrow H^{n+1}(\Sigma \Omega X),\]
where \(\bar{\text{ev}} : \Sigma \Omega X \rightarrow X\) is the (reduced) evaluation map.

We should be careful to use cohomology suspension when \(n = 0\) or \(X\) is not simply connected. In these cases, we may define the \(k\)-th component cohomology suspension of \(\sigma^*\) by
\[(B.9) \quad \sigma_k^* : H^{n+1}(X) \xrightarrow{i_k^*} H^n(\Omega \Sigma X),\]
where \(i_k : \Omega_k X \hookrightarrow \Omega X\) is the inclusion of the \(k\)-th component of \(\Omega X\) for \(k \in \pi_0(\Omega X)\). The other two equivalent definitions of \(\sigma_k^*\) can be easily obtained from (B.5) and (B.7).

\textbf{Example B.1.} Let us compute
\[\sigma^* : H^1(S^1) \rightarrow H^0(\Omega S^1),\]
which is equivalent to
\[\Omega : [S^1, S^1] \rightarrow \langle \Omega S^1, \Omega S^1 \rangle,\]
where \(\langle -, - \rangle\) denotes the set of homotopy classes of free maps. We notice that there are group isomorphisms
\[\langle \Omega S^1, \Omega S^1 \rangle \cong \text{Func}(\mathbb{Z}, \mathbb{Z}) \cong \prod_{k \in \mathbb{Z}} \text{Func}(k, \mathbb{Z}),\]
where \(\text{Func}(-, -)\) denotes the set of functions and the group structure of \(\prod_{k \in \mathbb{Z}} \text{Func}(k, \mathbb{Z})\) is defined pointwise and inherited from the targets \(\mathbb{Z}\). Further combining with Brown representation, \(H^0(\Omega_k(S^1))\) corresponds exactly to \(\text{Func}(k, \mathbb{Z})\). Since \(\Omega(\text{id}) = \text{id}\) corresponds to \(\prod_{k \in \mathbb{Z}} (\lambda_k : k \mapsto k)\), we see that
\[\sigma_k^*(s_1) = k.\]

The cohomology suspension has a “partial” inverse, known as cohomology transgression (e.g. see Section 6.2 of [35] or Section XIII.7 of [48]). For simplicity let us introduce it directly by Serre spectral sequence \((E_r^{s,t}, d_r)\) of any given orientable fibration \(F \xrightarrow{p} E \xrightarrow{p} B.\)

\textbf{Definition B.2.} The \textit{cohomology transgression} is the differential homomorphism
\[(B.10) \quad d_n : E^{0,n-1}_n \rightarrow E^{n,0}_n\]
for each \(n \geq 2\).
The cohomology transgression can be fitted into following commutative diagram

$$
\begin{array}{ccc}
H^{n-1}(E) & \xrightarrow{\tau} & H^{n-1}(F) \\
\downarrow & & \downarrow \delta \\
E_{n-1}^{0,0} & \xrightarrow{d_n} & E_{n}^{0,0} \\
\downarrow & & \downarrow \\
H^n(B,*) & \xrightarrow{j^*} & H^n(B)
\end{array}
$$

(B.11)

where the first line is part of the long exact sequence of the cohomology of the pair \((E, F)\), and the second row is exact by the definition of \(d_n\). Then it is easy to show that \(d_n\) can be described as a homomorphism

$$
\tau : H^{n-1}(F) \ni \delta^{-1}(\text{Im } p^*) \to H^n(B)/j^*(\text{Ker } p^*).
$$

(B.12)

To consider the connection to cohomology suspension, we specify the above argument to the loop fibration \(\Omega X \to PX \to X\). At this case both \(d_n\) and \(\delta\) is an isomorphism and the composition \(\delta^{-1} \circ p^* : H^n(X) \to H^{n-1}(\Omega X)\) is exactly the cohomology suspension \(\sigma^*\) by definition. Hence we see that \(\tau\) is a partial inverse of \(\sigma^*\).

**APPENDIX C. BLAKERS-MASSEY TYPE THEOREMS**

**Definition C.1.** Let \(f : X \to Y\) be a pointed map between pointed spaces \(X\) and \(Y\). Then \(f\) is \(n\)-connected if it induces isomorphisms on \(k\)-dimensional homotopy groups for \(k < n\) and an epimorphism for \(k = n\). The space \(X\) is \(m\)-connected if \(\pi_i(X) = 0\) for any \(i \leq m\). We use the convention that any space is \((-1)\)-connected.

It is then easy to check that \(f\) is \(n\)-connected is equivalent to any of the following:

1. the homotopy fibre of \(f\) is \((n-1)\)-connected;
2. the homotopy cofibre of \(f\) is \(n\)-connected;
3. \(f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z})\) is an isomorphism for each \(k < n\) and an epimorphism for \(k = n\);
4. \(f^* : H^k(Y;\mathbb{Z}) \to H^k(X;\mathbb{Z})\) is an isomorphism for each \(k < n\) and a monomorphism for \(k = n\).

**Theorem C.2** (An elegant form of Blakers-Massey Theorem; e.g., see Theorem 4.2.1 [38]). Let

\[
\begin{array}{ccc}
B & \xrightarrow{f} & A \\
\downarrow{g} & & \downarrow{h} \\
C & \xrightarrow{k} & X
\end{array}
\]

be a homotopy pushout diagram. Let

\[
\begin{array}{ccc}
Y & \xrightarrow{A} & A \\
\downarrow{f} & & \downarrow{h} \\
C & \xrightarrow{k} & X
\end{array}
\]

be the homotopy pullback diagram defining \(Y\). Suppose \(f\) is \(m\)-connected and \(g\) is \(n\)-connected. Then the induced map \(B \to Y\) is \((m+n-1)\)-connected.
Theorem C.3 (Dual Blakers-Massey Theorem of fibrations). Let
\[ F \to E \xrightarrow{p} B \]
be a fibration with the base \( B \) and the total space \( E \) path connected. Assume that \( B \) is \( m \)-connected and \( F \) is \( n \)-connected. Then there exists a partial long exact sequence
\[
0 \to H^0(B) \to H^0(E) \to H^0(F) \to H^1(B) \to \cdots \\
\cdots \to H^{m+n+1}(E) \to H^{m+n+1}(F) \to H^{m+n+2}(B) \to H^{m+n+2}(E);
\]
in other word, the fibration is a cofibration up to degree \((m + n + 2)\).

Proof. Let us define a homotopy commutative diagram of fibration
\[
\begin{array}{ccc}
Y & \xrightarrow{h} & Y \\
\downarrow{h} & & \downarrow{\ast} \\
F & \xrightarrow{i} & E \\
\downarrow{i} & & \downarrow{p} \\
Z & \xrightarrow{j} & X \\
\downarrow{j} & & \downarrow{f} \\
& B & \\
\end{array}
\]
(C.1)

where \( X \) is the homotopy cofibre of \( i \), \( Y \) and \( Z \) is the homotopy fibre of \( j \) and \( f \) respectively. In order to construct the exact sequence of the lemma, we only need to estimate the connectivity of the map \( f \), which is equivalent to that of the space \( Z \).

We then apply Theorem C.2 to the homotopy pushout and homotopy pullback diagrams
\[
\begin{array}{ccc}
F & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{j} & X, \\
\end{array} \quad \quad \begin{array}{ccc}
Y & \xrightarrow{\rho} & E \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{\iota} & X, \\
\end{array}
\]
to conclude that the induced map \( g : F \to Y \) is \((m + n)\)-connected (since \( F \to \ast \) is \((n + 1)\)-connected and \( i \) is \( m \)-connected). But we need to choose a nice \( g \). Indeed, we may apply the functor \([F, -]\) to Diagram C.1 to get a commutative diagram of exact sequences of pointed sets
\[
\begin{array}{ccc}
[F,Y] & \xrightarrow{h_*} & [F,Y] \\
\downarrow{h_*} & & \downarrow{\rho_*} \\
[F,F] & \xrightarrow{i_*} & [F,E] \\
\downarrow{0} & & \downarrow{j_*} \\
[F,Z] & \xrightarrow{f_*} & [F,X]. \\
\end{array}
\]
(C.2)

Then there exists a map \( g : F \to Y \) such that \( h \circ g = id \) and \( i = i_* h_* (g) = \rho_* (g) = \rho \circ g \).

This nice \( g \) as a section of \( h \) splits the long exact sequence of the homotopy groups of the fibration \( h \) to direct sums
\[
\pi_i (Y) \cong \pi_i (\Omega Z) \oplus \pi_i (F).
\]

Then \( g_* : \pi_i (F) \to \pi_i (Y) \) is indeed an isomorphism for each \( i \leq m + n \). Hence, \( \Omega Z \) is \((m + n)\)-connected. We should also notice that \( Z \) is 0-connected due to the commutative
implies that $\theta$ cohomology of the cofibration $F \rightarrow E \rightarrow X$ gives us the desired exact sequence in the lemma.

\[ \square \]

\section*{Appendix D. The Jacobi theta functions}

A general reference for this appendix is \cite{5}.

Let
\[ \text{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\} \]
as usual be the modular group. Let
\[ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
be the two generators of $\text{SL}_2(\mathbb{Z})$. Their actions on $\mathbb{H}$ are given by
\[ S : \tau \mapsto -\frac{1}{\tau}, \quad T : \tau \mapsto \tau + 1. \]

Let
\[ \Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2} \right\}, \]
\[ \Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}, \]
\[ \Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\} \]
be the three modular subgroups of $\text{SL}_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are $T, ST^2ST$, the generators of $\Gamma^0(2)$ are $STS, T^2STS$ and the generators of $\Gamma_\theta$ are $S, T^2$. (cf. \cite{5}).

The four Jacobi theta-functions (c.f. \cite{5}) defined by infinite multiplications are

\begin{align*}
\text{(D.1)} & \quad \theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1} \tau} q^j)(1 - e^{-2\pi \sqrt{-1} \tau} q^j)], \\
\text{(D.2)} & \quad \theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1} \tau} q^j)(1 + e^{-2\pi \sqrt{-1} \tau} q^j)], \\
\text{(D.3)} & \quad \theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1} \tau} q^{j-1/2})(1 - e^{-2\pi \sqrt{-1} \tau} q^{j-1/2})],
\end{align*}
\[ (D.4) \quad \theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1} q^j \tau})(1 + e^{-2\pi \sqrt{-1} q^j \tau})], \]

where \( q = e^{2\pi \sqrt{-1} \tau}, \tau \in \mathbb{H}. \)

They are all holomorphic functions for \((v, \tau) \in \mathbb{C} \times \mathbb{H}, \) where \( \mathbb{C} \) is the complex plane and \( \mathbb{H} \) is the upper half plane.

Let \( \theta'(0, \tau) = \frac{\partial}{\partial \tau} \theta(v, \tau) \big|_{v=0}. \) The \textit{Jacobi identity} [5],

\[ \theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau) \]

holds.

The theta functions satisfy the following transformation laws (cf. [5]),

\[ (D.5) \quad \theta(v, \tau + 1) = e^{\frac{2\pi \sqrt{-1} \tau}{\sqrt{-1}} v} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{1-\tau}} \left( \frac{\tau}{\sqrt{1-\tau}} \right)^{1/2} e^{\frac{2\pi \sqrt{-1} \tau^2 \tau}{\sqrt{-1}}} \theta(v, \tau); \]

\[ (D.6) \quad \theta_1(v, \tau + 1) = e^{\frac{2\pi \sqrt{-1} \tau}{\sqrt{-1}}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left( \frac{\tau}{\sqrt{1-\tau}} \right)^{1/2} e^{\frac{2\pi \sqrt{-1} \tau^2 \tau}{\sqrt{-1}}} \theta_2(v, \tau); \]

\[ (D.7) \quad \theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left( \frac{\tau}{\sqrt{1-\tau}} \right)^{1/2} e^{\frac{2\pi \sqrt{-1} \tau^2 \tau}{\sqrt{-1}}} \theta_1(v, \tau); \]

\[ (D.8) \quad \theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left( \frac{\tau}{\sqrt{1-\tau}} \right)^{1/2} e^{\frac{2\pi \sqrt{-1} \tau^2 \tau}{\sqrt{-1}}} \theta_1(v, \tau). \]

Let \( \Gamma \) be a subgroup of \( SL_2(\mathbb{Z}). \) A modular form over \( \Gamma \) is a holomorphic function \( f(\tau) \) on \( \mathbb{H} \cup \{ \infty \} \) such that for any

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \]

the following property holds

\[ f(g \tau) := f \left( \frac{a \tau + b}{c \tau + d} \right) = \chi(g)(c \tau + d)^k f(\tau), \]

where \( \chi : \Gamma \to \mathbb{C}^* \) is a character of \( \Gamma \) and \( k \) is called the weight of \( f. \)

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