On the spin orbit force in the Wilson loop context

V.M. Kustov *
Institute of Theoretical and Experimental Physics
117259, Moscow, Russia

Abstract

The Green function of the quark-antiquark system in the confining background field is analysed using the Feynman-Schwinger formalism. The Hamiltonian for the case of massive spinning quarks is obtained in the form containing essentially nonhermitian part. The eigenvalue problem for such type of the Hamiltonian is discussed, and it is shown that no complex eigenvalues arise. The corresponding nonunitary Foldy-Wouthuysen transformation is performed to obtain the hermitian Hamiltonian, and the standard spin-orbit interaction term is recovered.

*e-mail:kustov@vxitep.itep.ru
1 Introduction

The most natural way to discuss quark confinement at the constituent level is in terms of Wilson loop

\[ W(C) = \text{Tr} \, P \exp i g \oint A_\mu^a dz_\mu. \]  \hfill (1)

The area law asymptotics for the Wilson loop (1) averaged over the vacuum gluonic field

\[ <W(C)> \sim \exp(-\sigma S), \] \hfill (2)

leads to the linear potential between heavy quarks, while the essentially non-local and velocity- and spin-dependent interaction generated by nonperturbative QCD exhibits itself as corrections of order of \(1/m^2\) to the leading linear confinement potential term. These corrections, known as Eichten-Feinberg-Gromes relations were derived in several ways, see [1-3].

In the present paper we discuss the approach based on the most straightforward use of the Feynman-Schwinger representation for the quark Green function [3-6]. The Green function of the quark in the given external field is the product of the quadratic Dirac propagator and the linear Dirac operator:

\[ \frac{1}{m - \hat{D}} = (m + \hat{D}) \frac{1}{m^2 - \hat{D}^2} \] \hfill (3)

In the approach under discussion the Feynman-Schwinger representation is written out only for the quadratic part of the expression (3), and, as the result, the effective Hamiltonian of the \(q\bar{q}\) system contains the nonhermitian spin-dependent part. The energy eigenvalues, however, are real. Our aim is to study this unusual phenomenon, and to demonstrate that the procedure is quite legitimate at least in leading order in \(1/m^2\), and the resulting interaction is equivalent to the standard one [1, 3].

2 \(q\bar{q}\) Green function and the effective Lagrangian

First we consider the Green function of quark-antiquark system [4].

\[ G_{q\bar{q}} = \left\langle \Gamma < x \bigg| \frac{1}{m_1 - D_1} |y > \Gamma < y \bigg| \frac{1}{m_2 - D_2} |x > \right\rangle_A \] \hfill (4)
where $\hat{D}$ is Dirac operator and averaging is performed over a nonperturbative field $A$. In what follows we assume $m_2 \to \infty$. For finite $m_2$ the result is generalized without difficulties.

Each operator $\frac{1}{m^2 - D^2}$ is represented in the form $(m + \hat{D})\frac{1}{m^2 - D^2}$, to exponentiate the positively defined value of the denominator (Feynman-Schwinger method [5]). For the latter we have

$$<x|\frac{1}{m^2 - D^2}|y> = \int_0^\infty ds Dz\mu P \exp \left( - \int_0^s d\tau \left( m^2 + \frac{\dot{z}_2}{4} - g \Sigma F + igA_\mu \dot{z}_\mu \right) \right)$$

where $D_\mu = \partial_\mu - igA_\mu$, $P$ is an ordering operator and, in the Euclidean space,

$$\Sigma_{\mu\nu} = \frac{1}{4i}[\gamma_\mu, \gamma_\nu]; \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \quad (6)$$

Linear operator $\hat{D}(A)$ corresponds to $-\frac{1}{2}\dot{z}_\mu(s)\gamma_\mu$ in the functional integral, [4, 7]. Indeed, the matrix element $<x|\frac{-iD_\mu}{D^2}|y>$ can be represented as

$$\int_0^\infty ds <x|\frac{-iD_\mu}{D^2}e^{-\hat{D}^2s}|y> =$$

$$\int_0^\infty ds <x|(p_\mu - gA_\mu)e^{-\hat{p}^2s}|y>,$$

where the linear operator $-iD_\mu = p_\mu - gA_\mu$ acts on the final state $<x|$; introducing the functional integration over the momenta one has

$$\int_0^\infty ds \int Dz Dp(p_\mu(s) - gA_\mu(s))e^{-\frac{1}{2}\int^s_0 (p_\mu(s) - gA(s))^2 + ip\dot{z}}e^{-\frac{1}{2}\int^s_0 \delta_{\mu\nu}(s)\frac{1}{2}\delta p_{\mu}(\tau = s) e^{-\frac{1}{2}\int^s_0 (p_\mu - gA)^2 d\tau - \frac{1}{2}\int^s_0 g\Sigma F d\tau}}.$$

Integration by parts of this expression yields the replacement $D_\mu \to -\frac{1}{2}\dot{z}_\mu |_{\tau = s}$. So we are spared from the necessity to average preexponent factor $\hat{D}(A)$ at the cost of appearance of $\dot{z}_\mu$ in the preexponent in the functional integral.
Since this linear factor belongs to an ending point (x or y), we consider the exponent only.

Integration over $p$ gives the kinetic term $m^2 + \frac{\dot{z}^2}{4}$, Wilson loop $W(C) = \mathcal{P} \exp ig \int dz_\mu A_\mu$ and the spin-dependent term, [4].

Instead of integrating over the closed contour we integrate over the area, using the cluster expansion [4]:

$$\langle W(C) \rangle_A = \exp \sum_n \frac{1}{2n!} (ig)^n \int d\sigma_{\mu_1 \nu_1} ... d\sigma_{\mu_n \nu_n} < < F_{\mu_1 \nu_1}(1)...F_{\mu_n \nu_n}(n) >>$$

(7)

where $d\sigma_{\mu \nu} = a_{\mu \nu} d\Omega$ is the area element and we run over all the points with coordinates $w_\mu(\beta, t)$ inside the contour $w_\mu(\beta, \tau) = z_\mu(t)\beta + \bar{z}_\mu(t)(1 - \beta)$, $0 \leq \beta \leq 1$, $d\Omega = rd\beta dt = ra_{\mu \nu} d\mu d\nu$.

Since $\langle \Sigma F(x)W(C) \rangle_A = \sum_{\mu \nu} \delta a_{\mu \nu} \mid_{x}$, we have exponent $\exp\{\sum_{\mu \nu} \delta a_{\mu \nu}\}$ which is really a shift operator:

$$a_{\mu \nu} \to a_{\mu \nu} + \sum_{\mu \nu} \mid_{x}$$

(Operator $\Sigma_{\mu \nu}$ is defined on the trajectory $z$ (or $\bar{z}$)). For bilocal correlators we have Wilson loop average expression:

$$\langle W(C) \rangle = \exp\{-g^2 \int d\Omega d\Omega' < < F_{\mu \nu}(w)F_{\alpha \beta}(w') >> a_{\mu \nu}(w)a_{\alpha \beta}(w')d\Omega d\Omega'\},$$

and at $r \gg T_g, T \gg T_g$, $\langle W(C) \rangle := \exp\{-\sigma \int dt d\beta r \sqrt{a^2/2}\}$, where $\sigma$ is defined from $\sigma = C\pi g \int_0^\infty D(x^2) dx$ and the constant $C$ and function $D$ are taken from the expression for bilocal correlator [3]:

$$<< F_{\mu \nu}(w)F_{\alpha \beta}(w') >> = \text{const} \left( (\delta_{\mu \alpha}\delta_{\nu \beta} - \delta_{\mu \beta}\delta_{\nu \alpha})D(h^2) + 
+ 1/2(\partial_\alpha(h_\mu\delta_{\beta \nu} - h_\nu\delta_{\beta \mu}) + \partial_\beta(h_\nu\delta_{\alpha \mu} - h_\mu\delta_{\alpha \nu}))D_1(h^2) \right), h = w - w'$$

For spin-orbit term we obtain the expression

$$\frac{\sigma}{2\mu} \frac{\sum a}{\sqrt{a^2/2}} = \frac{i}{2\mu} \frac{\bar{a}\bar{n} + \bar{\sigma}\bar{\nu}_\perp}{\sqrt{1 + \bar{\nu}_\perp^2}}$$

(8)
in the Euclidean space (Instead of integration over \( z_0 \) we integrate over \( \mu \) in the functional integral, [6]: 
\[
\frac{1}{(2\mu(z_0))} = \frac{dz_0}{\pi}, \ v_i = \frac{dz_i}{dz_0},
\]
\( \mu \) plays the role of an effective mass.

In the Minkovsky space (8) has the form

\[
V_{SL} = \frac{\sigma}{2\mu} \frac{i\vec{\alpha}n + \vec{\sigma}v_\bot}{\sqrt{1 - v_\bot^2}}
\]

and the effective Lagrangian equals to

\[
L = -\frac{\mu}{2}(1 - \vec{v}^2) - \frac{m^2}{2\mu} - \sigma r \int d\beta \sqrt{1 - v_\bot^2} + \frac{\sigma}{2\mu} \frac{i\vec{\alpha}n + \vec{\sigma}v_\bot}{\sqrt{1 - v_\bot^2}}
\]

In the case of finite mass \( m_2 \) we get

\[
\left[v_\bot^{(1)} + v_\bot^{(2)}(1 - \beta)\right]^2
\]

instead of \( v_\bot^2 \) in the area law, [8], and should replace \( v_\bot \) by \( v_\bot^{(1,2)} \) for the spin-orbit terms.

How do these expression change if high order correlators are taken into account in (7)? At distances \( r \gg T_g \) the same expressions are valid and the only contribution from other correlators is reduced to the renormalization of the string tension \( \sigma \). Only Kronekker part of correlators (D for the bilocal one) gives the contribution to \( \sigma \) since the parts with derivatives (like \( D_1 \)) are suppressed at large distances. For small distances \( r \ll T_g \) high order correlators give corrections \( (r/T_g)^n \).

The spin-orbit expression also contains \( T_g/r \) terms (see for heavy masses [3])

### 3 Effective Hamiltonian

The expression (10) for the Lagrangian makes sense only in the context of the Feynman-Schwinger representation, because the last term in (10) contains \( \gamma \)-matrices. To pass to Hamiltonian formulation one has to define the canonical momentum as \( \vec{p} = \frac{\partial L}{\partial \vec{v}} \). Clearly, such procedure is possible only for heavy quark, when the last term in (10) is treated as perturbation; otherwise the expression for momentum would contain \( \gamma \)-matrices.

For heavy quark the area-law term becomes

\[
\sigma r \int d\beta \sqrt{1 - v_\bot^2} \approx \sigma r (1 - \frac{1}{6}v_\bot^2)
\]

integration over \( \mu \) gives \( \mu = m \), the Hamiltonian has a form
\[ H = m + \frac{p^2}{2m} + \sigma r - \frac{\sigma L^2}{6m^2r} + V_{SL}, \]  

and we are left with the nonhermitian part in the Hamiltonian. The appearance of it is not surprising, it is caused by the fact that we have exponentiated only the quadratic part of the Dirac operator, and the projective operator has not been exponentiated.

It appears, however, that the Hamiltonian (11) has real eigenvalues. There is nothing mystical in such situation, and the corresponding examples are given in the Appendix. In the Hamiltonian (11) \( V_{SL} \) is considered as the perturbation. Taking the wave function of zero approximation as that of a free massive particle \( \phi \), we get the matrix element of \( i\alpha \) in the form \( \phi^+ \frac{\vec{\sigma} \vec{L}}{mr} \phi \).

It is necessary to work carefully with new wave functions if we want to apply perturbation theory for new Hamiltonian: new metric norm \( M \) is introduced:

\[ E^{(1)} \int \psi^{+(0)} M \psi^{(0)} = \int \psi^{+(0)} HM \psi^{(0)}, \]

so that \( (HM)^+ = HM \) (the pseudohermitian condition [9]).

The Hamiltonians of such kind were considered in detail in the paper [9]. According to the theorem in [9] we can transform pseudohermitian Hamiltonian with real eigenvalues to the hermitian one, and vice versa, if we obtain a hermitian Hamiltonian by a nonunitary transformation we can expect real eigenvalues of the original Hamiltonian.

Using Foldy-Wouthuysen transformation, [8]

\[ \tilde{\psi} = \exp (iS) \psi, \quad \tilde{H} = \exp (iS)H \exp (-iS) \]

with

\[ S = \frac{1}{2m} \tilde{\alpha} \tilde{p}, \]

we get
\[
\hat{H} = m + \frac{p^2}{2m} + \sigma r - \frac{\sigma L^2}{6m^2 r} + \hat{V}_{SL}
\] (13)

\[
\hat{V}_{SL} = -\frac{\sigma}{2m} \left( \frac{\sigma \vec{L}}{mr} - \frac{\sigma \vec{L}}{2mr} + \frac{1}{2mr} \right) = \frac{\sigma}{4m^2 r} \sigma \vec{L} + \frac{\sigma}{4m^2 r},
\] (14)

which is well-known, [1-3,8]. The contribution from the electric field reduces the one from the magnetic field by the factor 2.

4 Discussion and conclusions

The nonunitary Foldy-Wouthuysen transformation which leads to equations (13), (14) should be compared with the usual one. If one first applies the usual unitary Foldy-Wouthuysen transformation for the quark in the given external field, writes out the Feynman-Schwinger representation for the transformed Green function and averages over the background field after that, as it was done in [3], no problems with nonhermitian Hamiltonian arises, and one arrives to the expressions (13), (14) straightforwardly. It should be noticed that the final Hamiltonian and the corresponding equations are not equivalent to Dirac equation for one particle in an external field. This is the case since after the averaging over nonperturbative field we get nonlocal interaction, and the real dynamical object is a string with quarks at the ends. The effective field is distributed between quarks and effective string. The spin-dependent interactions, on the contrary, are local and are defined along the contour: it is quark (antiquark) which has spin, and feels the dynamics. We have got area law and spin-orbit term (14) using successively Feynman-Schwinger formalism without introduction of the effective field for a particle. On the other hand, if one wishes to consider the problem of a Dirac particle in an external field, proceeding from the intermediate result - area law, and, in the case of a heavy quark, from the specific form of the string correction, which can be treated as external potential \( \vec{A} = 1/3 \vec{n} \times \vec{L} \), [10], doing in such a way one should obtain with necessity 1/6 in (14) instead of 1/4, [11]. At this point the question arises: why this external potential is treated as something that should be substituted into the linear Dirac equation? As the original Feynman-Schwinger representation was used only for the quadratic part of Dirac operator, this procedure is not well grounded at
We have demonstrated that it is possible to derive the spin-orbit force in the Wilson loop context, the results coincide with the well-known ones, and the Gromes relations are satisfied. We show, on the other hand, that if the particular form of Feynman-Schwinger representation is used, in which the projective operator is not included into the path integral, such procedure does not allow to go beyond the leading order in $1/m^2$.

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**Appendix. Some examples of pseudohermitian Hamiltonian operator**

The first example is Klein-Gordon equation for a free heavy-mass particle, [9]:

$$ (p_0 - p^2 - m^2)\phi = 0, $$

which can be transformed into:

$$ (\sqrt{p^2 + m^2} + E)(\sqrt{p^2 + m^2} - E)\phi = 0 $$

and after expanding the root becomes

$$ (m + \frac{p^2}{2m} + E)(m + \frac{p^2}{2m} - E)\phi - \frac{p^4}{4m^2}\phi = 0 $$

Defining $\chi = \frac{(m + \frac{p^2}{2m} - E)}{-i\frac{p^2}{2m}}\phi$, we obtain the equation $\hat{H}\psi = E\psi$ for the two-component wave function $\psi = (\phi, \chi)$, with the following Hamiltonian:

$$ \hat{H} = \begin{pmatrix} m + \frac{p^2}{2m} & 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} + i \frac{p^2}{2m} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (A.1) $$

$\hat{H}$ has real eigenvalues although it is nonhermitian operator. For such Hamiltonian another norm $M$ should be defined.
\[ \int \psi^+ \psi 
eq 1, \]
\[ \int \psi^+ M \psi = 1. \]

The second example is Klein-Gordon equation for a massive particle in an external field:

\[ \left( P_0^2 - \vec{P}^2 - m^2 \right) \phi = 0, \]

where \( P_0 = -i\partial_0 - A_0 \), \( P_i = p_i - A_i \).

For new wave function \( \tilde{\phi} = (P_0 + \Omega)\phi \), \( \Omega = \sqrt{\vec{P}^2 + m^2} \) we have the equation:

\[ \left( P_0 - \Omega + [\Omega, P_0] \frac{1}{P_0 + \Omega} \right) \tilde{\phi} = 0 \quad (A.2) \]

For heavy mass \( \Omega \approx m + \frac{\vec{P}^2}{2m} - \frac{\vec{P}^4}{8m^3} \)

and

\[ \tilde{H} \approx m + A_0 + \frac{\vec{P}^2}{2m} - \frac{\vec{P}^4}{8m^3} - \frac{1}{4m^2} [\vec{P}^2, A_0] + \frac{1}{16m^4} [\vec{P}^2, [\vec{P}^2, A_0]] + \frac{1}{8m^4} [\vec{P}^4, A_0] \quad (A.3) \]

The first order term is nonhermitian but after the nonunitary Foldy-Wouthuysen transformation \( \tilde{\phi} = \exp -iS\tilde{\phi} \), with \( S = -\frac{i}{4m^2} \vec{P}^2 \) the new Hamiltonian becomes a hermitian one up to the second order of \( 1/m \):

\[ \tilde{H} \approx m + A_0 + \frac{\vec{P}^2}{2m} - \frac{\vec{P}^4}{8m^3} + \frac{1}{32m^4} [\vec{P}^2, [\vec{P}^2, A_0]] \quad (A.4) \]

As we can see nonunitary transformation was required in order to represent the Hamiltonian in a hermitian form.

The third example is quadratic Dirac equation for a particle in the Coulomb field:

\[ (\vec{P}^2 - m^2)\psi = 0. \]

\[ (P_0^2 - \vec{p}^2 - m^2 - \Sigma F)\psi = 0. \quad (A.5) \]

Since \( P_0\psi = (E - A_0)\psi, (A_0 = \alpha/r, \alpha < \alpha_{crit}) \) we have the equation:
\[ (m + \varepsilon - A_0)^2 \psi = \left( p^2 + m^2 + \Sigma F \right)^2 \psi \]

So the final equation is \( \hat{H}\psi = E\psi \), where

\[ \hat{H} = m + A_0 + \frac{p^2}{2m} + \frac{\Sigma F}{2m} - \frac{(\varepsilon - A_0)^2}{2m}. \]

For a nonrelativistic particle it is possible to replace \( \varepsilon - A_0 \) by \( \frac{p^2}{2m} \), and

\[ \hat{H} = m + A_0 + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \frac{i\vec{\alpha}\vec{E}}{2m} \]  \hspace{1cm} (A.6)

The last term is nonhermitian although from the linear Dirac equation we have real eigenvalues for energy.

References

[1] E. Eichten, F. Feinberg, Phys. Rev. D 23 (1980), 2724.
[2] D. Gromes, Z. Phys. C.- Particles and Fields 26, 401(1984).
[3] Yu.A. Simonov, Nucl. Phys. B324(1989), 67.
[4] Yu.A. Simonov, Preprint ITEP 97-89 (unpublished).
[5] R.P. Feynman Phys.Rev. 80(1950), 440; J. Schwinger, Phys. Rev. 82(1951), 664
[6] A.Yu Dubin, A.B. Kaidalov and Yu.A. Simonov, Phys. Let. B 323(1994), 41.
[7] Yu.A. Simonov, Z. Phys. C53: 419, 1992.
[8] N. Brambilla, P. Consoli and G. M. Prosperi, IFUM 452/FT, 1993.
[9] R. A. Krajic and M. M. Nieto, Phys. Rev. D 13(1976), 2245.
[10] M. Olsson and K. Williams, Phys. Rev. D 48 (1993), 417.
[11] K. Williams, hep-ph/9607209 (1996).