Microscopic scattering theory for interacting bosons in weak random potentials

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Abstract. We develop a diagrammatic scattering theory for interacting bosons in a three-dimensional, weakly disordered potential. Based on a microscopic $N$-body scattering theory, we identify the relevant diagrams including elastic and inelastic collision processes that are sufficient to describe diffusive quantum transport. By taking advantage of the statistical properties of the weak disorder potential, we demonstrate how the $N$-body dynamics can be reduced to a nonlinear integral equation of Boltzmann type for the single-particle diffusive flux. Our theory reduces to the Gross–Pitaevskii mean field description in the limit where only elastic collisions are taken into account. However, even at weak interaction strength, inelastic collisions lead to energy redistribution between the bosons—initially prepared all at the same single-particle energy—and thereby induce thermalization of the single-particle current. In addition, we include also weak localization effects and determine the coherent corrections to the incoherent transport in terms of the coherent backscattering signal. We find that inelastic collisions lead to an enhancement of the backscattered cone in a narrow spectral window for increasing interaction strength.

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1. Introduction

In recent years, increasing interest has been devoted to the behavior of ultracold atoms in disordered potentials. Whereas the first experiments [1–3] concentrated on the realization of Anderson localization [4] in one dimension, this intriguing disorder effect—which leads to complete suppression of diffusive transport due to destructive interference—has now also been observed in three dimensions (3D) [5, 6]. The 3D case is especially interesting since it exhibits a transition from extended to localized single-particle eigenstates: in the absence of interactions, particles with low energy are localized, whereas those with higher energy (in comparison with the strength of the disorder potential) propagate diffusively in the random potential. Also in the latter case—on which we concentrate in the present paper—wave interference effects are relevant, though less pronounced: they lead to weak localization [7] (i.e. reduction of the diffusion constant instead of complete suppression of diffusion) and, associated with
that, coherent backscattering [8–10] (i.e. enhancement of backscattering), which has recently been observed also with atomic matter waves [11–13]. Beyond the scope of [11, 12] lies the investigation of the interplay between disorder and interactions, where it is not well understood, especially in the higher-dimensional case, to what extent interaction leads to a loss of coherence, i.e. to a breakdown of localization effects [14, 15]. Most theoretical works, e.g. [16–20], focus on the regime of—or close to—thermal equilibrium and examine, e.g. the effect of disorder on the condensate fraction, superfluid fraction or the sound velocity [17, 18, 20]. In this case, weak interactions can usually be treated perturbatively, e.g. by introducing Bogoliubov quasiparticles [21].

In contrast, the present paper investigates a stationary scattering setup far from thermal equilibrium. Here, bosonic atoms are continuously emitted from a coherent source (‘atom laser’ [22, 23]) and guided into the random potential until a stationary scattering state is reached. Theoretical studies of this scattering scenario so far either neglect the interparticle interaction [24, 25], treat it on the mean-field level [26–29], or apply a Hartree–Fock–Bogoliubov approach [30], which is appropriate in the case of a large condensate fraction. If all atoms enter the scattering region at fixed initial energy, the Gross–Pitaevski equation obtained within the mean-field approach predicts either a stationary regime with the same final energy for all scattered atoms, or a non-stationary, time-dependent behavior [26, 30]. In contrast, according to the microscopic scattering theory developed in the present paper for a fixed density \( \rho_0 \) of incident particles, atoms exchange energy with each other due to mutual collision events, leading to strong depletion of the condensate already for small interactions. As shown in [31], this finally leads to a stationary state with thermal Maxwell–Boltzmann distribution for those atoms which propagate deeply into the scattering region. The fact that, for a fixed density \( \rho_0 \) of particles, a stationary scattering state is always reached in the long-time limit \( t \to \infty \) can be traced back to the fact that the Schrödinger equation for the complete \( N \)-particle state is a linear equation. In contrast, if the Gross–Pitaevskii limit (\( \rho_0 \to \infty \) and, correspondingly, interaction strength \( \beta \to 0 \), see section 6 below) is taken before the limit \( t \to \infty \), a nonlinear equation for the Gross–Pitaevskii wavefunction is obtained. In other words, our results (stationary state for finite \( \rho_0 \) and \( t \to \infty \)) and the results of [26, 30] (non-stationary state for \( \rho_0 \to \infty \) and finite \( t \)) can be reconciled with each other if we assume that the time \( t \) needed to reach the stationary state becomes larger for increasing density \( \rho_0 \) of particles.

The present paper is devoted to a detailed presentation of the underlying stationary bosonic many-particle scattering theory. Starting from the \( N \)-particle Hamiltonian, we derive a nonlinear transport equation for the average particle density. Since this transport equation amounts to a stationary version of the Boltzmann equation [32], our approach is, in this respect, comparable to previous works on quantum kinetic equations [33–41]. In contrast to these works, however, the additional presence of a disorder potential (apart from the atom–atom interactions) in our setup allows us to quantify the regime of validity of the transport equation in a more rigorous way. In the regime of weak disorder, the disorder average enables us to neglect correlations between atoms induced by collisions, which in turn is the basic assumption required for reducing a many-particle problem to an effective single-particle description. Moreover—and again in contrast to the above works—we go beyond the case of purely diffusive transport, and also incorporate quantum interference corrections leading to coherent backscattering into our theory.

Correspondingly, the paper is structured as follows: in section 2, we set the stage by reviewing some important aspects of standard scattering theory for a single particle. The case of many interacting particles will be addressed in section 3: starting from the Hamiltonian...
including pairwise atom–atom interaction, we introduce a diagrammatic notation for the transition amplitudes of many particles, from which the scattered flux density can be calculated after taking the trace over the undetected particles. As we will see, this trace leads to a distinction of atom–atom collisions events into inelastic and elastic collisions, respectively, where the latter are shown to reproduce the mean-field description given by the Gross–Pitaevskii equation. Whereas the methods presented up to section 3 are generally valid for an arbitrary disorder scattering potential (as long as the disorder scattering mean free path is much larger than the range of the atom–atom interaction potential), we focus on the case of a weak random potential from section 4 on. The assumption of weak disorder \((k \ell_{\text{dis}} \gg 1)\) with wavenumber \(k\) and disorder scattering mean free path \(\ell_{\text{dis}}\) is crucial, since it allows to reduce the— in principle infinitely complicated—and hierarchy of many-particle diagrams to a tractable subclass of diagrams, i.e. ladder and crossed diagrams \([43]\), which are composed out of a small number of building blocks. As shown in section 4, the sum of all ladder diagrams amounts to a Boltzmann-like equation for diffusive transport eventually leading to complete thermalization due to inelastic atom–atom collisions in case of an infinitely large scattering region. Section 5 is devoted to the derivation of transport equations describing coherent backscattering based on crossed diagrams. Finally, in section 6 we present the results of numerical solutions of the ladder and crossed transport equations exemplifying the behavior of diffusive transport for a finitely large scattering region, and the effect of elastic and inelastic atom–atom collisions on coherent backscattering, respectively. Section 7 concludes the paper. Several technical aspects are relegated to appendices A–E.

2. Scattering theory for a single particle

We write the Hamiltonian for a single particle in the following form:

\[
\hat{H} = \hat{H}_0 + \hat{V},
\]

where \(\hat{H}_0\) denotes free propagation and \(\hat{V}\) the disorder potential. The eigenstates \(|k\rangle\) of \(\hat{H}_0\) are plane waves with wavevector \(k\)

\[
\hat{H}_0 = \int \frac{dk}{(2\pi)^3} E_k |k\rangle \langle k|,
\]

and energy

\[
E_k = k^2,
\]

where we set \(\hbar^2/(2m) \equiv 1\). The matrix elements of \(\hat{V}\) are given by the Fourier transform of the disorder potential \(V(r)\)

\[
\langle k_2 | \hat{V} | k_1 \rangle = \int dr \ V(r) \ e^{i(k_1-k_2)\cdot r}.
\]

In order to obtain a properly defined scattering scenario, we assume that \(V(r)\) is non-zero only inside a finite scattering region \(\mathcal{V}\). This allows us to define an asymptotically free initial state

\[
|i_1\rangle = \int \frac{dk}{(2\pi)^3} w(k) |k\rangle
\]

with normalized wavepacket \(w(k)\), i.e. \(\int dk |w(k)|^2 = (2\pi)^3\), which we assume to be a quasi-monochromatic wavepacket, i.e. sharply peaked around the initial wavevector \(k_i\) with energy.
\[ E_i = k_i^2, \text{ see equation (3).} \] Therefore, the spatial density resulting from the Fourier transform of \( w(k) \)

\[
|\hat{w}(r)|^2 = \left| \int \frac{dk}{(2\pi)^3} e^{ikr} w(k) \right|^2 \simeq \left| \int \frac{dk}{(2\pi)^3} w(k) \right|^2
\]

is approximately constant inside the scattering region, i.e. for \( r \in \mathcal{V} \). If the state \( \exp(-i\hat{H}_0T)|i_1\rangle \) is prepared at time \( T \to -\infty \), the wavepacket arrives at the scattering region at time \( t = 0 \), and a quasi-stationary scattering state

\[
|f_{+1}\rangle = \hat{\Omega}_+^{(V)}(E_i)|i_1\rangle
\]

is reached at that time. Here, the operator \( \hat{\Omega}_+^{(V)}(E) \) is defined by

\[
\hat{\Omega}_+^{(V)}(E) = 1 + \hat{G}_V(E)\hat{\nu},
\]

\[
\hat{G}_V(E) = \frac{1}{E - \hat{H}_0 - \hat{\nu} + i\epsilon}
\]

with infinitesimally small \( \epsilon > 0 \), denotes the (retarded) Green’s operator associated to the Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{\nu} \). The operators \( \hat{\Omega}_+^{(V)}(E) \) and \( \hat{G}_V(E) \) fulfill the following versions of the Lippmann–Schwinger equation:

\[
\hat{\Omega}_+^{(V)}(E) = 1 + \hat{G}_0(E)\hat{\nu}\hat{\Omega}_+^{(V)}(E),
\]

\[
\hat{G}_V(E) = \hat{G}_0(E) + \hat{G}_0(E)\hat{\nu}\hat{G}_V(E),
\]

where \( \hat{G}_0(E) \) denotes the vacuum Green’s operator

\[
\hat{G}_0(E) = \frac{1}{E - \hat{H}_0 + i\epsilon}.
\]

The operator \( \hat{\Omega}_+^{(V)}(E) \) is closely related to the Møller operator \( \hat{\Omega}_+^{(V)} = \lim_{T \to -\infty} \exp[i(\hat{H}_0 + \hat{\nu})T] \exp(-i\hat{H}_0T) \), as their action on an eigenstate \( |\psi\rangle \) of \( \hat{H}_0 \) with energy \( E \) is identical, i.e. \( \hat{\Omega}_+^{(V)}|\psi\rangle = \hat{\Omega}_+^{(V)}(E)|\psi\rangle \) if \( \hat{H}_0|\psi\rangle = E|\psi\rangle \). Since, in the following, we will apply \( \hat{\Omega}_+^{(V)}(E) \) only to such eigenstates—or quasi-eigenstates, as \( |i_1\rangle \) in equation (7)—we will henceforth refer also to \( \hat{\Omega}_+^{(V)}(E) \) as ‘Møller operator’. Finally, the expectation value of an arbitrary observable \( \hat{A} \) in the (quasi-)stationary scattering state results as \( \langle \hat{A} \rangle = \langle f_{+1}|\hat{A}|f_{+1}\rangle \).

Let us note that, instead of using the Møller operator, a scattering process can also be characterized by the \( S \)-matrix, \( \hat{S} = (\hat{\Omega}_+^{(V)})^\dagger \hat{\Omega}_+^{(V)} \) (where \( \hat{\Omega}_+^{(V)} \) is defined in the same way as \( \hat{\Omega}_+^{(V)} \), but with \( T \to +\infty \) instead of \( -\infty \)). We could formulate the following \( N \)-particle scattering theory equally well in terms of the \( S \)-matrix. However, since the \( S \)-matrix maps incoming onto outgoing asymptotically free states, it does not allow—in contrast to the Møller operator—to evaluate what is happening inside the scattering region, e.g. to calculate the (quasi-)stationary density or flux of particles inside \( \mathcal{V} \). For this reason, we prefer using the (quasi-)stationary scattering state \( |f_{+1}\rangle \), see equation (7) (or its \( N \)-particle counterpart \( |f_+\rangle \), see equation (22) below) in the following.
3. Scattering theory for many bosonic particles

3.1. Many-particle Hamiltonian

We add a term $\hat{U}$ to the Hamiltonian, equation (1), denoting the interaction between particles

$$\hat{H} = \hat{H}_0 + \hat{V} + \hat{U}. $$

As compared to equations (2), (4), the operators $\hat{H}_0$ and $\hat{V}$ are generalized as follows to the many-particle Hilbert space:

$$\hat{H}_0 = \int \frac{dk}{(2\pi)^3} E_k \hat{a}_k^\dagger \hat{a}_k, $$

$$\hat{V} = \int dr \ V(r) \hat{\psi}^\dagger (r) \hat{\psi} (r) $$

with creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ for particles with wavevector $k$, whereas the operators $\hat{\psi} (r) = \int dk \ exp(i k \cdot r) \hat{a}_k/(2\pi)^3$ and $\hat{\psi}^\dagger (r) = \int dk \ exp(-i k \cdot r) \hat{a}_k^\dagger/(2\pi)^3$ annihilate and create, respectively, a particle at position $r$.

In contrast to $\hat{H}_0$ and $\hat{V}$, the interaction $\hat{U}$ acts on two particles

$$\hat{U} = \frac{1}{2} \int dr_1 \ dr_2 \ U(r_1 - r_2) \hat{\psi}^\dagger (r_1) \hat{\psi}^\dagger (r_2) \hat{\psi} (r_2) \hat{\psi} (r_1) $$

with atom–atom interaction potential $U(r)$. Interactions between more than two particles will be neglected hereafter. In the following, a collision event between two particles will be described by the $T$-matrix [44]:

$$\hat{T}_U (E) = \hat{U} + \hat{U} \hat{G}_0 (E) \hat{U} + \hat{U} \hat{G}_0 (E) \hat{U} \hat{G}_0 (E) \hat{U} + \cdots. $$

According to equation (17), the matrix elements of $\hat{T}_U (E)$ with respect to two-particle states describe repeated application of the interaction $\hat{U}$ on the same pair of particles, interrupted by free propagation $\hat{G}_0 (E)$. Separating the center-of-mass from the relative coordinates, the two-body $T$ matrix fulfills momentum conservation

$$\langle k_3, k_4 | \hat{T}_U (E) | k_1, k_2 \rangle = (2\pi)^3 \delta(k_1 + k_2 - k_3 - k_4) \langle k_3, k_4 | \hat{T}_U^{(1)} (E) | k_1, k_2 \rangle, $$

where $\hat{T}_U^{(1)} (E_{12})$ is the $T$-matrix for a single particle (with reduced mass $m/2$) scattered by the potential $U(r)$ at energy $E_{12} = E - (k_1 + k_2)^2/2$, $|k_{12}\rangle = (|k_1 - k_2\rangle/2 + |(k_1 - k_2)/2\rangle)/\sqrt{2}$, and $|k_{34}\rangle = (|(k_3 - k_4)/2\rangle + |(k_3 - k_4)/2\rangle)/\sqrt{2}$. The single-particle $T$-matrix, in turn, fulfills the optical theorem [44]

$$\langle \hat{T}_U^{(1)} (E) \rangle^\dagger (\hat{G}_{0,m/2} (E) - \hat{G}_{0,m/2} (E)) \hat{T}_U^{(1)} (E) = \hat{T}_U^{(1)} (E) - \hat{T}_U^{(1)} (E) $$

expressing conservation of the particle and the energy flux (where $\hat{G}_{0,m/2}$ denotes the vacuum Green’s operator for a particle with mass $m/2$ and corresponding dispersion relation $E = 2k^2$).

Our many-particle scattering theory presented below, and in particular the transport equations in sections 4 and 5, are valid for an arbitrary interaction potential $U(r)$—as long as it is sufficiently weak in the sense specified below (mean distance between collision events larger than between disorder scattering events). Only for the numerical results presented in section 6, we will assume a short-range potential with corresponding s-wave scattering approximation, see equation (E.1).
Finally, we note that, in principle, the vacuum $T$-matrix as defined in equation (17) is modified by the presence of the disorder potential. To take this into account, the vacuum Green’s operator $\hat{G}_0(E)$ must be replaced by the disorder Green’s operator $\hat{G}_V(E)$, see equation (9), in equation (17). However, since the present paper assumes the case of a very weak disorder potential, we will neglect the disorder during each collision event in the following, and therefore use the vacuum $T$-matrix as introduced above. This approximation is valid if the disorder mean free path $\ell_{\text{dis}}$ introduced in section 4 is much larger than the range of the interaction potential $U(r)$.

3.2. Many-particle transition amplitudes

We now generalize the scattering scenario outlined in section 2 to the case of many particles. For this purpose, we assume that, both, the disorder and the particle–particle interaction are non-zero only inside a finite region $\mathcal{V}$ (which, for simplicity, we assume to be the same for $\hat{V}$ and $\hat{U}$). Note that the introduction of a finite interaction region in principle breaks translational invariance, and therefore the $\delta$-function expressing momentum conservation in equation (18) turns into an approximate $\delta$-function. Since, however, we assume the size $L$ of the scattering region $\mathcal{V}$ to be much larger than the disorder mean free path, i.e. $L \gg \ell_{\text{dis}} \gg k^{-1}$ (see below), we can safely neglect the associated small width ($\propto 1/L$) of this $\delta$-function, and still work with the $T$-matrix as given by equation (18).

Our initial state for $N$ particles reads

$$|i\rangle = \frac{1}{\sqrt{N!}} \int \frac{dk_1 \ldots dk_N}{(2\pi)^3N} w(k_1) \ldots w(k_N) |k_1, \ldots, k_N\rangle,$$

(20)

where all $N$ atoms are described by the same quasi-monochromatic single-atom wavepacket $w(k)$ as given in equation (5). Note that since, both, the Hamiltonian and the finally measured observable (i.e. the flux density, see section 3.3 below) conserve the particle number, our assumption of an initial state with fixed particle number $N$ imposes no loss of generality—the same measurement statistics would be obtained for an initial state corresponding to a superposition of different particle numbers (as long as the particle number is sufficiently well peaked around the mean value $\langle N \rangle$), which is the case, e.g. for a coherent state with $\langle N \rangle \gg 1$). The factor $1/\sqrt{N!}$ in equation (20) arises from the indistinguishability of bosonic particles. The corresponding density of particles reads

$$\rho_0 = \langle i | \hat{\psi}^{\dagger}(r) \hat{\psi}(r) | i \rangle \simeq N \left| \int \frac{dk}{(2\pi)^3} w(k) \right|^2.$$

(21)

As mentioned above, this density is approximately uniform within the whole scattering region $\mathcal{V}$ for a wavepacket sharply peaked around the initial wavevector $k_i$. Since, in this quasi-monochromatic limit, the density, equation (6), for $N = 1$ approaches zero (since the wavepacket is spread over an increasingly large region of space), the number $N$ of particles correspondingly must tend to infinity in order to obtain a finite density $\rho_0$.

The Møller operator, which yields the quasi-stationary $N$-particle scattering state

$$|f_i\rangle = \hat{\Omega}_+(N E_i)|i\rangle$$

(22)

is defined in the same way as above, see equations (8), (9) but with $\hat{V} + \hat{U}$ instead of $\hat{V}$. It therefore fulfills the Lippmann–Schwinger equation

$$\hat{\Omega}_+(E) = 1 + \hat{G}_0(E)(\hat{V} + \hat{U})\hat{\Omega}_+(E),$$

(23)
which, using equations (8), (11), can be rewritten as
\[ \hat{\Omega}_+(E) = \hat{\Omega}^{(V)}_+(E) + \hat{G}_V(E)\hat{U} \hat{\Omega}_+(E). \] (24)

Iteration of equation (24) yields an expansion in powers of \( \hat{U} \):
\[ \hat{\Omega}_+(E) = \hat{\Omega}^{(V)}_+(E) + \hat{G}_V(E)\hat{U} \hat{\Omega}^{(V)}_+(E) + \hat{G}_V(E)\hat{U} \hat{\Omega}_+(E)\hat{U} \hat{\Omega}^{(V)}_+(E) + \cdots. \] (25)

Remember that, according to equation (16), each operator \( \hat{U} \) annihilates and creates two particles. In contrast, the Green’s operator \( \hat{G}_V \) and the Møller operator \( \hat{\Omega}^{(V)}_+ \) act on all \( N \) particles. However, since these operators describe non-interacting particles, they can be factorized into single-particle operators. As an example, we give here the factorization formulae for the case \( N = 2 \):
\[
\langle k_3, k_4 | \hat{\Omega}^{(V)}_+(E_{k_1} + E_{k_2}) | k_1, k_2 \rangle = \langle k_3 | \hat{\Omega}^{(V)}_+(E_{k_1}) | k_1 \rangle \langle k_4 | \hat{\Omega}^{(V)}_+(E_{k_2}) | k_2 \rangle + \langle k_4 | \hat{\Omega}^{(V)}_+(E_{k_1}) | k_1 \rangle \langle k_3 | \hat{\Omega}^{(V)}_+(E_{k_2}) | k_2 \rangle
\] (26)

and
\[
\langle k_3, k_4 | \hat{G}_V(E) | k_1, k_2 \rangle = \frac{1}{(-2\pi i)} \int_{-\infty}^{\infty} dE' \left[ \langle k_3 | \hat{G}_V(E') | k_1 \rangle \times \langle k_4 | \hat{G}_V(E - E') | k_2 \rangle + \langle k_4 | \hat{G}_V(E') | k_1 \rangle \langle k_3 | \hat{G}_V(E - E') | k_2 \rangle \right].
\] (27)

As mentioned above, the energy argument of our Møller operator, equation (8), is always fixed to the energy of the state it acts on. In contrast, Green’s operators also act on states with different energies. Hence, the energy \( E \) of a two-particle Green’s operator has to be distributed among two one-particle Green’s operators according to equation (27).

Using the above factorization formulae—and analogous ones for \( N > 2 \) (see appendix A)—we obtain well-defined paths for individual particles between the two-particle interaction events \( \hat{U} \). Repeated interaction between the same pair of particles is included in the \( T \)-matrix, see equation (17) (and the discussion at the end of section 3.1). We hence replace two-particle matrix elements of \( \hat{U} \) by matrix elements of \( \hat{T}_V(E) \) (with appropriately defined two-particle energy \( E \), see below) in equation (25), and thereby obtain a sequence of collision events between different pairs of particles. An example of a three-particle scattering process is demonstrated in figure 1. As shown in appendix A, this diagram gives rise to the following contribution to the transition amplitude:
\[
\langle k_4, k_5, k_6 | \hat{\Omega}^{(figure \ 1)}_+(3E_i) | k_1, k_2, k_3 \rangle = \int_{-\infty}^{\infty} dE_4 dE_5 \int \frac{d\mathbf{p}_1 \ldots d\mathbf{p}_8}{(2\pi)^{24}} \langle k_6 | \hat{G}_V(3E_i - E_4 - E_5) | k_5 \rangle \langle k_5 | \hat{G}_V(E_5) | \mathbf{p}_5 \rangle \langle \mathbf{p}_5 | \hat{T}_U(3E_i - E_4) | \mathbf{p}_1 \rangle \langle \mathbf{p}_1 | \hat{G}_V(2E_i - E_4) | \mathbf{p}_7 \rangle \langle \mathbf{p}_7 | \hat{T}_U(2E_i) | \mathbf{p}_2 \rangle \langle \mathbf{p}_2 | \hat{\Omega}^{(V)}_+(E_i) | k_2 \rangle \langle k_2 | \hat{\Omega}^{(V)}_+(E_i) | k_3 \rangle
\] (28)
with \( E_{k_1} \sim E_{k_2} \sim E_{k_3} \sim E_i \), according to our above assumption of a quasi-monochromatic wavepacket.

In general, the rules for constructing an arbitrary \( N \)-particle scattering amplitude for a given diagram are as follows: (i) apply the disorder Møller operator \( \hat{\Omega}^{(V)}_+(E_i) \), see equation (8), to each initial single-particle state \( |k_1\rangle, \ldots, |k_N\rangle \). The energy associated to each initial particle is given by \( E_i \). (ii) Integrate over all intermediate particles \( \{\mathbf{p}_1, \ldots, \mathbf{p}_8\} \) in figure 1. (iii) Write down the
corresponding two-body $T$-matrix element, see equation (18), for any collision between two particles. The energy argument of $\hat{T}_U$ is given by the sum of the two incoming single-particle energies. (iv) For each $\hat{T}_U(E)$, write down an integral $\int_{-\infty}^{\infty} dE'/(-2\pi i)$ which determines the energy arguments of the Green’s operators $\hat{G}_V(E')$ and $\hat{G}_V(E - E')$, see equation (27), for the two particles after the collision. (v) These two particles may then collide with other particles, and so on.

The total transition amplitude defining the stationary scattering state $|f_+\rangle$, see equation (22), is then obtained by summing the contributions from all possible different diagrams. For example, in addition to the diagram shown in figure 1, eight more diagrams obtained by exchanging the initial and/or final wavevectors $(k_1, k_2, k_3)$ and $(k_4, k_5, k_6)$ also contribute to $|f_+\rangle$.

3.3. Scattered flux

As the finally measured quantity, we determine the expectation value of the flux density operator

$$\hat{J}(r) = 2\text{Im}(\hat{\psi}^\dagger(r) \nabla \hat{\psi}(r)) = \int \frac{dk dk'}{(2\pi)^6} \left( \frac{k + k'}{2} \right) e^{-i(k-k') \cdot r} \hat{a}^\dagger_k \hat{a}_{k'}$$

(29)

with respect to the stationary scattering state $|f_+\rangle$. Since $\hat{J}(r)$ is a one-particle operator, this implies a partial trace of the density matrix $|f_+\rangle\langle f_+|$ over $N - 1$ undetected particles:

$$\mathbf{J}(r) = \langle f_+ | \hat{J}(r) | f_+ \rangle$$

$$= \frac{N}{N!} \int \frac{dk dk'}{(2\pi)^6} \left( \frac{k + k'}{2} \right) e^{-i(k-k') \cdot r}$$

$$\times \int \frac{dk_1 \ldots dk_{N-1}}{(2\pi)^3(N-1)} \langle k_1, \ldots, k_{N-1}, k' | f_+ \rangle \langle f_+ | k_1, \ldots, k_{N-1}, k \rangle.$$  

(30)
Placing the detector at position \( \mathbf{R} \) in the far field of the scattering region (i.e., \( |\mathbf{R}| \gg |\mathbf{r}| \) for \( \mathbf{r} \in \mathcal{V} \)), the scattered flux is finally expressed as a dimensionless quantity (the so-called ‘bistatic coefficient’ [45]):

\[
\gamma(\hat{k}_d) = \lim_{R \to \infty} \left( \frac{\mathbf{R} \cdot \mathbf{J}(\mathbf{R})}{A \rho_0 \sqrt{E_i}} \right),
\]

(31)

normalized with respect to the incident flux \( A \rho_0 \sqrt{E_i} \), where \( A \) denotes the transverse area (with respect to the incident wave) of the scattering volume \( \mathcal{V} \), and \( \hat{k}_d = \mathbf{R}/|\mathbf{R}| \) is the direction of the detected particle’s wavevector. The limit \( R \to \infty \) is to be taken after the quasi-stationary limit \( N \to \infty \), see the discussion after equation (21). Apart from the total flux density \( \gamma(\hat{k}_d) \), we will also be interested in the spectral density \( \gamma_E(\hat{k}_d) \), i.e., the flux of particles scattered into direction \( \hat{k}_d \) with energy \( E \), which is given by

\[
\gamma_E(\hat{k}_d) = \lim_{R \to \infty} \int \frac{d\mathbf{k} d\mathbf{k}'}{16 \pi^5} (\mathbf{R} \cdot \mathbf{K}) e^{-i(\mathbf{k} - \mathbf{k}')} R \langle f_+ | \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} | f_+ \rangle \frac{1}{A \rho_0 \sqrt{E_i}/R} \delta(E - K^2),
\]

(32)

where \( \mathbf{K} = (\mathbf{k} + \mathbf{k}')/2 \), such that \( \int_0^\infty dE \gamma_E(\hat{k}_d) = \gamma(\hat{k}_d) \).

The factor \( 1/N! \) in equation (30) arises from the indistinguishability of the bosonic particles. It turns out, however, that this factor—together with the factors \( 1/\sqrt{N!} \) in equation (20)—is exactly counterbalanced once we sum the amplitudes of all processes where the initial and/or final particles are exchanged. In total, we get the same result as if the particles were distinguishable. This equivalence is generally valid if all particles are prepared in the same initial state, and if the Hamiltonian is symmetric under exchange of particles [46].

Remember that the number \( N \) of particles tends to infinity in the quasi-stationary limit, whereas, in case of a finite scattering region, only a finite number of particles will eventually interact with the finally detected particle. The evolution of the remaining particles (which do not interact with the detected particle) does not influence the result of the partial trace, equation (30). This follows from the factorization property, equation (26), and the left-unitarity, \((\hat{\Omega}_+)^\dagger \hat{\Omega}_+ = 1\) of the Möller operator. Consequently, in order to calculate the detection signal, we may disregard all scattering processes concerning those particles which do not interact (neither in \( |f_+\rangle \) nor in \( \langle f_+| \) with the detected particle. (The presence of these particles only leads to a prefactor giving rise to the correct dependence of a given scattering diagram on the density \( \rho_0 \), see the discussion at the end of appendix B.)

### 3.4. Trace over undetected particles

According to the recipe given above, the flux density for an arbitrary \( N \)-particle scattering process is obtained as follows: take a diagram contributing to \( |f_+\rangle \), a conjugate diagram contributing to \( \langle f_+| \), apply the observable \( \hat{J}(\mathbf{r}) \) to one of the final particles of both diagrams, and trace over the undetected particles. An example for two particles is shown in figure 2(a) (left-hand side). Since both conjugate diagrams (solid and dashed lines, respectively) exhibit a collision event, which redistributes the energy among the two particles according to the factorization formula, equation (27), the energy of the detected particle is different from the initial energy \( E_i \). For this reason, we call this scattering process ‘inelastic’. This means that the energies of the single particles change—although their sum remains conserved. In contrast, figure 2(b) shows an elastic scattering process. Here, the conjugate diagram (dashed lines) on the
Figure 2. Graphical equations exemplifying the trace over the undetected particles. Arrows and squares refer to single-particle propagators and two-body $T$-matrices, as defined in figure 1. Dashed arrows correspond to adjoint propagators $\hat{G}_V^\dagger$ and $(\hat{\Omega}_V^{(V)})^\dagger$. The half circle symbol denotes the detector, whereas the dots on the left-hand side of both equations represent the trace over the undetected particle. (a) Inelastic scattering of two particles. On the right-hand side, the trace has been performed using equation (33). This results in the dashed-solid double arrow representing the spectral function $[\hat{G}_V^\dagger(E) - \hat{G}_V(E)]/(2\pi i)$. The energy of the detected particle is $2E_i - E$. (b) Elastic scattering of two particles. The trace is performed using equation (34). The resulting diagram on the right-hand side is equivalent to a diagram obtained from the Gross–Pitaevski equation. Since, in contrast to (a), the conjugate particles (dashed lines) do not undergo a collision, the energy of the detected particle is unchanged ($E_i$).

left-hand side does not exhibit a collision event. As shown below, this implies that the energies of both particles remain unchanged.

We will now demonstrate how to perform the trace over the undetected particle for inelastic and elastic collisions, respectively. The result is represented on the right-hand side of figure 2.

3.4.1. Inelastic collisions. The complete expression for the inelastic scattering diagram, figure 2(a), is given in equation (B.1). Focusing on those terms which are relevant for the trace over the undetected particle, this trace can be written in the following general form:

$$\int_{-\infty}^{\infty} \frac{dE}{(2\pi i)^2} \int \frac{d\mathbf{k}}{(2\pi)^3} A^{(0)}(-E')\hat{G}_V^\dagger(E')|\mathbf{k}\rangle\langle \mathbf{k}|\hat{G}_V(E) A^{(0)}(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi i} A^{(0)}(-E)(\hat{G}_V^\dagger(E) - \hat{G}_V(E)) A^{(0)}(-E).$$

On the left-hand side of equation (33), $\mathbf{k}$ corresponds to the final state of the undetected particle, whereas $\hat{G}_V^\dagger(E)$ and $\hat{G}_V^\dagger(E')$ refer to the (single-particle) Green’s operators expressing propagation from the collision event to the final state. According to the rules given in section 3.2, the collision events are associated with integrals $\int dE/(-2\pi i)$ and $\int dE'/(2\pi i)$ which determine the energies of the undetected ($E$ and $E'$) and the detected particle ($2E_i - E$.)
Figure 3. Elastic scattering diagram where the conjugate undetected amplitude originates from a previous elastic scattering event. The sum of the two processes shown on the left-hand side reproduces the Gross–Pitaevskii diagram (cf [29]) on the right-hand side.

and $2E_i - E')$. The terms $A^{(l)}(-E')$ and $A^{(r)}(-E)$ denote all the remaining parts of the scattering diagram where the energy argument enters with a negative sign, see equation (B.1). Their precise form is irrelevant for equation (33)—except for the fact that $A^{(l)}(-E')$ is a complex analytic function with poles only in the lower half of the complex plane, and $A^{(r)}(-E)$ in the upper half. Due to the negative sign, this is in contrast to the respective contributions $\hat{G}_V(E')$ and $\hat{T}_U(E')$, as well as $\hat{G}_V(E)$ and $\hat{T}_U(E)$, which exhibit poles only in the upper (or lower) half plane.

Under these conditions—which do not only hold for the example shown in figure 2, but for all other inelastic scattering diagrams we will encounter in the following—the result of the trace is given on the right-hand side of equation (33). This general formula is proven in appendix C. Graphically, the result is depicted on the right-hand side of figure 2(a). As a consequence, the energies $E$ and $E'$ are set equal to each other, and the two conjugate Green’s functions $\hat{G}_V(E)$ and $\hat{G}_V(E)$ are replaced by their difference $[\hat{G}_V(E') - \hat{G}_V(E)]/(2\pi i)$ (which is also known as the ‘spectral function’, since the imaginary part of the Green’s function determines the density of states [47]).

3.4.2. Elastic collisions. In a similar way, the trace in the elastic scattering diagram, see equation (B.2), is performed as follows:

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi i} \int \frac{dk}{(2\pi)^3} \langle k_i | (\hat{\Omega}_+^{(V)}(E_i))^\dagger \langle k | \hat{G}_V(E) A(-E) = \langle k_i | (\hat{\Omega}_+^{(V)}(E_i))^\dagger A(-E_i),$$

see appendix C, where $A(-E)$ is a complex analytic function of $E$ with poles in the upper half plane. According to equation (34)—which is graphically depicted in figure 2(b)—the outgoing solid arrow emitted from the two-body collision event is replaced by an incoming dashed arrow with energy $E_i$. We note that precisely this diagram is the only interaction contribution generated by the Gross–Pitaevskii equation [29]. Thereby, we have shown that our $N$-particle scattering theory reproduces the Gross–Pitaevskii equation if only elastic scattering is taken into account.

In equation (34), the conjugate undetected particle originates directly from the initial state $\langle k_i |$ propagated in the disorder potential through the Møller operator $(\hat{\Omega}_+^{(V)}(E_i))^\dagger$. The formula can be generalized, however, to the case where the undetected particle undergoes previous collisions with other particles before colliding with the detected particle. An example is depicted in figure 3. Also in this case, the corresponding Gross–Pitaevskii diagram is reproduced (i.e. the outgoing solid arrow is replaced by an incoming dashed arrow). In a similar way, also
the inelastic trace formula, equation (33), is valid in the case where the undetected particle undergoes further collisions with other particles before the trace is taken—provided that none of these other particles, in turn, collides with the detected particle which, as discussed in section 4, is the case for a weak disorder potential and weak interactions. This allows us to take the trace over the undetected particles directly after their last collision with the detected particle—without being obliged to follow their further evolution before finally leaving the scattering region.

4. Incoherent transport

4.1. Ladder diagrams

The \( N \)-particle scattering formalism outlined above is valid for an arbitrary potential \( V(\mathbf{r}) \) with mean free path much larger than the range of the atom–atom interaction potential, as discussed at the end of section 3.1. Now, we consider \( V(\mathbf{r}) \) as a random potential, and calculate the corresponding average density matrix \( \langle f_+| f_+ \rangle \). For this purpose, we assume a Gaussian white noise potential, specified by the mean value \( \langle V(\mathbf{r}) \rangle = 0 \) and the two-point correlation function \( V(\mathbf{r}_1)V(\mathbf{r}_2) = 4\pi \ell_{\text{dis}} \delta(\mathbf{r}_1 - \mathbf{r}_2) \).

Furthermore, the disorder potential is assumed to be weak, i.e. \( \sqrt{E} \ell_{\text{dis}} \gg 1 \) for all relevant single-particle energies \( E \), see equation (3). Initially, this is the case if \( \sqrt{E_i} \ell_{\text{dis}} \gg 1 \). Due to inelastic collisions, the energies will change, but, as we will see later, their distribution will still be centered close to \( E_i \), with only a negligible fraction of particles that reach single-particle energies \( E \approx 0 \).

For the case of a single particle, the disorder average in the limit \( k \ell_{\text{dis}} \gg 1 \) is well known [48, 49]: firstly, the vacuum Green’s function \( \hat{G}_0 \), see equation (12), is replaced by the average single-particle Green’s function

\[
\langle \mathbf{k}' | \hat{G}(E) | \mathbf{k} \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') G_E(k),
\]

with

\[
G_E(k) = \frac{1}{k_E^2 - \mathbf{k}^2},
\]

where \( k_E = \sqrt{E} + i/(2 \ell_{\text{dis}}) \). In position representation, this leads to an exponential decay of the average density with \( 2\text{Im} k_E = 1/\ell_{\text{dis}} \) as the decay constant, see equation (39) below. This establishes \( \ell_{\text{dis}} \) as the mean free path, i.e. the average distance between subsequent disorder scattering events. Secondly, when calculating the average density matrix \( \langle f_+| f_+ \rangle \), and representing both \( |f_+\rangle \) and \( \langle f_+| \) as a sum of diagrams, only those combination of diagrams survive where both \( |f_+\rangle \) and \( \langle f_+| \) undergo the same sequence of disorder scattering events. Here, a disorder scattering event is induced by the correlation function, equation (35), where \( V(\mathbf{r}_1) \) acts in \( |f_+\rangle \) and \( V(\mathbf{r}_2) \) in \( \langle f_+| \) (or vice versa—whereas correlators with \( V(\mathbf{r}_1) \) and \( V(\mathbf{r}_2) \) both acting in \( |f_+\rangle \) or both in \( \langle f_+| \) are accounted for by the average Green’s function (37) [48]). These combinations of diagrams give rise to so-called ladder diagrams for the average density [49].

We now apply the same procedure to the \( N \)-particle scattering processes presented in section 3.2. First, we take a diagram contributing to \( |f_+\rangle \) and another one (called ‘conjugate diagram’ in the following) contributing to \( \langle f_+| \). Then, we replace all vacuum Green’s functions...
by average Green’s functions and correlate, using equation (35), each disorder scattering event with another one in the conjugate diagram such that both conjugate diagrams undergo the same sequence of disorder scattering events. Finally, we choose one of the final particles as detected particle, and trace over the remaining $N - 1$ particles, see equation (30). An example is shown in figure 4. In this figure, the trace over undetected particles is performed as soon as the corresponding particle (solid line) is re-united with its conjugate counterpart (dashed line) at a disorder scattering event. It turns out that the same result is obtained if the trace is performed before taking the disorder average according to figure 2. This leads to disorder-averaged trace formulae as depicted in figure 5, which we will use in the following to evaluate the trace over the undetected particles.
Among the $N$-particle ladder diagrams thus constructed, we neglect all those where two particles which interacted once meet again. Under the assumption that at least one disorder scattering event occurs between two collision events, this approximation is equivalent to the neglect of recurrent scattering [50] for a single particle, which, alike the neglect of non-ladder diagrams, is valid for $k\ell_{\text{dis}} \gg 1$. It allows us to trace away the undetected particles directly after their interaction with the detected particle. The assumption that at least one disorder scattering event occurs between two collision events is required in order to exclude complicated cascades of collision processes, possibly involving a large number of particles colliding and re-colliding with each other, between two disorder scattering events. This assumption is justified if $\ell_{\text{int}}^{(a)} \gg \ell_{\text{dis}}$ and $\ell_{\text{int}}^{(el)} \gg \ell_{\text{dis}}$, where $\ell_{\text{int}}^{(a)}$ and $\ell_{\text{int}}^{(el)}$ denote the length scales associated with inelastic or elastic collisions, respectively. The latter are given by

$$\ell_{\text{int}}^{(a)} = \frac{1}{\sigma \rho_0}, \quad \ell_{\text{int}}^{(el)} = \frac{\sqrt{E_i}}{8\pi |a(0)| \rho_0}$$  \hfill (38)

with $\sigma$ denoting the scattering cross section and $a(0)$ the forward scattering amplitude of the atom–atom interaction potential $U(\mathbf{r})$. For $s$-wave scattering, $\sigma = 8\pi a_s^2$ and $a(0) = a_s$, see equation (71).

4.2. Building blocks

The trace over the undetected particle allows us to decompose every ladder diagram (like the one shown in figure 4) into independent building blocks, where, as mentioned above, at least one disorder scattering event occurs after each elastic or inelastic collision. These building blocks are shown in figure 6. The first one, figure 6(a), represents a single average propagation step of a single particle with energy $E$ and corresponding wavevector $\mathbf{k}$ in the disordered potential from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$P_E(\mathbf{r}_1, \mathbf{r}_2) = \frac{4\pi}{\ell_{\text{dis}}} \left| \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} G_E(k) \right|^2 = \frac{e^{-|\mathbf{r}_1 - \mathbf{r}_2|/\ell_{\text{dis}}}}{4\pi \ell_{\text{dis}} |\mathbf{r}_1 - \mathbf{r}_2|^2},$$  \hfill (39)

for $E \geq 0$. Note that, for a white noise potential as defined in equation (35), the mean free path $\ell_{\text{dis}}$ is independent of $E$ [53]. For $E < 0$, the propagation is exponentially suppressed; in this case, equation (39) is multiplied by an additional factor $\exp(-2|\mathbf{r}_1 - \mathbf{r}_2|/\sqrt{|E|})$. Since the typical distance between two scattering events is given by the mean free path $\ell_{\text{dis}}$, we can neglect the occurrence of negative energies if $\sqrt{|E|} \ell_{\text{dis}} \gg 1$.

Figure 6. The three building blocks from which all ladder diagrams (see figure 4) are constructed. (a) Single-particle propagation in the disorder potential, see equation (39). (b) Elastic two-particle collision $g_{E_1, E_2}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, see equation (40). (c) Inelastic two-particle collision $f_{E_1, E_2, E_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, see equation (41).
The second building block, figure 6(b), represents an elastic collision event, where the energies of both particles are unchanged:

\[
g_{E_1,E_2}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = 2 \left( \frac{4\pi}{\ell_{\text{dis}}} \right)^2 \Re \left\{ \frac{1}{2} \int \frac{d\mathbf{k}_1 \ldots d\mathbf{k}_5}{(2\pi)^5} e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot \mathbf{r}_1 + (\mathbf{k}_2-\mathbf{k}_3)\cdot \mathbf{r}_2 + (\mathbf{k}_3-\mathbf{k}_4)\cdot \mathbf{r}_3} \langle \mathbf{k}_3, \mathbf{k}_4 | \tilde{T}_U(E_1 + E_2) | \mathbf{k}_1, \mathbf{k}_2 \rangle \times G_{E_1}(k_1)G_{E_2}(k_2)G_{E_3}(k_3)G_{E_4}^*(k_4)G_{E_5}^*(k_5) \right\}. \tag{40}
\]

The trace over the undetected particle was performed according to figure 5(b), giving rise to an average Green’s function \(G_{E_i}^*(k_4)\). For reasons of clarity, the wavevectors \(k_1, \ldots, k_5\) are not explicitly shown in figure 6. They can, however, be easily deduced from the phase factors \(\exp(\pm i \mathbf{k} \cdot \mathbf{r})\) describing annihilation or creation of a particle \(\mathbf{k}\) due to disorder scattering at \(\mathbf{r}\), see equations (4), (15), with the help of following rule: outgoing solid (dashed) arrows always contribute with negative (positive) sign, the opposite holds for incoming arrows. For example, \(k_5\)—with phase factor \(\exp[i \mathbf{k}_5 \cdot (\mathbf{r}_2 - \mathbf{r}_3)]\) in equation (40)—is associated with the dashed arrow pointing from \(\mathbf{r}_2\) to \(\mathbf{r}_3\) in figure 6(b).

The first factor 2 in equation (40) originates from the fact that the solid and dashed incoming amplitudes can be grouped together in two different ways. It can be shown that this accounts for fluctuations of the atomic density inside the disordered slab [28]. The factor 1/2 in front of the integral originates from the indistinguishability of particles, see the discussion at the end of appendix B.

Note that, since elastic collisions as described by figure 6(b) originate from interference between colliding and non-colliding particles, these events do not change the direction of propagation of the respective particles, but only contribute to forward scattering. At larger interaction strength, also scattering into other directions may be induced by elastic collisions (see, e.g. [59, chapter V]). However, these processes are negligible in the regime of weak interactions (i.e. \(\ell_{\text{int}} \gg \ell_{\text{dis}}\), see above) that we consider.

Finally, the third building block, figure 6(c), amounts to an inelastic collision event, where the energies of two particles \(E_1\) and \(E_2\) change to \(E_3\) and \(E_4 = E_1 + E_2 - E_3\):

\[
f_{E_1,E_2,E_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = 2 \left( \frac{4\pi}{\ell_{\text{dis}}} \right)^2 \int \frac{d\mathbf{k}_4}{(2\pi)^3} \frac{G_{E_1,E_2-E_3}^*(k_4) - G_{E_1+E_2-E_3}(k_4)}{2\pi i} e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot \mathbf{r}_1 + (\mathbf{k}_2-\mathbf{k}_3)\cdot \mathbf{r}_2 + (\mathbf{k}_3-\mathbf{k}_4)\cdot \mathbf{r}_3} \langle \mathbf{k}_3, \mathbf{k}_4 | \tilde{T}_U(E_1 + E_2) | \mathbf{k}_1, \mathbf{k}_2 \rangle \times G_{E_1}(k_1)G_{E_2}(k_2)G_{E_3}(k_3) \right\}^2, \tag{41}
\]

where figure 5(a) was used for the trace over the undetected particle.

4.3. Transport equation

The outgoing arrows of each building block may now be attached to the incoming arrows of the next building block, and so on. The sum of ladder diagrams resulting from all combinations of

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these building blocks is expressed by the following nonlinear integral equation:

\[
I_E(r) = I_0(r)\delta(E - E_i) + \int_V dr' P_E(r, r') I_E(r')
+ \int_0^\infty dE' \int_V dr' \int_V dr'' [g_{E', E}(r', r'', r) I_E(r'')]
+ \int_0^\infty dE'' f_{E', E''}(r', r'', r) I_{E''}(r'),
\]

where

\[
I_0(r) = \rho_0 e^{-z_{r, k_i}/\ell_{\text{dis}}}
\]

represents the incoming wave propagating to \(r\) without being scattered. Corresponingly, \(z_{r, k_i}\) denotes the distance from the surface of the scattering region \(V\) to \(r\) along a straight line parallel to the direction \(k_i\) of the incident wavepacket. The quantity \(I_E(r)\) can be interpreted as the average density of particles with energy \(E\) at position \(r\) (at least in the case \(\ell_{\text{dis}}\sqrt{E} \gg 1\) of weak disorder where the spectral function \(\{G_k(k') - G_k(k)\}/(2\pi i) \to \delta(E - k^2)\) approaches a \(\delta\)-function, such that a particle with wavevector \(k\) possesses a well defined energy). In particular, \(I(r) = \int dE I_E(r) = \langle \hat{\rho}(r)|\rho_i\rangle\) gives the disorder-averaged expectation value of the single-particle density operator \(\hat{\rho}(r) = \hat{\psi}^†(r)\hat{\psi}(r)\) with respect to the quasi-stationary scattering state \(|\rho_i\rangle\). From \(I_E(r)\), the diffuse flux \(\gamma^{(L)}(k_d)\) — i.e. the disorder average of \(\gamma(k_d)\), see equation (31), in ladder approximation—of particles scattered into direction \(k_d\) is finally obtained as

\[
\gamma^{(L)}(k_d) = \int_0^\infty dE \gamma^{(L)}_E(k_d),
\]

where

\[
\gamma^{(L)}_E(k_d) = \int_V \frac{dr}{A\ell_{\text{dis}}} e^{-z_{r, k_d}/\ell_{\text{dis}}} \sqrt{E E_i / \rho_0}
\]

denotes the ladder component of the average spectral flux density, i.e. the flux of particles scattered into direction \(k_d\) with energy \(E\), see equation (32). In equation (45), \(z_{r, k_d}\) denotes the distance from \(r\) to the surface of the scattering region in direction \(k_d\). Note that, in the far-field limit, only positive energies contribute to the scattered flux, equation (44). Within the scattering medium, negative energies are neglected in the transport equation (42) due to the exponential suppression mentioned after equation (39).

Since we assume that disorder scattering events—represented by the term \(P_E\) in equation (42)—are much more frequent than collision events, we may neglect the spatial dependence of the collision terms in equation (42) and approximate them by \(\delta\)-functions: \(g_{E, E}(r', r'', r) \simeq \delta(r' - r)\delta(r'' - r) g_{E, E}\) and \(f_{E, E', E}(r', r'', r) \simeq \delta(r' - r)\delta(r'' - r) f_{E, E', E}\), where

\[
g_{E, E} = \int dr' dr'' g_{E, E}(r', r'', r),
\]

\[
f_{E, E', E} = \int dr' dr'' f_{E, E', E}(r', r'', r).
\]
The transport equation (42) then reduces to

\[ I_E(r) = I_0(r)\delta(E - E_1) + \int_V dV' P_E(r, r') I_E(r') \]

\[ + \int_0^\infty dE' \left[ g_{E',E} I_E(r) + \int_0^\infty dE'' f_{E',E'',E} I_{E''}(r) \right] I_{E'}(r). \]  

(48)

As shown in [51], this equation can also be derived from the nonlinear Boltzmann transport equation. Due to the above collision approximation, the spatial transport of particles in equation (48) is solely governed by the propagation \( P \) in the disorder potential, whereas the collision terms \( g \) and \( f \) lead to a redistribution of energies. As compared to equations (40), (41), these terms simplify as follows:

\[ g_{E_1,E_2} = \left( \frac{4\pi}{\ell_{dis}} \right)^2 \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} |G_{E_1}(k_1)|^2 \frac{2 \Re \{ \langle k_{12} | \hat{T}_U^{(1)}(E_{12}) | k_{12} \rangle \times |G_{E_2}(k_2)|^2 G_{E_2}(k_2) \}}{2\pi i} \]

\[ f_{E_1,E_2,E_3} = 2 \left( \frac{4\pi}{\ell_{dis}} \right)^2 \frac{1}{4} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \frac{dk_3}{(2\pi)^d} \frac{G_{E_1+E_2-E_3}^*(k_4) - G_{E_1+E_2-E_3}(k_4)}{2\pi i} \]

\[ \times \left| \langle k_{34} | \hat{T}_U^{(1)}(E_{12}) | k_{12} \rangle \right|^2 \frac{|G_{E_1}(k_1)|^2 |G_{E_2}(k_2)|^2 |G_{E_3}(k_3)|^2}{2\pi i} \]

(49)

(50)

with \( k_4 = k_1 + k_2 - k_3 \), \( E_{12} = E_1 + E_2 - E_{k_1+k_2}/2 \), and \( |k_{12} \rangle, |k_{34} \rangle \) as defined after equation (18).

4.4. Thermalization

For a given form of the two-body \( T \)-matrix (e.g. s-wave scattering, see below), we can now calculate the collision terms \( g \) and \( f \) according to equations (49), (50), and then numerically solve the transport equation (48) by iteration. Before presenting the corresponding numerical results in section 6, however, we will discuss, in the remainder of this section, some general properties of \( g \) and \( f \), which, as shown below, lead to thermalization of the single-particle energies for an infinite system.

As shown in appendix D, the collision terms fulfill the following relations:

\[ \sqrt{E_2} g_{E_1,E_2} = -\int_0^\infty dE \sqrt{E} f_{E_1,E_2,E}, \]

(51)

\[ (E_1 + E_2)\sqrt{E_2} g_{E_1,E_2} = -\int_0^\infty dE \sqrt{E} 2E f_{E_1,E_2,E}. \]

(52)

Both relations follow from the fact that the \( T \)-matrix associated to the atom–atom interaction potential \( U(r) \) fulfills the optical theorem, equation (19), and express conservation of the particle and the energy flux, respectively. Moreover, from equation (41), one can show that

\[ \frac{f_{E_1,E_2}}{\sqrt{E_1 + E_2 - E_3}} = \frac{f_{E_1+E_2-E_3,E_1}}{\sqrt{E_3}}. \]

(53)

This equation expresses microscopic reversibility of the collision dynamics: given two particles with energy \( E_1 \) and \( E_2 \), the collision process \( E_1, E_2 \rightarrow E_3, E_4 \) occurs with the same probability as the reverse process \( E_3, E_4 \rightarrow E_1, E_2 \) given two particles with energy \( E_3 \) and \( E_4 \). The square roots in the denominators of (53) result from the traces over the undetected particle with energy \( E_4 = E_1 + E_2 - E_3 \) (left-hand side) or \( E_2 \) (right-hand side), respectively.
Using equations (51), (52)—and the fact that the linear propagator \( P_E(r, r') = P(r, r') \) is independent of \( E \)—it follows that the quantities \( J(r) = \int_0^\infty dE \ J_E(r) \) and \( K(r) = \int_0^\infty dE \ K_E(r) \), with

\[
J_E(r) = \sqrt{E} I_E(r), \quad K_E(r) = E \sqrt{E} I_E(r)
\]

(54)
corresponding to the particle and energy flux, respectively, both fulfill the same linear transport equation

\[
J(r) = J_0(r) + \int_V dr' P(r, r') J(r'),
\]

(55)
and

\[
K(r) = K_0(r) + \int_V dr' P(r, r') K(r'),
\]

(56)
where the source terms \( J_0(r) = \sqrt{E_i I_0(r)} \) and \( K_0(r) = E_i \sqrt{E_i} I_0(r) \) differ only by the constant factor \( E_i \). Due to equations (51), (52), the collision terms drop out from equation (48) when integrating over \( E \). Since the linear transport equation fulfills flux conservation, this, in turn, implies that, both, particle and energy flux are conserved. Furthermore, since \( K_0(r) = E_i J_0(r) \), the same relation holds for the solutions of the linear equations (55), (56)

\[
K(r) = E_i J(r).
\]

(57)
After these preparatory steps, we can now look for a solution of the transport equation (48) in case of a semi-infinite medium. Far away from its boundary, \( I_E(r) = I_E \) should become independent of \( r \), and \( I_0(r) \), equation (43), tends to zero. Hence, the constant solution \( I_E \) must fulfill

\[
\int_0^\infty dE' \left[ g_{E', E} I_E + \int_0^\infty dE'' f_{E', E'', E} I_{E''} \right] I_{E'} = 0.
\]

(58)
Using equations (51), (53), one can show that \( I_E = \sqrt{E} e^{-\gamma E} \) fulfills equation (58) for \( \gamma > 0 \). The constant \( \gamma \), in turn, is determined by equation (57) as \( \gamma = 2/E_i \). Hence the normalized particle flux distribution is given by

\[
\frac{J_E(r)}{J(r)} = \frac{4E}{E_i^2} e^{-2E/E_i}.
\]

(59)
This corresponds to a Maxwell–Boltzmann distribution the temperature of which is determined by the initial energy \( (E_i = k_B T/2) \). Thereby, we have demonstrated thermalization in case of a semi-infinite medium. In section 6, we will study the transport behavior predicted by equation (48) for a finite medium, and see how the thermal distribution is approached during propagation through a finite slab.

5. Coherent transport

5.1. Crossed diagrams

Before turning to the numerical results, however, we will extend the general formalism of section 4 in order to calculate the leading interference correction (in the weak disorder parameter \( 1/(k \ell_{\text{dis}}) \)) to the average scattered flux density. This correction is described by crossed
Figure 7. Example of a crossed diagram contributing to coherent backscattering in the case of three interacting particles. It is obtained from the ladder diagram shown in figure 4 by reversing the direction of propagation of the dashed propagators along the path \( r_1 \rightarrow r_3 \rightarrow r_4 \rightarrow r_7 \rightarrow r_8 \rightarrow r_9 \).

diagrams [52], which are obtained from the ladder diagrams by reversing the direction of propagation of a single amplitude. Starting from the ladder diagram shown in figure 4, we can construct, for example, the crossed diagram shown in figure 7. It amounts to an interference between two amplitudes where the detected atom is emitted from \( r_9 \) and \( r_1 \), respectively. For a given wavevector \( k_d \) of the detected atom, the backscattering angle \( \theta \) is defined by

\[
\cos \theta = -k_i \cdot k_d.
\]

Since annihilation and creation of atoms with wavevector \( k \) by the disorder potential at position \( r \) are associated with factors \( e^{\pm i k \cdot r} \), respectively, see equations (4), (15), this leads to a phase factor \( e^{i q (r_1 - r_9)} \) with respect to the ladder diagram, figure 4, where

\[
q = k_i + k_d.
\]  (60)

Since \( r_1 \) and \( r_9 \) refer to randomly chosen positions of scattering events, this phase factor vanishes on average unless \( q \simeq 0 \iff k_i \simeq -k_d \), corresponding to exact backscattering \( (\theta = 0) \). Therefore, this effect of interference between reversed amplitudes is called ‘coherent backscattering’ [8–10]. More precisely, one can show that the angular width of the coherent backscattering interference peak is approximately given by \( \Delta \theta \simeq 1/(k \ell_{\text{dis}}) \) [53]. For a single particle, the height of this peak at \( \theta = 0 \) equals the incoherent background as described by the ladder diagrams (except for single scattering which only contributes to the background), what amounts to an enhancement of the backscattered flux by a factor 2. We will show below how this enhancement factor changes as a consequence of elastic and inelastic atom–atom collisions.

For this purpose, we will derive a transport equation for the ‘crossed density’ which describes a pair of amplitudes propagating in opposite directions. In figure 7, the corresponding crossed scattering path is given by \( r_1 \rightarrow r_3 \rightarrow r_4 \rightarrow r_7 \rightarrow r_8 \rightarrow r_9 \) (where we define the direction of the path to be fixed by the solid arrows, whereas the dashed arrows propagate in the opposite sense). The remaining parts (\( r_2 \) and \( r_6 \)) correspond to ladder diagrams already treated in section 4. Due to energy conservation, the energies \( E \) and \( \tilde{E} \) associated with the two counterpropagating conjugate amplitudes always fulfill the following relation:

\[
\tilde{E} = E_i + E_d - E.
\]  (61)
5.2. Crossed building blocks

This leads us to the first crossed building block:

\[ P^{(C)}_E (\mathbf{r}_1, \mathbf{r}_2) = \frac{4\pi}{\ell_{\text{dis}}} \int \frac{d\mathbf{k}_1}{(2\pi)^6} \frac{d\mathbf{k}_2}{(2\pi)^6} \ e^{-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)} G_E(\mathbf{k}_1) G^*_E(\mathbf{k}_2) \]

\[ = \frac{e^{i|\mathbf{r}_1 - \mathbf{r}_2|/\ell_{\text{dis}}}}{4\pi \ell_{\text{dis}} |\mathbf{r}_1 - \mathbf{r}_2|^2}, \tag{62} \]

describing single-particle propagation with different energies \( E \) (wavevector \( \mathbf{k}_1 \)) and \( \tilde{E} \) (wavevector \( \mathbf{k}_2 \)) for the conjugate amplitudes, see figure 8(a). For \( E = \tilde{E} \), i.e. \( E = (E_1 + E_d)/2 \) due to equation (61), it reduces to the ladder propagator \( P_E \), see equation (39).

The following building block, figure 8(b)

\[ g^{(C)}_{E_1, E_2} = \left( \frac{4\pi}{\ell_{\text{dis}}} \right)^2 \int \frac{d\mathbf{k}_1}{(2\pi)^6} \frac{d\mathbf{k}_2}{(2\pi)^6} |G_{E_1}(\mathbf{k}_1)|^2 G_{E_2}(\mathbf{k}_2) G^{\ast}_{\tilde{E}_2}(\mathbf{k}_2) \]

\[ \times [G_{E_2}(\mathbf{k}_2)(\mathbf{k}_{12}[\hat{T}_U^{(1)}(E_{12})]\mathbf{k}_{12}) + G_{E_2}^{\ast}(\mathbf{k}_2)(\mathbf{k}_{12}[\hat{T}_U^{(1)}(E'_{12})]^\dagger)(\mathbf{k}_{12})], \tag{63} \]

where \( E_{12} = E_1 + E_2 - E_{k_1 + k_2}/2 \), \( E'_{12} = E_1 + \tilde{E}_2 - E_{k_1 + k_2}/2 \), and \( |\mathbf{k}_{12}\rangle \) as defined after equation (18), represents the crossed counterpart of the elastic collision \( g_{E_1, E_2} \), see figure 6(b). Again, it reduces to the corresponding crossed ladder term, equation (49), for \( \tilde{E}_2 = E_2 \). The wavevector \( \mathbf{k}_1 \) in equation (63) is associated with Green’s functions emitted from position \( \mathbf{r}_1 \), and \( \mathbf{k}_2 \) with those propagating between \( \mathbf{r}_2 \) and \( \mathbf{r}_3 \).
Similarly, figure 8(c)

\[ f_{E_1,E_2,E_3}^{(C)} = 4 \left( \frac{4\pi}{E_{\text{dis}}} \right)^2 \frac{1}{4} \int \frac{dk_1 dk_2 dk_3}{(2\pi)^6} \frac{G_{E_1+E_2-E_3}^*(k_4) - G_{E_1+E_2-E_3}(k_4)}{2\pi i} \]

\[ \times \langle k_{34} \hat{T}_{U}^{(1)}(E_{12}) | k_{12} \rangle \langle k_{13,+} \rangle \hat{T}_{U}^{(1)}(E_{13}^\prime) \rangle | k_{24,+} \rangle \]

\[ \times |G_{E_1}(k_1)|^2 G_{E_2}(k_2) G_{E_3}(k_3) G_{E_1}^*(k_3), \]  

(64)

where \( k_4 = k_1 + k_2 - k_3, \ E_{12} = E_1 + E_2 - E_{k_1+k_3}/2, \ E_{13} = E_1 + E_{k_1-k_3}/2, \) \( |k_{13,+}\rangle = (|(k_1+k_3)/2) + ((-k_1-k_3)/2))/\sqrt{2}, \)

\( |k_{24,+}\rangle = (|(k_2+k_4)/2) + ((-k_2-k_4)/2))/\sqrt{2}, \) and \( |k_{12}, k_{34}\rangle \) as defined after equation (50), represents the crossed counterpart of inelastic collision \( f_{E_1,E_2,E_3}^{(C)} \), see figure 6(c). It reduces to two times the corresponding ladder term, equation (50), for \( \bar{E}_2 = \bar{E}_3 = E_2 = E_3 \). The factor 2 originates from the fact that we can reverse the single-particle amplitudes of the ladder building block, figure 6(c), also in a different way (with the outgoing dashed arrow pointing to \( r_1 \) instead of \( r_2 \)) giving rise to an identical term. The wavevectors \( k_1, k_2 \) and \( k_3 \) in equation (63) are associated with Green’s functions emitted from (or pointing toward) positions \( r_1, r_2 \) and \( r_3 \), respectively.

Similarly, there also exist two different possibilities for reversing the ladder building block \( g_{E_1,E_2}^{(C)} \), figure 6(b). Apart from \( g_{E_1,E_2}^{(C)} \), see equation (63) and figure 8(b), this gives rise to a new building block, figure 8(d)

\[ h_{E_1,E_2}^{(C)} = \left( \frac{4\pi}{E_{\text{dis}}} \right)^2 \int \frac{dk_1 dk_2}{(2\pi)^6} G_{E_2}(k_2) G_{E_1}^*(k_1) \langle k_{11} \rangle \hat{T}_{U}^{(1)}(E_1+E_d)|k_{11}\rangle \times |G_{E_1}(k_1)|^2 G_{E_1}(k_1), \]

(65)

where \( |k_{11}\rangle = (|k_1| + |k_1|)/\sqrt{2} \). The corresponding conjugate diagram, see figure 8(e), is given by \( (h_{E_1,E_2}^{(C)})^\ast \). Note that the two colliding particles exhibit opposite wavevectors \( (k_1 \) and \(-k_1\)), and therefore the energy of the collision event is fixed to \( \bar{E}_1 + E_1 = E_1 + E_d \) due to equation (61).

Each of the above crossed building blocks exhibits an incoming and an outgoing crossed density (defined by the direction of the solid arrow, as mentioned above). Additionally, the two-particle building blocks, figures 8(b)–(e), exhibit an incoming ladder density. The latter is given by the solution \( I_E(r) \) of the ladder transport equation (48).

5.3. Crossed transport equation

Propagation of the crossed density can now be described by an integral equation accounting for all possible combinations of the above crossed building blocks (see figure 8). An example is displayed in figure 9(a). Here, the outgoing crossed density of the building block shown in figure 8(d) serves as the incoming crossed density for the building block shown in figure 8(e). The resulting combination, figure 9(a), exhibits the following remarkable property: if we look at the outgoing arrows (solid arrow pointing to \( r_2 \), dashed arrow pointing to \( r_1 \)) corresponding to the detected particle, we see that the detected particle exhibits no collision with the other particles involved in figure 9(a). The evolution of these undetected particles therefore has no impact on the detected particle and, consequently, as discussed at the end of section 3.2, the process shown in figure 9(a) may be disregarded when calculating the detection signal. The same remains true if—instead of attaching figure 8(e) directly to figures 8(d)—an arbitrary sequence.
of the remaining crossed building blocks, figures 8(a), (b) or (c), is inserted in between. In contrast, for any other combination of building blocks, e.g. figure 9(b), all involved particles turn out to be connected to each other (through collision events or partial traces), thus contributing to the propagation of the crossed density.

In order to exclude combinations of the former type from the transport equation, we split the crossed density into two parts, i.e. \( C_E(r) = C_E^{(1)}(r) + C_E^{(2)}(r) \). All combinations of the building blocks figures 8(a)–(c) and (e) are contained in \( C_E^{(1)}(r) \), and the remaining ones, i.e. those involving figure 8(d), in \( C_E^{(2)}(r) \). According to the rules mentioned above, the building block figure 8(e) is excluded from the transport equation for \( C_E^{(2)}(r) \). In total, the transport equations therefore read as follows:

\[
C_E^{(1)}(r) = I_0^{(C)}(r) \delta(E - E_i) + \int \mathcal{V} \, dE' \left[ I_E^{(C)} P_E^{(C)}(r, r') C_E^{(1)}(r') \right] \\
+ \int_0^\infty dE' \left[ g_{E', E}^{(C)} C_E^{(1)}(r) + \int_0^{E_i + E_d} dE'' f_{E', E''}^{(C)} C_E^{(1)}(r) \right] I_E^{(r)} \\
+ \int_0^{E_i + E_d} dE' \left[ h_{E', E}^{(C)} *) I_E^{(r)} C_E^{(1)}(r) \right] \tag{66}
\]

and

\[
C_E^{(2)}(r) = \int \mathcal{V} \, dE' P_E^{(C)}(r, r') C_E^{(2)}(r') + \int_0^\infty dE' \left[ g_{E', E}^{(C)} C_E^{(2)}(r) + \int_0^{E_i + E_d} dE'' f_{E', E''}^{(C)} C_E^{(2)}(r) \right] I_E^{(r)} \\
+ \int_0^{E_i + E_d} dE' \left[ h_{E', E}^{(C)} *) I_E^{(r)} (C_E^{(1)}(r''') + C_E^{(2)}(r)) \right] \tag{67}
\]

In equation (66), the incoming crossed density is given by

\[
I_0^{(C)}(r) = \rho_0 e^{iq \cdot (r - (r_c + r_e + r_i))} / (2\pi\hbar), \tag{68}
\]
where \( \mathbf{q} = \mathbf{k}_i + \mathbf{k}_d \), see equation (60), where the wavevector of the detected particle is determined by the energy \( E_d \) and the position \( \mathbf{R} \) of the detector (in the far field) as \( \mathbf{k}_d = \sqrt{E_d} \mathbf{R}/R \), and \( z_{r,-\mathbf{k}_i} \) (or \( z_{r,\mathbf{k}_i} \)) corresponds to the distance an incoming (or outgoing) particle travels inside the scattering region, as defined in equations (43), (45). Finally, the coherently backscattered flux density results as

\[
\gamma^{(C)}(\hat{\mathbf{k}}_d) = \int_0^\infty dE_d \gamma^{(C)}_{E_d}(\hat{\mathbf{k}}_d)
\]

with associated spectral density

\[
\gamma^{(C)}_{E_d}(\hat{\mathbf{k}}_d) = \int \frac{d\mathbf{r}}{A_{\text{dis}} \rho_0} \sqrt{\frac{E_d}{E_1}} e^{-i\mathbf{q} \cdot \mathbf{r} / (E_d - E_1)^{1/2} \ell_{\text{dis}} \delta + (C_{E_d}^{(1)}(\mathbf{r}) + C_{E_d}^{(2)}(\mathbf{r})) - \delta(E_d - E_1)I_0^{(C)}(\mathbf{r})},
\]

where the last term accounts for single scattering. The total average flux measured by a detector placed in direction \( \hat{\mathbf{k}}_d \), see equation (31), then corresponds to the sum of the ladder and the crossed component, \( \gamma(\hat{\mathbf{k}}_d) = \gamma^{(1)}(\hat{\mathbf{k}}_d) + \gamma^{(C)}(\hat{\mathbf{k}}_d) \).

6. Numerical solutions of the transport equations

After having developed the general scattering formalism valid for a large class of interaction potentials \( U(\mathbf{r}) \) (see the discussion at the end of section 3.1) and arbitrary shapes of the scattering region \( V \), we will now focus on the case of a short-range potential \( U(\mathbf{r}) \) and a slab geometry for \( V \). As explained in appendix E, the \( T \)-matrix is then described by a single parameter—the s-wave scattering length \( a_s \). The corresponding average distances between inelastic and elastic collision events, respectively, are given by

\[
\ell_{\text{int}}^{(\text{in})} = \frac{1}{8\pi a_s^2 \rho_0}, \quad \ell_{\text{int}}^{(\text{el})} = \frac{\sqrt{E_i}}{8\pi a_s \rho_0}.
\]

In the following, we measure the inelastic interaction strength in terms of the ratio between \( \ell_{\text{dis}} \) and \( \ell_{\text{int}}^{(\text{in})} \):

\[
\alpha = \frac{\ell_{\text{dis}}}{\ell_{\text{int}}^{(\text{in})}} = 8\pi a_s^2 \ell_{\text{dis}} \rho_0,
\]

which, as explained in section 4.1, should fulfill the condition \( \alpha \ll 1 \), and, due to equation (71), is proportional to \( a_s^2 \). Indeed, we see from equations (E.2), (E.3) that the ladder collision terms \( g \) and \( f \) both depend on \( \alpha \). The terms proportional to \( a_s \) in \( g \) drop out as a consequence of flux conservation, see equation (51). The same is not true for the crossed collision terms \( g^{(C)} \) and \( h^{(C)} \), see equations (E.4), (E.5), which depend on a second parameter proportional to \( a_s \):

\[
\beta = \frac{\ell_{\text{dis}}}{\ell_{\text{int}}^{(\text{el})}} = \frac{8\pi a_s \ell_{\text{dis}} \rho_0}{\sqrt{E_i}} = \frac{\alpha}{\sqrt{E_i} a_s}.
\]

Since \( \sqrt{E_i} a_s \ll 1 \) for s-wave scattering, it follows that \( \beta \gg \alpha \). The parameter \( \beta \) can also be expressed in terms of the healing length \( \xi = (8\pi \rho_0 a_s)^{-1/2} \) [55], i.e. \( \beta = \ell_{\text{dis}}/(\sqrt{E_i} \xi^2) \), or in...
terms of the interaction parameter \( g = 8\pi a_s \) appearing in the Gross–Pitaevskii equation, i.e. \( \beta = g\rho_0 \ell_{\text{dis}} / \sqrt{E} \). The Gross–Pitaevskii equation is valid in the limit \( a_s \to 0 \) and \( \rho_0 \to \infty \) such that \( a_s \rho_0 = \text{const} \). Since \( \alpha \to 0 \) in this limit, our previously derived transport equations for nonlinear coherent backscattering based on the Gross–Pitaevskii equation [27–29] must be recovered from equations (48), (66) and (67) for \( \alpha = 0 \), as it is indeed the case if we insert the s-wave expressions, equations (E.2)–(E.6), evaluated at \( \alpha = 0 \).

Concerning the geometry of the scattering medium, we choose a slab confined between two planes, \( z = 0 \) and \( L \), respectively, with perpendicular incident wavevector, i.e. \( \mathbf{k}_i = (0, 0, k_i) \). The thickness of the slab in units of the disorder mean free path defines its optical thickness \( b = L / \ell_{\text{dis}} \). The slab geometry is very convenient from a numerical point of view, since the integration over \( x \) and \( y \) can be performed analytically in equations (48), (66), (67), such that the resulting transport equations only depend on \( z \) [54]. Moreover, due to rotational symmetry around the \( z \)-axis, the backscattered flux \( g(\mathbf{k}_d) = g(\theta) \) depends only on the backscattering angle \( \theta \) defined by \( \mathbf{k}_i \cdot \mathbf{k}_d = -\cos \theta \), and the distances appearing in equations (43), (45), (68) and (70) simplify to \( z_{r,-k_i} = z \) and \( z_{r,k_d} = z / \cos \theta \), respectively. Finally, the integration over the scattering volume \( V \) in equations (45), (70) reduces to \( \int_0^L \text{d} r / A \to \int_0^L \text{d} z \). The one-dimensional versions of the transport equations (48), (66), (67) can now be solved numerically, e.g. by iteration.

### 6.1. Density inside the slab

Figure 10(a) shows the resulting flux density \( J(z) = \int_0^\infty \text{d} E \sqrt{E} J_E(z) \), see equation (54), as a function of the position \( z \) inside the slab, for weak interactions \( \alpha = 1/100 \), cf equation (72), and optical thickness \( b = 40 \). As already proven after equation (55), the total flux \( J(z) \) (black long-dashed line) equals the linear flux (red solid) as obtained from equation (48) with \( \alpha = 0 \). In contrast to the linear case, however, the flux \( J(z) \) splits into an elastic (green dashed) and an inelastic component (blue dotted), defined by \( J_E(z) = J_E^{(\text{el})}(z) \delta(E - E_i) + J_E^{(\text{inel})}(z) \). We see that, in spite of the weakness of the interaction (\( \alpha = 1/100 \)), the inelastic component dominates, especially deep inside the slab. This can be explained by the large number (\( \approx b^2 \)) of scattering events required to traverse a slab with thickness \( b \). The expected number of two-body collision events thus results as approximately \( \alpha b^2 = 16 \). Let us note that the inelastic component of the flux is associated with a non-condensed fraction of atoms, since an \( N \)-fold product of a single-particle state (as required from the formal definition of a condensate via the stationary one-particle density matrix [56]) with fixed total energy implies fixed energies also for the individual particles.

The normalized energy distribution \( J_E^{(\text{inel})}(z) \) of the inelastic component is shown in figure 10(b), for different positions \( z \) inside the slab. We see that, far inside the slab, i.e. at \( z = 10\ell_{\text{dis}} \) (green dashed) and \( z = 40\ell_{\text{dis}} \) (black long-dashed), the energy distribution approaches a Maxwell–Boltzmann distribution \( J_E^{(\text{MB})} / J = 4E \exp(-2E/E_i)/E_i^3 \) (red solid), see equation (59). Thereby, we confirm the analytical result derived in section 4.4 for an infinite slab. In contrast, at the beginning of the slab (blue dotted), the distribution is not yet thermalized, and lies between the Maxwell–Boltzmann distribution and the distribution \( \sqrt{E} f_{E_i,E_i,E} / (-\sqrt{E_i} \delta E_i) \) obtained after a single inelastic collision event (thin black line) and normalized according to equation (51).
Figure 10. (a) Different components of the average flux density $J(z)$, plotted as a function of position $z$ in the slab, for weak interaction $\alpha = 1/100$, and thickness $b = 40$. The linear flux density (red solid line) coincides with the total flux density for the case of many particles (black long-dashed). The latter splits into an elastic (green dashed) and inelastic component (blue dotted). In spite of the weak interaction, the transport is dominated by inelastically scattered particles, especially deep inside the slab. (b) Normalized energy distribution $J_{E}^{\text{inel}}(z)$ of inelastically scattered atoms for different positions $z = 0$ (blue dotted), $z = L/4$ (green dashed) and $z = L$ (black long-dashed) in the slab, and otherwise the same parameters as in (a). The thin black line displays $\sqrt{E} f_{E_{i},E_{i}}/(-\sqrt{E_{i}} g_{E_{i},E})$ (see equation (51) for the normalization) according to equation (E.3), i.e. the distribution after a single inelastic scattering event. The kink of this distribution is recovered at the beginning of the slab (i.e. for $z = 0$). Deep inside the slab (i.e. for $z = L/4$ and $z = L$), the spectrum collapses onto a thermal Maxwell–Boltzmann distribution with average energy $E_{av} = E_{i}$ (red solid).

6.2. Backscattered flux outside the slab

Figure 11(a) shows the background and interference contributions to the backscattered flux in exact backscattering direction $\theta = 0$ for $\ell_{\text{dis}} \sqrt{E_{i}} = 10$. Since, in general, the backscattered flux is dominated by scattering paths which do not penetrate very deeply into the slab, we restrict ourselves to the case of a moderate slab thickness $b = 4$ (as compared to $b = 40$ in figure 10). As already known from previous work on nonlinear coherent backscattering in the purely elastic Gross–Pitaevskii limit [27–29], the backscattered flux $\gamma^{(C)}(0)$ for $\alpha = 0$ (red line) exhibits a transition from constructive to destructive interference for increasing $\beta$. This transition can be explained by the fact that the nonlinearity effectively introduces dephasing between two reversed scattering paths [27–29]. For larger $\beta$, the phase difference accumulated along the
shortest scattering paths, i.e. those exhibiting only a few, but at least two scattering events (since, as mentioned above, single scattering does not contribute), may exceed π/2, and thus lead to destructive interference. This behavior changes if a finite amount of inelastic scattering, α = β/10 corresponding to \( a_s\sqrt{E_i} = 1/10 \), see equation (73), is taken into account (blue line): At first, the interference drops faster than for α = 0, since inelastic scattering changes the frequency and thus leads to an additional dephasing mechanism. At larger values β, however, the decrease of the backscattered flux is slowed down as compared to the purely elastic case, such that a transition to destructive interference is not observed in figure 11(a).

**Figure 11.** (a) Background and interference contributions \( \gamma^{(L)}(0) \) and \( \gamma^{(C)}(0) \) to the scattered flux in exact backscattering direction (θ = 0) for a slab with thickness \( b = 4 \) and \( \ell_{\text{dis}}\sqrt{E_i} = 10 \) as a function of the crossed collision strength \( \beta \), see equation (73). The Gross–Pitaevskii equation (α = 0, red line) predicts a crossover from constructive (\( \gamma^{(C)}(0) > 0 \)) to destructive interference (\( \gamma^{(C)}(0) < 0 \)) at \( \beta \approx 0.13 \). In presence of inelastic collisions (α = β/10, blue solid line), the decrease of \( \gamma^{(C)}(0) \) is initially faster, but for β > 0.04 slower than in the case α = 0, due to the steadily rising inelastic component \( \gamma^{(C,\text{inel}})(0) \) (blue dotted line). For comparison, the total background contribution \( \gamma^{(L)}(0) \) (green solid), which is independent of α and β, and its inelastic component \( \gamma^{(L,\text{inel}})(0) \) (for α = β/10, green dashed) are also shown. (b) Spectral distributions \( \gamma^{(C,\text{inel})}_{E_{\text{d}}}(0) \) (solid lines) and \( \gamma^{(L,\text{inel})}_{E_{\text{d}}}(0) \) (dashed lines) for the same parameters as in a) and β = 0.02 (red), 0.08 (green) and 0.2 (blue). In a narrow spectral region around \( E_{\text{d}} \approx E_i \) (but slightly shifted with respect to \( E_i \)), the interference contribution \( \gamma^{(C,\text{inel})}_{E_{\text{d}}}(0) \) exceeds the background \( \gamma^{(L,\text{inel})}_{E_{\text{d}}}(0) \), corresponding to a coherent backscattering enhancement factor larger than two.
Figure 12. Angular profile $\gamma^{(C)}(\theta)$ of the coherently backscattered flux for a slab with thickness $b = 4$ and $\ell_{\text{dis}} \sqrt{E_i} = 10$. The interaction strength increases from top to bottom from $\beta = 0$ (red line/open circles), 0.02, 0.04, 0.08 and 0.12–0.2 (green line/filled circles). The lines display the prediction of the Gross–Pitaevskii equation ($\alpha = 0$) whereas the symbols display the effect of inelastic collisions ($\alpha = \beta/10$). For small interactions, the flux is reduced mainly in exact backscattering direction (close to $\theta = 0$), but almost unchanged at larger angles ($\theta = 0.2$). For larger interactions, the Gross–Pitaevskii equation predicts a local minimum at $\theta = 0$ which is suppressed by inelastic collisions.

This behavior can be explained by examining the spectral distribution $\gamma_{E_d}^{(C, \text{inel})}$ of the inelastic interference component as a function of the detected frequency $E_d$ in figure 11(b). Since the amount of inelastic scattering is governed by $\alpha = \beta/10$, both $\gamma_{E_d}^{(L, \text{inel})}(0)$ (dashed lines) and $\gamma_{E_d}^{(C, \text{inel})}(0)$ (solid lines) increase as a function of $\beta$ (green, blue and red lines corresponding to $\beta = 0.02, 0.08$ and 0.2, respectively). The interference spectra $\gamma_{E_d}^{(C, \text{inel})}(0)$ exhibit narrow peaks close to the initial energy, $E_d \simeq E_i$. The relative width of these peaks approximately equals $\Delta E_d / E_i \simeq 1/(\ell_{\text{dis}} \sqrt{E_d}) = 1/10$, which can be understood as a consequence of frequency-induced dephasing given by equation (62) with $E \neq \tilde{E} = E_i + E_d - E$. Less intuitive is the fact that the maxima of the peaks are slightly shifted with respect to $E_i$. This could possibly be related to the asymmetry of the Maxwell–Boltzmann distribution. Please note, however, that the interference peaks exceed the background (most remarkably for larger values of $\beta$), corresponding to a coherent backscattering enhancement factor of the inelastic flux contribution $\eta_{E_d}^{(\text{inel})} = (\gamma_{E_d}^{(C, \text{inel})}(0) + \gamma_{E_d}^{(L, \text{inel})}(0)) / \gamma_{E_d}^{(L, \text{inel})}(0) > 2$ within a narrow spectral window around the energy $E_d$ where the crossed contribution is maximal. This enhancement is a consequence of the many-wave interference character of nonlinear coherent backscattering [57, 58] resulting from the fact that, as discussed in section 5.2, there are several ways of reversing the scattering paths when constructing crossed from ladder diagrams. The number of these possibilities increases with increasing number of inelastic scattering events. Although, due to the small width of these peaks, the total inelastic component $\gamma^{(C, \text{inel})}(0)$ (integrated over $E_d$) turns out to be smaller than the background $\gamma^{(L, \text{inel})}(0)$ (cf the dashed blue and dotted green lines in figure 11(a), this many-wave interference effect contributes to the above observed slowing down of the decrease of the backscattered flux.
Finally, figure 12 shows the impact of elastic and inelastic collisions on the angular profile of the coherently backscattered flux $\gamma^{(C)}(\theta)$, for the same parameters as in figure 11. We see that small interactions (i.e. $\beta = 0.02$ and 0.04) reduce the backscattered flux mainly around $\theta = 0$, whereas the wings of the angular profile (e.g. at $\theta = 0.2$) are hardly affected. This is due to the fact that long scattering paths (which only contribute at small backscattering angles [53]) are more strongly affected by interactions than short paths (which also contribute at larger angles). Therefore, small interactions reduce the curvature of $\gamma^{(C)}(\theta)$ at the top. (Consequently, the triangular cusp observed at $\theta = 0$ for a semi-infinite medium [53], i.e. in the limit $b \to \infty$, would be absent in the presence of interactions.) For larger interactions ($\beta \geq 0.08$ in figure 12), the Gross–Pitaevskii equation predicts a local minimum at $\theta = 0$, which is suppressed by inelastic collisions. In particular, in contrast to the Gross–Pitaevskii case, the coherently backscattered flux remains positive at all angles, as already observed in figure 11(a) for $\theta = 0$. Let us note, however, that inelastic collisions do not in principle forbid the occurrence of destructive coherent backscattering interference—this depends on the strength $\alpha$ of inelastic collisions, since the Gross–Pitaevskii prediction is recovered in the limit $\alpha \to 0$.

7. Conclusion

Within this paper, we have derived a microscopic $N$-body scattering theory for interacting particles in a weak disorder potential in 3D. We have applied this diagrammatic theory to a stationary scattering scenario for an asymptotically non-interacting quasi-plane matter wave incident on a 3D slab, with the disorder potential and inter-particle collisions confined to the slab region, and hereby verified the viability of our theory to address, on the one hand, very fundamental but, on the other hand, very timely questions of quantum transport for interacting particles in random environments. In a clear and precise manner, we demonstrated how one can bridge the gap between strictly unitary many-body evolution and its implications on the mesoscopic level governed by a transport equation similar to the nonlinear Boltzmann transport equation. Furthermore, we have determined the coherent corrections due to the wave nature of the particles leading to the effect of coherent backscattering. We have demonstrated that inelastic scattering slows down the decrease of the coherent backscattering peak as compared to the purely elastic case described by the Gross–Pitaevskii equation.

Let us briefly summarize the basic assumptions of our theory: firstly, we assume an optically thick scattering medium ($b = L/\ell_{\text{dis}} \gg 1$) allowing for multiple scattering in a weak disorder potential, where the mean free path is much larger than the wavelength of the incoming particles ($\ell_{\text{dis}} \sqrt{E_i} \gg 1$). Secondly, the interaction strength as, respectively, quantified by the parameters $\alpha$ and $\beta$ for ladder and crossed collisions, see equations (72), (73) for the case of s-wave scattering, should fulfill the condition $\alpha, \beta \ll 1$, such that atom–atom collisions occur less frequently than disorder scattering events. With view at future work, we expect that the condition $\beta \ll 1$ can be dropped by summing the corresponding diagrams, see figures 8(b), (d), (e), without intermediate disorder scattering [28]. If $\beta^2 \approx \ell_{\text{dis}} \sqrt{E_i}$, another type of collision process—corresponding to scattering induced by the fluctuating background density—sets in which is described by diagrams similar to our ladder diagrams [59]. Furthermore, relaxing the contact approximation for the collision terms, equations (46), (47), will allow to determine the effect of attractive or repulsive interactions onto the spatial atomic density profile. It will hence be a worthwhile task to extend our theory to stronger interactions, although we surely expect to encounter certain limits (e.g. the regime of superfluidity) where other methods will be required.
Concerning an experimental verification of our results, the application to a stationary scattering setup with matter waves constitutes, on the one hand, a very timely scenario, as, e.g. the developments of atom lasers and matter wave interferometers on atom chips [22, 23, 60] rapidly progress. On the other hand, many years of expertise have been gathered within the field of wavepacket spreading upon releasing the condensate from a trap into a new environment, where, e.g. the first experiments on coherent backscattering of (non-interacting) matter waves have been reported recently [11, 12]. Consequently, an extension of our theory to time-dependent scenarios based on recent progress in this field [51, 61, 62] presents a significant and feasible task. Similarly, also a finite correlation length of the disorder potential can be taken into account, see [24, 25] for the non-interacting case.

In conclusion, we are confident that our present theory and the rather straightforward extensions discussed above will substantially foster a more complete understanding of quantum transport under the interplay of disorder and inter-particle interaction and can contribute to a unifying picture from microscopic to macroscopic scales.

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Appendix A. Factorization of the transition amplitude

In this appendix, we show how an arbitrary \( N \)-particle scattering diagram can be factorized into single-particle propagators and two-body collisions. We first look at the example diagram shown in figure 1. As shown in figure A.1, this diagram can be split into four independent subdiagrams. The two subdiagrams connected to the initial state—(i) and (ii) in figure A.1—correspond to Möller operators, and the remaining ones—(iii) and (iv)—to Green’s operators. This gives rise to the following matrix elements:

\[
\Omega_+^{(i)}(E) = \langle p_9|\hat{\Omega}_+^{(V)}(E)|k_1\rangle, \tag{A.1}
\]
\[ \Omega_+^{(ii)}(E) = \frac{1}{2} \int \frac{dp_2 \, dp_3}{(2\pi)^6} \langle p_4, p_7 | \hat{T}_U(E) | p_2, p_3 \rangle \langle p_2, p_3 | \hat{\Omega}^{(iv)}(E) | k_2, k_3 \rangle, \]  
(A.2)

\[ G_+^{(iii)}(E) = \frac{1}{4} \int \frac{dp_1 \, dp_5 \, dp_6 \, dp_8}{(2\pi)^{12}} \langle k_5, k_6 | \hat{G}_V(E) | p_5, p_6 \rangle \times \langle p_5, p_6 | \hat{T}_U(E) | p_1, p_8 \rangle \langle p_1, p_8 | \hat{G}_V(E) | p_7, p_9 \rangle, \]  
(A.3)

\[ G_+^{(iv)}(E) = \langle k_4 | \hat{G}_V(E) | p_4 \rangle. \]  
(A.4)

Note that the diagrams (i) and (ii) are not connected to each other in figure A.1. The corresponding Møller operators can therefore be factorized as in equation (26). Likewise, the Green’s functions corresponding to (iii) and (iv) are factorized according to equation (27). The prefactors \(1/2\) and \(1/4\) in equations (A.2), (A.3) originate from symmetrization in the two-particle subspace (e.g. the states \( | p_2, p_3 \rangle \) and \( | p_3, p_2 \rangle \) are identical and therefore must not be summed over twice). It turns out that these factors are compensated, however, by the two possibilities to associate the initial and final single-particle states with each other in equations (26), (27).

The total transition amplitude results as

\[ \langle k_4, k_5, k_6 | \hat{\Omega}_+^{(\text{figure 1})} (3E_1) | k_1, k_2, k_3 \rangle = \frac{1}{2} \int \frac{dp_4 \, dp_7 \, dp_9}{(2\pi)^9} \Omega_+^{(i)}(E_1) \Omega_+^{(ii)}(2E_1) \]

\[ \times \int_{-\infty}^{\infty} \frac{dE_4}{(2\pi i)^3} G^{(iii)}(3E_1 - E_4) G^{(iv)}(E_4). \]  
(A.5)

Now, we again apply equations (26), (27) to factorize the two-particle Møller and Green’s operators on the right-hand side of equations (A.2), (A.3) into single-particle operators. In this way, we recover most of the terms in equation (28). The only ones which appear to differ from equation (28) are those associated to \( k_1, p_1, p_7 \) and \( p_8 \), which we reformulate as follows:

\[ \frac{1}{2} \int \frac{dp_8}{(2\pi)^3} \langle p_1, p_8 | \hat{G}_V(3E_1 - E_4) | p_7, p_9 \rangle \langle p_2 | \hat{\Omega}_+^{(iv)}(E_1) | k_1 \rangle \]

\[ = \int_{-\infty}^{\infty} \frac{dE_1}{(2\pi i)^3} \langle p_1 | \hat{G}_V(E_1) \hat{\Omega}_+^{(iv)}(E_1) | k_1 \rangle G(-E_1), \]  
(A.6)

where we again applied equation (27), used the completeness relation \( \int dp_9 | p_9 \rangle \langle p_9 | = (2\pi)^3 \), and defined

\[ G(-E_1) = \langle p_8 | \hat{G}_V(3E_1 - E_4 - E_1) | p_7 \rangle. \]  
(A.7)

Note that \( G(-E_1) \) is a complex analytic function of \( E_1 \) with poles only in the upper half of the complex plane. This, again, is due to the fact that \( \hat{G}_V(E) \) as a function of \( E \) exhibits poles only in the lower half, whereas \( E_1 \) enters with negative sign in the right hand side of equation (A.7).

We now reformulate some terms in equation (A.6) as follows:

\[ \hat{G}_V(E_1) \hat{\Omega}_+^{(iv)}(E_1) | k_1 \rangle = (\mathbb{1} + \hat{G}_V(E_1) \hat{V}) \hat{G}_0(E_1) | k_1 \rangle + \hat{G}_V(E_1) \hat{G}_V(E_1) \hat{V} | k_1 \rangle \]

\[ = \frac{1}{E_1 - E_i + i\epsilon} \left[ \mathbb{1} + (\hat{G}_V(E_1) + \hat{G}_V(E_1) - \hat{G}_V(E_1)) \hat{V} \right] | k_1 \rangle \]

\[ = \frac{1}{E_1 - E_i + i\epsilon} \hat{\Omega}_+^{(iv)}(E_1) | k_1 \rangle, \]  
(A.8)
where we used equation (8), the alternative but equivalent expression $\hat{G}_V(E) = \hat{G}_0(E) + \hat{G}_V(E) \hat{V} \hat{G}_0(E)$ with respect to equation (11), and the identity

$$\hat{G}_V(E_1)\hat{G}_V(E_i) = \frac{1}{E_1 - E_i} (\hat{G}_V(E_i) - \hat{G}_V(E_1))$$

$$= \frac{1}{E_1 - E_i + i\epsilon} (\hat{G}_V(E_i) - \hat{G}_V(E_1)), \quad (A.9)$$

resulting from $\frac{1}{ab} = \frac{1}{b-a} \left( \frac{1}{a} - \frac{1}{b} \right)$ (where we used the fact that $\hat{G}_V(E_1)$ and $\hat{G}_V(E_i)$ have the same set of eigenvectors). The second line of equation (A.9) is valid for infinitesimally small $\epsilon$, since equation (A.9) remains regular at $E_1 = E_i$. After inserting equation (A.8) into equation (A.6), we perform the integral over $E_1$ by closing the integration contour in the lower half of the complex plane. (Remember that $\hat{G}(-E_1)$ has no poles in the lower half!) Thereby, the energy $E_1$ is set to $E_i$, and we finally recover the missing terms in equation (28)

$$\int_{-\infty}^{\infty} \frac{dE_1}{(-2\pi i)} \langle p_1 | \hat{G}_V(E_1) \hat{\Omega}_j^{(V)}(E_i) | k_1 \rangle G(-E_1) = \langle p_1 | \hat{\Omega}_j^{(V)}(E_i) | k_1 \rangle G(-E_i). \quad (A.10)$$

The above procedure can be generalized to an arbitrary many-particle scattering diagram: we first divide the whole diagram into independent subdiagrams. Then, some of these subdiagrams turn out to be connected to each other by single-particle propagators. In the above example, figure A.1, this is the case for the subdiagram (i) and (iii), which are connected by the single-atom propagators from $k_1$ to $p_9$ with energy $E_i$ and from $p_9$ to $p_1$ with energy $E_1$. We have to show that these propagators merge into a single propagator (from $k_1$ to $p_1$ with energy $E_i$). For the case that one of the propagators is connected to the initial state (and thus corresponds to a Møller operator), the corresponding general identity is given by equation (A.10). If both propagators correspond to Green’s operators, e.g. $\hat{G}_V(E_1)$ and $\hat{G}_V(E_2)$ below, the required identity is proven as follows:

$$\int_{-\infty}^{\infty} \frac{dE_1 dE_2}{(-2\pi i)^2} \hat{G}_V(E_1)\hat{G}_V(E_2) G^{(1)}(-E_1) G^{(2)}(-E_2)$$

$$= \int_{-\infty}^{\infty} \frac{dE_1 dE_2}{(-2\pi i)^2} \frac{1}{E_1 - E_2 + i\epsilon} (\hat{G}_V(E_2) - \hat{G}_V(E_1)) G^{(1)}(-E_1) G^{(2)}(-E_2)$$

$$= \int_{-\infty}^{\infty} \frac{dE_2}{(-2\pi i)} \hat{G}_V(E_2) G^{(1)}(-E_2) G^{(2)}(-E_2), \quad (A.11)$$

where we again used equation (A.9) and the fact that $G^{(1)}(-E_1)$ and $G^{(2)}(-E_2)$ (which correspond to arbitrary other subdiagrams where the energy $E_1$ or $E_2$ enters with negative sign) exhibit no pole in the lower half of the complex plane. Note that the term with $\hat{G}_V(E_1)$ in the second line of equation (A.11) vanishes after integrating over $E_2$, since no pole remains in the lower half. In total, the concatenation of two Green’s operators, i.e. $\hat{G}_V(E_1)\hat{G}_V(E_2)$ on the left-hand-side of equation (A.11), reduces to a single Green’s operator, i.e. $\hat{G}_V(E_2)$ on the left-hand-side, whereas the energy $E_1$ is set equal to $E_2$. 

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Appendix B. Inelastic and elastic example diagrams

The inelastic diagram shown in figure 2(a) gives the following contribution to the flux density:

\[
J^{(\text{fig. 2(a))}}(r) = \left( \frac{1}{2} \right)^3 2^4 \int \frac{dk_1 dk_2 dk_3 dk_1' dk_2' dk_3' dp_1 dp_2 dp_3 dp_1' dp_2' dp_3'}{(2\pi)^{14}} 
\times \int_{-\infty}^{\infty} \frac{dE dE'}{2\pi |i|^2} \omega^*(k_1') \omega(k_2') \omega(k_1) \omega(k_2) \langle \hat{\Theta}^V(E_i) \rangle [p_1'] [k_1'] [\hat{\Theta}^V(E_i)] [p_2'] 
\times \langle p_1', p_2'| \hat{T}_U(2E_i) | p_3, p_4 \rangle \langle p_3' | \hat{\Theta}^V(E_i) | k_1 \rangle \langle p_2' | \hat{\Theta}^V(E_i) | k_2 \rangle. \quad (B.1)
\]

Here, we used the following labels for the wavevectors: the incoming solid arrows are called \( k_1 \) and \( k_2 \), whereas the detected and the traced out solid arrows are given by \( k_3 \) and \( k \), respectively. The intermediate solid arrows before and after the interaction event are labeled by \( p_1 \) and \( p_2 \) and by \( p_3 \) and \( p_4 \), respectively. The same notation holds for the dashed arrows, which are, however, denoted by an additional prime.

The trace formula, equation \((33)\), can now be applied after identifying \( A^{(d)}(-E') = \langle p_1' | (\hat{\Theta}^V(2E_i - E')) [k_3'] \rangle \) and \( A^{(v)}(-E) = \langle k_3 | \hat{G}_V(2E_i - E) | p_3 \rangle \).

For the elastic diagram, figure 2(b), we obtain

\[
J^{(\text{fig. 2(b))}}(r) = \left( \frac{1}{2} \right)^3 2^3 \int \frac{dk_1 dk_2 dk_3 dk_1' dk_2' dk_3' dp_1 dp_2 dp_3 dp_4}{(2\pi)^{13}} 
\times \int_{-\infty}^{\infty} \frac{dE dE'}{2\pi |i|^2} \omega^*(k_1') \omega(k_2') \omega(k_1) \omega(k_2) \langle \hat{\Theta}^V(E_i) \rangle [k_1'] [\hat{\Theta}^V(E_i)] [k_2'] 
\times \langle k_1' | (\hat{\Theta}^V(E_i)) [k] \rangle \langle k | \hat{\Theta}^V(E_i) | p_4 \rangle \langle k_3 | \hat{G}_V(2E_i - E) | p_3 \rangle 
\times \langle p_3, p_4 | \hat{T}_U(2E_i) | p_1, p_2 \rangle \langle p_1 | \hat{\Theta}^V(E_i) | k_1 \rangle \langle p_2 | \hat{\Theta}^V(E_i) | k_2 \rangle. \quad (B.2)
\]

Here, the labels of the wavevectors are identical to equation \((B.1)\), with the only difference, that the intermediate wavevectors \( p_1', \ldots, p_4' \) are not needed due to the missing interaction event for the dashed amplitudes. The trace formula, equation \((34)\), is applied with \( A(-E) = \langle k_3 | \hat{G}_V(2E_i - E) | p_1 \rangle \).

It is also insightful to take a look at the prefactors: the first factor 1/2 in equation \((B.1)\) results from \((1/\sqrt{2})^2\) in the initial states \( |i\rangle \) and \(|\bar{i}\rangle\), see equation \((20)\). The integration over the final states \( |k_3, k_4\rangle \) and \( |k_1', k_2'| \) (with \( k_1 = k_1' = k \) due to the trace) goes along with two more factors 1/2 (since both integrations must be performed in the symmetrized subspace). This, however, is counterbalanced by the fact that we may select either one of the two final particles as the detected particle. In equation \((B.1)\), we have selected \( |k_3\rangle \) and \( |k_1'\rangle \). Therefore, we have to include a factor 2\(^2\) to take into account the other possibilities. In equation \((B.2)\), we obtain an additional factor 2 due to the two possibilities in the factorization formula, equation \((26)\), for the dashed amplitudes: \( k_1' \) can be associated with \( k_3 \) and \( k_2' \) with \( k_4 = k \) – or vice versa.
Finally, the diagrams shown in figure 2 can be generalized to $N > 2$ particles. In this case, the remaining $N - 2$ particles are assumed not to interact with the detected particle. Hence, their evolution factorizes from the one of the detected particle and need not be taken into account. The prefactors are then generalized as follows: $1/2 \rightarrow N(N - 1)/4$ in equation (B.1) and $1 \rightarrow N(N - 1)/2$ in equation (B.2). Let us now compare these prefactors with the ones obtained from the iterative procedure based on the connection of building blocks in sections 4 and 5: The factors $N(N - 1) \simeq N^2$ (for $N \gg 1$, since $N \rightarrow \infty$ in the quasi-stationary limit) are accounted for by the source term $\rho_0$ in equation (43), which is proportional to $N$, see equation (21), and occurs two times for a two-particle process proportional to the density squared. What remains is a factor $1/2$ for each collision event (twice in figure 2(a) and once in figure 2(b)) which is included in the definition of the building blocks, equations (40), (41). The origin of this factor can be traced back to the indistinguishability of bosonic particles. Indeed, as argued at the end of section 3.2, all factors related to indistinguishability finally drop out in the case where all particles are initially in the same state. Since the $T$-matrix for indistinguishable particles, see equation (18), differs by a factor $2$ from the one for distinguishable particles, this must be counterbalanced by the above factor $1/2$.

Appendix C. Trace formulae

Here, we prove the trace formulae, equations (33), (34), for the trace over the undetected particle originating from an inelastic or an elastic collision. In both cases, we apply first the completeness relation $\int d|k\rangle\langle k| = (2\pi)^3$, and then the following identity for the product of two Green’s operators

$$\hat{G}_V(E')\hat{G}_V(E) = \frac{1}{E - E' + i\epsilon}(\hat{G}_{V}^\dagger(E') - \hat{G}_V(E)),$$

which is similar to equation (A.9). Thereby, equation (33) is proven as follows:

$$\int_{-\infty}^{\infty} \frac{dE \, dE'}{|2\pi i|^2} \int \frac{dk}{(2\pi)^3} A^{(i)}(-E')\hat{G}_{V}^\dagger(E')|k\rangle\langle k|\hat{G}_V(E)A^{(r)}(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE \, dE'}{|2\pi i|^2} A^{(i)}(-E')\hat{G}_V(E')\hat{G}_V(E)A^{(r)}(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE \, dE'}{|2\pi i|^2} \frac{1}{E - E' + i\epsilon} A^{(i)}(-E')\hat{G}_V(E')\hat{G}_V(E)A^{(r)}(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi i} A^{(i)}(-E)(\hat{G}_V^\dagger(E) - \hat{G}_V(E))A^{(r)}(-E).$$

In the last step, we have used the fact that $A^{(r)}(-E)$ is a complex analytic function without poles in the lower half of the complex plane. Similarly, $A^{(i)}(-E')$ exhibits no poles in the upper half. Thereby, considering the two terms $\hat{G}_V^\dagger(E')$ or $\hat{G}_V(E)$, respectively, we can perform the integral either over $E$ or over $E'$, closing the integration contour in the lower or upper half, respectively. In both cases, the term $1/(E - E' + i\epsilon)$ gives the only pole. This fixes $E' = E$, and we arrive at the final result, equation (C.2).
Concerning the trace formula for elastic collisions, equation (34), we proceed in a similar way as in equation (A.10). We use the definition of $\hat{\Omega}^{(V)}_+(E_i)$, equation (8), and the Lippmann–Schwinger equation (11) for $\hat{G}_V(E)$ as follows:

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi i} \int \frac{dk}{(2\pi)^3} \langle k_i | (\hat{\Omega}^{(V)}_+(E_i))^\dagger | k | \hat{G}_V(E)A(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi i} \langle k_i | [\hat{G}_V(E) + \hat{V}(\hat{G}_V(E) + \hat{G}_V(E))A(-E)$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi i} \frac{1}{E - E_i + i\epsilon} \langle k_i | [\mathbb{1} + \hat{V}(\hat{G}_V(E) + \hat{G}_V(E)) - \hat{G}_V(E))A(-E)$$

$$= \langle k_i | [\mathbb{1} + \hat{V} \hat{G}_V(E_i)]A(-E_i) = \langle k_i | (\hat{\Omega}^{(V)}_+(E_i))^\dagger A(-E_i). \tag{C.3}$$

This proves equation (34).

### Appendix D. Particle and energy flux conservation

In this appendix, we prove equations (51) and (52). Starting from equation (50) for $f_{E_1, E_2, E_3}$, we calculate $\int_0^\infty dE_3 \sqrt{E_3} f_{E_1, E_2, E_3}$. For this purpose, we first note that

$$\int_0^\infty dE_3 \sqrt{E_3} \left( \frac{G^*_{E_1+E_2-E_3}(k_4) - G_{E_1+E_2-E_3}(k_4)}{2\pi i} \right) |G_{E_3}(k_3)|^2$$

$$= \int_{-\infty}^{\infty} dE_3 \left( \frac{G^*_{E_1+E_2-E_3}(k_4) - G_{E_1+E_2-E_3}(k_4)}{2\pi i} \right) \left( \frac{G^*_{E_3}(k_3) - G_{E_3}(k_3)}{2i/\ell_{\text{dis}}} \right)$$

$$\approx \frac{\ell_{\text{dis}}}{2i} \left( \frac{1}{E_1 + E_2 - k_3^2 - k_4^2 - 2i\epsilon} - \frac{1}{E_1 + E_2 - k_3^2 - k_4^2 + 2i\epsilon} \right)$$

$$\approx \frac{\ell_{\text{dis}}}{2i} \left[ (G_{E_1}^{(0,m/2)}((k_3 - k_4)/2))^0 - G_{E_1}^{(0,m/2)}((k_3 - k_4)/2) \right] \tag{D.1}$$

with $E_{12} = E_1 + E_2 - E_{k_1+k_2}/2$, $k_1 + k_2 = k_3 + k_4$ and $|k_{34}|$ as defined after equation (18). Here, we have first used the identity $\sqrt{E_3}G_{E_3}(k_3)|^2 = \ell_{\text{dis}}[G^*_{E_3}(k_3) - G_{E_3}(k_3)]/(2i)$ for the average Green’s function, then replaced the average Green’s functions by vacuum Green’s functions (which is appropriate in the weak disorder limit), and evaluated the integral over $E_3$ using residual calculus. By setting the imaginary part $2\epsilon$ in the denominator to $\epsilon$ and using momentum conservation, i.e. $(k_3^2 + k_4^2)/2 \rightarrow E_{k_1+k_2}/2 - k_3k_4$ we arrived at equation (D.1).

Again, $G_{E_1}^{(0,m/2)}(k) = 1/(E - 2k^2 + i\epsilon)$ denotes the vacuum Green’s function for a particle with mass $m/2$, cf equation (19).

Inserting equation (D.1) into equation (50), and substituting the variable $k_3 \rightarrow k_{34} = (k_3 - k_4)/2$, the integration over $k_{34}$ reduces to

$$2 \int \frac{dk_{34}}{(2\pi)^3} \langle k_{34} | (\hat{G}_{0,m/2}(E_{12}) - \hat{G}_{0,m/2}(E_{12}))|k_{34}\rangle \left| \langle k_{12} | \hat{T}_U^{(1)}(E_{12})|k_{12}\rangle \right|^2$$

$$= 2 \langle k_{12} | \hat{T}_U^{(1)}(E_{12}) \rangle \langle E_{12} | \hat{G}_{0,m/2}(E_{12}) \rangle \hat{T}_U^{(1)}(E_{12}) \langle k_{12} \rangle. \tag{D.2}$$
Applying the optical theorem, equation (19), yields in total
\[
\int_{0}^{\infty} dE_3 \sqrt{E_3} \int dE_1, E_2, E_3 = \left(\frac{4\pi}{\ell_{\text{dis}}}\right)^2 \int d\mathbf{k}_1 d\mathbf{k}_2 (2\pi)^6 \text{Im}[(\mathbf{k}_{12}|\hat{T}^{(1)}_{U}(E_{12})|\mathbf{k}_{12})] |G_{E_1}(k_1)|^2 |G_{E_2}(k_2)|^2.
\]
(D.3)

On the other hand, the integral over \( r_1 \) and \( r_2 \) in equations (40), (46), see equation (49), together with the formula \( \sqrt{E_2} |G_{E_2}(k_2)|^2 = \ell_{\text{dis}}[G_{E_2}^*(k_2) - G_{E_2}(k_2)]/\sqrt{2} \), yields
\[
-\sqrt{E_2} g_{E_1, E_2} = \left(\frac{4\pi}{\ell_{\text{dis}}}\right)^2 \int d\mathbf{k}_1 d\mathbf{k}_2 (2\pi)^6 |G_{E_1}(k_1)|^2 \text{Im}[(\mathbf{k}_{12}|\hat{T}^{(1)}_{U}(E_{12})|\mathbf{k}_{12})] \times (G_{E_2}^*(k_2) - G_{E_2}(k_2)) G_{E_2}(k_2).
\]
(D.4)

The term \( |G_{E_2}(k_2)|^2 \) in the second line of equation (D.4) exactly reproduces equation (D.3), whereas the remaining term \( [G_{E_2}(k_2)]^2 \) gives a negligible contribution in the limit \( \sqrt{E_2} \ell_{\text{dis}} \gg 1 \). Moreover, one can show that this contribution is canceled by another diagram where an additional disorder correlation function is inserted just before and after the collision, see figures 1(e), (f) in [62]. This proves equation (51).

Equation (52) can be shown in almost the same way. When calculating \( \int_{0}^{\infty} dE_3 \sqrt{E_3} \int dE_1, E_2, E_3 \), equation (D.1) is replaced by
\[
\int_{0}^{\infty} dE_3 \sqrt{E_3} \int \left(\frac{G_{E_1 + E_2 - E_3}(k_4) - G_{E_1 + E_2 - E_3}(k_4)}{2\pi i}\right) |G_{E_3}(k_3)|^2 \approx \frac{\ell_{\text{dis}}}{2i} ([G_{E_12}^{0,m/2}(k_{34})]^* - G_{E_12}^{0,m/2}(k_{34}))(E_1 + E_2 + k_3^2 - k_4^2).
\]
(D.5)

When integrating over \( k_{34} \) as in equation (D.2), the term \( (k_3^2 - k_4^2) \) vanishes due to symmetry, but the factor \( (E_1 + E_2) \) remains. This proves equation (52).

### Appendix E. Collision terms for s-wave scattering

For a short-range interaction potential \( U(\mathbf{r}) \), the \( T \)-matrix (for a particle with mass \( m/2 \)) has the following form [49]:
\[
\langle \mathbf{k}' | \hat{T}^{(1)}_{U}(E) | \mathbf{k} \rangle = 8\pi a_s \left( 1 - i\sqrt{\frac{E}{2}} a_s + \mathcal{O}(\sqrt{E} a_s) \right)
\]
(E.1)
valid for arbitrary plane wave states \( |\mathbf{k} \rangle, |\mathbf{k}' \rangle \) in the limit \( \sqrt{E} a_s \ll 1 \), where \( a_s \) is the s-wave scattering length associated to the potential \( U(\mathbf{r}) \). The prefactor \( 8\pi a_s = 4\pi a_s h^2/m \) for \( h^2/(2m) \equiv 1 \) applies for a particle with mass \( m/2 \). Note that, due to the symmetrization of the states \( |\mathbf{k}_{12} \rangle \) and \( |\mathbf{k}_{34} \rangle \), an additional factor 2 appears when evaluating \( \langle \mathbf{k}_{34} | \hat{T}^{(1)}_{U}(E_{12}) | \mathbf{k}_{12} \rangle \) in equation (18), cf. the remark at the end of appendix B. Inserting equation (E.1) into the general expressions, equations (40), (41) and (63)–(65), yields the following results for the collision terms (in the limit \( \ell_{\text{dis}} \sqrt{E_{1,2,3}} \gg 1 \)):
\[
\begin{align*}
E_{E_1, E_2} &= -\frac{\alpha}{6\rho_0 \sqrt{E_1 E_2}} \left( (\sqrt{E_1} + \sqrt{E_2})^2 - |\sqrt{E_1} - \sqrt{E_2}|^2 \right), \\
E_{E_1, E_2, E_3} &= \frac{\alpha}{\rho_0 \sqrt{E_1 E_2 E_3}} \min(\sqrt{E_1}, \sqrt{E_2}, \sqrt{E_3}, \sqrt{E_1 + E_2 - E_3}).
\end{align*}
\]
(E.2) (E.3)
In the above expressions, equations (E.2)–(E.7), all energies appearing under a square root must be positive—otherwise, the corresponding expression is set to zero (e.g. \( f_{E_1,E_2,E_3} = 0 \) if \( E_3 > E_1 + E_2 \)). Furthermore, note that the density \( \rho_0 \) appearing in the denominators of equations (E.2)–(E.6) drops out when expressing the densities \( I_E(r) \), \( C_0^{(1)}(r) \) and \( C_0^{(2)}(r) \) in equations (48), (66), (67) in units of the incoming density \( \rho_0 \), see also equations (45), (70). Therefore, the effective strength of the collision terms, equations (E.2)–(E.6), is solely governed by the parameters \( \alpha \) and \( \beta \) introduced in equations (72), (73).

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