A new decomposition of portfolio return

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Abstract

For a functionally generated portfolio, there is a natural decomposition of the relative log-return into the log-change in the generating function and a drift process. In this note, this decomposition is extended to arbitrary stock portfolios by an application of Fisk-Stratonovich integration. With the extended methodology, the generating function is represented by a structural process, and the drift process is subsumed into a trading process that measures the profit and loss to the portfolio from trading.

Introduction

For $n > 1$, consider a stock market of positive, continuous, square-integrable semimartingales $X_1, \ldots, X_n$ that represent the capitalizations of the stocks. Let $\pi$ be a portfolio with weight processes $\pi_1, \ldots, \pi_n$, which are bounded measurable processes adapted to the underlying filtration, and which add up one. Denote the capitalizations of the stocks. Let $Z$ be the value process of the market portfolio, with

$$d\log(Z(t)/Z_\mu(t)) = d\log(S(\mu(t))) + d\Theta(t), \quad \text{a.s.,}$$

where the drift process $\Theta$ is of locally bounded variation. For an arbitrary portfolio $\pi$, we shall define a structural process $S_\pi$, which measures the efficacy of stock selection in the portfolio, and a trading process $T_\pi$, which measures the profit and loss from trading, such that

$$d\log(Z(\pi(t)/Z_\mu(t)) = d\log(S_\pi(t)) + dT_\pi(t), \quad \text{a.s.}$$

If the portfolio weights are continuous semimartingales, then $T_\pi$ will be of locally bounded variation, and for a functionally generated portfolio,

$$d\log(S_\pi(t)) = d\log(S(\mu(t))) \quad \text{and} \quad dT_\pi(t) = d\Theta(t), \quad \text{a.s.}$$

Let us first consider two types of stochastic integration.

Itô integrals and Fisk-Stratonovich integrals

We shall be considering both Itô integration and Fisk-Stratonovich integration, and details regarding the relationship between these two forms of stochastic integration can be found in [Protter (1990)]. Let $X$ and $Y$ be continuous, square-integrable semimartingales. Then the Itô integral satisfies

$$\int_0^T Y(t) dX(t) = \lim_{\Delta \to 0} \sum_{i=1}^{\nu-1} Y(t_i) (X(t_{i+1}) - X(t_i)), \quad \text{a.s.,}$$

where the limit is in quadratic mean and $\Delta$ is the mesh of the partition $\{0 = t_1 < t_2 < \cdots < t_{\nu} = T\}$. For the Fisk-Stratonovich integral, with the differential denoted by $\circ d$, this becomes

$$\int_0^T Y(t) \circ dX(t) = \lim_{\Delta \to 0} \sum_{i=1}^{\nu-1} \frac{Y(t_i) + Y(t_{i+1})}{2} (X(t_{i+1}) - X(t_i)), \quad \text{a.s.}$$

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where the limit is again in quadratic mean. The effect of this definition is to allow the integrand to extend past the usual filtration and be “half-way” into the future. It will be convenient to replace the integral in (4) with the differential notation
\[ Y'(t) \circ dX(t), \]
as is commonly done for the Itô integral.

The two integrals are related by
\[ Y'(t) \circ dX(t) = Y(t) \circ dX(t) + \frac{1}{2} d\langle X, Y \rangle_t, \quad \text{a.s.,} \]
where \( \langle X, Y \rangle_t \) is the cross-variation process for \( X \) and \( Y \). Indeed, this equation is sometimes used as the definition of the Fisk-Stratonovich integral (see Protter (1990), Chapter V). It follows from (5) that for continuous semimartingales the difference between an Itô integral and the corresponding Fisk-Stratonovich integral will be a process of locally bounded variation.

For a \( C^2 \) function \( F \) defined on the range of \( X \), the Fisk-Stratonovich integral satisfies the rules of standard calculus. By Itô’s rule we have
\[ dF(X(t)) = F'(X(t)) \circ dX(t) + \frac{1}{2} F''(X(t)) d\langle X \rangle_t, \quad \text{a.s.,} \]
where \( \langle X \rangle_t \) is the quadratic variation of \( X \), and since
\[ d\langle F', X \rangle_t = F''(X(t)) d\langle X \rangle_t, \quad \text{a.s.,} \]
it follows from (5) that
\[ dF(X(t)) = F'(X(t)) \circ dX(t), \quad \text{a.s.} \]
(see Protter (1990) Theorem V.20).

**Decomposition of portfolio return**

Let us consider the following thought experiment: Suppose we hold a large-capitalization stock index portfolio comprising the largest \( m < n \) stocks in the market at weights proportional to their market weights. Suppose now that the stock at rank \( m \) changes places with the stock at rank \( m + 1 \), and nothing else moves. In this case, we sell the former rank-\( m \) stock and buy the current one, and after the trade the portfolio is just as it was before, except that its value has decreased. The loss in portfolio value is due to the drop in price of the original rank-\( m \) stock, which was subsequently replaced by the new rank-\( m \) stock. If we had been able to hold both of these two stocks, each at its average weight over the period, then the loss would have vanished.

If we consider the portfolio log-return, the Fisk-Stratonovich integral (4) evaluates the log-return using the average weights, while the Itô integral (3) evaluates the log-return using the initial weights, so the difference between the values of these two integrals represents the effect of trading in our experiment. This motivates us to use the average-weight log-return to measure the efficacy of stock selection in the portfolio, and to use the difference between the actual log-return and the average-weight log-return to measure the effect of trading. Accordingly, we have

**Definition 1.** For a portfolio \( \pi \) with value process \( Z_\pi \), the **structural process** \( S_\pi \) is defined by
\[ d \log S_\pi(t) \overset{\Delta}{=} \sum_{i=1}^{n} \pi_i(t) \circ d \log \mu_i(t), \]
and the **trading process** \( T_\pi \) is defined by
\[ dT_\pi(t) \overset{\Delta}{=} d \log \left( Z_\pi(t)/Z_\mu(t) \right) - d \log S_\pi(t). \]

By construction, this definition is compatible with the return decomposition (2).
Proposition 1. If the portfolio weight processes $\pi_i$ are continuous semimartingales, then the trading process $T_\pi$ will be of locally bounded variation.

Proof. From Definition 1, we have

$$
dT_\pi(t) = d\log (Z_\pi(t)/Z_\mu(t)) - d\log S_\pi(t)
= \sum_{i=1}^{n} \pi_i(t) d\log \mu_i(t) + \gamma^*_\pi(t) dt - \sum_{i=1}^{n} \pi_i(t) \circ d\log \mu_i(t)
= \left( \sum_{i=1}^{n} \pi_i(t) d\log \mu_i(t) - \sum_{i=1}^{n} \pi_i(t) \circ d\log \mu_i(t) \right) + \gamma^*_\pi(t) dt, \text{ a.s.,}
$$

where $\gamma^*_\pi$ is the excess growth rate of $\pi$ (see Fernholz (2002)). If the portfolio weight processes $\pi_i$ are continuous semimartingales, then (5) implies that the term in the parentheses in (10) will be of locally bounded variation. Since the excess growth term is also of locally bounded variation, so will be $T_\pi$. \hfill \square

Decomposition of return for functionally generated portfolios

It was shown in Fernholz (2002) that a positive $C^2$ function $S$ defined on the unit simplex $\Delta^n$ such that for all $i$, $x_i D_i \log S(x)$ is bounded on $\Delta^n$, will generate a portfolio $\pi$ that satisfies (1) with portfolio weights

$$
\pi_i(t) = \left( D_i \log S(\mu_i(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t),
$$

and a drift process $\Theta$ defined by

$$
d\Theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^{n} D_{ij} S(\mu(t)) d\langle \mu_i, \mu_j \rangle_t
$$
(see also Karatzas and Ruf (2015)).

Proposition 2. Let $\pi$ be the portfolio generated by the positive $C^2$ function $S$. Then

$$
d\log S_\pi(t) = d\log S(\mu(t)), \text{ a.s.,}
$$

and

$$
dT_\pi(t) = d\Theta(t), \text{ a.s.}
$$

Proof. Under the rules of Fisk-Stratonovich integration,

$$
d\log S(\mu(t)) = \sum_{i=1}^{n} D_i \log S(\mu(t)) \circ d\mu_i(t)
= \sum_{i=1}^{n} D_i \log S(\mu(t)) \mu_i(t) \circ d\log \mu_i(t)
= \sum_{i=1}^{n} \pi_i(t) \circ d\log \mu_i(t)
= d\log S(t), \text{ a.s.,}
$$

by Definition 1, where (14) follows from (11) and the fact that

$$
\sum_{i=1}^{n} \mu_i(t) \circ d\log \mu_i(t) = \sum_{i=1}^{n} d\mu_i(t) = d \sum_{i=1}^{n} \mu_i(t) = 0, \text{ a.s.}
$$
With this established,
\[ d\mathcal{J}_\pi(t) = d\Theta(t), \text{ a.s.,} \]
follows from (1) and Definition 1.

\[ \square \]

**Discussion**

We see from (11) that the weight ratios \( \pi_i(t)/\mu_i(t) \) depend on the first derivatives \( D_i \log S(\mu(t)) \), and we see from (12) that the drift process \( \Theta \) depends on the second derivatives \( D_{ij} S(\mu(t)) \). Hence, when changes in the market weights induce changes in the weight ratios, the effect of the weight-ratio changes will be recorded in the drift process. When a weight ratio changes, this requires trading, so the drift process serves as a cumulative measure of the trading profit and loss. This measure is quantified by (13) of Proposition 2.

Let us now apply Proposition 2 to calculate \( \mathcal{J}_\pi \) for some of the portfolios included in Example 3.1.6 of Fernholz (2002). For the market portfolio, or for any buy-and-hold portfolio, there is no trading, and \( \mathcal{J}_\pi(t) = \Theta(t) \equiv 0 \). This is perhaps the minimal requirement for \( \mathcal{J}_\pi \) to be a measure of trading profit and loss — if there is no trading, there will be no trading profit or loss. For an equal-weighted or constant-weighted portfolio, we see that \( d\mathcal{J}_\pi(t) = d\Theta(t) = \gamma^*_\pi(t) dt \), the portfolio excess growth rate. Since the weights in these portfolios are constant, if the quadratic variation structure of the market is also constant, then \( \gamma^*_\pi \) will be constant, and so will be the rate of profit and loss from trading. Hence, at least in these simple cases, \( \mathcal{J}_\pi \) behaves in a manner consistent with expectations for a measure of trading profit and loss.

Fernholz (2001) introduces a class of portfolios that are generated by functions of the ranked market weights (see also Theorem 4.2.1 of Fernholz (2002)). Proposition 2 can be extended to these portfolios, at least with some additional regularity conditions imposed on the stock capitalization processes \( X_1, \ldots, X_n \), and the interpretation of the processes \( S_\pi \) and \( T_\pi \) will remain the same as for the cases we have studied. However, for portfolios generated by functions that use more information than the values of the current market weights \( \mu_i(t) \), as in Strong (2014) or Schied et al. (2016), Proposition 2 may fail, and the processes \( S_\pi \) and \( T_\pi \) may behave in a manner that no longer corresponds to the interpretation that we have given them here.

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