Light-Cone Quantization of the Schwinger Model

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Abstract

We consider constructing a canonical quantum theory of the light-cone gauge \((A_-=0)\) Schwinger model in the light-cone representation. Quantization conditions are obtained by requiring that translational generators \(P_+\) and \(P_-\) give rise to Heisenberg equations which, in a physical subspace, are consistent with the field equations. A consistent operator solution with residual gauge degrees of freedom is obtained by solving initial value problems on the light-cones. The construction allows a parton picture although we have a physical vacuum with nontrivial degeneracies in the theory.
§1. Introduction

We will use the following notation:

\[ g^{++} = g^{--} = 0, \quad g^{+-} = g^{-+} = 2, \quad g_{++} = g_{--} = 0, \quad g_{+-} = g_{-+} = \frac{1}{2}, \]
\[ x^+ = x^0 + x^1, \quad x^- = x^0 - x^1, \quad \partial_+ = \frac{1}{2}(\partial_0 + \partial_1), \quad \partial_- = \frac{1}{2}(\partial_0 - \partial_1), \]
\[ \gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma^5 = -\sigma_3 \]
\[ \Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}, \quad m = \frac{e}{\sqrt{\pi}}. \]

Previously, one of us (Nakawaki)\(^1\) gave the following operator solution to the Schwinger model in Landau gauge:

\[ \Psi_+ = Z e^{A_+} \sigma_-^* e^{A_+} \]  
\[ A_+ = -i\sqrt{\pi}(2\phi_+(x^-) + \tilde{\eta} + \tilde{\Sigma}(x^+, x^-)) \]  
\[ Z^2 = e^\gamma \frac{m\kappa}{8\pi^2} \]  
\[ \Psi_- = Z e^{A_-} \sigma_-^* e^{A_-} \]  
\[ A_- = -i\sqrt{\pi}(2\phi_-(x^+) - \tilde{\eta} - \tilde{\Sigma}(x^+, x^-)) \]  
\[ A_\mu = -m^{-1}\epsilon_{\mu\nu}\partial^\nu(\tilde{\eta} + \tilde{\Sigma}). \]  

Here, \(\phi_+(x^-)\) is the right moving component of a Klaiber\(^2\)-regulated (with parameter \(\kappa\)), free, massless scalar field composed of the fusion operators:

\[ \phi_+^{(+)}(x^-) = i(4\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} dk_1 k_1^{-1} c(k_1) \left( e^{-ik\cdot x} - \theta(\kappa - k_0) \right) \]  
\[ \phi_+^{(-)} = (\phi_+^{(+)})^*. \]  

Where \(c(k_1)\) are the fusion operators associated with Bosonizing\(^1\) the Fermi field\(^1\) (they satisfy the usual Boson commutation relations). \(\phi_-(x^+)\) is the left moving component of that field:

\[ \phi_-^{(+)}(x^+) = i(4\pi)^{-\frac{1}{2}} \int_{0}^{\infty} dk_1 k_1^{-1} c(k_1) \left( e^{-ik\cdot x} - \theta(\kappa - k_0) \right) \]  
\[ \phi_-^{(-)} = (\phi_-^{(+)})^*. \]  

\(\tilde{\eta}\) is a psuedoscalar ghost field given by:

\[ \tilde{\eta}^{(+)} = i(4\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dk_1 k_1^{-1} \tilde{\eta}(k_1) \left( e^{-ik\cdot x} - \theta(\kappa - k_0) \right), \]  
\[ \tilde{\eta}^{(-)} = (\tilde{\eta}^{(+)})^*. \]
where:
\[
[\eta(k_1), \eta^*(q_1)] = -k_0\delta(k_1 - q_1)
\] (1.12)

The field, $\hat{\Sigma}$ — the field associated with the physical Schwinger particles — is a massive psuedoscalar field of mass $m (= \frac{e}{\sqrt{\pi}})$. The spurions are given in terms of these modes as:

\[
\sigma_+^s = \exp\left[i\sqrt{\pi}\{Q_5 + Q\}(4m)^{-1} \right.
+ 2^{-1}\int_{-\kappa}^{\kappa} dk_1 k_1^{-1}\{\eta(k_1) - \eta^*(k_1)\} + \int_{-\kappa}^{0} dk_1 k_0^{-1}\{c(k_1) - c^*(k_1)\}\left.\right]
\] (1.13)

\[
\sigma_-^s = \exp\left[i\sqrt{\pi}\{Q_5 - Q\}(4m)^{-1} \right. 
- 2^{-1}\int_{-\kappa}^{\kappa} dk_1 k_1^{-1}\{\eta(k_1) - \eta^*(k_1)\} + \int_{-\kappa}^{0} dk_1 k_0^{-1}\{c(k_1) - c^*(k_1)\}\left.\right].
\] (1.14)

To be invariant under the large gauge transformations the vacuum must be chosen to be a theta-state, formed as:

\[
|\Omega(\theta)\rangle \equiv \sum_{n=-\infty}^{\infty} e^{iM\theta}|\Omega(M)\rangle ; \quad |\Omega(M)\rangle = (\sigma_+^s\sigma_-^s)^M|0\rangle.
\] (1.15)

The physical subspace is formed by applying all polynomials in $\hat{\Sigma}$ to $|\Omega(\theta)\rangle$.

Further details, and proof that the construction is an operator solution can be found in the reference. Here we wish to point out that while the above solution is representation independent, it is straightforward to formulate the problem as an initial value problem at $t = 0$, and thus find a solution in the equal-time representation. Gauge invariant point splitting (in a space-like direction) provides the necessary regulation for the operator products. If we contemplate the initial value problem on the characteristic surfaces, $x^+ = 0$ and $x^- = 0$, things are not so straightforward. The problem is that the Fermi products cannot be regulated by splitting in a light-like direction. The $\tilde{\eta}$ field is the sum of a function of $x^+$ and a function of $x^-$, so the operator products are not regulated by splitting in either light-like direction. Also, the $\hat{\Sigma}$ field suffers an apparent, but spurious, infrared singularity due to the fact that its massive character is not manifest at light-like separations. Note that these problems only have to do with the formulation of the theory as an initial value problem on the characteristics; the light-cone representation perfectly well exists — that is, modes of the fields along the characteristics provide all the operators necessary to generate the entire representation space. We do need modes along both characteristics but all the physical operators except those necessary to define the vacuum can be generated using modes
along $x^+ = 0$. To generate the vacuum we need both spurions and thus modes from both characteristics. We shall return to the problem of formulating an initial value problem on the characteristic surfaces below, but first we wish to consider the question of light-cone gauge.

§2. Light-cone gauge

We may attempt to reach the light-cone gauge by performing a nonlocal gauge transformation on the Landau gauge solution. If we use the gauge function:

$$\Theta = m^{-1}(\tilde{\eta} + \tilde{\Sigma}) \quad (2.1)$$

we find that $A^+ = 0$. The resulting construction almost works but it is not quite right. One problem is that the $x^-$-dependent parts of the $\tilde{\eta}$ and $\phi$ fields are not natural degrees of freedom in light-cone gauge. That is, standard quantization methods, whether in the equal-time representation or in the light-cone representation do not include those degrees of freedom in the representation space. Those degrees of freedom decouple, at least formally, and we shall simply remove them from the solution. The other problem is that we have performed the gauge transformation with the Klaiber-regulated $\tilde{\eta}$ field but have left the spurions, which contain the low frequency parts of that field, unchanged. The effect is that the Klaiber regulator, $\kappa$, does not disappear from physical matrix elements and the solution is no longer translationally invariant even in the physical subspace. To cure that problem we must modify the spurions in addition to making the gauge transformation specified by $\Theta$. The correct solution in light-cone gauge is then given by (we remove the tilde from the $\eta$ field since we keep only the $x^+$ dependent part):

$$\Psi_+ = Z_+ e^{\Lambda_+} \sigma_+ e^{A_+} \quad (2.2)$$

$$A_+ = -2i\sqrt{\pi}(\eta(x^+) + \tilde{\Sigma}(x^+, x^-)) \quad (2.3)$$

$$Z_+^2 = \frac{m^2 e^\gamma}{8\pi \kappa} \quad (2.4)$$

$$\Psi_- = Z_- e^{\Lambda_-} \sigma_- e^{A_-} \quad (2.5)$$

$$A_- = -2i\sqrt{\pi}\phi(x^+) \quad (2.6)$$

$$Z_-^2 = \frac{\kappa e^\gamma}{2\pi} \quad (2.7)$$

$$A_+ = \frac{2}{m} \partial_+(\eta + \tilde{\Sigma}) \quad (2.8)$$
\[ A_\pm = 0 \] (2.9)
\[
\sigma_+ = \exp \left[ i \sqrt{\pi} \{ Q_5 + Q \} (4m)^{-1} + \int_0^\infty dk_1 k_1^{-1} \{ \eta(k_1) - \eta^*(k_1) \} \right]
\] (2.10)
\[
\sigma_- = \exp \left[ i \sqrt{\pi} \{ Q_5 - Q \} (4m)^{-1} + \int_0^\infty dk_0 k_0^{-1} \{ c(k_1) - c^*(k_1) \} \right]
\] (2.11)
\[
Q = \int_{-\infty}^{\infty} m \partial_+ (\phi - \eta) dx^+
\] (2.12)
\[
Q_5 = \int_{-\infty}^{\infty} m \partial_+ (\phi + \eta) dx^+.
\] (2.13)

Again, the vacuum must be chosen to be of the form:
\[
|\Omega(\theta)\rangle \equiv \sum_{n=-\infty}^{\infty} e^{iM\theta} |\Omega(M)\rangle \quad ; \quad |\Omega(M)\rangle = (\sigma^* \sigma - \sigma \sigma^*) M |0\rangle.
\] (2.14)

Now we see that the field, \( \Psi_\pm \), is isomorphic to the left-moving component of a free, massless Fermi field and it has no dependence on the ghost field even through the spurion; both of these properties are to be expected in light-cone gauge\(^3\)). Even with the modifications of the spurions the physics contained in the light-cone gauge solution is the same as that in the Landau gauge solution. In particular, we find the anomaly:
\[
\partial^\mu J_\mu^5 = \frac{e}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu}
\] (2.15)
and the chiral condensate:
\[
\langle \Omega(\theta) | \bar{\Psi} \Psi | \Omega(\theta) \rangle = -\frac{m}{2\pi} e^\gamma \cos \theta
\] (2.16)

We believe that this construction is the correct light-cone gauge solution to the continuum Schwinger model but it does have some unexpected properties which we should discuss. Most striking is that the vacuum expectation of the spurion, \( \sigma_+ \), not only does not vanish, as it does in the Landau gauge solution, but diverges. That fact may cause one to wonder in what sense the equations of motion are satisfied in the physical subspace. The point is that the \( \Psi_\pm \) field is not a physical operator (since it carries a charge) and the only way the spurions enter physical operators is in the chargeless combinations, \( \sigma^*_+ \sigma_- \) and \( \sigma^*_- \sigma_+ \). The chargeless combinations of spurions simply add zero norm states to the state acted upon and, in particular, act as c-numbers in the physical subspace. With that in mind, it is easy to use the arguments in ref. 1 to show that the equations of motion are satisfied in the following sense: Take any physical operator and use the Lagrange equations of motion to derive an equation of motion for the physical operator. Then the derived equation of motion will be satisfied in the physical subspace. That is the common situation in QCD.
As with the Landau gauge solution, the light-cone gauge solution is straightforward to quantized on \( t = 0 \). Indeed, except for the spurions, it is very similar to the periodic solution found by Bassetto, Nardelli and Vianello.\(^4\) But it is not so straightforward to quantize the continuum solution on \( x^+ = 0 \) due to the fact that the current, \( \bar{\Psi} \gamma_5 \Psi \) is not regulated by splitting in the \( x^- \) direction. We notice that the problem was already present in the Landau gauge solution and is due to the initial value surface, not the gauge choice.

\[3\] Quantization in the light-cone representation

The Schwinger model is defined by the Lagrangian:

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda (A_0 - A_1) + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - e \bar{\Psi} \gamma^\mu \gamma^5 A_\mu = 2 F_{++} F_{+-} - 2 \lambda A_- + 2i \bar{\Psi}^* \partial_- \Psi_- + 2i \bar{\Psi}^* \partial_+ \Psi_+ - 2e \bar{\Psi}^* \Psi_- A_- - 2e \bar{\Psi}^* \Psi_+ A_+ \quad (3.1)
\]

where:

\[
A_\pm = \frac{1}{2} (A_0 \pm A_1), \quad F_{+-} = \partial_+ A_- - \partial_- A_+.
\]

The field equations and the gauge fixing condition are:

\[
2 \partial_- F_{+-} = -e \bar{\Psi}^* \Psi_+ = -J_-, \quad (3.3)
\]
\[
2 \partial_+ F_{+-} - \lambda = e \bar{\Psi}^* \Psi_- = J_+ , \quad (3.4)
\]
\[
i \partial_- \Psi_- = e \bar{\Psi} A_-, \quad (3.5)
\]
\[
i \partial_+ \Psi_+ = e \Psi A_+, \quad (3.6)
\]
\[
A_- = 0. \quad (3.7)
\]

From current conservation, \( \partial_+ J_- + \partial_- J_+ = 0 \), and Eqs.(3.3) and (3.4), we obtain the field equation of \( \lambda \):

\[
\partial_- \lambda = 0. \quad (3.8)
\]

The canonical energy-momentum tensor is given by:

\[
T_{\mu\nu} = i \bar{\Psi} \gamma_\nu \partial_\mu \Psi - F_{\nu\rho} \partial_\mu A^\rho + g_{\mu\nu} \left\{ \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} + \lambda (A_0 - A_1) \right\} \quad (3.9)
\]

where we have used the field equation of the Fermion field. Components in light-cone coordinates are given explicitly by:

\[
T_{++} = i \bar{\Psi}^* \partial_+ \Psi_+ - (F_{++}, \partial_+ A_+)_+ = i \bar{\Psi}^* \partial_+ \Psi_+ + (\partial_- A_+, \partial_+ A_+_+) , \quad (3.10)
\]
\[
T_{-+} = i \bar{\Psi}^* \partial_- \Psi_- - (F_{+-})^2 - (F_{+-}, \partial_- A_+)_+.
\]
\[ T_{++} = i\bar{\Psi}_+^* \partial_+ \Psi_+ - (F_{-+})^2 - (F_{++}, \partial_+ A_-)_+ \]
\[ = i\bar{\Psi}_+^* \partial_+ \Psi_+ - (\partial_- A_-)^2, \]  \hfill (3.12)
\[ T_{--} = i\bar{\Psi}_+^* \partial_- \Psi_+ + (F_{++}, \partial_- A_-)_+ \]
\[ = i\bar{\Psi}_+^* \partial_- \Psi_+ \] \hfill (3.13)

where we have used the gauge fixing condition.

From these expressions we can see some of the problems we will encounter when we apply the canonical formulation to the Schwinger model in the representation generated by modes of the fields along either light-cone surface. When we construct a light-cone-temporal gauge formulation, \(^5\) in which \(x^-\) is chosen to be the evolution parameter, we use \(T_{++}\) and \(T_{-+}\) as densities to calculate the translational generators. We see that \(\Psi_+\) and \(\bar{\Psi}_+_\) are not contained in the densities so that we can not treat \(\Psi_+\) as a degree of freedom. If we consider the standard light-cone gauge treatment (the light-cone-axial gauge), in which \(x^+\) is the evolution parameter and \(T_{++}\) and \(T_{-+}\) are the densities of the translational generators we see that we need zero mode fields (fields which are functions only of \(x^+\)). These fields will require special treatment.

We shall show in section 4 that we can find a light-cone-temporal gauge solution by expressing \(\Psi_+\) as a functional of \(A_+\) (it is done the other way around in the light-cone gauge). The problem of the zero-mode fields is principally one of recognizing them. In fact we can find those missing terms by defining the translational generators in the light-cone coordinate space by requiring that they are identical to those in ordinary coordinate space(\(x^0, x^1\)). \(^6\) From the divergence equation
\[ \partial^\nu T_{\mu \nu} = 0 \]  \hfill (3.14)
we obtain:
\[ \oint T_{\mu \nu} d\sigma^\nu = 0. \]  \hfill (3.15)

If we perform the integral over the closed surface shown in Fig.1, it is clear that the integral over the surface \(t = x^0\) is the negative of that over the light-cone surfaces. Thus we obtain
\[ \int_{-L}^{L} T_{\mu 0}(x^0, x^1) dx^1 = \int_{x^0 - L}^{x^0 + L} T_{\mu +}(x^+, x^- = x^0) dx^+ \]
\[ + \int_{x^0 - L}^{x^0} T_{\mu -}(x^+ = x^0 + L, x^-) dx^- + \int_{x^0}^{x^0 + L} T_{\mu -}(x^+ = x^0 - L, x^-) dx^-. \]
Hence in the limit $L \to \infty$ we obtain

$$P_\mu = \int_{-\infty}^{\infty} T_{\mu 0}(x^0, x^1)dx^1 = \int_{-\infty}^{\infty} T_{\mu +}(x^+, x^- = x^0)dx^+$$

$$+ \int_{-\infty}^{x^0} T_{\mu -}(x^+ = \infty, x^-)dx^- + \int_{x^0}^{\infty} T_{\mu -}(x^+ = -\infty, x^-)dx^-.$$  \hfill (3.16)

We show in section 4 that $T_{\mu -}$ is expressed solely in terms of a massive field so that $T_{\mu -}$ tends to 0 when $x^+ \to \pm \infty$. In that case we have:

$$P_+ = \int_{-\infty}^{\infty} \{ i\Psi_-^* \partial_+ \Psi_+ + (\partial_- A_+ + \partial_+ A_+) \} dx^+,$$  \hfill (3.17)

$$P_- = \int_{-\infty}^{\infty} (\partial_- A_+)^2 dx^+.$$  \hfill (3.18)

We see from this that in the temporal gauge formulation there are no missing degrees of freedom. We also see that the canonical momenta of $A_+$ and $\Psi_-$ are $2\partial_- A_+$ and $i\Psi_+^*$ respectively.

Similarly carrying out the contour integral shown in Fig.2, we obtain:

$$P_\mu = \int_{-\infty}^{\infty} T_{\mu 0}(x^0, x^1)dx^1 = \int_{-\infty}^{\infty} T_{\mu +}(x^+, x^- = x^0)dx^-$$

$$+ \int_{-\infty}^{x^0} T_{\mu +}(x^+ = \infty)dx^+ + \int_{x^0}^{\infty} T_{\mu +}(x^+, x^- = -\infty)dx^+.$$  \hfill (3.19)
We show in section 4 that if we choose nonvanishing initial values, then $T_{++}$ tends to them in the limit $x^- \to \pm \infty$. In that case we have

$$P_+ = \int_{-\infty}^{\infty} \{i\Psi_+^* \partial_+ \Psi_+ - (\partial_+ A_+)^2 \} dx^- + \int_{-\infty}^{\infty} T_{++}(x^+, x^- = \pm \infty) dx^+. \quad (3.20)$$

We also show that $T_{-+}$ vanishes in the limit $x^- \to \pm \infty$ so that we have:

$$P_- = \int_{-\infty}^{\infty} T_{-+}(x^+, x^-) dx^- = \int_{-\infty}^{\infty} i\Psi_+^* \partial_- \Psi_+ dx^-. \quad (3.21)$$

Now we derive quantization conditions for the canonical fields $A_+$, $\Psi_-$ and $\Psi_+$ by requiring that their commutation relations with $P_+$ and $P_-$ give rise to Heisenberg equations which are consistent with the field equations (the argument is similar to that of 6)). In the temporal gauge formulation $P_+$ in (3.17) is the kinematical operator so that we obtain the Heisenberg equation:

$$i[P_+, \Psi_-(x)] = \partial_+ \Psi_-(x) \quad (3.22)$$

if we require the equal-$x^-$ quantization conditions:

$$\{\Psi_-(x^+, x^-), \Psi_+^*(y^+, x^-)\}_+ = \delta(x^+ - y^+),$$

$$\{\Psi_-(x^+, x^-), \Psi_-(y^+, x^-)\}_+ = 0, \quad (3.23)$$
Furthermore, we obtain the Heisenberg equations:

\[ i[P_+, A_+(x)] = \partial_+ A_+(x), \quad i[P_+, \Psi_+(x)] = \partial_+ \Psi_+(x) \]  
(3.25)

if we require, in addition, the equal-\( x^- \) quantization conditions:

\[
[A_+(x^+, x^-), A_+(y^+, x^-)] = 0, \\
[\partial_- A_+(x^+, x^-), A_+(y^+, x^-)] = -i\frac{\delta}{2}(x^+ - y^+). \\
\{\Psi_+(x^+, x^-), \Psi_+(y^+, x^-)\} = 0, \\
\{\Psi_+(x^+, x^-), \Psi_-(y^+, x^-)\} = 0.
\]  
(3.26)

We remark that the second commutation relation in (3.28) is unusual in the canonical formalism and that at this point, nothing is known about the equal-\( x^- \) commutation relations between \( \Psi_+ \) and \( \Psi_+^* \). We also remark that, although \( A_+ \) obeys a field equation of the second order in the light-cone temporal gauge formulation, as is seen from (3.3), the commutator \( [\partial_- A_+(x^+, x^-), \partial_- A_+(y^+, x^-)] \) is not zero but has the following nonvanishing value

\[
[\partial_- A_+(x^+, x^-), \partial_- A_+(y^+, x^-)] = -i\frac{m^2}{16}\epsilon(x^+ - y^+). \\
\]  
(3.29)

This is because consistent operator solutions are obtained if and only if we regularize the Fermi products in a gauge invariant way. (In the Schwinger model, regularizing the current operators and the Fermionic kinetic terms gauge invariantly gives rise to the chiral anomaly.)

It is shown in section 4 that (3.24) combined with gauge invariant point splitting for the term \( i\Psi^* \partial_+ \Psi_- \) gives rise to \(-\frac{m^2}{4} A_+^2\) so that (3.29) is required to produce the Heisenberg equation

\[ i[P_+, \partial_- A_+(y^+, x^-)] = \partial_+ \partial_- A_+(x). \]  
(3.30)

Now that we have obtained the quantization conditions in the temporal gauge formulation, we can make use of them to obtain the Heisenberg equations which the dynamical \( P_- \) in (3.18) produces. Straightforward calculation gives

\[
i[P_-, \Psi_-(x)] = 0, \quad i[P_-, A_-(x)] = \partial_- A_-(x), \quad i[P_-, A_+(x)] = -\frac{m^2}{4}(\partial_+)^{-1}\partial_- A_+(x), \quad i[P_-, \Psi_+(x)] = -\frac{ie}{2}(\partial_+)^{-1}\partial_- A_+(x), \Psi_+(x)) = 0.
\]  
(3.31)
We see from the Heisenberg equation of \( \partial_- A_+ \) that \( \partial_- A_+ \) behaves like a free field of mass \( m \).

In the axial gauge formulation \( P_- \) in (3.21) is the kinematical operator so that we obtain the Heisenberg equations:

\[
i[P_-, \Psi_-(x)] = 0, \quad i[P_-, \Psi_+(x)] = \partial_- \Psi_+(x)
\] (3.32)

if we specify the following equal-\( x^+ \) commutation relations:

\[
\{\Psi_-(x^+, x^-), \Psi_+(x^+, y^-)\}_+ = 0, \quad \{\Psi_-(x^+, x^-), \Psi_+(x^+, y^-)\}_+ = 0,
\] (3.33)

\[
\{\Psi_+(x^+, x^-), \Psi_+(x^+, y^-)\}_+ = \delta(x^+ - y^-), \\
\{\Psi_+(x^+, x^-), \Psi_+(x^+, y^-)\}_+ = 0.
\] (3.34)

We can not obtain any other quantization conditions unless we solve Eqs. (3.3) and (3.6).

§4. Construction of operator solutions

Now that we have the algebra of the fields, we can proceed to construct the solution. We might consider the problem as an initial value problem on \( x^- = 0 \) or \( x^+ = 0 \), or, proceed in a more covariant way. Here, we shall consider the initial value problem on each characteristic. First we consider the initial value problem on the surface \( x^- = 0 \).

4.1. Light-cone temporal gauge solution

From (3.3) we see that when \( A_- = 0 \), the first component, \( \Psi_- \), is a free field depending only on \( x^+ \). Thus we specify \( \Psi_- \) as a free, massless Fermion field:

\[
\Psi_-(x) = \psi_-(x)
\] (4.1)

satisfying the anticommutation relations:

\[
\{\psi_-(x), \psi_-(y)\}_+ = \delta(x^+ - y^+), \quad \{\psi_-(x), \psi_-(y)\}_+ = 0.
\] (4.2)

Furthermore we make use of the fusion field defined by:

\[
:e^{\psi^*_-(x)\psi_-(x)} := m\partial_+ \phi(x)
\] (4.3)

to express \( \psi_- \) in the following equivalent bosonized form:

\[
\psi_-(x) = Z_- \exp[-2i\sqrt{\pi} \phi^-(x)] \sigma_- \exp[-2i\sqrt{\pi} \phi^+(x)].
\] (4.4)
Here, $Z_-$ is the finite normalization constant, $\sigma_-$ is the spurion operator and $\phi^{-}$ and $\phi^{+}$ are positive and negative frequency parts of $\phi$ regularized a la Klaiber (the construction is exactly the same as in (2.5), (2.7) and (2.11)). By construction $\phi$ satisfies the following commutation relation:
\[
[\phi(x), \phi(y)] = -\frac{i}{4} \epsilon(x^+ - y^+). \tag{4.5}
\]

Then, carrying out a gauge invariant point splitting procedure we find the following well-defined current:
\[
J_+(x) = \lim_{y^+ \to x^+} \frac{e}{2} \{ \psi_+(x^+) \psi_-(y^+) \exp[-ie \int_{y^+}^{x^+} A_+(z^+, x^-) dz^+] + h.c. \}
= m \partial_+ \phi(x) - \frac{m^2}{2} A_+(x). \tag{4.6}
\]

Furthermore, the kinetic term $i \psi^\dagger \partial_+ \psi_-$ is regularized as follows:
\[
\lim_{y^+ \to x^+} \frac{ie}{2} \{ \psi_+(x^+) \partial_+ \psi_-(y^+) \exp[-ie \int_{y^+}^{x^+} A_+(z, x^-) dz] + h.c. \}
= (\partial_+ \phi)^2 - \frac{m^2}{4} (A_+)^2. \tag{4.7}
\]
so that $P_+$ in (3-17) is given by
\[
P_+ = \int_{-\infty}^{\infty} \{ (\partial_+ \phi)^2 - \frac{m^2}{4} (A_+)^2 + (\partial_- A_+, \partial_+ A_+) \} dx^+. \tag{4.8}
\]

Now note that owing to (1-6), Eq.(3-4) can be written as
\[
(4 \partial_+ \partial_- + m^2) A_+(x) = 2 \{ \lambda(x) + m \partial_+ \phi(x) \}. \tag{4.9}
\]

Multiplying this by $\partial_-$ leads to
\[
(4 \partial_+ \partial_- + m^2) \partial_- A_+(x) = 0 \tag{4.10}
\]
due to the fact that $\phi(x)$ and $\lambda(x)$ depend only on $x^+$. Thus we see that $\partial_- A_+$ is a free field of mass $m$. Since $\partial_- A_+$ is gauge invariant and is equal to $-\frac{m}{2} \tilde{\Sigma}$ in the Landau gauge operator solution, our present result is in agreement with the earlier results. Here we set
\[
\partial_- A_+(x) = -\frac{m}{2} \tilde{\Sigma}. \tag{4.11}
\]
where the normalization is determined by (3-29). Then we notice that replacing $\partial_- A_+$ in (1-9) by $\tilde{\Sigma}$ enables us to express $A_+$ in terms of $\phi, \lambda$ and $\tilde{\Sigma}$ as
\[
A_+(x) = \frac{2}{m^2} \{ \lambda + m \partial_+ (\phi + \tilde{\Sigma}) \}. \tag{4.12}
\]
The commutation relation (3.29) is transcribed as that of $\tilde{\Sigma}$:

$$\left[ \tilde{\Sigma}(x^+,x^-), \tilde{\Sigma}(y^+,x^-) \right] = -\frac{i}{4}\epsilon(x^+ - y^+)$$  \hfill (4.13)

and the commutation relations of $\lambda$ are obtained by rewriting (4.9) for $\lambda$ as

$$\lambda = \frac{1}{2}\{ m^2A_+ + 4\partial_+(\partial_-A_+) - m\partial_+\phi \}$$  \hfill (4.14)

and by making use of the commutation relations (3.26), (3.29), (4.5) and

$$\left[ \phi(x), A_+(y) \right] = 0, \quad \left[ \phi(x), \partial_+A_+(y) \right] = 0,$$

which result from (3.24), (4.1) and (4.3). Combining these results we get

$$\left[ \lambda(x), \lambda(y) \right] = 0, \quad \left[ \lambda(x), \tilde{\Sigma}(y) \right] = 0, \quad \left[ \lambda(x), \phi(y) \right] = i\frac{m}{2}\delta(x^+ - y^+).$$  \hfill (4.16)

Now we see that $\lambda$ is a zero norm field and that Maxwell’s equations are recovered in a physical subspace formed by factoring the zero norm field $\lambda$ out of the representation space.

If we rewrite $\lambda$ as

$$\lambda(x) = m\partial_+(\eta(x) - \phi(x))$$  \hfill (4.17)

and take account of the third commutation relation of $\lambda$ in (4.16) we find that $\eta$ is a negative norm field depending only on $x^+$ and satisfies the following commutation relations

$$\left[ \eta(x), \eta(y) \right] = \frac{i}{4}\epsilon(x^+ - y^+), \quad \left[ \eta(x), \tilde{\Sigma}(y) \right] = 0, \quad \left[ \eta(x), \phi(y) \right] = 0.$$  \hfill (4.18)

As a result $A_+$ in (4.12) may be written as

$$A_+ = \frac{2m}{m}\partial_+(\tilde{\Sigma} + \eta).$$  \hfill (4.19)

In terms of these fields $P_+$ in (4.8) and $P_-$ in (3.18) are diagonalized as follows

$$P_+ = \int_{-\infty}^{\infty} \{ (\partial_+\phi)^2 - (\partial_+\eta)^2 + (\partial_+\tilde{\Sigma})^2 \} dx^+,$$

$$P_- = \int_{-\infty}^{\infty} \frac{m^2}{4}(\tilde{\Sigma})^2 dx^+,$$

which shows that $\phi, \eta$ and $\tilde{\Sigma}$ are constituent free fields of the light-cone temporal gauge Schwinger model.

Let us turn to specifying $\Psi_+$, which satisfies (3.3) and (3.6). Because in the temporal gauge formulation there is no dynamical equation which allows us to determine $\Psi_+$ as an initial value problem, we make use of the fact that using (4.19), we can write Eq. (3.19) as

$$J_- = e\Psi_+^*\Psi_+ = m\partial_-\tilde{\Sigma}.$$  \hfill (4.22)
We see from this that the $\tilde{\Sigma}$ field can be identified as a fusion field composed of $\Psi^*_+ \Psi_+ \Psi_+$ and in turn can be expressed in an equivalent bosonized form (this result will be obtained in the axial gauge formulation). As a matter of fact $A_+$ given in (4.19) is identical with the electromagnetic field given in (2.8) so that the Fermion operator (2.2) satisfies Eq. (3.6). To this end we also regularize the bilinear product $e\bar{\Psi}^* \Psi_+$ by the gauge-invariant point splitting procedure. We see immediately that if we split only in the $x^+$ direction, then the sum $\tilde{\Sigma} \eta$ in the exponent behaves like a zero norm operator so that the procedure does not work. We also see that if we split only in the $x^-$ direction, then the $\eta$ field gives rise to a divergence at high frequencies. Therefore we have to split in an other direction. The following two-step limit, with $\epsilon$ being a timelike vector, gives us the desired result:

$$J_-(x) = \frac{e}{2} \lim_{\epsilon^+ \to 0} \{ \lim_{\epsilon^- \to 0} \left( \Psi^*_+(x + \epsilon) \Psi_+(x) \exp[-ie \int_x^{x^+ + \epsilon} dz^\nu A_\nu(z)] + h.c. \right) \} = m \partial_- \tilde{\Sigma}(x).$$

(4.23)

The axial-vector current $J^5_{\mu} \equiv \epsilon_{\mu\nu} J^\nu$, where $\epsilon_{+-} = -\epsilon_{++} = \frac{1}{2}$ and $\epsilon_{--} = \epsilon_{++} = 0$, satisfies (2.17). In addition, a conserved axial-vector current

$$J^5_{\mu} \equiv \epsilon_{\mu\nu} (J^\nu + m^2 A^\nu),$$

(4.24)

is obtained by regularizing $e\bar{\Psi} \gamma_\mu \gamma^5 \Psi$, with $\Psi_+$ and $\Psi_+$ being $\psi_-$ and (2.2) respectively, in the same manner as the vector current:

$$J^5_{C+}(x) = -\frac{e}{2} \lim_{\epsilon^+ \to 0} \{ \psi^*_-(x^+ + \epsilon^+) \psi_-(x^+) \exp[i \epsilon \int_x^{x^+ + \epsilon^+} A_+(z^+, x^-) dz^+] + h.c. \}$$

$$= -\{ m \partial_+ \phi(x) + m^2 A_+(x) \} = -m \partial_+ (\phi + \eta + \tilde{\Sigma}),$$

(4.25)

$$J^5_{C-}(x) = \frac{e}{2} \lim_{\epsilon^- \to 0} \{ \lim_{\epsilon^+ \to 0} \left( \Psi^*_+(x + \epsilon) \Psi_+(x) \exp[i \epsilon \int_x^{x^+ + \epsilon} dz^\rho \varepsilon_{\rho\sigma} A^\sigma(z)] + h.c. \right) \}$$

$$= m \partial_- \tilde{\Sigma}(x).$$

(4.26)

It can be shown furthermore that the Fermion operator (2.2) satisfies the anticommutation relations in (3.34), if we define them as $y^+ \to x^+$ limit so as to avoid any divergences.

We end the temporal gauge construction by defining physical space $V$ by

$$V = \{ |\text{phys} \rangle | \lambda^{(+)}(x)|\text{phys} >= 0 \}$$

(4.27)

where $\lambda^{(+)}(x)$ denotes the positive frequency part of $\lambda$. 14
4.2. *Light-cone axial gauge solution*

In the axial gauge formulation $x^+$ is taken to be the evolution parameter so that we can solve Eq. (3.11) as an initial value problem on $x^+ = 0$. As an initial value of $\Psi_+$ we take a free Fermi field $\Psi_R(0,x^-)$ and define a fusion field $\tilde{\phi}$ by

$$e\Psi_R^*(x^-)\Psi_R(x^-) := m\partial_-\tilde{\phi}(x^-).$$

(4.28)

Then, by construction $\tilde{\phi}$ satisfies the commutation relation

$$[\tilde{\phi}(x^-), \tilde{\phi}(y^-)] = -\frac{i}{4}\epsilon(x^- - y^-)$$

(4.29)

and Eq. (3.3) becomes

$$J_-(0,x^-) = m\partial_-\tilde{\phi}(x^-) = -2\partial_-^2 A_+(0,x^-)$$

(4.30)

where point splitting in the $x^-$ direction has enabled us to utilize the fact that $A_0 = 0$. From (4.30) we obtain

$$\partial_- A_+(0,x^-) = -\frac{m}{2}\tilde{\phi}(x^-).$$

(4.31)

As a consequence if we neglect the second unspecified term of (3.20) for the moment, then we can express $P_+$ solely in terms of $\tilde{\phi}$ as

$$P_+ = \int_{-\infty}^{\infty} \{J_-(0,x^-)A_+(0,x^-) - (\partial_- A_+)^2\} dx^-$$

$$= \int_{-\infty}^{\infty} (\partial_- A_+)^2 dx^- = \frac{m^2}{4} \int_{-\infty}^{\infty} (\tilde{\phi}(x^-))^2 dx^-.$$  

(4.32)

Furthermore, by making use of the equivalent bosonized form of $\Psi_R$ we can express $P_-$ in (3.21) as

$$P_- = \int_{-\infty}^{\infty} i\Psi_R^*(0,x^-)\partial_-\Psi_R(0,x^-) dx^- = \int_{-\infty}^{\infty} (\partial_- \tilde{\phi})^2 dx^-.$$  

(4.33)

It follows from (4.32) and (4.33) that the fusion field $\tilde{\phi}$ is again constituent free field of mass $m$.

The temporal evolution of $\Psi_+(0,x^-)$ is defined by making use of $P_+$ in (4.32) by

$$\Psi_+(x^+,x^-) = e^{iP_+ x^+} \Psi_R(0,x^-) e^{-iP_+ x^+}.$$  

(4.34)

Then by making use of the equivalent bosonized form we can write

$$\Psi_+(x^+,x^-) = Z \exp[-2i\sqrt{\pi} \tilde{\phi}(-)(x)] \sigma_R \exp[-2i\sqrt{\pi} \tilde{\phi}(+)(x)]$$

(4.35)

where

$$Z^2 = \frac{\tilde{\kappa} \epsilon^\gamma}{2\pi},$$  

(4.36)
\[
\tilde{\phi}^{(+)}(x) = \frac{i}{\sqrt{4\pi}} \int_0^\infty \frac{dp_-}{p_-} c(p_-)(e^{-ip_+x} - \theta(\kappa - p_-)), \quad \tilde{\phi}^{(-)}(x) = (\tilde{\phi}^{(+)}(x))^* \tag{4.37}
\]

with \( p_+ = \frac{m^2}{4p_-} \) and \( \sigma_R \) is the spurion operator

\[
\sigma_R = \exp\int_0^\kappa \frac{dp_-}{p_-}(c(p_-) - c^*(p_-)). \tag{4.38}
\]

Here we have omitted the Klein transformation factor. It is evident that \( \Psi_+ \) satisfies the anticommutation relations \((3.34)\).

Now we notice that on the surface \( x^+ = \text{constant} \), \( \Psi_+ \) behaves like a free fermion field and that both \( P_+ \) and \( P_- \) are diagonalized in terms of the fusion field. Thus we see that the common hope in the light-cone quantization that the light-cone bare states are closer to partons than the ordinary equal-time bare states is realized in its strongest possible form. At the same time we also see that the initial value problem which we have considered also gives rise to the well-known problem common to axial gauge quantizations: we come have ill-defined equal-\( x^+ \) commutation relations of \( A_+ \) because \( A_+ \) is obtained from \((4.31)\) as

\[
A_+ = -\frac{m}{2}(\partial_-)^{-1}\tilde{\phi} = \frac{2}{m}\partial_+\tilde{\phi}. \tag{4.39}
\]

Note that the difficulty results from the fact that the antiderivative \((\partial_-)^{-1}\) is not well-defined in any positive definite Hilbert space.\(^8\) Therefore we introduce the \( \eta \) field as in \((4.19)\) to regularize \((4.33)\) although doing so obscures the parton picture. To obtain a consistent solution we also introduce the Fermi field \( \psi_-(x^-) \) as well as the fusion field \( \tilde{\phi}(x^+) \) and solve Eqs.\((3.4)\) and \((3.5)\) in the same manner as in the temporal gauge formulation. This enables us to identify the \( \tilde{\Sigma} \) field with the fusion field \( \tilde{\phi} \) and thus hereafter we denote \( \tilde{\phi} \) as \( \tilde{\Sigma} \).

Furthermore by assuming that the massive degrees of freedom of \( T^{++} \) contained in \((4.20)\) vanishes as \( x^- \to \pm \infty \), we obtain

\[
T^{++}(x^+, x^- = \pm \infty) = (\partial_+ \phi)^2 - (\partial_+ \eta)^2 \tag{4.40}
\]

so that \( P_+ \) in \((3.20)\) is fixed to be

\[
P_+ = \frac{m^2}{4} \int_{-\infty}^{\infty} \tilde{\Sigma}^2 dx^- + \int_{-\infty}^{\infty} \{- (\partial_+ \phi)^2 - (\partial_+ \eta)^2\} dx^+. \tag{4.41}
\]

In this way we can reconstruct, in the axial gauge formulation, the \( P_+ \) given in \((4.20)\).

Now that \( A_+ \) possesses zero mode fields (fields independent of \( x^- \)), we have to take this fact into account when we solve Eq.\((3.6)\) as the initial value problem on the surface \( x^+ = 0 \). As an alternative initial value satisfying the equal-\( x^+ \) anticommutation relations we choose

\[
\Psi_+(0, x^-) = \exp[-2i\sqrt{\pi}\eta(0)]\Psi_R(x^-). \tag{4.42}
\]
Note that (4.42) has a diverging vacuum expectation value, which is inevitable as long as we respect the equal-$x^+$ anticommutation relations. In fact when we rewrite the exponential function of $\eta(0)$ as the normal product, divergences appear at low frequencies and at high frequencies but the divergence at high frequencies is canceled by the zero from the $\Psi_R$, whereas divergence at low frequencies remains.

To investigate connection between (4.42) and (2.2), we rewrite the exponential function of $\eta(0)$ as follows

$$e^{-2i\sqrt{\pi}\eta(0)} = \exp\left[\frac{1}{2} \int_{-\infty}^{\kappa} \frac{dk_+}{k_+}\right]\exp[-2i\sqrt{\pi}\eta(0)]\sigma_+\exp[-2i\sqrt{\pi}\eta(+) (0)].$$

(4.43)

At the same time we rewrite the spurion operator $\sigma_R$ as the normal product form:

$$\sigma_R = \exp[-\frac{1}{2} \int_0^{\kappa} \frac{dp_-}{p_-}]\exp[-\int_0^{\kappa} \frac{dp_-}{p_-}c^*(p_-)]\exp\left[\int_0^{\kappa} \frac{dp_-}{p_-}c(p_-)\right].$$

(4.44)

Then we see that because

$$\int_0^{\kappa} \frac{dp_-}{p_-} = \int_{\frac{m^2}{4\kappa}}^{\infty} \frac{dp_+}{p_+}$$

(4.45)

the divergence from the former is canceled by one from the latter if we require

$$\kappa = \frac{m^2}{4\kappa}.$$  

(4.46)

In that case the normalization factor $Z$ in (4.36) is identical with the $Z_+$ in (2.4) so that (4.42) agrees exactly with (2.2) at $x^+ = 0$.

Now we notice that the Fermion operator (2.2) is obtained from the initial value (4.42) with $\kappa = \frac{m^2}{4\kappa}$ as a result of the temporal evolution

$$\Psi_+(x^+, x^-) = e^{iP_+x^+}\Psi'_+(0, x^-)e^{-iP_+x^+}$$

(4.47)

and that the zero mode fields do not prevent us from obtaining

$$J_- = m\partial \tilde{\Sigma}, \quad i\Psi'_+ \partial_- \Psi_+ = (\partial_- \tilde{\Sigma})^2$$

(4.48)

and hence both $P_+$ and $P_-$ are already diagonal so that the parton picture is realized even when there exist zero mode fields in the formulation. That is the main finding of this paper.

We end this section by pointing out that we can not change the order of integrations and differentiations in the evaluation of the commutation relations of $A_+$. In fact they are obtained unambiguously if we first evaluate the two-dimensional commutator $[\eta(x) + \tilde{\Sigma}(x), \eta(y) + \tilde{\Sigma}(y)] = iE(x - y)$ and then differentiate the $iE(x - y)$ with respect $\partial_+^\eta$ and $\partial_+^\eta$. It turns out that

$$E(x) = \frac{1}{2\pi} \int_0^\infty \frac{dk_+}{k_+}\{ \sin k_+x^+ - \sin(k_+x^+ + \frac{m^2}{4k_+}x^-)\}$$

$$= \frac{1}{4}\epsilon(x^+) - \frac{\epsilon(x^+) + \epsilon(x^-)}{4}J_0(m\sqrt{x^2})$$

(4.49)
where \( J_0 \) denotes the Bessel function of order 0.

§5. Concluding remarks

In this paper we have shown that an operator solution to the light-cone gauge Schwinger model is obtained by solving the initial value problem on \( x^- = 0 \) to specify \( \Psi_- \) and one on \( x^+ = 0 \) to specify \( \Psi_+ \) simultaneously. The solution turns out to be independent of any particular representation. We have discussed the formulation and solution of the problem in the equal-time representation and in the light-cone representation. The problem is straightforward to formulate at equal time, but solving it involves a nontrivial diagonalization of the Hamiltonian.

If we consider the initial value problem on \( x^+ = 0 \), there are two difficulties which we must overcome: we must include the zero-mode fields; and we must regularize the Fermi products on our initial value surface. As for the first problem, a substantial ability to find the necessary zero-mode fields has developed starting with the work of Bassetto, Soldati and Nardelli\(^9\). Once the necessary fields are recognized, putting them into the solution is not difficult. Looking at the full solution we can see how the Fermi products are regulated when the covariant solution is split in a timelike direction. There are four sources of singularity: there is a singularity due to the \( \hat{\Sigma} \) field at high frequency; if the splitting is not zero in the \( x^- \) direction it gives a factor proportional to \((x^-)^{-1}\)—that singularity is necessary for the point splitting procedure to work; there is a singularity due to the \( \hat{\Sigma} \) field at low frequencies; if the splitting is not zero in the \( x^+ \) direction it gives a factor proportional to \((x^+)^{-1}\) and is therefore not regulated by splitting in the \( x^- \) direction—that singularity is cancelled by a zero from the ghost field. There is a potential singularity due to the \( \eta \) field at high frequencies; actually, it turns out to be a zero; if the splitting is not zero in the \( x^+ \) direction it gives a factor proportional to \((x^+)^{-1}\) and is the zero which cancels the low frequency singularity from the \( \hat{\Sigma} \) field. There is a low frequency singularity from the \( \eta \) field—that singularity is absorbed into the spurion and gives rise to the infrared states, linear combinations of which form the \( \theta \) states.

In DLCQ the \( p^+ = 0 \) singularity is regulated with periodicity conditions. In that case the continuum answer is not recovered for physical matrix elements, even if the problem is solved exactly at finite \( L \) and the limit \( L \to \infty \) is then taken.\(^{10}\) It may well be that for many problems it is necessary to carefully regulate the theory prior to imposing periodicity conditions. The DLCQ grid would then be just a numerical device, not a regulator. Such procedures have been suggested in ref.\(^{11}\). We do not think the techniques necessary to carry out that procedure are known for all cases but the knowledge is growing.
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References
1) Y. Nakawaki, Prog. Theor. Phys. 72 (1984), 134
2) B. Klaiber, Boulder Lectures in Theoretical Physics (Gordon and Breach, 1967), vol. XA, p. 141.
3) G. McCartor and D. Robertson, Z. Phys. C68 (1995), 345
4) A. Bassetto, G. Nardelli and E. Vianello, Phys. Rev. D56 (1997), 3631
5) This gauge was discussed in an unpublished talk by K. Hornbostel,1992, Dallas, Texas; it has recently been studied in:
   M. Morara and R. Soldati, Phys. Rev. D58 (1998), 105011
6) G. McCartor, Z. Phys. C41 (1988), 271
7) G. McCartor and D. Robertson, Z. Phys. C62 (1994), 349
8) N. Nakanishi, Phys. Lett. 131B (1983), 381
   N. Nakanishi, Quantum Electrodynamics, ed. T. Kinoshita (World Scientific, Singapore, 1990), p. 36.
9) A. Bassetto, R. Soldati, and G. Nardelli, Yang-Mills Theories in Algebraic Non-Covariant Gauges (World Scientific, Singapore, 1991).
10) G. McCartor, Int. J. Mod. Phys. A12 (1997), 1091
11) S. Brodsky, J. Hiller and G. McCartor, Phys. Rev. D58 (1998), 25005