Reduced SL(2, R) WZNW Quantum Mechanics*

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Abstract

The SL(2, R) WZNW → Liouville reduction leads to a nontrivial phase space on the classical level both in 0 + 1 and 1 + 1 dimensions. To study the consequences in the quantum theory, the quantum mechanics of the 0 + 1 dimensional, point particle version of the constrained WZNW model is investigated. The spectrum and the eigenfunctions of the obtained—rather nontrivial—theory are given, and the physical connection between the pieces of the reduced configuration space is discussed in all the possible cases of the constraint parameters.

1 Introduction

In the past several years the Toda models have been studied intensively. In these field theories scalar fields are coupled to each other by certain special exponential terms, in a way that corresponds to a simple Lie algebra. The Toda models can be considered as generalizations of the Liouville theory, which is of particular interest since it appears in many problems of physics and mathematics. An interesting means of deriving and studying the remarkable properties—integrability, conformal

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invariance and W-algebraic symmetry—of the Toda models is offered by the ob-
servation that the Toda theories can be obtained by a suitable reduction of the
Wess-Zumino-Novikov-Witten (WZNW) model [1]. The WZNW model is a theory
of a field that takes its values from a Lie group G, and the reduction procedure - by
imposing appropriate (first class) constraints - associates it to a Toda theory that
corresponds to the Lie algebra of G. In the case $G = SL(2, R)$ the reduced theory
is nothing but the Liouville theory.

However, the connection between the WZNW and Toda models is more intri-
cate. A closer look at the reduction procedure shows that it yields not exactly the
Toda theory but a richer structure, the Toda theory arises only as a component, a
subsystem of it. This aspect was first noticed in [2]. To study the precise relation of
the Toda models to the WZNW ones a recent work examined the $SL(2, R)$ WZNW
→ Liouville reduction from the phase space point of view [3]. The authors con-
sidered the classical $SL(2, R)$ WZNW model, imposed the appropriate constraints
and described the reduction of the phase space under the constraints. They found
that the reduced phase space contains two subsystems (nonintersecting open sub-
manifolds) that admit a clear physical interpretation. On both subsets the reduced
WZNW theory leads to the Liouville theory locally, but these two copies of Liouville
theories are not independent. The connection between them comes from a 'border
line', a lower dimensional surface in the reduced phase space connecting them. For
a better understanding of the situation [3] carried out a similar analysis on the 0 + 1
dimensional, point mechanical analogue of the $SL(2, R)$ WZNW model. This can
be thought as the space independent version, the 'zero mode sector' of the WZNW
model. In [3] it was found that depending on the signs of the constraint parameters
one can arrive at two different types of reduced theories. In both cases the phase
space reduces into two locally independent parts. The difference is that when the
constraint parameters have equal signs the two halves are disconnected, there is no
'border line' between them. As a result a classical motion cannot touch both parts.
Actually, though the reduced Hamiltonian is not of the usual form of the sum of a
kinetic and a potential term, the system behaves as if the two halves of the reduced
one dimensional configuration space were separated by an infinitely high potential
barrier. On the other hand, if the signs of the constraint parameters are opposite
then an analogue of an infinitely deep potential well characterises the situation. In
this case the two 'half worlds' are connected, a motion can cross the point that
separates the two halves of the reduced configuration space, moreover, the negative
energy motions will oscillate between the two parts. [3] concludes that it is not
enough to give the (global) reduced theory in the local coordinates of the reduced
phase space. The arising two components of the reduced theory seem to be inde-
pendent while actually they may have a physical connection, a correlated behaviour
which can be discovered only from the global point of view. ( [3] also contains results
about the $SL(n, R)$ point particle model and discovers some similar properties for
$n > 2$ as well.)

These are the features of the classical theories. What happens on the quan-
tum level? Because of the expected difficulties of quantizing the $SL(2, R)$ WZNW model it is useful to examine the point particle version. Turning to the masspoint theory the following questions arise naturally: what kind of relation connects the two half-worlds in the quantum theory? Are there any oscillating motions quantum mechanically when the the constraint parameters are of opposite sign? Similarly we may ask if in the case of coinciding signs the separation remains or a tunneling is allowed. Moreover, in the first case one may expect negative energy bound states. Are they really present? In this paper we solve the quantum theory of the masspoint version, and thus we can answer whether our expectations based on the classical behaviour hold. While—as a consequence of its rather nontrivial characteristics—the point particle problem is interesting in itself, its properties are expected to shed a light on the field theoretical case, just as it happened on the classical level [3].

The paper is organized as follows. Sect. 2 and 3 present the classical mechanics of the $SL(2, R)$ masspoint before resp. after imposing the point mechanical form of the WZNW $\to$ Liouville constraints. The quantum mechanics of the unconstrained and the constrained systems are established in Sect. 4 and 5. The reduced system splits into two parts in a symmetric way, Sect. 6 gives the eigenfunctions on one such part. Sect. 7 investigates the orthogonality and the completeness properties of these 'half-eigenfunctions'. The connection between the two parts is examined in Sect. 8. Sect. 9 discusses the coordinate independence of the definition of the constrained system. Sect. 10 gives the conclusions of the paper. The larger proofs and calculations belonging to the statements of Sect. 7 are presented in two appendices, one concerning orthogonality and one concerning completeness.

2 The classical mechanics of the unconstrained theory

The $SL(2, R)$ WZNW model is defined by the following action:

$$S_0 = \frac{m}{4\pi} \int d\sigma d\tau \text{Tr} \left[ (g^{-1} \partial_\sigma g)^2 - (g^{-1} \partial_\tau g)^2 \right] + m' \int d^3 x \epsilon^{ijk} \text{Tr} \left( g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g \right),$$

where $\sigma$ and $\tau$ coordinate a two dimensional Minkowski space, $g$ is an $SL(2, R)$-valued function of $\sigma$ and $\tau$ being periodic in $\sigma$ with period $2\pi$. The coefficients of the first term, the action of the $SL(2, R)$ sigma model, and the second, topological term called Wess-Zumino term, are denoted by $m/4\pi$ and $m'$ respectively. The point particle version, i.e. the physics of the zero modes of the field theory (1) is defined by restricting the configurations to the space-independent ones $g = g(\tau)$ only. Then the action reduces to

$$S = \frac{m}{2} \int d\tau \text{Tr} \left[ (g^{-1} \partial_\tau g)^2 \right].$$
The motion of the point particle is a function $g : R \rightarrow SL(2, R)$; we can see that the Wess-Zumino term does not contribute to the masspoint version. The theory possesses left and right translation symmetries under the transformations $g \mapsto hg$, $g \mapsto gh^{-1}$, $h \in SL(2, R)$, the corresponding conserved quantities are $J = \dot{g}g^{-1}$ and $\tilde{J} := g^{-1}\dot{g}$, taking their values in $sl(2, R)$, the Lie algebra of $SL(2, R)$.

The equation of motion following from the action (2) is $\dot{g}g^{-1} = 0$ (or, equivalently, $(\dot{gg}^{-1}) = 0$). Its solution is $g(t) = g(0)e^{At}$, where $A \in sl(2, R)$ is a kind of initial data specified by the initial conditions as $A = g(0)^{-1}\dot{g}(0)$. Using the basis $T_1 = \sigma_1, T_2 = i\sigma_2, T_3 = \sigma_3$ in $sl(2, R)$ (where $\sigma_k$-s denote the Pauli-matrices) $A$ can be written as $A^k T_k$ (Einstein-summation). We will make use of the identity

$$e^{cb\sigma_k} = \cosh R \mathbf{1} + \sinh R/R c^k \sigma_k$$

where $R^2 = \frac{1}{2}\text{Tr}[(c^k \sigma_k)^2]$ (this formula also holds for $\text{Tr}[(c^k \sigma_k)^2] < 0$ with $R$ being imaginary and for $\text{Tr}[(c^k \sigma_k)^2] = 0$ with $\cosh R \rightarrow 1, \sinh R/R \rightarrow 1$). Based on (3) the solution of the equation of motion is

$$g(t) = g(0) \left( \begin{array}{cc} \cosh(rt) + A^3 \sinh(rt)/r & (A^1 + A^2) \sinh(rt)/r \\ (A^1 - A^2) \sinh(rt)/r & \cosh(rt) - A^3 \sinh(rt)/r \end{array} \right)$$

with $r^2 = \frac{1}{2}\text{Tr}[(A^k T_k)^2]$ (also allowing zero or imaginary values of $r$). If the trace is negative then the solution also can be expressed as

$$g(t) = g(0) \left( \begin{array}{cc} \cos(\omega t) + A^3 \sin(\omega t)/\omega & (A^1 + A^2) \sin(\omega t)/\omega \\ (A^1 - A^2) \sin(\omega t)/\omega & \cos(\omega t) - A^3 \sin(\omega t)/\omega \end{array} \right)$$

with $\omega = \sqrt{-r^2}$. For $r = 0$ the solution actually reads

$$g(t) = g(0) \left( \begin{array}{cc} 1 + A^3 t & (A^1 + A^2) t \\ (A^1 - A^2) t & 1 - A^3 t \end{array} \right).$$

A compact form of formula (3) is $g(t) = g(0)(\cosh(rt) \mathbf{1} + \sinh(rt)/r A)$. We can see that the $r^2 < 0$ solutions are closed because of their trigonometrical time dependence, the $r^2 \geq 0$ motions are open.

To study the canonical structure of the theory let us consider a parametrization of $SL(2, R) g(\xi^i), i = 1, 2, 3$. The Lagrangian reads as

$$L(\xi, \dot{\xi}) = \frac{m}{2} h_{kl}(\xi) \dot{\xi}^k \dot{\xi}^l,$$

where $h_{kl}(\xi) = \text{Tr}[g^{-1}\partial_k gg^{-1}\partial_l g]$ is the metric tensor on $SL(2, R)$. The canonical momenta are

$$p_k = \frac{\partial L}{\partial \dot{\xi}^k} = mh_{kl}\dot{\xi}^l,$$

from which we get the Hamiltonian

$$\mathcal{H}(\xi, p) = \frac{1}{2m} h^{kl} p_k p_l$$

\[4\]
\( \{h^{kl}\} \), the inverse of the matrix \( \{h_{kl}\} \) exists because \( SL(2, R) \) is simple.

The Hamiltonian can be expressed as the function of \( \xi \) and \( \dot{\xi} \) as well. One finds that

\[
\mathcal{H}(\xi, \dot{\xi}) = \frac{m}{2} h_{kl}(\xi) \dot{\xi}^k \dot{\xi}^l = L(\xi, \dot{\xi}).
\]  

(10)

Combining this result with the coordinate free form of the Lagrangian (cf. eq. (4)) and \( g(t) = g(0) e^{At} \), the value of the Hamiltonian on a solution of the equation of motion can be determined easily:

\[
\mathcal{H} = \frac{m}{2} \text{Tr}[A^2] = mr^2.
\]  

(11)

As a consequence we can see that the energy is negative for a closed motion and nonnegative for open motions.

Let us turn to a special parametrization, namely the one which is based on the Gauss decomposition:

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\delta & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.
\]  

(12)

This parametrization describes any \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, R) \) except those having \( \delta = 0 \).

With \( \delta = \xi^1 \), \( a = \xi^2 \) and \( c = \xi^3 \) \( \{h_{kl}\} \) is of the form

\[
\{h_{kl}\} = \begin{pmatrix}
2/\delta^2 & 0 & 0 \\
0 & 0 & \delta^2 \\
0 & 0 & 0
\end{pmatrix}.
\]  

(13)

The determinant of this metric is

\[
h = -2\delta^2,
\]  

(14)

and the inverse of the metric tensor, \( \{h^{kl}\} \) has the form

\[
\{h^{kl}\} = \begin{pmatrix}
\delta^2/2 & 0 & 0 \\
0 & 0 & 1/\delta^2 \\
0 & 1/\delta^2 & 0
\end{pmatrix}.
\]  

(15)

Thus we arrive at the following Hamiltonian in this parametrization:

\[
\mathcal{H}(\delta, a, c, p_\delta, p_a, p_c) = \frac{1}{4m} \delta^2 p_\delta^2 + \frac{1}{m} \frac{p_a p_c}{\delta^2}.
\]  

(16)

Expression (14) shows that \( a \) and \( c \) are cyclic coordinates since \( \mathcal{H} \) is independent of them. This is an advantage of using the parameters \( \delta, a, c \). Later we will see that this parametrization fits very well for our further considerations. That’s why in the following we will work in these coordinates.
Now we impose the constraints that reduce the $SL(2,R)$ WZNW theory to the Liouville one. For space independent configurations they read

$$\text{Tr} \left[ e_{12} \dot{g} g^{-1} \right] = \mu, \quad \text{Tr} \left[ e_{21} g^{-1} \dot{g} \right] = \nu,$$

(17)

where $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Expressing (17) in the $\delta, a, c$ parametrization gives

$$p_a = m\mu, \quad p_c = m\nu.$$  

(18)

As $p_a$ and $p_c$ are constants of the unrestricted motion we see that the constraints mean nothing else but a special choice of some of the initial conditions. This is a general feature of first class constraints, that these constants of motion happen to be canonical momenta is the real advantage of the parameters $\delta, a, c$.

The reduced phase space can be obtained by factorizing the complete phase space by the gauge transformations these first class constraints generate. The action of these gauge transformations is

$$g \mapsto e^{\theta_L} e_{12} g e^{\theta_R} e_{21} = \begin{pmatrix} 1 & \theta_L \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ \theta_R & 1 \end{pmatrix},$$

(19)

(see [3]) where $\theta_L$ and $\theta_R$ are the two parameters of the transformations. This symmetry transformation acts on $a$ and $c$ as $a \mapsto a + \theta_L$, $c \mapsto c + \theta_R$, while it leaves $\delta, p_\delta, p_a$ and $p_c$ invariant. Thus the factorization simply means that $\delta$ and $p_\delta$ parametrize the reduced phase space.

Any motion $\delta(t), p_\delta(t)$ allowed by the constrained dynamics corresponding to the constraint parameters $\mu$ and $\nu$ can be obtained by taking a solution $\delta(t), a(t), c(t), p_\delta(t), p_a, p_c$ of the unconstrained theory where $p_a = m\mu, p_c = m\nu$. Roughly speaking we just have to omit $a(t)$ and $c(t)$. The coordinate $\delta \in R$ survives the reduction, thus one may consider it as the coordinate of the one dimensional configuration space of the reduced theory. The constrained dynamics is governed by the Hamiltonian

$$H(\delta, p_\delta) = \frac{1}{4m} \delta^2 p_\delta^2 + ms \frac{1}{\delta^2},$$

(20)

where $s = \mu\nu$.

Until now we have found the Gauss decomposition a very appropriate way of introducing coordinates on the group $SL(2,R)$ to reach the canonical structure of the reduced system. Now let us face the problematic side of this approach.

The topology of $SL(2,R)$ is not $R^3$ but $R^2 \times S^1$, consequently it cannot be covered by a single parametrization. In the case of the Gauss decomposition the signal of this is that the Gauss decomposition works only for $\delta \neq 0$, the elements $\begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$ are left out. As a result the canonical coordinates $\delta, a, c, p_\delta, p_a$ and $p_c$
parametrize only two nonintersecting open submanifolds, two 'open halves' of the whole phase space, corresponding to the two regions $-\infty < \delta < 0$ and $0 < \delta < \infty$. In particular, the point $\delta = 0$ is left out from the reduced configuration space. In the meantime, extracted from (4), the time dependence of $\delta$ is of the form $\delta(t) = C_1 \sinh(rt)/r + C_2 \cosh(rt)$ where $C_1$ and $C_2$ are arbitrary constants. Thus we can see that there exist motions that cross the $\delta = 0$ surface in the whole phase space—for example for imaginary values of $r$ oscillations occur between the regions $\delta > 0$ and $\delta < 0$. Furthermore, for such a motion $p_\delta$ tends to infinity as the particle reaches $\delta = 0$. This can be seen from $p_\delta(t) = 2m\delta/\delta^2$, the connection between $p_\delta$ and the smoothly varying quantity $\dot{\delta}$ (cf. (8) and (13)). Hence the coordinates $\delta, p_\delta$ of the reduced phase space seem to be able to describe only those parts of a motion when $\delta(t) \neq 0$, they cannot give an account of how the particle moves across $\delta = 0$.

Fortunately we can overcome these difficulties. First, let us complete the reduced configuration space by mapping the points $\left( \frac{\alpha}{\gamma}, \frac{\beta}{0} \right)$ of the configuration space of the unconstrained system to the point $\delta = 0$ of the reduced configuration space, just as we have mapped the points $\left( \frac{\alpha}{\gamma}, \frac{\beta}{\delta \neq 0} \right)$ to $\delta$. Second, let the recipe to follow a motion through $\delta = 0$ be to fit the quantity $\delta^2 p_\delta$. This recipe is clear from the unconstrained point of view: we simply fit $\dot{\delta}$ here. By these tricks we can eliminate the disadvantage of working with only one parametrization instead of covering $SL(2,\mathbb{R})$ with more than one patches. Clearly, the canonical formalism of the unconstrained system also owes the problem of the missing $\delta = 0$. Nevertheless, the solution given here for the reduced system applies in a straightforward way for the unconstrained one, too.

We will see that the $\delta = 0$ problem also arises in the quantum theory. There it appears as an irregular singularity of the Hamiltonian at $\delta = 0$ and the challenge will be to define the quantum theory on the whole configuration space despite this singularity.

Finally let us introduce a canonical transformation which transforms the constrained Hamiltonian to a form of a sum of a kinetic and a potential term. This can be achieved by the following transformation:

$$x := \sqrt{2} \ln \delta,$$  \hspace{1cm} (21)

$$p_x := \frac{1}{\sqrt{2}} \delta p_\delta.$$  \hspace{1cm} (22)

The resulting Hamiltonian is

$$H_x(x, p_x) = \frac{1}{2m} p_x^2 + mse^{-\sqrt{2}x}.$$  \hspace{1cm} (23)

The price we have to pay for having such a nice Hamiltonian is that by (21) we restricted ourselves to $\delta > 0$ only. (Or, because of the $\delta \leftrightarrow -\delta$ symmetry of the
system, to $\delta < 0$, if writing $-\delta$ instead of $\delta$ in (21). Remarkably, the logarithmic connection between $x$ and $\delta$ is the point mechanical analogue of the one that relates the field of the reduced $SL(2, \mathbb{R})$ WZNW theory to the Liouville field $\phi$ in the field theoretical case [3].

With the aid of (23) it is easy to analyse the three qualitatively different situations arising. If $s > 0$ then the potential increases exponentially as we travel to the negative $x$ direction. Thus for all the allowed motions with positive energies there is a turning point when moving to the negative direction towards $x = -\infty$ ($\delta = 0$). In this case there is no possibility for the masspoint to cross the border $\delta = 0$. For $s = 0$ we have a free particle moving along the $x$ axis. Now the 'point' $x = -\infty$ ($\delta = 0$) cannot be reached in a finite time interval so the masspoint cannot cross the border even in this case. In the case $s < 0$ an exponentially deep potential valley attracts the particle towards the negative direction, what’s more, the time needed to reach $x = -\infty$ happens to be finite. This shows that for $s < 0$ the particle may cross the border $\delta = 0$.

4 The quantum theory of the unconstrained system

Let us define the quantum mechanics of the point particle $SL(2, \mathbb{R})$ WZNW theory via canonical quantization. We use the coordinates $\delta, a, c$ and work in the coordinate representation. The wave functions are then complex valued functions defined for all $\delta \in \mathbb{R} \setminus \{0\}, a \in \mathbb{R}$ and $c \in \mathbb{R}$. We define the scalar product as

$$(\Psi_1, \Psi_2) := \int \Psi_1^* \Psi_2 \sqrt{-h} d\delta da dc. \quad (24)$$

Here $h$ denotes the determinant of the matrix $\{h_{kl}\}$ in the $\delta, a, c$ parametrization (cf. (14)). The measure in this integral is the usual one used on curved manifolds. Moreover, in our case the metric tensor is invariant under the left and right transformations. Consequently, if we adopt the left and right transformations for the wave functions:

$$[D_L(h)\Psi](g) := \Psi(h^{-1}g), \quad [D_R(h)\Psi](g) := \Psi(gh), \quad g, h \in SL(2, \mathbb{R}), \quad (25)$$

then the scalar product is also invariant. This property is inevitable if we want the left and right symmetries of the classical theory to be present on the quantum level as well. Observe that these natural requirements led to the appearance of a nontrivial weight function $\rho(\delta) := \sqrt{-h} = \sqrt{2}|\delta|$ in the integral (24).

In coordinate representation the canonical momenta are defined as partial derivatives with respect to the corresponding coordinates if the configuration space is flat. For curved configuration spaces this definition does not give symmetric operators because of the presence of the weight function in the scalar product. Symmetricity
and the requirement $[\hat{\xi}_k, \hat{p}_l] = i\hbar \delta_{kl}$ meet in the definition $\hat{p}_k := \frac{\hbar}{i} (\partial_k + \frac{1}{2} \partial_k \ln \sqrt{-h})$ (see [4] for example). In our case this gives $\hat{p}_\delta := \frac{\hbar}{i} (\partial_\delta + \frac{1}{2})$, $\hat{p}_a := \frac{\hbar}{i} \partial_a$ and $\hat{p}_c := \frac{\hbar}{i} \partial_c$.

Now the stage is set for the definition of the Hamiltonian. This is also a step which needs some care. The problem with the naive quantum analogue of the classical Hamiltonian (16):

$$\hat{H}_n = \frac{1}{4m} \delta^2 \hat{p}_\delta^2 + \frac{1}{m} \hat{p}_a \hat{p}_c \frac{1}{\delta^2}$$

is that $\hat{H}_n$ is not symmetric (do not forget the weight function!). We need to find an appropriate ordering of $\hat{\delta}$ and $\hat{p}_\delta$. Moreover, symmetricity is not the only requirement since we would like the Hamiltonian to be invariant under the left-right symmetry transformations as well. That is why we turn to the Laplacian of $SL(2, \mathbb{R})$:

$$\Delta \Psi = \frac{1}{\sqrt{-h} \partial \xi} \left( \sqrt{-hh^{ij}(\xi)} \frac{\partial}{\partial \xi^j} \right) \Psi,$$

the usual definition for curved manifolds, which reads in the case of the $\delta, a, c$ parametrization

$$\Delta \Psi = \frac{1}{2|\delta|} \partial_b (|\delta|^3 \partial_b \Psi) + \frac{2}{\delta^2} \partial_a \partial_c \Psi.$$  

The definition (27) and the properties of the metric tensor guarantee that the Laplacian is invariant not only under reparametrizations but under left and right transformations as well. The reason why the Laplacian is interesting is that $-\frac{\hbar^2}{2m} \Delta$ is just a re-ordered version of $\hat{H}_n$. So we define the Hamiltonian as

$$\left( \hat{H} \Psi \right) (\delta, a, c) := -\frac{\hbar^2}{4m} \frac{1}{|\delta|} \partial_b (|\delta|^3 \partial_b \Psi(\delta, a, c)) - \frac{\hbar^2}{m} \frac{1}{\delta^2} \partial_a \partial_c \Psi(\delta, a, c).$$

A simple calculation shows that this $\hat{H}$ is symmetric (with respect to the weight function $\rho$). This choice of $\hat{H}$ is the usual and natural one for curved configuration spaces (see, for example, [4]).

We mention that we have not defined the wave functions at $\delta = 0$ and the operators $\hat{\delta}$, $\hat{p}_\delta$ and $\hat{H}$ are apparently ill-defined at $\delta = 0$. These singularities are only coordinate artifacts here. This won’t be the case for the reduced system as we will see soon.

5 The quantum mechanics of the reduced system

Being ready with the quantum mechanics of the unconstrained theory the next task is to consider the quantum analogue of the constraints and see what the reduction yields. Let us impose the constraints on the quantum level as

$$\hat{p}_a \Psi = m \mu \Psi, \quad \hat{p}_c \Psi = m \nu \Psi$$

(30)
It is very easy to find the wave functions that satisfy (30), they are of the form
\[ \Psi(\delta, a, c) = \psi(\delta) e^{i\frac{\hbar}{m}(ma + nc)}. \] (31)

Remarkably, the action of the operators \( \hat{\delta}, \hat{p}_\delta \) and \( \hat{H} \) on a wave function of this form touches only its \( \delta \)-depending part. This makes it possible to work with \( \psi(\delta) \) instead of \( \Psi(\delta, a, c) \) and to consider the one dimensional quantum mechanics driven by a Hamiltonian \( \hat{H} \):
\[ (\hat{H}\psi)(\delta) = -\frac{\hbar^2}{4m} \frac{1}{|\delta|} \partial_\delta \left( |\delta|^3 \partial_\delta \psi(\delta) \right) + ms \frac{1}{\delta^2} \psi(\delta) \] (32)

together with the scalar product
\[ (\psi_1, \psi_2) := \int \psi_1^* \psi_2 \rho(\delta) \, d\delta \] (33)

Therefore in the following we investigate the properties of this one dimensional quantum system. (The classical \( H \) corresponding to this \( \hat{H} \) is just (20) as we expect.) Here we witness how the decoupling of the variables \( a, c \) from the system happens on the quantum level.

The wave functions (31) are not square integrable in the \( SL(2, R) \) sense. This is a natural consequence of the constraints that decrease the degrees of freedom by two, we will require square integrability 'in the reduced sense', i.e. with respect to the scalar product (33). The situation is similar to the case of a free masspoint in a three dimensional Euclidean space with the constraints classically imposed as \( p_y = \text{const.}, p_z = \text{const.} \). When we consider the corresponding quantum theory it is obvious that the normalizability of the wave functions must be understood 'in the one dimensional sense'.

The one dimensional problem we arrived at is quite an unusual one. The Hamiltonian is not a
\[ \psi \mapsto -b^2 \psi'' + V\psi \] (34)
-type (with a real constant \( b \) and a potential function \( V \)) and a nontrivial weight function is present in the scalar product. It would be very convenient if our system could be transformed to an 'ordinary' one with a Hamiltonian of the form (34) and with no weight function. For this purpose let us consider a transformation of \( \delta \) to a new variable \( x \):
\[ x = g(\delta) \] (35)
accompanied by a change of \( \psi(\delta) \) to a new wave function \( \chi(x) \) which is related to \( \psi(\delta) \) as
\[ \psi(\delta) = f(\delta) \chi(g(\delta)), \] (36)
where \( f \) and \( g \) are arbitrary real functions. To see how \( \hat{H} \) transforms under such a transformation one has to examine the transformation of the \textit{(time dependent)} Schrödinger equation:
\[ i\hbar \frac{\partial \psi}{\partial t}(\delta, t) = (\hat{H}\psi)(\delta, t). \] (37)
After a simple calculation the result can be written in the form

\[
i\hbar \frac{\partial \chi}{\partial t}(x,t) = -\frac{\hbar^2}{4m}\delta^2(g')^2\chi'' - \frac{\hbar^2}{4m} \left(3\delta g' + 2\delta^2 \frac{f'}{f} g' + \delta^2 g''\right)\chi' + \\
\left[-\frac{\hbar^2}{4m} \left(3\delta \frac{f'}{f} + \delta^2 \frac{f''}{f}\right) + ms \frac{1}{\delta^2}\right] \chi.
\] (38)

Requiring eq. (38) to have the form given by eq. (34) gives a number of equations. First of all the coefficient of the second derivative must be a negative constant, for convenience we want to set it to \(-\frac{\hbar^2}{2m}\). This yields

\[
\delta^2(g')^2 = 2. \tag{39}
\]

The solution of this condition is

\[
g(\delta) = \sqrt{2} \ln(c_1\delta) \tag{40}
\]

where \(c_1\) is an arbitrary positive constant; for the moment let us restrict ourselves to the positive \(\delta\) half line. Secondly we require that the \(\chi'\) term must vanish. This and (40) together yield

\[
f(\delta) = \frac{c_2}{\delta} \tag{41}
\]

with a nonzero constant \(c_2\). Under the transformation (36) with such an \(f(\delta)\) and \(g(\delta)\) the weight function changes as

\[
\int_{\delta_1}^{\delta_2} \psi_1^*(\delta) \psi_2(\delta) \sqrt{2} |\delta| d\delta = \int_{g(\delta_1)}^{g(\delta_2)} \chi_1^*(x) \chi_2(x) c_2^2 dx. \tag{42}
\]

We want to transform away the weight function entirely. This can be reached simply by choosing \(c_2 = 1\). The parameter \(c_1\) does not have such an interesting specific value, we set \(c_1 = 1\). The transformation of the variable

\[
x = \sqrt{2} \ln \delta \tag{43}
\]

and the wave function

\[
\psi(\delta) = \frac{1}{\delta} \chi(x) = \frac{1}{\delta} \chi(\sqrt{2} \ln \delta) \tag{44}
\]

lead to the following \(\hat{H}_x\):

\[
(\hat{H}_x \chi)(x) = -\frac{\hbar^2}{2m} \chi''(x) + \left(\frac{\hbar^2}{4m} + mse^{-\sqrt{2}x}\right) \chi(x). \tag{45}
\]

This transformation is just the quantum analogue of (21) and (23). (The additional constant in the potential term of \(\hat{H}_x\) is due to the ordering procedure we maintained at the definition of the quantum Hamiltonian.) Unfortunately the problem is the same as well: it works only for the positive half of the configuration space (or the negative one, if exploiting the symmetry \(\delta \leftrightarrow -\delta\)). Nevertheless, \(\hat{H}_x\) will be very useful to understand the physics encoded in \(\hat{H}\).
6 Eigenfunctions on the half configuration space

With the aid of the transformed Hamiltonian \( \hat{H} \) one can have a rough picture of the reduced theory. In the cases \( s > 0, s < 0 \) it is more or less similar to a system with a potential infinitely increasing or decreasing for \( \delta \to 0 \), while for \( s = 0 \) the system is somehow a 'sum' of two free theories. The potential valley of the case \( s < 0 \) suggests to use the Bohr-Sommerfeld quantization to get a first impression about the spectrum of \( \hat{H} \). For this purpose let's consider a classical motion with energy \( E \) and express \( p_\delta \) as a function of \( \delta \) using (20):

\[
p_\delta = \pm \frac{\sqrt{4mE\delta^2 - 4m^2s}}{\delta^2}. \tag{46}
\]

For the Bohr-Sommerfeld quantization condition the phase space area \( \oint p_\delta d\delta \) is needed as a function of \( E \). The problem is that for \( \delta \approx 0 \) the behaviour of \( p_\delta \) is \( p_\delta \sim \frac{1}{\delta^2} \) (here and in the following \( \approx \) means asymptotical or approximate equality and \( \sim \) means proportionality), so the integral is infinite. Thus the Bohr-Sommerfeld quantization is impossible.

What is the reason behind this? The answer is that \( \hat{H} \) is not bounded from below if \( s < 0 \). To see this let us consider a square integrable wave function \( \psi \) and define

\[
\psi_\lambda(\delta) := \lambda \psi(\lambda \delta). \tag{47}
\]

The \( \psi_\lambda \)-s are normalized to 1; the expectation value of \( \hat{H} \) in a state \( \psi_\lambda \) is

\[
(\psi_\lambda, \hat{H} \psi_\lambda) = (\psi_\lambda, \hat{H}_1 \psi_\lambda) + (\psi_\lambda, \hat{H}_2 \psi_\lambda), \tag{48}
\]

where \( \hat{H}_1 \) and \( \hat{H}_2 \) denote the first and second term of \( \hat{H} \), respectively (cf. (32)). The scaling properties of \( \hat{H}_1 \) and \( \hat{H}_2 \) are such that

\[
(\psi_\lambda, \hat{H}_1 \psi_\lambda) = (\psi, \hat{H}_1 \psi) \tag{49}
\]

and

\[
(\psi_\lambda, \hat{H}_2 \psi_\lambda) = \lambda^2 (\psi, \hat{H}_2 \psi) = \lambda^2 ms \int_{-\infty}^{\infty} |\psi(\delta)|^2 \frac{1}{\delta^2} \sqrt{2} |\delta| d\delta. \tag{50}
\]

In the r.h.s. of (50) \( \lambda^2 \) is multiplied by a negative number. As a result if \( \lambda \) increases to \( \infty \) then \( (\psi_\lambda, \hat{H} \psi_\lambda) \) tends to \( -\infty \).

This lack of a ground state causes the failure of the Bohr-Sommerfeld quantization. Fortunately the eigenvalues and eigenfunctions of \( \hat{H} \) can be determined exactly in all the cases \( s > 0, s = 0, s < 0 \). To do this we have to solve the equation \( \hat{H} \psi = E \psi \) as a differential equation of second order. This equation has three singular points: \( \delta = \pm \infty \), which are regular singular points and \( \delta = 0 \), which is an irregular singular point. Consequently one has to solve this equation restricted to the domains \( \delta \in \mathbb{R}^+ \) and \( \delta \in \mathbb{R}^- \) respectively and then to fit together the obtained 'half-eigenfunctions'. Because of the symmetry \( \delta \leftrightarrow -\delta \) it is enough to work on \( \mathbb{R}^+ \), the restriction of \( \hat{H} \) to \( \mathbb{R}^+ \) will be denoted by \( \hat{H}_+ \).
In the case \( s < 0 \) the eigenvalue equation can be transformed to the Bessel equation
\[
z^2 w'' + z w' + (z^2 - \nu^2) w = 0 \quad (51)
\]
by the substitutions
\[
z = \frac{k}{\delta} \quad (52)
\]
and
\[
\psi(\delta) = \frac{1}{\delta} w \left( \frac{k}{\delta} \right), \quad (53)
\]
where
\[
\nu^2 = 1 - \frac{4mE}{\hbar^2}, \quad (54)
\]
and
\[
k = \sqrt{-\frac{4m^2s}{\hbar^2}}. \quad (55)
\]
The two linearly independent solutions of (51)—existing for any complex value of \( \nu \)—are \( J_\nu(z) \) and \( Y_\nu(z) \) (for the conventions and properties concerning the Bessel functions cf. [5]). Similarly, for \( s > 0 \) the transformations
\[
z = \frac{\kappa}{\delta}, \quad (56)
\]
\[
\psi(\delta) = \frac{1}{\delta} w \left( \frac{\kappa}{\delta} \right) \quad (57)
\]
lead to the modified Bessel equation
\[
z^2 w'' + z w' - (z^2 + \nu^2) w = 0 \quad (58)
\]
with \( \nu^2 \) being the same as in (54) and
\[
\kappa = \sqrt{\frac{4m^2s}{\hbar^2}}. \quad (59)
\]
Now the two solutions of (58) are the modified Bessel functions \( I_\nu(z) \) and \( K_\nu(z) \). In the case \( s = 0 \) the transformation (43) is the most useful. The eigenfunctions of \( \hat{H}_x \) are \( \exp(\pm iKx) \), where \( K = \sqrt{2mE/\hbar^2 - 1/2} \) (cf. (45)). In the variable \( \delta \) they read
\[
\frac{1}{\delta} e^{\pm i\sqrt{2}K \ln \delta}. \quad (60)
\]
In the cases \( s > 0, \ s = 0 \) \( \hat{H}_x \) is bounded from below. The corresponding condition on the energy eigenvalues is \( E \geq \frac{\hbar^2}{4m} \). For \( s = 0 \) it means that \( K \) is a nonnegative real number. In the case \( s > 0 \) the condition gives \( \nu^2 \leq 0 \) causing that only the functions \( I_{iu}(z) \) and \( K_{iu}(z), \ u \in \mathbb{R} \) mean energy eigenfunctions. For \( s < 0 \)
the Bessel functions with real indexes lead to $E \leq \frac{\hbar^2}{4m}$ and the imaginary indexes correspond to $E > \frac{\hbar^2}{4m}$.

To get more acquainted with the eigenfunctions let us carry out a simple check of our physical picture that is based on $\hat{H}_x$. In the cases $s > 0$, $s < 0$ the potential term of $\hat{H}_x$ decreases exponentially to zero as $x$ tends to $\infty$. Consequently we expect that for $x \to \infty$ the eigenfunctions with $E \geq \sqrt{\frac{\hbar^2}{4m}}$ correspond to plane waves. (For $s = 0$ this expectation is satisfied trivially.) To see whether this is the case we will make use of the $z \approx 0$ behaviour of $J_{iu}(z)$ and $I_{iu}(z)$. Using (96) up to a constant of proportionality they are of the form

$$J_{iu}(z) \approx I_{iu}(z) \sim z^{iu}. \quad (61)$$

From (43) and (52) the connection between the variables $x$ and $z$ is

$$z = ke^{-\frac{x}{\sqrt{2}}}, \quad x = \sqrt{2} \ln \left( \frac{k}{z} \right), \quad (62)$$

(or, for $s > 0$, the similar formulas with $\kappa$ instead of $k$) so we can see that in the variable $x$ $J_{iu}$ and $I_{iu}$ are asymptotically plane waves. The momentum corresponding to them is $p = -\hbar u/\sqrt{2}$. Considering that for $x \to \infty V(x)$ tends not to zero but to $\sqrt{\frac{\hbar^2}{4m}}$ and quoting the connection between $\nu = iu$ and $E$ we find that the expectation ‘kinetic energy = $p^2/2m$’ is satisfied as well. The two other eigenfunctions, $Y_{iu}$ and $K_{iu}$ are linear combinations of $J_{iu}$ and $J_{-iu}$ resp. $I_{iu}$ and $I_{-iu}$. Thus they also behave the way we expect from our physical picture.

7 Orthogonality and completeness

It will be important to form a complete orthogonal system from the half-eigenfunctions, an orthogonal basis in $L^2(\mathbb{R}^+, \rho)$. In the case $s = 0$ this is simple: the set $\{\exp(\pm iKx) \mid K \in [0, \infty)\}$ is a complete orthogonal system (in the variable $x \in (-\infty, \infty)$). Therefore the same can be said about the functions (60) in the variable $\delta$ in $L^2(\mathbb{R}^+, \rho)$. For $s < 0$ it is shown in the appendices that there exist several independent choices of a complete orthogonal system. The different bases can be indexed by a $p \in (0, 2]$, the corresponding eigenvectors (given in the transformed form $w(z)$) are

$$J_q(z), \quad q = p, p + 2, p + 4, \ldots,$$

$$\cos \left( \frac{\pi}{2}p \right) J_0(z) + \sin \left( \frac{\pi}{2}p \right) Y_0(z),$$

$$e^{-i\theta_p(u)} J_{iu}(z) + e^{i\theta_p(u)} J_{-iu}(z), \quad u \in (0, \infty),$$

where

$$e^{i\theta_p(u)} = \frac{\cos \left( \frac{\pi}{2}p \right) \sinh \left( \frac{\pi}{2}u \right) + i \sin \left( \frac{\pi}{2}p \right) \cosh \left( \frac{\pi}{2}u \right)}{\sqrt{\cos^2 \left( \frac{\pi}{2}p \right) \sinh^2 \left( \frac{\pi}{2}u \right) + \sin^2 \left( \frac{\pi}{2}p \right) \cosh^2 \left( \frac{\pi}{2}u \right)}}. \quad (64)$$
For the case $s > 0$ the appendices prove that only one complete orthogonal system can be built from the functions $I_{iu}(z)$ and $K_{iu}(z)$, namely, the set
\[
\{K_{iu}(z) \mid u \in [0, \infty)\} \tag{65}
\]

What makes the difference that in the cases $s = 0$ and $s > 0$ the eigenbasis is unique while for $s < 0$ there are infinitely many complete orthogonal systems? The answer is in the self-adjointness of $\hat{H}_+$. For this reason we determine the deficiency index of $\hat{H}_+$. $\hat{H}_+$ is a differential operator of second order with real coefficients and two singular points $\delta = 0$, $\delta = \infty$. Its deficiency index is equal to the number of its orthogonal square integrable eigenfunctions corresponding to a non-real eigenvalue (cf. [6]) (the deficiency index does not depend on the eigenvalue chosen). In the case $s > 0$ the deficiency index is zero as for a fixed non-real $\nu^2$ none of the two linearly independent eigenfunctions—$I_{\nu}(z)$ and $K_{\nu}(z)$ in the variable $z$—is square integrable (cf. Appendix A). In the case $s < 0$ $J_{\nu}$ is square integrable while $Y_{\nu}$ is not (we can choose $\text{Re} \nu > 0$ without loss of generality, see Appendix A again). Thus in this case the deficiency index is one. For $s = 0$ the deficiency index is zero, which can be seen most easily in the variable $x$.

Now, starting with the case $s < 0$, we recall a theorem of [6], which states that if the deficiency index is one then the operator has several self-adjoint extensions. [6] also gives a condition for the different domains of definition of the different self-adjoint extensions. For $\hat{H}_+$ this condition says that a function $\psi(\delta)$ lying in the domain of definition of a self-adjoint extension has (to be smooth enough, cf. [6], and) to satisfy
\[
\lim_{\delta \to 0} \left[ \delta^3 \left( \psi^* \frac{dU_\nu^\vartheta}{d\delta} - \frac{d\psi^*}{d\delta} U_\nu^\vartheta \right) \right] = \lim_{\delta \to \infty} \left[ \delta^3 \left( \psi^* \frac{dU_\nu^\vartheta}{d\delta} - \frac{d\psi^*}{d\delta} U_\nu^\vartheta \right) \right], \tag{66}
\]
where
\[
U_\nu^\vartheta(\delta) = \frac{1}{\delta} J_{\nu}\left(\frac{k}{\delta}\right) + e^{i\vartheta} \frac{1}{\delta} J_{\nu}^*\left(\frac{k}{\delta}\right), \tag{67}
\]
with $\vartheta \in [0, 2\pi)$ and $\nu \in C \setminus R$, $\text{Re} \nu > 0$. $\vartheta$ and $\nu$ together index the different self-adjoint extensions.

Then if one examines which eigenfunctions are included in the domain of definition of a self-adjoint extension indexed by an arbitrarily chosen value of $\vartheta$ and $\nu$, a straightforward if lengthy calculation shows that these eigenfunctions are exactly the ones that form one of the complete orthogonal systems (63). The number $p$ which characterises this system is expressed by $\vartheta$ and $\nu$ as
\[
p = \text{Re} \nu + \frac{2}{\pi} \arcsin \left( \frac{\sin(\frac{\vartheta}{2})}{\sqrt{\sinh^2(\frac{\vartheta}{2}) \sin^2\left(\frac{\nu}{2}\right) + \cosh^2\left(\frac{\nu}{2}\right) \cos^2\left(\frac{\nu}{2}\right)}} \right) \pmod{2} \tag{68}
\]
(For deriving this result one can make use of the asymptotics of the Bessel functions and their derivatives, for the asymptotics cf. Appendix A). Hence the multiplicity
of the eigenbases origins in the multiple self-adjoint extensions of the differential operator $H_+$. In the cases $s > 0$ and $s = 0$ the deficiency index is zero. The appropriate theorem of [3] states that then the operator is self-adjoint. Consequently, the domain of definition is unique. All the eigenfunctions (or, more precisely, all the wave packets superposed from the eigenfunctions—remember that for both $s > 0$ and $s = 0$ all the eigenfunctions are non-normalizable) are lying in the domain of definition so the eigenbasis is unique as well (up to linear equivalence, i.e. except from trivial phase factors or, in the case $s = 0$, choosing two linear combinations of $\exp(iKx)$ and $\exp(-iKx)$ instead of $\exp(iKx)$ and $\exp(-iKx)$).

8 The eigenfunctions on the whole configuration space

To investigate the eigenfunctions of the full system the task is to sew together the half-eigenfunctions and build up a complete orthogonal system of ‘whole-eigenfunctions’ (in the following: eigenfunctions). In usual quantum mechanical systems, i.e. with a Hamiltonian of the form (34) and with no weight function in the scalar product, the conditions for fitting parts of an eigenfunction together are the continuity of the eigenfunction and the continuity—or in special cases a given jump—of its (space) derivative. Now we cannot expect that such conditions work. In fact, the $\delta \to 0$ behaviour of the half-eigenfunctions proves to be $\delta^{-\frac{\epsilon}{2}} \cos(k/\delta + \text{const.})$ in the case $s < 0$, $\delta^{-\frac{\epsilon}{2}} \exp(\pm i\sqrt{2}K \ln \delta)$ if $s = 0$ and $\delta^{-\frac{\epsilon}{2}} \exp(-\kappa/\delta)$ if $s > 0$. Thus this kind of fitting together is impossible. The situation is not better in the variable $x$ either, the half-eigenfunctions tend to 0 in the limit $x \to -\infty$ in the cases $s < 0$ and $s > 0$, while for $s = 0$ they behave as $\exp(\pm iKx)$. This infinite growth or decrease and infinitely rapid oscillating behaviour of the half-eigenfunctions origins in the irregular singularity of the Hamiltonian at $\delta = 0$.

Fortunately, the probability current is finite at $\delta \to 0$, it is this quantity we are able to fit. However, in our case the probability current is not of the usual form though. By deriving the continuity equation for the probability density from the Schrödinger equation the probability current proves to be

$$\frac{\hbar}{2\sqrt{2mi}|\delta|^3} \left( \psi^* \frac{d\psi}{d\delta} - \frac{d\psi^*}{d\delta} \psi \right).$$

Any $\psi$ can be expressed as a linear combination of eigenfunctions $\varphi_k$, which makes it possible to decompose the probability current as a sum of

$$\frac{\hbar}{2\sqrt{2mi}|\delta|^3} \left( \varphi_k^* \frac{d\varphi_l}{d\delta} - \frac{d\varphi_k^*}{d\delta} \varphi_l \right).$$

It can be verified that in each cases $s < 0$, $s = 0$, $s > 0$ such a quantity has a well defined finite limit for $\delta \to 0$ so the probability current is also finite at $\delta \to 0$.  

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We do not fit the probability current directly but carry out an equivalent procedure. In fact, fitting the probability current of the half-eigenfunctions is to ensure that the norm of a whole wave function does not change in time. The latter is equivalent to the self-adjointness of the whole Hamiltonian. We know that the eigenfunctions of a self-adjoint Hamiltonian are orthogonal. Conversely, a complete orthogonal system of the eigenfunctions of the Hamiltonian as a differential operator defines a self-adjoint Hamiltonian from the differential operator on an everywhere dense set in $L^2(R, \rho)$, which is our purpose. That’s why it is enough—while it is more interesting as well—to build up complete orthogonal systems out of the eigenfunctions instead of fitting the probability current.

Let us start with the case $s < 0$. An eigenfunction $\Phi(\delta)$ is generally of the form

\[
\begin{cases}
\alpha \varphi(-\delta) & \text{if } \delta < 0, \\
\beta \varphi(\delta) & \text{if } \delta > 0,
\end{cases}
\]

where $\varphi$ is a half-eigenfunction, defined on $R^+$. From this it follows immediately that at most two linearly independent eigenfunctions can correspond to an eigenvalue in a (whole-)eigenbasis. Another important observation is that if a value $p$ corresponds to $\varphi$—the index of the half-eigenbasis $\varphi$ is a member of $-$, this $p$ characterizes $\Phi$ as well. Now let us suppose that a complete orthogonal system of eigenfunctions does not include two linearly independent eigenfunctions that correspond to the same eigenvalue and have the same value $p$. (Later we will examine the other case as well, i.e. when one can find two such eigenfunctions in the system.) In this case there must be at least one eigenfunction in this eigenbasis with a different $p$. Otherwise we do not have completeness: there exist functions that are orthogonal to any basis vector but are not identically zero; such an example is a whole-eigenfunction that is not included in the basis but has the same value $p$.

For two eigenfunctions having different $p$-s $(\quad)_+$ and $(\quad)_-$, the restriction of their scalar product to the positive resp. negative half of the configuration space are not zero, consequently they are orthogonal only if one of them is of the form

\[
\text{const.} \begin{cases}
\varphi(-\delta) & \text{if } \delta < 0, \\
\lambda \varphi(\delta) & \text{if } \delta > 0,
\end{cases}
\]

and the other is of the form

\[
\text{const.} \begin{cases}
-\lambda^* \varphi(-\delta) & \text{if } \delta < 0, \\
\varphi(\delta) & \text{if } \delta > 0,
\end{cases}
\]

(with a different $\varphi$ but) with the same complex $\lambda$ from the set $\{ |\lambda| \leq 1, \text{ if } |\lambda| = 1 \text{ then } \arg \lambda \in [0, \pi) \}$. Let $p_1$ denote the value $p$ of the eigenfunction of the first form and $p_2$ the $p$ of the other one. The other eigenfunctions with $p_1$ also must have the form \((72)\) (with the same $\lambda$) and the other eigenfunctions with $p_2$ also must have the form \((73)\), in order to be orthogonal to these two eigenfunctions. These forms ensure that the further eigenfunctions are orthogonal to each other as well. Orthogonality
also excludes the existence of any eigenfunctions in the eigenbasis having a $p$ other than $p_1$ or $p_2$.

To examine completeness first let us see whether an arbitrary function $\psi_1$ from $L^2(R, \rho)$ having the form (72) (where now $\varphi$ is not a half-eigenfunction but an arbitrary half-function) can be spanned by these eigenfunctions. It is easy to see that this requirement is equivalent to that the restriction of the eigenfunctions with $p_1$ to $R^-$ have to form a complete half-eigenbasis (a $\psi_1$ is orthogonal to the eigenfunctions with $p_2$, hence only the eigenfunctions with $p_1$ contribute to it). After a similar treatment of the $\psi_2$-s of the form (73) we conclude that a complete system must consist of each of the eigenfunctions with $p_1$ (that have the form (72)) and each of the eigenfunctions with $p_2$. Then if any $\psi \in L^2(R, \rho)$ can be given as a sum of a $\psi_1$ and a $\psi_2$ then completeness is reached. With the notation

$$\psi(\delta) = \begin{cases} \psi_+(-\delta), & \delta < 0, \\ \psi_+(\delta), & \delta > 0 \end{cases}$$

the sum of the functions

$$\psi_1(\delta) := \begin{cases} \frac{1}{1+|\lambda|^2} (\psi_-(-\delta) + \lambda^* \psi_+(-\delta)), & \delta < 0, \\ \frac{\lambda}{1+|\lambda|^2} (\psi_-(-\delta) + \lambda^* \psi_+(-\delta)), & \delta > 0 \end{cases}$$

and

$$\psi_2(\delta) := \begin{cases} \frac{-\lambda^*}{1+|\lambda|^2} (-\lambda \psi_-(-\delta) + \psi_+(\delta)), & \delta < 0, \\ \frac{1}{1+|\lambda|^2} (-\lambda \psi_-(-\delta) + \psi_+(\delta)), & \delta > 0 \end{cases}$$

is $\psi$, thus the completeness of the considered system of eigenfunctions—which we shall denote by $(p_1, p_2, \lambda)$—is proven. We remark that the above decomposition of $\psi$ is a generalization of the decomposition of a function to a sum of an even and an odd function, which is actually the case $\lambda = 1$.

Now let us turn to the other case, i.e. when the complete orthogonal system of whole-eigenfunctions includes two linearly independent eigenfunctions with a same value $p$ and corresponding to the same eigenvalue. In this case the other eigenfunctions must be of this $p$ as well, otherwise they cannot be orthogonal to both of these eigenfunctions. Furthermore, the eigenvalues of the eigenbasis must be identical with the eigenvalues of the half-eigenbasis $p$ and must be doubly degenerated: in the case of a simply degenerated or missing eigenvalue any (other) eigenfunction corresponding to this eigenvalue is orthogonal to each eigenfunction from the system, which is in contradiction with completeness. The constants $\alpha$ and $\beta$ (see (71)) for the eigenfunctions of the system can be arbitrary, the only requirement is that for each eigenvalue the corresponding two eigenfunctions be linearly independent. The concrete values of these $\alpha$-s and $\beta$-s are not important, they only embody a choice of two basis vectors in a two dimensional linear subspace. Remarkably, such an eigenbasis is linearly equivalent to a one which—in the spirit of our notation—can be denoted by $(p, p, \lambda)$ (the equivalence holds for an arbitrary $\lambda$). Based on this observation one can prove completeness the same way as for a system $(p_1, p_2, \lambda)$. 

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We see that in contrast to the eigenbases \((p_1, p_2, \lambda)\) considered earlier, these latter eigenbases are characterized by a single number \(p\). Each of the different eigenbases \((p_1, p_2, \lambda)\) and \(p\) means a different self-adjoint extension of the Hamiltonian as a differential operator.

For \(s > 0\) and \(s = 0\) the method to establish an eigenbasis is the same as for an eigenbasis with a single \(p\) in the case \(s < 0\). The difference is that now one starts with only one half-eigenbasis, consequently one arrives at only one eigenbasis (up to linear equivalence). As a result in these cases the self-adjoint Hamiltonian is unique.

That for \(s < 0\) two different half-eigenbases are needed in general for one eigenbasis may seem unusual. However, this situation is just an analogue of the case of the operator \(-\partial^2/\partial x^2\) on the interval \([-\pi, \pi]\). This simple example will help us to understand - at least to some extent - what happens here.

The complete orthogonal system of the eigenvectors of \(-\partial^2/\partial x^2\) corresponding to the conditions

\[
\Phi(-\pi) = \Phi(\pi) = 0
\]  

consists of the functions

\[
\sin\left(\frac{n}{2}(x + \pi)\right), \quad n = 1, 2, \ldots
\]

These eigenfunctions are either even or odd functions, the even ones can be written as

\[
\sin\left(\frac{k}{2}x\right), \quad k = 2, 4, \ldots,
\]

the odd ones are of the form

\[
\cos\left(\frac{l}{2}x\right), \quad l = 1, 3, \ldots
\]

Let us examine how one can get this eigenbasis by building it up from the half-eigenbases of \(-\partial^2/\partial x^2\), the eigenbases of the operator restricted to \([0, \pi]\) and \([-\pi, 0]\).

The eigenfunctions of \([0, \pi]\) corresponding to the conditions

\[
\varphi(0) = \varphi(\pi) = 0
\]

are

\[
\sin mx, \quad m = 1, 2, \ldots
\]

We see that these functions can be the building blocks of the functions \((78)\), via an antisymmetric (i.e. odd) extension from \([0, \pi]\) to \([-\pi, \pi]\). On the other side, there is no way to build up the functions \((80)\) from them as well. The solution is that we have to consider another half-eigenbasis, namely, the one corresponding to the conditions

\[
\varphi'(0) = \varphi(\pi) = 0.
\]
The eigenfunctions satisfying (83) are

\[ \cos jx, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, \]  \hspace{1cm} (84)

and clearly a symmetric (i.e. even) extension of them to \([-\pi, \pi]\) yields the functions (80).

The operators \(-\partial^2/\partial x^2\) and \(\hat{H}\) are common in that both commute with the space reflection operator, and in that both have several self-adjoint extensions. These are the aspects why the operator \(-\partial^2/\partial x^2\) is a good example to understand \(\hat{H}\). However, crucial differences can be observed between the two operators. One of them is that the proof that a Hamiltonian with space reflection symmetry must have even and odd eigenfunctions does not hold for \(\hat{H}\). The other, more serious difference is that in the case of \(-\partial^2/\partial x^2\) there exists a physical principle to choose one from the different self-adjoint extensions. This is possible because the operator \(-\partial^2/\partial x^2\) together with the interval \([-\pi, \pi]\) arises in physics as the Hamiltonian of the system characterized by the potential

\[ V(x) = \begin{cases} 0 & \text{if } |x| < \pi, \\ \infty & \text{if } |x| > \pi. \end{cases} \]  \hspace{1cm} (85)

Then the requirements that an eigenfunction be continuous and that it be zero where \(V(x) = \infty\) assign one of the possible boundary conditions that characterize the different self-adjoint extensions (this distinguished condition is just (77)). In this case there is a way to choose the 'physical' self-adjoint extension of the Hamiltonian. For \(\hat{H}\) there is no such principle, all the self-adjoint extensions prove to be equal. We have to face the fact that there is no unique quantum mechanics corresponding to the \(s<0\) classical system.

In spite of the non-usual form of the Hamiltonian and the presence of the nontrivial weight function, the \(\hat{H}_x\) form of the Hamiltonian enables us to give the physical interpretation of the results to some extent.

In the case \(s<0\) we expect that the two half-configuration spaces are in physical connection, the particle can cross the border \(\delta = 0\). Simple calculations show that this expectation is satisfied for the self-adjoint extensions \((p_1, p_2, \lambda)\), there is a probability flow from one half to the other one. Consider for example a wave function \(\psi = c_1 \psi_1 + c_2 \psi_2\) where \(\psi_1\) and \(\psi_2\) are eigenfunctions, one having \(p_1\) and the other having \(p_2\). Though \(\frac{d}{dt} (\psi, \psi) = 0\), \(\frac{d}{dt} (\psi, \psi)_+ = -\frac{d}{dt} (\psi, \psi)_- \neq 0\) for generic \(c_1\) and \(c_2\). Another transparent possibility to show the physical connectedness of the two halves is that one can easily find examples for a solution of the (time dependent) Schrödinger equation where the expectation value of the coordinate operator \(\hat{\delta}\) is oscillating in time between a positive and a negative value. However, in the case of the self-adjoint extensions \(p\) the two halves behave as two closed, independent subsystems. The reason is that for these eigenbases the restriction of the eigenfunctions on a half configuration space is a half-eigenbasis, causing that \(\hat{H}\) decouples to two self-adjoint half-operators.
The cases \( s > 0 \) and \( s = 0 \) are similar to the \( s < 0 \), \( p \) one. The Hamiltonian is simply a pair of two self-adjoint half-Hamiltonians, the two parts of the configuration space are physically independent. This result is in accord with the naive pictures of the \( s > 0 \) and \( s = 0 \) systems based on \( H_x \). For \( s > 0 \) we can think of an exponentially increasing and thus infinitely wide potential wall separating the two half-worlds, no wonder that we find no tunneling from one side to the other. The situation is similar to the quantum mechanics of the system with the potential

\[
V(x) = \begin{cases} 
0 & \text{if } |x| > a, \\
\infty & \text{if } |x| < a.
\end{cases}
\] (86)

where it is meaningful to speak about the quantum mechanics of the system on the whole configuration space, yet there is no physical connection between the two allowed parts. In the case \( s = 0 \) we have two free theories, both on an infinitely large configuration space (understood in the variable \( x \)). We may argue that under such circumstances a wave packet starting from one side (e.g. the \( \delta > 0 \) one) cannot reach the other side in a finite time period. We cannot say any stronger concerning interpretation: these are the limits we are forced into.

9 Coordinate independence

The reduced quantum system, as we saw, possesses several unusual properties. It is natural to ask whether these features are only artifacts, caused by the special coordinate system which was used for the definition of the reduced system. Therefore it is worth examining the possibility to define the system in a coordinate independent way.

The Hamiltonian of the unconstrained quantum theory—a multiple of the Laplacian of the manifold \( SL(2, \mathbb{R}) \)—and the scalar product (24) are in fact coordinate invariant. Consequently the question reduces to whether the constraints can be given a coordinate independent form. In Sect. 5 the constraints were imposed through the canonical momentum operators. The definition \( \hat{p}_k := \frac{\hbar}{i} (\partial_k + \frac{1}{2} \partial_k \ln \sqrt{-\hbar}) \) does not define a covariant quantity because \( \hbar \) is not a coordinate invariant scalar. That’s why it is recommended to impose the constraints independently of the canonical momentum operators.

In the spirit of the Lie derivative, let us introduce the following derivation operators

\[
(L_A \Psi)(g) = \left. \frac{d}{ds} \Psi(e^{As} g) \right|_{s=0}, \quad (R_A \Psi)(g) = \left. \frac{d}{ds} \Psi(g e^{As}) \right|_{s=0} \] (87)

for any \( A \in sl(2, \mathbb{R}) \). The definition of \( L_A \) and \( R_A \) does not need any coordinate system. Nevertheless, if expressing them using the coordinates \( \delta, a, c \) one finds that \( L_{e_{12}} = \partial_a \) and \( R_{e_{21}} = \partial_c \). Thus we obtained a coordinate independent reformulation of the constraints (31).
One can feel the need for checking whether the operators $\frac{\hbar}{im}L_{e_{12}}$ and $\frac{\hbar}{im}R_{e_{21}}$ are really the quantum equivalents of the classical quantities $\text{Tr} [e_{12} g g^{-1}]$ and $\text{Tr} [e_{21} g^{-1} g]$. The following heuristic argument makes this relation visible.

We consider a wave packet which is in some sense the most similar to a classical trajectory, determine the expectation value of $\frac{\hbar}{im}L_A$, and compare it to $\text{Tr} [A g g^{-1}]$ computed on the classical trajectory the wave packet is similar to (the relation will hold for any $A \in \text{sl}(2, R)$ in general; the analogous treatment for $R_A$ is straightforward).

In the case of the free quantum mechanics on a three dimensional Euclidean space the Gaussian wave packet

$$\text{const.} \int d^3 k e^{-\frac{1}{\sigma^2} (k-k_0)^2} e^{ik(x-x_0)}$$

(88)

is in some sense the best wave mechanical analogue of a classical trajectory. It is well-localized both in position and in momentum—around the position $x_0$ and the wave vector $k_0$. We wish to define the analogue of this wave function in the case of $SL(2, R)$. The differences in the two configuration spaces are that $SL(2, R)$ does not have a linear structure and the natural metric on it is not positive definite. Nevertheless, we 'flatten' $SL(2, R)$ around a chosen point $g_0$—which we want as the 'centre' of the wave packet—by characterizing a point $g$ with a Lie algebra element denoted by $X_g$ defined by the relation $g = e^{X_g} g_0$. This definition is correct, i.e. $X_g$ exists and is unique, if $g$ is in a small enough neighbourhood of $g_0$, and this is enough for our purposes because we want to define only a fairly well localized wave packet around $g_0$. $X_g$ is the analogue of $x-x_0$ in the Euclidean case. As far as the indefinite metric is concerned: that a wave packet is localized is actually a question involving topology rather than metric. That’s why we will not use the natural metric $\{h_{kl}\}$, or, more precisely, the corresponding Killing form $G$, to express that our wave packet is localized. ($G$ is the metric tensor at the unity element of $SL(2, R)$, acting on the Lie algebra of $SL(2, R)$, $G(U, V) = \text{Tr}[U V]$). Instead, we introduce an arbitrary positive definite inner product $\tilde{G}$ in $sl(2, R)$ and apply this $\tilde{G}$ in the Gaussian modulus of the integrand in (88). This procedure is similar to the case of the Minkowski space, where for example differentiability or other analytic properties are defined not by the use of the Lorentzian metric but by an arbitrarily chosen positive definite inner product; it turns out that these properties are independent of the choice of this artificial inner product. In the phase of the integrand of (88) we keep the Killing form.

All in all, let us work with the wave packet

$$\Psi_{g_0,K_0}(g) = \text{const.} \int d^3 K e^{-\frac{i}{\sigma^2} \tilde{G}(K-K_0,K-K_0)} e^{iG(K,X_g)}.$$  

(89)

The width $\sigma$ will not have to be specified. This wave packet is by definition localized around $K_0$, evaluating the integral shows that the result is a Gaussian in position (also in the sense of $\tilde{G}$), localized around $g_0$ as expected.
The function $g \mapsto \exp iG(K, X_g)$ is approximately an eigenfunction of $\frac{\hbar}{m} L_A$ with eigenvalue $\frac{\hbar}{m} G(K, A)$. By this we mean that the eigenvalue equation holds exactly at $g = g_0$ and is satisfied approximately at a neighbourhood of $g_0$. Consequently the expectation value of $\frac{\hbar}{m} L_A$ is $\frac{\hbar}{m} G(K_0, A)$—approximately, i.e. in the limit when the width of the wave packet in position tends to zero. Then what is left is to show that $\frac{\hbar}{m} K_0 = \dot{g} g^{-1}$ for the classical trajectory corresponding to (89). To obtain this we make use of the observation that $\Delta = (G^{-1})^{ij} L_i L_j = (G^{-1})^{ij} R_i R_j$ where $\{I_i\}$ is an arbitrary basis in $sl(2, R)$. The most straightforward verification of this formula is to prove it in coordinates. From this one can derive that $g \mapsto \exp iG(K, X_g)$ is approximately an eigenfunction of $\hat{H} = -\frac{\hbar^2}{2m} \Delta$ with eigenvalue $\frac{\hbar^2}{2m} G(K, K)$. With the aid of these pieces of information we are able to tell the time propagation of the wave function (89). If considering (89) as the state function at $t = 0$ then after a small time period the state function is approximately

$$\Psi_{g_0, K_0}(g, t) = \text{const.} \int d^3 K e^{-\frac{\hbar^2}{2m} G(K - K_0, K - K_0)} e^{iG(K, X_g)} e^{-\frac{i}{\hbar} \left(\frac{\hbar^2}{2m} G(K, K) t\right)}. \quad (90)$$

(90) is also a localized wave packet, but its centre is not $g_0$ but a $g(t)$. This $g(t)$ plays the role of the classical position of the masspoint at $t$. Using a stationary phase argument and taking into account that the modulus of the integrand of (90) is concentrated around $K_0$ the centre of the wave packet (90) is at $X_{g(t)} = \frac{\hbar}{m} K_0 t$. Consequently, $\dot{g} g^{-1} = \frac{\hbar}{m} K_0$, and this is what we wanted to show.

We saw that both the unconstrained system and the constraints are actually coordinate independent. After defining a system in a coordinate independent way, one can use concrete parametrizations to examine its properties. Turning to the concrete situation: all the properties explored in the coordinates $\delta, a, c$ are valid everywhere where $\delta \neq 0$. For example, the wave functions are scalars so the infinite growth and infinitely rapid oscillating of the eigenfunctions is a coordinate independent fact, since this is the behaviour of the eigenfunctions not at but around the invalid point $\delta = 0$.

10 Conclusions

We investigated the properties of the point particle version of the reduced $SL(2, R)$ WZNW model both on the classical and the quantum level, for all the possible values of the constraint parameters. We found that the quantum theory exhibits an analogous behaviour to the classical one. The cases where the two parts are disconnected classically lead to two independent systems on the quantum level as well, and in the cases where the half-systems have a physical connection, this connection can also be found in the quantum theory. The only exception is that there is a possibility for a classically connected case to be disconnected quantum mechanically. This is possible because not only one quantum theory corresponds to a classically connected case. Several self-adjoint extensions of the Hamiltonian exist, including special ones where the two half-systems turn out to be independent.
Classical mechanically bounded motions exist, with arbitrary large negative energies, in the connected cases. The disconnected cases do not allow bounded motions and energy is bounded from below. These properties are also reflected on the quantum level. It is remarkable that the quantum theory is formally consistent irrespective of the values of the constraint parameters, while in the connected cases it leads to systems with a Hamiltonian not bounded from below (no matter which self-adjoint extension is chosen). Recently a method was proposed to discuss quantum mechanical systems that exhibit such a behaviour [7]. The method implements the concept of Wilson renormalization. It would be interesting to carry out such an analysis for the system studied here. Nevertheless, the method of [7] means a kind of distortion of the system, which is not the purpose here as here we are interested in the properties of the original system for we want to obtain indications how the quantum theory of the corresponding field theory behaves. In Sect. 3 we have found classical space-independent configurations with arbitrary large negative energy in the connected cases. This and the quantum properties of the masspoint version make it quite possible that the energy is essentially not bounded from below in the quantum field theory.

That the energy is not bounded from below is not the only nontrivial property of the connected case. The most striking result of our analysis is the existence of several self-adjoint extensions corresponding to one classical system. There is no -- physical or mathematical -- principle to choose one out of them as the 'real' one. The origin of this behaviour is the strong singularity at the border which separates the two half-systems. This singularity is not present on the unconstrained level, it is a consequence of the characteristics of the constraints. As this singularity can also be observed in the classical reduced field theory version [4], we expect to face the problem of the non-unique self-adjoint energy operator on the quantum level, i.e. in the quantum field theory of the reduced \( SL(2, R) \) WZNW model as well.

The method applied here to present the quantum mechanics of the reduced system was canonical quantization (supplemented by a coordinate independent approach). Because of the nontrivial properties found it would be interesting to examine this system by using other tools, geometric quantization or functional integration, and see how these methods give account of the characteristics of the theory.

Additionally we remark that recently a paper carried out an analysis of the relativistic quantum mechanics of a free particle on the \( SL(2, R) \) manifold [8]. The problem studied there is independent from the one presented here. Clearly, in [8] the group \( SL(2, R) \) plays the role of the (curved) space-time the particle exists in while in our case \( SL(2, R) \) is the (configuration) space of the unconstrained system.

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A Orthogonality

Let us first collect the properties of the Bessel functions and modified Bessel functions we will need in the following. $J_\nu(z)$ and $Y_\nu(z)$ are analytic functions of $z$ on the whole complex plane except the negative real half line. For a fixed $z$ ($z \neq 0$) both are integer functions of $\nu$ for any $\nu \in C$. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent for any value of $\nu$ except $\nu = n$ (let $n$ denote an integer subsequently), when

$$J_{-n}(z) = (-1)^n J_n(z).$$

(91)

$J_\nu(z)$ and $Y_\nu(z)$ are always linearly independent. The following formula

$$Y_\nu(z) = \frac{\cos(\pi \nu) J_\nu(z) - J_{-\nu}(z)}{\sin(\pi \nu)}$$

(92)

establishes a connection between the $J$-s and the $Y$-s; for $\nu = n$ (92) is understood as a limit when $\nu \to n$.

For the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ all the statements mentioned above apply if we replace $J_\nu(z)$ by $I_\nu(z)$ and $Y_\nu(z)$ by $K_\nu(z)$, with the exceptions that $I_{-n}(z) = I_n(z)$ and that

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi \nu)}.$$

(93)

Under complex conjugation $J_\nu(z)^* = J_{\nu^*}(z)$

(94)

is valid if $z$ is real. This property holds for $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ as well.

The modified Bessel functions can be expressed in terms of the ordinary Bessel functions. We will make use of the identity

$$I_\nu(z) = e^{-i\pi \nu} J_\nu(e^{i\pi/2} z) \quad (-\pi < \arg z \leq \frac{\pi}{2}).$$

(95)

The asymptotic behaviour of these four functions at $z \approx 0$ is

$$J_\nu(z) \approx I_\nu(z) \approx \frac{1}{\Gamma(1+\nu)} \left( \frac{z}{2} \right)^\nu, \quad (\nu \in C \setminus \{-1, -2, -3, \ldots\}),$$

(96)

$$Y_\nu(z) \approx 2 \frac{\pi}{\nu} K_\nu(z) \approx -\frac{1}{\pi} \Gamma(\nu) \left( \frac{z}{2} \right)^{-\nu}, \quad (\text{Re } \nu > 0)$$

(97)

and

$$Y_0(z) \approx 2 \frac{\pi}{\nu} K_0(z) \approx \frac{2}{\pi} \ln z.$$

(98)
For $|z| \to \infty$ the ordinary and modified Bessel functions behave as
\[
J_\nu(z) \approx Y_\nu'(z) \approx \sqrt{\frac{2}{\pi z}} \cos \left[ z - \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right], \quad (|\arg z| < \pi), \tag{99}
\]
\[
J_\nu'(z) \approx -Y_\nu(z) \approx -\sqrt{\frac{2}{\pi z}} \sin \left[ z - \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right], \quad (|\arg z| < \pi), \tag{100}
\]
\[
I_\nu(z) \approx I_\nu'(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left( \frac{ez}{2\nu} \right)^\nu, \quad (|\arg z| < \frac{\pi}{2}), \tag{101}
\]
and
\[
K_\nu(z) \approx -K_\nu'(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left( \frac{3\pi}{2} \right). \tag{102}
\]

Expression (99) is valid not only for fixed $\nu$ and $z \approx 0$ but holds also if $z$ is fixed at an arbitrary (not necessarily small) value and $|\nu| \to \infty$, $|\arg \nu| < \pi$. In this case it can be combined with the Stirling formula giving the $|\nu| \to \infty$ asymptotics of the gamma function resulting
\[
J_\nu(z) \approx I_\nu(z) \approx \frac{1}{\sqrt{\nu 2\pi}} \left( \frac{ez}{2\nu} \right)^\nu. \tag{103}
\]

After these necessary pieces of information let us start finding the possible orthogonal systems of the eigenvectors of the case $s < 0$. The scalar product of $\psi_1(\delta)$ and $\psi_2(\delta)$ in $L^2(\mathbb{R}^+, \rho)$ is
\[
(\psi_1, \psi_2)_+ = \int_0^\infty \psi_1(\delta)^* \psi_2(\delta) \sqrt{2\delta} \, d\delta, \tag{104}
\]
here the notation $(\ , \ )_+$ reminds us that this scalar product is taken in $L^2(\mathbb{R}^+, \rho)$, i.e. on the positive half of the configuration space only. Instead of the variable $\delta$ it will be more suitable to work in $z$. Under the transformations (52) and (53) the integral (104) transforms to
\[
\sqrt{2} \int_0^\infty w_1(z)^* w_2(z) \frac{dz}{z}. \tag{105}
\]
We will study this integral by considering it between finite $a$ and $b$ and then take the limit $a \to 0$, $b \to \infty$.

If $w_1(z)$ and $w_2(z)$ are eigenfunctions of the Bessel equation (51) with indexes $\mu$ and $\nu$ respectively then such an integral can be easily evaluated due to a formula of [8]. This formula states that
\[
\int_a^b A_\mu(z) B_\nu(z) \frac{dz}{z} = \frac{1}{\nu^2 - \mu^2} \left[ z \left( A_\mu(z) B_\nu'(z) - A_\mu'(z) B_\nu(z) \right) \right]^b_a \tag{106}
\]
for $(A, B = J$ or $Y$, $0 < a \leq b < \infty)$. Applying it together with (94) the integral (105) between the limits $a$ and $b$ is
\[
\frac{\sqrt{2}}{\nu^2 - \mu^2} \left[ z \left( w_1(z)^* w_2'(z) - w_1'(z)^* w_2(z) \right) \right]^b_a. \tag{107}
\]
With the aid of (107) let us determine the scalar product \( (J_\mu, J_\nu)_+ \) (\( \mu, \nu \in C \)). Substituting them into (107) and using the asymptotic formulas (99) and (100), the contribution from the 'upper limit terms' (the terms depending on \( b \)) in the limit \( b \to \infty \) is

\[
\frac{2\sqrt{2}}{\pi(v^2 - \mu^*2)} \sin \left[ \frac{\pi}{2}(\nu - \mu^*) \right].
\] (108)

On the other hand, from (96) one can see that the 'lower limit terms' behave as a \( (\mu^* + \nu) \)-th power of \( a \) when \( a \to 0 \). If \( \Re(\mu^* + \nu) > 0 \) then the 'lower limit terms' tend to zero; so in this case

\[
(J_\mu, J_\nu)_+ = \frac{2\sqrt{2}}{\pi(v^2 - \mu^*2)} \sin \left[ \frac{\pi}{2}(\nu - \mu^*) \right].
\] (109)

If \( \Re(\mu^* + \nu) < 0 \) then the integral diverges for \( a \to 0 \). If \( \Re \mu > 0 \) then the \( \nu \to \mu \) limit gives the norm of \( J_\mu \):

\[
(J_\mu, J_\mu)_+ = \frac{1}{\sqrt{2} \Re \mu}.
\] (110)

Similarly we find that \( (J_\mu, Y_\nu)_+ \) is finite if \( \Re(\mu^* + \nu) > 0 \) and infinite if \( \Re(\mu^* + \nu) < 0 \), while the scalar product \( (Y_\mu, Y_\nu)_+ \) always diverges. As a special case of this latter we also see that the \( Y_\mu \)-s are non-normalizable, for any complex value of \( \mu \).

We want to build complete orthogonal systems out of the \( J_\mu \)-s and \( Y_\mu \)-s, where now only real and imaginary \( \mu \)-s are allowed (cf. Sect. 6). The role of the Bessel functions with real respectively imaginary index is to span the wave functions that have an energy expectation value \( E > \frac{\hbar^2}{4m} \) resp. \( E \leq \frac{\hbar^2}{4m} \). With a real \( \mu \) only the \( J_{\mu \geq 0} \)-s and \( Y_0 \) can be taken into account. In fact, a \( J_{\mu < 0} \) or an \( Y_{\mu \neq 0} \) cannot be orthogonal to a Bessel function with imaginary index as their scalar product diverges. From (109) it follows that a maximal, pairwise orthogonal set of \( J_{\mu > 0} \)-s is \( \{J_{p}, J_{p+2}, J_{p+4}, \ldots \} \), where \( p \in (0,2] \). With a \( p \) given, the only linear combination of a \( J_{iu}(z) \) and \( J_{-iu}(z) \) that is orthogonal to \( J_p(z) \) is the one given in (103), as one finds with the aid of (109). This linear combination proves to be orthogonal to the eigenfunctions \( J_{p+2}, J_{p+4}, \ldots \) as well. In the end, there exists one linear combination of \( J_0(z) \) and \( Y_0(z) \) that is orthogonal to \( J_p \) (and, as turns out, to the functions \( J_{p+2}, J_{p+4}, \ldots \) and \( \exp[-i\theta_p(u)]J_{iu}(z) + \exp[i\theta_p(u)]J_{-iu}(z) \), \( u \in (0, \infty) \) as well). This linear combination can be obtained either by using the scalar product or as the \( u \to 0 \) limit of the eigenfunctions \( \exp[-i\theta_p(u)]J_{iu}(z) + \exp[i\theta_p(u)]J_{-iu}(z) \).

What is left to check is the mutual orthogonality of the eigenfunctions with imaginary index. We will prove that

\[
\left( c(u) \left[ e^{-i\theta_p(u)}J_{iu} + e^{i\theta_p(u)}J_{-iu} \right], c(v) \left[ e^{-i\theta_p(v)}J_{iv} + e^{i\theta_p(v)}J_{-iv} \right] \right)_+ = \delta(u-v)
\] (111)

where

\[
c(u) = \frac{|\Gamma(1+iu)|}{\sqrt{2\sqrt{2}\pi}} = \sqrt{\frac{1}{2\sqrt{2}\sinh(\pi u)}}
\] (112)
(for the properties of the gamma function see for example [3]). As \( u \) and \( v \) run over the positive real numbers only and not on the whole real line, the Dirac delta distribution must be understood here to act on the test functions which are defined on the positive half of the real line and are smooth functions of compact support vanishing at the origin. In the following \( u \) will be treated as a variable—i.e. the variable of the test functions and the kernel functions—and \( v \) as a fixed parameter.

To inspect the scalar product let us consider (107) in our case. A simple calculation involving the use of (108) and some trigonometrical identities shows that the contribution of the upper limit tends to 0 when \( b \to \infty \). The lower limit terms are asymptotically

\[
-\sqrt{2}c(u)c(v) \left[ \frac{e^{-i(\theta_p(u)-\theta_p(v))}}{(u-v)\Gamma(1+iu)\Gamma(1-iv)} \left( \frac{a}{2} \right)^{i(u-v)} + \ldots \right]
\]

where the \ldots stands for three other terms which can be obtained from the first one by the substitutions \( u \to -u \), \( v \to -v \), or both, respectively. Introducing \( \Lambda = \ln \frac{2}{a} \), two of these terms have the form \( f(u,v) \sin \Lambda(u+v) + g(u,v) \cos \Lambda(u+v) \), where \( f \) and \( g \) are smooth functions of both \( u \) and \( v \). It is a well-known fact that the regular distributions \( \sin \Lambda x \) and \( \cos \Lambda x \) tend to zero if \( \Lambda \to \infty \). This property does not change if we multiply them by a smooth function, thus these terms give zero in the limit \( \Lambda \to \infty \) (\( a \to 0 \)).

With (112) the two other terms can be written in the following way

\[
-\frac{1}{2\pi i(u-v)} \left[ (e^{-i(\theta_p(u)-\theta_p(v))}+\text{arg}\Gamma(1+iu)-\text{arg}\Gamma(1+iv)} - 1) \left( \frac{a}{2} \right)^{i(u-v)} 
\]

\[
+ \left( \frac{a}{2} \right)^{i(u-v)} \right] - (u \leftrightarrow v).
\]

From the properties of \( \theta_p \) and the gamma function it follows that the function \( \exp(\ldots)-1)/(u-v) \) behaves smoothly even if \( u \to v \). Consequently the distributions coming from the first and the third terms also tend to zero. What remained is equal simply to \( \sin[\Lambda(u-v)]/\pi(u-v) \). It is well-known that \( \sin \Lambda x/\pi x \to \delta(x) \) in the limit \( \Lambda \to \infty \), so (111) is proven.

We close Appendix A by showing that in the case \( s > 0 \) \( \{K_{iu}(z) \mid u \in [0, \infty)\} \) is the only possible orthogonal system built from the eigenfunctions. For this let us consider the scalar product

\[
(c_+(u)I_{iu} + c_-(u)I_{-iu}, c_+(v)I_{iv} + c_-(v)I_{-iv})_+
\]

(here \( u, v > 0 \) again). In the case \( s < 0 \) we had formula (106) to evaluate the integral corresponding to this scalar product. Repeating the proof of (106) given in [8] one can obtain a corresponding result in the case of the modified Bessel functions. The formula one gets turns out to be exactly of the form of (106) with \( A \) and \( B \) denoting now \( I \) or \( K \). Thus we can study the scalar product similarly as we did in the case
s < 0. Let us evaluate (115) with the aid of (106), the 'upper limit terms' give asymptotically

\[
\frac{\sqrt{2}}{i\pi(u^2 - v^2)} \left[ c_+(u)^*c_+(v)(e^{-\pi u} - e^{-\pi v}) + c_+(u)^*c_-(v)(e^{-\pi u} - e^{-\pi v}) \\
+ c_-(u)^*c_+(v)(e^{\pi u} - e^{\pi v}) + c_-(u)^*c_-(v)(e^{\pi u} - e^{\pi v}) \right].
\] (116)

Using the asymptotics (96) we can see that the 'lower limit terms' behave as \((a/2)i(\pm u \pm v)\), just like in the case \(s < 0\). Two of them vanishes if \(a \to 0\). After a trick similar to (114) the nonzero contribution of the two other terms is a sum of a \(\sin \Lambda(u - v)/(u - v)\) term. In the limit \(\Lambda \to \infty\) the first of them leads to a Dirac delta. The second one is not, consequently its coefficient must be equal to zero. This gives the condition

\[
c_+(u)^*c_+(v) = c_-(u)^*c_-(v).
\] (117)

On the other hand, (116) is a smooth function for \(u \neq v\). Orthogonality requires that (116) must be equal to zero for any \(u \neq v\). These two conditions together yield

\[
[c_+(u)^* + c_-(u)^*] \left\{ c_+(v)e^{\pi u} + c_-(v)e^{-\pi u} - c_+(v)e^{\pi v} - c_-(v)e^{-\pi v} \right\} = 0.
\] (118)

(118) holds for any \(u, v, u \neq v\) only if \(c_+(u) + c_-(u) = 0\), which we wanted to prove (cf. (93)).

We note that by using (106) and the asymptotics (96)-(102) \(I_\nu(z)\) and \(K_\nu(z)\) prove to be not square integrable for any complex value of \(\nu\). Another remark is that the eigenfunctions in (63) and (65) are real (cf. (94)).

B Completeness

Here we prove the completeness of the orthogonal systems (63) and (65). Formulating the completeness of a system (63) in the variable \(x\) reads

\[
S(x_1, x_2) + I(x_1, x_2) = \delta(x_1 - x_2)
\] (119)

with

\[
S(x_1, x_2) = \sum_{q \in \mathbb{Z}_+} \sqrt{2} q J_q(z_1)J_q(z_2)
\] (120)

and

\[
I(x_1, x_2) = \int_0^\infty \frac{du}{2\sqrt{2}\sinh \pi u} \left[ e^{-i\theta u}(J_{iu}(z_1) + e^{i\theta u}J_{-iu}(z_1)) \times \left[ e^{-i\theta u}J_{iu}(z_2) + e^{i\theta u}J_{-iu}(z_2) \right] \right] \times
\] (121)

where

\[
z_1 = ke^{-\frac{x_1}{\sqrt{2}}}, \quad z_2 = ke^{-\frac{x_2}{\sqrt{2}}}.
\] (122)
The convergence of the infinite sum (120) is guaranteed by the asymptotics (103).

First let us prove (119) in the case when \( p = 2 \). We consider the integral (121) between 0 and \( \Lambda, \Lambda = N + 1/2, N \in \mathbb{Z}^+ \). The substitution \( \nu := iu \) transforms the integral to

\[
\int_0^{i\Lambda} \frac{d\nu \nu}{i2\sqrt{2} \sin \pi\nu} [J_\nu(z_1) + J_{-\nu}(z_1)] [J_\nu(z_2) + J_{-\nu}(z_2)].
\]  

(123)

As the integrand of (123) is invariant under the transformation \( \nu \rightarrow -\nu \), (123) can be written as

\[
\frac{1}{2} \int_{-i\Lambda}^{i\Lambda} \frac{d\nu \nu}{i2\sqrt{2} \sin \pi\nu} [J_\nu(z_1) + J_{-\nu}(z_1)] [J_\nu(z_2) + J_{-\nu}(z_2)].
\]  

(124)

Now we change the contour of this integral to a half circle (denoted by \( C' \)) starting from the point \(-i\Lambda\), running in the half plane \( \text{Re} \nu > 0 \) of the complex \( \nu \)-plane and ending at \( i\Lambda \). The difference of (124) and this new integral can be expressed by the residues of the poles of the integrand lying in the region bordered by the two contours. As the \( J \)-s behave analytically, poles arise only from \( \sin \pi\nu \), at the values \( \nu = 1, 2, 3, \ldots N \) (at \( \nu = 0 \) \( \nu/\sin(\pi\nu) \) is not singular). The contribution of the residues is

\[
- \frac{2\pi i}{i4\sqrt{2}} \sum_{n=1}^{N} \text{Res}_n
\]  

(125)

where, by using (91), \( \text{Res}_n \) turns out to be equal to \( \left(\frac{4n}{\pi}\right) J_n(z_1)J_n(z_2) \) if \( n \) is even and is zero if \( n \) is odd. We can see that (125) is just the opposite of (120) in the limit \( N \rightarrow \infty \) so from now on we have to prove that

\[
\int_{C} \frac{d\nu \nu}{i4\sqrt{2} \sin \pi\nu} [J_\nu(z_1) + J_{-\nu}(z_1)] [J_\nu(z_2) + J_{-\nu}(z_2)] =
\]

\[
\int_{C} \frac{d\nu \nu}{i4\sqrt{2} \sin \pi\nu} J_\nu(z_1)J_\nu(z_2) + \int_{C} \frac{d\nu \nu}{i4\sqrt{2} \sin \pi\nu} J_{-\nu}(z_1)J_{-\nu}(z_2) +
\]

\[
\int_{C} \frac{d\nu \nu}{i4\sqrt{2} \sin \pi\nu} [J_\nu(z_1)J_{-\nu}(z_2) + J_{-\nu}(z_1)J_\nu(z_2)]
\]  

(126)

tends to \( \delta(x_1 - x_2) \) when \( \Lambda \rightarrow \infty \).

We will perform the proof in three steps. We start by showing that the first and the second terms of the r.h.s. of (126) are equal. The second step proves that these terms are zero in the limit \( \Lambda \rightarrow \infty \). Thirdly it is shown that the third term tends to \( \delta(x_1 - x_2) \) as \( \Lambda \rightarrow \infty \).

To see that the first and the second terms are equal let us make the substitution \( \nu \rightarrow -\nu \) in the second term. The integrand of the resulting integral is the same as in the first term (up to a factor of \(-1\)), but the contour \( C' \) is a half circle starting from \( i\Lambda \), running through the half plane \( \text{Re} \nu < 0 \) and arriving at \( -i\Lambda \). Let us change the direction of \( C' \), this causes another factor of \(-1\) in the integral. Now we can change this contour \( C'' \) to \( C \), thus we arrive at the first term, plus the contribution
of the residues coming from the poles lying between \( C'' \) and \( C \). Fortunately this contribution

\[
\sum_{n=-N}^{N} \text{Res}_n (127)
\]

is zero because, as a consequence of (121), \( \text{Res}_n = -\text{Res}_{-n} \).

In the second step we are interested in the \( \Lambda \to \infty \) behaviour of the integral

\[
\int_{C} \frac{d\nu}{\sin \pi \nu} J_{\nu}(z_1) J_{\nu}(z_2). (128)
\]

Writing \( \nu \) in the form of \( \Lambda \exp(i\varphi) \) this integral can be expressed as

\[
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \frac{i\Lambda^2 e^{2i\varphi}}{\sin(\pi \Lambda e^{i\varphi})} J_{\Lambda e^{i\varphi}}(z_1) J_{\Lambda e^{i\varphi}}(z_2). (129)
\]

It will be enough if we show that the integral of the integrand’s modulus tends to zero.

We will use asymptotic expressions for analysing the \( \Lambda \to \infty \) behaviour. By doing so some care will be needed. That’s why we divide the domain of integration \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) into three parts: \( A := [-\frac{\pi}{2}, -\frac{\pi}{4}] \), \( B := [-\frac{\pi}{4}, \frac{\pi}{4}] \) and \( C := [\frac{\pi}{4}, \frac{\pi}{2}] \). It can be proved easily that for a \( \varphi \neq 0 \) and a large enough \( \Lambda \)

\[
|\sin[\pi \Lambda e^{i\varphi}]| \approx \frac{1}{2} e^{\pi \Lambda \sin|\varphi|}. (130)
\]

Let us apply (130) in the domains \( A \) and \( C \). Combining it with (103) the modulus of the integrand of (129) is asymptotically equal to

\[
\frac{\Lambda}{\pi} e^{-2\Lambda([\ln \Lambda - \zeta] \cos \varphi + (\frac{\pi}{2} - |\varphi|) \sin |\varphi|)}, (131)
\]

where \( \zeta = \frac{1}{2} \ln(z_1 z_2/4) + 1 \) is a quantity independent of \( \varphi \) and \( \Lambda \). We can see that the integral of (131) on \( A \) or \( C \) gives the same result. Thus we will consider this integral only on \( C \), for example.

If \( \varphi \in [\pi/4, \pi/2] \) then the inequalities \( \cos \varphi \geq 1 - (2/\pi)\varphi \) and \( (\pi/2 - \varphi) \sin \varphi \geq 0 \) hold, helping us to give an upper estimate of (131)

\[
\frac{\Lambda}{\pi} e^{-2\Lambda([\ln \Lambda - \zeta] + 1 - \frac{2}{\pi}\varphi)} (132)
\]

if \( \ln \Lambda > \zeta \). The integral of (132) on \( C \) can be calculated easily. The result is less then \( |4([\ln \Lambda - \zeta])^{-1} \), which is a quantity tending to zero if \( \Lambda \to \infty \).

In domain \( B \) we cannot use (130) but here it is enough to work with the inequality

\[
|\sin[\pi \Lambda e^{i\varphi}]| \geq 1. (133)
\]

For proving (133) it is not hard to show that \( |\sin[\pi \Lambda \exp(i\varphi)]| \) takes its minimum in \( \varphi = 0 \) as \( \varphi \) varies in \( B \) while \( \Lambda \) is fixed. (Remember that \( \Lambda = N + 1/2 \).) For the
Bessel functions \((103)\) is applicable in \(B\), too. Using \((133)\) the asymptotics of the absolute value of the integrand in \((129)\) is not greater than
\[
\frac{\Lambda}{2\pi} e^{-2\Lambda \left( [\ln \Lambda - \zeta] \cos \varphi - \varphi \sin \varphi \right)}.
\]
(134)

If \(\varphi \in B\) then \(\cos \varphi \geq 1/\sqrt{2}\) and \(\varphi \sin \varphi \leq \pi/4\sqrt{2}\) so an upper estimate of (134) is
\[
\frac{\Lambda}{2\pi} e^{-\sqrt{2}\Lambda (\ln \Lambda - \zeta - \frac{\pi}{4})}
\]
(135)

if \(\ln \Lambda > \zeta\). Integrating (135) on \(B\) means simply a factor of \(\pi/2\). We see that the value of the integral tends to zero in the limit \(\Lambda \to \infty\).

In the last step let us turn to the third term of the r.h.s. of (126). For great \(\Lambda\)-s this integral is asymptotically equal to
\[
\int_C \frac{d\nu}{4\sqrt{2}\sin \pi \nu} \left[ \frac{(\frac{x_1}{\sqrt{2}})^\nu}{\Gamma(1 + \nu)} \frac{(\frac{x_2}{\sqrt{2}})^{-\nu}}{\Gamma(1 - \nu)} \right] + \left[ \frac{(\frac{x_1}{\sqrt{2}})^{-\nu}}{\Gamma(1 - \nu)} \frac{(\frac{x_2}{\sqrt{2}})^\nu}{\Gamma(1 + \nu)} \right]
\]
(136)

(\(\Lambda\) is not an integer so (106) is applicable). Knowing that \(\Gamma(1+\nu)\Gamma(1-\nu) = \pi\nu/\sin \pi \nu\) and writing \(\nu\) as \(\Lambda \exp(i\varphi)\) we get the integral
\[
\frac{\Lambda}{4\pi \sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \ e^{i\varphi} \left[ e^{\frac{x_1}{\sqrt{2}}(x_1-x_2)e^{i\varphi}} + e^{-\frac{x_2}{\sqrt{2}}(x_1-x_2)e^{i\varphi}} \right]
\]
(137)

(cf. (122)). We determine this integral by expanding the exponentials in power series and integrating the terms individually. The sum of the results is
\[
\frac{\Lambda}{\pi \sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \left[ \frac{x_1 - x_2}{\sqrt{2}} \right]^{2k} \frac{(2k+1)!}{(2k+1)!},
\]
(138)

which is the power series of \(\sin \left[ \frac{x_1 - x_2}{\sqrt{2}} \right]/\pi(x_1 - x_2)\). This function tends to \(\delta(x_1 - x_2)\) if \(\frac{x_1 - x_2}{\sqrt{2}} \to \infty\), which we wanted to prove.

The method of the proof of the case \(p = 2\) can be applied in a straightforward way for any other values of \(p\) as well. If \(p = 1\) then a factor of \(-1\) appears at the first two terms of the r.h.s. of (126), which is of no significant importance in the proof. For the other possible values of \(p\) the remarkable difference is that poles come not only from \(\sin(\pi \nu)\) but also from the analytic continuation of \(\exp[ \pm i \theta_p(u) ]\). Nevertheless, the sum of all residues appearing in the proof gives exactly \(-S(x_1, x_2)\), which eliminates the first term in (113)—just as it happened in the case \(p = 2\). Besides these extra poles the proof needs no serious modification.

The proof given above also works for the orthogonal system (65) of the case \(s > 0\). Using (103) the residues turn out to be zero. What is left is just the same as we had in the case \(s < 0\), \(p = 1\), because of the common asymptotic behaviour of \(J_{\nu}(z)\) and \(I_{\nu}(z)\) (see (103)). This way the case \(s > 0\) can be treated with the same tools as the \(s < 0\) one.
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