Spectrum of Two-Dimensional (Super)Gravity

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Abstract

We review the BRST analysis of the system of a (super)conformal matter coupled to 2D (super)gravity. The spectrum and its operator realization are reported. In particular, the operators associated with the states of nonzero ghost number are given. We also discuss the ground ring structure of the super-Liouville coupled to $\hat{c} = 1$ matter. In appendices, hermiticities, states for $c < 1$ conformal matter coupled to gravity and the proof for the spectrum are discussed.

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1 Introduction

String theories, or more generally, 2D conformal field theories (CFTs) have been studied for their applications to 2D statistical systems and unified theories including gravity. In their long history, no one has ever been able to go beyond the perturbative treatment of strings. It is crucial, however, to understand nonperturbative aspects of the theories for their applications. The recent discovery of the double scaling limit in the matrix models [1] has opened a way to address this problem and has attracted much attention.

The 2D gravity coupled to matter is a system simple enough to be exactly solvable while retaining many of important features of physical interest. The key of its solvability is the fact that the model has very small number of degrees of freedom. The one-dimensional matrix model [2, 3] corresponds to interacting strings in “two-dimensional space-time”, where there is no transverse degree of freedom. One may then naively expect that they describe only the motion of “center of mass”. Surprisingly enough, it turns out that the system carries unexpectedly rich dynamical structure with infinite number of “extra discrete states” other than the “center of mass” degree of freedom [4, 5].

In the discrete (matrix model) approach, it is not clear what conditions characterize physical states and the origin of these “extra states” is obscure. Thus it is very important to understand these results in the continuum approach where physical state condition is well-defined. In the conformal gauge, the 2D gravity coupled to a CFT appears as Liouville theory [6, 7]. In this formulation physical states may be specified as the nontrivial cohomology classes of the BRST operator and the raison d’être of such “extra states” has been understood in the BRST formalism [8, 9]. However, their roles have not been fully clarified yet. Some of their known features are the following: they are the remnants of the higher massive modes of critical strings; the discrete states are responsible for the presence of a large algebra, $W_{\infty}$, in the system [10, 11].

In our earlier paper [12], we have reported our results on the spectrum of $N=1$ supersymmetric CFTs coupled to 2D supergravity (The same results are also announced in Ref. [13]). This study is also important for the following reason. Our present understand-
ing of bosonic 2D gravity is based on the results of two completely different approaches: discrete (matrix models, collective field theories) and continuum approaches. Both of them have brought us almost a consistent picture for the bosonic theory. However, we do not know even how to formulate supersymmetric theories in matrix models (some related works have been reported in [14]).

Our results are quite parallel to the bosonic case. In the spectrum, we find massless states in the Neveu-Schwarz (NS) and Ramond (R) sectors, and discrete states at the levels where we have “null” states in the minimal models. In the appendix, we summarize our proof emphasizing common features of two sectors. One may clearly understand these discrete states on the basis of two structures: quartet representations of the BRST formalism [15]; the singular and cosingular vectors of the Feigin-Fuchs realization [16, 17]. The appearance of the discrete states is due to “decomposition of quartets”. This has been reported in our paper and is not repeated here. In this article, however, we will give a comprehensive review of the subject together with some new results.

This paper is organized as follows. In the next section, we review how Liouville theory emerges when 2D gravity couples to a conformal matter. In sect. 3, we describe the spectrum of bosonic theories. The results are due to Refs. [8, 9]; a summary of the BRST analysis is given in appendix C. We introduce some new operators with nonzero ghost numbers to construct discrete states. The $N = 1$ supersymmetric case is reported in sect. 4, emphasizing the similarity to the bosonic case. Sect. 5 is devoted to discussions. In appendix A, we discuss hermiticity properties of the Fock space, which should be important in order to discuss the unitarity but is rarely paid attention. In appendix B, we show how to construct physical states for $c^M < 1$ CFTs coupled to gravity. Finally in appendix C, we give a description of our analysis of the spectrum which is relevant to sects. 3 and 4.
2 2D gravity coupled to CFT

In this section, we briefly describe the system of 2D gravity coupled to a CFT and argue that the gravity sector could be treated as Liouville theory with the free field measure \([7, 18]\).

The partition function of this system is given by

\[
Z = \int Dg Dg X e^{-S(X, g)} , \tag{2.1}
\]

where \(S(X, g)\) is an action for the matter (generically denoted as \(X\)) coupled to 2D metric \(g\). \(S(X, g)\) is invariant under the diffeomorphism as well as Weyl rescaling of the metric \(g \rightarrow e^{\sigma} g\)

\[
S(X, e^{\sigma} g) = S(X, g). \tag{2.2}
\]

However, the measure is not Weyl-invariant \([4, 19]\) and we find

\[
D_{e^{\sigma} g} X = e\left(\frac{c M}{48 \pi}\right) S_{L}(\sigma, g) D_g X , \tag{2.3}
\]

where \(c M\) is the central charge of the matter system and \(S_{L}\) is the Liouville action

\[
S_{L}(\sigma, g) = \int d^2 \xi \sqrt{g} \left( \frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R \sigma + \mu e^{\sigma} \right) , \tag{2.4}
\]

with \(R\) and \(\mu\) being the scalar curvature and cosmological constant, respectively.

Upon fixing the diffeomorphism invariance in the conformal gauge, we find the path integral measure in (2.1) becomes

\[
D_g \phi_0 D_g b D_g c D_g X , \tag{2.5}
\]

where \(b\) and \(c\) are Faddeev-Popov ghosts and \(\phi_0\) is a variable parametrizing the Weyl rescaling for a metric

\[
g = e^{\phi_0} \hat{g} . \tag{2.6}
\]

where \(\hat{g}\) is a reference metric parametrized by the moduli space. The integral over the moduli is suppressed. The integration measure over \(\phi_0\) is defined through the induced norm by the metric (2.6)

\[
||\delta \phi_0||^2 = \int d^2 \xi \sqrt{g} (\delta \phi_0)^2 = \int d^2 \xi \sqrt{\hat{g}} e^{\phi_0} (\delta \phi_0)^2 , \tag{2.7}
\]
which depends on $\phi_0$ itself in a complicated manner. The ingenious ansatz made by [7] and later proved in Ref. [18] is that we can shift the measure and make it independent of $\phi_0$. In doing this, one gets a Jacobian $J(\phi, \hat{g}) = e^{-S(\phi, \hat{g})}$,

$$
\mathcal{D}_g \phi_0 \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X = \mathcal{D}_g \phi \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X e^{-S(\phi, \hat{g})},
$$

where $\mathcal{D}_g \phi$ is the free field measure defined by the norm

$$
\int d^2 \xi \sqrt{\hat{g}}(\delta \phi)^2.
$$

The key assumption is that the most general renormalizable form of $S$ compatible with locality and diffeomorphism invariance is simply

$$
S(\phi, \hat{g}) = \frac{1}{8\pi} \int d^2 \xi \sqrt{\hat{g}}(\hat{g}^{ab} \partial_a \phi \partial_b \phi - 2Q \hat{R} \phi + 4\mu' e^{\alpha \phi}),
$$

$$
= \frac{1}{2\pi} \int d^2 z (\partial \phi \bar{\partial} \phi - \frac{1}{2} Q \sqrt{\hat{g}} \hat{R} \phi + \mu' \sqrt{\hat{g}} e^{\alpha \phi}),
$$

which is similar to the Liouville action (2.4).

The unknown coefficients $Q$ and $\alpha$ are determined if one notices that the original theory depends only on $g = e^{\alpha \phi} \hat{g}$ so that it is invariant under

$$
\hat{g} \rightarrow e^{\sigma} \hat{g}, \quad \phi \rightarrow \phi - \sigma/\alpha,
$$

which means

$$
\mathcal{D}_{e^{\sigma} \hat{g}}(\phi - \sigma/\alpha) \mathcal{D}_{e^{\sigma} \hat{g}} b \mathcal{D}_{e^{\sigma} \hat{g}} c \mathcal{D}_{e^{\sigma} \hat{g}} X e^{-S(\phi - \sigma/\alpha, e^{\sigma} \hat{g})} = \mathcal{D}_g \phi \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X e^{-S(\phi, \hat{g})}.
$$

However, $(\phi - \sigma/\alpha)$ on the left hand side is just an integration variable and we could have called it $\phi$ itself. Viewed this way, eq. (2.12) shows that the conformal anomaly for the total system should vanish [4]! This means that the central charges from three sectors should add up to zero

$$
c_{\text{total}} \equiv c^M + c^L - 26 = 0
$$

where $c^L = 1 + 12Q^2$. Note that the matter CFT no longer has conformal invariance by itself because of the conformal anomaly [3, 19], but that the inclusion of the Liouville
theory recovers the invariance. The other parameter $\alpha$ is determined by demanding that $g = e^{\alpha \phi} \hat{g}$ be invariant, or $e^{\alpha \phi}$ be a conformal tensor of dimension $(1, 1)$. Since the dimension of this operator is given by $-\frac{1}{2} \alpha (\alpha + 2Q)$, we obtain

$$\alpha = -Q + \sqrt{Q^2 - 2} = \frac{\sqrt{1 - c^M} - \sqrt{25 - c^M}}{2\sqrt{3}}$$

(2.14)
in agreement with Ref. [20]. Here an appropriate branch is chosen so as to agree with the semiclassical limit ($c^M \rightarrow -\infty$). Eq. (2.14) shows that there is a bound $c^M \leq 1$ in order for this approach to make sense.

The argument can be easily extended to supersymmetric case [22]. In the BRST formalism to be used in the present paper, the condition $c_{total} = 0$ is equivalent to the nilpotency of the BRST charge.

3 Bosonic non-critical strings

In this section, we discuss the spectrum of bosonic theory in the BRST formalism.

Let us first describe the free field realization which we will use for both the matter and gravity sectors. A scalar field $\phi$ has the following expansion

$$\phi(z) = q - i(p - \lambda) \ln z + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n},$$

(3.1)

with the commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}, \quad [q, p] = i.$$

(3.2)

The energy-momentum tensor of the system is

$$T = -\frac{1}{2}(\partial \phi)^2 - i\lambda \partial^2 \phi,$$

(3.3)

which satisfies Virasoro algebra with the central charge $c = 1 - 12\lambda^2$. (The Liouville theory described in sect. 2 corresponds to choosing $Q = i\lambda$.)

*Note that the relation (2.14) can be written as $Q = -(\frac{\alpha}{2} + \frac{1}{\alpha})$, a well-known relation in the quantum Liouville theory [2].
We use two scalar fields for the matter and gravity sectors, which will be distinguished by the superscripts $M$ and $L$.

By introducing the ghost fields $(b, c)$, we obtain the BRST charge \[ Q_B = \oint dz 2\pi i c(z)(T(z) + \frac{1}{2} T^{bc}(z)), \] (3.4)
where the energy momentum tensor $T$ contains the contributions from the matter and gravity sectors. The ghosts have the conformal dimensions, $\text{dim.}(b, c) = (2, -1)$. The nilpotency of the charge gives us a constraint that the total central charge is equal to zero, or equivalently
\[ (\lambda^M)^2 + (\lambda^L)^2 = -2. \] (3.5)
Note that $\lambda^M$ is real whereas $\lambda^L$ is pure imaginary. Accordingly the momenta $p^M$ and $p^L$ are real and pure imaginary, respectively.

Our problem is to find the quotient space $\text{Ker} Q_B / \text{Im} Q_B$ in the direct product space out of Fock modules for the matter, gravity and the ghost sectors; $\mathcal{F}(p^M, p^L) \equiv \mathcal{F}(p^M) \otimes \mathcal{F}(p^L) \otimes \mathcal{F}_{gh}$, where $\mathcal{F}(p)$ is a Fock module with momentum $p$. The ghost number is defined so that the physical vacuum $c_1|0\rangle_{gh}$ has $N_{FP} = 0$, where $c_n(n \geq 1)$ and $b_n(n \geq 0)$ vanish on $c_1|0\rangle_{gh}$. The relative cohomology is defined as the cohomology of $Q_B$ on $\mathcal{F}(p^M, p^L) \cap \text{Ker}(b_0)$. We denote them as $H^{(\ast)}_{rel}(*, Q_B)$ where the superscript indicates the ghost number. The absolute cohomology is defined on $\mathcal{F}(p^M, p^L)$.

We have extra physical states, i.e., discrete states when the momenta for both the matter and gravity sectors take the special values parametrized by two integers $j$ and $k (jk > 0)$ as
\[ p = t_{(j,k)} + \lambda = \frac{1}{2}(jt_+ + kt_-), \] (3.6)
where $t_{\pm} = -\lambda \pm \sqrt{\lambda^2 + 2}$. Note that $\lambda^M$ (or $c^M$) is a free parameter of our system except the constraint $c^M \leq 1$. $\lambda^L$ is determined by eq. (3.5); if we choose $\lambda^L = i\sqrt{(\lambda^M)^2 + 2}$, then $t_{\pm}^L = \mp it_{\pm}^M$.

We find the following results in Refs. [3, 9].

We have
Theorem 1

For given $\lambda^M$ (or $c^M$), we find the following nontrivial cohomology classes.

1. If $j = 0$ or $k = 0$, $H_{rel}^{(0)}(\mathcal{F}(p^M, p^L), Q_B) = C$,
2. If $j, k \in \mathbb{Z}_+$, $H_{rel}^{(0,1)}(\mathcal{F}(p^M, p^L), Q_B) = C$,
3. If $j, k \in \mathbb{Z}_-$, $H_{rel}^{(0,-1)}(\mathcal{F}(p^M, p^L), Q_B) = C$. ●

We have given a proof of this theorem in appendix C together with the super case. Some explanations of each case are in order.

The first case (1) corresponds to the “tachyon” field in two-dimensional space-time; the on-shell condition is simply $-p^+ p^- = \frac{1}{2} [(ip^L)^2 - (p^M)^2] = 0$, which implies the particle is “massless” in terms of these shifted momenta $p^{M,L} = t^{M,L} + \lambda^{M,L}$ ($p^\pm$ are defined by $p^\pm = \frac{1}{\sqrt{2}}(p^M \pm ip^L)$). The states are at the level zero. For this case, the physical state condition does not require $j(\neq 0)$ or $k(\neq 0)$ to be an integer, though the momenta take the form in (3.6) owing to the on-shell condition.

For cases (2) and (3), the states are called discrete states and at the level $jk$. They carry fixed values of momenta (3.6) for given integers $j$ and $k$ (with $-p^+ p^- = \frac{1}{2} [(ip^L)^2 - (p^M)^2] = jk$) but there are two states at the same levels with ghost number $N_{FP} = 0, 1$ for case (2) and $N_{FP} = 0, -1$ for (3). The states for (2) are associated with the singular vectors in $\mathcal{F}(p)$ while those for (2) are related to the cosingular vectors in $\mathcal{F}(p)$. This observation also explains the ghost numbers they carry [12].

When $c^M = 1$, the discrete states form $SU(2)$ multiplets, and hence the structure may be most easily studied. In the matter sector, we have $SU(2)$ current algebra with level $\kappa = 1$ generated by the following currents

\[
\begin{align*}
J^\pm(z) &= :e^{\pm \sqrt{2} \phi^M(z)}:\, \\
J^0(z) &= \frac{1}{\sqrt{2}}i \partial \phi^M(z),
\end{align*}
\]

1Recall that the Liouville momenta are pure imaginary. Thus the Liouville field may be considered like a time variable while the matter corresponds to space.
with the operator product expansions (OPEs)

\[ J^+(z) J^-(w) \sim \frac{\kappa}{(z-w)^2} + \frac{2}{z-w} J^0(w), \]
\[ J^0(z) J^\pm(w) \sim \frac{\pm 1}{z-w} J^\pm(w), \]
\[ J^0(z) J^0(w) \sim \frac{\kappa/2}{(z-w)^2}. \] (3.8)

In particular, their integrals satisfy $SU(2)$ algebra and the discrete states (including some charged vacuum states) form $SU(2)$ multiplets.

It has been shown [10, 11] that these states exhibit ground ring structure with the symmetry group of the area-preserving diffeomorphism [25].

For studying the ground ring as well as some other physical implications of the discrete states, it would be useful to have explicit representations which may be summarized as follows:

**Theorem 2**

1. For $j, k \in \mathbb{Z}_+$ and $N_{FP} = 0$,

\[ \Psi_{Jm}^-(z) = (J^0_0)^{-j-m} e^{i\sqrt{2} J\phi^M(z)} e^{\sqrt{2}(1+J)\phi^L(z)}, \] (3.9)

where

\[ J^0_0 \equiv \oint_{C_z} \frac{dz}{2\pi i} J^-(\zeta). \] (3.10)

2. For $j, k \in \mathbb{Z}_+$ and $N_{FP} = 1$,

\[ \tilde{\Psi}_{Jm}^-(z) = (J^0_0)^{-j-m-1} \oint_{C_z} \frac{dz}{2\pi i} K(z) e^{i\sqrt{2} J\phi^M(z)} e^{\sqrt{2}(1+J)\phi^L(z)}, \] (3.11)

where

\[ K(z) \equiv c(z) J^-(z). \] (3.12)

3. For $j, k \in \mathbb{Z}_-$ and $N_{FP} = 0$,

\[ \Psi_{Jm}^+(z) = (J^0_0)^{-j-m} e^{i\sqrt{2} J\phi^M(z)} e^{\sqrt{2}(1-J)\phi^L(z)}. \] (3.13)

4. For $j, k \in \mathbb{Z}_-$ and $N_{FP} = -1$,

\[ \tilde{\Psi}_{Jm}^+(z) = (J^0_0)^{-j-m-1} \oint_{C_z} \frac{dz}{2\pi i} L(z) e^{i\sqrt{2}(J-1/2)\phi^M(z)} e^{\sqrt{2}(3/2-J)\phi^L(z)}, \] (3.14)
where

\[ L(z) \equiv b(z)e^{-i\phi^M(z)/\sqrt{2}}e^{-i\phi^L(z)/\sqrt{2}}. \] (3.15)

The \( SU(2) \) quantum numbers are related to \((j,k)\) as
\[ J \equiv |\frac{j+k}{2}|, \quad m \equiv \frac{j-k}{2} \] (these relations will be explained later). •

The states representing the nontrivial cohomology classes are obtained by acting the above operators (with \( z = 0 \)) on the physical vacuum \(|\lambda\rangle \equiv |\lambda^M = 0\rangle \otimes |\lambda^L = \sqrt{2}i\rangle \otimes c_1|0\rangle_{bc} \). We have used the same notation as in Ref. [10] for eqs. (3.9) and (3.13). The corresponding states were given in terms of Schur Polynomials in [13] (see also [26]). Note also that we have introduced a BRST-invariant operator \( K(z) \), which carries \( N_{FP} = 1 \) and the same charge as \( J^-(z) \). It is easy to show that the above operators create the states with \( N_{FP} = \pm 1 \) in Ref. [13] written in terms of the Schur polynomials and that these states are BRST invariant.\(^1\) The operator \( L(z) \) with \( N_{FP} = -1 \) does create appropriate states, but its BRST transformation property is less obvious.

The relations of \( J \) and \( m \) to \((j,k)\) may be obtained as follows. First note a state created by (3.9) carries momenta \((p^M, p^L - i\sqrt{2}) = (\sqrt{2}m, -i\sqrt{2}(1 + J)) \). On the other hand, we have \( t^M_{\pm} = \pm \sqrt{2} \) and \( t^L_{\pm} = -i \sqrt{2} \). Combined with (3.6), we find \( J \) and \( m \) given in the theorem.

The above expressions clearly tell us that operators with \( J \) form \((2J + 1)\)-plet if \( N_{FP} = 0 \), while operators with the same \( J \) but with \( N_{FP} = \pm 1 \) form \((2J - 1)\)-plet with respect to \( SU(2) \).

Unfortunately the operators with \( N_{FP} = \pm 1 \) are not quite symmetric, but the corresponding states are dual to each other in the sense that they have the non-zero inner products defined in appendix A.

In appendix B we show how to relate the states given in theorem 2 to those for \( c^M < 1 \) CFTs coupled to gravity. This is a way to find representatives for the cohomology classes in theorem 1 for a general \( c^M < 1 \).

\(^1\)Using \( K(z) \), we find states in (2) associated with the absolute cohomology, which take the form as \( \psi^{(1)}_{J+m,J-m} + c_0\psi^{(0)}_{J+m,J-m} \) in the notation of Theorem 2.2 in Ref. [13].
4 \( N = 1 \) supersymmetric non-critical strings

The structure of the spectrum for \( N = 1 \) supersymmetric case is quite similar to the bosonic case, although the proof is slightly more involved. In appendix C, we give a short summary of the proof treating NS and R sectors at the same time.

Let us introduce some notation for a scalar supermultiplet \((\phi, \psi)\) to realize superconformal algebra. The scalar field \(\phi\) has the same properties as the previous section. The field \(\psi\) has the following expansion

\[
\psi(z) = \sum_n \psi_{n+\delta}z^{-n-\delta-\frac{1}{2}},
\]

with the commutation relations

\[
\{\psi_{n+\delta}, \psi_{m-\delta}\} = \delta_{n+m,0},
\]

where \(\delta = 1/2\) (0) for NS (R) sector. The super-stress tensors of the system are

\[
T = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\psi\partial\psi - i\lambda\partial^2\phi,
\]
\[
S = i\psi\partial\phi - 2\lambda\partial\psi,
\]

which form the \( N = 1 \) superconformal algebra with the central charge

\[
c = \frac{3}{2}\hat{c} = \frac{3}{2}(1-8\lambda^2).
\]

We use two sets of the supermultiplets for the matter and gravity sectors, which will be distinguished by the superscripts \(M\) and \(L\).

By introducing the ghost fields \((b, c)\) and \((\beta, \gamma)\), we obtain the BRST charge \(\[Q_B = \oint \frac{dz}{2\pi i} [c(z)(T(z) + \frac{1}{2}T^{gh}(z)) - \frac{1}{2}\gamma(z)(S(z) + \frac{1}{2}S^{gh}(z))],\]

where the super-stress tensors \(T\) and \(S\) contain the contributions from the matter and gravity sectors. The ghosts \((b, c)\) and \((\beta, \gamma)\) have the conformal dimensions \((2, -1)\) and \((\frac{3}{2}, -\frac{1}{2})\), respectively. In particular, \(\gamma_{n+\delta}^\dagger = \gamma_{-(n+\delta)}\), \(\beta_{n+\delta}^\dagger = -\beta_{-(n+\delta)}\) and \([\gamma_{n+\delta}, \beta_{m-\delta}] = \delta_{n+m,0}\). The nilpotency of the charge gives us a constraint that the total central charge is equal to zero, or

\[
(\lambda^M)^2 + (\lambda^L)^2 = -1.
\]

\[\text{\textsuperscript{\S}} We will use the same notation as the previous section if quantities are naturally extended to supersymmetric case.

The ghost number is defined so that the physical vacuum $c_1|0\rangle_{gh}$ has $N_{FP} = 0$, where $c_n(n \geq 1), b_n(n \geq 0), \gamma_n(n > 0)$ and $\beta_n(n \geq 0)$ vanish on $c_1|0\rangle_{gh}$. The relative cohomology $H^{(s)}_{rel}(\ast, Q_B)$ is defined as the cohomology of $Q_B$ on $\mathcal{F}(p^M, p^L) \cap \text{Ker}(b_0)[\cap \text{Ker}(\beta_0)]$ for NS [R] sector. The absolute cohomology is defined on $\mathcal{F}(p^M, p^L)$.

We have obtained the following results in Ref. [12] on the spectrum for super-Liouville theory coupled to $\hat{c} \leq 1$ superconformal matter.

**Theorem 3**

For a given $\lambda^M$ (or $c^M$), we find the following nontrivial cohomology classes:

1. If $j = 0$ or $k = 0$, $H^{(0)}_{rel}(\mathcal{F}(p^M, p^L), Q_B) = \mathbb{C}$,
2. If $j, k \in \mathbb{Z}_+$, $H^{(0,1)}_{rel}(\mathcal{F}(p^M, p^L), Q_B) = \mathbb{C}$,
3. If $j, k \in \mathbb{Z}_-$, $H^{(0,-1)}_{rel}(\mathcal{F}(p^M, p^L), Q_B) = \mathbb{C}$,

where $j - k \in 2\mathbb{Z}$ for NS and $\in 2\mathbb{Z} + 1$ for R sectors respectively.

The momenta, appearing in the theorem, take the same form as (3.6) with slight modifications in definitions of $t_\pm; t_\pm = -\lambda \pm \sqrt{\lambda^2 + 1}$. Again if we choose $\lambda^L = i \sqrt{\lambda^M)^2 + 1}$, then $t^L_\pm = \mp i t^M_\pm$. The fermionic vacuum in the R sector is determined by the physical state condition. This theorem is proved in appendix C.

The first case in the NS sector again corresponds to the “tachyon” (actually massless) field in two-dimensional space-time. The states in the R sector are expected to give space-time fermion. The states are at level zero for (1). The on-shell condition for both sectors is simply $-p^+ p^- = \frac{1}{2}[(ip^L)^2 - (p^M)^2] = 0$, which implies the particles are “massless” in terms of the shifted momenta $p = t + \lambda$. Given this degeneracy in the NS and R sectors, it would be interesting to address the question of space-time supersymmetry.

For cases (2) and (3), the states are called discrete states and at the level $\frac{1}{2}jk$. From the on-shell condition, we have $-p^+ p^- = \frac{1}{2}[(ip^L)^2 - (p^M)^2] = \frac{1}{2}jk$. The states for (2) is associated with singular vectors in $\mathcal{F}(p)$ while those for (3) is related to cosingular vectors in $\mathcal{F}(p)$. At this level of conceptual understanding, the reason why we have these extra states is the same for the bosonic and $N = 1$ supersymmetric theories.
When $c^M = 1$, the discrete states appear in $SU(2)$ multiplets. In the matter sector, we again have $SU(2)$ current algebra with level $\kappa = 2$ generated by the following currents:

$$
J^\pm(z) = \sqrt{2} \psi^M(z) e^{\pm i \phi^M(z)},
$$

$$
J^0(z) = i \partial \phi^M(z),
$$

(4.6)

with OPEs in (3.8) with $\kappa = 2$. Their integrals satisfy the $SU(2)$ algebra and the discrete states (including some charged vacuum states) form $SU(2)$ multiplets. A difference from the bosonic case is that the states belong to different sectors according to their spins; states in the NS sector have integer spins while those in the R sector have half-integer spins. Like the bosonic $c^M = 1$ case, we expect that there is a ground ring, which is expected to be generated by the states with $(j, k) = (-1, -2)$ and $(-2, -1)$ in the R sector.

For the NS sector, we find the following representatives in the $q = -1$ picture:

**Theorem 4**

1. For $j, k \in \mathbb{Z}_+$ and $N_{FP} = 0$,

$$
\Psi_{jm}(z) = (J_0^-)^{J-m} e^{i J \phi^M(z)} e^{(1+J)\phi^L(z)}. 
$$

(4.7)

2. For $j, k \in \mathbb{Z}_+$ and $N_{FP} = 1$,

$$
\tilde{\Psi}_{jm}(z) = (J_0^-)^{J-m-1} \int_{C_z} \frac{d\zeta}{2\pi i \zeta - z} K(\zeta) e^{i J \phi^M(z)} e^{(1+J)\phi^L(z)},
$$

(4.8)

where

$$
K(z) \equiv \frac{1}{2} \gamma(z) + c(z) \psi^M(z) e^{-i \phi^M(z)}.
$$

(4.9)

3. For $j, k \in \mathbb{Z}_-$ and $N_{FP} = 0$,

$$
\Psi_{jm}^+(z) = (J_0^-)^{J-m} e^{i J \phi^M(z)} e^{(1-J)\phi^L(z)}.
$$

(4.10)

Here $J \equiv |\frac{j+k}{2}|$ and $m \equiv \frac{j-k}{2}$ are integers.

At present, we do not have compact expressions for $j, k \in \mathbb{Z}_-$ and $N_{FP} = -1$ though their presence is guaranteed by theorem 3. The states representing the nontrivial cohomology classes are obtained by acting the above operators on the physical vacuum.
|λ⟩ ≡ |λ^M = 0⟩ ⊗ |λ^L = i⟩ ⊗ c_1 0⟩_b c_0 ⊗ |0⟩_β γ, with γ_r |0⟩_β γ = 0 for r ≥ 1/2. Note that we have introduced a BRST-invariant operator K(z), which carries N_{FP} = 1 and the same charge as J^-(z). These are the super extension of the operators reported in (3.9)–(3.13). Again from the above expressions, we see clearly that operators with J form (2J + 1)-plet if N_{FP} = 0, while operators with the same J but with N_{FP} = 1 form (2J − 1)-plet with respect to SU(2).

As for the R sector, one must be careful about the fermion zero-modes, which form two dimensional Clifford algebra. We use representations ψ^±_0 ≡ 1/√2(ψ^M_0 ± iψ^L_0) ≡ 1/2(σ_1 ± iσ_2): the vacua are two-dimensional spinors. From the condition F ≡ 2[β_0, Q_B] = 0, we find the spinor structure of the state \begin{pmatrix} 0 \\ 1 \end{pmatrix} for cases (1) and (2) and \begin{pmatrix} 1 \\ 0 \end{pmatrix} for case (3) in the theorem [see (C.9)] [12]. So we may take the vacuum \begin{pmatrix} |λ⟩ \\ β_0 |λ⟩ \end{pmatrix}, with β_0 |λ⟩ = 0, and create the representatives of nontrivial cohomology classes by using the same operators as in the NS sector given in theorem 4 for the cases (1) and (2) (but with half-odd-integers J and m). For case (3), we should take the vacuum \begin{pmatrix} |λ⟩ \\ 0 \end{pmatrix}. Of course, the mode expansions should be modified accordingly. It is interesting to find that the modified oscillators in Ref. [12] (cf. (C.10)) appear automatically from this procedure.

For the study of the ground ring as well as correlation functions for this system, it is desirable to find vertex operators to create the states in the R sector on the conformal vacuum by using spin fields, since we expect that the generating elements of the ground ring are in the R sector. Here we present a calculation of OPE among the operators in (4.10). This confirms our expectation that similar ground ring exists in the 2D supergravity theory. We will perform the calculation in the different (q = 0) picture in which the states in (4.10) are created by the operators

\[ \Psi_{Jm}^{(+)}(z) = (J^-_0)^{J-m}[Jψ^M(z) - i(1 - J)ψ^L(z)] e^{iJψ^M(z)} e^{(1-J)ψ^L(z)}. \] (4.11)

*These results are obtained in Ref. [30, 31].
The operator algebra for (4.11) can be written as

$$\Psi_{j_1,m_1}^{(+)}(z)\Psi_{j_2,m_2}^{(+)}(0) = \cdots + \frac{1}{z} \sum_{j_3,m_3} F_{j_1,m_1,j_2,m_2}^{j_3,m_3} \Psi_{j_3,m_3}^{(+)}(0) + \cdots,$$

(4.12)

As in the bosonic case, the $SU(2)$ invariance requires the structure constants $F$ to be of the form

$$F_{j_1,m_1,j_2,m_2}^{j_3,m_3} = C_{j_1,j_2,j_3,m_3}^{j_1+1,m_1+1,m_2} g(j_1,j_2),$$

(4.13)

where $C$ are the Clebsch-Gordan coefficients and $g(j_1,j_2)$ is a function of $j_1$ and $j_2$ only. For $j_3 = j_1 + j_2 - 1$, $C$ are given by

$$C_{j_1,j_2,j_3,m_3}^{j_1+1,j_2-1,m_1+1,m_2} = \frac{N(j_3,m_3) j_2 m_1 - j_1 m_2}{N(j_1,m_1) N(j_2,m_2) \sqrt{j_3(j_3+1)}}.$$

(4.14)

In order to determine $g(j_1,j_2)$, we consider the OPE for $m_1 = j_1 - 1$ and $m_2 = j_2$:

$$\Psi_{j_1-1,j_1}^{(+)}(z)\Psi_{j_2}^{(+)}(0) = \cdots + \frac{1}{z} F_{j_1-1,j_1,j_2}^{j_1,j_2-1} \Psi_{j_1+1,j_1,j_2}^{(+)}(0) + \cdots.$$

(4.15)

We then compute the left hand side to find

$$\sqrt{2j_1} \Psi_{j_1,j_1-1}^{(+)}(z)\Psi_{j_2,j_2}^{(+)}(0) = \int \frac{du}{2\pi i} \psi^M(u+z) e^{-i\phi^M(u+z)} : [j_1 \psi^M(z) - i(1-j_1)\psi^L(z)] e^{ij_1\phi^M(z)+(1-j_1)\phi^L(z)} : \times : [j_2 \psi^M(0) - i(1-j_2)\psi^L(0)] e^{ij_2\phi^M(0)+(1-j_2)\phi^L(0)} :$$

$$= - \int \frac{du}{2\pi i} \left[ j_1 u^{1-j_1}(u+z)^{-j_2} z^{j_1-1,j_1+j_2-1} \left( j_2 \psi^M(0) - i(1-j_2)\psi^L(0) \right) + j_2 u^{-j_1}(u+z)^{-1-j_2} z^{j_1,j_1+j_2-1} \left( j_1 \psi^M(z) - i(1-j_1)\psi^L(z) \right) + (j_1 + j_2 - 1) u^{-j_1}(u+z)^{-1-j_2} z^{j_1,j_1+j_2-2} \psi^M(u+z) \right] e^{i(j_1+j_2-1)\phi^M-(j_1+j_2-2)\phi^L}. \quad (4.16)$$

Changing the integration variable as $u = xz$, the integration can be performed to give

$$\sqrt{2j_1} \Psi_{j_1,j_1-1}^{(+)}(z)\Psi_{j_2,j_2}^{(+)}(0) = - \frac{1}{z} \frac{\Gamma(j_1+j_2)}{\Gamma(j_1)\Gamma(j_2)} \Psi_{j_1+1,j_1+1,j_2-1}^{(+)}(0),$$

(4.17)

which is very similar to the bosonic case \[1\]. Comparing this with eqs. (4.13) and (4.15), we find

$$g(j_1,j_2) = \frac{\sqrt{j_1+j_2(j_1+j_2-1)!}}{\sqrt{2j_1j_2(j_1-1)!(j_2-1)!}}. \quad (4.18)$$
After appropriate choice of normalization, this leads to the operator algebra isomorphic to the $W_{\infty}$ or area-preserving diffeomorphism:

$$\Psi_{j_1, m_1}^{(+)}(z) \Psi_{j_2, m_2}^{(+)}(0) = \frac{1}{z} (j_2 m_1 - j_1 m_2) \Psi_{j_1 + j_2 - 1, m_1 + m_2}^{(+)}(0).$$  \hspace{1cm} (4.19)

which is identical to the bosonic 2D gravity theory. We can similarly analyse the OPE for other operators in (4.7).

5 Discussions

There are couple of important questions to be addressed.

Our understanding of the relation between the matrix models and continuum approach has been improved recently, mainly because a method to calculate some correlation functions in a continuum approach has been found. However, there is still some gap in the identification of the scaling operators in the matrix models and the states in the continuum approach. In the latter, we encounter “non-decoupling of null states”. On the other hand, the presence of null states is responsible for discrete states with higher ghost numbers after applying the Felder’s cohomology. This aspect must also be related to the question what is the ground ring when $c^M < 1$. A thorough comparison of the results from two approaches will be important.

We would like to see the algebra associated with the discrete states in the supersymmetric case. As explicitly shown in sect. 4, the discrete states in theorem 4 satisfy the same area-preserving diffeomorphism as the bosonic case. This is an indication that the ground ring is identical to the bosonic case. It is, however, necessary to examine the structure in the R sector in order to identify the complete structure of the ground ring in the supersymmetric case.

In a study of the ground ring, Witten has identified the matrix model potential starting from a field theoretical approach. It would be interesting to see if we could find out a corresponding supersymmetric matrix model in this manner.

It would be appropriate to mention some results of related papers. In Ref., the
authors studied the BRST cohomology for the minimal models coupled to gravity. By imposing Felder’s cohomology [34], they recovered the results by Lian and Zuckerman [8]. In a recent paper [13], the same authors have reported the cohomology associated with the boundary states of conformal grids, which must be relevant to the identification of missing states in [8] in comparison with the matrix models. The states with $N_{FP} = \pm 1$ are also discussed in a different approach [35].

While preparing this report, we have received Ref. [36] which discusses the ground ring in the 2D supergravity theory.

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Appendix

A Hermiticity properties of Fock spaces

We would like to discuss the hermiticities and inner products in the charged Fock spaces assumed in our text implicitly. Some properties are well known and may be found in some other literature [34, 17]. However they have not been discussed clearly and in a coherent manner in the context of the present subject.

We will present our discussion on the bosonic case (see sect. 3 for the notation), but the extension to supersymmetric case is straightforward.

For a generic value of $\lambda$, we find that the Virasoro generators transform under the Fock-hermitian conjugation as

$$L_n^\dagger(\lambda) = L_{-n}(-\lambda^*),$$  \hspace{1cm} (A.1)

which is the right hermiticity only for representations with $\lambda \in i\mathbb{R}$ (pure imaginary). So for the gravity sector, the Fock space hermiticity is the right one. Obviously, we have to define another hermiticity for the matter sector where $\lambda$ is real, $c \leq 1$. This is provided by using the parity operator which changes signs of all the operators: $P(q,p,\phi_n)P = -(q,p,\phi_n)$; its action on a Fock space is defined as $P\mathcal{F}(p) = \mathcal{F}(-p)$. We then define $t$-operation by

$$L_n^t(\lambda) = P L_n(\lambda)\dagger P = L_{-n}(\lambda).$$  \hspace{1cm} (A.2)

This $t$-operation may be regarded as appropriate hermitian conjugation for the case of $c \leq 1$.

The inner products which respect the hermiticity properties are given by the following bilinear products defined over dual vector spaces [34] for $c \leq 1$,

$$\langle y', y \rangle \equiv |y'\rangle^t|y\rangle,$$  \hspace{1cm} (A.3)

\[\text{footnote}{\text{This possibility is pointed out to one of the authors (K.I.) by M. Kato.}}\]
where \( |y⟩ ∈ \mathcal{F}(p) \) and \( |y'⟩ ∈ \mathcal{F}(-p) \); for \( c ≥ 1 \)

\[
\langle y', y⟩ \equiv |y'⟩^\dagger |y⟩ \quad \text{(A.4)}
\]

where \( |y⟩ ∈ \mathcal{F}(p) \) and \( |y'⟩ ∈ \mathcal{F}(p^*) \). Note that the products are made compatible with \( ⟨µ|ν⟩ = δ_{µ∗,ν} \), which follows from \( p^\dagger = p \). Here \( |ν⟩ \) is a state with a momentum eigenvalue \( ν \).

Some comments are in order. As explained in the above, one may use either Fock space hermiticity or the t-operation for \( c = 1 \) case and both of them provides us with the desired hermiticity of Virasoro generators. If we use the t-operation as the hermitian conjugation, the \( SU(2) \) currents no longer have definite hermiticity. Rather, we find \( sl(2,R) \) current algebra satisfied by modified (anti-hermitian) currents, \( j^0(z) = -J^0(z) \) and \( j^\pm(z) = iJ^\pm(z) \).

\[B\] \( c^M < 1 \) CFTs coupled to gravity

In the text, we have presented expressions for operators which create states in \( KerQ_B/ImQ_B \) for \( c^M = 1 \) (bosonic) or \( \hat{c}^M = 1 \) (supersymmetric case). Here we will give a procedure to obtain the states for \( c^M < 1 \) bosonic theories starting from \( c^M = 1 \). One can easily extend the procedure to supersymmetric case.

The relations in the following discussion are most clearly seen if we compactify the bosons for the matter as well as gravity sectors. So let us first describe a compactified boson. For the boson on a torus with a radius \( R \), \( \phi_L + \phi_R \equiv \phi_L + \phi_R + 2πR\mathbf{Z} \), the momentum and winding modes are quantized. It has been noticed by many authors that the formula for chiral momenta may be compared with (3.6): \( p_L = t_{(j,k)} \) and \( p_R = t_{(i,-k)} \). Here we understand that \( t_+ = 2/R \) and \( t_- = -R \). From these relations, we obtain \( λ(R) = \frac{1}{2}(R - 2/R) \) and \( c(R) = 1 - 6(R/\sqrt{2} - \sqrt{2}/R)^2 \). We thus see that \( R = \sqrt{2} \) corresponds to \( c = 1 \) theory.

As the gravity sector, we take a compactified boson whose compactification radius is related to that of matter part by \( R^L = iR^M \). This relation is due to the anomaly
cancellation (3.5). For later convenience, let us parametrize the matter radius as \( R^M = \sqrt{2} e^\omega \), then \( t^M_\pm(\omega) = \pm it^L_\pm(\omega) = \pm \sqrt{2} e^{\mp \omega} \), and \( c^M(\omega) = 1 - 24 \sinh^2 \omega \).

Our strategy is to change the radius \( R^M \) (or \( \omega \)) and relate systems with different \( c^M \). Rather than changing the parameter by hand, we use a “Lorentz” transformation defined by

\[
G(\omega) \partial G(-\omega) = \Omega(\omega) \partial
\]

where

\[
\Omega(\omega) = \begin{pmatrix}
\cosh \omega & i \sinh \omega \\
-i \sinh \omega & \cosh \omega
\end{pmatrix}
\]

and \( \partial = q, p \) or \( \phi \), are two-dimensional vectors; for example, \( q \equiv \begin{pmatrix} q^M \\ q^L \end{pmatrix} \). The generator of the transformation is

\[
G(\omega) = e^{\omega \mathcal{G}}, \\
\mathcal{G} = q^M p^L - q^L p^M - i \sum_{n \neq 0} \frac{1}{n} \alpha^M_n \alpha^L_n.
\]

The BRST charge of the system with \( c^M(\omega) \) is obtained by

\[
Q_B(\omega) = G(\omega) Q_B(0) G(-\omega),
\]

which then implies the isomorphism;

\[
\text{Ker} Q_B(\omega)/\text{Im} Q_B(\omega) \sim \text{Ker} Q_B(0)/\text{Im} Q_B(0).
\]

Let us examine this isomorphism in more detail. For case (1) in theorem 1, we may conclude that our transformation provides us with the isomorphism from the following observations: the relation of the vacuum, \( |\lambda(\omega)\rangle \equiv |\lambda^M(\omega)\rangle \otimes |\lambda^L(\omega)\rangle \otimes c_1|0\rangle_{gh} = G(\omega)|\lambda(0)\rangle \); \( G(\omega)V(t^M_{(j,k)}(0), z)V(t^L_{(j,k)}(0), z)G(-\omega) = V(t^M_{(j,k)}(\omega), z)V(t^L_{(j,k)}(\omega), z) \). The states for \( c^M = 1 \), which represent nontrivial cohomology classes in (2) and (3) of the theorem 1, have nonvanishing bilinear products, discussed in appendix A, in the following combinations: the states with \( N_{FP} = 0 \) in (2) and (3); the states with \( N_{FP} = 1 \) in (2) and those with \( N_{FP} = -1 \) in (3). The states corresponding to (2) and (3) of theorem 1 for \( c^M(\omega) \) may be obtained from those in theorem 2 as follows. First use the operators listed in theorem 2 and obtain states for \( c^M = 1 \) system, and then act \( G(\omega) \) on them. The resultant states are in \( \text{Ker} Q_B(\omega) \) but not in \( \text{Im} Q_B(\omega) \), since the inner products are
nonzero. [Actually the inner products do not change under the transformation owing to \( P^M G(\omega)^\dagger P^M = G(-\omega) \).]

C Proof of Theorems 1 and 3

We are going to sketch the BRST analysis in our earlier paper [12], treating NS, R sectors and bosonic theory in as parallel as possible.

As a preparation, let us define some notations:

\[
P^\pm(n) = \frac{1}{\sqrt{2}}[(p^M + n\lambda^M) \pm ip^L + n\lambda^L)] \equiv \begin{cases} \frac{1}{\sqrt{2}}t^M_+(j - n) \\ \frac{1}{\sqrt{2}}t^M_-(k - n) \end{cases} ; \quad (C.1)
\]

and the light-cone variables

\[
p^\pm = P^\pm(0) = \frac{1}{\sqrt{2}}(p^M \pm ip^L), \quad q^\pm = \frac{1}{\sqrt{2}}(q^M \pm iq^L),
\]

\[
\alpha^\pm_n = \frac{1}{\sqrt{2}}(\alpha^M_n \pm i\alpha^L_n), \quad \psi^\pm_r = \frac{1}{\sqrt{2}}(\psi^M_r \pm i\psi^L_r). \quad (C.2)
\]

In eq. (C.1), we have used the linearity of \( P^\pm(n) \) in \( n \), and defined \( j \) and \( k \) as their zeros

\[
t^M_\pm = \begin{cases} -\lambda^M \pm \sqrt{(\lambda^M)^2 + 2} & \text{for bosonic theory} \\ -\lambda^M \pm \sqrt{(\lambda^M)^2 + 1} & \text{for supersymmetric theory} \end{cases} \quad (C.3)
\]

\( j \) and \( k \) are not necessarily integers at this stage. Note that (C.1) is rewritten as (3.6) for \( p^{M,L} \) and

\[
p^+p^- = P^+(0)P^-(0) = \frac{1}{2}t^M_+t^M_-jk = \begin{cases} -jk & \text{for bosonic theory} \\ -\frac{1}{2}jk & \text{for supersymmetric theory} \end{cases} \quad (C.4)
\]

The BRST charge has an expansion with respect to ghost zero modes as

\[
Q_B = c_0L - b_0M + d \left( -\frac{1}{2}\gamma_0F + 2\beta_0K - \frac{1}{4}b_0\gamma_0^2 \right) \quad (C.5)
\]

for NS sector and bosonic theory (for R sector, we add three other terms in the bracket).

Here

\[
L = p^+p^- + \hat{N} \quad (C.6)
\]
with $\hat{N}$ as the level counting operator.

Our physical states are defined as nontrivial states satisfying

$$Q_B|\text{phys}\rangle = 0.$$  \hfill (C.7)

Using the relation $L = \{b_0, Q_B\}$ on a physical state, we have

$$L|\text{phys}\rangle = Q_B b_0 |\text{phys}\rangle,$$  \hfill (C.8)

which implies that the state is $Q_B$-trivial unless it is on-shell $L = 0$.

The relative cohomology is defined as the cohomology of $Q_B$ on $F(p^M, p^L) \cap \text{Ker}(b_0)$ for the NS sector and bosonic theory (R sector). So the physical state condition becomes $L|\text{phys}\rangle = d|\text{phys}\rangle = 0$ for the NS sector and bosonic theory (R sector). Note that since $F^2 = L$, $V_F \equiv \{|\psi\rangle : F|\psi\rangle = 0\} \subset V_L \equiv \{|\psi\rangle : L|\psi\rangle = 0\}$ in the R sector. This means that any physical state is an on-shell state. In terms of the momenta $p^+$ and $p^-$, the on-shell condition for both sectors (bosonic case) is written as $p^+ p^- + \hat{N} = -jk/2 + \hat{N} (= -jk + \hat{N}) = 0$. Hence $jk$ must be a positive integer in order to satisfy the on-shell condition.

We may consider two possibilities depending on whether we have oscillator excitations. If we have no oscillator excitation, then $j = 0$ or $k = 0$, which corresponds to the case (1) of theorems 1 and 3. In the R sector, we find spinor structures of states as follows:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot |p^M, p^L > \text{ for } p^+ = 0; \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot |p^M, p^L > \text{ for } p^- = 0. \quad (C.9)$$

where we have used $\psi_0^\pm \equiv 1/\sqrt{2} (\psi_0^M \pm i\psi_0^L) \equiv 1/2 (\sigma_1 \pm i\sigma_2)$. These vacuum states satisfy $d = 0$ condition trivially.

Let us discuss the other case, $jk \neq 0$. Since $p^+ \neq 0$, we may define the following operators in the R sector:

$$\tilde{\alpha}_n^\pm \equiv \alpha_n^\pm + n\theta \psi_n^\pm, \quad \tilde{\psi}_n^\pm \equiv \psi_n^\pm - \theta \alpha_n^\pm,$$

$$\tilde{c}_n \equiv c_n - \theta \gamma_n, \quad \tilde{b}_n \equiv b_n + n\theta \beta_n,$$

$$\tilde{\gamma}_n \equiv \gamma_n + n\theta c_n, \quad \tilde{\beta}_n \equiv \beta_n - \theta b_n.$$  \hfill (C.10)
where \( \theta = \psi_0^+ / p^+ \), \( (\theta^2 = 0) \). In order to discuss the two sectors at the same time, let us use the same notation \( \tilde{\alpha}_n^\pm, \tilde{\psi}_n^\pm \) for \( \alpha_n^\pm, \psi_n^\pm \) in the NS sector, and for \( \tilde{\alpha}_n^\pm, \tilde{\psi}_n^\pm \) in the R sector. For the bosonic theory, we use the same notation as NS sector but, of course, without the fermions as well as super-ghosts.

We assign the degrees to the mode operators as follows:

\[
\text{deg} \left( \tilde{\alpha}_n^+, \tilde{\psi}_n^+, \tilde{c}_n, \tilde{\gamma}_n \right) = +1,
\]

\[
\text{deg} \left( \tilde{\alpha}_n^-, \tilde{\psi}_n^-, \tilde{b}_n, \tilde{\beta}_n \right) = -1,
\]

and 0 to ground states. All the states then carry definite degrees. The operator \( d \) is expanded according to the degree as \( d = d_0 + d_1 + d_2 \).

Our strategy is first to study \( d_0 \)-cohomology classes and then examine if they can be extended to \( d \)-cohomology classes. Let us explain the procedure in some details. The lowest degree term of \( d \) is given by (the second term is absent for bosonic theory)

\[
d_0 = \sum_{n \neq 0} P^+(n) \tilde{c}_{-n} \tilde{\alpha}_n^- - \frac{1}{2} \sum_{n+\delta \neq 0} P^+(2n + 2\delta) \tilde{\gamma}_{-(n+\delta)} \tilde{\psi}^-_{n+\delta}.
\]

We now define the operator

\[
K \equiv \sum_{n \neq 0} P^+(n) \tilde{\alpha}_{-n} \tilde{b}_n + \sum_{n+\delta \neq 0} P^+(2n + 2\delta) \tilde{\psi}^+_{-(n+\delta)} \tilde{\beta}_{n+\delta}
\]

where the primes imply that we have excluded the sum over \( n = j \) or/and \( n + \delta = j/2 \), since \( P^+(j) = 0 \).

The operator defined by \( \hat{N}' = \{ d_0, K \} \) is the number counting operator for \( \tilde{\alpha}_{-n}^+, \tilde{\alpha}_n^-, \tilde{b}_n, \tilde{c}_{-n}(n \neq j), \tilde{\psi}_{-(n+\delta)}^+, \tilde{\psi}_{n+\delta}^-, \tilde{\beta}_{n+\delta}, \tilde{\gamma}_{-(n+\delta)}, (2n + 2\delta \neq j) \) (The latter four oscillators are absent for the bosonic theory). We may conclude from this relation that a \( d_0 \)-closed state with any of the above-listed oscillators excited is \( d_0 \)-exact. This can be seen similarly to eq. (C.8). Therefore \( d_0 \)-nontrivial states must be created by the oscillators absent on the list, at the level of \( jk/2 \) \( (jk \text{ for bosonic}) \) to satisfy the on-shell condition. In particular, if \( j = 0 \) or \( j \) is not a nonzero integer, then we have no oscillators available and only the ground state is allowed. After choosing momentum to satisfy the on-shell condition, we
see that the ground state is already a representative of nontrivial $d$-cohomology. This is the case (1) of theorems 1 and 3.

In the following, we assume that $j \in \mathbb{Z}_+$ or $\mathbb{Z}_-$. Since $jk$ is a positive integer, $k \in \mathbb{Z}_+$ or $\mathbb{Z}_-$ accordingly. For the bosonic theory, we find the $d_0$-nontrivial states as

\[
\left(\tilde{\alpha}_j^+\right)^k | j, k \rangle \quad \tilde{c}_{-j} \left(\tilde{\alpha}_j^+\right)^{k-1} | j, k \rangle
\]

(C.14)

for $j, k > 0$ and

\[
\left(\tilde{\alpha}_j^-\right)^{-k} | j, k \rangle \quad \tilde{b}_j \left(\tilde{\alpha}_j^-\right)^{-k-1} | j, k \rangle
\]

(C.15)

for $j, k < 0$. Here the vacuum is $| j, k \rangle = |p^M = t_{(j,k)}^M + \lambda^M\rangle \otimes |p^L = t_{(j,k)}^L + \lambda^L\rangle \otimes c_1 |0\rangle_{gh}$. 

For the supersymmetric case, we find that there are four possible cases listed on table 1 depending on the values of $(j, k)$.

| $j$ | odd + $2\delta$ | even + $2\delta$ |
|-----|----------------|-----------------|
| even | eqs.(C.14,15) | (all parents or daughters) |
| odd | (no on-shell states) | eqs.(C.16,17) |

Table 1

When $(j, k) = (\text{odd} + 2\delta, \text{even})$, we find the $d_0$-nontrivial states as eqs. (C.14,15) with $k$ in the exponents replaced by $k/2$. The vacuum for the NS sector takes the same form as the bosonic case; but for the R sector it has a spinor structure, $| j, k \rangle = |p^M = t_{(j,k)}^M + \lambda^M\rangle \otimes |p^L = t_{(j,k)}^L + \lambda^L\rangle \otimes c_1 |0\rangle_{gh} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It is easy to see that no on-shell state can be constructed out of available oscillators if $(j, k) = (\text{odd} + 2\delta, \text{odd})$.

The situation is quite different from bosonic case if $j = \text{even} + 2\delta$. We may construct many states with different degrees and ghost numbers. The generic patterns are given on tables 2 for $k =$odd and 3 for $k =$even. On the tables, all the states $|\ast\rangle$ represent
nontrivial $d_0$-cohomology classes.

| $N_{FP}$ deg. | $k - 1$ | $k - 2$ |  
|---------------|--------|--------|
| $k$           | $|\text{daughter} \rangle$ | $|\text{daughter} \rangle$ |
| $k - 1$       | $|\text{parent} \rangle$ | $|\text{parent} \rangle$ |
| $|\text{nontrivial} \rangle$ | $|\text{parent} \rangle$ | $|\text{parent} \rangle$ |

**Table 2: $k =$odd**

| $N_{FP}$ deg. | $k - 1$ | $k - 2$ |  
|---------------|--------|--------|
| $k$           | $|\text{daughter} \rangle$ | $|\text{daughter} \rangle$ |
| $k - 1$       | $|\text{parent} \rangle$ | $|\text{parent} \rangle$ |
| $|\text{parent} \rangle$ | $|\text{parent} \rangle$ | $|\text{parent} \rangle$ |

**Table 3: $k =$even**

Now that we have finished the analysis of $d_0$-cohomology, we are in a position to extend the results to $d$-cohomology. In order to construct a state representing nontrivial cohomology class of $d$, we may start from a state nontrivial with respect to $d_0$ and add terms of higher degrees. This is due to the following Lemma [9]:

**Lemma 1**

The lowest degree term in a state nontrivial with respect to $d$ may always be chosen to represent a nontrivial cohomology of $d_0$. •

Furthermore, we have [3]:

**Lemma 2**
If, for each ghost number $N_{FP}$, the cohomology of $d_0$ is nontrivial for at most one fixed degree $k$ independent of $N_{FP}$, then the cohomologies of $d_0$ and $d$ are isomorphic. ●

For (bosonic as well as supersymmetric) critical strings and bosonic non-critical strings, the isomorphism has been proved. These lemmas are enough to derive the results in sect. 3.

In the super-Liouville theory, the assumption holds for $j = \text{odd} + 2\delta$ and we may construct $d$-nontrivial states starting from (C.14,15). However, when $j = \text{even} + 2\delta$, the assumption for the isomorphism does not hold as can be seen from the tables 2 and 3. However, we have in this case:

**Lemma 3**

If the action of $d$ on a $d_0$-nontrivial state produces another $d_0$-nontrivial state, those two states cannot give rise to $d$-nontrivial states. ●

If we find states related as $d|\zeta\rangle = |\xi\rangle$, obviously both of them do not contribute to the spectrum. The states related this way are called “parents” and “daughters” respectively in the context of the BRST formalism. By studying relations of $d_0$-nontrivial states under $d$, it is easy to show that they are classified as “parents”, “daughters” or $d$-nontrivial states in the pattern indicated in the tables 2 and 3; a “daughter” is created from a “parent” at the lower-right position on the tables. Only when $k =$odd, we may obtain $d$-nontrivial states starting from the following states:

\[
\tilde{\psi}_{-j/2}^+ (\tilde{\alpha}_{-j}^+)^{(k-1)/2} |j, k\rangle, \\
[(\tilde{\alpha}_{-j}^+)^{(k-1)/2} \tilde{\gamma}_{-j/2} - j(k - 1) \tilde{c}_{-j} \tilde{\psi}_{-j/2}^+ (\tilde{\alpha}_{-j}^+)^{(k-3)/2}] |j, k\rangle \tag{C.16}
\]

for $j, k > 0$ and

\[
\tilde{\psi}_{j/2}^- (\tilde{\alpha}_{j}^-)^{-(k+1)/2} |j, k\rangle, \\
[(\tilde{\alpha}_{j}^-)^{-(k+1)/2} \tilde{\beta}_{j/2} - \frac{1}{2} j \tilde{b}_{j} \tilde{\psi}_{j/2}^- (\tilde{\alpha}_{j}^-)^{-(k+3)/2}] |j, k\rangle \tag{C.17}
\]

for $j, k < 0$.  

25
The states for NS sector obtained above are in the $q = -1$ picture, and we have given in theorem 4 the explicit vertex representations of the states for $\hat{c} = 1$ theory in this picture. The operators (4.11), on the other hand, are written in the $q = 0$ picture.

This completes the proof of the results presented in sect. 4.
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