Bit-Vector Model Counting using Statistical Estimation

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Abstract—Approximate model counting for bit-vector SMT formulas (generalizing #SAT) has many applications such as probabilistic inference and quantitative information-flow security, but it is computationally difficult. Adding random parity constraints (XOR streamlining) and then checking satisfiability is an effective approximation technique, but it requires a prior hypothesis about the model count to produce useful results. We propose an approach inspired by statistical estimation to continually refine a probabilistic estimate of the model count for a formula, so that each XOR-streamlined query yields as much information as possible. We implement this approach, with an approximate probability model, as a wrapper around an off-the-shelf SMT solver or SAT solver. Experimental results show that the implementation is faster than the most similar previous approaches which used simpler refinement strategies. The technique also lets us model count formulas over floating-point constraints, which we demonstrate with an application to a vulnerability in differential privacy mechanisms.

Index Terms—model counting, bit-vectors, floating point, #SAT, randomized algorithms

I. INTRODUCTION

Model counting is the task of determining the number of satisfying assignments of a given formula. Model counting for Boolean formulas, #SAT, is a standard model-counting problem, and it is a complete problem for the complexity class #P in the same way that SAT is complete for NP. #P is believed to be a much harder complexity class than NP, and exact #SAT solving is also practically much less scalable than SAT solving. #SAT solving can be implemented as a generalization of the DPLL algorithm [15], and a number of systems such as Relsat [4], CDP [5], Cachet [32], sharpSAT [35], DSHARP [29] and countAntom [7] have demonstrated various optimization techniques. However, not surprisingly given the problem’s theoretical hardness, such systems often perform poorly when formulas are large and/or have complex constraints.

Since many applications do not depend on the model count being exact, it is natural to consider approximation algorithms that can give an estimate of a model count with a probabilistic range and confidence. Some approximate model counters include ApproxCount [38], SampleCount [22], MiniCount [25], ApproxMC [10], ApproxMC-p [37] and ApproxMC2 [11]. In this paper we build on the approximation technique of XOR streamlining [23], which reduces the number of solutions of a formula by adding randomly-chosen XOR (parity) constraints. In expectation, adding one constraint reduces the model count by a factor of 2, and $k$ independent constraints reduce the model count by $2^k$. If a formula with extra constraints has $n > 0$ solutions, the original formula likely had about $n \cdot 2^k$. If the model count after constraints is small, it can be found with a few satisfiability queries, so XOR streamlining reduces approximate model counting to satisfiability. However to have an automated system, we need an approach to choose $k$ values when the model count is not known in advance.

One application of approximate model counting is measuring the amount of information revealed by computer programs. For a deterministic computation, we say that the influence [30] is the base-two log of the number of distinct outputs that can be produced by varying the inputs, a measure of the information flow from inputs to outputs. Influence computation is related to model counting, but formulas arising from software are more naturally expressed as SMT (satisfiability modulo theories) formulas over bit-vectors than as plain CNF, and one wants to count values only of output variables instead of all variables. The theory of arithmetic and other common operations on bounded-size bit-vectors has the same theoretical expressiveness as SAT, since richer operations can be expanded (“bit-blasted”) into circuits. But bit-vector SMT is much more convenient for expressing the computations performed by software, and SMT solvers incorporate additional optimizations. We build a system for this generalized version of the problem which takes as input an SMT formula with one bit-vector variable designated as the output, and a specification of the desired precision.

Our algorithm takes a statistical estimation approach. It maintains a probability distribution that reflects an estimate of possible influence values, using a particle filter consisting of weighted samples from the distribution. Intuitively the mean of the distribution corresponds to our tool’s best estimate, while the standard deviation becomes smaller as its confidence grows. At each step, we refine this estimate by adding $k$ XOR constraints to the input formula, and then enumerating solutions under those constraints, up to a maximum of $c$ solutions (we call this enumeration process an exhaust-up-to-c query [30]). At a particular step, we choose $k$ and $c$ based on our previous estimate (prior), and then use the query result to update the estimate for the next step (posterior). The update from the query reweights the particle filter points according to a probability model of how many values are excluded by XOR constraints. We use a simple binomial-distribution model which would be exact if
each XOR constraint were fully independent. Because this model is not exact, a technique based only on it does not provide probabilistic soundness, even though it performs well practically. So we also give a variant of our technique which does produce a sound bound, at the expense of requiring more queries to meet a given precision goal.

We implement our algorithm in a tool SearchMC\footnote{The source code is available at https://github.com/seonmokim/SearchMC} that wraps either a bit-vector SMT solver compatible with the SMT-LIB 2 standard or a SAT solver, and report experimental results. SearchMC can be used to count solutions with respect to a subset of the variables in a formula, such as the outputs of a computation, the capability that Klebanov et al. call projected model counting [37], and Val et al. call subset model counting [36]. In our case the variables not counted need not be of bit-vector type. For instance this makes SearchMC to our knowledge the first tool that can be used to count models of constraints over floating-point numbers (counting the floating-point bit patterns individually, as contrasted with computing the measure of a subset of $\mathbb{R}^n$ as in the work of Chistikov et al. [13]). We demonstrate the use of this capability with an application to a security problem in the work of Chistikov et al. [13]). We demonstrate the use of this capability with an application to a security problem in the work of Chistikov et al. [13]).

In summary, the key attributes of our approach are as follows:

- Our approximate counting approach gives a two-sided bound with user-specified confidence.
- Our tool inherits the expressiveness and optimizations of SMT solvers.
- Our tool gives a probabilistically sound estimate if requested, or can give a result more quickly if empirical precision is sufficient.

II. BACKGROUND

**XOR Streamlining.** The main idea of XOR streamlining [23] is to add randomly chosen XOR constraints to a given input formula and feed the augmented formula to a satisfiability solver. One random XOR constraint will reduce the expected number of solutions in half. Consequently, if the formula is still satisfiable after the addition of $s$ XOR constraints, the original formula likely has at least $2^s$ models. If not, the formula likely has at most $2^s$ models. Thus we can obtain a lower bound or an upper bound with this approach. There are some crucial parameters to determine the bounds and the probability of the bounds and they need to be carefully chosen in order to obtain good bounds. However, early systems [23] did not provide an algorithm to choose the parameters.

**Influence.** Newsome et al. [30] introduced the terminology of “influence” for a specific application of model counting in quantitative information-flow measurement. This idea can capture the control of input variables over an output variable and distinguish true attacks and false positives in a scenario of malicious input to a network service. The influence of input variables over an output variable is the $\log_2$ of the number of possible output values.

**Exhaust-up-to-c query.** Newsome et al. also introduced the terminology of an “exhaust-up-to-c query”, which repeats a satisfiability query up to some number $c$ of solutions, or until there are no satisfying values left. This is a good approach to find a model count if the number of solution is small.

**Particle Filter.** A particle filter [17] is an approach to the statistical estimation of a hidden state from noisy observations, in which a probability distribution over the state is represented non-parametrically by a collection of weighted samples referred to as particles. The weights evolve over time according to observations; they tend to become unbalanced, which is corrected by a resampling process which selects new particles with balanced weights. A particle filtering algorithm with periodic resampling takes the following form:

1. Sample a number of particles from a prior distribution.
2. Evaluate the importance weights for each particle and normalize the weights.
3. Resample particles (with replacement) according to the weights.
4. The posterior distribution represented by the resampled particles becomes the prior distribution to next round and go to step 2.

III. DESIGN

This section describes the approach and algorithms used by SearchMC. It is implemented as a wrapper around an off-the-self bit-vector satisfiability solver that supports the SMT-LIB2 format [3]. It takes as input an SMT-LIB2 formula in a quantifier-free theory that includes bit-vectors (QF_BV, or an extension like QF_AUFBV or QF_FPBV) in which one bit-vector is designated as the output, i.e. the bits over which solutions should be counted. For ease of comparison with #SAT solvers, SearchMC also has a mode that takes a Boolean formula in CNF, with a list of CNF variables designated as the output.) SearchMC repeatedly queries the SMT solver with variations of the supplied input which add XOR constraints and/or “blocking” constraints that exclude previously-found solutions; based on the results of these queries, it estimates the total number of values of the output bit-vector for which the formula has a satisfying assignment.

SearchMC chooses fruitful queries by keeping a running estimate of possible values of the model count. We model the influence ($\log_2$ of model count) as if it were a continuous quantity, and represent the estimate as a probability distribution over possible influence values. In each iteration we use the current estimate to choose a query, and then update the estimate based on the query’s results. (At a given update, the most recent previous distribution is called the prior, and the new updated one is called the posterior.) As the algorithm runs, the confidence in the estimate goes up, and the best estimate changes less from query to query as it converges on...
the correct result. Each counting query SearchMC makes is parameterized by \( k \), the number of random XOR constraints to add, and \( c \), the maximum number of solutions to count. The result of the query is a number of satisfying assignments between 0 and \( c \) inclusive, where a result that stops at \( c \) means the real total is at least \( c \). Generally a low result leads to the next estimate being lower than the current one and a high result leads to the estimate increasing. Section III-A describes the process of updating the probability distribution, and then Section III-B gives the details of the algorithms that use it.

A. Updating distribution and confidence interval

We here explain the idea of how we compute a posterior distribution over influence, where both the prior and posterior are represented by particles. Suppose we have a formula \( f \) with a known influence \( \log_2 N \), and add \( k \) XOR random constraints to the formula. If we simulate checking the satisfiability of this augmented formula \( f' \) for different XOR constraints, we can estimate a probability of sat/unsat on \( f' \). We expand this idea by applying exhaust-up-to-\( c \) approach to \( f' \). We count the number of satisfying assignments \( n \) up to \( c \) and generate the distributions for each number of satisfying assignments (where \( n = c \) means that the number of satisfying assignments is in fact \( c \) or more). Thus under an assumption on the true influence of a formula, we can estimate the probabilities of each number of satisfying assignments based on \( k \). By collecting these probabilities across a range of influence, we obtain a probability distribution over influence for an unknown formula assumed to have less than a maximum bits of influence. Under the idealized assumption that each XOR constraint is completely independent, adding \( k \) XOR constraints will leave each satisfying assignment alive with probability \( 1/2^k \). For any particular set of \( n \geq 0 \) satisfying assignments remaining out of an original \( N \), the probability that exactly those \( n \) solutions will remain is the product of \( 1/2^k \) for each \( n \) and \( 1 - (1/2^k) \) for each of the other \( N - n \). Summing the total number of such sets with a binomial coefficient, we can approximately model the probability of exactly \( n \) solutions remaining as:

\[
Pr_{=n}(N, k) = \binom{N}{n} \left( \frac{1}{2^k} \right)^n (1 - \frac{1}{2^k})^{N-n}
\tag{1}
\]

For the case when the algorithm stops looking when there might still be more solutions, we also want an expression for the probability that the number of solutions is \( n \) or more. We compute this straightforwardly as one minus the sum of the probabilities for smaller values:

\[
Pr_{\geq n}(N, k) = 1 - \sum_{i=0}^{n-1} Pr_{=i}(N, k)
\tag{2}
\]

We use XOR constraints that contain each counted bit with probability one half, and are negated with probability one half. (This is the same family of constraints used in other recent systems [10], [37], [11]. Earlier work [23] suggested using constraints over exactly half of the bits, which have the same expected size, but less desirable independence properties.) Our binomial probability model is not precise in general, because these XOR constraints are 3-independent, but not \( r \)-independent for \( r \geq 4 \). When \( N \geq 4 \), some patterns among solutions (such as a set of four bitvectors whose XOR is all zeros) lead to correlations in the satisfiability of XOR constraints, and in turn to higher variance in the probability distribution without changing the expectation. This effect is relatively small, but we lack an analytic model of it, so we compensate by slightly increasing the confidence level our tool targets compared to what the user originally requested.

This probability model lets us simulate the probability of various query results as a function of the unknown formula influence. We use this model as a weighting function for each particle and resample particles based on each particle’s weight value. Then, we estimate a posterior distribution from sampled particles that have all equal weights. For instance, given a prior distribution over the influence sampled at 0.1 bit intervals, we can compute a sampled posterior distribution by counting and re-normalizing just the probability weights that correspond to a given query result value \( n \). From the estimated posterior distribution, the mean \( \mu \) and the standard deviation \( \sigma \) are computed. Hence, the \( \mu \) is our best possible answer as our algorithm iterates and \( \sigma \) shows how much we are close to the true answer. Sequentially, the posterior distribution will be the next round’s prior distribution and for use in the very first step of the algorithm we also implement a case of the prior distribution as uniform over influence.

Next we compute a confidence interval (lower bound and upper bound) symmetrically from the mean of the posterior distribution even though the distribution is not likely to be symmetrical. There are several ways to compute the confidence interval but the difference of the results is negligible as the posterior distribution gets narrower. Therefore, we used a simple way to compute the confidence interval: a half interval from the left side of the mean and another half from the right side.

B. Algorithm

We present our main algorithm SearchMC that runs automatically and always gives an answer with a given confidence interval. The pseudocode for algorithm SearchMC is given as Algorithm I. Our algorithm takes as input a formula \( f \), a desired confidence level \( CL \) (0 < \( CL < 1 \)), a confidence level adjustment \( \alpha \) (0 ≤ \( \alpha < 1 \)), a desired range size \( \text{thres} \) and an initial prior distribution \( \text{InitDist} \). \( f \) contains a set of bit-vector variables and bit-vector operators. We can obtain a confidence interval at a confidence level for a given mean and standard deviation. A confidence level \( CL \) is a fraction parameter specifying the probability with which the interval should contain the true answer, for example, 0.95 (95%) or 0.99 (99%). As we described above, the soundness of our model has not been proved hence a confidence level sometimes needs to be adjusted by the adjustment value \( \alpha \) to meet the actual confidence level requirement. If \( \alpha = 0 \), we do not adjust the input confidence level. In our experiments, we
used $\alpha = 0.5$, which empirically leads SearchMC to select large enough ranges. Finding the most appropriate value for $\alpha$ is future work. Our algorithm terminates when the length of our confidence interval is less than or equal to a given non-negative parameter $\text{thres}$. This parameter determines the amount of running time and there is a trade-off. If $\text{thres}$ value is small, it gives a narrow confidence interval, but the running time would be longer. If the value is large, it gives a wide confidence interval, but a shorter running time. Our tool can choose any initial prior distribution $\text{InitDist}$ represented by particles. For example, if we do not have any knowledge of input formula, it might be better to start with a uniform distribution over 0 to a number of output variables. If we have a small knowledge that the true influence is less than 64, a uniform distribution over 0 to 64 might perform better.

Algorithm 1 \text{SearchMC}(f, \text{thres}, CL, \alpha, \text{InitDist})

1: $\text{CL} \leftarrow \text{CL} + (1 - \text{CL}) \times \alpha$ \hfill \text{Confidence level adjustment}
2: \text{width} $\leftarrow \text{getWidth}(f)$ \hfill \text{The width of the output bit-vector of } f
3: $\text{prior} \leftarrow \text{InitDist}$ \hfill \text{Initial distribution}
4: $\delta \leftarrow \text{width}$
5: \textbf{while} $\delta \geq \text{thres}$ \textbf{do}
6: \hspace{1em} $c, k \leftarrow \text{ComputeCandK}(\text{prior}, \text{width})$
7: \hspace{1em} $n\text{Sat} \leftarrow \text{MBoundExhaustUpToC}(f, \text{width}, k, c)$
8: \hspace{1em} $\text{post}, \text{UB}, \text{LB} \leftarrow \text{Update}(\text{prior}, c, k, n\text{Sat}, \text{CL})$ \hfill \text{See Sec. III-A}
9: \hspace{1em} $\delta \leftarrow \text{UB} - \text{LB}$
10: \hspace{1em} \textbf{if} $k = 0$ \textbf{then}
11: \hspace{2em} \text{output} “Exact Count: ”, $n\text{Sat}$
12: \hspace{1em} \textbf{else}
13: \hspace{2em} $\text{prior} \leftarrow \text{post}$
14: \hspace{2em} \text{output} “Lower: ”, $\text{LB}$, “Upper: ”, $\text{UB}$
15: \hspace{2em} \textbf{end if}
16: \textbf{end while}

Algorithm 2 \text{ComputeCandK}(\text{prior}, \text{width})

1: $\mu, \sigma \leftarrow \text{getMuSigma}(\text{prior})$
2: $c \leftarrow [(2^\sigma + 1)/(2^\sigma - 1)]^2$
3: $k \leftarrow \lfloor \mu - \frac{1}{2} \log_2 c \rfloor$
4: \textbf{if} $k \leq 0$ \textbf{then}
5: \hspace{1em} $c \leftarrow 2^{\text{width}} + 1$ \hfill \text{In this case, } c \text{ is effectively infinite}
6: \hspace{1em} $k \leftarrow 0$ \hfill \text{No constraints}
7: \textbf{end if}
8: \textbf{return} $c, k$

Variables. There are several variables: $\text{prior}$, $\text{post}$, $\text{width}$, $k$, $c$, $n\text{Sat}$, $\text{UB}$, $\text{LB}$ and $\delta$. $\text{prior}$ represents a prior distribution by sampled particles with corresponding weights. In one iteration, we obtain the updated posterior distribution $\text{post}$ with resampled particles based on our probabilistic model as described in Section III-A. The posterior becomes the prior distribution for the next iteration. While our algorithm is in the loop, it keeps updating $\text{post}$, $\text{width}$ is the width of the output bit-vector of an input formula $f$, which is an initial upper bound for the influence since the influence cannot be more than the width of the output bit-vector. $c$ is a number of random XOR constraints and $k$ specifies the maximum number of solutions for the exhaust-up-to-$c$ query. We obtain $c$ and $k$ using the ComputeCandK function shown as Algorithm 2 and discussed below. $n\text{Sat}$ is a number of solutions from the exhaust-up-to-$c$ query. $\text{MBoundExhaustUpToC}$ runs until it finds the model count exactly or $c$ solutions from formula $f$ with $k$ random XOR constraints. $\text{UB}$ and $\text{LB}$ are variables to store an upper bound and a lower bound of the current model count approximation with a given confidence level as we describe in Section III-A. $\delta$ is the distance between the upper bound and lower bound. This parameter determines whether our algorithm terminates or not. If $\delta$ is less than or equal to our input value $\text{thres}$, our algorithm terminates. If not, it runs again with updated $\text{post}$ until $\delta$ reaches the desired range size $\text{thres}$. An extreme case $k = 0$ denotes that our guess is equivalent to the true model count. In this case, we print out the exact count and terminate the algorithm.

Functions. To motivate the definition of the function ComputeCandK, we view an exhaust-up-to-$c$ query as analogous to measuring influence with a bounded-length “ruler.” Suppose that we reduce the expected value of the model count by adding $k$ XOR constraints to $f$. Then, we can use the “length-(log2 $c$) ruler” to measure the influence starting at $k$ and this measurement corresponds to the result of an exhaust-up-to-$c$ query: the length-(log2 $c$) ruler has $c$ markings spaced logarithmically as illustrated in Figure 1. Each iteration of the algorithm chooses a location ($k$) and length ($c$) for the ruler, and gets a noisy reading on the influence as one mark on the ruler. Over time, we want to converge on the true influence value, but we also wish to lengthen the rule so that the finer marks give more precise readings. Based on this idea, we have the ComputeCandK function to choose the length and starting point of the ruler from a prior distribution. Then, we run an exhaust-up-to-$c$ query and call Update described in Section III-A to update the distribution based on the result of the query.

The pseudocode for algorithm update is described as Algorithm [3]. A prior distribution $\text{prior}$ and a posterior distribution $\text{post}$ are represented as a set of sampled particles (influences). We sampled 500 particles for each update function call. Once we have the updated distribution, we can find out the interval of a given confidence level.

Since we observe that our running $\sigma$ represents how much we are close to the true answer, we use a rational function to satisfy the condition that $c$ increases as $\sigma$ decreases (i.e., we get more accurate result as $c$ increases). The $k$ value denotes where to put the ruler. We want to place the ruler where the expected value of the prior distribution lies near the middle of the ruler hence our expected value is in the range of the ruler with high probability. Therefore, we subtract the half length of the ruler ($\frac{1}{2} \log_2 c$) from the expected value $\mu$ and then use the floor function to the
The idea is based on Lemma 2.10 and 2.13 from Klebanov et al.’s work [37]. They were inspired by ApproxMC [10] which is an \((\varepsilon,\delta)\) counter such that the true model count is within the interval \([|F|/(1+\varepsilon),|F|/(1+\varepsilon)]\) with a probability of at least \(1-\delta\). They transformed the definition of the result interval to \([|F| \cdot (1-\varepsilon),|F| \cdot (1+\delta)]\).

**Lemma 1.** Let \(n = |\Sigma|, \varepsilon \in [0, 1], k \in \mathbb{N}. k \leq \lfloor \log_2(|F| \cdot e^2/(\varepsilon \cdot r \cdot \sqrt{n})) \rfloor\) and \(h \in H(n,k,r)\) a randomly chosen strongly \(r\)-universal hash function. It holds:

\[
Pr \left( (1-\varepsilon) \cdot \frac{|F|}{2^k} \leq |F_h| \leq (1+\varepsilon) \cdot \frac{|F|}{2^k} \right) \geq 1 - e^{-r/2}
\]

They called pivot as what we call \(c\) from an exhaust-up-to-\(c\) query and pivot \(= \frac{2^r(1+\varepsilon) \cdot \sqrt{n}}{e}\). Since \(0 < \varepsilon < 1\) and \(r = 3, c\) (pivot) should be greater than 17 to make the lemma true with a probability of at least 0.86 \((1 - e^{-3/2}) \approx 0.86\). In SearchMC’s iterations, given \(c\) and \(k\), we can compute \(c\) value to estimate the bounds. Therefore, when \(c\) is greater than 17 from our tool’s iteration, we can compute a lower and upper bound such that the true influence is within the bounds with a probability of at least 0.86.

**IV. Experimental Results**

In this section, we present our experimental results. All our experiments were performed on a machine with an Intel Core i7 3.40Ghz CPU and 16GB memory. Our main algorithm is implemented with a Perl script and Update function is implemented in a C program called by the main script. Our algorithm can be applied to both SMT formulas and CNF formulas. We have tested a variety of SAT solvers and SMT solvers, and our current implementation specifically supports Cryptominisat2 [33], [34] for CNF formulas and Z3 [16] and MathSAT5 [14] for SMT formulas. We modified Cryptominisat2 to count the number of solutions over specified variables. For pure bit-vector SMT formulas, our tool also supports eagerly converting the formula to CNF first and then using CNF mode. (We implement the conversion using the first phase of the STP solver [21], [2] with optimizations disabled and a patch to output the SMT-to-CNF variable mapping.) Performing CNF translation eagerly gives up the benefit of some (e.g., word-level) optimizations performed by SMT solvers, but it can sometimes be profitable because it avoids repeating bit-blasting, and allows the tool to use a specialized multiple-solutions mode of Cryptominisat.

**A. Comparison with ApproxMC2 and ApproxMC-p**

We run our algorithm with a set of DQMR (Deterministic Quick Medical Reference) benchmarks [1] and ISCAS89 benchmarks [6] converted to CNF files by TG-Pro [12]. The DQMR benchmarks are medical diagnosis Bayesian networks and each problem is given by a two layer bipartite network graph for diseases and symptoms. There is an edge between the disease and the symptom if a disease may result a symptom. In these benchmarks, the numbers of diseases and symptoms are varied from 50 to 100 and each symptom...
is caused by four randomly chosen diseases. We want to compute the marginal probabilities for all the diseases given a set of observations when the size of observation set varies from 10% to 30% of all symptoms. The ISCAS89 benchmarks are digital sequential circuits that evaluate test pattern generation and TG-Pro can convert them into CNF files. The benchmarks have from 7 to 24,843 variables and from 8 to 56,487 clauses. and compare the results of the benchmarks with ApproxMC2 [11] and ApproxMC-p [37]. ApproxMC2 and ApproxMC-p are state-of-the-art approximate #SAT solvers which we describe in more detail in Section V. We used Cryptominisat2 as the back-end solver with all the tools for fair comparison. For the parameters for the tools, we set $CL = 0.86$ (confidence level), $\alpha = 0.5$ (confidence level adjustment factor) and $\text{thres} = 1.7$ (desired interval length). As described above, SearchMC-sound gives correct bounds with a probability of at least 0.86. Since the desired confidence level for ApproxMC2 is $1-\delta$, it can achieve a 86% confidence level by setting a parameter $\delta = 0.14$ which corresponds to our parameter $CL = 0.86$. The length of the interval for ApproxMC2 is computed as $\log_2(|f| \times (1+\epsilon)) - \log_2(|f| \times (1/(1+\epsilon))) = 1.7$ hence we can obtain the interval length 1.7 by setting a parameter $\epsilon = 0.8$, corresponding to our parameter $\text{thres} = 1.7$. Computing the interval for ApproxMC-p is a little different. The length of the interval for ApproxMC-p is $\log_2(|f| \times (1+\epsilon)) - \log_2(|f| \times (1-\epsilon)) = 1.7$ hence we can obtain the interval length 1.7 by setting a parameter $\epsilon = 0.53$. 

![Fig. 2. Comparison between SearchMC and ApproxMCs](image)
Note that SearchMC increases the $c$ value of an exhaust-up-to-$c$ query as it iterates while the corresponding ApproxMC2 and ApproxMC-p parameters are fixed as a function of $\epsilon$ \((72 \text{ and } 46, \text{ respectively})\) in this experiment. Also, we set an initial prior to be a uniform distribution over 0 to 64 bits for SearchMC. We tested 122 benchmarks (83 DQMRs and 39 ISCAS89s). SearchMC and ApproxMC-p were able to solve 106 benchmarks (83 DQMR and 23 ISCAS89) and ApproxMC2 was able to solve 50 benchmarks (35 DQMRs and 15 ISCAS89s) within 2 hours. The benchmarks that were solved completely by the other tools were also solved completely by SearchMC. Figures 2c and 2d are based on the benchmarks that were solved by all tools.

Figure 2a compares the quality of lower bounds and upper bounds computed by SearchMC-sound, ApproxMC-p and ApproxMC2. The influence bounds are the computed bounds minus the true influence. Filled markers and empty markers represent reported lower bounds and upper bounds, respectively. Since ApproxMC2 computes the bounds conservatively by using more queries, ApproxMC2’s intervals are more closely centered on the true influence, and it outperforms the requested 86% confidence level. By comparison, our tool and ApproxMC-p compute the bounds more aggressively, and in a few cases the true result is just outside the reported interval (visible as empty markers below the dotted line), though this still occurs somewhat less often than the 14% implied by the confidence level. SearchMC-sound tends to give tighter bounds than ApproxMCs since it stops when the interval length becomes less than $\text{threshold}$, while the interval lengths for ApproxMCs are fixed by a parameter $\epsilon$.

Figure 2b shows another perspective on the trade-off between performance and error. We selected a single benchmark and varied the parameter settings of each algorithm, measuring the absolute difference between the returned answer and the known exact result. We include results from running ApproxMC2 with parameter settings outside the range of its soundness proofs (shown as “disallowed” in the plot), since these settings are still empirically useful, and SearchMC makes no such distinction. From this perspective the tools are complementary depending on one’s desired performance-error trade-off. The results from all the tools improve similarly with configurations that use more queries, but SearchMC performs best at getting more precise results from a small number of queries. About 100 satisfiability queries is a minimum for ApproxMC2 to give any results. On the other hand our current SearchMC and SearchMC-sound implementation’s results improve as using more queries, the error varies over different runs compared to the stable error of ApproxMC2.

We also compare the running-time performance with ApproxMCs and show the running-time performance comparison on our 122 benchmarks in Figure 2c. Since ApproxMC-p refined the formulas of ApproxMC, it used a smaller number of queries than ApproxMC2. SearchMC can solve all the benchmarks faster than ApproxMCs with 86% confidence level. SearchMC-sound performs faster than ApproxMC-p even SearchMC-sound computes its confidence interval similarly to ApproxMC-p. The SearchMC’s and SearchMC-sound average running times are 41.79 and 84.89 seconds, compared to an average of 127.8 for ApproxMC-p. ApproxMC2 requires an average of 281.25 seconds just for the subset of benchmarks it can complete.

We also compare the number of SAT queries on the benchmarks for all the tools in Figure 2d. The average number of SAT queries for SearchMC, SearchMC-sound ApproxMC-p and ApproxMC2 is about 29.11, 79.75, 1257.19 and 4613.72 queries, respectively. Again this average for ApproxMC2 is based on 50 benchmarks and others are based on 106 benchmarks.

B. Floating Point / Differential Privacy Case Study

As an example of model counting with floating point constraints, we measure the security of a mechanism for differential privacy which can be undermined by unexpected floating-point behavior. The Laplace mechanism achieves differential privacy \([20]\) by adding exponentially-distributed noise to a statistic to obscure its exact value. For instance, suppose we wish to release a statistic counting the number of patients in a population with a rare disease, without releasing information that confirms any single patient’s status. In the worst case, an adversary might know the disease status of all patients other than the victim; for instance the attacker might know that the true count is either 10 or 11. If we add random noise from a Laplace distribution to the statistic before releasing it, we can leave the adversary relatively unsure about whether the true count was 10 or 11, while preserving the utility of an approximate result. A naive implementation of such a simple differentially private mechanism using standard floating-point techniques can be insecure because of a problem pointed out by Mironov \([28]\). For instance if we generate noise by dividing a random number in \([1, 2^{31}]\) by $2^{31}$ and taking the logarithm, the relative probability of particular floating point results will be quantized compared to the ideal probability, and many values will not be possible at all. If a particular floating point number could have been generated as $10 + \text{noise}$ but not as $11 + \text{noise}$ in our scenario, its release completely compromises the victim’s privacy.

To measure this danger using model counting, we translated the standard approach for generating Laplacian noise, including an implementation of the natural logarithm, into SMT-LIB 2 floating point and bit-vector constraints. (We followed the $\log$ function originally by SunSoft taken from the musl C library, which uses integer operations to reduce the argument to \([\sqrt{2}/2, \sqrt{2})\), followed by a polynomial approximation.) A typical implementation might use double-precision floats with an 11-bit exponent and 53-bit fraction, and 32 bits of randomness, which we abbreviate “53e11, 32"", but we tried a range of increasing sizes. We measured the total number of distinct values taken by $10 + \text{noise}$ as well as the size of the intersection of this set with the $11 + \text{noise}$ set.

The results and running time are shown in Table 4. We ran SearchMC with a confidence level of 80%, a confidence level adjustment of 0.5 and a threshold of 1.0; the SMT
TABLE I
RESULTS AND PERFORMANCE OF MODEL COUNTING (log₂ shown) OF NAIVE LAPLACIAN NOISE IN IEEE FLOATING POINT (SECTION IV-B)

| Problem size | Expected | All noise | Time | Intersection |
|--------------|----------|-----------|------|--------------|
| 9e7, 2⁰      | 5.977    | 5.997     | 107s | 3.170        |
| 10e7 2⁰      | 5.977    | 5.997     | 158s | 3.459        |
| 11e7, 2⁰     | 5.977    | 5.997     | 159s | 2.000        |
| 12e7, 2⁰     | 5.977    | 5.997     | 145s | 2.000        |
| 13e7, 2¹     | 6.989    | [6.903, 7.505] | 340s | 5.997        |
| 14e7, 2¹     | 6.989    | [6.709, 7.444] | 338s | 2.000        |
| 15e7, 2²     | 7.994    | [7.985, 8.813] | 330s | 1.000        |
| 16e7, 2²     | 8.997    | [8.678, 8.999] | 1088s | 3.322        |
| 16e8, 2¹⁰    | 9.999    | [9.695, 10.292] | 1087s | 4.754        |
| 18e8, 2¹⁰    | 9.999    | [9.362, 10.288] | 739s  | 1.000        |
| 19e8, 2¹¹    | 10.999   | [10.700, 11.446] | 1940s | 3.585        |

The running time increases steeply as the problem size increases, which matches the conventional wisdom that reasoning about floating-point constraints is challenging. But because floating-point SMT solving is a young area, there is significant scope for future solvers to improve the technique’s performance.

V. RELATED WORK

In this section, we summarize related work on model counting. We categorize model counting techniques into three areas: exact model counting, randomized approximate model counting and non-randomized approximate model counting. We also introduce some applications of model counting techniques for security and privacy purposes.

Exact model counters give the exact number of solutions for a formula but don’t perform well as the size of a problem increases. Approximate model counters have been proposed to resolve the scalability challenges. When a formula has a large number of solutions, it is often sufficient to provide rough estimates instead of the exact model counts, especially when this can be done much faster. Randomized approximate model counting techniques are likely to use random sampling and a SAT solver to produce a probabilistic result, which can often be argued to provide a lower bound and/or upper bound with high probability. In contrast, non-randomized approximate model counting techniques generally use approximations that are more efficient but do not provide probabilistic bounds.

A. Exact model counting

Some of the earliest Boolean model counters used the DPLL algorithm [15] for counting the exact number of solutions. Birnbaum et al. [5] formalized this idea and introduced an algorithm for counting models of propositional formulas. Based on this idea, Relsat [4], Cachet [32] sharpSAT [38] and DSHARP [29] showed improvements by using several optimizations. Relsat uses component analysis in which the model count of a formula is the product of the model count of each sub-formula (component). Cachet shows optimizations by combining component caching and clause learning. sharpSAT introduces an improved caching technique to reduce the space requirement compared to Cachet. DSHARP [29] is a CNF → d-DNNF compiler for efficient reasoning and uses sharpSAT as a back-end. The major contribution of countAntom [7] is techniques for parallelization, but it provides state-of-the-art performance even in single-threaded mode.

Phan et al. [31] encode a full binary search for feasible outputs in a bounded model checker. This approach is precise, but requires more than one call to the underlying solver for each feasible output. Specifically, it recursively calls the solver by adding a bit constraint for finding a single satisfying assignment as DPLL-based search. This is most similar to the exhaust-up-to-c approach that requires n + 1 queries to find n feasible outputs. This search tree approach is useful when the program verification system does not expose the underlying logical representation or when the used solver cannot generate models. Klebanov et al. [24] perform exact model counting for quantitative information-flow measurement, with an approach that converts C code to a CNF formula with bounded model checking and then uses exact #SAT solving. They explore both exhaustive enumeration and the existing DSHARP and sharpSAT tools, but only counting distinct values of the output variables. Val et al. [36] integrate a symbolic execution tool more closely with a SAT solver by using techniques from SAT solving to prune the symbolic execution search space, and then perform exact model counting restricted to an output variable.

| Problem size | Expected | All noise | Time | Intersection |
|--------------|----------|-----------|------|--------------|
| 9e7, 2⁰      | 5.977    | 5.997     | 107s | 3.170        |
| 10e7 2⁰      | 5.977    | 5.997     | 158s | 3.459        |
| 11e7, 2⁰     | 5.977    | 5.997     | 159s | 2.000        |
| 12e7, 2⁰     | 5.977    | 5.997     | 145s | 2.000        |
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| 16e8, 2¹⁰    | 9.999    | [9.695, 10.292] | 1087s | 4.754        |
| 18e8, 2¹⁰    | 9.999    | [9.362, 10.288] | 739s  | 1.000        |
| 19e8, 2¹¹    | 10.999   | [10.700, 11.446] | 1940s | 3.585        |
However, precisely counting solutions is a \#P-complete problem and these exact model counters typically only work well with small-sized problems or ones with only simple constraints. For many practical problems, it is infeasible to count the exact number of solutions in a reasonable amount of time.

B. Randomized approximate model counting

Randomized approximate model counting techniques perform well on many kinds of a formula for which finding single solutions is efficient. Also, there can sometimes be a smooth trade-off chosen between computational effort and the precision of results. However, this solving is still relatively expensive, so research to get the best precision for a given cost is still important. Wei and Selman [38] introduced ApproxCount which uses near-uniform sampling to estimate the true model count but it can significantly over-estimate or underestimate if the sampling is biased. SampleCount [22] improves this sampling idea and gives a lower bound with high probability by using a heuristic sampler. MiniCount [25] is based on a framework to compute an upper bound under statistical assumptions which is that counting the number of branching decisions (except unit propagations and failed branches) can be used to estimate the total number of solutions by setting a variable to true or false randomly. Specifically, they showed that the expected value of \( d \) is not lower than \( \log_2 \) of the true model count. By estimating the expected value, they can obtain the upper bound of the true model count. Also, they observed that \( d \) often has a distribution that is close to a normal distribution. Thus, the expected value can be easily computed under this assumption rather than computing low and high values of \( d \). This only guarantees the upper bound and they used a different method to compute a lower bound.

MBound [23] is an approximate model counting tool that gives probabilistic bounds on the model counts by adding randomly-chosen parity constraints as XOR streamlining. Chakraborty et al. [10] introduced ApproxMC, an approximate model counter for CNF formulas, which automated the choice of XOR streamlining parameters. The ApproxMC algorithm, in our terminology, starts by fixing \( c \) and a total number of iterations based on the desired precision and confidence of the results. In each iteration ApproxMC searches for an appropriate \( k \) value, adds \( k \) XOR random constraints, and then performs an exhaust-up-to-\( c \) query on the streamlined formula and multiplies the result by \( 2^k \). It stores all the individual estimates as a multiset and computes its final estimate of the model count as the median of the values. The original ApproxMC sequentially increases \( k \) in each iteration until it finds an appropriate \( k \) value. An improved algorithm ApproxMC2 [11] uses galloping binary search and saves a starting \( k \) value between iterations to make the selection of \( k \) more efficient. Other recent systems that build on ApproxMC include SMTApproxMC [9] and ApproxMC-p [37]. SMTApproxMC proposes word-level constraints based on modular arithmetic instead of bit-level XOR constraints, however these will not likely provide comparable performance until SMT solvers implement modular-arithmetic Gaussian elimination. ApproxMC-p implements projection (counting over only a subset of variables), and also gives more efficient formulas for parameter selection.

ApproxMC2, developed concurrently, is the system most similar to SearchMC: its binary search for \( k \) plays a similar role to our converging \( \mu \) value. However SearchMC also updates the \( c \) parameter over the course of the search, leading to fewer total queries. ApproxMC, ApproxMC2, and related systems choose the parameters of the search at the outset, and make each iteration either fully independent (ApproxMC) or dependent in a very simple way (ApproxMC2) on previous ones. These choices make it easier to prove the tool's probabilistic results are sound, but they require a conservative choice of parameters. By comparison SearchMC's approach of maintaining a probabilistic estimate explicitly at runtime means that its iterations are not at all independent: instead our approach is to extract the maximum guidance for future iterations from previous ones, to allow the search to converge more aggressively.

The runtime performance of SearchMC, like that of ApproxMC(2), is highly dependent on the performance of SAT solvers on CNF-XOR formulas. Some roots of the difficulty of this problem have been investigated by Dudek et al. [18], [19].

C. Non-randomized approximate model counting

Non-randomized approximate model counting using techniques similar to static program analysis is generally faster than randomized approximate model counting techniques, and such systems can give good approximations for some problem classes. However, they cannot provide a precision guarantee for arbitrary problems, and it is not possible to give more effort to have more refined results.

Castro et al. [8] compute an upper bound on the number of bits about an input that are revealed by an error report. They measure the entropy loss of an error report by computing the number of bits revealed by subsets of path conditions first and then combining these partial results to get the final result. Meng and Smith [27] use two-bit-pattern SMT entailment queries to calculate a propositional overapproximation and count its instances with a model counter from the computer algebra system Mathematica. Luu et al. [26] propose a model counting technique over an expressive string constraint language. Their tool computes the bounds on the cardinality of the valid string set and uses generating functions for reasoning about the cardinality of string sets.

D. Applications: Security and Privacy

Various applications of model counting have been proposed for security and privacy purposes. For instance Castro et al. [8] use model counting and symbolic execution approaches to measure leaking private information from bug reports. They compute an upper bound on the amount of private information leaked by a bug report and allow users to decide on whether to submit the report or not. Newsome et al. [30] show how an untrusted input affect a program
and introduce a family of technique for measuring influence which can be applicable to x86 binaries. They applied the XOR streamlining and the exhaust-up-to-c techniques separately, but did not combine the two techniques or describe an algorithm for choosing XOR streamlining parameters.

VI. FUTURE WORK

Closing the gap between the performance of SearchMC and SearchMC-sound is one natural direction for future research. On one hand, we would like to explore techniques for asserting sound probabilistic bounds which can take advantage of the results of all of SearchMC’s queries. At the same time, we would like to find a model of the number of solutions remaining after XOR streamlining that is more accurate than our current binomial model, which should improve the performance of SearchMC. Another future direction made possible by the particle filter implementation is to explore different prior distributions, including unbounded ones. For instance, using a negative exponential distribution over influence as a prior would avoid the any need to estimate a maximum influence in advance, while still starting the search process with low-k queries which are typically faster to solve.

VII. CONCLUSION

In sum, we have presented a new model counting approach SearchMC using XOR streamlining for SMT formulas with bit-vectors and other theories. We demonstrate our algorithm that adaptively maintains a probabilistic model count estimate based on the results of queries. Our tool computes a lower bound and an upper bound with a requested confidence level, and yields results more quickly than previous systems.

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