Regular irreducible characters  
of a hyperspecial compact group

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Abstract

A parametrization of irreducible unitary representations associated  
with the regular adjoint orbits of a hyperspecial compact subgroup of  
a reductive group over a non-dyadic non-archimedean local field is pre-  
sented. The parametrization is given by means of (a subset of) the char-  
acter group of certain finite abelian groups arising from the reductive  
group. Our method is based upon Clifford’s theory and Weil representa-  
tions over finite fields. It works under an assumption of the triviality of  
certain Schur multipliers defined for an algebraic group over a finite field.  
The assumption of the triviality has good evidences in the case of general  
linear groups and highly probable in general.

1 Introduction

Let \( F \) be a non-dyadic non-archimedean local field. The integer ring of \( F \) is  
denoted by \( O \) with the maximal ideal \( \mathfrak{p} \) generated by \( \varpi \). The residue class field  
\( F = O/\mathfrak{p} \) is a finite field of \( q \) elements. Fix a continuous unitary character \( \tau \) of  
the additive group \( F \) such that \( \{ x \in F \mid \tau(xO) = 1 \} = O \), and define an additive  
character \( \hat{\tau} \) of \( F \) by \( \hat{\tau}(x) = \tau(\varpi^{-1}x) \). For an integer \( l > 0 \) put \( O_l = O/\mathfrak{p}^l \) so  
that \( F = O_1 \).

If a connected reductive quasi-split linear group \( G \) over \( F \) is split over an  
unramified extension of \( F \), then there exists a smooth affine group scheme \( G \)  
over \( O \) such that \( G \otimes O F = G \) and \( G \otimes O F \) is a connected reductive group over \( F \).  
In this case the locally compact group \( G(F) = G(O) \) of the \( F \)-rational points  
has an open compact subgroup \( G(O) \) which is called a hyperspecial compact  
subgroup of \( G(F) \) [12 3.8.1]. An important problem in the harmonic analysis  
on \( G(F) \) is to determine the irreducible unitary representations of the compact  
group \( G(O) \). Such a representation \( \pi \) of \( G(O) \) factors through the canonical  
group homomorphism \( G(O) \to G(O_r) \) for some \( r > 0 \) since the canonical group  
homomorphism is surjective due to the smoothness of the group scheme \( G \) over  
\( O \) and Hensel’s lemma, and \( \pi \) is trivial on the kernel of the canonical group homomorphism  
for some \( r > 0 \). Hence the problem is reduced to determine the set  
\( \text{Irr}(G(O_r)) \) of the equivalence classes of the irreducible complex representations  
of the finite group \( G(O_r) \).

This problem in the case \( r = 1 \), that is the representation theory of the  
finite reductive group \( G(F) \), has been studied extensively, starting from Green  
[3] concerned with \( GL_n(F) \) to the decisive paper of Deligne-Lusztig [3].

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This paper treats the case $r > 1$ where the study of the representation theory of the finite group $G(O_r)$ is less complete. Shalika [14] treats the case $SL_2(O_r)$, Silberger [16] the case $PGL_2(O_r)$. Shintani [15] and Géradin [6] treat cuspidal representations of $GL_n(O_r)$ in order to construct supercuspidal representations of $GL_n(F)$. The last two papers use Clifford theory and Weil representations over finite fields. In the series of papers [9, 10, 11, 12], Hill treats the case $GL_n(O_r)$ systematically by means of Clifford theory, but different methods are used for representations associated with different type of adjoint orbits.

In this paper, we will establish a parametrization of the irreducible representations of $G(O_r)$ ($r > 1$) associated with the regular (more precisely smoothly regular) adjoint orbits. Taking a representative $\beta$ of the adjoint orbit, the parametrization is given by means of a subset of the character group of $G_{\beta}(O_r)$ where $G_{\beta}$ is the centralizer of $\beta$ in $G$ which is smooth commutative group scheme over $O$. Our theory is based on Clifford theory and Weil representations over finite fields, and it works well under an assumption of the triviality of certain Schur multiplier of a finite commutative group $G_{\beta}(F)$. We can verify the assumption in the case of $GL_n$ with $n \leq 4$, and the discussions in this paper show that the assumption is highly probable for the reductive groups in general.

The main result of this paper is Theorem 2.4.1. The situation is quite simple when $r$ is even, and almost all of this paper is devoted to treat the case of $r = 2l - 1$ being odd. In this case we need Weil representation over finite field to construct irreducible representations of $G_{\beta}(O_r) \cdot K_{l-1}(O_r)$, where $K_{l-1}(O_r)$ is the kernel of the canonical group homomorphism $G(O_r) \rightarrow G(O_{l-1})$, and at this point appears the Schur multiplier as an obstruction to the construction. Here we shall note that over a finite field, Weil representation is a representation of a symplectic group, not of the 2-fold covering group of it, and that the Schur multiplier is coming not from Weil representation but from certain twist which occurs en route of connecting $K_{l-1}(O_r)$ with the Heisenberg group over finite field (see section 3 for the details).

Several fundamental properties of the Schur multiplier will be discussed in section 4. These properties, combined with the results of [18] in the case of $G = GL_n$, shows that it is highly probable that the Schur multiplier is trivial for all reductive group schemes over $O$ provided that $\beta$ is regular and that the residue characteristic is big enough.

We will give some examples of classical groups where the characteristic polynomial of $\beta$ is irreducible modulo $p$. In this case the parametrization is given by a subset of the character group of unit groups of unramified extensions of the base field $F$. See propositions 5.1.4 for a general linear group, 5.2.4 for a group of symplectic similitudes, 5.3.4 and 5.3.5 for a general orthogonal group with respect to a quadratic form of even and odd variables respectively, and 5.4.4 for an unitary group associated with Hermitian form of odd variables.

2 Main results

2.1 Let $G \subset GL_n$ be a closed smooth $O$-group subscheme, and $\mathfrak{g}$ the Lie algebra of $G$ which is a closed affine $O$-subscheme of $\mathfrak{gl}_n$ the Lie algebra of $GL_n$. We may assume that the fibers $G \otimes_O K$ ($K = F$ or $K = \overline{F}$) are non-commutative algebraic $K$-group (that is smooth $K$-group scheme).

For any $O$-algebra $K$, the set of the $K$-valued points $\mathfrak{gl}_n(K)$ is identified
with the $K$-Lie algebra of square matrices $M_n(K)$ of size $n$ with Lie bracket $[X,Y] = XY - YX$, and the group of $K$-valued points $GL_n(K)$ is identified with the matrix group

$$GL_n(K) = \{ g \in M_n(K) \mid \det g \in K^\times \}$$

where $K^\times$ is the multiplicative group of $K$. Hence $\mathfrak{g}(K)$ is identified with a matrix Lie subalgebra of $\mathfrak{gl}_n(K)$ and $G(K)$ is identified with a matrix subgroup of $GL_n(K)$. Let

$$B : \mathfrak{gl}_n \times \mathfrak{gl}_n \to O = \text{Spec}(O[t])$$

be the trace form on $\mathfrak{gl}_n$, that is $B(X,Y) = \text{tr}(XY)$ for all $X,Y \in \mathfrak{gl}_n(K)$ with any $O$-algebra $K$. The smoothness of $G$ implies that we have a canonical isomorphism

$$\mathfrak{g}(O)/\mathfrak{w}^r \mathfrak{g}(O) \xrightarrow{\sim} \mathfrak{g}(O_r) = \mathfrak{g}(O) \otimes_O O_r$$

(\cite{H} Chap.II, §4, Prop.4.8) and that the canonical group homomorphism $G(O) \to G(O_r)$ is surjective due to Hensel’s lemma. Then for any $0 < l < r$ the canonical group homomorphism $G(O_r) \to G(O_l)$ is surjective whose kernel is denoted by $K_l(O_r)$.

For any $g \in G(O)$ (resp. $X \in \mathfrak{g}(O)$), the image under the canonical surjection onto $G(O_l)$ (resp. onto $\mathfrak{g}(O_l)$) with $l > 0$ is denoted by

$$g_l = g \pmod{p^l} \in G(O_l) \quad (\text{resp. } X_l = X \pmod{p^l} \in \mathfrak{g}(O_l)).$$

Since the reduction modulo $p$ plays a fundamental role in our theory, let us use the notation $\overline{g} = g \pmod{p} \in G(F)$ (resp. $\overline{X} = X \pmod{p} \in \mathfrak{g}(F)$) if $l = 1$.

We will pose the following three conditions:

I) $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ is non-degenerate,

II) for any integers $r = l + l'$ with $0 < l' \leq l < r$, we have a group isomorphism

$$\mathfrak{g}(O_r) \xrightarrow{\sim} K_l(O_r)$$

defined by $X \pmod{p^{l'}} \mapsto 1 + \mathfrak{w}^l X \pmod{p^r}$,

III) if $r = 2l - 1 \geq 3$ is odd, then we have a mapping

$$\mathfrak{g}(O) \to K_{l-1}(O_r)$$

defined by $X \mapsto (1 + \mathfrak{w}^{l-1}X + 2^{-1}\mathfrak{w}^{2l-2}X^{2l-2}) \pmod{p^r}$.

The condition I) implies that $B : \mathfrak{g}(O_l) \times \mathfrak{g}(O_l) \to O_l$ is non-degenerate for all $l > 0$, and so $B : \mathfrak{g}(O) \times \mathfrak{g}(O) \to O$ is also non-degenerate. The mappings of the conditions II) and III) from Lie algebras to groups can be regarded as truncations of the exponential mapping.

See section 5 for the examples of classical groups which satisfy these three fundamental conditions of our theory.

The character group of an finite abelian group $\mathcal{G}$ is denoted by $\mathcal{G}^\wedge$. 

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2.2 From now on we will fix an integer $r \geq 2$ and put $r = l + l'$ with the smallest integer $l$ such that $0 < l' \leq l$. In other word

$$l' = \begin{cases} l & : r = 2l, \\ l - 1 & : r = 2l - 1. \end{cases}$$

Take a $\beta \in \mathfrak{g}(O)$ and define a character $\psi_\beta$ of the finite abelian group $K_1(O_r)$ by

$$\psi_\beta((1 + x^tX) \ (\text{mod } p^r)) = \tau(\omega^{-1}B(X, \beta)) \quad (X \in \mathfrak{g}(O)).$$

Then $\beta(\text{mod } p^r) \mapsto \psi_\beta$ gives an isomorphism of the additive group $\mathfrak{g}(O_r)$ onto the character group $K_1(O_r)^\gamma$. For any $g_r = g(\text{mod } p^r) \in G_r$, we have

$$\psi_\beta(g_r^{-1}h g_r) = \psi_{\text{Ad}(g)\beta}(h) \quad (h \in K_1(O_r)).$$

So the stabilizer of $\psi_\beta$ in $G(O_R)$ is

$$G(O_r, \beta) = \{ g_r \in G(O_r) \mid \text{Ad}(g)\beta \equiv \beta \quad (\text{mod } p^r) \}$$

which is a subgroup of $G(O_r)$ containing $K_r(O_r)$.

Now let us denote by $\text{Irr}(G(O_r) \mid \psi_\beta)$ (resp. $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$) the set of the isomorphism classes of the irreducible complex representation $\pi$ of $G(O_r)$ (resp. $\sigma$ of $G(O_r, \beta)$) such that

$$\langle \psi_\beta, \pi \rangle_{K_1(O_r)} = \dim \mathbb{C} \text{Hom}_{K_1(O_r)}(\psi_\beta, \pi) > 0$$

(resp. $\langle \psi_\beta, \sigma \rangle_{K_1(O_r)} > 0$). Then Clifford’s theory says that

1) $\text{Irr}(G(O_r)) = \bigsqcup_{\beta(\text{mod } p^r)} \text{Irr}(G(O_r) \mid \psi_\beta)$ where $\bigsqcup$ is the disjoint union over the representatives $\beta(\text{mod } p^r)$ of the $\text{Ad}(G(O_r))-\text{orbits in } \mathfrak{g}(O_r)$,

2) a bijection of $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$ onto $\text{Irr}(G(O_r) \mid \psi_\beta)$ is given by

$$\sigma \mapsto \text{Ind}_{G(O_r, \beta)}^{G(O_r)} \sigma.$$

So our problem is to give a good parametrization of the set $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$ for $\beta \in \mathfrak{g}(O)$ which is regular enough.

For any $\beta \in \mathfrak{g}(O)$, let us denote by $G_\beta = Z_G(\beta)$ the centralizer of $\beta$ in $G$ which is a closed $O$-group subscheme of $G$. The Lie algebra $\mathfrak{g}_\beta = Z_\mathfrak{g}(\beta) = G_\beta$ is the centralizer of $\beta$ in $\mathfrak{g}$ which is a closed $O$-subscheme of $\mathfrak{g}$.

2.3 In this subsection we will define a Schur multiplier which is an obstruction to our theory.

Take a $\beta \in \mathfrak{g}(O)$ such that $\mathfrak{g}_\beta(\mathbb{F}) \subseteq \mathfrak{g}(\mathbb{F})$. Then non-zero $\mathbb{F}$-vector space $\mathcal{V}_\beta = \mathfrak{g}(\mathbb{F})/\mathfrak{g}_\beta(\mathbb{F})$ has a symplectic form

$$\langle \hat{X}, \hat{Y} \rangle_\beta = B([X, Y], \beta)$$
where $X = X \pmod{\mathfrak{g}_\beta(\mathbb{F})} \in \mathbb{V}_\beta$ with $X \in \mathfrak{g}_\beta(\mathbb{F})$. Then $g \in G_\beta(\mathbb{F})$ gives an element $\sigma_g$ of the symplectic group $Sp(\mathbb{V}_\beta)$ defined by

$$X \pmod{\mathfrak{g}_\beta(\mathbb{F})} \mapsto Ad(g)^{-1}X \pmod{\mathfrak{g}_\beta(\mathbb{F})}.$$  

Note that the group $Sp(\mathbb{V}_\beta)$ acts on $\mathbb{V}_\beta$ from right. Let $v \mapsto [v]$ be a $\mathbb{F}$-linear section on $\mathbb{V}_\beta$ of the exact sequence

$$0 \rightarrow \mathfrak{g}_\beta(\mathbb{F}) \rightarrow \mathfrak{g}(\mathbb{F}) \rightarrow \mathbb{V}_\beta \rightarrow 0. \quad (1)$$

For any $v \in \mathbb{V}_\beta$ and $g \in G_\beta(\mathbb{F})$, put

$$\gamma(v, g) = \gamma_g(v, g) = Ad(g)^{-1}[v] - [v\sigma_g] \in \mathfrak{g}_\beta(\mathbb{F}).$$

Take a $\rho \in \mathfrak{g}_\beta(\mathbb{F})$. Then there exists uniquely a $v_g \in \mathbb{V}_\beta$ such that

$$\rho(\gamma(v, g)) = \tilde{\tau}(v, v_g).$$

for all $v \in \mathbb{V}_\beta$. Note that $v_g \in \mathbb{V}_\beta$ depends on $\rho$ as well as the section $v \mapsto [v]$. Let

$$G_\beta(\mathbb{F})^{(c)} = \{g \in G(\mathbb{F}) \mid Ad(g)Y = Y \text{ for } \forall Y \in \mathfrak{g}_\beta(\mathbb{F})\}$$

be the centralizer of $\mathfrak{g}_\beta(\mathbb{F})$ in $G(\mathbb{F})$, which is a subgroup of $G_\beta(\mathbb{F})$. Then for any $g, h \in G_\beta(\mathbb{F})^{(c)}$, we have

$$v_{gh} = v_h\sigma_{gh}^{-1} + v_g \quad (2)$$

because $\gamma(v, gh) = \gamma(v, g) + \gamma(v\sigma_g^{-1}, h)$ for all $v \in \mathbb{V}_\beta$. Put

$$c_{\beta, \rho}(g, h) = \tilde{\tau}(2^{-1}v_g, v_{gh}).$$

for $g, h \in G_\beta(\mathbb{F})^{(c)}$. Then the relation $(2)$ shows that $c_{\beta, \rho} \in Z^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)$ is a 2-cocycle with trivial action of $G_\beta(\mathbb{F})^{(c)}$ on $\mathbb{C}^\times$. Moreover we have

**Proposition 2.3.1** The Schur multiplier $[c_{\beta, \rho}] \in H^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)$ is independent of the choice of the $\mathbb{F}$-linear section $v \mapsto [v]$.

**Proof** Take another $\mathbb{F}$-linear section $v \mapsto [v']$ with respect to which we will define $\gamma'(v, g) \in \mathfrak{g}_\beta$ and $v'_g \in \mathbb{V}_\beta$ as above. Then there exists a $\delta \in \mathbb{V}_\beta$ such that $\rho([v] - [v']) = \tilde{\tau}(\langle v, \delta \rangle)$. Then we have $v'_g = v_g + \delta - \delta\sigma_g$ for all $g \in G_\beta(\mathbb{F})^{(c)}$. So if we put $\alpha(g) = \tilde{\tau}(2^{-1}v'_g - v_{g-1}, \delta)g)$ for $g \in G_\beta(\mathbb{F})^{(c)}$, then we have

$$\tilde{\tau}(2^{-1}v'_g, v'_{gh}) = \tilde{\tau}(2^{-1}v_g, v_{gh}) \cdot \alpha(h)\alpha(gh)^{-1}\alpha(g)$$

for all $g, h \in G_\beta(\mathbb{F})^{(c)}$.  

**2.4** Now our main result is

**Theorem 2.4.1** Suppose that a $\beta \in \mathfrak{g}(O)$ satisfies the conditions

1) $G_\beta$ is commutative smooth $O$-group scheme, and

2) the Schur multiplier $[c_{\beta, \rho}] \in H^2(G_\beta(\mathbb{F}), \mathbb{C}^\times)$ is trivial for all characters $\rho \in \mathfrak{g}_\beta(\mathbb{F})$. 


Then we have a bijection $\theta \mapsto \sigma_{\beta,\theta}$ of the set

$$\{\theta \in G_{\beta}(O_r)^\sim \text{ s.t. } \theta = \psi_{\beta} \text{ on } G_{\beta}(O_r) \cap K_l(O_r)\}$$

onto $\text{Irr}(G(O_r,\beta) | \psi_{\beta})$.

The proof is given in subsection 2.5 for even $r$ and subsection 2.6 for odd $r$.

**Remark 2.4.2**  
1) A sufficient condition for the first condition of Theorem 2.4.1 is given by Theorem 2.7.1.

2) The smoothness of $G_{\beta}$ over $O$ implies that the canonical group homomorphism $G_{\beta}(O_r) \to G_{\beta}(O_{l'})$ is surjective. So we have

$$G(O_r,\beta) = G_{\beta}(O_r) \cdot K_{l'}(O_r), \quad l' = \begin{cases} l & : r = 2l, \\ l - 1 & : r = 2l - 1. \end{cases}$$ (3)

3) As presented in the following two subsections, the second condition in the theorem is required only in the case of $r$ being odd.

4) Since $G \otimes_{O} F$ and $G_{\beta} \otimes_{O} F$ are $F$-algebraic group, and the former is not commutative while the latter is, so we have

$$\dim_F g(F) = \dim G \otimes_{O} F > \dim G_{\beta} \otimes_{O} F = \dim_F g_{\beta}(F).$$

That is $g_{\beta}(F) \leq g(F)$.

5) Since $G_{\beta}$ is assumed to be commutative, we have $G_{\beta}(F)^{(c)} = G_{\beta}(F)$ in the definition of the Schur multiplier $[c_{\beta,\rho}]$.

6) Assume that $G_{\beta}(F)^{(c)}$ is commutative. Then the cohomology class $[c_{\beta,\rho}] \in H^2(G_{\beta}(F)^{(c)}, \mathbb{C}^\times)$ is trivial if and only if $c_{\beta,\rho}$ is symmetric, that is $c_{\eta,\rho}(g,h) = c_{\beta,\rho}(h,g)$ for all $g,h \in G_{\beta}(F)^{(c)}$. In fact, only if part is trivial. Let

$$1 \to T \to G \to G_{\beta}(F)^{(c)} \to 1$$ (4)

be the group extension associated with the 2-cocycle $c_{\beta,\rho} \in Z^2(B_{\beta}(F)^{(c)}, T)$ where $T$ is the subgroup of $z \in \mathbb{C}^\times$ such that $|z| = 1$. Then the groups are compact commutative group, and we have a group extension of the Pontryagin dual groups

$$1 \to G_{\beta}(F)^{(c)} \to T \to 1.$$ (5)

Since $T^\sim \simeq \mathbb{Z}$ is free the group extension (5) is trivial and so is the group extension (4).

2.5 Assume that $r = 2l$ is even so that $l' = l$. In this case the proof of Theorem 2.4.1 is quite easy. Let us suppose more generally that there exists a commutative subgroup $C$ of $G(O_r,\beta)$ such that

$$G(O_r,\beta) = C \cdot K_l(O_r).$$
Let us denote by $C_\beta^-$ the subset of the character group $C^-$ consisting of the $\theta \in C^-$ such that $\theta = \psi_{\beta}$ on $C \cap K_i(O_r)$. Then any $\theta \in C_\beta^-$ gives an one-dimensional representation $\sigma_{\beta, \theta}$ of $G(O_r, \beta)$ defined by

$$\sigma_{\beta, \theta}(gh) = \theta(g) \cdot \psi_{\beta}(h) \quad (g \in C, h \in K_i(O_r)).$$

Then we have a proposition of which our Theorem 2.4.1 is a special case;

**Proposition 2.5.1** \( \theta \mapsto \sigma_{\beta, \theta} \) gives a bijection of $C_\beta^-$ onto $\text{Irr}(G(O_r, \beta) | \psi_{\beta})$.

**[Proof]** Take any $\sigma \in \text{Irr}(G(O_r, \beta) | \psi_{\beta})$ with representation space $V_\sigma$. Then

$$V_\sigma(\psi_{\beta}) = \{ v \in V_\sigma \mid \sigma(g)v = \psi_{\beta}(g)v \text{ for } \forall g \in K_i(O_r) \}$$

is a non-trivial $G(O_r, \beta)$-subspace of $V_\sigma$ so that $V_\sigma = V_\sigma(\psi_{\beta})$. Then, for any one-dimensional representation $\chi$ of $G(O_r, \beta)$ such that $\chi = \psi_{\beta}$ on $K_i(O_r)$, we have $K_i(O_r) \subset \text{Ker}(\chi^{-1} \otimes \sigma)$. On the other hand $G(O_r, \beta)/K_i(O_r)$ is commutative, we have $\dim(\chi^{-1} \otimes \sigma) = 1$ and then $\dim \sigma = 1$. Put $\theta = \sigma|_{C} \in C_\beta^-$ and we have $\sigma = \sigma_{\beta, \theta}$. ■

2.6 Assume that $r = 2l - 1 \geq 3$ is odd so that $l' = l - 1 \geq 1$. We have a chain of canonical surjections

$$\triangledown : K_{l-1}(O_r) \to K_{l-1}(O_{r-1}) \hookrightarrow g(O_{l-1}) \twoheadrightarrow g(F) \quad (6)$$

defined by

$$1 + \omega^{l-1}X \pmod{p^r} \mapsto 1 + \omega^{l-1}X \pmod{p^{r-1}} \mapsto X \pmod{p^{l-1}} \mapsto X \equiv X \pmod{p}.$$

Let us denote by $Z(O_r, \beta)$ the inverse image under the surjection $\triangledown$ of $g_{\beta}(F)$. Then $Z(O_r, \beta)$ is a normal subgroup of $K_{l-1}(O_r)$ containing $K_i(O_r)$ as the kernel of $\triangledown$.

Let us denote by $Y_{\beta}$ the set of the group homomorphisms $\psi$ of $Z(O_r, \beta)$ to $\mathbb{C}^\times$ such that $\psi = \psi_{\beta}$ on $K_i(O_r)$. Then a bijection of $g_{\beta}(F)$ onto $Y_{\beta}$ is given by

$$\rho \mapsto \widetilde{\psi}_{\beta} \cdot (\rho \circ \triangledown),$$

where a group homomorphism $\widetilde{\psi}_{\beta} : Z(O_r, \beta) \to \mathbb{C}^\times$ is defined by

$$1 + \omega^{l-1}X \pmod{p^r} \mapsto \tau (\omega^{-1}B(X, \beta) - (2\omega)^{-1}B(X^2, \beta))$$

with $X = X \pmod{p} \in g_{\beta}(F)$.

Take a $\psi \in Y_{\beta}$. For two elements $x = 1 + \omega^{l-1}X \pmod{p^r}, \ y = 1 + \omega^{l-1}Y \pmod{p^r}$ of $K_{l-1}(O_r)$, we have $x^{-1} = 1 - \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r}$ so that we have

$$xyx^{-1}y^{-1} = 1 + \omega^{r-1}[X, Y] \pmod{p^r} \in K_{r-1}(O_r) \subset K_i(O_r)$$

and so $\psi_{\beta}(xyx^{-1}y^{-1}) = \tau (\omega^{-1}B(X, \text{ad}(Y)\beta))$. Hence we have

$$\psi(xyx^{-1}y^{-1}) = \psi_{\beta}(xyx^{-1}y^{-1}) = 1.$$
for all $x \in K_{l-1}(O_r)$ and $y \in Z(O_r, \beta)$ so that we can define

$$D_\psi : K_{l-1}(O_r)/Z(O_r, \beta) \times K_{l-1}(O_r)/Z(O_r, \beta) \to \mathbb{C}^\times$$

by

$$D_\psi(g, h) = \psi(ghg^{-1}h^{-1}) = \psi_\beta(ghg^{-1}h^{-1}) = \tau \left( \omega^{-1}B([X, Y], \beta) \right)$$

for $g = (1 + \omega^{l-1}X) \pmod{p^r}$, $h = (1 + \omega^{l-1}Y) \pmod{p^r} \in K_{l-1}(O_r)$, which is non-degenerate. Then Proposition 3.1.1 of [13] gives

**Proposition 2.6.1** For any $\psi = \psi_{\beta, \rho} \in Y_\beta$ with $\rho \in g_\beta(F)^\circ$, there exists unique irreducible representation $\pi_{\beta, \rho}$ of $K_{l-1}(O_r)$ such that $\langle \psi, \pi_{\beta, \rho} \rangle_{Z(O_r, \beta)} > 0$. Furthermore

$$\text{Ind}_{Z(O_r, \beta)}^{K_{l-1}(O_r)} \psi = \bigoplus_{\pi_{\beta, \rho}} \dim \pi_{\beta, \rho}$$

and $\pi_{\beta, \rho}(x)$ is the homothety $\psi(x)$ for all $x \in Z(O_r, \beta)$.

Fix a $\psi = \psi_{\beta, \rho} \in Y_\beta$ with $\rho \in g_\beta(F)^\circ$. Let $G_l(O_r, \beta)$ be a subgroup of $G(O_r, \beta)$ defined by

$$G_l(O_r, \beta)^{(c)} = \left\{ g \pmod{p^r} \in G(O_r) \left| \begin{array}{l}
\text{Ad}(g)\beta \equiv \beta \pmod{p^r}, \\
\text{Ad}(g)X = X \text{ for } \forall X \in g_\beta(F) \end{array} \right. \right\}.$$ 

Then, for any $g_r = g \pmod{p^r} \in G_l(O_r, \beta)^{(c)}$ and $x = (1 + \omega^{l-1}X) \pmod{p^r} \in Z(O_r, \beta)$, we have

$$g_r^{-1}xg_rx^{-1} = (1 + \omega^{l-1}g^{-1}Xg) \left( 1 - \omega^{l-1}X + 2\omega^{2l-2}X^2 \right) \pmod{p^r}$$

and

$$\psi(g_r^{-1}xg_rx^{-1}) = \psi_\beta(g_r^{-1}xg_rx^{-1}) = \tau \left( \omega^{-1}B(X, \text{Ad}(g)\beta - \beta) \right) = 1,$$

that is $\psi(g_r^{-1}xg_r) = \psi(x)$ for all $x \in Z(O_r, \beta)$. This means that, for any $g \in G_l(O_r, \beta)^{(c)}$, the $g$-conjugate of $\pi_{\beta, \rho}$ is isomorphic to $\pi_{\beta, \rho}$, that is, there exists a $U(g) \in GL(V_{\beta, \rho})$ ($V_{\beta, \rho}$ is the representation space of $\pi_{\beta, \rho}$) such that

$$\pi_{\beta, \rho}(g^{-1}xg) = U(g)^{-1} \circ \pi_{\beta, \rho}(x) \circ U(g)$$

for all $x \in K_{l-1}(O_r)$, and moreover, for any $g, h \in G_l(O_r, \beta)^{(c)}$, there exists a $c_U(g, h) \in \mathbb{C}^\times$ such that

$$U(g) \circ U(h) = c_U(g, h) \cdot U(gh).$$

Then $c_U \in Z^2(G_l(O_r, \beta)^{(c)}, \mathbb{C}^\times)$ is a $\mathbb{C}^\times$-valued 2-cocycle on $G_l(O_r, \beta)^{(c)}$ with trivial action on $\mathbb{C}^\times$, and the cohomology class $[c_U] \in H^2(G_l(O_r, \beta)^{(c)}, \mathbb{C}^\times)$ is independent of the choice of each $U(g)$.

In the next section, we will construct $\pi_{\beta, \rho}$ by means of Weil representations over the finite field $F$ (see Proposition 3.3.1), and will show that we can choose $U(g)$ so that we have

$$c_U(g, h) = c_{\beta, \rho}(g, h)$$

for all $g, h \in G_l(O_r, \beta)^{(c)}$, where $\overline{\pi} \in G_{\beta}(F)^{(c)}$ is the image of $g \in G_l(O_r, \beta)^{(c)}$ under the canonical surjection $G(O_r) \to G(F)$ (see subsection 3.4).

Let us assume
Assumption 2.6.2 There exists a commutative subgroup $\mathcal{C} \subset G_1(O_r, \beta)^{(c)}$ such that

1) $G(O_r, \beta) = \mathcal{C} \cdot K_{1-1}(O_r)$,

2) the cohomology class $[c_{\beta, \rho}|_{\mathcal{C} \times \mathbb{F}^*}] \in H^2(\mathcal{C}, \mathbb{C}^*)$ is trivial for all $\rho \in \mathfrak{g}_\beta(\mathbb{F})^-$, where $\mathcal{C} \subset G_\beta(\mathbb{F})^{(c)}$ is the image of $\mathcal{C}$ under the canonical surjection $G(O_r) \to G(\mathbb{F})$.

Under this assumption we have

Proposition 2.6.3 For any $\rho \in \mathfrak{g}_\beta(\mathbb{F})^-$, there exists a group homomorphism $U_{\beta, \rho} : \mathcal{C} \to GL_C(V_{\beta, \rho})$ such that

1) $\pi_{\beta, \rho}(g^{-1}xg) = U_{\beta, \rho}(g)^{-1} \circ \pi_{\beta, \rho}(x) \circ U_{\beta, \rho}(g)$ for all $g \in \mathcal{C}$ and $x \in K_{1-1}(O_r)$

and

2) $U_{\beta, \rho}(h) = 1$ for all $h \in \mathcal{C} \cap K_{1-1}(O_r)$.

[Proof] Because of 2) in Assumption 2.6.2 there exists a group homomorphism $U : \mathcal{C} \to GL_C(V_{\beta, \rho})$ such that $\pi_{\beta, \rho}(g^{-1}xg) = U(g)^{-1} \circ \pi_{\beta, \rho}(x) \circ U(g)$ for all $g \in \mathcal{C}$ and $x \in K_{1-1}(O_r)$. Then for any $h \in \mathcal{C} \cap K_{1-1}(O_r)$ there exists a $c(h) \in \mathbb{C}^*$ such that $U(h) = c(h) \cdot \pi_{\beta, \rho}(h)$. On the other hand we have

$G_1(O_r, \beta)^{(c)} \cap K_{1-1}(O_r) \subset Z(O_r, \beta)$

since $(1 + \omega^{-1}X)_r \in G_1(O_r, \beta)^{(c)} \cap K_{1-1}(O_r)$ means that

$$\beta \equiv (1 + \omega^{-1}X)\beta(1 + \omega^{-1}X)^{-1} \pmod{p}$$

$$\equiv [(\beta + \omega^{-1}X\beta)(1 - \omega^{-1}X)] \pmod{p}$$

and then $[X, \beta] \equiv 0 \pmod{p}$, that is $X \pmod{p} \in \mathfrak{g}_\beta(\mathbb{F})$. Then $\pi_{\beta, \rho}(h)$ is the homothety $\psi_{\beta, \rho}(h)$ for all $h \in \mathcal{C} \cap K_{1-1}(O_r)$. Extend the group homomorphism $h \mapsto \psi_{\beta, \rho}(h)$ of $\mathcal{C} \cap K_{1-1}(O_r)$ to a group homomorphism $\theta : \mathcal{C} \to \mathbb{C}^*$. Then $g \mapsto U_\psi(g) = \theta(g)^{-1}U(g)$ is the required group homomorphism. $\blacksquare$

Let us denote by $\mathcal{C}^* \times K_{1-1}(O_r) \mathfrak{g}_\beta(\mathbb{F})^-$ the set of $(\theta, \rho) \in \mathcal{C}^* \times \mathfrak{g}_\beta(\mathbb{F})^-$ such that $\theta = \psi_{\beta, \rho}$ on $\mathcal{C} \cap K_{1-1}(O_r)$. Then $(\theta, \rho) \in \mathcal{C}^* \times K_{1-1}(O_r) \mathfrak{g}_\beta(\mathbb{F})^-$ defines an irreducible representation $\sigma_{\theta, \rho}$ of $G(O_r, \beta) = \mathcal{C} \cdot K_{1-1}(O_r)$ by

$$\sigma_{\theta, \rho}(gh) = \theta(g) \cdot U_{\beta, \rho}(g) \circ \pi_{\beta, \rho}(h)$$

for $g \in \mathcal{C}$ and $h \in K_{1-1}(O_r)$. Then we have

Proposition 2.6.4 Under Assumption 2.6.3 a bijection of $\mathcal{C}^* \times K_{1-1}(O_r) \mathfrak{g}_\beta(\mathbb{F})^-$ onto $\text{Irr}(G(O_r, \beta) | \psi_\beta)$ is given by $(\theta, \rho) \mapsto \sigma_{\theta, \rho}$.

[Proof] Clearly $\pi_{\beta, \rho} \in \text{Irr}(G(O_r, \beta) | \psi_\beta)$ for all $(\theta, \rho) \in \mathcal{C}^* \times K_{1-1}(O_r) \mathfrak{g}_\beta(\mathbb{F})^-$. Take $\sigma \in \text{Irr}(G(O_r, \beta) | \psi_\beta)$. Then

$$\sigma = \text{Ind}_{K_{1}(O_r)}^{G(O_r, \beta)}(\psi_\beta) = \text{Ind}_{Z(O_r, \beta)}^{G(O_r, \beta)} \left( \text{Ind}_{K_{1}(O_r)}^{Z(O_r, \beta)}(\psi_\beta) \right)$$

$$= \bigoplus_{\rho \in \mathfrak{g}_\beta(\mathbb{F})^-} \text{Ind}_{Z(O_r, \beta)}^{G(O_r, \beta)}(\psi_{\beta, \rho})$$
so that there exists a $\rho \in \mathfrak{g}_\beta(F)\wedge$ such that

$$\sigma \hookrightarrow \text{Ind}^{G(O,\beta)}_{Z(O,\beta)} \psi_{\beta,\rho} = \text{Ind}^{G(O,\beta)}_{K_{l-1}(O)} \left( \text{Ind}^{K_{l-1}(O)}_{Z(O,\beta)} \psi_{\beta,\rho} \right) = \bigoplus_{\theta} \text{Ind}^{G(O,\beta)}_{K_{l-1}(O)} \pi_{\beta,\rho} = \bigoplus_{\theta} \bigoplus_{\psi} \sigma_{\theta,\psi},$$

where $\bigoplus$ is the direct sum over $\theta \in C'$ such that $\theta = \psi_{\beta,\rho}$ on $C \cap K_{l-1}(O)$. Then we have $\sigma = \sigma_{\theta,\rho}$ for some $(\theta, \rho) \in C' \times K_{l-1}(O), \mathfrak{g}_\beta(F)\wedge$.

Under the conditions of Theorem 2.4.1 we can put $C = G_\beta(O_r)$. We have the following proposition by which our Theorem 2.4.1 is given as a special case of Proposition 2.6.4.

**Proposition 2.6.5** If $G_\beta$ is commutative smooth over $O$, then $(\theta, \rho) \mapsto \theta$ gives a bijection of $G_\beta(F) \times K_{l-1}(O) \mathfrak{g}_\beta(F)\wedge$ onto the set

$$\{ \theta \in G_\beta(O_r)\wedge \text{ s.t. } \theta = \psi_{\beta,\rho} \text{ on } G_\beta(O_r) \cap K_{l-1}(O) \}.$$

**Proof** Take a $(\theta, \rho) \in G_\beta(F) \times K_{l-1}(O) \mathfrak{g}_\beta(F)\wedge$. The smoothness of $G_\beta$ over $O$ implies that the canonical mapping $\mathfrak{g}_\beta(O) \to \mathfrak{g}_\beta(F)$ is surjective. So take a $X \in \mathfrak{g}_\beta(F)$ with $X \in \mathfrak{g}_\beta(O)$. Then we have

$$g = 1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r} \in K_{l-1}(O_r) \cap G_\beta(O_r)$$

so that

$$\theta(g) = \psi_{\beta,\rho}(g) = \tau(1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r}) \cdot \rho(X) = \tau(1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r}) \cdot \rho(X).$$

Hence we have

$$\rho(X) = \tau(1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r}) \cdot \theta(1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r}) \cdot \rho(X).$$

This means that the mapping $(\theta, \rho) \mapsto \theta$ is injective. Take $X, X' \in \mathfrak{g}_\beta(O)$ such that $X \equiv X' \pmod{p}$. Then we have $X' = X + \omega T$ with $T \in \mathfrak{g}_\beta(O)$ and

$$1 + \omega^{l-1}X' + 2^{-1}\omega^{2l-2}X'^2 \pmod{p^r} = 1 + \omega^{-1}X + 2^{-1}\omega^{2l-2}X^2 + \omega^l T \pmod{p^r} = (1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2)(1 + \omega^l T) \pmod{p^r}.$$

where $1 + \omega^l T \pmod{p^r} \in K_{l-1}(O_r)$ and hence

$$\theta(1 + \omega^l T \pmod{p^r}) = \psi_{\beta}(1 + \omega^l T \pmod{p^r}) = \tau(1 + \omega^{l-1}B(T, \beta) \pmod{p^r}) \cdot \rho(X).$$

This and the commutativity of $G_\beta$ show that

$$\rho(X) = \tau(1 + \omega^{l-1}B(T, \beta) \pmod{p^r}) \cdot \rho(X) = \tau(1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r})$$

with $X \in \mathfrak{g}_\beta(F)$ with $X \in \mathfrak{g}_\beta(O)$ gives an well-defined group homomorphism of $\mathfrak{g}_\beta(F)$ to $C'$. Then $(\theta, \rho) \in G_\beta(O_r)\wedge K_{l-1}(O) \mathfrak{g}_\beta(F)\wedge$ and our mapping in question is surjective.
2.7 We will give a sufficient condition on $\beta \in \mathfrak{g}(O)$ under which $G_\beta$ is commutative and smooth over $O$.

Let us assume that the connected $O$-group scheme $G$ is reductive, that is, the fibers $G \otimes_O K$ ($K = F, \overline{F}$) are reductive $K$-algebraic groups. In this case the dimension of the maximal torus in $G \otimes_O K$ is independent of $K$ which is denoted by $\text{rank}(G)$. For any $\beta \in \mathfrak{g}(O)$ we have

$$\dim_K \mathfrak{g}_\beta(K) = \dim \mathfrak{g}_\beta \geq \dim G_\beta \otimes K \geq \text{rank}(G).$$

We say $\beta$ to be \textit{smoothly regular} with respect to $G$ over $K$ (or $\beta \in \mathfrak{g}(K)$ is smoothly regular with respect to $G \otimes O K$) if $\dim_K \mathfrak{g}_\beta(K) = \text{rank}(G)$ (see [17, 1.4]). In this case $G_\beta \otimes O K$ is smooth over $K$. If $\beta$ is smoothly regular with respect to $G$ over $F$ and over $\overline{F}$, then $\beta$ is said to be smoothly regular with respect to $G$.

We say $\beta$ to be \textit{connected} with respect to $G$ if the fibers $G_\beta \otimes O K$ ($K = F, \overline{F}$) are connected.

\textbf{Theorem 2.7.1} If $\beta \in \mathfrak{g}(O)$ is smoothly regular and connected with respect to $G$, then $G_\beta$ is commutative and smooth over $O$.

[Proof] Let $G_\beta'$ be the neutral component of $O$-group scheme $G_\beta$ which is a group functor of the category of $O$-scheme (see §3 of Exposé VI in [5]). The following statements are equivalent;

1) $G_\beta'$ is representable as an smooth open $O$-group subscheme of $G_\beta$,

2) $G_\beta$ is smooth at the points of unit section,

3) each fibers $G_\beta \otimes O K$ ($K = F, \overline{F}$) are smooth over $K$ and their dimensions are constant

(see Th. 3.10 and Cor. 4.4 of [5]). So if $\beta$ is smoothly regular with respect to $G$, then $G_\beta'$ is smooth open $O$-group subscheme of $G_\beta$. If further $\beta$ is connected with respect to $G$, then $G_\beta' = G_\beta$ is smooth over $O$. Let $\beta = \beta_s + \beta_n$ be the Jordan decomposition of $\beta \in \mathfrak{g}_\beta(F)$. Then the identity component $G$ of the centralizer $Z_{G \otimes O F}(\beta_s)$ is a reductive $F$-algebraic group and

$$G_\beta \otimes O F = Z_G(\beta_n)$$

because $G_\beta \otimes O F$ is connected. Then [13] shows that $G_\beta(F)$ is commutative ($\overline{F}$ is the algebraic closure of $F$), ans hence $G_\beta$ is a commutative $O$-scheme. \par

2.8 With the detailed discussion given in the section 4 and the results of [18, 2.8], the truth of the following statement is highly probable;

Assume that $G$ is connected smooth reductive $O$-group scheme. If $\beta \in \mathfrak{g}(O)$ is smoothly regular with respect to $G$ over $\overline{F}$, then the Schur multiplier $[c_{\beta, \rho}] \in \text{H}^2(G_\beta(\overline{F})^c, \mathbb{C}^\times)$ is trivial for all $\rho \in \mathfrak{g}_\beta(\overline{F})^c$ provided that the characteristic of $\overline{F}$ is big enough.

3 Weil representations over finite field

In this section we will use the notations of the preceding sections and will suppose $r = 2l - 1 \geq 3$ is odd so that $l' = l - 1 > 0$. Fix a $\beta \in \mathfrak{g}(O)$ such that $\beta \in \mathfrak{g}(F)$ is not in the center of $\mathfrak{g}(F)$. 

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3.1 A group extension

\[ 0 \to g(O_{l-1}) \xrightarrow{\phi} K_{l-1}(O_r) \xrightarrow{\psi} g(F) \to 0 \quad (7) \]

is given by the canonical surjection \([5]\), whose kernel is \( K_l(O_r) \), with the group isomorphism

\[ \phi : g(O_{l-1}) \to K_l(O_r) \]

defined by \( S(\text{mod } p^{l-1}) \mapsto (1 + \omega^l S)(\text{mod } p^r) \).

In order to determine the 2-cocycle of the group extension \( (7) \), choose any mapping \( \lambda : g(F) \to g(O) \) such that \( X = \lambda(X)(\text{mod } p) \) for all \( X \in g(F) \) and \( \lambda(0) = 0 \), and define a section

\[ l : g(F) \to K_{l-1}(O_r) \]

of \( (7) \) by \( X \mapsto 1 + \omega^l \lambda(X) + 2^{-1} \omega^{2l-2} \lambda(X)^2 \) (mod \( p^r \)). Then we have

\[ l(X)^{-1} = 1 - \omega^l \lambda(X) + 2^{-1} \omega^{2l-2} \lambda(X)^2 \quad (\text{mod } p^r) \]

for all \( X \in g(F) \) and

\[ l(X)(1 + \omega^l S)l(X)^{-1} \equiv 1 + \omega^l S \quad (\text{mod } p^r) \]

for all \( S_{l-1} \in g(O_{l-1}) \). Furthermore we have

\[ l(X)l(Y)(X + Y)^{-1} = 1 + \omega^l \left( \mu(X, Y) + 2^{-1} \omega^{2l-2}[\lambda(X), \lambda(Y)] \right) \quad (\text{mod } p^r) \]

for all \( X, Y \in g(F) \) where \( \mu : g(F) \times g(F) \to g(O) \) is defined by

\[ \lambda(X) + \lambda(Y) - \lambda(X + Y) = \omega \cdot \mu(X, Y) \]

for all \( X, Y \in g(F) \). Now we have two elements (2-cocycle)

\[ \mu = [(X, Y) \mapsto \mu(X, Y)_{l-1}], \quad c = [(X, Y) \mapsto 2^{-1} \omega^{l-2}[X, Y]_{l-1}] \]

of \( Z^2(g(F), g(O_{l-1})) \) with trivial action of \( g(F) \) on \( g(O_{l-1}) \).

We will consider two groups \( M \) and \( G \) corresponding to the two 2-cocycles \( \mu \) and \( c \) respectively. That is the group operation on \( M = g(F) \times g(O_{l-1}) \) is defined by

\[ (X, S_{l-1}) \cdot (Y, T_{l-1}) = (X + Y, (S + T + \mu(X, Y))_{l-1}) \]

and the group operation on \( G = g(F) \times g(O_{l-1}) \) is defined by

\[ (X, S_{l-1}) \cdot (Y, T_{l-1}) = ((X + Y)^c, (S + T + 2^{-1} \omega^{2l-2}[X, Y])_{l-1}). \]

Let \( G \times_{g(F)} M \) be the fiber product of \( G \) and \( M \) with respect to the canonical projections onto \( g(F) \). In other word

\[ G \times_{g(F)} M = \{ (X; S, T) = ((X, S), (X, T)) \in G \times M \} \]

is a subgroup of the direct product \( G \times M \). We have a surjective group homomorphism

\[ (*) : G \times_{g(F)} M \to K_{l-1}(O_r) \quad (8) \]

defined by

\[ (X; S_{l-1}, T_{l-1}) \mapsto l(X) \cdot (1 + \omega^l(S + T) \quad (\text{mod } p^r)) \]

\[ = 1 + \omega^l \lambda(X) + 2^{-1} \omega^{2l-2} \lambda(X)^2 + \omega^l(S + T) \quad (\text{mod } p^r). \]
3.2 A group homomorphism of additive groups \( B_\beta : g(O_{l-1}) \to O_{l-1} (X \mapsto B(X, \beta_{l-1})) \) induces a group homomorphism \( B_\beta^* : H^2(g(\mathbb{F}), g(O_{l-1})) \to H^2(g(\mathbb{F}), O_{l-1}). \) Let us denote by \( \mathcal{H}_\beta \) the group associated with the 2-cocycle

\[
e_{\beta} = B_\beta \circ c = \left[ (X, Y) \mapsto 2^{-1} \varpi^{l-2} B([X, Y], \beta)_{l-1} \right] \in Z^2(g(\mathbb{F}), O_{l-1}).
\]

That is \( \mathcal{H}_\beta = g(\mathbb{F}) \times O_{l-1} \) with a group operation \((X, s) : (Y, t) = ((X + Y)^\gamma, s + t + 2^{-1} \varpi^{l-2} B([X, Y], \beta)_{l-1}).\)

Then the center of \( \mathcal{H}_\beta \) is \( Z(\mathcal{H}_\beta) = g_\beta(\mathbb{F}) \times O_{l-1}, \) the direct product of two additive groups \( g_\beta(\mathbb{F}) \) and \( O_{l-1}. \)

The inverse image of \( Z(\mathcal{H}_\beta) \) with respect to the surjective group homomorphism

\[
\star : G \times g(\mathbb{F})_M \to \mathcal{H}_\beta ( (X; S_{l-1}, T_{l-1}) \mapsto (X, B(S, \beta)_{l-1}) )
\]

is \( (G \times g(\mathbb{F})_M)_\beta = \{ (X; S, T) \in G \times g(\mathbb{F})_M \mid X \in g(\mathbb{F})_M \} \) which is mapped onto \( Z(\mathcal{H}_\beta) \subset K_{l-1}(O_{l-1}) \) by the surjection \( \mathfrak{S}. \)

Take a \( \rho \in g_\beta(\mathbb{F})^* \) which defines group homomorphisms

\[
\chi_\rho = \rho \otimes [x_{l-1} \mapsto \tau(\varpi^{-(l-1)} x)] : Z(\mathcal{H}_\beta) = g(\mathbb{F})_\beta \times O_{l-1} \to \mathbb{C}^\times
\]

and

\[
\tilde{\chi}_\rho : (G \times g(\mathbb{F})_M)_\beta \to Z(\mathcal{H}_\beta) \xrightarrow{\chi_\rho} \mathbb{C}^\times.
\]

On the other hand we have a group homomorphism

\[
\tilde{\psi}_0 : G \times g(\mathbb{F})_M \to \mathbb{C}^\times
\]

defined by \( \tilde{\psi}_0(X; S_{l-1}, T_{l-1}) = \tau(\varpi^{-1} B(\lambda(X) + \varpi T, \beta)) \), and \( \tilde{\psi}_0 : \tilde{\chi}_\beta \) is trivial on the kernel of the surjection \( \mathfrak{S} \) and induces a group homomorphism \( \psi_{\beta, \rho} \in Y_\beta \) defined in subsection 2.6.

3.3 Fix a \( \rho \in g_\beta(\mathbb{F})^* \).

Let us determine the 2-cocycle of the group extension

\[
0 \to Z(\mathcal{H}_\beta) \to \mathcal{H}_\beta \xrightarrow{\star} V_\beta \to 0
\]

where \( \star : \mathcal{H}_\beta \to V_\beta \) is defined by \( (X, s) \mapsto \hat{X} \pmod{g_\beta(\mathbb{F})}. \) Fix a \( \mathbb{F} \)-linear section \( v \mapsto [v] \) of the exact sequence

\[
0 \to g_\beta(\mathbb{F}) \to g(\mathbb{F}) \to V_\beta \to 0
\]

\( V \)-vector spaces and define a section \( l : V_\beta \to \mathcal{H}_\beta \) of the group extension \( \mathfrak{10} \) by \( l(v) = ([v], 0). \) Then we have

\[
l(u)l(v)l(u + v)^{-1} = (0, 2^{-1} \varpi^{l-2} B([X, Y], \beta) \pmod{p^{l-1}})
\]

for \( u = \hat{X}, v = \hat{Y} \in V_\beta \) so that the 2-cocycle of the group extension \( \mathfrak{10} \) is

\[
\left[ (\hat{X}, \hat{Y}) \mapsto 2^{-1} \varpi^{l-2} B([X, Y], \beta) \pmod{p^{l-1}} \right] \in Z^2(V_\beta, O_{l-1}).
\]
Define a group operation on $\mathcal{H}_\beta = \mathcal{V}_\beta \times Z(\mathcal{H}_\beta)$ by

$$(X, z) \cdot (Y, w) = \left( X + Y, z + w + 2^{-1}\varepsilon^{-2}B([X, Y], \beta) \pmod{p^{-1}} \right).$$

Then $\mathcal{H}_\beta$ is isomorphic to $\mathcal{H}_\beta$ by $(v, Y, s) \mapsto ([v] + Y, s)$.

Let $H_\beta$ be the Heisenberg group of the symplectic $F$-space $\mathcal{V}_\beta$, that is $H_\beta = \mathcal{V}_\beta \times \mathbb{C}$ with a group operation

$$(u, s) \cdot (v, t) = (u + v, s \cdot t \cdot \pi^{-1}(u, v)),$$

Then we have a surjective group homomorphism $\mathcal{H}_\beta \to H_\beta$.

Proposition 3.3.1

Take a polarization $\beta$. Fix a polarization $\pi$. Let $\tilde{H}_\beta$ be the Heisenberg group of the symplectic $F$-space $\mathcal{V}_\beta$. Then we have a surjective group homomorphism $\mathcal{H}_\beta \to H_\beta$.

Define a group operation on $\tilde{H}_\beta = \mathcal{V}_\beta \times Z(\tilde{H}_\beta)$ by

$$(X, z) \cdot (Y, w) = \left( X + Y, z + w + 2^{-1}\varepsilon^{-2}B([X, Y], \beta) \pmod{p^{-1}} \right).$$

Then $\tilde{H}_\beta$ is isomorphic to $\mathcal{H}_\beta$ by $(v, Y, s) \mapsto ([v] + Y, s)$.

Let $H_\beta$ be the Heisenberg group of the symplectic $F$-space $\mathcal{V}_\beta$, that is $H_\beta = \mathcal{V}_\beta \times \mathbb{C}$ with a group operation

$$(u, s) \cdot (v, t) = (u + v, s \cdot t \cdot \pi^{-1}(u, v)).$$

Then we have a surjective group homomorphism $\tilde{H}_\beta \to H_\beta$.

Fix a polarization $\tilde{\mathcal{V}}_\beta = \mathcal{W}' \oplus \mathcal{W}$ of the symplectic $F$-space $\mathcal{V}_\beta$. Let us denote by $L^2(\mathcal{W}')$ the complex vector space of the complex-valued functions $f$ on $\mathcal{W}'$ with inner product $(f, f') = \sum_{w \in \mathcal{W}'} f(w)\overline{f}(w)$. The Schrödinger representation $(\pi^\beta, L^2(\mathcal{W}'))$ of $H_\beta$ associated with the polarization is defined for $(v, s) \in H_\beta$ and $f \in L^2(\mathcal{W}')$ by

$$(\pi^\beta(v, s)f)(w) = s \cdot \pi^{-1}(v, w)v + \langle w, v \rangle \pi^{-1}(v, w)v.$$}

where $v = v_+ + v_+ \in \mathcal{V}_\beta$ with $v_- \in \mathcal{W}', v_+ \in \mathcal{W}'$. Now an irreducible representation $(\pi^\beta, L^2(\mathcal{W}'))$ of $\tilde{H}_\beta$ is defined by $\pi^\beta(v, z) = \pi^\beta(v, \chi(z))$, and an irreducible representation $(\tilde{\pi}^\beta, L^2(\mathcal{W}'))$ of $\tilde{H}_\beta \times (\tilde{H}_\beta \times \mathbb{C})$ is defined by

$$\tilde{\pi}^\beta : \tilde{H}_\beta \times (\tilde{H}_\beta \times \mathbb{C}) \to \tilde{H}_\beta \to \tilde{H}_\beta \to GL_C(L^2(\mathcal{W}')).$$

Then $\tilde{\psi}_0 \cdot \tilde{\pi}^\beta$ is trivial on the kernel of $(*) : \tilde{H}_\beta \times (\tilde{H}_\beta \times \mathbb{C}) \to K_{t-1}(O_{r})$ so that it induces an irreducible representation $\pi^\beta \rho$ of $K_{t-1}(O_{r})$ on $L^2(\mathcal{W}')$.

**Proposition 3.3.1** Take a $g = 1 + \varepsilon^{-T}T \pmod{p^r} \in K_{t-1}(O_{r})$ with $T \in g_{O_{r}}(O)$. Then we have $T(\varepsilon^{-T}T \pmod{p^r})$.

Then $\tilde{T} = [v] + Y \in g(\mathcal{W})$ with $v \in \mathcal{V}_\beta$ and $Y \in \mathcal{V}_\beta$. In particular $\pi^\beta(h)$ is the homothety $\psi_{1, h}$ for all $h \in Z(\tilde{H}_\beta)$.}

**Proof** By the definition we have

$$\pi^\beta(v, s)(l(X)(1 + \varepsilon s) \pmod{p^r}))$$

$$= \psi_0(X, 0) \cdot \tilde{\pi}^\beta([v] + Y, S_{t-1}, 0)$$

$$= \mathcal{T}(l(X)(1 + \varepsilon s) \pmod{p^r}) \cdot \tau(\varepsilon^{-T}B(\lambda(X) - \lambda(Y), \beta)) \cdot \pi^\beta(v, 1)$$

$$= \tau(\varepsilon^{-T}B(S, \beta) + \varepsilon^{-T}B(\lambda(X), \beta)) \cdot \rho(Y) \cdot \pi^\beta(v, 1)$$

where $X = [v] + Y \in g(\mathcal{W})$ with $v \in \mathcal{V}_\beta$ and $Y \in g_{O_{r}}(\mathcal{W})$. Put $1 + \varepsilon^{-T}T \equiv l(X)(1 + \varepsilon s) \pmod{p^r}$ with $X \in g(\mathcal{W})$ and $S \in g(O)$. Then we have

$$1 + \varepsilon^{-T}T \equiv 1 + \varepsilon^{-T}l(X) \pmod{p^r}$$

14
so that we have $T \equiv X \in \mathfrak{g}(\mathbb{F})$ and

$$\varpi S \equiv T - \lambda(T) - 2^{-1} \varpi \lambda(T)^2 \pmod{p^\ell}.$$  

Then we have

$$\pi^\beta(g) = \pi^\gamma(l\mathbb{T})(1 + \varpi S) \in \mathfrak{g}(\mathbb{F})$$

$$= \tau(\varpi T(S, \beta) + \varpi B(\lambda(T), \beta)) \cdot \rho(Y) \cdot \pi^\beta(v, 1)$$

$$= \pi^\gamma(B(T, \beta) - 2^{-1} \varpi B(T^2, \beta)) \cdot \rho(Y) \cdot \pi^\beta(v, 1).$$

This proposition shows that the irreducible representation $(\pi^\beta, L^2(W))$ of $K_{l-1}(O_{\ell})$ is exactly the irreducible representation $\pi_{\beta, \rho}$ defined in Proposition 2.3.1.

3.4 Fix a $\rho \in \mathfrak{g}_\beta(\mathbb{F})$. In this subsection we will study the conjugate action of $g_{\ell} = g \pmod{p^\ell} \in G(O_{\ell}, \beta)$ on $K_{l-1}(O_{\ell})$ and on $\pi^\beta,\rho$. For any $X \in \mathfrak{g}(\mathbb{F})$, we have

$$g_{\ell}^{-1}l(X)g_{\ell} = l(\Ad(\mathbb{F})^{-1}X) + \varpi \nu(X, g) \pmod{p^\ell}$$

with $\nu(X, g) \in \mathfrak{g}(O)$ such that

$$\Ad(g)^{-1} \lambda(X) - \lambda(\Ad(g)^{-1}X) = \varpi \cdot \nu(X, g).$$

Then we have

$$g_{\ell}^{-1}l(X)(1 + \varpi (S + T))g_{\ell}$$

$$= l(\Ad(\mathbb{F})^{-1}X) (1 + \varpi (\Ad(g)^{-1}S + \Ad(g)^{-1}T + \nu(X, g))) \pmod{p^\ell}$$

and an action of $g_{\ell} \in G(O_{\ell}, \beta)$ on $(X; S_{l-1}, T_{l-1}) \in \mathcal{G} \times \mathfrak{g}(\mathbb{F})$ is defined by

$$(X; S_{l-1}, T_{l-1})g_{\ell} = (\Ad(\mathbb{F})^{-1}X; (\Ad(g)^{-1}S)_{l-1}, (\Ad(g)^{-1}T + \nu(X, g))_{l-1}).$$

(11)

The action (11) is compatible with the action

$$(X, s)^{g_{\ell}} = (\Ad(\mathbb{F})^{-1}X, s)$$

of $g_{\ell} \in G(O_{\ell}, \beta)$ on $(X, s) \in \mathbb{H}_\beta$ via the surjection (19). If we put $X = [v] + Y \in \mathbb{g}(\mathbb{F})$ with $v \in \mathcal{V}_\beta$ and $Y \in \mathfrak{g}_\beta(\mathbb{F})$, then we have

$$\Ad(\mathbb{F})^{-1}X = [v\sigma_{\mathbb{F}}] + \gamma(Y) + \Ad(\mathbb{F})^{-1}Y$$

in the notations of subsection 2.3. So $g_{\ell} \in G(O_{\ell}, \beta)$ acts on $(v, (Y, s)) \in \mathbb{H}_\beta$ by

$$(v, (Y, s))^{g_{\ell}} = (v\sigma_{\mathbb{F}}, (\Ad(\mathbb{F})^{-1}Y + \gamma(Y, s)), 0).$$

In particular $g_{\ell} \in G(\ell, \beta)^{(o)}$ acts on $(v, z) \in \mathbb{H}_\beta$ by

$$(v, z)^{g_{\ell}} = (v\sigma_{\mathbb{F}}, (\gamma(Y, s), 0) \cdot z).$$

There exists a group homomorphism $T : Sp(V_{\beta}) \rightarrow GL_{c}(L^2(W))$ such that

$$\pi^\beta(v, s) = T(\sigma)^{-1} \circ \pi^\beta(v, s) \circ T(\sigma)$$
for all $\sigma \in Sp(V,\beta)$ and $(v,s) \in H_\beta$ (see [7, Th.2.4]). Then we have
\[
\pi^\beta,\rho((v,z)\beta) = \pi^\beta(v\sigma_v, \rho(\gamma(v,F), 0) \cdot z) \\
= \pi^\beta(v\sigma_v, \rho(\gamma(v,F)) \cdot \chi_\rho(z)) \\
= \pi^\beta(v\sigma_v, \tau ((v, v\sigma_v) \cdot \chi_\rho(z)) \\
= T(\sigma_v)^{-1} \circ \pi^\beta(v, \tau ((v, v\sigma_v) \cdot \chi_\rho(z)) \circ T(\sigma_v) \\
= T(\sigma_v)^{-1} \circ \pi^\beta((v, v\sigma_v)^{-1}(v, \chi_\rho(z))(v, 1)) \circ T(\sigma_v) \\
= T(\sigma_v)^{-1} \circ \pi^\beta(v, 1)^{-1} \circ \pi^\beta(v, z) \circ \pi^\beta(v, 1) \circ T(\sigma_v)
\]
If we put
\[
U(g_r) = \pi^\beta(v, 1) \circ T(\sigma_v) \in GL_c(L^2(\mathbb{W}))
\]
for $g_r \in G_t(O_r, \beta)^{(c)}$ then we have
\[
U(g_r) \circ U(h_r) = c_{\beta,\rho}(\overline{\sigma}, \overline{\tau}) \cdot U((gh)_r)
\]
for all $g_r, h_r \in G_t(O_r, \beta)^{(c)}$, in fact
\[
U(g_r) \circ U(h_r) = \pi^\beta(v, 1) \circ T(\sigma_v) \circ \pi^\beta(v, 1) \circ T(\sigma_v) \\
= \pi^\beta(v, 1) \circ \pi^\beta(v, 1)^{-1} \circ T(\sigma_v) \circ T(\sigma_v) \\
= \pi^\beta(\overline{v} + v_{\overline{v}}^{-1} \cdot \overline{\tau} \cdot \gamma(2^{-1}(v, v_{\overline{v}}^{-1}))) \circ T(\sigma_v) \\
= c_{\beta,\rho}(\overline{\sigma}, \overline{\tau}) \cdot \pi^\beta(v, 1) \circ T(\sigma_v).
\]
On the other hand
\[
\tilde{\psi}_0((X; S_{t-1}, T_{t-1})^{\beta-1}) = \tau(\omega^{-i}B(\lambda(X) + \omega T, \operatorname{Ad}(g)\beta)) \\
= \tau(\omega^{-i}B(\lambda(X) + \omega T, \beta))
\]
for all $g_r \in G(O_r, \beta)^{(c)}$. That is $\tilde{\psi}_0$ is invariant under the conjugate action of $G_t(O_r, \beta)^{(c)}$. Hence we have
\[
\pi^\beta,\psi(g_r^{-1}hg_r) = U(g_r)^{-1} \circ \pi^\beta,\psi(h) \circ U(g_r)
\]
for all $g_r \in G_t(O_r, \beta)^{(c)}$ and $h \in K_{t-1}(O_r)$.

## 4 Properties of $c_{\beta,\rho}$

In this section we will present several properties of the Schur multiplier $[c_{\beta,\rho}] \in H^2(G_{\beta}(\mathbb{F})^{(c)}, \mathbb{C}^*)$ defined in subsection [23]. We will keep the notations and conventions of section [23] while only the structure of algebraic groups over finite field $\mathbb{F}$ is required to define and to discuss the Schur multiplier.

### 4.1 Let $\mathbb{K}/\mathbb{F}$ be a finite field extension. More specifically take the unramified extension $K/F$ with the integer ring $O_K \subset K$ such that $\mathbb{K} = O_K/\omega O_K$. Then $g \otimes O_K$ is the Lie algebra of smooth $O_K$-group scheme $G \otimes O_K$ and
\[
g(\mathbb{K}) = (g \otimes O_K)(\mathbb{K}) = g(\mathbb{F}) \otimes \mathbb{K}.
\]
So $B : g(K) \times g(K) \to K$ is non-degenerate.

We will assume that the degree of the extension $(K : F)$ is not divisible by the characteristic of $F$, and put $T_{K/F}' = (K : F)^{-1}T_{K/F}$. Then the additive character

$$\tilde{\tau} : K \xrightarrow{T_{K/F}'} F \xrightarrow{\tilde{\tau}} \mathbb{C}^\times$$

is an extension of $\tilde{\tau} : F \to \mathbb{C}^\times$.

Fix a $\beta \in g(O)$ such that $g_\beta(F) \subseteq g(F)$. Then for any $\rho \in g_\beta(F)$, the additive character

$$\tilde{\rho} : (g \otimes O_K)_{\beta}(K) = (g_\beta)(F) \otimes_{\mathbb{Z}} K \xrightarrow{1 \otimes T_{K/F}'} g_\beta(F) \to \mathbb{C}^\times$$

is an extension of $\rho : g_\beta(F) \to \mathbb{C}^\times$. Then we have

**Proposition 4.1.1** The Schur multiplier $[c_{\beta,\rho}] \in H^2(G(F)_{\beta}^{(c)}, \mathbb{C}^\times)$ is the image of $[c_{\beta,\rho}] \in H^2(G(K)_{\beta}^{(c)}, \mathbb{C}^\times)$ under the restriction mapping

$$\text{Res} : H^2(G(K)_{\beta}^{(c)}, \mathbb{C}^\times) \to H^2(G(F)_{\beta}^{(c)}, \mathbb{C}^\times).$$

**Proof** There exists a $F$-basis $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ of $K$ such that $T_{K/F}(\alpha_\lambda) = 1$ for all $\lambda \in \Lambda$. Let $v \mapsto [v]$ be a $F$-linear section of the exact sequence (1). Its $K$-linear extension gives a $K$-linear section of the exact sequence

$$0 \to g_\beta(F) \otimes_{\mathbb{F}} K \to g(F) \otimes_{\mathbb{F}} K \to V_\beta \otimes_{\mathbb{F}} K \to 0.$$

Take a $g \in G(F)_{\beta}^{(c)} \subseteq G(K)_{\beta}^{(c)}$ and

$$v = \sum_{\lambda \in \Lambda} v_\lambda \otimes \alpha_\lambda \in V_\beta \otimes_{\mathbb{F}} K \quad (v_\lambda \in V_\beta).$$

Then we have

$$\gamma(v, g) = \text{Ad}(g)^{-1}[v] - [v, \sigma_g] = \sum_{\lambda \in \Lambda} (\text{Ad}(g)^{-1}[v_\lambda] - [v_\lambda, \sigma_g]) \otimes \alpha_\lambda = \sum_{\lambda \in \Lambda} \gamma(v_\lambda, g) \otimes \alpha_\lambda,$$

hence

$$\tilde{\rho}(\gamma(v, g)) = \prod_{\lambda \in \Lambda} \rho \left( \gamma((K : F)^{-1}v_\lambda, g) \right) = \prod_{\lambda \in \Lambda} \tilde{\tau} \left( (K : F)^{-1}\langle v_\lambda, v_g \rangle_\beta \right)$$

$$= \tilde{\tau} \left( \sum_{\lambda \in \Lambda} \langle v_\lambda, v_g \rangle_\beta \alpha_\lambda \right) = \tilde{\tau} \left( \langle v, v_g \rangle_\beta \right).$$

So we have $c_{\beta,\rho}(g, h) = c_{\beta,\rho}(g, h)$ for all $g, h \in G(F)_{\beta}^{(c)} \subseteq G(K)_{\beta}^{(c)}$. ■
4.2 Let us assume that there exists a closed smooth $O$-group subscheme $H \subset GL_n$ of which our $G$ is a closed $O$-group subscheme and that the trace form

$$B : \mathfrak{h}(F) \times \mathfrak{h}(F) \to F$$

is non-degenerate where $\mathfrak{h}$ is the Lie algebra of $H$. Then we have $\mathfrak{h}(F) = g(F) \oplus g(F)^\perp$ where $g(F)^\perp = \{ X \in \mathfrak{h}(F) \mid B(X, g(F)) = 0 \}$ is the orthogonal complement of $g(F)$ in $\mathfrak{h}(F)$.

Take $\beta \in g(O)$ such that $g_\beta(F) \subseteq g(F)$. Then $\beta \in h(O)$ and $h_\beta(F) \subseteq h(F)$ where $h_\beta = Z_h(\beta)$ is the centralizer. We have decompositions $h_\beta(F) = g_\beta(F) \oplus (g(F)^\perp)_\beta$ where $(g(F)^\perp)_\beta = h_\beta(F) \cap g(F)^\perp$, and

$$\tilde{\mathbb{V}}_\beta = h(F)/h_\beta(F) = V_\beta \oplus \left( g(F)^\perp / (g(F)^\perp)_\beta \right)$$

is an orthogonal decomposition of symplectic spaces.

Let $v \mapsto [v]$ be a $F$-linear section of the exact sequence

$$0 \to h_\beta(F) \to h(F) \to \tilde{\mathbb{V}}_\beta \to 0$$

of $F$-vector space such that $[V_\beta] = g(F)$ and $[g(F)^\perp / (g(F)^\perp)_\beta] = g(F)^\perp$.

Take $\rho \in g_\beta(F)^\times$ and put

$$\tilde{\rho} : h_\beta = g_\beta \oplus (g(F)^\perp)_\beta \xrightarrow{\text{projection}} g_\beta \xrightarrow{\rho} \mathbb{C}^\times.$$

For any $g \in G_\beta(F) \subset H_\beta(F)$, there exists uniquely a $v_\beta \in V_\beta$ such that

$$\rho(\gamma_\beta(v, g)) = \tilde{\tau}(\langle v, v_\beta \rangle)$$

for all $v \in V_\beta$. Then we have

$$\tilde{\rho}(\gamma_\beta(v, v_\beta)) = \tilde{\tau}(\langle v, v_\beta \rangle)$$

for all $v \in \tilde{\mathbb{V}}_\beta$. In fact if we put $v = v' + v''$ with $v' \in V_\beta$ and $v'' \in g(F)^\perp / (g(F)^\perp)_\beta$, then we have $\gamma_\beta(v, g) = \gamma_\beta(v', g) + \gamma_\beta(v'', g)$ with $\gamma_\beta(v'', g) \in (g(F)^\perp)_\beta$, since

$$\text{Ad}(g)g(F)^\perp = g(F)^\perp, \quad \text{Ad}(g) (g(F)^\perp)_\beta = (g(F)^\perp)_\beta.$$

Then we have

$$\tilde{\rho}(\gamma_\beta(v, g)) = \rho(\gamma_\beta(v', g)) = \tilde{\tau}(\langle v', v_\beta \rangle)$$

because $\langle v'', v_\beta \rangle = 0$. Hence we have

**Proposition 4.2.1** If $G(F)^{(c)}_\beta \subset H(F)^{(c)}_\beta$ then the Schur multiplier $[c_{\beta, \rho}] \in H^2(G(F)^{(c)}_\beta, \mathbb{C}^\times)$ is the image under the restriction mapping

$$\text{Res} : H^2(H(F)^{(c)}_\beta, \mathbb{C}^\times) \to H^2(G(F)^{(c)}_\beta, \mathbb{C}^\times)$$

of the Schur multiplier $[c_{\beta, \rho}] \in H^2(H(F)^{(c)}_\beta, \mathbb{C}^\times)$.
4.3 Take a \( \beta \in \mathfrak{g}(O) \) such that \( g_\beta(\mathbb{F}) \subseteq g(\mathbb{F}) \).

Let \( \mathfrak{F} = \beta_s + \beta_n \) be the Jordan decomposition of \( \mathfrak{F} = \beta \) (mod \( \mathfrak{p} \)) \( \in g(\mathbb{F}) \) \( (\beta_s, \beta_n \in g(\mathbb{F}) \) are respectively the semisimple part and the nilpotent part of \( \beta \)).

Put

\[
L = Z_{G\beta(\mathbb{F})}(\beta_s), \quad I = \text{Lie}(L)(\mathbb{F}) = Z_{g(\mathbb{F})}(\beta_s).
\]

Let us assume the following assumption:

**Assumption 4.3.1** \( B : 1 \times 1 \to \mathbb{F} \) is non-degenerate.

Then we have \( g(\mathbb{F}) = I \oplus I^1 \) where

\[
I^1 = \{ X \in g(\mathbb{F}) \mid B(X, I) = 0 \}
\]

is an \( \text{Ad}(L(\mathbb{F})) \)-submodule of \( g(\mathbb{F}) \). Hence there exists a \( \mathbb{F} \)-vector subspace \( I^1 \subset V \subset g(\mathbb{F}) \) such that \( g(\mathbb{F}) = V \oplus g_\beta(\mathbb{F}) \). We can fix a \( \mathbb{F} \)-linear section \( v \mapsto [v] \) on \( V_\beta \) of the exact sequence

\[
0 \to g_\beta(\mathbb{F}) \to g(\mathbb{F}) \to V_\beta \to 0
\]

such that \( [v] \in V \) for all \( v \in V_\beta \). Then we have

**Proposition 4.3.2** Fix a \( \rho \in g_\beta(\mathbb{F})^\gamma \). Under the assumption 4.3.1, we have

1) \( X_\beta = [v_\beta] \in I \) for any \( g \in G_\beta(\mathbb{F})(^c) \),

2) for any extension \( \tilde{\rho} \in \Gamma \) of \( \rho \), we have

\[
ce_{\beta,\rho}(g, h) = \tilde{\rho} (2^{-1} \text{Ad}(g) X_h) \dot{\cdot} \tilde{\rho} (2^{-1} X_{gh})
\]

for all \( g, h \in G_\beta(\mathbb{F})(^c) \).

**Proof** 1) Take any \( X \in I^1 \) and put \( v = \bar{X} \in V_\beta = g(\mathbb{F})/g_\beta(\mathbb{F}) \). Then we have

\[
\gamma(v, g) = \text{Ad}(g)^{-1}[v] - [\sigma_g] = 0
\]

because \( [v] = X \) and \( \text{Ad}(g)^{-1}X \in I^1 \subset V \) for any \( g \in G_\beta(\mathbb{F})(^c) \). Then we have

\[
\hat{\tau}(B([X, X_\beta], \overline{\beta})) = \hat{\tau}(\langle v, v_\beta \rangle) = \rho(\gamma(v, g)) = 1
\]

and hence

\[
B(X, [X_\beta], \overline{\beta}) = B([X, X_\beta], \overline{\beta}) = 0
\]

for all \( X \in I^1 \). This means that \( [X_\beta, \overline{\beta}] = (I^1) = I \). Since \( [\beta_s, \beta] = 0 \) and \( I = Z_{g(\mathbb{F})}(\beta_s) \), we have

\[
[[\beta_s, X_\beta], \overline{\beta}] = [\beta_s, [X_\beta, \overline{\beta}]] = 0
\]

that is \( [\beta_s, X_\beta] \in g_\beta(\mathbb{F}) \subset I \). Finally

\[
B(Y, [\beta_s, X_\beta]) = B([Y, \beta_s], X_\beta) = 0
\]

for all \( Y \in I \), so we have \( [\beta_s, X_\beta] = 0 \), that is \( X_\beta \in I \).

2) Take \( g, h \in G_\beta(\mathbb{F})(^c) \). The relation 2 gives \( v_\beta \sigma_g = -v_{g^{-1}} \) and

\[
[v_\beta \sigma_g^{-1}] = X_{gh} - X_g \in I.
\]
Then we have
\[
c_{\beta,\rho}(g, h) = \tilde{\tau}(2^{-1}(v_h, v_{g^{-1}})_{\beta}) = \rho(2^{-1}\gamma(v_h, g^{-1}))
\]
\[
\hat{\rho}(2^{-1}\{\text{Ad}(g)X_h - (X_{gh} - X_g)\}).
\]

\[\Box\]

**Remark 4.3.3** Fix a $\beta \in \mathfrak{g}(O)$. If $B : \mathfrak{g}_{\beta}(F) \times \mathfrak{g}_{\beta}(F) \to F$ is non-degenerate, then we have $\mathfrak{g}(F) = \mathfrak{g}_{\beta}(F)^{\perp} \oplus \mathfrak{g}_{\beta}(F)$ where
\[
\mathfrak{g}_{\beta}(F)^{\perp} = \{X \in \mathfrak{g}(X) \mid B(X, \mathfrak{g}_{\beta}(F)) = 0\}
\]
is a $\text{Ad}(G_{\beta}(F))$-invariant $F$-subspace. So if we choose a $F$-linear section $v \mapsto [v]$ on $\mathbb{V}_{\beta} = \mathfrak{g}(F)/\mathfrak{g}_{\beta}(F)$ of the canonical exact sequence of $F$-vector space
\[
0 \to \mathfrak{g}_{\beta}(F) \to \mathfrak{g}(F) \to \mathbb{V} \to 0
\]
so that $[v] \in \mathfrak{g}_{\beta}(F)^{\perp}$ for all $v \in \mathbb{V}_{\beta}$, then we have $v_g = 0$ for all $g \in G_{\beta}(F)$. In this case have $c_{\beta,\rho}(g, h) = 1$ for all $g, h \in G_{\beta}(F)^{\circ}$.

4.4 In this subsection, we will consider the relation between the regularity of $\beta \in \mathfrak{g}(F)$ and the triviality of the Schur multiplier $[c_{\beta,\rho}] \in H^2(G_{\beta}(F)^{\circ}, \mathbb{C}^*)$.

Put $K = F$ or $K = F$. Let us suppose that $G_{\otimes O}K$ is connected reductive algebraic $K$-group and the characteristic of $K$ is not bad with respect to $G_{\otimes O}K$. The list of the bad primes is

| type of $G_{\otimes O}K$ | $A_r$ | $B_r, D_r$ | $C_r$ | $E_6, E_7, E_8$ | $G_2$ |
|--------------------------|-------|-----------|-------|----------------|-------|
| bad prime                | $\emptyset$ | 2 | 2 | 2, 3 | 2, 3, 5 | 2, 3 |

(see [2] p.178, I-4.3).

Take a $\beta \in \mathfrak{g}(O)$ and let $\beta = \beta_s + \beta_n$ be the Jordan decomposition of $\beta \in \mathfrak{g}(K)$ into the semi-simple part $\beta_s \in \mathfrak{g}(K)$ and the nilpotent part $\beta_n \in \mathfrak{g}(K)$ ($\beta \in \mathfrak{g}(K)$ is the image of $\beta \in \mathfrak{g}(O)$ under the canonical mapping $\mathfrak{g}(O) \to \mathfrak{g}(K)$). The identity component $L = Z_{G_{\otimes O}K}(\beta_s)^{\circ}$ of the centralizer of $\beta_s$ in $G_{\otimes O}K$ is a reductive group over $K$ and there exists a maximal torus $T$ of $G_{\otimes O}K$ such that $\beta_s \in \text{Lie}(T)(K)$ (see [1] Prop.13.19 and its proof). Then $T \subset L$ and rank($L$) = rank($G$). Put $l = \text{Lie}(L)$, then $l = Z_{G_{\otimes O}K}(\beta_s)$. So $\beta \in \mathfrak{g}(K)$ is smoothly regular with respect to $G_{\otimes O}K$ if and only if $\beta_n \in l(K)$ is smoothly regular with respect to $L$.

Now fix a system of positive roots $\Phi^+$ in the root system $\Phi(T, L)$ of $L$ with respect to $T$ such that
\[
\beta_n = \sum_{\alpha \in \Phi^+} c_{\alpha} \cdot X_{\alpha}
\]
where $X_{\alpha}$ is a root vector of the root $\alpha$. Then the result of [2] p.228,III-3.5] implies

**Proposition 4.4.1** $\beta \in \mathfrak{g}(K)$ is smoothly regular with respect to $G$ over $K$ if and only if $c_{\alpha} \neq 0$ for all simple $\alpha \in \Phi^+$.
Remark 4.4.2 Let us consider the case of $GL_n$ which is a connected smooth reductive $O$-group scheme. For $\beta \in gl_n(O)$, the following statements are equivalent:

1) $\beta \in gl_n(O)$ is smoothly regular with respect to $GL_n$ over $F$,
2) $\overline{\beta} \in M_n(F)$ is $GL_n(F)$-conjugate to

$$J_{n_1}(\alpha_1) \boxtimes \cdots \boxtimes J_{n_r}(\alpha_r) = \begin{bmatrix}
J_{n_1}(\alpha_1) & & \\
& \ddots & \\
& & J_{n_r}(\alpha_r)
\end{bmatrix},$$

where $\alpha_1, \ldots, \alpha_r$ are distinct elements of the algebraic closure $\overline{F}$ of $F$ and $J_m(\alpha) = \begin{bmatrix}
\alpha & 1 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
& & & \alpha
\end{bmatrix}$ is a Jordan block of size $m$,

3) the characteristic polynomial $\chi_\beta(t) = \det(tI_n - \overline{\beta}) \in F[t]$ is the minimal polynomial of $\overline{\beta} \in M_n(F)$,

4) $\{X \in M_n(F) \mid X\overline{\beta} = \overline{\beta}X\} = F[\overline{\beta}]$,

5) $\{X \in M_n(O_l) \mid X\beta \equiv \beta X \mod p^l\} = O_l[\beta]$ with $\beta_l = \beta \mod p^l$ for all $l > 0$,

6) $\{X \in M_n(O) \mid X\beta = \beta X\} = O[\beta]$.

In this case $\beta \in gl_n(O)$ is smoothly regular with respect to $GL_n$ over $F$ and the centralizer $GL_{n,\beta}$ is commutative and smooth over $O$.

The remark above and Proposition 4.3.2 combined with the results of [18, Examples 4.6.2–4], we have

Proposition 4.4.3 Take a $\beta \in gl_n(O)$ such that

1) $\beta$ is smoothly regular with respect to $GL_n$ over $F$,

2) the multiplicities of the roots of the characteristic polynomial $\chi_\beta(t)$ of $\overline{\beta} \in M_n(F)$ are at most 4.

Then the Schur multiplier $[c_{\beta,\rho}] \in H^2(GL_{n,\beta}(F)^{\text{o}}, C^\times)$ is trivial for all $\rho \in gl_{n,\beta}(F)^{\text{o}}$ provided that the characteristic of $F$ is big enough.

The author does not hesitate to present the following

Conjecture 4.4.4 The Schur multiplier $[c_{\beta,\rho}] \in H^2(GL_{n,\beta}(F)^{\text{o}}, C^\times)$ is trivial for all $\rho : gl_{n,\beta}(F)^{\text{o}}$, if $\beta \in gl_n(O)$ is smoothly regular with respect to $GL_n$ over $F$ and the characteristic of $F$ is big enough.
In general if $\beta \in \mathfrak{g}(O) \subset \mathfrak{gl}_n(O)$ is smoothly regular with respect to $GL_n$ over $\mathbb{F}$, then $\beta$ is also smoothly regular with respect to $GL_n$ over $F$, hence $GL_{n, \beta}$ is smooth commutative $O$-group scheme. So $G_\beta \subset GL_{n, \beta}$ is commutative, and so we have

$$G_\beta(\mathbb{F})^c = G_\beta(F) \subset GL_{n, \beta}(F) = GL_{n, \beta}(\mathbb{F})^c.$$ 

Then Proposition 4.4.5 says that the Schur multiplier $[c_{\beta, \rho}] \in H^2(G_\beta(F)^c, \mathbb{C}^\times)$ with $\rho \in \mathfrak{g}_\beta(F)^c$ is the image under the restriction mapping

$$\text{Res} : H^2(GL_{n, \beta}(F)^c, \mathbb{C}^\times) \to H^2(G_\beta(F)^c, \mathbb{C}^\times)$$

of the Schur multiplier $[c_{\beta, \rho}] \in H^2(GL_{n, \beta}(F)^c, \mathbb{C}^\times)$.

If $G \subset GL_n$ is a group symplectic similitude, a general orthogonal group with respect to a quadratic form of odd variables or an unitary group (see section 5), then Proposition 4.4.5 implies the following: $\beta \in \mathfrak{g}(O) \subset \mathfrak{gl}_n(O)$ is smoothly regular with respect to $G$ over $\mathbb{F}$ if and only if $\beta$ is smoothly regular with respect to $GL_n$ over $\mathbb{F}$. So Conjecture 4.4.4 implies the following conjectural proposition

**Proposition 4.4.5** If $G \subset GL_n$ is a group of symplectic similitudes, a general orthogonal group with respect to a quadratic form of odd variables or an unitary group, then the Schur multiplier $[c_{\beta, \rho}] \in H^2(G(F)^c, \mathbb{C}^\times)$ is trivial for all $\rho \in \mathfrak{g}_\beta(F)^c$ if $\beta \in \mathfrak{g}(O)$ is regular with respect to $G$ over $\mathbb{F}$ and the characteristic of $\mathbb{F}$ is big enough.

These arguments provide good reasons for the conjectural statement of subsection 2.3.

5 Examples

5.1 $G = GL_n$ is a connected smooth reductive $O$-group scheme which satisfies the conditions I), II) and III) of the subsection 2.1.

**Proposition 5.1.1** If $\beta \equiv \lambda \cdot 1_n + \beta_0 (\text{mod } p^r)$ for $\beta, \beta_0 \in \mathfrak{gl}_n(O)$ with $\lambda \in O$, then there exists a group homomorphism $\mu : O_r^c \to \mathbb{C}^\times$ such that

$$\pi \mapsto (\mu \circ \text{det}) \otimes \pi$$

gives a bijection of $\text{Irr}(G(O_r) \mid \psi_{\beta_0})$ onto $\text{Irr}(G(O_r) \mid \psi_{\beta})$.

**Proof** Let $\mu : O_r^c \to \mathbb{C}^\times$ be a group homomorphism such that $\mu((1 + \varpi^l)_r x) = \tau(\varpi^{-l} \lambda x)$ for all $x \in O$. Then we have

$$\psi_\beta(h) = \mu \circ \text{det}(h) \cdot \psi_{\beta_0}(h)$$

for all $h = 1_n + \varpi^l X (\text{mod } p^r) \in K_r(O_r) \ (X \in M_n(O))$, because

$$\text{det}(1_n + \varpi^l X) \equiv 1 + \varpi^l \text{tr}(X) \ (\text{mod } p^r).$$

A similar argument shows
Proposition 5.1.2 If $\beta \in \mathfrak{gl}_n(O)$ is central modulo $p$, then there exists a group homomorphism $\mu : O_r^\times \rightarrow \mathbb{C}^\times$ such that, for any $\pi \in \text{Irr}(G(O_r) \mid \psi_\beta)$, there exists a $\sigma \in \text{Irr}(G(O_{r-1})$ such that $\pi = (\mu \circ \text{det}) \otimes (\sigma \circ \text{proj})$ where proj : $G(O_r) \rightarrow G(O_{r-1})$ is the canonical surjection.

[Proof] Put $\beta \equiv \lambda \cdot 1_n (\text{mod } p)$ with $\lambda \in O$. Let $\mu : O_r^\times \rightarrow \mathbb{C}^\times$ be a group homomorphism such that $\mu((1 + \overline{\tau})x) = \overline{\tau}^{-r} \lambda x$ for all $x \in O$. Then, for any $\pi \in \text{Irr}(G(O_r) \mid \psi_\beta)$, the representation $(\mu \circ \text{det})^{-1} \otimes \pi$ of $G(O_r)$ is trivial on $K_{r-1}(O_r)$, in other word, it factors through $G(O_{r-1})$.

If $\beta \in \mathfrak{gl}_n(O)$ is smoothly regular with respect to $GL_n$, then $G_\beta$ is a smooth commutative $O$-group scheme (see Remark 4.3.2). If further $\beta \in \mathfrak{gl}_n(F)$ is semisimple, then $c_{\beta, \rho}(g, h) = 1$ for all $g, h \in G_\beta(F)$ and $\rho \in \mathfrak{gl}_n(F)^\times$ (see Remark 4.3.3). In this case Theorem 2.4.1 gives

Proposition 5.1.3 There exists a bijection $\theta \mapsto \text{Ind}_{G(O_{r-1})}^{G(O_r)} \sigma_{\beta, \theta}$ of the set

$$\{ \theta \in G_\beta(O_r) \mid \text{s.t. } \theta = \psi_\beta \text{ on } G_\beta(O_r) \cap K_1(O_r) \}$$

onto $\text{Irr}(G(O_r) \mid \psi_\beta)$.

There are $\frac{1}{n} \sum_{d|n} \mu(nd) \cdot q^d$ irreducible polynomials in $F[t]$ of degree $n$ ($\mu$ is the Möbius function). Take a polynomial

$$p(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n \in O[t]$$

such that $p(t)(\text{mod } p) \in F[t]$ is irreducible. Then

$$[p(t)] = \{ \beta \in \mathfrak{gl}_n(O) \mid \chi_\beta(t) \equiv p(t) \pmod{p} \}$$

is stable under the adjoint action of $GL_n(O)$. Take a $\beta \in [p(t)]$. Then $K = F[\beta]$ is an unramified field extension of $F$ of degree $n$, which is the splitting field of $p(t)$ over $F$, and $O_K = O_{F[\beta]}$ is the integer ring of $K$ with the maximal ideal $\mathfrak{p}_K = \overline{x} O_K$. If we identify $G_\beta(O_r)$ with $(O_K/\mathfrak{p}_K)^\times$, then we have

$$G_\beta(O_r) \cap K_{r}(O_r) = (1 + \mathfrak{p}_K^r)/(1 + \mathfrak{p}_K^\times) \hookrightarrow (O_K/\mathfrak{p}_K)^\times$$

and $\psi_\beta((1 + \overline{x})x) = T_{K/F}(\overline{x} \beta) \forall x \in O_K$. So Proposition 5.1.3 gives

Proposition 5.1.4 There exists a bijection $\theta \mapsto \text{Ind}_{G(O_{r-1})}^{G(O_r)} \sigma_{\beta, \theta}$ of the set

$$\{ \theta : O_K^\times \rightarrow (O_K/\mathfrak{p}_K)^\times \rightarrow \mathbb{C}^\times : \text{group homomorphism such that } \theta(1 + \overline{x})x = T_{K/F}(\overline{x} \beta) \forall x \in O_K \}$$

onto $\text{Irr}(G(O_r) \mid \psi_\beta)$.

This kind of parametrization of $\text{Irr}(G(O_r) \mid \psi_\beta)$ is given first by Shintani [15] and then Gérardin [6].
5.2 Let \( G = GSp_{2n} \) be the \( O \)-group scheme such that

\[
GSp_{2n}(L) = \{ g \in GL_{2n}(L) \mid gJ_n'g = \nu(g) \cdot J_n \text{ with } \nu(g) \in L^\times \}
\]

\((J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix})\) for all commutative \( O \)-algebra \( L \). Then \( G \) is a connected smooth reductive \( O \)-group scheme. The Lie algebra \( gsp_{2n} \) of \( G \) is an affine \( O \)-subscheme of \( gl_{2n} \) such that

\[
gsp_{2n}(L) = \{ X \in gl_{2n}(L) \mid XJ_n + J_n'X = \nu(X) \cdot J_n \text{ with } \nu(X) \in L \}
\]

for all commutative \( O \)-algebra \( L \). Assume that the characteristic of \( F \) does not divide \( n \). Then \( G \) satisfies the conditions I), II) and III) of the subsection \[5.1.2\]. The same arguments as Propositions \[5.1.1\] and \[5.1.2\] show

**Proposition 5.2.1** If \( \beta \equiv \lambda 1_n + \beta_0 (\mod p^r) \) for \( \beta, \beta_0 \in gsp_{2n}(O) \) with \( \lambda \in O \), then there exists a group homomorphism \( \mu : O^\times \to \mathbb{C}^\times \) such that

\[
\pi \mapsto (\mu \circ \det) \otimes \pi
\]

gives a bijection of \( \text{Irr}(G(\mathcal{O}_r) | \psi_{\beta_0}) \) onto \( \text{Irr}(G(\mathcal{O}_r) | \psi_{\beta}) \).

and

**Proposition 5.2.2** If \( \beta \in gsp_{2n}(O) \) is central modulo \( p \), then there exists a group homomorphism \( \mu : O^\times \to \mathbb{C}^\times \) such that, for any \( \pi \in \text{Irr}(G(\mathcal{O}_r) | \psi_{\beta}) \), there exists a \( \sigma \in \text{Irr}(G(\mathcal{O}_{r-1})) \) such that \( \pi = (\mu \circ \det) \otimes (\sigma \circ \text{proj}) \) where \( \text{proj} : G(\mathcal{O}_r) \to G(\mathcal{O}_{r-1}) \) is the canonical surjection.

If \( \beta \in gsp_{2n}(O) \) is smoothly regular with respect to \( G \) over \( F \) and over \( F \), then \( G \) is a smooth commutative \( O \)-group scheme. If further \( \overline{\beta} \in gsp_{2n}(\overline{F}) \) is semisimple, then \( c_{\beta,\rho}(g, h) = 1 \) for all \( g, h \in G_{\beta}(\overline{F}), G_{\beta}(\overline{F})/G_{\beta}(\overline{F}) \) and \( \rho \in gsp_{2n,\beta}(\overline{F}) \) (see Remark \[13.3\]). In this case Theorem \[24.1\] gives

**Proposition 5.2.3** There exists a bijection \( \theta \mapsto \text{Ind}^{G(O_r)}_{G(O_{r-1},\beta)} \sigma_{\beta,\theta} \) of the set

\[
\{ \theta \in G_{\beta}(O_r) \mid \text{s.t. } \theta = \psi_{\beta} \text{ on } G_{\beta}(O_r) \cap K_l(O_r) \}
\]

onto \( \text{Irr}(G(\mathcal{O}_r) | \psi_{\beta}) \).

Let us consider a \( \beta \in gsp_{2n}(O) \subset gl_{2n}(O) \) such that the characteristic polynomial \( \chi_{\beta}(t) \in O[t] \) is irreducible modulo \( p \). In this case \( F[\beta] \) is an unramified field extension of \( F \) of degree 2n such that \( O[\beta] \) is the integer ring of \( F[\beta] \) so that \( \beta \in O[\beta]^\times \).

Now fix an unramified field extension \( K \) of \( F \) of degree 2n. Let \( O_K \) be the integer ring of \( K \) with the maximal ideal \( p_K = \varpi O_K \). The finite field \( \overline{F} = O/p \) is identified with a subfield of \( K = O_K/p_K \). Then \( K/F \) is a Galois extension whose Galois group \( Gal(K/F) \) is isomorphic to \( \text{Gal}(\mathbb{F}/\overline{F}) \) by the mapping which sends \( \sigma \in Gal(K/F) \) to \( \overline{\sigma} \in Gal(\mathbb{F}/\overline{F}) \) where \( (x \mod p_K))^{\overline{\sigma}} = x^\sigma \mod p_K \).

Let \( \tau \in Gal(K/F) \) be the unique element of order 2. Fix an \( \varepsilon \in O_K^\times \) such that \( \varepsilon^2 + 1 = 0 \). Then the \( F \)-vector space \( K \) is a symplectic \( F \)-space with respect to the non-degenerate alternating form

\[
D_\varepsilon(x, y) = T_{K/F}(\varepsilon xy^\tau) \quad (x, y \in K)
\]
on $K$ with a polarization $K = K_+ \oplus K_+$ where

$$K_{\pm} = \{ x \in K \mid x^\pm = \pm x \},$$

and $F \subset K_+ \subset K$ is a subfield such that $(K_+:F) = n$. Let \( \{v_1, \ldots, v_n\} \) be an $O$-basis of $O_{K_+} = O_K \cap K_+$ the integer ring of $K_+$. Since $K/F$ is unramified, we have a $O$-basis of $O_{K_+} \{v_1^*, \ldots, v_n^*\}$ such that $T_{K/F}(v_i v_j^*) = \delta_{ij}$. Put $u_i = \varepsilon^{-1} v_i^* \subset K_+$. Then $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a symplectic $F$-basis of $K = K_+ \oplus K_+$ and an $O$-basis of $O_K = \varepsilon^{-1} O_{K_+} \oplus O_{K_+}$.

Now our $O$-group scheme $G = GSp_{2n}$ and its Lie algebra $\mathfrak{gsp}_{2n}$ are defined by

$$GSp_{2n}(L) = \left\{ g \in GL_n(O_K \otimes L) \mid D_{x}(gx, gy) = \nu(g)D_{x}(x, y) \quad \text{for } \forall x, y \in O_K \otimes L \text{ with } \nu(g) \in L^\times \right\}$$

and by

$$\mathfrak{gsp}_{2n}(L) = \left\{ X \in \text{End}_L(O_K \otimes L) \mid D_{x}(X, y) + D_X(x, Xy) = \nu(X)D_{x}(x, y) \quad \text{for } \forall x, y \in O_K \otimes L \text{ with } \nu(X) \in L \right\}$$

for all commutative $O$-algebra $L$.

Take a $\beta \in K_+ \cap O_K$ such that $O_K = O[\beta]$. Identify $\beta \in K$ with the element $x \mapsto x\beta$ of $GSp_{2n}(O) \subset gl_{2n}(O)$. Then the characteristic polynomial $\chi_\beta(t) \in O[t]$ of $\beta \in gl_{2n}(O)$ is irreducible modulo $p$. We have

$$G_\beta(O_\nu) = G(O_\nu) \cap (O_K/\mathfrak{p}_K^\times) = \{ \gamma \pmod{\mathfrak{p}_K^\times} \mid \gamma \in U_{K/F} \}$$

where

$$U_{K/F} = \{ \gamma \in O_K^\times \mid \gamma \cdot \gamma^\tau \in O_K^\times \}$$

is a subgroup of $O_K^\times$. In this case we have $\psi_\beta(h) = \tau(\varpi^{-\ell} T_{K/F}(\beta x))$ for all

$$h = 1 + \varpi^j x \pmod{\mathfrak{p}_K^\times} \in K_+(O_\nu) \cap G_\beta(O_\nu) \subset (O_K/\mathfrak{p}_K)^\times.$$

Then Proposition 5.2.3 gives

**Proposition 5.2.4** There exists a bijection $\theta : \text{Ind}^{G(O_\nu)}_{G_\beta(O_\nu)} \gamma_{\beta, \theta}$ of the set

$$\left\{ \theta : U_{K/F} \rightarrow (O_K/\mathfrak{p}_K^\times) \rightarrow \mathbb{C}^\times : \text{group homomorphism} \right\}$$

such that $\theta(\gamma) = \tau(\varpi^{-\ell} T_{K/F}(\varpi^{-\ell} \beta x)) \forall \gamma = 1 + \varpi^j x \in U_{K/F}$ onto $\text{Irr}(G(O_\nu) \mid \psi_\beta)$.

**5.3** Take a $S \in M_n(O)$ such that $^t S = S$ and $\det S \in O^\times$. Let $G = GO(S)$ be the $O$-group scheme such that

$$G(L) = \{ g \in GL_n(L) \mid gSg^t = \nu(g) \cdot S \text{ with } \nu(g) \in L^\times \}$$

for all commutative $O$-algebra $L$. Then $G$ is a connected smooth reductive $O$-group scheme. The Lie algebra $\mathfrak{g} = \mathfrak{g}(S)$ of $G$ is an affine $O$-subscheme of $gl_n$ such that

$$\mathfrak{g}(L) = \{ X \in gl_n(L) \mid XS + S^t X = \nu(X) \cdot S \text{ with } \nu(X) \in L \}$$

for all commutative $O$-algebra $L$. Assume that the characteristic of $F$ does not divide $n$. Then $G$ satisfies the conditions I), II) and III) of the subsection 2.1. The same arguments as Propositions 5.1.1 and 5.1.2 show
Proposition 5.3.1 If $\beta \equiv \lambda \cdot 1_n + \beta_0 (\mod \frak{p}^\ell)$ for $\beta, \beta_0 \in \frak{g}(O)$ with $\lambda \in O$, then there exists a group homomorphism $\mu : O^\times \rightarrow \mathbb{C}^\times$ such that

$$\pi \mapsto (\mu \circ \det) \circ \pi$$

gives a bijection of $\text{Irr}(G(O_r) \mid \psi_{\beta_0})$ onto $\text{Irr}(G(O_r) \mid \psi_{\beta})$.

and

Proposition 5.3.2 If $\beta \in \frak{g}(O)$ is central modulo $\frak{p}$, then there exists a group homomorphism $\mu : O^\times \rightarrow \mathbb{C}^\times$ such that, for any $\pi \in \text{Irr}(G(O_r) \mid \psi_{\beta})$, there exists a $\sigma \in \text{Irr}(G(O_{r-1}))$ such that $\pi = (\mu \circ \det) \circ (\sigma \circ \text{proj})$ where $\text{proj} : G(O_r) \rightarrow G(O_{r-1})$ is the canonical surjection.

If $\beta_\alpha \in \frak{g}(L)$ ($L = F$ or $L = \mathbb{F}$) is semisimple, then the centralizer $Z_{G \otimes \alpha L}(\beta_\alpha)$ is connected and its center is also connected. Hence if $\beta \in \frak{g}(O)$ is smoothly regular over $F$ and over $\mathbb{F}$, then $G_{\beta}$ is a smooth commutative $O$-group scheme. If further $\beta \in \frak{g}(\mathbb{F})$ is semisimple, then $c_{\beta, \rho}(g, h) = 1$ for all $g, h \in G_{\beta}(\mathbb{F})$ and all $\rho \in \frak{g}(\mathbb{F})^\vee$ (see Remark 4.3.3). In this case Theorem 2.4.1 gives

Proposition 5.3.3 There exists a bijection $\theta \mapsto \text{Ind}_{G(O_{r-1})}^{G(O_r)}(\theta)$ of the set

$$\{\theta \in G_{\beta}(O_r) \mid \theta \in G_{\beta}(O_r) \cap K_{i}(O_r)\}$$

onto $\text{Irr}(G(O_r) \mid \psi_{\beta})$.

Suppose $n = 2m$ is even. Let $K$ be an unramified field extension of $F$ of degree $n$ with the integer ring $O_K$. Then $\frak{p}_K = \omega O_K$ is the maximal ideal of $O_K$, and $\mathbb{F} = O/\frak{p}$ is canonically identified with a subfield of $K = O_K/\frak{p}_K$. Let $\tau \in \text{Gal}(K/F)$ be the unique element of order 2. Fix an $\varepsilon \in O_K^\times$ such that $\varepsilon^2 = \varepsilon$ and put

$$S_\tau(x, y) = T_{K/F}(\varepsilon x y^\tau) \quad (x, y \in K).$$

Then $S_\tau$ is a regular $F$-quadratic form on $K$. Fix a $O$-basis $\{u_1, \ldots, u_n\}$ of $O_K$ and put $B = (u_i \sigma_j)_{1 \leq i, j \leq n} \in M_n(O_K)$ where $\text{Gal}(K/F) = \{\sigma_1, \ldots, \sigma_n\}$. Then we have

$$(S_\tau(u_i, u_j))_{1 \leq i, j \leq n} = B \left[ \begin{array}{ccc} \varepsilon^{\sigma_1} & \cdots & \varepsilon^{\sigma_n} \end{array} \right] B^\tau$$

so that the discriminant of the quadratic form $S_\tau$ is

$$\det(S_\tau(u_i, u_j))_{1 \leq i, j \leq n} = (-1)^m (\det B)^2 N_{K/F}(\varepsilon).$$

Note that $(-1)^m (\det B)^2 = -\det B$ for a generator $\sigma$ of $\text{Gal}(K/F)$. Since $K/F$ is unramified extension, its discriminant $D(K/F) = (\det B^2)$ is trivial, in other word $\det(S_\tau(u_i, u_j))_{1 \leq i, j \leq n} \in O^\times$. So the $O$-group scheme $G = GO(S_\tau)$ and its Lie algebra $\frak{g} = \frak{go}(S_\tau)$ is defined by

$$G(L) = \left\{ g \in GL_L(O_K \otimes L) \mid S_\tau(xg, yg) = \nu(g)S_\tau(x, y) \right\}$$

for $x, y \in O_K \otimes GL$ with $\nu(g) \in L^\times$.
and by

$$g(L) = \left\{ X \in \text{End}_L(O_K \otimes L) \mid S_z(x, y) + S_z(x, yX) = \nu(X)S_z(x, y) \quad \text{for all } x, y \in O_K \otimes O \text{ with } \nu(X) \in L \right\}$$

for all commutative $O$-algebra $L$. Note that $\text{End}_F(K)$ acts on $K$ from the right side.

Take a $\beta \in O_K$ such that $O_K = O[\beta]$ and $\beta^r = -\beta$. Identify $\beta \in K$ with the element $x \mapsto x\beta$ of $g(O) \subset \text{End}_O(O_K) = g\mathfrak{l}_{2n}(O)$. Then the characteristic polynomial $\chi_\beta(t) \in O[t]$ of $\beta \in g\mathfrak{l}_{2n}(O)$ is irreducible modulo $p$. We have

$$G_\beta(O_r) = G(O_r) \cap (O_K/p^r) = \{ \gamma \pmod{p_K} \mid \gamma \in U_{K/F} \}$$

where

$$U_{K/F} = \{ \gamma \in O_K^* \mid \gamma \cdot \gamma^r \in O^* \}$$

is a subgroup of $O_K^*$. In this case we have $\psi_\beta(h) = \tau(\varpi^{-r}T_{K/F}(\beta x))$ for all $h = 1 + \varpi^r x \pmod{p^r_K} \in K_1(\theta) \cap G_\beta(\theta) \subset (O_K/p^r_K)^x$.

Then Proposition 5.3.3 gives

**Proposition 5.3.4** There exists a bijection $\theta \mapsto \text{Ind}^{G(O_r)}_{\theta(\beta)}$ of the set

$$\left\{ \theta : U_{K/F} \to (O_K/p^r_K)^x \to \mathbb{C}^x : \text{group homomorphism} \right\}$$

such that $\theta(\gamma) = \tau(\varpi^{-r}T_{K/F}(\varpi^{-r}\beta x))$ for all $\gamma = 1 + \varpi^r x \in U_{K/F}$

onto $\text{Irr}(G(O_r) \mid \psi_\beta)$.

Let us consider the case of $n = 2m + 1$ being odd. In this case det $\beta = 0$ for all $\beta \in g(O)$ so that the characteristic polynomial $\chi_\beta(t) \in O[t]$ of $\beta$ is of the form $\chi_\beta(t) = t \cdot p(t)$ with $p(t) \in O[t]$. We will give examples in which $p(t)$ is irreducible modulo $p$.

Let us use the notations used in the case of even $n$ and fix an $\eta \in O^*$. Then $F$-quadratic form on $V = K \times F$ is defined by

$$S_{z, \eta}((x, s), (y, t)) = S_z(x, y) + \eta \cdot st.$$ 

Then the $O$-group scheme $G = GO(S_{z, \eta})$ and its Lie algebra $g = gO(S_{z, \eta})$ is defined by

$$G(L) = \left\{ g \in GL((O_K \times O) \otimes L) \mid S_{z, \eta}(ug, v) = \nu(g)S_{z, \eta}(u, v) \quad \text{for all } u, v \in (O_K \times O) \otimes L \text{ with } \nu(g) \in L^x \right\}$$

and

$$g(L) = \left\{ X \in \text{End}_L((O_K \times O) \otimes L) \mid S_{z, \eta}(uX, v) + S_{z, \eta}(u, vX) = \nu(X)S_{z, \eta}(u, v) \quad \text{for all } u, v \in (O_K \times O) \otimes L \text{ with } \nu(X) \in L \right\}$$

for all commutative $O$-algebra $L$. An element $X \in \text{End}_L((O_K \times O) \otimes L)$ is denoted by

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{with} \quad A \in \text{End}_L(O_K \otimes O), \quad B \in \text{Hom}_L(O_K \otimes O, L), \quad C \in \text{Hom}_L(L, O_K \otimes O), \quad D \in \text{End}_L(L) = L.$$
Take a \( \beta_0 \in O_K^\times \) such that \( O_K = O[\beta_0] \) and \( \beta_0^2 = -\beta_0 \). Then the element 
\( (x,s) \mapsto (x\beta_0,0) \) of \( g(O) \) is denoted by 

\[
\beta = \begin{bmatrix} \beta_0 & 0 \\ 0 & 0 \end{bmatrix} \in g(O) \subseteq \text{End}_O(O_K \times O),
\]

where \( \beta_0 \in O_K^\times \) is identified with \( x \mapsto x\beta_0 \) an element of \( \text{End}_O(O_K) \). Now the characteristic polynomial of \( \beta \in g(O) \subset \text{End}_O(O_K \times O) \) is \( \chi_\beta(t) = t \cdot \chi_{\beta_0}(t) \) where \( \chi_{\beta_0}(t) \in \mathcal{O}[t] \) is the characteristic polynomial of \( \beta_0 \in \text{End}_O(O_K) \) which is irreducible modulo \( p \). In particular \( \beta \pmod{p} \in g(\mathbb{F}) \) is semisimple. We have 

\[
G_\beta(O_r) = G(O_r) \cap O_r[\beta \pmod{p^\tau}]^X = \left\{ \begin{bmatrix} \gamma \pmod{p^\tau_r} & 0 \\ 0 & a \pmod{p^\tau} \end{bmatrix} \mid \gamma \in U_{K/F}, a \in O^X \right\},
\]

where \( \gamma \pmod{p^\tau_r} \) is identified with \( \chi_\beta \). In particular \( \beta \pmod{p} \in g(\mathbb{F}) \) is semisimple. We have 

\[
G_\beta(O_r) = G(O_r) \cap O_r[\beta \pmod{p^\tau}]^X = \left\{ \begin{bmatrix} \gamma \pmod{p^\tau_r} & 0 \\ 0 & a \pmod{p^\tau} \end{bmatrix} \mid \gamma \in U_{K/F}, a \in O^X \right\}.
\]

In this case \( \psi_\beta(h) = \tau \left( \mathfrak{w}^{-\tau} T_{K/F}(\beta_0 x) \right) \) for all 

\[
h = \begin{bmatrix} 1 + \mathfrak{w}^r x \pmod{p^\tau_r} & 0 \\ 0 & 1 + \mathfrak{w}^r y \pmod{p^\tau} \end{bmatrix} \in K_i(O_r) \cap G_\beta(O_r).
\]

The Proposition 5.3.3 gives 

**Proposition 5.3.5** There exists a bijection \( \theta \mapsto \text{Ind}_{G(O_r, \beta)}^{G(O)} \sigma_{\beta, \theta} \) of the set 

\[
\left\{ \begin{array}{c} \theta : \text{U}_{K/F} \times \text{F} O^X \to (O_K/p^\tau_r)X O_r^X \to \mathbb{C}^X \\
\text{group homomorphism such that} \\
\theta(\gamma, a) = \tau(\mathfrak{w}^{-\tau} T_{K/F}(\mathfrak{w}^{-\tau} \beta x)) \\
\text{for } \forall (\gamma, a) = (1 + \mathfrak{w}^r x, 1 + \mathfrak{w}^r y) \in \text{U}_{K/F} \times \text{F} O^X \\
\end{array} \right\}
\]

onto \( \text{Irr}(G(O_r)) \mid \psi_\beta \). Here 

\[
\text{U}_{K/F \times \text{F} O^X} = \{ (\gamma, a) \in \text{U}_{K/F} \times O^X \mid \gamma \gamma^\tau \equiv a^2 \pmod{p^\tau} \}
\]

is a subgroup of \( \text{U}_{K/F} \times O^X \).

**5.4** Let \( K/F \) be the unramified quadratic field extension and put \( \text{Gal}(K/F) = \langle \tau \rangle \). Let us denote by \( O_K \) the integer ring of \( K \) and \( p_K = \mathcal{O}O_K \) the maximal ideal of \( O_K \). Fix a \( S \in M_n(O_K) \) such that \( \mathfrak{t} S^r = S \) and \( \det S \in O_K^\times \). Then the unitary group \( G = U(S) \) associated with the Hermitian form \( S \) is the \( O \)-group scheme defined by 

\[
G(A) = \{ g \in GL_n(O_K \otimes_O A) \mid g S^r g^\tau \otimes 1 = S \}
\]

for all commutative \( O \)-algebra \( A \). Its Lie algebra \( g(\mathbb{F}) = u(S) \) is an affine \( O \)-scheme defined by 

\[
g(A) = \{ X \in gl_n(O_K \otimes_O A) \mid XS + S^\mathfrak{t} X^\tau \otimes 1 = 0 \}.
\]

We can take an \( \epsilon \in O^X \) such that \( O_K = O[\sqrt{\epsilon}] \). If there exists an \( \eta \in A \) such that \( \epsilon 1_A = \eta^2 \), then we can identify \( O_K \otimes_O A \) with \( A \oplus A \) such a way
that \((x, y)^{\otimes 1} = (y, x)\). Then we have identifications \(G(A) = GL_n(A)\) and \(g(A) = gl_n(A)\). If we put
\[
K = \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \right| x, y \in F \right\}, \quad O_K = \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \right| x, y \in O \right\}
\]
then
\[
M_n(0_K) = \{ X \in M_{2n}(O_F) \mid I_n(\varepsilon)X = XI_n(\varepsilon) \}
\]
where
\[
I_n(\varepsilon) = \begin{bmatrix} I(\varepsilon) & \cdots & I(\varepsilon) \\ & \ddots & \\ & & I(\varepsilon) \end{bmatrix} \in M_{2n}(O_F) \text{ with } I(\varepsilon) = \begin{bmatrix} 0 & \varepsilon \\ 1 & 0 \end{bmatrix}.
\]
Then we have
\[
G(A) = \{ g \in GL_{2n}(A) \mid I_n(\varepsilon)g = gI_n(\varepsilon), gS^Tg^\tau = S \}
\]
and
\[
g(A) = \{ X \in gl_{2n}(A) \mid I_n(\varepsilon)X = XI_n(\varepsilon), XS + S^TX^\tau = 0 \}
\]
for all commutative \(O\)-algebra \(A\). In this way \(G\) is a closed \(O\)-group subscheme of \(GL_{2n}\). Then \(G\) satisfies the conditions I), II) and III) of the subsection \([2.1]\).

The center of \(g(O)\) is
\[
Z(g(O)) = \{ \lambda \cdot 1_n \in GL_n(O_K) \mid \lambda \in O_K \text{ s.t. } \lambda^\tau + \lambda = 0 \}.
\]
The similar arguments as Propositions \([5.1.1]\) and \([5.1.2]\) show

**Proposition 5.4.1** If \(\beta \equiv \lambda \cdot 1_n + \beta_0 \pmod{p^r}\) for \(\beta, \beta_0 \in g(O)\) with \(\lambda \in O_K\), then there exists a group homomorphism \(\mu : O_{K,r}^\times \to \mathbb{C}^\times\) such that
\[
\pi \mapsto (\mu \circ \det) \otimes \pi
\]
gives a bijection of \(\text{Irr}(G(O_r) \mid \psi_{\beta_0})\) onto \(\text{Irr}(G(O_r) \mid \psi_{\beta})\).

and

**Proposition 5.4.2** If \(\beta \in g(O)\) is central modulo \(p\), then there exists a group homomorphism \(\mu : O_{K,r}^\times \to \mathbb{C}^\times\) such that, for any \(\pi \in \text{Irr}(G(O_r) \mid \psi_{\beta})\), there exists a \(\sigma \in \text{Irr}(G(O_{r-1}))\) such that \(\pi = (\mu \circ \det) \otimes (\sigma \circ \text{proj})\) where \(\text{proj} : G(O_r) \to G(O_{r-1})\) is the canonical surjection.

If \(\beta_s \in g(L)\) (\(L = F\) or \(L = \mathbb{F}\)) is semisimple, then the centralizer \(Z_{G_{G_{O_r}}}(-\beta_s)\) is connected and its center is also connected. Hence if \(\beta \in g(O)\) is smoothly regular over \(F\) and over \(\mathbb{F}\), then \(G_{\beta}\) is a smooth commutative \(O\)-group scheme. If further \(\beta_{\mathbb{F}} \in g(\mathbb{F})\) is semisimple, then \(c_{\beta_{\mathbb{F}}}(g, h) = 1\) for all \(g, h \in G_{\beta}(\mathbb{F})\) and all \(\rho \in g_{\beta}(\mathbb{F})\) (see Remark \([1.3.3]\)). In this case Theorem \([2.4.1]\) gives

**Proposition 5.4.3** There exists a bijection \(\theta \mapsto \text{Ind}_{G(O_r,\beta)}^{G(O_r)}(\theta, \sigma_{\beta, \theta})\) of the set
\[
\{ \theta \in G_{\beta}(O_r) \mid \text{s.t. } \theta = \psi_{\beta} \text{ on } G_{\beta}(O_r) \cap K_t(O_r) \}
\]
ono to \(\text{Irr}(G(O_r) \mid \psi_{\beta})\).
Assume that \( n \) is odd. Let \( L/F \) be the unramified field extension of degree 2\( n \) so that \( K \) is a subfield of \( L \). There exists a \( \tau \in \text{Gal}(L/F) \) of order 2 and \( \text{Gal}(K/F) = \langle \tau|_K \rangle \). Fix an \( \varepsilon \in O_{K}^\times \) such that \( \varepsilon^T = \varepsilon \). Then

\[
S_\varepsilon(x, y) = T_{L/K}(\varepsilon xy^T) \quad (x, y \in L)
\]
is a non-degenerate Hermitian form over \( K \) on \( L \). Let \( \{u_1, \ldots, u_n\} \) be a \( O_K \)-basis of \( O_L \). Then we have

\[
(S_\varepsilon(u_i, u_j))_{1 \leq i, j \leq n} = B_{\varepsilon^{\sigma_1}} \cdots_{\varepsilon^{\sigma_n}} B^T
\]
where \( B = (u_i^T u_j)_{1 \leq i, j \leq n} \in M_n(O_L) \) with \( \text{Gal}(L/K) = \{\sigma_1, \ldots, \sigma_n\} \). So the discriminant of the Hermitian form \( S_\varepsilon \) is

\[
\det(S_\varepsilon(u_i, u_j))_{1 \leq i, j \leq n} = N_{L/K}(\varepsilon) \det B(\det B)^T \in O_{K}^\times
\]
because \( L/K \) is unramified. Then the \( O \)-group scheme \( G = U(S_\varepsilon) \) and its Lie algebra \( g = \text{u}(S_\varepsilon) \) are defined by

\[
G(A) = \{ g \in GL_A(O_L \otimes A) \mid S_\varepsilon(gx, gy) = S_\varepsilon(x, y) \quad \text{for } \forall x, y \in L \}
\]
and

\[
g(A) = \{ X \in \text{End}_A(O_L \otimes A) \mid S_\varepsilon(Xx, y) + S_\varepsilon(x, Xy) = 0 \quad \text{for } \forall x, y \in L \}
\]
for all commutative \( O \)-algebra \( A \).

Take a \( \beta \in O_L \) such that \( \beta^2 + \beta = 0 \) and \( O_L = O[\beta] \). Identify \( \beta \in L \) with the element \( x \mapsto x^2 + x \) of \( g(O) \subset \text{End}_O(O_L) = g_{2n}(O) \). Then the characteristic polynomial \( \chi_\beta(t) \) in \( O[t] \) is irreducible modulo \( p \). We have

\[
G_\beta(O_r) = G(O_r) \cap (O_L / p_L)^\times = \{ \gamma \pmod{p_L} \mid \gamma \in U_{L/F} \}
\]
where

\[
U_{L/F} = \{ \gamma \in O_{L}^\times \mid \gamma T_{L/F} \in O_{L}^\times \}.
\]
In this case \( \psi_\beta(h) = \tau(\varpi^{-r} T_{L/F}(\beta x)) \) for all

\[
h = 1 + \varpi^j x \pmod{p_L} \in K_1(O_r) \cap G_\beta(O_r) \subset (O_L / p_L)^\times.
\]
Then Proposition 5.4.3 gives

**Proposition 5.4.4** There exists a bijection \( \theta \mapsto \text{Ind}_{G(O_r, \beta) \sigma, \beta}^{G(O_r)} \) of the set

\[
\{ \theta : U_{L/F} \to (O_L / p_L)^\times \to \mathbb{C}^\times : \text{group homomorphism} \}
\]
such that \( \theta(\gamma) = \tau(\varpi^{-r} T_{L/F}(\varpi^{-r} \beta x)) \quad \forall \gamma = 1 + \varpi^j x \in U_{K/F} \)

onto \( \text{Irr}(G(O_r) \mid \psi_\beta) \).
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