Zero Lag Synchronization of Mutually Coupled Lasers in the Presence of Delays

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Abstract

We consider a line of three mutually coupled lasers with time delays and study chaotic synchronization of the outer lasers. Two different systems are presented: optoelectronically coupled semiconductor lasers and optically coupled fiber lasers. While the dynamics of the two systems are very different, robust synchronization of end lasers is obtained in both cases over a range of parameters. Here, we present analysis and numerical simulation to explain some of the observed synchronization phenomena. First, we introduce the system of three coupled semiconductor lasers and discuss the onset of oscillations that occurs via a bifurcation as the coupling strength increases. Next, we analyze the synchronization of the end lasers by examining the dynamics transverse to synchronized state. We prove that chaotic synchronization of the outer semiconductor lasers will occur for sufficiently long delays, and we make a comparison to generalized synchronization in driven dissipative systems. It is shown that the stability of synchronous state (as indicated by negative Lyapunov exponents transverse to the synchronization manifold) depends on the internal dissipation of the outer lasers. We next present numerical simulations for three coupled fiber lasers, highlighting some of the differences between the semiconductor and fiber laser systems. Due to the large number of coupled modes in fiber lasers, this is a good system for investigating spatiotemporal chaos. Stochastic noise is included in the fiber laser model, and synchrony of the outer lasers is observed even at very small coupling strengths.

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INTRODUCTION

When two or more systems are coupled, their interaction often leads to correlations in the dynamics. If the dynamics of these coupled systems are identical with respect to some measure, the correlated motion is considered synchronized. Synchronization has been studied since the time of Christian Huygens, and there now exist several reviews on the dynamics of synchrony in the literature, such as Refs. [2, 9, 22, 32]. It is now clear that synchronization appears in a wide range of applications from many fields of science, such as physics, engineering, biology and chemistry, as well as in various fields of social behavior.

In general, synchronization between two interacting systems may be quantified by examining and comparing the output time series from each dynamical system. Typically, a measure of correlation between signals may be used to classify the type of synchronization. Complete synchronization occurs in coupled phase oscillators [32] as well as in coupled chaotic oscillators [10, 21]. In this case, amplitudes and phases are identical, and the peak of the cross correlation between the signals is at zero time shift. A recent theoretical example of complete synchrony in a closed ring of three one dimensional Ikeda oscillators with delayed diffusive coupling has been seen in [3], and complete synchrony has been shown in two mutually delay coupled lasers with self feedback for models of both semiconductor lasers and fiber ring lasers [29].

Many dynamical phenomena beyond complete synchronized systems have been unveiled for two coupled systems. If the amplitudes are uncorrelated but the phases are locked, or entrained, between the two signals, then the systems are said to be in phase synchrony [23]. One other type of synchronization, called generalized synchronization, deals solely with the unidirectional coupling between two oscillators of drive and response type [26]. In generalized synchronization, there exists a functional relationship between the drive and response, where there exists a function $F$ such that $X_2(t) = F(X_1(t))$, where $X_1$ and $X_2$ are the time series for the driver and the response systems, respectively. In another setting, this may also be thought of as a generalized entrainment in dynamics, whereby one system is entrained functionally to another. Many examples of entrained systems occur in singularly perturbed problems, and specifically in systems with multiple time scales where dimension reduction forces a functional relationship to occur between dependent and independent coordinates [28]. Generalized synchronization also plays a crucial role in mutually coupled systems with long
delays [16], where the coupling term can be viewed as a driving signal over the interval of the round-trip time. This idea will be further elaborated in the present chapter in connection to chaotic synchronization of semiconductor lasers with long coupling delays.

One area used to explore interesting synchronization phenomena in delay coupled systems is that of nonlinear optics. Coupled lasers, both semiconductor lasers as well as spatiotemporal fiber lasers, have been used to study delay coupled dynamics experimentally as well as theoretically. In delay coupled systems, a time lag between the oscillators is typically observed in the cross correlation, with a leading time series followed by a lagging one. Such lagged systems are defined to exhibit achronal synchronization. Existence of achronal synchronization in a mutually delay-coupled semiconductor laser system was shown experimentally [12], and studied theoretically [39] in a single-mode semiconductor laser model. Anticipatory synchronization occurs when a response in a driven system’s state is not replicated simultaneously but instead anticipates the dynamics of the driver [36, 37]. An example of anticipatory synchronization in the presence of delays can be found in coupled semiconductor lasers [19, 31], and has also been observed in the presence of stochastic effects in models of excitable media [7]. Interestingly, anticipated synchronization was also observed in unidirectionally coupled systems with no delays [36] and was recently studied numerically and experimentally in coupled Rossler oscillators [24, 25]. The zero lag state, corresponding to complete synchronization, is generally unstable in the delay coupled systems where achronal or anticipatory behavior is observed. Moreover, when achronal synchronization occurs, the situation may be further complicated by switching between leader and follower [20, 30]. We are particularly interested in systems where the zero lag state, also known as isochronal state, is stable. The stability of this isochronal solution is often due to a coupling geometry that leads to complete synchronization in a variety of systems [17].

The present chapter focuses on optoelectronically coupled semiconductor lasers and fiber ring lasers as important examples of systems exhibiting isochronal synchronization when coupled in a certain way and in the presence of delays. This isochronal synchronization can be compared to the above discussed achronal synchronization of mutually coupled lasers, for which the solutions are identical, but shifted in time with respect to each other [5, 6]. Previously, there has been some investigation of chaotic synchronization of the outer lasers mutually coupled in a line [33, 42] in the absence of delays. Delays, however significantly complicate the analysis by possibly introducing an infinite number of degrees of freedom into
the system. In general, semiconductor lasers are considered low dimensional, since they are modeled by differential equations with no spatial component due to very short cavity lengths. In contrast, fiber lasers have light propagating over long distances through a length of optical fiber, forming a large number (on the order of $10^3$) of longitudinal modes. Thus even a single fiber ring laser exhibits high dimensional spatio-temporal dynamics. The chaotic dynamics of fiber ring lasers have been studied in the past. Experiments exploring the polarization mode dynamics in a single fiber ring laser were set up and modeled using a delay differential system in [41]. Other experiments on synchronization with coupled fiber lasers have been reported in [13, 34, 38], and noise-induced generalized synchronization in fiber ring lasers has been reported in [8]. Modeling the ring laser yields a system of equations which consists of coupled difference and differential delay equations. To obtain better agreement with experiment, it was found that inclusion of spontaneous emission effects was necessary in the modeling, which resulted in a stochastic difference-differential system of equations [40], and this approach was followed in [30].

In this chapter we explore complete or isochronal synchronization in mutually delay coupled systems. The coupling architecture is three lasers coupled in a line. The layout of the chapter is as follows. The first two sections explore a system of three semiconductor lasers with significant delays. We first introduce the model and show an instability of the steady state for sufficiently strong coupling. We then study chaotic synchronization that occurs for stronger coupling and prove the stability of the synchronized state for sufficiently long delays. The following section treats three fiber ring lasers with the same coupling architecture and shows synchrony of the end lasers in numerical simulation. We then conclude and discuss possible avenues for future work.

**SEMICONDUCTING LASER MODEL AND ONSET OF OSCILLATIONS**

In this and the following section we will investigate the dynamics of three semiconductor lasers coupled in a line with delays. Previously, the dynamics of two optoelectronically delay coupled lasers have been explored, showing lag synchronization between the two lasers and isochronal synchronization if self feedback is added [35].

A schematic of the set-up is shown in Fig. 1. The coupling is optoelectronic, where the coupling signal from one laser is transmitted to the other via fiber-optic cable and an
FIG. 1: A schematic showing how three lasers are coupled in a line. The outer two lasers (circles) are identical, while the middle laser (square) is detuned from the rest.

electronic circuit that introduce a time delay. In the absence of coupling, each laser is tuned so that it emits a stable constant light output. In the presence of coupling, there are fluctuations in the light intensity of the lasers that are converted to an electronic signal which controls the pump strength of the coupling term [4]. This type of coupling is called “incoherent coupling” since it does not contain phase information. Because the phase of the electric field is not used in coupling one laser to another, the equations of each laser can be modeled by only two variables; i.e., intensity and inversion.

The scaled equations of coupled semiconductor lasers have the following form [4, 14]:

\[
\begin{align*}
\frac{dy_1}{dt} &= x_1 (1 + y_1) \\
\frac{dx_1}{dt} &= -y_1 - \epsilon x_1 (a_1 + b_1 y_1) + \delta_2 y_2 (t - \tau) \\
\frac{dy_2}{dt} &= \beta x_2 (1 + y_2) \\
\frac{dx_2}{dt} &= \beta [-y_2 - \epsilon \beta x_2 (a_2 + b_2 y_2)] + \delta_1 [y_1 (t - \tau) + y_3 (t - \tau)] \\
\frac{dy_3}{dt} &= x_3 (1 + y_3) \\
\frac{dx_3}{dt} &= -y_3 - \epsilon x_3 (a_1 + b_1 y_1) + \delta_2 y_2 (t - \tau)
\end{align*}
\] (1)

Variables \(y_i\) and \(x_i\) denote scaled intensity and inversion of the \(i\)th laser, respectively, \(\{a_1, a_2, b_1, b_2\}\) are loss terms, and \(\epsilon\) is the dissipation. The subscript on \(\delta_2\) signifies that the
coupling is from the middle to the outer lasers. Similarly, $\delta_1 = \delta_3$ signifies the coupling strength from the outer to the middle lasers \cite{4,14}. Detuning of the middle laser from the outer ones is given by $\beta$: the ratio of the relaxation frequency of the middle to the outer lasers. If we focus only on linear terms in Eqn. (2), then we can easily recover the equation for a simple harmonic oscillator: $\ddot{y}_2 = -\beta^2 y_2$, which shows the correct dependence of force on the dimensionless frequency squared.

Equations (1)-(3) are scaled in such a way that $y_i > -1$, since the motion slows down asymptotically as $y_i \to -1$. (See [27] for details of the derivation from the original physical model.) It follows that the $\epsilon$ term in the above equations is always dissipative, provided $a_{(1,2)} > b_{(1,2)}$, and leads to a spiraling of the dynamics towards zero in the absence of mutual coupling. In the actual experiment, this zero state would correspond to some constant steady state output. The above equations are coupled via laser intensities, $y_i$, using optoelectronic incoherent coupling.

The delay in the coupling terms is fixed and given by $\tau$, and the strength of coupling from the center to the outer identical subsystems by $\delta_1$, while from the outer to the center by $\delta_2$. Variables $\{x_1, y_1\}$ and $\{x_3, y_3\}$ are symmetric with respect to interchange of variables. Due to this internal symmetry of the system, there exists an identical solution for the outer lasers: $x_1 = x_3; y_1 = y_3$. If this solution is stable then the outer lasers are synchronized. In the absence of dissipation ($\epsilon = 0$), the uncoupled system ($\delta_1 = \delta_2 = 0$) is a nonlinear conservative system, with behavior similar to a simple harmonic oscillator for small amplitudes, and becoming more pulse-like at high amplitudes \cite{27}. Dissipation, however, leads to energy loss, so that in the absence of coupling, the dynamics would settle into a steady state: $\{x_i = 0, y_i = 0\}$. Mutual coupling acts like a drive by pumping energy into the system, similar to a laser with injection. Recent studies of two mutually coupled semiconductor lasers with delay show explicitly in both theory and experiment how the amplitude of the intensity scales with coupling strength for the case of fixed delay \cite{14}. For most cases of physical interest, it can be assumed that dissipation is small: ($\epsilon \ll 1$). We assume small dissipation and $a_{(1,2)} > b_{(1,2)}$ throughout the rest of this chapter. It can be seen from Eqns. (1)-(3) that at low amplitudes the relaxation frequency is equal to one, so that the period of a single oscillation is given by $2\pi$, in the scaled variables used in the equations. In the typical experimental set-up, the relaxation oscillations are on the order of $2 - 3$ ns per cycle. Since the delay time is set to be about an order of magnitude higher than the period of
oscillation, we use about $\tau = 60$ as a typical delay time in many of the simulations, which corresponds to delays of about $20 - 30$ ns in an experimental set-up.

We now explore the onset of regular oscillations that occurs when the coupling strength between lasers is above a bifurcation value. Below this bifurcation value, the oscillations are damped out to steady state due to dissipation, $\epsilon$, in the lasers. It is clear from Eqns. (1) - (3) that the steady state, $\{x_i = 0, y_i = 0\}$, is a solution. To determine the stability of this solution, we linearize about the steady state, looking at time-evolution of small perturbations. It can be seen from the laser equations that at small amplitude the linear terms dominate, so that the dynamics are close to that of coupled simple harmonic oscillators. Since even at small amplitudes, we have a linear dissipation term, $-\epsilon a_1 x_i$ (see Eqns. (1) - (3)), the coupling will only induce oscillations if it contributes enough energy to each laser to overcome the dissipative terms. By linearizing around the zero solution, we obtain a characteristic equation whose eigenvalues determine the stability of the steady state. The actual form of the characteristic equation is not shown here due to a large number of terms, resulting from a $6 \times 6$ matrix corresponding to a 6-dimensional system obtained when Eqns. (1)- (3) are linearized. The delay term in the coupling introduces an exponential term $\exp(-2\tau \lambda)$ in the characteristic equation, where $\lambda$ is a complex eigenvalue. This transcendental function of eigenvalues in the characteristic equation is typical of delay differential equation systems and can result in an infinite number of roots. This is in contrast to systems without delays, where the number of eigenvalues (and hence roots of the characteristic equation) corresponds to the number of variables in the system.

As the coupling strengths $\delta_1$ and $\delta_2$ are increased, the system undergoes a Hopf bifurcation where the real parts of the eigenvalues change from negative to positive, leading to an onset of oscillations. To identify a point of bifurcation, we set the real part of $\lambda$ to zero: $\lambda = i\omega$. The transcendental function separates into real and imaginary parts: $\exp(-2i\tau \omega) = \cos(2\omega \tau) - i \sin(2\omega \tau)$. From the characteristic equation, we now obtain two equations, with the real part given by

$$-\omega^6 + A_r \omega^4 + B_r \omega^2 + C_r \omega + D_r = 0,$$

where $A_r = 2 + 2\epsilon^2 \beta^2 a_1 a_2 + \epsilon^2 a_1^2 + \beta^2$, $B_r = \beta^2 \delta^2 \cos(2\omega \tau) - 1 - 2\beta^2 - 2\epsilon^2 \beta^2 a_1 a_2 - \epsilon^2 a_1^2 \beta^2$; $C_r = -\epsilon a_1 \beta^2 \delta^2 \sin(2\omega \tau)$, $D_r = \beta^2 (1 - \delta^2 \cos(2\omega \tau))$, and $\delta^2 = 2\delta_1 \delta_2$. The imaginary part of the equation results in

$$A_i \omega^5 + B_i \omega^3 + D_i \omega^2 + F_i \omega + G_i = 0,$$
where \( A_i = 2\epsilon a_1 + \epsilon\beta^2 a_2, \) \( B_i = -2\epsilon\beta^2 a_2 - 2\epsilon a_1 - 2\beta^2\epsilon a_1 - \epsilon^3 a_1^2 \beta^2 a_2, \) \( D_i = -\beta^2 \delta^2 \sin (2\omega \tau), \) \( F_i = \beta^2 \epsilon (2a_1 + a_2) - \beta^2 \delta^2 \epsilon a_1 \cos (2\omega \tau), \) and \( G_i = \beta^2 \delta^2 \sin (2\omega \tau). \) For no detuning \( (\beta = 1), \) we assume \( \omega = 1. \) This assumption is not always valid but is justified for certain values of the delay, as we shall see shortly. Solving Eqns. (4) and (5) for \( \beta = 1, \) and \( \omega = 1, \) we obtain the bifurcation equation:

\[
\delta_1 \delta_2 \cos (2\tau) = -\frac{\epsilon^2 a_1 a_2}{2}.
\]

It follows that for values of the delay given by \( \tau = (\pi/2) + n\pi, \) where \( n \) is an integer, the onset of oscillations occurs when

\[
\delta_1 \delta_2 > \frac{\epsilon^2 a_1 a_2}{2}.
\]

In this case, the outer lasers are 180 degrees out of phase with the middle laser, and synchronized with each other, after the transients die out. This effect is plotted in Figure 2 where the slope of the line for the log intensity plots of the middle laser vs. the outer laser is negative, indicating that the two lasers are 180 degrees out of phase with each other. At the same time, the two outer lasers fall on a straight line of slope 1 in the log intensity plot, indicating complete synchronization. The circular dynamics around the straight line show the slow die out of transients as the amplitude of oscillation slowly increases from its initial conditions. It can be seen by substituting the parameters given in Fig. 2 into Eq. (7) that the coupling strengths are just above the bifurcation value, leading to low amplitude regular oscillations. This can be contrasted to much higher amplitude chaotic oscillations shown in Fig. 3 which will be treated in the following section.

The oscillations are regular at low amplitudes, since the dynamics are approximated by three coupled simple harmonic oscillators due to the dominance of linear terms when the equations are linearized about the steady state. The bifurcation condition in Eq. (7) can be understood as the point where the coupling strength, \( \delta_1 \delta_2, \) between the lasers is strong enough to overcome the internal dissipation, which is proportional to \( \epsilon a_1 \) and \( \epsilon a_2 \) for the outer and the inner lasers, respectively.

**THEORY OF CHAOTIC SYNCHRONIZATION OF SEMICONDUCTOR LASERS**

As the coupling strength between semiconductor lasers is increased further and the amplitude of oscillation grows, nonlinearities become important and the oscillations become chaotic. Chaotic oscillations are shown in Fig. 4 where the inversion, \( x, \) is plotted as a
FIG. 2: Synchronization of semiconductor lasers after the transients die out, low amplitude regular motion. Left: Intensity of Laser 2 vs. Laser 1. The inner laser is 180 degrees out-of-phase with the outer laser, as indicated by the negative slope of the line. Right: Laser 3 vs. Laser 1. Straight line with +1 slope indicates complete synchronization of outer lasers. \( \tau = \pi/2, \epsilon = \sqrt{0.001}, \delta_1 = \delta_2 = \epsilon \sqrt{2.1}, a_1 = a_2 = 2, b_1 = b_2 = 1. \)

function of time. The coupling strengths in Fig. 4 are well above the bifurcation value derived in the previous section, leading to relatively high amplitudes of oscillation. The system described by Eqs. (1)-(3) shows complete chaotic synchronization of outer lasers over a wide range of parameters. Figure 3 shows that while the outer lasers can become completely synchronized, there may be no apparent correlation between the middle and the outer lasers.

To understand why chaotic synchronization occurs without direct contact between the outer lasers and in the presence of significant delays, we need to consider the basic properties of the system. Equations (1)-(3) can be rewritten in a more general form as

\[
\frac{dz_1}{dt} = F(z_1(t)) + \delta_1 \cdot G(z_2(t - \tau)) \quad (8)
\]

\[
\frac{dz_2}{dt} = \tilde{F}(z_2(t)) + \delta_2 \cdot \tilde{G}(z_1(t - \tau), z_3(t - \tau)) \quad (9)
\]

\[
\frac{dz_3}{dt} = F(z_3(t)) + \delta_1 \cdot G(z_2(t - \tau)), \quad (10)
\]

where \( z_i = \{x_i, y_i\} \). Due to internal symmetry of the system, there exists an identical solution for the outer subsystems: \( z_1(t) = z_3(t) \equiv \{X(t), Y(t)\} \). Whether this symmetric solution is stable determines whether the lasers will become synchronized.
Before studying the stability of the synchronized state by linearizing about the \( \{X(t), Y(t)\} \) solution, let us consider a qualitative explanation for why complete chaotic synchronization would occur in the presence of long delays. For \( \delta_2 = 0 \), in Eq. (2), the dynamics of \( \mathbf{z}_{1,3} \) becomes that of a driven system, with \( \mathbf{z}_2 \) acting as the driver. Then, the synchronized dynamics correspond to generalized synchronization \([26]\), whereby the driven subsystem becomes a function of the driver. While the exact form of the function between the driver and the driven systems can be rather complicated and difficult to obtain, its existence can be inferred from the synchronization of identical systems when started from different initial conditions but exposed to the same drive. This method of detecting generalized synchronization using identical driven systems is known as the auxiliary systems approach \([1]\). In order for the driven subsystems, \( \mathbf{z}_{1,3} \), to become synchronized, their de-
dependence on initial conditions has to “wash out” as a function of time. This independence of later dynamics on initial conditions is necessary for synchronization. Otherwise systems that have different initial conditions will never settle into the same trajectory, which is necessary for complete synchrony. This “washing out” of initial conditions is provided by the dissipation in the system, which must therefore be present in either the coupling term or in the uncoupled dynamics of the system itself. For the case of semiconductor lasers that we are considering, this dissipation comes from the internal dissipation, $\epsilon$, in the lasers themselves. As was discussed in the previous section, due to this dissipation the dynamics would spiral down to $z_{1,3} = 0$ in the absence of any coupling. As will be shown shortly, this dissipation $\epsilon$ plays an important role in determining the Lyapunov exponents transverse to the synchronized state.

In addition to helping understand the unidirectionally driven case, this idea of generalized
synchronization between the driver and the driven systems leading to complete synchronization of identical driven systems is also useful in understanding complete chaotic synchronization of mutually coupled lasers with long delays \[16\]. In mutually coupled systems (i.e., $\delta_2 \neq 0$), the dynamics of $z_2$ are affected by $z_1$ and $z_3$. In this case, the synchronized state, $\{X(t), Y(t)\}$, will depend on the initial conditions of all of the three subsystems, $\{z_1, z_2, z_3\}$, so that $\{X(t), Y(t)\}$ can not be the result of generalized synchronization, in a strict sense. However, it takes a time interval of $2\tau$ for any change in the dynamics of systems $z_{1,3}$ to affect the trajectory of these systems via mutual coupling. During this time interval of length $2\tau$, $z_{1,3}$ can be viewed as driven by $z_2$, since the signal $z_{1,3}$ receives during that time interval is not affected by its dynamics on that interval. This idea that the dynamics of the outer lasers can be viewed as driven by the signal from the middle laser on the time interval $2\tau$ will be very useful when we linearize the dynamics about the synchronized state. Since dynamics transverse to the synchronized state will not affect the synchronous solution, $\{X(t), Y(t)\}$, over twice the coupling delay, we can separate the variables and obtain an analytic solution which is valid over that interval.

To linearize Eqns. (1) and (3) around the synchronized state, $\{X(t), Y(t)\}$, we introduce new variables defined as: $\Delta x_{1,3}(t) = x_{1,3}(t) - X(t)$ and $\Delta y_{1,3}(t) = y_{1,3}(t) - Y(t)$. If the conditional Lyapunov exponents calculated with respect to perturbation out of the synchronization manifold are all negative, then the outer lasers are synchronized. Calculating Lyapunov exponents is in general complicated due to the presence of time-delays in the equations. The coupling term containing delays, however, drops out of the equations if we take the difference of the outer variables: $\Delta y = y_1 - y_3; \Delta x = x_1 - x_3$. For simplicity, let us assume that we only perturb one of the outer lasers from the synchronous state. This will not affect the generality of the result, but allows us to identify the “synchronized state” with the dynamics of the other unperturbed laser, so that if $y_3 = Y(t)$ and $x_3 = X(t)$, then $\Delta y = \Delta y_{1,3}(t) = y_1 - Y(t)$ and $\Delta x = \Delta x_{1,3}(t) = x_1 - X(t)$. Using notation of Eqns. (8) - (10), the linearized dynamics transverse to the synchronization manifold are given by,

$$\frac{d\Delta z(t)}{dt} = J \cdot \Delta z(t) \quad (11)$$

where $\Delta z(t) = \{\Delta x, \Delta y\}$ and $J$ is a $2 \times 2$ Jacobian matrix of partial derivatives evaluated at $z = \{X(t), Y(t)\}$,

$$J = \frac{\partial F(z)}{\partial z} \quad (12)$$
Applying Eqns. (11) and (12) to Eqn. (1) (after comparing it to the more general form of Eqn. (8)) we obtain

\[
\begin{pmatrix}
\dot{\triangle x} \\
\dot{\triangle y}
\end{pmatrix}
= 
\begin{pmatrix}
-\epsilon (a_1 + b_1 Y(t)) - (1 + \epsilon b_1 X(t)) \\
1 + Y(t) & X(t)
\end{pmatrix}
\cdot 
\begin{pmatrix}
\triangle x(t) \\
\triangle y(t)
\end{pmatrix}
\tag{13}
\]

Both \(1 + Y(t)\) and \(1 + \epsilon b_1 X(t)\) terms in the matrix of Eqn. (13) are positive. The first because \(Y(t) > -1\), as follows from Eqns. (1)-(3), and the second because \(|\epsilon b_1 X(t)| < 1\) since \(\epsilon \ll 1\). It follows that the cross-terms in the matrix always have opposite signs, indicating a finite counter-clockwise rotation of a system with instantaneous frequency given by,

\[
\omega(t) = (1 + Y(t))^{1/2} (1 + \epsilon b_1 X(t))^{1/2}
\tag{14}
\]

The angular frequency in the above equation shows that the speed of rotation of the transverse dynamics varies as a function of time, but is always non-vanishing and counter-clockwise. We will come back to this property shortly in connection to proving the stability of synchronized state for sufficiently long delays.

If the dynamics of the outer laser are perturbed from the synchronized state at some time \(t = t_0\), then the perturbation will not affect the coefficient matrix in Eqn. (13) until \(t \geq t_0 + 2\tau\). It follows that during the time interval of \(2\tau\) the dynamics off the synchronization manifold can be viewed as driven by the time dependent coefficients: \(\{X(t), Y(t)\}\). Since on this timescale, the coefficients in the matrix are independent of the variables \(\{\triangle x, \triangle y\}\), Eqn. (13) can be solved over the interval. This is much like solving an equation of the form \(dh/dt = f(t)h\). As long as \(f(t)\) is only a function of time and independent of \(h\), we can easily obtain a solution: \(h = \exp \left(\int f(t)dt\right)\). The situation in Eqn. (13) is similar, but in two dimensions: as long as we are looking at the interval of \(2\tau\), the variables \(\{X(t), Y(t)\}\) can be viewed as some functions of time only, since they are independent of \(\{\triangle x, \triangle y\}\) over that interval. Thus we can choose any initial condition \(\{\triangle x_0, \triangle y_0\}\) at some time \(t = t_0\) for a small perturbation transverse to the synchronized state and solve for the dynamics on the time interval of \(t_0 \leq t \leq t_0 + 2\tau\). Of course we still have not solved for the matrix coefficients in Eqn. (13), since \(X(t)\) and \(Y(t)\) are solutions of a time-delayed differential equations and, in general, can not be easily obtained by analytic means. However, they have special properties, namely \(X(t)\), the inversion, is symmetric about zero, and \(Y(t)\), the intensity is always greater than \(-1\). As we shall see shortly, these properties will allow us
to make some conclusions about the stability of synchronized state without actually having to solve for \(\{X(t), Y(t)\}\).

A two dimensional equation with time-dependent coefficients can be solved using Abel’s formula, which relates the Wronskian of the linearized system to the trace of the matrix \([43]\)

\[
W(t) = \det \begin{vmatrix} \Delta x & \Delta y \\ \Delta x & \Delta y \end{vmatrix} = W_0 \exp \left( \int_{t_0}^{t} \{X(s) - \epsilon (a_1 + b_1 Y(s))\} \cdot ds \right) \tag{15}
\]

where \(W_0 > 0\) is a constant that depends on the magnitude of the initial perturbation: \(W_0 = W(t_0)\). The Wronskian gives the phase-space volume dynamics of the system \(\{\Delta x(t), \Delta y(t)\}\). Equation (15) is valid over the integration interval of twice the delay: \(t_0 < t < t_0 + 2\tau\). The term multiplied by \(\epsilon\) in the exponential is always negative, since \(a_1 > b_1\), and \(Y(s) > -1\), leading to the contraction of phase-space volume. The inversion term, \(X(s)\), on the other hand, is symmetric around zero, resulting in zero average over the oscillations. Since the fluctuations in inversion don’t have a preferred direction, their statistical average is zero. Thus for sufficiently long delays, the \(- \int \epsilon (a_1 + b_1 Y(s)) \, dt\) term will always dominate in the exponential of Eqn. (15). This term is always negative, assuming \(a_1 > b_1\), and monotonically decreasing as a function of delay (which determines the length of integration), resulting in shrinkage of the phase-space volume for sufficiently long delays.

Taking the determinant of the matrix in the above equation, \(W(t)\), can also be written as \(W(t) = |\Delta x \Delta y - \Delta y \Delta x|\). Substituting for \(\Delta x\) and \(\Delta y\) from Eqn. (13), we obtain

\[
W(t) = |(1 + Y(t)) \cdot (\Delta x)^2 + (1 + eb_1 X(t)) \cdot (\Delta y)^2 + [\epsilon (a_1 + b_1 Y(t)) + X(t)] \cdot \Delta x \Delta y| \tag{16}
\]

Since both \(1 + Y(t)\) and \(1 + eb_1 X(t)\) terms in the above equation are positive, the terms quadratic in \(\Delta x\) and \(\Delta y\) are positive as well. This is characteristic of the phase-space volume of rotating systems. Due to ever-present finite counter-clockwise rotation in the system (see discussion following Eqns. (13) and (14)), if the phase space volume, \(W(t)\), is shrinking over several rotations, then the distance from synchronized state, \(r = \sqrt{(\Delta x)^2 + (\Delta y)^2}\), has to shrink as well. This can be seen in the following way: suppose we draw a straight line, given by \(\Delta y = c \Delta x\), through the origin in the phase space plot of \(\{\Delta x, \Delta y\}\), where the slope, \(c\), is some arbitrary constant. Now, suppose we are interested in the value of \(W(t)\) whenever this line is crossed. To find \(W(t)\) at the point of crossing, we can substitute \(\Delta y = c \Delta x\) into Eqn. (16),

\[
W(t) = [1 + Y(t) + c [\epsilon (a_1 + b_1 Y(t)) + X(t)] + c^2 (1 + eb_1 X(t))] \Delta x^2 \tag{17}
\]
Since, as explained following Eqns. (13) and (14), there is a non-vanishing counter-clockwise finite rotation in the system, this line $\Delta y = c\Delta x$ will be crossed at each successive rotation, for any value of the slope, $c$. The first factor in the above equation depends on the variables $X(t)$ and $Y(t)$, which come from an arbitrarily chosen interval of $2\tau$ and therefore do not have any consistent time-dependent behavior within that interval. It follows that if $W(t)$ always decreases after a certain period of time, then $\{\Delta x, \Delta y\}$ have to decrease along any line $\Delta y = c\Delta x$ drawn from the origin. This is just shrinking of the radius, $r$, or distance transverse to the synchronized state. We have thus shown that monotonic shrinking of the phase-space volume of the perturbed dynamics corresponds to stability of synchronized state.

It remains for us to show that $W(t)$ always shrinks towards the end of twice the delay time for sufficiently long delays. We do this by showing that the integral in Eqn. (15) becomes negative for sufficiently long integration times. Since the variable $Y(t)$ is the scaled intensity of the laser, from Eqns. (1)-3) its minimum possible value is $-1$. Thus for $a_1 > b_1$ (a typical case), the contribution of the dissipation term to the Wronskian is always negative. The variable $X(t)$, on the other hand, is symmetric about zero, and thus averages out to zero when integrated over a single period of oscillation. It follows that $X(s)$ in Eqn. (15) averages out to zero if the integral is done over many periods of oscillation, while the dissipation term, multiplied by $\epsilon$, provides a continuous negative component. If that continuous negative component builds up sufficiently over the integration interval to overcome any fluctuations in $X(s)$, we then have a continuous shrinking of the phase-space of perturbed dynamics, which combined with rotation in the two dimensional system leads to synchronization.

We can now understand how synchronization is dependent on delays. The upper limit on the integration interval in Eqn. (15) is set at twice the delay time. If the delays are sufficiently long to overcome fluctuations in $X(t)$, then the phase-space dynamics transverse to synchronized state shrink. Depending on the value of dissipation, $\epsilon$, the integration time may have to be long to consistently get a negative exponent in Eqn. (15). This is due to the fact that $\epsilon \ll 1$, while the fluctuations in $X(s)$, although zero when integrated over a period, will introduce fluctuations of order one into the integral. The upper limit on the positive fluctuation in Eqn. (15) is given by $T|X(t)|_{max}/2$, where $|X(t)|_{max}$ is the maximum value of the intensity and $T$ is the corresponding period of oscillation. This is the maximum
value that the integral of any oscillation symmetric about zero of period $T$ and amplitude $|X(t)|_{max}$ can reach, leading to

$$\int_{t_0}^{t_0+2\tau} X(s) ds < T|X(t)|_{max}/2$$  \hspace{1cm} (18)

The dissipative term in the exponential of Eqn. (15) is equivalent to

$$\int_{t_0}^{t_0+2\tau} \epsilon (a_1 + b_1 Y(s)) = 2\tau \epsilon (a_1 + b_1 \bar{Y})$$  \hspace{1cm} (19)

where $\bar{Y}$ is the averaged intensity. Applying Eqns. (18) and (19) to Eqn. (15), we obtain a condition for the shrinking of transverse phase-space volume toward the end of the interval of twice the delay time,

$$4\epsilon (a_1 + b_1 \bar{Y}) \left( \frac{T}{T} \right) > |X(t)|_{max}$$  \hspace{1cm} (20)

From the above equation, it is clear that the ratio of the delay to the period of oscillation, $\tau/T$, plays an important role in synchronization. Since the dissipation, $\epsilon$, is small, we need rather long delays, compared to the period of oscillation, to guarantee the stability of synchronized state. When the fluctuations in intensity are sufficiently small so that the period $T$ is approximated well by $2\pi$ in our scaled equations, Eqn. (20) reduces to

$$2\epsilon \tau (a_1 + b_1 \bar{Y}) > \pi |X(t)|_{max}.$$

Since the solution in Eqn. (15) is no longer valid for integration times longer than twice the delay, we need to consider what happens at the end of that interval. At the beginning of the new interval at $t = t_0 + 2\tau$, our synchronized state has been affected by the dynamics of $\{\triangle x(t), \triangle y(t)\}$ over the previous interval. Let us call this new synchronized state $\{X(t)t, Y(t)t\}$. This is of course the same as saying that the perturbation of one of the outer lasers has finally reached the other, after a time of $2\tau$, and affected the “synchronized state.” What we are really interested in is the evolution of perturbation $\{\triangle x(t), \triangle y(t)\}$ from the altered dynamics, $\{X(t)t, Y(t)t\}$, of the other laser. This is because we are interested in whether the outer lasers will become synchronized (even if the “synchronized state” changes), not in whether they will come back to the same synchronized state that would have existed if the perturbation never happened. In addition, we have to consider that the time delay terms in the original Eqns. (11) - (13) are affected by a perturbation after a time of $2\tau$, so that we can no longer get the delay terms to drop out of the equations by linearizing around the same synchronized solution that would have existed in the absence of perturbations. Hence this is not quite the same as using linearization to find the divergence.
of two nearby trajectories in phase space that are governed by the same equation but have
different initial conditions. In our case, the two nearby trajectories affect each other and
thus can not be considered to evolve independently. In other words, the transverse dynamics
affect the synchronized state dynamics at a later time, so we can not just linearize around
the synchronized state the way we would linearize to find the divergence of two nearby
independent trajectories.

The perturbation from synchronized state at the beginning of the new time interval is
given by \( \{ \Delta x(t_0 + 2\tau), \Delta y(t_0 + 2\tau) \} \). Since the transverse dynamics are again independent
over the period of \( 2\tau \) from the synchronized dynamics \( \{ X(t), Y(t) \} \), we can again apply
Abel’s formula over that period, with the initial condition of \( \{ \Delta x(t_0 + 2\tau), \Delta y(t_0 + 2\tau) \} \).
However, \( \{ X(t), Y(t) \} \) have the same properties as the synchronized solution before,
namely, \( X(t) \) is symmetric about zero and \( Y(t) > -1 \). We have already shown that given
these properties, the distance from synchronized state, \( r = \sqrt{(\Delta x)^2 + (\Delta y)^2} \), will shrink
towards the end of twice the delay time. Applying the same argument to the next interval,
we can see that \( r(t_0) > r(t_0 + 2\tau) > r(t_0 + 4\tau) \ldots \) ad infinitum. We have thus proved that
the synchronized state of the outer lasers is stable for sufficiently long delays.

We have shown that the synchronized state is stable for sufficiently long delays. The
stability of the synchronized state indicates that all the transverse Lyapunov exponents
are negative. A negative Lyapunov exponent sum corresponds to the contraction of the
phase-space volume. In fact there is a simple relationship between the two, given by:

\[
\lambda_1 + \lambda_2 = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \ln \left( \frac{W(t_0 + \Delta t)}{W_0} \right),
\]

where \( \lambda_1 + \lambda_2 \) is the sum of transverse Lyapunov exponents. Based on Eqn. (15) and prior
discussion, it is clear that for sufficiently long delays, the phase space volume over each
interval of \( 2\tau \) contracts by a factor of approximately \( \exp \left( -2\tau \epsilon (a_1 + b_1 Y) \right) \), so that over \( n \)
intervals, the phase-space volume is approximated by

\[
\frac{W(t_0 + 2\tau n)}{W_0} \approx \exp \left( -2\tau n \epsilon (a_1 + b_1 Y) \right)
\]

Taking \( \Delta t = 2\tau n \) and \( n \to \infty \) for infinite times, we obtain after substituting Eqn. (22) into
Eq. (21),

\[
\lambda_1 + \lambda_2 \approx -\epsilon (a_1 + b_1 Y)
\]

The sum of Lyapunov exponents shows a negative linear dependence on dissipation whenever
the lasers synchronize. This intimate connection between the Lyapunov exponents and
dissipation is not accidental, since negative Lyapunov exponents determine how quickly the perturbed trajectory converges to the synchronized state, and the dissipation determines how quickly any differences in initial conditions of the outer lasers “wash out,” leading to synchronization.

Figure 5 shows the numerically computed sum of Lyapunov exponents, and corresponding correlations of the outer lasers, as a function of dissipation, $\epsilon$, for two values of the delay, $\tau = 120$ and $\tau = 240$. The fluctuations in the sum of Lyapunov exponents correspond well to the fluctuations in the correlation function of the outer lasers, with desynchronization when the Lyapunov sum increases above zero. As might be expected from Eqn. (20), longer delays mean synchronization at lower values of dissipation, since the dissipation term in the exponential in Eqn. (15) dominates for sufficiently long delays.

Figure 6a shows the sum of Lyapunov exponents as a function of delay. The Lyapunov exponents are negative for all $\tau > 170$, (corresponding to about 60 ns) resulting in complete synchronization of the outer lasers, as shown in Fig. 6b. At the same time, the outer lasers are not synchronized with the center one, Fig. 6c. The fluctuations in correlations of the outer lasers match well the fluctuations in the Lyapunov sum, with correlations increasing whenever the Lyapunov sum decreases. Figure 6 agrees well with the analysis in this section, since sufficiently long delays are needed for the Lyapunov exponents to become negative, leading to synchronization. After the onset of synchronization, Eqn. (23) becomes valid, so the Lyapunov sum becomes independent of delays. This is confirmed by the straight horizontal line in the figure, after the outer lasers synchronize. The degree of synchrony is given by the correlation function.

The amplitude of laser oscillations depends on the coupling strengths, $\delta_1$ and $\delta_2$, as well as the dissipation. It was shown in the last section that the product of the coupling strengths, $\delta_1 \delta_2$, has to be strong enough to overcome the dissipation to cause the onset of oscillations. Increasing the coupling strengths increases the role of nonlinearities in the system and the intensity of laser oscillations. Since increased coupling pumps more energy into the system, thereby increasing the effect of nonlinearities, the Lyapunov exponents may increase above zero, leading to desynchronization of the outer lasers. In this case, longer delays in coupling may be required in order for the outer lasers to synchronize. This effect is illustrated in Fig. 7 which shows the sum of Lyapunov exponents as a function of coupling strengths for two different delays, $\tau = 60$ and $\tau = 120$. There is an abrupt increase in Lyapunov
exponents above zero, due to increased nonlinearity, as the coupling strength is increased. Increasing the delay however to $\tau = 120$ leads to synchronization for a greater range of coupling strengths, as compared to $\tau = 60$. The corresponding correlations as a function of coupling strengths are shown in Fig. 8.

It is worthwhile to note that this loss of synchronization with increased coupling strengths may seem counter-intuitive, and is not found in the case of fiber lasers discussed in the following section. Nevertheless, desynchronization at higher coupling strengths, and the synchronizing effect of increased delays is in agreement with analytic results of this section. Since higher coupling strengths lead to greater fluctuations in $X(t)$, larger values of of $\tau$ or $\epsilon$ are needed in order to satisfy Eqn. (20). This means that increasing coupling strength may lead to desynchronization unless the values of delay or dissipation are increased accordingly.

The analysis in this section focuses on the local stability of synchronized state, rather than investigating the global properties of the system. However, from numerical simulation, it appears that local stability implies global stability, since the lasers synchronize, regardless of their initial conditions, whenever the synchronous state is locally stable. This suggests that the system investigated in the present section only has a single attractor, unlike the multiple attractor dynamics that can be induced in certain other systems that have delayed feedback [18]. In the presence of multiple attractors, local stability of synchronized state would not necessarily result in synchronization, since the initial condition can be such that the outer systems end up in different attractor basins. For synchronization in chaotic systems with coexisting attractors, see [25].
FIG. 5: a) Sum of Lyapunov exponents as a function of dissipation, $\epsilon$, for $\tau = 120$. b) Corresponding correlations between outer lasers, $\tau = 120$. c) Sum of Lyapunov exponents vs. $\epsilon$, for $\tau = 240$. d) Corresponding correlations between outer lasers, $\tau = 240$. In all cases, $a_1 = a_2 = 2$, $b_1 = b_2 = 1$, $\delta_1 = \delta_2 = 0.2$, $\beta = 0.5$. 
FIG. 6: a) Numerically computed sum of Lyapunov exponents as a function of delay, $\tau$. b) Corresponding correlations of outer lasers. c) Correlations of the middle and outer lasers, shifted by the delay time to maximize correlations. $\epsilon = \sqrt{0.001}$, $\delta_1 = \delta_2 = 7.5\epsilon$, $\beta = 0.5$. 
FIG. 7: a) Sum of Lyapunov exponents as a function of coupling strength, $\delta_1 = \delta_2$, for $\tau = 60$. b) $\tau = 120$. $\epsilon = \sqrt{0.001}$, $\beta = 0.5$. 
FIG. 8: Correlations corresponding to Fig. 7. a) Correlation between the middle and one of the outer lasers, \( \tau = 60 \). b) Correlations of outer lasers, \( \tau = 60 \). c) Correlation between the middle and one of the outer lasers, \( \tau = 120 \). d) Correlations of outer lasers, \( \tau = 120 \). Outer lasers synchronize for greater range of coupling strength as the delay is increased. The middle and the outer lasers show little correlation for all values of the coupling strengths.
SYNCHRONIZATION IN A SPATIO-TEMPORAL SYSTEM: FIBER LASERS

We next consider a different system with the same coupling geometry of three delay-coupled components arranged linearly. The components are fiber ring lasers, which are a more complicated system than semiconductor lasers. A fiber ring laser contains a ring of optical fiber, a portion of which is doped and can lase. Even a single, uncoupled ring laser is a time delayed system because of the time that light takes to travel around the ring (the round trip time), and a single ring laser can display spatio-temporal chaos [11]. In contrast to the semiconductor lasers, which have a much faster relaxation time, the relaxation time scale of a fiber ring laser is on the order of milliseconds to microseconds [15, 40]. Achieving a coupling delay that is long compared to the relaxation time would require many kilometers of optical fiber in an experiment and would be difficult to model because of computational limitations. Instead, coupled fiber ring lasers are often operated with a coupling delay that is short compared to the relaxation time of the laser, and this is the case we address here. We include independent noise sources on each laser to represent spontaneous emission. The outer lasers are assumed to be identical, while the center laser is detuned from the other two.

We model the fiber lasers via a system of ordinary differential equations for the population inversions coupled to a system of maps for the electric field. This model was used with other coupling geometries in [29, 30] and is a variation of that introduced in [40].

In each fiber ring laser, light circulates through a ring of optical fiber, part of which is doped for stimulated emission. A single polarization mode is modeled in each laser. Each laser is characterized by a total population inversion $W_j(t)$ (averaged over the length of the fiber amplifier) and an electric field $E_j(t)$. The time for light to circulate through the ring is the cavity round trip time $\tau_R$. Transit through the coupling lines takes a potentially different delay time $\tau_d$. The equations for the model dynamics are as follows:

$$E_j(t) = R \exp [\Gamma (1 - i\alpha_j) W_j(t) + i\Delta \phi] E_j^{\text{th}}(t) + \xi_j(t) \tag{24}$$

$$\frac{dW_j}{dt} = q - 1 - W_j(t)$$
$$- |E_j^{\text{th}}(t)|^2 \{ \exp [2\Gamma W_j(t)] - 1 \} \tag{25}$$

24
The electric field from earlier times which affects the field at time $t$ is

$$E_{fd}^{1}(t) = E_1(t - \tau_R) + \kappa E_2(t - \tau_d)$$

$$E_{fd}^{2}(t) = E_2(t - \tau_R) + \kappa E_1(t - \tau_d) + \kappa E_3(t - \tau_d)$$

$$E_{fd}^{3}(t) = E_3(t - \tau_R) + \kappa E_2(t - \tau_d).$$

$E_j(t)$ is the complex envelope of the electric field in laser $j$, measured at a given reference point inside the cavity. $E_{fdj}^{d}(t)$ is a feedback term that includes optical feedback within laser $j$ and optical coupling with the other lasers. Time is dimensionless, measured in units of the decay time of the atomic transition, $\gamma^{-1}_{||}$. The active medium is characterized by the dimensionless detuning $\alpha_j$ between the transition and lasing frequencies and by the dimensionless gain $\Gamma = \frac{1}{2}aL_aN_0$, where $a$ is the material gain, $L_a$ the active fiber length, and $N_0$ the population inversion at transparency. The ring cavity is characterized by its return coefficient $R$, which represents the fraction of light remaining in the cavity after one round trip, and the average phase change $\Delta\phi = 2\pi nL_p/\lambda$ due to propagation of light with wavelength $\lambda$ along the passive fiber of length $L_p$ and index of refraction $n$. Energy input is given by the pump parameter $q$, which is measured in units of the population decay rate $\gamma_{||}$. The electric field is perturbed by independent complex Gaussian noise sources $\xi_j$ with standard deviation $D$. Lasers 1 and 3 are each coupled mutually with Laser 2 with a coupling strength of $\kappa$, but Lasers 1 and 3 are not directly connected. Values of the parameters in the model, which are similar to those used in an experiment for two coupled fiber lasers with self feedback [30], are given in Table 1.

Eqns. 24-25 consist of a delay differential equation for $W(t)$ coupled to a map for $E(t)$. We integrated Eqn. 25 numerically using Heun’s method while propagating the map in Eqn. 24. The time step for integration was $\tau_R/N$, where $N = 600$. This step size corresponds to dividing the ring cavity into $N$ spatial elements.

Because of the feedback term $E_{fdj}^{d}(t)$ in Eqn. 24, one can think of Eqn. 24 as mapping the electric field on the time interval $[t - \tau_R, t]$ to the time interval $[t, t + \tau_R]$ in the absence of coupling ($\kappa = 0$). Equivalently, because the light is traveling around the cavity, Eqn. 24 maps the electric field at all points in the ring at time $t$ to the electric field at all points in the ring at time $t + \tau_R$. We can thus construct spatio-temporal plots for $E(t)$ or the intensity $I(t) = |E(t)|^2$ by unwrapping $E(t)$ into segments of length $\tau_R$.

To correspond with previous experiments in which the measured light intensity is passed
TABLE I: Parameters used in the coupled fiber laser model.

| Parameter | Value     | Units | Description                        |
|-----------|-----------|-------|------------------------------------|
| $R$       | 0.4       |       | output coupler return coefficient  |
| $a$       | $2.03 \times 10^{-23}$ | m$^2$ | material gain coefficient          |
| $L_a$     | 15        | m     | length of active fiber             |
| $L_p$     | 27        | m     | length of passive fiber            |
| $N_0$     | $10^{20}$ | m$^{-3}$ | transparency inversion            |
| $\Gamma$ | 0.0152    |       | dimensionless gain                 |
| $\alpha_1$ | 0.0202    |       | detuning factor, laser 1           |
| $\alpha_2$ | 0.0352    |       | detuning factor, laser 2           |
| $\alpha_3$ | 0.0202    |       | detuning factor, laser 3           |
| $n$       | 1.44      |       | index of refraction                |
| $\lambda$ | $1.55 \times 10^{-6}$ | m   | wavelength                         |
| $\Delta\phi$ | $1.58 \times 10^8$ |       | average phase change               |
| $D$       | 0.02      |       | standard deviation of noise        |
| $q$       | 100       |       | pump parameter                     |
| $\gamma_{||}$ | 100       | s$^{-1}$ | population decay rate              |
| $\tau_R$  | $201.6 \times 10^{-9}$ | s    | cavity round trip time             |
| $\tau_d$  | $45 \times 10^{-9}$ | s    | delay time between lasers          |
| $\kappa$  | 0-0.01    |       | coupling strength                  |

through a 125 MHz bandwidth photodetector [30], we computed intensities from model and applied a low pass filter with $f_0 = 125$ MHz, multiplying the Fourier transform by the transfer function

$$G = \left\{ \left( \frac{i f}{f_0} + 1 \right) \left[ - \left( \frac{f}{f_0} \right)^2 + i \frac{f}{f_0} + 1 \right] \right\}^{-1}. \tag{27}$$

All results presented here are based on the filtered intensity.

The coupled fiber laser model can exhibit several types of dynamics. Long time scale behavior is most easily seen through spatio-temporal plots, in which the time series is unwrapped into intervals of one round trip $\tau_R$, and subsequent round trips are stacked on top of each other. An example is shown in Fig. 9 for a coupling strength of $\kappa = 0.005$. The
FIG. 9: Spatio-temporal plots of intensity for three lasers coupled in a line with $\kappa = 0.005$. (a) Laser 1, (b) Laser 3, (c) Laser 2. All plots have the same color scale.

Dynamics approximately repeats from one round trip to the next but evolves gradually over tens or hundreds of round trips. At this coupling value, similarities between the outer lasers (1 and 3) can be seen. Other coupling strengths can produce behavior that appears to be noisy periodic.

To assess the type and extent of synchronization in the system, we shift the laser time series relative to each other and compute cross correlations between them. An example is given in Fig. 10. Lasers 1 and 3, the outer lasers, have a peak in the cross correlation at a time shift of zero, meaning they are isochronally synchronized. Other maxima occur at multiples of the round trip time $\tau_R$, 201.6 ns, because the laser time series approximately repeat every round trip. Laser 2 is not isochronally synchronized with the outer lasers; the peak at zero time shift is small. However, there are more significant peaks in the correlation between Laser 2 and the outer lasers at a shift of the coupling time $\tau_d$, i.e., $\pm 45$ ns, indicating partial delay synchrony. The cross correlation is approximately equal whether the lasers are compared with Laser 2 leading the others or with Laser 2 following them. This result is consistent with the reversible delay synchrony observed previously for two mutually delay
FIG. 10: Cross correlation between lasers vs. time shift between the laser time series. $\kappa = 0.005$, and averaging was done over 10 round trips. Time series were taken near the beginning of Fig. 9. (a) Laser 1 and Laser 3, (b) Laser 2 and Laser 1, (c) Laser 2 and Laser 3.

coupled fiber ring lasers, for which there is no clear leader and follower [30].

We next determine the long time synchronization behavior of the lasers. For each round trip, we compute the cross correlation between outer lasers without a time shift and between the center and outer lasers with a time shift of $\tau_d$. We shift so that Laser 2 leads the others, but because of the reversible synchronization, we would obtain similar results for cross correlations with Laser 2 following the others. To obtain good statistics, we average the round trip cross correlations over five separate intervals of 8 ms each, which are spaced apart by 100 ms. The standard deviation over all the round trips serves as an error estimate. Fig. 11 shows how the synchronization depends on the coupling strength. The outer lasers begin to synchronize isochronally at weak coupling and are well synchronized by the time $\kappa$ reaches 0.5%. Delay synchrony between center and outer lasers requires a stronger coupling, but the level of delay synchrony eventually saturates to the same level as that of the isochronal synchrony between outer lasers. It is likely that the delay synchrony arises more slowly because Laser 2 is detuned away from the others, while Lasers 1 and 3 are identical. Perfect
FIG. 11: Cross correlation vs. coupling for three lasers coupled in a line. Upper curve: Laser 1 and Laser 3 compared with zero time shift. Lower curve: Laser 1 compared to Laser 2 with a lag of the coupling delay time. Relationship between Lasers 3 and 2 is similar to that between 1 and 2.

synchrony is not achieved due to the noise in the system.

To relate the fiber laser system to the coupled semiconductor lasers discussed in the previous section, one might consider whether increasing coupling delay or increasing dissipation will improve synchrony in the fiber lasers. In the regime where the coupling delay is on the same order as the round trip time, and much less than the laser relaxation time, the delay has little effect on the synchronization. Increasing dissipation by increasing the decay rate $\gamma||$ does not improve synchronization either. The fiber lasers behave differently than the semiconductor lasers in several respects, although the coupling geometry with three lasers in a line leads to synchrony of the outer lasers in both cases.

Since fiber lasers had short coupling delays, with respect to relaxation frequency, mutual coupling might be expected to play an important role in synchronization. To compare the mutually coupled and the purely driven geometries, we simulated the case of Lasers 1 and 3 being driven by a common input, Laser 2, but with no feedback from Lasers 1 and 3 into Laser 2. Synchronization results are given in Fig. 12. These results were computed in the same way as for Fig. 11. Isochronal synchrony does begin to occur between Lasers 1 and 3 due to their common input, but they require a stronger coupling than in the previous case. It appears that generalized synchrony plays a role in the synchronization of the outer fiber lasers, but the mutual coupling between the outer and center lasers is also important.
FIG. 12: Cross correlation vs. coupling for unidirectionally driven lasers (Laser 2 driving Lasers 1 and 3). Solid line and filled points: Laser 1 and Laser 3 compared with zero time shift. Dashed line and open circles: Laser 1 compared to Laser 2 with a lag of the coupling delay time.

DISCUSSION

Using three mutually delay coupled lasers in a line, we have shown that synchronization exists in two different laser systems as a result of coupling architecture. The two types of lasers we have considered are semiconductor lasers and fiber ring lasers. Individually, the semiconductor lasers are low dimensional (as represented by intensity and population inversion), while the fiber ring lasers are considered spatio-temporal since there exist on the order of 1000 modes coupled within each laser.

Since the coupling is done using finite lengths of fiber, the lasers communicate with finite delay. Due to symmetric coupling, a solution exists where the dynamics of the outer lasers are identical. If this symmetric solution has global stability, then the outer lasers will become synchronized with zero lag. This architecture is in contrast to two lasers mutually coupled with delay, for which the solutions are synchronized but with a lag equal to the coupling delay.

In general, stability of the zero lag synchronized state is difficult to show for the case of nonlinear mutually coupled systems. However, we have been able to analyze the local stability of the synchronized state for the case of coupled semiconductor lasers with long delays, where the delay is typically long compared to the relaxation oscillation frequency. In this case, it can be demonstrated that synchronization is explicitly dependent on dissipation.
in the internal dynamics of the outer lasers, so that synchronization is due to “washing out” of the difference in initial conditions.

It is worthwhile to note that this sort of synchronization where the sum of Lyapunov exponents has a negative linear dependence on dissipation is also seen in the context of generalized synchronization in driven dissipative systems. In this case, there is unidirectional coupling from the driver to the response system with the onset of generalized synchronization whenever the dynamics of the response system become a function of the driver. While in the case of mutually coupled semiconductor lasers with long delays, it is clear that the signal the outer lasers receive from the center laser is affected by mutual coupling, the dependence of transverse Lyapunov stability on dissipation in the outer lasers indicates that the outer lasers synchronize due to a common signal from the middle laser. This explains the linear dependence of transverse Lyapunov exponents on dissipation, since two identical response systems will synchronize to the common signal from the driver, provided there is some internal dissipation in the response systems themselves to “wash out” any difference in initial conditions. It is perhaps not too surprising that for sufficiently long delays the chaotic synchronization phenomena may appear to be similar to driven systems, since on the time scale of the delay time, the dynamics of the outer lasers do not affect the mutual coupling term in their equations, just as the driven system cannot affect the signal it receives from the driver. It follows that for sufficiently long delays, the middle laser can be viewed as driving the dynamics of the outer lasers. Then, complete synchronization of the outer lasers is the result of generalized synchronization between the middle and the outer lasers. The cases where the outer lasers seem to anticipate the dynamics of the middle laser (due to the detuning values used), need not be excluded since anticipatory synchronization is seen in purely driven systems as well, as was discussed in the introduction to this chapter.

While for coupled semiconductor systems, chaotic synchronization seems in general to improve with longer delays, long delays are not necessary to obtain robust synchronization. There are many examples of synchronization in mutually coupled systems without delays, including semiconductor lasers. Fiber lasers serve as another example of a laser system that synchronizes for relatively short delays, where the coupling delay is short compared to the relaxation oscillation frequency. In contrast to semiconductor lasers, which will settle into a steady state in the absence of mutual coupling, fiber lasers will continue to oscillate, sometimes showing complicated dynamics even when left uncoupled. This is due to intrinsic
noise from spontaneous emission in each laser. Further research is needed to elucidate the phenomena behind fiber laser synchronization. One possible approach may be to model phase synchronization in the limit of weak coupling, in the regime where mutual coupling affects only the phase and not the amplitude of the lasers.

There are also a number of questions that remain to be answered in regard to synchronization in semiconductor lasers. One interesting question is the effect of delays on synchronization. While we have shown that for sufficiently long delays, optoelectronically coupled lasers will synchronize, it remains to be explained why shortening the delays sometimes leads to desynchronization. This issue becomes especially interesting when the delays are still long compared to the oscillation time but not long enough to result in synchronization. In this case, some non-trivial resonance-like phenomenon may be occurring, which requires further investigation. A related question that could be addressed is the decorrelation time of the signals as the coupling delay is increased. Thus for short coupling delays in semiconductor lasers, the dynamics are more regular, and synchronization may indeed hinge on the regularity of the signal and self-correlations in the dynamics. In this case, the generalized synchronization idea may not play an important role, since the “driving signal” from the middle laser adjusts itself quickly to the current dynamics of the outer lasers. At longer delays, however, the dynamics become decorrelated, and some measure of mutual information would be beneficial to quantify this phenomena.

Further study of both types of laser systems discussed here may lead to a better understanding of the key requirements for synchronization and the important role played by the linear coupling architecture. In addition, these ideas may help in designing coupling architectures to synchronize larger numbers of lasers.

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