A Remark on the Three Dimensional Baroclinic Quasigeostrophic Dynamics

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Abstract

We consider time-periodic patterns of the dissipative three dimensional baroclinic quasigeostrophic model in spherical coordinates, under time-dependent forcing. We show that when the forcing is time-periodic and the spatial square-integral of the forcing is bounded in time, the model has time-periodic solutions.

Keywords— quasigeostrophic fluid model, dissipative dynamics, time-periodic motion, nonlinear analysis.

Short running title:
Three Dimensional Quasigeostrophic Dynamics

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1 Introduction

The three dimensional baroclinic quasigeostrophic model is an approximation of the rotating Euler (inviscid case) or Navier-Stokes (viscous case) equations in the limit of zero Rossby number, i.e., at asymptotically high rotation rate; see, for example, [1], [2], [3], [4], [5], [6], and [7].

The well-posedness for the invisid three dimensional baroclinic quasigeostrophic model was studied in, for example, [8] and [4], and for the viscous three dimensional baroclinic quasigeostrophic model in [9]. The invisid baroclinic quasigeostrophic model is a Hamiltonian system ([10]). The work mentioned above and most other research about the baroclinic quasigeostrophic model has been on the Cartesian, $\beta-$plane version of this model.

Wang ([11]) discussed global attractors for the viscous three dimensional baroclinic quasigeostrophic model in spherical coordinates, under time-independent forcing.

In this paper, we consider the same viscous three dimensional baroclinic quasigeostrophic model, but under time-dependent forcing. We show that when the forcing is time-periodic and when the spatial square-integral of the forcing is bounded in time, the forced viscous three dimensional quasigeostrophic model has time-periodic solutions. We use a topological technique from nonlinear global analysis ([12]).

2 Dissipative Dynamics and Time-Periodic Motion

We consider the three dimensional baroclinic quasigeostrophic equation, in nondimensional form, for the atmospheric dynamics in spherical coordinates $\phi, \theta, \zeta$ as in [11]

\[
(A\psi)_t + J(\psi, Ro A\psi + 2\cos\theta) - \frac{1}{Re}A^2\psi = f(\phi, \theta, \zeta, t),
\]

where $\psi(\phi, \theta, \zeta, t)$ is the stream function, $f(\phi, \theta, \zeta, t)$ is the time-dependent forcing such as external source of heating, and $Ro, Re$ are the Rossby number, Reynolds number, respectively. The space domain for the equation is $D = S^2 \times (\zeta_0, 1)$ with $S^2$ the two dimensional unit sphere and $\zeta_0$ fixed: $1 > \zeta_0 \geq 0$. 
The operators appeared in the equation (1) are

\[ \Delta = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right], \]

\[ A = \Delta + \frac{\partial}{\partial \zeta} (N^2 \frac{\partial}{\partial \zeta}), \]

\[ J(p, q) = \frac{\partial p}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial q}{\partial \phi} - \frac{1}{\sin \theta} \frac{\partial p}{\partial \phi} \frac{\partial q}{\partial \theta}, \]

where \( N(\phi, \theta, \zeta) > 0 \) is a known smooth function related to the stratification frequency.

The equation (1) is supplemented by the following boundary and initial conditions

\[ \frac{\partial \psi}{\partial \zeta} = 0 \quad \text{if} \quad \zeta = \zeta_0, \]  
(2)
\[ \frac{\partial \psi}{\partial \zeta} + a \psi = \frac{\partial (A \psi)}{\partial \zeta} + a A \psi = 0 \quad \text{if} \quad \zeta = 1, \]  
(3)
\[ \psi|_{t=0} = \psi_0(\phi, \theta, \zeta), \]  
(4)

where \( a(\phi, \theta) > 0 \) is a known smooth function related to heat transfer between the atmosphere and the earth, and \( \psi_0 \) is initial data.

Denote \( L^2(D) \) as the space of square-integrable functions, with the standard norm \( \| \cdot \| \). The problem (1), (2), (3), (4) is well-posed (14) in \( L^2(D) \), with solution \( \psi(\phi, \theta, \zeta, t) \) at least continuous in time \( t \).

We address the issue of whether there are any time-periodic solutions in the nonlinear forced dissipative quasigeostrophic dynamics modeled by (1), (2), (3). To this end we further assume that the forcing \( f(t) := f(\phi, \theta, \zeta, t) \) is periodic in time with period \( T > 0 \), and the spatial square-integral of the forcing \( \| f(t) \| \) is bounded in time, i.e., \( \| f(t) \| \) is bounded by a time-independent constant. Then we can follow (14) exactly to show that there is a bounded absorbing set in \( L^2(D) \) (we omit this part). That is, all solutions \( \psi \) enter a bounded set \( \{ \psi : \| \psi \| \leq C(Re, Ro) \} \) as time goes to infinity. The system (1), (2), (3) is therefore a dissipative system as defined in (14) (also (13) or (14)).

We now recall a result from (14), page 235, that a \( T \)-time-periodic nonautonomous dissipative dynamical system in a Banach space has at least one \( T \)-time-periodic solution. This result follows from a Leray-Schauder topological degree argument and the Browder’s principle (14). So the system (1),
Theorem 1 Assume that the forcing \( f(\phi, \theta, \zeta, t) \) is time-periodic with period \( T > 0 \), and its spatial square-integral with respect to \( \phi, \theta, \zeta \) is bounded in time. Then the forced viscous three dimensional baroclinic quasigeostrophic model

\[
(A\psi)_t + J(\psi, Ro A\psi + 2\cos\theta) - \frac{1}{Re} A^2 \psi = f(\phi, \theta, \zeta, t),
\]

and

\[
\frac{\partial \psi}{\partial \zeta} = \frac{\partial (A\psi)}{\partial \zeta} = 0 \quad \text{if} \quad \zeta = \zeta_0,
\]

\[
\frac{\partial \psi}{\partial \zeta} + a\psi = \frac{\partial (A\psi)}{\partial \zeta} + aA\psi = 0 \quad \text{if} \quad \zeta = 1,
\]

has at least one time-periodic solution with period \( T > 0 \), for some square-integrable initial data \( \psi_0 \).

3 Discussions

Forced coherent structures in the two dimensional baroclinic model were studied in [15]. A wind forced two dimensional baroclinic model was also used in the study of multiple geophysical equilibria (e.g., [16]). In general, it is very difficult to show existence of periodic coherent structures in spatially extended physical systems. In this paper, we have shown that the three dimensional time-periodic quasigeostrophic patterns may form due to time-periodic forcing such as external source of heating.

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