AN EFFICIENT WEAK EULER-MARUYAMA TYPE APPROXIMATION SCHEME OF VERY HIGH DIMENSIONAL SDES BY ORTHOGONAL RANDOM VARIABLES

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ABSTRACT. We will introduce Euler-Maruyama approximations given by an orthogonal system in \( L^2[0,1] \) for high dimensional SDEs, which could be finite dimensional approximations of SPDEs. In general, the higher the dimension is, the more one needs to generate uniform random numbers at every time step in numerical simulation. The scheme proposed in this paper, in contrast, can deal with this problem by generating only single uniform random number at every time step. The scheme saves the time for simulation of very high dimensional SDEs. In particular, we will show that Euler-Maruyama approximation generated by the Walsh system is efficient in high dimensions.

1. Introduction

1.1. Simulation of high dimensional SDE. Let \( X \) be a unique solution of the following \( d \)-dimensional (time-homogeneous Markovian type) stochastic differential equation (SDE in short)

\[
(1.1) \quad dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad 0 \le t \le T, \quad X_0 = x_0 \in \mathbb{R}^d.
\]

Here, \( W \) is a \( d \)-dimensional Brownian motion starting at the origin, and the coefficients \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \) are sufficiently regular. Our purpose in this paper is to provide efficient weak approximations for the quantity \( \mathbb{E}[f(X_T)] \) in high dimensions for a given function \( f \). The quantity can mean, for example, a fair price of European type derivatives in financial market, where \( T > 0 \) is the maturity and \( f : \mathbb{R}^d \to \mathbb{R} \) is a pay-off function. In financial practice, its numerical value is of most importance. With an explicit finite dimensional expression, the problem reduces to a standard numerical analysis such as approximation of a finite — desirably less than three — dimensional integral using Riemann sum, but except for some simple cases such an expression is not available. A simplest but most frequently used way to reduce it to a finite dimensional integration is so-called Euler-Maruyama approximation (EM scheme in short), which is typically given by:

\[
(1.2) \quad X_0^{(n)} = x_0, \quad X_{\ell T/n}^{(n)} = X_{(\ell-1)T/n}^{(n)} + \sigma(X_{(\ell-1)T/n}^{(n)}) \cdot (W_{\ell T/n} - W_{(\ell-1)T/n}) + b(X_{(\ell-1)T/n}^{(n)}) \frac{T}{n}, \quad \ell \in \{1, 2, \ldots, n\},
\]

where the dimension is still very high, and in the limit it is infinite dimensional. The numerical integration is calculated, or should we say, simulated, by Monte-Carlo method.

It is widely-recognized that (cf. [5, 9]), apart from the numerical integration error, the finite dimensional reduction error is of order \( n^{-1/2} \) in the strong sense, and of order \( n^{-1} \) in the weak sense, under some regularity conditions on \( \sigma \) and \( b \). Moreover, even if the system \( [W_{\ell T/n} - W_{(\ell-1)T/n}]_{\ell=1}^n \) (called the Gaussian system hereinafter) is replaced with random variables “simulating Brownian increments”, by which we mean random variables sharing moments up to some order with the increments of standard Brownian motion, the weak error is still of order \( n^{-1} \). This implies that we can change and select the system in accordance with various purposes.

1.2. Contributions of the present paper. In this paper, we provide two systems generated by an orthogonal system of \( L^2[0,1] \) in order to deal with SDEs in “very high dimension”, by which we mean, let say, \( d \sim 2^{32} \). In such a high dimension, generating Gaussian system is heavily time consuming since we need to generate as many uniform random numbers at every time step. In contrast, our schemes only use a single uniform random number.

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It is true that, mathematically speaking, any Gaussian system could also be generated from a single uniform random variable if we were given an ideally uniform one from \([0, 1]\), which is equivalent to infinite binary distributed random numbers. In reality, however, a uniform random number is actually a finite sequence of binary random numbers, which is, by the dyadic expansion, equivalent to uniform random integers in a finite set. The Mersenne twister\(^*\), the most reliable pseudo random number generating algorithm offered by Matsumoto-Nishimura \([8]\), generates uniform 32-bit numbers, or equivalently, uniform integers over \([1, 2, \ldots, 2^{32}]\).

The very heart of our schemes is an algorithm to generate “simulating Brownian increment” in very high dimension out of a given set of uniform integers. We will present two distinct schemes; one is based on the Haar system of orthogonal functions in \(L^2[0, 1]\), and the other comes from the Walsh system. We shall show, both theoretically (Theorem 3.3) and experimentally (Tables 1-6), that the latter is more efficient than the former and the Gaussian scheme in very high dimensions. In the theoretical result, the error is estimated with a constant depending on the dimension, which enables us to evaluate the efficiency.

Our proposed Walsh-based scheme is composed of products of odd number of random signs (defined in section 2.3). We present a practical algorithm to output such random signs in Theorem 2.4, which is in itself interesting mathematically.

The scheme can be used to simulate SPDEs, though in this paper we will not present any theoretical results. In the following section we will give an illustration how it can be applied.

1.3. Simulation of SPDE. The present study of simulating such a very high dimensional SDE is motivated by an interest for numerical simulations for stochastic partial differential equations (SPDE in short), which is typically given as

\[
\frac{du}{dt} = Au + f(u, \nabla u) + \xi,
\]

where

- the solution \(u \equiv u(t, x)\) is desirably a differentiable function in \((t, x) \in (0, \infty) \times \mathbb{R}^k\), though often it is only guaranteed to be something much weaker,
- \(A\) is a differential operator,
- \(f\) is an elementary function, often a polynomial in \(u\) (and \(\nabla u\)), and
- \(\xi\) is so-called space-time white noise, meaning roughly that time derivative of space-time Gaussian system whose covariance may depend on \(u\).

Such a SPDE is often understood as an SDE in an infinite dimensional Hilbert space; by an identification \(u = X \in H\),

\[
dX_t = B(t, X_t)dt + F(t, X_t)dt, \quad 0 \leq t \leq T.
\]

Here, \(W\) is a \(U\)-valued Wiener process with covariance operator \(Q, F : [0, T] \times \Omega \times H \to H\) is a lift of \(Au + f(u, \nabla u)\), and \(B : [0, T] \times \Omega \times H \to L(Q^{1/2}U, H)\), the set of bounded linear operators from \(Q^{1/2}U\) to \(H\), such that \(B(t, X_t)dt\) behaves like \(\xi\). Here \(H\) and \(U\) are separable Hilbert spaces (Refer to \([6, 7, 10]\) for a more precise/detailed formulation).

As in the finite dimensional SDE case, we are concerned with obtaining an approximate quantity of \(\mathbb{E}[f(X_T)]\) for a test function \(f : H \to \mathbb{R}\). Two kinds of discretizations are required in order to build a practical algorithm. First, we apply the Galerkin finite element method \([11]\) to the space discretization: let \(d \in \mathbb{N}\) and \(P^{(d)} : H \to \text{span}(h_1, h_2, \ldots, h_d) \cong \mathbb{R}^d\) be the projection. Then the Galerkin approximation of the equation (1.3) is given by a solution of the \(d\)-dimensional Galerkin SDE,

\[
\frac{dq_{X_t}^{(d)}}{dt} = P^{(d)}B(t, X_t^{(d)})dt + \left(P^{(d)}F(t, X_t^{(d)})\right) dt, \quad 0 \leq t \leq T, \quad X_0^{(d)} = P^{(d)}X_0 \in \mathbb{R}^d.
\]

Then, we apply the Euler-Maruyama approximation of the equation (1.4) by time discretization.

Given the motivations as above, we are interested in how the computational time and the discretization error grow as the dimension \(n\) increases.

\(^*\)Many variants have been proposed, improving the original one, mainly by Makoto Matsumoto and his collaborators.
1.4. **Notations.** Throughout this paper, we use $d, n, m \in \mathbb{N}$ as the dimension of SDEs, the number of partitions of the closed interval $[0, T]$ (the number of the time step on the closed interval $[0, T]$ for the EM scheme) and the number of Monte-Carlo trials, respectively. $\delta_{i,j}$ denotes the Kronecker delta, $i, j \in \mathbb{N}$, i.e., if $i = j$ holds, then $\delta_{i,j} = 1$, otherwise, if $i \neq j$ holds, then $\delta_{i,j} = 0$. Components of a vector are denoted by superscripts without parentheses. Row vectors of a matrix are denoted by superscripts without parentheses. Column vectors of a matrix are denoted by subscripts. On the other hand, superscripts with parentheses and subscripts mean the dependence on their parameters.

The Euclidean inner product on $\mathbb{R}^d$ is denoted by $x \cdot y := \sum_{i=1}^d x_i y_i$, $x, y \in \mathbb{R}^d$. The Euclidean norm on $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ are denoted by $|x| := \sum_{i=1}^d |x_i|^2$, $x \in \mathbb{R}^d$ and $|x| := \sum_{i,j=1}^d |x_i|^2$, $x \in \mathbb{R}^d \otimes \mathbb{R}^d$, respectively.

Let $h \in \{1, 2, 3, 4\}$. The space of real valued polynomial growth bounded functions on $\mathbb{R}^d$ with polynomial growth bounded continuous derivatives up to $h$ is denoted by $C^h_{\mathbb{R}^d}$, that is

$$C^h_{\mathbb{R}^d} := \left\{ f \in C^h(\mathbb{R}^d); \quad |f(y)| \vee \max_{i, j, k \in [1, 2, \ldots, h]} \left| \frac{\partial^k f(y)}{\partial y^i_1 \partial y^i_2 \cdots \partial y^i_r} \right| \leq C \left(1 + |y|^{2r}\right) \right\}.$$

1.5. **Outline.** This article is divided as follows: In Section 2, we will introduce two Euler-Maruyama approximations generated by the Haar system and the Walsh system for high dimensional SDEs. Moreover, we will show that the 1st, 2nd and 3rd moments of these systems are the same ones of the Gaussian system. In Section 3, we will state the error estimate for the weak convergence of the EM scheme having the general system. This will imply that the EM schemes by the Haar system and the Walsh system have the same weak order of the EM scheme by the Gaussian system. As the same time, our estimate suggests that the error grows very rapidly as the dimension gets higher in the Haar case, while the scheme with the Walsh system behaves far more nicely. In Section 4, to confirm the theoretical result in the previous section, we will compare the EM schemes by the Gaussian system, the Haar system and the Walsh system through some numerical experiments. In particular, the results show that the EM scheme by the Walsh system is efficient. In Section 5 (Appendix), we will prove the error estimate stated in Section 3 using the Itô Taylor expansion (the Wagner-Platen expansion).

2. **Euler-Maruyama approximation for High Dimensional SDEs**

2.1. **Simulation by mimicking random variables.** In this section, we introduce two efficient algorithms for Euler-Maruyama approximation of a high dimensional SDE (1.1) by an orthogonal random variable after introducing the framework for the schemes to work on.

Let $n \in \mathbb{N}$ be the number of partitions of the closed interval $[0, T]$ and $(\Delta Z_{\ell}^{(n)})_{\ell=1}^n$ be $d$-dimensional i.i.d. random variables. We consider the following Euler-Maruyama approximation of the equation (1.1) given by $(\Delta Z_{\ell}^{(n)})_{\ell=1}^n$:

$$X_0^{(n)} = x_0, \quad X_{\ell T/n}^{(n)} = X_{(\ell-1)T/n}^{(n)} + \sigma(X_{(\ell-1)T/n}^{(n)}) \cdot \Delta Z_{\ell}^{(n)} + b(X_{(\ell-1)T/n}^{(n)}) \frac{T}{n}, \quad \ell \in \{1, 2, \ldots, n\}.$$

Here, $0 < T/n < 2T/n < \ldots < nT/n = T$ implies an equal time step on $[0, T]$. We suppose that the system $(\Delta Z_{\ell}^{(n)})_{\ell=1}^n$ satisfies that for any $\ell \in \{1, 2, \ldots, n\}$,

$$\mathbb{E}\left[(\Delta Z_{\ell}^{(n)})^j\right] = 0, \quad \forall j_1 \in \{1, 2, \ldots, d\}$$

and

$$\mathbb{E}\left[(\Delta Z_{\ell}^{(n)})^j(\Delta Z_{\ell}^{(n)})^{j_2}\right] = \frac{T}{n} \delta_{j_1, j_2}, \quad \forall j_1, j_2 \in \{1, 2, \ldots, d\},$$

and

$$\mathbb{E}\left[(\Delta Z_{\ell}^{(n)})^j(\Delta Z_{\ell}^{(n)})^{j_2}(\Delta Z_{\ell}^{(n)})^{j_3}\right] = 0, \quad \forall j_1, j_2, j_3 \in \{1, 2, \ldots, d\}.$$
Then the Euler-Maruyama approximation of (1.1) given by \((\Delta Z^{(n)}_{\ell})_{\ell=1}^n\) has weak order 1 even if \((\Delta Z^{(n)}_{\ell})_{\ell=1}^n\) is the Gaussian system, which will be made more precise and proven as a corollary to a more general theorem. Note that without (2.3) we can only prove that it has weak order 1/2 in Theorem 3.3 below.

Our objective in the following subsections is to find a system \((\Delta Z^{(n)}_{\ell})_{\ell=1}^n\) having the following conditions.

- \((\Delta Z^{(n)}_{\ell})_{\ell=1}^n\) satisfies the moment conditions (2.2), (2.3) and (2.4).
- The Euler-Maruyama approximation given by \((\Delta Z^{(n)}_{\ell})_{\ell=1}^n\) is more efficient than the standard Euler-Maruyama approximation in high dimensions. Here efficiency is measured in computational time balanced with its accuracy.

Our strategy is to generate good many random numbers out of a single uniform random number in order to reduce the time consumed to generate random numbers in high dimensions. Below we introduce two distinct constructions, one is from Haar functions, and the other is from Walsh functions. As is already discussed, the generating random variable is practically uniform on the set of integers, or equivalently, on a binary set \([0, 1]^K\) for some \(K \in \mathbb{N}\).

### 2.2. Mimicking by the Haar system

Let \(K \in \mathbb{N}\) with \(d \leq 2^{K-1}\) and

\[
h_{k,j}(x) = \begin{cases} 2^{(K-1)/2} & x = 2k - 1 \\ -2^{(K-1)/2} & x = 2k \\ 0 & \text{otherwise}, \end{cases} \quad x \in \{1, 2, \ldots, 2^K\}, \quad k \in \{1, 2, \ldots, 2^K\}.
\]

Define

\[(\Delta Y^{(n)})^j = h_{k,j}(U)\sqrt{\frac{T}{n}}, \quad j \in \{1, 2, \ldots, d\},
\]

where \(U\) is a random variable distributed uniformly over \(\{1, 2, \ldots, 2^K\}\).

**Proposition 2.1.** The \(d\)-dimensional random variable \(\Delta Y^{(n)}\) satisfies (2.2), (2.3) and (2.4).

**Proof.** Let \(p \in \mathbb{N}\) and \(j_1, j_2, \ldots, j_p \in \{1, 2, \ldots, d\}\). We obtain

\[
\mathbb{E} \left[ \prod_{k=1}^p h_{k,j}(U) \right] = \sum_{x=1}^{2^K} \left( \prod_{k=1}^p h_{k,j}(x) \right) \mathbb{P}(U = x) 
= \sum_{x=1}^{2^K} \left( \prod_{k=1}^p 2^{(K-1)/2} \mathbf{1}_{\{2k-1\}}(x) - 2^{(K-1)/2} \mathbf{1}_{\{2k\}}(x) \right) \frac{1}{2^K} 
= \begin{cases} 2^{p(K-1)/2 - K} \sum_{x=1}^{2^K} \left( \mathbf{1}_{\{2j_1-1\}}(x) - \mathbf{1}_{\{2j_1\}}(x) \right)^p & j_1 = j_2 = \cdots = j_p \\ 0 & \text{otherwise} \end{cases} 
= \begin{cases} 2^{(p-2)(K-1)/2} & \text{if } p \text{ is even and } j_1 = j_2 = \cdots = j_p \\ 0 & \text{otherwise}. \end{cases}
\]

Equation (2.6) implies (2.2), (2.3) and (2.4). \(\square\)

### 2.3. Mimicking by the Walsh system

Let \(K \in \mathbb{N}\) with \(d \leq 2^{K-1}\). We will denote by \(\tau = (\tau_1, \tau_2, \ldots, \tau_K)\) an element of the finite product set \([-1, 1]^K\) of a two-point set \([-1, 1] \subset \mathbb{R}\). Endowed with the uniform distribution on \([-1, 1]^K\), the free product of the coordinate maps \(\tau \mapsto \tau_i, i \in \{1, 2, \ldots, N\}\) will be called the **random sign**. Note that \(\tau_1, \tau_2, \ldots, \tau_K\) are mutually independent and distributed as \(\mathbb{P}(\tau_1 = \pm 1) = 1/2\). Moreover, elements of \(\mathcal{W}_K := \{\tau_S := \prod_{j \in S} \tau_j ; S \subset \{1, 2, \ldots, K\}\}\) are identically distributed, but they are not necessarily independent. However, if \(S\) and \(S'\) are disjoint, then \(\tau_S\) and \(\tau_{S'}\) are orthogonal (their covariance is 0, i.e., their correlation is 0). Indeed, we obtain

\[\tau_S \tau_{S'} = \tau_{S \Delta S'} \tau_{S \Delta S'}^2 = \tau_{S \Delta S'}^2.\]
Thus by the independence, we have

\[ \mathbb{E}[\tau_S \tau_{S'}] = \prod_{j \in S \cap S'} \mathbb{E}[\tau_j] = 0. \]

The system \( W_K \) apparently forms an orthonormal basis of the set of all functions on \([-1, 1]^K\) endowed with the uniform distribution. If we embed \([-1, 1]^K\) into \([0, 1]\) by the dyadic expansion, \( \bigcup_{K \in \mathbb{N}} W_K \) forms a complete orthogonal system of \( L^2[0, 1] \), which is often referred to as Walsh system.

To mimic the Brownian increments, we only use odd members of \( W_K \).

**Proposition 2.2.** Let \( \varphi \equiv \varphi^{(K)} : \{1, 2, \ldots, 2^{K-1}\} \to O_K := \{\tau_S ; S \subset \{1, 2, \ldots, K\}, \#S \text{ is odd}\} \) be a bijection, and set

\[ (\Delta Z^{(n)})^i = \varphi(i) \sqrt{n}, \quad j = 1, 2, \ldots, d. \]

Then the \( d \)-dimensional random variable \( \Delta Z^{(n)} \) satisfies (2.2), (2.3) and (2.4).

**Proof.** (2.2) is clear by the independence, and (2.3) also clear by (2.7). Let \( j_1, j_2, j_3 \in \{1, 2, \ldots, d\} \). Set \( \tau_{S_1} := \varphi(j_1) \), \( \tau_{S_2} := \varphi(j_2) \) and \( \tau_{S_3} := \varphi(j_3) \) for each corresponding \( S \subset \{1, 2, \ldots, K\} \) such that \( \#S \) is odd. Then we obtain

\[ \varphi(i_1) \varphi(i_2) \varphi(i_3) = \tau_{S_1 \cap S_2} \tau_{S_1 \cap S_3} \tau_{S_2 \cap S_3} = \tau_{S_1 \cap S_2} \tau_{S_3}. \]

On the other hand, \( S_1 \cap S_2 \) is never equal to \( S_3 \) since

\[ \#(S_1 \cap S_2) = \#S_1 - 2\#(S_1 \cap S_2) \]

is even and \( \#S_3 \) is odd. Hence (2.4) holds by (2.7). \( \square \)

**Remark 2.1.** We can also obtain higher order moments by considering the atom. For \( S \subset \{1, 2, \ldots, K\} \), we set \( S^1 := S \) and \( S^{-1} := S' \equiv \{1, 2, \ldots, K\} \setminus S \). Let \( p \in \mathbb{N} \). Then we obtain

\[ \prod_{k=1}^p \tau_{S_k} = \prod_{(i_1, i_2, \ldots, i_p) \in \{-1, 1\}^p} \tau_{\{i_1, i_2, \ldots, i_p\} \cap \{1, i_1, i_2, \ldots, i_p\}} \]

where \( \#(i_1, i_2, \ldots, i_p) = \sum_{k=1}^p \mathbf{1}[1](i_k) \) and \( \mathbf{1}[1] \equiv 1 \). Thus by the independence, we have

\[ \mathbb{E}\left[ \prod_{k=1}^p \tau_{S_k} \right] = \prod_{(i_1, i_2, \ldots, i_p) \in \{-1, 1\}^p} \mathbb{E}\left[ \tau_{\#(i_1, i_2, \ldots, i_p) \cap \{1, i_1, i_2, \ldots, i_p\}} \right] = \prod_{p} \mathbb{E}\left[ \tau_{\#(i_1, i_2, \ldots, i_p)} \right]. \]

As a practical scheme, the bijection \( \varphi \) should be algorithmically efficient in some sense, which we formulate mathematically as follows: let

\[ O_k := \{\tau_S ; S \subset \{1, 2, \ldots, k\}, \#S \text{ is odd}\}, \quad k \in \{1, 2, \ldots, K\}. \]

**Definition 2.3.** A map \( \varphi : \{1, 2, \ldots, 2^{K-1}\} \to O_K \) is called **odd-ordered** if it satisfies \( \varphi([1, 2, \ldots, 2^{k-1}]) = O_k \) for any \( k \in \{1, 2, \ldots, K\} \).

By the definition, an odd-ordered map is a bijection. Below we give an explicit odd-ordered map. We inductively define a map \( \varphi \) as follows.

\[ \varphi(k) := \begin{cases} 
\tau_1 & k = 1 \\
\varphi(k - 1) \tau_1 \tau_{\theta(k)} & k \in \{2, 3, \ldots, 2^{k-1}\}, 
\end{cases} \]
Moreover, this implies that \( \phi \{ k \} \) is bijective by (2.10). Furthermore, we can inductively show
\[
| \psi(2k) ; k \in \{ 1, 2, \ldots, 2^{k-1} \} | = O_k.
\]
This implies that \( \phi \) is odd-ordered by (2.9). \( \square \)
3. Error Estimates Depending on Dimension

In this section, we discuss the error estimate for the weak convergence of the Euler Maruyama approximation given by \((\Delta Z)^n\). In particular, we will see the weak order with respect to the number of the time step \(n\). This will imply that the EM scheme generated by the Haar system (2.5) and the Walsh system (2.8) have the same weak order of the EM scheme generated by the Gaussian system from Proposition 2.1 and 2.2 in Section 2. We suppose that the coefficients \(\sigma\) and \(b\) of the \(d\)-dimensional SDE \(dX_t = \sigma(X_t) dW_t + b(X_t) dt\) satisfy the following assumption.

**Assumption 3.1.** The coefficients \(\sigma\) and \(b\) satisfy the following conditions.

A1-1. There exists a positive constant \(C\) such that for any \(x, y \in \mathbb{R}^d\), \(|\sigma(y) - \sigma(x)| \vee |b(y) - b(x)| \leq C|y - x|\).

A1-2. There exists a positive constant \(C\) such that \(|b(y)| \vee |\sigma(y)| \leq C(1 + |y|)\).

A1-3. For any \(i, j \in [1, 2, \ldots, d]\), \(a^i_j, b^i \in C^4_b(\mathbb{R}^d)\).

The flow associated with the solution to the SDE is defined on the same filtered probability space since the Lipschitz condition A1-1 provides a unique strong solution to the SDE. The flow plays a very important role in considering separately for each partition of the time step. Note that this condition A1-1 is only used to guarantee a unique strong solution and not in the discussion of the error estimate.

Let \((X, W)\) be a solution of the \(d\)-dimensional SDE (1.1) with the initial value \(X_0 = x_0 \in \mathbb{R}^d\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \((\mathcal{F}_t)_{t \geq 0}\). Let \(T > 0, n \in \mathbb{N}\) be the number of partitions of the closed interval \([0, T]\) and \(t_\ell := \ell T/n, \ell \in \{1, 2, \ldots, n\}\) be an equal time step on \([0, T]\). Then we consider the Euler-Maruyama approximation \(X^{(n)}\) of the equation (1.1) given by \(d\)-dimensional random variables \((\Delta Z)^n\) which satisfy the following assumption.

**Assumption 3.2.** The \(d\)-dimensional random variables \((\Delta Z)^n\) satisfy the following conditions.

A2-1. \((\Delta Z)^n\) is \((\mathcal{F}_\ell)_{\ell = 1}^n\)-adapted.

A2-2. For any \(\ell \in \{1, 2, \ldots, n\}\), \(\Delta Z^{(n)}\) and \(\mathcal{F}_{\ell-1}\) are independent.

A2-3. For any \(j, j_1, j_2, j_3 \in \{1, 2, \ldots, d\}\) and \(\ell \in \{1, 2, \ldots, n\}\),

\[
    \mathbb{E} \left[ (\Delta Z^{(n)}_{\ell})^{i_1} \right] = 0, \quad \mathbb{E} \left[ (\Delta Z^{(n)}_{\ell})^{i_1} (\Delta Z^{(n)}_{\ell})^{i_2} \right] = \frac{T}{n} \delta_{j_1, j_2}, \quad \mathbb{E} \left[ (\Delta Z^{(n)}_{\ell})^{i_1} (\Delta Z^{(n)}_{\ell})^{i_2} (\Delta Z^{(n)}_{\ell})^{i_3} \right] = 0.
\]

Obviously, the Gaussian system satisfies Assumption 3.2. We see that the Haar system (2.5) and the Walsh system (2.8) also satisfy it from Proposition 2.1 and 2.2 in Section 2. Here, we do not assume that the independence between the elements of the system.

Moreover, we set for any \(p \in \mathbb{N}\),

\[
    M^{(n)}_p(Z) := \max_{\ell \in \{1, 2, \ldots, n\}} \mathbb{E} \left[ |\Delta Z^{(n)}_{\ell}|^p \right].
\]

Under these assumptions, we can get the following error estimate.

**Theorem 3.3.** Suppose that Assumptions A1 and A2 hold. Then for any \(f \in C^4_b(\mathbb{R}^d)\), there exists a positive constant \(C\) such that

\[
    \left| \mathbb{E}[f(X^{(n)}_T)] - \mathbb{E}[f(X_T)] \right| \leq C^{(n)}_r(Z) \left( n M^{(n)}_8(Z)^{1/2} + \frac{1}{n} \right),
\]

where

\[
    C^{(n)}_r(Z) = C \left( 1 + M^{(n)}_{4r}(Z)^{1/2} \right) \exp \left\{ C \left( 1 + n \left( M^{(n)}_2(Z) \vee M^{(n)}_{2(r+2)}(Z) \vee M^{(n)}_{2(4r+3)}(Z) \right) \right) \right\},
\]

and \(r := \min \left\{ r \in \mathbb{N}; \max_{i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, d\}, k \in \{1, 2, 3, 4\}} \left| \frac{\partial^k f(y)}{\partial y^{i_1} \partial y^{i_2} \cdots \partial y^{i_l}} \right| \vee \max_{i \in \{0, 1, \ldots, d\}} \max_{j \in \{1, 2, \ldots, d\}} \left| \frac{\partial^k \sigma^j(y)}{\partial y^{i_1} \partial y^{i_2} \cdots \partial y^{i_l}} \right| \leq C (1 + |y|^{2r}) \right\}, \sigma_0 := b.\)
Remark 3.1. We see that the Euler-Maruyama approximation generated by the Haar system (2.5) and the Walsh system (2.8) have the same weak order $n^{-1}$ as the Euler-Maruyama approximation generated by the Gaussian system from Proposition 2.1 and 2.2, Remark 2.1 and Theorem 3.3. However, the growth of the constant $C_r^{(n)}(Z)$ as $d$ gets larger will be different when we use (2.5) in Haar scheme and (2.8) in Walsh scheme, respectively, for the same $K \in \mathbb{N}$ such that $d \leq 2^{K-1}$. In Walsh scheme clearly we always have

$$M_r^{(n)}(Z) = \mathbb{E}\left[|\Delta Z^{(n)}_t|^2p\right] = d^p \left(\frac{T}{n}\right)^p,$$

while in Haar scheme, by (2.6),

$$M_r^{(n)}(Z) = d \left(\frac{T}{n}\right)^p (2^{K-1})^{p-1}.$$

They are equal only when $d = 2^{K-1}$, and other cases they are significantly different.

Remark 3.2. We assume that the coefficients $\sigma$ and $b$ and the test function $f$ are bounded, and they have bounded continuous derivatives up to 4. Then we can show that Theorem 3.3 with $r = 0$ holds.

Remark 3.3. Theorem 3.3 also holds even if we replace to the following conditions.

- the coefficients $b$ and $\sigma$ from time-independent coefficients to time-dependent coefficients
- the dimension of the Brownian motion from $d$ to a natural number that is truly smaller than $d$
- the initial value $X_0$ from a $\mathbb{R}^d$-valued constant to an integrable $\mathcal{F}_0$-measurable random variable
- the time step on the closed interval $[0, T]$ to an equal time step to a non-equal time step or a random time step generated by stopping times

This fact can be seen by replacing them in the proof of Theorem 3.3 (see Section 5).

4. Numerical experiments

In this section, we will compare the Euler-Maruyama approximations by the Gaussian system, the Haar system (2.5) and the Walsh system (2.8) through some numerical experiments. We will perform the following numerical experiments case 1 and case 2.

- case 1. In case 1, we consider the $d$-dimensional SDE (1.1). We will perform some numerical experiments for the quantity of $\mathbb{E}[f(X_T)]$ under the following conditions:
  - diffusion coefficient:
    $$\sigma^j(x^1, x^2, \ldots, x^d) = \begin{cases} x^{j-1} & j = i - 1 \\ x^{j} & j = i \\ x^{j+1} & j = i + 1 \\ 0 & \text{otherwise}, \end{cases} \quad (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d, \quad i, j \in \{1, 2, \ldots, d\},$$
  - drift coefficient: $b \equiv 0$,
  - initial value: $x_0 = (10, 10, \ldots, 10)$,
  - time horizon: $T = 1$,
  - test function:
    $$f(x^1, x^2, \ldots, x^d) = \left(\frac{1}{d}\sum_{i=1}^d x^i - 1\right) \vee 0, \quad (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d.$$

- case 2. In case 2, we consider the SPDE (1.4) called the semilinear stochastic heat equation with additive space-time white noise. This stochastic heat equation is formulated as follows:
  - separable Hilbert space: $H = L^2[0, 1]$ with the orthonormal basis $h_i = \sqrt{2} \sin(i\pi \cdot)$, $i \in \mathbb{N}$,
  - cylindrical Wiener process: $W_t = \sum_{i=1}^\infty h_i W^i_t$, $t \geq 0$, where $W^i$, $i \in \mathbb{N}$ are independent one-dimensional Brownian motions starting at zero,
  - $B \equiv 1$,
  - $F(h) = \Delta h + h/2$, $h \in \text{Dom}(\Delta) = H^2(0, 1) \cap H_0^1(0, 1)$, where $\Delta : \text{Dom}(\Delta) \to H$: the one-dimensional Laplace operator with Dirichlet boundary condition.
Further details about stochastic heat equations can be found in [3, 4, 6, 7, 10]. We will perform some numerical experiments for the quantity of $\mathbb{E}[f(X_T)]$ using the $d$-dimensional Galerkin SDE (1.4) under the following conditions:

- initial value: $X_0 = \sum_{i=1}^{\infty} h_i/i$,
- time horizon: $T = 1$,
- test function: $f(h) = \|h\|_H$, $h \in H$.

In particular, we fix $d \in \mathbb{N}$ and use the Linear-Implicit Euler-Maruyama approximation

$$(X^{(n)}_0)^i = \frac{1}{i}, \quad (X^{(n)}_{i+1})^\ell = \frac{1}{1 + i^2 \pi^2 T/n} \left( \left(1 + \frac{T}{2n}\right)(X^{(n)}_i)^\ell + (\Delta Z^{(n)}_\ell)^i \right), \quad \ell \in \{1, 2, \ldots, n\}, \quad i \in \{1, 2, \ldots, d\}$$

(cf. [4]) since the standard Euler-Maruyama approximation (2.1) is not appropriate (see Section 3.1 in [4]).

In the figures and tables below, we describe some numerical results about the sample mean, the sample variance and the processing time (sec). Here, $d$, $n$ and $m$ mean the dimension of the SDE, the number of the time step for the EM scheme and the number of Monte Carlo trials, respectively. The $x$-axis indicates the value of the dimension $d = 2^2, 2^3, \ldots, 2^{10}$ of the SDE in all the figures. The purple lines, the green lines and the blue lines are results about the EM schemes generated by the Gaussian system, the Haar system (2.5) and the Walsh system (2.8), respectively (In all the tables, they are abbreviated as Gaussian, Haar and Walsh, respectively). Here, we choose $\min(K \in \mathbb{N} ; d \leq 2^{K-1})$ as $K \in \mathbb{N}$ in Section 2.2 and 2.3 for all the results using the Haar system and the Walsh system to reduce the variance of the EM schemes by them as much as possible.

Tables 1-3 are results for case 1 and Tables 4-6 are results for case 2. Tables 1 and 4 show the results about the sample mean, Tables 2 and 5 show the results about the sample variance and Tables 3 and 6 show the results about the processing time (sec). In Table 4, the sample mean multiplied by $10^2$ is described, and in Table 5 the sample variance multiplied by $10^3$.

In case 1 and case 2, we can confirm that the EM scheme by the Haar system has the largest variance, and the variance of the EM schemes by the Gaussian system and the Haar system are almost the same. In case 1, the difference decreases as the dimension increases, and in case 2, it increases as the dimension increases. We can also confirm that the sample mean of the EM scheme by the Haar system for case 1 in the low dimension is not stable since the variance is large. In case 1, we can confirm that the processing time is faster in the order of the Haar system, the Walsh system and the Gaussian system. However, in case 2, the processing time of the Haar system and the Walsh system is almost the same.

5. Appendix: Proof of Theorem 3.3

In this section, we prove Theorem 3.3 in the same way as Theorem 14.5.2 in [5] using the Itô Taylor expansion (the Wagner-Platen expansion). We first introduce various notations.
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| d | n    | m    | Gaussian | Haar   | Walsh  |
|---|------|------|---------|--------|--------|
| 2^2 | 2^12 | 2^24 | 784.8221 | 12569.24 | 809.1716 |
| 2^3 | 2^11.75 | 2^23.5 | 533.2796 | 1318.584 | 531.5323 |
| 2^4 | 2^11.5 | 2^23 | 302.3776 | 489.3320 | 307.0605 |
| 2^5 | 2^11.25 | 2^22.5 | 166.7242 | 212.6647 | 163.5691 |
| 2^6 | 2^11 | 2^22 | 85.50918 | 96.03809 | 83.73009 |
| 2^7 | 2^10.75 | 2^21.5 | 43.45507 | 46.69187 | 43.25648 |
| 2^8 | 2^10.5 | 2^21 | 22.02093 | 22.78210 | 22.12202 |
| 2^9 | 2^10.25 | 2^20.5 | 11.11792 | 11.17828 | 11.12991 |
| 2^10 | 2^10 | 2^20 | 5.525419 | 5.547556 | 5.568346 |

Table 2. Sample variance for case 1

| d | n    | m    | Gaussian | Haar   | Walsh  |
|---|------|------|---------|--------|--------|
| 2^2 | 2^10 | 2^20 | 107.7026 | 19.83863 | 12.76452 |
| 2^3 | 2^10 | 2^20 | 234.2911 | 22.12289 | 21.04501 |
| 2^4 | 2^10 | 2^20 | 521.4949 | 38.91275 | 46.80439 |
| 2^5 | 2^10 | 2^20 | 999.3924 | 59.30260 | 91.72329 |
| 2^6 | 2^10 | 2^20 | 1838.725 | 109.3941 | 174.0424 |
| 2^7 | 2^10 | 2^20 | 3841.347 | 214.1789 | 335.8902 |
| 2^8 | 2^10 | 2^20 | 7332.045 | 424.1770 | 606.5542 |
| 2^9 | 2^10 | 2^20 | 13822.59 | 748.0165 | 1113.056 |
| 2^10 | 2^10 | 2^20 | 26175.23 | 1298.927 | 2099.693 |

Table 3. Time (sec) for case 1

| d | n    | m    | Gaussian | Haar   | Walsh  |
|---|------|------|---------|--------|--------|
| 2^2 | 2^12 | 2^24 | 7.476875 | 7.476218 | 7.477538 |
| 2^3 | 2^11.75 | 2^23.5 | 7.973660 | 7.973244 | 7.971283 |
| 2^4 | 2^11.5 | 2^23 | 8.196383 | 8.195982 | 8.198262 |
| 2^5 | 2^11.25 | 2^22.5 | 8.254288 | 8.251853 | 8.254530 |
| 2^6 | 2^11 | 2^22 | 8.230445 | 8.242443 | 8.236958 |
| 2^7 | 2^10.75 | 2^21.5 | 8.217783 | 8.198389 | 8.202224 |
| 2^8 | 2^10.5 | 2^21 | 8.159151 | 8.159707 | 8.172671 |
| 2^9 | 2^10.25 | 2^20.5 | 8.121781 | 8.112790 | 8.139412 |
| 2^10 | 2^10 | 2^20 | 8.083299 | 8.080273 | 8.088481 |

Table 4. Sample mean × 10^2 for case 2

5.1. Notations in this section.
Here, we see that Dom($\int_t^s f(u)du$) = $\{ f = (f(t))_{t \geq 0}; \text{ For any } t > 0, \int_0^t |f(u)|du < \infty, \}$.

$\int_t^s f(u)du = \left\{ \begin{array}{ll}
\int_s^t f(u)du & j = 0 \\
\int_s^t f(u)dW_u & j \in \{1, 2, \ldots, d\}
\end{array} \right.$, $f \in \text{Dom}(I_{s,t}^{(j)})$,

where

$$\text{Dom}(I_{s,t}^{(j)}) := \left\{ \begin{array}{ll}
f = (f(t))_{t \geq 0}; & \text{For any } t > 0, \int_0^t |f(u)|du < \infty,
\end{array} \right. \quad \text{for } j = 0$$

$$\left\{ \begin{array}{ll}
f = (f(t))_{t \geq 0}; & \text{f is a measurable process on } \Omega \text{ adapted to } (F_t)_{t \geq 0}
\end{array} \right. \quad \text{such that for any } t > 0, \int_0^t |f(u)|^2du < \infty \text{ a.s.}, \quad \text{for } j \in \{1, 2, \ldots, d\}.$$
The Itô coefficient functions is defined as follows: for any \( j \in \{0, 1, \ldots, d\} \), \( 0 \leq s \leq T \) and \( y \in \mathbb{R}^d \),

\[
\mathcal{L}^j f(s, y) := \begin{cases} 
\frac{\partial f(s, y)}{\partial s} + \sum_{r=1}^{d} b^r(y) \frac{\partial f(s, y)}{\partial y^r} + \frac{1}{2} \sum_{i, j, r, \ell} \sigma_i^r(y) \sigma_{t, \ell}^j (y) \frac{\partial^2 f(s, y)}{\partial y^i \partial y^\ell} & j = 0 \\
\sum_{i=1}^{d} \sigma_{t, \ell}^j (y) \frac{\partial f(s, y)}{\partial y^i} & j \in \{1, 2, \ldots, d\}
\end{cases}
\]

where

\[
\text{Dom}(\mathcal{L}^j) := \left\{ C^{1,2}([0, T] \times \mathbb{R}^d) \mid j = 0 \right\} \cap \left\{ C^0([0, T] \times \mathbb{R}^d) \mid j \in \{1, 2, \ldots, d\} \right\}
\]

The flow associated with the solution of the SDE is defined as follows: for any \( 0 \leq s \leq T \) and \( y \in \mathbb{R}^d \),

\[
X^{s,y}_t = y + \sum_{j=1}^{d} \int_s^t \sigma_j(X^{s,y}_{u})dW^j_u + \int_s^t b(X^{s,y}_u)d\ell_u, \quad s \leq t \leq T.
\]

Here, \( \sigma \) and \( b \) is the same coefficients as the SDE (1.1).

The functional associated with the flow for a fixed test function \( f \in C^1_p(\mathbb{R}^d) \) is defined as follows: for any \( 0 \leq s \leq T \) and \( y \in \mathbb{R}^d \),

\[
u(s, y) := \mathbb{E}[f(X^{s,y}_T)].
\]

We often use \( \int_{t_{i-1}}^t \cdot dW^j_s := \int_{t_{i-1}}^t \cdot ds \) and \( \sigma_0 := b \) to simplify the argument.

As is well known, we obtain the following two statements under Assumption 3.1. Theorem 5.1 states that the flow is a solution of the Kolmogorov backward equation. The expansion in Theorem 5.2 is called the (first order) Itô Taylor expansion or Wagner-Platen expansion.

**Theorem 5.1** (cf. Theorem 4.8.6 in [5]). Suppose that Assumption 3.1 holds. Then the functional \( u \) associated with the flow satisfies the following two statements.

- For any \( 0 \leq s \leq T \), \( u(s, \bullet) \in C^4_p(\mathbb{R}^d) \).
- For any \( 0 \leq s \leq T \), \( y \in \mathbb{R}^d \), \( \mathcal{L}^0 u(s, y) = 0 \).

**Theorem 5.2** (cf. Theorem 5.5.1 in [5]). Suppose that Assumption 3.1 holds. For any \( \ell \in \{1, 2, \ldots, n\} \),

\[
X^{\ell, t_{\ell-1}, \ell_{\ell-1}(n)}_{t_{\ell-1}} = \eta^{(n)}_{t_{\ell-1}}(t) + \sum_{j_1, j_2 \in \{0, 1, \ldots, d\}} I_{t_{\ell-1}, t}^{(j_1, j_2)} \mathcal{L}^{j_1} \sigma_{t_{\ell-1}, \ell}^{j_2}(X^{\ell, t_{\ell-1}, \ell_{\ell-1}(n)}_{t_{\ell-1}}), \quad t_{\ell-1} \leq t \leq t_\ell,
\]

where

\[
\eta^{(n)}_{t_{\ell-1}}(t) := X^{(n)}_{t_{\ell-1}} + \sum_{j=0}^{d} I_{t_{\ell-1}, t}^{(j)} \sigma_{t_{\ell-1}, \ell}^{j}(X^{(n)}_{t_{\ell-1}}), \quad t_{\ell-1} \leq t \leq t_\ell.
\]

Next, we give some lemmas to prove Theorem 3.3

### 5.2. Lemmas.

**Lemma 5.3.** For any \( p \in \mathbb{N} \), there exists a positive constant \( C \) such that for any \( j \in \{0, 1, \ldots, d\} \), \( \ell \in \{1, 2, \ldots, n\} \) and \( f \in \cap_{t \in [t_{\ell-1}, t_\ell]} \text{Dom}(I_{t_{\ell-1}, t}^{(j)}) \),

\[
\sup_{t \in [t_{\ell-1}, t_\ell]} \mathbb{E}\left[\left|I_{t_{\ell-1}, t}^{(j)} f\right|^{2p} \mid \mathcal{F}_{t_{\ell-1}}\right] \leq \begin{cases} 
\frac{C}{n^{2p-1}} \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\left[\left|f(s)\right|^{2p} \mid \mathcal{F}_{t_{\ell-1}}\right] ds & j = 0 \\
\frac{C}{n^{p-1}} \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\left[\left|f(s)\right|^{2p} \mid \mathcal{F}_{t_{\ell-1}}\right] ds & j \neq 0
\end{cases}
\]

**Proof.** The statement follows by using the Jensen’s inequality and Fubini’s theorem in \( j = 0 \) and using the Itô formula and Fubini’s theorem in \( j \neq 0 \). \qed
Lemma 5.4. Suppose that Assumption 3.1 holds. For any \( p \in \mathbb{N} \), there exists a positive constant \( C \) such that for any \( \ell \in \{1, 2, \ldots, n\} \),

\[
\sup_{t \in [t_{\ell-1}, t_\ell]} \mathbb{E} \left[ |X_t^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}}|^{2p} \mid \mathcal{F}_{t_{\ell-1}} \right] \leq C \left( 1 + \left| X_{t_{\ell-1}}^{(n)} \right|^2 \right)
\]

and

\[
\mathbb{E} \left[ \left| X_{t_{\ell-1}}^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} - X_{t_{\ell-1}}^{(n)} \right|^2 \mid \mathcal{F}_{t_{\ell-1}} \right] \leq \frac{C}{h^p} \left( 1 + \left| X_{t_{\ell-1}}^{(n)} \right|^2 \right).
\]

Proof. Let \( t \in [t_{\ell-1}, t_\ell] \). Using the Itô formula, we obtain

\[
\left| X_t^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2p} = \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + 2p \sum_{i \in \{1, 2, \ldots, d\}} \int_{t_{\ell-1}}^t \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-1)} b_s^i (X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{i} \, ds
\]

\[
+ 2p \sum_{i=1}^d \int_{t_{\ell-1}}^t \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-1)} \sigma_s^i (X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{i} \, ds
\]

\[
+ 2p(p-1) \sum_{i,j \in \{1, 2, \ldots, d\}} \int_{t_{\ell-1}}^t \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-2)} \sigma_s^i (X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{i} \sigma_s^j (X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{j} \, ds.
\]

Then by the Cauchy-Schwarz inequality, A2-1, the martingale property and Fubini’s theorem, we have

\[
\mathbb{E} \left[ \left| X_t^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2p} \mid \mathcal{F}_{t_{\ell-1}} \right] \leq \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + 2p \int_{t_{\ell-1}}^t \mathbb{E} \left[ \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-1)} \left| b_s^{(X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{i}} \right| \mid \mathcal{F}_{t_{\ell-1}} \right] \, ds
\]

\[
+ p(2p-1) \int_{t_{\ell-1}}^t \mathbb{E} \left[ \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-1)} \left| \sigma_s^{(X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})^{i}} \right|^2 \mid \mathcal{F}_{t_{\ell-1}} \right] \, ds.
\]

Thus by A1-2, we obtain

\[
\mathbb{E} \left[ \left| X_t^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2p} \mid \mathcal{F}_{t_{\ell-1}} \right] \leq \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + C \int_{t_{\ell-1}}^t \mathbb{E} \left[ \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2(p-1)} \left( 1 + \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^2 \right) \mid \mathcal{F}_{t_{\ell-1}} \right] \, ds
\]

\[
= \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + C(t - t_{\ell-1}) + C \int_{t_{\ell-1}}^t \mathbb{E} \left[ \left| X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2p} \mid \mathcal{F}_{t_{\ell-1}} \right] \, ds.
\]

Then by the Gronwall inequality, we have

\[
\mathbb{E} \left[ \left| X_t^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}} \right|^{2p} \mid \mathcal{F}_{t_{\ell-1}} \right] \leq \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + C(t - t_{\ell-1}) + C \int_{t_{\ell-1}}^t e^{C(t-s)} \left( \left| X_s^{(n)} \right|^{2p} + C(s - t_{\ell-1}) \right) \, ds
\]

\[
\leq \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + C(t - t_{\ell-1}) + C(t - t_{\ell-1}) e^{C(t-t_{\ell-1})} \left( \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} + C(t - t_{\ell-1}) \right)
\]

\[
= \left( 1 + C(t - t_{\ell-1}) e^{C(t-t_{\ell-1})} \right) \left( \left| X_{t_{\ell-1}}^{(n)} \right|^2 + C(t - t_{\ell-1}) \right)
\]

\[
\leq \left( 1 + CTe^{CT} \right) (1 \lor CT) \left( 1 + \left| X_{t_{\ell-1}}^{(n)} \right|^{2p} \right).
\]

Next, we obtain

\[
\left| \sum_{j=0}^d \int_{t_{\ell-1}}^{t_{\ell}} \sigma_j^{(X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})} \, dW_s^j \right|^{2p} = \left( \sum_{i=1}^d \sum_{j=0}^d \int_{t_{\ell-1}}^{t_{\ell}} \sigma_j^{(X_s^{t_{\ell-1}, X_{t_{\ell-1}}^{(n)}})} \, dW_s^j \right)^{2p}
\]
Thus by Lemma 5.5 A1-2 and the first statement, we have
\[
\mathbb{E} \left[ \left| X_{t_{1}}^{(n)} - X_{t_{1}^{-1}}^{(n)} \right|^{2p} \right| \mathcal{F}_{t_{1}^{-1}} \] 
\leq \frac{C}{n^{p-1}} \sum_{i=1}^{d} \sum_{j=0}^{d} \int_{t_{1}^{-1}}^{t} \mathbb{E} \left[ \left| \sigma_j^{(i, j)} (X_{s}^{(n)}) \right|^{2p} \right| \mathcal{F}_{t_{1}^{-1}} \] 
\leq \frac{C}{n^{p-1}} \int_{t_{1}^{-1}}^{t} \mathbb{E} \left[ 1 + \left| X_{s}^{(n)} \right|^{2p} \right| \mathcal{F}_{t_{1}^{-1}} \] 
\leq \frac{C}{n^{p-1}} \left( 1 + \left| X_{t_{1}^{-1}}^{(n)} \right|^{2p} \right).
\]

Lemma 5.5. Suppose that Assumption 3.1 and 3.2 hold. There exists a positive constant C such that for any \( k \in \{1, 2, 3\}, i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\} \) and \( \ell \in \{1, 2, \ldots, n\} \),
\[
\left| \mathbb{E} \left[ \prod_{j=1}^{k} (X_{t_{j_{1}}^{-1}}^{(n)} - X_{t_{j_{1}^{-1}}}^{(n)})^{y_{j}} - \prod_{j=1}^{k} (t_{j}^{(n)}(t_{\ell}) - X_{t_{j_{1}^{-1}}}^{(n)})^{y_{j}} \right| \mathcal{F}_{t_{1}^{-1}} \right| \leq \frac{C}{n^{2}} \left( 1 + \left| X_{t_{1}^{-1}}^{(n)} \right|^{6(r+1)} \right),
\]
where \( r := \min \left\{ r \in \mathbb{N} ; \max_{i_{1}, \ldots, i_{k} \in \{1, 2, \ldots, d\}} \max_{j_{1}, \ldots, j_{k} \in \{0, 1, \ldots, d\}} \left| \partial_{y_{j_{1}}^{y_{j_{1}}} \cdots y_{j_{k}}^{y_{j_{k}}}} \phi \right| \leq C \left( 1 + |y|^{2} \right) \right\} \).

Proof.

- We first show (5.1) for \( k = 1 \). By Theorem 5.2, the martingale property and Fubini’s theorem, we obtain
\[
\mathbb{E} \left[ (X_{t_{1}}^{(n)} - X_{t_{1}^{-1}}^{(n)})^{y_{1}} - (t_{1}^{(n)}(t_{\ell}) - X_{t_{1}^{-1}}^{(n)})^{y_{1}} \right| \mathcal{F}_{t_{1}^{-1}} \] 
\leq \frac{C}{n^{2}} \left( 1 + \left| X_{t_{1}^{-1}}^{(n)} \right|^{6(r+1)} \right).
\]

On the other hand, by the Cauchy-Schwarz inequality, A1-2 and A1-3, we have for any \( i \in \{1, 2, \ldots, d\}, j_{1}, j_{2} \in \{0, 1, \ldots, d\} \) and \( y \in \mathbb{R}^{d} \),
\[
\left| \mathbb{E} \left[ \prod_{j_{1}, j_{2} \in \{0, 1, \ldots, d\}} \right| \mathcal{F}_{t_{1}^{-1}} \right| \leq \frac{C}{n^{2}} \left( 1 + |y|^{4(r+1)} \right).
\]

Thus by Lemma 5.4, we obtain
\[
\left| \mathbb{E} \left[ (X_{t_{1}^{-1}}^{(n)} - X_{t_{1}^{-1}}^{(n)})^{y_{1}} - (t_{1}^{(n)}(t_{\ell}) - X_{t_{1}^{-1}}^{(n)})^{y_{1}} \right| \mathcal{F}_{t_{1}^{-1}} \right| \leq \frac{C}{n^{2}} \left( 1 + \left| X_{t_{1}^{-1}}^{(n)} \right|^{2(r+1)} \right).
\]
• Next, we show (5.1) for \( k = 2 \). We obtain

\[
\mathbb{E} \left[ \sum_{j=1}^{2} (X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}} - X_{t_{j-1}^{(n)}}^{(n)})^2 \mid \mathcal{F}_{t_{j-1}} \right]
\]

\[
= \mathbb{E} \left[ (\eta_{t_{j}^{(n)}}^{(n)}(t_{j}) - X_{t_{j}^{(n)}}^{(n)})^2 \mid \mathcal{F}_{t_{j-1}} \right] + \mathbb{E} \left[ (X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}} - \eta_{t_{j}^{(n)}}^{(n)}(t_{j}))^2 \mid \mathcal{F}_{t_{j-1}} \right] + \mathbb{E} \left[ (X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}} - \eta_{t_{j}^{(n)}}^{(n)}(t_{j}))^2 \mid \mathcal{F}_{t_{j-1}} \right].
\]

We estimate each term. By Theorem 5.2, the martingale property, Fubini’s theorem, the properties of the stochastic integral (cf. Proposition 2.1.1, 2.1.2 in [2]) and A2-1, we obtain

\[
\mathbb{E} \left[ (\eta_{t_{j}^{(n)}}^{(n)}(t_{j}) - X_{t_{j}^{(n)}}^{(n)})^2 \mid \mathcal{F}_{t_{j-1}} \right] = \sum_{j,j_{x} \in \{0,1,\ldots,d\}} \mathbb{E} \left[ f_{t_{j-1},t_{j}}^{(j)}(b_{j}^2 + \mathcal{N}_{t_{j-1},t_{j}}^{X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}}}) \mid \mathcal{F}_{t_{j-1}} \right]
\]

\[
= \sum_{j=1}^{d} \int_{t_{j-1}}^{t_{j}} \int_{t_{j-1}}^{t_{j}} \mathbb{E} \left[ f_{t_{j-1},t_{j}}^{(j)}(b_{j}^2 + \mathcal{N}_{t_{j-1},t_{j}}^{X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}}}) \mid \mathcal{F}_{t_{j-1}} \right] \, ds \, ds' \]

Thus by the Cauchy-Schwarz inequality, A2-1, A1-2, (5.2) and Lemma 5.4, we have

\[
\left| \mathbb{E} \left[ (\eta_{t_{j}^{(n)}}^{(n)}(t_{j}) - X_{t_{j}^{(n)}}^{(n)})^2 \mid \mathcal{F}_{t_{j-1}} \right] \right| \leq \frac{C}{n^2} \left( 1 + |X_{t_{j-1}^{(n)}}^{(n)}|^{2r+3} \right).
\]

On the other hand, by Theorem 5.2, the Cauchy-Schwarz inequality, Lemma 5.3, (5.2) and Lemma 5.4, we obtain

\[
\left| \mathbb{E} \left[ (X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}} - \eta_{t_{j}^{(n)}}^{(n)}(t_{j}))^2 \mid \mathcal{F}_{t_{j-1}} \right] \right| \leq \sum_{j_{1},j_{2} \in \{0,1,\ldots,d\}} \mathbb{E} \left[ f_{t_{j-1},t_{j}}^{(j_{1},j_{2})}(b_{j_{2}}^2 + \mathcal{N}_{t_{j-1},t_{j}}^{X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}}}) \mid \mathcal{F}_{t_{j-1}} \right]^{1/2} \mathbb{E} \left[ f_{t_{j-1},t_{j}}^{(j_{1},j_{2})}(b_{j_{2}}^2 + \mathcal{N}_{t_{j-1},t_{j}}^{X_{t_{j}^{(n)}}^{t_{j-1},X_{t_{j-1}^{(n)}}^{(n)}}}) \mid \mathcal{F}_{t_{j-1}} \right]^{1/2}.
\]
\[ \left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{4(r+1)}}. \]

Hence by (5.3)–(5.5), we have
\[
\mathbb{E} \left[ \sum_{i=1}^{2} \left( X_{t}^{(n)} - X_{t-1}^{(n)} \right)^{2} - \sum_{j=1}^{2} \left( \eta_{\ell}^{(n)}(t_{\ell}) - X_{t}^{(n)} \right)^{2} \right] \mathbb{E} \left( \left| \mathcal{F}_{t-1} \right| \right) \leq \left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{4(r+1)}}.
\]

Finally, we show (5.1) for \( k = 3 \). We obtain
\[
\left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{4(r+1)}}.
\]

We estimate each term. By Theorem 5.2, the Cauchy-Schwarz inequality, Lemma 5.3, A2-1, A1-2, 5.2 and Lemma 5.4, we have
\[
\left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{2(r+2)}}.
\]

Similarly, we obtain
\[
\mathbb{E} \left[ \left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{4(r+5)}} \right],
\]
\[
\mathbb{E} \left[ \left( \frac{C}{n^2} \right)^{1 + \left| X_{t-1}^{(n)} \right|^{6(r+1)}} \right].
\]
Thus by (5.6)-(5.8), we have
\[
\mathbb{E}
\left[
\prod_{j=1}^{3}
\left(n_{t-j+1}^{(n)}-n_{t-j}^{(n)}\right)^{i_{j}}
- \prod_{j=1}^{3}
\left(n_{t-j}^{(n)}-n_{t-j-1}^{(n)}\right)^{i_{j}} \right| F_{t-1}
\right]
\leq \frac{C}{n^{2}} \left(1 + X_{t-1}^{(n)} \right)\left(\sigma_{t}^{(n)} \right)^{2}.
\]

\[\square\]

**Lemma 5.6.** Suppose that Assumption 3.1 holds. For any \(\ell \in \{1,2,\ldots,n\},\)
\[
\mathbb{E}
\left[
\left|u(t_{\ell},\dot{X}_{t_{\ell}}^{(n)}-\dot{X}_{t_{\ell-1}}^{(n)}) - u(t_{\ell-1},\dot{X}_{t_{\ell-1}}^{(n)}) \right| F_{t_{\ell-1}}
\right]
= 0.
\]

**Proof.** By the Itô formula with Theorem 5.1 we obtain
\[
u(t_{\ell},\dot{X}_{t_{\ell}}^{(n)}-\dot{X}_{t_{\ell-1}}^{(n)}) - u(t_{\ell-1},\dot{X}_{t_{\ell-1}}^{(n)}) = \sum_{j=0}^{d}\int_{t_{\ell-1}}^{t_{\ell}} \mathcal{L}_{j}u(s,\bullet)|_{\dot{X}_{t_{\ell}}^{(n)}-\dot{X}_{t_{\ell-1}}^{(n)}} \ dW_{s}.
\]
Thus the statement follows by the martingale property and Theorem 5.1. \(\square\)

**Lemma 5.7.** Suppose that Assumption 3.1 and 3.2 hold. For any \(p \in \mathbb{N},\) there exists a positive constant \(C\) such that for any \(\ell \in \{1,2,\ldots,n\},\)
\[
\mathbb{E}
\left[
\left|X_{t_{\ell}}^{(n)}-X_{t_{\ell-1}}^{(n)}\right|^{2p} \right| F_{t_{\ell-1}} \right]
\leq C \left(1 + \left|X_{t_{\ell-1}}^{(n)}\right|^{2p} \right).
\]

**Proof.** By the Cauchy-Schwarz inequality, we obtain
\[
\sum_{i=1}^{d} a_{i}^{(n)}(X_{t_{\ell}}^{(n)})(\Delta Z_{\ell}^{(n)})^{i}
\leq \left|a(X_{t_{\ell}}^{(n)})\right|^{2} \left|\Delta Z_{\ell}^{(n)}\right|^{2}.
\]
Thus by A1-2, we have
\[
\left|X_{t_{\ell}}^{(n)}-X_{t_{\ell-1}}^{(n)}\right|^{2p}
\leq C \left(\left|a(X_{t_{\ell}}^{(n)})\right|^{2p} + \left|b(X_{t_{\ell}}^{(n)})\right|^{2p} \left|X_{t_{\ell-1}}^{(n)}\right|^{2p} \right)
\leq C \left(1 + \left|X_{t_{\ell-1}}^{(n)}\right|^{2p} \right).
\]
Hence by A2-1 and A2-2, we obtain
\[
\mathbb{E}
\left[
\left|X_{t_{\ell}}^{(n)}-X_{t_{\ell-1}}^{(n)}\right|^{2p} \right| F_{t_{\ell-1}} \right]
\leq C \left(1 + \left|X_{t_{\ell-1}}^{(n)}\right|^{2p} \right).
\]

Next, let \(k \in \{1,2,\ldots,\ell\}.\) By the Taylor expansion, there exists a \(d \times d\) diagonal matrix \(\theta_{k}\) such that for any \(i \in \{1,2,\ldots,d\},\) \((\theta_{k})_{ij} \in (0,1)\) and
\[
\left|X_{t_{k}}^{(n)}\right|^{2p} - \left|X_{t_{k-1}}^{(n)}\right|^{2p}
= \sum_{i=1}^{d} (X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{i}\left(X_{t_{k-1}}^{(n)}\right)^{i}
+ 2p(p-1) \left|X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)}\right|^{2p} \sum_{i=1}^{d} (X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{i}\left(X_{t_{k-1}}^{(n)}\right)^{i} + \theta(X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{2p} \sum_{i=1}^{d} (X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{i}\left(X_{t_{k-1}}^{(n)}\right)^{i} + \theta(X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{2p} \sum_{i=1}^{d} (X_{t_{k}}^{(n)} - X_{t_{k-1}}^{(n)})^{i}\left(X_{t_{k-1}}^{(n)}\right)^{i}.
\]
Then by A2-1-A2-3, the Cauchy-Schwarz inequality A1-2 and (5.9), we obtain
\[
\mathbb{E}
\left[
\left|X_{t_{k}}^{(n)}\right|^{2p} \right| F_{t_{k-1}} \right] - \left|X_{t_{k-1}}^{(n)}\right|^{2p}
\leq C \left(1 + \left|X_{t_{k-1}}^{(n)}\right|^{2p} \right).
\]
Hence by the discrete Gronwall inequality (cf. [1]), we obtain
\[
\ell
\]

Proof. We set \( (\text{Lemma } 5.8) \).

Thus we have
\[
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\]

5.3. Poor of \( \text{Theorem } 3.3 \).

\[
E \left[ |X_{t_\ell}^{(n)} - X_{t_{k-1}}^{(n)}|^2 \right] \leq C \left( \frac{1}{n} + M_2^{(n)}(Z) \right) \left( 1 + \left| X_{t_{k-1}}^{(n)} \right|^2 \right).
\]

Hence by the discrete Gronwall inequality (cf. [1]), we obtain
\[
E \left[ |X_{t_\ell}^{(n)}|^2 \right] \leq |x_0|^{2p} + C \left( 1 + n \left( M_2^{(n)}(Z) \right) \right) + C \left( \frac{1}{n} + M_2^{(n)}(Z) \right) \sum_{k=1}^{\ell} E \left[ |X_{t_{k-1}}^{(n)}|^2 \right].
\]

Lemma 5.8. Suppose that Assumption [3.1] and [3.2] hold. For any \( k \in \{1, 2, 3\} \) and \( i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\} \) and \( \ell \in \{1, 2, \ldots, n\} \),

\[
E \left[ \prod_{j=1}^{k} (X_{t_{j\ell}}^{(n)} - X_{t_{j\ell-1}}^{(n)})^{i_j} - \prod_{j=1}^{k} (\eta_{i\ell}^{(n)}(t_{\ell}) - X_{t_{j\ell-1}}^{(n)})^{i_j} \right] = 0.
\]

Proof. We set \( (\Delta Z_{t_\ell}^{(n)})^0 := T/n \) and \( (W_{t_{\ell}} - W_{t_{\ell-1}})^0 := T/n \) to simplify the argument. By A2-1-A2-3, we obtain

\[
E \left[ \prod_{j=1}^{k} (X_{t_{j\ell}}^{(n)} - X_{t_{j\ell-1}}^{(n)})^{i_j} - \prod_{j=1}^{k} (\eta_{i\ell}^{(n)}(t_{\ell}) - X_{t_{j\ell-1}}^{(n)})^{i_j} \right] = E \left[ \prod_{h=1}^{k} \alpha_h^{(n)}(X_{t_{h\ell}}^{(n)} - W_{t_{h\ell-1}}) \right] = 0.
\]

Next, we prove Theorem 3.3 using the above lemmas.

5.3. Poor of \( \text{Theorem } 3.3 \)

Proof. By Lemma 5.6 and the Taylor expansion with Theorem 5.1 we obtain

(5.10)

\[
E[ f(X_{t_\ell}^{(n)}) ] - E[ f(X_{t_{\ell-1}}^{(n)}) ] = E[ u(T, X_{t_{\ell}}^{(n)}) ] - E[ u(0, x_0) ]
\]

\[
= \left[ \sum_{t=1}^{n} \left( u(t_{\ell}, X_{t_{\ell}}^{(n)}) - u(t_{\ell-1}, X_{t_{\ell-1}}^{(n)}) \right) \right] = \left[ \sum_{t=1}^{n} \left( u(t_{\ell}, X_{t_{\ell}}^{(n)}) - u(t_{\ell}, X_{t_{\ell-1}}^{(n)}) \right) \right]
\]
We first estimate $I_1$. By A2-1 and Lemma 5.8, we obtain

$$I_1 = \sum_{\ell=1}^{n} \sum_{k=1}^{3} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}} \mathbb{E} \left[ \left. \frac{\partial^k u(t_\ell, y)}{\partial y^{i_1} \partial y^{i_2} \cdots \partial y^{i_k}} \right| y = X_{t_\ell}^{(n)} \right] \times \mathbb{E} \left[ \prod_{j=1}^{k} (X_{t_\ell}^{(n)} - X_{t_{\ell-1}}^{(n)})^{y_j} - \prod_{j=1}^{k} (\eta_{t_\ell}^{(n)}(t_\ell) - X_{t_{\ell-1}}^{(n)})^{y_j} \right] \mathcal{F}_{t_{\ell-1}}^{(n)}$$

$$= 0.$$

Next, we estimate $I_2$. By A2-1, Theorem 5.1, Lemma 5.5 and Lemma 5.7, we obtain

$$|I_2| \leq \sum_{\ell=1}^{n} \sum_{k=1}^{3} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}} \mathbb{E} \left[ \left. \frac{\partial^k u(t_\ell, y)}{\partial y^{i_1} \partial y^{i_2} \cdots \partial y^{i_k}} \right| y = X_{t_\ell}^{(n)} \right] \times \mathbb{E} \left[ \prod_{j=1}^{k} (\eta_{t_\ell}^{(n)}(t_\ell) - X_{t_{\ell-1}}^{(n)})^{y_j} - \prod_{j=1}^{k} (X_{t_\ell}^{(n)} - X_{t_{\ell-1}}^{(n)})^{y_j} \right] \mathcal{F}_{t_{\ell-1}}^{(n)}$$

$$\leq \frac{C}{n} \left( 1 + \mathbb{E} \left[ X_{t_{\ell-1}}^{(n)} \right]^{2(4r+3)} \right)$$

$$\leq \frac{C}{n} \exp \left( C \left( 1 + n \left( M_2^{(n)}(Z) \vee M_2^{(n)}(Z) \right) \right) \right).$$

Finally, we estimate $I_3$. Let $\ell \in \{1, 2, \ldots, n\}$. By the Cauchy-Schwarz inequality, Theorem 5.1, A2-1 and Lemma 5.7, we obtain

$$\mathbb{E} \left[ R_{t_\ell}^{(n)}(X_{t_\ell}^{(n)}) \right] \mathcal{F}_{t_{\ell-1}}^{(n)}$$
Thus by (5.13), (5.14) and Lemma 5.7, we obtain

\begin{equation}
\|I_3\| \leq \sum_{i=1}^{n} \left( \mathbb{E}\left[ \left| R_{T}^{(n)}(X_{t_i}^{(n)}) \right| \right] + \mathbb{E}\left[ \left| \hat{R}_{T}^{(n)}(X_{t_i}^{(n)}, X_{t_{i-1}}^{(n)}) \right| \right] \right) 
\leq C \left( 1 + M_{4r}(Z) \right) \exp \left\{ C \left( 1 + n \left( M_{2}^{(n)}(Z) \vee M_{2(2)}^{(n)}(Z) \right) \right) \right\} \left( nM_{8}^{(n)}(Z) \right)^{1/2} + \frac{1}{n} .
\end{equation}

Hence by (5.10), (5.11), (5.12) and (5.15), we have

\[ \mathbb{E}[f(X_{T}^{(n)})] - \mathbb{E}[f(X_{T})] \leq C_{r}^{(n)}(Z) \left( nM_{8}^{(n)}(Z) \right)^{1/2} + \frac{1}{n} . \]

\[ \Box \]

**Conclusions**

We provided two Euler-Maruyama approximations generated by the Haar system and the Walsh system in Section 2. We theoretically showed that they have the same weak order \( n^{-1} \) as the Euler-Maruyama approximation generated by the Gaussian system in Section 3 and 5 noting some arguments in Section 4. Their superiority or inferiority about the variance and the processing time were shown by numerical experiments in Section 4. In particular, we showed that the EM scheme by the Walsh system is efficient in high dimensions.

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