To the question of the shape of the spatial curve of the fastest descent in the conditions of its rotation

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Abstract. The non-linear system of dynamic equations of the movement of the ball, sliding down on rotating with frequency $\omega$ on one-dimensional gutter of arbitrary form, is obtained. From the minimum sliding time, defined by the form of this curve. Taking into account the forces of dry and viscous friction, the results of the analytical and numerical solution of the received system of equations are presented.

1. Introduction
In this work, we will continue the topic of research that we have outlined in the works [1-3]. It’s about the next problem. Suppose there is an infinitely thin and completely flexible hose on which the ball rolls down. The diameters of the hose and the ball are the same. The task is to find its form, to roll inside it under the influence of competitions of gravity, centripetal force and resistance force, the time of rolling down was minimal. It would be a typical setting of a brachistochron task (see [1]) if the chute (flexible hose) were stationary in the plane $xy$. We will solve the same problem, but in a more complex three-dimensional case, provided that the chute rotates with a constant angular speed $\omega$, directed along the axis of rotation $z$. At the same time, the trajectory of the ball becomes two-dimensional, in contrast to the one-dimensional classical brachistochron. It is quite clear that the glide of the ball takes place along the surface of this figure of rotation on a very defined two-dimensional trajectory, the shape of which will be dictated by the competition of the above three forces. Our task will be precisely to use the equations of the dynamics of the curvilinear movement to find the only possible spatial shape of the trough, provided that the time of sliding in the set conditions was minimal, as in the case of the classical brachistochron. It is clear that the solution of this problem in the extreme case $\omega = 0$, when it should lead to the parametric equations of the cycloids. The geometry of the formulated problem is illustrated in the figure 1, where the use of single orthogonal orthotics will be, in which we will solve the problem. However we should notice that in our previous papers, related to the problems of brachistochrone [4-8], the method of the mobile basis of the curve has never been applied.

2. Materials and methods
As we have already said, the solution to the problem is very convenient to bring in the mobile instant base of the right coordinate system \( \mathbf{n}, \mathbf{\tau}_1, \mathbf{\tau}_2 \), which is elementary laid out on fixed orts \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) (as well as in the works of the [1-3]) according to the transformations

\[
\begin{align*}
\mathbf{\tau}_2 &= \mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi, \\
\mathbf{\tau}_1 &= \mathbf{i} \cos \alpha \sin \varphi - \mathbf{j} \cos \alpha \cos \varphi + \mathbf{k} \sin \alpha, \\
\mathbf{n} &= \mathbf{i} \sin \alpha \sin \varphi - \mathbf{j} \sin \alpha \cos \varphi - \mathbf{k} \cos \alpha,
\end{align*}
\]

where angle \( \alpha \) is the outer angle between the tangent to the shape of the gutter and the projection of the gutter on the plane \( x-y \), and angle \( \varphi \) is an polar angle between axis \( x \) and a projection of a gutter on a plane \( x-y \) (see figure 1).

![Figure 1. The geometry of the problem. See text for the abbreviations.](image)

Let’s present a metric in the cylindrical coordinate system for our task in the form of

\[
dl^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2.
\] (2)

As you can see, this metric is written in the basis \( \mathbf{\tau}_2, \mathbf{n}, \mathbf{k} \). That is,

\[
dl = d\rho \mathbf{n}_1 + \rho d\varphi \mathbf{\tau}_2 + dz \mathbf{k}.
\] (3)

To go to the basis \( \mathbf{\tau}_2, \mathbf{\tau}_1, \mathbf{n} \) we need to find the appropriate transformation. Indeed, because

\[
\mathbf{n}_1 = -i \sin \varphi + j \cos \varphi,
\] (4)

then with the help (1) of (3) we have

\[
dl = d\rho (-i \sin \varphi + j \cos \varphi) + \rho d\varphi (i \cos \varphi + j \sin \varphi) + dz \mathbf{k} =
\]

\[
= (\rho \cos \varphi d\varphi - d\rho \sin \varphi)i + (\rho \sin \varphi d\varphi + d\rho \cos \varphi)j + dz \mathbf{k}.
\] (5)
Indeed, because $\det \hat{T} = 1$, where the matrix transformation (1) is

$$
\hat{T} = \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
\cos \alpha \sin \varphi & -\cos \alpha \cos \varphi & \sin \alpha \\
\sin \alpha \sin \varphi & -\sin \alpha \cos \varphi & -\cos \alpha
\end{pmatrix}.
$$

(6)

easily get the opposite as

$$
\hat{T}^{-1} = \hat{T} = \begin{pmatrix}
\cos \varphi & \cos \alpha \sin \varphi & \sin \alpha \sin \varphi \\
\sin \varphi & -\cos \alpha \cos \varphi & -\sin \alpha \cos \varphi \\
0 & \sin \alpha & -\cos \alpha
\end{pmatrix}.
$$

(7)

Therefore, the transformation we are interested will be

$$
i = n \sin \alpha \sin \varphi + \tau_1 \cos \alpha \sin \varphi + \tau_2 \cos \varphi,
$$

$$
j = -n \sin \alpha \cos \varphi - \tau_1 \cos \alpha \cos \varphi + \tau_2 \sin \varphi,
$$

$$
k = -n \cos \alpha + \tau_1 \sin \alpha.
$$

(8)

In the result from (5) we get

$$
dl = (\rho \cos \varphi d\varphi - \rho \cos \alpha d\varphi) i + (\rho \sin \varphi d\varphi + \rho \cos \varphi) j + dz k =
$$

$$
= (\rho \cos \varphi d\varphi - \rho \cos \alpha d\varphi) (n \sin \alpha \sin \varphi + \tau_1 \cos \alpha \sin \varphi + \tau_2 \cos \varphi) +
$$

$$
+ (\rho \sin \varphi d\varphi + \rho \cos \varphi) (-n \sin \alpha \cos \varphi - \tau_1 \cos \alpha \cos \varphi + \tau_2 \sin \varphi) +
$$

$$
+ dz (-n \cos \alpha + \tau_1 \sin \alpha) =
$$

$$
= n \sin \alpha \left[ -\cos \varphi (\rho \cos \varphi d\varphi + \rho \cos \varphi) + \sin \varphi (\rho \cos \varphi d\varphi - \rho \sin \varphi) - dz \tau_2 \right] +
$$

$$
+ \tau_1 \cos \alpha \left[ -\cos \varphi (\rho \cos \varphi d\varphi + \rho \cos \varphi) + \sin \varphi (\rho \cos \varphi d\varphi - \rho \sin \varphi) + dz \tau_2 \right] +
$$

$$
+ \tau_2 \left[ \sin \varphi (\rho \sin \varphi d\varphi + \rho \cos \varphi) + \cos \varphi (\rho \cos \varphi d\varphi - \rho \sin \varphi) \right] =
$$

$$
= n \sin \alpha \left[ -\rho \cos \alpha (\varphi - \tau_2 \alpha) \right] - \tau_1 \cos \alpha \left[ \rho \cos \alpha (\varphi - \tau_2 \alpha) \right] + \tau_2 \rho \varphi.
$$

(9)

Therefore, the speed in the basis $\tau_2, \tau_1, n$ is

$$
v = n \sin \alpha (-\dot{\rho} - \tau_1 \dot{\varphi} - \tau_2 \omega) - \tau_1 \cos \alpha (\dot{\rho} - \tau_2 \omega) + \tau_2 \rho \omega,
$$

(10)

and her square, as easy to check, will

$$
v^2 = \sin^2 \alpha (\dot{\rho} - \tau_2 \omega)^2 + \cos^2 \alpha (\dot{\rho} - \tau_2 \omega)^2 + \rho^2 \omega^2 = \dot{\rho}^2 + \dot{\varphi}^2 + \rho^2 \omega^2.
$$

(11)

There is nothing surprising in this result, as the speed module in any basis and in $\tau_2, \tau_1, n$, and in $\tau_2, n, k$ should be the equal. Differentiating now (10) in time, we find a common expression for the full acceleration of the ball in the new base $\tau_2, \tau_1, n$

$$
a = n \sin \alpha (-\ddot{\rho} - \tau_1 \ddot{\tau} - \tau_2 \dot{\omega}) + n \dot{\alpha} \cos \alpha (-\ddot{\rho} - \tau_1 \ddot{\tau} + \frac{\dot{\varphi}^2}{\sin^2 \alpha}) -
$$

$$
- \tau_1 \dot{\alpha} \cos \alpha (\dot{\rho} - \tau_2 \omega) + \tau_1 \dot{\alpha} \sin \alpha (\dot{\rho} - \tau_2 \omega) - \tau_1 \cos \alpha \left( \dot{\rho} - \tau_2 \omega - \frac{\dot{\varphi}^2}{\cos^2 \alpha} \right) + \tau_2 \rho \ddot{\omega} + \tau_1 \dot{\alpha} \omega.
$$
Because $\dot{\mathbf{t}}_2 = \omega \mathbf{n}_1$, $\dot{\mathbf{n}}_1 = -\omega \mathbf{t}_2$, $\mathbf{n} = \omega \left[ \mathbf{n}_1 \times \mathbf{t}_1 \right] + \frac{v_1}{R} \left[ \mathbf{n} \times \mathbf{t}_2 \right] = \omega \left[ \mathbf{n}_1 \times \mathbf{t}_1 \right]$, and $\dot{\mathbf{t}}_1 = \frac{v_1}{R} \mathbf{n}$, where $R$ is an instant curvature radius, from here we have

$$a = \omega \left[ \mathbf{n}_1 \times \mathbf{t}_1 \right] \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha \right) + \mathbf{n} \dot{\alpha} \cos \alpha \left( -\rho - \dot{\varphi} \cos \alpha \right) + \mathbf{n} \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha + \frac{\dot{\varphi}}{\sin \alpha} \right) - \frac{v_1}{R} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) \mathbf{n} + \mathbf{t}_1 \dot{\alpha} \sin \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) - \mathbf{t}_1 \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha - \frac{\dot{\varphi}}{\cos \alpha} \right) + \rho \omega^2 \mathbf{n}_1 + \mathbf{t}_2 \rho \omega.$$  \hfill (12)

Therefore, according to (12), the full force acting on the ball can be found thanks to the ratio

$$F = m \left[ \omega \left[ \mathbf{n}_1 \times \mathbf{t}_1 \right] \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha \right) - \mathbf{n} \dot{\alpha} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) + \mathbf{n} \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha + \frac{\dot{\varphi}}{\sin \alpha} \right) - \frac{v_1}{R} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) \mathbf{n} + \mathbf{t}_1 \dot{\alpha} \sin \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) - \mathbf{t}_1 \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha - \frac{\dot{\varphi}}{\cos \alpha} \right) + \rho \omega^2 \mathbf{n}_1 + \mathbf{t}_2 \rho \omega \right].$$  \hfill (13)

Its projection on the normal $\mathbf{n}$ to the gutter will be defined as a scalar work $\mathbf{F} \cdot \mathbf{n}$. Therefore for the force of reaction $N = F_n$ taking into account gravity we have out (13)

$$N = m \left( \omega^2 \rho \mathbf{n}_1 \mathbf{n} - \dot{\alpha} \cos \alpha \left( \dot{\rho} + \dot{\varphi} \cos \alpha \right) + \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha + \frac{\dot{\varphi}}{\sin \alpha} \right) - \frac{v_1}{R} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) - g \cos \alpha \right).$$

And since $\mathbf{n} \cdot \mathbf{n}_1 = \sin \alpha$, we have

$$N = m \left( \omega^2 \rho \sin \alpha - \dot{\alpha} \cos \alpha \left( \dot{\rho} + \dot{\varphi} \cos \alpha \right) + \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha + \frac{\dot{\varphi}}{\sin \alpha} \right) - \frac{v_1}{R} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) - g \cos \alpha \right).$$

Because the same $v_1 = -Ra$, from here after simple transformations we will find

$$N = m \left( \omega^2 \rho \sin \alpha - \dot{\alpha} \cos \alpha \left( \dot{\rho} + \dot{\varphi} \cos \alpha \right) + \sin \alpha \left( -\rho - \dot{\varphi} \cos \alpha + \frac{\dot{\varphi}}{\sin \alpha} \right) + \dot{\alpha} \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha - g \cos \alpha \right) \right) = m \left( \omega^2 \rho \sin \alpha - g \cos \alpha - \left( \dot{\rho} \sin \alpha + \dot{\varphi} \cos \alpha \right) \right).$$  \hfill (14)

Similarly, it is possible to design (12) on mobile orts $\mathbf{t}_1$ and $\mathbf{t}_2$. The system of dynamic motion equations according to the general expression (12) will record how

$$a = \rho \omega \mathbf{t}_2 + \mathbf{g} + \frac{F_p}{m}.$$  \hfill (15)

By designing the equation (15) on the axis $\mathbf{t}_1$ with the help (12) and taking into account the strength of resistance we get an equation

$$-g \sin \alpha + \dot{\alpha} \sin \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha \right) - \cos \alpha \left( \dot{\rho} - \dot{\varphi} \cos \alpha - \frac{\dot{\varphi}}{\cos \alpha} \right) + \omega^2 \rho \cos \alpha + \frac{F_p}{m} =$$

$$= -g \sin \alpha - \frac{d}{dt} \left( \rho \cos \alpha - \dot{\varphi} \sin \alpha \right) + \omega^2 \rho \cos \alpha + \frac{F_p}{m} = 0,$$  \hfill (16)

where the resistance force [9-10] is
It is chosen as a form proportional to the speed of the ball with the proportionality factor \( k_1 \). Dry friction is described by the second component and characterized by friction factor \( k_2 \). Note also that because \( F_\mu = -F_\mu \tau_1 \), in expression (16) the resistance force comes with a sign of "plus." Finding further a projection of the equation (15) on the axis \( \tau_2 \) using the general expression (12), we get

\[
\omega_2 \cdot [n_1 \times \tau_1] \sin \alpha (-\dot{\rho} - z \tan \alpha) = 0.
\]

But because \( \tau_2 \cdot [n_1 \times \tau_1] \neq 0 \) it follows the condition

\[
\dot{\rho} \sin \alpha + \dot{z} \cos \alpha = 0.
\]

The parametric solution to this equation has the only possible form

\[
\dot{\rho} = -v_1 \cos \alpha, \quad \dot{z} = v_1 \sin \alpha.
\]

As a result, we find that

\[
\dot{\rho} \sin \alpha + \dot{z} \cos \alpha = v_1 \alpha = -\frac{v_1^2}{R}.
\]

Consequently, the strength of the reaction according to (14) is defined as

\[
N = m \left( \omega^2 \rho \sin \alpha - g \cos \alpha + \frac{v_1^2}{R} \right).
\]

And the equation (16) according to (20) (since \( \dot{\rho} \cos \alpha - \dot{z} \sin \alpha = -v_1 \)) will

\[
\ddot{v}_1 = g \sin \alpha - \omega^2 \rho \cos \alpha - \frac{F_\mu}{m}.
\]

A prerequisite for the shape of the trough to be a brachistochron, as has been proven in ref. [1] is the condition of inequality of zero reaction force. The condition that the trajectory is a brachistochron means that in the equation (21) it is necessary to choose a "minus" sign, that is, to assume that in the ratio \( v_1 = -R \dot{\alpha} \) you should take the plus sign, namely, put \( v_1 = R \dot{\alpha} \). It must be said that the condition of moving from the "minus" to the "plus" sign before \( R \dot{\alpha} \) means that there is a kind of qualitative and quantitative geometric jump, which leads to the splitting of the trajectory into two different parts. If \( v_1 = -R \dot{\alpha} \), that runs up to some point in time \( t \leq t_s \), the movement will take place on a cycloid. In the moment \( t = t_s \), there is a "switch" of the trajectory from cycloid to brachistochron, and at \( t > t_s \) will move along the brachistochron at condition \( N \neq 0 \). Such a rather curious physical picture is manifested only in the conditions of a purely analytical solution of the problem within the equations of dynamics, which is fundamentally impossible to detect in its solution by methods of variation calculus or theory of optimal management. In the light of the foregoing, as well as in the work [1] of the

\[
\frac{v_1^2}{R} = \omega^2 \rho \sin \alpha - g \cos \alpha.
\]

Therefore, the reaction force for a rotating brachistochron can be calculated according to the expression
Note that angle $\alpha$, as it’s see from figure 1, varies from $\pi$ to $\frac{\pi}{2}$. It would be more correct, of course, to write the module of the right part of the expression (22), since the force of the reaction is non-negative ($N \geq 0$), but since, as we have just noted, the cos is negative, and the sinus is positive, it is not necessary.

Equations (20), (23) - (25) are a complete system of the motion dynamic equations that are being used, taking into account the dry and viscous forces of resistance for the rotating brachistoron. Let’s write down these equations taking into account the identity $v_1 = R\dot{\alpha}$ in the form of a single system

\[
\begin{align*}
N &= 2m\left(\omega^2 \rho \sin \alpha - g \cos \alpha\right), \\
\dot{v}_1 &= g \sin \alpha - \omega^2 \rho \cos \alpha - \frac{F_r}{m}, \\
v_1 \dot{\alpha} &= -g \cos \alpha - \omega^2 \rho \sin \alpha, \\
\dot{\rho} &= -v_1 \cos \alpha, \\
\dot{z} &= v_1 \sin \alpha.
\end{align*}
\]

As you can see from here, in the case $\omega = 0$ of the system (26) goes into the equations of work.

3. Results and discussion

3.1. Lack of resistance
Let’s analyze the solution of equations (26) in a simpler case where both types of friction are absent, that is $k_1 = k_2 = 0$, provided. As a result, we have

\[
\begin{align*}
v_1 &= g \sin \alpha - \omega^2 \rho \cos \alpha, \\
v_1 \dot{\alpha} &= -g \cos \alpha - \omega^2 \rho \sin \alpha, \\
\dot{\rho} &= -v_1 \cos \alpha, \\
\dot{z} &= v_1 \sin \alpha.
\end{align*}
\]

Entering here a sizeless argument $\tau = \frac{gt}{v_0}$ and functions $y_1 = \frac{v_1}{v_0}$, $y_2 = \alpha$, $y_3 = \frac{\rho g}{v_0^2}$, $y_4 = \frac{zg}{v_0^2}$, get the following also immeasurable equations

\[
y_1' = \sin y_2 - \lambda y_3 \cos y_2, \quad y_1 y_3' = -\cos y_2 - \lambda y_3 \sin y_2, \quad y_1' = -y_1 \cos y_2, \quad y_4' = y_1 \sin y_2.
\]

where parameter $\lambda = \left(\frac{v_0 \omega}{g}\right)^2$, and “prim” it means differentiated on dimensionless time $\tau$. To solve them, we will set the following initial conditions

\[
y_1(0) = 1, \quad y_2(0) = \pi, \quad y_3(0) = 0, \quad y_4(0) = 0.
\]

The numerical solution of equations (28) depending on the parameter $\lambda$ is illustrated by figures 2 - 3, where its specific values are specified.
Figure 2. The computational solution of the system of equations (31) with no rotation of $\lambda = 0$, $k_1 = k_2 = 0$, that ignoring the resistance forces of $y_4(y_3)$ was shown as a form of the dependence. As it must be, we have got classical brachistichrone [1].

Figure 3. The solution of the equations (31), where assuming that $k_1 = k_2 = 0$ and frequency of rotation is $\lambda = 5$, it’s shown as an dependence $y_4(y_3)$. As we can see, in this case the form of the brachistichrone would change a lot.

As to the dependences $x(t)$, $y(t)$, $z(t)$, their connect with the parameters $v_1, \rho$ and $\alpha$ it’s easy find due to the reverse transformation (8). Indeed, then we get that

$$\dot{x} = v_{ix} = (v_i,\hat{i}) = v_i \cos \alpha \sin \varphi, \quad \dot{y} = v_{iy} = (v_i,\hat{j}) = -v_i \cos \alpha \cos \varphi, \quad \dot{z} = v_{iz} = (v_i,\hat{k}) = v_i \sin \alpha. \quad (29)$$

Putting here $\varphi = \omega t$, after integration we find

$$x(t) = \int_0^t v_i(t) \cos \alpha(t) \sin \omega t dt, \quad y(t) = -\int_0^t v_i(t) \cos \alpha(t) \cos \omega t dt, \quad z(t) = \int_0^t v_i(t) \sin \alpha(t) \omega t dt.$$
As you can see from here, in the absence of rotation $\omega=0$, that is, at we will come to the occasion of the classic flat brachistochron, which was considered by us in the [1]. Their numerical integration allows you to find projections of spatial brachistochrons on the plane $x-y$, $y-z$, $x-z$.

3.2. Accounting for friction
Let's consider now a more complex task, and take into account the forces of resistance. According to the equations (27) we will have

$$\begin{aligned}
v_1 &= \frac{g \sin \alpha - \omega^2 \rho \cos \alpha - \frac{k_1}{m} v_1 - 2 k_2 \left( \omega^2 \rho \sin \alpha - g \cos \alpha \right)}{m}, \quad \dot{\rho} = -v_1 \cos \alpha, \\
v_1 \dot{\alpha} &= -g \cos \alpha - \omega^2 \rho \sin \alpha, \quad \dot{z} = v_1 \sin \alpha.
\end{aligned} \tag{30}$$

By introducing the same immeasurable functions as the system (28), we get

$$\begin{aligned}
y'_1 &= \sin y_2 - \lambda y_1 \cos y_2 - \mu y_1 - 2 k_2 \left( \lambda y_1 \sin y_2 - \cos y_2 \right), \\
y_1 y'_2 &= -\cos y_2 - \lambda y_1 \sin y_2,
\end{aligned} \tag{31}$$

where the new dimensionless coefficient $\mu = \frac{k_1 v_0}{mg}$.

The numerical solution of equations (31) under the same initial conditions leads to very interesting dependencies illustrated in the figures 4–5. As can be seen, when you consider the rotation, the possible forms of the gutter become quite curious.

\textbf{Figure 4. The solution of the system the equations (36) at $\lambda = 0$, but accounted the resistance forces in the case when $k_2 = 0.01$.}
Figure 5. The drawing of the form the channel in conditions of the rotation $y_4(y_3)$, at $\lambda = 1$ and the resistance forces with coefficient $\mu = 0.01$ and dry friction coefficient is $k_2 = 0.3$.

4. Conclusion
In conclusion of the work we note a few key points of the above study.
1. The basic dynamic motion equations for the case of rotating brachistochron are obtained, taking into account the forces of dry and viscous friction acting on the moving ball;
2. The number of equations found was numerically integrated, both in the absence of friction and when it was taken into account, and the solutions obtained were illustrated in the figures 2 – 5.

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