SUPER-TETRAHEDRA AND SUPER-ALGEBRAS

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Abstract. In this paper we give a detailed classification scheme for three-dimensional quantum zero curvature representation and tetrahedron equations. This scheme includes both even and odd parity components, the resulting algebras of observables are either Bose $q$-oscillators or Fermi oscillators. Three-dimensional $R$-matrices intertwining variously oriented tensor products of Bose and Fermi oscillators and satisfying tetrahedron and super-tetrahedron equations are derived. The $3d \to 2d$ compactification reproduces $\mathcal{B}_q(\mathfrak{gl}(n|m))$ super-algebras and their representation theory.

Introduction

The $q$-oscillator solution of the quantum tetrahedron equation was derived in [1] as an interwiner of quantum local Yang-Baxter equation with a specific Ansatz for auxiliary matrices. However, a more fundamental zero curvature representation of three-dimensional models is based on an auxiliary linear problem [2]. Namely, the $q$-oscillator model can be viewed as the quantization [3, 4] of discrete three-wave equations and their linear problem [5, 6]. In the first section of this paper we discuss in details this type’s zero curvature representation and the tetrahedron equation from the quantum mechanical point of view, and formulate a classification problem for algebras of observables.

The $q$-oscillator algebra is the solution of the classification problem for even algebras, Theorem 1 of the second section. The linear problem provides a natural way to introduce also odd algebras and extend the classification problem to mixed case of even and odd algebras, corresponding classification Theorem 2 is the subject of the second section as well. The result of classification theorem is the existence of four distinct automorphisms for even algebras $\mathcal{A}(q^{\pm 1})$ and odd algebras $\mathcal{F}(q^{\pm 1})$ and eight (super-)tetrahedra for them.

Bose and Fermi oscillators are “evaluation representations” of formal algebras $\mathcal{A}(q^{\pm 1})$ and $\mathcal{F}(q^{\pm 1})$. These representations are fixed in the third section. In the fourth section we construct explicitly all corresponding quantum intertwiners ($R$-matrices) and in the fifth section we discuss briefly all eight tetrahedron equations.

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Any solution of the tetrahedron equation produces a series of solutions of the Yang-Baxter equation. In section six we remind this scheme for the obtained Bose/Fermi inhomogeneous solutions of the tetrahedron equations. The resulting quantum groups are in general $\mathcal{U}_q(\hat{gl}(n|m))$. This statement is detailed in the last section for the illustrative case of $\mathcal{U}_q(\hat{gl}(2|1))$.

1. **Zero Curvature Representation.**

The primary concept of integrability is an auxiliary linear problem. The simplest form of local auxiliary linear problem in the theory of quantum integrable systems in wholly discrete 2 + 1 dimensional space-time is the pair of linear relations [7]

\begin{align*}
\psi'_{\alpha} &= a_j \psi_{\alpha} + b_j \psi_{\beta}, \\
\psi'_{\beta} &= c_j \psi_{\alpha} + d_j \psi_{\beta},
\end{align*}

where $a_j, b_j, c_j, d_j$ are elements of some local algebra $C_j$,

\begin{equation}
C_j = C[a_j, b_j, c_j, d_j],
\end{equation}

and $\psi_{\alpha}, \psi'_{\alpha}, \psi_{\beta}, \psi'_{\beta}$ are auxiliary linear elements from a formal left module of a tensor power of local algebras.

Geometrically, a collection of local linear problems is associated with a 2d “space-like” section of a three-dimensional graph, see Fig. 1. Auxiliary variables are associated with the edges of auxiliary plane, while the elements of $C_j$ are associated with the $j$th vertex of auxiliary plane which corresponds to a “time-like” edge of the 3d graph (bold edge in Fig. 1). It is convenient to rewrite the local linear problem in a matrix form,

\begin{equation}
\begin{pmatrix}
\psi'_{\alpha} \\
\psi'_{\beta}
\end{pmatrix} = X_{\alpha\beta}[C_j] \begin{pmatrix}
\psi_{\alpha} \\
\psi_{\beta}
\end{pmatrix},
\end{equation}

where $X_{\alpha\beta}[C]$ is a true 3d analogue of a Lax operator. Collection of local linear problems along an auxiliary 2d graph relates the outer auxiliary variables. For the triangle graph in

\begin{center}
\begin{tikzpicture}[scale=0.7]
\node (c) at (0,0) {$C_j$};
\node (a) at (1,1) {$\psi_{\alpha}$};
\node (b) at (1,-1) {$\psi_{\beta}$};
\node (c) at (1,1) {$\psi'_{\alpha}$};
\node (d) at (1,-1) {$\psi'_{\beta}$};
\draw (a) -- (c);
\draw (b) -- (d);
\end{tikzpicture}
\end{center}

\textbf{Figure 1. 3d geometry of auxiliary linear problem.}
The local linear problems allow one to express $\psi''_\alpha$, $\psi''_\beta$ and $\psi''_\gamma$ as linear combinations of $\psi_\alpha$, $\psi_\beta$ and $\psi_\gamma$. These expressions can be written in the matrix form as

$$
\begin{pmatrix}
\psi''_\alpha \\
\psi''_\beta \\
\psi''_\gamma
\end{pmatrix} = X_{\alpha\beta} [C_1] X_{\alpha\gamma} [C_2] X_{\beta\gamma} [C_3] \cdot
\begin{pmatrix}
\psi_\alpha \\
\psi_\beta \\
\psi_\gamma
\end{pmatrix},
$$

where $X_{\alpha\beta}$ is the two-by-two matrix in the block $\alpha \oplus \beta$ completed by the unity in the block $\gamma$, etc.

One can consider an “opposite” graph to Fig. 2 shown in Fig. 3. The opposite graph has the same external data as the initial one: the collection of linear problems also allows to express $\psi''_\alpha$, $\psi''_\beta$ and $\psi''_\gamma$ as linear combinations of $\psi_\alpha$, $\psi_\beta$ and $\psi_\gamma$:

$$
\begin{pmatrix}
\psi''_\alpha \\
\psi''_\beta \\
\psi''_\gamma
\end{pmatrix} = X_{\beta\gamma} [C'_3] X_{\alpha\gamma} [C'_2] X_{\alpha\beta} [C'_1] \cdot
\begin{pmatrix}
\psi_\alpha \\
\psi_\beta \\
\psi_\gamma
\end{pmatrix},
$$

with some $C'_j = C[a'_j, b'_j, c'_j, d'_j]$.

Zero curvature condition is the complete independence of linear problem of a choice of graphs, Fig. 2 and Fig. 3

$$
X_{\alpha\beta} [C_1] X_{\alpha\gamma} [C_2] X_{\beta\gamma} [C_3] = X_{\beta\gamma} [C'_3] X_{\alpha\gamma} [C'_2] X_{\alpha\beta} [C'_1].
$$

This equation resembles the local Yang-Baxter equation, however (6) is the equation in tensor sum of spaces $\alpha, \beta$ and $\gamma$. Equation (6) for $C$-valued fields $C$ was studied in details by I.
Korepanov in [7], some earlier applications of (6) to functional tetrahedron equation can be found in [8].

In classics, when \( \mathcal{C}_j \) are Abelian algebras, equation (6) is just an equation of motion for \( \mathbb{C} \)-valued fields \( a_j, b_j, c_j, d_j \) since the zero curvature condition is just a self-consistency condition for the linear problems. In the conventional classical approach the gauge symmetry of auxiliary fields, \( \psi_\alpha \rightarrow G_\alpha \psi_\alpha \), etc., is used to reduce the number of independent fields:

\[
(7) \quad a_j = 1, \quad b_j = -A_j, \quad c_j = A_j^*, \quad d_j = 1 - A_j A_j^*,
\]

what corresponds to auxiliary relations \( \psi_\alpha - \psi'_\alpha = A_j \psi_\beta \) and \( \psi'_\beta - \psi_\beta = A_j^* \psi'_\alpha \) of the discrete three-wave system [9].

Our aim, however, is a quantum theory where the gauge transformations affecting quantum structure must be considered more carefully. Yet the algebras \( \mathcal{C}_j \) and \( \mathcal{C}'_j \) are uncertain. What we expect from the foundations of quantum theories: quantum equations of motion are conjugations by a discrete time evolution operator and therefore the Heisenberg quantum equations of motion in discrete space-time are sequences of automorphisms.

Following the foundations of quantum theories, suppose now that algebras \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \) are local and equivalent: the triplet \([\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3]\) is the tensor product of three independent copies of the same algebra,

\[
(8) \quad [\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3] = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 = \mathcal{C}^{\otimes 3}
\]

so that the index \( j \) stands for the component of tensor product. Then one comes to a

**Problem:** What is an algebra \( \mathcal{C} \) such that equation (6) defines uniquely an automorphism \( \mathcal{C}^{\otimes 3} \rightarrow \mathcal{C}^{\otimes 3} \),

\[
(9) \quad \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \rightarrow \mathcal{C}'_1 \otimes \mathcal{C}'_2 \otimes \mathcal{C}'_3 .
\]

Natural extension of this problem is the case of non-equivalent algebras \( \mathcal{C}_j \). We will prove the classification theorems for this problem in the next section.

Our final aim is the explicit construction of intertwining operators for quantum equation (6). If algebras \( \mathcal{C}_j \) and their irreducible representations are chosen properly, then the automorphism (9) is internal one: there exists an uniquely defined operator \( R_{123} \) such that

\[
(10) \quad \mathcal{C}'_j = R_{123} \mathcal{C}_j R_{123}^{-1}, \quad j = 1, 2, 3 .
\]

\(^1\)formally, \( \mathcal{C}_j \) now stands for an enveloping algebra of \([1, a_j, a_j^{-1}, b_j, c_j, d_j, d_j^{-1}]\).
Indices of $R_{123}$ refer to the components of tensor product $V_1 \otimes V_2 \otimes V_3$ of representation spaces of $C_1 \otimes C_1 \otimes C_3$. Equation (11) takes the form

$$X_{\alpha \beta} [C_1] X_{\alpha \gamma} [C_2] X_{\beta \gamma} [C_3] R_{123} = R_{123} X_{\beta \gamma} [C_3] X_{\alpha \gamma} [C_2] X_{\alpha \beta} [C_1] ,$$

see Fig. 4 for the graphical representation of this intertwining relation. Equation (11) can be viewed as one of possible 3d extensions of Quantum group’s intertwining of co-products $R \Delta = \Delta' R$.

![Figure 4](image_url)

**Figure 4.** Graphical representation of Equation (11): auxiliary triangles from Figs. 2 and 3 with the solid vertex standing for the intertwining operator $R_{123}$. Auxiliary planes are sections of the solid vertex of three-dimensional lattice. Equation (11) has the structure of tetrahedron equation in $V_1 \otimes V_2 \otimes V_3 \otimes (\alpha \oplus \beta \oplus \gamma)$.

Operators $R_{ijk}$ satisfy an associativity condition following from equivalence

$$X_{\alpha \beta} [C_1] X_{\alpha \gamma} [C_2] X_{\beta \gamma} [C_3] X_{\alpha \delta} [C_4] X_{\beta \delta} [C_5] X_{\gamma \delta} [C_6] = X_{\gamma \beta} [C''_6] X_{\beta \delta} [C''_5] X_{\alpha \delta} [C''_4] X_{\beta \gamma} [C''_3] X_{\alpha \gamma} [C''_2] X_{\alpha \beta} [C''_1]$$

This automorphism of sixth tensor power can be decomposed into elementary automorphisms in two different ways:

$$T_L = R_{123} R_{145} R_{246} R_{356} \quad \text{and} \quad T_R = R_{356} R_{246} R_{145} R_{123}.$$ 

Due to the uniqueness of automorphisms, both ways coincide:

$$T_L = T_R ,$$

what is the tetrahedron equation – the three-dimensional generalization of the Yang-Baxter (triangle) equation. Graphical representation of the tetrahedron equation is given in Fig. 5.
Figure 5. Graphical representation of the tetrahedron equation (14) in $V_1 \otimes \cdots \otimes V_6$: equivalence of two three-dimensional graphs.

2. Classification theorems

2.1. Even case. The answer to the problem above is

**Theorem 1.** Equation (6) defines an automorphism of tensor cube of $C$ is and only if

$$C[a, b, c, d] = A(q; a, b, c, d)$$

where $A(q)$ is defined by

$$ad = da, \quad ab = qba, \quad ca = qac, \quad db = qbd, \quad cd = qdc, \quad bc - cb = (q^{-1} - q)ad.$$ 

Algebra $A(q)$ has two centers,

$$\eta = ad^{-1} \quad \text{and} \quad \xi = q^{-1}ad - bc.$$ 

The inverse relation for $A(q)$ is

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)^{-1} = \frac{1}{\xi} \left(\begin{array}{cc} q^{-1}d & -b \\ -c & qa \end{array}\right).$$

We consider the centers $\eta_j$ and $\xi_j$ of $A_j(q)$ as $\mathbb{C}$-numerical parameters of $A_j(q)$. Under this assumption $A(q)$ becomes the $q$-oscillator algebra from [1].

**Sketch proof.** Whichever $C_j$ are taken,

- the non-primed elements of $C_j$ with different $j$ commute since $j$ stands for a component of tensor product, $C_1 = C \otimes 1 \otimes 1$, etc.
- we are looking for an automorphism. This means in particular, primed elements of $C_j'$ with different $j$ also must commute.
This is called (ultra-)locality, the starting point of the proof is its test. Matrix equation (6) in components reads:

\begin{equation}
\begin{aligned}
    a_2'a_1' &= a_1a_2, \\
    d_3'd_2' &= d_2d_3,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
    a_2'b_1' &= b_1a_3 + a_1b_2c_3, \\
    c_1a_2 &= a_3c_1' + b_3'c_2'a_1', \\
    b_2' &= b_1b_3 + a_1b_2d_3, \\
    c_2 &= c_3'c_1' + d_3'c_2a_1', \\
    b_3'd_2' &= d_1b_3 + c_1b_2d_3, \\
    d_2c_3 &= c_3'd_1' + d_3'c_2b_1'
\end{aligned}
\end{equation}

and finally,

\begin{equation}
    a_3'd_1' + b_3'c_2b_1' = d_1a_3 + c_1b_2c_3.
\end{equation}

Note, the left column in (20) gives almost explicit expressions for primed \( b_j \).

The locality of \( C_1' \) and \( C_2' \) implies in particular

\begin{equation}
    [b_2', a_1'^{-1}b_1'] = [b_1b_3 + a_1b_2d_3, (a_1a_2)^{-1}(b_1a_3 + a_1b_2c_3)] = 0
\end{equation}

Expanding the commutator, we have

\begin{equation}
\begin{aligned}
    & (b_1a_1^{-1}b_3a_3 - a_1^{-1}b_1a_3b_3) b_1a_2^{-1} \\
    & + (b_2a_2^{-1}d_3c_3 - a_2^{-1}b_2c_3d_3) a_1b_2 \\
    & + b_1b_2a_2^{-1}d_3a_3 - a_1^{-1}b_1a_1a_2^{-1}b_2a_3d_3 + b_1a_2^{-1}b_2[b_3, c_3] = 0
\end{aligned}
\end{equation}

Here the locality of \( C_j \) is taken into account. Check now the structure of three lines here in \( C_2 \). The first line has \( a_2^{-1} \), the second line has \( a_2^{-1}b_2^2 \), the third line has \( a_2^{-1}b_2 \). Since one can hardly expect \( ab \sim ba \sim a \), we can conclude that the whole expression is zero if each line is zero. The first line gives

\begin{equation}
    b_1a_1^{-1}b_3a_3 = a_1^{-1}b_1a_3b_3 \implies ab = qba
\end{equation}

for some \( q \). The second line gives

\begin{equation}
    b_2a_2^{-1}d_3c_3 = a_2^{-1}b_2c_3d_3 \implies cd = qdc
\end{equation}

with the same \( q \). Finally, the third line gives

\begin{equation}
    [b, c] = q^{-1}ad - qda.
\end{equation}

In a similar way one can check

\begin{equation}
    [b_2', d_3'^{-1}b_3'] = 0
\end{equation}

and get

\begin{equation}
    db = q'bd, \quad ca = q'ac, \quad [b, c] = q'^{-1}da - q'ad
\end{equation}
for some $q'$. Comparing two variants of $[b,c]$, we have

$$qda = q'ad.$$ 

All the other locality tests provide no additional information. Thus, the locality test gives us the most general form of $\mathcal{C}$:

$$\mathcal{C}(q,q') : \begin{cases} 
  ab = qba, & cd = qdc, & db = q'bd, & ca = q'ac, \\
  [b, c] = (q^{-1} - q')ad, & qda = q'ad 
\end{cases}$$

One can verify further, for $C_j = \mathcal{C}(q,q')$ the system of relations (15) is self-consistent: not only locality but relations (30) for primed elements do not contradict the system (6). However, we have no uniqueness yet. Nine equations (10) for twelve elements do not produce a unique map in general. The map is unique only if $\mathcal{C}(q,q')$ has centers and the map conserves them (centers of algebras commute with their intertwiner, Eq. (10)). It is easy to verify the following

**Lemma:** The algebra $\mathcal{C}(q,q')$ has centers compatible with (6) if and only if $q = q'$. This lemma can be proven in quasiclassical limit $q = e^h \to 1$ and $q' = e^{h'} \to 1$, so that $h/h' = \delta$ as a parameter of Poisson algebra. Equation for a center of Poisson algebra is a system of differential equation with trivial solution unless $\delta \neq \pm 1$. Conservation of centers for the case $\delta = -1$, what is $q' = q^{-1}$ and $[b, c] = 0$, contradict with (6). Thus $q = q'$ is the only choice.

Final step: when the centers (17) conserve,

$$\text{centers of } \mathcal{A}_j = \text{centers of } \mathcal{A}_j' \text{ for all } j,$$

we can solve (19, 20, 21) with respect to all primed generators (equation (18) is of exceptional use) and verify directly that this is the automorphism of $\mathcal{A}(q)^{\otimes 3}$. \[\hfill \blacksquare \hfill\]

Remark: in this proof we initially considered $\mathcal{C}^{\otimes 3}$ framework – the tensor cube of the same algebra. However, the answer is the same without this assumption; the equivalence of $C_j$ follows from analysis of all possible locality relations.

Another remark concerns the case $q' \neq q$. In general, $\mathcal{C}(q,q') = Qsc_{q,q'} \otimes \text{Weyl}_{q/q'}$, the Weyl subalgebra is generated by $\eta$ and $\xi$, $\eta \cdot \xi = \left(q/q'\right)^2 \xi \cdot \eta$. For instance, the algebras $\mathcal{C}(q^2, 1)$ and $\mathcal{C}(1, q^2)$ appear in the three-dimensional approach to spectral equations [9]. Due to ambiguity, equation (11) does not define intertwiner $R$ for $\mathcal{C}(q,q')^{\otimes 3}$ uniquely. However, such intertwiner satisfying the tetrahedron equation can be constructed by dressing the constant $q$-oscillator $R$-matrix (Eq. (64) below) by non-commutative exponential factors.

2.2. **Odd case.** Theorem [11] is based on the locality principle: elements in different components of a tensor product commute. This is the feature of even algebras; for odd algebras their odd elements in different components of a tensor product anti-commute.
A natural (and only possible) way to introduce odd algebras is to assign a parity to auxiliary variables, for instance to modify the linear problem as follows:

\[ \psi'_\alpha = a_j \psi_\alpha + b_j \psi_\beta, \quad \psi'_\beta = c_j \psi_\alpha + d_j \psi_\beta. \]

Here the under-line symbol marks the odd parity, relations (32) correspond to the change of parity of auxiliary \( \beta \)-line. Thus, the odd algebras correspond to parity changes of some of the lines \( \alpha, \beta, \gamma \) in equation (6); in general there are eight different parity patterns of \( \alpha, \beta, \gamma \). This way is the only possible one since it takes into account the parity conservation in equation (6). Now we are ready to extend Theorem 1:

**Theorem 2.** Given \( X_{\alpha\beta} = X_{\alpha\beta}[A(q)] \), equations (6) provide automorphisms if and only if

\[ X_{\alpha\beta} = X_{\alpha\beta}[F(q)], \quad X_{\alpha\beta} = X_{\alpha\beta}[F(q^{-1})], \quad X_{\alpha\beta} = X_{\alpha\beta}[A(q^{-1})] \]

for all parity patterns, where \( F(q) = F(q; a, b, c, d) \) (odd elements are underlined) is defined by

\[ ad = da, \quad ab = qba, \quad ca = qac, \quad bd = qdb, \quad dc = qcd, \quad bc + cb = (q - q^{-1})ad. \]

For the odd elements of \( F(q^{\pm 1}) \) we have

\[ b^2 = c^2 = 0. \]

Algebra \( F(q) \) has two centers \( \xi \) and \( \eta \) as well,

\[ \xi = ad, \]

and \( \eta \) is defined by

\[ qad - bc = \eta d^2, \quad qad - cb = \eta^{-1}a^2. \]

The inversion relation for \( F(q) \) is

\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \frac{q}{\xi} \left( \begin{array}{cc} \eta^{-1}a & -b \\ -c & \eta d \end{array} \right). \]

We consider the centers \( \eta_j \) and \( \xi_j \) of \( F_j(q) \) as \( \mathbb{C} \)-numerical parameters of \( F_j(q) \). Note also the existence of two orthogonal projectors in \( F(q) \):

\[ P_1 = qa - \eta d, \quad P_2 = qd - \eta^{-1}a, \]

such that

\[ P_1 P_2 = b P_1 = P_1 c = c P_2 = P_2 b = 0. \]
Sketch proof of Theorem 2. Consider as an example the parity pattern $\alpha \beta \gamma$. By the condition of the theorem, $C_2 = A_2(q)$ while $C_1$ and $C_3$ are uncertain, but $c_{1,3}$ and $b_{1,3}$ are odd elements.

The locality test: whichever $C_1$ and $C_3$ are taken, the ultra-locality demands

\[(a_2 b_1 b_2 - q b_2 a_2 b_1)' = (b_2 d_2 b_2 - q b_2 b_2 d_2)' = (a_2 b_1 b_2 d_2 + b_2 d_2 a_2 b_1)' = 0 .\]

Here we take into account $C_2' = A_2'(q)$ and the parity of $C_1'$ and $C_3'$. Immediate consequence of (41) is

\[(a_1 b_1) = q b_1 a_1 , \quad c_1 a_1 = q a_1 c_1 , \quad c_1 b_1 + b_1 c_1 = (q - q^{-1}) a_1 d_1 \]

and

\[(c_3 d_3) = q d_3 c_3 , \quad d_3 b_3 = q b_3 d_3 , \quad c_3 b_3 + b_3 c_3 = (q^{-1} - q) a_3 d_3 .\]

Therefore, $C_1 = F_1(q^{-1})$ and $C_3 = F_3(q)$ according to the definition (34).

The final step of the proof: when the centers (36) and (37) conserve,

\[(44) \quad \text{centers of } C_j = \text{centers of } C_j' \quad \text{for all } j \]

then equations (6) can be solved with respect to all primed elements (again with an intensive use of inversion relation (38)) and one can verify directly that (6) provide an unique automorphism of

\[(45) \quad \text{centers of } C_j = \text{centers of } C_j' \quad \text{for all } j .\]

All the other parity patterns can be considered similarly. ■

3. “Evaluation representations” of $A(q^{\pm 1})$ and $F(q^{\pm 1})$

Before the derivation of intertwining operators we must choose firstly irreducible representations of $A(q^{\pm 1})$ and $F(q^{\pm 1})$. Algebra $A$ is equivalent to Bose $q$-oscillator extended by two $C$-valued parameters (centers of $A(q)$). Algebra $F$ is equivalent to Fermi oscillator with two extra $C$-valued parameters. In all considerations below we imply $0 < |q| < 1$.

3.1. Bose $q$-oscillator. We define the Bose oscillator by

\[(46) \quad a^+ a^- = 1 - q^{2N} , \quad a^- a^+ = 1 - q^{2N+2} , \quad [N, a^\pm] = \pm a^\pm .\]

For shortness we use

\[(47) \quad q^N = k .\]
The Fock space representations are defined either by

\begin{equation}
\mathbf{a}^{-}|0\rangle = 0, \quad \text{Spectrum}(n) = 0, 1, 2, 3, \ldots
\end{equation}

where $|0\rangle$ is the Fock vacuum, or by

\begin{equation}
\mathbf{a}^+|-1\rangle = 0, \quad \text{Spectrum}(n) = -1, -2, -3, -4, \ldots
\end{equation}

where $|-1\rangle$ is an “anti-vacuum”. These two representations are related by the external automorphism $\iota$,

\begin{equation}
\iota(k) = q^{-1}k^{-1}, \quad \iota(a^+) = a^-k^{-1}, \quad \iota(a^-) = -k^{-1}a^+.
\end{equation}

Here we prefer not to fix scales of $\mathbf{a}^\pm$ and use $\mathbf{a}^\pm \rightarrow \xi^\pm \mathbf{a}^\pm$ invariant formalism.

3.2. Fermi oscillator. Fermi oscillator is defined by

\begin{equation}
[f^+, f^-]_+ = (1 - q^2), \quad [M, f^\pm] = \pm f^\pm, \quad (f^+)^2 = (f^-)^2 = 0,
\end{equation}

where $[\, , \, ]_+$ stands for anti-commutator. Fields $f^\pm$ have odd parity. For instance, the locality of two copies $\mathcal{F}_1$ and $\mathcal{F}_2$ of Fermi oscillators means

\begin{equation}
f_1 f_2 + f_2 f_1 = 0, \quad \text{etc.}
\end{equation}

The Fock vacuum is annihilated by $f^-$. Since Spectrum($M$) = 0 and 1, the Fock space representation of Fermi oscillator implies in addition $g(M) = g(0)(1 - M) + g(1)M$,

\begin{equation}
M^2 = M, \quad Mf^- = f^+M = 0, \quad f^-M = f^-, \quad MF^+ = f^+,
\end{equation}

and

\begin{equation}
f^+ f^- = (1 - q^2)M, \quad f^- f^+ = (1 - q^2)(1 - M).
\end{equation}

Let for the shortness again

\begin{equation}
k = q^M = 1 - (1 - q)M.
\end{equation}

Automorphism \ref{eq:50} for Fermi oscillator, $\iota(M) = 1 - M$ and $\iota(f^\pm) = f^\mp$, is the internal one and therefore there is no necessity to consider it separately.

3.3. Representation of $\mathcal{A}(q)$. We choose the following form of matrix $X[\mathcal{A}(q)]$:

\begin{equation}
X[\mathcal{A}(q)] = \begin{pmatrix} \lambda k & \mathbf{a}^+ \\ \lambda \mu \mathbf{a}^- & -q \mu k \end{pmatrix}, \quad X[\mathcal{A}(q)]^{-1} = \begin{pmatrix} k/\lambda & \mathbf{a}^+ / \lambda \mu \\ \mathbf{a}^- & -q k / \mu \end{pmatrix},
\end{equation}

where $q$-oscillator is either in \ref{eq:48} or in \ref{eq:49} representation. In this parameterization $\eta = \frac{-\lambda}{\mu\eta}$ and $\xi = -\lambda \mu$. 

3.4. Representation of $\mathcal{A}(q^{-1})$. We choose

$$X[\mathcal{A}(q^{-1})] = \begin{pmatrix} q\lambda k & a^- \\ \lambda \mu a^+ & -\mu k \end{pmatrix}, \quad X[\mathcal{A}(q^{-1})]^{-1} = \begin{pmatrix} q\frac{k}{\lambda} & a^-/\lambda \mu \\ a^+ & -k/\mu \end{pmatrix}. \tag{57}$$

3.5. Representation of $\mathcal{F}(q)$. We choose

$$X[\mathcal{F}(q)] = \begin{pmatrix} \lambda k & f^+ \\ \lambda \mu f^- & -q\mu k^{-1} \end{pmatrix}, \quad X[\mathcal{F}(q)]^{-1} = \begin{pmatrix} k/\lambda & f^+/\lambda \mu \\ f^- & -qk^{-1}/\mu \end{pmatrix}. \tag{58}$$

In this parameterization $\eta = -\frac{\lambda}{\mu}$ and $\xi = -q\lambda \mu$.

3.6. Representation of $\mathcal{F}(q^{-1})$. We choose

$$X[\mathcal{F}(q^{-1})] = \begin{pmatrix} q^{-1}\lambda k & q^{-1}f^- \\ -q^{-1}\lambda \mu f^+ & -\mu k^{-1} \end{pmatrix}, \quad X[\mathcal{F}(q^{-1})]^{-1} = \begin{pmatrix} q^{-1}k/\lambda & -q^{-1}f^-/\lambda \mu \\ -q^{-1}f^+ & -k^{-1}/\mu \end{pmatrix}. \tag{59}$$

3.7. Remarks. Linear equation (13) for matrix $X[\mathcal{A}(q)]$ (56) may be rewritten identically as

$$\begin{pmatrix} \psi'_\beta \\ \psi_\alpha \end{pmatrix} = \overline{X}_{\beta\alpha} \begin{pmatrix} \psi_\beta \\ \psi'_\alpha \end{pmatrix}, \quad \overline{X}_{\beta\alpha} = \begin{pmatrix} -\mu q^{-1}k_1^{-1} & \mu a^-k_1^{-1} \\ -\lambda^{-1}k_1^{-1}a^+ & \lambda^{-1}k_1^{-1} \end{pmatrix}, \tag{60}$$

so that $\overline{X}_{\beta\alpha} \sim \iota(X_{\alpha\beta})$ up to a change of spectral parameters. Thus, a change of orientation of linear problem (see Fig. 1) is equivalent to $\iota$-automorphisms (50).

4. Eight intertwining relations

In this section we give the list of intertwining operators (11) for all parity patterns of $\alpha, \beta, \gamma$ for representations of $\mathcal{A}(q^{\pm 1}), \mathcal{F}(q^{\pm 1})$ chosen above. For shortness, the symbols $\mathcal{A}$ and $\mathcal{F}$ as the indices of three-dimensional $R$-matrices will stand for corresponding representation Fock spaces.

The following eight intertwining relations hold.

4.1. Configuration $\alpha\beta\gamma$. The basic intertwining relation is

$$X_{\alpha\beta}[\mathcal{A}_1(q)]X_{\alpha\gamma}[\mathcal{A}_2(q)]X_{\beta\gamma}[\mathcal{A}_3(q)] R = R X_{\beta\gamma}[\mathcal{A}_3(q)]X_{\alpha\gamma}[\mathcal{A}_2(q)]X_{\alpha\beta}[\mathcal{A}_1(q)]. \tag{61}$$

The intertwiner here is

$$R = R_{\mathcal{A}_1(q),\mathcal{A}_2(q),\mathcal{A}_3(q)}(u,v,w) = v^N_2 R_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3} u^N_1 w^N_3, \tag{62}$$
where for brevity
\begin{equation}
    u = \frac{\lambda_3}{\lambda_2}, \quad v = \lambda_1 \mu_3, \quad w = \frac{\mu_1}{\mu_2}.
\end{equation}
This definition of parameters \( u, v, w \) is used in all cases below. Constant matrix \( R \) can be written as a power series
\begin{equation}
    R_{A_1 A_2 A_3} = R_0(N_1, N_2, N_3) + \sum_{k=1}^{\infty} \left( R_k(N_1, N_2, N_3)(a_1^+ a_2^+ a_3^-)^k + (a_1^- a_2^- a_3^+)^k R_k(N_1, N_2, N_3) \right),
\end{equation}
where the coefficients \( R_k(N_1, N_2, N_3) \) are given by expansion of three equivalent generating functions:
\begin{equation}
\begin{aligned}
    \text{Tr}_{A_1} \left( z^{N_1} a_1^{-k} a_1^{+k} R_k(N_1, N_2, N_3) \right) &= z^{N_2-k} \frac{(q^{N_1-N_2+k+2z-1}; q^2)_\infty}{(q^{N_3-N_2-k+2z-1}; q^2)_\infty} \frac{(q^{N_3-N_2+k}; q^2)_\infty}{(q^{N_3-N_2-k}; q^2)_\infty}, \\
    \text{Tr}_{A_2} \left( z^{N_2} a_2^{-k} R_k(N_1, N_2, N_3) a_2^{+k} \right) &= q^{N_2} \frac{(-q^{1-N_1-N_3}; q^2)_\infty (-q^3+q^{N_3+2k}; q^2)_\infty}{(-q^1+q^{N_1-N_3}; q^2)_\infty (-q^1+q^{N_1-N_3}; q^2)_\infty}, \\
    \text{Tr}_{A_3} \left( z^{N_3} a_3^{-k} a_3^{+k} R_k(N_1, N_2, N_3) \right) &= z^{N_2-k} \frac{(q^{N_1-N_2+k+2z-1}; q^2)_\infty (q^{N_1-N_2+k+2}; q^2)_\infty}{(q^{N_1-N_2+k}; q^2)_\infty (q^{N_1-N_2+k}; q^2)_\infty}.
\end{aligned}
\end{equation}
Here we use the standard \( q \)-hypergeometric notations [10]:
\begin{equation}
    (x; q^2)_n = \prod_{k=0}^{n-1} (1 - xq^{2k}), \quad (x; q^2)_\infty = \prod_{k=0}^{\infty} (1 - xq^{2k}).
\end{equation}
These generating functions are valid for both representations (48) and (49) for any of \( A_1, A_2, A_3 \). Note, for finite integer \( n_j \) and \( k \) all generating functions are rational functions regular at \( z = 0 \) and at \( z = \infty \) with finite number of poles in \( z \)-plane, a generation function \( F(z) = \text{Tr}_{A} z^N f(n) \) gives \( f(n) \) as expansion near \( z = 0 \) or near \( z = \infty \) for representations (48) or (49) respectively. The expressions for \( R_k(N_1, N_2, N_3) \) in terms of \( q \)-hypergeometric function can be obtained by Cauchy integrals of generating functions, in particular the matrix elements of (64) for representation (48) in all \( A_1, A_2, A_3 \) are given in [11] [4].

Note a few symmetry properties. Firstly, generating functions provide the definition of \( R_k \) for negative \( k \),
\begin{equation}
    R_{-k}(n_1 + k, n_2 - k, n_3 + k) = \frac{(q^2; q^2)_{n_1+k}(q^2; q^2)_{n_2}(q^2; q^2)_{n_3+k}}{(q^2; q^2)_{n_1}(q^2; q^2)_{n_2-k}(q^2; q^2)_{n_3}} R_k(n_1, n_2, n_3).
\end{equation}
Power series (64) can be formally rewritten as
\begin{equation}
    R_{A_1 A_2 A_3} = \sum_{k=-\infty}^{\infty} R_k(N_1, N_2, N_3)(a_1^+ a_2^+ a_3^-)^k = \sum_{k=-\infty}^{\infty} (a_1^+ a_2^+ a_3^-)^k R_k(N_1, N_2, N_3),
\end{equation}
where \( R_k \equiv 0 \) at states where negative powers of creation and annihilation operators are not defined.

Operator (64) is the square root of unity,

\[
R_{A_1 A_2 A_3}^{-1} = R_{A_1 A_2 A_3}
\]

for any choice of representations (48,49). Remarkably, an analytic proof of this statement involves the Ramanujan summation formula [10]. Also, expression (64) has the evident symmetry with respect to an anti-involution \( N \rightarrow N \) and \( a^\pm \rightarrow a^{\mp} \). This anti-involution is the Hermitian conjugation for real \( q \) and unitary representation (48).

Operator (64) is the unique solution of (61) provided the integer spectra of \( N_{\alpha j} \).

4.2. Configuration \( \alpha\beta\gamma \). Relation

\[
X_{\alpha\beta}[A_1(q^{-1})] X_{\alpha\gamma}[A_2(q^{-1})] X_{\beta\gamma}[A_3(q^{-1})] \ R = R \ X_{\beta\gamma}[A_3(q^{-1})] X_{\alpha\gamma}[A_2(q^{-1})] X_{\alpha\beta}[A_1(q^{-1})]
\]

provides

\[
R = R_{A_1 A_2 A_3}(q^{-1}) \left(u, v, w\right) = v^{-N_2} R_{A_1 A_2 A_3} \left(u, v, w\right)
\]

where \( u, v, w \) and \( R_{A_1 A_2 A_3} \) are given by (63) and (64).

4.3. Fermionic configurations and intertwiners. Intertwining operators for the fermions can be presented in two ways. One is the way of matrix elements for the basis of fermionic states, even vacuum \( |0\rangle \) and odd one-fermion state \( |1\rangle = \frac{f_+}{\sqrt{1-q^2}} |0\rangle \); all equations in matrix elements will have then unpleasant sign factors taking into account the odd parity of the state \( |1\rangle \) and ordering of fermions. Another way avoiding the sign factors is to consider fermionic operators directly as expressions in terms of odd fermionic creation and annihilation operators like in [11]. We choose the second way here. Besides, the expressions for the fermionic intertwiners below do not need integer spectra of Bose oscillators and therefore they are valid also for the modular representation of \( q \)-oscillator [4].

4.4. Configuration \( \alpha\beta\gamma \). Relation

\[
X_{\alpha\beta}[F_1(q^{-1})] X_{\alpha\gamma}[F_2(q^{-1})] X_{\beta\gamma}[A_3(q)] \ R = R \ X_{\beta\gamma}[A_3(q)] X_{\alpha\gamma}[F_2(q^{-1})] X_{\alpha\beta}[F_1(q^{-1})]
\]

yields uniquely

\[
R = R_{F_1 F_2 A_3}(q^{-1}) \left(u, v, w\right) = v^{-M_2} R_{F_1 F_2 A_3} \left(u, v, w\right)
\]

where the constant matrix \( R \) is

\[
R_{F_1 F_2 A_3} = (1-M_1)(1-M_2) + q M_1 (1-M_2) k_3 - (1-M_1) M_2 k_3 - M_1 M_2 + \frac{f_+^2 a_3^+ - f_+^2 f_2^+ a_3^+}{1-q^2}
\]
and $u, v, w$ are given by (63). Constant $R$-matrix is the root of unity, 

$$R_{F_1,F_2,A_3} = R_{F_1,F_2,A_3}^{-1},$$

and expression (76) is symmetric with respect to $f^\pm \rightarrow f^\mp, a^\pm \rightarrow a^\mp$ anti-involution.

### 4.5. Configuration $\alpha\beta\gamma$. Relation

(78) \[ X_{\alpha\beta}[F_1(q)]X_{\alpha\gamma}[F_2(q)]X_{\beta\gamma}[A_3(q^{-1})] \quad R = R \quad X_{\beta\gamma}[A_3(q^{-1})]X_{\alpha\gamma}[F_2(q)]X_{\alpha\beta}[F_1(q)] \]

yields

(79) \[ R = R_{F_1,F_2,A_3}(q^{-1})(u, v, w) = v^{M_2} R_{F_1,F_2,A_3} u^{M_1} w^{-N_3} \]

where the constant matrix $R_{F_1,F_2,A_3}$ is given by (76).

### 4.6. Configuration $\alpha\beta\gamma$. Relation

(80) \[ X_{\alpha\beta}[A_1(q)]X_{\alpha\gamma}[F_2(q)]X_{\beta\gamma}[F_3(q)] \quad R = R \quad X_{\beta\gamma}[F_3(q)]X_{\alpha\gamma}[F_2(q)]X_{\alpha\beta}[A_1(q)] \]

provides uniquely

(81) \[ R = R_{A_1,F_2,F_3}(q^{-1})(u, v, w) = v^{M_2} R_{A_1,F_2,F_3} u^{M_1} w^{M_3} \]

where

(82) \[ R_{A_1,F_2,F_3} = (1-M_2)(1-M_3) - qk_1M_2(1-M_3) + k_1(1-M_2)M_3 - M_2M_3 + \frac{a_1^-f_2^{-} f_3^{-} - a_1^+ f_2^{+} f_3^{+}}{1-q^2} \]

Operator (82) is the square root of unity symmetric with respect to the anti-involution.

### 4.7. Configuration $\alpha\beta\gamma$. Relation

(83) \[ X_{\alpha\beta}[A_1(q^{-1})]X_{\alpha\gamma}[F_2(q^{-1})]X_{\beta\gamma}[F_3(q^{-1})] \quad R = R \quad X_{\beta\gamma}[F_3(q^{-1})]X_{\alpha\gamma}[F_2(q^{-1})]X_{\alpha\beta}[A_1(q^{-1})] \]

gives

(84) \[ R = R_{A_1,F_2,F_3}(q^{-1})(u, v, w) = v^{-M_2} R_{A_1,F_2,F_3} u^{-N_1} w^{-M_3} \]

where $R_{A_1,F_2,F_3}$ is given by (82).
4.8. **Configuration** $\alpha \beta \gamma$. Relation

\[ X_{\alpha \beta}[F_1(q)]X_{\alpha \gamma}[A_2(q)]X_{\beta \gamma}[F_3(q^{-1})] = R X_{\beta \gamma}[F_3(q^{-1})]X_{\alpha \gamma}[A_2(q)]X_{\alpha \beta}[F_1(q)] \]

gives uniquely

\[ R = R_{F_1(q),A_2(q),F_3(q^{-1})} = v^{N_2} R_{F_1,A_2,F_3} u^M w^{-M} \]

where

\[ R_{F_1,A_2,F_3} = (-q)^{-N_2} \left( (1 - M_1)k_2(1 - M_3) + q^{-1}M_1(1 - M_3) + (1 - M_1)M_3 + M_1k_2M_3 \right. \]

\[ \left. + \frac{q^{-1}f_1^+ a_2 f_3^- - f_1^- a_2^+ f_3^+}{1 - q^2} \right) \]

Operator (87) is the root of unity but it is not symmetric with respect to the anti-involution.

4.9. **Configuration** $\underline{\alpha \beta \gamma}$. Relation

\[ X_{\underline{\alpha \beta}}[F_1(q^{-1})]X_{\underline{\alpha \gamma}}[A_2(q^{-1})]X_{\beta \gamma}[F_3(q)] = R X_{\beta \gamma}[F_3(q)]X_{\underline{\alpha \gamma}}[A_2(q^{-1})]X_{\underline{\alpha \beta}}[F_1(q^{-1})] \]

produces

\[ R = R_{F_1(q^{-1}),A_2(q^{-1}),F_3(q)}(u,v,w) = v^{-N_2} R_{F_1,A_2,F_3} u^{-M} w^M \]

where $R_{F_1,A_2,F_3}$ is given by (87).

4.10. **Remarks.** All constant $R$-matrices are roots of unity since for $\lambda = \mu = 1$

\[ X[\mathcal{C}] = X[\mathcal{C}]^{-1} \]

for all $X$-matrices (56–59). Also, matrices $X[A(q)]$, $X[A(q^{-1})]$, $X[F(q)]$ at $\lambda = \mu = 1$ and matrix $X[F(q^{-1})]$ at $\lambda = 1, \mu = -1$ are symmetric with respect to the anti-involution $a^\pm \to a^{\mp}$ and $f^\pm \to f^{\mp}$ accompanied by the matrix transposition. Recall, this anti-involution is the Hermitian conjugation for $0 < q < 1$ for Fermi oscillators and Bose oscillator in representation (48). Representation (49) of Bose oscillator admits $(a^\pm)^\dagger = -a^{\mp}$. Thus, the unitarity of intertwiners is an extra condition fixing a proper choice of representations (48) or (49). For instance, $R_{F_1,F_2,A_3}$ and $R_{A_1,F_2,F_3}$ are unitary for representation (48) for $A_1$ and $A_3$ while $R_{F_1,A_2,F_3}$ is unitary for representation (49) of $A_2$. Matrix $R_{A_1,A_2,A_3}$ is unitary if representation (49) is chosen in even number of components $A_1,A_2,A_3$. 
5. Examples of tetrahedra

We have four constant R-matrices, spectral parameters enter as simple exponential fields:

\[ R_{C_1 C_2 C_3} (u, v, w) = v^{-N_2} R_{C_1 C_2 C_3} u^{-N_1} w^{-N_3} , \]

where

| N  | A(q) | A(q⁻¹) | F(q) | F(q⁻¹) |
|----|------|--------|------|--------|
| N  | N    | -N     | M    | -M     |

The only difference between \( C(q) \) and \( C(q⁻¹) \) is the sign of a field exponent.

Note, any matrix \( R_{C_1 C_2 C_3} \) commutes with \( N_1 + N_2 \) and \( N_2 + N_3 \). Therefore, the spectral parameters can be removed from any tetrahedron equation and finally we have only eight constant tetrahedron equations

\[ R_{C_1 C_2 C_3} R_{C_1 C_4 C_5} R_{C_2 C_4 C_6} R_{C_3 C_5 C_6} = R_{C_3 C_5 C_6} R_{C_2 C_4 C_6} R_{C_1 C_4 C_5} R_{C_1 C_2 C_3} \]

with

| Variant | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) | \( C_5 \) | \( C_6 \) |
|---------|--------|--------|--------|--------|--------|--------|
| 1       | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) |
| 2       | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( F_4 \) | \( F_5 \) | \( F_6 \) |
| 3       | \( A_1 \) | \( F_2 \) | \( F_3 \) | \( A_4 \) | \( A_5 \) | \( F_6 \) |
| 4       | \( F_1 \) | \( A_2 \) | \( F_3 \) | \( A_4 \) | \( F_5 \) | \( A_6 \) |
| 5       | \( F_1 \) | \( F_2 \) | \( F_3 \) | \( F_4 \) | \( F_5 \) | \( A_6 \) |
| 6       | \( F_1 \) | \( A_2 \) | \( F_3 \) | \( F_4 \) | \( A_5 \) | \( F_6 \) |
| 7       | \( F_1 \) | \( A_2 \) | \( A_3 \) | \( F_4 \) | \( F_5 \) | \( F_6 \) |
| 8       | \( F_1 \) | \( F_2 \) | \( A_3 \) | \( A_4 \) | \( F_5 \) | \( F_6 \) |

5.1. Tetrahedron \( AAAFFF \). The quadrilateral configuration

\[ X_{\alpha\beta}[A_1] X_{\alpha\gamma}[A_2] X_{\beta\gamma}[A_3] X_{\alpha\delta}[F_4] X_{\beta\delta}[F_5] X_{\gamma\delta}[F_6] \]

provides the tetrahedron equation

\[ R_{A_1 A_2 A_3} R_{A_1 F_4 F_5} R_{A_2 F_4 F_6} R_{A_3 F_5 F_6} = R_{A_3 F_5 F_6} R_{A_2 F_4 F_6} R_{A_1 F_4 F_5} R_{A_1 A_2 A_3} , \]

number 2 from the table. Being written in matrix elements in fermionic spaces, \(|1\rangle \sim f^+|0\rangle\),

\[ R_{A_1 F_2 F_3} id \otimes |j_2\rangle \otimes |j_3\rangle = \sum_{i_1, i_2} |i_2\rangle \otimes |i_3\rangle (-)^p(i_1) p(i_2) L_{i_1, i_2}^{j_1, j_2}[A_1], \]

where fermionic occupation numbers \( i, j = 0, 1 \) and the parity is \( p(i) = i \), equation (94) is equivalent to \( RLLL \) relation (4) from [1].
5.2. Tetrahedron \( AFFFFA \). The quadrilateral configuration

\[
X_{\alpha \beta} [A_1] X_{\alpha \gamma} [F_2] X_{\beta \gamma} [F_3] X_{\alpha \delta} [F_4] X_{\beta \delta} [F_5] X_{\gamma \delta} [A_6]
\]

provides the constant tetrahedron equation

\[
R_{A_1 F_2 F_3} R_{A_1 F_4 F_5} R_{F_2 F_4 A_6} R_{F_3 F_5 A_6} = R_{F_2 F_4 A_6} R_{A_1 F_4 F_5} R_{A_1 F_2 F_3}
\]

number 6 from the table above. Fermionic lines in this equation are not planar, hence sign parity factors are irremovable from a matrix form of (97). However, the parity factors can be absorbed by a re-definition of \( A_6 \), result is the “strange” \( LLMM \) relation (34) from [1].

6. Yang-Baxter equation and transfer matrices

6.1. \( R \)-matrices of Yang-Baxter equation. An \( R \)-matrix of the Yang-Baxter equation can be defined by

\[
R_{C_1, C_2} = \text{Trace}_{C_3} \left( \prod_{\ell} R_{C_1, \ell} C_{2, \ell} C_3 \right),
\]

where semicolon in indices separates the orientation index \( j = 1, 2, 3 \) and the coordinate indices, the ordered product stands for

\[
\prod_{\ell} f_{\ell} = f_1 f_2 \cdots f_L,
\]

and the spaces of two-dimensional \( R \)-matrix are

\[
C_1 = \bigotimes_{\ell=1} L C_{1, \ell}, \quad C_2 = \bigotimes_{\ell=1} L C_{2, \ell}.
\]

The sequence of tetrahedron equations

\[
\left( \prod_{\ell} R_{C_1, C_2, C_3} R_{C_1, C_4, C_5} R_{C_2, C_4, C_5} R_{C_3, C_5, C_6} \right) = R_{C_3, C_5, C_6} \left( \prod_{\ell} R_{C_2, C_4, C_5} R_{C_1, C_4, C_5} R_{C_1, C_2, C_3} \right)
\]

provide the Yang-Baxter equation for (98),

\[
R_{C_1 C_2} R_{C_1 C_3} R_{C_2 C_4} = R_{C_2 C_4} R_{C_1 C_3} R_{C_1 C_2}.
\]

The “third” space of three-dimensional \( R \)-matrix is chosen as the hidden space in (98), in general it can be “first” or “second” orientation spaces as well.

Locally, the spectral parameters enter (98) as

\[
R_{C_1, \ell} C_{2, \ell} C_3 = v_{\ell}^{\nu_1} w_{\ell}^{\nu_2} R_{C_1, C_2, \ell} C_3 w_{\ell}^{\nu_3} v_{\ell}^{\nu_4},
\]
see (91), where

\begin{equation}
(104) \quad u_\ell = \frac{\lambda_3}{\lambda_2}, \quad v_\ell = \lambda_1; \mu_3, \quad w_\ell = \frac{\mu_1}{\mu_2}.
\end{equation}

Since \( N_1 + N_2 \) and \( N_2 + N_3 \) commute with \( R_{C_1C_2C_3} \), the expression for (98) can be rewritten identically as

\begin{equation}
(105) \quad R_{C_1C_2} = U_1^{-1} U_2^{-1} \left( \prod_\ell v_\ell^{N_2, \ell} \right) R_{C_1C_2}(\mu_1/\mu_2) \left( \prod_\ell u_\ell^{N_1, \ell} \right) U_1 U_2,
\end{equation}

where the gauge factors are

\begin{equation}
(106) \quad U_1 = \prod_{\ell=1}^L \mu_1; \ell = 0 \cdots N_1; \ell, \quad U_2 = \prod_{\ell=1}^L \mu_2; \ell = 0 \cdots N_2; \ell,
\end{equation}

the simplified \( R \)-matrix in the right hand side of (105) is

\begin{equation}
(107) \quad R_{C_1C_2}(w) = \text{Trace}_{C_3} \left( w^{N_3} \prod_\ell \hat{R}_{C_1C_2C_3, \ell} \right),
\end{equation}

and its single spectral parameter is given by

\begin{equation}
(108) \quad \frac{\mu_1}{\mu_2} = \prod_\ell w_\ell, \quad \mu_1 = \prod_\ell \mu_1; \ell, \quad \mu_2 = \prod_\ell \mu_2; \ell.
\end{equation}

\( R \)-matrix (98) commutes with \( N_1; \ell + N_2; \ell \) for any \( \ell \) and with

\begin{equation}
(109) \quad N_1; s = \sum_\ell N_1; \ell \quad \text{and} \quad N_2; s = \sum_\ell N_2; \ell,
\end{equation}

what corresponds to arbitrariness of parameters \( \lambda_3, \mu_3 \) of hidden space. Cancelling then all gauges and fields in the general Yang-Baxter equation (102), we come to the standard Yang-Baxter equation with multiplicative spectral parameter for simplified \( R \)-matrix (107)

\begin{equation}
(110) \quad R_{C_1C_2}(\mu_1/\mu_2)R_{C_1C_4}(\mu_1/\mu_4)R_{C_2C_4}(\mu_2/\mu_4) = R_{C_2C_4}(\mu_2/\mu_4)R_{C_1C_4}(\mu_1/\mu_4)R_{C_1C_2}(\mu_1/\mu_2).
\end{equation}

6.2. Layer-to-layer transfer matrices. For a given set of quantum spaces \( C_{1; \ell,m} \) and for any suitable sequences of auxiliary spaces \( C_{2; \ell} \) and \( C_{3; m} \), the layer-to-layer transfer matrix is

\begin{equation}
(111) \quad T = \text{Trace}_{C_{2; \ell}, C_{3; m}} \left( \prod_\ell \prod_m \hat{R}_{C_{1; \ell,m}, C_{2; \ell}, C_{3; m}} \right),
\end{equation}

where the ordered products are defined by

\begin{equation}
(112) \quad \prod_\ell f_\ell = f_1 f_2 \cdots f_L, \quad \prod_m g_m = g_1 g_2 \cdots g_M.
\end{equation}

\(^2\)Due to the factor \( w^{N_3} \), there are no necessity to use super-trace for the case of \( C_3 = \mathcal{F}_3 \).
Transfer matrix (111) can be understood as a two-dimensional transfer matrix for a length \( M \) chain of \( R \)-matrices (98) with hidden “third” spaces, or as a two-dimensional transfer matrix for a length \( L \) chain of \( R \)-matrices with hidden “second” spaces. The tetrahedron equation provides the commutativity of any two transfer matrices with identical sets of \( C_{1:1:1} \). Transfer matrices can differ by spectral parameters in auxiliary spaces and by a choice of statistics of auxiliary spaces.

Locally, the spectral parameters enter (111) as

\[
R_{c_{1:1:1}c_{2:2:2}c_{3:3:3}} = (\lambda_{1:1:1} \mu_{3:3:3})^{N_{2:2:2}} R_{c_{1:1:1}c_{2:2:2}c_{3:3:3}} \left( \lambda_{3:3:3} \mu_{2:2:2} \right)^{N_{3:3:3}},
\]

see (91). Since \( N_{1} + N_{2} \) and \( N_{2} + N_{3} \) commute with \( R_{c_{1:1:1}c_{2:2:2}} \), the spectral parameters in auxiliary spaces can be pushed to boundary, transfer matrix (111) can be rewritten as

\[
T = \text{Trace}_{c_{2:2:2}c_{3:3:3}} \left( \prod_{\ell} v_{\ell}^{N_{2:2:2}} \prod_{m} w_{m}^{N_{3:3:3}} \prod_{\ell} \prod_{m} R_{c_{1:1:1}c_{2:2:2}c_{3:3:3}} \left( \lambda_{3:3:3} \mu_{2:2:2} \right)^{N_{3:3:3}} \right),
\]

where

\[
v_{\ell} = \prod_{m} \lambda_{1:1:1} \mu_{3:3:3}, \quad w_{m} = \prod_{\ell} \mu_{1:1:1} \mu_{2:2:2}.
\]

This most general expression corresponds to inhomogeneous spectral parameters \( v_{\ell} \neq v_{\ell}', w_{m} \neq w_{m}' \). The most right external factor in (114) depends only on auxiliary spectral parameters, hence

\[
N_{\ell} = \sum_{m} N_{1:1:1} \quad \text{and} \quad N_{m} = \sum_{\ell} N_{1:1:1},
\]

commute with transfer matrix and therefore this factor is inessential for spectral problem.

The choice \( \lambda_{1:1:1} = \mu_{1:1:1} \) gives the homogeneous transfer matrix,

\[
T(v, w) = \text{Trace}_{c_{2:2:2}c_{3:3:3}} \left( v^{N_{2:2:2}} w^{N_{3:3:3}} \prod_{\ell} \prod_{m} R_{c_{1:1:1}c_{2:2:2}c_{3:3:3}} \left( \lambda_{3:3:3} \mu_{2:2:2} \right)^{N_{3:3:3}} \right),
\]

where for shortness

\[
N_{2} = \sum_{\ell} N_{2:2:2} \quad \text{and} \quad N_{3} = \sum_{m} N_{3:3:3}.
\]

The tetrahedron equations and related effective Yang-Baxter equations provide the commutativity of transfer matrices,

\[
[T(v, w), T(v', w')] = 0.
\]
7. Classification of $R$-matrices in terms quantum groups

7.1. General case. Consider effective two-dimensional $R$-matrix (107) with $C_3 = A_3(q)$. Let all Bose $q$-oscillators are at unitary representation (48). According to the patterns (62) and (75), possible choice of $C_1: \ell$ and $C_2: \ell$ are

$$C_1: \ell \otimes C_2: \ell = \begin{cases} A_{1: \ell}(q) \otimes A_{2: \ell}(q) \\
F_{1: \ell}(q^{-1}) \otimes F_{2: \ell}(q^{-1}) \end{cases}$$

Thus, we can define the following space of the Yang-Baxter equation:

$$Q_j = \bigotimes_{\ell=1}^{L} \left[ A_{j: \ell}(q) \text{ or } F_{j: \ell}(q^{-1}) \right], \quad \#A_{j: \ell} = L_1, \quad \#F_{j: \ell} = L_2.$$  

Alternative space can be defined by

$$A_j = \bigotimes_{\ell=1}^{L} \left[ F_{j: \ell}(q) \text{ or } A_{j: \ell}(q^{-1}) \right] \text{ respectively.}$$

Any of choices $C_j = Q_j$ or $A_j$ in the Yang-Baxter equation (102) is valid. Matrix $R_{Q_1,Q_2}$ has the hidden space $A_3(q)$, matrix $R_{Q_1,A_2}$ has the hidden space $F_3(q)$, matrix $R_{A_1,A_2}$ has the hidden space $A_3(q^{-1})$, matrix $R_{A_1,Q_2}$ has hidden space $F_3(q^{-1})$.

Our statement is the following: For given $L_1$ and $L_2$, all the $R$-matrices reproduce the $R$-matrices and $L$-operators for quantum super-algebra $\mathcal{U}_q(\hat{gl}(L_1|L_2))$. Spaces $Q$ and $A$ are two types of reducible evaluation representations of $\mathcal{U}_q(\hat{gl}(L_1|L_2))$ [12].

If $L_2 = 0$, these $R$-matrices correspond to $\mathcal{U}_q(\hat{sl}_L)$. It is shown in [1], the space $Q_j$ is the direct infinite sum of all symmetric tensor representations of $sl_L$; the space $A_j$ is the direct sum of all antisymmetric tensor (fundamental) representations of $sl_L$. Note, in a case of a mixture of representations (18) and (19), infinite dimensional evaluation representations of $\mathcal{U}_q(\hat{sl}_L)$ appear; we do not discuss this possibility in details here.

Remarks:

- For given $L_1$ and $L_2$, there are different choices of particular ordering of $A$ and $F$. This corresponds to different choices of Cartan matrix and Dynkin diagram for $gl(L_1|L_2)$. All such choices are equivalent due to the tetrahedron equation (providing $Z$-invariance of three-dimensional lattice).

- Transfer matrix (111) has the structure of simple square lattice in auxiliary orientation spaces “2” and “3”. More complicated structure of auxiliary configuration provides more reach class of evaluation representations, e.g. multicomponent Bose/Fermi gases.

The case of massless free fermions $L_1 = L_2 = 1$ is too primitive. Below we argue our statement for more illustrative case $L_1 = 2, L_2 = 1$. 


7.2. \textit{R-matrices for }$\mathcal{Y}_q(\hat{\mathfrak{gl}}(2|1))$. For $L_1 = 2$ and $L_2 = 1$ the “quantum” and “auxiliary” spaces (121,122) are respectively

\begin{equation}
Q_j = A_{j;1}(q) \otimes A_{j;2}(q) \otimes \mathcal{F}_{j;3}(q^{-1}), \quad A_j = \mathcal{F}_{j;1}(q) \otimes \mathcal{F}_{j;2}(q) \otimes A_{j;3}(q^{-1}).
\end{equation}

These spaces have the following invariants (109):

\begin{equation}
\begin{array}{c}
\mathcal{N}_Q = N_{j;1} + N_{j;2} - M_{j;3}, \\
\mathcal{N}_A = M_{j;1} + M_{j;2} - N_{j;3}.
\end{array}
\end{equation}

Both $Q$ and $A$ are infinite dimensional spaces. Spectrum of $\mathcal{N}_Q$ is $-1, 0, 1, 2, 3, \ldots$. Let $Q(N)$ be a subspace of $Q$ with $\mathcal{N}_Q = N$. Elementary combinatorics gives

\begin{equation}
\dim Q(N) = 2N + 3.
\end{equation}

We identify $Q(-1)$ – scalar representation, $Q(0)$ – vector representation, $Q(N)$ with $N \geq 1$ – highest atypical representations of $gl(2|1)$.

In its turn, spectrum of $\mathcal{N}_A$ is $2, 1, 0, -1, \ldots$. Let again $A(N)$ be a subspace of $A$ with $\mathcal{N}_A = N$. Then

\begin{equation}
\begin{array}{c}
\dim A(2) = 0, \\
\dim A(1) = 3, \\
\dim A(N) = 4 \text{ for } N \leq 0.
\end{array}
\end{equation}

We identify $A(2)$ – scalar representation, $A(1)$ – vector representation, $A(N)$ with $N \leq 0$ – typical representation of $gl(2|1)$.

Below we consider two $R$-matrices,

\begin{equation}
R_{A_1,Q_2}(w) = (-q)^{-\mathcal{N}_Q} \text{ Trace}_{\mathcal{F}_3} \left( w^{M_3} R_{\mathcal{F}_{1;1},A_{2;1},A_3} R_{\mathcal{F}_{1;2},A_{2;2},A_3} R_{A_{1;3}A_{2;2}A_3} \right)
\end{equation}

and

\begin{equation}
R_{A_1,A_2}(w) = \text{ Trace}_{A_3} \left( w^{N_3} R_{\mathcal{F}_{1;1},A_{2;2}A_3} R_{\mathcal{F}_{1;2},A_{2;2}A_3} R_{A_{1;3}A_{2;2}A_3} \right)
\end{equation}
Explicit expression for $R_{A_1, Q_2}(w)$ is

\begin{equation}
R_{A_1, Q_2}(w) =
\end{equation}

\begin{align*}
& [(1 - M_{11})k_{21} + q^{-1}M_{11}] [(1 - M_{12})k_{22} + q^{-1}M_{12}] [(1 - M_{23}) + k_{13}M_{23}] \\
& + w [(1 - M_{11}) + M_{11}k_{21}] [(1 - M_{12}) + M_{12}k_{22}] k_{13}(1 - M_{23}) + q^{-1}M_{23}
\end{align*}

\begin{align*}
& + \frac{q^{-1}}{1 - q^2} f_{11}^+ f_{12} a_{21} a_{22}^+ [(1 - M_{23}) + k_{13}M_{23}] + w \frac{q^{-1}}{1 - q^2} f_{11}^+ f_{12} a_{21} a_{22}^+ k_{13}(1 - M_{23}) + q^{-1}M_{23}
\end{align*}

\begin{align*}
& + \frac{q^{-1}}{1 - q^2} f_{11}^+ a_{21}^- [(1 - M_{12}) + M_{12}k_{22}] a_{13}^+ f_{23}^- - w \frac{q^{-1}}{1 - q^2} f_{11}^+ a_{21}^- [(1 - M_{12})k_{22} + q^{-1}M_{12}] a_{13}^+ f_{23}^-
\end{align*}

\begin{align*}
& + \frac{q^{-1}}{1 - q^2} [(1 - M_{11})k_{21} + q^{-1}M_{11}] f_{12}^+ a_{22} a_{13}^+ f_{23}^- - w \frac{q^{-1}}{1 - q^2} [(1 - M_{11}) + M_{11}k_{21}] f_{12}^+ a_{22} a_{13}^+ f_{23}^-
\end{align*}

In what follows, it is more convenient to change notations to tensor product form:

\begin{equation}
C_{1,j} C_{2,k} = C_j \otimes C_k
\end{equation}

Consider now the following basis of $A_1 = A \otimes 1$ with $\gamma_A = -n$ (recall, we consider now the unitary representation $U$ of Bose oscillator, $|0\rangle$ is the total even Fock vacuum annihilated by all $f^-$ and $a^-$ operators):

\begin{align*}
|e_0\rangle &= \frac{a_3^{+n}}{\sqrt{(q^2;q^2)_n}} |0\rangle = |0, 0, n\rangle , \\
\frac{f_1^+ }{\sqrt{1 - q^2}} \frac{a_3^{+(n+1)}}{\sqrt{(q^2;q^2)_{n+1}}} |0\rangle = |1, 0, n + 1\rangle , \\
|e_2\rangle &= \frac{f_2^+ }{\sqrt{1 - q^2}} \frac{a_3^{+(n+1)}}{\sqrt{(q^2;q^2)_{n+1}}} |0\rangle = |0, 1, n + 1\rangle , \\
\frac{f_1^+ f_2^+ }{1 - q^2} \frac{a_3^{+(n+2)}}{\sqrt{(q^2;q^2)_{n+2}}} |0\rangle = |1, 1, n + 2\rangle .
\end{align*}

Parity of states are $p(e_0) = p(e_3) = 0$ and $p(e_1) = p(e_2) = 1$. Matrix units $E_{jk}$ can be introduced by

\begin{equation}
E_{jk} |e_k\rangle = |e_j\rangle
\end{equation}
Using definition of matrix units in $A \otimes 1$ space, we rewrite operator $R_{A \otimes Q}$ as

\begin{align*}
R_{A \otimes Q} &= E_{00} \otimes \left( q^{N_1+N_2+nM_3} + wq^{n-(1+n)M_3} \right) + E_{11} \otimes \left( q^{N_2+(n+1)M_3-1} + wq^{n+1+N_1-(n+2)M_3} \right) \\
&+ E_{22} \otimes \left( q^{N_1+(n+1)M_3-1} + wq^{n+1+N_2-(n+2)M_3} \right) + E_{33} \otimes \left( q^{(n+2)M_3-2} + wq^{n+2+N_1+N_2-(n+3)M_3} \right) \\
&+ E_{12} \otimes a_1^+ a_2^+ q^{(n+1)M_3-1} - wE_{21} \otimes a_1^+ a_2^+ q^{n-(n+2)M_3}
\end{align*}

\begin{align*}
= &+ q^{-1} \sqrt{\frac{1-q^{2(n+1)}}{1-q^2}} E_{10} \otimes a_1 f_3 - q^{-1} \sqrt{\frac{1-q^{2(n+2)}}{1-q^2}} E_{32} \otimes a_1 q^{N_2} f_3 \\
&+ q^{-1} w \sqrt{\frac{1-q^{2(n+1)}}{1-q^2}} E_{01} \otimes a_1^+ q^{N_2} f_3^+ - q^{-2} w \sqrt{\frac{1-q^{2(n+2)}}{1-q^2}} E_{23} \otimes a_1^+ f_3^+ \\
&+ q^{-1} \sqrt{\frac{1-q^{2(n+1)}}{1-q^2}} E_{20} \otimes q^{N_1} a_2 f_3 + q^{-2} \sqrt{\frac{1-q^{2(n+2)}}{1-q^2}} E_{31} \otimes a_2 f_3 \\
&+ q^{-1} w \sqrt{\frac{1-q^{2(n+1)}}{1-q^2}} E_{02} \otimes a_2^+ f_3^+ + q^{-1} w \sqrt{\frac{1-q^{2(n+2)}}{1-q^2}} E_{13} \otimes q^{N_1} a_2^+ f_3^+ 
\end{align*}

When $n = -1$, $E_{00}$ component factors out and one has three-dimensional representation in $A$-space:

\begin{align*}
L_{A \otimes Q}(u) = qR_{A \otimes Q}(w = -wq^{N_1+N_2+1-M_3}) = \sum_{j,k=1}^{3} E_{jk} \otimes A_{kj}(u)
\end{align*}

where

\begin{align*}
A_{11} = q^{N_2} - wq^{-N_2}, \quad A_{22} = q^{N_1} - wq^{-N_1}, \quad A_{33} = q^{M_3-1} - wq^{1-M_3},
\end{align*}

and

\begin{align*}
A_{12} = wq^{-(N_1+N_2+1)}a_1^+ a_2, \quad A_{21} = -wq^{-N_2}a_2^+ f_3^+, \quad A_{31} = -wq^{-N_2}a_2^+ f_3^+,
\end{align*}

\begin{align*}
A_{32} = wq^{-(N_1+N_2+1)}a_1^+ f_3^+, \quad A_{13} = q^{-1}a_2 f_3, \quad A_{23} = -q^{N_2}a_1 f_3.
\end{align*}

This is definitely the $L$-operator for $U_q(\widehat{gl}(2|1))$ with vector representation in auxiliary space $A$ and oscillator evaluation representation $[13]$ in quantum space $Q$. A few exchange relations...
for $A_{ij}$,

\[ [A_{12}, A_{21}] = u(q^{-1} - q)(q^{N_2-N_1} - q^{N_1-N_2}), \]

\[ [A_{31}, A_{13}]_+ = u(q^{-1} - q)(q^{N_2+1-M_3} - q^{-1}N_2+1+M_3), \]

\[ [A_{32}, A_{23}]_+ = u(q^{-1} - q)(q^{N_1+1-M_3} - q^{-1}N_1+1+M_3), \]

fix the Cartan elements of $\text{gl}(2|1)$

\[ h_1 = n_2 + 1 - M_3, \quad h_2 = n_1 - N_2, \quad h_3 = h_1 + h_2. \]

In its turn, all sixteen matrix elements of operator $R_{A_1,A_2}(w)$

\[ R_{A_1,A_2}(w)|e_{j_1}\rangle \otimes |e_{j_2}\rangle = \sum_{k_1,k_2} |e_{k_1}\rangle \otimes |e_{k_2}\rangle (-)^{p(k_1)p(k_2)} R^{j_1,j_2}_{k_1,k_2}(w) \]

can be calculated explicitly with the help of generating functions \[67\]

\[ \langle n_1+k, n_2 | Tr_{A_1} \left( v^{N_1} a_3^k R_{A_1,A_2} \right) | n_1, n_2+k \rangle = v^{n_2} \sqrt{\frac{(q^2;q^2)_{n_1+k,n_2+k}(q^{n_1-n_2+2}v^{-1};q^2)_{n_2}}{(q^2;q^2)_{n_1,n_2}(q^{n_1-n_2}v;q^2)_{n_2+k+1}}} \]

\[ \langle n_1, n_2+k | Tr_{A_1} \left( v^{N_1} a_3^{+k} R_{A_1,A_2} \right) | n_1+k, n_2 \rangle = v^{n_2+k} \sqrt{\frac{(q^2;q^2)_{n_1+k,n_2+k}(q^{n_1-n_2+2}v^{-1};q^2)_{n_2}}{(q^2;q^2)_{n_1,n_2}(q^{n_1-n_2}v;q^2)_{n_2+k+1}}} \]

Parameters $n_1 = -\delta A_1$ and $n_2 = -\delta A_2$ are the additional spectral parameters for $R_{A_1,A_2}$ with no difference property. Matrix \[139\] coincides with that from \[14, 15\].

8. Conclusion

In the algebraic approach, a two-dimensional quantum integrable model is defined by a quantum group and by its evaluation representation. Contrary to the two-dimensional case, such choice for three-dimensional models is rather limited, instead of a rich representation theory one has locally just the choice of statistics: Bose or Fermi. Note however, three-dimensional $R$-matrices intertwine even number of fermions, there is no three-fermions intertwiners \[16, 17\] in our scheme. The transfer matrix in $3d$ is the layer-to-layer transfer matrix, it has not a structure of a one-dimensional quantum chain but the structure of a two-dimensional quantum lattice. It is shown in this paper, the simple square quantum lattice reproduces a collection of effective two-dimensional models for at least quantum super-algebras of $\widehat{\text{gl}}$ type and certain set of their representations. More complicated quantum lattices \[18\] produces more complicated evaluation representations of quantum groups, for instance the lattice Bose and Fermi gases, their multi-component generalizations, etc., as well as $q$-deformed Toda chain.
and quantum Liouville theory [20]. Moreover, there are specific quantum lattices with twisted boundary that have no quantum group interpretation at all [21].

Classification theorems in this paper are essentially based on the ultra-locality test. A three-dimensional analogue of fusion gives a simple example when this test breaks down: the matrix

\begin{equation}
X_{\alpha\beta} = X_{\alpha_1,\beta_1} [A_{11}] X_{\alpha_1,\beta_2} [A_{12}] X_{\alpha_2,\beta_1} [A_{21}] X_{\alpha_2,\beta_2} [A_{22}]
\end{equation}

is the block matrix, its elements \(a, b, c, d\) are two-by-two blocks with matrix elements from \(A_{11} \otimes A_{12} \otimes A_{21} \otimes A_{22}\). The corresponding intertwiner of equation (61) is a product of eight elementary intertwiners. Ultra-locality test does not work for block matrices \(a_j, b_j, c_j, d_j\) and thus the classification scheme is essentially enlarged. A classification method for the case when ultra-locality test is not applicable or when involved algebras are not ultra-local is an open problem.

It worth noting here, the “edge-type” linear problem of Fig. 1 considered here is not only possible one; there is a distinct quantum “face-type” linear problem providing the local Weyl algebra of observables [22, 18]; a classification scheme for mixed quantum auxiliary linear problems is not known either.

An inclusion of \(B, C, D\) series into the tetrahedral scheme is not yet known. However, there is no doubt that this is possible. For instance, the spinor representations of rotation groups have dimension \(2^n\) what is an evident criterion of the hidden third dimension.

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References

[1] Bazhanov, V. V. and Sergeev, S. M. Zamolodchikov’s tetrahedron equation and hidden structure of quantum groups. J. Phys. A 39 (2006) 3295–3310.
[2] Zaharov, V. E. and Manakov, S. V. Generalization of the method of the inverse scattering problem. Teoret. Mat. Fiz. 27 (1976) 283–287.
[3] Sergeev, S. Quantization of three-wave equations. J. Phys. A: Math. Theor. 40 (2007) 12709–12724.
[4] Bazhanov, V. V., Mangazeev, V. V., and Sergeev, S. M. Quantum geometry of 3-dimensional lattices. arXiv:0801.0129, 2008.
[5] Bogdanov, L. V. and Konopelchenko, B. G. Lattice and \(q\)-difference Darboux-Zakharov-Manakov systems via \(\delta\)-dressing method. J. Phys. A 28 (1995) L173–L178.
[6] Doliwa, A. and Santini, P. M. Multidimensional quadrilateral lattices are integrable. Phys. Lett. A 233 (1997) 365–372.
[7] Korepanov, I. G. Algebraic integrable dynamical systems, 2 + 1 dimensional models on wholly discrete space-time, and inhomogeneous models on 2-dimensional statistical physics. arXiv:solv-int/9506003, 1995.

[8] Kashaev, R. M., Korepanov, I. G., and Sergeev, S. M. The functional tetrahedron equation. Teoret. Mat. Fiz. 117 (1998) 370–384.

[9] Sergeev, S. Quantum curve in q-oscillator model. Int. J. Math. Math. Sci. (2006) Art. ID 92064, 31.

[10] Gasper, G. and Rahman, M. Basic Hyper-geometric Series. Cambridge University Press, Cambridge, 1990.

[11] Umeno, Y., Shiroishi, M., and Wadati, M. Fermionic R-operator for the fermion chain model. J. Phys. Soc. Japan 67 (1998) 1930–1935.

[12] Kac, V. Representations of classical Lie superalgebras. In Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), volume 676 of Lecture Notes in Math., pages 597–626. Springer, Berlin, 1978.

[13] Chaichian, M. and Kulish, P. Quantum Lie superalgebras and q-oscillators. Phys. Lett. B 234 (1990) 72–80.

[14] Bracken, A. J., Gould, M. D., Zhang, Y. Z., and Delius, G. W. Solutions of the quantum Yang-Baxter equation with extra non-additive parameters. J. Phys. A 27 (1994) 6551–6561.

[15] Delius, G. W., Gould, M. D., Links, J. R., and Zhang, Y.-Z. Solutions of the Yang-Baxter equation with extra non-additive parameters. II. $U_q(\mathfrak{gl}(m|n))$. J. Phys. A 28 (1995) 6203–6210.

[16] Bazhanov, V. V. and Stroganov, Y. G. On commutativity conditions for transfer matrices on multidimensional lattice. Theor. Math. Phys. 52 (1982) 685–691.

[17] Ambjorn, J., Khachatryan, S., and Sedrakyan, A. Simplified tetrahedron equations: Fermionic realization. Nucl. Phys. B734 (2006) 287–303.

[18] Sergeev, S. Quantum integrable models in discrete 2+1 dimensional space-time: auxiliary linear problem on a lattice, zero curvature representation, isospectral deformation of the Zamolodchikov-Bazhanov-Baxter model. Particles and Nuclei 35 (2004) 1051–1111.

[19] Kharchev, S., Lebedev, D., and Semenov-Tian-Shansky, M. Unitary representations of $U_q(\mathfrak{sl}(2, R))$, the modular double, and the multiparticle q-deformed Toda chains. Commun. Math. Phys. 225 (2002) 573–609.

[20] Faddeev, L. D., Kashaev, R. M., and Volkov, A. Y. Strongly coupled quantum discrete Liouville theory. I. Algebraic approach and duality. Commun. Math. Phys. 219 (2001) 199–219.

[21] Sergeev, S. Ansatz of Hans Bethe for a two-dimensional lattice Bose gas. J. Phys. A 39 (2006) 3035–3045.

[22] Sergeev, S. M. Quantum 2 + 1 evolution model. J. Phys. A: Math. Gen. 32 (1999) 5693–5714.