Abstract

This is the second of a series of papers in which we develop a “discretization approach” for the rigorous realization of the non-Abelian Chern-Simons path integral for manifolds $M$ of the form $M = \Sigma \times S^1$ and arbitrary simply-connected compact structure groups $G$. More precisely, we introduce, for general links $L$ in $M$, a rigorous version $\text{WLO}_{\text{rig}}(L)$ of (the expectation values of) the corresponding Wilson loop observable $\text{WLO}(L)$ in the so-called “torus gauge” by Blau and Thompson (Nucl. Phys. B408(1):345–390, 1993). For a simple class of links $L$ we then evaluate $\text{WLO}_{\text{rig}}(L)$ explicitly in a non-perturbative way, finding agreement with Turaev’s shadow invariant $|L|$.

1 Introduction

Recall from [29] that our goal is to find, for manifolds $M$ of the form $M = \Sigma \times S^1$, a rigorous realization of the non-Abelian Chern-Simons path integral in the torus gauge. We want to achieve this with the help of a suitable “discretization approach”. In [29] we introduced such an approach but we omitted the proof of the main result, i.e. Theorem 6.4 in [29]. In the present paper we will finally prove Theorem 6.4 of [29] (= Theorem 3.4 in Sec. 3.9 below).

The paper is organized as follows: In Sec. 2 we will recall some of the notation from [29] and we will restate the basic heuristic formula from [29], cf. Eq. (2.7) below. In Sec. 3 we recall the discretization approach of [29] and restate the main result, Theorem 3.4. In Sec. 4 we study “oscillatory Gauss-type measures” on Euclidean spaces and discuss some of their properties. In Sec. 5 we prove Theorem 3.4. In Sec. 6 we make some remarks regarding the case of general (simplicial ribbon) links. In Sec. 7 we then give an outlook on some promising further directions within the framework of $BF_3$-theory before we conclude the main part of this paper with a short discussion of our results in Sec. 8.

The present paper has an appendix consisting of five parts: part A contains a list of the Lie theoretic notation which will be relevant in the present paper (this list is a continuation of the one in Appendix A in [29]). In part B we recall the definition of Turaev’s shadow invariant $|L|$ for links $L$ in 3-manifolds $M$ of the type $M = \Sigma \times S^1$. In part C we recall the definition of $BF$-theory in 3 dimensions and we briefly comment on the relationship between $BF_3$-theory and CS theory.

In part D we sketch a reformulation/modification of the discretization approach in Sec. 7.4. Finally, in part E we make some general remarks on the so-called “simplicial program” for CS/$BF_3$-theory.
Comment 1 Let us briefly comment on the changes that have been made in the present version of this paper in comparison with the last version (v2) of September 2013:

Some straightforward changes have been made in Sec. 1 and Sec. 2. What is now Sec. 3 contains some of the material from the “old” Sec. 3 and the “old” Sec. 4 but also some of the material from the current version of [29]. The “new” Sec. 4 contains the material that originally appeared in Sec. 3 of version (v3) of [29]. Many important changes have been made in Sec. 5, Sec. 7, and Sec. 8.

We made some minor changes in Appendix B, in Appendix C, and in the “new” Appendix E (= “old” Appendix F). Several important changes have been made in the “new” Appendix D (= “old” Appendix E). Since the content of the “old” Appendix D now appears in Sec. 4 in [29] it has been eliminated from the present version of this paper.

2 The basic heuristic formula in [29]

2.1 Basic spaces

As in [29] we fix a simply-connected compact Lie group \( G \) and a maximal torus \( T \) of \( G \). By \( g \) and \( t \) we will denote the Lie algebras of \( G \) and \( T \) and by \( \langle \cdot, \cdot \rangle_g \) or simply by \( \langle \cdot, \cdot \rangle \) the unique \( \text{Ad} \)-invariant scalar product on \( g \) satisfying the normalization condition \( \langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2 \) for every short coroot \( \tilde{\alpha} \) w.r.t. \((g,t)\), cf. part A of the Appendix. For later use let us also fix a Weyl chamber \( C \subset t \).

Moreover, we will fix a compact oriented 3-manifold \( M \) of the form \( M = \Sigma \times S^1 \) where \( \Sigma \) is a (compact oriented) surface, and an ordered oriented link \( L = (l_1, \ldots, l_m) \), \( m \in \mathbb{N} \), in \( M = \Sigma \times S^1 \). Each \( l_i \) is “colored” with a finite-dimensional representation \( \rho_i \) of \( G \).

As in [29] we will use the following notation

\[
\begin{align*}
\mathcal{B} &= C^\infty(\Sigma, t) = \Omega^0(\Sigma, t) \quad (2.1a) \\
\mathcal{A} &= \Omega^1(M, g) \quad (2.1b) \\
\mathcal{A}_\Sigma &= \Omega^1(\Sigma, g) \quad (2.1c) \\
\mathcal{A}_{\Sigma,t} &= \Omega^1(\Sigma, t) \quad (2.1d) \\
\mathcal{A}^\perp &= \{ A \in \mathcal{A} | A(\partial/\partial t) = 0 \} \quad (2.1e) \\
\int \mathcal{A}^\perp(t)dt &\subseteq \mathcal{A}_{\Sigma,t} \quad (2.1f) \\
\mathcal{A}^\perp_c &= \{ A^\perp \in \mathcal{A}^\perp | A^\perp \text{ is constant and } \mathcal{A}_{\Sigma,t}\text{-valued} \} \quad (2.1g)
\end{align*}
\]

Here \( \mathfrak{t} \) is the orthogonal complement of \( t \) in \( g \) w.r.t. \( \langle \cdot, \cdot \rangle \), \( dt \) is the normalized (translation-invariant) volume form on \( S^1 \), \( \partial/\partial t \) is the vector field on \( M = \Sigma \times S^1 \) obtained by “lifting” the standard vector field \( \partial/\partial t \) on \( S^1 \) and in Eqs. \((2.1c)\) and \((2.1g)\) we used the “obvious” identification (cf. Sec. 2.3.1 in [29])

\[
\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma) \quad (2.2)
\]

where \( C^\infty(S^1, \mathcal{A}_\Sigma) \) is the space of maps \( f : S^1 \to \mathcal{A}_\Sigma \) which are “smooth” in the sense that \( \Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_\sigma) \in g \) is smooth for every smooth vector field \( X \) on \( \Sigma \). It follows from the definitions above that

\[
\mathcal{A}^\perp = \mathcal{A}^\perp_c \oplus \mathcal{A}^\perp_c \quad (2.3)
\]

\[\text{recall that } \Omega^p(N,V) \text{ denotes the space of } V\text{-valued } p\text{-forms on a smooth manifold } N\]

\[\text{or, equivalently, the normalized Haar measure}\]
2.2 The heuristic Wilson loop observables

Recall that in the special case when $G$ is simple, the Chern-Simons action function $S_{CS} : \mathcal{A} \to \mathbb{R}$ associated to $M, G$, and the “level” $k \in \mathbb{Z} \setminus \{0\}$ is given by

$$S_{CS}(A) = - k \pi \int_M \langle A \wedge dA \rangle + \frac{1}{2} \langle A \wedge [A \wedge A] \rangle, \quad A \in \mathcal{A}$$  \hspace{1cm} (2.4)

where $[\cdot \wedge \cdot]$ denotes the wedge product associated to the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and where $\langle \cdot, \cdot \rangle$ denotes the wedge product associated to the scalar product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Recall also that the heuristic Wilson loop observable $WLO(L)$ of a link $L = (l_1, l_2, \ldots, l_m)$ in $M$ with “colors” $(\rho_1, \rho_2, \ldots, \rho_m)$ is given by the informal expression

$$WLO(L) := \int_{A} \prod_{l} \text{Tr}_{\rho_l} (\text{Hol}_l(A)) \exp(iS_{CS}(A)) dA$$  \hspace{1cm} (2.5)

where $\text{Hol}_l(A)$ is the holonomy of $A \in \mathcal{A}$ around the loop $l \in \{l_1, \ldots, l_m\}$. The following explicit formula for $\text{Hol}_l(A)$ proved to be useful in [29]:

$$\text{Hol}_l(A) = \lim_{n \to \infty} \prod_{k=1}^{n} \exp \left( \frac{1}{n} A(l'(t)) \right) |_{t=k/n}$$  \hspace{1cm} (2.6)

where $\exp : \mathfrak{g} \to G$ is the exponential function of $G$.

**Remark 2.1** One can assume without loss of generality that $G$ is a closed subgroup of $U(N)$ for some $N \in \mathbb{N}$. In the special case where $G$ is simple we can then rewrite Eq. (2.1) as

$$S_{CS}(A) = k \pi \int_{M} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

with $\text{Tr} := c \cdot \text{Tr}_{\text{Mat}(\mathbb{N}, \mathbb{C})}$ where $c \in \mathbb{R}$ is chosen such that $\langle A, B \rangle = - \text{Tr}(A \cdot B)$ for all $A, B \in \mathfrak{g} \subset u(N) \subset \text{Mat}(\mathbb{N}, \mathbb{C})$. Clearly, making this assumption is a bit inelegant but it has some practical advantages, which is why we made use of it in [29]. In the present paper we will use this assumption only at a later stage, namely in part C of the Appendix below (with $G$ replaced by $\hat{G}$).

**Remark 2.2** Recall from Remark 2.2 in [29] that if $G$ is a general simply-connected compact Lie group then $G$ will be of the form $G = \prod_{i=1}^{r} G_i$, $r \in \mathbb{N}$, where each $G_i$ is a simple simply-connected compact Lie group. We can generalize the definition of $S_{CS}$ to this general situation by setting – for any fixed sequence $(k_i)_{i \leq r}$ of non-zero integers –

$$S_{CS}(A) := \sum_{i=1}^{r} S_{CS,i}(A_i) \quad \forall A \in \mathcal{A}$$

where $S_{CS,i}$ is the Chern-Simons action function associated to $M, G_i$, and $k_i$ and where $(A_i)_i$ are the components of $A$ w.r.t. to the decomposition $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ ($\mathfrak{g}_i$ being the Lie algebra of $G_i$).

In the present paper only two special cases will play a role, namely the case $r = 1$ (i.e. $G$ simple) and the case $r = 2$, $G_2 = G_1$ and $k_2 = -k_1$, cf. Sec. 7 below.

2.3 The basic heuristic formula

The starting point for the main part of [29] was a second heuristic formula for $WLO(L)$ which one obtains from Eq. (2.5) above by applying “torus gauge fixing”, cf. Sec. 2.2.4 in [29].
Let $\sigma_0 \in \Sigma$ be fixed. Then we have (cf. Eq. (2.53) in $[29]$)

$$
WLO(L) \sim \sum_{y \in I} \int_{A^+ \times B} \left\{ 1_{C^\infty(\Sigma_{\text{reg}})}(B) \right\} \text{Det}_{FP}(B) \times \left[ \int_{A^+} \prod_i \text{Tr}_i \left( \text{Hol}_i(\tilde{A}^+ + A^+_c, B) \right) \exp(i S_{CS}(\tilde{A}^+, B)) D\tilde{A}^+ \right]
$$

$$
\times \exp\left( -2\pi i k \langle y, B(\sigma_0) \rangle \right) \exp(i S_{CS}(A^+_c, B))(D A^+_c \otimes DB) \quad (2.7)
$$

where $I := \text{ker}(\exp_{|_{\mathfrak{t}}}) \subset \mathfrak{t}$ and where for each $B \in \mathcal{B}$, $A^+ \in A^+$ we have set

$$
S_{CS}(A^+, B) := S_{CS}(A^+ + B \mathcal{D}) \quad (2.8)
$$

$$
\text{Hol}_i(A^+, B) := \text{Hol}_i(A^+ + B \mathcal{D}) \quad (2.9)
$$

Here $\mathcal{D}$ is the real-valued 1-form on $M = \Sigma \times S^1$ obtained by pulling back the 1-form $\mathcal{D}$ on $S^1$ in the obvious way. Moreover, $\text{Det}_{FP}(B)$ is the informal expression given by

$$
\text{Det}_{FP}(B) := \text{det}(1_{\mathfrak{t}} - \exp(ad(B))_{|_{\mathfrak{t}}}) \quad (2.10)
$$

For the rest of this paper we will now fix an auxiliary Riemannian metric $g$ on $\Sigma$. After doing so we obtain a scalar product $\ll \cdot, \cdot \gg \,_{A^+}$ on $A^+ \cong C^\infty(S^1, A^+_\Sigma)$ in a natural way. Moreover, we then have a well-defined Hodge star operator $\star : A^+_\Sigma \to A^+_\Sigma$ which induces an operator $\star : C^\infty(S^1, A^+_\Sigma) \to C^\infty(S^1, A^+_\Sigma)$ in the obvious way. According to Eq. (2.48) in $[29]$ we then have the following explicit formula

$$
S_{CS}(A^+, B) = \pi k \ll A^+, \star \left( \frac{\partial}{\partial t} + \text{ad}(B) \right) A^+ + 2\pi k \ll \star A^+, dB \gg A^+ \quad (2.11)
$$

for all $B \in \mathcal{B}$ and $A^+ \in A^+$, which implies

$$
S_{CS}(\tilde{A}^+, B) = \pi k \ll \tilde{A}^+, \star \left( \frac{\partial}{\partial t} + \text{ad}(B) \right) \tilde{A}^+ + 2\pi k \ll \star \tilde{A}^+, dB \gg \tilde{A}^+ \quad (2.12)
$$

$$
S_{CS}(A^+_c, B) = 2\pi k \ll \star A^+_c, dB \gg A^+_c \quad (2.13)
$$

for $B \in \mathcal{B}$, $\tilde{A}^+ \in \tilde{A}^+$, and $A^+_c \in A^+_c$.

**Rem. 2.3** In view of Sec. 3.1 below we recall that – according to Remark 2.8 in $[29]$ – we can replace in Eq. (2.7) above the space $B$ appearing in the outer integral $\int_{A^+ \times B} \cdots (DA^+_c \otimes DB)$ by the space

$$
\mathcal{B}^{\text{loc}}_{\sigma_0} := \{ B \in \mathcal{B} | B \text{ is locally constant around } \sigma_0 \}
$$

In fact, we could replace $B$ even by the much smaller space

$$
\mathcal{B}_L := \{ B \in \mathcal{B} | B \text{ is constant on each connected component of } \Sigma \setminus \bigcup_j \text{arc}(l^j_{\Sigma}) \}
$$

with $l^j_{\Sigma} := \pi_\Sigma \circ l_j$ where $\pi_\Sigma : \Sigma \times S^1 \to \Sigma$ is the canonical projection.

### 3 Simplicial realization of WLO($L$)

In the present section we briefly recall the definition of the rigorous simplicial analogue $\text{WLO}_{\text{rig}}(L)$ for the RHS of Eq. (2.7) above which we gave in $[29]$ and we recall the main result of $[29]$, namely Theorem 6.4 (= Theorem 3.4 below).

Anyway, the reader will probably find it useful to have a look at Sec. 4 in $[29]$ where we explain in much more detail the motivation of our constructions.
3.0 Review of the simplicial setup in Sec. 4 in [29]

Recall from Sec. 4.1 in [29] that for a finite oriented polyhedral cell complex $\mathcal{P}$ we denote by $\mathfrak{F}_p(\mathcal{P})$, $p \in \mathbb{N}_0$, the set of $p$-faces of $\mathcal{P}$, and – for every fixed real vector space $V$ – we denoted by $C^0(\mathcal{P}, V)$ the space of maps $\mathfrak{F}_p(\mathcal{P}) \to V$ ("$V$-valued p-cochains of $\mathcal{P}$"). The elements of $\mathfrak{F}_0(\mathcal{P})$ (resp. $\mathfrak{F}_1(\mathcal{P})$) will be called the “vertices" (resp. “edges") of $\mathcal{P}$. Instead of $C^0(\mathcal{P}, \mathbb{R})$ we will often write $C_0(\mathcal{P})$. By $d\mathcal{P}$ we will denote the usual coboundary operator $C^0(\mathcal{P}, V) \to C^{b+1}(\mathcal{P}, V)$.

In [29] we actually only considered the special situation $\mathcal{P} \in \{\mathbb{Z}_N, \mathcal{K}, \mathcal{K}', q\mathcal{K}, \mathcal{K} \times \mathbb{Z}_N, \mathcal{K}' \times \mathbb{Z}_N, q\mathcal{K} \times \mathbb{Z}_N\}$ where $\mathbb{Z}_N$, $\mathcal{K}$, $\mathcal{K}'$, and $q\mathcal{K}$ are given as follows:

Recall that in Sec. 4.4 in [29] we fixed $N \in \mathbb{N}$ and used the finite cyclic group $\mathbb{Z}_N$ with the “obvious" (oriented) graph structure as a discrete analogue of the Lie group $S^1$.

Moreover, we fixed a finite oriented smooth polyhedral cell decomposition $\mathcal{C}$ of $\Sigma$. By $\mathcal{C}$ we denoted the dual polyhedral cell decomposition, equipped with an orientation. By $\mathcal{K}$ and $\mathcal{K}'$ we denoted the corresponding (oriented) polyhedral cell complexes, i.e. $\mathcal{K} := (\Sigma, \mathcal{C})$ and $\mathcal{K}' := (\Sigma, \mathcal{C}')$.

Instead of $\mathcal{K}$ (resp. $\mathcal{K}'$) we usually wrote $K_1$ (resp. $K_2$) and we set $K := (K_1, K_2)$.

We then introduced a joint sub division $q\mathcal{K} := (\Sigma, q\mathcal{C})$ of $\mathcal{K} = K_1$ and $\mathcal{K}' = K_2$ which can be characterized by the conditions

$$\mathfrak{F}_0(q\mathcal{K}) = \mathfrak{F}_0(b\mathcal{K}), \quad \mathfrak{F}_1(q\mathcal{K}) = \mathfrak{F}_1(b\mathcal{K}) \setminus \{e \in \mathfrak{F}_1(b\mathcal{K}) \mid \text{both endpoints of } e \text{ lie in } \mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2)\}$$

where $b\mathcal{K}$ is the barycentric sub division of $\mathcal{K}$. The set $\mathfrak{F}_2(q\mathcal{K})$ is uniquely determined by $\mathfrak{F}_0(q\mathcal{K})$ and $\mathfrak{F}_1(q\mathcal{K})$. Observe that each $F \in \mathfrak{F}_2(q\mathcal{K})$ is a tetragon. We introduced the notation

$$\mathfrak{F}_0(K_1|K_2) := \mathfrak{F}_0(q\mathcal{K}) \setminus (\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2)) \quad (3.1)$$

Recall that also the faces of $q\mathcal{K}$ were equipped with an orientation. For convenience we chose the orientation on the edges of $q\mathcal{K}$ to be “compatible" with the orientation on the edges of $K_1$ and $K_2$.

Recall from Sec. 4.2 in [29] that a “simplicial curve" in $\mathcal{P}$ is a finite sequence $x = (x(k))_{k \leq n}$, $n \in \mathbb{N}$, of vertices in $\mathcal{P}$ such that for every $k \leq n$ the two vertices $x(k)$ and $x(k+1)$ either coincide or are the two endpoints of an edge $e(k) \in \mathfrak{F}_1(\mathcal{P})$. We call $n$ the “length" of the simplicial curve $x$. If $x(n) = x(1)$ we will call $x = (x(k))_{k \leq n}$ a “simplicial loop" in $\mathcal{P}$.

Recall also that every simplicial curve $x = (x(k))_{k \leq n}$ with $n > 1$ induces a sequence $(e(k))_{k \leq n-1}$ of “generalized edges", i.e. elements of $\mathfrak{F}_1(\mathcal{P}) \cup \{0\} \cup (-\mathfrak{F}_1(\mathcal{P})) \subset C_1(\mathcal{P})$ in a canonical way. Unless $x = (x(k))_{k \leq n}$ is constant, we can reconstruct $x = (x(k))_{k \leq n}$ from $(e(k))_{k \leq n-1}$.

We write $\bullet e(k)$ instead of $x(k)$ (for $k \leq n-1$).

Recall from Sec. 4.3 in [29] that a “(closed) simplicial ribbon" in $\mathcal{P}$ is a finite sequence $R = (F_i)_{i \leq n}$ of 2-faces of $\mathcal{P}$ such that every $F_i$ is a tetragon and such that $F_i \cap F_j = \emptyset$ unless $i = j$ or $j = i \pm 1$ (mod $n$). In the latter case $F_i$ and $F_j$ intersect in a (full) edge.

**Convention 1** Let $V$ be a fixed finite-dimensional real vector space.

i) We set $C_1(\mathcal{K}) := C_1(K_1) \oplus C_1(K_2)$ and $C^1(\mathcal{K}, V) := C^1(K_1, V) \oplus C^1(K_2, V)$.

ii) Let $\psi : C_1(\mathcal{K}) \to C_1(q\mathcal{K})$ be the (injective) linear map given by

$$\psi(e) = e_1 + e_2 \quad \text{for all} \quad e \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)$$

---

4i.e. the set of edges is given by $\{(t, t+1) \mid t \in \mathbb{Z}_N\}$

5more precisely, for each $e \in \mathfrak{F}_1(q\mathcal{K})$ we choose the orientation which is induced by orientation of the unique edge $e' \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)$ which contains $e$

6we will often omit the word “closed"
where \( e_1 = e_1(\epsilon), e_2 = e_2(\epsilon) \in \mathfrak{g}_1(qK) \) are the two edges of \( qK \) "contained" in \( \epsilon \). In the following we will identify \( C_1(K) \) with the subspace \( \psi(C_1(K)) \) of \( C_1(qK) \). Moreover, using the identifications \( C^1(qK, V) \cong C_1(qK) \otimes R V \) and \( C^1(K, V) \cong C_1(K) \otimes R V \) we naturally obtain the linear map

\[
\psi^V := \psi \otimes \text{id}_V : C^1(K, V) \to C^1(qK, V)
\]

We will identify \( C^1(K, V) \) with the subspace \( \psi^V(C^1(K, V)) \) of \( C^1(qK, V) \).

3.1 The basic spaces

As in Sec. 5.1 in [29] we introduce the following discrete analogues of the spaces \( B, A^\Sigma \) and \( A^\perp \) in Sec. 2.1 above:

\[
B(qK) := C^0(qK, t) \quad (3.2a)
\]

\[
A^\Sigma(qK) := C^1(qK, g) \quad (3.2b)
\]

\[
A^\perp(qK) := \text{Map}(Z_N, A^\Sigma(qK)) \quad (3.2c)
\]

Clearly, the scalar product \( \langle \cdot, \cdot \rangle_g \) on \( g \) induces scalar products \( \ll \cdot, \cdot \gg_B(qK) \) and \( \ll \cdot, \cdot \gg_{A^\Sigma(qK)} \) on \( B(qK) \) and \( A^\Sigma(qK) \) in the standard way. We introduce a scalar product \( \ll \cdot, \cdot \gg_{A^\perp(qK)} \) on \( A^\perp(qK) = \text{Map}(Z_N, A^\Sigma(qK)) \) by

\[
\ll A_1^\perp, A_2^\perp \gg_{A^\perp(qK)} := \frac{1}{N} \sum_{t \in Z_N} \ll A_1^\perp(t), A_2^\perp(t) \gg_{A^\Sigma(qK)} \quad (3.3)
\]

for all \( A_1^\perp, A_2^\perp \in A^\perp(qK) \).

**Convention 2** We identify \( A^\Sigma(qK) \) with the subspace \( \{ A^\perp \in \text{Map}(Z_N, A^\Sigma(qK)) \mid A^\perp \text{ is constant} \} \) of \( A^\perp(qK) \) in the obvious way.

For technical reasons\(^7\) we will not only work with the full spaces \( A^\Sigma(qK) \) and \( A^\perp(qK) \) but also their subspaces (cf. Convention \(^8\) above) \( A^\Sigma(K) \) and \( A^\perp(K) \) given by

\[
A^\Sigma(K) := C^1(K_1, g) \oplus C^1(K_2, g) \subset A^\Sigma(qK) \quad (3.4)
\]

\[
A^\perp(K) := \text{Map}(Z_N, A^\Sigma(K)) \subset A^\perp(qK) \quad (3.5)
\]

**The decomposition** \( A^\perp(K) = A^\perp(K) \oplus A^\perp_c(K) \)

In order to obtain a discrete analogue of the decomposition \( A^\perp = A^\perp \oplus A^\perp_c \) in Eq. (2.3) above let us introduce the following spaces:

\[
A^\Sigma_{\perp}(K) := C^1(K_1, t) \oplus C^1(K_2, t) \quad (3.6a)
\]

\[
A^\Sigma_{\perp}(t) := C^1(K_1, t) \oplus C^1(K_2, t) \quad (3.6b)
\]

\[
\tilde{A}^\perp(K) := \{ A^\perp \in A^\perp(K) \mid \sum_{t \in Z_N} A^\perp(t) \in A^\Sigma_{\perp}(K) \} \quad (3.6c)
\]

\[
A^\perp_c(K) := \{ A^\perp \in A^\perp(K) \mid A^\perp(\cdot) \text{ is constant and } A^\Sigma_{\perp}(K)-\text{valued} \} \cong A^\Sigma_{\perp}(K) \quad (3.6d)
\]

Observe that we have

\[
A^\perp(K) = \tilde{A}^\perp(K) \oplus A^\perp_c(K) \quad (3.7)
\]

which is indeed a discrete analogue of the decomposition \( A^\perp = \tilde{A}^\perp \oplus A^\perp_c \) in Eq. (2.3) above.

\(^7\)namely, in order to obtain a nice simplicial analogue of the Hodge star operator, cf. Sec. 3.3 below
The space $\mathcal{B}_0(qK)$

In the following we will replace the space $\mathcal{B}(qK) = C^0(qK, t)$ by a smaller subspace $\mathcal{B}_0(qK)$ (chosen as naturally as possible) such that

$$\ker(\pi \circ (d_qK)|_{\mathcal{B}_0(qK)}) = \mathcal{B}_c(qK)$$

holds where $\pi : C^1(qK, t) \to C^1(K, t)$ is the orthogonal projection w.r.t. $\ll \cdot, \cdot \gg_{A^+(qK)}$ and

$$\mathcal{B}_c(qK) := \{ B \in C^0(qK, t) \mid B \text{ constant} \}$$

(3.8)

We remark that Eq. (3.8) will play a crucial role in the proof of Theorem 3.4. (We also remark that we cannot choose simply $\mathcal{B}_0(qK) = \mathcal{B}(qK)$ because $\ker(\pi \circ d_qK) \neq \mathcal{B}_c(qK)$).

**Choice 1** In view of Remark 2.3 above it is justified to make the choice $\mathcal{B}_0(qK) := \mathcal{B}_L(qK)$ where

$$\mathcal{B}_L(qK) := \{ B \in \mathcal{B}(qK) \mid B \text{ is constant on the closure of each conn. comp. of } \Sigma \setminus \bigcup_j \text{Image}(R^J) \}$$

where $\text{Image}(R^J)$ is given as in Sec. 3.8 below.

Choice 1 is quite drastic, since the space $\mathcal{B}_L(qK)$ is very small. In the main part of the present paper we will work with the following choice:

**Choice 2** In view of Remark 2.3 above it is also justified to choose $\mathcal{B}_0(qK) := \mathcal{B}^{loc}_{\sigma_0}(qK)$ where

$$\mathcal{B}^{loc}_{\sigma_0}(qK) := \{ B \in C^0_{\text{aff}}(qK, t) \mid B \text{ is constant on } U(\sigma_0) \cap \mathcal{F}_0(qK) \}$$

where for fixed $\sigma_0 \in \mathcal{F}_0(qK)$ we have set

$$U(\sigma_0) := \bigcup_{F \in \mathcal{F}(qK), \sigma_0 \in F} F \subset \Sigma$$

(3.10)

(This was the choice we made in [29], cf. (Mod2) and (Mod3) of Sec. 5.10 in [29]). Above we have introduced – for technical reasons – the space

$$C^0_{\text{aff}}(qK, t) := \{ B \in C^0(qK, t) \mid B \text{ is affine on each } F \in \mathcal{F}_2(qK) \}$$

where by “$B$ is affine on $F$” we mean that we have

$$B(p_1) + B(p_2) = B(p_3) + B(p_4)$$

(3.11)

where $p_1, p_2, p_3, p_4$ are the four vertices of $F$ and numbered in such a way that $p_1$ is diagonal to $p_4$ and therefore $p_2$ is diagonal to $p_3$.

Choice 2 is less drastic than Choice 1 but the space $C^0_{\text{aff}}(qK, t)$ is still quite restrictive. If instead of “half ribbons” we decide to work with “full ribbons” (cf. Remark 3.2 below) as we will do in Sec. 7 below (and in [34]) the following choice will be superior:

**Choice 3** $\mathcal{B}_0(qK) := \psi(\mathcal{B}(K))$ where $\mathcal{B}(K) := C^0(K, t)$ and where $\psi : \mathcal{B}(K) \to \mathcal{B}(qK)$ is the linear injection which associates to each $B \in \mathcal{B}(K)$ the extension $\bar{B} \in \mathcal{B}(qK)$ given by

$$\bar{B}(x) = \text{mean}_{y \in C(x)} B(y) \quad \text{for all } x \in \mathcal{F}_0(qK)$$

where “mean” denotes the arithmetic mean and where $C(x)$ is the set of all $y \in \mathcal{F}_0(K)$ which lie in the closure of the unique $qK$-cell containing $x$.

---

8 here “conn. comp.” is an abbreviation of “connected component”
9 More explicitly:

$$\bar{B}(x) = \begin{cases} B(x) & \text{if } x \in \mathcal{F}_0(K_1) \\ \frac{1}{2}[B(y_1) + B(y_2)] & \text{if } x \in \mathcal{F}_0(K_1|K_2) \\ \frac{1}{|v(F)|} \sum_{y \in v(F)} B(y) & \text{if } x \in \mathcal{F}_0(K_2) \end{cases}$$

where in the second line $y_1$ and $y_2$ are the two endpoints of the edge $e \in \mathcal{F}_1(K_1)$ containing $x$ and in the third line $v(F)$ is the set of vertices of the unique 2-face $F \in \mathcal{F}_2(K_1)$ containing $x$. 


In the following\注10\ we will work with Choice 2.

### 3.2 Discrete analogue of the operator \( \frac{\partial}{\partial t} + \text{ad}(B) : \mathcal{A} \to \mathcal{A} \)

Let us recall the definition of the operator \( L(N)(b) : \mathcal{A}(K) \to \mathcal{A}(K) \) which we introduced in \([29]\) as the discrete analogue of the continuum operator \( \frac{\partial}{\partial t} + \text{ad}(B) : \mathcal{A} \to \mathcal{A} \) in Eq. (2.11) above.

Let \( \tau_x \), for \( x \in \mathbb{Z}_N \), denote the translation operator \( \text{Map}(\mathbb{Z}_N, g) \to \text{Map}(\mathbb{Z}_N, g) \) given by \((\tau_x f)(t) = f(t + x)\) for all \( t \in \mathbb{Z}_N \). Instead of \( \tau_0 \) we will simply write 1 in the following.

In \([29]\) we introduced, for fixed \( b \in t \), the following natural discrete analogues \( L(N)(b) : \text{Map}(\mathbb{Z}_N, g) \to \text{Map}(\mathbb{Z}_N, g) \) of the continuum operator \( L(b) := \frac{\partial}{\partial t} + \text{ad}(b) : C^\infty(S^1, g) \to C^\infty(S^1, g) \):

\[
\begin{align*}
\hat{L}^{(N)}(b) &:= N(1 - e^{-\text{ad}(b)/N}) & \text{if } N \text{ is even} \\
\hat{L}^{(N)}(b) &:= N(1 - \tau_1 e^{-\text{ad}(b)/N}) & \text{if } N \text{ is odd}
\end{align*}
\]

Let \( B \in B(q\mathbb{K}) \). The operator \( L(N)(b) : \mathcal{A}(K) \to \mathcal{A}(K) \) mentioned above is the linear operator which, under the identification

\[
\mathcal{A}(K) \cong \text{Map}(\mathbb{Z}_N, C^1(K_1, g)) \oplus \text{Map}(\mathbb{Z}_N, C^1(K_2, g))
\]

is given by

\[
L(N)(b) = \begin{pmatrix}
\hat{L}^{(N)}(b) & 0 \\
0 & \hat{L}^{(N)}(b)
\end{pmatrix}
\]

Here the linear operators \( \hat{L}^{(N)}(b) : \text{Map}(\mathbb{Z}_N, C^1(K_1, g)) \to \text{Map}(\mathbb{Z}_N, C^1(K_1, g)) \) and \( \hat{L}^{(N)}(b) : \text{Map}(\mathbb{Z}_N, C^1(K_2, g)) \to \text{Map}(\mathbb{Z}_N, C^1(K_2, g)) \) are given by

\[
\hat{L}^{(N)}(b) \cong \oplus_{\epsilon \in \mathfrak{g}_0(K_1|K_2)} \hat{L}^{(N)}(B(\bar{\epsilon}))
\]

where \( \mathfrak{g}_0(K_1|K_2) \) is as in Eq. (3.1) above. In Eqs. (3.14a) and (3.14b) we used the obvious identification

\[
\text{Map}(\mathbb{Z}_N, C^1(K_j, g)) \cong \oplus_{\epsilon \in \mathfrak{g}_0(K_j)} \text{Map}(\mathbb{Z}_N, g) \cong \oplus_{\epsilon \in \mathfrak{g}_0(K_1|K_2)} \text{Map}(\mathbb{Z}_N, g)
\]

We remark that \( L(N)(b) \) leaves the subspace \( \mathcal{A}(K) \) of \( \mathcal{A}(K) \) invariant. The restriction of \( L(N)(b) \) to \( \mathcal{A}(K) \) will also be denoted by \( L(N)(b) \) in the following.

### 3.3 Definition of \( \mathcal{S}^\text{disc}_S(\mathcal{A}, B) \)

Recall that in \([29]\) we introduced discrete Hodge operators \( \ast_{K_1} : C^1(K_1, g) \to C^1(K_2, g) \) and \( \ast_{K_2} : C^1(K_2, g) \to C^1(K_1, g) \), cf. Sec. 4.5 in \([29]\). Moreover, we introduced two different operators denoted by \( \ast_K \) (cf. Sec. 4.5 and Sec. 5.3 in \([29]\)). Firstly, the operator \( \ast_K : \mathcal{A}_\Sigma(K) \to \mathcal{A}_\Sigma(K) = C^1(K_1, g) \oplus C^1(K_2, g) \) given by

\[
\ast_K := \begin{pmatrix}
0 & \ast_{K_2} \\
\ast_{K_1} & 0
\end{pmatrix}
\]

and, secondly, the operator \( \ast_K : \mathcal{A}(K) \to \mathcal{A}(K) \) given by

\[
(\ast_K \mathcal{A})(t) = \ast_K(\mathcal{A}(t)) \quad \forall \mathcal{A} \in \mathcal{A}(K), t \in \mathbb{Z}_N
\]

\[\text{in fact this will be relevant from Sec.}\not\in\text{on. In Sec.}\not\in\text{ and}\not\in\text{ we can work with arbitrary } B \in B(q\mathbb{K})]
As the discrete analogues of the continuum expression $S_{CS}(A^\perp, B)$ in Eq. (2.11) above we use the expression

$$S_{CS}^{\text{disc}}(A^\perp, B) := \pi k \left[ \ll A^\perp, *_K L^{(N)}(B) A^\perp \gg_{A^\perp(qK)} + 2 \ll *_K A^\perp, d_q K B \gg_{A^\perp(qK)} \right]$$  \hspace{1cm} (3.17a)

for $B \in B(qK)$, $A^\perp \in A^\perp(K) \subset A^\perp(qK)$. Observe that this implies

$$S_{CS}^{\text{disc}}(A^\perp, B) = \pi k \ll A^\perp, *_K L^{(N)}(B) A^\perp \gg_{A^\perp(qK)}$$  \hspace{1cm} (3.17b)

$$S_{CS}^{\text{disc}}(A^\perp_c, B) = 2 \pi k \ll *_K A^\perp_c, d_q K B \gg_{A^\perp(qK)}$$  \hspace{1cm} (3.17c)

for $B \in B(qK)$, $A^\perp_c \in \tilde{A}^\perp(K)$, $A^\perp_c \in A^\perp_c(K)$.

Recall that we have (cf. Proposition 5.3 in [29]):

**Proposition 3.1** The operator $*_K L^{(N)}(B) : A^\perp(K) \to A^\perp(K)$ is symmetric w.r.t to the scalar product $\ll \cdot, \cdot \gg_{A^\perp(qK)}$.

### 3.4 Definition of $\text{Hol}_{R}^{\text{disc}}(A^\perp, B)$

Let $A^\perp \in A^\perp(K) \subset A^\perp(qK)$ and $B \in B(qK)$. Moreover, let $K = (K_k)_{k \leq n}, n \in \mathbb{N}$, be a closed simplicial ribbon in $qK \times Z_N$. According to Remark 4.3 in [29] $R$ induces a pair $(l, l')$ of simplicial loops $l = (l^k)_{k \leq n}$ and $l' = (l'^k)_{k \leq n}$ in $qK \times Z_N$ in the obvious way ($l$ and $l'$ are simply the two loops “on the boundary” of $R$). Let $l_S, l'_S, l_{S_1}, l'_{S_1}$ denote the corresponding “projected” simplicial loops in $qK$ and $Z_N$, cf. Sec. 4.4.4 in [29].

The simplicial analogue of the continuum expression $\text{Hol}_l(A^\perp, B)$ we used in [29] was

$$\text{Hol}_{R}^{\text{disc}}(A^\perp, B) := \prod_{k=1}^n \exp \left( \frac{1}{2} (A^\perp(l_{S_1}^k)) (l_{S_1}^k) + \frac{1}{2} (A^\perp(l'^{S_1}_1)) (l'^{S_1}_1) \right)$$

$$+ \frac{1}{2} B(l_{S_1}^k) \cdot dt^{(N)}(l_{S_1}^k) + \frac{1}{2} B(l'^{S_1}_1) \cdot dt^{(N)}(l'^{S_1}_1)$$  \hspace{1cm} (3.18)

where $dt^{(N)} \in C^1(Z_N, \mathbb{R}) \cong \text{Hom}_R(C_1(Z_N), \mathbb{R})$ is given by

$$dt^{(N)}(e) = \frac{1}{N} \quad \forall e \in \mathfrak{g}_1(Z_N)$$  \hspace{1cm} (3.19)

and where we have made the identification $A_{\Sigma}(qK) = C^1(qK, \mathfrak{g}) \cong \text{Hom}(C_1(qK), \mathfrak{g})$.

**Remark 3.2** In view of Remark 3.5 below let us point out that instead of working with simplicial ribbons in $qK \times Z_N$ (“half ribbons” \footnote{observe that every simplicial ribbon $R$ in $K \times Z_N$ can be considered as the union of two simplicial ribbons $R_+$ and $R_-$ in $qK \times Z_N$ in a natural way}) one could also work with simplicial ribbons in $K \times Z_N$ (“full ribbons”). Observe that every closed simplicial ribbon $R$ in $K \times Z_N$ induces three loops $l^+, l^-$, and $l$ in $qK \times Z_N$ in a natural way, $l^+$ and $l^-$ being the two boundary loops and $l$ being the loop “inside” \footnote{More precisely, the loop $l$ is the loop “lying on the intersection” of the two associated half ribbons $R_+$ and $R_-$} $R$. We now define $\text{Hol}_{R}^{\text{disc}}(A^\perp, B)$ by

$$\text{Hol}_{R}^{\text{disc}}(A^\perp, B) := \prod_{k=1}^n \exp \left( \sum_{\pm} \frac{1}{4} (A^\perp(l_{S_1}^{\pm k})) (l_{S_1}^{\pm k}) \right)$$

$$+ \left( \sum_{\pm} \frac{1}{4} B(l_{S_1}^{\pm k}) \cdot dt^{(N)}(l_{S_1}^{\pm k}) \right) + \frac{1}{2} B(l_{S_1}^{\pm k}) \cdot dt^{(N)}(l_{S_1}^{\pm k})$$  \hspace{1cm} (3.20)

where we use the notation $\sum_{\pm} : \cdots$ in the obvious way.\footnote{eg. $\sum_{\pm} \frac{1}{4} A^\perp(l_{S_1}^{\pm k}) (l_{S_1}^{\pm k})$ is a short form of $\frac{1}{4} A^\perp(l_{S_1}^{++}(k)) (l_{S_1}^{++}(k)) + \frac{1}{4} A^\perp(l_{S_1}^{+-}(k)) (l_{S_1}^{+-}(k))$}
3.5 Definition of \( \text{Det}^\text{disc}_{FP}(B) \)

In Sec. 5.10 in [29] we made the following ansatz for the discrete analogue \( \text{Det}^\text{disc}_{FP}(B) \) of the heuristic expression \( \text{Det}_{FP}(B) = \det(1_t - \exp(\text{ad}(B)))_{|t} \) given by Eq. (2.10) above:

\[
\text{Det}^\text{disc}_{FP}(B) := \prod_{x \in \tilde{\mathcal{O}}(q\mathcal{K})} \det(1_t - \exp(\text{ad}(B(x)))_{|t})^{1/2}
\]

(3.21)

for every \( B \in \mathcal{B}(q\mathcal{K}) \).

**Remark 3.3** It will turn out later (cf. Sec. 5.5 below) that the inclusion of the exponent 1/2 on the RHS of Eq. (3.21) is necessary if we want to obtain the correct values for the WLOs. It is not clear whether it is possible to give a good additional justification for the inclusion of this 1/2-exponent at this stage. We remark that after making the transition to the BF-theoretic setting as explained in Sec. 7 the chances of obtaining such an additional justification in the expression analogous to (3.21) (cf. Eq. (7.23) below) seem to be better.

3.6 Discrete version of \( 1_{C^\infty((\Sigma, t_{reg}))(B)} \)

Let us fix a family \((t_{reg}^{(s)})_{s>0}\) of elements of \( C^\infty_{\mathbb{R}}(t) \) with the following properties:

- \( \text{Image}(t_{reg}^{(s)}) \subset [0, 1] \) and \( \text{supp}(t_{reg}^{(s)}) \subset t_{reg} \) for each \( s > 0 \),

- \( t_{reg}^{(s)} \rightarrow t_{reg} \) pointwise as \( s \rightarrow 0 \),

- Each \( t_{reg}^{(s)} \), \( s > 0 \), is invariant under the operation of the affine Weyl group \( \mathcal{W}_{\text{aff}} \) on \( t \).

For fixed \( s > 0 \) and \( B \in \mathcal{B}(q\mathcal{K}) \) we will now take the expression

\[
\prod_{x} t_{reg}^{(s)}(B(x)) := \prod_{x \in \tilde{\mathcal{O}}(q\mathcal{K})} t_{reg}^{(s)}(B(x))
\]

(3.22)

as the discrete analogue of \( 1_{C^\infty((\Sigma, t_{reg}))(B)} \). Later we will let \( s \rightarrow 0 \).

3.7 Discrete versions of the two Gauss-type measures in Eq. (2.7)

**Convention 3** In the following we will always consider \( \mathcal{B}_0(q\mathcal{K}) \), \( \mathcal{A}^\perp(K) \), \( \mathcal{A}^\perp(K) \), and the direct sum \( \mathcal{B}_0(q\mathcal{K}) \oplus \mathcal{A}^\perp(K) \) as Euclidean spaces in the “obvious” way.

i) Let \( D\tilde{\mathcal{A}}^\perp \) denote the (normalized) Lebesgue measure on \( \tilde{\mathcal{A}}^\perp(K) \). According to Eq. (3.17) the complex measure

\[
\exp(iS_{\mathcal{CS}}^\text{disc}(\tilde{\mathcal{A}}^\perp, B))D\tilde{\mathcal{A}}^\perp
\]

(3.23)

is a centered oscillatory Gauss-type measure on \( \tilde{\mathcal{A}}^\perp(K) \) in the sense of Definition 4.1 in Sec. 4 below.

ii) Let \( D\mathcal{A}^\perp_c \) denote the (normalized) Lebesgue measure on \( \mathcal{A}^\perp_c(K) \) and \( DB \) the (normalized) Lebesgue measure on \( \mathcal{B}_0(q\mathcal{K}) \).

According to Eq. (3.17) above, the complex measure

\[
\exp(iS_{\mathcal{CS}}^\text{disc}(\mathcal{A}^\perp_c, B))(D\mathcal{A}^\perp_c \otimes DB)
\]

(3.24)

is a centered oscillatory Gauss-type measure on \( \mathcal{A}^\perp_c(K) \oplus \mathcal{B}_0(q\mathcal{K}) \) in the sense of Definition 4.1 in Sec. 4 below.

---

\(^{14}\)More precisely, we will assume that the space \( \mathcal{B}_0(q\mathcal{K}) \) is equipped with the restriction of the scalar product \( \ll ?, \cdot ? \gg_{\mathcal{B}_0(q\mathcal{K})} \) on \( \mathcal{B}(q\mathcal{K}) \) and the spaces \( \mathcal{A}^\perp(K) \) and \( \mathcal{A}^\perp(K) \) are equipped with the restrictions of the scalar product \( \ll ?, \cdot ? \gg_{\mathcal{A}^\perp(K)} \), introduced in Sec. 3.5 above.
3.8 Definition of $WLO_{rig}^{disc}(L)$ and $WLO_{rig}(L)$

For the rest of this paper we will fix a simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $qK \times Z_N$ with “colors” $(\rho_1, \rho_2, \ldots, \rho_m)$, $m \in N$.

Using the definitions of the previous subsections we then arrive at the following simplicial analogue $WLO_{rig}^{disc}(L)$ of the heuristic expression $WLO(L)$ in Eq. 2.7

\[
WLO_{rig}^{disc}(L) := \lim_{s \to 0} \sum_{y \in I} [\prod_x I_{treg}^{(s)}(B(x))] \text{Det}_{FP}^{disc}(B) \\
\times \left[ \prod_{i=1}^{m} \text{Tr}_{\rho_i} (\text{Hol}_{R_i}^{disc}(A^\perp_i + A^\perp_c, B)) \exp(iS_{CS}^{disc}(A^\perp_i, B)) D A^\perp_i \right] \\
\times \exp(-2\pi i k(y, B(\sigma_0))) \exp(iS_{CS}^{disc}(A^\perp_c, B))(D A^\perp_c \otimes DB) \quad (3.25)
\]

where we use the notation $\int_{\sim} \cdots$ as defined in Definition 4.2 in Sec. 4 below and where $\sigma_0$ is an arbitrary fixed point of $\mathfrak{g}_0(qK)$ which does not lie in $\bigcup_{\leq m} \text{Image}(R_i^\perp)$. Here $R_i^\perp$ is the (“reduced”) $\Sigma$-projection of the closed simplicial ribbon $R_i$ (cf. Sec. 4.4.4 in [29]) and we consider each $R_i^\perp$ as a map $[0, 1] \times S^1 \to \Sigma$, cf. Remark 4.3 in [29].

Finally, we set\footnote{At this stage we do not yet claim that $WLO_{rig}^{disc}(L)$ and $WLO_{rig}(L)$ are actually well-defined} \footnote{So $WLO_{rig}^{disc}(\emptyset)$ is a discrete analogue of the partition function $Z(\Sigma \times S^1)$. Explicitly, $WLO_{rig}^{disc}(\emptyset)$ is given by the expression which we get from the RHS of Eq. (3.25) after omitting the product $\prod_{i=1}^{m} \text{Tr}_{\rho_i} (\text{Hol}_{R_i}^{disc}(A^\perp_i + A^\perp_c, B))$} \footnote{I.e. these maps have pairwise disjoint images and each $R_i^\perp$ considered as a continuous map $[0, 1] \times S^1 \to \Sigma$ is an embedding} \footnote{The situation $0 < k < c_\mathfrak{g}$ is not interesting since in this case the set $\Lambda_i^k$ appearing in Eq. (12.3) of part [12] of the Appendix below is empty, cf. Remark 12.4 below. Accordingly, $|L| = |\emptyset| = 0$. It turns out that we then also have $WLO_{rig}^{disc}(L) = WLO_{rig}^{disc}(\emptyset) = 0$}.

\[
WLO_{rig}(L) := \frac{WLO_{rig}^{disc}(L)}{WLO_{rig}^{disc}(\emptyset)} \quad (3.26)
\]

where $\emptyset$ is the “empty” link.\footnote{Recall that in [29] we stated the following theorem which will be proven in Sec. 5 below:}

3.9 The main result

From now on we will assume that the simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $qK \times Z_N$ fixed in Sec. 3.8 above fulfills the following two conditions:

(NCP)’ The maps $R_i^\perp$ neither intersect each other nor themselves.\footnote{Note that in [29] we stated the following theorem which will be proven in Sec. 5 below:} \footnote{We require that $L$ fulfills conditions (NCP)’ and (NH)’ above. Assume also that $k \geq c_\mathfrak{g}$} \footnote{where $c_\mathfrak{g}$ is the dual Coxeter number of $\mathfrak{g}$. Then $WLO_{rig}(L)$ is well-defined and we have}$

(NH)’ Each of the maps $R_i^\perp$, $i \leq m$ is null-homotopic.

where $R_i^\perp$ is as in Sec. 3.8 above and where we consider each $R_i^\perp$ as a map $[0, 1] \times S^1 \to \Sigma$, cf. Remark 4.3 in [29].

Recall that in [29] we stated the following theorem which will be proven in Sec. 5 below:

**Theorem 3.4** Assume that the (colored) simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $qK \times Z_N$ fixed in Sec. 3.8 above fulfills conditions (NCP)’ and (NH)’ above. Assume also that $k \geq c_\mathfrak{g}$ where $c_\mathfrak{g}$ is the dual Coxeter number of $\mathfrak{g}$. Then $WLO_{rig}(L)$ is well-defined and we have

\[
WLO_{rig}(L) = \frac{|L|}{|\emptyset|} \quad (3.27)
\]

where $\emptyset$ is the “empty link” and where $|\cdot|$ is the shadow invariant associated to $\mathfrak{g}$ and $k$, cf. part [12] of the Appendix.
Remark 3.5 As mentioned in Remark 3.2 above, instead of working with simplicial ribbons in $qK \times \mathbb{Z}_N$ ("half ribbons") one could try to work with simplicial ribbons in $K \times \mathbb{Z}_N$ ("full ribbons"). This would have several important advantages, cf. Example 3 above and Remark 5.4 in Sec. 5.5 below. On the other hand the use of “full ribbons” instead of “half ribbons” would also have an important disadvantage, cf. again Remark 5.4. This is why in Theorem 3.4 we only consider the case of half ribbons. We will come back to the case of full ribbons in Sec. 7 below and in [30].

4 Oscillatory Gauss-type measures on Euclidean spaces

In the present section we will recall the definitions introduced in Sec. 3 in [29] and then derive several elementary results which will play an important role in the proof of Theorem 3.4.

4.1 Basic Definitions

Let us fix a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ and set $d := \dim(V)$.

Definition 4.1 An “oscillatory Gauss-type measure” on $(V, \langle \cdot, \cdot \rangle)$ is a complex Borel measure $d\mu$ on $V$ of the form

$$d\mu(x) = \frac{1}{Z} e^{-\frac{i}{2} \langle x - m, S(x - m) \rangle} dx$$

with $Z \in \mathbb{C}\{0\}$, $m \in V$, and where $S$ is a symmetric endomorphism of $V$ and $dx$ the normalized Lebesgue measure on $V$. Note that $Z$, $m$ and $S$ are uniquely determined by $d\mu$ so we can use the notation $Z_\mu$, $m_\mu$ and $S_\mu$ in order to denote these objects.

i) We call $d\mu$ “centered” iff $m = 0$.

ii) We call $d\mu$ “degenerate” iff $S$ is not invertible

iii) We call $d\mu$ “normalized” iff $Z = \left( \frac{2\pi}{\det(S')}^{d/2} \right)$ where $S' := S|_{\ker(S)^\perp}$. (See Example 4.4 below for the definition of $\det(S')$ and a motivation for the term “normalized”).

Definition 4.2 Let $d\mu$ be an oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$. A (Borel) measurable function $f : V \to \mathbb{C}$ will be called improperly integrable w.r.t. $d\mu$ if

$$\int f d\mu := \int f(x) d\mu(x) := \lim_{\epsilon \to 0} \left( \frac{\epsilon}{\pi} \right)^{n/2} \int f(x) e^{-\epsilon \|x\|^2} d\mu(x)$$

exists. Here we have set $n := \dim(\ker(S_\mu))$. Note that if $d\mu$ is non-degenerate we have $n = 0$ so the factor $\left( \frac{\epsilon}{\pi} \right)^{n/2}$ is then trivial.

Most of the time we will consider non-degenerate oscillatory Gauss-type measures, the exception being Proposition 4.12 below.

Using a simple analytic continuation argument and the corresponding explicit formulas for Gaussian probability measures we can easily prove the existence of $\int f d\mu$ and compute the corresponding value explicitly for a large class of functions $f$. Let us illustrate this by looking at some simple examples:

19 i.e. unit hyper-cubes have volume 1 w.r.t. $dx$

20 Observe that $\int_{\ker(S_\mu)} e^{-\epsilon \|x\|^2} d\mu(x) = \left( \frac{\epsilon}{\pi} \right)^{-n/2}$. In particular, the factor $\left( \frac{\epsilon}{\pi} \right)^{n/2}$ in Eq. (4.2) above ensures that also for degenerate oscillatory Gauss-type measure the improper integrals $\int f d\mu$ exists, cf. Example 4.13 below.
Example 4.3 Consider the special case where $V = \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the scalar product given by $(x, y) = x \cdot y$, and where $d\mu(x) = \exp(i(x, x))dx = \exp(ix^2)dx$. Then the improper integrals

$$\int_{-\infty}^{\infty} 1 \, d\mu(x), \quad \int_{-\infty}^{\infty} x \, d\mu(x), \quad \int_{-\infty}^{\infty} x^2 \, d\mu(x), \quad \int_{-\infty}^{\infty} e^{ix} \, d\mu(x), \quad c \in \mathbb{C}$$

exist and are given explicitly by

- $\int_{-\infty}^{\infty} 1 \, d\mu(x) = \sqrt{\pi}$
- $\int_{-\infty}^{\infty} x \, d\mu(x) = 0$
- $\int_{-\infty}^{\infty} x^2 \, d\mu(x) = \frac{i}{2} \sqrt{\pi}$
- $\int_{-\infty}^{\infty} e^{ix} \, d\mu(x) = e^{i\pi}$

where $\sqrt{\cdot}: \mathbb{C}\setminus(-\infty,0) \to \mathbb{C}$ denotes the standard square root.

In order to show the existence (and to compute the explicit value) of $\int_{-\infty}^{\infty} d\mu(x)$ we consider the analytic function $F : \{z \mid \text{Re}(z) > 0\} \to \mathbb{C}$ given by $F(z) := \int \exp(-zx^2)dx$. According to a well-known formula we have $F(a) = \sqrt{\pi/a}$ for all $a \in (0,\infty)$. The obvious uniqueness argument for analytic functions now implies that $F(z) = \sqrt{\pi/z}$ for all $z \in \mathbb{C}$ with $\text{Re}(z) > 0$. Thus $\int_{-\infty}^{\infty} d\mu = \lim_{\epsilon \to 0} F(\epsilon - i) = \lim_{\epsilon \to 0} \sqrt{\pi/(\epsilon - i)} = \sqrt{i\pi} = \sqrt{\pi} e^{i\pi}$

The other three integrals can be dealt with in a similar way.

In the next example $(V, \langle \cdot, \cdot \rangle)$ is again an arbitrary Euclidean space.

Example 4.4 Let $d\mu$ be a non-degenerate oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$ with $S$, $m$, and $Z$ given as in Eq. (4.1).

i) We have\footnote{we remark that if $d\mu$ is degenerate then an analogous statement will hold with $S$ replaced by $S' := S|_{\ker(S)^\perp}$}

$$\int_{-\infty}^{\infty} d\mu = \frac{(2\pi)^{d/2}}{Z} \det^\frac{1}{2}(iS)$$

where we have set $\det^\frac{1}{2}(iS) := \prod_k \sqrt{i\lambda_k} = e^{\frac{i}{4} \sum_k \text{sgn}(\lambda_k) (\prod_k |\lambda_k|^{1/2})}$ where $(\lambda_k)_k$ are the (real) eigenvalues of the symmetric matrix $S$. In particular, $d\mu$ is normalized in the sense of Definition 4.1 above iff $\int_{-\infty}^{\infty} d\mu = 1$.

ii) In the special case when $d\mu$ is normalized we have for all $v, w \in V$

$$\int_{-\infty}^{\infty} \langle v, x \rangle \, d\mu(x) = \langle v, m \rangle, \quad \int_{-\infty}^{\infty} \langle v, x \rangle \langle w, x \rangle \, d\mu(x) = \frac{1}{4} \langle v, S^{-1}w \rangle + \langle m, m \rangle$$

We will not try to identify the largest possible class of functions $f$ for which $\int_{-\infty}^{\infty} f \, d\mu$ exists. For our purposes the function algebra $P_{\exp}(V)$ defined in the next definition will be sufficient.

Definition 4.5

i) Let $W$ be a finite-dimensional associative $\mathbb{R}$-algebra (with the standard topology). By $P_{\exp}(V, W)$ we will denote the subalgebra of $\text{Map}(V, W)$ which is generated by the affine maps $\varphi : V \to W$ and their “exponentials” $\exp_W \circ \varphi$. Here $\exp_W : W \to W$ denotes the exponential map of $W$.

ii) By $P_{\exp}(V)$ we denote the subalgebra of $\text{Map}(V, \mathbb{C})$ which is generated by the functions of the form $\theta \circ f$ with $f \in P_{\exp}(V, W)$ and $\theta \in \text{Hom}_\mathbb{R}(W, \mathbb{S})$ where $W$ is any finite-dimensional associative $\mathbb{R}$-algebra.
Using analytic continuation arguments, some explicit formulas for Gaussian probability measures, and suitable growth estimates one can prove the following result (which is easy to believe):

**Proposition 4.6** Let \( d\mu \) be a non-degenerate oscillatory Gauss-type measure on \((V,\langle \cdot, \cdot \rangle)\). Then for every \( f \in \mathcal{P}_{\exp}(V) \) the improper integral \( \int f \, d\mu \in \mathbb{C} \) exists.

### 4.2 Three propositions

Let us fix for a while a normalized non-degenerate oscillatory Gauss-type measure \( d\mu \) on \((V,\langle \cdot, \cdot \rangle)\) and introduce the notation

\[
\mathbb{E}_\sim[X] := \int X d\mu \in \mathbb{C} \quad \text{(4.5a)}
\]

\[
\text{cov}_\sim(X, X') := \mathbb{E}_\sim[XX'] - \mathbb{E}_\sim[X]\mathbb{E}_\sim[X'] \in \mathbb{C} \quad \text{(4.5b)}
\]

for maps \( X, X' \in \mathcal{P}_{\exp}(V) \) (in analogy to the case of (Gaussian or non-Gaussian) probability measures on \( V \)).

**Observation 4.7** Let \( d\mu \) be as above and let \( X_1, X_2, \ldots, X_n \) be a sequence of affine maps \( V \to \mathbb{R} \).

i) If \( \mathbb{E}_\sim[X_i] = 0 \) for every \( i \leq n \) then

\[
\mathbb{E}_\sim[\prod_j X_j] = \begin{cases} \frac{1}{\prod_{n/2!^{2n/\sigma}} \prod_{i=1}^{n/2} \text{cov}_\sim(X_{\sigma(2i-1)}, X_{\sigma(2i)})} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{(4.6)}
\]

ii) If \( \text{cov}_\sim(X_i, X_j) = 0 \) for \( i, j \leq n \) with \( i \neq j \) we have

\[
\mathbb{E}_\sim[\prod_j X_j] = \prod_j \mathbb{E}_\sim[X_j] \quad \text{(4.7)}
\]

Here Eq. (4.6) follows from the analogous formula for the moments of a Gaussian probability measure and a suitable analytic continuation argument. Clearly, in the special case where \( \text{cov}_\sim(X_i, X_j) = 0 \) for \( i, j \leq n \) with \( i \neq j \) Eq. (4.6) reduces to \( \mathbb{E}_\sim[\prod_j X_j] = 0 \). By applying the latter equation to the subsequences of the sequence \( X'_1, X'_2, \ldots, X'_n \) given by \( X'_j := X_j - \mathbb{E}[X_j] \), we arrive at Eq. (4.7).

Let us now consider the case where \( Y : V \to \mathbb{R} \) is an affine map with \( \text{cov}_\sim(Y, Y) = 0 \) and let us consider the trivial sequence \((X_i)_{i=1}^n\) where \( X_i = Y \) for each \( i \leq n \). Since \( \text{cov}(Y, Y) = 0 \) we trivially have \( \text{cov}_\sim(X_i, X_j) = 0 \) for \( i \neq j \) (and even for \( i = j \)) so according to Eq. (4.7) we have \( \mathbb{E}_\sim[Y^n] = \mathbb{E}_\sim[Y]^n \), from which we conclude, for example, that

\[
\mathbb{E}_\sim[\exp(Y)] = \mathbb{E}_\sim[\sum_n \frac{Y^n}{n!}] = \sum_n \mathbb{E}_\sim[\frac{Y^n}{n!}] = \sum_n \frac{\mathbb{E}_\sim[Y]^n}{n!} = \exp(\mathbb{E}_\sim[Y]) \quad \text{(4.8)}
\]

In order to see that step \((*)\) above holds observe that for every fixed \( \epsilon > 0 \) we have

\[
\int \sum_n \left| \frac{{Y(x)}^n}{n!} e^{-|x|^2} \frac{1}{2 \pi} e^{-\frac{1}{2} (x-m, S(x-m))} \right| dx \leq \int \sum_n \frac{|Y(x)|^n}{n!} e^{-\epsilon |x|^2} dx < \infty
\]

Note that the condition \( \text{cov}_\sim(Y, Y) = 0 \) does not imply that \( Y \) is a constant map on \( V \). This is in sharp contrast to the situation for (Gaussian or non-Gaussian) probability measures where the relation \( \text{cov}(Y, Y) = 0 \) always implies \( Y = \mathbb{E}[Y] \, d\mu\)-a.s.
and we can therefore conclude from the dominated convergence theorem\textsuperscript{24} that
\[
\int \sum_n \frac{Y(x)^n}{n!} e^{-\epsilon|x|^2} d\mu(x) = \sum_n \int \frac{Y(x)^n}{n!} e^{-\epsilon|x|^2} d\mu(x)
\]
for each $\epsilon > 0$. Accordingly, in order to prove step (*) it is enough to prove that $\lim_{\epsilon \to 0} \sum_n \frac{I(n, \epsilon)}{n!} = \sum_n \lim_{\epsilon \to 0} \frac{I(n, \epsilon)}{n!}$ where we have set $I(n, \epsilon) := \int Y(x)^n e^{-\epsilon|x|^2} d\mu(x)$. The latter claim can easily be proven by computing the integrals $I(n, \epsilon)$ explicitly.

More generally, we obtain\textsuperscript{25} for every $\Phi \in \mathcal{P}_{\exp}(\mathbb{R})$
\[
\mathbb{E}_\sim[\Phi(Y)] = \Phi(\mathbb{E}_\sim[Y])
\]
(4.9)
since every such $\Phi$ is necessarily entire analytic and the coefficients $(c_n)_n$, given by $\Phi(x) = \sum_{n=0}^{\infty} c_n x^n$ for all $x \in \mathbb{R}$, have the property that $c_n \to 0$ rapidly enough so that

1. we can again apply the dominated convergence theorem in a similar way as above and prove that the two limit procedures $\int \cdots dx$ and $\sum_n$ can be interchanged

2. we can prove again that the two limit procedures $\lim_{\epsilon \to 0}$ and $\sum_n$ can be interchanged

Finally, we can generalize Eq. (4.9) to the case where we have a $\Phi \in \mathcal{P}_{\exp}(\mathbb{R}^n)$, $n \in \mathbb{N}$, and where $(Y_k)_{k \leq n}$ is a sequence of affine maps $V \to \mathbb{R}$ such that $\text{cov}_\sim(Y_i, Y_j) = 0$ holds for all $i, j \leq n$. We then arrive at the following result, which will be the key argument in Sec. 5.1 below. In order to make the application of Proposition 4.8 in Sec. 5.1 more transparent we avoid the use of the notation $\mathbb{E}_\sim[\cdot]$ and $\text{cov}_\sim(\cdot, \cdot)$ from now on.

**Proposition 4.8** Let $d\mu$ be a normalized non-degenerate oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$ and let $(Y_k)_{k \leq n}$, $n \in \mathbb{N}$, be a sequence of affine maps $V \to \mathbb{R}$ such that
\[
\int_Y Y_i Y_j d\mu = (\int_Y Y_i d\mu) (\int_Y Y_j d\mu)
\]
holds for all $i, j \leq n$. Then for every $\Phi \in \mathcal{P}_{\exp}(\mathbb{R}^n)$ we have
\[
\int_Y \Phi((Y_k)_k) d\mu = \Phi((\int_Y Y_k d\mu)_k)
\]
(4.11)

**Remark 4.9** In Sec. 5.1 below we will actually apply a reformulation of Proposition 4.8 where the sequence $(Y_k)_{k \leq n}$, $n \in \mathbb{N}$, is replaced by a family $(Y_{i,a})_{k \leq n, i \leq m, a \leq D}$ of affine maps fulfilling the obvious analogue of Eq. (4.10) above and the function $\Phi \in \mathcal{P}_{\exp}(\mathbb{R}^n)$ is replaced by a function $\Phi \in \mathcal{P}_{\exp}(\mathbb{R}^{m \times n \times D})$.

**Example 4.10** Consider the (non-degenerate normalized centered) oscillatory Gauss-type measure $d\mu(x) := \frac{1}{2\pi} e^{-x_1^2 - x_2^2} dx_1 dx_2$ on $V = \mathbb{R}^2$. For every $f \in \mathcal{P}_{\exp}(\mathbb{R})$ we have
\[
\int_Y f(x_1) d\mu(x) = f(0)
\]
(4.12)
This follows by applying Proposition 4.8 with $\Phi = f$ and $Y_1(x) := x_1$. Observe that Eq. (4.10) is indeed fulfilled since according to Example 4.4 we have $\int_Y Y_1 d\mu = \int_Y \langle x, e_1 \rangle d\mu(x) = 0$ and $\int_Y Y_1 Y_1 d\mu = \frac{1}{2} \langle e_1, S_\mu^{-1} e_1 \rangle = 0$ where we have set $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and used that $S_\mu = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.\textsuperscript{24} applied to the positive measure $dx$ “appearing” in $d\mu$
\textsuperscript{25}Observe that $\Phi(Y) \in \mathcal{P}_{\exp}(V)$ so the existence of the LHS of Eq. (4.9) is guaranteed by Proposition 4.6
Using a different argument one can easily prove that Eq. (4.12) holds for arbitrary continuous bounded functions $f$. Moreover, we can include an additional “exponential factor”, and we can in fact consider more general oscillatory Gauss-type measures $d\mu$, cf. Proposition 4.12 below, which will play a key role in Sec. 5.2 below.

As a preparation for Proposition 4.12 let us first consider the special case where the oscillatory Gauss-type measure $d\mu$ is non-degenerate:

**Proposition 4.11** Assume that $V = V_1 \oplus V_2$ where $V_1$ and $V_2$ are two subspaces of $V$ which are orthogonal to each other. For each $j = 1, 2$ we denote the $V_j$-component of $x \in V$ by $x_j$. Moreover, let $d\mu$ be a (non-degenerate centered) normalized oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$ of the form $d\mu(x) = \frac{1}{Z} \exp(i\langle x_2, Mx_1 \rangle) dx$ for some linear isomorphism $M : V_1 \to V_2$. Then for every bounded continuous function $f : V_1 \to \mathbb{C}$ and every fixed $v \in V_2$ we have

$$\int_{V_1} f(x_1) \exp(i\langle x_2, v \rangle) d\mu(x) = f(-M^{-1}v)$$

(In particular, the LHS of Eq. (4.13) exists; note that the present situation is not covered by Proposition 4.10 above).

**Proof.** Let $dx_1$ resp. $dx_2$ be the normalized Lebesgue measure on $V_1$ resp. $V_2$ (equipped with the scalar product induced by the one on $V$). We have

$$\int_{V_1} f(x_1) \exp(i\langle x_2, v \rangle) d\mu(x)$$

$$= \frac{1}{Z} \lim_{\epsilon \to 0} \int_{V_1} \int_{V_2} e^{-\epsilon(|x_1|^2 + |x_2|^2)} f(x_1) \exp(i\langle x_2, v + Mx_1 \rangle) dx_2 dx_1$$

$$= \frac{(2\pi)^{d_2}}{Z} \lim_{\epsilon \to 0} \int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1$$

(4.14)

where $d_2 := \dim(V_2) = d/2$ and where $\delta_\epsilon : V_2 \to \mathbb{R}$ is given by

$$\delta_\epsilon(w) := \frac{1}{(2\pi)^{d_2} 2^{d_2/2}} \int_{V_2} e^{-\epsilon|x_2|^2} \exp(i\langle x_2, w \rangle) dx_2 = \frac{1}{(2\pi)^{d_2} 2^{d_2/2}} e^{-\frac{1}{\epsilon} |w|^2 - \frac{1}{2} |w|^2} = \frac{1}{(4\pi)^{d_2} 2^{d_2/2}} e^{-\frac{|w|^2}{4\epsilon}}$$

(4.15)

for all $w \in V_2$. Let us now fix an (arbitrary) isometry $\psi : V_2 \to V_1$. Clearly, the pushforward $(\psi)_* dx_2$ of $dx_2$ coincides with $dx_1$ so we have

$$\int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = \int_{V_2} e^{-\epsilon|\psi x_2|^2} f(\psi x_2) \delta_\epsilon(v + M\psi x_2) dx_2$$

(4.16)

Making the change of variable $M\psi x_2 + v \to y_2$ on the RHS of Eq. (4.16) we obtain

$$\int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = |\det(M\psi)|^{-1} \int_{V_2} e^{-\epsilon|M^{-1}(y_2 - v)|^2} f(M^{-1}(y_2 - v)) \delta_\epsilon(y_2) dy_2$$

(4.17)

where $dy_2$ is the normalized Lebesgue measure on $V_2$. Since $(\delta_\epsilon)_\epsilon$ is an “approximation to the identity” (i.e. converges weakly to the Dirac distribution $\delta_0$) it is therefore clear that

$$\lim_{\epsilon \to 0} \int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = f(-M^{-1}v) \cdot |\det(M\psi)|^{-1}$$

(4.18)

The assertion of the proposition now follows from Eq. (4.14), Eq. (4.18) and the following equation:

$$\frac{(2\pi)^{d_2}}{Z} = \frac{(2\pi)^{d/2}}{Z}$$

(4.19)

---

26 A formal proof of Eq. (4.18) can be obtained after a suitable change of variable and the application of the dominated convergence theorem, cf. the proof of Eq. (4.20) below which generalizes Eq. (4.18).
where the normalized Lebesgue measure on \( V \) is provided that we can show Eq. (4.22) above and the equality just mentioned will therefore imply the assertion of the proposition provided that we can make the identification \( V_2 \cong \psi V_1 \) we have \( S_\mu = - \begin{pmatrix} 0 & M \psi \\ (M \psi)^t & 0 \end{pmatrix} \).

\[ \square \]

**Convention 4** For a continuous function \( f : V_0 \to \mathbb{C} \) on a \( d_0 \)-dimensional Euclidean space \( V_0 \) we set
\[
\int_{V_0} f(x_0)dx_0 := \frac{1}{\pi^{d_0/2}} \lim_{\epsilon \to 0} \int_{V_0} e^{-\epsilon \|x_0\|^2} f(x_0)dx_0
\]
provided that the expression on the RHS of the previous equation is well-defined. Here \( dx_0 \) is the normalized Lebesgue measure on \( V_0 \).

**Proposition 4.12** Assume that \( V = V_0 \oplus V_1 \oplus V_2 \) where \( V_0, V_1, V_2 \) are pairwise orthogonal subspaces of \( V \). For each \( j = 0, 1, 2 \) we denote the \( V_j \)-component of \( x \in V \) by \( x_j \). Moreover, let \( d\mu \) be a (centered) normalized oscillatory Gauss-type measure on \( (V, \langle \cdot, \cdot \rangle) \) of the form \( d\mu(x) = \frac{1}{Z} \exp(i \langle x_2, Mx_1 \rangle)dx \) for some linear isomorphism \( M : V_1 \to V_2 \). Then for every fixed \( v \in V_2 \) and every bounded uniformly continuous function \( F : V_0 \oplus V_1 \to \mathbb{C} \) the LHS of the following equation exists iff the RHS exists and in this case we have
\[
\int_{V_0} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) d\mu(x) = \int_{V_0} F(x_0 - M^{-1}v)dx_0
\]
where \( dx_0 \) is the normalized Lebesgue measure on \( V_0 \).

**Proof.** We set \( d_j := \text{dim}(V_j) \) for \( j = 0, 1, 2 \). Similarly as in Eq. (4.14) and with \( \delta_\epsilon : V_2 \to \mathbb{R} \) as above we obtain
\[
\int_{V_0} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) d\mu(x)
\]
\[
= \lim_{\epsilon \to 0} \left( \frac{2\pi}{\epsilon} \right)^{d_0/2} \int_{V_0} \left[ \int_{V_1} \left[ \int_{V_2} e^{-\epsilon \|x_0\|^2 + \|x_1\|^2 + \|x_2\|^2} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) \times \right. \right.
\]
\[
\times \left. \left. \exp(i \langle x_2, Mx_1 \rangle)dx_2 \right] dx_1 \right] dx_0
\]
\[
= \frac{1}{\pi^{d_0/2}} \lim_{\epsilon \to 0} \int_{V_0} dx_0 e^{-\epsilon \|x_0\|^2} \left[ \frac{(2\pi)^d}{Z} \int_{V_1} dx_1 e^{-\epsilon \|x_1\|^2} F(x_0 + x_1) \delta_\epsilon(v + Mx_1) \right].
\]

Let \( \psi : V_2 \to V_1 \) be a fixed isometry. From the assumption that \( d\mu \) was normalized it follows – using the same argument as in Eq. (4.19) above – that \( \frac{(2\pi)^d}{Z} \delta_\epsilon(v) = \det(\frac{1}{Z} \langle i(S_\mu)_{V_1 \oplus V_2} \rangle) = |\det(M\psi)| \).

Eq. (4.22) above and the equality just mentioned will therefore imply the assertion of the proposition provided that we can show
\[
\lim_{\epsilon \to 0} T(\epsilon) = 0
\]
where
\[
T(\epsilon) := \epsilon^{d_0/2} \int_{V_0} dx_0 e^{-\epsilon \|x_0\|^2} \left[ F(x_0 - M^{-1}v) - |\det(M\psi)| \int_{V_1} dx_1 e^{-\epsilon \|x_1\|^2} F(x_0 + x_1) \delta_\epsilon(v + Mx_1) \right]
\]
\[27\]this can be seen, e.g., from the equation \( J^{-1}S_\mu J = -S_\mu \) where \( J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \) denoting both the identity in \( \text{End}(V_1) \) and \( \text{End}(V_2) \).
In order to prove (4.23) recall that \( \psi(dx_2) = dx_1 \) and that
\[
1 = \frac{1}{\pi^d 2^d} \int_{V_2} e^{-|y_2|^2} dy_2
\]
so we obtain
\[
T(\epsilon) = \epsilon^{d_0/2} \int_{V_0} dx_0 e^{-|x_0|^2} \left[ \frac{1}{\pi^d 2^d} \int_{V_2} dy_2 e^{-|y_2|^2} F(x_0 - M^{-1} v) \right. \\
- \left. \frac{1}{|\det (M\psi)|} \int_{V_2} dy_2 e^{-|y_2|^2} F(x_0 + \psi x_2) \right]
\]
\[
= \frac{1}{\pi^d 2^d} \int_{V_0 \oplus V_2} dy_0 dy_2 e^{-|y_0|^2 - |y_2|^2} \left[ F\left( \frac{y_0}{\sqrt{\epsilon}} - M^{-1} v \right) - \gamma_\epsilon(y_2) F\left( \frac{y_0}{\sqrt{\epsilon}} - M^{-1} v + \sqrt{2} M^{-1} y_2 \right) \right]
\]
(4.25)

In Step (*) we have made the changes of variable \( \sqrt{\epsilon} x_0 \to y_0 \) and \( \frac{1}{\sqrt{\epsilon}} (v + M\psi x_2) \to y_2 \) and we have set \( \gamma_\epsilon(y_2) := e^{-\epsilon|M^{-1}(2\sqrt{\epsilon} y_2 - v)|^2} \). Relation (4.23) now follows by applying the dominated convergence theorem to the last expression in Eq. (4.25) and taking into account that for all fixed \( y_0 \in V_0 \) and \( y_2 \in V_2 \) we have
\[
\lim_{\epsilon \to 0} \left[ F\left( \frac{y_0}{\sqrt{\epsilon}} - M^{-1} v \right) - \gamma_\epsilon(y_2) F\left( \frac{y_0}{\sqrt{\epsilon}} - M^{-1} v + \sqrt{2} M^{-1} y_2 \right) \right] = 0
\]
and, by assumption, \( F \) is bounded and uniformly continuous.

The following remark will be useful in Sec. 5.2 and Sec. 5.4 below.

**Remark 4.13** If \( \Gamma \) is a lattice in \( V \) and \( f : V \to \mathbb{C} \) a \( \Gamma \)-periodic continuous function then \( \int_V f(x) dx \) exists and we have
\[
\int_V f(x) dx = \frac{1}{\text{vol}(Q)} \int_Q f(x) dx
\]
(4.26)

with \( Q := \{ \sum_i a_i e_i \mid 0 \leq a_i \leq 1 \forall i \leq d \} \) where \( (e_i)_{i \leq d} \) is any fixed basis of the lattice \( \Gamma \) and where \( \text{vol}(Q) \) denotes the volume of \( Q \). Observe that Eq. (4.26) implies
\[
\forall y \in V : \int_V f(x) dx = \int_V f(x + y) dx
\]
(4.27)

## 5 Proof of Theorem 3.4

Recall that Theorem 3.4 states that in the special situation described above \( \text{WLO}_{rig}(L) \) is well-defined and has the value \( |L|/|0| \). In the following we will concentrate on the “computational half” of this statement. That \( \text{WLO}_{rig}(L) \) is well-defined in the first place will also become clear during the computations\[28\] even though we will rarely make explicit statements in this directions.

### 5.0 Some preparations

**a) Computation of \( \det(L(N))(B) \)**

Recall that in Sec. 4 above we used the notation \( L(N)(B) \) both for the linear operator \( A^\perp(K) \to A^\perp(K) \) and the restriction of this operator to the invariant subspace \( \hat{A}^\perp(K) \). From now on the notation \( L(N)(B) \) will always refer to the restricted operator.

---

\[28\] in order to check well-definedness we should, of course, reverse the order of our considerations/computations: we first check that the expressions appearing in Step 6 are well-defined. Based on this we can verify that also the expressions in Step 5 must be well-defined and so on until we arrive at the expressions in Step 1.
Proposition 5.1 For $B \in \mathcal{B}(q\mathcal{K})$ we have

$$\det(L^{(N)}(B)) = N^d \prod_{e \in \mathfrak{g}(K_1 K_2)} \det \left( 1_t - \exp(\text{ad}(\vec{e})) |_{\bar{\mathfrak{g}}} \right)^2$$  \hspace{1cm} (5.1)

where $d := \text{dim}(\mathcal{A}^\perp(K))$. In particular, if

$$B \in \mathcal{B}_{\text{reg}}(q\mathcal{K}) := \{ B \in \mathcal{B}(q\mathcal{K}) \mid B(x) \in t_{\text{reg}} \text{ for all } x \in \mathfrak{g}(q\mathcal{K}) \}$$  \hspace{1cm} (5.2)

then we have $\det(L^{(N)}(B)) \neq 0$.

According to Eqs. (3.13), (3.14a), (3.14b) in order to prove Proposition 5.1 it will be enough to prove Lemma 1 below.

Recall the definition of the two linear operators $\hat{L}^{(N)}(b)$ and $\tilde{L}^{(N)}(b)$ on $\text{Map}(\mathbb{Z}_N, \mathfrak{g})$ for fixed $b \in t$, cf. Eqs. (3.12) above.

In the following we will consider the restriction of each of these two operators to the orthogonal complement of its kernel. The restrictions will again be denoted by $\hat{L}^{(N)}(b)$ and $\tilde{L}^{(N)}(b)$.

Lemma 1 We have

$$\det(\hat{L}^{(N)}(b)) = \pm \det \left( 1_t - \exp(\text{ad}(b)) |_{\bar{\mathfrak{g}}} \right) \cdot N^d,$$  \hspace{1cm} (5.3)

$$\det(\tilde{L}^{(N)}(b)) = \pm \det \left( 1_t - \exp(\text{ad}(b)) |_{\bar{\mathfrak{g}}} \right) \cdot N^d,$$  \hspace{1cm} (5.4)

where $d := \text{dim}(\text{Map}(\mathbb{Z}_N, \mathfrak{g})) = N \text{dim}(\mathfrak{g})$, $r := \text{dim}(t)$.

Proof. Let us prove Eq. (5.3). The proof of Eq. (5.4) is similar. First observe that

$$\det(\hat{L}^{(N)}(b)) = \det(N(\tau_1 e^{\text{ad}(b)/N} - 1)) = N^{d'} \det(\tau_1 - e^{-\text{ad}(b)/N})$$  \hspace{1cm} (5.5)

where $d' := \text{dim}(\text{Map}(\mathbb{Z}_N, \mathfrak{g})) = d - \text{dim}(t)$ and where we have used that $e^{\text{ad}(b)/N}$ is orthogonal. The complexified operator $(\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_C$ is diagonalizable with eigenvalues

$$\lambda_{k,\alpha} := e^{\frac{2\pi i k}{N} - \frac{\alpha(b)}{N}}, \quad \text{for each } k \in \mathbb{Z}_N, \alpha \in \mathcal{R}_C$$

$$\mu_{k,\alpha} := e^{\frac{2\pi i k}{N} - 1}, \quad \text{for each } k \in (\mathbb{Z}_N \setminus \{0\}), \alpha \in \{1, 2, \ldots, r\}$$

where $\mathcal{R}_C$ denotes the set of complex roots of $\mathfrak{g}$ w.r.t. $t$ (cf. part A1 of the Appendix). Using the two polynomial equations $x^N - 1 = \prod_{k=0}^{N-1} (x - e^{2\pi i k/N}) = (-1)^N \prod_{k=1}^{N-1} (e^{2\pi i k/N} - x)$ and $x^{N-1} + x^{N-2} + \ldots + 1 = \prod_{k=1}^{N-1} (x - e^{2\pi i k/N}) = (-1)^{N-1} \prod_{k=1}^{N-1} (e^{2\pi i k/N} - x)$ we therefore obtain

$$\det(\tau_1 - e^{-\text{ad}(b)/N}) = \det_C \left( (\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_C \right)$$

$$= \left( \prod_{\alpha \in \mathcal{R}_C} \prod_k \left( e^{\frac{2\pi i k}{N} - \frac{\alpha(b)}{N}} \right) \right) \left( \prod_{\alpha = 1}^r \prod_{k \neq 0} \left( e^{\frac{2\pi i k}{N} - 1} \right) \right)$$

$$= \left( \prod_{\alpha \in \mathcal{R}_C} (-1)^N \left( e^{-\alpha(b)} - 1 \right) \right) \left( \prod_{\alpha = 1}^r (-1)^{N-1} \left\{ N \right\} \right) = (-1)^{r(N-1)} N^r \prod_{\alpha \in \mathcal{R}_C} \left( e^{-\alpha(b)} - 1 \right)$$

The assertion now follows by combining the last equation with Eq. (5.5) above and by taking into account the relations $d = d' + r$ and $\prod_{\alpha \in \mathcal{R}_C} (e^{-\alpha(b)} - 1) = \prod_{\alpha \in \mathcal{R}_C} (1 - e^{\alpha(b)}) = \det_C \left( (1_t - \exp(\text{ad}(b)) |_{\bar{\mathfrak{g}}} \right) \otimes \text{id}_C \right) = \det \left( 1_t - \exp(\text{ad}(b)) |_{\bar{\mathfrak{g}}} \right)$.
b) Some consequences of conditions (NCP)' and (NH)'

Let \( L = (R_1, R_2, \ldots, R_m) \) be the simplicial ribbon link in \( qK \times \mathbb{Z}_N \) fixed in Sec. 3.8 above. Recall that each \( R_i \) is \((F_k)_{k \in \mathbb{N}}, n_i \in \mathbb{N}, \) “induces”\(^{32} \) two simplicial loops \( l_i \) and \( l'_i \) in \( qK \times \mathbb{Z}_N \).

In the following we use the short notation \( l^i_\Sigma := (l_i)_\Sigma, l'^i_\Sigma := (l'_i)_\Sigma, l^i_{\Sigma 1} := (l_i)_{\Sigma 1}, \) and \( l'^i_{\Sigma 1} := (l'_i)_{\Sigma 1} \) for the \( \Sigma \)- or \( S^1 \)-projections of these loops and we will often consider \( l^i_\Sigma \) and \( l'^i_\Sigma \) as (piecewise smooth) maps \( S^1 \to \Sigma \). Clearly, \( \text{arc}(l^i_\Sigma) \) and \( \text{arc}(l'^i_\Sigma) \) can then be considered as subsets of \( \Sigma \).

It is not difficult to see that the two conditions (NCP)’ and (NH)’ on our simplicial ribbon link \( L \) imply the following conditions:

1. **(FC1)** For all \( i, j \leq m \) we have \( \text{arc}(l^i_\Sigma) \cap \text{arc}(l'^j_\Sigma) = \emptyset \) and we also have \( \text{arc}(l^i_\Sigma) \cap \text{arc}(l'^j_\Sigma) = \emptyset \) if \( i \neq j \).

2. **(FC2)** For each \( i \leq m \) the open region \( O_i \subset \Sigma \) “between”\(^{33} \) \( \text{arc}(l^i_\Sigma) \) and \( \text{arc}(l'^i_\Sigma) \) does not contain an element of \( \mathfrak{S}_0(qK) \).

3. **(FC3)** For each \( i \leq m \) and every \( F \in \mathfrak{S}_2(qK) \) with \( F \subset \text{Image}(R^i_\Sigma) \) exactly one of the four sides of (the tetragon) \( F \) will lie on \( \text{arc}(l^i_\Sigma) \) and exactly one side will lie on \( \text{arc}(l'^i_\Sigma) \).

4. **(FC4)** \( l^i_{\Sigma 1} = l'^i_{\Sigma 1} \) is fulfilled for each \( i \leq m \).

In order to simplify the notation in Secs 5.1, 5.3 below we will set

\[
n := \max_{i \leq m} n_i
\]

and we will extend each simplicial loop \( l^i_\Sigma, l'^i_\Sigma, l^i_{\Sigma 1}, l'^i_{\Sigma 1} \) to a simplicial loop of length \( n \) in a trivial way, i.e. by “adding” \( n - n_i \) empty edges. For the extended simplicial loops we will use the same notation.

Finally we set, for each \( i \leq m \) and \( k \leq n \)

\[
\tilde{l}^i(k) := \pi(l^i(k)), \quad \text{and} \quad \tilde{l}'^i(k) := \pi(l'^i(k))
\]

(5.6)

where \( \pi : C_1(qK) \to C_1(K)(\subset C_1(qK)) \) is the orthogonal projection. Observe that for each \( i \leq m \) we have

\[
\forall k \leq n : \tilde{l}^i(k) \in C_1(K_1) \quad \text{and} \quad \forall k \leq n : \tilde{l}'^i(k) \in C_1(K_2)
\]

(5.7a)

or

\[
\forall k \leq n : \tilde{l}^i(k) \in C_1(K_2) \quad \text{and} \quad \forall k \leq n : \tilde{l}'^i(k) \in C_1(K_1)
\]

(5.7b)

From (FC1) and Eqs (5.7) it easily follows that for all \( i_1, i_2 \leq m, \) and \( k_1, k_2 \leq n \) we have\(^{34} \)

\[
\star_K \tilde{l}^{i_1(k_1)}_\Sigma \neq \pm \tilde{l}^{i_2(k_2)}_\Sigma, \quad \star_K \tilde{l}'^{i_1(k_1)}_\Sigma \neq \pm \tilde{l}'^{i_2(k_2)}_\Sigma, \quad \star_K \tilde{l}'^{i_1(k_1)}_\Sigma \neq \pm \tilde{l}'^{i_2(k_2)}_\Sigma
\]

(5.8)

provided that \( \tilde{l}^{i_1(k_1)}_\Sigma \neq 0 \) and \( \tilde{l}'^{i_1(k_1)}_\Sigma \neq 0 \). Here \( \star_K \) is the linear isomorphism on \( C_1(K) = C^1(K, \mathbb{R}) \) which is defined exactly in the same way as the operator \( \star_K \) on \( A_\Sigma(K) = C^1(K, \mathfrak{g}) \) which we introduced in Sec. 3.3 above.

\(^{32}\) \( l_i \) and \( l'_i \) are just the two loops on the boundary of \( R_i \).

\(^{33}\) more precisely, \( O_i \) is the interior of \( \text{Image}(R^i_\Sigma) \).

\(^{34}\) or, more precisely, both \( \star_K \tilde{l}^{i_1(k_1)}_\Sigma \neq \pm \tilde{l}'^{i_2(k_2)}_\Sigma \) and \( \star_K \tilde{l}'^{i_1(k_1)}_\Sigma \neq \pm \tilde{l}^{i_2(k_2)}_\Sigma \) etc.
c) Two conventions

**Convention 5** In the following \( \sim \) will denote equality up to a multiplicative non-zero constant. This “constant” may depend on \( G, \ N, \ K, \) and \( k \) but it will never depend on the (simplicial ribbon) link \( L \).

**Convention 6** From now on we will simply write \( B(qK) \) instead of \( B_0(qK) \).

### 5.1 Step 1: Performing the \( \int_\sim \cdots \exp(iS_{CS}^{\text{disc}}(\tilde{A}^\perp, B)) \, D\tilde{A}^\perp \) integration in Eq. (3.25)

**Lemma 2** Under the assumptions on the simplicial ribbon link \( L = (R_1, \ldots, R_m) \) made above we have for every fixed \( A_c^\perp \in A_c^\perp(K) \) and \( B \in B_{\text{reg}}(qK) \)

\[
\int_\sim \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{R_i}^{\text{disc}}(\tilde{A}^\perp + A_c^\perp, B)) \exp(iS_{CS}^{\text{disc}}(\tilde{A}^\perp, B)) \, D\tilde{A}^\perp = Z_B^{\text{disc}} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{R_i}^{\text{disc}}(A_c^\perp, B))
\]

where \( Z_B^{\text{disc}} := \int_\sim \exp(iS_{CS}^{\text{disc}}(\tilde{A}^\perp, B)) \, D\tilde{A}^\perp \)

**Proof.** Let \( A_c^\perp \in A_c^\perp(K) \) and \( B \in B_{\text{reg}}(qK) \) be as in the assertion of the lemma. In order to prove the lemma we will apply Proposition 1.8 (and Remark 1.9 above) to the special situation where (cf. Convention 3 in Sec. 3.7)

- \( V = \tilde{A}^\perp(K) \),
- \( d\mu = d\nu_B^{\text{disc}} \) with \( d\nu_B^{\text{disc}} = \frac{1}{Z_B^{\text{disc}}} \exp(iS_{CS}^{\text{disc}}(\tilde{A}^\perp, B)) \, D\tilde{A}^\perp , \)
- \((Y_{i,a}^{i,a})_{i \leq m, k \leq n, a \leq \dim(\mathfrak{g})}\) is the family of maps \( Y_{i,a}^{i,a} : \tilde{A}^\perp(K) \to \mathbb{R} \) given by\[^{35}\]

\[
Y_{i,a}^{i,a}(\tilde{A}^\perp) := \left( T_a, (\tilde{A}^\perp(\bullet l_{S_1}^{i(k)})) + A_c^{\perp} \left( \frac{1}{2} l_{S_1}^{i(k)} + \frac{1}{2} l_{S_1}^{j(k)} \right) \right)
\]

where \( (T_a)_{a \leq \dim(\mathfrak{g})} \) is an arbitrary \( \langle \cdot, \cdot \rangle \)-ONB of \( \mathfrak{g} \) (which will be kept fixed in the following), and

- \( \Phi : \mathbb{R}^{m \times n \times \dim(\mathfrak{g})} \to \mathbb{C} \) is given by

\[
\Phi((x_{i,k,a}^{i,a})_{i,k,a}) = \prod_{i=1}^m \text{Tr}_{\rho_i}(\prod_{k=1}^n \exp(\sum_{a=1}^{\dim(\mathfrak{g})} T_a x_{i,k,a}^{i,a})) \quad \text{for all } (x_{i,k,a}^{i,a})_{i,k,a} \in \mathbb{R}^{m \times n \times \dim(\mathfrak{g})}
\]

Observe that

i) \( d\nu_B^{\text{disc}} \) is a well-defined normalized non-degenerate centered oscillatory Gauss-type measure. Since by assumption \( B \in B_{\text{reg}}(qK) \) this follows from Eq. (3.17b) and Proposition 3.1 in Sec. 3.3 above and from Proposition 5.1 in Sec. 5.4.

For later use let us mention that according to Example 4.4 above and Eq. (5.11) in Proposition 5.1 above we have

\[
Z_B^{\text{disc}} = \int_\sim \exp(iS_{CS}^{\text{disc}}(\tilde{A}^\perp, B)) \, D\tilde{A}^\perp \sim \det(L^{(N)}(B))^{-1/2} \sim \prod_{\ell \in \mathfrak{g}_0(K_1|K_2)} \det(1_\ell - \text{ad}(B(\ell)))^{-1} \quad (5.12)
\]

\[^{35}\text{observe that in view of condition (FC4) above the RHS of Eq. (5.10) is very closely related to the RHS of Eq. (3.18) in Sec. 3.4.}\]
ii) \( \Phi \in \mathcal{P}_{\exp}(\mathbb{R}^{m \times n \times \dim(g)}) \) since

\[
\Phi((x_k^{i,a})_{i,k}) = \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T.ax_k^{i,a})) = \prod_i \text{Tr}_{\text{End}(V_i)}(\prod_k \rho_i(\exp(\sum_a T.ax_k^{i,a})))
\]

\[= \prod_i \text{Tr}_{\text{End}(V_i)}(\prod_k \exp(\text{End}(V_i))(\sum_a ((\rho_i)_* T.a)x_k^{i,a})) \quad (5.13)\]

where \( \exp(\text{End}(V_i)) \) is exponential map of the associative algebra \( \text{End}(V_i) \) and \((\rho_i)_*: g \rightarrow \text{gl}(V_i)\), for \( i \leq m \), is the Lie algebra representation induced by \( \rho_i \).

iii) For all \( i \leq m, k \leq n, a \leq \dim(g) \) we have

\[
\int_{\sim} Y^{i,a}_k \, d\nu_B^{\text{disc}} = Y^{i,a}_k(0) \quad (5.14)
\]

In order to see this let us introduce \( j : \mathbb{Z}_N \rightarrow C_1(qK) \) by

\[
j(t) := \begin{cases} \frac{1}{2} l^{i(k)} + \frac{1}{2} l^{j(k)} & \text{if } t = \bullet l^{i(k)} \\ 0 & \text{if } t \neq \bullet l^{i(k)} \end{cases}
\]

and set \( \tilde{j}_a := p(Taj) \) where \( p : A^\perp(qK) \rightarrow \bar{A}^\perp(K) \) is the \( \ll \cdot, \cdot \gg A^\perp(qK) \)-orthogonal projection onto the subspace \( \bar{A}^\perp(K) \) of \( A^\perp(qK) \) and where \( Taj := j \otimes T_a \in A^\perp(qK) \cong \text{Map}(\mathbb{Z}_N, C_1(qK)) \otimes g \). Then

\[
Y^{i,a}_k(\bar{A}^\perp) - Y^{i,a}_k(0) = (T_a, ((\bar{A}^\perp)(\bullet l^{i(k)})(\frac{1}{2} l^{i(k)} + \frac{1}{2} l^{j(k)})) = \ll \bar{A}^\perp, Taj \gg A^\perp(qK) = \ll \bar{A}^\perp, \tilde{j}_a \gg A^\perp(qK)
\]

On the other hand since \( d\nu_B^{\text{disc}} \) is centered Example 4.4 above implies that \( \int_{\sim} \ll \cdot, \tilde{j}_a \gg A^\perp(qK) \) \( d\nu_B^{\text{disc}} = 0 \). Since \( d\nu_B^{\text{disc}} \) is also normalized we obtain Eq. (5.14).

iv) For all \( i, i' \leq m, k, k' \leq n, a, a' \leq \dim(g) \) we have

\[
\int_{\sim} \int Y^{i,a}_k Y^{i',a'}_{k'} \, d\nu_B^{\text{disc}} = \int_{\sim} Y^{i,a}_k \, d\nu_B^{\text{disc}} \int_{\sim} Y^{i',a'}_{k'} \, d\nu_B^{\text{disc}} \quad (5.15)
\]

This follows from Eq. (5.14) above and

\[
\int_{\sim} (Y^{i,a}_k - Y^{i,a}_k(0))(Y^{i',a'}_{k'} - Y^{i',a'}_{k'}(0)) \, d\nu_B^{\text{disc}} = \int_{\sim} \ll \cdot, \tilde{j}_a \gg A^\perp(qK) \ll \cdot, \tilde{j}'_{a'} \gg A^\perp(qK) \, d\nu_B^{\text{disc}} \quad (5.16)
\]

where \( \tilde{j}_a \) is as in point iii) above and where \( \tilde{j}'_{a'} \) is defined in a completely analogous way with \( i, k, \) and \( a \) replaced by \( i', k', \) and \( a' \). Here step (*) follows from Example 4.4 above and step (**) follows from the inequalities \( (5.8) \) appearing at the end of Sec. 5.8 above.

Thus the assumptions of Proposition 4.8 above are fulfilled and we obtain

\[
\frac{1}{Z_B^{\text{disc}}} \int_{\sim} \prod_i \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}(A_c^\perp + A^\perp, B)) \exp(i\mathcal{S}^{\text{disc}}(A_c^\perp, B)) \, d\tilde{A}^\perp
\]

\[
(+) \int_{\sim} \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T.a Y_k^{i,a})) \, d\nu_B^{\text{disc}} = \int_{\sim} \Phi((Y_k^{i,a})_{i,k,a}) \, d\nu_B^{\text{disc}} \equiv \Phi((\int_{\sim} Y_k^{i,a} \, d\nu_B^{\text{disc}})_{i,k,a})
\]

\[
= \Phi((Y_k^{i,a}(0))_{i,k,a}) = \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T.a Y_k^{i,a}(0))) = \prod_i \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}(A_c^\perp, B)) \quad (5.17)
\]

Observe that the vector spaces underlying \( \text{End}(V_i) \) and \( \text{gl}(V_i) \) coincide
Here step (+) follows from the definitions and condition (FC4) in Sec. 5.0 above and step (*) follows from Proposition 4.8 and Remark 4.9 above.

Using Lemma 2 and taking into account the implication

\[ \prod_x 1_{ \mathcal{B}_{\text{reg}} }^{(s)}(B(x)) \neq 0 \Rightarrow B \in \mathcal{B}_{\text{reg}}(qK) \]

for all \( s > 0 \) we now obtain from Eq. (3.25)

\[ \text{WLO}_{\text{reg}}^{\text{disc}}(L) = \lim_{s \to 0} \sum_{y \in I} \int_{\sim} \left( \prod_x 1_{ \mathcal{B}_{\text{reg}} }^{(s)}(B(x)) \right) \prod_i \text{Tr}_{\rho_i} \left( \text{Hol}_{\mathcal{R}_i}^{\text{disc}}(A_{\mathcal{C}}^+, B) \right) \text{Det}_{\text{disc}}^{\text{reg}}(B) \]

\times \exp \left( -2\pi ik\langle y, B(\sigma_0) \rangle \right) \exp(iS_{\mathcal{CS}}^{\text{disc}}(A_{\mathcal{C}}^+, B)) (DA_{\mathcal{C}}^+ \otimes DB) \quad (5.18) \]

where we have set

\[ \text{Det}_{\text{disc}}^{\text{reg}}(B) := \text{Det}_{\mathcal{F} \mathcal{P}}^{\text{disc}}(B) Z_{B}^{\text{disc}} \quad (5.19) \]

5.2 Step 2: Performing the \( \int_{\sim} \cdots \exp(iS_{\mathcal{CS}}^{\text{disc}}(A_{\mathcal{C}}^+, B)) (DA_{\mathcal{C}}^+ \otimes DB) \)-integration in (5.18)

With the help of Proposition 4.12 above let us now evaluate the \( \int_{\sim} \cdots \exp(iS_{\mathcal{CS}}^{\text{disc}}(A_{\mathcal{C}}^+, B)) (DA_{\mathcal{C}}^+ \otimes DB) \)-integral appearing in Eq. (5.18). In order to do so we first rewrite the integrand in Eq. (5.18) in such a way that it assumes the form of the integrand on the LHS of the formula appearing in Proposition 4.12 above. In order to achieve this we will now exploit the fact that all of the remaining fields \( A_{\mathcal{C}}^+ \) and \( B \) in Eq. (5.18) take values in the Abelian Lie algebra \( \mathfrak{t} \). For fixed \( A_{\mathcal{C}}^+ \) and \( B \) we can therefore rewrite \( \text{Hol}_{\mathcal{R}_i}^{\text{disc}}(A_{\mathcal{C}}^+, B) \) as an exponential of a sum, namely as

\[ \text{Hol}_{\mathcal{R}_i}^{\text{disc}}(A_{\mathcal{C}}^+, B) = \exp(\Phi_i(B) + \sum_k A_{\mathcal{C}}^+(\frac{1}{2} l_i^{(k)} + \frac{1}{2} l_i^{(k)})) \quad (5.20) \]

(cf. Eq. (3.18) above) where we have set for each \( i \leq m \)

\[ \Phi_i(B) := \sum_k \left( \frac{1}{2} B(\bullet l_i^{(k)}) + \frac{1}{2} B(\bullet l_i^{(k)}) \right) \cdot dt(N)(l_i^{(k)}) \quad (5.21) \]

Moreover, since \( \text{Hol}_{\mathcal{R}_i}^{\text{disc}}(A_{\mathcal{C}}^+, B) \in T \) we can replace in Eq. (5.18) the characters \( \chi_i := \text{Tr}_{\rho_i}, \quad i \leq m \), by their restrictions \( \chi_i|_{\mathfrak{t}} \). But \( \chi_i|_{\mathfrak{t}} \) is just a linear combination of global weights, more precisely, for every \( b \in \mathfrak{t} \) we have

\[ \text{Tr}_{\rho_i}(\exp(b)) = \chi_i|_{\mathfrak{t}}(\exp(b)) = \sum_{\alpha \in \Lambda} m_{\chi_i}(\alpha) e^{2\pi i(\alpha, b)} \quad (5.22) \]

where \( m_{\chi_i}(\alpha) \) the multiplicity of \( \alpha \in \Lambda \) as a weight in \( \chi_i \) (here \( \Lambda \subset \mathfrak{t}^* \cong \mathfrak{t} \) denotes the lattice of the real weights associated to the pair \((\mathfrak{g}, \mathfrak{t})\), cf. part A of the Appendix below). Combining Eqs. (5.20) - (5.22) we obtain

\[ \prod_i \text{Tr}_{\rho_i} \left( \text{Hol}_{\mathcal{R}_i}^{\text{disc}}(A_{\mathcal{C}}^+, B) \right) = \prod_i \left( \sum_{\alpha \in \Lambda} m_{\chi_i}(\alpha_i) \cdot \exp(2\pi i(\alpha_i, \Phi_i(B))) \cdot \exp(2\pi i \sum_k (\alpha_i, A_{\mathcal{C}}^+(\frac{1}{2} l_i^{(k)} + \frac{1}{2} l_i^{(k)}))) \right) \]

\[ = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \Lambda} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \left( \prod_i \exp(2\pi i(\alpha_i, \Phi_i(B))) \right) \exp(2\pi i \ll A_{\mathcal{C}}^+, \sum_i \alpha_i \cdot l_i^{\Sigma} \gg A_{\mathcal{C}}^+(qK)) \quad (5.23) \]

\[^{37}\]here we are a bit sloppy and use the letter \( i \) both for the multiplication index and the imaginary unit
where we have set
\[ l_{\Sigma}^{(i)} := \sum_k \frac{1}{2} (l_{\Sigma}^{(i)(k)} + i l_{\Sigma}^{(i)(k)}) \in C_1(qK) \] (5.24)

Let us now set for each \( s > 0, y \in I, (\alpha_i)_i := (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Lambda^m, \) and \( B \in B(qK):\)

\[ F_{(\alpha_i)_i}(y) := (\prod_{x} \mathbb{1}_{t_{reg}}(B(x))) (\prod_{i} \exp(2\pi i \langle \alpha_i, \Phi_i(y) \rangle)) \text{Det}^{\text{disc}}(B) \]
\[ \times \exp(-2\pi ik(y, B(\sigma_0))) \] (5.25)

Then, according to Eq. (5.23), we can rewrite Eq. (5.18) as

\[ \text{WLO}^{\text{rig}}_{\text{rig}}(L) = \lim_{s \to 0} \sum_{(\alpha_i)_i \in \Lambda^m} (\prod_{i} m_{\chi_i}(\alpha_i)) \sum_{y \in I} \times \int_{\sim} F_{(\alpha_i)_i}(y) (2\pi i \ll A_{\Sigma}^{\perp}, \sum_{i} \alpha_i \cdot l_{\Sigma}^{(i)} \gg A_{\Lambda}^{\perp}(qK)) \exp(iS_{CS}^{\text{disc}}(A_{\Sigma}^{\perp}, B))(DA_{\Sigma}^{\perp} \otimes DB) \] (5.26)

(Observe that there are only finitely many \((\alpha_i)_i \in \Lambda^m\) for which the product \(\prod_i m_{\chi_i}(\alpha_i)\) does not vanish. Accordingly, the summation \(\sum_{(\alpha_i)_i \in \Lambda^m} (\prod_i m_{\chi_i}(\alpha_i)) \cdots\) above is a finite and we can interchange it with \(\sum_{y \in I}\).)

Let us now fix for a while \( s > 0, y \in I, \) and \((\alpha_i)_i \in \Lambda^m\) and evaluate the corresponding \(\int_{\sim} \cdots\)-integral in Eq. (5.26). In order to do so we will apply Proposition 4.12 above to the special situation where

\begin{itemize}
  \item \(V := A_{\Sigma}^{\perp}(K) \oplus B(qK)\) (cf. Convention 3 and Convention 6)
  \item \(d\mu := d\mu^{\text{disc}}\) where
    \[ d\mu^{\text{disc}} := \frac{1}{Z^{\text{disc}}} \exp(iS_{CS}^{\text{disc}}(A_{\Sigma}^{\perp}, B))(DA_{\Sigma}^{\perp} \otimes DB) \]
    \[ = \frac{1}{Z^{\text{disc}}} \exp(i \ll A_{\Sigma}^{\perp}, -2\pi k(\pi \circ d_{qK} \circ \pi) \gg A_{\Lambda}^{\perp}(qK))(DA_{\Sigma}^{\perp} \otimes DB) \]
    where we have set \(Z^{\text{disc}} := \int_{\sim} \exp(iS_{CS}^{\text{disc}}(A_{\Sigma}^{\perp}, B))(DA_{\Sigma}^{\perp} \otimes DB)\) and \(\pi : A_{\Sigma,1}(qK) \to A_{\Lambda,1}(qK) \cong A_{\Sigma}^{\perp}(K)\) is the orthogonal projection (cf. also Convention 2 above).
  \item \(V_1 := \ker(\pi \circ d_{qK})^{\perp} = \ker(\pi \circ d_{qK})^{\perp} \subset B(qK)\) where in (*) we used Eq. (5.33) in Sec. 3.1 above (cf. also Convention 6).
  \item \(V_2 := \text{Image}(\pi \circ d_{qK}) \subset A_{\Sigma}^{\perp}(K),\)
  \item \(V_0 := B_{qK} \oplus (V_2)^{\perp}\)
    where \((V_2)^{\perp}\) is the orthogonal complement of \(V_2\) in \(A_{\Sigma}^{\perp}(K)\) (cf. Convention 3).
  \item \(F := F_{(\alpha_i)_i}(y) \circ p\) where \(p : V_0 \oplus V_1 = B(qK) \oplus (V_2)^{\perp} \to B(qK)\) is the obvious projection.
  \item \(v := 2\pi \sum_{i} \alpha_i \cdot l_{\Sigma}^{(i)}\)
\end{itemize}

The following remarks show that the assumptions of Proposition 4.12 above are indeed fulfilled:

i) \(d\mu^{\text{disc}}\) is a normalized centered oscillatory Gauss type measure on \(A_{\Sigma}^{\perp}(K) \oplus B(qK)\) which has the form as in Proposition 4.12 with \(V_0, V_1, \) and \(V_2\) given as above and where \(M : V_1 \to V_2\) is the well-defined linear isomorphism given by
\[ M = -2\pi k(\pi \circ d_{qK})|_{V_1} \] (5.27)

ii) The function \(F\) is bounded and uniformly continuous.
iii) In order to see that \( v \) is an element of \( V_2 \) we consider the linear map \( m_{\mathbb{R}} : C^0(qK, \mathbb{R}) \rightarrow C^1(qK, \mathbb{R}) \) which is given by

\[
m_{\mathbb{R}} := \alpha_K \circ \pi \circ d_{qK}
\]
where \( d_{qK} : C^0(qK, \mathbb{R}) \rightarrow C^1(qK, \mathbb{R}) \), \( \pi : C^1(qK, \mathbb{R}) \rightarrow C^1(K, \mathbb{R}) \), and \( \alpha_K : C^1(K, \mathbb{R}) \rightarrow C^1(K, \mathbb{R}) \) are the “real analogues” of the three maps appearing on the RHS of Eq. (5.27) above. From Lemma 4.13 above and a “periodicity argument”, which will be given in Step 4 below

Apart from the remaining

\[
\sum_c (\mathbb{R}) = \sum_c (\mathbb{R}^a) \sum_c (\mathbb{R}^b)
\]

Applying Proposition 4.12 above to the present situation we therefore obtain

\[
\frac{1}{C_{\text{disc}}} \int \sim F_{(\alpha_i),y}^{(s)}(B) \exp(2\pi i \ll A_{c}^s, \sum \alpha_i \cdot \mathbb{I}_c \gg A_{qK}) \exp(iS_{CS}^{\text{disc}}(A_{c}^s, B)) (DA_{c}^s) \otimes DB)
\]

where the two improper integrals \( \sim \int \cdots dx_0 \) and \( \sim \int \cdots db \) are defined according to Convention 4 in Sec. 4 above. In Step (+) we have applied Proposition 4.12. Step (+) above follows because \( F(x_0 - M^{-1}v) \) depends only on the \( B_c(qK) \)-component of \( x_0 \in V_0 = B_c(qK) \oplus V_2^\perp \) and because \( B_c(qK) = \{ B \in B(qK) \mid B \text{ is constant } \} \subset t \).

Recall that \( f_i \) as introduced above is uniquely determined up to an additive constant. We can fix this constant by demanding that the normalization condition

\[
\sum_{x \in \tilde{\sigma}_0(qK)} f_i(x) = 0
\]

is fulfilled. With this normalization condition we obtain \( \sum_i \alpha_i \cdot f_i \in V_1 = (B_c(qK))^\perp \). From this and Eq. (5.29) above we see that

\[
M^{-1}v = -\frac{1}{\mathcal{K}} \sum_i \alpha_i \cdot f_i
\]

Combining Eqs. (5.25), (5.26), (5.30), and (5.32) we obtain

\[
\text{WLO}_{\text{rig}}^\text{disc} (L) \sim \lim_{s \rightarrow 0} \sum_{(\alpha_i), \in \Lambda} \sum_{y \in I} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \times \int_{\sim} \cdots db \left[ \exp(-2\pi i k(y, B(\sigma_0))) \left( \prod_{x} 1^{(s)}_{\nu_{xy}}(B(x)) \right) \times \left( \prod_i \exp(2\pi i \langle \alpha_i, \Phi_i(B) \rangle) \right) \right]_{B=b+\frac{1}{\mathcal{K}} \sum_i \alpha_i f_i}
\]

Apart from the remaining \( \int \cdots db \)-integration (which will be taken care of in Step 4 below) we have now completed the evaluation of the \( \int \cdots \exp(iS_{CS}^{\text{disc}}(A_{c}^s, B)) (DA_{c}^s) \otimes DB \)-integral in Eq. (5.18).
Remark 5.2 It is not difficult to see that Eq. (5.33) also holds if we (re)define \( f_i \) using the following normalization condition (instead of the normalization condition (5.31) above):

\[
f_i(\sigma_0) = 0
\]  

(5.34)

Since condition (5.34) is technically more convenient than (5.31) we will use the latter normalization condition in the following, i.e. we assume that \( f_i \) is defined as in (5.28) above in combination with (5.34).

5.3 Step 3: Some simplifications

The next lemma will prove a claim made in Sec. 5.2 above and it will also allow us to simplify the RHS of Eq. (5.33) above.

Lemma 3 Assume that \( \tilde{l}_i \in C_1(qK) \cong C^1(qK, \mathbb{R}) \) as in Eq. (5.24) above. Then we have:

i) There is a \( f \in C^0_{aff}(qK, \mathbb{R}) \) such that \( f_i = m_{\mathbb{R}} \cdot f \) and such that \( f \) is constant on \( U(\sigma_0) \cap \mathfrak{F}_0(qK) \) (cf. Eq. (3.10) above). Moreover, these properties determine \( f \) uniquely up to an additive constant.

ii) If \( f \) is as in part i) of the present lemma then the map \( f : \mathfrak{F}_0(qK) \ni \sigma \mapsto f(\sigma) \in \mathbb{R} \) is constant on \( \text{arc}(\tilde{l}_i) \cap \mathfrak{F}_0(qK) \) and on \( \text{arc}(\tilde{l}_i') \cap \mathfrak{F}_0(qK) \) for all \( j \leq m \).

Proof. From conditions (NCP)' and (NH)' on the simplicial ribbon link \( L \) it follows that there are three connected components \( C_0, C_1 \) and \( C_2 \) of \( \Sigma \setminus (\text{arc}(\tilde{l}_i) \cup \text{arc}(\tilde{l}_i')) \). In the following we assume that these three connected components are given as in Fig. 1 below.

![Figure 1](image)

It follows from Condition (FC2) in Sec. 5.1 that \( \mathfrak{F}_0(qK) \cap C_0 = \emptyset \), or, in other words, that \( \mathfrak{F}_0(qK) \subset \overline{C_1} \cup \overline{C_1} \). Accordingly, the map \( f : \mathfrak{F}_0(qK) \to \mathbb{R} \) given by

\[
f(p) := \begin{cases} 
  c & \text{if } p \in \overline{C_1} \\
  c \pm 1 & \text{if } p \in \overline{C_2}
\end{cases}
\]  

(5.35)

for all \( p \in \mathfrak{F}_0(qK) \) is well-defined. Here \( c \in \mathbb{R} \) is an arbitrary constant which will be kept fixed in the following and the sign \( \pm \) is “+” if for any \( t \in [0, 1] \) in which \( \tilde{l}_i \) is differentiable the normal vector \( n = \ast(\frac{d}{dt}\tilde{l}_i(t)) \) “points into” \( C_0 \) and “−” otherwise.

---

Footnote: this follows from Eq. (4.27) in Remark 4.13 above and the periodicity properties of the integrand in \( \int_{\cdots} db \) (for fixed \( y \) and \( \alpha_1, \ldots, \alpha_m \)), cf. Step 4 below.
Let us first verify that \( f \in C^0_{\text{aff}}(qK, \mathbb{R}) \) and let \( p_1, p_2, p_3, p_4 \) be the four vertices of \( F \) enumerated such that \( p_1 \) is diagonal to \( p_4 \) (and therefore \( p_2 \) is diagonal to \( p_3 \)). We have to show that \( f(p_1) + f(p_4) = f(p_2) + f(p_3) \) (cf. Eq. (6.11)). If \( F \subset C_1 \) or \( F \subset C_2 \) this is obvious. If \( F \subset C_0 \) (for example \( F = F_0 \) where \( F_0 \) is as in Fig. 1 above) then Condition (FC3) implies that \( f(p_1) + f(p_4) = c + (c \pm 1) = f(p_2) + f(p_3) \).

Recall from Sec. [3.3] that \( \sigma_0 \in \mathcal{F}_0(qK) \) was chosen such that \( \sigma_0 \notin \text{Image}(R_{\Sigma}^i) \). This implies that \( \sigma_0 \in C_1 \) or \( \sigma_0 \in C_2 \) and from the definition of \( U(\sigma_0) \) we obtain \( U(\sigma_0) \subset C_1 \) or \( U(\sigma_0) \subset C_2 \) so Eq. (5.35) implies that \( f \) is constant on \( U(\sigma_0) \cap \mathcal{F}_0(qK) \).

The uniqueness part of the assertion follows by combining the definition of \( m_{K} \) with the real analogue of Eq. (3.8) in Sec. [3] above and the fact that \( \ast_K : C^1(K, \mathbb{R}) \rightarrow C^1(K, \mathbb{R}) \) is a bijection.

In order to conclude the proof of part i) of Lemma 3 we have to show that

\[
I_\Sigma = m_{K} \cdot f = \ast_K(\pi(d_{qK} f)) \tag{5.36}
\]

Observe first that \( (d_K f)(e) = 0 \) unless the interior of \( e \in \mathcal{F}_1(qK) \) is contained in \( C_0 \). In the latter case we have \( (d_K f)(e) = \text{sgn}(e) \) where \( \text{sgn}(e) = 1 \) if the (oriented) edge \( e \) “points from” the region \( C_1 \) to the region \( C_2 \) and \( \text{sgn}(e) = -1 \) otherwise.

Next observe that for every \( e \in \mathcal{F}_1(qK) \) whose interior is contained in \( C_0 \) there exists an index \( k \leq n \) such that 40

\[
\psi(e) := \ast_K(\pi(\text{sgn}(e) \cdot e)) = \begin{cases} f_{i_\Sigma}^{i(k)} & \text{if } e \text{ has an endpoint in } \mathcal{F}_0(K_2) \\ f_{\bar{i}_\Sigma}^{i(k)} & \text{if } e \text{ has an endpoint in } \mathcal{F}_0(K_1) \end{cases} \tag{5.37}
\]

and, in fact, the map 41

\[
\psi : \{ e \in \mathcal{F}_1(qK) \mid \text{the interior of } e \text{ is contained in } C_0 \} \rightarrow (\{ f_{i_\Sigma}^{i(k)} \mid k \leq n \} \cup \{ f_{\bar{i}_\Sigma}^{i(k)} \mid k \leq n \}) \setminus \{0\}
\]

is a bijection.

Finally, observe that the sum of the elements of the finite subset \( \{ f_{i_\Sigma}^{i(k)} \mid k \leq n \} \cup \{ f_{\bar{i}_\Sigma}^{i(k)} \mid k \leq n \} \) of \( C_1(K) \subset C_1(qK) \) equals \( I_\Sigma \), cf. Eq. (5.24) above. From this Eq. (5.36) follows.

It remains to show part ii) of Lemma 3. Recall that according to Eq. (5.35), \( f \) is constant on \( \mathcal{F}_0(qK) \cap C_1 \) and also on \( \mathcal{F}_0(qK) \cap C_2 \). From condition (FC1) and condition (FC2) it follows that for each \( j \neq i \) the set \( \text{arc}(l_{\Sigma}^{i}) \) (considered as a subset of \( \Sigma \)) either lies entirely in \( C_1 \) or entirely in \( C_2 \). Similarly, it follows that for each \( j \neq i \) the set \( \text{arc}(l_{\Sigma}^{j}) \) either lies entirely in \( C_1 \) or in \( C_2 \). Since we also have \( \text{arc}(l_{\Sigma}^{i}) = \partial C_2 \subset C_2 \) and \( \text{arc}(l_{\Sigma}^{j}) = \partial C_1 \subset C_1 \) part ii) of Lemma 3 now follows.

\[\square\]

**Corollary 5.3** Let \( B \in \mathcal{B}(qK) \) be of the form

\[
B = b + \frac{1}{k} \sum_{i=1}^{m} \alpha_i f_i \tag{5.38}
\]

with \( b \in t \), \( \alpha_i \in \Lambda \) and where \( f_i \) is given by Eq. (5.28) above in combination with (5.34). Then the map \( \mathcal{F}_0(qK) \ni \sigma \rightarrow B(\sigma) \in t \) is constant on \( \text{arc}(l_{\Sigma}^{i}) \cap \mathcal{F}_0(qK) \) and on \( \text{arc}(l_{\Sigma}^{j}) \cap \mathcal{F}_0(qK) \) for all \( j \leq m \).

40 from the definition of \( qK \) it follows that for every edge \( e \) in \( qK \) exactly one endpoint is in \( \mathcal{F}_0(K_1|K_2) \) and the other endpoint is either in \( \mathcal{F}_0(K_1) \) or in \( \mathcal{F}_0(K_2) \).

41 here \( 0 \) is the zero element of \( C_1(K) \).
Let us set

\[ \sigma_i := \bullet l_{\Sigma}^{i(1)} \in \mathcal{F}_0(q \mathcal{K}), \quad \sigma'_i := \bullet l_{\Sigma}^{i(1)} \in \mathcal{F}_0(q \mathcal{K}) \]  \tag{5.39}

According to Corollary 5.3 we have

\[ B(\sigma_i) = B(\bullet l_{\Sigma}^{i(k)}) \quad \forall k \leq n \]  \tag{5.40a}

\[ B(\sigma'_i) = B(\bullet l_{\Sigma}^{i(k)}) \quad \forall k \leq n \]  \tag{5.40b}

for every \( B \) of the form in Eq. 5.38. Next observe that

\[ \epsilon_i := \text{wind}(l_{\Sigma}^{i}) = \sum_k dt(N)(l_{\Sigma}^{i(k)}) \]  \tag{5.41}

where wind\( (l_{\Sigma}^{i}) \) is the winding number of \( l_{\Sigma}^{i} \).

Combining Eq. (5.33) and Eq. (5.21) with Eqs. (5.40a) – (5.41) and taking into account the normalization condition (5.44) appearing at the end of Sec. 5.2 we obtain

\[
\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i) \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \sum_{y \in I} \right.
\]

\[
\times \left( \int_t \sum_{\gamma_{reg}} \exp(-2\pi ik\langle y, b \rangle) \left( \prod_x 1^{(s)}(\gamma_{reg}(B(x))) \right) \right.
\]

\[
\times \left( \prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \right) \left. \right|_{B=b+\frac{1}{\mathcal{K}} \sum_i \alpha_i f_i} \]  \tag{5.42}

Setting

\[
F_{(\alpha_i)}^{(s)}(b) := \left[ \left( \prod_x 1^{(s)}(\gamma_{reg}(B(x))) \right) \right.
\]

\[
\times \left( \prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \right) \left. \right|_{B=b+\frac{1}{\mathcal{K}} \sum_i \alpha_i f_i} \]  \tag{5.43}

we can rewrite Eq. (5.42) as

\[
\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i) \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \sum_{y \in I} \int_t \sum_{\gamma_{reg}} e^{-2\pi ik\langle y, b \rangle} F_{(\alpha_i)}^{(s)}(b) \right) \]  \tag{5.44}

### 5.4 Step 4: Performing the remaining limit procedures \( \int_t \cdots \), \( \sum_{y \in I} \), and \( s \to 0 \) in Eq. (5.44)

i) Let us first rewrite the \( \int_t \cdots \) integral. The crucial observation is that for fixed \( y \in I, s > 0, \) and \( (\alpha_i) \in \Lambda^m \) the function \( t \mapsto e^{-2\pi ik\langle y, b \rangle} F_{(\alpha_i)}^{(s)}(b) \in \mathbb{C} \) is invariant under all translations of the form \( b \mapsto b + x \) where \( x \in I = \ker(\exp|_t) \cong \mathbb{Z}^{\dim(t)} \).

Indeed, for all \( b \in t \) and \( x \in I \) we have

\[ 1^{(s)}_{\gamma_{reg}}(b + x) = 1^{(s)}_{\gamma_{reg}}(b) \]  \tag{5.45a}

\[ e^{2\pi i \epsilon(\alpha, b+x)} = e^{2\pi i \epsilon(\alpha, b)} \quad \text{for all } \alpha \in \Lambda, \epsilon \in \mathbb{Z} \]  \tag{5.45b}

\[ \det(1_t - \exp(ad(b + x))|_t) = \det(1_t - \exp(ad(b))|_t) \]  \tag{5.45c}

\[ e^{-2\pi ik\langle y, b+x \rangle} = e^{-2\pi ik\langle y, b \rangle} \quad \text{for all } y \in I \]  \tag{5.45d}

The first of these four equations follows from the assumptions in Sec. 3.6. The second equation follows because the assumption that \( G \) is simply-connected implies that

\[ I = \Gamma \]  \tag{5.46}
where \( \Gamma \subset \mathfrak{t} \) is the lattice generated by the real coroots and, by definition, \( \Lambda \) is the lattice dual to \( \Gamma \). The third equation follows from the second equation and the relations \( \mathcal{R} \subset \Lambda \) and \( \mathbb{R} \)

\[
\det \left( 1_t - \exp(\text{ad}(b))t \right) = \prod_{\alpha \in \mathcal{R}} (1 - e^{2\pi i \langle \alpha, b \rangle})
\]

where \( \mathcal{R} \) is the set of real roots of \((\mathfrak{g}, \mathfrak{t})\). Finally, in order to see that the fourth equation holds, observe that because of (5.46) it is enough to show that

\[
\langle \hat{\alpha}, \hat{\beta} \rangle \in \mathbb{Z} \quad \text{for all coroots } \hat{\alpha}, \hat{\beta}
\]

(5.47)

According to the general theory of semi-simple Lie algebras the entries of the so-called “Cartan matrix” are integers, i.e. \( 2 \frac{\langle \hat{\alpha}, \hat{\beta} \rangle}{\langle \hat{\alpha}, \hat{\alpha} \rangle} \in \mathbb{Z} \). Moreover, there are at most two different (co)roots lengths and the quotient between the square lengths of the long and short coroots is either 1, 2, or 3. Since the normalization of \( \langle \cdot, \cdot \rangle \) was chosen such that \( \langle \hat{\alpha}, \hat{\alpha} \rangle = 2 \) holds if \( \hat{\alpha} \) is a short coroot we therefore have \( \langle \hat{\alpha}, \hat{\alpha} \rangle / 2 \in \{1, 2, 3\} \) and (5.47) follows.

From Eqs. (5.43) and (5.45) (cf. also Eqs. (5.19), (3.21), and (5.12)) we conclude that \( t \ni b \mapsto e^{-2\pi i k(y, b)} F_{(\alpha_i)}^{(s)}(b) \subset \mathbb{C} \) is indeed \( I \)-periodic and we can therefore apply Eq. (4.26) in Remark 4.13 above and obtain

\[
\int \cdots \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} e^{-2\pi i k(y, b)} F_{(\alpha_i)}^{(s)}(b) \sim \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} e^{-2\pi i k(y, b)} F_{(\alpha_i)}^{(s)}(b)
\]

(5.48)

where on the RHS \( \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \) is now an ordinary integral and where we have set

\[
Q := \left\{ \sum_i \lambda_i e_i \mid \lambda_i \in (0, 1) \text{ for all } i \leq m \right\} \subset \mathfrak{t},
\]

(5.49)

Here \( (e_i)_{i \leq m} \) is an (arbitrary) fixed basis of \( I \).

According to Eq. (5.48) we can now rewrite Eq. (5.44) as

\[
\text{WLO}^{\text{disc}}_{\text{rig}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i)_{i \in \Lambda^m}} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{y \in I} \int_{\mathbb{R}} e^{-2\pi i k(y, b)} F_{(\alpha_i)}^{(s)}(b)
\]

(5.50)

ii) We can now perform the infinite sum \( \sum_y \) and the \( \int \cdots \int_{\mathbb{R}} \) in Eq. (5.50):

First recall that, due to Eq. (5.40) above and the definition of \( \Lambda, \Lambda \) is dual to \( I \). According to the (rigorous) Poisson summation formula for distributions we therefore have

\[
\sum_{y \in I} e^{-2\pi i k(y, b)} = c_{\Lambda} \sum_{x \in \mathbb{R}} \delta_x(b)
\]

(5.51)

where \( \delta_x \) is the delta distribution in \( x \in \mathfrak{t} \) and \( c_{\Lambda} \) a constant depending on the lattice \( \Lambda \). Let us now apply Eq. (5.51) to the RHS of Eq. (5.50) above. In order to see that this is possible note first that not only \( F_{(\alpha_i)}^{(s)}(b) \) is smooth but also the product \( 1_Q F_{(\alpha_i)}^{(s)} \) because \( \partial Q \subset \mathfrak{t} \backslash \mathfrak{t}_{\text{reg}} \) and because \( F_{(\alpha_i)}^{(s)} \) vanishes on an open neighborhood of the set \( \mathfrak{t} \backslash \mathfrak{t}_{\text{reg}} \) (cf. the condition supp\(1_{\text{reg}}^{(s)}(b) \subset \mathfrak{t}_{\text{reg}} \) in Sec. 3.6) so, according to the definition of \( F_{(\alpha_i)}^{(s)} \) and Eq. (5.54), there is a factor \( 1_{\text{reg}}^{(s)}(b) \) appearing in \( F_{(\alpha_i)}^{(s)}(b) \). Moreover, since \( Q \) is bounded \( 1_Q F_{(\alpha_i)}^{(s)} \) has compact support. Thus we can indeed apply Eq. (5.51) to the RHS of Eq. (5.50) above and we then obtain

\[
\text{WLO}^{\text{disc}}_{\text{rig}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i)_{i \in \Lambda^m}} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{b \in \mathbb{R}} 1_{\Lambda}(b) 1_Q(b) F_{(\alpha_i)}^{(s)}(b)
\]

(5.52)

\[\text{observe that in contrast to the proof of Lemma 4.1 in Sec. 5.0 above we now work with the set } \mathcal{R} \text{ of real roots instead of the set of complex roots } \mathcal{R}_c = \{2\pi i \alpha \mid \alpha \in \mathbb{R}\}\]
iii) Finally, let us also perform the \( s \to 0 \) limit. Taking into account that \( 1_{\text{reg}}(x) \to 1_{\text{reg}} \) pointwise we obtain from Eq. (5.52) and Eq. (5.43) after the change of variable \( b \to kb =: \alpha_0 \)

\[
\text{WLO}_{\text{rig}} (L) \sim \sum_{\alpha_0, \alpha_1, \ldots, \alpha_m \in \Lambda} \exp \left( \alpha_0 \left( \prod_{i=1}^{m} m_{i}^\alpha \right) \right) \\
\times \left[ \left( \prod_{x} 1_{\text{reg}}(B(x)) \right) \prod_{j=1}^{m} \exp \left( \pi i e_j(x_j, B(\sigma_j) + B(\sigma_j')) \right) \right] \det_{\text{disc}} (B) \bigg|_{B = \frac{1}{k} (\alpha_0 + \sum_i \alpha_i f_i)}
\]

(5.53)

### 5.5 Step 5: Rewriting \( \det_{\text{disc}} (B) \) in Eq. (5.53)

In Steps 1–4 we have reduced the original “path integral” expression for \( \text{WLO}_{\text{rig}} (L) \) to a “combinatorial expression”, i.e. an expression which does not involve any limit procedure. Let us now have a closer look at \( \det_{\text{disc}} (B) \), cf. Eq. (5.19) in Step 1 above.

From assumptions (NCP)' and (NH)' above it follows that the set of connected components of \( \Sigma \setminus \left( \bigcup_j \text{arc}(L_j^\Sigma) \right) \) has exactly \( m + 1 \) elements, which we will denote by \( Y_0, Y_1, \ldots, Y_m \) in the following. Moreover, it follows that also the set of connected components of \( \Sigma \setminus \bigcup_j \text{Image}(R^j_\Sigma) = \Sigma \setminus \bigcup_j (O_j \cup \text{arc}(L_j^\Sigma) \cup \text{arc}(L_j^\Sigma)) \) has exactly \( m + 1 \) elements, which we will denote by \( Z_0, Z_1, \ldots, Z_m \).

(Here \( O_j \subset \Sigma, j \leq m \), denotes the open “region between” \( \text{arc}(L_j^\Sigma) \) and \( \text{arc}(L_j^\Sigma) \), cf. condition (FC2) in Sec. 5.0 above.)

In the following we assume without loss of generality that the numeration of the \( Z_i, i \leq m \), was chosen such that

\[
Z_i \subset Y_i \quad \forall i \in \{0, 1, \ldots, m\}
\]

holds. For later use we remark that \( \{Z_i | 0 \leq i \leq m\} \cup \{O_i | i \leq m\} \), is a partition of \( \Sigma \). Thus condition (FC2) implies that

\[
\mathfrak{F}_0 (qK) = \bigcup_{i=0}^{m} \left( \mathfrak{F}_0 (qK) \cap Z_i \right)
\]

(5.54)

Observe also that

\[
\mathfrak{F}_0 (qK) = \mathfrak{F}_0 (K_1) \sqcup \mathfrak{F}_0 (K_1 \setminus K_2) \cup \mathfrak{F}_0 (K_2)
\]

(5.55)

(Here and in the following we use the notation \( S \sqcup T \) for the disjoint union of two sets \( S \) and \( T \)).

In the following we assume that \( B \in \mathcal{B}(qK) \) is of the form

\[
B = \frac{1}{k} (\alpha_0 + \sum_i \alpha_i f_i), \quad \text{with } \alpha_0, \ldots, \alpha_m \in \Lambda
\]

(5.56)

and with \( f_i \) as in Eq. (5.28) above in combination with (5.34).

**Lemma 4** If \( B \in \mathcal{B}(qK) \) is of the form (5.56) then the restriction of \( B : \mathfrak{F}_0 (qK) \to t \) to \( \mathfrak{F}_0 (qK) \cap Z_i \) is constant for each \( i \).

**Proof.** This lemma is a generalization of Corollary 5.3 in Sec. 5.3 above and follows easily from (the proof of) Lemma 5 above.

In the following we set \( B(Y_i) := B(Z_i) \) for any \( x \in Z_i \cap \mathfrak{F}_0 (qK) \) (according to Lemma 4 the value of \( B(Y_i) = B(Z_i) \) does not depend on the choice of \( x \in Z_i \subset Y_i \)).

**Lemma 5** For every \( B \in \mathcal{B}(qK) \) of the form (5.56) and fulfilling \( \prod_x 1_{\text{reg}} (B(x)) \neq 0 \) we have

\[
\det_{\text{disc}} (B) \sim \prod_{i=0}^{m} \det (1_t - \exp (\text{ad}(B(Y_i))))^{\chi (Y_i) / 2}
\]

(5.57)

where \( \chi (Y_i) \) is the Euler characteristic of \( Y_i \).

\(^{43}\)here \( Z_i \) denotes the closure of \( Z_i \).
Proof. From the definition of $\text{Det}^{\text{disc}}(B)$ in \eqref{definition:det_disc} and Eqs. \eqref{eq:det_disc} and \eqref{eq:det_disc_greater_0} in Sec. 8 above it follows that

$$\text{Det}^{\text{disc}}(B) \sim \left( \frac{\prod_{x \in \mathfrak{F}(K_1)} \det(1_t - \exp(\text{ad}(B(x))))|_{t} \prod_{x \in \mathfrak{F}(K_2)} \det(1_t - \exp(\text{ad}(B(x))))|_{t}}{\prod_{x \in \mathfrak{F}(K_1) \cap K_2} \det(1_t - \exp(\text{ad}(B(x))))|_{t}} \right)^{1/2}$$

(\text{5.58})

(observe that the expression on the RHS is well-defined since by assumption $\prod_x 1_{\text{tr}reg}(B(x)) \neq 0$, which implies that the denominator is non-zero).

According to Lemma 4 and Eqs. \eqref{eq:det_disc} and \eqref{eq:det_disc_greater_0} it is enough to prove that for each $i \in \{0, 1, \ldots, m\}$ we have

$$\chi(Y_i) = \#(\mathfrak{F}_0(K_1) \cap \overline{Z}_i) - \#(\mathfrak{F}_0(K_1|K_2) \cap \overline{Z}_i) + \#(\mathfrak{F}_0(K_2) \cap \overline{Z}_i)$$

(\text{5.59})

Clearly, $\chi(Y_i) = \chi(Y_i)$ where $Y_i$ is the closures of $Y_i$. Moreover, $Y_i$ is a subcomplex of the CW complex $K_1 = K = (\Sigma, C)$ so setting $\text{Cell}_p(Y_i) := \{ \sigma \in \text{Cell}_p(K_1) \mid \sigma \subset Y_i \}$ where $\text{Cell}_p(K_1)$ is the set of (open) p-cells of $K_1$ we obtain

$$\chi(Y_i) = \sum_{p=0}^{2} (-1)^p \# \text{Cell}_p(Y_i)^{(s)} = \#(\mathfrak{F}_0(K_1) \cap Y_i) - \#(\mathfrak{F}_0(K_1|K_2) \cap Y_i) + \#(\mathfrak{F}_0(K_2) \cap Y_i)$$

(\text{5.60})

(step (*) follows by taking into account the natural 1-1-correspondences $\text{Cell}_0(K_1) \leftrightarrow \mathfrak{F}_0(K_1)$, $\text{Cell}_1(K_1) \leftrightarrow \mathfrak{F}_0(K_1|K_2)$, and $\text{Cell}_2(K_1) \leftrightarrow \mathfrak{F}_0(K_2)$).

In order to complete the proof of Lemma 5 it is therefore enough to show that the RHS of Eq. \eqref{eq:det_disc} and the RHS of Eq. \eqref{eq:det_disc_greater_0} coincide.

In order to see this observe that for each $0 \leq i \leq m$ there is $J \subset \{0, 1, \ldots, m\}$ such that

$$Y_i = \overline{Z}_i \cup \bigcup_{j \in J} (O_j \cup \text{arc}(l_{j}'_{Z}))$$

so our claim follows from

$$\#(\mathfrak{F}_0(K_1) \cap \text{arc}(l_{Z}')_{J}) - \#(\mathfrak{F}_0(K_1|K_2) \cap \text{arc}(l_{Z}')_{J}) + \#(\mathfrak{F}_0(K_2) \cap \text{arc}(l_{Z}')_{J})$$

$$= -\#(\mathfrak{F}_0(K_1|K_2) \cap \text{arc}(l_{Z}')_{J}) + \#(\mathfrak{F}_0(K_2) \cap \text{arc}(l_{Z}')_{J}) = 0$$

(\text{5.61a})

and from

$$\#(\mathfrak{F}_0(K_1) \cap O_J) - \#(\mathfrak{F}_0(K_1|K_2) \cap O_J) + \#(\mathfrak{F}_0(K_2) \cap O_J) = \#\emptyset - \#\emptyset + \#\emptyset = 0$$

(\text{5.61b})

(cf. condition (FC2) and Eq. \eqref{eq:det_disc}).

\[ \square \]

**Lemma 6** For every $B \in \mathcal{B}(qK)$ of the form \eqref{eq:det_disc}, we have

$$\prod_{x \in \mathfrak{F}(qK)} 1_{\text{tr}reg}(B(x)) = \prod_{i=0}^{m} 1_{\text{tr}reg}(B(Y_i))$$

(\text{5.62})

**Proof.** The assertion follows from Lemma 4 and Eq. \eqref{eq:det_disc} above.

\[ \square \]

Combining Eq. \eqref{eq:disc_wlo} with Lemma 5 and Lemma 6 we arrive at

$$\text{WLO}^{\text{disc}}(L) \sim \sum_{\alpha_0, \alpha_1, \ldots, \alpha_m} 1_{kq(\alpha_0)}(\prod_{i=1}^{m} m_{\chi_i}(\alpha_i))$$

$$\times \left[ \prod_{i=0}^{m} 1_{\text{tr}reg}(B(Y_i)) \det(1_t - \exp(\text{ad}(B(Y_i)))|_{t}) \chi(Y_i)/2 \right]$$

$$\times \prod_{j=1}^{m} \exp(\pi i e_j(\alpha_j, B(\sigma_j) + B(\sigma_j')))$$

(\text{5.63})
Remark 5.4 What would happen if we had worked with “full ribbons” $\bar{R}_j$ instead of “half ribbons” $R_j$ (cf. Remark 3.2 and Remark 3.5 above)? In this case the RHS of Eq. (7.24) above and therefore also the elements $f_i$ of $C^0(gk, \mathbb{R})$ given by Eq. (5.28) would have different values, which would make it necessary to redefine the sets $Z_i$ appearing above in a suitable way.\footnote{recall that the “old” sets $Z_i$ are the connected components of $\Sigma \setminus \bigcup \text{Image}(R_{ij})$ where each $R_{ij}$ is the (reduced) projection of the “half ribbon” $\bar{R}_j$ in $gk \times \mathbb{Z}_0$. The “new” sets $\bar{Z}_i$ will be the connected components of $\Sigma \setminus \bigcup \text{Image}(\bar{R}_{ij})$ where each $\bar{R}_{ij}$ is the (reduced) projection of the “full ribbon” $\bar{R}_j$ in $k \times \mathbb{Z}_0$}

i) Lemma 6 above would then still be true. In fact, the proof would be simpler. Even more importantly, the structure of the proof of the new version of Lemma 6 (or rather, its BF-theoretic analogue) is exactly what is needed when trying to obtain a result like Eq. (7.25) below\footnote{recall that we do not expect that Theorem 4.4 can be generalized successfully to the case of general ribbon links, cf. the beginning of Sec. 7.1 below} for general ribbon links.

ii) On the other hand, for the “new” definition of the sets $Z_i$ mentioned above, Eq. (5.51) would no longer be true and Lemma 6 would no longer hold unless we insert additional indicator functions on the RHS of Eq. (5.62). It is still possible that – by exploiting suitable algebraic identities – one can recover Eq. (5.63) after all (in spite of the additional indicator functions appearing in Eq. (5.62)). If Eq. (5.63) cannot be recovered, which is likely, then one can bypass this complication by using an additional regularization procedure. Since at the moment it is not clear whether such an additional regularization procedure is indeed necessary or not we decided to work only with half ribbons until now. We will begin to work with full ribbons in Sec. 7 below.

5.6 Step 6: Comparison of $\text{WLO}_{\text{rig}}(L)$ with the shadow invariant $|L|$

From the computations in Sec. 5 in \cite{iwaniec} it follows that the RHS of Eq. (5.63) above coincides with the shadow invariant $|L|$ (associated to $g$ and $k$) up to a multiplicative constant. For the convenience of the reader we will briefly sketch this derivation. In the following we will use the notation of part B of the Appendix.

For $\alpha_1, \ldots, \alpha_m \in \Lambda$ and $\alpha_0 \in \Lambda \cap kQ$ set

$$B := \frac{1}{k}(\alpha_0 + \sum_i \alpha_if_i)$$ (5.64)

and introduce the function $\varphi : \{Y_0, Y_1, \ldots, Y_m\} \to \Lambda$ by

$$\varphi(Y) := kB(Y) - \rho \quad \forall Y \in \{Y_0, Y_1, \ldots, Y_m\}$$ (5.65)

One can show that then (cf. Sec. 5 in \cite{iwaniec})

$$\det(1_t - \exp(\text{ad}(B(Y)))|_t) \sim \dim(\varphi(Y))^2$$ (5.66a)

$$\prod_j \exp(\pi i Y_j \cdot \langle \alpha_j, B(\sigma_j) - B(\sigma_j') \rangle) = \prod_Y \exp(\pi i \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)\text{gleam}(Y)$$ (5.66b)

Let $P$ be the unique Weyl alcove which is contained in the Weyl chamber $\mathcal{C}$ fixed in Sec. 2.4 and which has $0 \in t$ on its boundary. Moreover, let $\Lambda_k$ and $W_k \cong W_{\text{aff}}$ be as in part B of the Appendix and let $\text{col}(L) = \Lambda_+ = \{Y_0, Y_1, \ldots, Y_m\}$ (the set of “area colorings”). From the relation $\Lambda_+ = \Lambda \cap (pk - \rho)$, the bijectivity of the map $\theta : P \times W_{\text{aff}} \ni (b, \sigma) \mapsto (b, \sigma) \in t_{\text{reg}}$ and the fact that for a suitable finite subset $W$ of $W_{\text{aff}}(\cong W_k)$ we have $\theta(P \times W) = Q \cap t_{\text{reg}}$ it follows that there is a natural 1-1-correspondence between the set $\text{col}(L) \times W \ni \{Y_1, \ldots, Y_m\}$ and the set of those $B$ which are of the form in Eq. (5.64) above (with $\alpha_0 \in \Lambda \cap kQ$ and $\alpha_1, \ldots, \alpha_m \in \Lambda$) and which have the extra property that $\prod_Y 1_{t_{\text{reg}}}(B(Y)) = 1$.\footnote{recall that we do not expect that Theorem 4.4 can be generalized successfully to the case of general ribbon links, cf. the beginning of Sec. 7.1 below}
Using this and Eq. (B.7) below plus a suitable symmetry argument based on the group $W_k$ (cf. the proof of Theorem 5.1 in [18]) one then arrives at

$$WLO_{rig}^\text{disc}(L) \sim \sum_{\varphi \in \text{cod}(L)} \left( \prod_{i} N_{\varphi(Y_i)}^{\varphi(Y_i)} \gamma'(l_i) \right)$$

$$\times \left( \prod_{Y} \dim(\varphi(Y)) \gamma(Y) \exp(\frac{4i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle) \gamma(\text{gleam}(Y)) \right) = |L| \quad (5.67)$$

If we apply Eq. (5.67) to the empty link $\emptyset$ instead of $L$ and take into account that $WLO_{rig}^\text{disc}(\emptyset) \neq 0$ if $k \geq c_0$ and that the symbol \( \sim \) denotes equality up to a multiplicative non-zero constant independent of $L$ we see that also $WLO_{rig}^\text{disc}(\emptyset) \neq 0$ for $k \geq c_0$. Accordingly, $WLO_{rig}(L)$ is then well-defined and Eq. (5.67) implies that indeed

$$WLO_{rig}(L) = \frac{WLO_{rig}^\text{disc}(L)}{WLO_{rig}^\text{disc}(\emptyset)} = \frac{|L|}{|\emptyset|}$$

6 Some comments regarding general simplicial ribbon links

Let us make some comments regarding the question if it is possible to generalize the computations above to general simplicial ribbon links. The crucial step will be the evaluation of the integral

$$\int \prod_{i=1}^{m} \text{Tr}_{\rho_i}(\text{Hol}^\text{disc}_{K_i}(A_{c_1}^\perp + A_{c_2}^\perp, B)) \exp(iS^\text{disc}_{CS}(A_{c_1}^\perp, B)) D\tilde{A}^\perp$$

for given $B$ and $A_{c_1}^\perp$, cf. Eq. (5.9) above. For general simplicial ribbon links this is considerably more difficult than in Sec. 5.1 above. The good news is that the evaluation of the integral (6.1) can be reduced to the computation of the “2-clusters”

$$\int \prod_{i=1}^{m} \{ \rho_i(\exp(\sum_{a} T_a Y_{i,a}^{i,a})) \sigma_i(\exp(\sum_{a'} T_{a'} Y_{i,a'}^{i,a'})) \} \exp(iS^\text{disc}_{CS}(A_{c_1}^\perp, B)) D\tilde{A}^\perp \in \text{End}(V_i) \otimes \text{End}(V_i')$$

for the few $i, i' \leq m$ and $k, k' \leq n$ for which $\ast K_l^{\Sigma}(k) = \pm l^{\Sigma}(k')$. Here $Y_{i,a}^{i,a}$ and $Y_{i,a'}^{i,a'}$ are as in Eq. (5.10) above and $V_i, V_i'$ are the representation spaces of $\rho_i$ and $\rho_i'$, cf. Sec. 5.1. The integral in (6.1) above can be expressed by these “2-clusters” by a similar formula as Eq. (6.4) in [24] (cf. also Sec. 5.3 in [27] and [38, 16]).

The explicit formula for $WLO_{rig}(L)$ for general $L$ which one obtains in this way should again be a sum over the set of “area coloring” $\varphi$, but this time every summand will contain an extra factor involving a product $\prod_{x \in V(L)} \cdots$. One could hope that this factor coincides with the factor $|L|^c$ (cf. part B of the Appendix for the notation used here).

In order to evaluate the chances for this being the case we can consider the case of Abelian structure group $G = U(1)$. The computations are then analogous to those appearing in the continuum setting in Secs 5.1 and 6.1 in [27] (which led to the correct result). However, in these computations there is one crucial difference in comparison to the computations in the continuum computations: there are several factors of $1/2$, coming from the RHS of Eq. (5.13) above, which “spoil” the final result. So ultimately we do not recover the (correct) expressions which appeared

\[46\] the multiplicities $m_{\chi_i}(a_i), i \leq m$, appearing in Eq. (5.63) lead to the fusion coefficients $N_{\rho_{i'}}^A$, appearing in Eq. (5.67) below, cf. the RHS of Eq. (5.7).

\[47\] this follows easily from Eq. (5.9) below after taking into account that the set $\Lambda^k_+$ is not empty if $k \geq c_0$, cf. Remark B.1.

\[48\] in fact, the heuristic equation Eq. (2.7) was only derived for simply-connected compact groups $G$ and therefore does not include the case $G = U(1)$ or $G = U(1) \times U(1)$. However, it is not difficult to see that for $G = U(1) \times U(1)$ and $(k_1, k_2) = (k, -k)$ an analogue of Eq. (6.8) below can be derived, cf. Remark B.1 below.
in the continuum setting. This complication can be resolved\(^{49}\) by making the transition to the “\(BF_3\)-theory point of view”.

7 Transition to the “\(BF_3\)-theory point of view”

7.1 Motivation

The simplicial program for Abelian CS theory (cf. part \(\mathbb{A}\) of the Appendix) was completed successfully by D.H. Adams, see \([1, 2]\). A crucial step in \([1, 2]\) was the transition to the “BF-theory point of view”, which can be divided into two steps, namely “field doubling”\(^{50}\) followed by a suitable linear change of variables, cf. part \(\mathbb{C}\) of the Appendix below.

Adams’ results seem to suggest that – if one wants to have a chance of carrying out the simplicial program successfully also for Non-Abelian CS theory – then a similar strategy will have to be used.

So far we have worked with the original CS point of view because this helped us to reduce the lengths of many formulas considerably and because for Theorem 3.4 (which deals only with a special class of simplicial ribbon links \(L\)) the original CS point of view is sufficient. On the other hand, the Abelian “test situation” which we considered at the end of Sec. 6 showed us that we can not expect to obtain correct results within the original CS point of view when dealing with general simplicial ribbon links. This is why from now on we will work with the “\(BF_3\)-theory point of view”.

7.2 The “\(BF_3\)-theory point of view”

In the following we will make the transition from non-Abelian CS theory in the torus gauge to the corresponding “\(BF_3\)-theory point of view” at a heuristic level.

Step 1: “Group doubling”

Let us now consider the version of Eq. (2.7) in the special case where \(G = \tilde{G} \times \tilde{G}\) where \(\tilde{G}\) is a simple, simply-connected compact Lie group and where \((k_1, k_2)\) fulfills \(k_1 = -k_2\), cf. Remark 2.2 above. We set \(k := k_1 = -k_2\).

For simplicity, let us consider the special case where each of the representations \(\rho_i\) appearing in Eq. (2.7) is of the form \(\rho_i(\tilde{g}_1, \tilde{g}_2) = \tilde{\rho}_i(\tilde{g}_1), \tilde{g}_1, \tilde{g}_2 \in \tilde{G}\), for some \(\tilde{G}\)-representation \(\tilde{\rho}_i\). In this situation we should have\(^{51}\)

\[
WLO(L) \sim WLO_{\tilde{G}}(L)WLO_{\tilde{G}}(\emptyset) \sim |L| \cdot |\emptyset|^{(s)} \sim |L| \cdot |\emptyset| \quad (7.1)
\]

where \(WLO_{\tilde{G}}(L)\) on the RHS is defined as \(WLO(L)\) in Sec. 2.1 for the group \(\tilde{G}\) instead of \(G\) and where \(| \cdot |\) is now the shadow invariant for \(\tilde{g}\) and \(k\). In step (*) we used the fact that \(|\emptyset|\) is a real number.

Let us fix a maximal torus \(\tilde{T}\) of \(\tilde{G}\) and take \(T = \tilde{T} \times \tilde{T}\). Let \(B, A^\perp, \tilde{A}^\perp, \tilde{A}^\perp\), and \(\ll \cdot, \cdot \gg_{A^\perp}\) be defined as in Sec. 2.1 and Sec. 2.3 for the group \(G = \tilde{G} \times \tilde{G}\).

In the following \(B_1, B_2\) (resp. \(A^\perp_1\) and \(A^\perp_2\)) will denote the two components of \(B \in C^\infty(\Sigma, \tilde{t}) = C^\infty(\Sigma, \tilde{t}) \oplus C^\infty(\Sigma, \tilde{t})\) (resp. \(A^\perp \in C^\infty(S^1, A_{\Sigma, \emptyset}) = C^\infty(S^1, A_{\Sigma, \emptyset}) \oplus C^\infty(S^1, A_{\Sigma, \emptyset})\)). Moreover, we denote by \(\tilde{I}\) the kernel of \(\exp_{\tilde{g}} : \tilde{t} \to \tilde{T}\).

\(^{49}\)observe that in contrast to the RHS of Eq. 3.18 above not all of the summands in the exponential on the RHS of Eq. 3.18 below are multiplied with a weight factor \(1/2\)

\(^{50}\)which can come in the form of “group doubling” (see Step 1 below) or “base manifold doubling”; observe that the word “doubling” is slightly misleading because it ignores a sign change: we have \(k_2 = -k_1\) where \(k_1, k_2\) are as in the first paragraph of “Step 1” below

\(^{51}\)the first “\(\sim\)” follows from a short heuristic computation
Step 2: Linear change of variable

As we explain in part C of the Appendix, CS theory with group $G = \hat{G} \times \hat{G}$ and $(k_1, k_2) = (k, -k)$ is equivalent to $BF_3$-theory with group $\hat{G}$ and “cosmological constant” $\Lambda$ given by $\Lambda = \frac{1}{k}$. More precisely, at the heuristic level, these two theories are related by a simple linear change of variables, cf. Eqs. (C.5) or Eqs. (C.10) in part C of the Appendix depending on whether we are dealing with the non-gauge fixed path integral or the path integral in the torus gauge.

In order to simplify the notation a bit (and to avoid the appearance of multiple $k$-factors) we will work with the following simplified change of variable $A^\perp \rightarrow \hat{A}^\perp$, $B \rightarrow \hat{B}$ instead of the one in Eq. (C.10):

$$\hat{A}^\perp := \left( \frac{A^\perp_1 + A^\perp_2}{2}, \frac{A^\perp_1 - A^\perp_2}{2} \right), \quad \hat{B} := \left( \frac{B_1 + B_2}{2}, \frac{B_1 - B_2}{2} \right) \quad (7.2a)$$

By applying this linear change of variable to the RHS of Eq. (2.7) (in the special case $G = \hat{G} \times \hat{G}$, $(k_1, k_2) = (k, -k)$) we arrive at

$$WLO(L) \sim \sum_{(y_1, y_2) \in I^2} \int_{\hat{A}_1^\perp \times \hat{B}} 1_{C^\infty(\Sigma, \text{reg} \times \text{reg})}((\hat{B}_1 + \hat{B}_2, \hat{B}_1 - \hat{B}_2)) \det_{FP}((\hat{B}_1 + \hat{B}_2, \hat{B}_1 - \hat{B}_2))$$

$$\times \left[ \int_{\hat{A}_1^\perp} \prod_k \text{Tr}_{\hat{p}_i}((\text{Hol}_{c\ast}((\hat{A}_1^\perp + \hat{A}_2^\perp)_1 + (\hat{A}_1^\perp + \hat{A}_2^\perp)_2, \hat{B}_1 + \hat{B}_2)) \exp(iS(\hat{A}_1^\perp, \hat{B}))D\hat{A}_1^\perp \right]$$

$$\times \exp(-2\pi ik((y_1, y_2), ((\hat{B}_1 + \hat{B}_2)(\sigma_0), (\hat{B}_1 - \hat{B}_2)(\sigma_0)))) \exp(iS(\hat{A}_1^\perp, \hat{B}))((\hat{A}_1^\perp \otimes D\hat{B})) \quad (7.3)$$

where for reasons of notational consistency, we have written $\hat{A}_1^\perp$ instead of $\hat{A}_1^\perp$, $\hat{A}_c^\perp$ instead of $A_c^\perp$, and $\hat{B}$ instead of $B$ and where we have set

$$S(\hat{A}_1^\perp, \hat{B}) := S_{CS}(\hat{A}_1^\perp, B)$$

$$S(\hat{A}_c^\perp, \hat{B}) := S_{CS}(\hat{A}_c^\perp, B)$$

More explicitly, we have

$$S(\hat{A}_1^\perp, \hat{B}) = S_{CS}((\hat{A}_1^\perp + \hat{A}_2^\perp, \hat{A}_1^\perp - \hat{A}_2^\perp), (\hat{B}_1 + \hat{B}_2, \hat{B}_1 - \hat{B}_2))$$

$$= S_{CS}(\hat{A}_1^\perp + \hat{A}_2^\perp, \hat{B}_1 + \hat{B}_2) - S_{CS}(\hat{A}_1^\perp - \hat{A}_2^\perp, \hat{B}_1 - \hat{B}_2)$$

$$= S_{CS}(\hat{A}_1^\perp + \hat{A}_2^\perp + (\hat{B}_1 + \hat{B}_2)dt) - S_{CS}(\hat{A}_1^\perp - \hat{A}_2^\perp + (\hat{B}_1 - \hat{B}_2)dt)$$

$$= \pi k \ll (\hat{A}_1^\perp, \hat{A}_2^\perp), \left( \ast \text{ad}(\hat{B}_1) \ast (\frac{\partial}{\partial t} + \text{ad}(\hat{B}_1)) \right) \cdot (\hat{A}_1^\perp, \hat{A}_2^\perp) \gg \hat{A}_1^\perp \quad (7.4)$$

and

$$S(\hat{A}_c^\perp, \hat{B}) = S_{CS}(((\hat{A}_c^\perp)_1 + (\hat{A}_c^\perp)_2, (\hat{A}_c^\perp)_1 - (\hat{A}_c^\perp)_2), (\hat{B}_1 + \hat{B}_2, \hat{B}_1 - \hat{B}_2))$$

$$= \ldots$$

$$= 4\pi k \ll \ast \cdot ((\hat{A}_c^\perp)_2, (\hat{A}_c^\perp)_1), (d\hat{B}_1, d\hat{B}_2) \gg A_{\Sigma, i=1} \quad (7.5)$$

where $\hat{A}_1^\perp = (\hat{A}_1^\perp, \hat{A}_2^\perp)$, $\hat{A}_c^\perp = (\hat{A}_c^\perp, (\hat{A}_c^\perp)_1)$, and $\hat{B} = (\hat{B}_1, \hat{B}_2)$.

---

52 Observe that $k = \frac{1}{k}$ in Eqs. (C.5) and (C.10).

53 Here we have used that $\rho_1(\hat{g}_1, \hat{g}_2) = \hat{\rho}(\hat{g}_1)$, for all $\hat{g}_1, \hat{g}_2 \in \hat{G}$, which implies $\text{Tr}_{\rho_i}(\text{Hol}_i(A^\perp, B)) = \text{Tr}_{\rho_i}(\text{Hol}_i(A^\perp, B)) = \text{Tr}_{\rho_i}(\text{Hol}_i(A^\perp, B))$.

54 The first appearance of $S_{CS}$ on the RHS of the following equation is a shorthand for $S_{CS}(M, \hat{G} \times \hat{G}, (k_1, k_2))$ while the other appearances are a shorthand for $S_{CS}(M, \hat{G}, k_1) = S_{CS}(M, \hat{G}, -k_2)$. 
Remark 7.1 Recall that above we have assumed that $\tilde{G}$ is a simple, simply-connected compact (and therefore non-Abelian) Lie group. For sake of completeness (and in view of the discussion at the end of Sec. 6 above and Remark 7.2 and Remark 7.4 below) let us mention that, in fact, one can derive (an analogue of) Eq. (7.3) also if $\tilde{G}$ is an Abelian compact Lie group. Of course, in this case the RHS of Eq. (7.3) simplifies drastically.

Remark 7.2 It might seem surprising that the second of the aforementioned two steps, i.e. the linear change of variable, really makes an essential difference. Clearly, the original heuristic path integral and the heuristic path integral after the application of the change of variable are equivalent. However, once the problem of discretizing the corresponding path integral is considered the difference really matters. A detailed look at [2] will convince the reader that this is indeed the case at least in the Abelian situation.

That a linear change of variables is useful also for the discretization of non-Abelian CS (with doubled group) is less obvious. Observe, for example, that there is a $\star$-operator on the main diagonal of the $2 \times 2$-matrix appearing in Eq. (7.1) above. Because of this we cannot hope to be able to find a discretized version of the path integral on the RHS of Eq. (7.3) where each of the two components $\tilde{A}_1^+$ and $\tilde{A}_2^+$ “lives” either on $K_1 \times \mathbb{Z}_N$ or on $K_2 \times \mathbb{Z}_N$. Instead, each component $\tilde{A}_1^+$ and $\tilde{A}_2^+$ must be implemented in a “mixed” fashion (which is what we did in the discretization approach of the present paper, cf. Sec. 3). This is a crucial difference compared to the Abelian situation where it was indeed possible to find a non-mixed discretization for the relevant simplicial fields. This difference is one of the reasons why we decided to postpone the transition to the $BF_3$-theory point of view until now.

Digression: Alternative to Step 2

As we observed in Remark 7.2 above there is a $\star$-operator on the main diagonal of the $2 \times 2$-matrix appearing in Eq. (7.4) above. Because of this we cannot hope to be able to find a discretized version of the path integral on the RHS of Eq. (7.3) where each of the two components $\tilde{A}_1^+$ and $\tilde{A}_2^+$ “lives” either on $K_1 \times \mathbb{Z}_N$ or on $K_2 \times \mathbb{Z}_N$.

It turns out that by using a more sophisticated change of variable $A^+ \to \tilde{A}^+$ instead of the $\star$-operator on the main diagonal can be eliminated after all, cf. Eq. (7.10) below. This new change of variable, which we will introduce below, is based on the “root space decomposition”

$$
\tilde{\mathfrak{g}} = \mathfrak{t} \oplus (\oplus_{\alpha \in \tilde{R}_+} \tilde{\mathfrak{g}}_\alpha)
$$

of $\tilde{\mathfrak{g}}$ where $\tilde{R}_+$ is the set of positive real roots of $\tilde{\mathfrak{g}}$ w.r.t. $\mathfrak{t}$ and $\tilde{\mathfrak{g}}_\alpha \cong \mathbb{R}^2$ is the root space corresponding to $\alpha \in \tilde{R}_+$. For simplicity let us consider only the special case $\tilde{G} = SU(2)$ and $T = \{\exp(\theta \tau) \mid \theta \in \mathbb{R}\}$ (with $\tau$ given below). In this case we can rewrite Eq. (7.6) as

$$
\tilde{\mathfrak{g}} = su(2) = \mathbb{R} \cdot \tau \oplus (\mathbb{R} \cdot X_+ \oplus \mathbb{R} \cdot X_-)
$$

where

$$
\tau := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
$$

Using the concrete basis $(\tau, X_+, X_-)$ we can identify $\tilde{\mathfrak{g}} = su(2)$ with $\mathbb{R}^3$ in the obvious way, which in turn leads to the identification

$$
C^\infty(S^1, A_{\Sigma, \tilde{g}}) \cong C^\infty(S^1, A_{\Sigma, \mathbb{R}})^3
$$

\footnote{we then have $\tilde{T} = \tilde{G}$, $\mathfrak{t}_{\text{reg}} = \mathfrak{t}$, $\text{ad}(\tilde{B}_1) = 0$, and the sum $\sum_{y_1, y_2}$ is trivial. So we obtain: WLO(L) $\sim \int \prod \text{Tr}_{\rho_i} \text{Hol}_{\lambda_i} ((\tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+)_{2,3} + (\tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+)_{2,3} + (\tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+ + \tilde{\mathcal{A}}^+)_{2,3}) \exp(i\tilde{\delta}(\tilde{A}^+, \tilde{B}))D\tilde{A}^+ \exp(i\tilde{\delta}(\tilde{A}^+, \tilde{B}))(D\tilde{A}^+ \otimes D\tilde{B})$}
The new change of variable $A^\perp \rightarrow \tilde{A}^\perp$ mentioned above is defined by

$$\tilde{A}^\perp := \left( \frac{Q}{2}(A_1^\perp + A_2^\perp), \frac{Q}{2-\xi}(A_1^\perp - A_2^\perp) \right),$$

where $Q$ is the operator on $C^\infty(S^1, A_{\Sigma,\bar{g}}) \cong C^\infty(S^1, A_{\Sigma,R})^3$ given by

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 1 & -* \end{pmatrix}$$

Here $1$ denotes the identity operator on $C^\infty(S^1, A_{\Sigma,\bar{g}})$ and $*$ the Hodge star operator on $C^\infty(S^1, A_{\Sigma,R})$ which is induced by the auxiliary Riemannian metric $g$ fixed in Sec. 2.3 above.

Using the changes of variable (7.9) and (7.21) we arrive at the following modification of Eq. (7.4) above

$$\mathcal{S}(\tilde{A}^\perp, \tilde{B}) = \pi k \ll (\tilde{A}^\perp, \tilde{A}_2^\perp), \left( J \text{ad}(\tilde{B}_2) \ast \frac{\partial}{\partial t} + J \text{ad}(\tilde{B}_1) \right) \cdot (\tilde{A}^\perp, \tilde{A}_2^\perp) \gg \tilde{A}^\perp$$

where $J$ is the linear operator on $C^\infty(S^1, A_{\Sigma,\bar{g}})$ which is induced in the obvious way by the linear operator $J_0$ on $\mathfrak{g} = \mathfrak{su}(2)$ given by $J_0 \cdot \tau = 0, J_0 \cdot X_{\pm} = X_{\mp}$.

Observe that in contrast to Eq. (7.4) above, there is no $\ast$-operator appearing on the main diagonal of the matrix operator in Eq. (7.10). It turns out, however, that when trying to discretize the analogue of Eq. (7.3) which is obtained after applying the change of variable (7.9) instead of (7.2a) it is still necessary to use a “mixed” implementation in the sense of Remark 7.2. This time we have more freedom in choosing the implementation. It remains to be seen how useful this is for our purposes. We plan to study this issue in the near future.

**7.3 Simplification of some of the notation in Sec. 7.2**

Before we discretize the expression on the RHS of Eq. (7.3) let us first simplify the notation somewhat:

Firstly, we will drop all the $\tilde{\cdot}$-signs appearing in the previous subsection, for example will write $G$ instead of $\tilde{G}$, $B$ instead of $\tilde{B}$, $I$ instead of $\tilde{I}$ and so on. Clearly, we then have

$$B = C^\infty(\Sigma, t \oplus t)$$

$$A^\perp = C^\infty(S^1, A_{\Sigma,\bar{g} \oplus \bar{g}})$$

$$\tilde{A}^\perp = \{ A^\perp \in A^\perp \mid \int A^\perp(t)dt \in A_{\Sigma, t \oplus t} \}$$

$$A^\perp = \{ A^\perp \in A^\perp \mid A^\perp \text{ is constant and } A_{\Sigma, t \oplus t} \text{-valued} \}$$

Moreover, we will set

$$B_\pm := B_1 \pm B_2 \quad \text{for } B = (B_1, B_2) \in B$$

$$A^\perp_\pm := A^\perp_1 \pm A^\perp_2 \quad \text{for } A^\perp = (A^\perp_1, A^\perp_2) \in A^\perp$$

and use the notation $y_+$ instead of $y_1$ and $y_-$ instead of $y_2$. Then we can rewrite Eq. (7.3) in the following way

$$\text{WLO}(L) \sim \sum_{y_+, y_- \in I} \int_{A^\perp_+ \times B} \left\{ \prod_{k} \left[ 1_{C^\infty(\Sigma, \bar{g} \oplus \bar{g})(B_\pm)} \text{Det}_F \text{P}(B_\pm) \right] \right\}$$

$$\times \left[ \int_{A^\perp_+} \prod_{t} \text{Tr}_{\mathcal{L}_t} \left( \text{Hol}_t((\tilde{A}^\perp + A^\perp_+), B_+) \exp(i\mathcal{S}(\tilde{A}^\perp, B))D\tilde{A}^\perp \right) \right.$$}

$$\times \exp(-2\pi i k \sum \langle y_\pm, B_\pm(\sigma_0) \rangle) \left\{ \exp(i\mathcal{S}(A^\perp_+, B)) \right\} (DA^\perp_+ \otimes DB)$$

(7.12)
where $\prod_{\pm} \cdots$ (resp. $\sum_{\pm} \cdots$) is the obvious two term product (resp. sum). Above we have set (cf. Eq. (7.4) and Eq. (7.5) above)

$$
S(\mathcal{A}, B) := \pi k \ll (\tilde{A}^1, \tilde{A}^2), \ast \left( \frac{\partial}{\partial t} + \text{ad}(B_1) \right) \cdot (\tilde{A}^1, \tilde{A}^2) \gg \mathcal{A}^\perp \quad (7.13)
$$

$$
S(A^\perp, B) := 4\pi k \ll \ast \cdot ((A^\perp_1)_2, (A^\perp_1)_1), (dB_1, dB_2) \gg A^\perp \quad (7.14)
$$

where $\ast : \mathcal{A}^\perp \to \mathcal{A}^\perp$ is the Hodge star operator induced by the auxiliary Riemannian metric $g$.

**Remark 7.3** Observe that for each $l \in \{l_1, l_2, \ldots, l_m\}$ we have (cf. Eqs. (2.6) and (2.9) in Sec. 2.1 above and Eq. (2.4) in [29])

$$
\text{Hol}_l(A^\perp, B_{\perp}) = \lim_{n \to \infty} \prod_{k=1}^n \exp\left( \frac{1}{n} (A^\perp + B_{\perp} dt)(l^l(t)) \right) \bigg|_{t=k/n}
$$

$$
= \lim_{n \to \infty} \prod_{k=1}^n \exp\left( A^\perp_1 (l_{S^1}(t))(\frac{1}{n} l^l_1(t)) + A^\perp_2 (l_{S^1}(t))(\frac{1}{n} l^l_2(t))
$$

$$
+ B_1(l_{S^1}(t))dt(\frac{1}{n} l^l_1(t)) + B_2(l_{S^1}(t))dt(\frac{1}{n} l^l_2(t)) \right) \bigg|_{t=k/n} \quad (7.15)
$$

for $A^\perp \in \mathcal{A}^\perp$ and $B \in \mathcal{B}$.

### 7.4 Discretization of Eq. (7.12)

We will now sketch how – using a suitable discretization of the expression on the RHS of Eq. (7.12) – a rigorous definition of WLO$(L)$ appearing in Eq. (7.12) can be obtained. (In part D of the Appendix we will sketch an alternative way of discretizing the RHS of Eq. (7.12)).

In view of what we have learned in Sec. 5.5 (cf. Remark 5.4) we will now work with simplicial ribbons in $K \times \mathbb{Z}_N$ (“full ribbons”) instead of simplicial ribbons in $qK \times \mathbb{Z}_N$ (“half ribbons”).

We set

$$
\mathcal{B}(qK) := C^0(qK, t \oplus t) \quad (7.16a)
$$

$$
\mathcal{A}_\Sigma(qK) := C^1(qK, g \oplus g) \quad (7.16b)
$$

$$
\mathcal{A}^\perp(qK) := \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(qK)) \quad (7.16c)
$$

and introduce the scalar product $\ll \cdot, \cdot \gg_{A^\perp(qK)}$ on $A^\perp(qK)$ in an analogous way as in Sec. 3.1 above.

For technical reasons we will again introduce the subspaces $\mathcal{A}_\Sigma(K)$ and $\mathcal{A}^\perp(K)$ of $\mathcal{A}_\Sigma(qK)$ and $\mathcal{A}^\perp(qK)$ given by\(^{57}\)

$$
\mathcal{A}_\Sigma(K) := \mathcal{A}_{\Sigma, g \oplus g}(K) \subset \mathcal{A}_\Sigma(qK) \quad (7.17a)
$$

$$
\mathcal{A}^\perp(K) := \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(K)) \subset A^\perp(K) \quad (7.17b)
$$

Moreover, we will introduce the subspace $\mathcal{B}_0(qK) := \psi(\mathcal{B}(K))$ of $\mathcal{B}(qK)$ where $\mathcal{B}(K) := C^0(K, t \oplus t)$ and where $\psi : \mathcal{B}(K) \to \mathcal{B}(qK)$ is given exactly like in Choice 3 in Sec. 3.1 above with $t$ replace by $t \oplus t$.

As in Sec. 3.3 we have a well-defined operator $\ast_K : \mathcal{A}^\perp(K) \to A^\perp(K)$ and as in Sec. 3.1 we have a decomposition

$$
\mathcal{A}^\perp(K) = \mathcal{A}^\perp(K) \ominus \mathcal{A}^\perp(K)
$$

\(^{57}\)Here and in the following we use again the notation $\mathcal{A}_{\Sigma, V}(K) := C^1(K_1, V) \oplus C^1(K_2, V)$ for a finite-dimensional real vector space $V$. 
where

\[ \mathcal{A}^\perp(K) := \{ A^\perp \in \mathcal{A}^\perp(K) \mid \sum_{t \in \mathbb{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma: \mathbb{R}^d}(K) \} \]  
\[ A_c^\perp(K) := \{ A^\perp \in \mathcal{A}^\perp(K) \mid A^\perp(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma: \mathbb{R}^d}(K)-\text{valued} \} \cong \mathcal{A}_{\Sigma: \mathbb{R}^d}(K) \]  

Furthermore, we set

\[ B_\pm := B_1 \pm B_2 \quad \text{for } B = (B_1, B_2) \in B(q\mathcal{K}) \]
\[ A_\pm := A_1^\perp \pm A_2^\perp \quad \text{for } A^\perp = (A_1^\perp, A_2^\perp) \in \mathcal{A}^\perp(K) \]

As the discrete analogues of Eq. (7.13) and Eq. (7.14) above we now take

\[ S^{\text{disc}}(\mathcal{A}^\perp, B) := \pi k \ll (\mathcal{A}_1^\perp, \mathcal{A}_2^\perp) \cdot \ast \mathcal{R}^{(N)}(B) \cdot (\mathcal{A}_1^\perp, \mathcal{A}_2^\perp) \gg A^\perp(q\mathcal{K}) \]  
\[ S^{\text{disc}}(A_c^\perp, B) := 4\pi k \ll \ast \mathcal{R} \cdot ((A_1^\perp)_1, (A_2^\perp)_1) \cdot (d_{q\mathcal{K}} B_1, d_{q\mathcal{K}} B_2) \gg A^\perp(q\mathcal{K}) \]  

where

\[ \mathcal{R}^{(N)}(B) := \left( \frac{\mathcal{L}^{(N)}(B_+) - \mathcal{L}^{(N)}(B_-)}{2} \right) \left( \frac{\mathcal{L}^{(N)}(B_+) + \mathcal{L}^{(N)}(B_-)}{2} \right) \]

As mentioned above we will now work with “full ribbons”, i.e. closed simplicial ribbons \( R = (F_k)_{k \leq n}, \ n \in \mathbb{N}, \ K \times Z_N \). Recall from Remark 3.22 that \( R \) induces three simplicial loops \( l_\Sigma, l_\Sigma^+, \text{ and } l_\Sigma^- \) in \( q\mathcal{K} \) and three simplicial loops \( l_{S_1}^+, l_{S_1}^-, \text{ and } l_{S_1}^- \) in \( Z_N \). As the discrete analogue \( \text{Hol}^{\text{disc}}_R(A_+, B_+) \) of the continuum expression \( \text{Hol}_R(A_+, B_+) \) in Eq. (7.15) above we now take

\[ \text{Hol}^{\text{disc}}_R(A_+, B_+) := \prod_{k=1}^n \exp \left( \sum_{\pm} \frac{1}{2} A_+^\perp(\bullet l_{S_1}^\pm(k)(l_{S_1}^\pm(k))) + A_2^\perp(\bullet l_{S_1}^\pm(k)(l_{S_1}^\pm(k))) \right. \\
\left. + \sum_{\pm} \frac{1}{2} B_1(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) + B_2(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) \right) \]

In view of Eq. (3.20) above and the list of replacements in Sec. 5.4.1 in [29] the ansatz in Eq. (7.22) is quite natural. The only point that requires an explanation is why the field component \( A_2^\perp \) “interacts” only with the loops \( l_\Sigma \) while \( A_1^\perp \) “interacts” with the two loops \( l^+ \) and \( l^- \). We will give this explanation in part D of the Appendix, cf. “Insight 1”.

**Remark 7.4** In the special case where \( G \) is Abelian (cf. Remark 7.1 above) there are several simplifications. For example, in the special case where the (complex) representation \( \rho \) of \( G \) is irreducible (and therefore 1-dimensional) we have from Eq. (7.22)

\[ \text{Tr}_\rho(\text{Hol}^{\text{disc}}_R(A_+, B_+)) = \rho \left( \prod_{k=1}^n \exp \left( \sum_{\pm} \frac{1}{7} A_+^\perp(\bullet l_{S_1}^\pm(k)(l_{S_1}^\pm(k))) + \frac{1}{2} B_1(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) \right) \right. \\
\left. \times \rho \left( \prod_{k=1}^n \exp \left( A_2^\perp(\bullet l_{S_1}^\pm(k)(l_{S_1}^\pm(k))) + B_2(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) \right) \right) \right) \]

58Recall that \( L^{(N)}(B_0) \) is a discrete “approximation” of \( \partial_t + \text{ad}(B_0) \) so \( \frac{1}{2} (L^{(N)}(B_+) + L^{(N)}(B_-)) \) is a discrete analogue of \( \frac{1}{2}(\partial_t + \text{ad}(B_+) + \partial_t + \text{ad}(B_-)) = \frac{1}{2}(\partial_t + \text{ad}(B_+ + B_-)) + \partial_t + \text{ad}(B_1 - B_2) \); similarly \( \frac{1}{2} (L^{(N)}(B_1) - L^{(N)}(B_-)) \) is a discrete approximation of \( \frac{1}{2}(\partial_t + \text{ad}(B_1 + B_-) - \partial_t + \text{ad}(B_1 - B_2)) = \text{ad}(B_2) \). The latter question can easily be avoided since it turns out that the value of WLO$_{rig}(L)$ as defined in Eq. (7.21) below does not change if in Eq. (7.22) we replace the expression \( \left( \sum_{\pm} \frac{1}{7} B_1(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) + B_2(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) \right) \) by the expression \( \sum_{j=1}^2 \left( \sum_{\pm} \frac{1}{7} B_j(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) + B_2(\bullet l_{S_1}^\pm(k)) dt^{(N)}(l_{S_1}^\pm(k)) \right) \). Clearly, in the last expression both field components \( B_1 \) and \( B_2 \) interact with all three loops.
We can work with a generalization of the last expression where the second “$\rho$” appearing above is replaced by another finite-dimensional representation $\rho'$ of $G$. By doing so we obtain a kind of torus gauge “analogue” of the results in [1, 2].

As the discrete version for the two expressions $\text{Det}_{FP}(B_{\pm})$ appearing in Eq. (7.12) we choose again (as in Eq. (3.21) in Sec. 3.5 above)

$$\text{Det}_{FP}^{\text{disc}}(B_{\pm}) := \prod_{x \in \mathbb{Z}_0(qK)} \text{det}(1 - \exp(\text{ad}(B_{\pm}(x))))^{1/2}$$

(7.23)

The remaining steps for discretizing the RHS of Eq. (7.12) can be carried out easily (as in in Secs 3.6–3.8 above). There is only one exception: since we are now working with “full ribbons” it will be necessary to use an additional regularization procedure, cf. Remark 5.4 in Sec. 5.5 above.

We then arrive at a rigorous version $\text{WLO}^{\text{disc}}_{\text{rig}}(L)$ of $\text{WLO}(L)$ and its normalization

$$\text{WLO}_{\text{rig}}(L) := \frac{\text{WLO}^{\text{disc}}_{\text{rig}}(L)}{\text{WLO}^{\text{disc}}_{\text{rig}}(\emptyset)}$$

(7.24)

In view of the heuristic formula Eq. (7.1) we expect that for $k \geq c_{\mathfrak{g}}$ we have

$$\text{WLO}_{\text{rig}}(L) = \frac{|L||\mathbb{E}|}{|\mathbb{E}||\mathbb{E}|} = \frac{|L|}{|\mathbb{E}|}$$

(7.25)

and in contrast to the situation in Theorem 3.4 there is now a reasonable chance that Eq. (7.25) even holds for general simplicial ribbon links $L$ in $\mathcal{K} \times \mathbb{Z}_N$.

## 8 Discussion & Outlook

In [29] we proposed a “discretization approach” for making rigorous sense of the torus-gauge-fixed non-Abelian CS/BF$_3$-path integral for manifolds $M$ of the form $M = \Sigma \times S^1$. In the present paper we proved the main result of [29], Theorem 3.4 above, which deals with a special class of simplicial ribbon links. During the proof of Theorem 3.4 (and in Sec 6) it became clear that in order to have a reasonable chance of generalizing our computations successfully to the case of general simplicial ribbon links it seems to be necessary to make (at least) the following two modifications of our approach:

- we should work with simplicial ribbons in $\mathcal{K} \times \mathbb{Z}_N$ (“full ribbons”) instead of simplicial ribbons in $q\mathcal{K} \times \mathbb{Z}_N$ (“half ribbons”),
- we should make the “transition to the BF-theory point of view” (cf. “Step 1” and “Step 2” in Sec. 7.2 above).

We sketched such a modification of our approach in Sec. 7.4 (and a suitable reformulation of it in part 1 of the Appendix). In [30] we will study this new approach in full detail. There we will also study in more detail the alternative approach mentioned in the “Digression” at the end of Sec. 7.2.

**Acknowledgements:** I want to thank the anonymous referee of my paper [27] whose comments motivated me to look for an alternative approach for making sense of the RHS of (the original

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Let us emphasize that we do not get a strict analogue: in the case of Abelian $G$ our approach is less general than [1, 2]. We remark that for an Abelian $G$ one actually can construct a strict torus gauge analogue of the approach in [1, 2] by modifying our approach in a suitably way. However, this modified approach will not be useful for dealing with the case of non-Abelian $G$.

Here $| \cdot |$ is the shadow invariant for $\mathfrak{g}$ and $k$; recall that we write the group $\tilde{G}$ now simply as $G$, i.e. without the $\sim$.
version of) Eq. (2.7), which is less technical than the continuum approach in [25, 27, 28]. This eventually led to [29] and the present paper.

I am also grateful to Jean-Claude Zambrini for several comments which led to improvements in the presentation of the present paper.

Finally, it is a great pleasure for me to thank Benjamin Himpe1 for many useful and important comments and suggestions which had a major impact on the presentation and overall structure of the present paper.

A Appendix: Lie theoretic notation II

The following two lists extend the two lists in Appendix A in [29].

A.1 List of notation in the general case

Recall that in Sec. 2.1 we fixed a simply-connected compact Lie group $G$ (with Lie algebra $\mathfrak{g}$), a maximal $T$ of $G$ (with Lie algebra $\mathfrak{t}$), and a Weyl chamber $C \subset \mathfrak{t}$.

Apart from the notation given in Appendix A of [29] we also use the following Lie theoretic notation in the present paper:

- $\langle \cdot, \cdot \rangle$: the unique Ad-invariant scalar product on $\mathfrak{g}$ such that $\langle \check{\alpha}, \check{\alpha} \rangle = 2$ holds for every short real coroot $\check{\alpha}$ associated to $(\mathfrak{g}, \mathfrak{t})$. Using $\langle \cdot, \cdot \rangle$ we now make the identification $\mathfrak{t} \cong \mathfrak{t}^*.$

- $\mathcal{R}_C$: the set of complex roots $\mathfrak{t} \to \mathbb{C}$ associated to $(\mathfrak{g}, \mathfrak{t})$

- $\mathcal{R} \subset \mathfrak{t}^*$: the set $\{ \frac{1}{2\pi} \alpha_c | \alpha_c \in \mathcal{R}_C \}$ of real roots associated to $(\mathfrak{g}, \mathfrak{t})$

- $\mathcal{R}_+ \subset \mathcal{R}$: the set of positive (real) roots corresponding to $C$

- $\Gamma \subset \mathfrak{t}$: the lattice generated by the set of real coroots associated to $(\mathfrak{g}, \mathfrak{t})$, i.e. by the set $\{ \check{\alpha} | \alpha \in \mathcal{R} \}$ where $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{t}^* \cong \mathfrak{t}$ is the coroot associated to the root $\alpha \in \mathcal{R}$.

- $I \subset \mathfrak{t}$: the kernel of $\exp|_I : \mathfrak{t} \to T$. From the assumption that $G$ is simply-connected it follows that $I = \Gamma$.

- $\Lambda \subset \mathfrak{t}^*(\cong \mathfrak{t})$: the real weight lattice associated to $(\mathfrak{g}, \mathfrak{t})$, i.e. $\Lambda$ is the lattice which is dual to $\Gamma$.

- $\Lambda_+ \subset \Lambda$: the set of dominant weights corresponding to $C$, i.e. $\Lambda_+ := \bar{C} \cap \Lambda$

- $\rho$: half sum of positive roots ("Weyl vector")

- $\theta$: unique long root in the Weyl chamber $C$.

- $c_\mathfrak{g} = 1 + \langle \theta, \rho \rangle$: the dual Coxeter number of $\mathfrak{g}$.

- $P \subset \mathfrak{t}$: a fixed Weyl alcove

- $Q \subset \mathfrak{t}$: a subset of $\mathfrak{t}$ of the form $Q = \{ \sum i \lambda_i e_i | 0 < \lambda_i < 1 \ \forall i \leq \dim(\mathfrak{t}) \}$ where $(e_i)_{i \leq \dim(\mathfrak{t})}$ is a fixed basis of $\Gamma = I.$

- $\mathcal{W} \subset \text{GL}(\mathfrak{t})$: the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$

- $\mathcal{W}_{\text{aff}} \subset \text{Aff}(\mathfrak{t})$: the “affine Weyl group of $(\mathfrak{g}, \mathfrak{t})$”, i.e. the subgroup of $\text{Aff}(\mathfrak{t})$ generated by $\mathcal{W}$ and the set of translations $\{ \tau_x | x \in \Gamma \}$ where $\tau_x : \mathfrak{t} \ni b \mapsto b + x \in \mathfrak{t}.$
• $W_k \subset \text{Aff}(t)$, $k \in \mathbb{N}$: the subgroup of $\text{Aff}(t)$ given by $\{\psi_k \circ \sigma \circ \psi_k^{-1} \mid \sigma \in W_{\text{aff}}\}$ where $\psi_k : t \mapsto b \cdot k - \rho \in t$ (the “quantum Weyl group corresponding to the level $l := k - c_g$”)

• $\Lambda^k_+ \subset \Lambda$, $k \in \mathbb{N}$: the subset of $\Lambda_+$ given by $\Lambda^k_+ := \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_g\}$ (the “set of dominant weights which are integrable at level $l := k - c_g$”).

In the main text, the number $k \in \mathbb{N}$ appearing above will be the integer $k$ fixed in Sec. 2.1 (which later is assumed to fulfill $k \geq c_g$).

### A.2 List of notation in the special case $G = SU(2)$

Let us now consider the special group $G = SU(2)$ with the standard maximal torus $T = \{\exp(\theta \tau) \mid \theta \in \mathbb{R}\}$ where

$$\tau := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Then $g = su(2)$ and $t = \mathbb{R} \cdot \tau$. There are two Weyl chambers, namely $C_+$ and $C_-$ where $C_\pm := \pm (0, \infty) \tau$. Let us fix $C := C_+$ in the following.

• $\langle \cdot, \cdot \rangle$ is the scalar product on $g$ given by

$$\langle A, B \rangle = -\frac{1}{4\pi^2} \text{Tr}_{\text{Mat}(2, \mathbb{C})}(AB) \quad \text{for all} \ A, B \in g \subset \text{Mat}(2, \mathbb{C})$$

• $\mathcal{R}_C = \{\alpha_c, -\alpha_c\}$ where $\alpha_c : t \to \mathbb{C}$ is given by $\alpha_c(\tau) = 2i$

• $\mathcal{R} = \{\alpha, -\alpha\}$ where $\alpha := \frac{1}{2\pi} \alpha_c$. A short computation shows that $\alpha = 2\pi \tau \in t$ (recall that we made the identification $t \cong t^*$).

• $\mathcal{R}_+ = \{\alpha\}$

• $I = \Gamma = \mathbb{Z} \cdot \hat{\alpha}$ where $\hat{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} = \alpha$

• $\Lambda = \mathbb{Z} \cdot \frac{\alpha}{2}$

• $\Lambda_+ = \mathbb{N}_0 \cdot \frac{\alpha}{2}$

• $\rho = \frac{\alpha}{2}$

• $\theta = \alpha$

• $c_g = 2$

• possible choices for $P$ and $Q$ are $P = (0, \frac{1}{2})\alpha$ and $Q = (0, 1)\alpha$

• $\mathcal{W} = \{1, \sigma\}$ where $1 = \text{id}_t$ and $\sigma(b) = -b$ for $b \in t$; using this and the explicit description of $I = \Gamma$ above one easily obtains an explicit description of $W_{\text{aff}}$ and $W_k$

• $\Lambda_+^k = \{0, \frac{1}{2} \alpha, \ldots, \frac{k-2}{2} \alpha\}$ for $k \in \mathbb{N}$

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63 in view of the formula $\hat{\alpha} = \alpha = 2\pi \tau$ below we see that $\langle \cdot, \cdot \rangle$ indeed fulfills the normalization condition $\langle \hat{\alpha}, \hat{\alpha} \rangle = 2$
Appendix: Turaev’s shadow invariant

Let us briefly recall the definition of Turaev’s shadow invariant in the situation relevant for us, i.e. for base manifolds \( M \) of the form \( M = \Sigma \times S^1 \) where \( \Sigma \) is an oriented surface.

Let \( L = (l_1, l_2, \ldots, l_m) \), \( m \in \mathbb{N} \), be a framed piecewise smooth link \(^{64}\) in \( M = \Sigma \times S^1 \). For simplicity we will assume that each \( l_i \), \( i \leq m \) is equipped with a “horizontal” framing, cf. Remark 4.5 in Sec. 4.3 in \(^{20}\). Let \( V(L) \) denote the set of points \( p \in \Sigma \) where the loops \( l_{\Sigma_i}^i \), \( i \leq m \), cross themselves or each other (the “crossing points”) and \( E(L) \) the set of curves in \( \Sigma \) into which the loops \( l_{\Sigma_1}^1, l_{\Sigma_2}^2, \ldots, l_{\Sigma_m}^m \) are decomposed when being “cut” in the points of \( V(L) \). We assume that there are only finitely many connected components \( Y_0, Y_1, Y_2, \ldots, Y_{m'} \), \( m' \in \mathbb{N} \) (“faces”) of \( \Sigma \)\( \setminus \bigcup_i \operatorname{arc}(l_{\Sigma_i}^i) \) and set

\[
F(L) := \{Y_0, Y_1, Y_2, \ldots, Y_{m'}\}.
\]

As explained in \(^{47}\) one can associate in a natural way a half integer gleam \((Y) \in 1\mathbb{Z} \), called “gleam” of \( Y \), to each face \( Y \leq F(L) \). In the special case where the two conditions (NCP) and (NH) appearing in Sec. 6.1 in \(^{20}\) are fulfilled \(^{65}\) we have the explicit formula

\[
gleam(Y) = \sum_{i \text{ with } \operatorname{arc}(l_{\Sigma_i}^i) \subset \partial Y} \operatorname{wind}(l_{\Sigma_i}^i) \cdot \operatorname{sgn}(Y; l_{\Sigma_i}^i) \in \mathbb{Z}
\]

where \( \operatorname{wind}(l_{\Sigma_i}^i) \) is the winding number of the loop \( l_{\Sigma_i}^i \) and where \( \operatorname{sgn}(Y; l_{\Sigma_i}^i) \) is given by

\[
\operatorname{sgn}(Y; l_{\Sigma_i}^i) := \begin{cases} 1 & \text{if } Y \subset R_i^+ \\ -1 & \text{if } Y \subset R_i^- \end{cases}
\]

Here \( R_i^+ \) (resp. \( R_i^- \)) is the unique connected component \( R \) of \( \Sigma \setminus \operatorname{arc}(l_{\Sigma_i}^i) \) such that \( l_{\Sigma_i}^i \) runs around \( R \) in the “positive” (resp. “negative”) direction.

Let \( G \) be a simply-connected and simple compact Lie group with maximal torus \( T \). In the following we will use the notation from part \([\text{A}]\) of the Appendix. In particular, we have

\[
\Lambda_+^k = \{ \lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_\theta \}
\]

where \( k \in \mathbb{N} \) is as in Sec. \(2 \) above and where \( c_\theta = 1 + \langle \theta, \rho \rangle \) is the dual Coxeter number of \( \theta \).

Remark B.1 Observe that for \( k < c_\theta \) the set \( \Lambda_+^k \) is empty so \( |L| \) as defined in Eq. \((B.4)\) below will then vanish. If \( k = c_\theta \) then \( |L| \) will in general not vanish but will still be rather trivial. Not surprisingly, the definition of \( |L| \) in the literature often excludes the situation \( k < c_\theta \) so our definition of \( |L| \) in Eq. \((B.4)\) below is more general than usually.

Assume that each loop \( l_i \) in the link \( L \) is equipped with a “color” \( \rho_i \), i.e. a finite-dimensional complex representation of \( G \). By \( \gamma_i \in \Lambda_+ \) we denote the highest weight of \( \rho_i \) and set \( \gamma(e) := \gamma_i \) for each \( e \in E(L) \) where \( i \leq n \) denotes the unique index such that \( \operatorname{arc}(e) \subset \operatorname{arc}(l_i) \). Finally, let \( \operatorname{col}(L) \) be the set of all mappings \( \varphi : \{Y_0, Y_1, Y_2, \ldots, Y_{m'}\} \rightarrow \Lambda_+^k \) (“area colorings”).

We can now define the “shadow invariant” \( |L| \) of the (colored and “horizontally framed”) link \( L \) associated to the pair \((g, k)\) by

\[
|L| := \sum_{\varphi \in \operatorname{col}(L)} |L|_\varphi \cdot |L|_\varphi^2 \cdot |L|_\varphi^3 \cdot |L|_\varphi^4
\]

\(^{64}\)this includes the case of simplicial ribbon links in \( qK \times \mathbb{Z}_N \) as a special case, cf. Remark \([B.2]\) below

\(^{65}\)cf. Remark \([B.2]\) below for the relevance of these two conditions for the present paper
with

\[ |L|_1^\varphi = \prod_{Y \in F(L)} \dim(\varphi(Y)) \chi(Y) \] (B.5a)

\[ |L|_2^\varphi = \prod_{Y \in F(L)} \exp(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle) \] (B.5b)

\[ |L|_3^\varphi = \prod_{e \in E(L)} N_{\gamma(e)} \] (B.5c)

\[ |L|_4^\varphi = \prod_{x \in V(L)} T(x, \varphi) \] (B.5d)

Here \( Y_+ \) (resp. \( Y_- \)) denotes the unique face \( Y \) such that \( \text{arc}(e) \subset \partial Y \) and, additionally, the orientation on \( \text{arc}(e) \) described above coincides with (resp. is opposite to) the orientation which is obtained by restricting the orientation on \( \partial Y \) to \( e \). Moreover, we have set (for \( \lambda, \mu, \nu \in \Lambda_k^+ \))

\[ \dim(\lambda) := \prod_{\alpha \in R_+} \frac{\sin \frac{\pi(\lambda + \rho, \alpha)}{k}}{\sin \frac{\pi(\rho, \alpha)}{k}} \] (B.6)

\[ N^\lambda_{\mu\nu} := \sum_{\tau \in W_k^\nu} \text{sgn}(\tau) m_\mu(\nu - \tau(\lambda)) \] (B.7)

where \( m_\mu(\beta) \) is the multiplicity of the weight \( \beta \) in the unique (up to equivalence) irreducible representation \( \rho_\mu \) with highest weight \( \mu \) and \( W_k^\nu \) is as in part A of the Appendix.

The explicit expression for \( T(x, \varphi) \) appearing in the formula for \( |L|_4^\varphi \) involves the so-called “quantum 6j-symbols” (cf. Chap. X, Sec. 1.2 in [46]) associated to \( U_q(g_C) \) where \( q \) is the root of unity

\[ q := \exp\left(\frac{2\pi i}{k}\right) \] (B.8)

We omit the explicit formula for \( T(x, \varphi) \) since it is irrelevant for the present paper: for links \( L \) fulfilling the aforementioned conditions (NCP) and (NH) of Sec. 6.1 in [29] the set \( V(L) \) is empty and Eq. (B.9) above then reduces to

\[ |L| = \sum_{\varphi \in \text{col}(L)} \left( \prod_{i=1} N_{\gamma_l(\varphi, Y_+^i)} \right) \left( \prod_{Y} \dim(\varphi(Y)) \chi(Y) \exp(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle) \right) \] (B.9)

where we have set \( Y_+^\pm := Y_{1\pm} \).

**Remark B.2** The shadow invariant can be defined in a straightforward for every (colored) simplicial ribbon link \( L \) in \( qK \times \mathbb{Z}_N \) by setting

\[ |L| := |L_f| \]

where \( L_f \) is the (colored and horizontally framed) piecewise smooth link associated to \( L \). Observe that if \( L \) fulfills the conditions (NCP)' and (NH)' of Sec. 3.9 above then \( L_f \) will fulfill the conditions (NCP) and (NH) mentioned above, so in this case Eq. (B.9) above will again hold.

**C Appendix: \( BF_3 \)-theory in the torus gauge**

**C.1 \( BF_3 \)-theory**

Let \( M \) be a closed oriented 3-manifold, let \( \tilde{G} \) be a simple simply-connected compact Lie group with Lie algebra \( \tilde{\mathfrak{g}} \), and let \( \tilde{G} := C^\infty(M, \tilde{G}) \).

\[ 66 \text{We remark that there are different conventions for the definition of } U_q(g_C). \text{ Accordingly, one finds different formulas for } q \text{ in the literature. For example, using the convention in } [45] \text{ one would be led to the formula } q := e^{\frac{D}{2k}} \text{ where } D \text{ is the quotient of the square lengths of the long and the short roots of } \mathfrak{g} \]
For $\tilde{A} \in \tilde{A} := \Omega^1(M, \tilde{g})$ and $\tilde{C} \in \tilde{C} := \Omega^1(M, \tilde{g})$ and $\Lambda \in \mathbb{R}$ (the “cosmological constant”) we define\(^{67}\)

$$S_{BF}(\tilde{A}, \tilde{C}) := \frac{1}{\pi} \int_M \text{Tr}(F^\tilde{A} \wedge \tilde{C} + \frac{\Lambda}{3} \tilde{C} \wedge \tilde{C} \wedge \tilde{C})$$ \hspace{1cm} (C.1)

where $F^\tilde{A} := d\tilde{A} + \tilde{A} \wedge \tilde{A}$. Let us assume in the following that $\Lambda \in \mathbb{R}_+$ and set

$$\kappa := \sqrt{\Lambda}$$ \hspace{1cm} (C.2)

Note that $S_{BF} : \tilde{A} \times \tilde{C} \to \mathbb{C}$ is $\tilde{G}$-invariant under the $\tilde{G}$-operation on $\tilde{A} \times \tilde{C}$ given by $(A, C) \cdot \tilde{\Omega} = (\tilde{\Omega}^{-1} A \tilde{\Omega} + \tilde{\Omega}^{-1} d\tilde{\Omega}, \tilde{\Omega}^{-1} C \tilde{\Omega})$.

It is well-known that in the situation $\kappa \neq 0$ the relation

$$S_{BF}(\tilde{A}, \tilde{C}) = S_{CS}(\tilde{A} + \kappa \tilde{C}) - S_{CS}(\tilde{A} - \kappa \tilde{C})$$ \hspace{1cm} (C.3)

holds with $S_{CS} = S_{CS}(M, G, k)$ where $G := \tilde{G}$ and $k := \frac{1}{\kappa}$. Using the change of variable $(\tilde{A}, \tilde{C}) \to (A_1, A_2)$ given by

$$A_1 := \tilde{A} + \kappa \tilde{C}, \quad A_2 := \tilde{A} - \kappa \tilde{C}$$ \hspace{1cm} (C.4)

or, equivalently,

$$\tilde{A} := \frac{1}{\kappa}(A_1 + A_2), \quad \tilde{C} := \frac{1}{\kappa}(A_1 - A_2)$$ \hspace{1cm} (C.5)

we therefore obtain, informally, for every $\tilde{\chi} : \tilde{A} \times \tilde{C} \to \mathbb{C}$

$$\int \int \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C}$$

$$\sim \int \int \chi((A_1, A_2)) \exp(iS_{CS}(A_1)) \exp(-iS_{CS}(A_2)) DA_1 DA_2$$ \hspace{1cm} (C.6)

where $\chi : A_1 \times A_2 \to \mathbb{C}$ with $A_j := \Omega^1(M, g)$, $j = 1, 2$, is the function given by $\chi((A_1, A_2)) = \tilde{\chi}(\tilde{A}, \tilde{C})$.

If — instead of setting $S_{CS} := S_{CS}(M, G, k)$ with $G := \tilde{G}$ and $k := \frac{1}{\kappa}$ — we use $S_{CS} := S_{CS}(M, G, (k_1, k_2))$ with $G = \tilde{G} \times \tilde{G}$ and $(k_1, k_2) = (1/\kappa, -1/\kappa)$ (cf. Remark 2.2 in\(^{29}\)) then we can\(^{68}\) rewrite Eq. (C.6) as

$$\int \int_{\tilde{A} \times \tilde{C}} \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C} \sim \int_{\tilde{A}} \chi(\tilde{A}) \exp(iS_{CS}(\tilde{A})) DA$$ \hspace{1cm} (C.7)

Thus we see that $BF_3$-theory on $M$ with group $\tilde{G}$ and $\kappa \neq 0$ is essentially equivalent to CS theory on $M$ with group $G = \tilde{G} \times \tilde{G}$ and $(k_1, k_2) = (1/\kappa, -1/\kappa)$.

### C.2 $BF_3$-theory on $M = \Sigma \times S^1$ “in the torus gauge”

Let us now consider the special case where $M = \Sigma \times S^1$, $\kappa \neq 0$ and $1/\kappa \in \mathbb{N}$, and where $\tilde{\chi} : \tilde{A} \times \tilde{C} \to \mathbb{C}$ is of the form

$$\tilde{\chi}(\tilde{A}, \tilde{C}) = \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{\rho_i}(\tilde{A} + \kappa \tilde{C}, \tilde{A} - \kappa \tilde{C}))$$ \hspace{1cm} (C.8)

(with $(l_1, l_2, \ldots, l_m)$ and $(\rho_1, \rho_2, \ldots, \rho_m)$ as in Sec.\(^{21}\)).

Let us now apply “torus gauge fixing”\(^{69}\) to the expression

$$\int \int_{\tilde{A} \times \tilde{C}} \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C}$$ \hspace{1cm} (C.9)

More precisely, we will perform the following three steps:

---

\(^{67}\)Here we assume for simplicity (cf. Remark\(^{24}\) above) that $\tilde{G}$ is Lie subgroup of $U(\tilde{N})$ for some $\tilde{N} \in \mathbb{N}$ and we set $\text{Tr} := \oint_{\text{Mat}(N, \mathbb{C})}$ where $\oint \in \mathbb{R}$ is chosen suitably

\(^{68}\)using $A := A_1 \oplus A_2$ and $DA = DA_1 DA_2$

\(^{69}\)observe that the field $\tilde{C}$ does not transform like a 1-form, so the use of the notion “gauge” in the present context is somewhat misleading
• we make a change of variable from “BF-variables” to “CS-variables” (Step 1)
• we apply torus gauge fixing (Step 2)
• we change back to “BF-variables” (Step 3)

Concretely, these three steps are given as follows:

**Step 1:** We replace the expression (C.9) by the RHS of Eq. (C.7)

**Step 2:** We perform torus gauge-fixing on the RHS of Eq. (C.7), i.e. we replace the RHS of Eq. (C.7) by the RHS of Eq. (2.7) in Sec. 2.3 above (in the situation $G = \tilde{G} \times \tilde{G}$, $T := \tilde{T} \times \tilde{T}$, and $(k_1, k_2) = (1/\kappa, -1/\kappa)$ where $\tilde{T}$ is a fixed maximal torus of $\tilde{G}$)

**Step 3:** We apply the change of variable $(A^\perp, B) \to (\tilde{A}^\perp, \tilde{B})$ given by

\[
\tilde{A}^\perp := \left( \frac{A^\perp + A^\perp_{\text{even}}}{2}, \frac{A^\perp - A^\perp_{\text{even}}}{2\kappa} \right),
\]

\[
\tilde{B} := \left( \frac{B_1 + B_2}{2}, \frac{B_1 - B_2}{2\kappa} \right)
\]

to the RHS of Eq. (2.7) (in the situation $G = \tilde{G} \times \tilde{G}$, $T := \tilde{T} \times \tilde{T}$, and $(k_1, k_2) = (1/\kappa, -1/\kappa)$)

The expression which we obtain after performing the three steps above is the analogue of the RHS of Eq. (7.2) above where instead of the change of variable (7.2) the change of variable (C.10) is used.

**D The spaces $A^\perp_{2N}(K)$, $A^\perp_{\text{altern},1}(K)$, and $A^\perp_{\text{altern},2}(K)$**

We will now reformulate/modify the discretization approach in Sec. 7.4 in a suitable way. This will not only lead to certain stylistic improvements but also to several insights which should be useful in [30]. Let us consider the space

\[
A^\perp_{2N}(K) := \text{Map}(\mathbb{Z}_{2N}, A_{\Sigma, g}(K)) = \text{Map}(\mathbb{Z}_{2N}, C^1(K_1, g) \oplus C^2(K_1, g))
\]

Clearly, we have

\[
A^\perp_{2N}(K) \cong A^\perp_{\text{altern},1}(K) \oplus A^\perp_{\text{altern},2}(K)
\]

where

\[
A^\perp_{\text{altern},1}(K) := \text{Map}(\mathbb{Z}_{2N}^{\text{even}}, C^1(K_1, g)) \oplus \text{Map}(\mathbb{Z}_{2N}^{\text{odd}}, C^1(K_2, g))
\]

\[
A^\perp_{\text{altern},2}(K) := \text{Map}(\mathbb{Z}_{2N}^{\text{even}}, C^1(K_2, g)) \oplus \text{Map}(\mathbb{Z}_{2N}^{\text{odd}}, C^1(K_1, g))
\]

with

\[
\mathbb{Z}_{2N}^{\text{even}} := \{ \pi(t) \mid t \in \{2, 4, 6, \ldots, 2N\}\} \subset \mathbb{Z}_{2N}
\]

\[
\mathbb{Z}_{2N}^{\text{odd}} := \{ \pi(t) \mid t \in \{1, 3, 5, \ldots, 2N - 1\}\} \subset \mathbb{Z}_{2N}
\]

$\pi : \mathbb{Z} \to \mathbb{Z}_{2N}$ being the canonical projection.

Let us now make the connection with the constructions of Sec. 7.4. It is convenient to use the notation $A^\perp_{\text{double}}(K)$ for what was denoted by $A^\perp(K)$ in Sec. 7.4. Observe that

\[
A^\perp_{\text{double}}(K) = \text{Map}(\mathbb{Z}_N, A_{\Sigma, g \oplus g}(K)) \cong A^\perp_1(K) \oplus A^\perp_2(K)
\]
where we have set

\[ A_{j}^{\pm}(K) := \text{Map}(\mathbb{Z}_N, A_{\Sigma,g}(K)) \cong \text{Map}(\mathbb{Z}_N, C^1(K_1, g)) \oplus \text{Map}(\mathbb{Z}_N, C^1(K_2, g)) \]  \hspace{1cm} (D.6)

We now make the identifications

\[ \mathbb{Z}^{\text{even}}_{2N} \cong \mathbb{Z}_N \cong \mathbb{Z}^{\text{odd}}_{2N} \]  \hspace{1cm} (D.7)

which are induced by the bijections \( j_{\text{even}} : \mathbb{Z}_N \to \mathbb{Z}^{\text{even}}_{2N} \) and \( j_{\text{odd}} : \mathbb{Z}_N \to \mathbb{Z}^{\text{odd}}_{2N} \) which are given by

\[ j_{\text{even}}(\pi(t)) = \pi(2t), \quad j_{\text{odd}}(\pi(t)) = \pi(2t - 1) \quad \forall t \in \{1, 2, \ldots, N\} \]

Clearly, the identifications \( (D.7) \) give rise to the identifications

\[ A_{\text{altern},j}^{\pm}(K) \cong A_{j}^{\pm}(K), \quad j = 1, 2, \quad \text{and} \quad A_{\text{double}}^{\pm}(K) \cong A_{2N}^{\pm}(K) \]  \hspace{1cm} (D.8)

Recall that in Sec. \( 7.4 \) we worked with “full ribbons” \( R \), i.e. closed simplicial ribbons in \( K \times \mathbb{Z} = K_1 \times \mathbb{Z} \) and recall also that such a \( R \) induces three loops \( l^+ \), \( l^- \), and \( l \) in a natural way, \( l^\pm \) being simplicial loops in \( K_1 \times \mathbb{Z} \) and \( l \) being a simplicial loop in \( K_2 \times \mathbb{Z} \). In view of the identification \( \mathbb{Z}_N \cong \mathbb{Z}^{\text{even}}_{2N} \) we will now consider \( R \) as a closed simplicial ribbon in \( K_1 \times \mathbb{Z}^{\text{even}}_{2N} \) and the loops \( l^\pm \) as simplicial loops in \( K_1 \times \mathbb{Z}^{\text{even}}_{2N} \) and \( l \) as a simplicial loop in \( K_2 \times \mathbb{Z}^{\text{even}}_{2N} \) (here we have equipped \( \mathbb{Z}^{\text{even}}_{2N} \) with the polyhedral cell complex structure inherited from \( \mathbb{Z}_N \)). After these preparations we can now reinterpret/reformulate the discretization approach of Sec. \( 7.4 \) above in the obvious way. By doing so we obtain the following “Insights”:

**Insight 1:** Recall that on the RHS of Eq. \( (7.22) \) the field component \( A_{j}^{\pm} \) “interacts” only with the loops \( l^+ \) and \( l^- \) while \( A_{j}^{\pm} \) “interacts” with the loop \( l \). In view of the reformulation we just made this ansatz is very natural: If \( A_{1}^{\pm} \) and \( A_{2}^{\pm} \) are given as the components w.r.t. the decomposition \( (D.2) \) then \( A_{1}^{\pm} \) and \( A_{2}^{\pm} \) will “live” on different edges. More precisely, for \( t \in \mathbb{Z}^{\text{even}}_{2N} \) we have \( A_{1}^{1}(t) \in C^1(K_1, g) \) and \( A_{2}^{1}(t) \in C^1(K_2, g) \). Since the \( l^\pm \) and \( l^- \) are simplicial loops in \( K_1 \times \mathbb{Z}^{\text{even}}_{2N} \) and \( l \) is a simplicial loop in \( K_2 \times \mathbb{Z}^{\text{even}}_{2N} \) \( A_{1}^{1} \) will indeed only interact with \( l^+ \) and \( l^- \) and \( A_{2}^{1} \) will interact only with \( l \).

**Insight 2:** Recall that the operators \( L^{(N)}(B_{\pm}) \) appearing on the RHS of Eq. \( (7.21) \) are given by

\[ L^{(N)}(B_{\pm}) := \begin{pmatrix} \hat{L}^{(N)}(B_{\pm}) & 0 \\ 0 & \bar{L}^{(N)}(B_{\pm}) \end{pmatrix} \]  \hspace{1cm} (D.9)

with \( \hat{L}^{(N)}(B_{\pm}) \) and \( \bar{L}^{(N)}(B_{\pm}) \) as in Eq. \( (8.14a) \) and Eq. \( (8.14b) \) in Sec. \( 8.2 \) above (with \( B \) replaced by \( B_{\pm} \) and where the matrix notation refers to the decomposition appearing in Eq. \( (D.6) \) above.

It is natural to ask whether it is possible to rewrite or redefine (if not \( L^{(N)}(B_{\pm}) \) itself then at least) the product \( *_{K} L^{(N)}(B_{\pm}) \) by a formula which involves the (anti-symmetrized) operators \( \hat{L}^{(2N)}(b) \), \( b \in t \), as in Eq. \( (8.12c) \) instead of the operators \( \hat{L}^{(N)}(b) \) and \( \bar{L}^{(N)}(b) \) appearing in Eq. \( (8.12a) \) and Eq. \( (8.12b) \). And indeed, using the aforementioned reformulation of the discretization approach of Sec. \( 7.4 \) this is possible, and we will now explain.

Let \( *_{K} : A_{2N}^{1}(K) \to A_{2N}^{2}(K) \) be the operator defined totally analogously as the operator \( *_{K} : A^{1}(K) \to A^{2}(K) \) in Sec. \( 3.3 \) above but with \( 2N \) playing the role of \( N \). Moreover, let \( L^{(2N)}(B_{\pm}) \) be the operator on

\[ A_{2N}^{1}(K) \cong \langle \oplus_{\epsilon \in \tilde{b}_0(K_1|K_2)} \text{Map}(\mathbb{Z}_{2N}, g) \rangle \oplus \langle \oplus_{\epsilon \in \tilde{b}_0(K_1|K_2)} \text{Map}(\mathbb{Z}_{2N}, g) \rangle \]  \hspace{1cm} (D.10)

which is given by

\[ L^{(2N)}(B_{\pm}) := \langle \oplus_{\epsilon \in \tilde{b}_0(K_1|K_2)} \hat{L}^{(2N)}(B_{\pm}(\epsilon)) \rangle \oplus \langle \oplus_{\epsilon \in \tilde{b}_0(K_1|K_2)} \bar{L}^{(2N)}(B_{\pm}(\epsilon)) \rangle \]  \hspace{1cm} (D.11)
where each $\tilde{L}^{(2N)}(b)$ with $b = B_{\pm}(\bar{e})$ is defined totally analogously as in Eq. (3.12) above (with $N$ replaced by $2N$). Observe that, even though neither of the two operators $*_{K}^{(2N)}$ nor $L^{(2N)}(B_{\pm})$ leaves the two subspaces $A^{\perp}_{\text{altern},j}(K), j = 1, 2$, of $A_{2N}^{1}(K)$ invariant the composition $*_{K}^{(2N)} L^{(2N)}(B_{\pm}) : A_{2N}^{1}(K) \rightarrow A_{2N}^{1}(K)$ does. It turns out that under the identifications (D.8) the operator $*_{K}^{(2N)} L^{(2N)}(B_{\pm})$ is similar but does not quite coincide with the operator $*_{K} L^{(N)}(B_{\pm})$ where $L^{(N)}(B_{\pm})$ is as in Eq. (D.9) above. In [30] we will work with the “new” operators $*_{K}^{(2N)} L^{(2N)}(B_{\pm})$.

E Some remarks on the simplicial program for CS-theory/BF$_{3}$-theory

The goal of what is called the “simplicial program” for CS-theory/BF$_{3}$-theory in [39] (cf. also [5]) is to find a discretized and rigorous version of the non-gauged fixed CS or BF$_{3}$ path integral for the WLOs associated to links in arbitrary oriented closed 3-manifolds $M$ such that the values of these discretized path integrals coincide with the values of the corresponding Reshetikhin-Turaev-Witten invariants. This discretization is supposed to involve only finite triangulations (or, more generally, finite polyhedral cell decompositions) of $M$ and no continuum limit.

As mentioned in Sec. 7.1 above the simplicial program for CS/BF$_{3}$-theory was completed successfully for Abelian structure groups in [1, 2]. For non-Abelian structure groups there are partial results in the case of vanishing cosmological constant, cf. [20, 21, 7, 39]. The case of non-Abelian structure groups with non-vanishing cosmological constant is an important open problem.

One possible strategy for making progress in the simplicial program for non-Abelian CS/BF$_{3}$-theory could be to try to complete, successively, the following three “projects”:

Project 1 Find a simplicial definition of the WLOs associated to general links for non-Abelian CS-theory/BF$_{3}$-theory on the special base manifold $M = \Sigma \times S^{1}$ in the torus gauge.

Clearly, Project 1 is exactly what we are dealing with in [29], the present paper, and its sequel [30]. Theorem 3.4 above can be seen as a first step towards the completion of Project 1. In order to complete Project 1 successfully will have to prove that Eq. (7.25) (or a suitably modified version of it) holds for general simplicial ribbon links in $M = \Sigma \times S^{1}$.

Project 2 Find a simplicial definition of the WLOs associated to general links for the non-gauge fixed non-Abelian CS-theory/BF$_{3}$-theory on $M = \Sigma \times S^{1}$.

Observe that there is a natural “discrete” analogue of the torus gauge fixing procedure. So if one can complete Project 1 successfully there might be a quick way to complete also Project 2. In order to do so we could look at the simplicial definition of the WLOs used in Project 1 and then try to “reverse engineer” from it a non-gauge fixed “version”, i.e. a suitable simplicial expression which – after applying “discrete” torus gauge fixing – leads to the simplicial expression used in Project 1.

Project 3 Generalize the simplicial expressions for the WLOs at which we arrive in Project 2 to arbitrary $M$ and evaluate these expressions explicitly.

One can speculate that if Project 2 could be carried out successfully then the chances for a successful completion of Project 3 would be quite good. All we would have to do then is to find a rigorous implementation of Witten’s surgery procedure, see [31] for some ideas in this direction.

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73 more precisely: if $G$ and $k$ are as in Sec. 2.1 $M$ is a general oriented closed 3-manifold, $L$ a link in $M$, and $\tau_{q}(M, L), q = \exp(2\pi i/k)$ the corresponding Reshetikhin-Turaev-Witten invariant then we want that the discretized rigorous version of WLO($L$) coincides with $\tau_{q}(M, L)$ up to a suitable multiplicative constant.
At the moment it is completely open whether these three “projects” can indeed be carried out successfully. For Project 2 it seems to be necessary to find a way to bypass/eliminate the following “obstructions” or “complications”:

**Complication (C1)** Recall the definition of $\text{Det}^{\text{disc}}_{FP}(B)$ in Eq. (3.21) (cf. also Eq. (7.23)). We had
\[
\text{Det}^{\text{disc}}_{FP}(B) := \prod_{x \in \mathcal{S}_0(qK)} \det(1_t - \exp(\text{ad}(B(x)))|_t)^{1/2}
\]
Discrete torus gauge fixing (cf. the second paragraph in Project 2 above) is related to the map $q : G/T \times T \to G$ given by $q(g,t) = \text{Ad}(g)t$ for $t \in T$ and $gT \in G/T$. Since $(\bar{g},t) \mapsto \det(1_t - \text{Ad}(t)|_\bar{g})$ is the “Jacobian”\textsuperscript{74} of the map $q$ it is easy to see how factors of the form $\det(1_t - \exp(\text{ad}(B(x)))|_t) = \det(1_t - \text{Ad}(\exp(\text{ad}(B(x))))|_t)$ in Eq. (3.21) above can arise from discrete torus gauge fixing.

However, it is not clear how the 1/2-exponents in Eq. (3.21) and Eq. (7.23) above can arise from such a procedure.

**Complication (C2)** In the standard formulation of lattice gauge theory (LGT) the traces of the holonomies associated to “single” (=unframed) loops are gauge-invariant functions. However, in [29] and in the present paper we used closed simplicial ribbons for reasons explained at the end of Sec. 6.1 in [29]. In contrast to the traces of the holonomies associated to single loops the traces of the holonomies associated to closed simplicial ribbons will no longer be gauge-invariant functions. In other words: even if we can resolve Complication (C1) above and we can work within a non-gauged-fixed LGT setting there will not be a totally natural candidate for a “non-gauge fixed version” of the expressions $\text{Tr}_\rho(\text{Hol}^{\text{disc}}_R(A^\perp, B))$ and $\text{Tr}_\rho(\text{Hol}^{\text{disc}}_R(A^\perp, B^+))$, where $\text{Hol}^{\text{disc}}_R(A^\perp, B)$ and $\text{Hol}^{\text{disc}}_R(A^\perp, B^+)$ are as in Eq. (3.18) and Eq. (7.22) above (and where $\rho$ is a fixed finite-dimensional complex representation of $G$).

One possible approach for resolving (C2) could be to use the LGT-analogue of the strategy sketched in the remark at the end of Appendix B in [29] for resolving a similar complication in the continuum setting.

**Complication (C3)** Recall that the sum $\sum_{y \in \mathcal{I}} \cdots$ and the factor $\exp(-2\pi i k \langle y, B(\sigma_0) \rangle)$ in the continuum equation Eq. (2.7) above are the result of certain topological obstructions which arose when applying (continuum) torus gauge fixing, cf. Sec. 2.2.4 in [29]. On the other hand, for discrete torus gauge fixing there are no topological obstructions. Accordingly, it is not clear how – by applying discrete torus gauge fixing one can obtain the sum $\sum_{y \in \mathcal{I}} \cdots$ and the factors $\exp(-2\pi i k \langle y, B(\sigma_0) \rangle)$ appearing in Eq. (3.23) above. Similar remarks apply to the BF-theoretic situation in Sec. 7.3 and Sec. 7.4.

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\textsuperscript{74}more precisely, we have $q^*(\nu_G) = \det(1_t - \text{Ad}(\sigma_2(\cdot)))|_\nu\pi_1^*(\nu_G/T \wedge \pi_2^*(\nu_T))$ where $\nu_G, \nu_T$, and $\nu_G/T$ are the normalized left-invariant volume forms on $G$, $T$, and $G/T$ and $\pi_1 : G/T \times T \to G/T$ and $\pi_2 : G/T \times T \to T$ the canonical projections
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