Discrete Green functions of the SDFEM on Shishkin triangular meshes

Jin Zhang\textsuperscript{*}\dagger

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Abstract

We present bounds of discrete Green functions in the energy norm for the standard (or modified) streamline diffusion finite element method (SDFEM) on Shishkin triangular meshes.

1 Problem

We consider the singularly perturbed boundary value problem

\[-\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f \quad \text{in} \quad \Omega = (0,1)^2,\]
\[u = 0 \quad \text{on} \quad \partial \Omega,\]

where $\varepsilon \ll |\mathbf{b}|$ is a small positive parameter, $\mathbf{b} = (b_1, b_2)^T$ is a constant vector with $b_1 > 0, b_2 > 0$ and $c > 0$ is constant. It is also assumed that $f$ is sufficiently smooth. The solution of (1.1) typically has two exponential layers of width $O(\varepsilon \ln(1/\varepsilon))$ at the sides $x = 1$ and $y = 1$ of $\Omega$.

2 The SDFEM on Shishkin meshes

2.1 Shishkin meshes

When discretizing (1.1), we use Shishkin meshes, which are piecewise uniform. See [3, 7] for a detailed discussion of their properties and applications.

First, we define two mesh transition parameters, which are to be used to specify the mesh changes from coarse to fine in $x-$ and $y-$direction,

\[
\lambda_x := \min \left\{ \frac{1}{2}, \frac{\rho \varepsilon}{\beta_1} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{2}, \frac{\rho \varepsilon}{\beta_2} \ln N \right\},
\]

\textsuperscript{*}Email: jinzhangalex@hotmail.com
\textsuperscript{1}School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China
Figure 1: Dissection of $\Omega$ and triangulation $T_N$.  

Figure 2: $K_{i,j}^1$ and $K_{i,j}^2$

where $\beta_1$ and $\beta_2$ are defined as in [3] Assumption 2.1. For technical reasons, we set $\rho = 2.5$ in our analysis which is the same as ones in [9] and [8]. The domain $\Omega$ is dissected into four parts as $\bar{\Omega} = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$ (see Figure 1), where

$$
\Omega_s := [0, 1 - \lambda_x] \times [0, 1 - \lambda_y], \quad \Omega_x := [1 - \lambda_x, 1] \times [0, 1 - \lambda_y],
$$
$$
\Omega_y := [0, 1 - \lambda_x] \times [1 - \lambda_y, 1], \quad \Omega_{xy} := [1 - \lambda_x, 1] \times [1 - \lambda_y, 1].
$$

Assumption 1. We assume that $\varepsilon \leq N^{-1}$, as is generally the case in practice. Furthermore we assume that $\lambda_x = \rho \varepsilon \beta_1^{-1} \ln N$ and $\lambda_y = \rho \varepsilon \beta_2^{-1} \ln N$ as otherwise $N^{-1}$ is exponentially small compared with $\varepsilon$.

Next, we define the set of mesh points $\{(x_i, y_j) : (x_i, y_j) \in \Omega, i = 0, \cdots, N\}$

$$
x_i = \begin{cases} 
2i(1 - \lambda_x)/N & \text{for } i = 0, \cdots, N/2, \\
1 - 2(N - i)\lambda_x/N & \text{for } i = N/2 + 1, \cdots, N
\end{cases}
$$

and

$$
y_j = \begin{cases} 
2j(1 - \lambda_y)/N & \text{for } j = 0, \cdots, N/2, \\
1 - 2(N - j)\lambda_y/N & \text{for } j = N/2 + 1, \cdots, N
\end{cases}
$$

By drawing lines through these mesh points parallel to the $x$-axis and $y$-axis the domain $\Omega$ is partitioned into rectangles. Each rectangle is divided into two triangles by drawing the diagonal which runs from $(x_i, y_{j+1})$ to $(x_{i+1}, y_j)$. This yields a triangulation of $\Omega$ denoted by $\mathcal{T}_N$ (see Fig. 1). The mesh sizes $h_{x,i} := x_{i+1} - x_i$ and $h_{y,j} := y_{j+1} - y_j$ satisfy

$$
h_{x,i} = \begin{cases} 
H_x := \frac{1 - \lambda_x}{N/2} & \text{for } i = 0, \cdots, N/2 - 1, \\
h_x := \frac{\lambda_x}{N/2} & \text{for } i = N/2, \cdots, N - 1
\end{cases}
$$
and
\[
    h_{y,j} = \begin{cases} 
        H_y := \frac{1 - \lambda_y}{N/2} & \text{for } j = 0, \ldots, N/2 - 1, \\
        h_y := \frac{\lambda_y}{N/2} & \text{for } j = N/2, \ldots, N - 1.
    \end{cases}
\]

The mesh sizes \( h_{x,i} \) and \( h_{y,j} \) satisfy
\[
    N^{-1} \leq H_x, H_y \leq 2N^{-1} \quad \text{and} \quad C_1 \varepsilon N^{-1} \ln N \leq h_x, h_y \leq C_2 \varepsilon N^{-1} \ln N.
\]

For convenience, we shall use some notations: \( K_{i,j}^1 \) for the mesh triangle with vertices \( (x_i, y_j), (x_{i+1}, y_j) \) and \( (x_i, y_{j+1}) \); \( K_{i,j}^2 \) for the mesh triangle with vertices \( (x_i, y_{j+1}), (x_{i+1}, y_j) \) and \( (x_{i+1}, y_{j+1}) \) (see Fig. 2); \( \tau \) or \( K \) for a generic mesh triangle.

2.2 The streamline diffusion finite element method

On the above Shishkin meshes we define a \( C^0 \) linear finite element space
\[
    V^N := \{ v^N \in C(\bar{\Omega}) : v^N|_{\partial \Omega} = 0 \text{ and } v^N|_K \in P_1(K), \forall K \in T_N \}.
\]

Now we are in a position to state the standard SDFEM for (1.1) which reads:
\[
    \begin{cases} 
        \text{Find } u^N \in V^N \text{ such that for all } v^N \in V^N, \\
        a_{SD}(u^N, v^N) = (f, v^N) + \sum_{K \subset \Omega} (f, \delta_K b \cdot \nabla v^N)_K, 
    \end{cases}
\]
(2.1)

where
\[
    a_{SD}(u^N, v^N) = \varepsilon (\nabla u^N, \nabla v^N) + (b \cdot \nabla u^N, v^N) + (c u^N, v^N) \\
    + \sum_{K \subset \Omega} (-\varepsilon \Delta u^N + b \cdot \nabla u^N + c u^N, \delta_K b \cdot \nabla v^N)_K.
\]

Note that \( \Delta (u^N|_K) = 0 \) for \( u^N|_K \in P_1(K) \). Following usual practice \[7\], the parameter \( \delta_K := \delta|_K \) is defined as follows
\[
    \delta := \delta(x, y) = \begin{cases} 
        C^* N^{-1} & \text{if } (x, y) \in \Omega_s, \\
        0 & \text{otherwise,}
    \end{cases}
\]
(2.2)

where \( C^* \) is referred to \[7\] Lemma 3.25].

We set
\[
    b := \sqrt{b_1^2 + b_2^2}, \quad \beta := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}/b, \quad \eta := \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix}/b \quad \text{and} \quad \nu_\zeta := \zeta^T \nabla v
\]
for any vector \( \zeta \) of unit length. By an easy calculation one shows that
\[
    (\nabla w, \nabla v) = (w_\beta, v_\beta) + (w_\eta, v_\eta).
\]
We rewrite (2.1) as
\[
\varepsilon(u_{\beta}^N, v_{\eta}^N) + \varepsilon(u_{\eta}^N, v_{\beta}^N) + (bu_{\beta}^N + u^N, v^N) + \sum_{K \subset \Omega} (bu_{\beta}^N + cu^N, \delta_K bv_{\beta}^N)_K \\
= (f, v^N) + \sum_{K \subset \Omega} (f, \delta_K bv_{\beta}^N)_K.
\]

For technical reasons in the later analysis, we increase the crosswind diffusion (see [1, 4, 6]) by replacing \(\varepsilon(u_{\beta}^N, v_{\eta}^N)\) by \(\hat{\varepsilon}(u_{\beta}^N, v_{\eta}^N)\) where \(\varepsilon \leq \hat{\varepsilon} \leq N^{-1}\) and \(\hat{\varepsilon}\) is constant on each of subdomains including \(\Omega_s, \Omega_s \setminus \Omega_s\). For convenience, we denote \(\hat{\varepsilon}|_{\Omega_s}\) by \(\hat{\varepsilon}_s\). We will consider two cases: (1) \(\hat{\varepsilon} = \varepsilon\); (2) \(\hat{\varepsilon}\) defined as in [3, pg 463], that is
\[
\hat{\varepsilon} = \begin{cases} 
\hat{\varepsilon} & \text{if } x \in \Omega_s, \\
\varepsilon & \text{if } x \in \Omega \setminus \Omega_s
\end{cases}
\]
where \(\hat{\varepsilon} := \max(\varepsilon, N^{-3/2})\).

We now state a streamline diffusion method with artificial crosswind diffusion (ACD), also called a modified SDFEM:
\[
\begin{cases}
\text{Find } u^N \in V^N \text{ such that for all } v^N \in V^N \\
a_{MSD}(u^N, v^N) = (f, v^N + \delta bv_{\beta}^N),
\end{cases}
\]
with
\[
a_{MSD}(u^N, v^N) := \varepsilon(u_{\beta}^N, v_{\eta}^N) + \hat{\varepsilon}(u_{\eta}^N, v_{\beta}^N) + (bu_{\beta}^N + u^N, v^N) + \sum_{K \subset \Omega} (bu_{\beta}^N + u^N, \delta_K bv_{\beta}^N)_K. \tag{2.3}
\]

Finally, we define a special energy norm associated with \(a_{MSD}(. , .)\):
\[
\|v^N\|_{MSD}^2 := \varepsilon\|v_{\beta}^N\|^2 + \hat{\varepsilon}\|v_{\eta}^N\|^2 + \|v^N\|^2 + \sum_{K \subset \Omega} \delta_K \|bv_{\beta}^N\|^2_K, \quad \forall v^N \in V^N.
\]

For brevity, we often write \(\varepsilon(u_{\beta}^N, v_{\eta}^N) + \sum_{K \subset \Omega} (bu_{\beta}^N, \delta_K bv_{\beta}^N)_K\) as \((\varepsilon + b^2\delta)(u_{\beta}^N, v_{\eta}^N)\).

### 3 The discrete Green function

Let \(x^*\) be a mesh node in \(\Omega\). The discrete Green’s function \(G \in V^N\) associated with \(x^*\) is defined by
\[
a_{MSD}(v^N, G) = v^N(x^*) \quad \forall v^N \in V^N. \tag{3.1}
\]

The weight function \(\omega\) is defined by
\[
\omega(x) := g \left( \frac{(x-x^*) \cdot \beta}{\sigma_{\beta}} \right) g \left( \frac{(x-x^*) \cdot \eta}{\sigma_{\eta}} \right) g \left( \frac{(x-x^*) \cdot \eta}{\sigma_{\eta}} \right)
\]
where
\[
g(r) = \frac{2}{1 + e^r}
\]
for \( r \in (-\infty, \infty) \).

Now, we are to derive a global estimate on \( G \) in the weighted energy norm
\[
\|G\|_2^2 = \varepsilon \|\omega^{-1/2} G_\beta\|_2^2 + \varepsilon \|\omega^{-1/2} G_\eta\|_2^2 + \frac{b}{2} \|\omega^{-1/2} G\|_2^2
\]
\[
+ c \|\omega^{-1/2} G\|_2^2 + \sum_K b^2 \delta_K \|\omega^{-1/2} G_\beta\|_K^2.
\]

From (2.3) and (3.2), we have
\[
\|G\|_2^2 = a_{\text{MSD}}(\omega^{-1} G, G) - \varepsilon (\omega^{-1} \beta G, G_\beta) - \varepsilon (\omega^{-1} G_\eta G, G_\eta)
\]
\[
- \sum_K (b(\omega^{-1}) G + c \omega^{-1} G, \delta_K b G_\beta)_K.
\]

Considering (3.4) we also have
\[
a_{\text{MSD}}(\omega^{-1} G, G) = a_{\text{MSD}}(E, G) + a_{\text{MSD}}((\omega^{-1} G)^t, G)
\]
\[
= a_{\text{MSD}}(E, G) + (\omega^{-1} G)(x^*)
\]
where \( E := \omega^{-1} G - (\omega^{-1} G)^t \).

**Lemma 3.1.** If
\[
\sigma_\beta \geq k(\varepsilon + \delta_M) \quad \text{and} \quad \sigma_\eta \geq k \varepsilon_M^{1/2},
\]
where \( \delta_M := \max_{x \in \Omega} \delta \) and \( \varepsilon_M := \max_{x \in \Omega} \varepsilon \), then for \( k > 1 \) sufficiently large and independent of \( N \) and \( \varepsilon \), we have
\[
a_{\text{MSD}}(\omega^{-1} G, G) \geq \frac{1}{4} \|G\|_2^2.
\]

**Proof.** Hölder inequalities, Cauchy inequalities and [3] Lemma 4.1 give
\[
(\varepsilon + b^2 \delta) \|((\omega^{-1}) G, G_\beta)\|
\leq C \varepsilon^{1/2} \sigma_\beta^{-1/2} \|((\omega^{-1}) G)\|^{1/2} \|\varepsilon^{1/2} \|\omega^{-1/2} G_\beta\|
\]
\[
+ \sum_K C b \delta_K^{1/2} \sigma_\beta^{-1/2} \|((\omega^{-1}) G)\|^{1/2} \|\varepsilon^{1/2} \|\omega^{-1/2} G_\beta\|_K
\]
\[
\leq \frac{1}{4} \varepsilon \|\omega^{-1/2} G_\beta\|_2^2 + C \varepsilon \sigma_\beta^{-1} \|\omega^{-1/2} G\|_2^2 + \frac{1}{4} \sum_K b^2 \delta_K \|\omega^{-1/2} G_\beta\|_K^2
\]
\[
+ C \sum_K \delta_K \sigma_\beta^{-1} \|((\omega^{-1}) G)\|^{1/2} \|\omega^{-1/2} G_\beta\|_2^2
\]
\[
\leq \frac{1}{4} \varepsilon \|\omega^{-1/2} G_\beta\|_2^2 + \frac{1}{4} \sum_K b^2 \delta_K \|\omega^{-1/2} G_\beta\|_K^2 + C (\varepsilon + \delta) \sigma_\beta^{-1} \|((\omega^{-1}) G)\|^{1/2} \|\omega^{-1/2} G_\beta\|_2^2.
\]
Similarly, we have
\[
\tilde{\varepsilon} \|(\omega^{-1})_\beta G, G_\eta)\| \leq C \varepsilon^{1/2} \sigma_-^{-1} \cdot \|\omega^{-1/2} G\| \cdot \varepsilon^{1/2} \|\omega^{-1/2} G_\eta\|
\leq \frac{1}{2} \tilde{\varepsilon} \|\omega^{-1/2} G_\eta\|^2 + \frac{1}{2} C \varepsilon^{1/2} \sigma_- \|\omega^{-1/2} G\|^2
\]
and
\[
\sum_K (\omega^{-1} G, \delta K, b G_\beta)_K \leq \sum_K c^{1/2} \delta K \cdot c^{1/2} \|\omega^{-1/2} G\|_K \cdot b \delta K \cdot \|\omega^{-1/2} G_\beta\|_K
\leq \frac{1}{4} \|\omega^{-1/2} G\|^2 + \sum_K c \delta K \cdot b^2 \delta K \|\omega^{-1/2} G_\beta\|_K
\leq \frac{1}{4} \|\omega^{-1/2} G\|^2 + \frac{1}{2} \sum_K b^2 \delta K \|\omega^{-1/2} G_\beta\|_K.
\]

If (3.3) holds true and \( k \) is taken sufficiently large and independent of \( N \) and \( \varepsilon \), then we have
\[
(\varepsilon + b^2 \delta)|((\omega^{-1})_\beta G, G_\beta)| + \varepsilon|((\omega^{-1})_\gamma G, G_\eta)| + \sum_K |((\omega^{-1} G, \delta K, b G_\beta)_K| \leq \frac{3}{4} \|G\|_\omega^2.
\]
Considering (3.3), we are done.

\[\Box\]

Lemma 3.2. If \( 1 \geq \sigma_\beta \geq k(\varepsilon + \delta_M) \), with \( k > 0 \) sufficiently large and independent of \( N \) and \( \varepsilon \). Then for each mesh point \( \mathbf{x}^* \in \Omega \setminus \Omega_{xy} \), we have
\[
|((\omega^{-1})_\beta G)(\mathbf{x}^*)| \leq \frac{1}{16} \|G\|_\omega^2 + \begin{cases} 
CN^2 \sigma_\beta & \text{if } \mathbf{x}^* \in \Omega_x, \\
CN \ln N & \text{if } \mathbf{x}^* \in \Omega_x \cup \Omega_y
\end{cases}
\]
where \( C \) is independent of \( N, \varepsilon \) and \( \mathbf{x}^* \).

Proof. First let \( \mathbf{x}^* \in \Omega_x \). Let \( \tau^* \) be the unique triangle that has \( \mathbf{x}^* \) as its north-east corner. Then
\[
|((\omega^{-1} G)(\mathbf{x}^*))| \leq CN \|G\|_{\tau^*} \leq CN \max_{\tau^*} \left|((\omega^{-1})_\beta)^{-1/2}\right| \cdot \|((\omega^{-1})_\beta)^{1/2} G\|_{\tau^*}.
\]
Calculating \( ((\omega^{-1})_\beta)^{-1}(\mathbf{x}) \) explicitly, we see that
\[
((\omega^{-1})_\beta^{-1}(\mathbf{x})) \leq C \sigma_\beta \quad \forall \mathbf{x} \in \tau^*.
\]
Thus the arithmetic-geometric mean inequality gives
\[
|((\omega^{-1} G)(\mathbf{x}^*))| \leq CN \sigma_\beta + \frac{1}{16} \|G\|_\omega^2.
\]
Next, let \( \mathbf{x}^* \in \Omega_x \). (The case \( \mathbf{x}^* \in \Omega_y \) is similar.) Write \( \mathbf{x}^* = (x_i, y_j) \). Then
\[
|((\omega^{-1} G)(\mathbf{x}^*))| = |G(\mathbf{x}^*)| = \left| \int_{x_i}^{x_{i+1}} G_x(t, y_j)dt \right| \leq CH_y^{-1} \int_{x_i}^{x_{i+1}} |G_x(t, y_j)| dy dt 
\leq CN(\varepsilon \ln N \cdot N^{-1})^{1/2} \|G_x\|_{\Omega_x} \leq CN^{1/2} \ln^{1/2} N \|G\|
\leq CN \ln N + \frac{1}{16} \|G\|_\omega^2.
\]
Thus, we are done.

Lemma 3.3. If
\[
(\varepsilon + b^2\delta)^{1/2}\|\omega^{1/2}E_\beta\| + \delta_s^{1/2}\|\omega^{1/2}E_\beta\| + \delta_s^{-1/2}\|\omega^{1/2}E\|_{\Omega_s} + \varepsilon^{-1/2}\|\omega^{1/2}E\|_{\Omega_s} \leq Ck^{-1/2}\|G\|_\omega.
\]
where \(k > 1\) sufficiently large and independent of \(N\) and \(\varepsilon\), then
\[
a_{MSD}(E, G) \leq \frac{1}{16}\|G\|^2_\omega
\]
where \(E = (\omega^{-1}G)^I - \omega^{-1}G\).

Proof. Hölder inequality gives
\[
|a_{MSD}(E, G)| \leq (\varepsilon + b^2\delta)^{1/2}\|\omega^{1/2}E_\beta\| \cdot (\varepsilon + b^2\delta)^{1/2}\|\omega^{-1/2}G_\beta\| + \varepsilon^{1/2}\|\omega^{1/2}E_\eta\| \cdot \varepsilon^{1/2}\|\omega^{-1/2}G_\eta\| + C\|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G_\beta\| + \|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G\|.
\]
\[
\leq \frac{1}{16}\|G\|^2_\omega, \text{ apply (3.6) and take } k \text{ sufficiently large.}
\]

Set \(h_\tau = \max\{h_{x, \tau}, h_{y, \tau}\}\). For \(\forall \tau \in T_N\) we have
\[
\|\omega^{1/2}E\|_\tau \leq C(\max_{\tau} \omega)^{1/2} \cdot h_\tau^2 \left(\|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\eta\|_\tau\right)
\]
and
\[
\|\omega^{1/2}E_\beta\|_\tau \leq C(\max_{\tau} \omega)^{1/2} \cdot h_\tau \left(\|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\eta\|_\tau\right),
\]
\[
\|\omega^{1/2}E_\eta\|_\tau \leq C(\max_{\tau} \omega)^{1/2} \cdot h_\tau \left(\|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\eta\|_\tau\right),
\]
where we have used [3] Corollary 3.1. Clearly, the following estimate holds
\[
\|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\eta\|_\tau \leq C \left(\|(\omega^{-1}G)_\beta\|_\tau + \|(\omega^{-1}G)_\beta\|_\tau\right) + C \left(\|(\omega^{-1})\beta\|_\beta G_\tau + \|(\omega^{-1})\beta\|_\beta G_\tau + \|(\omega^{-1})\beta\|_\beta G_\tau\right) + C \left(\|(\omega^{-1})\beta\|_\beta G_\tau + \|(\omega^{-1})\beta\|_\beta G_\tau\right).
\]

Set \(\alpha_\omega := (\min_{\tau} \alpha_\omega)^{-1/2}\). Lemma 4.1 in [3] yields
\[
\|(\omega^{-1})\beta\|_\tau \leq C \alpha_\omega \cdot \sigma^{-3/2}_{\beta} \cdot \|(\omega^{-1})\beta\|_\tau,
\]
\[
\|(\omega^{-1})\beta\|_\tau \leq C \alpha_\omega \cdot \sigma^{-1}_{\beta} \cdot \|(\omega^{-1})\beta\|_\tau,
\]
\[
\|(\omega^{-1})\beta\|_\tau \leq C \alpha_\omega \cdot \sigma^{-1}_{\beta} \cdot \|(\omega^{-1})\beta\|_\tau,
\]
\[
\|(\omega^{-1})\beta\|_\tau \leq C \alpha_\omega \cdot \sigma^{-1}_{\beta} \cdot \|(\omega^{-1})\beta\|_\tau,
\]
\[
\|(\omega^{-1})\beta\|_\tau \leq C \alpha_\omega \cdot \sigma^{-2}_{\beta} \cdot \|(\omega^{-1})\beta\|_\tau.
\]
\[
(3.7)
\]
Similarly, for $\tau \subset \Omega_s$, we have
\[
\|(\omega^{-1})_{\eta}G_\eta\|_\tau \leq C\alpha_\omega \cdot \begin{cases} 
\sigma_{\eta}^{-1} N \cdot \|\omega^{-1/2}G\|_\tau & \text{if } \hat{\varepsilon}_{s} \leq N^{-2} \\
\sigma_{\eta}^{-1} \hat{\varepsilon}_{s}^{-1/2} \cdot \hat{\varepsilon}_{s}^{1/2} \|\omega^{-1/2}G_\eta\|_\tau & \text{if } \hat{\varepsilon}_{s} \geq N^{-2}
\end{cases}
\] (3.8)
and for $\tau \subset \Omega \Omega_s$,
\[
\|(\omega^{-1})_{\eta}G_\eta\|_\tau \leq C\alpha_\omega \cdot \varepsilon^{-1/2} \sigma_{\eta}^{-1} \cdot \varepsilon^{1/2} \|\omega^{-1/2}G_\eta\|_\tau.
\] (3.9)

For $\tau \subset \Omega \Omega_s$, we have
\[
\|(\omega^{-1})_{\beta}G_\eta\|_\tau \leq C \max(\omega^{-1})_{\beta} \cdot |G_\eta|_\tau \leq C \max(\omega^{-1})_{\beta} \cdot N \cdot |G|_\tau \\
\leq CN \left( \max(\omega^{-1})_{\beta} \right)^{1/2} \left( \min(\omega^{-1})_{\beta} \right)^{1/2} \cdot |G|_\tau \\
\leq C\alpha_\omega \cdot N\sigma_{\beta}^{-1/2} \cdot |(\omega^{-1})_{\beta}^{1/2}G|_\tau
\] (3.10)
where we have used \[3,\] Lemma 4.1 (vii). For $\tau \subset \Omega \Omega_s$, we have
\[
\|(\omega^{-1})_{\beta}G_\eta\|_\tau \leq C\alpha_\omega \cdot \varepsilon^{-1/2} \sigma_{\beta}^{-1} \cdot \varepsilon^{1/2} \|\omega^{-1/2}G_\eta\|_\tau.
\] (3.11)

Set
\[
\sigma_\eta(\hat{\varepsilon}_{s}) = \begin{cases} 
\sigma_{\eta}^{-1} N & \text{if } \hat{\varepsilon}_{s} \leq N^{-2} \\
\sigma_{\eta}^{-1} \hat{\varepsilon}_{s}^{-1/2} & \text{if } \hat{\varepsilon}_{s} \geq N^{-2}
\end{cases}
\] (3.12)

From (3.7)–(3.11) and \[3,\] Lemma 4.1 (iii), we have
\[
\|\omega^{1/2}E\|_{\Omega_s} + N^{-1} \left( \|\omega^{1/2}E_{\beta}\|_{\Omega_s} + \|\omega^{1/2}E_{\eta}\|_{\Omega_s} \right) \\
\leq CN^{-2} \left( \sigma_{\beta}^{-3/2} + \sigma_{\beta}^{-1} \delta_{s}^{-1/2} + \sigma_{\beta}^{-1/2} \sigma_{\eta}^{-1} + \sigma_{\eta}^{-1} \delta_{s}^{-1/2} + \sigma_{\eta}^{-2} + \sigma_\eta(\hat{\varepsilon}_{s}) + \sigma_{\beta}^{-1/2} N \right) \|G\|_{\omega,\Omega_s} \\
\|\omega^{1/2}E\|_{\Omega,\Omega_s} + N^{-1} \left( \|\omega^{1/2}E_{\beta}\|_{\Omega,\Omega_s} + \|\omega^{1/2}E_{\eta}\|_{\Omega,\Omega_s} \right)
\] (3.13)

To make sure that (3.6) holds, we should have
\[
\|\omega^{1/2}E_{\beta}\|_{\Omega_s} \leq Ck^{-1/2} (\varepsilon + \delta_s)^{-1/2} \|G\|_{\omega,\Omega_s},
\] (3.14)
\[
\|\omega^{1/2}E_{\eta}\|_{\Omega_s} \leq Ck^{-1/2} \hat{\varepsilon}_s^{-1/2} \|G\|_{\omega,\Omega_s},
\] (3.15)
\[
\|\omega^{1/2}E_{\beta}\|_{\Omega,\Omega_s} + \|\omega^{1/2}E_{\eta}\|_{\Omega,\Omega_s} \leq Ck^{-1/2} \varepsilon^{-1/2} \|G\|_{\omega,\Omega,\Omega_s},
\] (3.16)
\[
\|\omega^{1/2}E\|_{\Omega_s} \leq Ck^{-1/2} \delta_s^{1/2} \|G\|_{\omega,\Omega_s},
\] (3.17)
and
\[
\|\omega^{1/2}E\|_{\Omega,\Omega_s} \leq Ck^{-1/2} \varepsilon^{1/2} \|G\|_{\omega,\Omega,\Omega_s}.
\] (3.18)

To ensure that (3.14) holds true, we can set
\[
\sigma_{\beta} \geq kN^{-1},
\]
\[
\sigma_{\eta} \geq kN^{-3/4}, \quad \sigma_{\eta} \geq \sigma_{\eta}^* := \begin{cases} 
kN^{-1/2} & \text{if } \hat{\varepsilon}_{s} \leq N^{-2} \\
\hat{\varepsilon}_{s}^{-1/2} N^{-3/2} & \text{if } \hat{\varepsilon}_{s} \geq N^{-2}
\end{cases}
\] (3.19)
Assume that (3.19) holds true, then (3.14)–(3.17) hold and we have
\[
\|\omega^{1/2}E\|_{1,\Omega_x} \leq Ck^{-1}\varepsilon^{-1/2}N^{-1}\|G\|_{\omega}.
\] (3.20)

Next, we are to obtain sharper bounds for \[\omega_{1}^{2}\] \(E_{\Omega_{z}}\Omega_{s}\). Following the techniques of (see [3, Lemma 4.4]), we have
\[
(\omega^{1/2}E)(x) = \int_{x}^{\Gamma(x)} (\omega^{1/2}E)_{\eta}ds
\] (3.21)
where \(x \in \Omega \setminus \Omega_{s}\), \(\Gamma(x) \in \partial \Omega\) satisfies \((x - \Gamma(x)) \cdot \beta = 0\) and the following condition:
\[
|x - \Gamma(x)| = \min_{y} |x - y|, \text{ where } y \in \partial \Omega \text{ and } (x - y) \cdot \beta = 0,
\]
From (3.21), we have
\[
\|\omega^{1/2}E\|^{2}_{1,\Omega_x \cup \Omega_{s}} = \int_{1-\lambda}^{1} \int_{0}^{\lambda} \left[ \int_{x}^{\Gamma(x)} (\omega^{1/2}E)_{\eta}ds \right]^{2} dy dx
\]
\[
\leq \lambda^{2} \left\{ \|\omega^{1/2}E\|^{2}_{2,\Omega_x} + \|\omega^{1/2}E_{\eta}\|^{2}_{2,\Omega_{s}} \right\}
\]
\[
\leq C\varepsilon^{2}\ln^{2}N \left( \sigma_{\beta}^{-2}\|\omega^{1/2}E\|^{2}_{2,\Omega_x} + \|\omega^{1/2}E_{\eta}\|^{2}_{2,\Omega_{s}} \right).
\]
Similar argument holds for \(\Omega_{y}\). Thus, we have
\[
\|\omega^{1/2}E\|^{2}_{2,\Omega_x} \leq C\varepsilon^{2}\ln^{2}N \left( \sigma_{\beta}^{-2}\|\omega^{1/2}E\|^{2}_{2,\Omega_x} + \|\omega^{1/2}E_{\eta}\|^{2}_{2,\Omega_{s}} \right). \tag{3.22}
\]
To make sure that (3.18) holds, according to (3.22) the following estimates should hold
\[
\varepsilon^{2}\ln^{2}N \sigma_{\beta}^{-2}\|\omega^{1/2}E\|^{2}_{2,\Omega_x} \leq Ck^{-1}\varepsilon\|G\|_{\omega}^{2,\Omega_{x}}
\]
\[
\|\omega^{1/2}E_{\eta}\|^{2}_{2,\Omega_{s}} \leq Ck^{-1}\varepsilon^{-1}\ln^{-2}N \|G\|_{\omega}^{2,\Omega_{s}} \tag{3.23}
\]
From (3.22) and (3.20), we can set
\[
\sigma_{\beta} \geq kN^{-1}\ln N, \quad \sigma_{\eta} \geq kN^{-1}\ln N, \quad \sigma_{\eta} \geq k\varepsilon^{1/4}N^{-1/2}\ln^{1/2}N. \tag{3.24}
\]
Assume that (3.19) and (3.24) hold true, substituting (3.20) and (3.23) into the right-hand side of (3.22), we have
\[
\|\omega^{1/2}E\|^{2}_{1,\Omega_{x}} \leq Ck^{-1}\varepsilon\|G\|_{\omega}^{2} \tag{3.25}
\]
Thus, we have the following lemma.
Thus we are done.

Assume that Theorem 1.
and \( x \) is chosen so that Lemmas 3.1, 3.2 and 3.3 hold. Then for \( N \) sufficiently large and independent of Lemma 3.4.
Assume that \( \sigma \) where \( \beta \).

\( \epsilon \)

If \( \epsilon \) Now, we consider the case of \( \hat{\epsilon} \). From (3.5), (3.19) and (3.24) we have

\[
\begin{align*}
\| \omega^{1/2} E_{\beta} \|_{\Omega_x} &\leq C k^{-1/2} N^{-1/2} \| G \|_{\omega, \Omega_x}, \\
\| \omega^{1/2} E_{\eta} \|_{\Omega_x} &\leq C k^{-1/2} \epsilon^{-1/2} \| G \|_{\omega, \Omega_x}.
\end{align*}
\]

and

\[
\begin{align*}
\| \omega^{1/2} E_{\beta} \|_{\Omega_x} + &\| \omega^{1/2} E_{\eta} \|_{\Omega_x} \leq C k^{-1/2} N^{1/2} \| G \|_{\omega, \Omega_x}, \\
\| \omega^{1/2} E_{\beta} \|_{\Omega_x} + &\| \omega^{1/2} E_{\eta} \|_{\Omega_x} \leq C k^{-1/2} \epsilon^{-1/2} \ln^{-1} N \| G \|_{\omega, \Omega_x}.
\end{align*}
\]

Theorem 1. Assume that \( \sigma_{\beta} \) and \( \sigma_{\eta} \) satisfy (3.5), (3.19) and (3.24), where \( k \) is chosen so that Lemmas 3.1, 3.2 and 3.3 hold. Then for \( x^* \in \Omega \setminus \Omega_{xy} \), we have

\[
\| G \|_{MSD} \leq \sqrt{8} \| G \|_{\omega} \leq \begin{cases} 
CN \sigma_{\beta}^{1/2} & \text{if } x^* \in \Omega_x \\
CN^{1/2} \ln^{1/2} N & \text{if } x^* \in \Omega_x \cup \Omega_y
\end{cases}
\]

Proof. The readers are referred to [3] Theorem 4.1] for the estimate \( \| G \|_{MSD} \leq \sqrt{8} \| G \|_{\omega} \). Considering (3.3), (3.4) and Lemmas 3.1, 3.3 we obtain

\[
\begin{align*}
\frac{1}{4} \| G \|_{\omega}^2 \leq & a_{MSD}(\omega^{-1} G, G) = a_{MSD}(E, G) + (\omega^{-1} G)(x^*) \\
\leq & \frac{1}{8} \| G \|_{\omega}^2 + \begin{cases} 
CN^{2} \sigma_{\beta} & \text{if } x^* \in \Omega_x \\
CN \ln N & \text{if } x^* \in \Omega_x \cup \Omega_y
\end{cases}
\end{align*}
\]

Thus we are done. \( \square \)

From (3.5), (3.19) and (3.24) we have

\[
\sigma_{\beta} \geq k N^{-1} \ln N, \quad \sigma_{\eta} \geq k \max\{\hat{\epsilon}^{1/2}, N^{-3/4}, \sigma_{\beta}^*, N^{-1} \ln N, \epsilon^{1/4} N^{-1/2} \ln^{1/2} N\}
\]

where \( \sigma_{\beta}^* = \begin{cases} 
k N^{-1/2} & \text{if } \hat{\epsilon}_s \leq N^{-2} \\
k \hat{\epsilon}_s^{-1/2} N^{-3/2} & \text{if } \hat{\epsilon}_s \geq N^{-2}
\end{cases} \). Note that if \( N \geq 4 \), then \( N^{1/2} \geq \ln N \)

\[
N^{-1/2} \geq N^{-1} \ln N, \quad N^{-1/2} \geq N^{-1/2} \ln^{1/2} N \quad \text{for } N \geq 4.
\]

Now, we consider the case of \( \hat{\epsilon} = \epsilon \). Clearly, \( \hat{\epsilon}_s = \hat{\epsilon} = \epsilon \).

- If \( \epsilon \leq N^{-2} \), then \( \sigma_{\eta}^* = k N^{-1/2} \) and we have

\[
N^{-1/2} \geq \epsilon^{1/2}, \quad N^{-1/2} \geq N^{-3/4}, \quad N^{-1/2} \geq N^{-1} \ln N \quad \text{for } N \geq 4,
\]

\[
N^{-1/2} \geq \epsilon^{1/4} N^{-1/2} \ln^{1/2} N \quad \text{for } N \geq 4.
\]
• If $N^{-2} < \varepsilon \leq N^{-3/2}$, then $\sigma^*_H = k\varepsilon^{-1/2}N^{-3/2}$ and we have
  $$\varepsilon^{-1/2}N^{-3/2} \geq \varepsilon^{1/2}, \quad \varepsilon^{-1/2}N^{-3/2} \geq N^{-3/4}.$$  
  It means that $\sigma_H \geq k\max\{\varepsilon^{-1/2}N^{-3/2}, N^{-1}\ln N, \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N\}$. Note that if $N^{-2} < \varepsilon \leq N^{-3/2}$,
  $$N^{-1/2} > \varepsilon^{-1/2}N^{-3/2}$$
  $$N^{-1/2} > N^{-1}\ln N \quad \text{for} \ N \geq 4,$n  $$N^{-1/2} > \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N \quad \text{for} \ N \geq 4.$$

• If $N^{-3/2} < \varepsilon \leq N^{-1}$, then $\sigma^*_H = k\varepsilon^{-1/2}N^{-3/2}$ and we have
  $$\varepsilon^{1/2} \geq \varepsilon^{-1/2}N^{-3/2}, \quad \varepsilon^{1/2} \geq N^{-3/4}.$$  
  It means that $\sigma_H \geq k\max\{\varepsilon^{1/2}, N^{-1}\ln N, \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N\}$. Note that if $N^{-3/2} < \varepsilon \leq N^{-1}$,
  $$N^{-1/2} > \varepsilon^{1/2}$$
  $$N^{-1/2} > N^{-1}\ln N \quad \text{for} \ N \geq 4,$n  $$N^{-1/2} > \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N \quad \text{for} \ N \geq 4.$$

These mean that in the case of $\hat{\varepsilon} = \varepsilon$ we can set $\sigma_H \geq kN^{-1/2}$.

If $\hat{\varepsilon} = \begin{cases} \bar{\varepsilon} & \text{if } \varepsilon \in \Omega_s, \\ \varepsilon & \text{otherwise}, \end{cases}$ where $\bar{\varepsilon} := \max\{\varepsilon, N^{-3/2}\}$. Clearly, $\hat{\varepsilon}_M = \hat{\varepsilon}_s = \bar{\varepsilon}$.

Note that $\sigma^*_H = k\bar{\varepsilon}^{-1/2}N^{-3/2}$ for $\hat{\varepsilon}_s \geq N^{-3/2}$.

• If $\varepsilon \leq N^{-3/2}$, then $\hat{\varepsilon}_M = \bar{\varepsilon} = N^{-3/2}$ and we have
  $$\varepsilon^{1/2}_M = \bar{\varepsilon}^{1/2} = N^{-3/4}, \quad \varepsilon^{-1/2}N^{-3/2} = N^{-3/4}.$$  
  It means that $\sigma_H \geq k\max\{\varepsilon^{1/2}, N^{-1}\ln N, \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N\}$. Note that
  $$\varepsilon^{1/2}\ln^{1/2}N > N^{-1}\ln N \quad \text{for} \ N \geq 4,$n  $$\varepsilon^{1/2}\ln^{1/2}N > \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N \quad \text{for} \ N \geq 4.$$

• If $N^{-3/2} < \varepsilon \leq N^{-1}$, then $\bar{\varepsilon} = \varepsilon > N^{-3/2}$ and we have
  $$\varepsilon^{1/2} > \varepsilon^{-1/2}N^{-3/2},$$
  $$\varepsilon^{1/2}\ln^{1/2}N > N^{-1}\ln N \quad \text{for} \ N \geq 4,$n  $$\varepsilon^{1/2}\ln^{1/2}N > \varepsilon^{1/4}N^{-1/2}\ln^{1/2}N \quad \text{for} \ N \geq 4.$
These mean that we can take $\sigma_\eta \geq k\tilde{\varepsilon}^{1/2} \ln^{1/2} N$.

From the above derivations, we have the following remark.

**Remark 3.1.** If we set $\hat{\varepsilon} = \varepsilon$, we can take

$$\sigma_\beta = kN^{-1} \ln N, \quad \sigma_\eta = kN^{-1/2}.$$  

If we set $\hat{\varepsilon}$ as in [3, pg 463], i.e.,

$$\hat{\varepsilon} = \begin{cases} \tilde{\varepsilon} & \text{if } x \in \Omega_s, \\
\varepsilon & \text{otherwise,} \end{cases}$$

where $\tilde{\varepsilon} := \max(\varepsilon, N^{-3/2})$, we can define

$$\sigma_\beta = kN^{-1} \ln N, \quad \sigma_\eta = k\tilde{\varepsilon}^{1/2} \ln^{1/2} N.$$  

It seems that the definition of $\sigma_\beta$ in [3] is not proper.

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