Super-Radiance and the Unstable Photon Oscillator

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I. INTRODUCTION

In many theoretical treatments of condensed matter systems, the Hamiltonian consists of a sum of kinetic energy and Coulomb potential energy. In reality, the transverse electromagnetic (radiation) fields are also important. If the transverse electromagnetic field is included in the description of condensed matter, then phenomena such as the Casimir effect \[1\], the Lamb shift \[2–5\] and the long ranged inter-molecular interaction forces \[6–8\] can be explained employing standard low order quantum electrodynamic perturbation theory. If the coupling of the transverse electromagnetic field with matter is strong enough to produce an instability in the photon oscillators, then a transition occurs from a normal radiative phase to a super-radiant phase. The Dicke model \[9\] has been studied as a prime example of this phase transition. In the Dicke model, many two level molecules interact with a single photon mode of the radiation field. The Dicke model has been extensively studied by many workers \[10–13\], each adopting a somewhat different method to explain the physical properties of the system. Hepp and Lieb \[10\], analyzed the thermodynamic properties of the Dicke maser model and elucidated the mathematical structure of the super-radiant phase transition.

In subsequent works of Wang and Hoe \[11\] and Hepp and Lieb \[12\], the coherent states of Glauber \[14\] were employed for multi-photon oscillators in the Dicke model and the resulting phase transition has been investigated in great detail in these papers. The existence of such phase transitions in condensed matter systems has been investigated in a series of papers by Preparata and coworkers \[15–20\]. Preparata has proposed several examples of super-radiant phases in a recent review \[18\]. Water \[18–20\] is a particularly interesting example.

The purpose of this work is to present a very direct method for computing the thermodynamic region wherein super-radiant phase exists. When a condensed matter system interacts with the electromagnetic field, the field degrees of freedom may be described by a renormalized thermodynamic potential. For small oscillator displacements (in the normal radiation phase) the effective potential is parabolic. As the thermodynamic parameters change, the effective oscillator frequency may turn imaginary. The resulting oscillator instability is the signature of the phase transition into a super-radiant state.

In Sec. II, the statistical thermodynamic method for calculating the effective potential of a photon oscillator mode will be discussed. The conditions for the stability of the oscillator will be exhibited. In Sec. III, the dynamics in real time of the oscillator will be discussed in terms of the photon mode propagator. The conditions for dynamic real time stability are shown to be the same as for thermodynamic stability. In Sec. IV, the example of a mesoscopic object placed in an optical cavity is considered. If the mesoscopic object has a sufficiently high polarizability, then a cavity mode can become unstable. In the Sec. V, the nature of the non-linear forces which stabilize the oscillator mode in the super-radiant phase are explored. The shift in the equilibrium position required to stabilize the oscillator in the low temperature super-radiant phase is exhibited. In the concluding Sec. VI, the notion of super-radiance with electric dipole coupling is related to the notion of coherent ferro-electricity. A material grain with a high electric polarizability, but otherwise without a net dipole moment, can be induced into a ferro-electric phase by an interaction with transverse electric fields. The transverse fields may induce a coherence in the electronic states then yielding a net electric dipole moment.

II. STATISTICAL THERMODYNAMICS

Let us consider a simple harmonic oscillator with momentum $P$, displacement $Q$ and frequency $\omega_{\infty}$. The oscillator is coupled to a thermal bath described by the Hamiltonian $H$. The free energy $\mathcal{F}_0$ of the thermal bath is then determined by
where (without loss of generality) one may assume (for displacement, Eq.(10) implies
\[ e^{-\mathcal{F}_0/k_BT} = \text{Tr}_{\text{bath}} e^{-\mathcal{H}/k_BT}. \] (1)

The coupling between the thermal bath and the oscillator is described by a force \( f \). The total Hamiltonian is then
\[ \mathcal{H} = \frac{1}{2} \left( P^2 + \omega_\infty^2 Q^2 \right) + H - fQ. \] (2)

which determines the total free energy \( \mathcal{F} \) via
\[ e^{-\mathcal{F}/k_BT} = \text{Tr} e^{-\mathcal{H}/k_BT}. \] (3)

If \( Q \) and \( P \) are classical to a sufficient degree of accuracy, then the total trace may be approximated by
\[ \text{Tr}(...) = \int \int \left( \frac{dPdQ}{2\pi\hbar} \right) \text{Tr}_{\text{bath}}(...). \] (4)

From Eqs.(1-4) one obtains
\[ e^{-\mathcal{F}_0/k_BT} = \int \int \left( \frac{dPdQ}{2\pi\hbar} \right) e^{-P^2/2kB_T} e^{-\mathcal{G}(Q,T)/kB_T}, \] (5)
i.e.
\[ e^{-\mathcal{F}_0/k_BT} = \frac{\sqrt{k_BT}}{2\pi\hbar} \int dQe^{-\mathcal{G}(Q,T)/kB_T}, \] (6)
where
\[ \mathcal{G}(Q,T) = \frac{\omega_\infty^2 Q^2}{2} - k_BT \ln \left\{ \frac{\text{Tr}_{\text{bath}} e^{-(H-fQ)/kB_T}}{\text{Tr}_{\text{bath}} e^{-\mathcal{H}/kB_T}} \right\}. \] (7)

Finally, if the force \( f \) is also classical to a sufficient degree of accuracy, then
\[ \left\{ \text{Tr}_{\text{bath}} e^{-(H-fQ)/kB_T} \right\} \rightarrow \langle e^{fQ/k_BT} \rangle, \] (8)
yielding
\[ \mathcal{G}_{\text{classical}}(Q,T) = \frac{\omega_\infty^2 Q^2}{2} - k_BT \ln \langle e^{fQ/k_BT} \rangle. \] (9)

The “effective” thermally induced potential of the oscillator is given by
\[ \mathcal{G}(Q,T) = \omega_\infty^2 Q^2/2 - k_BT \ln \langle e^{fQ/k_BT} \rangle, \] (10)
where (without loss of generality) one may assume (for \( Q = 0 \)) that \( \langle f \rangle = 0 \). For small values of the oscillator displacement, Eq.(10) implies
\[ \lim_{Q \to 0} \frac{\mathcal{G}(Q,T)}{Q^2} = (\omega_0^2/2), \] (11)
where, for a classical force \( f \), the shifted frequency \( \omega_0 \) of the oscillator is given by
\[ \omega_0^2 = \omega_\infty^2 - \langle f^2 \rangle/k_BT. \] (12)

The frequency shift (from \( \omega_\infty \) downward to \( \omega_0 \)) of the oscillator due to the interaction of the oscillator with the thermal bath is given by
\[ \omega_\infty^2 - \omega_0^2 = \langle f^2 \rangle/k_BT. \] (13)

The oscillator is “stable” if the shifted frequency is real; i.e. \( \omega_0^2 > 0 \) or equivalently
\[ k_BT \omega_\infty^2 > \langle f^2 \rangle \] (stable). (14)

The oscillator is “unstable” if the shifted frequency is imaginary; i.e. \( \omega_0^2 < 0 \) or equivalently
\[ k_BT \omega_\infty^2 < \langle f^2 \rangle \] (unstable). (15)

One may appreciate the dynamical significance of the shifted frequency by considering the spectral density \( S_{ff}(\omega) \) of the force fluctuations \([23,24]\) defined by
\[ \langle f(t)f(t') \rangle = \int_{-\infty}^{\infty} S_{ff}(\omega) \cos(\omega(t-t')) d\omega. \] (16)
For equal times
\[ \langle f^2 \rangle = 2 \int_0^{\infty} S_{ff}(\omega) d\omega \] (17)

The classical fluctuation response theorem asserts that
\[ S_{ff}(\omega) = \left( \frac{k_B T}{\pi} \right) \Re \Gamma(\omega + i0^+) \] (18)
where \( \Gamma(\zeta) \) for \( \Re \zeta > 0 \) is the dynamic damping coefficient for the oscillator. \( \Gamma(\zeta) \) will be discussed in detail in Sec. III below. Here we note that Eqs.(13), (17) and (18) imply that
\[ \omega_\infty^2 - \omega_0^2 = \left( \frac{2}{\pi} \right) \int_0^{\infty} \Re \Gamma(\omega + i0^+) d\omega. \] (19)

If the integrated (over frequency) dissipative damping on the right hand side of Eq.(19) is sufficiently large, the oscillator becomes unstable. The oscillator will be stable for
\[ \omega_\infty^2 > \left( \frac{2}{\pi} \right) \int_0^{\infty} \Re \Gamma(\omega + i0^+) d\omega \] (stable). (20)

Eq.(20) has been here deduced by statistical thermodynamic reasoning. However, it is perhaps more simple to deduce Eq.(20) directly from the simple dynamical equations of motion for the oscillator.
III. REAL TIME OSCILLATOR DYNAMICS

Let us now consider a simple harmonic oscillator with a canonical momentum $P$, displacement $Q$ and frequency $\omega_{\infty}$. The oscillator is driven by a small time varying external force $f_{\text{ext}}(t)$. The total Hamiltonian, including the external force is then

$$H_{\text{total}} = H - \delta f_{\text{ext}}(t)Q$$  \hspace{1cm} (21)

where $H$ is defined in Eq.(2). In a phenomenological linear response theory, the small mean deviation from thermal equilibrium in the oscillator coordinate obeys the equation of motion

$$\delta \ddot{Q}(t) + \omega_{\infty}^2 \delta Q(t) = \delta f(t) + \delta f_{\text{ext}}(t)$$  \hspace{1cm} (22)

where $\delta f(t)$ represents the induced change in the frictional damping force exerted on the coordinate by the thermal bath.

For simple engineering purposes, one often considers that the frictional damping force has a simple local (in time) form $\delta f(t) = -\gamma \dot{Q}(t)$. More generally, the damping force will be non-local in time; i.e.

$$\delta f(t) = -\int_0^\infty \mathcal{G}(s) \delta \dot{Q}(t-s) ds.$$  \hspace{1cm} (23)

For complex frequency $\zeta$, with $\Im \zeta > 0$, if

$$\delta f_{\text{ext}}(t) = \Re \{ \delta f_{\text{ext}}(0) e^{-i\zeta t} \},$$

then

$$\delta Q(t) = \Re \{ D(\zeta) \delta f_{\text{ext}}(0) e^{-i\zeta t} \}$$  \hspace{1cm} (25)

defines the dynamical oscillator propagator $D(\zeta)$. From Eqs.(23) and (25)

$$\delta f(t) = \Re \{ \Pi(\zeta) D(\zeta) \delta f_{\text{ext}}(0) e^{-i\zeta t} \},$$

where

$$\Pi(\zeta) = i\zeta \int_0^\infty \mathcal{G}(s) e^{i\zeta s} ds. = i\zeta \Gamma(\zeta).$$  \hspace{1cm} (27)

From Eqs.(22), (25) and (26), one finds that

$$D(\zeta) = \left( \frac{1}{\omega_{\infty}^2 - \zeta^2 - \Pi(\zeta)} \right);$$  \hspace{1cm} (28)

i.e. $\Pi(\zeta)$ is the “self energy” of the harmonic oscillator propagator.

From the retarded (causal) nature of Eqs.(23) and (27) follows the dispersion relation

$$\Pi(\zeta) = \left( \frac{2}{\pi} \right) \int_0^\infty \omega \Im \Pi(\omega + i0^+) \frac{d\omega}{(\omega^2 - \zeta^2)}$$  \hspace{1cm} (29)

Equivalently, in terms of the frequency dependent damping coefficient $\Gamma(\omega + i0^+)$,

$$\Im \Pi(\omega + i0^+) = \omega \Re \Gamma(\omega + i0^+)$$  \hspace{1cm} (30)

Eq.(29) reads

$$\Pi(\zeta) = \left( \frac{2}{\pi} \right) \int_0^\infty \omega \Re \Gamma(\omega + i0^+) \frac{d\omega}{(\omega^2 - \zeta^2)}$$  \hspace{1cm} (31)

As $\zeta \rightarrow 0$, Eq.(25) reads as the thermal (static) response function

$$D(0) = \left( \frac{\partial Q}{\partial f_{\text{ext}}} \right)_T = \frac{1}{\omega_0^2}$$  \hspace{1cm} (32)

where

$$\omega_0^2 = \omega_{\infty}^2 - \Pi(0).$$  \hspace{1cm} (33)

Equivalently, employing Eqs.(31) and (33)

$$\omega_{\infty}^2 - \omega_0^2 = \left( \frac{2}{\pi} \right) \int_0^\infty \Re \Gamma(\omega + i0^+) d\omega$$  \hspace{1cm} (34)

as previously derived in Eq.(19) from purely thermodynamic reasoning.

If the dynamical oscillator propagator is presumed to be retarded, it too must satisfy a dispersion relation

$$D(\zeta) = \left( \frac{2}{\pi} \right) \int_0^\infty \omega \Im m D(\omega + i0^+) \frac{d\omega}{(\omega^2 - \zeta^2)}.$$  \hspace{1cm} (35)

If the oscillator is stable, then causality is obeyed with the dissipative condition $\Im m D(\omega + i0^+) \geq 0$. The response function $D(\zeta)$ is then analytic in the upper half $\Im \zeta > 0$ of the complex frequency plane.

On the other hand if the oscillator is unstable, then at least one pole at $\zeta = i\varpi$ appears in the upper-half plane and the dispersion relation Eq.(35) for the dynamical oscillator propagator is no longer valid. The occurrence of an instability in the system can be viewed as inducing the non-analyticity of the dynamical oscillator propagator given in Eq.(28). To locate an unstable oscillator pole in the upper half plane at $\zeta = i\varpi$, we look for a zero in the denominator of Eq.(28); i.e.

$$\omega_{\infty}^2 - (i\varpi)^2 - \Pi(i\varpi) = 0.$$  \hspace{1cm} (36)

From Eqs.(31) and (36) we seek a solution for $\varpi > 0$ of the equation

$$\omega_{\infty}^2 + \varpi^2 = \left( \frac{2}{\pi} \right) \int_0^\infty \omega^2 \Re \Gamma(\omega + i0^+) \frac{d\omega}{(\omega^2 + \varpi^2)}.$$  \hspace{1cm} (37)

As a function of imaginary frequency in the range $0 < \varpi < \infty$, it is evident that the right hand side of Eq.(36) is smoothly and monotonically decreasing from $\Pi(0)$ to zero, while the left hand side of Eq.(36) is smoothly and monotonically increasing from $\omega_{\infty}^2$ to infinity. If $\Pi(0) > \omega_{\infty}^2$, then the two smooth curves intersect at precisely one value of $\varpi$. It is easily proved that $\zeta = i\varpi$ is an isolated zero in the denominator of Eq.(28). Thus $D(\zeta)$ has an isolated pole in the upper-half plane. The condition for this pole to exist, i.e.

$$\omega_{\infty}^2 < \left( \frac{2}{\pi} \right) \int_0^\infty \Re \Gamma(\omega + i0^+) d\omega,$$  \hspace{1cm} (38)

is the same as the existence of the oscillator instability.
IV. OPTICAL CAVITY INSTABILITY

Consider an electromagnetic mode in an empty optical cavity \([27–32]\). The electric field of the mode \(E_\infty(r)\) may be related to the oscillator displacement \(Q\) of the mode by equating the mean energy stored in the electric field to the potential energy of the oscillator; i.e.

\[
\frac{1}{8\pi} \int_{\text{cavity}} |E_\infty(r)|^2 d^3r = \left(\frac{1}{2}\right) \omega_\infty^2 Q^2. \tag{39}
\]

In order to implement Eq.(39), one may choose

\[
E_\infty(r) = \sqrt{4\pi} \omega_\infty Q e_\infty(r) \tag{40}
\]

where \(e_\infty(r)\) is normalized according to

\[
\int_{\text{cavity}} |e_\infty(r)|^2 d^3r = 1. \tag{41}
\]

Let us further suppose that cavity has been designed so that the mode electric field is localized in a spatial region of length scale \(L\) about the origin. Then

\[
e_\infty(0) = \left(\frac{n}{L^3/2}\right), \tag{42}
\]

where \(n\) is a unit vector; \((n \cdot n) = 1\).

Let us finally suppose that a mesoscopic object is placed within the mode neighborhood. The mesoscopic object interacts with the electric field via the electric dipole moment \(p\). The interaction Hamiltonian between the mesoscopic object and the optical mode oscillator then takes the form

\[
H_{\text{int}} = -p \cdot E_\infty(0) = -fQ. \tag{43}
\]

so that the force on the oscillator coordinate is due to the electric dipole moment. The force is

\[
f = \sqrt{(4\pi/L^3)} \omega_\infty (n \cdot p) = \sqrt{4\pi/L^3} \omega_\infty p. \tag{44}
\]

Eq.(44) follows from Eqs.(40), (42) and (43).

If the oscillator were driven by an external force at frequency \(\omega\), then from Eqs.(25) and (26) the damping force and displacement at frequency \(\omega\) obey

\[
\delta f_\omega = \Pi(\omega + i0^+) \delta Q_\omega. \tag{45}
\]

The electric dipole moment response to the electric field defines the polarizability \(\alpha(\zeta)\) of the mesoscopic object via

\[
\delta p_\omega = \alpha(\omega + i0^+) n \cdot \delta E_\infty(0). \tag{46}
\]

Equivalently,

\[
\delta p_\omega = \alpha(\omega + i0^+) \sqrt{4\pi/L^3} \omega_\infty \delta Q_\omega. \tag{47}
\]

where Eqs.(40), (42) and (46) have been employed. From Eqs.(44), (45) and (47), one can relate the self energy part of the mode propagator to the polarizability of the mesoscopic object; The relationship is

\[
\Pi(\zeta) = \left(\frac{4\pi \omega_\infty^2}{L^3}\right) \alpha(\zeta). \tag{48}
\]

The condition for the oscillator stability

\[
\omega_0^2 = \omega_\infty^2 - \Pi(0) > 0, \tag{49}
\]

may be written in terms of the static polarizability

\[
4\pi \alpha(0) < L^3 \quad \text{(stable oscillator)}.
\]

For a very highly polarizable mesoscopic object, say a single domain ferroelectric grain \([21,33,34]\), the photon oscillator can exhibit an instability. The general nature of the non-linear forces which eventually stabilize the oscillator must come into play.

V. SUPER-RADIANT PHASE

The problem of how an oscillator reaches a new thermodynamic equilibrium position in the “unstable” regime can be discussed only in the context of a specific microscopic environmental Hamiltonian \(H\) which enters into Eq.(7). An often discussed model for a photon oscillator employs \(N\) identical “two level atoms” \([25–28]\) in the thermal environment. The Hamiltonian is

\[
H = -\left(\frac{\epsilon}{2}\right) \sum_{j=1}^{N} \sigma_{zj} = -\epsilon S_z, \tag{51}
\]

where \(\epsilon\) is the energy difference between the single atom ground state and the excited state, \(\sigma_j = (\sigma_{xj}, \sigma_{yj}, \sigma_{zj})\) are the Pauli matrices for the \(j\)th two level atom and \(S = (1/2) \sum_{j=1}^{N} \sigma_j\) is formally the “total spin” of the mesoscopic object. The object is placed in the region of the cavity where the electric fields of the photon oscillator are concentrated. The electric field is presumed to interact with the atomic system via the total electric dipole moment component in the \(n\) direction; i.e.

\[
p = \mu \sum_{j=1}^{N} \sigma_{xj} = 2\mu S_x, \tag{52}
\]

where \(\mu\) is the dipole moment matrix element. Using Eqs.(44) and (52) the force exerted on the photon oscillators by the dipole moment of the atomic system is

\[
f = \sqrt{16\pi} \left(\omega_\infty \mu S_x \right). \tag{53}
\]

One is now in a position to compute the free energy \(G(Q,T)\) for all domains (stable or unstable) of the (now non-linear) oscillator.
Substituting Eqs (51) and (53) into Eq.(7) and taking the trace over the formal spin degrees of freedom yields

$$G(Q, T) = \omega_\infty^2 (Q^2/2) + k_B T \ln \cosh \left[ \frac{\epsilon}{2k_B T} \right] - k_B T \ln \cosh \left[ \frac{1}{2k_B T} \sqrt{\epsilon^2 + \left( \frac{16\pi \omega_\infty^2 \mu^2 Q^2}{L^3} \right)} \right].$$

Using Eqs (11) and (54), the shift in the frequency of the photon oscillators from $\omega_\infty$ down to the temperature dependent $\omega_\infty$ is found to be

$$\omega_\infty(T) = \omega_\infty \sqrt{1 - \frac{8\pi N \mu^2}{L^3 \epsilon}} \tanh \left( \frac{\epsilon}{2k_B T} \right).$$

The stability of the oscillator requires $\omega_\infty(T) > 0$, which in virtue of Eq.(55) reads

$$T > T_c \quad \text{stable oscillator},$$

where the critical temperature for stability obeys

$$\tanh \left( \frac{\epsilon}{2k_B T_c} \right) = \left( \frac{L^3 \epsilon}{8\pi N \mu^2} \right).$$

The shift in the equilibrium position required to stabilize the oscillator when $T < T_c$ is found from the free energy minimum condition $(\partial G/\partial Q)_T = 0$. The resulting equilibrium position $Q_e$ obeys the transcendental equation

$$\tanh \left( \frac{\epsilon}{2k_B T_c} \right) = \left( \frac{L^3 \epsilon}{8\pi N \mu^2} \right).$$

In Fig. 1, with $y = (\epsilon/k_B T_c) = 0.1$, we show the following: (i) Above the critical temperature $x = (T/T_c) > 1$ the renormalized frequency of the stable oscillator $\omega_\infty(T) = \nu(x, y) \omega_\infty$ is lowered as the temperature is lowered. (ii) Below the critical temperature $x = (T/T_c) < 1$, the oscillator position shift $Q_e(T) = (\sqrt{Nk_B T_c/\omega_\infty}) \xi(x, y)$ (required to restore stability) increases as the temperature is lowered. Thus as the temperature is lowered from above the critical temperature, the oscillator frequency decreases to zero and the “soft mode” has vanishing frequency at the critical temperature. The transition to an ordered phase begins as the temperature is lowered below the critical temperature. The oscillator seeks a new equilibrium position $Q_e(T < T_c)$ displaced from the old equilibrium position of $Q = 0$. The further the temperature is lowered, the more growth is exhibited in $Q_e$.

In Fig. 2 the free energy is plotted as a function of the displacement of the oscillator for two values of the temperature. For $T > T_c$, the potential has a single minimum at $Q = 0$. For $T < T_c$, the potential has a double minimum at $Q = \pm Q_e(T)$. The oscillator will choose one of these minima in the super-radiant phase.

**VI. CONCLUSIONS**

It has been shown that any simple harmonic oscillator subject to a random force (producing damping) can be made unstable if the induced damping of the oscillator is sufficiently strong. If $\omega_\infty$ is the frequency of the unperturbed oscillator and if $\Pi(\xi)$ in Eq.(31) is the self energy...
induced by the damping coefficient \( \Re \Gamma(\omega + i0^+) \), then
the precise condition for the instability is that \( \Pi(0) > \omega_c^2 \).

It is possible to observe this instability for a mesoscopic grain which is highly polarizable. The grain must be
placed in an optical cavity with a mode of spatial extent \( L \) large on the scale of the grain size. The condition for
an optical cavity mode to go unstable is that

\[
4\pi \sigma(0) > L^3 \quad (\text{unstable cavity mode}), \quad (59)
\]

where \( \sigma(\omega + i0^+) \) is the dynamic polarizability of the grain. The inequality in Eq.(59) can surely be reached
with a single domain highly polarizable ferroelectric grain.

The manner in which the oscillator is eventually stabili-
zated (when Eq.(59) holds true) is not a trivial prob-
lem. The example of a grain which consists of “two-level
atoms” was worked out in detail. The normal and super-
radiant free energies were exhibited in Fig. 2. When the
frequency of the oscillator mode softens towards zero as
\( T \to T_c + 0^+ \), the equilibrium position of the oscilla-
tor shifts to a nonzero value \( Q_c(T < T_c) \). For two level
atoms, the details of the super-radiant transition can be
computed.

It is to be hoped that the theoretical considerations
here put forth for ferroelectric grains will soon be put to
laboratory tests.