ON THE GAUSS MAP WITH VANISHING BIHARMONIC STRESS-ENERGY TENSOR

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ABSTRACT. We study the biharmonic stress-energy tensor $S_2$ of Gauss map. Adding few assumptions, the Gauss map with vanishing $S_2$ would be harmonic.

1. INTRODUCTION

Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Assume $M$ compact and define the energy of $\phi$ to be:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$$

Call a map harmonic if it is a critical point of $E$, and this is characterized by:

$$\tau(\phi) = \text{trace} \nabla d\phi = 0$$

where $\tau$ is called the tension field. The vanishing of this field is used to define the harmonic map in noncompact case.

As a natural generalization, the biharmonic map is the critical point of the bienergy (see details in [MO]):

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g$$

it’s associated bitension field is:

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi)) d\phi$$

Described by Hilbert [H1], the stress-energy tensor associates to a variational problem is a symmetric 2-covariant tensor $S$ conservative at critical points, i.e. $\text{div} S = 0$.

Mathematics Classification Primary(2000): 58E20.
Keywords: biharmonic maps, Gauss map, stress-energy tensor, Grassmannian, pseudo-umbilical.
Thank Prof. Dong and Prof. Ji for helpful discussion.
In the context of harmonic maps, the stress-energy tensor was studied by Baird and Eells in details (BE):

\[ S = \frac{1}{2} |d\phi|^2 g - \phi^* h \]

s.t \( \text{div} S = - < \tau(\phi), d\phi > \).

The biharmonic stress-energy tensor \( S_2 \) corresponding to \( E_2(\phi) \) was introduced by Jiang in [J] and studied by Loubeau et.al [LMO1]:

\[
S_2(X, Y) = \frac{1}{2} |\tau(\phi)|^2 < X, Y > + < d\phi, \nabla \tau(\phi) > < X, Y >
- < d\phi(X), \nabla_Y \tau(\phi) > - < d\phi(Y), \nabla_X \tau(\phi) >
\]

It can be easily drawn out: \( S = 0 \Rightarrow \tau = 0 \Rightarrow S_2 = 0 \Rightarrow \tau_2 = 0 \).

A natural question is when the reverse be true. For example, when a biharmonic map be harmonic? Readers could confer [MO] for the results in this direction.

In this article, we focus on how would \( S_2 = 0 \) imply \( \tau = 0 \). Jiang had proved:

**Theorem 1 ([J]).** A map \( \phi: (M, g) \rightarrow (N, h) \), \( m \neq 4 \), with \( S_2 = 0 \), \( M \) compact and orientable, is harmonic

**Proof.** Trace of \( S_2 \) is

\[
0 = \text{trace} S_2 = \frac{m}{2} |\tau(\phi)|^2 + m < d\phi, \nabla \tau(\phi) > -2 < d\phi, \nabla \tau(\phi) >
\]

and integrating over \( M \):

\[
0 = \frac{4 - m}{2} \int_M |\tau(\phi)|^2 v_g
\]

hence, \( \phi \) is harmonic when \( m \neq 4 \). \( \square \)

And

**Theorem 2 ([J]).** A non-minimal Riemannian immersion \( \phi: (M^4, g) \rightarrow (N, h) \) satisfies \( S_2 = 0 \) if and only if it is pseudo-umbilical.

The 4-folds have somehow exotic behaviors. At another hand, if the map \( \phi \) arise from a submanifold’s Gauss map, under few assumptions, the map would regain part of “rigidity”, i.e. \( S_2 = 0 \) implying harmonic. It is the theme of this paper:

**Theorem 3.** \( M^4 \) is a compact pseudo-umbilical submanifold of \( R^n \), if its Gauss map has vanishing \( S_2 \), then \( M^4 \) has constant mean curvature.
And

**Theorem 4.** $M^4$ is a compact analytic hypersurface of $R^5$, if it is strictly convex and its Gauss map has vanishing $S_2$, then $M$ is a hypersphere, i.e the Gauss map is identity.

2. PRELIMINARY

Form now on, $M^m$ always denotes an oriented submanifold of $R^n$, and $G$ shorts for the Gauss map.

In order to study the biharmonic stress-energy tensor of Gauss map, we have to understand $G^*T(G(n, m))$ well, especially the connection on it.

In [RX], Ruh and Vilms had shown: the pull back of the tangent bundle of Grassmannian via the Gauss map is isomorphic to $T^*(M) \otimes N(M)$, i.e. $T(M) \otimes N(M)$ after the musician transformation, where $T(M)$, $T^*(M)$, $N(M)$ are the tangent, cotangent and normal bundle respectively.

There is a more explicit way to see this([LMO2]):

Choose $\{e_i\}_{i=1}^m$ an oriented geodesic basis centered around $p \in M$. In the neighborhood $U \ni p$, the Gauss map can be written as:

$$G(q) = e_1(q) \wedge \cdots \wedge e_m(q), \forall q \in U$$

Since

$$dG_q(e_i) = \sum_{j=1}^m e_1(q) \wedge \cdots \wedge e_j(q) \wedge (\nabla_{e_i} e_j)(q) \wedge e_{j+1}(q) \wedge \cdots \wedge e_m(q),$$

restricting at $p$, we have:

$$dG_p(e_i) = \sum_{j=1}^m e_1(p) \wedge \cdots \wedge e_j(p) \wedge B_p(e_i, e_j) \wedge e_{j+1}(p) \wedge \cdots \wedge e_m(p)$$

where $B$ is the second fundamental form of $M$, taking value in $N(M)$. Now $dG_p(e_i)$ can be identified with $\sum_j e_j^*(p) \otimes B_p(e_i, e_j)$, i.e. $\sum_j e_j(p) \otimes B_p(e_i, e_j)$. Thus the bundles are isomorphic.

For latter utility, we’d better explain the relationship between above invariant method and the moving frames method. See [IX] or [X], complete $\{e_i(p)\}_{i=1}^m$ into an orthonormal basis $\{e_\alpha(p)\}_{\alpha=1}^n$ of $R^n$. $\{w_\alpha\}$ is the dual frame. The Riemannian connection on $R^n$ is uniquely determined by the equation:

$$dw_\alpha = w_{\alpha\beta} \wedge w_\beta$$
\begin{align*}
\omega_{\alpha\beta} + \omega_{\beta\alpha} &= 0
\end{align*}

The canonical metric on \( G(m,n) \) is:

\begin{equation}
 ds^2 = \sum_{i,a} w_{ia}^2
\end{equation}

where \( a = m + 1, \ldots, n \).

\( \{w_{ia}\} \) can be thought as the dual frame of \( \{e_i \otimes e_a\} \), which means nothing but \( e_j \otimes e_a \) is orthonormal basis of \( T_{G(p)} G(n, m) \) with respect to the canonical metric on Grassmannian, i.e.

\[
g_{\text{can}}(dG_p(e_i), dG_p(e_k)) = \sum_j < B_p(e_i, e_j), B_p(e_k, e_j) >
\]

The connection due to metric given by equation (2) is:

\[
w_{iajb} = \delta_{ab}w_{ij} + \delta_{ij}w_{ab}
\]

Pull back these forms to \( M \), we have:

\textbf{Lemma 5 (X).} The connection on \( G^* T(G(n, m)) \) is the connection \( \nabla^M \otimes \nabla^\perp \) on \( T(M) \otimes N(M) \)

3. Proof of main theorem

In [RV], it had been showed that the tension field of the Gauss map is identical with \( \nabla H, \nabla_{e_i} H \otimes e_i \) in our setting, where \( H \) is the mean curvature.

Let \( G \) be the Gauss map of \( M^4 \). Take the trace of its biharmonic stress-energy tensor \( S_2 \):

\[
\frac{1}{2} \text{trace} S_2 = |\tau(G)|^2 + < \nabla \tau(G), dG > = \nabla_{e_i} < \tau(G), dG >
\]

\begin{equation}
 = \nabla_{e_i} < \nabla_{e_j} H \otimes e_j, B_{ik} \otimes e_k > = \nabla_{e_i} < \nabla_{e_j} H, B_{ij} >
\end{equation}

\[
= \nabla_{e_i} \nabla_{e_j} < H, B_{ij} > - \nabla_{e_i} < H, \nabla_{e_i} B_{jj} >
\]

If \( M^4 \) is pseudo-umbilical and \( S_2 \) vanishes, equation (3) become:

\[
0 = \frac{1}{4} \nabla_{e_i} \nabla_{e_i} |H|^2 - \frac{1}{2} \nabla_{e_i} \nabla_{e_i} |H|^2 = -\frac{1}{4} \Delta |H|^2
\]

\(|H|^2 \) is a harmonic function, by the maximum principle, it must be constant. This ends the proof of the theorem 3.

\textbf{Remark:} By theorem 2, we know that in the case \( m=4 \), if an embedding and its Gauss map are both with vanishing biharmonic stress-energy tensor, then the submanifold has constant mean curvature.

For the sequel, we need a reformulation of \( S_2 = 0 \):
Lemma 6 ([LMO1]). Let $\phi : (M^4, g) \to (N, h)$, then $S_2 = 0$ if and only if

\begin{equation}
\frac{1}{2} |\tau(\phi)|^2 < X, Y > + < d\phi(X), \nabla_Y \tau(\phi) >
+ < d\phi(Y), \nabla_X \tau(\phi) > = 0
\end{equation}

$\forall X, Y$

Now, we are in the position to prove theorem 4.

**Proof.** Denote $h$ the scalar value of $H$, by equation (4) we have:

\begin{equation}
0 = \frac{1}{2} |\text{grad} h|^2 < X, Y > + < B(X, e_j) e_j, \nabla_Y \text{grad} h >
+ < B(Y, e_j) e_j, \nabla_X \text{grad} h >
= \frac{1}{2} |\text{grad} h|^2 < X, Y > + B(X, \nabla_Y \text{grad} h) + B(Y, \nabla_X \text{grad} h)
\end{equation}

For $M$ is compact, assume $h$ achieves its maximum at point $p$. Choose local patch around $p \{U, x_i\}$, s.t $\{\frac{\partial}{\partial x_i}\}$ is orthonormal at $p$ and diagonalizing the second fundamental form. Replace $X, Y$ with $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$, write equation (5) in local coordinates:

\begin{equation}
\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} B_{ij} + \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} B_{ii} = - \frac{\partial h}{\partial x_i} \Gamma_{jk} B_{ik} - \frac{\partial h}{\partial x_i} \Gamma_{il} B_{jk} - \frac{1}{2} \frac{\partial h}{\partial x_i} g_{ik} \frac{\partial h}{\partial x_k} g_{ij}
\end{equation}

If $B$ has eigenvalues $\{\lambda_i\}$, equation (6) at point $p$ is:

\begin{equation}
\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} (\lambda_i + \lambda_j) = - \frac{1}{2} \frac{\partial h}{\partial x_i} g_{ik} \frac{\partial h}{\partial x_k} g_{ij} - \frac{\partial h}{\partial x_i} \Gamma_{jk} B_{ik} - \frac{\partial h}{\partial x_i} \Gamma_{ij} B_{jk}
\end{equation}

$M$ is analytic, so we treat every thing in the analytic category.

Expend $h$ as a convergent polynomial series in neighborhood of $p$:

\[ h(x_1, \ldots, x_4) = c + h_{ij} x_i x_j + h_{ijk} x_i x_j x_k + \cdots \]

where $h_{ij}$ means $h_{ij}(p)$ in effect.

It has no one order terms.

Expend $B$ either, we have:

\begin{equation}
\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} (\lambda_i + \lambda_j) + O(x) F(h_{kl})
= - \frac{1}{2} \frac{\partial h}{\partial x_i} g_{ik} \frac{\partial h}{\partial x_k} g_{ij} - \frac{\partial h}{\partial x_i} \Gamma_{jk} B_{ik} - \frac{\partial h}{\partial x_i} \Gamma_{ij} B_{jk}
\end{equation}

where $F$ is a linear combination if $h_{ij}$. 
Since:

\[
\frac{\partial h}{\partial x_i \partial x_j} = h_{ij} + \sum_k h_{ijk} x_k + O(x^2)
\]

\[
\frac{\partial h}{\partial x_i} = \sum_k h_{ik} x_k + O(x^2)
\]

Comparing the lowest order terms in two sides of equation (7).

For \(\lambda_i > 0\), the left hand side may contain zero terms, while the left hand side has order no less than one, so \(h_{ij}\) must vanish, \(\forall i, j\).

Using the boot-strap argument, it is easy to see all the derivative of \(h\) at \(p\) must be zero.

Thus \(h\) must be constant for the anality. This means the Gauss map is harmonic.

In fact we have more stronger conclusion. Hsiung had shown in \[Hs\] that a strictly convex hypersurfaces with constant mean curvature must be hypersphere.

\[\square\]

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