Critical behavior of Born-Infeld dilaton black holes

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We explore the critical behavior of \((n+1)\)-dimensional topological Born-Infeld-dilaton black holes in an extended phase space. We treat the cosmological constant and the Born-Infeld (BI) parameter as the thermodynamic pressure and BI vacuum polarization which can vary. We obtain thermodynamic quantities of the system such as pressure, temperature, Gibbs free energy, and investigate the behaviour of these quantities. We also study the analogy of the van der Waals liquid-gas system with the Born-Infeld-dilaton black holes in canonical ensemble in which we can treat the black hole charge as a fixed external parameter. Moreover, we show that the critical values of pressure, temperature and volume are physical provided the coupling constant of dilaton gravity is less than one and the horizon is finite. Finally, we calculate the critical exponents and show that although thermodynamic quantities depend on the dilaton coupling constant, BI parameter and the dimension of the spacetime, they are universal and are independent of metric parameters.

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I. INTRODUCTION

Recent works on the thermodynamics of black holes have shown that one may enlarge the thermodynamic space to include the effective cosmological constant and other parameters of the gravitational theory as thermodynamic variables. For instance, in the case of Reissner-Nordstrom (RN) black hole \(^1\), considering the cosmological constant as a thermodynamic variable proportional to the pressure: \(P = -\Lambda/8\pi\), its conjugate quantity will be the geometrical volume. In this case the black hole mass \(M\) determines the enthalpy: \(H = U + PV\) which includes a contribution from the energy of formation of the system \(^2\). Also, from the point of view of consistency of Smarr formula and first law of thermodynamics, one should extend the thermodynamic space to include the cosmological constant as a thermodynamic variable \(^3\). The idea of including cosmological constant as the thermodynamic pressure have been studied in many papers \(^4\). Although in the case of RN black holes the conjugate quantity \(V\) is the geometrical volume, it is not necessarily needed to be a geometrical volume as it was revealed in the case of rotating black holes \(^5\).

In addition to the extension of the thermodynamic space to include the cosmological constant and its conjugate volume, one may extend this space to include other parameters of a black hole provided the mass is equated to the enthalpy. For instance, one may associate the non-geometrical thermodynamic volumes to Taub–NUT, Taub–Bolt \(^6\) and Kerr-bolt \(^7\) spacetimes. Also, considering nonlinear electrodynamics, one may extend the thermodynamic space for the consistency of first law of thermodynamics with the corresponding Smarr relation \(^5\). For instance for Born-Infeld (BI) black holes, one should also consider the variation of the minimal field strength \(\beta\) in the first law to be consistent with the corresponding Smarr relation \(^6\).

In this new context with extended thermodynamic space, one may study the analogy between charged black holes in AdS space and the van der Waals fluid and investigate the critical behaviour of the system. Phase transition and critical behaviour in Einstein gravity have been investigated by many authors \(^1\,3\,4\,6\,9\,11\). For the case of Myers-Perry black holes, these have been investigated in \(^1\). Also, the critical behavior of higher order gravities such as Gauss-Bonnet \(^12\,13\) and Lovelock gravity coupled to BI electrodynamics have been investigated \(^14\). The studies on the critical behavior of charged black holes were also extended to dilaton gravity \(^15\). In this regards, the critical behavior of charged black holes of Einstein-Maxwell-dilaton gravity in the presence of two Liouville-type potentials which make the solution asymptotically neither flat nor AdS has been explored in Ref. \(^16\). It was found that the critical exponents are universal and are independent of the details of the system although the thermodynamic quantities depend on the dilaton coupling constant \(^17\).

In this paper we further generalize the studies on the extended thermodynamic space and critical behavior of dilaton black holes by investigating the critical behavior of the \((n+1)\)-dimensional dilaton black holes coupled to nonlinear BI electrodynamics \(^18\) in an extended phase space with fixed charge. Due to the fact that the BI Lagrangian coupled to a dilaton field appears frequently in string theory, it is important to investigate various properties of black hole solutions in this theory. As we shall see the presence of the dilaton field affects the thermodynamic properties of black holes. As in Ref. \(^19\), we consider the BI parameter as a thermodynamic phase space variable to satisfy the Smarr relation and introduce its conjugate quantity as polarization. We also calculate the critical exponents and show that they are universal and are independent of the nonlinearity parameter as well as the dilaton- electromagnetic coupling
constant.

This paper is organized as follows. In the next section we present the basic field equations and consider a class of 
(n + 1)-dimensional topological black hole solutions in Einstein-Born-Infeld dilaton (EBId) gravity and review their
thermodynamic properties. In Sec. III we study the phase structure of the solution and present the generalized Smarr
relation in the presence of dilaton field. In Sec. IV we obtain the equation of state, study the critical behavior of
the solutions and compare them with van der Waals fluid. Gibbs free energy is investigated in Sec. V while critical
exponents is considered in Sec. VI. The last section is devoted to summery and conclusion.

II. TOPOLOGICAL BORN-INFELD-DILATON BLACK HOLES IN (n + 1) DIMENSIONS

The (n + 1)-dimensional action in which gravity is coupled to a dilaton field and nonlinear BI electrodynamics can be
written as [21]

\[
S = \frac{1}{16} \int d^{n+1}x \sqrt{-g} \left( R - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) + L(F, \Phi) \right),
\]

where \( R \) is the Ricci scalar, \( \Phi \) is the dilaton field, \( V(\Phi) \) is a potential for \( \Phi \) and the BI Lagrangian \( L(F, \Phi) \) is given by

\[
L(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-1)} \mathcal{L}(Y)
\]

\[
\mathcal{L}(Y) = 1 - \sqrt{1 + Y},
\]

\[
Y = \frac{e^{-8\alpha \Phi/(n-1)} F^2}{2\beta^2}.
\]

In the above equations \( F^2 = F^{\mu \nu} F_{\mu \nu} \) with \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor with electromagnetic
vector potential \( A_\mu \) and \( \alpha \) is the the coupling constant of the scalar and electromagnetic field. As the BI parameter
\( \beta \) goes to infinity, \( L(F, \Phi) \) reduces to the standard Maxwell field coupled to dilaton.

The equations of motion can be obtained by varying the action (1) with respect to the gravitational field \( g_{\mu \nu} \), the
dilaton field \( \Phi \) and the gauge field \( A_\mu \) which yields the following field equations

\[
\mathcal{R}_{\mu \nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu \nu} V(\Phi) \right) - 4e^{-4\alpha \Phi/(n-1)} \partial_\nu \mathcal{L}(Y) F_{\mu \eta} F^{\nu \eta}
\]

\[
+ \frac{4\beta^2}{n-1} e^{4\alpha \Phi/(n-1)} \left[ 2Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y) \right] g_{\mu \nu},
\]

\[
\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} + 2\alpha \beta^2 e^{4\alpha \Phi/(n-1)} \left[ 2Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y) \right],
\]

\[
\nabla_\mu \left( e^{-4\alpha \Phi/(n-1)} \partial_\nu \mathcal{L}(Y) F^{\mu \nu} \right) = 0.
\]

In particular, in the case of the linear electrodynamics with \( \mathcal{L}(Y) = -\frac{1}{2} Y \), the system of equations (5)-(7) reduce to
the well-known equations of Einstein-Maxwell dilaton gravity [22, 23].

The most general \((n + 1)\)-dimensional static metric with constant curvature \((t = \text{const.}, \ r = \text{const.})\)-boundary may be written as

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) d\Omega^2,
\]

where \( d\Omega^2 \) is an \((n - 1)\)-dimensional hypersurface with constant curvature \((n - 1)(n - 2) k \) and volume \( \omega_{n-1} \). In
general, one can set \( k = 0, 1, -1 \). The action (1) admits a static black hole solution with metric (8) provided \( V(\Phi) \) is
taken as \( V(\Phi) = 2\Lambda_0 e^{2\zeta_0 \Phi} + 2\Lambda e^{2\zeta \Phi} \) and \( R(r) = e^{2\alpha \Phi/(n-1)} \), where \( \Lambda_0, \ \Lambda, \ \zeta_0 \) and \( \zeta \) are the following constants [21]

\[
\zeta_0 = \frac{2}{\alpha (n-1)}, \quad \zeta = \frac{2\alpha}{n-1}, \quad \Lambda_0 = \frac{k (n-1) (n-2) \alpha^2}{2\beta^2 (1-\alpha^2)}.
\]
With these assumptions the equations of motion (5)-(7) admit the following solution

$$A_t(r) = \frac{gb^{3-n}\gamma}{16\pi r} {}_2F_1\left(\left[\frac{3}{2}, \frac{\alpha^2 - 2 + n}{2n - 2}\right], \left[\frac{\alpha^2 + 3n - 4}{2n - 2}\right], -\eta\right),$$  

$$\Phi(r) = \frac{(n-1)\alpha}{2(1 + \alpha^2)} \ln \left(\frac{b}{r}\right),$$  

$$f(r) = -\frac{k(n - 2)(\alpha + 1) b^{-2\gamma}}{(n-1)(1 - \gamma) - \gamma} r^2 - \frac{m}{1 - \gamma} + 2\Lambda (\alpha + 1)^2 b^{2\gamma} + \frac{4\beta^2 (\alpha + 1)^2 b^{2\gamma} r^{(1 - \gamma)}}{(n-1)(\alpha + 2) - \gamma} \left\{1 - 2{}_2F_1\left(\left[\frac{1}{2}, \frac{\alpha^2 - n}{2n - 2}\right], \left[\frac{\alpha^2 + n - 2}{2n - 2}\right], -\eta\right)\right\},$$

where \(b\) is an arbitrary nonzero positive constant, \(\Lambda\) is a free parameter which plays the role of the cosmological constant, \(\gamma = \alpha^2/(\alpha + 1)\) and

$$\Upsilon = (n - 3)(1 - \gamma) + 1 = \frac{n - 2 + \alpha^2}{1 + \alpha^2}$$

$$\eta = \frac{q^2 b^{2\gamma}(1 - n)}{\beta^2 (n-1)(1 - \gamma)}.$$

In the above equations \(m\) and \(q\) are the mass and charge parameters, respectively.

The ADM mass of the black hole is

$$M = \frac{b^{(n-1)\gamma}(n-1)\omega_{n-1}}{16\pi (\alpha + 1)} m,$$

where the mass parameter \(m\) may be written in term of the horizon radius as

$$m(r_+) = \frac{k(n - 2)(\alpha + 1) b^{-2\gamma}}{(1 - \alpha^2) T} r^T - \frac{2\Lambda (\alpha + 1)^2 b^{2\gamma} r^{(1 - \gamma)}}{(n-1)(\alpha + 2) - \gamma} + \frac{4\beta^2 (\alpha + 1)^2 b^{2\gamma} r^{(1 - \gamma)}}{(n-1)(\alpha + 2) - \gamma} \left\{1 - 2{}_2F_1\left(\left[\frac{1}{2}, \frac{\alpha^2 - n}{2n - 2}\right], \left[\frac{\alpha^2 + n - 2}{2n - 2}\right], -\eta_+\right)\right\}.$$

In the above equation, \(r_+\) denotes the radius of the event horizon which is the largest root of \(f(r_+) = 0\) and \(\eta_+\) is the value of \(\eta\) at \(r_+\). The temperature of the topological black hole on outer horizon \(r_+\) can be written as

$$T = f'(r_+) = -\frac{(\alpha + 1) b^{2\gamma} r^{1 - 2\gamma}}{2\pi (n - 1)} \left(\frac{k(n - 2)(\alpha + 1)^2 b^{-4\gamma}}{2(\alpha + 2) - \gamma} r^{4\gamma - 2} + \Lambda - 2\beta^2 \left(1 - \sqrt{1 + \eta_+}\right)\right).$$

Using the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area, one can obtain

$$S = \frac{b^{(n-1)\gamma} r^{(n-1)(1 - \gamma)} \omega_{n-1}}{4}.$$

Using Gauss law, the charge can be obtained as

$$Q = \frac{q^2 \omega_{n-1}}{4\pi}.$$

The electric potential \(U\), measured at infinity with respect to the horizon, is defined by

$$U = A_\mu \chi^\mu \mid_{r \to \infty} - A_\mu \chi^\mu \mid_{r = r_+},$$

where \(\chi = \partial_t\) is the null generator of the horizon. One obtains

$$U = \frac{gb^{3-n}\gamma}{Yr_+} {}_2F_1\left(\left[\frac{3}{2}, \frac{\alpha^2 - 2 + n}{2n - 2}\right], \left[\frac{\alpha^2 + 3n - 4}{2n - 2}\right], -\eta\right).$$
III. PHASE STRUCTURE

In this section, we would like to investigate thermodynamics of BI-dilaton black holes in an extended phase space in which the cosmological constant and BI parameter and their conjugate quantities are treated as thermodynamic variables. The conjugate quantity of the cosmological constant, which is proportional to pressure, is volume. Using the fact that the entropy of black hole is a quarter of the area of the horizon, the thermodynamic volume $V$ is obtained as

$$V = \int 4Sdr_+ = \frac{b(n-1)\gamma r_+^{n-\gamma(n-1)}}{n-\gamma(n-1)}\omega_{n-1}. \tag{19}$$

In the extended phase space $M$ should be a function of the extensive quantities: entropy and charge, and intensive quantities: pressure, and Born-Infeld parameter. Hence, defining $B$ as an extensive quantity conjugate to $\beta$

$$B = \left(\frac{\partial M}{\partial \beta}\right), \tag{20}$$

the first law takes the form

$$dM = TdS + UdQ + VdP + Bd\beta. \tag{21}$$

It is easy to show that the conjugate quantities of the thermodynamic volume and Born-Infeld parameter are

$$P = -\frac{\Lambda}{8\pi n} \frac{n-\gamma(n-1)}{n-\gamma(n+1)} \left(\frac{b}{r_+}\right)^{2\gamma} = -\frac{(n+\alpha^2)}{(n-\alpha^2)} \left(\frac{b}{r_+}\right)^{2\gamma} \frac{\Lambda}{8\pi}, \tag{22}$$

$$B = \frac{(1+\alpha^2)\beta b^{2\gamma(n+1)}\omega_{n-1}}{2\pi(n-\alpha^2)r^{\gamma(n+1)-n}} \left\{1 - 2F_1\left[\left[\frac{1}{2} - \frac{1}{2n-1}\right], \left[\frac{\alpha^2 + n - 2}{2n-1}\right], -\eta_+\right]\right\}, \tag{23}$$

which shows that $B$ is an extensive quantity. One may note that pressure is proportional to the cosmological constant $\Lambda$, while the constant of proportionality depends on the dilaton parameter. We can see that the above $P$ reduces to the pressure for Reissner-Nordstrom black hole [13] or BI-AdS black hole [9] in the absence of dilaton ($\gamma = 0 = \alpha$). One may also note that the above expression for the pressure is the same as that of Einstein-Maxwell dilaton black holes [10]. Also, it is clear that the pressure is positive provided $\alpha < \sqrt{n}$. This is consistent with the argument given in [21], which state that the topological BI-dilaton black hole solutions exist provided $\alpha < \sqrt{n}$. The Smarr relation may be obtained from the values of thermodynamic variables and mass as

$$M = \frac{n-1}{n-2+\alpha^2}TS - \frac{1}{n-2+\alpha^2}(2VP + \beta B) + UQ. \tag{24}$$

One may note that the above generalized Smarr formula reduces to those of Refs. [13, 16] in the absence of dilaton field ($\alpha = 0$)

$$M = \frac{n-1}{n-2}S \left(\frac{\partial M}{\partial S}\right) + Q \left(\frac{\partial M}{\partial Q}\right) - \frac{2}{n-2}P \left(\frac{\partial M}{\partial P}\right) - \frac{\beta}{n-2} \left(\frac{\partial M}{\partial \beta}\right). \tag{25}$$

In what follows, we study the phase transition of the charged BI-dilatonic black hole system in the extended phase space in canonical ensemble. Indeed, we treat the black hole charge $Q$ as a fixed external parameter, not a thermodynamic variable.

IV. EQUATION OF STATE

Using Eq. [22] and regarding the charge $Q$ as a fixed parameter, Eq. [14] can be written as

$$P = \frac{\Gamma T}{\tau_+} - \frac{k(n-2)(1+\alpha^2)\Gamma}{4\pi(1-\alpha^2)b^{2\gamma}\tau_+^{2\gamma}}$$

$$\quad + \frac{\beta^2(n+\alpha^2)b^{2\gamma}}{4\pi(n-\alpha^2)\tau_+^{2\gamma}} \left(\sqrt{1+\eta_+ - 1}\right), \tag{26}$$
where
\[
\Gamma = \frac{(n-1)(n+\alpha^2)}{4(n-\alpha^2)(\alpha^2+1)}.
\]

Taking into account the fact that \( r_+ \) is a function of the thermodynamic volume \( V \), as one may see from Eq. (19), the above equation can be regarded as the equation of state \( P(V, T, \beta) \). Before proceeding further, we translate the ‘geometric’ equation of state (20) to a physical one by performing a dimensional analysis. Noting that the physical pressure and temperature are given by
\[
\mathcal{P} = \frac{hc}{l_p^2} P, \quad T = \frac{hc}{\kappa} T,
\]
where the Planck length is \( l_p = \sqrt{\frac{\hbar G}{c^5}} \) and \( \kappa \) is the Boltzmann constant, Eq. (20) can be written as
\[
\mathcal{P} = \frac{\kappa \Gamma}{l_p^2} \frac{\beta^2 (n+\alpha^2) b^2}{4 \pi (n-\alpha^2) r_+^2} \left( \frac{1}{1 + \eta_+} \right).
\]

Now, comparing the above physical equation of state with the van der Walls equation (13)
\[
\mathcal{P} = \frac{T}{v} + ..., \quad v = \frac{l_p^2 r_+}{\Gamma},
\]
we understand that the specific volume \( v \) of the fluid in terms of the horion radius should be written as,
\[
v = \frac{l_p^2 r_+}{\Gamma}.
\]

Returning to the geometrical units \( (G = \hbar = c = 1 \implies l_p^2 = 1) \), the equation of state (26) can be written as
\[
P = \frac{T}{v} - \frac{k(n-2)(\alpha^2+1)}{4 \pi (1-\alpha^2) b^2 \gamma v^2 - \gamma} + \frac{\beta^2}{4 \pi (n-\alpha^2) (v \Gamma)^{-\gamma}} \left( \sqrt{1 + \frac{(v \Gamma)^{(2(n-1)(\gamma-1))q^2}}{b^2 (n-1) \beta^2}} - 1 \right).
\]

In order to compare the critical behavior of the system with van der Waals gas, we should plot isotherm diagrams. The corresponding \( P - v \) diagrams are displayed in Figs. (30).

The behavior of the isotherms diagrams depend on how deep we are in BI nonlinear regime. The critical point can be obtained by solving the following equations
\[
\left. \frac{\partial P}{\partial v} \right|_{T_c} = 0, \quad \left. \frac{\partial^2 P}{\partial v^2} \right|_{T_c} = 0.
\]

**A. Large \( \beta \)**

First, we consider the the case of large \( \beta \). In this case, Eq. (31) leads to
\[
v_c = \left( \frac{X}{b^2 (3-n) \gamma} \right)^{-1/(2 \gamma)} q^{1/\gamma} \frac{\Gamma}{T} \left\{ 1 - \frac{(\alpha^2 + 1) (\alpha^2 + 4n - 5)(\alpha^2 + 2n - 2)}{4k(n+\alpha^2 - 2)(n-1)(n-2)\beta^2} q^{6(\gamma-1)-4(n-1)\gamma} + O(\frac{1}{\beta^2}) \right\},
\]
\[
P_c = \frac{\Gamma X (\gamma-\gamma+1)^\gamma}{2 \pi (n-1)} \left\{ \frac{(2n-3+\alpha^2)(n-2+\alpha^2)}{b^2 \gamma q^{(1-\gamma)/\gamma}} + \frac{(\alpha^2 + 4n - 5) X (\gamma-2)(\gamma+1)^\gamma}{4 \beta^2 b^2 \gamma q^{(2-3\gamma)/\gamma}} + O(\frac{1}{\beta^2}) \right\},
\]
\[
T_c = \frac{(\alpha^2 + 2n - 2) k(n-2) X (1-2\gamma)^2}{\pi (1-\alpha^2)(n-3+\alpha^2) b^2 q^{(1-\gamma)/\gamma}} + \frac{(\alpha^2 + 2n - 2) X (2n-3)(\gamma+1) + 2\gamma}{4 \pi (n-1) b^2 \gamma q^{(1-\gamma)/\gamma}} + O(\frac{1}{\beta^2}),
\]
FIG. 1: $P-v$ diagram of BID black holes for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $\beta = 1$ and $\alpha = 0.3$.

FIG. 2: $P-v$ diagram of BID black holes for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $\beta = 0.25$ and $\alpha = 0.3$.

where

$$X = \frac{k(n-1)(n-2)}{2(2n-3+\alpha^2)(\alpha^2+n-1)}.$$  \hspace{1cm} (35)

It is easy to show the above critical quantities reduces to [19] as $\beta \to \infty$. Using the above critical values, $\rho_c$ is obtained as

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{(1-\alpha^2)(2n-3+\alpha^2)}{4(n-1+\alpha^2)} \left\{ 1 - \frac{(n-1)(1+\alpha^2)}{4\beta^2 (\alpha^2+n-2)(\alpha^2+n-1)} q^{2(1-\gamma)/\gamma} + O\left(\frac{1}{\beta^2}\right) \right\}. \hspace{1cm} (36)$$

As one expects, the above $\rho_c$ reduces to that of Ref. [19] as $\beta \to \infty$,

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{(1-\alpha^2)(2n-3+\alpha^2)}{4(n-1+\alpha^2)}. \hspace{1cm} (37)$$
and in the absence of the dilaton field ($\alpha = 0 = \gamma$) in four dimensions ($n = 3$), it reduces to $3/8$ which is the characteristic of van der Waals fluid. One should note that $p_c$ is positive and real provided $\alpha < 1$.

## B. Small $\beta$

In deep BI nonlinear regime, Eq. (30) can be written as

$$P = \frac{T}{v} - \frac{k(n - 2)(\alpha^2 + 1)\Gamma^{\gamma - 1}}{4\pi (1 - \alpha^2)b^{2\gamma}v^{2-2\gamma}} + \frac{(n + \alpha^2)\beta q(\Gamma v)^{-n+1+(n-3)\gamma}}{4\pi (n - \alpha^2)b^{(n-3)\gamma}}. \quad (38)$$
Using Eq. (31) one can obtain the critical values as
\[ v_c = \frac{b\alpha^2(3-n)/(n-3+2a^2)}{1/Z(1+c^2)/(n-3+2a^2)} \beta(1+c^2)/(n-3+2a^2), \]  \tag{39}
\[ P_c = \frac{(2\alpha^2 + n - 3)(n + 2\alpha^2)(\alpha^2 + n - 2)qZ}{8\pi(n - \alpha^2)(1 + c^2)b^{4\alpha^2}/(n-3+2a^2)\beta^2/(n-3+2a^2)}, \]  \tag{40}
\[ T_c = \frac{(2\alpha^2 + n - 1)(2\alpha^2 + n - 3)qZ}{\pi(n - 1)(1 - \alpha^2)b^{4\alpha^2}/(n-3+2a^2)\beta(1-c^2)/(n-3+2a^2)}, \]  \tag{41}
where
\[ Z = \frac{k(n - 1)(n - 2)}{2q(\alpha^2 + n - 2)(2\alpha^2 + n - 1)} \]  \tag{42}
which leads to the following value for \( \rho_c \)
\[ \rho_c = \frac{(n - 2 + \alpha^2)(1 - \alpha^2)}{2(n - 1 + 2a^2)}. \]  \tag{43}

One should note that \( \rho_c \) in deep Born-Infeld regime is different from that in the presence of Maxwell field. Also as in the case of large \( \beta \), the first order approximation of \( \rho_c \) does not depend on \( \beta \), while upon inclusion of higher power of \( \beta \) we expect this to become \( \beta \) dependent. Also, one should note that as \( \beta \) goes to zero \( T_c \) or \( P_c \) goes to infinity. Indeed, at \( \beta = 0 \) the third term in Eq. (38) vanishes and no critical behavior will occur.

V. GIBBS FREE ENERGY

In the canonical ensemble with fixed charge, the potential, which is the free energy of the system presents the thermodynamic behaviour of a system in a standard approach. But, since we are considering an extended phase space, we associate it with the Gibbs free energy \( G = M - TS \). The Gibbs free energy can be obtained as
\[ G = G(T, P) = \frac{(1 + \alpha^2)b^{(n-1)^\gamma}\omega_{n-1}}{4\pi r_+^{n(\gamma-1)-\gamma}} \left\{ \frac{k(n - 2)b^{-2\gamma}}{4(n - 2 + \alpha^2)r_+^{2(1-\gamma)}} - \frac{4\pi P(1 - \alpha^4)}{(n - 1)(n + \alpha^2)} \right\} + \frac{\beta^2 b^{2\gamma}}{(n - 1)r_+^{2\gamma}} \left( \sqrt{1 + \eta_+} - 1 \right), \]  \tag{44}
where \( r_+ \) is understood as a function of pressure and temperature via equation of state (20). If we expand the Gibbs free energy for large \( \beta \), we arrive at
\[ G(T, P) = \frac{b^{(n-1)^\gamma}\omega_{n-1}}{4\pi r_+^{n(\gamma-1)-\gamma}} \left\{ \frac{k(n - 2)(n - 2 + \alpha^2)}{4(1 + \alpha^2)b^{2\gamma}r_+^{2(1-\gamma)}} - \frac{4\pi P(1 - \alpha^4)}{(n - 1)(n + \alpha^2)} \right\} + \frac{\beta^2 (2n - 3 + \alpha^2)(\alpha^2 + 1)b^{-2(n-2)^\gamma}}{2(n - 2 + \alpha^2)(n - 1)r_+^{2(1-\gamma)+1}} + O \left( \frac{1}{\beta^2} \right), \]  \tag{45}
which reduces to the result obtained for black holes in EMd gravity as \( \beta \) goes to infinity (19). Although the hyper-geometrical series expression in Eq. (44) is convergent only for \( \eta_+ < 1 \), one may use the integral representation of hypergeometrical function for any value \( \eta_+ \). In this case one can obtain the limit of Gibbs free energy as \( \beta \) goes to zero as
\[ \lim_{\beta \to 0} G = \frac{(1 + \alpha^2)b^{(n-1)^\gamma}\omega_{n-1}}{4\pi r_+^{n(\gamma-1)-\gamma}} \left\{ \frac{k(n - 2)b^{-2\gamma}}{4(n - 2 + \alpha^2)r_+^{2(1-\gamma)}} - \frac{4\pi P(1 - \alpha^4)}{(n - 1)(n + \alpha^2)} \right\}, \]
FIG. 5: Gibbs free energy versus $\beta$ for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $p=0.1$, $r=1$, and $\alpha = 0.3$.

Thus, the Gibbs free energy starts from the above value and increases to

$$
\lim_{\beta \to \infty} G = \frac{b^{(n-1)}\sqrt{n-1}}{4\pi r_{+}^{n(\gamma-1)-\gamma}} \left\{ \frac{k(n-2)(n-2+\alpha^{2})}{4(1+\alpha^{2})b^{2}\gamma r_{+}^{2(1-\gamma)}} - \frac{4\pi P(1-\alpha^{4})}{(n-1)(n+\alpha^{2})} + \frac{q^{2}(2n-3+\alpha^{2})(\alpha^{2}+1)b^{-2(n-2)\gamma}}{2(n-2+\alpha^{2})(n-1)r_{+}^{2(1-\gamma)+1}} \right\}
$$

as $\beta$ goes from zero to infinity. This can be seen in Fig. 5.

The behavior of the Gibbs free energy in terms of temperature is shown in Figs. 6-10. From these figures we see that there is a swallowtail behavior for Gibbs free energy as a function of temperature, which means that we have a first order small-large black hole transition for the system.

VI. CRITICAL EXPO NENTS

The behavior of physical quantities in the vicinity of critical point can be characterized by the critical exponents. So, following the approach of [9], one can calculate the critical exponents $\alpha'$, $\beta'$, $\gamma'$ and $\delta'$ for the phase transition of
FIG. 7: Gibbs free energy versus $T$ for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $\beta = 0.25$ and $\alpha = 0.5$.

FIG. 8: Gibbs free energy versus $T$ for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $\beta = 0.5$ and $\alpha = 0.5$.

FIG. 9: Gibbs free energy versus $T$ for $b = 1$, $n = 3$, $q = 1$, $k = 1$, $\beta = 0.1$ and $\alpha = 0.2$. 
Expanding Eq. (47) near the critical point we define the reduced thermodynamic variables as

\[ p = \frac{P}{P_c}, \quad \nu = \frac{\nu}{\nu_c}, \quad \tau = \frac{T}{T_c}. \]

So, equation of state (30) translates into the law of corresponding state,

\[
p = \frac{1}{\rho_c \nu} - \frac{k(n-2)(\alpha^2+1)\Gamma^{\gamma-1}b^{2\gamma-2}}{4\pi P_c (1-\alpha^2) b^{2\gamma-2}\gamma} \\
+ \frac{(\nu\Gamma\nu_c)^{-2\gamma}b^{2\gamma}\beta^2 (n+\alpha^2)}{4\pi P_c (n-\alpha^2)} \left( 1 + \frac{(\nu\Gamma\nu_c)^{2(n-1)(\gamma-1)}}{b^{2\gamma(n-1)\beta^2}} - 1 \right),
\]

which reduces to

\[
p = \frac{1}{\rho_c \nu} - \frac{k(n-2)(\alpha^2+1)\Gamma^{\gamma-1}}{4\pi b^{2\gamma} P_c (1-\alpha^2) (\nu\nu_c)^{2-2\gamma}} + \frac{(n+\alpha^2)q^2(\nu\Gamma\nu_c)^{2(n-1)(\gamma-1)}}{8\pi P_c (n-\alpha^2) b^{2\gamma(n-3)}(\nu\Gamma)^{2\gamma}}
\]

and

\[
p = \frac{1}{\rho_c \nu} - \frac{k(n-2)(\alpha^2+1)\Gamma^{\gamma-1}}{4\pi b^{2\gamma} P_c (1-\alpha^2) (\nu\nu_c)^{2-2\gamma}} + \frac{(n+\alpha^2)q(\nu\Gamma\nu_c)^{-n+1+(n-3)\gamma}}{4\pi P_c (n-\alpha^2) b^{(n-3)\gamma}}
\]

for large and small \( \beta \), respectively. Although this law depends on parameter \( \gamma \), this doesn’t affect the behavior of the critical exponents as we will see below. To calculate the critical exponent \( \alpha' \), we consider the entropy \( S \) as a function of \( T \) and \( V \). Using Eq. (19) we have

\[ S = S(T,V) = \frac{b^{(n-1)\gamma}\omega_{n-1}}{4} \left\{ \frac{((n-1)(1-\gamma) + 1) V}{(n-1)\gamma\omega_{n-1}} \right\}^{(n-1)/(n+\alpha^2)}. \]

Obviously, this is independent of \( T \) and therefore the specific heat vanishes, \( C_V = T (\partial S/\partial T)_V = 0 \). Since the exponent \( \alpha' \) governs the behavior of the specific heat at constant volume \( C_V \propto |\tau - 1|^{\alpha'} \), hence the exponent \( \alpha' = 0 \). Expanding Eq. (17) near the critical point

\[ \tau = t + 1, \quad \nu = (\omega + 1)^{\frac{1}{\gamma}}, \]

where \( \varepsilon \) is a positive parameter defined as \( \varepsilon = n - \gamma (n - 1) = (n + \alpha^2)/(1 + \alpha^2) \) and following the method of Ref. [9], we obtain

\[ p = 1 + At - Bt\omega - C\omega^3 + O(t\omega^2,\omega^4), \]

an \((n + 1)\)-dimensional charged dilatonic black hole in the presence of BI field. To obtain the critical exponents, we define the reduced thermodynamic variables as

\[ \frac{P}{P_c}, \quad \frac{\nu}{\nu_c}, \quad \frac{T}{T_c}. \]

and

\[ \frac{S}{S_c}. \]

FIG. 10: Gibbs free energy versus \( T \) for \( b = 1, n = 4, q = 1, k = 1, \beta = 1 \) and \( \alpha = 0.2 \).
where

\[ A = \frac{1}{\rho_c}, \quad B = \frac{1}{\varepsilon \rho_c}, \quad C = \frac{2(n - 1 + \alpha^2)}{3(1 + \alpha^2)^2 \varepsilon^3} - \frac{(n - 1)(\alpha^2 + 4n - 5)b^{-2\gamma(n-1)/\gamma}X^{(n-1)(1-\gamma)/\gamma}}{(\alpha^2 + n - 2)(\alpha^2 + 1)^2 \beta^2 \varepsilon^3 \eta^{(1-\gamma)/\gamma}}, \]

and

\[ C = \frac{(n - 1 + 2\alpha^2)}{3(1 + \alpha^2)^2 \varepsilon^3} \]

for large and small \( \beta \), respectively. Differentiating Eq. (49) at a fixed \( t < 0 \) with respect to \( \omega \), we get

\[ dP = -P_c Bt + 3C \omega^3 \frac{d\omega}{d\omega}. \]  

(51)

Now, we apply the Maxwell’s equal area law [13]. Denoting the volume of small and large black holes with \( \omega_s \) and \( \omega_l \), respectively, we obtain

\[ p = 1 + At - Bt \omega_l - C \omega_l^3 = 1 + At - Bt \omega_s - C \omega_s^3 \]

\[ 0 = \int_{\omega_s}^{\omega_l} \omega dP. \]

(52)

Equation (52) leads to the unique non-trivial solution

\[ \omega_l = -\omega_s = \sqrt{-\frac{Bt}{C}}, \]

which gives the order parameter \( \eta = V_c (\omega_l - \omega_s) \) as

\[ \eta = 2V_c \omega_l = 2 \sqrt{-\frac{B}{C} t^{1/2}}. \]

(54)

Thus, the exponent \( \beta' \) which describes the behavior of the order parameter \( \eta \) near the critical point is \( \beta' = 1/2 \). To calculate the exponent \( \gamma' \), we may determine the behavior of the isothermal compressibility near the critical point. Differentiating Eq. (49) with respect to \( V \), one obtains

\[ \frac{\partial V}{\partial P} \bigg|_T = -\frac{V_c}{BP_c} \frac{1}{t} + O(\omega). \]

Hence, the isothermal compressibility near the critical point may be written as

\[ \kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T \propto -\frac{V_c}{BP_c} \frac{1}{t} \implies \gamma' = 1. \]

(55)

Finally, the shape of the critical isotherm \( t = 0 \) is given by (49)

\[ p - 1 = -C \omega^3 \implies \delta' = 3. \]

(56)

So we have shown that for BI-dilaton black hole in \( (n + 1) \) dimensions, we obtain the same critical exponents as in the linear Maxwell case [13] and in the dilatonic black holes [19].

VII. SUMMARY AND CONCLUSIONS

In this paper, we investigated the critical behavior of \((n + 1)\)-dimensional BI-dilaton black holes in the presence of two Liouville type potentials. While one of the Liouville type potential guarantees the existence of the solution, the second one contains a constant \( \Lambda \), which has the role of cosmological constant. We enlarged the phase space by considering the constant \( \Lambda \) and the BI parameter \( \beta \) to be treated as thermodynamic quantities. By calculating the
thermodynamic quantities, we obtained the generalized Smarr relation, which reduces to the Smarr relation in the absence of dilaton field given in [15, 16]. After constructing the Smarr relation, we used the pressure and Hawking temperature to build the equation of state and plot $P-v$ isotherm diagrams. These figures show the analogy between our system and the van der Walls fluid, with the same phase transition. We also found that the critical behavior can occurred only for black holes with spherical horizon ($k = 1$) provided $\alpha < 1$. Then, we obtained the critical pressure, volume and temperature both for large and small BI parameter $\beta$. We found that the critical temperature and pressure go to infinity as $\beta$ goes to zero. Indeed, at $\beta = 0$ the third term in Eq. (35) vanishes and no critical behavior will occur. In the absence of dilaton field ($\alpha = 0 = \gamma$), our results reduce to those of BI black holes [8], while for sufficiently large $\beta$ we recovered the critical quantities of Einstein-Maxwell dilaton black hole [19]. Moreover, we considered the behavior of the Gibbs free energy and found that there is a swallowtail behavior for Gibbs free energy as a function of temperature. This shows that there is a first order small-large black hole transition in the system. Finally, we calculated the critical exponents and found that the results are the same as van der Waals system. This, implies that the inclusion of nonlinear electrodynamics, dilaton field or extra dimensions do not change the critical exponents. In the present work we only considered the effects of dilaton field on the critical behavior of the black holes in BI nonlinear electrodynamics. It is worth to investigate the effects of dilaton on the critical behaviour of black holes in the presence of other nonlinear electrodynamics theories.

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