On the Large $N$ Limit of Conformal Field Theory

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Following recent advances in large $N$ matrix mechanics, I discuss here the free (Cuntz) algebraic formulation of the large $N$ limit of two-dimensional conformal field theories of chiral adjoint fermions and bosons. One of the central results is a new affine free algebra which describes a large $N$ limit of $\mathfrak{su}(N)$ affine Lie algebra. Other results include the associated free-algebraic partition functions and characters, a free-algebraic coset construction, free-algebraic construction of $\mathfrak{osp}(1|2)$, free-algebraic vertex operator constructions in the large $N$ Bose systems and a provocative new free-algebraic factorization of the ordinary Koba–Nielsen factor.

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1 Introduction

Large $N$ matrix mechanics [1-6] is the adaptation of Heisenberg’s matrix mechanics [7] to study systems of $\text{su}(N)$ adjoint matter at large $N$. For both Hamiltonian [1-4,6] and action [5,6] theories, this approach gives a simpler “reduced” formulation of the adjoint sector of the large $N$ theory in terms of so-called free algebras. The simplest free algebra is the Cuntz algebra, but many generalizations [4,5] have also been found in large $N$ matrix mechanics, including the interacting Cuntz algebras and various fermionic and supersymmetric versions of the Cuntz algebra. In all cases, the free algebras are associated to new, hidden conserved quantities at large $N$ – and corresponding simplifications of the large $N$ theory.

Recently, Halpern and Thorn [8] showed for any matrix light-cone Hamiltonian field theory that the large $N$ reduced Hamiltonian of the theory can be expressed in terms of the generators of the Cuntz algebra itself, with correspondingly enhanced simplifications. As simple examples of this result, I consider here the large $N$ limit of 2-$d$ matrix conformal field theory – which always has such a light-cone formulation.

More precisely I will follow an historical development, limiting the discussion here to the matrix analogues of the earliest and simplest 2-$d$ conformal and superconformal field theories – namely those composed of free chiral adjoint world-sheet fermions and bosons. In spite of the apparent simplicity of these theories at finite values of $N$, we shall see that their large $N$ limit shows a wealth of new phenomena, all succinctly described in terms of free algebras.

For example, the reduced, Cuntz-algebraic description of large $N$ chiral adjoint fermions is studied in Subsec.2.1 and, following the independent discovery of affine Lie algebra in physics [9], reduced currents are constructed as reduced fermion bilinears in Subsec.2.2. The reduced current modes satisfy a new affine free algebra at reduced level $\hat{x}$

\[
J(m \geq 0)J(n \leq 0) = -J(m + n) + \hat{x}m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}
\]  

(1.1)

which describes the large $N$ limit of $\text{su}(N)$ affine Lie algebra in the adjoint sector. The representation theory of this new algebra is surprisingly rich because, as we will see, many new affine primary states occur in the large $N$ limit which are not present at any finite value of $N$.

Other large $N$ results include the corresponding free-algebraic partition functions and characters (which exhibit a surprising 2-$d$ limiting tempera-
an extended affine free algebra and a free algebraic coset construction, as well as a free-algebraic construction of $\text{osp}(1|2)$. In the last section, I consider a set of more exotic free-algebraic constructions in the large $N$ Bose systems – including free-algebraic vertex operators and free-algebraic vertex operator constructions of new excitations at large $N$. This leads us finally to a provocative new free-algebraic factorization of the familiar Koba-Nielsen factor [10] and hence to the question of new free-algebraic strings.

2 Chiral Adjoint Fermions

It is an historical curiosity that the discovery of half-integral moded chiral world-sheet fermions [9] and integral-moded world sheet fermions [11] appeared as consecutive articles in the same issue of Physical Review. In the large $N$ formulations of this section, I will follow the development of Bardakci and Halpern [9,12], who used the world-sheet fermions to construct the so-called dual quark model, including the independent discovery of affine Lie algebra in physics and the first examples of the affine-Sugawara and coset constructions. See the Appendix of Ref. [13] for a short history of affine Lie algebra and associated early developments in conformal field theory.

2.1 Cuntz-algebraic description of the large $N$ limit

To avoid the complexities of fermionic zero modes, I will begin with a single half-integral moded chiral world-sheet fermion in the adjoint of $\text{u}(N)$

\[
[H_a(p), H_b(q)]_+ = \delta_{ab} \delta_{p+q,0}, \quad H_a(p)^\dagger = H_a(-p)
\] (2.1a)

\[
H_a(p > 0)|0.\rangle = \langle .0|H_a(p < 0) = 0
\] (2.1b)

\[
\langle .0.|.0.\rangle = 1, \quad a, b = 1 \ldots N^2, \quad p, q \in \mathbb{Z} + \frac{1}{2}
\] (2.1c)

which approximates the adjoint of $\text{su}(N)$ at large $N$. Systems of many chiral adjoint fermions will be considered later. The development of this subsection is an infinite-dimensional generalization of the discussion given for quantum-mechanical adjoint fermions in Ref.[4].

For fields in the adjoint, the first step is always to introduce matrix fields

\[
H(p)_{rs} = H_a(p)<T_a>_{rs}, \quad H(p)_{rs}^\dagger = H(-p)_{sr}
\] (2.2a)
\[ [T_a, T_b] = i f_{abc} T_c, \quad T_a^\dagger = T_a, \quad \text{Tr}(T_a T_b) = \delta_{ab} \quad (2.2b) \]

\[ (T_a)_{rs} (T_a)_{uv} = \delta_{rv} \delta_{su}, \quad r, s, u, v = 1 \ldots N \quad (2.2c) \]

\[ [H_{rs}(p), H_{uv}(q)]_+ = \delta_{rs} \delta_{su} \delta_{p+q,0} \quad (2.2d) \]

\[ H_{rs}(p > 0)|0.\rangle = \langle 0|H_{rs}(p < 0) = 0 \quad (2.2e) \]

where \( \{T_a\} \) is the fundamental representation of \( \mathfrak{u}(N) \) and I have chosen root length squared \( \alpha^2 = 2 \) for \( \mathfrak{su}(N) \). In large \( N \) matrix mechanics [1-6] we restrict the Hilbert space to the adjoint sector at large \( N \), which includes only the singlet ground state and the dominant adjoint eigenstates \( |\ldots\rangle_{rs} \) which saturate the traced Wightman functions \( \langle 0|\text{Tr}(H(p_1) \ldots H(p_n))|0.\rangle \) of the large \( N \) theory. The large \( N \) dynamics is then mapped by a generalization of Bardakci’s [14] reduced matrix elements (Wigner-Eckart for \( \mathfrak{su}(N) \)) onto reduced states and reduced operators [4]

\[ \left( \frac{H(p)}{\sqrt{N}} \right)_{rs} \to H(p), \tilde{H}(p) \quad (2.3a) \]

\[ \langle 0|\text{Tr} \left( \frac{H(p_1)}{\sqrt{N}} \ldots \frac{H(p_n)}{\sqrt{N}} \right) |0.\rangle = N \langle 0|H(p_1) \ldots H(p_n)|0 \rangle \quad (2.3b) \]

where \( |0\rangle \) is the reduced ground state. Note that each reduced operator \( H \) is accompanied by a reduced tilde partner \( \tilde{H} \), and that all \( r, s \) indices are modded out in the reduction procedure. The relation in (2.3b) is only a sample, the full list of reduction formulae being given in Ref. [4]. Next, one finds from Eq. (2.29) of Ref. [4] that the reduced operators satisfy the quasi-canonical algebra

\[ [H(p), \tilde{H}(q)]_+ = |0\rangle \langle 0|\delta_{p+q,0}, \quad H^\dagger(p) = H(-p), \quad \tilde{H}^\dagger(p) = \tilde{H}(-p) \quad (2.4a) \]

\[ H(p > 0)|0 \rangle = \tilde{H}(p > 0)|0 \rangle = 0 \langle 0|H(p < 0) = \langle 0|\tilde{H}(p < 0) = 0 \quad (2.4b) \]

\[ \langle \tilde{H}(p < 0) - H(p < 0)|0 \rangle = \langle 0|\tilde{H}(p > 0) - H(p > 0) = 0 \quad (2.4c) \]
As discussed in Ref.[4], a basis for a complete set of reduced states can be constructed by acting on the reduced vacuum with the creation operators \( \{ H(p < 0) \} \) or equivalently by acting with the tilde creation operators \( \{ \tilde{H}(p < 0) \} \). Finally, the action of the algebra \( (2.4) \) on either basis gives the further free-algebraic relations

\[
H(p > 0)H(q < 0) = \delta_{p+q,0}, \quad \sum_{p>0} H(-p)H(p) = 1 - |0\rangle\langle 0| \quad (2.5a)
\]

\[
\tilde{H}(p > 0)\tilde{H}(q < 0) = \delta_{p+q,0}, \quad \sum_{p>0} \tilde{H}(-p)\tilde{H}(p) = 1 - |0\rangle\langle 0| \quad (2.5b)
\]

which, together with the relations in \( (2.4) \), form an infinite-dimensional version of a so-called symmetric Fermi/Cuntz algebra [4]. Further applications of the tilde operators are discussed in Refs. [4-6] and the Appendix.

In this paper I will focus primarily on the \( H \)-subalgebra \( (2.5a) \) of the symmetric Fermi/Cuntz algebra, which is a single infinite-dimensional Cuntz algebra

\[
b(p) \equiv H(p), \quad b^\dagger(p) = H(-p), \quad p > 0 \quad (2.6a)
\]

\[
b(p)b^\dagger(q) = \delta_{p,q}, \quad \sum_{p>0} b^\dagger(p)b(p) = 1 - |0\rangle\langle 0| \quad (2.6b)
\]

\[
b(p)|0\rangle = \langle 0|b^\dagger(p) = 0 \quad (2.6c)
\]

and it should be emphasized that, up to a relabelling, this fermionic Cuntz algebra is the same Cuntz algebra which is obtained for chiral Bose systems at large \( N \) (see Sec.3). Indeed it is well known [4] that the Pauli principle is lost at large \( N \) and all systems, Fermi or Bose, satisfy the same classical or Boltzmann statistics (with no relations) at large \( N \).

Finally, I will use standard word notation for the reduced basis states

\[
b_w \equiv b(p_1)\ldots b(p_n), \quad w \equiv p_1 \ldots p_n \quad (2.7a)
\]

\[
[w] = n, \quad \{w\} = \sum_{i=1}^{n} p_i \quad (2.7b)
\]

\[
|w\rangle = b^\dagger_w|0\rangle = b^\dagger(p_n)\ldots b^\dagger(p_1)|0\rangle = H(-p_n)\ldots H(-p_1)|0\rangle \quad (2.7c)
\]

\[
\langle w'|w\rangle = \langle 0|b_w^\dagger b_w|0\rangle = \delta_{w',w} \quad (2.7d)
\]
where \([w]\) and \(\{w\}\) are respectively the length and the weight of the word \(w\). Note that, in the present application, the weight of \(w\) is the (mode) level of the state \(|w\rangle\). It will also be helpful to recall [4,8] that the reduced basis states (2.7) correspond at large \(N\) to the unreduced ground state and (dominant) unreduced adjoint basis states

\[
|w.\rangle_{rs} \equiv \sqrt{N} \left( \frac{b(p_n)\dagger}{\sqrt{N}} \ldots \frac{b(p_1)\dagger}{\sqrt{N}} \right)_{rs} |0.\rangle
\]  

(2.8)

whose norm is \(O(N^0)\) at large \(N\).

2.2 Affine free algebra

Let us now return to the unreduced theory, where we may follow Bardakci and Halpern [9] to consider the modes \(\{J(m)\}\) of the \(\mathfrak{su}(N)\) currents formed from the unreduced chiral adjoint fermion:

\[
J_a(m) \equiv \frac{1}{2} \sum_p : H_b(p)(\mathbb{T}_a^{\text{adj}})_{bc} H_c(m-p) : \quad (2.9a)
\]

\[
(\mathbb{T}_a^{\text{adj}})_{bc} = -i f_{abc}, \quad J_a(m)\dagger = J_a(-m), \quad a,b,c = 1 \ldots N^2 \quad (2.9b)
\]

\[
J(m)_{rs} \equiv J_a(m)(\mathbb{1}_a)_{rs} = - \sum_p : H(p)_r H(m-p)_s : = J(-m)_{sr}\dagger \quad (2.9c)
\]

\[
: H(p)H(q) : \equiv -\theta(p > 0)H(q)H(p) + \theta(p < 0)H(p)H(q). \quad (2.9d)
\]

These currents satisfy the algebra of affine \(\mathfrak{su}(N)\) at invariant affine level \(N\), whose matrix form is:

\[
[J(m)_{rs}, J(u)_{uv}] = \delta_{rv} J(m+n)_{us} - \delta_{us} J(m+n)_{rv} + N m \delta_{rv}\delta_{us}\delta_{m+n,0} \quad (2.10a)
\]

\[
[J(m)_{rs}, H(p)_{uv}] = \delta_{rv} H(p+m)_{us} - \delta_{us} H(p+m)_{rv}. \quad (2.10b)
\]

To find the reduced form of these currents at large \(N\), we will need the more explicit forms:

\[
J(0)_{rs} = \sum_{p>0} (H(-p)_t H(p)_r - H(-p)_r H(p)_t) \quad (2.11a)
\]
These expressions show that the non-zero modes are densities (matrix products), while the zero mode involves a term which is a twisted density.

A key result of Ref. [8] is that twisted densities are subleading under large $N$ reduction, and so we obtain the explicit form of the modes of the reduced currents

$$\frac{1}{N}J(m)_{rs} \to J(m)$$

$J(0) = -\sum_{p>0} H(-p)H(p) = |0\rangle \langle 0| - 1, \quad J(0)^2 = -J(0)$

in terms of the reduced fermion modes. The second form of $J(0)$ in (2.12b) follows from (2.5b). Tilde partners of the reduced current modes can also be obtained, but I will not study these here.

The reduced current modes (2.12) can be written in the more useful form:

$$J(m \geq 0) = -\sum_{p>0} H(m-p)H(p)$$

$$J(m \leq 0) = -\sum_{p>0} H(-p)H(m+p)$$

$$J(m)\dagger = J(-m), \quad J(m \geq 0)|0\rangle = \langle 0|J(m \leq 0) = 0$$

$$J(0)|\alpha\rangle = -|\alpha\rangle, \quad \langle \alpha|J(0) = -\langle \alpha|, \quad \forall |\alpha\rangle \neq |0\rangle.$$  

To obtain (2.13a,b) from (2.12c), use the fermionic Cuntz algebra (2.5a) to verify the identities

$$\sum_{p>m} H(p)H(m-p) = 0 \text{ for } m > 0$$

$$\sum_{p m} H(p)H(m-p) = 0 \text{ for } m < 0$$

and use the second form of $J(0)$ in (2.12b) to verify (2.13d).

Because the reduced currents (2.13) satisfy only free-algebraic relations, I will sometimes refer to them as free currents. For example, the free algebra of the free currents with the reduced fermions

$$J(m \geq 0)H(p < 0) = H(p > 0)J(m \leq 0) = -H(p + m)$$

8
follows from (2.13) and the fermionic Cuntz algebra.

Moreover, we may use (2.13) and (2.15) to obtain the free algebra of the free current modes with themselves, which we call affine free algebra at reduced level $\hat{x} = 1$:

$$J(m \geq 0)J(n \leq 0) = -J(m + n) + m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}. \quad (2.16)$$

The affine free algebra (2.16), which describes the large $N$ limit of the su($N$) affine Lie algebra (2.10a) in the adjoint sector, is one of the central results of this paper. It should be emphasized that this result is an example in the class of so-called interacting Cuntz algebras, which appear quite generally in interacting Bose systems [4-6] at large $N$.

To obtain a realization of affine free algebra at higher integer level, begin with $\hat{x} \in \mathbb{Z}_+$ chiral fermions in the adjoint. For brevity, I give only the reduced results for this case:

$$H_i(p > 0)H_j(q < 0) = \delta_{ij}\delta_{p+q,0}, \quad i, j = 1 \ldots \hat{x} \quad (2.17a)$$

$$\sum_{i=1}^{\hat{x}} \sum_{p>0} H_i(-p)H_i(p) = 1 - |0\rangle\langle 0| \quad (2.17b)$$

$$J_i(m \geq 0) \equiv -\sum_{p>0} H_i(m - p)H_i(p), \quad J_i(m \leq 0) \equiv -\sum_{p>0} H_i(-p)H_i(m + p) \quad (2.17c)$$

$$J_i(m) \dagger = J_i(-m), \quad J(m) \dagger = J(-m), \quad J(0) = |0\rangle\langle 0| - 1 \quad (2.17d)$$

$$J(m \geq 0)|0\rangle = J_i(m \geq 0)|0\rangle = 0, \quad \langle 0|J(m \leq 0) = \langle 0|J_i(m \leq 0) = 0 \quad (2.17f)$$

$$J_i(m \geq 0)H_j(p < 0) = H_i(p > 0)J_j(m \leq 0) = -\delta_{ij}H_i(p + m) \quad (2.17g)$$

$$J(m \geq 0)H_i(p > 0) = H_i(p > 0)J(m \leq 0) = -H_i(p + m) \quad (2.17h)$$

$$J(m \geq 0)J(n \leq 0) = -J(m + n) + \hat{x}m\delta_{m+n,0}. \quad (2.17i)$$

At reduced level $\hat{x}$, the affine free algebra (2.17i) is the result quoted in the Introduction.

The representation theory of affine free algebra, including the possibility of non-integer reduced level $\hat{x}$, is discussed in the following subsections.
2.3 Free affine primary states and modules

Following the usual convention, I will define a free affine primary state $|FAP\rangle$ of the affine free algebra (2.17i) at arbitrary reduced level $\hat{x}$ as any state which satisfies

$$J(m \geq 0)|FAP\rangle = -\delta_{m,0}|FAP\rangle, \quad t = 0 \text{ or } 1$$

(2.18)

and I will assume in what follows that each free affine primary state is normalized to 1. Only the indicated values of the eigenvalue $t$ are allowed because $-J(0) = J(0)^2$ is a projection operator. We may always introduce a free affine ground state $|0\rangle$ which is a free affine primary state with $t = 0$. For the fermionic realizations at positive integer $\hat{x}$ we find with (2.13c,d) that

$$t = \begin{cases} 
0 & \text{for } |0\rangle \\
1 & \text{for all others}
\end{cases}$$

(2.19)

so that, in these realizations, the fermion ground state $|0\rangle$ is the only free affine primary state with $t = 0$.

The free affine module of any free affine primary state is constructed as usual with the negative modes of the reduced currents, and the affine free algebra tells us that all the free affine secondaries of each module have $t = 1$

$$(J(0) + 1)J(-m) = 0, \ m \geq 0$$

(2.20a)

$$(J(0) + 1)J(-m_1) \cdots J(-m_n)|FAP\rangle = 0, \ n \geq 1$$

(2.20b)

independent of the free affine primary state.

Using (2.20) and the affine free algebra (2.17i) at arbitrary reduced level $\hat{x}$, it is straightforward to compute the norms of all the states in each free affine module

$$\|J(-m_1) \cdots J(-m_n)|FAP\rangle\|^2 = (t + m_n\hat{x}) \prod_{i=1}^{n}(1 + m_i\hat{x}) > 0 \text{ for } \hat{x} > 0.$$  

(2.21)

This surprisingly simple result shows that all the norms are positive and there are no null states in the free affine modules so long as the level $\hat{x}$ is strictly greater than zero. Although I will not pursue this subject here, the result (2.21) suggests that unitary realizations of the affine free algebra exist for all $\hat{x} > 0$. 
As concrete examples, I list here the lowest (normalized) free affine primary states at reduced level \( \hat{x} = 1 \)

\[
|\Delta = 0, 0\rangle = |0\rangle, \quad |\Delta = 1/2, 1\rangle = H(-1/2)|0\rangle \tag{2.22a}
\]

\[
|\Delta = 3/2, 1\rangle = \frac{1}{\sqrt{2}}(H(-1/2)^3 - H(-3/2))|0\rangle \tag{2.22b}
\]

and the lowest free affine primary states at reduced level \( \hat{x} \in \mathbb{Z}_+ \)

\[
|\Delta = 0\rangle = |0\rangle, \quad |\Delta_i = 1/2, 1\rangle = H_i(-1/2)|0\rangle, \quad i = 1 \ldots \hat{x} \tag{2.23a}
\]

\[
|\Delta_{i\neq j} = 1, 1\rangle = H_i(-1/2)H_j(-1/2)|0\rangle, \quad 1 \leq i \neq j \leq \hat{x} \tag{2.23b}
\]

\[
|\Delta_{ii} = 1, 1\rangle = \frac{1}{\sqrt{2}}(H_i(-1/2)^2 - H_{i+1}(-1/2)^2)|0\rangle, \quad i = 1 \ldots \hat{x} - 1. \tag{2.23c}
\]

The labelling of these states will be explained in the following subsections, where we will also study the partition functions and characters associated to positive integer level of the free affine algebra.

As a final topic in this subsection, I want to distinguish between two types of free affine primary states which I will call “ordinary” and “extraordinary”. The distinction is based on the fact that, as we shall see, we are finding more free affine primary states than there are adjoint affine primary states in the unreduced theory

\[
\left( \frac{J(m > 0)}{N} \right)_{rs} C(p_1, \ldots, p_n) \left\{ \sqrt{N} \frac{H(-p_1)}{\sqrt{N}} \ldots \frac{H(-p_n)}{\sqrt{N}} \right\}_{ts} |0\rangle = 0 \tag{2.24}
\]

at any finite value of \( N \). Examples of adjoint affine primary states at finite \( N \) are easily found in the case of one adjoint fermion:

\[
\left( \frac{J(m > 0)}{N} \right)_{rs} |0\rangle = \left( \frac{J(m > 0)}{N} H(-1/2) \right)_{rs} |0\rangle = 0 \tag{2.25a}
\]

\[
\left( \frac{J(m \geq 2)}{N} \right)_{rs} \sqrt{N} \left\{ \left( \frac{H(-1/2)}{\sqrt{N}} \right)^3 - H(-3/2) \right\} |0\rangle = 0. \tag{2.25b}
\]
Then the free affine primary states in (2.22a) are ordinary in the sense that they are in one-to-one correspondence ($H \leftrightarrow H_{rs}$) with these adjoint affine primary states at all finite values of $N$. Similarly, one finds that the adjoint states which correspond to the free affine primary states in Eq. (2.23) are in fact affine primary states at finite $N$, so these free affine primary states are also ordinary.

On the other hand, we find after some algebra that the unreduced adjoint state which corresponds to the free affine primary state (2.22b) satisfies

$$\left( \frac{J(m \geq 2)}{N} \sqrt{N} \left\{ \left( \frac{H(-1/2)}{\sqrt{N}} \right)^3 - H(-3/2) \right\} \right)_{rs} |0,\rangle = 0 \quad (2.26a)$$

$$\left( \frac{J(1)}{N} \sqrt{N} \left\{ \left( \frac{H(-1/2)}{\sqrt{N}} \right)^3 - H(-3/2) \right\} \right)_{rs} |0,\rangle = \left\{ \frac{1}{N^2} H(-1/2)_{rs} - \frac{\delta_{rs}}{\sqrt{N}} \text{Tr} \left( \frac{H(-1/2)}{\sqrt{N}} \right) \right\} |0,\rangle \quad (2.26b)$$

so that this unreduced state is not an affine primary state at any finite $N$. The trace term in (2.26) is an artifact of our inclusion of the extra $u(1)$ fermion, but the first term would appear even for a traceless fermion. Of course the norm of this state goes to zero smoothly at large $N$, in fact as $O(N^{-1})$, so it is no surprise that the reduced state (2.22b) is a free affine primary state – albeit an extraordinary one which does not correspond to an adjoint affine primary state at any finite value of $N$.

The lesson here is that new extraordinary free affine primary states can and do occur in the large $N$ limit (accompanied in fact by new null states such as the limit of (2.26)), and these extraordinary states are included on an equal footing with the ordinary free affine primary states in our free-algebraic formulation.

### 2.4 Reduced fermionic $sl(2)$

I turn next to study the fate of the Virasoro algebra in large $N$ matrix mechanics.

Before considering any particular theory, some general remarks will be helpful. In the first place, the Virasoro generators of general matrix conformal field theories are $su(N)$-invariant operators which comprise particular
examples of so-called trace class operators in each theory. It is known [4] that algebraic relations among trace class operators and algebraic relations of trace class operators with the densities of the theory are preserved under large $N$ reduction— but the composite structure of reduced trace class operators is not obtained directly in the reduction procedure. This is called the opacity problem in Ref. [4]. To obtain the composite structure of a given reduced trace class operator, it is necessary to solve [4,8] the algebraic relations of that operator with the reduced fundamental densities of the theory.

As a concrete example I will consider first the theory of a single adjoint fermion, which has conformal weight $1/2$ under the Virasoro algebra. It follows that, if the reduced Virasoro generators $\{L_F(m)\}$ are to exist at all, they must solve the reduced commutation relations

$$[L_F(m), H(p)] = -(\frac{m}{2} + p)H(p + m), \quad \forall \ p \in \mathbb{Z} + 1/2$$ (2.27)

which have the same form as the corresponding unreduced relations.

It is clear on general grounds however that the Virasoro generators and hence their reduced counterparts cannot be well-defined for $|m| > 1$, so that something will prevent us from solving (2.27) beyond the $\mathfrak{sl}(2)$ subalgebra. The reason is that the central charge of the unreduced theory is $c = N^2/2$, and more generally the central charge of any matrix conformal field theory grows as $O(N^2)$ at large $N$. This means that the Virasoro generators with $m \leq -2$ would create states of infinite norm at large $N$—and this is of course quite intolerable in a rigorous formulation such as large $N$ matrix mechanics.

In fact, it is easy to find the implied inconsistency in our example. Use the reduced commutation relations (2.27) to compute the commutator of $L_F(m)$ with the fermion Cuntz algebra (2.5a):

$$0 = [L_F(m), H(p > 0)H(q < 0)]$$ (2.28a)

$$= (m/2 + p)H(p + m)H(q) + (\frac{m}{2} + q)H(p)H(q + m).$$ (2.28b)

These relations are consistent with the Cuntz algebra when $|m| \leq 1$. For example one finds

$$0 = (p + q)H(p)H(q) = (p + q)\delta_{p+q,0}$$ (2.29)

when $m = 0$. But the relations (2.28) are inconsistent with Cuntz for $|m| > 1$— for example, we find the inconsistent relation

$$0 = \frac{1}{2}H(p)H(3/2), \quad p \in \mathbb{Z} + \frac{1}{2}$$ (2.30)
when \( m = 2 \) and \( q = -1/2 \).

So we must be satisfied at large \( N \) to construct reduced generators only for the \( \mathfrak{sl}(2) \) subalgebra of the original Virasoro algebra. To proceed in this example, it is helpful to recall [4] the following lemma

\[
A(B) \equiv \sum_{w} b_w^\dagger B b_w = B + \sum_{p>0} b^\dagger(p) B b(p) + \ldots \tag{2.31a}
\]

\[
[A(B), b(p)] = -b(p) B, \quad [A(B), b^\dagger(p)] = B b^\dagger(p) \tag{2.31b}
\]

where \( B \), called the kernel of \( A \), is an arbitrary function of the Cuntz operators \( b, b^\dagger \). The operation \( \sum_w b_w^\dagger (...) b_w \) in (2.31a) is called the dressing of the kernel \( B \). Then one finds the explicit form of the reduced \( \mathfrak{sl}(2) \) generators

\[
L_F(m) = \sum_{w} b_w^\dagger B_F(m) b_w, \quad |m| \leq 1 \tag{2.32a}
\]

\[
B_F(m) = \sum_{p>0} \left( \frac{m}{2} + p \right) H(-p) H(p + m) \tag{2.32b}
\]

as the solution of (2.27) for \( |m| \leq 1 \).

Using the fermionic Cuntz algebra (2.5a), we also verify the following properties of the kernels

\[
B_F(m)^\dagger = B_F(-m), \quad B_F(m)|0\rangle = \langle 0| B_F(m) = 0 \tag{2.33a}
\]

\[
H(p > 0)B_F(m) = -B_F(m) H(p < 0) = \left( \frac{m}{2} + p \right) H(p + m) \tag{2.33b}
\]

\[
[B_F(m), B_F(n)] = (m - n) B_F(m + n) \tag{2.33c}
\]

\[
[L_F(m), B_F(n)] = (m - n) B_F(m + n) \tag{2.33d}
\]

and these properties allow us to check that our reduced operators generate the expected \( \mathfrak{sl}(2) \)

\[
[L_F(m), L_F(n)] = \sum_{w} b_w^\dagger [L_F(m), B_F(n)] b_w \tag{2.34a}
\]

\[
= (m + n) L_F(m + n), \quad |m|, |n| \leq 1 \tag{2.34b}
\]
\[ L_F(m) = L_F(-m), \quad L_F(m)|0\rangle = 0|L_F(m) = 0. \quad (2.34c) \]

Note that the reduced fermionic ground state is an \( \mathfrak{sl}(2) \)-invariant state, as in the unreduced theory, so that we may derive \( \mathfrak{sl}(2) \) Ward identities (see the Appendix) for the reduced ground state averages – which are the large \( N \) limit of the unreduced traces.

Moreover, we may use (2.27) for \( |m| \leq 1 \) and the explicit form (2.13) of the free current modes to show that the free currents are \( (1, 0) \) operators under the reduced \( \mathfrak{sl}(2) \)

\[ [L_F(m), J(n)] = -nJ(m + n) \quad (2.35) \]

as they are in the unreduced theory. The action of \( L_F(0) \) on the reduced basis states (2.7c) is also not surprising

\[ [L_F(0), b^\dagger(p)] = pb^\dagger(p) \quad (2.36a) \]

\[ (L_F(0) - \Delta(w))|w\rangle = \langle w|(L_F(0) - \Delta(w)) = 0, \quad \Delta(w) = \{w\} \quad (2.36b) \]

where \( \{w\} \) in (2.7b) is the weight (sum of the mode numbers) of the word \( w \). Since \( L_F(0) \) commutes with \( J(0) \), we may then label the free affine primary states as follows:

\[ J(m \geq 0)|\Delta, t\rangle = |\Delta, t\rangle \delta_{m,0}(-t) \quad (2.37a) \]

\[ L_F(0)|\Delta, t\rangle = |\Delta, t\rangle \Delta. \quad (2.37b) \]

This is in fact the labelling employed in the examples (2.22) and (2.23). Of course, the eigenvalues \( \Delta \) are real because \( L_F(0) \) is hermitian, and, similarly, Schur’s lemma guarantees that we may take all distinct free affine primary states (and their modules) to be orthogonal.

It is instructive to consider the action of the reduced \( \mathfrak{sl}(2) \) generator \( L_F(1) \) on a free affine primary state. Using (2.35-37) and the reduced \( \mathfrak{sl}(2) \) algebra, it is not difficult to see that

\[ J(m \geq 0)(L_F(1)|\Delta, t\rangle) = (L_F(1)|\Delta, t\rangle) \delta_{m,0}(-t) \quad (2.38a) \]
\[ L_F(0)(L_F(1)|\Delta, t\rangle) = (L_F(1)|\Delta, t\rangle)(\Delta - 1). \quad (2.38b) \]

These relations tell us either that a) the state \( L_F(1)|\Delta, t\rangle \) vanishes, in which case the free affine primary state \( |\Delta, t\rangle \) was an \( \mathfrak{sl}(2) \) primary state, or b) the state \( L_F(1)|\Delta, t\rangle \) is another (lower) free affine primary state, and the original free affine primary state \( |\Delta, t\rangle \) was not \( \mathfrak{sl}(2) \) primary. In fact, both cases can and do occur, as seen for the examples at \( \hat{x} = 1 \) in Eq. (2.22):

\[ L_F(1)|\Delta = t = 0\rangle = L_F(1)|\Delta = 1/2, t = 1\rangle = 0 \quad (2.39a) \]

\[ L_F(1)|\Delta = 3/2, t = 1\rangle = -\frac{1}{\sqrt{2}}|\Delta = 1/2, t = 1\rangle. \quad (2.39b) \]

Of course, any free affine primary state which is not \( \mathfrak{sl}(2) \) primary must be one of the extraordinary free affine primary states (see Subs. 2.3) – which does not correspond to an unreduced affine primary state at any finite value of \( N \). The reason is that, up to the extra \( \mathfrak{u}(1) \) fermion, the unreduced free-fermion Virasoro generators are equal to those of the affine-Sugawara construction \[9,12,15-17\] on the unreduced fermionic currents (2.9) at level \( N \) of \( \mathfrak{su}(N) \) – and for the affine-Sugawara construction all affine primary states are also Virasoro primary.

The generalization of the reduced \( \mathfrak{sl}(2) \) generators to the case of an arbitrary number \( \hat{x} \) of adjoint fermions is

\[ L_F(m) = \sum_w b_w^\dagger B_F(m) b_w, \quad |m| \leq 1 \quad (2.40a) \]

\[ B_F(m) = \hat{x} \sum_{i=1}^m \sum_{p>0} \frac{m}{2} + p)H_i(-p)H_i(p + m) \quad (2.40b) \]

\[ b_w = H_{i_1}(p_1) \ldots H_{i_n}(p_n), \quad w = (i_1p_1, \ldots, i_np_n) \quad (2.40c) \]

\[ [w] = n, \quad \{w\} = \sum_{i=1}^n p_i \quad (2.40d) \]

and, except for an extra label \( i = 1 \ldots \hat{x} \) on the \( b^\dagger \)’s in (2.36a), there is no change in any of the results and conclusions above. In particular, we know that the ordinary free affine primary states in (2.23) are primary states under this \( \mathfrak{sl}(2) \).
2.5 Free-algebraic partition functions and characters

Starting with \( \hat{x} \in \mathbb{Z}_+ \) reduced (Cuntz-algebraic) chiral fermions and the corresponding reduced \( \mathfrak{sl}(2) \) generator \( L_F(0) \) in (2.40), one can define and evaluate the reduced fermionic partition function

\[
Z_F(z) \equiv \text{Tr}(z^{L_F(0)}) = \sum_w z^\{w\} = \frac{1}{1 - \hat{x}} \sum_{q>0} z^q
\]

\[
= \frac{1 - u^2}{1 - \hat{x}u - u^2} = \sum_{n=0}^\infty f_n(\hat{x}) u^n, \quad z = u^2
\]

where the positive numbers \( f_n(\hat{x}) \) (see below) count the reduced fermionic states at mode level \( L_F(0) = n/2 \). The radius of convergence \( u_0(\hat{x}) \) of this power series and the corresponding 2-d limiting (Hagedorn) temperature \( \beta_0(\hat{x})^{-1} \) of this system are

\[
z = e^{-\beta}, \quad u = e^{-\beta/2}
\]

\[
uo(\hat{x}) = e^{-\beta_o(\hat{x})/2} = \frac{1}{2}(\sqrt{\hat{x}^2 + 4} - \hat{x}) < 1
\]

\[
e^{\beta_o(\hat{x})/2} = \frac{1}{2}(\sqrt{\hat{x}^2 + 4} + \hat{x})
\]

and the ratio in (2.41b) provides an analytic continuation to temperatures beyond the limiting temperature.

The radius of convergence in (2.42) is less than the usual radius of convergence (at \( u = z = 1 \)) of conventional string partition functions (say for any finite number of abelian chiral world-sheet fermions). This tells us that we are dealing here with a great many more states than are present in the conventional string partition functions, and it is this fact that gives us a finite 2-d limiting temperature – instead of the infinite 2-d limiting temperature obtained in the conventional case. Relative to the counting of states for conventional chiral world-sheet fermions, the large number of reduced fermionic states in the reduced large \( N \) theory is due to the classical or Boltzmann statistics of the reduced (Cuntz-algebraic) fermions, that is, the loss of the Pauli principle at large \( N \). (The large number of states is not associated to the \( N^2 \) degrees of freedom in the \( r, s \) labels of the unreduced matrix fermions, since these degrees of freedom are modded out in the reduction procedure.)
To evaluate the integers \( \{ f_n(\hat{x}) \} \) in (2.41b), consider first the generalized Fibonacci numbers \( F_n(\hat{x}) \)

\[
(1 - \hat{x}u - u^2)^{-1} = \sum_{n=0}^{\infty} F_n(\hat{x})u^n \quad (2.43a)
\]

\[
F_0(\hat{x}) = 1, \quad F_1(\hat{x}) = \hat{x}, \quad F_{n+2}(\hat{x}) = \hat{x}F_{n+1}(\hat{x}) + F_n(\hat{x}), \quad n \geq 0 \quad (2.43b)
\]

\[
F_n(\hat{x}) = \frac{1}{\sqrt{\hat{x}^2 + 4}} \left( \frac{\hat{x} + \sqrt{\hat{x}^2 + 4}}{2} \right)^{n+1} - \left( \frac{\hat{x} - \sqrt{\hat{x}^2 + 4}}{2} \right)^{n+1} \quad (2.43c)
\]

\[
F_n(\hat{x}) \sim e^{n\beta_0(\hat{x})/2} = e^{n\ln \left( \frac{\hat{x} + \sqrt{\hat{x}^2 + 4}}{2} \right)} \quad (2.43d)
\]

which are defined so that \( F_n(1) \) is the nth Fibonacci number. Then one finds from (2.41b) that

\[
f_0(\hat{x}) = 1, \quad f_{n+1}(\hat{x}) = \hat{x}F_n(\hat{x}), \quad n \geq 0 \quad (2.44a)
\]

\[
f_{n+1}(1) = F_n(1), \quad f_n(\hat{x}) \sim e^{n\beta_0(\hat{x})/2} \quad (2.44b)
\]

and this \( O(\exp(an)) \) growth at large \( n \) should be compared with the familiar \( O(\exp(b\sqrt{n})) \) growth found in conventional string partition functions. The same qualitative behavior is found (see Sec.3) in the partition functions of reduced chiral bosons.

I turn next to the free affine character

\[
\chi_{\Delta = \frac{n}{2}}(z) \equiv \text{Tr}_{\Delta = \frac{n}{2}}(z^{L_F(0)}) = \frac{u^n(1 - u^2)}{1 - \hat{x}u - u^2} \quad (2.45)
\]

which sums over the states of the module of any free affine primary state at mode level \( \Delta = n/2 \). One may expand the reduced fermionic partition functions (2.41) in terms of the free affine characters

\[
Z_F(z) = \sum_{n=0}^{\infty} a_n(\hat{x})\chi_{\frac{n}{2}}(z), \quad \frac{1 - 2u^2}{1 - \hat{x}u - u^2} = \sum_{n=0}^{\infty} a_n(\hat{x})u^n \quad (2.46)
\]
so that the integers \( \{a_n(\hat{x})\} \) count the number of free affine primary states at mode level \( n/2 \) in the reduced fermionic Hilbert space. From the last part of (2.46) one finds that

\[
a_0(\hat{x}) = 1, \quad a_1(\hat{x}) = \hat{x}, \quad a_2(\hat{x}) = \hat{x}^2 - 1 \quad (2.47a)
\]

\[
a_{n+2}(\hat{x}) = (\hat{x}^2 - 1)F_n(\hat{x}) + \hat{x}F_{n-1}(\hat{x}) = \hat{x}a_{n+1}(\hat{x}) + a_n(\hat{x}), \quad n \geq 1 \quad (2.47b)
\]

\[
a_{n+2}(1) = F_{n-1}(1), \quad a_n(\hat{x}) \sim F_n(\hat{x}) \quad (2.47c)
\]

and these numbers are in agreement with the low-lying examples in Eqs. (2.22) and (2.23).

Further discussion of affine free algebra is found in Sec. 4.

### 3 Chiral Adjoint Bosons

In this section I will develop the corresponding large \( N \) theory of chiral adjoint bosons which, up to a point and except for some subtlety with the zero modes, is simpler than that given above for the large \( N \) fermions.

#### 3.1 Cuntz algebra and reduced bosonic partition functions

I will begin with the unreduced theory of a single chiral adjoint boson in sector \( k \)

\[
[\pi_a(m), \pi_b(n)] = \delta_{ab} m \delta_{m+n,0}, \quad a, b = 1 \ldots N^2 \quad (3.1a)
\]

\[
\pi_a(m)^\dagger = \pi_a(-m), \quad m \in \mathbb{Z} \quad (3.1b)
\]

\[
(\pi_a(m \geq 0) - \delta_{m,0} k_a) |k\rangle = \langle k| (\pi_a(m \leq 0) - \delta_{m,0} k_a) = 0 \quad (3.1c)
\]

where \( \{k_a\} \) are the eigenvalues of the zero modes \( \{\pi_a(0)\} \). The matrix forms of these operators are defined as usual, and we find:

\[
\pi(m)_{rs} = \pi_a(m) (T_a)_{rs}, \quad k_{rs} = k_a(T_a)_{rs}, \quad r, s = 1 \ldots N \quad (3.2a)
\]

\[
[\pi(m)_{rs}, \pi(n)_{uv}] = \delta_{rv} \delta_{su} m \delta_{m+n,0} \quad (3.2b)
\]
\[ [\pi(m)_{rs}, k_{uv}] = 0, \quad \pi(m)_{rs}^\dagger = \pi(-m)_{sr} \quad (3.2c) \]

\[ \{\pi(m \geq 0) - \delta_{m,0} k\}_{rs}\ket{k} = \langle k| \{\pi(m \leq 0) - \delta_{m,0} k\}_{rs} = 0. \quad (3.2d) \]

The large \( N \) reduction of this system follows closely the development in Refs.\[4,8\].

In particular, Eq.(2.29) of Ref.\[4\] gives us the quasicanonical algebra of the reduced operators in \textit{reduced sector} \( k \):

\[ \frac{\pi(m)_{rs}}{\sqrt{N}} \rightarrow \pi(m), \quad \frac{k_{rs}}{\sqrt{N}} \rightarrow \tilde{k} = k \quad (3.3a) \]

\[ [\pi(m), \overline{\pi}(n)] = m\delta_{m+n,0}\ket{k}\bra{k}, \quad m, n \in \mathbb{Z} \quad (3.3b) \]

\[ \pi(m)^\dagger = \pi(-m), \quad \overline{\pi}(m)^\dagger = \overline{\pi}(-m) \quad (3.3c) \]

\[ \overline{\pi}(0) = \pi(0), \quad [\pi(m \neq 0), \pi(0)] = [\overline{\pi}(m \neq 0), \pi(0)] = 0 \quad (3.3d) \]

\[ \langle \pi(m \geq 0) - \delta_{m,0} k\rangle \rangle = 0 \quad (3.3e) \]

\[ \langle k| \pi(m \leq 0) - k\delta_{m,0} \rangle = 0 \quad (3.3f) \]

\[ \langle \pi(m \leq 0) - \pi(m \leq 0) \rangle \rangle = 0. \quad (3.3g) \]

Here I have chosen to scale the unreduced eigenvalues so that the reduced eigenvalues \( k \) and orthonormal reduced eigenstates \( \ket{k} \) are finite in the large \( N \) limit. I will also assume that the reduced momenta are defined on some lattice, so that the reduced eigenstates can be taken orthonormal. Again, a basis for sector \( k \) can be formed by the action on \( \ket{k} \) of the negative modes of either the untilded operators or the tilded operators. Finally (see Eq.(3.7) of Ref.\[4\]) operation on either basis with both sets of positive modes shows that the reduced bosonic modes also satisfy the free-algebraic relations

\[ \pi(m > 0)\pi(n < 0) = m\delta_{m+n,0}, \quad \sum_{m>0} \frac{\pi(-m)\pi(m)}{m} = 1 - |k\rangle\langle k| \quad (3.4a) \]

\[ \overline{\pi}(m > 0)\overline{\pi}(n < 0) = m\delta_{m+n,0}, \quad \sum_{m>0} \frac{\overline{\pi}(-m)\overline{\pi}(m)}{m} = 1 - |k\rangle\langle k|. \quad (3.4b) \]
Together, Eqs. (3.3) and (3.4) form an infinite-dimensional version of a so-called symmetric Bose/Cuntz algebra [4]. See Refs. [4-6] and the Appendix for further applications of the tilde operators.

As for the fermions above, I will confine our discussion here primarily to the subalgebra (3.4a) of the untilde operators, which is in fact an infinite-dimensional Cuntz algebra:

\[
a(m) \equiv \frac{\pi(m)}{\sqrt{m}}, \quad a^\dagger(m) = \frac{\pi(-m)}{m}, \quad m > 0
\]  \hspace{1cm} (3.5a)

\[
a(m)a^\dagger(n) = \delta_{m,n}, \quad \sum_{m>0} a^\dagger(m)a(m) = 1 - |k\rangle\langle k|
\]  \hspace{1cm} (3.5b)

\[
a(m)|k\rangle = \langle k|a^\dagger(m) = 0
\]  \hspace{1cm} (3.5c)

\[
|w,k\rangle = a_w^\dagger |k\rangle, \quad \langle w',k'|w,k\rangle = \delta_{w',w}\delta_{k',k}.
\]  \hspace{1cm} (3.5d)

The reduced bosonic words \(w\) are formed as in (2.7) with \(b,b^\dagger \to a,a^\dagger\). I emphasize that, up to a relabeling, this Cuntz algebra is that same as we found above for the reduced fermions – so that all reduced states, fermionic or bosonic, satisfy the same classical or Boltzmann statistics at large \(N\).

Following the logic of Section 2, let us move quickly in this case to find the explicit form of the reduced “Hamiltonian” \(L_B(0)\) of the large \(N\) system. This trace class operator must satisfy the reduced commutation relation

\[
[L_B(0), \pi(m)] = -m\pi(m), \quad L_B^\dagger(0) = L_B(0)
\]  \hspace{1cm} (3.6)

which is the reduced image of the familiar commutator in the unreduced theory. The solution of (3.6) is

\[
L_B(0) - \frac{\pi^2(0)}{2} = \sum_w a_w^\dagger \left( \sum_{m>0} \pi(-m)\pi(m) \right) a_w
\]  \hspace{1cm} (3.7a)

\[
(L_B(0) - \Delta(w,k))|w,k\rangle = \langle w,k|(L_B(0) - \Delta(w,k)) = 0
\]  \hspace{1cm} (3.7b)

\[
\Delta(w,k) = \frac{k^2}{2} + \{w\}_B, \quad \{w\}_B \equiv \sum_{j=1}^n m_{j} \hspace{1cm} (3.7c)
\]

where the weight \(\{w\}_B\) is again the mode level of the basis state \(|w,k\rangle\). We will also need the generalization of these results to the case of \(D\) chiral adjoint bosons

\[
\pi_i(m > 0)\pi_j(n < 0) = \delta_{ij}m\delta_{m+n,0}, \quad i,j = 1 \ldots D
\]  \hspace{1cm} (3.8a)
\[ \sum_{i=1}^{D} \sum_{m > 0} \pi_i(-m) \pi_i(m) = 1 - |k\rangle\langle k|, \quad \pi_i^\dagger(m) = \pi_i(-m) \quad (3.8b) \]
\[ L_B(0) - \frac{\pi^2}{2} = \sum_w a_w \sum_{m > 0} \pi_i(-m) \pi_i(m) a_w \quad (3.8d) \]
\[ L_B^\dagger(0) = L_B(0), \quad \pi_i^\dagger(0),\pi_i(0) = -m \pi_i^\dagger(m) \quad (3.8e) \]

where the words of the dressing in (3.8d) are now \( w = (m_1, i_1...m_n, i_n) \). With the substitution \( k^2 \to \vec{k}^2 \), the action of \( L_B(0) \) on the basis states is the same as that given in (3.7c).

We are now prepared to define and evaluate the Cuntz-algebraic reduced bosonic partition functions

\[ Z_B(z) \equiv \text{Tr}_k(z^{L_B(0)}) = z^{\frac{\pi^2}{2}} \sum_w z\{w\}_B \]
\[ = z^{\frac{\pi^2}{2}} \frac{1}{1 - D \sum_{m > 0} z^m} = z^{\frac{\pi^2}{2}} \frac{1 - z}{1 - (D + 1)z} \quad (3.9b) \]
\[ = z^{\frac{\pi^2}{2}} \left( 1 + \sum_{n=1}^{\infty} d_n(D) z^n \right) \quad (3.9c) \]
\[ d_n(D) = D^{[D + 1)^{n-1}], \quad n \geq 1 \quad (3.9d) \]

which provide simpler examples of the phenomena discussed above for the reduced fermions. In particular, one finds the radius of convergence \( z_0(D) \), the 2-d limiting temperature \( \beta_0(D)^{-1} \) and the apparently universal \( O(\exp(an)) \) growth in the number of states

\[ z_0(D) = \frac{1}{D + 1} < 1, \quad \beta_0(D) = \ln(D + 1), \quad d_n(D) \sim e^{n\beta_0(D)} \quad (3.10) \]

each of which is a consequence of the Boltzmann statistics associated to the Cuntz algebra. We remind the reader that the conventional fermionic and bosonic string partition functions show an infinite 2-d limiting temperature associated to growth of order \( \exp(b\sqrt{n}) \).

### 3.2 Reduced bosonic \( \mathfrak{s}(2) \)

Following the discussion above for the fermions I turn next to complete the reduced bosonic \( \mathfrak{s}(2) \) generators \( L_B(m) \), starting from the reduced commutation relations

\[ [L_B(l), \pi(n)] = -n \pi(n + l). \quad (3.11) \]
Using these relations and the Cuntz algebra (3.4a), we find the consistency relations
\[ m\pi(m + l)\pi(n) + n\pi(m)\pi(n + l) = 0, \quad m > 0, \; n < 0 \] (3.12)
and, as argued above on general grounds, these relations are easily seen to be inconsistent beyond the \(\mathfrak{sl}(2)\) subalgebra. For example, the inconsistency
\[ \pi(m)\pi(1) = 0, \quad m > 0 \] (3.13)
is obtained after using the Cuntz algebra for the case \(l = 2, n = -1\).

But there is also something new here. The choice \(l = 1, n = -1\) gives in the same way the further relations
\[ \pi(m)\pi(0) = 0, \quad m > 0 \] (3.14)
which can only be satisfied by setting the reduced momenta to zero
\[ \pi(0) = k = 0 \] (3.15)
and keeping only the Cuntz module \(a_w^\dagger|0\rangle\) based on the reduced zero momentum ground state \(|0\rangle\). I do not have a deeper understanding of this phenomenon, although it is presumably traceable to the rescaling of the unreduced momenta which was necessary to obtain finite reduced momenta in the first place.

With the proviso (3.15), it is not difficult to solve the commutation relations (3.11) to obtain the explicit form of the reduced bosonic \(\mathfrak{sl}(2)\) generators
\[ L_B(|m| \leq 1) = \sum a_w^\dagger B_B(m)a_w \] (3.16a)
\[ B_B(m) = \sum_{n>0} \pi^n(-n)\pi(n + m) \] (3.16b)
and this agrees with our previous result for \(L_B(0)\) when \(\pi(0) = 0\).

Continuing to compute with the Cuntz algebra, one verifies the additional properties of the kernels
\[ B_B(m)^\dagger = B_B(-m), \quad B_B(m)|0\rangle = |0\rangle B_B(m) = 0 \] (3.17a)
\[ \pi(m > 0)B_B(n) = -B_B(n)\pi(m < 0) = m\pi(m + n) \] (3.17b)
and finally one verifies the reduced $\mathfrak{sl}(2)$ algebra

\[
[L_B(m), L_B(n)] = \sum w^w a_w^\dagger [L_B(m), B_B(n)] a_w = (m - n)L_B(m + n) \tag{3.18a}
\]

\[
L_B(m)^\dagger = L_B(-m), \quad L_B(m)|0\rangle = \langle 0|L_B(m) = 0. \tag{3.18b}
\]

Note that the reduced bosonic ground state $|0\rangle$ is $\mathfrak{sl}(2)$ invariant, as it is in the unreduced bosonic theory. To generalize these results to the Cuntz algebra (3.8a,b) of $D$ reduced bosons, use the kernels

\[
B_B(|m| \leq 1) = \sum_{i=1}^{D} \sum_{n>0} \pi_i(-n)\pi_i(n + m) \tag{3.19}
\]

and construct the dressing with the full words $(m_1, i_1...m_n, i_n)$ as above.

This concludes my discussion of reduced chiral fermions and bosons separately, and I turn next to some applications which, in a variety of ways, combine reduced chiral fermions and bosons in the same system.

\section{Extended Affine Free Algebra}

\subsection{Short and long currents}

Returning to the reduced chiral fermions of Sec. 2, I want to discuss here an extended affine free algebra which contains both the affine free algebra and a surprising new copy of the Cuntz algebra among its free subalgebras. In turn, all these algebras are subalgebras of the algebra of all normal-ordered products of Cuntz operators (see Eqs. (88-90) of Ref. [8] with $a \rightarrow b$), but we shall not need the explicit form of this much larger algebra. Although it is straightforward to extend this construction to any number of reduced fermions, I will confine the discussion for simplicity to the case of a single reduced fermion, and hence reduced level $\hat{x} = 1$ of the affine free algebra.

To begin, let us decompose the free currents (2.13) into the following \textit{short currents} $J_{++}, J_{--}$ and \textit{long currents} $J_{+-}$:

\[
J(m \geq 0) = J_{++}(m \geq 0) + J_{+-}(m \geq 0) \tag{4.1a}
\]
\begin{align}
J_{++}(m \geq 0) &\equiv - \sum_{0 < p < m} H(m - p)H(p) \quad \text{(4.1b)} \\
J_{--}(m \geq 0) &\equiv - \sum_{p > m} H(-p)H(p + m) \quad \text{(4.1c)} \\
J(m \leq 0) &= J_{--}(m \leq 0) + J_{-+}(m \leq 0) \quad \text{(4.2a)} \\
J_{--}(m \leq 0) &\equiv - \sum_{0 < p < -m} H(-p)H(m + p) \quad \text{(4.2b)} \\
J_{-+}(m \leq 0) &\equiv - \sum_{p > -m} H(-p)H(m + p). \quad \text{(4.2c)}
\end{align}

The short currents are a sum of a finite number of terms, and the subscripts are such that, for example, \( J_{-+} \) has negative (positive) reduced fermionic modes on the left (right).

Simple properties of the short and long currents include:

\begin{align}
J_{++}(m \geq 0) &\dagger = J_{--}(-m), \quad J_{--}(m \geq 0) &\dagger = J_{-+}(-m) \quad \text{(4.3a)} \\
J_{++}(0) = J_{--}(0) &= 0, \quad J_{-+}(0) = J(0) = |0\rangle\langle 0| - 1 \quad \text{(4.3b)} \\
J_{++}(m \geq 0)|0\rangle &= J_{--}(m \geq 0)|0\rangle = J_{-+}(m \leq 0)|0\rangle = 0 \quad \text{(4.3c)} \\
\langle 0|J_{--}(m \leq 0) &= \langle 0|J_{-+}(m \geq 0) = \langle 0|J_{-+}(m \geq 0) = 0. \quad \text{(4.3d)}
\end{align}

We will also need the relations of these currents with the reduced fermions

\begin{align}
H(p > 0)J_{--}(m \leq 0) = J_{-+}(m \geq 0)H(p < 0) &= -\theta(p + m < 0)H(p + m) \quad \text{(4.4a)} \\
H(p > 0)J_{-+}(m \leq 0) = J_{++}(m \geq 0)H(p < 0) &= -\theta(p + m > 0)H(p + m) \quad \text{(4.4b)}
\end{align}

which follow from the fermion Cuntz algebra (2.5a).

Using (4.4), we then obtain the \textit{extended affine free algebra}

\begin{align}
J_{++}(m \geq 0)J_{--}(n \leq 0) &= m\delta_{m+n,0} \quad \text{(4.5a)}
\end{align}
\[ J_{++}(m \geq 0)J_{-+}(n \leq 0) = -\theta(m + n \geq 0)J_{++}(m + n) \quad (4.5b) \]

\[ J_{-+}(m \geq 0)J_{--}(n \leq 0) = -\theta(m + n \leq 0)J_{--}(m + n) \quad (4.5c) \]

\[ J_{-+}(m \geq 0)J_{-+}(n \leq 0) = -J_{-+}(m + n) \quad (4.5d) \]

\[ J_{-+}(m \geq 0)J_{-+}(n \geq 0) = J_{-+}(m \leq 0)J_{-+}(n \leq 0) = -J_{-+}(m + n) \quad (4.5e) \]

whose properties I will discuss below. There are also three further relations

\[ J_{++}(m \geq 0)J_{--}(n \geq 0) = \sum_{0 < p < m} H(m - p)H(p + n) \quad (4.6a) \]

\[ J_{-+}(m \leq 0)J_{--}(n \leq 0) = \sum_{0 < p < -n} H(m - p)H(p + n) \quad (4.6b) \]

\[ J_{-+}(m \leq 0)J_{-+}(n \geq 0) = \sum_{p > -m} H(-p)H(m + n + p) \]

\[ = \sum_{p > 0} H(m - p)H(p + n). \quad (4.6c) \]

which open into new normal-ordered fermion bilinears. This exhausts the relations among the short and long currents, which are all of the form \( A_x B_y = C_{xy} \).

Let us discuss the free subalgebras of the extended affine free algebra (4.5). In the first place, we see that the long currents \( J_{-+} \) form a closed free subalgebra of the extended affine free algebra. Second, the first four relations in (4.5) and the definitions (4.2a),(4.3a) can be used to verify that the affine free algebra (2.16) is also a free subalgebra of the extended affine free algebra.

Finally note that the short currents form a third short free subalgebra which is recognized as another infinite-dimensional “bosonic” Cuntz algebra

\[ a_S(m) \equiv \frac{J_{++}(m > 0)}{\sqrt{m}} = -\frac{1}{\sqrt{m}} \sum_{0 < p < m} b(m - p)b(p) \quad (4.7a) \]

\[ a_S^\dagger(m) = \frac{J_{--}(-m)}{\sqrt{m}} = -\frac{1}{\sqrt{m}} \sum_{0 < p < m} b^\dagger(p)b^\dagger(m - p) \quad (4.7b) \]
whose generators \( a_S, a_S^\dagger \) we have constructed as \textit{bilinears} in the original fermionic Cuntz operators \( b, b^\dagger \). Since the bosonic Cuntz algebra (4.7c) is isomorphic to the fermionic Cuntz algebra (2.6b), the relations in (4.7a,b) define what can be called an automorphism of the Cuntz algebra.

It is also instructive to rewrite some of the relations above in the Cuntz notation:

\[
 b(p)a_S^\dagger(m) = -\theta(m > p) \frac{1}{\sqrt{m}} b^\dagger(m - p) \quad (4.8a) \\
 a_S(m)b^\dagger(p) = -\theta(m > p) \frac{1}{\sqrt{m}} b(m - p) \quad (4.8b) \\
 a_S(m)J_+(n) = -\theta(m \geq n) \sqrt{\frac{m - n}{m}} a_S(m - n), \quad n \geq 0 \quad (4.8c) \\
 J_+(m)a_S^\dagger(n) = -\theta(n \geq m) \sqrt{\frac{n - m}{n}} a_S^\dagger(n - m), \quad m \geq 0. \quad (4.8d)
\]

In this form we see clearly that the “bosonic” Cuntz generators (short currents) are raising and lowering operators for the the fermionic Cuntz generators, and moreover that the long currents are raising and lowering operators for the bosonic Cuntz generators.

Applications of the extended affine free algebra are discussed in the following two subsections.

### 4.2 Free-current form of the fermionic \( \mathfrak{sl}(2) \) kernels

I have mentioned above that the Virasoro operators of one unreduced adjoint fermion are equal to those of the affine-Sugawara construction [9,12,15-17] on the unreduced fermionic currents (2.9a) at invariant level \( N \) of affine \( \mathfrak{su}(N) \). Since our extra \( u(1) \) fermion is negligable at large \( N \), this suggests that it may be possible to find what could be called a \textit{free affine-Sugawara construction} – in which the reduced fermionic \( \mathfrak{sl}(2) \) generators (2.32) are re-expressed entirely in terms of free currents.
Indeed, using the extended affine free algebra (4.5), I have been able to find an equivalent free-current form of the kernels $B_F(m)$ of the reduced fermionic $\mathfrak{sl}(2)$ generators

$$B_F(0) = \sum_{p>0} pH(-p)H(p) = \sum_{m \geq 1} J_+(\text{negative } m)J_+(m) + \frac{1}{2}J_+(0)^2 \quad (4.9a)$$

$$B_F(1) = \sum_{p>0} (p + \frac{1}{2})H(-p)H(p+1) = \sum_{m \geq 0} J_+(\text{negative } m)J_+(m+1) \quad (4.9b)$$

$$B_F(-1) = \sum_{p>0} (p + \frac{1}{2})H(-p-1)H(p) = \sum_{m \geq 0} J_+(\text{negative } m-1)J_+(m) \quad (4.9c)$$

but I have not yet found a free-current form of the operation $\sum_w b_w^\dagger(...)b_w$ which is necessary to dress the kernels.

4.3 A free-algebraic coset construction

The first examples of coset constructions were given implicitly in Ref. [9] and explicitly in Ref. [12], and the general coset construction was given later in Ref. [18]. In this subsection I will present a free-algebraic coset construction based on the “bosonic” or short free subalgebra of the extended affine free algebra.

To begin this discussion, it is convenient to define a unified form of the short currents:

$$J_S(m) \equiv \theta(m > 0)J_+(m) + \theta(m < 0)J_-(m), \quad \forall \ m \in \mathbb{Z} \quad (4.10a)$$

$$J_S(0) = 0, \quad J_S^\dagger(m) = J_S(-m) \quad (4.10b)$$

$$J_S(m > 0)J_S(n < 0) = m\delta_{m+n,0}. \quad (4.10c)$$

Then we find with (2.27) that the long and short currents are both $\mathbf{(1,0)}$ operators under the reduced fermionic $\mathfrak{sl}(2)$

$$[L_F(m), J_S(n)] = -nJ_S(n + m), \quad \forall \ n \in \mathbb{Z} \quad (4.11a)$$
\[ [L_{F}(m), J_{-}^{\pm}(n)] = -n J_{-}^{\pm}(n), \quad \forall \ n \in \mathbb{Z} \]  
(4.11b)

where the explicit form of \( \{L_{F}(m)\} \) is given in Eq. (2.32). This is of course consistent with our earlier observation in (2.35) that the full free current \( J = J_{S} + J_{-}^{\pm} \) is a \((1,0)\) operator.

Next, I can construct a new short reduced \( \mathfrak{sl}(2) \) using only the short currents

\[ L_{S}(m) \equiv \sum_{w} (a_{S})_{w}^{\dagger} \left( \sum_{n>0} J_{S}(-n)J_{S}(n + m) \right) (a_{S})_{w} \]  
(4.12a)

\[ [L_{S}(m), L_{S}(n)] = (m - n)L_{S}(m + n) \]  
(4.12b)

\[ [L_{S}(m), J_{S}(n)] = -n J_{S}(n + m) \]  
(4.12c)

in complete analogy with the reduced bosonic \( \mathfrak{sl}(2) \) in Eq. (3.16). We do not need to check these relations explicitly because the Cuntz algebra (4.10c) of the short currents is isomorphic to the bosonic Cuntz algebra of the reduced bosonic modes \( \{\pi(m)\} \). Note however that the kernels of the short reduced \( \mathfrak{sl}(2) \) are quartic in the reduced fermion modes.

We are now ready for the free-algebraic coset construction—in which we mod out the reduced fermion theory by the short or bosonic free subalgebra (4.7) of the extended affine free algebra (4.5). The key of course is that the short currents are \((1,0)\) operators under both sets \( \{L_{F}(m)\} \) and \( \{L_{S}(m)\} \) of reduced \( \mathfrak{sl}(2) \) generators. Then the short currents commute with the differences \( \{L_{F/S}(m)\} \) of the generators

\[ L_{F/S}(m) \equiv L_{F}(m) - L_{S}(m), \quad |m| \leq 1 \]  
(4.13a)

\[ = \sum_{w} \left\{ b_{w}^{\dagger} \sum_{p>0} (\frac{m}{2} + p)H(-p)H(p + m)b_{w} \right\} \]  
(4.13b)

\[ -(a_{S})_{w}^{\dagger} \sum_{n>0} J_{S}(-n)J_{S}(n + m)(a_{S})_{w} \]  
(4.13b)

\[ [L_{F/S}(m), J_{S}(n)] = 0 \]  
(4.13c)

and we may follow the familiar steps

\[ [L_{F/S}(m), L_{S}(n)] = 0 \]  
(4.14a)
\[ [L_{FS}(m), L_{FS}(n)] = (m - n)L_{FS}(m + n) \] (4.14b)
to show that the operators \( \{L_{FS}(m)\} \) generate a new reduced coset \( \mathfrak{sl}(2) \).
So far as I can tell, this free-algebraic coset construction has no analogue at finite values of \( N \).

5 Free-Algebraic Construction of \( \mathfrak{osp}(1|2) \)

Historically, the next step was taken by Neveu and Schwarz [19], who noticed that the idea of Ramond’s superconformal symmetry [11] could be combined with half-integral modeled world sheet fermions of the Bardakci-Halpern type to find the so-called NS superconformal construction. In this section I will consider the large \( N \) limit of the simplest matrix generalization of the NS construction, namely that composed of one chiral boson-fermion pair in the adjoint. This construction uses our previous knowledge of reduced chiral fermions and reduced chiral bosons, as well as the technology developed for the construction of various reduced supersymmetries in Ref. [4].

In this case, we know from the discussion above that only the reduced \( \mathfrak{osp}(1|2) \) subalgebra of the superconformal algebra
\[
\begin{align*}
\{G(r), r = \pm 1/2; \ L(m), m = 0, \pm 1\} \\
[G(r), \pi(m)] &= -mH(r + m), \ \ [G(r), H(p)]_+ = \pi(r + p) \\
[G(r), G(s)]_+ &= 2L(r + s), \ [L(m), L(n)] = (m - n)L(m + n) \\
[L(m), \pi(n)] &= -n\pi(n + m), \ [L(m), H(p)] = -(\frac{m}{2} + p)H(p + m) \\
[L(m), G(r)] &= (\frac{m}{2} - r)G(r + m)
\end{align*}
\] (5.1a)
will be well defined in the large \( N \) limit. The reduced fermion modes \( \{H(p)\} \) and reduced boson modes \( \{\pi(m)\} \) with \( \pi(0) = 0 \) satisfy:
\[
\begin{align*}
\pi(m > 0)\pi(n < 0) &= m\delta_{m+n,0}, \ \ H(p > 0)H(q < 0) = \delta_{p+q,0} \quad (5.2a)
\end{align*}
\]
\[
\pi(m > 0)H(p < 0) = H(p > 0)\pi(m < 0) = 0 \quad (5.2b)
\]

\[
\sum_{m > 0} \frac{1}{m}\pi(-m)\pi(m) + \sum_{p > 0} H(-p)H(p) = 1 - |0\rangle\langle 0| \quad (5.2c)
\]

\[
\pi(m > 0)|0\rangle = H(p > 0)|0\rangle = \langle 0|\pi(m < 0) = \langle 0|H(p < 0) = 0. \quad (5.2d)
\]

Derivation of such systems from the unreduced theory is discussed in Ref. [4]. As an important intermediate step one also obtains a tilde copy of each of the relations in (5.2) plus the following relations between the tilde and untilded operators

\[
[\pi(m), \tilde{\pi}(n)] = \delta_{m+n,0}|0\rangle\langle 0| \quad (5.3a)
\]

\[
[H(p), \tilde{H}(q)]_+ = \delta_{p+q,0}|0\rangle\langle 0| \quad (5.3b)
\]

\[
[\pi(m), \tilde{H}(p)] = [\tilde{\pi}(m), H(p)] = 0 \quad (5.3c)
\]

and all these relations together form an infinite-dimensional version of a so-called symmetric Bose/Fermi/Cuntz algebra [4]. Again, the tilde operators will not be needed in the discussion below.

Note that the relations (5.2) can be written as an infinite-dimensional Cuntz superalgebra [4]

\[
S(m) \equiv \left[\begin{array}{c}
ap(m) \\
bp(m - \frac{1}{2})
\end{array}\right], \quad S^\dagger(m) = [a^\dagger(m), b^\dagger(m - \frac{1}{2})], \quad m = 1, 2, \ldots \quad (5.4a)
\]

\[
S_i(m)S_j^\dagger(n) = \delta_{ij}\delta_{m,n}, \quad i, j = 1, 2 \quad (5.4b)
\]

\[
\sum_{m} S^\dagger(m)S(m) = 1 - |0\rangle\langle 0|, \quad S(m)|0\rangle = \langle 0|S^\dagger(m) = 0 \quad (5.4c)
\]

in which the reduced fermions and reduced bosons are on equal footing. This gives us the useful superword notation

\[
S_w = S_{i_1}(m_1) \ldots S_{i_n}(m_n), \quad w = (i_1m_1, \ldots, i_n m_n) \quad (5.5a)
\]

\[
|w\rangle = S_w^\dagger|0\rangle = S_{i_n}^\dagger(m_n) \ldots S_{i_1}^\dagger(m_1)|0\rangle \quad (5.5b)
\]
for describing the general basis state in the combined system.

The total reduced $\mathfrak{sl}(2)$ operators of the system are obtained as above by solving the simultaneous reduced algebraic relations in (5.1d). Following the discussion of Ref. [4], one finds that the kernels of these operators are additive, but that the dressing of the kernels must be a superdressing

$$L(m) = \sum_w S_w^\dagger B(m) S_w, \quad |m| \leq 1 \quad (5.6a)$$

$$B(m) = B_F(m) + B_B(m) \quad (5.6b)$$

$$L(m)|0\rangle = \langle 0|L(m) = 0 \quad (5.6c)$$

constructed with the generators $S, S^\dagger$ of the Cuntz superalgebra (5.4). The fermionic and bosonic kernels $B_F$ and $B_B$ are given respectively in Eqs. (2.32b) and (3.16b). It is not difficult to check as above that these operators generate the reduced $\mathfrak{sl}(2)$ in (5.1c), and moreover that the Fermi and Bose terms of $L(m)$

$$\sum_w S_w^\dagger B_F(m) S_w, \quad \sum_w S_w^\dagger B_B(m) S_w \quad (5.7)$$

commute to generate a reduced $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

The strategy to find the realization of the reduced supercharges $\{G(r)\}$ is to solve the simultaneous algebraic relations in (5.1b), following Ref. [4]. The answer is

$$G(r) = \sum_w (S \tau_3)_w^\dagger K(r) S_w = \sum_w S_w^\dagger K(r) (\tau_3 S)_w \quad (5.8a)$$

$$K(r) = \sum_{m>0} \{H(r - m)\pi(m) + \pi(-m)H(m + r)\} \quad (5.8b)$$

$$K(r)^\dagger = K(-r), \quad G(r)^\dagger = G(-r) \quad (5.8c)$$

$$G(r)|0\rangle = \langle 0|G(r) = 0 \quad (5.8d)$$

where $\tau_3$ is the third Pauli matrix, which operates in the $2 \times 2$ space of the Cuntz superalgebra. More explicitly, this factor means that there is an extra minus sign associated with each reduced fermion pair in the superdressing

$$L(m) = B(m) + \sum_{n>0} a_n^\dagger B(m) a(n) - \sum_{p>0} b_p^\dagger B(m) b(p) + \ldots \quad (5.9)$$
To check the anticommutation relations among the reduced supercharges, we need the identities

\[
\begin{align*}
\{G(r), K(s)\}_+ &= 2B(r + s), \\
\{L(m), K(r)\} &= \left(\frac{m}{2} - r\right)K(r + m)
\end{align*}
\]

which follow straightforwardly from (5.1b,d). Then one finds for example that

\[
\{G(r), G(s)\}_+ = \sum_w S_w^\dagger \{G(r), K(s)\}_+ S_w = 2L(r + s)
\]

and similarly for the commutator in (5.1e).

\section{Exotic Constructions at Large $N$}

Using the vertex operator [20] of the bosonic string, the next developments in conformal field theory were the vertex operator construction of fermions [21-23] and the vertex operator construction of level one of $su(N)$ [21-23], followed some years later by the vertex operator construction of level one of simply laced $g$ [24]. Qualitatively, we do not expect to find such constructions directly in the large $N$ bosonic systems because chiral adjoint matter lives at unreduced affine levels which are multiples of $N$. Quantitatively, it appears at first that we are denied any analogue of the vertex operators by the fact that the Cuntz operators do not satisfy commutation relations.

But as we shall see below, one can use the bosonic Cuntz algebra to construct a new set of creation and annihilation operators whose algebra, although not canonical, allows us to find new free-algebraic vertex operators and constructions.

\subsection{Dressed Cuntz operators}

We return in this subsection to the infinite-dimensional bosonic Cuntz algebra

\[
\begin{align*}
a(m)a^\dagger(n) &= \delta_{m,n}, \\
\sum_{m > 0} a^\dagger(m)a(m) &= 1 - |0\rangle \langle 0| \\
a(m)|0\rangle &= \langle 0|a^\dagger(m) = 0, \quad m = 1, 2, \ldots
\end{align*}
\]

which governs (see Sec.3) the large $N$ limit of the conformal field theory of one chiral adjoint boson. The results of this subsection hold as well for either
a) any reduced bosonic sector $k$ (with $|0\rangle \rightarrow |k\rangle$) or b) any finite dimensional Cuntz algebra.

In terms of these operators, let us define the following new dressed Cuntz operators:

$$A(m) \equiv \sum_w a_w^\dagger a(m)a_w, \quad A^\dagger(m) = \sum_w a_w^\dagger a^\dagger(m)a_w$$  \hspace{1cm} (6.2a)

$$A(m)|0\rangle = \langle 0|A^\dagger(m) = 0.$$  \hspace{1cm} (6.2b)

The reader will recall that I have used this Cuntz dressing $\sum_w a_w^\dagger(...)a_w$ a number of times above in the construction of reduced trace class operators (see also Refs.[4,8]), but this is the first time it has been used to dress reduced densities such as the Cuntz operators themselves. It will be helpful to have a few examples of the action of the dressed operators on the ground state

$$A^\dagger(m)|0\rangle = a^\dagger(m)|0\rangle, \quad \langle 0|A(m) = \langle 0|a(m)$$  \hspace{1cm} (6.3a)

$$A^\dagger(m)A^\dagger(n)|0\rangle = A^\dagger(m)a^\dagger(n)|0\rangle$$  \hspace{1cm} (6.3b)

$$= (a^\dagger(m)a^\dagger(n) + a^\dagger(n)a^\dagger(m))|0\rangle$$  \hspace{1cm} (6.3c)

which are easily checked from the definitions in (6.2).

Using the lemma (2.31) with $b \rightarrow a$, we find after some algebra that the dressed Cuntz operators satisfy the commutation relations:

$$[a(m), A^\dagger(n)] = [A(m), a^\dagger(n)] = \delta_{m,n}$$  \hspace{1cm} (6.4a)

$$[A(m), A(n)] = [A^\dagger(m), A^\dagger(n)] = 0.$$  \hspace{1cm} (6.4b)

These commutation relations tell us that the dressed Cuntz operators are “canonical” to the original Cuntz operators, and moreover that all states created by the dressed creation operators

$$|\text{Bose}, W\rangle = A_W^\dagger|0\rangle, \quad W = (m_1, \ldots, m_n)$$  \hspace{1cm} (6.5)

are Bose symmetric in the letters of $W$, as illustrated in (6.3).

In spite of the simple commutation relations (6.4), the pairs $(a, A^\dagger)$ and/or $(A, a^\dagger)$ are by no means canonical in the ordinary sense. This is clear already
because $a$ and $a^\dagger$ satisfy only the Cuntz algebra (6.1), and, moreover, I find after some algebra the following additional commutation relations:

\[
[a(m), A(n)] = a(m)a(n), \quad [A^\dagger(m), a^\dagger(n)] = a^\dagger(m)a^\dagger(n) \quad (6.6a)
\]

\[
[A(m), A^\dagger(n)] = \sum_w a_w^\dagger(\delta_m^n + a^\dagger(n)a(m))a_w. \quad (6.6b)
\]

Verification of the last relation in (6.6) is somewhat involved, and the reader may find helpful the following intermediate steps

\[
A^\dagger(m) = a^\dagger(m) + \sum_{n>0} a^\dagger(n)A^\dagger(m)a(n) \quad (6.7a)
\]

\[
[A(m), A^\dagger(n)] = \delta_{m,n} + a^\dagger(n)a(m) + \sum_{l>0} a^\dagger(l)[A(m), A^\dagger(n)]a(l) \quad (6.7b)
\]

\[
A = B + \sum_{l>0} a^\dagger(l)Aa(l) = \sum_w a_w^\dagger Ba_w, \quad \forall B. \quad (6.7c)
\]

Finally, I have checked all the Jacobi identities for the new algebra (6.4),(6.6).

### 6.2 Quasi-fermionic excitations in the large $N$ Bose system

In this subsection, I will use the new dressed bosonic Cuntz operators (6.2) to construct a free-algebraic analogue of the vertex operator construction of one complex world-sheet fermion.

To begin this discussion, I will revert to our earlier form (see Subs. (3.1)) of the reduced bosonic theory in which we allowed the zero-mode $\pi(0)$ of the reduced boson to be non-zero. Recall that in this case the reduced theory does not possess a full set of reduced $\mathfrak{sl}(2)$ generators, but only the reduced “Hamiltonian” $L_B(0)$ in (3.6). Then, following the conventional intuition, we may consider the states

\[
|BH\rangle_\pm = |k = \pm 1\rangle, \quad |CR\rangle_\pm = |k = \pm 1/2\rangle \quad (6.8a)
\]

\[
L_B(0)|BH\rangle_\pm = \frac{1}{2}|BH\rangle_\pm, \quad L_B(0)|CR\rangle_\pm = \frac{1}{8}|CR\rangle_\pm \quad (6.8b)
\]
as candidates for Bardakci-Halpern (BH) fermion/antifermion states and complex Ramond (CR) ground states in the reduced bosonic theory.

To see if this interpretation makes sense, we need a local description of these excitations. In particular, let us consider using the dressed creation operators \( \{ A^\dagger(m) \} \) and the Cuntz annihilation operators \( \{ a(m) \} \) to construct the following free-algebraic analogue of the conventional bosonized form of a complex chiral world-sheet fermion:

\[
\bar{\psi}(z) \equiv e^{iq(e^{\pi\hbar(0)\ln z} - \sum_{n>0} A^\dagger(n) \sqrt{n} z^n)} e^{-\sum_{n>0} a(n) \sqrt{n} z^{-n}} \quad (6.9a)
\]

\[
\psi(z) \equiv e^{-iq(e^{-\pi\hbar(0)\ln z} - \sum_{n>0} A^\dagger(n) \sqrt{n} z^n)} e^{\sum_{n>0} a(n) \sqrt{n} z^n} \quad (6.9b)
\]

\[
[a(m), A^\dagger(n)] = \delta_{m,n}, \quad [A^\dagger(m), A^\dagger(n)] = 0, \quad [q, \pi(0)] = i \quad (6.9c)
\]

\[
a(m)|k\rangle = \langle k|A^\dagger(m) = 0, \quad |k\rangle = e^{ikq}|0\rangle. \quad (6.9d)
\]

Here I have also introduced a canonical coordinate \( q \) which commutes with all operators except the zero mode \( \pi(0) \). As a consequence, the operators \( \bar{\psi}, \psi \) and all the relations in (6.9c-d) look quite ordinary, except that there are no relations among the Cuntz annihilation operators \( \{ a(m) \} \).

These free-algebraic vertex operators are constructed to have many of the desirable properties of ordinary vertex operators. Note first that \( \bar{\psi}, \psi \) are interpolating fields for the candidate BH states in (6.8)

\[
\bar{\psi}(0)|0\rangle = |k = 1\rangle, \quad \psi(0)|0\rangle = |k = -1\rangle \quad (6.10)
\]

and multiple applications of \( \bar{\psi}, \psi \) on the reduced ground state \( |0\rangle \) generates an integer-valued lattice of momenta and hence a “BH sector” in which the new operators are half-integral moded:

\[
\mathcal{O}(z) = \sum_{p \in \mathbb{Z}+1/2} \mathcal{O}(p) z^{-p-1/2}, \quad \mathcal{O} = \bar{\psi}, \psi \quad (6.11a)
\]

\[
\bar{\psi}(-1/2)|0\rangle = |k = 1\rangle, \quad \psi(-1/2)|0\rangle = |k = -1\rangle. \quad (6.11b)
\]

Similarly, \( \bar{\psi}, \psi \) generate a “CR sector” when acting on the candidate CR ground states in (6.8). Second, it is not difficult to verify the following operator identities

\[
[L_B(0), a(m)] = -ma(m), \quad [L_B(0), A^\dagger(m)] = mA^\dagger(m) \quad (6.12a)
\]
\[(z \partial_z + 1/2) \mathcal{O}(z) = [L_B(0), \mathcal{O}(z)], \quad \mathcal{O} = \bar{\psi}, \psi \quad (6.12b)\]

which tell us that $\bar{\psi}, \psi$ boost with ”conformal weight” $1/2$, as expected for chiral world-sheet fermions.

But we must also study the algebra of $\bar{\psi}, \psi$. Using the commutators in (6.9c), one may compute in the usual way the exact operator products

\[
\bar{\psi}(z) \psi(\omega) = \frac{1}{z - \omega} (z/\omega)^{\pi(0)} e^{\sum_{n>0} \frac{a(n)^{\dagger}}{\sqrt{n}} (z^n - \omega^n)} e^{-\sum_{n>0} \frac{a(n)}{\sqrt{n}} \omega^{-n} e^{-\sum_{n>0} \frac{a(n)}{\sqrt{n}} \omega^{-n}} \pi(0)} \quad (6.13a)
\]

\[
\psi(\omega) \bar{\psi}(z) = -\frac{1}{z - \omega} (z/\omega)^{\pi(0)} e^{\sum_{n>0} \frac{a(n)^{\dagger}}{\sqrt{n}} (z^n - \omega^n)} e^{-\sum_{n>0} \frac{a(n)}{\sqrt{n}} \omega^{-n} e^{-\sum_{n>0} \frac{a(n)}{\sqrt{n}} \omega^{-n}} \pi(0)} \quad (6.13b)
\]

for $|z| > |\omega|$ and $|\omega| > |z|$ respectively. Comparing the right sides to the conventional result, the only difference is that, because there are no relations among the Cuntz operators, we cannot combine or commute the last two ‘a factors’ in any simple way. This difference does not contribute, however, to the leading term of the operator product expansions

\[
\bar{\psi}(z) \psi(\omega) = \frac{1}{z - \omega} + O(z - \omega)^0 \quad (6.14a)
\]

\[
\simeq -\psi(\omega) \bar{\psi}(z) \quad (6.14b)
\]

\[
\psi(z) \psi(\omega) = \bar{\psi}(z) \bar{\psi}(\omega) = O(z - \omega)^0 \quad (6.14c)
\]

where $\simeq$ means analytic continuation. These familiar relations tempt us to conclude that our new operators $\bar{\psi}, \psi$ are ordinary complex chiral fermion fields.

But such a conclusion would be premature (and incorrect). The reason is that the right sides of the exact relations (6.13a,b) are not the same – even by analytic continuation – which blocks the familiar contour trick needed to obtain the mode algebra from this system!

The operator products (6.13) can however be written in the following more useful form

\[
\bar{\psi}(z) \psi(\omega) = \frac{1}{z - \omega} (z/\omega)^{\pi(0)} e^{\sum_{n>0} \frac{a(n)^{\dagger}}{\sqrt{n}} (z^n - \omega^n)} \Omega^{-1}(z) \Omega(\omega) \quad (6.15a)
\]

\[
\simeq -\psi(\omega) \bar{\psi}(z) R(z, \omega) \quad (6.15b)
\]
\[ \Omega(z) \equiv e^{\sum_{n>0} \frac{a(n)}{\sqrt{n}} z^{-n}}, \quad R(z, \omega) \equiv \Omega(z)\Omega^{-1}(\omega)\Omega^{-1}(z)\Omega(\omega) \]  
(6.15c)

where \( R(z, w) \) is the commutant of the operator \( \Omega \). In this form, we can use for example the BH mode expansions (6.11) and the usual contour trick to obtain the commutant-dependent mode relations

\[ \bar{\psi}(p)\psi(q) + \sum_{m,n=0}^{\infty} \psi(q-n)\bar{\psi}(p-m)R_{mn}(a) = \delta_{q+p,0}, \quad p, q \in \mathbb{Z} + 1/2 \]  
(6.16a)

\[ R(z, \omega) = \sum_{m,n=0}^{\infty} R_{mn}(a)z^{-m}\omega^{-n} \]  
(6.16b)

in the BH sector, and similarly for the CR sector with integer \( p \) and \( q \). Using heat kernel methods, it is straightforward to work out the explicit form of the modes \( R_{mn}(a) \) of the commutant, beginning with

\[ R_{00}(a) = 1, \quad R_{01}(a) = R_{10}(a) = 0, \ldots \]  
(6.17)

but I will confine myself here to some remarks about the full commutant.

Using Eqs. (6.9c) and (6.9d), one finds after some algebra that the commutant satisfies the simple relations

\[ [A^\dagger(m), R(z, \omega)] = 0, \quad R(z, \omega)|k\rangle = |k\rangle \]  
(6.18a)

\[ R(z, \omega)A_W^\dagger|k\rangle = A_W^\dagger|k\rangle \]  
(6.18b)

and so it follows that \( \bar{\psi}, \psi \) behave as ordinary chiral complex fermions when acting on the Bose symmetric states \( A_W^\dagger|k\rangle \):

\[ ([\bar{\psi}(p), \psi(q)]_+ - \delta_{p+q,0})A_W^\dagger|k\rangle = 0 \]  
(6.19a)

\[ [\psi(p), \bar{\psi}(q)]_+ A_W^\dagger|k\rangle = [\bar{\psi}(p), \psi(q)]_+ A_W^\dagger|k\rangle = 0 \]  
(6.19b)

\[ p, q \in \begin{cases} \mathbb{Z} + 1/2 & \text{for BH} \\ \mathbb{Z} & \text{for CR} \end{cases}, \quad k \in \begin{cases} \mathbb{Z} & \text{for BH} \\ \mathbb{Z} + 1/2 & \text{for CR} \end{cases} \]  
(6.19c)

On the other hand, we cannot expect to find this simplification in general because the Bose symmetric states are not generic in the Cuntz Hilbert space.
Recall that the set \( \{ A_W^\dagger | k > \} \) includes the low-lying states
\[
\{|k\rangle, \ a^\dagger(m) |k\rangle, \ [a^\dagger(m), a^\dagger(n)]_+ |k\rangle, \ldots \}
\] (6.20)
so that the lowest states with non-trivial commutant \( R(z, w) \) are the anti-symmetric states \([a^\dagger(m), a^\dagger(n)]|k\rangle\). As an example, I have worked out the explicit form taken by the mode relations (6.16a)
\[
\{R(z, \omega) - 1\} [a^\dagger(m), a^\dagger(n)]|k\rangle = \frac{2}{\sqrt{mn}} (z^{-m} \omega^{-n} - \omega^{-m} z^{-n})
\] (6.21a)
\[
\{[\bar{\psi}(p), \psi(q)]_+ - \delta_{p+q,0}\} [a^\dagger(m), a^\dagger(n)]|k\rangle = \frac{2}{\sqrt{mn}} \{\psi(q - m) \bar{\psi}(p - n) - \psi(q - n) \bar{\psi}(p - m)\}|k\rangle
\] (6.21b)
when acting on these states.

In this subsection I have begun the discussion of an unexpected new excitation or soliton in the large \( N \) limit of the conformal field theory of a single chiral adjoint boson. One might call the new excitation a complex chiral quasi-fermion because it acts like a complex chiral fermion in a certain sector of the Cuntz Hilbert space, and we have found both a half-integral moded BH sector and an integral-moded CR sector for the chiral quasi-fermion. Finally, it is not difficult to find the free-algebraic intertwiners or spin fields
\[
\sigma_{\pm}(z) = e^{\pm i q/2} e^{\pm \frac{1}{2} \pi(0) \ln z} e^{\pm \frac{1}{2} \sum_{n>0} a^\dagger(n) z^{n}} e^{\pm \frac{1}{2} \sum_{n>0} a(n) z^{-n}}
\] (6.22a)
\[
\sigma_{\pm}(0)|0\rangle = |k = \pm 1/2\rangle
\] (6.22b)
which connect the BH and the CR sectors of the chiral quasi-fermion.

### 6.3 Free-algebraic vertex operators

The free-algebraic vertex operators in Eqs.(6.9),(6.22) are easily generalized as follows.

We know that the large \( N \) limit of \( D \) chiral adjoint bosons is described by a set of \( D \) reduced chiral boson modes \( \{ \pi_i(m) \} \) which satisfy the bosonic
Cuntz algebra (3.8a). Following Subsec.(6.1), we may then construct \( D \) pairs of dressed creation and annihilation operators

\[
A_i(m) = \sum_\omega a_\omega^\dagger a_i(m) a_\omega, \quad A_i^\dagger(m) = \sum_\omega a_\omega^\dagger a_i^\dagger(m) a_\omega \quad (6.23a)
\]

\[
i = 1 \ldots D, \ m \in \mathbb{Z} \quad (6.23b)
\]

\[
[A_i(m), a_j^\dagger(n)] = [a_i(m), A_j^\dagger(n)] = \delta_{ij} \delta_{m,n} \quad (6.23c)
\]

\[
[A_i(m), A_j(n)] = [A_i^\dagger(m), A_j^\dagger(n)] = 0 \quad (6.23d)
\]

\[
[a_i(m), A_j(n)] = a_i(m) a_j(n), \quad [A_i^\dagger(m), a_j^\dagger(n)] = a_i^\dagger(m) a_j^\dagger(n) \quad (6.23e)
\]

\[
[A_i(m), A_j^\dagger(n)] = \sum_\omega a_\omega^\dagger [a_i(m), a_j^\dagger(n)] + a_\omega = \sum_\omega a_\omega^\dagger [a_i(m), a_j^\dagger(n)] + a_\omega \quad (6.23f)
\]

\[
A_i(m) |k\rangle = \langle k | A_i^\dagger(m) = 0 \quad (6.23g)
\]

where the dressing includes the appropriate sums over \( i \). The states constructed from the dressed creation operators

\[
A_W^\dagger |k\rangle = A_{i_1}^\dagger(m_1) \ldots A_{i_n}^\dagger(m_n) |k\rangle, \ W = \{i_1 m_1, \ldots, i_n m_n\} \quad (6.24)
\]

are Bose symmetric, as above, in the letters of \( W \).

Then we may consider the more general free-algebraic vertex operator

\[
U(\alpha, z) \equiv e^{i\alpha \cdot q z^\alpha \cdot \pi(0)} e^{\alpha \sum_{n>0} a_\omega^\dagger \frac{z^n}{\sqrt{n}}} e^{-\alpha \sum_{n>0} a_\omega \frac{z^{-n}}{\sqrt{n}}} \quad (6.25a)
\]

\[
[q_i, \pi_j(0)] = i \delta_{ij}, \quad [L_B(0), U(\alpha, z)] = (z \partial_z + \frac{\alpha^2}{2}) U(\alpha, z) \quad (6.25b)
\]

\[
U(\alpha, 0) |0\rangle = |\alpha\rangle, \quad (L_B(0) - \frac{\alpha^2}{2}) |\alpha\rangle = 0 \quad (6.25c)
\]
where $\alpha$ is an arbitrary $D$-vector. The reduced “Hamiltonian” $L_B(0)$ of this system is given in Eq.(3.8d). The operator products of these more general constructions satisfy

$$U(\alpha, z)U(\beta, \omega) = (z - \omega)^{\alpha \cdot \beta} e^{i(\alpha + \beta) \cdot q} z^{\alpha \cdot \pi(0)} \omega^{\beta \cdot \pi(0)} \times e^{\sum_{n>0} \frac{A_n^{(1)}}{2n} (\alpha z^n + \beta \omega^n)} \Omega(-\alpha, z) \Omega(-\beta, \omega)$$ (6.26a)

$$\simeq U(\beta, \omega)U(\alpha, z)(-1)^{\alpha \cdot \beta} R(\alpha, z; \beta, \omega)$$ (6.26b)

$$\Omega(\alpha, z) \equiv e^{\sum_{n>0} \frac{a_n^{(2)}}{\sqrt{n}} z^{-n}}$$ (6.26c)

$$R(\alpha, z; \beta, \omega) \equiv \Omega(\alpha, z) \Omega(\beta, \omega) \Omega(-\alpha, \omega) \Omega(-\beta, \omega)$$ (6.26d)

and the commutants $\{R(\alpha, z; \beta, w)\}$ satisfy

$$[R(\alpha, z; \beta, \omega), A_i^\dagger(m)] = 0, \quad R(\alpha, z; \beta, \omega)|\gamma\rangle = |\gamma\rangle$$ (6.27a)

$$R(\alpha, z; \beta, \omega)A_W^\dagger|\gamma\rangle = A_W^\dagger|\gamma\rangle$$ (6.27b)

in parallel with the quasi-fermionic example of the previous subsection.

The relations (6.26),(6.27) guarantee the free-algebraic vertex operator construction of many complex chiral quasi-fermions

$$\bar{\psi}_i(z) = U(e_i, z)\bar{K}_i, \quad \psi_i(z) = K_i U(-e_i, z)$$ (6.28a)

$$(e_i)_j = \delta_{ij}, \quad i, j = 1 \ldots D$$ (6.28b)

$$\bar{K}_i = \prod_{j=1}^i e^{-i\pi\pi_j(0)}, \quad K_i = \prod_{j=1}^i e^{i\pi\pi_j(0)}$$ (6.28c)

where $\bar{K}, K$ are the Klein transformations used in the conventional vertex operator construction of many complex chiral fermions [21-23]. In particular, the relations (6.27) guarantee that, as above, these operators behave as a set of $D$ ordinary anticommuting complex chiral fermions when acting on any of the Bose symmetric states $A_W^\dagger|\gamma\rangle$.

One may also consider the free-algebraic analogue of other vertex operator constructions, for example by choosing $\alpha, \beta \in \Delta(slg)$ with $\alpha^2 = \beta^2 = 2$ to
be roots of a simply laced Lie algebra. But it seems (see however Subsec. 6.4) that this construction will not directly reproduce the simply laced affine algebra, even on the Bose symmetric states. The reason is the complexity of the non-leading terms in OPEs such as:

\[ U(\alpha, z)U(-\alpha, \omega) = \frac{1}{(z - \omega)^2} + \frac{\alpha \cdot P(\omega)}{z - \omega} + O(z - \omega) \]  

(6.29a)

\[ zP_i(z) = \pi_i(0) + \sum_{n>0} \sqrt{n} \left\{ A_i^\dagger(n)z^n + \int_0^1 dt \Omega(-\alpha t, z)a_i(n)\Omega(\alpha t, z)z^{-n} \right\}. \]  

(6.29b)

In particular, the modes of the operators \( \{P_i(z)\} \) do not satisfy an abelian current algebra.

### 6.4 Cuntz-algebraic factorization of the Koba-Nielsen factor

As a final topic, let us consider the reduced ground state expectation value of a product of an arbitrary number of free-algebraic vertex operators. The zero mode factor is the same as usual, and so we proceed by moving say the “\( A^\dagger \) factors” to the left until we reach the configuration:

\[ \langle 0 | e^{\sum_{I=1}^M \alpha_I \sum_{n>0} \frac{A_I^\dagger(n)}{\sqrt{n}} z_I^{-n}} \Omega(-\alpha_1, z_1) \cdots \Omega(-\alpha_M, z_M) | 0 \rangle = 1. \]  

(6.30)

Since the algebra of the \( a \)'s with the \( A^\dagger \)'s is also normal, our result is a new Cuntz-algebraic factorization of the ordinary [10] Koba-Nielsen factor

\[ \langle 0 | U(\alpha_1, z_1) \cdots U(\alpha_M, z_M) | 0 \rangle = \delta^D \left( \sum_{I=1}^M \alpha_I \right) \prod_{I<J} (z_I - z_J)^{\alpha_I \cdot \alpha_J} \]  

(6.31)

and hence a new Cuntz-algebraic factorization of conventional string correlators! As a simple example the reduced ground state expectation values of the quasi-fermions (6.28) are exactly the same as those of the corresponding set of conventional BH fermions.

At first sight this result is quite surprising because, owing to the Boltzmann statistics of the Cuntz algebra, there are many more Cuntz states \( \{a_w^\dagger | 0 \} \) in the completeness sums of each channel than there are bosonic
states in the factorization of the conventional bosonic string. It is not difficult however to check that

\[ U(\alpha_i, z_i) \ldots U(\alpha_M, z_M)|0\rangle = \sum_w C_W(\{\alpha\}, \{z\}) A_w^\dagger |\alpha_1 + \cdots + \alpha_M\rangle \] (6.32)

so that in fact only the Bose-symmetric subset \( A_w^\dagger |\gamma\rangle \) of the Cuntz states are coupling in each channel of the free-algebraic factorization – and these states are in one-to-one correspondence with the Bose states of the conventional bosonic string. Moreover, the general result (6.31) defines a subspace of the reduced Hilbert space in which one finds effectively all the usual string constructions, including the conventional bosonic Virasoro generators and the conventional vertex operator construction of level one of simply laced g.

The story does not end here however because, in the free-algebraic factorization, the rest of the Cuntz states can also couple. As an example, the amplitudes for scattering into the lowest non-Bose symmetric Cuntz states are proportional to:

\[ \langle 0 | U(\alpha_1, z_1) \ldots U(\alpha_M, z_M) | a_i^\dagger(m), a_j^\dagger(n) | \gamma \rangle \neq 0. \] (6.33)

The explicit form of these new amplitudes is beyond the scope of this paper. I should also mention the existence of another set of free-algebraic vertex operators using the pair \((a^\dagger, A)\) instead of \((A^\dagger, a)\).

I am not sanguine about the viability of such new free-algebraic strings, even when the free-algebraic degrees of freedom are limited to compactified dimensions. The reason is the very large number of Cuntz states, whose partition functions (see Subsecs. 2.5 and 3.1) would be expected to cause new singularities in free-algebraic string loops. Similarly, formulations of space-time canonical ensembles are apparently ruled out by the rapid growth

\[ O(e^{an}) \rightarrow O(e^{aE^2}) \] (6.34)

in the number of high-mass Cuntz states.

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After submission of this paper, I was reminded that the large $N$ limit of $Z_N$ parafermions was studied by different methods in Ref. [25].

A  Local reduced fields and operator products

Although they are not used in the text, one may define “local” reduced fields as usual

\[ H(z) \equiv \sum_p H(p) z^{-p/2}, \quad J(z) \equiv \sum_m J(m) z^{-m} \]

in terms of the reduced modes $H, J$ and $\pi$ of the text. These reduced fields serve as interpolating fields in the usual way

\[ \lim_{z \to 0} H(z) |0\rangle = H(-1/2) |0\rangle, \quad \lim_{z \to 0} J(z) |0\rangle = J(-1) |0\rangle \]

\[ \lim_{z \to 0} \pi(z) |0\rangle = \pi(-1) |0\rangle \]

and the vev’s of the reduced fields are not difficult to evaluate from the free algebras, for example:

\[ \langle 0 | H(z) H(\omega) | 0 \rangle = \frac{1}{z - \omega}, \quad \langle 0 | \pi(z) \pi(\omega) | 0 \rangle = \frac{1}{(z - \omega)^2} \]

\[ \langle 0 | J(z) J(\omega) | 0 \rangle = \frac{\hat{x}}{(z - \omega)^2}, \quad \langle 0 | J(z_1) J(z_2) J(z_3) | 0 \rangle = \frac{-\hat{x}_{12} z_{13} z_{23}}{z_1 z_2 z_3} \]

I remind the reader (see Eq.(2.3b)) that the reduced vev’s are proportional to the corresponding traced Wightman functions at large $N$ in the unreduced theories.
The commutation relations of the reduced $\mathfrak{sl}(2)$ generators with the local reduced fields

\[
[L_F(m), H(z)] = z^m (z\partial_z + \frac{1}{2}(m + 1))H(z) \quad (A.4a)
\]

\[
[L_F(m), J(z)] = z^m (z\partial_z + (m + 1))J(z) \quad (A.4b)
\]

\[
[L_B(m), \pi(z)] = z^m (z\partial_z + (m + 1))\pi(z) \quad (A.4c)
\]

also follow from the corresponding mode commutators of the text. These relations and the fact that the reduced fermionic ground state is $\mathfrak{sl}(2)$ invariant gives the usual $\mathfrak{sl}(2)$ Ward identities

\[
A(z) \equiv \langle 0 | H(z_1) \ldots H(z_M) | 0 \rangle \quad (A.5a)
\]

\[
\sum_{j=1}^{M} z_j^m (z_j\partial_j + \frac{1}{2}(m + 1))A(z) = 0, \quad |m| \leq 1 \quad (A.5b)
\]

for the reduced fermionic vev's, and similarly for reduced vev's including the reduced local currents and bosons. Reduced $\mathfrak{osp}(1|2)$ Ward identities can also be obtained for the reduced vev's of the NS sector discussed in Sec.5.

Let us also introduce the local tilde fields

\[
\tilde{H}(z) \equiv \sum_p \tilde{H}(p)z^{-p-1/2}, \quad \tilde{\pi}(z) \equiv \sum_m \tilde{\pi}(m)z^{-m-1} \quad (A.6a)
\]

\[
(\tilde{H}(z) - H(z))|0\rangle = (\tilde{\pi}(z) - \pi(z))|0\rangle = 0 \quad (A.6b)
\]

\[
0 |(\tilde{H}(z) - H(z)) = 0 |(\tilde{\pi}(z) - \pi(z)) = 0 \quad (A.6c)
\]

where the tilde modes are defined in Eqs.(2.4) and (3.3). The untilded modes satisfy commutation and/or anticommutation relations with the tilde modes, so we may compute in more or less the usual fashion the operator products of the untilded fields with the tilde fields. For example, the operator products

\[
H(z)\tilde{H}(\omega) = \frac{|0\rangle\langle 0|}{z - \omega} + :H(z)\tilde{H}(\omega): \quad (A.7a)
\]

\[
J(z)\tilde{H}(\omega) = \frac{|0\rangle\langle 0|\tilde{H}(\omega) - \tilde{H}(\omega)|0\rangle\langle 0|}{z - \omega} + :J(z)\tilde{H}(\omega): \quad (A.7b)
\]

\[
= \frac{|0\rangle\langle 0|H(\omega) - H(\omega)|0\rangle\langle 0|}{z - \omega} + :J(z)\tilde{H}(\omega): \quad (A.7c)
\]
\[ \pi(z)\bar{\pi}(\omega) = \frac{|0\rangle\langle 0|}{(z - \omega)^2} + :\pi(z)\bar{\pi}(\omega): \quad (A.7d) \]

are obtained for \(|z| > |w|\), where I have used the conventional fermionic and/or bosonic normal ordering for untilded/tilded mode pairs.

Finally, the reduced fields satisfy the expected Hamiltonian equations of motion on the cylinder \((t, \xi)\), for example

\[
H(\xi, t) \equiv \sum_p H(p, t)e^{-i\xi p} = \sum_p H(p, 0)e^{-i(\xi+t)p} \quad (A.8a)
\]

\[
\partial_t H(p, t) = i[L_F(0), H(p, t)] = -ipH(p, t) \quad (A.8b)
\]

\[
\partial_\xi H(\xi, t) = i[L_F(0), H(\xi, t)] = \partial_\xi H(\xi, t) \quad (A.8c)
\]

so that the reduced fermion is a chiral field.

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