q-oscillators, (non-)Kähler manifolds and constrained dynamics

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Abstract

It is shown that \( q \)-deformed quantum mechanics (\( q \)-deformed Heisenberg algebra) can be interpreted as quantum mechanics on Kähler manifolds, or as a quantum theory with second (or first)-class constraints.

1. In the present letter, a classical limit of multimode \( q \)-deformed Heisenberg-Weyl algebras [1],[2], meaning \( \hbar \to 0 \) (rather than \( q \to 1 \)), is analyzed. As was show in [3], a non-commutative phase space does not necessarily emerge from \( q \)-deformed quantum mechanics in the classical limit if one assumes \( q \) to be a function of the Plank constant and a certain dimensional constant (its possible physical interpretation is discussed in [3]). In this approach, the \( q \)-deformed Heisenberg-Weyl algebra produces a quadratic symplectic structure in the classical limit. For example, the one-mode \( q \)-deformed Heisenberg-Weyl algebra

\[
\hat{b}\hat{b}^+ - q^2\hat{b}^+\hat{b} = \hbar,
\]

where \( \hat{b} \) and \( \hat{b}^+ \) are creation and destruction operators, turns into the following symplectic structure [3]

\[
\{b, b^*\} = -i(1 - b^*b/\beta),
\]

where \( \beta \) is a constant and \( b \) and \( b^* \) are commutative complex coordinates on the phase space. One can easily be convinced that (2) follows from (1) when taking the classical limit \([ , ]/i\hbar \to \{ , \}\) as \( \hbar \to 0 \) and \( \hat{b}, \hat{b}^+ \) are changed by classical holomorphic variables \( b, b^* \), respectively. An assumption of the existence of this limit actually yields \( 1 - q^2 = \hbar/\beta + O(\hbar^2) \) because \([\hat{b}, \hat{b}^+]/i\hbar = -i(1 - (1 - q^2)\hat{b}\hat{b}^+)/\hbar \) in accordance with (1). Notice that the formal limit \( \hbar \to 0 \) in the right-hand side of (1) would lead to a non-commutative phase space \( bb^* = q^2b^*b \). Yet, a canonical quantization of the symplectic structure (2) \(([ , ] = i\hbar\{ , \})\) results in the "deformed" Heisenberg-Weyl algebra (1) [3], [4].

The modification of the symplectic structure (2) obviously leads to a modification of Hamiltonian equations of motion. If the Hamiltonian is assumed to be proportional to \( b^*b \) (a harmonic \( q \)-oscillator), then the frequency of oscillations becomes a function of the oscillator energy. Thus, one can regard a harmonic \( q \)-oscillator as a familiar anharmonic oscillator [3].

*Supported by an MRT grant of the government of France and Russian Foundation of Fundamental Research, grant 93-02-3827.

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Known examples of the $q$-deformation of the Heisenberg algebra ($q$-particles) \cite{3,4} can also be treated as systems with a "deformed" symplectic structure. Consider the following symplectic structure

$$\{x, p\} = 1 - xp/\beta$$

(3)

where $x$ and $p$ are the coordinate and momentum of a particle. A canonical quantization of (3) gives $q$-deformed Heisenberg algebra discussed in \cite{3}

$$\hat{p}\hat{x} - q\hat{x}\hat{p} = -i\hbar q^{1/2}, \quad \hat{p}^+ = \hat{p}, \quad \hat{x}^+ = \hat{x}$$

(4)

with $q = e^{i\theta}$ and $\theta = \theta(h/\beta)$ \cite{4}. Indeed, by virtue of the canonical quantization rule, we have from (3) $[\hat{x}, \hat{p}] = i\hbar(1 - (\hat{x}\hat{p} + \hat{p}\hat{x})/2\beta)$. Relation (4) is obtained then by renormalizing the operators $\hat{p}$ and $\hat{x}$ with the coefficient $|1 - i\hbar/2\beta|^{1/2}$, and $q = (1 + i\hbar/2\beta)/(1 - i\hbar/2\beta)$.

The Hamiltonian equation of motion $\dot{x} = \{x, H\}$ of a free particle $H = p^2/2$ induced by (3) coincides with the equation of motion for a particle with friction. The friction coefficient depends on the deformation parameter $\beta$ and the particle generalized momentum $p$. Notice that $\hat{p} = \{p, H\} = 0$ therefore $p = \text{const}$.

A lattice quantum mechanics \cite{3,4} appearing upon a deformation of the Heisenberg algebra with a real $q$ \cite{3}

$$\hat{p}\hat{x} - q\hat{x}\hat{p} = -i\hbar, \quad \hat{x}\hat{p}^+ - q\hat{p}^+\hat{x} = i\hbar, \quad \hat{p}^+\hat{p} = q\hat{p}\hat{p}^+, \quad \hat{x}^+ = \hat{x}$$

(5)

can be obtained by quantizing a degenerate symplectic structure

$$\{x, p\} = 1 - ixp/\beta, \quad \{x, p^*\} = 1 + ixp^*/\beta, \quad \{p^*, p\} = ipp^*/\beta.$$ 

(6)

The degeneracy is associated with the existence of an absolute integral of motion

$$C = pp^*x/\beta - i(p - p^*)$$

(7)

which commutes with all symplectic coordinates $\{C, x\} = \{C, p\} = \{C, p^*\} = 0$. Therefore, the system never leaves the surface $C = \text{const}$ in due course. A phase space of the system is a two-dimensional surface $C = \text{const}$. In quantum theory, eigen values of the Casimir operator $\hat{C}$ determine irreducible representation of the algebra (5) \cite{3}.

A straightforward application of the canonical quantization rule to (6) meets the operator ordering ambiguity in the right-hand side of the commutation relation $[,] = i\hbar\{,\}$ (see also a Remark in p.3). The operator ordering should be chosen so that the Jacobi identity is fulfilled on the quantum level \cite{3}. It is remarkable that any operator ordering consistent with the Jacobi identity results in the algebra (5). Different choices of the operator ordering correspond to variations of terms $O(\hbar^2)$ in $q = q(h, \beta)$ \cite{3}.

To get (5) from (6), one can, for instance, postulate the first commutation relation as follows $[\hat{x}, \hat{p}] = i\hbar(1 - i\hat{x}\hat{p}/\beta)$, then $[\hat{x}, \hat{p}^+]$ is obtained by the Hermitian conjugation of the first one, assuming $\hat{x} = \hat{x}^+$, while the operator ordering in the last commutation relation in (6) is fixed by the Jacobi identity and reads $[\hat{p}^+, \hat{p}] = -\hbar\hat{p}\hat{p}^+/\beta$. So, $q = 1 - \hbar/\beta$.

So, the $q$-deformed Heisenberg algebra can appear as a result of quantizing a quadratic symplectic structure

$$\{\theta^j, \theta^k\} = \omega^{jk} + \omega^{ik}\theta^j\theta^k$$

(8)
where $\theta^j$ is a set of real phase-space coordinates, $\omega^{jk}$ is the canonical symplectic structure and $c_{in}^{jk}$ are "deformation" constants chosen so that the Jacobi identity for (8) is satisfied.

Below we shall demonstrate that the symplectic structure associated with the $SU_q(n)$-covariant deformation of the Heisenberg-Weyl algebra is related to quantum mechanics on Kähler manifolds. We shall also show that physical phase-space variables in constrained quantum mechanics may naturally form a $q$-deformed Heisenberg-Weyl algebra.

2. The following $q$-deformed commutation relations remain untouched under the action of the quantum group $SU_q(n)$ [3]

$$\hat{a}_i \hat{a}_j = q \hat{a}_j \hat{a}_i, \quad \hat{a}_i^+ \hat{a}_j^+ = \frac{1}{q} \hat{a}_j^+ \hat{a}_i^+, \quad i < j$$

(9)

$$\hat{a}_i \hat{a}_j^+ = q \hat{a}_j^+ \hat{a}_i, \quad i \neq j$$

(10)

$$\hat{a}_i \hat{a}_i^+ - q^2 \hat{a}_i^+ \hat{a}_i = h + (q^2 - 1) \sum_{k<i} \hat{a}_k^+ \hat{a}_k.$$  

(11)

To obtain a corresponding symplectic structure in a classical theory, one may use the rule $[,]/ih \rightarrow \{ , \}$ as $h \rightarrow 0$ and $\hat{a}_i$, $\hat{a}_j^+$ are simultaneously to be changed by classical holomorphic variables $a_i$, $a_j^*$. However, this is just a formal rule which sometimes helps to guess a correct classical limit of a given quantum theory (see a rigorous consideration of the classical limit in [3]). To ensure that this rule works for the algebra (9)-(11), we notice that by means of a transformation proposed in [3] the commutation relations (9)-(11) can be "diagonalized", be transformed to the form (1) for each oscillator mode, while operators of different modes commute amongst each other. For the commutation relation (1), the validity of the rule $[,]/ih \rightarrow \{ , \}$ can be rigorously established in the framework of the path integral formalism [3]. Therefore the above mentioned formal approach gives a correct classical mechanics in our case. Assuming again $1 - q = h/\beta + O(h^2)$ (otherwise there is no commutative phase space in the classical theory) we arrive at the following Poisson bracket structure

$$\{a_k, a_j\} = ia_k a_j / \beta, \quad \{a_k^*, a_j^*\} = -ia_k a_j^* / \beta, \quad k < j$$

(12)

$$\{a_k, a_j^*\} = ia_k a_j^* / \beta, \quad k \neq j ;$$

(13)

$$\{a_j, a_j^*\} = -i \left( 1 - \frac{2}{\beta} \sum_{k=1}^{j} a_k^* a_k \right),$$

(14)

where $a_j, a_j^*$ are holomorphic coordinates on a phase space.

Let us recall now a basic definition of the Kähler manifold [3]. Let $z^i$ and $z^{k*}$ are complex coordinates on a manifold $\mathcal{M}$ and $g_{ik}(z, z^*)$ is a metric tensor on it such that the interval on $\mathcal{M}$ has the form $ds^2 = g_{ik} dz^i dz^{k*}$ and

$$g_{ik} = \partial^2 \phi / \partial z^i \partial z^{k*} ;$$

(15)

the scalar function $\phi$ is called the Kähler potential, and $\mathcal{M}$ is called the Kähler manifold. A Kähler manifold turns into a symplectic manifold if the following symplectic structure is introduced on it

$$\{A, B\} = -ig^{jk} \left( \frac{\partial A}{\partial z^k} \frac{\partial B}{\partial z^{j*}} - \frac{\partial A}{\partial z^{j*}} \frac{\partial B}{\partial z^k} \right)$$

(16)
for any two functions $A$ and $B$ of $z$, $z^*$, where $g^{ki}$ is a matrix inverse to (15). The Poisson bracket thus defined obey the Jacobi identity due to the property (15) [3].

It is easy to see that the Poisson brackets (12)-(14) are not of the Kählerian type because of (12). However, they can be transformed to the form (16). Indeed, the algebra (12)-(14) admits the following representation

$$a_i = z^i \prod_{k=1}^{i-1} (1 - 2z^k z^{k*}/\beta)^{1/2}$$

(17)

and $a_i^*$ is obtained by a complex conjugation of (17), where

$$\{z^j, z^{k*}\} = -i(1 - 2z^j z^{j*}/\beta)\delta^{jk}$$

(18)

and $\{z^j, z^k\} = \{z^{k*}, z^{j*}\} = 0$. Therefore, the Kähler metric reads

$$g_{ik} = \delta_{ik}(1 - 2z^k z^{k*}/\beta)^{-1}.$$  

(19)

Representing the Kähler potential in the form

$$\phi = \frac{\beta}{2} \sum_i \varphi \left( \frac{2z^i z^{i*}}{\beta} \right)$$

(20)

and substituting (20) and (19) into (15) we obtain

$$\varphi(x) = Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \quad |x| < 1,$$

(21)

with $Li_2$ being the Euler dilogarithm.

So, the $SU_q(n)$—covariant deformation of the Heisenberg-Weyl algebra describes a quantum theory on a Kähler manifold with the potential (20), (21).

The phase-space manifold with the metric (19) is curved. The scalar curvature corresponding to the metric (19),

$$R = \sum_i \frac{8}{\beta} \left(1 - \frac{2z^i z^{i*}}{\beta}\right)^{-1},$$

(22)

tends to infinity as any of variables $z^i$ approaches the circle $|z^i|^2 = \beta/2$, assuming $\beta > 0$. Therefore, for positive $\beta$ the phase space turns out to be compact, while for negative $\beta$ the function (22) is regular on the entire complex plane.

3. Another more general "interpretation" of the $q$-deformation of the Heisenberg-Weyl algebra can be achieved in the framework of the constrained dynamics. It is known since long time ago that a non-trivial symplectic structure may occur through second-class constraints [11] in dynamical systems. Let $\varphi_a(\xi) = 0, \; a = 1, 2, \ldots, 2M$, are second-class constraints on a phase space spanned by coordinates $\xi^i$, i.e. the matrix $\{\varphi_a, \varphi_b\} = \Delta_{ab}$ is not degenerate, where $\{\xi^i, \xi^j\} = \tilde{\omega}^{ij}$ is the canonical symplectic structure. Let $\xi^i = \xi^i(\theta)$ be a solution of constraints where physical variables $\theta^\alpha, \; \alpha = 1, 2, \ldots, 2(N - M)$ are
coordinates of a physical phase space, \( \varphi_a(\xi(\theta)) \equiv 0 \). Then the symplectic structure on the physical phase space is induced by the Dirac bracket

\[
\{ A, B \}_D = \{ A, B \} - \{ A, \varphi_a \} \Delta^{ab} \{ \varphi_b, B \}
\]

projected on the surface \( \xi^i = \xi^i(\theta) \), here \( \Delta^{ab} \Delta_{bc} = \delta^a_c \).

The induced symplectic structure might not coincide with the canonical one, i.e. it might turn out to be "deformed". Therefore, one can ask the question: is it possible to find such second-class constrained system that the symplectic structure induced by the Dirac bracket on the physical phase space has the quadratic form (8)? The answer is positive for the simplest \( q \)-deformed systems considered in p.1 [7]. A generalization is rather trivial.

Let \( \omega^{ij}(\theta) \) be a non-constant symplectic structure and \( \omega_{ij}\omega^{jk} = \delta_j^k \). Let us extend the initial phase space spanned by \( \theta^i \) by adding new variables \( \pi_j \) and postulate the canonical symplectic structure on the extended phase space, \( \{ \theta^j, \theta^k \} = \{ \pi_i, \pi_k \} = 0 \) and \( \{ \theta^j, \pi_k \} = \delta^j_k \), i.e. the initial phase space is a configuration space in the extended theory. Following [12] we introduce second class constraints as

\[
\varphi_i(\pi, \theta) = \pi_i + \bar{\omega}_{ij}(\theta) \theta^j = 0
\]

where

\[
\bar{\omega}_{ij}(\theta) = (\theta^i \frac{\partial}{\partial \theta^i} + 2)^{-1} \omega_{ij}(\theta) = \frac{1}{\int_0^1 d\alpha \alpha \omega_{ij}(\alpha \theta)}.
\]

Then [12]

\[
\{ \theta^i, \theta^j \}_D = -\{ \theta^i, \varphi_k \} \Delta^{kn} \{ \varphi_n, \theta^j \} = \omega^{ij}(\theta).
\]

For the symplectic structures (18) or (12)-(14), the integral (25) can be taken explicitly. Thus, \( q \)-deformed quantum mechanics can appear upon a quantization of second-class constrained systems.

Remark. Quantization of the quadratic symplectic structure (8) (induced by the Dirac bracket (23)) is not obvious because of the operation ordering. A naive application of the formal rule \([ , , ] = i\hbar \{ , , \}\) can break down the Jacobi identity in the quantum theory (or the associativity of the quantum algebra). One way to obtain an associative quantum theory is to assume that the coefficients \( c_{ij}^{kn} \) are also to be changed upon quantizing, \( c_{ij}^{kn} \to \tilde{c}_{ij}^{kn}(\hbar) \) so that \( \tilde{c}_{ij}^{kn}(\hbar = 0) = c_{ij}^{kn} \) and \( \tilde{c}_{ij}^{kn} \) provide the associativity of the quantum theory. The latter yields some algebraic equations for \( \tilde{c}_{ij}^{kn} \) of the Yang-Baxter-Hecke type to be solved.

Another way to manage the operator ordering problem is to convert the second-class constrained dynamics (24) into the first-class ones with a sequent quantization [13], [12]. A curious observation in this approach is that \( q \)-deformed commutation relations can also occur through reducing a first-class constrained (gauge) quantum system to physical (gauge-invariant) variables. One should point out that a quadratic symplectic structure is

\[\text{This procedure is equivalent to a quantization via the Darboux variables for a given symplectic form. However, the goal of the conversion method is that it allows us to avoid an explicit construction of the Darboux variables (which is a rather hard problem in general).}\]
just a particular case in the framework of the generalized canonical quantization of curved phase spaces developed in [12].

4. For any symplectic matrix $\omega_{ij}(\theta)$ obeying the Jacobi identity $\partial_k \omega_{ij} + \text{cycle}(k, i, j) = 0$, there exist local Darboux coordinates in which the symplectic structure has the canonical form [14]. The Darboux variables for the Poisson bracket (18) and, hence, for (12)-(14) (due to the relation (17)) can be explicitly found [3]. Therefore, the quadratic "deformation" locally looks like a special non-canonical transformation of the ordinary (Darboux) phase-space coordinates [3]. From the mathematical point of view, all phase-space coordinate systems are to be treated on equal footing. In a physical theory, phase-space coordinates are associated with observables, which makes some particular coordinates to be dynamically distinguished. For example, the excitations of various physical systems can be modelled through $q$-oscillators [15], which means that all complicated interactions in a physical system can be accumulated into $q$-deformed commutation relations. So, from this point of view the $q$-deformed quantum mechanics appears to be an effective theory for describing physical excitations. It has been, actually, illustrated in p.1 with examples of the harmonic $q$-oscillator and $q$-particle (which are dynamically equivalent to an anharmonic oscillator and a particle with friction, respectively).

In contrast with the above said, the interpretation within constrained dynamics does not imply, in general, any non-trivial interaction leading to "$q$-deformed" excitations. The $q$-deformation of the algebra of observables appears kinematically upon eliminating all unphysical (gauge) degrees of freedom (i.e. after solving constraints).

Acknowledgement

The author is kindly grateful to Prof. I.A. Batalin for useful discussions.

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