Light Spanners for High Dimensional Norms via Stochastic Decompositions

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Abstract

Spanners for low dimensional spaces (e.g. Euclidean space of constant dimension, or doubling metrics) are well understood. This lies in contrast to the situation in high dimensional spaces, where except for the work of Har–Peled, Indyk and Sidiropoulos (SODA 2013), who showed that any $n$-point Euclidean metric has an $O(t)$-spanner with $\tilde{O}(n^{1+1/t^2})$ edges, little is known. In this paper we study several aspects of spanners in high dimensional normed spaces. First, we build spanners for finite subsets of $\ell_p$ with $1 < p \leq 2$. Second, our construction yields a spanner which is both sparse and also light, i.e., its total weight is not much larger than that of the minimum spanning tree. In particular, we show that any $n$-point subset of $\ell_p$ for $1 < p \leq 2$ has an $O(t)$-spanner with $n^{1+\tilde{O}(1/t^p)}$ edges and lightness $n^{\tilde{O}(1/t^p)}$. In fact, our results are more general, and they apply to any metric space admitting a certain low diameter stochastic decomposition. It is known that arbitrary metric spaces have an $O(t)$-spanner with lightness $O(n^{1/t})$. We exhibit the following tradeoff: metrics with decomposability parameter $\nu = \nu(t)$ admit an $O(t)$-spanner with lightness $\tilde{O}(\nu^{1/t})$. For example, metrics with doubling constant $\lambda$, graphs of genus $g$, and graphs of treewidth $k$, all have spanners with stretch $O(t)$ and lightness $\tilde{O}(\lambda^{1/t}), \tilde{O}(g^{1/t}), \tilde{O}(k^{1/t})$ respectively. While these families do admit a $(1+\epsilon)$-spanner, its lightness depend exponentially on the dimension (resp. log $g, k$). Our construction alleviates this exponential dependency, at the cost of incurring larger stretch.

A preliminary version of this paper appeared in proceedings of 26th European Symposium on Algorithms (ESA 2018) [37]. This full version contains a new result on light (sub-graph) spanners for graphs in general (Theorem 4), and for graph of bounded treewidth (Corollary 7) in particular.

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1 Introduction

1.1 Spanners

Given a metric space \((X, d_X)\), a weighted graph \(H = (X, E)\) is a \(t\)-spanner of \(X\), if for every pair of points \(x, y \in X\), \(d_X(x, y) \leq d_H(x, y) \leq t \cdot d_X(x, y)\) (where \(d_H\) is the shortest path metric in \(H\)). The factor \(t\) is called the stretch of the spanner. Two important parameters of interest are: the sparsity of the spanner, i.e. the number of edges, and the lightness of the spanner, which is the ratio between the total weight of the spanner and the weight of the minimum spanning tree (MST).

The tradeoff between stretch and sparsity/lightness of spanners is the focus of an intensive research effort, and low stretch spanners were used in a plethora of applications, to name a few: Efficient broadcast protocols [2, 3], network synchronization [2, 3, 10, 61, 63], data gathering and dissemination tasks [15, 27, 74], routing [63, 64, 71, 75], distance oracles and labeling schemes [60, 66, 72], and almost shortest paths [24, 29, 31, 34, 67]. The construction of spanners were studied in various computational models [8, 13, 14, 28, 35, 62]. We refer to [4] for an extensive survey.

Spanners for general metric spaces are well understood. The seminal paper of [6] showed that for any parameter \(k \geq 1\), any metric admits a \((2k - 1)\)-spanner with \(O(n^{1+1/k})\) edges, which is conjectured to be best possible. For light spanners, improving \([19, 30]\), it was shown in \([25]\) that for every constant \(\epsilon > 0\) there is a \((2k - 1)(1 + \epsilon)\)-spanner with lightness \(O(n^{1/k})\) and \(O(n^{1+1/k})\) edges.

There is an extensive study of spanners for restricted classes of metric spaces, most notably subsets of low dimensional Euclidean space\(^1\), and more generally doubling metrics.\(^2\) For such low dimensional metrics, much better spanners can be obtained. Specifically, for \(n\) points in \(d\)-dimensional Euclidean space, \([26, 56, 68, 73]\) showed that for any \(\epsilon \in (0, \frac{1}{2})\) there is a \((1 + \epsilon)\)-spanner with \(n \cdot \epsilon^{-O(d)}\) edges and lightness \(\epsilon^{-O(d)}\) (further details on Euclidean spanners could be found in \([59]\)). This result was recently generalized to doubling metrics by \([18]\), with \(\epsilon^{-O(\text{ddim})}\) lightness and \(n \cdot \epsilon^{-O(\text{ddim})}\) edges (improving \([40, 42, 69]\)). Such low stretch spanners were also devised for metrics arising from certain graph families. For instance, \([6]\) showed that any planar graph admits a \((1 + \epsilon)\)-spanner with lightness \(O(1/\epsilon)\). This was extended to graphs with small genus\(^3\) by \([43]\), who showed that every graph with genus \(g > 0\) admits a spanner with stretch \((1 + \epsilon)\) and lightness \(O(g/\epsilon)\). A long sequence of works for other graph families, concluded recently with a result of \([17]\), who showed \((1 + \epsilon)\)-

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\(^1\) That is a set of points \(X \subset \mathbb{R}^d\) equipped with the Euclidean metric \(\ell_2\), for small \(d\).

\(^2\) A metric space \((X, d)\) has doubling constant \(\lambda\) if for every \(x \in X\) and radius \(r > 0\), the ball \(B(x, 2r)\) can be covered by \(\lambda\) balls of radius \(r\). The doubling dimension is defined as \(\text{ddim} = \log_2 \lambda\). A \(d\)-dimensional \(\ell_p\) space has \(\text{ddim} = \Theta(d)\), and every \(n\) point metric has \(\text{ddim} = O(\log n)\).

\(^3\) The genus of a graph is minimal integer \(g\), such that the graph could be drawn on a surface with \(g\) “handles”.
spanners for graphs excluding $K_r$ as a minor, with lightness $\approx O(r/\epsilon^3)$. See also [21, 47] for subset spanners.

In all these results there is an exponential dependence on a certain parameter of the input metric space (the dimension, the logarithm of the genus/minor-size), which is unfortunately unavoidable for small stretch (for all $n$-point metric spaces the dimension/parameter is at most $O(\log n)$, while spanner with stretch better than 3 requires in general $\Omega(n^2)$ edges [72]). So when the relevant parameter is small, light spanners could be constructed with stretch arbitrarily close to 1. However, in metrics arising from actual data, the parameter of interest may be moderately large, and it is not known how to construct light spanners avoiding the exponential dependence on it. In this paper, we devise a tradeoff between stretch and sparsity/lightness that can diminish this exponential dependence. To the best of our knowledge, the only such tradeoff is the recent work of [45], who showed that $n$-point subsets of Euclidean space (in any dimension) admit a $O(t)$-spanner with $\tilde{O}(n^{1+1/t^2})$ edges (without any bound on the lightness).

1.2 Stochastic Decompositions

In a (stochastic) decomposition of a metric space, the goal is to find a partition of the points into clusters of low diameter, such that the probability of nearby points to fall into different clusters is small. More formally, for a metric space $(X, d_X)$ and parameters $t \geq 1$ and $\delta = \delta(|X|, t) \in [0, 1]$, we say that the metric is $(t, \delta)$-decomposable, if for every $\Delta > 0$ there is a probability distribution over partitions of $X$ into clusters of diameter at most $t \cdot \Delta$, such that every two points of distance at most $\Delta$ have probability at least $\delta$ to be in the same cluster.

Such decompositions were introduced in the setting of distributed computing [11, 54], and have played a major role in the theory of metric embedding [1, 12, 33, 36, 38, 48, 52, 65], distance oracles and routing [5, 57], multi-commodity flow/sparsest cut gaps [50, 53] and also were used in approximation algorithms and spectral methods [16, 23, 49]. We are not aware of any direct connection of these decompositions to spanners (except spanners for general metrics implicit in [5, 57]).

Note that our definition is slightly different than the standard one. The probability $\delta$ that a pair $x, y \in X$ is in the same cluster may depend on $|X|$ and $t$, but unlike previous definitions, it does not depend on the precise value of $d_X(x, y)$ (rather, only on the fact that it is bounded by $\Delta$). This simplification suits our needs, and it enables us to capture more succinctly the situation for high dimensional normed spaces, where the dependence of $\delta$ on $d_X(x, y)$ is non-linear. These stochastic decompositions are somewhat similar to Locality Sensitive Hashing (LSH), that were used by [45] to construct spanners. The main difference is that in LSH, far away points may be mapped to the same cluster with some small probability, and more focus was given to efficient computation of the hash function. It is implicit in [45] that existence of good LSH imply sparse spanners.

A classic tool for constructing spanners in normed and doubling spaces is WSPD (Well Separated Pair Decomposition, see [22, 46, 70]). Given a set of points $P$, a WSPD is a set of pairs $\{(A_i, B_i)\}_i$ of subsets of $P$, where the diameters of $A_i$ and $B_i$
are at most an $\epsilon$-fraction of $d(A_i, B_i),$ and such that for every pair $x, y \in P$ there is some $i$ with $(x, y) \in A_i \times B_i.$ A WSPD is designed to create a $(1 + O(\epsilon))$-spanner, by adding an arbitrary edge between a point in $A_i$ and a point in $B_i$ for every $i$ (as opposed to our construction, based on stochastic decompositions, in which we added only inner-cluster edges). An exponential dependence on the dimension is unavoidable with such a low stretch, thus it is not clear whether one can use a WSPD to obtain very sparse or light spanners in high dimensions.

### 1.3 Our Results

Our main result is exhibiting a connection between stochastic decompositions of metric spaces, and light spanners. Specifically, we show that if an $n$-point metric is $(t, \delta)$-decomposable, then for any constant $\epsilon > 0$, it admits a $(2 + \epsilon) \cdot t$-spanner with $\tilde{O}(n^{\epsilon})$ edges and lightness $\tilde{O}(1/\delta)$. \(^5\) (Abusing notation, $\tilde{O}$ hides polylog$(n)$ factors.)

It can be shown that Euclidean metrics are $(t, n^{-O(1/t^2)})$-decomposable, thus our results extend \(^\[45\] \) by providing a smaller stretch $(2 + \epsilon) \cdot t$-spanner, which is both sparse—with $\tilde{O}(n^{1+O(1/t^2)})$ edges—and has lightness $\tilde{O}(n^{1+O(1/t^2)})$. For $d$-dimensional Euclidean space, where $d = o(\log n)$ we can obtain $\tilde{O}(n \cdot 2^{O(1/t^2)})$ edges and lightness $\tilde{O}(2^{O(1/t^2)})$. We also show that $n$-point subsets of $\ell_p$ spaces for any fixed $1 < p < 2$ are $(t, n^{-O(\log^2 1/p)})$-decomposable, which yields light spanners for such metrics as well.

In addition, metrics with doubling constant $\lambda$ are $(t, \lambda^{-O(1/t)})$-decomposable \(^\[1, 41\] \) which enables us to alleviate the exponential dependence on $d$dim in the sparsity/lightness by increasing the stretch. Two additional interesting families are graphs with genus $g$ which are $(t, g^{-O(1/t)})$-decomposable \(^\[7, 55\] \), and graph with treewidth $k$ which are $(t, (k \cdot \log n)^{-O(1/t)})$-decomposable \(^\[7\] \). Here as well we can increase the stretch and avoid exponential dependence on the family parameter. See Table 1 for more details. (We remark that for graphs excluding $K_r$ as a minor, the current best decomposition achieves probability only $2^{-O(t/r)}$ \(^\[7\] \) (see also \(^\[32\] \)); if this can be improved to the conjectured $r^{-O(1/t)}$, then our results would provide interesting spanners for this family as well.)

One can view our result as an extension of light spanners results for decomposable metrics. In \(^\[19, 25, 30\] \) it was shown that any $n$-point metric (or graph) admits an $O(t)$-spanner with lightness $O(n^{1/t})$. In this work, for a $(t, \delta)$-decomposable metric, we achieve a similar result, replacing the dependence on number of points $n$, by a decomposability parameter $\nu = \delta^{-1}$ (up to polylog factors).

For example, consider an $n$-point metric with doubling constant $\lambda = 2^{\sqrt{\log n}}$. No spanner with stretch $o(\log n/\log \log n)$ and lightness $\tilde{O}(1)$ for such a metric was known. Our result implies such a spanner, with stretch $O(\sqrt{\log n})$.

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\(^4\) $d(A_i, B_i) = \max\{d(x, y) \mid x \in A_i, y \in B_i\}$ is the maximum pairwise distance between $A_i$ to $B_i$.

\(^5\) Note that there is no explicit dependence between the stretch parameter $t$, and the sparsity/lightness. Nonetheless, these dependence is implicit in the decomposition parameters, as for smaller stretch $t$, the inclusion parameter $\delta$ becomes smaller as well, and thus the sparsity/lightness grow.

\(^6\) Originally this fact was observed by James R. Lee and Anastasios Sidiropoulos. A proof sketch could be found in the full version of \(^\[7\] \).
We also remark that the existence of light spanners does not imply decomposability. For example, consider the shortest path metrics induced by bounded-degree expander graphs. Even though these metrics have the (asymptotically) worst possible decomposability parameters (they are only \(t, n^{-\Omega(1/\ell)}\))-decomposable \([51]\), they nevertheless admit 1-spanners with constant lightness (the spanner being the expander graph itself).

## 2 Preliminaries

Given a metric space \((X, d_X)\), we will treat it as a complete weighted graph over \(X\), where the weight of the edge \([u, v]\) is simply \(d_X(u, v)\). Let \(T\) denote its minimum spanning tree (MST) of weight \(L\). For a set \(A \subseteq X\), the diameter of \(A\) is \(\operatorname{diam}(A) = \max_{x, y \in A} d_X(x, y)\). We will assume that the minimal distance in \(X\) is 1. Due to scaling, this is without loss of generality.

By \(O_\varepsilon\) we denote asymptotic notation which hides polynomial factors of \(\frac{1}{\varepsilon}\), that is \(O_\varepsilon(f) = \Theta(f)/e^{O(1)}\). Unless explicitly specified otherwise, all logarithms are in base 2.

### Nets

For \(r > 0\), a set \(N \subseteq X\) is an \(r\)-net, if (1) for every \(x \in X\) there is a point \(y \in N\) with \(d_X(x, y) \leq r\), and (2) every pair of net points \(y, z \in N\) satisfy \(d_X(y, z) > r\). It is well known that nets can be constructed in a greedy manner. For \(0 < r_1 \leq r_2 \leq \cdots \leq r_s\), a hierarchical net is a collection of nested sets \(X \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_s\), where each \(N_i\) is an \(r_i\)-net. Since \(N_{i+1}\) satisfies the second condition of a net with respect to radius \(r_i\), one can obtain \(N_i\) from \(N_{i+1}\) by greedily adding points until the first condition is satisfied as well.

In the following claim we argue that nets are sparse sets with respect to the MST weight.

**Claim 1** Consider a metric space \((X, d_X)\) with MST of weight \(L\), let \(N\) be an \(r\)-net, then \(|N| \leq \frac{2L}{r}\).
Let \( T \) be the MST of \( X \), note that for every \( x, y \in N \), \( d_T(x, y) \geq d_X(x, y) > r \). For a point \( x \in N \), \( B_T(x, b) = \{ y \in X \mid d_T(x, y) \leq b \} \) is the ball of radius \( b \) around \( x \) in the metric of \( T \). We say that an edge \( \{ y, z \} \) of \( T \) is cut by the ball \( B_T(x, b) \) if \( d_T(x, y) < b < d_T(x, z) \). Consider the set \( \mathcal{B} \) of balls of radius \( \frac{r}{2} \) around the points of \( N \). We can subdivide the edges of \( T \) until no edge is cut by any of the balls of \( \mathcal{B} \), see Fig. 1 for an illustration. Note that the subdivisions do not change the total weight of \( T \) nor the distances between the original points of \( X \).

If both the endpoints of an edge \( e \) belong to the ball \( B \), we say that the edge \( e \) is internal to \( B \). By the second property of nets, and since \( B_T(x, b) \subseteq B_X(x, b) \), the set of internal edges corresponding to the balls \( B \) are disjoint. On the other hand, as the tree is connected, the weight of the internal edges in each ball must be at least \( \frac{r}{2} \). This is because there must be a path \( P \) for the center vertex \( c \) to some vertex not in the ball. In particular, as no edge is cut, there is a sub-path of \( P \) fully contained in the ball of weight at least \( \frac{r}{2} \). As this balls are disjoint, the weight of all these internal edges is at least \( |N| \cdot \frac{r}{2} \). As the total weight is bounded by \( L \), it follows that \( |N| \leq L \cdot \frac{2}{r} \).

\[ \square \]

**Stochastic Decompositions** Consider a partition \( \mathcal{P} \) of \( X \) into disjoint clusters. For \( x \in X \), we denote by \( \mathcal{P}(x) \) the cluster \( P \in \mathcal{P} \) that contains \( x \). A partition \( \mathcal{P} \) is \( \Delta \)-bounded if for every \( P \in \mathcal{P} \), \( \text{diam}(P) \leq \Delta \). If a pair of points \( x, y \) belong to the same cluster, i.e. \( \mathcal{P}(x) = \mathcal{P}(y) \), we say that they are clustered together by \( \mathcal{P} \). Given a distribution \( \mathcal{D} \), \( \text{supp}(\mathcal{D}) \) denotes the support of the distribution.

Intuitively, a stochastic decomposition is a random partition of the points of a metric space into diameter bounded clusters, such that close-by points are likely to be clustered together. In our algorithm, we will create many stochastic decompositions so to insure that each nearby pair of vertices will be clustered together in one of them. Our spanner will be constructed by adding a “star” for each such cluster.

**Definition 1** For metric space \( (X, d_X) \) and parameters \( t \geq 1, \Delta > 0 \) and \( \delta \in [0, 1] \), a distribution \( \mathcal{D} \) over partitions of \( X \) is called a \((t, \Delta, \delta)\)-decomposition, if it fulfills the following properties.

- Every \( \mathcal{P} \in \text{supp}(\mathcal{D}) \) is \( t \cdot \Delta \)-bounded.

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![Fig. 1](image-url) An edge \( \{y, z\} \) is cut by the ball \( B_T(x, b) \) if \( d_T(x, y) < b < d_T(x, z) \). The edge that being cut are depicted in orange (on the left). To subdivide an edge \( e = \{x, y\} \) of weight \( w \) the following steps are taken:

1. Delete the edge \( e \).
2. Add a new vertex \( v_e \).
3. Add two new edges \( \{x, v_e\}, \{v_e, y\} \) with weights \( \alpha \cdot w \) and \((1 - \alpha) \cdot w \) for some \( \alpha \in (0, 1) \).

On the right we subdivide the edges, so that no edge is cut (Color figure online)
• For every $x, y \in X$ such that $d_X(x, y) \leq \Delta$, $\Pr_D[\mathcal{P}(x) = \mathcal{P}(y)] \geq \delta$.

A metric is $(t, \delta)$-decomposable, where $\delta = \delta(|X|, t)$, if it admits a $(t, \Delta, \delta)$-decomposition for any $\Delta > 0$. A family of metrics is $(t, \delta)$-decomposable if each member $(X, d_X)$ in the family is $(t, \delta)$-decomposable.

We observe that if a metric $(X, d_X)$ is $(t, \delta(|X|, t))$-decomposable, then also every sub-metric $Y \subseteq X$ is $(t, \delta(|X|, t))$-decomposable. In some cases $Y$ is also $(t, \delta(|Y|, t))$-decomposable (we will exploit these improved decompositions for subsets of $\ell_p$). The following claim argues that sampling $O(\log n/\delta)$ partitions suffices to guarantee that every pair is clustered at least once.

Claim 2 Let $(X, d_X)$ be a metric space which admits a $(t, \Delta, \delta)$-decomposition, and let $N \subseteq X$ be of size $|N| = n$. Then there is a set $\{\mathcal{P}_1, \ldots, \mathcal{P}_\varphi\}$ of $t \cdot \Delta$-bounded partitions of $N$, where $\varphi = \frac{2\ln n}{\delta}$, such that every pair $x, y \in N$ at distance at most $\Delta$ is clustered together by at least one of the $\mathcal{P}_i$.

Proof Let $\{\mathcal{P}_1, \ldots, \mathcal{P}_\varphi\}$ be i.i.d partitions drawn from the $(t, \Delta, \delta)$-decomposition of $X$. Consider a pair $x, y \in N$ at distance at most $\Delta$. The probability that $x, y$ are not clustered in any of the partitions is bounded by

\[
\Pr[\forall i, \ \mathcal{P}_i(x) \neq \mathcal{P}_i(y)] \leq (1 - \delta)^{\frac{2\ln n}{\delta}} \leq \frac{1}{n^2}.
\]

The claim now follows by the union bound.

3 Light Spanner Construction

In this section we present a generalized version of the algorithm of [45], depicted in Algorithm 1. The differences in execution and analysis are: (1) Our construction applies to general decomposable metric spaces—we use decompositions rather than LSH schemes. (2) We analyze the lightness of the resulting spanners. (3) We achieve stretch $t \cdot (2 + \epsilon)$ rather than $O(t)$.

The basic idea is as follows. For every weight scale $\Delta_i = (1 + \epsilon)^i$, construct a sequence of $t \cdot \Delta_i$-bounded partitions $\mathcal{P}_1, \ldots, \mathcal{P}_\varphi$ such that every pair $x, y$ at distance $\leq \Delta_i$ will be clustered together at least once. Then, for each $j \in [\varphi]$ and every cluster $P \in \mathcal{P}_j$, we pick an arbitrary root vertex $v_P \in P$, and add to our spanner edges from $v_P$ to all the points in $P$. This ensures stretch $2t \cdot (1 + \epsilon)$ for all pairs with $d_X(x, y) \in [(1 - \epsilon)\Delta_i, \Delta_i]$. Thus, repeating this procedure on all scales $i = 1, 2, \ldots$ provides a spanner with stretch $2t \cdot (1 + \epsilon)$.

However, the weight of the spanner described above is unbounded. In order to address this problem at scale $\Delta_i$, instead of taking the partitions over all points, we partition only the points of an $\epsilon \Delta_i$-net. The stretch is still small: $x, y$ at distance $\Delta_i$ will have nearby net points $\hat{x}, \hat{y}$. Then, a combination of newly added edges with older ones will produce a short path between $x$ to $y$. The bound on the lightness will follow from the observation that the number of net points is bounded with respect to the MST weight.
Theorem 1 Let \((X, d_X)\) be a \((t, \delta)\)-decomposable \(n\)-point metric space. Then for every \(\epsilon \in (0, \frac{1}{8})\), there is a \(t \cdot (2 + \epsilon)\)-spanner for \(X\) with \(O_\epsilon\left(\frac{n}{8} \cdot \log n \cdot \log t\right)\) edges, and lightness \(O_\epsilon\left(\frac{2}{8} \cdot \log^2 n\right)\).

Algorithm 1 \(H = \text{Spanner-From-Decompositions}((X,d_X), t, \epsilon)\)

\begin{enumerate}
\item \textbf{input}: \((t, \delta)\)-decomposable \(n\)-point metric space \((X,d_X)\).
\item \textbf{output}: \((t, \delta)\)-decomposable \(n\)-point metric space \((X,d_X)\).
\end{enumerate}

1. Let \(N_0 \supseteq N_1 \supseteq \ldots \supseteq N_{\log 1+\epsilon} L\) be a hierarchical net, where \(N_i\) is \(\epsilon \cdot \Delta_i = \epsilon \cdot (1 + \epsilon)^i\)-net of \((X,d_X)\).

2. for \(i \in [0,1,\ldots,\log_1+\epsilon L]\) do
   
   // Recall that \(L\) is the weight of the MST, and that we assume that the minimal distance is 1.

   3. For parameters \(\Delta = (1+2\epsilon)\Delta_t\) and \(t\), let \(\mathcal{P}_1, \ldots, \mathcal{P}_{\psi_1}\) be the set of \(t \cdot \Delta\)-bounded partitions guaranteed by Claim 2 on the set \(N_i\).

   4. for \(j \in [1,\ldots,\psi_1]\) and \(P \in \mathcal{P}_j\) do

   5. Let \(v_P \in P\) be an arbitrarily point.

   6. Add to \(H\) an edge from every point \(x \in P\backslash\{v_P\}\) to \(v_P\).

   7. return \(H\)

Proof We will prove stretch \(t \cdot (2 + O(\epsilon))\) instead of \(t \cdot (2 + \epsilon)\). This is good enough, as post factum we can scale \(\epsilon\) accordingly.

Stretch Bound Let \(c > 1\) be a constant (to be determined later). Consider a pair \(x, y \in X\) such that \((1 + \epsilon)^{i-1} < d_X(x, y) \leq (1 + \epsilon)^i\). We will assume by induction that every pair \(x', y'\) at distance at most \((1 + \epsilon)^{i-1}\) already enjoys stretch at most \(\alpha = t \cdot (2 + c \cdot \epsilon)\) in \(H\). Set \(\Delta_i = (1 + \epsilon)^i\) and let \(\tilde{x}, \tilde{y} \in N_i\) be net points such that \(d_X(\tilde{x}, \tilde{y}) \leq \epsilon \cdot \Delta_i\). By the triangle inequality \(d_X(\tilde{x}, \tilde{y}) \leq (1 + 2\epsilon) \cdot \Delta_i = \Delta\). Therefore there is a \(t \cdot \Delta\)-bounded partition \(P\) constructed at round \(i\) such that \(P(\tilde{x}) = P(\tilde{y})\). In particular, there is a center vertex \(v = v_{P(\tilde{x})}\) such that both \([\tilde{x}, v]\) and \([\tilde{y}, v]\) were added to the spanner \(H\). Using the induction hypothesis on the pairs \([x, \tilde{x}]\) and \([y, \tilde{y}]\), we conclude

\[
\begin{align*}
d_H(\tilde{x}, \tilde{y}) &\leq d_H(x, \tilde{x}) + d_H(\tilde{x}, v) + d_H(v, \tilde{y}) + d_H(\tilde{y}, y) \\
&\leq \alpha \cdot \epsilon \cdot \Delta_i + (1 + 2\epsilon) t \cdot \Delta_i + (1 + 2\epsilon) t \cdot \Delta_i + \alpha \cdot \epsilon \cdot \Delta_i \\
&\leq \frac{\alpha}{1 + \epsilon} \cdot \Delta_i \leq \alpha \cdot d_X(\tilde{x}, \tilde{y}),
\end{align*}
\]

where the inequality \((\ast)\) follows as \(2(1 + 2\epsilon) t < \alpha(\frac{1}{1+\epsilon} - 2\epsilon)\) for large enough constant \(c\), using that \(\epsilon < \frac{1}{8}\).

Sparsity bound For a point \(x \in X\), let \(s_x\) be the maximal index such that \(x \in N_{s_x}\). Note that the number of edges in our spanner is not affected by the choice of “cluster centers” in line 5 in Algorithm 1. Therefore, the edge count will be still valid if we assume that \(v_P \in P\) is the vertex \(y\) with maximal value \(s_y\) among all vertices in \(P\). Consider an edge \([x, y]\) added during the \(i\)’s phase of the algorithm. Necessarily \(x, y \in N_i\), and \(x, y\) belong to the same cluster \(P\) of a partition \(\mathcal{P}_j\). W.l.o.g., \(y = v_P\).
in particular $s_x \leq s_y$. The edge $\{x, y\}$ will be charged upon $x$. Since the partitions at level $i$ are $t \cdot \Delta$ bounded, we have that $d_X(x, y) \leq t \cdot \Delta = t \cdot (1 + 2\epsilon) \cdot (1 + \epsilon)^i$. Hence, for $i'$ such that $\epsilon \cdot (1 + \epsilon)^i > t \cdot (1 + 2\epsilon) \cdot (1 + \epsilon)^i$, i.e. $i' > i + O_\epsilon(\log t)$, the points $x, y$ cannot both belong to $N_{i'}$. As $s_x \leq s_y$, it must be that $x \not\in N_{i'}$. We conclude that $x$ can be charged in at most $O_\epsilon(\log t)$ different levels. As in level $i$ each vertex is charged for at most $\varphi_i \leq O_\epsilon(\log n \delta)$ edges, the total charge for each vertex is bounded by $O_\epsilon(\log n \cdot \log t \delta)$.

**Lightness bound** Consider the scale $\Delta_i = (1 + \epsilon)^i$. As $N_i$ is an $\epsilon \cdot \Delta_i$-net, Claim 1 implies that $N_i$ has size $n_i \leq 2L \epsilon \cdot \Delta_i$, and in any case at most $n$. In that scale, we constructed $\varphi_i = \frac{2}{\delta} \log n_i \leq \frac{2}{\delta} \log n$ partitions, adding at most $n_i$ edges per partition. The weight of each edge added in this scale is bounded by $O_\epsilon(t \cdot \Delta_i)$.

Let $H_1$ consist of all the edges added in scales $i \in \{\log \frac{L}{n}, \ldots, \log \frac{L}{1+\epsilon}\}$, while $H_2$ consist of edges added in the lower scales. Note that $H = H_1 \cup H_2$.

$$w(H_1) \leq \sum_{i \in \{\log \frac{L}{1+\epsilon}, \ldots, \log \frac{L}{1+\epsilon}\}} O(t \cdot \Delta_i) \cdot n_i \cdot \varphi_i$$

$$= O\left(\frac{t}{\delta} \cdot \log n \cdot \sum_{i \in \{\log \frac{L}{1+\epsilon}, \ldots, \log \frac{L}{1+\epsilon}\}} \Delta_i \cdot \frac{L}{\epsilon \cdot \Delta_i}\right) = O_\epsilon\left(\frac{t}{\delta} \cdot \log^2 n\right) \cdot L.$$

$$w(H_2) \leq \sum_{\Delta_i \in \frac{L}{\delta} \cdot \{(1+\epsilon)^{-1}, (1+\epsilon)^{-2}, \ldots\}} O(t \cdot \Delta_i) \cdot n_i \cdot \varphi_i$$

$$= O\left(\frac{t}{\delta} \cdot \log n \cdot \sum_{i \geq 1} \frac{1}{(1+\epsilon)^i}\right) \cdot L = O_\epsilon\left(\frac{t}{\delta} \cdot \log n\right) \cdot L.$$

The bound on the lightness follows. 

**4 Corollaries and Extensions**

In this section we describe some corollaries of Theorem 1 for certain metric spaces, and show some extensions, such as improved lightness bound for normed spaces, and discuss graph spanners.

### 4.1 High Dimensional Normed Spaces

Here we consider the case that the given metric space $(X, d)$ satisfies that every sub-metric $Y \subseteq X$ of size $|Y| = n$ is $(t, \delta)$-decomposable for $\delta = n^{-\beta}$, where $\beta = \beta(t) \in (0, 1)$ is a function of $t$. In such a case we are able to shave a log $n$ factor in the lightness.
Theorem 2 Let \((X, d_X)\) be an \(n\)-point metric space such that every \(Y \subseteq X\) is \((t, |Y|^{-\beta})\)-decomposable. Then for every \(\epsilon \in (0, \frac{1}{8})\), there is a \(t \cdot (2 + \epsilon)\)-spanner for \(X\) with \(O_\epsilon \left( n^{1+\beta} \cdot \log n \cdot \log t \right)\) edges, and lightness \(O_\epsilon \left( \frac{t}{\beta} \cdot n^\beta \cdot \log n \right)\).

Proof Using the same Algorithm 1, the analysis of the stretch and sparsity from Theorem 1 is still valid, since the number partitions taken in each scale is smaller than in Theorem 1. Recall that in scale \(i\) we set \(\Delta_i = \left(1 + \frac{\epsilon}{\log_1 \beta} \right)^i\), and the size of the \(\epsilon \cdot \Delta_i\)-net \(N_i\) is \(n_i \leq \max\{\frac{2L}{\epsilon \Delta_i}, n\}\). The difference from the previous proof is that \(N_i\) is \((t, n_i^{-\beta})\)-decomposable, so the number of partitions taken is \(\varphi_i = O(n_i^\beta \log n_i)\). In each partition we might add at most one edge per net point, and the weight of this edge is \(O(\frac{t}{\epsilon \Delta_i})\).

We divide the edges of \(H\) to \(H_1\) and \(H_2\), and bound the weight of \(H_2\) as above (using that \(n_i \leq n\)). For \(H_1\) we get,

\[
\begin{align*}
  w(H_1) &\leq \sum_{i \in \left\{ \log_1 \epsilon, \ldots, \log_1 \epsilon L \right\}} O(t \cdot \Delta_i) \cdot n_i \cdot \varphi_i \\
  &= O\left(t \cdot \sum_{i \in \left\{ \log_1 \epsilon, \ldots, \log_1 \epsilon L \right\}} \Delta_i \cdot \frac{L}{\epsilon \cdot \Delta_i} \cdot \left( \frac{L}{\epsilon \cdot \Delta_i} \right)^\beta \log \frac{L}{\epsilon \cdot \Delta_i} \right) \\
  &= O_\epsilon \left(t \cdot \sum_{i \in \left\{ \log_1 \epsilon, \ldots, \log_1 \epsilon L \right\}} \left( \frac{L}{\Delta_i} \right)^\beta \cdot \log \frac{L}{\Delta_i} \right) \cdot L \\
  &= O_\epsilon \left(t \cdot \sum_{i \in \left\{ 0, \ldots, \log_1 \epsilon n \right\}} (i + 1) \cdot \left( (1 + \epsilon)^\beta \right)^i \right) \cdot L.
\end{align*}
\]

Set the function \(f(x) = \sum_{i=0}^k (i + 1) \cdot x^i\), on the domain \((1, \infty)\), with parameter \(k = \log_1 \epsilon n\). Then,

\[
\begin{align*}
f(x) &= \left( \int f(x) \, dx \right)' = \left( \sum_{i=0}^k x^{i+1} \right)' = \left( \frac{x^{k+2} - x}{x - 1} \right)' \\
  &= \frac{(k + 2) x^{k+1} - 1}{(x - 1)^2} \cdot (x - 1) - \frac{x^{k+2} - x}{x - 1} \leq \frac{(k + 2) x^{k+1}}{x - 1}.
\end{align*}
\]

Hence,

\[
\begin{align*}
w(H_1) &= O_\epsilon \left(t \cdot f \left( (1 + \epsilon)^\beta \right) \right) \cdot L \\
  &= O_\epsilon \left(t \cdot \frac{\log_1 \epsilon n \cdot (1 + \epsilon)^\beta \cdot \log_1 \epsilon n}{(1 + \epsilon)^\beta - 1} \right) \cdot L = O_\epsilon \left( \frac{t}{\beta} \cdot n^\beta \cdot \log n \right) \cdot L.
\end{align*}
\]
We conclude that the lightness of $H$ is bounded by $O_\epsilon \left( \frac{1}{\beta} \cdot n^\beta \cdot \log n \right)$. \hfill \Box

In Sect. 5 we will show that any $n$-point Euclidean metric is $(t, n^{-O(1/\beta)})$-decomposable, and that for fixed $p \in (1, 2)$, any $n$-point subset of $\ell_p$ is $(t, n^{-O(\log^2 1/\beta)})$-decomposable. The following corollaries are implied by Theorem 2 (rescaling $t$ by a constant factor allows us to remove the $O(\cdot)$ term in the exponent of $n$, while obtaining stretch $O(t)$).

**Corollary 3** For a set $X$ of $n$ points in Euclidean space, $t > 1$, there is an $O(t)$-spanner with $O(n^{1+4/p} \cdot \log n \cdot \log t)$ edges, and lightness $O(1/2 \cdot n^{4/p} \cdot \log n)$.

**Corollary 4** For a constant $p \in (1, 2)$ and a set $X$ of $n$ points in $\ell_p$ space, there is an $O(t)$-spanner with and $O(n^{1+4/p} \cdot \log n \cdot \log t)$ edges, and lightness $O(1/2 \cdot n^{4/p} \cdot \log n)$.

**Remark 1** Corollary 3 applies for a set of points $X \subseteq \mathbb{R}^d$, where the dimension $d$ is arbitrarily large. If $d = o(\log n)$ we can obtain improved spanners. Specifically, $n$-point subsets of $d$-dimensional Euclidean space are $(O(t), 2^{-d})$-decomposable (see Sect. 6). Applying Theorem 1 we obtain an $O(t)$-spanner with $O_\epsilon \left(n \cdot 2^{4/p} \cdot \log n \cdot \log t\right)$ edges, and lightness $O_\epsilon \left(t \cdot 2^{4/p} \cdot \log^2 n\right)$.

### 4.2 Doubling Metrics

It was shown in [1] that metrics with doubling constant $\lambda$ are $(t, \lambda^{-O(1/\beta)})$-decomposable (the case $t = \Theta(\log \lambda)$ was given by [41]). Therefore, Theorem 1 implies:

**Corollary 5** For every metric space $(X, d_X)$ with doubling constant $\lambda$, and $t \geq 1$, there exist an $O(t)$-spanner with $O(n \cdot \lambda^{1/\beta} \cdot \log n \cdot \log t)$ edges, and lightness $O(t \cdot \log^2 n \cdot \lambda^{1/\beta})$.

### 4.3 Graph Spanners

In the case where the input is a graph $G$, it is natural to require that the spanner will be a graph-spanner, i.e., a subgraph of $G$. Given a (metric) spanner $H$, one can define a graph-spanner $H'$ by replacing every edge $\{x, y\} \in H$ with the shortest path from $x$ to $y$ in $G$. It is straightforward to verify that the stretch and lightness of $H'$ are no larger than those of $H$ (however, the number of edges may increase).

Consider a graph $G$ with genus $g$. In [7] it was shown that (the shortest path metric of) $G$ is $(t, g^{-O(1/\beta)})$-decomposable. Furthermore, graphs with genus $g$ have $O(n + g)$ edges [44], so any graph-spanner will have at most so many edges. By Theorem 1 we have:

**Corollary 6** Let $G$ be a weighted graph on $n$ vertices with genus $g$. Given a parameter $t \geq 1$, there exist an $O(t)$-graph-spanner of $G$ with $O(n + g)$ edges, and lightness $O(t \cdot \log^2 n \cdot g^{1/\beta})$. 
For general graphs, the transformation to graph-spanners described above may arbitrarily increase the number of edges (in fact, it will be bounded by $O(\sqrt{|E_H|} \cdot n)$, [20]). Nevertheless, if we have a strong-decomposition, we can modify Algorithm 1 to produce a sparse spanner. In a graph $G = (X, E)$, the strong-diameter of a cluster $A \subseteq X$ is $\max_{u,v \in A} d_{G[A]}(u, v)$, where $G[A]$ is the induced graph by $A$ (as opposed to weak diameter, which is computed w.r.t the original metric distances). A partition $\mathcal{P}$ of $X$ is $\Delta$-strongly-bounded if the strong diameter of every $P \in \mathcal{P}$ is at most $\Delta$. A distribution $\mathcal{D}$ over partitions of $X$ is $(t, \Delta, \delta)$-strong-decomposable, if it is $(t, \Delta, \delta)$-decomposition and in addition every partition $P \in \text{supp}(\mathcal{D})$ is $\Delta$-strongly-bounded. A graph $G$ is $(t, \delta)$-strongly-decomposable, if for every $\Delta > 0$, the graph admits a $(\Delta, t \cdot \Delta, \delta)$-strong-decomposition.

**Theorem 3** Let $G = (V, E, w)$ be a $(t, \delta)$-strongly-decomposable, $n$-vertex graph with aspect ratio $\Lambda = \frac{\max_{e \in E} w(e)}{\min_{e \in E} w(e)}$. Then for every $\epsilon \in (0, 1/\delta)$, there is a $t \cdot (2 + \epsilon)$-graph-spanner for $G$ with $O_\epsilon(n^{1-\epsilon} \cdot \log n \cdot \log \Lambda)$ edges, and lightness $O_\epsilon \left(\frac{\epsilon^2}{\delta} \cdot \log^2 n\right)$.

**Proof** We will execute Algorithm 1 with several modifications:

1. The for loop (in Line 2) will go over scales $i \in \{0, \ldots, \log_{1+\epsilon} \Lambda\}$ (instead $\{0, \ldots, \log_{1+\epsilon} L\}$).
2. We will use strong-decompositions instead of regular (weak) decompositions.
3. The partitions created in Line 3 will be over the set of all vertices $X$, rather then only net points $N_i$ (as otherwise it will be impossible to get strong diameter).
   However, the requirement from close pairs to be clustered together (at least once), is still applied to net points only. Similarly to Claim 2, $\varphi_i = (2 \ln n_i) / \delta$ repetitions will suffice.
4. In Line 6, we will no longer add edges from $v_P$ to all the net points in $P \in \mathcal{P}_j$.
   Instead, for every net point $x \in P \cap N_i$, we will add a shortest path in $G[P]$ from $v_P$ to $x$. Note that all the edges added in all the clusters constitute a forest. Thus we add at most $n$ edges per partition.

We now prove the stretch, sparsity and lightness of the resulting spanner.

**Stretch** By the triangle inequality, it is enough to show small stretch guarantee only for edges (that is, only for $x, y \in V$ s.t. $\{x, y\} \in E$.) As we assumed that the minimal distance is 1, all the weights are within $[1, \Lambda]$. In particular, every edge $\{x, y\} \in E$ has weight $(1 + \epsilon)^i - 1 < w \leq (1 + \epsilon)^i$ for $i \in \{0, \ldots, \log_{1+\epsilon} \Lambda\}$. The rest of the analysis is similar to Theorem 1, with the only difference being that we use a path from $v_P$ to $\tilde{x}$ rather than the edge $\{\tilde{x}, v_P\}$. This is fine since we only require that the length of this path is at most $(t \cdot (1 + 2\epsilon) \cdot \Delta)$, which is guaranteed by the strong diameter of clusters.

**Sparsity** We have $O_\epsilon(\log \Lambda)$ scales. In each scale we had at most $\varphi_i \leq \frac{2}{\delta} \log n$ partitions, where for each partition we added at most $n$ edges. The bound on the sparsity follows.

**Lightness** Consider scale $i$. We have $n_i$ net points. For each net point we added at most one shortest path of weight at most $O(t \cdot \Delta_i)$ (as each cluster is $O(t \cdot \Delta_i)$ strongly bounded). As the number of partitions is $\varphi_i$, the total weight of all edges added at scale
is bounded by $O(t \cdot \Delta_i) \cdot n_i \cdot \varphi_i$. The rest of the analysis follows by similar lines to Theorem 1 (noting that $\Delta < L$).

Next we consider graph families which are closed under edge contractions. Given a weighted graph $G = (X, E, w)$, contracting an edge $\{u, v\} \in E$ creates a new graph $G'$ where the vertices $v, u$ are replaced by a new vertex $w$, and every edge $\{v, x\}$ or $\{u, x\}$ is replaced by an edge $\{w, x\}$ of the same weight. If duplicates are created, we keep only the edge of smaller weight. A graph family $\mathcal{F}$ is closer under edge contractions, if for every graph $G \in \mathcal{F}$ and edge $e \in G$, the graph created by contracting $e$ in $G$ also belongs to $\mathcal{F}$. Examples of such families being graph excluding a fixed minor, bounded treewidth graph etc. For such families we are able to remove the dependency on the aspect ration in the sparsity.

**Theorem 4** Let $\mathcal{F}$ be a graph family closed under contractions, such that every $G \in \mathcal{F}$ is $(t, \delta)$-strongly-decomposable. Then for every $n$-vertex graph $G \in \mathcal{F}$ and $\epsilon \in (0, 1/8)$, there is a $t \cdot (2 + \epsilon)$-graph-spanner with $O_\epsilon(n^{3} \cdot \log^{2} n)$ edges, and lightness $O_\epsilon(n^{3} \cdot \log^{2} n)$.

**Proof** In Algorithm 2 we describe a modified algorithm for the families closed under contraction case. We assume here that there is a unique shortest path between every pair of vertices. If this is not the case, we can introduce negligible perturbations on the weights to achieve such a state.

**Algorithm 2**

$H = \text{Graph-Spanner-From-Decompositions}((X, E, w), t, \epsilon)$

| input | $(t, \delta)$-decomposable $n$-point weighted graph $G$. |
|-------|----------------------------------------------------------|
| output| $t \cdot (2 + \epsilon)$-spanner $H$. |

1. Let $N_0 \supseteq N_1 \supseteq \cdots \supseteq N_{\log_2 L}$ be a hierarchical net, where $N_i$ is $\epsilon \cdot \Delta_i = \epsilon \cdot (1 + \epsilon)^i$-net of $X$.
2. For a vertex $v \in V$, let $s_v$ be the maximal index such that $v \in N_{s_v}$.
3. Let $v_1, \ldots, v_n$ be an order of the vertices such that $i \geq j \Rightarrow s_{v_i} \geq s_{v_j}$.
4. For $i \in \{0, 1, \ldots, \log_2 L \}$ do
   
   // Recall that $L$ is the weight of the MST, and we assume that the minimal distance is 1.
   
   Contract all edges of weight below $\frac{t \cdot \Delta_i}{n^2}$. Associate each super node $\hat{x}$ with the vertex $v_i \in \hat{x}$ with maximal index. Denote the new graph by $\hat{G}_i$.
   
   For parameters $\Delta = (1 + 2\epsilon)\Delta_i$ and $t$, let $\mathcal{P}_1, \ldots, \mathcal{P}_{\varphi_i}$ be the set of $t \cdot \Delta$-strongly-bounded partitions guaranteed by Claim 2 on the set $N_i$ in $\hat{G}_i$.
5. For $j \in \{1, \ldots, \varphi_i\}$ and $P \in \mathcal{P}_j$ do
   
   Let $\hat{v}_P \in P$ be the point with highest index in $P$.
   
   Add to $H$ the shortest path (in $P \subseteq \hat{G}_i$) from every point $x \in P \cap N_i \setminus \{\hat{v}_P\}$ to $\hat{v}_P$ (i.e. for every edge $[\hat{x}, \hat{y}] \in \hat{G}_i$ in a shortest path we want to add, add some edge $[x, y] \in G$ such that $x \in \hat{x}$ and $y \in \hat{y}$).

return $H$.

---

We now prove the stretch, sparsity and lightness of the resulting spanner.

**Stretch** Consider a pair $x, y \in X$ such that $(1 + \epsilon)^{i-1} < d_G(x, y) \leq (1 + \epsilon)^i$. Similarly to the proof of Theorem 1, we assume by induction that every pair $x', y'$ at distance at
most \((1+\epsilon)^{i-1}\) already enjoys stretch at most \(\alpha = t \cdot (2+c \cdot \epsilon)\) in \(H\) (for \(c > 1\) a constant to be determined later). Set \(\Delta_i = (1 + \epsilon)^i\), and let \(\hat{x}, \hat{y} \in N_I\) be net points such that \(d_G(x, \hat{x}), d_G(y, \hat{y}) \leq \epsilon \cdot \Delta_i\). By the triangle inequality \(d_G(\hat{x}, \hat{y}) \leq (1 + 2\epsilon) \cdot \Delta_i = \Delta\).

Let \(\hat{x}, \hat{y} \in \hat{G}_i\) be the nodes containing \(x, y\), respectively. Note that contractions might only decrease distances, therefore \(d_{\hat{G}_i}(\hat{x}, \hat{y}) \leq d_G(\hat{x}, \hat{y}) \leq \Delta\). In particular, there is a \(t \cdot \Delta\)-bounded partition \(P\) constructed for scale \(i\) such that \(P(\hat{x}) = \hat{P}(\hat{y})\).

There is a center vertex \(\hat{v}_j\) such that we added to \(H\) paths (in \(\hat{G}_i\)) from both \(\hat{x}, \hat{y}\) to \(\hat{v}_j\). More formally, there is a path \(\hat{x} = \hat{v}_0\), \(\hat{v}_1\), \ldots, \(\hat{v}_k = \hat{y}\) in \(\hat{G}_i\) of weight at most \(2 \cdot t \cdot \Delta\), such that for every \(0 \leq j \leq k - 1\) we added an edge from \(v_j^2 \in \hat{v}_j\) to \(v_{j+1}^2\) in \(\hat{v}_j\) to the spanner \(H\). In particular, \(\sum_{j=0}^{k-1} w\left(\{v_j^2, v_{j+1}^2\}\right) \leq 2 \cdot t \cdot \Delta\). Set \(v_0^1 = \hat{x}\) and \(v_2^1 = \hat{y}\).

For every\( j\), as both \(v_j^1, v_j^2\) belong to \(\hat{v}_j\), a node that created by contracting at most \(n - 1\) edges of weight at most \(\epsilon \cdot \Delta_i\), we have \(d_G(v_j^1, v_j^2) \leq \epsilon \cdot \Delta_i < (1 + \epsilon)^{i-1}\). By our induction hypothesis, \(d_H(v_j^1, v_j^2) \leq \alpha \cdot \epsilon \cdot \Delta_i\). Summing over all these paths we get

\[
d_H(\hat{x}, \hat{y}) \leq \sum_{j=0}^{k-1} d_H\left(v_j^1, v_j^2\right) + \sum_{j=0}^{k-1} d_H\left(v_j^2, v_{j+1}^1\right) \\
\leq \alpha \cdot \epsilon \cdot \Delta_i + 2 \cdot t \cdot \Delta.
\]

Using the induction hypothesis on the pairs \(\{x, \hat{x}\}\) and \(\{y, \hat{y}\}\), we conclude

\[
d_H(x, y) \leq d_H(\hat{x}, \hat{y}) + d_H(\hat{x}, \hat{y}) + d_H(\hat{y}, y) \\
\leq \alpha \cdot \epsilon \cdot \Delta_i + \alpha \cdot \epsilon \cdot \Delta_i + 2 \cdot t \cdot \Delta + \alpha \cdot \epsilon \Delta_i \\
\leq \frac{\alpha}{1 + \epsilon} \cdot \Delta_i \leq \alpha \cdot d_G(x, y),
\]

where the inequality (*) holds for large enough constant \(c\), using that \(\epsilon < \frac{1}{8}\).

**Sparsity** For a vertex \(v_s\), denote by \(D(v_s) = \min_{x' > s} d_G(v_s, v_{x'})\) the distance from \(v_s\) to the closest vertex of higher index (\(D(v_n) = \infty\)). We say that \(v_s\) is active at scale \(i\) if \(\frac{\epsilon \cdot \Delta_i}{n^2} < D(v_s) \leq 2 \cdot t \cdot \Delta_i\). As \(D(v_s)\) is fixed, each vertex \(v_s\) is active in at most \(\log_{1 + \epsilon} \frac{2t \cdot n^2}{\epsilon} = O(\epsilon \log n)\) scales. We denote by \(a_i\) the number of active vertices at scale \(i\).

Consider \(\hat{G}_i\), we will abuse notation and denote each node \(\hat{x} \in \hat{G}_i\) by \(v_s \in \hat{x}\), the vertex with highest index in \(\hat{x}\). Note that for every \(v_s \in \hat{G}\), necessarily \(D(v_s) \geq \frac{\epsilon \cdot \Delta_i}{n^2}\). As otherwise \(v_s\) would’ve been contracted with a vertex of higher index). Next, consider a partition \(P_j\) of \(\hat{G}_i\) drawn at scale \(i\). For each cluster \(P \in P_j\), \(\hat{v}_P \in P\) is the vertex with highest index in \(P\). For every node \(v_s \in P \setminus \hat{v}_P\), \(d_G(\hat{v}_P, v_s) \leq t \cdot (1 + 2\epsilon) \cdot \Delta_i\).

In other words, there is a path \(v_s = \hat{v}_0, \hat{v}_1, \ldots, \hat{v}_k = \hat{v}_P\) in \(\hat{G}_i\) such that for every \(0 \leq j \leq k - 1\), there is an edge \(\{v_j^2, v_{j+1}^2\} \in G\) where \(v_j^2 \in \hat{v}_j\) and \(v_{j+1}^2 \in \hat{v}_{j+1}\) such that \(\sum_{j=0}^{k-1} w\left(\{v_j^2, v_{j+1}^2\}\right) \leq t \cdot (1 + 2\epsilon) \cdot \Delta_i\). Set \(v_0^1 = v_s\) and \(v_k^2 = \hat{v}_P\). For every \(j\), as both \(v_j^1, v_j^2\) belong to \(\hat{v}_j\), a node that created by contracting at most \(n - 1\) edges of
weight at most \( \frac{\epsilon \Delta_i}{n^2} \), we have \( d_G(v_j^1, v_j^2) \leq \frac{\epsilon \Delta_i}{n} \). Therefore

\[
d_G(v_x, v^P) \leq \sum_{j=0}^{k} d_G(v_j^1, v_j^2) + \sum_{j=0}^{k-1} d_G(v_j^2, v_j^{i+1}) \\
\leq \epsilon \cdot \Delta_i + t \cdot (1 + 2\epsilon) \cdot \Delta_i \leq 2 \cdot t \cdot \Delta_i.
\]

In particular, \( D(v_x) \leq 2 \cdot t \cdot \Delta_i \). We conclude that \( P \) includes at least \(|P| - 1\) active vertices.

For each cluster \( P \), note that the \( \hat{G} \) edges added to \( H \) in Line 16 are all contained in the shortest path tree rooted in \( v_P \). In particular, we added to \( H \) at most \(|P| - 1\) edges due to \( P \). We conclude that the number of edges added to \( H \) for the partition \( \mathcal{P}_j \) is bounded by \( \sum_{P \in \mathcal{P}_j} |P| - 1 \leq a_i \) (as \( P \) contains at least \(|P| - 1\) active vertices). In particular, the total number of edges added in scale \( i \) is bounded by \( \varphi_i \cdot a_i = O(a_i \cdot \frac{\log n}{\delta}) \).

As each vertex \( v_x \) is active in at most \( O(\epsilon \log n) \) scales we conclude:

\[
|H| \leq \sum_{i=0}^{\log_{1+\epsilon} L} O \left( a_i \cdot \frac{\log n}{\delta} \right) = O \left( n \cdot \frac{\log^2 n}{\delta} \right).
\]

**Lightness** The same reasoning as in the proof of Theorem 3 works here as well (as the total weight of all edges added at scale \( i \) is bounded by \( O(t \cdot \Delta_i \cdot n \cdot \varphi_i) \)). \( \square \)

Consider the family of graphs with treewidth \( k \). In [7] (see footnote 6) it was shown that such graphs are \((t, (k \cdot \log n)^{-O(1/\delta)})\)-strongly-decomposable. By Theorem 4 we have:

**Corollary 7** Let \( G \) be a weighted graph on \( n \) vertices with treewidth \( k \). Given a parameter \( t \geq 1 \), there exist an \( O(t) \)-graph-spanner of \( G \) with \( O(n \cdot k^4 \cdot \log^{2+1/\delta} n) \) edges, and lightness \( O \left( k^{1/\delta} \cdot t \cdot \log^{2+1/\delta} n \right) \).

### 5 LSH Induced Decompositions

In this section, we prove that LSH (locality sensitive hashing) induces decompositions. In particular, using the LSH schemes of [9, 58], we will get decompositions for \( \ell_2 \) and \( \ell_p \) spaces, \( 1 < p < 2 \).

**Definition 2** (Locality-Sensitive-Hashing) Let \( H \) be a family of hash functions mapping a metric \( (X, d_X) \) to some universe \( U \). We say that \( H \) is \((r, cr, p_1, p_2)\)-sensitive if for every pair of points \( x, y \in X \), the following properties are satisfied:

1. If \( d_X(x, y) \leq r \) then \( \Pr_{h \in H} [h(x) = h(y)] \geq p_1 \).
2. If \( d_X(x, y) > cr \) then \( \Pr_{h \in H} [h(x) = h(y)] \leq p_2 \).

Given an LSH, its parameter is \( \gamma = \frac{\log^{1/p}}{\log^{1/p}} \). We will implicitly always assume that \( p_1 \geq n^{-\gamma} (n = |X|) \), as indeed will occur in all the discussed settings. Andoni and Indyk [9] showed that for Euclidean space \((\ell_2)\), and large enough \( t > 1 \), there is an
LSH with parameter $\gamma = O\left(\frac{1}{t^2}\right)$. Nguyen [58], showed that for constant $p \in (1, 2)$, and large enough $t > 1$, there is an LSH for $\ell_p$, with parameter $\gamma = O\left(\frac{\log t}{t^p}\right)$. We start with the following claim.

**Claim 8** Let $(X, d_X)$ be a metric space, such that for every $r > 0$, there is an $(r, t \cdot r, p_1, p_2)$-sensitive LSH family with parameter $\gamma$. Then there is an $(r, t \cdot r, n^{-O(\gamma)}, n^{-2})$-sensitive LSH family for $X$.

**Proof** Set $k = \left\lceil \log_{\frac{1}{p_2}} n^2 \right\rceil \leq \frac{O(\log n)}{\log \frac{1}{p_2}}$, and let $H$ be the promised $(r, t \cdot r, p_1, p_2)$-sensitive LSH family. We define an LSH family $H'$ as follows. In order to sample $h \in H'$, pick $h_1, \ldots, h_k$ uniformly and independently at random from $H$. The hash function $h$ is defined as the concatenation of $h_1, \ldots, h_k$. That is, $h(x) = (h_1(x), \ldots, h_k(x))$.

For $x, y \in X$ such that $d_X(x, y) \geq t \cdot r$ it holds that

$$\Pr[h(x) = h(y)] = \prod_i \Pr[h_i(x) = h_i(y)] \leq p_2^k \leq n^{-2}.$$ 

On the other hand, for $x, y \in X$ such that $d_X(x, y) \leq r$, it holds that

$$\Pr[h(x) = h(y)] = \prod_i \Pr[h_i(x) = h_i(y)] \geq p_1^k = 2^{\frac{-\log 1}{\log \frac{1}{p_2}} \cdot \frac{O(\log n)}{\log \frac{1}{p_2}}} = n^{-O(\gamma)}.$$

□

**Lemma 9** Let $(X, d_X)$ be a metric space, such that for every $r > 0$, there is a $(r, t \cdot r, p_1, p_2)$-sensitive LSH family with parameter $\gamma$. Then $(X, d_X)$ is $(t, n^{-O(\gamma)})$-decomposable.

**Proof** Let $H'$ be an $(r, t \cdot r, n^{-O(\gamma)}, n^{-2})$-sensitive LSH family, given by Claim 8. We will use $H'$ in order to construct a decomposition for $X$. Each hash function $h \in H'$ induces a partition $\mathcal{P}_h$, by clustering all points with the same hash value, i.e. $\mathcal{P}_h(x) = \mathcal{P}_h(y) \iff h(x) = h(y)$. However, in order to ensure that our partition will be $t \cdot r$-bounded, we modify it slightly. For $x \in X$, if there is a $y \in \mathcal{P}_h(x)$ with $d_X(x, y) > t \cdot r$, remove $x$ from $\mathcal{P}_h(x)$, and create a new cluster $\{x\}$. Denote by $\mathcal{P}'_h$ the resulting partition. $\mathcal{P}'_h$ is clearly $t \cdot r$-bounded, and we argue that every pair $x, y$ at distance at most $r$ is clustered together with probability at least $n^{-O(\gamma)}$. Denote by $\chi_x$ (resp., $\chi_y$) the probability that $x$ (resp., $y$) was removed from $\mathcal{P}_h(x)$ (resp., $\mathcal{P}_h(y)$).

By the union bound on the at most $n$ points in $\mathcal{P}_h(x)$, we have that both $\chi_x, \chi_y \leq \frac{1}{n}$.

We conclude

$$\Pr_{\mathcal{P}'_h}[\mathcal{P}'_h(x) = \mathcal{P}'_h(y)] \geq \Pr_{h \sim H}[h(x) = h(y)] - \Pr_h[\chi_x \lor \chi_y] \geq n^{-O(\gamma)} - \frac{2}{n} = n^{-O(\gamma)}.$$ 

□

Using [9], Lemma 9 implies that $\ell_2$ is $(t, n^{-O(\ell_2)})$-decomposable. Moreover, using [58] for constant $p \in (1, 2)$, Lemma 9 implies that $\ell_p$ is $(t, n^{-O(\log^2 p)})$-decomposable.
6 Decomposition for $d$-Dimensional Euclidean Space

In Sect. 5, using a reduction from LSH, we showed that $\ell_2$ is $(t, n^{-O(\epsilon/d)})$-decomposable. Here, we will show that for dimension $d = o(\log n)$, using a direct approach, better decomposition could be constructed.

Denote by $B_d(x, r)$ the $d$ dimensional ball of radius $r$ around $x$ (w.r.t $\ell_2$ norm). $V_d(r)$ denotes the volume of $B_d(x, r)$ (note that the center here is irrelevant). Denote by $C_d(u, r)$ the volume of the intersection of two balls of radius $r$, the centers of which are at distance $u$ (i.e. for $\|x - y\|_2 = u$, $C_d(u, r)$ denotes the volume of $B_d(x, r) \cap B_d(y, r)$). We will use the following lemma which was proved in [9] (based on a lemma from [39]).

Lemma 10 ([9]) For any $d \geq 2$ and $0 \leq u \leq r$

$$\Omega \left( \frac{1}{\sqrt{d}} \right) \cdot \left( 1 - \left( \frac{u}{r} \right)^2 \right)^{\frac{d}{2}} \leq \frac{C_d(u, r)}{V_d(r)} \leq \left( 1 - \left( \frac{u}{r} \right)^2 \right)^{\frac{d}{2}}.$$

Using Lemma 10, we can construct better decompositions:

Lemma 11 For every $d \geq 2$ and $2 \leq t \leq 2 - \frac{2\ln d}{\epsilon}$, $\ell_d$ is $O(t, 2^{-O(\epsilon/d)})$-decomposable.

Proof Consider a set $X$ of $n$ points in $\ell_d^d$, and fix $r > 0$. Let $B$ be some box which includes all of $X$ and such that each $x \in X$ is at distance at least $t \cdot r$ from the boundary of $B$. We sample points $s_1, s_2, \ldots$ uniformly at random from $B$. Set $P_i = B_X(s_i, \frac{t \cdot r}{2}) \setminus \bigcup_{j=1}^{i-1} B_X(s_j, \frac{t \cdot r}{2})$. We sample points until $X = \bigcup_{i \geq 1} P_i$. Then, the partition will be $\mathcal{P} = \{P_1, P_2, \ldots\}$ (dropping empty clusters).

It is straightforward that $\mathcal{P}$ is $t \cdot r$-bounded. Thus it will be enough to prove that every pair $x, y$ at distance at most $r$, has high enough probability to be clustered together. Let $s_i$ be the first point sampled in $B_d \left( x, \frac{t \cdot r}{2} \right) \cup B_d \left( y, \frac{t \cdot r}{2} \right)$. By the minimality of $i$, $x, y \notin \bigcup_{j=1}^{i-1} B_d \left( s_j, \frac{t \cdot r}{2} \right)$ and thus both are yet un-clustered.

If $s_i \in B_d \left( x, \frac{t \cdot r}{2} \right) \cap B_d \left( y, \frac{t \cdot r}{2} \right)$ then both $x, y$ join $P_i$ and thus clustered together. Using Lemma 10 we conclude,

$$\Pr_{\mathcal{P}} \left[ \mathcal{P}(x) = \mathcal{P}(y) \right] = \Pr \left[ s_i \in B_d \left( x, \frac{t \cdot r}{2} \right) \cap B_d \left( y, \frac{t \cdot r}{2} \right) \right] = \Pr \left[ s_i \text{ is first in } B_d \left( x, \frac{t \cdot r}{2} \right) \cup B_d \left( y, \frac{t \cdot r}{2} \right) \right] \geq \frac{C_d(\|x - y\|_2, \frac{t \cdot r}{2})}{2 \cdot V_d \left( \frac{t \cdot r}{2} \right)} = \Omega \left( \frac{1}{\sqrt{d}} \right) \left( 1 - \left( \frac{\|x - y\|_2}{\frac{t \cdot r}{2}} \right)^2 \right)^{\frac{d}{2}}$$

$$\geq \Omega \left( \frac{1}{\sqrt{d}} \right) \left( 1 - \frac{4}{t^2} \right)^{\frac{d}{2}} = \Omega \left( e^{-\frac{2d}{t^2} - \frac{4}{t^2} \ln d} \right) = 2^{-O(\epsilon/d)}.$$

$\square$
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