Research Article

Notion of Complex Spherical Fuzzy Graph with Application

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A complex spherical fuzzy set (CSFS) is a generalization of a spherical fuzzy set (SFS). CSFS handles vagueness more explicitly, and its range is expanded from the real subset to the complex with unit disc. The major goal of this research is to present the foundation of a complex spherical fuzzy graph (CSFG) due to the limitation of the complex neutral membership function in a complex Pythagorean fuzzy graph (CPFG). Complex spherical fuzzy models have more flexibility as compared to complex fuzzy models, complex intuitionistic fuzzy models, and complex Pythagorean fuzzy models due to their coverage in three directions: complex membership functions, neutral membership functions, and complex non-membership functions. Firstly, we present the motivation for CSFG. Furthermore, we define the order, degree of a vertex, size, and total degree of a vertex of CSFG. We elaborate on primary operations, including complement, join, and the union of CSFG. This research study introduces some operations, namely, strong product, composition, Cartesian product, and semi-strong product, on CSFG. Moreover, we present the application of CSFG, which ensures the ability to deal with problems in three directions.

1. Introduction

Zadeh [1] proposed the fuzzy set (FS) as an extension of the crisp set, and it has numerous applications in networking, decision making, telecommunication, artificial intelligence, management sciences, computer science, social science, and the chemical industry. The FS theory was created in order to deal with problems involving a lack of information and imprecision. FS can deal with uncertainty and vague problems whose membership functions lie in the interval [0, 1]. Atanassov [2] extended the FS to the intuitionistic fuzzy set (IFS), which consists of truthness $\mu$ and falsity degree $\upsilon$ and fulfills the condition $\mu + \upsilon \leq 1$ involving hesitancy part.

IFSs provide valuable scope in various fields of the world. Yager [3, 4] proposed the concept of the Pythagorean fuzzy set (PFS), which consists of truthness $\mu$ and falsity degree $\upsilon$ with the condition $\mu^2 + \upsilon^2 \leq 1$ due to a lack of both grades with the condition $\mu^2 + \upsilon^2 \leq 1$. For example, truthness grade 0.8 and falsity grade 0.3 fail to fulfill the condition $\mu + \upsilon \leq 1$, i.e., $0.8 + 0.3 = 1.1$ is larger than 1. Both grades satisfy the $\mu^2 + \upsilon^2 \leq 1$ condition, i.e., $0.82 + 0.32 = 0.71$. Therefore, a CPFS is much better than IFS. References [5–8] demonstrated decision making using the PFS. Later, the PFS was extended to include interval-valued PFSs. Cuong [9] proposed the picture fuzzy set (Pic-FS), which deals with neutrality, truth, and false membership grade.

Graph models have many applications in networking, sciences, technology, and operational management. To analyze world problems, boundedness between the nodes exists due to a few relations. Many researchers [10–22] participated in improving fuzzy concepts in different fields. Thirunavukarasu et al. [23] initiated the concept of complex fuzzy graphs. A FS can be defined by a complex-valued truth membership function, which is a mix of a traditional truth membership function and an extra term known as the phase term. In this study, the motivation for a fuzzy graph is expanded to include a complex fuzzy graph. A complex fuzzy theory is important in mathematics because it provides
greater flexibility, accuracy, and comparability to the system than a fuzzy model. The complexity of the FS originates in the range of values that its membership function may achieve. In contrast to a traditional fuzzy membership function, its range is expanded to the complex plane’s unit circle rather than \([0, 1]\).

Naveed et al. [24] studied a complex neutrosophic graph. Shoaib et al. [34] defined new operations briefly on PFzG. Known operations on bipolar fuzzy graphs. We apply these properties of spherical fuzzy averaging operator. CSFS was proposed by Mahmood et al. [28]. Poulik et al. [32] worked on fuzzy graph theory. Poulik et al. [33] described the Randic index of bipolar fuzzy graphs and its application in network system. Poulik and Ghor [29–31] worked on fuzzy graph theory. Poulik et al. [32] described different concepts and definitions, along with intuitive knowledge.

Our main contributions are given as follows:

The notion of CSFG is initiated.
With the help of an example, the order and size of CSFG are defined. The complement of CSFG is determined.
The join, union, and ring sum of CSFG are defined.
CSFG’s degree and total degree are discussed.
Strong product, composition, Cartesian product, and semi-strong product of CSFG are established.
A three-dimensional problem is elaborated in the application of CSFG.

The advantages of our concept are mentioned below:
CSFG deals with a three-dimensional phenomenon for intuitive knowledge.
The phase term of CSFG avoids any loss of knowledge.
CSFG accommodates a large amount of data and is more applicable in different fields.
CSFG has both the properties of a SFG and a complex fuzzy graph.
The concept of CSFGs is elaborated in the CSFS environment. Different concepts and definitions, along with operations, are presented for CSFGs. The complex spherical fuzzy environment is more adaptive and generalized than the spherical fuzzy environment due to phase terms.

### 2. Preliminaries

**Definition 1** (see [35]). The score function of complex spherical fuzzy number \(Q = (\mu_Q, \lambda_Q, \nu_Q, \delta_Q)\) is defined as follows:

\[
S(Q) = (\mu_Q - \lambda_Q - \nu_Q - \delta_Q)^{-1} - 1/4\pi^2 (\alpha_Q^2 - \gamma_Q^2 - \beta_Q^2).
\]

**Definition 2** (see [36]). The accuracy function of complex spherical fuzzy number \(Q = (\mu_Q, \lambda_Q, \nu_Q, \delta_Q)\) is defined as follows:

\[
S(Q) = (\mu_Q + \lambda_Q + \nu_Q + \delta_Q)^{1/4\pi^2} (\alpha_Q^2 + \gamma_Q^2 + \beta_Q^2).
\]

**Definition 3** (see [36]). For the comparison of two complex spherical fuzzy numbers \(Q_1 = (\mu_{Q_1}, \lambda_{Q_1}, \nu_{Q_1}, \delta_{Q_1})\) and \(Q_2 = (\mu_{Q_2}, \lambda_{Q_2}, \nu_{Q_2}, \delta_{Q_2})\):

1. If \(S(Q_1) > S(Q_2)\), then \(Q_1 > Q_2\) (\(Q_1\) is superior to \(Q_2\)).
2. If \(S(Q_1) = S(Q_2)\), then \(A(Q_1) > A(Q_2)\), then \(Q_1 > Q_2\) (\(Q_1\) is superior to \(Q_2\)).
3. If \(A(Q_1) = A(Q_2)\), then \(Q_1 \sim Q_2\) (\(Q_1\) is equivalent to \(Q_2\)).

**Definition 4** (see [36]). A CPFS \(Q\) on a universe \(Y\) is represented as \(Q = \{x, \mu_Q(x)e^{i\alpha_Q(x)}, \nu_Q(x)e^{i\beta_Q(x)}| x \in Y\}\), where \(\mu_Q(x), \nu_Q(x)\) are real-valued and both belong to \([0, 1]\) such that \(0 \leq \mu_Q(x) + \nu_Q(x) \leq 1\) and \(\alpha_Q(x), \beta_Q(x)\) such that \(0 \leq \alpha_Q(x)/2\pi + \beta_Q(x)/2\pi \leq 1\), for all \(x \in Y\). Note that \(\mu_Q(x), \nu_Q(x)\) are called amplitude terms and \(\alpha_Q(x), \beta_Q(x)\) are called phase terms.

Degree of refusal is given by \(t_Q(x) = (1 - \mu_Q^2(x) - \nu_Q^2(x))^{1/2}\).

**Definition 5** (see [36]). SFS \(Q\) on a universe \(Y\) is represented as \(Q = \{x, \mu_Q(x), \lambda_Q(x), \nu_Q(x)| x \in Y\}\), where \(\mu_Q(x), \lambda_Q(x), \nu_Q(x)\) are positive membership, neutral membership, and negative membership which are restricted to the unit interval \([0, 1]\).

\[
0 \leq \mu_Q^2(x) + \lambda_Q^2(x) + \nu_Q^2(x) \leq 1, \quad \forall x \in Y.
\]

Degree of refusal is given by \(t_Q(x) = (1 - \mu_Q^2(x) - \lambda_Q^2(x) - \nu_Q^2(x))^{1/2}\).

**Definition 6** (see [36]). CSFS \(Q\) on a universe \(Y\) is represented as \(Q = \{x, \mu_Q(x)e^{i\alpha_Q(x)}, \lambda_Q(x)e^{i\beta_Q(x)}, \nu_Q(x)e^{i\gamma_Q(x)}| x \in Y\}\), where \(\mu_Q(x), \lambda_Q(x), \nu_Q(x)\) are real-valued and all belong to \([0, 1]\) such that \(0 \leq \mu_Q^2(x) + \lambda_Q^2(x) + \nu_Q^2(x) \leq 1\) and \(\alpha_Q(x), \beta_Q(x), \gamma_Q(x)\) such that \(0 \leq \alpha_Q^2(x)/2\pi + \beta_Q^2(x)/2\pi + \gamma_Q^2(x)/2\pi \leq 1\), for all \(x \in Y\). Note that \(\mu_Q(x), \lambda_Q(x), \nu_Q(x)\) are known as amplitude terms and \(\alpha_Q(x), \beta_Q(x), \gamma_Q(x)\) are known as phase terms.

Degree of refusal is given by \(t_Q(x) = (1 - \mu_Q^2(x) - \lambda_Q^2(x) - \nu_Q^2(x))^{1/2}\).

**Definition 7** (see [25]). CPFG on a universe \(Y\) with underlying set \(A\) is an ordered pair \(\mathcal{G} = (Q, Z)\) is a CPFS on \(A\), and \(Z\) is CSFS on \(B \subseteq A \times A\) such that

\[
\begin{align*}
\mu_Z(xy)e^{i\alpha_Z(xy)} &\leq \min\{\mu_Q(x), \mu_Q(y)\}e^{i\min\{\alpha_Q(x), \alpha_Q(y)\}}, \\
\nu_Z(xy)e^{i\beta_Z(xy)} &\leq \max\{\nu_Q(x), \nu_Q(y)\}e^{i\max\{\beta_Q(x), \beta_Q(y)\}},
\end{align*}
\]

where \(0 \leq \mu_Q^2(x) + \nu_Q^2(x) \leq 1\) and \(\alpha_Q(x), \beta_Q(x) \in [0, 2\pi]\ \forall x, y \in A\).

### 3. Basic Concepts and Operation of CSFGs

This section defines and illustrates CSFG. We demonstrate with examples how to define the order, size, complement, union, join, and ring sum of CSFGs. On CSFGs, we define four distinct types of operations: Cartesian products, semi-strong products, compositions, and strong products.
Definition 8. CSFG on a universe $Y$ with underlying set $A$ is an ordered pair $G = (Q, Z)$, $Q$ is CSFS on $A$, and $Z$ is CSFS on $B = A \times A$ such that

$$
\mu_Z(xy) e^{\mu_y(xy)} \leq \min\{\mu_Q(x), \mu_Q(y)\} e^{\min\{\alpha_Q(x), \alpha_Q(y)\}},
$$
$$
\lambda_Z(xy) e^{\lambda_y(xy)} \leq \min\{\lambda_Q(x), \lambda_Q(y)\} e^{\min\{\gamma_Q(x), \gamma_Q(y)\}},
$$
$$
\nu_Z(xy) e^{\mathbf{\nu}_y(xy)} \leq \max\{\nu_Q(x), \nu_Q(y)\} e^{\max\{\beta_Q(x), \beta_Q(y)\}},
$$
where $0 \leq \mu_Z(x) + \lambda_Z(x) + \nu_Z(x) \leq 1$ and $\alpha_Q(x), \gamma_Q(x), \beta_Q(x) \in [0, 2\pi]$ for all $x, y \in A$.

Definition 9. Let $Q = \{x, \mu_Q(x) e^{\mu_0(x)}, \lambda_Q(x) e^{\lambda_0(x)}, \nu_Q(x) e^{\nu_0(x)} | x \in Y\}$, $Q_1 = \{x, \mu_Q(x) e^{\mu_0(x)}, \lambda_Q(x) e^{\lambda_0(x)}, \nu_Q(x) e^{\nu_0(x)} | x \in Y\}$, and $Q_2 = \{x, \mu_Q(x) e^{\mu_0(x)}, \lambda_Q(x) e^{\lambda_0(x)}, \nu_Q(x) e^{\nu_0(x)} | x \in Y\}$ be three CSFSs in $Y$; then,

(i) $Q_1 \subseteq Q_2$ if and only if $\mu_Q \leq \mu_Q$, $\lambda_Q \leq \lambda_Q$, $\nu_Q \leq \nu_Q$ for amplitude terms and $\alpha_Q \leq \alpha_Q$, $\gamma_Q \leq \gamma_Q$, $\beta_Q \leq \beta_Q$ for phase terms, for all $x \in Y$.

(ii) $Q_1 = Q_2$ if and only if $\mu_Q = \mu_Q$, $\lambda_Q = \lambda_Q$, $\nu_Q = \nu_Q$ for amplitude terms and $\alpha_Q = \alpha_Q$, $\gamma_Q = \gamma_Q$, $\beta_Q = \beta_Q$ for phase terms, for all $x \in Y$.

(iii) $Q = \{x, \nu_Q(x) e^{\mathbf{\nu}_0(x)}, \lambda_Q(x) e^{\lambda_0(x)}, \mu_Q(x) e^{\mu_0(x)} | x \in Y\}$.

For simplicity, the triplet $(\mu e^{\mu_0}, \lambda e^{\lambda_0}, \nu e^{\nu_0})$ is known as the complex spherical fuzzy number.

Example 1. Let $Y$ be a fixed set with just one element $x$, $\mu_Q(x) = 0.6$, $\alpha_Q(x) = 0.8$, $\lambda_Q(x) = 0.4$, $\gamma_Q(x) = 0.6$, $\beta_Q(x) = 0.2$. Then, $Q = \{(x, 0.6 e^{2\pi i/4}, 0.4 e^{2\pi i/3}, 0.2 e^{2\pi i/1})\}$ is a complex spherical fuzzy number.

Conclusion 1 is valid on $A \times A$ such that $0 \leq \mu^2 + \lambda^2 + \nu^2 \leq 1$ and $\alpha, \gamma, \beta \in [0, 2\pi]$ such that $0 \leq \mu^2 + \lambda^2 + \nu^2 \leq 2\pi$.

$$
Q = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0.3 e^{0.4 i} & 0.5 e^{0.5 i} & 0.4 e^{0.6 i} & 0.4 e^{0.7 i} & 0.4 e^{0.8 i} & 0.4 e^{0.9 i} \\
0.5 e^{0.2 i} & 0.2 e^{0.3 i} & 0.4 e^{0.4 i} & 0.4 e^{0.5 i} & 0.3 e^{0.6 i} & 0.4 e^{0.7 i} \\
\end{pmatrix},
$$

$$
Z = \begin{pmatrix}
0.2 e^{0.2 i} & 0.3 e^{0.3 i} & 0.4 e^{0.4 i} & 0.5 e^{0.5 i} \\
0.2 e^{0.2 i} & 0.3 e^{0.3 i} & 0.4 e^{0.4 i} & 0.5 e^{0.5 i} \\
0.2 e^{0.2 i} & 0.3 e^{0.3 i} & 0.4 e^{0.4 i} & 0.5 e^{0.5 i} \\
\end{pmatrix}.
$$

In order to compare CSFGs and SFGs, we must convert their vertices and edges from complex spherical fuzzy numbers to spherical fuzzy numbers by treating the phase terms of each complex spherical fuzzy value as zero, as shown in Figure 2, which is the vertex set and edge set of the CSFG shown in Figure 1.
When employing the proposed extended fuzzy graph, known as the CSFG, it is more fair to incorporate facts into the decision-making process. There are three sorts of degrees available in an SFG: membership degrees, neutral degrees, and non-membership degrees. The amplitude term is the sole term that is investigated, resulting in information loss. In addition, CSFG is an extension of existing theories such as fuzzy graphs, complex fuzzy graphs, and SFGs in that it takes into account increasing quantities of information about vertices and relations and works with two-dimensional data in a single set.

**Definition 12.** Let \( Q = \{ a, \mu_Q(a)e^{i\rho_Q(a)}, \lambda_Q(a)e^{i\gamma_Q(a)}, \nu_Q(a)e^{i\phi_Q(a)} \mid a \in A \} \) and \( Z = \{ (ab), \mu_Z((ab))e^{i\rho_Z(ab)}, \lambda_Z((ab))e^{i\gamma_Z(ab)}, \nu_Z((ab))e^{i\phi_Z(ab)} \mid ab \in B \} \) be the vertex set and edge set of CSFG \( G \); then, the order of CSFG is defined as

\[
O(G) = \left( \sum_{a \in A} \mu_Q(a)e^{i\rho_Q(a)} \right) + \left( \sum_{a \in A} \lambda_Q(a)e^{i\gamma_Q(a)} \right) + \left( \sum_{a \in A} \nu_Q(a)e^{i\phi_Q(a)} \right).
\]

The size of CSFG \( G \) is defined as

\[
S(G) = \left( \sum_{a \in A} \mu_Z(ab)e^{i\rho_Z(ab)} \right) + \left( \sum_{a \in A} \lambda_Z(ab)e^{i\gamma_Z(ab)} \right) + \left( \sum_{a \in A} \nu_Z(ab)e^{i\phi_Z(ab)} \right).
\]

**Example 4.** The order and size of CSFG given in Figure 1 are \( O(G) = (2.4e^{3.5}, 1.8e^{1.1}, 2.2e^{1.8}) \) and \( S(G) = (1.2e^{3.8}, 1.2e^{1.8}, 2.2e^{1.8}) \), respectively.
Definition 13. The complement of CSFG $G = (Q, Z)$ on the underlying graph $G = (A, B)$ is CSFG $\overline{G} = (\overline{Q}, \overline{Z})$ defined by

(1) $\mu_{\overline{Q}}(x)e^{\lambda_{\overline{Q}}(x)} = \mu_{Q}(x)e^{\lambda_{Q}(x)}$, $\lambda_{\overline{Q}}(x)e^{\lambda_{\overline{Q}}(x)} = \lambda_{Q}(x)e^{\lambda_{Q}(x)}$ and $\nu_{\overline{Q}}(x)e^{\nu_{\overline{Q}}(x)} = \nu_{Q}(x)e^{\nu_{Q}(x)}$.

(2) $\mu_{\overline{Z}}(x)e^{\lambda_{\overline{Z}}(x)} = \begin{cases} \min\{\nu_{Q}(x), \nu_{Q}(y)\}e^{\min\{\lambda_{Q}(x), \lambda_{Q}(y)\}} & \text{if } \nu_{Q}(x)e^{\nu_{Q}(x)} > 0, \\
\max\{\nu_{Q}(x), \nu_{Q}(y)\}e^{\min\{\lambda_{Q}(x), \lambda_{Q}(y)\}} & \text{if } 0 < \nu_{Q}(x)e^{\nu_{Q}(x)} \leq 1,
\end{cases}$

Example 5. Consider CSFG $G = (Q, Z)$ on $A = \{s_1, s_2, s_3\}$, as shown in Figure 3, defined by

$$\lambda_{Z}(xy)e^{\nu_{Z}(xy)} = \begin{cases} \min\{\nu_{Q}(x), \nu_{Q}(y)\}e^{\min\{\lambda_{Q}(x), \lambda_{Q}(y)\}} & \text{if } \nu_{Q}(x)e^{\nu_{Q}(x)} = 0, \\
\nu_{Q}(x)e^{\nu_{Q}(x)} & \text{if } 0 < \nu_{Q}(x)e^{\nu_{Q}(x)} \leq 1.
\end{cases}$$

Using Definition 13, complement of CSFG can be obtained, as given in Figure 4, and defined as

$$\overline{Q} = \begin{pmatrix} s_1 & s_2 & s_3 \\ 0.3e^{0.4m} & 0.5e^{0.5m} & 0.4e^{0.6m} \\ 0.4e^{0.3m} & 0.3e^{0.4m} & 0.3e^{0.5m} \\ 0.2e^{0.2m} & 0.3e^{0.3m} & 0.4e^{0.4m} \\ 0.1e^{0.1m} & 0.3e^{0.4m} & 0.4e^{0.4m} \end{pmatrix},$$

$$\overline{Z} = \begin{pmatrix} s_1 & s_2 & s_3 \\ 0.3e^{0.4m} & 0.5e^{0.5m} & 0.4e^{0.6m} \\ 0.4e^{0.3m} & 0.3e^{0.4m} & 0.3e^{0.5m} \\ 0.2e^{0.2m} & 0.3e^{0.3m} & 0.4e^{0.4m} \\ 0.1e^{0.1m} & 0.3e^{0.4m} & 0.4e^{0.4m} \end{pmatrix}.$$
\[
\begin{align*}
(\mu_{Q_1} \cup \mu_{Q_2})(x) e^{a_{Q_1} \cup a_{Q_2}}(x) &= \begin{cases} 
\mu_{Q_1}(x) e^{a_{Q_1}}(x) & \text{if } x \in A_1 - A_2, \\
\mu_{Q_2}(x) e^{a_{Q_2}}(x) & \text{if } x \in A_2 - A_1, \\
\max\{\mu_{Q_1}(x), \mu_{Q_2}(x)\} e^{\max\{a_{Q_1}, a_{Q_2}\}}(x) & \text{if } x \in A_1 \cap A_2,
\end{cases} \\
(\lambda_{Q_1} \cup \lambda_{Q_2})(x) e^{\lambda_{Q_1} \cup \lambda_{Q_2}}(x) &= \begin{cases} 
\lambda_{Q_1}(x) e^{\lambda_{Q_1}}(x) & \text{if } x \in A_1 - A_2, \\
\lambda_{Q_2}(x) e^{\lambda_{Q_2}}(x) & \text{if } x \in A_2 - A_1, \\
\max\{\lambda_{Q_1}(x), \lambda_{Q_2}(x)\} e^{\max\{\lambda_{Q_1}, \lambda_{Q_2}\}}(x) & \text{if } x \in A_1 \cap A_2,
\end{cases} \\
(\nu_{Q_1} \cup \nu_{Q_2})(x) e^{\nu_{Q_1} \cup \nu_{Q_2}}(x) &= \begin{cases} 
\nu_{Q_1}(x) e^{\nu_{Q_1}}(x) & \text{if } x \in A_1 - A_2, \\
\nu_{Q_2}(x) e^{\nu_{Q_2}}(x) & \text{if } x \in A_2 - A_1, \\
\min\{\nu_{Q_1}(x), \nu_{Q_2}(x)\} e^{\min\{\nu_{Q_1}, \nu_{Q_2}\}}(x) & \text{if } x \in A_1 \cap A_2,
\end{cases} \\
(\mu_{Z_1} \cup \mu_{Z_2})(xy) e^{a_{Z_1} \cup a_{Z_2}}(xy) &= \begin{cases} 
\mu_{Z_1}(xy) e^{a_{Z_1}}(xy) & \text{if } xy \in B_1 - B_2, \\
\mu_{Z_2}(xy) e^{a_{Z_2}}(xy) & \text{if } xy \in B_2 - B_1, \\
\max\{\mu_{Z_1}(xy), \mu_{Z_2}(xy)\} e^{\max\{a_{Z_1}, a_{Z_2}\}}(xy) & \text{if } xy \in B_1 \cap B_2,
\end{cases}
\end{align*}
\]
Definition 15. The ring sum $G_1 \oplus G_2 = (Q_1 \oplus Q_2, Z_1 \oplus Z_2)$ of two CSFGs $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ of the graphs $G_1$ and $G_2$, respectively, is defined as follows:

$$(\mu_{Q_1} \oplus \mu_{Q_2})(x)e^{(a_{Q_1} \oplus a_{Q_2})(x)} = (\mu_{Q_1} \cup \mu_{Q_2})(x)e^{(a_{Q_1} \cup a_{Q_2})(x)},$$

$$((\lambda_{Q_1} + \lambda_{Q_2})(x)e^{(\gamma_{Q_1} \oplus \gamma_{Q_2})(x)} = (\lambda_{Q_1} \cup \lambda_{Q_2})(x)e^{(\gamma_{Q_1} \cup \gamma_{Q_2})(x)},$$

$$(\nu_{Q_1} + \nu_{Q_2})(x)e^{(b_{Q_1} \oplus b_{Q_2})(x)} = (\nu_{Q_1} \cup \nu_{Q_2})(x)e^{(b_{Q_1} \cup b_{Q_2})(x)},$$

if $x \in A_1 \cup A_2$, for all $x \in A_1 \cup A_2$,

\[ (\lambda_{Q_1} \oplus \lambda_{Q_2})(x)e^{(\gamma_{Q_1} \oplus \gamma_{Q_2})(x)} = \min\{\lambda_{Q_1}(x), \lambda_{Q_2}(y)\}e^{(\min\{\gamma_{Q_1}(x), \gamma_{Q_2}(y)\})(x)} \]

\[ (\nu_{Q_1} \oplus \nu_{Q_2})(x)e^{(b_{Q_1} \oplus b_{Q_2})(x)} = \min\{\nu_{Q_1}(x), \nu_{Q_2}(y)\}e^{(\min\{b_{Q_1}(x), b_{Q_2}(y)\})(x)}, \]

if $x \in A_1 \cup A_2$, for all $x \in A_1 \cup A_2$.

Definition 16. Let $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ be two CSFGs of $G_1$ and $G_2$, respectively. The join $G_1 + G_2 = (Q_1 + Q_2, Z_1 + Z_2)$ of $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ is defined as

\[ (\mu_{Q_1} + \mu_{Q_2})(x)e^{(a_{Q_1} + a_{Q_2})(x)} = (\mu_{Q_1} \cup \mu_{Q_2})(x)e^{(a_{Q_1} \cup a_{Q_2})(x)}, \]

\[ (\lambda_{Q_1} + \lambda_{Q_2})(x)e^{(\gamma_{Q_1} + \gamma_{Q_2})(x)} = (\lambda_{Q_1} \cup \lambda_{Q_2})(x)e^{(\gamma_{Q_1} \cup \gamma_{Q_2})(x)}, \]

\[ (\nu_{Q_1} + \nu_{Q_2})(x)e^{(b_{Q_1} + b_{Q_2})(x)} = (\nu_{Q_1} \cup \nu_{Q_2})(x)e^{(b_{Q_1} \cup b_{Q_2})(x)}, \]

if $x \in A_1 \cup A_2$, for all $x \in A_1 \cup A_2$.

Definition 17. The degree of a vertex $x \in A$ in CSFG $G$ is denoted by $d_0(x)$ and is defined as

\[ d_0(x) = (d_{\text{new}}(x), d_{\text{old}}(x), d_{\text{seq}}(x)), \]

\[ d_{\text{new}}(x) = \sum_{y \neq x} \mu_Z(x)e^{(a_{Q_1} + a_{Q_2})(x)}, \]

\[ d_{\text{old}}(x) = \sum_{y \neq x} \lambda_Z(x)e^{(\gamma_{Q_1} + \gamma_{Q_2})(x)}, \]

\[ d_{\text{seq}}(x) = \sum_{y \neq x} \nu_Z(x)e^{(b_{Q_1} + b_{Q_2})(x)}. \]

Definition 18. The total degree of a vertex $x \in A$ in CSFG $G$ is denoted by $td_0(x)$ and is defined as

\[ td_0(x) = (td_{\text{new}}(x), td_{\text{old}}(x), td_{\text{seq}}(x)), d_{\text{new}}(x), d_{\text{old}}(x), d_{\text{seq}}(x)) \],
Definition 19. Let $G_1$ and $G_2$ be two CSFGs. For any vertex $x$ in $x \in A_1 \cup A_2$, there are three cases to consider.

Case 1:
Either $x \in A_1 - A_2$ or $x \in A_2 - A_1$. Then, there is no edge incident at $x$ which lies in $B_1 \cap B_2$. Thus, for $c \in C_1 - C_2$,

\[
(d_{mc})_{G_1 \cup G_2}(x) = \sum_{u \in B_1} \mu_{Z_1}(x) e^{\sum_{u \in A_2 \cup A_2} a_{z_2}(x)} (d_{mc})_{G_1}(x),
\]

\[
(d_{lc})_{G_1 \cup G_2}(x) = \sum_{u \in B_1} \lambda_{Z_1}(x) e^{\sum_{u \in A_2 \cup A_2} b_{z_2}(x)} (d_{lc})_{G_1}(x),
\]

\[
(d_{rc})_{G_1 \cup G_2}(x) = \sum_{u \in B_1} \nu_{Z_1}(x) e^{\sum_{u \in A_2 \cup A_2} b_{z_2}(x)} (d_{rc})_{G_1}(x),
\]

\[
(td_{mc})_{G_1 \cup G_2}(x) = (td_{mc})_{G_1}(x), \quad (td_{lc})_{G_1 \cup G_2}(x) = (td_{lc})_{G_1}(x), \quad (td_{rc})_{G_1 \cup G_2}(x) = (td_{rc})_{G_1}(x).
\]

For $x \in A_1 - A_1$,

\[
(d_{mc})_{G_1 \cup G_2}(x) = \sum_{u \in B_2} \mu_{Z_2}(x) e^{\sum_{u \in A_1 \cup A_1} a_{z_2}(x)},
\]

\[
(d_{lc})_{G_1 \cup G_2}(x) = \sum_{u \in B_2} \lambda_{Z_2}(x) e^{\sum_{u \in A_1 \cup A_1} b_{z_2}(x)},
\]

\[
(d_{rc})_{G_1 \cup G_2}(x) = \sum_{u \in B_2} \nu_{Z_2}(x) e^{\sum_{u \in A_1 \cup A_1} b_{z_2}(x)},
\]

\[
(td_{mc})_{G_1 \cup G_2}(x) = (td_{mc})_{G_1}(x), \quad (td_{lc})_{G_1 \cup G_2}(x) = (td_{lc})_{G_1}(x), \quad (td_{rc})_{G_1 \cup G_2}(x) = (td_{rc})_{G_1}(x).
\]

Case 2:
$x \in A_1 \cap A_2$, but there is no edge incident at $x$ which lies in $B_1 \cap B_2$. Then, any edge incident at $x$ is either $B_1 - B_2$ or $B_2 - B_1$.

\[
(d_{mc})_{G_1 \cup G_2}(x) = \sum_{u \in B_1 \cup B_2} \left( \mu_{Z_1} \cup \mu_{Z_2} \right)(x),
\]

\[
= \sum_{u \in B_1 - B_2} \mu_{Z_1}(x) e^{\sum_{u \in A_2 \cup A_2} a_{z_2}(x)} + \sum_{u \in B_2 - B_1} \mu_{Z_2}(x) e^{\sum_{u \in A_1 \cup A_1} a_{z_2}(x)},
\]

\[
+ \sum_{u \in B_1 \cap B_2} \max\{\mu_{Z_1}(x), \mu_{Z_2}(x)\} e^{\max\{a_{z_1}(x), a_{z_2}(x)\}},
\]

\[
\sum_{u \in B_1 \cup B_2} \left( \mu_{Z_1} \cup \mu_{Z_2} \right)(x) + \sum_{u \in B_1 \cup B_2} \sum_{u \in A_2 \cup A_2} \max\{\mu_{Z_1}(x), \mu_{Z_2}(x)\} e^{\max\{a_{z_1}(x), a_{z_2}(x)\}}.
\]
Example 6. Suppose that $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ are two CSFGs on $A_1 = \{s_1, s_2, s_3, s_4\}$ and $A_2 = \{s_1, s_2, s_3, s_4\}$, respectively, as shown in Figures 5 and 6. Moreover, $G_1 \cup G_2$ is shown in Figure 7. Since $s_3 \in A_1 \setminus A_2$, then
Figure 5: $G_1$.

Figure 6: $G_2$.

Figure 7: $G_1 \cup G_2$. 
Therefore, $d_{Q_1 \cup Q_2}(s_j) = d_{Q_1}(s_j) = 0.5e^{0.7ni}$.

Therefore, $td_{Q_1 \cup Q_2}(s_j) = td_{Q_1}(s_j) = 1.1e^{1.4ni}$.

Since $s_4 \in A_1 \cap A_2$, there is no edge incident at $s_4$ that lies in $B_1 \cap B_2$.

Therefore, $d_{Q_1 \cup Q_2}(s_4) = d_{Q_1}(s_4) = 0.5e^{1.1ni}$.

Therefore, $td_{Q_1 \cup Q_2}(s_4) = td_{Q_1}(s_4) = 1.1e^{0.9ni}$.

Since $s_2 \in A_1 \cap A_2$ and $s_1, s_2 \in B_1 \cap B_2$,

(i)

\[
(\mu_{Q_1} \times \mu_{Q_2})(x_1, x_2)e^{(q_{0_1}, q_{0_2})(x_1, x_2)} = \min\{\mu_{Q_1}(x_1), \mu_{Q_2}(x_2)\}e^{\min\{q_{0_1}(x_1), q_{0_2}(x_2)\}},
\]

\[
(\lambda_{Q_1} \times \lambda_{Q_2})(x_1, x_2)e^{(q_{0_1}, q_{0_2})(x_1, x_2)} = \min\{\lambda_{Q_1}(x_1), \lambda_{Q_2}(x_2)\}e^{\min\{q_{0_1}(x_1), q_{0_2}(x_2)\}},
\]

\[
(\nu_{Q_1} \times \nu_{Q_2})(x_1, x_2)e^{(q_{0_1}, q_{0_2})(x_1, x_2)} = \min\{\nu_{Q_1}(x_1), \nu_{Q_2}(x_2)\}e^{\min\{q_{0_1}(x_1), q_{0_2}(x_2)\}},
\]

\[
\forall (x_1, x_2) \in (A_1 \times A_2).
\]

(ii)

\[
(\mu_{Q_1} \times \mu_{Q_2})(m, x_2)(m, y_2)e^{(q_{0_1}, q_{0_2})(m, x_2)(m, y_2)} = \min\{\mu_{Q_1}(m), \mu_{Q_2}(x_2)\}e^{\min\{q_{0_1}(m), q_{0_2}(x_2)\}},
\]

\[
(\lambda_{Q_1} \times \lambda_{Q_2})(m, x_2)(m, y_2)e^{(q_{0_1}, q_{0_2})(m, x_2)(m, y_2)} = \min\{\lambda_{Q_1}(m), \lambda_{Q_2}(x_2)\}e^{\min\{q_{0_1}(m), q_{0_2}(x_2)\}},
\]

\[
(\nu_{Q_1} \times \nu_{Q_2})(m, x_2)(m, y_2)e^{(q_{0_1}, q_{0_2})(m, x_2)(m, y_2)} = \min\{\nu_{Q_1}(m), \nu_{Q_2}(x_2)\}e^{\min\{q_{0_1}(m), q_{0_2}(x_2)\}},
\]

\[
\forall m \in A_1 \text{ and } x_2 y_2 \in B_2.
\]
Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs of graph \( G \), respectively. Then, the Cartesian product \( G_1 \times G_2 \) of \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively, can be proved. Let \((x_1, x_2) (y_1, y_2) \in B_1 \times B_2 \).

(i) It is trivial.

(ii) If \( x_1 = y_1 = m \),

Example 7. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs on \( A_1 = [a, b] \) and \( A_2 = [c, d] \), respectively, which are shown in Figures 8 and 9. Also, the Cartesian product is shown in Figure 10.

**Proposition 1.** Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs of graph \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. Then Cartesian product \( G_1 \times G_2 \) of \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) is CSFG.
(iii) If $x_2 = y_2 = z$, 

\[
\begin{align*}
&\left(\mu_{Z_1} \times \mu_{Z_2}\right)((x_1, z) (y_1, z)) \mathcal{E} e^{i \lambda_{z_1} \times \lambda_{z_2}} ((x,z)(y,z)) \\
&= \min\{\mu_{Z_1}(x_1, y_1), \mu_{Z_2}(z)\} e^{i \min\{\lambda_{z_1}(x, y_1), \lambda_{z_2}(z)\}} \\
&\leq \min\{\mu_{Z_1}(x_1, y_1), \mu_{Z_2}(z)\} e^{i \min\{\lambda_{z_1}(x, y_1), \lambda_{z_2}(z)\}},
\end{align*}
\]

(26)

\[
\begin{align*}
&\left(\lambda_{Z_1} \times \lambda_{Z_2}\right)((x_1, z) (y_1, z)) \mathcal{E} e^{i \lambda_{z_1} \times \lambda_{z_2}} ((x,z)(y,z)) \\
&= \min\{\lambda_{z_1}(x_1, y_1), \lambda_{z_2}(z)\} e^{i \min\{\gamma_{z_1}(x_1, y_1), \gamma_{z_2}(z)\}} \\
&\leq \min\{\lambda_{z_1}(x_1, y_1), \lambda_{z_2}(z)\} e^{i \min\{\gamma_{z_1}(x_1, y_1), \gamma_{z_2}(z)\}},
\end{align*}
\]

(27)
(\gamma_{Z_1} \times \gamma_{Z_2})((x_1, z)(y_1, z)) = 
\min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}}, 
\leq \min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}}, 
\leq \min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}} \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}, 
\min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}}, 
\min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}}, 
\min \{\gamma_{Z_1}(x_1, y_1), \gamma_{Z_2}(z)\} e^{i \min \{\beta_{Z_1}(x_1, y_1), \beta_{Z_2}(z)\}}. 

(28)

Hence, \( G_1 \times G_2 \) is a CSFG.

Definition 21. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are CSFGs defined on \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. The composition of \( G_1 \) and \( G_2 \) is represented by \( G_1^\circ G_2 = (Q_1, Q_2, Z_1, Z_2) \). The composition of \( G_1 \) on \( dG_2 \) is defined as follows:

(i) \( (\mu_{G_1} \circ \mu_{G_2})((x_1, x_2)) e^{i (a_{Q_1} \times a_{Q_2})((x_1, x_2))} = \)
\min \{\mu_{G_1}(x_1), \mu_{G_2}(x_2)\} e^{i \min \{a_{Q_1}(x_1), a_{Q_2}(x_2)\}},
\min \{\lambda_{G_1}(x_1), \lambda_{G_2}(x_2)\} e^{i \min \{a_{Q_1}(x_1), a_{Q_2}(x_2)\}},
\min \{\nu_{Q_1}(x_1), \nu_{Q_2}(x_2)\} e^{i \min \{a_{Q_1}(x_1), a_{Q_2}(x_2)\}},
\max \{\nu_{Q_1}(x_1), \nu_{Q_2}(x_2)\} e^{i \max \{a_{Q_1}(x_1), a_{Q_2}(x_2)\}},
\forall (x_1, x_2) \in (A_1 \times A_2).

(ii) \( (\mu_{G_1} \circ \mu_{G_2})((m, x_2)(m, y_2)) e^{i (a_m \times a_{Q_2})(m, y_2)(m, x_2)(m, y_2)) = \)
\min \{\mu_{G_1}(m), \mu_{G_2}(x_2, y_2)\} e^{i \min \{a_{Q_1}(m), a_{Q_2}(x_2, y_2)\}},
\min \{\lambda_{G_1}(m), \lambda_{G_2}(x_2, y_2)\} e^{i \min \{a_{Q_1}(m), a_{Q_2}(x_2, y_2)\}},
\min \{\nu_{Q_1}(m), \nu_{Q_2}(x_2, y_2)\} e^{i \min \{a_{Q_1}(m), a_{Q_2}(x_2, y_2)\}},
\max \{\nu_{Q_1}(m), \nu_{Q_2}(x_2, y_2)\} e^{i \max \{a_{Q_1}(m), a_{Q_2}(x_2, y_2)\}},
\forall m \in A_1 \text{ and } x_2, y_2 \in B_2.

(iii) \( (\mu_{G_1} \circ \mu_{G_2})((x_1, z)(y_1, z)) e^{i (a_{Q_1} \times a_{Q_2})(x_1, y_1)(y_1, z)) = \)
\min \{\mu_{G_1}(x_1, y_1), \mu_{G_2}(z)\} e^{i \min \{a_{Q_1}(x_1, y_1), a_{Q_2}(z)\}},
\min \{\lambda_{G_1}(x_1, y_1), \lambda_{G_2}(z)\} e^{i \min \{a_{Q_1}(x_1, y_1), a_{Q_2}(z)\}},
\min \{\nu_{Q_1}(x_1, y_1), \nu_{Q_2}(z)\} e^{i \min \{a_{Q_1}(x_1, y_1), a_{Q_2}(z)\}},
\max \{\nu_{Q_1}(x_1, y_1), \nu_{Q_2}(z)\} e^{i \max \{a_{Q_1}(x_1, y_1), a_{Q_2}(z)\}},
\forall (x_1, y_1, z) \in B_1 \times B_2.

Example 8. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs on \( A_1 = [a, b] \) and \( A_2 = [c, d] \), respectively, which are shown in Figures 11 and 12. Also, composition is shown in Figure 13.

Proof. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs of graph \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. Then, the composition \( G_1^\circ G_2 \) of \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) can be proved. Let

(i) It is trivial.
(ii) If \( x_1 = y_1 = m, \)
\((\lambda_{Z_1} \circ \lambda_{Z_2})((m, x_2) (m, y_2)) e^{i \min \{\gamma_{0_1}, \gamma_{0_2}, \gamma_{Q_1}, \gamma_{Q_2}\}}\) \\
= \min \{\lambda_{Q_1}(m), \lambda_{Z_1}(x_2, y_2)\} e^{i \min \{\gamma_{0_1}, \gamma_{0_2}, \gamma_{Q_1}, \gamma_{Q_2}\}}.
\[\text{(30)}\]

\((\gamma_{Z_1} \circ \gamma_{Z_2})((m, x_2) (m, y_2)) e^{i \min \{\beta_{0_1}, \beta_{0_2}, \beta_{Q_1}, \beta_{Q_2}\}}\) \\
= \min \{\gamma_{Q_1}(m), \gamma_{Z_1}(x_2, y_2)\} e^{i \min \{\beta_{0_1}, \beta_{0_2}, \beta_{Q_1}, \beta_{Q_2}\}}.
\[\text{(31)}\]

(iii) If \(x_2 = y_2 = z\),
(\mu_{Z_1 \circ \mu_{Z_0}})((x_1, z) (y_1, z)) e^{i a_z (x_1, z) (y_1, z)}
= \min \{\mu_{Z_1}(x_1, y_1), \mu_{Q_1}(z)\} e^{i \min \{a_{x_1} (x_1, y_1) a_{Q_1}(z)\}},
\leq \min \{\min \{\mu_{Z_1}(x_1, y_1), \mu_{Q_1}(z)\}, \min \{\mu_{Z_0}(x_1, y_1) a_{Q_1}(z)\}\},
= \min \{\min \{\mu_{Z_1}(x_1, y_1), \mu_{Q_1}(z)\}, \min \{\mu_{Q_1} (x_1, z), \mu_{Q_1} (y_1, z)\} e^{i \min \{a_{Q_1} (x_1, z) a_{Q_1} (y_1, z)\}\},
= \min \{\mu_{Q_1} \circ \mu_{Q_1})(x_1, z), (\mu_{Q_1} \circ \mu_{Q_1})(y_1, z)\} e^{i \min \{a_{Q_1} a_{Q_1}\} (x_1, z) (a_{Q_1} a_{Q_1}) (y_1, z)\},
(\lambda_{Z_1 \circ \lambda_{Z_0}})((x_1, z) (y_1, z)) e^{i (\gamma_{x_1} + \gamma_{y_1})(x_1, z) (y_1, z)}
= \min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\} e^{i \min \{\gamma_{x_1} (x_1, y_1) \gamma_{Q_1}(z)\}}
\leq \min \{\min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\}, \min \{\lambda_{Q_1} (x_1, z), \lambda_{Q_1} (y_1, z)\} e^{i \min \{\gamma_{x_1} (x_1, y_1) \gamma_{Q_1}(z)\}}\},
= \min \{\min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\}, \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\} e^{i \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\}\},
(\lambda_{Z_1 \circ \lambda_{Z_0}})((x_1, z) (y_1, z)) e^{i (\gamma_{x_1} + \gamma_{y_1})(x_1, z) (y_1, z)}
= \min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\} e^{i \min \{\gamma_{x_1} (x_1, y_1) \gamma_{Q_1}(z)\}}
\leq \min \{\min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\}, \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\} e^{i \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\}\},
= \min \{\min \{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_1}(z)\}, \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\} e^{i \min \{\gamma_{Q_1} (x_1, z), \gamma_{Q_1} (y_1, z)\}\},
(\nu_{Z_1 \circ \nu_{Z_0}})((x_1, z) (y_1, z)) e^{i (\beta_{x_1} + \beta_{y_1})(x_1, z) (y_1, z)}
= \min \{\nu_{Z_1}(x_1, y_1), \nu_{Q_1}(z)\} e^{i \min \{\beta_{x_1} (x_1, y_1) \beta_{Q_1}(z)\}}
\leq \min \{\min \{\nu_{Z_1}(x_1, y_1), \nu_{Q_1}(z)\}, \min \{\nu_{Q_1} (x_1, z), \nu_{Q_1} (y_1, z)\} e^{i \min \{\beta_{x_1} (x_1, y_1) \beta_{Q_1}(z)\}\},
= \min \{\min \{\nu_{Z_1}(x_1, y_1), \nu_{Q_1}(z)\}, \min \{\nu_{Q_1} (x_1, z), \nu_{Q_1} (y_1, z)\} e^{i \min \{\beta_{x_1} (x_1, y_1) \beta_{Q_1}(z)\}\},
(iv) For all \(x_2, y_2 \in A_2, x_2 \neq y_2\) and \(x_1 y_1 \in B_1,\)

(\mu_{Z_1 \circ \mu_{Z_0}})((x_1, x_2) (y_1, y_2)) e^{i (a_{x_1} + a_{x_2})(x_1, x_2) (y_1, y_2)}
= \min \{\mu_{Q_1}(x_2), \mu_{Q_1}(y_2), \mu_{Z_1}(x_1, y_1)\} e^{i \min \{a_{x_1} (x_1, x_2) a_{x_2} (y_1, y_2)\}},
\leq \min \{\min \{\mu_{Q_1}(x_2), \mu_{Q_1}(y_2)\}, \min \{\mu_{Q_1} (x_1, y_1)\} e^{i \min \{a_{x_1} (x_1, y_1) a_{Q_1}(y_1)\}},
= \min \{\min \{\mu_{Q_1}(x_2), \mu_{Q_1}(y_2)\}, \min \{\mu_{Q_1} (x_1, y_1)\} e^{i \min \{a_{x_1} (x_1, y_1) a_{Q_1}(y_1)\}},
= \min \{\mu_{Q_1} \circ \mu_{Q_1})(x_1, x_2), (\mu_{Q_1} \circ \mu_{Q_1})(y_1, y_2)\} e^{i \min \{a_{Q_1} a_{Q_1}\} (x_1, x_2) (a_{Q_1} a_{Q_1}) (y_1, y_2)\},
= \min \{\mu_{Q_1} \circ \mu_{Q_1})(x_1, x_2), (\mu_{Q_1} \circ \mu_{Q_1})(y_1, y_2)\} e^{i \min \{a_{Q_1} a_{Q_1}\} (x_1, x_2) (a_{Q_1} a_{Q_1}) (y_1, y_2)\},
\[(\lambda_{Z_1} \circ \lambda_{Z_2})(x_1, x_2)(y_1, y_2)e^{(\gamma_{y_0} \circ \gamma_{y_1})(x_1, x_2)(y_1, y_2)}\]

\[= \min\{\lambda_{Q_1}(x_1), \lambda_{Q_2}(y_1), \lambda_{Z_1}(x_1, y_1)\}e^{\min\{\gamma_{y_0}(x_1), \gamma_{y_1}(y_1), \gamma_{y_1}(x_1, y_1)\}},\]

\[\leq \min\{\lambda_{Q_1}(x_1), \lambda_{Q_2}(y_1), \min\{\lambda_{Q_1}(x_1)\lambda_{Q_2}(y_1)\}\}e^{\min\{\gamma_{y_0}(x_1), \gamma_{y_1}(y_1), \min\{\gamma_{y_1}(x_1), \gamma_{y_1}(y_1)\}\}},\]

\[= \min\{\lambda_{Q_1}(x_1), \lambda_{Q_2}(y_1), \lambda_{Q_1}(x_1, y_1)\}e^{\min\{\gamma_{y_1}(x_1, y_1)\}}\]

\[= \min\{\lambda_{Q_1}(x_1)\lambda_{Q_2}(y_1), \lambda_{Q_1}(x_1, y_1)\}e^{\min\{\gamma_{y_1}(x_1, y_1)\}},\]

\[\text{(36)}\]

\[(\nu_{Z_1} \circ \nu_{Z_2})(x_1, x_2)(y_1, y_2)e^{(\beta_{y_1} \circ \beta_{y_2})(x_1, x_2)(y_1, y_2)}\]

\[= \min\{\nu_{Q_1}(x_1), \nu_{Q_2}(y_1), \nu_{Z_1}(x_1, y_1)\}e^{\min\{\beta_{y_0}(x_1), \beta_{y_0}(y_1), \beta_{y_0}(x_1, y_1)\}},\]

\[\leq \min\{\nu_{Q_1}(x_1), \nu_{Q_2}(y_1), \max\{\nu_{Q_1}(x_1)\nu_{Q_2}(y_1)\}\}e^{\min\{\beta_{y_0}(x_1), \beta_{y_0}(y_1), \min\{\beta_{y_0}(x_1), \beta_{y_0}(y_1)\}\}},\]

\[= \min\{\nu_{Q_1}(x_1), \nu_{Q_2}(y_1), \nu_{Q_1}(x_1, y_1)\}e^{\min\{\beta_{y_1}(x_1, y_1)\}}\]

\[= \min\{\nu_{Q_1}(x_1)\nu_{Q_2}(y_1), \nu_{Q_1}(x_1, y_1)\}e^{\min\{\beta_{y_1}(x_1, y_1)\}},\]

\[\text{(37)}\]

Hence, \(G_1 \ast G_2\) is a CSFG.

\textbf{Definition 22.} Suppose that \(G_1 = (Q_1, Z_1)\) and \(G_2 = (Q_2, Z_2)\) are CSFGs defined on \(G_1 = (A_1, B_1)\) and \(G_2 = (A_2, B_2)\), respectively. The strong product of \(G_1\) and \(G_2\) is represented by \(G_1 \otimes G_2 = (Q_1 \otimes Q_2, Z_1 \otimes Z_2)\). The strong product of \(G_1\) and \(G_2\) is defined as follows:

\[(\mu_{Q_1} \otimes \mu_{Q_2})(x_1, x_2)e^{(\beta_{y_0} \circ \beta_{y_0})(x_1, x_2)(y_1, y_2)}\]

\[= \min\{\mu_{Q_1}(x_1)\mu_{Q_2}(x_2)\}e^{\min\{\beta_{y_0}(x_1), \beta_{y_0}(x_2)\}},\]

\[(\lambda_{Q_1} \otimes \lambda_{Q_2})(x_1, x_2)e^{(\gamma_{y_0} \circ \gamma_{y_0})(x_1, x_2)(y_1, y_2)}\]

\[= \min\{\lambda_{Q_1}(x_1)\lambda_{Q_2}(x_2)\}e^{\min\{\gamma_{y_0}(x_1), \gamma_{y_0}(x_2)\}},\]

\[(\nu_{Q_1} \circ \nu_{Q_2})(x_1, x_2)e^{(\beta_{y_0} \circ \beta_{y_0})(x_1, x_2)(y_1, y_2)}\]

\[= \min\{\nu_{Q_1}(x_1)\nu_{Q_2}(x_2)\}e^{\min\{\beta_{y_0}(x_1), \beta_{y_0}(x_2)\}},\]

\[\forall (x_1, x_2) \in (A_1 \times A_2).
\]

\[(\mu_{Z_1} \otimes \mu_{Z_2})(m, x_2)(m, y_2)e^{(\beta_{y_0} \circ \beta_{y_0})(m, x_2)(m, y_2)}\]

\[= \min\{\mu_{Z_1}(m)\mu_{Z_2}(x_2)\}e^{\min\{\beta_{y_0}(m), \beta_{y_0}(x_2)\}},\]

\[(\lambda_{Z_1} \otimes \lambda_{Z_2})(m, x_2)(m, y_2)e^{(\gamma_{y_0} \circ \gamma_{y_0})(m, x_2)(m, y_2)}\]

\[= \min\{\lambda_{Z_1}(m)\lambda_{Z_2}(x_2)\}e^{\min\{\gamma_{y_0}(m), \gamma_{y_0}(x_2)\}},\]

\[(\nu_{Z_1} \circ \nu_{Z_2})(m, x_2)(m, y_2)e^{(\beta_{y_0} \circ \beta_{y_0})(m, x_2)(m, y_2)}\]

\[= \min\{\nu_{Z_1}(m)\nu_{Z_2}(x_2)\}e^{\min\{\beta_{y_0}(m), \beta_{y_0}(x_2)\}},\]

\[\forall m \in A_1 an dx_2y_2 \in B_2.
\]

\[(\mu_{Z_1} \otimes \mu_{Z_2})(x_1, z)(y_1, z)e^{(\beta_{y_0} \circ \beta_{y_0})(x_1, z)(y_1, z)}\]

\[= \min\{\mu_{Z_1}(x_1, y_1)\mu_{Z_2}(z)\}e^{\min\{\beta_{y_0}(x_1, y_1), \beta_{y_0}(z)\}},\]

\[(\lambda_{Z_1} \otimes \lambda_{Z_2})(x_1, z)(y_1, z)e^{(\gamma_{y_0} \circ \gamma_{y_0})(x_1, z)(y_1, z)}\]

\[= \min\{\lambda_{Z_1}(x_1, y_1)\lambda_{Z_2}(z)\}e^{\min\{\gamma_{y_0}(x_1, y_1), \gamma_{y_0}(z)\}},\]

\[(\nu_{Z_1} \circ \nu_{Z_2})(x_1, z)(y_1, z)e^{(\beta_{y_0} \circ \beta_{y_0})(x_1, z)(y_1, z)}\]

\[= \min\{\nu_{Z_1}(x_1, y_1)\nu_{Z_2}(z)\}e^{\min\{\beta_{y_0}(x_1, y_1), \beta_{y_0}(z)\}},\]

\[\forall z \in A_1 an dx_1y_1 \in B_1.
\]
Example 9. Suppose that $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ are two CSFGs on $A_1 = \{a, b\}$ and $A_2 = \{c, d\}$, respectively, which are shown in Figures 14 and 15. Also, strong product is shown in Figure 16.

Proposition 2. Suppose that $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ are two CSFGs of graph $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then, the strong product $G_1 \otimes G_2$ of $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is CSFG.

Proof. Suppose that $G_1 = (Q_1, Z_1)$ and $G_2 = (Q_2, Z_2)$ are two CSFG of graph $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then, the strong product $G_1 \otimes G_2$ of $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ can be proved. Let $(x_1, x_2) (y_1, y_2) \in B_1 \times B_2$.

(i) It is trivial.

(ii) If $x_1 = y_1 = m$,
\[(\mu_{Z_1} \otimes \mu_{Z_2})((x_1, z), (y_1, z)) e^{j\pi e_{z_1} a_{z_2}}((x_1, z), (y_1, z))\]

\[= \min \{\mu_{Z_1}(x_1, y_1), \mu_{Q_2}(z)\} e^{i\min\{a_{z_1}(x_1, y_1), a_{z_2}(z)\}},\]

\[\leq \min\{\min\{\mu_{Z_1}(x_1, y_1), \mu_{Q_2}(z)\}, \min\{\mu_{Q_1}(y_1), \mu_{Q_2}(z)\}\} e^{i\min\{\min\{a_{z_1}(x_1, y_1), a_{z_2}(z)\}, \min\{a_{z_1}(y_1), a_{z_2}(z)\}\}},\]

\[= \min\{\mu_{Q_1}(x_1), \mu_{Q_2}(z)\}, \min\{\mu_{Q_1}(y_1), \mu_{Q_2}(z)\}\} e^{i\min\{\min\{\min\{a_{z_1}(x_1), a_{z_2}(z)\}, \min\{a_{z_1}(y_1), a_{z_2}(z)\}\}, \min\{a_{z_1}(y_1), a_{z_2}(z)\}\}},\]

\[(\lambda_{Z_1} \otimes \lambda_{Z_2})((x_1, z), (y_1, z)) e^{j\pi e_{z_1} a_{z_2}}((x_1, z), (y_1, z))\]

\[= \min\{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_2}(z)\} e^{i\min\{\gamma_{z_1}(x_1, y_1), \gamma_{z_2}(z)\}},\]

\[\leq \min\{\min\{\lambda_{Z_1}(x_1, y_1), \lambda_{Q_2}(z)\}, \min\{\lambda_{Q_1}(y_1), \lambda_{Q_2}(z)\}\} e^{i\min\{\min\{\gamma_{z_1}(x_1, y_1), \gamma_{z_2}(z)\}, \min\{\gamma_{z_1}(y_1), \gamma_{z_2}(z)\}\}},\]

\[= \min\{\lambda_{Q_1}(x_1), \lambda_{Q_2}(z)\}, \min\{\lambda_{Q_1}(y_1), \lambda_{Q_2}(z)\}\} e^{i\min\{\min\{\min\{\gamma_{z_1}(x_1), \gamma_{z_2}(z)\}, \min\{\gamma_{z_1}(y_1), \gamma_{z_2}(z)\}\}, \min\{\gamma_{z_1}(y_1), \gamma_{z_2}(z)\}\}},\]
(\nu_{Z_1} \otimes \nu_{Z_2})((x_1,z),(y_1,z))e^{i(\beta_{Z_1} \otimes \beta_{Z_2})((x_1,z),(y_1,z))}
= \max\{\nu_{Z_1}(x_1,y_1),\nu_{Z_2}(z)\}e^{i\max\{\beta_{Z_1}(x_1,y_1)\beta_{Z_2}(z)\}}
\leq \max\{\max\{\nu_{Z_1}(x_1,y_1),\nu_{Z_2}(z)\}e^{i\max\{\max\beta_{Z_1}(x_1,y_1)\beta_{Z_2}(z)\}}\},
= \max\{\max\{\nu_{Q_1}(x_1),\nu_{Q_2}(z)\},\max\{\nu_{Q_1}(y_1),\nu_{Q_2}(z)\}\}e^{i\max\{\max\beta_{Q_1}(x_1)\beta_{Q_2}(z)\}}
= \max\{\nu_{Q_1} \otimes \nu_{Q_2})(x_1,z),(\nu_{Q_1} \otimes \nu_{Q_2})(y_1,z)\}e^{i\max\{\beta_{Q_1} \otimes \beta_{Q_2})((x_1,z),(y_1,z))\}}.

(iv)

(\mu_{Z_1} \otimes \mu_{Z_2})((x_1,x_2),(y_1,y_2))e^{i(\alpha_{Z_1} \otimes \alpha_{Z_2})((x_1,x_2),(y_1,y_2))}
= \min\{\mu_{Z_1}(x_1,y_1),\mu_{Z_2}(x_2,y_2)\}e^{i\min\{\alpha_{Z_1}(x_1,y_1)\alpha_{Z_2}(x_2,y_2)\}}
\leq \min\{\min\{\mu_{Q_1}(x_1),\mu_{Q_2}(y_2)\},\{\mu_{Q_2}(x_2),\mu_{Q_2}(y_2)\}\}e^{i\min\{\min\{\alpha_{Q_1}(x_1)\alpha_{Q_2}(y_1)\},\{\alpha_{Q_2}(x_2)\alpha_{Q_2}(y_2)\}\}}
= \min\{\min\{\mu_{Q_1}(m),\mu_{Q_2}(x_2)\},\min\{\mu_{Q_2}(m),\mu_{Q_2}(y_2)\}\}e^{i\min\{\min\{\alpha_{Q_1}(m)\alpha_{Q_2}(x_2)\},\{\alpha_{Q_2}(m)\alpha_{Q_2}(y_2)\}\}}
= \min\{\mu_{Q_1} \otimes \mu_{Q_2})(x_1,x_2),(\mu_{Q_1} \otimes \mu_{Q_2})(y_1,y_2)\}e^{i\min\{\min\{\alpha_{Q_1} \otimes \alpha_{Q_2}(x_1,x_2)\alpha_{Q_2} \otimes \alpha_{Q_2}(y_1,y_2)\}\}}
\text{for all } x_1,x_2 \in B_1 \text{ and for all } x_2,y_2 \in B_2.

(\lambda_{Z_1} \otimes \lambda_{Z_2})(x_1,x_2),(y_1,y_2) = \min\{\lambda_{Z_1}(x_1,y_1),\lambda_{Z_2}(x_2,y_2)\}e^{i\min\{\lambda_{Z_1}(x_1,y_1)\lambda_{Z_2}(x_2,y_2)\}}
\leq \min\{\min\{\lambda_{Q_1}(x_1),\lambda_{Q_2}(y_2)\},\{\lambda_{Q_2}(x_2),\lambda_{Q_2}(y_2)\}\}e^{i\min\{\min\{\lambda_{Q_1}(x_1)\lambda_{Q_2}(y_1)\},\{\lambda_{Q_2}(x_2)\lambda_{Q_2}(y_2)\}\}}
= \min\{\min\{\lambda_{Q_1}(m),\lambda_{Q_2}(x_2)\},\min\{\lambda_{Q_2}(m),\lambda_{Q_2}(y_2)\}\}e^{i\min\{\min\{\lambda_{Q_1}(m)\lambda_{Q_2}(x_2)\},\{\lambda_{Q_2}(m)\lambda_{Q_2}(y_2)\}\}}
= \min\{\lambda_{Q_1} \otimes \lambda_{Q_2})(x_1,x_2),(\lambda_{Q_1} \otimes \lambda_{Q_2})(y_1,y_2)\}e^{i\min\{\min\{\lambda_{Q_1} \otimes \lambda_{Q_2}(x_1,x_2)\lambda_{Q_2} \otimes \lambda_{Q_2}(y_1,y_2)\}\}}
\text{for all } x_1,x_1 \in B_1 \text{ and for all } x_2,y_2 \in B_2.

(\nu_{Z_1} \otimes \nu_{Z_2})(x_1,x_2),(y_1,y_2) = \max\{\nu_{Z_1}(x_1,y_1),\nu_{Z_2}(x_2,y_2)\}e^{i\max\{\beta_{Z_1}(x_1,y_1)\beta_{Z_2}(x_2,y_2)\}}
\leq \max\{\max\{\nu_{Q_1}(x_1),\nu_{Q_2}(y_2)\},\max\{\nu_{Q_2}(x_2),\nu_{Q_2}(y_2)\}\}e^{i\max\{\max\{\beta_{Q_1}(x_1)\beta_{Q_2}(y_2)\},\{\beta_{Q_2}(x_2)\beta_{Q_2}(y_2)\}\}}
= \max\{\max\{\nu_{Q_1}(x_1),\nu_{Q_2}(y_2)\},\max\{\nu_{Q_1}(m),\nu_{Q_2}(y_2)\}\}e^{i\max\{\max\{\beta_{Q_1}(m)\beta_{Q_2}(x_2)\},\{\beta_{Q_1}(m)\beta_{Q_2}(y_2)\}\}}
= \max\{\nu_{Q_1} \otimes \nu_{Q_2})(x_1,x_2),(\nu_{Q_1} \otimes \nu_{Q_2})(y_1,y_2)\}e^{i\max\{\min\{\beta_{Q_1}(x_1)\beta_{Q_2}(x_2)\},\{\beta_{Q_1}(y_1)\beta_{Q_2}(y_2)\}\}}
\text{for all } x_1,x_1 \in B_1 \text{ and for all } x_2,y_2 \in B_2.

Hence, \(G_1 \otimes G_2\).
Definition 23. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are CSFGs defined on \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. The semi-strong product of \( G_1 \) and \( G_2 \) is represented by \( G_1 \circ G_2 = (Q_1 \circ Q_2, Z_1 \circ Z_2) \). The semi-strong product of \( G_1 \) and \( G_2 \) is defined as follows:

\[
\begin{align*}
\text{(i)} & \quad (\mu_{Q_1} \circ \mu_{Q_2})(((x_1, x_2), e^{(a_{y_1} \circ a_{y_2})}((x_1, x_2))) = \min \{\mu_{Q_1}(x_1), \mu_{Q_2}(x_2)\} e^{\min \{a_{y_1}(x_1), a_{y_2}(x_2)\}}, \\
& \quad (\lambda_{Q_1} \circ \lambda_{Q_2})(((x_1, x_2), e^{(y_{y_1} \circ y_{y_2})}((x_1, x_2))) = \min \{\lambda_{Q_1}(x_1), \lambda_{Q_2}(x_2)\} e^{\min \{y_{y_1}(x_1), y_{y_2}(x_2)\}}, \\
& \quad (\nu_{Q_1} \circ \nu_{Q_2})(((x_1, x_2), e^{(\nu_{y_1} \circ \nu_{y_2})}((x_1, x_2))) = \max \{\nu_{Q_1}(x_1), \nu_{Q_2}(x_2)\} e^{\max \{\nu_{y_1}(x_1), \nu_{y_2}(x_2)\}},
\end{align*}
\]
\( \forall (x_1, x_2) \in (A_1 \times A_2) \).

\[
\begin{align*}
\text{(ii)} & \quad (\mu_{Z_1} \circ \mu_{Z_2})(((m, x_2), e^{(a_{y_1} \circ a_{y_2})}((m, x_2))) = \min \{\mu_{Z_1}(m), \mu_{Z_2}(x_2)\} e^{\min \{a_{y_1}(m), a_{y_2}(x_2)\}}, \\
& \quad (\lambda_{Z_1} \circ \lambda_{Z_2})(((m, x_2), e^{(y_{y_1} \circ y_{y_2})}((m, x_2))) = \min \{\lambda_{Z_1}(m), \lambda_{Z_2}(x_2)\} e^{\min \{y_{y_1}(m), y_{y_2}(x_2)\}}, \\
& \quad (\nu_{Z_1} \circ \nu_{Z_2})(((m, x_2), e^{(\nu_{y_1} \circ \nu_{y_2})}((m, x_2))) = \max \{\nu_{Z_1}(m), \nu_{Z_2}(x_2)\} e^{\max \{\nu_{y_1}(m), \nu_{y_2}(x_2)\}},
\end{align*}
\]
\( \forall m \in A_1 \) and \( x_2 \in A_2 \).

\[
\begin{align*}
\text{(iii)} & \quad (\mu_{Z_1} \circ \mu_{Z_2})(((x_1, x_2, y_1), e^{(a_{y_1} \circ a_{y_2})}((x_1, x_2, y_1))) = \min \{\mu_{Z_1}(x_1 y_1), \mu_{Z_2}(x_2 y_2)\} e^{\min \{a_{y_1}(x_1 y_1), a_{y_2}(x_2 y_2)\}}, \\
& \quad (\lambda_{Z_1} \circ \lambda_{Z_2})(((x_1, x_2, y_1), e^{(y_{y_1} \circ y_{y_2})}((x_1, x_2, y_1))) = \min \{\lambda_{Z_1}(x_1 y_1), \lambda_{Z_2}(x_2 y_2)\} e^{\min \{y_{y_1}(x_1 y_1), y_{y_2}(x_2 y_2)\}}, \\
& \quad (\nu_{Z_1} \circ \nu_{Z_2})(((x_1, x_2, y_1), e^{(\nu_{y_1} \circ \nu_{y_2})}((x_1, x_2, y_1))) = \max \{\nu_{Z_1}(x_1 y_1), \nu_{Z_2}(x_2 y_2)\} e^{\max \{\nu_{y_1}(x_1 y_1), \nu_{y_2}(x_2 y_2)\}},
\end{align*}
\]
for all \( x_1 y_1 \in B_1 \) and for all \( x_2 y_2 \in B_2 \).

Example 10. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs on \( A_1 = [a, b] \) and \( A_2 = [c, d] \), respectively, which are shown in Figures 17 and 18. Also, semi-strong product is shown in Figure 19.

Proposition 3. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFGs of graph \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. Then, semi-strong product \( G_1 \circ G_2 \) of \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) is CSFG.

\[
(\mu_{Z_1} \circ \mu_{Z_2})(((m, x_2), e^{(a_{y_1} \circ a_{y_2})}((m, x_2)))) = \min \{\mu_{Q_1}(m), \mu_{Q_2}(x_2)\} e^{\min \{a_{y_1}(m), a_{y_2}(x_2)\}},
\]
\[
\leq \min \{\mu_{Q_1}(m), \min \{\mu_{Q_1}(x_2), \mu_{Q_2}(y_2)\}\} e^{\min \{a_{y_1}(m), \min \{a_{y_1}(x_2), a_{y_2}(y_2)\}\}},
\]
\[
= \min \{\mu_{Q_1}(m), \min \{\mu_{Q_1}(x_2), \mu_{Q_2}(y_2)\}\} \min \{\nu_{Q_1}(x_1 y_1), \nu_{Q_2}(x_2 y_2)\} e^{\max \{\nu_{y_1}(x_1 y_1), \nu_{y_2}(x_2 y_2)\}},
\]
\[
= \min \{\mu_{Q_1}(m), \mu_{Q_2}(x_2)\} e^{\min \{a_{y_1}(m), a_{y_2}(x_2)\} e^{\max \{\nu_{y_1}(x_1 y_1), \nu_{y_2}(x_2 y_2)\}}}
\]

Proof. Suppose that \( G_1 = (Q_1, Z_1) \) and \( G_2 = (Q_2, Z_2) \) are two CSFG of graph \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \), respectively. Then, the semi-strong product \( G_1 \circ G_2 \) of \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) can be proved. Let \( (x_1, x_2, y_1, y_2) \in B_1 \times B_2 \).

(i) It is trivial.

(ii) If \( x_1 = y_1 = m \),
\[(a, c) \quad (0.2 e^{0.2\pi i}, 0.2 e^{0.3\pi i}, 0.5 e^{0.2\pi i})
(a, d) \quad (0.2 e^{0.2\pi i}, 0.2 e^{0.3\pi i}, 0.5 e^{0.3\pi i})
(b, c) \quad (0.2 e^{0.2\pi i}, 0.3 e^{0.2\pi i}, 0.4 e^{0.3\pi i})
(b, d) \quad (0.2 e^{0.2\pi i}, 0.3 e^{0.3\pi i}, 0.5 e^{0.4\pi i})
\]

\[\{\lambda_{Q_1} \odot \lambda_{Q_2}\}(m, x_2)(m, y_2) e^{i\{y_{Q_1} \odot y_{Q_2}\}(m, y_2)}
= \min\{\lambda_{Q_1}(m), \lambda_{Q_2}(x_2)\} e^{i\min\{y_{Q_1}(m), y_{Q_2}(x_2)\}}
\leq \min\{\lambda_{Q_1}(m), \min\{\lambda_{Q_1}(x_2), \lambda_{Q_2}(y_2)\}\} e^{i\min\{y_{Q_1}(m), \min\{y_{Q_1}(x_2), y_{Q_2}(y_2)\}\}}
= \min\{\lambda_{Q_1}(m), \lambda_{Q_2}(x_2)\}, \min\{\lambda_{Q_1}(m), \lambda_{Q_2}(y_2)\} e^{i\min\{y_{Q_1}(m), \min\{y_{Q_1}(x_2), y_{Q_2}(y_2)\}\}}
= \min\{\lambda_{Q_1}(m), \lambda_{Q_2}(x_2)\}(m, x_2), (\lambda_{Q_1}(m), \lambda_{Q_2}(y_2)) e^{i\min\{y_{Q_1}(m), \min\{y_{Q_1}(m), y_{Q_2}(y_2)\}\}}
\]

(55)
create a few assumptions that are physically beneficial. In the application, we

4. Application of CSFG

SFs are a useful extension of FSs, IFs, and PFSs. CSFG is useful in decision-making problems. In the application, we create a few assumptions that are physically beneficial.

The International Monetary Fund (IMF) wants to select countries in the world that have a target to deliver funds for minimum number of countries in the world. In such a way, most of the people in a country can take advantage of this project. The following are some of the criteria that were taken into consideration for this purpose:

(1) Elimination of price controls.

(2) Ceiling on government borrowing.

(3) Minimum level of the general government primary balance.

(4) Improving financial sector operations.

(5) Budget consistent with fiscal framework.

(6) Minimum level of international reserves.

(7) Building up social safety nets.

(8) Strengthening public financial management.

The conditionality notion encompasses all of the activities connected with establishing IMF-supported programmes, which include macroeconomic and structural policies, as well as the precise methodologies that are used to monitor progress toward objectives set by a nation in conjunction with the IMF. A government that is experiencing balance of payment difficulties may benefit from using conditionality to help them resolve the problem without resorting to actions that are harmful to the national or international economy. The measures are also meant to preserve IMF resources by ensuring that the country’s
balance of payments stays strong enough to enable it to repay the loan.

Members of a team should choose four countries in which they are planning to release funds so that they may provide assistance to countries that meet the requirements of the IMF. They focus on the following two important situations:

In the first scenario, we proceed as follows. Let \( P = \{ \text{Afghanistan, Pakistan, Bangladesh, Iran} \} \) be the set of countries where the team wishes to deliver the fund to country as a node set. Let 70 percent of the team’s specialists feel that Afghanistan should be chosen, 5 percent of the specialists feel neutral, and 20 percent of the specialists feel that there is no need to release the fund to Afghanistan after carefully analyzing the parameters. Therefore, we can determine the terms of all membership, neutral, and non-membership functions. It is necessary to compute the phase term, which defines the period, in order to do this. Let 30 percent of the specialists feel that in a particular time Afghanistan will fulfill the conditions, 10 percent of the specialists be neutral, and 20 percent of the experts have the opposite opinion. We will make model of this information as

\[
S(\text{Afghanistan}) = (0.7)^2 - (0.05)^2 - (0.2)^2 + \frac{1}{4(\pi)^2}((0.3\pi)^2 - (0.1\pi)^2 - (0.2\pi)^2) = 0.4575,
\]

\[
S(\text{Pakistan}) = (0.9)^2 - (0.4)^2 - (0.1)^2 + \frac{1}{4(\pi)^2}((0.4\pi)^2 - (0.05\pi)^2 - (0.1\pi)^2) = 0.6769,
\]

\[
S(\text{Bangladesh}) = (0.4)^2 - (0.2)^2 - (0.7)^2 + \frac{1}{4(\pi)^2}((0.2\pi)^2 - (0.1\pi)^2 - (0.5\pi)^2) = -0.425,
\]

\[
S(\text{Iran}) = (0.6)^2 - (0.3)^2 - (0.5)^2 + \frac{1}{4(\pi)^2}((0.4\pi)^2 - (0.4\pi)^2 - (0.3\pi)^2) = -0.0025.
\]

Pakistan is the best choice for IMF. Complex spherical fuzzy graph with no edge is shown in Figure 20.

Take \( P = \{ \text{Afghanistan, Pakistan, Bangladesh, Iran} \} = \{ R_1, R_2, R_3, R_4 \} \).

Now the team of IMF looks into situation two as follows.

If analyze the relationship between \( R_1 \) and \( R_2 \) for arc \( R_1 R_2 \):

Suppose the model \( \langle R_1 R_2: 0.6e^{0.5\pi i}, 0.3e^{0.2\pi i}, 0.1e^{0.1\pi i}, 0.4e^{0.4\pi i} \rangle \).

All possible relationships of four countries are given below:

\[
F = \begin{cases} 
\langle R_1 R_2: 0.7e^{0.3\pi i}, 0.05e^{0.05\pi i}, 0.2e^{0.2\pi i} > \text{undefined} \rangle
\end{cases}
\]

Score function of edges for optimal solution is given by

\[
\langle \text{Afghanistan: } 0.7e^{0.3\pi i}, 0.05e^{0.05\pi i}, 0.2e^{0.2\pi i} \rangle. \text{ So, this is their final opinion. Now, the team wants to go to Pakistan. Suppose that the model of this information is } \langle \text{Pakistan: } 0.9e^{0.4\pi i}, 0.4e^{0.05\pi i}, 0.1e^{0.1\pi i} \rangle. \text{ After this, they visit Bangladesh for their valuable mission. Suppose model information about Bangladesh is } \langle \text{Bangladesh: } 0.4e^{0.2\pi i}, 0.2e^{0.1\pi i}, 0.7e^{0.5\pi i} \rangle, \text{ and finally they visit Iran, and the model of this information is } \langle \text{Iran: } 0.6e^{0.4\pi i}, 0.3e^{0.4\pi i}, 0.5e^{0.3\pi i} \rangle. \text{ We represent this model as } \langle H \rangle = \langle H \rangle.
\]

We will use the score function \( S(Q) = \mu_{Q}(\text{Afghanistan}) - \lambda_{Q}(\text{Pakistan}) - \gamma_{Q}(\text{Bangladesh}) - \phi_{Q}(\text{Iran}) \).

We will use the score function of four values to decide which option is the best one.
\[
S(R_1, R_2) = (0.7)^2 - (0.05)^2 - (0.2)^2 + \frac{1}{4(\pi)^2} \left[ (0.3\pi)^2 - (0.05\pi)^2 - (0.2\pi)^2 \right] = 0.4594,
\]
\[
S(R_1, R_3) = (0.4)^2 - (0.05)^2 - (0.7)^2 + \frac{1}{4(\pi)^2} \left[ (0.2\pi)^2 - (0.1\pi)^2 - (0.5\pi)^2 \right] = 0.3875,
\]
\[
S(R_1, R_4) = (0.6)^2 - (0.05)^2 - (0.5)^2 + \frac{1}{4(\pi)^2} \left[ (0.3\pi)^2 - (0.1\pi)^2 - (0.3\pi)^2 \right] = 0.105,
\]
\[
S(R_2, R_3) = (0.4)^2 - (0.2)^2 - (0.7)^2 + \frac{1}{4(\pi)^2} \left[ (0.2\pi)^2 - (0.05\pi)^2 - (0.5\pi)^2 \right] = 0.4575 = -0.4231,
\]
\[
S(R_2, R_4) = (0.6)^2 - (0.3)^2 - (0.5)^2 + \frac{1}{4(\pi)^2} \left[ (0.4\pi)^2 - (0.05\pi)^2 - (0.3\pi)^2 \right] = 0.4575 = 0.0369,
\]
\[
S(R_3, R_4) = (0.4)^2 - (0.2)^2 - (0.7)^2 + \frac{1}{4(\pi)^2} \left[ (0.2\pi)^2 - (0.1\pi)^2 - (0.5\pi)^2 \right] = -0.425.
\]
$S(R_1 R_2)$ is the largest value and therefore more suitable for funding. Complex spherical fuzzy graph with edge is shown in Figure 21.

4.1. Comparative Analysis. In a spherical fuzzy graphical model, there exist just three standard membership grades of each vertex and edge. CSFG can be used to get better approximation. In this work, different types of degrees of vertices have been used. The degree of vertices in SFG gives the total contribution of the amplitude in the system. However, the degree of vertices in CSFG gives the total information and contribution of the amplitude and phase terms.

4.2. Advantages and Limitations. The main advantages of the proposed method are as follows:

- The communication relationship between a few countries has been examined in this article. This method can be used to explain IMF-country communication in the world when dealing with complex spherical fuzzy information.

Some of the limitations of this work are as follows:

- The focus of this research was just on CSFGs and their related network systems.

- In a connected complex spherical fuzzy graphical system, this method is only applicable if three kinds of directional thinking exist.

- It is not always possible to collect real data.

5. Conclusion and Future Works

CSFG is more flexible than the SFG and CPFG. CSFG is the extension of CPFG due to fulfilling the criteria of complex neutrality grade. We have defined the order, size, union, join, and ring sum of CSFG. The degree and the total degree of CSFG are determined. We also discussed four properties on CSFG known as a strong product, composition, Cartesian product, and semi-strong product of CSFG. In the end, we have established the application of CSFG. In the future, we will apply some new operations to CSFG. We will check edge regularity of CSFGs. Furthermore, we will introduce complex Dombi spherical fuzzy graph.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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