Some notes on Goodman’s marginal-free
correspondence analysis

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Résumé

In his seminal paper Goodman (1996) introduced marginal-free
correspondence analysis; where his principal aim was to reconcile
Pearson correlation measure with Yule’s association measure in the
analysis of contingency tables. We show that marginal-free correpon-
dence analysis is a particular case of correspondence analysis with
prespecified weights studied in the beginning of the 1980s by Benzécri
and his students. Furthermore, we show that it is also a particular
first-order approximation of logratio analysis with uniform weights.

Key words: Marginal-free correspondence analysis; logratio ana-
lysis; interactions; scale invariance; taxicab singular value decompo-
sition.

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1 Introduction

Correspondence analysis (CA) and logratio analysis (LRA) are two popu-
lar methods for the analysis and visualization of a contingency table (two-way
frequency counts data having \( I \) rows and \( J \) columns) or a compositional data
set (\( I \) individuals, also named samples, of \( J \) compositional parts). The refer-
ence book on CA is Benzécri (1973); Beh and Lombardo (2014) present a
panoramic review of CA and its variants.

LRA includes two independently well developed methods: RC association
models for the analysis of contingency tables by Goodman (1979, 1981a,
1981b, 1991, 1996) and compositional data analysis (CoDA) by Aitchison (1986). CA and LRA are based on three different principles: CA on Benzécri’s distributional equivalence principle, RC association models on Yule’s scale invariance principle, and CoDA on Aitchison’s subcompositional coherence principle.

From a statistical point of view there is a fundamental difference between the structures of a two-way contingency table $N = (n_{ij})$ and a compositional data set $X = (x_{ij})$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$; while from a mathematical point of view the form of the resulting statistical equations arising from different departure assumptions may be identical in Goodman’s RC association models and Aitchison’s CoDA.

Goodman (1996, equation (46)) in his seminal paper introduced marginal-free correspondence analysis (mfCA); where his principal aim was to reconcile Pearson correlation measure with Yule’s association measure in the analysis of contingency tables. In this paper, we show that mfCA is a particular case of CA with prespecified weights, which has been studied in the beginning of 1980s under the direction of Benzécri. In Benzécri’s edited journal Les Cahiers de l’Analyse des Données, the following papers appeared [Madre (1980), Cholakian (1980, 1984), Benzécri (1983a, 1983b), Benzécri et al. (1980) and Moussaoui (1987)]. Furthermore, we show that mfCA is also a particular first-order approximation of LRA analysis with uniform weights.

This paper is organized as follows: Section 2 presents three different basic ways of representing the concept of interaction in a contingency table; section 3 discusses the the important consequences of Yule’s scale invariance association index; section 4 presents Goodman’s marginal-free CA; section 5 discusses an example; section 6 presents the R code to do the computations; finally we conclude in section 7.

2 Preliminaries on analysis of contingency tables

Let $P = N/n = (p_{ij})$ of size $I \times J$ be the associated correspondence matrix (probability table) of a contingency table $N$. We define as usual $p_{i+} = \sum_{j=1}^{J} p_{ij}$, $p_{+j} = \sum_{i=1}^{I} p_{ij}$, the vector $r = (p_{i+}) \in \mathbb{R}^I$, the vector $c = (p_{+j}) \in \mathbb{R}^J$, and $M_I = \text{Diag}(r)$ the diagonal matrix having diagonal elements $p_{i+}$, and similarly $M_J = \text{Diag}(c)$. We suppose that $M_I$ and $M_J$ are positive definite metric matrices of size $I \times I$ and $J \times J$, respectively; this means that the diagonal elements of $M_I$ and $M_J$ are strictly positive.
2.1 Independence of the row and column categories

a) The \( I \) row categories are independent of the \( J \) column categories,

\[
\sigma_{ij} = p_{ij} - p_i p_{+j} = 0,
\]

where \( \sigma_{ij} \) is the residual matrix of \( p_{ij} \) with respect to the independence model \( p_{i+} p_{+j} \).

\textbf{Remark 1 :} The contingency table \( N = (n_{ij}) \) can also be represented (coded) as an indicator matrix \( Z = \begin{bmatrix} Z^I & Z^J \end{bmatrix} = [(z_{ai}) \ (z_{aj})] \) of size \( n \) by \( I + J \), where \( z_{ai} = 0 \) if individual \( \alpha \) does not have level \( i \) of the row variable, \( z_{ai} = 1 \) if individual \( \alpha \) has level \( i \) of the row variable; \( z_{aj} = 0 \) if individual \( \alpha \) does not have level \( j \) of the column variable, \( z_{aj} = 1 \) if individual \( \alpha \) has level \( j \) of the column variable. Note that \( N = (Z^I)^T Z^J \) and \( \sigma_{ij} = p_{ij} - p_{i+} p_{+j} \) is the covariance between the \( i \)-th column of \( Z^I \) and the \( j \)-th column of \( Z^J \).

b) The independence assumption \( \sigma_{ij} = 0 \) can also be interpreted in another way as

\[
\Delta_{ij} = \left( \frac{p_{ij}}{p_{i+} p_{+j}} - 1 \right) = 0 \quad (2)
\]

\[
= \frac{1}{p_{i+} p_{+j}} \left( p_{ij} - p_{i+} \right) = 0
\]

\[
= \frac{1}{p_{+j}} \left( \frac{p_{ij}}{p_{i+}} - p_{+j} \right);
\]

this is the column and row homogeneity models. Benzécri (1973, p.31) named the conditional probability vector \((\frac{p_{ij}}{p_{i+} p_{+j}})\) for \( i = 1, \ldots, I \) and \( j \) fixed) the profile of the \( j \)-th column; and the element \( \frac{p_{ij}}{p_{i+} p_{+j}} \) the density function of the probability measure \( (p_{ij}) \) with respect to the product measure \( p_{i+} p_{+j} \). The element \( \frac{p_{ij}}{p_{i+} p_{+j}} \) is named Pearson ratio in Goodman (1996) and Beh and Lombardo (2014, p.123).

c) A third way to represent the independence assumption \( \sigma_{ij} = 0 \) and the row and column homogeneity models \( \Delta_{ij} = 0 \) is via the \((w^R_i, w^C_j)\) weighted loglinear formulation, equation (3), assuming \( p_{ij} > 0 \) and defining \( G_{ij} = \log(p_{ij}) \),

\[
\lambda_{ij} = 0
\]

\[
= G_{ij} - G_{i+} - G_{+j} + G_{++}, \quad (3)
\]
where \( G_{i+} = \sum_{j=1}^{J} G_{ij} w_j^C \), \( G_{+j} = \sum_{i=1}^{I} G_{ij} w_i^R \) and \( G_{++} = \sum_{j=1}^{J} \sum_{i=1}^{I} G_{ij} w_j^C w_i^R \); \( w_j^C > 0 \) and \( w_i^R > 0 \), satisfying \( \sum_{j=1}^{J} w_j^C = \sum_{i=1}^{I} w_i^R = 1 \), are a priori fixed probability weights. Two popular weights are marginal \( (w_j^C = p_{+j}, w_i^R = p_{i+}) \) and uniform \( (w_j^C = 1/J, w_i^R = 1/I) \). This is implicit in equation 7 in Goodman (1996) or equation 2.2.6 in Goodman (1991); and explicit in Egozcue et al. (2015).

Equation (3) is equivalent to the logratios

\[
\log \left( \frac{p_{ij}}{p_{i_{1}j_{1}}} \right) = 0 \quad \text{for} \quad i \neq i_1 \quad \text{and} \quad j = j_1,
\]

which Goodman (1979, equation 2.2) names it "null association" model.

Equation (3) is also equivalent to

\[
p_{ij} = \frac{\exp(G_{i+}) \exp(G_{+j})}{\exp(G_{++})},
\]

from which we deduce that: under the independence assumption the marginal row probability vector \((p_{i+})\) is proportional to the vector of weighted geometric means \((\exp(G_{i+}))\); and a similar property is true also for the columns; see for instance Egozcue et al. (2015).

### 2.2 Interaction factorization

Suppose the independence-homogeneity-null association models are not true, then each of the three equivalent model formulations (1,2,3) can be generalized to explain the nonindependence-nonhomogeneity-association, named interaction, among the \( I \) rows and the \( J \) columns by adding \( k \) bilinear terms, where \( k = \text{rank}(N) - 1 \). We designate any one of the interaction indices (1,2,3) by \( \tau_{ij} \).

Benzécri (1973, Vol.1, p. 31-32) emphasized the importance of row and column weights or metrics in multidimensional data analysis; this is the reason in the french data analysis circles any study starts with a triplet \((X, M_I, M_J)\), where \( X \) represents the data set, \( M_I = (\text{Diag}(m_i^r)) \) is the metric defined on the rows and \( M_J = (\text{Diag}(m_j^c)) \) the metric defined on the columns. We follow the same procedure where :

- a) In covariance analysis, \( X = (\tau_{ij}) = (\sigma_{ij}) \) and \( (M_I, M_J) = (\text{Diag}(1/I), \text{Diag}(1/J)) \);
- b) In CA, \( X = (\tau_{ij}) = (\Delta_{ij}) \) and \( (M_I, M_J) = (\text{Diag}(p_{i+}), \text{Diag}(p_{+j})) \);
c) In LRA, $X = (\tau_{ij}) = (\lambda_{ij})$ and $(M_I, M_J) = (Diag(w^R_i), Diag(w^C_j))$ with $\sum_{j=1}^{I} w^C_j = \sum_{i=1}^{I} w^R_i = 1$.

We factorize the interactions in (1,2,3) by singular value decomposition (SVD) or taxicab SVD (TSVD) as

$$\tau_{ij} = \sum_{\alpha=1}^{k} f^{(i)}(\alpha)g^{(j)}(\alpha)/\delta^{(\alpha)}.$$  (4)

In the SVD case the parameters $(f^{(i)}(\alpha), g^{(j)}(\alpha), \delta^{(\alpha)})$ satisfy the conditions : for $\alpha, \beta = 1, ..., k$

$$\delta^{2}_{\alpha} = \sum_{\alpha=1}^{k} f^{(i)}(\alpha)m^{r}_{i} = \sum_{\alpha=1}^{k} g^{2}_{\alpha}(j)m^{c}_{j}$$

$$0 = \sum_{\alpha=1}^{k} f^{(i)}(\alpha)m^{r}_{i} = \sum_{\alpha=1}^{k} g^{(j)}(\alpha)m^{c}_{j}$$

$$0 = \sum_{\alpha=1}^{k} f^{(i)}(\alpha)f^{(i)}(\beta)m^{r}_{i} = \sum_{\alpha=1}^{k} g^{(j)}(\alpha)g^{(j)}(\beta)m^{c}_{j}$$

In the TSVD case the parameters $(f^{(i)}(\alpha), g^{(j)}(\alpha), \delta^{(\alpha)})$ satisfy the conditions : for $\alpha, \beta = 1, ..., k$

$$\delta_{\alpha} = \sum_{\alpha=1}^{k} |f^{(i)}(\alpha)m^{r}_{i} = \sum_{\alpha=1}^{k} |g^{(j)}(\alpha)m^{c}_{j}$$

$$0 = \sum_{\alpha=1}^{k} f^{(i)}(\alpha)m^{r}_{i} = \sum_{\alpha=1}^{k} g^{(j)}(\alpha)m^{c}_{j}$$

$$0 = \sum_{\alpha=1}^{k} f^{(i)}(\alpha) \text{sign}(f^{(i)}(\beta)m^{r}_{i} = \sum_{\alpha=1}^{k} g^{(j)}(\alpha) \text{sign}(g^{(j)}(\beta)) m^{c}_{j} \text{ for } \alpha > \beta.$$  

A description of TSVD can be found, among others, in Choulakian (2006, 2016).

**Remark 2**
a) In the case $(\tau_{ij}) = (\sigma_{ij})$, the bilinear decomposition (4) is also named interbattery analysis first proposed by Tucker (1958); later on, Tenenhaus
and Augendre (1996) reintroduced it within correspondence analysis circles, where they showed that the Tucker decomposition by SVD produced on some correspondence tables more interesting structure, more interpretable, than CA. 

b) In the case $\tau_{ij} = (\Delta_{ij})$, the CA decomposition has many interpretations. Essentially, for data analysis purposes Benzécri (1973) interpreted it as weighted principal components analysis of row and column profiles. Another useful interpretation of CA, comparable to Tucker interbattery analysis, is Hotelling (1936)’s canonical correlation analysis, see Lancaster (1958) and Goodman (1991, 1996).

3 Yule’s principle of scale invariance

We start by quoting Goodman (1996, section 10) to really understand Yule’s principle of scale invariance: “Pearson’s approach to the analysis of cross-classified data was based primarily on the bivariate normal. He assumed that the row and column classifications arise from underlying continuous random variables having a bivariate normal distribution, so that the sample contingency table comes from a discretized bivariate normal; and he then was concerned with the estimation of the correlation coefficient for the underlying bivariate normal. On the other hand, Yule felt that, for many kinds of contingency tables, it was not desirable in scientific work to introduce assumptions about an underlying bivariate normal in the analysis of these tables; and for such tables, he used, to a great extent, coefficients based on the odds-ratios (for example, Yule’s Q and Y), coefficients that did not require any assumptions about underlying distributions. The Pearson approach and the Yule approach appear to be wholly different, but a kind of reconciliation of the two perspectives was obtained in Goodman (1981a)”.

An elementary exposition of these ideas with examples can also be found in Mosteller (1968).

In the notation of our paper, Goodman’s reconciliation is based on defining the a priori weights in the association index (3), $\lambda_{ij} = \lambda(p_{ij}, w^C_j, w^R_i)$, where by its decomposition into bilinear terms, mWlRA will correspond to
Pearson’s approach, while uwLRA to Yule’s approach. Because log-odds
\[
\log\left(\frac{p_{ij}p_{i1}j}{p_{j1}p_{ij1}}\right) = \lambda_{ij} + \lambda_{i1}j - \lambda_{i1} - \lambda_{ij1}
\]
\[
= \sum_{\alpha=1}^{k} (f_\alpha(i) - f_\alpha(i_1))(g_\alpha(j) - g_\alpha(j_1))/\delta_\alpha.
\] (5)

To have a clear picture of LRA with general a priori prescribed weights \((w_j^C, w_i^R)\), we first study the properties of the association index \(\lambda_{ij}\), that distinguishes it from interaction indices \((2,3)\).

### 3.1 Scale invariance of an interaction index

We are concerned with the property of scale dependence or independence of the three interaction indices \((1,2,3)\). We note that in \((1,2,3)\), \(p_{ij}\) depends on \(n_{ij}, p_{ij} = n_{ij}/\sum_{i,j} n_{ij}\). To emphasize this dependence, we express an interaction index by \(\tau_{ij}(n_{ij}) = \tau(p_{ij}, m_i^R, m_j^C)\) where: in the case of the association index \(\tau_{ij}(n_{ij}) = \lambda_{ij}\) is defined in \((3)\), in the case of the nonhomogeneity index \(\tau_{ij}(n_{ij}) = \Delta_{ij}\) is defined in \((2)\), and in the case of the nonindependence index \(\tau_{ij}(n_{ij}) = \sigma_{ij}\) is defined in \((1)\). Following Yule (1912), we state the following

**Definition 1**: An interaction index \(\tau_{ij}(n_{ij})\) is scale invariant if \(\tau_{ij}(n_{ij}) = \tau_{ij}(a_i n_{ij}, b_j)\) for scales \(a_i > 0\) and \(b_j > 0\).

It is important to note that Yule’s principle of scale invariance concerns a function of four interaction terms, see equation \((5)\); while in Definition 1 the invariance concerns each interaction term.

It is evident that the interaction indices \((1\ and\ 2)\) are not scale invariant: because they are marginal-dependent.

Concerning the association index \((3)\) we have

**Lemma 1**: The association index \((3)\) is scale invariant.
Proof: Let \( n^* = \sum_{i,j} a_i n_{ij} b_j \), then

\[
\tau_{ij}(a_i n_{ij} b_j) = \lambda(a_i n_{ij} b_j / n^*, w^C_j, w^R_i)
\]

\[
= \log(a_i n_{ij} b_j / n^*) - \sum_{j=1}^{J} w^C_j \log(a_i n_{ij} b_j / n^*)
\]

\[
- \sum_{i=1}^{I} w^R_i \log(a_i n_{ij} b_j / n^*) + \sum_{j=1}^{J} \sum_{i=1}^{I} w^C_i w^R_i \log(a_i n_{ij} b_j / n^*)
\]

\[
= \lambda(n_{ij}, w^C_j, w^R_i) = \lambda(p_{ij}, w^C_j, w^R_i) = \tau_{ij}(n_{ij})
\]

\[
= \lambda(a_i p_{ij} b_j, w^C_j, w^R_i).
\]

(6)

Lemma 2: To a first-order approximation, \( \lambda_{ij} \approx \frac{p_{ij}}{w^C_j w^R_i} - \frac{p_{i+}}{w^R_i} - \frac{p_{+j}}{w^C_j} + 1 \).

Proof: The average value of the density function \( \frac{p_{ij}}{w^C_j w^R_i} \) with respect to the product measure \( w^C_j w^R_i \) is 1; so the \( IJ \) values \( \frac{p_{ij}}{w^C_j w^R_i} \) are distributed around 1. By Taylor series expansion of \( \log x \) in the neighborhood of \( x = 1 \), we have to a first-order \( \log x \approx x - 1 \).

Putting \( a_i = 1/w^R_i \) and \( b_j = 1/w^C_j \) in (6), and by using \( \log\left(\frac{p_{ij}}{w^C_j w^R_i}\right) \approx \frac{p_{ij}}{w^C_j w^R_i} - 1 \), which is the required result.

Remark 3: Lemma 2 provides a first order approximation to mwTLRA and uwTLRA, where we see that both first-order approximations are marginal-dependent but in different ways.

a) In the case \( (a_i, b_j) = (1/p_{i+}, 1/p_{+j}) \) and \( (w^C_j, w^R_i) = (p_{+j}, p_{i+}) \) in Lemma 2, \( \lambda_{ij} = \lambda(p_{ij}, p_{+j}, p_{i+}) \approx \frac{p_{ij}}{p_{i+p} p_{+j}} - 1 \); which implies that CA (or TCA)
is a first-order approximation of mwLRA (or mwTLRA), a result stated in Cuadras et al. (2006).

b) In the case \((a_i, b_j) = (I, J)\) and \((w^C_j, w^R_i) = \frac{1}{I}, \frac{1}{J}\) in Lemma 2, 
\[ \lambda_{ij} = \lambda(p_{ij}1/I, 1/J) \approx IJp_{ij} - Ip_{iJ} + Jp_{iJ} + 1; \]
which implies that the bilinear expansion of the right side by TSVD (or SVD) is a first-order approximation of uwTLRA (or uwLRA).

In this subsection, we discussed the approximation of LRA to CA related methods. Greenacre (2009) posed the reciprocal question: when CA related methods converge to LRA? And he stated two results; which we provide a proof in the following subsection.

### 3.2 Box-Cox transformation

Theoretically CA and LRA have been presented in a unified mathematical framework via Box-Cox transformation by Goodman (1996), where the bilinear terms have been estimated by SVD. Goodman’s framework was further considered, among others, by Cuadras et al. (2006), Greenacre (2009, 2010), and Cuadras and Cuadras (2015).

Consider the triplet \((X, Q, D)\), where \(X = (x_{ij})\) with \(x_{ij} > 0\) represents the data set, and \((Q, D) = (Diag(w_i^R), Diag((w_j^C)))\) with \(\sum_{j=1}^{J} w_j^C = \sum_{i=1}^{I} w_i^R = 1\). Let \(\alpha\) be a nonnegative real number. Following Goodman (1996, equations (3,4,5)), we define the interaction index,

\[
Int\left(\frac{x_{ij}^\alpha}{\alpha}, w^C_j, w^R_i\right) = \frac{x_{ij}^\alpha}{\alpha} - \sum_{j=1}^{J} w^C_j \frac{x_{ij}^\alpha}{\alpha} - \sum_{i=1}^{I} w^R_i \frac{x_{ij}^\alpha}{\alpha} + \sum_{j=1}^{J} \sum_{i=1}^{I} w^C_j w^R_i \frac{x_{ij}^\alpha}{\alpha}
\]

\[
= \left(\frac{x_{ij}^\alpha}{\alpha} - 1\right) - \sum_{j=1}^{J} w^C_j \left(\frac{x_{ij}^\alpha}{\alpha} - 1\right)
\]

\[
- \sum_{i=1}^{I} w^R_i \left(\frac{x_{ij}^\alpha}{\alpha} - 1\right) + \sum_{j=1}^{J} \sum_{i=1}^{I} w^C_j w^R_i \left(\frac{x_{ij}^\alpha}{\alpha} - 1\right).
\]

Using the well-known result based on Hopital’s rule, \(\lim_{\alpha \to 0} \left(\frac{x_{ij}^\alpha}{\alpha} - 1\right) = \log(x_{ij})\),
We consider two cases of (7, 8):

a) $\lambda(x_{ij}, w_j^C, w_i^R) = \lambda(p_{ij}, p_{+j}, p_{+i}) = \lambda(p_{ij}/(p_{+j}p_{+i}), p_{+j}, p_{+i})$, which is the interaction term of mwLRA, and equivalent to Result 2 in Greenacre (2010).

b) $\lambda(x_{ij}, w_j^C, w_i^R) = \lambda(p_{ij}, 1/J, 1/I) = \lambda(IJp_{ij}, 1/J, 1/I) = \lambda(p_{ij}/(p_{+j}p_{+i}), 1/J, 1/I)$, which is the interaction term of uwLRA; this is similar to Result 1 in Greenacre (2010).

Equation (7) can also be applied differently, where:

In (7) we replace $w_j^C$ by $w_j^C(\alpha) = \frac{\sum_{i=1}^{J} x_{ij}^0}{\sum_{i=1}^{J} \sum_{j=1}^{J} x_{ij}^0}$, $w_i^R$ by $w_i^R(\alpha) = \frac{\sum_{i=1}^{J} x_{ij}^0}{\sum_{i=1}^{J} \sum_{j=1}^{J} x_{ij}^0}$;
we see that $\lim_{\alpha \to 0} w_j^C(\alpha) = \frac{I}{J} = 1/J$; similarly $\lim_{\alpha \to 0} w_i^R(\alpha) = 1/I$. Then we get

$\lim_{\alpha \to 0} Int(x_{ij}^0, w_j^C(\alpha), w_i^R(\alpha)) = \lambda(x_{ij}, 1/J, 1/I)$.

In particular $\lim_{\alpha \to 0} Int(x_{ij}^0, w_j^C(\alpha), w_i^R(\alpha)) = \lambda(p_{ij}, 1/J, 1/I) = \lambda(IJp_{ij}, 1/J, 1/I) = \lambda(p_{ij}/(p_{+j}p_{+i}), 1/J, 1/I)$.

4 CA with prescribed weights and Goodman’s mfCA

CA with prescribed weights is done in two steps in the following way: We observe a probability table $P = (p_{ij})$ of size $I$ by $J$. Let $Q$ of size $I$ by $J$ be an unknown probability table with known marginals $q_{+i}$ and $q_{+j}$. The two steps are:

Step1: We construct $Q$ which is in a sense ’nearest to $P$’. Two general criteria are: $Int(q_{ij}, q_{+j}, q_{+i}) = \lambda(q_{ij} = a_ip_{ij}b_j, q_{+j}, q_{+i})$ based on (3) and

$\min_{q_{ij}} \sum_{i,j} \left( \frac{q_{ij} - q_{ij+q_{+j}}}{q_{ij+q_{+j}}} \right)^2 q_{ij+q_{+j}}$ based on (2).

Step 2: We apply CA to the constructed probability $Q$

$\frac{q_{ij} - q_{ij+q_{+j}}}{q_{ij+q_{+j}}} = \sum_{\alpha=1}^{k} f_{\alpha}(i)g_{\alpha}(j)/\delta_{\alpha},$
which represents CA of $P$ with prespecified weights $(q_{i+}, q_{+j})$. Cholakian (1980) presents an example, where both criteria have been applied and similar results have been obtained.

In the particular case, where $\text{Int}(q_{ij}, 1/J, 1/I) = \lambda(q_{ij} = a_i p_{ij} b_j, 1/J, 1/I)$, we get Goodman’s mfCA, see Goodman (1996, equation (46)). $Q = (q_{ij})$ is related to $P = (p_{ij})$ via the strictly positive scales $(a_i, b_j)$, that keeps the association between the $i$-th row and the $j$-th column unchanged. The famous iterative proportional fitting algorithm (IPF) is used to construct $Q$. That is, the constructed probability table $(q_{ij})$ has uniform marginals $q_{+j} = 1/J$ and $q_{i+} = 1/I$. So in Step 2, CA representation is

$$IJq_{ij} - 1 = \sum_{\alpha=1}^{k} f_\alpha(i)g_\alpha(j)/\delta_\alpha,$$  

which represents a first-order approximation to both uwLRA and m wLRA by Remark 3. Furthermore, by Remark 2 we see that mfCA can be interpreted both as Tucker and Hotelling decompositions.

5 Examples

We present the analysis of two datasets for comparative purposes.

5.1 Example 1

This dataset is contrived and taken from Goodman (1991, Table 10(1), that we reproduce below

$$
\begin{bmatrix}
4 & 10 & 1 \\
10 & 50 & 10 \\
1 & 10 & 4 \\
\end{bmatrix}
$$

According to Goodman, LRA has one principal dimension, while CA has 2 principal dimensions. Here we compare the dispersion results of the 4 methods: CA, TCA, mfCA and mfTCA.

In CA : $\delta_1 = \text{corr}(f_1(i), g_1(j)) = 0.20$ and $\delta_2 = \text{corr}(f_2(i), g_2(j)) = 0.048$

In mfCA : $\delta_1 = \text{corr}(f_1(i), g_1(j)) = 0.41$ and $\delta_2 = \text{corr}(f_2(i), g_2(j)) = 0.050$

In TCA : $\delta_1 = 0.070$ and $\delta_2 = 0.034$

In mfTCA : $\delta_1 = 0.285$ and $\delta_2 = 0.043$
5.2 Example 2

We consider the *rodent* data set of size 28 by 9 found in TaxicabCA in R package. This is an abundance data set of 9 kinds of rats in 28 cities in California. It can be considered both a contingency table and a compositional data set. Choulakian (2017) analyzed it by comparing the CA and TCA maps; furthermore Choulakian (2021) showed that it has quasi-2-blocks structure. Here we compare the dispersion results of the first 2 principal dimensions in the 4 methods: CA, TCA, mfCA and mfTCA:

In CA: \( \delta_1 = \text{corr}(f_1(i), g_1(j)) = 0.864 \) and \( \delta_2 = \text{corr}(f_2(i), g_2(j)) = 0.678 \).

In mfCA: \( \delta_1 = \text{corr}(f_1(i), g_1(j)) = 0.827 \) and \( \delta_2 = \text{corr}(f_2(i), g_2(j)) = 0.679 \).

In TCA: \( \delta_1 = 0.478 \) and \( \delta_2 = 0.422 \).

In mfTCA: \( \delta_1 = 0.743 \) and \( \delta_2 = 0.541 \).

The curious reader can apply the R code below to compare the 4 maps: CA, mfCA, TCA and mfTCA.

6 R code

```r
# # install packages
install.packages(c("ipfr", "ca", "TaxicabCA"))
#
library(TaxicabCA)
dataMatrix = as.matrix(rodent)
nRow <- nrow(dataMatrix)
nCol <- ncol(dataMatrix)
ssize <- sum(dataMatrix)
#
# Computation of Q matrix of rodent
library(ipfr)
mtx <- dataMatrix
row_targets <- rep(ssize/nRow, nRow)
column_targets <- rep(ssize/nCol, nCol)
QMatrix <- ipu.matrix(mtx, row_targets, column_targets)
rownames(QMatrix) <- paste("", 1:nRow, sep="")
colnames(QMatrix) <- paste("C", 1:nCol, sep="")
```

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# CA map of rodent dataset
library(ca)
plot(ca(dataMatrix))

# mfCA map of rodent
plot(ca(QMatrix))

# TCA map of rodent

tca.Data <- tca(dataMatrix, nAxes=2,algorithm = "exhaustive")
plot(tca.Data,
    axes = c(1, 2),
    labels.rc = c(1, 1),
    col.rc = c("blue", "red"),
    pch.rc = c(5, 5, 0.3, 0.3),
    mass.rc = c(F, F),
    cex.rc = c(0.6, 0.6),
    jitter = c(F, T))

# mfTCA map of rodent dimensions 1-2

tca.Data <- tca(QMatrix, nAxes=2,algorithm = "exhaustive")
plot(tca.Data,
    axes = c(1, 2),
    labels.rc = c(1, 1),
    col.rc = c("blue", "red"),
    pch.rc = c(5, 5, 0.3, 0.3),
    mass.rc = c(F, T),
    cex.rc = c(0.6, 0.6),
    jitter = c(F, T))

# mfTCA map of rodent dimensions 2-3

tca.Data <- tca(QMatrix, nAxes=2,algorithm = "exhaustive")
plot(tca.Data,
    axes = c(2, 3),
    labels.rc = c(1, 1),
    col.rc = c("blue", "red"),
pch.rc = c(5, 5, 0.3, 0.3),
mass.rc = c(F, T),
cex.rc = c(0.6, 0.6),
jitter = c(F, T)
)

7 Conclusion

In his seminal paper Goodman (1996) introduced marginal-free correspondence analysis; where his principal aim was to reconcile Pearson correlation measure with Yule’s association measure in the analysis of contingency tables. We showed that marginal-free correspondence analysis is a particular case of correspondence analysis with prespecified weights studied in the beginning of the 1980s by Benzécri and his students. Furthermore, we showed that it is also a particular first-order approximation of logratio analysis with uniform weights.

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