CREMONA MAPS AND INVOLUTIONS

JULIE DÉSERTI

ABSTRACT. We deal with the following question of Dolgachev: is the Cremona group generated by involutions? Answer is yes in dimension 2 (see [6]). We give an upper bound of the minimal number $n_\varphi$ of involutions we need to write a birational self map $\varphi$ of $\mathbb{P}^2$.

We prove that de Jonquières maps of $\mathbb{P}^3$ and maps of small bidegree of $\mathbb{P}^3$ can be written as a composition of involutions of $\mathbb{P}^3$ and give an upper bound of $n_\varphi$ for such maps $\varphi$. We get similar results in particular for automorphisms of $(\mathbb{P}^1)^n$, automorphisms of $\mathbb{C}^n$, monomial maps of $\mathbb{P}^2$, and elements of the subgroup generated by the standard involution of $\mathbb{P}^n$ and $\text{PGL}(n + 1, \mathbb{C})$.

1. INTRODUCTION

This article is motivated by the following question:

**Question** (Dolgachev). Is the $n$ dimensional Cremona group generated by involutions?

Answer is yes in dimension 2; more precisely:

**Proposition 1.1** ([6]). *For any $\varphi$ in Bir($\mathbb{P}^2$) there exist $A_0, A_1, \ldots, A_k$ in Aut($\mathbb{P}^2$) such that*

$$\varphi = \left( A_0 \circ \sigma_2 \circ A_0^{-1} \right) \circ \left( A_1 \circ \sigma_2 \circ A_1^{-1} \right) \circ \cdots \circ \left( A_k \circ \sigma_2 \circ A_k^{-1} \right)$$

*where $\sigma_2$ denotes the standard involution of $\mathbb{P}^2$

$$\sigma_2: (z_0 : z_1 : z_2) \rightarrow (z_1 z_2 : z_0 z_2 : z_0 z_1).$$

Let us note that since Bir($\mathbb{P}^2$) is generated by $\text{PGL}(3, \mathbb{R})$ and some involutions ([21]), any element of Bir($\mathbb{P}^2$) can be written as a composition of involutions.

If $\varphi$ is an element of $G$, then $n(\varphi, H)$ is the minimal number of involutions of $H \supset G$ we need to write $\varphi$. In dimension 2 we get the following result:

**Theorem A.** *If $\varphi$ is an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$, then $n(\varphi, \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)) \leq 4$.

If $\varphi$ is an automorphism of $\mathbb{P}^2$, then $n(\varphi, \text{Aut}(\mathbb{P}^2)) \leq 8$.

If $\varphi$ belongs to the Jonquières subgroup $J_2 \subset \text{Bir}(\mathbb{P}^2)$, then $n(\varphi, J_2) \leq 10$.

If $\varphi$ is a birational self map of $\mathbb{P}^2$ of degree $d$, then $n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 10d - 2$.*

One can be more precise for the well-known subgroup Aut($\mathbb{C}^2$) of polynomial automorphisms of $\mathbb{C}^2$ of Bir($\mathbb{P}^2$):

**Theorem B.** *Let $\varphi$ be an element of Aut($\mathbb{C}^2$) of degree $d$. Then $n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 44 + \frac{2d}{4}.$

More precisely,

- if $\varphi$ is affine, then $n(\varphi, \text{Aut}(\mathbb{P}^2)) \leq 8$;
- if $\varphi$ is elementary, then $n(\varphi, J_2) \leq 10$;
- if $\varphi$ is generalized Hénon map, then either it is of jacobian 1 and $n(\varphi, \text{Aut}(\mathbb{C}^2)) = 2$ or $n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 11$;*

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• If \( d \) is prime, then \( n(\varphi, \text{Bir}(\mathbb{P}^2_C)) \leq 26 \).

What happens in higher dimension? A first result is the following:

**Theorem C.** Let \( n \geq 3 \). Every automorphism of \( (\mathbb{P}^1_C)^n \), then \( \varphi \) can be written as a composition of involutions of \( (\mathbb{P}^1_C)^n \), and \( n(\varphi, \text{Aut}(\mathbb{P}^1_C)^n) \leq 2n \).

• If \( \varphi \) is an automorphism of \( \mathbb{P}^n_C \), then \( \varphi \) can be written as a composition of involutions of \( \mathbb{P}^n_C \), and \( n(\varphi, \text{Aut}(\mathbb{P}^n_C)) \leq 2(n + 1) \).

Since any element of

\[
G_n(C) = \langle \sigma_n = \left( \prod_{i \neq 0}^n z_i : \prod_{i \neq 1}^n z_i : \ldots : \prod_{i \neq n}^n z_i \right), \text{Aut}(\mathbb{P}^n_C) \rangle
\]

can be written as a composition of conjugate involutions ([7]) one gets that:

**Theorem D.** Let \( n \geq 3 \). Any element of the normal subgroup generated by \( G_n(C) \) in \( \text{Bir}(\mathbb{P}^n_C) \) can be written as a composition of involutions of \( \mathbb{P}^n_C \).

Furthermore one can give an upper bound of \( n(\varphi, \text{Bir}(\mathbb{P}^n_C)) \) when \( \varphi \) belongs to the subgroup of tame automorphisms of \( C^n \):

**Theorem E.** Let \( n \geq 3 \). Let \( \varphi \) be a tame automorphism of \( C^n \) of degree \( d \). Then \( \varphi \) can be written as a composition of involutions of \( \mathbb{P}^n_C \). Moreover,

- if \( \varphi \) is affine, then \( n(\varphi, \text{Aut}(\mathbb{P}^n_C)) \leq 2n + 4 \);
- if \( \varphi \) is elementary, then \( n(\varphi, \text{Bir}(\mathbb{P}^n_C)) \leq 2n + 10 \);
- otherwise \( n(\varphi, \text{Bir}(\mathbb{P}^n_C)) \leq \frac{4}{3}(2n + 7) + 10n + 32 \).

Let us recall (see [15]) that the Jonquières subgroup \( J_0(1, \mathbb{P}^3_C) \) of \( \text{Bir}(\mathbb{P}^3_C) \) is given in the affine chart \( z_3 = 1 \) by

\[
\{ \varphi = (\varphi_0(z_0, z_1, z_2), \psi(z_1, z_2)) | \varphi_0 \in \text{PGL}(2, C[z_1, z_2]), \psi \in \text{Bir}(\mathbb{P}^2_C) \}
\]

Denote by \( \text{Mon}(n, C) \) the group of monomial maps of \( \mathbb{P}^n_C \), and finally set

\[
J_n = \text{PGL}(2, C(z_1, z_2, \ldots, z_{n-1})) \times \text{PGL}(2, C(z_2, z_3, \ldots, z_{n-1})) \times \ldots \times \text{PGL}(2, C(z_{n-1})) \subset \text{Bir}(\mathbb{P}^n_C).
\]

**Theorem F.** Assume that \( 2 \leq \ell \leq 4 \), and \( n \geq 3 \).

- If \( \varphi \in \text{Bir}(\mathbb{P}^3_C) \) is of bidegree \( (2, \ell) \), then \( \varphi \) can be written as a composition of involutions of \( \mathbb{P}^3_C \), and \( n(\varphi, \text{Bir}(\mathbb{P}^3_C)) \leq 9 + 7\ell \).
- Any element \( \varphi \) of \( J_0(1; \mathbb{P}^3_C) \) of degree \( d \) can be written as a composition of involutions of \( \mathbb{P}^3_C \), and \( n(\varphi, \text{Bir}(\mathbb{P}^3_C)) \leq 10d + 6 \).
- If \( \varphi \) belongs to \( \text{Mon}(n, C) \), then \( \varphi \) can be written as a composition of involutions of \( \mathbb{P}^n_C \), and \( n(\varphi, \text{Bir}(\mathbb{P}^n_C)) \leq 3n + 9 \).
- Any element \( \varphi \) of \( J_n \) can be written as a composition of involutions of \( \mathbb{P}^n_C \), and \( n(\varphi, J_n) \leq 4(2n - 1) \).

If \( H \) is a subgroup of \( G \) let us denote by \( N(H; G) \) the normal subgroup generated by \( H \) in \( G \).

**Corollary G.** Any element of

\[
\langle N(\text{PGL}(4, C); \text{Bir}(\mathbb{P}^3_C)), N(J_0(1; \mathbb{P}^3_C); \text{Bir}(\mathbb{P}^3_C)), N(\text{Mon}(3, C); \text{Bir}(\mathbb{P}^3_C)),
N(G_3(C); \text{Bir}(\mathbb{P}^3_C)), N(\langle \varphi_1, \ldots, \varphi_k \rangle; \text{Bir}(\mathbb{P}^3_C)) | \varphi_i \in \text{Bir}(\mathbb{P}^3_C) \text{ of bidegree } (2, \ell), 2 \leq \ell \leq 4 \rangle
\]

can be written as a composition of involutions of \( \mathbb{P}^3_C \).
For any \( n \geq 4 \), any element of
\[
(N[\text{PGL}(n+1, \mathbb{C}); \text{Bir}(\mathbb{P}^n_2)], N[\text{Inn}(\text{Bir}(\mathbb{P}^n_2)), N(\text{Mon}(n, \mathbb{C}); \text{Bir}(\mathbb{P}^n_2)), N(G_n(\mathbb{C}); \text{Bir}(\mathbb{P}^n_2))]
\]
can be written as a composition of involutions of \( \mathbb{P}^n_2 \).

**Remark 1.2.** An other motivation for studying birational maps of \( \mathbb{P}^n_2 \) that can be written as a composition of involutions is the following. The group of birational maps of \( \mathbb{P}^n_2 \) that can be written as a composition of involutions is a normal subgroup of \( \text{Bir}(\mathbb{P}^n_2) \). So if the answer to Dolgachev Question is no, we can give a negative answer to the following question asked by Mumford ([12]): is \( \text{Bir}(\mathbb{P}^n_2) \) a simple group?

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## 2. Recalls and Definitions

### 2.1. Polynomial automorphisms of \( \mathbb{C}^n \).
A polynomial automorphism \( \varphi \) of \( \mathbb{C}^n \) is a bijective map from \( \mathbb{C}^n \) into itself of the type
\[
(z_0, z_1, \ldots, z_{n-1}) \mapsto (\varphi_0(z_0, z_1, \ldots, z_{n-1}), \varphi_1(z_0, z_1, \ldots, z_{n-1}), \ldots, \varphi_{n-1}(z_0, z_1, \ldots, z_{n-1})
\]
with \( \varphi_i \in \mathbb{C}[z_0, z_1, \ldots, z_{n-1}] \). The set of polynomial automorphisms of \( \mathbb{C}^n \) form a group denoted \( \text{Aut}(\mathbb{C}^n) \).

Let \( A_n \) be the group of affine automorphisms of \( \mathbb{C}^n \), and let \( E_n \) be the group of elementary automorphisms of \( \mathbb{C}^n \). In other words \( A_n \) is the semi-direct product of \( \text{GL}(n, \mathbb{C}) \) with the commutative unipotent subgroup of translations. Furthermore \( E_n \) is formed with automorphisms \( (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \) of \( \mathbb{C}^n \) where
\[
\varphi_i = \varphi_i(z_i, z_{i+1}, \ldots, z_{n-1})
\]
depends only on \( z_i, z_{i+1}, \ldots, z_{n-1} \). The subgroup \( \text{Tame}_n \) of \( \text{Aut}(\mathbb{C}^n) \), called the group of tame automorphisms of \( \mathbb{C}^n \), is the group generated by \( A_n \) and \( E_n \). For \( n = 2 \) one has \( \text{Tame}_2 = \text{Aut}(\mathbb{C}^2) \), more precisely:

**Theorem 2.1 ([11]).** The group \( \text{Aut}(\mathbb{C}^2) \) has a structure of amalgamated product
\[
\text{Aut}(\mathbb{C}^2) = A_2 \ast_{S_2} E_2
\]
with \( S_2 = A_2 \cap E_2 \).

Nevertheless \( \text{Tame}_3 \neq \text{Aut}(\mathbb{C}^3) \) (see [19]).

### 2.2. Birational maps of \( \mathbb{P}^n_2 \).
A rational self map of \( \mathbb{P}^n_2 \) is a map of the type
\[
(z_0 : z_1 : \ldots : z_n) \mapsto (\varphi_0(z_0, z_1, \ldots, z_n) : \varphi_1(z_0, z_1, \ldots, z_n) : \ldots : \varphi_n(z_0, z_1, \ldots, z_n))
\]
where the \( \varphi_i \)'s denote homogeneous polynomials of the same degree without common factor (of positive degree).

A birational self map \( \varphi \) of \( \mathbb{P}^n_2 \) is a rational map of \( \mathbb{P}^n_2 \) such that there exists a rational self map \( \psi \) of \( \mathbb{P}^n_2 \) with the following property \( \varphi \circ \psi = \psi \circ \varphi = \text{id} \) where id: \( (z_0 : z_1 : \ldots : z_n) \mapsto (z_0 : z_1 : \ldots : z_n) \).

The degree of \( \varphi \in \text{Bir}(\mathbb{P}^n_2) \) is the degree of the \( \varphi_i \)'s. For \( n = 2 \), one has \( \text{deg} \varphi = \text{deg} \varphi^{-1} \); for \( n = 3 \) such an equality does not necessary hold, we thus speak about the bidegree of \( \varphi \) which is \( (\text{deg} \varphi, \text{deg} \varphi^{-1}) \).

The group of birational self maps of \( \mathbb{P}^n_2 \) is denoted \( \text{Bir}(\mathbb{P}^n_2) \) and called Cremona group.

The groups \( \text{Aut}(\mathbb{P}^n_2) = \text{PGL}(n+1, \mathbb{C}) \) and \( \text{Aut}(\mathbb{C}^n) \) are subgroups of \( \text{Bir}(\mathbb{P}^n_2) \).

Let us mention that contrary to \( \text{Aut}(\mathbb{C}^2) \) the Cremona group in dimension 2 does not decompose as a non-trivial amalgam (appendix of [4]).
2.3. Birational involutions in dimension 2. Let us first describe some involutions:

- Consider an irreducible curve \( \mathcal{C} \) of degree \( v \geq 3 \) with a unique singular point \( p \); assume furthermore that \( p \) is an ordinary multiple point with multiplicity \( v - 2 \). To \((\mathcal{C}, p)\) we can associate a birational involution \( \mathcal{I}_f \) which fixes pointwise \( \mathcal{C} \) and preserves lines through \( p \) as follows. Let \( m \) be a generic point of \( P^2_{\mathcal{C}} \setminus \mathcal{C} \); let \( r_m, q_m \) and \( p \) be the intersections of the line \((mp)\) with \( \mathcal{C} \); the point \( \mathcal{I}_f(m) \) is defined by: the cross ratio of \( m, \mathcal{I}_f(m), q_m \) and \( r_m \) is equal to \(-1\). The map \( \mathcal{I}_f \) is a de Jonquières involution of \( P^2_{\mathcal{C}} \). A birational involution is of de Jonquières type if it is birationally conjugate to a de Jonquières involution of \( P^2_{\mathcal{C}} \).

- Let \( p_1, p_2, \ldots, p_8 \) be eight points of \( P^2_{\mathcal{C}} \) in general position. Consider the set of sextics \( \mathcal{S} = \mathcal{S}(p_1, p_2, \ldots, p_8) \) with double points at \( p_1, p_2, \ldots, p_8 \). Take a point \( m \) in \( P^2_{\mathcal{C}} \). The pencil given by the elements of \( \mathcal{S} \) having a double point at \( m \) has a tenth base double point point \( m^\prime \). The involution which switches \( m \) and \( m^\prime \) is a \text{Bertini involution}. A birational involution is of Bertini type if it is birationally conjugate to a Bertini involution.

- Let \( p_1, p_2, \ldots, p_7 \) be seven points of \( P^2_{\mathcal{C}} \) in general position. Denote by \( L \) the linear system of cubics through the \( p_i \)'s. Consider a generic point \( p \) in \( P^2_{\mathcal{C}} \) and define by \( L_p \) the pencil of elements of \( L \) passing through \( p \). The involution which switches \( p \) and the ninth base-point of \( L_p \) is a \text{Geiser involution}. A birational involution is of Geiser type if it is birationally conjugate to a Geiser involution.

Birational involutions of \( P^2_{\mathcal{C}} \) have been classified:

**Theorem 2.2** ([2]). A non-trivial birational involution of \( P^2_{\mathcal{C}} \) is either of de Jonquières type, or of Bertini type, or of Geiser type.

2.4. Birational involutions in higher dimension. There are no classification in higher dimension; in [17] the author gives a first nice step toward a classification in dimension 3. Let us give some examples:

- the involution
  \[
  \sigma_n = \left( \prod_{i=0}^{n} z_i : \prod_{i=0}^{n} z_i : \ldots : \prod_{i=0}^{n} z_i \right)
  \]

- the involutions of \( \text{PGL}(n + 1, \mathbb{C}) \);
- the involutions of \( \text{Mon}(n, \mathbb{C}) \) induced by the involutions of \( \text{GL}(n, \mathbb{Z}) \);
- the de Jonquières involutions: consider a reduced hypersurface \( H \) of degree \( v \) in \( P^n_{\mathbb{C}} \) that contains a linear subspace of points of multiplicity \( v - 2 \). Assume that \( p \) is a singular point of \( H \) of multiplicity \( v - 2 \). Take a general point \( m \) of \( H \). Denote by \( \ell_p \) the line passing through \( p \) and \( m \). The intersection of \( \ell_p \) with \( H \) contains \( p \) with multiplicity \( v - 2 \), and the residual intersection is a set of two points \( r_m \) and \( q_m \) in \( \ell_p \). Define \( \mathcal{I}_f(m) \) to be the point on \( \ell_p \) such that the cross ratio of \( m, \mathcal{I}_f(m), q_m \) and \( r_m \) are equal to \(-1\). The map \( \mathcal{I}_f \) is a de Jonquières involution of \( P^n_{\mathbb{C}} \).

3. Automorphisms of \((P^1_{\mathbb{C}})^n\) and of \( P^n_{\mathbb{C}} \)

**Lemma 3.1.** Any non-trivial homography is either an involution, or the composition of two involutions of \( \text{PGL}(2, \mathbb{C}) \).
In particular if \( \varphi \) belongs to \( \text{Aut}(\mathbb{P}^1_n \times \mathbb{P}^1_n \times \ldots \times \mathbb{P}^1_n) \), then

\[
n(\varphi, \text{Aut}(\mathbb{P}^1_n \times \mathbb{P}^1_n \times \ldots \times \mathbb{P}^1_n)) \leq 2n.
\]

**Remark 3.2.** The homography \( v \in \text{PGL}(2, \mathbb{C}) \) is a non-trivial involution if and only if there exists \( p \in \mathbb{P}^1 \setminus \text{Fix}(v) \) such that \( v^2(p) = p \), where \( \text{Fix}(v) \) denotes the set of fixed points of \( v \).

Indeed assume that there exists \( p \in \mathbb{P}^1 \setminus \text{Fix}(v) \) such that \( v^2(p) = p \), then \( \text{Fix}(v) \neq \mathbb{P}^1 \) and so \( v \neq \text{id} \). If \( m \in \{p, v(p)\} \), then \( v^2(m) = m \). If \( p \notin \{p, v(p)\} \), the cross ratio of \( p, v(p), v(m), m \) is equal to the cross ratio of \( p, v(p), v(m) \) and \( v^2(m) \). This implies that \( v^2(m) = m \).

**Lemma 3.3.** Let \( v \in \text{PGL}(2, \mathbb{C}) \) be an homography. Consider three points \( a, b, c \) of \( \mathbb{P}^1 \) such that \( a, b, c \) are distinct, \( a \notin \text{Fix}(v) \), \( b \notin \text{Fix}(v) \) and \( b \neq v(a) \).

There exist two involutions \( t_1, t_2 \in \text{PGL}(2, \mathbb{C}) \) such that \( v = t_2 \circ t_1 \).

**Proof.** Let us first prove that there exists two unique homographies \( t_1, t_2 \in \text{PGL}(2, \mathbb{C}) \) such that

\[
\begin{cases}
t_1(a) = v(b), & t_1(b) = v(a), & t_1(v(a)) = b; \\
t_2(v(a)) = v(b), & t_2(v(b)) = v(a), & t_2(t_1(c)) = v(c).
\end{cases}
\]

Note that by assumptions \( a, b, v(a) \) (resp. \( v(b), v(a), b \)) are pairwise distinct. Hence there exists a unique homography \( t_1 \in \text{PGL}(2, \mathbb{C}) \) that sends \( a, b, v(a) \) onto \( v(b), v(a), b \).

The points \( v(a), v(b) \) and \( t_1(c) \) are distinct. Assume by contradiction that \( t_1(c) = v(a) \), then \( t_1(c) = t_1(b) \). By injectivity of \( t_1 \), one has \( c = b \); contradiction. Similarly \( t_1(c) \neq v(b) = t_1(a) \) and \( v(a) \neq v(b) \). Since \( a, b, c \) are distinct, \( v(a), v(b) \) and \( v(c) \) also. As a consequence there exists a unique homography \( t_1 \in \text{PGL}(2, \mathbb{C}) \) that sends \( v(a), v(b), t_1(c) \) onto \( v(b), v(a), v(c) \).

By assumption \( b \) and \( v(a) \) are distinct so \( b \) does not belong to \( \text{Fix}(t_1) \). But \( t_1^2(b) = t_1(v(a)) = b \). According to Remark 3.2 the homography \( t_1 \) is thus an involution. Similarly \( v(a) \) and \( v(b) \) are distinct but \( v(a) \) and \( v(b) \) are switched by \( t_2 \) hence \( t_2 \) is an involution (Remark 3.2).

Since \( v(p) = t_2 \circ t_1(p) \) for \( p \in \{a, b, c\} \) one gets \( v = t_2 \circ t_1 \).

**Proof of Lemma 3.1.** Let \( v \) be an homography. If \( v = \text{id} \), then \( v = t \circ i \) for any involution \( t \). Assume now that \( v \neq \text{id} \); then \( v \) has at most two fixed points. Let us choose \( a, b \) in \( \mathbb{P}^1 \setminus \text{Fix}(v) \). If \( a \neq v(b) \) or if \( b \neq v(a) \), then \( v \) can be written as a composition of two involutions (Lemma 3.3). If \( b = v(a) \) and \( a = v(b) \), then \( v^2(a) = a \) with \( a \notin \text{Fix}(v) \); Remark 3.2 thus implies that \( v \) is an involution.

**Lemma 3.4.** Let \( n \geq 2 \) be an integer.

1. Let \( k \) be a commutative ring of any characteristic. If \( \varphi \) is an element of \( \text{SL}(n, k) \), then \( n(\varphi, \text{SL}(n, k)) \leq 2(n + 1) \).
2. Assume that \( k \) is an algebraically closed field, and that \( \varphi \) belongs to \( \text{PGL}(n, k) \). Then \( n(\varphi, \text{PGL}(n, k)) \leq 2(n + 1) \).
3. If \( \varphi \) is an element of \( \text{PGL}(2, \mathbb{C}|z_1, z_2, \ldots, z_{n-1}) \), then \( n(\varphi, \text{Bir}(\mathbb{P}^n)) \leq 8 \).

**Proof.**

1. Let us recall that an element of \( \text{SL}(n, k) \) can be written as a composition of \( \leq n + 1 \) transvections ([16]). But a transvection is a composition of two involutions so any element of \( \text{SL}(n, k) \) can be written as a composition of \( \leq 2(n + 1) \) involutions.
2. If \( k \) is algebraically closed, then \( \text{PSL}(n, k) \cong \text{PGL}(n, k) \) and one gets the result.
(3) Let \( g \) be an element of \( \text{PGL}(2, \mathbb{C}[z_1, z_2, \ldots, z_{n-1}]) \); denote by \( P(z_1, z_2, \ldots, z_{n-1}) \) its determinant and by \( h \) a scaling of scale factor \( \frac{1}{P(z_1, z_2, \ldots, z_{n-1})} \). Then \( h \circ g \) belongs to \( \text{SL}(2, \mathbb{C}[z_1, z_2, \ldots, z_{n-1}]) \) and hence, according to the first assertion, can be written as a composition of \( \leq 6 \) involutions. But \( h \) is as a composition of two involutions:

\[
\frac{1}{z_0 P(z_1, z_2, \ldots, z_{n-1})} \circ \frac{1}{z_0}
\]

As a result \( n(\varphi, \text{Bir}(\mathbb{P}_C^2)) \leq 8. \)

\( \square \)

4. Dimension 2

4.1. The real Cremona group. There is an analogue to Proposition 1.1 for the real Cremona group.

**Theorem 4.1.** Any element of \( \text{Bir}(\mathbb{P}_R^2) \) can be written as a composition of involutions of \( \mathbb{P}_R^2 \).

Theorem 4.1 directly follows from the simplicity of \( \text{PGL}(3, \mathbb{R}) \) and the following statement:

**Proposition 4.2** ([21]). The group \( \text{Bir}(\mathbb{P}_R^2) \) is generated by \( \text{PGL}(3, \mathbb{R}) \), the set of standard quintic involutions and the two following quadratic involutions

\[
(z_1 z_2 : z_0 z_2 : z_0 z_1) \quad \quad (z_0 z_2 : z_1 z_2 : z_0^2 + z_1^2).
\]

4.2. The de Jonquières subgroup. An element of \( \text{Bir}(\mathbb{P}_C^2) \) is a de Jonquières map if it preserves a rational fibration, i.e. if it is conjugate to an element of

\[
J_2 = \text{PGL}(2, \mathbb{C}(z_1)) \rtimes \text{PGL}(2, \mathbb{C}).
\]

We will denote by \( \tilde{J}_2 \) the subgroup of birational maps that preserves fiberwise the fibration \( z_1 = \text{constant} \), i.e. \( \tilde{J}_2 = \text{PGL}(2, \mathbb{C}(z_1)) \).

**Lemma 4.3.** If \( \varphi = (\varphi_0, \varphi_1) \) belongs to \( \tilde{J}_2 \), then \( n(\varphi, \tilde{J}_2) \leq 8 \).

Furthermore if \( \det \varphi_0 = \pm 1 \), then \( n(\varphi, \tilde{J}_2) \leq 4 \).

**Proof.** Lemma 3.4 implies the first assertion, and the last assertion follows from [9]. \( \square \)

**Corollary 4.4.** Any de Jonquières map of \( \mathbb{P}_C^2 \) can be written as a composition of \( \leq 10 \) Cremona involutions of \( \mathbb{P}_C^2 \).

**Proof.** Let us remark that any Jonquières map \( \varphi \) of \( \mathbb{P}_C^2 \) can be written as \( \psi \circ j \circ \psi^{-1} \) where \( \psi \) denotes an element of \( \text{Bir}(\mathbb{P}_C^2) \) and \( j \) an element of \( J_2 \). But

\[
j = \left( \frac{\alpha(z_1) z_0 + b(z_1)}{c(z_1) z_0 + d(z_1)}, \frac{\alpha z_1 + \beta}{z_1 + \delta} \right) = \left( z_0, \frac{\alpha z_1 + \beta}{z_1 + \delta} \right) \circ \left( \frac{\alpha(z_1) z_0 + b(z_1)}{c(z_1) z_0 + d(z_1)}, z_1 \right)
\]

As a consequence

\[
\varphi = \left( \psi \circ \left( z_0, \frac{\alpha z_1 + \beta}{z_1 + \delta} \right) \circ \psi^{-1} \right) \circ \left( \psi \circ \left( \frac{\alpha(z_1) z_0 + b(z_1)}{c(z_1) z_0 + d(z_1)}, z_1 \right) \circ \psi^{-1} \right)
\]

Then one concludes with Lemmas 3.1 and 4.3. \( \square \)
4.3. **Subgroup of polynomial automorphisms of** $\mathbb{C}^2$. Note that there is no analogue to Proposition 1.1 in the context of polynomial automorphisms of $\mathbb{C}^2$. For instance the automorphism $(2z_0,3z_1)$ cannot be written as a composition of involutions in $\text{Aut}(\mathbb{C}^2)$.

According to Lemma 3.4 and Corollary 4.4 one has the following result:

**Lemma 4.5.** Let $\varphi$ be a polynomial automorphism of $\mathbb{C}^2$.

If $\varphi$ is an affine automorphism, then $n(\varphi, \text{Aut}(\mathbb{P}_\mathbb{C}^2)) \leq 8$.

If $\varphi$ is an elementary automorphism, then $n(\varphi, J_2) \leq 10$.

An element $\varphi \in \text{Aut}(\mathbb{C}^2)$ is a generalized Hénon map if

$$\varphi = (z_1, P(z_1) - \delta z_0)$$

where $\delta$ belongs to $\mathbb{C}^*$ and $P$ is an element of $\mathbb{C}[z_1]$ of degree $\geq 2$. Note that $\delta = \text{jac}(\varphi)$.

**Lemma 4.6.** Let $\varphi \in \text{Aut}(\mathbb{C}^2)$ be a generalized Hénon map.

- If $\varphi$ has jacobian 1, then $n(\varphi, \text{Aut}(\mathbb{C}^2)) \leq 2$.
- Otherwise $n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq 11$.

**Proof.** Any generalized Hénon map of jacobian 1 can be written $(z_1, P(z_1) - z_0)$ and so is the composition of two involutions: $(z_1, P(z_1) - z_0) = (z_1, z_0) \circ (P(z_1) - z_0, z_1)$.

Let $\varphi$ be a generalized Hénon map; then

$$\varphi = (z_1, P(z_1) - \delta z_0) = (z_1, z_0) \circ (P(z_1) - \delta z_0, z_1).$$

Note that $(P(z_1) - \delta z_0, z_1)$ is an elementary automorphism; therefore $n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq 1 + 10 = 11$ (Lemma 4.5).

Friedland and Milnor proved that any polynomial automorphism of degree $d$ with $d$ prime is conjugate via an affine automorphism either to a generalised Hénon map or to an elementary automorphism ([8, Corollary 2.7]). Since any generalised Hénon map is the composition of $(z_1, z_0) \in A_2$ with an elementary map one gets that any polynomial automorphism of degree $d$ with $d$ prime can be written as $a_1 e a_2$ with $a_i \in A_2$ and $e \in E_2$. Lemmas 4.5 and 4.6 thus imply:

**Lemma 4.7.** If $\varphi \in \text{Aut}(\mathbb{C}^2)$ is of degree $d$ with $d$ prime, then $n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq 26$.

A sequence $(\varphi_1, \varphi_2, \ldots, \varphi_k)$ of length $k \geq 1$ is a reduced word, representing the group element $\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$ if

- each factor $\varphi_i$ belongs to either $A_2$ or $E_2$ but not to the intersection $A_2 \cap E_2$,
- and no two consecutive factors belong to the same subgroup $A_2$ or $E_2$.

It follows from Theorem 2.1 that every element of $\text{Aut}(\mathbb{C}^2)$ can be expressed as such a reduced word, unless it belongs to the intersection $S_2 = A_2 \cap E_2$. The degree of any reduced word $\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$ is equal to the product of the degree of the factor $\varphi_i$ (see [8, Theorem 2.1]). Hence take $\varphi \in \text{Aut}(\mathbb{C}^2)$ of degree $d \geq 2$, then $\varphi$ is a reduced word $\varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$ and

- either there exists only one $\varphi_i$ of degree $> 1$, then $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$ with $\deg \varphi_2 > 1$ and $\deg \varphi_1 = \deg \varphi_3 = 1$; as a result $n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq 26$ (Lemma 4.5),
- or there exits at least two $\varphi_i$’s of degree $> 1$, then $n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq \frac{4d}{3} + 44$. Indeed let $(a_0, e_1, a_1, e_2, a_2, \ldots, e_k, a_k)$ be a reduced word representing $\varphi$. Any $e_i$ has degree $\geq 2$ and $\deg \varphi = \deg e_1 \deg e_2 \prod_{i=3}^k \deg e_i$; hence $\prod_{i=3}^k \deg e_i \leq \frac{d}{4}$ and so $2(k-2) \leq \frac{d}{4}$.

As a result $k \leq \frac{d}{8} + 2$ and

$$n(\varphi, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq (k + 1)n(a_i, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) + kn(e_i, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) \leq \left(\frac{d}{8} + 3\right)n(a_i, \text{Bir}(\mathbb{P}_\mathbb{C}^2)) + \left(\frac{d}{8} + 2\right)n(e_i, \text{Bir}(\mathbb{P}_\mathbb{C}^2)).$$
One can thus state

**Theorem 4.8.** Let \( \varphi \) be a polynomial automorphism of \( \mathbb{C}^2 \) of degree \( d \).

- If \( \varphi \) is affine, \( n(\varphi, \text{Aut}(\mathbb{P}^2)) \leq 8 \);
- if \( \varphi \) is elementary, then \( n(\varphi, 1_2) \leq 10 \);
- if \( \varphi \) is generalized Hénon map, then either it is of jacobian 1 and \( n(\varphi, \text{Aut}(\mathbb{C}^2)) = 2 \) or \( n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 11 \);
- if \( d \) is prime, then \( n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 26 \);
- otherwise \( n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq \frac{9d}{4} + 44 \).

**Corollary 4.9.** If \( \varphi \) is a polynomial automorphism of \( \mathbb{C}^2 \) of degree \( d \), then \( n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq \frac{9d}{4} + 44 \).

### 4.4 Birational maps

**Theorem 4.10.** If \( \varphi \in \text{Bir}(\mathbb{P}^2) \) is of degree \( d \), then \( n(\varphi, \text{Bir}(\mathbb{P}^2)) \leq 10d - 2 \).

Before proving Theorem 4.10 let us give a first and "bad" bound. Let \( \varphi \) be a birational self map of \( \mathbb{P}^2 \) of degree \( d \). The number of base points of \( \varphi \) is \( \leq 2d - 1 \) and the map \( \varphi \) can be written with \( \leq 2(2d - 1) \) blow ups. Since a blow up can be written as \( A_1 \circ \sigma_2 \circ A_2 \circ \sigma_2 \circ A_3 \) with \( A_i \in \text{PGL}(3, \mathbb{C}) \) the map \( \varphi \) can be written with \( 4(2d - 1) \) involution \( \sigma_2 \) and \( 4(2d - 1) + 1 \) elements of \( \text{PGL}(3, \mathbb{C}) \). As a consequence \( \varphi \) can be written as a composition of \( \leq 4(2d - 1) + 8(4(2d - 1) + 1) = 72d - 28 \) involutions.

**Proof of Theorem 4.10.** Let us recall that if \( \varphi \) is a birational self map of \( \mathbb{P}^2 \) of degree \( d \), then there exists a de Jonquières map \( \psi \) of \( \mathbb{P}^2 \) such that \( \deg(\varphi \circ \psi) < d \) (see [5], [1, Theorem 8.3.4]).

As a result any \( \varphi \in \text{Bir}(\mathbb{P}^2) \) of degree \( d \geq 1 \) can be written as follows

\[
A \circ (\psi_1 \circ j_1 \circ \psi_1^{-1}) \circ (\psi_2 \circ j_2 \circ \psi_2^{-1}) \circ \ldots \circ (\psi_k \circ j_k \circ \psi_k^{-1})
\]

with \( A \) in \( \text{PGL}(3, \mathbb{C}) \), \( \psi_\ell \in \text{Bir}(\mathbb{P}^2) \), \( j_\ell \in 1_2 \) and \( k \leq d - 1 \).

The statement follows from Lemma 3.4 and Corollary 4.4. \(\square\)

### 5. Dimension 3

#### 5.1. de Jonquières maps in dimension 3

Let us recall that a de Jonquières map \( \varphi \) of \( \mathbb{P}^2 \) of degree \( d \) is a plane Cremona map satisfying one of the following equivalent conditions:

- there exists a point \( \sigma \in \mathbb{P}^2 \) such that the restriction of \( \varphi \) to a general line passing through \( \sigma \) maps it birationally to a line passing through \( \sigma \);
- \( \varphi \) has homaloidal type \((d; d - 1, 1, 2d - 2)\), i.e. \( \varphi \) has \( 2d - 1 \) base points, one of multiplicity \( d - 1 \) and \( 2d - 2 \) of multiplicity \( 1 \);
- \( \varphi \) is of jacobian 1 and the map \( \varphi \) can be written with \( \leq 2(2d - 1) \) blow ups.

In [15] Pan and Simis propose suitable generalizations of de Jonquières maps to higher dimensional space \( \mathbb{P}^n \), \( n \geq 3 \). More precisely they study elements of the Cremona group \( \text{Bir}(\mathbb{P}^n) \) satisfying a condition akin to the first alternative above: for a point \( \sigma \in \mathbb{P}^n \) and a positive integer \( k \) they consider the Cremona transformations that map a general \( k \)-dimensional linear subspace passing through \( \sigma \) onto another such subspace. Fixing the point \( \sigma \) these maps form a subgroup \( J_\varphi(k; \mathbb{P}^n) \) of \( \text{Bir}(\mathbb{P}^n) \). For any \( k \leq \ell \) the following inclusion holds ([15])

\[
J_\varphi(\ell; \mathbb{P}^n) \subset J_\varphi(k; \mathbb{P}^n)
\]
Let us recall the following characterization of elements of $I_G(1;\mathbb{P}_C^n)$:

**Proposition 5.1** ([13]). Fix $\theta = (0:0: \ldots :0:1)$. A Cremona map $\varphi \in \text{Bir}(\mathbb{P}_C^n)$ belongs to $I_G(1;\mathbb{P}_C^n)$ if and only if

$$\varphi = \left( z_0 g_{d-1} + g_d : (z_0 q_{\ell-1} + q_\ell) t_1 : (z_0 q_{\ell-1} + q_\ell) t_2 : \ldots : (z_0 q_{\ell-1} + q_\ell) t_n \right)$$

where

- $g_d, g_{d-1}, q_\ell, q_{\ell-1}, t_1, \ldots, t_n \in \mathbb{C}[z_1, z_2, \ldots, z_n]$, 
- $\deg g_{d-1} = d - 1$, $\deg g_d = d$, $\deg q_{\ell-1} = \ell - 1$, $\deg q_\ell = \ell$, 
- $\deg t_i = d - \ell$ for $i \in \{1, \ldots, n\}$, 
- $(t_1 : t_2 : \ldots : t_n) \in \text{Bir}(\mathbb{P}_C^{n-1})$.

**Theorem 5.2.** Let $\varphi$ be an element of $I_G(1,\mathbb{P}_C^3)$ of degree $d$; then $n(\varphi, \text{Bir}(\mathbb{P}_C^3)) \leq 10d + 6$.

If $H$ is a subgroup of $G$ let us denote by $N(H; G)$ the normal subgroup generated by $H$ in $G$.

**Corollary 5.3.** Any birational map of $N(I_G(1,\mathbb{P}_C^3); \text{Bir}(\mathbb{P}_C^3))$ is a composition of involutions of $\text{Bir}(\mathbb{P}_C^3)$.

**Proof of Theorem 5.2.** Any $\varphi$ in $I_G(1,\mathbb{P}_C^3)$ can be written in the affine chart $z_3 = 1$

$$\varphi = \left( \frac{z_0 A(z_1, z_2) + B(z_1, z_2)}{z_0 C(z_1, z_2) + D(z_1, z_2)}, \psi(z_1, z_2) \right)$$

where

$$\frac{z_0 A(z_1, z_2) + B(z_1, z_2)}{z_0 C(z_1, z_2) + D(z_1, z_2)} \in \text{PGL}(2, \mathbb{C}[z_1, z_2]), \quad \psi \in \text{Bir}(\mathbb{P}_C^2).$$

Let us note that

$$\varphi = (z_0, \psi(z_1, z_2)) \circ \left( \frac{z_0 A(z_1, z_2) + B(z_1, z_2)}{z_0 C(z_1, z_2) + D(z_1, z_2)}, z_1, z_2 \right).$$

The map $\psi$ can be written as a composition of $\leq 10d - 2$ involutions (Theorem 4.10) and $\frac{z_0 A(z_1, z_2) + B(z_1, z_2)}{z_0 C(z_1, z_2) + D(z_1, z_2)} \in \text{PGL}(2, \mathbb{C}[z_1, z_2])$ can be written as a composition of $\leq 8$ involutions (Lemma 3.4). As a result $\varphi$ is a composition of $10d + 6$ or fewer involutions. \hfill \Box

### 5.2. Maps of small bidegrees.

If $\varphi$ is a birational self map of $\mathbb{P}_C^3$, then the bidegree of $\varphi$ is the pair $(\deg \varphi, \deg \varphi^{-1})$. Let us recall that $\deg \varphi^{-1} \leq (\deg \varphi)^2$. The left-right conjugacy is the following one

$$\text{PGL}(4, \mathbb{C}) \times \text{Bir}(\mathbb{P}_C^3) \times \text{PGL}(4, \mathbb{C}) \quad (A, \varphi, B) \mapsto A \varphi B^{-1}.$$

Pan, Ronga and Vust give birational self maps of $\mathbb{P}_C^3$ of bidegree $(2, \cdot)$ up to left-right conjugacy, and show that there are only finitely many bimclasses ([14, Theorems 3.1.1, 3.2.1, 3.2.2, 3.3.1]). In particular they show that the smooth and irreducible variety of birational self maps of $\mathbb{P}_C^3$ of bidegree $(2, \cdot)$ has three irreducible components of dimension 26, 28, 29. More precisely the component of dimension 26 (resp. 28, resp. 29) corresponds to birational maps of bidegree $(2, 4)$ (resp. $(2, 3)$, resp. $(2, 2)$). Let us denote by $\Theta(\varphi)$ the orbit of $\varphi$ under the left-right conjugacy.

**Proposition 5.4.** Let $\varphi$ be a birational self map of $\mathbb{P}_C^3$ of bidegree $(2, 2)$. Then $\varphi$ can be written as a composition of involutions of $\mathbb{P}_C^3$. Furthermore $n(\varphi, \text{Bir}(\mathbb{P}_C^3)) \leq 23$. 

Proof. If \( \varphi \) is a birational self map of \( \mathbb{P}^3_\mathbb{C} \) of bidegree (2, 2), then up to left-right conjugacy \( \varphi \) is one of the following \((14)\)

\[
\begin{align*}
f_1 &= (z_0 z_1 : z_1 z_3 : z_2 : z_3 : z_0^2 - z_1 z_2) & f_2 &= (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_1 z_2)
f_3 &= (z_0 z_1 : z_1 z_3 : z_2 : z_3^2) & f_4 &= (z_0 z_1 : z_1 z_3 : z_2^2 : z_1 z_3 - z_0^2 + z_1 z_2)
f_5 &= (z_0 z_1 : z_2 z_3 : z_1 z_3 : z_1 z_2) & f_6 &= (z_0 z_1 : z_2 z_3 : z_1^2 : z_1 z_3 - z_2^2) \\
f_7 &= (z_0 z_1 - z_1 z_2 : z_1 z_3 : z_2 z_3 : z_3^2) & f_8 &= (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_3^2)
\end{align*}
\]

Note that \( f_8 = \text{id} \), and that \( f_1, f_2, f_3 \) are involutions. Any element \( \psi \) in \( \Theta(f_i) \), \( i \in \{1, 2, 3, 8\} \), satisfies \( n(\psi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 21 \). The other \( f_i \) are de Jonquières maps of \( \mathbb{P}^3_\mathbb{C} \) so according to Theorem 5.2 can be written as compositions of involutions. Nevertheless to find a better bound for \( n(\varphi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \) we will give explicit decompositions.

First

\[
f_4 = (z_0 : z_2 + z_3 : -z_2 : z_1) \circ (z_0 z_2 : z_1 z_3 : z_2^2 : z_2 z_3) \circ (z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3) \circ (z_0 z_2 : z_1 z_3 : z_2 z_3 : z_0 + z_3)
\]

hence for any \( \psi \in \Theta(f_4) \) one has \( n(\psi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 23 \) (which corresponds to two elements in \( \text{PGL}(4, \mathbb{C}) \) and three involutions).

Second

\[
f_5 = (z_0 : z_3 - z_2 : z_2 : z_1) \circ (z_0 z_2 : z_1 z_3 : z_2^2 : z_2 z_3) \circ (z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3) \circ (z_0 : z_1 : -z_3 : z_2 + z_3)
\]

As a consequence \( n(\psi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 22 \) for any \( \psi \in \Theta(f_5) \).

Third

\[
f_6 = (z_0 : z_1 : z_3 : -z_2) \circ (z_0 z_3 : z_1 z_3 : z_1^2 - z_3 z_2 : z_3^2) \circ (z_0 : z_1 : z_3 - z_2 : z_3).
\]

Therefore for any \( \psi \in \Theta(f_6) \) one has the inequality \( n(\psi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 21 \).

Last

\[
f_7 = ( -z_0 : z_1 : z_2 : z_3) \circ ( -z_0 z_3 + z_1 z_2 : z_1 z_3 : z_2 z_3 : z_3^2)
\]

So \( n(\psi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 21 \) for any \( \psi \in \Theta(f_7) \). \( \square \)

**Proposition 5.5.** Any birational self map \( \varphi \) of \( \mathbb{P}^3_\mathbb{C} \) of bidegree (2, 3) can be written as a composition of involutions of \( \mathbb{P}^3_\mathbb{C} \); moreover \( n(\varphi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 30 \).

**Proof.** If \( \varphi \) is a birational self map of \( \mathbb{P}^3_\mathbb{C} \) of bidegree (2, 3), then \( \varphi \in \Theta(f_i) \) where \( f_i \) is one of the following map \((14)\)

\[
\begin{align*}
f_1 &= ( -z_0 z_1 + z_0 z_2 : z_0 z_3 : -z_0 z_1 + z_1 z_2 : z_1 z_3) & f_2 &= ( z_0 z_1 : z_0 z_2 - z_1 z_2 : z_0 z_3 : z_1 z_3)
f_3 &= ( z_0 z_1 - z_0 z_2 : z_0 z_3 : z_1 z_2 : z_1 z_3) & f_4 &= ( z_0 z_1 : z_0 z_2 : z_0 z_3 - z_1 z_3 : z_1^2)
f_5 &= ( z_0 z_2 : z_0 z_3 : z_1^2 : z_1 z_3) & f_6 &= ( z_0 z_1 : z_0 z_2 - z_1^2 : z_0 z_3 : z_1 z_3)
f_7 &= ( z_0 z_1 : z_0 z_2 - z_1 z_3 : z_0 z_3 : z_1^2) & f_8 &= ( z_0 z_2 - z_1^2 : z_0 z_3 : z_1 z_2 : z_1 z_3)
f_9 &= ( z_0^2 : z_0 z_1 : z_1 z_2 : z_0 z_3 - z_1^2) & f_{10} &= ( z_0^2 - z_1^2 : z_0 z_2 : z_0 z_3 : z_1^2)
f_{11} &= ( z_0 z_2 + z_1^2 : z_0 : z_0 z_1 : z_0 z_3 - z_1 z_2) &
\end{align*}
\]
Let us give for any of these maps a decomposition with involutions and elements of $\text{PGL}(4, \mathbb{C})$:

\[
\begin{align*}
\phi_1 &= \left(-z_2 + z_0 : -z_1 + z_3 : z_0 : z_3 \right) \circ \left( z_0 z_1 : z_3^2 : z_2 z_3 : z_1 z_3 \right) \circ \left( z_0 z_1 : z_3^2 : z_1 z_2 : z_1 z_3 \right) \\
& \circ \left( z_0 : z_1 - z_2 : z_3 \right) \circ \left( z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3 \right) \circ \left( z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3 \right) \\
& \circ \left(-z_0 + z_2 : z_1 : z_2 : z_3 \right) \circ \left( z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3 \right) \circ \left( z_0 z_2 : z_1 z_3 : z_3^2 : z_2 z_3 \right) \\
& \circ \left(x_2^2 : z_1 z_3 : z_0 z_2 : z_0 z_3 \right) \circ \left(x_2^2 : z_1 z_3 : z_0 z_2 : z_0 z_3 \right) \\
\phi_2 &= \left(z_1 : z_2 : z_0 : z_3 \right) \circ \left( z_0 z_3 : z_0 z_2 : z_2 z_3 : x_2^2 \right) \circ \left(z_0 + z_1 : z_2 : z_3 \right) \\
& \circ \left(x_2^2 : z_1 z_3 : z_0 z_2 : z_0 z_3 \right) \circ \left(z_0 z_1 : z_3^2 : z_2 z_3 : z_1 z_3 \right) \circ \left(z_0 z_1 : z_3^2 : z_2 z_3 : z_1 z_3 \right) \\
& \circ \left(z_0 z_1 : z_3^2 : z_1 z_2 : z_1 z_3 \right) \circ \left(z_0 : z_1 : z_2 : z_3 \right) \\
\phi_3 &= \left(-z_0 : z_1 : z_2 : z_3 \right) \circ \left(-z_0 z_3 + z_1 z_2 : z_1 z_3 : z_2 z_3 : z_3^2 \right) \\
& \circ \left(x_2^2 : z_1 z_3 : z_0 z_2 : z_0 z_3 \right) \circ \left(x_2^2 : z_1 z_3 : z_0 z_2 : z_0 z_3 \right) \\
\phi_4 &= \left(z_0 + z_1 : z_2 : z_3 : z_1 \right) \circ \left(z_0 z_1 : z_3^2 : z_2 z_3 : z_1 z_3 \right) \circ \left(z_0 : z_1 : z_2 : z_3 \right) \\
& \circ \left(z_0 : z_1 : z_2 : z_3 \right) \circ \left(z_0 z_1 : z_3^2 : z_2 z_3 : z_1 z_3 \right) \\
\phi_5 &= \left(z_2 : z_0 : z_1 : z_3 \right) \circ \left(z_2^2 : z_0 z_2 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_0 z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_2 : z_1 z_3 \right) \circ \left(z_2 z_3 : z_1 z_2 : z_1 z_3 \right) \\
\phi_6 &= \left(z_1 : z_2 : z_0 : z_3 \right) \circ \left(z_0 z_2 : z_1 z_3 : z_2 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
\phi_7 &= \left(z_0 : z_1 : z_0 : z_2 \right) \circ \left(z_0 z_2 : z_1 z_3 : z_2 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
\phi_8 &= \left(-z_2 : z_0 : z_1 : z_2 : z_3 \right) \circ \left(-z_0 z_3 : z_0 z_2 : z_2 z_3 : z_3^2 \right) \\
& \circ \left(z_0 z_1 : z_0 z_2 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
\phi_9 &= \left(z_0 : z_1 : z_2 : z_3 : z_1 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
\phi_{10} &= \left(z_2 : z_0 : z_1 : z_2 : z_3 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \\
\phi_{11} &= \left(-z_2 : z_0 : z_1 : z_2 : z_3 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2 z_3 : z_2^2 \right) \\
& \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right) \circ \left(z_2 z_3 : z_1 z_3 : z_2 z_3 : z_2^2 \right)
\end{align*}
\]

\[\square\]

**Proposition 5.6.** Let $\varphi$ be a birational self map of $\mathbb{P}^3_\mathbb{C}$ of bidegree $(2, 4)$. Then $\varphi$ can be written as a composition of involutions of $\mathbb{P}^3_\mathbb{C}$. Furthermore $\text{deg}(\varphi, \text{Bir}(\mathbb{P}^3_\mathbb{C})) \leq 37$.

**Proof.** If $\varphi$ is a birational self map of $\mathbb{P}^3_\mathbb{C}$ of bidegree $(2, 4)$, then $\varphi \in \mathcal{O}(f_i)$ where $f_i$ is one of the following maps (14):

\[
\begin{align*}
\phi_1 &= (z_1 z_2 : z_2 z_3 : z_2 z_3 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_2 &= (z_1^2 - z_1 z_3 : z_1 z_2 z_3 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_3 &= (z_0 z_2 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2) \\
\phi_4 &= (z_1^2 : z_1 z_3 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_5 &= (z_1 z_2 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2) \\
\phi_6 &= (z_1^2 : z_1 z_3 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_7 &= (z_0 z_2 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2) \\
\phi_8 &= (z_1^2 : z_1 z_3 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_9 &= (z_0 z_2 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2) \\
\phi_{10} &= (z_1^2 : z_1 z_3 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2 - z_2^2) \\
\phi_{11} &= (z_0 z_2 : z_2 z_3 : z_1^2 : z_0 z_2 - z_1^2 - z_2^2) \\
\end{align*}
\]
Let us give for any of these maps a decomposition with involutions and elements of $\text{PGL}(4, \mathbb{C})$:

\[
\begin{aligned}
    f_1 &= (z_3 : z_2 : z_1 : -z_0) \circ (-z_0 z_2 z_1 + z_2^2 z_1 + z_2^2 z_1 - z_1 z_2 z_3) \\
    &= (z_0 z_2 z_1 : z_2 z_1 ^2 : z_1 z_2 z_3) \circ (z_2 z_1 ^2 : z_1 z_2 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_2 &= (z_3 : z_2 : z_1 : z_0) \circ (z_0 z_2 : z_1 z_2 : z_2^2 z_1) \circ (z_0 : z_1 : z_2 : z_2 z_3) \\
    &= (z_0 z_2 : z_1 z_2 : z_2^2 z_1) \circ (z_0 z_2 : z_1 z_2 : z_1 z_3) \circ (z_0 z_2 : z_1 z_2 : z_2 z_3) \circ (z_0 z_1 : z_2^2 z_1 : z_1 z_2 z_3) \circ (z_0 z_2 : z_1 z_2 : z_2 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_3 &= (z_3 : z_2 : z_1 : -z_0) \circ (z_0 z_2 : z_1 z_2 : z_2^2 z_1 : z_1 z_2) \circ (-z_0 : z_1 : z_2 : z_3) \\
    &= (z_0 z_2 : z_1 z_2 : z_2^2 z_1 : z_1 z_2) \circ (z_0 z_3 : z_1 z_3 : z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_4 &= (z_1 : z_2 : z_3 : z_2 : z_2^2 : z_3 : z_1 - z_3) \circ (z_0 z_3 : z_1 z_3 : z_1 z_2 : z_2 z_3) \circ (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_1 z_3) \\
    &= (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_1 z_3) \circ (z_0 z_2 : z_1 z_3 : z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_5 &= (z_2 : z_2 : z_1 : z_0) \circ (z_0 z_2 : z_1 z_2 : z_2^2 z_1 : z_2 z_3) \circ (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_1 z_3) \\
    &= (z_0 z_1 : z_1 z_3 : z_2 z_3 : z_1 z_3) \circ (z_0 z_2 : z_1 z_3 : z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_6 &= (z_1 : z_2 : z_3 : -z_0 + z_1 z_3 : z_2^2 z_1 : z_2 z_3) \circ (z_0 z_2 : z_1 z_3 : z_2 z_3 : z_1 z_3) \\
    &= (z_0 z_2 : z_1 z_3 : z_2 z_3 : z_1 z_3) \circ (-z_0 z_3 + z_1 z_3 : z_1 z_3 : z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_7 &= (z_2 : z_1 : z_0) \circ (z_0 z_2 : z_1 z_2 : z_2^2 z_1 : z_2 z_3) \\
    &= (z_0 z_2 : z_1 z_2 : z_2^2 z_1 : z_2 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_8 &= (z_1 : z_2 : z_0) \circ (z_0 z_1 : z_2^2 : z_1 z_3 : z_1 z_2 z_3) \circ (z_0 z_2 : z_1 z_2 z_3 : z_1 z_2 z_3) \\
    &= (z_0 z_2 : z_1 z_2 z_3 : z_1 z_2 z_3) \circ (z_0 z_1 : z_2^2 : z_1 z_3 : z_1 z_2 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_9 &= (z_1 : z_2 : z_0) \circ (z_0 z_3 : z_1 z_3 : z_2 z_3 : z_1 z_3) \circ (-z_0 z_3 + z_2^2 : z_1 z_3 : z_2 z_3 : z_1 z_3) \\
    &= (z_0 z_3 : z_1 z_3 : z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_{10} &= (z_3 - z_1 : z_2 + z_3 : z_2 : z_0) \circ (z_0 z_3 : z_1 z_2 : z_2 z_3 : z_1 z_3) \\
    &= (z_0 z_1 : z_2^2 : z_2 z_3 : z_1 z_2 z_3) \circ (z_0 z_2 : z_1 z_2 z_3 : z_1 z_3)
\end{aligned}
\]

\[
\begin{aligned}
    f_{11} &= (z_2 : z_1 : z_3 : -z_0) \circ (z_0 z_2 : z_1 z_2 : z_2^2 : z_2 z_3) \circ (z_0 z_2 : z_1 z_2 : z_2 z_3 : z_0) \\
    &= (z_0 z_2 : z_1 z_2 : z_2 z_3 : z_0)
\end{aligned}
\]
6. Dimension $\geq 3$

6.1. The group generated by the automorphisms of $\mathbb{P}^n$ and the Cremona involution.

Pan has proved that, as soon as $n \geq 3$, the subgroup generated by $\text{Aut}(\mathbb{P}^n)$ and the involution $\sigma_n$ is a strict subgroup $G_n(C)$ of Bir$(\mathbb{P}^n_C)$. This subgroup has been studied in [3, 7], and in particular:

**Proposition 6.1** ([7]). For any $\varphi$ in $G_n(C)$ there exist $A_0, A_1, \ldots, A_k$ in $\text{Aut}(\mathbb{P}^n_C)$ such that

$$\varphi = \left(A_0 \circ \sigma_n \circ A_0^{-1}\right) \circ \left(A_1 \circ \sigma_n \circ A_1^{-1}\right) \circ \cdots \circ \left(A_k \circ \sigma_n \circ A_k^{-1}\right)$$

**Corollary 6.2.** Any element of $N(G_n(C); \text{Bir}(\mathbb{P}^n_C))$ can be written as a composition of involutions of $\mathbb{P}^n_C$.

6.2. The group of tame automorphisms. As we already mentioned it, $\text{Tame}_3$ does not coincide with $\text{Aut}(C^3)$ (see §2.1): the Nagata automorphism

$$N = (z_0 + 2z_1(z_0z_2 - z_1^2) + z_2(z_0z_2 - z_1^2)^2, z_1 + z_2(z_0z_2 - z_1^2), z_2)$$

is not tame ([19]). Note that since the Nagata automorphism is contained in $G_3(C)$ (see [3]), it can also be written as a composition of involutions (Proposition 6.1). Since $G_n(C)$ contains the group of tame polynomial automorphisms of $C^n$ (see [7]) one gets that

**Proposition 6.3.** Any element of $N(\text{Tame}_n, \text{Aut}(C^n))$ is a composition of involutions of $\mathbb{P}^n_C$.

Can we give an upper bound for $\text{n}(\varphi, \text{Bir}(\mathbb{P}^n_C))$ when $\varphi \in \text{Tame}_n$?

Set

$$H_1 = \left\{ (a z_0 + p(z_1), \sum_{i=1}^{n-1} a_{1,i} z_i + \gamma_1, \sum_{i=1}^{n-1} a_{2,i} z_i + \gamma_2, \ldots, \sum_{i=1}^{n-1} a_{n-1,i} z_i + \gamma_{n-1}) \right\} \quad p \in C[z_1], a, a_{i,j}, \gamma_i \in C, a \det(a_{i,j}) \neq 0,$$

$$H_2 = \left\{ (a z_0 + \beta z_1 + \gamma, \delta z_0 + \sum_{i=1}^{n-1} a_{1,i} z_i + \gamma_1, \sum_{i=1}^{n-1} a_{2,i} z_i + \gamma_2, \ldots, \sum_{i=1}^{n-1} a_{n-1,i} z_i + \gamma_{n-1}) \right\}$$

$$\alpha, \beta, \gamma, \delta, a_{i,j}, \gamma_i \in C, \det M(\alpha, \beta, \gamma, \delta, a_{i,j}) \neq 0$$

where

$$M(\alpha, \beta, \gamma, \delta, a_{i,j}) = \begin{pmatrix} \alpha & \beta & \gamma & 0 & \ldots & 0 \\ \delta & 0 & a_{i,j} & 0 & \ldots & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & & & & \ddots & \ddots \\ \end{pmatrix}$$

One can check that

$$H_1 \cap H_2 = \left\{ (a z_0 + \beta z_1 + \gamma, \sum_{i=1}^{n-1} a_{1,i} z_i + \gamma_1, \sum_{i=1}^{n-1} a_{2,i} z_i + \gamma_2, \ldots, \sum_{i=1}^{n-1} a_{n-1,i} z_i + \gamma_{n-1}) \right\}$$

$$\alpha, \beta, \gamma, \delta, a_{i,j}, \gamma_i \in C, \det M(\alpha, \beta, \gamma, \delta, a_{i,j}) \neq 0$$
where
\[
M(\alpha, \beta, \gamma, a_{i,j}) = \begin{pmatrix}
\alpha & \beta & \gamma & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

**Proposition 6.4.** Let \( \varphi = \varphi_k \circ \varphi_{k-1} \circ \ldots \circ \varphi_1 \) be a reduced word in the amalgamated product \( H_1 \ast_{H_1 \cap H_2} H_2 \).

The degree of \( \varphi \) is equal to the product of the degree of the factors \( \varphi_i \).

**Proof.** We follow the proof of [8, Theorem 2.1].

Let \( \psi = (\psi_0, \psi_1, \ldots, \psi_{n-1}) \) be an element of \( H_1 \ast_{H_1 \cap H_2} H_2 \) which satisfy the condition degrees \( d_0 = \deg \psi_1 \geq \deg \psi_i \) for any \( 0 \leq i \leq n-1 \).

Now consider \( \varphi = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \) an element of \( H_1 \setminus (H_1 \cap H_2) \) of degree \( d \); in particular \( \varphi_0 = a \varphi_0 + p(z_1) \) with \( d = \deg p \geq 2 \). Denote by \( \bar{\varphi}_i \) the components of \( \varphi \circ \psi \). One has \( \bar{d}_0 = \deg \bar{\varphi}_0 > \deg \bar{\varphi}_i \) for any \( 1 \leq i \leq n-1 \).

Take \( \phi \) in \( H_2 \setminus (H_1 \cap H_2) \), and set \( \phi \circ \varphi \circ \psi = (\bar{\varphi}_0, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n-1}) \). Then \( \bar{d}_0 = \deg \bar{\varphi}_1 = \deg \bar{\varphi}_i \) for any \( i \geq 0 \).

As a result whenever we compose with an element of \( H_1 \setminus (H_1 \cap H_2) \) followed with an element of \( H_2 \setminus (H_1 \cap H_2) \) the degree will be multiply by \( d \). The statement follows by induction. \( \Box \)

Let us now remark that \( (H_1, H_2) \) contains both \( A_n \) and \( (z_0 + z_1^2, z_1, z_2, \ldots, z_{n-1}) \). Since \( \text{Tame}_n = \langle A_n, (z_0 + z_1^2, z_1, z_2, \ldots, z_{n-1}) \rangle \) (see [20, Chapter 5.2]) any tame automorphism is a reduced word in \( H_1 \ast_{H_1 \cap H_2} H_2 \). Following what we did in §4.3 one obtains:

**Theorem 6.5.** Let \( \varphi \) be a tame automorphism of \( \mathbb{C}^n, n \geq 3 \), of degree \( d \).

- If \( \varphi \) is affine, then \( n(\varphi, \text{Aut}(\mathbb{P}^n_\mathbb{C})) \leq 2n + 4 \);
- if \( \varphi \) is elementary, then \( n(\varphi, \text{Bir}(\mathbb{P}^n_\mathbb{C})) \leq 2n + 10 \);
- otherwise \( n(\varphi, \text{Bir}(\mathbb{P}^n_\mathbb{C})) \leq \frac{d}{2}(2n + 7) + 10n + 32 \).

**Remark 6.6.** We cannot use this strategy to get a more precise statement for \( G_n(\mathbb{C}) \). Indeed using similar arguments as in the appendix of [4] one can prove that \( G_n(\mathbb{C}) \) has property \((FR)\); in particular, according to [18] one has:

**Proposition 6.7.** The group \( G_n(\mathbb{C}) \) does not decompose as a non-trivial amalgam.

More precisely if \( G_n(\mathbb{C}) \) is contained in an amalgam \( G_1 \ast_A G_2 \), then \( G_n(\mathbb{C}) \) is contained in a conjugate of either \( G_1 \) or \( G_2 \) (see [18]).

### 6.3. Monomial maps in any dimension.

Let \( \mathbb{A}^n_\mathbb{C} \) be the affine space of dimension \( n \). The multiplicative group \( GL_\mathbb{C}^n \) can be identified to the Zariski open subset \( (\mathbb{A}^1_\mathbb{C} \setminus \{0\})^n \) of \( P^m_\mathbb{C} \).

Hence \( \text{Bir}(\mathbb{P}^m_\mathbb{C}) \) contains the group of all algebraic automorphisms of the group \( G^n_\mathbb{C} \) \( i.e. \) the group \( \text{Mon}(n, \mathbb{C}) \) of monomial maps \( GL(n, \mathbb{Z}) \).

**Theorem 6.8 ([10]).** Let \( n \geq 3 \) be an integer. Any element \( \varphi \) of \( GL(n, \mathbb{Z}) \) can be written as a composition of involutions of \( GL(n, \mathbb{Z}) \), and \( n(\varphi, GL(n, \mathbb{Z})) \leq 3n + 9 \).

**Corollary 6.9.** Let \( \varphi \) be an element of \( \text{Mon}(n, \mathbb{C}) \), with \( n \geq 3 \). Then \( \varphi \) can be written as a composition of involutions of \( \text{Mon}(n, \mathbb{C}) \), and \( n(\varphi, \text{Mon}(n, \mathbb{C})) \leq 3n + 9 \).

**Remark 6.10.** If \( n \) is even, then \( \text{Mon}(n, \mathbb{C}) \subset G_n(\mathbb{C}) \) (see [3]) ; Proposition 6.1 thus already says that any monomial map of \( \text{Bir}(\mathbb{P}^m_\mathbb{C}) \) can be written as a composition of involutions but here we get two more informations:
• a bound for the minimal number of involutions,
• and the fact that the involutions belong to $\text{Mon}(n, C)$.

Furthermore Proposition 6.1 gives nothing for $\text{Mon}(n, C)$ for $n$ odd since $\text{Mon}(n, C) \neq C_n$ as soon as $n$ is odd ([3]).

**Corollary 6.11.** Any element of $N[\text{Mon}(n, C); \text{Bir}(P^n_C)]$ is a composition of involutions of $P^n_C$.

### 6.4. Subgroups $J_n$.

Let us introduce $J_n$ the subgroup of $\text{Bir}(P^n_C)$ formed by the maps of the type

$$
\left( \varphi_0, \varphi_1, \ldots, \varphi_{n-2}, \frac{\alpha z_{n-1} + \beta}{\gamma z_{n-1} + \delta} \right)
$$

with

$$
\varphi_i = \left( \frac{z_i A_i(z_{i+1}, z_{i+2}, \ldots, z_{n-1}) + B_i(z_{i+1}, z_{i+2}, \ldots, z_{n-1})}{z_i C_i(z_{i+1}, z_{i+2}, \ldots, z_{n-1}) + D_i(z_{i+1}, z_{i+2}, \ldots, z_{n-1})} \right) \in \text{PGL}(2, C(z_{i+1}, z_{i+2}, \ldots, z_{n-1}))
$$

and

$$
\frac{\alpha z_{n-1} + \beta}{\gamma z_{n-1} + \delta} \in \text{PGL}(2, C).
$$

According to the proof of Lemma 4.3 one gets:

**Proposition 6.12.** Let $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1})$ be an element of $J_n$.

Assume $0 \leq i \leq n - 2$. If $\det \varphi_i = \pm 1$, then $n(\varphi_i, \text{PGL}(2, C(z_{i+1}, z_{i+2}, \ldots, z_{n-1})) \leq 4$ otherwise $n(\varphi_i, \text{PGL}(2, C(z_{i+1}, z_{i+2}, \ldots, z_{n-1})) \leq 8$.

In particular $n(\varphi, J_n) \leq 4(2n - 1)$.

**Corollary 6.13.** Any element of $N(J_n; \text{Bir}(P^n_C))$ can be written as a composition of involutions of $P^n_C$.

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