ABELIAN QUOTIENTS OF MAPPING CLASS GROUPS OF HIGHLY CONNECTED MANIFOLDS

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Abstract. We compute the abelianisations of the mapping class groups of the manifolds $W_{g}^{2n} = g(S^n \times S^n)$ for $n \geq 3$ and $g \geq 5$. The answer is a direct sum of two parts. The first part arises from the action of the mapping class group on the middle homology, and takes values in the abelianisation of the automorphism group of the middle homology. The second part arises from bordism classes of mapping cylinders and takes values in the quotient of the stable homotopy groups of spheres by a certain subgroup which in many cases agrees with the image of the stable $J$-homomorphism. We relate its calculation to a purely homotopy theoretic problem.

1. Introduction

Let $W_{g}^{2n} = g(S^n \times S^n)$ denote the $g$-fold connected sum and choose a fixed closed disc $D_{g}^{2n} \subset W_{g}^{2n}$. Let $\text{Diff}^{+}(W_{g}^{2n})$ be the topological group of orientation preserving diffeomorphisms of $W_{g}^{2n}$, and $\text{Diff}(W_{g}^{2n}, D_{g}^{2n})$ be the subgroup of those diffeomorphisms which fix an open neighbourhood of the disc. Define the mapping class groups

$$
\Gamma_{g}^{n,1} = \pi_0(\text{Diff}(W_{g}^{2n}, D^{2n})) \quad \Gamma_{g}^{n} = \pi_0(\text{Diff}^{+}(W_{g}^{2n})).
$$

There is a homomorphism $\gamma : \Gamma_{g}^{n,1} \to \Gamma_{g}^{n}$, which simply forgets that diffeomorphisms fix a disc. We will construct two abelian quotients of these groups, one coming from arithmetic properties of the intersection form of $W_{g}^{2n}$, and one coming from a cobordism theoretic construction. Together, these will give the abelianisation of either group.

Construction 1.1. Recall that Wall [Wal62] has constructed for each $(n-1)$-connected $2n$-manifold $W$ a certain quadratic form $Q_{W}$, which we shall describe later, whose underlying bilinear form is the intersection form on $H_{n}(W; \mathbb{Z})$. Diffeomorphisms of the manifold act by automorphisms of this quadratic form, so there is a group homomorphism

$$
\hat{f} : \Gamma_{g}^{n} \to \text{Aut}(Q_{W_{g}^{2n}}),
$$

from which we can construct the map $f : \Gamma_{g}^{n,1} \to H_{1}(\text{Aut}(Q_{W_{g}^{2n}}))$ to an abelian group. We will also write $\hat{f}$ for the composition $\hat{f} \circ \gamma : \Gamma_{g,1}^{n} \to \Gamma_{g}^{n} \to \text{Aut}(Q_{W_{g}^{2n}})$, and similarly with $f$.

Construction 1.2. Let $\varphi \in \text{Diff}(W_{g}^{2n}, D^{2n})$ be a diffeomorphism of $W_{g}^{2n}$ which is the identity on a fixed disc $D_{g}^{2n} \subset W_{g}^{2n}$. We may form the mapping torus

$$
T_{\varphi} = W_{g}^{2n} \times [0,1]/(x,0) \sim (\varphi(x), 1),
$$

which is a $(2n+1)$-dimensional manifold fibering over $S^1$, and contains an embedded $D^{2n} \times S^1$ given by the disc fixed by $\varphi$. The $(n-1)$-connected manifold obtained by surgery along this embedded $D^{2n} \times S^1$ shall be denoted $T'_{\varphi}$. By obstruction theory, a
map \( \tau : T^\prime_\varphi \to BO \) classifying its stable normal bundle admits a lift \( \ell : T^\prime_\varphi \to BO(n) \), unique up to homotopy; where \( BO(n) \to BO \) denotes the \( n \)-connected cover. The pair \( (T^\prime_\varphi, \ell) \) represents an element of \( \Omega^{2n+1}_n \), the cobordism theory associated to the map \( BO(n) \to BO \), and one easily verifies that the function

\[
t : \Gamma_{g,1}^n \to \Omega^{(n)}_{2n+1} \\
\varphi \mapsto [T^\prime_\varphi, \ell]
\]

is a group homomorphism.

Our main theorem, proved in Sections 3–5 below, is that these two homomorphisms combine to give the maximal abelian quotient of the group \( \Gamma_{g,1}^n \).

**Theorem 1.3.** For all \( n \) and \( g \) (except we require \( g \geq 2 \) if \( n = 2 \)) the map

\[
t \oplus f : \Gamma_{g,1}^n \to \Omega^{(n)}_{2n+1} \oplus H_1(\text{Aut}(Q_{W^g_s}))
\]

is surjective, and for \( n \neq 2 \) and \( g \geq 5 \) it is the abelianisation. Furthermore, in this range

\[
H_1(\text{Aut}(Q_{W^g_s})) \cong \begin{cases} 
(\mathbb{Z}/2)^2 & n \text{ even} \\
0 & n = 1, 3 \text{ or } 7 \\
\mathbb{Z}/4 & \text{otherwise.}
\end{cases}
\]

We obtain the following table describing \( H_1(\Gamma_{g,1}^n; \mathbb{Z}) \) for small \( n \), using known calculations of \( \Omega^{(3)}_n = \Omega^{(3)}_n \text{Spin} \) and \( \Omega^{(7)}_n = \Omega^{(7)}_n \text{String} \).

| \( n \) | \( H_1(\Gamma_{g,1}^n; \mathbb{Z}) \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) |
|-------|-----------------|----|----|----|----|----|----|----|
| \( 1 \) | \( \mathbb{Z}/2 \oplus ? \) | \( 0 \) | \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) | \( \mathbb{Z}/4 \) | \( \mathbb{Z}/2 \oplus \mathbb{Z}/3 \) | \( \mathbb{Z}/2 \) |

1.1. **The cobordism groups** \( \Omega^{(n)}_{2n+1} \). In light of Theorem 1.3 it is of interest to describe the cobordism group \( \Omega^{(n)}_{2n+1} \) in terms of more familiar objects. There is a homomorphism

\[
\rho : \Omega^r_{2n+1} \to \Omega^{(n)}_{2n+1}
\]

from framed cobordism obtained by simply remembering that a stably tangentially framed manifold in particular has a \( BO(n) \)-structure. The cobordism theoretic interpretation of the \( J \)-homomorphism

\[
J : \pi_{2n+1}(SO) \to \pi^s_{2n+1} = \Omega^r_{2n+1}
\]

is that it sends a map \( f : S^{2n+1} \to SO \) to the stably framed manifold obtained by taking the \((2n+1)\)-sphere with its usual—bounding—stable framing, and changing the framing using \( f \). The resulting stable framing need not extend over \( D^{2n+2} \), but the \( BO(n) \)-structure does always extend (as the map \( BO(n) \to BO \) is \( n \)-connected), so \( \rho \circ J \) is trivial. Thus there is an induced map

\[
\rho' : \text{Coker}(J)_{2n+1} \to \Omega^{(n)}_{2n+1}.
\]

It follows from work of Stolz that this map is an isomorphism in many cases.

**Theorem 1.4** (Stolz). The map \( \rho' \) is surjective, and is an isomorphism if either

(i) \( n + 1 \equiv 2 \mod 8 \) and \( n + 1 \geq 18 \),
(ii) \( n + 1 \equiv 1 \mod 8 \) and \( n + 1 \geq 113 \),
(iii) \( n + 1 \not\equiv 0, 1, 2, 4 \mod 8 \).
In the cases not covered by this theorem, the kernel of \( \rho' \) is at most \( \mathbb{Z}/2 \) if \( n+1 \equiv 1,2 \mod 8 \), and cyclic if \( n+1 \equiv 0 \mod 4 \). We give more detailed information in \[11\].

1.2. Closed manifolds. We can also use Theorem [13] to calculate the abelianisation of the mapping class group \( \Gamma^n_g \), of orientation preserving diffeomorphisms of the closed manifolds \( W^{2n}_g \).

**Corollary 1.5.** For \( n \geq 3 \) and \( g \geq 5 \), the map \( \gamma_* : H_1(\Gamma^n_{g,1}) \rightarrow H_1(\Gamma^n_g) \) is an isomorphism.

**Proof.** Suppose that \( n \geq 3 \), so the homotopy fibre sequences

\[
\text{Fr}^+(W^{2n}_g) \rightarrow BD\text{iff}(W^{2n}_g, D^{2n}_g) \rightarrow BD\text{iff}^+(W^{2n}_g)
\]

and

\[
SO(2n) \rightarrow \text{Fr}^+(W^{2n}_g) \rightarrow W^{2n}_g
\]

give an exact sequence

\[
\cdots \rightarrow \pi_1(\text{Diff}^+(W^{2n}_g)) \rightarrow \pi_1(SO(2n)) = \mathbb{Z}/2 \xrightarrow{\tau} \Gamma^n_{g,1} \xrightarrow{\tau_1} \Gamma^n_g \rightarrow 1.
\]

We claim that the map \( \mathbb{Z}/2 \xrightarrow{\tau} \Gamma^n_{g,1} \rightarrow H_1(\Gamma^n_{g,1}) \) is zero for \( g \geq 5 \).

The composition \( \mathbb{Z}/2 \xrightarrow{\tau} \Gamma^n_{g,1} \xrightarrow{f} H_1(\text{Aut}(Q^{2n}_{W_g})) \) is clearly zero, as \( \gamma \circ \tau \) is trivial. The composition

\[
\mathbb{Z}/2 \xrightarrow{\tau} \Gamma^n_{g,1} \xrightarrow{f} \Omega^{(n)}_{2n+1} \text{ is also zero: a choice of relative bordism from } (W^{2n}_g, D^{2n}) \text{ to } (S^{2n}, D^{2n}) \text{ as } BO(n)-\text{manifolds gives a natural nullbordism of } t \circ \tau(1).
\]

1.3. Perfection. Recall that a group is called perfect if it is equal to its derived subgroup, or equivalently if its abelianisation is trivial. Table [11] shows that \( \Gamma^n_{g,1} \) (or, by Corollary [13], \( \Gamma^n_g \)) is perfect for \( n = 1 \) or \( n = 3 \) and \( g \geq 5 \), but the fact that \( H_1(\text{Aut}(Q^{2n}_{W_g})) \) is trivial only for \( n = 1,3,7 \) and the fact that \( \Omega^{(7)}_{15} \neq 0 \) means that these are the only examples.

**Corollary 1.6.** For \( g \geq 5 \), the groups \( \Gamma^n_{g,1} \) and \( \Gamma^n_g \) are perfect if and only if \( n \) is 1 or 3.

If we denote by \( \mathring{W}^{2n}_{g,1} \) the complement of the chosen disc \( D^{2n} \) in \( W^{2n}_g \), then the group \( \text{Diff}(W^{2n}_g, D^{2n}_g) \) is isomorphic to \( \text{Diff}_{c}(W^{2n}_{g,1}) \), the group of compactly supported diffeomorphisms of \( W^{2n}_{g,1} \). Thurston [Thur] has proved that for any manifold \( M \) without boundary the identity component \( \text{Diff}_{c}(M)^0_{g,1} \), considered as a discrete group, is perfect (in fact, it is simple). Thus the extension of discrete groups

\[
1 \rightarrow \text{Diff}_{c}(\mathring{W}^{2n}_{g,1})^\delta_{g,1} \rightarrow \text{Diff}(W^{2n}_g, D^{2n}_g)^{\delta} \rightarrow \Gamma^n_{g,1} \rightarrow 1
\]

shows that the discrete group \( \text{Diff}(W^{2n}_g, D^{2n}_g)^{\delta} \) is perfect if and only \( \Gamma^n_{g,1} \) is, and more generally that the abelianisation of the discrete group \( \text{Diff}(W^{2n}_g, D^{2n}_g)^{\delta} \) is also described by Theorem [13]. Similarly, the abelianisation of the discrete group \( \text{Diff}_{c}^+(W^{2n}_g)^{\delta} \) is also described by Theorem [13]. See [Far14] for more information about the homology of \( \text{Diff}(W^{2n}_g, D^{2n}_g)^{\delta} \).

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2. Wall’s quadratic form

The fibration \( S^n \to BO(n) \to BO(n + 1) \) gives a long exact sequence on homotopy groups

\[
\cdots \to \pi_{n+1}(BO(n + 1)) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\tau} \pi_n(BO(n)) \xrightarrow{s} \pi_n(BO(n + 1)) \to 0,
\]

and we let \( \Lambda_n = \text{Im}(\partial) \subset \mathbb{Z} \). The map \( \tau \) sends 1 to the map which classifies the tangent bundle of the \( n \)-sphere, so \( \Lambda_n \) is trivial if \( n \) is even, \( \mathbb{Z} \) if \( n = 1, 3 \) or 7, and \( 2\mathbb{Z} \) otherwise, by the Hopf invariant 1 theorem. The data \( ((-1)^n, \Lambda_n) \) is a form parameter in the sense of Bak \[\text{Bak69, Bak81}\].

Suppose that \( n \geq 4 \), and let \( W \) be an \((n-1)\)-connected \( 2n \)-manifold which is stably parallelizable. We will describe how to associate to it a non-degenerate quadratic form \( Q_W \) having form parameter \( ((-1)^n, \Lambda_n) \), following Wall \[\text{Wal62}\].

The \( \mathbb{Z} \)-module

\[
\pi_n(W) \cong H_n(W; \mathbb{Z})
\]

has a \((-1)^n\)-symmetric bilinear form

\[
\lambda : H_n(W; \mathbb{Z}) \otimes H_n(W; \mathbb{Z}) \to \mathbb{Z}
\]

given by the intersection form, which is non-degenerate by Poincaré duality. If \( x = [f] \in \pi_n(W) \), then by a theorem of Haefliger \[\text{Hae61}\] as \( n \geq 4 \) we may represent it uniquely up to isotopy by an embedding \( f : S^n \hookrightarrow W \), which has an \( n \)-dimensional normal bundle which is stable trivial. This represents an element

\[
\alpha(x) \in \mathbb{Z}/\Lambda_n = \text{Ker}(\pi_n(BO(n)) \xrightarrow{s} \pi_n(BO(n + 1))
\]

and Wall has shown that this satisfies

(i) \( \alpha(a \cdot x) = a^2 \cdot \alpha(x) \), for \( a \in \mathbb{Z} \),

(ii) \( \alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y) \), where \( \lambda(x, y) \) is reduced modulo \( \Lambda_n \).

Thus the data \( (\pi_n(W), \lambda, \alpha) \) is a quadratic form with form parameter \( ((-1)^n, \Lambda_n) \).

Remark 2.1. This construction above does not quite work for \( n \leq 3 \), as Haefliger’s theorem does not apply, but we can proceed anyway. When \( n = 1 \) or 3 we have \( \mathbb{Z}/\Lambda_n = \{0\} \) and so a quadratic form with parameter \((-1, \Lambda_n)\) should be a module with antisymmetric bilinear form. We take \( H_n(W_0; \mathbb{Z}) \) with its intersection form.

When \( n = 2 \) we have \( \mathbb{Z}/\Lambda_2 = \mathbb{Z} \) and so a quadratic form with parameter \((1, \Lambda_2)\) should be an even symmetric bilinear form. The intersection form on \( H_2(W_0; \mathbb{Z}) \) is even, so we can take this.

By construction, it is clear that if \( \varphi : W_0 \to W_1 \) is a diffeomorphism then \( \varphi_* : H_n(W_0; \mathbb{Z}) \to H_n(W_1; \mathbb{Z}) \) is a morphism of quadratic forms. The most elementary quadratic form is the hyperbolic form

\[
H = \left( \mathbb{Z}^2 \right. \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \alpha(e) = \alpha(f) = 0 \right).
\]

The manifold \( W_0^{2n} = g(S^n \times S^n) \) has associated quadratic form \( H^{\otimes g} \), the direct sum of \( g \) copies of the hyperbolic form, and so we have a homomorphism

\[
f : \Gamma_g^{n} \to \text{Aut}(H^{\otimes g}).
\]

Kreck \[\text{Kre79}\] has shown that this map is surjective for \( n \geq 3 \), Wall \[\text{Wal64}\] has shown it is surjective for \( n = 2 \) as long as \( g \geq 5 \), and it is well-known to be surjective for \( n = 1 \) and all \( g \). We obtain an abelian quotient

\[
(2.1) \quad f : \Gamma_g^{n} \to H_1(\text{Aut}(H^{\otimes g}); \mathbb{Z}).
\]
Proposition 2.2. There are isomorphisms

\[ H_1(\text{Aut}(H^g)); \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^2 & n \text{ even} \\ 0 & n = 1, 3 \text{ or } 7 \\ \mathbb{Z}/4 & \text{otherwise.} \end{cases} \]

as long as \( g \geq 5 \).

Proof. By Charney’s stability theorem [Cha87] for the homology of automorphism groups of quadratic forms over a PID, the group \( H_1(\text{Aut}(H^g)); \mathbb{Z}) \) is independent of \( g \) as long as \( g \geq 5 \). In fact, the statement in [Cha87] claims this only for \( g \geq 6 \), but using the slightly improved connectivity for the necessary poset / simplicial complex which is established in [GRW14a] Theorem 3.2 this can be improved to \( g \geq 5 \) (the poset \( HU_g = HU(H^g) \) of [Cha87] is the face poset of the simplicial complex \( K^a(H^g) \) of [GRW14a], so they have homeomorphic geometric realisations).

If \( n \) is even, then \( \text{Aut}(H^g) = O_{g, g}(\mathbb{Z}) \) is the indefinite orthogonal group over the integers. This is a subgroup of \( O_{g, g}(\mathbb{R}) \), which has maximal compact subgroup \( O_g(\mathbb{R}) \times O_g(\mathbb{R}) \); the determinants of these two factors provides a surjective homomorphism \( a : O_{g, g}(\mathbb{Z}) \to (\mathbb{Z}/2)^2 \). In [GHS09] Theorem 1.7 it is shown that a certain index 4 normal subgroup \( \text{SO}_{g, g}(\mathbb{Z}) \) of \( O_{g, g}(\mathbb{Z}) \), for the definition of which we refer to that paper, has trivial abelianisation. Thus the homomorphism \( a \) is the abelianisation.

If \( n = 1, 3 \) or 7 then a quadratic form with parameter \((-1, A_n)\) is nothing but an antisymmetric bilinear form, so \( \text{Aut}(H^g) = \text{Sp}_{2g}(\mathbb{Z}) \) is the symplectic group over the integers. This is well-known to have trivial abelianisation, as long as \( g \geq 3 \).

For the remaining odd \( n \), \( \text{Aut}(H^g) = \text{Sp}_{2g}^+(\mathbb{Z}) \subset \text{Sp}_{2g}(\mathbb{Z}) \) is the subgroup of those symplectic matrices which stabilise the quadratic form \( \alpha(e_i) = \alpha(f_i) = 0 \). The abelianisation of this group has been computed in [JM90] Theorem 1.1 to be \( \mathbb{Z}/4 \) as long as \( g \geq 3 \).

Remark 2.3. The argument above can be used to strengthen the “only if” part of Corollary 1.6 for \( n \neq 1, 3 \), the mapping class groups are not perfect for any \( g \geq 1 \).

For \( n = 7 \) this is in fact the case for \( g \geq 0 \), as the generator of \( \Omega^2_{g, g} = \mathbb{Z}/2 \) can be hit by a diffeomorphism supported inside a disc. For \( n \neq 7 \) we argue as follows.

The matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) defines an element of Aut(\( H \)) for all \( n \), and is easily seen to be realised by an element of \( \Gamma^a_{g, 1} \). For \( n \) even this maps to a non-trivial element of \( (\mathbb{Z}/2)^2 \), and for \( n \) odd apart from 1, 3, 7, it follows from the formula [JM90, p. 147] that it maps to the order two element of \( \mathbb{Z}/4 \).

3. Low-dimensional cases

The cases \( n < 3 \) of Theorem 1.3 require special treatment, so let us dispense with them first. We will then focus on the generic case \( n \geq 3 \).

3.1. \( n = 1 \). The relevant bordism group is \( \Omega^{(1)}_{2, g} \), third oriented bordism, which is well-known to be zero. Thus the first part of Theorem 1.3 states that \( \Gamma^a_{0, 1} \) surjects onto the trivial group, which is certainly true, and the second part states that the abelianisation of \( \Gamma^a_{0, 1} \) is zero as long as \( g \geq 5 \). This is [Pow78 Theorem 1] (which in fact only requires \( g \geq 3 \)).

3.2. \( n = 2 \). The relevant bordism group is \( \Omega^{(2)}_{3, g} \), fifth Spin bordism, which is zero by the results of [ABP66]. Thus in this case Theorem 1.3 just says that the map \( f : \Gamma^a_{g, 1} \to H_1(\text{Aut}(QW^g_2)) \) is surjective for \( g \geq 2 \). But the homomorphism \( \hat{f} : \Gamma^a_{g, 1} \to \text{Aut}(QW^g_2) \) is already surjective in this case, by [Wal64 Theorem 2].
4. Nontriviality of the mapping torus construction

Using the stabilisation maps we have a homomorphism

\[ d : \Gamma^n_{g,1} \rightarrow \Gamma^n_{g,1} \]

for any \( g \), and the group \( \Gamma^n_{0,1} \)—the mapping class group of the sphere relative to a disc—is isomorphic to the group \( \Theta_{2n+1} \) of exotic \((2n+1)\)-spheres via the clutching construction.

**Lemma 4.1.**

(i) The image of \( \Theta_{2n+1} \) in \( \Gamma^n_{g,1} \) is central.

(ii) The composition \( \Theta_{2n+1} \xrightarrow{d} \Gamma^n_{g,1} \xrightarrow{\iota} \Omega^{(n)}_{2n+1} \) is surjective, so in particular \( t \) is surjective.

(iii) The composition \( \Theta_{2n+1} \xrightarrow{d} \Gamma^n_{g,1} \xrightarrow{\hat{f}} \text{Aut}(H^\oplus g) \) is trivial.

**Proof.** Let \( f \) be a diffeomorphism of \( W_g \) fixing a neighbourhood \( U \) of \( D^2n \), and \( g \) be a diffeomorphism supported in a disc disjoint from the marked one. Then \( g \) is isotopic to a diffeomorphism \( g' \) supported in \( U \) but still disjoint from \( D^2n \), and now \( g' \) commutes with \( f \). Thus \( \text{Im}(d) \subset \Gamma^n_{g,1} \) is central.

The map \( t \circ d \) sends an exotic \((2n+1)\)-sphere to its \( BO(n) \)-bordism class (such an exotic sphere has a canonical \( BO(n) \)-structure by virtue of being highly-connected), so we must show that any \((2n+1)\)-dimensional manifold with \( BO(n) \)-structure \((W,\ell_W)\) is cobordant to a \( n \)-connected manifold (as it is then \( 2n \)-connected by Poincaré duality).

This follows from the methods of Kervaire and Milnor, specifically \([KM63, \text{Theorem 6.6}]\). They work with manifolds which are stably parallelisable, but this is only used in two ways: to show that homotopy classes of dimension \( * \leq n \) can be represented by framed embeddings, and to show that the trace of the surgery is stably parallelisable. A \( BO(n) \)-structure still allows one to represent homotopy classes of dimension \( * \leq n \) by framed embeddings, and a \( BO(n) \)-structure can be induced on the trace of the surgery, too.

Finally, a mapping class in the image of \( d \) is supported in a small disc, and so acts trivially on the homology of \( W_g \), so \( \hat{f} \circ d \) is trivial. \( \square \)

This lemma has the following implication regarding the kernel of the mapping torus construction \( t \).

**Corollary 4.2.** The kernel of the homomorphism \( t : H_1(\Gamma^n_{g,1}) \rightarrow \Omega^{(n)}_{2n+1} \) has cardinality at least 4 if \( n \neq 1, 3 \) or 7 and \( g \geq 5 \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
\Theta_{2n+1} & \xrightarrow{d} & H_1(\Gamma^n_{g,1}) \\
\downarrow & & \downarrow t \\
\text{Ker}(t) & \rightarrow & \Omega^{(n)}_{2n+1}
\end{array}
\]

where the middle row is exact. Diagram chasing shows that the dashed arrow is surjective, and so \( \text{Ker}(t) \rightarrow H_1(\Gamma^n_{g,1}) \rightarrow H_1(\text{Aut}(H^\oplus g)) \) is surjective. The target has cardinality 4 in these cases by Proposition 2.2. \( \square \)
5. A Refinement of the Mapping Torus Construction

From now on we suppose that $n \geq 3$. The proof of the remainder of Theorem 1.3 uses two more involved theorems proved recently by the authors, which concern not the mapping class groups but the entire diffeomorphism groups of the manifolds $W_{2n}$. There are continuous homomorphisms

$$\text{Diff}(W_{2n}^g, D_{2n}) \longrightarrow \text{Diff}(W_{g+1}^2, D^{2n})$$

given by connect-sum with $W_{2n}$ inside the marked disc, and extending diffeomorphisms by the identity. In [GRW14a] Theorem 1.2 we showed that for $n \geq 3$ the maps on classifying spaces

$$B\text{Diff}(W_{2n}^g, D^{2n}) \longrightarrow B\text{Diff}(W_{g+1}^2, D^{2n})$$

induce homology isomorphisms in degrees $2* \leq q - 3$. In particular, as long as $q \geq 5$ they induce isomorphisms on first homology. The map $H_1(B\text{Diff}(W_{2n}^g, D^{2n}); \mathbb{Z}) \longrightarrow H_1(\Gamma_{g+1}; \mathbb{Z})$ is also an isomorphism, which shows that the stabilisation map

$$H_1(\Gamma_{g+1}; \mathbb{Z}) \longrightarrow H_1(\Gamma_{g+1}; \mathbb{Z})$$

is an isomorphism for $g \geq 5$.

Secondly, we showed how to identify the stable homology, that is, the homology of $\text{hocolim}_{g \rightarrow \infty} B\text{Diff}(W_{2n}^g, D^{2n})$, as follows. Let $\theta_n : BO(2n)(n) \rightarrow BO(2n)$ denote the $n$-connected cover, and $\theta_n^{\ast} \gamma_{2n}$ denote the pullback of the tautological $2n$-dimensional vector bundle. Write $\text{MT}\theta_n$ for the Thom spectrum of the virtual bundle $-\theta_n^{\ast} \gamma_{2n} \rightarrow BO(2n)(n)$. Parametrised Pontryagin–Thom theory provides maps

$$\alpha_g : B\text{Diff}(W_{2n}^g, D^{2n}) \longrightarrow \Omega^\infty_0 \text{MT}\theta_n$$

which assemble to a map $\alpha_\infty : \text{hocolim}_{g \rightarrow \infty} B\text{Diff}(W_{2n}^g, D^{2n}) \rightarrow \Omega^\infty_0 \text{MT}\theta_n$ which we show in [GRW14b] Theorem 1.1 induces an isomorphism on homology as long as $n \geq 3$. Given these two theorems, we are reduced to calculating $H_1(\Omega^\infty_0 \text{MT}\theta_n)$.

Recall that $\text{MT}\theta_n = \text{Th}(-\theta_n^{\ast} \gamma_{2n} \rightarrow BO(2n)(n))$. Let us write $\text{MO}(n)$ for the Thom spectrum of the tautological bundle over $BO(n)$, so the stabilisation map induces a spectrum map

$$s : \text{MT}\theta_n \longrightarrow \Sigma^{-2n} \text{MO}(n).$$

**Lemma 5.1.** The composition

$$H_1(\Gamma_{g+1}) \longrightarrow H_1(B\text{Diff}(W_{g}^{2n}, D^{2n})); \theta_g \longrightarrow H_1(\Omega^\infty_0 \text{MT}\theta_n) \cong \pi_1(\text{MT}\theta_n) \cong \Omega^\infty_0 \text{MT}\theta_n$$

agrees with the mapping torus construction $t$.

**Proof.** Both apply the Pontryagin–Thom construction to the mapping torus. \qed

5.1. A long exact sequence in stable homotopy. Let us write $F_n$ for the homotopy fibre of the spectrum map $s$. There is a commutative diagram

$$\begin{array}{ccc}
SO/\text{SO}(2n) & \longrightarrow & BO(2n)(n) & \longrightarrow & BO(2n) \\
\ast & \longrightarrow & BO(n) & \longrightarrow & BO,
\end{array}$$

(5.1)

where both squares are homotopy pullback. The left square induces a map of homotopy cofibres $\Sigma(SO/\text{SO}(2n)) \rightarrow (BO(n))/(BO(2n)(n))$ which we see from the Serre spectral sequence to be $(3n+1)$-connected. The whole diagram maps to $BO$, and may be Thomified. The map of cofibres, desuspended $(2n+1)$ times, gives a map

$$\Sigma^{-2n}SO/\text{SO}(2n) \longrightarrow F_n.$$
which is n-connected.

We may therefore rewrite the long exact sequence in stable homotopy for the map \( s : \text{MT}_\theta \to \Sigma^{-2n}\text{MO}(n) \) in the following way:

\[
\cdots \xrightarrow{s_*} \pi_{2n+2}(\text{MO}(n)) \xrightarrow{\partial_*} \pi_{2n+1}(\text{SO}/\text{SO}(2n)) \xrightarrow{s_*} \pi_{2n+1}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n-1}(\text{SO}/\text{SO}(2n)) \xrightarrow{s_*} \pi_{2n-2}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n-3}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n-4}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n-5}(\text{MO}(n)) \xrightarrow{s_*} \pi_{2n-6}(\text{MO}(n)) \xrightarrow{s_*} \cdots
\]

We first compute the stable homotopy groups of \( \text{SO}/\text{SO}(2n) \) in the first two non-trivial degrees, and to do this we first compute its \( \mathbb{F}_2 \)-cohomology as a module over the Steenrod algebra in low degrees.

**Lemma 5.2.** The space \( \text{SO}/\text{SO}(2n) \) is \((2n-1)\)-connected, and as long as \( 2n \geq 4 \) the cohomology groups

\[
H^i(\text{SO}/\text{SO}(2n); \mathbb{F}_2)
\]

are all 1-dimensional. If we write \( x_i \in H^i(\text{SO}/\text{SO}(2n); \mathbb{F}_2) \) for the non-trivial class in these degrees, then there are Steenrod operations \( \text{Sq}^i(x_{2n+1}) = x_{2n+2} \) and

\[
\text{Sq}^2(x_{2n}) = \begin{cases} x_{2n+2} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}
\]

Furthermore, the integral homology of \( \text{SO}/\text{SO}(2n) \) is \( \mathbb{Z}, \mathbb{Z}/2, \) and 0 in degrees \( 2n, 2n+1, \) and \( 2n+2 \).

**Proof.** We consider the Serre spectral sequence for the fibration

\[
\text{SO}/\text{SO}(2n) \to \text{BSO}(2n) \to \text{BSO}.
\]

The \( \mathbb{F}_2 \)-cohomology of \( \text{BSO} \) is the polynomial algebra on the Stiefel–Whitney classes \( w_i \in H^i(\text{BSO}; \mathbb{F}_2) \) for \( 2 \leq i \), and the \( \mathbb{F}_2 \)-cohomology of \( \text{BSO}(2n) \) is the polynomial algebra on the Stiefel–Whitney classes \( w_i \in H^i(\text{BSO}; \mathbb{F}_2) \) for \( 2 \leq i \leq 2n \). Thus for each \( i \geq 2n \) there must be a class

\[
x_i \in H^i(\text{SO}/\text{SO}(2n); \mathbb{F}_2)
\]

which transgresses to \( w_{i+1} \), and \( H^*(\text{SO}/\text{SO}(2n); \mathbb{F}_2) \) is isomorphic as a vector space to the exterior algebra on the classes \( x_i \). (This could also be computed using the Eilenberg–Moore spectral sequence.) As we have assumed \( 2n \geq 4 \), it follows that in degrees \( 2n \leq i \leq 2n+2 \) the \( i \)-th cohomology of \( \text{SO}/\text{SO}(2n) \) is 1-dimensional and is generated by \( x_i \).

To compute Steenrod operations on the \( x_i \) we use the Kudo transgression theorem along with the Wu formulae for \( \text{Sq}^i(w_k) \), namely that

\[
\text{Sq}^1(w_{2n+2}) \equiv w_{2n+3} \quad \text{and} \quad \text{Sq}^2(w_{2n+1}) \equiv \begin{cases} w_{2n+3} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}
\]

modulo decomposables.

Finally, we may compute the \( \mathbb{Z}(2) \)-homology from our knowledge that the Bockstein

\[
\beta = (\text{Sq}^1)^* : H_{2n+2}(\text{SO}/\text{SO}(2n); \mathbb{F}_2) \to H_{2n+1}(\text{SO}/\text{SO}(2n); \mathbb{F}_2)
\]

is an isomorphism, to be \( \mathbb{Z}(2), \mathbb{Z}/2, \) and 0 in degrees \( 2n, 2n+1, \) and \( 2n+2 \). To go from this to the integral statement, we can repeat the above calculation with
coefficients in $\mathbb{Z}[\frac{1}{2}]$, where the cohomology of $BSO$ and $BSO(2n)$ is well-known, to find that the $\mathbb{Z}[\frac{1}{2}]$-homology is $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z}$, and $0$ in degrees $2n, 2n + 1$, and $2n + 2$. □

**Lemma 5.3.** We have $\pi^*_{2n}(SO/SO(2n)) \cong \mathbb{Z}$, and $\pi^*_{2n+1}(SO/SO(2n))$ has cardinality $4$.

**Proof.** It follows from the Hurewicz theorem that the Hurewicz map

$$\pi^*_{2n}(SO/SO(2n)) \rightarrow H_{2n}(SO/SO(2n); \mathbb{Z}) \cong \mathbb{Z}$$

is an isomorphism, which proves the first part. To see that the homotopy group $\pi^*_{2n+1}(SO/SO(2n))$ has cardinality $4$, we will use the Atiyah–Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(SO/SO(2n), \ast; \pi^*_q) \Longrightarrow \pi^*_p(SO/SO(2n)).$$

On the $E^2$ page the only two relevant groups are

$$E^2_{2n,1} = H_{2n}(SO/SO(2n); \pi^*_1) = H_{2n}(SO/SO(2n); \mathbb{Z}/2) = \mathbb{Z}/2$$

and

$$E^2_{2n+1,0} = H_{2n+1}(SO/SO(2n); \pi^*_0) = H_{2n+1}(SO/SO(2n); \mathbb{Z}) = \mathbb{Z}/2.$$

Since

$$E^2_{2n+2,0} = H_{2n+2}(SO/SO(2n); \pi^*_0) = H_{2n+2}(SO/SO(2n); \mathbb{Z}) = 0,$$

the differential $d^2 : E^2_{2n+2,0} \rightarrow E^2_{2n,1}$ is zero, so both copies of $\mathbb{Z}/2$ survive to $E^\infty$. □

**Lemma 5.4.** The group $\pi^*_{2n+1}(SO/SO(2n))$ is isomorphic to $\mathbb{Z}/4$ when $n$ is odd and to $(\mathbb{Z}/2)^2$ when $n$ is even.

**Proof.** To detect whether the group $\pi^*_{2n+1}(SO/SO(2n))$ is cyclic or not, we use the mod $2$ Moore spectrum $S^0/2$. Smashing the cofibre sequence

$$S^0 \rightarrow S^0 \rightarrow S^0/2$$

with $SO/SO(2n)$ and taking homotopy groups, it follows that

$$\pi_{2n+1}(SO/SO(2n) \wedge (S^0/2)) \cong \begin{cases} \mathbb{Z}/2 & \text{if } \pi^*_{2n+1}(SO/SO(2n)) = \mathbb{Z}/4 \\ (\mathbb{Z}/2)^2 & \text{if } \pi^*_{2n+1}(SO/SO(2n)) = (\mathbb{Z}/2)^2. \end{cases}$$

Thus to determine whether $\pi^*_{2n+1}(SO/SO(2n))$ is cyclic we will compute this homotopy group.

The mod $2$ Moore spectrum has first two homotopy groups $\pi_0(S^0/2) = \mathbb{Z}/2$ and $\pi_1(S^0/2) = \mathbb{Z}/2$, and the $k$-invariant connecting these two groups is $Sq^2$. There is an Atiyah–Hirzebruch spectral sequence for computing $S^0/2$-homology, which in this case has the form

$$E^2_{p,q} = H_p(SO/SO(2n), \ast; \pi_q(S^0/2)) \Longrightarrow \pi^*_p(SO/SO(2n) \wedge (S^0/2)).$$

On the $E^2$ page, the groups which may contribute to $\pi^*_{2n+1}(SO/SO(2n) \wedge (S^0/2))$ are again

$$E^2_{2n,1} = H_{2n}(SO/SO(2n), \ast; \pi_1(S^0/2)) = H_{2n}(SO/SO(2n), \ast; \mathbb{Z}/2) = \mathbb{Z}/2$$

and

$$E^2_{2n+1,0} = H_{2n+1}(SO/SO(2n), \ast; \pi_0(S^0/2)) = H_{2n+1}(SO/SO(2n), \ast; \mathbb{Z}/2) = \mathbb{Z}/2,$$

but this time we have

$$E^2_{2n+2,0} = H_{2n+2}(SO/SO(2n); \pi_0(S^0/2)) = H_{2n+2}(SO/SO(2n); \mathbb{Z}/2) = \mathbb{Z}/2,$$

and the differential $d^2 : E^2_{2n+2,0} \rightarrow E^2_{2n,1}$ is the linear dual of

$$Sq^2 : H^{2n}(SO/SO(2n); \mathbb{Z}/2) \rightarrow H^{2n+2}(SO/SO(2n); \mathbb{Z}/2).$$
We have shown in Lemma 5.2 that this is an isomorphism when \( n \) is odd and zero when \( n \) is even, so \( \pi_{2n+1}(SO/SO(2n) \wedge (S^0/2)) \) is \( \mathbb{Z}/2 \) if and only if \( n \) is odd, which finishes the proof. \( \square \)

**Lemma 5.5.** The map \( \pi^*_n(SO/SO(2n)) \to \pi_0(MT\theta_n) \) is injective.

**Proof.** The class \( e \cdot u_{-2n} \in H^0(MT\theta_n; \mathbb{Z}) \) gives a map \( E : MT\theta_n \to H\mathbb{Z} \), and the composition

\[
\Sigma^{-2n}(SO/SO(2n)) \to MT\theta_n \xrightarrow{E} H\mathbb{Z}
\]

is isomorphic precisely 4, as does \( \partial \). On the other hand, the exact sequence shows it has cardinality at most 4, so it has cardinality at least 4, as follows from the surjectivity of \( \partial \).

The long exact sequence above thus simplifies to

\[
\cdots \to \Omega^{(n)}_{2n+2} \xrightarrow{\partial} \pi^*_n(SO/SO(2n)) \to \pi_1(MT\theta_n) \xrightarrow{s_n} \Omega^{(n)}_{2n+1} \to 0,
\]

and we recall that under the isomorphism

\[
H_1(\Gamma_n^{SO}) \xrightarrow{\sim} H_1(BDiff(W^{2n}_g, D^{2n}_g)) \xrightarrow{\sim} H_1(\Omega^{(n)}_0 MT\theta_n) \cong \pi_1(MT\theta_n)
\]

the map \( s_n \) coincides with the map \( t \).

**Lemma 5.6.** If \( n \neq 3 \) or 7 then the map \( \partial_* : \Omega^{(n)}_{2n+2} \to \pi^*_n(SO/SO(2n)) \) is zero.

**Proof.** By Corollary 4.2 the kernel of the map \( t \), and hence \( s_\ast \), has cardinality at least 4, and so the kernel of \( s_\ast : \pi_1(MT\theta_n) \to \Omega^{(n)}_{2n+1} \) also has cardinality at least 4. On the other hand, the exact sequence shows it has cardinality at most 4, so it has cardinality precisely 4, as does \( \pi_{2n+1}(SO/SO(2n)) \) by Lemma 5.3 so \( t \) is zero. \( \square \)

**Lemma 5.7.** If \( n = 3 \) or 7 then \( \partial_* : \Omega^{(n)}_{2n+2} \to \pi^*_n(SO/SO(2n)) \) is surjective (so \( s_* \) is injective).

**Proof.** Using Lemma 5.4, we consider the diagram

\[
\begin{array}{ccc}
\pi_{2n+2}(MO(n)) & \xrightarrow{\partial} & \pi^*_n(SO/SO(2n)) \cong \mathbb{Z}/4 \\
\downarrow h & & \downarrow h \\
H_{2n+2}(MO(n); \mathbb{F}_2) & \xrightarrow{\partial} & H_{2n+1}(SO/SO(2n); \mathbb{F}_2) \cong \mathbb{Z}/2,
\end{array}
\]

where \( h \) denotes the Hurewicz map, and the surjectivity on the right follows from the AHSS in the proof of Lemma 5.3. The isomorphism \( H_{2n+1}(SO/SO(2n); \mathbb{F}_2) \cong \mathbb{Z}/2 \) is given by the class \( x_{2n+1} \in H^{2n+1}(SO/SO(2n); \mathbb{F}_2) \), which under \( \partial^\ast \) corresponds to \( w_{2n+2} \). Thus the composition \( \partial_* \circ h \) can be identified with the functional \( \Omega^{(n)}_{2n+2} \to \mathbb{Z}/2 \) given by \( [W^{2n+2}] \mapsto \langle [W], w_{2n+2}(TW) \rangle \).

The manifolds \( \mathbb{H}P^2 \in \Omega^{(3)}_8 \) and \( [\mathbb{O}P^2] \in \Omega^{(7)}_{16} \) have Euler characteristic 3, so non-trivial top Stiefel–Whitney class. Thus \( \partial_* \circ h = h \circ \partial_* \) is surjective in these cases, but it follows that \( \partial_* \) must then be surjective. \( \square \)

5.2. **Proof of Theorem 1.3**. For the surjectivity part of the statement, we have already explained how Kreck’s result (Kreck) implies the surjectivity of the homomorphism \( f : \Gamma_{g,1} \to H_1(\text{Aut}(Q_{W^g})) \). To see surjectivity of \( t \circ f \) it suffices to see that the restriction \( t|_{\text{Ker}(f)} : \text{Ker}(f) \to \Omega^{(n)}_{2n+1} \) is surjective, but that follows from Lemma 4.1.

It remains to see that \( t \circ f : H^1(\Gamma_{g,1}^{SO} \to \Omega^{(n)}_{2n+1} \oplus H_1(\text{Aut}(Q_{W^g})) \) is injective for \( n \neq 2 \) and \( g \geq 5 \). For \( n = 3 \) or 7, the second summand vanishes, so it suffices
to prove that \( t \) is injective, which we did in Lemma 5.7. In the remaining cases, Lemma 5.6 gives a short exact sequence fitting into the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_{2n+1}^s(SO/SO(2n)) & \rightarrow & \pi_1(M\theta_n) & \rightarrow & \Omega_{2n+1}^{(n)} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & H_1(Aut(QW_1)) & \rightarrow & H_1(\Gamma^n_{2n+1}).
\end{array}
\]

By the proof of Corollary 1.2, the map \( \pi_{2n+1}^s(SO/SO(2n)) \rightarrow H_1(Aut(QW_1)) \) is an isomorphism, giving a splitting of the exact sequence in the top row of the diagram. This proves Theorem 1.3 in these cases. (And for \( n = 2 \), it establishes the direct sum splitting implicitly asserted in Table 1.)

6. A filtration of the sphere spectrum

In this section we shall describe and study a filtration of the sphere spectrum, and a resulting filtration of the stable homotopy groups of spheres. This plays a role in computing the cobordism groups \( \Omega_{2n+1}^{(n)} \) in terms of the stable homotopy groups of spheres and the \( J \)-homomorphism.

Recall that \( BO(n) \rightarrow BO \) denotes the \( n \)-connected cover, and there is an associated Thom spectrum \( MO(n) \). Thus \( MO(0) = MO \), the spectrum representing unoriented cobordism theory, \( MO(1) = MSpin \), \( MO(2) = MO(3) = MSpin \), etc. There are maps

\[
MO = MO(0) \leftarrow MO(1) \leftarrow MO(2) \leftarrow MO(3) \leftarrow \cdots
\]

with inverse limit \( S \), the sphere spectrum. We write \( \iota_n : S \rightarrow MO(n) \), and define a filtration of the stable homotopy groups of spheres by

\[
F^n\pi_k(S) = \text{Ker } (\pi_k(\iota_n) : \pi_k(S) \rightarrow \pi_k(MO(n))).
\]

Let us write \( \overline{MO(n)} \) for the homotopy cofibre of \( S \rightarrow MO(n) \).

Lemma 6.1.

(i) \( F^n\pi_k(S) = 0 \) for \( k < n \).

(ii) \( F^n\pi_k(S) \) contains the image of \( J : \pi_k(O) \rightarrow \pi_k(S) \) for \( k \geq n \).

(iii) \( F^n\pi_k(S) \) is equal to the image of \( J : \pi_k(O) \rightarrow \pi_k(S) \) for \( 2n \geq k \geq n \).

Proof. The spectrum \( MO(n) \) is \( n \)-connected, and so \( \pi_k(S) \rightarrow \pi_k(MO(n)) \) is injective for \( k < n \); this establishes (i).

In the cobordism-theoretic interpretation of the homotopy groups of spheres, \( J(\alpha : S^k \rightarrow O) \) is given by the manifold \( S^k \) with the framing given by twisting the standard (bounding) framing of \( S^k \) using \( \alpha \) to obtain a new framing \( \xi_\alpha \).

\[
\begin{array}{cccccc}
S^k & \xrightarrow{\xi_\alpha} & EO & \xrightarrow{} & BO(n) \\
D^{k+1} & \downarrow & & \downarrow & \\
& & BO & &
\end{array}
\]

While this framing cannot necessarily be extended to \( D^{k+1} \), the associated \( BO(n) \)-structure can be extended as long as \( k \geq n \), as the right-hand map is \( n \)-co-connected; this establishes (ii).

Let \( (M^k, \xi) \) be a framed cobordism class representing an element of \( F^n\pi_k(S) \), so considered as a \( BO(n) \)-manifold \( M \) bounds a \( BO(n) \)-manifold \( W \). Now \( k + 1 \leq 2n + 1 \) so (similarly to the proof of Lemma 1.1) by the techniques of [KM63] Theorems 5.5 and 6.6 we may perform surgery on the interior of \( W \) to obtain a new \( BO(n) \)-manifold \( W' \) which is \( [k/2] \)-connected, with the same framed boundary \( M \).
We may then find a handle structure on $W'$ having no handles of index between 1 and $[k/2]$, and so $W' \setminus D^{k+1}$ is a $BO(n)$-cobordism from $M$ to $S^k$ which may be obtained from $M$ by attaching handles of index at most $k - [k/2] \leq n$. As $EO \to BO(n)$ is $n$-connected, it follows that the framing $\xi$ on $M$ may be extended to $W' \setminus D^{k+1}$, and so $(M, \xi)$ is framed cobordant to $(S^k, \zeta)$ for some framing $\zeta$ of the sphere. But those cobordism classes represented by spheres with some framing are precisely the image of the $J$-homomorphism; this establishes \(\text{(iii)}\). □

We wish to understand the group $\Omega^{(n)}_\infty = \pi_{2n+1}(MO(n))$, which is related to $F^n\pi_{2n+1}(S)$ and lies just outside of the range treated in Lemma 6.1 \(\text{(ii)}\). However, the groups $F^n\pi_k(S)$ for $k > 2n$ have been studied, though not quite expressed in this form, by Stolz \cite{Sto85}. Let us explain his technique.

For $n \geq 2$ the universal (virtual) bundle $\gamma(n)$ over $BO(n)$ is Spin, and so the Thom spectrum $MO(n)$ has a KO-theory Thom class, $\lambda_n$. There is thus a KO-theory class $\gamma(n) \cdot \lambda_n \in KO^0(MO(n))$, which we represent by a map $\alpha_n : MO(n) \to KO$ to the connective KO-theory spectrum. As the bundle $\gamma(n) \in KO^0(BO(n))$ becomes trivial when restricted to a point, the class lifts to $\gamma(n) \in KO^0(BO(n), *)$, and $\alpha_n$ factors through a map $\alpha_n' : MO(n) \to ko$, and as $MO(n)$ is $n$-connected this lifts further to a map

$$\pi_n : \overline{MO(n)} \to ko(n).$$

Stolz defines $A[n+1]$ to be the homotopy fibre of $\pi_n$. Under the Thom isomorphism we have

$$H^*(\overline{MO(n)}) \cong H^*(BO(n),*) = H^*(\Omega^\infty(ko(n)),*),$$

and using the known cohomology of $BO(n)$ and $ko(n)$ as modules over the Steenrod algebra Stolz establishes the following.

**Theorem 6.2** (Stolz \cite{Sto85}). The spectrum $A[n+1]$ is $(2n+1)$-connected, and

$$\pi_{2n+2}(A[n+1]) = \begin{cases} \mathbb{Z} & n+1 \equiv 0, 4 \mod 8 \\ \mathbb{Z}/2 & n+1 \equiv 1, 2 \mod 8 \\ 0 & \text{otherwise}. \end{cases}$$

Let us write $J : \pi_k(O) \to \pi_k(S)$ for the $J$-homomorphism. For $\alpha \in \pi_k(O)$, $J(\alpha)$ is given by the stably framed manifold obtained by changing the bounding framing on $S^k$ using $\alpha$. As explained in the proof of Lemma 6.1 \(\text{(i)}\), the associated $BO(n)$-structure extends canonically over $D^{k+1}$ as long as $k \geq n$, which gives a map

$$\overline{J} : \pi_k(O) \to \pi_{k+1}(\overline{MO(n)})$$

such that $\partial \circ \overline{J} = J$. The composition

$$\pi_k(O) \to \pi_{k+1}(\overline{MO(n)}) \xrightarrow{\overline{J}} \pi_{k+1}(ko(n))$$

is an isomorphism (cf. [Sto85, Lemma 3.7]), and it follows from the commutative diagram

\[
\begin{array}{c}
\pi_{2n+2}(\text{MO}(n)) \\
\downarrow \\
\pi_{2n+2}(A[n+1]) \\
\downarrow \\
\pi_{2n+2}(\text{MO}(n)) \\
\downarrow \\
\pi_{2n+2}(\text{ko}(n)) \\
\downarrow 7 \\
\pi_{2n+1}(S) \\
\downarrow \\
\pi_{2n+1}(O) \\
\downarrow \\
\pi_{2n+1}(\text{MO}(n)) \\
\downarrow \\
0
\end{array}
\]

that there is an exact sequence

\[\pi_{2n+2}(A[n+1]) \overset{\sigma}{\longrightarrow} \text{Coker}(J)_{2n+1} \longrightarrow \pi_{2n+1}(\text{MO}(n)) \longrightarrow 0.\]

Hence, given the description of \(\pi_{2n+2}(A[n+1])\) in Theorem 6.2, it follows that the quotient \(F^n\pi_{2n+1}(S)/\text{Im}(J)\) is cyclic. Stolz finds various conditions under which the quotient \(F^n\pi_{2n+1}(S)/\text{Im}(J)\) is in fact trivial, i.e. the map \(\sigma\) is zero.

**Theorem 6.3** (Stolz [Sto85]). If either

(i) \(n + 1 \equiv 2 \mod 8\) and \(n + 1 \geq 18\),

(ii) \(n + 1 \equiv 1 \mod 8\) and \(n + 1 \geq 113\),

(iii) \(n + 1 \not\equiv 0, 1, 2, 4 \mod 8\),

then \(F^n\pi_{2n+1}(S) = \text{Im}(J)_{2n+1}\).

If \(n + 1 = 4\ell\) then \(F^n\pi_{2n+1}(S)/\text{Im}(J)\) is generated by the exotic sphere \(\Sigma\) which is the boundary of the manifold obtained by plumbing together two copies of the linear \(4\ell\)-dimensional disc bundle over \(S^{4\ell}\) having trivial Euler class and representing a generator of \(\pi_{4\ell}(\text{BO}) \cong \mathbb{Z}\).

**Proof.** By [Sto85, Theorem B (i) and (ii)], in the case \(\pi_{2n+2}(A[n+1]) = \mathbb{Z}/2\), these map to zero in \(\text{Coker}(J)_{2n+1}\) under the conditions given in the statement of the proposition.

It follows from [Sto85, Lemma 10.3] that when \(n + 1 = 4\ell\) a generator of \(\mathbb{Z} = \pi_{2n+2}(A[n+1])\) in \(\pi_{2n+2}(\text{MO}(n))\) is given by the class of the plumbing described in the statement of the proposition. \(\square\)

In the case \(n+1=4\ell\), it seems to be a difficult problem to obtain any information about the order, or indeed the nontriviality, of \([\Sigma] \in \text{Coker}(J)_{8\ell-1}\). All calculations we have attempted are consistent with the following conjecture.

**Conjecture A.** \([\Sigma] = 0 \in \text{Coker}(J)_{8\ell-1}\).

This conjecture would imply that \(\sigma\) is zero in these cases too, and so \(\Omega_{8\ell-1}^{(4\ell-1)} \cong \text{Coker}(J)_{8\ell-1}\). The most promising approach to this conjecture seems to be as follows. By the discussion above, the map

\[\text{Coker}(J)_{2n+1} \longrightarrow \pi_{2n+1}(\text{MO}(n + 1))\]

is an isomorphism, so Conjecture A is equivalent to

**Conjecture B.** The map \(\pi_{8\ell-1}(\text{MO}(4\ell)) \rightarrow \pi_{8\ell-1}(\text{MO}(4\ell-1))\) is injective.
For example, when \( \ell = 1 \) this asks if \( \Omega_{7}^{\text{String}} \to \Omega_{15}^{\text{String}} \) is injective, which it is as both groups are zero. When \( \ell = 2 \) this asks if \( \text{Coker}(J_{15}) \to \Omega_{15}^{\text{String}} \) is injective, which it is as \( \pi_{15}(\text{MO}(8)) = \text{Coker}(J_{15}) = \mathbb{Z}/2, \Omega_{15}^{\text{String}} = \mathbb{Z}/2, \) and generators of either group may be represented by an exotic sphere \([\text{KM63, Gia71}]\).

**Corollary 6.4.** If \( n \) satisfies one of the conditions of Theorem 6.3, then the cobordism group \( 
abla_{2n+1}^{(n)} \) occurring in Theorem 6.3 is isomorphic to \( \text{Coker}(J_{2n+1}) \).

### 7. Relation to the Work of Kreck

In [Kre79], Kreck gives a description of the mapping class groups of \( (n - 1) \)-connected \( 2n \)-manifolds, up to two extension problems. Applied to our situation, he gives extensions [Kre79, Proposition 3]

\[
1 \to \Theta_{2n+1} \to T_{g,1}^{n} \to \Gamma_{g,1}^{n} \to \text{Aut}(Q_{W_{g}}) \to 1
\]

and

\[
1 \to \Omega_{2n+1} \to T_{g,1}^{n} \xrightarrow{\chi} \text{Hom}(H_{n}(W_{g}), S\pi_{n}(\text{SO}(n))) \to 1
\]

where \( S\pi_{n}(\text{SO}(n)) = \text{Im}(\pi_{n}(\text{SO}(n)) \to \pi_{n}(\text{SO}(n + 1))) \). These groups are given, for \( n \geq 3 \), by Table 2 (except that \( S\pi_{6}(\text{SO}(6)) = 0 \).) The map \( \chi \) may be described as follows: to a diffeomorphism \( \varphi : W_{g,1}^{2n} \to W_{g,1}^{2n} \) which acts as the identity on homology, and a class \( x \in H_{n}(W_{g,1}; \mathbb{Z}) \cong \pi_{n}(W_{g,1}^{2n}) \) represented by an embedding \( x : S^{n} \to W_{g,1}^{2n} \), the sphere \( \varphi \circ x \) is isotopic to \( x \) and so by the isotopy extension theorem we may suppose that \( \varphi \circ x = x \). Then \( \epsilon^{1} \oplus \nu_{2} \cong \epsilon^{n+1} \) and the differential \( D_{\varphi}|_{\epsilon^{(S^{n})}} \) gives an automorphism of this bundle, corresponding to a map \( \chi(\varphi)(x) : S^{n} \to \text{SO}(n + 1) \). It can be checked that this map lies in \( S\pi_{n}(\text{SO}(n)) \).

As the manifolds \( W_{g,1}^{2n} \) bound the parallelisable manifolds \( g^{n} S^{n} \times D^{n+1} \), the \( g \)-fold boundary connected sum of \( S^{n} \times D^{n+1} \), by [Kre79, Lemma 3 (ii)] certain characteristic elements \( \Sigma_{W_{g,1}}^{2n} \) are trivial, and it then follows from [Kre79, Theorem 2] that \( \Gamma_{g}^{n} \) fits into the same extensions. Thus the natural map \( \Gamma_{g,1}^{n} \to \Gamma_{g}^{n} \) is an isomorphism.

The calculations of this paper can be used to shed light on some of these extension problems, especially for those \( n \) such that \( S\pi_{n}(\text{SO}(n)) = 0 \).

**Theorem 7.1.** The map

\[
t \times f : \Gamma_{g,1}^{6} \to \Omega_{13}^{(6)} \times \text{O}_{g,1}(\mathbb{Z})
\]

is an isomorphism, and \( \Omega_{13}^{(6)} = \Omega_{13}^{\text{String}} \cong \mathbb{Z}/3 \).

**Proof.** We have \( S\pi_{6}(\text{SO}(6)) = 0 \), and so the two extensions reduce to

\[
1 \to \Theta_{13} \to \Gamma_{g,1}^{6} \xrightarrow{f} \text{O}_{g,1}(\mathbb{Z}) \to 1.
\]

The group \( bP_{13} \) is trivial \([\text{KM63}]\) and so \( \Theta_{13} \cong \text{Coker}(J_{13}) \), which is \( \mathbb{Z}/3 \), and is isomorphic to \( \Omega_{13}^{(6)} \) by Theorem 6.3. Thus \( \Theta_{13} \to \Gamma_{g,1}^{6} \to \Omega_{13}^{(6)} \) is an isomorphism, which shows that the extension is trivial. \( \square \)
Theorem 7.2. If \( n \equiv 5 \mod 8 \) then there is an extension

\[
1 \longrightarrow bP_{2n+2} \longrightarrow \Gamma^n_{g,1} \longrightarrow \Omega^n_{2n+1} \longrightarrow \text{Sp}^q_{2g}(\mathbb{Z}) \longrightarrow 1,
\]

where we write \( \text{Sp}^q_{2g}(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z}) \) for the subgroup of those automorphisms of the symplectic space \( \mathbb{Z}^{2g} \) which preserve the standard quadratic function. The subgroup \( bP_{2n+2} \trianglelefteq \Gamma^n_{g,1} \) consists of commutators, and \( \Omega^n_{2n+1} \cong \text{Cok}(J)_{2n+1} \).

Proof. We have \( S\pi_n(SO(n)) = 0 \) and so Kreck’s exact sequences reduce to

\[
1 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma^n_{g,1} \longrightarrow \text{Sp}^q_{2g}(\mathbb{Z}) \longrightarrow 1.
\]

Furthermore, in this dimension we have an exact sequence

\[
(7.1) \quad 1 \longrightarrow bP_{2n+2} \longrightarrow \Theta_{2n+1} \longrightarrow \text{Coker}(J)_{2n+1} \longrightarrow 1
\]

and \( n + 1 \equiv 6 \mod 8 \) so by Theorem [13] the map \( \text{Coker}(J)_{2n+1} \rightarrow \Omega^n_{2n+1} \) is an isomorphism. Thus the kernel of \( t \times \tilde{f} \) is precisely the subgroup \( bP_{2n+2} \trianglelefteq \Theta_{2n+1} \trianglelefteq \Gamma^n_{g,1} \). Furthermore, we know \( t \times \tilde{f} \) induces an isomorphism on abelianisations, so \( bP_{2n+2} \) consists of commutators.

For each \( n \equiv 5 \mod 8 \) the extension in the theorem can be pulled back to an extension

\[
(7.2) \quad 1 \longrightarrow bP_{2n+2} \longrightarrow E(g, n) \longrightarrow \text{Sp}^q_{2g}(\mathbb{Z}) \longrightarrow 1,
\]

which in turn fits into an extension

\[
1 \longrightarrow E(g, n) \longrightarrow \Gamma^n_{g,1} \longrightarrow \Omega^n_{2n+1} \longrightarrow 1.
\]

Brumfiel has shown [Bru68] that \( \text{Coker}(J)_{2n+1} \) is split (for \( n \neq 2^k - 2 \), which is satisfied as we are supposing that \( n \equiv 5 \mod 8 \)), and choosing a splitting \( s : \text{Coker}(J)_{2n+1} \rightarrow \Theta_{2n+1} \), we have that the composition

\[
\text{Coker}(J)_{2n+1} \xrightarrow{s} \Theta_{2n+1} \longrightarrow \Gamma^n_{g,1} \longrightarrow \Omega^n_{2n+1}
\]

is an isomorphism. As \( \Theta_{2n+1} \) lies in the centre of \( \Gamma^n_{g,1} \), this gives an isomorphism

\[
\Gamma^n_{g,1} \cong E(g, n) \times \text{Cok}(J)_{2n+1}.
\]

It seems a rather interesting problem to understand the extension (7.2), which must be highly nontrivial (as \( bP_{2n+2} \trianglelefteq E(g, n) \) again consists of commutators).

References

[ABP66] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, Spin cobordism, Bull. Amer. Math. Soc. 72 (1966), 256–260.

[Bak69] Anthony Bak, On modules with quadratic forms, Algebraic K-Theory and its Geometric Applications (Conf., Hull, 1969), Springer, Berlin, 1969, pp. 55–66.

[Bak81] ________, K-theory of forms, Annals of Mathematics Studies, vol. 98, Princeton University Press, Princeton, N.J., 1981.

[Bru68] G. Brumfiel, On the homotopy groups of BPL and PL/O, Ann. of Math. (2) 88 (1968), 291–311.

[Bru69] ________, On the homotopy groups of BPL and PL/O. II, Topology 8 (1969), 305–311.

[Bru70] ________, On the homotopy groups of BPL and PL/O. III, Michigan Math. J. 17 (1970), 217–224.

[Cha87] Ruth Charney, A generalization of a theorem of Vogtmann, Proceedings of the Northwestern conference on cohomology of groups (Evanson, Ill., 1985), vol. 44, 1987, pp. 107–125.

[GHS09] V. Gritsenko, K. Hulek, and G. K. Sankaran, Abelianisation of orthogonal groups and the fundamental group of modular varieties, J. Algebra 322 (2009), no. 2, 463–478.

[Gia71] V. Giambalvo, On (8)-cobordism, Illinois J. Math. 15 (1971), 533–541.

[GRW14a] Søren Galatius and Oscar Randal-Williams, Homological stability for moduli spaces of high dimensional manifolds. I, arXiv:1403.2334, 2014.
Stable moduli spaces of high dimensional manifolds, Acta Math. 212 (2014), no. 2, 257–377.

André Haefliger, Plongements différentiables de variétés dans variétés, Comment. Math. Helv. 36 (1961), 47–82.

Dennis Johnson and John J. Millson, Modular Lagrangians and the theta multiplier, Invent. Math. 100 (1990), no. 1, 143–165.

Michel A. Kervaire and John W. Milnor, Groups of homotopy spheres. I, Ann. of Math. (2) 77 (1963), 504–537.

Matthias Kreck, Isotopy classes of diffeomorphisms of \((k−1)\)-connected almost-parallelizable \(2k\)-manifolds, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 643–663.

Sam Nariman, Homological stability and stable moduli of flat manifold bundles, arXiv:1406.6416, 2014.

Jerome Powell, Two theorems on the mapping class group of a surface, Proc. Amer. Math. Soc. 68 (1978), no. 3, 347–350.

Stephan Stolz, Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder, Lecture Notes in Mathematics, vol. 1116, Springer-Verlag, Berlin, 1985, With an English introduction.

William Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974), 304–307.

C. T. C. Wall, Classification of \((n−1)\)-connected \(2n\)-manifolds, Ann. of Math. (2) 75 (1962), 163–189.

Diffeomorphisms of 4-manifolds, J. London Math. Soc. 39 (1964), 131–140.

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