PAINTING A GRAPH WITH COMPETING RANDOM WALKS

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Let $X_1, X_2$ be independent random walks on $\mathbb{Z}_n^d$, $d \geq 3$, each starting from the uniform distribution. Initially, each site of $\mathbb{Z}_n^d$ is unmarked and, whenever $X_i$ visits such a site, it is set irreversibly to $i$. The mean of $|A_i|$, the cardinality of the set $A_i$ of sites painted by $i$ once all of $\mathbb{Z}_n^d$ has been visited, is $n^d/2$ by symmetry. We prove the following conjecture due to Pemantle and Peres: for each $d \geq 3$ there exists a constant $\alpha_d$ such that $\lim_{n \to \infty} \text{Var}(|A_i|)/h_d(n) = \frac{1}{4} \alpha_d$ where $h_3(n) = n^4$, $h_4(n) = n^4 \log(n)$, and $h_d(n) = n^d$ for $d \geq 5$. We will also identify $\alpha_d$ explicitly. This is a special case of a more general theorem which gives the asymptotics of $\text{Var}(|A_i|)$ for a large class of transient, vertex transitive graphs; other examples include the hypercube and the Caley graph of the symmetric group generated by transpositions.

1. Introduction. Suppose that $X_1, X_2$ are independent random walks on a graph $G = (V, E)$ starting from stationarity. Initially, each vertex of $G$ is unmarked and, whenever $X_i$ visits such a site, it is marked $i$ irreversibly. If both $X_1$ and $X_2$ visit a site for the first time simultaneously, then the mark is chosen by the flip of an independent fair coin. Let $A_i$ be the set of sites marked $i$ once every vertex of $G$ has been visited. By symmetry, it is obvious that $\mathbb{E}|A_i| = \frac{1}{2}|V|$. The purpose of this manuscript is to derive precise asymptotics for $\text{Var}(|A_i|)$ for many families of graphs.

The study of the statistical properties of $A_i$ was first proposed in the work of S.R. Gomes Junior et. al. in [8] as a source of interesting mathematical problems which serve as models for the technical challenges associated with physical problems involving interacting random walks. Their main result is an estimate of the growth of $\mathbb{E}|B|$ where $B$ is the interface separating $A_1$ from $A_2$ in the special case of the one-cycle $\mathbb{Z}_n^1$. As with $\mathbb{E}|A_i|$, computing $\mathbb{E}|B|$ for $\mathbb{Z}_n^d$ becomes trivial starting with $d = 3$ since it is easy to see that, with probability strictly between 0 and 1, for any pair of adjacent vertices $x, y$, $X_2$ will hit $y$ before $X_1$ conditional on the event that $X_1$ hits $x$ first.

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On the other hand, estimating \( \text{Var}(|A_i|) \) in this setting poses a much greater challenge since its expansion in terms of correlation functions exhibits significant cancellation which, when ignored, leads to bounds that are quite imprecise. We will develop this point further at the end of the introduction.

We remark that the problem we consider was communicated to us by Hilhorst [5] in a slightly different setting. Rather than considering the sets of sites \( A_1, A_2 \) \textit{first painted} by \( X_1, X_2 \), respectively, it is also natural to study the sets \( \tilde{A}_1, \tilde{A}_2 \) of sites \textit{most recently painted} by \( X_1, X_2 \), respectively. In other words, in the latter formulation the constraint that the marks are irreversible is removed. It turns out that these two classes of problems are equivalent, which is to say \((A_1, A_2) \overset{d}{=} (\tilde{A}_1, \tilde{A}_2)\). This helpful observation, which follows from the time-reversibility of random walk, was made and communicated to us by Comets [2].

We restrict our attention to \textit{lazy} walks \( X_1, X_2 \) to avoid issues of periodicity. That is, the one-step transition kernel is given by

\[
p(x, y; G) = \begin{cases} 
\frac{1}{2} & \text{if } x = y, \\
\frac{1}{2\text{deg}(x)} & \text{if } x \sim y, \\
0 & \text{otherwise},
\end{cases}
\]

where \( x \sim y \) means that \( x \) is adjacent to \( y \) in \( G \). The particular choice of holding probability \( \frac{1}{2} \) is not important for the proof; indeed, any \( \lambda \in (0, 1) \) would suffice. Our proofs also work in the setting of continuous time walks. Let \( p^t(\cdot, \cdot; G) \) be the \( t \)-step transition kernel of lazy random walk on \( G \) and \( \pi(\cdot; G) \) its unique stationary distribution.

Our main result is the precise asymptotics for \( \text{Var}(|A_i|) \) on tori of dimension at least three, thus verifying a conjecture due to Pemantle and Peres [11].

**Theorem 1.1.** Suppose that \( G_n = \mathbb{Z}^d_n, \ d \geq 3 \). There exists a finite constant \( \alpha_d > 0 \) such that

\[
\lim_{n \to \infty} \frac{\text{Var}(|A_i|)}{h_d(n)} = \frac{1}{4} \alpha_d
\]

where

\[
h_d(n) = \begin{cases} 
n^4 & \text{if } d = 3, \\
 n^4(\log n) & \text{if } d = 4, \text{ and} \\
n^d & \text{if } d \geq 5.
\end{cases}
\]
Our proof allows us to identify $\alpha_d$ explicitly and is given as follows. Let

$$
G(x; \mathbb{Z}^d) = \mathbb{E}_x \sum_{t=0}^{\infty} 1_{\{X(t)=x\}}
$$

be the Green’s function for lazy random walk on $\mathbb{Z}^d$. For $d \geq 5$,

$$
\alpha_d = \frac{1}{G^2(0; \mathbb{Z}^d)} \sum_{y \in \mathbb{Z}^d} G^2(y; \mathbb{Z}^d).
$$

It is not difficult to show that $\alpha_d \to 1$ as $d \to \infty$, so that $\text{Var}(|A_1|) \approx \frac{1}{4} \text{Var}(\text{range})$ for $d$ and $n$ large is close to the variance of an iid marking. For $d=4$,

$$
\alpha_4 = \lim_{T \to \infty} \frac{1}{G^2(0; \mathbb{Z}^4) \log n} \sum_{y \in \mathbb{Z}^4: |y| \leq n} G^2(y; \mathbb{Z}^4);
$$

we will explain why this limit exists and is positive and finite in Proposition 2.1. The definition of $\alpha_3$ is slightly more involved. Let $T^3$ denote the three-dimensional continuum torus, $p^t(\cdot, \cdot; T^3)$ the kernel for Brownian motion on $T^3$, and

$$
g^T(x, y; T^3) = \int_0^T p^t(x, y; T^3) dt.
$$

Now set

$$
\alpha_3 = \frac{1}{G^2(0; \mathbb{Z}^3)} \int_{T^3} \int_{T^3} (g^{T/2}(x, y; T^3) - \frac{1}{2} T)^2 dx dy,
$$

The reason that the limit exists and is positive and finite is that $p^t(x, y; T^3)$ converges to the uniform density exponentially fast in $t$ (see Proposition 3.1 for a discrete version of this statement).

One interesting remark is that the variance for $d=3, 4$ is significantly higher than that for an iid marking. The results of Theorem 1.1 should also be contrasted with behavior of the variance of the range $R$ of random walk on $\mathbb{Z}^d$ run up to the cover time $T_{\text{cov}}(\mathbb{Z}^d_n)$ of $\mathbb{Z}^d_n$, which is the expected amount of time it takes for a single random walk to visit every site. When $d \geq 3$, $T_{\text{cov}}(\mathbb{Z}^d_n) \sim c_d n^d (\log n)$ (see [10]). For $d \geq 5$, it follows from work of Jain and Orey [6] that $\text{Var}(|R|) = \Theta(n^d (\log n))$. For $d = 3, 4$, it follows from work on Jain and Pruitt [7] that $\text{Var}(|R|)$ is $\Theta(n^3 (\log n)^2)$ and $\Theta(n^4 (\log n))$, respectively.

This work opens the doors to many other problems involving two random walks. Natural next steps include CLTs for the fluctuations of $|A_i|$ and for
the number of sites painted by $i$ at time $t$, as well as the development of an understanding of the geometrical properties of the clusters of $A_i$. The latter seem to be connected to the theory of interlacements, which arise in the process of disconnection of a discrete cylinder by a random walk [3], [12], [13].

Theorem 1.1 is a special case of a much more general result, which gives the asymptotics of $\text{Var}(\left|A_i\right|)$ for many other graphs, such as the hypercube and the Caley graph of the symmetric group generated by transpositions. We will now review some additional terminology which is necessary to give a precise statement of the result. Recall that the uniform mixing time is

$$T_{\text{mix}}(G) = \min \left\{ t \geq 0 : \max_{x,y} \left| \frac{p^t(x,y;G)}{\pi(y;G)} - 1 \right| \leq \frac{1}{4} \right\}$$

and the Green’s function for $G$ is

$$g(x,y;G) = \sum_{t=0}^{T_{\text{mix}}(G)} p^t(x,y;G),$$

i.e. the expected amount of time $X_i$ spends at $y$ up until $T_{\text{mix}}(G)$ when started from $x$. Let $\tau_i(x) = \min\{t \geq 0 : X_i(t) = x\}$; we will omit $i$ if there is only one random walk under consideration.

**Assumption 1.2.** $(G_n)$ is a sequence of vertex transitive graphs with $|V_n| \to \infty$ such that

1. $T_{\text{mix}}(G_n) = o(|V_n|/(\log |V_n|)^2)$ and $\lim_{n \to \infty} T_{\text{mix}}(G_n) = \infty$,
2. $\sum_{y \neq x} g^2(x,y;G_n) = o(T_{\text{mix}}(G_n)/\log |V_n|)$, and
3. there exists $\rho_0 < 1$ so that $\mathbf{P}_x[\tau(y) \wedge \tau(z) \leq T_{\text{mix}}(G_n)] \leq \rho_0$ uniformly in $n$ and $x, y, z \in V_n$ distinct.

Note that vertex transitivity implies that $g$ is constant along the diagonal and, combined with part (3), implies the existence of $g_0 > 0$ such that $g(x,x;G_n) \leq g_0$ uniformly in $x$ and $n$.

The general theorem is:

**Theorem 1.3.** Suppose that $(G_n)$ satisfies Assumption 1.2. Let $f_{n,c}(x,y) = \mathbf{P}_x[\tau(y) \leq cT_{\text{mix}}(G_n)]$, $\mathcal{F}_{n,c} = \sum_y f_{n,c}(x,y)\pi(y;G_n)$, and

$$F_{n,c} = \sum_{x,y} (f_{n,c}(x,y) - \mathcal{F}_{n,c})^2.$$
There exists $\gamma > 0$ so that for every $c \geq 2$ we have

$$\text{(1.5)} \quad \text{Var}(|A_i|) = \left( \frac{1}{4} + O(\Delta_n) \right) F_{n,c} + O(e^{-\gamma c}(T_{\text{mix}}(G_n))^2)$$

as $n \to \infty$ where $\Delta_n = (T_{\text{mix}}(G_n) \log |V_n|)/|V_n|$.

Applying this to the special cases of the hypercube and the Caley graph of $S_n$ generated by transpositions leads to the following corollary.

**Corollary 1.4.** Suppose that $G_n = (V_n, E_n)$ is either the hypercube $\mathbb{Z}_2^n$ or the Caley graph of $S_n$ generated by transpositions. Then

$$\text{Var}(|A_i|) = \frac{1}{4} (1 + o(1)) |V_n|.$$  

In particular, the first-order asymptotics of the variance are exactly the same as for an iid marking.

Throughout the remainder of the article, all graphs under consideration shall satisfy Assumption 1.2. In most examples, it will be that $T_{\text{mix}}^2(G_n) = o(F_{n,c})$ so that the second term in (1.5) is negligible. In this case, taking $c = 2$ in (1.5) provides a means to compute not only the magnitude but also the constant in the first order asymptotics of the variance. In some cases, such as $G_n = \mathbb{Z}_n^d$, the constant can even be computed when $F_{n,c} = \Theta((T_{\text{mix}}(G_n))^2)$.

The challenge in obtaining Theorems 1.1 and 1.3 is that the cancellation in the expansion of the variance is quite significant which, when ignored, yields only an upper bound that can be off by as much as a multiple of $T_{\text{mix}}(G_n)$. We will now illustrate this point in the case of $\mathbb{Z}_n^d$ for $d \geq 3$. It will turn out that the contribution to the variance from the sites visited by both $X_1, X_2$ simultaneously is negligible, hence we will ignore this possibility in the present discussion. Up to negligible error,

$$\text{Var} \left( \sum_x 1_{\{\tau_1(x) < \tau_2(x)\}} \right)$$

is equal to

$$2 \sum_{x,y,z} \left( P_{x,z}[\tau_1(y) < \tau_2(y)] - \frac{1}{2} \right) \tilde{\pi}(z;x,y)P[H(x,y)].$$

Here, $P_{x,z}$ denotes the joint law of $X_1, X_2$ with $X_1(0) = x$ and $X_2(0) = z$ and $\tilde{\pi}(.;x,y)$ is the law of $X_2(\tau_1(x))$ conditional on the event $H(x,y) = \{\tau_1(x) < \tau_1(y), \tau_2(x), \tau_2(y)\}$. The reason we do not have equality is that
\[ \mathbb{P}[H(x,y)] \neq \frac{1}{4} \] since it could be that \( X_1, X_2 \) hit either \( x \) or \( y \) at the same time. We will, however, argue in Lemma 4.3 that replacing \( \mathbb{P}[H(x,y)] \) by \( \frac{1}{4} \) introduces negligible error. Thus we need to estimate

\[ \frac{1}{2} \sum_{x,y,z} \left( \mathbb{P}_{x,z} \left[ \tau_1(y) < \tau_2(y) \right] - \frac{1}{2} \right) \tilde{\pi}(z; x, y). \tag{1.6} \]

At this point, one is tempted to insert absolute values and then work on each of the summands separately. An application of Bayes’ rule shows that there exists \( C_0 > 0 \) so that \( \tilde{\pi}(z; x, y) \leq C_0 \pi(z; Z_d^n) \) (see Theorem 4.1 for a much finer estimate), hence the expression in (1.6) is bounded from above by

\[ C_1 \sum_{x,y} \left| \mathbb{P}_{x,x} \left[ \tau_1(y) < \tau_2(y) \right] - \frac{1}{2} \right|, \tag{1.7} \]

where \( \mathbb{P}_{x,x} \) denotes the law of \( X_1, X_2 \) with \( X_1(0) = x \) and \( X_2(0) \sim \pi \). We can analyze \( \mathbb{P}_{x,x} \left[ \tau_1(y) < \tau_2(y) \right] \) as follows. First, the probability that \( X_2 \) hits \( y \) before \( t_c \equiv cn^2 \) is of order \( n^{2-d} \) by a union bound. Second, by the transience of random walk, the probability that \( X_1 \) hits \( y \) before \( t_c \) is, up to a multiplicative constant, well approximated by \( g(x, y; Z_d^n) \). By time \( t_c \), \( X_1 \) will have mixed, which means that if neither \( X_1 \) nor \( X_2 \) has hit \( y \) by this time, then the probability that either one hits first is close to 1/2. The careful reader who wishes to see precise, quantitative versions of these statements will find such in the lemmas we use to prove Theorem 1.3. Thus it is not difficult to see that there exists \( C_2 > 0 \) so that \( \left| \mathbb{P}_{x,x} \left[ \tau_1(y) < \tau_2(y) \right] - 1/2 \right| \leq C_2 g(x, y; Z_d^n) \), which leads to an upper bound of

\[ C_3 \sum_{x,y} g(x, y; Z_d^n) \leq C_4 n^{d+2}. \]

A slightly more refined analysis leads to a lower bound of (1.7) with the same growth rate. As we will show in the next section, in every dimension this estimate is typically quite far from being sharp. The reason that it is so poor is that by moving the absolute value into the sum in (1.7) we are unable to take advantage of the cancellation that arises as \( \mathbb{P}_{x,x} \left[ \tau_1(y) < \tau_2(y) \right] > 1/2 \) when \( x \) is close to \( y \) and \( \mathbb{P}_{x,x} \left[ \tau_1(y) < \tau_2(y) \right] < 1/2 \) when \( x \) is far from \( y \).

**Outline.** The remainder of this article is structured as follows. In the next section, we will deduce Theorem 1.1 and Corollary 1.4 from Theorem 1.3. In Section 3, we introduce some notation that will be used throughout in addition to collecting several basic random walk estimates. Next, in Section
4, we give a precise estimate of the Radon-Nikodym derivative of \( \tilde{\pi}(\cdot; x, y) \) with respect to \( \pi \). In Section 5, we prove Theorem 1.3 and end in Section 6 with a list of related problems and discussion.

### 2. Proof of of Theorem 1.1 and Corollary 1.4.

The following Proposition will be important for the proof of Theorem 1.1.

**Proposition 2.1.** For each \( c > 1 \), the limit

\[
\lim_{n \to \infty} \frac{1}{h_d(n)} \sum_{x,y} (f_{n,c}(x, y) - \tilde{f}_{n,c})^2
\]

exists. When \( d \geq 4 \), it is \( \alpha_d \) as in (1.2), (1.3). When \( d = 3 \), it is given by \( \alpha_3^c \) where \( \alpha_3^T \) is as in (1.4).

The first step in the proof of the proposition is to reduce the existence of the limit to a computation involving Green’s functions. Recall from (1.1) that \( G(y; Z^d) \) is the Green’s function for lazy random walk on \( Z^d \).

**Lemma 2.2.** For each \( c > 1 \), we have that

\[
\lim_{n \to \infty} \frac{1}{h_d(n)} \sum_{x,y} (f_{n,c}(x, y) - G^{-1}(0; Z^d)g_c(x, y; Z^d_n))^2 = 0.
\]

**Proof.** Let \( g_c(x, y; Z^d_n) = \mathbb{E}_x \sum_{t=0}^{cT_{\text{mix}}} 1_{\{X(t) = y\}} \) and observe

\[
g_c(x, y; Z^d_n) \leq \mathbb{P}_x [\tau(y) \leq cT_{\text{mix}}]g_c(y, y; Z^d_n).
\]

We shall now prove a matching lower bound. Fix \( 0 < \tilde{c} < c \). Then we have that

\[
g_c(x, y; Z^d_n) \geq \mathbb{E}_x \left[ \sum_{t=\tau(y)}^{cT_{\text{mix}}} 1_{\{X(t) = y\}} \mathbb{1}_{\{\tau(y) \leq (c-\tilde{c})T_{\text{mix}}\}} \right] \geq f_{n,c-\tilde{c}}(x, y)g_c(y, y; Z^d_n).
\]

Assuming \( c - \tilde{c} > 1 \), by mixing considerations (see Proposition 3.1) we have that

\[
f_{n,c-\tilde{c}}(x, y) = f_{n,c}(x, y) + O(\tilde{c} n^{-d}).
\]

We also have

\[
g_c(y, y; Z^d_n) = g_c(y, y; Z^d_n) - \sum_z \tilde{p}^{T_{\text{mix}}}(y, z; Z^d_n)g_{c-\tilde{c}}(z, y; Z^d_n) = g_c(y, y; Z^d_n) + O(n^{-d})
\]

Therefore, the limit exists and is equal to \( 0 \).
since $\tilde{c} > 0$. Combining (2.1), (2.2), and (2.3), we have thus proved the lower bound

$$g_c(x, y; Z^d_n) \geq f_{n,c}(x, y)g_c(y, y; Z^d_n) + O((\tilde{c} + g(x, y; Z^d_n))n^{2-d}).$$

Theorem 1.5.4 of [9] implies $g_c(x, y; Z^d_n) = \Theta(|x - y|^{2-d})$ (it is actually stated for walks on $\mathbb{Z}^d$ which are not lazy, but the generalization is straightforward).

Consequently,

$$\sum_y g^2_c(x, y; Z^d_n) = \begin{cases} 
\Theta(n) & \text{if } d = 3, \\
\Theta(\log n) & \text{if } d = 4, \\
\Theta(1) & \text{if } d \geq 5.
\end{cases}$$

Hence,

$$\sum_{x,y} (f_{n,c}(x, y)g_c(y, y; Z^d_n) - g_c(x, y; Z^d_n))^2 = \sum_{x,y} O((\tilde{c} + g(x, y; Z^d_n))n^{2-d})^2 = O((\tilde{c}^2 + o(1))n^4).$$

Dividing both sides by $h_d(n)$, taking a limsup as $n \to \infty$, then as $\tilde{c} \to 0$ yields

$$\lim_{n \to \infty} \frac{1}{h_d(n)} \sum_{x,y} (f_{n,c}(x, y)g_c(y, y; Z^d_n) - g_c(x, y; Z^d_n))^2 = 0.$$ 

By (2.3) we know that $|g_c(y, y; Z^d_n) - g_1(y, y; Z^d_n)| = o(1)$ and, by transience, it is not hard to see that $\lim_{n \to \infty} g_1(y, y; Z^d_n) = G(0; Z^d)$. □

**Proof of Proposition 2.1.** Letting $\overline{g}_{n,c} = cT_{mix}n^{-d}$, Lemma 2.2 implies that it suffices to prove the existence of the limit

$$(2.4) \quad \lim_{n \to \infty} \frac{1}{h_d(n)} \sum_{x,y} (g_c(x, y; Z^d_n) - \overline{g}_{n,c})^2.$$ 

We will divide the proof into the cases $d \geq 4$ and $d = 3$. 

**Case 1: $d \geq 4$.** As $\overline{g}_{n,c} = O(n^{2-d})$, we have

$$\frac{1}{h_d(n)} \sum_{x,y} (\overline{g}_{n,c}^2 + 2\overline{g}_{n,c}g(x, y; Z^d_n)) = o(1).$$
Thus it suffices to show in this case that \( \lim_{n \to \infty} \tilde{h}_d^{-1}(n) \sum_y g_c^2(0, y; Z^d_n) \) exists, where \( \tilde{h}_d(n) = \log n \) and \( \tilde{h}_d(n) = 1 \) for \( d \geq 5 \). This will be a consequence of two observations. First, as

\[
\sum_{|y| \geq \ell} g_c^2(0, y; Z^d_n) = \sum_{|y| \geq \ell} O(|y|^{4-2d}) = \sum_{m=\ell}^n O(m^{4-2d} \cdot m^{d-1})
\]

it suffices to show that, for each \( \epsilon > 0 \) and for \( \ell = \ell(n, \epsilon) = n^{1-\epsilon} \), the limit \( \lim_{t \to 0} \lim_{n \to \infty} \tilde{h}_d^{-1}(n) \sum_{|y| \leq \ell} g_c^2(0, y; Z^d_n) \) exists (we can even restrict to finite \( \ell \) if \( d \geq 5 \)). Our second observation is that \( g_c(0, y; Z^d_n) - G(y; Z^d) = O(n^{2-d}) \) for \( |y| \leq \ell \). This follows since we can couple the walks on \( Z^d_n \) and \( Z^d \) starting at 0 such that they are the same until the first time \( \tau_0 \) they have reached distance \( n/2 \) from 0, then move independently thereafter. The expected number of visits each walk makes to 0 after time \( \tau_0 \) is easily seen to be \( O(n^{2-d}) \), which proves our claim. Thus,

\[
\sum_{|y| \leq \ell} (g_c(0, y; Z^d_n) - G(y; Z^d))^2 = o(1).
\]

Therefore if \( d \geq 5 \) we have

\[
\lim_{n \to \infty} \frac{1}{h_d(n)} \sum_{x,y} g_c^2(x, y; Z^d_n) = \sum_{y \in Z^d} G^2(y; Z^d).
\]

For \( d = 4 \),

\[
\lim_{n \to \infty} \frac{1}{h_4(n)} \sum_{x,y} g_c^2(x, y; Z^d_n) = \lim_{n \to \infty} \frac{1}{\log n} \sum_{y \in Z^d, |y| \leq n} G^2(y; Z^d).
\]

Note that the limit on the right side exists since by Theorem 1.5.4 of [9] (generalized to lazy walks), \( G(y; Z^d) = a_d|y|^{2-d} + o(|y|^{-\alpha}) \) if \( \alpha \in (0, d) \) is fixed.

Case 2: \( d = 3 \). The thrust of the previous argument was that random walk on \( Z^d_n \) for \( d \geq 4 \) is sufficiently transient so that pairs of points of distance \( \Omega(n^{1-\epsilon}) \) make a negligible contribution to the variance, which in turn allowed us to make an accurate comparison between the Green’s function for random walk on \( Z^d_n \) with that on \( Z^d \). The situation for \( d = 3 \) is more delicate since
the opposite is true: pairs of distance $O(n^{1-\epsilon})$ do not measurably affect the variance.

Theorem 1.2.1 of [9] (extended to the case of lazy random walk, see also Corollary 22.3 of [1]) implies the existence of constants $\beta_3, \gamma_3 > 0$ such that with $p^t(x, y; Z^3) = p^t_{n^{\beta_3/2}} \exp \left( -\frac{2\|x-y\|^2}{t} \right)$, we have the estimate

$$|p^t(x, y; Z^3) - p^t(x, y; Z^3)| = |x - y|^{-2} O(t^{-3/2}).$$

Hence letting $p^t(x, y; Z^3_n) = \sum_{k \in \mathbb{Z}} p^t(x, y + kn; Z^3)$, one can easily show that with

$$\Delta(x, y) = \sum_{t=0}^{cT_{\text{mix}}} \left| p^t(x, y; Z^3_n) - p^t(x, y; Z^3) \right|$$

we have that

$$(2.5) \quad \frac{1}{h_3(n)} \sum_{x,y} \Delta^2(x, y) = o(1).$$

By differentiating $p$ in $t$, we see that for $1 \leq t \leq s \leq t + 1$, we have

$$|p^t(0, y; Z^3_n) - p^s(0, y; Z^3_n)| = O \left( \frac{1}{t} + \frac{|y|^2}{t^2} \right) p^t(0, y; Z^3_n).$$

We are now going to prove that

$$(2.6) \quad \sum_{y \in Z^3_n} \left( \int_1^{cT_{\text{mix}}} \left| p^t(0, y; Z^3_n) - p^{|t|}(0, y; Z^3_n) \right| dt \right)^2 = O(1).$$

It suffices to bound

$$A \equiv \sum_{y \in Z^3_n} \left( \int_1^{cT_{\text{mix}}} \frac{1}{t} p^t(0, y; Z^3_n) dt \right)^2,$$

$$B \equiv \sum_{y \in Z^3_n} \left( \int_1^{cT_{\text{mix}}} \frac{|y|^2}{t^2} p^t(0, y; Z^3_n) dt \right)^2.$$

For $A$, we apply Cauchy-Schwarz to the integral and invoke the integrability of $1/t^2$ over $[1, \infty)$ to arrive at

$$A \leq C_2 \sum_{y \in Z^3_n} \int_1^{cT_{\text{mix}}} |p^t(0, y; Z^3_n)|^2 dt = O(1).$$
For $B$, we insert the formula for $p$ into the integral, make the substitution $u = t/|y|^2$, and then compute to see

$$B \leq C_3 \sum_{y \in \mathbb{Z}_3} \frac{1}{|y|^6 + 1} = O(1).$$

This proves (2.6). Recall that $T^3$ is the three-dimensional continuum torus.

For $x, y \in T^3$, let

$$g_c(x, y; T^3) = \int_0^{cT_{\text{mix}}} p_t(nx, ny; Z_n^3) dt = \frac{1}{n} \int_0^{cT} p^u(x, y; T^3) du,$$

where $T = T_{\text{mix}}/n^2$. By (2.5), (2.6), we have that

$$\frac{1}{h_3(n)} \sum_{x, y \in \mathbb{Z}_n^3} (g_c(x, y; Z_n^3) - g_c(x/n, y/n; T^3))^2 = o(1).$$

Therefore we may replace $g_c(x, y; Z_n^3)$ in (2.4) with $g_c(x/n, y/n; T^3)$. Note that $g_c(\cdot, \cdot; T^3)$ is the product of $n^{-1}$ and the Green’s function for $B_{1/2}$, where $B_t$ is a Brownian motion on $T^3$; roughly, the reason that the Brownian motion moves at $1/2$-speed is that lazy random walk moves at $1/2$ the speed of simple random walk. It is left to bound

$$n^2 \int_{T^3} \int_{T^3} (g_c([nx]/n, [ny]/n; T^3) - g_c(x, y; T^3))^2 \, dx \, dy;$$

the reason for the pre-factor $n^2$ is that we need to multiply by $(n^3)^2$ in order to make the double integral comparable to the double summation and we also divide by the normalization $h_3(n)$. From (2.7), we see that $g_c(x, y; T^3)$ is $O(n^{-1})$-Lipschitz away from the diagonal $D_\epsilon = \{(x, y) \in T^3 \times T^3 : |x - y| \leq \epsilon\}$. Thus since $|(x, y) - ([nx]/n, [ny]/n)| = O(n^{-1})$, the integrand is $O(n^{-4})$ on $D_\epsilon^c$, hence the integral over $D_\epsilon^c$ is $O(n^{-2})$. Since both $ng_c([nx]/n, [ny]/n; T^3)$ and $ng_c(x, y; T^3)$ are uniformly $L^2$-integrable over $T^3 \times T^3$, it follows that the contribution coming from $D_\epsilon$ can be made uniformly small in $n$ by first fixing $\epsilon > 0$ small enough.

We now deduce Theorem 1.1 from Theorem 1.3.

**Proof of Theorem 1.1.** Suppose $G_n = \mathbb{Z}_n^d$ for $d \geq 3$. Then $T_{\text{mix}}(\mathbb{Z}_n^d) = \Theta(n^2)$ (see [10]) and there exists $c_d > 0$ so that $g(x, y; \mathbb{Z}_n^d) \leq c_d|x - y|^{2-d} \wedge 1$ (see [9]). Consequently, the hypotheses of Theorem 1.3 are obviously satisfied, except for possibly (3). This is easy to see if $x$ is sufficiently far from
y, z so that \( g(x, y; \mathbb{Z}_n^d) + g(x, z; \mathbb{Z}_n^d) \leq 1/2 \). If x is close to y, z then we can combine the Green’s function estimate with the trivial bound that \( X \) starting at x with distance \( r \) to y, z will get to distance \( r + s \) without hitting y, z with probability at least \( (4d)^{-s} \).

Proposition 2.1 implies that \( F_{n,c} \sim \frac{1}{4} \alpha_{d,c} h_d(n) \) as \( n \to \infty \). This is enough to dominate \( T_{\text{mix}}^2(\mathbb{Z}_n^d) = \Theta(n^4) \) except if \( d = 3 \). We shall now argue that nevertheless, \( F_{n,c} \) is still the dominant term in this case. Note that

\[
\mathcal{F}_{n,c} \leq \frac{1}{n^3} \sum_y c g(x, y; \mathbb{Z}_n^3) \leq A_0 cn^{-1}
\]

for some \( A_0 > 0 \) and \( c \geq 2 \) fixed. Also, the transience of random walk on \( \mathbb{Z}_n^3 \) implies that there exists \( A_1 > 0 \) so that \( f_{n,c}(x, y) \geq A_1 |x - y|^{-1} \wedge 1 \). Thus for

\[
|x - y| \leq \left( \frac{A_1}{2A_0c} \right) n \equiv A_2 n
\]

we have that \( f_{n,c}(x, y) - \mathcal{F}_{n,c} \geq \frac{A_1}{2} |x - y|^{-1} \wedge 1 \). Consequently,

\[
F_{n,c} \geq \frac{A_1^2}{4} \sum_{|x - y| \leq A_2 n} |x - y|^{-2} \wedge 1 = c^{-1} \Theta(n^4).
\]

A matching upper bound, up to a multiplicative factor, is also not difficult to see.

Our lower bound for \( F_{n,c} \) depends on \( c \) by a multiplicative factor of \( 1/c \) while the second term in (1.5) decays exponentially in \( c \). Thus by taking \( c \geq 2 \) large enough we see that \( F_{n,c} \) is still dominant for \( d = 3 \).

We now turn to the proof of Corollary 1.4.

**Proof of Corollary 1.4 for the Hypercube.** For \( \mathbb{Z}_2^n \), it is easier to work with the continuous time random walk (CTRW) since the types of estimates we require easily translate over to the corresponding lazy walk.

The transition kernel of the CTRW is

\[
p^t(x, y; \mathbb{Z}_2^n) = \frac{1}{2^n} (1 + e^{-2t/n})^{n-|x-y|}(1 - e^{-2t/n})^{|x-y|},
\]

where \(|x - y|\) is the number of coordinates in which \( x \) and \( y \) differ. The spectral gap is \( 1/n \) (see Example 12.15 of [10]) which implies \( \Omega(n) = T_{\text{mix}}(\mathbb{Z}_2^n) = O(n^2) \) (see Theorem 12.3 of [10]). Consequently, the first hypothesis of Theorem 1.3 holds. If \(|x - y| = r\), then it is easy to see there exists \( C_c, \rho_c > 0 \)
so that
\[ p^t(x, y; Z_2^n) \leq \begin{cases} 
(Ct/n)^r \exp[-(t/C\epsilon)(n - r)] & \text{if } t \leq \epsilon n, \\
e^{-\rho n} & \text{if } t > \epsilon n,
\end{cases} \]
provided \( \epsilon > 0 \) is sufficiently small. Thus it is not difficult to see that 
\[ g(x, y; Z_2^n) \leq C'n^{-r}. \]
Trivially, 
\[ |\{ y \in Z_2^n : |x - y| = r \}| = \binom{n}{r} \leq n^r. \]
Thus
\[ \sum_{y \neq x} g^2(x, y; Z_2^n) \leq O\left( \sum_{r=1}^{n} n^{-2r} \cdot n^r \right) = O\left( \frac{1}{n} \right), \]
so the second hypothesis of Theorem 1.3 is satisfied. The final hypothesis is obviously also satisfied. Now, a union bound implies that 
\[ f_{n,c}(x, x) - f_{n,c}(x, y) = o(2^n). \]
On the other hand,
\[ \sum_{|x - y| \geq 1} f_{n,c}(x, y) = O\left( \sum_{r=1}^{n} n^{-2r} \cdot n^r \right) = o(2^n). \]
Putting everything together, Theorem 1.3 implies
\[ \text{Var}(|\mathcal{A}_i|) = \frac{1}{4} (1 + o(1)) 2^n. \]

**Proof of Corollary 1.4 for the Caley graph of \( S_n \).** Let \( G_n \) be the Caley graph of \( S_n \) generated by transpositions. By work of Diaconis and Shashahani [4], the total variation mixing time of \( S_n \) is \( \Theta(n \log n) \), which by Theorem 12.3 of [10] implies 
\[ T_{mix}(G_n) = O(n \log n) \] 
implies 
\[ T_{mix}(G_n) = O(n \log n)! = O(n^2 \log n)^2. \]
We are now going to give a crude estimate of \( p^t(\sigma, \tau; S_n) \). By applying an automorphism, we may assume without loss of generality that \( \sigma = \text{id} \). Suppose that \( d(\text{id}, \tau) = r \) and that \( \tau_1, \ldots, \tau_r \) are transpositions such that \( \tau_r \cdots \tau_1 = \tau \). Then \( \tau_1, \ldots, \tau_r \) move at most \( 2r \) of the \( n \) elements of \{1, \ldots, n\}, say, \( k_1, \ldots, k_{2r} \). Suppose \( k'_1, \ldots, k'_{2r} \) are distinct from \( k_1, \ldots, k_{2r} \) and \( \alpha \in S_n \) is such that \( \alpha(k_i) = k'_i \) for \( 1 \leq i \leq r \). Then the automorphism of \( G_n \) induced by conjugation by \( \alpha \) satisfies \( \alpha \tau \alpha^{-1} \neq \tau \). Therefore the size of the set of elements \( \tau' \) in \( S_n \) such that there exists a graph automorphism \( \varphi \) of
$G_n$ satisfying $\varphi(\tau) = \tau'$ and $\varphi(\text{id}) = \text{id}$ is at least $\left( \frac{n}{2r} \right) \geq 2^{-2r} n^{2r} (2r)!^{-1}$ assuming $n \geq 4r$. Therefore

\begin{equation}
(2.8) \quad p^f(e, \tau; G_n) \leq \frac{2^{2r}(2r)!}{n^{2r}} \quad \text{and} \quad g(e, \tau; G_n) \leq C(2^{2r}(2r)!)(\log n)^2 n^{2-2r}.
\end{equation}

This bound is good enough for $r \geq 2$ but does not quite suffice when $r = 1$. This case is not difficult to handle, however, since it is easy to see that the random walk has distance 3 from $e$ with probability $1 - O(1/n)$ after its first three moves, hence with distance at least 2 from any permutation with distance 1 from $e$. Combining this with (2.8) gives the desired bound. From this is it clear that $(G_n)$ satisfies the hypotheses of Theorem 1.3 and, arguing as in the case of the hypercube, that

$$\text{Var}(|A_i|) = \frac{1}{4}(1 + o(1))n!.$$

\[\square\]

3. Preliminaries.

3.1. Notation. For $x, y \in V_n$, let $\tau_i(x, y) = \tau_i(x) \wedge \tau_i(y)$ and $\tau(x, y) = \tau_1(x, y) \wedge \tau_2(x, y)$. Let $H(x, y) = \{\tau_1(x) < \tau_1(y), \tau_2(x, y)\}$,

$$\tilde{\pi}(z; x, y) = \mathbf{P}[X_2(\tau_1(x, y)) = z|H(x, y)],$$

and let $\pi$ be the uniform measure on $V_n$. Throughout, $\mathbf{P}_z[\cdot]$ denotes the law of random walk initialized at $z$ (and the initial distribution is stationary whenever $z$ is omitted). We will often not emphasize the dependence of various quantities on $G_n$ since it will be clear from the context. Let

$$\Gamma_n = c_0 T_{\text{mix}} \log |V_n|, \quad \Upsilon_n = \frac{T_{\text{mix}}}{|V_n|}, \quad \Delta_n = \Upsilon_n \log |V_n|, \quad S_n = \sum_{x \neq y} g^2(x, y)$$

where $c_0$ will be determined later. The proofs in this article will involve probabilities of complicated events. To keep the formulas succinct, it will be helpful for us to introduce the following notation: let

$$G_{ij}(x) = \{\tau_i(x) < \tau_j(x)\},$$

$$G_{ij}(x, y) = \{\tau_i(x, y) < \tau_j(x, y)\},$$

and

$$G_i(x, y) = \{\tau_i(x) < \tau_i(y)\}.$$
3.2. Elementary Estimates. Recall that the total variation distance of measures $\mu, \nu$ on $V$ is
\[ \|\mu - \nu\|_{TV} = \max_{A \subseteq V} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in V} |\mu(\{x\}) - \nu(\{x\})|. \]

The following provides a bound on the rate of decay of the distance of $p^t(x, \cdot)$ to stationarity.

**Proposition 3.1.** For every $s, t \in \mathbb{N}$,
\[
\begin{align*}
\max_x \|p^{t+s}(x, \cdot) - \pi\|_{TV} &\leq 4 \max_{x,y} \|p^t(x, \cdot) - \pi\|_{TV} \|p^s(y, \cdot) - \pi\|_{TV} \tag{3.1} \\
\max_{x,y} \left| \frac{p^{t+s}(x,y)}{\pi(y)} - 1 \right| &\leq \max_{x,y} \frac{p^s(x,y)}{\pi(y)} \max_x \|p^t(x, \cdot) - \pi\|_{TV}. \tag{3.2}
\end{align*}
\]

**Proof.** The first part is a standard result; see, for example, Lemmas 4.11 and 4.12 of [10]. The second part is a consequence of the semigroup property:
\[
\begin{align*}
\frac{1}{\pi(z)}p^{t+s}(x,z) &= \frac{1}{\pi(z)} \sum_y p^t(x,y)p^s(y,z) \\
&= \frac{1}{\pi(z)} \sum_y [p^t(x,y) - \pi(y) + \pi(y)]p^s(y,z) \\
&\leq \left( \max_{y,z} \frac{p^s(y,z)}{\pi(z)} \right) \|p^t(x, \cdot) - \pi\|_{TV} + 1.
\end{align*}
\]

Note that (3.1) and (3.2) give
\[
\max_{x,y} \left| \frac{p^t(x,y)}{\pi(y)} - 1 \right| \leq \gamma e^{-\gamma \alpha} \text{ for } t \geq \alpha T_{\text{mix}} \tag{3.3}
\]
where $\gamma > 0$ is a universal constant. We will often use (3.3) without reference.

Throughout the article, it will be important for us to have precise estimates of the Radon-Nikodym derivative of the law of random walk conditioned on various events with respect to the uniform measure. In the following, we are interested in the case of a random walk conditioned not to have hit a particular point. Let $T_k = kT_{\text{mix}}$.

**Lemma 3.2.** There exists $\gamma, p_0 > 0$ so that for all $k \geq 1$ satisfying $k\Upsilon_n \leq p_0$ and $c \geq 2$ we have
\[
P_x[X(cT_k) = z|\tau(y) \geq cT_k] = [1 + O(e^{-\gamma ck} + k\Upsilon_n + g(y,z))]\pi(z).
\]
Note that by part (1) of Assumption 1.2, this lemma applies if \( k = O((\log \|V_n\|)^2) \).

**Proof.** Using that \( P_x[X(cT_k) = z] = (1 + O(e^{-\gamma c}))\pi(z) \), an application of Bayes’ formula yields

\[
P_x[X(cT_k) = z | \tau(y) \geq cT_k] = \frac{\sum_{\{y \in [0,cT_k]\}} P_x[\tau(y) \geq cT_k] X(cT_k) = z] \pi(y)}{\sum_{\{y \in [0,cT_k]\}} P_x[\tau(y) \geq cT_k] X(cT_k) = z] (1 + O(e^{-\gamma c}))\pi(z)}.
\]

The idea of the rest of the proof is to show that it is unlikely that \( X \) hits \( y \) close to time \( cT_k \), in which case we can use a mixing argument to show that conditioning on \( X(cT_k) = z \) has little effect. For \( 1 \leq \tilde{c} \leq \tilde{c} + 1 \leq c \), we have

\[
P_x[\tau(y) \geq \tilde{c}T_k | X(cT_k) = z] = P_x[\tau(y) \geq \tilde{c}T_k | X(cT_k) = z] - P_x[\tau(y) \geq \tilde{c}T_k | \tau(y) \geq \tilde{c}T_k | X(cT_k) = z].
\]

By time-reversal, a union bound, and mixing considerations, it is easy to see that the second term on the right hand side is bounded by

\[
P_x[\tau(y) \leq \tilde{c}T_k | X(cT_k) = x] = O(g(y,z) + kY_n).
\]

Applying Bayes’ formula, observe

\[
P_x[\tau(y) \geq \tilde{c}T_k | X(cT_k) = z] = \frac{P_x[X(cT_k) = z | \tau(y) \geq \tilde{c}T_k] P_x[\tau(y) \geq \tilde{c}T_k]}{\sum_{\{y \in [0,cT_k]\}} P_x[\tau(y) \geq \tilde{c}T_k] X(cT_k) = z] (1 + O(e^{-\gamma c}))\pi(z)}.$
\]

Similarly,

\[
P_x[\tau(y) \geq cT_k] = P_x[\tau(y) \geq \tilde{c}T_k] - P_x[\tau(y) \geq \tilde{c}T_k | \tau(y) \geq \tilde{c}T_k],
\]

where the second term on the right hand side is of order \( O(kY_n) \). We now take \( \tilde{c} = c/2 \) and \( \gamma = \tilde{\gamma}/2 \). By part (3) of Assumption 1.2, we have that

\[
P_x[\tau(y) \geq kT_{\text{mix}}] \geq 1 - \rho_0 - O(kY_n)
\]

uniformly in \( n \). In particular, there exists \( \rho_0 > 0 \) so that if \( kY_n \leq \rho_0 \) then \( P_x[\tau(y) \geq kT_{\text{mix}}] + O(kY_n) \) is uniformly positive in \( n \). Putting everything together, for such \( k \) we thus have

\[
P_x[X(cT_k) = z | \tau(y) \geq cT_k] = \frac{(1 + O(e^{-\gamma c}))P_x[\tau(y) \geq \tilde{c}T_k] + O(g(y,z) + kY_n)}{(1 + O(e^{-\gamma c}))\pi(z)}(1 + O(e^{-\gamma c}))\pi(z)
\]

as desired. \( \square \)
In the following lemma, we will show that the difference in the probability that a random walk hits points \( y, z \) when started from \( x \) before time \( \Gamma_n = c_0 (\log n) T_{\text{mix}} \) is essentially determined by the corresponding difference except up to time \( c T_{\text{mix}} \). The reason for the cancellation is that the previous lemma implies that conditional on not hitting a given point up to time \( c T_{\text{mix}} \), the walk is well mixed and has long forgotten its starting point. Recall that \( f_c(x, y) = \mathbb{P}_x[\tau(y) \leq c T_{\text{mix}}] \).

**Lemma 3.3.** There exists \( \gamma > 0 \) such that for all \( c \geq 2 \),

\[
\mathbb{P}_x[\tau(y) \leq \Gamma_n] - \mathbb{P}_x[\tau(z) \leq \Gamma_n] = f_c(x, y) - f_c(x, z) + O(e^{-\gamma c} \mathbb{Y}_n + \Delta_n[g(x, y) + g(x, z)]).
\]

**Proof.** We observe,

\[
\mathbb{P}_x[\tau(y) \leq \Gamma_n] = f_c(x, y) + \sum_k \mathbb{P}_x[c T_k \leq \tau(y) < c T_{k+1} | \tau(y) \geq c T_k](1 - \mathbb{P}_x[\tau(y) < c T_k]),
\]

where here and throughout the rest of this proof the summation over \( k \) is from 1 to \( c_0 c \log |V_n| \). As the previous lemma is applicable for such choices of \( k \), we thus have

\[
\mathbb{P}_x[c T_k \leq \tau(y) < c T_{k+1} | \tau(y) \geq c T_k] = \mathbb{P}_\pi[c T_k \leq \tau(y) < c T_{k+1} | \tau(y) \geq c T_k] + O \left( \sum_w g(y, w)(e^{-\gamma c k} + k \mathbb{Y}_n + g(y, w)) \pi(w) \right).
\]

Performing the summation, we see that the latter term is of order

\[
O(\mathbb{Y}_n e^{-\gamma c} + k \mathbb{Y}_n^2 + \mathcal{S}_n |V_n|^{-1}).
\]

Recall from part (2) of Assumption 1.2 that \( \mathcal{S}_n = o(T_{\text{mix}} / \log |V_n|) \), hence \( (\log |V_n|) \mathcal{S}_n |V_n|^{-1} = o(\mathbb{Y}_n) \). Consequently, summing (3.4) over \( k \) from 1 to \( c_0 \log |V_n| \) gives an error of \( O(\mathbb{Y}_n e^{-\gamma c} + \Delta_n^2) \). By part (1) of Assumption 1.2 it is clear that \( \Delta_n^2 = o(\mathbb{Y}_n) \), hence the former is of order \( O(\mathbb{Y}_n e^{-\gamma c}) \). This leaves

\[
\sum_k \mathbb{P}_x[c T_k \leq \tau(y) < c T_{k+1} | \tau(y) \geq c T_k] \mathbb{P}_x[\tau(y) < c T_k] = O \left( \sum_k \sum_z \mathbb{P}_z[\tau(y) < c T_{\text{mix}}] \pi(z)(g(x, y) + k \mathbb{Y}_n) \right).
\]
Here, we used the bound

\[ P_x[\tau(y) < cT_k] = O(g(x, y) + k\Upsilon_n) \]

along with the previous lemma to get the crude estimate

\[ P_x[X(cT_k) = z|\tau(y) \geq cT_k] \leq C\pi(z) \]

for some \( C > 0 \). Summing everything up gives us an error of order

\[ O(\Delta_n g(x, z) + \Delta_n^2) \]

coming from the corresponding estimate of \( P_x[\tau(z) \leq \Gamma_n] \). Therefore our total error is \( O(\Upsilon_n e^{-\gamma c} + \Delta_n[g(x, y) + g(x, z)]) \), which proves the lemma.

4. The Radon-Nikodym Derivative. Let \( \tilde{\pi}(z; x, y) = P[X_2(\tau_1(x, y)) = z|H(x, y)] \). The purpose of this section is to prove the following estimate of the Radon-Nikodym derivative of \( \tilde{\pi}(z; x, y) \) with respect to \( \pi(z) \). Recall \( f_c(x, y) = P_x[\tau(y) < cT_{\text{mix}}] \) and \( \tilde{f}_c = \sum_y f_c(x, y)\pi(y) \); we are omitting the dependence on \( n \) to keep the notation simple.

**Theorem 4.1.** There exists a constant \( \gamma > 0 \) so that for all \( c \geq 2 \) we have

\[
\frac{\tilde{\pi}(z; x, y)}{\pi(z)} = 1 + \left( 1 + O(\Delta_n) \right) \left( 2\tilde{f}_c - f_c(x, z) - f_c(y, z) \right) + O(e^{-\gamma c\Upsilon_n}) + O(\Delta_n[g(x, z) + g(y, z) + \Gamma_n[g(x, y) + \Delta_n])
\]

In particular,

\[
\frac{\tilde{\pi}(z; x, y)}{\pi(z)} = 1 + O(g(x, y) + g(y, z) + g(x, z)).
\]

(4.1)

Let \( Y_2 = X_2(\tau_1(x, y)) \). The idea of the proof is to observe that

\[
\tilde{\pi}(z; x, y) = P[Y_2 = z|H(x, y)] = \frac{P[H(x, y)|Y_2 = z]\pi(z)}{P[H(x, y)]}
\]

since \( X_1, X_2 \) are independent and the initial distribution of \( X_2 \) is stationary, then estimate the effect of conditioning on \( Y_2 = z \) on the probability of \( H(x, y) \). We will divide the proof into three lemmas. Let

\[ A(x, y) = \{ \tau_2(x, y) > \tau_1(x, y) - \Gamma_n, G_1(x, y) \} \setminus H(x, y) \]

\[ = \{ \tau_1(x, y) \geq \tau_2(x, y) > \tau_1(x, y) - \Gamma_n, G_1(x, y) \}. \]

**Lemma 4.2.** Uniformly in \( x, y, z, n \),

\[
\frac{\tilde{\pi}(z; x, y)}{\pi(z)} = 1 + \frac{P[A(x, y)] - P[A(x, y)|Y_2 = z]}{P[H(x, y)]} + O(|V_n|^{-100}).
\]

(4.2)
In Theorem 4.1, we give a precise estimate of the Radon-Nikodym derivative of the law of $Y_2 = X_2(\tau_1(x,y))$ with respect to the uniform measure on $V_n$ conditional on the event that $H(x,y) = \{\tau_1(x,y) \leq \tau_2(x,y)\}$, i.e. that the first point in $\{x, y\}$ hit by $X_1, X_2$ is $x$ by $X_1$. The open circles indicate the starting points of $X_1, X_2$ and the shaded circle is $Y_2$.

**Proof.** Letting $R(x,y) = \{\tau_2(x,y) > \tau_1(x,y) - \Gamma_n, G_1(x,y)\}$, observe

$$P[H(x,y)|Y_2 = z] = P[R(x,y)|Y_2 = z] - P[A(x,y)|Y_2 = z].$$

We will now manipulate the first term on the right hand side. Let $\tilde{R}(x,y) = G_1(x,y) \setminus R(x,y)$. We have,

$$(4.3) \quad P[R(x,y)|Y_2 = z] = P[G_1(x,y)|Y_2 = z] - P[\tilde{R}(x,y)|Y_2 = z]$$

and, since $Y_2 \sim \pi$, Bayes’ rule implies

$$P[\tilde{R}(x,y)|Y_2 = z] = \frac{1}{\pi(z)} P[Y_2 = z]\tilde{R}(x,y)P[\tilde{R}(x,y)].$$

Since the conditional probability on the right hand side involves conditioning on the behavior of $X_2$ before $\tau_1(x,y) - \Gamma_n$, mixing considerations imply that this is equal to

$$(4.4) \quad [1 + O(|V_n|^{-\gamma_0})]P[\tilde{R}(x,y)] = P[\tilde{R}(x,y)] + O(|V_n|^{-\gamma_0}).$$

As $P[G_1(x,y)|Y_2 = z] = P[G_1(x,y)]$, combining (4.3) with (4.4) we thus have

$$P[H(x,y)|Y_2 = z] = P[R(x,y)] - P[A(x,y)|Y_2 = z] + O(|V_n|^{-\gamma_0})$$

$$= P[H(x,y)] + P[A(x,y)] - P[A(x,y)|Y_2 = z] + O(|V_n|^{-\gamma_0}).$$
Assume that \( \gamma c_0 > 100 \). Putting everything together, we see that

\[
\frac{\pi(z; x, y)}{\pi(z)} = \frac{P[H(x, y)] + P[A(x, y)] - P[A(x, y)|Y_2 = z]}{P[H(x, y)]} + O(|V_n|^{-100}),
\]

uniformly in \( x, y, z, n \).

Note that if \( V_n = \mathbb{Z}_n^d \) for \( d \geq 3 \) then \( P[G_1(x, y), G_{12}(x, y)] = P[H(x, y)] \)
does not change when \( x \) is swapped with \( y \) nor when \( 1 \) is swapped with \( 2 \) and, moreover, is equal to \( \frac{1}{4} \) up to negligible error. This holds more generally if for every \( x, y \in V_n \) distinct there exists an automorphism \( \varphi \) of \( G_n \) such that \( \varphi(x) = y \) and \( \varphi(y) = x \). The weaker hypothesis of vertex transitivity implies that we can always find an automorphism \( \varphi \) of \( G_n \) such that \( \varphi(x) = y \) but not necessarily so that \( \varphi(y) = x \) as well. Nevertheless, it is still true in this case that \( P[H(x, y)] \approx \frac{1}{4} \).

**Lemma 4.3.** We have that

\[
P[H(x, y)] = \frac{1}{4} + o \left( \frac{\Gamma_n}{\log |V_n|} \right) + O \left( \frac{1}{|V_n|} \right).
\]

**Proof.** Let \( \tilde{A}(x, y) = \{ \tau_1(x, y) \geq \tau_2(x, y) \geq \tau_1(x, y) - \Gamma_N \} \) and \( \mu(z; x, y) = P[Y_2 = z|\tau_1(x, y) \leq \tau_2(x, y)] \). Using exactly the same proof as the previous lemma, we have

\[
\frac{\mu(z; x, y)}{\pi(z)} = 1 + O(P[\tilde{A}(x, y)] + P[\tilde{A}(x, y)|Y_2 = z] + |V_n|^{-100}).
\]
Consequently, using the trivial bound
\[ P[\tilde{A}(x,y)|Y_2 = z] = O(\Delta_n + g(x,z) + g(y,z)) \]
(and similarly for \( P[\tilde{A}(x,y)] \)), we have that
\[ \sum_z g(x,z)\mu(z;x,y) = \Upsilon_n + \frac{1}{|V_n|} \sum_z g(x,z)O(\Delta_n + g(x,z) + g(y,z)) = \Upsilon_n + \frac{1}{|V_n|}O(S_n + \Delta_n T_{\text{mix}}). \]

By parts (1) and (2) of Assumption 1.2, we have that \( S_n + \Delta_n T_{\text{mix}} = o(T_{\text{mix}}/(\log |V_n|)) \). Consequently, the above is equal to
\[ \Upsilon_n + o\left(\frac{\Upsilon_n}{\log |V_n|}\right). \]

Let \( p_x = P[G_1(x,y), G_{12}(x,y)] \) and \( p_y = P[G_1(y,x), G_{12}(x,y)] \). Note that
\[ p_x + p_y = P[G_{12}(x,y)] = \frac{1}{2} + O\left(\frac{1}{|V_n|}\right) \]
since \( P[\tau_1(z) = \tau_2(w)] \leq P[X_2(\tau_1(z)) = w] = |V_n|^{-1} \) for any \( z, w \in V_n \).

Define stopping times as follows. Let 
\[ \tau_1 = \min\{t \geq 1 : X_1(t) \in \{x,y\} \text{ or } X_2(t) \in \{x,y\}\} = \tau_1(x,y). \]

For \( j \geq 1 \), inductively set 
\[ \tau_{j+1} = \min\{t \geq \tau_j + T_{\text{mix}} + 1 : X_1(t) \in \{x,y\} \text{ or } X_2(t) \in \{x,y\}\}. \]

Let \( T_{j,z} = \sum_{t=\tau_j}^{\tau_{j+1}} 1_{\{X(t) = z\}} \) and, for \( E \subseteq V_n \), set \( A_{ij}(E) = \{X_i(\tau_j) \in E\} \).

Note that the average amount of time spent at \( x \) by \( X_1 \) through time \( \tau_k + T_{\text{mix}} \) is given by the expression
\[ \frac{1}{\tau_k + T_{\text{mix}}} \sum_{j=1}^{k} \left( 1_{A_{1j}(x)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{1j}(y)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{2j}(x,y)} T_{j,x} \right). \]

It is not difficult to see that the above quantity converges to \( \pi(x) \) as \( k \to \infty \).

We can also define a similar quantity but replacing \( x \) with \( y \) which will converge to \( \pi(y) \) as \( k \to \infty \). Taking the ratio of these two quantities, we arrive at
\[ 1 = \lim_{k \to \infty} \frac{\frac{1}{k} \sum_{j=1}^{k} \left( 1_{A_{1j}(x)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{1j}(y)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{2j}(x,y)} T_{j,x} \right)}{\frac{1}{k} \sum_{j=1}^{k} \left( 1_{A_{1j}(x)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{1j}(y)} 1_{A_{2j}(x,y)} T_{j,x} + 1_{A_{2j}(x,y)} T_{j,x} \right)} \]
since $\pi(x) = \pi(y)$. It is not difficult to see that, almost surely,

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} 1_{A_{1j}(x)} 1_{A_{2j}(x,y)} T_{j,x} = p_x g(x,x),
$$

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} 1_{A_{1j}(y)} 1_{A_{2j}(x,y)} T_{j,x} = p_y g(y,x),
$$

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} 1_{A_{2j}(x,y)} T_{j,x} = q_{xy} \sum_{z} g(z,x) \mu(z;x,y),
$$

where $q_{xy} = 1 - p_x - p_y$. Analogous formulae hold for the terms in the denominator. Combining this with (4.5), we thus have

$$
1 = p_x g(x,x) + p_y g(y,x) + q_{xy} \sum_{z} g(z,x) \mu(z;x,y)
$$

$$
+ \frac{p_y g(y,y) + p_x g(x,y) + q_{xy} \Upsilon_n}{p_y g(y,y) + p_x g(x,y) + q_{xy} \Upsilon_n + o \left( \frac{\Upsilon_n}{\log |V_n|} \right)}.
$$

Rearranging and using that $g(x,y) = g(y,x)$ and $g(x,x) = g(y,y)$, this implies that

$$
p_x = p_y + o \left( \frac{\Upsilon_n}{\log |V_n|} \right).
$$

Combining this with (4.6) proves the lemma. \hfill \Box

In order to complete the proof of Theorem 4.1 we need to estimate $P[A(x,y)|Y_2 = z]$, which is this purpose of the following lemma. Though the proof will be computationally intensive, the basic idea is fairly simple. The main goal is to eliminate the conditioning on $Y_2 = z$. The first step is to perform a time reversal, which converts the terminal condition to an initial condition at the cost of making the event whose probability we are to compute a bit more complicated. The latter is easily mitigated, however, since the event can be greatly simplified at the cost of negligible error.

**Lemma 4.4.** There exists $\gamma > 0$ so that for all $c \geq 2$ we have

$$
P[A(x,y)|Y_2 = z] = \left( \frac{1}{4} + O(\Delta_n) \right) \left[ f_c(x,z) + f_c(y,z) \right] + E_c(x,y) +
$$

$$
O(e^{-\gamma c \Upsilon_n} + [g(x,z) + g(y,z) + \Delta_n][g(x,y) + \Delta_n]),
$$

where $E_c(x,y)$ is some constant which does not depend on $z$. 
Note that the lemma implies
\[ P[A(x, y)] - P[A(x, y)|Y_2 = z] = O(g(x, y) + g(y, z) + g(x, z) + e^{-γc}Υ_n). \]

**Proof.** Let
\[ B(x, y) = \{X_2(t) \notin \{x, y\} \text{ for all } t \in (Γ_n, τ_1(x, y)], G_1(x, y) \} \]
and let \( P_{π, z} \) be the law of \((X_1, X_2)\) where \( X_1(0) \sim π \) and \( X_2(0) = z \). We compute,
\[ P[A(x, y)|Y_2 = z] = \frac{1}{π(z)} P[A(x, y), Y_2 = z] = \sum_w P_{π, w}[τ_1(x, y) ≥ τ_2(x, y) > τ_1(x, y) - Γ_n, G_1(x, y), Y_2 = z] \]

By reversing the time of \( X_2 \), we see that this is equal to
\[ \sum_w P_{π, z}[τ_2(x, y) ≤ Γ_n ∧ τ_1(x, y), B(x, y), Y_2 = w] \]
(4.7) \[ = P_{π, z}[τ_2(x, y) ≤ Γ_n ∧ τ_1(x, y), B(x, y)]. \]

We will now work towards simplifying the event, at the cost of some error. We begin by eliminating the “minimum” operation using the observation that it is unlikely that both \( X_1, X_2 \) hit quickly. Indeed, as \( P_{π, z}[τ_1(x, y) ≤ Γ_n, τ_2(x, y) ≤ Γ_n] = O([g(x, z) + g(y, z) + Δ_n]Δ_n) \)
we see by setting \( B(x, y) = B(x, y) \cap \{τ_1(x, y) > Γ_n\} \) that (4.7) is equal to
\[ P_{π, z}[τ_2(x, y) ≤ Γ_n, B(x, y)] + O([g(x, z) + g(y, z) + Δ_n]Δ_n). \]

We would now like to eliminate the dependence of the probability on \( z \), the starting point of \( X_2 \). We accomplish this by considering two possible cases. Either \( X_2 \) hits \( x \) or \( y \) within some multiple of the mixing time or it does not and, conditional on the latter, the walk will have mixed, so the relevant probability does not depend on \( z \). We implement this strategy as follows:
\[ P_{π, z}[τ_2(x, y) ≤ Γ_n, B(x, y)] = P_{π, z}[τ_2(x, y) < cT_{mix}, B(x, y)] + P_{π, z}[τ_2(x, y) ≤ Γ_n, B(x, y)|τ_2(x, y) ≥ cT_{mix}] \]

Using the same proof as Lemma 3.2 except in the case that random walk is conditioned not to hit two points rather than just one implies \( \mu(w; x, y, z) = \)
\[ P_z[X_2(cT_{\text{mix}}) = w|\tau_2(x, y) \geq cT_{\text{mix}}] \leq C\pi(w) \text{ for some constant } C > 0. \]

Consequently,

\[ P_{\pi,z}[\tau_2(x, y) \leq \Gamma_n, \tilde{B}(x, y)|\tau_2(x, y) \geq cT_{\text{mix}}] P_z[\tau_2(x, y) < cT_{\text{mix}}] \]
\[ \leq C P_{\pi}[\tau_2(x, y) \leq \Gamma_n] P_z[\tau_2(x, y) < cT_{\text{mix}}] = O((g(x, z) + g(y, z)\Delta_n)). \]

We are left with two terms to estimate:

\begin{align*}
(4.8) \quad & P_{\pi,z}[\tau_2(x, y) < cT_{\text{mix}}, \tilde{B}(x, y)], \\
(4.9) \quad & P_{\pi,z}[\tau_2(x, y) \leq \Gamma_n, \tilde{B}(x, y)|\tau_2(x, y) \geq cT_{\text{mix}}].
\end{align*}

We will first deal with (4.8) which, using the independence of \( \tau_1(x, y) \) and \( \tau_2(x, y) \), we can rewrite as

\[ P_{\pi,z}[B(x, y)|\tau_1(x, y) > \Gamma_n, \tau_2(x, y) < cT_{\text{mix}}] P_z[\tau_2(x, y) < cT_{\text{mix}}] P[\tau_1(x, y) > \Gamma_n] \]

Since \( B(x, y) \) depends on \( X_2(t) \) only for \( t \geq \Gamma_n \), from mixing considerations it is easy to see that

\[ P_{\pi,z}[B(x, y)|\tau_1(x, y) > \Gamma_n, \tau_2(x, y) < cT_{\text{mix}}] = P[B(x, y)|\tau_1(x, y) > \Gamma_n] + O(|V_n|^{-100}). \]

Consequently, (4.8) is equal to

\[ P_z[\tau_2(x, y) < cT_{\text{mix}}] P[\tilde{B}(x, y)] + O(|V_n|^{-100}). \]

Note that

\[ \tilde{B}(x, y) = (H(x, y) \cap \{\tau(x, y) > \Gamma_n\}) \cup (\tilde{B}(x, y) \cap \{\tau(x, y) \leq \Gamma_n\}). \]

Using \( P[\tau(x, y) \leq \Gamma_n] = O(\Delta_n) \), the previous lemma thus implies

\[ P[\tilde{B}(x, y)] = P[H(x, y)] + O(\Delta_n) = \frac{1}{4} + O(\Delta_n). \]

Observe

\[ P_z[\tau_2(x, y) < cT_{\text{mix}}] = P_z[\tau_2(x) < \tau_2(y) < cT_{\text{mix}}] + P_z[\tau_2(y) < \tau_2(x) < cT_{\text{mix}}] \]
\[ = O(g(x, z)g(x, y) + g(y, z)g(x, y)). \]

Consequently,

\[ P_z[\tau_2(x, y) < cT_{\text{mix}}] = f_c(x, z) + f_c(y, z) - P_z[\tau_2(x), \tau_2(y) < cT_{\text{mix}}] \]
\[ = f_c(x, z) + f_c(y, z) + O([g(x, z) + g(y, z)]g(x, y)). \]
The final part of the lemma follows since, arguing as in the proof of Lemma 3.3, we can estimate (4.9) as follows:
\[
\begin{align*}
P_{\pi,z}[\tau_2(x,y) &\leq \Gamma_n, \tilde{B}(x,y) | \tau_2(x,y) \geq cT_{\text{mix}}] \\
&= P[\tau_2(x,y) \leq \Gamma_n, \tilde{B}(x,y) | \tau_2(x,y) \geq cT_{\text{mix}}] \\
&\quad + O(e^{-\gamma c \Upsilon_n} + (g(x,z) + g(y,z)) \Delta_n).
\end{align*}
\]

Taking \( E_c(x,y) = P[\tau_2(x,y) \leq \Gamma_n, \tilde{B}(x,y) | \tau_2(x,y) \geq cT_{\text{mix}}] \) finishes the proof of the lemma.

Combining the three lemmas immediately gives Theorem 4.1.

5. The Variance. We will complete the proof of Theorem 1.3 in this section. The general theme is to eliminate asymmetry wherever possible. We first apply this idea by considering
\[
B = \sum_x 1_{G_{12}(x)} - \sum_x 1_{G_{21}(x)}
\]
in place of \(|A_1|\). In addition to being symmetric in \(X_1, X_2\), note that \(B\) also differs from \(|A_1|\) in that we have eliminated those sites whose marking is determined by the flip of a fair coin. These, however, do not make a significant contribution since it is a rare event that both walks hit a particular point for the first time simultaneously. In particular, we will show in Lemma 5.1 that \(\text{Var}(B) \approx 4\text{Var}(|A_1|)\), up to negligible error, so to prove Theorem 1.3 it suffices to show that
\[
\text{Var}(B) = \sum_{x,y} (f_c(x,y) - \bar{f}_c)^2 + O(e^{-\gamma c} (T_{\text{mix}})^2).
\]

It is more convenient to work with \(B\) as
\[
\text{Var}(B) = 2 \sum_{x,y} \left( P[G_{12}(x), G_{12}(y)] - P[G_{12}(x), G_{21}(y)] \right)
\]
\[
= 4 \sum_{x,y,z} \left( P_{x,z}[G_{12}(y)] - P_{x,z}[G_{21}(y)] \right) \bar{\pi}(z;x,y) P[H(x,y)]
\]

Applying Lemma 4.3, we can rewrite the above as
\[
(5.1) \sum_{x,y,z} \left( P_{x,z}[G_{12}(y)] - P_{x,z}[G_{21}(y)] \right) \bar{\pi}(z;x,y) +
\]
\[
(5.2) \sum_{x,y,z} |P_{x,z}[G_{12}(y)] - P_{x,z}[G_{21}(y)]| \left[ o \left( \frac{T_{\text{mix}}}{|V_n|^2 \log |V_n|} \right) + O \left( \frac{1}{|V_n|^2} \right) \right].
\]
We will show at the end of this section that \((5.2)\) is negligible. Note that
\[
P_{x,z}[G_{ij}(y)] = P_{x,z}[\tau_i(y) < \tau_j(y) \leq \Gamma_n] + P_{x,z}[\tau_i(y) \leq \Gamma_n, \tau_j(y) > \Gamma_n] \\
+ P_{x,z}[\Gamma_n < \tau_i(y) < \tau_j(y)] \equiv A + B + C.
\]

In subsection 5.2, we break the sum in \((5.1)\) into three different cases based on the time decomposition above and bound each in a given lemma. It will turn out that the error coming from the terms corresponding to \(A\) and \(C\) is negligible (Lemmas 5.3, 5.4). The reason for the former is that it is unlikely that both \(X_1, X_2\) hit \(y\) quickly and the latter follows as, conditional on having not hit \(y\) by time \(\Gamma_n\), both walks have long forgotten their initial conditions and are well mixed. This leaves \(B\), which dominates the variance. Its asymptotics will be computed (Lemma 5.2) by reducing the estimate to a computation involving \(\tilde{\pi}(z; x, y)\), whose Radon-Nikodym derivative with respect to the uniform measure has already been estimated quite precisely in Theorem 4.1.

5.1. Symmetrization.

**Lemma 5.1.** We have
\[
\text{Var}(B) = 4\text{Var}(|A_1|) + O \left( \sqrt{T_{\text{mix}} \text{Var}(|A_1|) + T_{\text{mix}}} \right) \text{ as } n \to \infty.
\]

**Proof.** Let \((\xi_n(x) : x \in V_n)\) be iid random variables independent of \(X_1, X_2\) with \(P[\xi_n(x) = 1] = P[\xi_n(x) = 2] = \frac{1}{2}\) and let \(A(x, i) = \{\tau_1(x) = \tau_2(x), \xi_n(x) = i\}\). By definition,
\[
\text{Var}(B) = \text{Var}(2|A_1| + \sum_x (1_{A(x, 2)} - 1_{A(x, 1)}))
\]
\[
= \text{Var}
\left(2|A_1| + \sum_x (1_{A(x, 2)} - 1_{A(x, 1)})\right).
\]

Observe,
\[
E \left(\sum_x 1_{A(x, 1)}\right)^2 \leq \sum_{x,y} P[\tau_1(x) = \tau_2(x), \tau_1(y) = \tau_2(y)].
\]

By the strong Markov property and independence of \(X_1, X_2\), the above is
bounded by twice

\[
\sum_{x,y} P_{x,x}[\tau_1(y) = \tau_2(y)] P[\tau_1(x) = \tau_2(x)]
\]

\[
\leq \sum_{x,y} \sum_t (P_x[\tau(y) = t])^2 P[X_2(\tau_1(x)) = x].
\]

Using that \( P_x[\tau(y) = t] \leq P_x[X(t) = y] \) and \( X_2(\tau_1(x)) \sim \pi \) when \( X(0) \sim \pi \), we have the further bound

\[
(5.4) \quad \frac{1}{|V_n|} \sum_{x,y} \left( \sum_{t=1}^{4T_{\text{mix}}} P_x[X(t) = y] + \sum_{t>4T_{\text{mix}}} (P_x[\tau(y) = t])^2 \right).
\]

Summing the first term over \( x,y,t \) plainly yields \( 4T_{\text{mix}} \). For the second term, note there exists \( C > 0 \) so that for \( t > 4T_{\text{mix}} \) we have

\[
P_x[\tau(y) = t] \leq P_x[X(t) = y] \leq C |V_n|
\]

hence

\[
\sum_{t>4T_{\text{mix}}} (P_x[\tau(y) = t])^2 \leq \frac{C}{|V_n|} \sum_{t>4T_{\text{mix}}} P_x[\tau(y) = t] \leq \frac{C}{|V_n|}.
\]

Therefore the second term in the summation in (5.4) is \( O(1) \). The lemma now follows from Cauchy-Schwarz. \( \square \)

5.2. Conditioning. We begin by estimating the part of (5.1) corresponding to “B” from (5.3).

**Lemma 5.2.** We have

\[
\sum_{x,y,z} \left( P_{x,z}[\tau_1(y) \leq \Gamma_n, \tau_2(y) > \Gamma_n] - P_{x,z}[\tau_2(y) \leq \Gamma_n, \tau_1(y) > \Gamma_n] \right) \bar{\pi}(z;x,y) = (1 + O(\Delta_n)) \sum_{x,y} (f_c(x,y) - f_c)^2 + O(e^{-\gamma c}(T_{\text{mix}})^2).
\]

**Proof.** Note that

\[
P_{x,z}[\tau_1(y) \leq \Gamma_n, \tau_2(y) > \Gamma_n] - P_{x,z}[\tau_2(y) \leq \Gamma_n, \tau_1(y) > \Gamma_n] = P_x[\tau_1(y) \leq \Gamma_n] - P_z[\tau_2(y) \leq \Gamma_n].
\]
Let $\delta_1(x, y, z) = O(e^{-\gamma c}Y_n + (g(x, z) + g(y, z) + \Delta_n)(g(x, y) + \Delta_n))$ be the error term from Theorem 4.1 and $\delta_2(x, y, z) = O(e^{-\gamma c}Y_n + \Delta_n(g(x, y) + g(x, z)))$ be the error from Lemma 3.3. Then we can rewrite the summation in the statement of the lemma as

$$
\sum_{x, y, z} \left( P_{x}(\tau_1(y) \leq \Gamma_n) - P_{z}(\tau_2(y) \leq \Gamma_n) \right) \tilde{\pi}(z; x, y)
$$

$$
= \frac{1}{|V_n|} \sum_{x, y, z} \left( P_{x}[\tau_1(y) \leq \Gamma_n] - P_{z}[\tau_2(y) \leq \Gamma_n] \right) (1 + \epsilon(x, y, z)) +
$$

$$
= \frac{1}{|V_n|} \sum_{x, y, z} \left( |f_c(x, y) - f_c(y, z)| + \delta_2(x, y, z) \right) \delta_1(x, y, z) \equiv B_1 + B_2
$$

where, by Theorem 4.1,

$$
\epsilon(x, y, z) = (1 + O(\Delta_n)) \left( 2f_c - f_c(x, z) - f_c(y, z) \right).
$$

Applying Assumption 1.2 repeatedly, it is tedious but not difficult to see that $B_2 = O(e^{-\gamma c}T_{\text{mix}}^2)$. By Lemma 3.3,

$$
B_1 = (1 + O(\Delta_n)) \left( \frac{1}{|V_n|} \sum_{x, y, z} \left( f_c(x, y) - f_c(y, z) + \delta_2(x, y, z) \right) \right)
$$

$$
\left( 2f_c - f_c(x, z) - f_c(y, z) \right).
$$

Multiplying through, using the symmetry of $f$ in its arguments, and canceling many terms, this becomes

$$
(1 + O(\Delta_n)) \sum_{x, y} (f_c(x, y) - \bar{f}_c)^2 + O(e^{-\gamma c}(T_{\text{mix}}^2)).
$$

We will now show that the part of (5.1) coming from “A” of (5.3) is negligible. Roughly, the reason for this is that it is unlikely that both walks hit $y$ quickly, though in order to get a sufficiently good bound we will need to take advantage of some of the cancellation inherent in the situation. This will require us to invoke (4.1), which is a rough estimate of the Radon-Nikodym derivative of $\pi$ with respect to $\tau$.

**Lemma 5.3.** Uniformly in $n$ we have

$$
\sum_{x, y, z} \left( P_{x, z}[\tau_1(y) < \tau_2(y) \leq \Gamma_n] - P_{x, z}[\tau_2(y) < \tau_1(y) \leq \Gamma_n] \right) \tilde{\pi}(z; x, y)
$$

$$
= o(T_{\text{mix}}^2)
$$
Proof. By (4.1), the summation in the statement of the lemma is equal to

\[
\frac{1}{|V_n|} \sum_{x,y,z} \left( P_{x,z}[\tau_1(y) < \tau_2(y) \leq \Gamma_n] - P_{x,z}[\tau_2(y) < \tau_1(y) \leq \Gamma_n] \right)
\]

\[
(1 + O(g(x,y) + g(y,z) + g(x,z))).
\]

By symmetry, we see that this is equal to

\[
\frac{1}{|V_n|} \sum_{x,y,z} \left( P_{x,z}[\tau_1(y) < \tau_2(y) \leq \Gamma_n] - P_{x,z}[\tau_2(y) < \tau_1(y) \leq \Gamma_n] \right)
\]

\[
(O(g(x,y) + g(y,z) + g(x,z))).
\]

Using

\[
P_{x,z}[\tau_1(y) < \tau_2(y) \leq \Gamma_n] \leq P_{x,z}[\tau_1(y) \leq \Gamma_n, \tau_2(y) \leq \Gamma_n]
\]

and the independence of \( X_1, X_2 \), we have the further bound

\[
\frac{1}{|V_n|} \sum_{x,y,z} O((g(x,y) + \Delta_n)(g(y,z) + \Delta_n))O(g(x,y) + g(y,z) + g(x,z)).
\]

By the symmetry of \( g \) in its arguments, we can rewrite this as

\[
\frac{1}{|V_n|} \sum_{x,y,z} O((g(x,y) + \Delta_n)(g(y,z) + \Delta_n))O(g(x,y) + g(y,z) + g(x,z) + g(x,y)g(x,z)g(y,z)).
\]

Repeated applications of Assumption 1.2 and a tedious but easy calculation give the lemma. \( \square \)

We complete this subsection by proving that “\( C \)” from (5.3) is also negligible. The intuition for this is that by time \( \Gamma_n \) both walks are very well mixed hence given that both have not hit \( y \), the difference in the probability that one hits before the other is of smaller order than any negative power of \( |V_n| \) (though we chose to write \(-100\)). The proof will be in a slightly different spirit than the previous lemmas since estimating the Radon-Nikodym derivative of the conditioned laws with respect to \( \pi \) will not quite suffice.

**Lemma 5.4.** We have

\[
P_{x,z}[\Gamma_n < \tau_1(y) < \tau_2(y)] - P_{x,z}[\Gamma_n < \tau_2(y) < \tau_1(y)] = O(|V_n|^{-100}).
\]
PROOF. The idea of the proof is to use a standard coupling argument to show that, conditional on \( \{\tau_1(y), \tau_2(y) \geq \Gamma_n\} \), the laws of \( X_1(\Gamma_n) \) and \( X_2(\Gamma_n) \) have total variation distance \( O(|V_n|^{-100}) \) independent of \( x, z \). To this end, we set \( \mu(z; x, y) = P_x[X(\Gamma_n) = z|\tau(y) \geq \Gamma_n] \). Let \( Y(t) \) be the process given by \( X(t) \) conditioned on the event \( \{\tau(y) \geq \Gamma_n\} \). Then \( Y(t) \) is Markov (though time-inhomogeneous) as

\[
P[Y(t) = z|Y(1) = z_1, \ldots, Y(t-1) = z_{t-1}]
\]

\[
P[X(t) = z|X(1) = z_1, \ldots, X(t-1) = z_{t-1}, \tau(y) \geq \Gamma_n]
\]

\[
P[X(t) = z, \tau(y) \geq \Gamma_n|X(1) = z_1, \ldots, X(t-1) = z_{t-1}]
\]

\[
\frac{P\{\tau(y) \geq \Gamma_n - (t-1)\}}{P\{\tau(y) \geq \Gamma_n - (t-1)\}}
\]

depends only on \( z, z_{t-1} \). Recall that \( T_k = kT_{mix} \). For \( t = cT_k \), note that

\[
\nu(z; t, x) = P_x[Y(t) = z] = \frac{P_x[X(t) = z, \tau(y) \geq \Gamma_n|\tau(y) \geq t]}{P_x[\tau(y) \geq \Gamma_n|\tau(y) \geq t]}
\]

\[
= \frac{P_x[\tau(y) \geq \Gamma_n - t]P_x[X(t) = z|\tau(y) \geq t]}{P_x[\tau(y) \geq \Gamma_n|\tau(y) \geq t]}
\]

Combining part (3) of Assumption 1.2 with Lemma 3.2, we have that

\[
\frac{P_x[\tau(y) \geq \Gamma_n - t]}{P_x[\tau(y) \geq \Gamma_n|\tau(y) \geq t]} = \Theta(1) \text{ for } z \neq y.
\]

Also, since \( \sum_z g(y, z) = T_{mix} \) and \( T_{mix} = o(|V_n|) \), it follows that for each \( \epsilon > 0 \) fixed, with \( A = \{z : g(y, z) \leq \epsilon\} \) we have \(|A|/|V_n| = 1 - o(1)\). Lemma 3.2 also implies that \( P_x[X(t) = z|\tau(y) \geq t] = \Theta(1)\pi(z) \) on \( A \) uniformly in \( n \) large provided that provided \( k \) is large enough and \( \epsilon > 0 \) is sufficiently small. This implies that we can couple together the laws \( Y_u(cT_k), Y_v(cT_k) \) starting at \( u, v \) distinct so that with probability \( \rho_0 > 0 \) we have \( Y_u(cT_k) = Y_v(cT_k) \).

If we iterate this procedure \( c_2 = \frac{\alpha}{\eta} \log |V_n| \) times, \( \eta = \eta(c, k, \rho_0) \), we get that with probability \( 1 - O(|V_n|^{-c_2}) \) we have \( Y_u(\Gamma_n) = Y_v(\Gamma_n) \). Consequently, we may assume that \( c_0 \) is sufficiently large so that

\[
\max_{u,v} \|\nu(\cdot; \Gamma_n, u) - \nu(\cdot, \Gamma_n, v)\|_{TV} = O(|V_n|^{-500}).
\]

Let \( \nu \) be a measure so that \( \max_u \|\nu(\cdot; \Gamma_n, u) - \nu\|_{TV} = O(|V_n|^{-500}) \). Let
\[ D = \{ \tau_1(y), \tau_2(y) \geq \Gamma_n \} \]. Then we have that
\[
P_{x,z}[\Gamma_n < \tau_1(y) < \tau_2(y)] - P_{x,z}[\Gamma_n < \tau_2(y) < \tau_1(y)]
\]
\[ = \left( P_{x,z}[G_{12}(y)|D] - P_{x,z}[G_{21}(y)|D] \right) P_{x,z}[D]
\]
\[ = \sum_{u,v} \left( P_{u,v}[G_{12}(y)] - P_{u,v}[G_{21}(y)] \right) \nu(u; \Gamma_n, x) \nu(v; \Gamma_n, z) P_{x,z}[D] + O(|V_n|^{-200})
\]
\[ = O(|V_n|^{-200}). \]

\[ \square \]

**Proof of Theorem 1.3.** To finish the proof of Theorem 1.3 we just need to take care of the term in (5.2). The previous lemma implies
\[
\sum_{x,y,z} |P_{x,z}[G_{12}(y)] - P_{x,z}[G_{21}(y)]|
\]
\[ = \sum_{x,y,z} |P_{x,z}[G_{12}(y), \tau_1(y) \leq \Gamma_n] - P_{x,z}[G_{21}(y), \tau_2(y) \leq \Gamma_n]| + O(|V_n|^{-50}).
\]

Observe \( \{G_{ij}(y), \tau_i(y) \leq \Gamma_n\} \subseteq \{\tau_i(y), \tau_j(y) \leq \Gamma_n\} \cup \{\tau_i(y) \leq \Gamma_n < \tau_j(y)\}\).

Thus we can bound from above the previous expression by
\[
\sum_{x,y,z} \left( 2P_{x,z}[\tau_1(y), \tau_2(y) \leq \Gamma_n] + |P_x[\tau_1(y) \leq \Gamma_n] - P_z[\tau_2(y) \leq \Gamma_n]| \right) \equiv E_1 + E_2.
\]

The term corresponding to \( E_1 \) can be bounded in a similar manner as “A” in the proof of Lemma 5.3. Indeed, by the independence of \( X_1, X_2 \), we have that
\[
P_{x,z}[\tau_1(y), \tau_2(y) \leq \Gamma_n] \leq (g(x, y) + \Delta_n)(g(y, z) + \Delta_n)
\]

which, when summed over \( x, y, z \), is of order \( O((\log |V_n|)^2|V_n|T_{\text{mix}}^2) \). We can estimate \( E_2 \) using techniques similar to the proof of Lemma 5.2 since by Lemma 3.3,
\[
|P_x[\tau_1(y) \leq \Gamma_n] - P_z[\tau_2(y) \leq \Gamma_n]| = O(g(x, y) + g(y, z) + \delta_2(x, y, z)),
\]

where, as in the proof of Lemma 5.2, \( \delta_2(x, y, z) \) corresponds to the error from Lemma 3.3. When summed over \( x, y, z \), this is of order \( O(|V_n|^2T_{\text{mix}}) \).

Therefore
\[
(E_1 + E_2) \left( o \left( \frac{T_{\text{mix}}}{|V_n|^2 \log |V_n|} \right) + O \left( \frac{1}{|V_n|^2} \right) \right) = o(T_{\text{mix}}^2),
\]

as desired. \( \square \)
**6. Further Questions.**

1. The first step in proving a sequence of random variables \((X_n)\) has a Gaussian limit after appropriate normalization is the determination of the asymptotic mean and variance. We remarked in the beginning that, in our case, the expected number of sites colored blue is \(|V_n|/2\) and Theorem 1.3 gives the limiting variance. Figure 3 shows Q-Q plots of the empirical distribution of the number of sites painted 1 in the final coloring against an appropriately fitted normal for three different base graphs. Based on these plots, we conjecture that

\[
\frac{|A_i| - E|A_i|}{\sqrt{\text{Var}(|A_i|)}}
\]

has a normal limit for all graphs satisfying Assumption 1.2.

2. Our derivation of the variance ignores the time aspect of the problem in the sense that it gives no indication of at what point in the process of coverage the variance is “created.” Does it come in bursts or continuously? Does it come sooner than any multiple of the cover time or perhaps in \([\epsilon T_{\text{cov}}, T_{\text{cov}}]\)? More generally, when normalized appropriately, does the the process \(t \mapsto \sum_{x} 1\{\tau_1(x) < \tau_2(x) \leq t\}\) have a scaling limit?

3. We make repeated used of the symmetry afforded by the fact that we consider two random walks moving at the same speed on vertex transitive graph. It would be interesting to see if a similar result holds when the various degrees of symmetry are broken. Starting points for exploring this problem include considering continuous time walks moving at various speeds, multiple walks, and graphs which are not vertex transitive.
4. Theorem 1.1 only holds for tori of dimension \(d \geq 3\) as the case \(d = 2\) falls just outside of the scope of Theorem 1.3. It would be interesting to see a more refined analysis carried out to handle this case.

5. That the variance computed in Theorem 1.1 for \(d = 3, 4\) is significantly larger than in the iid case suggests that the clusters which have an unusually large number of sites painted a given color are either larger or more dense than in an iid marking. How large and frequent are such clusters? What is their geometric structure?

6. Another interesting quantity is the size \(B\) of the boundary separating the sites painted 1 and 2, as studied in [8]. It is not difficult to see that there exists a constant \(\beta_d > 0\) such that \(\mathbf{E}|B| \sim \beta_d n^d\) when \(d \geq 3\) as \(n \to \infty\). Indeed, this follows since the probability that \(\{\tau_1(y) < \tau_2(y)\}\) for \(y \sim x\) given \(\{\tau_1(x) < \tau_2(x)\}\) converges to a limit \(p_d \in (0, 1)\). Note that this is of the same order of magnitude as \(\mathbf{E}|A_1|\). Is it also true that \(\text{Var}(|B|) = \Theta(\text{Var}(|A_1|))\) or do these quantities differ significantly?

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