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Regular Article

Rotary dynamics of the rigid body electric dipole under the radiation reaction

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Abstract. Rotation of a permanently polarized rigid body under the radiation reaction torque is considered. Dynamics of the spinning top is derived from a balance condition of the angular momentum. It leads to the non-integrable nonlinear 2nd-order equations for angular velocities, and then to the reduced 1st-order Euler equations. The example of an axially symmetric top with the longitudinal dipole is solved exactly, with the transverse dipole analyzed qualitatively and numerically. Physical solutions describe the asymptotic power-law slowdown to stop or the exponential drift to a residual rotation; this depends on initial conditions and a shape of the top.

1 Introduction

It was reported recently that silica particles of mass about 1 fg and size 100 nm were spinned up in the optical trap to the frequency above 1 GHz [1]. This corresponds to the orbital quantum number of order \( \ell \gtrsim 10^{10} \), i.e., the rotational motion is quite classical. If such a particle possesses an electric dipole moment then it emits the electromagnetic radiation and receives the reaction torque which slows the particle rotation down. Of course, this relativistic effect is very small for the aforementioned particle whose constituents move no faster than \( 10^{-6} \) of light speed. But if the spinning particle is kept free for a long time, its radiative slowdown can be measurable and deserves a theoretical study.

The classical dynamics of a single point-like charge is governed by the relativistic Lorentz-Dirac equation or its slow-motion predecessor, the Abraham-Lorentz equation [3,4]. Both include the radiation reaction terms which depend on 3rd-order derivatives and give rise to redundant runaway solutions. To get rid of these solutions one uses frequently the approximated 2nd-order reduction of the Abraham-Lorentz-Dirac equations in which higher derivatives in r.-h.s. are eliminated by means of the same but truncated equation (i.e., without a radiation reaction term) [5–7].

For a composite spinning particle the dynamics is complemented with rotary degrees of freedom and complicated by further electrodynamical effects, such as an interaction between different charges, interference of radiation outgoing from them etc. As a result, the external and radiation reaction forces are woven in a complex way together in a complete set of equations on motion [8] whose cumbersome form leaves little feasibility for their analysis.

In the present paper we consider a free non-relativistic composite particle with an electric dipole moment. In this case the translational motion is inertial while of interest is a nontrivial 3D-rotary dynamics under the radiation reaction torque. We utilize the rigid body kinematics and the angular momentum balance condition in slow-motion approximation [5, Sect. 75], and derive the Euler-type equations of motion. Two specific examples of axially-symmetric top are considered: with longitudinal and transverse dipole moments. The first example is solvable, the second one is not, thus it is analyzed qualitatively and numerically. Our methodology is illustrated by the simple case of a plane rotator.

2 Dynamics of a single charge plane rotator

The slowdown of a charged plane rotator can be deduced easily from the energy balance condition, by means of the Larmor formula [3–5]. Instead, we proceed by methodological purpose from the Abraham-Lorentz equation:

\[
\dot{m} \dot{v} = F + \frac{2q^2}{3c^3} \ddot{v},
\]

where \( \dot{v} \) and \( \ddot{v} \equiv d\dot{v}/dt \) are velocity and acceleration of particle of mass \( m \) and charge \( q \), and \( c \) is the speed of light. The last term in r.-h.s. of equation (1) describes the radiation reaction force; it causes physical effect only in the presence of other forces \( F \).

Let \( F \) be a reaction of holonomic constraint which admits a circular motion, i.e., turns the particle into the
plane rotator. Then the equation (1) reduces to the form:

\[ \dot{\Omega} = \tau_0 (\Omega - \Omega^3), \]

where \( \tau_0 = \frac{2q^2}{3mc^3} \),

which henceforth will be referred to as the rotator Lorentz equation. This is the 2nd-order equation for the angular velocity \( \Omega \). Moreover, similarly to the Abraham-Lorentz equations (1), the equation (2) is singularly perturbed, i.e., there is a small time-scale parameter \( \tau_0 \) in r.-h.s. at the higher-order derivative \( \ddot{\Omega} \). Thus this equation possesses redundant set of solutions which must be separated out from the physical solutions.

Exact solutions of (2) are unknown. In Appendix the equation (2) is analyzed qualitatively and numerically, and selection rules for the physical solutions are formulated.

Here we use in the r.-h.s. of the equation (2) the truncated form of this equation, \( \dot{\Omega} = O(\tau_0) \), and arrive at the approximately reduced 1st-order equation:

\[ \dot{\Omega} = -\tau_0 \Omega^3. \]

It admits a standard Cauchy problem and possesses the following solution:

\[ \Omega(t) = \frac{\Omega_0}{\sqrt{1 + 2\tau_0 \Omega_0^2 t}}, \quad \Omega(0) = \Omega_0. \]

The characteristic quantity \( T = 1/(\tau_0 \Omega_0^2) \) is a time during which a rotatory braking is most intense. Asymptotically, at \( t \gg T \), the angular velocity decreases by the power law \( \Omega \sim 1/\sqrt{2\tau_0 t} \) which does not depend on the initial value \( \Omega_0 \). In appendix this result is obtained by the asymptotical analysis of the rotator Lorentz equation (2).

3 Equation of motion of a polarized spinning top

A plane rotator is the simplest spinning system. Here we consider a composite particle consisting of point-like charges \( q_a \) (\( a = 1, 2, \ldots \)) with masses \( m_a \) located at positions \( r_a \). We proceed from the slow-motion balance equation [5, Sect. 75]:

\[ \ddot{L} = \frac{2}{3c^4} \ddot{\Omega} \times \ddot{\Omega}, \]

for the angular momentum \( L = \sum_{a} m_a r_a \times v_a \), where \( \ddot{\Omega} \equiv \sum_{a} q_a r_a \) is the dipole moment of the system.

The composite particle is considered as a free rigid body, i.e., a top. A rotational motion of an arbitrary point \( r(t) \) of the top can be presented as follows: \( r(t) = O(t) \rho \), where \( O(t) \in SO(3) \) is a rotation matrix, and \( \rho \) is a constant position of the point in the proper reference frame of the top. We will use the following kinematic relations:

\[ v \equiv r = \dot{O} \rho = O(\Omega \times \rho), \]
\[ \dot{\rho} = O(\Omega \times (\Omega \times \rho) + \dot{\Omega} \times \rho), \]
\[ \dot{\Omega} = O((\Omega \times (\Omega \times \rho)) + \dot{\Omega} \times \rho). \]

where \( \Omega \) is the angular velocity vector, dual to the skew matrix \( \Omega = O^T \dot{\Omega} \).

Using the relations (6) in equation (5) yields the equation of rotary motion of a top:

\[ \ddot{\Omega} + \Omega \times \dot{\Omega} = \frac{2}{3c^4} \ddot{\rho} \times \{d \times (\Omega^2 \Omega - \dot{\Omega}) + (d \cdot \dot{\Omega}) \Omega \}
+ 2(d \cdot \Omega) \dot{\Omega}; \]

here \( l = ||l_{ij}|| \) (\( i, j = 1, 2, 3 \)) is the inertia tensor, \( d \equiv \Omega^T \ddot{\rho} \) is a constant dipole moment of the top in the proper reference frame, and \( \Omega \equiv \{O \Omega \} \).

The 1st-order reduction of the equation (7) implies the elimination the 2nd-order derivative in r.-h.s. by means of the truncated equation and its differential consequence:

\[ \dot{\Omega} = -l^{-1}(\Omega \times \dot{\Omega}), \]
\[ \ddot{\Omega} = -l^{-1}(\dot{\Omega} \times \Omega + \Omega \times \dot{\Omega}) = l^{-1}(\{\dot{\Omega} \times (\Omega \times \Omega) - (\Omega \times \Omega) \times \dot{\Omega} \}). \]

General explicit form of the reduced equation is cumbersome and omitted here.

3.1 Dynamics of an axially symmetric spinning top with the longitudinal dipole

Here we limit ourself by the case of spinning top with an axially-symmetric inertia ellipsoid (i.e., spheroid) and the longitudinal dipole moment:

\[ l_{ij} = l_i \delta_{ij}, \quad l_2 = l_1; \quad d_1 = d_2 = 0, \quad d_3 \equiv d. \]

Then the equation (7) splits into the following set:

\[ l_1 \Omega_1 = (l_1 - l_3) \Omega_2 \Omega_3 + \frac{2d}{3c^4} \left[ \Omega_1 - (\Omega_2^2 - \Omega_3^2 + \Omega_2 \Omega_3 - 2m_1 \Omega_2 - 2m_3 \Omega_3 \right], \]
\[ l_1 \Omega_2 = (l_1 - l_3) \Omega_2 \Omega_3 + \frac{2d}{3c^4} \left[ \Omega_2 - (\Omega_1^2 - \Omega_3^2 + \Omega_1 \Omega_3 + 2m_3 \Omega_1 \right], \]
\[ l_3 \Omega_3 = 0. \]

It follows from equation (12) that \( \Omega_3 = \text{const} \) provided \( l_3 \neq 0 \). Otherwise \( \Omega_3 \) remains an undetermined function of time. From the physical viewpoint, in the case \( l_3 \to 0 \) a rigid body degenerates to an infinitely thin rod along the \( \rho_1 \) axis, and a rotation in this direction is indefinite.

The remaining set of equations (10)–(11) is not solved exactly. Here we consider the 1st-order reduction of these equations:

\[ l_1 \Omega_1 = (l_1 - l_3) \Omega_2 \Omega_3 - \frac{2d}{3c^4} \left[ \Omega_1^2 + \Omega_2^2 + \frac{2l_2}{l_1} \Omega_3^2 \right] \Omega_1, \]
\[ l_1 \Omega_2 = (l_3 - l_1) \Omega_2 \Omega_3 - \frac{2d}{3c^4} \left[ \Omega_2^2 + \Omega_3^2 + \frac{2l_2}{l_1} \Omega_1^2 \right] \Omega_2. \]

If \( l_3 \neq 0 \), the equation (12) yields \( \Omega_3 = \text{const} \). The remaining nonlinear equations (13)–(14) determine components \( \Omega_1, \Omega_2 \) which form the vector \( \Omega_{\perp} = \{\Omega_1, \Omega_2, 0\} \).

Then one multiplies the equation (13) by \( \Omega_3 \), the equation (14) by \( \Omega_2 \), so their sum yields an integrable equation for \( \Omega_{\perp}^2 \equiv \Omega_{\perp}^2 \). Substituting this integral back into
the set (13)–(14) reduces the latter to a linear set. A final integration yields the solution:

\[ \Omega_1 = \Omega_2 = -\Omega_3 \sin \hat{\Omega}_3 t, \]
\[ \Omega_\perp = |\Omega_\perp| = \frac{(I_3/I_1)|\Omega_3|}{\sqrt{(1 + \frac{\delta_1^2 I_3^2}{I_1^2 R_3^2}) \exp\left\{ \frac{4d^2 I_3^2 I_3^2}{4d^2 I_3^2 I_3^2} t \right\} - 1} \]

where \( \hat{\Omega}_3 \equiv (1 - I_3/I_1) \Omega_3 \) and \( \Omega_{\perp,0} = \Omega_{\perp}|_{t=0} \). (Here one choice of a reference frame provides that \( \Omega_{1,0} > 0, \Omega_{2,0} < 0 \).

It follows from (15) that the precession of \( \Omega_\perp \) has opposite directions for a prolate \( (I_3 < I_1) \) and oblate \( (I_3 > I_1) \) top, and is absent for a spherical top.

In the limit \( \Omega_3 \to 0 \) one obtains for \( \Omega_\perp \):

\[ \Omega_\perp = \frac{\Omega_{\perp,0}}{1 + \frac{4d_1^2 I_3^2}{I_1^2 R_3^2} t}. \]

Besides, this expression is true in the case \( I_3 \to 0 \) of spinning rod when the variable \( \Omega_3 \) and a direction of the vector \( \Omega_\perp \) (but not its length \( \Omega_{\perp} \)) become indefinite, and equation (15) become meaningless.

4 Dynamics of an axially symmetric spinning top with the transverse dipole

The case of the spinning top with the dipole moment perpendicular to a symmetry axis,

\[ I_{ij} = I_i \delta_{ij}, \quad I_2 = I_1; \quad d_1 \equiv d, \quad d_2 = d_3 = 0. \]

is more cumbersome than that with parallel moment. The reduced equation of motion splits into the following ones:

\[ \dot{\Omega}_3 = (I_3 - I_1) \Omega_1 \Omega_2, \]
\[ \dot{\Omega}_2 = (I_3 - I_1) \Omega_1 \Omega_3 - \frac{2d^2 d_1}{3d^2} \left( \Omega_2^2 + \frac{(I_3 - I_1) (I_3 - I_1)}{I_1^2} \right) \Omega_3. \]
\[ \dot{\Omega}_3 = -\frac{2d^2 d_1}{3d^2} \left( \Omega_2^2 + \frac{I_3 - I_1}{I_1^2} (2\Omega_1^2 - \Omega_3^2) \right) \Omega_1. \]

One finds easily two partial solutions of the set (19)–(21).

\[ \Omega_1 = \Omega_2 = 0. \]
Equations (19), (20) become identities while (21) reduces to the equation

\[ \dot{\Omega}_3 = -\frac{2d^2}{3I_1 c^3} \Omega_3^3 \]

which, upon redefinition of parameters, coincides with the flat rotator equation (3). The power-law solution of the type (4) tends to zero, \( \Omega_3 \to 0 \), at \( t \to \infty \).

\[ \Omega_3 = 0. \]
Then \( \Omega_1 = \text{const} \) by (19) while (20) reduces to the equation

\[ \dot{\Omega}_2 = -\frac{2d^2}{3I_1 c^3} \left( \Omega_1^2 + \Omega_2^2 \right) \Omega_2. \]

It possesses solution on the type (16) with the exponential asymptotics \( \Omega_2 \to 0 \).

Asymptotics of both solutions lead to a set of fixed points \( \Omega_{\infty} = \{ \Omega_{\infty,0}, 0, 0 \} \), \( \Omega_{\infty} \in \mathbb{R} \). There are no other points which are fixed for the set (19)–(21).

General exact solution of the set (19)–(21) is unknown. Further some qualitative analysis is undertaken. For this purpose let us introduce the dimensionless quantities:

\[ \tau_0 = \frac{2d^2}{3I_1 c^3}, \quad \tau = \frac{t}{\tau_0}, \quad \omega = \tau_0 \Omega, \]

and change Cartesian components \( \omega_1, \omega_2, \omega_3 \) by cylindrical ones \( \omega_\perp, \varphi \):

\[ \omega_1 = \omega_\perp \cos \varphi, \quad \omega_2 = \omega_\perp \sin \varphi. \]

here the angle \( \varphi \) determines a direction of the vector \( \omega_\perp = \{ \omega_1, \omega_2, 0 \} \), and \( \omega_\perp = |\omega_\perp| \). In these terms the equations (19)–(21) take the form:

\[ \dot{\omega}_\perp = -(\omega_1^2 + (1 + \delta + \delta^2) \omega_3^2) \omega_\perp \sin^2 \varphi, \]
\[ \dot{\omega}_3 = -\frac{1}{1 - \delta} \left( (1 - 3\cos^2 \varphi) \omega_\perp^2 + \omega_3^2 \right) \omega_3, \]
\[ \dot{\varphi} = -\delta \omega_3 - \frac{1}{2} \left( \omega_\perp^2 + (1 + \delta + \delta^2) \omega_3^2 \right) \sin 2\varphi, \]

where \( \delta = 1 - I_3/I_1, -1 < \delta < 1 \).

\( \delta \geq 0 \) corresponds to the prolate (oblate) inertia spheroid which becomes the sphere at \( \delta = 0 \) and the disc at \( \delta = -1 \). The case \( \delta = 1 \) of thin rod has no meaning since the variables \( \omega_3, \varphi \) and the direction of dipole are indefinite.

Note some general properties of the equations (26)–(28).

It follows from equation (27) that \( \dot{\omega}_3 / \delta \tau \leq 0 \) provided \( \omega_3 \geq 0 \), i.e., \( |\omega_3| \to 0 \) monotonously at \( \tau \to \infty \).

It follows from equation (26) that \( \omega_\perp \to 0 \) monotonously at \( \tau \to \infty \) with undetermined final angle \( \varphi_{\infty} \). Otherwise one may occur \( \omega_\perp \to |\omega_\perp| > 0 \) provided \( \sin \varphi \to 0 \). But actual behavior of the angle \( \varphi \) is not evident from the equation (28) and needs some analysis.

4.1 The averaged dynamics and the linearized dynamics

It is noteworthy that the r.h.s of the set (26)–(28) is a \( \pi \)-periodic function of \( \varphi \). Averaging this set yields the equations for averaged variables \( u \equiv \omega_\perp^2, \quad v \equiv \omega_3^2, \quad \bar{\varphi} \):

\[ \dot{u} = -\{ u + (1 + \delta + \delta^2) v \} u, \]
\[ \dot{v} = -\frac{1}{1 - \delta} \{ (2 + \delta) u + 2v \} v, \]
\[ \dot{\varphi} = -\delta \omega_3. \]

This closed set of equations possesses the exact solution in a parametric form:

\[ u = \xi v = C \xi^{1-v/3} (1 + 24 \xi + 1 + \delta^3)^{-v/3}, \]
\[ \tau - \tau_0 = \frac{1 - \delta}{c} \int_{\tau_0}^{\xi} d\xi \xi^{v-1} (1 + 2\delta \xi + 1 + \delta^3)^{-v/3-1}, \]
\[ \varphi - \varphi_0 = \pm \frac{\delta(1 - \delta)}{\sqrt{C}} \int_0^t \Delta \xi \xi \frac{1}{(1 + 2\delta \xi + 1 + \delta^2)} d\xi, \quad (35) \]

where \( q = \frac{2}{1 + \delta^2}, \) \( p = \frac{\delta (1 - \delta)(3 + 3\delta + \delta^3)}{(1 + 2\delta)(1 + \delta)}, \) \( \delta > 0 \) is an integration constant. Quadratures (34)–(35) can be expressed in terms of incomplete beta-functions, but these expressions will not be needed further.

Instead, let us consider asymptotics of the solution (33)–(35) at \( \tau \to \infty: \)

- \( 1 \leq \delta \leq (1 - \frac{1}{2}) \) \( \omega_\perp \sim |\omega_3| \sim \tau^{-1/2}, \) \( |\varphi| \sim \tau^{1/2}; \) \( \omega_3 = 0, \) \( \varphi = k\pi \) with \( k = 0, \pm 1, \pm 2, \ldots. \) The linearization of (26)–(28) in a neighborhood of fixed points yields the set of equations

\[
\begin{align*}
\dot{\varphi} &= 0, \\
\dot{\omega}_3 &= -\frac{1 + 2\delta}{1 - \delta} \omega_\perp^2 \omega_3, \\
\dot{\psi} &= -\delta \omega_3 - \omega_\perp^2 \varphi
\end{align*}
\]

The deviation \( \nu = \omega_\perp - \omega_\infty, \omega_3 = \omega_3 - 0, \varphi = \varphi - k\pi. \) This set possesses exponential solutions \( \propto e^{1/3} \) with the characteristics \( \lambda_1 = 0, \lambda_2 = -\frac{1 + 2\delta}{1 - \delta} \omega_\infty^2, \lambda_3 = -\omega_\infty^2. \) The root \( \lambda_1 = 0 \) generates the change of the fixed value \( \omega_\perp \sim \omega_\infty + \nu_0 \) which corresponds to a nearby fixed point. Fixed points with \( \omega_\infty \neq 0 \) are stable (due to the linear approximation) for \( \delta > -1/2 \) and unstable for \( \delta < -1/2. \) Once the system gets into the neighborhood of a stable fixed point, the latter will be reached necessarily.

### 4.2 Analytical vs numerical results

The analysis of the original (26)–(28), the averaged (30)–(32) and the linearized (39)–(41) sets of equations shows that there are different cases, (1). If \( 0 \leq \delta < 1, \) i.e., the spinning top is prolate or spherical \((0 < I_3 \leq I_1),\) the fixed point \( \Omega_\infty = \{ \Omega_\infty, 0, 0 \} \) with any \( \Omega_\infty \neq 0 \) is stable. Thus, starting from arbitrary \( \Omega_0 = \{ \Omega_{10}, \Omega_{20}, \Omega_{30} \}, \) after few revolutions, the vector \( \Omega \) tends to \( \Omega_\infty \) exponentially with the characteristic braking time \( T = (3I_1/I_3 - 2)/\tau_0 \) \( \Omega_\infty \)^{−1}. It is noteworthy that the asymptotic components arise \( \Omega_1|_{t \to \infty} = \Omega_\infty \neq 0 \) even if \( \Omega_{10} = 0. \) (2). For the markedly oblate top of \(-1 \leq \delta \leq -1/2 (i.e., \( \frac{3}{2}I_1 \leq I_3 \leq 2I_1) \) this fixed point \( \Omega_\infty = \{ \Omega_\infty, 0, 0 \} \) with \( \Omega_\infty \neq 0 \) is unstable. Thus, by general properties of the set (26)–(28) and of (33)–(35), the vector \( \Omega \) continues to decrease by the asymptotic power law \( \Omega \sim 1/\sqrt{\tau_0 t} \) and reaches the value \( \Omega = 0 \) after an infinite number of revolutions around the axis \( O_3, (3). \) In the case \( -1/2 < \delta < 0 \) of a weakly oblate top \( (i.e., I_1 < I_3 < I_2/I_1) \) both scenarios are possible, and the final state depends on the starting point.

These cases are summarized in Figure 1.

Numerical integration of the set (26)–(28) with initial conditions chosen randomly confirms this picture. There are shown in Figure 2 two examples of evolution of the top with \( \delta = -1/5 \) for the same dimensionless initial values \( \omega_{\perp 0} = \omega_{3 0} = 1/5 \) but with different \( \varphi_0 = \pi/30 \) and \( \varphi_0 = \pi/3. \) The example (a) is typical for the prolate top, the (b) – for the marked oblate top.

### 5 Conclusions

The equation of a rotationary motion of the rigid body with the electric dipole moment (7) is derived from the Landau-Lifshitz angular momentum balance condition [5]. This equation of the 2nd order with respect to the angular velocity \( \Omega \) can be reduced approximately to the 1st-order Euler-type equation.

The specific case of a body with the axially-symmetric inertia ellipsoid is considered. The Euler equations for the spinning top with the longitudinal dipole is axially-symmetric and integrable. The longitudinal component \( \Omega_3 \) of the angular velocity \( \Omega \) is conserved while the transverse component \( \Omega_{\perp} \) decreases exponentially at \( t \to \infty \) with the braking time \( T = 1/(\tau_0 \Omega_3^2 I_3^2/I_1^2) \) where \( \tau_0 \) is defined by (24). If \( \Omega_3 = 0, \) then \( \Omega \) decreases asymptotically (at \( t \ll T \)) by the power law \( \Omega \sim 1/\sqrt{\tau_0 t} \) independently of the initial value of \( \Omega, \) similarly to the case of a plane rotator.

Absence of a symmetry may complicate the dynamics. Here it is considered the example of an axially-symmetric top but with transverse dipole which breaks the symmetry. In this case the dynamics is not integrable and has been analyzed qualitatively and numerically. It turned out that a final state of the top depends notably on its shape. If...
the spinning top is prolate, i.e., $I_3 < I_1$, some residual component of the angular velocity $\Omega$ along the dipole survives (even if this component was absent initially) while other components decrease exponentially. The markedly oblate top (i.e., if $I_3 > \frac{3}{2}I_1$) stops asymptotically by the power law $\Omega \sim 1/\sqrt{\tau_0 t}$. For a weakly oblate top (i.e., if $I_1 < I_3 < \frac{3}{2}I_1$) both scenarios are possible, and the final state depends the initial conditions.

The most realistic case is the asymmetric top. Its dynamics is more complicated. Preliminary calculations reveal exponential drift to some residual rotation if the top is polarized along a stable principal axis. Otherwise, the top slows down to a complete stop. Study of this braking in detail would be desirable.

Finally, let us consider some numerical estimates relevant to the experiment [1] with the silica particle mentioned in the Introduction. The inertia moment of a spherical particle is $I = \frac{2}{5}mR^2$, where $m = 1$ fg and $R = 50$ nm. Being ionized once up to the elementary charge $q = e = 4.8 \cdot 10^{-10}$ CGSE, the particle acquires the dipole moment $d = eR \approx 2400$ D. Then the scale time $\tau_0 \approx 10^{-35}$ s is extremely small, but the characteristic braking time for $\Omega_0 = 2\pi$ GHz is astronomical: $T \approx 2 \cdot 10^{35}$ s $\sim$ 0.5 $\times$ $10^8$ years. Hypothetically, these figures may have relevance to the interstellar dust which consists of silicate-graphite grains of size 50–500 nm [9]. Grains can be ionized by cosmic rays [10] and spun up by circularly polarized radiation from relativistic sources such as a black hole [11]. But GHz rotation of grains seems unlikely.

More realistic is a laboratory spin up (and subsequent slowdown) of artificial nanoparticles which may carry a large permanent electric dipole moment. The examples are Janus-like particles [12] or nanocrystals CdSe and CdS which at the size $\sim$5 nm may acquire up to 100 D of a dipole moment [13]. Of special interest are organic nanocrystals. Cellulose $\sim$300 nm$\times$30 nm–elongated nanocrystals are reported to possess the moment $\gtrsim 4000$ D [14]. But DAST-nanoparticles may appear the record holders: such a 100 nm$\times$100 nm$\times$50 nm–crystal is estimated to have the moment $\gtrsim 2.8 \cdot 10^7$ D [15, pp. 387–390]. The characteristic braking time for this last example with the initial $\Omega_0 = 2\pi$ GHz is $T \sim$ 0.3 year which is of order of a storage time in Penning trap [16]. Since the idea of a similar trap for neutral dipole particles is developing at present [17], the rotational braking of such particles by radiation reaction can be of interest.
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Author contribution statement

The author himself contributed to a formulation of the problem, to all analytical and numerical calculations and to the writing of the article.

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Appendix A: Analysis of the plane rotator

In terms of dimensionless variables \( \tau = t/\tau_0 \), \( \omega = \tau_0 \Omega \) the nonlinear differential 2nd-order equation (2) becomes free of any parameter:

\[
\ddot{\omega} - \dot{\omega} - \omega^3 = 0; \quad (A.1)
\]

here the dot “\( \dot{} \)” denotes differentiation by \( \tau \). The equation (A.1) is invariant under time translations \( \tau \to \tau + \lambda \), \( \lambda \in \mathbb{R} \) but the corresponding integral of motion is unknown. The change of variable \( \tau \to \theta = e^\tau \) reduces the equation (A.1) to the form:

\[
d^2 \omega / d\theta^2 = \omega^3 / \theta^2, \quad (A.2)
\]

which is of the Emden-Fowler type equation \( d^2 y / dx^2 = x^n y^m \). The pair of indices \( n = -2 \), \( m = 3 \) does not correspond to integrable cases of this equation [18].

Let us study the asymptotic behavior of solutions [19] of the equations (2) or (A.1) at \( t \to \pm \infty, 0 \). We suppose a power-law or exponential asymptotic behavior of solutions.

\( t \to +\infty \). Substituting the power-law anzatz \( \omega = A \tau^n [1 + O(\tau^{-1})] \) with the constants \( A \) and \( n \) to be found into equation (A.1) leads to the equality:

\[
A \alpha (\alpha - 1) \tau^{\alpha - 2} - A \alpha \tau^{\alpha - 1} - A^3 \tau^{3\alpha - 3} = O(\tau^{\alpha - 3}) + O(\tau^{\alpha - 2}) + O(\tau^{3\alpha - 3}).
\]

\[
(A.3)
\]

The 1st term in 1-h.s. is negligibly small as to the 2nd term which, in turn, can be canceled by the 3rd term, provided \( \alpha = -1/2 \) and \( A^2 = 1/2 \). In dimensional terms this can be summarized as follows:

\[
\dot{\Omega} \sim -\tau_0 \Omega^3 \Rightarrow \Omega(t) = \pm \frac{1}{\sqrt{2\tau_0 t}} [1 + O(t^{-1})], \quad t \to +\infty. \quad (A.4)
\]

\( t \to 0 \). Similarly, one obtains the asymptotics: \( \omega(\tau) = \pm \sqrt{2\tau}, [1 + O(\tau)] \). Since the equation (2) is invariant under the time translation \( t \to t - t_1 \) by the arbitrary \( t_1 \), this asymptotics can be presented in dimensional terms as follows:

\[
\dot{\Omega} \sim \Omega^3 \Rightarrow \Omega(t) = \pm \frac{\sqrt{2}}{t - t_1} [1 + O((t - t_1)^{-1})], \quad t \to t_1, \quad \forall t_1 \in \mathbb{R}. \quad (A.5)
\]

The equations (2) or (A.1) does not admit a power-law asymptotics, but the Emden-Fowler equation (A.2) does: \( \omega \sim A \theta = Ae^\tau, \theta \to +0 \iff \tau \to -\infty \), where \( A \) is an arbitrary real constant. In dimensional terms we have:

\[
\dot{\Omega} \sim \tau_0 \Omega \Rightarrow \Omega(t) \sim A \exp(t/\tau_0)/\tau_0, \quad t \to -\infty. \quad (A.6)
\]

The asymptotics (A.5) describes unlimited self-acceleration of a circulating particle during a finite time. Remarkably, the small parameter \( \tau_0 \) drops out from this asymptotics. Thus the corresponding solution must be regarded as obviously nonphysical.

The asymptotics (A.6) is obviously non-analytical in \( \tau_0 \). It is a segment of non-physical solution of equation (2) which, in turn, is similar (at \( t \to -\infty \)) to the runaway solution of the original Lorentz equation (1).

The only asymptotics (A.4) is physically meaningful since it correlates completely with the solution (4) of the reduced equation (3).

More details on solutions of the equation (A.1) can be seen in the phase portrait of the rotator. Let us recast for this purpose the 2nd order (with respect to \( \omega \)) equation (A.1) into the dynamical system:

\[
\dot{\omega} = \varpi, \quad (A.7)
\]

\[
\dot{\varpi} = \varpi + \omega^3. \quad (A.8)
\]

It follows from this the Abel equation for phase trajectories of the system:

\[
\frac{d\varpi}{d\omega} = 1 + \frac{\omega^3}{\varpi}. \quad (A.9)
\]

By now this equation is not solvable, but the asymptotics (A.4)–(A.6) suggest corresponding asymptotics of phase trajectories:

\[
\begin{align*}
\tau & \to +\infty \quad \Rightarrow \omega \sim (2\tau)^{3/2 - 1/2}, \quad \varpi \sim (2\tau)^{-3/2} \to 0 \quad \Rightarrow \varpi \sim -\omega^3; \quad (A.10) \\
\forall \tau_1 \quad \tau \approx \tau_1 \to 0 \quad \Rightarrow \omega \sim \sqrt{2}/\sqrt{\tau \approx \tau_1} \quad \Rightarrow \varpi \sim \omega^2/\sqrt{\tau} \quad (A.11) \\
\tau & \to -\infty \quad \Rightarrow \omega \sim Ae^{\tau}, \quad \varpi \sim Ae^{-\tau} \quad \Rightarrow \varpi \sim -\omega. \quad (A.12)
\end{align*}
\]

The system possesses a fixed point \( O = (\omega = 0, \varpi = 0) \), which is unstable. This follows from the behavior in neighborhood of this point of the following Lyapunov function [20]:

\[
V(\omega, \varpi) = \omega(\varpi - \frac{1}{2}\varpi^2): \quad (A.13)
\]

\[
V(0, 0) = 0; \quad V(0 \leq \omega \leq 2\varpi) > 0; \quad \dot{V} = \varpi^2 + \omega^4 > 0. \quad (A.14)
\]

A phase portrait of a charged plane rotator derived by means of the above analysis and a numerical integration
of the equation (A.9) is presented in Figure A.1. It is divided by four domains by two separatrices \(AOC\) and \(BOD\) crossing in the fixed point \(O\). In the neighborhood of \(O\) the separatrix \(AOC\) is described by the asymptotics (A.10) while \(BOD\) by the asymptotics (A.12). Opposite i.e. infinite asymptotics of these separatrices as well as the asymptotics of other phase trajectories are described by equation (A.11), and are reachable in a finite time. Thus all the phase trajectories are non-physical, except the separatrix \(AOC\).

It is obvious from Figure A.1 that the set (curve) \(AOC\) is a repeller, or, following the Spoon’s terminology, a critical manifold [6]. Every solution passing through an arbitrary point beyond the curve \(AOC\) goes to infinity in a finite time, so it is non-physical.

Every physical solution passes points of the curve \(AOC\). Thus it satisfies both the 2st-order equation (A.1) as well as the 1st-order equation

\[
\dot{\omega} = f(\omega), \quad \text{(A.15)}
\]

where the function \(f(\omega)\) determines the curve \(AOC\) by the equation \(\varpi = f(\omega)\) and thus meets the following conditions:

\[
f(\omega) = -\omega^3 + f(\omega) \frac{df(\omega)}{d\omega}, \quad \text{(A.16)}
\]

\[
f(\omega) \underset{\omega \rightarrow 0}{\sim} -\omega^3. \quad \text{(A.17)}
\]

In other words, the equations (A.15)–(A.17) is the exact 1st-order reduction of the 2nd-order equation (A.1). Since the equation (A.16) or (A.9) is not integrable, one can use an analytic approximation for \(f(\omega)\). The approximation \(f(\omega) \approx -\omega^3\) represents in equation (A.15) a dimensionless form of the reduced equation (2). It is precise in the neighborhood of \(\omega \rightarrow 0\) and yields the asymptotics (A.10). Both asymptotics (A.10) and (A.11) can be taken into account in the following approximation \(f(\omega) \approx -\omega^3/\sqrt{1 + 2\omega^2}\) with which the equation (A.15) is integrable analytically.

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