Principal Components of CMB non-Gaussianity

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ABSTRACT
The skew-spectrum statistic introduced by Munshi & Heavens (2010) has recently been used in studies of non-Gaussianity from diverse cosmological data sets including the detection of primary and secondary non-Gaussianity of Cosmic Microwave Background (CMB) radiation. Extending previous work, focussed on independent estimation, here we deal with the question of joint estimation of multiple skew-spectra from the same or correlated data sets. We consider the optimum skew-spectra for various models of primordial non-Gaussianity as well as secondary bispectra that originate from the cross-correlation of secondaries and lensing of CMB: coupling of lensing with the Integrated Sachs-Wolfe (ISW) effect, coupling of lensing with thermal Sunyaev-Zeldovich (tSZ), as well as from unresolved point-sources (PS). For joint estimation of various types of non-Gaussianity, we use the PCA to construct the linear combinations of amplitudes of various models of non-Gaussianity, e.g. $f^{loc}_{NL}$, $f^{eq}_{NL}$, $f^{ortho}_{NL}$ that can be estimated from CMB maps. Bias induced in the estimation of primordial non-Gaussianity due to secondary non-Gaussianity is evaluated. The PCA approach allows one to infer approximate (but generally accurate) constraints using CMB data sets on any reasonably smooth model by use of a lookup table and performing a simple computation. This principle is validated by computing constraints on the DBI bispectrum using a PCA analysis of the standard templates.

1 INTRODUCTION
The cosmic microwave background (CMB) radiation is the most important probe of the very earliest stages of the Universe. In standard inflationary models, the early Universe should be very close to random Gaussian field. Deviations from pure Gaussian statistics can provide direct clues regarding inflationary dynamics (Bartolo et al. 2004; Komatsu 2010; Liguori et al. 2010).

Analysis of temperature maps from the nominal Planck mission\(^1\) (Planck Collaboration (2013)) has set unprecedented constraints on various models of primordial non-Gaussianity by improving earlier results from WMAP\(^2\) (Bennett et al. (2013)). The Planck team also reported detections of secondary non-Gaussianity generated by coupling of the Integrated Sachs Wolfe (ISW) effect with lensing of the CMB, as well as from residual unresolved point sources. Following the data release from the Planck team cross-correlation of thermal Sunyaev-Zeldovich (tSZ) effect and lensing of CMB has also been reported by Hill & Spergel (2014).

Study of non-Gaussianity is typically performed using multiple techniques for cross validation. The techniques include the optimal KSW (Komatsu, Spergel & Wandelt (2005)) estimator, skew-spectrum (Munshi & Heavens (2010)), modal decomposition (see Fergusson, Liguori, Shellard (2010) and references therein) as well as the sub-optimal Minkowski functional (see Ducont et al. (2013) and references therein). The skew-spectrum was designed to address the dual challenge of estimation of primary or secondary non-Gaussianity without compressing all the available information into a single number in an optimal way. It has the power to differentiate among various possible sources of non-Gaussianity as well as contamination from unknown systematics. Initially proposed for the detection of non-Gaussianity via the bispectrum, the method has now been extended to include non-Gaussianity at the level of tripsectrum in Munshi et al. (2011). The skew-spectrum, being a (pseudo) power-spectrum associated with the underlying bispectrum, can provide an estimate of non-Gaussianity at each harmonic mode $\ell$. However, not only

\(^1\) http://sci.esa.int/planck/
\(^2\) http://map.gsfc.nasa.gov/
are the individual skew-spectrum modes generally correlated for a given underlying model of bispectrum, different skew-spectra estimated from the same data may be correlated too. Based on a Fisher matrix analysis, in this paper we are primarily interested in finding independent linear combinations of different modes of the same skew-spectrum as well as among harmonics associated with different skew-spectra. In doing so we will adopt the well established technique of principal components analysis (PCA) in our study.

In recent years there has been a renewed interest in applying principal component analysis (PCA) techniques to various cosmological data sets, a technique pioneered by Efstathiou & Bond (1999). Extending previous work here we consider the joint estimation of different types of non-Gaussianity from the same data set. This method can reveal the detailed statistical structure of parameter space which is lacking in an one-dimensional confidence level presentation. Efstathiou (2002) studied PCA in the context of the tensor degeneracy in CMB. For a recent of non-Gaussianity from the same data set. This method can reveal the detailed statistical structure of parameter space which is lacking in an one-dimensional confidence level presentation. Efstathiou (2002) studied PCA in the context of the tensor degeneracy in CMB. For a recent work see Rocha et al. (2004), where the possibility of measurement of the fine-structure constant \( \alpha \) has been explored in the context of CMB data with analysis based on Fisher matrix and PCA. Munshi & Kilbinger (2006) studied optimization of weak-lensing surveys using PCA. Hu & Keeton (2002) applied this technique to map the density distribution along the radial direction from weak lensing surveys. Jarvis & Jain (2004) used PCA to correct for the point spread function (PSF) variation in weak lensing surveys. In the context of SN Ia observations to constrain the dark energy equation of state, Huterer & Starkman (2003) and Huterer & Cooray (2005) employed PCA and its variants (see Wang & Tegmark (2005); Crittenden, Pogosian& Zhao (2005) for more recent results). Tegmark et al. (1998) used PCA for decorrelating the power spectrum of galaxies. This idea was initially proposed by Hamilton (1997) and further discussed in the context of galaxy surveys by Hamilton & Tegmark (2000). More recently the PCA has also been applied to study of reconstruction of reionization history Mortonson & Hu (2008) as well as inflationary potential reconstruction Dvorkin & Hu (2010). In the context of primordial non-Gaussianity, PCA was applied to a modal decomposition of an effective field theory description of single-field inflation in Anderson, Regan & Seery (2014), demonstrating that only four linearly independent combinations may be constrained using WMAP data. The approach allows a certain independence from the primordial templates used for comparison to the data, with the principal directions corresponding to those shapes that may be best constrained. In this work, the PCA approach is utilised in the context of the skew-spectrum, and extended to include secondary shapes.

This paper is organised as follows: In §2 we present a brief review of various models of non-Gaussianity primary and secondary. We introduce our estimators in §3. After discussing the process involved in performing a joint estimation of these estimators in §4, we introduce the principal components in §5. We apply our techniques to Planck-like data in §6. §7 is devoted to concluding remarks.

2 PRIMARY AND SECONDARY CMB NON-GAUSSIANITY

The harmonic transform \( a_{\ell m} \) of the temperature fluctuation \( \delta T(\hat{\Omega}) \) is defined on the surface of the sky as a function of the angular coordinate \( \hat{\Omega} \equiv (\theta, \phi) \):

\[
a_{\ell m} \equiv \int d\hat{\Omega} \ Y_{\ell m}(\hat{\Omega}) \frac{\delta T(\hat{\Omega})}{T}.
\]

The angular bispectrum \( B_{\ell_1 \ell_2 \ell_3} \) is the three-point correlation function defined in the harmonic domain:

\[
\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}.
\]

\[
B_{\ell_1 \ell_2 \ell_3} = b_{\ell_1 \ell_2 \ell_3} h_{\ell_1 \ell_2 \ell_3}; \quad h_{\ell_1 \ell_2 \ell_3} \equiv \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

This form preserves the the rotational invariance of the three-point correlation function in the harmonic domain. The quantity in parentheses is the Wigner 3j-symbol, which is non-zero only for triplets \( (\ell_1, \ell_2, \ell_3) \) which satisfy the triangle rule, including that the sum \( \ell_1 + \ell_2 + \ell_3 \) is even, ensuring the parity invariance of the bispectrum (see Munshi et al. (2013) for a discussion regarding odd-parity skew-spectrum). The reduced bispectrum \( b_{\ell_1 \ell_2 \ell_3} \) was introduced by Komatsu & Spergel (2001) which will be helpful for separating the purely geometrical factor from the dependence on underlying physics (see Bahch, Creminelli & Zaldarriaga (2004) for a more detailed discussion).

2.1 Primary Non-Gaussianity

The single-field slow-roll model of inflation provides a very small level of departure from Gaussianity, far below present experimental detection limits (Maldacena (2003); Acquaviva et al. (2003)). Many other variants, however, will allow for a much higher-level of non-Gaussianity (Komatsu (2010)). Various models of primordial non-Gaussianities are known as local, equilateral, orthogonal or folded models in the literature. Different aspects of the physics of the primordial Universe appear in different shapes of the three- and four-point functions.

- The “local” model appears in multi-field models of inflation due to interactions which operate on superhorizon scales (Salopek & Bond (1990); Gangui et al. (1994); Verde et al. (2000); Komatsu & Spergel (2001)).
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The radial functions $\alpha$ transfer function $f$ These estimates however ignore the cross-correlation among the templates is that they may be expressed in a separable form, i.e. in the form $\Phi(k_1)\Phi(k_2)\delta(k_3) + 5 \text{cyc.perm.}$ (Chen & Wang (2009); Holman & Tolley (2008); Moss & Graham (2007)). A feature of the standard $\delta$ inflation (Armendariz-Picon, Damour, Mukhanov (1999); Chen et al. (2007)) or Dirac-Born-Infield (DBI) inflation (Silverstein & Tong (2004); Alishahiha, Silverstein & Tong (2004)) models characterized by more general higher-derivative interactions of the inflaton field, such as ghost inflation (Arkani-Hamed et al. (2004)), and models arising from effective field theories (Chung et al. (2008)).

Examples of the class of “folded” (or flattened) NG include: single-field models with non-Bunch-Davies vacuum (Chen et al. (2007); Holman & Tolley (2008)) and models with general higher-derivative interactions (Senatore, Smith, Zaldarriaga (2010); Bartolo, Matarrese, Riotto (2010)); “Orthogonal” NG may be generated in single- field models of inflation with a non-canonical kinetic term (Renaux-Petel (2011); Ribeiro & Seery (2011)), or with general higher-derivative interactions. The orthogonal form is constructed in such a way that it is nearly orthogonal to both local and equilateral forms (Senatore, Smith, Zaldarriaga (2010); Chung et al. (2008); Meierburg et al. (2009)).

The extensions to these models to take into account isocurvature modes was considered by Hikage et al. (2010). The primordial bispectrum, $B_{\Phi}$ is defined by the three-point function of the gravitational potential, $\Phi$, via

$$\langle \Phi(k_1)\Phi(k_2)\Phi(k_3) \rangle = (2\pi)^3 \delta_{3D}(k_1 + k_2 + k_3) B_{\Phi}(k_1, k_2, k_3);$$

with the Dirac delta function $\delta_{3D}$ - arising as a consequence of statistical homogeneity - imposing the triangle condition $k_1 + k_2 + k_3 = 0$. The harmonic transform of the CMB temperature map, $a_{\ell m}$ and the primordial gravitational potential, $\Phi$, are related in linear perturbation theory through the correspondence

$$a_{\ell m} = 4\pi (-i)^{\ell} \int \frac{d^3 k}{(2\pi)^3} \Delta(k) \Phi(k) Y_{\ell m}^*(k);$$

where $\Delta(k)$ is known as the transfer function. This relationship may be used to relate the primordial bispectrum with its CMB counterpart. The three standard templates used for CMB analysis are the local, equilateral and orthogonal bispectra, given by:

$$b^{loc}_{\ell_1\ell_2\ell_3} = 2f^{loc}_{NL} \int r^2 dr [\alpha_{\ell_1}(r)\beta_{\ell_2}(r)\beta_{\ell_3}(r) + 2 \text{cyc.perm.}];$$

$$b^{eq}_{\ell_1\ell_2\ell_3} = -6f^{eq}_{NL} \int r^2 dr [\alpha_{\ell_1}(r)\beta_{\ell_2}(r)\gamma_{\ell_3}(r) + (2 \text{cyc.perm.}) - \beta_{\ell_1}(r)\gamma_{\ell_2}(r)\delta_{\ell_3}(r) + (5 \text{cyc.perm.}) + 2\delta_{\ell_1}(r)\delta_{\ell_2}(r)\delta_{\ell_3}(r)];$$

$$b^{ortho}_{\ell_1\ell_2\ell_3} = -6f^{ortho}_{NL} \int r^2 dr [\alpha_{\ell_1}(r)\beta_{\ell_2}(r)\delta_{\ell_3}(r) + (2 \text{cyc.perm.}) - \beta_{\ell_1}(r)\gamma_{\ell_2}(r)\delta_{\ell_3}(r) + (5 \text{cyc.perm.}) + 4\delta_{\ell_1}(r)\delta_{\ell_2}(r)\delta_{\ell_3}(r)].$$

The radial functions $\alpha_{\ell}(r)$, $\beta_{\ell}(r)$, $\gamma_{\ell}(r)$ and $\delta_{\ell}(r)$ depend on the power-spectrum of the primordial potential fluctuation $P_{\Phi}(k)$ and the radiation transfer function $\Delta_{\ell}(k)$:

$$\alpha_{\ell}(r) \equiv \frac{2}{\pi} \int k^2 dk \Delta_{\ell}(k) j_{\ell}(kr); \quad \beta_{\ell}(r) \equiv \frac{2}{\pi} \int k^2 dk P_{\Phi}(k) \Delta_{\ell}(k) j_{\ell}(kr);$$

$$\gamma_{\ell}(r) \equiv \frac{2}{\pi} \int k^2 dk P^{1/3}_{\Phi}(k) \Delta_{\ell}(k) j_{\ell}(kr); \quad \delta_{\ell}(r) \equiv \frac{2}{\pi} \int k^2 dk P^2_{\Phi}(k) \Delta_{\ell}(k) j_{\ell}(kr).$$

These functions are computed using publicly-available Boltzmann solvers such as CMBFAST Seljak & Zaldarriaga (1996) or CAMB Lewis, Challinor, Lasenby (2000).

Any of estimates of the parameters $f^{loc}_{NL}$, $f^{eq}_{NL}$ and $f^{ortho}_{NL}$ are bound to be correlated. One of the aim of this study is to investigate using PCA linear combinations of these parameters that can be estimated with minimum error-bars.

The current limits from nominal Planck mission are $f^{loc}_{NL} = 2.7 \pm 5.8$, $f^{eq}_{NL} = -42 \pm 75$, and $f^{ortho}_{NL} = -25 \pm 39$ (68 % CL statistical). These estimates however ignore the cross-correlation among the $f_{NL}$ parameters (Planck Collaboration (2013)).

These models do not exhaust all options and indeed there are other forms which would probe different aspects of the inflationary physics (Chen & Wang (2009); Holman & Tolley (2008); Moss & Xiong (2007); Huang (2008); Moss & Graham (2007)). A feature of the standard templates is that they may be expressed in a separable form, i.e. in the form $f(k_1)g(k_2)h(k_3)$, which allows for the estimation to performed in a much more computationally efficient manner. For general - and possibly non-separable shapes -, the bispectrum may be decomposed into a sum of separable basis functions (see Fergusson, Liguori, Shellard (2010) and references therein, as well as Regan, Shellard & Fergusson (2010) for a similar treatment of the trispectrum):

$$(k_1k_2k_3)^2 B_{\Phi}(k_1, k_2, k_3) = \sum_n \alpha_n^Q Q_n(k_1, k_2, k_3) \equiv \sum_n \alpha_n^Q \{ q_{n1}(k_1)q_{n2}(k_2)q_{n3}(k_3) + 5 \text{cyc.perm.} \};$$

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where \( n \) is an dummy index representing the triplet \( \{n_1, n_2, n_3\} \), and \( q_{n_1}(k) \) denotes a one dimensional basis function (often chosen to be a polynomial of degree \( n_1 \)). The CMB bispectrum is then given by

\[
b_{\ell_1\ell_2\ell_3} = f_{NL} \sum_n \alpha_n^Q \int dr \, r^2 \left[ q_{n_1}^Q(r) q_{n_2}^Q(r) q_{n_3}^Q(r) + \text{5 cyc. perms.} \right],
\]

where \( q_{n_1}^Q(r) = (2/\pi) \int dk k^2 q_{n_1}(k) \Delta_i(k) j_i(kr) \). Alternatively the CMB bispectrum may be written in the form

\[
b_{\ell_1\ell_2\ell_3} = \sum_n \alpha_n^Q Q_n(\ell_1, \ell_2, \ell_3),
\]

where the late-time coefficients \( \alpha_n^Q \) may be related to the primordial coefficients \( \alpha_n^Q \) via a ‘transfer matrix’, which accounts for the integration over the line of sight (for more details see, for example, Regan, Mukherjee & Seery (2013)). This simple prescription allows for primordial models to be efficiently mapped to their CMB counterparts, and for the analysis of non-separable shapes to be performed. This decomposition is utilised in this work for the analysis of the aforementioned models, as well as the non-separable DBI model given by the primordial bispectrum,

\[
B^\text{BI}_b = \frac{1}{(k_1 k_2 k_3)^3 (\sum_i k_i)^2} \left( \sum_i k_i^3 + \sum_{i \neq j} (2k_i^2 k_j - 3k_i^2 k_j^2) + \sum_{i \neq j \neq l} (k_i^2 k_j k_l - 4k_i^2 k_j^2 k_l) \right),
\]

For reference, the flattened bispectrum shape is given by,

\[
B^\text{flat}_b = \frac{1}{2} (B^\text{equil}_b - B^\text{orthog}_b).
\]

The different primordial models may give an insight into different microphysical mechanisms at work during the inflationary epoch. As such the standard approach of measuring the bispectrum using a single number, \( f_{NL} \), appears insufficient. Any possible detection of non-Gaussianity must be accompanied with an analysis of the possible mechanism which may induce it. In this respect the PCA approach described in this paper may prove particularly useful, by identifying the orthogonal directions in the data. Each model may be correlated with each direction in order to identify which model corresponds most with which feature.

We have focussed on the temperature anisotropy, mainly for simplicity. The constraints from the Planck satellite on \( f_{NL} \) are dominated by temperature information and are not expected to improve drastically with the inclusion of polarization data.

2.2 Secondary Non-Gaussianity

The secondary non-Gaussianities are generated at late time. An important type of secondary is generated at the level of bispectrum results from cross-talks of secondaries such as the Integrated Sachs-Wolfe’s (ISW) effect and lensing of CMB by large-scale-structure. Secondary non-Gaussianity of a similar form is expected also from coupling of point source (PS) and lensing as well between the thermal Sunyaev-Zeldovich (tSZ) effect and lensing. \( \hat{b}_{\ell_1\ell_2\ell_3}^{\text{ISW-lens}} = -\frac{1}{2} \left\{ C_{\ell_3}^{\phi T} C_{\ell_1\ell_2}^{TT} [\Pi_{\ell_3} - \Pi_{\ell_1} - \Pi_{\ell_2}] + \text{cyc. perm.} \right\}; \quad \Pi_{\ell} = \ell(\ell + 1) \)

(see Goldberg & Spergel (1999a),Goldberg & Spergel (1999b) for a derivation). The long wavelength modes of ISW contribution couples with the short-wavelength modes of fluctuations generated due to lensing hence the origin of the cross-spectra \( C_{\ell}^{\phi T} \). The reduced bispectrum above is denoted \( \hat{b}_{\ell_1\ell_2\ell_3}^{\text{ISW-lens}} \). The cross-spectrum \( C_{\ell}^{\phi T} \) introduced above represents the cross-correlation between the projected lensing potential \( \phi \) and the secondary contribution. \( C_{\ell}^{TT} \) denotes the temperature power-spectrum. The cross-spectra \( C_{\ell}^{\phi T} \) take different forms for ISW-lensing, PS-lensing or SZ-lensing correlation (Munshi et al. (2011)). The skew-spectrum statistic has already been applied to WMAP 5 year data release by Calabrese et al. (2010) to probe the correlation of CMB lensing potential and the secondary anisotropies.

The bispectrum for unresolved point sources takes the following form:

\[
\hat{b}_{\ell_1\ell_2\ell_3}^{\text{PS}} = b_{\ell_1\ell_2\ell_3}^{\text{PS}}.
\]

It is derived assuming the point sources are distributed randomly. The exact value of the parameter \( b_{\text{PS}} \) however depends on the flux limit as well as the mask used in a particular survey. The accuracy of such an approximation can indeed be extended by adding contributions from correlation terms.

The overlap between a secondary source of non-Gaussianity and a primordial source may result in biased estimates for parameters as shall be discussed further in §4.

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3 OPTIMUM SKEW-SPECTRA AND RELATED FISHER MATRICES

Many studies involving primordial non-Gaussianity have used the bispectrum, motivated by the fact that it contains all the information about \( f_{NL} \) (Babich (2005)). It has been extensively studied (Komatsu, Spergel & Wandelt (2005); Creminelli (2003); Creminelli et al. (2006); Medeiros & Contaldo (2006); Cabella et al. (2006); Liguori et al. (2007); Smith, Senatore & Zaldarriaga (2009)), with most of these measurements providing convolved estimates of the bispectrum. Optimised 3-point estimators were introduced by Heavens (1998), and have been successively developed (Komatsu, Spergel & Wandelt (2005); Creminelli et al. (2006); Creminelli, Senatore, & Zaldarriaga (2007); Smith, Zahn & Dore (2000); Smith & Zaldarriaga (2006)) to the point where an estimator for \( f_{NL} \) which saturates the Cramer-Rao bound exists for partial sky coverage and inhomogeneous noise (Smith, Senatore & Zaldarriaga (2009)). The skew spectrum was devised in Munshi & Heavens (2010) as a method to constrain the bispectrum without reduction to a single parameter.

The optimum skew-spectrum \( S_{opt}^{\ell} \) can be expressed in terms of the bispectrum \( B_{\ell_1 \ell_2 \ell_3} \) as (Munshi & Heavens (2010)):

\[
S_{opt}^{\ell} \equiv \frac{1}{6} \sum_{l_1 l_2 l_3} \frac{b_{l_1}^2 b_{l_2}^2 b_{l_3}^2 B_{l_1 l_2 l_3}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_1} C_{l_2} C_{l_3}} ,
\]

where \( C_{l}^{tot} \) is the total CMB temperature plus noise power spectrum: \( C_{l}^{tot} = b_{l}^2 C_{l} + n_{\ell} \), and we simplify notation by setting \( C_{l}^{tot} = C_{l}^{tot}/b_{l}^2 \). Here \( b_{l} \) represents the beam function and \( n_{\ell} \) is the noise power spectrum.

The covariance of the optimum skew-spectrum \( S_{opt}^{\ell} \) is represented by the Fisher matrix \([F]_{\ell \ell'}\) and is expressed as:

\[
[F]_{\ell \ell'} \equiv \langle (\delta S_{opt}^{\ell}) (\delta S_{opt}^{\ell'}) \rangle = \frac{1}{18} \delta_{\ell \ell'} \sum_{l_1 l_2 l_3} \frac{B_{l_1 l_2 l_3}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_1} C_{l_2} C_{l_3}} ,
\]

For one point estimator \( \langle (\delta S_{opt}^{\ell})^2 \rangle = S_{opt}^{\ell \ell} \). The Fisher matrix for the one-point estimator collapses to a single number and can be constructed from the Fisher matrix introduced above: \( F = \sum_{\ell_1 \ell_2} F_{\ell_1 \ell_2} \). Thus in our notation \( F = S_{opt}^{\ell \ell} \).

For joint estimation of two-different types of skew-spectrum of type X and Y we can write:

\[
\langle \delta S^{X}(X,Y) \delta S^{Y}(Y,Y) \rangle = \frac{1}{18} \delta_{\ell \ell'} \sum_{l_1 l_2 l_3} \frac{B_{l_1 l_2 l_3}^{X}(X) B_{l_1 l_2 l_3}^{Y}(Y)}{C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot} C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot}} + \frac{9}{3} \delta_{\ell \ell'} \sum_{l_1 l_2 l_3} \frac{B_{l_1 l_2 l_3}^{X}(X) B_{l_1 l_2 l_3}^{X}(X)}{C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot} C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot}} + \frac{9}{3} \delta_{\ell \ell'} \sum_{l_1 l_2 l_3} \frac{B_{l_1 l_2 l_3}^{Y}(Y) B_{l_1 l_2 l_3}^{Y}(Y)}{C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot} C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot}} .
\]

We have introduced the following notations above:

\[
S_{opt}^{\ell} (X,Y) \equiv \sum_{l_1 l_2 l_3} \frac{B_{l_1 l_2 l_3}^{X}(X) B_{l_1 l_2 l_3}^{Y}(Y)}{C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot} C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot}} ,
\]

similarly for \( S_{opt}^{\ell} (X,X) \) and \( S_{opt}^{\ell} (Y,Y) \). The joint covariance matrix can be written as \([F]_{\ell \ell'}^{XY} = \langle (\delta S_{opt}^{X}(X,Y) \delta S_{opt}^{Y}(Y,Y) \rangle \).

4 JOINT ESTIMATION OF MULTIPLE SKEW-SPECTRA

In this section we will consider the problem of simultaneous estimation of the multiple amplitudes \( f_{NL} \) from a given data. We will assume that the total non-Gaussianity is a sum of contributions from individual components.

\[
B_{\ell_1 \ell_2 \ell_3} = \sum_{X} f_{NL}^{X} [B_{\ell_1 \ell_2 \ell_3}^{X}]_{\ell_1 \ell_2 \ell_3} .
\]

Here we defined a \( B_{\ell_1 \ell_2 \ell_3}^{X} = f_{NL}^{X} [B_{\ell_1 \ell_2 \ell_3}^{X}]_{\ell_1 \ell_2 \ell_3} \) where \([B_{\ell_1 \ell_2 \ell_3}^{X}]_{\ell_1 \ell_2 \ell_3} \) is the bispectrum of type (X) evaluated at \( f_{NL} = 1 \). The inverse Fisher matrix \([F^{-1}]^{XY} = \langle (\delta f_{NL}^{X}(X,Y) \delta f_{NL}^{Y}(Y)) \rangle \) defines error-covariance matrix for the \( f_{NL} \) parameters,

\[
[F^{-1}]^{XY} = \langle (\delta f_{NL}^{X} \delta f_{NL}^{Y}) \rangle .
\]

In particular, the expected error bar for each parameter \( f_{NL} \) is given by \( \delta f_{NL}^{X} = \sqrt{[F^{-1}]^{XX}} \). The Fisher matrix \([F]^{XY} \) is defined in terms of \([B]^X\) and \([B]^Y\):

\[
[F]^{XY} = \frac{1}{6} S_{opt}^{\ell} (X,Y) \equiv \frac{1}{6} \sum_{\ell_1 \ell_2 \ell_3} \frac{[B_{\ell_1 \ell_2 \ell_3}^{X}]_{\ell_1 \ell_2 \ell_3} [B_{\ell_1 \ell_2 \ell_3}^{Y}]_{\ell_1 \ell_2 \ell_3}}{C_{l_1}^{tot} C_{l_2}^{tot} C_{l_3}^{tot}} ,
\]

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while the correlation between models \((X)\) and \((Y)\) is given by 

\[
\text{Corr}(X, Y) = \frac{\mathbf{F}^{(XX)}/\sqrt{\mathbf{F}^{(XX)}\mathbf{F}^{(YY)}}}{\mathbf{F}^{(XY)}}.
\]

This allows one to define a maximum likelihood estimator as:

\[
f_{NL}^{(X)} = \sum_Y \mathbf{F}^{-1}^{(XY)} S_Y^{(Y)}; \quad S_Y^{(Y)} = \frac{1}{6} \sum_{\ell_1 \ell_2 \ell_3} \frac{[B]^{(Y)}_{\ell_1 \ell_2 \ell_3} \hat{B}^{\ell_1 \ell_2 \ell_3}}{c_{\ell_1} c_{\ell_2} c_{\ell_3}}.
\]

(26)

Here the bispectrum \(\hat{B}^{\ell_1 \ell_2 \ell_3}\) is estimated from the data. This formalism is easily extended to the case of the skew-spectrum with the Fisher matrix for the cross-spectrum, \([F^{(XY)}]_{\ell_1 \ell_2}\) in this case given by \(\text{Eq.}(20)\). We note that the Fisher matrix given by \(\text{Eq.}(25)\) may be evaluated in the form \(F^{(XY)} = \sum_{\ell_1 \ell_2} F^{(XY)}_{\ell_1 \ell_2}\). We may then define the maximum likelihood estimator at each scale, \(\ell\), in the form,

\[
[f_{NL}]_\ell = \sum_Y \sum_{\ell'} \mathbf{F}^{-1}^{(XY)}_{\ell \ell'} S_{\ell'}^{(Y)}; \quad S_{\ell'}^{(Y)} = \frac{1}{6} \sum_{\ell_2 \ell_3} \frac{[B]^{(Y)}_{\ell_2 \ell_3} \hat{B}^{\ell_2 \ell_3}}{c_{\ell_2} c_{\ell_3} c_{\ell_3}}.
\]

(27)

The inversion of the block diagonal matrix \(\mathbf{F}^{(XY)}\) becomes very numerically challenging for large numbers of models. Instead in this paper we will restrict to the case of using observations of the skew-spectrum, \(S_I^{(Y)}\), for a single model \(Y\), in order to infer the quantity \(f_{NL}^{(X)}\) for (a possibly distinctly) model \(X\). However, as we shall see, for the case of the PCA components, the Fisher matrix becomes block diagonal, making inversion relatively trivial.

In the case of a secondary bispectrum, the overlap with a primordial bispectrum results in an expected bias, which must be corrected for in measurements of the estimator, \(f_{NL}\). The expected bias for a primordial model \((P)\) due to a secondary model \((S)\) is given by

\[
\delta b_{f_{NL}^{(X)}} = [F^{-1}]^{(PP)} [F]^{(PS)}.
\]

(28)

A joint estimation of primary and secondary non-Gaussianity allows us to marginalise over the presence of secondaries in order to provide accurate estimates for the bias as well as their impact on the error bars.

## 5 TRUTHS AND MYTHS CONCERNING PRINCIPAL COMPONENT ANALYSIS

A principal component analysis is an effective tool to probe the extent and dimensionality of the error ellipsoid. The standard deviation of the \(i\)th parameter, \(\delta \Theta_i = [(\Theta_i^2 - \langle \Theta_i \rangle^2)]^{1/2}\), may be obtained from the Fisher matrix, \(F_{ij}^{-1} = \langle \delta \Theta_i \delta \Theta_j \rangle\). The individual modes of the skew-spectra are correlated. The diagonalisation of the Fisher matrix can be performed to obtain uncorrelated linear combinations:

\[
F_{ij} = W_{ik}^T \Lambda_{kj} W_{lj}.
\]

(29)

Any real matrix \(W\) is called a decorrelation matrix if \(A\) is a diagonal matrix. The new quantities, \(\delta \Phi_i = W_{ij} \delta \Theta_j\), are uncorrelated because their covariance matrix is diagonal Hamilton & Tegmark (2000):

\[
\langle \delta \Phi_i \delta \Phi_j \rangle = W_{ik} \langle \delta \Theta_k \delta \Theta_j \rangle W_{lj}^T = \Lambda_{ij}^{-1}.
\]

(30)

By multiplying \(W\) with the square root of the diagonal matrix \(\Lambda\) the quantities \(\Phi\) can be scaled to unit variance. Without the loss of generality we can write: \(F = \mathbf{W}^T \mathbf{W}\). Where \(\mathbf{W} = \mathbf{A}^{1/2}\). The choice of \(\mathbf{W}\) is not unique. Any orthogonal rotation \(\mathbf{OW}\) with \(O \in \mathbf{SO}(n)\) can also work as decorrelation matrix thus there are infinitely many decorrelation matrices. If \(W\) is an orthogonal matrix, its rows are the eigenvectors of \(p_{ij}\) of \(F\) and \(\mathbf{A} = \text{diag}(\lambda_i)\) of the corresponding eigenvalues. In this case \(\mathbf{F} = \mathbf{W}^T \mathbf{A} \mathbf{W}\) is called principal component analysis. The eigenvectors or the principal components of \(F\) determine the principal axes of the \(n\)-dimensional error ellipsoid in parameter space. The eigenvectors represent orthogonal linear combination of the physical parameters. The accuracy with which the linear parameters can be determined is quantified by the variance \(\sigma_i \equiv \sigma(P_i) = \delta \Phi_i = \Lambda_{ii}^{-1/2} = \lambda_i^{-1/2}\). Since the eigenvalues are in descending order, the first eigenvector \(P_1\) having the smallest variance correspond to the best constrained parameter combination. The last eigenvector \(P_n\) is the direction with the largest uncertainty. Using Eq.(29) we can reconstruct the errors of individual physical parameters:

\[
\delta \Theta_i = \left[ \sum_{j=1}^n W_{ji}^2 / \lambda_j \right]^{1/2}.
\]

(31)

More concretely, for a joint analysis of multiple bispectra, \(\{B_i\}\), the \((i,j)\) Fisher entry is the overlap between models \(i\) and \(j\), i.e. \(F_{ij} = \langle B_i B_j \rangle\). Constructing the combination \(P_i = W_{ij} B_j\) we form the orthogonal (or decorrelated) combinations with Fisher matrix \([F]^{(P)} = \lambda_i \delta_{ij}\).

While Planck has produced constraints on the range of primordial models listed already, one may wish to estimate the error bars and expected bias for another model, \(M\). Rather than redo the entire analysis for this particular model, one can perform an approximate (but generally quite accurate) analysis in the following way:
Fisher Matrices for Primary and Secondary Skew-Spectra

Using a primordial measure between two bispectra, $B_{\Phi}^{(X)}$ and $B_{\Phi}^{(Y)}$, $\langle\langle B_{\Phi}^{(X)} B_{\Phi}^{(Y)} \rangle\rangle$ \(^3\), the correlation $\text{Corr}(X, Y)$ between two models may be approximated as

$$\text{Corr}(X, Y) \approx \mathcal{C}(X, Y) \equiv \frac{\langle B_{\Phi}^{(X)} B_{\Phi}^{(Y)} \rangle}{\sqrt{\langle B_{\Phi}^{(X)} B_{\Phi}^{(X)} \rangle \langle B_{\Phi}^{(Y)} B_{\Phi}^{(Y)} \rangle}}.$$  

(32)

In this manner one may find the correlation between the model under consideration, $M$, and the set of templates already constrained, $\{B_i\}$.

- From the set of templates $\{B_i\}$ we consider the set of orthogonal shapes $P_i = \sum_j W_{ij} B_j$, where $\langle P_i P_j \rangle = \lambda_i \delta_{ij}$. We may express model $M$ in terms of these orthogonal shapes in the form

$$M \approx \sum_i A_i P_i,$$

where $A_i = \langle M P_i \rangle / \langle P_i P_i \rangle \equiv \frac{1}{\lambda_i} \langle M | P_i \rangle^{\text{MP}_i}$.

(33)

The accuracy of this approximation is dependent on the set templates $\{P_i\}$ forming a complete basis. We may further simplify and use the approximation for the correlation measure to express, $A_i = (\langle M M | / \lambda_i \rangle)^{1/2} \text{Corr}(M, P_i) \approx (\langle M M | / \lambda_i \rangle)^{1/2} \mathcal{C}(M, P_i)$.

- The maximum likelihood estimator given by equation (26) may then be expressed solely in terms of the estimators for the principal components, with

$$f_{\delta_{\text{NL}}}^\text{(M)} \approx \left[ \sum_i A_i S_i^{(P_i)} \right] \left[ \delta f_{\delta_{\text{NL}}}^\text{(M)} \right]^2 \equiv \left[ \sum_i \sum_j A_i W_{ij} S_j^{(B_j)} \right] \left[ \delta f_{\delta_{\text{NL}}}^\text{(M)} \right]^2.$$

(34)

- The error bar for the model may also be approximated using the principal components by $\delta f_{\delta_{\text{NL}}}^\text{M} = \langle |M M| \rangle^{1/2} = ((M M))^{-1/2} \approx$

For example, one might use the measure $\langle B_{\Phi}^{(X)} B_{\Phi}^{(Y)} \rangle = \int_{2\max(k_i)} dk_1 dk_2 dk_3 S_{\Phi}^{(X)}(k_1, k_2, k_3) S_{\Phi}^{(Y)}(k_1, k_2, k_3)$, where $S_{\Phi}^{(X)} = (k_1 k_2 k_3)^2 B_{\Phi}^{(X)}$. 

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Primordial and Secondary Skew-Spectra

Figure 2. The skew-spectra associated with primordial (top panels) and secondary (bottom panels) non-Gaussianities are depicted. The skew-spectra is defined in Eq.(19). A Planck-realistic experiment at frequency 143GHz; was used with parameters $\theta_b = 7.1\prime$, $\sigma_{\text{pix}} = 2.2 \times 10^{-6}$ and $\Omega_{\text{pix}} = 0.0349$.

($\sum_i A_i^2 \lambda_i^{1/2}$, while the expected bias due to a secondary model, $S$, given by Eq.(28), may be approximated as $\delta_b f_{NL} = \sum_{ij} A_i W_{ij} \langle B_j S \rangle / (\sum_i A_i^2 \lambda_i)$.

As described in §4, performing a full Fisher analysis of the skew-spectra of several models becomes numerically very challenging for large numbers of modes. However, one may use the simpler analysis for joint estimation, describing the models using their skewness parameters (c.f. Eq.(26)), in order to identify the orthogonal shapes, $P_i = W_{ij} B_j$. Considering these orthogonal combinations, one need only consider the Fisher matrix, $[F]_{P_i P_j \ell \ell'}$, (since $[F]_{P_i P_j \ell \ell'}$ may be set to zero for $i \neq j$). One then computes the quantities (c.f. Eq.(27))

$$[f_{NL}^{P_i \ell}]_\ell = \sum_{\ell'} [F]_{P_i \ell \ell'}^{-1} \langle P_i^{\ell'} \rangle \langle S^{\ell'} \rangle.$$  \hspace{2cm} (35)

One may relate the quantities $[f_{NL}^{P_i \ell}]_\ell$ to individual templates. The advantage of performing a PCA analysis is thus perhaps even more apparent. Having identified the orthogonal directions, a more complete analysis of each orthogonal mode in terms of its skew-spectrum may be performed, taking full advantage of the data available to more clearly identify features that may be present at different scales, $\ell$, therefore allowing for a more exacting analysis of the consistency of the data with particular models.

6 RESULTS: IMPLICATIONS FOR PLANCK-TYPE EXPERIMENTS

We apply the methodology described in the previous sections to the case of Planck-like data. For this purpose we require the noise power spectrum and beam function, as described in §3. As described in Baumann et al. (2009) the beam $b_\ell$ and noise $n_\ell$ may be characterised by the parameters $\sigma_{\text{beam}}$ and $\sigma_{\text{rms}}$, respectively, and

$$b_\ell(\theta_b) = \exp(-\ell(\ell + 1)\sigma_{\text{beam}}^2); \quad \sigma_{\text{beam}} = \frac{\theta_b}{\sqrt{8 \ln 2}}; \quad n_\ell = \sigma_{\text{pix}}^2 \Omega_{\text{pix}}; \quad \Omega_{\text{pix}} = \frac{4\pi}{N_{\text{pix}}};$$  \hspace{2cm} (36)
where \( \theta_b \) describes the resolution of the beam, \( N_{\text{pix}} \) represents the number of pixels (of area \( \Omega_{\text{pix}} \)) required to cover the sky, and \( \sigma_{\text{pix}}^2 \) describes the variance per pixel. For a Planck-realistic experiment at frequency 143 GHz we use the parameters \( \theta_b = 7.1' \), \( \sigma_{\text{pix}} = 2.2 \times 10^{-6} \) and \( \Omega_{\text{pix}} = 0.0349 \). We perform our computations out to \( \ell_{\text{max}} = 1500 \), beyond which the signal is increasingly noise dominated.

We shall consider in our PCA analysis the three standard templates for the primary sources of non-Gaussianity, i.e. local, equilateral and orthogonal bispectra. We shall, in addition, discuss results applied to the DBI model as an example of the usefulness and efficacy of the PCA analysis described. We also analyse four secondary templates - three due to the cross correlation of lensing with the ISW, SZ and point source signal, respectively, and the other due to point sources only. With regard to nomenclature we will use the \( \delta f_{\text{NL}} \) parameter to signify the amplitude of the primary and secondary signals. For the latter case \( \delta f_{\text{NL}} \) may be understood as representing the inverse of the signal to noise. In the case of the point source only bispectrum, as given by Eq. (18), we characterise the amplitude in units of \( 10^{-29} \), i.e. \( f_{\text{NL}} = b^2/10^{-29} \).

### 6.1 Results for Individual Skew-Spectra

The skew spectrum, \( S_i \), is a useful statistic with which to estimate the signal from a primary or secondary source of non-Gaussianity without compressing all the information down to a single number. While the bispectrum contains up to \( \ell_{\text{max}}^2 \) independent numbers (due to the triangle condition), the skew spectrum - described with up to \( \ell_{\text{max}} \) numbers - offers a useful data compression, while retaining the power to differentiate between different models. A principal component analysis formalises this power through the identification of orthogonal modes. In this section we consider the skew-spectra for various models of primordial non-Gaussianity and for secondary sources of non-Gaussianity. In Figure 1 we plot the “off-diagonal” contribution to the Fisher matrix given by the second term on the right hand side of Eq. (20). This gives an indication of the correlation between different modes in the various models. In Figure 2 we plot the skew spectra for three primordial models - local, equilateral and orthogonal - as well as for three secondary models - arising from the cross correlation of ISW and lensing (ISW x Lens), thermal Sunyaev-Zeldovich with lensing (SZ x Lens) and residual point sources (PS). As a visual diagnostic it is clear that the skew spectrum allows for the models to be distinguished. However, we are interested here in what information the skew spectrum carries for the individual models. One may expect that the local model requires fewer modes for its description that the others. Having performed the PCA, in Figure 3 we plot the quantity \( 1/\sqrt{3 \sum_{n=1}^i \lambda_n} \) in order to establish the number of modes required for convergence (for \( i = \ell_{\text{max}} \) this quantity is \( \delta f_{\text{NL}} \)). In this respect one should have in mind the partial-wave decomposition utilised in the Planck non-Gaussianity analysis, whereby approximately \( \sim 600 \) partial waves are required for convergence to be achieved. One may hope that the skew-spectrum gives independent information at each harmonic scale \( \ell \), thus ensuring a clearer diagnostic tool to differentiate between different models.

From Figure 3 it is apparent that while convergence to within \( \mathcal{O}(10\%) \) of the signal to noise is achievable with approximately 400 modes, the full range of modes is generally required for full convergence. Hence the skew-spectrum \( S_i \) generally provides up to \( \ell_{\text{max}} \) independent measures for each model considered. One should caution that a visual inspection of this plot does not distinguish between the different models, since the eigenvectors associated with the eigenvalues in each case are likely to differ. To do so would necessitate a joint analysis as described in §4, or in a more robust and efficient manner, to follow the prescription at the end of §5 to identify the orthogonal directions and extract the skew spectra for these directions, before quantifying the results in terms

---

**Figure 3.** Plot of \( i \) versus \( 1/\sqrt{3 \sum_{n=1}^i \lambda_n} \) for the eigenvalues for the primary skew-spectra (left panel), the secondary skew-spectra (middle panel), and the overlap between various models and the equilateral model (right panel) are shown. The sum converges to \( \delta f_{\text{NL}} \) as we include more eigenvalues.
of the individual models. Nevertheless, it is instructive to consider the Fisher matrix of the cross-skew-spectrum between models $X$ and $Y$, i.e. $[F_{\ell\ell'}^{XY}]$, and perform a principal component analysis to identify the number of modes required for its accurate measurement. For this purpose we consider the cross spectrum between various primary and secondary templates with the equilateral model of non-Gaussianity. In Figure 3 we observe that an accurate identification of the overlap between the skew spectra between the equilateral and the local and orthogonal models of primordial non-Gaussianity again requires almost all modes, while the spectral overlap between the equilateral model and the ISW x Lens or SZ x Lens bispectra may be identified with as few as $\sim 100$ modes.

This analysis indicates that, as with the partial wave analysis of Planck data, one may expect the measurement of individual skew-spectra to require measuring $\mathcal{O}(\ell_{\text{max}})$ modes. An advantage of the approach adopted here is that the analysis is performed in multipole space. As such features in the data may be identified with specific multipoles.

### 6.2 Results for Multiple Skew-Spectra

A principal component analysis is most useful for the identification of orthogonal directions, ranking them according to those which may be best measured using the data. We consider the case firstly of only primordial models, specifically the local, equilateral and orthogonal models. These three templates are generally regarded as identifying the most distinct shapes that may be constrained. As we show however, the orthogonal directions are given by different combinations of these three shapes. In Table 1 we list the eigenvectors corresponding to these eigenvectors, along with the expected error bars (we also list the error bars for each of the templates). We also list the bias induced by each of these secondaries on these primordial models, namely the secondary bispectra ISW x Lens, PS x Lens and SZ x Lens, respectively. We compute also the bias (labelled $\delta^{\ell}_{\ell'} f_{NL}^{P_\ell}$) for each of the principal components, $P_\ell$, due to each of the secondaries, $S_j$.

As an example of the advantages of using the PCA analysis, we use these results to estimate the error bar for the DBI model using the prescription described in §5, comparing to the error bars computed using the Fisher matrix computed for the DBI model itself. We express the model under consideration in terms of the principal components of the three shapes using equation (33). The results using the approximation using the PCA analysis are listed with comparison to the exact results. The estimates for $\hat{f}_{NL}^{\text{DBI}}$ are compared to those given in the Planck results paper, Planck Collaboration (2013), i.e. $\hat{f}_{NL}^{\text{DBI}} = 11$, with the PCA values computed using those for the templates considered. From Table 3 we
Principal Components of CMB non-Gaussianity

| Shape     | $p_0$  | $p_1$  | $p_2$  | $\delta f_{NL}^0$ | $\delta f_{NL}^1$ | $\delta f_{NL}^2$ | $\delta f_{NL}^3$ |
|-----------|--------|--------|--------|-------------------|-------------------|-------------------|-------------------|
| Local     | 0.9939 | 0.0981 | -0.0503| 8.14              | 10.2             | 0.46              | 4.88              |
| Equil     | 0.0256 | 0.2384 | 0.9708 | 76.2              | 2.08             | 1.88              | 17.8              |
| Orthog    | -0.1072| 0.9662 | -0.2344| 40.9              | -33.1            | -6.99             | -71.6             |
| $\alpha$  | 8.09   | 47.4   | 82.9   |                   |                   |                   |                   |

Table 1. Eigenvectors of the Fisher matrix chosen from three primordial templates (local, equilateral, orthogonal). The principal directions reassuringly agree well with these templates - especially for the local and flattened models. However it is apparent that results quoted for the equilateral and orthogonal models are not independent (though this overlap is to expected). Also listed are error bars for each of the templates and for the principal directions. We also include the bias parameters for each model due to overlap between these primordial models and the ISW x Lens, PSxLens and SZ x Lens secondary bispectra (labelled $S_1$ to $S_3$, respectively).

| Bias      | $p_0$  | $p_1$  | $p_2$  | $\delta f_{NL}^{0}$ | $\delta f_{NL}^{1}$ | $\delta f_{NL}^{2}$ | $\delta f_{NL}^{3}$ |
|-----------|--------|--------|--------|---------------------|---------------------|---------------------|---------------------|
| $\delta f_{NL}^{NL}$ | 10.2   | -8.86  | -19.0  |                     |                     |                     |                     |
| $\delta f_{NL}^{PS}$   | 0.48   | -7.40  | 6.53   |                     |                     |                     |                     |
| $\delta f_{NL}^{SZ}$   | 5.10   | -75.2  | 64.1   |                     |                     |                     |                     |

Table 2. The bias for each of the principal components due to the secondaries labelled as $S_1$, $S_2$, $S_3$ that denotes ISW x Lens, PSxLens and SZ x Lens respectively are shown.

observe that the approximation to the DBI model using the principal components gives a very accurate approximation to the bias parameters (accurate to within 0.05$\sigma$) for each of the parameters except for the estimate $f_{NL}^{NL}$ - although we caution, that this value was obtained using central values using the reported Planck results for each of the templates (and for the DBI model itself). The value obtained for the PCA approximation to $f_{NL}^{NL}$ appears to be driven by the degree of overlap between the DBI and equilateral model (98.3% correlation).

Next we combine the primary and secondary models in order to more accurately characterise the orthogonal directions that may be constrained using the data. In Table 4 we again list the corresponding eigenvectors. It is apparent that there is a high degree of overlap between the ISW x Lens and SZ x Lens bispectra as expected. There is also a relatively large overlap between features in the equilateral and orthogonal models. The parameter $\delta f_{NL}$ for the secondaries should be interpreted for the ISW x Lens and the SZ x Lens as representing the inverse of the signal to noise, e.g. for the ISW x Lens model the signal to noise expected at $\ell_{\text{max}} \sim 1500$ for the Planck satellite is expected to be $\sim 5$. For the residual point source bispectrum $b_{NL}$, $b_{NL}^{PS}$, $b_{NL}^{SZ}$.

We now consider the skew-spectra of the principal directions which may be best constrained from the data as listed in Table 4. In Figure 4 we plot the associated skew spectra, while in Figure 3 we plot the inverse of the eigenvalues obtained from a decomposition of these spectra. The initial principal component analysis of the six shapes proved useful in identifying the orthogonal directions, while the subsequent decomposition allows for a simpler comparison of data to the skew spectrum of each principal component using the eigenvalues associated with each skew spectrum, $S_i^{PS}$. In Figure 5 we plot the inverse of the square root of the sum of eigenvalues of this subsequent decomposition for the two best and two worst constrained shapes $P_i$. Again it is apparent that each shape contains up to $\ell_{\text{max}}$ independent pieces of information. The analysis becomes simpler, since each component is no longer correlated. From the list of eigenvectors, and eigenvalues one may compute the bias simply. For example, the local bispectrum, $B_0$, satisfies $B_0 = \sum_i W_{i0} P_i$ and the ISW x Lens model, $B_3$, satisfies $B_3 = \sum_i W_{i3} P_i$. Thus the bias due to ISW x Lens is given by $\delta_b f_{NL} = \sum_i W_{i0} W_{i3} \lambda_i^2 / \sum_i W_{i0}^2 \lambda_i^2$. Since the primordial directions are not impacted heavily by the presence of secondaries (compare the eigenvectors of Table 1 and $P_2$, $P_4$ and $P_5$ of Table 4) one may compute the approximate form of a primordial model using the the PCA analysis of only the three primordial templates using Table 1 in order to write the model as in the form $M = \sum_{i=0}^3 a_i B_i$ and compute the expected bias by reference to Table 4.

| Shape       | $f_{NL}$ | $\delta f_{NL}$ | $\delta f_{NL}^{PS}$ | $\delta f_{NL}^{SZ}$ | $\delta f_{NL}^{PS}$ |
|-------------|----------|-----------------|----------------------|----------------------|----------------------|
| DBI (exact) | 11       | 68.2            | 9.7                  | 4.0                  | 39.5                 |
| DBI (PCA approx) | -31     | 68.5            | 6.7                  | 3.6                  | 35.8                 |

Table 3. Comparison of the parameters associated with the DBI model computed using the DBI model itself, compared to the approximated form computed from use of the principal components. The values for $f_{NL}$ are computed using values reported by the Planck team, but all other results are computed using the simplified Planck-like data described in this section.
7 DISCUSSION AND CONCLUSIONS

A formalism has been developed in this paper to exploit the use of a principal component analysis to assist in the study of both primary and secondary sources of non-Gaussianity. The PCA analysis is used in conjunction with the skew spectrum statistic with which the bispectrum is represented with a pseudo-Cℓ representation, and so in terms of ℓmax numbers. The advantage of the skew spectrum, Sℓ, is that, while the statistic remains an optimal method with which to measure non-Gaussianity, the information is not compressed to a single number, fNL. In addition one may associate features with a harmonic scale, ℓ. However, there remains the possibility that perhaps a sufficient statistic with which to accurately identify a source of non-Gaussianity may require far fewer numbers. PCA formalises into a clearly defined problem.

We apply the principal component analysis to skew spectra associated with individual bispectra (both primary and secondary), to the skew spectra associated with the three Minkowski functionals and to the joint analysis of multiple bispectra. The PCA may be described as a method with which to identify the uncorrelated components in each of the cases. In this paper we apply our analysis to the case of Planck-like data, with realistic estimates of the beam and noise.

First we considered the case of individual skew-spectra associated with the three standard primordial templates (local, equilateral and orthogonal) along with the three Minkowski functionals and to the joint analysis of multiple bispectra. The PCA may be described as a method with which to identify the uncorrelated components in each of the cases. In this paper we apply our analysis to the case of Planck-like data, with realistic estimates of the beam and noise.

Next, we consider the joint estimation of primary and secondary bispectra - again analysing the three standard primordial templates, along with the ISW x Lens, SZ x Lens and the residual point source bispectra. Considering firstly the three primordial models in isolation, we identify the orthogonal directions. We utilise the results of the earlier sections to make estimates of the bias due to the secondaries. We describe how the PCA may be used to relate the results reported using Planck data on the standard primordial templates to infer the constraints on another

| Shape    | P₀   | P₁   | P₂   | P₃   | P₄   | P₅   | δfNL |
|----------|------|------|------|------|------|------|------|
| Local    | 0.0041| −0.0040| 0.9942| −0.0210| 0.0889| −0.0568| 8.14 |
| Equil    | 0.0001| 0.0001| 0.0270| −0.0048| 0.3050| 0.9520| 76.2 |
| Ortho    | −0.0012| −0.0006| −0.1038| −0.1311| 0.9396| −0.2987| 40.8 |
| ISW x Lens | 0.0188| −0.7855| −0.0057| −0.0018| −0.0005| 0.0003| 0.20 |
| SZ x Lens | 0.7855| 0.6188| 0.0008| 0.0051| 0.0018| −0.0006| 0.18 |
| Point Source | 0.0030| 0.0048| −0.0074| −0.9911| −0.1276| 0.0361| 20.1 |

Table 4. Eigenvectors of the Fisher matrix for 6 shapes chosen from three primordial templates (local, equilateral, orthogonal) and three secondary shapes (ISW x Lens, SZ x Lens and residual point sources).

Figure 5. The inverse of the square root of the sum of eigenvalues (1/√∑nλn) for the two best (left panel) and two worst (right panel) eigenvectors are shown.
primordial model without requiring a full analysis to be carried out in that case (all that is required is the calculation of the Fisher matrix of that model along with a measurement of the correlation between this model and the standard templates - this correlation may be approximated using a simple primordial measure). We have applied this approximation to the case of the DBI bispectrum to obtain constraints on this model as well estimates of the bias induced by three secondary models. Having identified the orthogonal directions, we next considered the skew spectra associated with each of the principal components. Performing a principal component decomposition on these skew spectra allows one again to identify the (up to) \( \ell_{\text{max}} \) independent pieces of information that may be measured.

The formalism described allows for an efficient method with which to analyse the skew-spectra associated with a joint analysis of primary and secondary bispectra. The application to active sources of non-Gaussianity such as cosmic strings Hindmarsh, Ringleval & Suyama (2009); Regan & Shellard (2009) is straightforward. The use of a principal component analysis may prove useful in identifying any underlying source of non-Gaussianity. This may be particularly useful in the context of recent work on the possibility of using the skew spectrum of the bispectrum associated with the ISW x Lens signal to probe for modified theories of gravity Munshi et al. (2014).

We have ignored iscurvature perturbations in this paper but the PCA analysis of joint estimates using adiabatic and isocurvature bispectrum can be performed in a similar manner to the work presented here. The PCA results presented here can also be extended to higher-order i.e. to the kurt-spectra. It will also be possible to include skew-spectra associated with mixed bispectra involving polarization and temperature anisotropies.

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