FUNCTIONAL INTEGRAL REPRESENTATION OF
THE PAULI-FIERZ MODEL WITH SPIN 1/2

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1 The Pauli-Fierz model

In quantum mechanics the Hamiltonian with the electric field \( a = (a_1, a_2, a_3) \) of the form

\[
h(a) = \frac{1}{2} (p - a)^2 + V
\]

can be represented in terms of the Wiener measure \( dP^x \) on the set of continuous paths \( W = C([0, \infty); \mathbb{R}^3) \) by

\[
(f, e^{-th(a)}g) = \int dx \mathbb{E}^x \left[ \tilde{f}(B_0)g(B_t) e^{-\int_0^t V(B_s)ds} e^{-i \int_0^t a(B_s) \circ dB_s} \right], \tag{1.1}
\]

where \( \mathbb{E}^x[\cdots] = \int_W \cdots dP^x \) and \( B_t = B_t(w) = w(t) \) for \( w \in W \) is the Brownian motion. From \( |e^{-i \int_0^t a(B_s) \circ dB_s}| = 1 \), one can derive the so-called diamagnetic inequality

\[
|\langle f, e^{-th(a)}g \rangle| \leq \langle |f|, e^{-th(0)}|g| \rangle. \tag{1.2}
\]

In this talk we can extend (1.1) and (1.2) to some quantum field model in nonrelativistic quantum electrodynamics. Let \( B = B(x, t) \) and \( E = (x, t) \) be the magnetic field and the electric field, respectively. \( q_j = q_j(t) \in \mathbb{R}^3 \), \( j = 1, ..., N \), denotes the position of the \( j \)th electron at time \( t \). Then the Maxwell equations are given by

\[
\dot{B} = -\nabla \times E, \quad \dot{E} = \nabla \times B - e \sum_{j=1}^N \varphi(\cdot - q_j) \dot{q}_j,
\]

\[
\nabla \cdot B = 0, \quad \nabla \cdot E = e \sum_{j=1}^N \varphi(\cdot - q_j).
\]

Here \( \varphi \) denotes a charge distribution, i.e., \( e \int \varphi(x) dx = e \). The vector potential \( A = A(x, t) \) is introduced by \( B = \nabla \times A \) and a scalar potential \( \phi = \phi(x, t) \) by \( E = -\dot{A} - \nabla \phi \).

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Take the Coulomb gauge, $\nabla \cdot A = 0$, the Maxwell equation is reduced by

$$\square A = \nabla \phi - e \sum_{j=1}^{N} \varphi(\cdot - q_j) \partial_j = -e \sum_{j=1}^{N} \varphi(\cdot - q_j).$$

By the Legendre transformation we have the Hamiltonian

$$H_{cl} = \frac{1}{2} \sum_{j=1}^{N} \left( p_j - e \int A(x) \varphi(x - q_j) dx \right)^2 + \frac{1}{2} \int (\dot{A}^2 + (\nabla \times A)^2) dx + \frac{1}{e} \sum_{j=1}^{N} \int \phi(x) \varphi(x - q_j) dx.$$

Here $p_j$ denotes the $j$th canonical momentum and the scalar part above can be computed by $\Delta \phi = -e \sum_{j=1}^{N} \varphi(\cdot - q_j)$ as

$$\frac{1}{2} e \sum_{j=1}^{N} \int \phi(x) \varphi(x - q_j) dx = \frac{1}{2} e^2 \sum_{\substack{i,j=1 \atop i \neq j}}^{N} \int \int \frac{\varphi(q_i - y) \varphi(q_j - y')}{|y - y'|} dydy'. \tag{1.3}$$

The Pauli-Fierz Hamiltonian is defined by secondquantizing $H_{cl}$. For simplicity we set $N = 1$. We secondquantize $A$. Then $A$ is defined as the time zero field in the momentum representation;

$$A_{\mu}(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int e_{\mu}(k, j) \left( e^{-ikx} a^*(k, j) \frac{\dot{\phi}(-k)}{\sqrt{\omega(k)}} + e^{ikx} a(k, j) \frac{\dot{\phi}(k)}{\sqrt{\omega(k)}} \right) dk,$$

where $e(k, 1)$ and $e(k, 2)$ denote polarization vectors, $\omega(k) = |k|$ and $a^*(k, j)$ satisfies the canonical commutation relations $[a(k, j), a^*(k', j')] = \delta(k - k')\delta_{jj'}$. Then the field energy turns to be

$$\frac{1}{2} \int (\dot{A}^2 + (\nabla \times A)^2) dx \rightarrow \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk.$$

$A_{\mu}(x)$ and $H_{f}$ are the symmetric operator on the Boson Fock space $\mathcal{F}_b = \bigoplus_{n=0}^{\infty} \otimes_{s=n} L^2(\mathbb{R}^3)$. Then the Pauli-Fierz Hamiltonian $H$ is defined as the symmetric operator on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_b$$

by

$$H = \frac{1}{2} (-i \nabla \otimes 1 - e A(x))^2 + V \otimes 1 + 1 \otimes H_{f}. \tag{1.4}$$

In this talk we do not specify $V$ as (1.3). The Functional integral representation of $H$ without any spin is established. The net result [Hir07] is

$$(F, e^{-tH}G) = \int dx^2 \int_{Q_E} d\mu_E J_0 F(B_0) J_t G(B_t) e^{-\int_0^t V(B_s) ds - ie \int \Lambda_E (j_s, \varphi(-B_s)) dB_s}.$$

Here $A_E(f)$ denotes the Euclidean field in $L^2(Q_E, \mu_E)$, which is labeled by $f \in L^2(\mathbb{R}^{3+1})$, and $J_s: \mathcal{F}_b \rightarrow L(Q_E, \mu_E)$ and $j_s: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^{3+1})$, $s \in \mathbb{R}$, the family of isometries. Note that

$$\int_{Q_E} d\mu_E A_{E, \mu}(j_s f) A_{E, \mu}(j_t g) = \frac{1}{2} \int \tilde{f}(k) \tilde{g}(k) e^{-|s-t|\omega(k)} dk. \tag{1.6}$$

By (1.5), the diamagnetic inequality, the ergodic property of $e^{-tH}$, a spacial exponential decay of bound states, etc. have been derived. See [Hir04, Spo04] for other applications.
2 Spin and Poisson point processes

Including spin 1/2 of the electron, \( H \) is modified as the Hamiltonian on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}_b \):

\[
H_S := \frac{1}{2} \left( \sum_{\mu=1}^{3} \sigma_\mu (-i \nabla_\mu \otimes 1 - e A_\mu(x)) \right)^2 + V \otimes 1 + 1 \otimes H_T = H - \frac{e}{2} \sum_{\mu=1}^{3} \sigma_\mu B_\mu. \tag{2.1}
\]

Here

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

denote the \( 2 \times 2 \) Pauli matrices and \( B_\mu = (\nabla \times A)_\mu \) is the quantized magnetic field.

We also consider the functional integral representation of \( e^{-tH_S} \). Let \( F = \begin{bmatrix} F(+1) \\ F(-1) \end{bmatrix} \in L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}_b \). Then we have

\[
H_S F(\sigma) = H F(\sigma) - e \frac{\sigma B_3(x)}{2} F(\sigma) - e \frac{1}{2} (B_1 - i \sigma B_2) F(-\sigma), \quad \sigma = \pm 1. \tag{2.2}
\]

By (2.2) we can regard \( H_S \) as the operator in \( L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes \mathcal{F}_b \), where \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) denotes the additive group with degree two. In addition to the Wiener measure and the Euclidean field, we need a Poisson point process \( (p(s))_{s \geq 0} \) on some probability space \((\Omega, \mathcal{S}, \mathcal{P})\) valued in measurable space \((\mathcal{M}, \mathcal{B}_\mathcal{M})\) to construct the functional integral representation of \((F, e^{-tH}G)\). Let \( N(t, \tau, U) \) denote the counting measure associated with \( p(s) \),

\[
N(t, \tau, U) := \# \{ s \in D(p) | p(s, \tau) \in U, 0 < s \leq t \}
\]

for \( U \in \mathcal{B}_\mathcal{M}, \tau \in \Omega \). Let \( N_t = N_t(\tau) = N(t, \tau, \mathcal{M}) \), where \( D(p) = D(p(\cdot)) = D(p(\cdot, \tau)) \) denotes the domain of \( p(\cdot, \tau) \), which is countable set in \([0, \infty)\) for each \( \tau \in \Omega \). We fix \((p(s))_{s \geq 0}\) such that

\[
P(N_t = n) = e^{-t \ln n!}.
\]

Define the random process \( \sigma_t : \Omega \times \mathbb{Z}_2 \ni (\tau, \sigma) \mapsto \sigma \times (-1)^{N_t(\tau)} \in \mathbb{Z}_2 \) and

\[
\xi_t = (B_t, \sigma_t) : W \times \Omega \rightarrow \mathbb{R}^3 \times \mathbb{Z}_2, \quad \xi_0 = (x, \sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2. \tag{2.3}
\]

It is seen directly that

\[
ev^t \sum_{\sigma \in \mathbb{Z}_2} \int_{x \in \mathbb{R}^3} \int_{\mathcal{E}} dx \mathbb{E}^{x, \sigma} \left[ \int_{Q_E} d\mu_{E} J_0 F(\xi_0) J_t G(\xi_t) \right] = (F, e^{-tH_S(0)}G), \tag{2.4}
\]

where

\[
H_S(0) = -\frac{1}{2} \Delta \otimes 1 + 1 \otimes H_T + \sigma_F,
\]

and \( \sigma_F = (1/2)(\sigma_3 + i \sigma_2)(\sigma_3 - i \sigma_2) - 1/2 = -\sigma_1 \) is the Fermionic harmonic oscillator.
Theorem 2.1 [HL07]

\[
(F, e^{-tH_S} G) = \lim_{\epsilon \to 0} e^{t} \sum_{\sigma \in \mathbb{Z}^2} \int d\mathbf{x} \omega_{x,\sigma} \left[ \int_{Q_E} d\mu_E J_0 F(\xi_0) J_t G(\xi_t) e^{X_\epsilon} \right],
\]

(2.5)

where \(X_\epsilon = X_\epsilon(w, \tau, t)\) is given by

\[
X_\epsilon = -\int_0^t \left( V(B_s) ds - ie \int_0^t A_E(j_s \varphi(\cdot - x)) dB_s \right) = \text{external potential} - \text{quantized radiation field}
\]

\[
- \int_0^t H_{D,E}(B_s, \sigma_s, s) ds + \int_0^{t+} \log(-H_{OD,E}^\epsilon(B_s, -\sigma_s, -s)) dN_s.
\]

Here

\[
H_{D,E}(B_s, \sigma_s, s) = -(e/2)\sigma_s B_{3,E}(j_s \varphi(\cdot - B_s)),
\]

\[
H_{OD,E}(B_s, \sigma_s, s) = -(e/2) (B_{1,E}(j_s \varphi(\cdot - B_s)) - i\sigma_s B_{2,E}(j_s \varphi(\cdot - B_s))) + i\epsilon \psi_\epsilon(H_{OD,E}(B_s, \sigma_s, s));
\]

\[
B_E \text{ is the Euclidean quantum field version of } B \text{ and } \psi_\epsilon(x) = \begin{cases} 1, & |x| < \epsilon/2, \\ 0, & |x| \geq \epsilon/2. \end{cases}
\]

Note that \(|H_{OD,E}^\epsilon(X, Y, Z)| > \epsilon/2\). Hence \(|\log(-H_{OD,E}^\epsilon(X, Y, Z))| \neq \infty\). By making use of Theorem 2.1 we have the comparison energy inequality, which is an extension of the diamagnetic inequality. Let \(E(A, B_1, B_2, B_3) := \inf \sigma(H_S).\) We regard that \(A\) and \(B\) are independent of each other.

Corollary 2.2 We have

\[
\max \left\{ E(0, \sqrt{B_1^2 + B_2^2}, 0, B_3), E(0, \sqrt{B_2^2 + B_1^2}, 0, B_2), E(0, \sqrt{B_3^2 + B_1^2}, 0, B_1) \right\} \leq E(A, B_1, B_2, B_3).
\]

See [HH07] for applications of Poisson point processes to spin-boson models.

References

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