TOPOLOGICAL PROPERTIES OF P.A. RANDOM GRAPHS WITH EDGE-STEP FUNCTIONS

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Abstract. In this work we investigate a preferential attachment model whose parameter is a function \( f : \mathbb{N} \to [0,1] \) that drives the asymptotic proportion between the numbers of vertices and edges of the graph. We investigate topological features of the graphs, proving general bounds for the diameter and the clique number. Our results regarding the diameter are sharp when \( f \) is a \textit{regularly varying function at infinity} with strictly negative index of regular variation \(-\gamma\). For this particular class, we prove a characterization for the diameter that depends only on \(-\gamma\). More specifically, we prove that the diameter of such graphs is of order \( 1/\gamma \) with high probability, although its vertex set order goes to infinity polynomially. Sharp results for the diameter for a wide class of \textit{slowly varying functions} are also obtained. The almost sure convergence for the properly normalized logarithm of the clique number of the graphs generated by slowly varying functions is also proved.

Keywords: complex networks; cliques; preferential attachment; concentration bounds; diameter; scale-free; small-world

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1. Introduction

P. Erdos and A. Rényi in their seminal paper \cite{ErdosRenyi1959} introduced the random graph model that now carries their name in order to solve combinatorial problems. However, the theory of Random Graphs as a whole has proven to be a useful tool for treating concrete problems as well. Any discrete set of entities whose elements interact in a pairwise fashion may be seen as a graph: the vertices represent the entities, and the edges, the possible interactions. This approach is nowadays intuitive and very fruitful. In the scenario where there exists some randomness on the interactions among the entities, random graphs became the natural tool to represent abstract or real phenomena.

From a mathematical/statistical point of view, the Erdos-Rényi model – and many others related to it – is \textit{homogeneous}, in the sense that its vertices are \textit{statistically indistinguishable}. However, the empirical findings of the seminal work of A. Bárabasi and R. Albert \cite{BarabasiAlbert1999}...
suggested that many real-world networks are non-homogeneous. They observed that such graphs were scale-free, i.e., their degree sequence had a power-law distribution. The authors proposed a mechanism – known as preferential attachment – that could explain the emergence of such highly skewed distributions. Roughly speaking, the idea is that some sort of popularity drives the interaction among the entities.

Motivated mainly by these empirical findings, nowadays Preferential Attachment models (PA-models for short) constitutes a well known class of random graph models investigated from both theoretical and applied perspectives. Recently, the preferential attachment mechanism has been generalized in many ways and combined with other rules of attachment, such as spatial proximity [19] and fitness of vertices [14]. It also arises naturally even in models where it is not entirely explicit such as the deletion-duplication models [4, 26], in which vertices’ degree still evolve according to the PA-rule. Furthermore, the PA-models provide an interesting and natural environment for other random processes, such as bootstrap percolation, contact process and random walks, see [3, 11, 20] for recent examples of random processes whose random media is sampled from some PA-model.

When dealing with PA-models, there exists a set of natural questions that arises. They concern the empirical degree distribution, the order of the diameter and the robustness of the network. Their interest relies on modeling purposes and on the implications for the graph’s combinatorial structures.

In this paper we address the two latter topics on a PA-model which is a modification of the BA-model. The kind of result we pursuit is to show that some graph properties hold asymptotically almost surely (a.a.s). Given a sequence of random graphs \( \{G_t\}_{t \in \mathbb{N}} \), we say that a graph property \( \mathcal{P} \) holds a.a.s, if
\[
P(G_t \in \mathcal{P}) = 1 - o(1)
\]
i.e., the probability of observing such property increases to 1 as \( t \) goes to infinity. For instance, \( \mathcal{P} \) may be the set of graphs having diameter less than the logarithm of the total number of vertices.

In order to offer a clearer discussion of our results, we introduce the model in the next subsection, then we discuss separately the properties which we want the graph to satisfy a.a.s, as well as the associated motivation.

1.1. \textbf{Preferential attachment model with an edge-step function.} The model we investigate here in its generality was proposed in [2] and combines the traditional preferential attachment rule with a function called edge-step function that drives the growth rate of the vertex set.

The model has one parameter \( f \) which is a real non-negative function with domain given by \( \mathbb{N} \) and bounded by one on the \( L_\infty \)-norm. Without loss of generality and to simplify the expressions we deal with, we start the process from an initial graph \( G_1 \) consisting in one vertex and one loop. The model evolves inductively and at each step the next graph is
obtained by performing one of the two stochastic operations defined below on the previous one:

- **Vertex-step** - Add a new vertex $v$ and add an edge $\{u, v\}$ by choosing $u \in G$ with probability proportional to its degree. More formally, conditionally on $G$, the probability of attaching $v$ to $u \in G$ is given by
  \[ P(v \rightarrow u|G) = \frac{\text{degree}(u)}{\sum_{w \in G} \text{degree}(w)}. \]

- **Edge-step** - Add a new edge $\{u_1, u_2\}$ by independently choosing vertices $u_1, u_2 \in G$ according to the same rule described in the vertex-step. We note that both loops and parallel edges are allowed.

The model alternates between the two types of operations according to a sequence $\{Z_t\}_{t \geq 1}$ of independent random variables such that $Z_t \overset{d}{=} \text{Ber}(f(t))$. We then define inductively a random graph process $\{G_t(f)\}_{t \geq 1}$ as follows: start with $G_1$. Given $G_t(f)$, obtain $G_{t+1}(f)$ by either performing a **vertex-step** on $G_t(f)$ when $Z_t = 1$ or performing an **edge-step** on $G_t(f)$ when $Z_t = 0$.

Given $f$, its partial sum is an important quantity for us and we reserve the letter $F$ to denote it, i.e., $F$ is a function defined as

\[ F(t) := 1 + \sum_{s=2}^{t} f(s). \]

Observe that the edge-step function $f$ is intimately related to the growth of the vertex set. If we let $V_t$ denote the number of vertices added up to time $t$, then

\[ V_t = 1 + \sum_{s=2}^{t} Z_s \approx F(t), \]

since the sequence of random variables $(Z_s)_{s \geq 1}$ is independent. Thus, abusing from the notation for a brief moment, we may write

\[ \frac{dV_t}{dt} = f(t). \]

When the proper machinery has been settled, we will discuss in Section 8 that some regularity should be imposed on $f$ in order to avoid some pathological behaviors. For now, we define a list of conditions we may impose on $f$ at different points of the paper in order to get the proper results. For instance, we say $f$ satisfies condition (D) if it is non-increasing. We define the further conditions:

\[ f \text{ is non-increasing and } \lim_{t \to \infty} f(t) = 0; \]

\[ \sum_{s=1}^{\infty} \frac{f(s)}{s} < \infty; \]
\( (L_\kappa) \quad \sum_{s=t/13}^t \frac{f(s)}{s} < (\log t)^\kappa, \) for all \( t \in \mathbb{N} \) and some \( \kappa \in (0,1) \);

\( (RV_\gamma) \quad \exists \gamma, \) such that \( \forall a > 0, \lim_{t \to \infty} \frac{f(at)}{f(t)} = \frac{1}{a^\gamma}. \)

We must point out that for modeling purposes, conditions (D) and (D\(_0\)) may be desirable. For instance, in the context of social networks, these conditions assure that the rate at which new individuals join the network is decreasing as the size of the network increases. Whereas, conditions (S) and (L\(_\kappa\)) are related to the order of the maximum degree of \( G_t(f) \). In \([2]\), the authors point out that the maximum degree at time \( t \) should be of order \( t \cdot \exp \left\{ -\frac{1}{2} \sum_{s=2}^t \frac{f(s)}{s-1} \right\} \).

A function satisfying condition (RV\(_\gamma\)) is called regularly varying at infinity and the exponent \( \gamma \) is called the index of regular variation. Functions in this class are well-studied in mathematics in many contexts and a variety of asymptotic results for them and their integrals is known due mainly to the theory developed by Karamata, see \([8]\) for a complete reference.

In general, we may say that this paper investigates how sensitive some graph observables are to changes of \( f \) and aims at a general characterization of such observables for a class of functions \( f \) that is as wide as possible.

1.2. Robustness and large cliques. As said before, one of the main questions concerns the robustness of the network or how vulnerable the network is to spread of a disease \([6]\) or to deliberate/random attacks \([10]\). Regarding the spread of rumors of diseases some graph substructures play important roles, such as stars, triangles and cliques (complete subgraphs). The latter is also related to the robustness of the network, since the existence of a large clique may let the network less vulnerable to attacks aiming at edge-deletion.

In \([7]\), the authors give lower and upper bounds for the clique number in a uncorrelated PA-model. Their bounds are polynomial on the number of vertices. In the same direction, in \([21]\) the authors prove some sort of phase transition on the clique number of a random graph \( G(n, \alpha) \) on \( n \) vertices and degree sequence obeying a power-law with exponent \( \alpha \). For their model, when \( \alpha > 2 \) the clique number is of constant size, whereas for heavy tails, when \( 0 < \alpha < 2 \), \( \omega(G(n, \alpha)) \) is a power of \( n \).

In this matter, we prove the existence \( a.a.s \) of large cliques whose order depends essentially on the definite integral of \( f \). The result is formally stated on the theorem below

**Theorem 1** (Large cliques). Let \( f \) be an edge-step function satisfying condition (D\(_0\)). Then, for every \( \delta \in (0,1) \), there exist a positive constant \( C \) depending on \( \delta \) only such that

\[
\mathbb{P} \left( \exists K_n \subset G_t(f), \text{ such that } n \geq CF(t^{\frac{1}{\delta^2}}) \right) = 1 - o(1),
\]
where $K_n$ denotes a subgraph isomorphic to the complete graph with $n$ vertices.

For edge-step functions which are also slowly varying varying the above theorem can provide the right order of largest clique of $G_t(f)$, i.e., the so-called clique number, denoted by $\omega(G_t(f))$. In the more general case of a regularly varying function $f$, a non-sharp result can be obtained. We summarize this in the Corollary below

**Corollary 1.4 (Clique number for regular varying functions).** Let $f$ be an edge-step function satisfying conditions $[D_0]$ and $[RV_\gamma]$, for $\gamma \in [0,1)$. Then,

(a) for any $\varepsilon > 0$ and $t$ sufficiently large

$$P\left(t^{(1-\gamma)\frac{1-\varepsilon}{2}} \leq \omega(G_t(f)) \leq 7t^{\frac{1}{2}}\right) \geq 1 - \frac{1}{\log(t)};$$

(b) For $f$ under $(RV_0)$, we have

$$\lim_{t \to \infty} \frac{\log \omega(G_t(f))}{\log F(t^{1/2})} = 1, \text{ almost surely.}$$

1.3. **Shaping the diameter.** Another topological property of graphs which is also related to spread of rumors and connectivity of networks is the diameter, that is, the maximal graph distance between two vertices of said graph. Originally, investigating the diameter of real-world networks, the authors in [25] observed that, although coming from different contexts, those networks usually have diameter of order less than the logarithm of the number of vertices, the so-called small-world phenomena.

In this paper we also address the issue of determining the order of the diameter. Our main goals in this subject are to obtain a characterization for the diameter imposing conditions on $f$ as weak as possible and also to obtain regimes for the diameter arbitrarily small but still preserving the scalefreeness of the graph.

In order to slow the growth of the diameter of PA-models, two observables play important roles: the maximum degree and the proportion of vertices with low degree. The former tends to concentrate connections on vertices with very high degree which acts in the way of shortening the diameter, since they attract connections to them. Whereas the latter, acts in the opposite way. In [28] and [18], the authors have shown that in the configuration model with power-law distribution the diameter order is extremely sensitive to the proportion of vertices with degree 1 and 2.

One way to reduce the effect of low degree vertices on the diameter is via affine preferential attachment rules, i.e., introducing a parameter $\delta$ and choosing vertices with probability proportional to their degree plus $\delta$. In symbols, conditionally on $G_t$, we connect a new vertex $v_{t+1}$ to an existing one $u$ with probability

$$P(v_{t+1} \to u|G_t) = \frac{\text{degree}(u) + \delta}{\sum_{w \in G_t}(\text{degree}(w) + \delta)}.$$
By taking a negative $\delta$, the above rule increases the influence of high degree vertices and indeed decreases drastically the diameter’s order. For instance, for positive $\delta$ the diameter of $G_t$ is at least $\log(t)$, whereas for $\delta < 0$ the diameter of $G_t$ is at most $\log \log t$. See [15] for several results on the diameter of different combinations for the affine preferential attachment rule.

Diminishing the effect of low degree vertices is not enough to break the growth of the diameter completely. The reason for that is, despite their low degree, these vertices exist in large amount. Even the existence of a vertex with degree close to $t$ at time $t$ may not be enough to freeze the diameter’s growth. In [23] the authors have proven that the maximum degree of a modification of the BA-model is of order $t$ at time $t$. However, the authors believe that this is not enough to obtain a diameter of order $\log \log t$, the reason being that this large hub still has to compete with a large number of low degree vertices.

1.3.1. General bounds for the diameter. As said before, our goal is to develop bounds for the diameter of $G_t(f)$ with $f$ as general as possible. Under the condition of monotonicity, we prove the following lower bound

**Theorem 2** (Lower bound on the diameter). Let $f$ be an edge-step function satisfying condition (D). Then

\[
\mathbb{P} \left( \text{diam}(G_t(f)) \geq \frac{1}{3} \left( \frac{\log t}{\log \log t} \wedge \frac{\log t}{-\log f(t)} \right) \right) = 1 - o(1).
\]

For the upper bound, by a coupling argument, we are able to prove that the diameter of the Barabási-Albert random tree is the ceiling for the diameter generated by any $f$. Requiring more information on $f$, we prove upper bounds that, for a broad class of functions, are of the same order of the lower bounds given by the previous theorem. This is all summarized in the Theorem below.

**Theorem 3** (Upper bound on the diameter). Let $f$ be an edge-step function. Then

(a) $\text{diam}(G_t(f))$ is at most the diameter of the Barabási-Albert random tree, i.e.,

\[
\mathbb{P} \left( \text{diam}(G_t(f)) \leq \log t \right) = 1 - o(1).
\]

(b) if $f$ also satisfies condition (S) then there exists a positive constant $C_1$ such that

\[
\mathbb{P} \left( \text{diam}(G_t(f)) \leq 2 + 6 \left( \frac{\log t}{-\log \left( \sum_{s=t+1}^{t+\log \log t} \frac{f(s)}{s-1} \right)} \wedge \frac{\log t}{\log \log t} \right) \right) \geq 1 - C_1 t^{-144};
\]

(c) if $f$ satisfies condition (L$\kappa$) then there exists a positive constant $C_2$ such that

\[
\mathbb{P} \left( \text{diam}(G_t(f)) \leq 2 + \frac{6}{1 - \kappa \log \log t} \log t \right) \geq 1 - C_2 t^{-144}.
\]
1.3.2. The class regularly varying functions. In [2], the authors prove a characterization of the empirical degree distribution of graphs generated by \( f \) satisfying condition (RV\(_\gamma\)), for \( \gamma \in [0,1) \). More specifically, they prove that the degree distribution of such graphs obeys a power law distribution whose exponent depends only on the index of regular variation \( -\gamma \).

A byproduct of our general bounds is a similar characterization for the diameter. For edge-step functions satisfying conditions (D\(_0\)) and (RV\(_\gamma\)) for \( \gamma \in (0,\infty) \) the graphs generated by such functions have constant diameter and its order depends only on the index of regular variation \( -\gamma \). We state this result in the theorem below

\textbf{Theorem 4} (Diameter of regularly varying functions). Let \( f \) be an edge-step function satisfying conditions (D\(_0\)) and (RV\(_\gamma\)), for \( \gamma \in (0,\infty) \). Then,

\[ \mathbb{P} \left( \frac{1}{4\gamma} \leq \text{diam}(G_t(f)) \leq \frac{100}{\gamma} + 2 \right) = 1 - o(1). \]

1.3.3. The class of slowly varying functions. The case when \( \gamma = 0 \) is richer in terms of possible orders of the diameter and does not admit a nice characterization as the one we obtain for positive \( \gamma \). In this settings, we present another consequence of our bounds for particular subclasses of the class of slowly varying functions. Let us first define the subclass of functions and later state how our results fit these specific classes.

\begin{align*}
(1.6) \quad L & := \left\{ \text{f is an edge-step function such that } \frac{1}{\log^\alpha(t)}, \text{ for some } \alpha > 0 \right\}; \\
(1.7) \quad E & := \left\{ \text{f is an edge-step function such that } e^{-\log^\alpha(t)}, \text{ for some } \alpha \in (0,1) \right\}.
\end{align*}

It is straightforward to verify that functions belonging to the set above defined are slowly varying. For functions belonging to the two subclasses \( L \) and \( E \), our results have the following consequences, verifiable through elementary calculus,

\textbf{Corollary 1.8.} Let \( f \) be an edge-step function.

(a) if \( f \) belongs to \( L \), with \( \alpha \leq 1 \), then

\[ \mathbb{P} \left( \frac{1}{3\log \log t} \leq \text{diam}(G_t(f)) \leq \frac{8}{\alpha \log \log t} \right) = 1 - o(1); \]

(b) if \( f \) belongs to \( L \), with \( \alpha > 1 \), then

\[ \mathbb{P} \left( \frac{1}{3\alpha \log \log t} \leq \text{diam}(G_t(f)) \leq \frac{7}{\alpha - 1 \log \log t} \right) = 1 - o(1); \]

(c) if \( f \) belongs to \( E \), then

\[ \mathbb{P} \left( C_\alpha^{-1}(\log t)^{1-\alpha} \leq \text{diam}(G_t(f)) \leq C_\alpha(\log t)^{1-\alpha} \right) = 1 - o(1), \]

for some \( C_\alpha \geq 1 \).
1.4. Comparing $G_t(f)$ and $G_t(h)$. If two edge-step functions are “close” to each other in some sense, then one should expect that the random processes they generate should be “close” as well. We make precise this intuition in the theorem below. In words, if $f$ and $h$ are close in the $L_1(N)$-norm (denoted by $\|\cdot\|_1$), then $\text{Law}(\{G_t(f)\}_{t \geq 1})$ is close to $\text{Law}(\{G_t(h)\}_{t \geq 1})$ in the total variation distance, denoted by $\text{dist}_{TV}(\cdot, \cdot)$.

**Theorem 5.** Consider $f$ and $h$ two edge-step functions. We have

$$\text{dist}_{TV}(\text{Law}(\{G_t(f)\}_{t \geq 1}), \text{Law}(\{G_t(h)\}_{t \geq 1})) \leq \|f - h\|_1.$$  

In particular, if $(f_n)_{n \in \mathbb{N}}$ is a sequence of edge-step functions, then

$$f_n \xrightarrow{L_1(n)} f \implies \text{Law}((G_t(f_n))_{t \geq 1}) \xrightarrow{\text{dist}_{TV}} \text{Law}((G_t(f))_{t \geq 1}).$$

The above theorem may be read as a perturbative statement. It assures that we may add a small noise $\epsilon = \epsilon(t)$ to an edge-step function $f$ and still obtain the same process up to an error of at most $\|\epsilon\|_1$ in total variation distance.

1.5. Main technical ideas. In order to prove the existence of some given subgraph in the (affine) BA-random graphs a key ingredient is usually to use the fact that two given vertices $v_i$ and $v_j$ may be connected only at one specific time-step, since (assuming $i < j$) the model’s dynamic only allows $v_j$ to connect to $v_i$ at the moment in which $v_j$ is created. This property facilitates the computation of the probability of the occurrence of a given subgraph and decreases the combinatorial complexity of the arguments. In [9, 16] the authors estimate the number of triangles and cherries (paths of length 3) on the (affine) BA-model and their argument relies heavily on this feature of the model. In our case, however, the edge-step prevents an application of such arguments, since a specific subgraph may appear at any time after the vertices have been added.

Another difficulty in our setup is the degree of generality we work with. Our case replaces the parameter $p \in (0, 1]$ in the models investigated in [1, 12, 13] by any non-negative real function $f$ with $\|f\|_{\infty} \leq 1$. The introduction of such function naturally increases the complexity of any analytical argument one may expect to rely on and makes it harder to discover threshold phenomena. This is the reason why in our work the Karamata’s Theory of regularly varying functions is crucial in order to prove sharper results.

In order to overcome the issues presented above, more specifically to prove Theorem 4 and Theorem 5 we construct an auxiliary process that we call the doubly-labeled random tree process, $\{T_t\}_{t \geq 1}$. In essence, this process is a realization of the traditional BA-model (obtained in our settings choosing $f \equiv 1$) where each vertex has two labels attached to it. We then show in Proposition 2.1 how to generate $G_t(f)$ from $T_t$ using the information on those labels. This procedure allows us to generate $G_t(f)$ and $G_t(h)$, for two distinct functions $f$ and $h$, from exactly the same source of randomness. The upshot is that an edge-step function may be seen as a map from the space of doubly-labeled trees to the space of (multi)graphs – therefore it makes sense to use the notation $f(T_t)$. Furthermore, this map has the crucial
property of being *monotonic* (in a way we make precise latter). Roughly speaking, if \( f \leq h \), then certain monotonic graph observables respect this order, so if \( \zeta \) is such an observable than \( \zeta(f(T_t)) \leq \zeta(h(T_t)) \). In Proposition 2.5 we give important examples of suitable monotonic graph observables, the diameter being one of them. Our machinery then allows us to transpose some results about graphs generated for functions in a particular regime to another just by comparing the functions themselves. We use known results about cliques when \( f \) is taken to be a constant less than one to propagate this result down other regimes of functions.

For the proof of Theorem 2, we apply the second moment method on the number of *isolated paths*. This approach demands correlation estimations for the existence of two such paths in \( G_t(f) \), which we do only under the assumption of \( f \) being monotonic. For Theorem 3 we apply a lower bound for the degree of earlier vertices which is obtained by estimation of negative moments of a given vertex’s degree, and then show that, under conditions \([L_\kappa]\) or \([L_\kappa]\), long paths of younger vertices are unlikely and older vertices are all very close in graph distance. Finally, using results from the Karamata’s theory of regularly varying functions, we verify that this broad class of functions satisfies our assumptions, proving Theorem 4.

1.6. Organization. In Section 2 we introduce the doubly-labeled random tree process and prove the main results about it and as a consequence of this results, we obtain Theorem 5. In Section 3 we explore the theory developed in the previous section to prove the existence of large cliques, i.e., Theorem 1. Then, in Section 4 we prove technical estimates for the degree of a given vertex, which is needed for the upper bound on the diameter. Section 5 is devoted to the general lower bound for the diameter, i.e., for the proof of Theorem 2. We prove the upper bound for the diameter, Theorem 3 in Section 6. Finally, in Section 7 we show how our results fit on the class of the regularly varying functions and subclasses of slowly varying functions. We end the paper at Section 8 with some comments on the affine version of our model and a brief discussion on what may happen to the model if some regularity conditions are dropped.

1.7. Notation. We let \( V(G_t(f)) \) and \( E(G_t(f)) \) denote the set of vertices and edges of \( G_t(f) \), respectively. Given a vertex \( v \in V(G_t(f)) \), we will denote by \( D_t(v) \) its degree in \( G_t(f) \). We will also denote by \( \Delta D_t(v) \) the *increment* of the discrete function \( D_t(v) \) between times \( t \) and \( t+1 \), that is,

\[
\Delta D_t(v) = D_{t+1}(v) - D_t(v).
\]

When necessary in the context, we may use \( D_G(v) \) to denote the degree of \( v \) in the graph \( G \).

Given two sets \( A, B \subseteq V(G_t(f)) \), we let \( \{A \leftrightarrow B\} \) denote the event where there exists an edge connecting a vertex from \( A \) to a vertex from \( B \). We denote the complement of this event by \( \{A \not\leftrightarrow B\} \). We let \( \text{dist}(A,B) \) denote the graph distance between \( A \) and \( B \), i.e. the minimum number of edges that a path that connects \( A \) to \( B \) must have. When one of these
subsets consists of a single vertex, i.e. \( A = \{ v \} \), we drop the brackets from the definition and use \( \{ v \leftrightarrow B \} \) and \( \text{dist}(v, B) \), respectively.

For \( t \in \mathbb{N} \), we let \([t]\) denote the set \( \{1, \ldots, t\} \).

Regarding constants, we let \( C_1, C_2, \ldots \) and \( c, c_1, c_2, \ldots \) be positive real numbers that do not depend on \( t \) whose values may vary in different parts of the paper. The dependence on other parameters will be highlighted throughout the text.

Since our model is inductive, we use the notation \( \mathcal{F}_t \) to denote the \( \sigma \)-algebra generated by all the random choices made up to time \( t \). In this way we obtain the natural filtration \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \) associated to the process.

2. The doubly-labeled random tree process

In this section we introduce a stochastic process \( \{ \mathcal{T}_t \}_{t \geq 1} \) that provides a grand coupling between the random graphs \( \{ G_t(f) \}_{t \geq 1} \) for every edge-step function \( f \).

The process \( \{ \mathcal{T}_t \}_{t \geq 1} \) is essentially a realization of the Barabási-Albert random tree where each vertex has two labels: an earlier vertex chosen according to the preferential attachment rule and an independent uniform random variable. The label consisting in the earlier vertex can be seen as a “ghost directed edge”, we later use these random labels to collapse subsets of vertices into a single vertex in order to obtain a graph with the same distribution as \( G_t(f) \) for any prescribed function \( f : \mathbb{N} \to [0, 1] \).

We begin our process with a graph \( \mathcal{T}_1 \) consisting as usual in a single vertex and a single loop connecting said vertex to itself. We then inductively construct the labeled graph \( \mathcal{T}_{t+1} \) from \( \mathcal{T}_t \) in the following way:

![Diagram of the process \( \{ \mathcal{T}_t \}_{t \geq 1} \) up to time 6 without the uniform labels. The dashed lines indicate the label \( \ell(v_j) \) taken by each vertex \( v_j \).](image)
(i) We add to $T_t$ a vertex $v_{t+1}$;
(ii) We tag $v_{t+1}$ with a random label $\ell(v_{t+1})$ chosen from the set $V(T_t)$ with the preferential attachment rule, that is, with probability of choosing $u \in V(T_t)$ proportional to the degree of $u$ in $T_t$;
(iii) Independently from the step above, we add an edge $\{w,v_{t+1}\}$ to $E(T_t)$ where $w \in V(T_t)$ is also randomly chosen according to the preferential attachment rule, see Figure 1.

We then finish the construction by tagging each vertex $v_j \in V(T_t)$ with a second label consisting in an independent random variable $U_j$ with uniform distribution on the interval $[0,1]$, as shown in Figure 2. We note that only the actual edges contribute to the degree taken in consideration in the preferential attachment rule, the tags are not considered.

![Figure 2](image)

**Figure 2.** The graph $T_6$, now each vertex receives an independent uniform random variable.

We now make precise the notation $f(T_t)$ which indicates that an edge-step function $f$ may also be seen as a function that maps a doubly-labeled tree to a (multi)graph. Our goal is to define this map in such way that $f(T_t) \xrightarrow{d} G_t(f)$.

In order to do so, let us fix such a function $f$. Given $v_j \in V(T_t)$, we compare $U_j$ to $f(j)$. If $U_j \leq f(j)$, we do nothing. Otherwise, we collapse $v_j$ onto its label $\ell(v_j)$, that is, we consider the set $\{v_j, \ell(v_j)\}$ to be a single vertex with the same labels as $\ell(v_j)$. We then update the label of all vertices $v$ such that $\ell(v) = v_j$ to $\{v_j, \ell(v_j)\}$. This procedure is associative in the sense that the order of the vertices on which we perform this operation does not affect the
final resulting graph, as long as we perform it for all the vertices of \( T_t \). We let then \( f(T_t) \) be the (multi)graph obtained when this procedure has run over all the \( t \) vertices of \( T_t \). We refer to Figure 3 as an illustration of the final outcome.

In the next proposition we prove that \( f(T_t) \) is indeed distributed as \( G_t(f) \).

**Figure 3.** The figure shows how one can sample the distribution of \( G_6(f) \) using the labeled graph \( T_6 \).

**Proposition 2.1.** Let \( T_t \) be the doubly-labeled tree above defined and \( f \) an edge-step function. Then,

\[
f(T_t) \overset{d}{=} G_t(f).
\]

**Proof.** We first observe that the associativity of the collapsing operation and the independence of the sequence \( (U_j)_{j \geq 1} \) from the previous operations imply that we can glue together the vertex \( v_{t+1} \) to \( \ell(v_{t+1}) \) whenever \( U_{t+1} > f(t+1) \) right after we complete step (iii) of the above construction by induction. The resulting graph has either a new vertex \( v_{t+1} \) with an edge \( \{w(v_{t+1}), v_{t+1}\} \) or an edge \( \{w(v_{t+1}), \ell(v_{t+1})\} \) with the exact same probability distribution as the \((t + 1)\)-th step in the construction of the graph \((G_t(f))_{t \geq 1}\). By induction, both random graphs have the same distribution.

In the light of the above discussion and Proposition 2.1, from now on, we will tacitly assume that all process \( \{G_t(f)\}_{t \geq 1} \) for all edge-step functions are on the same probability space provided by our previous results.

A straightforward consequence of Proposition 2.1 is Theorem 5 which roughly speaking states that edge-step functions close to each other in the \( L_1(\mathbb{N}) \)-norm generates essentially the same processes. Once we have the machinery provided by Proposition 2.1, the proof of this fact becomes a simple application of the union bound.
Proof of Theorem 5. Fixed two edge-step functions $f$ and $h$, by Proposition 2.1 we have that

\begin{equation}
\text{Law}(\{G_t(f)\}_{t \geq 1}) = \text{Law}(\{f(T_t)\}_{t \geq 1}).
\end{equation}

Thus

\[
\text{dist}_{TV}(\text{Law}(\{G_t(f)\}_{t \geq 1}), \text{Law}(\{G_t(h)\}_{t \geq 1})) \leq \sum_{i=1}^{\infty} P(U_i \in (f(i) \land h(i), f(i) \lor h(i)))
\]

\[
= \sum_{i=1}^{\infty} |f(i) - g(i)|
\]

\[
= \|f - g\|_1,
\]

and the above equation immediately implies \((1.10)\).

The next step is to use the coupling provided by the auxiliary process \(\{T_t\}_{t \geq 1}\) to compare graph observable of graphs generated by different edge-step functions. This method will allow us to transport results we have obtained for a fixed $f$ to other edge-step functions by comparing them with $f$. To do this we introduce the notion of increasing (decreasing) graph observable.

Definition 1 (Monotone graph observable). We say that a (multi)graph observable $\zeta$ is increasing if, given two edge-step functions $f$ and $h$, we have that

\begin{equation}
f(s) \leq h(s), \forall s \leq t \implies \zeta(f(T_t)) \leq \zeta(h(T_t)) \text{ a.s.}
\end{equation}

When the second inequality in \((2.4)\) holds with “\(\geq\)”, we say $\zeta$ is decreasing.

A first example of an increasing observable is the total number of vertices. Indeed, since every vertex $v_j$ of $T_t$ remain preserved under $f$ whenever its $U_j$ label is less than $f(j)$, it follows that every $f$-preserved $v_j$ is $h$-preserved as well.

The next proposition states that the maximum degree is a decreasing observable whereas the diameter, in which we are interested in, is an increasing one.

Proposition 2.5. Let $f$ and $h$ be two edge-step functions satisfying the relation $f(s) \leq h(s)$ for all $s \in (1, \infty)$. Then,

(a) the maximum degree is a decreasing graph observable. More precisely, if we let $D_{\text{max}}(G)$ denote the maximum degree of a (multi)graph $G$, then

\[
P(\forall t \in \mathbb{N}, D_{\text{max}}(G_t(f)) \geq D_{\text{max}}(G_t(h))) = 1;
\]

(b) the diameter is an increasing graph observable. More precisely,

\[
P(\forall t \in \mathbb{N}, \text{diam}(G_t(f)) \leq \text{diam}(G_t(h))) = 1
\]
Proof. The proofs follow from an analysis of the action of \( f \) over \( T_t \).

Proof of part (a): We first point out that if \( v_s \) is a vertex of \( T_t \) whose \( U_s \)-label is less than \( f(s) \), then
\[
D_{G_t(f)}(v_s) \geq D_{G_t(h)}(v_s).
\]
(2.6)
To see why the above inequality is true, notice that the degree of \( v_s \) in \( G_t(f) \) is the total number of solid edges (see Figure 2) incident on \( v_s \) in \( T_t \) plus the total number of vertices whose \( \ell \)-label points to \( v_s \) and whose \( U \)-label is greater than \( f \). Formally,
\[
D_{G_t(f)}(v_s) = \sum_{r=1}^{t} \{ v_r \rightarrow v_s \text{ in } T_t \} + \{ \ell(v_r) = v_s, U_r \geq f(r) \}.
\]
(2.7)
The contribution of the first term in the RHS of the above identity remains stable under \( f \) and \( h \), however, being \( f(r) \leq h(r) \) for all \( r \leq t \), we automatically have
\[
\{ \ell(v_r) = v_s, U_r \geq f(r) \} \geq \{ \ell(v_r) = v_s, U_r \geq h(r) \},
\]
which implies that \( D_{G_t(f)}(v_s) \geq D_{G_t(h)}(v_s) \).

To finish the proof of this part, notice that, if \( v_s \) is truly a vertex in \( G_t(h) \) but is glued to its \( \ell \)-label under \( f \), this operation just increases the maximum degree in the sense that the degree of \( \ell(v_s) \) inherits all the contributions of \( v_s \). Thus, if \( v_{\max} \) is a vertex that achieves \( D_{\max}(G_t(h)) \), then if \( v_{\max} \) is also a vertex of \( G_t(f) \) by the above discussion, its degree is at least \( D_{\max}(G_t(h)) \). On the other hand, if it is identified to \( \ell(v_{\max}) \), then the degree of \( \ell(v_{\max}) \) is also at least the maximum degree of \( G_t(h) \).

Proof of part (b): Observe that if \( u \) and \( w \) are vertices whose \( U \)-label is less than \( f \) and such that \( u \leftrightarrow w \) in \( G_t(f) \), then \( u \) and \( w \) are possibly not connected by a single edge in \( G_t(h) \), in other words, if \( \text{dist}_{G_t(f)}(u, w) = 1 \) then \( \text{dist}_{G_t(h)}(u, w) \geq 1 \). Therefore, if \( v_0 = u, v_1, \ldots, v_k = w \) is a minimal path in \( G_t(f) \), by the previous observation, this path induces a path in \( G_t(h) \) whose length is equal to or greater than \( k \).

On the other hand, if \( u = v_0, v_1, \ldots, v_k = w \) is a path realizing the minimal distance between \( u \) and \( w \), in \( G_t(h) \) we have that this path induces another path in \( G_t(f) \) of same length or less. To see this, just notice that, if \( U_j > f(j) \) and \( \ell(v_j) \) points to a vertex outside the path for some \( j \in \{2, 3, \ldots, k-1\} \), then, after identifying \( v_j \) to its \( \ell \)-label, this operation does not increase the path length. This is enough to conclude the proof.

\[\square\]

3. LARGE CLIQUES: proof of Theorem 1

In this section we keep exploring the machinery developed in previous sections. Here, we transpose the existence of large cliques using the edge-step function, i.e., if \( h \) is such that \( G_t(h) \) has a clique of order \( K \) and \( f \) another function satisfying \( f(s) \leq h(s) \) for all \( s \in (1, \infty) \), then the clique existence propagates to \( G_t(f) \) too but with a possibly \( f \)-dependent order.
Our strategy to prove Theorem \[ \text{I} \] will be to apply the results obtained for the function identically equal to \( p \), with \( p \in (0, 1) \) (which we denote by \( \text{cons}_p \) in \[ \text{I} \]) and propagate them to smaller functions. For the sake of the reader’s convenience, we state and comment the aforementioned results here. Given \( m \in \mathbb{Z}_+ \), we order the vertices of \( G_t(\text{cons}_p) \) from oldest to earliest and then divide them into blocks of size \( m \). We then denote by \( d_{t,m}(j) \) the sum of all degrees of the vertices from the \( j \)-th block. The following theorem has a very involved notation, but in essence it provides with high probability an explicit polynomial lower bound for \( d_{t,m}(j) \).

**Theorem** (Theorem 2 from \[ \text{I} \]). Given \( p \in (0, 1] \), let \( \xi \in (0, 2(2-p)^{-1}-1) \) and fix \( m \in \mathbb{Z}_+ \) sufficiently large. Define \[ \zeta_m := \frac{(1-p)}{2(2-p)m^\xi}. \]

and let \[ 1 < R < m(1-p/2)(1-\zeta_m). \]

There exists a positive constant \( c = c(m, R, p) \) such that, for \( \beta \in (0, (1-p/2)(1-\zeta_m)) \) and \( j \geq m^{2-R} + 1 \), we have

\[
\mathbb{P}(d_{t,m}(j) < t^3) \leq c \frac{j^R}{t^{R - \beta R(1-p/2)-1(1-\zeta_m)}} + \frac{m}{([j-1]m)^{99}}.
\]

Using the above bound one can then prove

**Theorem** (Theorem 1 from \[ \text{I} \]). For any \( \varepsilon > 0 \) and every \( p \in (0, 1) \), it follows that

\[
\mathbb{P}(\exists K_n \subset G_t(\text{cons}_p), \text{ such that } n = t(1-\varepsilon)^{(1-p)/(2-p)}) = 1 - o(1).
\]

We can now use the doubly-labeled tree together with the above results in order to prove Theorem \[ \text{I} \]. It will be useful to recall our special notation to the expected number of vertices,

\[
F(t) := \mathbb{E}(V(G_t(f))) = 1 + \sum_{s=2}^t f(s).
\]

**Proof of Theorem \[ \text{I} \]** Given \( \delta > 0 \) sufficiently small, let \( \varepsilon, p = \delta/6 \) so that

\[
\alpha = \alpha(p, \varepsilon) := \frac{1-p}{2-p}(1-\varepsilon) > \frac{1}{2} \left(1 - \frac{\delta}{2}\right).
\]

In the proof of Theorem 1 of \[ \text{I} \] one uses Theorem 2 of \[ \text{I} \] to show that there exist a fixed integer \( m = m(\varepsilon, p) > 0 \) and a small number \( \varepsilon' \in (0, \alpha) \) with the following property: if one divides the set of vertices born between times \( t^{\varepsilon'} \) and \( t^\alpha \) into disjoint subsets of \( m \) vertices born consecutively, then with high probability (at least 1 minus a polynomial function of \( t \)) one can choose a vertex from each of these subsets in such a way that the subgraph induced by the set of chosen vertices is a complete subgraph of \( G_t(\text{cons}_p) \).
Having the above in mind and letting \( m \) be the auxiliary \( m \) that appears in proof of Theorem 1 of [1], define the following event

\[
A_{p,\varepsilon',m,k} := \{ \exists \{t_1, \ldots, t_k\} \subseteq [t] \text{ such that } t_j \in (t^\varepsilon' + (j-1)m, t^\varepsilon' + jm] \text{ and } v_{t_j} \leftrightarrow v_i \text{ in } G_t(\text{cons}_p) \text{ for all } i, j \in [k] \}.
\]

In the context of the doubly labeled tree process, the event \( A_{p,\varepsilon',m,k} \) says that we may find \( k \) vertices of \( T_t \) with the property that the \( j \)-th vertex \( v_{t_j} \) was added by the doubly-labeled tree process sometime in the interval \( (t^\varepsilon' + (j-1)m, t^\varepsilon' + jm] \) and its \( U_{t_j} \)-label is less than \( p \). Moreover, when we apply \( \text{cons}_p \) on \( T_t \) in the event \( A_{p,\varepsilon',m,k} \), all these \( k \) vertices form a complete graph in the resulting graph \( G_t(\text{cons}_p) \). Theorem 1 of [1] states that setting \( k = t^\alpha - t^\varepsilon' \), the event \( A_{p,\varepsilon',m,k} \) occurs with probability at least \( 1 - t^{-\eta} \), for some positive small \( \eta \) depending on \( p \), which, in our case, is a function of \( \delta \). Thus, for \( t \) large enough, we may simply use that

\[
\mathbb{P}(A_{p,\varepsilon',m,t^\alpha}) \geq 1 - \frac{1}{\log t^\delta},
\]

this bound will be useful for the proof of Corollary 1.4. This is why we are exchanging a polynomial decay by a log one, to get rid off the dependency on \( \delta \) and consequently simplify latter arguments.

Now, let \( t \) be large enough so that \( f(t^\varepsilon') < p \). This is possible since \( f \) decreases to zero. Also notice that if \( u \) and \( v \) are vertices of \( G_t(f) \) added after time \( t^\varepsilon' \) and are connected in \( G_t(\text{cons}_p) \), then they are connected in \( G_t(f) \) as well. Moreover, since \( f \) is non-increasing and the \( U \)-labels are assigned independently and according to a uniform distribution on \([0, 1]\), we have for all \( j \leq k \)

\[
\mathbb{P}(U_{t_j} \leq f(t_j) \mid A_{p,\varepsilon',m,k}) \geq p^{-1} f(t^\varepsilon' + jm).
\]

Setting \( k = (t^\alpha - t^\varepsilon')/m \) and using that \( f \) is non-increasing, we have that

\[
m \sum_{j=0}^{k} f(t^\varepsilon' + jm) \geq F(t^\alpha) - F(t^\varepsilon' + m) \geq F(t^\alpha)/2
\]

for large enough \( t \).

By the independence of the \( U \)-labels, it follows that conditioned on \( A_{p,\varepsilon',m,k} \), the random variable that counts how many vertices of the clique in \( G_t(\text{cons}_p) \) remain vertices of \( G_t(f) \) as well is a sum of \( k \) independent random variables taking values on \([0, 1]\) and whose expected value is greater than \( F(t^\alpha)/2mp \). Finally, by Chernoff bounds, this random variable is at least half its expected value with probability at least \( 1 - \exp\{-F(t^\alpha)/16mp\} \). This and (3.5) gives us that there exists a constant \( C_\delta \) such that

\[
\mathbb{P}\left( \hat{\mathcal{K}}_n \subset G_t(f), \text{ such that } n \geq C_\delta F(t^{1-\delta}) \right) \leq \frac{1}{\log t} + e^{-F(t^\alpha)/16mp},
\]

which proves the Theorem.
4. Technical estimates for the degree

In this section we develop technical estimates related to the degree of a given vertex. We begin by stating one of the most fundamental identities in the study of preferential attachment models: the conditional distribution of the increment of the degree of a given vertex. Given \( v \in V(G_t(f)) \), we have

\[
\begin{align*}
\mathbb{P}(\Delta D_t(v) = 0|\mathcal{F}_t) &= f(t+1) \left( 1 - \frac{D_t(v)}{2t} \right) + (1 - f(t+1)) \left( 1 - \frac{D_t(v)}{2t} \right)^2, \\
\mathbb{P}(\Delta D_t(v) = 1|\mathcal{F}_t) &= f(t+1) \frac{D_t(v)}{2t} + 2(1 - f(t+1)) \frac{D_t(v)}{2t} \left( 1 - \frac{D_t(v)}{2t} \right), \\
\mathbb{P}(\Delta D_t(v) = 2|\mathcal{F}_t) &= (1 - f(t+1)) \frac{D_t(v)^2}{4t^2}. 
\end{align*}
\]

(4.1)

To see why the above identities hold true, observe for example that in order for \( \Delta D_t(v) = 0 \), either a vertex step was taken, and the vertex did not connect to \( v \), or an edge step was taken and neither of the endpoints of the new edge connected to \( v \). The other equations follow from analogous reasonings. As a direct consequence, we obtain

\[
\mathbb{E}[\Delta D_t(v)|\mathcal{F}_t] = 1 \cdot f(t+1) \cdot \frac{D_t(v)}{2t} + 1 \cdot 2(1 - f(t+1)) \frac{D_t(v)}{2t} \left( 1 - \frac{D_t(v)}{2t} \right) + 2 \cdot (1 - f(t+1)) \frac{D_t(v)^2}{4t^2}.
\]

(4.2)

Using the above equation repeatedly one obtains, conditioned on the event where the vertex \( v \) is born at time \( t_0 \),

\[
\mathbb{E}[D_t(v)] = \mathbb{E}[\mathbb{E}[D_t(v)|\mathcal{F}_{t-1}]] = \left( 1 + \frac{1}{t-1} - \frac{f(t)}{2(t-1)} \right) \mathbb{E}[D_{t-1}(v)] = \prod_{s=t_0}^{t-1} \left( 1 + \frac{1}{s} - \frac{f(s+1)}{2s} \right). 
\]

(4.3)

We state a lower bound for the degree, whose proof comes from a direct application of Theorem 2 of [1] and Proposition 2.5, which assures that the maximum degree is a decreasing graph observable.

**Lemma 1.** Let \( f \) be an edge-step function such that \( f(t) \to 0 \) as \( t \) goes to infinity. Then, for any fixed \( \varepsilon > 0 \) we have

\[
\mathbb{P}(D_{\text{max}}(G_t(f)) < t^{1-\varepsilon}) \leq t^{-2}.
\]
Proof. By Theorem 2 of [1], w.h.p, \( D_{\text{max}}(G_t(p)) \) is at least \( t^{(1-\varepsilon)(1-p/2)} \), for fixed \( \varepsilon > 0 \) and \( p \in (0, 1) \). Thus, taking \( p_0 \) small enough, we have that \( D_{\text{max}}(G_t(p_0)) \) is at least \( t^{1-\varepsilon} \), w.h.p. Then, using Proposition 2.5, we have that for large enough \( t \), \( D_{\text{max}}(G_t(f)) \geq D_{\text{max}}(G_t(p_0)) \) which proves the Lemma.

Our objective now is to obtain a polynomial lower bound for the degree of older vertices, which will be important in the proof of the upper bound for the diameter in Theorem 3. We begin with an upper bound for the expectation of the multiplicative inverse of the degree. Recall the definition of the process \((Z_t)_{t \geq 1}\), consisting of independent Bernoulli variables that dictate whether a vertex-step or an edge-step is performed at time \( t \).

**Lemma 2.** Given any edge-step function \( f \), consider the process \( \{G_t(f)\}_{t \geq 1} \). Denote by \( v_i \) the vertex born at time \( i \in \mathbb{N} \). We have

\[
\mathbb{E}\left[(D_t(v_i))^{-1}Z_i\right] \leq f(i) \left(\frac{t-1}{i}\right)^{-\frac{1}{6}}.
\]

And consequently

\[
\mathbb{P}\left(D_t(v_i) \leq \left(\frac{t-1}{i}\right)^{\frac{1}{12}} Z_i = 1\right) \leq \left(\frac{t-1}{i}\right)^{-\frac{1}{12}}.
\]

Proof. If \( Z_i = 1 \), then for every \( s \geq 1 \) we have that \( \Delta D_s(v_i) \geq 0 \), that \( D_{s+1}(v_i) \leq D_s(v_i)+2 \leq 3D_s(v_i) \), and that \( D_s(v_i) \) is \( \mathcal{F}_s \) measurable. Together with (4.2), these facts imply. on the event \( \{Z_i = 1\},
\]

\[
\mathbb{E}\left[\frac{1}{D_{s+1}(v_i)} - \frac{1}{D_s(v_i)}\right| \mathcal{F}_s] = \mathbb{E}\left[-\frac{\Delta D_s(v_i)}{D_{s+1}(v_i)D_s(v_i)}\right| \mathcal{F}_s]
\]

\[
\leq -\frac{1}{3(D_s(v_i))^2} \left(1 - \frac{f(s+1)}{2}\right) \frac{D_s(v_i)}{s}
\]

\[
\leq -\frac{1}{6sD_s(v_i)},
\]

since \( f(k) \leq 1 \) for very \( k \in \mathbb{N} \). Therefore,

\[
\mathbb{E}\left[Z_i \cdot (D_{s+1}(v_i))^{-1}\right| \mathcal{F}_s] \leq Z_i (1 - (6s)^{-1})(D_s(v_i))^{-1}.
\]

Iterating the above argument from \( i \) until \( t \), we obtain

\[
\mathbb{E}\left[Z_i \cdot (D_t(v_i))^{-1}\right] \leq f(i) \prod_{s=i}^{t-1} \left(1 - \frac{1}{6s}\right) \leq f(i) \exp\left\{-\frac{1}{6} \sum_{s=i}^{t-1} \frac{1}{s}\right\} \leq f(i) \left(\frac{t-1}{i}\right)^{-\frac{1}{6}},
\]
proving (4.4). Equation (4.5) is then obtained by an elementary application of the Markov inequality:

\[
P(D_t(v_i) \leq \left(\frac{t - 1}{i}\right)^{\frac{1}{12}} \middle| Z_i = 1) = P((D_t(v_i))^{-1} \geq \left(\frac{t - 1}{i}\right)^{-\frac{1}{12}} \middle| Z_i = 1) \leq \left(\frac{t - 1}{i}\right)^{\frac{1}{12}} E[(D_t(v_i))^{-1}\middle| Z_i = 1] \leq \left(\frac{t - 1}{i}\right)^{-\frac{1}{12}}.
\]

(4.9)

We now provide an elementary consequence of the above result, which uses the union bound in order to show that, with high probability, every vertex born before time \(t^{\frac{1}{12}}\) has degree at least \(t^{\frac{1}{15}}\) by time \(t\).

**Lemma 3.** Using the same notation as in Lemma 2 we have, for every edge-step function \(f\) and for sufficiently large \(t \in \mathbb{N}\),

\[
P(\exists i \in \mathbb{N}, 1 \leq i \leq t^{\frac{1}{12}}, \text{ such that } Z_i = 1 \text{ and } D_t(v_i) \leq t^{\frac{1}{15}}) \leq Ct^{-\frac{1}{144}}.
\]

(4.10)

**Proof.** By the union bound and equation (4.5), we have that the probability in the left hand side of (4.10) is smaller than or equal to

\[
\sum_{i=1}^{t^{\frac{1}{12}}} P(Z_i = 1, D_t(v_i) \leq t^{\frac{1}{15}}) \leq \sum_{i=1}^{t^{\frac{1}{12}}} f(i) P(D_t(v_i) \leq \left(\frac{t - 1}{i}\right)^{\frac{1}{12}} \middle| Z_i = 1) \leq \sum_{i=1}^{t^{\frac{1}{12}}} \left(\frac{t - 1}{i}\right)^{-\frac{1}{12}} \\
\leq C t^{-\frac{1}{12} + \frac{1}{12}(1 - \frac{1}{12})} \\
\leq C t^{-\frac{1}{144}},
\]

finishing the proof of the Lemma.

\[
\]

5. General lower bound for the diameter: proof of Theorem 2

The proof of Theorem 2 follows a second moment argument. The idea is to count the number of “long” (the specific size depending on \(f\) and \(t\)) isolated paths in \(G_t(f)\). We begin by showing that the expected number of isolated paths goes to infinity with \(t\) in Lemma 4. We then show in Lemma 5 that the presence of a specific isolated path is almost independent from the presence of some other given isolated path whenever said paths are disjoint. This
“almost independence” makes the second moment of the number of such paths very close to the first moment squared. The proof is then completed via the Payley-Zygmund inequality.

We start by defining precisely what we mean by an isolated path.

**Definition 2 (Isolated path).** Let \( l \) be a positive integer. Let \( \vec{t} = (t_1, \ldots, t_l) \) be a vector of distinct positive integers. We say that this vector corresponds to an **isolated path** \( \{v_{t_1}, \ldots, v_{t_l}\} \) in \( G_t(f) \) if and only if:

1. \( t_l \leq t; \)
2. \( t_i < t_j \) whenever \( 1 \leq i < j \leq l; \)
3. during each time \( t_i, i = 1, \ldots, l, \) a vertex-step is performed;
4. for every integer \( k \leq l, \) the subgraph induced by the vertices \( \{v_{t_i}\}_{1 \leq i \leq k} \) is connected in \( G_{t_k}(f); \)
5. for \( i = 1, \ldots, l - 1, \) the degree of \( v_{t_i} \) in \( G_t(f) \) is 2. The degree of \( v_{t_l} \) in \( G_t(f) \) is 1.

In other words, an isolated path \( \{v_{t_i}\}_{1 \leq i \leq l} \) is a path where each vertex \( v_{t_i}, \) for \( i = 2, \ldots, l, \) is born at time \( t_i \) and makes its first connection to its predecessor \( v_{t_{i-1}}. \) Other than that, no other vertex or edge gets attached to \( \{v_{t_i}\}_{1 \leq i \leq l} \). We will denote \( \{v_{t_i}\}_{1 \leq i \leq l} \) by \( v_{\vec{t}}. \)

Given \( \xi \in (0, 1), \) denote by \( S_{t,\xi}(t) \) the set of all isolated paths in \( G_t(f) \) of size \( l \) whose vertices were created between times \( \xi t \) and \( t. \) Our first goal is to obtain lower bounds for \( \mathbb{E} [||S_{t,\xi}(t)||] \):

**Lemma 4.** Let \( f \) be a non-increasing edge-step function, then, for any \( 0 < \xi < 1 \) and any integer \( l, \) the following lower bound holds:

\[
(5.1) \quad \mathbb{E} [||S_{t,\xi}(t)||] \geq \left( \frac{(1 - \xi)t}{l} \right) \frac{f(t)^l}{(2t)^{l-1}} \left( 1 - \frac{2l}{\xi t} \right)^t.
\]

Furthermore, for

\[
(5.2) \quad l \leq \frac{1}{3} \left( \frac{\log t}{\log \log t} \wedge \frac{\log t}{-\log f(t)} \right),
\]

we have that, for sufficiently large \( t, \)

\[
(5.3) \quad \mathbb{E} [||S_{t,\xi}(t)||] \geq t^{\frac{t}{4}}.
\]

**Proof.** The random variable \( |S_{t,\xi}(t)| \) can be written as

\[
(5.4) \quad |S_{t,\xi}(t)| = \sum_{t_1 < t_2 < \cdots < t_l} \mathbbm{1} \{v_\vec{t} \in S_{t,\xi}(t)\} \implies \mathbb{E} [||S_{t,\xi}(t)||] = \sum_{t_1 < t_2 < \cdots < t_l} \mathbb{P} (v_\vec{t} \in S_{t,\xi}(t)).
\]

So it will be important to obtain a proper lower bound for \( \mathbb{P} (v_\vec{t} \in S_{t,\xi}(t)). \) Given a time vector of an isolated path \( \vec{t} = (t_1, \ldots, t_l) \) such that \( \xi t \leq t_1 < t_2 < \cdots < t_l \leq t, \) it follows that

\[
(5.5) \quad \mathbb{P} (v_\vec{t} \in S_{t,\xi}(t)) \geq \frac{f(t)^l}{(2t)^{l-1}} \left( 1 - \frac{2l}{\xi t} \right)^t,
\]
since in order for $v_t$ to be in $S_{l,\xi}(t)$, we need to assure that $l$ vertices are born exactly at times $t_1, \ldots, t_l$ (which happens with probability greater than $f(t)^l$, by the monotonicity of $f$), that $v_{t_i}$ connects to $v_{t_{i-1}}$ for every $i = 2, \ldots, l$ (which happens with probability greater than $(2t)^{-(l-1)}$), and that no other vertex or edge connects to $v_t$ until time $t$ (which happens with probability greater than $\left(1 - \frac{2l}{t}\right)^t$).

Finally, by counting the number of possible ways to choose $t_1 < t_2 < \cdots < t_l$ so that $t_i \in [\xi t, t]$ for $1 \leq i \leq l$, we obtain 5.1.

We now assume $l$ to be such that 5.2 holds. Stirling’s formula gives us

\[
\log \left( \left( \frac{(1-\xi)t}{l} \right) \right) \geq c + (1-\xi)t \log((1-\xi)t) - (1-\xi)t + \frac{\log((1-\xi)t)}{2} - l \log l + \frac{\log l}{2} - ((1-\xi)t - l) \log((1-\xi)t - l) + (1-\xi)t - l - \frac{\log((1-\xi)t - l)}{2},
\]

Since $l \ll t$, we obtain

(5.6) \quad \log \left( \left( \frac{(1-\xi)t}{l} \right) \right) \geq l \log t - l \log l - cl.

Equation (5.5) then implies, again for sufficiently large $t$,

(5.7) \quad \log \left( \mathbb{P}(v_t \in S_{l,\xi}(t)) \right) \geq l \log f(t) - (l-1) \log t - cl.

Combining the above inequality with (5.1), (5.2), and (5.6) gives us that, for large enough $t$,

\[\mathbb{E} \|S_{l,\xi}(t)\| \geq \exp \{l \log t - l \log l + l \log f(t) - (l-1) \log t - cl\} \geq t^{\frac{1}{2}},\]

since

\[l \log l \leq \frac{1}{3} \log \log t \log \log t \left(1 - \frac{\log \log t + \log 3}{\log \log t}\right) \leq \frac{\log t}{3},\]

which finishes the proof of the lemma.

Some new notation will be useful throughout the proof of Theorem 2.

**Definition 3** (Degree of an isolated path). Given an isolated path $v_t = \{v_{t_i}\}_{1 \leq i \leq t}$, we denote by $D_r(v_t)$ the sum of the degrees of each of its vertices at time $r$, i.e. :

\[D_r(v_t) = \sum_{t_i \leq r} D_r(v_{t_i}),\]

where we assumed that $D_r(v_{t_i}) = 0$ if $t_i < r$. 

Note that if \( r > t \) and \( u_r \) has size \( l \), then \( D_s(u_r) = 2l - 1 \). Furthermore, by the same reasoning as in (4.1),

\[
\mathbb{P}(\Delta D_s(v_r) = 0 | \mathcal{F}_s) = f(s + 1) \left( 1 - \frac{D_s(v_r)}{2s} \right) + (1 - f(s + 1)) \left( 1 - \frac{D_s(v_r)}{2s} \right)^2
\]

(5.8)

\[
= 1 - \left( 1 - \frac{f(s + 1)}{2} - (1 - f(s + 1)) \frac{D_s(v_r)}{4s} \right) \frac{D_s(v_r)}{s}.
\]

Paley-Zigmund’s inequality (see e.g. section 5.8 of [22]) assures us that, for any \( 0 \leq \theta \leq 1 \),

\[
\mathbb{P}(\|S_{t,\xi}(t)\| > \theta \mathbb{E}[\|S_{t,\xi}(t)\|]) \geq (1 - \theta)^2 \left( \frac{\mathbb{E}[\|S_{t,\xi}(t)\|]}{\mathbb{E}[\|S_{t,\xi}(t)\|^2]} \right).
\]

(5.9)

If we are able to guarantee that

(i) \( \mathbb{E}[\|S_{t,\xi}(t)\|] \to \infty \);

(ii) \( (\mathbb{E}[\|S_{t,\xi}(t)\|])^2 = (1 - o(1)) \mathbb{E}[\|S_{t,\xi}(t)\|^2] \);

then by choosing \( \theta = \theta(t) = \mathbb{E}[\|S_{t,\xi}(t)\|]^{-1/2} \), we see that

\[
\mathbb{P}\left(\|S_{t,\xi}(t)\| > \mathbb{E}[\|S_{t,\xi}(t)\|]^{1/2}\right) \geq \left(1 - \mathbb{E}[\|S_{t,\xi}(t)\|]^{-1/2}\right)^2 (1 - o(1)) = 1 - o(1),
\]

thus guaranteeing with probability \( 1 - o(1) \) the existence of many isolated paths of length \( l \), finishing the proof of the theorem.

By Lemma 4, we know that item (i) is true. Therefore from now on we will focus on proving item (ii).

In order for the isolated path \( v_r \) to appear the following must happen:

- a vertex \( v_{t_1} \) must be be created at time \( t_1 \), which happens with probability \( f(t_1) \);
- between times \( t_1 + 1 \) and \( t_2 - 1 \) there can be no new connection to \( v_{t_1} \), which, by (5.8), happens with probability

\[
\prod_{r_1 = t_1 + 1}^{t_2 - 1} \left( 1 - \left( 1 - \frac{f(r_1)}{2} - (1 - f(r_1)) \frac{1}{4(r_1 - 1)} \right) \frac{1}{r_1 - 1} \right);
\]

- in general, at time \( t_k \) a vertex \( v_{t_k} \) is created and makes its first connection to \( v_{t_k-1} \), no new connection is then made to \( \{v_j\}_{1 \leq j \leq k} \) between times \( t_k + 1 \) and \( t_{k+1} - 1 \) for every \( k = 2, \ldots, l - 1 \), all this happens with probability equal to

\[
f(t_k) \frac{1}{2(t_k - 1)} \prod_{r_k = t_{k+1}}^{t_{k+1} - 1} \left( 1 - \left( 1 - \frac{f(r_k)}{2} - (1 - f(r_k)) \frac{D_{r_k}(v_{t_k})}{4(r_k - 1)} \right) \frac{D_{r_k}(v_{t_k})}{r_k - 1} \right)
\]

\[
= f(t_k) \frac{1}{2(t_k - 1)} \prod_{r_k = t_{k+1}}^{t_{k+1} - 1} \left( 1 - \left( 1 - \frac{f(r_k)}{2} - (1 - f(r_k)) \frac{2k - 1}{4(r_k - 1)} \right) \frac{2k - 1}{r_k - 1} \right);
\]
• finally, a vertex $v_t$ is born at time $t_l$, connects to $v_{t_{l-1}}$ and no new connection is made to $\{v_{t_j}\}_{1 \leq j \leq l}$ between times $t_l + 1$ and $t$.

This implies

$$P(v_{\vec{t}} \in S_{t,\xi}(t)) = f(t_1) \prod_{r_i = t_1 + 1}^{t_2 - 1} \left(1 - \left(1 - \frac{f(r_1)}{2} - (1 - f(r_1)) \frac{1}{4(r_1 - 1)} \right) \frac{1}{r_1 - 1}\right)$$

$$\times \cdots \times f(t_k) \prod_{r_k = t_1 + 1}^{t_k - 1} \left(1 - \left(1 - \frac{f(r_k)}{2} - (1 - f(r_k)) \frac{2k - 1}{4(r_k - 1)} \frac{1}{r_k - 1}\right) \frac{2k - 1}{r_k - 1}\right)$$

$$\times \cdots \times f(t_l) \prod_{r_l = t_1 + 1}^{t_l - 1} \left(1 - \left(1 - \frac{f(r_l)}{2} - (1 - f(r_l)) \frac{2l - 1}{4(r_l - 1)} \frac{1}{r_l - 1}\right) \frac{2l - 1}{r_l - 1}\right).$$

Given two time vectors $\vec{r}$ and $\vec{t}$, we note that $P(v_{\vec{t}}, v_{\vec{r}} \in S_{t,\xi}(t))$ is only nonzero if $\vec{r}$ and $\vec{t}$ have either disjoint or identical sets of entries. Our focus now is on proving the following lemma:

**Lemma 5.** Let $l$ be such that (5.2) is satisfied. For two isolated paths with disjoint time vectors $\vec{t}$ and $\vec{r}$, we have

$$\frac{P(v_{\vec{t}} \in S_{t,\xi}(t)) P(v_{\vec{r}} \in S_{t,\xi}(t))}{P(v_{\vec{t}}, v_{\vec{r}} \in S_{t,\xi}(t))} = 1 + o(1).$$

**Proof.** To prove the above result we will make a comparison between the two probabilities terms $P(v_{\vec{t}}, v_{\vec{r}} \in S_{t,\xi}(t))$ and $P(v_{\vec{t}} \in S_{t,\xi}(t)) \cdot P(v_{\vec{r}} \in S_{t,\xi}(t))$. We can write both these terms as products in the manner of (5.10). We can then compare the terms from these products associated to each time $s \in [\xi, t]$. There are two cases we must study.

**Case 1:** $s \in \vec{t}$ but $s \notin \vec{r}$ ($s \notin \vec{t}$ but $s \in \vec{r}$).

The product term related to time $s$ in $P(v_{\vec{t}}, v_{\vec{r}} \in S_{t,\xi}(t))$ is

$$f(s) \frac{1}{2(s - 1)},$$

since a new vertex is created and then makes its first connection specifically to the latest vertex of $\vec{t}$. On the other hand, the term related to time $s$ in $P(v_{\vec{t}} \in S_{t,\xi}(t)) \cdot P(v_{\vec{r}} \in S_{t,\xi}(t))$ is

$$f(s) \frac{1}{2(s - 1)} \left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_{\vec{r}})}{4(s - 1)} \frac{1}{s - 1}\right) \frac{D_{s-1}(v_{\vec{r}})}{s - 1}\right),$$
since the term related to \(s\) in the product form of \(\mathbb{P}(v_\tilde{\xi} \in S_{l,\xi}(t))\) continues to be equal to (5.12), but the related term in \(\mathbb{P}(v_{\tilde{\xi}} \in S_{l,\xi}(t))\) is

\[
(5.13) \quad \left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_\tilde{\xi}) + D_{s-1}(v_{\tilde{\xi}})}{4(s-1)} \right) \frac{D_{s-1}(v_\tilde{\xi})}{s-1} \right).
\]

The above expression is the term that will appear regarding the time \(s\) in the fraction in the left hand side of (5.11). This case occurs \(2l\) times since the isolated paths are disjoint. Thus, recalling that \(s \in [\xi t, t]\), \(l \leq 3^{-1} \log(t)/\log(\log(t))\), and that the degree of each isolated path is at most \(2l - 1\), we obtain that there exist constants \(c_1, c_2 > 0\) such that we can bound the product of all the terms of the form (5.13) from above by

\[
\left(1 - \frac{c_1}{t}\right)^{2l},
\]

and from below by

\[
\left(1 - \frac{c_2}{t}\right)^{2l}.
\]

Observe that both products go to 1 as \(t\) goes to infinity.

**Case 2:** \(s \not\in \tilde{\xi}\) and \(s \not\in \tilde{\xi}\).

In \(\mathbb{P}(v_\tilde{\xi} \in S_{l,\xi}(t))\) as well as in \(\mathbb{P}(v_{\tilde{\xi}} \in S_{l,\xi}(t))\) we see terms of the form (5.13), since we must avoid the isolated paths in both events. But in the term related to \(\mathbb{P}(v_\tilde{\xi}, v_{\tilde{\xi}} \in S_{l,\xi}(t))\) we actually observe

\[
\left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_\tilde{\xi}) + D_{s-1}(v_{\tilde{\xi}})}{4(s-1)} \right) \frac{D_{s-1}(v_\tilde{\xi})}{s-1} \right),
\]

since we must guarantee that neither isolated path receives a connection. We note however that

\[
\left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_\tilde{\xi})}{4(s-1)} \right) \frac{D_{s-1}(v_\tilde{\xi})}{s-1} \right)
\]

\[
\times \left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_{\tilde{\xi}})}{4(s-1)} \right) \frac{D_{s-1}(v_{\tilde{\xi}})}{s-1} \right)
\]

\[
= \left(1 - \left(1 - \frac{f(s)}{2} - (1 - f(s)) \frac{D_{s-1}(v_\tilde{\xi}) + D_{s-1}(v_{\tilde{\xi}})}{4(s-1)} \right) \frac{D_{s-1}(v_\tilde{\xi}) + D_{s-1}(v_{\tilde{\xi}})}{s-1} \right)
\]

\[
\times \left(1 + O\left(\frac{l^2}{t^2}\right)\right),
\]

since \(D_{s-1}(v_\tilde{\xi}), D_{s-1}(v_{\tilde{\xi}}) \leq 2l - 1\) and \(s \geq \xi t\). In the fraction in the left hand side of (5.11), we will then have \(\Theta(t)\) terms of the form

\[
\left(1 + O\left(\frac{l^2}{t^2}\right)\right),
\]

But again, as in Case 1, their product goes to 1 as \(t \to \infty\) since \(l^2 = o(t)\). This finishes the proof of the Lemma. \(\square\)
We can finally finish the proof of the theorem.

**Proof of Theorem 2.** Let again $l$ be such that (5.2) is satisfied, and consider $\xi \in (0, 1)$. Since it is impossible for two non disjoint and non equal isolated paths to exist at the same time, we have that

$$\mathbb{E} \left[ |S_{l, \xi}(t)|^2 \right] = \mathbb{E} \left[ \left( \sum_{\tilde{t}} 1 \{ v_{\tilde{t}} \in S_{l, \xi}(t) \} \right) \left( \sum_{\tilde{r}} 1 \{ v_{\tilde{r}} \in S_{l, \xi}(t) \} \right) \right]$$

(5.14)

$$= \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}}, v_{\tilde{r}} \in S_{l, \xi}(t)) + \mathbb{E} \left[ |S_{l, \xi}(t)| \right],$$

and that

$$(\mathbb{E} \left[ |S_{l, \xi}(t)| \right])^2 = \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}} \in S_{l, \xi}(t)) \mathbb{P} (v_{\tilde{r}} \in S_{l, \xi}(t)) + \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}} \in S_{l, \xi}(t)) \mathbb{P} (v_{\tilde{r}} \in S_{l, \xi}(t)).$$

Therefore, by lemmas 3 and 4 we obtain

$$\frac{\mathbb{E} \left[ |S_{l, \xi}(t)|^2 \right]}{(\mathbb{E} \left[ |S_{l, \xi}(t)| \right])^2} = \frac{\sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}}, v_{\tilde{r}} \in S_{l}(t))}{(\mathbb{E} \left[ |S_{l, \xi}(t)| \right])^2} + \frac{\mathbb{E} \left[ |S_{l, \xi}(t)| \right]}{(\mathbb{E} \left[ |S_{l, \xi}(t)| \right])^2} \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}} \in S_{l, \xi}(t)) + \frac{1}{\mathbb{E} \left[ |S_{l, \xi}(t)| \right]}$$

(5.15)

$$\leq \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} \mathbb{P} (v_{\tilde{t}} \in S_{l, \xi}(t)) \mathbb{P} (v_{\tilde{r}} \in S_{l, \xi}(t)) + \mathbb{E} \left[ |S_{l, \xi}(t)| \right]$$

$$\leq \sum_{\tilde{t}, \tilde{r} \text{ disjoint}} (1 + o(1)) \mathbb{P} (v_{\tilde{t}} \in S_{l, \xi}(t)) \mathbb{P} (v_{\tilde{r}} \in S_{l, \xi}(t)) + \mathbb{E} \left[ |S_{l, \xi}(t)| \right]$$

$$\leq 1 + o(1),$$

which proves the desired result.

---

6. **General upper bound for the diameter: proof of Theorem 3**

In this section we provide a proof for Theorem 3. The main idea is to use Lemmas 1 and 3 to show that, with high probability, all vertices born up to time $t^{1/2}$ are in a connected component with diameter 2. We then use a first moment estimate (Lemma 7 below) to show that the lengths of the paths formed by newer vertices have the desired upper bound.

Given $k \in \mathbb{N}$ and $k$ times $s_1, \ldots, s_k \in \mathbb{N}$ such that $s_1 < \cdots < s_k$, we say that the vector of times $\vec{s} = (s_1, \ldots, s_k)$ is a *vertex path* if in the process $\{G_t(f)\}_{t \geq 1}$, at each time $s_j$,
Lemma 6. Using the notation above, we have, for each vector \( \vec{s} = (s_1, \ldots, s_k) \) and step function \( f \),

\[
P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_k) \leq f(s_1)^{s_k - 1} \prod_{m=2}^{k} \frac{f(s_m)}{s_1 + 1} \frac{f(s_m)}{2(s_m - 1)}.
\]

Proof. Consider the events \( \{ s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1} \} \) and \( \{ s_{k-1} \leftarrow s_k \} \), defined analogously as the event in the above equation, but for the vectors \( (s_1, \ldots, s_{k-1}) \) and \( (s_{k-1}, s_k) \) respectively. We have

\[
P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_k) = \mathbb{E} \left[ \mathbb{1}\{ s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1} \} P(s_{k-1} \leftarrow s_k | \mathcal{F}_{s_{k-1}}) \right]
\]

\[
= \mathbb{E} \left[ \mathbb{1}\{ s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1} \} f(s_k) \cdot \frac{D_{s_{k-1}}(v_{s_{k-1}})}{2(s_k - 1)} \right].
\]

But, crucially, conditioned on the event where a vertex is born at time \( s_{k-1} \), the degree of said vertex at time \( s_k - 1 \) depends only on the connections made after time \( s_{k-1} \), and is therefore independent of the indicator function above. We then obtain, by (4.3),

\[
P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_k)
= \mathbb{E} \left[ \mathbb{1}\{ s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1} \} f(s_k) \cdot \frac{D_{s_{k-1}}(v_{s_{k-1}})}{2(s_k - 1)} \bigg| Z_{s_{k-1}} = 1 \right] f(s_{k-1})
= f(s_{k-1}) f(s_k) P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1} | Z_{s_{k-1}} = 1) \mathbb{E} \left[ \frac{D_{s_{k-1}}(v_{s_{k-1}})}{2(s_k - 1)} \bigg| Z_{s_{k-1}} = 1 \right]
= P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1}) \frac{f(s_k)}{2(s_k - 1)} \prod_{m=s_{k-1}}^{s_k-2} \left( 1 + \frac{1}{m} - \frac{f(m+1)}{2m} \right)
\]

\[
\leq P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_{k-1}) \frac{f(s_k)}{2(s_k - 1)} \exp \left\{ \sum_{m=s_{k-1}}^{s_k-2} \left( \frac{1}{m} - \frac{f(m+1)}{2m} \right) \right\},
\]

by elementary properties of the exponential function. Iterating the above argument and recalling that the vertex \( s_1 \) is born with probability \( f(s_1) \), we obtain

\[
P(s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_k) \leq f(s_1) \exp \left\{ \sum_{m=s_1}^{s_k-2} \left( \frac{1}{m} - \frac{f(m+1)}{2m} \right) \right\} \prod_{m=2}^{k} \frac{f(s_m)}{2(s_m - 1)}
\]

\[
\leq f(s_1)^{s_k - 1} \prod_{m=2}^{k} \frac{f(s_m)}{2(s_m - 1)},
\]
finishing the proof of the lemma.

Given $k, t_0, t \in \mathbb{N}$, denote by $V_{k,t_0}(t)$ the set of all vertex-paths of length $k$ whose vertices were born between times $t_0$ and $t$. Our goal now is to bound from above the expectation of $|V_{k,t_0}(t)|$, this will be accomplished in the next lemma.

**Lemma 7.** Using the notation above defined, we have, for $k, t_0, t \in \mathbb{N}$,

$$\mathbb{E} [|V_{k,t_0}(t)|] \leq C_1 \exp \left\{ 2 \log t - (k - 2) \left( \log(k - 2) + C_2 + \log \left( \sum_{j=t_0}^t \frac{f(j)}{j-1} \right) \right) \right\}. \quad (6.5)$$

**Proof.** We will use Lemma 6 and the an application of the union bound. First, fix $s_1, s_k \in \mathbb{N}$ such that $t_0 \leq s_1 < \ldots < s_k \leq t$. We have, by Stirling’s approximation formula and the positivity of the terms involved,

$$\sum_{s_2, s_3, \ldots, s_{k-1}}^{s_1 < s_2 < \cdots < s_{k-1} < s_k} \frac{\prod_{m=2}^{k-1} f(s_m)}{s_m - 1} \leq \frac{1}{(k-2)!} \left( \sum_{j=t_0}^t \frac{f(j)}{j-1} \right)^{k-2} \leq C \exp \left\{ -(k - 2) \left( \log(k - 2) - 1 - \log \left( \sum_{j=t_0}^t \frac{f(j)}{j-1} \right) \right) \right\}. \quad (6.6)$$

We can then show, by the above equation, Lemma 6 and the union bound,

$$\mathbb{E} [|V_{k,t_0}(t)|] \leq \sum_{s_1, \ldots, s_k}^{s_1, \ldots, s_k} \prod_{m=2}^k \frac{f(s_m)}{s_m - 1} \prod_{m=2}^k f(s_m) \frac{s_k - 1}{s_1 + 1} \leq C \exp \left\{ - (k - 2) \left( \log(k - 2) + C_2 - \log \left( \sum_{j=t_0}^t \frac{f(j)}{j-1} \right) \right) \right\} \sum_{s_1, \ldots, s_k} \frac{f(s_1) f(s_k)}{s_1 + 1} \quad (6.7)$$

thus concluding the proof of the result.
We can finally finish the proof of Theorem 3.

Proof of Theorem 3. We prove part (a) first.

Proof of part (a): This part follows immediately from the fact that the B-A random tree has diameter bounded by $\log t$ a.a.s (see Theorem 7.1 of [27]) combined with Proposition 2.5 which states that the diameter is an increasing graph observable.

Proof of part (b): Recall that, in this part of the theorem, $f$ is under condition $[S]$, which holds if $\sum_{s=2}^{\infty} f(s)/s$ is finite.

For $t_0, t \in \mathbb{N}$ and $\delta \in (0, 1)$, let $A_\delta(t_0, t)$ be the event where every vertex born before time $t_0$ has degree at least $t_0^\delta$ in $G_t(f)$. For $\varepsilon \in (0, 1)$, let $B_\varepsilon(t)$ be the event where there exists a vertex $v$ in $V(G_t(f))$ such that $D_v(t) \geq t^{1-\varepsilon}$. Then, by Lemmas 1 and 3, we have

$$P(A_{1/15}(t_0, t)) \geq 1 - C t^{-144-1}, \quad P(B_{1/30}(t)) \geq 1 - t^{-2}. \quad (6.8)$$

We have, on the event where two vertices $u_1, u_2 \in V(G_t(f))$ are such that $D_{u_1}(t) \geq t^{15-1}$ and $D_{u_2}(t) \geq t^{1-30-1}$,

$$P(u_1 \leftrightarrow u_2 \text{ in } G_{2t}(f)) \leq \prod_{s=t+1}^{2t} \left( 1 - \frac{f(s)}{2(s-1)} \right) \left( 1 - \frac{t^{15-1} t^{1-30-1}}{4(s-1)^2} \right)$$

$$\leq \exp \left\{ -\frac{t^{15-1} t^{1-30-1}}{16 t^2} \cdot t \right\}$$

$$= \exp \left\{ -\frac{t^{30-1}}{16} \right\}. \quad (6.9)$$

Recall the notation $v_i$ symbolizing the vertex born at time $i \in \mathbb{N}$. Together with (6.8) and the union bound, the above equation implies

$$P\left( \exists i, j \in \mathbb{N}, \text{ such that } 1 \leq i < j \leq t^{12-1}, Z_i = Z_j = 1, \text{ and } \text{dist}(v_i, v_j) > 2 \text{ in } G_{2t}(f) \right)$$

$$\leq C t^{-144-1} + t^{-2} + \sum_{1 \leq i < j \leq t^{12-1}} P\left( A_{1/15}(t_0, t), B_{1/30}(t), \text{dist}(v_i, v_j) > 2 \text{ in } G_{2t}(f) \right)$$

$$\leq C t^{-144-1} + t^{-2} + t^{-6-1} \exp \left\{ -\frac{t^{30-1}}{16} \right\}$$

$$\leq C t^{-144-1}. \quad (6.10)$$

This implies the existence of a constant $C_1 > 0$ such that the probability of there existing two vertices born before time $C_1 t^{12-1}$ such that the distance between said vertices is larger than 2 in $G_t(f)$ is polynomially small in $t$. We now turn our attention to vertices born after $t^{12-1}$.

We will use Lemma 7 in order to bound the probability of there existing long vertex-paths...
formed by vertices born after $t^{\frac{1}{13}}$, the notion of a “long” path being $f$-dependent. Let $V_t(t^{\frac{1}{13}})$ denote the set of all vertices of $V(G_t(f))$ born before time $t^{\frac{1}{13}}$, let $d_{\text{max}}(t^{\frac{1}{13}}, t)$ be the length of a maximal vertex-path of vertices born between times $t^{\frac{1}{13}}$ and $t$. Let $u_1, u_2 \in V(G_t(f))$. Since $G_t(f)$ is connected,

$$
\text{dist}(u_1, u_2) \leq \text{dist}(u_1, V_t(t^{\frac{1}{13}})) + \text{diam}(V_t(t^{\frac{1}{13}})) + \text{dist}(u_2, V_t(t^{\frac{1}{13}}))
$$

(6.10)

But we know that $\text{diam}(V_t(t^{\frac{1}{13}})) \leq 2$ with high probability. Bounding $d_{\text{max}}(t^{\frac{1}{13}}, t)$ then gives us an a.a.s. upper bound for the diameter of $G_t(f)$.

Now, given $t \in \mathbb{N}$, if

$$
k \geq 3 \left( \frac{\log t}{-\log \left( \sum_{s=t^{\frac{1}{13}}}^{t} \frac{f(s)}{s-1} \right)} \wedge \frac{\log t}{\log \log t} \right),
$$

(6.11)

then, by Lemma 7, and since $\log \left( \sum_{j=t^{\frac{1}{13}}}^{t} \frac{f(j)}{j-1} \right)$ is eventually negative for large $t$ (recall again that $f$ satisfies (S)), we have,

$$
P(d_{\text{max}} > k) \leq \mathbb{E}[|\mathcal{V}_{k,t^{1/13}}(t)|]
$$

(6.12)

$$
\leq C_1 \exp \left\{ 2 \log t - (k - 2) \left( \log(k - 2) + C_2 - \log \left( \sum_{j=t^{\frac{1}{13}}}^{t} \frac{f(j)}{j-1} \right) \right) \right\}
$$

$$
\leq C t^{-\frac{1}{2}}.
$$

The above upper bound and (6.10) proves part (b).

Proof of part (c): Recall that in this part, we are under condition \((L_\kappa)\), which holds if, for $\kappa \in (0, 1)$ and every $t \in \mathbb{N}$ sufficiently large, one has

$$
\sum_{s=t^{\frac{1}{13}}}^{t} \frac{f(s)}{s} < (\log t)^\kappa.
$$

(6.13)

Then, let $k$ be so that

$$
k \geq \frac{3 \log t}{1 - \kappa \log \log t}.
$$
We then obtain, again by Lemma $\mathbb{7}$ for sufficiently large $t$,
\[
\mathbb{P}(d_{\text{max}} > k) \\
\leq E[|\mathcal{V}_{k,t1/13}(t)|] \\
\leq C_1 \exp \left\{ 2 \log t - (k - 2) \left( \log(k - 2) + C_2 - \log \left( \sum_{j=t^{1/13}}^t \frac{f(j)}{j - 1} \right) \right) \right\} \\
\leq C_1 \exp \left\{ 2 \log t - \frac{3}{1 - \kappa} \log \log t \right\} \\
\leq C t^{-\frac{1}{2}}.
\]
Finally, the above upper bound together with (6.10) finishes the proof of the Theorem.

7. THE FAMILY OF REGULARLY VARYING FUNCTIONS

In this section we explore our results when more information on $f$ is provided. In particular, we assume that $f$ satisfies condition (RV$_\gamma$) for $\gamma \in [0, 1)$. Recall that this condition holds whenever

$$
\exists \gamma, \text{ such that } \forall a > 0, \lim_{t \to \infty} \frac{f(at)}{f(t)} = \frac{1}{a^\gamma}.
$$

Also recall that when $f$ satisfies the above condition for $\gamma > 0$ we say it is a regular varying function with index of regular variation $-\gamma$. The case $\gamma = 0$ is said to be slowly varying.

To prove the results under the assumption of regular variation our arguments rely on the theorems from Karamata’s theory. In particular, the Representation Theorem (Theorem 1.4.1 of [8]) and Karamata’s theorem (Proposition 1.5.8 of [8]). The former states that if $f$ is a regularly varying function with index $-\gamma$, then there exists a slowly varying function $\ell$ such that $f$ is of the form

$$
f(t) = \frac{\ell(t)}{t^\gamma}.
$$

Whereas, the latter states that if $\ell$ is a slowly varying function, then, for any $a < 1$

$$
\int_1^t \frac{\ell(x)}{x^a} \, dx \sim \frac{\ell(t)t^{1-a}}{1-a},
$$

and for $a > 1$, we have

$$
\int_t^\infty \frac{\ell(x)}{x^a} \, dx \sim \frac{\ell(t)}{(a-1)t^{a-1}}.
$$

We begin by proving Corollary 1.4 which gives lower and upper bounds for the clique number $\omega(G_t(f))$. 
Proof of Corollary 1.4. We prove part (a) first.

Proof of part (a): The lower bound follows from (7.1), when \( f \) is regularly varying, (7.2) and Theorem 1. Just observe that, if \( f \) is regularly varying, then \( F(t) = \Theta(t^{1-\gamma}) \), whereas if it is slowly, we have \( F(t) = \Theta(f(t)t) \). In the latter case, we appeal to another result from Karamata’s Theory (Corollary A.6 of [2]) which assures that for all \( \varepsilon > 0 \),

\[
(7.4) \quad \frac{f(t)}{t^{\varepsilon}} \xrightarrow{t \to \infty} 0.
\]

Then, by Theorem 1, \( G_t(f) \) has, with probability \( 1 - o(1) \), a clique of order \( C_{\varepsilon}F\left(t^{\frac{1}{2}}(1-\varepsilon)\right) \), which gives the desired lower bound for \( \omega(G_t(f)) \). The upper bound comes from the deterministic bound that says that a graph with \( t \) (simple) edges has at most \( t^2 \) triangles (see e.g. Theorem 4 of [24]), and the fact that a complete subgraph with \( k \) vertices has \( 6^{-1}k(k-1)(k-2) \) triangles.

Proof of part (b): First observe that for any small \( \varepsilon > 0 \) by (7.2), \( F(t^{\frac{1}{2}(1-\varepsilon)}) \gg \log \log t \) for large enough \( t \). Thus, by (3.8), we have that

\[
(7.5) \quad \mathbb{P}\left( \omega(G_t(f)) \geq C_{\varepsilon}F(t^{\frac{1}{2}+\varepsilon}) \right) \geq 1 - \frac{2}{\log t}.
\]

Now, define the following sequence of deterministic times

\[
(7.6) \quad t_k := (1 + \varepsilon)^k.
\]

By the first Borel-Cantelli Lemma and (7.5) we have that,

\[
(7.7) \quad \liminf_{k \to \infty} \frac{\log \omega(G_t(f))}{\log(C_{\varepsilon}F(t_k^{\frac{1}{2}}))} \geq 1, \text{ a.s.}
\]

Using that \( f \) is slowly varying, Karamata’s Theorem (7.2), and (7.4), we also have that

\[
(7.8) \quad \lim_{t \to \infty} \frac{\log(C_{\varepsilon}F(t^{\frac{1}{2}+\varepsilon}))}{\log(F(t^{\frac{1}{2}}))} = 1 - \frac{\varepsilon}{2}.
\]

For the same reason, it also follows that

\[
(7.9) \quad \lim_{t \to \infty} \frac{\log F(t^{\frac{1}{2}})}{\log(7t^{\frac{1}{2}})} = 1.
\]

Then (7.7) yields

\[
(7.10) \quad \liminf_{k \to \infty} \frac{\log \omega(G_{t_k}(f))}{\log(F(t_k^{\frac{1}{2}}))} \geq 1 - \varepsilon, \text{ a.s.}
\]

And using the upper bound given in part (a) and (7.9) we also obtain

\[
(7.11) \quad \limsup_{k \to \infty} \frac{\log \omega(G_{t_k}(f))}{\log(F(t_k^{\frac{1}{2}}))} \leq 1, \text{ a.s.}
\]
Now, note that \( \omega(G_t(f)) \leq \omega(G_{t+1}(f)) \), since \( G_t(f) \) is contained in \( G_{t+1}(f) \). Thus, for any time \( t \in (t_k, t_{k+1}) \) it follows

\[
\frac{\log(\omega(G_{t_k}(f)))}{\log F(t_{k+1}^{\frac{1}{2}})} \leq \frac{\log(\omega(G_t(f)))}{\log F(t^\frac{1}{2})} \leq \frac{\log(\omega(G_{t+1}(f)))}{\log F(t_{k+1}^{\frac{1}{2}})}.
\]

Using (7.10), (7.11) and that

\[
\lim_{k \to \infty} \frac{\log F(t_{k+1}^{\frac{1}{2}})}{\log F(t_k^{\frac{1}{2}})} = 1
\]

we finally obtain that

\[
1 - \varepsilon \leq \liminf_{t \to \infty} \frac{\log(\omega(G_t(f)))}{\log F(t^\frac{1}{2})} \leq \limsup_{t \to \infty} \frac{\log(\omega(G_t(f)))}{\log F(t^\frac{1}{2})} \leq 1, \text{ a.s.}
\]

Since \( \varepsilon \) was arbitrarily chosen, we conclude the proof.

The next step is to prove the constant order of the diameter of the graphs generated by regularly varying functions and how it depends on the index of regular variation on infinity. This result is stated on Theorem 4 and we provide a proof for it below.

**Proof of Theorem 4.** We begin proving the lower bound.

**Lower bound:** By the Representation Theorem (7.1) we have that there exists a slowly varying function \( \ell \) such that \( f(t) = \ell(t)/t^\gamma \). Moreover, (7.4) implies that

\[
\log \ell(t)/t^\gamma \to 0.
\]

Thus, we have that

\[
- \frac{\log t}{\log f(t)} = \frac{\log t}{\gamma \log(t) + \log \ell(t)} = \frac{1 - o(1)}{\gamma}.
\]

Applying Theorem 2 gives us a lower bound of order \( \gamma^{-1} \).

**Upper bound:** By the Representation Theorem and (7.4), we have that \( f \) also satisfies condition (S). Just notice that

\[
\sum_{s=1}^{\infty} \frac{f(s)}{s^{1+\gamma}} < \infty.
\]

And by Karamata’s Theorem (Equation (7.3) in particular) we have that

\[
\sum_{s=t^{\frac{1}{1+\gamma}}}^{t} \frac{\ell(s)}{s^{1+\gamma}} \sim \frac{\ell(t)}{t^{\gamma \frac{1}{1+\gamma}}} \Rightarrow - \frac{\log t}{\log \left( \sum_{s=t^{\frac{1}{1+\gamma}}}^{t} \frac{\ell(s)}{s^{1+\gamma}} \right)} \sim \frac{13(1 - o(1))}{\gamma}.
\]

And finally, applying Theorem 3 we prove the result.
8. Final comments

We end this paper with a brief discussion on the affine version of our model and how dropping some regularity conditions on \( f \) may produce a sequence of graphs \( \{G_t\}_{t \in \mathbb{N}} \) that has a subsequence which is essentially of complete graphs and another one which is close to the BA-model.

**Affine version.** At the introduction, we have discussed the affine version of the PA-rule, which we recall below.

\[
\mathbb{P}(v_{t+1} \rightarrow u | G_t) = \frac{\text{degree}(u) + \delta}{\sum_{w \in G_t} (\text{degree}(w) + \delta)}.
\]

In [2], the authors showed that the effect of the affine term \( \delta \) vanishes in the long run when one is dealing with the empirical degree distribution. I.e., their results show that \( \delta \) has no effect on the degree sequence of the graphs, what is not observed in the affine version of the BA-model, for which the exponent of the power-law distribution depends on \( \delta \), see [13]. However, regarding the diameter, we believe \( \delta \) may have an increasing/decreasing effect on the diameter’s order. One also may find interesting to consider \( \delta = \delta(t) \) and investigate which one takes over: is it the edge-step function or the affine term?

**Dropping regularity conditions.** In this part we illustrate that dropping some assumptions on \( f \) may produce a somewhat pathological sequence of graphs. For instance, if we drop the assumption of \( f \) being non-increasing, we may obtain a sequence of graphs whose diameter sequence oscillates between 1 and \( \log t \). More generally, the sequence of graphs oscillates between graphs similar to the BA-random tree, \( \{BA_t\}_{t \in \mathbb{N}} \), and graphs close to complete graphs.

Let \( (t_k)_{k \in \mathbb{N}} \) be the following sequence: \( t_0 = 1 \) and \( t_{k+1} := \exp\{t_k\} \), for \( k > 1 \). Now, let \( h \) be the edge-step function defined as follows

\[
h(t) = \begin{cases} 
1 & \text{if } t \in [t_{2k}, t_{2k+1}], \\
0 & \text{if } t \in (t_{2k+1}, t_{2k+2}). 
\end{cases}
\]

The idea behind such \( h \) is that between times \( [t_{2k}, t_{2k+1}] \) the process behaves essentially as the traditional BA-model, whereas at interval \( (t_{2k+1}, t_{2k+2}) \) the process “messes things up” connecting almost all vertices by only adding new edges. Moreover, in both regimes the process has time enough to “forget about what was built in the past”.

Using Lemma 3 and reasoning as in (6.9), one may prove that

\[
\text{diam}_{G_{t_{2k+1}}(h)} G_{t_{2k+1}}(h) \leq 2, \text{ a.a.s.}
\]

However, the process does not add any new vertex in the interval \( (t_{2k+1}, t_{2k+2}) \). Therefore, \( \text{diam} G_{t_{2k+2}}(h) \leq 2, \text{ a.a.s.} \)
On the other hand, if we sample $G_{t_{2k+3}}(h)$ and $BA_{t_{2k+3}}$ from the doubly-labeled tree $T_{t_{2k+3}}$, it follows that the quantity
\[
\max_{u > t_{2k+2}} \text{dist}(u, T_{t_{2k+2}})
\]
remains the same under $h$ and under $f \equiv 1$, since paths using vertices added after $t_{2k+2}$ belong to the graphs generated by both functions. Finally, noting that $T_{t_{2k+2}}$ has diameter at most $\log t_{2k+2} = t_{2k+1}$ w.h.p, which means it is a very small graph when compared to $T_{t_{2k+3}}$, it is not hard to see that the diameter of $G_{t_{2k+3}}(h)$ has the same order the diameter from $BA_{t_{2k+3}}$.

Observe that the function $h$ may be constructed considering it equal to any other edge-step function $f$ instead of the constant case 1. Roughly speaking, when we sample both processes from the doubly-labeled random tree, up to a very small subgraph, the graph $G_{t_{2k+1}}(h)$ is similar to $G_{t_{2k+1}}(f)$, whereas, $G_{t_{2k}}(h)$ is a graph of diameter at most 2 and whose vertices have degree at least a subpolynomial of $t_{2k}$.

The conclusion is that if we drop some monotonicity assumption on $f$, we may obtain a sequence of graphs having at least two subsequences that are completely different as graphs.

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