On (almost) 2-Y-homogeneous distance-biregular graphs

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Abstract
Let \( \Gamma \) denote a bipartite graph with vertex set \( X \) and color partitions \( Y, Y' \). For a nonnegative integer \( i \) and a vertex \( x \in X \), let \( \Gamma_i(x) \) denote the set of vertices in \( X \) that are at distance \( i \) from \( x \). For \( x \in Y \), \( y \in \Gamma_2(x) \) and \( z \in \Gamma_i(x) \cap \Gamma_j(y) \), let \( \gamma_i(x, y, z) \) denote the number of common neighbors of \( x \) and \( y \) which are at distance \( i-1 \) from \( z \) (i.e., let \( \gamma_i(x, y, z) := |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| \)). For the moment assume that every vertex in \( Y \) has eccentricity \( D \geq 3 \). Graph \( \Gamma \) is almost 2-Y-homogeneous whenever for all \( i \) (\( 1 \leq i \leq D - 2 \)) and for all \( x \in Y \), \( y \in \Gamma_2(x) \) and \( z \in \Gamma_i(x) \cap \Gamma_j(y) \), the number \( \gamma_i(x, y, z) \) is independent of the choice of \( x, y \) and \( z \). In addition, if the above condition holds also for \( i = D - 1 \), then we say that \( \Gamma \) is 2-Y-homogeneous. In this paper, we study the combinatorial structure of distance-biregular graphs. We give sufficient and necessary conditions under which a distance-biregular graph is (almost) 2-Y-homogeneous. Moreover, for a 2-Y-homogeneous distance-biregular graph we write the intersection numbers of the color class \( Y \) in terms of three parameters.

Keywords Distance-biregular graph · Equitable partition · Intersection array

Mathematics Subject Classification 05C75 · 05E30

1 Introduction

The reader is referred to Sect. 2 for formal definitions.
Let $\Gamma$ denote a graph with the vertex set $X$. For $z \in X$ and an integer $i$, let $\Gamma_i(z)$ denote the set of all vertices which are at distance $i$ from $z$. A vertex $x \in X$ is said to be distance-regularized if for each $y \in \Gamma_i(x)$, the numbers $c_i(x) := |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i(x) := |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i(y) := |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ depend only on the distance $i$ between $x$ and $y$, and are independent of the choice of $y \in \Gamma_i(x)$. For the moment assume that $x$ is a distance-regularized vertex of $\Gamma$. Then, the array
\[
(b_0(x), b_1(x), \ldots, b_{d-1}(x); 1, c_2(x), \ldots, c_d(x))
\]
is called the intersection array for $x$ (or, intersection array of $\Gamma$ with respect to $x$). A connected graph in which every vertex is distance-regularized is called a distance-regular graph. A special case of such graphs is distance-regular graphs in which all vertices have the same intersection array. Other examples are bipartite graphs in which vertices in the same color class have the same intersection array, but which are not distance-regular. We call these graphs distance-biregular. GODSIL and SHAWETAYLOR [18] proved that every distance-regularized graph is either distance-regular or distance-biregular. The family of distance-biregular graphs is quite rich, and examples can be obtained from several algebraic and geometric objects [10, Section 4]. Moreover, these objects can be seen as non-symmetric association schemes [10, Section 1]. There are a number of excellent articles on distance-biregular graphs, e.g., [16–18, 32, 35, 42].

In this paper, we study the combinatorial structure of distance-biregular graphs. We give an answer to the following problem.

**Problem 1.1** Let $\Gamma$ denote a distance-biregular graph with vertex set $X = Y \cup Y'$, color partitions $Y$, $Y'$, and let $D$ denote the eccentricity of vertices in $Y$. For vertices $x, y \in X$, let $\Gamma_{i,j}(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$. Find necessary and sufficient conditions on the intersection arrays of $\Gamma$ for which the graph has one of the following two combinatorial structures:

(a) for all $i$ $(1 \leq i \leq D - 2)$, and for all $x \in Y \cup Y'$ and $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$ the number $|\Gamma_{i-1}(z) \cap \Gamma_1(x, y)|$ is independent of the choice of $x, y$ and $z$. If $\Gamma$ has this combinatorial structure, then we say that $\Gamma$ is almost $2$-$Y$-homogeneous.

(b) for all $i$ $(1 \leq i \leq D - 1)$ and for all $x \in Y \cup Y'$ and $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$ the number $|\Gamma_{i-1}(z) \cap \Gamma_1(x, y)|$ is independent of the choice of $x, y$ and $z$. If $\Gamma$ has this combinatorial structure, then we say that $\Gamma$ is $2$-$Y$-homogeneous.

This paper is motivated by a desire to find a combinatorial characterization of bipartite graphs $\Gamma = (Y \cup Y', R)$ which are (almost) $2$-$Y$-homogeneous and to classify all such graphs. If $\Gamma$ is distance-regular, then this situation occurs if and only if the intersection array of $\Gamma$ is at least one of the types given in [36, Theorem 1.2]. However, for an arbitrary graph $\Gamma$ it seems that the above problem is difficult, and we will not consider it in this paper. Here, we study distance-biregular graphs. We are also interested in the (almost) $2$-$Y$-homogeneous distance-biregular graphs because they arise as a natural family in the study of the Terwilliger algebra $T$ of a distance-biregular graph. One of our long-term goals is to study and classify all distance-biregular graphs which have a unique irreducible $T$-module (up to isomorphism) with endpoint 2.
which is thin, and this paper is one of the steps for solving this problem. See [14] for the results about the structure of irreducible $T$-modules, when the endpoint is 1 (also for the case when $\Gamma$ is distance-biregular). The study of thin irreducible $T$-modules with endpoint 2 of distance-regular graphs has been a very active area of research for the last several years; see for example [13, 23–26, 29]. The second long-term goal for better understanding combinatorial properties of such graphs, is finding an algorithm which will draw such a graph (i.e., explicitly give all edges between vertices) from its intersection arrays (if such graph exists)—a problem that because of its difficulty does not have enough attention inside the mathematical community. A “simple” combinatorial structure of the graph can contribute to more easily find such an algorithm, and the combinatorial structure of 2-$Y$-homogeneous distance-biregular graphs it seems promising. Let us also mention that MOHAR [32] showed that $\Gamma$ is a distance-biregular graph with vertices of valency 2 if and only if $\Gamma$ is either a complete bipartite graph $K_{2,n}$ for some $n \geq 1$, or the subdivision graph of a $(\kappa, g)$-cage (see [32, Corollary 3.5] and Sect. 2.5 for more details). It turns out that if $\Gamma$ is a $(\kappa, g)$-cage with vertex set $X$, then the subdivision graph $S(\Gamma)$ is a 2-$X$-homogeneous distance-biregular graph (see Sect. 4). Our main result is the following theorem.

Theorem 1.1 Let $\Gamma$ denote a 2-$Y$-homogeneous distance-biregular graph with color classes $Y, Y'$, and let $D$ denote the eccentricity of vertices in the color class $Y$. Assume that $D \geq 3$, and let $c_2'$ denote the intersection number $|\Gamma_1(x) \cap \Gamma_1(y)|$ ($x \in Y', y \in \Gamma_2(x)$) of the color class $Y'$.

(i) If $c_2' = 1$, then the intersection array of the color class $Y$ is one of the following two types

$(k, k' - 1, k - 1, k' - 1, k - 1, \ldots, k' - 1, k - 1; 1, 1, 1, 1, \ldots, 1, k')$, \hspace{1cm} for odd $D$,

$(k, k' - 1, k - 1, k' - 1, k - 1, \ldots, k' - 1, k - 1; 1, 1, 1, 1, \ldots, 1, k)$, \hspace{1cm} for even $D$,

for some integers $k$ and $k' \geq 2$.

(ii) If $c_2' = 2$, then the first three intersection numbers of the color class $Y$ are

$$\left( k, \frac{k - 1}{c - 1}, k - c, \frac{k - 1}{c - 1} - c, \ldots; 1, c, c + 1, \ldots \right)$$

for some integers $k$ and $c \geq 2$.

(iii) If $c_2' \geq 3$, then $D \leq 5$. Moreover, we have

(a) If $D = 3$, then the intersection array of the color class $Y$ is of the form

$(k, c, k - c; 1, c, c + 1)$ for some integers $k$ and $c$, where $k > c \geq 2$.

(b) If $D = 4$, then the intersection array of the color class $Y$ is of the form

$(k, k' - 1, k - c, k' - 1 - \frac{c(c - 1)}{\gamma}; 1, c, \frac{c(c - 1)}{\gamma} + 1, k)$ for some positive integers $k$, $k'$ and $c$, where $k > c \geq 2, k' > 2, c' = \frac{(k' - 1)(c - 1)}{k - 1} + 1$ and $\gamma = \frac{(c - 1)(c' - 2)}{k' - 2} + 1$.

(c) If $D = 5$, then the intersection array of the color class $Y$ is of the form

$(k, k' - 1, k - c, b_3, b_4; 1, c, c_3, c_4, k')$ for some positive integers $k$, $k'$ and
c, where $k > c \geq 2$, $k' > 2$, $c_3 = k' - 2\gamma + 1 - \frac{c(c'-1)}{\gamma}$, $c' = \frac{(k'-1)(c'-1)}{k-1} + 1$,

$$\gamma = \frac{(c-1)(c'-2)}{k-2} + 1, \quad b_3 = k' - c_3, \quad c_4 = \frac{k(k'-1)-c(b_3-c)(k-1)}{c_3} \quad \text{and} \quad b_4 = k - c_4.$$ 

By [18, Lemma 2.3], the intersection numbers of the color class $Y'$ can be computed in terms of the intersection numbers of the color class $Y$.

In the end of this section we summarize our main results (we use the same notation as in Theorem 1.1). In Sect. 2, we give notation, formal definitions and we recall some basic properties of distance-biregular graphs. In Sect. 3, for an (almost) $2$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$, we define certain scalars $\gamma$, and we compute some equalities in the case when $c_2' \geq 2$, which we use later. We also show that $c_2' \geq 3$ implies $D \leq 5$. In Sect. 4, we prove that the subdivision graph of a $(\kappa, g)$-cage graph with vertex set $X$ is $2$-$X$-homogeneous. For a given $(Y, Y')$-distance-biregular graph $\Gamma$, in Sect. 5 we study certain scalars $\Delta_i (2 \leq i \leq \min\{D-1, D'-1\})$ where $D'$ denotes the eccentricity of vertices in the color class $Y'$. These scalars can be computed from the intersection array of a given distance-biregular graph and they play an important role: from their values, we can determine if a given distance-biregular graph is (almost) $2$-$Y$-homogeneous or not. In Sect. 6, we focus on the combinatorial structure of an (almost) $2$-$Y$-homogeneous distance-biregular graph, setting up deeper knowledge of a combinatorial structure of $\Gamma$ for easier understanding the rest of the paper. Among else, we prove that the scalars $\gamma_i$ are nonzero, when $k' \geq 3$ and $D \geq 3$. In Sect. 7, we show that $\Gamma$ is $2$-$Y$-homogeneous if and only if $\Delta_i = 0$ ($2 \leq i \leq \min\{D-1, D'-1\}$); and $\Gamma$ is almost $2$-$Y$-homogeneous if and only if $\Delta_i = 0$ ($2 \leq i \leq D-2$). In Sect. 8, we study distance-biregular graphs with $c_2' = 1$ and we show that $\Gamma$ is almost $2$-$Y$-homogeneous with $c_2 = 1$ if and only if $c_i = 1$ for every integer $i$ ($1 \leq i \leq D-1$). In Sects. 9 and 10, we give possible types for the intersection array for a $2$-$Y$-homogeneous distance-biregular graph, when $D \in \{3, 4, 5\}$. The proof of Theorem 1.1 is in Sect. 10. We finish the paper giving simple examples and some open problems which could be of interest for further research.

## 2 Preliminaries

An (undirected) graph $\Gamma$ with vertex set $X$ and edge set $\mathcal{R}$ is a pair $\Gamma = (X, \mathcal{R})$, where $X$ is a nonempty set and $\mathcal{R}$ is a collection of one or two element subsets of $X$. The elements of $X$ are called the vertices of $\Gamma$, and the elements of $\mathcal{R}$ are called the edges of $\Gamma$. A one element subset of $X$ in $\mathcal{R}$ is an edge which starts and ends at the same vertex—it is called a loop. When $xy \in \mathcal{R}$ ($x \neq y$), we say that vertices $x$ and $y$ are adjacent, or that $x$ and $y$ are neighbors. Adjacency between vertices $x$ and $y$ will be denoted by $x \sim y$. A graph is finite if both its vertex set and edge set are finite. A graph is simple if it has no loops and no two of its edges join the same pair of vertices.

Let $\Gamma = (X, \mathcal{R})$ be a graph. For any two vertices $x, y \in X$, a walk of length $h$ from $x$ to $y$ is a sequence $[x_0, x_1, x_2, \ldots, x_h]$ ($x_i \in X$, $0 \leq i \leq h$) such that $x_0 = x$, $x_h = y$, and $x_i$ is adjacent to $x_{i+1}$ ($0 \leq i \leq h - 1$). We say that $\Gamma$ is connected if for any $x, y \in X$, there is a walk from $x$ to $y$. A path is a walk such that all vertices of the walk are distinct. From now on, assume that $\Gamma$ is finite, simple and connected.
For any \( x, y \in X \), the distance between \( x \) and \( y \), denoted by \( \partial(x, y) \), is the length of a shortest walk from \( x \) to \( y \). The diameter \( d = d(\Gamma) \) of \( \Gamma \) is defined to be

\[
d = \max\{\partial(u, v) \mid u, v \in X\}.
\]

The eccentricity of \( x \), denoted by \( \varepsilon = \varepsilon(x) \), is the maximum distance between \( x \) and any other vertex of \( \Gamma \). Note that the diameter of \( \Gamma \) equals \( \max\{\varepsilon(x) \mid x \in X\} \).

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. The complete graph on \( n \) vertices is denoted by \( K_n \). A bipartite (or \((Y,Y')\)-bipartite) graph is one whose vertex set can be partitioned into two subsets \( Y \) and \( Y' \), so that each edge has one endpoint in \( Y \) and another endpoint in \( Y' \). The vertex sets \( Y \) and \( Y' \) in such a partition are called a color partition (or bipartition) of the graph. A complete bipartite graph is a simple bipartite graph with color partitions \( Y \) and \( Y' \) in which each vertex of \( Y \) is joined to each vertex of \( Y' \); if \( |Y| = m \) and \( |Y'| = n \), such a graph is denoted by \( K_{m,n} \). A graph \( \Gamma \) is regular with valency \( k \) if each vertex in \( \Gamma \) has exactly \( k \) neighbors.

### 2.1 Distance-regularized vertices and distance-regular graphs

The concept of distance appears in all of science. In the study of graphs, distances play an important role, and one of the main reasons for it is maybe in their wide applicability. The distance between two vertices in a graph is simple and useful notion. It leads to the definition of several graph parameters such as the diameter, the radius, the average distance, the eccentricity of a vertex, the adjacency matrix, the distance-\( i \) matrix, and in help of some additional restrictions to the definition of several graphs classes (distance-balanced, strongly distance-balanced, nicely distance-balanced, distance-regular, distance-biregular, distance-transitive, distance-hereditary). Some interesting articles regarding different uses of distances are [1, 2, 7, 19, 21, 30, 33, 45, 46].

Let \( \Gamma \) denote a graph with vertex set \( X \) and diameter \( d \). For a vertex \( x \in X \) and any nonnegative integer \( i \) not exceeding \( d \), let \( \Gamma_i(x) \) denote the subset of vertices in \( X \) that are at distance \( i \) from \( x \). Let \( \Gamma_{-1}(x) = \Gamma_{d+1}(x) = \emptyset \). For any two vertices \( x \) and \( y \) in \( X \) at distance \( i \), let \( C_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma_1(y) \), \( A_i(x, y) = \Gamma_i(x) \cap \Gamma_1(y) \), \( B_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma_1(y) \). We say that a vertex \( x \in X \) is distance-regularized (or that \( \Gamma \) is distance-regular around \( x \)) if the numbers \( |A_i(x, y)|, |B_i(x, y)| \) and \( |C_i(x, y)| \) do not depend on the choice of \( y \in \Gamma_i(x) \) (\( 0 \leq i \leq d \)); in this case, the numbers \( |A_i(x, y)|, |B_i(x, y)| \) and \( |C_i(x, y)| \) are simply denoted by \( a_i(x) \), \( b_i(x) \) and \( c_i(x) \), respectively, and are called the intersection numbers of \( x \). A graph \( \Gamma \) is called distance-regular if there are integers \( b_i, c_i \) (\( 0 \leq i \leq d \)) which satisfy \( c_i = |C_i(x, y)| \) and \( b_i = |B_i(x, y)| \) for any two vertices \( x \) and \( y \) in \( X \) at distance \( i \). Clearly, such a graph is regular of valency \( k := b_0 \) and \( a_i := |A_i(x, y)| = k - b_i - c_i \) (\( 0 \leq i \leq d \)) is the number of neighbors of \( y \) in \( \Gamma_i(x) \) for \( x, y \in X (\partial(x, y) = i) \). Note that a graph is distance-regular if each of its vertices is distance-regularized, and if all of its vertices have the same intersection numbers. For more information about distance-regular graphs we refer the reader to [4–6, 15, 39, 44].
2.2 Equitable partitions and intersection diagrams

A partition of a graph $\Gamma$ is a collection $\{P_1, P_2, \ldots, P_s\}$ of nonempty subsets of the vertex set $X$, such that $X = \bigcup_{i=1}^{s} P_i$ and $P_i \cap P_j = \emptyset$ for all $i, j$ ($1 \leq i, j \leq s$, $i \neq j$).

An equitable partition of a graph $\Gamma$ is a partition $\{P_1, P_2, \ldots, P_s\}$ of its vertex set, such that for all integers $i, j$ ($1 \leq i, j \leq s$) the number $c_{ij}$ of neighbors, which a vertex in the cell $P_i$ has in the cell $P_j$, is independent of the choice of the vertex in $P_i$.

Assume that $\Gamma$ is a graph of diameter $d$, and pick arbitrary vertices $x$ and $y \in \Gamma_h(x)$, for some fixed $h$ ($0 \leq h \leq \varepsilon(x)$). The collection of all nonempty subsets $\Gamma_{i,j}(x, y)$ ($0 \leq i, j \leq d$) is a partition of the vertex set of $\Gamma$ which is called the intersection diagram of $\Gamma$ of rank $h$.

2.3 Distance-biregular graphs (DBG)

A natural continuation in understanding combinatorial and algebraic properties of distance-regular graphs is to study the family of graphs known as distance-biregular graphs. Let $\Gamma$ denote a graph (not necessarily regular) with the property that each vertex is distance-regularized (i.e., $\Gamma$ is distance-regular around every vertex). Such a graph is said to be distance-regularized. A distance-regularized graph is called distance-biregular if the following (i)–(iii) hold: (i) it is bipartite; (ii) the vertices in the same color partition have the same intersection array; and (iii) the vertices in different color classes have different intersection arrays. A well-known result by GODSIL and SHAWE-TAYLOR [18] states that if $\Gamma$ is a connected graph that is distance-regular around every vertex, then $\Gamma$ is distance-regular or distance-biregular. The complete bipartite graph $K_{m,n}$ is an example of a distance-biregular graph. The subdivision graph of the Petersen graph is an example of a 2-$Y$-homogeneous distance-biregular graph (here $Y$ is the set of vertices of the Petersen graph). We refer to [10, 13, 16, 17, 32, 35] for further reading on distance-biregular graphs.

Let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with vertex set $X$. Pick $x \in X$ and let $\varepsilon(x)$ denote the eccentricity of $x$. Since $\Gamma$ is bipartite, we have $a_i(x) = 0$ for $0 \leq i \leq \varepsilon(x)$. Note that all vertices from $Y$ ($Y'$, respectively) have the same eccentricity. We denote this common eccentricity by $D$ ($D'$, respectively). Since we have two color classes, observe that $|D - D'| \leq 1$ and the diameter of $\Gamma$ equals $\max\{D, D'\}$. In addition, all vertices from $Y$ ($Y'$, respectively) have the same valency $k$ ($k'$, respectively). Observe that for $0 \leq i \leq D$, $c_i + b_i = k$ if $i$ is even, while $c_i + b_i = k'$ if $i$ is odd.

Note that, if $D = 2$ then $\Gamma$ is a complete bipartite graph with $D' \in \{1, 2\}$ (if $D' = 3$, then for any $x \in Y'$, $\Gamma_3(x) \subseteq Y$, which yields $D \geq 3$, a contradiction). Moreover, if $D = 2$ then $\Gamma$ is 2-$Y$-homogeneous by definition. For the rest of the paper, we assume that $D \geq 3$ (which also yields $D' \geq 3$ — just use the same argument as in the previous sentence). From now on, until the end of the paper, we use the following notation.
Notation 2.1 Let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with vertex set $X$, and color partitions $Y$ and $Y'$. Let $D (D'$, respectively) denote the eccentricity of vertices from $Y (Y'$, respectively), and assume that $D \geq 3$. For vertices $x_1, x_2, \ldots, x_n \in X$ and nonnegative integers $i_1, i_2, \ldots, i_n \ (0 \leq i_1, i_2, \ldots, i_n \leq d$, where $d = \max (D, D')$ we define $\Gamma_{i_1,i_2,\ldots,i_n}(x_1,x_2,\ldots,x_n) = \bigcap_{\ell=1}^n \Gamma_{i_\ell}(x_\ell)$. For $x \in Y$ and $z \in \Gamma_i(x) \ (0 \leq i \leq D)$, we abbreviate $c_0 = 0$, $c_i = |\Gamma_{i=1}(x,z)|$, $b_i = |\Gamma_{i+1}(x,z)|$, $b_D = 0$, $k_i := |\Gamma_i(x)|$, and write $(k, b_1, \ldots, b_{D-1}; 1, c_2, \ldots, c_D)$ for the intersection array of the color partition Y. For $u \in Y'$ and $v \in \Gamma_i(u) \ (0 \leq i \leq D')$ we abbreviate $c'_0 = 0$, $c'_i = |\Gamma_{i=1}(u,v)|$, $b'_i = |\Gamma_{i+1}(u,v)|$, $b'_D = 0$, $k'_i := |\Gamma_i(u)|$ and write $(k', b'_1, \ldots, b'_{D-1}; 1, c'_2, \ldots, c'_D)$ for the intersection array of the color partition $Y'$. 

Note that, for $0 \leq i \leq D'$, $c'_i + b'_i = k'$ if $i$ is even, while $c'_i + b'_i = k$ if $i$ is odd (see Fig. 1). Using the same intersection diagram, for any $x \in Y$ and $u \in Y'$, we also get

$$|\Gamma_i(x)| = \frac{b_0b_1 \cdots b_{i-1}}{c_1c_2 \cdots c_i} \ (1 \leq i \leq D), \quad |\Gamma_i(u)| = \frac{b'_0b'_1 \cdots b'_{i-1}}{c'_1c'_2 \cdots c'_i} \ (1 \leq i \leq D'). \quad (1)$$

In Lemma 2.2, we recall some relations between the intersection numbers of a distance-biregular graph. Both of these lemmas will be used later in the paper.

Lemma 2.2 ([10, Proposition 2]) With reference to Notation 2.1, the following hold.

(i) $c'_i \leq c_{i+1}$ (1 $\leq i \leq D - 1$) and $c_i \leq c'_{i+1}$ (1 $\leq i \leq D' - 1$).
(ii) $b_i \leq b'_{i-1}$ (1 $\leq i \leq D - 1$) and $b'_i \leq b_{i-1}$ (1 $\leq i \leq D' - 1$).
Lemma 2.3 With reference to Notation 2.1, if \( i + j \leq D \) and \( i + j \) is an even number, then \( c_i \leq b_j \).

Proof Fix \( u \in Y, v \in \Gamma_i(u), w \in \Gamma_{i+j}(u,v) \) and note that \( c_i = |\Gamma_{i-1,1}(u,v)| = |\Gamma_{i-1,1,j+1}(u,v,w)| \leq |\Gamma_{j,1+1}(v,w)| = b_j \).

\[ \Box \]

2.4 The intersection diagram of rank 1 of a DBG

For the moment, pick an odd \( i \) (\( 3 \leq i \leq \min(D, D') \)), \( x \in Y \) and \( z \in \Gamma_i(x) \). There are \( c_i c_{i-1} \cdots c_1 \) different paths of length \( i \) between \( x \) and \( z \). Since \( z \in \Gamma_i(x) \subset Y' \), the number of \( xz \)-paths of length \( i \) is also equal to \( c_i' c_{i-1}' \cdots c_1' \). Thus,

\[ c_1 c_2 \cdots c_i = c_i' c_2' \cdots c_i' \quad \text{for any odd } i \, (3 \leq i \leq \min(D, D')) \tag{2} \]

Similarly, if we count the number of ordered pairs \((x, y)\) such that \( x \in Y \) and \( y \in \Gamma_i(x) \) in two different ways, we have \( |Y| k_i = |Y'| k_i' \). In particular, \( |Y| k = |Y'| k' \). Using (1) and (2), we get

\[ b_1 b_2 \cdots b_{i-1} = b_i' b_2' \cdots b_{i-1}' \quad \text{for any odd } i \, (3 \leq i \leq \min(D, D')) \tag{3} \]

(see also [10, Proposition 3]).

Assume now that \( x \in Y \) and let \( y \in \Gamma_1(x) \). From the intersection diagram of rank 1 (see Fig. 2), every vertex from \( \Gamma_{1,2}(x, y) \) has exactly \( c_2' - 1 \) neighbors in \( \Gamma_{2,1}(x, y) \), and every vertex in \( \Gamma_{2,1}(x, y) \) is adjacent to \( c_2 - 1 \) vertices in \( \Gamma_{1,2}(x, y) \) (for more properties of combinatorial structure of \( \Gamma \) obtained from intersection diagram of rank 1, see [13, Lemma 4.3, Lemma 4.4, Lemma 4.5]). This yields

\[ b_1(c_2 - 1) = b_i'(c_2' - 1). \tag{4} \]

Note that \( \Gamma_{1,0}(x, y) = \{y\} \) and so \( |\Gamma_{1,0}(x, y)| = 1 \). Moreover, notice that for \( 1 \leq i \leq D \), every vertex from \( \Gamma_{i,i-1}(x, y) \) has exactly \( b_i \) neighbors in \( \Gamma_{i+1,i}(x, y) \), and
every vertex in $\Gamma_{i+1,i}(x, y)$ is adjacent to $c'_i$ vertices in $\Gamma_{i,i-1}(x, y)$. This shows $b_i \cdot |\Gamma_{i,i-1}(x, y)| = c'_i \cdot |\Gamma_{i+1,i}(x, y)|$ for every $1 \leq i \leq D - 1$. Therefore, by a straightforward induction argument it follows that
\[ |\Gamma_{i+1,i}(x, y)| = \frac{b_1 b_2 \cdots b_i}{c'_1 c'_2 \cdots c'_i} \neq 0 \quad (1 \leq i \leq D - 1). \tag{5} \]

Analogously, using the intersection diagram (of rank 1) from Fig. 2 (or from [13, Section 4]), it is routine to compute
\[ |\Gamma_{0,1}(x, y)| = 1, \quad |\Gamma_{i,i+1}(x, y)| = \frac{b'_1 b'_2 \cdots b'_i}{c_1 c_2 \cdots c_i} \neq 0 \quad (1 \leq i \leq D' - 1). \tag{6} \]

From (5) and (6), we see that $\Gamma_{i+1,i}(x, y)$ ($0 \leq i \leq D - 1$) and $\Gamma_{i,i+1}(x, y)$ ($0 \leq i \leq D' - 1$) are all nonempty. All six equations (1)–(6) we use later in the paper.

**Proposition 2.4** With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $D \geq 3$. For every integer $i$ ($1 \leq i \leq \min \{D - 1, D' - 1\}$) the following hold:
\[ c'_{i+1} = c_i \iff c_{i+1} = c'_i, \tag{7} \]
\[ c'_{i+1} > c_i \iff c_{i+1} > c'_i. \tag{8} \]

**Proof** Pick $x \in Y, z \in \Gamma_1(x)$ and an integer $i$ ($1 \leq i \leq \min \{D - 1, D' - 1\}$). By (5) and (6) the sets $\Gamma_{i,i+1}(x, z)$ and $\Gamma_{i+1,i}(x, z)$ are nonempty. Using the intersection diagram (of rank 1) from Fig. 2, it follows that every vertex in $\Gamma_{i,i+1}(x, z)$ has exactly $c'_{i+1} - c_i$ neighbors in $\Gamma_{i+1,i}(x, z)$, and similarly, every vertex in $\Gamma_{i+1,i}(x, z)$ is adjacent to exactly $c_{i+1} - c'_i$ neighbors in $\Gamma_{i,i+1}(x, z)$ (see also [13, Lemma 4.3, Lemma 4.4, Lemma 4.5] for deeper insight in the combinatorial structure). Thus, $(c'_{i+1} - c_i)|\Gamma_{i,i+1}(x, z)| = |\Gamma_{i+1,i}(x, z)|(c_{i+1} - c'_i)$ and the claim follows. \( \Box \)

**Corollary 2.5** With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $D \geq 3$. If there exists an integer $j$ ($2 \leq j \leq \min \{D - 1, D' - 1\}$) such that $c_i = 1$ for all $1 \leq i \leq j$ then $c'_i = 1$ for all $1 \leq i \leq j$.

**Proof** Using mathematical induction on $i$, the result follows immediately from (7). \( \Box \)

For more relations between intersection numbers of the color partitions, see [10, 13].

### 2.5 A $(\kappa, g)$-cage

A $(\kappa, g)$-cage is a $\kappa$-regular graph of girth $g$ and minimum possible number of vertices $v = v(\kappa, g)$ consistent with the valency $\kappa$ and the girth $g$. The problem of determining $v(\kappa, g)$ was posed in 1959 by F. Kátezsi who noticed that $v(3, 5) = 10$ was realized by the Petersen graph. For a $(\kappa, g)$-cage with diameter $d$, the best-known upper bound for the scalar $v(\kappa, g)$ is $1 + \kappa(\kappa - 1) + \cdots + \kappa(\kappa - 1)^{d-1}$, and determining
Fig. 3  Distance-biregular graph with intersection array $(3, 1, 2, 1, 2; 1, 1, 1, 1, 2)$ for one color class (the left-hand side), and $(2, 2, 1, 2, 1; 1, 1, 1, 1, 2, 2)$ for the other color class (the right-hand side).

$v(\kappa, g)$ as well as the construction of $(\kappa, g)$-cages have been studied by many authors (for more information we refer to [11, 12, 20, 22, 34, 40, 41]). In fact if a regular graph of valency $\kappa$ and diameter $d$ has $v$ vertices then $v \leq 1 + \kappa(\kappa - 1) + \cdots + \kappa(\kappa - 1)^{d-1}$. Graphs with $v = 1 + \kappa(\kappa - 1) + \cdots + \kappa(\kappa - 1)^{d-1}$ are called Moore graphs. Damerell [9] proved that a Moore graph of valency $\kappa \geq 3$ has diameter 2. The subdivision graph of a graph $\Gamma$ is the graph obtained from $\Gamma$ by replacing each of its edges by a path of length 2 (in Fig. 3 the subdivision graph of the Petersen graph is given). The following complete list of $(\kappa, g)$-cages is known (cf. [3, 32]): (ci) cycles of length $g$ ($g \geq 3$); (cii) complete graphs $K_{\kappa + 1}$ ($\kappa \geq 3$); (ciii) complete bipartite graphs $K_{\kappa,\kappa}$ ($\kappa \geq 3$); (civ) the Petersen graph; (cv) the Hoffman–Singleton graph; (cvi) the $(57, 5)$-cage graph; (cvii) some $(\kappa, g)$-cages with $g \in \{6, 8, 12\}$, for some values of $\kappa \geq 3$ (not yet completely classified). Note that the subdivision of the complete graph $K_{\kappa + 1}$ ($\kappa \geq 3$) has diameter 3 and the subdivision of the complete bipartite graph $K_{\kappa,\kappa}$ ($\kappa \geq 3$) has diameter 4.

In Theorem 2.6 and Lemma 2.7, we recall some well-known results of one particular subfamily of distance-biregular graphs which we use later.

**Theorem 2.6 ([32, Lemma 3.3, Theorem 3.4])** Let $\Gamma = (X, R)$ denote a $(\kappa, g)$-cage of diameter $d$. The subdivision graph $S(\Gamma)$ of $\Gamma$ is $(X, R)$-distance-biregular with the following intersection arrays.

(a) Assume that $g$ is odd. The eccentricity of $x \in X$ in $S(\Gamma)$ is $2d + 1$, and the intersection array for $x \in X$ in $S(\Gamma)$ is

$$\left(\begin{array}{ccccccc}
\kappa & 1 & \kappa - 1 & 1 & \kappa - 1 & \cdots & 1 \\
\frac{1}{b_0} & \frac{1}{b_1} & \frac{1}{b_{2d-1}} & \frac{1}{b_{2d}} & \frac{1}{c_{2d+1}} & \cdots & \frac{1}{1} \end{array}\right)$$
The eccentricity of $e \in R$ in $S(\Gamma)$ is $2d + 2$, and the intersection array for $e \in R$ in $S(\Gamma)$ is

\[
\left( \frac{2}{b_0'}, \kappa - 1, 1, \kappa - 1, 1, \ldots, \kappa - 1, \frac{1}{b_1'}, \kappa - 2; 1, 1, 1, 1, \ldots, 1, 1, 2, \frac{2}{c_{2d'}} \right).
\]

(b) Assume that $g$ is even. The eccentricity of $x \in X$ in $S(\Gamma)$ is $2d$, and the intersection array for $x \in X$ in $S(\Gamma)$ is

\[
\left( \frac{\kappa}{b_0}, \frac{1}{b_1}, \kappa - 1, 1, \kappa - 1, 1, \ldots, \kappa - 1, \frac{1}{b_{2d - 2}}, \frac{1}{b_{2d - 1}}, 1, 1, 1, 1, \ldots, 1, 1, \frac{\kappa}{c_{2d}} \right).
\]

The eccentricity of $e \in R$ in $S(\Gamma)$ is $2d$, and the intersection array for $e \in R$ in $S(\Gamma)$ is

\[
\left( \frac{2}{b_0'}, \kappa - 1, 1, \kappa - 1, 1, \ldots, \frac{1}{b_1'}, \kappa - 1; 1, 1, 1, 1, \ldots, 1, 1, 2 \right).
\]

**Lemma 2.7** ([32, Corollary 3.5]) With reference to Notation 2.1, a graph $\Gamma$ with vertices of valency 2 is distance-biregular if and only if $\Gamma$ is either a complete bipartite graph $K_{2,n}$ (for $n \geq 1$) or the subdivision graph of a $(\kappa, g)$-cage.

### 2.6 The intersection diagram of rank 2 and the scalars $p_{2i,i}^1$ and $p_{i,i}^2$

For a $(Y, Y')$-distance-biregular graph $\Gamma$ with $D \geq 2$, pick $x \in Y, y \in \Gamma_2(x)$ and consider the intersection diagram from Fig. 4. Using the structure of $\Gamma$ from this intersection diagram, and the fact that $\Gamma$ is distance-biregular, for every $i$ ($1 \leq i \leq D$) we have

\[
\Gamma_i(x) = \Gamma_{i,i+2}(x, y) \cup \Gamma_{i,i}(x, y) \cup \Gamma_{i,i-2}(x, y), \quad (9)
\]

\[
z \in \Gamma_{i,i-2}(x, y) \cup \Gamma_{i-2,i}(x, y) \Rightarrow \quad \left| \Gamma_1(z) \cap \Gamma_{i-1,i-1}(x, y) \right| = c_i - c_{i-2} = b_{i-2} - b_i, \quad (10)
\]

\[
\Gamma_{1,1}(x, y) \neq \emptyset \quad \text{and} \quad \Gamma_{D-1,D-1}(x, y) \neq \emptyset. \quad (11)
\]

For example, if $\Gamma_{D-1,D-1}(x, y) = \emptyset$ then vertices in $\Gamma_{D-2,D}(x, y)$ have no neighbors in $\Gamma_{D-1}(x)$, that is $b_{D-2} = 0$, a contradiction with the definition of a distance-biregular graph. From the same diagram, it is routine to compute

\[
|\Gamma_{2,0}(x, y)| = 1, \quad |\Gamma_{i,i-2}(x, y)| = \frac{b_2 b_3 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-2}} \neq 0 \quad (3 \leq i \leq D), \quad (12)
\]
Fig. 4 Intersection diagram of rank 2 of a distance-biregular graph with respect to $x \in X$ and $y \in \Gamma_2(x)$, where $\Gamma_{i,j} = \Gamma_{i,j}(x, y) (0 \leq i, j \leq D)$

$$|\Gamma_{0,2}(x, y)| = 1, \quad |\Gamma_{i,i+2}(x, y)| = \frac{b_2 b_3 \cdots b_{i+1}}{c_1 c_2 \cdots c_i} \neq 0 \quad (1 \leq i \leq D - 2), \quad (13)$$

$$|\Gamma_{i,i}(x, y)| = \frac{1}{c_2} (b_0 b_1 - b_2 b_3 - c_2) = \frac{1}{c_2} (c_2 (b_1 - 1) + b_2 (c_3 - 1)) \quad (14)$$

$$|\Gamma_{i,i}(x, y)| = |\Gamma_i(x)| - |\Gamma_{i,i-2}(x, y)| - |\Gamma_{i,i+2}(x, y)| = \frac{b_2 b_3 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} (k b_1 - b_i b_{i+1} - c_{i-1} c_i) \quad (3 \leq i \leq D - 2), \quad (15)$$

$$|\Gamma_{D-1,D-1}(x, y)| = |\Gamma_{D-1}(x)| - |\Gamma_{D-1,D-3}(x, y)| = \frac{b_2 b_3 \cdots b_{D-2}}{c_1 c_2 \cdots c_{D-1}} (k b_1 - c_{D-2} c_{D-1}), \quad (16)$$

$$|\Gamma_{D,D}(x, y)| = |\Gamma_D(x)| - |\Gamma_{D,D-2}(x, y)| = \frac{b_2 b_3 \cdots b_{D-1}}{c_1 c_2 \cdots c_D} (k b_1 - c_{D-1} c_D). \quad (17)$$

Now fix $i$ ($1 \leq i \leq D$), and pick $z \in \Gamma_i(x)$. We define the numbers $p_{2,i}^i$ and $p_{1,i}^2$ as follows:

$$p_{2,i}^i = p_{2,i}^i(x, z) := |\Gamma_2(x) \cap \Gamma_i(z)|, \quad p_{1,i}^2 = p_{1,i}^2(x, y) := |\Gamma_i(x) \cap \Gamma_i(y)|.$$

Fixing $x \in Y$, and counting the number of ordered pairs $(y, z) \in X \times X$, where $y \in \Gamma_2(x), z \in \Gamma_i(x)$ and $\partial(y, z) = i$, it is routine to show that

$$k_2 p_{1,i}^2 = k_i p_{2,i}^i. \quad (18)$$
Note that $p^2_{2,1} = b_1$, $p^2_{1,1} = c_2$, and note that $p^2_{i,i} = 0$ if and only if $p^i_{2,i} = 0$. From (1), (14)–(18), we have that the numbers $p^i_{2,i}(x, z)$ and $p^2_{i,i}(x, y)$ do not depend on the choices of $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_1(x)$, but only on the value of $i$.

**Lemma 2.8** With reference to Notation 2.1, pick $i$ ($2 \leq i \leq D - 2$). Then,

$$p^2_{i,i} = 0 \quad \text{if and only if} \quad kb_1 - b_ib_{i+1} - c_{i-1}c_i = 0. \quad (19)$$

Furthermore, $p^2_{D-1,D-1} \neq 0$, and $p^2_{D,D} = 0$ if and only if either $D$ is even and $b_{D-1} = 1$, or $D$ is odd and $b'_{D-1} = 1$ (in the case when $D$ is odd, we also have $D = D'$).

**Proof** Note that $p^2_{i,i} = |\Gamma_{i,i}(x, y)|$ ($2 \leq i \leq D$). Equation (19) follows immediately from (15). The claim $p^2_{D-1,D-1} \neq 0$ follows from (11). The proof for the third claim we split it in two cases.

**Case 1.** Assume that $D$ is even. By (17), $p^2_{D,D} = 0$ if and only if $kb_1 = c_{D-1}c_D$. Since $c_D = k$ we have $p^2_{D,D} = 0$ if and only if $b_1 = c_{D-1}$. On the other hand $D - 1$ is odd, and we have $c_{D-1} + b_{D-1} = k' = b_1 + 1 = c_{D-1} + 1$. The result follows.

**Case 2.** Assume that $D$ is odd. Pick $x \in Y$, and note that $\Gamma_D(x) \subseteq Y'$. This yields $D' \geq D$. Since $\min\{D, D'\} = D$, by (2) we have $c_{D-1}c_D = c'_{D-1}c'_D$. Now (17) yields $p^2_{D,D} = 0$ if and only if $kb_1 = c'_{D-1}c'_D$. Note that $b'_{D-1} + c'_{D-1} = k' = b_1 + 1$, and with it $b'_{D-1} - 1 = b_1 - c'_{D-1}$. As $c'_D + b'_D = k$ we have $p^2_{D,D} = 0$ if and only if $c'_D(b_1 - c'_{D-1}) + b'_Db_1 = 0$. If $D' = D + 1$ then $c'_D(b_1 - c'_{D-1}) + b'_Db_1 = c'_D(b'_{D-1} - 1) + b'_Db_1 = 0$, a contradiction (the numbers $b_1, b'_{D-1}$ and $b'_D$ are positive).

Thus, we have $D' = D$ and $b'_D = 0$. With it $p^2_{D,D} = 0$ if and only if $b_1 = c'_{D-1}$, and the result follows. □

3 The scalars $\gamma_i$ and the case when $c'_2 \geq 2$

Let $\Gamma$ denote an almost $2$-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$. In this section, we define scalars $\gamma_i$ in the following way. Pick $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$. Then,

$$\gamma_i := |\Gamma_{i-1}(z) \cap \Gamma_{1,1}(x, y)| \quad (1 \leq i \leq D - 2).$$

In addition, if $\Gamma$ is $2$-$Y$-homogeneous, for $z \in \Gamma_{D-1,D-1}(x, y)$, we also define $\gamma_{D-1}$ by

$$\gamma_{D-1} := |\Gamma_{D-2}(z) \cap \Gamma_{1,1}(x, y)|.$$

Since we are working with an (almost) $2$-$Y$-homogeneous DBG, we are not interested in defining $\gamma_D$. Clearly, we have $\gamma_1 = 1$. In the case when $D - 1 = D'$, we have $\gamma_{D-1} = c_2$ (see the proof of Corollary 7.3). In Sect. 6, we show that the scalars $\gamma_i$ are nonzero, when $k' \geq 3$. 

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In this section, we compute some equalities for the case $c'_2 \geq 2$, and we show that, if $c'_2 \geq 3$ then $D \leq 5$.

**Lemma 3.1** With reference to Notation 2.1, let $\Gamma$ denote a 2-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$. Pick $i$ $(1 \leq i \leq \min\{D - 1, D' - 1\})$, and assume that $\gamma_2$ and $\gamma_i$ are nonzero.

(i) If $c'_2 \geq 2$, then $(k' - 2)(\gamma_2 - 1) = (c_2 - 1)(c'_2 - 2)$.

(ii) If $i$ is even, then $\gamma_i(c_{i+1} - 1) = c_i(c'_2 - 1)$.

(iii) If $i$ is odd, then $\gamma_i(c'_{i+1} - 1) = c'_i(c'_2 - 1)$.

**Proof** Our proof is along the lines of the proof of [36, Lemma 5.1], where the author studied bipartite distance-regular graphs.

(i) Note that, by assumption we have $k' \geq 3$ (since $k' = c'_2 + b'_2$ and $D' \geq 3$). Pick $u \in Y$, $v \in \Gamma_1(u)$ and $w \in \Gamma_{2,1}(u, v)$. We count the number $N$ of ordered pairs $(x, y)$ with $x \in \Gamma_{1,2,1}(u, v, w)$ and $y \in \Gamma_{2,1,2}(u, v, w, w')$ in two different ways. Since $w \in \Gamma_2(u)$, there are precisely $c_2 - 1$ vertices $x \in \Gamma_{1,1}(u, w)$ with $x \neq v$. Fix such a vertex $x$. Since $x \in \Gamma_2(v) \subseteq Y'$, there are precisely $c'_2 - 2$ vertices $y \in \Gamma_{1,1}(x, v)$ with $y \neq u$, $y \neq w$. So we have $N = (c_2 - 1)(c'_2 - 2)$. On the other hand, there are precisely $k' - 2$ vertices $y \in \Gamma_{2,1}(u, v)$ with $y \neq w$. Fix such a vertex $y$ and note that $y \in \Gamma_2(w)$. Since $\partial(y, w) = 2$ and $u \in \Gamma_{2,1}(y, w, w')$ and $y$ have precisely $\gamma_2 - 1$ common neighbors $x \in \Gamma_1(u)$ with $x \neq v$. So we obtain $N = (k' - 2)(\gamma_2 - 1)$.

(ii) Assume that $i$ is even. Pick $u \in Y$, $v \in \Gamma_i(u)$ and $w \in \Gamma_{i+1,1}(u, v)$. We will count the number $N$ of ordered pairs $(x, y)$ with $x \in \Gamma_{i-1,1}(u, v)$, $y \in \Gamma_{i,2,1}(u, v, w, w')$ and $\partial(x, y) = 1$ in two different ways. Firstly, since $v \in \Gamma_i(u) \subseteq Y$ we notice that $v$ has exactly $c_i$ neighbors $x$ at distance $i - 1$ from $u$. Fix such a vertex $x$. Since $w \in Y'$ and $\partial(x, w) = 2$, we have that $w$ has precisely $c'_2 - 1$ neighbors $y \in \Gamma_i(x)$ with $y \neq v$. Hence, it follows that $N = c_i(c'_2 - 1)$. On the other hand, since $\partial(u, w) = i + 1$, vertex $w$ has precisely $c_{i+1} - 1$ neighbors $y \in \Gamma_i(u)$ with $y \neq v$. Fix such a vertex $y$. By the triangle inequality in $vwy$, we have $\partial(v, y) \in \{0, 1, 2\}$, and an easy analysis show that cases $\partial(v, y) \in \{0, 1\}$ are not possible. Now since $v, y \in \Gamma_i(u)$ and $\partial(v, y) = 2$, vertices $v$ and $y$ have precisely $\gamma_i$ common neighbors $x$ in $\Gamma_{i-1}(u)$. So we get $N = \gamma_i(c_{i+1} - 1)$. The result follows.

(iii) Assume that $i$ is odd. Pick $u \in Y'$, $v \in \Gamma_i(u)$ and $w \in \Gamma_{i+1,1}(u, v)$. Note that $w \in Y'$. The result follows by counting the number $N$ of ordered pairs $(x, y)$ with $x \in \Gamma_{i-1,1}(u, v)$, $y \in \Gamma_{i,2,1}(u, v, w, w')$ and $\partial(x, y) = 1$ in two different ways (similarly as in (ii)). \qed

**Remark 3.2** Note that all three claims of Lemma 3.1 hold also for an almost 2-$Y$-homogeneous distance-biregular graph, under the assumption that $D \geq 4$ and $1 \leq i \leq D - 2$.

**Lemma 3.3** With reference to Notation 2.1, let $\Gamma$ denote a 2-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$. If $c'_2 \geq 2$, then the following hold.

(i) $k' \geq 3$ and $\gamma_2 \geq 1$.

(ii) $c_3 = \frac{c_2(c'_2 - 1)}{\gamma_2} + 1$.

(iii) $(k' - 2)(c_2 - \gamma_2) = (k - c_2)(c'_2 - 1)$. 

\[ \square \] Springer
Proof (i) If $D \geq 3$ then the case $D' = 2$ is not possible (note that for any $u \in Y$ we have $\Gamma_3(u) \subseteq Y'$, which yields $D' \geq 3$). The fact $D' \geq 3$ implies $b_2' \geq 1$. Since $k' = c_2' + b_2'$ we have $k' \geq 3$. Now pick $u \in Y$ and $v \in \Gamma_2(u)$. If there are no edges between vertices of $\Gamma_{1,1}(u,v)$ and $\Gamma_{2,2}(u,v)$ we would get that all vertices in $\Gamma_{1,1}(u,v)$ are of valency 2, a contradiction (vertices in $\Gamma_{1,1}(u,v)$ are of valency $k' \geq 3$; for an illustration see Fig. 4). This implies that $\Gamma_{2,2}(u,v)$ is nonzero and that $\gamma_2 \geq 1$.

(ii) Pick $u \in Y$, $v \in \Gamma_2(u)$ and $w \in \Gamma_{3,1}(u,v)$. By (i), $\gamma_2$ is nonzero. By counting the number $N$ of ordered pairs $(x,y)$ with $x \in \Gamma_{1,1}(u,v)$, $y \in \Gamma_{2,1}(u,v,w)$ and $d(x,y) = 1$ in two different ways, we get $c_2(c_2'-1) = N = \gamma_2(c_3-1)$ (see also the proof of Lemma 3.1(ii)). Since $c_2' \geq 2$, by Proposition 2.4 we have $c_2 \geq 2$, and the product $c_2(c_2'-1)$ is nonzero. Since $c_2(c_2'-1) = \gamma_2(c_3-1)$, we have $c_3-1 = \frac{c_2(c_2'-1)}{\gamma_2}$, and the result follows.

(iii) Pick $x \in Y$ and $y \in \Gamma_2(x)$. By (14) and (ii) above,

$$|\Gamma_{2,2}(x,y)| = p_2^2 = k' - 2 + \frac{b_2(c_2'-1)}{\gamma_2}. \quad (20)$$

On the other hand, since every vertex from $\Gamma_{2,2}(x,y)$ has exactly $\gamma_2$ neighbors in $\Gamma_{1,1}(x,y)$, we have

$$|\Gamma_{2,2}(x,y)| = \frac{c_2(k'-2)}{\gamma_2}. \quad (21)$$

Using (20) and (21), the result follows. $\square$

Theorem 3.4 With reference to Notation 2.1, let $\Gamma$ denote a $2$-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$. If $c_2' \geq 3$, then $D \leq 5$.

Proof From Lemma 3.3(i) we have $\gamma_2 \geq 1$. As $b_1$ and $b_1'$ are positive integers, (4) yields $c_2 = 1$ if and only if $c_2' = 1$. By assumption $c_2' \geq 3$ which implies $c_2 \geq 2$ (otherwise the left-hand (and the right-hand) side of (4) is equal to zero, a contradiction). We have that $(c_2-1)(c_2'-2)$ is nonzero, and that $c_2 + b_2' = k' \geq 3$. Now by Lemma 3.1(i), $\gamma_2 - 1 > 0$, which yields $\gamma_2 \geq 2$.

Our proof is by contradiction. Assume that $D \geq 6$. Lemma 2.3 yields $c_3 \leq b_3$; that is, $2c_3 \leq k'$, or expressed differently, $2(c_3 - 1) \leq k' - 2$. By Lemma 3.1(i)(ii), this yields

$$2 \cdot \frac{c_2(c_2'-1)}{\gamma_2} \leq \frac{(c_2-1)(c_2'-2)}{\gamma_2 - 1}.$$

Multiplying both sides by $\gamma_2(\gamma_2 - 1)$, we get

$$2c_2(c_2'-1)(\gamma_2 - 1) \leq (c_2-1)(c_2'-2)\gamma_2.$$

Since $(c_2-1)(c_2'-2)\gamma_2 < c_2(c_2'-1)\gamma_2$, we have $2(\gamma_2 - 1) < \gamma_2$, that is, $\gamma_2 < 2$ a contradiction. $\square$
Remark 3.5 Let $\Gamma$ denote a $2$-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D \geq 3$. From the proof of Theorem 3.4, we also have that $c'_2 \geq 3$ yields $b_3 < c_3$ (assumption $c_3 \leq b_3$ yields $\gamma_2 < 2$, a contradiction).

4 Distance-biregular graphs with $k' = 2$

By Lemma 2.7, a graph $\Gamma$ with $2$-valent vertices is distance-biregular if and only if either $\Gamma = K_{2,r}$, or $\Gamma$ is the subdivision graph of a $(\kappa, g)$-cage. In this section, we show that a $(Y, Y')$-distance-biregular graph with $k' = 2$ is $2$-$Y$-homogeneous, and we give combinatorial properties of such graphs. As a corollary, we get that the subdivision graph of a $(\kappa, g)$-cage $\Gamma$ (with vertex set $X$) is $2$-$X$-homogeneous. The combinatorial structure of distance-biregular graphs with $k' = 2$ plays an important role later in the paper.

Proposition 4.1 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph. If $D = 3$, then the following are equivalent.

(i) $\Gamma$ is the subdivision graph of a complete graph $K_n$ ($n \geq 3$).
(ii) $k' = 2$.

Moreover, if (i), (ii) hold with $Y$ as the set of vertices of the complete graph $K_n$, then $\Gamma$ is $2$-$Y$-homogeneous, and the intersection array of the color class $Y$ is

$$(n - 1, 1, n - 2; 1, 1, 2).$$

Proof (i)$\Rightarrow$(ii) Immediate from the definition of subdivision graph.

(ii)$\Rightarrow$(i) Assume that $\Gamma$ is a distance-biregular graph with vertices in $Y'$ of valency $2$. Since $D = 3$, by Lemma 2.7, $\Gamma$ is the subdivision graph of a $(\kappa, g)$-cage $\Gamma^\circ = (X^\circ, R^\circ)$. By Theorem 2.6, the girth $g$ of $\Gamma^\circ$ is odd as otherwise vertices of $\Gamma^\circ$ have even eccentricity. Even more, all vertices of eccentricity $3$ are those which lie in $X^\circ$. This shows that diameter $d$ of $\Gamma^\circ$ is equal to $1$ (as every vertex in $X^\circ$ has eccentricity $2d + 1$). Thus, $\Gamma^\circ$ is a complete graph $K_n$ ($n \geq 3$), and the result follows.

Assume now that (i), (ii) hold. By Theorem 2.6, vertices in $\Gamma$ of the color partition $Y$ have intersection array $(n - 1, 1, n - 2; 1, 1, 2)$. Pick $x \in Y$, $y \in \Gamma_2(x)$ and consider the combinatorial structure of $\Gamma$ from Fig. 4. Since $c_2 = 1$ and $k' = 2$, the unique vertex of $\Gamma_{1,1}(x, y)$ does not have neighbors in $\Gamma_{2,2}(x, y)$. Therefore, for every $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{2,2}(x, y)$, $|\Gamma_1(z) \cap \Gamma_{1,1}(x, y)| = 0$. The result follows.

Theorem 4.2 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph. If $D \geq 4$, then the following are equivalent.

(i) Either $\Gamma$ is the subdivision graph of a $(\kappa, g)$-cage $\Gamma^\circ = (X^\circ, R^\circ)$ (where $\Gamma^\circ$ has diameter $d \geq 2$, valency $\kappa \geq 3$ and girth $g \geq 3$) with $Y = X^\circ$ or it is an even-length cycle.
(ii) For all $x \in Y$ and $y \in \Gamma_2(x)$, the sets $\Gamma_{i,i}(x, y)$ ($2 \leq i \leq D - 2$) are empty.
(iii) There exists $i$ ($2 \leq i \leq D - 2$) such that $p^2_{i,i} = 0$.
(iv) $k' = 2$. 

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Moreover, if (i)–(iv) hold with \( Y \) as the set of vertices of a \((\kappa, g)\)-cage, then \( \Gamma \) is 2-\( Y \)-homogeneous, and the intersection array of the color class \( Y \) is one of the follow two types:

\[
(k, 1, k - 1, 1, k - 1, \ldots, 1, k - 1; 1, 1, 1, 1, \ldots, 1, 1, 2), \quad \text{for odd } D
\]
\[
(k, 1, k - 1, 1, k - 1, \ldots, k - 1, 1; 1, 1, 1, 1, \ldots, 1, k), \quad \text{for even } D.
\]

**Proof**

(i)⇒(ii) If \( \Gamma^o \) is an even-length cycle the result immediately follows. Let \( \Gamma^o = (X^o, R^o) \) denote a \((\kappa, g)\)-cage with diameter \( d \geq 2, \kappa \geq 3 \) and \( g \geq 3 \). Assume that \( \Gamma \) is the subdivision graph of \( \Gamma^o \) and recall that by assumption \( D \geq 4 \). The color partitions of \( \Gamma \) are \( Y = X^o \) and \( Y' = R^o \). Moreover, the intersection arrays of \( \Gamma \) are given in Theorem 2.6 and they depend on the parity of \( g \). Thus, these two cases must be considered. Pick \( x \in X^o = Y \), and note that \( D = 2d + 1 \), if \( g \) is odd; while \( D = 2d \), if \( g \) is even. Then, for every \( i \) (\( 2 \leq i \leq D - 2 \)) we have \( b_i b_{i+1} = \kappa - 1 \) and \( c_{i-1} c_i = 1 \). Since \( b_1 = 1 \), it follows that \( \kappa b_1 - b_i b_{i+1} - c_{i-1} c_i = 0 \) (\( 2 \leq i \leq D - 2 \)). The result follows from Lemma 2.8.

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Let \( x \in Y, y \in \Gamma_2(x) \), pick \( i \) (\( 2 \leq i \leq D - 2 \)) and assume that \( \Gamma_{i,i}(x, y) = \emptyset \) if and only if \( kb_1 - b_i b_{i+1} - c_{i-1} c_i = 0 \). Since \( b_i + c_i = k \), we have

\[
k b_1 - b_i b_{i+1} - c_{i-1} c_i = c_i (b_1 - c_i - 1) + b_i (b_1 - b_{i+1}) = 0.
\]  (22)

Note that \( b_1 = k' - c_1 = k' - 1 \geq k' - b_{i-1} = c_{i-1} \). Also, by Lemma 2.2, we have \( b_{i+1} \leq b'_i \leq b_{i-1} \) which yields \( b_1 \geq b_{i+1} \). Equation (22) now yields \( b_{i+1} = b_1 = c_{i-1} \). On the other hand, since \( b_1 = k' - 1 \), we have

\[
c_{i-1} = b_{i+1} = k' - 1.
\]  (23)

Note that, as \( i \) is even, the integers \( i - 1 \) and \( i + 1 \) are odd, and we have \( c_{i-1} + b_{i-1} = k' \) and \( c_{i+1} + b_{i+1} = k' \). Now by (23), we get \( b_{i-1} = c_{i+1} = 1 \), and since \( c_{i-1} \leq c'_i \leq c_{i+1} \) and \( b_{i+1} \leq b'_i \leq b_{i-1} \) (see Lemma 2.2) we have

\[
c_{i-1} = b_{i+1} = 1.
\]  (24)

The result now follows from (23) and (24).

**Case 2.** Assume that \( i \) is odd. By (10), note that every vertex \( z \in \Gamma_{i+1,i-1}(x, y) \) has exactly \( c_{i+1} - c_{i-1} \) neighbors in \( \Gamma_{i,i}(x, y) \). Since by our assumption \( \Gamma_{i,i}(x, y) \) is empty, it must be \( c_{i+1} = c_{i-1} \). Now, \( k = b_{i+1} + c_{i+1} = b_{i+1} + c_{i-1} \) (since \( i + 1 \) is even) and we have

\[
k b_1 - b_i b_{i+1} - c_{i-1} c_i = c_{i-1} (b_1 - c_i) + b_{i+1} (b_1 - b_i) = 0.
\]  (25)

Since \( b_1 + c_1 = k' = b_i + c_i, b_1 = k' - 1 \geq k' - b_i = c_i \). Now, similarly as in Case 1 (from Lemma 2.2) it is not hard to see that \( b_1 \geq b_i \). Using the last two facts together
with (25), we have $b_1 = c_i = b_i$. In the end, since $k' - 1 = b_1 = b_i = k' - c_i$ and $k' - 1 = b_1 = c_i = k' - b_i$ we get $c_i = 1 = b_i$, and the result follows.

(iv)$\Rightarrow$(i) By assumption $D \geq 4$, so $\Gamma$ is not a complete bipartite graph $K_{2,n}$ for some $n \geq 1$. By Lemma 2.7 it follows that $\Gamma$ is either the subdivision graph of a $(\kappa, g)$-cage (with diameter $d \geq 2$, valency $\kappa \geq 3$ and girth $g \geq 3$) or it is an even-length cycle. The result follows.

\begin{corollary}
Let $\Gamma^o$ denote a $(\kappa, g)$-cage for integers $\kappa \geq 2$ and $g \geq 3$, with vertex set $X^o$. Then, the subdivision graph of $\Gamma^o$ is 2-$X^o$-homogeneous.
\end{corollary}

\begin{proof}
Let $\Gamma^o = (X^o, R^o)$ denote a $(\kappa, g)$-cage with $\kappa \geq 2$ and $g \geq 3$. By Theorem 2.6, the subdivision graph $\Gamma = S(\Gamma^o)$ (of $(\kappa, g)$-cage $\Gamma^o$) is distance-biregular with $k' = 2$ and color partitions $X^o$ and $R^o$. Pick $x \in X^o$ from $\Gamma = S(\Gamma^o)$, and consider two cases:

\textbf{Case 1.} Assume that $k = 2$. Then, $\Gamma$ is an even-length cycle, and therefore $\Gamma$ is 2-$X^o$-homogeneous. \textbf{Case 2.} Assume that $k \geq 3$. Then, $D = 2$ is not possible. So, if $D = 3$ then $\Gamma^o$ is a complete graph $K_{\kappa+1}$ for some $\kappa \geq 3$, which, by Proposition 4.1, implies that $\Gamma$ is 2-$X^o$-homogeneous. If $D \geq 4$, then by Theorem 4.2, for every $y \in \Gamma_2(x)$ and $i \ (2 \leq i \leq D - 2)$ the set $\Gamma_{i,i}(x, y)$ is empty. Thus, for all $i \ (2 \leq i \leq D - 1)$ and for all $x \in X^o$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$, the number $|\Gamma_{i-1}(z) \cap \Gamma_{1,1}(x, y)|$ equals 0. The result follows.
\end{proof}

Note that if $k' = 2$ then $|\Gamma_2(x)| = \deg(x)$ (where $\deg(x)$ denotes the number of neighbors of $x$). Moreover, as we will see in Corollary 9.2, for the general case when $D = 3$, we have that graph $\Gamma$ is 2-$Y$-homogeneous if and only if $|\Gamma_2(x)| = \deg(x)$.

\begin{remark}
Assume that $\Gamma$ is a $(Y, Y')$-distance-biregular graph with $k' = 2$. Since $c'_2 + b'_2 = k'$, we have $c'_2 = 1$. In Sect. 8, we study distance-biregular graphs with $k' \geq 3$ and $c'_2 = 1$.
\end{remark}

5 The scalars $\Delta_i$, part I

In this section, we define certain scalars $\Delta_i \ (2 \leq i \leq \min\{D - 1, D' - 1\})$, which can be computed from the intersection array of a given distance-biregular graph. These scalars play an important role, since from their values we can decide if a given distance-biregular graph is (almost) 2-$Y$-homogeneous or not. The main ideas for both defining the scalar $\Delta_i$ and proving some related results are taken from the theory of bipartite distance-regular graphs (in particular [7, 27, 28, 31, 38, 39]).

\begin{definition}
Let $\Gamma$ denote a distance-biregular graph $\Gamma$ with color partitions $(Y, Y')$, pick $i \ (1 \leq i \leq \min\{D - 1, D' - 1\})$, and define the scalar $\Delta_i = \Delta_i(Y)$ in the following way:

$$\Delta_i = \begin{cases} (b_{i-1} - 1)(c_{i+1} - 1) - p_{2,i}^i(c'_2 - 1) & \text{if } i \text{ is even}, \\ (b'_{i-1} - 1)(c'_{i+1} - 1) - p_{2,i}^i(c'_2 - 1) & \text{if } i \text{ is odd}. \end{cases}$$

\end{definition}

\begin{remark}
If $D$ is odd, since for any $x \in Y$, $\Gamma_D(x) \subseteq Y'$, we have $D' \geq D$, and with it for odd $D$, $\min\{D - 1, D' - 1\} = D - 1$. This yields that, if $D$ is even and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Illustration of the scalar $\Delta_i$.}
\end{figure}
If \( D' = D - 1 \) then the numbers \( \Delta_i \) are defined for \( 2 \leq i \leq D - 2 \), while in all other cases the scalars are defined for \( 2 \leq i \leq D - 1 \). We already mentioned that, in the case when \( D' = D - 1 \) we have \( \gamma_{D-1} = c_2 \) (see the proof of Corollary 7.3), which yields that we do not need the definition of \( \Delta_{D-1} \) for this case. Since we are working with (almost) 2-Y-homogeneous DBG, we are not interested in the case when \( i = D \).

The next inequality will be important and useful later.

**Lemma 5.3** Let \( s_i, t_i \in \mathbb{R} \) (1 \( \leq i \leq n \)) and \( c \in \mathbb{R} \). If \( s_i + t_i = c \) for all \( i \) (1 \( \leq i \leq n \)), then

\[
\left( \sum_{i=1}^{n} s_i \right) \left( \sum_{i=1}^{n} t_i \right) \geq n \left( \sum_{i=1}^{n} s_i t_i \right).
\]

Moreover, the equality holds if and only if all the numbers \( s_i \) (1 \( \leq i \leq n \)) are equal to their arithmetic mean.

**Proof** The idea for the proof we found in [8, Chapter 4].

Let \( \bar{s} \) denote the average value of the numbers \( s_i \) for \( 1 \leq i \leq n \). Note that \( \sum_{i=1}^{n} s_i^2 = \bar{s} \sum_{i=1}^{n} s_i \), and compute \( \sum_{i=1}^{n} (s_i - \bar{s})^2 \).

**Lemma 5.4** With reference to Notation 2.1, let \( \Gamma \) denote a distance-biregular graph with \( k' \geq 3 \) and \( D \geq 3 \). Then, the number \( \Delta_i \) is nonnegative for every integer \( i \) (1 \( \leq i \leq \min \{ D - 1, D' - 1 \} \)).

**Proof** The idea for the proof we found in [7]. Pick \( x \in Y \) and fix an integer \( i \) (1 \( \leq i \leq \min \{ D - 1, D' - 1 \} \)). Let \( z \in \Gamma_i(x) \), consider the set \( \Gamma_{2,i}(x, z) \) and note that \( |\Gamma_{2,i}(x, z)| = p_{2,i}^j \). By (11), (18) and (19), the scalars \( p_{2,1}^j \) and \( p_{2,D' - 1}^j \) are nonzero. Moreover, if \( D \geq 4 \), since \( k' > 2 \), Theorem 4.2 yields that the scalar \( p_{2,j}^j \) is nonzero for \( 2 \leq j \leq D - 2 \). By (18), the scalar \( p_{2,j}^j \) is nonzero for all \( j \), so the set \( \Gamma_{2,i}(x, z) \) is not empty. For any \( y \in \Gamma_{2,i}(x, z) \), \( |\Gamma_{1,1,i-1}(x, y, z)| + |\Gamma_{1,1,i+1}(x, y, z)| = c_2 \), so Lemma 5.3 yields

\[
\left( \sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, y, z)| \right) \left( \sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i+1}(x, y, z)| \right) \geq |\Gamma_{2,i}(x, z)| \left( \sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, y, z)| \cdot |\Gamma_{1,1,i+1}(x, y, z)| \right) .
\]

The sum \( \sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, y, z)| \) represents the number of ordered pairs \((y, w)\) such that \( y \in \Gamma_{2,i}(x, z) \) and \( w \in \Gamma_{1,1,i-1}(x, y, z) \). We can compute this number in another way, that is, by counting first the number of vertices \( w \in \Gamma_{1,1,i-1}(x, z) \), and for every such \( w \) counting the number of vertices \( y \in \Gamma_{2,1,i}(x, w, z) \). For that purpose, we consider separately the cases depending on the parity of \( i \), because for example, if \( i \) is even then \( z \in Y \), and if \( i \) is odd then \( z \in Y' \). If \( i \) is even, since \( x \in \Gamma_i(z) \), \( x \)
has exactly $c_i$ neighbors $w$ at distance $i - 1$ from $z$, and if $i$ is odd, $x$ has exactly $c_i'$ neighbors $w$ at distance $i - 1$ from $z$. Next, for a given $w \in \Gamma_{1,i-1}(x, z)$ we count the number of vertices $y \in \Gamma_{2,i}(x, z)$ which are neighbors of $w$. Note that every vertex in $\Gamma_{1,i}(w, z)$ is either $x$ or is at distance 2 from $x$. Since $w \in \Gamma_{i-1}(z)$, we therefore have $b_{i-1}' - 1$ possibilities for the choice of $y$ if $i$ is even or $b_{i-1}' - 1$ possibilities for the choice of $y$ if $i$ is odd. Thus,

$$\sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, y, z)| = \begin{cases} c_i (b_{i-1}' - 1) & \text{if } i \text{ is even,} \\ c_i' (b_{i-1}' - 1) & \text{if } i \text{ is odd.} \end{cases} \quad (27)$$

Using the same technique as above, it is routine to compute

$$\sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{i+1}(z) \cap \Gamma_{1,1}(x, y)| = \begin{cases} b_i (c_i+1 - 1) & \text{if } i \text{ is even,} \\ b_i' (c_i'+1 - 1) & \text{if } i \text{ is odd,} \end{cases} \quad (28)$$

and

$$\sum_{y \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, y, z)||\Gamma_{1,1,i+1}(x, y, z)| = \begin{cases} c_i b_i (c_i' - 1) & \text{if } i \text{ is even,} \\ c_i' b_i' (c_i' - 1) & \text{if } i \text{ is odd.} \end{cases} \quad (29)$$

The result follows. \hfill \Box

Note that, in the proof of Lemma 5.4, in two different ways we counted the number of ordered pairs $(y, w)$ which are at certain distances from three fixed vertices $x$, $y$ and $z$. For a “different” and more advanced technique of counting the number of ordered tuples, we recommend [30, 33]. Next, we consider the case when the equality in (26) holds (the idea for Proposition 5.5 we found in [7], where the author studied bipartite distance-regular graphs).

**Proposition 5.5** With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. For any $i$ $(2 \leq i \leq \min \{ D - 1, D' - 1 \})$, the following are equivalent.

(i) The scalar $\Delta_i = 0$.

(ii) For all $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$, the number $|\Gamma_{1,1,i-1}(x, y, z)|$ is independent of the choice of $x$, $y$ and $z$. In addition, we have

$$\gamma_i := |\Gamma_{1,1,i-1}(x, y, z)| = \begin{cases} c_i (b_{i-1}' - 1)/p_{2,i}' & \text{if } i \text{ is even,} \\ c_i' (b_{i-1}' - 1)/p_{2,i}' & \text{if } i \text{ is odd.} \end{cases}$$

(iii) There exist $x \in Y$ and $z \in \Gamma_i(x)$ such that for all $y \in \Gamma_{2,i}(x, z)$ the number $|\Gamma_{1,1,i-1}(x, y, z)|$ is independent of the choice of $y$.

**Proof** Pick some integer $i$ $(2 \leq i \leq \min \{ D - 1, D' - 1 \})$. As in the proof of Lemma 5.4, it can be shown that the scalar $p_{2,i}'$ is nonzero.
(i)⇒(ii) In the proof of Lemma 5.4, using Lemma 5.3, we showed that for any \( x \in Y \) and \( z \in \Gamma_i(x) \), if \( i \) is even then
\[
(b_{i-1} - 1)(c_{i+1} - 1) \geq p^j_{2,i}(c'_2 - 1),
\]
and if \( i \) is odd then
\[
(b'_{i-1} - 1)(c'_{i+1} - 1) \geq p^j_{2,i}(c'_2 - 1).
\]
Note that equality holds if and only if all the numbers \(|\Gamma_{1,1,i-1}(x, y, z)|\) (where \( y \in \Gamma_{2,i}(x, z) \)) are equal to their arithmetic mean, that is, for every \( y \in \Gamma_{2,i}(x, z) \),
\[
|\Gamma_{1,1,i-1}(x, y, z)| = \frac{1}{|\Gamma_{2,i}(x, z)|} \sum_{w \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, w, z)|.
\]
Since by assumption \( \Delta_i = 0 \) the result follows from (27).

(ii)⇒(iii) Immediate.

(iii)⇒(i) Pick \( x \in Y \) and let \( z \in \Gamma_i(x) \). Since the number \( p^j_{2,i} \) is nonzero, the set \( \Gamma_{2,i}(x, z) \) is nonempty. Assume that for all \( y \in \Gamma_{2,i}(x, z) \) the number \(|\Gamma_{1,1,i-1}(x, y, z)|\) is independent of \( y \). Since these numbers do not depend on the choice of \( y \in \Gamma_{2,i}(x, z) \), the numbers \(|\Gamma_{1,1,i-1}(x, y, z)|\) are all equal to their average value, that is, for \( y \in \Gamma_{2,i}(x, z) \),
\[
|\Gamma_{1,1,i-1}(x, y, z)| \cdot |\Gamma_{2,i}(x, z)| = \sum_{w \in \Gamma_{2,i}(x, z)} |\Gamma_{1,1,i-1}(x, w, z)|.
\]
Observe \(|\Gamma_{1,1,i-1}(x, y, z)| + |\Gamma_{1,1,i+1}(x, y, z)| = c_2\) for all \( y \in \Gamma_{2,i}(x, z) \). Therefore, by Lemma 5.3, equality holds in (26). Now, the equality \( \Delta_i = 0 \) follows from (27), (28) and (29).

\[\square\]

6 Positivity of the scalars \( \gamma_i \) and equitable partitions

Let \( \Gamma \) denote a \((Y, Y')\)-distance-biregular graph with \( k' \geq 3 \) and \( D \geq 3 \). In this section, we show that for a \(2-Y\)-homogeneous graph \( \Gamma \), for every integer \( i \) \((1 \leq i \leq \min \{D - 1, D' - 1\})\) and for all \( x \in Y \), \( y \in \Gamma_2(x) \) and \( z \in \Gamma_{i,i}(x, y) \) the number \( \gamma_i = |\Gamma_{1,1,i-1}(x, y, z)| \) is nonzero. We also show that the collection of all the nonempty sets \( \Gamma_{i,j}(x, y) \) \((0 \leq i, j \leq D)\) (for any \( x \in Y \) and \( y \in \Gamma_2(x) \)) is an equitable partition of \( \Gamma \) if and only if \( \Gamma \) is \(2-Y\)-homogeneous.

**Lemma 6.1** With reference to Notation 2.1, let \( \Gamma \) denote a \((Y, Y')\)-distance-biregular graph with \( k' \geq 3 \) and \( D \geq 3 \). Pick \( x \in Y \), \( y \in \Gamma_2(x) \) and an integer \( i \) \((2 \leq i \leq \min \{D - 1, D' - 1\})\). Then, for \( z \in \Gamma_{i,i}(x, y) \) the following holds:
\[
\sum_{v \in \Gamma_{i-1,i-1}(x, y, z)} |\Gamma_{1,1,i-2}(x, y, v)| = c'_{i-1} |\Gamma_{1,1,i-1}(x, y, z)|. \tag{30}
\]
Proof We count, in two different ways, the number $N$ of ordered pairs $(u, v)$ with $\partial(u, v) = i - 2, u \in \Gamma_{i+1,i-1}(x, y, z)$ and $v \in \Gamma_{i-1,i+1}(x, y, z)$. Observe that for every fixed vertex $v \in \Gamma_{i-1,i-1}(x, y, z)$ we have $|\Gamma_{i+1,i-1}(x, y, z)|$ possible choices for $u$. For any such vertex $u$, $\partial(u, z) = i - 1$. So,

$$N = \sum_{v \in \Gamma_{i-1,i-1}(x, y, z)} |\Gamma_{i+1,i-1}(x, y, z)|.$$ 

We next fix $u \in \Gamma_{i+1,i-1}(x, y, z)$ and observe that, for such $u$ we have $|\Gamma_{i+1,i-1}(u, z)|$ possible choices for $v$. Applying the triangle inequality in triangles $xuv, xuy, uvy$ and $zvy$, we get $v \in \Gamma_{i-1,i-1}(x, y)$. Note also that $u \in Y'$ and $z \in \Gamma_{i-1}(u)$. Since $\Gamma$ is distance-biregular, $N = c_i'|\Gamma_{i+1,i-1}(x, y, z)|$. The claim follows. 

**Theorem 6.2** With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. Suppose that $\Gamma$ is 2-$Y$-homogeneous. Then, for every integer $i$ $(1 \leq i \leq \min\{D - 1, D' - 1\})$ and for all $x \in Y, y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_{i,i}(x, y)$ the number $\gamma_i = |\Gamma_{i+1,i-1}(x, y, z)|$ is nonzero.

**Proof** Since $\Gamma$ is 2-$Y$-homogeneous, for all $i$ $(1 \leq i \leq \min\{D - 1, D' - 1\})$ and for all $x \in Y, y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_{i,i}(x, y)$ the number $\gamma_i = |\Gamma_{i+1,i-1}(x, y, z)|$ does not depend on the choice of $x, y,$ and $z$. Note that, if $D = 3$ then (11) yields $|\Gamma_{2,2}(x, y)| \neq 0$, and with it $p_{2,2}^2$ is not zero. If $D \geq 4$ then by Theorem 4.2 we have that the number $p_{2,2}^2$ is not zero. From the definition of $\gamma_i$, it follows that $\gamma_1 = 1$. For $i \geq 2$, we will proceed by contradiction. Suppose first $\gamma_2 = 0$. Then, since $c_2$ is a positive integer, by Proposition 5.5 we have $b_1 = 1$. Therefore, $k' = b_1 + c_1 = 2$ which is a contradiction. If $D = 3$ we are done. Otherwise, assume that there exists $\ell$ $(3 \leq \ell \leq \min\{D - 1, D' - 1\})$ such that $\gamma_\ell = 0$. Without loss of generality, we can pick an integer $i$ $(3 \leq i \leq \min\{D - 1, D' - 1\})$ such that $\gamma_{i-1} \neq 0$ and $\gamma_i = 0$. By Theorem 4.2 and (18), $p_{2,i}'$ is not zero. As $c_i$ and $c_i'$ are positive, by Proposition 5.5 either $b_{i-1} = 1$ (if $i$ is even) or $b_{i-1}' = 1$ (if $i$ is odd). Now, by Lemma 6.1, for every $x \in Y$ with $y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_{i,i}(x, y)$,

$$c_{i-1}'\gamma_i = \gamma_{i-1}|\Gamma_{i+1,i-1}(x, y, z)|$$

and since the number $\gamma_{i-1}$ is nonzero, we have $|\Gamma_{i+1,i-1}(x, y, z)| = 0$. We next consider two cases.

**Case 1.** Suppose that $i$ is even. By Lemma 2.2, we have $b_{i-1} \geq b_{i+1}$ and so $b_{i+1} = 1$. Now, from (10), for every vertex $w \in \Gamma_{i+1,i-1}(x, y) \cup \Gamma_{i-1,i+1}(x, y)$ we have $|\Gamma_{i,i}(x, y, w)| = |\Gamma_{i-1,i+1}(x, y, z)| = 0$ for every $z \in \Gamma_{i,i}(x, y)$. This yields the set $\Gamma_{i,i}(x, y)$ is empty, contradicting Theorem 4.2.

**Case 2.** Assume next that $i$ is odd. Since $b_{i-1}' = 1$, by Lemma 2.2 we have $b_i = 1$. Recall that arbitrary $z \in \Gamma_{i,i}(x, y)$ does not have neighbors in $\Gamma_{i-1,i-1}(x, y)$. Considering the intersection diagram of rank 2 (see Fig. 4) this implies that $z \in \Gamma_{i,i}(x, y)$ has exactly $b_i - c_i$ neighbors in $\Gamma_{i+1,i+1}(x, y)$, which yields $c_i = 1$. Thus, $k' = b_i + c_i = 2$ contradicting our assumption $k' \geq 3$. 

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There exist

Moreover, by Lemma 6.1, if the following two claims are equivalent.

Let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. The following are equivalent.

(i) $\Gamma$ is 2-$Y$-homogeneous.
(ii) For all integers $i (1 \leq i \leq \min\{D - 1, D' - 1\})$ and for all $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_i(x, y)$, the scalar $\delta_i = |\Gamma_{i-1,i-1,1}(x, y, z)|$ is nonzero and it is independent of the choice of $x, y$, and $z$.

Proof The idea for the proof we found in [7, Theorem 16], where the author studied bipartite distance-regular graphs.

(i)⇒(ii) Assume that $\Gamma$ is 2-$Y$-homogeneous. By Theorem 6.2, for every $i (1 \leq i \leq \min\{D - 1, D' - 1\})$ and for all $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_i(x, y)$, $\gamma_i = |\Gamma_{i,1,i-1}(x, y, z)|$ is nonzero and does not depend on the choice of $x, y$ and $z$.

Moreover, by Lemma 6.1,

$$c'_{i-1} \gamma_i = \gamma_i - 1 |\Gamma_{i-1,i-1,1}(x, y, z)|.$$

Since $\gamma_i, \gamma_{i-1}, c'_{i-1} (2 \leq i \leq \min\{D - 1, D' - 1\})$ are nonzero, we have $|\Gamma_{i-1,i-1,1}(x, y, z)|$ is nonzero and does not depend on the choice of $x, y$ and $z$.

(ii)⇒(i) Assume that for all integers $h (1 \leq h \leq \min\{D - 1, D' - 1\})$ and for all $x \in Y$, $y \in \Gamma_2(x)$, $z \in \Gamma_{h,h}(x, y)$, the scalar $\delta_h = |\Gamma_1(z) \cap \Gamma_{h-1,h-1}(x, y)|$ is independent of the choice of $x, y$ and $z$, and it is nonzero. Note that $\gamma_1 = 1$ and $\gamma_2 = \delta_2$. Setting $i = 2, 3, \ldots, \ell$ (where $\ell \leq \min\{D - 1, D' - 1\}$) in (30), and using mathematical induction we get

$$\gamma_\ell = |\Gamma_{\ell-1}(z) \cap \Gamma_{1,1}(x, y)| = \frac{1}{c'_{\ell-1}} \gamma_\ell \delta_\ell,$$

and with it

$$\gamma_i = \frac{\delta_2 \delta_3 \cdots \delta_i}{c'_2 c'_3 \cdots c'_{i-1}} (1 \leq i \leq \min\{D - 1, D' - 1\}).$$

The result follows.

Remark 6.4 Let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. Fix $i (2 \leq i \leq \min\{D - 1, D' - 1\})$. From the results of this paper, we do not know if the following two claims are equivalent.

(i) The scalar $\Delta_i = 0$.
(ii) There exist $x \in Y$ and $y \in \Gamma_2(x)$ such that for all $z \in \Gamma_{i,i}(x, y)$ the number $|\Gamma_{i-1,i-1,1}(x, y, z)|$ is independent of the choice of $z$.

Corollary 6.5 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. The following are equivalent.

(i) The collection of all nonempty sets $\Gamma_{i,j}(x, y) (0 \leq i, j \leq D)$ is an equitable partition of $\Gamma$ (for any $x \in Y$ and $y \in \Gamma_2(x)$).
(ii) Γ is 2-Y-homogeneous.

Proof (i)⇒(ii) Immediate.
(i)⇒(i) Immediate from (10), Theorem 4.2 and Proposition 6.3 (see also Fig. 4).

□

7 The scalars Δ_1, part II

The main result of this section is somehow unexpected: for a fixed \( i \), the scalar \( Δ_i = 0 \) if and only if there exist \( x \in \Gamma \) and \( y \in \Gamma_2(x) \) such that for all \( z \in \Gamma_{i,i}(x, y) \) the number \(|Γ_{1, i-1}(x, y, z)|\) is independent of the choice of \( z \). The proof is tedious and time consuming. The main technique is counting certain number of ordered pairs in two different ways. As a corollary we get \( Γ \) is almost 2-Y-homogeneous if and only if \( Δ_i = 0 \) (\( 2 \leq i \leq D - 2 \)). We start with Lemma 7.1, which we use in the proof of Theorem 7.2.

Lemma 7.1 With reference to Notation 2.1, let \( Γ \) denote a \((Y, Y')\)-distance-biregular graph with \( k' \geq 3 \) and \( D \geq 3 \). For any \( i \) (\( 2 \leq i \leq \min\{D - 1, D' - 1\}\)), the following holds.

(i) Pick \( x \in Y, y \in \Gamma_2(x) \) and \( w \in \Gamma_{1,1}(x, y) \). Then,

\[
|Γ_{i,i-1}(x, y, w)| = \begin{cases} 
\frac{c_i k_i (b_{i-1} - 1)}{b_0 b_1} & \text{if } i \text{ is even}, \\
\frac{c_i' k_i' (b_{i-1} - 1)}{b_0 b_1} & \text{if } i \text{ is odd}.
\end{cases}
\]

(ii) Pick \( x \in Y, y \in \Gamma_2(x) \) and \( w \in \Gamma_{1,1}(x, y) \). Then,

\[
|Γ_{i,i+1}(x, y, w)| = \begin{cases} 
\frac{b_i k_i (c_{i+1} - 1)}{b_0 b_1} & \text{if } i \text{ is even}, \\
\frac{b_i' k_i' (c_{i+1} - 1)}{b_0 b_1} & \text{if } i \text{ is odd}.
\end{cases}
\]

(iii) Pick \( x \in Y, y \in \Gamma_2(x) \) and \( u, v \in \Gamma_{1,1}(x, y) \). If \( \partial(u, v) = 2 \), then

\[
|Γ_{i-1,i+1}(u, v)| = |Γ_{i,i-1,i+1}(x, y, u, v)| = \begin{cases} 
\frac{k_i b_i c_i}{b_0 b'_1} & \text{if } i \text{ is even}, \\
\frac{k_i' b'_i c'_i}{b_0 b'_1} & \text{if } i \text{ is odd}.
\end{cases}
\]

Proof (i) We have \( |Γ_{i,i-1}(x, w)| = |Γ_{i,i-2,i-1}(x, y, w)| + |Γ_{i,i,i-1}(x, y, w)| = |Γ_{i,i-2}(x, y)| + |Γ_{i,i,i-1}(x, y, w)| \), that is

\[
|Γ_{i,i-1}(x, w)| = |Γ_{i,i-1}(x, w)| - |Γ_{i,i-2}(x, y)|. \tag{31}
\]

By (5), \( |Γ_{i,i-1}(x, w)| = \frac{b_i b_{i-2} \cdots b_{i-1}}{c_i' c_{i-2} \cdots c_{i-1}} \), and by (12), \( |Γ_{i,i-2}(x, y)| = \frac{b_2 b_{i-3} \cdots b_{i-1}}{c_{i-2} c_{i-3} \cdots c_{i-3}} \). Now we consider two cases.
Case 1. Assume that \( i \) is even. Because \( i - 1 \) is odd, by (1), (2), (5), (12) and (31) we have

\[
|\Gamma_{i,i-1}(x,y,w)| = \frac{b_2 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-2}} \cdot \frac{b_1 - c_{i-1}}{c_{i-1}} = \frac{k_i c_i}{b_0 b_1} (b_1 - c_{i-1}).
\]

Since \( k' = b_{i-1} + c_{i-1} = b_1 + c_1 \) we have \( b_1 - c_{i-1} = b_{i-1} - 1 \), and the result follows.

Case 2. Assume that \( i \) is odd. By (2) and (5),

\[
|\Gamma_{i,i-1}(x,y,w)| = \frac{b_2 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-2}} \cdot \frac{b_1 - c'_{i-1}}{c'_{i-1}}.
\]

Since \( k' = b'_{i-1} + c'_{i-1} = b_1 + c_1 \) we have \( b_1 - c'_{i-1} = b'_{i-1} - 1 \), and the result follows.

(ii) Recall that \( |\Gamma_i(x)| = k_i \) and pick \( z \in \Gamma_{i,i+1}(x,w) \). Applying the triangle inequality in triangles \( zxy \) and \( zw_y \), we get \( \partial(y,z) \in \{i,i+2\} \). Even more, \( \Gamma_{i,i+2,i+1}(x,y,w) = \Gamma_{i,i+2}(x,y) \). This yields

\[
|\Gamma_{i,i,i+1}(x,y,w)| = |\Gamma_{i,i+1}(x,w)| - |\Gamma_{i,i+2}(x,y)|.
\]

Assume \( D' \geq D \). Then, it is easy to see that the result holds for \( i = D - 1 \). So, suppose that \( 2 \leq i \leq D - 2 \). By (1), (6) and (12), we have

\[
|\Gamma_{i,i+1}(x,w)| = \frac{b'_1 b'_2 \cdots b'_{i-1}}{b_1 b_2 \cdots b_{i-1}} \cdot \frac{b'_{i} k_i}{b_0} \quad \text{and} \quad |\Gamma_{i,i+2}(x,y)| = \frac{b_{i+1} b_{i} k_i}{b_0 b_1}.
\]

We next consider two cases. Assume first that \( i \) is even. Then, \( b_1 - b_{i+1} = c_{i+1} - 1 \).

Note that \( b_1 \neq 0 \) and by (3), since \( i + 1 \) is odd, \( b_1 b_2 \cdots b_i = b'_1 b'_2 \cdots b'_i \). Suppose now that \( i \) is odd. Then, \( b_1 - b'_{i+1} = c'_{i+1} - 1 \). We also have the products \( b_1 b_2 \cdots b_{i-1} = b'_1 b'_2 \cdots b'_{i-1} \) and \( b_{i+1} b_i = b'_{i+1} b'_i \). The result follows immediately from (32), (33) and the above comments.

(iii) Note that \( u, v \in Y' \) and \( \partial(u,v) = 2 \). Similarly as in (13), it is routine to get

\[
|\Gamma_{i-1,i+1}(u,v)| = \frac{b'_i b'_1 b'_2 \cdots b'_{i-1}}{c'_1 c'_2 \cdots c'_{i-1}} = \frac{1}{b'_1 c'_1 c'_2 \cdots c'_{i-1}}.\]

Splitting into two cases, depending on the parity of \( i \), the result now immediately follows from (1), (2) and (3).

\[\Box\]

**Theorem 7.2** With reference to Notation 2.1, let \( \Gamma \) denote a \((Y,Y')\)-distance-biregular graph with \( k' \geq 3 \) and \( D \geq 3 \). For any \( i \) \((2 \leq i \leq \min\{D - 1, D' - 1\})\), the following are equivalent.

(i) The scalar \( \Delta_i = 0 \).

(ii) There exist \( x \in Y \) and \( y \in \Gamma_2(x) \) such that for all \( z \in \Gamma_{i,i}(x,y) \) the number

\[
|\Gamma_{1,i-1}(x,y,w)|
\]

is independent of the choice of \( z \).

**Proof** Pick \( x \in Y \) and \( y \in \Gamma_2(x) \). Recall that, since \( k' \geq 3 \), by Theorem 4.2, \( p_i^2 \) is nonzero. Assume that for all \( z \in \Gamma_{i,i}(x,y) \) the number \( |\Gamma_{1,i-1}(x,y,z)| \) does not
depend on the choice of \( z \). With it the numbers \( |\Gamma_{1,i-1}(x, y, z)| \) are all equal to their average value. Since \( |\Gamma_{1,i-1}(x, y, z)| + |\Gamma_{1,i+1}(x, y, z)| = c_2 \), Lemma 5.3 yields

\[
\left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| \right) \left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i+1}(x, y, z)| \right) = |\Gamma_{i,j}(x, y)| \left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| \cdot |\Gamma_{1,i+1}(x, y, z)| \right). \tag{34}
\]

Next, we consider the case when \( c_2 > 1 \) (the case \( c_2 = 1 \) can be done in a similar way — note that \( c_2 = 1 \) iff \( c'_2 = 1 \), simplify expression for \( \Delta_i \), and consider the parity of \( i \)). By (4) and (18), (34) becomes

\[
\frac{b_0b_1}{c_2k_i} \left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| \right) \frac{b_0b_1}{c_2k_i} \left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i+1}(x, y, z)| \right) = p_{i,j} (c'_2 - 1)b_0b'_1 c_2(c_2 - 1)k_i \left( \sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| \cdot |\Gamma_{1,i+1}(x, y, z)| \right) \tag{35}
\]

(note that we used \( b_1 = \frac{b'_1(c'_2 - 1)}{c_2 - 1} \).

As in the proof of Lemma 5.4, for a fixed \( x \in Y \) and \( y \in \Gamma_2(x) \), counting the number of ordered pairs \((z, w)\) (where \( z \in \Gamma_{i,j}(x, y) \) and \( w \in \Gamma_{1,i-1}(x, y, z) \)) in two different ways, using Lemma 7.1(i) it follows

\[
\sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| = \begin{cases} \frac{c_2k_i c_i(b_i - 1)}{b_0b_1} & \text{if } i \text{ is even}, \\ \frac{c_2k_i c'_i(b'_i - 1)}{b_0b'_1} & \text{if } i \text{ is odd}. \end{cases} \tag{36}
\]

Similarly, using Lemma 7.1(ii), it is routine to compute

\[
\sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i+1}(x, y, z)| = \begin{cases} \frac{c_2b_1k_i(c_i+1)}{b_0b_1} & \text{if } i \text{ is even}, \\ \frac{c_2b'_1k'_i(c'_i+1)}{b_0b'_1} & \text{if } i \text{ is odd}, \end{cases} \tag{37}
\]

and for a fixed \( x \in Y \) and \( y \in \Gamma_2(x) \), counting the numbers of triples \((u, v, z)\) (where \( u, v \in \Gamma_{1,1}(x, y), d(u, v) = 2, z \in \Gamma_{i,i-1,i+1}(x, y, u, v) \)) in two different ways, we get

\[
\sum_{z \in \Gamma_{i,j}(x,y)} |\Gamma_{1,i-1}(x, y, z)| \cdot |\Gamma_{1,i+1}(x, y, z)| = \begin{cases} \frac{c_2(c_2 - 1)k_i b_i c_i}{b_0b_1} & \text{if } i \text{ is even}, \\ \frac{c_2(c_2 - 1)k'_i b'_i c'_i}{b_0b'_1} & \text{if } i \text{ is odd}. \end{cases} \tag{38}
\]

Since the integers \( c_i, b_i, c'_i \) and \( b'_i \) are positive, the equality \( \Delta_i = 0 \) follows from (35)–(38) and the definition of \( \Delta_i \). \( \Box \)
Corollary 7.3 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. Then, the following are equivalent.

(i) $\Delta_i = 0$ ($2 \leq i \leq \min\{D - 1, D' - 1\}$).
(ii) For every $i$ ($2 \leq i \leq D - 1$), there exist $x \in Y$ and $y \in \Gamma_2(x)$ such that for all $z \in \Gamma_{1,i}(x, y)$ the number $|\Gamma_{1,1,i-1}(x, y, z)|$ is independent of the choice of $z$.
(iii) $\Gamma$ is 2-$Y$-homogeneous.

Proof Recall that $D' \in \{D - 1, D, D + 1\}$.

Case 1. Assume that $D - 1 \neq D'$. Then, $\min\{D - 1, D' - 1\} = D - 1$. The result now follows immediately from Proposition 5.5 and Theorem 7.2.

Case 2. Assume that $D - 1 = D'$. This implies that $D$ is even and $\min\{D - 1, D' - 1\} = D - 2$. We prove that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii) Assume that $\Delta_i = 0$ for $2 \leq i \leq D - 2$. By Proposition 5.5 and Theorem 7.2, for all $x \in Y$ and $y \in \Gamma_2(x)$ the number $\Gamma_{1,1,i-1}(x, y, z)$ is independent of the choice of $z \in \Gamma_{1,i}(x, y)$ ($2 \leq i \leq D - 2$). Fix $x \in Y$ and $y \in \Gamma_2(x)$.

We need to show that $\Gamma_{1,1,D-2}(x, y, z)$ is independent of $z \in \Gamma_{D-1,Y-1}(x, y)$. We show that $|\Gamma_{1,1,D-2}(x, y, z)| = c_2$. Fix $z \in \Gamma_{D-1,Y-1}(x, y)$. For any $w \in \Gamma_{1,1}(x, y)$ we have $\partial(z, w) \in \{D - 2, D\}$.

Note that, since $D$ is even, $D - 1$ is odd which yields that $z \in Y'$. Now, because $D' < D$ we have $\partial(z, w) = D - 2$. The result follows.

(ii)$\Rightarrow$(iii)$\Rightarrow$(i) Follows immediately from Proposition 5.5 and Theorem 7.2.

Corollary 7.4 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $k' \geq 3$ and $D \geq 3$. The following are equivalent.

(i) $\Delta_i = 0$ ($2 \leq i \leq D - 2$).
(ii) For every $i$ ($2 \leq i \leq D - 2$), there exist $x \in Y$ and $y \in \Gamma_2(x)$ such that for all $z \in \Gamma_{1,i}(x, y)$ the number $|\Gamma_{1,1,i-1}(x, y, z)|$ is independent of the choice of $z$.
(iii) $\Gamma$ is almost 2-$Y$-homogeneous.

Proof Since $D - 2 \leq \min\{D - 1, D' - 1\}$, the result follows immediately from Proposition 5.5 and Theorem 7.2.

8 Distance-biregular graphs with $c'_2 = 1$ and $k' \geq 3$

Let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $D \geq 3$ and $k' \geq 3$. In this section, we show that $\Gamma$ is almost 2-$Y$-homogeneous with $c'_2 = 1$ if and only if $c_i = 1$ for every integer $i$ ($1 \leq i \leq D - 1$).

Theorem 8.1 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $D \geq 3$ and $k' \geq 3$. The following are equivalent.

(i) $\Gamma$ is almost 2-$Y$-homogeneous and $c_2 = 1$.
(ii) $c_i = 1$ for every integer $i$ ($1 \leq i \leq D - 1$).

Proof (i)$\Rightarrow$(ii) By (4), $c'_2 = 1$. To obtain a contradiction we assume that there exists $t$ ($2 \leq t \leq D - 2$) such that $c_j = 1$ for all $1 \leq j \leq t$ and $c_{t+1} > 1$. With this assumption it follows from Corollary 2.5 that $c'_1 = \cdots = c'_t = 1$. Moreover, since $c_{t+1} > c'_t = 1$,
(8) yields $c'_{i+1} > c_i = 1$. Since $\Gamma$ is almost 2-$Y$-homogeneous and $c'_2 = 1$, by Corollary 7.4, $\Delta_t = 0$, which implies that we have either $(b_i - 1)(c_{i+1} - 1) = 0$ or $(b'_{i+1} - 1)(c'_i + 1 - 1) = 0$, depending on the parity of $t$. Since both $c_{i+1} > 1$ and $c'_{i+1} > 1$, we have either $b_{i-1} = 1$ or $b'_{i-1} = 1$. By Lemma 2.2, if $t$ is even then $b'_t = 1$, and if $t$ is odd then $b_t = 1$. Note that $k' = b'_i + c'_i$ for even $t$, and $k' = b_i + c_t$ for odd $t$. Therefore, since $c'_i = 1$ (for even $t$) or $c_t = 1$ (for odd $t$) we get that $k' = 2$, a contradiction. Since both cases contradict $k' > 2$, the result follows.

(ii)$\Rightarrow$(i) Assume that $c_i = 1$ for every $i$ $(1 \leq i \leq D - 1)$. Then, Corollary 2.5 implies $c'_i = 1$ for every $i$ $(1 \leq i \leq D - 1)$. Thus, $\Delta_i = 0$ for every $i$ $(1 \leq i \leq D - 2)$. The result now follows from Corollary 7.4. \hfill \Box

9 Distance-biregular graphs with $D = 3$

Note that any $(Y, Y')$-distance-biregular graph with $D = 3$ is almost 2-$Y$-homogeneous, by definition. In this section, we show that a $(Y, Y')$-distance-biregular graph with $D = 3$ is 2-$Y$-homogeneous if and only if $|\Gamma_2(x)| = \deg(x)$. Note that $k' = 2$ automatically yields $|\Gamma_2(x)| = \deg(x)$ (by Proposition 4.1 any such graph is 2-$Y$-homogeneous). In Theorem 9.1, we consider the case when $k' \geq 3$.

Theorem 9.1 With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph $\Gamma$ with $k' \geq 3$ and $D = 3$. The following are equivalent.

(i) $\Delta_2 = 0$.

(ii) $\Gamma$ is 2-$Y$-homogeneous.

(iii) $|\Gamma_2(x)| = \deg(x)$.

Proof First note that, if $D$ is odd then $D' \geq D$. Considering the intersection diagram of rank 0 (see Fig. 1) we have $|\Gamma_2(x)| = \frac{b_0b_1}{c_2}$. Now, it is not hard to see that $|\Gamma_2(x)| = \deg(x)$ holds if and only if $b_1 = c_2$.

(i)$\Rightarrow$(ii) Since $D = 3$ is odd we have $D' \geq D$, and with it min{$D - 1, D' - 1$} = 2. The claim now follows immediately from Corollary 7.3.

(ii)$\Rightarrow$(iii) Assume that $\Gamma$ is 2-$Y$-homogeneous. By Corollary 7.3, the scalar $\Delta_2 = 0$, and since $D = 3$, we have $c_3 - 1 = b_1$. This yields

$$b_1(b_1 - 1) = p^2_{2,2}(c'_2 - 1).$$

(39)

Suppose first that $c'_2 = 1$. Equation (4) yields $c_2 = 1$. Moreover, the right-hand side of (39) is equal to 0, and with it $b_1 = 1$. This shows that $b_1 = c_2 = 1$ and the result follows. Assume now that $c'_2 \geq 2$. Note that $k - 1 = b'_1 = b_0 - 1$. Now from (4) and (39) we have

$$p^2_{2,2} = \frac{b_1(b_1 - 1)}{c'_2 - 1} = \frac{(b_0 - 1)(b_1 - 1)}{c_2 - 1}.$$  

(40)

On the other hand, since for any $x \in Y$ and $y \in \Gamma_2(x)$, $p^2_{2,2} = |\Gamma_2(x, y)|$ and $|\Gamma_2,4(x, y)| = 0$, by (1) and (9) this yields

$$p^2_{2,2} = |\Gamma_2(x)| - 1 = \frac{b_0b_1 - c_2}{c_2}.$$  

(41)
By (40) and (41) we have $c_2(b_0 - 1)(b_1 - 1) = (b_0b_1 - c_2)(c_2 - 1)$. Replacing $b_0$ by $b_2 + c_2$ in the last equality, after simplification we get $b_2(b_1 - c_2) = 0$, which yields $c_2 = b_1$. Hence, $|\Gamma_2(x)| = \deg(x)$.

(iii)$\Rightarrow$(i) Assume that $|\Gamma_2(x)| = \deg(x)$. Note that $b_1' = b_0 - 1$ and so

$$b_1' = \deg(x) - 1 = |\Gamma_2(x)| - 1 = p_2^{2,2}.$$ 

Now, by (4) we have

$$b_1(c_2 - 1) = p_2^{2,2}(c_2' - 1).$$

(42) After replacing $c_3$ by $b_1 + c_1$ in the definition of $\Delta_2$, and applying (42) we have $\Delta_2 = b_1(b_1 - c_2)$. Since $b_1 = c_2$ we get $\Delta_2 = 0$. The result now follows from Proposition 5.5.

**Corollary 9.2** With reference to Notation 2.1, a $(Y, Y')$-distance-biregular graph with $D = 3$ is 2-$Y$-homogeneous if and only if $|\Gamma_2(x)|$ is equal to the number of neighbors of $x$. Moreover, if $\Gamma$ is a 2-$Y$-homogeneous distance-biregular graph with $D = 3$, then the intersection array of the color class $Y$ is of the following type

$$(k, c, k - c; 1, c, c + 1)$$

for some integers $k$ and $c$, where $k > c \geq 2$.

**Proof** The equivalence follows immediately from Proposition 4.1 and Theorem 9.1. For the intersection array, note that $|\Gamma_2(x)| = \deg(x)$ holds if and only if $b_1 = c_2$. On the other hand $c_3 = k' = b_1 + c_1 = c_2 + 1$, and the result follows. \[\square\]

**10 DBGs with $D = 4, D = 5$ and proof of Theorem 1.1**

In this section, we give possible types for the intersection array of a 2-$Y$-homogeneous $(Y, Y')$-distance-biregular graph with $D = 4$ and $D = 5$, written in terms of three parameters.

**Lemma 10.1** With reference to Notation 2.1, let $\Gamma$ denote a $(Y, Y')$-distance-biregular graph with $D = 4$. If $\Gamma$ is 2-$Y$-homogeneous with $c_2' \geq 2$, then the intersection array of the color class $Y$ is of the following type:

$$(k, k' - 1, k - c, k' - 1 - \frac{c(c' - 1)}{\gamma}; 1, c, \frac{c(c' - 1)}{\gamma} + 1, k)$$

for some positive integers $k$, $k'$ and $c$, where $k > c \geq 2, k' > 2, c' = \frac{(k' - 1)(c' - 1)}{k - 1} + 1$ and $\gamma = \frac{(c - 1)(c' - 2)}{k - 2} + 1$.

**Proof** By assumption $c_2' \geq 2$, which yields $k' \geq 3$. Equation (4), Lemma 2.2(i) and Lemma 3.1(i)(ii) yield $c_2 \geq 2$ and $\gamma_2 \geq 1$. Note that $D = 4$ implies $c_4 = k$, and by (4) we have $c_2' - 1 = \frac{(k' - 1)(c_2' - 1)}{k - 1}$. The result follows from Lemma 3.1(i) and Lemma 3.3(i). \[\square\]
Lemma 10.2 With reference to Notation 2.1, let \( \Gamma \) denote a \((Y, Y')\)-distance-biregular graph with \( D = 5 \). If \( \Gamma \) is 2-\( Y \)-homogeneous with \( c'_2 \geq 2 \), then the intersection array of the color class \( Y \) is of the following type:

\[
(k, k' - 1, k - c, 1 + \frac{c(c' - 1)}{\gamma}, b_4; 1, c, k' - 1 - \frac{c(c' - 1)}{\gamma}, c_4, k')
\]

for some positive integers \( k, k' \) and \( c \), where \( k > c \geq 2, k' > 2, c' = \frac{(k' - 1)(c - 1)}{\gamma} + 1,\)

\[
\gamma = \frac{(c - 2)(c' - 2)}{k' - 2} + 1, \quad c_4 = \frac{k(k' - 1) - \frac{(b_3 - 1)(k' - 1)}{c'_2 - 1}}{\gamma}, \quad b_4 = k - c_4.
\]

Proof The result for the first three intersection numbers follow immediately from the proof of Lemma 10.1. It is only left to compute \( c_4 \). First note that \( D \geq 5 \) yields \( D' \geq 5 \). From the definition of \( \Delta_i \) and Corollary 7.3, \( p_4', p_2 = \frac{(b_3 - 1)(k' - 1)}{c'_2 - 1} \). Now from (1), (16) and (18), we have \( kb_1 - c_3 = \frac{k_2 b_3 - 1}{c_2 - 1} \), and the result follows. \( \square \)

Proof of Theorem 11.1 If \( c_2 \geq 2 \), then from (4) we have \( b_1 = k' - 1 = \frac{(c'_2 - 1)(k - 1)}{c_2 - 1} \). Just for the moment assume that \( c'_2 = 2 \). Then, by Lemmas 3.3(i) and 3.1(i), \( \gamma_2 = 1 \), which together with Lemma 3.3(ii) yield \( c_3 = c_2 + 1 \). The rest of the theorem follows immediately from Proposition 4.1, Theorems 3.4, 4.2, 8.1, Corollary 9.2 and Lemmas 10.1, 10.2. \( \square \)

11 Simple examples and suggestions for further research

In this section, we give simple examples of (almost) 2-\( Y \)-homogeneous distance-biregular graphs and present some open problems for future research.

Example 11.1 Let \( \Gamma^o = (X^o, R^o) \) denote the Petersen graph. The Petersen graph is a \((3, 5)\)-cage with diameter 2 and odd girth. The subdivision graph of the Petersen graph is 2-\( X^o \)-homogeneous but it is not almost 2-\( R^o \)-homogeneous (see also Fig. 3).

Example 11.2 The Heawood graph \( \Gamma^o = (X^o, R^o) \) is a \((3, 6)\)-cage with diameter 3 and even girth. The subdivision graph of the Heawood graph is 2-\( X^o \)-homogeneous, it is almost 2-\( R^o \)-homogeneous but it is not 2-\( R^o \)-homogeneous.

Example 11.3 Consider the set \( \mathcal{P} = \{1, 2, 3, 4, 5, 6, 7, 8\} \) and let \( \mathcal{B} \) be the collection of the following nonempty subsets of \( \mathcal{P} \):

\[
\{1, 3, 7, 8\}, \{1, 2, 4, 8\}, \{2, 3, 5, 8\}, \{3, 4, 6, 8\}, \{4, 5, 7, 8\}, \{1, 5, 6, 8\}, \{2, 6, 7, 8\}, \{1, 2, 3, 6\}, \{1, 2, 5, 7\}, \{1, 3, 4, 5\}, \{1, 4, 6, 7\}, \{2, 3, 4, 7\}, \{2, 4, 5, 6\}, \{3, 5, 6, 7\}.
\]

Let \( \Gamma \) denote the bipartite graph with color classes \( \mathcal{P} \) and \( \mathcal{B} \) where \( \{p, B\} \) is an edge of \( \Gamma \) (\( p \in \mathcal{P}, B \in \mathcal{B} \)) if and only if \( p \in B \). It is easy to check that \( \Gamma \) is distance-biregular. Moreover, every vertex in \( \mathcal{P} \) has eccentricity equal to 3 and intersection array \((7, 3, 4; 1, 3, 4)\). Pick \( p \in \mathcal{P} \). By (1), \( |\Gamma_2(p)| = 7 = \deg(p) \). Corollary 9.2 yields that \( \Gamma \) is 2-\( \mathcal{P} \)-homogeneous.
Example 11.4 For $n \in \mathbb{N}$, an $n$-grid is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ where $\mathcal{P} = \{x_{ij} \mid 0 \leq i, j \leq n\}$ and $\mathcal{B} = \{L_0, \ldots, L_n, M_0, \ldots, M_n\}$ such that $x_{ij}$ lies on $L_k$ if and only if $i = k$ and $x_{ij}$ lies on $M_k$ if and only if $j = k$. Generalized quadrangles were introduced by J. Tits [43]. We follow [37] for the standard concepts in generalized quadrangles. It is easy to see that an $n$-grid is a generalized quadrangle with parameters $s = n$ and $t = 1$. For $n \geq 2$, let $\Gamma(n) = (X, \mathcal{R})$ denote the incidence graph of the $n$-grid (that is, the graph with $X = \mathcal{P} \cup \mathcal{B}$ where two vertices $p \in \mathcal{P}, \ell \in \mathcal{B}$ are adjacent if and only if $p$ lies on $\ell$). As the $n$-grid is a generalized quadrangle, it is well known that $\Gamma(n)$ is characterized by being a connected, bipartite graph with diameter 4 and girth 8. In addition, we observe $\Gamma(n)$ is distance-biregular. The intersection array of every vertex $p \in \mathcal{P}$ is $(2, n, 1, n; 1, 1, 1, 2)$ while every vertex $\ell \in \mathcal{B}$ has intersection array $(n + 1, 1, n, n; 1, 1, 1, n + 1)$. Since the numbers $c_i = c'_i = 1$ $(1 \leq i \leq 3)$ it follows from Theorem 8.1 that $\Gamma(n)$ is both almost $2-\mathcal{P}$-homogeneous and almost $2-\mathcal{B}$-homogeneous.

In Sect. 4, it is proven that if $\Gamma^o$ is a $(\kappa, g)$-cage with vertex set $X^o$, then the subdivision graph of $\Gamma^o$ is $2$-$X^o$-homogeneous. In addition, in Proposition 11.5 we show that, if $g$ is even then the subdivision graph of $\Gamma^o$ is almost $2-\mathcal{R}^o$-homogeneous.

Proposition 11.5 Let $\Gamma^o = (X^o, \mathcal{R}^o)$ denote a $(\kappa, g)$-cage with $\kappa \geq 3$ and $g \geq 3$ and let $\Gamma$ denote the subdivision graph of $\Gamma^o$. If $g$ is even then $\Gamma$ is almost $2-\mathcal{R}^o$-homogeneous.

Proof The subdivision graph $\Gamma = S(\Gamma^o)$ with vertex set $X = Y \cup Y'$ is distance-biregular with bipartite parts $Y = X^o$ and $Y' = \mathcal{R}^o$ (see Theorem 2.6). Moreover, their intersection numbers depend on the parity of the girth $g$ of $\Gamma^o$. Pick $x \in Y$ and $x' \in Y'$. Note $k = \deg(x) = \kappa$ and $k' = \deg(x') = 2$. Assume that $g$ is even. Since $x$ and $x'$ have both eccentricity $2d$, we can compute $\Delta_i(\mathcal{R}^o)$ for every integer $i$ $(1 \leq i \leq 2d - 1)$. Recall that $c_2 = 1$ and that $\deg(x) = \kappa > 2$. Then, by Theorem 2.6 we have $\Delta_i = \Delta'_i(\mathcal{R}^o) = 0$ for all $i$ $(1 \leq i \leq 2d - 2)$. Hence, by Corollary 7.4, the subdivision graph $\Gamma$ is almost $2-\mathcal{R}^o$-homogeneous. □

A problem which is beyond our reach is to find an algorithm for constructing a distance-biregular graph from its two intersection arrays (under the assumption that such graph exists). As far as we know, this is not known in the literature and we kindly ask the reader to contact us about any results in this direction. To explain what we want here, let us give a concrete example: we know there exists a distance-biregular graph with intersection arrays $(3, 1, 2, 1, 2; 1, 1, 1, 1, 2)$ and $(2, 2, 1, 2, 1, 1; 1, 1, 1, 1, 2, 2)$. We would like to find an algorithm which as input has these two intersection arrays, and which will give us as output the adjacency matrix of a corresponding graph (note that, in general case, it can happen that such a graph is not unique). Moreover, the same problem can be set up for a bipartite distance-regular graph but the problem is not easier.

Problem 11.1 Let $\Gamma$ denote a distance-biregular graph. Find an algorithm, if possible, to construct $\Gamma$ from its two intersection arrays.
Now, let $\Gamma$ denote a distance-biregular graph with color partitions $(Y, Y')$ and $k' \geq 3$. Recall that in Theorem 9.1 we showed that if $D = 3$ then $\Gamma$ is 2-$Y$-homogeneous if and only if $|\Gamma_2(x)| = \deg(x)$, and we have examples of such graphs (see Example 11.3). Thus, the following problem is also interesting.

**Problem 11.2** Let $\Gamma$ denote a distance-biregular graph with color partitions $Y$ and $Y'$ such that the following (i)–(iii) hold:

(i) $D \geq 4$ and $k' \geq 3$,
(ii) $\Gamma$ is 2-$Y$-homogeneous,
(iii) $|\Gamma_2(x)| > \deg(x)$ for every $x \in Y$.

Prove or disprove that such a graph $\Gamma$ exists.

Let’s mention two more open problems that we did not manage to solve in this paper: (a) with reference to Theorem 1.1(ii), find the intersection array of a 2-$Y$-homogeneous distance-biregular graph for the case when $c_2' = 2$; and (b) prove (or disprove) that the two claims from Remark 6.4 are equivalent.

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**Conflicts of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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