Ordinal patterns in long-range dependent time series

Annika Betken¹ | Jannis Buchsteiner¹ | Herold Dehling¹ | Ines Münker² | Alexander Schnurr² | Jeannette H.C. Woerner³

¹Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
²Department of Mathematics, University of Siegen, Germany
³Department of Mathematics, TU Dortmund, Germany

Correspondence
Alexander Schnurr, Department of Mathematics, University of Siegen, 57068 Siegen, Germany.
Email: schnurr@mathematik.uni-siegen.de

Abstract

We analyze the ordinal structure of long-range dependent time series. To this end, we use so called ordinal patterns which describe the relative position of consecutive data points. We provide two estimators for the probabilities of ordinal patterns and prove limit theorems in different settings, namely stationarity and (less restrictive) stationary increments. In the second setting, we encounter a Rosenblatt distribution in the limit. We prove more general limit theorems for functions with Hermite rank 1 and 2. We derive the limit distribution for an estimation of the Hurst parameter $H$ if it is higher than 3/4. Thus, our theorems complement results for lower values of $H$ which can be found in the literature. Finally, we provide some simulations that illustrate our theoretical results.

Keywords
Hurst index, limit theorems, long-range dependence, ordinal patterns

1 INTRODUCTION

Originally, ordinal patterns have been introduced to analyze long and noisy time series. They have proved to be useful in various contexts such as sunspot numbers (Bandt & Shiha, 2007), EEG...
data (Keller, Maksymenko, & Stolz, 2015), speech signals (Bandt, 2005) and chaotic maps which appear in the theory of dynamical systems (Bandt & Pompe, 2002). Further applications include the approximation of the Kolmogorov-Sinai entropy (Sinn, Ghodsi, & Keller, 2012). Recently, ordinal patterns have been used to detect and to model dependence structures between time series; see Schnurr (2014). Limit theorems for the parameters under consideration have been proved in the short-range dependent setting in Schnurr and Dehling (2017).

In the present paper we will investigate ordinal patterns in the long-range dependent setting. To the best of our knowledge, Sinn and Keller (2011) is the only article which explicitly deals with the interplay between ordinal patterns and the Hurst parameter $H$. The authors estimate this parameter of a fractional Brownian motion restricting their considerations to $H < \frac{3}{4}$. An overview and a comparison of various other techniques for estimating the Hurst parameter is given in Taqqu, Teverovsky, and Willinger (1995) and Rea, Oxley, Reale, and Brown (2009). None of the therein considered methods requires a restriction on the range of admissible values for $H$. Nonetheless, graphical methods that are used to estimate the Hurst parameter such as the aggregated variance method or the R/S method (Mandelbrot, 1975; Mandelbrot & Taquq, 1979; Mandelbrot & Wallis, 1969) are known to be biased. Estimators operating in the frequency domain of time series, such as the Whittle estimator, which are usually based on an estimation of the spectral density by the periodogram, often make parametric assumptions on the spectral density of the data-generating process. Semiparametric alternatives such as the GPH estimator (Geweke & Porter-Hudak, 1983) and the local Whittle estimator (Künsch, 1987; Robinson, 1995) require the choice of a bandwidth parameter denoting the number of Fourier frequencies incorporated in the estimation of the spectral density by the periodogram. The choice of this tuning parameter is crucial to the performance of semiparametric estimates, but difficult to select in practice. For the local Whittle estimator the selection of the bandwidth has been addressed by several authors; see for example, Henry (2001), Delgado and Robinson (1996) and Henry and Robinson (1996). A different approach to estimate the Hurst parameter is to apply variational methods and techniques from stochastic analysis as for example derived in Coeurjolly (2001) and Istas and Lang (1997). For an ordinal-pattern based estimation of the Hurst parameter, the asymptotic distribution of the estimator is derived on the basis of limit theorems for short-range dependent time series in Sinn and Keller (2011). Complementing the results of Sinn and Keller (2011), we derive the limit distribution for the estimator if $H > \frac{3}{4}$.

In Fischer, Schumann, and Schnurr (2017) the authors used ordinal patterns in the context of hydrological data. It is a well-known fact that hydrological data is often long-range dependent. In this case, the limit theorems presented in Schnurr and Dehling (2017) are no longer valid. In the present paper we close this gap and provide limit theorems in the long-range dependent setting.

For $h \in \mathbb{N}$ let $S_h$ denote the set of permutations of $\{0, \ldots, h\}$, which we write as $(h+1)$-tuples containing each of the numbers $0, \ldots, h$ exactly one time. By the ordinal pattern of order $h$ we refer to the permutation

$$\Pi(x_0, \ldots, x_h) = (\pi_0, \ldots, \pi_h) \in S_h,$$

which satisfies

$$x_{\pi_0} \geq \cdots \geq x_{\pi_h}.$$

Given a time series $(\xi)_n \geq 0$, we consider the relative frequency

$$\hat{q}_n(\pi) := \frac{1}{n} \sum_{i=0}^{n-1} 1_{\{\Pi(\xi_i, \xi_{i+1}, \ldots, \xi_{i+h}) = \pi\}}.$$
of an ordinal pattern \( \pi \in S_h \) as a natural estimator for the probability

\[
p(\pi) := P(\Pi(\xi_0, \ldots, \xi_h) = \pi).
\]

Sinn and Keller (2011) show that Rao–Blackwellization leads to an estimator \( \hat{p}_n(\pi) \) with lower risk and therefore better statistical properties.

In this article, both estimators are studied. Confirming the results of Sinn and Keller (2011), we show that \( \hat{q}_n(\pi) \) and \( \hat{p}_n(\pi) \) are consistent estimators; see Proposition 1. We consider separately the case of a stationary time series and the case of a time series with stationary increments. While the asymptotic distribution of \( \hat{q}_n(\pi) \) can be derived from a limit theorem for functions with Hermit rank 1, the limit behavior of \( \hat{p}_n(\pi) \) is derived from corresponding results for functions with Hermite rank 2. Along the way we explicitly calculate the asymptotic distribution of partial sums of the form \( \sum_{i=1}^{n} f(X_i, \ldots, X_{i+p-1}) \) where \( f \) has Hermite rank 1 or Hermite rank 2 and \((X_i)_{i \geq 1}\) is a stationary long-range dependent Gaussian process.

The paper is organized as follows: in the next section we introduce the mathematical framework. In Section 3 we present the main results, namely the asymptotic properties of two estimators of ordinal pattern probabilities. In Section 4, on the basis of these considerations, the asymptotic distribution of an estimator for the Hurst parameter based on ordinal patterns is derived. The detailed proofs of more general limit theorems for functions with Hermite ranks 1 and 2, that constitute the theoretical background of the results in Sections 3 and 4, are given in Section 5. In the final section a simulation study is presented.

## 2 Mathematical Framework

Let \((X_j)_{j \geq 0}\) be a stationary standard Gaussian process with autocovariance function

\[
r(k) := \text{Cov}(X_0, X_k) = L(k)k^{-D}, \quad k \geq 1,
\]

where \( L \) is a function, slowly varying at infinity (see Bingham, Goldie, and Teugels, 1987, p. 6), and \( 0 < D < 1 \). Such a process is called long-range dependent. For \( p \in \mathbb{N} \) we consider the \( \mathbb{R}^p \)-valued process \((X_j)_{j \geq 0}\) given by

\[
X_j := (X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(p)}) \quad \text{with} \quad X_j^{(i)} := X_{j+i-1},
\]

that is, we consider overlapping finite sequences of the original process. For \( 1 \leq l, m \leq p, p \in \mathbb{N} \), the corresponding cross-covariance function satisfies

\[
r^{(l,m)}(k) = \text{E}X_0^{(l)}X_k^{(m)} = L(|k + m - l|)|k + m - l|^{-D}, \quad k \geq 1,
\]

and, since \( L \) is a slowly varying function, we thus obtain

\[
\lim_{k \to \infty} \frac{k^D r^{(l,m)}(k)}{L(k)} = 1,
\]

for all \( l, m \in \mathbb{N} \). Consequently, \((X_j)_{j \geq 0}\) is multivariate long-range dependent in the sense of Arcones (1994), Section 3, if \( 0 < D < 1 \). If \( D > 1 \), we speak of short-range dependence.

We recall the concept of Hermite expansion. Let \( H_k \) denote the Hermite polynomial of order \( k \) given by
\[ H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad x \in \mathbb{R}, \]

and define the multivariate Hermite polynomial \( H_{l_1, \ldots, l_p} \) by

\[ H_{l_1, \ldots, l_p}(x) = \prod_{i=1}^{p} H_{l_i}(x_i), \quad x \in \mathbb{R}^p. \quad (1) \]

The collection \( (H_{l_1, \ldots, l_p})_{l_1, \ldots, l_p \geq 0} \) forms an orthogonal basis of \( L^2(\mathcal{N}(0, E_p)) \), where \( \mathcal{N}(0, E_p) \) denotes the \( p \)-dimensional standard normal distribution; see section 3.2 in Beran, Feng, Ghosh, and Kulik (2013). Thus, for any square-integrable \( G : \mathbb{R}^p \rightarrow \mathbb{R} \) the following \( L^2 \)-identity holds:

\[ G(U) - E G(U) = \sum_{k=m(G,E_p)}^{\infty} \sum_{l_1+\ldots+l_p=k} \frac{J_{l_1,\ldots,l_p}}{l_1! \ldots l_p!} H_{l_1,\ldots,l_p}(U), \quad (2) \]

where \( U \sim \mathcal{N}(0, E_p) \). The Hermite coefficients are given by the inner product, that is \( J_{l_1,\ldots,l_p} = E(G(U)H_{l_1,\ldots,l_p}(U)) \). The starting index

\[ m(G, E_p) := \min \left\{ \sum_{i=1}^{p} l_i : J_{l_1,\ldots,l_p} \neq 0 \right\}, \]

is called the Hermite rank of \( G \). Since the left-hand side in (2) is centered, we have \( m \geq 1 \). In contrast to (1) the definition of multivariate Hermite polynomials with respect to \( \mathcal{N}(0, \Sigma) \) is more complicated; see Beran et al. (2013), section 3.2. The Hermite rank is defined analogously

\[ m(G, \Sigma) := \min \left\{ \sum_{i=1}^{p} l_i : E(G(X)H_{l_1,\ldots,l_p}(X)) \neq 0 \right\}, \]

where \( X \sim \mathcal{N}(0, \Sigma) \).

The Hermite expansion in (2) is crucial to determining the asymptotics of partial sums of the type

\[ \sum_{i=1}^{n} \{f(X_i, \ldots, X_{i+p-1}) - E f(X_1, \ldots, X_p)\}, \quad (3) \]

where \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) satisfies \( E(f(X_1, \ldots, X_p))^2 < \infty \).

## 3 | ORDINAL PATTERNS

In this section we introduce the concept of ordinal pattern analysis and present asymptotic distributions for estimators of ordinal pattern probabilities, where functions with different Hermite ranks show up. We also provide examples for the calculation of the coefficients specifying the limiting distributions of these estimators for certain ordinal patterns. For detailed proofs of the given theorems the reader is referred to Section 5.
Definition 1. Let \( S_h \) denote the set of permutations of \( \{0, \ldots, h\} \), which we write as \((h + 1)\)-tuples containing each of the numbers 0, …, \( h \) exactly one time. By the ordinal pattern of order \( h \) we refer to the permutation

\[
\Pi(x_0, \ldots, x_h) = (\pi_0, \ldots, \pi_h) \in S_h,
\]

which satisfies

\[
x_{\pi_0} \geq \cdots \geq x_{\pi_h},
\]

and \( \pi_{i-1} > \pi_i \) if \( x_{\pi_{i-1}} = x_{\pi_i} \) for \( i = 1, \ldots, h-1 \).

The latter is introduced in order to deal with ties which do not occur in our simulation study, but which might occur when dealing with real data.

Remark 1. Naturally, ordinal patterns are closely linked to the ranks of observations. Given observations \( \xi_0, \ldots, \xi_h \), we define the rank \( R_i \) of \( \xi_i \) by

\[
R_i := \sum_{j=0}^{h} 1_{\{\xi_j \leq \xi_i\}}.
\]

Note that if \( \xi_i \neq \xi_j \) for all \( i, j = 0, \ldots, h, i \neq j \) then

\[
R_i = j + 1 \Leftrightarrow \pi_j = i.
\]

Thus, ranks provide a complete description of the order structure of the vector \((\xi_1, \ldots, \xi_n)\) equivalent to the description by ordinal patterns.

In this paper we are interested in estimating the probability \( p(\pi) := P(\Pi(\xi_0, \ldots, \xi_h) = \pi) \) for a given time series \( \xi = (\xi_t)_{t \geq 0} \) and therefore define the estimator

\[
\hat{p}_n(\pi) := \frac{1}{n} \#\{0 \leq i \leq n-1 : \Pi(\xi_i, \ldots, \xi_{i+h}) = \pi\} = \frac{1}{n} \sum_{i=0}^{n-1} 1_{\{\Pi(\xi_i, \ldots, \xi_{i+h}) = \pi\}},
\]

for a time series \( (\xi_t)_{t \geq 0} \).

We will see later (Remark 3) that the assumption that the time series \((\xi_t)_{t \geq 0}\) is stationary yields trivial limits. Therefore, we relax this assumption and use a helpful relation that was derived in Sinn and Keller (2011). They have shown that the estimator above is uniquely determined by the increments of this process. Let us consider \( X_t := \xi_t - \xi_{t-1} \) for \( t \geq 1 \).

For a vector \( x = (x_0, \ldots, x_h) \in \mathbb{R}^{h+1} \) define

\[
\Pi(x_0, \ldots, x_h) := \Pi(0, x_0, x_0 + x_1, \ldots, x_0 + \cdots + x_h).
\]

(4)

Then, it holds that

\[
\Pi(x_1 - x_0, \ldots, x_h - x_{h-1}) = \Pi(0, x_1 - x_0, \ldots, x_h - x_0) = \Pi(x_0, \ldots, x_h),
\]

since ordinal patterns are not affected by monotone transformations.
In terms of random vectors we hence arrive at

\[ \Pi(\xi_t, \xi_{t+1}, \ldots, \xi_{t+h}) = \tilde{\Pi}(X_{t+1}, \ldots, X_{t+h}), \quad t \geq 0. \]

In the following, we will study under which assumptions on the underlying time series we can derive an asymptotic result for the estimator \( \hat{q}_n(\pi) \). Since regarding \( (\xi_t)_{t \geq 0} \) as a stationary time series is not interesting due to the degenerate limit, we relax this assumption as follows: let \( \xi = (\xi_t)_{t \geq 0} \) be a (possibly non-stationary) stochastic process and let \( X = (X_t)_{t \geq 1} \) denote the corresponding increment process given by \( X_t := \xi_t - \xi_{t-1} \) for \( t \geq 1 \). We assume that \( X \) is a stationary standard Gaussian process with autocovariance function

\[ r(k) = L(k)k^{-D}, \quad k \geq 1, \]

where \( L \) is a function, slowly varying at infinity, and \( 0 < D < 1 \).

We now rewrite the estimator \( \hat{q}_n(\pi) \) in terms of the increment variables following the considerations in (4):

\[ \hat{q}_n(\pi) = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}\{\tilde{\Pi}(X_{i+1}, \ldots, X_{i+h}) = \pi\} = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}\{\tilde{\Pi}(X_{i+1}, \ldots, X_{i+h}) = \pi\}. \]

We will show that the relative frequency of any ordinal pattern is a consistent estimator for the corresponding probability.

**Theorem 1.** Suppose that \( (X_t)_{t \geq 1} \) is a stationary ergodic process. Then, \( \hat{q}_n(\pi) \) is a consistent estimator of \( p(\pi) := P(\tilde{\Pi}(X_1, \ldots, X_h) = \pi) \). More precisely,

\[ \lim_{n \to \infty} \hat{q}_n(\pi) = p(\pi), \]

almost surely.

### 3.1 Limit distribution of \( \hat{q}_n(\pi) \)

At first we need to determine the Hermite rank of the estimator. Here, and in what follows, proofs are postponed to Section 5.

**Lemma 1.** Let \( (X_k)_{k \geq 1} \) be a stationary standard normal Gaussian process and let \( h \in \mathbb{N} \). Then, for any \( \pi \in S_h \), the Hermite rank of

\[ 1\{\tilde{\Pi}(X_1, \ldots, X_h) = \pi\} - P(\tilde{\Pi}(X_1, \ldots, X_h) = \pi), \]

is equal to 1.

We now give the asymptotic distribution of the estimator and in doing so, we will take a closer look at the Hermite coefficients which determine the limit variance and hence the limit distribution.
**Theorem 2.** Let $\xi = (\xi_t)_{t \geq 0}$ be a stochastic process and let $X = (X_t)_{t \geq 1}$ denote the increment process of $\xi$ given by $X_t := \xi_t - \xi_{t-1}$ for $t \in \mathbb{Z}$. Assume that $X$ is a stationary, long-range dependent standard Gaussian process with autocovariance function $r(k) = L(k)k^{-D}$ and write $\Sigma_h := (r(i-j))_{1 \leq i,j \leq h}$. Then,

$$n^{D/2}L^{-1/2}(n)(\hat{q}_n(\pi) - \operatorname{P}(\tilde{\Pi}(X_1, \ldots, X_h) = \pi)) \to^D \mathcal{N}
\left(0, c_D \left(\sum_{j=1}^{h} \alpha_j\right)^2\right),$$

where $c_D = \frac{2}{(1-D)(2-D)}$ and where the vector $\alpha = (\alpha_1, \ldots, \alpha_h)^t$ is given by

$$\alpha := \Sigma_h^{-1} c$$

with $c = (c_1, \ldots, c_h)^t$ defined by

$$c_k = \mathbb{E}\{1_{\{\tilde{\Pi}(X_1, \ldots, X_h) = \pi\}}X_k\}, \quad 1 \leq k \leq h.$$

Thus, in order to compute the limit variance of $\hat{q}_n(\pi)$, we have to calculate the constants $c_k$ for $k = 1, \ldots, h$. We can reduce the number of calculations by making use of the time and space symmetry of stationary multivariate normal random vectors. For a normal random vector $(X_1, \ldots, X_h)$ these are given by

$$(X_1, \ldots, X_h) \overset{D}{=} (-X_1, \ldots, -X_h),$$

$$(X_1, \ldots, X_h) \overset{D}{=} (X_h, \ldots, X_1).$$

Following p. 1784 of Sinn and Keller (2011), we define two mappings (Figure 1):

$$S : S_h \to S_h, (\pi_0, \ldots, \pi_h) \mapsto (\pi_h, \ldots, \pi_0),$$

$$T : S_h \to S_h, (\pi_0, \ldots, \pi_h) \mapsto (h - \pi_0, \ldots, h - \pi_h).$$

Graphically, the mapping $S$ can be considered as space reversal, that is, as the reflection of $\pi$ on a horizontal line, while $T$ can be considered as time reversal, that is, as the reflection of $\pi$ on a vertical line.
For each \( \pi \in S_h \), we define

\[
\bar{\pi} := \{ \pi, S(\pi), T(\pi), T \circ S(\pi) \}.
\] (5)

It is easily seen that the set \( \bar{\pi} \) is closed under \( S \) and \( T \), since \( S \circ S(\pi) = T \circ T(\pi) = \pi \) and \( T \circ S(\pi) = S \circ T(\pi) \). This yields a partition of \( S_h \) into sets each having either two or four elements, depending on whether \( T(\pi) = S(\pi) \) holds for the considered \( \pi \).

In p. 1786 and lemma 1 of Sinn and Keller (2011), it is shown that with respect to ordinal patterns the above considerations yield

\[
E(X_k^1|\Pi(X_1, \ldots, X_h) = \pi) = -E(X_k^1|\Pi(X_1, \ldots, X_h) = S(\pi)), \quad k = 1, \ldots, h,
\] (6)

\[
E(X_k^1|\Pi(X_1, \ldots, X_h) = \pi) = -E(X_{h+1-k}^1|\Pi(X_1, \ldots, X_h) = T(\pi)), \quad k = 1, \ldots, h.
\] (7)

Both equations follow from the space and time symmetry of the multivariate normal distribution. More precisely, (7) holds since ordinal patterns are not affected by monotone transformations. For \( \pi \in S_h \) we have

\[
\{ \Pi(X_1, \ldots, X_h) = T(\pi) \} = \{ T(\Pi(X_1, \ldots, X_h)) = \pi \}
\]

\[
= \{ T(\Pi(0, X_1, X_1 + X_2, \ldots, X_1 + \cdots + X_h)) = \pi \}
\]

\[
= \{ \Pi(X_1 + \cdots + X_h, \ldots, X_1 + X_2, X_1, 0) = \pi \}
\]

\[
= \{ \Pi(0, -X_h, -(X_h + X_{h-1}), \ldots, -(X_1 + \cdots + X_h)) = \pi \}
\]

\[
= \{ \Pi(-X_h, -X_{h-1}, \ldots, -X_2, -X_1) = \pi \}.
\]

We compute the limit variance for ordinal patterns of lengths \( p = 2 \) and \( p = 3 \), that is, we need to study increments of length \( h = 1 \) and \( h = 2 \). As it is common in the literature, we restrict ourselves to small \( h \) in the present article. Unfortunately, the computations for larger values of \( h \) exceed the computing capacity of Mathematica.

Given the symmetry relations in (6) and (7), we only need to calculate the Hermite coefficients of the estimator \( \hat{q}_n(\pi) \) for one pattern \( \pi \) of each reversion group. Regarding \( S_1 = \{(0,1),(1,0)\} \) it is sufficient to choose \((1,0)\). Regarding \( S_2 \) we can partition this set into the two subsets \( \{(2,1,0),(0,1,2)\} \) and \( \{(2,0,1),(0,2,1),(1,2,0),(1,0,2)\} \). In the following we will study the Hermite coefficients of \( \hat{q}_n(\pi) \) for \( \pi = (2,1,0) \) and \( \pi = (2,0,1) \) so that we can reduce the number of lengthy calculations since we only need to consider two ordinal patterns instead of six.

**Example 1** (Ordinal patterns of length \( p = 2 \)). In the case \( h = 1 \) there are only two possible patterns: \( \pi = (0,1) \) and the corresponding spatial (or time) reverse \( \pi = (1,0) \). We focus on \( \pi = (1,0) \). This pattern corresponds to the event \( \{ \Pi(\xi_0, \xi_1) = (1,0) \} = \{ \xi_1 \geq \xi_0 \} = \{ X_1 \geq 0 \} \). Hence, we consider

\[
c_1 = E(X_1^1|X_1 \geq 0) = \int_0^\infty y_1 \varphi(y_1)dy_1 = \varphi(0).
\]

Correspondingly, we obtain \( c_1 = -\varphi(0) \) for \( \pi = (0,1) \) since this is the spatial reversion of \((1,0)\). Thus, for these two ordinal patterns we arrive at a limit distribution of \( q_n(\pi) \) given by

\[
\mathcal{N}(0, c_D \varphi^2(0)), \quad \text{where } c_D = \frac{2}{(1-D)(2-D)}.
\]
We continue with the calculation of the limit variances in the case \( p = 3 \). The integrals under consideration were solved by using Mathematica as well as a lengthy calculations that make use of the Cholesky decomposition (cf. Appendix).

**Example 2** (Ordinal patterns of length \( p = 3 \)). First, we study the limit variance for \( \pi = (2, 1, 0) \). In this case, \( \bar{\pi} \) has two elements. Note that \( \{ \Pi(\xi_0, \xi_1, \xi_2) = (2, 1, 0) \} = \{ \xi_2 \geq \xi_1 \geq \xi_0 \} = \{ X_2 \geq 0, X_1 \geq 0 \} \). Due to the symmetry of the bivariate normal distribution, we obtain \( c_1 = c_2 \), so that we only need to calculate

\[
c_1 = \mathbb{E}(X_11_{\{X_2 \geq 0, X_1 \geq 0\}}) = \int_0^\infty \int_0^{\infty} y_1 \varphi(X_1, X_2)(y_1, y_2)dy_1dy_2 = \frac{\varphi(0)}{2}(1 + r(1)),
\]

where \( \varphi(X_1, X_2) \) denotes the joint density of \((X_1, X_2)\). Hence,

\[
\sum_{j=1}^{2} \alpha_j = 2c_1(g_{1,1} + g_{2,1}) = 2c_1 \frac{1 - r(1)}{1 - (r(1))^2} = \varphi(0),
\]

where \( g_{ij} \) are the entries of \( \Sigma^{-1} \) given by

\[
\Sigma^{-1} = \frac{1}{1 - (r(1))^2} \begin{pmatrix} 1 & -r(1) \\ -r(1) & 1 \end{pmatrix}.
\]

Again, we obtain the limit variance \( c_D \varphi^2(0) \) which is here more surprising than in the case \( h = 1 \) because the result is independent of \( r(1) \). For the space reverse pattern \( \pi_2 = (0, 1, 2) \) we apply (6) and obtain \( c_1 = -\varphi(0) \) leading to the same limit variance. It is an interesting question whether it is just a coincidence that this variance is independent of the covariance between the increments. The answer turns out to be yes, since the dependence is reflected in the limit variance of the pattern \( \pi = (2, 0, 1) \).

Note that \( \{ \Pi(\xi_0, \xi_1, \xi_2) = (2, 0, 1) \} = \{ \xi_1 \leq \xi_0 \leq \xi_2 \} = \{ X_1 \leq 0, X_1 + X_2 \geq 0 \} = \{ X_1 \leq 0, X_2 \geq -X_1 \} \). As a result, we have

\[
c_1 = \mathbb{E}(X_11_{\{X_1 \leq 0, X_2 \geq -X_1\}}) = \int_{-\infty}^{0} \int_{-y_1}^{\infty} y_1 \varphi(X_1, X_2)(y_1, y_2)dy_2dy_1
\]

\[
= \frac{\varphi(0)}{2} \left( \frac{\sqrt{1 + r(1)}}{\sqrt{2}} - 1 \right),
\]

\[
c_2 = \mathbb{E}(X_21_{\{X_1 \leq 0, X_2 \geq -X_1\}}) = \int_{-\infty}^{0} \int_{-y_1}^{\infty} y_2 \varphi(X_1, X_2)(y_1, y_2)dy_2dy_1
\]

\[
= \frac{\varphi(0)}{2} \left( \frac{\sqrt{1 + r(1)}}{\sqrt{2}} - r(1) \right),
\]

where \( \varphi(X_1, X_2) \) denotes the joint density of \((X_1, X_2)\). As a result, we obtain

\[
\sum_{j=1}^{2} \alpha_j = (c_1 + c_2)(g_{1,1} + g_{2,1}) = \frac{\varphi(0)}{2} \left( \frac{\sqrt{2(1 + r(1))} - (1 + r(1))}{1 + r(1)} \right)
\]
\[ = \frac{\varphi(0)}{2} \left( \frac{\sqrt{2}}{\sqrt{1 + r(1)}} - 1 \right). \]

The above expression depends on \( r(1) \). Due to space and time symmetry discussed in (6) and (7) all permutations that belong to the reversion group of \( \pi = (2, 0, 1) \), that is, \((1,0,2),(0,2,1)\) and \((1,2,0)\), lead to the same limit distribution for \( \hat{q}_n(\pi) \), namely

\[ \mathcal{N} \left( 0, c_d \left( \frac{\varphi(0)}{2} \left( \frac{\sqrt{2}}{\sqrt{1 + r(1)}} - 1 \right) \right)^2 \right). \]

### 3.2 Limit distribution of an improved estimator based on Rao–Blackwellization

In the previous section we considered the natural estimator for the frequency of a certain ordinal pattern. However, in Sinn and Keller (2011) it is shown that the estimator which results from averaging the estimates of the same reversion class has better statistical properties. The corresponding estimator is therefore defined by

\[ \hat{p}_n(\pi) := \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\#\pi} 1\{\Pi(\xi_t, \xi_{t+1}, \ldots, \xi_{t+h}) \in \pi\}, \]

where \( \#\pi \) denotes the cardinality of the set \( \pi \).

Recalling that \( \Pi(\xi_t, \xi_{t+1}, \ldots, \xi_{t+h}) = \tilde{\Pi}(X_{t+1}, \ldots, X_{t+h}) \), we are, in particular, interested in the function \( f : \mathbb{R}^h \rightarrow \mathbb{R} \) defined by

\[ f(x_1, \ldots, x_h) := \frac{1}{\#\pi} 1\{\tilde{\Pi}(x_1, \ldots, x_h) \in \pi\} - \frac{1}{\#\pi} P(\tilde{\Pi}(x_1, \ldots, x_h) \in \tilde{\pi}). \quad (8) \]

In order to specify the limit distribution of \( \hat{p}_n(\pi) \), we need to determine the Hermite rank of this function. For this, note that p. 1786 of Sinn and Keller (2011), show that \( f \) has Hermite rank \( m \geq 2 \).

For a multivariate random vector \( (X_1, \ldots, X_h) \sim \mathcal{N}(0, \Sigma_h) \) define

\[ c_{ii}^\pi := E[(X_i^2 - 1) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = \pi\}] = E[(X_i^2 - 1) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = S(\pi)\}] = E[(X_{h+1-i}^2 - 1) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = T(\pi)\}], \quad i = 1, \ldots, h. \]

Analogously, we obtain

\[ c_{ij}^\pi := E[(X_i X_j - E(X_i X_j)) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = \pi\}] = E[(X_i X_j - E(X_i X_j)) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = S(\pi)\}] = E[(X_{h+1-i} X_{h+1-j} - E(X_{h+1-i} X_{h+1-j})) 1\{\tilde{\Pi}(x_1, \ldots, x_h) = T(\pi)\}], \quad i, j = 1, \ldots, h, i \neq j, \]

so that altogether we derive
\[ c_{ij}^\pi = c_{ij}^{S(\pi)} = c_{h+1-i,h+1-j}^\tau = c_{h+1-i,h+1-j}^{S(\pi)} \quad i, j = 1, \ldots, h. \]

With this result we can simplify the second-order Hermite coefficients for the improved estimator

\[ c_{i,j} := E[(X_i X_j - E(X_i X_j)) f(X_1, \ldots, X_h)] = \frac{1}{\# \pi} \sum_{x \in \pi} E[(X_i X_j - E(X_i X_j)) 1_{\{\tilde{\pi}(X_i, \ldots, X_h) = \pi\}}] \]

\[ = \frac{1}{\# \pi} \sum_{x \in \pi} c_{i,j}^\pi \]

\[ = \frac{1}{2} (c_{i,j}^{\pi} + c_{h+1-i,h+1-j}^{\pi}). \]

Analogously, we obtain

\[ c_{i,i} = \frac{1}{2} (c_{i,i}^{\pi} + c_{h+1-i,h+1-i}^{\pi}). \]

Hence, we can uniquely determine the second order Hermite coefficients of the improved estimator by calculating the second order Hermite coefficients for only one pattern \( \pi \) that belongs to the considered reversion group \( \tilde{\pi} \). By following the symmetry properties discussed above we derive for the special case \( T \circ S(\pi) = \pi \)

\[ c_{i,j} = c_{i,j}^{\pi}, \quad \pi \in \tilde{\pi}, \]

for all \( i, j = 1, \ldots, h \).

The second-order Hermite coefficients of the improved estimator \( \hat{q}_n(\pi) \) are equal to the second-order Hermite coefficients of \( \hat{p}_n(\pi) \).

We use this result to determine the Hermite rank of the function \( f \) defined in (8), for details see Section 5, and to simplify the calculations concerning the parameters determining the variance in the next Theorem 3.

**Lemma 2.** The function \( f(x_1, \ldots, x_h) := \frac{1}{\# \pi} 1_{\{\tilde{\pi}(x_1, \ldots, x_h) \in \tilde{\pi}\}} - \frac{1}{\# \pi} P(\tilde{\pi}(x_1, \ldots, x_h) \in \tilde{\pi}) \) has Hermite rank \( m(f, \Sigma_h) = 2 \).

Following the above Lemma, we derive the asymptotic distribution of the new estimator:

**Theorem 3.** Let \( \xi = (\xi_t)_{t \geq 0} \) be a stochastic process and let \( X_t = (X_i)_{t \geq 1} \) denote the process of increments of \( \xi \) given by \( X_t := \xi_t - \xi_{t-1} \) for \( t \geq 1 \). Assume that \( X_t \) is a stationary, long-range dependent standard Gaussian process with auto-covariance function \( r(k) = L(k)k^{-D} \). Then, if \( D \in (0, \frac{1}{2}) \),

\[ n^D(2!C_2)^{-\frac{1}{2}} L^{-1}(n)(\hat{p}_n(\pi) - P(\tilde{\pi}(X_1, \ldots, X_h) = \pi)) \overset{D}{\rightarrow} Z_{2,(1-D)/2}(1) \sum_{j=1}^h \sum_{k=1}^h \alpha_{j,k}, \]

with \( C_2 = ((1 - 2D)(2 - D))^{-1} \), \( (\alpha_{i,k})_{1 \leq i \leq k \leq h} = \Sigma_h^{-1} C \Sigma_h^{-1} \) and

\[ C = E \left( (X_1, \ldots, X_h) \frac{1}{\# \pi} \left[ 1_{\{\tilde{\pi}(X_1, \ldots, X_h) \in \tilde{\pi}\}} - P(\tilde{\pi}(X_1, \ldots, X_h) \in \tilde{\pi}) \right] (X_1, \ldots, X_h)^t \right). \]
Remark 2. For \( D > \frac{1}{2} \), the asymptotic distribution of \( \hat{p}_h(\pi) \) is derived in theorem 7 of Keller and Sinn (2005). In this case, it is Gaussian.

For small \( h \) we calculate the matrix of coefficients \( (a_{l,k})_{1 \leq l,k \leq h} \) explicitly:

**Example 3** (The case \( h = 1 \)). Since we are interested in increments with length \( h = 1 \), we have to study ordinal patterns of length \( p = 2 \). Regarding \( \pi = (1,0) \) we derive the event \( \{\Pi(\xi_0, \xi_1) = (1,0)\} = \{\xi_0 \leq \xi_1\} = \{X_1 \geq 0\} \) and therefore

\[
c_{1,1} = E[(X_1^2 - 1)1_{\{X_1 \geq 0\}}] = \int_0^\infty \int_0^\infty (y_1^2 - 1)\varphi(y_1)dy_1 = 0.
\]

So in the trivial case (only one increment variable) we derive a degenerate limit distribution again.

For increments of length \( h = 2 \), we used Mathematica to calculate the Hermite coefficients.

**Example 4** (The case \( h = 2 \)). First, we consider the pattern \( \pi = (2,1,0) \) and the corresponding event \( \{\Pi(\xi_0, \xi_1, \xi_2) = (2,1,0)\} = \{\xi_2 \geq \xi_1 \geq \xi_0\} = \{X_1 \geq 0, X_2 \geq 0\} \). We know that \( c_{ij} = c_{ij}^\pi \), \( i,j = 1,2 \), and by (9) that \( c_{1,1} = c_{2,2} \) since \( T o S(2,1,0) = (2,1,0) \). We have

\[
c_{1,1} = E[(X_1^2 - 1)1_{\{X_1 \geq 0, X_2 \geq 0\}}] = \int_0^\infty \int_0^\infty (y_1^2 - 1)\varphi(y_1,y_2)dy_1dy_2
\]

\[
= \varphi^2(0)r(1)\sqrt{1 - (r(1))^2}
\]

and

\[
c_{1,2} = E[(X_1X_2 - E(X_1X_2))1_{\{X_1 \geq 0, X_2 \geq 0\}}]
\]

\[
= \int_0^\infty \int_0^\infty y_1y_2\varphi(y_1,y_2)dy_1dy_2 - r(1)\int_0^\infty \int_0^\infty \varphi(y_1,y_2)dy_1dy_2
\]

\[
= \varphi^2(0)\sqrt{1 - (r(1))^2}.
\]

This yields

\[
\sum_{ij=1}^2 \alpha_{ij} = 2(g_{1,2} + g_{2,2})^2(c_{1,1} + c_{1,2})
\]

\[
= 2 \frac{c_{1,1} + c_{1,2}}{1 + r(1))^2}
\]

\[
= 2\varphi^2(0)\sqrt{\frac{1 - r(1)}{1 + r(1)}}.
\]

For \( \pi = (2,1,0) \) the left-hand side in (10) converges in distribution to \( 2\varphi^2(0)\sqrt{\frac{1 - r(1)}{1 + r(1)}}Z_{2,HT}(1) \).

Consider the pattern \( \pi = (2,0,1) \) and the corresponding event \( \{\Pi(\xi_0, \xi_1, \xi_2) = (2,0,1)\} = \{\xi_1 \leq \xi_0 \leq \xi_2\} = \{X_1 \leq 0, X_1 + X_2 \geq 0\} \). It holds that

\[
c_{1,1}^\pi = E[(X_1^2 - 1)1_{\{X_1 \leq 0, X_2 \geq -X_1\}}]
\]

\[
= \int_{-\infty}^0 \int_{-y_1}^\infty (y_1^2 - 1)\varphi(y_1,y_2)dy_2dy_1
\]

\[
= -\varphi^2(0)\frac{\sqrt{1 - (r(1))^2}}{2}.
\]
and
\[
c_{1,2}^\pi = E[(X_1X_2 - E(X_1X_2))1_{X_1 \leq 0, X_2 \geq -X_1}]
\]
\[
= \int_{-\infty}^{0} \int_{-y_1}^{\infty} y_1y_2 \varphi(x_1,x_2)(y_1,y_2)dy_2dy_1 - r(1) \int_{-\infty}^{0} \int_{-y_1}^{\infty} \varphi(x_1,x_2)(y_1,y_2)dy_2dy_1
\]
\[
= -\varphi^2(0) \sqrt{1 - (r(1))^2}.
\]

Since the reversion group of this pattern has four elements we also need to calculate
\[
c_{2,2}^\pi = E[(X_2^2 - 1)1_{X_1 \leq 0, X_2 \geq -X_1}]
\]
\[
= \int_{-\infty}^{0} \int_{-y_1}^{\infty} (y_2^2 - 1) \varphi(x_1,x_2)(y_1,y_2)dy_2dy_1
\]
\[
= -\varphi^2(0) \sqrt{1 - (r(1))^2(2r(1) - 1)}.
\]

Altogether we arrive at
\[
\sum_{i,j=1}^{2} \alpha_{ij} = \frac{1}{(1 + r(1))^2}(c_{1,1} + 2c_{1,2} + c_{2,2})
\]
\[
= \frac{1}{(1 + r(1))^2}(c_{1,1} + 2c_{1,2} + c_{2,2})
\]
\[
= -\varphi^2(0) \sqrt{1 - (r(1))^2(2r(1) - 1)}.
\]

For \(\pi = (2, 0, 1)\) the left-hand side in (10) converges in distribution to \(-\varphi^2(0)\sqrt{1 - (r(1))^2}Z_{2,H}(1)\).

**Remark 3.** The reader might wonder which limit theorems one can derive in the special case that it is not only the increment process which is stationary but the time series itself. We have to determine the Hermite rank of the estimator \(\hat{q}_n(\pi)\) in this setting and we obtain that for any \(\pi \in S_h\) the Hermite rank of the function \(f : \mathbb{R}^{h+1} \to \mathbb{R}\), defined by
\[
f(x_0,x_1, \ldots, x_h) := 1_{\{\Pi(x_0, \ldots, x_h) = \pi\}} - P(\Pi(x_0, \ldots, x_h) = \pi),
\]
is equal to 1 (for details see Section 5).

We get the following asymptotic result concerning the ordinal pattern probability estimator \(\hat{q}_n(\pi)\) in this modified setting:
\[
n^{D/2}L^{-1/2}(n)(\hat{q}_n(\pi) - P(\Pi(x_0, \ldots, x_h) = \pi)) \to^D \delta_0,
\]
where \(\delta_0\) denotes the Dirac measure in 0. In this special case, the limit distribution for \(\hat{q}_n(\pi)\) is trivial.
However, taking the classical rate of convergence $n^{1/2}$, we will get a nontrivial Gaussian central limit theorem as explained in Section 2.1.

### 4 ESTIMATION OF THE HURST PARAMETER

Sinn and Keller (2011) derive an estimator for the Hurst parameter based on the improved estimator for ordinal pattern probabilities $\hat{p}_n(\pi)$. They show asymptotic normality of this estimator in the case $H < \frac{3}{4}$. In order to obtain the asymptotic distribution for $H > \frac{3}{4}$, we briefly describe the setting that was developed in that article. The idea is to determine the probability of changes in the “up-and-down” behavior of the process $\xi$. Since we need to use orthant probabilities of the normal distribution, we restrict ourselves to the case $h = 2$ here (Figure 2).

To capture this mathematically, we define

$$W(i) := 1_{\{\Pi(\xi_i, ..., \xi_{i+2}) \in \Pi\}},$$

with $\Pi = \{(2, 0, 1), (1, 0, 2), (0, 2, 1), (1, 2, 0)\}$.

Therefore, we obtain

$$c := P(W(i) = 1) = 2P(X_{i+1} \geq 0, X_{i+2} \leq 0) = \frac{1}{2} - \frac{1}{\pi} \arcsin(r(1)),$$

where $r$ is the covariance function of the stationary and long-range dependent increment process $X=(X_k)_{k \geq 1}$ of $\xi$ as defined above; see p. 92 of Kotz, Balakrishnan, and Johnson (2004). Since $r$ depends on the long-range dependence parameter $D$, which we can express as $D = 2 - 2H$ in terms of the Hurst parameter, we will write $c = c(H)$ in the following.

In order to estimate this probability, we choose the relative frequency as an estimator:

$$\hat{c}_n := \frac{1}{n} \sum_{i=0}^{n-1} W(i) = 4\hat{p}_n(\pi),$$

with $\pi \in \{(2, 0, 1), (1, 0, 2), (0, 2, 1), (1, 2, 0)\}$. We want to estimate the Hurst parameter $H$ in the case that $X$ is fractional Gaussian noise and hence $\xi$ is fractional Brownian motion. The correlation function of fractional Gaussian noise is given by

$$r_H(k) = \frac{1}{2}[(k + 1)^{2H} - 2k^{2H} + (k - 1)^{2H}],$$

such that $r_H(1) = 2^{2H-1} - 1$. Therefore, we obtain
\[ c(H) = 1 - \frac{2}{\pi} \arcsin(2^{H-1}), \quad H \in (0, 1), \]

since \( \arcsin(x) = 2 \arcsin(\sqrt{\frac{1+x}{2}}) - \frac{\pi}{2} \) for \( x \in [-1,1] \). The probability of changes in the up-and-down-behavior gets smaller if the Hurst parameter gets larger, as expected intuitively due to the persistent behavior of long-range dependent time series. We calculate the inverse of \( c \) by

\[ g(x) := \max \left\{ 0, \log_2 \left( \cos \left( \frac{\pi x}{2} \right) \right) + 1 \right\}, \quad x \in [0,1], \]

so that \( H = g(c(H)) \) is satisfied.

The Zero-Crossing estimator of the Hurst Parameter \( H \) is then defined by

\[ \hat{H}_n := g(\hat{c}_n). \]

In corollary 11 of Sinn and Keller (2011), it is shown that \( \hat{H}_n \) is a strongly consistent and asymptotically unbiased estimator of the Hurst Parameter, as well as it is asymptotically normal if \( H < \frac{3}{4} \). Using Theorem 3 we can complement their result by the following theorem.

**Theorem 4.** If \( H > \frac{3}{4} \),

\[ n^{2-2H} \frac{\sqrt{4H-3}}{\sqrt{H(2H-1)}} (\hat{H}_n - H) \rightarrow^D Z_{2,H}(1) \left( \frac{2\pi}{2\log 2} \tan \left( \frac{\pi c(H)}{2} \right) \right) \phi^2(0) \sqrt{2^2-2H-1}. \]

**Proof.** Since \( \hat{c}_n = 4\hat{p}_n \), it follows by Theorem 3 and Example 4 that

\[ n^D (2C_2)^{-1} L^{-1}(n)(\hat{c}_n - c(H)) \rightarrow^D Z_{2,H}(1) \left( 4 \sum_{k=1}^{2} \sum_{l=1}^{2} \alpha_{l,k} \right), \]

where \( (\alpha_{l,k})_{1 \leq l,k \leq h} = C\Sigma_{h+1}^{-1} \) with

\[ C = E \left( (X_1, \ldots, X_h) \frac{1}{\#\bar{\pi}} [1_{\{\bar{\pi}(X_1, \ldots, X_h) \in \bar{\pi}\}} - P(\bar{\pi}(X_1, \ldots, X_h) \in \bar{\pi})] (X_1, \ldots, X_h)^t \right) \]

for \( \bar{\pi} = \{(2,0,1), (1,0,2), (0,2,1), (2,0,1)\} \). Therefore, and according to Example 4, we arrive at

\[ \sum_{k=1}^{2} \sum_{l=1}^{2} \alpha_{l,k} = -\phi^2(0) \sqrt{2^2-2H-1} = -\phi^2(0) \sqrt{2^2-2H-1}. \]

We also know that \( 2C_2 = 2((1 - 2D)(2 - D))^{-1} = (H(4H - 3))^{-1} \) and since

\[ r_H(k) \sim H(2H - 1)k^{2H-2}, \]

we get \( L(n) \sim H(2H - 1) \) (see Beran et al., 2013, p. 34) with \( f(k) \sim g(k) \) meaning that \( \lim_{k \to \infty} \frac{f(k)}{g(k)} = 1 \). All in all, it follows that
\[
\frac{n^{2-H} \sqrt{4H-3}}{\sqrt{H(2H-1)}} (\hat{c}_n - c(H)) \rightarrow^D Z_{2, H}(1)(-4\varphi^2(0)\sqrt{2^{2-2H} - 1}).
\]

We have \( H = g(c(H)) \) and \( \hat{H}_n = g(\hat{c}_n) \). Due to \( c(H) \in \left( 0, \frac{2}{3} \right) \) for \( H \in (0, 1) \), \( g'(c(H)) = -\frac{x}{2\log^2 \tan \left( \frac{x}{2} \right) } \) exists and does not equal zero for \( H \in (0, 1) \). Applying Theorem 3 in Van der Vaart (2000) we arrive at the above limit.

\section{Proofs}

In this section the proofs of the results derived in Section 3 are presented. We are able to give these results in a more general way than needed in the context of ordinal patterns and therefore consider a larger class of functions. In the following we consider the asymptotic behavior of the partial sums

\[
\sum_{i=1}^{n} \{ f(X_i, \ldots, X_{i+p-1}) - Ef(X_1, \ldots, X_p) \},
\]

where \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) satisfies \( E(f(X_1, \ldots, X_p))^2 < \infty \). A first result on the asymptotic behavior of these partial sums, which includes the statement of Theorem 1, is given by the following proposition that can be derived from Birkhoff’s ergodic theorem; see also Sinn and Keller (2011).

\textbf{Proposition 1.} Suppose that \( (X_i)_{i \geq 1} \) is a stationary ergodic process, and that \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) is a measurable function such that \( f(X_1, \ldots, X_p) \in L_1 \). Then,

\[
\frac{1}{n} \sum_{i=1}^{n} f(X_i, \ldots, X_{i+p-1}) \rightarrow Ef(X_1, \ldots, X_p)
\]

almost surely, as \( n \rightarrow \infty \).

\textbf{Proof.} Ergodicity of the process \( (X_i)_{i \geq 1} \) means that the shift operator

\( S : \mathbb{R}^N \rightarrow \mathbb{R}^N \),

defined by \( (x_i)_{i \geq 1} \mapsto (x_{i+1})_{i \geq 1} \), is an ergodic transformation on the sequence space \( \mathbb{R}^N \), equipped with the product \( \sigma \)-field and the probability measure \( \mu = \mathcal{L}((X_i)_{i \geq 1}) \). Thus, by Birkhoff’s ergodic theorem, we obtain for any integrable function \( g : \mathbb{R}^N \rightarrow \mathbb{R} \)

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} g(S^k(\omega)) = \int g(\omega) d\mu(\omega),
\]

almost surely. We now apply the ergodic theorem to the function \( g : \mathbb{R}^N \rightarrow \mathbb{R} \), defined by

\[
g(x_i)_{i \geq 1} := f(x_1, \ldots, x_p).
\]
With this choice of \( g \), we obtain \( g(S_k^k((x_i)_{i \geq 1})) = f(x_{k+1}, \ldots, x_{k+p}) \) and \( \int g(\omega) d\mu(\omega) = Eg((X_i)_{i \geq 1}) = Ef(X_1, \ldots, X_p) \), and thus

\[
\frac{1}{n} \sum_{k=1}^{n} f(x_{k+1}, \ldots, x_{k+p}) \to Ef(X_1, \ldots, X_p),
\]

for \( \mu \)-almost every sequence \((x_i)_{i \geq 1}\). Thus, by definition of \( \mu \), we find

\[
\frac{1}{n} \sum_{k=1}^{n} f(X_{k+1}, \ldots, X_{k+p}) \to Ef(X_1, \ldots, X_p),
\]

almost surely. \( \blacksquare \)

**Remark 4.** A stationary Gaussian process \((X_k)_{k \geq 0}\) with auto-covariance function \( r(k) = \text{Cov}(X_0, X_k) \) such that \( r(k) \to 0 \) if \( k \to \infty \) is mixing and hence ergodic; see Samorodnitsky (2007), pp. 43, 46. Thus, we may apply the above results to such Gaussian processes.

In general, in order to derive the limit distributions of partial sums given in (11), a careful analysis of the Hermite rank of the considered function \( f \) is crucial. First, we present two lemmas that are helpful tools to determine the Hermite rank of a function \( f \) and later on we give the proofs of Lemma 1 and Lemma 2, in which we deal with the heuristic estimator of ordinal pattern probabilities and the improved estimator based on the Rao–Blackwellization, respectively. Furthermore, we give the justification for the Hermite rank of the estimator considered in Remark 3.

It is well known that for \( \Sigma_p = AA' \) we have \( m(f, \Sigma_p) = m(f \circ A, E_p) \) and that \( m(f, \Sigma_p) \neq m(f, E_p) \) in general; see Beran et al. (2013), lemma 3.7. The last fact is disadvantageous since determining \( m(f \circ A, E_p) \) is usually much more complicated than determining \( m(f, E_p) \). However, it is possible to show that under a mild additional assumption \( m(f \circ A, E_p) \) is bounded by \( m(f, E_p) \); see Lemma 4.

**Lemma 3.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions in \( L^2(\mathcal{N}(0, E_p)) \), and let \( f \) be another function in this space such that \( f_n \to f \) in the metric of this space. Furthermore, let \( \Sigma_p \) be a positive definite matrix such that \( (\Sigma_p^{-1} - E_p) \) is positive semidefinite. In this case, \((f_n)_{n \in \mathbb{N}}\) converges to \( f \) in \( L^2(\mathcal{N}(0, \Sigma_p)) \).

**Proof.** Let \( h \) be the Radon–Nikodym density of \( \mathcal{N}(0, \Sigma_p) \) with respect to \( \mathcal{N}(0, E_p) \) such that we have

\[
\int_{\mathbb{R}^d} [f_n - f]^2 \, d\mathcal{N}(0, \Sigma_p) = \int_{\mathbb{R}^d} [f_n - f]^2 h \, d\mathcal{N}(0, E_p).
\]

Hence, proving Lemma 3 boils down to the boundedness of \( h \) which is obtained in the above setting by an elementary calculation: let \( \varphi \) and \( \tilde{\varphi} \) denote the density of \( \mathcal{N}(0, E_p) \) and \( \mathcal{N}(0, \Sigma_p) \), respectively. Then,

\[
h(x) = \frac{\varphi(x)}{\tilde{\varphi}(x)} = \frac{\det(E_p)}{\det(\Sigma_p)} \exp \left( -\frac{1}{2}(x' \Sigma_p^{-1} x - x'E_p x) \right) = \frac{1}{\det(\Sigma_p)} \exp \left( -\frac{1}{2}(x' (\Sigma_p^{-1} - E_p) x) \right).
\]

\( \blacksquare \)
Given the above result, we arrive at the following upper bound for $m(f \circ A, E_p)$:

**Lemma 4.** Let $f : \mathbb{R}^p \to \mathbb{R}$ be square-integrable with respect to $\mathcal{N}(0, E_p)$ and let $\Sigma_p = AA^t$ be a $(p) \times (p)$ positive definite covariance matrix such that $(\Sigma_p^{-1} - E_p)$ is positive semidefinite. Then,

$$m(f \circ A, E_p) \leq m(f, E_p).$$

**Remark 5.** Note that, for $\rho \neq 0$,

$$x^t(\rho \Sigma_p^{-1} - E_p)x > 0 \iff \frac{x^t \Sigma_p^{-1} x}{x^t x} > \frac{1}{\rho}.$$  

With $\lambda_{\min}(\Sigma^{-1})$ denoting the smallest eigenvalue of $\Sigma^{-1}$, we have

$$\frac{x^t \Sigma_p^{-1} x}{x^t x} > \frac{1}{\rho} \geq \lambda_{\min}(\Sigma^{-1}).$$

Given that $\Sigma_p$, and thus $\Sigma_p^{-1}$, are positive definite matrices, $\lambda_{\min}(\Sigma^{-1}) > 0$ so that we can choose $\rho$ such that $\rho \Sigma_p^{-1} - E_p$ is positive semidefinite. Since ordinal patterns are not affected by scaling, we may for this reason assume that $(\Sigma_p^{-1} - E_p)$ is positive semidefinite.

**Proof.** Expanding both $f$ and $f \circ A$ in Hermite polynomials with respect to $\mathcal{N}(0, E_p)$ yields

$$f(U) = \sum_{k=m_1}^{\infty} \sum_{l_1+\cdots+l_p=k} \frac{J_{l_1,\ldots,l_p}^1}{l_1! \cdots l_p!} H_{l_1,\ldots,l_p}(U)$$

$$f(A \cdot U) = \sum_{k=m_1}^{\infty} \sum_{l_1+\cdots+l_p=k} \frac{J_{l_1,\ldots,l_p}^2}{l_1! \cdots l_p!} H_{l_1,\ldots,l_p}(U),$$  

(12)

where $m_1 = m(f, E_p)$ and $m_2 = m(f, \Sigma_p)$. Using Lemma 3 we can replace $U$ by $A \cdot U$ in (12) such that

$$f(A \cdot U) = \sum_{k=m_1}^{\infty} \sum_{l_1+\cdots+l_p=k} \frac{J_{l_1,\ldots,l_p}^3}{l_1! \cdots l_p!} (H_{l_1,\ldots,l_p} \circ A)(U).$$  

(13)

Each polynomial $H_{l_1,\ldots,l_p} \circ A$ can be represented by a linear combination of multivariate Hermite polynomials of degree less than or equal to $l_1 + \cdots + l_p$. Therefore, we can rewrite (13) to

$$f(A \cdot U) = \sum_{k=m_3}^{\infty} \sum_{l_1+\cdots+l_p=k} \frac{J_{l_1,\ldots,l_p}^3}{l_1! \cdots l_p!} H_{l_1,\ldots,l_p}(U),$$

with $m_3 \leq m_1$. By uniqueness of the Hermite decomposition we have $m_2 = m_3$, which completes the proof.■
5.1 Proof of Lemma 1

Proof. Since ordinal patterns are not affected by scaling, we may assume that \((\Sigma_h^{-1} - E_h)\) is positive semidefinite. According to Lemma 4 it suffices to show \(E(Y_k^1(\tilde{\Pi}(Y_1,...,Y_h) = \pi_1)) \neq 0\) for some independent standard normal random variables \(Y_1,...,Y_h\) and some \(1 \leq k \leq h\). For simplicity, we regard the pattern \(\pi = (h, \ldots, 0)\) which corresponds to the event \(\{Y_i \geq 0, i = 1, \ldots, h\}\). Hence, we arrive at

\[
E(Y_1 \mathbb{1}_{Y_1 \geq 0, \ldots, Y_h \geq 0}) = \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} y_1 \varphi(y_1) \varphi(y_2) \cdots \varphi(y_h) dy_1 dy_2 \cdots dy_h
\]

\[
= \left(\frac{1}{2}\right)^{h-1} \varphi(0) \neq 0.
\]

It follows by the same reasoning that none of the expected values that correspond to the other ordinal patterns equals zero. □

5.2 Proof of Lemma 2

Proof. For \(\Sigma_h = AA'\) we have \(m(f, \Sigma) = m(f \circ A, E_h)\); see lemma 3.7 of Beran et al. (2013). According to Lemma 4, we have

\[m(f \circ A, E_h) \leq m(f, E_h).\]

As a result, it is sufficient to show that \(m(f, E_h) \leq 2\), such that we may conclude \(m(f, \Sigma_h) = 2\).

To this end, let \(Y_1 = (Y_1, \ldots, Y_h)\) be a standard Gaussian random vector (i.e., with auto-covariance matrix \(E_h\)). Following the arguments above, we only need to consider the second order Hermite coefficients of \(\hat{q}_n(\pi)\) for a fixed pattern \(\pi \in \Pi:\)

\[b^x_{jk} = E(Y_k Y_j \mathbb{1}_{(\tilde{\Pi}(Y_1, \ldots, Y_h) = \pi)}), \quad 1 \leq k < j \leq h, \quad \text{and}\]

\[b^x_{jj} = E((Y_j^2 - 1) \mathbb{1}_{(\tilde{\Pi}(Y_1, \ldots, Y_h) = \pi)}), \quad j = 1, \ldots, h.
\]

For simplicity we regard \(\pi = (h, h - 1, \ldots, 0)\). Note that for this pattern it suffices to show \(b^x_{jk} \neq 0\) to prove \(m(f, E_h) \leq 2\), since in this case \(c_{ij} = c^x_{ij}\) for \(i, j = 1, \ldots, h\). For \(j \neq k\)

\[
b^x_{jk} = \int_{0}^{\infty} \cdots \int_{0}^{\infty} y_j y_k \varphi(y_1) \cdots \varphi(y_h) dy_1 \cdots dy_h
\]

\[
= \frac{1}{2^{h-2}} \int_{0}^{\infty} y_j \varphi(y_j) dy_j \int_{0}^{\infty} y_k \varphi(y_k) dy_k
\]

\[
= \frac{\varphi^2(0)}{2^{h-2}}.
\]

Remark 6. By a similar calculation we obtain that \(b^x_{jj} = 0\) for all \(j = 1, \ldots, h\) for the fixed pattern in the setting above.
Proof of the determination of the Hermite rank of the considered estimator in Remark 3:

**Proof.** Let \((X_k)_{k \geq 0}\) be a stationary, long-range dependent, standard normal Gaussian process and let \(h \in \mathbb{N}\). By Lemma 4 it is enough to show that \(E(Y_0 1_{\{\Pi(Y_0, \ldots, Y_h) = \pi\}}) \neq 0\) for some independent standard normal random variables \(Y_0, \ldots, Y_h\) and some \(0 \leq k \leq h\). Without loss of generality let \(\pi = \text{id}\) and set \(k = 0\). This yields

\[
E(Y_0 1_{\{Y_0 \leq \ldots \leq Y_h\}}) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \varphi(x_2) \varphi(x_3) \ldots \varphi(x_{h+1})dx_2 \ldots dx_{h+1}
\]

since we integrate a strictly positive function. Hence, for any \(\pi \in S_h\) the Hermite rank of the function \(f : \mathbb{R}^{h+1} \rightarrow \mathbb{R}\), defined by

\[
f(x_0, x_1, \ldots, x_h) := 1_{\{\Pi(x_0, \ldots, x_h) = \pi\}} - P(\Pi(x_0, \ldots, x_h) = \pi),
\]

is equal to 1.

We have finished our preparations and are now able to turn to the limit theorems for the partial sums in (11) for function \(f\) with Hermite rank 1 and 2. In the following, we will assume without loss of generality that

\[
E(f(X_1, \ldots, X_p)) = 0.
\]

We want to apply the results of Arcones (1994) which hold for partial sums of functions of \(\mathbb{R}^p\)-valued random vectors \(Y_i\) that have a multivariate standard normal distribution; see also Major (2019) for an alternative approach. Thus, we need to transform the vector \(X_i := (X_i, \ldots, X_{i+p-1})\) accordingly. Let \(\Sigma_p\) denote the covariance matrix of the vector \((X_1, \ldots, X_p)\). Observe that \(\Sigma_p\) is a Toeplitz matrix whose entries are determined by the autocovariance function \(r(i) = E(X_1X_{1+i})\) of the process \((X_i)_{i \geq 0}\), that is,

\[
\Sigma_p = (r(i-j))_{1 \leq i,j \leq p}.
\]

The Cholesky decomposition yields

\[
\Sigma_p = AA^t,
\]

where \(A\) is an upper triangular matrix. Thus, there exists a standard normally distributed random vector \(Y_i\) such that

\[
X_i = A Y_i.
\]
We can rewrite the partial sum in (3) in terms of the random vectors $Y_i$ as follows:

$$\sum_{i=1}^{n} f(X_i, \ldots, X_{i+p-1}) = \sum_{i=1}^{n} f(AY_i) = \sum_{i=1}^{n} g(Y_i),$$  \hspace{1cm} (14)

where $g : \mathbb{R}^p \to \mathbb{R}$ is defined by

$$g(y) = f(Ay).$$

In order to characterize the asymptotic distribution of the considered partial sum process, we apply theorem 6 of Arcones (1994). Employing the special structure of $Y_i$ we obtain explicit representations of the limit distributions for the cases Hermite rank equal to 1 and 2.

5.3 Limit theorems for functions with Hermite rank 1

First we consider the asymptotic behavior of function $f$ with Hermite rank 1. Note that this theorem implies the statement of Theorem 2.

**Theorem 5.** Let $(X_j)_{j \geq 0}$ be a stationary, long-range dependent standard Gaussian process with auto-covariance function $r(k) = L(k) k^{-D}$ and let $f : \mathbb{R}^p \to \mathbb{R}$ be a function with Hermite rank 1 satisfying $E(f(X_1, \ldots, X_p))^2 < \infty$. Then,

$$\frac{1}{n^{1-D/2}L^{1/2}(n)} \sum_{i=1}^{n} (f(X_i, \ldots, X_{i+p-1}) - Ef(X_1, \ldots, X_p)) \to \mathcal{N} \left( 0, c_D \left( \sum_{j=1}^{p} \alpha_j \right)^2 \right),$$

where $c_D = \frac{2}{(1-D)(2-D)}$ and where $\alpha = (\alpha_1, \ldots, \alpha_p)^t = \Sigma_p^{-1} c$ with $c = (c_1, \ldots, c_p)^t = E(f(X_1, \ldots, X_p)X_1)$.

**Proof.** Given that the function $f$ has Hermite rank $m(f, \Sigma_p) = 1$, the limit behavior corresponds to the asymptotic behaviour of the first-order term in the Hermite expansion of $f$. The Hermite rank $m(f, \Sigma_p)$ of $f$ with respect to $X_i$ is the same as the Hermite rank $m(g, E_p) = m(f \circ A, E_p)$ of $g$ with respect to $Y_i$; see lemma 3.7 of Beran et al. (2013). Since $f(X_i, \ldots, X_{i+p-1}) = g(Y_i)$ this first-order term is given by

$$\sum_{j=1}^{p} (E(g(Y_i)Y_{i+j-1})) Y_{i+j-1},$$  \hspace{1cm} (15)

with $1 \leq j \leq p$, since $Y_i = (Y_{i_1}, \ldots, Y_{i+p-1})$. It follows by stationarity and by definition of the process $(Y_i)_{i \geq 0}$ that the coefficient in the Hermite expansion (15) corresponds to

$$b_j := E(g(Y_i)Y_{i+j-1}) = E(g(Y_1)Y_j) = E(f(X_1, \ldots, X_{1+h})Y_j).$$

We can thus express the vector of coefficients $b := (b_1, \ldots, b_p)^t$ as follows:

$$b = E(f(X_1, \ldots, X_p)Y_1)$$
\[
E(f(X_1, \ldots, X_p)A^{-1}X_1) = A^{-1} \cdot E(f(X_1, \ldots, X_p)X_1) = A^{-1} \cdot c,
\]

where \( c = (c_1, \ldots, c_p)' \) is the vector of inner products of the random variables \( X_1, \ldots, X_p \) with \( f(X_1, \ldots, X_p) \), that is,
\[
c_k = E(f(X_1, \ldots, X_p)X_k), \quad 1 \leq k \leq p.
\]

According to the results of Arcones (1994), we know that the partial sums \( \sum_{i=1}^{n} g(Y_i) \) are dominated by the corresponding partial sums of the first-order term in the Hermite expansion, that is, that
\[
\sum_{i=1}^{n} g(Y_i) = \sum_{i=1}^{n} \left( \sum_{j=1}^{p} (E(g(Y_i)Y_{i+j-1})) Y_{i+j-1} \right) + o_p(n^{1-D/2}L^{1/2}(n)), \tag{16}
\]

where for a sequence of random variables \((X_n)_{n \in \mathbb{N}}\) we write \( X_n = o_p(n) \) if \( \frac{X_n}{n} \to 0 \).

With the notations introduced above, we obtain
\[
\sum_{j=1}^{p} E(g(Y_i))Y_{i+j-1} \cdot Y_{i+j-1} = b'Y_i = b'A^{-1}X_i = \sum_{j=1}^{p} a_jX_{i+j-1},
\]

where the vector \( \alpha = (\alpha_1, \ldots, \alpha_p)' \) is given by
\[
\alpha := (b'A^{-1})' = (A^{-1})'b = (A^{-1})'A^{-1}c = \Sigma_p^{-1}c.
\]

Thus, we obtain
\[
\sum_{i=1}^{n} g(Y_i) = \sum_{i=1}^{n} \left( \sum_{j=1}^{p} a_jX_{i+j-1} \right) + o_p(n^{1-D/2}L^{1/2}(n))
\]
\[
= \sum_{j=1}^{p} \left\{ a_j \left( \sum_{i=1}^{n} X_{i+j-1} \right) \right\} + o_p(n^{1-D/2}L^{1/2}(n))
\]
\[
= \left( \sum_{j=1}^{p} a_j \right) \sum_{i=1}^{n} X_i + o_p(n^{1-D/2}L^{1/2}(n)).
\]

The distribution of the partial sum \( \sum_{i=1}^{n} X_i \) on the right-hand side can be calculated exactly, as this is a partial sum of normal random variables.

In the following, we study partial sums of functions of increments of a stationary long-range dependent Gaussian process of the following type
\[
\sum_{i=1}^{n} f(X_{i+1} - X_i, \ldots, X_{i+p-1} - X_{i+p-2}).
\]

This is a special case of partial sums of the type \( \sum_{i=1}^{n} f(X_i, \ldots, X_{i+p-1}) \), where
\[
f(x_1, \ldots, x_p) = \tilde{f}(x_2 - x_1, \ldots, x_p - x_{p-1}).
\]
Functions of this kind appear, for example, when studying ordinal patterns, compare Section 3, considerations in (4). Therefore the following lemma combined with Theorem 5 yield the justification for the asymptotic result derived in Remark 3.

**Lemma 5.** If $f$ can be written as a function of the increments, we have

$$\sum_{i=1}^{p} \alpha_i = 0.$$ 

**Proof.** We use a well-known fact about Gaussian random variables: Let $Y = (Y_1, \ldots, Y_p)'$ be a vector of independent standard normally distributed random variables $Y_i$, $1 \leq i \leq p$, and let $C_1 \in \mathbb{R}^{k \times p}$ and $C_2 \in \mathbb{R}^{l \times p}$ be two matrices. Then, the random vectors $C_1Y$ and $C_2Y$ are independent, if and only if each of the rows of $C_1$ is orthogonal to each of the rows of $C_2$, that is, when $C_1C_2^t = 0$.

We then use the representation of $\alpha$ that we derived in the course of the proof of Theorem 5, namely

$$\alpha = \Sigma_p^{-1/2} E\{\tilde{f}(X_2 - X_1, \ldots, X_p - X_{p-1})'(X_1, \ldots, X_p)'\}$$

$$= E\{\tilde{f}(X_2 - X_1, \ldots, X_p - X_{p-1})'\Sigma_p^{-1/2}(X_1, \ldots, X_p)\}'$$

$$= E\{\tilde{f}(UX)\Sigma_p^{-1/2}X\},$$

where $X = (X_1, \ldots, X_p)'$, and where $U$ is the $(p - 1) \times p$ matrix defined by

$$U = \begin{pmatrix}
-1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 1
\end{pmatrix}.$$

Let $\Sigma_p^{1/2}$ be a positive definite symmetric matrix such that $\Sigma_p^{1/2}\Sigma_p^{1/2} = \Sigma_p$ and let $\Sigma_p^{-1/2}$ be its inverse. Then, $Y := \Sigma_p^{-1/2}X$ has a $p$-variate standard normal distribution. With this notation, we can rewrite the above expression for $\alpha$ as follows:

$$\alpha = E\{\tilde{f}(UX)\Sigma_p^{-1/2}X\} = E\{\tilde{f}(U\Sigma_p^{1/2}Y)\Sigma_p^{-1/2}Y\}.$$ 

If $\|p\| = (1, 1, \ldots, 1, 1)$,

$$\sum_{i=1}^{p} \alpha_i = \|p\| \alpha = E\{\tilde{f}(U\Sigma_p^{1/2}Y)\Sigma_p^{-1/2}Y\}.$$ 

Now we can apply the initial remark to the vectors $U\Sigma_p^{1/2}Y$ and $\Sigma_p^{-1/2}Y$. We have

$$(U\Sigma_p^{1/2})(\Sigma_p^{-1/2})' = U\Sigma_p^{1/2}\Sigma_p^{-1/2}\Sigma_p^{-1/2} = U\Sigma_p^{-1/2} = 0,$$

and thus the vectors $U\Sigma_p^{1/2}Y$ and $\Sigma_p^{-1/2}Y$ are independent. Hence,
\[ \sum_{i=1}^{p} \alpha_i = \mathbb{I}_{p}^{T} \alpha = \mathbb{E} \{ \tilde{f}(U \Sigma_{p}^{1/2} Y) \mathbb{I}_{p}^{T} \Sigma_{p}^{-1/2} Y \} = \mathbb{E} \{ \tilde{f}(U \Sigma_{p}^{1/2} Y) \} \mathbb{E} \{ \mathbb{I}_{p}^{T} \Sigma_{p}^{-1/2} Y \} = 0, \]

since \( \mathbb{E}(Y) = 0 \).

**Remark 7.** Lemma 5 implies that the limit in theorem 2.3 is trivial if the function \( f \) can be considered as a function of the increment process of a stationary, long-range dependent Gaussian process \( (X_j)_{j \geq 0} \). An explanation for this phenomenon results from the observation that the increments of long-range dependent time series do not display characteristic features of long-range dependence. To see this, let \( g \) denote the spectral density of the time series \( (X_j)_{j \geq 0} \), that is, \( g \) is a nonnegative function satisfying

\[ r(k) := \text{Cov}(X_0, X_k) = \int_{-\pi}^{\pi} e^{ik\lambda} g(\lambda) d\lambda. \]

By assumption, we have

\[ r(k) = k^{-D} L(k), \]

where \( L \) is a function that is slowly varying at infinity. If, additionally, \( L \) is quasi-monotone, it follows that

\[ g(\lambda) = |\lambda|^{D-1} L_{\tilde{g}}(\lambda), \]

for some function \( L_{\tilde{g}} \) that is slowly varying at zero; see Pipiras and Taqqu (2017), p. 19. For the increment process \( (Z_j)_{j \geq 1} \) defined by \( Z_j := X_j - X_{j-1} \), it then holds that

\[ \text{Cov}(Z_1, Z_{k+1}) = \int_{-\pi}^{\pi} e^{ik\lambda} 2(1 - \cos(\lambda)) f(\lambda) d\lambda. \]

For this reason, \( \tilde{g}(\lambda) := 2(1 - \cos(\lambda)) g(\lambda) \) corresponds to the spectral density of the process \( (Z_j)_{j \geq 1} \). Note that

\[ \tilde{g}(\lambda) = |\lambda|^{D+1} L_{\tilde{g}}(\lambda), \]

with

\[ L_{\tilde{g}}(\lambda) := L_{\tilde{g}}(\lambda) \frac{2(1 - \cos(\lambda))}{\lambda^2}, \]

slowly varying at zero. It follows that

\[ \sum_{k=-\infty}^{\infty} r(k) = 0, \]

that is, the increment process is antipersistent and, in particular, short-range dependent; see p. 31 of Pipiras and Taqqu (2017).
This finding coincides with results on limit theorems for discretely observed processes based on fractional Brownian motion, where the application of linear difference filters leads to a smaller exponent in the auto-covariance function, compare Coeurjolly (2001) and Istas and Lang (1997). In our setting this would mean that considering differences of the stationary, long-range dependent process would lead to a short-range dependent process and hence to a Gaussian central limit theorem with a different normalizing constant, namely \( n^{-1/2} \).

### 5.4 Limit theorems for functions with Hermite rank 2

We continue to study the asymptotic behavior of the partial sums in (3) for a function \( f \) with Hermite rank 2 and therefore obtain the statement of Theorem 3 along the way.

**Theorem 6.** Let \( (X_i)_{i \geq 1} \) be a stationary, long-range dependent standard Gaussian process with auto-covariance function \( r(k) = L(k)k^{-D}, k \geq 1 \), and let \( f: \mathbb{R}^p \to \mathbb{R} \) be a function with Hermite rank 2 satisfying \( E(f(X_1, \ldots, X_p))^2 < \infty \). Then, if \( D \in \left(0, \frac{1}{2}\right) \),

\[
\frac{n^{D-1}(2!C_2)^{-\frac{1}{2}}L^{-1}(n) \sum_{i=1}^{n} (f(X_i, \ldots, X_{i+p-1}) - Ef(X_1, \ldots, X_p))}{Z_{2,(1-D/2)}(1) \sum_{l=1}^{p} \sum_{k} \alpha_{l,k}} \to D \text{ in } L^2.
\]

with \( (\alpha_{l,k})_{1 \leq l,k \leq p} := \Sigma_{p}^{-1} C \Sigma_{p}^{-1} \), \( C = (c_{l,k})_{1 \leq l,k \leq p} = Ef(X_1) - Ef(X_1)X_1 \) and \( C_2 = ((1 - 2D)(2 - D))^{-1} \).

**Remark 8.** The extension of the theorem above for \( D > \frac{1}{2} \) is given in theorem 4 of Arcones (1994).

**Proof.** Recall that \( X_i = A Y_i \) for an upper triangular matrix \( A \) with \( AA^T = \Sigma_p \) and \( \Sigma_p = (r(i - j))_{1 \leq i,j \leq p} \). and a multivariate standard normally distributed vector \( Y_i \). Since the Hermite rank of \( g \), defined by \( g(y) := f(Ay) \), equals 2, the partial sums \( \sum_{i=1}^{n} g(Y_i) \) are dominated by the corresponding partial sums of the second-order term in the Hermite expansion, that is,

\[
\sum_{i=1}^{n} g(Y_i) = \sum_{l_1+l_2+\ldots+l_p=2} \left( E(g(Y_i)H_{l_1,l_2,\ldots,l_p}(Y_i)) \right) H_{l_1,l_2,\ldots,l_p}(Y_i) + o_p(n^{1-D}L(n));
\]

see theorem 6 in Arcones (1994). Note that

\[
\sum_{l_1+l_2+\ldots+l_p=2} \left( E(g(Y_i)H_{l_1,l_2,\ldots,l_p}(Y_i)) \right) H_{l_1,l_2,\ldots,l_p}(Y_i)
\]

\[
= \sum_{j=1}^{p} \left( E(g(Y_i)(Y_{i+j-1}^2 - 1)) \right) (Y_{i+j-1}^2 - 1) + \sum_{1 \leq j,k \leq p, j \neq k} \left( E(g(Y_i)Y_{i+j-1}Y_{i+k-1}) \right) Y_{i+j-1}Y_{i+k-1}
\]

\[
= \sum_{j=1}^{p} \left( E(g(Y_i)(Y_{i+j-1}^2 - 1)) \right) (Y_{i+j-1}^2 - 1) + \sum_{1 \leq j,k \leq p, j \neq k} \left( E(g(Y_i)Y_{i+j-1}Y_{i+k-1}) \right) Y_{i+j-1}Y_{i+k-1}
\]

\[
= \sum_{1 \leq j,k \leq p} E(g(Y_i)Y_{i+j-1}Y_{i+k-1}) Y_{i+j-1}Y_{i+k-1} - \sum_{j=1}^{p} E(g(Y_i)Y_{i+j-1}^2).
\]
Since the left-hand side of the above equality is centered to mean zero,
\[
E \left( \sum_{1 \leq j, k \leq p} E(g(Y_i)Y_{i+j-1}Y_{i+k-1}) \right) Y_{i+j-1}Y_{i+k-1} = \sum_{j=1}^{p} E(g(Y_i)Y_{i+j-1}^2).
\]

With \( B = (b_{j,k})_{1 \leq j, k \leq p} \), where \( b_{j,k} = E(g(Y_i)Y_{i+j-1}Y_{i+k-1}) = E(g(Y_1)Y_jY_k) \), it follows that
\[
\sum_{1 \leq j, k \leq p} E(g(Y_i)Y_{i+j-1}Y_{i+k-1})Y_{i+j-1}Y_{i+k-1} = Y_i^tBY_i = X_i^t(A^{-1})^tBA^{-1}X_i,
\]

where \( Y_{i+j-1} \) denotes the \( j \)-th entry of the vector \( Y_i \), \( 1 \leq j \leq p \), since \( Y_i = (Y_i, \ldots, Y_{i+p-1}) \).

Note that \( B = E(Y_i^t g(Y_i)Y_i^t) = E(A^{-1}X_i^t g(Y_i)(A^{-1}X_i)^t) = A^{-1}E(X_i^t f(X_i)X_i^t)(A^{-1})^t \). As a result,
\[
X_i^t(A^{-1})^tBA^{-1}X_i = X_i^t(AA^{-1})^tE(X_i^t f(X_i)X_i^t)(AA^{-1})^{-1}X_i = X_i^t \sum_p^{-1} E(X_i^t f(X_i)X_i^t) \sum_p^{-1}X_i.
\]

With \( A := (\alpha_{jk})_{1 \leq j, k \leq p} := \sum_p^{-1} E(X_i^t f(X_i)X_i^t) \sum_p^{-1} \) it follows that
\[
X_i^t(A^{-1})^tBA^{-1}X_i = \sum_{1 \leq j, k \leq p} X_{i+j-1}X_{i+k-1} \alpha_{jk}.
\]

All in all, we arrive at
\[
\sum_{i=1}^{n} \sum_{1 \leq j, k \leq p} E(g(Y_i)Y_{i+j-1}Y_{i+k-1})Y_{i+j-1}Y_{i+k-1} = \sum_{i=1}^{n} \sum_{1 \leq j, k \leq p} X_{i+j-1}X_{i+k-1} \alpha_{jk}
\]
\[
= \sum_{1 \leq j, k \leq p} \sum_{i=1}^{n} X_{i+j-1}X_{i+k-1} \alpha_{jk}.
\]

Note that
\[
\sum_{1 \leq j, k \leq p} \sum_{i=1}^{n} X_{i+j-1}X_{i+k-1} \alpha_{jk} = \sum_{1 \leq j, k \leq p} \sum_{i=1}^{n+j-1} X_iX_{i+j-k} \alpha_{jk}
\]
\[
= \sum_{k=2}^{p} \sum_{i=1}^{n-k+1} \sum_{i=k-l}^{n+l-1} X_iX_{i+l} + \sum_{k=1}^{p} \sum_{i=k-1}^{n-k-l} \sum_{i=k-l}^{n-l} X_iX_{i+l}.
\]

Define the sample covariance at lag \( l \) by
\[
\hat{r}_n(l) := \frac{1}{n} \sum_{i=0}^{n-l} X_iX_{i+l}.
\]

Considering both summands separately, we arrive at
\[
\sum_{l=1}^{k-1} \alpha_{k-l} \sum_{i=l}^{n-k-l} X_iX_{i+l} = \sum_{l=1}^{k-1} \alpha_{k-l} \left( \frac{1}{n} \sum_{i=n-l+1}^{n} X_iX_{i+l} + \hat{r}_n(l) - \frac{1}{n} \sum_{i=0}^{n} X_iX_{i+l} \right),
\]
and
\[
\sum_{l=k-p}^{0} \sum_{i=k-l}^{n+k-1} X_i X_{i+l} = \sum_{l=k-p}^{0} \sum_{i=k-l}^{n+k-1} X_i X_{i-l}
\]
\[
= \sum_{l=0}^{p-k} \sum_{i=k}^{n+k-1} X_{i+l} X_i
\]
\[
= \sum_{l=0}^{p-k} \sum_{i=k}^{n+k-1} \hat{r}_n(l) - \frac{1}{n} \sum_{i=0}^{k-1} X_{i+l} X_i
\]

All in all, it follows that
\[
n^{D-1}(2!)^{-\frac{1}{2}} L^{-1}(n) \sum_{i=1}^{n} \left( \sum_{1 \leq j, k \leq p} E(g(Y_j) Y_{i+j-1} Y_{i+k-1}) Y_{i+j-1} Y_{i+k-1} - p \sum_{j=1}^{p} E(g(Y_j) Y_{i+j-1}^2) \right)
\]
\[
= n^{D-1}(2!)^{-\frac{1}{2}} L^{-1}(n) \sum_{1 \leq j, k \leq p} \alpha_{jk} \sum_{i=1}^{n} (X_{i+j-1} X_{i+k-1} - E(X_{i+j-1} X_{i+k-1}))
\]
\[
= n^D (2!)^{-\frac{1}{2}} L^{-1}(n) \sum_{k=2}^{p} \sum_{l=1}^{k-1} \alpha_{k-l,k} (\hat{r}_n(l) - r(l))
\]
\[
+ n^D (2!)^{-\frac{1}{2}} L^{-1}(n) \sum_{k=1}^{p} \sum_{l=0}^{p-k} \alpha_{k+l,k} (\hat{r}_n(l) - r(l)) + o_p(1).
\]

If \( D \in (0, \frac{1}{2}) \), it follows by section 4.4.1.3 in Beran et al. (2013) that
\[
n^D (2!C_2)^{-\frac{1}{2}} L^{-1}(n) (\hat{r}_n(0) - r(0), \ldots, \hat{r}_n(p) - r(p)) \overset{D}{\to} (Z_{2,H}(1), \ldots, Z_{2,H}(1)),
\]
where \( Z_{2,H}(\cdot) \) is a Rosenblatt process with parameter \( H = 1 - \frac{D}{2} \) and \( C_2 = ((1 - 2D)(2 - D))^{-1} \).

Therefore, the considered expression converges in distribution to
\[
Z_{2,H}(1) \left( \sum_{k=2}^{p} \sum_{l=1}^{k-1} \alpha_{k-l,k} + \sum_{k=1}^{p} \sum_{l=0}^{p-k} \alpha_{k+l,k} \right) = Z_{2,H}(1) \sum_{k=1}^{p} \sum_{l=1}^{p} \alpha_{l,k}.
\]

Hence, we are able to characterize the limit distribution of the partial sums in (11) for functions \( f \) with Hermite rank 1 and 2.

6 | SIMULATION STUDY

We simulate \( N = 10,000 \) paths of fractional Gaussian noise (by the command “simFGN0” from the RPackage “longmemo”) with sample size \( n = 1,000,000 \) for different values of \( H \) to compare the distribution of the estimators \( \hat{q}_n(x) \), \( \hat{p}_n(x) \), and \( \hat{H}_n \) with the theoretical results derived above. We standardized the estimators following the normalization constants given in Theorems 2 and 3. The results depending on the long-range dependent parameter \( H \) are shown in Figures 3 and 4.
FIGURE 3  Histogram, kernel density estimation and qqplot of the estimators $\hat{q}_n(\pi)$ (blue) and $\hat{p}_n(\pi)$ (red) for $n = 1,000,000$ and $\pi = (2, 1, 0)$ in the case $H = 0.8$ ($D = 0.4$)

FIGURE 4  Histogram, kernel density estimation and qqplot of the estimators $\hat{q}_n(\pi)$ (blue) and $\hat{p}_n(\pi)$ (red) for $n = 1,000,000$ and $\pi = (2, 1, 0)$ in the case $H = 0.9$ ($D = 0.2$)
In Figure 5, the histograms and kernel density estimations of the estimator of the Hurst parameter are given, standardized by the normalizing constants we derived in Theorem 4.

ACKNOWLEDGEMENTS
We would like to thank two anonymous referees for their helpful comments. This research was supported in part by the German Research Foundation (DFG) through Collaborative Research Center SFB 823 Statistical Modelling of Nonlinear Dynamic Processes, Research Training Group RTG 2131 High-dimensional Phenomena in Probability - Fluctuations and Discontinuity and the project Ordinal-Pattern-Dependence: Grenzwertsätze und Strukturbrüche im langzeitabhängigen Fall mit Anwendungen in Hydrologie, Medizin und Finanzmathematik (SCHN 1231/3-1).

ORCID
Alexander Schnurr https://orcid.org/0000-0002-3749-3991

REFERENCES
Arcones, M. A. (1994). Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. The Annals of Probability, 22(4), 2242–2274.
Bandt, C. (2005). Ordinal time series analysis. Ecological Modelling, 182(3-4), 229–238.
Bandt, C., & Pompe, B. (2002). Permutation entropy: a natural complexity measure for time series. Physical Review Letters, 88(17), 174102–1–174102–4.
Bandt, C., & Shiha, F. (2007). Order patterns in time series. Journal of Time Series Analysis, 28(5), 646–665.
Beran, J., Feng, Y., Ghosh, S., & Kulik, R. (2013). Long-memory processes. Berlin Heidelberg, Germany: Springer-Verlag.
Bingham, N. H., Goldie, C. M., & Teugels, J. L. (1987). Regular variation. Cambridge, MA: Cambridge University Press.
Coeurjolly, J.-F. (2001). Estimating the parameters of a fractional brownian motion by discrete variations of its sample paths. Statistical Inference for Stochastic Processes, 4(2), 199–227.
Delgado, M. A., & Robinson, P. M. (1996). Optimal spectral bandwidth for long memory. Statistica Sinica, 6, 97–112.
Fischer, S., Schumann, A., & Schnurr, A. (2017). Ordinal pattern dependence between hydrological time series. Journal of Hydrology, 548, 536–551.
Geweke, J., & Porter-Hudak, S. (1983). The estimation and application of long memory time series models. Journal of Time Series Analysis, 4(4), 221–238.
Henry M., & Robinson PM. 1996 Bandwidth choice in Gaussian semiparametric estimation of long range dependence. Paper presented at: Proceedings of the Athens Conference on Applied Probability and Time Series Analysis (pp. 220–232). Springer.
Henry, M. (2001). Robust automatic bandwidth for long memory. Journal of Time Series Analysis, 22(3), 293–316.
Istas, J., & Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. In *Annales de l’Institut Henri Poincare (B) probability and statistics* (Vol. 33, pp. 407–436).

Keller K. Maksymenko S., & Stolz I. 2015 Entropy determination based on the ordinal structure of a dynamical system. arXiv preprint: 1502.01309.

Keller, K., & Sinn, M. (2005). Ordinal analysis of time series. *Physica A: Statistical Mechanics and its Applications, 356*(1), 114–120.

Kotz, S., Balakrishnan, N., & Johnson, N. L. (2004). *Continuous multivariate distributions, Volume 1: Models and applications*. Vol. 1. Hoboken: John Wiley & Sons.

Künsch H. R. 1987 *Statistical aspects of self-similar processes*. Paper presented at: Proceedings of the 1st World Congress of the Bernoulli Society (vol 1, pp. 67–74). Utrecht, The Netherlands, VNU Science Press.

Major P. 2019 Non-central limit theorem for non-linear functionals of vector valued Gaussian stationary random fields. arXiv preprint: 1901.04086.

Mandelbrot B. B., & Taqqu M. S. 1979 *Robust R/S analysis of long run serial correlation*. Paper presented at: Proceedings of the 42nd Session of the International Statistical Institute (pp. 4–14). Manila.

Mandelbrot, B. B. (1975). Limit theorems on the self-normalized range for weakly and strongly dependent processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 31*(4), 271–285.

Mandelbrot, B. B., & Wallis, J. R. (1969). Computer experiments with fractional Gaussian noises: Part 1, averages and variances. *Water Resources Research, 5*(1), 228–241.

Pipiras, V., & Taqqu, M. S. (2017). *Long-range dependence and self-similarity* (Vol. 45). Cambridge, MA: Cambridge University Press.

Rea W., Oxley L., Reale M., & Brown J. 2009 Estimators for long range dependence: An empirical study. arXiv preprint arXiv:0901.0762.

Robinson, P. M. (1995). Gaussian semiparametric estimation of long range dependence. *The Annals of Statistics, 25*(5), 1630–1661.

Samorodnitsky, G. (2007). *Long range dependence*. Foundations and Trends in Stochastic Systems, 1(3), 163–257.

Schnurr, A. (2014). An ordinal pattern approach to detect and to model leverage effects and dependence structures between financial time series. *Statistical Papers, 55*(4), 919–931.

Schnurr, A., & Dehling, H. (2017). Testing for structural breaks via ordinal pattern dependence. *Journal of the American Statistical Association, 112*(518), 706–720.

Sinn M, Ghodsi A, Keller K. *Detecting change-points in time series by maximum mean discrepancy of ordinal pattern distributions*. Paper presented at: Proceedings of the 28th Conference on Uncertainty in Artificial Intelligence UAI’12; 2012:786–794.

Sinn, M., & Keller, K. (2011). Estimation of ordinal pattern probabilities in Gaussian processes with stationary increments. *Computational Statistics & Data Analysis, 55*(4), 1781–1790.

Taqqu, M. S., Teverovsky, V., & Willinger, W. (1995). Estimators for long-range dependence: an empirical study. *Fractals, 3*(04), 785–798.

Van der Vaart, A. W. (2000). *Asymptotic statistics* (Vol. 3). Cambridge, MA: Cambridge University Press.

**How to cite this article:** Betken A, Buchsteiner J, Dehling H, Münker I, Schnurr A, Woerner JH. Ordinal patterns in long-range dependent time series. *Scand J Statist*. 2021;48:969–1000. https://doi.org/10.1111/sjos.12478

**APPENDIX**

Calculation of the Hermite coefficients of \( \hat{q}_n(\pi) \) for \( h = 2 \) for the pattern \( \pi = (2, 1, 0) \); compare Example 2.
Since we look at $h = 2$, the covariance matrix of $X_1 = (X_1, X_2)^\prime$ is given by

$$\Sigma_2 = \begin{pmatrix} 1 & r(1) \\ r(1) & 1 \end{pmatrix}.$$ 

The Cholesky decomposition $\Sigma = AA^\prime$ has the following form:

$$A = \begin{pmatrix} 1 & 0 \\ r(1) & \sqrt{1 - (r(1))^2} \end{pmatrix}$$

Note that $X_1 = AY_1$, where $Y_1 = (Y_1, Y_2)^\prime$ has a bivariate standard normal distribution. Following Theorem 2, we need to calculate $a = (A^{-1})^\prime b$, where $b = \mathbb{E}(Y_1 1_{\{\bar{\Pi}(X_1, X_2) = \pi\}})$. Since

$$(A^{-1})^\prime = \begin{pmatrix} 1 & -\frac{r(1)}{\sqrt{1 - (r(1))^2}} \\ 0 & \frac{1}{\sqrt{1 - (r(1))^2}} \end{pmatrix},$$

we need to determine $b$ to calculate the variance in the limit distribution.

We consider $\pi = (2, 1, 0)$. From the Cholesky decomposition it follows that $X_1 = Y_1$ and $X_2 = r(1)Y_1 + \sqrt{1 - (r(1))^2}Y_2$ and therefore $c_1 = \mathbb{E}(X_1 1_{\{\bar{\Pi}(X_1, X_2) = \pi\}}) = b_1$ and $c_2 = r(1)b_1 + \sqrt{1 - (r(1))^2}b_2$. For this choice of $\pi$ we also know by (6) and (7) that $c_1 = c_2$ and hence we arrive at

$$b_1 = \frac{\sqrt{1 - (r(1))^2}}{1 - r(1)}b_2.$$ 

Therefore, it is sufficient to only determine $b_2$. For this, we rewrite

$$\{\bar{\Pi}(X_1, X_2) = (2, 1, 0)\} = \{X_1 \geq 0, X_2 \geq 0\} = \{Y_1 \geq 0, r(1)Y_1 + \sqrt{1 - (r(1))^2}Y_2 \geq 0\}$$

$$= \{Y_1 \geq 0, Y_2 \geq -\frac{r(1)}{\sqrt{1 - (r(1))^2}}Y_1\}.$$ 

Hence, we need to determine

$$b_2 = \mathbb{E}(Y_2 1_{\{\bar{\Pi}(X_1, X_2) = \pi\}})$$

$$= \int_0^\infty \int_0^\infty \frac{r(1)}{\sqrt{1 - (r(1))^2}Y_1} y_2 \varphi(y_2) \varphi(y_1) dy_2 dy_1$$

$$= \int_0^\infty \varphi \left( \frac{r(1)}{\sqrt{1 - (r(1))^2}}y_1 \right) \varphi(y_1) dy_1$$

$$= \frac{1}{2\pi} \int_0^\infty \exp \left( -\frac{\left( 1 + \frac{(r(1))^2}{1 - (r(1))^2} \right) y_1^2}{2} \right) dy_1.$$
\[\begin{align*}
&= \frac{1}{2\pi} \int_0^\infty \exp \left( -\frac{\left( \frac{1}{1-(r(1))^2} \right) y_1^2}{2} \right) dy_1 \\
&= \frac{1}{2\sqrt{2\pi}} \sqrt{1 - (r(1))^2}.
\end{align*}\]

Finally, we obtain

\[\begin{align*}
\sum_{j=1}^2 a_j &= b_1 + \frac{1 - r(1)}{\sqrt{1 - (r(1))^2}} b_2 \\
&= \frac{\sqrt{1 - (r(1))^2}}{1 - r(1)} b_2 + \frac{1 - r(1)}{\sqrt{1 - (r(1))^2}} b_2 \\
&= \left( \sqrt{\frac{1 + r(1)}{1 - r(1)}} + \sqrt{\frac{1 - r(1)}{1 + r(1)}} \right) b_2 \\
&= \frac{1}{2\sqrt{2\pi}} \frac{2}{\sqrt{1 - (r(1))^2}} \sqrt{1 - (r(1))^2} \\
&= \frac{1}{\sqrt{2\pi}}.
\end{align*}\]

As a result, we confirm the result from Example 2 for the pattern \(\pi = (2, 1, 0)\). For \(\pi = (2, 0, 1)\), the analytical calculations work analogously.