No fixed-point guarantee of Nash equilibrium in quantum games

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The theory of quantum games permits players to choose strategies that prepare and measure quantum states. Whereas conventional game theory provides guarantees for fixed-point stability in non-cooperative games, so-called Nash equilibria, we find this guarantee is not provided for quantum games. In particular, we show the conditions for Glickberg’s fixed-point theorem do not apply to pure quantum games when the payoff is a physical observable. We further show that Nash equilibrium can be guaranteed when the payoff is defined with respect to state preparation.

PACS numbers: 42.50.Ar,03.67.Bg,42.79.Sz

Any quantum physical process modeled as a non-cooperative game describes a quantum game of the same type. The first instance of non-cooperative game-theoretic modeling of quantum physical processes appears to be the 1980 work of A. Blaquiere, where wave mechanics are considered as a two player, zero-sum (strictly competitive) differential game and a minimax result is established for certain quantum physical aspects. The more recent and more sustained game theoretic treatment of quantum physical processes was initiated in 1999 with the work of Meyer. Meyer’s work only considered quantum computational and quantum algorithmic aspects of quantum physics as games, making the game theoretic analysis of these quantum systems more straightforward using finite-dimensional linear algebra instead of differential equations.

The year 1999 also saw the publication of the paper by Eisert et al. in which a quantum informational model for the informational component of two players games was considered. This consideration was in the same spirit as the consideration of stochastic or randomization model for information in a game which produces the so-called mixed game. The quantum informational model of Eisert et al. produces a quantized game. The inspiration for considering extensions of the informational aspect of games to larger domains comes from John Nash’s famous theorem which not only innovates the solution concept of non-cooperative games as an equilibrium problem but, for the right kind of extension of the informational component of finite non-cooperative games, also guarantees its existence. The relative simplicity of the proof of Nash’s theorem relies entirely on Kakutani’s fixed-point theorem.

As for the motivation behind Meyer’s gaming the quantum approach, it lies in the fact that game theory has been fruitfully applied to several disciplines with the goal of constrained optimization of aspects of these disciplines, with economics and evolutionary biology being those that can be argued to have benefitted the most. Along similar lines, Meyer’s goal was to gain insights into the constrained optimal behavior of quantum algorithms, such as when some components of the algorithms were constrained to be classical.

It is important to note that even though both quantized games and gaming the quantum share a common game theoretic scaffolding, the former is game theoretically meaningful while the latter holds quantum physical meaning.

An $n$ player, non-cooperative game in normal form is a function $\Gamma$

$$\Gamma : \prod_{i=1}^{n} S_i \rightarrow O$$

with the additional feature of the notion of non-identical preferences over the elements of $O$ for every “player” of the game. The factor $S_i$ in the domain of $\Gamma$ is said to be the strategy set of player $i$, and a play of $\Gamma$ is a $n$-tuple of strategies, one per player, producing a payoff to each player in terms of his preferences over the elements of $O$ in the image of $\Gamma$. These preferences are typically quantified numerically for the ease of calculation of the payoffs. In this numerical context, $\Gamma$ can be considered to be
composed of component functions $\Gamma_i$ which act as the payoff functions for each player $i$.

A Nash equilibrium is a play of $\Gamma$ in which every player employs a strategy that is a best reply, with respect to his preferences over the outcomes, to the strategic choice of every other player. In other words, unilateral deviation from a Nash equilibrium by any one player in the form of a different choice of strategy will produce an outcome which is less preferred by that player than before. Following Nash, we say that a play $p'$ of $\Gamma$ counters another play $p$ if $\Gamma_i(p') > \Gamma_i(p)$ for all players $i$, and that a self-countering play is an (Nash) equilibrium.

Let $C_p$ denote the set of all the plays of $\Gamma$ that counter $p$. Denote $\prod_{i=1}^{n} S_i$ by $S$ for notational convenience, and note that $C_p \subset S$ and therefore $C_p \in 2^S$. Further note that the game $\Gamma$ can be factored as

$$\Gamma : S \stackrel{\Gamma_C}{\rightarrow} 2^S \stackrel{E}{\rightarrow} O$$  \hspace{1cm} (2)

where to any play $p$ the map $\Gamma_C$ associates its countering set $C_p$ via the payoff functions $\Gamma_i$. The set-valued map $\Gamma_C$ may be viewed as a pre-processing stage where players seek out a self-countering play, and if one is found, it is mapped to its corresponding outcome in $O$ by the function $E$. The condition for the existence of a self-countering play, and therefore of a Nash equilibrium, is that $\Gamma_C$ have a fixed point, that is, an element $p^* \in S$ such that $p^* \in \Gamma_C(p^*) = C_{p^*}$.

In a general set-theoretic setting for non-cooperative games, the map $\Gamma_C$ may not have a fixed point. Hence, not all non-cooperative games will have a Nash equilibrium. However, according to Nash's theorem, when the $S_i$ are finite and the game is extended to its mixed version, that is, the version in which randomization via probability distributions is allowed over the elements of all the $S_i$, as well as over the elements of $O$, then $\Gamma_C$ has at least one fixed point and therefore at least one Nash equilibrium.

Formally, given a game $\Gamma$ with finite $S_i$ for all $i$, its mixed version is the product function

$$\Lambda : \prod_{i=1}^{n} \Delta(S_i) \longrightarrow \Delta(O)$$  \hspace{1cm} (3)

where $\Delta(S_i)$ is the set of probability distributions over the $i^{th}$ player's strategy set $S_i$, and the set $\Delta(O)$ is the set of probability distributions over the outcomes $O$. Payoffs are now calculated as expected payoffs, that is, weighted averages of the values of $\Gamma_i$, for each player $i$, with respect to probability distributions in $\Delta(O)$ that arise as the product of the plays of $\Lambda$. Denote the expected payoff to player $i$ by the function $\Lambda_i$. Also, note that $\Lambda$ restricts to $\Gamma$. In such $n$-player games, at least one Nash equilibrium play is guaranteed to exist as a fixed point of $\Lambda$ via Kakutani's fixed-point theorem.

Kakutani’s fixed-point theorem: Let $S \subset \mathbb{R}^n$ be nonempty, bounded, closed, and convex, and let $F : S \rightarrow 2^S$ be an upper semi-continuous set-valued mapping such that $F(s)$ is non-empty, closed, and convex for all $s \in S$. Then there exists some $s^* \in S$ such that $s^* \in F(s^*)$.

To see this, make $S = \prod_{i=1}^{n} \Delta(S_i)$. Then $S \subset \mathbb{R}^n$ and $S$ is non-empty, bounded, and closed because it is a finite product of finite non-empty sets. The set $S$ is also convex because its the convex hull of the elements of a finite set. Next, let $C_p$ be the set of all plays of $\Lambda$ that counter the play $p$. Then $C_p$ is non-empty, closed, and convex. Further, $C_p \subset S$ and therefore $C_p \in 2^S$. Since $\Lambda$ is a game, it factors according to

$$\Lambda : S \stackrel{\Lambda_C}{\rightarrow} 2^S \stackrel{E_\Pi}{\rightarrow} \Delta(O)$$  \hspace{1cm} (4)

where the map $\Lambda_C$ associates a play to its countering set via the payoff functions $\Lambda_i$. Since $\Lambda_i$ are all continuous, $\Lambda_C$ is continuous. Further, $\Lambda_C(s)$ is non-empty, closed, and convex for all $s \in S$ (we will establish the convexity of $\Lambda_C(s)$ below; the remaining conditions are also straightforward to establish). Hence, Kakutani’s theorem applies and there exists an $s^* \in S$ that counters itself, that is, $s^* \in \Lambda_C(s^*)$, and is therefore a Nash equilibrium. The function $E_\Pi$ simply maps $s^*$ to $\Delta(O)$ as the product probability distribution from which the Nash equilibrium expected payoff is computed for each player.

The convexity of the $\Lambda_C(s) = C_p$ is straightforward to show. Let $q, r \in C_p$. Then

$$\Lambda_i(q) \geq \Lambda_i(p) \quad \text{and} \quad \Lambda_i(r) \geq \Lambda_i(p)$$  \hspace{1cm} (5)

for all $i$. Now let $0 \leq \mu \leq 1$ and consider the convex combination $\mu q + (1 - \mu)r$ which we will show to be in $C_p$. First note that $\mu q + (1 - \mu)r \in S$ because $S$ is the product of the convex sets $\Delta(S_i)$. Next, since the $\Lambda_i$ are all linear, and because of the inequalities
in (6) and the restrictions on the values of \( \mu \),
\[
\Lambda_i(\mu q + (1 - \mu) r) = \mu \Lambda_i(q) + (1 - \mu) \Lambda_i(r) \geq \Lambda_i(p)
\]
whereby \( \mu q + (1 - \mu) r \in C_p \) and \( C_p \) is convex.

Going back to the game \( \Gamma \) in (1) defined in the general set-theoretic setting, certainly Kakutani’s theorem would apply to \( \Gamma \) if the conditions are right, such as when the image set of \( \Gamma \) is pre-ordered and \( \Gamma \) is linear.

Kakutani’s fixed-point theorem can be generalized to include subsets \( S \) of convex topological vector spaces, as was done by Glicksberg in [6]. The following is a paraphrased but equivalent statement of Glicksberg’s fixed-point theorem (the term “linear space” in the original statement of Glicksberg’s theorem is equivalent to the term vector space):

**Glicksberg’s fixed-point theorem:** Let \( H \) be nonempty, compact, convex subset of a convex Hausdorff topological vector space and let \( \Phi : H \to 2^H \) be an upper semi-continuous set-valued mapping such that \( \Phi(h) \) is non-empty and convex for all \( h \in H \). Then there exists some \( h^* \in H \) such that \( h^* \in \Phi(h^*) \).

Using Glicksberg’s fixed-point theorem, one can show that Nash equilibrium exists in games where the strategy sets are infinite or possibly even uncountably infinite. In the next section, we contextualize the guarantee of Nash equilibrium in quantum games via Glicksberg’s theorem.

**QUANTUM GAMES AND NASH EQUILIBRIUM**

An \( n \)-player quantum game in normal form arises from (1) when one introduces quantum physically relevant restrictions. We declare a quantum game to be any quantum physically meaningful function
\[
Q : \prod_{i=1}^{n} \mathcal{H}_i \to \mathcal{H}
\]
where \( \mathcal{H}_i \) is a complex Hilbert space acting as the set of strategies of player \( i \), and \( \mathcal{H} \) is the complex Hilbert space of outcomes. By analogy with mixed game extensions, or more generally, stochastic games where players’ strategies are probability distributions over the elements of some set, the strategies of each player in a quantum game consist of quantum superpositions over the elements of a set of observables in \( \mathcal{H}_i \). These strategic choices are then mapped by \( Q \) into elements of \( \mathcal{H} \) over which the players have non-identical preferences defined.

Set \( H = \prod_{i=1}^{n} \mathcal{H}_i \) and \( C_h \subset 2^H \) as the set of all countering plays of a play \( h \) of \( H \). Then
\[
Q : H \xrightarrow{Q_C} 2^H \xrightarrow{E_Q} \mathcal{H}
\]
with the map \( Q_C \) taking a play to its countering set via payoff functions \( Q_i \) for each \( i \), and the function \( E_Q \) takes a self-countering play, if it exists, to the Nash equilibrium quantum superposition in \( \mathcal{H} \). In the search for a guarantee of Nash equilibrium in a quantum game, Kakutani’s theorem cannot be invoked as \( H \subset \mathbb{R}^n \) for any \( n \). Hence, one looks to the more general Glicksberg’s theorem. To this end, first note that \( H \subset \bigotimes_{i=1}^{n} \mathcal{H}_i \) with the latter being a Hausdorff topological vector space. All other requirements of Glicksberg’s theorem are satisfied in a quantum game set up except for convexity of the countering set \( Q_c(h) = C_h \) (for any \( h \)). The convexity of the countering sets may arise from the properties of the payoff functions \( Q_i \). In particular, if these functions are linear or even semi-linear and if their images can be pre-ordered, then convexity of the countering sets follows immediately from (5) and (6) with the \( \Lambda_i \) replaced with \( Q_i \). But this may not be the case in general.

As a first example, we consider a quantum game in which the set of preferred outcomes lie in the Hilbert space of the underlying quantum system. In particular, consider the payoff to be defined by the overlap of the prepared quantum state relative to a preferred quantum state, or in other words, the inner-product of the two states. For a pure quantum game, this payoff function is
\[
Q_i = \langle \psi_i, \phi \rangle
\]
with \( \psi_i \in \mathcal{H} \) the preferred state of the \( i \)-th player and \( \phi \in \mathcal{H} \) the state prepared by \( Q \). For a mixed quantum game, a similar payoff can be defined with respect to the trace operator. In both the pure and mixed settings, the payoff function is linear in the prepared state. The fact that the complex numbers can be pre-ordered allows one to invoke (5) and (6) to establish the convexity of \( C_h \). By Glicksberg’s theorem, we conclude that this game has fixed point guarantee of Nash equilibrium.
In contrast, we also consider the case of a quantum game in which the payoff is defined with respect to a physical observable. In quantum theory, a physical observable is represented by a linear Hermitian operator whose eigenstates define possible outcomes. We may define the expectation value of such an operator with respect to a prepared quantum state as the corresponding payoff. For example, consider the quantum game \( Q \) of these observable states or the prepared states in superposition and the payoff function \( Q \). In such finite quantum games, it is straightforward to define non-identical preferences of the players over the finite number of observable states \( b_j \) of \( H \), and then induce preferences over arbitrary quantum superpositions of these observable states or the prepared states in \( H \). The quantum game \( Q \) maps a play to a quantum superposition \( q = \sum_{j=1}^{n} \alpha_j b_j \in H \) with \( \sum |\alpha_j|^2 = 1 \) and the payoff function \( Q_i \) calculates the expected value of \( q \) to player \( i \) via

\[
Q_i(q) = \sum_{j=1}^{n} e_j |\alpha_j|^2
\]

where the \( e_j \) are real numbers that numerically reflect the preferences of player \( i \) over observables states \( b_j \) of \( H \) and \( |\alpha_j|^2 \) is the probability with which \( q \) measures as \( b_j \). The payoff function \( Q_i \) in this case is not linear in general since for \( 0 \leq \mu \leq 1 \) and another quantum superposition \( p = \sum_{j=1}^{n} \beta_j b_j \)

\[
Q_i(\mu q + (1 - \mu)p) = \sum_{j=1}^{n} a_j |\mu \alpha_j + (1 - \mu)\beta_j|^2
\]

whereas

\[
Q_i(\mu q) + Q_i((1-\mu)p) = \sum_{j=1}^{n} a_j (|\mu \alpha_j|^2 + |(1 - \mu)\beta_j|^2).
\]

Linearity of \( Q_i \) occurs when the expressions in equations (11) and (12) are equal. This equality is assured when

\[
\mu \text{Re}(\alpha_j)\text{Re}(\beta_j)(1 - \mu) + \text{Im}(\alpha_j)\text{Im}(\beta_j)(1 + \mu) = 0
\]

for all \( j \). Hence, the convexity of \( C_h \) does not follow in general and neither does a fixed point guarantee for the existence of Nash equilibrium.

However, in other settings for quantum games, it may happen that the \( C_h \) are convex nonetheless and that a Nash equilibrium exists in \( Q \). For example, the subset of generalized quantum measurements on finite dimensional systems known as local operations and classical communication (LOCC) is compact and convex. Because LOCC holds special significance in many quantum information processes as the natural class of operations, constructing a non-cooperative finite quantum game models for it would be a worthwhile effort given its compact and convex structure.

For mixed extensions of finite quantum games and stochastic finite quantum games in general, the payoff functions are defined in terms of probability distributions over pure quantum states and are therefore linear. Hence, the counting sets will be convex in this case and Glicksberg fixed point theorem will apply, guaranteeing at least one Nash equilibrium in the mixed finite quantum game. Lee et al. study certain types of mixed extensions of finite quantum games they refer to as "static quantum games which have the guarantee of a Nash equilibrium via Kakutani’s theorem rather than Glicksberg’s theorem.

There exist extensions of \( \Gamma \) beyond the mixed one known as “mediated communication”. Mediated communication corresponds to the situation where a probability distribution in \( \Delta(\mathcal{O}) \) is desired so as to increase the value of the payoff functions, but which is not available in the image of the mixed extension. This is achieved as follows. A neutral referee starts with a non-product probability distribution \( P \) over the outcomes of a finite non-cooperative game \( \Gamma \) and advises each player to engage in a play \( \Pi \) of \( \Gamma \) with respect to \( P \). If each player agrees with the referee’s advice after having evaluated his expected payoff with respect to \( P \), then the play \( \Pi \) is said to be a correlated equilibrium and the larger game with mediation is called a correlated game. A more formal mathematical discussion of correlated games appears in [4]. Quantum entanglement maybe considered as a form of mediated quantum communication and an extension of a quantum game’s image in \( \mathcal{H} \) so as to include non-factorisable quantum superpositions. Quantum entanglement does not appear to offer any immediate insights into the existence of fixed points in correlated quantum games, such as the linearity of the payoff functions.

What does quantum game theory offer quantum
information processing? For quantum processes modeled as appropriate quantum games, such as the state preparation quantum game discussed here, Glicksberg’s theorem guarantees a Nash equilibrium outcome which can be viewed as an optimal solution to an optimization problem with constraints defined over the outcomes of the process. Another example would be quantum communication or stochastic quantum processes, cast as stochastic quantum games where a coalition of players (Alice and Bob) engage in a non-cooperative way with the eavesdropper (Eve). Alice and Bob want to amplify privacy of the communication whereas Eve does not, and in fact may want to decrease it. Or, Alice and Bob may wish to reconcile the information in such protocols whereas Eve would not want to. If the Alice and Bob coalition and Eve try to achieve their respective outcomes via random quantum processes, then Glicksberg’s theorem will guarantee an optimal solution, with respect to the preference constraints, in the form of a Nash equilibrium. With this guarantee in place, mechanism design methods can be adopted to find this equilibrium.

In a quantum computational setting, quantum search algorithms or state amplification algorithms can be viewed as zero-sum games between a player and Nature and the payoffs defined with respect to an observable as in (10). The item being searched and Nature and the payoffs defined with respect to can be viewed as zero-sum games between a player search algorithms or state amplification algorithms in place, mechanism design methods can be adopted to find this equilibrium. Such quantum games, cast as stochastic quantum games where a coalition of players (Alice and Bob) engage in a non-cooperative way with the eavesdropper (Eve). Alice and Bob want to amplify privacy of the communication whereas Eve does not, and in fact may want to decrease it. Or, Alice and Bob may wish to reconcile the information in such protocols whereas Eve would not want to. If the Alice and Bob coalition and Eve try to achieve their respective outcomes via random quantum processes, then Glicksberg’s theorem will guarantee an optimal solution, with respect to the preference constraints, in the form of a Nash equilibrium. With this guarantee in place, mechanism design methods can be adopted to find this equilibrium.

Adiabatic quantum computing, and the many NP problems that it has the potential to solve, can potentially benefit from the quantum game model. In a nutshell, an adiabatic quantum computation starts with a system of $n$ qubits in its lowest energy state. A Hamiltonian $H_f$ is constructed that corresponds to this lowest energy state, and another Hamiltonian $H_i$ is used to encode an objective function the solution of which is the minimum energy state of $H_f$. Finally, the actual adiabatic computation occurs as the interpolating Hamiltonian

$$H(s(t)) = s(t)H_f + (1 - s(t))H_i \quad (14)$$

which is expected to adiabatically transform the lowest energy state of $H_f$ to that of $H_f$ as a function of the interpolating path $s(t)$ with respect to time $t$. For large enough $t$ values, adiabaticity holds; on the other hand, $t$ should be much smaller than its corresponding value in classical computational processes for $H(s(t))$ to constitute a worthwhile effort. In the current quantum game theoretic context, note that

$$H(s(t)) : \prod_{i=1}^{n} H_i \rightarrow \otimes_{i=1}^{n} H_i \quad (15)$$

can be viewed as a zero-sum quantum game, where a notional player I most prefers the element of $H$ which corresponds to the lowest energy state of $H_f$. Player II or Nature, prefers anything but that element. Players I and II can be given access to any division of qubits to manipulate respectively via “quantum strategies”. A Nash equilibrium play would be one which is optimal under the constraint of players’ non-identical preferences over the outcomes, and therefore correspond to optimal values of the probability of success of the computation versus the time it takes to perform it. As discussed in this letter, the existence will depend on the nature of $H(s(t))$, but will be guaranteed if randomization is introduced in the quantum game.

Acknowledgments Faisal Shah Khan is indebted to Davide La Torre from the department of Mathematics at Nazarbayev University for helpful discussion on the topic of fixed-point theorems. This manuscript has been authored by UT-Battelle, LLC under Contract No. DE-AC05-00OR22725 with the U.S. Department of Energy. The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes. The Department of Energy will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (http://energy.gov/downloads/doe-public-access-plan).
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