Noncommutative Brownian motion

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We investigate the classical Brownian motion of a particle in a two-dimensional noncommutative (NC) space. Using the standard NC algebra embodied by the sympletic Weyl-Moyal formalism we find that noncommutativity induces a non-vanishing correlation between both coordinates at different times. The effect stands out as a signature of spatial noncommutativity and thus could offer a way to experimentally detect the phenomena. We further discuss some limiting scenarios and the trade-off between the scale imposed by the NC structure and the parameters of the Brownian motion itself.

Keywords: Noncommutative geometry; classical mechanics; Brownian motion.

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1. Introduction

How does the space-time structure look like as we gradually shift towards smaller scales, say, the Planck scale? Is there some sort of minimum length? Those are long-standing fundamental questions in physics and has been the core of theories that attempt to join gravity and quantum mechanics. In particular, there has been a growing interest in investigating the role of noncommutative (NC) geometries in nature, namely when spatial coordinates do not commute. This statement may sound a bit striking as such property would limit our knowledge about the exact lo-
cution of a given particle in the space-time manifold, analogously to the Heisenberg’s uncertainty principle which imposes a fundamental limit on the measurement precision of position and momentum variables. Thereby, when assuming a NC algebraic framework, one adds a minimum-length constraint into the problem.

The assumption that space-time is not continuous but, instead, a quantized object goes back from Snyder’s seminal paper where it was argued that a NC geometry allows for regularized quantum field theories. The overall idea was placed back again into the spotlight by strong arguments from the string theory side a while ago. In this way, spatial noncommutativity indeed seems to play an essential role at the Planck’s length scale, where quantum effects of gravity might not be negligible.

Although noncommutativity embodies a puzzle piece in the high-energy scenario, great interest has also been addressed to its implications on condensed-matter physics. In particular, the NC version of quantum mechanics has been extensively explored over the past few years. Traces of noncommutativity have been studied, for instance, in the hydrogen atom, quantum Hall effect, Aharonov-Bohm effect, graphene, and even in quantum information theory. Overall, the main motivation turns out to be searching for observable signatures of NC effects in more accessible platforms, as well as setting experimental bounds on the NC scale itself. Interestingly, despite many proposals, there is still no actual experimental evidence that holds the assumption of a NC structure neither there is a way to prove it does not exist at all. Here, we go along this direction and suggest another platform which can possibly enable its verification.

In particular, we take the classical limit of NC quantum mechanics and deal with a well-known model in statistical physics featuring a single particle going through random displacements in a two-dimensional manifold, namely the Brownian motion. This model poses a fundamental importance in statistical physics by describing the macroscopic picture of the particle due to microscopic effects. This phenomena thus embodies how small-scale physics can have a major influence at larger scales and then it becomes natural to ask whether spatial noncommutativity plays any significant role on it. Therefore, our aim is to place the two-dimensional Brownian motion as a conceivable testbed to detect signatures of spatial noncommutativity. Although dealing with the tiny scale where NC effects might emerge is physically demanding, the effect we describe here depends on many properties of the Brownian motion itself which, in principle, could be manipulated so as to overcome this issue.

It is worth mentioning that noncommutativity has also been explored in the classical domain. In order to do so, one can assume that the underlying algebra has a sympletic structure compatible with that of NC quantum mechanics (Dirac’s correspondence principle). For instance, a NC version of Newton’s second law of motion was derived in.

In the following, we set the theoretical ground for investigating the Brownian motion in a two-dimensional NC manifold. This is done by using the framework of NC algebra in the classical limit. In particular, we solve the Langevin equation
and find that noncommutativity induces a time-dependent variance of the correlation between spatial coordinates. For larger measurement-time differences, this quantity saturates to about a constant value that depends on the Brownian motion parameters such as particle’s size and density, temperature, and fluid viscosity. The key point here is that the very fact of having such a non-zero correlation implies in the existence of spatial noncommutativity.

2. NC formalism

In NC quantum mechanics, the position \( \hat{x}_i \) and momentum \( \hat{p}_i \) operators obey the following commutation rules

\[
[\hat{x}_i, \hat{x}_j] = i\hbar \Theta_{ij}, \\
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \\
[\hat{p}_i, \hat{p}_j] = 0,
\]

with \( i = 1, 2 \), where \( \hbar \Theta_{ij} \) is a antisymmetric matrix with dimension of area and denotes the NC parameter. Several studies have established bounds on the scale of \( \Theta \) based on experimental data [11, 15, 20, 28, 37, 38] for instance, measurements of the Lamb shift of the hydrogen atom gives \( \Theta \lesssim (6 \text{ GeV})^{-2} \).

The mathematical framework for dealing with a NC space is implemented via the Weyl-Moyal correspondence, where any arbitrary function of the position operators \( f(\hat{x}) \) is associated with a Weyl symbol \( f(x) \) defined in the commutative scenario. Hence, the usual product of two given functions \( f(\hat{x})g(\hat{x}) \) is replaced by the so-called Weyl star product \( f(x) \star g(x) \) satisfying

\[
(f \star g)(x) = \exp \left( \frac{i}{\hbar} \Theta_{ij} \partial_i \partial_j \right) f(x)g(y)|_{x=y},
\]

where \( f \) and \( g \) are infinitely differentiable functions.

In the classical limit the commutators must be replaced with Poisson brackets via the correspondence principle

\[
[\hat{A}, \hat{B}] \rightarrow i\hbar \{A, B\},
\]

where \( A \) and \( B \) are two arbitrary functions. Thereby, the relations in Eq. (1) are rewritten as

\[
\{x_i, x_j\} = \Theta_{ij}, \\
\{x_i, p_j\} = \delta_{ij}, \\
\{p_i, p_j\} = 0.
\]

Note that, in the classical limit, \( \Theta \) must have dimension of \([\text{time/mass}]\).

The general form of the Poisson brackets on the NC space are simply worked out as

\[
\{A, B\} = \left( \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_i} \right) + \Theta_{ij} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j}.
\]
Now, consider the Hamiltonian
\[
H = \frac{p_1^2 + p_2^2}{2m} + V(x_1, x_2)
\]  
(6)
describing a particle of mass \(m\) in two dimensions subjected to an external potential \(V\). The equations of motion in the NC space are then given by\[1\]
\[
\dot{x}_i = \{x_i, H\} = \frac{p_i}{m} + \Theta_{ij} \frac{\partial V}{\partial x_j},
\]
(7)
\[
\dot{p}_i = \{p_i, H\} = -\frac{\partial V}{\partial x_i},
\]
(8)
and thus
\[
m\ddot{x}_i = -\frac{\partial V}{\partial x_i} + m\Theta_{ij} \frac{\partial^2 V}{\partial x_j \partial x_k} \dot{x}_k,
\]
(9)
which represents Newton’s second law in NC space.\[2\] Note that the spatial noncommutativity induces another force denoted by the last term of the above equation. This correction can be seen as an effective potential defined on the NC background.

3. Brownian motion

Let us first introduce the Brownian motion in its standard commutative version, usually described by the Langevin formalism.\[3\] One can actually find many theoretical frameworks to deal with it (see\[4\] for a review). The Langevin equation, however, stands out as a simple and straightforward stochastic model which takes into account most the relevant physics associated to the phenomena. Let us consider a particle going through random displacements due to collisions with (much smaller) molecules of a fluid and subjected to a viscous resistance force. The Langevin equation accurately describes the macroscopic dynamics of the Brownian particle in a much longer time scale compared with the collision time and is written as
\[
\frac{d\vec{v}(t)}{dt} = -\gamma \frac{m}{m} \vec{v}(t) + \vec{\xi}(t)
\]
(10)
where \(m\) is the particle’s mass, \(\gamma \vec{v}(t)\) denotes the viscous force with coefficient \(\gamma\), and \(\vec{\xi}(t)\) is the noise term forces arising from collisions in the fluid. The latter satisfies the time-correlation functions
\[
\langle \xi_i(t) \rangle = 0,
\]
\[
\langle \xi_i(t) \xi_j(t') \rangle = g \delta_{ij} \delta(t - t'),
\]
(11)
where \(g\) defines the force strength, \(\delta_{ij}\) and \(\delta(t - t')\) are, respectively, the Kronecker and Dirac delta functions, and \(\langle ... \rangle\) denotes the average with respect to realizations of the random forces. The correlations above define a second-order stochastic process in which the random forces are given by a Gaussian distribution. Those are completely uncorrelated at different times, thus yielding a Markovian (memoryless) source of noise.
Now, let us move on to the NC algebra set in Eq. (4). Taking $\xi_j = -\partial V/\partial x_j$ as an external random force, from Eq. (7) we get

$$\frac{dx_i}{dt} = v_i(t) - \Theta_{ij} \xi_j(t).$$  \hspace{1cm} (12)

Integrating Eq. (10), we note that the solution for the particle’s velocity remains the same as in the usual commutative framework,

$$v_i(t) = v_{0i}e^{-\frac{2\mu}{m\gamma} t} + \frac{1}{m} \int_0^t ds \ e^{-\frac{2\mu}{m\gamma} (t-s)} \xi_i(s),$$  \hspace{1cm} (13)

where alongside Eq. (11), the expectation value of the quadratic velocity can be obtained:

$$\langle v_i^2(t) \rangle = \frac{g}{2m\gamma} + \left( v_{0i}^2 - \frac{g}{m\gamma} \right) e^{-\frac{2\mu}{m\gamma} t}. \hspace{1cm} (14)$$

In the long-time regime ($t \gg 1$), the above equation reduces to

$$\langle v_i^2(t) \rangle = \frac{g}{2m\gamma}, \hspace{1cm} (15)$$

thus yielding the so-called fluctuation-dissipation theorem. For long times ($t \gg 1$), the system is driven towards a thermal equilibrium state, balancing out fluctuation and dissipation effects. In this scenario, the equipartition theorem becomes valid so that

$$\frac{1}{2} m \langle v_i^2(t) \rangle_{eq} = \frac{g}{4\gamma} = \frac{k_B T}{2}, \hspace{1cm} (16)$$

where $T$ denotes temperature and $k_B$ is the Boltzmann’s constant.

At this point, it is convenient to address the diffusion coefficient, which can be extracted from the particles’ equations of motion. Equation (12) leads to

$$x_i(t) = x_{0i} + \frac{mv_{0i}}{\gamma} \left( 1 - e^{-\frac{2\mu}{m\gamma} t} \right)$$

$$+ \frac{1}{\gamma} \int_0^t ds \ \left( 1 - e^{-\frac{2\mu}{m\gamma} (t-s)} \right) \xi_i(s) - \Theta_{ij} \int_0^t \xi_j(s) ds, \hspace{1cm} (17)$$

where along with Eq. (11), the variance can be obtained,

$$\sigma_i^2(t) = \langle x_i^2(t) \rangle - \langle x_i(t) \rangle^2$$

$$= g \left( \frac{1}{\gamma^2} + \Theta_{ij}^2 \right) t - \frac{3mg}{2\gamma^3} \left( \frac{g}{2\gamma^3} - 4e^{-\frac{2\mu}{m\gamma} t} \right). \hspace{1cm} (18)$$

In the long-time regime, the above expression turns into

$$\sigma_i^2(t) = 2D_it - \frac{3mg}{2\gamma^3}, \hspace{1cm} (19)$$

with

$$D_x = D_y \equiv \frac{g}{2\gamma^2} \left( 1 + \Theta^2\gamma^2 \right) = \frac{k_B T}{\gamma} \left( 1 + \Theta^2\gamma^2 \right) \hspace{1cm} (20)$$
being the diffusion coefficient along the $i$-axis (note that the system is in thermal equilibrium). According to Eq. (20), it is clear that noncommutativity induces an extra term to the diffusion coefficient, which is proportional to the square of $\Theta$. If $\Theta = 0$, we fully recover the usual commutative description. It is worth to mention that this correction itself is rather small to be detected experimentally and that is not what we want to highlight. The most intriguing feature in considering the Brownian motion in a NC manifold is shown in the following.

From Eqs. (11) and (17) we find that the NC nature of space induces a non-vanishing variance of the correlation between different coordinates at different times (say, $t_1 > t_2$), in contrast with the commutative case:

$$\sigma_{xy}(t_1, t_2) = \langle x(t_1)y(t_2) \rangle - \langle x(t_1) \rangle \langle y(t_2) \rangle = \frac{\Theta xy gm}{\gamma^2} \left( 1 - e^{-\frac{\gamma}{m}(t_1-t_2)} \right) + \frac{\Theta yx mg}{\gamma^2} \left( e^{-\frac{\gamma}{m}t_2} - e^{-\frac{\gamma}{m}t_1} \right).$$

(21)

In thermal equilibrium ($t_1 \gg 1$ and $t_2 \gg 1$), and considering that $\frac{\gamma}{m}t_1 \gg 1$ and $\frac{\gamma}{m}t_2 \gg 1$, the above equation reduces to

$$\sigma_{xy}(t_1, t_2) = \frac{2m \Theta k_B T}{\gamma} \left( 1 - e^{-\frac{\gamma}{m}|t_1-t_2|} \right).$$

(22)

Note that we can, in a similar way, take $t_2 > t_1$ [cf. Eqs. (11) and (17)], what makes the above equation more general.

We now address two limiting cases for that. First, for $\frac{\gamma}{m}|t_1 - t_2| \gg 1$, Eq. (22) reads

$$\sigma_{xy}(t_1, t_2) = \frac{2m \Theta k_B T}{\gamma}.$$  

(23)

On the other hand, for $\frac{\gamma}{m}|t_1 - t_2| \ll 1$ we have (expanding the exponential and dropping out higher-order terms)

$$\sigma_{xy}(t_1, t_2) = 2\Theta k_B T|t_1 - t_2|.$$  

(24)

4. Discussion

Equations (23) and (24) are the key results of this work. The first thing we note is that spatial noncommutativity allows for the emergence of correlations not seen in the standard commutative framework, as expected. Naturally, this very scenario is recovered when $\Theta$ is null. Interestingly, the NC correction in Eq. (24) features a time dependence for short measurement-time differences, further saturating to Eq. (23) for later times. In order to be able to make it physically attainable, one must find a way to overcome the scale imposed by $\Theta$. In our case, it implies in setting $\gamma/m$ as low as possible. That would work for both regimes given by Eqs. (23) and (24). Most importantly, the time difference in Eq. (21) can (and ideally should) be made larger as long as we keep $|t_1 - t_2| \ll m/\gamma$. That would ultimately permit the observation of the time-dependence of the variance.
In summary, the feasibility of probing NC effects depends upon the trade-off between the properties of Brownian particle along with its substrate and the NC parameter. Considering a spherical Brownian particle with radius $a$ such that $m = \frac{4}{3} \pi a^3 \rho$, with $\rho$ being the particle’s density and using Stokes’ formula $\gamma = 6 \pi \eta a$ for a given fluid with viscosity $\eta$, Eq. (23) turns into

$$\sigma_{xy}(t_1, t_2) = \frac{4 \Theta k_B T \rho a^2}{9 \eta}.$$  

(25)

The above equation tells us that the particle’s size and density plays a significant role in setting the scale of the effect. Also, it is desired to have very weak viscous forces acting on it.

The Brownian motion is a well-established subject and has been studied within various physical platforms with a high degree of precision and control\cite{43-51} Still, there remains the challenge of meeting the constraints imposed by the NC parameter. Nevertheless, the advantages of searching for signatures of noncommutativity on the space-time structure of the Brownian motion are many: (i) it is an exactly solvable model and finds a handful of applications; (ii) it is free of decoherence effects, unlike quantum systems; (iii) recent advances in optical devices and nanotechnology have increased the accuracy level in trajectory analysis as well as in Brownian particle sizing\cite{44-51} thus providing means to perform the experiment with a high degree of resolution.

Another crucial aspect is that the Brownian motion shows self similarity at any length and time scales thus establishing a valuable platform to carry out studies on NC phenomena. Furthermore, from a theoretical point of view, it is highly relevant to explore other aspects of the motion itself, e.g., its trace\cite{52, 53} A deep look at it could unveil solutions to bypass the stringent range of parameters required to extract macroscopic signatures of noncommutativity. Our work further motivates the search for NC signatures in other stochastic models, generalizing what we have found so far.

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