A disturbance tradeoff principle for incompatible quantum observables

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We demonstrate a fundamental principle of disturbance tradeoff for incompatible observables in quantum theory, along the lines of the celebrated uncertainty principle. Given a set of observables which do not have any common eigenvector, not all disturbances associated with measurements of such observables – measured on distinct yet identically prepared copies of a system in a pure state – can be arbitrarily small. Indeed, we show that the average of the disturbances associated with such a set of observables is strictly greater than zero. We show an exact equivalence between disturbance tradeoff and the entropic uncertainty principle formulated in terms of the Tsallis entropy ($T_2$). We also derive an optimal $T_2$ entropic uncertainty relation for a pair of qubit observables and thereby arrive at an optimal disturbance tradeoff bound for such observables.

The uncertainty principle, which is one of the cornerstones of quantum theory, has had a long history. In its original formulation by Heisenberg for canonically conjugate variables [1], the uncertainty principle was stated as an effect of the disturbance caused due to a measurement of one observable on a succeeding measurement of another. However, the subsequent mathematical formulation due to Robertson [2] and Schrödinger [3] in terms of variances departed from this original interpretation. Rather, they obtained a non-trivial lower bound on the product of the variances associated with the measurement of a pair of incompatible observables, performed on distinct yet identically prepared copies of a given system. The Robertson-Schrödinger inequality thus expresses a fundamental constraint on these disturbances.

The more recent entropic formulations of the uncertainty principle [4] also demonstrate the existence of a fundamental tradeoff for the uncertainties associated with independent measurements of incompatible observables on identically prepared ensemble of systems. Entropic uncertainty relations (EURs) have been obtained for specific classes of observables for both the Rényi class of entropies [7–14], as well as for the Tsallis entropies [13, 17].

Here, we prove the existence of a similar fundamental principle of tradeoff for the disturbances associated with the measurements of a set of observables. The disturbance measure we use is based on the standard notion of fidelity between quantum states, and belongs to a class of measures used recently in the context of quantifying incompatibility of a pair of observables [18]. We demonstrate the existence of a fundamental tradeoff principle for the disturbances associated with a set of incompatible observables, when they are measured on distinct yet identically prepared copies of a pure state. Specifically, we show that the average disturbance associated with a set of observables is strictly greater than zero when the observables do not have any common eigenvector.

Furthermore, we prove an exact equivalence between the disturbance due to the measurement of an observable on a pure state and the Tsallis entropy ($T_2$) of the probability distribution over the outcomes of such a measurement. Our work thus provides a new operational significance to the $T_2$ entropy in the context of quantum information theory. We make use of some of the known EURs to obtain disturbance tradeoff relations for certain classes of observables. Finally, we prove an optimal disturbance tradeoff relation for a pair of qubit observables, which is based on a new, tight $T_2$ EUR.

Uncertainty relations have also been studied in the successive measurement scenario, both in the form of entropic relations [19, 20], and in the form of error-disturbance relations [1, 4, 21, 23] that are in line with Heisenberg’s original interpretation of the uncertainty principle. In contrast, here we look at the disturbances associated with distinct measurements of incompatible observables on identically prepared ensembles of systems. Our work thus brings to light a completely novel aspect of measurement-induced-disturbance: quantum theory places a fundamental constraint on these disturbances even when the corresponding measurements are made on distinct copies of a state.

Disturbance measures: We work within the framework of standard quantum theory, and restrict our discussion to observables that are self-adjoint operators with purely discrete spectra. Any such observable $A$ has a spectral resolution $A = \sum a_i P^A_i$, where $\{P^A_i\}$ are orthogonal projectors. Measurement of such an observable $A$ transforms the state $\rho$, as per the von Neumann-Lüders collapse postulate, to $E^A(\rho) = \sum_i P^A_i \rho P^A_i$. The post-measurement state is thus described via the action of a completely positive trace-preserving (CPTP) map $E^A$, often called the measurement channel. We use the standard measure of fidelity between the states $\rho$ and $E^A(\rho)$...
to quantify the disturbance caused to state $\rho$ by a measurement of $A$. Recall that the fidelity $F(\rho, \sigma)$ is defined as \[ F(\rho, \sigma) \equiv \text{tr} \sqrt{\sqrt{\rho} \sqrt{\sigma} \sqrt{\rho} \sqrt{\sigma}}. \] The disturbance due to a measurement of $A$ on state $\rho$ is then defined as

\[ D_F(A; \rho) \equiv 1 - F^2[\mathcal{E}^A(\rho), \rho]. \]  

(1)

Note that $D_F(A; \rho)$ satisfies $0 \leq D_F(A; \rho) \leq 1$.

Recently, such a measure of disturbance was used to quantify incompatibility of a pair of observables in a sequential measurement scenario [18]. It was also shown that, in $d$-dimensions, the maximal disturbance due to a measurement of observable $A$, defined as $D_F^{\text{max}}(A) = \sup_\rho D_F(A; \rho)$, is given by,

\[ D_F^{\text{max}}(A) = 1 - \frac{1}{r}, \] where $r \leq d$ is the number of distinct eigenvalues of $A$. If $A$ is a totally non-degenerate observable, then, $D_F^{\text{max}}(A) = (1 - \frac{1}{d})$, with the maximum being attained for a state that is mutually unbiased with respect to the eigenstates of $A$.

Here, we will focus on the disturbance associated with measurements on pure states. Then we have an explicit expression for the disturbance due to a measurement of observable $A = \sum_i p_i^A |i\rangle |i\rangle$, in terms of the probability distribution over the outcomes of the measurement. From Eq. (1), we have,

\[ D_F^A(|\psi\rangle) = 1 - F^2[\mathcal{E}^A(|\psi\rangle \langle\psi|), |\psi\rangle \langle\psi|] = 1 - \sum_i \left( p_i^A(i) \right)^2, \] where $p_i^A(i) \equiv \langle i | p_i^A |i\rangle$ is the probability of obtaining outcome $i$. The least disturbance is of course attained only for eigenstates of the observable $A$, as we note in the following Lemma.

Lemma 1. For any observable $A$ measured in a pure state $|\psi\rangle$, $D_F(A; |\psi\rangle) = 0$ if and only if $|\psi\rangle$ is an eigenstate of $A$.

Proof. By definition, $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$. Therefore, $D_F(A; |\psi\rangle) = 0$ if and only if $\mathcal{E}^A(|\psi\rangle \langle\psi|) = |\psi\rangle \langle\psi|$. This in turn implies that $\sum_i (\langle i | p_i^A |i\rangle)^2 = 1$, which proves the Lemma.

The Disturbance Tradeoff Principle: Using the disturbance measure defined above, we now prove the tradeoff principle for the disturbances associated with a set of incompatible observables. We first note the following property of the average disturbance of a pair of observables.

Lemma 2. For a pair of observables $A$ and $B$ with purely discrete spectra, define the quantity

\[ c(A, B) \equiv \inf_{|\psi\rangle} \frac{1}{2} [D_F(A; |\psi\rangle) + D_F(B; |\psi\rangle)]. \] Then, $0 \leq c(A, B) \leq 1$, with $c(A, B) = 0$ if and only if $A$ and $B$ have a common eigenvector.

Proof. The first part simply follows from the fact that $0 \leq D_F(X; |\psi\rangle) \leq 1$ for $X = A, B$. Next, we know from Lemma 1 that the individual disturbances vanish only for eigenstates of the corresponding observables. Therefore the average vanishes if and only if there exists a common eigenvector of $A$ and $B$.

This implies the following disturbance tradeoff principle: For any two observables $A$ and $B$ with purely discrete spectra which do not have any common eigenvector, there exists a quantity $c(A, B) > 0$, such that for any pure state $|\psi\rangle$, the average of the disturbances due to measurements of $A$ and $B$ (performed independently, on identically prepared copies of $|\psi\rangle$) is greater than or equal to $c(A, B)$.

In other words, the disturbances due to measurements of incompatible observables $A$ and $B$ on any pure state cannot both be made arbitrarily small; if one is small, the other must necessarily be of the order of $c(A, B)$, even though the measurements are performed independently, on identically prepared copies of any given pure state. Mathematically, this may be stated as,

\[ \frac{1}{2} [D_F(A; |\psi\rangle) + D_F(B; |\psi\rangle)] \geq c(A, B) > 0, \] for all pure states $|\psi\rangle$ and observables $A$ and $B$ which do not have any common eigenvector. From Eq. (2) it follows that, in $d$-dimensions, $c(A, B) \leq (1 - \frac{1}{d})$, with the maximum value being obtained when $A, B$ are non-degenerate observables with mutually unbiased bases.

More generally, a disturbance relation for a set of observables $\{A_1, A_2, \ldots, A_N\}$ is a state-independent lower bound of the form

\[ \frac{1}{N} \sum_{i=1}^{N} D_F(A_i; |\psi\rangle) \geq c(A_1, \ldots, A_N), \forall |\psi\rangle. \]

A simple extension of Lemma 2 proves that $c(A_1, \ldots, A_N) > 0$ for a set of observables that do not have any common eigenvector.

It should be noted that the above disturbance tradeoff principle holds only for pure state ensembles. If we take into consideration mixed states as well, then we have for instance the maximally mixed state $\frac{1}{d}$ in $d$-dimensions, which is not disturbed by the measurement of any observable!

Apart from the fidelity-based measure, the trace-distance can also be used to quantify disturbance. For instance, the quantity $D_1(A; |\psi\rangle) = \frac{1}{d} \text{tr} |\mathcal{E}^A(|\psi\rangle \langle\psi|) - |\psi\rangle \langle\psi||$ is a valid measure of the disturbance associated with a measurement of $A$ in state $|\psi\rangle$ [18]. Since for pure states, it is well known [24] that $D_1(A; |\psi\rangle) \geq D_F(A; |\psi\rangle)$, the lower bound (5) on the average fidelity-based disturbance is also a lower bound on the average $D_1$ disturbance. Thus, the disturbance tradeoff relations
for $D_F$ are stronger: they imply tradeoff relations for the associated $D_1$ quantities.

Equivalence between Disturbance Tradeoff and the Uncertainty Principle: For pure states $|\psi\rangle$, there is a mathematical relation between the disturbance caused by a measurement of observable $A$ on $|\psi\rangle$ and the spread of the probability distribution over the outcomes of a measurement of $A$ on $|\psi\rangle$. Recall that the Tsallis entropy $T_\alpha(p_i)$ of a discrete probability distribution $\{p_i\}$, for any real $\alpha > 0 (\alpha \neq 1)$, is defined as:

$$T_\alpha(p_i) = \frac{1}{1-\alpha} \left( \sum_i p_i^\alpha - 1 \right). \quad (6)$$

For $d$-dimensional distributions the Tsallis entropies satisfy,

$$0 \leq T_\alpha(p_i) \leq \log_\alpha d,$$

where, the $\alpha$-logarithm function is defined as

$$\log_\alpha x = x^{1-\alpha} - 1 \quad \frac{1}{1-\alpha}.$$

In the limit $\alpha \to 1$ we recover the usual logarithm function $\log x$, and the Tsallis entropy $T_1$ is the same as the Shannon entropy: $T_1\{p_i\} = -\sum_i p_i \log p_i$.

The $T_\alpha$ entropy defined in Eq. (6) is a non-extensive entropy originally developed by Tsallis in the context of statistical physics [1]. The Tsallis entropies are concave for the entire range of $\alpha \in [0, \infty)$ [22]. Similar to the Rényi entropies, the Tsallis entropies of the probability distribution $p_\psi^A(i)$ over the outcomes of a measurement of an observable $A$ can also be used to quantify the uncertainty associated with the outcome of a measurement of $A$ in state $|\psi\rangle$. For instance, the uncertainty principle for canonically conjugate observables has been formulated in terms of the Tsallis entropies [11]. More recently, Tsallis entropic uncertainty relations have been obtained for specific classes of observables in finite dimensions [17, 26-28].

It is easy to see that the disturbance measure $D_F(A; |\psi\rangle)$ defined in Eq. (1), is indeed the same as the $T_2$-entropy of the probability distribution of the outcomes of a measurement of observable $A$ on state $|\psi\rangle$:

$$D_F(A; |\psi\rangle) = 1 - \sum_i (p_\psi^A(i))^2 = T_2(A; |\psi\rangle) \quad (7)$$

Both the disturbance $D_F(A; |\psi\rangle)$ and the uncertainty $T_2(A; |\psi\rangle)$ vanish when $|\psi\rangle$ is an eigenstate of $A$. For a non-degenerate observable $A$ in $d$-dimensions, both the disturbance and the uncertainty attain the maximum value $\left(1 - \frac{1}{d}\right)$ when $|\psi\rangle$ is mutually unbiased with respect to the eigenstates of $A$.

It may however be noted that the interesting equivalence (7) between the fidelity-based measure of disturbance and the uncertainty measure given by the $T_2$ Tsallis entropy holds only for pure states.

Disturbance Tradeoffs for specific classes of observables: Eq. (7) shows that there is indeed an equivalence between the disturbance tradeoff principle and entropic uncertainty relations formulated in terms of the Tsallis entropy $T_2$. We can thus directly obtain disturbance tradeoff inequalities for those classes of observables for which $T_2$ EURs are known. Here we use the known results in the literature to show disturbance tradeoff relations for (a) a set of $N$ mutually unbiased bases (MUBs), and, (b) a set of dichotomic anti-commuting observables.

(a) Let $B_m = \{|m\}; m = 1, \ldots, N\}$ denote a set of $N$ MUBs in $d$-dimensions. Recall that two bases $B_m, B_n$ are said to be mutually unbiased if their respective basis vectors satisfy

$$\langle i| j \rangle = \frac{1}{d}, \quad \forall i, j.$$

Let $p^B_\psi(i) = |\langle i| \psi \rangle|^2$ denote the probability of obtaining the $i$th outcome when measuring $B_m$ on a pure state $|\psi\rangle$. Then, Wu et al. show that [29],

$$\sum_{i=1}^d \sum_{m=1}^N \left( p^B_\psi(i) \right)^2 \leq 1 + \frac{N-1}{d}. \quad (8)$$

This immediately implies the following disturbance tradeoff relation for a set of $N$ MUBs in $d$-dimensions:

$$\frac{1}{N} \sum_{m=1}^N D_F(B_m; |\psi\rangle) \geq \left( 1 - \frac{1}{N} \right) \left( 1 - \frac{1}{d} \right). \quad (9)$$

In dimensions where a complete set of $(d+1)$ MUBs exist, it is known that Eq. (8) is in fact an exact equality for the full set of $(d+1)$ MUBs [10, 30]. Correspondingly, the disturbance tradeoff relation in Eq. (9) becomes an exact equality for a complete set of $(d+1)$ MUBs.

(b) Let $\{A_1, A_2, \ldots, A_N\}$ be a set of pairwise anticommuting operators with eigenvalues $\{\pm 1\}$:

$$\{A_j, A_k\} = 0 \quad \forall j \neq k, \quad (A_i)^2 = I, \quad \forall i = 1, \ldots, N. \quad (10)$$

Let $P^+_i$ and $P^-_i$ denote the projectors onto the positive and negative eigenspaces respectively, of observable $A_i$. Correspondingly, $p^A_\psi(\pm) = \langle \psi | P^{\pm}_i | \psi \rangle$ are the probabilities of obtaining values $\pm 1$ while measuring $A_i$ in state $|\psi\rangle$. We obtain a disturbance tradeoff relation for such a set of dichotomic, anticommuting observables. We merely state the relation here and refer to the appendix for the proof.

Theorem 3. For a set of $N$ anticommuting observables $\{A_1, A_2, \ldots, A_N\}$ defined in Eq. (11),

$$\frac{1}{N} \sum_{i=1}^N D_F(A_i; |\psi\rangle) \geq \frac{1}{2} \left( 1 - \frac{1}{N} \right). \quad (11)$$
As a particular instance of Eq. (9) and Eq. (11), we obtain an elegant tradeoff relation for the disturbances associated with measurements of the spin components $\sigma_X$, $\sigma_Y$ and $\sigma_Z$ in $d = 2$:

$$D_F(\sigma_X; |\psi\rangle) + D_F(\sigma_Y; |\psi\rangle) + D_F(\sigma_Z; |\psi\rangle) = 1.$$  (12)

The corresponding $T_2$ EUR has been derived in [13].

**Optimal Disturbance Tradeoff Relation for a pair of Qubit Observables:** Finally, we demonstrate the following optimal disturbance tradeoff relation for any pair of observables in a two-dimensional Hilbert space.

**Theorem 4.** For a pair of qubit observables $A, B$ with discrete spectra $A = \sum_{i=1}^2 a_i |a_i\rangle\langle a_i|$, $B = \sum_{j=1}^2 b_j |b_j\rangle\langle b_j|$, and any pure state $|\psi\rangle \in \mathbb{C}^2$,

$$\frac{1}{2} [D_F(A; |\psi\rangle) + D_F(B; |\psi\rangle)] \geq \frac{1}{2}(1 - c^2),$$  (13)

where $c \equiv \max_{i,j=1,2} |\langle a_i|b_j\rangle|$.

The problem of finding the lower bound on the average disturbance simplifies considerably once we use the Bloch sphere representation for qubit observables. In other words, we parameterize $A$ and $B$ in terms of unit vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ and real parameters $\{\alpha_i, \beta_i\}$ as follows:

$$A = \alpha_1 \mathbb{1} + \alpha_2 \vec{a} \cdot \vec{\sigma} \quad \text{and} \quad B = \beta_1 \mathbb{1} + \beta_2 \vec{b} \cdot \vec{\sigma}. $$

The quantity $c$ is then given by

$$c = \sqrt{\frac{1 + \sqrt{1 + 4c^2}}{2}}, \quad (\vec{a} \cdot \vec{b} > 0); \quad c = \sqrt{\frac{1 - \sqrt{1 + 4c^2}}{2}}, \quad (\vec{a} \cdot \vec{b} < 0).$$

A similar approach has been used to obtain optimal entropic uncertainty relations for a pair of qubit observables, both in the case of the Shannon entropy $H_2$ [12, 31] and the collision entropy $H_{1/2}$ [32]. Further details of our proof can be found in the appendix.

We also show that the bound in Eq. (13) is tight. When $(\vec{a} \cdot \vec{b})^2 = 1$, $c = 0$; $A$ and $B$ commute and the RHS of (13) reduces to 0. This lower bound is attained for the common eigenstates of $A, B$. When $\vec{a} \cdot \vec{b} = 0$, $c = \frac{1}{\sqrt{2}}$; $A$ and $B$ are mutually unbiased. The bound in (13) is $\frac{1}{2}$, which is attained for any eigenstate of $A$ or $B$. For any other value of $\vec{a} \cdot \vec{b}$, the lower bound is attained for the states whose Bloch vectors bisect the angle between $\vec{a}$ and $\vec{b}$. The minimizing states are thus given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[ I + \frac{\vec{a} \pm \vec{b}}{2c} \cdot \vec{\sigma} \right]|\psi\rangle.$$  (14)

Since our disturbance measure $D_F(A; |\psi\rangle)$ is in fact the same as the entropy $T_2(A; |\psi\rangle)$, the tradeoff relation in Eq. (13) is nothing but a tight entropic uncertainty relation for the $T_2$ entropy:

$$\frac{1}{2} [T_2(A; |\psi\rangle) + T_2(B; |\psi\rangle)] \geq \frac{1}{2}(1 - c^2),$$  (15)

Our result for $T_2$ assumes importance in the light of the fact that such exact analytical bounds are known only for a handful of entropic functions, namely, the Rényi entropies $H_2$ [32], $H_{1/2}$, and the Tsallis entropy $T_{1/2}$ [28]. For the Shannon entropy, there is in general only a numerical estimate of the bound [12, 31].

**Concluding Remarks:** To summarize, we demonstrate a fundamental principle of disturbance tradeoff for incompatible quantum observables. The existence of such a tradeoff principle implies that quantum theory places a fundamental restriction on the disturbances associated with a set of incompatible observables, even when they are measured on distinct, identically prepared copies of a pure state. Our results formally place disturbance measures on an equal footing with uncertainty measures, affirming a long standing folklore to this effect.

We in fact prove a mathematical equivalence between the fidelity-based disturbance measure and the $T_2$ Tsallis entropy for pure states. Our work thus provides a new operational significance to the Tsallis entropy in the context of quantum foundations and quantum information theory.

While we have focused on projective measurements, it will be interesting to see if similar tradeoffs can be obtained for more general quantum measurements. It also remains to be investigated whether the lower bounds on the sum of disturbances for specific sets of observables assume further operational significance in the context of quantum cryptography.

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Proof of optimal $T_2$ EUR for qubit observables

The proof of Theorem 4 is based on the following optimal entropic uncertainty relation (EUR) in terms of the Tsallis entropy $T_2$, for a pair of qubit observables. Optimal EURs for the Tsallis entropies $T_\alpha$ have only been obtained for $\alpha = \frac{1}{2}, 1$ [12, 28, 31] thus far.

**Theorem 5.** For a pair of qubit observables $A, B$ with discrete spectra $A = \sum_{i=1}^{2} a_i |a_i\rangle \langle a_i|$, $B = \sum_{j=1}^{2} b_j |b_j\rangle \langle b_j|$, and any state $\rho$, $\inf_{\rho} \frac{1}{2} \left[ T_2(A; \rho) + T_2(B; \rho) \right] \geq \frac{1}{2} (1 - c^2), \quad (16)$

where $c \equiv \max_{i,j=1,2} |\langle a_i|b_j\rangle|$. 

**Proof.** Since the $T_2$ entropy is concave, it suffices to minimize the sum of entropies over pure states of qubits. Thus, the quantity we seek to evaluate is:

$$c(A, B) = \frac{1}{2} \inf_{|\psi\rangle} \left[ T_2(A; |\psi\rangle) + T_2(B; |\psi\rangle) \right]$$

$$= \inf_{|\psi\rangle} \left[ 1 - \frac{1}{2} \left( \sum_i \left( p_i^A \right)^2 + \sum_i \left( p_i^B \right)^2 \right) \right]$$

Let $A$ and $B$ associated Bloch sphere representation in terms of unit vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$. Specifically,

$$A = \alpha_1 \mathbb{I} + \alpha_2 \vec{a} \vec{\sigma}; \quad B = \beta_1 \mathbb{I} + \beta_2 \vec{b} \vec{\sigma},$$

where, $\{\alpha, \beta\}$ are real parameters and $\vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ denote the Pauli matrices and $\mathbb{I}$ denotes the $2 \times 2$ identity matrix. Any pure state can similarly be denoted in terms of a unit vector $\vec{r} \in \mathbb{R}^3$, as $|\psi\rangle \langle \psi| = \frac{1}{2} (\mathbb{I} + \vec{r} \vec{\sigma})$.

Rewriting the probabilities in terms of the vectors $\vec{a}, \vec{b}$ and $\vec{r}$, the quantity in Eq. (17) becomes,

$$c_{\vec{a}, \vec{b}} = \frac{1}{\vec{r}} \inf \left[ 1 - \frac{1}{2} \left( (\vec{a} \vec{r})^2 + (\vec{b} \vec{r})^2 \right) \right] = \frac{1}{\vec{r}} \inf \mathcal{F}_{\vec{a}, \vec{b}}(\vec{r})$$

Thus, the average disturbance function we seek to minimize is of the form

$$\mathcal{F}_{\vec{a}, \vec{b}}(\vec{r}) = \left[ 1 - \frac{1}{2} \left( (\vec{a} \vec{r})^2 + (\vec{b} \vec{r})^2 \right) \right]$$

Closely following the earlier work of Ghirardi et al. [31] and Bosyk et al. [32], we first argue that the minimizing vector $\vec{r}$ must be coplanar with $\vec{a}$ and $\vec{b}$. Let $\mathcal{P}$ denote the plane determined by the vectors $\vec{a}$ and $\vec{b}$. Given any unit vector $\vec{v}_\perp$ in a plane $\mathcal{P}_\perp$ perpendicular to $\mathcal{P}$, there exists a vector $\vec{v}_\perp$ in the intersection of $\mathcal{P}$ and $\mathcal{P}_\perp$ such that $|\langle a | \vec{v}_\perp \rangle| \leq |\langle a | \vec{v} \rangle|$. Now, note that the function $f(x) = 1 - \frac{x^2}{2}$ is monotonically decreasing for $x \in [0, 1]$. Thus, for every vector $\vec{v}_\perp \in \mathcal{P}_\perp$ in a plane perpendicular to $\mathcal{P}$, there exists a coplanar vector $\vec{v}_\perp \in \mathcal{P} \cap \mathcal{P}_\perp$ such that

$$1 - \frac{(\vec{a} \vec{v}_\perp)^2}{2} \leq 1 - \frac{(\vec{a} \vec{v})^2}{2}.$$

Making a similar argument for the vector $\vec{b}$, we see that the minimum value $\mathcal{F}_{\vec{a}, \vec{b}}(\vec{r})$ is attained for a vector $\vec{r} \in \mathcal{P}$.

Coplanarity of $\vec{a}, \vec{b}$ and $\vec{r}$, implies that if $\theta = \cos^{-1}(\vec{a} \vec{b})$ and $\alpha = \cos^{-1}(\vec{a} \vec{r})$, then $\vec{b} \vec{r} = \cos(\theta - \alpha)$. Therefore, we can rewrite the function $\mathcal{F}_{\vec{a}, \vec{b}}(\vec{r})$ as

$$\mathcal{F}_{\theta}(\alpha) = 1 - \frac{1}{2} \left[ \cos^2 \alpha + \cos^2(\theta - \alpha) \right]. \quad (18)$$

reducing the problem to a minimization over a single variable $\alpha$. Since $\mathcal{F}_{\theta}(\alpha)$ is periodic in $\alpha$, it suffices to minimize over the interval $\alpha \in [0, \pi]$. Differentiating with respect to $\alpha$ and setting the first derivative to zero, we see that the extremizing values of $\alpha$ satisfy

$$\sin(2\alpha_*) = \sin 2(\theta - \alpha_*) \Rightarrow \alpha_* = \frac{\theta}{2} + k\frac{\pi}{2}, \quad k = 0, 1, 2, \ldots$$

Thus, $\alpha_+ = \frac{\theta}{2}$ and $\alpha_- = \frac{\theta}{2} + \frac{\pi}{2}$, are the two relevant solutions in the interval $\alpha \in [0, \pi]$. By explicitly evaluating the function $\mathcal{F}_{\theta}(\alpha)$ at $\alpha_{\pm}$, we can check that $\alpha_{\pm}$ indeed corresponds to a minimum for $0 \leq \theta \leq \frac{\pi}{2}$ and $\alpha_{\pm}$ corresponds to a minimum for $\frac{\pi}{2} \leq \theta \leq \pi$.

The minimum value of the average disturbance function is therefore,

$$c_{\theta} = \frac{1}{\alpha} \inf_{\alpha} \mathcal{F}_{\theta}(\alpha) = \left\{ \begin{array}{ll}
\frac{1}{2} \left( 1 - \cos^2 \frac{\theta}{2} \right) & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\
\frac{1}{2} \left( 1 - \sin^2 \frac{\theta}{2} \right) & \text{for } \frac{\pi}{2} \leq \theta \leq \pi
\end{array} \right\} \quad (20)$$
To realize the minimum value in terms of the overlap between the eigenstates of $A$ and $B$, recall that $A = \alpha_1 \mathbb{1} + \alpha_2 \vec{a} \cdot \vec{d}$; $B = \beta_1 \mathbb{1} + \beta_2 \vec{b} \cdot \vec{d}$.

Therefore, in terms of the angle $\theta$ between $\vec{a}$ and $\vec{b}$, we have

$$|\langle a_i | b_j \rangle|^2 = \begin{cases} \frac{1+\alpha_2 \beta_2}{2} & \text{for } i = j \\ \frac{1-\alpha_2 \beta_2}{2} & \text{for } i \neq j \end{cases}$$

(20)

Therefore, in terms of the angle $\theta$ between $\vec{a}$ and $\vec{b}$, we see that

$$|\langle a_i | b_j \rangle| = \begin{cases} \cos \frac{\theta}{2} & \text{for } i = j \\ \sin \frac{\theta}{2} & \text{for } i \neq j \end{cases}$$

(21)

In particular, defining $c \equiv \max_{i,j} |\langle a_i | b_j \rangle|$, we see that

$$c = \max_{i,j} |\langle a_i | b_j \rangle| = \begin{cases} \cos \frac{\theta}{2} & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ \sin \frac{\theta}{2} & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

(22)

Putting together equations Eq. (20) and Eq. (22), we see that

$$\frac{1}{2} [D_F(A; |\psi\rangle) + D_F(B; |\psi\rangle)] \geq c_0 = \frac{1}{2}(1 - c^2).$$

(23)

The minimizing Bloch vector $\vec{r}_+$ corresponding to the solution $\alpha_+ = \frac{\theta}{2}$ satisfies

$$\vec{a} \cdot \vec{r}_+ = \vec{b} \cdot \vec{r}_+ = \cos \frac{\theta}{2}.$$

Thus, $\vec{r}_+ = \vec{a} + \vec{b}$, up to normalization, and is the vector that bisects the interior angle between $\vec{a}$ and $\vec{b}$. The vector $\vec{r}_-$ corresponding to the other solution $\alpha_- = \frac{\theta}{2} + \frac{\pi}{2}$ satisfies

$$\vec{a} \cdot \vec{r}_- = \cos \left(\frac{\theta}{2} + \frac{\pi}{2}\right), \vec{b} \cdot \vec{r}_- = \cos \left(\frac{\theta}{2} - \frac{\pi}{2}\right).$$

Therefore, $\vec{r}_- = \vec{a} - \vec{b}$, up to normalization; it is the exterior angle bisector and is perpendicular to $\vec{r}_+$. Both minimizing vectors satisfy $|\vec{r}_+| = |\vec{r}_-| = 2c$. In summary, the Bloch vectors corresponding to the minimizing states are:

$$\vec{r}_\pm = \begin{cases} \frac{\vec{a} + \vec{b}}{2} & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ \frac{\vec{a} - \vec{b}}{2} & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

Proof of Theorem 3

Recall that we seek to prove the disturbance tradeoff relation in Eq. (11) for a set of pairwise anticommuting observables $\{A_1, A_2, \ldots, A_N\}$ with eigenvalues $\{\pm 1\}$:

$$\{A_j, A_k\} = 0 \forall j \neq k, (A_i)^2 = \mathbb{1} \forall i = 1, \ldots, N.$$

Proof. For any pure state $|\psi\rangle$, the expectation values of such a set of $N$ dichotomic anticommuting observables are known to satisfy the following meta uncertainty relation [14, 27]:

$$\sum_{i=1}^{N} (\langle \psi | A_i | \psi \rangle)^2 \leq 1.$$  

(24)

Corresponding to a measurement of observable $A_i$ in state $|\psi\rangle$, the probabilities of obtaining outcomes $\pm 1$ are related to the expectation value $\langle \psi | A_i | \psi \rangle$, as follows:

$$p_{\pm}^{A_i}(\pm 1) = \langle \psi | P_{\pm}^{A_i} | \psi \rangle = \frac{1 \pm \langle \psi | A_i | \psi \rangle}{2}.$$  

(25)

Thus the meta uncertainty relation above implies the following upper bound on the sums of the probabilities:

$$\sum_{j=1}^{N} \left[ (p_{\psi}^{A_{i}(+)}(+) + (p_{\psi}^{A_{i}(-)}(-))^2 \right] = \sum_{j=1}^{N} \frac{1 + (\langle \psi | A_i | \psi \rangle)^2}{2} \leq \frac{N + 1}{2}.$$  

(26)

The disturbance tradeoff relation follows immediately.