A Venn diagram for supersymmetric, exactly solvable, shape invariant, and Infeld-Hull factorizable potentials

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Abstract. Supersymmetry, shape invariance, exact solubility, and the factorization method are often studied together in the literature. At the dawn of these topics confusion was present in regards to their scope of applicability and the relation among them. Considerable work have been put to study and resolve the relation among two or more of these topics. These works are scattered over the literature. While looking at the literature, one can not overlook the number of places where authors confuse these terms, and concluding implications depending on wrong assumptions of the relation between two or more of these topics. In this letter we define supersymmetry, and shape invariance, and show the relations which connects them to exact solubility and the factorization method, referring to the literature for the respective detailed work and proofs. At last we conclude our letter with a Venn diagram which illustrates those relations.

1. Supersymmetric Quantum Mechanics

Witten, in his seminal paper \cite{1}, defined the algebra of a supersymmetric quantum system, which he derived from the algebra of supersymmetry in field theory. In supersymmetric quantum systems, there are charge operators $Q_i$, which commute with the Hamiltonian

$$[Q_i, \mathcal{H}] = 0, \quad i = 1, 2, ..., N \quad (1)$$

and they obey the algebra

$$\{Q_i, Q_j\} = \delta_{ij} \mathcal{H} \quad (2)$$

where $\mathcal{H}$ is called the Supersymmetric Hamiltonian. Witten stated that the simplest quantum mechanical system has $N = 2$, it was later shown that the case where $N = 1$, if it is supersymmetric, it is equivalent to an $N = 2$ supersymmetric quantum system \cite{2}. In the case where $N = 2$ we can define

$$Q = \left(\frac{1}{\sqrt{2}}\right)(Q_1 + iQ_2), \quad Q^\dagger = \left(\frac{1}{\sqrt{2}}\right)(Q_1 - iQ_2) \quad (3)$$
these operators will obey the algebra
\[ \mathcal{H} = \{Q, Q^\dagger\}, \quad Q^2 = Q^{\dagger 2} = 0 \] (4)
consequently,
\[ [Q, \mathcal{H}] = [Q^\dagger, \mathcal{H}] = 0 \] (5)

We can represent our charge operators as 2 × 2 matrices
\[
Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \tag{6a}
\]
\[
Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \tag{6b}
\]

where \( A \) is a linear operator, and \( A^\dagger \) is its adjoint. Now the supersymmetric Hamiltonian has the representation
\[ \mathcal{H} = \{Q, Q^\dagger\} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \] (7)

where in the last step we designated the hermitian operators \( A^\dagger A \) and \( AA^\dagger \) as the Hamiltonians \( H_- \) and \( H_+ \), respectively. It is important to notice that \( H_- \) and \( H_+ \) are positive semi-definite. for if \( |\alpha\rangle \) is an eigenstate in the Hilbert space of any of these Hamiltonians, say \( H_- \) then
\[ \langle \alpha | H_- | \alpha \rangle = \alpha \langle \alpha | \alpha \rangle = \alpha \] (8)
where \( \alpha \) is the eigenvalue of \( |\alpha\rangle \). At the same time
\[ \langle \alpha | H_- | \alpha \rangle = \langle \alpha | A^\dagger A | \alpha \rangle = |A | \alpha \rangle|^2 \] (9)

where \(|.|\) is the norm of our Hilbert space, whose range is always real. Since the right hand side of (8) must be equal to the right hand side of (9), and by the positivity axiom of our metric space (Hilbert space), \(||x \rangle| \geq 0\) for all the vectors in our metric space \([3, \text{pp.14}]\), so it follows that \( H_- \) is positive semi-definite. It also follows that \( H_- \) has eigenvalue zero iff \( A \) annihilates a vector, which must be the ground state. The same argument can be applied to \( H_+ \). For an alternative proof see \([4, \text{Th. X25, pp 180}]\).

It is straightforward to show that if \( \psi^- \) is an eigenstate for \( H_- \) with a positive eigenvalue \( E^- \)
\[ H_- \psi^- = A^\dagger A \psi^- = E^- \psi^- \] (10)
then \( \psi^+ = (E^-)^{-1/2}(A \psi^-) \) is a normalized eigenstate for \( H_+ \) with an eigenvalue \( E^+ = E^- \)
\[ H_+ \psi^+ = \frac{1}{\sqrt{E^-}} AA^\dagger A \psi^- = E^- \psi^+ \] (11)
in the same manner $A^\dagger$ transforms an eigenstate of $H_+$ to an eigenstate of $H_-$ with the same energy [5, 6]. This can be written as

$$Q \begin{pmatrix} \psi^- \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi^- \end{pmatrix}$$  

$$Q^\dagger \begin{pmatrix} 0 \\ \psi^+ \end{pmatrix} = \begin{pmatrix} A^\dagger \psi^+ \\ 0 \end{pmatrix}$$  

This result comes with no surprise for us, since we constructed our charge operators [1] to obey our algebra (4), and we saw that this algebra implies that the charges commute with the Hamiltonian (5), this commutativity is responsible for this degeneracy.

In literature, the linear operators $A$ and $A^\dagger$ are usually considered in the following form

$$A = \frac{d}{dx} + w(x)$$  

$$A^\dagger = -\frac{d}{dx} + w(x)$$  

where $w(x)$ is an arbitrary function of $x$ and it is called the superpotential. Now, we can look at the differential form of our partner Hamiltonians

$$H_- = A^\dagger A = -\frac{d}{dx^2} + w^2(x) - w'(x)$$  

$$H_+ = AA^\dagger = -\frac{d}{dx^2} + w^2(x) + w'(x)$$

the prime is $d/dx$, and we set $\hbar = 2m = 1$. The potentials in the above Hamiltonians are called the supersymmetric partner potentials or superpartners, and they are given by

$$V_\pm = w^2(x) \pm w'(x)$$  

this nonlinear differential equation is known as Ricatti equation. A node-less solution to the Hamiltonians [1] can written in terms of the superpotential

$$\psi_0^\pm = N \exp \left( \pm \int^x w(t)dt \right)$$  

where $\psi_0^-$ is a solution of $H_-$ and $\psi_0^+$ is a solution of $H_+$, the subscript is the number of nodes in the function, zero in this case. $N$ is a normalization constant. By direct substitution we see that $A$ annihilates $\psi_0^-$, $A^\dagger$ annihilates $\psi_0^+$, consequently they are solutions with eigenvalue zero

$$H_- \psi_0^- = A^\dagger (A\psi_0^-) = 0$$  

$$H_+ \psi_0^+ = A(A^\dagger \psi_0^+) = 0$$
but, these wave functions are not normalizable, i.e. square integrable, at the same time, actually they need not be normalizable at all, but in the case when one of them is normalizable the supersymmetry is said to be unbroken. If none of them is normalizable, then none of the partner Hamiltonians has a zero eigenvalue and the supersymmetry is broken \[7, 8, 9\].

From now on we will consider the case where the supersymmetry is unbroken, i.e. one of $\psi_0^-$ and $\psi_0^+$ is normalizable, say $\psi_0^-$. In this case $H_-$ has an eigenvalue zero. We saw earlier that if $H_-$ has an eigenstate with a positive eigenvalue then $H_+$ also has the same eigenvalue and vice versa. Since the lowest eigenvalue for $H_-$ is zero, then the lowest eigenvalue of $H_+$ is the first positive eigenvalue of $H_-$. Moreover, the operators $A$ and $A^\dagger$ connect the eigenstates of $H_-$ and $H_+$ except for the ground state of $H_-$. This can be summarized in the following equations

\begin{align}
H_- \psi_n^- &= E_n^- \psi_n^- & n = 0, 1, 2, ... \\
H_+ \psi_n^+ &= E_n^+ \psi_n^+ & n = 0, 1, 2, ... \\
E_{n+1}^- &= E_n^+ & n = 0, 1, 2, ... \quad \text{and} \quad E_0^- = 0 \\
\psi_n^+ &= \frac{1}{\sqrt{E_{n+1}^-}} A \psi_{n+1}^- & n = 0, 1, 2, ... \\
\psi_{n+1}^- &= \frac{1}{\sqrt{E_n^+}} A^\dagger \psi_n^+ & n = 0, 1, 2, ... \quad (18e)
\end{align}

We notice here that the operators $A$ and $A^\dagger$ annihilate and create nodes in wave functions, respectively, and this is a result of unbroken supersymmetry. They are not ladder operators though, because ladder operators move up and down the states of the same Hilbert space in contrast to these operators which shift us between different Hilbert spaces, for this reason some authors call them shift operators \[10\].

All the development so far depended on having a superpotential $w(x)$ which generates all the previous results. But the unbroken supersymmetric structure which has been considered so far can be constructed for any one dimensional quantum system with at least one bound state, i.e.

**Every one-dimensional potential with bound states admits SUSY:**

Given a one dimensional quantum system with at least one bound state one can find a partner Hamiltonian which has exactly the same discrete spectrum except for the ground state energy of $H_-$. \[11, 12\].

Let us consider having a one dimensional potential $V_-$ which admits bound state wave functions $\psi_n^-$, and the ground state energy is $E_0^-$, its Hamiltonian $H_-$ given by

\[ H_- = -\frac{d^2}{dx^2} + V_-(x) - E_0^- \quad (19) \]

can be factorized and be written as

\[ H_- = A^\dagger A \quad (20) \]
where $A$ and $A^\dagger$ are defined as in (13a) and (13b), respectively. The superpotential is

$$w(x) = -\frac{d}{dx} \ln \psi^\pm_0$$  \hspace{1cm} (21)

where $\psi^\pm_0$ is the ground state wave function. The ground state energy of $H_-$ is zero, and we can construct the supersymmetric partner Hamiltonian $H_+$

$$H_+ = AA^\dagger$$  \hspace{1cm} (22)

All the results of (18) are also true for our constructed partner Hamiltonians. This show us how to construct a supersymmetric system from a one dimensional potential. In fact this technique can be repeated again and again to obtain a sequence of Hamiltonians where all the adjacent pairs of Hamiltonians are supersymmetric partners, the cardinality of this sequence is the same as the dimensionality of the Hilbert space of $H_-$, which is the number of bound states it possess [5, 11, 13].

2. Shape Invariance

We saw that supersymmetry allows us to construct and solve a hierarchy of Hamiltonians provided that we know the solution to one of the Hamiltonians in the sequence. The question now is, Can supersymmetry help us solve a supersymmetric system? The answer to this question was provided in 1983 by Gendenshtein [14]. Having supersymmetry is not a sufficient condition for the system to be exactly solvable, because as we saw we can construct a supersymmetric Hamiltonian for any potential with bound state(s). Gendenshtein discovered another symmetry which if the supersymmetric system satisfies it will be an exactly solvable system, this symmetry is known as shape invariance. If our potential satisfies shape invariance we can readily write down its bound state spectrum, and with the help of the charge operators we can find the bound state wave functions [7, 14, 15]. It turned out that all the potentials which were known to be exactly solvable until then have the shape invariance symmetry.

If the supersymmetric partner potentials have the same dependence on $x$ but differ in a parameter, in such a way that they are related to each other by a change of of that parameter, then they are said to be shape invariant. Gendenshtein stated this condition in this way,

$$V_+(x, a_0) = V_-(x, a_1) + R(a_1)$$  \hspace{1cm} (23)

where $a_0$ is a parameter in our original potential whose ground state energy is zero. $a_1 = f(a_0)$ where $f$ is assumed to be an arbitrary function for the time being, and $a_n = f^n(a_0)$ where $f^n$ is the composition of $f$ with itself $n$ times. The remainder $R(a_1)$ can be dependent on the parametrization variable $a_0$ but never on $x$. In this case $V_-$ is said to be shape invariant, and we can readily find its spectrum, take a look at $H^n$ and $H^{n+1}$,

$$H^n = -\frac{d^2}{dx^2} + V_-(x, a_n) + \sum_{k=1}^{n} R(a_k)$$  \hspace{1cm} (24a)
\[ H^{n+1} = -\frac{d^2}{dx^2} + V_-(x, a_{n+1}) + \sum_{k=1}^{n+1} R(a_k) \]  
\[ = -\frac{d^2}{dx^2} + V_+(x, a_n) + \sum_{k=1}^{n} R(a_k) \]  
(24b)

where in the last step we applied (23). Here we see that \( H^n \) and \( H^{n+1} \) are supersymmetric, so they have identical spectrum except for the ground state energy of \( H^n \) which is,

\[ E_0^n = \sum_{k=1}^{n} R(a_k) \]

but from the supersymmetry arguments discussed in section (11) we know that \( H^n \) has the same spectrum of \( H_- \equiv H^0 \) except for the first \( n \) levels of \( H_- \) which are missing from \( H^n \), so the ground state energy of the \( n \)th Hamiltonian \( H^n \) is equal to the \( n \)th energy \( E_0^- \) of our original Hamiltonian \( H_- \). So, the spectrum of \( H_- \) is

\[ E^-_n = \sum_{k=1}^{n} R(a_k), \quad E^-_0 = 0 \]  
(25)

Supersymmetry together with shape invariance give us a formal expression for the wave functions \( \psi^-_n \) of \( H_- \), for if we know the ground state wave function \( \psi_0^- (x, a_0) \) of \( H_-(x, a_0) \), then the ground state wave function of \( H^n(x, a_n) \) is \( \psi_0^- (x, a_n) \), but the operator \( A^\dagger(x, a_{n-1}) \) move us from the Hilbert space of \( H^n \) to the Hilbert space of \( H^{n-1} \), thus we can apply this operator repetitively till we reach the Hilbert space of \( H_- = H^0 \), we need to remember here that \( A^\dagger \) creates a node in the wave function each time we apply it, so applying it \( n \) times we will reach \( \psi^-_n (x, a_0) \), hence

\[ \psi^-_n (x, a_0) = A^\dagger(x, a_0)A^\dagger(x, a_1) \ldots A^\dagger(x, a_{n-2})A^\dagger(x, a_{n-1})\psi_0^- (x, a_n) \]  
(26)

where \( A^\dagger(x, a_n) \) is defined as

\[ A^\dagger(x, a_n) = \frac{d}{dx} + w(x, a_n) = \frac{d}{dx} - \frac{d}{dx}\ln \psi_0^- (x, a_n) \]  
(27)

This formal expression for the wave functions of shape invariant potentials were first presented in [16], the explicit expressions for the wave functions of all the well known shape invariant wave potentials have been worked out in [17]. Supersymmetry together with shape invariance can also be exploited to obtain the scattering matrices [18, 19].

Different types of shape invariance have been studied in the literature, they are classified depending on the relation among the parameters which creates the shape invariant sequence of potentials. The first and the most dominant type is known as translational shape invariance, where the potential parameter is shifted, \( a_1 \equiv f(a_0) = a_0 + \alpha \). This is the most well studied type of shape invariance, and many of the
generalizations of the shape invariance concept have been worked specifically for this type. In fact it turned out that all of the textbook examples of exactly solvable nonrelativistic one dimensional shape invariant quantum systems have this type of shape invariance. Complete lists of translational shape invariant potentials together with their wave functions and scattering matrices have been prepared \[7, 15, 17, 18\].

Translational shape invariance was the only known shape invariance type for a long time, and some authors hypothesized that it is a necessary condition for shape invariance \[20\]. In 1993, as new types of shape invariance emerged, it was clear that translational shape invariance is not the only type. The first of the new types to emerge is the scaling shape invariance \[21, 22\], \(a_1 = qa_0\), \(0 < q < 1\). There are also two nonlinear transformations which have been introduced together with the previous one in \[22\], \(a_1 = qa_0^p\), \(p \in \mathbb{Z}\), \(0 < q < 1\), and \(a_1 = qa_0/(1 + pa_0)\), \(0 < q,p < 1\), \(pa_0 \ll 1\). The eigenvalues, eigenfunction functions and transmission coefficients have been obtained algebraically for potentials with such shape invariance \[21, 22\].

Cyclic shape invariance has been introduced in the following years \[23, 24\]. In this type the parameters repeat themselves after cycle of \(p\) elements, \(a_p = a_0\).

In all the types, except for the translational shape invariance, the potentials are not obtained in closed form, i.e in terms of elementary functions, they are obtained in series form.

Another important generalization of the ordinary shape invariance relation \(23\) is to have shape invariance in multi-steps.

Shape invariance has also been generalized in another direction. In 1987, in an attempt to classify shape invariant potentials, Cooper et al. \[25\] introduced the idea of translational shape invariance in an “\(n\)” arbitrary but finite number of parameters, they were not successful at finding a solution for such a system, they were even pessimistic about the existence of a solution. It was an open problem for more than a decade, later on, by skillfully exploiting some properties of the Ricatti equation a solution was shown to exist and worked out \[26\].

Many successful efforts have been put to show the underlying algebraic structure of the shape invariance symmetry, the associated Lie algebras have also been identified \[27\], and in the case of the non translational shape invariance nonlinear generalizations of Lie algebras have been obtained \[28, 29\].

The technique of factorizing the Hamiltonian and using shape invariance to solve the schrödinger equation is not at all new, Schrödinger himself used this technique to solve the hydrogen atom, Dirac also used it to solve the harmonic oscillator. Of course the ideas of supersymmetry and the supergroups were not known back then.

Later on, the technique of Schrödinger and Dirac was rejuvenated by Infeld and Hull and later they summarized their work in their infamous review \[30\] were they gave the name for the method, “The Factorization Method”. They summarized the procedure of this technique to solve second order differential equations, they studied its range of applicability and classified the factorization types into six types (transformation between the types exist), they also prepared a table for these six possible factorizations
which can be used to solve the differential equations simply by identifying to which
factorization type it belongs [30]. For the historical progress of this method see [31].

Every Infeld and Hull factorizable potential is shape invariant but the converse is not true:
As far as solving differential equations is concerned, the factorization technique offers
almost the same power as the supersymmetry with translational shape invariance approach. The supersymmetry and shape invariance offer physical insight and more group theoretic ideas of why such systems are exactly solvable. In fact shape invariance, in
general, offers more than the factorization method, since the factorization method treats
only the translational shape invariance. The equivalence of the factorization method to
supersymmetry with translational shape invariance is easy to see, equation (3.1.2) in
[30] is exactly equivalent to (23) with translation of the shape invariance parameter
[12, 32].

The table prepared by Infeld and Hull, though complete for most purposes, is not
the most general table possible, as they did not consider the most general solution of
Ricatti equations. The most general solution is worked out in [32].

Shape invariance is a sufficient but not a necessary condition for exact solubility:
Gendenshtein suggested that all exactly solvable potentials must be shape invariant
[14], but now the relation among exact solubility, shape invariance and supersymmetry
is much clearer. One counter example is enough to refute Gendenshtein suggestions,
and many such examples have been constructed [13 25, 33], the most important of
these counter examples is Natanzon class of potentials which are, in general, not shape
invariant [25, 33].

3. Conclusion
In this letter we defined, supersymmetric quantum mechanics, and shape invarinace. We
also showed the relation which connects them to exact solubility and the factorization
method. Fig.1 summarizes these relations.

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Figure 1. A Venn diagram showing the relation among, supersymmetric, exactly solvable, shape invariant, and Infeld and Hull factorizable potentials.
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