Instability of QBC systems to Topological Anderson Insulating phases

Nicolau Sobrosa$^1$, Miguel Gonçalves$^2$, Eduardo V. Castro$^{1,3}$

$^1$Centro de Física das Universidades do Minho e Porto, Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal

$^2$CeFEMA, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal and Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal

$^3$Beijing Computational Science Research Center, Beijing 100084, China

Here we study the instabilities of a quadratic band crossing system to Chern insulating states and uncorrelated disorder. We determine the phase diagram in the plane of topological mass versus disorder strength, characterizing the system with respect to spectral, localization and topological properties. In the clean limit, the system has two gapped Chern insulating phases with Chern numbers $C = \pm 2$, and a trivial phase with $C = 0$. For finite disorder, the quadratic band crossing points are unstable to emergent gapless Chern insulating phases with $C = \pm 1$, not present in the clean limit. These phases occupy a considerable region of the phase diagram for intermediate disorder and show features of topological Anderson insulators: it is possible to reach them through disorder-driven transitions from trivial phases.

I. INTRODUCTION

Topological insulators are a remarkable state of electronic matter. They show quantized responses that are proportional to topological invariants and, as a consequence, typically very robust to perturbations and system's details [1–4]. Topological band insulators, as paradigmatic examples of systems with non-trivial topology, have been extensively studied and are fairly well understood [3]. Non-trivial topology, however, also manifests in systems with broken translational invariance that are not described by topological band theory. Among these systems, disordered topological insulators are a popular sub-group [5].

Topological phases are robust to disorder regarding that no symmetry protecting the topological properties is broken [5, 6]. In quantum Hall insulators, disorder even plays a fundamental role for the observation of a quantized Hall conductance. More generally, it is now well established that in the case of Chern insulators, where time-reversal symmetry is broken, uncorrelated disorder localizes every eigenstate except at specific energies $\frac{\pi}{2}$ [7–12]. The extended eigenstates at these energies carry finite Chern numbers and are therefore responsible for a quantized Hall response as long as the Fermi level lies between them (the localized states cannot change this response). Topological phase transitions in disordered Chern insulators occur when the extended states merge and annihilate at the Fermi level (through the so-called “levitation and annihilation” mechanism), becoming localized [10].

The localization properties of non-interacting topological systems can be understood within a low energy description in terms of random, massive Dirac Hamiltonians [13]. Generic phase diagrams in the plane of Dirac mass versus disorder strength have been obtained for all ten symmetry classes from the tenfold way [14]. For class A, to which Chern insulators belong, the phase diagram consists of multiple localized phases which can be distinguished by their Chern number and are separated by phase boundaries at which the localization length diverges. However, not all Chern insulators can be derived from massive Dirac Hamiltonians at low energies. A well known example are quadratic band crossing (QBC) systems.

Systems with quadratic band crossings in two-dimensions (2D) are very interesting because, contrary to conventional band degeneracy points, they are associated with a finite Berry phase of $\pm 2\pi$. Due to the finite density of states at the QBC, these systems are unstable to interactions [15]: originating nematic phases with two Dirac cones, each carrying half of the QBC's Berry phase; or gap openings that may give rise to topological insulating phases precisely due to the non-trivial Berry phase of the QBC [16–19]. The fate of interaction induced topological insulating phases in the presence of disorder has been examined in Refs. [20, 21] within the one-loop RG approach. The suppression of topological phases under increasing disorder and possible transition to trivial phases was predicted.

Quite surprisingly, there are systems which show topological phase transitions from trivial to topological phases with increasing disorder. This disorder-driven topological phase is now known as topological Anderson insulator (TAI) [22, 23]. This phase has been observed in many models, including paradigmatic models of topological insulators in 2D such as the Kane-Mele model [24–25] and the disordered Haldane model [26–28], in the 1D SSH model [29], in quasiperiodic systems [30, 31], and more recently in non-Hermitian models [32]. Whether TAI phases may be realized for Chern insulators derived from QBC systems is still an open question. Moreover, the fate of the QBC itself in the presence of disorder has received little attention.
In this work we study the interplay between the instability of a 2D QBC system to a Chern insulating state and disorder of the Anderson type. The full phase diagram in the plane of the gap opening coupling parameter $B$ and disorder strength $W$ is shown in Fig. 1. For null disorder, QBC points (QBCP) occur for $B = \pm 2, 0$, while the system is a gapped trivial insulator or a Chern insulator with Chern number $C = \pm 2$ for any other $B$ value. These phases are also present for finite disorder, but new gapless Chern insulating phases also emerge. Besides gapless and gapped phases with $C = \pm 2$, new gapless topological phases with $C = \pm 1$, not present in the clean limit, arise. In fact, the most important result of our work is that the QBCP are unstable to the formation of these phases, for any finite disorder. Finally, the existence of TAI phenomena is also clear: it is possible to undergo a transition between the trivial phase and the disorder-induced topological phases with $C = \pm 1$ by increasing disorder.

The paper is organized as follows: In Sec. II we introduce the tight-binding model used to describe the electronic properties of the disordered QBC system and the methods to analyze its properties. The topological, spectral, and localization properties are discussed in Sec. III. A thorough discussion of the obtained results is given in Sec. IV. In Sec. V the key results are summarized and some conclusions are drawn. We also include four appendices: in Appendix A we provide the phase diagram for binary disorder; in Appendix B we discuss the criterion used to distinguish gapped and gapless regimes; an example of a possible k-dependent self-energy that would lead to a $C = 1$ phase in the self consistent Born approximation is given in Appendix C; the robustness of the obtained phase diagram when the QBC is split into two Dirac cones is discussed in Appendix D.

II. MODEL AND METHODS

We study a QBC system realized on the square lattice with two orbitals per site. The model considers first-neighbor hoppings between the same orbitals and second-neighbor hoppings coupling different orbitals. The Hamiltonian for the disorder-free model can be written in the reciprocal space as

$$H_0 = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \mathcal{H}_k \Psi_{\mathbf{k}},$$

where $\Psi_{\mathbf{k}}^\dagger = (c_{\mathbf{k}1}^\dagger, c_{\mathbf{k}2}^\dagger)$ is the two component spinor in the space of the two orbitals, where $c_{\mathbf{k}1}$ creates an electron with Bloch momentum $\mathbf{k}$ in orbital $\alpha$, and

$$\mathcal{H}_k = \mathbf{h} \cdot \sigma,$$

with $\sigma$ the Pauli vector and $\mathbf{h}$ the vector given by:

$$h_x = 2t_x \sin k_x \sin k_y,$$

$$h_y = 0,$$

$$h_z = 2t_z (\cos k_x - \cos k_y).$$

In the following, we set $t_x = t_z = t$ and $t$ to unity.

This model has two QBCPs: at $\Gamma = (0,0)$ and $M = (\pm \pi, \pm \pi)$. By adding a finite $h_y$, it is possible to open a gap. For a constant $h_y$, the system is a trivial insulator. Similarly to the Haldane Model, we may add a $k$-dependent component which allows to tune independently the gap at each QBCP. In the following we use a simple choice which depends on a single parameter $B$:

$$h_y = 1 + \frac{B + 1}{2} (\cos k_x + \cos k_y).$$

This type of $k$-dependence implies the breaking of time-reversal symmetry since, as seen from Eq. (2), we have $\mathcal{H}_k \neq \mathcal{H}_{-k}$. As shown below, the system will become a Chern insulator for some intervals of $B$ values.

In real-space, the constant term corresponds to an intracell complex hopping between different orbitals and the term with $k$-modulation to a nearest neighbor complex hopping between different orbitals. As the modification in Eq. (4) does not change $h_x$ and $h_z$, a QBCP still exists when $h_y = 0$ as seen before. For $B = 0$, there is a QBCP at $M$ and for $B = -2$, there is a QBCP at $\Gamma$.

Adding the disorder potential, the Hamiltonian reads:

![Diagram](image-url)
\[ H = H_0 + \sum_i \sum_{\alpha=1,2} \xi_{i\alpha} \hat{c}_{i\alpha} \hat{c}_{i\alpha}^\dagger, \tag{5} \]

where \( \xi_{i\alpha} \) are site-dependent potentials that follow the uniform distribution (Anderson disorder),

\[ P_W(\xi_{i\alpha}) = \frac{1}{W} \Theta \left( |\xi_{i\alpha}| - \frac{W}{2} \right), \tag{6} \]

and \( W \) defines the disorder strength. In Appendix A, we present the phase diagram obtained for the case of binary disorder. The result is very similar to that of Fig. 1 obtained for Anderson disorder, and the conclusions are qualitatively the same.

We carried out a complete study of the phase diagram of the model in Eq. (5), characterizing topological, spectral, and localization properties. The density of states (DOS) was calculated for finite systems containing more than \( 10^6 \) sites using the Kite [34] quantum transport software that has a very efficient implementation of the kernel polynomial method (KPM). The topological phase diagram was obtained by computing the Chern number through the Coupling Matrix Method introduced in Refs. [35] and [36]. We note that, even though the Chern number computed for each disorder realization is an integer within numerical accuracy, the averaged Chern number may change continuously and present non-quantized values due to finite size effects. The localization properties were characterized through the Transfer Matrix Method (TMM) [37–39], that also allowed to cross-check the Chern number results. This method considers a finite system with a large longitudinal length \( L \) and a transverse width \( M \) which we varied in order to find the localization length for a given \( M \), \( \lambda_M \). The localization properties were studied through the normalized localization length, \( \lambda_M = \lambda_M/M \), in the following way: if \( \lambda_M \) decreases with \( M \) the states are localized in the thermodynamic limit which corresponds to an insulating behaviour; on the other hand, if \( \lambda_M \) increases with \( M \) the states are extended; a constant \( \lambda_M \) is characteristic of critical states that appear at transition points between different phases. We choose \( L \) so that \( \lambda_M \) is calculated with an error below 1%. We note that the behaviour of \( \lambda_M \) can capture topological phase transitions at finite disorder. As mentioned in the introduction, the spectrum of finite-disorder Chern insulators consists of localized states, except at specific energies where critical states live. Topological phase transitions occur when these states cross (and merge at) the Fermi level. Therefore, for a disordered Chern insulator, \( \lambda_M \) should always decrease with \( M \) except at the topological phase transitions, where it becomes \( M \)-independent.

\[ \lambda_M = \lambda_M/M, \]

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C

c was more than 5% away from its integer value. 

The error bars that appear in Fig. 1 are determined through the analysis of curves as the ones shown in Fig. 2: the length of the error bars indicates the range over which the Chern number is expected to disappear in the thermodynamic limit, where transitions between different Chern numbers should become sharp. The error in the transition from gapped to gapless regions corresponds to a variation of ±25% of \( \rho_{\text{cut}} \). This error corresponds to the thickness of the gapped-gapless transition lines in Fig. 1. The small error shows that variations in the criterion do not affect significantly the results, especially in the regions of the topological phase transitions.

C. Localization properties

We studied the normalized localization length \( \Lambda_M \) obtained through the TMM, at the Fermi level (\( E = 0 \)). We recall that in Chern insulating systems, we expect \( \Lambda_M \) to always decrease with \( M \), except at specific points. This means that, as expected, all states at the Fermi level are localized except when a topological phase transition occurs accompanied by the merging of critical states carrying opposite Chern numbers. Example results are shown in Fig. 6. The results in Fig. 6(a) are for \( W = 5 \) and a range of \( B \) containing the \( C = 2 \) to \( C = 1 \) and \( C = 1 \) to \( C = 0 \) transitions. Fig. 6(b) corresponds to a cut at \( W = 8 \), for a range of \( B \) containing all the topological phase transitions in the system. The phase transition points were considered to be those with constant \( \Lambda_M \). The uncertainty in these points was considered to be the range of \( B \) over which there is no clear decrease of \( \Lambda_M \) with \( M \). This uncertainty is represented in Fig. 7 by the shaded red bars, being perfectly bounded by the error bars in the Chern number.

B. Gapped/Gapless regions

In order to study the existence of a spectral gap at the Fermi level (\( E = 0 \) at half-filling) we computed the DOS using KPM as implemented in Kite [34]. In Fig. 1 we present the gapped/gapless transitions with thicker lines, and shaded gapped regions in grey. Examples of the DOS are shown in Fig. 4 for systems with parameters corresponding to the points marked with A, B, and C in the phase diagram of Fig. 1. As seen in the inset, system A is clearly gapped, with zero DOS at and around \( E = 0 \). System C is gapless, as it presents a finite DOS at and around \( E = 0 \). Although system B also presents a finite DOS, its value at \( E = 0 \) is small. In such cases, we used a criterion to distinguish gapless from gapped systems, as explained next.

The system was considered gapped when the DOS at the Fermi Level was below a certain threshold, \( \rho_{\text{cut}} \), that was determined by exact diagonalization of the Hamiltonian in Eq. (5). In Appendix B we detail the analysis we performed and show the results that support our choice. The error in the transition from gapped to gapless regions corresponds to a variation of ±25% of \( \rho_{\text{cut}} \). This error corresponds to the thickness of the gapped-gapless transition lines in Fig. 1. The small error shows that variations in the criterion do not affect significantly the results, especially in the regions of the topological phase transitions.

To obtain the transition lines, the disorder strength \( W \) was fixed at some value and the gap opening parameter \( B \) was varied continuously. For each disorder strength we performed an average over 120 different disorder configurations in a lattice with 35 \( \times \) 35 sites. Some examples of the obtained curves are represented in Fig. 3, including the values for the clean limit. The continuous variation of the averaged Chern number is expected to disappear in the thermodynamic limit, where transitions between different Chern numbers should become sharp. The error bars that appear in Fig. 1 are determined through the analysis of curves as the ones shown in Fig. 2: the length of the bar indicates the range over which the Chern number was more than 5% away from its integer value.

From the phase diagram in Fig. 1 it is not clear if the finite-disorder Chern insulating phases with \( C = \pm 1 \) exist for any infinitesimal disorder. In order to shed light on whether this is the case, we fixed \( B = 0 \) (QBC system in the clean limit) and computed the Chern number as a function of \( W \) for different system sizes. The results are shown in Fig. 3: where it is clear that the \( C = -1 \) phase occurs for disorder strengths that approach \( W = 0 \) as the system size is increased. In the inset of Fig. 3 we plot, for fixed \( C \) values in the apparent transition region, the corresponding disorder strength \( W_N \) for each size \( N \). The fits clearly indicate that the transition should occur discontinuously at \( W = 0 \) in the thermodynamic limit. Although we are limited numerically to reach larger system sizes, these results support the conjecture that in the thermodynamic limit the \( B = 0 \) QBCP is unstable to the formation of the \( C = -1 \) phase for any infinitesimal disorder. The same is expected for the QBCP at \( B = -2 \), as suggested by the symmetry of the phase diagram around \( B = -1 \).
M and disorder dashed lines indicate the critical points. The gap opening parameter $B$ explored models: the existence of new disorder-driven topological insulators such as the existence of robust finite-disorder gapped disordered topological insulators, however, there is an important difference from the conventional TAI in the present case: the $C = \pm 1$ TAI phases have a Chern number that does not exist in any zero-disorder topological phase of the model. Therefore, these TAI phases do not evolve smoothly from the clean-limit phases as disorder is increased, in contrast to conventional TAI.

For conventional TAs, a self-consistent, low-order Born approximation is usually enough to capture the topological phase transition. In the present case, since the $C = \pm 1$ phases are not present at zero-disorder, a perturbative approach is not well justified. Moreover, within the Born approximation a $k$-independent self-energy is obtained, which translates into a renormalization of the parameters of the original model. For the Hamiltonian in Eq. (2), no $k$-independent self-energy of the general form $\Sigma = \Sigma(k)$, with $\Sigma = (\Sigma_x, \Sigma_y, \Sigma_z)$, is able to induce a $C = \pm 1$ phase. Nevertheless, a $k$-dependent self-energy $\Sigma(k) = \Sigma(k) \cdot \sigma$ may give rise to the $C = \pm 1$ phase, as shown in the Appendix C for a particular example with $\Sigma(k) = (\Sigma_x(k), 0, 0)$. In Appendix C we even show that the clean limit model with the proposed $k$-dependent self-energy is adiabatically connected to the disorder induced $C = \pm 1$ TAI we have found for our QBC system. However, for the uncorrelated disorder used here, no $k$-dependent self-energy is allowed which invalidates such an approach.

A possible variation of the QBC system is to split the QBC system into two Dirac cones with a suitable perturbation (see Appendix D). Even in this case, a finite disorder gives rise to the $C = \pm 1$ phases which could be an argument against the importance of the QBCP for their existence. Nonetheless, the topological information carried by the split Dirac cones and the QBCP is the same since no gap is opened in the splitting process. The topological properties of the new system can then be traced back to the possibility of creating a QBCP without closing the gap.

Finally, the results obtained here are not restricted to disorder of the Anderson type. We also obtained the phase diagram for binary disorder, being qualitatively similar to the phase diagram for Anderson disorder (see App. A). This indicates that our conclusion on the instability of QBCP to TAI phases is robust to model details.

### IV. DISCUSSION

The phase diagram of the QBC system here unveiled shows many features that are characteristic of well-known disordered topological insulators such as the existence of robust finite-disorder gapped and gapless topological insulating phases. However, there is a feature that distinguishes it from the previously explored models: the existence of new disorder-driven topological phases with $C = \pm 1$, absent in the clean limit.

Particularly interesting is the plateaus transitions $C = \pm 1 \rightarrow \pm 0$ at finite disorder strength as the topological gap parameter $B$ is changed shown in Fig. 2. This shows that an equivalent plateaus sequence is possible with increasing disorder. This possibility is conjectured to be ruled out in the quantum Hall systems and other Chern insulators derived from Dirac (linear band crossings) Hamiltonians. In those systems, starting with $|C| \geq 2$, a plateaus transition $\Delta C = \pm 1$ is never observed with increasing disorder due to ensemble averaging over disorder realizations.

Our results suggest that the existence of a QBCP is a key ingredient for the formation of these phases. In particular, the finite-size scaling analysis in Fig. 3 is a strong indication that they are instabilities of the QBCPs: in the thermodynamic limit, any infinitesimal amount of disorder should drive the QBC system to one of these phases. Furthermore, from the phase diagram in Fig. 1 it is clear that for low disorder the new topological phases are located around the clean-limit QBCP.

The new phases with $C = \pm 1$ are also TAI since they can be reached by increasing disorder from a trivial phase. TAI phenomena were by now observed in a multitude of disordered topological insulators. However, there is an important difference from the conventional TAI in the present case: the $C = \pm 1$ TAI phases have a Chern number that does not exist in any zero-disorder topological phase of the model. Therefore, these TAI phases do not evolve smoothly from the clean-limit phases as disorder is increased, in contrast to conventional TAI.

For conventional TAs, a self-consistent, low-order Born approximation is usually enough to capture the topological phase transition. In the present case, since the $C = \pm 1$ phases are not present at zero-disorder, a perturbative approach is not well justified. Moreover, within the Born approximation a $k$-independent self-energy is obtained, which translates into a renormalization of the parameters of the original model. For the Hamiltonian in Eq. (2), no $k$-independent self-energy of the general form $\Sigma = \Sigma(k) \cdot \sigma$, with $\Sigma = (\Sigma_x, \Sigma_y, \Sigma_z)$, is able to induce a $C = \pm 1$ phase. Nevertheless, a $k$-dependent self-energy $\Sigma(k) = \Sigma(k) \cdot \sigma$ may give rise to the $C = \pm 1$ phase, as shown in the Appendix C for a particular example with $\Sigma(k) = (\Sigma_x(k), 0, 0)$. In Appendix C we even show that the clean limit model with the proposed $k$-dependent self-energy is adiabatically connected to the disorder induced $C = \pm 1$ TAI we have found for our QBC system. However, for the uncorrelated disorder used here, no $k$-dependent self-energy is allowed which invalidates such an approach.

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Finally, the results obtained here are not restricted to disorder of the Anderson type. We also obtained the phase diagram for binary disorder, being qualitatively similar to the phase diagram for Anderson disorder (see App. A). This indicates that our conclusion on the instability of QBCP to TAI phases is robust to model details.

### V. CONCLUSIONS

We have studied a model of a QBC system under gap-opening and disorder-inducing couplings. A complete

| Chart 5: Normalized localization length $\Lambda_M$ as a function of the gap opening parameter $B$, for systems with varying width $M$ and disorder $W = 5$ (a) and $W = 8$ (b). The vertical dashed lines indicate the critical points. |
spectral, topological and localization analysis was carried out in order to obtain a detailed phase diagram. We not only found that the topological phases existing in the clean limit were robust to disorder but also that new gapless topological phases were formed. Most importantly, we found a new instability of the QBCPs: a disorder-induced instability to gapless topological phases with Chern numbers \( C = \pm 1 \), that are absent in the clean limit. The possibility of emulating quantum Hamiltonians using ultracold gases of atoms in an optical lattices [50, 51] opens interesting prospects to realize the observed phases experimentally. In particular, the ability to realize disordered or quasiperiodic potentials into the system to induce localization phenomena has been recently achieved [52, 53].

An interesting question for future work is whether instabilities of the QBCPs to electron-electron interactions can give rise to topological phases with similar properties to the gapless topological insulators here uncovered. The full phase diagram capturing the interplay between disorder and interactions would then be a natural follow-up.

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Appendix A: Qualitative Phase Diagram for Binary Disorder

In this Appendix, we present the phase diagram for binary disorder. In this case, the on-site potentials \( \epsilon_i \) are randomly generated according to the following distribution:

\[
P_V(\epsilon_i) = \frac{1}{2} (\delta(\epsilon_i) + \delta(V - \epsilon_i)).
\]

The Chern number results are shown in Fig. 6. Qualitatively, the phase diagram is very similar to the one obtained for Anderson disorder in Fig. 1. In particular, the disorder-induced \( C = \pm 1 \) topological phases are still present and the reentrant TAI behaviour is also observed. However, a smaller degree of disorder is needed to destroy all the non-trivial phases. Besides this, for large \( B \) (absolute value), the \( C = \pm 1 \) phases are much narrower with binary disorder which means that they are more robust to Anderson disorder.

In this Appendix, we present the phase diagram for binary disorder. All points were averaged over 200 disorder configurations and systems of size \( 15 \times 15 \) were used.

Appendix B: Gapped/gapless regions

In this Appendix, we present details on the criterion used in the main text to distinguish gapped and gapless regimes.

Due to the finite resolution of KPM, it is typically challenging to find the transition point between gapped and gapless regimes. In particular, the DOS obtained through KPM may have a finite spectral weight at energies within gaps if the system size and the number of polynomials are not large enough. We therefore use finite-size scaling results from exact diagonalization to find a suitable \( \rho_{\text{cut}} \) below which the KPM DOS should be considered null. For the system to be gapless, the energy difference \( \Delta E \) between the two states closest to the Fermi level, immediately above and below, must converge to zero in the thermodynamic limit, \( \Delta E \to 0 \). To define \( \rho_{\text{cut}} \), we set the parameter \( B = -1 \). Results for \( \Delta E \) as a function of inverse system size \( 1/N \) are shown in Fig. 7. After fitting the finite-size scaling results to a cubic function and extrapolating to \( N \to \infty \), we observe that for a disorder around \( W \approx 4.2 \) the system must be gapless. It was, then, just a matter of computing the DOS with Kite [34] for \( B = -1 \) with that critical disorder and then fix the \( \rho(E = 0) \) obtained with Kite to be \( \rho_{\text{cut}} \).

Appendix C: Self Consistent Born Approximation

Here we provide an example of a possible \( k \)-dependent self-energy term that leads to a \( C = 1 \) phase:

\[
\Sigma_x(k) = d_x \sin(k_x + k_y).
\]  

(C1)

Results for the topological phases as a function of \( B \) and \( d_x \) are found in Fig. 8. To see that these phases are adiabatically connected to the \( C = \pm 1 \) disorder-induced phases we compute the
FIG. 7. Energy difference \( \Delta E \) between the two states closest to the Fermi level, directly above and below, computed using exact diagonalization for systems with different disorder strength \( W \) and varying number of sites \( N \).

FIG. 8. Example of a perturbation that leads to a C=1 phase without disorder.

Chern number in a parameter path \( \mathcal{P}_1 \) given by the pair \((d_x, W)\):

\[
\mathcal{P}_1 : (d_x, W) = \begin{cases} 
(0.7, 12\lambda), & 0 < \lambda \leq \frac{1}{7} \\
(1.4(1 - \lambda), 6), & \frac{1}{7} < \lambda \leq 1,
\end{cases}
\]

where we have fixed \( B \) to \( B = 0 \). For \( \lambda = 0 \), the model has the self-energy given in Eq. (C1) and is in the \( C = -1 \) phase of the phase diagram shown in Fig. 8. When \( \lambda = 1 \), we have \( d_x = 0 \) which is a point of the phase diagram in Fig. 1 with \( C = -1 \). It can be seen in Fig. 9 that the system has a \( C = -1 \) for all points along the \( \mathcal{P}_1 \) path. The \( \mathcal{P}_2 \) path, which is defined as

\[
\mathcal{P}_2 : (d_x, W) = (0, 6\lambda), \quad 0 < \lambda \leq 1,
\]

shows that the \( C = -2 \) phase may then be reached by decreasing disorder \( (\lambda = 1 \to 0) \), in total agreement with the phase diagram of Fig. 1.

Appendix D: Evolution of Phase Diagram with \( B_x \) constant perturbation

To study the robustness of the QBCP we introduce the term \( b \cdot \sigma \) in the Hamiltonian written in the reciprocal space where \( b \) can be a constant vector with three components, \( b = (b_x, b_y, b_z) \). The \( b_x \) perturbation lifts the degeneracy of the QBCP, splitting it into two Dirac Cones. In fig. 10, we show the value of the Chern number in the plane \( B \) vs \( b_x \) with increasing disorder. For a small \( b_x \) the phase diagram of the fig. 2 is practically unchanged. It is also clear from the figure that the \( C = \pm 1 \) phases exist even when \( b_x \neq 0 \).

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FIG. 10. Evolution of phase diagram in the $B - b_x$ plane. The black lines correspond to the phase transition for null disorder corresponding to the points where there are Dirac Cones or QBCPs in the spectrum. All points were averaged with 100 disorder configurations for systems of size $15 \times 15$.

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