Nonextensive aspects of small-world networks

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Abstract

Nonextensive aspects of the degree distribution in Watts-Strogatz (WS) small-world networks, $P_{SW}(k)$, have been discussed in terms of a generalized Gaussian (referred to as $Q$-Gaussian) which is derived by the three approaches: the maximum-entropy method (MEM), stochastic differential equation (SDE), and hidden-variable distribution (HVD). In MEM, the degree distribution $P_Q(k)$ in complex networks has been obtained from $Q$-Gaussian by maximizing the nonextensive information entropy with constraints on averages of $k$ and $k^2$ in addition to the normalization condition. In SDE, $Q$-Gaussian is derived from Langevin equations subject to additive and multiplicative noises. In HVD, $Q$-Gaussian is made by a superposition of Gaussians for random networks with fluctuating variances, in analogy to superstatistics. Interestingly, a single $P_Q(k)$ may describe, with an accuracy of $|P_{SW}(k) - P_Q(k)| \lesssim 10^{-2}$, main parts of degree distributions of SW networks, within which about 96-99 percent of all $k$ states are included. It has been demonstrated that the overall behavior of $P_{SW}(k)$ including its tails may be well accounted for if the $k$-dependence is incorporated into the entropic index in MEM, which is realized in microscopic Langevin equations with generalized multiplicative noises.

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1 INTRODUCTION

In the last several years, there is an increased interest in two subjects of statistical physics: (1) nonextensive statistical mechanics [1]-[4] and (2) complex networks [5]-[10]. As for the first subject (1), Tsallis has developed the nonextensive statistics (NES), proposing the generalized entropy given by [1][2]

\[ S_q = \frac{\sum_i p_i^q - 1}{1-q}, \]  

where \( q \) denotes the entropic index and \( p_i \) the probability distribution of the state \( i \). It is noted that in the limit of \( q = 1 \), \( S_q \) reduces to the entropy in the Boltzmann-Gibbs statistics given by

\[ S_1 = -\sum_i p_i \ln p_i. \]  

The NES has been successfully applied to a wide range of nonextensive systems including physics, chemistry, mathematics, astronomy, geophysics, biology, medicine, economics, engineering, linguistics, and others [4].

The subject (2) has been initiated by the seminal paper of Watts and Strogatz [5]. Since then considerable study has been made on complex networks [9][10]. Existing and proposed networks are classified into two types:

(i) Networks where the number of nodes is constant and where the degree distribution \( P(k) \) for a given node to be connected to \( k \) neighbors, has a maximum at or near \( <k> \) with finite width where \( <\cdot> \) denotes the average over \( P(k) \). This type includes random [11] and small-world networks [5][6].

(ii) Networks where the number of nodes is not stationary and where the distribution varies between the exponential and power forms. This type includes scale-free (SF) networks, which were originally proposed by Barabási and Albert [8] with a growth of nodes and their preferential attachments. Many models and mechanisms have been proposed not only for growing but also for non-growing SF networks with geographical and non-geographical structures.

It is possible that the two subjects (1) and (2) are intimately related [12]-[16]. By using the nonextensive information entropy as given by Eq. (1), Wilk and Wlodarczyk have shown that degree distribution \( P(k) \) of SF networks which belongs to type (ii), is given by [12][13]

\[ P_{SF}(k) \propto \left[ 1 - (1 - q) \left( \frac{k}{\mu} \right)^{\frac{1}{1-q}} \right]. \]  

(3)
where $\mu = \langle k \rangle$ and $\langle \cdot \rangle$ denotes the average over $P(k)$. The expression (3) is universal because in the limit of $q = 1$ it reduces to the exponential distribution given by $P(k) \propto e^{-k/\mu}$, and for $k \gg \mu/(q - 1)$ it yields the power law given by $P(k) \propto k^{-q/(q-1)}$. Thus the expression (3) successfully accounts for degree distribution in SF networks. Soares et al. [14] have proposed the two-dimensional growth model for SF networks where the degree distribution is given by Eq. (3) with $\mu$ and $q$ expressed in terms of model parameters. Thurner and Tsallis [15] have discussed scale-free gas-like networks, showing their nonextensive aspects. It has been pointed out by Abe and Thurner [16] that degree distribution in scale-free networks may be generated by superimposing Poisson distributions relevant to random networks [11] with a proper hidden-variable distribution, just as in superstatistics [17].

As for the type (i) networks, degree distribution $P(k)$ of random network is given by the binomial distribution [11]:

$$P_B(k) = C^N_{N-k} c^k (1-c)^{N-k},$$

(4)

where $c = \mu/N$ and $C^m_n = m!/n!(m-n)!$. In the limit of large $N$, Eq. (4) becomes the Poisson distribution given by

$$P_P(k) = P_P(k; \mu) = \frac{\mu^k}{k!} e^{-\mu}. \quad (5)$$

On the contrary, $P(k)$ of Watts-Strogatz (WS) small-world (SW) networks [5] is given by [18]

$$P_{SW}(k) = \sum_{n=0}^{f(k, \mu)} C^\mu_{n/2} p^{\mu/2-n}(1-p)^n P_P(k - \mu/2 - n; p\mu/2),$$

(6)

(for $k \geq \mu/2$ and large $N$)

where $f(k, \mu) = \min(k - \mu/2, \mu/2)$ and $p = 2N_r/N\mu$ denotes a randomness parameter for random rewirings of $N_r$ edges from a regular network [5]. Figure 1 shows $P_{SW}(k)$ with $N = 100$ and $\mu = 10$ for various randomness $p$ [19]. Results shown in Fig. 1 are obtained by simulations with 1000 trials after Ref.[5], because the analytic expression given by Eq. (6) is valid only for $k \geq \mu/2$ and large $N$. For $p = 0$ (regular networks), $P_{SW}(k)$ becomes a delta-function at $k = \mu$. With more increasing $p$, $P_{SW}(k)$ has a peak at $k = \mu$ with a wider width. As a comparison, we show by open circles in Fig. 1, $P_B(k)$ of random networks given by Eq. (4). Note that even for $p = 1$, $P_{SW}(k)$ of the WS-SW network is different from $P_B(k)$ of random networks because the former is not locally equivalent to the latter with some memory of the starting regular network [18].
We suppose that $P_{SW}(k)$ shown in Fig. 1 may be described or approximated by a simple function like Eq. (3) if the nonextensive property of SW networks is properly taken into account. The purpose of present paper is to investigate such a possibility. This is indeed possible in an approximate sense, as will be demonstrated in the present paper.

The paper is organized as follows. By using the maximum-entropy method (MEM) with constraints on $<k>=\mu$ and $<k^2>=\rho$ besides the normalization condition, we get in Sec. 2, the generalized Gaussian with a maximum at $k=\mu$ (hereafter referred to as $Q$-Gaussian). In contrast, a conventional $q$-Gaussian distribution which has been widely employed in many subjects such as self-gravitating stellar systems [20] and ideal gas [21], has a peak at the zero value. In Sec. 3, we discuss the stochastic-differential equation (SDE) by using Langevin equation with additive and multiplicative noises. In Sec. 4, SW networks are constructed by hidden-variable distribution (HVD). An analysis of degree distributions of SW networks is made in terms of $Q$-Gaussian in Sec. 5. Section 6 is devoted to more detailed analysis, including the $k$-dependence in the entropic index in MEM, which is shown to be realized in Langevin equations with generalized multiplicative noises. Our conclusions are presented in Sec. 7.

Before going to Sec. 2, we summarize various distributions used in this paper as follows. $P_{SF}(k)$, SF networks; $P_{SW}(k)$, SW networks; $P_B(k)$, binomial; $P_p(k)$, Poisson; $P_G(k)$, Gaussian; $p^Q_k$, $Q$-Gaussian; $P_Q(k)$, an escort probability of $p^Q_k$.

## 2 Maximum-entropy method

### 2.1 Shannon information entropy

We have adopted a network with $N$ nodes whose adjacent matrix is given by $c_{ij} = c_{ji} = 1$ for a coupled pair $(i, j)$ and zero otherwise. The degree distribution $P(k)$ for a given node to have $k$ neighbors is defined by

$$P(k) = \frac{1}{N} \sum_i <\delta(k - \sum_j c_{ij})>_G,$$

where $<>_G$ denotes the average over graphs. The averaged coordination number $\mu$ is given by

$$\mu = \frac{1}{N} \sum_i \sum_j <c_{ij}>_G,$$

$$= \sum_k P(k) k.$$
The *coupling connectivity* $R$ is defined by [19]

$$R = \frac{1}{N\mu^2} \sum_i \sum_j \sum_{\ell} < c_{ij} c_{i\ell} >_G,$$

$$= \frac{1}{\mu^2} \sum_k P(k) k^2.$$

(10)

The coupling connectivity $R$ expresses a factor for a cluster where the two sites $j$ and $\ell$ are coupled to the third site $i$, but the sites $j$ and $\ell$ are not necessarily coupled. In contrast, the clustering coefficient $C$ defined by

$$C = \frac{1}{N\mu^2} \sum_i \sum_j \sum_{\ell} < c_{ij} c_{j\ell} c_{i\ell} >_G,$$

expresses a factor forming a cluster where the three sites $i$, $j$ and $\ell$ are mutually coupled [22]. Unfortunately $C$ cannot be expressed in terms of $P(k)$.

First we consider the Shannon entropy given by

$$S = - \sum_i P(k) \ln P(k).$$

(13)

Assuming that $\mu$ and $R$ of the adopted networks are given, we obtain $P(k)$ by MEM with the three constraints given by

$$\sum_k P(k) = 1,$$

$$\sum_k P(k) k = \mu,$$

$$\sum_k P(k) k^2 = \mu^2 R \equiv \rho^2.$$

(14)

(15)

(16)

We get

$$P(k) \propto e^{-(\beta k + \gamma k^2)},$$

(17)

where $\beta$ and $\gamma$ are Lagrange multipliers relevant to the constraints given by Eqs. (15) and (16), respectively.

### 2.2 Nonextensive information entropy

Next we extend our discussion by employing the generalized entropy first proposed by Tsallis [1, 2]:

$$S_q = \frac{\sum_k P_k^q - 1}{1 - q},$$

(18)
which reduces to the Shannon entropy in the limit of $q = 1$. Here $p_k$ denotes the probability distribution for couplings of $k$ neighbors whose explicit form will be given shortly [Eq. (24)]. The three constraints corresponding to Eqs. (14)-(16) are given by

\begin{align*}
\sum_k p_k &= 1, \quad (19) \\
\sum_k P(k) k &= \mu, \quad (20) \\
\sum_k P(k) k^2 &= \mu^2 R = \rho^2, \quad (21)
\end{align*}

with

\begin{align*}
P(k) &= \frac{p_k^q}{c_q}, \quad (22) \\
c_q &= \sum_k p_k^q, \quad (23)
\end{align*}

where $P(k)$ is the escort probability expressing the degree distribution [2]. By maximizing the entropy with the three constraints, we get

$$
p_k \propto \left[ 1 - \left( \frac{1-q}{c_q} \right) \left[ \beta (k - \mu) + \gamma (k^2 - \rho^2) \right] \right]^{\frac{1}{1-q}}. \quad (24)
$$

Equation (24) may be rewritten as

$$
p_k = \frac{1}{Z_q} \exp_q \left[ -\zeta (k - \eta)^2 \right], \quad (25)
$$

with

\begin{align*}
Z_q &= \sum_k \exp_q \left[ -\zeta (k - \eta)^2 \right], \quad (26) \\
\zeta &= \left( \frac{\gamma}{c_q} \right) \left[ 1 + \left( \frac{1-q}{c_q} \right) \left( \frac{\beta^2}{4\gamma} + \beta \mu + \gamma \rho^2 \right) \right]^{-1}, \quad (27) \\
\eta &= -\frac{\beta}{2\gamma}, \quad (28)
\end{align*}

where $\exp_q(x)$ expresses the $q$-exponential function defined by

\begin{align*}
\exp_q(x) &= \left[ 1 + (1-q)x \right]^{\frac{1}{1-q}}, \quad \text{for } 1 + (1-q)x > 0 \quad (29) \\
&= 0, \quad \text{for } 1 + (1-q)x < 0 \quad (30)
\end{align*}

Parameters of $\zeta$ and $\eta$, newly introduced in place of $\beta$ and $\gamma$, are determined by the constraints given by Eqs. (20) and (21) as

\begin{align*}
\mu &= \sum_k P(k) k, \quad (31) \\
\rho^2 &= \sum_k P(k) k^2, \quad (32)
\end{align*}
with
\[ P(k) = \frac{(\exp_q[-\zeta(k-\eta)^2])^q}{\sum_k \exp_q[-\zeta(k-\eta)^2])^q}, \] (33)
which are numerically evaluated in general.

### 2.3 \( Q \)-Gaussian

In order to get some analytical results, we hereafter assume that \( k \) is a continuous variable varying from \(-\infty\) to \( \infty \), which may be justified for \( \mu \gg \sigma \). Equations (31)-(33) yield
\[
\begin{align*}
\mu &= \eta, \quad (34) \\
\sigma^2 &= \rho^2 - \mu^2, \quad (35)
\end{align*}
\]
\[
\begin{align*}
&= \left[ \frac{1}{(1-q)\zeta} \right] \frac{B(\frac{3}{2}, \frac{q}{1-q} + 1)}{B(\frac{1}{2}, \frac{q}{1-q} + 1)}, \quad \text{for } q < 1 \quad (36) \\
&= \frac{1}{2\zeta}, \quad \text{for } q = 1 \quad (37) \\
&= \left[ \frac{1}{(q-1)\zeta} \right] \frac{B(\frac{3}{2}, \frac{q}{q-1} - \frac{3}{2})}{B(\frac{1}{2}, \frac{q}{q-1} - \frac{1}{2})}, \quad \text{for } q > 1 \quad (38)
\end{align*}
\]
where \( B(x, y) \) denotes the beta function. Equations (35)-(38) lead to
\[ \zeta = \frac{1}{2\nu\sigma^2}, \] (39)
with
\[ \nu = \frac{3-q}{2}. \] (40)

From Eqs. (25), (34) and (39), we get
\[ p_k = p_k^Q \equiv \frac{1}{\sqrt{2\pi} \sigma D(q, 1)} \exp_q \left[ -\frac{(k-\mu)^2}{2\nu\sigma^2} \right], \] (41)
with
\[ D(q, r) = \sqrt{\frac{\nu}{1-q}} \left( \frac{2r}{2r+1-q} \right) \frac{\Gamma\left(\frac{r}{1-q}\right)}{\Gamma\left(\frac{r}{1-q} + 1/2\right)}, \quad \text{for } q < 1 \quad (42) \\
= 1, \quad \text{for } q = 1 \quad (43) \\
= \sqrt{\frac{\nu}{q-1}} \frac{\Gamma\left(\frac{r}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{r}{q-1}\right)}, \quad \text{for } q > 1 \quad (44)
\]
\( \Gamma(x) \) being the gamma function [23]. From Eqs. (22), (23) and (41), we get degree distribution given by
\[ P(k) = P_Q(k) \equiv \frac{1}{\sqrt{2\pi} \sigma D(q, q)} \left( \exp_q \left[ -\frac{(k-\mu)^2}{2\nu\sigma^2} \right] \right)^q. \] (45)
\( p_k^Q \) in Eq. (41) expresses the generalized Gaussian which is called \( Q\text{-Gaussian} \) in this paper. It is easy to see that \( D(q, r) \) reduces to unity in the limit of \( q = r = 1 \) because

\[
\lim_{|z| \to \infty} \frac{\Gamma(z + a)}{\Gamma(z) z^a} = 1 \tag{24}
\]

then both \( p_k^Q \) and \( P_Q(k) \) reduce to Gaussian:

\[
P_G(k) = P_G(k; \mu, \sigma) \equiv \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}, \tag{46}
\]

with the average \( \mu \) and variance \( \sigma^2 \). Equations (39) and (40) show that \( q \leq 3 \) because \( \nu \geq 0 \) for the positive definiteness of fluctuations. We note from Eqs. (29) and (30) that

\[
P_Q(k) \propto (k-\mu)^{-\frac{2q}{q-1}}, \quad \text{for } |k-\mu|/\sigma \gg r_c, \; q > 1
\]

\[
= 0, \quad \text{for } |k-\mu|/\sigma > r_c, \; q < 1 \tag{47}
\]

with

\[
r_c = \sqrt{\frac{3-q}{|q-1|}} \tag{48}
\]

It is worthwhile to note the difference between \( Q\text{-Gaussian} \) and the conventional \( q\text{-Gaussian} \) which is given by \( p(u) \propto \exp\left[-\beta'u^2\right] \) derived from Eq. (3) with a replacement of \( k \to (u^2/2) \) \cite{20,21}, \( u \) standing for the generalized velocity and \( \beta' \) the relevant Lagrange multiplier. Alternatively, \( q\text{-Gaussian} \) is obtained from \( Q\text{-Gaussian} \) with \( \mu = 0 \) in Eq. (41).

### 2.4 Model calculations

In order to first get a broad insight to the \( Q\text{-Gaussian} \), we show in Figs. 2(a) and 2(b), some numerical results of \( p_k^Q \) and \( P_Q(k) \) calculated by using Eqs. (41)-(45). Figure 2(a) shows \( p_k^Q \) with \( \mu = 10 \) and \( \sigma = 1 \) for various \( q \) values of \( q = 0.5, 1.0, 1.5, 2.0 \) and 2.5. The \( Q\text{-Gaussian} \) \( p_k^Q \) is fat-tailed for \( q > 1 \) and compact support for \( q < 1 \). Fig. 2(b) shows \( P_Q(k) \) with \( \mu = 10 \) and \( \sigma = 1 \) for various \( q \) values corresponding to Fig. 2(a). We note that a profile of \( P_Q(k) \) is rather different from that of \( p_k^Q \): both are normalized by \( \int dk \; p_k^Q = \int dk \; P_Q(k) = 1 \). For \( q = 1.0 \), \( p_k^Q \) and \( P_Q(k) \) reduce to the Gaussian distribution located at \( k = \mu = 10 \). With more increasing \( q \) above unity, \( P_Q(k) \) has a sharper peak with narrower width. In contrast, when \( q \) is more decreased below unity, \( P_Q(k) \) has a wider width but has no magnitudes at \( |k-\mu| \geq r_c \sigma \), for example, \( |k-\mu| \geq 2.236 \) for \( q = 0.5 \).
3 Stochastic differential equation approach

3.1 Formulation

Recently Anteneodo and Tsallis [25] have studied the Langevin equation subject to additive and multiplicative noises, which lead to nonextensive distributions. We here assume that Langevin equations are given by

$$\frac{dk_i}{dt} = -\lambda (k_i - \mu) + g(k_i) \xi_i(t) + \eta_i(t), \quad (i = 1 - N)$$  \hspace{1cm} (49)$$

where $k_i(t)$ denotes a real variable expressing the number of couplings of the node $i$: $\lambda$ the relaxation rate: $g(k)$ an arbitrary function whose explicit form will be given later [Eqs. (52), (88) and (89)]: $\eta_i(t)$ and $\xi_i(t)$ are additive and multiplicative white noises, respectively, as given by

$$<\eta_i(t) \eta_j(t')> = 2A \delta_{ij} \delta(t - t'),$$  \hspace{1cm} \hspace{1cm} (50)$$

$$<\xi_i(t) \xi_j(t')> = 2M \delta_{ij} \delta(t - t'),$$  \hspace{1cm} \hspace{1cm} (51)$$

means of noises being vanishing. Equation (49) shows that the number of couplings $k_i$ is influenced by both additive and multiplicative noises. In particular, effects of multiplicative noises given by $g(k_i)$ depend on the present state of node $k_i(t)$.

When $g(k)$ is given by

$$g(k) = k - \mu,$$  \hspace{1cm} \hspace{1cm} (52)$$

we get the Fokker-Planck equation for $p(k_i,t)$ ($\equiv p(k,t)$) given by [25]

$$\frac{\partial p(k,t)}{\partial t} = \lambda \frac{\partial}{\partial k} [g(k) p(k,t)] + M \frac{\partial}{\partial k} \left( g(k) \frac{\partial}{\partial k} [g(k) p(k,t)] \right) + A \frac{\partial^2 p(k,t)}{\partial k^2}. \hspace{1cm} (53)$$

Its stationary solution is given by [25][26]

$$p_k = p(k, \infty) \propto \left[ 1 + \left( \frac{M}{A} \right) (k - \mu)^2 \right]^{-\frac{M+M}{2}}. \hspace{1cm} (54)$$

Equation (54) may be rewritten in two ways as

$$p_k \propto \left[ 1 - (1 - q) \frac{(k - \mu)^2}{2\sigma^2} \right]^{-\frac{1}{(1-q)}}, \hspace{1cm} \hspace{1cm} (55)$$

$$\propto \left[ 1 - (1 - q) \frac{(k - \mu)^2}{2\nu \sigma^2} \right]^{-\frac{1}{(1-q)}}, \hspace{1cm} \hspace{1cm} (56)$$
with
\[ q = \frac{\lambda + 3M}{\lambda + M}, \tag{57} \]
\[ \sigma^2 = \frac{A}{\lambda + M}, \tag{58} \]
\[ \sigma^2 = \frac{A}{\lambda}, \tag{59} \]

where \( \nu \) is given by Eq. (40). Note the difference between \( \sigma \) in Eq. (58) and \( \sigma \) in Eq. (59). The Q-Gaussian given by Eq. (55) with \( \mu = 0 \) is adopted in [25]. Equation (56) agrees with the result obtained by MEM given by Eq. (41) for \( q \geq 1 \).

Equation (57) shows that the nonextensivity arises from multiplicative noises. In the case of no multiplicative noises (\( M = 0 \)), we get \( q = 1 \), leading to the Gaussian distribution:
\[ p_k = P_G(k) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}. \tag{60} \]

### 3.2 Model calculations

We show model calculations, simulating Langevin equation given by Eq. (49) with \( N = 100 \): the Heun method is employed with a time step of 0.0001. Figures 3(a) and 3(b) show time courses of \( k_i(t) \) \( \equiv k(t) \) for \( (A, M) = (1.0, 0.0) \) and \( (1.0, 0.5) \), respectively, with \( g(k) = k - \mu \), \( \lambda = 1.0 \) and \( \mu = 10 \). The number of \( k \), starting from an initial value of \( k = 10 \), fluctuates around \( k = \mu \) by effects of noises. A comparison between Figs. 3(a) and 3(b) shows that fluctuations are much increased by effects of multiplicative noises. Circles in Figs. 4(a) and 4(b) show degree distributions \( P(k) \) for \( (A, M) = (1.0, 0.0) \) and \( (1.0, 0.5) \), respectively, obtained by averages over \( N = 100 \) for \( 50 < t \leq 100 \) [27]. Solid curves in Figs. 4(a) and 4(b) express \( P_Q(k) \) given by Eq. (45) for \( (q, \sigma) = (1.0, 1.008) \) and \( (1.667, 0.984) \), respectively: adopted \( q \) values are calculated by Eq. (57) and \( \sigma \) are evaluated by using \( P_Q(k) \) obtained by simulations [27]. Degree distribution for \( M = 0 \) in Figs. 4(a) shows Gaussian. When the multiplicative noises with \( M = 0.5 \) are introduced, degree distribution \( P(k) \) has tails expressed by the power law as shown in Fig. 4(b). Results of stochastic differential equation are in good agreement with those of Q-Gaussian shown by solid curves.
4 Hidden-parameter distribution approach

Employing Poisson distribution \( P_p(k) \) for random networks, Abe and Thurner [16] proposed the hidden-parameter expansion to get

\[
\int_0^\infty d\mu' \ P_p(k; \mu') \ \Pi_1(\mu') = P(k),
\]

(61)

where \( \Pi_1(\mu) \) stands for distribution of the hidden variable \( \mu \). For scale-free distribution \( P_{SF}(k) \) given by

\[
P_{SF}(k) = A (k + k_0)^{-\gamma},
\]

(62)

they obtained [16]

\[
\Pi_1(\mu) \propto \mu^{-\delta}, \quad \text{for large } \mu
\]

(63)

with

\[
\delta = \gamma,
\]

(64)

where \( \gamma > 1 \), and \( A \) and \( k_0 \) denote constants.

It should be noted that degree distribution of random networks may be described not only by Poisson \( P_p(k) \) but also by Gaussian distribution \( P_G(k) \). \( P_p(k) \) is obtained for large \( N \) from binomial distribution \( P_B(k) \) as shown by Eq. (5). Alternatively, \( P_G(k) \) may be derived from binomial distribution when \( k \) in \( P_B(k) \) is assumed to be a real variable as follows. Expanding \( P_B(k) \) around \( k = \mu = Nc \) where \( P_B(k) \) takes the maximum value, we get

\[
P_B(k) = e^{\ln P_B(k)} \approx e^{\ln P_B(\mu) - \frac{(k-\mu)^2}{2\sigma^2}},
\]

(65)

\[
= \frac{1}{\sqrt{2\pi}\sigma} \ e^{-\frac{(k-\mu)^2}{2\sigma^2}} = P_G(k),
\]

where \( \sigma^2 = Nc(1 - c) \).

When generalizing the hidden-parameter method proposed by Abe and Thurner [16], we may assume that \( P(k) \) is expressed as a superposition of Gaussians \( P_G(k) \) for random networks, as

\[
\int_0^\infty d\mu' \int_0^\infty d\sigma' \ P_G(k; \mu', \sigma') \ \Pi(\mu', \sigma') = P(k),
\]

(66)

where \( \Pi(\mu, \sigma) \) denotes distribution for hidden variables \( \mu \) and \( \sigma \). When \( P_{SW}(k) \) obtained by simulations or an analytical expression given by Eq. (6) is substituted to \( P(k) \) in Eq.
\( \Pi(\mu, \sigma) \) will be determined by inversely solving Eq. (66). This would be interesting but seems rather difficult.

When we assume
\[
\Pi(\mu', \sigma') = \Pi_2(\sigma') \delta(\mu' - \mu),
\] (67)
Eq. (66) becomes
\[
\int_0^\infty d\sigma' P_G(k; \mu, \sigma') \Pi_2(\sigma') = P(k),
\] (68)
where \( \mu = <k> \). We may get \( \Pi_2(\sigma) \) for \( P_{SW}(k) \) given by Eq. (6), in principle, from Eq. (68), which is, however, problematic because \( P_{SW}(k) \) is valid only for \( k \geq \mu/2 \).

Instead, we have obtained \( \Pi_2(\sigma) \) for Q-Gaussian \( p^Q_k \) from Eq. (68) by
\[
\int_0^\infty d\sigma' \frac{1}{\sqrt{2\pi}\sigma'} e^{-\frac{(k-\mu)^2}{2\sigma'^2}} \Pi_2(\sigma') = \exp_q \left[ -\frac{(k-\mu)^2}{2\sigma^2} \right],
\] (69)
where \( \nu = (3-q)/2 \) [Eq. (40)], and normalization factor in \( p^Q_k \) is neglected for a simplicity of calculation. By using the change of variables: \( 1/2\sigma'^2 = \alpha, 1/2\nu\sigma^2 = \alpha_0, \) and \( (k-\mu)^2 = x \), we get
\[
\int_0^\infty d\alpha e^{-\alpha x} f(\alpha) = \exp_q(-\alpha_0 x),
\] (70)
with
\[
f(\alpha) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{2\alpha} \right) \Pi_2 \left( \left( \frac{1}{2\alpha} \right)^{1/2} \right),
\] (71)
\[
= \frac{1}{\Gamma \left( \frac{n}{2} \right)} \left( \frac{n}{2\alpha_0} \right)^{\frac{n}{2}} \alpha^{\frac{n}{2}-1} e^{-\frac{\alpha\alpha_0}{2\alpha_0}},
\] (72)
\[
\alpha_0 = \frac{1}{2\nu\sigma^2} = E(\alpha),
\] (73)
\[
q - 1 = \frac{2}{n} = \frac{E(\alpha^2) - E(\alpha)^2}{E(\alpha)^2},
\] (74)
where \( f(\alpha) \) in Eq. (72) denotes the \( \Gamma \)- (or \( \chi^2 \)-) distribution function of the order \( n \), \( E(\alpha) \) the average of the variable \( \alpha \) over \( f(\alpha) \), and \( E(\alpha^2) \) its variance [28][29]. Equation (70) is nothing but the Laplace transformation. By using the relation derived from Eq. (71):
\[
\Pi_2(\sigma) = \sqrt{2\pi} \left( \frac{1}{\sigma^2} \right) f \left( \frac{1}{2\sigma^2} \right),
\] (75)
we get
\[
\Pi_2(\sigma) \propto \sigma^{-\delta}, \quad \text{for large } \sigma
\] (76)
with
\[
\delta = \gamma = \frac{2}{q - 1},
\] (77)
where $\gamma$ denotes the index for the expression given by

$$p_k^Q \propto (k - \mu)^{-\gamma}. \quad \text{for large } k$$

Eqs. (76)-(78) should be compared to the result of Ref. [16] given by Eqs. (62)-(64), related discussion being given in Sec. 6.

Figure 5 shows Log plots of $\Pi_2(\sigma)$ for $Q$-Gaussian with various $q$ values against $\sigma/(\sigma_0\sqrt{\nu})$ where $\sigma_0 = 1/\sqrt{2\alpha_0}$ for $q = 1.0$. In the limit of $q \to 1.0 \ (n \to \infty)$, $\Pi_2(\sigma)$ becomes the delta function at $\sigma = \sigma_0$. As $q$ is more increased than unity, $\Pi_2(\sigma)$ has a wider width, expressing larger fluctuations in the variable $\sigma$. For $q = 1.5, 2.0$ and 2.5, the index $\delta$ given by Eq. (77) is 4, 2 and 1.33, whose slopes are shown by chain curves in Fig. 5.

If we express Gaussian in Eq. (66) as

$$P_G(k; \mu', \sigma') = \psi\left(\frac{k - \mu'}{\sigma'}\right),$$

we get

$$P(k) = \int_0^\infty d\mu' \int_0^\infty d\sigma' \psi\left(\frac{k - \mu'}{\sigma'}\right) \Pi(\mu', \sigma'),$$

which is similar to the wavelet transformation [30]. It is interesting to obtain analytical expression of $\Pi(\mu, \sigma)$ for a general $P(k)$.

5 Analysis of degree distribution in SW networks

We will discuss an application of the $Q$-Gaussian to an analysis of $P_{SW}(k)$. The coupling connectivity $R$ has been calculated with the use of Eq. (10) by simulations (1000 trials) for SW networks with $N = 100$ and $\mu = 10$ as a function of $p$ [19]. We have obtained $\sigma$ by using $\sigma = \mu\sqrt{R - 1}$ derived from Eqs. (21) and (35). Chain and dashed curves in Fig. 6 show the $p$ dependence of $R$ and $\sigma$, respectively. With increasing $p$, both $R$ and $\sigma$ are monotonously increased.

We have tried to fit $P_Q(k)$ in Eqs. (41)-(45) to $P_{SW}(k)$ obtained by simulations, choosing a proper $q$ value with $\mu$ and $\sigma$ calculated for a given $p$ shown in Fig. 6. When we have chosen the $q$ value such that maxima of the two distributions agree, their whole shapes are in good agreement. Figures 7(a)-7(f) show such fittings for $p = 0.1, 0.2, 0.4, 0.6, 0.8$ and 1.0. In the case of $p = 0.1$ shown in Fig. 7(a), $P_Q(k)$ with $q = 1.35, \sigma = 0.97$ and $\mu = 10$ well reproduces the relevant degree distribution plotted by circles. For a
comparison, we show also results with the use of \( q = 1.5 \) and 1.2 whose agreements are worse than that of \( q = 1.35 \). Gaussian \( (q = 1.0) \) for \( \mu = 10 \) and \( \sigma = 0.97 \) is in significant disagreement with \( P_{SW}(k) \) at \( 9 \leq k \leq 11 \). In the case of \( p = 0.2 \) shown in Fig. 7(b), \( P_Q(k) \) with \( q = 1.2, \sigma = 1.34 \) and \( \mu = 10 \) well explains \( P_{SW}(k) \) obtained by simulations, results with \( q = 1.0 \) and 1.4 being also plotted for a comparison. In cases of \( p = 0.4, 0.6 \) and 0.8 shown in Figs. 7(c), 7(d) and 7(e), we have obtained a good agreement between \( P_{SW}(k) \) and \( P_Q(k) \) with \( q = 1.0 \) expressing the Gaussian distribution. In contrast, in the case of \( p = 1.0 \) shown in Fig. 7(f), the result of \( q = 0.95 \) yields a better fit than that of \( q = 1.0 \).

At a glance at Figs. 7(a)-7(f), we have an impression that \( P_{SW}(k) \) is well described by \( P_Q(k) \). It is, however, not true in a strict sense, because at \( |k - \mu|/\sigma \gg r_c \), \( P_{SW}(k) \) follows the exponential function while \( P_Q(k) \) obeys the power law \( (q > 1) \), as Eq. (47) shows. This is clearly seen in Fig. 8(a)-8(f) where results in Figs. 7(a)-7(f) are shown in the Log plots. For example, in the case of \( p = 0.1 \), the agreement of \( P_Q(k) \) with \( P_{SW}(k) \) is good for \( |k - \mu| \sim 3 \) as demonstrated in Fig. 7(a), but it becomes worse for \( |k - \mu| \sim 3 \). Nevertheless, it should be stressed that main parts of \( P_{SW}(k) \) are well described by a single \( P_Q(k) \) as shown in Fig. 7(a)-7(f). When \( P_{SW}(k) \) and \( P_Q(k) \) are in good agreement at \( k \in [\mu - k_a, \mu + k_a] \) with an accuracy \( |P_{SW}(k) - P_Q(k)| \leq 10^{-2} \), the number of \( k \) states included in this region is given by

\[
\int_{\mu - k_a}^{\mu + k_a} dk \, P(k),
\]

which yields \( n_a = 0.992 \) for \( k_a = 3 \) in the case of \( p = 0.1 \). Similarly, we get \((p, k_a, n_a) = (0.2, 4, 0.993), (0.4, 4, 0.975), (0.6, 5, 0.983), (0.8, 5, 0.970), \) and \((1.0, 5, 0.959)\).

6 Discussions

We will present more detailed analysis, trying to reduce the discrepancy between \( P_{SW}(k) \) and \( P_Q(k) \) at their tails in this section. \( P_{SW}(k) \) calculated by simulations for SW networks with \( p = 0.1, \mu = 10 \) and \( N = 100 \) is again shown by filled circles in Fig. 9(a), where the analytical result valid for large \( N \) given by Eq. (6) is also shown by filled triangles for a comparison. We note a deviation of \( P_{SW}(k) \) from \( P_Q(k) \) shown by solid curve at \( |k - \mu| \sim 3 \), where \( P_{SW}(k) \) obeys the exponential (Gaussian) law. This suggests that \( p_k \) in SW networks behaves as

\[
p_k \propto \exp_q \left[ -\frac{(k - \mu)^2}{(3-q)\sigma^2} \right], \quad \text{for small } |k - \mu|.
\]
\[
e^{-\frac{(k-\mu)^2}{2\sigma^2}}.
\] for large \( |k - \mu| \) \hspace{1cm} (82)

If we incorporate the \( k \)-dependence into the entropic index, the entropy in Eq. (18) is generalized as
\[
S_q = \frac{\sum_k p_k^{q_k} - 1}{1 - q_k},
\] where \( q_k \) denotes the \( k \)-dependent entropic index. MEM with the constraints given by Eqs. (19)-(21) with a replacement of \( q \rightarrow q_k \) yields
\[
p_k = \frac{1}{Z_q} \exp_{q_k} \left( \frac{(k - \mu)^2}{(3 - q_k)\sigma^2} \right),
\] with
\[
Z_q = \int dk \exp_{q_k} \left( \frac{(k - \mu)^2}{(3 - q_k)\sigma^2} \right),
\] from which \( P(k) \) is given by Eqs. (22) and (23) with a replacement of \( q \rightarrow q_k \). We expect that the result given by Eq. (82) may be realized in Eq. (84) if \( q_k = q \) for small \( |k - \mu| \) and \( q_k = 1 \) for large \( |k - \mu| \).

Bearing these considerations in mind, we have assumed the \( k \)-dependent entropic index given by
\[
q_k = 1 + (q - 1) \Theta(k - \mu + k_1, \theta) \Theta(\mu + k_1 - k, \theta),
\] with
\[
\Theta(k, \theta) = \frac{1}{e^{-k/\theta} + 1},
\] where \( k_1 \) denotes the characteristic \( k \) value. Equation (86) shows that \( q_k = q \) for \( |k - \mu| \lesssim k_1 \) and \( q_k = 1 \) for \( |k - \mu| \gtrsim k_1 \). Open squares in Fig. 9(a) show \( P(k) \) calculated by using \( q_k \) given by Eqs. (86) and (87) with \( k_1 = 4 \) and \( \theta = 1 \). The \( k \)-dependence of \( P(k) \) changes from \( Q \)-Gaussian to Gaussian with increasing \( |k - \mu| \), and it is in better agreement with \( P_{SW}(k) \) than \( P_Q(k) \).

In order that the \( k \)-dependence in the entropic index given by Eq. (86) is realized in the Langevin model, we have tried to modify a function \( g(k) \) for multiplicative noises. Open circles in Fig. 9(b) express the results of simulations of Langevin equation (49) with \( g(k) = k - \mu \) given by Eq. (52) for \( \lambda = 1 \), \( A = 0.941 \) and \( M = 0.212 \), which are chosen from Eqs. (57) and (59) with \( q = 1.35 \) and \( \sigma = 0.97 \) for \( p = 0.1 \) [27]. In contrast, open squares in Fig. 9(b) denote the result of simulations of the Langevin model with \( g(k) \) given by
\[
g(k) = (k - \mu) \Theta(k - \mu + k_2, \theta) \Theta(\mu + k_2 - k, \theta),
\]
for $\lambda = 1$, $\mu = 10$, $A = 0.941$, $M = 0.212$, $k_2 = 4$ and $\theta = 1$. The result shown by open squares in Fig. 9(b) deviates from that shown by open circles at $|k - \mu| \sim 4$ where effects of multiplicative noises do not work for $g(k)$ given by Eq. (88). It is shown that an agreement of $P_{SW}(k)$ with $P(k)$ expressed by open squares is better than that with $P_Q(k)$ expressed by the solid curve (and open circles).

A similar analysis has been made for the case of $p = 0.2$, whose results are shown in Figs. 10(a) and 10(b). Filled circles and triangles in Fig. 10(a) denote $P_{SW}(k)$ obtained by simulations and analytic method, respectively. Solid curve, expressing $P_Q(k)$ with $q = 1.20$ and $\sigma = 1.34$, deviates from $P_{SW}(k)$ at $|k - \mu| \sim 4$. In contrast, open squares express the result calculated with the use of $q_k$ given by Eq. (86) for $k_1 = 5$ and $\theta = 1$, which are in better agreement with $P_{SW}(k)$ than $P_Q(k)$.

SDE has been applied to the case of $p = 0.2$. Open circles in Fig. 10(b) express the result of simulations for Langevin equations with $g(k) = k - \mu$, $\lambda = 1$, $\mu = 10$, $A = 1.788$ and $M = 0.111$, which are chosen from Eqs. (57) and (59) with $q = 1.20$ and $\sigma = 1.34$ for $p = 0.2$ [27]. In contrast, open squares in Fig. 10(b) denote $P(k)$ calculated by simulation with $g(k)$ given by Eq. (88) for $\lambda = 1$, $\mu = 10$, $A = 1.788$, $M = 0.111$, $k_2 = 5$ and $\theta = 1$. An agreement of $P_{SW}(k)$ with $P(k)$ is much better than that with $P_Q(k)$ shown by the solid curve (and open circles).

We have obtained the optimum entropic index of $q = 1.0$ for $p = 0.4$, $0.6$ and $0.8$ [Fig. 7(c)-7(e)], and $q = 0.95$ for $p = 1.0$ [Fig. 7(f)]. Log plots in Figs. 8(c)-8(f) show that there are deviations between $P_{SW}(k)$ and $P_Q(k)$ at $|k - \mu| \sim 5$. A Log plot of results of $p = 1.0$ are again shown in Fig. 11, where $P_{SW}(k)$ calculated by simulations and analytical method are shown by filled circles and triangles, respectively. We note that at $|k - \mu| \sim 4$, $P_{SW}(k)$ becomes gradually larger than $P_Q(k)$ for $q = 0.95$ and $\sigma = 6.0$ shown by solid curve. In order to reduce the discrepancy at $|k - \mu| \sim 4$, we adopt a function of $g(k)$ for multiplicative noises given by

$$g(k) = \Theta(k - \mu - k_3, \theta), \quad (89)$$

which has a magnitude of unity at $k \sim \mu + k_3$ and zero at $k \sim \mu + k_3$. This expression with Eq. (49) may be alternatively interpreted such that magnitude of additive noises is increased by an amount of $M$ at $k \sim \mu + k_3$. Open circles in Fig. 11 show $P(k)$ calculated by using SDE with $g(k) = k - \mu$, $\lambda = 1$, $A = 6.0$ and $M = 0$, which are chosen from Eqs. (57) and (59) with $q = 1.0$ and $\sigma = 2.45$ [27]. In contrast, open squares express $P(k)$ calculated with $g(k)$ given by Eq. (89) with $\lambda = 1$, $A = 6.0$, $M = 2.0$, $k_3 = 4$ and $\theta = 1,$
which is in better agreement with $P_{SW}(k)$ than $P_Q(k)$ shown by the solid curve.

It is necessary to point out that the normalization factor of $1/\sqrt{2\pi \sigma'}$ in Eq. (69) plays an important role in determining $\Pi_2(\sigma')$ as shown below. When $Q$-Gaussian of the right-hand side in Eq. (69) is replaced by

$$P(k) = (k - \mu)^{-\gamma}, \quad (\gamma > 1),$$

we get

$$\int_0^\infty d\alpha \, e^{-\alpha x} f(\alpha) = x^{-\gamma/2},$$

with $1/2\sigma'^2 = \alpha, (k - \mu)^2 = x$, and

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{2\alpha} \right) \Pi_2 \left( \left( \frac{1}{2\alpha} \right)^{1/2} \right),$$

$$= \frac{\alpha^{\gamma/2-1}}{\Gamma(\gamma/2)},$$

which lead to

$$\Pi_2(\sigma) \propto \sigma^{-\gamma}, \quad \text{for large } \sigma$$

as given by Eqs. (76) and (77).

On the contrary, if the normalization factor of $1/\sqrt{2\pi \sigma'}$ is neglected in Eq. (76), we get Eqs. (91) and (93) with

$$f(\alpha) = \left( \frac{1}{2\alpha} \right)^{3/2} \Pi_2 \left( \left( \frac{1}{2\alpha} \right)^{1/2} \right).$$

By using the relation obtained from Eq. (95):

$$\Pi_2(\sigma) = \left( \frac{1}{\sigma^2} \right) f \left( \frac{1}{2\sigma^2} \right),$$

we get

$$\Pi_2(\sigma) \propto \sigma^{-(\gamma+1)}, \quad \text{for large } \sigma$$

The difference between indices of $\sigma$ in Eqs. (94) and (97) arises from the normalization factor $1/\sqrt{2\pi \sigma'}$ of Gaussian in Eq. (69).

7 Conclusions

In summary, degree distribution of WS small-world networks has been discussed in terms of $Q$-Gaussian obtained by the three approaches: MEM, SDE, and HVD. It is interesting
that with an accuracy of $|P_{SW}(k) - P_Q(k)| \lesssim 10^{-2}$, a single $P_Q(k)$ may describe main parts of $P_{SW}(k)$ which is expressed as a superposition of Poissons in Eq. (6). In order to account for the overall behavior in $P_{SW}(k)$ including its main part and tails, we have assumed the $k$-dependent entropic index in MEM with the entropy given by Eq. (83), which is nonextensive unless $q_k = 1$ for all $k$.

This $k$-dependent entropic index is realized in SDE, by adopting multiplicative noises given by Eq. (88) for $p = 0.1$ and 0.2, and those given by Eq. (89) for $0.4 \lesssim p \lesssim 1.0$. Degree distributions of scale-free and random networks are obtained for $g(k) = k - \mu$ and $g(k) = 0$, respectively, in SDE. We expect that $P(k)$ of SW networks has the combined property of scale-free and random networks because $g(k)$ is $k - \mu$ and 0 for small $|k - \mu|$ and large $|k - \mu|$, respectively. It is necessary to understand the relation between rewiring processes in making WS-SW networks [5] and the two types of multiplicative noises in Langevin equation.

Although the three approaches, MEM, SDE, and HVD, are conceptually rather different, they lead to equivalent results. There are, however, some differences among them. (i) The entropic index $q$ is given by $q = 1 + 2M/()$ and $q = 1 + 2/n$ in SDE and HVD, respectively, while that in MEM is a free parameter with the constraint of $q \leq 3$. (ii) The escort probability $P(k)$ is obtained from $p_k$ as given by Eqs. (22) and (23) in MEM, while there are no ways in SDE and HVD. Although no analyses have been made for $P_{SW}(k)$ with the use of HVD, Gaussian-based expression (66) or (81) is expected to be promising. This subject is left as our future study.

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What is customarily called the clustering coefficient corresponds to $C/R$ where $C$ and $R$ are defined by Eqs. (10) and (12), respectively, in this paper [9].

The root-mean value of $k^2$ averaged over $p_k^Q$ is $\sqrt{(3-q)/(5-3q)} \sigma$ for $0 < q < 5/3$ and diverges for $5/3 \leq q < 3$, while that averaged over $P_Q(k)$ is $\sigma$, where $p_k^Q$ and $P_Q(k)$ are given by Eqs. (41) and (45), respectively.

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Figure 1: Degree distribution $P_{SW}(k)$ (filled marks) of the WS small-world model with $N = 100$ and $\mu = 10$ for various $p$ values, calculated by simulations with 1000 trials, solid lines being shown for a guide of the eye. As a comparison, $P(k)$ for random networks with the same parameters is plotted by open circles (see text).

Figure 2: (a) The $Q$-Gaussian $p^Q_k$ and (b) $P_Q(k)$ with $\mu = 10$ and $\sigma = 1.0$ for various $q$ values.

Figure 3: Time courses of $k(t)$ obtained by dynamical simulations of stochastic differential equations [Eqs. (49)-(52)] for (a) $A = 1.0$ and $M = 0.0$, and (b) $A = 1.0$ and $M = 0.5$ with $\lambda = 1.0$ and $\mu = 10$. Results are simulations of a single trial by using the Heun method with the time step of 0.0001.

Figure 4: (color online) Log-plots of $P(k)$ calculated by simulations (circles) of stochastic differential equations [Eqs. (49)-(52)] for (a) $A = 1.0$ and $M = 0.0$, and (b) $A = 1.0$ and $M = 0.5$ with $\lambda = 1.0$ and $\mu = 10$, and those of $P_Q(k)$ [Eq. (45)] (solid curves): $q = 1.0$ and $\sigma = 1.008$ for (a) and $q = 1.667$ and $\sigma = 0.984$ for (b). Results of simulations are averages of $N = 100$ for $50 < t \leq 100$ by using the Heun method with the time step of 0.0001.

Figure 5: Log plots of $\Pi_2(\sigma)$ for $Q$-Gaussian against $\sigma / (\sigma_0 \sqrt{\nu})$ with various $q$ values, their slopes for $q = 1.5, 2.0$ and 2.5 being shown by chain curves with $\delta = 4, 2$ and 1.33, respectively ($\sigma_0 = 1/\sqrt{2\alpha_0}$ for $q = 1.0$).

Figure 6: The $p$ dependence of $R$ (chain curve), $\sigma$ (dashed curve) and $q$ (solid curve) adopted for a fitting of $P_Q(k)$ to $P_{SW}(k)$ of simulations for SW networks with $N = 100$ and $\mu = 10$, $R - 1$ being multiplied by a factor of ten.

Figure 7: (color online) $P_{SW}(k)$ with $N = 100$ and $\mu = 10$ obtained by simulations (circles), and $P_Q(k)$ (solid curves) with adopted $q$ values for (a) $p = 0.1$, (b) $p = 0.2$, (c) $p = 0.4$, (d) $p = 0.6$, (e) $p = 0.8$, and (f) $p = 1.0$. Dashed lines are shown for a guide of the eye.

Figure 8: (color online) Log-plots of Fig. 7.

Figure 9: (color online) (a) Log plots of $P_{SW}(k)$ for $p = 0.1$ with $\mu = 10$ and $N = 100$ calculated by simulations (filled circles) and analytical method (filled triangles), $P_Q(k)$ (solid curve), and $P(k)$ with $q_k$ given by Eq. (86) for $k_1 = 4$ and $\theta = 1$ (open squares). (b) $P(k)$ in Langevin method with $g(k) = k - \mu$, $\lambda = 1$, $A = 0.941$ and $M = 0.212$ (open circles), and that with $g(k)$ given by Eq. (88), $\lambda = 1$, $A = 0.941$, $M = 0.212$, $k_2 = 4$ and $\theta = 1$ (open squares) (see text).
Figure 10: (color online) (a) Log plots of $P_{SW}(k)$ for $p = 0.2$ with $\mu = 10$ and $N = 100$ calculated by simulations (filled circles) and analytical method (filled triangles), $P_Q(k)$ (solid curve), and $P(k)$ with $q_k$ given by Eq. (86) for $k_1 = 5$ and $\theta = 1$ (open triangles). (b) $P(k)$ in Langevin method with $g(k) = k - \mu$, $\lambda = 1$, $A = 1.788$ and $M = 0.111$ (open circles), and that with $g(k)$ given by Eq. (88), $\lambda = 1$, $A = 1.788$, $M = 0.111$, $k_2 = 5$ and $\theta = 1$ (open squares) (see text).

Figure 11: (color online) Log plots of $P_{SW}(k)$ for $p = 1.0$ with $\mu = 10$ and $N = 100$ obtained by simulations (filled circles) and analytical method (filled triangles), $P_Q(k)$ with $q = 0.95$ and $\sigma = 2.45$ (solid curve), $P(k)$ in Langevin method with $g(k) = k - \mu$, $\lambda = 1$, $A = 6.0$ and $M = 0$ and (open circles), and that with $g(k)$ given by Eq. (89), $\lambda = 1$, $A = 6.0$, $M = 2.0$, $k_3 = 4$ and $\theta = 1$ (open squares) (see text).