Neutrinos do oscillate, which up to our best knowledge implies that they are massive particles. As such, neutrinos should interact with gravitational fields. As their masses are tiny, the gravitational fields must be extremely strong. In this paper we study the influence of black holes described by non-trivial topologies on the neutrino oscillations. We present approximate analytical and numerical solutions of certain specific cases.

1. Introduction

Neutrino oscillations are one of the most interesting phenomena in particle physics. They were anticipated long time ago but their detection was complicated due to very weak interaction of these particles. Nowadays, these phenomena have been detected and studied in the case of solar neutrinos, reactor neutrinos, and atmospheric interaction of cosmic rays. Up to our best knowledge massless particles cannot oscillate, and so neutrinos (at least two out of three) must have mass, which is estimated from the supernova and other astrophysical observations to be roughly 0.3 eV.

In this paper we are going to discuss the (very weak) interaction between massive neutrinos and a gravitational field. We focus here on the change of oscillation rate for neutrinos propagating close to black holes. In principle, the strong gravitational field should modify the vacuum oscillation results, introducing additional phase shift. The quantum mechanical phase of neutrinos in the Schwarzschild spacetime was presented in Ref. where the authors discussed propagation including the possible effect of interaction with matter (MSW effect). The non-radial propagation of neutrinos in the aforementioned background was elaborated in Ref., while the critical examination of the gravitationally induced quantum mechanical phases in neutrino oscillations was given in Ref.

We have found the results of Ref. particularly interesting, as the expressions for the oscillation phase \( \Phi_k \), gained by neutrinos during propagation in a gravitational field, contained terms proportional to \( M \), the mass of the gravitating source. This observation suggests that for certain astrophysical objects, like super-massive black
holes for example, the contribution to $\Phi_k$ may be substantial. By performing a more carefull analysis we show that unfortunately this is not the case.

In our work we use a general form of the metric, which allows us to discuss not only the flat Schwarzschild background, but also the case of certain topological defects (like a black hole pierced by a cosmic string and a black hole-global monopole system).

2. Neutrino oscillations in the vicinity of a black hole

In this section we derive the formula for the quantum mechanical phase, acquired by a neutrino which propagates in a strong gravitational field, like in the vicinity of a black hole. Let us start with the following line element,

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + \tilde{C}(r)r^2d\theta^2 + C(r)r^2\sin^2\theta d\phi^2,$$

where $B(r)$, $\tilde{C}(r)$, and $C(r)$ are functions of the radial coordinate $r$. In our case we are interested in the motion of a neutrino in the space-time around a black hole described by Eq. (1). Because of the spherical symmetry we may always confine this motion to the plane $\theta = \pi/2$. This simplifies the line element to the following form:

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + C(r)r^2d\phi^2.$$

Moreover, we assume that the coefficient functions $B(r)$ and $C(r)$ do not depend on time nor on the $\phi$ coordinate. It follows that the canonical momenta $p_t^{(k)}$ and $p_\phi^{(k)}$ will play the role of constants of motion: the energy $E_k$ and angular momentum $J_k$ of the neutrino $k$-th mass eigenstate, as seen by an observer at infinity.

The phase, gained by a neutrino during propagation from a space-time point $A$ to a space-time point $B$, written in a covariant form, reads

$$\Phi_k = \int_A^B p_\mu^{(k)} dx^\mu,$$

where $p_\mu^{(k)}$ is the four-momentum of the $k$-th mass eigenstate of the neutrino, characterized by the mass $m_k$, $p_\mu^{(k)} = m_k g_{\mu\nu} \frac{dx^\nu}{ds}$. These quantities are related to each other and to the mass $m_k$ by the mass-shell relation $m_k^2 = g_{\mu\nu} p_\mu^{(k)} p_\nu^{(k)}$.

Following closely the method presented in Ref. [6] (a detailed presentation is beyond the scope of this contribution) we find, that the metric parameter $C(r)$, describing the possible topological non-triviality, will contribute only if the relativistic expansion in the neutrino mass to energy ratio squared parameter, $(m_k^2/E_k^2)$, will be performed up to the second order. Consequently, after a rather lengthy computation, we finish with

$$\Phi_k = \int_A^B p_\mu^{(k)} dx^\mu,$$

\begin{align*}
\Phi_k &= -\int_{r_A}^{r_B} \frac{E_k dr}{\sqrt{1 - B(r)} C(r) r^2} \left[ \frac{m_k^2}{2E_k^2} \frac{d^2}{C(r) r^2} \left( 1 + 2B(r) \right) \right] \right],
\end{align*}
where $d$ denotes the impact parameter. We notice that when the $(m_2^2/E_k^2)^2$ is neglected, the result of Ref.\[6\] is reproduced. Also, the minus sign is irrelevant and we will drop it for simplicity.

3. Flavour changing probability in neutrino oscillations

Knowing the phase gained by neutrinos during their propagation one may ask about the probability of a neutrino to change its flavour as a function of the distance from the source. Let us for simplicity limit ourselves to the two-neutrino case. Then the flavour changing probability in its text-book form is

$$P = \sin^2(2\theta) \sin^2\left(\frac{\Delta E}{2}t\right),$$

(5)

$\theta$ being the mixing angle. In our case, the time-dependent phases have to be modified by the phase coming from the gravitational field. The latter, however, depends on the impact parameter $d$, which represents the distance in which the neutrino passes the black hole. Therefore we recognize two sources of possible interference among neutrino states: between different mass eigenstates going along the same path, and between mass eigenstates going along slightly different paths. We call the different paths “long” (L) and “short” (S) and rewrite the standard definitions of the time-dependent fields as

$$|\nu_e(t)\rangle = \frac{\cos \theta}{2} \left( e^{-i(\Phi_{L1} + \Phi_{L2} - \Phi_{S1} - \Phi_{S2} + \Delta Et)} - e^{-i(\Phi_{L1} - \Phi_{S1} + \Delta Et)} \right) |\nu_1\rangle$$

$$+ \frac{\sin \theta}{2} \left( e^{-i(\Phi_{L2} - \Phi_{S2})} + e^{-i(\Phi_{L1} - \Phi_{S1})} \right) |\nu_2\rangle,$$

(6)

$$|\nu_\mu(t)\rangle = -\frac{\sin \theta}{2} \left( e^{-i(\Phi_{L1} + \Phi_{L2} - \Phi_{S1} - \Phi_{S2} + \Delta Et)} - e^{-i(\Phi_{L1} - \Phi_{S1} + \Delta Et)} \right) |\nu_1\rangle$$

$$+ \frac{\cos \theta}{2} \left( e^{-i(\Phi_{L2} - \Phi_{S2})} + e^{-i(\Phi_{L1} - \Phi_{S1})} \right) |\nu_2\rangle.$$

(7)

The usual expression for the probability now turns into

$$P_{\text{grav}} = \frac{\sin^2(2\theta)}{8} \left[ 2 + \cos(\Phi_{L1} - \Phi_{S1}^S) + \cos(\Phi_{L2} - \Phi_{S2}^S) \right. $$

$$\left. - \cos(\Phi_{L1} - \Phi_{L2} + \Delta Et) - \cos(\Phi_{S1}^L - \Phi_{S2}^S + \Delta Et) \right] - \cos(\Phi_{S1}^L - \Phi_{S2}^L + \Delta Et) - \cos(\Phi_{L1} - \Phi_{S1}^S + \Delta Et),$$

(8)

which represents the probability of $\nu_e \rightarrow \nu_\mu$ transition in the background of the gravitational field. As a check one notices, that for $\Phi = 0$ the usual probability Eq. (5) is recovered. In what follows we will attempt to estimate the full probability $P_{\text{grav}}$ as well as the difference $P_{\text{grav}} - P$ for some special cases.

4. Approximate solution for the phase $\Phi_k$

The integral Eq. (4) is not solvable exactly in terms of elementary or special functions. We will attempt, however, to give an estimate of the solution for two extreme cases: a super-massive and a micro black hole.
An approach that has been used in Ref. 6 was the so-called weak field approximation, in which it is globally assumed that $GM \ll r$. This has lead to the solution $\Phi_k \sim m_k^2 GM/E_0$, cf. Eq. (59) in Ref. 6, which increases with the mass of the source of the gravitational field. As this approximation is valid in certain cases, we will not use it here. One example for which it cannot be used is the super-massive black hole which is believed to reside in the center of our galaxy. Its mass is estimated to be around $10^{37}$ kg. One may easily check that neither for $r$ close to its event horizon $\sim 10^{10}$ m, nor for $r$ being approximately the distance between the Earth and the center of the Milky Way $\sim 10^{20}$ m, this approach is not justified.

A few words about possible metrics describing topological defects are in order. For example, a black hole pierced by a cosmic string is described by the metric (2) with $B(r) = 1 - R/r$, $C(r) = 1 - 4\mu$, while a black hole with a global monopole has $B(r) = 1 - 8\pi G\eta^2 - R/r$, $C(r) = 1 - 8\pi G\eta^2$. Both of these cases, although physically different, are mathematically equivalent, with $B(r)$ being a function of the black hole’s mass $M$ and the distance $r$, and $C(r) = \text{const}$. The parameters $\mu$ and $\eta$ are purely theoretical and only rough bounds for them can be formulated, but they are generically very small. In order not to violate existing observations, $C(r)$ is believed to be of the order $1 - 10^{-\alpha}$ with $\alpha = 6 - 15$.

For the asymptotic case $r \to \infty$ one may replace both $B(r)$ and $C(r)$ by 1. This yields in the leading order

$$\Phi_{\text{far}}^k \approx \frac{m_k^2}{2E_k} \left[ \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{d^2}{r^2}}} \right] \approx \frac{m_k^2}{2E_k} (r_B - r_A). \quad (9)$$

For the close limit, $r \approx d \geq R$, we approximate $B(r) \approx 1 - \frac{R}{d}$. This simplification results in

$$\Phi_{\text{close}}^k \approx \frac{m_k^2}{2E_k} \left[ \int_{r_A}^{r_B} \sqrt{1 - \left(1 - \frac{R}{d}\right) \frac{d^2}{C r^2}} \right]^{d'}. \quad (10)$$

One may check by solving the inequality $r^3 - d^2 r/C + d^2 R/C > 0$, that if only $r > R$ there is always a $C$ such that $\Phi_k$ is real. An example is presented in Fig. 1 in which the Schwarzschild radius has been taken to be $10^{10}$ m. This value corresponds to the super-heavy ($M \sim 10^{37}$ kg) black hole that is anticipated to reside in the center of our galaxy. On the other hand, for a micro black hole ($M \sim 10^{-27}$ kg) that may appear in the LHC experiments, Fig. 1 has to be rescaled such that $R \sim 10^{-54}$ m.

The generic approximation presented above may be reformulated in some special numerical cases by taking the leading terms, which dominate significantly over the others. For instance, in the super massive case, the close limit up to the second order in relativistic expansion is given by

$$\Phi_{\text{close}}^k \approx \frac{E_k}{\sqrt{2GMd^2} \frac{m_k^2}{2E_k}} \left[ \frac{2}{5} \left(1 + \frac{m_k^2}{4E_k^2} \right) r^\frac{d}{2} + \frac{1}{9} \frac{m_k^2}{E_k^2} d^2 r^\frac{d}{2} \right]^{d'}. \quad (11)$$
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Fig. 1. Distance \( r \) as a function of the metric parameter \( C \), that corresponds to real phases \( \Phi_k \) in the case of a super massive black hole from the center of the Milky Way. The shaded region represents physically acceptable solutions.

On the other hand, the close limit for a micro black hole takes the form

\[
\Phi_k^{\text{close}} \approx E_k \frac{m_1^2}{2E_k^2} \left[ \frac{m_2^2}{2E_k^2} \frac{r^2 - d^2}{\sqrt{r^2 - d^2}} - \frac{5m_2^2}{2E_k^2} \frac{d'}{d} \right]. \tag{12}
\]

The far limits are basically unaffected, as all metrics we may be interested in are asymptotically flat.

The oscillation probabilities for the super-massive black hole are depicted in Fig. 2. We recall here, that the Earth, thus our would-be observation point, is roughly at the distance \( r \approx 2.5 \times 10^{20} \text{m} \). The actual numbers used in the calculations were \( d_S = R, d_L = 100R, m_1 = 0.30 \text{ eV}, m_2 = 0.29 \text{ eV}, \) and \( \Delta E = 1 \text{ eV} \). In Fig. 3 we have collected the functions \( P_{\text{grav}} - P \) calculated for different energies of the neutrinos. In all the cases the oscillatory behaviour with interference patterns is eminent.

Another possible case of interest is a micro black hole. Given that certain theories which assume the existence of additional spatial dimensions are true, at the energy scales which will be reached in the Large Hadron Collider in CERN some extradimensional black holes should appear. Their mass is expected to be of the order of \( 10^{-27} \text{ kg} \), which corresponds to the Schwarzschild radius \( R \sim 10^{-54} \text{ m} \). These extremely tiny objects would almost instantly evaporate, however, some models predict similar objects to be created and travel almost freely through the Universe. As such, they may act as gravitational lenses (in the same way as does regular black holes and other massive dark objects). The results are presented in Fig. 4 from which one sees that the difference in transition probabilities may reach the order of \( 10^{-6} \) and more on the distance of at least 1500 km. This may hypothetically

Fig. 2. Flavour changing probability \( P_{\text{grav}} \) in neutrino oscillations as a function of the distance to the Milky Way black hole. The result based on the standard formula \( P \) is shown for reference (right panel). Neutrino energy \( E_k = 1 \text{ GeV} \). See also Fig. 3.
5. Conclusions

Basing on the results presented in Ref. 6, in which the phase $\Phi_k$ is proportional to $M$, one may expect that the black holes should generate a gravitational field strong enough, to give a substantial contribution to the neutrino oscillations. The conclusion from our calculations is quite contrary. Even though the general formula is known, see Eq. 4, its solution depends on the actual metric parameters, and in most cases requires pure numerical treatment. We have managed to formulate approximate analytical solutions for two distinct
examples of a super-massive and a micro black hole with topological defects. Numerical illustrations of the flavour changing probability as a function of the distance and neutrino energies have also been presented. In the case of the super-massive black hole, the interference patterns are quite different for different energies. It is, however, difficult to imagine how one would be able to use this knowledge in practice. A more promising case is the micro black hole. Firstly, using accelerators like the LHC or even more powerful which will be built in the future, one may pretty well localize the spot in which such a (hypothetical) black hole will be produced. The difference in flavour changing probabilities, $10^{-6}$ for 1500 km up to almost $10^{-5}$ for the Earth diameter, are still not possible to detect now, but such sensitivity may be probably reached in the future.

References

1. B. Pontecorvo, Sov. Phys. JETP 26, 984 (1968); V.N. Gribov, B. Pontecorvo, Phys. Lett. B28, 493 (1969).
2. S. Fukuda et al. (Super-Kamiokande Collaboration), Phys. Rev. Lett. 81, 1562 (1998); Y. Ashie et al. (Super-Kamiokande Collaboration), Phys. Rev. Lett. 93, 101801 (2004); Phys. Rev. D 71, 112005 (2005); K. Abe et al. (Super-Kamiokande Collaboration), Phys. Rev. Lett. 97, 171801 (2006); Phys. Rev. D 77, 052001 (2008); T. Araki et al. (KamLAND Collaboration), Phys. Rev. Lett. 94, 081801 (2005); S. Abe et al. (KamLAND Collaboration), Phys. Rev. Lett. 100, 221803 (2008); Q.R. Ahmed et al. (SNO Collaboration), Phys. Rev. Lett. 87, 071301 (2001); Phys. Rev. Lett. 89, 011301 (2002); Phys. Rev. Lett. 89, 011302 (2002); B. Aharmin et al. (SNO Collaboration), Phys. Rev. C 72 055502 (2005); Phys. Rev. C 75 045502 (2007); M. Apollonio et al. (CHOOZ Collaboration), Phys. Lett. B 466, 415 (1999); Eur. Phys. J. C 27, 331 (2003); G.L. Fogli et al., Phys. Rev. D 66, 093008 (2002).
3. S.M. Bilenky, Phys. Scr. T 121, 17 (2005); R.N. Mohapatra, A.Y. Smirnov, Ann. Rev. Nucl. Part. Sci. 56, 569 (2006); D.P. Roy, Phys. News 39, 51 (2009).
4. G.L. Fogli et al., Phys. Rev. D 78, 033010 (2008).
5. C.Y. Cardall, G.M. Fuller, Phys. Rev. D 55, 7960 (1997).
6. N. Fornengo, C. Giunti, C.W. Kim, J. Song, Phys. Rev. D 56, 1895 (1997).
7. T. Bhattacharya, S. Habib, E. Mottola, Phys. Rev. D 59, 067301 (1999).
8. L. Stodolsky, Gen. Relativ. Gravit. 11, 391 (1979).
9. A.M. Ghez et al., Astrophys. J. 620, 744 (2005).
10. A. Villenkin, E.P.S. Shellard, Cosmic Strings and other Topological Defects, Cambridge (1994); W.H.C. Freire, V.B. Bezerra, J.A.S. Lima, Braz. J. Phys. 30, 398 (2000).