Detection of the gravitomagnetic clock effect

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Abstract

The essence of the gravitomagnetic clock effect is properly defined showing that its origin is in the topology of world lines with closed space projections. It is shown that, in weak field approximation and for a spherically symmetric central body, the loss of synchrony between two clocks counter-rotating along a circular geodesic is proportional to the angular momentum of the source of the gravitational field. Numerical estimates are presented for objects within the solar system. The less unfavorable situation is found around Jupiter.

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I. Introduction

Recently Bonnor and Steadman (BS) published a paper on the so called gravitomagnetic clock effect. The subject is important because it could hint at an opportunity to verify a general relativistic effect caused by the angular momentum of a source of gravitational field. The present work is intended to clarify the nature and the origin of the effect and contains also some new calculations of the synchrony defect between identical clocks moving in opposite senses around a spinning massive body; the order of magnitude of this defect is within the range of the observations which can be made in the solar system.

The gravitomagnetic clock effect BS speak of is the difference in time shown after one revolution by two identical clocks co-rotating and counter-rotating on a circular orbit around a spinning mass. The above authors show that the amount of the time difference depends on the reference frame with respect to which the phenomenon is described and discuss two special cases: a rotating massive infinitely long dust cylinder and a Kerr black hole.

In fact however, with a slight difference in perspective, one could better speak of a gravitomagnetic effect directly comparing the proper times read on two oppositely rotating clocks when their trajectories intersect each other at a point. The difference in the readings will so be an absolute quantity, whose value is a measure of the angular momentum of the source. Other “clock effects” have been considered in the literature depending on the nature of the clock and related for instance to the coupling of the spin with the angular momentum of the gravity source: what we are treating here is the purely geometrical effect, perfectly insensitive to the nature of the clock.

We shall first remind and show that the special relativistic root of the phenomenon is in the behaviour of clocks in rotary motion (sect. II), then in sect. III it will be shown how the same situation is described in a gravitational non rotating field. Sect. IV deals with the drag effect due to the presence of a rotating source. In sect. V the van Stockum and Kerr metrics are considered. Sect. VI contains some estimates valid for the solar system environment and finally, in sect. VII, we shall draw some conclusions.
II. Effect in the absence of a gravitational field

The standard situation is the one where the rotation is considered from a viewpoint far away from the gravitational source, where the space time is flat, and supposing that the observer is at rest with respect to the rotation axis. The, so to speak, zero level case is the one where the two clocks perform their rotational motion in the absence of any gravitational field; of course no closed space geodesics exist in this case, a closed circuit then requires some constraint. When the two clocks move with the same speed in opposite directions along a circular path the situation is trivial: whenever they cross each other at a point they turn out to be synchronous. The situation is different when the angular speeds are different, no matter whether they are in the same sense or in opposite senses. This is indeed a peculiarity of special relativity and a special case of the so called Sagnac effect. The Sagnac effect is usually expressed in terms of the propagation of light emitted by a rotating source but its essence can be reduced precisely to the lack of synchrony of clocks moving along closed space paths (for a comprehensive review on the Sagnac effect see ). The phenomenon is a purely geometrical property of Minkowski space time. In fact the four dimensional trajectory of a steadily rotating object is a timelike helix; the proper times to compare, shown by two identical clocks rotating with different angular speeds along the same circumference, are the lengths of the helixes arcs intercepted between two successive intersection events: these lengths are necessarily different when the speeds are different.

Let us call $R$ the radius of the circular path, $\phi$ the rotation angle, $\omega$ the angular speed with respect to an inertial static observer and $t$ the time of the inertial observer. The world line of a rotating clock may then be written

$$\begin{cases} t = \frac{\phi}{\omega} \\ r = R \end{cases} \quad (1)$$

Consider now two identical clocks rotating respectively with angular speeds $\omega_1$ and $\omega_2$ at the same $R$, and synchronous in $t = \phi = 0$; the next encounter between them will happen when

$$\frac{\phi}{\omega_1} = \frac{\phi \pm 2\pi}{\omega_2} \quad (2)$$

where $\phi$ is the angular coordinate of the first clock; the $+$ sign holds when $\omega_2 > \omega_1$, the $-$ one when $\omega_2 < \omega_1$. 
Solving (2) for \( \phi \) (or, to say better, \( \phi_1 \)) gives the angular span between two successive intersections of the helixes:

\[
\phi_1 = 2\pi \frac{\omega_1}{|\omega_2 - \omega_1|} \tag{3}
\]

For the other clock it would be:

\[
\phi_2 = 2\pi \frac{\omega_2}{|\omega_2 - \omega_1|} \tag{4}
\]

The length of the helix arc spanned by the angle (3) or (4), divided by \( c \), is the proper time \( \tau \) shown by the corresponding clock:

\[
\tau_1 = \int_{\text{helix}} \sqrt{dt^2 - \frac{R^2 d\phi^2}{c^2}} = \sqrt{1 - \frac{R^2 \omega_1^2}{c^2}} \frac{2\pi}{|\omega_2 - \omega_1|}
\]

\[
\tau_2 = \sqrt{1 - \frac{R^2 \omega_2^2}{c^2}} \frac{2\pi}{|\omega_2 - \omega_1|}
\]

The lack of synchrony at the first intersection event is:

\[
\Delta \tau = |\tau_2 - \tau_1| = \frac{2\pi}{|\omega_2 - \omega_1|} \left( \sqrt{1 - \frac{R^2 \omega_1^2}{c^2}} - \sqrt{1 - \frac{R^2 \omega_2^2}{c^2}} \right) \tag{5}
\]

It must be remarked that \( \Delta \tau \) is not, strictly speaking, registered after one revolution of any of the clocks: the two objects meet again after more or less than one revolution, according to which of them is being considered. When \( \omega_1 > 0 \) and \( \omega_2 < 0 \) the intersection corresponding to (5) actually happens a bit after or before a half revolution.

The quantity (5) is of course zero when \( \omega_2 = -\omega_1 \). Suppose one of the clocks (be it the number 1) is stationary in the inertial reference frame; now its revolution time is infinite, but this does not mean that \( \Delta \tau \) too is infinite or is undefined. The intersections between the world lines of the two clocks are still perfectly recognizable; simply it is \( \omega_1 = 0 \) and \( \Delta \tau = \frac{2\pi}{\omega_2} \sqrt{1 - \frac{R^2 \omega_2^2}{c^2}} \).

Formula (5) expresses an absolute, i.e. invariant, quantity. This means that the result is the same no matter what the reference frame is. We can in particular consider the case of an observer steadily rotating along the same circumference at an (inertial) angular speed \( \omega_o \). The synchrony defect between the two moving clocks is still (5) but now it should be expressed
in terms of the parameters relevant for the new non inertial observer. The revolution time of a clock as measured by him is indeed:

\[ T = \sqrt{1 - \frac{R^2\omega_o^2}{c^2}} \frac{2\pi}{|\omega - \omega_o|} \]  

(6)

From (6) it is possible to find the relation between the angular speeds as seen by the moving observer and the ones measured in the inertial frame. One obtains

\[ \omega' = \frac{2\pi}{T} = \frac{\omega - \omega_o}{\sqrt{1 - \frac{R^2\omega_o^2}{c^2}}} \]  

(7)

Inverting (7) to find \( \omega \), then substituting in (5) we express the synchrony defect in terms of the parameters of the moving observer:

\[ \Delta \tau = \frac{2\pi}{(\omega'_2 - \omega'_1)} \left( \sqrt{1 - \frac{R^2}{c^2} \left( \omega'_1 + \frac{\omega_o}{\sqrt{1 - \frac{R^2\omega_o^2}{c^2}}} \right)^2} - \sqrt{1 - \frac{R^2}{c^2} \left( \omega'_2 + \frac{\omega_o}{\sqrt{1 - \frac{R^2\omega_o^2}{c^2}}} \right)^2} \right) \]  

(8)

The invariant character of the synchrony defect means also that if it is non zero in one reference frame it will be so in any other too; in other words no \( \omega_o \) value is able to bring \( \Delta \tau \) to 0 when \( \omega_1 \neq -\omega_2 \) (this inequality corresponds to \( \omega'_1 \neq -\omega'_2 - \frac{2\omega_o}{\sqrt{1 - \frac{R^2\omega_o^2}{c^2}}} \)).

III. Effect of the gravitational field

The special relativistic analysis in the previous section is the basis of the evaluation of the effect when a gravitational field is present. In fact the structure of the phenomenon is not different; what is different is the metric of space time. A point to be stressed is that the relevant events are those in which the two clocks coincide in space; using these events frees from the need of a third party deciding when a complete revolution has been performed. Also the fact that the trajectory is a geodesic turns out to be irrelevant, being reduced to a special case of the general situation. Of course the presence of a gravitational field renders closed space geodesics available, which was not the case in Minkowski space time.

Let us consider first a static spherically symmetric metric. The world line of an object steadily moving along a circular trajectory in an equatorial
plane of the source is still a helix. The typical line element in flat space time polar coordinates is: 

\[ ds^2 = c^2 e^{2\lambda} dt^2 - e^{2\nu} dr^2 - e^{2\nu} r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

where \( \lambda, \mu, \nu \) are functions of \( r \) only. The corresponding equation of the helix is

\[ \phi = \omega t \]

\[ r = \text{constant} \]

and the helix arc, in terms of proper time, spanned by the angle \( d\phi \) is

\[ d\tau = \sqrt{e^{2\lambda} \omega^2 - e^{2\nu} \frac{r^2}{c^2}} d\phi \]

On these bases the synchrony defect of two identical clocks rotating with speeds \( \omega_1 \) and \( \omega_2 \) at the same radius \( r \) is

\[ \Delta \tau = \frac{2\pi e^\lambda}{\omega_2 - \omega_1} \left( \sqrt{1 - e^{2(\nu - \lambda) \frac{\omega_1^2 r^2}{c^2}}} - \sqrt{1 - e^{2(\nu - \lambda) \frac{\omega_2^2 r^2}{c^2}}} \right) \quad (9) \]

Again (and trivially because of the symmetry) it can be \( \Delta \tau = 0 \) only when \( \omega_2 = -\omega_1 \).

Formula (9) is expressed in terms of parameters of an inertial static observer; it is however an invariant quantity. If we want the result from the viewpoint of a steadily rotating observer on the same circumference as the clocks, we can proceed as in the flat space time case. The revolution time of a clock as seen by the rotating observer is

\[ T = \frac{2\pi e^\lambda}{|\omega - \omega_o|} \sqrt{1 - e^{2(\nu - \lambda) \frac{\omega_o^2 r^2}{c^2}}} \]

and the corresponding angular speed is

\[ \omega' = \frac{(\omega - \omega_o) e^{-\lambda}}{\sqrt{1 - e^{2(\nu - \lambda) \frac{\omega_o^2 r^2}{c^2}}} \sqrt{1 - e^{2(\nu - \lambda) \frac{\omega_o^2 r^2}{c^2}}}} \]

Solving for \( \omega \) and substituting into (9), then posing \( e^{\nu - \lambda} r = R \) leads to

\[ \Delta \tau = \frac{2\pi}{(\omega_2' - \omega_1')} \left( \sqrt{1 - \frac{R^2}{\omega_o^2 c^2}} - \frac{R^2}{c^2} \left( \omega_1' e^\lambda + \frac{\omega_o}{\sqrt{1 - \frac{R^2}{\omega_o^2 c^2}}} \right) \right)^2 \]
\[-\sqrt{\frac{1}{1 - \omega R^2 e^\lambda} - \frac{R^2}{c^2} \left( \frac{\omega e^\lambda}{\sqrt{1 - \omega R^2 e^\lambda}} \right)^2}\]

The presence of the term $e^\lambda$, now to be considered as a function of $R$, manifests the presence of the field.

If the clocks are orbiting along a circular space geodesic the angular speeds are necessarily the same but in opposite directions, consequently it is trivially $\Delta \tau = 0$.

### IV. Angular momentum effect

A more interesting situation is found when the source of the gravitational field possesses an angular momentum. In this case the metric is in general axisymmetric; assuming it is also stationary and has a full cylindrical space symmetry we can write, using appropriate coordinates:

$$ds^2 = f c^2 dt^2 - 2Kcdt d\phi - ld\phi^2 - H \left( dr^2 + dz^2 \right)$$  \hspace{1cm} (10)

where $f$, $K$, $l$, $H$ are all functions of $r$ only.

We limit our analysis to the plane $z = 0$. A constant speed circular space trajectory in that plane will correspond again to a helical world line in space time ($\phi$ is proportional to $t$, $\omega = d\phi/dt$). The helix arc spanned by an angle $\phi$ and expressed as the proper time of a clock is now:

$$\tau = \sqrt{\frac{f}{\omega^2} - 2 \frac{K}{c\omega} - \frac{l}{c^2} \phi}$$ \hspace{1cm} (11)

Formulae (2), (3) and (4) continue to hold, consequently it is:

$$\Delta \tau = \left| \frac{2\pi}{\omega_2 - \omega_1} \left( \sqrt{f - 2 \frac{K}{c} \omega_1 - \frac{l}{c^2} \omega_1^2} - \sqrt{f - 2 \frac{K}{c} \omega_2 - \frac{l}{c^2} \omega_2^2} \right) \right|$$ \hspace{1cm} (12)

This result differs from the one obtained by Bonnor and Steadman in that theirs considers revolution times as seen by an inertial static observer, whereas here $\Delta \tau$ is the lack of synchrony obtained directly comparing two clocks when they coincide in space time.
Now the effect is zero, excluding the trivial solution $\omega_2 = \omega_1$, when

$$2 \frac{K}{c} \omega_1 + \frac{l}{c^2} \omega_1^2 = 2 \frac{K}{c} \omega_2 + \frac{l}{c^2} \omega_2^2$$

i.e. for

$$\omega_2 = -\omega_1 - 2 \frac{K}{l} c$$  \hspace{1cm} (13)

When the angular momentum of the source (which is in general proportional to $K$) is zero the solution comes back to the known $\omega_2 = -\omega_1$.

Again we can express (12) in terms of the variables seen by a non inertial observer rotating at an angular speed $\omega_o$ along the same circumference as the clocks. The revolution time measured by the observer is now

$$T = \frac{2\pi}{\omega - \omega_o} \sqrt{f - 2 \frac{K}{c} \omega_o - \frac{l}{c^2} \omega_o^2}$$

The angular speed he sees is then

$$\omega' = \frac{\omega - \omega_o}{\sqrt{f - 2 \frac{K}{c} \omega_o - \frac{l}{c^2} \omega_o^2}}$$

Solving for $\omega$, then introducing the result into (12) one ends up with a rather messy expression. The operation is simpler in the translation of (13), which becomes

$$\omega'_2 = -\omega'_1 - 2 \frac{\omega_o + \frac{K}{c} c}{\sqrt{f - 2 \frac{K}{c} \omega_o - \frac{l}{c^2} \omega_o^2}}$$

The results found up to now can be specialized to the case of geodesic circular trajectories with the same radius $r$. For such geodesics the condition must be satisfied:

$$c^2 \frac{df}{dr} - 2c \frac{dK}{dr} - \frac{dl}{dr} \omega^2 = 0$$

Two solutions exist for $\omega$, provided $\left(\frac{dK}{dr}\right)^2 + \frac{dl}{dr} \frac{df}{dr} \geq 0$. Posing $\frac{dr}{dr} = f'$,

$$\frac{dK}{dr} = K', \quad \frac{dl}{dr} = l'$$

one has

$$\omega_{1,2} = \frac{c}{l'} \left(-K' \pm \sqrt{K'^2 + l' f'}\right)$$  \hspace{1cm} (14)
Introducing (14) into (12) leads to

$$\Delta \tau = \frac{\pi l'}{c \sqrt{K'^2 + l' f'}} \left( \sqrt{f - \frac{2}{l'} \left(-K' + \sqrt{K'^2 + l' f'} \right)} \right) - \frac{1}{l'} \left(-K' + \sqrt{K'^2 + l' f'} \right)^2 +$$

$$- \sqrt{f - \frac{2}{l'} \left(-K' - \sqrt{K'^2 + l' f'} \right)} - \frac{1}{l'} \left(-K' - \sqrt{K'^2 + l' f'} \right)^2 \right)$$

The condition for our two freely falling clocks to stay synchronous is (13), which, together with (14) gives

$$\frac{K'}{l'} = \frac{K}{l} \quad (15)$$

This equation is identically satisfied whenever it is $K$ proportional to $l$; otherwise it gives the special values of $r$, if they exist, such that the synchrony condition is maintained.

The result is again independent from any peculiar reference frame.

**V. Special cases**

The treatment of the problem has been quite general up to this moment. We can of course specialize to some particular cases. Bonnor and Steadman considered both the van Stockum space time and the Kerr metric. They however focused on a quantity depending on the reference frame because they compared different revolution times seen by an observer, rather than proper time lapses between absolute events such as the coincidence of two clocks in space and time.

The calculation of the absolute synchrony defect for a constant radius and constant angular speed geodesic orbit around a rigidly rotating dust cylinder (van Stockum metric) is straightforward but rather tedious and not particularly useful. We may just explicit the condition for maintaining synchrony, recovering from the following expressions for the exterior of the cylinder (whose radius is $R$) and adapting them to our notations:

$$K = \frac{a R^2}{c} \left[ \frac{1}{4n} \left(2n + 1\right) \left(\frac{r}{R}\right)^{2n+1} + \frac{1}{2n} \left(\frac{r}{R}\right)^{1-2n} \right]$$
\[ l = \frac{R^2}{16n} \left[ (2n + 1)^3 \left( \frac{r}{R} \right)^{2n+1} + (2n - 1)^3 \left( \frac{r}{R} \right)^{1-2n} \right] \]

\[ n^2 = \frac{1}{4} - \frac{a^2 R^2}{c^2} \]

The parameter \( a \) can be interpreted as the angular velocity on the axis of the cylinder.

It is

\[ K' = \frac{a R^2}{c} \frac{1}{4n} \left[ \frac{(2n + 1)^2}{r} \left( \frac{r}{R} \right)^{2n+1} - \frac{(2n - 1)^2}{r} \left( \frac{r}{R} \right)^{1-2n} \right] \]

\[ l' = \frac{1}{16} \frac{R^2}{n} \left[ \frac{(2n + 1)^4}{r} \left( \frac{r}{R} \right)^{2n+1} - \frac{(2n - 1)^4}{r} \left( \frac{r}{R} \right)^{1-2n} \right] \]

and (15) reduces to

\[ \frac{(2n + 1)^2 \left( \frac{r}{R} \right)^{2n+1} - (2n - 1)^2 \left( \frac{r}{R} \right)^{1-2n}}{(2n + 1)^4 \left( \frac{r}{R} \right)^{2n+1} - (2n - 1)^4 \left( \frac{r}{R} \right)^{1-2n}} = \frac{(2n + 1) \left( \frac{r}{R} \right)^{2n+1} + (2n - 1) \left( \frac{r}{R} \right)^{1-2n}}{(2n + 1)^3 \left( \frac{r}{R} \right)^{2n+1} + (2n - 1)^3 \left( \frac{r}{R} \right)^{1-2n}} \]

or \( X = \left( \frac{r}{R} \right)^{4n} \)

\[ \frac{(2n + 1)^2 X - (2n - 1)^2}{(2n + 1)^4 X - (2n - 1)^4} = \frac{(2n + 1) X + (2n - 1)}{(2n + 1)^3 X + (2n - 1)^3} \]

The formal solutions are

\[ r = R \left( \frac{2n - 1}{2n + 1} \right)^{1/4n} \]

\[ r = R \left( \frac{1 - 2n}{2n + 1} \right)^{3/4n} \]

Provided \( n \) is real, the upper solution holds when \(|n| > \frac{1}{2}\), the other one when \(|n| < \frac{1}{2}\). It is however \( r < R \) in any case, this means that no exterior circular geodesic orbit exists along which two clocks can stay synchronous.

Let us come now to the Kerr metric. A problem similar to the present one, where instead of the two clocks we found a couple of light beams, has been treated in [13]. Using the same notations as there and considering an
equatorial circular orbit, the elementary arc length of the four-dimensional helix of a clock is

\[ d\tau = \sqrt{\frac{1}{\omega^2} - \frac{a^2}{c^4} - \frac{r^2}{c^2} - 2GM \frac{1}{c^2r} \left( \frac{1}{\omega} - \frac{a}{c^2} \right)^2} d\phi \]

The parameter \( a = J/M \) is the ratio between the angular momentum \( J \) and the mass \( M \) of the source.

Using the general formulae (3) and (4) we obtain for the proper times marked by the two clocks between two successive rendezvous:

\[ \tau_1 = \sqrt{1 - \left( \frac{a^2}{c^2} + r^2 \right) \frac{\omega_1^2}{c^2} - 2GM \frac{M}{c^2r} \left( 1 - \frac{a}{c} \omega_1 \right)^2} \frac{2\pi}{|\omega_2 - \omega_1|} \]

\[ \tau_2 = \sqrt{1 - \left( \frac{a^2}{c^2} + r^2 \right) \frac{\omega_2^2}{c^2} - 2GM \frac{M}{c^2r} \left( 1 - \frac{a}{c} \omega_2 \right)^2} \frac{2\pi}{|\omega_2 - \omega_1|} \]

which amounts to have an absolute synchrony defect

\[ \Delta \tau = \frac{2\pi}{|\omega_2 - \omega_1|} \left| \sqrt{1 - \left( \frac{a^2}{c^2} + r^2 \right) \frac{\omega_1^2}{c^2} - 2GM \frac{M}{c^2r} \left( 1 - \frac{a}{c} \omega_1 \right)^2} - \sqrt{1 - \left( \frac{a^2}{c^2} + r^2 \right) \frac{\omega_2^2}{c^2} - 2GM \frac{M}{c^2r} \left( 1 - \frac{a}{c} \omega_2 \right)^2} \right| \]  \hspace{1cm} (16)

The \( \omega \)'s are the ones seen by a far away inertial observer. To maintain synchrony (\( \Delta \tau = 0 \)) it must be

\[ \omega_2 = -\omega_1 + \frac{4GMac^2}{a^2c^2r + r^3c^4 + 2GMa^2} \]  \hspace{1cm} (17)

The last term on the right is twice the angular velocity of a "locally non rotating observer" (LNRO)\(^{11}\). Actually such an observer would indeed find that two counter-rotating clocks remain synchronous in his reference frame: converting (17) into the variables of the LNRO one obtains \( \omega'_2 = -\omega'_1 \).

It is possible to specialize the result to circular geodesics in the equatorial plane. It is then enough to substitute into (16) the angular speeds\(^{11}\):

\[ \omega_{1,2} = \frac{c}{a \pm c^2 \sqrt{\frac{r^3}{GM}}} \]  \hspace{1cm} (18)
The explicit results are not particularly enlightening though.

The solution of the problem in the Kerr metric lends itself to the study of its weak field limit suitable for clocks moving around ordinary astronomical spinning bodies such as the earth or the sun.

Let us assume for simplicity that

$$\frac{a}{cr} \sim \frac{\omega r}{c} \sim \frac{GM}{c^2 r} < < 1$$

From (16) we have consequently

$$\Delta \tau \simeq 4\pi \frac{GM}{c^4 r} a - \frac{\pi r^2}{c^2} (\omega_1 + \omega_2) \quad (19)$$

The second term is nothing but the first order approximation of (5). For a circular geodesic (18) gives approximately

$$\omega_1 + \omega_2 \simeq -2 \frac{GM}{c^2 r^3} a$$

The final expression for (19) is then

$$\Delta \tau \simeq 6\pi \frac{GM}{c^4 r} a = 6\pi \frac{GI}{c^4} \Omega \quad (20)$$

This result coincides with the one obtained in ref. [12] provided one remembers that here we refer to the first conjunction of the clocks after the origin (a half revolution) whereas there the second one is considered (a full revolution).

In the last term of (20) the non relativistic form for \(a\) appears, being \(I\) the classical moment of inertia of the source and \(\Omega\) its angular speed (if a solid object is considered).

VI. Solar system estimates

Considering for simplicity solid, homogeneous, spherical, non relativistic objects, it is [10],

$$a = \frac{2}{5} R^2 \Omega \quad (21)$$

where \(R\) is the radius of the body and \(\Omega\) its angular velocity.
The numerical values of \( (21) \) respectively for the Earth, the Sun and Jupiter are

\[
\begin{align*}
a_{\text{Earth}} &= 1.2 \times 10^9 \text{ m}^2/\text{s} \\
a_{\text{Sun}} &= 8.9 \times 10^{11} \text{ m}^2/\text{s} \\
a_{\text{Jupiter}} &= 3.6 \times 10^{11} \text{ m}^2/\text{s}
\end{align*}
\]

Correspondingly the orders of magnitude of the synchrony defects at the first conjunction (after a half revolution) will be

\[
\begin{align*}
\Delta \tau_{\text{Earth}} &\sim 10^{-16} \text{ s} \\
\Delta \tau_{\text{Sun}} &\sim 10^{-11} \text{ s} \\
\Delta \tau_{\text{Jupiter}} &\sim 10^{-12} \text{ s}
\end{align*}
\]

The radii of the orbits of the clocks have been assumed to be respectively \( \sim 10^7 \text{ m}, \sim 10^{10} \text{ m}, \sim 10^8 \text{ m} \).

Time differences like these are extremely small but, at least for the Sun and Jupiter they are in the range of measurability and their detection would be a test of the influence that the angular momentum exerts on the pace of the clocks. It must however be considered that the clocks to be compared should be on board of orbiting spacecrafts and should be appropriately stable for times sufficiently long.

A good parameter to measure the need for stability is obtained comparing the synchrony defect at the first conjunction with the duration of one orbit. For a typical satellite orbiting the Earth, this ratio is \( \varepsilon \sim 10^{-19} \).

In the case of Sun orbits of the order of the one of Mercury (period of a couple months) correspond to \( \varepsilon \sim 10^{-17} \).

The situation would be better for an artificial satellite of Jupiter (10 hours or so period) where \( \varepsilon \sim 10^{-16} \).

VII. Conclusion

We have shown that the most convenient definition of a gravitomagnetic effect on clocks is based on two fiducial absolute events which are two successive crossings of the four dimensional orbits of the clocks, when they rotate about an axis. Basically the effect is topological in origin and it is appropriate to refer to gravity only when the source of the field possesses an angular
momentum. Actually the Sagnac effect and its general relativistic counterpart are but a special case of the phenomenon we have described, when the two "clocks" are light beams.

Comparison of the readings of two clocks oppositely orbiting along circular equatorial geodesics would be a means to reveal the dragging effect of the spinning central body and to measure its angular momentum.

The numerical estimates we have made show that this effect within the solar system is extremely weak, however it is also cumulative. Considering both the size of the effect and the stability requirements the best condition would be attained for a couple of clocks orbiting around Jupiter, also because of the environment less hostile than around the Sun.
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