Compositional ADAM: An Adaptive Compositional Solver

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Abstract
In this paper we present C-ADAM, the first adaptive solver for compositional problems involving a non-linear functional nesting of expected values. We proof that C-ADAM converges to a stationary point in $O(\delta^{-2.25})$ with $\delta$ being a precision parameter. Moreover, we demonstrate the importance of our results by bridging, for the first time, model-agnostic meta-learning (MAML) and compositional optimisation showing fastest known rates for deep network adaptation to-date. Finally, we validate our findings in a set of experiments from portfolio optimisation, and meta-learning. Our results manifest significant sample complexity reductions compared to both standard and compositional solvers.

1. Introduction
The availability of large data-sets coupled with improvements in optimisation algorithms and the growth in computing power has led to an unprecedented interest in machine learning and its applications in, for instance, medical imaging (Faes et al., 2019), autonomous self-driving (Bojarski et al., 2016), fraud detection (Wang et al., 2020), computer games (Mnih et al., 2015; Silver et al., 2017; Tian et al., 2018), and many other fields.

Fueling these successes are novel developments in non-convex optimisation, which is, by now, a wide-reaching field with algorithms involving zero (Hu et al., 2016; Shamir, 2017; Gabillon et al., 2019), first (Sun et al., 2019; Beck, 2017; Menghan, 2019), and second-order methods (Boyd & Vandenberghe, 2004; Nesterov & Polyak, 2006; Tutunov et al., 2016). Albeit such diversity, first-order algorithms are still highly regarded as the go-to technique when it comes to large-scale machine learning applications due to their ease of implementation, and memory and computational efficacy. First-order optimisers can further be split into gradient-based, momentum, and adaptive solvers, each varying in the process by which a decent direction is computed. On one hand, momentum techniques (Nesterov, 2014; Li & Lin, 2015) modify learning rates, while adaptive designs handle the direction itself (Duchi et al., 2011; Zeiler, 2012; Mukkamala & Hein, 2017). In spite of numerous theoretical developments (Liu et al., 2018; Fang et al., 2018; Allen-Zhu & Hazan, 2016), ADAM – an adaptive optimiser originally proposed in (Kingma & Ba, 2014), and then theoretically grounded in (Zaheer et al., 2018; Reddi et al., 2019) – (arguably) retains state-of-the-art status in machine learning practice.

Although this first wave of machine learning applications led to major breakthroughs and paradigm shifts, experts and practitioners are rapidly realising limitations of current models in that they require a large amount of training data and are slow to adapt to novel tasks even when these tasks involve minor changes in the data distribution. The quest for having fast and adaptable models re-surfaced interest in general and model-agnostic machine learning forms, such as meta (Finn et al., 2017; Fallah et al., 2019), lifelong (Thrun & Pratt, 1998; Ruvolo & Eaton, 2013; Bou-Ammar et al., 2014; 2015a;b), and few-shot learning (Wang & Yao, 2019).

When investigated closer, one comes to realise that such key problems typically involve non-linear functional nesting of expected values. For instance, in both meta and lifelong learning, a learner attempts to acquire a model across multiple tasks, each exhibiting randomness in its data generating process. In other words, the goal of the agent is to find a set of deep network parameters that minimise an expectation over tasks, where each task’s loss function involves another expectation over training data. Such a bridge, however, has not been formally established before, partially due to the discrepancy between communities and the lack of stochastic and adaptive

1Please note that in model-agnostic meta-learning such an expectation can involve more than two layers. In this paper, however, we focus on the original presentation (Finn et al., 2017) that performs one gradient step, and as such only includes two nesting. Generalising to an $m$ nesting is an interesting avenue for future research.
compositional solvers that allow for scalable and effective algorithms (see Section 2)\(^2\). With meta-learning as our motivation, we propose compositional ADAM (C-ADAM), the first, to the best of our knowledge, adaptive compositional optimiser that provably converges to a first-order stationary point. Along the way, we also extend standard ADAM’s proof from (Zaheer et al., 2018) to a more realistic setting of hyper-parameters closer to these used by practitioners (i.e., none of the \(\beta\)’s are set to zero). Apart from theoretical rigour, C-ADAM is also stochastic allowing for mini-batching, and only executes one-loop easing implementation.

Unlocking major associations, we further derive a novel connection between model-agnostic meta-learning (MAML) and compositional optimisation allowing, for the first time, the application of compositional methods in large-scale machine learning problems beyond portfolio mean-variance trade-offs, and discrete state (or linear function approximation) reinforcement learning (Wang et al., 2014). Here, we show that MAML can be written as a nested expectation problem and derive corollaries from our results shedding light on MAML’s convergence guarantees. Such an analysis reveals that our method achieves best known convergence bounds to-date\(^3\) for problems involving fast adaptation of deep networks. Validating our theoretical discoveries, we, finally, conduct an in-depth empirical study demonstrating the C-ADAM outperforms others by significant margins.

2. Related Work

Since (Wang et al., 2014) proposed a stochastic compositional gradient optimiser with \(O(\delta^{-4})\) query complexity, much attention has been devoted towards more efficient and scalable solvers. Efforts in (Wang et al., 2017), for instance, led to further improvements achieving \(O(\delta^{-3.5})\) complexity by adopting an accelerated version of the approach in (Wang et al., 2014). Building on these results, further developments employed Nesterov acceleration techniques to arrive to an \(O(\delta^{-2.25})\) for gradient query complexities (Wang et al., 2016). Concurrently, other authors considered a relaxation of the problem by studying a finite-sum alternative, i.e., a Monte-Carlo approximation to the stochastic objectives\(^4\). Here, Liu et al. (2017) examined variance reduction techniques to achieve a total complexity of \(O((m + n)^{0.8}\delta^{-1})\). Succeedingly, Huo et al. (2017) adapted variance reduction to proximal algorithms acquiring \(O((m + n)^{1.5}\delta^{-1})\) bound. A similar complexity bound was obtained in (Liu et al., 2018) by applying two kinds of variance reduction techniques. Such bound has been later improved to \(O((m + n)^{2}\delta^{-1})\) with recursive gradient methods in (Hu et al., 2019)\(^5\).

3. Compositional ADAM

3.1. Problem Definition, Notation & Assumptions

We focus on the following two-level nested optimisation problem:

\[
\min_{x \in \mathbb{R}^p} J(x) = E_{\nu}[f_\nu (E_\omega [g_\omega (x)])],
\]

where \(\nu\) and \(\omega\) are random variables sampled from \(\nu \sim P_{\nu}(\cdot)\) and \(\omega \sim P_{\omega}(\cdot)\) with \(P_{\nu}(\cdot)\) and \(P_{\omega}(\cdot)\) being unknown. Furthermore, for any \(\nu\) and \(\omega\), \(f_\nu(\cdot) : \mathbb{R}^q \rightarrow \mathbb{R}\) is a function mapping to real-values, while \(g_\omega (\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^q\) represents a map transforming the \(p\)-dimensional optimisation variable to a \(q\)-dimensional space. We make no restrictive assumptions on the probabilistic relationship between \(\nu\) and \(\omega\), which can be either dependent or independent.

As our method utilises gradient information in performing updates, it is instructive to clearly state assumptions we exploit in designing compositional ADAM. Defining \(J(x) = f(g(x))\), with \(f(y) = E_\nu[f_\nu(y)]\), and \(g(x) = E_\omega[g_\omega(x)]\), gradients can be written as: \(\nabla_x J(x) = \nabla_x g(x)^T \nabla_y f(g(x))\). Analogous to most optimisation techniques, we prefer if gradient are Lipschitz continuous by that introducing regularity conditions that can aid in designing better behaved algorithms. Rather than restricting overall objectives, however, we realise that it suffices for gradients of \(f_\nu(\cdot)\) and \(g_\omega(\cdot)\) to be Lipschitz and bounded so as to achieve such results. As such, we assume:

**Assumption 1.** We make the following assumptions:

1. We assume that \(f_\nu(\cdot)\) is bounded above by \(B_f\) and its gradients by \(M_f\), i.e., \(\forall y \in \mathbb{R}^q\) and for any \(\nu\), \(|f_\nu(y)| \leq B_f\), and \(||\nabla f_\nu(y)|| \leq M_f\).

2. We assume that \(f_\nu(\cdot)\) is \(L_f\)-smooth, i.e., for any \(y_1, y_2 \in \mathbb{R}^q\) and for any \(\nu\), we have: \(||\nabla f_\nu(y_1) - \nabla f_\nu(y_2)||_2 \leq L_f ||y_1 - y_2||_2\).

3. We assume that the mapping \(g_\omega(x)\) is \(M_g\) Lipschitz continuous, i.e., for any \((x_1, x_2) \in \mathbb{R}^p\) and for any \(\omega\), we have: \(||g_\omega(x_1) - g_\omega(x_2)||_2 \leq M_g ||x_1 - x_2||_2\).
4. We assume that the mapping \( g_\omega(x) \) is \( L_\theta \)-smooth, i.e., for any \((x_1, x_2) \in \mathbb{R}^p\) and for any \( \omega \), we have:
\[
\|\nabla g_\omega(x_1) - \nabla g_\omega(x_2)\|_2 \leq L_\theta \|x_1 - x_2\|_2.
\]

Given the above assumptions, we can now prove Lipschitz smoothness of \( \mathcal{J}(x) \):

**Lemma 1.** If Assumption 1 holds, then \( \mathcal{J}(x) \) is \( L \)-Lipschitz smooth, i.e., \( \|\nabla \mathcal{J}(x_1) - \nabla \mathcal{J}(x_2)\|_2 \leq L \|x_1 - x_2\|_2 \), for all \((x_1, x_2) \in \mathbb{R}^p\), with \( L = M_2^2 L_f + L_\theta M_f \).

We now focus on the process by which gradients are evaluated. As we assume that distributions \( P_\nu(\cdot) \) and \( P_\omega(\cdot) \) are unknown, we introduce two first-order oracles that can be queried to return gradients and function values. We presume \( \text{Oracle}_f(\cdot, \cdot) \) and \( \text{Oracle}_g(\cdot, \cdot) \), such that, at a time instance \( t \), for any two fixed vectors \( z_t \in \mathbb{R}^p \), \( y_t \in \mathbb{R}^q \), and for any two integers, representing batch sizes, \( K_f^{(1)} \) and \( K_f^{(2)} \), these oracles return the following collection:

\[
\text{Oracle}_f \left( y_t, K_f^{(1)} \right) = \left\{ \langle \nu_t, \nabla f_{\nu_t}(y_t) \rangle \right\}_{i=1}^{K_f^{(1)}},
\]

\[
\text{Oracle}_g \left( z_t, K_g^{(2)} \right) = \left\{ \langle \omega_t, g_{\omega_t}(z_t), \nabla g_{\omega_t}(z_t) \rangle \right\}_{i=1}^{K_g^{(2)}},
\]

with \( \nu_t \) and \( \omega_t \) being identically independently distributed (i.i.d.).

We report our convergence complexity results in terms of the total number of calls to these first-order oracles. Similar to (Wang et al., 2014), we introduce one final set of assumptions needed to understand the randomness of the sampling process:

**Assumption 2.** For any time step \( t \), \( \text{Oracle}_f(\cdot, \cdot) \) and \( \text{Oracle}_g(\cdot, \cdot) \) satisfy the following conditions for any \( z \in \mathbb{R}^p \) and \( y \in \mathbb{R}^q \):

1. Independent sample collections:

\[
\left( \{\nu_{t_1}\}_{i=1}^{K_f^{(1)}}, \{\omega_{t_1}\}_{i=1}^{K_g^{(2)}} \right), \ldots, \left( \{\nu_{t_i}\}_{i=1}^{K_f^{(1)}}, \{\omega_{t_i}\}_{i=1}^{K_g^{(2)}} \right).
\]

2. Oracles return unbiased gradient estimates for any \( t, i, \) and \( j \):

\[
\mathbb{E}_{\nu_{t_1}} \left[ g_{\omega_{t_1}}(z) \right] = \mathbb{E}_\omega \left[ g_\omega(z) \right],
\]

\[
\mathbb{E}_{\omega_{t_1}, \nu_{t_j}} \left[ \nabla g_{\omega_{t_1}}(z) \nabla f_{\nu_{t_j}}(y) \right] = \mathbb{E}_\omega \left[ \nabla g_\omega(z) \right] \times \mathbb{E}_\nu \left[ \nabla f_\nu(y) \right].
\]

\footnote{Due to space constraints, we refrain proofs to the appendix.}

\footnote{Please notice that i.i.d. assumption here should be interpreted as follows: at each iteration \( t \), samples \( \nu_t \) and \( \nu_{t_j} \) (respectively \( \omega_t \) and \( \omega_{t_j} \)) for \( i, j \in 1, \ldots, K_f^{(1)} \) (respectively \( i, j \in 1, \ldots, K_g^{(2)} \)) are i.i.d., but \( \omega_t \) and \( \nu_{t_i} \) might not be independent.}

3. Variance-bounded stochastic gradients for any \( t, i, \) and \( j \):

\[
\mathbb{E}_{\nu_{t_1}} \left[ ||\nabla f_{\nu_{t_1}}(y) - \nabla f_\nu(y)||_2 \right] \leq \sigma_1^2,
\]

\[
\mathbb{E}_{\omega_{t_1}} \left[ ||\nabla g_{\omega_{t_1}}(z) - \nabla g_\omega(z)||_2 \right] \leq \sigma_2^2,
\]

\[
\mathbb{E}_{\omega_{t_1}} \left[ ||g_{\nu_{t_1}}(z) - g_\omega(z)||_2 \right] \leq \sigma_3^2.
\]

### 3.2. Algorithmic Development & Theoretical Results

We now propose our algorithm providing its convergence properties. We summarise our solver, titled C-ADAM, in the pseudo-code of Algorithm 1.

On a high level, C-ADAM shares similarities with original ADAM (Kingma & Ba, 2014) in that it exhibits both main, \( x_t \), and auxiliary variables \( m_t \) and \( v_t \). Similar to ADAM, at each iteration, we compute two parameters corresponding to a weighted combination of historical and current gradients, and a weighted combination of squared components that relate to variances. When performing parameter updates, however, we deviate from ADAM by introducing additional auxiliary variables essential for an extrapolation smoothing scheme that allows for fast estimation of the variances of \( g_\omega(x_t) \).

In addition to precision parameters and an initialisation, our algorithm acquires a schedule of learning rates and minibatches as inputs. Given such inputs, we then execute a loop for \( \mathcal{O}(\delta^{-\frac{5}{4}}) \) steps to return a \( \delta \)-first-order stationary point to the problem in Equation 1. The loop operates in two main phases. In the first, necessary variables needed for parameter updates are computed according to lines 5-6, while the second updates each of \( x_t, y_t \), and \( z_t \). Similar to ADAM, \( x_t \) is updated with no additional calls to the oracle. Contrary to standard ADAM, on the other hand, our method makes another call to the oracle to sample \( K_g^{(3)} \) function values before revising the value of \( y_t \), see lines 8-9.

The remainder of this section is dedicated to proving theoretical guarantees for Algorithm 1. We, next, prove that our algorithm converges to a \( \delta \)-first-order stationary point after \( T = \mathcal{O}(\delta^{-\frac{5}{4}}) \) iterations:

**Theorem 1 (Main Result).** Consider Algorithm 1 with a parameter setup given by: \( \alpha_t = C_{\alpha} \delta^{-\frac{5}{4}}, \beta_t = C_{\beta} K_f^{(1)} K_g^{(2)} C_t = C_2 \delta^{-\frac{3}{4}}, \gamma_t = C_3 \delta^{\frac{3}{4}}, \gamma_t^{(1)} = C_4 \mu^t, \gamma_t^{(2)} = 1 - C_5 \mu^t (1 - C_6 \mu^t)^2, \) for some positive constants \( C_{\alpha}, C_{\beta}, C_1, C_2, C_3, C_4, \gamma_t \) such that \( C_{\beta} < 1 \) and \( \mu \in (0, 1) \). For any \( \delta \in (0, 1) \), Algorithm 1 outputs, in expectation, a \( \delta \)-approximate first-order stationary point \( \hat{x} \) of \( \mathcal{J}(x) \). That is:

\[
\mathbb{E}_{\text{total}} \left[ \|\nabla \mathcal{J}(\hat{x})\|_2^2 \right] \leq \delta,
\]

with “total” representing all incurred randomness. More-
Compositional ADAM (C-ADAM)

We can easily adapt the above lemma to the specifics of our algorithm bounding expected norm differences between successive updates. Throughout our proof, we also realise the need to study two additional components that we derive by generalising lemmas originally presented in (Wang et al., 2014). We note that our proof is not just a mere application of the results in (Zaheer et al., 2018) to a compositional setting. In fact, our derivations are novel in that they follow alternative directions combining and generalising results from (Wang et al., 2014; 2017) with these from (Zaheer et al., 2018). Due to space constraints, all proofs can be found in the Appendix A.

3.2.1. Preliminary Lemmas:

We, first, detail two essential lemmas that are adjusted from the work in (Wang et al., 2014) to deal with our ADAM solver.

Lemma 2 (Auxiliary Variable Properties). Consider auxiliary variable updates in lines 8 and 9 in Algorithm 1. Let $E_{\text{total}}[\cdot]$ denote the expectation with respect to all incurred randomness. For any $t$, the following holds:

$$E_{\text{total}} \left[ \| g(x_{t+1}) - y_{t+1} \|^2 \right] \leq \frac{L^2}{2} E_{\text{total}} \left[ D_{t+1}^2 \right] + 2E_{\text{total}} \left[ \| E_{t+1} \|^2 \right],$$

with $g(x_{t+1}) = E_{\omega}[g_{\omega}(x_{t+1})]$, and $D_{t+1}$, $\| E_{t+1} \|^2$ satisfy the following recurrent inequalities:

$$D_{t+1} \leq (1 - \beta_t) D_t + \frac{2M^2 g^2 M^2}{\epsilon^2} \frac{\alpha_t^2}{\beta_t} + \beta_t F_t,$$

$$E_{\text{total}} \left[ \| E_{t+1} \|^2 \right] \leq (1 - \beta_t)^2 E_{\text{total}} \left[ \| E_t \|^2 \right] + \frac{\beta_t^2}{K_t} \sum_{i=1}^{K_t} \| y_{\omega_i}(z_{t+1}) \|^2,$$

with $F_t \leq (1 - \beta_{t-1}) F_{t-1} + \frac{4M^2 g^2 M^2}{\epsilon^2} \frac{\alpha_{t-1}^2}{\beta_{t-1}}$, and $D_1 = 0$, $E_{\text{total}} \left[ \| E_1 \|^2 \right] = \| g(x_1) \|_2^2$, $F_1 = 0$.

Having established the first preliminary lemma depicting relationships between auxiliary variable updates, we now present a second one essential in our proof.

Lemma 3 (Recursion Property). Let $\eta_t = \frac{C_\eta}{\gamma t}$, $\zeta_t = \frac{C_\zeta}{\tau t}$, where $C_\eta > 1 + b - a$, $C_\zeta > 0$, $(b-a) \notin [-1,0]$ and $a \leq 1$. Consider the following recurrent inequality: $A_{t+1} \leq (1 - \eta_t + C_1 \eta_t^2) A_t + C_2 \zeta_t$, where $C_1, C_2 > 0$. Then, there is a constant $C_A > 0$ such that $A_t \leq \frac{C_A}{1 - \eta_t}$. We can easily adapt the above lemma to the specifics of our algorithm bounding expected norm differences between $g(x_t)$ and $y_t$.

Corollary 1. (Recursion & Auxiliary Variables in Algorithm 1) Consider Algorithm 1 with step sizes given by:
With the above lemmas established, we now detail essential steps needed to arrive at the statement of Theorem 1. As mentioned previously, our main analysis relies on the variables being $\alpha_t, \beta_t, t^a, t^b, t^c, t^d, t^e$ for some constants $C_\alpha, C_\beta, C_\gamma, a, b, c, d, e > 0$ such that $(2a - 2b) \notin [-1, 0]$, and $b \leq 1$. For $CD, CE > 0$, we have:

$$E_{\text{total}} \left[ \|g(x_t) - y_t\|^2 \right] \leq \frac{L_0^2 C_0^2}{2} \frac{1}{t^{4a-4b}} + 2C_2^2 \frac{1}{t^{b+c}}.$$  

3.2.2. Steps in Main Proof:

With the above lemmas established, we now detail essential steps needed to arrive at the statement of Theorem 1. As mentioned previously, our main analysis relies on the study of the change in the value of the compositional loss between two successive iterations. Previously in Lemma 1, we have shown that under our assumptions, the loss function is Lipschitz with a constant $L$. Hence, using Corollary 1 and re-grouping yields:

$$J(x_{t+1}) \leq J(x_t) + \nabla^T J(x_t) \Delta x_{t+1} + \frac{L}{2} ||\Delta x_{t+1}||^2,$$

with $\Delta x_{t+1} = x_{t+1} - x_t$.

Due to randomness induced by first-order oracles, such a change, has to be analysed in expectation with respect to all randomness incurred at iteration $t$ given a fixed $x_t$. We define such an expectation as $E_t[\cdot] = E_{K^{(1)}_t,K^{(2)}_t,K^{(3)}_t} \cdot x_t$ to consider all sampling performed at the $t^{th}$ iteration. With this, we note that gradients, and all primary and auxiliary variables are $t$-measurable, leading us to:

$$E_t[J(x_{t+1})] \leq J(x_t) + \alpha_t \nabla^T J(x_t) E_t \left[ \frac{m_t}{\sqrt{t_1 + \epsilon}} \right] + \frac{L \alpha_t^2}{2} E_t \left[ ||\frac{m_t}{\sqrt{t_1 + \epsilon}} ||^2 \right].$$

To achieve the convergence result, we bound each of the terms on the right-hand-side of the above equation. Using Assumptions 1 and 2 with the proposed setup of free-parameters, and applying the law of total expectation $E_{\text{total}}[E_t[\cdot]] = E_t[\cdot]$, we get:

$$E_{\text{total}}[J(x_{t+1}) - J(x_t)] \leq O(t^{-a+c+3}) + O(\mu t^{-a}) - O(t^{-a}) \left( E_{\text{total}}[||\nabla J(x_t)||^2] - E_{\text{total}}[||g(x_t) - y_t||^2] \right).$$

Hence, using Corollary 1 and re-grouping yields:

$$E_{\text{total}} \left[ ||\nabla J(x_t)||^2 \right] \leq t^{-5a-5b} + t^{-a+b+c} + t^{-a+c} + \mu t^{-a} + O(t^{2a} E_{\text{total}}[J(x_t) - J(x_{t+1})]).$$

Considering the average over all iterations $T$, and using first-order concavity conditions for $f(t) = t^a$ (when $a \in (0, 1)$) with the following setup for the constants: $a = 0.2; b = 0; c = e = 0.8$, eventually yields:

$$\frac{1}{T} \sum_{t=1}^T E_{\text{total}} \left[ ||\nabla J(x_t)||^2 \right] \leq \delta,$$

with a total gradient sample complexity given by $O((\delta^{-9/4})$ (i.e., result in Theorem 1).

4. Use Case: Model-Agnostic Meta-Learning

In this section, we present a novel connection mapping model-agnostic meta-learning (MAML) to compositional optimisation. MAML, introduced in (Finn et al., 2017), is an algorithm aiming at efficiently adapting deep networks to unseen problems by considering meta-updates performed on a set of tasks.

In its original form, MAML solves a meta-optimisation problem of the form:

$$\min_x \mathcal{L}(x) = \min_x \sum_{T_t \sim P_{\text{tasks}}(\cdot)} \mathcal{L}_{T_t}(\Phi(x; D_{T_t})) ; D_{T_t},$$

where $T_t$ represents the $T_th$ task, $P_{\text{tasks}}(\cdot)$ an unknown task distribution, and $D_{T_t}$ the data set for task $T_t$. Furthermore, $\Phi$ is a map operating on neural network parameters denoted by $x$ to define a function typically encoded by a neural network or NN(\cdot) in the above equation. Various algorithms from meta-learning have considered a variety of maps $\Phi$ see (Finn et al., 2018; Kim et al., 2018; Grant et al., 2018; Vuorio et al., 2018) and references therein. In this paper, we follow the original presentation in (Finn et al., 2017) that defines $\Phi$ as a one-step gradient update, i.e., $\Phi(x; D_{T_t}) = x - \alpha \nabla \mathcal{L}_{T_t}(\Phi(x; D_{T_t})).$ The remaining ingredient needed for fully defining the problem in Equation 2 is the loss function for a task $T_t$, i.e., $\mathcal{L}_{T_t}(\cdot; D_{T_t})$. Such a loss, however, is task dependent, e.g., logistic loss in classification, and mean-squared error in regression. Keeping with the generality of the exposition, we will map to a compositional form, while assuming generic losses\(^8\) in Equation 2.

4.1. Compositional Meta-Learning: C-MAML

4.1.1. Stochastic Problem Formulation:

To map MAML to compositional optimisation, we first, rewrite the problem in Equation 2 in an equivalent stochastic form. To do so, we start by considering Tasks $= \{T_1, T_2, \ldots\}$ to be a collection of tasks (e.g., a set of regression or classification tasks) that can be observed by the learner. Furthermore, we assume a distribution, $P_{\text{tasks}} = \Delta (\{T_1, T_2, \ldots\})$, from which tasks are sampled over rounds. To evaluate loss functions and perform updates of Equation 2, we also assume a deterministically available data-set, $D_{T_t}$, for each task $T_t \sim \Delta (\{T_1, T_2, \ldots\})$. As typical in machine learning, we suppose that an agent can not access all data for a task $T_t$ but rather needs to employ a mini-batching scheme in performing updates. Signifying

\(^8\)Specifics on various machine learning tasks can be found in the appendix.
such a process, we introduce an additional random variable \( \xi \) used for performing data sub-sampling according to an unknown distribution \( P_{T_1}^{(\text{data})}(\cdot) \) over \( D_{T_1} \), with \( T_1 \) being the \( i^{th} \) task. With these notations, we can write the gradient step update \( \Phi(x; D_{T}) \) for a given \( T \sim \Delta \{ \{ T_1, T_2, \ldots \} \) as:

\[
\Phi(x; D_{T}) = \mathbb{E}_{\xi \sim P_{T_1}^{(\text{data})}(\cdot)} \left[ x - \alpha \nabla L_t (\eta_\Phi(x); \xi) \mid t = T \right].
\]

Given a stochastic form of the update \( \Phi(x; D_{T}) \), we now focus on the problem in Equation 2. In (Finn et al., 2017), the authors re-sample a new data-set from the same data distribution to compute the outer-loss. Hence, we introduce one more random variable \( \xi' \) to write:

\[
\min_x \mathbb{E}_{\xi' \sim P_{T_1}^{(\text{data})}(\cdot)} \left[ L_t (\eta_\Phi(x; D_{T}); \xi') \mid t = T \right].
\]

In words, the stochastic formulation in Equation 3 simply states that MAML attempts to minimise an expected (over tasks) loss, which itself involves a nested expectation over data mini-batches used to compute per-task errors and updates.

Given such a stochastic formulation of MAML, we now consider two cases. The first, involves one task, while the second handles a meta-learning scenario as we detail next.

4.1.2. CASE I – SINGLE TASK WITH LARGE DATA-SET:

Here, we assume the availability of only one task \( T_1 \) whose data-set is large to motivate a data-streaming scenario. As such, the outer expectation in Equation 3 is non-existent, and the gradient map \( \Phi \) can be written as:

\[
\Phi(x; D_{T_1}) = \mathbb{E}_{\xi \sim P_{T_1}^{(\text{data})}(\cdot)} \left[ x - \alpha \nabla L_t (\eta_\Phi(x); \xi) \mid t = T_1 \right].
\]

Noting that the original form in Equation 1 can be rewritten as \( J(x) = f(g(x)) \) (Section 3.1), we first define \( g(x) = \mathbb{E}_{\xi \sim P_{T_1}^{(\text{data})}(\cdot)} \left[ x - \alpha \nabla L_t (\eta_\Phi(x); \xi) \mid t = T_1 \right] \). Subsequently, the objective in Equation 3, when considering one task \( T_1 \), can be rewritten as:

\[
\min_x \mathbb{E}_{\xi \sim P_{T_1}^{(\text{data})}(\cdot)} \left[ f_{\xi'} \left( \mathbb{E}_{\xi \sim P_{T_1}^{(\text{data})}(\cdot)} [g_{\xi}(x)] \right) \right],
\]

where \( g_{\xi}(x) = x - \alpha \nabla L_t (\eta_\Phi(x); \xi) \), and \( f_{\xi'}(y) = L_{T_1} (\eta_\Phi(x); \xi') \). Clearly, the problem in Equation 4 is a special case of that in 1 and thus, one can employ Algorithm 1 for finding a stationary point (Section 5.2).

4.1.3. CASE II – MULTIPLE TASKS & META-UPDATE:

After showing single task MAML as a special case of compositional optimisation, we now generalise previous results to \( K > 1 \) tasks \( T_1, \ldots, T_K \). Sampling according to \( P_{\text{tasks}}(\cdot) \), we can write a Monte-Carlo estimator of the loss in Equation 3 to arrive at the following optimisation problem:

\[
\min_x \hat{L}(x) = \min_x \frac{1}{K} \sum_{j=1}^{K} \mathbb{E}_{\xi_j \sim \sigma T_j} \left[ \gamma_j^{(j)}(x; \xi_j) \mid t = T_j \right],
\]

with \( \gamma_j^{(j)}(x; \xi_j) = L_t (\eta_\Phi(x; D_{T_j}); \xi_j) \), and \( \Phi(\cdot) = \Phi(x; D_{T_j}) \) defined as:

\[
\Phi(\cdot) = \mathbb{E}_{\xi \sim P_{T_j}^{(\text{data})}(\cdot)} \left[ x - \alpha \nabla L_t (\eta_\Phi(x); \xi) \mid t = T_j \right].
\]

Now, consider \( G_{\xi_1, \ldots, \xi_K}(x) \in \mathbb{R}^{p \times K} \) to be a matrix with dimension of neural network parameters \( p \) rows and total number of tasks \( K \) columns, such that each column \( j \in \{1, \ldots, K\} \) is defined as:

\[
G_{\xi_1, \ldots, \xi_K}(x)[; j] = x - \alpha \nabla L_t (\eta_\Phi(x); \xi_j).
\]

Denoting \( \xi = (\xi_1, \ldots, \xi_K) \), where \( \xi_j \sim P_{T_j}^{(\text{data})}(\cdot) \) for all \( j \in \{1, \ldots, K\} \), we define \( G(x) = \mathbb{E}_\xi [G_\xi(x)] \) as an expected map such that each of its \( j \) columns is given by:

\[
G(x)[; j] = \mathbb{E}_{\xi_j \sim \sigma T_j} \left[ x - \alpha \nabla L_t (\eta_\Phi(x); \xi_j) \mid t = T_j \right].
\]

Hence, one can write the gradient update map in Equation 6 for any task \( j \) simply as: \( \Phi(x; D_{T_j}) = G(x) e_j \) with \( e_j \) being the \( j^{th} \) basis vector in a standard orthonormal basis in \( \mathbb{R}^K \). Applying our newly introduced notions to the loss in Equation 5, we get:

\[
\hat{L}(x) = \mathbb{E}_\xi \left[ \frac{1}{K} \sum_{j=1}^{K} L_t (\eta_\Phi(x; D_{T_j}); \xi_j) \mid t = T_j \right] = \mathbb{E}_\xi \left[ f_{\xi'} (\mathbb{E}_\xi [G_\xi(x)]) \right] \iff \text{Compositional Form}
\]

with \( \xi' = (\xi_1', \ldots, \xi_K') \), and \( f_{\xi'}(A) = \frac{1}{K} \sum_{j=1}^{K} L_{T_j} (\eta_\Phi(x; D_{T_j}); \xi_j') \), with \( A[; j] \) being the \( j^{th} \) column of matrix \( A \in \mathbb{R}^{p \times K} \). It is again clear that multi-task MAML is another special case of compositional optimisation for which we can employ Algorithm 1 to determine a stationary point.

C-MAML Implications: Connecting MAML and compositional optimisation sheds light on interesting theoretical insights for meta-learning – a topic gaining lots of attention.

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\[^{10}\text{Please note that for ease of notation we have denoted } E_\xi[\cdot] = E_{\xi_1, \ldots, \xi_K} = E_{\xi_1}[E_{\xi_2}[\ldots E_{\xi_K}[\cdot]]].\]
We consider three large 100-portfolio data-sets ($m = 13781$, $n = 100$) and 18 region-based medium-sized sets with $m = 7240$, and 25 assets as collected by CRSP.

in recent literature (Fallah et al., 2019). Importantly, we realise that upon the application of C-ADAM to the bridge made in Section 4, we achieve the fastest known complexity bound for meta-learning. Table 1 depicts these results demonstrating that the closest results to ours are recent bounds derived in (Fallah et al., 2019). In the table, we also note that methods proposed in (Finn et al., 2017; Zaheer et al., 2018) have no provided convergence guarantees (marked by N/A in the table) leaving such an analysis as an interesting avenue for future research.

5. Experiments

We present an in-depth empirical study demonstrating that C-ADAM (Algorithm 1) outperforms state-of-the-art methods from both compositional and standard optimisation. We split this section in two. In the first, we experiment with more conventional portfolio optimisation tasks as presented in (Wang et al., 2017), while in the second, we focus on model-agnostic meta-learning building on our results from Section 4. In the portfolio scenario, algorithms tasked to determine stationary points for sparse mean-variance trade-offs on real-world data-sets gathered by the center for research in security prices (CRSP) are studied. In MAML’s case, on the other hand, we benchmark against regression tasks originally introduced in (Finn et al., 2017) demonstrating that C-ADAM achieves new state-of-the-art performance.

5.1. Portfolio Mean-Variance

We consider three large 100-portfolio data-sets ($m = 13781$, $n = 100$) and 18 region-based medium-sized sets with $m = 7240$, and 25 assets as collected by CRSP to demonstrate book-to-market (BM), operating profitability (OP), and investment (Inv.). Given $n$ assets and reward vectors at $m$ time instances, the goal of sparse mean-variance optimisation (Ravikumar et al., 2007) is to maximise returns and control risk; a problem that can be mapped to a compositional form (see Appendix B.1). We compared C-ADAM with a line of existing algorithms for compositional optimisation: ASCVRG (Lin et al., 2018), VRSC-PG (Huo et al., 2017), ASC-PG (Wang et al., 2016), and SCGD (Wang et al., 2014). Implementation of these baselines were provided by respective authors and optimised for mean-variance tasks. As a comparative metric between algorithms, we measured the optimality gap $\mathcal{J}(x) - \mathcal{J}^*$, versus number of samples used-per asset. Free parameters in Algorithm 1 were set to $C_\alpha = 0.01, C_\beta = 0.01, K_t^{(1)} = K_t^{(2)} = K_t^{(3)} = 1$, and $C_j = 1$ unless otherwise specified. Figure 1(a, b, c) shows that C-ADAM outperforms other algorithms on large data-sets, while Figure 1(d, e, f) demonstrates that C-ADAM outperforms others on sample results from BM, OP, and Inv data-sets.

Ablation Study: Algorithm 1 introduces free-parameters that need to be tuned for successful execution. To assess their importance, we conducted an ablation study on batch-sizes $K^{(1)}_i, K^{(2)}_t, K^{(3)}_j$ and step-sizes $C_\alpha, C_\beta$. We demonstrate the effects of varying these parameters on an asset data-set in Figure 1(g, h). It is clear that C-ADAM performs best with a batch-size of 1 (Figure 1(g)), and a small step size of 0.01 (Figure 1(h)). This, in turn, motivated our choice of these parameters in the portfolio experiments.

5.2. Compositional MAML

To validate theoretical guarantees achieved for MAML, we conduct two sets of experiments on regression problems originally introduced in (Finn et al., 2017). After observing a set of tasks, the goal is to perform few-shot supervised regression on novel unobserved tasks when only a few-data points are available. Tasks vary in their data distribution, e.g., changing parameters for data generation distribution.

We use a neural network regressor with two hidden layers each of size 40 with ReLU non-linearities. For all experiments, we use one gradient update with ten examples, i.e., 10-shot regression, and adopt an $\alpha = 0.01$ learning rate for all algorithms. We compare C-ADAM with both compositional (ASC-PG and SCGD) and non-compositional optimisers (ADAM, Adagrad (Duchi et al., 2011), ASGD (Polyak & Juditsky, 1992), RMSprop (Mukkamala & Hein, 2017), SGD) and demonstrate the performance in Figure 1. When

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Table 1. Theoretical sample complexity bounds comparing our solver C-MAML to state-of-the-art in current literature to achieve $||\nabla \mathcal{L}(x)||^2 \leq \delta$. It is clear that C-MAML achieves the best rate known so-far.

| Algorithms | Complexities |
|------------|--------------|
| C-MAML (This paper) | $O(\delta^{-2.25})$ |
| (Fallah et al., 2019) – Alg. 1 | $O(\delta^{-3})$ |
| (Fallah et al., 2019) – Alg. 3 | $O(\delta^{-3})$ |
| (Kingma & Ba, 2014) in MAML | N/A |
| (Finn et al., 2017) | N/A |

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11Furthermore, we note that other theoretical attempts aiming at understanding MAML (Finn et al., 2019; Golmant, 2019) were not explicitly mentioned as these consider convex assumptions on loss functions abide using deep network models.

12https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

13Please note that due to space constraints full results (all demonstrating that C-ADAM outperforms others) on all 18 data-sets can be found in Appendix B.1.

14Due to space constraints all hyper-parameters we used in our experiments can be found in Appendix B.2.
C-ADAM: An Adaptive Compositional Solver

Figure 1. (a, b, c): C-ADAM’s performance versus other compositional optimisation methods on the three large 100-portolio data-sets demonstrating that our method significantly outperforms others in terms of convergence speeds. (d, e, f): C-ADAM’s performance versus other composition methods on one data-set from BM. (e): C-ADAM’s performance on OP, and [f]: C-ADAM’s performance on a data-set from Inv. In all case, we see that C-ADAM outperforms other techniques significantly. (g, h): An ablation study demonstrating the effect of modifying the batch sizes – $K^{(1)} = K^{(2)} = K^{(3)} = K$, and [h:] Effect of step-size – $C_0 = C_\beta = C$. (i, j): Training loss curves of single-task compositional MAML compared to methods from compositional and standard optimisation. These results again demonstrate that C-ADAM outperforms others. (k): The meta test loss of multi-task compositional MAML compared to others demonstrating C-ADAM’s performance. (l): Quantitative regression results showing the learning curve after training with 40000 iterations on meta test-time. It is again clear that C-ADAM outperforms others.

dealing with only one task, i.e., streaming data points, C-ADAM achieves best performance in training and test loss with respect to the number of samples, as shown in Figures 1(i) and (j).

Evaluating few-shot regression (multi-task scenario), we fine-tune the meta-learnt model using the same optimiser (SGD) on $M = 10$ examples for each method. We also compare performance on two additional baseline models: (1) pre-training on all of the tasks, which entails training a single neural net to deal with multiple tasks at the same time (Transfer in Figure 1(k, l)), and (2) random initialization (Random in Figure 1(k, l)).

During fine-tuning, each gradient step is computed using the same $M = 10$ data-points. For each evaluation point, we report the result with 100 test tasks and show the meta test loss (i.e., test loss on novel unseen tasks during training) after $M = 10$ gradient steps in Figure 1(k). The meta-learnt model using C-ADAM optimiser is able to quickly adapt during meta test. While models learnt with baseline optimisers can adapt to the meta test set after training for 40000 iterations as shown in Figures 1(i, l), C-ADAM’s model still achieves best convergence performance among all baselines, which demonstrates both the advantage of our compositional formulation of MAML and the adaptive nature of C-ADAM.

6. Conclusions and Future Work

We proposed C-ADAM the first adaptive compositional solver. We validated our method both theoretically and empirically and provided the first connection between model-agnostic deep learning and compositional optimisation that
attained best-known convergence bounds to-date.

In future, we plan to investigate further applications of compositional optimisation, and propose adaptive algorithms for problems involving nesting of more than two expected values.

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A. Theoretical Results

In this part of Appendix we present proofs for all statements made in the main paper:

A.1. Proof of Lemma 1

**Lemma 4.** If Assumptions 1.1, 1.3, and 1.4 hold, then $\mathcal{J}(x)$ is $L$-Lipschitz smooth, i.e.,

$$||\nabla_x \mathcal{J}(x_1) - \nabla_x \mathcal{J}(x_2)||_2 \leq L ||x_1 - x_2||_2,$$

for all $(x_1, x_2) \in \mathbb{R}^p$, with $L = M_f^2 L_f + L_g M_f$.

**Proof.** Assumption 1 implies that $||\nabla g_w(x)||_2 \leq M_g$ for any $x \in \mathbb{R}^p$. Hence, using Jensen inequality as well as property of the norm we have:

$$||\nabla \mathcal{J}(x_1) - \nabla \mathcal{J}(x_2)||_2 =$$

$$||E_w [\nabla g_w^T (x_1)] E_v [\nabla f_v (E_w [g_w(x_1)])] - E_w [\nabla g_w^T (x_2)] E_v [\nabla f_v (E_w [g_w(x_2)])]||_2 \leq$$

$$||E_w [\nabla g_w^T (x_1)] E_v [\nabla f_v (E_w [g_w(x_1)])] - E_w [\nabla g_w^T (x_1)] E_v [\nabla f_v (E_w [g_w(x_1)])]||_2 +$$

$$||E_w [\nabla g_w^T (x_2)] E_v [\nabla f_v (E_w [g_w(x_2)])] - E_w [\nabla g_w^T (x_2)] E_v [\nabla f_v (E_w [g_w(x_2)])]||_2 \leq$$

$$E_w [||\nabla g_w^T (x_1)||_2] E_w [\nabla f_v (E_w [g_w(x_1)])] - E_w [\nabla f_v (E_w [g_w(x_2)])]||_2 +$$

$$||E_w [\nabla g_w^T (x_2)] - E_w [\nabla g_w^T (x_2)]||_2 E_v [\nabla f_v (E_w [g_w(x_2)])]||_2 \leq$$

$$M_g E_w [||\nabla f_v (E_w [g_w(x_1)]) - \nabla f_v (E_w [g_w(x_2)])||_2] +$$

$$M_f E_w [||\nabla g_w^T (x_1) - \nabla g_w^T (x_2)||_2] \leq M_g L_f ||E_w [g_w(x_1)] - E_w [g_w(x_2)]||_2 +$$

$$M_f E_w [||\nabla g_w^T (x_1) - \nabla g_w^T (x_2)||_2] \leq M_g L_f ||g_w(x_1) - g_w(x_2)||_2 +$$

$$M_f E_w [||\nabla g_w^T (x_1) - \nabla g_w^T (x_2)||_2] \leq M_g L_f ||g_w(x_1) - g_w(x_2)||_2 + M_f L_g ||x_1 - x_2||_2 \leq$$

$$M_f^2 L_f ||x_1 - x_2||_2 + M_f L_g ||x_1 - x_2||_2 = (M_f^2 L_f + M_f L_g) ||x_1 - x_2||_2 = L ||x_1 - x_2||_2,$$

which finishes the proof of the claim. \qed

A.2. Proof of Lemma 2

**Lemma 5.** Consider auxiliary variable updates in lines 8 and 9 in Algorithm 1. Let $E_{\text{total}}[\cdot]$ denote the expectation with respect to all incurred randomness. For any $t$, the following holds:

$$E_{\text{total}} \left[ ||g(x_{t+1}) - y_{t+1}||_2^2 \right] \leq \frac{L_f^2}{2} E_{\text{total}} [D_{t+1}^2] + 2 E_{\text{total}} \left[ ||\mathcal{E}_{t+1}||_2^2 \right],$$

with $g(x_{t+1}) = E_w [g_w(x_{t+1})]$, and $D_{t+1}$, $||\mathcal{E}_{t+1}||_2^2$ satisfy the following recurrent inequalities:

$$D_{t+1} \leq (1 - \beta_t) D_t + \frac{2 M_f^2 M_f^2}{\epsilon^2} \alpha_{t}^2 + \beta_t \mathcal{F}_t^2,$$

$$E_{\text{total}} \left[ ||\mathcal{E}_{t+1}||_2^2 \right] \leq (1 - \beta_t)^2 E_{\text{total}} \left[ ||\mathcal{E}_t||_2^2 \right] + \frac{\beta_t^2}{K_t^{(3)} \sigma^2} \mathcal{F}_t^2,$$

$$\mathcal{F}_t^2 \leq (1 - \beta_{t-1}) \mathcal{F}_{t-1}^2 + \frac{4 M_f^2 M_f^2}{\epsilon^2} \alpha_{t-1}^2 + \frac{\beta_{t-1}^2}{\beta_{t-1}},$$

and $D_1 = 0$, $E_{\text{total}} [||\mathcal{E}_1||_2^2] = ||g(x_1)||_2^2$, $\mathcal{F}_1 = 0$.

**Proof.** Let us introduce the sequence of coefficients $\{\theta^t\}_{j=0}^t$ such that

$$\theta^t_j = \begin{cases} \beta_j \prod_{i=j+1}^t (1 - \beta_i) & \text{if } j < t, \\ \beta_j & \text{if } j = t. \end{cases}$$
and we assume \(\beta_0 = 1\) for simplicity. Denote \(S_t = \sum_{j=0}^{t} \theta_j(t)\), then:

\[
S_t = \sum_{j=0}^{t} \theta_j(t) = \\
\beta_t + (1 - \beta_t) \beta_{t-1} + (1 - \beta_t)(1 - \beta_{t-1}) \beta_{t-2} + \cdots + (1 - \beta_t)(1 - \beta_{t-1}) \cdots (1 - \beta_1) \beta_0 = \\
\beta_t + (1 - \beta_t) \beta_{t-1} + (1 - \beta_t)(1 - \beta_{t-1}) \beta_{t-2} + \cdots + (1 - \beta_t)(1 - \beta_{t-1}) \cdots (1 - \beta_1) \beta_0 = \beta_t + (1 - \beta_t)S_{t-1}
\]

and \(S_1 = \beta_1 + (1 - \beta_1)\beta_0\). Since \(\beta_0 = 1\) it implies \(S_1 = S_2 = \ldots = S_t = 1\). By assuming \(g_0(z_1) = 0\), one can represent \(x_{t+1}\) and \(y_{t+1}\) as a convex combinations of \(\{z_j\}_{j=1}^{t+1}\) and \(\{g_j(z_j+1)\}_{j=0}^{t}\) respectively:

\[
x_{t+1} = \sum_{j=0}^{t} \theta_j(t) z_{j+1}, \quad \text{and} \quad y_{t+1} = \sum_{j=0}^{t} \theta_j(t) g_j(z_{j+1})
\]

(7)

Hence, using Taylor expansion for function \(g(z_{j+1})\) around \(x_{t+1}\) we have:

\[
y_{t+1} = \sum_{j=0}^{t} \theta_j(t) g_j(z_{j+1}) = \sum_{j=0}^{t} \theta_j(t) \left[g(z_{j+1}) - g(x_{t+1}) + \nabla g(x_{t+1}) (z_{j+1} - x_{t+1}) + O(||z_{j+1} - x_{t+1}||^2_2)\right] = \sum_{j=0}^{t} \theta_j(t) g(z_{j+1}) + \\
\sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right] = \sum_{j=0}^{t} \theta_j(t) \left[g(x_{t+1}) + \nabla g(x_{t+1}) \left[\sum_{j=0}^{t} \theta_j(t) z_{j+1} - \sum_{j=0}^{t} \theta_j(t) x_{k+1}\right] + \\
\sum_{j=0}^{t} \theta_j(t) \left[||z_{j+1} - x_{t+1}||^2_2\right] + \sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right] = \sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right] + \\
\sum_{j=0}^{t} \theta_j(t) \left[||z_{j+1} - x_{t+1}||^2_2\right]
\]

Therefore, using that \(g(\cdot)\) is \(L_g\)-smooth function:

\[
||y_{t+1} - g(x_{t+1})||_2 \leq \frac{L_g}{2} \sum_{j=0}^{t} \theta_j(t) ||z_{j+1} - x_{t+1}||^2_2 + \left|\sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right]\right|_2
\]

and, applying \((a + b)^2 \leq 2a^2 + 2b^2:\)

\[
||y_{t+1} - g(x_{t+1})||_2^2 \leq \frac{L_g^2}{2} \sum_{j=0}^{t} \theta_j(t) ||z_{j+1} - x_{t+1}||^2_2 + 2 \left|\sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right]\right|_2^2
\]

Taking expectation \(E_{\text{total}}\) from both sides gives:

\[
E_{\text{total}} \left[||y_{t+1} - g(x_{t+1})||^2_2\right] \leq \frac{L_g^2}{2} E_{\text{total}} \left[||D_{t+1}||^2_2\right] + 2 E_{\text{total}} \left[||E_{t+1}||^2_2\right]
\]

(8)

where

\[
D_{t+1} = \sum_{j=0}^{t} \theta_j(t) ||z_{j+1} - x_{t+1}||^2_2, \quad E_{t+1} = \sum_{j=0}^{t} \theta_j(t) \left[g_j(z_{j+1}) - g(z_{j+1})\right],
\]
C-ADAM: An Adaptive Compositional Solver

It is easy to see that $\mathcal{D}_t = 0$ and $\mathcal{E}_t = g(z_t)$ (using notational assumptions $g_0(z_0) = 0$ and $\beta_0 = 1$). Let us bound both terms in expression (8). Due to $\theta_j^t = (1 - \beta_t)\theta_j^{(t-1)}$ for $j < t$ for the expression $\mathcal{D}_{t+1}$ we have:

$$\mathcal{D}_{t+1} = \sum_{j=0}^{t} \theta_j^t \|z_{j+1} - x_{t+1}\|^2 =$$

$$\sum_{j=0}^{t-1} \theta_j^t \|z_{j+1} - x_{t+1}\|^2 + \beta_t \|z_{t+1} - x_{t+1}\|^2 = (1 - \beta_t) \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|z_{j+1} - x_{t+1}\|^2 + \beta_t \|z_{t+1} - x_{t+1}\|^2 =$$

$$(1 - \beta_t) \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|z_{j+1} - x_{t+1}\|^2 + \frac{(1 - \beta_t)^2}{\beta_t} \|x_{t+1} - x_t\|^2 = (1 - \beta_t) \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|z_{j+1} - x_{t+1}\|^2 +$$

$$\frac{(1 - \beta_t)^2}{\beta_t} \|x_{t+1} - x_t\|^2 + (1 - \beta_t) \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|z_{j+1} - x_{t+1}\|^2 =$$

$$\|z_{j+1} - x_{t+1}\|^2 \leq (1 - \beta_t) \mathcal{D}_t + \frac{(1 - \beta_t)^2}{\beta_t} \|x_{t+1} - x_t\|^2 +$$

$$\sum_{j=0}^{t-1} \theta_j^{(t-1)} \|z_{j+1} - x_{t+1}\|^2 + \|x_{t+1} - x_t\|^2 + 2 \|x_t - z_{j+1}\|^2 = (1 - \beta_t) \mathcal{D}_t + \frac{(1 - \beta_t)^2}{\beta_t} \|x_{t+1} - x_t\|^2 +$$

$$\sum_{j=0}^{t-1} \theta_j^{(t-1)} \|x_t - z_{j+1}\|^2 = (1 - \beta_t) \mathcal{D}_t +$$

$$\frac{1 - \beta_t}{\beta_t} \|x_{t+1} - x_t\|^2 + 2(1 - \beta_t) \|x_{t+1} - x_t\|^2 \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|x_t - z_{j+1}\|^2$$

Applying $2ab \leq \frac{1}{\alpha^2} a^2 + \beta b^2$:

$$\mathcal{D}_{t+1} \leq$$

$$(1 - \beta_t) \mathcal{D}_t + \frac{1 - \beta_t}{\beta_t} \|x_{t+1} - x_t\|^2 + (1 - \beta_t) \left[ \frac{\|x_{t+1} - x_t\|^2}{\beta_t} + \beta_t \left( \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|x_t - z_{j+1}\|^2 \right) \right]$$

$$(1 - \beta_t) \mathcal{D}_t + 2 \frac{1 - \beta_t}{\beta_t} \|x_{t+1} - x_t\|^2 + (1 - \beta_t) \beta_t \left( \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|x_t - z_{j+1}\|^2 \right)^2 \leq$$

$$(1 - \beta_t) \mathcal{D}_t + \frac{2}{\beta_t} \|x_{t+1} - x_t\|^2 + \beta_t \left( \sum_{j=0}^{t-1} \theta_j^{(t-1)} \|x_t - z_{j+1}\|^2 \right)^2$$

Applying the primal variable update with $\|\nabla_x \bar{f}()\|^2 \leq M_\alpha M_f$ gives:

$$\mathcal{D}_{t+1} \leq$$

$$(1 - \beta_t) \mathcal{D}_t + \frac{2M_\alpha^2}{\beta_t} \left\| \frac{m_t}{\sqrt{v_t} + \epsilon} \right\|^2 + \beta_t \bar{f}_t^2 \leq (1 - \beta_t) \mathcal{D}_t + \frac{2M_\alpha^2 M_f^2}{\epsilon^2} \beta_t + \beta_t \bar{f}_t^2 $$

(9)
Next, for expression $F_t$ we have (using $\theta_j^{t-1} = (1 - \beta_{t-1})\theta_j^{(t-2)}$ for $j < t - 1$):

$$F_t = \sum_{j=0}^{t-1} \theta_j^{(t-1)}||x_t - z_{j+1}||_2 =$$

$$\sum_{j=0}^{t-2} \theta_j^{(t-1)}||x_t - z_{j+1}||_2 + \theta_t^{(t-1)}||x_t - z_t||_2 = \sum_{j=0}^{t-2} \theta_j^{(t-1)}||x_t - z_{j+1}||_2 + \beta_{t-1}||x_t - z_t||_2 =$$

$$(1 - \beta_{t-1})\sum_{j=0}^{t-2} \theta_j^{(t-2)}||x_t - z_{j+1}||_2 + \beta_{t-1}||x_t - z_t||_2 \leq (1 - \beta_{t-1})||x_t - x_{t-1}||_2 +$$

$$(1 - \beta_{t-1})\sum_{j=0}^{t-2} \theta_j^{(t-2)}||x_{t-1} - z_{j+1}||_2 + ||x_t - x_{t-1}||_2 = (1 - \beta_{t-1}) (F_{t-1} + 2||x_t - x_{t-1}||_2)$$

and $F_1 = 0$. Hence, applying $(a + b)^2 \leq (1 + \alpha)a^2 + (1 + \frac{1}{\alpha})b^2$ for $\alpha = \beta_{t-1} > 0$ and using primal variable update with $||\nabla_{x_t} F_t||_2 \leq M_g M_f$:

$$F_t^2 \leq$$

$$(1 + \beta_{t-1})(1 - \beta_{t-1})^2 F_{t-1}^2 + 4\left(1 + \frac{1}{\beta_{t-1}}\right)(1 - \beta_{t-1})^2||x_t - x_{t-1}||_2^2 \leq$$

$$(1 - \beta_{t-1})2F_{t-1}^2 + \frac{4}{\beta_{t-1}}||x_t - x_{t-1}||_2^2 = (1 - \beta_{t-1})2F_{t-1}^2 + \frac{4\eta^2_{t-1}}{\beta_{t-1}}\left\|\frac{m_{t-1}}{\sqrt{\nu_{t-1}}} + \epsilon\right\|_2^2 \leq$$

$$(1 - \beta_{t-1})2F_{t-1}^2 + \frac{4M_g^2M_f^2\alpha_t^2}{\epsilon^2}\beta_{t-1}$$

Finally, for $E_{t+1}$ we have (using $\theta_j^t = (1 - \beta_t)\theta_j^{(t-1)}$ for $j < t$):

$$E_{t+1} =$$

$$\sum_{j=0}^{t-1} \theta_j^{(t)} \left[ g_j(z_{j+1}) - g(z_{j+1}) \right] = \sum_{j=0}^{t-1} \theta_j^{(t)} \left[ g_j(z_{j+1}) - g(z_{j+1}) \right] + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] =$$

$$\sum_{j=0}^{t-1} (1 - \beta_t)\theta_j^{(t-1)} \left[ g_j(z_{j+1}) - g(z_{j+1}) \right] + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] =$$

$$(1 - \beta_t)\sum_{j=0}^{t-1} \theta_j^{(t-1)} \left[ g_j(z_{j+1}) - g(z_{j+1}) \right] + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] =$$

$$(1 - \beta_t)E_t + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right]$$

Due to the fact, that all samplings done at iteration $t_1$ are independent from samplings done at iteration $t_2 \neq t_1$, then consider expectation with all randomness induced at iteration $t$ (with fixed iterative value $x_t$):

$$E_t [\cdot] = E_{K_1^{(t)}, K_2^{(t)}, K_3^{(t)}} [\cdot | x_t]$$

for any $t$. Using that $E_t$ is independent from the randomness induced at iteration $t$ we have:

$$E_t \left[ ||E_{t+1}||_2^2 \right] =$$

$$E_t \left[ \left(1 - \beta_t\right)E_t^T + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right]^T \right] \left(1 - \beta_t\right)E_t + \beta_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] =$$

$$(1 - \beta_t)^2 ||E_t||_2^2 + 2\beta_t(1 - \beta_t)E_t E_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] + \beta_t^2 E_t \left[ \left|\left| g_t(z_{t+1}) - g(z_{t+1}) \right|\right|^2_2 \right] =$$

$$(1 - \beta_t)^2 ||E_t||_2^2 + \beta_t^2 E_t \left[ \left|\left| g_t(z_{t+1}) - g(z_{t+1}) \right|\right|^2_2 \right]$$
where \( E_t \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] = E_{K_t^{(1)}, K_t^{(2)}, K_t^{(3)}} \left[ g_t(z_{t+1}) - g(z_{t+1}) \right] = E_{K_t^{(1)}, K_t^{(2)}} [0] = 0_q \) due to Assumption 2.2.

Assumption 2.3 implies \( E_t \left[ \left| g_t(z_{t+1}) - g(z_{t+1}) \right| \right] \leq \frac{1}{K_t^{(3)}} \sigma_3^2 \), therefore,

\[
E_t \left[ \| \mathbf{e}_{t+1} \|_2^2 \right] \leq (1 - \beta_t)^2 \| \mathbf{e}_t \|_2^2 + \frac{\beta_t^2}{K_t^{(3)}} \sigma_3^2
\]

Taking expectation \( E_{total} \) from both sides of the above inequality and using (11) and the law of total expectation, we have:

\[
E_{total} \left[ \| \mathbf{e}_{t+1} \|_2^2 \right] \leq (1 - \beta_t)^2 E_{total} \left[ \| \mathbf{e}_t \|_2^2 \right] + \frac{\beta_t^2}{K_t^{(3)}} \sigma_3^2
\]

(12)

Combining (8), (9), (10), and (12) gives the statement of the Lemma.

\[
\square
\]

A.3. Proof of Lemma 3

**Lemma 6.** Let \( \eta_t = \frac{C_\eta}{t^a} \), \( \zeta_t = \frac{C_\zeta}{t^a} \), where \( C_\eta > 1 + b - a \), \( C_\zeta > 0 \), \( (b - a) \notin [-1, 0] \) and \( a \leq 1 \). Consider the following recurrent inequality:

\[
A_{t+1} \leq (1 - \eta_t + C_1 \eta_t^2) A_t + C_2 \zeta_t, \]

where \( C_1, C_2 \geq 0 \). Then, there is a constant \( C_A > 0 \) such that \( A_t \leq \frac{C_A}{t^{p+a+b}} \).

**Proof.** We adopted this proof from (Wang et al., 2017) and added to appendix to make the narration of the paper self-contained.

Let us introduce constant \( C_A \) such that

\[
C_A = \max_{t \leq (C_1 C_\eta^2)^{1/2} + 1} C_A t^{b-a} + \frac{C_2 C_\zeta}{C_\eta - 1 - b + a}
\]

The claim will be proved by induction. Consider two cases here:

1. **If** \( t \leq (C_1 C_\eta^2)^{1/2} \): Then, by the definition of constant \( C_A \) it follows immediately:

\[
A_t \leq C_A t^{a-b} = \frac{C_A}{t^{b-a}}
\]

2. **If** \( t > (C_1 C_\eta^2)^{1/2} \): Assume that \( A_s \leq \frac{C_A}{t^{b-a}} \) for some \( t > (C_1 C_\eta^2)^{1/2} \). Hence:

\[
A_{t+1} \leq (1 - \eta_t + C_1 \eta_t^2) A_t + C_2 \zeta_t = \left( 1 - \frac{C_\eta}{t^a} + C_1 \frac{C_\eta^2}{t^{2a}} \right) A_t + C_2 \zeta_t \leq
\]

\[
\left( 1 - \frac{C_\eta}{t^a} + C_1 \frac{C_\eta^2}{t^{2a}} \right) C_A t^{b-a} + C_2 C_\zeta t^{b-a} + C_2 C_\zeta = \frac{C_A}{t^{b-a}} - \frac{C_A C_\eta}{t^{b-a}} + \frac{C_2 C_\zeta}{t^{b-a}} + \frac{C_2 C_\zeta}{t^{b-a}} =
\]

\[
\frac{C_A}{(t+1)^{b-a}} - C_A \Delta_{t+1} + \frac{C_2 C_\zeta}{t^{b-a}} = \frac{C_A}{(t+1)^{b-a}} - \Delta_{t+1} \left( C_A - \frac{C_2 C_\zeta}{\Delta_{t+1} t^{b-a}} \right)
\]

where \( \Delta_{t+1} = \frac{C_\eta}{t^{b-a}} - \frac{C_\eta}{t^{b-a}} \).
Since function $f(t) = \frac{1}{t^c}$ is convex for $c \notin [-1, 0]$ and $t > 0$ one can apply the first order condition of convexity:

$$f(t + 1) \geq f(t) + f'(t) \implies \frac{1}{(t + 1)^c} \geq \frac{1}{t^c} - \frac{c}{t^{c+1}}$$

Hence, using $a \leq 1$ and $t > (C_1C_\eta^2)\frac{1}{2}$ for $\Delta_{t+1}$ we have:

$$\Delta_{t+1} = \frac{1}{(t + 1)^{b-a}} - \frac{1}{t^{b-a}} + C_\eta \frac{1}{t^b} - C_1C_\eta^2 \geq -(b-a) \frac{1}{t^{b-a+1}} + C_\eta \frac{1}{t^b} - C_1C_\eta^2 \geq - b - a + C_\eta \frac{1}{t^b} \geq - b + a + 1 \geq 0$$

Moreover,

$$\frac{C_2C_\zeta}{\Delta_{t+1}t^b} \leq \frac{C_2C_\zeta}{C_\eta - 1 - b + a} \leq C_A$$

Combining these two result in (13) gives:

$$A_{t+1} \leq \frac{C_A}{(t + 1)^{b-a}} - \Delta_{t+1} \left( C_A - \frac{C_2C_\zeta}{\Delta_{t+1}t^b} \right) \leq \frac{C_A}{(t + 1)^{b-a}}$$

which proves the induction step.

\[ \square \]

**A.4. Proof of Corollary 1**

**Corollary 2.** Consider Algorithm 1 with step sizes given by: $\alpha_t = \frac{C_\alpha}{t^a}$, $\beta_t = \frac{C_\beta}{t^b}$, and $K^{(3)}_t = C_3t^e$, for some constants $C_\alpha, C_\beta, C_3, a, b, e > 0$ such that $(2a - 2b) \notin [-1, 0]$, and $b \leq 1$. For $C_\Delta, C_\xi > 0$, we have:

$$\mathbb{E}_{\text{total}} \left[ \|g(x_t) - y_t\|^2 \right] \leq \frac{L_g^2C^2_\Delta}{2} \frac{1}{t^{4a-4e}} + 2C_\xi \frac{1}{t^{b+e}},$$

for some constants $C_\Delta, C_\xi > 0$.

**Proof.** Using $\alpha_t = \frac{C_\alpha}{t^a}$, $\beta_t = \frac{C_\beta}{t^b}$ in the recurrent inequalities for $\mathcal{F}_t^2, \mathcal{D}_t$ and $\mathbb{E}_{\text{total}} \left[ \|\mathcal{E}_{t+1}\|^2 \right]$ gives:

1. For $\mathcal{F}_{t+1}^2$:

$$\mathcal{F}_{t+1}^2 \leq (1 - \beta_t)\mathcal{F}_t^2 + \frac{4M_g^2M_f^2\alpha_t^2}{\epsilon^2} = \left(1 - \frac{C_\beta}{t^b}\right)\mathcal{F}_t^2 + \frac{4M_g^2M_f^2C_\alpha^2}{\epsilon^2C_\beta} \frac{1}{t^{2a-b}}$$

and applying Lemma 6 gives

$$\mathcal{F}_t^2 \leq \frac{C_\mathcal{F}}{t^{2a-2b}} \quad \text{where} \quad C_\mathcal{F} = \frac{4M_g^2M_f^2C_\alpha^2}{\epsilon^2C_\beta(C_\beta - 1 - 2a + 2b)} \quad (14)$$

2. For $\mathcal{D}_{t+1}$:

$$\mathcal{D}_{t+1} \leq \left(1 - \frac{C_\beta}{t^b}\right)\mathcal{D}_t + \frac{2M_g^2M_f^2C_\alpha^2}{\epsilon^2C_\beta} \frac{1}{t^{2a-b}} + C_\beta C_\mathcal{F} \frac{1}{t^{2a-b}} = \left(1 - \frac{C_\beta}{t^b}\right)\mathcal{D}_t + \left[\frac{2M_g^2M_f^2C_\alpha^2}{\epsilon^2C_\beta} + C_\beta C_\mathcal{F}\right] \frac{1}{t^{2a-b}}$$
and applying Lemma 6 gives

\[ D_t \leq \frac{C_D}{t^{2a-2b}} \quad \text{where} \quad C_D = \frac{2M^2M_F^2C^2_a + e^2C^2_3C_F\epsilon}{\epsilon^2C_\beta(1 - 2a + 2b)} \]  \hspace{1cm} (15)

3. For \( \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_{t+1} \|_2^2 \right] \):

\begin{align*}
\mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_{t+1} \|_2^2 \right] &\leq (1 - \beta_t)^2 \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right] + \frac{\beta_t^2}{K_t(3)} \sigma_3^2 = \\
\left( 1 - \frac{2C_\beta}{t^b} + \frac{C^2_\beta}{t^{2b}} \right) \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right] + \frac{C^2_\beta \sigma_3^2}{C_3} \frac{1}{t^{2b+e}} = \\
\left( 1 - \frac{\tilde{C}_\beta}{t^b} + \frac{\tilde{C}^2_\beta}{4t^{2b}} \right) \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right] + \frac{C^2_\beta \sigma_3^2}{C_3} \frac{1}{t^{2b+e}}
\end{align*}

and applying Lemma 6 gives:

\[ \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right] \leq \frac{C_\mathcal{E}}{t^{b+e}} \quad \text{where} \quad C_\mathcal{E} = \max_{t \leq (C_2^3)^{1/3} + 1} \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right]^{b+e} + \frac{C^2_\beta \sigma_3^2}{C_3(2C_\beta - 1 - b - e)} \]  \hspace{1cm} (16)

Next, combining results (15) and (16) in the bound of Lemma 5 gives:

\[ \mathbb{E}_{\text{total}} \left[ \| \mathcal{g}(x_t) - y_t \|_2^2 \right] \leq \]

\[ \frac{L^2}{2} \mathbb{E}_{\text{total}} \left[ D_t^2 \right] + 2 \mathbb{E}_{\text{total}} \left[ \| \mathcal{E}_t \|_2^2 \right] \leq \frac{L^2C_D^2}{2} \frac{1}{t^{4a-4b}} + 2C^2_\mathcal{E} \frac{1}{t^{b+e}} \]

\[ \square \]

A.5. Proof of Main Theorem 1

**Theorem 2.** Consider Algorithm 1 with a parameter setup given by: \( \alpha_t = C_{\alpha}/t^{\delta} \), \( \beta_t = C_\beta, K_t(1) = \frac{1}{4} t^{\delta}, K_t(2) = C_2 t^{\delta}, K_t(3) = C_3 t^{\delta}, \gamma_t(1) = C_4 t^{\delta}, \gamma_t(2) = 1 - C_{\gamma}/t^{\delta}(1 - C_{\gamma}/t^{\delta})^2 \), for some positive constants \( C_{\alpha}, C_\beta, C_1, C_2, C_3, C_4, C_{\gamma} \) such that \( C_\beta < 1 \) and \( \mu \in (0, 1) \). For any \( \delta \in (0, 1) \), Algorithm 1 outputs, in expectation, a \( \delta \)-approximate first-order stationary point \( \hat{x} \) of \( \mathcal{J}(x) \). That is:

\[ \mathbb{E}_{\text{total}} \left[ \| \nabla_{\mathcal{J}}(\hat{x}) \|_2^2 \right] \leq \delta, \]

with “total” representing all incurred randomness. Moreover, Algorithm 1 acquires \( \hat{x} \) with an overall oracle complexity for Oracle\(_f(\cdot, \cdot)\) and Oracle\(_g(\cdot, \cdot)\) of the order \( O \left( \delta^{-\frac{2}{3}} \right) \).

**Proof.** Let us study the change of the function between two consecutive iterations. Using, that function \( \mathcal{J}(\cdot) \) is Lipschitz continuous (see Lemma 4):

\[ \mathcal{J}(x_{t+1}) \leq \mathcal{J}(x_t) + \nabla \mathcal{J}(x_t)^T(x_{t+1} - x_t) + \frac{L}{2} \| x_{t+1} - x_t \|_2^2 = \]  \hspace{1cm} (17)

\[ \mathcal{J}(x_t) - \alpha_t \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \frac{[m_i]^2}{\sqrt{[v_i]^2} + \epsilon} + \frac{L\alpha_t^2}{2} \sum_{i=1}^p \frac{[m_i]^2}{\sqrt{[v_i]^2} + \epsilon} \]

Now, let us introduce mathematical expectation with respect to all randomness at iteration \( t \) given a fixed iterative value \( x_t \) as \( \mathbb{E}_t[\cdot] = \mathbb{E}_{K_t(1), K_t(2), K_t(3)}[\cdot | x_t] \). This expectation taking into account all samplings which is done on iteration \( t \) of Algorithm 1. Then, it is easy to see that the following variables will be \( t \) – measurable:
\( \mathcal{J}(x_t), m_t, v_t, x_{t+1}, z_{t+1}, y_{t+1} \). On the other hand, the following variables will be independent from randomness introduced at iteration \( t \): \( \nabla \mathcal{J}(x_{t-1}), m_{t-1}, v_{t-1}, x_{t}, z_{t}, y_{t} \). Hence, taking expectation \( E_t \) from the both sides of equation (17) gives:

\[
E_t [\mathcal{J}(x_{t+1})] \leq \mathcal{J}(x_t) - \alpha_t \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\|v_t\|_i + \epsilon}} \right] + \frac{L \alpha_t^2}{2} \sum_{i=1}^p \mathbb{E}_t \left[ \frac{\|m_i\|^2}{\sqrt{\|v_t\|_i + \epsilon}^2} \right]
\]

(18)

Now, let us focus on the second term in the above expression:

\[
\sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\|v_t\|_i + \epsilon}} \right] = \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\|v_t\|_i + \epsilon}} - \frac{m_i}{\sqrt{\gamma_t(1)|v_{t-1}|_i + \epsilon}} \right] + \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\|v_t\|_i + \epsilon}} - \frac{m_i}{\sqrt{\gamma_t(2)|v_{t-1}|_i + \epsilon}} \right]
\]

Please notice, from Assumptions 1.3 and 1.4 it follows immediately:

\[
\|\nabla \mathcal{J}(x_t)\|_2 \leq E_w \left[ \|\nabla g_w(x_t)^T\|_2 \right] \mathbb{E}_v \left[ \|\nabla f_v(E_w[g_w(x_t)])\|_2 \right] \leq M_g M_f
\]

(19)

and applying induction:

\[
\|m_t\|_2 \leq \gamma_t(1) M_g M_f + \left( 1 - \gamma_t(1) \right) M_g M_f = M_g M_f, \quad \forall t
\]

\[
\|v_t\|_2 \leq \gamma_t(2) M_g^2 M_f^2 + \left( 1 - \gamma_t(2) \right) M_g^2 M_f^2 = M_g^2 M_f^2, \quad \forall t
\]

Now, let us apply (19) for the expression \( A \):

\[
A = \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\gamma_t(1)|v_{t-1}|_i + \epsilon}} \right] = \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\gamma_t(2)|v_{t-1}|_i + \epsilon}} \right] + \sum_{i=1}^p \left[ \nabla \mathcal{J}(x_t) \right]_i \mathbb{E}_t \left[ \frac{m_i}{\sqrt{\gamma_t(2)|v_{t-1}|_i + \epsilon}} \right]
\]

Please notice, from Assumptions 1.3 and 1.4 it follows immediately:
Using Assumption 2.2 we have:

\[
A_1 = \mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla J(x_t) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] + \mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla g_t(x_t)^T \nabla f_t(y_t) - \nabla g_t(x_t)^T \nabla f_t(y_t) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] =
\]

\[
\mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] = \mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla f_t(g(x_t)) - \nabla f_t(g(x_t)) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] =
\]

\[
\mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla g_t(x_t)^T \nabla f_t(y_t) - \nabla g_t(x_t)^T \nabla f_t(y_t) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] = \mathbb{E}_t \left[ \frac{\nabla J(x_t)^T \left( \nabla f_t(y_t) - \nabla f_t(y_t) \right)}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] =
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

\[
\mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla g_t(x_t)^T \nabla f_t(g(x_t)) - \nabla g_t(x_t)^T \nabla f_t(g(x_t)) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] \leq \frac{1}{2} \mathbb{E}_t \left[ \sum_{i=1}^p \frac{\left| \nabla J(x_t) \right|^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} \right] +
\]

Hence, we arrive at the following expression for $A$:

\[
- A = -\gamma_t^{(1)} \nabla J(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} - (1 - \gamma_t^{(1)}) \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} + \left( 1 - \gamma_t^{(1)} \right) A_1 \leq \quad (20)
\]

\[
- \gamma_t^{(1)} \nabla J(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} - (1 - \gamma_t^{(1)}) \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} +
\]

\[
\left( 1 - \gamma_t^{(1)} \right) \left[ \frac{1}{2} \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} + \frac{1}{2} \epsilon M_g L_j^2 \| g(x_t) - y_t \|_2 \right] =
\]

\[
- \gamma_t^{(1)} \nabla J(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} - (1 - \gamma_t^{(1)}) \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} +
\]

\[
\left( 1 - \gamma_t^{(1)} \right) \left[ \frac{1}{2} \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} + \frac{1}{2} \epsilon M_g L_j^2 \| g(x_t) - y_t \|_2 \right] =
\]

\[
- \gamma_t^{(1)} \nabla J(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} - (1 - \gamma_t^{(1)}) \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} +
\]

\[
\left( 1 - \gamma_t^{(1)} \right) \left[ \frac{1}{2} \sum_{i=1}^p \frac{[\nabla J(x_t)]^2}{\sqrt{\gamma_t^2 v_{t-1} + \epsilon}} + \frac{1}{2} \epsilon M_g L_j^2 \| g(x_t) - y_t \|_2 \right] =
\]
Now, let us consider more carefully the second term:

\[-B = \sum_{i=1}^{p} [\nabla J(x_i)]_i E_t \left[ \frac{|m_i|}{\sqrt{|v_i| + \epsilon}} - \frac{|m_i|}{\sqrt{\gamma_t(2)|v_{i-1}| + \epsilon}} \right] \leq \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{B^2} \left| \frac{|m_i|}{\sqrt{|v_i| + \epsilon}} - \frac{|m_i|}{\sqrt{\gamma_t(2)|v_{i-1}| + \epsilon}} \right| \]

For the expression \(B1\) we have:

\[B1 = \frac{|m_i|}{\sqrt{|v_i| + \epsilon}} \leq \frac{1}{\sqrt{|v_i| + \epsilon}} \leq \frac{(1 - \gamma_t(2))|\nabla J(x_i)|^2}{\sqrt{\gamma_t(2)|v_{i-1}| + \sqrt{\gamma_t(2)|v_{i-1}|}}}
\]

\[\leq \frac{1 - \gamma_t(2)}{\sqrt{\gamma_t(2)|v_{i-1}| + \sqrt{\gamma_t(2)|v_{i-1}|}} \leq \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \leq \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \]

where in the last step we use \(|ab| \leq \alpha a^2 + \frac{1}{\alpha} b^2\) for \(\alpha = \frac{1 - \gamma_t(1)}{\gamma_t(1)}\). Therefore, for the term \(B\) we have:

\[-B \leq \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{B^2} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]

\[\leq \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{B^2} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]

Using that \(|v_t| \geq 0\) for any \(t\) and applying (19) we get immediately:

\[-B \leq \frac{(\gamma_t(1))^2}{\epsilon(1 - \gamma_t(1))} \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{B^2} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]

\[\leq \frac{M_g M_f (\gamma_t(1))^2}{\epsilon(1 - \gamma_t(1))} \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{\gamma_t(2)|v_{i-1}| + \epsilon} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]

\[\leq \frac{M_g M_f (\gamma_t(1))^2}{\epsilon(1 - \gamma_t(1))} \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{\gamma_t(2)|v_{i-1}| + \epsilon} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]

\[+ \frac{M_g M_f (1 - \gamma_t(1))^2}{\epsilon} \sum_{i=1}^{p} \frac{||\nabla J(x_i)||}{B^2} \left[ \frac{2}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{1 - \gamma_t(1)}{\epsilon \left( \sqrt{\gamma_t(2)|v_{i-1}| + \epsilon} \right)} \frac{|m_{i-1}|}{\gamma_t(1)} \right] \]
Finally, we have the bound for the second term in the expression (18):

\[
- \sum_{i=1}^{p} \left[ \nabla J(x_t) \right]_{i} \mathbb{E}_t \left[ \frac{[m_{t,i}]}{[\sqrt{v_{t,i}}] + \epsilon} \right] = - A - B \leq (22)
\]

\[
- \gamma_t^{(1)} \nabla J(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma^{(2)}_t [v_{t-1}]} + \epsilon} \frac{1 - \gamma_t^{(1)}}{2} \sum_{i=1}^{p} \left[ \nabla J(x_t) \right]_{i}^2 + \frac{(1 - \gamma_t^{(1)})}{2\epsilon} M_d^2 L_f^2 \|g(x_t) - y_t\|_2^2
\]

\[
M_g M_f \frac{(\gamma_t^{(1)})^2}{\epsilon (1 - \gamma_t^{(1)})} \sqrt{1 - \gamma_t^{(2)}} \sum_{i=1}^{p} \left[ \frac{[\nabla J(x_t)]_{i} [m_{t-1,i}]}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon} \right] + \frac{M_g M_f (1 - \gamma_t^{(1)})}{\epsilon} \sqrt{1 - \gamma_t^{(2)}} \sum_{i=1}^{p} \mathbb{E}_t \left[ \frac{[\nabla J(x_t)]_{i}^2}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon} \right]
\]

Now, we can focus on the third term in the expression (18). Applying that $[v_t]_i \geq \gamma_t^{(2)} [v_{t-1}])$, we have:

\[
\sum_{i=1}^{p} \mathbb{E}_t \left[ \frac{[m_{t,i}]}{[\sqrt{v_{t,i}}] + \epsilon} \right] \leq \sum_{i=1}^{p} \mathbb{E}_t \left[ \frac{[m_{t,i}]}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon} \right] = \sum_{i=1}^{p} \mathbb{E}_t \left[ [m_{t,i}^2] \right] = \sum_{i=1}^{p} \mathbb{E}_t \left[ [\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon \right] 
\]

\[
2 \left( \gamma_t^{(1)} \right)^2 \sum_{i=1}^{p} \frac{[m_{t-1,i}^2]}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon} \leq 2 \left( \gamma_t^{(1)} \right)^2 \sum_{i=1}^{p} \frac{[\nabla J(x_t)]_{i}^2}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon}
\]

\[
2 \left( \gamma_t^{(1)} \right)^2 \sum_{i=1}^{p} \frac{[m_{t-1,i}^2]}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon} \leq \frac{2 \left( \gamma_t^{(1)} \right)^2}{\epsilon} \sum_{i=1}^{p} \frac{[\nabla J(x_t)]_{i}^2}{[\sqrt{\gamma^{(2)}_t [v_{t-1}]}] + \epsilon}
\]

Hence, we arrive at the following expression for change in the function between two consecutive interaction:

\[
\mathbb{E}_t \left[ J(x_{t+1}) - J(x_t) \right] \leq \mathbb{E}_t \left[ J(x_{t+1}) - J(x_t) \right] \leq ((23))
\]
where in the last step we use (24). Combining this result with (25) gives:

$$\begin{align*}
\sum_{i=1}^{p} \frac{2}{\sqrt{\gamma_t^{(2)} [v_{i-1}]} + \epsilon} & + \alpha_t M_g M_f (1 - \gamma_t^{(1)}) \sqrt{1 - \gamma_t^{(2)}} \epsilon \\
\sum_{i=1}^{p} \frac{2}{\sqrt{\gamma_t^{(2)} [v_{i-1}]} + \epsilon} & + \alpha_t M_g M_f (1 - \gamma_t^{(1)}) \sqrt{1 - \gamma_t^{(2)}} \epsilon
\end{align*}$$

Please notice, from Assumption 2.3 we immediately have the following bounds on the variances of \( \nabla g_t(x) \) and \( \nabla f_t(y) \):

$$
E_t \left[ \left| \nabla g_t(x) - \nabla g(x) \right|_2^2 \right] \leq \frac{1}{K_t^{(2)}} \sigma_1^2, \quad E_t \left[ \left| \nabla f_t(y) - \nabla f(y) \right|_2^2 \right] \leq \frac{1}{K_t^{(2)}} \sigma_1^2
$$

(24)

Now, let us focus on the term \( E_t \left[ \left\| \nabla f_t(y) \right\|_2^2 \right] \). Using Assumption 2.2 we have:

$$
E_t \left[ \left( \nabla f_t(x) \right)_i - \left[ \nabla g(x)^T \nabla f(y) \right]_i \right]^2 = E_t \left[ \left\| \nabla f_t(y) \right\|_2^2 \right] - \left[ \nabla g(x)^T \nabla f(y) \right]_i
$$

(25)

From the other hand, using the properties of matrix \( ||\cdot||_2 \) and Assumptions 1.1, 1.3 and 2.3:

$$
E_t \left[ \left( \nabla f_t(x) \right)_i - \left[ \nabla g(x)^T \nabla f(y) \right]_i \right]^2 = E_t \left[ \left( \nabla g_t(x)^T \nabla f_t(y) \right)_i - \left[ \nabla g(x)^T \nabla f(y) \right]_i \right]^2
$$

$$
E_t \left[ \left. \left( \nabla g_t(x)^T \nabla f_t(y) - \nabla g(x)^T \nabla f_t(y) + \nabla g(x)^T \nabla f_t(y) - \nabla g(x)^T \nabla f(y) \right) \right|_i \right]^2
$$

$$
E_t \left[ \left( \nabla g_t(x)^T - \nabla g(x)^T \right) \left( \nabla f_t(y) \right)_i \right]^2 + 2E_t \left[ \left( \nabla g(x)^T \left( \nabla f_t(y) - \nabla f(y) \right) \right)_i \right]^2
$$

$$
E_t \left[ \left( \sum_{i=1}^{q} \left[ \nabla g_t(x)^T - \nabla g(x)^T \right]_{ir} \left( \nabla f_t(y) \right)_i \right)^2 \right] + 2E_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qE_t \left[ \left( \sum_{i=1}^{q} \left[ \nabla g_t(x)^T - \nabla g(x)^T \right]_{ir} \left( \nabla f_t(y) \right)_i \right)^2 \right] + 2qE_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qE_t \left[ \left( \sum_{i=1}^{q} \left[ \nabla g_t(x)^T - \nabla g(x)^T \right]_{ir} \left( \nabla f_t(y) \right)_i \right)^2 \right] + 2qE_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qE_t \left[ \left( \nabla g(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 \left( \nabla f_t(y) \right)_i \right]^2 + 2qE_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qE_t \left[ \left( \nabla g_t(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 \left( \nabla f_t(y) \right)_i \right]^2 + 2qE_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qE_t \left[ \left( \nabla g_t(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 \left[ \nabla f_t(y) \right]^2 + 2qE_t \left[ \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

$$
2qM_f^2 E_t \left[ \left( \nabla g_t(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 + 2qE_t \left[ \left( \nabla f_t(y) \right)^2 \left( \nabla g(x)^T \right)^2 \right]
$$

$$
2qM_f^2 E_t \left[ \left( \nabla g_t(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 + 2qM_f^2 E_t \left[ \left( \nabla f_t(y) \right)^2 \left( \nabla g(x)^T \right)^2 \right]
$$

$$
2qM_f^2 E_t \left[ \left( \nabla g_t(x)^T \right)_i - \left[ \nabla g_t(x)^T \right]_i \right]^2 + 2qM_f^2 E_t \left[ \left( \nabla f_t(y) \right)^2 \left( \nabla g(x)^T \right)^2 \right]
$$

$$
2qM_f^2 \sum_{i=1}^{q} \left[ \nabla g(x)^T \right]_{ir} \left[ \nabla f_t(y) - \nabla f(y) \right]_{ir} \right]^2
$$

where in the last step we use (24). Combining this result with (25) gives:

$$
E_t \left[ \left\| \nabla f_t(y) \right\|_2^2 \right] \leq 2qM_f^2 \left[ \frac{1}{K_t^{(1)}} \sigma_1^2 \right] + 2qM_g^2 \left[ \frac{1}{K_t^{(2)}} \sigma_1^2 \right] + \left( \nabla g(x)^T \nabla f(y) \right)_{i_r}^2
$$

\[15\] Particularly, we use that for any matrix \( A \in \mathbb{R}^{p \times q} \): \( ||A||_2^2 = \sup_{v, ||v|| = 1} v^T A A^T v \geq \sum_{i=1}^{q} ||A||_{i_r}^2 \) for any \( i = 1, \ldots, p \). To see this, just take \( v = e_i \).
Moreover,
\[
\left[\nabla g(x_t)^T \nabla f(y_t)\right]^2 = \left[\nabla g(x_t)^T \nabla f(y_t) - \nabla g(x_t)^T \nabla f(g(x_t)) + \nabla g(x_t)^T \nabla f(g(x_t))\right]^2 = \\
\left[\nabla g(x_t)^T \nabla f(y_t) - \nabla g(x_t)^T \nabla f(g(x_t))\right] = \left[\nabla g(x_t)^T \nabla f(g(x_t))\right]^2 \leq 2 \left[\nabla g(x_t)^T \nabla f(g(x_t))\right] \leq 2 \left[\nabla \mathcal{J}(x_t)\right]_i^2 + 2 \left[\nabla g(x_t)^T\right]_i^2 \left[\nabla f(y_t) - \nabla f(g(x_t))\right]_i^2 \leq 2 \left[\nabla \mathcal{J}(x_t)\right]_i^2 + 2 M_g^2 L_f^2 \left\| y_t - g(x_t) \right\|_2^2
\]

Hence, we have
\[
\sum_{i=1}^{p} \frac{\mathbb{E}_t \left[\left\| \nabla \mathcal{J}(x_t)\right\|_i^2\right]}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon} \leq \sum_{i=1}^{p} \frac{2 q M_f^2 1 \gamma^{(1)}_t \sigma_2^2 + 2 q M_g^2 \frac{1}{K^{(1)}} \sigma_1^2 + \left[\nabla g(x_t)^T \nabla f(y_t)\right]^2}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon} \leq 2 \left[\nabla \mathcal{J}(x_t)\right]_i^2 + 2 M_g^2 L_f^2 \left\| y_t - g(x_t) \right\|_2^2
\]

Hence, expression (23) can be simplified as follows:
\[
\mathbb{E}_t \left[\mathcal{J}(x_{t+1})\right] - \mathcal{J}(x_t) \leq -\alpha_t \frac{1 - \gamma_t^{(1)}}{2} \sum_{i=1}^{p} \frac{\left[\nabla \mathcal{J}(x_t)\right]_i^2}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon} + \alpha_t \frac{1 - \gamma_t^{(1)}}{2 \epsilon} M_g^2 L_f^2 \left\| g(x_t) - y_t \right\|_2^2 + \\
\left[\frac{L \alpha_t^2 \left(1 - \gamma_t^{(1)}\right)^2}{\epsilon} + \frac{\alpha_t M_g M_f \left(1 - \gamma_t^{(1)}\right) \sqrt{1 - \gamma_t^{(2)}}}{\epsilon} \right] \sum_{i=1}^{p} \frac{\mathbb{E}_t \left[\left\| \nabla \mathcal{J}(x_t)\right\|_i^2\right]}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon} - \alpha_t \gamma_t^{(1)} \nabla \mathcal{J}(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon}
\]

where we used (19):
\[
-\alpha_t \gamma_t^{(1)} \nabla \mathcal{J}(x_t)^T \frac{m_{t-1}}{\sqrt{\gamma_t^{(2)}[v_{t-1}]}} \leq \frac{\alpha_t \gamma_t^{(1)}}{\epsilon} \left\| \nabla \mathcal{J}(x_t) \right\|_2 \left\| m_{t-1} \right\|_2 \leq \frac{\alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\epsilon},
\]

and
\[
\frac{L \alpha_t^2 \left(\gamma_t^{(1)}\right)^2}{\epsilon} \sum_{i=1}^{p} \frac{\left[m_{t-1}\right]_i^2}{\sqrt{\gamma_t^{(2)}[v_{t-1}]_i} + \epsilon} \leq \frac{L \alpha_t^2 \left(\gamma_t^{(1)}\right)^2}{\epsilon} M_g^2 M_f^2,
\]
Grouping the terms in (28) gives:

\[
\alpha_t M_g M_f \left( \gamma_t^{(1)} \right)^2 \frac{\sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i ||\mathbf{m}_{t-1}||_i}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} \leq \alpha_t M_g M_f \left( \gamma_t^{(1)} \right)^2 \frac{\sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \sum_{i=1}^{p} ||\nabla J(x_i)||_i ||\mathbf{m}_{t-1}||_i \leq \alpha_t M_g M_f \left( \gamma_t^{(1)} \right)^2 \frac{\sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} ||\nabla J(x_i)||_i^2 + ||\mathbf{m}_{t-1}||_i^2} \leq \alpha_t M_g^3 M_f^3 \left( \gamma_t^{(1)} \right)^2 \frac{\sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})}
\]

Hence, applying (26) in (27) gives:

\[
\mathbb{E}_t [J(x_{t+1})] - J(x_t) \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} + \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) M_g^2 L_f^2 \left| |g(x_t) - y_t||_2 \right| + \frac{\alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{L_{\alpha_t} \gamma_t^{(1)}}{\varepsilon} + \frac{M_g M_f \gamma_t^{(1)} \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} \\
- \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} + \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) M_g^2 L_f^2 \left| |g(x_t) - y_t||_2 \right| + \frac{\alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{L_{\alpha_t} \gamma_t^{(1)}}{\varepsilon} + \frac{M_g M_f \gamma_t^{(1)} \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} \\
- \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} + \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) M_g^2 L_f^2 \left| |g(x_t) - y_t||_2 \right| + \frac{\alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{L_{\alpha_t} \gamma_t^{(1)}}{\varepsilon} + \frac{M_g M_f \gamma_t^{(1)} \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}}
\]

Grouping the terms in (29) gives:

\[
\mathbb{E}_t [J(x_{t+1})] - J(x_t) \leq - \frac{\alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right)}{\varepsilon} \left[ 1 - \frac{4 \left[ L_{\alpha_t} \left( \frac{1 - \gamma_t^{(1)}}{2} \right) + M_g M_f \sqrt{1 - \gamma_t^{(2)}} \right]}{\varepsilon} \right] \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}} + \frac{\alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{\left. \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \right.}{\varepsilon} + \frac{M_g M_f \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}}
\]

\[
\alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) M_g^2 L_f^2 \left[ 1 + \frac{4p \left[ L_{\alpha_t} \left( \frac{1 - \gamma_t^{(1)}}{2} \right) + M_g M_f \sqrt{1 - \gamma_t^{(2)}} \right]}{\varepsilon} \right] \left| |g(x_t) - y_t||_2 \right| + \frac{2p \alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{\left( \frac{1 - \gamma_t^{(1)}}{2} \right)}{\varepsilon} + \frac{M_g M_f \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}}
\]

\[
\alpha_t \gamma_t^{(1)} M_g^2 M_f^2 \left[ 1 + \frac{\left( \frac{1 - \gamma_t^{(1)}}{2} \right)}{\varepsilon} + \frac{M_g M_f \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \left| |g(x_t) - y_t||_2 \right| + \frac{2p \alpha_t \gamma_t^{(1)} M_g^2 M_f^2}{\varepsilon} \left[ 1 + \frac{\left( \frac{1 - \gamma_t^{(1)}}{2} \right)}{\varepsilon} + \frac{M_g M_f \sqrt{1 - \gamma_t^{(2)}}}{\varepsilon (1 - \gamma_t^{(1)})} \right] \leq \alpha_t \left( \frac{1 - \gamma_t^{(1)}}{2} \right) \sum_{i=1}^{p} \frac{||\nabla J(x_i)||_i^2}{\sqrt{\gamma_t^{(2)} ||\mathbf{v}_{t-1}||_i + \varepsilon}}
\]
where we use notation  

\[ C_t = \frac{L_{\alpha t}(1-\gamma_t^{(1)}) + M_g M_f \sqrt{1-\gamma_t^{(2)}}}{\epsilon} \]  

and \(|v_{t-1}| \leq M_g^2 M_f^2\).

Next, let us denote \(E_t \text{ total} \[ \] \) be the expectation with respect to all randomness induced in all \( T \) iterations of Algorithm 1. Using the low of total expectation:

\[
E_{\text{total}}[E_t[\zeta_t]] = E_{\text{total}}[E_{K_t^{(1)}, K_t^{(2)}, K_t^{(3)}}[\zeta_t|x_t]] = E_{\text{total}}[\zeta_t]
\]

for any \( t \)-measurable\(^{16} \) random variable \( \zeta_t \). Hence, taking expectation \( E \) from both sides of (29) gives:

\[
E_{\text{total}}[\mathcal{J}(x_{t+1}) - \mathcal{J}(x_t)] \leq \]

\[
- \alpha_t \left(1 - \gamma_t^{(1)} \right) \left[ 1 - 4C_t \sum_{i=1}^{p} \left[ \frac{\sqrt{\mathcal{J}(x_i)}^2}{\epsilon} \right] + \frac{\alpha_t \left(1 - \gamma_t^{(1)} \right) M_g^2 L_f^2}{2\epsilon} \right] + \frac{\alpha_t \left(1 - \gamma_t^{(1)} \right) M_g^2 L_f^2}{2\epsilon} \leq \]  

\[
+ \frac{2pq\alpha_t \left(1 - \gamma_t^{(1)} \right) \left( M_g^2 \sigma_2^2 + M_g^2 \sigma_1^2 \right)}{\epsilon} \left[ \gamma_t^{(1)} C_t \left(1 - \gamma_t^{(1)} \right) \right]
\]

Now, let \( \alpha_t = \frac{C_t}{t^a}, \beta_t = \frac{C_t}{t^b}, K_t^{(1)} = C_1 t^e, K_t^{(2)} = C_2 t^e \) and \( K_t^{(3)} = C_3 t^e \) for some constants \( C_\alpha, C_\beta, C_1, C_2, C_3, a, b, c, e > 0 \) such that \((2a - 2b) \notin [-1, 0], b \leq 1\). Following Corollary 2 we have:

\[
E_{\text{total}}[||g(x_t) - y_t||_2^2] \leq \frac{L_g^2 C_D^2}{2} \frac{1}{4 a - 4 b} + 2 C_F^2 \frac{1}{b+e}
\]

for some constants \( C_D, C_F > 0 \), and (30) can be written as:

\[
E_{\text{total}}[\mathcal{J}(x_{t+1}) - \mathcal{J}(x_t)] \leq \]

\[
- C_\alpha \left(1 - \gamma_t^{(1)} \right) \left[ 1 - 4C_t \sum_{i=1}^{p} \left[ \frac{\sqrt{\mathcal{J}(x_i)}^2}{\epsilon} \right] + \frac{C_\alpha \left(1 - \gamma_t^{(1)} \right) M_g^2 L_f^2}{2t^a \epsilon} \right] + \frac{C_\alpha \left(1 - \gamma_t^{(1)} \right) M_g^2 L_f^2}{2t^a \epsilon} \leq \]  

\[
+ \frac{2pqC_\alpha C_t \left(1 - \gamma_t^{(1)} \right) \left( M_g^2 \sigma_2^2 + M_g^2 \sigma_1^2 \right)}{t^a \epsilon} \left[ \gamma_t^{(1)} C_t \left(1 - \gamma_t^{(1)} \right) \right]
\]

with \( C_t = \frac{L_{\alpha t}(1-\gamma_t^{(1)}) + M_g M_f \sqrt{1-\gamma_t^{(2)}}}{\epsilon} \). By choosing \( \gamma_t^{(2)} = 1 - \frac{C_F^2}{t^b} \left(1 - \gamma_t^{(1)} \right)^2 \) such that \( \sqrt{1 - \gamma_t^{(2)}} = \frac{C_F}{t^b} \left(1 - \gamma_t^{(1)} \right) \) we have \( C_t = \frac{C_\alpha (L+M_g M_f) \left(1 - \gamma_t^{(1)} \right)}{t^a \epsilon} \). By choosing \( C_\alpha \leq \frac{C_F}{t^b} \sqrt{s} (L+M_g M_f) \) we have \( \Delta C_t \leq 1 \), hence, (31) can be

\(^{16}\)Random variable is called \( t \)-measurable if its affected by the randomness induced in the first \( t \) rounds.
simplified as:
$$\mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_{t+1}) - \mathcal{J}(\mathbf{x}_t)] \leq$$
$$- \frac{C_\alpha}{2\alpha(M_g M_f + \epsilon)} \mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] + \frac{C_\alpha (1 - \gamma_t^{(1)}) M_g^2 L_f^2}{t^a} \left[ \frac{L_g^2 C_\alpha^2}{2} \frac{1}{t^{4a-4b}} + 2C_\epsilon^2 \frac{1}{t^{b+c}} \right] +$$
$$qC_\alpha \left( C_1 M_g^2 \sigma_1^2 + C_2 M_f^2 \sigma_2^2 \right) \left( 1 - \gamma_t^{(1)} \right) \frac{1}{t^{a+c}} + \frac{C_\alpha M_g^2 M_f^2}{\epsilon} \left[ 1 + \frac{\gamma_t^{(1)}}{4p (1 - \gamma_t^{(1)})} \right] \frac{\gamma_t^{(1)}}{t^a} \leq$$
$$- \frac{C_\alpha}{4p (M_g M_f + \epsilon)} \mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] + \frac{C_\alpha (1 - \gamma_t^{(1)}) M_g^2 L_f^2}{t^a} \left[ \frac{L_g^2 C_\alpha^2}{2} \frac{1}{t^{4a-4b}} + 2C_\epsilon^2 \frac{1}{t^{b+c}} \right] +$$
$$qC_\alpha \left( C_1 M_g^2 \sigma_1^2 + C_2 M_f^2 \sigma_2^2 \right) \left( 1 - \gamma_t^{(1)} \right) \frac{1}{t^{a+c}} + \frac{C_\alpha M_g^2 M_f^2}{\epsilon} \left[ 1 + \frac{\gamma_t^{(1)}}{4p (1 - \gamma_t^{(1)})} \right] \frac{\gamma_t^{(1)}}{t^a} \leq$$

Assuming that $\gamma_t^{(1)} = C_\gamma \mu t$ for some $C_\gamma \in (0, \frac{1}{2})$ and $\mu \in [0, 1)$, then expression (32):
$$\mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_{t+1}) - \mathcal{J}(\mathbf{x}_t)] \leq$$
$$- \frac{C_\alpha}{8p (M_g M_f + \epsilon)} \mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] + \frac{C_\alpha M_g^2 L_f^2}{t^a} \left[ \frac{L_g^2 C_\alpha^2}{2} \frac{1}{t^{4a-4b}} + 2C_\epsilon^2 \frac{1}{t^{b+c}} \right] +$$
$$qC_\alpha \left( C_1 M_g^2 \sigma_1^2 + C_2 M_f^2 \sigma_2^2 \right) \left( 1 - \gamma_t^{(1)} \right) \frac{1}{t^{a+c}} + \frac{C_\alpha M_g^2 M_f^2}{\epsilon} \left[ 1 + \frac{\gamma_t^{(1)}}{4p (1 - \gamma_t^{(1)})} \right] \frac{\gamma_t^{(1)}}{t^a} \leq$$

Finally, choosing
$$C_\alpha = \min \left\{ \frac{\epsilon}{8p(L + M_g M_f)}, \frac{\epsilon}{2M_g C_\epsilon}, \frac{2C_\alpha C_\epsilon}{L_g^2 C_\epsilon^2}, \frac{2\epsilon}{M_f C_\epsilon}, \frac{\epsilon}{2M_f C_\epsilon} \right\}$$
we have:
$$\mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_{t+1}) - \mathcal{J}(\mathbf{x}_t)] \leq$$
$$- \frac{C_\alpha}{t^a} \mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] + \frac{1}{t^{5a-4b}} + \frac{1}{t^{a+c}} + \frac{1}{t^a} + \frac{\mu t}{t^a}$$

where $C_\alpha = \frac{C_\alpha}{8p(M_g M_f + \epsilon)}$. Therefore,
$$\mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] \leq$$
$$\frac{t^a}{C_\alpha} \mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_t) - \mathcal{J}(\mathbf{x}_{t+1})] + \frac{1}{C_\alpha t^{4a-4b}} + \frac{1}{C_\alpha t^{b+c}} + \frac{1}{C_\alpha t^a} + \frac{\mu t}{C_\alpha}$$

Taking summation in (33) over $t = 1, \ldots, T$ and dividing the result by $T$ gives:
$$\sum_{t=1}^T \mathbb{E}_{\text{total}} [\|
abla \mathcal{J}(\mathbf{x}_t)\|_2^2] \leq$$
$$\frac{t^a}{C_\alpha} \mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_t) - \mathcal{J}(\mathbf{x}_{t+1})] + \frac{1}{C_\alpha T^{4a-4b}} + \frac{1}{C_\alpha T^{b+c}} + \frac{1}{C_\alpha T^a} + \frac{\mu t}{C_\alpha}$$

Notice, using first order concavity condition for function $f(t) = t^a$ (if $a \in (0, 1)$) and Assumption 1.1:
$$\sum_{t=1}^T t^a \mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_t) - \mathcal{J}(\mathbf{x}_{t+1})] =$$
$$\mathcal{J}(\mathbf{x}_1) + \sum_{t=1}^{T} (t+1)^a - t^a \mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_t)] \leq B_f + \sum_{t=2}^{T} at^{a-1} \mathbb{E}_{\text{total}} [\mathcal{J}(\mathbf{x}_t)] \leq \mathcal{J}(\mathbf{x}_1) + B_f \sum_{t=1}^{T} at^{a-1} \leq$$
$$B_f + B_{\alpha d} T^a$$
Hence, for (34) we have (for $4a - 4b \leq 1$):

$$
\sum_{t=1}^{T} \mathbb{E}_{\text{total}} \left[ \| \nabla J(x_t) \|^2 \right] \leq \frac{B_f}{C_0} \frac{1}{T} + \frac{aB_f}{C_0} \frac{1}{T^{1-a}} + \frac{1}{T} \mathcal{O} \left( T^{4b - 4a + 1} \| b + e = 1 \| \right) + \frac{1}{T} \mathcal{O} \left( \log T \| b = 1 \| \right)
$$  \tag{35}

implies

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\text{total}} \left[ \| \nabla J(x_t) \|^2 \right] \leq \mathcal{O} \left( \frac{1}{\log T} \right) \leq \delta
$$

and for the first oracles complexity we have the following expression

$$
\Psi(a, b, c, e) = \min_{a, b, c, e} \left\{ \min \left( 1, \frac{1 + \max\{c, e\}}{\delta^{\frac{1}{\delta^2}}} \right) \right\}
$$

where $\|\text{condition}\| = 1$ if condition is satisfied, and 0 otherwise. Notice, that oracle complexity per iteration of Algorithm 1 is given by $\mathcal{O}(\delta^{\max\{c, e\}})$. Hence, after $T$ iterations, the total first order oracles complexity is given by $\mathcal{O}(T^{1 + \max\{c, e\}})$.

Let us denote $\phi(a, b, c, e) = \min\{1 - a, 4a - 4b, b + e, c\}$. Then, ignoring logarithmic factors $\log T$ we have:

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\text{total}} \left[ \| \nabla J(x_t) \|^2 \right] \leq \mathcal{O} \left( \frac{1}{\phi(a, b, c, e)} \right)
$$

implies $T = \mathcal{O}\left( \frac{1}{\phi(a, b, c, e)} \right)$, and for the first oracles complexity we have the following expression $\Psi(a, b, c, e) = \frac{1}{\delta^{\frac{1}{\delta^2}}}$. Hence, we have the following optimisation problem to find the optimal setup of parameters $a, b, c, e$:

$$
\min_{a, b, c, e} \min \left\{ 1, \frac{1 + \max\{c, e\}}{\delta^{\frac{1}{\delta^2}}} \right\}
$$

s.t. $a > b$

$0 \leq a \leq 1$

$0 \leq b \leq 1$

$4a - 4b \leq 1$

$c \geq 0$

$e \geq 0$

Considering all possible cases (8 in total) it is easy to see that the optimal setup is given by: $a^* = \frac{1}{5}, b^* = 0, c^* = e^* = \frac{1}{5}$, and overall complexity is given by:

$$
\Psi(a^*, b^*, c^*, e^*) = \delta^{-\frac{2}{\delta}}
$$

This implies that Algorithm 1 in expectation outputs $\delta-$approximate first order stationary point of function $J(x)$ and requires $\mathcal{O}\left( \delta^{-\frac{2}{\delta}} \right)$ calls to the oracles $\mathcal{F}O\mathcal{O}_{J}[\cdot, \cdot]$ and $\mathcal{F}O\mathcal{O}_{g}[\cdot, \cdot]$.

B. Additional Experiment Details

B.1. Portfolio Mean-Variance

In this section, we present numerical results of sparse mean-variance optimization problems on real-world portfolio datasets from CRSP\footnote{https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html}, which are formed on size and: 1) Book-to-Market (BM), 2) Operating Profitability (OP), and 3) Investment (INV). We consider three large 100-portfolio datasets ($m = 13781, n = 100$), and eighteen region-based medium 25-portfolio datasets ($m = 7240, n = 25$).

Given $n$ assets and the reward vectors at $m$ time points, the goal of sparse mean-variance optimization is to maximize the
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Figure 2. Performance on 3 large 100-portfolio datasets (m = 13781, n = 100).

Figure 3. Performance on 6 region-based Operating Profitability datasets (m = 7240, n = 25).

return of the investment as well as to control the investment risk:

$$\min_{x \in X} \frac{1}{m} \sum_{i=1}^{m} \left( r_i^T x - \frac{1}{m} \sum_{i=1}^{m} r_i^T x \right)^2 - \frac{1}{m} \sum_{i=1}^{m} r_i^T x,$$

which is in the form of Eq. (1) with $p = n, q = n + 1$,

$$g_\omega(x) = g_j(x) = \left[ \begin{array}{c} x \\ - r_j^T x \end{array} \right], \quad f_\nu(g_\omega(x)) = f_\nu(g_j(x)) = (r_j^T x) - [r_j^T, 0] f_j(x).$$

for $x \in \mathbb{R}^p$ and a bounded set $X$.

Figure 2 shows that C-ADAM outperforms other algorithms on large datasets. AGD performs the worst because of the highest per-iteration sample complexity on large datasets. Figures 3, 4, 5 show that C-ADAM outperforms other algorithms on Operating Profitability, Investment, and Book-to-Market datasets, respectively. This is consistent with the better complexity bound of C-ADAM. Overall, C-ADAM has the potential to be a benchmark algorithm for convex composition optimization.
Figure 4. Performance on 6 region-based Investment datasets ($m = 7240, n = 25$).

Figure 5. Performance on 6 region-based Book-to-Market datasets ($m = 7240, n = 25$).
In few-shot supervised regression problem, the goal is to predict the outputs of a sine wave $T_k$ from only a few datapoints $\xi_{obs}$ sampled from $P_{T_k}^{(data)}(\cdot)$, after training on many functions $T_k$ using observations of the form $\xi = (\xi_{obs}, \xi_{target})$. The amplitude and phase of the sinusoid are varied between tasks. Specifically, for each task, the underlying function is

$$\xi_{target} = a \sin(\xi_{obs} + b),$$

where $a \in [0.1, 5.0]$ and $b \in [0, 2\pi]$. The goal is to learn to find $\xi_{target}$ given $\xi_{obs}$ based on $M = 10$ $(\xi_{obs}, \xi_{target})$ pairs. The loss for task $T_k$ is represented by the mean-squared error between the model’s output for $\xi_{obs}$ and the corresponding target values $\xi_{target}$:

$$L_{T_k}(\text{NN}(x); \xi) = \sum_{(\xi_{obs}, \xi_{target}) \sim P_{T_k}^{(data)}(\cdot)} \| \text{NN}_x(\xi_{obs}) - \xi_{target} \|_2^2.$$

During training and testing, datapoints $\xi_{obs}$ are sampled uniformly from $[-5.0, 5.0]$. Free parameters in Algorithm 1 were set to $C_\alpha = 0.001, C_\beta = 0.99, K_1^{(1)} = K_1^{(2)} = K_1^{(3)} = 10$, and $C_\gamma = 1$. All models were trained on a single NVIDIA GeForce GTX 1080 Ti GPU.