Optimization of one-step hybrid method for direct solution of fifth order ordinary differential equations of initial value problems

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Abstract
This paper focuses on the derivation, analysis and implementation of a hybrid method by optimizing the order of the method by introduction of six-hybrid points for direct solution of fifth order ordinary differential equations of initial value problems (IVPs). Power series was used as the basis function for the solution of the IVP. The basis function was interpolated at some selected hybrid points whereas the fifth derivative of the approximate solution was collocated at all the interval of integration of the method to generate a system of linear equations for the determination of the unknown parameters. The derived method was tested for consistency, zero stability, convergence and absolute stability. The method was tested with two linear test problems to confirm its accuracy and usability. The comparison of the results with some existing methods shows the superiority of the accuracy of the method.

AMS Subject Classification: 65L05, 65L06, 65L10, 65L12

Keywords: Hybrid Method; Fifth Order ODEs; Initial Conditions; Linear Fifth Order Problems

1. Introduction
In this paper, we consider fifth-order ordinary differential equations, ODEs which we often encounter in the field of sciences, engineering and dynamic systems. They are generally written as

\[ y^{(v)}(x) = f(x, y, y', y'', y''', y''''), \]  \hspace{1cm} (1)

with the following initial conditions

\[ y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \quad y'''(x_0) = y_3, \quad y''''(x_0) = y_4 \]

In literature, it has been shown that attention has been directed at solving higher order ordinary differential equations with Linear Multistep Methods [LMMs] or One-step method without following the conventional method of reduction to systems of first order problems, see [1-10].

The essence of this work is to avoid the inherent drawbacks associated with such methods and to improve accuracy. The many research activities along this line have produced numerical schemes that are not sufficient enough in handling fifth order problems directly without reducing them to lower order problems. [11] proposed an order six multi-derivative implicit (closed) methods was proposed to solve fifth order ordinary differential Equations. In the work of
[12], an explicit method of order six was used as a main predictor for its implementation was presented. The setback associated with such method lies in the fact that they have low order of accuracy; hence the performances of their methods are not good enough. In this present work, we presents an optimization of one-step hybrid method with order of accuracy of eight for direct solution of fifth order ordinary differential equations for the purpose of enhancing accuracy. The proposed method is zero stable, A-stable, consistent and convergent.

2. Derivation of the Method

The exact solution $y(x)$ to (1) is approximated by form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j$$

(2)

with the fifth derivative given as

$$y^{(5)}(x) = \sum_{i=5}^{c+i-1} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}$$

(3)

where $c$ is the number of collocation points and $i$ is the number of interpolation points. (2) is called interpolation equation while (3) is called collocation equation.

Collocating (3) at the points $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and $x_{n+6}$ and interpolating (2) at the points $x = x_{n+7}, x_{n+8}, x_{n+9}, x_{n+10}, x_{n+11}$ leads to a system of twelve equations which is solved by a Computer Aided Software such as Maple to obtain the required parameters. For simplicity, we compute $a = \frac{1}{7}, b = \frac{2}{7}, c = \frac{3}{7}, d = \frac{4}{7}, e = \frac{5}{7}$

$$a_0 = -\frac{1873}{67765824} h^5 f_{n+1} - \frac{522}{203297472} h^5 f_{n+4} + \frac{1}{67765824} h^5 f_n + 5y_{n+\frac{1}{7}} - 10y_{n+\frac{3}{7}} + 10y_{n+\frac{5}{7}} - 5y_{n+\frac{6}{7}} + y_{n+\frac{8}{7}} - \frac{1}{203297472} h^5 f_{n+1} - \frac{11}{67765824} h^5 f_{n+4} + \frac{1}{22586608} h^5 f_n - \frac{1831}{67765824} h^5 f_{n+\frac{1}{7}} - \frac{439}{203297472} h^5 f_{n+\frac{4}{7}}$$

(4)

$$a_1 = -\frac{1}{19168047360} \frac{1}{h^2} \begin{pmatrix} 635h^5 f_n + 467h^5 f_{n+1} + 1297121h^5 f_{n+\frac{1}{7}} + 8539731h^5 f_{n+\frac{2}{7}} + 7707593h^5 f_{n+\frac{3}{7}} + 672593h^5 f_{n+\frac{4}{7}} + 15075h^5 f_{n+\frac{5}{7}} - 4543h^5 f_{n+\frac{6}{7}} - 860964793920y_{n+\frac{1}{7}} + 23928125440y_{n+\frac{3}{7}} - 261643846440y_{n+\frac{5}{7}} + 1364126037120y_{n+\frac{7}{7}} - 279534924000y_{n+\frac{9}{7}} \\ 997331h^5 f_n + 24745h^5 f_{n+1} + 5810207h^5 f_{n+\frac{1}{7}} + 210880995h^5 f_{n+\frac{2}{7}} + 164992235h^5 f_{n+\frac{3}{7}} + 13625545h^5 f_{n+\frac{4}{7}} + 645969h^5 f_{n+\frac{5}{7}} - 204895h^5 f_{n+\frac{6}{7}} - 1190814942200y_{n+\frac{1}{7}} + 39582017798400y_{n+\frac{2}{7}} - 49309801833600y_{n+\frac{3}{7}} + 27506147961600y_{n+\frac{4}{7}} - 5870214504000y_{n+\frac{5}{7}} \end{pmatrix}$$

(5)

$$a_2 = -\frac{1}{19168047360} \frac{1}{h^2} \begin{pmatrix} 997331h^5 f_n + 24745h^5 f_{n+1} + 5810207h^5 f_{n+\frac{1}{7}} + 210880995h^5 f_{n+\frac{2}{7}} + 164992235h^5 f_{n+\frac{3}{7}} + 13625545h^5 f_{n+\frac{4}{7}} + 645969h^5 f_{n+\frac{5}{7}} - 204895h^5 f_{n+\frac{6}{7}} - 1190814942200y_{n+\frac{1}{7}} + 39582017798400y_{n+\frac{2}{7}} - 49309801833600y_{n+\frac{3}{7}} + 27506147961600y_{n+\frac{4}{7}} - 5870214504000y_{n+\frac{5}{7}} \end{pmatrix}$$

(6)
Substituting (4 – 16) into (2) gives a continuous coefficient of the form:
\[ y(t) = \alpha_1(t)y_{n+\frac{\tau}{7}} + \alpha_2(t)y_{n+\frac{\tau}{7}} + \alpha_3(t)y_{n+\frac{\tau}{7}} + \alpha_4(t)y_{n+\frac{\tau}{7}} + \alpha_5(t)y_{n+\frac{\tau}{7}} + \]
\[ h^5 \left( \beta_0(t) + \beta_1(t) + \beta_2(t) + \beta_3(t) + \beta_4(t) + \beta_5(t) + \beta_6(t) \right) \]  
(17)

where \( \alpha_i(t), \alpha_j(t), \ldots, \alpha_k(t) \) and \( \beta_i(t), \beta_j(t), \ldots, \beta_k(t) \) are continuous coefficients. The continuous method (17) is used to generate the required method for solving (1). That is, we evaluate at \( t = 1 \)

\[ y_{n+1} = 5y_{n+\frac{\tau}{7}} - 24y_{n+\frac{\tau}{7}} + 45y_{n+\frac{\tau}{7}} - 40y_{n+\frac{\tau}{7}} + 15y_{n+\frac{\tau}{7}} - \frac{1}{50824368} h^5 f_n + \frac{1}{12706092} h^5 f_{n+\frac{\tau}{7}} + \frac{209}{16941456} h^5 f_{n+\frac{\tau}{7}} + \]
\[ \frac{3523}{25412184} h^5 f_{n+\frac{\tau}{7}} + \frac{8341}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{137}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{1}{25412184} h^5 f_{n+1} \]  
(18)

3. Analysis of the method

3.1. Order and error Constants of the Methods

According to ([9-17]), the order of the new method (18) is obtained by using the Taylor series. The procedure is shown below

**Theorem 1:** The linear operator and the associated method are said to be of order \( p \) if

\[ C_0 = C_1 = \ldots C_p = C_{p+1} = 0, C_{p+2} = 0, C_{p+3} = 0, C_{p+4} = 0, C_{p+5} = 0 \] 
\( C_{p+6} \) is called the error constant.

\[ y_{n+1} = \left( \begin{array}{c} 5y_{n+\frac{\tau}{7}} - 24y_{n+\frac{\tau}{7}} + 45y_{n+\frac{\tau}{7}} - 40y_{n+\frac{\tau}{7}} + 15y_{n+\frac{\tau}{7}} - \frac{1}{50824368} h^5 f_n + \frac{1}{12706092} h^5 f_{n+\frac{\tau}{7}} + \frac{209}{16941456} h^5 f_{n+\frac{\tau}{7}} + \\
\frac{3523}{25412184} h^5 f_{n+\frac{\tau}{7}} + \frac{8341}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{137}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{1}{25412184} h^5 f_{n+1} \end{array} \right) \]  
(19)

Carrying out Taylor series expansion on (19) gives

\[ \sum_{j=0}^{\infty} \left( \frac{1}{j!} \right)^5 y_{j+1} = \sum_{j=0}^{\infty} \left( \frac{1}{j!} \right)^5 y_{j+1} - 24 \sum_{j=0}^{\infty} \left( \frac{1}{j!} \right)^5 y_{j+1} - 40 \sum_{j=0}^{\infty} \left( \frac{1}{j!} \right)^5 y_{j+1} - 15 \sum_{j=0}^{\infty} \left( \frac{1}{j!} \right)^5 y_{j+1} - \frac{1}{50824368} h^5 f_n + \frac{1}{12706092} h^5 f_{n+\frac{\tau}{7}} + \frac{209}{16941456} h^5 f_{n+\frac{\tau}{7}} + \\
\frac{3523}{25412184} h^5 f_{n+\frac{\tau}{7}} + \frac{8341}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{137}{50824368} h^5 f_{n+\frac{\tau}{7}} + \frac{1}{25412184} h^5 f_{n+1} \]  
(20)

Comparing the values in \( h \) gives

\[ C_0 = \left( 1 \right)^5 - \frac{1}{50824368} \left( 5 \right)^5 \left( \frac{1}{7} \right)^0 - 24. \left( \frac{2}{7} \right)^5 + 45. \left( \frac{3}{7} \right)^5 - 40. \left( \frac{4}{7} \right)^5 + 15. \left( \frac{5}{7} \right)^5 \right) = 0 \]

\[ C_1 = \left( \frac{1}{11} \right)^5 - \frac{1}{50824368} \left( 5 \right)^5 \left( \frac{1}{7} \right)^1 - 24. \left( \frac{2}{7} \right)^5 + 45. \left( \frac{3}{7} \right)^5 - 40. \left( \frac{4}{7} \right)^5 + 15. \left( \frac{5}{7} \right)^5 \right) = 0 \]
\[ C_2 = \left( \frac{(1)^2}{2!} - \frac{1}{6!} \right) \left( 5 \left( \frac{1}{7} \right)^2 - 24 \left( \frac{2}{7} \right)^2 + 45 \left( \frac{3}{7} \right)^2 - 40 \left( \frac{4}{7} \right)^2 + 15 \left( \frac{5}{7} \right)^2 \right) = 0 \]

\[ C_3 = \left( \frac{(1)^3}{3!} - \frac{1}{3!} \right) \left( 5 \left( \frac{1}{7} \right)^3 - 24 \left( \frac{2}{7} \right)^3 + 45 \left( \frac{3}{7} \right)^3 - 40 \left( \frac{4}{7} \right)^3 + 15 \left( \frac{5}{7} \right)^3 \right) = 0 \]

\[ C_4 = \left( \frac{(1)^4}{4!} - \frac{1}{4!} \right) \left( 5 \left( \frac{1}{7} \right)^4 - 24 \left( \frac{2}{7} \right)^4 + 45 \left( \frac{3}{7} \right)^4 - 40 \left( \frac{4}{7} \right)^4 + 15 \left( \frac{5}{7} \right)^4 \right) \]

\[ C_5 = \left( \frac{(1)^5}{5!} - \frac{1}{5!} \right) \left( 5 \left( \frac{1}{7} \right)^5 - 24 \left( \frac{2}{7} \right)^5 + 45 \left( \frac{3}{7} \right)^5 - 40 \left( \frac{4}{7} \right)^5 + 15 \left( \frac{5}{7} \right)^5 \right) - \left( \frac{1}{5!} \left( \frac{0}{7} \right)^0 + \frac{1}{12706092} \left( \frac{1}{7} \right)^0 + \frac{209}{16941456} \left( \frac{2}{7} \right)^0 \right) \]

\[ + \left( \frac{1}{25412184} \left( \frac{3}{7} \right)^0 + \frac{8341}{50824368} \left( \frac{4}{7} \right)^0 + \frac{83}{2117682} \left( \frac{5}{7} \right)^0 \right) = 0 \]

\[ C_6 = \left( \frac{(1)^6}{6!} - \frac{1}{6!} \right) \left( 5 \left( \frac{1}{7} \right)^6 - 24 \left( \frac{2}{7} \right)^6 + 45 \left( \frac{3}{7} \right)^6 - 40 \left( \frac{4}{7} \right)^6 + 15 \left( \frac{5}{7} \right)^6 \right) \]

\[ C_7 = \left( \frac{(1)^7}{7!} - \frac{1}{7!} \right) \left( 5 \left( \frac{1}{7} \right)^7 - 24 \left( \frac{2}{7} \right)^7 + 45 \left( \frac{3}{7} \right)^7 - 40 \left( \frac{4}{7} \right)^7 + 15 \left( \frac{5}{7} \right)^7 \right) \]

\[ C_8 = \left( \frac{(1)^8}{8!} - \frac{1}{8!} \right) \left( 5 \left( \frac{1}{7} \right)^8 - 24 \left( \frac{2}{7} \right)^8 + 45 \left( \frac{3}{7} \right)^8 - 40 \left( \frac{4}{7} \right)^8 + 15 \left( \frac{5}{7} \right)^8 \right) \]

\[ C_9 = \left( \frac{(1)^9}{9!} - \frac{1}{9!} \right) \left( 5 \left( \frac{1}{7} \right)^9 - 24 \left( \frac{2}{7} \right)^9 + 45 \left( \frac{3}{7} \right)^9 - 40 \left( \frac{4}{7} \right)^9 + 15 \left( \frac{5}{7} \right)^9 \right) \]
\[
\frac{-1}{4!} \left( -\frac{1}{50824368}(0)^4 + \frac{1}{23523}(\frac{3}{7})^4 + \frac{1}{25412184}(\frac{6}{7})^4 - \frac{1}{50824368}(1)^4 \right) + \frac{1}{8341}(\frac{1}{7})^4 + \frac{209}{16941456}(\frac{2}{7})^4 = 0
\]

\[
C_{10} = \left( \frac{(1)^{10}}{10!} - \frac{1}{10!} \right) \left( 5.\left(\frac{1}{7}\right)^{10} - 24.\left(\frac{2}{7}\right)^{10} + 45.\left(\frac{3}{7}\right)^{10} - 40.\left(\frac{4}{7}\right)^{10} + 15.\left(\frac{5}{7}\right)^{10} \right)
\]

\[
\frac{-1}{5!} \left( -\frac{1}{50824368}(0)^5 + \frac{1}{23523}(\frac{3}{7})^5 + \frac{1}{25412184}(\frac{6}{7})^5 - \frac{1}{50824368}(1)^5 \right) + \frac{1}{8341}(\frac{1}{7})^5 + \frac{209}{16941456}(\frac{2}{7})^5 = 0
\]

\[
C_{11} = \left( \frac{(1)^{11}}{11!} - \frac{1}{11!} \right) \left( 5.\left(\frac{1}{7}\right)^{11} - 24.\left(\frac{2}{7}\right)^{11} + 45.\left(\frac{3}{7}\right)^{11} - 40.\left(\frac{4}{7}\right)^{11} + 15.\left(\frac{5}{7}\right)^{11} \right)
\]

\[
\frac{-1}{6!} \left( -\frac{1}{50824368}(0)^6 + \frac{1}{23523}(\frac{3}{7})^6 + \frac{1}{25412184}(\frac{6}{7})^6 - \frac{1}{50824368}(1)^6 \right) + \frac{1}{8341}(\frac{1}{7})^6 + \frac{209}{16941456}(\frac{2}{7})^6 = 0
\]

\[
C_{12} = \left( \frac{(1)^{12}}{12!} - \frac{1}{12!} \right) \left( 5.\left(\frac{1}{7}\right)^{12} - 24.\left(\frac{2}{7}\right)^{12} + 45.\left(\frac{3}{7}\right)^{12} - 40.\left(\frac{4}{7}\right)^{12} + 15.\left(\frac{5}{7}\right)^{12} \right)
\]

\[
\frac{-1}{7!} \left( -\frac{1}{50824368}(0)^7 + \frac{1}{23523}(\frac{3}{7})^7 + \frac{1}{25412184}(\frac{6}{7})^7 - \frac{1}{50824368}(1)^7 \right) + \frac{1}{8341}(\frac{1}{7})^7 + \frac{209}{16941456}(\frac{2}{7})^7 = 0
\]

\[
C_{13} = \left( \frac{(1)^{13}}{13!} - \frac{1}{13!} \right) \left( 5.\left(\frac{1}{7}\right)^{13} - 24.\left(\frac{2}{7}\right)^{13} + 45.\left(\frac{3}{7}\right)^{13} - 40.\left(\frac{4}{7}\right)^{13} + 15.\left(\frac{5}{7}\right)^{13} \right)
\]
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\[
\begin{align*}
\left(-\frac{1}{81} \cdot \frac{1}{50824368} \cdot (0)^8 + \frac{1}{81} \cdot \frac{1}{12706092} \cdot (1)^8 + \frac{209}{16941456} \cdot (2)^8\right) + \\
\left(\frac{3}{7} \cdot \frac{1}{25412164} \cdot (3)^8 + \frac{8341}{690223592} \cdot (4)^8 + \frac{83}{2117682} \cdot (5)^8\right) + \\
\left(\frac{137}{50824368} \cdot (6)^8 - \frac{1}{7} \cdot \frac{1}{12706092} \cdot (7)^8\right) \\
= \frac{31}{117196498830720}
\end{align*}
\]

Hence the method is of order eight with Error constant \[ \frac{31}{117196498830720} \]

3.2. Consistency

Definition 3.1: The hybrid method (18) is said to be consistent if it has an order more than or equal to one i.e. \( P \geq 1 \). Therefore, the method is consistent ([15, 18, 20]).

3.3. Zero Stability

Definition 3.2: The hybrid method (18) is said to be zero stable if the first characteristic polynomial \( \pi(r) \) having roots such that \( |r_1| \leq 1 \) and if \( |r_1| = 1 \), then the multiplicity of \( r_1 \) must not be greater than two ([15, 16, 23]). In order to find the zero-stability of hybrid method (18), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

\[ \rho(z) = z - 5z^7 + 24z^{10} - 45z^{14} + 40z^{17} - 15z^{21} \] (21)

Setting equation (21) equal to zero and solving for \( z \) gives \( z=1 \), hence the method is zero stable.

3.4. Convergence

Theorem (2): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (7) is consistent and zero stable, it implies the method is convergent for all point ([15, 16, 19]).

3.5. Regions of Absolute Stability (RAS)

The absolute stability region of the new method is found according to ([16, 20, 21]). The region of its periodicity lies between (0, 24.5). Hence It is A-stable in nature. See [20]

4. Numerical experiments

The method (18) was employed to solve (1) with the help of Taylor Series to provide starting values. The method is tested on two linear fifth order problems to test the accuracy of the proposed methods and our results are compared with the results obtained in the cited papers.

4.1. Numerical examples

The following problems are taken as test problems:

Example I: Consider a Linear fifth order problem
\[ y'' = 32y + \cos x - 32\sin x, \]
\[ y(0) = 1, y'(0) = 3, y''(0) = 4, y'''(0) = 7, y''''(0) = 16, h = 0.1 \]
\[ y(x) = \sin x + e^{2x} \]
Source: [22]

Example II: Consider a Linear fifth order problem
\[ y'' = 5y'' - 4y', \]
\[ y(0) = 3, y'(0) = -5, y''(0) = 11, y'''(0) = -23, y''''(0) = 47, h = 0.1 \]
\[ y(x) = 1 - e^{-x} + 3e^{-2x} \]
Source: [24]
The following Notations were used in the tables

- **X-val**: Value of the independent variables where numerical value is taken
- **Exact-solution**: Exact solution at X-val
- **Computed-solution**: Computed solution at X-val
- **Error**: \( |\text{Exact Solution} - \text{Computed Solution}| \)

**Table 1** Showing the exact solution, the computed solution and the absolute error in the developed method using Problem 1

| x-values   | Exact solution | Computed solution | Error in our Method |
|------------|----------------|-------------------|---------------------|
| 0.100000   | 1.321236174806998100 | 1.321236174806998100 | 0.000000e+000       |
| 0.200000   | 1.690494028436331700  | 1.690494028436331200 | 4.440892e-016       |
| 0.300000   | 2.117639007051848500  | 2.117639007051781400 | 6.705747e-014       |
| 0.400000   | 2.614959270801118200  | 2.614959270800171400 | 9.467982e-013       |
| 0.500000   | 3.197707367063248500  | 3.197707367057542000 | 5.706546e-012       |
| 0.600000   | 3.884759396131582500  | 3.884759396108925100 | 2.265743e-011       |
| 0.700000   | 4.699417654082365600  | 4.699417654012514800 | 6.985079e-011       |
| 0.800000   | 5.670388515294636300  | 5.670388515113154800 | 1.814815e-010       |
| 0.900000   | 6.832974374040428100  | 6.832974373623768500 | 4.166596e-010       |
| 1.000000   | 8.230527083738545400  | 8.23052708267489200  | 8.710561e-010       |

**Table 2** Showing the exact solution, the computed solution and the absolute error in the developed method using Problem 2

| x-values   | Exact solution | Computed solution | Error in our Method |
|------------|----------------|-------------------|---------------------|
| 0.100000   | 2.551354841197985800 | 2.551354841197985800 | 2.664535e-015       |
| 0.200000   | 2.192229385028936500 | 2.192229385029126100 | 1.896261e-013       |
| 0.300000   | 1.905616687600363100 | 1.905616687602535600 | 2.172484e-012       |
| 0.400000   | 1.677666846316027400 | 1.677666846328145400 | 1.211808e-011       |
| 0.500000   | 1.497107663801695600 | 1.497107663847459900 | 4.576428e-011       |
| 0.600000   | 1.354770999642579600 | 1.354770999777845900 | 1.352662e-010       |
| 0.700000   | 1.243205580334085000 | 1.243205583713731000 | 3.379645e-010       |
| 0.800000   | 1.156360589866742400 | 1.156360509614063900 | 7.473215e-010       |
| 0.900000   | 1.089327004924157900 | 1.089327006430507400 | 1.506349e-009       |
| 1.000000   | 1.038126408538393000 | 1.038126411362454900 | 2.824062e-009       |
Table 3 Showing the comparison of error in the developed method and [22] using Problem 3

| x-values | Error in our Method | Error in [22] |
|----------|---------------------|--------------|
| 0.100000 | 0.0000000e+000      | 2.000e-009   |
| 0.200000 | 4.440892e-016       | 9.000e-009   |
| 0.300000 | 6.705747e-014       | 1.000e-009   |
| 0.400000 | 9.467982e-013       | 3.900e-008   |
| 0.500000 | 5.706546e-012       | 3.700e-007   |
| 0.600000 | 2.265743e-011       | 1.966e-007   |
| 0.700000 | 6.985079e-011       | 9.373e-006   |
| 0.800000 | 1.814815e-010       | 3.632e-005   |
| 0.900000 | 4.166596e-010       | 1.203e-004   |
| 1.000000 | 8.710561e-010       | 3.523e-004   |

Table 4 Showing the comparison of error in the developed method and [24] using Problem 4

| x-values | Error in our Method P=8, k=1 | Error in [24] (2019), p=8, k=6 |
|----------|-----------------------------|--------------------------------|
| 0.100000 | 2.664535e-015               | 3.108624468950438e-015          |
| 0.200000 | 1.896261e-013               | 2.39808173190338e-014          |
| 0.300000 | 2.172484e-012               | 6.894929072132072e-012         |
| 0.400000 | 1.211808e-011               | 1.453641651494309e-010         |
| 0.500000 | 4.576428e-011               | 1.681371930573050e-009         |
| 0.600000 | 1.352662e-010               | 1.226418588906597e-008         |
| 0.700000 | 3.379645e-010               | 6.574431687944582e-008         |
| 0.800000 | 7.473215e-010               | 2.808988770475196e-007         |
| 0.900000 | 1.506349e-009               | 1.009537685892070e-006         |
| 1.000000 | 2.824062e-009               | 3.165450833897410e-006         |

5. Conclusion

The optimization of One-step Hybrid Method proposed in this work was applied to solve fifth-order linear problems. This method has been shown to be efficient in terms of its applicability. The numerical Results of Problem 1 and Problem 2 are presented in Table 1 and Table 2. The comparison of this method with other existing ones namely [22] and [24] are shown in Table 3 and Table 4. It shows that the proposed method compares favorably and it’s thus recommended for solution of linear fifth order ODEs.

Compliance with ethical standards

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Disclosure of conflict of interest

The authors declare that they have no conflicts of interests.
References

[1] Awoyemi, D. O., Kayode, S. J. and Adoghe, L. O. A six-step continuous multistep method for the solution of general fourth order initial value problems of ordinary differential equations. Journal of Natural Science Research. 2015; 5(5).

[2] Fasasi, K. M., Adesanya, A. O. and Adee, S. O. One Step Continuous Hybrid Block Method for the Solution of Third Order Ordinary Differential Equations. Journal of Natural Sciences Research. 2014; 4(10): 55 – 62.

[3] Bun, R. A. and Vasil'yer, Y. D. A numerical methods for solving differential equations of any order. Computational Mathematics and Mathematical Physics. 1992; 32: 317 – 330.

[4] Anake, T. A., Awoyemi, D. O. and Adesanya, A. O. One Step Implicit Hybrid Block Methods for the Direct Solution of General Second Order Ordinary Differential Equations. IAENG International Journal of Applied Mathematics. 2012; 42(4): 42 – 44.

[5] Adoghe, L. O., Ogunware, B. G. and Omole, E. O. A family of symmetric implicit higher order methods for the solution of third order initial value problems in ordinary differential equations: Journal of Theoretical Mathematics & Applications. 2016; 6(3): 67-84.

[6] Ismail, F., Ken, Y. L. and Othman, M. Explicit and Implicit 3-point Block Methods for Solving Special Second Order Differential Equations Directly. International Journal of Math. Analysis. 2009; 3(5): 239 – 254.

[7] Fatunla, S. O. “Block Method for Second Order IVPs”. International Journal of Computer Mathematics. 1991; 41: 55-63.

[8] Ogunware, B. G., Omole, E. and Olayemi, 0. O. Hybrid and Non-Hybrid Schemes for Solving Third Order ODEs using Block Methods as Predictors. Mathematical Theory and Modeling. 2015; 5(3): 10 – 23.

[9] Henrici, P. Discrete variable method in ordinary differential equations, John Wiley and Sons. New York. 1962.

[10] Familua, A. B. and Omole E. O. Five points Mono Hybrid point Linear Multistep Method for solving nth Order Ordinary Differential Equations Using Power Series function”. Asian Journal of Research and Mathematics. 2007; 3(1):1-17.

[11] Jator, S N and Lee, L. Implementing a seventh-order linear multistep method in a predictor- corrector mode or block mode: which is more efficient for the general second order initial value problem. SpringerPlus. 2011; 3(1): 1–8.

[12] Kayode, S. J. and Awoyemi, D. O. A multiderivative collocation method for fifth order ordinary differential equations. J. Math. Stat. 2010; 6(1): 60-63.

[13] Kayode, S. J. An order seven continuous explicit method for direct solution of general fifth order ordinary differential equations, International Journal of Differential Equations and Applications. 2014; 13(2): 71-80.

[14] Kayode, S. J. Symmetric implicit multiderivative numerical integrators for direct solution of fifth-order differential equations, Thammasat International Journal of Science and Technology. 2014; 19(2): 1-8.

[15] Kayode, S. J. A Zero Stable Method for Direct Solution of Fourth Order Ordinary Differential Equation. American Journal of Applied Sciences. 2008; 5: 1461-1466.

[16] Adeyeye, O. and Omar, Z. A new algorithm for developing block methods for solving fourth order ordinary differential equations. Global Journal of Pure and Applied Mathematics. 2016; 12(2): 1465-1471.

[17] Olusola, K. J. Block methods for direct solution of higher order ordinary differential equations using interpolation and collocation approach (Doctoral dissertation, University Utara Malaysia). 2015.

[18] Lee, K. Y and Fudziah, I. Block Hybrid Collocation method with application to fourth order differential equations, Journal of Mathematical problems in Engineering, Hindawi Publishing Corporation. 2015.

[19] Waeleh, N., Majid, Z. A. and Ismail, F. A New Algorithm for Solving Higher Order IVPs of ODEs, Journal of Applied Mathematical Sciences. 2011; 5(56): 2795 – 2805.

[20] Areo, E. A. and Omole E. O. Half-Step symmetric continuous hybrid block method for the numerical solutions of fourth order ordinary differential equations: Archives of Applied Science Research. 2015; 7(10): 39-49.

[21] Olabode, B. T. and Alabi, T. J. Direct Block Predictor-Corrector Method for the solution of general fourth order ODEs, Journal of Mathematics Research. 2013; 5(1): 26 – 33.
[22] Salkuyeh, D. K. Convergence of the variational iteration method for solving linear systems of ODEs with constant coefficients, *Computers and Mathematics with Applications*. 2008; 56: 2027–2033.

[23] Ogunware, B. G. and Omole, E. O. A Class of Irrational Linear Multistep Block Method for the Direct Numerical Solution of Third Order Ordinary Differential Equations. *Turkish Journal of Analysis and Number Theory*. 2020; 8(2): 21-27.

[24] Jena, S. R. and Mohanty, M. Numerical Treatment for ODE (Fifth order). *International Journal on Emerging Technologies*. 2019; 10(4): 191–196.