Regular product vague graphs and product vague line graphs

Ganesh Ghorai and Madhumangal Pal

Abstract: Vague graph is a generalized structure of fuzzy graph which gives more precision, flexibility, and compatibility to a system when compared with systems that are designed using fuzzy graphs. In this paper, we introduced the notion of regular, totally regular product vague graphs, and product vague line graph. We proved that under some conditions regular and totally regular product vague graph becomes equivalent. Some properties of product vague line graph are investigated. We showed that a product vague graph is isomorphic to its corresponding product vague line graph under some conditions.

1. Introduction

Nowadays, most mathematical models are developed using fuzzy sets to handle various types of systems containing elements of uncertainty. In 1993, Gau and Buehrer (1993), introduced the notion of vague set theory as a generalization of Zadeh fuzzy set theory (1965). Vague sets are higher order...
fuzzy sets. Application of higher order fuzzy sets makes the solution-procedure more complex, but if the complexity on computation-time, computation-volume, or memory-space are not matters of concern, then we can achieve better results. In a fuzzy set, each element is associated with a point-value selected from the unit interval $[0, 1]$, which is termed as the grade of membership in the set. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. The interval-based membership in vague sets is more expressive in capturing vagueness of data. There are some interesting features for handling vague data that are unique to vague sets. For example, vague sets allow for a more intuitive graphical representation of vague data, which facilitates significantly better analysis in data relationships, incompleteness, and similarity measures.

Considering the fuzzy relations between fuzzy sets, Rosenfeld (1975) first introduced the concept of fuzzy graphs in 1975 and developed the structure of fuzzy graphs, obtaining analogous of several graph concepts. Ramakrishna (2009) introduced the concept of vague graphs. Akram et al. studied some properties of vague graphs (Akram, Chen, & Shu, 2013), vague hypergraphs (Akram, Gani, & Saeid, 2014), regularity in vague intersection graphs (Akram, Dudek, & Yousof, 2014), irregular and highly irregular vague graphs (Akram, Feng, Sarvar, & Jun, 2014), and vague cycles and vague trees (Akram, Feng, Sarvar, & Jun, 2015). Talebi, Rashmanlou, and Mehdiipoor (2013), Talebi, Mehdiipoor, and Rashmanlou (2014) studied isomorphism and operations on vague graphs. Borzooei and Rashmanlou (2015a, 2015b, 2015c, 2016), Rashmanlou and Borzooei (2015, 2016, 2015) introduced many new concepts of vague graphs. Samanta et al. introduced fuzzy competition graphs (Samanta, Akram, & Pal, 2013; Samanta & Pal, 2015), fuzzy planar graphs (Samanta & Pal, 2015), bipolar fuzzy intersection graphs (Samanta & Pal, 2014), and strength of vague graphs (Samanta, Pal, Rashmanlou, & Borzooei, 2016). Later on, Ghorai and Pal studied some properties of generalized $m$-polar fuzzy graphs (2016a), defined operations and density of $m$-polar fuzzy graphs (2015), introduced $m$-polar fuzzy planar graphs (2016b), and defined faces and dual of $m$-polar fuzzy planar graphs (2016c). In this paper, the concept of regular, totally regular product vague graphs, and product vague line graphs are introduced. Necessary and sufficient condition is established under which regular and totally regular product vague graph becomes equivalent and a product vague graph is isomorphic to its corresponding product vague line graph.

2. Preliminaries

In this section, we point out some basic definitions of graphs. The readers are encouraged to see these references (Balakrishnan, 1997; Harary, 1972; Mordeson & Nair, 2000) for further study.

**Definition 2.1** Harary (1972) A graph is an ordered pair $G = (V, E)$, where $V$ is the set of vertices of $G$ and $E$ is the set of all edges of $G$. Two vertices $x$ and $y$ in an undirected graph $G$ are said to be adjacent in $G$ if $xy$ is an edge of $G$. A simple graph is an undirected graph that has no loops and not more than one edge between any two different vertices.

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.

We write $xy \in E$ to mean $(x, y) \in E$, and if $e = xy \in E$, we say $x$ and $y$ are adjacent. Formally, given a graph $G = (V, E)$, two vertices $x, y \in V$ are said to be neighbors or adjacent nodes, if $xy \in E$. The neighborhood of a vertex $v$ in a graph $G$ is the induced subgraph of $G$ consisting of all vertices adjacent to $v$ and all edges connecting two such vertices. The neighborhood of $v$ is often denoted by $N(v)$. The degree $\deg(v)$ of vertex $v$ is the number of edges incident on $v$. The open neighborhood for a vertex $v$ in a graph $G$ consists of all vertices adjacent to $v$ but not including $v$, i.e. $N(v) = \{u \in V : uv \in E\}$. If $v$ is included in $N(v)$, then it is called closed neighborhood for $v$ and is denoted by $N[v]$, i.e. $N[v] = N(v) \cup \{v\}$. A regular graph is a graph where each vertex has the same open neighborhood degree. A complete graph is a simple graph in which every pair of distinct vertices has an edge.
An isomorphism \( \phi \) of the graphs \( G_1 \) and \( G_2 \) is a bijection between the vertex sets of \( G_1 \) and \( G_2 \) such that any two vertices \( v_1 \) and \( v_2 \) of \( G_1 \) are adjacent in \( G_1 \) if and only if \( \phi(v_1) \) and \( \phi(v_2) \) are adjacent in \( G_2 \). If \( G_1 \) and \( G_2 \) are isomorphic, then we denote it by \( G_1 \cong G_2 \).

The line graph \( L(G^*) \) of a simple graph \( G^* \) is a graph which represents the adjacency between edges of \( G^* \). For a graph \( G^* \), its line graph \( L(G^*) \) is a graph such that:

(i) Each vertex of \( L(G^*) \) represents an edge of \( G^* \), and

(ii) Two vertices of \( L(G^*) \) are adjacent if and only if their corresponding edges have a common end point in \( G^* \).

Let \( G^* = (V,E) \) be a graph where \( V = \{v_1,v_2,...,v_n\} \). Let \( S_j = \{v_i,x_{i1},...,x_{ij}\} \) where \( x_{ij} \in E \) and \( x_{ij} \) has \( v_j \) as a vertex, \( j = 1,2,...,q; \ i = 1,2,...,n \). Let \( S = \{S_i,S_j,...,S_n\} \). Let \( T = \{S_i:S_j,S_j \in S,S_j \cap S_j \neq \emptyset,i \neq j\} \). Then \( P(S) = (S,T) \) is an intersection graph and \( P(S) = G^* \). The line graph \( L(G^*) \) of \( G^* \), is defined by the intersection graph \( P(E) \). That is, \( L(G^*) = (Z,W) \) where \( Z = \{\{x\} \cup \{u_x,v_x\}:x \in E,u_x,v_x \in V,x = u_x,v_x\} \) and \( W = \{S_i:S_j:S_x \cap S_y \neq \emptyset,x,y \in E,x \neq y\} \) and \( S_* = \{x\} \cup \{u_x,v_x\},x \in E \).

**Definition 2.2**  Gau and Buehrer (1993) A vague set on a non-empty set \( X \) is a pair \((t_A,f_A)\), where \( t_A:X \rightarrow [0,1] \) and \( f_A:X \rightarrow [0,1] \) are true and false membership functions, respectively, such that \( t_A(x) + f_A(x) \leq 1 \) for all \( x \in X \).

In the above definition, \( t_A(x) \) is considered as the lower bound for degree of membership of \( x \) in \( A \) (based on evidence for), and \( f_A(x) \) is the lower bound for negation of membership of \( x \) in \( A \) (based on evidence against). Therefore, the degree of membership of \( x \) in the vague set \( A \) is characterized by the interval \([t_A(x),1-f_A(x)]\). So, a vague set is a special case of interval-valued sets studied by many mathematicians and applied in many branches of mathematics. Vague sets also have many applications. The interval \([t_A(x),1-f_A(x)]\) is called the vague value of \( x \) in \( A \) and is denoted by \( V_A(x) \).

We denote zero vague and unit vague value by \( 0 = [0,0] \) and \( 1 = [1,1] \), respectively.

It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the "evidence for" only, without considering "evidence against". In vague sets both are independently proposed by the decision-maker. This makes a major difference in the judgment about the grade of membership.

A vague relation is a further generalization of a fuzzy relation.

**Definition 2.3** Ramakrishna (2009) Let \( X \) and \( Y \) be ordinary finite non-empty sets. We call a vague relation a vague subset of \( X \times Y \), that is an expression \( R \) defined by \( R = \{(x,y)\in X \times Y:\exists x \in X,y \in Y\} \), where \( t_R:X \times Y \rightarrow [0,1] \) and \( f_R:X \times Y \rightarrow [0,1] \) which satisfies the condition \( 0 \leq t_R(x,y) + f_R(x,y) \leq 1 \), for all \((x,y) \in X \times Y \).

**Definition 2.4** Ramakrishna (2009) A vague relation \( B \) on a set \( V \) is a vague relation from \( V \) to \( V \). If \( A \) is a vague set on a set \( V \), then a vague relation \( B \) on \( A \) is a vague relation which satisfies \( t_B(x,y) \leq \min\{t_A(x),t_B(y)\} \) and \( f_B(x,y) \geq \max\{t_A(x),t_B(y)\} \) for all \( x,y \in V \).

**Definition 2.5** Ramakrishna (2009) Let \( G^* = (V,E) \) be a crisp graph. A pair \( G = (V,A,B) \) is called a vague graph of \( G^* \), where \( A = (t_A,f_A) \) is a vague set on \( V \) and \( B = (t_B,f_B) \) is a vague set on \( E \subseteq V \times V \) such that \( t_B(x,y) \leq \min\{t_A(x),t_B(y)\} \) and \( f_B(x,y) \geq \max\{t_A(x),t_B(y)\} \) for each \((x,y) \in E \).

We call \( A \) the vague vertex set of \( G \) and \( B \) as the vague edge set of \( G \), respectively.
A vague graph $G$ is said to be strong if $t_b(u,v) = \min(t_a(u),t_a(v))$ and $f_b(u,v) = \max(f_a(u),f_a(v))$ for all $(u,v) \in E$.

A vague graph $G$ is said to be complete if $t_b(u,v) = \min(t_a(u),t_a(v))$ and $f_b(u,v) = \max(f_a(u),f_a(v))$ for all $u, v \in V$.

### 3. Regular and totally regular product vague graphs

Throughout the paper, $G^*$ represents a crisp graph and $G$ is a product vague graph of $G^*$. Rashmanlou and Borzooei (2015) defined the product vague graphs as follows.

Hereafter we use $xy \in E$ to denote $(x,y) \in E$ throughout the paper.

**Definition 3.1** A product vague graph of a graph $G^* = (V,E)$ is a pair $G = (V,A,B)$ where $A = (t_a, f_a)$ is an vague set in $V$ and $B = (t_b, f_b)$ is a vague set on $V^2$ such that $t_b(xy) \leq t_a(x) \times t_a(y)$ and $f_b(xy) \geq f_a(x) \times f_a(y)$ for all $x, y \in V$.

Note that, every product vague graph is also a vague graph.

**Example 3.2** Let us consider the graph $G^* = (V,E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4\}$. A product vague graph $G$ of $G^*$ is shown in Figure 1.

**Definition 3.3** A product vague graph $G = (V,A,B)$ of $G^* = (V,E)$ is said to be strong if $t_b(xy) = t_a(x) \times t_a(y)$ and $f_b(xy) = f_a(x) \times f_a(y)$ for all $xy \in E$.

The product vague graph $G$ of Figure 1 is not strong.

**Definition 3.4** Let $G = (V,A,B)$ be a product vague graph of $G^* = (V,E)$. The open neighborhood degree of a vertex $v$ in $G$ is defined by $\deg(v) = (\deg^o(v), \deg^t(v))$, where $\deg^o(v) = \sum_{uv \in E} t_o(uv)$ and $\deg^t(v) = \sum_{uv \in E} f_o(uv)$. If all the vertices of $G$ have the same open neighborhood degree $(t_1, t_2)$, then $G$ is called $(r_1, r_2)$-regular product vague graph.

**Definition 3.5** Let $G = (V,A,B)$ be a regular product vague graph of $G^* = (V,E)$. The order of $G$ is defined as $O(G) = (O^o(G), O^t(G))$ where $O^o(G) = \sum_{v \in V} t_o(v)$ and $O^t(G) = \sum_{v \in V} f_o(v)$. The size of $G$ is defined as $S(G) = (S^o(G), S^t(G))$ where $S^o(G) = \sum_{uv \in E} t_o(uv)$ and $S^t(G) = \sum_{uv \in E} f_o(uv)$.

**Definition 3.6** Let $G = (V,A,B)$ be a product vague graph of $G^* = (V,E)$. The closed neighborhood degree of a vertex $v$ is defined by $\deg[v] = (\deg^o[v], \deg^t[v])$, where $\deg^o[v] = \deg^t[v] + t_o(v)$ and $\deg^t[v] = \deg^o[v] + f_o(v)$. If each vertex of $G$ has the same closed neighborhood degree $(g_1, g_2)$, then $G$ is called $(g_1, g_2)$-totally regular product vague graph.

Now, we give some examples which show that product vague graphs may be both regular and totally regular, neither totally regular nor regular and totally regular but not regular. In other words, there is no relation between regular and totally regular product vague graphs.
First we give an example of a product vague graph which is both regular and totally regular (see Figure 2).

**Example 3.7** Let us consider the graph $G^* = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_4, v_3v_4, v_1v_3\}$ and consider the product vague graph $G = (V, A, B)$ of $G^*$ (see Figure 2). Here, $\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) = (0.3, 0.4)$ and $\deg[v_1] = \deg[v_2] = \deg[v_3] = \deg[v_4] = (0.8, 0.8)$. Hence, $G$ is both $(0.3,0.4)$-regular and $(0.8,0.8)$-totally regular product vague graph.

Now, we give an example of a product vague graph which is neither regular nor totally regular (see Figure 3).

**Example 3.8** Let us consider a product vague graph $G = (V, A, B)$ of $G^* = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_1v_3\}$ (see Figure 3). We have, $\deg(v_1) = (0.4, 0.45)$, $\deg(v_2) = (0.2, 0.2)$, $\deg(v_3) = (0.2, 0.25)$ and $\deg[v_1] = (0.8, 0.95)$, $\deg[v_2] = (0.7, 0.6)$, $\deg[v_3] = (0.8, 0.85)$. Hence, $G$ is neither regular nor totally regular product vague graph.

The following example shows that a product vague graph may be totally regular but not regular (see Figure 4).

**Example 3.9** Consider the product vague graph $G$ given in Figure 4. Since $\deg(v_1) = (0.27, 0.08) \neq \deg(v_2) = (0.24, 0.08)$ and $\deg[v_1] = \deg[v_2] = \deg[v_3] = (0.67, 0.28)$, therefore $G$ is $(0.67, 0.28)$-totally regular and but not regular product vague graph.

Similarly, we can give example of a product vague graph which is regular but not totally regular (see Figure 5).

We now state the following propositions without proof.

**Proposition 3.10** Let $G = (V, A, B)$ be a $(r_1, r_2)$-regular product vague graph of $G^* = (V, E)$. Then $S(G) = \sum (r_1, r_2)$ where $|V| = n$. 

---

**Figure 2.** $G$ is $(0.3,0.4)$-regular and $(0.8,0.8)$-totally regular product vague graph.

**Figure 3.** $G$ is neither regular nor totally regular product vague graph.
Proposition 3.11 Let \( G = (V, A, B) \) be a \((g_1, g_2)\)-totally regular product vague graph of \( G^* = (V, E) \). Then \( 25(G) + O(G) = n(g_1, g_2) \) where \( |V| = n \).

Proposition 3.12 Let \( G = (V, A, B) \) be a \((r_1, r_2)\)-regular and \((g_1, g_2)\)-totally regular product vague graph of \( G^* = (V, E) \). Then \( O(G) = n(g_1 - r_1, g_2 - r_2) \) where \( |V| = n \).

Theorem 3.13 Let \( G = (V, A, B) \) be a product vague graph of \( G^* = (V, E) \). Then \( A = (t_A, f_A) \) is constant function if and only if the following are equivalent:

(i) \( G \) is \((r_1, r_2)\)-regular vague graph,
(ii) \( G \) is \((g_1, g_2)\)-totally regular vague graph.

Proof Let us assume that \( A = (t_A, f_A) \) is constant function. Therefore, let \( t_A(v) = a_1 \) and \( f_A(v) = a_2 \) for all \( v \in V \), where \( a_1, a_2 \in [0, 1] \).

We will now show that the statements (i) and (ii) are equivalent.

(i) \( \Rightarrow \) (ii): Let \( G \) be a \((r_1, r_2)\)-regular product vague graph. Therefore, \( \deg(v) = (\deg^1(v), \deg^2(v)) = (r_1, r_2) \) for all \( v \in V \).

Now, \( \deg(v) = (\deg^1(v), \deg^2(v)) = (\deg^1(v) + t_A(v), \deg^2(v) + f_A(v)) = (r_1 + a_1, r_2 + a_2) \) for all \( v \in V \). Hence, \( G \) is \((r_1 + a_1, r_2 + a_2)\)-totally regular product vague graph.

(ii) \( \Rightarrow \) (i): Let \( G \) be a \((g_1, g_2)\)-totally regular product vague graph.

Then \( \deg(v) = (\deg^1(v), \deg^2(v)) = (g_1, g_2) \) for all \( v \in V \).

i.e. \( \deg^1(v) + t_A(v) = g_1 \) and \( \deg^2(v) + f_A(v) = g_2 \) for all \( v \in V \), or \( \deg^1(v) = g_1 - a_1 \) and \( \deg^2(v) = g_2 - a_2 \) for all \( v \in V \). Hence, \( G \) is \((g_1 - a_1, g_2 - a_2)\)-regular product vague graph.

Conversely, let (i) and (ii) are equivalent. Suppose \( A \) is not constant function. This means there exist at least two vertices \( u, v \in V \) such that \( t_A(u) \neq t_A(v) \) and \( f_A(u) \neq f_A(v) \).

Let \( G \) be a \((r_1, r_2)\)-regular product vague graph. Then,
$\deg[u] = (\deg^1(u) + t_A(u), \deg^f(u) + f_A(u)) = (r_1 + t_A(u), r_2 + f_A(u))$ and

$\deg[v] = (\deg^1(v) + t_A(v), \deg^f(v) + f_A(v)) = (r_1 + t_A(v), r_2 + f_A(v))$.

This shows that $\deg[u] \neq \deg[v]$ since $t_A(u) \neq t_A(v)$ and $f_A(u) \neq f_A(v)$. Hence, $G$ is not totally regular which is a contradiction to the assumption that (i) and (ii) are equivalent. Therefore, $A$ must be constant.

In a similar way, we can show that if $A$ is not constant function, then $G$ totally regular does not imply $G$ is regular. \hfill $\square$

**Proposition 3.14** Let $G = (V, A, B)$ be a product vague graph which is both regular and totally regular. Then $A = (t_A, f_A)$ is constant.

**Proof** Let $G$ be a $(r_1, r_2)$-regular and $(g_1, g_2)$-totally regular product vague graph. Now, $\deg^1[v] = \deg^1(v) + t_A(v) = r_1 + t_A(v) = g_1$ and $\deg^f[v] = \deg^f(v) + f_A(v) = r_2 + f_A(v) = g_2$ for all $v \in V$. Hence, $t_A(v) = g_1 - r_1$ and $f_A(v) = g_2 - r_2$ for all $v \in V$. This shows that $A(v) = (t_A(v), f_A(v)) = (g_1 - r_1, g_2 - r_2)$ for all $v \in V$, i.e. $A$ is constant. \hfill $\square$

**Remark 3.15** The converse of the Proposition 3.14 is not true always. For example, consider the product vague graph $G = (V, A, B)$ of $G^* = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1\}$ (see Figure 6). Here, $A(v) = (t_A(v), f_A(v)) = (0.4, 0.2)$ for all $v \in V$, i.e. $A$ is constant but $G$ is neither regular nor totally regular.

**Theorem 3.16** Let $G = (V, A, B)$ be a product vague graph of $G^* = (V, E)$ where $G^*$ is an odd cycle. Then $G$ is regular product vague graph of $G^*$ if and only if $B = (t_b, f_b)$ is constant.

**Proof** Let $G$ be a $(r_1, r_2)$-regular product vague graph. Let $e_1, e_2, \ldots, e_{2n+1}$ be the edges of $G^*$ such that $e_i = v_{i-1}v_i \in E$, $v_{0i}, v_i \in V$, $i = 1, 2, \ldots, 2n + 1$ and $v_0 = v_{2n+1}$. Let $t_b(e_i) = k_1$ and $f_b(e_i) = k_2$ where $k_1, k_2 \in [0, 1]$. $G$ is $(r_1, r_2)$-regular implies that $\deg^1(v_1) = r_1$ and $\deg^f(v_1) = r_2$.

This means, $\deg^1(v_2) = t_b(e_2) + t_b(e_2) = r_1$ and $\deg^f(v_2) = f_b(e_1) + f_b(e_2) = r_2$

i.e. $k_1 + t_b(e_1) = k_2 + r_2$

This implies $t_b(e_1) = r_1 - k_1$ and $f_b(e_2) = r_2 - k_2$.

Again, $\deg^f(v_2) = t_b(e_2) + t_b(e_2) = r_1$ and $\deg^f(v_2) = f_b(e_1) + f_b(e_2) = r_2$.

This implies, $t_b(e_1) = r_1 - (r_2 - k_2) = k_1$ and $f_b(e_2) = r_2 - (r_1 - k_2) = k_2$, and so on.

Therefore, $t_b(e_1) = \begin{cases} k_1, & \text{if } i \text{ is odd} \\ (r_1 - k_2), & \text{if } i \text{ is even} \end{cases}$ and $f_b(e_2) = \begin{cases} k_2, & \text{if } i \text{ is odd} \\ (r_2 - k_2), & \text{if } i \text{ is even} \end{cases}$

---

**Figure 6.** A is constant but $G$ is neither regular nor totally regular.
Therefore, \( t_B(e_1) = t_B(e_{2n+1}) = k_1 \) and \( f_B(e_1) = f_B(e_{2n+1}) = k_2 \).

Since \( e_1 \) and \( e_{2n+1} \) are incident on the vertex \( v_0 \) and \( \deg(v_0) = (r_1, r_2) \), we have, \( t_B(e_1) + t_B(e_{2n+1}) = r_1 \) and \( f_B(e_1) + f_B(e_{2n+1}) = r_2 \).

i.e. \( 2k_1 = r_1 \) and \( 2k_2 = r_2 \) i.e. \( k_1 = \frac{r_1}{2} \) and \( k_2 = \frac{r_2}{2} \).

Therefore, \( t_B(e_i) = \frac{r_i}{2} \) and \( f_B(e_i) = \frac{r_i}{2} \) for all \( i = 1, 2, \ldots, 2n + 1 \). Hence \( B \) is constant.

Conversely, let \( B \) be a constant function. Let \( B(uv) = (t_B(uv), f_B(uv)) = (k, k) \) for all \( uv \in E \), where \( k, k \in [0, 1] \).

Then \( \deg(v) = (\deg^1(v), \deg^2(v)) = (\sum_{uv \in E} t_B(uv), \sum_{uv \in E} f_B(uv)) = (2k_1, 2k_2) \) for all \( v \in V \).

Consequently, \( G \) is \((2k_1, 2k_2)\)-regular product vague graph.

\[ \square \]

4. Product vague line graphs

In this section, we first define a product vague intersection graph of a product vague graph. Finally we define the product vague line graphs.

**Definition 4.1**  Let \( P(S) = (S, T) \) be an intersection graph of a simple graph \( G^* = (V, E) \). Let \( G = (V, A, B) \) be a product vague graph of \( G^* \). We define a product vague intersection graph \( P(G) = (A_T, B_T) \) of \( P(S) \) as follows:

(i) \( A_i \) and \( B_i \) are vague subsets of \( S \) and \( T \) respectively,

(ii) \( t_{A_i}(S) = t_{A_i}(v_j), f_{A_i}(S) = f_{A_i}(v_j) \),

(iii) \( t_{B_i}(S, S_j) = t_{B_i}(v_j, v_j), f_{B_i}(S, S_j) = f_{B_i}(v_j, v_j) \) for all \( S, S_j \in S \), \( S, S_j \in T \). In other words, any product vague graph of \( P(S) \) is called a product vague intersection graph.

The following proposition is immediate.

**Proposition 4.2**  Let \( G = (V, A, B) \) be a product vague graph of \( G^* = (V, E) \) and \( P(G) = (A_T, B_T) \) be a product vague intersection graph of \( P(S) \). Then the following holds:

(a) \( P(G) \) is a product vague graph of \( P(S) \),

(b) \( G \cong P(G) \).

**Proof**

(a) Since \( G = (V, A, B) \) is a product vague graph we have by Definition 4.1, \( t_B(S, S_j) = t_B(v_j, v_j) \leq t_A(S) \times t_A(S_j) \) and \( f_B(S, S_j) = f_B(v_j, v_j) \geq f_A(S) \times f_A(S_j) \).

Hence, \( P(G) \) is a product vague graph.

(b) Let us define a mapping \( \phi: V \to S \) by \( \phi(v_j) = S_i \) for \( i = 1, 2, \ldots, n \). Then clearly \( \phi \) is one to one mapping of \( V \) onto \( S \). Now \( v_j \in E \) if and only if \( S_i, S_j \in T \) and \( T = \{ \phi(v_j), \phi(v_j) : v_j \in E \} \). Also, \( t_A(v_j) = t_A(S_i) = t_A(\phi(v_j)) \) and \( f_A(v_j) = f_A(S_i) = f_A(\phi(v_j)) \) for all \( v_j \in V \).

This proposition shows that any product vague graph is isomorphic to a product vague intersection graph. The product vague line graph of a product vague graph is defined as below.
Definition 4.3. Let \( L(G') = (Z, W) \) be a line graph of a simple graph \( G' = (V, E) \). Let \( G = (V, A, B) \) be a product vague graph of \( G' \). Then a product vague line graph \( L(G) = (A_1, B_1) \) of \( G \) is defined as follows:

(i) \( A_1 \) and \( B_1 \) are vague subsets of \( Z \) and \( W \), respectively,
(ii) \( t_{A_1}(S_x) = t_{A_1}(x) = t_{A_1}(x_2) = 0.15, \)
(iii) \( f_{A_1}(S_x) = f_{A_1}(x) = f_{A_1}(x_2) = 0.25, \)
(iv) \( t_{B_1}(S_x) = t_{B_1}(x) = t_{B_1}(x_2) = 0.07, \)
(v) \( f_{B_1}(S_x) = f_{B_1}(x) = f_{B_1}(x_2) = 0.14, \)

\( \forall S_x, S_y \in Z \) and \( S_x, S_y \in W \).

Example 4.4. Consider now a graph \( G = (V, E) \) where \( V = \{v_1, v_2, v_3, v_4\} \) and \( E = \{x_1 = v_1v_2, x_2 = v_2v_3, x_3 = v_3v_4, x_4 = v_4v_1\} \). Let \( G = (V, A, B) \) be a product vague graph of \( G' \) (see Figure 7).

Now, consider a line graph \( L(G') = (Z, W) \) such that \( Z = \{S_x, S_x, S_x, S_x\} \) and \( W = \{S_x, S_x, S_x, S_x, S_x, S_x\} \).

Let \( A_1 \) and \( B_1 \) be vague subsets of \( Z \) and \( W \), respectively. Then we have

\[
\begin{align*}
\text{Let } t_{A_1}(S_x) &= t_{A_1}(x_1) = 0.15, t_{A_1}(S_x) &= t_{A_1}(x_2) = 0.14, \\
t_{A_1}(S_x) &= t_{A_1}(x_3) = 0.25, t_{A_1}(S_x) &= t_{A_1}(x_4) = 0.25, \\
f_{A_1}(S_x) &= f_{A_1}(x_1) = 0.07, f_{A_1}(S_x) &= f_{A_1}(x_2) = 0.15, \\
f_{A_1}(S_x) &= f_{A_1}(x_3) = 0.17, f_{A_1}(S_x) &= f_{A_1}(x_4) = 0.09, \\
t_{B_1}(S_x, S_x) &= t_{B_1}(x_1) = t_{B_1}(x_2) = 0.15 \times 0.14 = 0.021, \\
t_{B_1}(S_x, S_x) &= t_{B_1}(x_3) = t_{B_1}(x_4) = 0.14 \times 0.25 = 0.035, \\
t_{B_1}(S_x, S_x) &= t_{B_1}(x_1) = t_{B_1}(x_2) = 0.25 \times 0.25 = 0.0625, \\
t_{B_1}(S_x, S_x) &= t_{B_1}(x_3) = t_{B_1}(x_4) = 0.25 \times 0.15 = 0.0375, \\
f_{B_1}(S_x, S_x) &= f_{B_1}(x_1) = f_{B_1}(x_2) = 0.07 \times 0.15 = 0.0105, \\
f_{B_1}(S_x, S_x) &= f_{B_1}(x_3) = f_{B_1}(x_4) = 0.15 \times 0.17 = 0.0255, \\
f_{B_1}(S_x, S_x) &= f_{B_1}(x_1) = f_{B_1}(x_2) = 0.17 \times 0.09 = 0.0153, \\
f_{B_1}(S_x, S_x) &= f_{B_1}(x_3) = f_{B_1}(x_4) = 0.07 \times 0.09 = 0.0063.
\end{align*}
\]
Hence, \( L(G) = (A_1, B_1) \) is the product vague line graph of \( G \). We see that \( L(G) \) is neither regular nor totally regular product vague line graph.

**Proposition 4.5** A product vague line graph is a strong product vague graph.

**Proof** Follows from the definition of product vague line graph. \( \square \)

**Proposition 4.6** If \( L(G) \) is a product vague line graph of the product vague graph \( G \), then \( L(G^*) \) is the line graph of \( G^* \).

**Proof** Since \( G = (V, A, B) \) is a product vague graph and \( L(G) = (A_1, B_1) \) is a product vague line graph, therefore \( t_{A_1}(S_x) = t_{A_1}(x) \) and \( f_{A_1}(S_x) = f_{A_1}(x) \) for all \( x \in E \) and so \( S_x \subseteq Z \) if \( x \in E \).

Also, \( t_{B_1}(S_y, S_y) = t_{B_1}(x) \times t_{B_1}(y) \) and \( f_{B_1}(S_y, S_y) = f_{B_1}(x) \times f_{B_1}(y) \) for all \( S_x, S_y \in Z \) and so \( W = \{ S_x, S_y, S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y \} \). This completes the proof. \( \square \)

**Proposition 4.7** \( L(G) = (A_1, B_1) \) is a product vague line graph of some product vague graph \( G = (V, A, B) \) if and only if \( t_{B_1}(S_x, S_y) = t_{B_1}(S_x) \times t_{B_1}(S_y) \) and \( f_{B_1}(S_x, S_y) = f_{B_1}(S_x) \times f_{B_1}(S_y) \) for all \( S_x, S_y \in W \).

**Proof** Suppose that \( t_{B_1}(S_x, S_y) = t_{B_1}(S_x) \times t_{B_1}(S_y) \) and \( f_{B_1}(S_x, S_y) = f_{B_1}(S_x) \times f_{B_1}(S_y) \) for all \( S_x, S_y \in W \) and \( \inf \{ t_{B_1}(x) \times t_{B_1}(y), f_{B_1}(x) \times f_{B_1}(y) \} \) and \( \sup \{ t_{B_1}(x) \times t_{B_1}(y), f_{B_1}(x) \times f_{B_1}(y) \} \) will suffice. The converse part follows from the Definition 4.3. \( \square \)

**Proposition 4.8** \( L(G) = (A_1, B_1) \) is a product vague line graph of some product vague graph if and only if \( L(G^*) = (Z, W) \) is a line graph satisfying \( t_{B_1}(uv) = t_{B_1}(u) \times t_{B_1}(v) \) and \( f_{B_1}(uv) = f_{B_1}(u) \times f_{B_1}(v) \) for all \( uv \in W \).

**Proof** Follows from the Propositions 4.6 and 4.7. \( \square \)

**Definition 4.9** Let \( G_1 = (V_1, A_1, B_1) \) and \( G_2 = (V_2, A_2, B_2) \) be two product vague graphs of the graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), respectively. A homomorphism between \( G_1 \) and \( G_2 \) is a mapping \( \phi: V_1 \to V_2 \) such that

(i) \( t_{A_1}(x) \leq t_{A_2}(\phi(x)) \) and \( f_{A_1}(x) \geq f_{A_2}(\phi(x)) \) for all \( x \in V_1 \)

(ii) \( t_{A_1}(xy) \leq t_{A_2}(\phi(x)) \) and \( f_{A_1}(xy) \geq f_{A_2}(\phi(x)) \phi(y) \) for all \( xy \in V_1^2 \)

A bijective homomorphism with the property that \( t_{A_1}(x) = t_{A_2}(\phi(x)) \) and \( f_{A_1}(x) = f_{A_2}(\phi(x)) \) for all \( x \in V_1 \) is called a (weak) vertex-isomorphism.

A bijective homomorphism with the property that \( t_{A_1}(xy) = t_{A_2}(\phi(x)) \phi(y) \) and \( f_{A_1}(xy) = f_{A_2}(\phi(x)) \phi(y) \) for all \( xy \in V_1^2 \) is called a (weak) line-isomorphism.

If \( \phi \) is both (weak) vertex isomorphism and (weak) line-isomorphism, then \( \phi \) is called a (weak) isomorphism of \( G_1 \) onto \( G_2 \). If \( G_1 \) is isomorphic to \( G_2 \), then we write \( G_1 \cong G_2 \).

**Proposition 4.10** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two product vague graphs of the graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), respectively. If \( \phi \) is a weak isomorphism of \( G_1 \) onto \( G_2 \), then \( \phi \) is an isomorphism of \( G_1 \) onto \( G_2 \).

**Proof** Obvious. \( \square \)
Proposition 4.11  Let \( L(G) = (A_1, B_1) \) be the product vague line graph corresponding to the product vague graph \( G = (V, A, B) \) of \( G^* = (V, E) \). Suppose that \( G^* \) is connected. Then the following hold:

(i) There exists a weak isomorphism of \( G \) onto \( L(G) \) if and only if \( G^* \) is a cycle and for all \( v \in V, x \in E \),
\[
\begin{align*}
t_s(v) &= t_A(x) + t_B(v) = f_A(x) = f_B(v), \\
&= f_A(x) = f_B(v), \text{ i.e. } A = (t_A, f_A) \text{ and } B = (t_B, f_B) \text{ are constant functions on } V \text{ and } E, \\
& \text{respectively, taking on the same value.}
\end{align*}
\]

(ii) If \( \phi \) is a weak isomorphism of \( G \) onto \( L(G) \), then \( \phi \) is a homomorphism.

Proof  Suppose that \( \phi \) is a weak isomorphism of \( G \) onto \( L(G) \). By Proposition 4.10, \( \phi \) is an isomorphism of \( G^* \) onto \( L(G^*) \). By Proposition 4.6, \( G^* \) is a cycle, (by Harary, 1972), Theorem 8.2).

Let \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{v_1, v_2, v_2 = v_3, \ldots, v_n = v_1\} \), where \( v_1, v_2 \ldots v_n \) is a cycle. Let us define the vague sets \( t_s(v_i) = s_i \) and \( t_B(v_i) = t_i \), \( t_A(v_i) = t_i \), \( i = 1, 2, \ldots, n \). \( v_n = v_1 \).

Then for \( s_{n+1} = s_1 \), \( s_{n+1} = s_1 \),
\[
\begin{align*}
t_i &\leq s_i \times s_{i+1} \\
t_i &\geq s_i \times s_{i+1}, i = 1, 2, \ldots, n.
\end{align*}
\]

Now, \( Z = G_{x_1}, x_2, \ldots, x_{n_1} \) and \( W = G_{x_1}, x_2, \ldots, x_{n_1} \).

Also for \( t_{n+1} = t_1 \) and \( t_{n+1} = t_1 \),
\[
\begin{align*}
t_1 &\leq t_1 \times t_{i+1} \\
t_1 &\geq t_1 \times t_{i+1}, i = 1, 2, \ldots, n.
\end{align*}
\]

Since \( \phi \) is an isomorphism of \( G^* \) onto \( L(G^*) \), \( \phi \) maps \( V \) one-to-one onto \( Z \). Also \( \phi \) preserves adjacency. Hence, \( \phi \) induces a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) such that \( \phi(v_i) = s_{x_{i+1}} \times s_{x_{i+1}}, x_{i+1} \) and \( x_i = v_1 v_{i+1} \to \phi(v_i) = s_{x_{i+1}} \times s_{x_{i+1}}, x_{i+1} \), \( i = 1, 2, \ldots, (n-1) \).

Now,
\[
s_i = t_s(v_i) \leq t_A(\phi(v_i)) = t_A(s_{x_{i+1}}, x_{i+1}) = t_{x_{i+1}} \]
\[
\hat{s}_i = f_A(v_i) \geq f_A(\phi(v_i)) = f_A(s_{x_{i+1}}, x_{i+1}) = t_{x_{i+1}} \]
\[
t_i = f_B(v_i) \leq f_B(\phi(v_i)) = f_B(s_{x_{i+1}}, x_{i+1}) \]
\[
= f_B(s_{x_{i+1}}, x_{i+1}) \times f_B(s_{x_{i+1}}, x_{i+1}) = t_{x_{i+1}} \times t_{x_{i+1}} \]
\[
t_i = f_B(\phi(v_i)) \geq f_B(\phi(v_i)) = f_B(s_{x_{i+1}}, x_{i+1}) \]
\[
= f_B(s_{x_{i+1}}, x_{i+1}) \times f_B(s_{x_{i+1}}, x_{i+1}) = t_{x_{i+1}} \times t_{x_{i+1}}
\]

for \( i = 1, 2, \ldots, n \). That is, \( s_i \leq t_{x_{i+1}} \hat{s}_i \geq \hat{t}_{x_{i+1}} \) and
\[
\begin{align*}
t_i &\leq t_{x_{i+1}} \times t_{x_{i+1}} \\
\hat{t}_i &\leq \hat{t}_{x_{i+1}} \times \hat{t}_{x_{i+1}}, i = 1, 2, \ldots, n.
\end{align*}
\]

By (2), we have \( t_i \leq t_{x_{i+1}} \hat{t}_i \geq \hat{t}_{x_{i+1}} \) for \( i = 1, 2, \ldots, n \) and so \( t_{x_{i+1}} \leq t_{x_{i+1}} \hat{t}_{x_{i+1}} \hat{t}_{x_{i+1}} \) for \( i = 1, 2, \ldots, n \). Continuing, we have
\[ t_1 \leq t_{s(i)} \leq \cdots \leq t_{s(n)} \leq t_1, t_j = t_{s(i)} \geq \cdots \geq t_{s(n)} \geq t_j \]

and so \( t_i = t_{s(i)} i = 1, 2, \ldots, n \) where \( s^{-1} \) is the identity map. Again by (2), we have
\[ t_i = t_{s(i)} = t_{s(i+1)} = \cdots = t_{s(n)} = t_i \]
\[ t_i = t_{s(i+1)} = t_{s(i+2)} = t_{s(i+3)} = \cdots = t_{s(n)} = \cdots = t_i \]

Hence by (1) and (2), we have \( t_1 = \cdots = t_n = s_1 = \cdots = s_n \to t_1 = s_1 = \cdots = s_n \)

Thus we have not only proved the conclusion about A and B being constant functions, but also we have shown that (ii) holds.

Conversely, suppose that \( G' \) is a cycle and for all \( v \in V \), \( x \in E \), \( t_1(v) = t_2(x) \), \( f_1(v) = f_2(x) \). By Proposition 4.6, \( L(G') \) is the line graph of \( G' \). Since \( G' \) is a cycle, \( G' \cong L(G') \) by (Harary (Harary (1972)), Theorem 8.2). This isomorphism induces an isomorphism of \( G \) onto \( L(G) \) since \( t_1(v) = t_2(x) \), \( f_1(v) = f_2(x) \) for all \( v \in V \), \( x \in E \) and so \( A = B = A_1 = B_1 \) on their respective domains.

**Proposition 4.12** Let \( G_1 = (V_1, A_1, B_1) \) and \( G_2 = (V_2, A_2, B_2) \) be two product vague graphs of the graphs \( G'_1 = (V_1, E_1) \) and \( G'_2 = (V_2, E_2) \), respectively, such that \( G'_1 \) and \( G'_2 \) is connected. Let \( L(G_1) = (A_1, B_1) \) and \( L(G_2) = (A_2, B_2) \) be the product vague line graphs corresponding to \( G_1 \) and \( G_2 \), respectively. Suppose that it is not the case that one of \( G_1 \) and \( G_2 \) is complete graph \( K_2 \) and other is bipartite complete graph \( K_{1, n} \). If \( L(G_1) \cong L(G_2) \), then \( G_1 \) and \( G_2 \) are line isomorphic.

**Proof** Since \( L(G_1) \cong L(G_2) \), therefore by Proposition 4.10, \( L(G'_1) \cong L(G'_2) \). Since \( L(G'_1) \) and \( L(G'_2) \) are the line graphs of \( G'_1 \) and \( G'_2 \), respectively, by Proposition 4.6, we have that \( G'_1 \cong G'_2 \) by (Harary (1972), Theorem 8.3).

Let \( \psi \) be the isomorphism of \( L(G_1) \) onto \( L(G_2) \) and \( \phi \) be the isomorphism of \( G'_1 \) onto \( G'_2 \). Then \( t_1(S_1) = t_2(S_1) = t_3(S_1) = \cdots = t_n(S_1) \). Hence by (1) and (2), we have \( t_1 = s_1 = \cdots = s_n \)

Thus we have not only proved the conclusion about A and B being constant functions, but also we have shown that (ii) holds.

Conversely, suppose that \( G' \) is a cycle and for all \( v \in V \), \( x \in E \), \( t_1(v) = t_2(x) \), \( f_1(v) = f_2(x) \). By Proposition 4.6, \( L(G') \) is the line graph of \( G' \). Since \( G' \) is a cycle, \( G' \cong L(G') \) by (Harary (Harary (1972)), Theorem 8.2). This isomorphism induces an isomorphism of \( G \) onto \( L(G) \) since \( t_1(v) = t_2(x) \), \( f_1(v) = f_2(x) \) for all \( v \in V \), \( x \in E \) and so \( A = B = A_1 = B_1 \) on their respective domains.

**Conversely, suppose that G’ is a cycle and for all v ∈ V, x ∈ E, t1(v) = t2(x), f1(v) = f2(x). By Proposition 4.6, L(G’) is the line graph of G’. Since G’ is a cycle, G’ ∼= L(G’) by (Harary (1972)), Theorem 8.2). This isomorphism induces an isomorphism of G onto L(G) since t1(v) = t2(x), f1(v) = f2(x) for all v ∈ V, x ∈ E and so A = B = A1 = B1 on their respective domains.**

**Proposition 4.12** Let G1 = (V1, A1, B1) and G2 = (V2, A2, B2) be two product vague graphs of the graphs G’1 = (V1, E1) and G’2 = (V2, E2), respectively, such that G’1 and G’2 is connected. Let L(G1) = (A1, B1) and L(G2) = (A2, B2) be the product vague line graphs corresponding to G1 and G2, respectively. Suppose that it is not the case that one of G1 and G2 is complete graph K2 and other is bipartite complete graph K1,n. If L(G1) ∼= L(G2), then G1 and G2 are line isomorphic.

**Proof** Since L(G1) ∼= L(G2), therefore by Proposition 4.10, L(G’1) ∼= L(G’2). Since L(G’1) and L(G’2) are the line graphs of G’1 and G’2, respectively, by Proposition 4.6, we have that G’1 ∼= G’2 by (Harary (1972), Theorem 8.3).

Let ψ be the isomorphism of L(G1) onto L(G2) and φ be the isomorphism of G’1 onto G’2. Then t1(S1) = t2(S1) = t3(S1) = ... = tn(S1). Hence by (1) and (2), we have t1 = s1 = ... = sn → t1 = s1 = ... = sn.

**5. Conclusions**

Graph theory has several interesting applications in system analysis, operations research, computer applications, and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy systems. It is known that fuzzy graph theory has numerous applications in modern sciences and technology, especially in the fields of neural networks, artificial intelligence, and decision-making. In this paper, we defined the notions of regular, totally regular product vague graphs, and product vague line graphs. We investigated some properties of them. In our future work we will focus on categorical properties on product vague graphs, edge regular, and irregular product vague graph product vague graph competition graph.
Balakrishnan, V. K. (1997). Graph theory. McGraw-Hill.
Borzooei, R. A., & Rashmanlou, H. (2015a). Domination in vague graphs and its applications. Journal of Intelligent and Fuzzy Systems, 29, 1933–1940.
Borzooei, R. A., & Rashmanlou, H. (2015b). New concepts of vague graphs. International Journal of Machine Learning and Cybernetics. doi:10.1007/s13042-015-0475-x.
Borzooei, R. A., & Rashmanlou, H. (2015c). Degree of vertices in vague graphs. Journal of applied mathematics and informatics, 33, 545–557.
Borzooei, R. A., & Rashmanlou, H. (2016). Semi global domination sets in vague graphs with application. Journal of Intelligent and Fuzzy Systems, 30, 3645–3652.
Gau, W. L., & Buehrer, D. L. (1993). Vague sets. IEEE Transaction on Systems, Man and Cybernetics, 23, 610–614.
Ghorai, G., & Pal, M. (2013). On some operations and density of m-polar fuzzy graphs. Pacific Science Review A: Natural Science and Engineering, 17, 14–22.
Ghorai, G., & Pal, M. (2016). Some properties of m-polar fuzzy graphs. Pacific Science Review A: Natural Science and Engineering. doi:10.1016/j.psrsa.2016.06.004.
Ghorai, G., & Pal, M. (2016a). A study on m-polar fuzzy planar graphs. International Journal of Computational Science and Engineering, 7, 283–292.
Ghorai, G., & Pal, M. (2016b). Faces and dual of m-polar fuzzy planar graphs. Journal of Intelligent and Fuzzy Systems. doi:10.3233/JIFS-16433.
Harary, F. (1972). Graph theory (3rd ed.). Reading, MA: Addison-Wesley.
Mordeson, J. N., & Noir, P. S. (2000). Fuzzy graphs and hypergraphs. Physica Verlag.
Ramakrishna, N. (2009). Vague graphs. International Journal of Computational Cognition, 7, 51–58.
Rashmanlou, H., & Borzooei, R. A. (2015). A Note on vague graphs. Journal of Algebraic Structures and Their Applications, 2, 9–19.
Rosenfeld, A. (1975). Fuzzy graphs. In L. A. Zadeh, K. S. Fu, & M. Shimura (Eds.), Fuzzy sets and their applications (pp. 77–95). New York: Academic Press.
Rashmanlou, H., & Borzooei, R. A. (2015). Product vague graphs and its applications. Journal of Intelligent and Fuzzy Systems, 30, 371–382.
Rashmanlou, H., & Borzooei, R. A. (2016). More results on vague graphs. University Politehnica of Bucharest Scientific Bulletin-Series A, 78, 109–122.
Samanta, S., Akram, M., & Pal, M. (2015). M-step fuzzy competition graphs. Journal of Applied Mathematics and Computing, 47, 461–472.
Samanta, S., & Pal, M. (2013). Fuzzy k-competition graphs and p-competitions fuzzy graphs. Fuzzy Information and Engineering, 5, 191–204.
Samanta, S., & Pal, M. (2014). Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs. The Journal of Fuzzy Mathematics, 22(2), 1–10.
Samanta, S., & Pal, M. (2015). Fuzzy planar graphs. IEEE Transactions on Fuzzy Systems, 23, 1936–1942.
Samanta, S., Pal, M., Rashmanlou, H., & Borzooei, R. A. (2016). Vague graphs and strengths. Journal of Intelligent and Fuzzy Systems, 30, 3675–3680.
Talebi, A. A., Mehdipoor, N., & Rashmanlou, H. (2014). Some operations on vague graphs. Journal of Advanced Research in Pure Mathematics, 6, 61–77.
Talebi, A. A., Rashmanlou, H., & Mehdipoor, N. (2013). Isomorphism on vague graphs. Annals of Fuzzy Mathematics and Informatics, 6, 575–588.
Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8, 338–353.