MOST(?) THEORIES HAVE BOREL COMPLETE REDUCTS

MICHAEL C. LASKOWSKI AND DOUGLAS S. ULRICH

Abstract. We prove that many seemingly simple theories have Borel complete reducts. Specifically, if a countable theory has uncountably many complete one-types, then it has a Borel complete reduct. Similarly, if \( Th(M) \) is not small, then \( M^{eq} \) has a Borel complete reduct, and if a theory \( T \) is not \( \omega \)-stable, then the elementary diagram of some countable model of \( T \) has a Borel complete reduct.

§1. Introduction. In their seminal paper [1], Friedman and Stanley define and develop a notion of Borel reducibility among classes of structures with universe \( \omega \) in a fixed, countable language \( L \) that are Borel and invariant under permutations of \( \omega \). It is well known (see, e.g., [3] or [2]) that such classes are of the form \( \text{Mod}(\Phi) \), the set of models of \( \Phi \) whose universe is precisely \( \omega \) for some sentence \( \Phi \in L_{\omega_1, \omega} \), but here we concentrate on first-order, countable theories \( T \). For countable theories \( T, S \) in possibly different language, a Borel reduction is a Borel function \( f : \text{Mod}(T) \rightarrow \text{Mod}(S) \) that satisfies \( M \cong N \) if and only if \( f(M) \cong f(N) \). One says that \( T \) is Borel reducible to \( S \) if there is a Borel reduction \( f : \text{Mod}(T) \rightarrow \text{Mod}(S) \). As Borel reducibility is transitive, this induces a quasi-order on the class of all countable theories, where we say \( T \) and \( S \) are Borel equivalent if there are Borel reductions in both directions. In [1], Friedman and Stanley show that among Borel invariant classes (hence among countable first-order theories) there is a maximal class with respect to \( \leq_B \). We say \( \Phi \) is Borel complete if it is in this maximal class. Examples include the theories of graphs, linear orders, groups, and fields.

The intuition is that Borel complexity of a theory \( T \) is related to the complexity of invariants that describe the isomorphism types of countable models of \( T \). Given an \( L \)-structure \( M \), one naturally thinks of the reducts \( M_0 \) of \( M \) to be ‘simpler objects’, hence the invariants for a reduct ‘should’ be no more complicated than for the original \( M \), but we will see that this intuition is incorrect. As a paradigm, let \( T \) be the theory of ‘independent unary predicates’ i.e., \( T = Th(2^{\omega}, U_n) \), where each \( U_n \) is a unary predicate interpreted as \( U_n = \{ \eta \in 2^\omega : \eta(n) = 1 \} \). The countable models of \( T \) are rather easy to describe. The isomorphism type of a model is specified by which countable, dense subset of ‘branches’ is realized, and how many elements realize each of those branches. However, with Theorem 3.2, we will see that \( T \) has a Borel complete reduct.

To be precise about reducts, we have the following definition.

Definition 1.1. Given an \( L \)-structure \( M \), a reduct \( M' \) of \( M \) is an \( L' \)-structure with the same universe as \( M \), and for which the interpretation in every atomic \( L' \)-formula \( \alpha(x_1, ..., x_k) \) is an \( L \)-definable subset of \( M^k \) (without parameters). An \( L' \)-theory
$T'$ is a reduct of an $L$-theory $T$ if $T' = Th(M')$ for some reduct $M'$ of some model $M$ of $T$.

In the above definition, it would be equivalent to require that the interpretation in $M'$ of every $L'$-formula $\theta(x_1, \ldots, x_k)$ is a 0-definable subset of $M'^k$.

§2. An engine for Borel completeness results. This section is devoted to proving Borel completeness for a specific family of theories. All of the theories $T_h$ are in the same language $L = \{E_n : n \in \omega\}$ and are indexed by strictly increasing functions $h : \omega \to \omega \setminus \{0\}$. For a specific choice of $h$, the theory $T_h$ asserts that

- Each $E_n$ is an equivalence relation with exactly $h(n)$ classes; and
- The $E_n$'s cross-cut, i.e., for all nonempty, finite $F \subseteq \omega$, $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$ is an equivalence relation with precisely $\Pi_{n \in F} h(n)$ classes.

It is well known that each of these theories $T_h$ is complete and admits elimination of quantifiers. Thus, in any model of $T_h$, there is a unique one-type. However, the strong type structure is complicated.\(^1\) So much so, that the whole of this section is devoted to the proof of:

**Theorem 2.1.** For any strictly increasing $h : \omega \to \omega \setminus \{0\}$, $T_h$ is Borel complete.

**Proof.** Fix a strictly increasing function $h : \omega \to \omega \setminus \{0\}$. We begin by describing representatives $B$ of the strong types and a group $G$ that acts faithfully and transitively on $B$. As notation, for each $n$, let $[h(n)]$ denote the $h(n)$-element set $\{1, \ldots, h(n)\}$ and let $\text{Sym}([h(n)])$ be the (finite) group of permutations of $[h(n)]$. Let

$$B = \{f : \omega \to \omega : f(n) \in [h(n)] \text{ for all } n \in \omega\},$$

and let $G = \Pi_{n \in \omega, \text{Sym}([h(n)])}$ be the direct product. As notation, for each $n \in \omega$, let $\pi_n : G \to \text{Sym}([h(n)])$ be the natural projection map. Note that $G$ acts coordinate-wise on $B$ by: For $g \in G$ and $f \in B$, $g \cdot f$ is the element of $B$ satisfying $g \cdot f(n) = \pi_n(g)(f(n))$.

Define an equivalence relation $\sim$ on $B$ by:

$$f \sim f' \text{ if and only if } \{n \in \omega : f(n) \neq f'(n)\} \text{ is finite.}$$

For $f \in B$, let $[f]$ denote the $\sim$-class of $f$ and, abusing notation somewhat, for $W \subseteq B$

$$[W] := \bigcup \{|f| : f \in W\}.$$

Observe that for every $g \in G$, the permutation of $B$ induced by the action of $g$ maps $\sim$-classes onto $\sim$-classes, i.e., $G$ also acts transitively on $B/\sim$.

We first identify a countable family of $\sim$-classes that are ‘sufficiently indiscernible’. Our first lemma is where we use the fact that the function $h$ defining $T_h$ is strictly increasing.

**Lemma 2.2.** There is a countable set $Y = \{f_i : i \in \omega\} \subseteq B$ such that whenever $i \neq j$, $\{n \in \omega : f_i(n) = f_j(n)\}$ is finite.

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\(^1\) Recall that in any structure $M$, two elements $a, b$ have the same strong type, $\text{stp}(a) = \text{stp}(b)$, if $M \models E(a, b)$ for every 0-definable equivalence relation. Because of the quantifier elimination, in any model $M \models T_h$, $\text{stp}(a) = \text{stp}(b)$ if and only if $M \models E_n(a, b)$ for every $n \in \omega$. 

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Proof. We recursively construct \( Y \) in \( \omega \) steps. Suppose \( \{ f_i : i < k \} \) have been chosen. Choose an integer \( N \) large enough so that \( h(N) > k \) (hence \( h(n) > k \) for all \( n \geq N \)). Now, construct \( f_k \in B \) to satisfy \( f_k(n) \neq f_i(n) \) for all \( n \geq N \) and all \( i < k \).

Fix an enumeration \( \{ f_i : i \in \omega \} \) of \( Y \) for the whole of the argument. The ‘indiscernibility’ of \( Y \) alluded to above is formalized by the following definition and lemma.

Definition 2.3. Given a permutation \( \sigma \in \text{Sym}(\omega) \), a group element \( g \in G \) respects \( \sigma \) if \( g \cdot [f_i] = [f_{\sigma(i)}] \) for every \( i \in \omega \).

Lemma 2.4. For every permutation \( \sigma \in \text{Sym}(\omega) \), there is some \( g \in G \) respecting \( \sigma \).

Proof. Note that since \( h \) is increasing, \( h(n) \geq n \) for every \( n \in \omega \). Fix a permutation \( \sigma \in \text{Sym}(\omega) \) and we will define some \( g \in G \) respecting \( \sigma \) coordinate-wise. Using Lemma 2.2, choose a sequence

\[
0 = N_0 \ll N_1 \ll N_2 \ll \ldots
\]

of integers such that for all \( i \in \omega \), both \( f_i(n) \neq f_j(n) \) and \( f_{\sigma(i)}(n) \neq f_{\sigma(j)}(n) \) hold for all \( n \geq N_i \) and all \( j < i \).

Since \( \{ N_i \} \) are increasing, it follows that for each \( i \in \omega \) and all \( n \geq N_i \), the subsets \( \{ f_j(n) : j \leq i \} \) and \( \{ f_{\sigma(j)}(n) : j \leq i \} \) of \( [h(n)] \) each have precisely \( (i + 1) \) elements. Thus, for each \( i < \omega \) and for each \( n \geq N_i \), there is a permutation \( \delta_n \in \text{Sym}([h(n)]) \) satisfying

\[
\bigwedge_{j \leq i} \delta_n(f_j(n)) = f_{\sigma(j)}(n).
\]

(Simply begin defining \( \delta_n \) to meet these constraints, and then complete \( \delta_n \) to a permutation of \( [h(n)] \) arbitrarily.) Using this, define \( g := \{ \delta_n : n \in \omega \} \), where each \( \delta_n \in \text{Sym}([h(n)]) \) is constructed as above. To see that \( g \) respects \( \sigma \), note that for every \( i \in \omega \), \( (g \cdot f_i)(n) = f_{\sigma(i)}(n) \) for all \( n \geq N_i \), so \( (g \cdot f_i) \sim f_{\sigma(i)} \).

Definition 2.5. For distinct integers \( i \neq j \), let \( d_{i,j} \in B \) be defined by:

\[
d_{i,j}(n) := \begin{cases} f_i(n) & \text{if } n \text{ even;} \\ f_j(n) & \text{if } n \text{ odd.} \end{cases}
\]

Let \( Z := \{ d_{i,j} : i \neq j \} \).

Note that \( d_{i,j} \not\sim f_k \) for all distinct \( i, j \) and all \( k \in \omega \), hence \( \{[f_i] : i \in \omega \} \) and \( \{[d_{i,j}] : i \neq j \} \) are disjoint.

Lemma 2.6. For all \( \sigma \in \text{Sym}(\omega) \), if \( g \in G \) respects \( \sigma \), then \( g \cdot [d_{i,j}] = [d_{\sigma(i),\sigma(j)}] \) for all \( i \neq j \).

Proof. Choose \( \sigma \in \text{Sym}(\omega) \), \( g \) respecting \( \sigma \), and \( i \neq j \). Choose \( N \) such that \( (g \cdot [f_i])(n) = [f_{\sigma(i)}](n) \) and \( (g \cdot [f_j])(n) = [f_{\sigma(j)}](n) \) for every \( n \geq N \). Since \( d_{i,j}(n) = f_i(n) \) for \( n \geq N \) even,

\[
(g \cdot d_{i,j})(n) = \pi_n(g)(d_{i,j}(n)) = \pi_n(g)(f_i(n)) = (g \cdot f_i)(n) = f_{\sigma(i)}(n).
\]

Dually, \( (g \cdot d_{i,j})(n) = f_{\sigma(j)}(n) \) when \( n \geq N \) is odd, so \( g \cdot d_{i,j} \sim d_{\sigma(i),\sigma(j)} \).
With the combinatorial preliminaries out of the way, we now prove that $T_h$ is Borel complete. We form a highly homogeneous model $M^* \models T_h$ and thereafter, all models we consider will be countable, elementary substructures of $M^*$. Let $A = \{a_f : f \in B\}$ and $B = \{b_f : f \in B\}$ be disjoint sets and let $M^*$ be the $L$-structure with universe $A \cup B$ and each $E_n$ interpreted by the rules:

- For all $f, \, f' \in B$ and $n \in \omega$, $E_n(a_f, b_{f'})$ and
- For all $f, \, f' \in B$ and $n \in \omega$, $E_n(a_f, a_{f'})$ iff $f(n) = f'(n)$,

with the other instances of $E_n$ following by symmetry and transitivity. For any finite $F \subseteq \omega$, $\{f | F : f \in B\}$ has exactly $\Pi_{n \in F} h(n)$ elements, hence $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$ has $\Pi_{n \in F} h(n)$ classes in $M^*$. Thus, the $\{E_n : n \in \omega\}$ cross cut and $M^*$ is Borel complete.

Let $E_\infty(x, y)$ denote the (type definable) equivalence relation $\bigwedge_{n \in \omega} E_n(x, y)$. Then, in $M^*$, $E_\infty$ partitions $M^*$ into two-element classes $\{a_f, b_f\}$, indexed by $f \in B$. Note also that every $g \in G$ induces an $L$-automorphism $g^* \in \text{Aut}(M^*)$ by

$$g^*(x) := \begin{cases} a_{(g, f)} & \text{if } x = a_f \text{ for some } f \in B; \\ b_{(g, f)} & \text{if } x = b_f \text{ for some } f \in B. \end{cases}$$

Recall the set $Y = \{f_i : i \in B\}$ from Lemma 2.2, so $[Y] = \{[f_i] : i \in \omega\}$. Let $M_0 \subseteq M^*$ be the substructure with universe $\{a_f : f \in [Y]\}$. As $T_h$ admits elimination of quantifiers and as $[Y]$ is dense in $B$, $M_0 \preceq M^*$. Moreover, every substructure $M$ of $M^*$ with universe containing $M_0$ will also be an elementary substructure of $M^*$, hence a model of $T_h$.

To show that $\text{Mod}(T_h)$ is Borel complete, we define a Borel mapping from \{irreflexive graphs $\mathcal{G} = (\omega, R)$\} to $\text{Mod}(T_h)$ as follows: Given $\mathcal{G}$, let $Z(R) := \{d_{i,j} \in Z : \mathcal{G} \models R(i, j)\}$, so $[Z(R)] = \bigcup \{[d_{i,j}] : d_{i,j} \in Z(R)\}$. Let $M_G \preceq M^*$ be the substructure with universe $M_0 \cup \{a_d, b_d : d \in [Z(R)]\}$.

That the map $\mathcal{G} \mapsto M_G$ is Borel is routine, given that $Y$ and $Z$ are fixed throughout.

Note that in $M_G$, every $E_\infty$-class has either one or two elements. Specifically, for each $d \in [Z(R)]$, the $E_\infty$-class $[d_d]_\infty = \{a_d, b_d\}$, while the $E_\infty$-class $[a_f]_\infty = \{a_f\}$ for every $f \in [Y]$.

We must show that for any two graphs $\mathcal{G} = (\omega, R)$ and $\mathcal{H} = (\omega, S)$, $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if and only if the $L$-structures $M_G$ and $M_H$ are isomorphic.

To verify this, first choose a graph isomorphism $\sigma : (\omega, R) \to (\omega, S)$. Then $\sigma \in \text{Sym}(\omega)$ and, for distinct integers $i \neq j$, $d_{i,j} \in Z(R)$ if and only if $d_{\sigma(i), \sigma(j)} \in Z(S)$. Apply Lemma 2.4 to get $g \in G$ respecting $\sigma$ and let $g^* \in \text{Aut}(M^*)$ be the $L$-automorphism induced by $g$. By Lemma 2.6 and Definition 2.3, it is easily checked that the restriction of $g^*$ to $M_G$ is an $L$-isomorphism between $M_G$ and $M_H$.

Conversely, assume that $\Psi : M_G \to M_H$ is an $L$-isomorphism. Clearly, $\Psi$ maps $E_\infty$-classes in $M_G$ to $E_\infty$-classes in $M_H$. In particular, $\Psi$ permutes the one-element $E_\infty$-classes $\{\{a_f\} : f \in [Y]\}$ of both $M_G$ and $M_H$, and maps the two-element $E_\infty$-classes $\{\{a_d, b_d\} : d \in [Z(R)]\}$ of $M_G$ onto the two-element $E_\infty$-classes $\{\{a_d, b_d\} : d \in [Z(S)]\}$ of $M_H$. That is, $\Psi$ induces a bijection $F : [Y \sqcup Z(R)] \to [Y \sqcup Z(S)]$ that permutes $[Y]$. 
As well, by the interpretations of the $E_n$’s, for $f, f' \in [Y \sqcup Z(R)]$ and $n \in \omega$, 

\[ f(n) = f'(n) \quad \text{if and only if} \quad F(f)(n) = F(f')(n). \]

From this it follows that $F$ maps $\sim$-classes onto $\sim$-classes. As $F$ permutes $[Y]$ and as $[Y] = \bigcup\{[f_i] : i \in \omega\}$, $F$ induces a permutation $\sigma \in \text{Sym}(\omega)$ given by $\sigma(i)$ is the unique $i^* \in \omega$ such that $F([f_i]) = [f_{i^*}]$.

We claim that this $\sigma$ induces a graph isomorphism between $G = (\omega, R)$ and $H = (\omega, S)$. Indeed, choose any $(i, j) \in R$. Thus, $d_{i,j} \in Z(R)$. As $F$ is $\sim$-preserving, choose $N$ large enough so that $F(f_i)(n) = F(f_{\sigma(i)})(n)$ and $F(f_j)(n) = F(f_{\sigma(j)})(n)$ for every $n \geq N$. By definition of $d_{i,j}$, $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even, so $F(d_{i,j})(n) = F(f_i)(n) = f_{\sigma(i)}(n)$ for such $n$. Dually, for $n \geq N$ odd, $F(d_{i,j})(n) = F(f_j)(n) = f_{\sigma(j)}(n)$. Hence, $F(d_{i,j}) \sim d_{\sigma(i), \sigma(j)} \in [Z(S)]$. Thus, $(\sigma(i), \sigma(j)) \in S$. The converse direction is symmetric (i.e., use $\Psi^{-1}$ in place of $\Psi$ and run the same argument).

**Remark 2.7.** If we relax the assumption that $h : \omega \to \omega \setminus \{0\}$ is strictly increasing, there are two cases. If $h$ is unbounded, then the proof given above can easily be modified to show that the associated $T_h$ is also Borel complete. Conversely, with Theorem 6.2 of [6] the authors prove that if $h : \omega \to \omega \setminus \{0\}$ is bounded, then $T_h$ is not Borel complete. The salient distinction between the two cases is that when $h$ is bounded, the associated group $G$ has bounded exponent. However, even in the bounded case $T_h$ has a Borel complete reduct by Lemma 3.1 below.

**§3. Applications to reducts.** We begin with one easy lemma that, when considering reducts, obviates the need for the number of classes to be strictly increasing.

**Lemma 3.1.** Let $L = \{E_n : n \in \omega\}$ and let $f : \omega \to \omega \setminus \{0, 1\}$ be any function. Then every model $M$ of $T_f$, the complete theory asserting that each $E_n$ is an equivalence relation with $f(n)$ classes, and the $\{E_n\}$ cross-cut, has a Borel complete reduct.

**Proof.** Given any function $f : \omega \to \omega \setminus \{0, 1\}$, choose a partition $\omega = \bigcup\{F_n : n \in \omega\}$ into non-empty finite sets for which $\Pi_{k \in F_n} f(k) < \Pi_{k \notin F_n} f(k)$ whenever $n < m < \omega$. For each $n$, let $h(n) := \Pi_{k \in F_n} f(k)$ and let $E^*_n(x, y) := \bigwedge_{k \in F_n} E_k(x, y)$. Then, as $h$ is strictly increasing and $\{E^*_n\}$ is a cross-cutting set of equivalence relations with each $E^*_n$ having $h(n)$ classes.

Now let $M \models T_f$ be arbitrary and let $L' = \{E^*_n : n \in \omega\}$. As each $E^*_n$ described above is 0-definable in $M$, there is an $L'$-reduct $M'$ of $M$. It follows from Theorem 2.1 that $T' = Th(M')$ is Borel complete, so $T_f$ has a Borel complete reduct.

**Theorem 3.2.** Suppose $T$ is a complete theory in a countable language with uncountably many one-types. Then every model $M$ of $T$ has a Borel complete reduct.

**Proof.** Let $M \models T$ be arbitrary. As usual, by the Cantor–Bendixon analysis of the compact, Hausdorff–Stone space $S_1(T)$ of complete one-types, choose a set $\{\varphi_\eta(x) : \eta \in 2^{<\omega}\}$ of 0-definable formulas, indexed by the tree $(2^{<\omega}, \leq)$ ordered by initial segment, satisfying:

1. $M \models \exists x \varphi_\eta(x)$ for each $\eta \in 2^{<\omega}$;
2. For $\nu \leq \eta, M \models \forall x (\varphi_\eta(x) \to \varphi_\nu(x));$
3. For each $n \in \omega$, $\{\varphi_\eta(x) : \eta \in 2^n\}$ are pairwise contradictory.

By increasing these formulas slightly, we can additionally require
4. For each \( n \in \omega \), \( M \models \forall x (\bigvee_{\eta \in 2^n} \varphi_\eta(x)) \).

Given such a tree of formulas, for each \( n \in \omega \), define

\[
\delta_n^0(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \rightarrow \varphi_\eta^0(x)] \quad \text{and} \quad \delta_n^1(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \rightarrow \varphi_\eta^1(x)].
\]

Because of (4) above, \( M \models \forall x (\delta_n^0(x) \lor \delta_n^1(x)) \) for each \( n \). Also, for each \( n \), let

\[
E_n(x, y) := [\delta_n^0(x) \leftrightarrow \delta_n^0(y)].
\]

From the above, each \( E_n \) is a 0-definable equivalence relation with precisely two classes.

**Claim:** The equivalence relations \( \{E_n : n \in \omega \} \) are cross-cutting.

**Proof.** It suffices to prove that for every \( m > 0 \), the equivalence relation \( E_m^*(x, y) := \bigwedge_{n < cm} E_n(x, y) \) has \( 2^m \) classes. So fix \( m \) and choose a subset \( A_m = \{a_\eta : \eta \in 2^m\} \subseteq M \) forming a set of representatives for the formulas \( \{\varphi_\eta(x) : \eta \in 2^m\} \).

It suffices to show that \( M \models \neg E_m^*(a_\eta, a_\tau) \) whenever \( \eta \neq \nu \) are from \( 2^m \). But this is clear. Fix distinct \( \eta \neq \nu \) and choose any \( k < m \) such that \( \eta(k) \neq \nu(k) \). Then \( M \models \neg E_k(a_\eta, a_\tau) \), hence \( M \models \neg E_m^*(a_\eta, a_\tau) \).

Thus, taking the 0-definable relations \( \{E_n\} \), \( M \) has a reduct that is a model of \( T_f \) (where \( f \) is the constant function 2). As reducts of reducts are reducts, it follows from Lemma 3.1 and Theorem 2.1 that \( M \) has a Borel complete reduct.

We highlight how unexpected Theorem 3.2 is with two examples. First, the theory of ‘Independent unary predicates’ mentioned in the Introduction has a Borel complete reduct.

Next, we explore the assumption that a countable, complete theory \( T \) is not small, i.e., for some \( k \) there are uncountably many \( k \)-types. We conjecture that some model of \( T \) has a Borel complete reduct. If \( k = 1 \), then by Theorem 3.2, every model of \( T \) has a Borel complete reduct. If \( k > 1 \) is least, then it is easily seen that there is some complete \((k - 1)\)-type \( p(x_1, \ldots, x_{k-1}) \) with uncountably many complete \( q(x_1, \ldots, x_k) \) extending \( p \). Thus, if \( M \) is any model of \( T \) realizing \( p \), say by \( \bar{a} = (a_1, \ldots, a_{k-1}) \), the expansion \((M, a_1, \ldots, a_{k-1})\) has a Borel complete reduct, also by Theorem 3.2. Similarly, we have the following result.

**Corollary 3.3.** Suppose \( T \) is a complete theory in a countable language that is not small. Then for any model \( M \) of \( T \), \( M^{eq} \) has a Borel complete reduct.

**Proof.** Let \( M \) be any model of \( T \) and choose \( k \) least such that \( T \) has uncountably many complete \( k \)-types consistent with it. In the language \( L^{eq} \), there is a sort \( U_k \) and a definable bijection \( f : M^k \rightarrow U_k \). Hence \( Th(M^{eq}) \) has uncountably many one-types consistent with it, each extending \( U_k \). Thus, \( M^{eq} \) has a Borel complete reduct by Theorem 3.2.

Finally, recall that a countable, complete theory is not \( \omega \)-stable if, for some countable model \( M \) of \( T \), the Stone space \( S_1(M) \) is uncountable. From this, we immediately obtain our final corollary.

**Corollary 3.4.** If a countable, complete \( T \) is not \( \omega \)-stable, then for some countable model \( M \) of \( T \), the elementary diagram of \( M \) in the language \( L(M) = L \cup \{c_m : m \in M\} \) has a Borel complete reduct.
Proof. Choose a countable $\mathcal{M}$ so that $S_1(\mathcal{M})$ is uncountable. Then, in the language $L(M)$, the theory of the expanded structure $M_M$ in the language $L(\mathcal{M})$ has uncountably many one-types, hence it has a Borel complete reduct by Theorem 3.2.

The results above are by no means characterizations. Indeed, there are many Borel complete $\omega$-stable theories. In [5], the first author and Shelah prove that any $\omega$-stable theory that has eni-DOP or is eni-deep is not only Borel complete, but also $\lambda$-Borel complete for all $\lambda$.\footnote{Definitions of eni-DOP and eni-deep are given in Definitions 2.3 and 6.2, respectively, of [5], and the definition of $\lambda$-Borel complete is recalled in Section 4 of this paper.} As well, there are $\omega$-stable theories with only countably many countable models that have Borel complete reducts. To illustrate this, we introduce three interrelated theories. The first, $T_0$, in the language $L_0 = \{U, V, W, R\}$ is the paradigmatic DOP theory. $T_0$ asserts that:

- $U, V, W$ partition the universe;
- $R \subseteq U \times V \times W$;
- $T_0 \models \forall x \forall y \exists^\infty z R(x, y, z)$ [more formally, for each $n$, $T_0 \models \forall x \forall y \exists^n z R(x, y, z)$]; and
- $T_0 \models \forall x \forall x' \forall y \forall y' \forall z[R(x, y, z) \land R(x', y', z) \rightarrow (x = x' \land y = y')]$.

$T_0$ is both $\omega$-stable and $\omega$-categorical and its unique countable model is rather tame. The complexity of $T_0$ is only witnessed with uncountable models, where one can code arbitrary bipartite graphs in an uncountable model $\mathcal{M}$ by choosing the cardinalities of the sets $R(a, b, \mathcal{M})$ among $(a, b) \in U \times V$ to be either $\aleph_0$ or $|\mathcal{M}|$.

To get bad behavior of countable models, we expand $T_0$ to an $L = L_0 \cup \{f_n : n \in \omega\}$-theory $T \supseteq T_0$ that additionally asserts:

- Each $f_n : U \times V \rightarrow W$;
- $\forall x \forall y R(x, y, f_n(x, y))$ for each $n$; and
- for distinct $n \neq m$, $\forall x \forall y (f_n(x, y) \neq f_m(x, y))$.

This $T$ is $\omega$-stable with eni-DOP and hence is Borel complete by Theorem 4.12 of [5].

However, $T$ has an expansion $T^*$ in a language $L^* := L \cup \{c, d, g, h\}$ whose models are much better behaved. Let $T^*$ additionally assert:

- $U(c) \land V(d)$;
- $g : U \rightarrow V$ is a bijection with $g(c) = d$;
- Letting $W^* := \{z : R(c, d, z)\}, h : U \times V \times W^* \rightarrow W$ is an injective map that is the identity on $W^*$ and, for each $(x, y) \in U \times V$, maps $W^*$ onto $\{z \in W : R(x, y, z)\}$; and moreover
- $h$ commutes with each $f_n$, i.e., $\forall x \forall y(h(x, y, f_n(c, d))) = f_n(x, y))$.

Then $T^*$ is $\omega$-stable and two-dimensional (the dimensions being $|U|$ and $|W^* \setminus \{f_n(c, d) : n \in \omega\}|$). Hence $T^*$ has only countably many countable models. However, $T^*$ visibly has a Borel complete reduct, namely $T$.

§4. Observations about the theories $T_h$. In addition to their utility in proving Borel complete reducts, the theories $T_h$ in Section 2 illustrate some novel behaviors. First off, model theoretically, these theories are extremely simple. More precisely,
each theory $T_h$ is weakly minimal with the geometry of every strong type trivial (such theories are known as mutually algebraic in [4]).

Additionally, the theories $T_h$ are the simplest known examples of theories that are Borel complete, but not $\lambda$-Borel complete for all cardinals $\lambda$. For $\lambda$ any infinite cardinal, $\lambda$-Borel completeness was introduced in [5]. Instead of looking at $L$-structures with universe $\omega$, we consider $X^L_\lambda$, the set of $L$-structures with universe $\lambda$.

We topologize $X^L_\lambda$ analogously: namely a basis consists of all sets

$$U_{\varphi(\alpha_1, \ldots, \alpha_n)} := \{ M \in X^L_\lambda : M \models \varphi(\alpha_1, \ldots, \alpha_n) \}$$

for all $L$-formulas $\varphi(x_1, \ldots, x_n)$ and all $(\alpha_1, \ldots, \alpha_n) \in \lambda^n$. Define a subset of $X^L_\lambda$ to be $\lambda$-Borel if it is the smallest $\lambda^+$-algebra containing the basic open sets, and call a function $f : X^L_{\lambda_1} \to X^L_{\lambda_2}$ to be $\lambda$-Borel if the inverse image of every basic open set is $\lambda$-Borel. For $T, S$ theories in languages $L_1, L_2$, respectively, we say that $\text{Mod}_\lambda(T)$ is $\lambda$-Borel reducible to $\text{Mod}_\lambda(S)$ if there is a $\lambda$-Borel $f : \text{Mod}_\lambda(T) \to \text{Mod}_\lambda(S)$ preserving back-and-forth equivalence in both directions (i.e., $M \equiv_{\infty, \omega} N \iff f(M) \equiv_{\infty, \omega} f(N)$).

As back-and-forth equivalence is the same as isomorphism for countable structures, $\lambda$-Borel reducibility when $\lambda = \omega$ is identical to Borel reducibility. As before, for any infinite $\lambda$, there is a maximal class under $\lambda$-Borel reducibility, and we say a theory is $\lambda$-Borel complete if it is in this maximal class. All of the ‘classical’ Borel complete theories, e.g., graphs, linear orders, groups, and fields, are $\lambda$-Borel complete for all $\lambda$. However, the theories $T_h$ are not.

**Lemma 4.1.** If $T$ is mutually algebraic in a countable language, then there are at most $\beth_2$ pairwise $\equiv_{\infty, \omega}$-inequivalent models (of any size).

**Proof.** We show that every model $M$ has an $(\infty, \omega)$-elementary substructure of size $2^{|\omega_0|}$, which suffices. So, fix $M$ and choose an arbitrary countable $M_0 \subseteq M$. By Proposition 4.4 of [4], $M \setminus M_0$ can be decomposed into countable components, and any permutation of isomorphic components induces an automorphism of $M$ fixing $M_0$ pointwise. As there are at most $2^{|\omega_0|}$ non-isomorphic components over $M_0$, choose a substructure $N \subseteq M$ containing $M_0$ and, for each isomorphism type of a component, $N$ contains either all of copies in $M$ (if there are only finitely many) or else precisely $|\omega_0|$ copies if $M$ contains infinitely many copies. It is easily checked that $N \preceq_{\infty, \omega} M$. \qed

**Corollary 4.2.** No mutually algebraic theory $T$ in a countable language is $\lambda$-Borel complete for $\lambda \geq \beth_2$. In particular, $T_h$ is Borel complete, but not $\lambda$-Borel complete for large $\lambda$.

**Proof.** Fix $\lambda \geq \beth_2$. It is readily checked that there is a family of $2^\lambda$ graphs that are pairwise not back and forth equivalent. As there are fewer than $2^\lambda \equiv_{\infty, \omega}$-classes of models of $T$, there cannot be a $\lambda$-Borel reduction of graphs into $\text{Mod}_\lambda(T)$. \qed

In [7], another example of a Borel complete theory that is not $\lambda$-Borel complete for all $\lambda$ is given (it is dubbed $TK$ there) but the $T_h$ examples are cleaner. In order to understand this behavior, in [7] we call a theory $T$ grounded if every potential canonical Scott sentence $\sigma$ of a model of $T$ (i.e., in some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, $\sigma$ is a canonical Scott sentence of some model), then $\sigma$ is a canonical Scott...
Proposition 4.3. If $T$ is Borel complete with a cardinal bound on the number of $\equiv_{\infty,0}$-classes of models, then $T$ is not grounded. In particular, $T_h$ is not grounded.

Proof. Let $\kappa$ denote the number of $\equiv_{\infty,0}$-classes of models of $T$. If $T$ were grounded, then $\kappa$ would also bound the number of potential canonical Scott sentences. As the class of graphs has a proper class of potential canonical Scott sentences, it would follow from Theorem 3.10 of [7] that $T$ could not be Borel complete. \hfill \dasharrow

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