HIGHER ORDER REDUCTION THEOREMS
FOR GENERAL LINEAR CONNECTIONS

JOSEF JANYŠKA

Abstract. The reduction theorems for general linear and classical connections are generalized for operators with values in higher order gauge-natural bundles. We prove that natural operators depending on the $s_1$-jets of classical connections, on the $s_2$-jets of general linear connections and on the $r$-jets of tensor fields with values in gauge-natural bundles of order $k \geq 1$, $s_1 + 2 \geq s_2$, $s_1, s_2 \geq r - 1 \geq k - 2$, can be factorized through the $(k - 2)$-jets of both connections, the $(k - 1)$-jets of the tensor fields and sufficiently high covariant differentials of the curvature tensors and the tensor fields.

Introduction

It is well known that natural operators of classical (linear and symmetric) connections on manifolds and of tensor fields with values in natural bundles of order one can be factorized through the curvature tensors, the tensor fields and their covariant differentials. These theorems are known as the first (operators on classical connections only) and the second reduction, [8, 10], or replacement, [12, 13], theorems. In [6] the reduction theorems are proved by using methods of natural bundles and operators, [7, 9, 11].

In [4] the reduction theorems were generalized for general linear connections on vector bundles. In this gauge-natural situation we need auxiliary classical connections on the base manifolds. It is proved that natural operators with values in gauge-natural bundles of order $(1,0)$ defined on the space of general linear connections on a vector bundle, on the space of classical connections on the base manifold and on certain tensor bundles can be factorized through the curvature tensors of linear and classical connections, the tensor fields and their covariant differentials with respect to both connections.

In [5] another generalization of the classical reduction theorems was presented. Namely, the reduction theorems were proved for operators with values in higher order natural bundles. It was proved that an $r$-th order natural operator on classical connections with values in natural bundles of order $k \geq 1$, $r + 2 \geq k$, can be factorized through the $(k - 2)$-jets of connections and sufficiently high covariant differentials of the curvature tensor.

In this paper we combine both possible generalizations of the reduction theorems and we prove the reduction theorems for general linear connections on vector bundles for operators with values in higher order gauge-natural bundles.

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All manifolds and maps are assumed to be smooth. The sheaf of (local) sections of a fibered manifold \( p : Y \to X \) is denoted by \( C^\infty(Y) \), \( C^\infty(Y, \mathbb{R}) \) denotes the sheaf of (local) functions.

1. Gauge-natural bundles

Let \( \mathcal{M}_m \) be the category of \( m \)-dimensional \( C^\infty \)-manifolds and smooth embeddings. Let \( \mathcal{FM}_m \) be the category of smooth fibered manifolds over \( m \)-dimensional bases and smooth fiber manifold maps over embeddings of bases and \( \mathcal{P}\mathcal{B}_m(G) \) be the category of smooth principal \( G \)-bundles with \( m \)-dimensional bases and smooth \( G \)-bundle maps \( (\varphi, f) \), where the map \( f \in \text{Mor}\mathcal{M}_m \).

**Definition 1.1.** A G-gauge-natural bundle is a covariant functor \( F \) from the category \( \mathcal{P}\mathcal{B}_m(G) \) to the category \( \mathcal{FM}_m \) satisfying

i) for each \( \pi : P \to M \in \text{Ob}\mathcal{P}\mathcal{B}_m(G) \), \( \pi_P : FP \to M \) is a fibered manifold over \( M \),

ii) for each map \( (\varphi, f) \) in \( \text{Mor}\mathcal{P}\mathcal{B}_m(G) \), \( F\varphi = F(\varphi, f) \) is a fibered manifold morphism covering \( f \),

iii) for any open subset \( U \subseteq M \), the immersion \( \iota : \pi^{-1}(U) \to P \) is transformed into the immersion \( F\iota : \pi_P^{-1}(U) \to FP \).

Let \( (\pi : P \to M) \in \text{Ob}\mathcal{P}\mathcal{B}_m(G) \) and \( W^rP \) be the space of all \( r \)-jets \( j_{(0,e)}^r\varphi \), where \( \varphi : \mathbb{R}^m \times G \to P \) is in \( \text{Mor}\mathcal{P}\mathcal{B}_m(G) \), \( 0 \in \mathbb{R}^m \) and \( e \) is the unit in \( G \). The space \( W^rP \) is a principal fiber bundle over the manifold \( M \) with the structure group \( W^rG \) of all \( r \)-jets \( j_{(0,e)}^r\Psi \) of principal fiber bundle isomorphisms \( \Psi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G \) covering the diffeomorphisms \( \psi : \mathbb{R}^m \to \mathbb{R}^m \) such that \( \psi(0) = 0 \). The group \( W^rG \) is the semidirect product of \( G^m = \text{inv} \, J_0(\mathbb{R}^m, \mathbb{R}^m) \) and \( T^rG = J_0^r(\mathbb{R}^m, G) \) with respect to the action of \( G^m \) on \( T^rG \) given by the jet composition, i.e. \( W^rG = G^m \rtimes T^rG \). If \( (\varphi : P \to P) \in \text{Mor}\mathcal{P}\mathcal{B}_m(G) \), then we can define the principal bundle morphism \( W^r\varphi : W^rP \to W^rP \) by the jet composition. The rule transforming any \( P \in \text{Ob}\mathcal{P}\mathcal{B}_m(G) \) into \( W^rP \in \text{Ob}\mathcal{P}\mathcal{B}_m(W^rG) \) and any \( \varphi \in \text{Mor}\mathcal{P}\mathcal{B}_m(G) \) into \( W^r\varphi \in \text{Mor}\mathcal{P}\mathcal{B}_m(W^rG) \) is a G-gauge-natural bundle.

The gauge-natural bundle functor \( W^r \) plays a fundamental role in the theory of gauge-natural bundles. We have, \([4,6]\).

**Theorem 1.2.** Every gauge-natural bundle is a fiber bundle associated to the bundle \( W^r \) for a certain order \( r \).

The number \( r \) from Theorem 1.2 is called the order of the gauge-natural bundle. So if \( F \) is an \( r \)-order gauge-natural bundle, then

\[
FP = (W^rP, S_F), \quad F\varphi = (W^r\varphi, \text{id}_{S_F}),
\]

where \( S_F \) is a left \( W^rG \)-manifold called the standard fiber of \( F \).

If \( (x^\lambda, z^s) \) is a local fiber coordinate chart on \( P \) and \( (y^i) \) a coordinate chart on \( S_F \), then \( (x^\lambda, y^i) \) is the fiber coordinate chart on \( FP \) which is said to be adapted.

Let \( F \) be a G-gauge-natural bundle of order \( s \) and let \( r \leq s \) be a minimal number such that the action of \( W^sG = G^s \rtimes T^sG \) on \( S_F \) can be factorized through the canonical projection \( \pi_r^s : T^sG \to T^rG \). Then \( r \) is called the gauge-order of \( F \) and we say that \( F \) is of order \((s, r)\). We shall denote by \( W^{(s,r)}G = G^s \rtimes T^rG \) the Lie group acting on the standard fiber of an \((s, r)\)-order G-gauge-natural bundle. Then there is a one-to-one, up to equivalence,
correspondence of smooth left $W_m^{(s,r)} G$-manifolds and $G$-gauge-natural bundles of order $(s,r)$, \cite{1}. So any $(s,r)$-order $G$-gauge-natural bundle can be represented by its standard fiber with an action of the group $W_m^{(s,r)} G$.

If $F$ is an $(s, r)$-order $G$-gauge-natural bundle, then $J^k F$ is an $(s + k, r + k)$-order $G$-gauge-natural bundle with the standard fiber $T_m^k S_F = J^k_0 (\mathbb{R}^m, S_F)$.

The class of $G$-gauge-natural bundles contains the class of natural bundles in the sense of \cite{3, 7, 9, 11}. Namely, if $F$ is an $r$-order natural bundle, then $F$ is the $(r, 0)$-order $G$-gauge-natural bundle with trivial gauge structure.

Let $F$ be a $G$-gauge-natural bundle and $(\varphi, f) : P \to \bar{P}$ be in the category $\mathcal{PB}_n(G)$. Let $\sigma$ be a section of $F_P$. Then we define the section $\varphi_f \sigma = F \varphi \circ \sigma \circ f^{-1}$ of $F_P$. Let $H$ be another $G$-gauge-natural bundle.

**Definition 1.3.** A natural differential operator from $F$ to $H$ is a collection $D = \{D(P), P \in \text{Ob} \mathcal{PB}_n(G)\}$ of differential operators from $C^\infty (F_P)$ to $C^\infty (H_P)$ satisfying $D(P) \circ \varphi_f^* = \varphi_{H_P}^* \circ D(P)$ for each map $(\varphi, f) \in \text{Mor} \mathcal{PB}_n(G), \varphi : P \to \bar{P}$.

$D$ is of order $k$ if all $D(P)$ are of order $k$. Let $D$ be a natural differential operator of order $k$ from $F$ to $H$. For any $P \in \text{Ob} \mathcal{PB}_n(G)$ we have the associated map $D(P) : J^k F_P \to H_P$, over $M$, defined by $D(P)(j^k_x \sigma) = D(P)\sigma(x)$ for all $x \in M$ and any section $\sigma : M \to F_P$. From the naturality of $D$ it follows that $D = \{D(P), P \in \text{Ob} \mathcal{PB}_n(G)\}$ is a natural transformation of $J^k F$ to $H$. The following theorem is due to Eck, \cite{1}.

**Theorem 1.4.** Let $F$ and $H$ be $G$-gauge-natural bundles of order $\leq (s, r), s \geq r$. Then we have a one-to-one correspondence between natural differential operators of order $k$ from $F$ to $H$ and $W_n^{(s+k,r+k)} G$-equivariant maps from $T_m^k S_F$ to $S_H$.

So according to Theorem 1.4 a classification of natural operators between $G$-gauge-natural bundles is equivalent to the classification of equivariant maps between standard fibers. Very important tool in classifications of equivariant maps is the orbit reduction theorem, \cite{3, 7}.

Let $p : G \to H$ be a Lie group epimorphism with the kernel $K$, $M$ be a left $G$-space, $Q$ be a left $H$-space and $\pi : M \to Q$ be a $p$-equivariant surjective submersion, i.e., $\pi(gx) = p(g)\pi(x)$ for all $x \in M, g \in G$. Having $p$, we can consider every left $H$-space $N$ as a left $G$-space by $gy = p(g)y, g \in G, y \in N$.

**Theorem 1.5.** If each $\pi^{-1}(q), q \in Q$ is a $K$-orbit in $M$, then there is a bijection between the $G$-maps $f : M \to N$ and the $H$-maps $\varphi : Q \to N$ given by $f = \varphi \circ \pi$.

2. **Linear connections on vector bundles**

In what follows let $G = GL(n, \mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^n$ with coordinates $(a_{ij}^j)$. Let us consider the category $\mathcal{VB}_{m,n}$ of vector bundles with $m$-dimensional bases, $n$-dimensional fibers and local fibered linear diffeomorphisms. Then any vector bundle $(p : E \to M) \in \text{Ob} \mathcal{VB}_{m,n}$ can be considered as a zero order $G$-gauge-natural functor $\mathcal{PB}_m(G) \to \mathcal{VB}_{m,n}$.

Local linear fiber coordinate charts on $E$ will be denoted by $(x^\lambda, y^i)$. The induced local bases of sections of $TE$ or $T^*E$ will be denoted by $(\partial_\lambda, \partial_i)$ or $(d^\lambda, d^i)$, respectively.
We define a linear connection on $E$ to be a linear splitting

$$K : E \to J^1 E.$$  

Considering the contact morphism $J^1 E \to T^* M \otimes TE$ over the identity of $TM$, a linear connection can be regarded as a $TE$-valued 1-form

$$K : E \to T^* M \otimes TE$$

projecting onto the identity of $TM$.

The coordinate expression of a linear connection $K$ is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K^i_\lambda y^j \partial_i), \quad \text{with} \quad K^i_\lambda \in C^\infty(M, \mathbb{IR}).$$

Linear connections can be regarded as sections of a (1,1)-order $G$-gauge-natural bundle $\text{Lin} E \to M$, \cite{[1],[2]}. The standard fiber of the functor $\text{Lin}$ will be denoted by $R = \mathbb{IR}^{n*} \otimes \mathbb{IR}^n \otimes \mathbb{IR}^{m*}$, elements of $R$ will be said to be formal linear connections, the induced coordinates on $R$ will be said to be formal symbols of formal linear connections and will be denoted by $(K^i_\lambda)$. The action $\beta : W^{(1,1)}_m G \times R \to R$ of the group $W^{(1,1)}_m G = G^1_m \times T^1_m G$ on the standard fiber $R$ is given in coordinates by

$$(K^i_\lambda) \circ \beta = a^i_p (K^p_q \rho_\lambda \rho_q - \rho^p_\lambda),$$

where $(a^\lambda_\mu, a^i_\lambda, a^i_\lambda)$ are coordinates on $W^{(1,1)}_m G$ and $\sim$ denotes the inverse element.

**Note 2.1.** Let us note that the action $\beta$ gives, in a natural way, the action $\beta^r : W^{(r+1,r+1)}_m G \times T^r_m R \to T^r_m R$ determined by the jet prolongation of the action $\beta$.

**Remark 2.2.** Let us consider the group epimorphism $\pi^{r+1,r+1}_{r,r} : W^{(r+1,r+1)}_m G \to W^{(r,r)}_m G$ and its kernel $G^{r+1,r+1}_{r,r} := \text{Ker} \pi^{r+1,r+1}_{r,r}$. On $G^{r+1,r+1}_{r,r}$ we have the induced coordinates $(a^\lambda_\mu, a^i_\mu, a^i_{\mu \rho})$. Then the restriction $\beta^r$ of the action $\beta^r$ to $G^{r+1,r+1}_{r,r}$ has the following coordinate expression

$$
(2.1) \quad (K^i_{\mu_1, \ldots, \mu_r} \mu_{1, \ldots, \mu_r}) \circ \beta^r
\quad = (K^i_{\mu_1, \ldots, \mu_r} \mu_{1, \ldots, \mu_r}, K^i_{\mu_1, \ldots, \mu_r} \mu_{\rho, \rho + 1} - \rho^i_{\mu_1, \ldots, \mu_r + 1}),
$$

where $(K^i_{\mu_1, \ldots, \mu_r} \mu_{1, \ldots, \mu_r}, K^i_{\mu_1, \ldots, \mu_r} \mu_{\rho, \rho + 1})$ are the induced jet coordinates on $T^r_m R$.

The curvature of a linear connection $K$ on $E$ turns out to be the vertical valued 2–form

$$R[K] = -[K, K] : E \to VE \otimes \bigwedge^2 T^* M,$$

where $[,]$ is the Froelicher-Nijenhuis bracket. The coordinate expression is

$$R[K] = R[K]^i_{j \lambda} y^j \partial_i \otimes d^\lambda \wedge d^\mu$$

$$= -2(\partial_\lambda K^i_{j \mu} + K^p_{j \lambda} K_p^i_{j \mu}) y^j \partial_i \otimes d^\lambda \wedge d^\mu.$$
If we consider the identification \( VE = E \times E \) and linearity of \( R[K] \), the curvature \( R[K] \) can be considered as the curvature tensor field \( R[K] : M \to E^* \otimes E \otimes \wedge^2 T^* M \) and

\[
R[K] : C^\infty(\text{Lin} E) \to C^\infty(E^* \otimes E \otimes \bigwedge^2 T^* M)
\]

is a natural operator which is of order one, i.e., we have the associated \( W_{m}^{(2,2)} G \)-equivariant map, called the formal curvature map of formal linear connections,

\[
\mathcal{R}_L : T^1_m R \to \mathcal{U}
\]

with the coordinate expression

\[
(2.2) \quad (u_j^i \lambda_\mu) \circ \mathcal{R}_L = K_j^i \lambda_\mu - K_j^p \mu_\lambda K_p^i \lambda - K_j^r \lambda K_r^i \mu,
\]

where \((u_j^i \lambda_\mu)\) are the induced coordinates on the standard fiber \( \mathcal{U} := B^m \otimes \mathbb{R} \otimes \bigwedge^2 B^m \) of \( E^* \otimes E \otimes \bigwedge^2 T^* M \).

We define a classical connection on \( M \) to be a linear symmetric connection on the tangent vector bundle \( p_M : TM \to M \) with the coordinate expression

\[
\Lambda = d^\lambda \otimes (\partial_\lambda + \Lambda_\nu^\mu \lambda \partial^\nu \partial_\mu), \quad \Lambda_\mu^\lambda \nu \in C^\infty(M, \mathbb{R}), \quad \Lambda_\mu^\lambda \nu = \Lambda_\nu^\lambda \mu.
\]

Classical connections can be regarded as sections of a 2nd order natural bundle \( \text{Cl}a M \to M \), \( [6] \). The standard fiber of the functor \( \text{Cl}a \) will be denoted by \( Q = B^m \otimes S^2 B^m \), elements of \( Q \) will be said to be formal classical connections, the induced coordinates on \( Q \) will be said to be formal Christoffel symbols of formal classical connections and will be denoted by \( (\Lambda_\mu^\lambda \nu) \). The action \( \alpha : G^2_m \times Q \to Q \) of the group \( G^2_m \) on \( Q \) is given in coordinates by

\[
(\Lambda_\mu^\lambda \nu) \circ \alpha = \alpha_\rho^\mu (\Lambda_\sigma^\rho \tau \alpha_\mu^\sigma \alpha_\nu^\tau - \alpha_{\mu \nu}^\tau).
\]

Note 2.3. Let us note that the action \( \alpha \) gives, in a natural way, the action

\[
\alpha^r : G^{r+2}_m \times T^r_m Q \to T^r_m Q
\]

determined by the jet prolongation of the action \( \alpha \).

Remark 2.4. Let us consider the group epimorphism \( \pi_{r+1}^{r+2} : G^{r+2}_m \to G^{r+1}_m \) and its kernel \( B^m_{r+1} := \text{Ker} \pi_{r+1}^{r+2} \). We have the induced coordinates \((a_\mu^\lambda \nu \alpha_{\mu \nu}^\tau)\) on \( B_{r+2}^m \). Then the restriction \( \tilde{\alpha}^r \) of the action \( \alpha^r \) to \( B_{r+2}^m \) has the following coordinate expression

\[
(2.3) \quad (\Lambda_\mu^\lambda \nu \alpha_{\mu \nu}^\tau) \circ \tilde{\alpha}^r = (\Lambda_\mu^\lambda \nu \alpha_{\mu \nu}^\tau),
\]

where \((\Lambda_\mu^\lambda \nu \alpha_{\mu \nu}^\tau)\) are the induced jet coordinates on \( T^r_m Q \).

The curvature tensor of a classical connection is a natural operator

\[
R[\Lambda] : C^\infty(\text{Cl}a M) \to C^\infty(T^* M \otimes TM \otimes \bigwedge^2 T^* M)
\]

which is of order one, i.e., we have the associated \( G^3_m \)-equivariant map, called the formal curvature map of formal classical connections,

\[
\mathcal{R}_C : T^1_m Q \to S_{T^* \otimes T^* \otimes \bigwedge^2 T^*}
\]
with the coordinate expression
\[(w, \rho, \lambda, \mu) \circ \mathcal{R}_C = \Lambda^\rho_{\mu, \lambda} - \Lambda^\rho_{\mu, \lambda} + \Lambda^\mu_{\sigma, \mu, \lambda} - \Lambda^\sigma_{\lambda, \lambda} \Lambda^\rho_{\mu} ,\]
where \((w, \rho, \lambda, \mu)\) are the induced coordinates on the standard fiber \(\mathcal{W} := S_{\text{T}(\mathcal{W})}^2 \mathcal{W} = \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^{m^2} \).

Let us denote by \(E^{p,r}_{q,s} := \otimes^p \mathcal{E} \otimes \otimes^q \mathcal{E}^* \otimes \otimes^r \mathcal{T} \mathcal{M} \otimes \otimes^s \mathcal{T}^* \mathcal{M}\) the tensor product over \(\mathcal{M}\) and recall that \(E^{p,r}_{q,s}\) is a vector bundle which is a \(G\)-gauge-natural bundle of order \((1,0)\).

A classical connection \(\Lambda\) on \(\mathcal{M}\) and a linear connection \(K\) on \(\mathcal{E}\) induce the linear tensor product connection \(K^p_q \otimes \Lambda^r_s : = \otimes^p K \otimes \otimes^q \Lambda^* \otimes \otimes^r \mathcal{M} \otimes \otimes^* \mathcal{M}\) on \(E^{p,r}_{q,s}\)
\[K^p_q \otimes \Lambda^r_s : E^{p,r}_{q,s} \rightarrow T^* \mathcal{M} \otimes T_{\mathcal{E}} E^{p,r}_{q,s}\]
which can be considered as a linear splitting
\[K^p_q \otimes \Lambda^r_s : E^{p,r}_{q,s} \rightarrow J^1 E^{p,r}_{q,s}.\]

Then we define, \(3\), the covariant differential of a section \(\Phi : \mathcal{M} \rightarrow E^{p,r}_{q,s}\) with respect to the pair of connections \((K, \Lambda)\) as a section of \(E^{p,r}_{q,s} \otimes T^* \mathcal{M}\) given by
\[\nabla^{(K, \Lambda)} \Phi = j^1 \Phi - (K^p_q \otimes \Lambda^r_s) \circ \Phi.\]

In what follows we set \(\nabla = \nabla^{(K, \Lambda)}\) and \(\phi^{i_1 \ldots i_r \lambda_1 \ldots \lambda_r}_{j_1 \ldots j_q \mu_1 \ldots \mu_s} = \nabla_i \phi^{i_1 \ldots i_r \lambda_1 \ldots \lambda_r}_{j_1 \ldots j_q \mu_1 \ldots \mu_s} = \nabla \phi^{i_1 \ldots i_r \lambda_1 \ldots \lambda_r}_{j_1 \ldots j_q \mu_1 \ldots \mu_s}.\)

We have the following relations between the covariant differentials and the curvatures, \(3\).

**Proposition 2.5.** The curvature
\[R[K^p_q \otimes \Lambda^r_s] := -[K^p_q \otimes \Lambda^r_s, K^p_q \otimes \Lambda^r_s] : E^{p,r}_{q,s} \rightarrow E^{p,r}_{q,s} \otimes \bigwedge^2 T^* \mathcal{M}\]
is determined by the curvatures \(R[K]\) and \(R[\Lambda]\).

**Theorem 2.6.** (The generalized Bianchi identity) We have
\[R[K]_{ij}^l \lambda_{\mu; \nu} + R[K]_{ij}^l \mu_{\nu; \lambda} + R[K]_{ij}^l \nu_{\lambda; \mu} = 0.\]

**Theorem 2.7.** Let \(\Phi \in C^\infty (E^{p,r}_{q,s})\). Then we have
\[\text{Alt} \nabla^2 \Phi = -\frac{1}{2} R[K^p_q \otimes \Lambda^r_s] \circ \Phi \in C^\infty (E^{p,r}_{q,s} \otimes \bigwedge^2 T^* \mathcal{M})\],
where \text{Alt} is the antisymmetrization.

**Remark 2.8.** From the above Theorem 2.7 and the expression of \(R[K^p_q \otimes \Lambda^r_s], \) \(3,\) it follows, that \text{Alt} \nabla^2 \Phi is a \(E^{p,r}_{q,s}\)-valued 2-form which is a quadratic polynomial in \(R[K], R[\Lambda], \Phi.\) Especially, we have
\[\text{Alt} \nabla^2 R[K] : \mathcal{M} \rightarrow E^* \otimes E \otimes \bigwedge^2 T^* \mathcal{M} \otimes \bigwedge^2 T^* \mathcal{M},\]
given in coordinates by
\[
\text{Alt } \nabla^2 R[K] = -\frac{1}{2} \left( R[K]_p^i \nu_1 \nu_2 R[K]_j^p \lambda_\mu - R[K]_j^p \nu_1 \nu_2 R[K]_p^i \lambda_\mu \right.
\]
\[
- R[\Lambda]^\omega_{\nu_1 \nu_2} R[K]_j^i \omega_\mu - R[\Lambda]^\omega_{\nu_1 \nu_2} R[K]_j^i \lambda_\omega \big)
\]
\[
b^j \otimes b_1 \otimes d^2 \wedge d^\mu \otimes d^{i_1} \wedge d^{i_2}.
\]

**Remark 2.9.** Let us note that for classical connections we have the first and the second Bianchi identities
\[
R[\Lambda]_{\nu (\rho \lambda \mu)} = 0 \quad \text{and} \quad R[\Lambda]_{\nu (\rho \lambda \mu) \sigma} = 0,
\]
respectively, where (... ) denotes the cyclic permutation. Moreover, we have the antisymmetrization of the second order covariant differential of the curvature tensor which is a quadratic polynomial of the curvature tensor.

### 3. The First k-th Order Reduction Theorem for Linear and Classical Connections

Let us introduce the following notations.

Let \( W_0 M := WM = T^* M \otimes T M \otimes \wedge^2 T^* M \), \( W_i M = WM \otimes \wedge^i T^* M \), \( i \geq 0 \). Let us put \( W^{(k,r)} M = \bigcup M \), \( i \geq 0 \). Let us set \( W^{(r)} M = W^{0 \ldots r} M \). Then \( W_i M \) and \( W^{(k,r)} M \) are natural bundles of order one and the corresponding standard fibers will be denoted by \( W_i \) and \( W^{(k,r)} \), where \( W_0 := W = \mathbb{R}^{m \times} \otimes \mathbb{R}^{m \times} \otimes \wedge^2 \mathbb{R}^{m \times} \), \( W_i = W \otimes \wedge^i \mathbb{R}^{m \times} \), \( i \geq 0 \), and \( W^{(k,r)} = \bigcup M \). Let us denote by \((w_0^\rho_{\lambda \mu .. \sigma}, \sigma)\) the coordinates on \( W_i \).

We denote by
\[
\mathcal{R}_{C,i} : T^{i+1} Q \rightarrow W_i
\]
the \( G_m \)-equivariant map associated with the \( i \)-th covariant differential of the curvature tensors of classical connections
\[
\nabla^i R[\Lambda] : C^\infty (\text{C} M) \rightarrow C^\infty (W_i M).
\]
The map \( \mathcal{R}_{C,i} \) is said to be the **formal curvature map of order** \( i \) of classical connections.

Let \( C_{C,i} \subset W_i \) be a subset given by identities of the \( i \)-th covariant differentials of the curvature tensors of classical connections, i.e., by covariant differentials of the Bianchi identities and the antisymmetrization of the second order covariant differentials, see Remark 2.9. So \( C_{C,i} \) is given by the following system of equations
\[
w_{\nu (\rho \lambda \mu) \sigma_1 ... \sigma_i} = 0,
\]
\[
w_{\nu (\lambda \mu \sigma)}_{\sigma_2 ... \sigma_i} = 0,
\]
\[
w_{\nu (\lambda \mu \sigma_1 ...) \sigma_j} + \text{pol}(W^{(i-2)}) = 0,
\]
where \( j = 2, ..., i \) and \([..]\) denotes the antisymmetrization.

Let us put \( C^{(r)}_C = C_{C,0} \times ... \times C_{C,r} \) and denote by \( C^{(k,r)}_{C, k=1} \), \( k \leq r \), the fiber in \( C^{(k-1)}_C \) of the canonical projection \( \text{pr}_r^{k-1} : C^{(r)}_C \rightarrow C^{(k-1)}_C \). For \( r < k \) we put \( C^{(k,r)}_{C, r=1} = \emptyset \). Let us note that there is an affine structure on the fibres of the projection \( \text{pr}_r^{k-1} : C^{(r)}_C \rightarrow C^{(r-1)}_C \).
Really, $C_C^{(r)}$ is a subbundle in $C_C^{(r-1)} \times W_r$ given by the solution (for $i = r$) of the system of nonhomogeneous equations (3.31) – (3.33).

Then we put

$$R_C^{(k,r)} := (\mathcal{R}_{C,k}, \ldots, \mathcal{R}_{C,r}) : T_{m+1}^{r+1} Q \to W^{(k,r)}, \quad \mathcal{R}_C^{(r)} := \mathcal{R}_C^{(0,r)},$$

which has values, for any $j_0^{r+1} \gamma \in T_{m+1}^{r+1} Q$, in $C_C^{(k,r)} \mathcal{R}_{C,\mathcal{R}_{C,0^{(k+r)}}}$. In [5] it was proved that $C_C^{(r)}$ is a submanifold in $W^{(r)}$ and the restriction of $\mathcal{R}_C^{(r)}$ to $C_C^{(r)}$ is a surjective submersion. Then we can consider the fiber product $T_m^k Q \times C(C^{(k-1)} C_C^{(r)}$ which will be denoted by $T_m^k Q \times C_C^{(k,r)}$. In [5] it was proved that the mapping

$$(\pi_k^{r+1}, \mathcal{R}_C^{(k,r)} ) : T_{m+1}^{r+1} Q \to T_m^k Q \times C^{(k,r)}$$

is a surjective submersion.

Similarly let $U_0 E := U E = E^* \otimes E \otimes \bigwedge^2 T^* M$, $U_r E = U E \otimes \bigwedge^2 T^* M$, $i \geq 0$, $U^{(k,r)} E = U_k E \times \ldots \times U_r E$. Especially, $U^{(r)} E := (0^{(r)}) E$. Then $U_r E$ and $U^{(k,r)} E$ are $G$-bundle of order $(1, 0)$ and the corresponding standard fibers will be denoted by $U_i$ and $U^{(k,r)}$, where $U_0 := U = \mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \bigwedge^2 \mathbb{R}^{m*}$, $U_1 = U \otimes \bigwedge^2 \mathbb{R}^{m*}$, $i \geq 0$, and $U^{(k,r)} = U_k \times \ldots \times U_r$. Let us denote by $(u_j^{ \lambda \mu \sigma_1 \ldots \sigma_i})$ the coordinates on $U_i$.

We denote by

$$R_{L,i} : T_{m+1}^{i-1} Q \times T_{m+1}^i \to U_i$$

the $W^{(i+2)}_m G$-equivariant map associated with the $i$-th covariant differential of the curvature tensors of linear connections

$$\nabla^i R[K] : C^\infty (\text{Cl} M \times \text{Lin} E) \to C^\infty (U_i E).$$

The map $R_{L,i}$ is said to be the formal curvature map of order $i$ of general linear connections.

Let $C_{L,i} \subset U_i$ be a subset given by identities of the $i$-th covariant differentials of the curvature tensors of linear connections, i.e., by covariant differentials of the Bianchi identity and the antisymmetrization of the second order covariant differentials, see Theorem [2.0] and Remark [2.8]. So $C_{L,i}$ is given by the following system of equations

$$(3.5) \quad u_j^{ \lambda \mu \sigma_1 \ldots \sigma_i} = 0,$$

$$(3.6) \quad u_j^{ \lambda \mu \sigma_1 \ldots \sigma_j \ldots \sigma_i} \in \mathcal{P}(C^{(i-2)}_C \times U^{(i-2)}),$$

$j = 2, \ldots, i$, where $\mathcal{P}(C^{(i-2)}_C \times U^{(i-2)})$ are some polynomials on $C^{(i-2)}_C \times U^{(i-2)}$.

Let us put $C_L^{(r)} = C_{L,0} \times \ldots \times C_{L,r}$ and denote by $C^{(k,r)}_{L,r}$, $k \leq r$, the fiber in $r_{L_{C^{(k-1)}}} \subset C_{L_{C^{(k-1)}}}$ of the canonical projection $r_{L_{C^{(k-1)}}} : C^{(r)}_L \to C^{(k-1)}_L$. For $r < k$ we put $C^{(k,r)}_{L,r} = \emptyset$. Let us note that there is an affine structure on the projection $r_{C^{(r-1)}} : C^{(r)}_L \to C^{(r-1)}_L$. Really, $C^{(r)}_L$ is a subbundle in $C^{(r-1)}_L \times U_r$ given as the solution (for $i = r$) of the system of nonhomogeneous equations (3.3) – (4.0).

Then we set

$$R_L^{(k,r)} := (\mathcal{R}_{L,k}, \ldots, \mathcal{R}_{L,r}) : T_{m+1}^{r+1} Q \times T_{m+1}^i \to U^{(k,r)}, \quad \mathcal{R}_L^{(r)} := \mathcal{R}_L^{(0,r)}.$$
which has values, for any \((j^r_{0-1} \lambda, j^{r+1}_0 \gamma) \in T^{r-1}_m Q \times T^{r+1}_m R\), in \(C^{(k,r)}_{L,R^{(k-1)}_L(j^{k-2}_0 \lambda, j^k_0 \gamma)}\).

In [4] it was proved that \(C^{(s)}_C \times C^{(r)}_L, s \geq r - 2, r \geq 0\), is a submanifold of \(W^{(s)} \times U^{(r)}\) and the restriction

\[
(R^{(s)}_C, R^{(r)}_L) : T^{s+1}_m Q \times T^{r+1}_m R \rightarrow C^{(s)}_C \times C^{(r)}_L
\]

is a surjective submersion. Then we can consider the fiber product

\[
(T^{k_1}_m Q \times T^{k_2}_m R)_{\text{C}(k_1-1)_{C}(k_2-1)} (C^{(s)}_C \times C^{(r)}_L),
\]

\(k_1 \geq k_2 - 2\), and denote it by \(T^{k_1}_m Q \times T^{k_2}_m R \times C^{(k_1,s)}_C \times C^{(k_2,r)}_L\).

Now we shall prove the technical

**Lemma 3.1.** If \(s \geq r - 2, k_1 \geq k_2 - 2, s + 1 \geq k_1, r + 1 \geq k_2\), then the restricted map

\[
(R^{(s)}_C, R^{(r)}_L) : T^{s+1}_m Q \times T^{r+1}_m R \rightarrow T^{k_1}_m Q \times T^{k_2}_m R \times C^{(k_1,s)}_C \times C^{(k_2,r)}_L
\]

is a surjective submersion.

**Proof.** In [5] it was proved that

\[
(R^{(s)}_C, R^{(r)}_L) : T^{s+1}_m Q \rightarrow T^{k_1}_m Q \times C^{(k_1,s)}_C
\]

is a surjective submersion. The mapping of Lemma 5.1 is then a surjective submersion if and only if the mapping \((\pi^{r+1}_k, R^{(k_2,r)}_L(j^{s+1}_0 \lambda, -)) : T^{r+1}_m R \rightarrow T^{k_2}_m R \times C^{(k_2,r)}_L\) is a surjective submersion for any \(j^{s+1}_0 \lambda \in T^{s+1}_m Q\). Let us assume \(i = k_2, \ldots, r\). By [4] the mapping \(R^{(i)}_L(j^{s+1}_0 \lambda, -) : T^{i+1}_m R \rightarrow C^{(i)}_L\) is a surjective submersion and we have the commutative diagram

\[
\begin{array}{ccc}
T^{i+1}_m R & \xrightarrow{\pi^{(i)}_L(j^{s+1}_0 \lambda, -)} & C^{(i)}_L \\
\pi^{i+1}_i \downarrow & & \downarrow \text{pr}^{i-1}_i \\
T^{i}_m R & \xrightarrow{\pi^{(i-1)}_L(j^{s+1}_0 \lambda, -)} & C^{(i-1)}_L
\end{array}
\]

All morphisms in the above diagram are surjective submersions which implies that the mapping \((\pi^{i+1}_i, R^{(i)}_L(j^{s+1}_0 \lambda, -)) : T^{i+1}_m R \rightarrow T^{i}_m R \times C^{(i)}_L\) is a surjection over \(R^{(i-1)}_L(j^{s+1}_0 \lambda, -)\) given by \((\pi^{i+1}_i, R^{(i)}_L(j^{s+1}_0 \lambda, -))\). But the mapping \(R^{(i)}_L(j^{s+1}_0 \lambda, -)\) is affine morphisms over \(R^{(i-1)}_L(j^{s+1}_0 \lambda, -)\) (with respect to the affine structures on \(\pi^{i+1}_i : T^{i+1}_m R \rightarrow T^{i}_m R\) and \(\text{pr}^{i-1}_i : C^{(i)}_L \rightarrow C^{(i-1)}_L\) which has a constant rank. So the surjective morphism \((\pi^{i+1}_i, R^{(i)}_L(j^{s+1}_0 \lambda, -))\) has a constant rank and hence is a submersion. \((\pi^{k_2+1}_k, R^{(k_2,r)}_L(j^{s+1}_0 \lambda, -))\) is then a composition of surjective submersions

\[
(\pi^{k_2+1}_k, R^{(k_2,r)}_L(j^{s+1}_0 \lambda, -), \text{id}_{C^{(k_2+1,r)}_L}) \circ \ldots \\
\ldots \circ (\pi^{r-1}_r, R^{(r)}_L(j^{s+1}_0 \lambda, -), \text{id}_{C^{(r)}_L}) \circ (\pi^{r+1}_r, R^{(r)}_L(j^{s+1}_0 \lambda, -))
\]

Let \(F\) be a \(G\)-gauge-natural bundle of order \(k\), i.e., \(S_F\) is a \(W^{(k,k)}_m G\)-manifold.
Theorem 3.2. Let $s \geq r - 2$, $r + 1, s + 2 \geq k \geq 1$. For every $W_{m(s+2,r+1)}^s$ $G$-equivariant map $f : T_m^s Q \times T_m^r R \to S_F$ there exists a unique $W_{m}^{(k,k)} G$-equivariant map $g : T_m^{k-2} Q \times T_m^{k-1} R \times C_C^{(k-2,s-1)} \times C_L^{(k-1,r-1)} \to S_F$ satisfying

$$ f = g \circ \left( \pi_{k-2}^s \times \pi_{k-1}^r, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)} \right). $$

Proof. Let us consider the space $S_{C,s} := \mathbb{R}^{m} \otimes S^s \mathbb{R}^{m*}$ or $S_{L,r} := \mathbb{R}^{m*} \otimes \mathbb{R}^n \otimes S^n \mathbb{R}^{m*}$ with coordinates $(s^j_{\mu_1 \mu_2 \ldots \mu_s})$ or $(s^i_{j_{\mu_1 \ldots \mu_r}})$, respectively. Let us consider the action of $G_m s$ on $S_{C,s}$ and the action of $W_{m}^{(r,c)} G$ on $S_{L,r}$ given by

$$ (3.7) \quad s^j_{\mu_1 \mu_2 \ldots \mu_r} = s^j_{\mu_1 \mu_2 \ldots \mu_s} - \tilde{a}^j_{\mu_1 \ldots \mu_s}, \quad s^i_{j_{\mu_1 \ldots \mu_r}} = s^i_{j_{\mu_1 \ldots \mu_r}} - \tilde{a}^i_{j_{\mu_1 \ldots \mu_r}}. $$

From (2.1), (2.3) and (3.7) it is easy to see that the symmetrization maps

$$ \sigma_{C,s} : T_m^s Q \to S_{C,s+2}^s, \quad \sigma_{L,r} : T_m^r R \to S_{L,r+1} $$

given by

$$ (s^r_{\mu_1 \mu_2 \ldots \mu_{r+1}}) \circ \sigma_{C,s} = \Lambda^r_{\mu_1 \mu_2 \mu_3 \ldots \mu_{r+2}}, \quad (s^i_{j_{\mu_1 \ldots \mu_{r+1}}}) \circ \sigma_{L,r} = \Lambda^i_{j \mu_1 \mu_2 \ldots \mu_{r+1}} $$

determine equivariant maps.

We have the $G_{m}^{s+2}$-equivariant map

$$ \varphi_{C,s} := \left( \sigma_{C,s}, \pi_{s-1}^s, \mathcal{R}_{C,s-1} \right) : T_m^s Q \to S_{C,s+2}^s \times T_m^{s-1} Q \times \mathcal{W}_{s-1}. $$

On the other hand we define the $G_{m}^{s+2}$-equivariant map

$$ \psi_{C,s} : S_{C,s+2}^s \times T_m^{s-1} Q \times \mathcal{W}_{s-1} \to T_m^s Q $$

by the following coordinate expression

$$ (3.8) \quad \Lambda^r_{\mu \nu_1 \ldots \nu_s} = s^r_{\mu \nu_1 \ldots \nu_s} + \text{lin}(w_{\mu \nu_1 \ldots \nu_s} - \text{pol}(T_m^{s-1} Q)), $$

where $\text{lin}$ denotes the linear combination with real coefficients which arises in the following way. We recall that $\mathcal{R}_{C,s-1}$ gives the coordinate expression

$$ (3.9) \quad \Lambda^r_{\mu \nu_1 \ldots \nu_s} = \Lambda^r_{\rho_1 \nu_2 \ldots \nu_s} = w_{\mu \nu_1 \ldots \nu_s} - \text{pol}(T_m^{s-1} Q). $$

We can write

$$ \Lambda^r_{\mu \nu_1 \ldots \nu_s} = s^r_{\mu \nu_1 \ldots \nu_s} + \left( \Lambda^r_{\mu \nu_1 \ldots \nu_s} - \Lambda^r_{\mu \nu_1 \ldots \nu_s} \right). $$

Then the term in brackets can be written as a linear combination of terms of the type

$$ \Lambda^r_{\mu \nu_1 \ldots \nu_s} = \lambda^r_{\rho_1 \nu_2 \ldots \nu_s - \lambda^r_{\mu \nu_1 \ldots \nu_s}}. $$

We have the identity

$$ \psi_{C,s} \circ \varphi_{C,s} = \text{id}_{T_m^s Q}. $$
Similarly we have the $W^{(r+1,r+1)}_m G$-equivariant map

$$\varphi_{L,r} := (\sigma_{L,r}, \text{id}_{T_{m}^{-2}}Q \times \pi_{r-1}^{r}, \mathcal{R}_{L,r-1})$$

$$: T_{m}^{r-2}Q \times T_{m}^{r}R \rightarrow S_{L,r+1} \times T_{m}^{r-2}Q \times T_{m}^{r-1}R \times \mathcal{U}_{r-1}$$

and we define the $W^{(r+1,r+1)}_m G$-equivariant map

$$\psi_{L,r} : S_{L,r+1} \times T_{m}^{r-2}Q \times T_{m}^{r-1}R \times \mathcal{U}_{r-1} \rightarrow T_{m}^{r-2}Q \times T_{m}^{r}R$$

over the identity of $T_{m}^{r-2}Q \times T_{m}^{r-1}R$ by the following coordinate expression

$$(3.10) \quad K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} = s_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} + \text{lin}(u_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} - \text{pol}(T_{m}^{r-2}Q \times T_{m}^{r-1}R)),$$

where $\text{lin}$ denotes the linear combination with real coefficients which arises in the following way. We recall that $\mathcal{R}_{L,r-1}$ gives the coordinate expression

$$(3.11) \quad K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} - K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} = u_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} - \text{pol}(T_{m}^{r-2}Q \times T_{m}^{r-1}R).$$

We can write

$$K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} = s_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} + (K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}} - K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r}}).$$

Then the term in brackets can be written as a linear combination of terms of the type

$$K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r-1} \rho_{r+1} \ldots \rho_{r}} - K_{j}^{i} \lambda_{\rho_{1} \ldots \rho_{r-1} \rho_{r+1} \ldots \rho_{r}};$$

$i = 1, \ldots, r$, and from (3.11) we get (3.10).

Moreover,

$$\psi_{L,r} \circ \varphi_{L,r} = \text{id}_{T_{m}^{r-2}Q \times T_{m}^{r}R}.$$

Now we have to distinguish three possibilities.

A) Let $s = r - 1$. We have the same orders of groups $G_{m}^{r+1}$ and $W^{(r+1,r+1)}_m G$ acting on $T_{m}^{r-1}Q$ and $T_{m}^{r}R$.

Let us denote by

$$A^{r} := T_{m}^{r-2}Q \times T_{m}^{r-1}R \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1}.$$

Then the map $f \circ (\psi_{C,r-1}, \psi_{L,r}) : S_{C,r+1} \times S_{L,r+1} \times A^{r} \rightarrow S_{F}$ satisfies the conditions of the orbit reduction Theorem for the group epimorphism $\pi^{r+1,r+1}_{T} : W^{(r+1,r+1)}_m G \rightarrow W^{(r,r)}_m G$ and the surjective submersion $\pi_{3} : S_{C,r+1} \times S_{L,r+1} \times A^{r} \rightarrow A^{r}$. Indeed, the space $S_{C,r+1} \times S_{L,r+1}$ is a $B^{r+1,r+1}_{r,r} G$-orbit. Moreover, (3.7) implies that the action of $B^{r+1,r+1}_{r,r} G$ on $S_{C,r+1} \times S_{L,r+1}$ is simply transitive. Hence there exists a unique $W^{(r,r)}_m G$-equivariant map

$$g_{r} : A^{r} = T_{m}^{r-2}Q \times T_{m}^{r-1}R \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \rightarrow S_{F}$$

such that the following diagram

$$\begin{array}{ccc}
S_{C,r+1} \times S_{L,r+1} \times A^{r} & \xrightarrow{\psi_{C,r-1}, \psi_{L,r}} & T_{m}^{r-1}Q \times T_{m}^{r}R \xrightarrow{f} S_{F} \\
\downarrow \text{pr}_{3} & & \downarrow \text{id}_{S_{F}} \\
A^{r} & \xrightarrow{\text{id}_{A^{r}}} & A^{r} \xrightarrow{g_{r}} S_{F}
\end{array}$$
commutes. So \( f \circ (\varphi_{C,r-1}, \psi_{L,r}) = g_r \circ \text{pr}_3 \) and if we compose both sides with \((\varphi_{C,r-1}, \varphi_{L,r})\), by considering \( \text{pr}_3 \circ (\varphi_{C,r-1}, \varphi_{L,r}) = (\pi^{-1}_{r-2} \times \pi^{-1}_r, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-1}) \), we obtain

\[
    f = g_r \circ (\pi^{-1}_{r-2} \times \pi^{-1}_r, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-1})
\]

In the second step we consider the same construction for the map \( g_r \) and obtain the commutative diagram

\[
\begin{array}{ccc}
S_{C,r} \times S_{L,r} \times A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} & \xrightarrow{(\psi_{C,r-2}, \psi_{L,r-1}, \text{id}_{\mathcal{U}_{r-2} \times \mathcal{U}_{r-1}})} & A^r \\
\downarrow \text{pr}_{3,4,5} & \quad & \downarrow \text{id}_{SF} \\
A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} & \xrightarrow{\text{id}_{A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1}}} & A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \\
\end{array}
\]

So that there exists a unique \( W_m^{(r-1, r-1)} \mathcal{G}\)-equivariant map \( g_{r-1} : A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \to SF \) such that \( g_r = g_{r-1} \circ (\pi^{-1}_{r-3} \times \pi^{-1}_{r-2}, \mathcal{R}_{C,r-3}, \mathcal{R}_{L,r-2}, \text{id}_{\mathcal{W}_{r-2} \times \mathcal{U}_{r-1}}) \), i.e.

\[
    f = g_{r-1} \circ (\pi^{-1}_{r-3} \times \pi^{-1}_{r-2}, \mathcal{R}_{C,r-3}, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-2}, \mathcal{R}_{L,r-1})
\]

Proceeding in this way we get in the last step a unique \( W_m^{(k,k)} \mathcal{G}\)-equivariant map

\[
    g_k : T_m^{k-2}Q \times T_m^{k-1}R \times \mathcal{W}(k-2, r-2) \times \mathcal{U}(k-1, r-1) \to SF
\]

such that

\[
    f = g_k \circ (\pi^{-1}_{k-2} \times \pi^{-1}_{k-1}, \mathcal{R}_{C}^{(k-2, r-2)}, \mathcal{R}_{L}^{(k-1, r-1)}).\]

B) Let \( s = r - 2 \). We have the action of the group \( \mathcal{G}_m \) on \( T_m^{r-2}Q \) and the action of the group \( W_m^{(r+1, r+1)} \mathcal{G} \) on \( T_m^{r}R \).

Then the map \( f \circ (\text{id}_{T_m^{r-2}Q}, \psi_{L,r}) : S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to SF \) satisfies the conditions of the orbit reduction theorem \[1.3\] for the group epimorphism \( \pi_{r+1}^{r+1} : W_m^{(r+1, r+1)} \mathcal{G} \to W_m^{(r,r)} \mathcal{G} \) and the surjective submersion \( \text{pr}_{2,3,4} : S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \). Indeed, the space \( S_{L,r+1} \) is a \( B^{r+1,r+1} \mathcal{G}\)-orbit. Let us note that the action of \( B^{r+1,r+1} \mathcal{G} \) on \( S_{L,r+1} \) is transitive, but not simple transitive. Hence there exists a unique \( W_m^{(r,r)} \mathcal{G}\)-equivariant map \( g_r : T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to SF \) such that the following diagram

\[
\begin{array}{ccc}
S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} & \xrightarrow{(\text{id}_{T_m^{r-2}Q}, \psi_{L,r})} & T_m^{r-2}Q \times T_m^{r-1}R \\
\downarrow \text{pr}_{2,3,4} & \quad & \downarrow \text{id}_{SF} \\
T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} & \xrightarrow{\text{id}_{T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1}}} & T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \\
\end{array}
\]

commutes. So \( f \circ (\text{id}_{T_m^{r-2}Q}, \psi_{L,r}) = g_r \circ \text{pr}_{2,3,4} \) and if we compose both sides with \( (\text{id}_{T_m^{r-2}Q}, \varphi_{L,r}) \), by considering \( \text{pr}_{2,3,4} \circ (\text{id}_{T_m^{r-2}Q}, \varphi_{L,r}) = (\text{id}_{T_m^{r-2}Q} \times \pi^{-1}_{r-1}, \mathcal{R}_{L,r-1}) \), we obtain

\[
    f = g_r \circ (\text{id}_{T_m^{r-2}Q} \times \pi^{-1}_{r}, \mathcal{R}_{L,r-1})
\]

Further we proceed as in the second step in A) and we get a unique \( W_m^{(k,k)} \mathcal{G}\)-equivariant map

\[
    g_k : T_m^{k-2}Q \times T_m^{k-1}R \times \mathcal{W}(k-2, r-3) \times \mathcal{U}(k-1, r-1) \to SF
\]
such that
\[ f = g_k \circ (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,r-3)}, \mathcal{R}_L^{(k-1,r-1)}) . \]

C) Let \( s > r - 1 \). We have the action of the group \( W_m^{(s+2,r+1)} G \) on \( T_m^s Q \times T_m^r R \).

By [5] there exists a \( W_m^{(r+1,r+1)} G \)-equivariant mapping
\[ g_{r+1} : T_m^{r-1} Q \times T_m^r R \times \mathcal{W}^{(r-1,s-1)} \rightarrow S_F \]
such that
\[ f = g_{r+1} \circ (\pi_{r-1} \times \text{id}_{T_m^r R}, \mathcal{R}_C^{(r-1,s-1)}) . \]

\( g_{r+1} \) is then the mapping satisfying the condition A), i.e. there is a unique \( W_m^{(k,k)} G \)-equivariant map
\[ g_k : T_m^{k-2} Q \times T_m^{k-1} R \times \mathcal{W}^{(k-2,r-2)} \times \mathcal{U}^{(k-1,r-1)} \rightarrow S_F \]
such that
\[ g_{r+1} = g_k \circ (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,r-2)}, \mathcal{R}_L^{(k-1,r-1)}) , \]
i.e.,
\[ f = g_k \circ (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)}) , \]
summarizing all cases we have
\[ f = g_k \circ (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)}) \]
for any \( s \geq r - 2 \) and the restriction of \( g_k \) to \( T_m^{r-2} Q \times T_m^{r-1} R \times C_C^{(k-2,s-1)} \times C_L^{(k-1,r-1)} \) is uniquely determined map \( g \) we wished to find.

In the above Theorem 3.2 we have found a map \( g \) which factorizes \( f \), but we did not prove, that
\( (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)}) : T_m^s Q \times T_m^r R \rightarrow T_m^{k-2} Q \times T_m^{k-1} R \times C_C^{(k-2,s-1)} \times C_L^{(k-1,r-1)} \)
satisfy the orbit conditions, namely we did not prove that
\( (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)})^{-1} (j_0^{k-2\gamma}, j_0^{k-1\gamma}, r_C^{(k-2,s-1)}, r_L^{(k-1,r-1)}) \)
is a \( B_{k,k}^{s+2+r+1} G \)-orbit for any \( (j_0^{k-2\lambda}, j_0^{k-1\gamma}, r_C^{(k-2,s-1)}, r_L^{(k-1,r-1)}) \) \( \in T_m^{k-2} Q \times T_m^{k-1} R \times C_C^{(k-2,s-1)} \times C_L^{(k-1,r-1)} \). Now we shall prove it.

**Lemma 3.3.** If \( (j_0^{k-2\lambda}, j_0^{k-1\gamma}, j_0^{k-2\gamma}) \in T_m^s Q \times T_m^r R \) satisfy
\( (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)}) (j_0^{k-2\lambda}, j_0^{k-1\gamma}) = (\pi_{k-2} \times \pi_{k-1}, \mathcal{R}_C^{(k-2,s-1)}, \mathcal{R}_L^{(k-1,r-1)}) (j_0^{k-2\gamma}, j_0^{k-1\gamma}) \),
then there is an element \( h \in B_{k,k}^{s+2+r+1} G \) such that \( h . (j_0^{k-2\lambda}, j_0^{k-1\gamma}) = (j_0^{k-2\gamma}, j_0^{k-1\gamma}) \).

**Proof.** Consider the orbit set \( T_m^s Q \times T_m^r R / B_{k,k}^{s+2+r+1} G \). This is a \( W_m^{(k,k)} G \)-set. Clearly the factor projection
\( p : T_m^s Q \times T_m^r R \rightarrow (T_m^s Q \times T_m^r R) / B_{k,k}^{s+2+r+1} G \)
is a \( W_m^{(s+2,r+1)} G \)-map. By Theorem 3.2 there is a \( W_m^{(k,k)} G \)-equivariant map
\[ g : T_m^{k-2} Q \times T_m^{k-1} R \times C_C^{(k-2,s-1)} \times C_L^{(k-1,r-1)} \rightarrow (T_m^s Q \times T_m^r R) / B_{k,k}^{s+2+r+1} G \]
satisfying \( p = g \circ (\pi^s_{k-2} \times \pi^r_{k-1}) \), \( R_C^{(k-2,s-1)} \), \( R_L^{(k-1,r-1)} \). If
\[
(\pi^s_{k-2} \times \pi^r_{k-1}) \circ (j^0_\lambda, j^0_\gamma) = (\pi^s_{k-2} \times \pi^r_{k-1}) \circ (j^s_\lambda, j^r_\gamma) = (j^0_\lambda, j^r_\gamma) = (j^0_\lambda, j^0_\gamma)
\]
then
\[
p(j^0_\lambda, j^0_\gamma) = g(j^0_\lambda, j^0_\gamma) = (j^0_\lambda, j^r_\gamma) \quad \text{and}
\]
i.e. \((j^0_\lambda, j^0_\gamma), (j^0_\lambda, j^0_\gamma)\) are in the same \( B_{k,k}^{s+2,r+1} \)-orbit, which proves Lemma 3.3 \( \Box \)

The space \( T^m \times T^m R \times C^{(k-2,s-1)} \) is a left \( W^{(k,k)}_M \)-space corresponding to the \( G \)-gauge-natural bundle \( J^{k-2} \text{Cl}_M \times J^{k-1} \text{Lin} E \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L \) \( \text{E} \). Setting \( \nabla^{(k,s)} = (\nabla^k, \ldots, \nabla^s) \), then, as a direct consequence of Theorem 3.3, we obtain the \( k \)-th order first reduction theorem for linear and classical connections.

**Theorem 3.4.** Let \( s \geq r - 2, r + 1, s + 2 \geq k \geq 1 \). Let \( F \) be a \( G \)-gauge-natural bundle of order \( k \). All natural differential operators
\[
f : C^\infty(M \times \text{Lin} E) \to C^\infty(F \text{E})
\]
which are of order \( s \) with respect to classical connections and of order \( r \) with respect to linear connections are of the form
\[
f(j^s_\Lambda, j^r_K) = g(j^s_\Lambda, j^k_K, \nabla^{(k-2,s-1)} \text{R}[\Lambda], \nabla^{(k-1,r-1)} \text{R}[K])
\]
where \( g \) is a unique natural operator
\[
g : J^{k-2} \text{Cl}_M \times J^{k-1} \text{Lin} E \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L \text{E} \to F \text{E}
\]

**Remark 3.5.** From the proof of Theorem 3.3 it follows that the operator \( g \) is the restriction of a zero order operator defined on the \( k \)-th order \( G \)-gauge-natural bundle \( J^{k-2} \text{Cl}_M \times J^{k-1} \text{Lin} E \times W^{(k-2,s-1)}_M \times \text{U}^{(k-1,r-1)}_E \).

4. THE SECOND \( k \)-TH ORDER REDUCTION THEOREM FOR LINEAR AND CLASSICAL CONNECTIONS

Write \( (E^{p_1,q_2})_i := E^{p_1,q_2} \otimes \otimes^i T^* M, \ i \geq 0 \), and set
\[
(E^{p_1,q_2})^{(k,r)}_M := (E^{p_1,q_2})_k \times \ldots \times (E^{p_1,q_2})_r, \quad (E^{p_1,q_2})^{(r)} := (E^{p_1,q_2})^{(0,r)}.
\]
The \( i \)-th order covariant differential of sections of \( E^{p_1,q_2} \) with respect to \( (\Lambda, K) \) is a natural operator
\[
\nabla^i : C^\infty(M \times \text{Lin} E \times E^{p_1,q_2}) \to C^\infty((E^{p_1,q_2})_i)
\]
which is of order \((i-1)\) with respect to classical and linear connections and of order \( i \) with respect to sections of \( E^{p_1,q_2} \). Let us note that \( E^{p_1,q_2} \) is a \((1,0)\)-order \( G \)-gauge-natural bundle and let us denote by \( V := \otimes^n R^n \otimes \otimes^q R^q \otimes \otimes^r R^r \) \( \text{standard fiber with coordinates} (v^A) = \left(v_{j_1 \ldots j_n q_1 \ldots q_k}^A \right). \) By \( V_i \) \( V_0 \) \( V \) \( V(r) \) respectively, we denote the standard fibers of \((E^{p_1,q_2})_i \) \( (E^{p_1,q_2})^{(k,r)} \), respectively.
Hence we have the associated $W^{(i+1,j+1)}_m G$-equivariant map, denoted by the same symbol,

$$\nabla^i : T^{i-1}_m Q \times T^{i-1}_m R \times T^{i-1}_m V \to V_i.$$ 

If $(v^A, v^A_{\lambda_1}, \ldots, v^A_{\lambda_i})$ are the induced jet coordinates on $T^i_m V$ (symmetric in all subscripts) and $(V^A_{\lambda_1, \ldots, \lambda_i})$ are the canonical coordinates on $V_i$, then $\nabla^i$ is of the form

$$(4.1) \quad (V^A_{\lambda_1, \ldots, \lambda_i}) \circ \nabla^i = v^A_{\lambda_1, \ldots, \lambda_i} + \text{pol}(T^{i-1}_m Q \times T^{i-1}_m R \times T^{i-1}_m V),$$

where pol is a polynomial on $T^{i-1}_m Q \times T^{i-1}_m R \times T^{i-1}_m V$.

We define the $k$-th order formal Ricci equations, $k \geq 2$, as follows. For $k = 2$ we have by Remark 2.8

$$(E_2) \quad V^A_{[\lambda \mu]} - \text{pol}(C^{(0)}_C \times C^{(0)}_L \times V) = 0.$$ 

For $k > 2$, $(E_k)$ is obtained by the formal covariant differentiating of $(E_2) - (E_{k-1})$ and antisymmetrization of the last two formal covariant differentials. They are of the form

$$(E_k) \quad V^A_{\lambda_1 \cdots [\lambda_i, \lambda_{i+1}, \ldots, \lambda_k]} - \text{pol}(C^{(k-2)}_C \times C^{(k-2)}_L \times V^{(k-2)}) = 0,$$

$i = 1, \ldots, k - 1$.

**Definition 4.1.** The $k$-th order formal Ricci subspace $Z^{(k)} \subset C^{(k-2)}_C \times C^{(k-2)}_L \times V^{(k)}$ is defined by equations $(E_2), \ldots, (E_k), k \geq 2$. For $k = 0, 1$ we set $Z^{(0)} = V$ and $Z^{(1)} = V^{(1)}$.

In [4] it was proved that $Z^{(k)}$ is a submanifold of $C^{(k-2)}_C \times C^{(k-2)}_L \times V^{(k)}$ and the restricted morphism

$$(\mathcal{R}^{(k-2)}_C, \mathcal{R}^{(k-2)}_L, \nabla^{(k)} : T^{k-1}_m Q \times T^{k-1}_m R \times T^{k}_m V \to Z^{(k)}$$

is a surjective submersion. Let us consider the projection $\text{pr}_r^r : Z^{(r)} \to Z^{(k)}$. We have an affine structure on fibres of the projection $\text{pr}_{r-1}^r : Z^{(r)} \to Z^{(r-1)}$. It follows from the fact that $Z^{(r)}$ is a subbundle in $Z^{(r-1)} \times (C^{(r-2)}_{C,r-2} \times C^{(r-2)}_{L,r-2} \times V_r)$ given as the space of solutions of the system of nonhomogeneous equations $(E_r)$. Let us denote by $Z^{(k-1)}_{z^{(k-1)}}$ the fiber in $z^{(k-1)} \in Z^{(k-1)}$ of the projection $\text{pr}_{k-1}^r : Z^{(r)} \to Z^{(k-1)}$. Then we can consider the fiber product over $Z^{(k-1)}$

$$T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k)}_{Z^{(k-1)}}$$

and denote it by

$$T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)}.$$

**Lemma 4.2.** If $r + 1 \geq k \geq 1$, then the restricted morphism

$$(\pi^{r-1}_{k-2} \times \pi^{r-1}_{k-2} \times \pi^{r}_{k-1}) \times (\mathcal{R}^{(k-2, r-2)}_C, \mathcal{R}^{(k-2, r-2)}_L, \nabla^{(k,r)} : T^{r-1}_m Q \times T^{r-1}_m R \times T^{r}_m V \to T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)}$$

is a surjective submersion.
Proof. The proof of Lemma 4.2 follows from the commutative diagram
\[
\begin{array}{ccc}
T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V & \overset{(\mathcal{R}^{(r-2)}_C, \mathcal{R}^{(r-2)}_L, \nabla^{(r)})}{\longrightarrow} & Z^{(r)} \\
\pi^{r-1}_k \times \pi^{r-1}_k \times \pi^{r}_k \downarrow & & \downarrow \text{pr}^{r-1}_k \\
T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V & \overset{(\mathcal{R}^{(k-3)}_C, \mathcal{R}^{(k-3)}_L, \nabla^{(k-1)})}{\longrightarrow} & Z^{(k-1)}
\end{array}
\]
where all morphisms are surjective submersions. Hence
\begin{equation}
(4.2) \quad (\pi^{r-1}_k \times \pi^{r-1}_k \times \pi^{r}_k) \times (\mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-2,r-2)}_L, \nabla^{(k,r)})
\end{equation}
is surjective. For \(k = r\) the map \((\mathcal{R}^{(r-2,r-2)}_C, \mathcal{R}^{(r-2,r-2)}_L) = \mathcal{R}_{C,r-2}, \mathcal{R}^{(r-2,r-2)}_L = \mathcal{R}_{L,r-2}, \nabla^{(r,r)} = \nabla^{r}\) is affinely morphism over \((\mathcal{R}^{(r-3)}_C, \mathcal{R}^{(r-3)}_L, \nabla^{(r-1)})\) with constant rank, i.e. \((\pi^{r-1}_2 \times \pi^{r-1}_2 \times \pi^{r}_2) \times (\mathcal{R}^{(r-2,r-2)}_C, \mathcal{R}^{(r-2,r-2)}_L, \nabla^{r})\) is a submersion. The mapping \((4.2)\) is then a composition of surjective submersions. \(\square\)

**Theorem 4.3.** Let \(S_F\) be a left \(W^{(k)}_m G\)-manifold. For every \(W^{(r+1,r+1)}_m G\)-equivariant map \(f : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to S_F\) there exists a unique \(W^{(k)}_r G\)-equivariant map \(g : T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \to S_F\) such that
\[f = g \circ (\pi^{r-1}_k \times \pi^{r-1}_k \times \pi^{r}_k, \mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-2,r-2)}_L, \nabla^{(k,r)})\]

Proof. Consider the map
\[(\text{id}_{T^{r-1}_m Q} \times \text{id}_{T^{r-1}_m R} \times \pi^{r}_k, \nabla^{(k,r)}) : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \times V^{(k,r)} \]
and denote by \(\widetilde{V}^{(k,r)} \subset T^{r-1}_m Q \times T^{r-1}_m R \times T^{k-1}_m V \times V^{(k,r)}\) its image. By \((4.1)\), the restricted morphism
\[
\widetilde{\nabla}^{(k,r)} : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to \widetilde{V}^{(k,r)}
\]
is bijective for every \((j_0^{r-1}, j_0^{r-1}) \in T^{r-1}_m Q \times T^{r-1}_m R\), so that \(\widetilde{\nabla}^{(k,r)}\) is an equivariant diffeomorphism. Define
\[
(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L) : \widetilde{V}^{(k,r)} \to T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)}
\]
by
\[
(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)(j_0^{r-1}, j_0^{r-1}) \mu, v) = (j_0^{r-2}, j_0^{r-2}, j_0^{r-1}, j_0^{r-1}, \mu, v,
(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)(j_0^{r-1}, j_0^{r-1}) \mu, v) \in \widetilde{V}^{(k,r)}.
\]
By Lemma 3.1 \((\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)\) is a surjective submersion.

Thus, Lemma 3.1 and Lemma 3.3 imply that \((\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)\) satisfies the orbit conditions for the group epimorphism \(\pi^{r+1,r+1}_k : W^{(r+1,r+1)}_m G \to W^{(k)}_m G\) and there exists a
unique $W_m^{(k,k)} G$-equivariant map $g : T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V \times Z^{(k,r)} \to S_F$ such that the diagram

$$
\begin{array}{ccc}
\widetilde{V}(k,r) & \xrightarrow{(\nabla(k,r))^{-1}} & T_m^{r-1}Q \times T_m^{r-1}R \times T_m^r V \xrightarrow{f} S_F \\
(\widetilde{R}_C^{(k-2,r-2)}, \widetilde{R}_L^{(k-2,r-2)}) & \xrightarrow{(\pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r, \nabla(k,r))} & T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V \times Z^{(k,r)} \xrightarrow{g} S_F \\
\end{array}
$$

commutes. Hence $f \circ (\nabla(k,r))^{-1} = g \circ (\widetilde{R}_C^{(k-2,r-2)}, \widetilde{R}_L^{(k-2,r-2)})$. Composing both sides with $\widetilde{\nabla}(k,r)$, by considering

$$(\widetilde{R}_C^{(k-2,r-2)}, \widetilde{R}_L^{(k-2,r-2)}) \circ \widetilde{\nabla}(k,r) = (\pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r, \nabla(k,r)),
$$

we get

$$f = g \circ (\pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r, \nabla(k,r)).$$

$T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V \times Z^{(k,r)}$ is closed with respect to the action of the group $W_m^{(k,k)} G$. The corresponding natural bundle is $J^{k-2} \text{Cl}a M \times J^{k-2} \text{Lin} E \times J^{k-1} E_{q_1,q_2}^{p_1,p_2} \times Z^{(k,r)} M$. Then the second $k$-order reduction theorem can be formulated as follows.

**Theorem 4.4.** Let $F$ be a $G$-gauge-natural bundle of order $k \geq 1$ and let $r + 1 \geq k$. All natural differential operators $f : C^\infty (\text{Cl}a M \times \text{Lin} E \times E_{q_1,q_2}^{p_1,p_2}) \to C^\infty (F \text{Lin} E)$ of order $r$ with respect sections of $E_{q_1,q_2}^{p_1,p_2}$ are of the form

$$f(j^{r-1} \Lambda, j^{r-1} K, j^{r} \Phi) = g(j^{k-2} \Lambda, j^{k-2} K, j^{k-1} \Phi, \nabla(k-2,r-2) R[\Lambda], \nabla(k-2,r-2) R[K], \nabla(k,r) \Phi)$$

where $g$ is a unique natural operator

$$g : J^{k-2} \text{Cl}a M \times J^{k-2} \text{Lin} E \times J^{k-1} E_{q_1,q_2}^{p_1,p_2} \times Z^{(k,r)} E \to F \text{Lin} E.$$

**Remark 4.5.** The order $(r - 1)$ of the above operators with respect to linear and classical connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operators of orders $s_1$ or $s_2$ with respect to connections $\Lambda$ or $K$, respectively, where $s_1 \geq s_2 - 2, s_1, s_2 \geq r - 1$. Then

$$f(j^{s_1} \Lambda, j^{s_2} K, j^{r} \Phi) = g(j^{k-2} \Lambda, j^{k-2} K, j^{k-1} \Phi, \nabla(k-2,s_1-1) R[\Lambda], \nabla(k-2,s_2-1) R[K], \nabla(k,r) \Phi).$$

**Remark 4.6.** It is easy to see that the second reduction theorem can be generalized for any number of fields $\Phi, i = 1, \ldots, m$, of order $(1,0)$ and that any finite order operator

$$f(j^{s_1} \Lambda, j^{s_2} K, j^{r} \Phi), \quad s_1, s_2 \geq \max(r_i) - 1, s_1 \geq s_2 - 2,$$

factorizes through $j^{k-2} \Lambda, j^{k-2} K, j^{k-1} \Phi$ and sufficiently high covariant differentials of $R[\Lambda], R[K], \Phi$. 

References

[1] D. E. Eck: *Gauge-natural bundles and generalized gauge theories*, Mem. Amer. Math. Soc. 33 No. 247 (1981).

[2] J. Janýška: *Natural and gauge-natural operators on the space of linear connections on a vector bundle*, in: Diff. Geom. and Its Appl., Proc. Conf. Brno 1989, World Scientific, Singapore, 1990, 58–68.

[3] J. Janýška: *On the curvature of tensor product connections and covariant differentials*, to appear in the Proc. of the 23rd Winter School Geometry and Physics, Srní (Czech Republic) 2003.

[4] J. Janýška: *Reduction theorems for general linear connections*, Diff. Geom. and its Appl. 20 (2004) 177–196.

[5] J. Janýška: *Higher order reduction theorems for classical connections and natural (0,2)-tensor fields on the cotangent bundle*, preprint 2004, arXiv: math.DG/0405218.

[6] I. Kolár, P. W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer–Verlag 1993.

[7] D. Krupka, J. Janýška: Lectures on Differential Invariants, Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis, Brno, 1990.

[8] G. Lubczonok: *On reduction theorems*, Ann. Polon. Math. 26 (1972) 125–133.

[9] A. Nijenhuis: *Natural bundles and their general properties*, Diff. Geom., in honour of K. Yano, Kinokuniya, Tokyo 1972, 317–334.

[10] J. A. Schouten: *Ricci calculus*, Berlin-Göttingen, 1954.

[11] C. L. Terng: *Natural vector bundles and natural differential operators*, Am. J. Math. 100 (1978) 775–828.

[12] T. Y. Thomas: *The replacement theorem and related questions in the projective geometry of paths*, Ann. Math. 28 (1927) 549–561.

[13] T. Y. Thomas, A. D. Michal: *Differential invariants of affinely connected manifolds*, Ann. Math. 28 (1927) 196–236.

Department of Mathematics, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail: janyska@math.muni.cz