Asymptotics of the quantization errors for Markov-type measures with complete overlaps

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Abstract
Let $G$ be a directed graph with vertices $1, 2, \ldots, 2N$. Let $\mathcal{T} = (T_{i,j})_{(i,j) \in G}$ be a family of contractive similitudes. For every $1 \leq i \leq N$, let $i^\# := i + N$. For $1 \leq i, j \leq N$, we define $M_{i,j} = \{(i, j), (i, j^\#), (i^\#, j), (i^\#, j^\#)\} \cap G$. We assume that $T_{i,j} = T_{i,j}$ for every $(i, j) \in M_{i,j}$. Let $K$ denote the Mauldin-Williams fractal determined by $\mathcal{T}$. Let $\chi = (\chi_i)_{i=1}^{2N}$ be a positive probability vector and $P$ a row-stochastic matrix which serves as an incidence matrix for $G$. We denote by $\nu$ the Markov-type measure associated with $\chi$ and $P$. Let $\Omega = \{1, \ldots, 2N\}$ and $G_\infty = \{\sigma \in \Omega^\mathbb{N} : (\sigma_i, \sigma_{i+1}) \in G, i \geq 1\}$. Let $\pi$ be the natural projection from $G_\infty$ to $K$ and $\mu = \nu \circ \pi^{-1}$. We consider the following two cases: 1. $G$ has two strongly connected components consisting of $N$ vertices; 2. $G$ is strongly connected. With some assumptions for $G$ and $\mathcal{T}$, for case 1, we determine the exact value $s_r$ of the quantization dimension $D_r(\mu)$ for $\mu$ and establish a necessary and sufficient condition for the upper quantization coefficient for $\mu$ to be finite; for case 2, we determine $D_r(\mu)$ in terms of a pressure-like function and prove that $D_r(\mu)$-dimensional upper and lower quantization coefficient are always positive and finite.

1 Introduction
Let $\nu$ be a Borel probability measure on $\mathbb{R}^d$. The quantization problem for $\nu$ is concerned with the approximation of $\nu$ by discrete measures of finite support in $L_r$-metrics. This problem has a deep background in information theory (cf. [1, 10]). Rigorous mathematical foundations of quantization theory can be found in Graf and Luschgy’s book [4].

In the past decades, asymptotics of the quantization errors have been studied both for absolutely continuous distributions and for several classes of fractal measures, including self-similar measures, $F$-conformal measures, self-affine
measures, Markov-type measures and some in-homogeneous self-similar measures (cf. [5, 6, 8, 9, 13, 14, 15, 19, 20, 21, 23, 25]).

In the above-mentioned research on the quantization for fractal measures, certain kind of separation properties, such as the open set condition (OSC) or the strong separation condition, are required. Up to now, little is known about the asymptotic properties of the quantization errors for those measures associated with an IFS with overlaps.

In this paper, we study the asymptotics of the quantization errors for the image measures of Markov-type measures associated with a class of graph-directed IFS with complete overlaps. Our work is partially motivated by our previous study on a class of in-homogenous self-similar measures $\nu$ supported on self-similar sets (cf. [21, 24] and Example 2.7).

1.1 The quantization error and its asymptotics

Let $r \in (0, \infty)$ and $k \in \mathbb{N}$. Let $d$ denote the Euclidean metric on $\mathbb{R}^q$. For every $k \geq 1$, let $D_k := \{\alpha \subset \mathbb{R}^q : 1 \leq \text{card}(\alpha) \leq k\}$. For $x \in \mathbb{R}^q$ and $\emptyset \neq A, B \subset \mathbb{R}^q$, let $d(x, A) := \inf_{a \in A} d(x, a)$ and $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$. The $k$th quantization error for $\nu$ of order $r$ can be defined by

$$e_{k,r}^{r}(\nu) = \inf_{\alpha \in D_k} \int d(x, \alpha)^r d\nu(x). \quad (1.1)$$

One may see [4] for various equivalent definitions of the quantization error and interesting interpretations in different contexts. For $r \in [1, \infty)$, $e_{k,r}(\nu)$ is equal to the minimum error when approximating $\nu$ by discrete probability measures supported on at most $k$ points in the $L_r$-metrics.

The asymptotics of the quantization errors can be characterized by the upper (lower) quantization coefficient and the upper (lower) quantization dimension. For $s \in (0, \infty)$, the $s$-dimensional upper and lower quantization coefficient for $\nu$ of order $r$ can be defined by

$$Q_s^u(\nu) := \limsup_{k \to \infty} k^{-s} e_{k,r}^{r}(\nu), \quad Q_s^l(\nu) := \liminf_{k \to \infty} k^{-s} e_{k,r}^{r}(\nu).$$

The upper (lower) quantization dimension for $\nu$ of order $r$ is exactly the critical point at which the upper (lower) quantization coefficient jumps from zero to infinity (cf. [4, 18]), which are given by

$$\overline{D}_r(\nu) = \limsup_{k \to \infty} \frac{\log k}{-\log e_{k,r}(\nu)}; \quad \underline{D}_r(\nu) = \liminf_{k \to \infty} \frac{\log k}{-\log e_{k,r}(\nu)}.$$ 

If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, then we denote the common value by $D_r(\nu)$.

Next, we recall a classical result by Graf and Luschgy. Let $(f_i)_{i=1}^N$ be a family of contractive similarity mappings on $\mathbb{R}^q$ with contraction ratios $(c_i)_{i=1}^N$. According to [12], there exists a unique non-empty compact set $E$ satisfying the equation $E = \bigcup_{i=1}^N f_i(E)$. The set $E$ is called the self-similar set determined by
(f_i)^N_{i=1}. Given a probability vector (p_i)^N_{i=1}, there exists a unique Borel probability measure satisfying \( \nu = \sum^N_{i=1} p_i \nu \circ f_i^{-1} \). We call \( \nu \) the self-similar measure associated with \((f_i)^N_{i=1}\) and \((p_i)^N_{i=1}\). We say that \((f_i)^N_{i=1}\) satisfies the OSC if there exists a bounded non-empty open set \( U \) such that \( f_i(U), 1 \leq i \leq N, \) are pairwise disjoint and \( f_i(U) \subset U \) for all \( 1 \leq i \leq N \). Let \( \xi_r \) be implicitly defined by \( \sum^N_{i=1} (p_i c_i^r) \nu \bigcirc f_i = 0 \). Assuming the OSC for \((f_i)^N_{i=1}\), Graf and Luschgy proved that \( 0 < Q_r^\nu(\nu) \leq \mathcal{G}_r^\nu(\nu) < \infty \) (cf. \([5, 6]\)).

1.2 A class of Mauldin-Williams fractals

Mauldin-Williams (MW) fractals were introduced and studied in detail in \([17]\). Multi-fractal decompositions for such fractals were accomplished by Edgar and Mauldin (cf. \([2]\)). Next, we describe a class of MW-fractals with complete overlaps, which we will work with in the remainder of the paper.

Fix an integer \( N \geq 2 \). Let \( G \) be a directed graph with vertices \( 1, 2, \ldots, 2N \). We assume that there exists at most one directed edge from a vertex \( i \) to another vertex \( j \), and that there exists at least one edge leaving a vertex \( i \). Let \((c_{i,j})^N_{i,j=1}\) be an incidence matrix for \( G \). Thus, \( c_{i,j} \geq 0 \) for all \( 1 \leq i, j \leq 2N \), and \( c_{i,j} > 0 \) if and only if there exists one directed edge from the vertex \( i \) to \( j \). We write

\[
G_1 = \Omega := \{1, \ldots, 2N\}; \quad \Psi := \{1, \ldots, N\}; \quad \Psi^* := \bigcup_{n=1}^{\infty} \Psi^n; \quad \Omega^* := \bigcup_{n=1}^{\infty} \Omega^n.
\]

\[
G_k := \{(\sigma_1, \ldots, \sigma_k) \in \Omega^k : c_{\sigma_i,\sigma_{i+1}} > 0 \quad 1 \leq i \leq k-1\}; \quad k \geq 2;
\]

\[
G_\infty := \{(\sigma_1, \ldots, \sigma_k, \ldots) \in \Omega^\infty : c_{\sigma_i,\sigma_{i+1}} > 0, \quad i \geq 1\}; \quad G^* := \bigcup_{k=1}^{\infty} G_k.
\]

For \( i \in \Psi \), we define \( i^+ := i + N \) and for \((i, j) \in \Psi^2\), we define

\[
\mathcal{N}_{i,j} := \{(i, j), (i, j^+), (i^+, j), (i^+, j^+)\}; \quad \mathcal{M}_{i,j} := \mathcal{N}_{i,j} \cap G_2. \tag{1.2}
\]

Let \( J_i, i \in \Psi \), be non-empty, pairwise disjoint compact subsets of \( \mathbb{R}^q \) with \( \text{int}(J_i) = \overline{A}, i \in \Psi \), where \( \overline{A} \) and \( \text{int}(A) \) denote the closure and interior of a subset \( A \) of \( \mathbb{R}^q \). We define \( J_i^+ := J_i \) for \( i \in \Psi \). For every \( \sigma \in \Psi^* \cup \Psi^N \), we define

\[
\mathcal{T}(\sigma) := \left\{ \begin{array}{ll}
\{\tilde{\sigma} \in \Omega^* : \tilde{\sigma}_i = \sigma_i, \text{ or } \sigma_i^+, 1 \leq i \leq |\sigma|\} & \text{if } \sigma \in \Psi^* \\
\{\tilde{\sigma} \in \Omega^N : \tilde{\sigma}_i = \sigma_i, \text{ or } \sigma_i^+, i \geq 1\} & \text{if } \sigma \in \Psi^N.
\end{array} \right. \tag{1.3}
\]

We will use the following sets in the construction of MW-fractals with overlaps:

\[
S_1 := \Psi, \quad S_k = \{\sigma \in \Psi^k : \mathcal{T}(\sigma) \cap G_k \neq \emptyset\}, \quad k \geq 2; \tag{1.4}
\]

\[
S_\infty = \{\sigma \in \Psi^\infty : \mathcal{T}(\sigma) \cap G_\infty \neq \emptyset\}; \quad S^* = \bigcup_{k \geq 1} S_k.
\]

Let \( T_{i,j}, (i, j) \in S_2 \), be contractive similarity mappings on \( \mathbb{R}^q \) of similarity ratios \( s_{i,j}, (i, j) \in S_2 \). Note that for every \((i, j) \in G_2\), there exists a unique
$(i, j) \in S_2$ such that $(\tilde{i}, j) \in M_{i,j}$. We assume that
\[ T_{i,j} := T_{i,j}, \quad \text{for every } (\tilde{i}, j) \in M_{i,j}. \] (1.5)
Thus, we obtain an IFS $\mathcal{T} = \{T_{i,j} : (i, j) \in G_2\}$. $\mathcal{T}$ has complete overlaps if for some $(i, j) \in \Psi^2$, $\text{card}(M_{i,j}) \geq 2$; for example, if $(i, j), (i, j^+) \in G_2$, then
\[ J_{i,j^+} := T_{i,j^+}(J_{i,j}) = T_{i,j}(J_j) = J_{i,j}. \]

We just write $J_{i,j}$ for $J_{i,(j)}$. Now we further assume that, for every $i \in \Psi$, $\bigcup_{j : (i,j) \in S_2} T_{i,j}(J_j) \subseteq J_i$; $T_{i,j}(J_j), (i, j) \in S_2$, are pairwise disjoint. (1.6)

For each $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n \cup G_n$, we define
\[ T_\sigma = \left\{ \begin{array}{ll} id_{\mathbb{R}^d}, & \text{if } n = 1; \\ T_{\sigma_1,\sigma_2} \circ T_{\sigma_2,\sigma_3} \circ \cdots \circ T_{\sigma_{n-1},\sigma_n}, & \text{if } n > 1; \end{array} \right. \quad J_\sigma = T_\sigma(J_{\sigma_n}). \] (1.7)

For every $n \geq 1$, we call the sets $J_\sigma, \sigma \in S_n \cup G_n$, cylinders of order $n$. Then we obtain a MW-fractal $K$ ([4], Theorem 1]):
\[ K = \bigcap_{k \geq 1} \bigcup_{\sigma \in G_k} J_\sigma = \bigcap_{k \geq 1} \bigcup_{\sigma \in S_k} J_\sigma. \]

For and $\tilde{\sigma} \in G_n \cup S_n$, we define
\[ s_{\tilde{\sigma}} := \left\{ \begin{array}{ll} 1, & \text{if } n = 1; \\ s_{\tilde{\sigma}_1} s_{\tilde{\sigma}_2} \cdots s_{\tilde{\sigma}_{n-1}} & \text{if } n > 1. \end{array} \right. \]

Remark 1.1 In general, the sets $S_n, n \geq 1$, can be very complicated and the fractal set $K$ cannot be generated by a reduced graph-directed IFS (see Example 2.8). We will impose some conditions for the incidence matrix such that $S_n$ can be well tracked and $K$ remains a typical MW-fractal.

1.3 Markov-type measures with complete overlaps

Let $\theta$ denote the empty word. For every $\tilde{\sigma} \in G^* \cup S^*$, we denote by $|\tilde{\sigma}|$ the length of $\tilde{\sigma}$; we define $|\tilde{\sigma}| := \infty$ if $\tilde{\sigma} \in G_\infty \cup S_\infty$. For $\tilde{\sigma} \in G^* \cup G_\infty$, or $\tilde{\sigma} \in S^* \cup S_\infty$ and $1 \leq h \leq |\tilde{\sigma}|$, we write
\[ \tilde{\sigma}|_h := (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_h); \quad \tilde{\sigma}^h := \left\{ \begin{array}{ll} \theta, & \text{if } |\tilde{\sigma}| = 1; \\ \tilde{\sigma}|_{|\tilde{\sigma}|-1}, & \text{if } |\tilde{\sigma}| > 1. \end{array} \right. \]

For every $h \geq 1$, we call $\tilde{\sigma}$ a descendant of $\tilde{\sigma}|_h$, and $\tilde{\sigma}|_h$ a predecessor of $\tilde{\sigma}$.

Let $\pi : G_\infty \to K$ be defined by $\pi(\tilde{\sigma}) := \cap_{k=1}^\infty J_{\tilde{\sigma}|_k}$. We define
\[ \Gamma(\sigma) := G_n \cap T(\sigma), \quad \sigma \in S_n; \quad [\tilde{\sigma}] := \{\tilde{\tau} \in G_\infty : \tilde{\tau}|_n = \tilde{\sigma}\}, \quad \tilde{\sigma} \in G^*. \]
Then for every \( \overline{\sigma} \in \Gamma(\sigma) \), by (1.3), we have \( J_{\overline{\sigma}} = J_\sigma \). By (1.6), we have
\[
\pi^{-1}(J_\sigma) = \pi^{-1}(J_\sigma \cap K) = \bigcup_{\overline{\sigma} \in \Gamma(\sigma)} [\overline{\sigma}], \sigma \in S^*.
\]

Let \( (\chi_i)_{i=1}^{2N} \) be a positive probability vector. Let \( P = (p_{i,j})_{i,j=1}^{2N} \) be a row-stochastic matrix (transition matrix), with \( p_{i,j} > 0 \) if and only if \((i, j) \in G_2\). For every \( n \geq 1 \) and \( \overline{\sigma} \in G_n \), we define
\[
p_{\overline{\sigma}} := \begin{cases} 
1, & \text{if } n = 1 \\
\overline{p}_{\overline{\sigma}_1 \overline{\sigma}_2 \cdots \overline{p}_{\overline{\sigma}_{k-1} \overline{\sigma}_k} & \text{if } n > 1.
\end{cases}
\]
By Kolmogorov consistency theorem, there exists a unique Borel probability measure \( \nu \) on \( G_\infty \) such that for every \( k \geq 1 \) and \( \overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_k) \in G_k \),
\[
\nu([\overline{\sigma}]) = \chi_{\overline{\sigma}_1} p_{\overline{\sigma}} = \chi_{\overline{\sigma}_1} \overline{p}_{\overline{\sigma}_2} \cdots \overline{p}_{\overline{\sigma}_{k-1} \overline{\sigma}_k}.
\]
(1.8)
We define \( \mu = \nu \circ \pi^{-1} \). We call \( \mu \) a Markov-type measure with complete overlaps if \( \text{card}(M_{i,j}) \geq 2 \) for some \((i, j) \in S_2\). We have
\[
\mu(J_\sigma) = \nu \circ \pi^{-1}(J_\sigma) = \sum_{\overline{\sigma} \in \Gamma(\sigma)} \chi_{\overline{\sigma}} p_{\overline{\sigma}}, \sigma \in S^*.
\]
(1.9)

1.4 Statement of the main results

Let \( P \) be the \( 2N \times 2N \) transition matrix as above. We write
\[
P = \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} \quad \text{with} \quad P_1 = (p_{i,j})_{i,j=1}^{N}.
\]
Let \( 0 \) denote a zero matrix. In the present paper, we consider two cases:

**Case I:** \( P \) is reducible. We assume:
(A1) \( P_1, P_2 \) are both irreducible and \( P_3 = 0 \);
(A2) \( \text{card}(\{j \in \Psi : p_{i,j} > 0\}) \geq 2 \); \( i \in \Psi \);
(A3) for \((i, j) \in \Psi^2\), either \( M_{i,j} = \emptyset \) or \( M_{i,j} = \{(i, j), (i, j^+), (i^+, j^+)\} \).

**Case II:** \( P \) is irreducible. We will assume (A2) and (A4) for \((i, j) \in \Psi^2\), either \( M_{i,j} = \emptyset \) or \( M_{i,j} = N_{i,j} \) (see (1.2)).
(A5) \( P_1 \) is irreducible.

For every \( s \in (0, \infty) \), we define
\[
A_1(s) := ((p_{i,j} s_{i,j}^{s^+})_{i,j=1}^{N} ; \quad A_2(s) := ((p_{i,j} s_{i,j}^+)^s)_{i,j=1}^{N} ;
A_3(s) := ((p_{i,j} s_{i,j}^{s^+})_{i,j=1}^{N} ; \quad A_4(s) := ((p_{i,j} s_{i,j}^+)^s)_{i,j=1}^{N}.
\]
Let \( \psi(s) \) denote the spectral radius of \( A_i(s) \) and \( \rho_i(s) := \psi_i(s_{i,j}) \). Define
\[
A(s) = \begin{pmatrix} A_1(s) & A_3(s) \\ A_4(s) & A_2(s) \end{pmatrix}.
\]
Let \( \psi(s) \) denote the spectral radius of \( A(s) \) and define \( \rho(s) := \psi(s_{i,j}) \).
Remark 1.2 By [17, Theorem 2], $\psi_i(s), \psi(s)$ are continuous and strictly decreasing. Assuming (A2), we have $\psi_i(0) \geq 2$ and $\psi_i(1) < 1$ for $i = 1, 2$. Thus, there exists a unique $s \in (0, 1)$ such that $\psi_i(s) = 1$. Further, there exists a unique positive number $s_i$ such that $\rho_i(s_i) = 1$. Also, there exists a unique $s_r$ with $\rho(s_r) = 1$. When $P_4 = 0$, we have $s_r = \max\{s_{1,r}, s_{2,r}\}$.

Let $\pi_1 : S_\infty \rightarrow K$ be defined by $\pi_1(\sigma) := \bigcap_{k=1}^{\infty} J_\sigma|_k$. By (1.6), $\pi_1$ is a bijection. We say that $\mu$ is reducible if $\mu = \nu_1 \circ \pi_1^{-1}$ for the Markov-type measure $\nu_1$ associated with some transition matrix $\tilde{P}_{N \times N}$ and some initial probability vector $(\tilde{\chi}_i)_{i=1}^{N}$. Whenever $\mu$ is reducible, the asymptotics of $(e_{n,r}(\mu))_{n=1}^{\infty}$ is characterized by [13, Theorem 1.1].

For every $n \geq 1$ and $\sigma \in S_n(i)$, we split $\mu(J_\sigma)$ into two parts (see (1.9)):

$I_{1,\sigma} := \sum_{\tilde{\sigma} \in \Gamma(\sigma), \tilde{\sigma}_n = i} \chi_{\tilde{\sigma}} p_{\tilde{\sigma}}; \quad I_{2,\sigma} := \sum_{\tilde{\sigma} \in \Gamma(\sigma), \tilde{\sigma}_n = i^+} \chi_{\tilde{\sigma}} p_{\tilde{\sigma}}.$

As our second result, for Case I, we prove that $D_r(\mu) = s_r$ and establish a necessary and sufficient condition for $Q_r^{s_r}(\mu)$ and $\tilde{Q}_r^{s_r}(\mu)$ to be both positive and finite. That is,

Theorem 1.4 Assume that (A1)-(A3) hold. We have

(i) $D_r(\mu) = s_r, \quad Q_r^{s_r}(\mu) > 0$;

(ii) $\tilde{Q}_r^{s_r}(\mu) < \infty$ if and only if $s_{1,r} \neq s_{2,r}$;

(iii) there exists some $r_0 > 0$, such that $\tilde{Q}_r^{s_r}(\mu) < \infty$ for every $r \in (0, r_0)$. 

For the proof of Theorem 1.3, we will construct some auxiliary measures by applying some ideas of Mauldin and Williams [17]. These measures will allow us to estimate the quantization error for μ in a more accurate and concise way. It seems somewhat surprising that Theorem 1.3(i) and (ii) are shared by Markov-type measures in non-overlapping cases (see [13]). When P is irreducible, T is even more overlapped. Our third result shows that the measure μ exhibits quite different properties and Theorem 1.4 can fail. Assuming (A2), (A4) and (A5), we will show that there exists a unique positive number t_r satisfying

$$\lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma \in \mathcal{S}_n} \left( \sum_{\hat{\sigma} \in \Gamma(\sigma)} \chi_{\hat{\sigma}} p_{\hat{\sigma}} s_{\hat{\sigma}}^r \right)^{-1} = 0. \quad (1.10)$$

We will consider the following two cases which might help to illustrate Case II:

(g1) \( p_{i,j} + p_{i+1,j} = p_{i+1,j} + p_{i+1,j+1} \) for every \((i,j) \in \mathcal{S}_2;\)

(g2) \( p_{i,j} + p_{i+1,j} = p_{i,j+1} + p_{i+1,j+1} \) for every \((i,j) \in \mathcal{S}_2.\)

**Theorem 1.5** Assume that \((A2), (A4)\) and \((A5)\) hold. Then we have

(i) \( D_r(\mu) = \mathcal{O}_r(\mu) = t_r \leq s_r, \) and \( 0 < \mathcal{O}_r(\mu) \leq \mathcal{O}_r(\mu) < \infty;\)

(ii) if \((g1)\) or \((g2)\) holds, then we have \( t_r < s_r.\)

For the proof of Theorem 1.5 we will apply a Helley-type theorem (cf. [16, Theorem 1.23]) and some ideas contained in the proof of [13, Theorem 5.1] to construct some auxiliary measures. These measures are closely connected with the upper and lower quantization coefficient for μ and enable us to prove Theorem 1.3(i) in a convenient way.

## 2 Proof of Theorem 1.3 and some examples

Let \( \Psi = \{1, 2, \ldots, N\} \) as before. For each \( n \geq 1 \) and \( i \in \Psi, \) we write

\[
\begin{align*}
H_{1,n} & := G_n \cap \Psi^n; \quad H^* := G^* \cap \Psi^*; \quad H^\infty := G^\infty \cap \Psi^N; \\
H_{1,n}(i) & := \{ \sigma \in H_{1,n} : \sigma_1 = i \}; \quad H^*(i) := \{ \sigma \in H^* : \sigma_1 = i \}; \\
H^\infty(i) & := \{ \sigma \in H^\infty : \sigma_1 = i \}.
\end{align*}
\]

Let \( \mathcal{Y} := \{1^+, 2^+, \ldots, N^+\}. \) Let \( H_{2,n}, H^2, H^\infty, H_{2,n}(i^+), H^2(i^+), H^\infty(i^+) \) be defined in the same manner by replacing \( \Psi \) with \( \mathcal{Y}. \)

For \( \sigma, \tau \in G^* \) or \( \sigma, \tau \in \mathcal{S}^*, \) we denote by \( \sigma * \tau \) the concatenation of \( \sigma \) and \( \tau. \)

For every \( \sigma \in \Psi^*, \) we write \( \sigma^+ := (\sigma_1^+, \ldots, \sigma_1^{|\sigma|}). \) We define

\[
\mathcal{L}(\sigma) := \{ \sigma, \sigma^+ \} \cup \{ \sigma|_h * (\sigma_{h+1}^+, \sigma_{h+2}^+, \ldots, \sigma_n^+) : 1 \leq h \leq n - 1 \}; \quad \sigma \in \mathcal{S}_n.
\]

The subsequent two lemmas shows that with \((A1)-(A3),\) or \((A2)\) and \((A4),\)

The sets \( \Gamma(\sigma) \) and \( \mathcal{S}_n \) can be well tracked.
Lemma 2.1 Assume that (A1)-(A3) hold. We have
\[ \Gamma(\sigma) = \mathcal{L}(\sigma), \quad \sigma \in \mathcal{S}_n; \quad \mathcal{S}_n = H_{1,n}, \quad n \geq 1; \quad \mathcal{S}^* = H_1^*. \]

Proof. Let \( \sigma \in \mathcal{S}_n \) and \( \tilde{\sigma} \in \Gamma(\sigma) \) be given. By (A1), for every \( 1 \leq i \leq n-1 \), we have \( \nu_{\sigma_i^+,\sigma_{i+1}^+} = 0 \). Hence, if \( \tilde{\sigma}_{i+1} = \sigma_{i+1}^+ \) for some integer \( h \geq 0 \), then \( \tilde{\sigma}_i = \sigma_i^+ \) for all \( h+1 \leq i \leq n \). This implies that either \( \tilde{\sigma} = \sigma \) or \( \sigma^+ \), or for some \( 1 \leq h \leq n-1 \), \( \tilde{\sigma} = \sigma_{h}^+ \cdot (\sigma_{h+1}^+, \sigma_{h+2}^+, \ldots, \sigma_n^+) \). It follows that \( \Gamma(\sigma) \subseteq \mathcal{L}(\sigma) \).

For every \( \sigma \in \mathcal{S}_n \), by (1.3) and (A1), we have \( \sigma \in \mathcal{G}_n \), or \( \sigma^+ \in \mathcal{G}_n \), or for some \( 1 \leq h \leq n-1 \), we have \( \sigma_{h}^+ \cdot (\sigma_{h+1}^+, \sigma_{h+2}^+, \ldots, \sigma_n^+) \subseteq \mathcal{G}_n \). Using this and (A3), we obtain that \( \mathcal{L}(\sigma) \subseteq \mathcal{G}_n \). It follows that \( \mathcal{L}(\sigma) \subseteq \Gamma(\sigma) \). Thus the first part of the lemma holds. In particular, we have \( \sigma \in \mathcal{G}_n \), which implies that \( \mathcal{S}_n \subset H_{1,n} \). Since \( H_{1,n} \subset \mathcal{S}_n \), we obtain that \( \mathcal{S}_n = H_{1,n} \) and \( \mathcal{S}^* = H_1^* \).

Let \( \mathcal{T}(\sigma) \) be as defined in (1.3). We have

Lemma 2.2 Assume that (A2) and (A4) hold. Then
\[ \Gamma(\sigma) = \mathcal{T}(\sigma), \quad \sigma \in \mathcal{S}_n; \quad \mathcal{S}_n = H_{1,n}, \quad n \geq 1; \quad \mathcal{S}^* = H_1^*. \]

Proof. This can be proved analogously to the proof of Lemma 2.1.

Let \( n \geq 1 \) and \( i \in \Psi \), we write
\[ B_n(i) := \{ \sigma \in H_{1,n} : \sigma_n = i \}, \quad B_n(i^+) := \{ \sigma^+ \in H_{2,n} : \sigma_n^+ = i^+ \}. \]

We clearly have \( B_n(i) \subset \mathcal{S}_n(i) \). If (A1)-(A3) hold, or (A2) and (A4) hold, then by Lemmas 2.1 and 2.2 we have \( B_n(i) = \mathcal{S}_n(i) \).

Remark 2.3 Assume that (A5) holds. By induction, one can easily see that \( B_n(i) \neq \emptyset \) for every \( i \in \Psi \) and \( n \geq 1 \).

Proof of Theorem 1.3 (1)

For every \( (i,j) \in \mathcal{S}_2 \), we define
\[ \bar{\pi}_{i,j} := \frac{\mu(J_{i,j})}{\mu(J_i)} = \frac{\chi_i (p_{i,j} + p_{i,j}^+)}{\chi_i + \chi_i^+}. \quad (2.1) \]

Then we have \( \mu(J_{i,j}) = \mu(J_i) \cdot \bar{\pi}_{i,j} \) for every \( (i,j) \in \mathcal{S}_2 \).

\( \Rightarrow \) First we assume that \( \mu = \nu \circ \pi_1^{-1} \) for some markov-type measure \( \nu \).

Then \( \nu \) is the Markov-type measure associated with \( \bar{\nu} = \overline{(\bar{\pi}_{i,j})}_{i,j=1}^{N} \) and \( \bar{\chi} = (\chi_i + \chi_i^+)^{N}_{i=1} \). For every \( i \in \Psi \) and \( \sigma \in \mathcal{S}^*(i) \) and \( j \) with \( (i,j) \in \mathcal{S}_2 \), we have
\[ \Delta_{\sigma,j} := \mu(J_{\sigma,j}) - \mu(J_{\sigma}) \cdot \bar{\pi}_{i,j} = 0. \quad (2.2) \]

Note that \( I_{1,\sigma} + I_{2,\sigma} = \mu(J_\sigma) \). It follows that
\[ \Delta_{\sigma,j} = I_{1,\sigma}(p_{i,j} + p_{i,j}^+) + I_{2,\sigma}(p_{i+j} + p_{i+j}^+) - \mu(J_\sigma) \cdot \bar{\pi}_{i,j} = I_{1,\sigma}(p_{i,j} + p_{i,j}^+) + I_{2,\sigma}(p_{i+j} + p_{i+j}^+) - \bar{\pi}_{i,j} \]
\[ = \frac{\chi_i I_{1,\sigma} - \chi_i I_{2,\sigma}}{\chi_i + \chi_i^+}(p_{i,j} + p_{i,j}^+) = 0. \quad (2.3) \]

8
By (2.2) and (2.3), for every \( i \in \Psi \), we have the following two cases:

Case (1): for every \( j \) with \((i, j) \in S_2\), we have
\[
p_{i,j} + p_{i,j}^+ = p_{i,j}^+ + p_{i,j}^++.
\]
(2.4)

Case (2): there exists some \( j_0 \in \Psi \) with \((i, j_0) \in S_2\) such that (2.4) fails. Then by (2.2) and (2.3), we obtain that
\[
\chi_i + I_{1,\sigma} = \chi_i I_{2,\sigma}, \quad \text{for every } \sigma \in S^*(i),
\]
(2.5)

Otherwise, there would be some \( \sigma \in S^*(i) \) such that (2.2) fails for \( j_0 \). This contradicts the assumption that \( \mu \) is reducible.

\(\Rightarrow\) Assume that, for every \( i \in \Psi \), (a) or (b) holds. Then from (2.2), we know that (2.2) holds for every \( \sigma \in S^*(i) \) and every \( j \) with \((i, j) \in S_2\), which implies that \( \mu = \nu \circ \pi^{-1} \) for the Markov-type measure associated with \( P = (\bar{p}_{i,j})_{i,j=1}^N \) and \( \bar{\chi} = (\chi_i + \chi_i^+)_{i=1}^N \).

Next, we give sufficient conditions such that \( \mu \) is reducible or non-reducible.

**Corollary 2.4** Assume that (A2), (A4) and (A5) hold and that

(b1) \( \chi_i = \chi_i^+ \) for every \( i \in \Psi \);

(b2) for every \( i \in \Psi \), there exists some \( l \in \Psi \) such that \( p_{l,i} + p_{l,i}^+ \neq p_{l,i}^+ + p_{l,i}^+ \).

Then \( \mu \) is reducible if and only if \( (2.2) \) holds for every \((i, j) \in S_2\).

**Proof.** By (b1) and (b2), for \( \sigma = (l, i) \in S_2(i) \), we have
\[
\frac{I_{1,\sigma}}{I_{2,\sigma}} = \frac{\chi_l p_{l,i} + \chi_i p_{l,i}^+}{\chi_l p_{l,i}^++ \chi_i p_{l,i}^+} = \frac{p_{l,i} + p_{l,i}^+}{p_{l,i}^+ + p_{l,i}^+} \neq 1 = \frac{\chi_i}{\chi_i^+}.
\]
(2.6)
The corollary follows immediately from Theorem 1.3 (1).

**Corollary 2.5** Assume that (A2), (A4), (A5) and (b1) hold and that for every \((l, i) \in S_2\), we have \( p_{l,i} + p_{l,i}^+ \neq p_{l,i}^+ + p_{l,i}^+ \). Then \( \mu \) is reducible.

**Proof.** We first will show that

Claim 1: \( \chi_i^+ I_{1,\sigma} = \chi_i I_{2,\sigma} \) holds for every \( i \in \Psi \), every \( n \geq 1 \) and \( \sigma \in S_n(i) \).

By the hypothesis, we know that \( \chi_l = \chi_l^+ \) for all \( l \in \Psi \). Also, for \( n = 1 \), we have \( S_1(i) = \{i\} \). For \( \sigma = i \in S_1(i) \), we have \( I_{1,\sigma} = \chi_i = \chi_i^+ = I_{2,\sigma} \). For \( n = 2 \) and every \( \sigma = (l, i) \in S_2(i) \), by considering (2.6), we have \( I_{1,\sigma} = I_{2,\sigma} \). Thus, Claim 1 holds for every \( i \in \Psi \) and \( n = 1, 2 \), and every \( \sigma \in S_n(i) \).

Now we assume that Claim 1 holds for \( n = k \geq 2 \) and every \( i \in \Psi \) and \( \sigma \in S_k(i) \). Let \( n = k + 1, i \in \Psi \) and \( \sigma \in S_n(i) \). Let \( \tau := \sigma^\delta \). By Lemma 2.2
\[
\Gamma(\sigma) = \{\tau \ast i, \; \tau \ast i^+: \tau \in \Gamma(\tau)\}.
\]

Note that \( \tau \in S_k(\tau_k) \). By the inductive assumption, we have \( I_{1,\tau} = I_{2,\tau} \). Using the hypothesis of the corollary, we deduce
\[
I_{1,\sigma} = \sum_{\tau \in \Gamma(\tau)} \chi_{\tilde{\tau}} p_{\tau} \cdot p_{\tau_k,i} = I_{1,\tau} p_{\tau_k,i} + I_{2,\tau} p_{\tau_k,i}^+ = I_{1,\tau} (p_{\tau_k,i} + p_{\tau_k,i}^+) = I_{2,\tau}.
\]

9
By induction, Claim 1 holds. Thus, by Theorem 1.4 (1), \( \mu \) is reducible.

For two variables \( X, Y \) taking values in \((0, \infty)\), we write \( X \precsim Y \) (\( X \succeq Y \)), if there exists some constant \( C > 0 \), such that the inequality \( X \leq CY \) (\( X \geq CY \)) always holds. We write \( X \equiv Y \), if we have both \( X \precsim Y \) and \( X \succeq Y \). We define

\[
\chi := \min_{1 \leq i \leq 2N} \chi_i, \quad \overline{\chi} := \max_{1 \leq i \leq 2N} \chi_i.
\]

To complete the proof for Theorem 1.3 (2), we need one more lemma.

**Lemma 2.6** Assume that \((A1) - (A3)\) hold. For every \( i \in \Psi \), there exists some integer \( k_0 \) such that for every \( n \geq k_0 \) and some \( \sigma \in S_n(i) \) such that

\[
I_{1, \sigma} < \chi_i \chi_{i+1}^{-1} I_{2, \sigma}.
\]  

(2.7)

**Proof.** Let \( \rho_1, \rho_2 \) denote the spectral radius of \( P_1, P_2 \). We first show that, for every \( i \in \Psi \), we have

\[
\sum_{\sigma \in B_n(i)} p_\sigma \simeq \rho_1^{n-1}, \quad \sum_{\sigma^+ \in B_n(i^+)} p_{\sigma^+} \simeq 1.
\]  

(2.8)

By the assumption \((A3)\), we see that \( \sum_{j=1}^{N} p_{i,j} < 1 \) for every \( i \in \Psi \). By [11, Theorem 8.1.22], this implies that \( \rho_1 < 1 \). By \((A1)\), we have \( P_4 = 0 \). Thus, \( \sum_{j=1}^{N} p_{i,j} + = 1 \), for every \( i \in \Psi \), which implies that \( \rho_2 = 1 \). Let \( (c_{j,i})_{j=1}^{N} \) and \( (c_{j,i}^+)_{j=1}^{N} \) denote the \( i \)th column of \( P_1^{n-1} \) and \( P_2^{n-1} \) respectively. Then

\[
\sum_{\sigma \in B_n(i)} p_{\sigma} = \sum_{j=1}^{N} c_{j,i}; \quad \sum_{\sigma^+ \in B_n(i^+)} p_{\sigma^+} = \sum_{j=1}^{N} c_{j,i}^+.
\]

From \((A1)\), we know that \( P_1, P_2 \) are both non-negative and irreducible. Thus, \((2.8)\) is a consequence of Corollary 8.1.33 of [11].

Note that \( \rho_1^n \to 0 \) as \( n \to \infty \). By \((2.8)\), for all large \( n \), we have

\[
\sum_{\sigma \in B_n(i)} p_{\sigma} < \chi_i \chi_{i+1}^{-1} \chi \sum_{\sigma^+ \in B_n(i^+)} p_{\sigma^+},
\]  

(2.9)

By Lemma 2.1, for every \( \sigma \in B_n(i) \), we have

\[
\{ \overline{\sigma} \in \Gamma(\sigma) : \overline{\sigma}_n = i \} = \{ \sigma \}, \quad \{ \overline{\sigma} \in \Gamma(\sigma) : \overline{\sigma}_n = i^+ \} \supset \{ \sigma^+ \}. \quad \text{(2.10)}
\]

Combining \((2.9)\) and \((2.10)\), we deduce

\[
\sum_{\sigma \in B_n(i)} I_{1, \sigma} = \sum_{\sigma \in B_n(i)} \chi_{\sigma} p_{\sigma} < \chi_i \chi_{i+1}^{-1} \sum_{\sigma^+ \in B_n(i^+)} \chi_{\sigma^+}^{-1} \sum_{\sigma \in B_n(i)} I_{2, \sigma}.
\]

It follows that there exists some \( \sigma \in B_n(i) \) fulfilling \((2.7)\).

**Proof of Theorem 1.3 (2)**
We assume that (A1)-(A3) hold. By Lemma 2.6 for every \( i \in \Psi \), there exists some \( \sigma \in S^*(i) \) such that \( \chi_{i+1}I_{i+1,\sigma} < \chi_{i}I_{i,\sigma} \). Thus, by Theorem 1.3 (1), \( \mu \) is reducible if and only if \( (f_{i})_{i=1}^{N} \) holds for every \((i, j) \in \mathcal{S}_2 \). Note that by (A1), we have that \( p_{i+1,j} = 0 \) for every \((i, j) \in \mathcal{S}_2 \). Therefore, \( \mu \) is reducible if and only if \( p_{i+1,j} + p_{i+1,j^+} = p_{i+1,j^+} \) for every \((i, j) \in \mathcal{S}_2 \).

Next, we construct some examples to illustrate our results and assumptions. Our first example shows that for a suitable transition matrix \( P \) and some initial probability vector, the measure \( \mu \) coincides with the in-homogeneous self-similar measure that is studied in [21, 24].

**Example 2.7** Let \( f_i, 1 \leq i \leq N \), be contractive similarity mappings on \( \mathbb{R}^d \). Let \( E \) be the self-similar set determined by \( (f_i)_{i=1}^{N} \). Let \( (q_i)_{i=0}^{N} \) and \( (t_i)_{i=1}^{N} \) be two positive probability vectors. Let \( \nu_0 \) denote the self-similar measure associated with \( (f_i)_{i=1}^{N} \) and \( (t_i)_{i=1}^{N} \). We define

\[
P = \begin{pmatrix}
q_1 & \cdots & q_N & q_0t_1 & \cdots & q_0t_N \\
\vdots & & \vdots & \vdots & & \vdots \\
q_1 & \cdots & q_N & q_0t_1 & \cdots & q_0t_N \\
0 & \cdots & 0 & t_1 & \cdots & t_N \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & t_1 & \cdots & t_N
\end{pmatrix}
\]

\[
\chi = \begin{pmatrix}
q_1 \\
\vdots \\
q_N \\
q_0t_1 \\
\vdots \\
q_0t_N
\end{pmatrix}
\]

For \( 1 \leq i, j \leq N \), we define \( T_{i,j} = T_{i,j^+} = T_{i^+,j} = f_i \). Then the MW-fractal \( K \) agrees with \( E \). Let By Lemma 2.1 and (1.9), for \( \sigma \in \Psi^n \), one easily gets

\[
\mu(J_\sigma) = \prod_{h=1}^{n} q_{\sigma_h} + q_0 \prod_{h=1}^{n} t_{\sigma_h} + q_0 \sum_{h=1}^{n-1} \left( \prod_{l=1}^{h} q_{\sigma_l} \cdot \prod_{l=h+1}^{n} t_{\sigma_l} \right).
\]

Thus, \( \mu \) agrees with the in-homogeneous self-similar measure in [24]. That is, the unique probability measures satisfying \( \mu = q_0\nu_0 + \sum_{i=1}^{N} q_i \circ \mu \circ f_i^{-1}. \) As noted in [24] Remark 1.4, for fixed \( r > 0 \) and suitably selected \( P \), it indeed can happen that \( s_{1,r} > s_{2,r}, \) \( s_{1,r} < s_{2,r}, \) or \( s_{1,r} = s_{2,r} \).

Our second example shows that, if (A3) and (A4) are not satisfied, it can happen that \( \inf_{\sigma \in S^*} \frac{\mu(J_{\sigma})}{\mu(J_{\sigma})} = 0 \), regardless of whether \( P \) is reducible. In addition, it is possible that \( \sigma, \tau \in S^* \) and \( (\sigma_{[\sigma]}, \tau_1) \in S_2 \), but \( \sigma \circ \tau \notin S^* \). This might cause major difficulties in the estimation for the quantization errors.

**Example 2.8** For \( N = 3 \), we assume that (1.5) holds. We define

\[
P(1) = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}, \quad P(2) = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]
Note that $P(1)$ is reducible, while $P(2)$ is irreducible, since $1 \to 4 \to 6 \to 5 \to 1 \to 2 \to 3 \to 1$ forms a cycle in the graph $\mathcal{G}$. Let $\chi_1 = \ldots = \chi_6 = \frac{1}{6}$. For every $n \geq 1$, let $\sigma(n) = (1, 1, \ldots, 1) \in S_n$. Either $P = P(1)$ or $P = P(2)$, one can see that $P_1$ and $P_2$ are both irreducible, and $p_{1,1} > 0$ and $p_{1,5} = p_{4,2} = p_{4,5} = p_{4,1} = 0$, implying that $(1, 2) \in G_2$, but $(1, 5), (4, 2), (4, 5), (4, 1) \notin G_2$. Thus,

$$\Gamma(\sigma(n)) = \{(1, \ldots, 1), (1, 4, 4, \ldots, 4), (1, 1, 4, 4, \ldots, 4), \ldots, (4, 4, \ldots, 4)\},$$

but $\Gamma(\sigma(n) \ast 2) = \{\sigma(n) \ast 2\}$. As one can see, $\frac{\mu(J_{\chi(n)} \ast 2)}{\mu(J_{\chi(n)})} \to 0$ as $n \to \infty$. This also happens in case that $P = P(1)$, even if we add $(1, 5)$ to $G_2$ by adjusting the first row of $P$.

Now let $P = P(2)$ and $\sigma = (1), \tau = (2, 1)$. Since $(5, 1) \in G_2$, we have, $(2, 1) \in S_2$. However, $(1, 5), (2, 1), (2, 4), (4, 5) \notin G_2$. This implies that $(1, 2, 1) \notin S_3$.

Our third example shows that even if (g2) holds, the measure $\mu$ may not be reducible, when both (b1) and the condition in (g1) fail.

**Example 2.9** Let $N = 3$. Let $P$ and $\chi = (\chi_i)_{i=1}^6$ be defined by

$$P = \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix}.$$

By Lemma 2.2 one can see that

$$S_2 = H_2^2 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}.$$

For every $(l, i) \in S_2$, we have

$$p_{l,i} + p_{l+i} = p_{l,i} + p_{l+i}, \quad p_{2,3} + p_{2,6} \neq p_{5,3} + p_{5,6}, \quad \chi_2 \neq \chi_5.$$

Thus, both (g1) and (b1) fail, but the condition in (g2) is fulfilled. Next, we show that $\mu$ is not reducible. Let $\sigma = (1, 2, 3)$. The set $\Gamma(\sigma)$ is exactly given by

$$\{(1, 2, 3), (2, 2, 3), (1, 5, 3), (4, 5, 3), (1, 2, 6), (4, 2, 6), (1, 5, 6), (4, 5, 6)\}.$$

By (1.9) and (2.10), we easily get $\mu(J_\sigma) \neq \chi_1 \tilde{p}_{1,2} \tilde{p}_{2,3}$. Hence, $\mu$ is not reducible.

### 3 Some estimates for the quantization error for $\mu$

In this section, we assume that either (A1)-(A3) hold, or, (A2), (A4) and (A5) hold. For $\sigma, \tau \in S^\alpha \cup S_\infty$, We say that $\sigma$ is comparable with $\tau$ and write $\sigma \prec \tau$.
if $|\sigma| \leq \tau$ and $\sigma = \tau|_{|\sigma|}$. If we have neither $\sigma \prec \tau$ nor $\tau \prec \sigma$, then we say that $\sigma, \tau$ are incomparable. Write

$$p := \min_{(i,j) \in G_2} p_{i,j}; \quad \bar{p} := \max_{(i,j) \in G_2} p_{i,j}; \quad s := \min_{(i,j) \in G_2} s_{i,j}; \quad \bar{s} := \max_{(i,j) \in G_2} s_{i,j}.$$ 

**Remark 3.1** Let $|A|$ denote the diameter of a set $A \subset \mathbb{R}^q$. Without loss of generality, we assume that $|J_i| = 1$ for all $i \in \Psi$. Then using (1.5), we have

$$|J_\sigma| = s_\sigma, \quad \sigma \in \mathcal{S}^*; \quad s_\bar{\sigma} = s_\sigma, \text{ for all } \bar{\sigma} \in \Gamma(\sigma).$$

We define $E_\tau(\theta) := 1$. For every $\sigma \in \mathcal{S}^*$, we define

$$E_\tau(\sigma) := \mu(J_\sigma)s_\tau^\gamma = \sum_{\bar{\sigma} \in \Gamma(\sigma)} (\chi_{\bar{\sigma}}p_{\bar{\sigma}}s_\bar{\sigma}). \quad (3.1)$$

Using the following lemmas, we present some basic facts about the cylinder sets and the measure $\mu$, so that Lemma 3 of [14] is applicable.

**Lemma 3.2** There exist some constants $c_{1,r}, c_{2,r} \in (0,1)$ such that

$$c_{1,r}E_\tau(\sigma^\delta) \leq E_\tau(\sigma) \leq c_{2,r}E_\tau(\sigma^\delta), \quad \sigma \in \mathcal{S}^*.$$ 

**Proof.** Note that $N \geq 2$. By (A2) and (A3), or, (A2) and (A4), we have

$$\bar{c} := \max_{1 \leq i \leq N} \left(\chi_i + \chi_i^+\right) < 1; \quad \min_{(i,j) \in \mathcal{S}_2} \left(p_{i+,j} + p_{i+,j^+}\right) \geq \bar{p};$$

$$\bar{d} = \max_{(i,j) \in \mathcal{S}_2} \max\left\{(p_{i,j} + p_{i,j^+}), (p_{i+,j} + p_{i+,j^+})\right\} < 1.$$ 

For $n \geq 2$ and $\sigma \in \mathcal{S}_n$, we write $\sigma = \tau * j$. Then $\sigma^\delta = \tau$. By Lemma 2.1 and (1.10), $\mu(J_\sigma) = \chi_i + \chi_i^+$ for $i \in \Psi$; and for $\sigma \in \mathcal{S}_n$ with $n \geq 2$, we have

$$\mu(J_\sigma) = I_{1,\tau}(p_{\sigma_{n-1},j} + p_{\sigma_{n-1},j^+}) + I_{2,\tau}(p_{\sigma_{n-1}^+,j} + p_{\sigma_{n-1}^+,j^+}).$$

The lemma follows by defining $c_{1,r} := \min\{\bar{p}, 2\chi\}s_\tau^\gamma$ and $c_{2,r} := \max\{\bar{p}, \bar{d}\}s_\bar{\tau}^\gamma$.

**Lemma 3.3** Let $L \in \mathbb{N}$. There exist a constant $\delta > 0$ and a number $D_L > 0$ which is independent of $\sigma \in \mathcal{S}^*$ and $\alpha \subset \mathbb{R}^d$ such that

1. $d(J_\sigma, J_\tau) \geq \delta \max\{|J_\sigma|, |J_\tau|\}$ for incomparable words $\sigma, \tau \in \mathcal{S}^*$;
2. $\int_{J_\sigma} d(x, \alpha)^\tau d\mu(x) \geq D_L E_\tau(\sigma)$ for $\alpha \subset \mathbb{R}^q$ of cardinality $L$ and $\sigma \in \mathcal{S}^*$.

**Proof.** (c1) By (1.6) and our assumption for $J_i, i \in \Psi$, for some constant $\delta > 0$,

$$d(J_i, J_j) \geq \delta \max\{|J_i|, |J_j|\}, \quad 1 \leq i \neq j \leq N; \quad (3.2)$$

$$d(J_{i,j}, J_{i,l}) \geq \delta \max\{|J_{i,j}|, |J_{i,l}|\}, \quad j \neq l, \quad (i,j), (i,l) \in \mathcal{S}_2.$$ 

13
Let $\sigma, \tau \in S^*$ be incomparable words. Let $l = \min\{i \geq 1 : \sigma_i \neq \tau_i\}$. We have that $J_\sigma \subset J_{\sigma|i|$ and $J_\tau \subset J_{\tau|i|}$. If $l = 1$, then (c2) follows by (3.2). For $l \geq 2$, we write $\sigma|l = \rho \ast i$ and $\tau|l = \rho \ast j$, for some $1 \leq i \neq j \leq N$. Then

$$d(J_\sigma, J_\tau) \geq d(J_{\sigma|i|}, J_{\tau|i|}) \geq s_0 \delta \max\{|J_{\rho_{i-1} \ast i}|, |J_{\rho_{i-1} \ast j}|\} \geq \delta \max\{|J_\sigma|, |J_\tau|\}.$$  

(c2) Let $\sigma \in S^*$ be given. By (A2), one can see that

$$\text{card}(|\tau \in S^*: \sigma < \tau, |\tau| = |\sigma| + h|) \geq 2^h.$$  

Thus, (c2) can be obtained from the proof of [22, Lemma 4].

For every $k \geq 1$ and $s \in (0, \infty)$, we define

$$
\Lambda_{k,r} := \{\sigma \in S^*: \mathcal{E}_r(\sigma) < \mathcal{E}_r(\sigma^*)\}; \\
\phi_{k,r} := \text{card}(\Lambda_{k,r}), \ l_{1k} := \min_{\sigma \in \Lambda_{k,r}} |\sigma|; \ l_{2k} := \max_{\sigma \in \Lambda_{k,r}} |\sigma|; \\
\bar{P}_s^r(\mu) := \limsup_{k \to \infty} \phi_{k,r}^{e_{\phi_{k,r}}(r)}(\mu), \ \overline{P}_s^r(\mu) := \limsup_{k \to \infty} \phi_{\Lambda_{k,r}}^{e_{\phi_{k,r}}(r)}(\mu).
$$

**Remark 3.4** The following facts will be useful in the proof of our main result:

(d1) $l_{1k}, l_{2k} \asymp k$; This can be seen from the following facts:

$$c_1^{l_{1k}} < e_{1,r}^k, \text{ and } c_2^{l_{2k}-1} \geq e_{1,r}^k.$$

(d2) For $s \in (0, \infty)$, $Q_s^\sigma(\mu) > 0$ if and only if $\bar{P}_s^r(\mu) > 0$, and $\overline{P}_s^r(\mu) < \infty$ if and only if $\overline{P}_s^r(\mu) < \infty$ (cf. [23, Lemma 2.4]).

**Lemma 3.5** Assume that (A1)-(A3) hold, or, (A2), (A4), (A5) hold. Then

$$e_{\phi_{k,r},r}^r(\mu) \asymp \sum_{\sigma \in \Lambda_{k,r}} \mathcal{E}_r(\sigma).$$

**Proof.** This follows from (3.3), Lemma 3.2 and [14, Lemma 3].

For $s \in (0, \infty)$ and $k \geq 1$, we define

$$F_{k,r}^s(\mu) := \sum_{\sigma \in \Lambda_{k,r}} (\mathcal{E}_r(\sigma))^{\rho_{k,r}^{e_{\phi_{k,r},r}}}, \ F_s^r(\mu) = \liminf_{k \to \infty} F_{k,r}^s(\mu), \ \overline{F}_s^r(\mu) = \limsup_{k \to \infty} F_{k,r}^s(\mu).$$

Using techniques from [4, Proposition 14.5, 14.11], we are able to reduce the asymptotics of the quantization errors to those for the sequence $(F_{k,r}^s(\mu))_{k=1}^\infty$. That is,

**Lemma 3.6** For every $s > 0$, we have

1. $Q_s^\sigma(\mu) > 0$ if and only if $F_s^r(\mu) > 0$;
2. $\overline{Q}_s^r(\mu) < \infty$ if and only if $\overline{F}_s^r(\mu) < \infty$.  

1
Proof. Let $s > 0$ be given. We only show (1), and (2) can be proved analogously. Assume that $F^s(\mu) := \xi > 0$. Then there exists some $k_1 > 0$ such that for every $k \geq k_1$, we have $F^s_{k,r}(\mu) > \frac{\xi}{2}$. Using Lemma 3.3 and Hölder’s inequality with exponent less than one, we deduce

$$\phi^\tau_{k,r} e^{\phi_{k,r}}(\mu) \gtrsim \phi^\tau_{k,r} \sum_{\sigma \in \Lambda_{k,r}} \mathcal{E}_\tau(\sigma) \gtrsim \phi^\tau_{k,r} (F^s_{k,r}(\mu)) \frac{\xi}{2} \phi^\tau_{k,r} \gtrsim \frac{\xi}{2} \frac{\xi}{2} .$$

It follows that $P^s(\mu) > 0$. This and Remark 3.3 (d2) yield that $Q^s(\mu) > 0$.

Now we assume that $F^s(\mu) = 0$. Then for every $\epsilon \in (0,1)$, there exists a subsequence $(k_i)_{i=1}^\infty$ of positive integers, such that $F^s_{k_i,r}(\mu) < \epsilon$ for every $i \geq 1$.

By Lemma 3.2, for every $\sigma \in \Lambda_{k_i,r}$, we have $\mathcal{E}_\tau(\sigma) \geq c_{1,r}^{k_i+1}$. It follows that

$$\phi_{k_i,r}(\mathcal{E}_\tau(c_{1,r})^{(k_i+1)s/(s+r)}) \leq F^s_{k_i,r}(\mu) < \epsilon .$$

Using this, (3.3) and Lemma 3.3, we deduce

$$\phi^\tau_{k_i,r} e^{\phi_{k_i,r}}(\mu) \lesssim \phi^\tau_{k_i,r} \sum_{\sigma \in \Lambda_{k_i,r}} (\mathcal{E}_\tau(\sigma)) \frac{k_i}{r} c_{1,r}^{k_i/r} < c_{1,r}^{k_i/r} \epsilon .$$

It follows that $P^s(\mu) = 0$. By Remark 3.4 (d2), we conclude that $Q^s(\mu) = 0$.

4 Proof of Theorem 1.4

In this section, we assume that either (A1)-(A3) hold. A subset $\Gamma$ of $S^*$ is called a finite anti-chain if $\Gamma$ is finite and words in $\Gamma$ are pairwise incomparable. A finite anti-chain $\Gamma$ is called maximal if for every word $\tau \in S^\infty$, there exists some $\sigma \in \Gamma$ such that $\sigma \prec \tau$. We define a finite (maximal) anti-chain in $G^*$, $H^*_i$, $i = 1, 2$, or $H^*_i(j)$, $H^*_i(j^\dagger)$ with $j \in \Psi$ analogously. The following lemma provides us with a useful tool to estimate $F^s_{k,r}(\mu)$, which can be seen as a generalization of [13, Lemma 3.1].

Lemma 4.1 Let $\Gamma_1$ be an arbitrary finite maximal anti-chain in $H^*_1$, or in $H^*_2(j)$ for some $j \in \Psi$ and $\Gamma_2$ a finite maximal an-chain in $H^*_1$ or in $H^*_2(j^\dagger)$. Let $l(\Gamma_i) := \min_{\sigma \in \Gamma_i} |\sigma|$ and $L(\Gamma_i) := \max_{\sigma \in \Gamma_i} |\sigma|$, $i = 1, 2$. Then for $s > 0$, there exist positive numbers $c_5(s), c_6(s)$, which are independent of $\Gamma_i$, such that

$$\begin{cases} c_5(s) \rho_i(s)^{l(\Gamma_i)} \leq \sum_{\sigma \in \Gamma_i} (p_{\sigma} s_{\sigma}^r)^{\frac{k_i}{r}} \leq c_6(s) \rho_i(s)^{L(\Gamma_i)} & \text{if } s \leq s_{i,r} \\ c_5(s) \rho_i(s)^{L(\Gamma_i)} \leq \sum_{\sigma \in \Gamma_i} (p_{\sigma} s_{\sigma}^r)^{\frac{k_i}{r}} \leq c_6(s) \rho_i(s)^{l(\Gamma_i)} & \text{if } s > s_{i,r} \end{cases} . \quad (4.1)$$

Proof. It suffices to give the proof for $i = 1$. Note that the spectral radius of $\rho_1(s)^{-1} A_1(\frac{s}{s_{i,r}})$ equals 1. Since $A_1(\frac{s}{s_{i,r}})$ is nonnegative and irreducible, by
Perron-Frobenius theorem, there exists a unique positive normalized right eigenvector \((\xi_l)_{l=1}^N\) of \(\rho_1(s)^{-1}A_1(\frac{s}{s+r})\) with respect to 1:

\[
\sum_{j=1}^N \rho_1(s)^{-1}(p_{l,j}s_{l,j})^{\frac{s}{s+r}}\xi_j = \xi_l, \ 1 \leq l \leq N.
\]

For \(k \geq 2\) and \(\sigma \in H_1^k\), we define \(\nu_1([\sigma]) := \rho_1(s)^{-k}(p_\sigma s_\sigma^r)^{\frac{s}{s+r}}\xi_{\sigma_k}\). We have

\[
\sum_{j=1}^N \nu_1([\sigma * j]) = \rho_1(s)^{-k}(p_\sigma s_\sigma^r)^{\frac{s}{s+r}}\rho_1(s)^{-1}\sum_{j=1}^N (p_{\sigma h,j}s_{\sigma h,j}^r)^{\frac{s}{s+r}}\xi_j
\]

\[
= \rho_1(s)^{-k}(p_\sigma s_\sigma^r)^{\frac{s}{s+r}}\xi_{\sigma_k} = \nu([\sigma]).
\]

Thus, \(\nu_1\) extends a measure on \(H_1^\infty\). We distinguish the following two cases.

1. \(\Gamma_1\) is a finite maximal anti-chain in \(H_1^\ast\). We have

\[
\sum_{\sigma \in \Gamma_1} \nu_1(\Gamma_1) = \sum_{\sigma \in \Gamma_1} \rho_1(s)^{-|\sigma|}(p_\sigma s_\sigma^r)^{\frac{s}{s+r}}\xi_{\sigma[\sigma]} = \nu_1(H_1^\infty) = \rho_1(s)^{-1}. \quad (4.2)
\]

2. \(\Gamma_1\) is a finite maximal anti-chain in \(H_1^\ast(j)\) for some \(j \in \Psi\). We have

\[
\sum_{\sigma \in \Gamma_1} \nu_1(\Gamma_1) = \sum_{\sigma \in \Gamma_1} \rho_1(s)^{-|\sigma|}(p_\sigma s_\sigma^r)^{\frac{s}{s+r}}\xi_{\sigma[\sigma]} = \rho_1(s)^{-1}\xi_j. \quad (4.3)
\]

We define \(\xi := \min_{1 \leq l \leq N} \xi_l\) and \(\bar{\xi} := \max_{1 \leq l \leq N} \xi_l\). Then by (4.2), (4.3), one can see that (4.1) is fulfilled with \(c_5(s) := \bar{\xi}^{\mu_1} \xi_1 \rho_1(s)^{-1}\) and \(c_6(s) := \underline{\xi}^{-1} \rho_1(s)^{-1}\).

For the proof of Theorem 1.4, we define

\[
\Lambda_{k,r} := \bigcup_{\sigma \in \Lambda_{k,r}} \left\{ \sigma | h \ast (\sigma_{h+1}^+, \ldots, \sigma_{|\sigma|}^+) \mid 1 \leq h \leq |\sigma| - 1 \right\}; \quad (4.4)
\]

\[
ak_{k,r}(s) := \sum_{\sigma \in \Lambda_{k,r}} \sum_{h=1}^{|\sigma| - 1} (p_{\sigma|h\ast(\sigma_{h+1}^+, \ldots, \sigma_{|\sigma|}^+)^r})^{\frac{s}{s+r}}, \ k \geq 1.
\]

**Proof of Theorem 1.4 (i)**

By Lemma 2.1, \(\{\sigma, \sigma^+\} \subset \Gamma(\sigma)\). Thus,

\[
F_{k,r}^\ast(\mu) \geq \bar{\xi}^{-\mu} \max \left\{ \sum_{\sigma \in \Lambda_{k,r}} (p_{\sigma} s_\sigma^r)^{\frac{s}{s+r}}, \sum_{\sigma \in \Lambda_{k,r}} (p_{\sigma^+} s_{\sigma^+}^r)^{\frac{s}{s+r}} \right\}.
\]

Note that \(\Lambda_{k,r}\) is a finite maximal anti-chain in \(H_1^\ast\). By (A3), \(\{\sigma^+ : \sigma \in \Lambda_{k,r}\}\) is a maximal anti-chain in \(H_2^\ast\). By Lemma 1.1 one easily gets \(F_{k,r}^\ast(\mu) \geq 1\). This and Lemma 3.6 yield that \(Q_{k,r}^\ast(\mu) > 0\) and \(D_r(\mu) \geq s_r\). Next, we prove that \(D_r(\mu) \leq s_r\). For \(1 \leq h \leq l_{2k} - 1\), we define

\[
B(\omega) := \{ \tau^+ \in H_2^\ast : \omega * \tau^+ \in \Lambda_{k,r} \}, \ \omega \in H_1^h.
\]
For every \( \omega \in H_1^h \), either \( B(\omega) = \emptyset \) or \( B(\omega) \) is an anti-chain in \( H_2^r \). In fact, suppose that \( \tau^+, \rho^+ \in B(\omega) \) are distinct words with \( \tau^+ < \rho^+ \); by (1.4), we would have \( \omega * \tau, \omega * \rho \in \Lambda_{k,r} \) which are comparable, a contradiction. We have

\[
A_{k,r} \subset \bigcup_{h=1}^{l_{2k}-1} \bigcup_{\omega \in H_1^h} \{ \omega * \tau^+ : \tau^+ \in B(\omega) \}. \tag{4.5}
\]

For \( s > s_r \), we have, \( \rho_i(s) < 1 \) for \( i = 1, 2 \). By (4.5) and Lemma 4.1,

\[
F_{k,r}^s(\mu) \leq \sum_{\sigma \in \Lambda_{k,r}} (p_\sigma s_\sigma^r)^{r+s} + \sum_{\sigma \in \Lambda_{k,r}} (p_\sigma s_\sigma^r)^{r+s} + a_{k,r}(s) \tag{4.6}
\]

\[
\lesssim 2 + \sum_{h=1}^{l_{2k}-1} \sum_{\omega \in H_1^h} \sum_{\tau^+ \in B(\omega)} (p_\omega s_\omega^r)^{r+s} (p_\tau s_\tau^r)^{r+s} \lesssim 2 + \frac{\rho_1(s)}{1 - \rho_1(s)} \tag{4.7}
\]

This and Lemma 4.6 yield that \( \underline{Q}_r(\mu) < \infty \) and \( \overline{Q}_r(\mu) \leq s_r \). This completes the proof of Theorem 4.1 (i).

Next, we are going to prove Theorem 4.1 (ii). For \( \tau^+ \in H_2^r \), we define

\[ B(\tau^+) := \{ \omega \in H_1^r : \omega * \tau^+ \in A_{k,r} \} = \{ \omega \in H_1^r : \omega * \tau \in \Lambda_{k,r} \}. \]

Clearly, \( B(\tau^+) \) might be empty for some \( \tau^+ \in H_2^r \). Also, it can happen that two words \( \omega, \rho \in B(\tau^+) \) are comparable while \( \omega * \tau, \rho * \tau \) are incomparable. In fact, this happens if for some \( \nu \in S^* \),

\[ E_r(\omega * \nu * \tau^+) > E_r(\omega * \nu * \tau) \geq c_{1,r} E_r(\omega * \tau) > E_r(\omega * \nu * \tau) \]

The following Lemma 4.2 are devoted to the case that \( s_{1,r} > s_{2,r} \). Our next lemma will enable us to estimate the difference \( ||\omega| - |\nu|| \) for any two comparable words \( \omega, \rho \in B(\tau^+) \).

**Lemma 4.2** Let \( \omega \in H_{1,n}, \tau \in H_{1,m}, \nu \in H_{1,1} \) and \( \omega * \tau, \omega * \nu * \tau \in S^* \). Then

\[
E_r(\omega * \nu * \tau) \leq (l + 1)(p_{\nu}^{r'})^{l+1}(p_{\nu}^{r'})^{-1}E_r(\omega * \tau). \]

**Proof.** By Lemma 2.1 and (1.9), with \( \nu|_0 := \theta \), we have

\[
\mu(J_{\omega*\tau}) = \chi_{\omega_1} p_{\omega*\tau} + \sum_{h=1}^{n-1} \chi_{\omega_h} p_{|\omega|_{h+1} * (\omega_{h+1}^+ \cdots \omega_n^+)} + \chi_{\omega_1} p_{\omega*\tau^+} + \sum_{h=1}^{m-1} \chi_{\omega_h} p_{|\omega|_{h+1} * (\nu_{h+1}^+ \cdots \nu_m^+)} + \chi_{\omega_1} p_{\omega*\nu^+}. \tag{4.10}
\]
Similarly, using Lemma 2.1 and (1.9), we have

\begin{align}
\mu(J_{\omega*\tau}) &= \chi_{\omega_1} P_{\omega*\tau} + \sum_{h=1}^{n-1} \chi_{\omega_1} P_{\omega[h(a_{h+1}^{+} \ldots \omega_n^{+})*\tau]} + \sum_{h=0}^{m-1} \chi_{\omega_1} P_{\omega[v_{h+1}^{+} \ldots v_{1}^{+})*\tau]} + \sum_{h=1}^{n} \chi_{\omega_1} P_{\omega[v_{h+1}^{+} \ldots v_{1}^{+})*\tau]} + \chi_{\omega_1} P_{\omega[v_{1}^{+} \ldots v_{1}^{+})*\tau]}.
\end{align}

We denote the sum in (4.8), (4.9), (4.10), by \( I_1 \), \( I_2 \), \( I_3 \), and denote the sum in (4.11), (4.12), (4.13), by \( I_4 \), \( I_5 \), \( I_6 \). Then

\begin{align}
I_1 &= \chi_{\omega_1} P_{\omega} \cdot P_{\omega_1^+} \cdot \tau_1 \cdot \rho \\
&+ \sum_{h=1}^{n-1} \chi_{\omega_1} P_{\omega[h(a_{h+1}^{+} \ldots \omega_n^{+})] \cdot P_{\omega_1^+} \cdot \tau_1^+} \cdot \rho \\
I_4 &= \chi_{\omega_1} P_{\omega} \cdot P_{\omega_1 \cdot v_1} \cdot P_{\psi} \cdot P_{\psi^{+} \cdot \tau_1} \cdot \rho \\
&+ \sum_{h=1}^{n-1} \chi_{\omega_1} P_{\omega[h(a_{h+1}^{+} \ldots \omega_n^{+})] \cdot P_{\omega_1^+} \cdot v_1^+ \cdot \tau_1^+} \cdot \rho \\

\text{We compare the preceding two equations and obtain} \quad I_4 \leq (p^{d+1} \cdot p^{-1}) \cdot I_1. 
\end{align}

In an similar manner, one can see that

\begin{align}
I_6 \leq (p^{d+1} \cdot p^{-1}) \cdot I_3.
\end{align}

Next, we compare \( I_5 \) and \( I_2 \). We have

\begin{align}
I_2 &= \chi_{\omega_1} P_{\omega+\tau}^+ = \chi_{\omega_1} \cdot P_{\omega} \cdot P_{\omega_1^+} \cdot \rho \\
&+ \sum_{h=0}^{m-1} \chi_{\omega_1} P_{\omega} \cdot P_{\omega_1^{+} \cdot v_1} \cdot P_{\psi[\omega_h^{+} \ldots \omega_n^{+}) \cdot \tau_1^+} \cdot \rho \\
&\leq (l + 1) p^{d+1} \chi_{\omega_1} \cdot P_{\omega} \cdot P_{\tau^+} \\
&\leq (l + 1) p^{d+1} p^{-1} I_2.
\end{align}

Combining (4.14)-(4.16), we deduce

\begin{align}
\mu(J_{\omega*\tau}) &= I_4 + I_5 + I_6 \leq (l + 1) p^{d+1} p^{-1} (I_1 + I_2 + I_3) \\
&\leq (l + 1) p^{d+1} p^{-1} \mu(J_{\omega*\tau}).
\end{align}

Note that \( s_{\omega*\tau} \leq p^{d+1} p^{-1} s_{\omega*\tau}. \) We obtain

\begin{align}
E_r(\omega*\psi*\tau) = \mu(J_{\omega*\psi*\tau}) s_{\omega*\psi*\tau}^r \leq (l + 1)(p^{d+1}(p^{d})^{-1}) E_r(\omega*\psi).
\end{align}

This completes the proof of the lemma.
Remark 4.3 For $\tau^+ \in H_2^*$, we define
\[ B^\tau(\tau^+) := \{ \omega \in B(\tau^+) : \rho \neq \omega \text{ for every } \rho \in B(\tau^+) \setminus \{\omega\} \}. \]

It is easy to see that $B^\tau(\tau^+)$ is an anti-chain in $H_1^*$.

For every $\omega \in B^\tau(\tau^+)$, we define $B_\omega(\tau^+) := \{ \rho \in B(\tau^+) : \omega < \rho \}$. Next, we give an estimate for the size of $B_\omega(\tau^+)$ by using Lemma 12. Let $k_2$ be the smallest integer such that
\[ (k_2 + 1)(\overline{p}^\tau)^{k_2 + 1} \overline{p}^{\tau - 1} < c_{1,r}. \]

Lemma 4.4 Let $C := \sum_{k=0}^{k_2} N^k$. For every $\omega \in B^\tau(\tau^+)$, we have
\[ \sum_{\rho \in B_\omega(\tau^+)} (p_{\rho s_\omega^r})^{\frac{k_2}{\overline{p}^\tau}} \leq C (p_{\omega s_\omega^r})^{\frac{k_2}{\overline{p}^\tau}}. \quad (4.17) \]

Proof. Suppose that there exists some $v$ with $|v| > k_2$ such that $\omega * v \in B(\tau^+)$. Then by Lemma 12 we have $\xi_r(\omega * v * \tau) < c_{1,r} \xi_r(\omega * \tau)$. This contradicts \(\text{(13)}\), because both $\omega * \tau$ and $\omega * v * \tau$ are elements of $A_{k,r}$. Thus,
\[ B_\omega(\tau^+) \subset \{ \rho \in H_1^* : \omega < \rho, |\rho| \leq |\omega| + k_2 \}; \text{ card}(B_\omega(\tau^+)) \leq C. \]

For every $\rho \in B_\omega(\tau^+)$, we have $p_{\rho s_\omega^r} \leq p_{\omega s_\omega^r}$. Thus, (4.17) is fulfilled.

Proof of Theorem 1.4 (ii) and (iii)
For Theorem 1.4 (ii), we need to treat the following three cases.

Case 1: $s_{1,r} < s_{2,r}$. In this case, we have $s_r = s_{2,r}$ and $\rho_1(s_{2,r}) < 1$. One can see that (4.17) remains valid for $s = s_{2,r} > s_{1,r}$. Thus, we have $Q_r^\tau(\mu) < \infty$.

Case 2: $s_{1,r} > s_{2,r}$. By (4.14), $\rho * \tau^+ \in A_{k,r}$ if and only if $\rho * \tau \in \Lambda_{k,r}$. We have
\[ A_{k,r} \subset \bigcup_{h=1}^{l_{2k}} \bigcup_{\tau^* \in H_2^*} \bigcup_{\omega \in B^\tau(\tau^+) \omega} \{ \rho * \tau^+ : \rho \in B_\omega(\tau^+) \}. \]

For $s = s_{1,r} > s_{2,r}$, by (1.6), Lemmas 4.1, 4.3 and Remark 4.3 we deduce
\[ F_{k,r}^s(\mu) \leq 2 + \sum_{h=1}^{l_{2k} - 1} \sum_{\tau^* \in \Lambda_{k,r}} \sum_{\omega \in B^\tau(\tau^+) \omega} \sum_{\rho \in B_\omega(\tau^+) \rho} (p_{\rho s_{\omega}^r} s_{\rho s_{\omega}^r} + p_{\rho s_{\omega}^r} s_{\rho s_{\omega}^r})^{\overline{p}^\tau}. \]
\[ \leq 2 + C \sum_{h=1}^{l_{2k} - 1} \sum_{\tau^* \in H_2^*} (p_{\tau s_{\tau}^r} s_{\tau s_{\tau}^r} + p_{\tau s_{\tau}^r} s_{\tau s_{\tau}^r})^{\overline{p}^\tau} \sum_{\omega \in B^\tau(\tau^+ \omega)} \overline{p}^{\tau - 1}. \]
\[ \leq 2 + \frac{p_2(s)}{1 - p_2(s)}. \]

It follows that $F_r(s) < \infty$. This and Lemma 3.6 yield that $Q_r^\tau(\mu) < \infty$. 

19
Case 3: \( s_{1,r} = s_{2,r} \). For every \( 1 \leq h \leq l_{1k} - 1 \), \( \Lambda_{k,r}(h) := \{ \sigma | h : \sigma \in \Lambda_{k,r} \} \) is a maximal anti-chain in \( H_1^* \). Fix an arbitrary \( \omega \in \Lambda_{k,r}(h) \). By (A2), the set \( D_\omega := \{ \tau \in H_1^* : \omega \ast \tau \in \Lambda_{k,r} \} \) contains a maximal anti-chains in \( H_1^*(j_1) \) for some \( j_1 \in \Psi \). Hence, by (A3), the set \( D_\omega^+ := \{ \tau^+ : \omega \ast \tau \in \Lambda_{k,r} \} \) contains some maximal anti-chains \( A(\omega) \) in \( H_2^*(j_1^+) \), and \( \{ \omega \ast \tau^+ : \tau^+ \in D_\omega^+ \} \subset \Lambda_{k,r} \). Thus,

\[
\Lambda_{k,r} \supset \bigcup_{h=1}^{l_{1k}-1} \bigcup_{\omega \in A(\omega)} \{ \omega \ast \tau^+ : \tau^+ \in A(\omega) \}.
\]  

(4.18)

By Lemma \[2.1\] and Hölder’s inequality with exponent less than one, we have

\[
F_{k,r}^\ast(\mu) \geq \sum_{\sigma \in \Lambda_{k,r}} \left( \sum_{\lambda \in \Gamma(\sigma) \backslash \{ \sigma, \sigma^+ \}} \chi_{\bar{\sigma}}p_\sigma s_\sigma^r \right)^{\frac{1}{1+r}}.
\]

\[
\geq \sum_{\sigma \in \Lambda_{k,r}} \sum_{\lambda \in \Gamma(\sigma) \backslash \{ \sigma, \sigma^+ \}} \left( \chi_{\bar{\sigma}}p_\sigma s_\sigma^r \right) \left| \frac{\sigma}{\lambda} \right|^{-\frac{r}{1+r}}.
\]

\[
\geq (\chi l_{2k})^{-\frac{sr}{1+r}} \sum_{\sigma \in \Lambda_{k,r}} (p_\sigma s_\sigma^r)^{\frac{sr}{1+r}}.
\]

Using this, (4.18), Lemma \[4.1\] and Remark \[3.4\] (d1), we deduce

\[
F_{k,r}^\ast(\mu) \geq l_{2k}^{-\frac{sr}{1+r}} l_{1k}^{-\frac{sr}{1+r}} \sum_{h=1}^{l_{1k}-1} \sum_{\omega \in A(\omega)} \omega (p_\omega s_\omega^r)^{\frac{sr}{1+r}} \sum_{\tau^+ \in A(\omega)} (p_{\tau^+} s_{\tau^+}^r)^{\frac{sr}{1+r}}
\]

\[
\geq l_{2k}^{-\frac{sr}{1+r}} l_{1k} \approx k^{\frac{sr}{1+r}}.
\]

Thus, by Lemma \[3.6\], we conclude that \( Q_{k,r}(\mu) = \infty \).

(iii) Let \( G(i^+) := \{ j^+ : (i^+, j^+) \in G_2 \} \). By (A2), for \( i \in \Psi \), we have \( \text{card}(G(i^+)) \geq 2 \). Let \( t_{i,r} \), \( i \in \Psi \), be implicitly defined by

\[
\sum_{j^+ \in G(i^+)} (p_{i^+, j^+} s_{i^+, j^+}^r)^{\frac{1}{1+r}} = 1.
\]

By (A1), we have \( P_3 = 0 \). Thus, for every \( i \in \Psi \), \( (p_{i^+, j^+})_{j^+ \in G(i^+)} \) is a probability vector. By Theorem 14.14 of [4], \( t_{i,r} = D_r(\lambda_{i^+}) \) for the self-similar measure \( \lambda_{i^+} \) associated with \( (p_{i^+, j^+})_{j^+ \in G(i^+)} \) and some IFS \( (f_{i^+, j^+})_{j^+ \in G(i^+)} \) with similarity ratios \( (s_{i^+, j^+})_{j^+ \in G(i^+)} \). Thus, for every \( r > 0 \), we apply [4] Theorem 11.6 and obtain \( t_{i,r} \geq \frac{\log 2}{\log 3} =: \kappa > 0 \). Let \( \zeta_r := \min_{1 \leq i \leq N} t_{i,r} \). By Theorem 8.1.22 of [11],

\[
\rho_2(s) \geq \min_{1 \leq i \leq N} \sum_{j=1}^{N} (p_{i^+, j^+} s_{i^+, j^+}^r)^{\frac{1}{1+r}} = \min_{1 \leq i \leq N} \sum_{j^+ \in G(i^+)} (p_{i^+, j^+} s_{i^+, j^+}^r)^{\frac{1}{1+r}}.
\]
It follows that $s_{2,r} \geq \zeta_r \geq \kappa$. For every $i \in \Psi$, we have
\[
\limsup_{r \to 0} \sum_{j=1}^{N} (p_{i,j}s_{i,j}^r)^{\frac{\omega_s}{\omega_s + \tau}} \leq \sum_{j=1}^{N} \limsup_{r \to 0} p_{i,j}^r (s_{i,j}^r)^{\frac{\omega_s}{\omega_s + \tau}} = \sum_{j=1}^{N} p_{i,j}^r.
\]

By (A2) and (A3), we know that $\sum_{j=1}^{N} p_{i,j}^r < 1$ for every $i \in \Psi$. Thus, there exists some $r_0 > 0$ such that for every $r \in (0, r_0)$, we have $\rho_1(s_{2,r}) < 1$. It follows that $s_{2,r} > s_{1,r}$. This and Theorem 1.4 (ii) yield Theorem 1.4 (iii).

### 5 Proof of Theorem 1.5

In this section, we always assume that (A2), (A4) and (A5) hold. For every $\sigma \in S^*$, let $E_r(\sigma)$ be as defined in (3.1). We have

**Lemma 5.1** Let $\sigma \ast \tau \in S^*$. For $s > 0$, we have
\[
(p_s \chi^{-1})^s E_r(\sigma)^s E_r(\tau)^s \leq E_r(\sigma \ast \tau)^s \leq (p_s \chi^{-1})^s E_r(\sigma)^s E_r(\tau)^s.
\]

**Proof.** By Lemma 2.2, for every $\sigma \in S^*$, we have
\[
\Gamma(\sigma \ast \tau) = T(\sigma \ast \tau) = \{ \bar{\sigma} \ast \bar{\tau} : \bar{\sigma} \in T(\sigma), \bar{\tau} \in T(\tau) \}.
\]

By (1.3), for every $\rho \in S^*$ and $\bar{\rho} \in \Gamma(\rho)$, we have $s_{\bar{\rho}} = s_\rho$. Hence,
\[
(E_r(\sigma \ast \tau))^s = \left( \sum_{\bar{\sigma} \in T(\sigma)} \sum_{\bar{\tau} \in T(\tau)} \chi_{\bar{\sigma}}, (p_s \chi_{\bar{\sigma}}), (p_s \chi_{\bar{\tau}}) \right)^s
\geq \left( \sum_{\bar{\sigma} \in T(\sigma)} (\chi_{\bar{\sigma}}, p_s \chi_{\bar{\sigma}}) \sum_{\bar{\tau} \in T(\tau)} (\chi_{\bar{\tau}} p_s \chi_{\bar{\tau}}) \right)^s
= (p_s \chi^{-1})^s (E_r(\sigma))^s (E_r(\tau))^s.
\]

The remaining part of the lemma can be obtained similarly.

For every $s > 0$ and $n \geq 1$, we define
\[
T_n(s) := \sum_{\sigma \in S_n} (E_r(\sigma))^s = \sum_{\sigma \in S_n} \left( \sum_{\bar{\rho} \in \Gamma(\sigma)} \chi_{\bar{\rho}} p_s \chi_{\bar{\rho}} \right)^s.
\]

(5.1)

Next, we are going to show that $(T_n(s))_{n=1}^\infty$ is sub-multiplicative up to a constant factor. For $h, l \geq 1$ and $\sigma \in S_h$, we write
\[
\Lambda(\sigma, l) := \{ \rho \in S_l : \rho \ast \sigma \in S_{h+l} \}, \quad S_{l,i} := \{ \tau \in S_l : \tau_1 = i \}, \quad i \in \Psi.
\]

For $s > 0, l \geq 1$ and $i \in \Psi$, we define
\[
T_{i,l}(s) := \sum_{\omega \in S_{l,1}} (E_r(\omega))^s, \quad h(s) := (Nc_{2,r})^s N(p_s \chi^{-1})^{-s} (c_{1,r})^N p_s \chi^{-1}.
\]
Lemma 5.2 Let $s > 0$ be given. There exist positive numbers $g_1(s), g_2(s)$ such that for every pair $n, l \in \mathbb{N}$, we have

$$g_1(s)T_n(s)T_l(s) \leq T_{n+l}(s) \leq g_2(s)T_n(s)T_l(s).$$

(5.2)

Proof. We first show the following claim:

Claim: for $1 \leq i \neq j \leq N$, we have $T_{i,j}(s) \geq h(s)T_{i,j}(s)$. By (A5), $P_1$ is irreducible, so the sub-graph $G_1$ of $G$ with vertex set $\Psi$ is strongly connected. Hence, there exists a word $\gamma$ with $|\gamma| < N$, such that $i * \gamma * j \in H_1^s = S^s$. Note that $\{i * \gamma * \tau : \tau \in S_{i,j}\} \subset S_{i+|\gamma|+1,i}$. Using Lemmas 3.2 and 5.1, we deduce

$$T_{i+|\gamma|+1,i}(s) \geq (c_1,r)^N_s \left( \frac{p_s \chi^{-1}}{s} \right)^s T_{i,j}(s).$$

This completes the proof of the claim.

Now let $g_1(s) := N^{-1}h(s)\left( \frac{p_s \chi^{-1}}{s} \right)^s$ and $g_2(s) := \left( \frac{p_s \chi^{-1}}{s} \right)^s$. For every $\sigma \in S_n$ and $l \geq 1$, we have, $\Lambda(\sigma,l) \subset S_l$. Let $j_1 \in \Psi$ such that $S_{i,j_1} \subset \Lambda(\sigma,l)$. Using Lemma 5.1 and the claim, we deduce

$$\sum_{\omega \in \Lambda(\sigma,l)} (\mathcal{E}_r(\sigma * \omega))^s \leq \sum_{\tau \in S_1} (\mathcal{E}_r(\sigma * \tau))^s \geq g_1(s)(\mathcal{E}_r(\sigma))^s T_l(s).$$

(5.3)

Since $S_{n+l} = \bigcup_{\sigma \in S_n} \{\sigma * \omega : \omega \in \Lambda(\sigma,l)\}$, (5.2) follows from (5.3).

Remark 5.3 Let $\sigma \in S_n, j \in \Psi$ with $\sigma_n * j \in S_2$. If (A4) is not satisfied, then it can happen that $S_{i,j} \notin \Lambda(\sigma,l)$. This can be seen from Example 2.8. Let $P = P(2)$ and $\sigma = (1,1), l = 2, j_1 = 2, r = (2,1)$. As we have noted, $\sigma \in S_2$ and $\sigma_2 * j_1 = (1,2) \in S_2$. Also, we have $\tau \in S_2$ and $\tau \in S_{2,2}, \text{but } s * \tau \notin S_1$. This means that $\tau \notin \Lambda(\sigma,2)$.

Lemma 5.4 For $s > 0$, the limit $\lim_{n \to \infty} \frac{1}{n} \log T_n(s) =: \Phi(s)$ exists. Moreover,

(f1) $\Phi$ is continuous and strictly decreasing; there exists a unique $s_0 \in (0,1)$ and a unique $t_r > 0$ such that $\Phi(s_0) = 0$ and $\Phi(\frac{1}{t_r + r}) = 0$;

(f2) for $b := g_2(s_0)/g_1(s_0)$ and for every pair $m, n \in \mathbb{N}$, we have

$$b^{-1}T_m(s_0) \leq T_m(s_0) \leq bT_m(s_0).$$

Proof. The lemma can be proved by using (A2), Lemmas 3.2, 5.2 and Corollary 1.2 along the line of Lemma 5.2.

Remark 5.5 For $s > 0$ and $n \geq 1$, we define

$$M(\sigma) := \sum_{\sigma \in \mathcal{T}(\sigma)} p_s s_{\sigma}^r.$$

(5.4)
For every $\sigma \in \mathcal{S}_m$ and $1 \leq h \leq n - 1$, we define

$$E_{\sigma, h+1} := \begin{pmatrix} p_{\sigma, h, h+1}^r s_{\sigma, h, h+1}^r & p_{\sigma, h, h+1}^+ s_{\sigma, h, h+1}^+ \\ p_{\sigma, h, h+1}^+ s_{\sigma, h, h+1}^r & p_{\sigma, h, h+1}^+ s_{\sigma, h, h+1}^+ \end{pmatrix}. $$

Let $V := (1, 1)$ and $U := V^T$. Let $\|x\|_1$ denote the $l_1$-norm for $x \in \mathbb{R}^2$. We have

$$M(\sigma) = \left\| \prod_{h=1}^{n-1} E_{\sigma, h+1} U \right\|_1 = \left\| V \prod_{h=1}^{n-1} E_{\sigma, h+1} \right\|_1. \quad (5.5)$$

Using some ideas in [3, Theorem 5.1], we are now able to obtain an auxiliary measure. As we will see, this measure is closely connected with the quantization errors for $\mu$.

**Lemma 5.6** There exists a probability measure $\lambda$ supported on $K$, such that

$$\lambda(J_\sigma) = (E_r(\sigma))^{\frac{r}{r+\rho}}, \quad \sigma \in \mathcal{S}^*.$$

**Proof.** For $m \geq 1$ and $\sigma \in \mathcal{S}_m$, let $x_\sigma$ be an arbitrary point of $J_\sigma \cap K$ and denote by $\delta_\sigma$ the Dirac measure at the point $x_\sigma$. For every $m \geq 1$, we define $\lambda_m := \frac{1}{\Lambda_m(s_0)} \sum_{\sigma \in \mathcal{S}_m} (E_r(\sigma))^{\frac{r}{r+\rho}} \delta_\sigma$. Then $(\lambda_m)_{m=1}^\infty$ is a sequence of probability measures. By [16, Theorem 1.23], there exist a sub-sequence $(\lambda_{m_k})_{k=1}^\infty$ and a measure $\lambda$ such that $\lambda_{m_k} \to \lambda$ (weak convergence) as $k \to \infty$. One can see that $\lambda$ is a probability measure supported on $K$. Now let $n \geq 1$ and $\sigma \in \mathcal{S}_n$ be given. For every $m > n$,

$$\lambda_m(J_\sigma) = \sum_{\rho \in \Lambda(\sigma, m-n)} \lambda_m(\sigma \ast \rho) = \sum_{\rho \in \Lambda(\sigma, m-n)} \frac{1}{\Lambda_m(s_0)} (E_r(\sigma \ast \rho))^{\frac{1}{r+\rho}}.$$

Using (5.3) and Lemma 5.4 (f2), one can easily obtain

$$b^{-1} g_1(s_0)(E_r(\sigma))^{\frac{1}{r+\rho}} \leq \lambda_m(J_\sigma) \leq bg_2(s_0)(E_r(\sigma))^{\frac{1}{r+\rho}}.$$

This implies the assertion of the lemma.

For the proof for Theorem 1.23 (ii), we define

$$\tilde{p}_{i,j} = \begin{cases} p_{i,j} + p_{i,j+1} & \text{in Case (g1)} \\ p_{i,j} + p_{i+1,j} & \text{in Case (g2)} \end{cases}, \quad (i,j) \in \Psi^2.$$

For every $i \in \Psi$, $(\tilde{p}_{i,j})_{j=1}^N$ is a probability vector. In fact, we have

$$\sum_{j=1}^N \tilde{p}_{i,j} = \begin{cases} \sum_{j=1}^N (p_{i,j} + p_{i,j+1}) = \sum_{j=1}^{2N} p_{i,j} = 1, & \text{Case (g1)} \\ \frac{1}{2} \sum_{j=1}^N (p_{i,j} + p_{i+1,j} + p_{i,j+1} + p_{i+1,j+1}) = 1 & \text{Case (g2)} \end{cases}.$$

Let $B(s) := ((\tilde{p}_{i,j}, s_{i,j})^2)_{i,j=1}^N$. We denote by $\xi(s)$ the spectral radius of $B(s)$. By (A2), there exists a unique $a_r > 0$ such that $\xi(\frac{a_r}{a_{r+\rho}}) = 1$. We have
Lemma 5.7 We have $t_r = a_r$.

Proof. Using (5.6) and the conditions in (g1) and (g2), we deduce

$$
(E_r(\sigma))^\frac{t_r}{s_r} \lesssim \begin{cases} 
\| \prod_{h=1}^{n-1} E_{\theta_h, \sigma_{h+1}} U \|_1^{t_r/s_r} = 2^{\frac{t_r}{s_r}} (\tilde{p}_\sigma s_\sigma^r)^{\frac{t_r}{s_r}} & \text{Case (g1)} \\
\| V \prod_{h=1}^{n-1} E_{\theta_h, \sigma_{h+1}} U \|_1^{t_r/s_r} = 2^{\frac{t_r}{s_r}} (\tilde{p}_\sigma s_\sigma^r)^{\frac{t_r}{s_r}} & \text{Case (g2)}
\end{cases}.
$$

Therefore, $t_r$ satisfies $\Upsilon(t_r) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma \in S_n} (\tilde{p}_\sigma s_\sigma^r)^{\frac{t_r}{s_r}} = 0$. As we did for $\Phi(s)$, one can see that the solution of $\Upsilon(s) = 0$ is unique. By applying Lemma 4.1, we know that $\sum_{\sigma \in S_n} (\tilde{p}_\sigma s_\sigma^r)^{\frac{t_r}{s_r}} \asymp 1$, implying that $a_r = t_r$.

Proof of Theorem 1.5
(i) For every $k \geq 1$, by Lemma 5.6 we have

$$
F_{k,r}(\mu) = \sum_{\sigma \in A_{k,r}} (E_r(\sigma))^\frac{t_r}{s_r} \lesssim \sum_{\sigma \in A_{k,r}} \lambda(J_\sigma) = 1.
$$

This and Lemma 3.6 yield that $0 < Q^r_t(\mu) \leq Q^r_t(\mu) < \infty$ and $D_r(\mu) = t_r$. Note that $\bigcup_{\sigma \in S_n} \Gamma(\sigma) = G_n$ is a finite maximal anti-chain in $G^*$ and that $P, A(\frac{a_r}{s_r + t_r})$ are irreducible. By Lemma 4.1 we deduce

$$
T_n(\frac{s_r}{s_r + r}) = \sum_{\sigma \in S_n} \left( \sum_{\tilde{\sigma} \in \Gamma(\sigma)} (\chi_{\tilde{\sigma}} p_{\tilde{\sigma}} s_{\tilde{\sigma}}^r) \right)^\frac{t_r}{s_r} \lesssim \sum_{\tilde{\sigma} \in G_n} (p_{\tilde{\sigma}} s_{\tilde{\sigma}}^r)^\frac{t_r}{s_r} \asymp 1.
$$

Hence, $\Phi(\frac{s_r}{s_r + r}) \leq 0$ and $s_r \geq t_r$. This completes the proof of Theorem 1.5 (i).

(ii) Assume that (g1) or (g2) holds. By Lemma 5.7 we have $D_r(\mu) = t_r = a_r$. Next, we show that $a_r < s_r$. Since $P_1$ is irreducible, so is the matrix $B(\frac{a_r}{s_r + t_r})$. There exists a positive right eigenvector $v = (v_1, \ldots, v_N)^T$ of $B(\frac{a_r}{s_r + t_r})$ in case (g1) and a positive left eigenvector $w = (w_1, \ldots, w_N)$ of $B(\frac{s_r}{s_r + t_r})$ in case (g2), with respect to eigenvalue 1:

$$
\begin{cases} 
\sum_{j=1}^{N} (\tilde{p}_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} v_j = v_i & \text{Case (g1)} \\
\sum_{j=1}^{N} (\tilde{p}_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} w_j = w_i & \text{Case (g2)}
\end{cases}, \quad 1 \leq i \leq N.
$$

Let $\tilde{v} = (v_1, \ldots, v_N, v_1, \ldots, v_N)^T$ and $\tilde{w} = (w_1, \ldots, w_N, w_1, \ldots, w_N)$. Then $\tilde{v}, \tilde{w}$ are positive vectors. For every $1 \leq i \leq 2N$, let $R_i$ denote the ith row of the matrix $A(\frac{a_r}{s_r + t_r})$ and $C_i$ its ith column. By (1.3), for every $(i, j) \in S_2$, we have $s_{i,j} = s_{i,j}^+ = s_{i,j} = s_{i,j}^+$. Hence,

$$
\begin{cases} 
R_i \tilde{v} = \sum_{j=1}^{N} (p_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} v_j + \sum_{j=1}^{N} (p_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} v_j > v_i & \text{Case (g1)} \\
\tilde{w} C_i = \sum_{j=1}^{N} (p_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} w_j + \sum_{j=1}^{N} (p_{i,j} s_{i,j}^r)^{\frac{s_r}{s_r + t_r}} w_j > w_i & \text{Case (g2)}
\end{cases}.
$$

Using (g1) and (g2), one can also see that $(R_i^+ \tilde{v}) > v_i$ and $(\tilde{w} C_i^+) > w_i$, for every $i \in \Psi$. It follows that $A(\frac{a_r}{s_r + t_r}) \tilde{v} > \tilde{v}$ in Case (g1) and $\tilde{w} A(\frac{a_r}{s_r + t_r}) > \tilde{w}$ in Case (g2). Thus, by applying [11] Corollary 8.1.29, we obtain that $\rho(a_r) > 1$ and $s_r > a_r$. This completes the proof of Theorem 1.5 (ii).
Remark 5.8 In Case (g1), \( \mu \) agrees with the Markov-type measure \( \bar{\mu} \) associated with \( (\bar{p}_{i,j})_{i,j=1}^{N} \) and \( \bar{\chi} = (\chi_{i} + \chi_{i+}) \). As Example 2.4 shows, in Case (g2), it may happen that \( \mu \neq \bar{\mu} \); however, \( \mu \) is equivalent to \( \bar{\mu} \), in the sense that \( \mu(A) \asymp \bar{\mu}(A) \) for all Borel sets \( A \). In fact, we have
\[
\mu(J_{\sigma}) = \sum_{\hat{\sigma} \in \Gamma(\sigma)} \chi_{\hat{\sigma}} p_{\hat{\sigma}} \asymp \sum_{\hat{\sigma} \in \Gamma(\sigma)} p_{\hat{\sigma}} = 2p_{\sigma} \asymp \bar{\mu}(J_{\sigma}).
\]

Let \( \bar{p}_{i,j}, i, j \in \Psi \), be as defined in (2.1). Let \( \bar{\mu} \) be the Markov-type measure associated with \( (\bar{p}_{i,j})_{i,j=1}^{N} \) and \( \bar{\chi} = (\chi_{i} + \chi_{i+}) \). As the following example shows, when \( \mu \) is not reducible, \( \mu \) and \( \bar{\mu} \) are, in general, not equivalent.

Example 5.9 Let \( N = 3 \). Let \( P \) be defined by
\[
P = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}.
\]

Then (A2) and (A5) hold, but for \( i = 1 \), neither (a) nor (b) in Theorem 1.3 (1) holds. In fact, \( (1, 1), (3, 1) \in S^{*}(1) \), and \( (1, 1), (1, 4), (4, 1), (4, 4) \in G_{2} \), but
\[
p_{1,1} + p_{1,4} \neq p_{4,1} + p_{4,4}; \quad \frac{p_{1,1} + p_{4,1}}{p_{1,4} + p_{4,4}} \neq \frac{p_{3,1} + p_{6,1}}{p_{3,4} + p_{6,4}}.
\]

Thus, the measure \( \mu \) is not reducible, regardless of the choice of \( \chi \). Next, we show that \( \mu \) is not equivalent to the Markov-type measure \( \bar{\mu} \). We consider
\[
\sigma^{(n)} = (1, 1, \ldots, 1) \in S_{n}; \quad \tau^{(n)} = (1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3, 1) \in S_{3n+1}.
\]

Let \( R_{1} \) denote the spectral radius of the following matrix:
\[
M_{11} := \begin{pmatrix}
p_{1,1} & p_{1,4} \\
p_{4,1} & p_{4,4}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{6} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

We have, \( R_{1} = \frac{1}{30}(14 + \sqrt{772}) \approx 0.43526 \). Let \( \alpha_{i,j}^{(n)} \) denote the \((i,j)\)-entry of the matrix \( M_{11}^{n-1} \). By Corollary 8.1.33 of [11], \( \alpha_{1,1}^{(n)} + \alpha_{1,2}^{(n)} \asymp R_{1}^{n-1} \) and \( \alpha_{2,1}^{(n)} + \alpha_{2,2}^{(n)} \asymp R_{1}^{n-1} \). We deduce
\[
\mu(J_{\sigma^{(n)}}) \asymp \sum_{\hat{\sigma} \in \Gamma(\sigma^{(n)})} p_{\hat{\sigma}} = \sum_{i,j=1,2} \alpha_{i,j}^{(n)} \asymp R_{1}^{n-1}, \quad \bar{\mu}(\tau^{(n)}) \asymp \bar{p}_{1,1}^{n-1}.
\]

Let \( R_{2} \) denote the spectral radius of the following matrix:
\[
M_{1231} := \begin{pmatrix}
\frac{1}{4} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}.
\]
We have $R_2 > \frac{4\omega_1}{3\omega_2} > 0.1426$. By (2.1), we have $\bar{p}_{2,3} = \bar{p}_{1,3} = 2^{-1}$. Thus, 

$$\mu(\tau^{(n)}) = \|M_{1231}^n\|_1 \asymp R_2^n, \quad \bar{\mu}(\tau^{(n)}) \asymp (\bar{p}_{1,2} \bar{p}_{2,3} \bar{p}_{3,1})^n \asymp 4^{-n} \bar{p}_{1,2}^n. \quad (5.7)$$

Now we assume that $\mu$ and $\bar{\mu}$ are equivalent. Then by (5.6) and (5.7), we obtain $\bar{p}_{1,1} = R_1$ and $\bar{p}_{1,2} = 4R_2$. Setting $\zeta := \chi_1/(\chi_1 + \chi_4)$, we have 

$$\frac{1}{2} \zeta + \frac{3}{8} (1 - \zeta) = R_1; \quad \frac{1}{2} \zeta + \frac{5}{8} (1 - \zeta) = 4R_2.$$ 

However, we have $8(R_1 - 0.375) > 0.482$, but $8(0.625 - 4R_2) < 0.437$, a contradiction. Therefore, $\mu, \bar{\mu}$ are not equivalent, regardless of the choice of $\chi$.

References

[1] J. A. Bucklew, G.L. Wise, Multidimensional asymptotic quantization with $r$th power distortion measures. IEEE Trans. Inform. Theory 28 (1982), 239-247.

[2] G. A. Edgar and R.D. Mauldin, Multifractal decompositions of digraph recursive fractals. Proc. London Math. Soc. (3) 65 (1992), 604-628.

[3] K. Falconer, Techniques in fractal geometry. John Wiley & Sons, 1997.

[4] S. Graf and H. Luschgy, Foundations of quantization for probability distributions. Lecture Notes in Math., Vol. 1730, Springer-Verlag, 2000.

[5] S. Graf and H. Luschgy, The asymptotics of the quantization errors for self-similar probabilities, Real Anal. Exchange 26 (2000/2001), 795-810.

[6] S. Graf and H. Luschgy, The quantization dimension of self-similar probabilities, Math. Nachr., 241 (2002), 103-109.

[7] S. Graf and H. Luschgy, Quantization for probability measures with respect to the geometric mean error. Math. Proc. Camb. Phil. Soc. 136 (2004), 687-717.

[8] S. Graf, H. Luschgy and G. Pagès, Distortion mismatch in the quantization of probability measures. ESAIM Probab. Stat. 12 (2008), 127-153.

[9] S. Graf, H. Luschgy and G. Pagès, The local quantization behavior of absolutely continuous probabilities. Ann. Probab. 40 (2012), 1795-1828.

[10] R. Gray, D. Neuhoff, Quantization. IEEE Trans. Inform. Theory 44 (1998), 2325-2383.

[11] R. A. Horn, C. R. Johnson. Matrix analysis. Cambridge University Press, Second edition, 2013.

[12] J.E. Hutchinson, Fractals and self-similarity. Indiana Univ. Math. J. 30, 713-747 (1981).
[13] M. Kesseböhmer and S. Zhu, M. Kesseböhmer and S. Zhu, The upper and lower quantization coefficient for Markov-type measures. Math. Nachr. 290 (2017), 827-839.

[14] M. Kesseböhmer and S. Zhu, On the quantization for self-affine measures on Bedford-McMullen carpets. Math. Z. 283 (2016), 39-58.

[15] L.J. Lindsay and R.D. Mauldin, Quantization dimension for conformal iterated function systems, Nonlinearity 15 (2002), 189-199

[16] P. Mattila, Geometry of sets and measures in Euclidian spaces. Cambridge University press, 1995.

[17] R.D. Mauldin and S.C. Williams, Hausdorff dimension in graph-directed constructions. Trans. Amer. Math. Soc. 309 (1998), 811-829.

[18] K. Pötzelberger, The quantization dimension of distributions. Math. Proc. Camb. Phil. Soc. 131 (2001), 507-519.

[19] K. Pötzelberger, The quantization error of self-similar distributions. Math. Proc. Camb. Phil. Soc. 137 (2004), 725-740.

[20] M. Roychowdhury, Quantization dimension estimate of inhomogeneous self-similar measures. Bull. Pol. Acad. Sci. Math. 61 (2013), 35-45

[21] S. Zhu, Quantization dimension for condensation systems. Math. Z. 259 (2008), 33-43.

[22] S. Zhu, Quantization dimension of probability measures supported on Cantor-like sets. J. Math. Anal. Appl. 338 (2008), 742-750.

[23] S. Zhu, On the upper and lower quantization coefficient for probability measures on multiscale Moran sets. Chaos, Solitons & Fractals 45 (2012), 1437-1443

[24] S. Zhu, Asymptotics of the quantization errors for in-homogeneous self-similar measures supported on self-similar sets. Sci. China Math. 59 (2016), 337-350

[25] S. Zhu, Asymptotic order of the quantization errors for a class of self-affine measures, Proc. Amer. Math. Soc. 146 (2018), 637-651.