Linear Matrix Inequalities for Ultimate Boundedness of Dynamical Systems with Conic Uncertain/Nonlinear Terms

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Abstract
This note introduces a sufficient Linear Matrix Inequality (LMI) condition for the ultimate boundedness of a class of continuous-time dynamical systems with conic uncertain/nonlinear terms.

1 Introduction
This note introduces an LMI [1] result for the ultimate boundedness of dynamical systems with conic uncertain/nonlinear terms. Earlier research developed necessary and sufficient conditions for quadratic stability for systems with similar characterizations for uncertainties and nonlinearities [2, 3, 4, 5, 6, 7]. Incremental version of these characterizations are used in the synthesis of nonlinear observers [8, 9, 10, 11, 12, 13, 14] and to design robust Model Predictive Control (MPC) algorithms [15, 16, 17, 18]. The following results first appeared in [17].

Notation: The following is a partial list of notation used in this paper: $Q = Q^T > (\geq) 0$ implies $Q$ is a positive-(semi-)definite matrix; $Co\{G_1, \ldots, G_N\}$ represents the convex hull of matrices $G_1, \ldots, G_N$; $\mathbb{Z}^+$ is the set of non-negative integers; $\|v\|$ is the 2-norm of the vector $v$; $\lambda_{\text{max}}(P)$ and $\lambda_{\text{min}}(P)$ are maximum and minimum eigenvalues of symmetric
matrix $P$; $\mathcal{E}_P := \{ x : x^T P x \leq 1 \}$ is an ellipsoid (possibly not bounded) defined by $P = P^T \geq 0$; for a bounded signal $w(\cdot)$, $\|w\|_{[t_1,t_2]} := \sup_{\tau \in [t_1,t_2]} \|w(\tau)\|$; for a compact set $\Omega$, $\text{diam}(\Omega) := \max_{x,y \in \Omega} \|x - y\|$; and, for $V : \mathbb{R}^N \to \mathbb{R}$, $\nabla V = [\partial V/\partial x_1 \ldots \partial V/\partial x_n]$. A set $\Omega$ is said to be invariant over $[t_0, \infty)$ for $\dot{x} = f(x,t)$ if: $x(t_0) \in \Omega$ implies that $x(t) \in \Omega$, $\forall t \geq t_0$. $\Omega$ is also attractive if for every $x(t_0)$, $\lim_{t \to \infty} \text{dist}(x(t), \Omega) = 0$.

2 A General Analysis Result on Ultimate Boundedness

The following lemma gives a Lyapunov characterization for the ultimate boundedness of a nonlinear time-varying system, which is used in the proof of main result.

**Lemma 1.** Consider a system with state $\eta$ and input $\sigma$ described by

$$\dot{\eta} = \phi(t, \eta, \sigma), \quad t \geq t_0. \quad (1)$$

Suppose there exists a positive definite symmetric matrix $P$ with, $V(\eta) = \eta^T P \eta$, and a continuous function $W$ such that for all $\eta, \sigma$ and $t \geq 0$

$$\dot{V} = 2 \eta^T P \phi(t, \eta, \sigma) \leq -W(\eta) < 0 \quad \text{when} \quad \eta^T P \eta > \|\sigma\|^2. \quad (2)$$

Then for every bounded continuous input signal $\sigma(\cdot)$, the ellipsoid $\mathcal{E} := \{ \eta : \eta^T P \eta \leq \|\sigma(\cdot)\|^2_{[t_0,\infty)} \}$ is invariant and attractive for system (1). Furthermore, for any solution $\eta(\cdot)$ we have

$$\limsup_{t \to \infty} [\eta(t)^T P \eta(t)] \leq \|\sigma(\cdot)\|^2_{[t_0,\infty)}. \quad (3)$$

See [19] for a proof of the above lemma.

3 Analysis of Systems with Conic Uncertainty/Nonlinearity

In this section we consider the following system

$$\dot{x} = Ax + Ep(t, x) + Gw \quad (4)$$
where $x$ is the state, $p$ represents the uncertain/nonlinear terms, and $w$ is a bounded disturbance signal, and $p \in F(M)$ with

$$q = C_q x + Dp. \quad (5)$$

To define $p \in F(M)$, let

$$F(M) := \{ \phi : \mathbb{R}^{n_q+1} \to \mathbb{R}^{n_p} : \phi \text{ satisfies QI (7)} \}.$$

(6)

where the following QI (Quadratic Inequality) is satisfied

$$\begin{bmatrix} q \phi(t,v) \\ \phi(t,v) \end{bmatrix}^T M \begin{bmatrix} q \\ \phi(t,v) \end{bmatrix} \geq 0, \quad \forall M \in M, \quad \forall v \in \mathbb{R}^{n_q}, \quad \text{and} \quad \forall t. \quad (7)$$

where $M$ is a set of symmetric matrices.

The following condition, which is instrumental in the control synthesis, is assumed to hold for the incrementally-conic uncertain/nonlinear terms.

**Condition 1.** There exist a nonsingular matrix $T$ and a convex set $\mathcal{N}$ of matrix pairs $(X, Y)$ with $Y \in \mathbb{R}^{n_p \times n_p}$ and $X, Y$ symmetric and nonsingular such that for each $(X, Y) \in \mathcal{N}$, the matrix

$$M = T^T \begin{bmatrix} X^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix} T \in M \quad \text{with} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (8)$$

where $T_{22} + T_{21}D$ is nonsingular, $T_{21} \in \mathbb{R}^{n_p \times n_q}$ and $T_{22} \in \mathbb{R}^{n_p \times n_p}$. Furthermore, the set $\mathcal{N}$ can be parameterized by a finite number of LMIs.

It is also assumed that the set of multipliers $\mathcal{M}$ satisfies Condition 1. The following theorem, the main result of this note, presents an LMI condition guaranteeing ultimate boundedness of all the trajectories of the system (4).

**Theorem 1.** Consider the system given by (4) with $p \in F(M)$ where the multiplier set $\mathcal{M}$ satisfies Condition 1. Suppose that there exist $Q = Q^T > 0$, $(X, Y) \in \mathcal{N}$, $\lambda > 0$, and $R = R^T > 0$ such that the following matrix inequality holds

$$\begin{bmatrix} (A - E \Gamma^{-1} T_{21} C_q) Q + Q (A - E \Gamma^{-1} T_{21} C_q)^T + \lambda Q + R & E \Gamma^{-1} Y & Q C_q \Sigma \Sigma^T & G \\ Y \Gamma^{-T} E^T & -Y & Y \Lambda^T & 0 \\ \Sigma C_q Q & \Lambda Y & -X & 0 \\ G^T & 0 & 0 & -\lambda I \end{bmatrix} \leq 0 \quad (9)$$

3
where

\[ \Gamma = T_{21}D + T_{22}, \quad \Lambda = (T_{11}D + T_{12})\Gamma^{-1}, \quad \Sigma = T_{11} - (T_{11}D + T_{12})\Gamma^{-1}T_{21}. \]

Then, letting \( V(x) := x^TQ^{-1}x \), we have

\[ \dot{V}(x) + x^TQ^{-1}RQ^{-1}x \leq 0, \quad \forall V(x) \geq \|\omega\|^2. \tag{10} \]

**Proof.** First pre- and post-multiply (9) by

\[
\begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & I \\
0 & 0 & I & 0
\end{bmatrix}
\]

and then pre- and post-multiply the resulting matrix inequality with \( \text{diag}(Q^{-1}, Y^{-1}, I, I) \) to obtain

\[
\begin{bmatrix}
Q^{-1}(A - ET^{-1}T_{21}C_q) + (A - ET^{-1}T_{21}C_q)^TQ^{-1} & Q^{-1}E \Gamma^{-1} & 0 & C_q\Sigma^T \\
\Gamma^{-T}E^TQ^{-1} & -Y^{-1} & 0 & \Lambda^T \\
G^TQ^{-1} & 0 & -\lambda I & 0 \\
\Sigma C_q & \Lambda & 0 & -X
\end{bmatrix} \leq 0
\]

By using Schur complements the above inequality implies that

\[
\begin{bmatrix}
Q^{-1}(A - ET^{-1}T_{21}C_q) + (A - ET^{-1}T_{21}C_q)^TQ^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E \Gamma^{-1} & 0 & Q^{-1}G \\
\Gamma^{-T}E^TQ^{-1} & -Y^{-1} & 0 & \Lambda^T \\
G^TQ^{-1} & 0 & -\lambda I & 0 \\
\Sigma C_q & \Lambda & 0 & -X
\end{bmatrix} \leq 0
\]

which then implies that

\[
\begin{bmatrix}
Q^{-1}(A - ET^{-1}T_{21}C_q) + (A - ET^{-1}T_{21}C_q)^TQ^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E \Gamma^{-1} & 0 & Q^{-1}G \\
\Gamma^{-T}E^TQ^{-1} & 0 & 0 & \Lambda^T \\
G^TQ^{-1} & 0 & -\lambda I & 0 \\
\Sigma C_q & \Lambda & 0 & -X
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
\Sigma C_q & \Lambda & 0 \\
0 & I & 0
\end{bmatrix}^T \begin{bmatrix}
X^{-1} & 0 \\
0 & -Y^{-1}
\end{bmatrix} \begin{bmatrix}
\Sigma C_q & \Lambda & 0 \\
0 & I & 0
\end{bmatrix} \leq 0.
\]

4
Now post- and pre-multiply the earlier matrix inequality with the following matrix and its transpose

$$
\begin{bmatrix}
I & 0 & 0 \\
T_{21}C_q & \Gamma & 0 \\
0 & 0 & I
\end{bmatrix}
$$

to obtain

$$
\begin{bmatrix}
Q^{-1}A + A^TQ^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E & Q^{-1}G \\
E^TQ^{-1} & 0 & 0 \\
G^TQ^{-1} & 0 & -\lambda I
\end{bmatrix}\
\begin{bmatrix}
C_q & 0 & 0 \\
T_{21}C_q & \Gamma & 0 \\
0 & 0 & I
\end{bmatrix}^T
\begin{bmatrix}
\Sigma & \Lambda \\
0 & I
\end{bmatrix}^T
\begin{bmatrix}
X^{-1} & 0 \\
0 & -Y^{-1}
\end{bmatrix}\
\begin{bmatrix}
\Sigma & \Lambda \\
0 & I
\end{bmatrix}
\begin{bmatrix}
C_q & 0 & 0 \\
T_{21}C_q & \Gamma & 0
\end{bmatrix} \leq 0,
$$

where

$$
\begin{bmatrix}
\Sigma & \Lambda \\
0 & I
\end{bmatrix}^T
\begin{bmatrix}
C_q & 0 & 0 \\
T_{21}C_q & \Gamma & 0
\end{bmatrix} =
\begin{bmatrix}
T_{11}C_q & T_{11}D + T_{12} & 0 \\
T_{21}C_q & T_{21}D + T_{22} & 0
\end{bmatrix} = T
\begin{bmatrix}
C_q & D & 0 \\
0 & I & 0
\end{bmatrix}.
$$

By using Condition $\Pi$,

$$
M = T^T
\begin{bmatrix}
X^{-1} & 0 \\
0 & -Y^{-1}
\end{bmatrix} \in \mathcal{M}
$$

This implies that, for some $M \in \mathcal{M}$, we have

$$
\begin{bmatrix}
Q^{-1}A + A^TQ^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E & Q^{-1}G \\
E^TQ^{-1} & 0 & 0 \\
G^TQ^{-1} & 0 & -\lambda I
\end{bmatrix} +
\begin{bmatrix}
C_q & D & 0 \\
0 & I & 0
\end{bmatrix}^T
\begin{bmatrix}
C_q & D & 0 \\
0 & I & 0
\end{bmatrix} \leq 0.
$$

Pre- and post-multiplying the above inequality with $[x^T \ p^T \ w^T]$ and its transpose and using $V = x^TQ^{-1}x$, we obtain

$$
2x^TQ^{-1}(Ax + Ep + Gw) + x^TQ^{-1}RQ^{-1}x + \lambda(V - ||w||^2) +
\begin{bmatrix}
q \\
p
\end{bmatrix}^T
\begin{bmatrix}
q \\
p
\end{bmatrix} \leq 0, \text{ for all } \begin{bmatrix}
x \\
p \\
w
\end{bmatrix}.
$$

Since $p \in \mathcal{F}(\mathcal{M})$ with $q = Cx + Dp$, by using the S-procedure $\Pi$, the above inequality implies that the system $\mathcal{H}$ satisfies: $\dot{V} \leq -x^TQ^{-1}RQ^{-1}x < 0, \ \forall V \geq ||w||^2$. $\blacksquare$
The following corollary gives a matrix inequality condition for the quadratic stability of the system (10) (when $w = 0$), that is, existence of a quadratic Lyapunov function $V = x^T P x$ proving the exponential stability by establishing

$$
\dot{V} + x^T Q^{-1} R Q^{-1} x \leq 0 \quad (11)
$$

for all trajectories of the system (10). The proof of the lemma follows from a straight adaption of the proof of Theorem 1.

**Corollary 1.** Consider the system given by (4) with $w \equiv 0$ and $p \in \mathcal{F}(\mathcal{M})$ where the multiplier set $\mathcal{M}$ satisfies Condition 1. Suppose that there exist $Q = Q^T > 0$, $(X, Y) \in \mathcal{N}$ and $\lambda > 0$ such that the following matrix inequality holds

$$
\begin{bmatrix}
(A - E \Gamma^{-1} T_{21} C_q) Q + Q (A - E \Gamma^{-1} T_{21} C_q)^T + R & E \Gamma^{-1} Y & Q C_q^T \Sigma^T \\
Y \Gamma^{-T} E^T & -Y & Y \Lambda^T \\
\Sigma C_q Q & \Lambda Y & -X
\end{bmatrix} \leq 0 \quad (12)
$$

where $\Gamma$, $\Sigma$, $\Lambda$ are as given in Theorem 1. Then the system (4) is quadratically stable with a Lyapunov function $V = x^T Q^{-1} x$ and all the trajectories satisfy

$$
V(x(t)) \leq V(x(t_0)), \quad \forall t \geq t_0, \quad (13)
$$

$$
\dot{V}(x) + x^T Q^{-1} R Q^{-1} x \leq 0, \quad \forall x. \quad (14)
$$

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