Semi-supervised Local Cluster Extraction by Compressive Sensing

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Abstract

Local clustering problem aims at extracting a small local structure inside a graph without the necessity of knowing the entire graph structure. As the local structure is usually small in size compared to the entire graph, one can think of it as a compressive sensing problem where the indices of target cluster can be thought of as a sparse solution to a linear system. In this paper, we propose a new semi-supervised local cluster extraction approach by applying the idea of compressive sensing based on two pioneering works under the same framework. Our approach improves the existing works by making the initial cut to be the entire graph and hence overcomes a major limitation of existing works, which is the low quality of initial cut. Extensive experimental results on multiple benchmark datasets demonstrate the effectiveness of our approach.

1 Background and Motivation

Being able to learn from the data by investigating its underlying pattern, and separate the data into different groups or clusters based on their latent similarity and differences is one of the main interests in machine learning and data mining. There are many applications of clustering phenomena in different disciplines such as social science, health science, and engineering. Through the past few decades, traditional clustering problem has been studied a lot and many algorithms have been developed, such as k-means clustering [42], hierarchical clustering [47], and density based clustering [19]. For graph structured data or the data which can be converted into a graph structure by applying some techniques (e.g. K-NN auxiliary graph), it is natural to consider the task as a graph clustering problem.

Traditional graph clustering problem assumes the underlying data structure as a graph where data points are the nodes and the connections between data points are the edges. It assigns each node into a unique cluster or group of nodes, assuming there are no multi-class assignments. For nodes with high connection density, they are considered in the same cluster, and for nodes with low connection density, they are considered in different clusters. Since the task is to learn the clustering patterns by investigating the underlying graph structure, it is an unsupervised learning problem. Many unsupervised graph clustering algorithms have been developed through decades. For example, spectral clustering [45], which is based on the eigen-decomposition of Laplacian matrices of either weighted or unweighted graphs. Based on this, many variants of spectral clustering algorithms have been proposed, such as [60] and [28]. Another category is the graph partition based method such as finding the optimal cut [13], [14]. It is worthwhile to note that spectral clustering and graph partition have the same essence, see [36]. Spectral clustering has become one of the popular modern clustering algorithms since it enjoys the advantage of exploring the intrinsic data structures. It is simple to implement, and it often outperforms the traditional algorithm such as k-means. However, one of the main drawbacks of spectral clustering is its high computational cost, so it is usually not applicable to very large datasets. Meanwhile, it does not perform well if the data points distribution have certain shapes, e.g., the elongated band shape data and the moon shape data. In addition, many other clustering methods have been developed, such as the low rank and sparse representations based methods [41], [29], deep embedding based methods [59], and graph neural network based methods [50], [54]. Besides the unsupervised way, some semi-supervised graph clustering methods have also been proposed [31], [52], [49].

These clustering algorithms, whether unsupervised or semi-supervised, are all global clustering algorithms, which means that the algorithms output all the clusters simultaneously. However, it is often to people’s interests in only finding the single target cluster which contains the given labels, without worried too much about how the remaining part of the graph will be clustered. Such idea is very useful in detecting small-scale structure in a large-scale graph. This type of problem is referred as local clustering or local cluster extraction. Most of the current local clustering algorithms aim at finding the best cut from the graph, but the approaches they take to achieve this vary, for example [56], [23] and [48]. It is worth pointing out that Fountoulakis et. al. [21] kindly

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put several methods of local graph clustering into software, including both spectral methods \[4\] \[22\] and flow methods \[35\] \[55\] \[57\]. More recently, new approaches for making the cut of graph based on the idea of compressive sensing are proposed in \[37\] \[38\], where they took a complete different perspective by considering the way of finding the optimal cut as an improvement from an initial cut through solving a sparse solution to a linear system. However, their approaches require an initial cut from graph, and the clustering performance will heavily depend on the quality of the initial cut.

In this paper, we propose a new approach for semi-supervised local cluster extraction. Our approach improves the quality of initial cut used in existing work and overcomes the issue that missing vertices in the target cluster are not recoverable. Experimental results on benchmark datasets show that our approach outperforms their counterparts \[37\] \[38\], and it obtains comparable results than many other state-of-the-art semi-supervised clustering algorithms as well.

The subsequent sections of the paper are structured as follows. In Section 2, we introduce the problem setup and the idea of compressive sensing. In Section 3, we motivate the ideas of our approach, propose main algorithm, and show their asymptotic correctness of finding the clusters. In Section 4, we evaluate our algorithm on various datasets and compare its performance with baselines. In Section 5, we draw conclusions and discuss potential future research directions.

2 Preliminaries

2.1 Graph Notations and Concepts We adopt the standard notations for graph \(G = (V, E)\), where \(V\) is the set of all vertices and \(E\) is the set of all edges. In the case that the size of graph equals to \(n\), we assign \(V = \{1, 2, \cdots , n\} = [n]\). For a graph \(G\) with \(k\) underlying clusters \(C_1, C_2, \cdots , C_k\), we use \(n_i\) to indicate the size of \(C_i\) for \(i = 1, 2, \cdots , k\). Without loss of generality, let us assume \(n_1 \leq n_2 \leq \cdots \leq n_k\). Furthermore, we use matrix \(A\) to denote the adjacency matrix of graph \(G\), and use \(D\) to denote the diagonal matrix where each diagonal entry in \(D\) is the degree of each corresponding vertex.

**Definition 2.1.** The unnormalized graph Laplacian is defined as \(L = D - A\). The symmetric graph Laplacian is defined as \(L_{sym} := I - D^{-1/2} A D^{-1/2}\) and the random walk graph Laplacian is defined as \(L_{rw} := I - D^{-1} A\). For the scope of our problem, we will only focus on \(L_{rw}\) for the rest of discussion and we will use \(L\) to denote \(L_{rw}\) for the concise of notation. Recall the following elementary but important result from spectral graph theory. We omit the proof by referring to \[10\] and \[30\].

**Lemma 2.1.** Let \(G\) be an undirected graph with non-negative weights. The multiplicity \(k\) of the eigenvalue zero of \(L\) equals to the number of connected components \(C_1, C_2, \cdots , C_k\) in \(G\), and the indicator vectors \(1_{C_1}, \cdots , 1_{C_k} \in \mathbb{R}^n\) on these components span the kernel of \(L\).

For a graph \(G\) with underlying structure which separates vertices into different clusters, we can write \(G = G^{in} \cup G^{out}\), where \(G^{in} = (V, E^{in})\) and \(G^{out} = (V, E^{out})\). Here \(E^{in}\) is the set of all intra-connection edges within the same cluster, \(E^{out}\) is the set of all inter-connection edges between different clusters. We use \(A^{in}\) and \(A^{out}\) to denote the adjacency matrices associated with \(G^{in}\) and \(G^{out}\), respectively, and use \(L^{in}\) and \(L^{out}\) to denote the Laplacian matrices associated with \(G^{in}\) and \(G^{out}\), respectively.

Furthermore, for a set \(S\), we will use \(|S|\) to denote its size. For a matrix \(M\) or vector \(v\), we will use \(|M|\) or \(|v|\) to denote the entrywise absolute value operation. For a matrix \(M \in \mathbb{R}^{n \times n}\) and a set \(S \subset V\), we will use \(M_S\) to denote the submatrix of \(M\) where the columns of \(M_S\) consists of only those indices in \(S\).

2.2 Compressive Sensing Recall that \(|\cdot|_0\) counts the number of nonzero components in a vector. The idea of compressive sensing comes from solving the optimization problem:

\[
\min \|x\|_0 \quad \text{s.t.} \quad \|\Phi x - y\|_2 \leq \epsilon,
\]
where \(\Phi \in \mathbb{R}^{m \times n}\) is called sensing matrix, \(y \in \mathbb{R}^n\) is called measurement vector. The goal is to recover the sparse solution \(x \in \mathbb{R}^n\) under some constraints. It can be reformulated as solving:

\[
\arg \min \|\Phi x - y\|_2 \quad \text{s.t.} \quad \|x\|_0 \leq s.
\]
Its idea was first introduced by Dohono \[16\] and Candès, Romberg, Tao \[3\]. Since then, many algorithms have been developed to solve (2.1) or (2.2), including the greedy based approaches such as Orthogonal Matching Pursuit (OMP) \[53\] and its variants, Quasi-orthogonal Matching Pursuit (QOMP) \[20\] thresholding based approaches such as Iterative Hard Thresholding \[7\] and its variants, Compressive Sensing Matching Pursuit (CoSAMP) \[41\], Subspace Pursuit \[12\], etc. Note that (2.1) is NP-hard because of the appearance of the zero norm, and therefore it is sometimes convenient to solve its convex relaxation:

\[
\min \|x\|_1 \quad \text{s.t.} \quad \|\Phi x - y\|_2 \leq \epsilon.
\]
Algorithms such as LASSO \[51\], CVX \[21\] and Reweighted \(\ell_1\)-minimization \[9\] fall into this category.
We do not analyze further here. The book [30] gives a comprehensive summary of these methods. Let us also introduce the thresholding operator $L_s(\cdot)$ which will be used later:

$$L_s(v) := \{i \in [n] : v_i \text{ among } s \text{ largest-in-magnitude entries in } v\}.$$ 

It is worthwhile to mention one of the key concepts in compressive sensing, Restricted Isometry Property (RIP), which guarantees a good recovery of the solution to (2.1).

**Definition 2.2.** Let $\Phi \in \mathbb{R}^{m \times n}, 1 \leq s \leq n$ be an integer. Suppose there exists a constant $\delta_s \in (0, 1)$ such that

$$\tag{2.4} (1 - \delta_s)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$$

for all $x \in \mathbb{R}^n$ with $\|x\|_0 \leq s$. Then the matrix $\Phi$ is said to have the Restricted Isometry Property (RIP). The smallest constant $\delta_s$ which makes (2.4) hold is called the Restricted Isometry Constant (RIC).

**Remark 2.1.** The RIP condition is a sharp and sufficient condition for the matrix to have possible sparse vector recovery, but it is very hard to verify. There are other conditions such as mutual incoherence [27] which is simple and sufficient, but not sharp. Null space property [11], which is necessary and sufficient, but also very difficult to verify.

Another very important aspect which makes compressive sensing very useful is its robustness to noise. Suppose we try to solve the linear system $y = \Phi x$ given the measurement $y$ and sensing matrix $\Phi$. It is possible that we only have access to a noise version of $\Phi$, say $\tilde{\Phi} = \Phi + \epsilon_1$, and also only have access to a noise version of $y$, say $\tilde{y} = y + \epsilon_2$. Therefore, instead of solving $\tilde{y} = \tilde{\Phi} x$, what we solve in reality is $\tilde{y} = \Phi \tilde{x}$. However, if $\epsilon_1, \epsilon_2$ are both small in some sense, and the sensing matrix $\Phi$ satisfies certain conditions, then we will have $\tilde{x} \approx x$. There are multiple ways of solving $\tilde{x}$ given $\tilde{\Phi}$ and $\tilde{y}$. For the scope of our problem, we will focus on applying subspace pursuit [12]. Theorem 2.5 in [27] and Corollary 1 in [10] gives a result about how close $\tilde{x}$ and $x$ can be based on the conditions of $\Phi$, $\tilde{\Phi}$, $\tilde{y}$, $y$ which we will apply later in our theoretical analysis part.

## 3 Compressive Sensing of Local Clustering

### 3.1 Main Algorithm

The local clustering problem can be considered as a compressive sensing problem in the following way. Suppose there is a matrix $L_{-1}^i$ obtained from $L^i$ by deleting some column(s) from the target cluster. In our illustration below, the target cluster is the first diagonal block, and the deleted column is the first column.

**Algorithm 1 Compressive Sensing of Local Cluster Extraction (CS-LCE)**

**Require:** Adjacency matrix $A$, a random walk threshold parameter $\epsilon \in (0, 1)$, a set of seed vertices $\Gamma \subset C_1$, estimated size $\hat{n}_1 \approx |C_1|$, and depth of random walk $t \in \mathbb{Z}^+$, least squares threshold parameter $\gamma \in [0.1, 0.5]$, and rejection parameter $0.1 \leq R \leq 0.9$.

- Compute $P = AD^{-1}, v^0 = D1_\Gamma$, and $L = I - D^{-1}A$.
- Compute $v(t) = P^i v(0)$.
- Define $\Omega = L_{(1+\epsilon)\hat{n}_1}(v(t))$.
- Let $T$ be the set of column indices of $\gamma \cdot |\Omega|$ smallest components of the vector $|L_{1\Omega}^T| \cdot |L_{1\Omega}|$.
- Set $y := L_{1\Omega \setminus T}$. Let $x^\#$ be the solution to

$$\tag{3.5} \min_{x \in \mathbb{R}^{|V|-|T|}} \|L_{1\Omega \setminus T} x - y\|_2 : \|x\|_0 \leq (1 - \gamma)\hat{n}_1$$

obtained by using $O(\log n)$ iterations of Subspace Pursuit [12].
- Let $W^\# = \{i : x^\#_i > R\}$.

Ensure: $C_1^\# = W^\#$. 

Let $y^m$ be the row sum vector of $L_{-1}^i$. Then the desired solution to the constrained optimization problem

$$\tag{3.7} \min_{x} \|x\|_0 \text{ s.t. } L_{-1}^i x = y^m$$

is $x^* = (1, 1, 0, \cdots, 0)'$. The nonzero components in $x^*$ correspond to the indices of vertices which belong to the target cluster. In reality, we do not have access to $L^i$ or $L_{-1}^i$, what we have access to are $L$ and $L_{-1}$. Also we can relax the exact equality condition to approximately equal to, so the problem becomes

$$\tag{3.8} \min_{x} \|x\|_0 \text{ s.t. } L_{-1} x \approx y,$$
where $y$ is the row sum vector of $L_{-1}$. Let $x^#$ be the solution to (3.8). Suppose our graph has a good underlying clustered structure, i.e., $L_{-1} \approx L_{-1}^{\infty}$ and consequently $y \approx y^{\infty}$, then the difference between $x^#$ and $x^*$ should be small. This gives the intuitive idea of how to apply the idea of compressive sensing to solve local clustering problem.

In a more general setting, we aim at solving the compressive sensing problem

\begin{equation}
\min \|x\|_0 \quad \text{s.t.} \quad \|L_{V\setminus T}x - y\| \leq \epsilon,
\end{equation}

after acquiring the index set $T$ for removal in a smart way. Or equivalently, we can solve

\begin{equation}
\arg\min_{x \in \mathbb{R}^{|V|-|T|}} \{ \|L_{V\setminus T}x - y\|_2 : \|x\|_0 \leq s \}
\end{equation}

for some sparsity $s$, where the vector $y$ is the row sum vector of $L_{V\setminus T}$. The removal set $T$ is selected based on some heuristic criterion from a candidate set $\Omega$ which is obtained from a random walk originates from a set of seed vertices $\Gamma$ with known labels. We will give some more explanation of the reason and the way to choose $T$ in Remark 3.1.

We summarize our idea as Algorithm 1. For convenience, we will assume the target cluster is the first cluster $C_1$ for the rest of discussion. More generally, we can iteratively apply Algorithm 1 and extract all the clusters one at a time.

We want to point out the differences between CS-LCE with its counterparts CP+RWT in [37] and LSC in [38]. The key difference is that the latter two methods only be able to extract target cluster from the initial unperturbed problem:

\begin{equation}
\min \|x\|_0 \quad \text{s.t.} \quad \|L_{V\setminus T}x - y\| \leq \epsilon,
\end{equation}

for some sparsity $s$.

3.2 Theoretical Analysis Our ultimate goal is to make sure the output $C_1^\#$ from Algorithm 1 is close to the true cluster $C_1$. In order to investigate more to this aspect, let us use $x^*$ to denote the solution to the unperturbed problem:

\begin{equation}
x^* := \arg\min_{x \in \mathbb{R}^{|V|-|T|}} \{ \|L_{V\setminus T}x - y^{\infty}\|_2 : \|x\|_0 \leq 0.6n_1 \}
\end{equation}

where $y^{in} = L_{in}1_{V\setminus T}$, and $x^#$ is the solution to (2.5).

Let us now establish the correctness of having $x^*$ equals to an indicator vector as the solution to the noiseless case (3.11), and then conclude that $x^\# \approx x^*$ if $M := L - L^{in}$ is small. Once this is established, we will be able to conclude $C_1^\# \approx C_1$. These results are summarized in the following as a series of theorems and lemmas.

**Theorem 3.1.** Suppose $T \subset C_1$, then $x^* = 1_{C_1 \setminus T} \in \mathbb{R}^{|V|-|T|}$ is the unique solution to (3.11).

**Proof.** Note that for $y^{in} = L_{in}1_{V\setminus T}$, we can rewrite it as $y^{in} = L_{in}1_{V\setminus T}$ where $1 \in \mathbb{R}^{|V|-|T|}$. It is straightforward to check $x^* = 1_{C_1 \setminus T}$ is a solution to (3.11). It remains to show it is unique. Suppose otherwise, then since $L_{V\setminus T}1_{C_1 \setminus T} = y^{in}$, we want to find $x \in \mathbb{R}^{|V|-|T|}$ and $x \neq 1_{C_1 \setminus T}$ such that $L_{V\setminus T}(x - 1) = 0$. Without loss of generality, we can assume the columns of $L$ are permuted such that $L$ is in the block diagonal form. In other words, we have:

\begin{equation}
L_{V\setminus T} = \begin{pmatrix}
L_{C_1 \setminus T} & & \\
& L_{C_2} & \\
& & \ddots
\end{pmatrix}.
\end{equation}

So we have $x - 1 \in Ker(L_{V\setminus T}^{in})$. Let us now show that $L_{C_1 \setminus T}^{in}$ is of full column rank, i.e., the columns of $L_{C_1 \setminus T}^{in}$ is linearly independent.

To show the full rankness, let us first observe the following fact. By Lemma 2.1, each of $L_{C_i \setminus T}^{in}$ has $\lambda = 0$ as an eigenvalue with multiplicity one, and the corresponding eigenspace is spanned by $1_{C_i}$. Now suppose by contradiction that the columns of $L_{C_1 \setminus T}^{in}$ are linearly dependent, so there exists $v \neq 0$ such that $L_{C_1 \setminus T}^{in}v = 0$, or $L_{C_1 \setminus T}^{in}v + L_{T}^{in} \cdot 0 = 0$. This means that $u = (v, 0)$ is an eigenvector associated to eigenvalue zero, which contradicts the fact that the eigenspace is spanned by $1_{C_1}$. Therefore $L_{C_1 \setminus T}^{in}$ is of full column rank.
Now since \(L_{C_1}^n\) is of full column rank, and \(\text{Ker}(L_{C_1}^n) = \text{Span}\{1_{C_1}\}\) for \(i \geq 2\). We conclude that \(x - 1 \in \text{Ker}(L_{V_1}) = \text{Span}\{1_{C_2}, \ldots, 1_{C_n}\}\). Therefore in order to satisfy \(|x| < 0.6n_1\), it is easy to see \(x = 1 - 1_{C_2} - 1_{C_3} - \cdots - 1_{C_n} = 1_{C_1}\), which results in a contradiction by our assumption. □

Next let us show \(x^*\) and \(x^\#\) are close to each other.

**Theorem 3.2.** Suppose \(T \subset C_1\), \(|M|\) \(= o(n^{1/2})\) and \(\delta_{1.8n_1}(L) = o(1)\). Then

\[
(3.12)
\]

\[
\frac{\|x^\# - x^*\|}{\|x^*\|} = o(1).
\]

**Proof.** Recall that \(x^\#\) is the output to (2.5) after \(O(\log n)\) iterations of Subspace Pursuit. By our assumption on \(M\), we have:

\[
\|y - y^m\|_2 = \|L_1 x^m - L^n 1_{V_1 T}\|_2 \leq \|(L - L^n) 1_{V_1 T}\|_2 = \|M\| \|1_{V_1 T}\|_2 \leq o(n^{1/2}). \sqrt{n} = o(1).
\]

Then applying Theorem 2.5 in [37], we get the desired result. □

**Lemma 3.1.** Consider \(K \subset [n]\), any \(v \in \mathbb{R}^n\), and \(W^\# = \{i : v_i > R\}\). If \(||v| - v| \leq D\), then \(|K \Delta W^\#| \leq \min\{(1-R)^2, R^2\}\).

**Proof.** Let \(U^\# = [n] \setminus W^\#\), we can write \(v = v_{U^\#} + v_{W^\#}\) where \(v_{U^\#}\) and \(v_{W^\#}\) are the components of \(v\) supported on \(U^\#\) and \(W^\#\) respectively. Then we have

\[
||1_K - v| |^2 = ||1_K - v_{U^\#} - v_{W^\#}| |^2
\]

\[
= ||1_K \setminus W^\# - v_{U^\#}| |^2 + ||v_{W^\#} \setminus T| |^2
\]

\[
+ ||1_K \setminus W^\# - v_{K \setminus W^\#}| |^2
\]

\[
\geq \min\{(1-R)^2, R^2\}(|K \setminus W^\#| + R^2 \cdot |W^\# \setminus K|)
\]

Therefore \(||1_K - v| | \leq D\) implies \(|K \Delta W^\#| \leq \min\{(1-R)^2, R^2\}\) as desired. □

**Theorem 3.3.** Suppose \(T \subset C_1\), then

\[
(3.13)
\]

\[
\frac{|C_1 \Delta C_1^\#|}{|C_1|} \leq o(1)
\]

**Proof.** It is equivalent to show \(|C_1 \Delta C_1^\#| \leq o(n_1)|\). Note that \(x^* = 1_{C_1 \setminus T}\). We apply Lemma 3.1 with \(K = C_1 \setminus T\), \(W^\# = C_1^\#\) and \(v = x^\#\). By Theorem 3.2, we get \(|1_{C_1 \setminus T} - v| \leq o(\sqrt{n_1})\). Hence by Lemma 3.1 we get \(|(C_1 \setminus T) \Delta C_1^\#| \leq o(n_1)|\). Therefore \(|C_1 \Delta C_1^\#| \leq o(n_1)|\). □

4 Experimental Results

In this section, we evaluate our CS-LCE algorithm on various synthetic and real datasets and compare the performance of our method with baselines. For the datasets, we use simulated stochastic block model, simulated geometric data with three particular shapes, network data on political blogs [3], Optdigits [4], AT&T Database of Faces [5], MNIST data [6] as our benchmark datasets. For the experiments, we select LSC [38], CP+RWT [37], HK-Grow [33], PPR [5], ESSC [59] and LBSA [50] as our baseline methods. For our experiments of stochastic block model, the only target cluster is the most dominant cluster, i.e., the cluster with the highest connection probability. For all other experiments, all the clusters are considered as our target clusters, and we apply Algorithm 2 iteratively to extract all of them. We use the Jaccard index to measure the performance of one cluster task and use accuracy to measure the performance of multiple clusters task. For all experiments, we perform 100 individual runs and take the average. A more detailed descriptions of the experimental setup are given in the supplement.

4.1 Simulated Data

4.1.1 Symmetric Stochastic Block Model

The stochastic block model is a generative model for random graphs with certain edge densities within and between underlying clusters, where the edges within clusters are

Figure 1: Average Jaccard Index on SSBM.
more dense than the edges between clusters. In the case of each cluster has the same size and the intra- and inter-connection probability are the same among all the vertices, we have the symmetric stochastic block model $SSBM(n,k,p,q)$. The parameter $n$ is the size of the graph, $k$ is the number of clusters, $p$ is the probability of intra-connectivity, and $q$ is the probability of inter-connectivity. In our experiments, we fix $k = 3$, let $n$ vary from 600, 1200, 1800, 2400, 3000, and choose $p = 5 \log n/n$, $q = \log n/n$. With five labeled vertices as seeds, we achieve the performances shown in Fig. 1. We can see CS-LCE algorithm clearly outperforms all other baselines, especially when the size of target cluster is relatively small.

### 4.1.2 Non-symmetric Stochastic Block Model

In a more general stochastic block model $SBM(n,k,P)$, where $n$ and $k$ are the same as symmetric case, matrix $P$ indicates the connection probability within each individual cluster and between different clusters. It is worthwhile to note that the information theoretical bound for exact cluster recovery in $SBM$ are given in [1] and [2]. In our experiments, we fix $k = 3$, and the size of clusters are chosen as $n = (n_1, 2n_1, 3n_1)$ where $n_1$ is chosen from $\{200, 400, 600, 800, 1000\}$. We set the connection probability matrix $P = (p,q,q,p,q,q,q,p,q)$ where $p = \log^2(8n_1)/(8n_1)$ and $q = 5 \log(8n_1)/(8n_1)$. With five labeled vertices as seeds, we achieve comparable results than the best baseline as shown in Fig. 2.

### 4.1.3 Geometric Data

We also simulated three high dimensional datasets in Euclidean space where the projections of the clusters onto two dimensional plane look like three lines, three circles, or three moons. See Fig. 3 for an illustration of them. These datasets are often used as benchmark for data clustering and they are also described in [43] with slightly different parameters. Because of the shape of underlying clusters, traditional $k$-means clustering or spectral clustering fail on these contrived datasets. The average accuracy of CS-LCE compared with LSC [38] and CP+RWT [37] are given in Table 1. A more detailed description of this simulated dataset is given in the supplement.

### 4.2 Network Data

“The political blogosphere and the 2004 US Election” [3] dataset contains a list of political blogs that were classified as liberal or conservative and links between the blogs. There are two underlying clusters in this dataset. The state-of-the-art result on this dataset is given by Abbe and Sandon in [2]. Their simplified algorithm gave a successful classification 37 times out of 40 trials, and each of the successful trials correctly classified all but 56 to 67 of the 1222 vertices in the graph main component. In our experiments, given one labeled seed, Algorithm 1 succeeds 35 trials out of a total of 40 trials. Among these 35 successful trials, the average number of misclassified node in the graph main component is 49, which is comparable to the state-of-the-art result. We note that LSC in [38] also succeeds 35 out of 40 trials, but the average number of misclassified node equals to 55. We also note that CP+RWT in [37] fails on this dataset.
Table 2: Clustering Accuracy on AT&T Data (%).

| Labeled Ratio % | 10  | 20  | 30  |
|-----------------|-----|-----|-----|
| CS-LCE          | 98.4| 100 | 100 |
| LSC [38]        | 96.5| 97.5| 98.2|
| CP+RWT [37]     | 92.2| 95.7| 97.1|
| SC              | 95.8| 95.8| 95.8|

We test CS-LCE, LCS, CP+RWT, and some other modern semi-supervised methods on these two datasets, the results are show in Table 2, Table 3, and Table 4. Note that in Table 5 we compare CS-LCE with several other constraint clustering algorithms. In each constrained clustering algorithms, the total number of pairwise constraints are set to equal to the total data points. Therefore in order to have a fair comparison, we choose a certain amount of labeled data in CS-LCE such that the total pairwise constraints are the same.

4.4 Digits Data

4.4.1 OptDigits This dataset contains grayscale images of handwritten digits from 0 to 9 of size 8×8. There are a total of 5620 images and each cluster has approximately 560 images. The average accuracy of Algorithm 1 compared with several other algorithms are shown in Fig. 6, we exclude PPR and ESSC in the comparison as they either run too slow or the accuracy is too low.

4.4.2 MNIST and USPS The MNIST dataset consists of 70000 grayscale images of the handwritten digits 0-9 of size 28 × 28 with approximately 7000 images of each digit. The USPS data set contains 9298 grayscale images, obtained from the scanning of handwritten digits from envelopes by the U.S. postal service.

Table 3: Clustering Accuracy on MNIST Data (%).

| Label Ratio % | CS-LCE | LSC [38] | CP+RWT [37] |
|---------------|--------|----------|-------------|
| 0.02          | 83.6   | 71.5     | 69.2        |
| 0.04          | 88.3   | 79.2     | 76.3        |
| 0.06          | 92.5   | 87.7     | 83.0        |

Table 4: Clustering Accuracy on USPS Data (%).

| Label Ratio % | CS-LCE | LSC [38] | CP+RWT [37] |
|---------------|--------|----------|-------------|
| 0.2           | 78.6   | 74.4     | 60.0        |
| 0.3           | 79.9   | 76.7     | 65.1        |
| 0.4           | 84.4   | 80.6     | 64.6        |

5 Conclusion

In this work, we proposed a new semi-supervised local clustering algorithm based on the idea of compressive sensing. The proposed algorithm improves the disadvantages in the two prior works under the same framework and are shown to be asymptotically correct under certain assumptions of graph structure. The experiments on various simulated and real datasets validate our results. We hope this work will draw people’s interests and bring attentions to this new framework. Potential research directions in the future could be done on developing a more calibrated way of choosing the removal set and investigating on methods which can incorporate the compressive sensing technique into some modern architecture such as deep neural network.

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Table 5: Clustering Results Measured by Accuracy (%)

| Data     | CS-LCE | KM-cst | AE+KM [12] | AE+KM-cst | DEC [59] | IDEC [25] | SDEC [49] |
|----------|--------|--------|------------|-----------|--------|--------|--------|
| MNIST    | 96.02  | 54.27  | 74.09      | 75.98     | 84.94  | 83.85  | 86.11  |
| USPS     | 82.10  | 68.18  | 70.28      | 71.87     | 75.81  | 75.86  | 76.39  |

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