Non-occurrence of gap for one-dimensional non-autonomous functionals

Carlo Mariconda

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Abstract
Let $F(y) = \int_T^T L(s, y(s), y'(s)) \, ds$ be a positive functional defined on the space $W^{1,p}(I, T; \mathbb{R}^n)$ ($p \geq 1$) of Sobolev functions with, possibly, one or both end point conditions. It is important, especially for the applications, to be able to approximate the infimum of $F$ with the values of $F$ along a sequence of Lipschitz functions satisfying the same boundary condition(s). Sometimes this is not possible, i.e., the so called Lavrentiev phenomenon occurs. This is the case of the seemingly innocent Manià’s Lagrangian $L(s, y, y') = (y^3 - s)^2 (y')^6$ and boundary data $y(0) = 0$, $y(1) = 1$; nevertheless in this situation the phenomenon does not occur with just the end point condition $y(1) = 1$. The paper focuses about the different set of conditions that are needed to avoid the Lavrentiev phenomenon for problems depending on the number of end point conditions that are considered. Under minimal assumptions on the (possibly) extended value, Lagrangian, we ensure the non-occurrence of the Lavrentiev phenomenon with just one end point condition, thus extending a milestone result by Alberti and Serra Cassano to non-autonomous case. We then introduce an additional hypothesis, satisfied when the Lagrangian is bounded on bounded sets, in order to ensure the non-occurrence of the phenomenon when dealing with both end point conditions; the result gives some new light even in the autonomous case.

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Carlo Mariconda

carlo.mariconda@unipd.it

Dipartimento di Matematica “Tullio Levi-Civita”, Università degli Studi di Padova, Via Trieste 63, 35121 Padua, Italy
1 Introduction

The state of the art

We consider here a one-dimensional, vectorial functional of the calculus of variations

\[ F(y) = \int_t^T L(s, y(s), y'(s)) \, ds \]

defined on the space of Sobolev functions \( W^{1,p}(I; \mathbb{R}^n) \) on \( I := [t, T] \) with values in \( \mathbb{R}^n \), for some \( p \geq 1 \). The Lagrangian \( L(s, y, u) \) is Borel and is assumed to have values in \([0, +\infty[ \cup \{+\infty\}\). Following the terminology of control theory we will refer to \( s \) as the time variable, to \( y \) as the state variable, and to \( u \) as the velocity variable. The space of absolutely continuous functions provides the correct framework in order to find a minimizer of \( F \): Tonelli’s Theorem ensures its existence for any given boundary data when \( L(s, \cdot, \cdot) \) is lower semicontinuous, \( L \) has a superlinear growth, and \( L(s, y, \cdot) \) is convex. In order to approximate the value of the infimum of \( F \) (e.g., by means of numerical methods), one may be tempted to approximate in the space of absolutely continuous functions, a given minimizer \( y \in W^{1,p}(I; \mathbb{R}^n) \) with a sequence of Lipschitz functions \((y_h)_h\) sharing the same boundary data of \( y \), in such a way that \( \lim_{h \to 0} F(y_h) = F(y) \). It turns out however, that this may not be possible: we say that the Lavrentiev phenomenon occurs. This is the case, for instance, for the innocent-looking Manià’s [20] problem

\[ \min F(y) := \int_0^1 (y^3 - s)^2 (y')^6 \, ds : y \in W^{1,1}(I), \quad y(0) = 0, \quad y(1) = 1 \]

described in Example 2.5: clearly the minimum is obtained for \( y_+(s) := s^{1/3} \), however it can be shown that there is \( \varepsilon > 0 \) such that \( F(z) > \varepsilon \) whenever \( z : [0, 1] \to \mathbb{R} \) is Lipschitz and \( z(0) = y_+(0) = 0, \quad z(1) = y_+(1) = 1 \). As is pointed out in [6], it is interesting to note that things change drastically if one allows the approximating sequence to have a different initial datum: indeed the sequence \((y_h)_h\) of Lipschitz functions given by

\[ y_h(s) := \begin{cases} \frac{1}{h^{1/3}} & \text{if } s \in [0, 1/h], \\ s^{1/3} & \text{otherwise,} \end{cases} \]

converges to \( y \) and \( F(y_h) \to F(y) \). Another celebrated example, due to Ball and Mizel [2], is provided with a polynomial Lagrangian in \((s, y, u)\) that fully satisfies Tonelli’s existence conditions (whereas Manià’s is not coercive).

When a minimizer exists, the most direct way to exclude the Lavrentiev phenomenon is to provide conditions that ensure the Lipschitz continuity of the minimizer itself. In the autonomous case it turns out, starting from the work [15] of Clarke and Vinter, up to its refinement by Dal Maso and Frankowska [16], that the hypotheses of Tonelli’s existence theorem (even without convexity) ensure that the minimizers are Lipschitz. Actually, weaker growth conditions than superlinearity ensure both existence and that they are Lipschitz: we refer to the pioneer work [13] of Clarke and to the subsequent papers of Cellina and his school (see [8, 10, 21]). In the non-autonomous case, there are functionals satisfying Tonelli’s condition whose minimizers are not Lipschitz or worse, that exhibit the Lavrentiev phenomenon. Condition (S), an additional local Lipschitz continuity assumption on the “time” variable \( s \) of \( L(s, y, u) \) (thus always satisfied in the autonomous case), formulated in Sect. 2.2, is enough to ensure they are Lipschitz: this fact was established by Clarke [13] and generalized.
by Bettiol and Mariconda [4]. Property (S) is known to be a sufficient condition for the validity of the Du Bois-Reymond equation; we refer to [11] for the smooth case, to [13] for the nonsmooth convex case under weak growth assumptions, and to [3, 4] by Bettiol and the author in the general case.

Many functionals arising from applications do not fulfill, however, known existence criteria and the previous regularity approach may thus not be pursued. In that case, looking for the non-occurrence of the Lavrentiev phenomenon is even more challenging and interesting. It is easy if one requires some “natural growth assumptions” from above and continuous Lipschitz functions converging to a datum. Therefore, in particular, the Lavrentiev phenomenon does not occur. Instead, the validity of the Du Bois-Reymond equation; we refer to [11] for the smooth case, to [13] for the nonsmooth case, and to [4] by Bettiol and Mariconda [24] for a general class of non-autonomous Lagrangians.

Here again, the autonomous case stands on its own: Alberti and Serra Cassano [1, Theorem 2.4] state that if \( L(s, y, u) = \Lambda(y, u) \) is just Borel and satisfies

\[
\forall K > 0 \exists r_K > 0 \quad \Lambda \text{ is bounded on } B_K \times B_{r_K}, \quad (B_A)
\]

then, given \( y \in W^{1,p}(I; \mathbb{R}^n) \) such that \( \Lambda(y, y') \in L^1(I; \mathbb{R}^n) \), there is a sequence \((y_h)_h\) of Lipschitz functions converging to \( y \) in \( W^{1,p}(I; \mathbb{R}^n) \) such that \( \Lambda(y_h) \to \Lambda(y) \) and \( y_h(t) = y(t) \) for all \( h \); we say in this case that there is no Lavrentiev gap at \( y \) (here from now on \( B_r \) denotes the closed ball with the origin as center and radius \( r \) in \( \mathbb{R}^n \)) for the initial prescribed datum. Therefore, in particular, the Lavrentiev phenomenon does not occur. Instead, the violation of Assumption (B_A), may lead to the occurrence of the phenomenon, as shown in Example 3.6. Though [1, Remark 2.8] conjectures that the result does still hold for the variational problem with both end point conditions, the proof of [1, Theorem 2.4] does not ensure that the resulting Lipschitz sequence preserves both boundary data: Example 3.5 (Alberti, personal communication) actually shows that this is not just a technical issue. The importance of the boundary datum, and the difficulty of preserving it, was noticed while exploring the recent literature concerning the multidimensional case by Bousquet et al. [5], Treu and Mariconda [22, 23].

In the non-autonomous case, the examples (see Manià’s) show that some additional conditions have to be added: To the author’s knowledge, starting from Lavrentiev himself in [17], most of the criteria for the avoidance of the Lavrentiev phenomenon require some regularity in the state variable (say locally Lipschitz or Hölder) (see [18, 25, 26]). As was pointed out by Carlson [7], many of them can actually be obtained as a consequence of a property introduced by Cesari and Angell [12]. There are however a few exceptions: the non-occurrence of the Lavrentiev phenomenon was established for two end point conditions without the above regularity hypotheses on the state variable by:

- Cellina et al. [9] for a class of real valued Lagrangians of the form \( L(s, y, u) = \Lambda(y, u)\Psi(s, y) \) assuming the continuity of \( \Lambda(y, u) \), the positivity condition \( \inf \Psi \geq m > 0 \), and the convexity of \( u \mapsto \Lambda(y, u) \). Note that the Lagrangian in Manià’s example, the product of \( \Lambda(y, u) = u^6 \) with \( \Psi(s, y) = (y - s^2)^2 \), violates the positivity condition since \( \Psi \) vanishes (actually along the minimizer).
- Mariconda [24] for a general class of non-autonomous Lagrangians \( L(s, y, u) \), assuming the local Lipschitz Condition (S) on the time variable \( s \), radial convexity in the velocity variable, and a linear growth from below in the velocity variable. Condition (S) for \( \Lambda \) is not a technical option: the lack of its validity may lead to the Lavrentiev phenomenon, as shown by the example of Ball and Mizel (see Example 2.2). In the extended valued case, it is assumed, moreover, that \( \Lambda \) tends with some uniformity to \( +\infty \) as the distance to the boundary of the effective domain \( \text{Dom}(\Lambda) \) (i.e., the set where \( \Lambda \) is finite) tends to \( 0 \). As in [1], the lack of a priori regularity of the Lagrangian is compensated by requiring some
kind of local boundedness conditions, including \((B_{\Lambda})\), adapted to the non-autonomous case.

The main results

Some questions arise naturally:

1. Is it true, as suggested by [1, Remark 2.8], that the Lavrentiev phenomenon for the problem with two end point conditions does not occur in the case of a real valued, positive, autonomous Lagrangian that is bounded on bounded sets?

2. Find a set of assumptions, possibly less restrictive than those presented in [9, 24], that guarantee, even in the autonomous case, the non-occurrence of the Lavrentiev phenomenon for the two end points conditions problem.

3. May one extend the results of [1] and give sufficient conditions for the non-occurrence of the Lavrentiev phenomenon with just one end point constraint for non-autonomous Lagrangians?

The main result formulated here, Theorem 3.1, sheds some light on all the quoted problems, and applies to the wider class of Lagrangians of the form

\[
L(s, y, u) = \Lambda(s, y, u)\Psi(s, y), \quad \Lambda, \Psi \geq 0, \tag{1.1}
\]

with different sets of hypotheses for \(\Lambda\) and for \(\Psi\).

All of the results in the present paper assume Condition (S) on \(s \mapsto \Lambda(s, y, u)\), and the continuity of \(s \mapsto \Psi(s, y)\). Note that this class of product functions is strictly larger than the one of Lagrangians satisfying (S): For instance, the Lagrangian \(L(s, y, u) := \Lambda(s, y, u)\Psi(s, y)\), with

\[
\forall s \in [0, 1], \forall y, u \in \mathbb{R} \quad \Lambda(s, y, u) := u^2, \quad \Psi(s, y) = \sqrt{s}
\]

does not satisfy (S), whereas \(\Lambda\) does.

In Sect. 3 we extend the result of Alberti and Serra Cassano [1, Theorem 2.4] to the wider class of Lagrangians (1.1), for the variational problem with (just) one end point constraint

\[
\min\{F(y) : y \in W^{1, p}(I; \mathbb{R}^n), y(t) = X \in \mathbb{R}^n\}. \tag{PX}
\]

Answering Question 3, Corollary 3.7 shows the non-occurrence of the Lavrentiev phenomenon once \(\Lambda\), other than Condition \((B_{\Lambda})\) (obviously adapted to the non-autonomous case), satisfies Condition (S). Regarding \(\Psi\), it is enough that:

\((P_{\Psi})\) \(\inf \Psi > 0\).

\((B_{\Psi})\) \(\Psi\) is bounded on bounded sets.

Concerning Question 2, we show that, given \(y \in W^{1, p}(I; \mathbb{R}^n)\) such that \(F(y) < +\infty\), there is no Lavrentiev gap for the two end point conditions problem once one assumes, moreover, that:

\((B_{y, \Lambda}^+)\) There is a neighbourhood \(O_y\) of \(y(I)\) such that, for all \(r > 0\), \(\Lambda\) is bounded on \(I \times O_y \times B_r\).

Hypothesis \((B_{y, \Lambda}^+)\) was conjectured by Alberti in a personal communication for the autonomous case. As shown in Example 3.5, its lack may indeed lead to the gap. Note that Hypothesis \((B_{y, \Lambda}^+)\) subsumes the fact that \(\Lambda\) is real valued on an infinite strip. Thus, in order to be satisfied for any admissible trajectory \(y\), we require in Corollary 3.7 that \(\Lambda\) is
bounded on bounded sets (and thus a fortiori, that $\Lambda$ is not extended valued). This gives, in any case, a reassuring positive answer to Question 1, confirming the statement in [1, Remark 2.8].

When $\Psi \equiv 1$, the conclusions of Theorem 3.1 and Corollary 3.7 do not overlap with those obtained by the author in [24, Corollary 6.7]. Indeed, some extra hypotheses are assumed there, e.g., the radial convexity of $0 < r \mapsto \Lambda(s, z, ru)$ and the uniform limit to $+\infty$ at the boundary of the effective domain; as a counterpart, unlike the results of this paper, the Lipschitz approximating sequence built in [24] may preserve a prescribed state constraint, an outcome that is out of reach with the methods of the present paper.

The proof of Theorem 3.1 follows the path of that of [1, Theorem 2.4], especially for the case with one end point constraint. Given $y \in W^{1,p}(I; \mathbb{R}^n)$ such that $F(y) < +\infty$, we build a sequence $(y_h)_h$ of Lipschitz functions satisfying the required boundary conditions/state constraints, and:

1. $\lim_{h \to +\infty} F(y_h) = F(y)$ (approximation in “energy”);
2. $y_h \to y$ in $W^{1,p}(I; \mathbb{R}^n)$ (approximation in norm).

We begin by considering a standard approximating sequence $(z_h)_h$ of Lipschitz functions such that $z_h' = y'$ everywhere except at most an open subset $A_h$ whose measure tends to 0 as $m \to +\infty$. We then reparametrize each $z_h$, i.e., we set $y_h := z_h \circ \psi_h$ for a suitable Lipschitz, strictly increasing function $\psi_h : I \to I$. A delicate point here is that, in general, $(B_A)$ and the assumptions on $\Psi$ alone do not ensure that $\psi_h(I) = I$, i.e., $\psi_h(T)$ may be strictly less than $T$, and hence $y_h(T)$ may differ from $y(T)$. This is where, for the two end point conditions problem, Hypothesis $(B^+_{y, \Lambda})$ plays a role. It allows to “increase” the slope of $\psi_h$ on a sufficiently small subset of the complement of $A_h$, where $y'$ is bounded, in such a way that $\psi_h(T) = T$.

2 Notation and basic assumptions

2.1 Notation

We introduce the main recurring notation:

- The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$, the closed ball of $\mathbb{R}^n$ of radius $K \geq 0$ centered at the origin is denoted by $B_K$;
- The Lebesgue measure of a subset $A$ of $I = [t, T]$ is $|A|$ (no confusion can occur with the Euclidean norm);
- If $y : I \to \mathbb{R}^n$ is a function, we denote by $y(I)$ its image, by $\|y\|_\infty$ its sup norm, and by $\|y\|_p$ its norm in $L^p(I; \mathbb{R}^n)$;
- The characteristic function of a set $A$ is $\chi_A$.
- If $x \in \mathbb{R}$, we denote by $x^+$ its positive part, and by $x^-$ its negative part;
- $\text{Lip}(I; \mathbb{R}^n) = \{y : I \to \mathbb{R}^n, y \text{ Lipschitz}\}$; if $n = 1$ we simply write $\text{Lip}(I)$;
- For $p \geq 1$, $W^{1,p}(I; \mathbb{R}^n)$ is the Sobolev space
  
  $W^{1,p}(I; \mathbb{R}^n) = \{y : I \to \mathbb{R}^n : y, y' \in L^p(I; \mathbb{R}^n)\}$;

  if $n = 1$ we simply write $W^{1,p}(I)$. 

\[ \text{Springer} \]
2.2 Basic assumptions and condition (S)

Let $p \geq 1$, $n \geq 1$ in $\mathbb{N}$, $I = [t, T]$, $t < T$. The functional $F$ (sometimes referred to as the “energy”) is defined by

$$\forall y \in W^{1,p}(I; \mathbb{R}^n) \quad F(y) := \int_I \Lambda(s, y(s), y'(s)) \Psi(s, y(s)) \, ds.$$  

**Basic Assumptions** We assume the following conditions.

- $\Lambda: I \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[ \cup \{+\infty\}$, $(s, y, u) \mapsto \Lambda(s, y, u)$ is Lebesgue–Borel measurable in $(s, (y, u))$, i.e., measurable with respect to the $\sigma$-algebra generated by the products of Lebesgue measurable subsets of $I$ (for $t$) and Borel measurable subsets of $\mathbb{R}^n \times \mathbb{R}^n$ (for $(y, u)$): this guarantees that if $y, u: I \to \mathbb{R}^n$ are measurable then $s \mapsto \Lambda(s, y(s), u(s))$ is measurable (see [14, Proposition 6.34]);
- $\Psi: I \times \mathbb{R}^n \to [0, +\infty[ \cup \{+\infty\}$ is Borel and continuous in its first variable, i.e., $s \mapsto \Psi(s, z)$ is continuous for all $z \in \mathbb{R}^n$.
- The effective domain of $\Lambda$, given by
  $$\text{Dom}(\Lambda) := \{(s, y, u) \in I \times \mathbb{R}^n \times \mathbb{R}^n : \Lambda(s, y, u) < +\infty\}$$
  is of the form $\text{Dom}(\Lambda) = I \times D_\Lambda$, with $D_\Lambda \subseteq \mathbb{R}^n \times \mathbb{R}^n$: thus, for all $s' \in I$, $(y, u) \in \mathbb{R}^n \times \mathbb{R}^n$, $\Lambda(s', y, u) < +\infty$ whenever $\Lambda(s, y, u) < +\infty$ for some $s \in I$.

We consider the following local Lipschitz condition (S) on the first variable of $\Lambda$: it is automatically fulfilled whenever $\Lambda$ is autonomous, i.e., if $\Lambda$ depends just on $(y, u)$.

**Condition (S)** For every $K \geq 0$ there are $\kappa, \beta \geq 0$, $\gamma \in L^1(I)$, $\varepsilon_* > 0$ satisfying, for a.e. $s \in I$

$$|\Lambda(s_2, z, u) - \Lambda(s_1, z, u)| \leq (\kappa \Lambda(s, z, u) + \beta |u|^p + \gamma(s)) |s_2 - s_1| \quad (2.1)$$

whenever $s_1, s_2 \in [s - \varepsilon_*, s + \varepsilon_*] \cap I$, $z \in B_K$, $u \in \mathbb{R}^n$, $(s, z, u) \in \text{Dom}(\Lambda)$.

**Remark 2.1** Condition (S) is fulfilled if $\Lambda = \Lambda(y, u)$ is autonomous. In the smooth setting, Condition (S) ensures the fulfillment of the Erdmann–Du Bois-Reymond (EDBR) condition. In this more general framework it plays a key role in Lipschitz regularity under slow growth conditions [4, 13, 24] and ensures the validity of the EDBR for real valued Borel, possibly nonsmooth, Lagrangians [3, 4].

**Example 2.2** (Ball–Mizel’s celebrated Lagrangian does not satisfy (S)) Ball and Mizel [2] introduce a Lagrangian of the form

$$\Lambda(s, y, u) = (s^4 - y^6)^2 u^{28} + cu^2, \quad c > 0.$$  

The polynomial $\Lambda$ is convex in the variable $u$ and has a superlinear growth (with $p = 2$). It provides an example of various pathologies: for specific boundary data it exhibits the Lavrentiev phenomenon, minima do not satisfy neither the Euler nor the Du Bois-Reymond equation (see also [4]). Notice that $\Lambda$ does not satisfy (S). Indeed, assume the contrary. Then there are $\varepsilon_* > 0$, $\kappa, \beta \geq 0$, $\gamma \in L^1(I)$ such that (2.1) holds for every $s \in [0, 1]$, $y \in [0, 1]$. $s_1, s_2 \in [s - \varepsilon_*, s + \varepsilon_*] \cap [0, 1]$. It is not restrictive to assume that $\varepsilon_* < 1/4$. By taking $s \in [1/2, 3/4]$, $s_1 = s$, $s_2 = s + \varepsilon_*$, $y = y(s) = s^{2/3}$ and $K = 1$ in (2.1) we obtain, for all $u$

$$c_1 u^{28} \leq (c_2 u^2 + \gamma(s)) \varepsilon_*$$

for suitable positive constants $c_1, c_2$, whence a contradiction.
2.3 Two variational problems

We shall consider different variational problems associated to the functional $F$, with different end point conditions and/or state constraints. Let $X, Y \in \mathbb{R}^n$. We consider the following end point conditions.

- **Free end points** We consider the free end point variational problem
  \[
  \text{Minimize } \{ F(y) : y \in W^{1,p}(I; \mathbb{R}^n) \}, \quad (P)
  \]
  and assume that $\inf(P) < +\infty$.

- **One end point** We set $\Gamma_X := \{ y \in W^{1,p}(I; \mathbb{R}^n) : y(t) = X \}$, and the corresponding variational problem
  \[
  \text{Minimize } \{ F(y) : y \in \Gamma_X \}, \quad (P_X)
  \]
  and assume that $\inf(P_X) < +\infty$.

- **Both end points** We set $\Gamma_{X,Y} := \{ y \in W^{1,p}(I; \mathbb{R}^n) : y(t) = X, y(T) = Y \}$, and the corresponding variational problem
  \[
  \text{Minimize } \{ F(y) : y \in \Gamma_{X,Y} \}, \quad (P_{X,Y})
  \]
  and assume that $\inf(P_{X,Y}) < +\infty$.

There is no privilege in considering the initial condition $y(t) = X$ instead of the final one $y(T) = Y$ for the one end point conditions problem: any result obtained here can be reformulated for a final end point condition variational problem, with the same set of assumptions.

2.4 Lavrentiev gap at a function and Lavrentiev phenomenon

In this paper we consider different boundary data for the same integral functional.

**Definition 2.3** (Lavrentiev gap) Let $X, Y \in \mathbb{R}^n$, $\Gamma \in \{ W^{1,p}(I; \mathbb{R}^n), \Gamma_X, \Gamma_{X,Y} \}$ and $y \in \Gamma$ be such that $F(y) < +\infty$. We say that the **Lavrentiev gap** does not occur at $y$ for the variational problem corresponding to $\Gamma$ if there exists a sequence $(y_h)_{h \in \mathbb{N}}$ of functions in $\text{Lip}(I, \mathbb{R}^n)$ satisfying:

1. $\forall h \in \mathbb{N}, y_h \in \Gamma$;
2. $\lim_{h \to +\infty} F(y_h) = F(y)$;
3. $y_h \to y$ in $W^{1,p}(I; \mathbb{R}^n)$.

We say that the **Lavrentiev phenomenon** does not occur for the variational problem corresponding to $\Gamma$ if

\[
\inf_{y \in W^{1,p}(I; \mathbb{R}^n)} F(y) = \inf_{y \in \text{Lip}(I; \mathbb{R}^n)} F(y). \quad (2.2)
\]

**Remark 2.4** (Lavrentiev gap versus Lavrentiev phenomenon)

- Let $y \in W^{1,p}(I; \mathbb{R}^n)$. The non-occurrence of the Lavrentiev gap at $y$ ensures that, given $\varepsilon > 0$, there is a Lipschitz $\varepsilon$ better competitor, i.e., a Lipschitz function $\overline{y} \in \Gamma$ such that $F(\overline{y}) \leq F(y) + \varepsilon$. 
The non-occurrence of the Lavrentiev gap along a minimizing sequence implies the non-occurrence of the Lavrentiev phenomenon for the same variational problem.

The non-occurrence of the gap at \( y \) for \((\mathcal{P}_X, Y)\) for any choice of \( Y \in \mathbb{R}^n \) implies the non-occurrence of the gap at \( y \) for \((\mathcal{P}_X)\). Similarly, the non-occurrence of the gap for \((\mathcal{P}_X)\) as \( X \) varies in \( \mathbb{R}^n \) implies the same for \((\mathcal{P})\).

The following celebrated example motivates the need to distinguish problems with just one end point condition from problems with conditions at both end points.

**Example 2.5** (Manià [20]) Consider the problem of minimizing

\[
F(y) = \int_0^1 (y^3 - s)^2(y')^6 \, ds : y \in W^{1,1}([0, 1]), \ y(0) = 0, \ y(1) = 1. \quad (\mathcal{P}_{0,1})
\]

Then \( y_*(s) := s^{1/3} \) is a minimizer and \( F(y_*) = 0 \). Not only is \( y_* \) not Lipschitz, it turns out (see [6, §4.3]) that the Lavrentiev gap occurs at \( y_* \), and

\[
0 = \min F = F(y_*) < \inf \{ F(y) : y \in \text{Lip}([0, 1]), \ y(0) = 0, \ y(1) = 1 \}.
\]

However, as is noted in [6], the situation changes drastically if one takes into account just the final end point condition. Indeed it turns out that the sequence \((y_h)_h\), where each \( y_h \) is obtained by truncating \( y_* \) at \( 1/h, h \in \mathbb{N}_{\geq 1} \), as follows,

\[
y_h(s) := \begin{cases} \frac{1}{h} & \text{if } s \in [0, 1/h], \\
\frac{s^{1/3}}{h} & \text{otherwise}, \end{cases}
\]

is a sequence of Lipschitz functions satisfying (Fig. 1)

![Fig. 1](image-url) The function \( y_h \) in Example 2.5
\[ y_h(1) = y(1) = 1, \quad F(y_h) \rightarrow F(y_*), \quad y_h \rightarrow y_* \text{ in } W^{1,1}([0, 1]). \]

Therefore, unlike the initial two boundary data constraint case, no Lavrentiev phenomenon occurs for the variational problem

\[
\min F(y) = \int_0^1 (y^3 - s)^2(y')^6 \, ds : y \in W^{1,1}([0, 1]), \quad y(1) = 1.
\]

3 Non-occurrence of the Lavrentiev gap/phenomenon for \((P_X)\) and for \((P_{X,Y})\)

Let \(X, Y \in \mathbb{R}^n\). We consider here both problems \((P_X)\) and \((P_{X,Y})\).

3.1 Non-occurrence of the Lavrentiev gap at \(y \in W^{1,p}(I; \mathbb{R}^n)\)

Theorem 3.1 gives some sufficient conditions that ensure the non-occurrence of the gap at a specific function.

**Theorem 3.1** (Non-occurrence of the Lavrentiev gap) Suppose the validity of the Basic Assumptions, that \(\Lambda\) satisfies Condition (S) and let \(y \in W^{1,p}(I; \mathbb{R}^n)\) with \(y(t) = X, y(T) = Y\) be such that \(F(y) < +\infty\). Assume

\((P_{y,\Psi})\) \(\exists m_{y,\Psi} > 0 \quad \Psi(s, z) \geq m_{y,\Psi} \text{ for all } s \in I, z \in y(I),\)

and that there is a neighbourhood \(O_y\) of \(y(I)\) in \(\mathbb{R}^n\) such that:

\((B_{y,\Psi})\) \(\Psi\) is bounded on \(I \times O_y;\)
\((B_{y,\Lambda})\) There is \(r_y > 0\) such that \(\Lambda\) is bounded on \(I \times O_y \times B_{r_y}\).

Then:

1. There is no Lavrentiev gap for \((P_X)\) at \(y\).
2. There is no Lavrentiev gap for \((P_{X,Y})\) at \(y\) assuming, instead of \((B_{y,\Lambda})\):

\((B_{y,\Lambda}^+)\) For all \(r > 0\), \(\Lambda\) is bounded on \(I \times O_y \times B_r\).

**Remark 3.2** Notice that Condition \((P_{y,\Psi})\) and the fact that \(F(y) < +\infty\) imply that

\[ \Lambda(s, y(s), y'(s)) \in L^1(I). \]

Indeed, if \(\Psi \geq m_{y,\Psi}\) on \(I \times y(I)\), then

\[
\int_I^T \Lambda(s, y(s), y'(s)) \, ds \leq \frac{1}{m_{y,\Psi}} F(y) < +\infty. \tag{3.1}
\]

The proof of Theorem 3.1 shows actually that in order to obtain the conclusion of Claim 1 it is enough, instead of \((P_{y,\Psi})\), that \(\Lambda(s, y(s), y'(s)) \in L^1(I)\) holds (see Step ix of the proof).

**Remark 3.3** Note that Hypothesis \((B_{y,\Lambda}^+)\), required for the two end point problem, has an impact on the geometry of the effective domain, which must therefore contain the unbounded strip \(I \times O_y \times \mathbb{R}^n\). Theorem 3.1 extends the known literature on the subject in several directions.
1. In the autonomous case with $\Lambda(s, y, u) = \Lambda(y, u)$ and $\Psi \equiv 1$, Claim 1 was proven by Alberti and Serra Cassano [1, Theorem 2.4]. The need of Assumption $(B_{y, \Lambda}^+)$ in order to obtain Claim 2 was conjectured by Alberti in a personal communication.

2. In the non-autonomous case with $\Lambda(s, y, u) = \Lambda(y, u)$ and $\Psi(s, y)$ both real valued, continuous and the additional convexity assumption on $u \mapsto \Lambda(y, u)$, Cellina, Ferriero and Marchini prove in [9, Theorem 1] a weaker version of Claim 2, namely the existence of a Lipschitz sequence $(y_h)_h$ for $(P_{x, y})$ such that $\lim_{h \to} F(y_h) = F(y)$.

3. In the non-autonomous case, with $\Psi \equiv 1$ and $\Lambda$ not necessarily autonomous, the above weaker version of Claim 2 (convergence in energy but not in norm) was proven by the author in [24, Theorem 5.1, Corollary 5.7] assuming $(B_{y, \Lambda}^+)$ and an additional set of assumptions: radial convexity of $u \mapsto \Lambda(s, y, u)$, boundedness on the relatively compact subsets of the effective domain, and a uniform limit to $+\infty$ approaching the boundary of the domain. The proof there is based on building a suitable sequence of Lipschitz reparametrizations $y_h$ of $y$: this has the advantage, with respect to the technique used in the proof of Theorem 3.1, of preserving the image of $y$ in $\mathbb{R}^n$ (and thus, possibly, a given state constraint) along the required sequence.

Example 2.5 shows that Condition $(P_{x, \Psi})$ in Theorem 3.1 is not a technical one.

**Example 3.4** Consider Manià’s Example 2.5. The Lagrangian may be written as

$$L(s, y, u) = (y^3 - s)^2 u^6 = \Lambda(u)\Psi(s, y), \quad \Psi(s, y) = (y^3 - s)^2, \quad \Lambda(u) = u^6.$$ 

Consider the minimizer $y_*(s) := s^{1/3}$. All of the assumptions of Theorem 3.1 are satisfied, except $(P_{x, \Psi})$, since $\Psi$ vanishes along $(s, y_*(s))$. As a matter of fact, the Lavrentiev gap occurs at $y_*$, for the variational problem with $(just)$ the initial prescribed datum

$$\min F(y) := \int_0^1 (y^3 - s)^2(y')^6 \, ds, \quad y(0) = 0.$$ 

Indeed, if $y_h$ is a sequence of Lipschitz functions that converges to $y$ in $W^{1,1}$, then $y_h(1)$ belongs definitely in any given neighborhood of $y_*(1) = 1$, say in $[3/4, 3/2]$; the arguments of [6, §4.3] show that there is an $\eta > 0$ such that $F(y_h) \geq \eta$ for $h$ big enough.

### 3.2 Examples

The next examples concern the autonomous case, with $\Lambda = \Lambda(z, u)$ and $\Psi \equiv 1$. Example 3.5 below shows that Hypothesis $(B_{y, \Lambda}^+)$ is essential for the validity of Claim 2 in Theorem 3.1, when $\Lambda$ is extended valued. It is a slight modification of an example of G. Alberti (personal communication).

**Example 3.5** (*Occurrence of the Lavrentiev phenomenon in an autonomous problem with both endpoint constraints*) Let $y_* \in W^{1,1}([0, 1]; \mathbb{R}) \cap C^2([0, 1])$ be such that

- $y_*(0) = 0$, $y_*(1) = 1$;
- $y'_* > 0$, $y''_* > 0$ on $[0, 1]$;
- $y'_*(1) := \lim_{s \to 1} y'_*(s) = +\infty$.

Such a function exists, e.g., $y_*(s) := 1 - \sqrt{1 - s}$, $s \in [0, 1]$. For every $y \in [0, 1]$ set $q(y) := y'_*(y^{-1}(y))$. Let, for $(y, u) \in \mathbb{R} \times \mathbb{R}$,

$$\Lambda(y, u) := \begin{cases} 0 & \text{if } y \in [0, 1] \text{ and } u \leq q(y) \text{ or } y \notin [0, 1], \\ +\infty & \text{otherwise}, \end{cases}$$

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and set \( F(y) := \int_0^1 \Lambda(y(s), y'(s)) \, ds \) for every \( y \in W^{1,1}([0, 1]; \mathbb{R}) \). Clearly \( F(y_*) = \min F = 0 \). Note that:

1. \( \Lambda \) is autonomous;
2. \( \Lambda \) is lower semicontinuous on \( \mathbb{R}^2 \) and \( \Lambda(y, \cdot) \) is convex for all \( y \in \mathbb{R} \);
3. \( \Lambda \) is bounded on \( \mathbb{R} \times [-y_*'(0), y_*'(0)] \);

The effective domain of \( \lambda \) is \((\mathbb{R}[0, 1] \times \mathbb{R}) \cup \{(y, u) \in [0, 1] \times \mathbb{R} : u \leq q(y)\} \) (see Fig. 2).

Thus \( \Lambda, \Psi \) satisfy all the assumptions of Claim 1 in Theorem 3.1 at \( y_* \). In particular, there is no Lavrentiev gap at \( y_* \) (and thus phenomenon) for the problem with just one end point condition: either \( y(0) = 0 \) or \( y(1) = 1 \).

Claim \( F(y) = +\infty \) for every Lipschitz \( y : [0, 1] \to \mathbb{R} \) satisfying \( y(0) = 0, y(1) = 1 \).

Indeed, assume the contrary: let \( y \) be such a function and suppose \( F(y) < +\infty \). Let \( 0 \leq t_1 < t_2 \leq 1 \) be such that \( y(t_1) = 0, y(t_2) = 1 \) and \( y([t_1, t_2]) = [0, 1] \). Since \( F(y) < +\infty \), then

\[
y'(s) \leq q(y(s)) \text{ a.e. on } [t_1, t_2]. \tag{3.2}
\]

Note that, since \( \lim_{s \to t_2} q(y(s)) = +\infty \) and \( y' \) is bounded, then necessarily (3.2) is strict on a non-negligible set. It follows that

\[
\int_{t_1}^{t_2} \frac{y'(s)}{q(y(s))} \, ds < \int_{t_1}^{t_2} ds = t_2 - t_1 \leq 1.
\]
However the change of variable $\zeta = y(s)$ (which is justified, for instance, by the chain rule [19, Theorem 1.74]), gives
\[
\int_{t_1}^{t_2} \frac{y'(s)}{q(y(s))} ds = \int_0^1 \frac{1}{q(\zeta)} d\zeta
\]
\[
= \int_0^1 \frac{1}{y_s'(\zeta)} d\zeta
\]
\[
= (\tau = y_s^{-1}(\zeta)) \int_0^1 \frac{y_s'(\tau)}{y_s'(\tau)} d\tau = 1,
\]
a contradiction, proving the claim.

**Question** Can one build a counterexample in the spirit of Example 3.5 with an autonomous Lagrangian that is finite valued instead of extended valued?

The violation of Condition (B$_y$, $\Lambda$) in Theorem 3.1 may cause the occurrence of the phenomenon for one end point constraint problems.

**Example 3.6** (Occurrence of the phenomenon in autonomous scalar problems with one end-point constraint) Let $y_s$ and $q$ be as in Example 3.5. Define, for $(y, u) \in \mathbb{R} \times \mathbb{R}$,
\[
\Lambda(y, u) := \begin{cases} 
0 & \text{if } y \in [0, 1] \text{ and } u \geq q(y), \\
+\infty & \text{otherwise},
\end{cases}
\]
and set $F(y) := \int_0^1 \Lambda(y(s), y'(s)) ds$ for every $y \in W^{1,1}([0, 1]; \mathbb{R})$. Note that $\Lambda$ is autonomous, lower semicontinuous and $\Lambda(y, \cdot)$ is convex for every $y \in \mathbb{R}$. The Lagrangian here is of the form $\Lambda(y, y')\Psi(s, y)$ with $\Psi \equiv 1$ and the effective domain of $\lambda$ is $\{(y, u) \in [0, 1] \times \mathbb{R} : u \leq q(y)\}$ (see Fig. 3). Clearly $\Psi$ satisfies the assumptions of Theorem 3.1 at $y_s$, whereas, on every rectangle $y_s(I) \times B_r, \Lambda$ takes the value $+\infty$ and thus violates (B$_y$, $\Lambda$) in Theorem 3.1.

**Claim** The Lavrentiev phenomenon occurs for $F$ with the end point condition $y(1) = 1$.

Clearly $\Lambda(y_s, y'_s) = 0$ a.e. in $[0, 1]$, so that $F(y_s) = \min F = 0$. However, $F(y) = +\infty$ for every Lipschitz function $y$ satisfying $y(1) = 1$. Indeed, let $y(\cdot)$ be a Lipschitz function on $[0, 1]$ satisfying $y(1) = 1$. Let $M \geq 0$ be such that $y' \leq M$ a.e. on $[0, 1]$ and $t_s \in [0, 1]$ be such that $y_s'(t_s) > M$. Since $y_s(t_s) < 1 = y(1)$, there is, by continuity, an $\varepsilon > 0$ such that
\[
\forall s \in [1 - \varepsilon, 1] \quad y_s'(t_s) < y(s).
\]
Now, if $s \in [1 - \varepsilon, 1]$, the monotonicity of $y'_s$ implies
\[
y'(s) \leq M < y'_s(t_s) < y'_s(y_s^{-1}(y(s))) = q(y(s)),
\]
Then $\Lambda(y(s), y'(s)) = +\infty$ on $[1 - \varepsilon, 1]$. The claim follows.

### 3.3 Non-occurrence of the Lavrentiev phenomenon

Theorem 3.1 provides a sufficient condition for the non-occurrence of the Lavrentiev gap. The hypotheses of Corollary 3.7 ensure that the conditions of Theorem 3.1 are satisfied for every $y \in W^{1,p}(I; \mathbb{R}^n)$ with $F(y) < +\infty$. 
Corollary 3.7 (Non-occurrence of the Lavrentiev phenomenon) Suppose the validity of the Basic Assumptions, that \( \Lambda \) satisfies Condition (S) and that:
\[ (P_\Psi) \quad \text{For all } K > 0, \inf \{ \Psi(s, z) : s \in I, z \in B_K \} > 0; \]
\[ (B_\Psi) \quad \text{For all } K > 0, \Psi \text{ is bounded on } I \times B_K, \text{i.e., } \Psi \text{ is bounded on bounded sets}; \]
\[ (B_\Lambda) \quad \text{For all } K > 0 \text{ there is an } r_K > 0 \text{ such that } \Lambda \text{ is bounded on } I \times B_K \times B_{r_K}. \]

Let \( X, Y \in \mathbb{R}^n \). Then:
1. The Lavrentiev phenomenon does not occur for \((P_X)\).
2. Moreover, assume the following reinforced version of \((B_\Lambda)\):
\[ (B_\Lambda^+) \quad \text{For all } K, r > 0, \Lambda \text{ is bounded on } I \times B_K \times B_r, \text{i.e., } \Lambda \text{ is bounded on bounded sets}. \]

Then the Lavrentiev phenomenon does not occur for \((P_{X,Y})\).
3. Assume, moreover, that:
\[ (P_\Psi^+) \quad \inf \Psi = m_\Psi > 0; \]
\[ (G_\Lambda) \quad \exists \alpha > 0, d \geq 0, \text{for almost all } s \in I \text{ and every } z \in \mathbb{R}^n, u \in \mathbb{R}^n, \]
\[ \Lambda(s, z, u) \geq \alpha |u| - d. \]

Then there is \( \overline{K} > 0 \) such that Claims 1 and 2 still hold if \((B_\Psi)\), \((B_\Lambda)\), and \((B_\Lambda^+)\) are valid for just one arbitrary value of \( K > \overline{K} \).

\textbf{Proof} 1. Let \((\overline{y}_h)_h\) be a minimizing sequence for \((P_X)\) such that
\[ \forall h \in \mathbb{N} \quad F(\overline{y}_h) \leq \inf (P_X) + \frac{1}{h + 1}. \]
Fix \( h \in \mathbb{N} \) and choose \( K = K_h > 0 \) in such a way that \( B_{K_h} \) contains a neighborhood of \( \overline{y}_h(I) \); the validity of hypotheses (P\( \Psi \)), (B\( \Psi \)), and (B\( \Lambda \)) imply, respectively, that of (P\( \Psi \)), (B\( \Psi \)), and (B\( \Psi \)), with \( O_h := B_{K_h} \), in Theorem 3.1. The application of Claim 1 of Theorem 3.1 yields the existence of a function \( y_h \in \text{Lip}(I; \mathbb{R}^n) \) satisfying the boundary condition \( y_h(t) = X \) and
\[
F(y_h) \leq F(\overline{y}_h) + \frac{1}{h + 1} \leq \inf (P_X) + \frac{2}{h + 1}.
\]
The claim follows.

2. Consider a minimizing sequence \( (\overline{y}_h) \) for \((P_X, y)\) such that
\[
\forall h \in \mathbb{N} \quad F(\overline{y}_h) \leq \inf (P_X, y) + \frac{1}{h + 1}.
\]
We proceed as above, replacing in the proof of Claim 1 Hypothesis \((B\Lambda)\) with \((B\Lambda)\), and Hypothesis \((B\Lambda)\) with \((B\Lambda)\). It is enough to apply Claim 2 of Theorem 3.1.

3. Note first that if \( y \in W^{1,p}(I; \mathbb{R}^n) \) and \( F(y) < +\infty \), then
\[
\|y'\|_1 \leq \frac{F(y) + m\psi d(T-t)}{m\psi \alpha}.
\]
Indeed, from \((P\psi)\) and \((G\Lambda)\) we obtain
\[
F(y) = \int_t^T \Lambda(s, y(s), y'(s))\Psi(s, y(s)) \, ds \\
\geq m\psi \int_t^T \Lambda(s, y(s), y'(s)) \, ds \geq m\psi \alpha \int_t^T |y'(s)| \, ds - m\psi d(T-t),
\]
whence (3.3). Suppose now that \((B\Psi)\) and \((B\Lambda)\) hold for some \( K > \overline{K} := |X| + \inf (P_X) + m\psi d(T-t) \).

In the proof of Claim 1 choose \( h_1 \) such that
\[
\frac{1}{(h_1 + 1)m\psi \alpha} < K - \overline{K}
\]
and set \( K_h := K \) for all \( h \geq h_1 \). It follows from (3.3) that, for all \( h \geq h_1 \),
\[
\|\overline{y}_h\|_\infty \leq |X| + \|\overline{y}'_h\|_1 \\
\leq |X| + \frac{F(\overline{y}_h) + m\psi d(T-t)}{m\psi \alpha} \\
\leq |X| + \frac{\inf (P_X) + \frac{1}{\alpha} + m\psi d(T-t)}{m\psi \alpha} \\
\leq |X| + \frac{\inf (P_X) + m\psi d(T-t)}{m\psi \alpha} + \frac{1}{(h + 1)m\psi \alpha} \\
< \overline{K} + (K - \overline{K}) = K.
\]

Thus \((B\Psi, \Psi)\) and \((B\Psi, \Lambda)\) hold with \( O_h := B_K \); the conclusion follows as in the proof of Claim 1. If \((B\Lambda)\) holds, Claim 2 follows similarly by choosing
\[
K > \overline{K} := |X| + \frac{\inf (P_X, y) + m\psi d(T-t)}{m\psi \alpha}.
\]

\( \square \)
In the real valued, continuous case, many of the assumptions of Corollary 3.7 are satisfied whenever $\Lambda$ is bounded on bounded sets. In the autonomous case ($\Lambda = \Lambda(y, u)$, $\Psi \equiv 1$), the result, formulated in [1], has now a complete proof.

**Corollary 3.8** (Non-occurrence of the Lavrentiev phenomenon—real valued case) *Suppose the validity of the Basic Assumptions, that $\Lambda$ satisfies Condition (S) and, moreover, that:

- $\Lambda$ and $\Psi$ are real valued and bounded on bounded sets;
- $\inf \Psi > 0$.

Then the Lavrentiev phenomenon does not occur for $(P_{X, Y})$.*

**4 Proof of Theorem 3.1**

Lemma 4.1 is a slight extension of [1, Lemma 2.6].

**Lemma 4.1** *Let $g \in L^1(I)$ be positive, $(\psi_h)_h$ be a sequence of measurable, positive, equi-bounded functions converging a.e. in $I$ to $\psi$. Let $(E_h)_h$ be a sequence of measurable subsets of $I$ such that $|E_h| \to |I|$. Then

$$\int_{E_h} g\psi_h \, ds \to \int_I g\psi \, ds \quad h \to +\infty.$$*

**Proof** Possibly passing to a subsequence, we may assume that the characteristic functions of $E_h$ converge to 1 a.e. in $I$. Fatou’s lemma then yields

$$\liminf_h \int_{E_h} g\psi_h \, ds \geq \int_I g\psi \, ds.$$  

Moreover, let $M \geq 0$ be such that $\psi_h \leq M$ for all $h$. Since $g \geq 0$ and $E_h \subset I$ for every $h$, we have

$$\int_{E_h} g\psi_h \, ds \leq \int_I g\psi_h \, ds, \quad g\psi_h \leq Mg \in L^1(I).$$

Fatou’s lemma gives

$$\limsup_h \int_{E_h} g\psi_h \, ds \leq \limsup_h \int_I g\psi_h \, ds \leq \int_I g\psi \, ds.$$  

Thus, from any sequence $(E_h)_h$ with $|E_h| \to |I|$, we may extract a subsequence $(E_{hk})_k$ such that

$$\int_{E_{hk}} g\psi_{hk} \, ds \to \int_I g\psi \, ds \quad k \to +\infty,$$

the conclusion follows. \qed

**Proof of Theorem 3.1** 1. The first steps of the proof follow the path of the proof of [1, Theorem 2.4], which are recalled and adapted to the more general Lagrangian considered here.

(i) It follows from Assumptions $(B_y, \Psi)$ and $(B_y, \Lambda)$ that there are $M, r_y > 0$ and a neighbourhood $O_y$ of $y(I)$ such that:

$$\Lambda(s, z, u) \leq M, \quad \Psi(s, z) \leq M \text{ whenever } z \in O_y, |u| \leq r_y.$$  

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(ii) For every $h \in \mathbb{N}$ there are a Lipschitz function $z_h : I \to \mathbb{R}^n$ and an open subset $A_h$ of $I$ such that (see Fig. 4):

- $z_h(t) = y(t), \quad z_h(T) = y(T),$
- $z_h = y, \quad z'_h = y'$ in $I \setminus A_h,$
- $z_h$ is affine in each connected component of $A_h,$
- $|A_h| \leq \frac{T - t}{2(h + 1)} \to 0, \quad h \to +\infty.$

(iii) Writing each $A_h$ as a countable union of open intervals $I_{h,k} := (a_{h,k}, b_{h,k})$ of $I,$ $k \in J_h \subseteq \mathbb{N},$ set

$$\alpha_{h,k} := |I_{h,k}| = b_{h,k} - a_{h,k}, \quad \beta_{h,k} := z_h(b_{h,k}) - z_h(a_{h,k}) = y(b_{h,k}) - y(a_{h,k}).$$

Then

$$\sum_{k \in J_h} |\beta_{h,k}| \leq \sum_{k \in J_h} \int_{I_{h,k}} |y'| \, ds = \int_{A_h} |y'| \, ds \to 0 \quad h \to \infty.$$

(iv) Note that

$$\int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \vee 1 \right) \, ds \to 0 \quad h \to +\infty. \quad (4.2)$$

Fig. 4 The functions $y_h$ and $z_h$ in the proof of Theorem 3.1
indeed,
\[
\int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) \, ds \leq \frac{1}{r_y} \int_{A_h} |z'_h(s)| \, ds + |A_h| \\
\leq \frac{1}{r_y} \sum_{k \in J_h} \int_{I_{h,k}} |z'_h(s)| \, ds + |A_h| \\
\leq \frac{1}{r_y} \sum_{k \in J_h} |\beta_{h,k}| + |A_h| \to 0 \quad h \to +\infty.
\]

(v) For every $h \in \mathbb{N}$ define $\varphi_h \in W^{1,1}(I)$ by
\[
\varphi_h(t) = t, \quad \varphi'_h = \begin{cases} \frac{1}{r_y} & \text{in } I \setminus A_h, \\ |z'_h| & \text{in } A_h. \end{cases}
\]

(vi) As in [1], it turns out that $\varphi_h$ is injective, $I \subseteq \varphi_h(I)$. $(\varphi_h)_h$ converges uniformly to $Id(s):=s$ on $I$ and $|\varphi_h(I)| \to |I|$.

(vii) From (4.2) of Step iv) we have
\[
|\varphi_h(A_h)| = \int_{\varphi_h(A_h)} 1 \, ds = \int_{A_h} |\varphi'_h(\tau)| \, d\tau \\
\leq \int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) \, ds \to 0 \quad h \to +\infty.
\]

(viii) Let $t_h \in I$ be such that $\varphi_h(t_h) = T$. Denoting by $\psi_h$ the inverse $\varphi_h$, restricted to $I$, we have $\psi_h(I) = [t, t_h]$. Note that $t_h \to T$: indeed,
\[
|T - t_h| = |\varphi_h(t_h) - t| \to 0.
\]

Define $y_h := z_h(\psi_h)$ (see Fig. 4). Then $y_h$ is Lipschitz, $y_h(t) = y(t)$ and
\[
y'_h = z'_h(\psi_h) = \begin{cases} y'(\psi_h) & \text{in } I \setminus \varphi_h(A_h), \\ \left(\frac{r_y}{|z'_h(\psi_h)|}\right) \wedge z'_h(\psi_h) & \text{in } I \cap \varphi_h(A_h). \end{cases}
\]

Moreover, as in [1], $(y_h)_h$ converges to $y$ in $W^{1,p}(I; \mathbb{R}^n)$. We may thus assume that $y_h(I) \subset O_y$ for every $h$. We point out here that $y_h(T) = z_h(\psi_h(T)) = z_h(t_h)$ may differ from $y(T)$.

(ix) It remains to show that $(F(y_h))_h$ converges to $F(y)$ as $h \to +\infty$; this is where the proof differs from the one given in [1] for the autonomous case. We write
\[
F(y_h) = \int_{I \setminus \varphi_h(A_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds + \int_{I \cap \varphi_h(A_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds.
\]
\[
\begin{align*}
(P_{1,h}) & \quad F(y_h) = \int_{I \setminus \varphi_h(A_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds + \int_{I \cap \varphi_h(A_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds. \\
(P_{2,h}) & \quad F(y_h) = \int_{I \setminus \varphi_h(A_h)} \Lambda(s, y(\psi_h), y'(\psi_h)) \Psi(s, y(\psi_h)) \, ds.
\end{align*}
\]

(a) Study of the convergence of $(P_{1,h})_h$. From the definition of $y_h$, recalling that $z_h = y$ and $\varphi'_h = 1$ outside of $A_h$, we get
\[
P_{1,h} = \int_{I \setminus \varphi_h(A_h)} \Lambda(s, y(\psi_h), y'(\psi_h)) \Psi(s, y(\psi_h)) \, ds.
\]
The change of variable \( \tau = \psi_h(s) \) and the fact that \( \psi_h' = 1 \) on \( I \setminus \varphi_h(A_h) \) give

\[
P_{1,h} = \int_{\psi_h(I) \setminus A_h} \Lambda(\varphi_h(\tau), y(\tau), y'(\tau)) \Psi(\varphi_h(\tau), y(\tau)) \, d\tau.
\]

Choose \( h \) so large that \( \|\varphi_h - Id\|_\infty < \varepsilon_\ast \), with \( \varepsilon_\ast \) as in Condition (S). Condition (S) (with \( K:=\|y\|_\infty \)) implies that, a.e. in \( I \),

\[
|\Lambda(\varphi_h, y, y') - \Lambda(\tau, y, y')| \leq \left( \kappa \Lambda(\tau, y, y') + \beta |y'|^p + \gamma(\tau) \right) \|\varphi_h - Id\|_\infty.
\]

From (4.1), for all \( h \in \mathbb{N} \) big enough,

\[
\varepsilon_h := \int_{I \setminus \varphi_h(A_h)} |\Lambda(\varphi_h, y, y') - \Lambda(\tau, y, y')| \Psi(\varphi_h, y) \, d\tau \leq C_Y \|\varphi_h - Id\|_\infty,
\]

(4.3)

for a suitable constant \( C_Y \), possibly depending on \( y \). It follows from (4.3) that

\[
P_{1,h} \leq \int_{I \setminus \varphi_h(A_h)} |\Lambda(\varphi_h, y, y') - \Lambda(\tau, y, y')| \Psi(s, y(\psi_h)) \, ds +
\]

\[
+ \int_{I \setminus \varphi_h(A_h)} \Lambda(\tau, y, y') \Psi(s, y(\psi_h)) \, ds
\]

\[
\leq \int_{\psi_h(I) \setminus A_h} \Lambda(\tau, y, y') \Psi(\varphi_h, y) \, d\tau + \varepsilon_h, \quad \varepsilon_h \to 0.
\]

Set \( g(\tau):=\Lambda(\tau, y(\tau), y'(\tau)), \xi_h(\tau):=\Psi(\varphi_h(\tau), y(\tau)), E_h:=\psi_h(I) \setminus A_h \). Condition \( (P_Y, \Psi) \) implies that \( g \in L^1(I) \) (see (3.1)). Moreover, \( g, \xi_h \geq 0 \), the continuity of \( \Psi(\cdot, y) \) implies that \( \xi_h(\tau) \to \xi(\tau):=\Psi(\tau, y(\tau)) \) a.e. on \( I \) and

\[
|I| \geq |E_h| = |\psi_h(I) \setminus A_h| = |[t, t_h] \setminus A_h| \geq (t_h - t) - |A_h|,
\]

so that, from Step \( \text{vii} \), \( |E_h| \to |I| \). Assumption \( (B_Y, \Psi) \) ensures that the functions \( \xi_h \) are uniformly bounded. It follows from Lemma 4.1 that

\[
\lim_{h \to \infty} P_{1,h} = \lim_{h \to +\infty} \int_{E_h} g(\tau) \xi_h(\tau) \, d\tau
\]

\[
= \int_I g(\tau) \xi(\tau) \, d\tau = \int_I \Lambda(\tau, y(\tau), y'(\tau)) \Psi(\tau, y(\tau)) \, d\tau.
\]

(b) **Study of the convergence of** \( (P_{2,h}) \). We know that for all \( h, y_h(I) \subseteq O_Y \) and \( |y_h' | \leq r_y \)
a.e. in \( \varphi_h(A_h) \cap I \). It follows from (4.1) that

For a.e. \( \tau \in \varphi_h(A_h) \) \( 0 \leq \Lambda(\tau, y_h(\tau), y'_h(\tau)) \Psi(\tau, y_h(\tau)) \leq M^2 \).

Therefore Step \( \text{vii} \) implies that

\[
0 \leq P_{2,h} \leq M^2 |\varphi_h(A_h)| \to 0.
\]

2. We proceed as above, with some additional care with the definition of \( \varphi_h \), since we want now that

\[
\varphi_h(T) - \varphi_h(t) = \int_t^T \varphi_h'(s) \, ds = T - t,
\]

in order to ensure, at Step \( \text{viii} \) of the above construction of \( y_h \), that \( y_h(T) = y(T) \). Referring to the proof of Claim 1, we proceed as in Steps i, ii, iii, iv and modify the subsequent steps, Steps \( \text{v} - \text{ix} \), as follows:
Choose $\overline{h}$ in such a way that
\[
|A_h| \leq \frac{|I|}{10} \quad h \geq \overline{h}.
\] (4.4)

Since $y$ coincides with $z_\overline{h}$ outside of $A_\overline{h}$, there is an $\ell > 0$ satisfying
\[
\|y\|_\infty \leq \ell \text{ on } I \setminus A_\overline{h}.
\]

Note that, from Step ii and (4.2) of Step iv,
\[
0 \leq \int_{A_h} \left[ \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) - 1 \right] ds = \int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) ds - |A_h| \rightarrow 0:
\]
we may assume, without loss of generality, that $\overline{h}$ is large enough that
\[
\forall h \geq \overline{h} \quad 0 \leq \int_{A_h} \left[ \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) - 1 \right] ds \leq \frac{|I|}{5}.
\]

Since, from (4.4),
\[
\forall h \geq \overline{h} \quad |I \setminus (A_h \cup A_\overline{h})| \geq \frac{4|I|}{5},
\]
we deduce that for any $h \geq \overline{h}$, we may choose a measurable subset
\[
\Sigma_h \subset I \setminus (A_h \cup A_\overline{h})
\]
whose measure equals
\[
|\Sigma_h| = 2 \int_{A_h} \left[ \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) - 1 \right] ds,
\]
so that
\[
\int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) ds = \frac{|\Sigma_h|}{2} + |A_h|.
\] (4.5)

For each $h \geq \overline{h}$, define the absolutely continuous function $\varphi_h$ as follows:
\[
\varphi_h(t) = t, \quad \varphi'_h = \begin{cases} 
1 & \text{in } I \setminus (A_h \cup \Sigma_h), \\
\frac{|z'_h|}{r_y} \lor 1 & \text{in } A_h, \\
1/2 & \text{in } \Sigma_h.
\end{cases}
\]

From (4.5) of $v'$ we now have
\[
\int_t^T \varphi_h'(s) \, ds = |I \setminus (A_h \cup \Sigma_h)| + \int_{A_h} \left( \frac{|z'_h(s)|}{r_y} \lor 1 \right) ds + \frac{|\Sigma_h|}{2}
\]
\[
= |I| - |A_h| - |\Sigma_h| + \left( \frac{|\Sigma_h|}{2} + |A_h| \right) + \frac{|\Sigma_h|}{2} = |I|.
\]

Thus $\varphi_h : I \rightarrow I$ is bijective.

We proceed as in Step vii.
(viii') Let $\psi_h$ be the inverse of $\varphi_h$. Define $y_h := z_h(\psi_h)$. Then $y_h$ is Lipschitz, $y_h(t) = y(t)$, $y_h(T) = y(T)$ and

$$y'_h = z'_h(\psi_h)\psi'_h = \begin{cases} y'(\psi_h) & \text{in } I \setminus \varphi_h(A_h \cup \Sigma_h), \\ \left(\frac{\int y'_h(\psi_h)}{\|z'_h(\psi_h)\|} \right) \wedge z'_h(\psi_h) & \text{in } \varphi_h(A_h), \\ 2z'_h(\psi_h) = 2y'(\psi_h) & \text{in } \varphi_h(\Sigma_h). \end{cases}$$

As in Claim 1, it turns out that $(y_h)_h$ converges to $y$ in $W^{1,p}(I; \mathbb{R}^n)$. Indeed, it follows from the proof of Claim 1 that

$$\int_{I \setminus \varphi_h(\Sigma_h)} |y'_h - y'|^p \, ds \rightarrow 0.$$ 

It remains to prove that $\|y'_h - y\|_{L^p(\varphi_h(\Sigma_h))} \rightarrow 0$. Since $|y'| \leq \ell$ on $\Sigma_h$, we have

$$\int_{\varphi_h(\Sigma_h)} |y'_h - y'|^p \, ds \rightarrow 2^p \int_{\varphi_h(\Sigma_h)} |y'(s)|^p \, ds + 2^p \int_{\varphi_h(\Sigma_h)} |y'(s)|^p \, ds \rightarrow 2^p \int_{\varphi_h(\Sigma_h)} |y'(s)|^p \, ds \rightarrow 0,$$

since $|\varphi_h(\Sigma_h)| \rightarrow 0$, proving the claim. We may thus assume, as in the proof of Claim 1, that $y_h(I) \subset O_y$ for every $h$ big enough.

(ix') It remains to show that $(F(y_h))_h$ converges to $F(y)$ as $h \rightarrow +\infty$. Write

$$F(y_h) = \int_{\varphi_h(A_h \cup \Sigma_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds + \int_{\varphi_h(A_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds + \int_{\varphi_h(\Sigma_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds,$$

where here $\ast$ stands for $\Lambda(s, y_h, y'_h) \Psi(s, y_h)$. As in the proof of Claim 1 we get

$$P_{1,h} \rightarrow F(y), \quad P_{2,h} \rightarrow 0.$$ 

It remains to prove that $P_{3,h} \rightarrow 0$. Since $\varphi'_h \equiv 1/2$ on $\Sigma_h$, the change of variables $s = \varphi_h(\tau)$ gives

$$P_{3,h} = \int_{\varphi_h(\Sigma_h)} \Lambda(s, y_h, y'_h) \Psi(s, y_h) \, ds = \int_{\varphi_h(\Sigma_h)} \Lambda(s, y(\psi_h), y'(\psi_h)) \Psi(s, y(\psi_h)) \, ds = \frac{1}{2} \int_{\Sigma_h} \Lambda(\varphi_h, y, 2y') \Psi(\varphi_h, y) \, d\tau.$$

Now $y(I) \subset O_y$ and $\Sigma_h \subset I \setminus A_h$ for $h \geq \tilde{h}$. From Step vii' we have $2|y'(\tau)| \leq 2\ell$ on $I \setminus A^n_h$. Hypotheses (B$^+_\lambda$, A) and (B$^+_\lambda$, y, y) imply that $\Lambda(\varphi_h, y, 2y') \Psi(\varphi_h, y)$ is bounded on $I \setminus A^n_h$; the conclusion follows from the fact that $|\Sigma_h| \rightarrow 0$.

$\square$
Remark 4.2 Note that in the proof of Claim 1 of Theorem 3.1, every element of the sequence \((z_h)_h\) satisfies the boundary conditions \(z_h(t) = y(t), \ z_h(T) = y(T)\). However, since \(\psi_h' \geq 1\) on \(I\), it may happen that \(\psi_h(T) > T\), and this yields \(\psi_h(T) < T\); as a consequence \(y_h(T) = z_h(\psi_h(T))\) may differ from \(z_h(T) = y(T)\).

In the presence of a final end point condition of the form \(y(T) = Y\), instead of an initial one, the proof goes as above, replacing the definition of the change of variables \(\psi_h\) by

\[
\varphi_h(T) = T, \quad \varphi_h' = \begin{cases} 
1 & \text{in } I \setminus A_h, \\
\frac{|z_h|}{r_y} \lor 1 & \text{in } A_h.
\end{cases}
\]

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