Indestructibly productively Lindelöf and Menger function spaces

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Abstract

For a Tychonoff space $X$ and a family $\lambda$ of subsets of $X$, we denote by $C_{\lambda}(X)$ the $T_1$-space of all real-valued continuous functions on $X$ with the $\lambda$-open topology.

A topological space is productively Lindelöf if its product with every Lindelöf space is Lindelöf. A space is indestructibly productively Lindelöf if it is productively Lindelöf in any extension by countably closed forcing. A Menger space is a topological space in which for every sequence of open covers $U_1, U_2, \ldots$ of the space there are finite sets $F_1 \subset U_1, F_2 \subset U_2, \ldots$ such that family $F_1 \cup F_2 \cup \ldots$ covers the space.

In this paper, we study indestructibly productively Lindelöf and Menger function spaces. In particular, we proved that the following statements are equivalent for a $T_1$-space $C_{\lambda}(X)$:

1. $C_{\lambda}(X)$ is indestructibly productively Lindelöf;
2. $C_{\lambda}(X)$ is metrizable Menger;
3. $C_{\lambda}(X)$ is metrizable $\sigma$-compact;
4. $X$ is pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$ and the family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is the countable set of all isolated points of $X$;
5. $C_{\lambda}(X)$ is homeomorphic to $C^*_p(\mathbb{N})$.

Keywords: Menger property, Hurewicz property, set-open topology, $\sigma$-compact, function space, indestructibly Lindelöf, productively Lindelöf, indestructibly productively Lindelöf, selection principles

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1. Introduction

A space $X$ is said to be Menger [16] (or, [31]) ($X$ satisfies $S_{fin}(O,O)$) if for every sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there are finite subfamilies $V_n \subset U_n$ such that $\bigcup\{V_n : n \in \mathbb{N}\}$ is a cover of $X$.

Note that every $\sigma$-compact space is Menger, and a Menger space is Lindelöf. The Menger property is closed hereditary, and it is preserved by continuous maps. It is well known that the Baire space $\mathbb{N}^\mathbb{N}$ (hence, $\mathbb{R}^\omega$) is not Menger.

Menger conjectured that in ZFC every Menger metric space is $\sigma$-compact. Fremlin and Miller [12] proved that Menger’s conjecture is false, by showing that there is, in ZFC, a set of real numbers that is Menger but not $\sigma$-compact.

For a function space $C_p(X)$: Velichko proved that $C_p(X)$ is $\sigma$-compact iff $X$ is finite and Arhangel’skii proved that $C_p(X)$ is Menger iff $X$ is finite [6].

For a function space $C_\lambda(X)$ the situation is more interesting.

**Theorem 1.1.** (Nokhrin) (Theorem 5.13 in [21]) For a space $X$ the following statements are equivalent:

1. $C_\lambda(X)$ is $\sigma$-compact;
2. $X$ is pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$ and the family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is the set of all isolated points of $X$.

**Theorem 1.2.** (Osipov) (Theorem 3.4 in [24]) A space $C_\lambda(X)$ is Menger iff it is $\sigma$-compact.

Various properties between $\sigma$-compactness and the Menger property are investigated in the papers [11, 33, 35] and others. We continue to study these properties on a function $T_1$-space $C(X)$ with the set-open topology.

2. Main definitions and notation

Throughout this paper $X$ will be a Tychonoff space. Let $\lambda$ be a family nonempty subsets of $X$ and let $C(X)$ be a set of all continuous real-valued functions on $X$. Denote by $C_\lambda(X)$ the set $C(X)$ is endowed with the $\lambda$-open topology. The elements of the standard subbases of the set-open topology:
\[ [F, U] = \{ f \in C(X) : f(F) \subseteq U \}, \text{ where } F \in \lambda, \ U \text{ is an open subset of the real line } \mathbb{R}. \]

Note that if \( \lambda \) consists of all finite (compact) subsets of \( X \) then the \( \lambda \)-open topology coincides with the topology of pointwise convergence (the compact-open topology), that is \( C_\lambda(X) = C_p(X) \) \( (C_\lambda(X) = C_k(X)) \). The set-open topology was first introduced by Arens and Dugundji in [2] and studied over the last years by many authors. We continue to study the different topological properties of the space \( C(X) \) with the set-open topology (see [21-27]).

For a topological property \( P \), A.V. Arhangel’skii calls \( X \) projectively \( P \) if every second countable image of \( X \) is \( P \). Arhangel’skii consider projective \( P \) for \( P = \sigma \)-compact, analytic and other properties [3]. The projective selection principles were introduced and first time considered in [18]. Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in \( \mathbb{R}^\omega \).

**Theorem 2.1.** (Kočinac) A space is Menger if and only if it is Lindelöf and projectively Menger.

Recall that, if \( X \) is a topological space and \( \mathcal{P} \) is a topological property, we say that \( X \) is \( \sigma \)-\( \mathcal{P} \) if \( X \) is the countable union of subspaces with the property \( \mathcal{P} \). So a space \( X \) is called \( \sigma \)-compact (\( \sigma \)-pseudocompact, \( \sigma \)-bounded), if \( X = \bigcup_{i=1}^{\infty} X_i \), where \( X_i \) is a compact (pseudocompact, bounded) for every \( i \in \mathbb{N} \).

A subset \( A \) of a space \( X \) is said to be bounded in \( X \) if for every continuous function \( f : X \mapsto \mathbb{R} \), \( f|A : A \mapsto \mathbb{R} \) is a bounded function. Every \( \sigma \)-bounded space is projectively Menger (Proposition 1.1 in [3]).

Recall that a family \( \lambda \) of nonempty subsets of a topological space \( (X, \tau) \) is called a \( \pi \)-network for \( X \) if for any nonempty open set \( U \in \tau \) there exists \( A \in \lambda \) such that \( A \subseteq U \).

By Theorem 4.1 in [21], the space \( C_\lambda(X) \) is a \( T_1 \)-space (=Hausdorff space) iff \( \lambda \) is a \( \pi \)-network of \( X \).

Throughout this paper, a family \( \lambda \) of nonempty subsets of the set \( X \) is a \( \pi \)-network.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let \( \mathcal{A} \) and \( \mathcal{B} \) be sets consisting of families of subsets of an infinite set \( X \). Then:
$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$, and \{b_n : n \in \mathbb{N}\} is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $U_1, U_2, \ldots \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of $X$ if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space $X$ is:
- an $\omega$-cover if every finite subset of $X$ is contained in a member of $\mathcal{U}$;
- a $\gamma$-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space $X$ we denote:
- $\mathcal{O}$ — the family of open covers of $X$;
- $\Gamma$ — the family of open $\gamma$-covers of $X$;
- $\Omega$ — the family of open $\omega$-covers of $X$.

Many equivalences hold among these properties, and the surviving ones appear in the following Diagram (where an arrow denotes implication), to which no arrow can be added except perhaps from $U_{fin}(\Gamma, \Gamma)$ or $U_{fin}(\Gamma, \Omega)$ to $S_{fin}(\Gamma, \Omega)$ [15].
Definition 2.2. A topological space $X$ is
- Indestructibly Lindelöf if it is Lindelöf in every countably closed forcing extension [32, 7].
- Productively Lindelöf if $X \times Y$ is Lindelöf for any Lindelöf space $Y$ [8].
- Indestructibly productively Lindelöf if it is productively Lindelöf in every extension by countably closed forcing [7].
- Powerfully Lindelöf if its $\omega$th power is Lindelöf [1, 7].
- Alster if every cover by $G_{\delta}$ sets that covers each compact set finitely includes a countable subcover [1].
- Rothberger ($X$ satisfies $S_{1}(\mathcal{O}, \mathcal{O})$) if for each sequence of open covers $\{U_{n}\}_{n<\omega}$, there are $\{U_{n}\}_{n<\omega}$, $U_{n} \in \mathcal{U}_{n}$, such that $\{U_{n}\}_{n<\omega}$ is a cover [28].
- Hurewicz ($X$ satisfies $U_{\text{fin}}(\mathcal{O}, \Gamma)$) if for each sequence $\{U_{n} : n < \omega\}$ of $\gamma$-covers, there is for each $n$ a finite $\mathcal{V}_{n} \subset \mathcal{U}_{n}$ such that either $\bigcup \mathcal{V}_{n} : n < \omega$ is a $\gamma$-cover, or else for some $n$, $\mathcal{V}_{n}$ is a cover [16, 17].

Definition 2.3. We play a Menger game ($M$-game) in which ONE chooses in the $n$th inning an open cover $\mathcal{U}_{n}$ and TWO choses a finite $\mathcal{V}_{n} \subset \mathcal{U}_{n}$. TWO wins if $\bigcup \mathcal{V}_{n} : n < \omega$ covers $X$.

Hurewicz proved $X$ is Menger if and only if ONE has no winning strategy [16].

Figure 1. The Scheepers Diagram for Lindelöf spaces.
The Tall’s Diagram in Figure 2 (see Diagram in [35]) below shows the relationships among the properties we have discussed in this article.

![Diagram showing relationships among properties]

**Figure 2.** The Tall’s Diagram for Lindelöf spaces.

**Theorem 2.4.** For a space $X$ the following statements are equivalent:

1. $C_\lambda(X)$ is $\sigma$-compact;
2. $C_\lambda(X)$ is Alster;
3. (CH) $C_\lambda(X)$ is productively Lindelöf;
4. "TWO wins M-game" for $C_\lambda(X)$;
5. $C_\lambda(X)$ is projectively $\sigma$-compact and Lindelöf;
6. $C_\lambda(X)$ is Hurewicz;
7. $C_\lambda(X)$ is Menger;
8. $X$ is a pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$ and family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is the set of all isolated points of $X$;
9. $(C_\lambda(X))^n$ is Menger for every $n \in \mathbb{N}$;
10. $C_\lambda(X)$ satisfies $S_{\text{fin}}(\Omega, \Omega)$;
11. $C_\lambda(X)$ is $\sigma$-countably compact and Lindelöf;
12. $C_\lambda(X)$ is $\sigma$-pseudocompact and Lindelöf;
13. $C_\lambda(X)$ is $\sigma$-bounded and Lindelöf;
14. $C_\lambda(X)$ is homeomorphic to $\bigcup_{i=1}^{\infty} [-i, i]^{D(X)}$. 

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Proof. (1) ⇒ (2). It is obvious that every σ-compact space is Alster.

(1) ⇒ (4). It is obvious that every σ-compact space has "TWO wins M-game".

(2) ⇒ (3). Every Alster space is productively Lindelöf [1].

(2) ⇒ (5). Since Alster metrizable spaces are σ-compact [1], then every Alster space is projectively σ-compact and Lindelöf.

(4) ⇒ (5). By Theorem 18 in [35].

(7) ⇒ (1). By Theorem 1.2.

(9) ⇔ (10). By Theorem 3.9 in [15].

(13) ⇒ (1). Every σ-bounded space is projectively Menger (Proposition 1.1 in [3]). By Theorem 2.4, \( C_\lambda(X) \) is Menger. By Theorem 1.2, \( C_\lambda(X) \) is σ-compact.

(1) ⇔ (14). By Theorem 5.5 in [21].

The remaining implications are trivial and follows from the definitions [35]. □

Corollary 2.5. If \( C_\lambda(X) \) is Menger. Then \( C_\lambda(X) \) is powerfully (productively) Lindelöf.

Proof. If \( C_\lambda(X) \) is Menger, then, by Theorem 2.4, \( C_\lambda(X) \) is σ-compact and hence it is Alster. But Alster spaces are powerfully (productively) Lindelöf [1]. □

Corollary 2.6. \( C_k(X) \) is Menger (σ-compact, Hurewicz, Alster) if and only if \( X \) is finite.

For the selection properties of the space \( C_\lambda(X) \) (see Fig. 2) we have next trivial corollaries.

Corollary 2.7. If \( C_\lambda(X) \) is Rothberger (in particular, \( S_1(\Omega, \Gamma) \) or \( S_1(\Omega, \Omega) \)). Then \( X = \emptyset \).

Proof. If \( C_\lambda(X) \) is Rothberger, then it is Menger and, by Theorem 2.4, \( X \) contains an isolated point. Hence, the real line \( \mathbb{R} \subset C_\lambda(X) \). But every Rothberger subset of the real line has strongly measure zero [29]. It follows that \( X = \emptyset \). □

Corollary 2.8. If \( C_\lambda(X) \) has the property \( S_1(\Gamma, \mathcal{O}) \) (in particular, \( S_1(\Gamma, \Gamma) \) or \( S_1(\Gamma, \Omega) \)). Then \( X = \emptyset \).
Proof. If $C_\lambda(X)$ has the property $S_1(\Gamma, \mathcal{O})$, then it is Menger and, by Theorem 2.4, $X$ contains an isolated point. Hence, the Cantor set $2^\omega \subset C_\lambda(X)$. But the Cantor set, $2^\omega$, is not in the class $S_1(\Gamma, \mathcal{O})$ [15]. It follows that $X = \emptyset$.

Note that every $\sigma$-compact topological space is a member of both the class $S_{\text{fin}}(\Omega, \Omega)$ and $U_{\text{fin}}(\Gamma, \Gamma)$ (Theorem 2.2 in [15]). It follows that if $C_\lambda(X)$ is Menger then $C_\lambda(X)$ has the properties $S_{\text{fin}}(\Omega, \Omega)$ and $U_{\text{fin}}(\mathcal{O}, \Gamma)$ (in particular, $S_{\text{fin}}(\Gamma, \Omega)$ and $U_{\text{fin}}(\mathcal{O}, \Omega)$).

Remark 2.9. If $X$ is compact and $C_\lambda(X)$ is Menger, then $X$ is homeomorphic to $\beta(D)$, where $\beta(D)$ is Stone-\v{C}ech compactification of a discrete space $D$, and $\lambda = [D]^{<\aleph_0}$.

A Lindelöf space $Y$ is called a Michael space, if $\omega^\omega \times Y$ is not Lindelöf.

- (Repovs-Zdomskyy) If there exists a Michael space (this follows from $b = \aleph_1$ or $\mathfrak{d} = \text{Cov}(\mathcal{M})$), then every productively Lindelöf spaces has the Menger property (Proposition 3.1 in [13]).
- (Repovs-Zdomskyy) If $\text{Add}(\mathcal{M}) = \mathfrak{d}$, then every productively Lindelöf space has the Hurewicz property (Theorem 1.1 in [14]).
- (Zdomskyy) If $u = \aleph_1$, then every productively Lindelöf space has the Hurewicz property.
- (Tall) $\mathfrak{d} = \aleph_1$ implies productively Lindelöf spaces are Hurewicz. (Theorem 10 in [35]).
- (Tall) $b = \aleph_1$ implies every productively Lindelöf space is Menger (Theorem 7 in [35]).

Proposition 2.10. If $b = \aleph_1$ (or $\text{Add}(\mathcal{M}) = \mathfrak{d}$ or $u = \aleph_1$ or $\mathfrak{d} = \aleph_1$), then every productively Lindelöf space $C_\lambda(X)$ is $\sigma$-compact.

Proof. If $b = \aleph_1$ (or $\text{Add}(\mathcal{M}) = \mathfrak{d}$ or $u = \aleph_1$ or $\mathfrak{d} = \aleph_1$) and $C_\lambda(X)$ is productively Lindelöf, then $C_\lambda(X)$ is Menger. By Theorem 1.2, $C_\lambda(X)$ is $\sigma$-compact. \qed

Denote by $C_p^*(\mathbb{N})$ the set of all bounded continuous real-valued functions on $\mathbb{N}$ with the topology of pointwise convergence.

Theorem 2.11. For a space $X$ the following statements are equivalent:

1. $C_\lambda(X)$ is indestructibly productively Lindelöf;
2. $C_\lambda(X)$ is metrizable $\sigma$-compact;
3. $C_\lambda(X)$ is metrizable Menger;
4. $X$ is a pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$, family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is the countable set of all isolated points of $X$;
5. $C_\lambda(X)$ is homeomorphic to $C^*_p(\mathbb{N})$.

**Proof.** (1) $\Rightarrow$ (4). By Theorem 9 in [35], indestructibly productively Lindelöf spaces are projectively $\sigma$-compact and hence Hurewicz and Menger. By Theorems 1.1 and 1.2, $X$ is a pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$ and family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is the set of all isolated points of $X$.

Assume that $\kappa = |D(X)| > \aleph_0$. Then $C_\lambda(X, \mathbb{I})$ is homeomorphic to the space $\mathbb{I}^\kappa$ (Theorem 5.5 in [21]) where $\mathbb{I} = [-1, 1]$. It follows that the compact space $C_\lambda(X, \mathbb{I})$ includes a copy of $2^{\omega_1}$. Note that every indestructibly productively Lindelöf space is indestructibly Lindelöf. By Lemma 4.7 and Corollary 4.4 in [10], $C_\lambda(X)$ is destructibility. It follows that $|D(X)| \leq \aleph_0$.

(4) $\Rightarrow$ (3). By Theorem 2.4, $C_\lambda(X)$ is Menger. Clearly that, if $|D(X)| \leq \aleph_0$ then $w(C_\lambda(X)) = \aleph_0$ and hence $C_\lambda(X)$ is metrizable.

(3) $\Rightarrow$ (2). By Theorem 1.2.

(2) $\Rightarrow$ (1). By Theorem 7 in [7], a metrizable space is indestructibly productively Lindelöf if and only if it is $\sigma$-compact.

(5) $\Leftrightarrow$ (2). By Theorem 5.5 in [21], if $C_\lambda(X)$ is $\sigma$-compact then $C_\lambda(X)$ is homeomorphic to the space $\bigcup_{i=1}^{\kappa} [-i, i]^\kappa$ where $\kappa = |D(X)|$. Since $C_\lambda(X)$ is metrizable then $\kappa \leq \aleph_0$. It follows that $C_\lambda(X)$ is homeomorphic to $\bigcup_{i=1}^{\aleph_0} [-i, i]^\aleph_0$. Note that $\bigcup_{i=1}^{\aleph_0} [-i, i]^\aleph_0$ is homeomorphic to $C^*_p(\mathbb{N})$.

**Corollary 2.12.** $C_k(X)$ is indestructibly productively Lindelöf iff $X$ is finite.

**Remark 2.13.** If $X$ is compact and $C_\lambda(X)$ is indestructibly productively Lindelöf, then $X$ is homeomorphic to $\beta\mathbb{N}$, where $\beta\mathbb{N}$ is Stone-Čech compactification of the natural numbers $\mathbb{N}$, and $\lambda = [\mathbb{N}]^{<\aleph_0}$.

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