INFINITE DIVISIBILITY OF RANDOM FIELDS ADMITTING AN INTEGRAL REPRESENTATION WITH AN INFINITELY DIVISIBLE INTEGRATOR

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ABSTRACT. Let $\Lambda$ be an infinitely divisible random measure. We consider random fields of the form

$$X(t) = \int_{\mathbb{R}^d} f_t(x) \Lambda(dx), \quad t \in \mathbb{R}^q, \quad d, q \geq 1,$$

where $f_t : \mathbb{R}^d \to \mathbb{R}$ is $\Lambda$-integrable for all $t \in \mathbb{R}^d$. We show that $X$ is an infinitely divisible random field, that is the law of the random vector $(X(t_1), ..., X(t_n))$ is an infinitely divisible probability measure on $\mathbb{R}^n$ for all $t_1, ..., t_n \in \mathbb{R}^q$.

1. Preliminaries

We start with the definition of an infinitely divisible random measure (see [1], [2] and [3]). Following [1], let $R$ be a Borel subset of $\mathbb{R}^d$, $\mathcal{B}(R)$ be the Borel sets contained in $R$, and $\mathcal{S}$ be the $\delta$-ring (a ring closed under countable intersections) of bounded subsets of $R$. Let $\Lambda = \{\Lambda(A), A \in \mathcal{S}\}$ be a stochastic process with the following three properties.

- $\Lambda$ is independently scattered: If $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ is a sequence of disjoint sets, then the random variables $\Lambda(A_n)$, $n \in \mathbb{N}$, are independent.

- $\Lambda$ is $\sigma$-additive: If $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ is a sequence of disjoint sets and $\bigcup A_n \in \mathcal{S}$, then

$$\Lambda\left(\bigcup A_n\right) = \sum_n \Lambda(A_n) \quad a.s.$$

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\( \Lambda(A) \) is an infinitely divisible random variable for each \( A \in \mathcal{S} \), i.e. \( \Lambda(A) \) has the law of the sum of \( n \) independent identically distributed random variables for any \( n \geq 1 \).

Then \( \Lambda \) is called *infinitely divisible random measure*.

We now consider the cumulant function \( C_{\Lambda(A)}(t) = \ln(\mathbb{E}e^{it\Lambda(A)}) \) of \( \Lambda(A) \) for a set \( A \) in \( \mathcal{S} \) which is given by the Lévy-Khintchine representation

\[
C_{\Lambda(A)}(t) = ita(A) - \frac{1}{2}t^2b(A) + \int_{\mathbb{R}} \left( e^{itx} - 1 - it\tau(x) \right) F(dx, A),
\]

where \( a \) is a \( \sigma \)-additive set function on \( \mathcal{S} \), \( b \) is a measure on \( \mathcal{B}(\mathbb{R}) \), and \( F(dr, A) \) is a measure on \( \mathcal{B}(\mathbb{R}) \) for fixed \( dr \) and a Lévy measure on \( \mathcal{B}(\mathbb{R}) \) for each fixed \( A \in \mathcal{B}(\mathbb{R}) \), that is \( F(\{0\}, A) = 0 \) and \( \int_{\mathbb{R}} \min\{1, r^2\} F(dr, A) < \infty \), and \( \tau(r) = r\mathbb{1}_{[-1,1]}(r) \). \( F \) is a measure and referred to as the *generalized Lévy measure* and \( (a, b, F) \) is called *characteristic triplet* (see [1], [2] and [3]).

Let \( |a| = a^+ + a^- \). The measure \( \lambda \) with

\[
\lambda(A) := |a|(A) + b(A) + \int_{\mathbb{R}} \min\{1, r^2\} F(dr, A), \quad A \in \mathcal{S},
\]

is called *control measure* of the infinitely divisible random measure \( \Lambda \).

Let \( f_t : \mathbb{R}^d \to \mathbb{R}, \ d \geq 1 \), be \( \Lambda \)-integrable for all \( t \in \mathbb{R}^q, \ q \geq 1 \), that is there exists a sequence of simple functions \( \{\tilde{f}_t^{(n)}\}_{n \in \mathbb{N}}, \tilde{f}_t^{(n)} : \mathbb{R}^d \to \mathbb{R} \) for all \( t \in \mathbb{R}^q \), such that

(a) \( \tilde{f}_t^{(n)} \to f_t \ \lambda - \text{a.e.} \),

(b) for every Borel set \( B \in \mathcal{B}(\mathbb{R}) \), the sequence \( \{\int_B \tilde{f}_t^{(n)}(x) \Lambda(dx)\}_{n \in \mathbb{N}} \) converges in probability.

For each \( t \in \mathbb{R}^q \), we define

\[
\int_{\mathbb{R}^d} f_t(x) \Lambda(dx) := \text{plim}_{n \to \infty} \int_{\mathbb{R}^d} \tilde{f}_t^{(n)}(x) \Lambda(dx),
\]

where plim means convergence in probability (see [1], [2] and [3]), and consider random fields of the form

\[
X(t) = \int_{\mathbb{R}^d} f_t(x) \Lambda(dx), \quad t \in \mathbb{R}^q.
\]
2. Main result

**Theorem 2.1.** The random field $X$ is infinitely divisible, that is the law of the random vector $(X(t_1), ..., X(t_n))$ is an infinitely divisible probability measure on $\mathbb{R}^n$ for all $t_1, ..., t_n \in \mathbb{R}^q$.

**Proof.** Let $\varphi_{(t_1, ..., t_n)}$ be the characteristic function of $(X(t_1), ..., X(t_n))$. It is enough to show that $\varphi_{(t_1, ..., t_n)}^\gamma$ is a characteristic function for all $\gamma > 0$, cf. [4].

Due to the linearity of the integral (1) and the fact that any linear combination of $\Lambda$-integrable functions is $\Lambda$-integrable (cf. [3], p. 81), we have

$$
\sum_{j=1}^n x_j X(t_j) = \int_{\mathbb{R}^d} \left( \sum_{j=1}^n x_j f_{t_j}(s) \right) \Lambda(ds)
$$

and the characteristic function $\varphi_{(t_1, ..., t_n)}$ is given by

$$
\varphi_{(t_1, ..., t_n)}(x) = \varphi_{\sum_{j=1}^n x_j X(t_j)}(1)
$$

$$
= \exp \left\{ i a \sum_{j=1}^n x_j f_{t_j} - \frac{1}{2} b \sum_{j=1}^n x_j f_{t_j} + \int_{\mathbb{R}^d} \int_{\mathbb{R}} c \sum_{j=1}^n x_j f_{t_j}(s, y) F(ds, dy) \right\},
$$

cf. [1], where

$$
a \sum_{j=1}^n x_j f_{t_j} = \int_{\mathbb{R}^d} \left( \sum_{j=1}^n x_j f_{t_j}(s) \right) a(ds),
$$

$$
b \sum_{j=1}^n x_j f_{t_j} = \int_{\mathbb{R}^d} \left( \sum_{j=1}^n x_j f_{t_j}(s) \right)^2 b(ds),
$$

$$
c \sum_{j=1}^n x_j f_{t_j}(s, y) = e^{i \sum_{j=1}^n x_j f_{t_j}(s)y} - 1 - i \sum_{j=1}^n x_j f_{t_j}(s)\tau(y).
$$

Let $\gamma > 0$. Then

$$
\varphi_{(t_1, ..., t_n)}^\gamma(x) = \exp \left\{ i \gamma a \sum_{j=1}^n x_j f_{t_j} - \frac{1}{2} \gamma b \sum_{j=1}^n x_j f_{t_j} + \int_{\mathbb{R}^d} \int_{\mathbb{R}} c \sum_{j=1}^n x_j f_{t_j}(s, y) \gamma F(ds, dy) \right\}
$$
with
\[
\gamma a \sum x_j f_{t_j} = \int_{\mathbb{R}^d} \left( \sum_{j=1}^n x_j f_{t_j}(s) \right) \gamma a(ds),
\]
\[
\gamma b \sum x_j f_{t_j} = \int_{\mathbb{R}^d} \left( \sum_{j=1}^n x_j f_{t_j}(s) \right)^2 \gamma b(ds).
\]

Since \( a^* := \gamma a \) is a \( \sigma \)-additive set function on \( \mathcal{S} \), \( b^* := \gamma b \) is a measure on \( \mathcal{B}(\mathbb{R}) \), and \( F^*(dr, A) := \gamma F(dr, A) \) is a measure on \( \mathcal{B}(\mathbb{R}) \) for fixed \( dr \) and a Lévy measure on \( \mathcal{B}(\mathbb{R}) \) for each fixed \( A \in \mathcal{B}(\mathbb{R}) \), there exists an infinitely divisible random measure \( \Lambda^* \) with characteristic triplet \( (a^*, b^*, F^*) \), cf. Proposition 2.1.(b) in [2]. Therefore, \( \varphi_{\gamma}^{\gamma}(t_1, ..., t_n) \) is the characteristic function of \( (Y(t_1), ..., Y(t_n)) \) with
\[
Y(t) = \int_{\mathbb{R}^d} f_t(x) \Lambda^*(dx).
\]

\[\square\]

**References**

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