Some variational properties of the weighted $\sigma_r$--curvature for submanifolds in Riemannian manifolds

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Abstract

The objet of this paper is the study of the variations of a functional whose integrant is the $r$-th weighted curvature on the hypersurface of a closed Riemannian manifold. Some applications to hypersurfaces of the Euclidean space and the unit round sphere are given.

1 Introduction

The study of Riemannian geometry seems to be based essentially on the study of certain operators such as the shape operator, the Ricci tensor, the Schouten operator etc...Some functions constructed from these operators play a fundamental role in Riemannian geometry. Particularly the algebraic invariants of these operators such as the $r$-th symmetric functions $\sigma_r$, associated with the shape operator and the Newton transformations $T_r$. The following articles can be consulted on this subject ([1], [2], [3], [4], [5], [6], [7], [8], [9]). Reilly (see [7]) has considered the variations where the integrant of the functional is a function of the $r$-th mean curvatures $\sigma_r$. In a recent paper Case (see [3]) introduced and studied the notion of $r$-th weighted curvatures. The aim of this paper is the study of the variations of a functional whose integrant is $r$-th weighted curvature on the hypersurface of a closed Riemannian manifold. Some applications to hypersurfaces of the Euclidean space and the unit round sphere are given.

2 Preliminaries

We associate with any endomorphism $A$ on an $n$--dimensional vector space $V$ a family of endomorphisms $(T_r)_r$ on $V$ defined recurrently by:

$$T_0 = id_V$$

$$T_r = \sigma_r id_V - AT_{r-1}, r = 1, 2, ...$$
or under a condensed formula

\[ T_r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j \]

with \( T_r = 0 \), for all \( r \geq n \).

First we recall (see [3]) the notion of the weighted elementary symmetric polynomials.

Let \( r \in \mathbb{N} \cup \{\infty\} \). The \( r \)-th weighted elementary symmetric polynomial \( \sigma^\infty_r : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), is inductively given by

\[
\begin{align*}
\sigma^\infty_0(\mu_0, \mu) &= 1 \\
\sigma^\infty_r(\mu_0, \mu) &= \sigma^\infty_{r-1}(\mu_0, \mu) \sum_{j=1}^{n} \mu_j + \sum_{i=1}^{n-1} \sum_{j=1}^{r} (-1)^i \sigma^\infty_{r-i-1}(\mu_0, \mu) \mu^i_j, \text{ for } r \geq 1 \\
\text{and } \mu &= (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n.
\end{align*}
\]

(1)

\( \sigma^\infty_r(\mu_0, \mu) \) is expressed in terms of \( \mu_0 \) and \( 1, \sigma_1(\mu), \ldots, \sigma_r(\mu) \) (see [3] page 6):

**Proposition 1** Given \( r \in \mathbb{N} \cup \{\infty\}, \mu_0 \in \mathbb{R} \) and \( \mu \in \mathbb{R}^n \), we have

\[ \sigma^\infty_r(\mu_0, \mu) = \sum_{j=0}^{r} \frac{\mu_0^j}{j!} \sigma_{r-j}(\mu). \]  

(2)

**Definition 2** Since a symmetric matrix is diagonalizable one can consider the symmetric elementary polynomials associated with a real number \( \mu_0 \) and a symmetric matrix \( A \) with eigenvalues \( \mu = (\mu_1, \ldots, \mu_n) \) like being the polynomials acting on that real number \( \mu_0 \) and on the vector \( \mu = (\mu_1, \ldots, \mu_n) \), so the weighted elementary symmetric polynomial of \( A \) is defined by \( \sigma^\infty_r(\mu_0, A) = \sigma^\infty_r(\mu_0, \mu) \).

The weighted Newton transformations are then defined as follows (see [3])

Let \( A \) be a symmetric \( n \times n \) real-valued matrix and \( \mu_0 \in \mathbb{R} \). The \( r \)-th weighted Newton transformation function \( T_r^\infty(\mu_0, A) \) of \( A \) and \( \mu_0 \) is

\[ T_r^\infty(\mu_0, A) = \sum_{j=0}^{r} (-1)^j \sigma^\infty_{r-j}(\mu_0, A) A^j. \]  

(3)

The weighted elementary symmetric polynomials can be expressed in terms of the weighted Newton transformations in the following way (see [3])

**Proposition 3** Let \( A \) be a symmetric \( n \times n \) real-valued matrix and \( \mu_0 \in \mathbb{R} \). It follows that

\[ \text{trace}(AT_r^\infty(\mu_0, A)) = (r + 1) \sigma^\infty_{r+1}(\mu_0, A) - \mu_0 \sigma^\infty_r(\mu_0, A). \]  

(5)
Proposition 4

\[ T_r^\infty(\mu_o, A) = \sum_{j=0}^{r} \frac{\mu_o^j}{j!} T_{r-j}(A) \]

where \( T_r(A) \) is the classical Newton transformation.

Proof. From the formulae (2) and (3), we get

\[ T_r^\infty(\mu_o, A) = \sum_{l=0}^{r} (-1)^l \sigma_r^{r-l}(\mu_o, A) A^l \]

\[ = \sum_{l=0}^{r} (-1)^l \sum_{j=0}^{r-l} \frac{\mu_o^j}{j!} \sigma_{r-l-j}(A) A^j \]

\[ = \sum_{l=0}^{r} \frac{\mu_o^l}{l!} T_{r-l}(A). \]

As it is well known that the classical Newton transformations are of free-divergence i.e.

\[ T_{r,j}^{ij}(A) = 0 \]

we deduce that:

Proposition 5

\[ T_r^\infty(\mu_o, A)^j_{i,j} = \sum_{l=0}^{r-1} \frac{\mu_o^l}{l!} (T_{r-1-l})^j_{i} (A) \mu_o^{j_i}. \]

From the formula (2) i.e.

\[ \sigma_r^{\infty}(\mu_o, A) = \sum_{j=0}^{r+1} \frac{\mu_o^j}{j!} \sigma_{r+1-j}(A) \]

we obtain by differentiation with respect to \( t \) that

\[ \frac{\partial}{\partial t} \sigma_r^{\infty}(\mu_o, A) = \sum_{j=0}^{r+1} \frac{\mu_o^j}{j!} \frac{\partial}{\partial t} \sigma_{r+1-j}(A) + \sum_{j=1}^{r+1} \frac{\mu_o^{j-1}}{(j-1)!} \sigma_{r+1-j} \frac{\partial \mu}{\partial t} \]

\[ = \sum_{j=0}^{r+1} \frac{\mu_o^j}{j!} \frac{\partial}{\partial t} \sigma_{r+1-j} + \sum_{j=0}^{r} \frac{\mu_o^j}{j!} \sigma_{r-j} \frac{\partial \mu}{\partial t}. \]

Under the following formula (see [7] Lemma A page 467)

\[ \frac{\partial \sigma_r^{\infty}}{\partial t} = \text{trace} \left( \frac{\partial A}{\partial t} T_r \right) \]

(7)
and the Newton’s formula (see [7] formula (1) page 467)

\[(r + 1) \sigma_{r+1} = \text{trace}(AT_r), \quad (8)\]

we get

\[
\begin{aligned}
\frac{\partial}{\partial t} \sigma^\infty_{r+1} (\mu_o, A) &= \sum_{j=0}^{r+1} \frac{\mu_o^j}{j!} \text{trace} \left( \frac{\partial A}{\partial t} T_{r-j} \right) + \sum_{j=0}^{r} \frac{\mu_o^j}{j!} \sigma_{r-j} \frac{\partial \mu_o}{\partial t} \\
&= \sum_{j=0}^{r+1} \frac{\mu_o^j}{j!} \text{trace} \left( \frac{\partial A}{\partial t} T_{r-j} \right) + \sigma^\infty_r (\mu_o, A) \frac{\partial \mu_o}{\partial t} \\
&= \text{trace} \left( \frac{\partial A}{\partial t} T^\infty_r (\mu_o, A) \right) + \sigma^\infty_r (\mu_o, A) \frac{\partial \mu_o}{\partial t}.
\end{aligned}
\]

Consider a one family of parameter \( \psi_t : M^m \to \overline{M}^n \) of immersions of an \( m \)--dimensional closed manifold \( M^m \) into an \( n \)--Riemannian manifold \((\overline{M}^n, \langle \cdot, \cdot \rangle)\). Denote by \( X \) the deformation vector field and by \( \nu \) the normal vector field to \( \overline{M}^n \). Put \( \lambda = \langle X, \nu \rangle, \mu = X^\top \) the tangential component of \( X \) and \( dV \) the volume form on \( M^m \). Consider the following variational problem

\[
\delta \left( \int_M \sigma^\infty_r dV \right) = 0
\]

**Theorem 6** With the above notations and assumptions the first variation of the global \( \sigma^\infty_r \)--curvature is given by:

\[
\begin{aligned}
\frac{d}{dt} \int_M \sigma^\infty_r dV &= \int_M \left\{ \lambda \left( - (r + 1) \sigma^\infty_{r+1} + \mu_o \sigma^\infty_r - \sigma^\infty_{r-1} \mu \mu_{ol} \right) + \lambda_{ij} (T^\infty_{r})^{ij} \\
&\quad - g^{jm} \left( R^{\overline{M}} \right) \left( \nu, \frac{\partial \psi}{\partial x_i}, X, \frac{\partial \psi}{\partial x_m} \right) (T^\infty_{r-1})^{ij} \\
&\quad + g^{jm} \left( R^{\overline{M}} \right)^{ij} \left( \frac{\partial \psi}{\partial x_j}, \mu \right) \frac{\partial \psi}{\partial x_m} (T^\infty_{r-1})^{ij} + \sigma^\infty_r (\mu_o, A) \frac{\partial \mu_o}{\partial t} \right\} dV.
\end{aligned}
\]

On the other hand

\[
\text{trace} \left( \frac{\partial A}{\partial t} T^\infty_k \right) = \frac{\partial A^j}{\partial t} (T^\infty_k)^{ij}
\]

with

\[
A^j = g^{jk} A_{ik}
\]

where

\[
A_{ik} = \left\langle \nabla \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k} \right\rangle = - \left\langle \nabla \frac{\partial \psi}{\partial x_i} \nu, \frac{\partial \psi}{\partial x_k} \right\rangle
\]

Hence

\[
\frac{\partial A^j}{\partial t} = \frac{\partial g^{jk}}{\partial t} A_{ik} + g^{jk} \frac{\partial A_{ik}}{\partial t}.
\]
Obviously
\[ \frac{\partial g_{jk}}{\partial t} = -g^{jl} \frac{\partial g_{pl}}{\partial t} g^{pk} \]

Now, if we consider the calculations in a normal coordinates that is at a point \( x \in M \) where the metric tensor fulfills \( g_{ij}(x) = \left\langle \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial x} \right\rangle = \delta_{ij} \) and \( \Gamma^k_{ij}(x) = 0 \), where \( \Gamma^k_{ij} \) stand for the Christoffel symbols corresponding to the metric connection \( \nabla \) on \( M \), we get

\[
\frac{\partial g_{pl}}{\partial t} = \left\langle \nabla \frac{\partial \psi}{\partial x_p}, \frac{\partial \psi}{\partial x_l} \right\rangle
= \left\langle \nabla \frac{\partial \psi}{\partial x_p} X, \frac{\partial \psi}{\partial x_l} \right\rangle
= \left\langle \nabla \frac{\partial \psi}{\partial x_p} \left( \lambda \nu + \mu^m \frac{\partial \psi}{\partial x_m} \right), \frac{\partial \psi}{\partial x_l} \right\rangle
+ \left\langle \frac{\partial \psi}{\partial x_p}, \nabla \frac{\partial \psi}{\partial x_l} \left( \lambda \nu + \mu^m \frac{\partial \psi}{\partial x_m} \right) \right\rangle
= \mu^p_{\cdot l} + \mu^l_{\cdot p} - 2\lambda A_{pl}.
\]

hence
\[
\frac{\partial g_{jk}}{\partial t} = -g^{jl} g^{pk} \left( \mu^p_{\cdot l} + \mu^l_{\cdot p} - 2\lambda A_{pl} \right).
\]

(10)

We have also
\[
\frac{\partial \nu}{\partial t} = \left\langle \frac{\partial \nu}{\partial t}, \frac{\partial \psi}{\partial x_i} \right\rangle \frac{\partial \psi}{\partial x_k}
= -\left\langle \nu, \nabla \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k} \right\rangle
= -\left\langle \nu, \nabla \frac{\partial \psi}{\partial x_i} X \right\rangle \frac{\partial \psi}{\partial x_k}
= -h^i_k \left( \lambda_{ij} + \mu^l A_{jl} \right) \frac{\partial \psi}{\partial x_k}.
\]

Now we compute
\[
\frac{\partial A_{ik}}{\partial t} = -\left\langle \nabla \frac{\partial \psi}{\partial x_i}, \nabla \frac{\partial \psi}{\partial x_k} \right\rangle
= -R^{\kappa\nu}_{\gamma\delta} \left( \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k}, X \right)
- \left\langle \nabla \frac{\partial \psi}{\partial x_i} \nabla \frac{\partial \psi}{\partial x_k}, \nu \right\rangle
- \left\langle \nabla \frac{\partial \psi}{\partial x_i}, \nabla \frac{\partial \psi}{\partial x_k} X \right\rangle
\]

By formula (10), we get
\[
\nabla \frac{\partial \psi}{\partial x_i} \nabla \frac{\partial \psi}{\partial x_i} \nu = -g^{jm} \left( \lambda_{ij} + \mu^l A_{jl} + \mu^l A_{jl,i} \right) \frac{\partial \psi}{\partial x_m} + \left( \lambda_{ij} + \mu^l A_{jl} \right) \nabla \frac{\partial \psi}{\partial x_i} \nabla \frac{\partial \psi}{\partial x_m}
\]
so
\[
\left\langle \nabla \frac{\partial \psi}{\partial x_i} \nabla \frac{\partial \psi}{\partial x_i} \nu, \frac{\partial \psi}{\partial x_k} \right\rangle = -g^{jm} \left( \lambda_{ij} + \mu^l A_{jl} + \mu^l A_{jl,i} \right) g_{mk}
- \left( \lambda_{ij} + \mu^l A_{jl} \right) g^{jm} \left\langle \nabla \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_m}, \frac{\partial \psi}{\partial x_k} \right\rangle.
\]
In the same manner, we have

\[
\langle \nabla_{\partial x_i, \nu}, \nabla_{\partial x_k, \nu} \rangle = \langle \nabla_{\partial x_i, \nu}, \nabla_{\partial x_k, \nu} (\lambda \nu + \mu^m \frac{\partial \psi}{\partial x_m}) \rangle \\
= \langle \nabla_{\partial x_i, \nu}, \lambda \nabla_{\partial x_k, \nu} \nu + \mu^m k \frac{\partial \psi}{\partial x_m} + \mu^m \nabla_{\partial x_k, \nu} \frac{\partial \psi}{\partial x_m} \rangle \\
= \lambda \langle \nabla_{\partial x_i, \nu}, \nabla_{\partial x_k, \nu} \rangle - \mu^m k A_{im} + \mu^m \langle \nabla_{\partial x_i, \nu}, \nabla_{\partial x_k, \nu} \rangle. 
\]

By noticing that

\[
\langle \nabla_{\partial x_i, \nu}, \nabla_{\partial x_k, \nu} \rangle = \langle \nabla_{\partial x_i, \nu}, \frac{\partial \psi}{\partial x_j} \rangle \langle \nabla_{\partial x_k, \nu}, \frac{\partial \psi}{\partial x_j} \rangle = A_{ik}^2,
\]
we get

\[
\frac{\partial A_{ik}}{\partial t} = -R^M(\nu, \frac{\partial \psi}{\partial x_j}, \frac{\partial \psi}{\partial x_j}, X) + g^{jm} \left( \lambda j j + \mu^{l,i} A_{jl} + \mu^l A_{jl,i} \right) g_{mk} - \lambda A_{ik}^2 + \mu^{l,k} A_{il}.
\]

Hence

\[
g^{jk} \frac{\partial A_{ik}}{\partial t} = -g^{jk} R^M(\nu, \frac{\partial \psi}{\partial x_j}, \frac{\partial \psi}{\partial x_j}, X) + g^{jk} g^{pm} \left( \lambda p i + \mu^{l,i} A_{pl} + \mu^l A_{pl,i} \right) g_{mk} - \lambda g^{jk} A_{ik}^2 + g^{jk} \mu^{l,k} A_{il}
\]

Taking into account formula (10), we get

\[
\frac{\partial A_{ij}}{\partial t} = -g^{jk} R^M(\nu, \frac{\partial \psi}{\partial x_j}, \frac{\partial \psi}{\partial x_j}, X) + g^{jk} \left( \lambda, k i + \mu^{l,i} A_{kl} + \mu^l A_{kl,i} \right)
\]

\[
\quad - \lambda g^{jk} A_{ik}^2 + g^{jk} \mu^{l,k} A_{il} - g^{jl} g^{pk} \left( \mu^{l p} + \mu^{l p, i} - 2 \lambda A_{p l} \right) A_{ik}.
\]

To compute \( \frac{\partial \sigma_{r+1}}{\partial t} \) we multiply both sides of (1) by \((T_r^\infty)_j^i\) and sum and add the term \(\sigma_r^\infty (\mu_o, A) \frac{\partial \mu_o}{\partial t}\).

First, we have

\[
\lambda A_{i}^j A_{p}^p (T_r^\infty)_j^i = \lambda \text{trace} \left( A^2 T_r^\infty \right) = \lambda \text{trace} \left( \sigma_{r+1}^\infty A - AT_{r+1}^\infty \right)
\]

and by the formulas (2) and (11), we get

\[
\lambda A_{i}^j A_{p}^p (T_r^\infty)_j^i = \lambda \left( \sigma_1^\infty \sigma_{r+1}^\infty - (r + 2) \sigma_{r+2}^\infty \right).
\]

We also write

\[
g^{jk} \lambda, k i (T_r^\infty)_j^i = (T_r^\infty)_j^i \lambda, ij.
\]
By the Codazzi formula, we have

\[ g^{jm} \mu^l \alpha_m, i \left( T^\infty_r \right)_j = g^{jm} \mu^l \alpha_m, i \left( T^\infty_r \right)_j + g^{jm} \left( R_M \right)_l \left( \frac{\partial \psi}{\partial x_i} \right) \frac{\partial \psi}{\partial x_m} \left( T^\infty_r \right)_j \]

By formula (11) we get

\[ g^{jm} \mu^l \alpha_m, i \left( T^\infty_r \right)_j = g^{jm} \mu^l \alpha_m, i \left( T^\infty_r \right)_j + g^{jm} \left( R_M \right)_l \left( \frac{\partial \psi}{\partial x_i} \right) \frac{\partial \psi}{\partial x_m} \left( T^\infty_r \right)_j. \]

Also, we have

\[ g^{jl} g^{pm} (\mu^l, p + \mu^l, p) A_{im} = 2 \mu^j \nu A_{ip} \]  
(16)

and

\[ g^{jk} (\mu^l, i A_{kl} + \mu^l, k A_{il}) = 2 \mu^j \nu A_{jl}. \]  
(17)

Hence

\[ \frac{\partial A^j}{\partial t} = -g^{jk} \left( R_M \left( \nu, \frac{\partial \psi}{\partial x_k}, X \right) + g^{jk} \lambda, k + g^{jk} \mu^l \alpha_{kl}, i \right. 
\left. - \lambda g^{jk} \left( \frac{\nabla}{\partial x_i} \nu, \frac{\nabla}{\partial x_k} \nu \right) + 2 \lambda g^{jl} g^{pk} A_{pi} A_{ik} \right). \]  
(18)

and since

\[ \lambda \left( \frac{\nabla}{\partial x_i} \nu, \frac{\nabla}{\partial x_k} \nu \right) = \lambda \left( \frac{\nabla}{\partial x_i} \nu, \frac{\nabla}{\partial x_k} \nu \right) = \lambda A^{(2)} \]

we infer that

\[ \frac{\partial A^j}{\partial t} = -g^{jk} \left( R_M \left( \nu, \frac{\partial \psi}{\partial x_k}, X \right) + g^{jk} \lambda, k + g^{jk} \mu^l \alpha_{kl}, i \right. 
\left. + \lambda g^{jl} g^{pk} A_{pi} A_{ik} \right). \]  
(19)

Hence, by virtue of the formula (11), we deduce

\[ \frac{\partial}{\partial t} \sigma_{r+1} = -g^{jm} \left( R_M \left( \nu, \frac{\partial \psi}{\partial x_m}, \frac{\partial \psi}{\partial x_i} \right) \left( T^\infty_r \right)_j + \lambda, ij \right) 
\left( T^\infty_r \right)^{ij} 
+ \mu^l \sigma_{r+1, l} - \sigma_r \left( \mu, A \right) \mu^l \mu_{o, l} 
+ g^{jm} \left( R_M \right)_l \left( \frac{\partial \psi}{\partial x_i} \right) \frac{\partial \psi}{\partial x_m} \left( T^\infty_r \right)_j 
+ \lambda \left( \sigma^1 \sigma^\infty - (r + 2) \sigma^\infty \right) 
- \lambda \mu, o \sigma_{r+1} + \sigma_r \left( \mu, A \right) \frac{\partial \mu}{\partial t}. \]  
(21)

The expression of \( \frac{\partial dV}{\partial t} \) is standard and it is given by

\[ \frac{\partial dV}{\partial t} = \left( -\lambda \sigma^1 + \mu, l \right) dV \]
and by \((2)\), we get
\[
\frac{\partial dV}{\partial t} = \left(-\lambda (\sigma_1^\infty - \mu_o) + \mu_1^l\right) dV.
\] (22)

Combining the expressions \((1)\) and \((22)\) we obtain the expression of the integrand in Theorem 6.

In the particular case where \(M\) is of constant curvature \(c\), we have
\[
g^{jm} R^M (\nu, \frac{\partial \psi}{\partial x_m}, \frac{\partial \psi}{\partial x_i}, X) (T^\infty)_r^i = \lambda \text{trace} (T^\infty)_r^i
\]
\[
= \lambda c \left( (n - r) \sigma_r^\infty + \sigma_o^\infty \right)
\]
and in addition, we know that the covariant derivative of the second fundamental form satisfies the relation
\[
A_{ml,i} = A_{mi,l}
\]
consequently:

**Corollary 7** If the ambient manifold \(M\) is of constant curvature \(c\), we have: for any \(r \geq 2\)
\[
\frac{d}{dt} \int_M \sigma_r^\infty dV = \int_M \left\{ \lambda \left( -(r + 1) \sigma_{r+1}^\infty + \mu_o \sigma_r^\infty - \sigma_{r-1}^\infty \mu_o \right) + \lambda \sigma_{i,j} (T^\infty)_r^i \right\} dV.
\] (23)

**Remark 8** In the particular case \(\mu_o = 0\), i.e. \(\sigma_r^\infty = \sigma_r\) we recover the result in [7].

3 Hypersurfaces in Euclidean space

We restrict ourselves to the case \(\mu_o\) is a constant

**Definition 9** A hypersurface in an Euclidean space is said \(\sigma_r^\infty\)-minimal if \((r + 1) \sigma_{r+1}^\infty - \mu_o \sigma_r^\infty\) vanishes identically.

As in the paper of Reilly (see [7]) we will express the minimality of an hypersurface in terms of partial differential equations. Let \(\psi = (\psi_1, ..., \psi_{n+1})\) be the position vector of the hypersurface \(M\) in the Euclidean space \(E^{n+1}\) and \(\psi_{,ij} = (\psi_{1,ij}, ..., \psi_{,n+1,ij})\) the second covariant derivative of \(x\) on \(M\). We have
\[
\psi_{,ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} - d\psi (\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j})
\]
\[
= \frac{\partial^2 \psi}{\partial x_i \partial x_j} - d\psi (\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j})
\]
\[
= \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \nabla \psi_{,i} \frac{\partial}{\partial x_j}
\]
\[
= \left\{ \nabla \psi_{,i} \frac{\partial}{\partial x_j} \nabla \psi \frac{\partial}{\partial x_j} \right\} \psi
\]
where $\nabla$ is the covariant derivative on $M$ and $\nu$ denotes the unit normal vector field to $M$ so

\[
(T_r^\infty)^{ij}_\psi,_{ij} = (T_r^\infty)^{ij}_A_{ij}\nu = \text{trace}(AT_r^\infty)\nu = (r + 1)\sigma_{r+1}^\infty - \mu_o\sigma_{\infty}^r.
\]

Observe that in this case ($\mu_o = \text{constant}$) the Newton tensor $(T_r^\infty)^{ij}_\psi$ is of free divergence so we have established

**Proposition 10** In case $\mu_o = \text{constant}$, a hypersurface in an Euclidean space is $\sigma_{r+1}^\infty$-minimal if and only if each component of its position vector field satisfies the partial differential equation $(T_r^\infty)^{ij}_\psi,_{ij} = 0$ or in its divergence form

\[
\left((T_r^\infty)^{ij}_\psi,_{i}\right)_j = 0
\]

Suppose $\mu_o = \text{constant}$ and consider the function $\phi : M \rightarrow \mathbb{R}$, given by $\phi(x) = \langle B(x), \nu(x) \rangle$ where $B(x)$ and $\nu(x)$ are respectively any vector in $E^{n+1}$ and the normal one.

\[
B = \phi\nu + \left(B, \frac{\partial\psi}{\partial x_k}\right) \frac{\partial\psi}{\partial x_k}
\]

or briefly

\[
B = \phi\nu + B_k \frac{\partial\psi}{\partial x_k}
\]

so

\[
\phi,_{i} = \langle B, \nabla_{i}\nu \rangle = -B_k A_{ik}
\]

and

\[
\phi,_{ij} = -\phi A_{ik}^2 - B_k A_{ik,j}.
\]  \hspace{1cm} (26)

Multiplying both sides of (26) by $(T_r^\infty)_j^i$, and thanks to the Codazzi formula, proposition (3) and formula (1) we obtain

\[
\phi,_{ij} (T_r^\infty)_j^i = -\phi \text{trace} \left(A^2 T_r^\infty\right) - B_k A_{ij,k} (T_r^\infty)_j^i = -\phi \left(\sigma_{r+1}^\infty - (r + 2)\sigma_{r+2}^\infty\right) - (r + 1) B_k \sigma_{r+1,k} + \mu_o B_k \sigma_{r,k}^\infty.
\]

So, we established

**Proposition 11** The $\sigma_{r+1}^\infty$-mean curvature of a hypersurface in an Euclidean space is constant if and only if the components of the normal vector field satisfy the partial differential equation

\[
\phi,_{ij} (T_r^\infty)_j^i = -\phi \left(\sigma_{r+1}^\infty - (r + 2)\sigma_{r+2}^\infty\right) + \mu_o B_k \sigma_{r,k}^\infty.
\]
4 Hypersurfaces in the round unit sphere

We consider hypersurfaces of the unit round sphere $S^n \subset E^{n+1}$. From the Corollary 7 the Euler-Lagrange equation is expressed by:

$$2\sigma^\infty_r - \mu_o \sigma^\infty_1 - n = 0$$

for $r = 1$, and

$$(r + 1) \sigma^\infty_{r+1} - (n - r + 1) \sigma^\infty_{r-1} - \mu_o (\sigma^\infty_r - \sigma^\infty_{r-2}) = 0$$

(27)

for $r \geq 2$.

Definition 12 A hypersurface in the unit round sphere is said $\sigma^\infty_r$-minimal with $r \geq 2$ if

$$(r + 1) \sigma^\infty_{r+1} - (n - r + 1) \sigma^\infty_{r-1} - \mu_o (\sigma^\infty_r - \sigma^\infty_{r-2}) = 0.$$

Let $\psi = (\psi_1, ..., \psi_{n+2})$ be the position vector of the hypersurface $M$ in the unit round sphere $S^{n+1} \subset E^{n+2}$ and $\psi_{ij} = (\psi_{1,ij}, ..., \psi_{n+2,ij})$ the second covariant derivative of $x$ on $M$. If $\nabla$ denotes the covariant derivative on $M$ induced by the covariants derivative $\nabla^{S^{n+1}}$ on the unit sphere. We have

$$\psi_{ij} = \nabla_\omega \nabla \frac{\partial \psi}{\partial \omega} = \nabla_\omega \frac{\partial d\psi}{\partial x_i}$$

$$= \left\langle \nabla^{S^{n+1}}_\omega \frac{\partial d\psi}{\partial x_i}, \nu \right\rangle + \left\langle \nabla^{S^{n+1}}_\omega \frac{\partial d\psi}{\partial x_i}, \psi \right\rangle$$

$$= \left\langle \nabla^{S^{n+1}}_\omega \frac{\partial d\psi}{\partial x_i}, \nu \right\rangle - \left\langle \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_j} \right\rangle$$

(28)

where $\nu$ is the unit normal vector field to $M$. In order to characterize the $\sigma^\infty_r$-minimality of the sub-manifolds of the unit sphere, we multiply both sides of (1) by $(r - 1) (T_r^\infty)^{ij} - (n - r + 1) (T_{r-2}^\infty)^{ij}$ and sum to infer

$$\left( (r - 1) (T_r^\infty)^{ij} + (n - r + 1) (T_{r-2}^\infty)^{ij} \right) \psi_{ij} =$$

$$- \left( (n - r) \sigma^\infty_r + \mu_o \sigma^\infty_{r-1} - (n - r + 1) \left( (n - r + 2) \sigma^\infty_{r-2} + \mu_o \sigma^\infty_{r-3} \right) \right) \psi$$

$$+ n \mu_o \sigma^\infty_{r-1} \nu$$

Hence we proven:

Proposition 13 A hypersurface in the unit round sphere is $\sigma^\infty_r$-minimal $r \neq 1$ if and only if each component of the position vector field is solution of the partial differential equation

$$\left( (r - 1) (T_r^\infty)^{ij} + (n - r + 1) (T_{r-2}^\infty)^{ij} \right) \psi_{ij} = n \mu_o \sigma^\infty_{r-1} \nu$$

$$- \left( (n - r) \sigma^\infty_r + \mu_o \sigma^\infty_{r-1} - (n - r + 1) \left( (n - r + 2) \sigma^\infty_{r-2} + \mu_o \sigma^\infty_{r-3} \right) \right) \psi.$$
Remark 14 In case \( r = 1 \), a hypersurface in the unit round sphere is \( \sigma_1^\infty \)-minimal if and only if
\[
2\sigma_2^\infty - \mu_\sigma \sigma_1^\infty - n = 0 \tag{29}
\]
or equivalently
\[
\psi,_{ij} T_{ij}^{1} = n \nu - ((n - 1) \sigma_1^\infty + \mu_\sigma) \psi
\]
onobtained by multiplying both sides of (7) by \( T_1^\infty \) and summing.

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