COUNTERTERMS, HOLONOMY AND SUPERSYMMETRY

K.S. STELLE

Theoretical Physics Group,
Imperial College London,
Prince Consort Road,
London SW7 2AZ, UK
E-mail: k.stelle@imperial.ac.uk

The divergence structure of supergravity has long been a topic of concern because of the theory’s non-renormalizability. In the context of string theory, where perturbative finiteness should be achieved, the supergravity counterterm structures remain nonetheless of importance because they still occur, albeit with finite coefficients. The leading nonvanishing supergravity counterterms have a particularly rich structure that has a bearing on the preservation of supersymmetry in string vacua in the presence of perturbative string corrections. Although the holonomy of such manifolds is deformed by the corrections, a Killing spinor structure nevertheless can persist. The integrability conditions for the existence of such Killing spinors remarkably remain consistent with the perturbed effective field equations.

1. Supergravity Counterterms

The ultraviolet divergences of quantized general relativity and its various matter couplings have posed a key problem for the reconciliation of quantum mechanics and relativity. The potential for ultraviolet trouble with gravity was apparent already since the 1930’s from rudimentary power counting, in consequence of Newton’s constant having dimensionality [length]$^2$. When detailed calculations of gravitational Feynman diagrams became possible in the 1970’s, this became a reality with the first calculations of divergence structures that are not present in the original second-order action. As ever, in the key issues involving gravity and its quantization, Stanley Deser played a major rôle in this development.

As disastrous as the ultraviolet problem was for quantized field theories containing gravity, there was nonetheless some hope that a clever combina-

---

*Research supported in part by the EC under RTN contract HPRN-CT-2000-00131 and by PPARC under rolling grant PPA/G/O/2002/00474.
tion of fields might save the day by arranging for the divergences to marvelously cancel. The prime candidate for an organizing principle that might engineer this was supersymmetry, and when supergravity came forth\(^4\), there was palpable hope that it might enable the construction of some jewel-like theory that could resolve (maybe uniquely) the ultraviolet problems. This hope was encouraged by the development of non-renormalization theorems for chiral supermatter\(^5\) and by initial calculations showing that supergravity also had better-than-generic ultraviolet behavior. For one-loop Feynman diagrams, the divergences cancel in pure \(N = 1\) supersymmetry, as one can see by summing the contributions of the different field species occurring in the loop. A range of differing arguments was advanced on formal grounds to why these cancellations occur and why they could be expected to persist at the two-loop level (despite the prohibitive difficulty of actually performing such calculations). One approach\(^6\) that has much current resonance focused on helicity conservation properties.

In many of these early developments, the lively scientific atmosphere at Brandeis yielded important understanding of these ultraviolet problems. For me, as a graduate student there at the time, it was a marvellous training ground for learning the way physics should really be done, but one with a decidedly European flavor. Stan Deser was without doubt the leader in these matters, and it summons pleasant recollections to think back to how these fundamental issues were grappled with. Given Stan’s status as doyen of the canonical formalism, another natural development we got into at the time was the canonical formulation of supergravity.\(^7\) Although not directly related to the issue of infinities, this revealed a number of essential duality properties of the theory and it also provides, \(via\) the duality-related form of the constraints, a link to the Ashtekar variable program for quantum gravity.

The clearest reason for ultraviolet cancellations was the requirement that the counterterms preserve local supersymmetry. This was given a clear expression in the detailed analysis of \(N = 1\) supergravity counterterms that we performed together with Stanley and John Kay in Ref.\(^8\) The result was not ultimately encouraging for the prospects of finiteness, but it was intriguing nonetheless. The first relevant \(N = 1\) supergravity counterterm occurs at the three-loop level, at which order power counting leads one to expect an expression quartic in curvatures, since at one loop the leading logarithmic divergences are of fourth order in derivatives and each loop adds two more to this count. However, at one and two loops, the possible counterterm structures happen to vanish subject to the classical
field equations, so they can be eliminated by field redefinitions renormalizations. This is in striking contrast to the situation obtaining in pure General Relativity, where there is an available \( \int d^4x R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta\mu\nu} \) counterterm which moreover has a definite nonzero infinite coefficient, as found in the heroic calculations of Ref.\(^9\).

The most intriguing aspect of the three-loop supergravity counterterm was its geometrical structure: the purely gravitational part is the contracted square of the Bel-Robinson tensor\(^{10}\) \( T_{\mu\nu\alpha\beta} = -R^\lambda_{\alpha\mu} R^\rho_{\lambda\beta\nu} + R^\lambda_{\alpha\rho} R^\mu_{\lambda\beta\nu} \). Subject to the Einstein field equations, this tensor is co-

variantly divergence-free on any index, totally symmetric and totally trace-

less. Thus, it is a higher-order analogue of the stress tensor, whose contracted square occurs in the the (nonrenormalizable) one-loop divergences of the gravity plus Yang-Mills system\(^3\). Similarly, in \( N = 1 \) supergravity plus super Yang-Mills, one encounters the stress-tensor supermultiplet \( (T_{\mu\nu}, J_{\mu\alpha}, C_{\mu}) \), where \( J_{\mu\alpha} \) is the matter supersymmetry current and \( C_{\mu} \) is the matter axial current. These come together in the counterterm \( \int d^4x (T_{\mu\nu} T_{\mu\nu} + i \bar{J}_{\mu\alpha\beta} \gamma^\rho \partial_\rho J_{\mu\alpha} - \frac{3}{2} C_{\mu} \Box C_{\mu}) \).

In extended supergravities, the gravitational and lower-spin contributions give expressions that manage to vanish subject to the classical field equations at the one- and two-loop levels, as Stanley and John Kay found already in the \( N = 2 \) case.\(^{11}\) In the early days, it was hoped that this situation might continue on to higher orders, but the added constraints of local supersymmetry (and this for all degrees of extension) prove to be exhausted at the next, three-loop, order. The corresponding \( N = 1 \) counterterm, whose structure continues to figure importantly in quantum gravity discussions in the string era, is a natural generalization of the one-loop matter divergence structure:

\[
\Delta_3 I = \int d^4x \left( (T_{\mu\nu\alpha\beta} + H_{\mu\nu\alpha\beta}) (T_{\mu\nu\alpha\beta} + H_{\mu\nu\alpha\beta}) + i J_{\mu\alpha\beta} \bar{J}_{\mu\alpha\beta} - \frac{3}{2} C_{\mu} \Box C_{\mu} \right)
\]

(1)

where \( H_{\mu\nu\alpha\beta} = -\frac{i}{2} F^\lambda_{\alpha\beta} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) f_{\lambda\beta}, J_{\mu\alpha\beta} = \frac{i}{2} R^\lambda_{\alpha\beta\rho\sigma} \sigma^{\rho\sigma} \gamma_\mu f_{\lambda\beta}, C_{\mu\alpha\beta} = -\frac{i}{2} \bar{f}^\lambda_{\alpha\beta} \gamma_5 f_{\lambda\beta} \) in which \( f_{\alpha\beta} = \partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha \) is the Rarita-Schwinger field strength for the gravitino field, \( H_{\mu\nu\alpha\beta} \) plays the role of the ‘matter’ contribution to the Bel-Robinson ‘stress’, while \( J_{\mu\alpha\beta} \) and \( C_{\mu\alpha\beta} \) are similarly higher-order analogues of the supersymmetry current \( J_{\mu} \) and axial current \( C_{\mu} \).

Although direct Feynman diagram calculations of the divergent coefficients of such higher-loop counterterms remain out of reach, other tech-
Niches for evaluating such divergences have progressed immensely since the late 1970’s. Clever use of unitarity cutting rules plus dimensional regularization have yielded the result that analogues of the \( D = 4 \) three-loop counterterm do indeed occur with an infinite coefficient for all \( N \leq 6 \) extended supergravities, but that the \( N = 8 \) theory (which is the same as \( N = 7 \)) manages to remain finite until it too succumbs at five loops (by which time the ordinary Feynman diagram approach would involve something like \( 10^{30} \) terms . . . ). Similar considerations also apply to supergravity divergences in higher dimensions, where the corresponding divergences occur at lower loop orders, e.g. in \( D = 11 \) one has divergences already at the two loop level. In the nonlocal parts of the four-graviton amplitude in such cases one again finds analogues of the \((\text{Bel} - \text{Robinson})^2\) counterterm.

The special circumstance of the \( N = 8 \) theory remains highly intriguing. It is likely to be analogous to that which obtains for maximal (i.e. 16-supercharge, corresponding to \( N = 4 \) in \( D = 4 \)) super Yang-Mills theory in 5 dimensions, which also becomes nonrenormalizable, but later than previously expected. The previous SYM expectation was based on power counting in ordinary superspace, leading to an anticipated divergence at the four-loop level in \( D = 5 \) or at three loops in \( D = 6 \). This was in agreement with explicit two-loop calculations in \( D = 6 \) showing finitess at that level. Using the unitarity cutting rule techniques, it is now known, however, that the onset of divergences is in fact delayed until 6 loops in \( D = 5 \) although it does occur at the previously expected 3 loops in \( D = 6 \). This late onset of the \( D = 5 \) SYM divergences can be understood using superspace power counting together with a more powerful 12-supercharge harmonic superspace formalism. Although the story remains incomplete, it may be anticipated that something similar is going on in maximal supergravity, perhaps with an \( N = 6 \) harmonic superspace formalism. Since these field-theory divergences, sooner or later, ineluctably arrive, the SYM situation does not, unfortunately, revive the hope for finiteness of supergravity itself, but it does have a bearing on the sorts of finite radiative corrections to be expected in superstring theories, to which we now turn.

2. String corrections

String theory may be viewed as a ‘physical’ regulator for the divergent supergravity theories. Instead of a Feynman integral cutoff, one has the string length \( \sqrt{\alpha'} \). Counterterms that would have occurred with divergent coefficients in a supergravity field theory now occur with finite \( \alpha' \)-dependent
coefficients in quantum string corrections. In particular, the \((\text{curvature})^4\) counterterms of \(D = 4\), 3-loop supergravity now are present at the \(\alpha'^3\) level in superstring theories. Generally, these string corrections have been calculated using the Neveu-Schwarz-Ramond (NSR) formalism in string light-cone gauge. As in the analogous field theory case, the first corrections not vanishing subject to the classical supergravity field equations (which are now removable by nonlinear redefinitions of the background fields that the string propagates on) occur at the \(\alpha'^3\) level. One may write the string tree-level correction in the general form

\[
\Delta I = \xi \alpha'^3 \int d^{10}x \sqrt{-g} e^{-2\phi} Y \tag{2}
\]

where the dependence on the dilaton \(\phi\) is appropriate to string tree level. A similar form is obtained at one string loop, but without any \(e^\phi\) factor, as is appropriate for one-loop order in string perturbation theory.

From the string light-cone gauge calculations, the integrand \(Y\) in (2) can be written in terms of a Berezin integral over an anticommuting spinor field \(\psi_{L,R}\). The \((\text{curvature})^4\) correction thus takes the form \(^{20,21}\)

\[
Y = \int d^{8}\psi_{L}d^{8}\psi_{R}\exp(\bar{\psi}_{L}\Gamma^{ij}\psi_{L}\bar{R}_{ijkl}\bar{\psi}_{R}\Gamma^{kl}\psi_{R}) \tag{3}
\]

where \(i,j = 1, \ldots, 8\) are light-cone transverse indices, \(\Gamma^{ij} = \frac{1}{2}(\Gamma^{i}\Gamma^{j} - \Gamma^{j}\Gamma^{i})\) are SO(8) Gamma matrices and \(\psi_{L}\) and \(\psi_{R}\) are left- and right-handed SO(8) chiral spinors. From the fact that Berezin integration gives zero except when a linear expression in each spinor field is integrated over, one sees immediately that (3) produces exclusively \((\text{curvature})^4\) corrections.

Letting \(\alpha\) and \(\dot{\alpha}\) be 8-valued \(R, L\) spinor indices, one has, up to a proportionality constant,

\[
Y = e^{\alpha_{1i}a_{12} \cdots a_{8} \dot{\alpha}_{1i} \dot{a}_{1j} \cdots \dot{a}_{8}} R_{t_{1i}t_{2j}t_{3k}t_{4l}} R_{t_{5i}t_{6j}t_{7k}t_{8l}} R_{t_{1i}t_{2j}t_{3k}t_{4l}} R_{t_{5i}t_{6j}t_{7k}t_{8l}} \tag{4}
\]

Working this out in more detail, one finds \(Y = Y_0 - Y_2\), where

\[
Y_0 = \frac{1}{512} e^{t_{1i} \cdots t_{8i} t_{1j} \cdots t_{8j}} R_{t_{1i}t_{2j}t_{3k}t_{4l}} R_{t_{5i}t_{6j}t_{7k}t_{8l}} R_{t_{1i}t_{2j}t_{3k}t_{4l}} R_{t_{5i}t_{6j}t_{7k}t_{8l}} \tag{5}
\]

in which \(t_{i1i} \cdots i_{8i}\) is defined by

\[
t_{1i} \cdots i_{8i} M_{i1i} \cdots M_{i8i} = 24 M_{1j} M_{j} M_{k} M_{l} = 6(M_{ij} M_{ij})^2. \tag{6}
\]

In making a light-cone gauge choice for the string variables in order to derive the form of these \((\text{curvature})^4\) corrections, one has in fact to restrict
the background curvature to the transverse 8 coordinates $i_1 \ldots i_8$, so in fact the term $Y_2$ contributes a total derivative here, since it becomes the Euler density in $D = 8$. Since the string tree-level correction (2) multiplies this by $e^{-2\phi}$, there still is a contribution to the Einstein equation, but this becomes proportional to $\partial \phi$, so it vanishes if one is considering corrections to background field solutions that have an initially constant dilaton $\phi$. The $D = 8$ Euler density $Y_2$ comes in very usefully in the equation for the dilaton itself, since the combination $Y_0 - Y_2$ actually vanishes for all spaces that are endowed with a Killing spinor. In consequence, at order $\alpha'^3$, the dilaton’s contribution to the correction for supersymmetric spaces can be integrated out explicitly, leaving one with an expression that is purely gravitational, and which is a direct generalization of the $D = 4$ (Bel-Robinson) supergravity counterterm (1).

For spaces with suitable initial supersymmetry (so that there is at least one holonomy singlet among the spinors coupling to ‘front’ and ‘rear’ indices of the curvature in the exponent of (3)), the variation of the quantum correction (2) simplifies further in that the ‘explicit’ metric variations also vanish. This is the case for $D \leq 8$ spaces with special holonomies such as $\text{SU}(3), \text{SU}(4), G_2$ or $\text{Spin}_7$. Consequently, the contributions to the Einstein equations arise purely from the ‘implicit’ metric variations coming from the spin connections. One obtains in each case a corrected Einstein equation that to order $\alpha'^3$ becomes

$$R_{ij} + 2\nabla_i \nabla_j \phi - \alpha'^3 X_{ij} = 0$$

in which the correction $X_{ij}$ arises from the connection variations, giving a correction of the form

$$X_{ij} = \nabla^k \nabla^\ell X_{ikj\ell}$$

where $X_{ikj\ell}$ is an expression cubic in curvatures with symmetries similar to those of the curvature tensor: $[ik], [j\ell]$ antisymmetric but $[ik] \leftrightarrow [j\ell]$ symmetric under pair interchange. Tracing the corrected gravitational equation and combining it with the dilaton equation one obtains to this order

$$2\Box \phi + \alpha'^3 X = 0$$

where the $X = g^{ij}X_{ij}$ correction arises purely from the gravitational equation trace, since the dilaton equation itself does not have order $\alpha'^3$ corrections for initially supersymmetric spaces, as we have seen above. Moreover, for the special holonomy manifolds in question, one finds

$$g^{ij}X_{ikj\ell} = g_{k\ell}Z.$$
Thus, $X = \square Z$ and consequently one can solve explicitly for the dilaton correction: for $\phi = \text{const} + \phi_1$, where $\phi_1$ is the correction to the initially constant dilaton. One finds $\phi_1 = -\frac{1}{2} \alpha' Z$ so the corrected Einstein equation becomes

$$R_{ij} = \alpha' \mathbf{3} (\nabla_i \nabla_j Z + \nabla^k \nabla^\ell X_{ikj\ell}) . \tag{11}$$

3. Special Holonomy

To see how the corrected form (11) of the Einstein equation influences the background field solutions that initially have special holonomy, consider first the case of spaces with structure $M_8 = \mathbb{R} \times K_7$ where $K_7$ is, at order $\alpha'0$, a 7-manifold with holonomy $G_2$. Similar conclusions are obtained for 8-manifolds of Spin$\,7$ holonomy.\(^{25}\) To study the $G_2$ case, pick the following basis for the SO(8) Dirac $\Gamma$ matrices:

$$\tilde{\Gamma}^i = \sigma^2 \otimes \Gamma^i \quad i = 1, \ldots, 7; \quad \tilde{\Gamma}^8 = -\sigma^1 \otimes 1_l , \tag{12}$$

where the $\Gamma^i$ are antisymmetric imaginary $8 \times 8$ SO(7) $\Gamma$-matrices; signs are chosen such that $i \Gamma^1 \cdots \Gamma^7 = 1_l$. Chiral SO(8) spinors are eigenspinors of $\tilde{\Gamma}_9 \equiv \tilde{\Gamma}^1 \cdots \tilde{\Gamma}^8 = \sigma_3 \otimes 1_8$, so $\Psi = \begin{pmatrix} \Psi_+ \\ \psi_+ \end{pmatrix}$ where $\Psi_+$ and $\psi_+$ are real 8-component SO(7) spinors. Consequently, for manifolds of $G_2 \subset \text{SO}(7)$ holonomy, the $8_\pm$ representations decompose as $8_\pm \rightarrow 7 \oplus 1$. Accordingly, the (curvature)$^4$ correction

$$Y \propto \int d^8 \psi_+ d^8 \psi_- \exp \left[ \left( \psi_+ \Gamma^i_+ \psi_+ \right) \left( \bar{\psi}_- \Gamma^j_- \bar{\psi}_- \right) R_{ijk\ell} \right] \tag{13}$$

satisfies the requirements for vanishing of ‘explicit’ metric variations in (3), and the resulting corrections to the Einstein equations arising solely from the connection variations are of the form (11).

The value of $Y$ in (13) is zero for manifolds of initial $G_2$ holonomy (i.e. before the effects of $\alpha'$ corrections are included), owing to the presence of the holonomy singlets in both the $\Psi_\pm$ decompositions, together with the rules of Berezin integration, which give a vanishing result for $\int d\theta$ integrals without a corresponding $\theta$ in the integrand. This accounts for the absence of direct $\alpha'^3$ corrections to the dilaton equation, as we have noted.

The vanishing of $Y$ for such spaces does not, however, imply the vanishing of its full variation. This has to be performed without restriction to spaces of any particular holonomy, although the initial holonomy is subsequently used in evaluating the result after variation. The only surviving
terms in the variation of (13) are those where the singlets in the $8\pm$ decompositions go onto the ‘front’ and ‘back’ of the same varied curvature, since the only way one can get a nonvanishing result is to keep the singlet products from contracting with unvaried curvatures. This observation gives a way to write the variation in a nice fashion (where now $i, j = 1, \ldots, 7$):

$$\delta Y \propto \epsilon^{m_1 \ldots i_6} \epsilon^{n_1 \ldots j_6} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} R_{i_5 i_6 j_5 j_6} c^{i_7 m} c^{j_7 n} \nabla_i \nabla_k \delta g_{j_7 k} ,$$

(14)

where $c_{ijk} = i \bar{\eta} \Gamma_{ijk} \eta$ is the covariantly constant 3-form that characterizes a $G_2$ holonomy manifold.

The variation (14) thus takes the general form (11) with

$$X_{ijk\ell} = c_{ikm} c_{j\ell n} Z_{mn} ,$$

(15)

where

$$Z_{mn} \equiv \frac{1}{32} \epsilon^{m_1 \ldots i_6} \epsilon^{n_1 \ldots j_6} R_{i_1 i_2 j_1 j_2} \cdots R_{i_5 i_6 j_5 j_6} , \quad Z = g_{mn} Z^{mn} .$$

(16)

i.e. the corrected Einstein equation is now

$$R_{ij} = c\alpha' \left[ \nabla_i \nabla_j Z + c_{ikm} c_{j\ell n} \nabla^k \nabla^\ell Z^{mn} \right] .$$

(17)

The corrected Einstein equation (17) modifies the curvature at order $\alpha' \bar{c}$ so as to give an apparently generic SO(7) holonomy, i.e. the initial $G_2$ special holonomy is lost as a result of the $\alpha'$ corrections. To see this, note that the integrability condition for the existence of an ordinary Killing spinor $\eta$ satisfying $\nabla_i \eta = 0$ is

$$R_{ijk\ell} c_{k\ell mn} = 2 R_{ij mn}$$

where $c_{ijk\ell} \equiv \frac{1}{6} \epsilon_{ijk\ell mnop} c_{mnp} = \bar{\eta} \Gamma_{ijk\ell} \eta$ is the Hodge dual of $c_{ijk}$ in $D = 7$. Taking the trace of this integrability condition, one finds that $G_2$ holonomy requires Ricci flatness, so the corrected Ricci tensor (17) definitely takes the metric out of the class of $G_2$ holonomy manifolds.

This should be contrasted with the more familiar case of Kähler manifolds, where the corrected Ricci form is required to be a cohomologically trivial (1,1) form, but is not required to vanish. One may see this explicitly is by considering an initial 7-manifold $K_7 = \mathbb{R} \times K_6$, where at order $\alpha' \bar{c}$, $K_6$ is Kähler and Ricci flat. This fits into the above $G_2$ holonomy discussion when one recognises that the only non-zero component of $Z_{mn}$ in this case is $Z_{77} = Z$, while $c_{77} = J_{ij}$ is the Kähler 2-form. In this Kähler case, the corrected Einstein equation becomes

$$R_{ij} = c\alpha' \left[ \nabla_i \nabla_j + J_{ik} J_{j\ell} \nabla^k \nabla^\ell \right] Z .$$

(18)

Going over to a Darboux complex coordinate basis $i, j = 1, \ldots, 6 \to a, \bar{a} = 1, 2, 3$, one then has the standard Calabi-Yau result

$$R_{ab} = c\alpha' \nabla_a \nabla_b Z ,$$

(19)
which is a cohomologically trivial (1,1) form, but which does not destroy the Kähler structure (which depends on the vanishing of the Nijenhuis tensor, which is not disturbed).

4. Corrected Killing equations

Despite the fact that the string $\alpha'$ corrections perturb manifolds of initially special holonomy into manifolds of generic Riemannian holonomy, another remarkable property of the Bel-Robinson-descendant string corrections is that a manifold’s initial supersymmetry can nonetheless be preserved. This can happen because the Killing spinor equation can itself be modified in such a way that its corrected integrability condition reproduces precisely the corrected Einstein equations. To see how this can happen, seek a condition $\hat{\nabla}_i \eta = 0$, where $\hat{\nabla}_i = \nabla_i + \alpha' \lambda Q_i$. We need to choose $Q_i$ such that the integrability condition $[\hat{\nabla}_i, \hat{\nabla}_j] \eta = 0$ yields the corrected Einstein equation.

One has directly the integrability condition

$$\frac{1}{4} R_{ijkl} \Gamma^{kl} \eta + c \alpha' \lambda Q_{ij} \eta = 0,$$

where

$$Q_{ij} \equiv \nabla_i Q_j - \nabla_j Q_i.$$  \hspace{1cm} (20)

In the case of a manifold of initial $\text{G}_2$ holonomy, one can use the Fierz identity $\Gamma_i \eta \bar{\eta} \Gamma_i + \eta \bar{\eta} = 1$ to find

$$R_{ijkl} c^{kl} \eta + 4c \alpha' \lambda \bar{\eta} \Gamma_m Q_{ij} \eta = 0,$$

where $\bar{\eta} Q_{ij} \eta = 0$. Multiplying by $\Gamma_i$ and using the Fierz identity, one obtains a supersymmetry integrability condition involving the corrected Ricci tensor

$$R_{ij} = 2c \alpha' \lambda \bar{\eta} \Gamma_{(jk} Q_{ij)}^k \eta.$$  \hspace{1cm} (23)

The condition (23) must then be consistent with the corrected Einstein equations for some choice of the Killing spinor correction $Q_i$.

In principal, one should be able to find the Killing spinor correction $Q_i$ by an exhaustive study of the supersymmetry properties of the (curvature)$^4$ counterterm. This would require first determining the structure of the superpartners to the pure (curvature)$^4$ part by varying it subject to the original $\alpha'^0$ supersymmetry transformations but subject to the $\alpha'^0$ field equations, then relaxing the latter and calculating the required corrections to the gravitino supersymmetry transformation. This is a long process which has not been carried out for the maximal $D = 11$ and $D = 10$
supergravities. However, the requirements for \( Q_i \) nonetheless allow one to find out its structure. The answer, \( i.e. \) the solution to (20) is

\[
Q_i = -\frac{1}{2} c_{ijk} \nabla^j Z^{k\ell} \Gamma_\ell .
\]  

(24)

The integrability condition for the modified Killing spinor condition then reproduces precisely the corrected Einstein equation (17).

The Killing spinor correction (24) seemingly depends on special properties of the order \( \alpha'^0 \) manifold, since it involves the \( G_2 \) manifold’s covariantly constant 3-form \( C_{ijk} \). However, another remarkable structural feature emerges here. The Killing spinor correction (24) can be rewritten in a form that does not make use of any special tensors on the manifold:

\[
Q_i = -\frac{3}{4} (\nabla^j R_{ikm_1 m_2}) R_{jlm_3 m_4} R^{k\ell m_5 m_6} \Gamma^{m_1 \cdots m_6} .
\]  

(25)

Moreover, this is precisely the same expression as one finds from the study of corrections to \( D = 6 \) Kähler manifolds, so there is a strong argument for the universality of the result (25).

5. Conclusion

The quantum field theoretic approach to quantum gravity, of which Stan Deser is a key pioneer continues to yield important insights into a theory of which we still have only glimpses. The main approach to quantum gravity has changed from the canonical formulation to supergravity and on to superstrings, but there is considerable continuity in certain central elements of the story. The ultraviolet problem for gravity, which gave rise to much soul-searching about the nature of the entire perturbative quantum gravity program, has now more or less been solved. Accordingly, one can now begin to actually look at the perturbatively finite theory that lies behind. Despite the evolution in dynamical details, the analytical approach of focusing on symmetries and their consequences remains an important strategy.

In the examples that we have looked at, ultraviolet counterterms that spelled the end of supergravity as a fundamental theory in its own right remain nonetheless of keen interest as finite local contributions to the supergravity effective action for superstrings or M-theory. They have a set of ‘miraculous’ properties that appear to make them precisely tailored to preserving the integrity of the underlying, still incompletely known, string or M-theory. In particular, they may lead to important insights into the structure of M-theory, for which we still have no full microscopic formulation.
An example of this is the link between the $C_3 \wedge R^4$ coupling in M-theory and the $R^4$ terms that arise in type IIA string theory at the one loop level, which are in turn related to M-theory (curvature) terms by dimensional oxidation. These two types of terms are related by on-shell supersymmetry; the relation is also crucial for the way in which the supersymmetry of an initially SU(5) holonomy Kähler manifold can be preserved despite the fact that the (curvature) terms in this case destroy the Kähler structure, yielding a general complex $D = 10$ manifold. Via a sequence of ‘miracles’ analogous to those we have sketched here for the string tree level $C_2$ case, the initial supersymmetry of such a background turns out to be preserved thanks to interrelated corrections to the Einstein and 4-form field equations. The $C_3 \wedge R^4$ terms play a key rô le in this mechanism, because they force the turning on of a necessary amount of 4-form flux. The same terms are also crucial for the elimination of the sigma-model anomalies of the M5 brane and for duality between M2 and M5 branes. The quartic curvature corrections thus are deeply related to the internal consistency of our best chance for a fundamental theory of quantum gravity.

References

1. G. ‘t Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* 20, 69 (1974).
2. S. Deser and P. van Nieuwenhuizen, *Phys. Rev.* D10, 401;411 (1974).
3. S. Deser, H.-S. Tsao and P. van Nieuwenhuizen, *Phys. Rev.* D10, 3337 (1974).
4. D.Z. Freedman, S. Ferrara and P. van Nieuwenhuizen, *Phys. Rev.* D13, 3214 (1976); S. Deser and B. Zumino, *Phys. Lett.* B62, 335 (1976).
5. D. Capper and G. Leibbrandt, *Nucl. Phys.* B85, 492 (1975); K. Fujikawa and W. Lang, *Nucl. Phys.* B88, 61 (1975); R. Delbourgo, *Nuovo Cimento* 25A, 646 (1975); S. Ferrara and O. Piguet, *Nucl. Phys.* B93, 261 (1975); P.C. West, *Nucl. Phys.* B106, 219 (1976); M.T. Grisaru, W. Siegel and M. Roček, *Nucl. Phys.* B159, 42 (1976).
6. M.T. Grisaru, P. van Nieuwenhuizen and J.A.M. Vermaseren, *Phys. Rev. Lett.* 37, 1662 (1976); M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, *Phys. Rev.* D15, 996 (1977); M.T. Grisaru and H.N. Pendleton, *Nucl. Phys.* B124, 81 (1977); S.M. Christensen, S. Deser, M.J. Duff and M.J. Grisaru, *Phys. Lett.* B84, 411 (1979).
7. S. Deser, J.H. Kay and K.S. Stelle, *Phys. Rev.* D16, 2448 (1977).
8. S. Deser, J.H. Kay and K.S. Stelle, *Phys. Rev. Lett.* 38, 527 (1977).
9. M.H. Goroff and A. Sagnotti, *Nucl. Phys.* B266, 709 (1986); A.E.M. van de Ven, *Nucl. Phys.* B378, 309 (1992).
10. I. Robinson, unpublished; L. Bel, C. R. Acad. Sci. 247, 1094 (1958).
11. S. Deser and J.H. Kay, Phys. Lett. B76, 400 (1978).
12. Z. Bern, L. Dixon, D. Dunbar, M. Perelstein and J.S. Rosowsky, Nucl. Phys. B530, 401 (1998); Class. Quantum Grav. 17, 979 (2000); Z. Bern, L. Dixon, D. Dunbar, A.K. Grant, M. Perelstein and J.S. Rosowsky, Nucl. Phys. Proc. Suppl. 88, 194 (2000).
13. S. Deser and D. Seminara, Phys. Rev. D62, 084010 (2000), hep-th/0002241.
14. M.T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. B159, 429 (1979).
15. P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B236, 125 (1984).
16. P.S. Howe and K.S. Stelle, Phys. Lett., 175 B137 (1984).
17. N. Marcus and A. Sagnotti, Phys. Lett. B135, 85 (1984); Nucl. Phys. B256, 77 (1985).
18. P.S. Howe and K.S. Stelle, Phys. Lett. B554, 190 (2003), hep-th/0211279.
19. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quantum Grav. 2,155 (1985).
20. D.J. Gross and E. Witten, Nucl. Phys. B277, 1 (1986).
21. M.D. Freeman and C.N. Pope, Phys. Lett. B174, 48 (1986).
22. P. Candelas, M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, Phys. Lett. B177, 341 (1986).
23. H. Lü, C.N. Pope and K.S. Stelle, JHEP 0407, 072 (2004), hep-th/0311018.
24. H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, JHEP 0410, 019 (2004), hep-th/0312002.
25. H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, hep-th/0410176.
26. D. Constantin, Nucl. Phys. B 706, 221 (2005), hep-th/0410157.
27. M.B. Green and M. Gutperle, Nucl. Phys. B498 195 (1997), hep-th/9701093; M.B. Green, M. Gutperle and P. Vanhove, Phys. Lett. B409 177 (1997), hep-th/9706175.
28. K. Becker and M. Becker, Nucl. Phys. B477, 155 (1996), hep-th/9605053; K. Becker and M. Becker, JHEP 0107, 038 (2001), hep-th/0107044.
29. M.J. Duff, J.T. Liu and R. Minasian, Nucl. Phys. B452, 261 (1995).