Time functions as utilities

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based on

- Time functions as utilities Archive: 0909.0890
- K-causality coincides with stable causality - *Commun. Math. Phys.* 290 (2009) 239-248
Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

**Preorder**

$R \subseteq M \times M$ is a (reflexive) *preorder* on $M$ if it is

- **reflexive**: $\Delta \subseteq R$,
- **transitive**: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, 

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A preorder which is
- total: $(x, y) \in R$ or $(y, x) \in R$

Every two elements are comparable.
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Causal Relations

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They can be regarded as relations on \( M \) i.e. as subsets of \( M \times M \)

\[
I^+ = \{ (p, q) \in M \times M : p \ll q \}, \quad \text{chronology relation}
\]

\[
J^+ = \{ (p, q) \in M \times M : p \leq q \}, \quad \text{causal relation}
\]

\[
E^+ = \{ (p, q) \in M \times M : p \rightarrow q \} = J^+ \setminus I^+, \quad \text{horismos relation}
\]

\( I^+ \) and \( J^+ \) are transitive. \( I^+ \) is open but \( J^+ \) and \( E^+ \) are not necessarily closed.
Closed and transitive relations

Stable Causality

\((M, g)\) is stably causal if there is \(g' > g\) with \((M, g')\) causal.

Here \(g' > g\) if the light cones of \(g\) are everywhere strictly larger than those of \(g\).
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**Seifert’s relation**

\[ J_S^+ = \bigcap_{g' > g} J_g^+ \quad (1971) \]

\(J_S^+\) is closed, transitive and contains \(J^+\).
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Sorkin and Woolgar’s relation \(K^+\) (1996)

The smallest closed and transitive relation which contains \(J^+\). A spacetime is \(K\)-causal if \(K^+\) is antisymmetric. It is difficult to work with \(K^+\).
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**Sorkin and Woolgar’s relation** $K^+$ (1996)

The smallest closed and transitive relation which contains $J^+$. A spacetime is $K$-causal if $K^+$ is antisymmetric. It is difficult to work with $K^+$.

By definition $K^+ \subset J^+_S$; do they coincide?

No, but

- $K$-causality is equivalent to stable causality and in this case $K^+ = J^+_S$. 
Time and stable causality

Semi-time function: a continuous real function such that $p \ll q \Rightarrow t(p) < t(q)$.

Time function: a continuous real function such that $p < q \Rightarrow t(p) < t(q)$.

Temporal function: a $C^1$ time function with timelike gradient.

Relation with stable causality

Hawking 1968: Temporal function $\Rightarrow$ stable causality

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Bernal and Sánchez 2004: Time function $\Rightarrow$ temporal function

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### Time functions and temporal functions

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Hawking’s averaging method

Geroch’s time $\mu(I^{-}(x))$ is only lower semi-continuous.

- Stable causality $\Rightarrow$ time function.

Let $g_\lambda = (1 - \frac{\lambda}{2})g + \frac{\lambda}{2}\tilde{g}$ with $\tilde{g} > g$, define

$$t(x) = \int_{0}^{1} \mu(I^{-}_{(M,g_\lambda)}(x))d\lambda$$
Other route: prove directly

(i) The existence of a time function implies $K$-causality (skip smoothability),
(ii) $K$-causality implies the existence of a time function (skip Hawking’s averaging technique).

(i): Is possible and somewhat technical.
(ii): The idea behind (ii) is that the result holds because $K^+$ is closed.
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Concerning (i): Typical strategy to work with $K^+$

Theorem: If $(x, y) \in K^+$ and $\mathcal{I}$ then $\mathcal{I}$.
Proof: Define $B^+ = \{(x, y) \in K^+ : \text{if } \mathcal{I} \text{ then } \mathcal{T}\}$ and prove

- $J^+ \subset B^+ \subset K^+$,
- $B^+$ is transitive,
- $B^+$ is closed,

then $B^+ = K^+$ and the thesis follows.
1728 Bernoulli models the preferences of an individual (an apple over an orange) as an abstract space of alternatives $A$ (prospects space) endowed with a total preorder $R$. Bernoulli introduces the concept of utility:

$$x \leq_R y \iff u(x) \leq u(y),$$

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1944 Von Neumann and Morgenstern in “Theory of games and economic behavior” had the insight of considering the preference over lotteries as more fundamental. A lottery is a probability distribution on the (pure) prospects space. The idea is that often the agent is faced with some risky alternatives, in each alternative the final outcomes in the prospect space being given only with certain probabilities. The total preorder is over the lotteries not over the prospects space. They prove that the utility function $u$ exists on the basis of consistency conditions on the preorder (no mention of continuity).

$$\mu_1 \leq \mu_2 \text{ iff } \int u \, d\mu_1 \leq \int u \, d\mu_2$$
Utility theory: two admissible points of view

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The formalisms

The mathematics in Bernoulli’s approach is that of topological ordered spaces. The mathematics in Von Neumann and Morgenstern’s approach is that of (convex subsets of) topological ordered vector spaces.
Eilenberg and Debreu prove that the continuous utility exist provided $R^\pm(x)$ are closed (recall: for total preorders).

1954 Ward notes that for total preorders the condition "$R^\pm(x)$ are closed" is equivalent to $R$ is closed.

1962 Aumann removes the totality axiom from the model. The individual cannot always choose between alternatives (indecisiveness). The utility is defined by

\[ x \sim_R y \Rightarrow u(x) = u(y) \]

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Does a continuous utility exist?

1970 Peleg proves his theorem for open strict partial orders. According to Peleg a strict partial order $S$ is separable if there is a countable subset $C$ of $X$ such that for any $(x,y) \in S$ the diamond $S^+ (x) \cap S^- (y)$ contains some element of the subset $C$, and spacious if for $(x,y) \in S$, $S^- (x) \subset S^- (y)$.

**Theorem (Peleg)**

Let $S$ be a strict partial order on a topological space $X$. Suppose that (a) $S^- (x)$ is open for every $x \in X$, (b) $S$ is separable, and (c) $S$ is spacious, then there is a function $u : X \to \mathbb{R}$ such that $(x,y) \in S \Rightarrow u(x) < u(y)$.
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Sondermann tries to construct the utility from a finite measure on the alternative space (not being aware of Geroch's idea) but obtains only lower semi-continuity (as expected).

Levin proves that the closure of $R$ suffices to prove the existence of a continuous utility, and that from the utilities $R$ can be recovered.
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Utility theory II: subsequent history of Bernoulli’s approach

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Utilities in spacetime

Utilities for $I^+$

In a chronological spacetime the utilities of the relation $I^+$ are the semi-time functions.

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Given this correspondences Levin’s and Peleg’s theorems of utility theory lead to the following results

Theorem

A spacetime is $K$-causal if and only if it admits a time function. In this case, denoting with $\mathcal{A}$ the set of time functions we have that the partial order $K^+$ can be recovered from the time functions, that is

$$(x, y) \in K^+ \iff \forall t \in \mathcal{A}, \ t(x) \leq t(y).$$

Theorem

A chronological spacetime in which $\overline{J}^+$ is transitive admits a semi-time function.
Conclusions in short

From causality to time

Stable causality (antisymmetry of $J^+S$) implies the existence of time. This is the analog of Szpilrajn order extension principle: every partial order can be extended to a total order. (But here continuity comes into play!)

From time to causality

In a stably causal spacetime the time functions on spacetime allow us to recover $J^+S$ (whose antisymmetry is equivalent to stable causality). This is the analog of the result which states that: every partial order is the intersection of the total orders which extend it.

Considerations about time suggest to regard $J^+S$ (or $K^+S$) as more fundamental than $J^+K$.
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Lemma

Let \((M, g)\) be non-total imprisoning. Let \((p, q) \in K^+\) then either \((p, q) \in J^+\) or for every relatively compact open set \(B \ni p\) there is \(r \in \hat{B}\) such that \(p < r\) and \((r, q) \in K^+\).

Define the relation

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\]

it is immediate \(J^+ \subset B^+ \subset K^+\), so we have only to prove closure and transitivity.