GENERALIZATION OF ARNOLD—VIRO INEQUALITIES
FOR REAL SINGULAR ALGEBRAIC CURVES

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§1. INTRODUCTION

1.1. Petrovskii and Arnold inequalities. The subject of the present paper is related to the 16th Hilbert problem, namely, to the question of possible distribution of ovals of plane real algebraic curves. Recall that the particular Hilbert’s question of possibility of a non-singular sextic having 11 ovals (the maximal possible number by Harnack theorem) lying separately was answered negatively. The prohibition follows from the Petrovskii inequality, which states that

\[ \chi(W) \geq -\frac{3}{2}k(k-1), \]

where \( W = \{ [x : y : z] \in \mathbb{RP}^2 \mid f(x, y, z) \geq 0 \} \) and \( f \) is a real form of degree \( 2k \) defining a non-singular curve; we denote by \( A \) the complex point set of the latter and by \( A_\mathbb{R} \) its real part, \( A \cap \mathbb{RP}^2 \). The change of the sign of \( f \) substitutes \( W \) for the complementary region; thus we get the other Petrovskii inequality,

\[ \chi(W) \leq \frac{3}{2}k(k-1) + 1. \]

The modern approach for studying the topology of \( A_\mathbb{R} \) in \( \mathbb{RP}^2 \), developed in works of V. A. Rokhlin and V. I. Arnold, is to use its close relationship with the topology of the double plane, \( \pi: X \to \mathbb{CP}^2 \), branched along \( A \). One can define \( X \) by the equation \( f(x, y, z) = t^2 \) in a quasi-homogeneous complex projective 3-space, so it is a real algebraic surface; we denote by conj: \( X \to X \) the involution of complex conjugation in \( X \) and by \( X_\mathbb{R} \) the fixed point set of conj, which is obviously projected by \( \pi \) into \( W \). Arnold [A] noticed that the Petrovskii inequality can be obtained by analyzing the negative inertia indices of the intersection form restricted to the eigenspaces of the involution conj*: \( H_2(X; \mathbb{R}) \to H_2(X; \mathbb{R}) \). This idea and its original proof [A] can be interpreted in terms of the quotient manifold \( Y = X/\text{conj} \) as follows. We subtract the Hirzebruch signature formula from the Riemann–Hurwitz formula applied to the branched coverings \( X \to Y \) and \( X \to \mathbb{CP}^2 \)

\[ 2\chi(Y) - \chi(X_\mathbb{R}) = \chi(X) = 2\chi(\mathbb{CP}^2) - \chi(A) = 4k^2 - 6k + 6 \]
\[ 2\sigma(Y) - X_\mathbb{R} \circ X_\mathbb{R} = \sigma(X) = 2\sigma(\mathbb{CP}^2) - A \circ A = 2 - 2k^2, \]

\( (X_\mathbb{R} \circ X_\mathbb{R} \text{ and } A \circ A \text{ are self-intersection numbers in } X) \) and take into account that \( b_1(X) = b_1(Y) = 0 \), and \( X_\mathbb{R} \circ X_\mathbb{R} = -\chi(X_\mathbb{R}) \) (the latter is due to the anti-isomorphism between the normal and the tangent bundle to \( X_\mathbb{R} \)). We obtain

\[ b_2^+(Y) - \frac{1}{2}\chi(X_\mathbb{R}) = \frac{1}{2}(b_2^-(X) - 1) = \frac{3}{2}k(k-1). \]
Furthermore, \( \chi(X_\mathbb{R}) = 2\chi(W) \), and \( 0 \leq b_2^+(Y) = \frac{3}{2}k(k - 1) + \chi(X_\mathbb{R}) \) can be seen as the Petrovskii inequality.

This point of view resulted about in a strengthening of Petrovskii inequalities, namely the Arnold inequalities, which can be read (after \( b_2^+(Y) \) is similarly expressed) as

\[
\sigma_+ \leq b_2^+(Y) \\
\sigma_+ + \sigma_0 \leq b_2^+(Y) + 1,
\]

where \( \sigma_+, \sigma_- \), and \( \sigma_0 \) are the numbers of the connected components of \( X_\mathbb{R} \) with negative, positive and zero Euler characteristic respectively.

1.2. **Generalizations of the Arnold inequalities for nodal curves.** V.I. Zvonilov [Z] extended the Arnold inequalities to non-singular curves \( A \) of odd degrees by applying a version of these inequalities to the curve obtained from \( A \) by adding a line. This involved consideration of real curves with nodal singularities. The extension of the Arnold inequalities to arbitrary plane real nodal curves was formulated by O.Ya.Viro [V1]. In the case of nodal curves, the double plane \( X \) is no longer a manifold, but is a rational homology manifold, so the approach of Arnold still works. However, the numbers of components, \( \sigma_+ \), \( \sigma_- \), should be replaced by the inertia indices \( \sigma_+(q), \sigma_-(q) \), of the form induced on \( H_2(X_\mathbb{R}) \) from the intersection form in \( X \). Viro described this form in combinatorial terms and determined the upper estimates in the generalized Arnold inequalities. These estimates were further improved by V. M. Kharlamov and O. Ya. Viro. A modification of the inequalities treating nodal surfaces was also formulated by Kharlamov [Kh]. The proof of Viro [V2] was not published, but a slightly simplified version of the proof, in which the original constructions are descended into the quotient \( Y \), can be found in [F1].

1.3. **The results.** In this paper, I present a generalization of the Arnold–Viro inequalities to the case of a curve with the other isolated singularities with which \( X \) is still a rational homology manifolds. The detailed proofs, some applications and further generalizations and refinements (for curves with more general singularities on non-singular surfaces and for singular surfaces, estimates of the sharpness of the inequalities) are to appear in [F2].

§2. **Generalized Arnold–Viro inequalities**

2.1. **Linking forms.** Assume that \( U \) is a sufficiently small compact regular, thus cone-like, neighborhood of an isolated surface singularity at point \( p \); we call it \( Q \)-singularity if \( M = \partial U \) is a rational homology sphere. It is well known (and trivial) that a singularity is \( Q \)-singularity if and only if its resolution graph is a tree with all vertices represented by spheres. For singularities of functions \( Q \)-property is equivalent to non-degeneracy of the Milnor form.

Assume now that the singularity at \( p \) is real and \( U \) is invariant under the complex conjugation, \( \text{conj} \). Denote by \( U_\mathbb{R} \) the fixed point set of \( \text{conj} \) in \( U \). We obtain a link \( L = \partial U_\mathbb{R} = U_\mathbb{R} \cap M \) in \( M \), which has a natural framing given by the outward normal vector field to \( L \) in \( U_\mathbb{R} \) multiplied by \( i \). Thus we obtain the linking form of a real \( Q \)-singularity, \( l_p: H_1(L) \to \mathbb{Q} \). Denote by \( l_p + 1 \) the form obtained from \( l_p \) by adding 1 to the self-linking numbers of the components of \( L \) and preserving the linking coefficients between distinct components.
2.2. The forms $q_p$ for real curve singularities. Consider now a curve $A \subset \mathbb{C}^2$ defined by a polynomial $f(x, y)$, with real coefficients. Its singularity at $p = (x_0, y_0) \in \mathbb{C}^2$ will be called $Q$-singularity if its suspension (that is the corresponding singularity at $\tilde{p} = (x_0, y_0, 0)$ on the surface $\{f(x, y) = z^2\} \subset \mathbb{C}^3$) is $Q$-singularity.

Assume now that $p \in \mathbb{R}^2$ and consider a neighborhood $U \ni \tilde{p}$, $U = \{(x, y, z) \in \mathbb{C}^2 | |x - x_0|^2 + |y - y_0|^2 \leq \varepsilon, |z|^2 \leq \varepsilon_1\}$, for $0 < \varepsilon < \varepsilon_1 < 1$. The link $L$ defined above is projected by $\pi: \mathbb{C}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$, into $S_p^+ = S_p \cap \{(x, y) \in \mathbb{R}^2 | f(x, y) \geq 0\}$, where $S_p$ is the circle of radius $\varepsilon$ around $p$. Fix an orientation of $S_p$; then split $L$ into halves, $L^+ = \{(x, y, z) \in L | \pm z \geq 0\}$, and orient $L$ so that $\pi$ preserves (reverses) the orientation of $L^+$ ($L^-$). Denote by $q_p: H_p \to \mathbb{Q}$ the quadratic form on $H_p = H^0(S_p^+)$ obtained as the pull back of $l_p + 1$ via the product of the homomorphism $\pi^*: H^0(S_p^+) \to H^0(L)$ and the Poincare duality $H^0(L) \to H_1(L)$. Note that $q_p$ is independent of the orientation of $S_p$. If $p \notin Cl(\text{Int} W)$ (i.e., $\rho = 0$ and $f$ is negative around $p$), then $S_p^+ = \emptyset$, and we put $H_p = 0$, $q_p = 0$.

Assume that a simple closed curve $\omega \subset A_{\mathbb{R}}$ contains $p$; then $S_p$ is split by $\omega \cap S_p$ in two arcs. Fix one of them and change the above orientation of $L$ on the components which are projected into this arc. The new fundamental class $[L] \in H_1(L)$ changes the duality $H^0(L) \to H_1(L)$, so the pull back of $l_p + 1$ is another quadratic form, $q_p, \omega: H_p \to \mathbb{Q}$, which is independent of the choice of the arc and the orientation of $S_p$.

2.3. Partition forms. Consider now a real curve $A \subset \mathbb{CP}^2$ of degree $2k$ defined by a homogeneous polynomial $f(x, y, z)$. For odd $k$ denote by $W_1, \ldots, W_n$ the partition components of $W = \{[x : y : z] \in \mathbb{RP}^2 | f(x, y, z) > 0\}$ (the closures of the connected components of the interior, $\text{Int}(W)$). For even $k$ denote by $W_1, \ldots, W_n$ the orientable partition components of $W$. Put $W^\circ = \bigcup_{i=1}^n \text{Int} W_i$. If $k$ is even, then choose a simple closed piecewise smooth non-contractible in $\mathbb{RP}^2$ curve $\omega \subset \mathbb{RP}^2 - W^\circ$. Such a curve can be chosen inside the non-orientable component of $W$ if such a component exists; otherwise the inclusion homomorphism $H_1(A_{\mathbb{R}}) \to H_1(\mathbb{RP}^2)$ is non-trivial and we can choose $\omega \subset A_{\mathbb{R}}$. For odd $k$ put $\omega = \emptyset$. Denote by $S$ the set of all singularities of $A$ and put $S_{\mathbb{R}} = S \cap \mathbb{RP}^2$.

Assume for the rest of this section that all the real singular points of $A$, $p \in S_{\mathbb{R}}$, are $Q$-singular. Denote by $w^i \in H^0(W^\circ)$ ($w^i_p \in H_p$), $i = 1, \ldots, n$, the classes represented by the cochains equal to 1 on $\text{Int} W_i$ ($W_i \cap S_p$) and to 0 on the rest. Note that $w^i_p = 0$ if $p \notin W_i$.

We define the partition form $q: H^0(W^\circ) \to \mathbb{Q}$ as follows

\[
q(w^i, w^j) = \sum_{p \in S_{\mathbb{R}} - \omega} q_p(w^i_p, w^j_p) + \sum_{p \in S_{\mathbb{R}} \cap \omega} q_p, \omega(w^i_p, w^j_p), \quad \text{if } 1 \leq i, j \leq n, i \neq j
\]

\[
q(w^i, w^i) = \sum_{p \in S_{\mathbb{R}} - \omega} q_p(w^i_p, w^i_p) + \sum_{p \in S_{\mathbb{R}} \cap \omega} q_p, \omega(w^i_p, w^i_p) - 2\chi(\text{Int} W_i), \quad \text{if } 1 \leq i \leq n.
\]

2.4. Generalized Arnold–Viro inequalities. Let $r$ denote the number of irreducible components of $A$, $\nu = 0$ if all irreducible components of $A$ have even degree and $\nu = 1$ otherwise; $m(p)$ the Milnor form of the singularity at $p \in S$, and $\mu_{\pm} = \sigma_{\pm}(m)$, where $m = \oplus_{p \in S} (m(p))$ is the total Milnor form of $A$, $\sigma_{\pm}$ denote the inertia indices of quadratic forms and $\sigma_0$ denotes the nullity.
Theorem A. (Generalized Arnold–Viro inequalities)

\[ \sigma_+(q) \leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu_+ \]

\[ \sigma_+(q) + \sigma_0(q) \leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu_+ + (r - \nu) \]

\[ \sigma_-(q) \leq \frac{3}{2}k(k-1) + \frac{1}{2}\chi(X_\mathbb{R}) - \frac{1}{2}\mu_- \]

\[ \sigma_-(q) + \sigma_0(q) \leq \frac{3}{2}k(k-1) + \frac{1}{2}\chi(X_\mathbb{R}) - \frac{1}{2}\mu_- + (r - \nu) \]

2.5. Scheme of the proof. By Edmonds theorem the fixed point set of an involution on a spin manifold carries a naturally defined semi-orientation (a pair of the opposite orientations), provided the involution preserves the orientation and the spin structure. For odd \( k \) the complement \( X - S_X \) of the singularity, \( S_X \subset X \), is a spin simply connected manifold, thus, \( X_\mathbb{R} - S_X \) obtains a semi-orientation. If \( k \) is even, then we can define the relative with respect to \( \omega \) spin semi-orientation of the union of the components \( \Gamma_i = \pi^{-1}(W_i) \subset X_\mathbb{R}, i = 1, \ldots, n \), (outside the singularity) cf. [F1]. Let us fix any of the two orientations provided by the spin (relative spin) semi-orientation and denote by \( [\Gamma_i] \) the fundamental classes of \( \Gamma_i \).

Note that the quotient \( Y \) is a rational homology manifold since \( X \) is.

Denote by \( \langle \cdot, \cdot \rangle_X \) and \( \langle \cdot, \cdot \rangle_Y \) the intersection forms in \( X \) and \( Y \).

The following lemmas imply Theorem A.

Lemma 1. For any \( 1 \leq i, j \leq n \)

\[ q(w^i, w^j) = \langle [\Gamma_i], [\Gamma_j] \rangle_X = \frac{1}{2} \langle [\Gamma_i], [\Gamma_j] \rangle_Y \]

Lemma 2.

\[ b_2^+(Y) = \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu_+ \]

\[ b_2^-(Y) = \frac{3}{2}k(k-1) + \frac{1}{2}\chi(X_\mathbb{R}) - \frac{1}{2}\mu_- \]

Lemma 3. \( \dim \ker(H_2(X_\mathbb{R}; \mathbb{R}) \to H_2(Y; \mathbb{R})) \leq r - \nu \)

The versions of the above lemmas for non-singular and for nodal curves \( A \) are well known. To prove Lemma 3 in our more general setting we do not need any changes in the arguments, which can be found for instance in [F1]. Proofs of Lemmas 1–2 are straightforward calculations, although rather less trivial then for nodal curves.

§3. Computation of the forms \( q_\rho \)

3.1. The local partition forms. Assume that a real polynomial \( f(x, y) \) defines an isolated singularity at \( p \in \mathbb{R}^2 \); denote by \( \mu \) its Milnor number and by \( \rho \) the number of real branches at \( p \). Consider a small real deformation (morsification) \( \tilde{f} \), of \( f \) such that the singularity at \( p \) produces the maximal possible number, \( \delta = \frac{1}{2}(\mu + \rho - 1) \), of real hyperbolic nodes on the curve \( \{ \tilde{f} = 0 \} \), cf. [M], which all lie inside a closed \( \varepsilon \)-disk \( B_p \subset \mathbb{R}^2, 0 < \varepsilon \ll 1 \), around \( p \) (recall that a hyperbolic node
is a singularity of type $A_1$, whose branches are real). The method for constructing such a deformation can be found in [AC], [GZ]. We put $\tilde{W}_p = \{ f(x, y) \geq 0 \} \cap B_p$, and denote by $\tilde{W}_1, \ldots, \tilde{W}_m$ the closures of the connected components of Int($\tilde{W}$). Let $\tilde{H}_p = H^0(\text{Int}(\tilde{W}))$. Assume that $\tilde{W}_1, \ldots, \tilde{W}_m$ lie in the interior of $B_p$ and $\tilde{W}_{n+1}, \ldots, \tilde{W}_m$ intersect $\partial B_p$. If $\rho \geq 1$, then $m = n + \rho$. If $\rho = 0$, then $m = n + 1$ if $f$ is positive around $p$ and $m = n$ if negative. To define the following quadratic form, $\tilde{q}: \tilde{H}_p \rightarrow \mathbb{Q}$, we use the local versions of the formulas in 2.3, which are apparently the local versions of Viro formulas [V1], since $\tilde{f}$ has no other singularities except hyperbolic nodes. Namely,

$$\tilde{q}(w^i, w^j) = \frac{1}{2} \#(\tilde{W}_i \cap \tilde{W}_j); \quad \text{for } 1 \leq i, j \leq m, i \neq j$$

$$\tilde{q}(w^i, w^j) = \frac{1}{2} \#(\tilde{W}_i \cap \text{Cl}(\tilde{W} - \tilde{W}_i)) - 2\chi(\tilde{W}_i) \quad \text{for } 1 \leq i \leq m$$

where $\#$ denotes the number of points and $w^i \in \tilde{H}_p$ are the generators representing Int($\tilde{W}_i$).

Denote by $E$ the subspace of $\tilde{H}_p$ generated by $w^1, \ldots, w^n$. Note that $\tilde{H}_p$ introduced in 2.2 can be naturally identified with the subspace of $\tilde{H}_p$ generated by $w^{n+1}, \ldots, w^m$.

**Theorem B.** Assume that we have a $Q$-singularity at $p$. Then

1. the restriction of $\tilde{q}$ to $E$ is non-degenerated; in particular, we have a decomposition $\tilde{H}_p = E \oplus E^\perp$, where $E^\perp$ is the orthogonal complement to $E$ with respect to $\tilde{q}$, and the orthogonal projection $\text{pr}: \tilde{H}_p \rightarrow E^\perp$ is an isomorphism;
2. if $\rho \geq 1$ or $\rho = 0$ and $f$ is positive around $p$, then the form $q_p$ introduced in 2.2 is equal to $\tilde{q} \circ \text{pr} + 2I$ in $\tilde{H}_p \subset \tilde{H}_p$, where $I$ denotes the quadratic form defined by the identity matrix in the basis $w^{n+1}, \ldots, w^m$ (if $\rho = 0$ and $f$ is negative around $p$, then obviously $\tilde{H}_p = E \perp = 0$ and $q_p = \tilde{q} \circ \text{pr} = 0$).

Note that this theorem really gives an explicit method to calculate the forms $q_p$, because $\tilde{q} \circ \text{pr}$ is obtained from $\tilde{q}$ by the usual algorithm of completing the squares of $w^1, \ldots, w^n$ and taking the rest, which is a form in variables $w^{n+1}, \ldots, w^m$.

The forms $q_{p, \omega}$ are obviously determined by $q_p$.

**3.2. The case of simple singularities.** As an illustration we present below the matrices $M = M(q_p)$ of the forms $q_p$ for the simple real surface singularities, which can be easily computed applying the above algorithm to the well known (cf. [AC], [GZ]) real morsifications of these singularities.

Note that we may not specify the correspondence between variables, $w^i$, and the entries of matrices due to the symmetry in all the cases except tree: for $D_{2n+2}$, $D_{2n+3}$ and $E_7$. The correspondence in the exceptional cases is determined by the rule: the lesser diagonal entry, $q_p(w^i, w^i)$, corresponds to the angle 0 between the real branches at $(0, 0)$ which embrace the corresponding region $W_i$. 


\[ A_{2n-1} : \quad f(x, y) = \begin{cases} 
-x^{2n} + y^2, & M = \left( \frac{n}{2}, \frac{n}{2} \right) \\
x^{2n} - y^2; & M = \left( \frac{2n-1}{2n}, \frac{1}{2n} \right) \\
x^{2n} + y^2; & M = \left( 2n \right) \\
-x^{2n} - y^2; & M = \left( 0 \right)
\]

\[ A_{2n} : \quad f(x, y) = \begin{cases} 
\pm x^{2n+1} + y^2; & M = \left( 2n \right) \\
\pm x^{2n+1} - y^2; & M = \left( \frac{2n}{2n+1} \right)
\]

\[ D_{2n+2} : \quad f(x, y) = \begin{cases} 
\pm x(x^{2n} - y^2); & M = \left( 1, \frac{n+1}{2}, \frac{n}{2} \right) \\
\pm x(x^{2n} + y^2); & M = \left( 4n-2, \frac{1}{2} \right) \\
x(x^{2n+1} \pm y^2); & M = \left( 2n+1, 1, 1 \right) \\
-x(x^{2n+1} \pm y^2); & M = \left( \frac{2n+3}{4}, \frac{2n+1}{4}, \frac{2n+3}{4} \right)
\]

\[ E_6 : \quad f(x, y) = \begin{cases} 
x^4 \pm y^3; & M = \left( 6 \right) \\
-x^4 \pm y^3; & M = \left( 2 \right)
\]

\[ E_7 : \quad f(x, y) = \pm y(x^3 \pm y^2); \quad M = \left( \frac{7}{3}, \frac{3}{2}, \frac{3}{2} \right)
\]

\[ E_8 : \quad f(x, y) = \pm x^5 \pm y^3; \quad M = \left( 8 \right)
\]

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