AN INVERSE PROBLEM FOR MOORE–GIBSON–THOMPSON EQUATION ARISING IN HIGH INTENSITY ULTRASOUND

R. ARANCIBIA, R. LECAROS, A. MERCADO, AND S. ZAMORANO

Abstract. In this article we study the inverse problem of recovering a space-dependent coefficient of the Moore–Gibson–Thompson (MGT) equation, from knowledge of the trace of the solution on some open subset of the boundary. We obtain the Lipschitz stability for this inverse problem, and we design a convergent algorithm for the reconstruction of the unknown coefficient. The techniques used are based on Carleman inequalities for wave equations and properties of the MGT equation.

1. Introduction

Let \( \Omega \subseteq \mathbb{R}^N \) be a nonempty bounded open set (for \( N = 2 \) or \( N = 3 \)), with a smooth boundary \( \Gamma \), and let \( T > 0 \). We consider the MGT equation

\[
\begin{cases}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f, & \Omega \times (0, T) \\
u = 0, & \Gamma \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{cases}
\]

where \( \alpha \in L^\infty(\Omega) \), \( c \in \mathbb{R} \) and \( \tau, b > 0 \).

This equation arises as a linearization of a model for wave propagation in viscous thermally relaxing fluids. In that cases, the space-dependent coefficient \( \alpha \) depends on a viscosity of the fluid [16]. This third order in time equation has been studied by several authors from various points of view. We can mentioned, among others, the works [15, 17, 18, 21, 23, 24, 22] for a variety of problems related to this equation.

In particular, one interesting characteristic of this equation is that the structural damping \( b \) plays a crucial role for the well-posedness, contrary of second order equations with damping (\( \tau = 0 \) and \( \alpha > 0 \) in (1.1)). For instance, in [17] it is proved that, if \( b = 0 \) and \( \alpha \) a positive constant, there does not exist an infinitesimal generator of a semigroup, in contrast with second order equations, where the structural damping does not affect the well-posedness of the equation. The parameter \( \gamma := \alpha - \frac{\tau c^2}{b} \) gives relevant information regarding the stability of the system. If \( \gamma > 0 \), the group associated to the equation is exponentially stable, and for \( \gamma = 0 \), the group is conservative, see for instance [23]. On the other hand, Conejero, Lizama and Rodenas [11] proved that the one-dimensional equation exhibits a chaotic behavior if \( \gamma < 0 \). Also, for the case in which \( \alpha \) is given by a function depending on space and time, the well posedness and the exponential decay was proved by Kaltenbacher and Lasiecka in [16].

Key words and phrases. Carleman inequalities, Bukhgeim–Klibanov method, hidden regularity, Moore–Gibson–Thompson equation.

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Concerning the well posedness of the system (1.1), it is known (see [16, Theorem 2.2]) that, given a coefficient \( \alpha \in L^\infty(\Omega) \) and data satisfying
\[
(u_0, u_1, u_2) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^0_0(\Omega) \times L^2(\Omega),
\]
the system (1.1) admits a unique weak solution \((u, u_t, u_{tt})\) satisfying
\[
(u, u_t, u_{tt}) \in C([0, T]; H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)).
\]

In this article, we study the inverse problem of recovering the unknown space-dependent coefficient \( \alpha = \alpha(x) \), the frictional damping term, from partial knowledge of some trace of the solution \( u \) at the boundary, namely,
\[
\frac{\partial u}{\partial n} \text{ on } \Gamma_0 \times (0, T),
\]
where \( \Gamma_0 \subset \Gamma \) is a relatively open subset of the boundary, called the observation region, and \( n \) is the outward unit normal vector on \( \Gamma \). We will often write \( u(\alpha) \) to denote the dependence of \( u \) on the coefficient \( \alpha \).

More precisely, in this paper we study the following properties of the stated inverse problem:

- **Uniqueness:**
  \[
  \frac{\partial u(\alpha_1)}{\partial n} = \frac{\partial u(\alpha_2)}{\partial n} \text{ on } \Gamma_0 \times (0, T) \implies \alpha_1 = \alpha_2 \text{ in } \Omega.
  \]

- **Stability:**
  \[
  \|\alpha_1 - \alpha_2\|_{X(\Omega)} \leq C \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|_{Y(\Gamma_0)},
  \]
  for some appropriate spaces \( X(\Omega) \) and \( Y(\Gamma_0) \).

- **Reconstruction:** Design an algorithm to recover the coefficient \( \alpha \) from the knowledge of \( \frac{\partial u(\alpha)}{\partial n} \) on \( \Gamma_0 \).

The first part of this work is concerned with the uniqueness and stability issues of the inverse problem. We obtain a stability result, which directly implies a uniqueness one, under certain conditions for \( \alpha \), \( \Gamma_0 \) and the time \( T \). We use the Bukheim–Klibanov method, which is based on the so-called Carleman estimates. We prove a Carleman estimate for MGT equation, which will be based on the Carleman inequality for wave operator given in [4].

The second part of this work is focused on giving a constructive and iterative algorithm which allows us to find the coefficient \( \alpha \) from the knowledge of the additional data \( \frac{\partial u}{\partial n} \) on the observation domain \( \Gamma_0 \). For that, we study an appropriate functional, and we show that this functional admits a unique minimizer on a suitable domain. Using this results, we will prove the convergence of an iterative algorithm. We refer to Section 5 for details. This algorithm is adapted from [4], where it was introduced an algorithm for recovering zero-order terms in the wave equation. We can also mention the works of Beilina and Klibanov [6, 7], where the authors studied the reconstruction of a coefficient in a hyperbolic equation using the Carleman weight.

The remaining of this paper is organized as follows. In Section 2 we present our main results: Theorem 2.1, which establishes the stabilization property of our inverse problem and a Carleman type estimate which is contained in Theorem 2.6. In section 3, we present some auxiliary results of the MGT equation which are needed for the inverse problem. Besides, we prove the hidden regularity for the MGT equation. In section 4, we prove the main results of our work, namely Theorems 2.1 and 2.6. Finally, in section 5, we focus on the algorithm for the reconstruction of coefficient \( \alpha \) and we prove the convergence of this Algorithm.
2. Statement of the main results

In this section we state our main results concerning the inverse problem proposed in the Introduction. In order to state the precise result that we obtain, we consider the following set of admissible coefficients:

\[ A_M = \left\{ \alpha \in L^\infty(\Omega), \quad \frac{c^2}{b} \leq \alpha(x) \leq M \quad \forall x \in \Omega \right\} , \]  

and the geometrical assumptions, sometimes referred to as the Gamma–condition of Lions or the multiplier condition:

\[ \exists x_0 \notin \Omega \text{ such that } \Gamma_0 \supset \{ x \in \Gamma : (x - x_0) \cdot n \geq 0 \} , \]  

and

\[ T > \sup_{x \in \Omega} |x - x_0| . \]  

Henceforth we will set \( \tau = 1 \) for simplicity. Our main result concerns the stability of the inverse problem:

**Theorem 2.1.** For \( \Gamma_0 \subset \Gamma, M > 0 \) and \( T > 0 \) satisfying (2.2) - (2.3), suppose there exists \( \eta > 0 \) such that

\[ |u_2| \geq \eta > 0 \quad \text{a.e. in } \Omega , \]  

and \( \alpha_2 \in A_M \) is such that the unique solution \( u(\alpha_2) \) of (1.1) satisfies

\[ u(\alpha_2) \in H^3(0, T; L^\infty(\Omega)) . \]  

Then there exists a constant \( C > 0 \) such that

\[ C^{-1} \| \alpha_1 - \alpha_2 \|^2_{L^2(\Omega)} \leq \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|^2_{H^2(0, T; L^2(\Gamma_0))} \leq C \| \alpha_1 - \alpha_2 \|^2_{L^2(\Omega)} \]  

for all \( \alpha_1 \in A_M \).

**Remark 2.2.** The hypothesis \( u(\alpha_2) \in H^3(0, T; L^\infty(\Omega)) \) in Theorem 2.1 is satisfied if more regularity is imposed on the data. For instance, taking \( m > \frac{N}{2} + 1 \), it is enough to take \( (u_0, u_1, u_2) \in (H^{m+2}(\Omega) \times H^{m+1}(\Omega) \times H^m(\Omega)) \), \( \alpha_2 \in H^{m-1}(\Omega) \), \( f \equiv 0 \) and appropriate boundary compatibility conditions. Indeed, by Theorem 2.2 in [10], we obtain

\[ u = u(\alpha_2) \in C([0, T]; H^{m+2}(\Omega)) \cap C^1([0, T]; H^{m+1}(\Omega)) \cap C^2([0, T]; H^m(\Omega)) . \]  

Then, from equation (1.1) and taking into account that, for \( s > \frac{N}{2} \), the Sobolev space \( H^s(\Omega) \) is an algebra, we have that \( (u_{ttt}(\cdot, 0), u_{tt}(\cdot, 0), u_{ttt}(\cdot, 0)) \in (H^m(\Omega) \times H^{m-1}(\Omega) \times H^{m-2}(\Omega)) \). Therefore, using again Theorem 2.2 in [10], we deduce that \( u_{tt} \in C([0, T]; H^m(\Omega)) \cap C^1([0, T]; H^{m-1}(\Omega)) \cap C^2([0, T]; H^{m-2}(\Omega)) \).

Hence

\[ u_{tt} \in C([0, T]; H^{m-1}(\Omega)) \cap C^1([0, T]; H^{m-2}(\Omega)) , \]  

and using Sobolev’s embedding theorem, we get that \( u_{tt} \in L^2(0, T; L^\infty(\Omega)) \).

**Remark 2.3.** The inverse problem studied in this paper was previously considered by Liu and Triggiani [21, Theorem 15.5]. They considered \( \alpha \in H^m(\Omega) \) and initial data \( (u_0, u_1, u_2) \in (H^{m+2}(\Omega) \times H^{m+1}(\Omega) \times H^m(\Omega)) \) with \( m > \frac{N}{2} + 2 \). By using Carleman estimates for a general hyperbolic equation, the authors proved global uniqueness of any damping coefficient \( \alpha \) with boundary measurement given by

\[ \frac{c^2}{b} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial n}, \quad \text{on } \Gamma_0 \times [0, T] , \]
and the initial data is supposed to satisfy \((2.4)\) and
\[
\frac{c^2}{b} u_0(x) + u_1(x) = 0, \quad x \in \Omega.
\]

In this paper, using an appropriate Carleman inequality and the method of Bukhgeim–Klibanov, we obtain stability around any regular state, under hypothesis \(m > \frac{N}{2} + 1\) and without the additional assumption \((2.4)\).

**Remark 2.4.** The hypotheses \((2.2)\) and \((2.3)\) on \(\Gamma_0\) and \(T\) typically arise in the study of stability or observability inequalities for the wave equation, see \([12, 30]\) where the multiplier method is used, or \([12, 30]\) where some observability inequalities are obtained from Carleman estimates. These hypotheses provide a particular case of the geometric control condition stated in \([2]\).

**Remark 2.5.** The assumption of the positivity for \(|u_2|\) appearing in Theorem \(2.1\) is classical when applying the Bukhgeim-Klibanov method and Carleman estimates for inverse problems with only one boundary measurement, see \([3, 20, 26]\).

As we mentioned before, in order to study the stated inverse problem, we use global Carleman estimates and the method of Bukhgeim–Klibanov, introduced in \([10]\). To state our Carleman estimates precisely, we shall need the following notations.

Assume that \(\Gamma_0\) satisfies \((2.2)\) for some \(x_0 \in \mathbb{R}^N \setminus \overline{\Omega}\). For \(\lambda > 0\), we define the weight function
\[
\varphi_\lambda(x, t) = e^{\lambda \phi(x, t)}, \quad (x, t) \in \Omega \times (-T, T),
\]
where
\[
\phi(x, t) = |x - x_0|^2 - \beta t^2 + M_0
\]
for some \(\beta \in (0, 1)\) to be chosen later, and for some \(M_0\) such that \(\phi \geq 1\), for example any constant satisfying \(M_0 \geq \beta T^2 + 1\). To prove Theorem \(2.1\) we shall use the following Carleman estimate.

**Theorem 2.6.** Suppose that \(\Gamma_0\) and \(T\) satisfy \((2.2), (2.3)\). Let \(M > 0\) and \(\alpha \in A_M\). Let \(\beta \in (0, 1)\) such that
\[
\beta T > \sup_{x \in \Omega} |x - x_0|.
\]

Then, there exists \(s_0 > 0\), \(\lambda > 0\) and a positive constant \(C\) such that
\[
\sqrt{s} \int_\Omega e^{2s \varphi_\lambda(\cdot, 0)}|y(t, \cdot)|^2 \, dx + s\lambda e^4 \int_\Omega \int_0^T e^{2s \varphi_\lambda(\cdot, t)} |y(t)|^2 + |\nabla y|^2 \, dx \, dt
\]
\[
\quad + s^3 \lambda^3 e^4 \int_\Omega \int_0^T e^{2s \varphi_\lambda(\cdot, t)} |y(t)|^2 \, dx \, dt + s\lambda \int_\Omega \int_0^T e^{2s \varphi_\lambda(\cdot, t)} |y(t)|^2 + |\nabla y|^2 \, dx \, dt
\]
\[
\quad + s^3 \lambda^3 \int_0^T \int_\Omega e^{2s \varphi_\lambda(\cdot, t)} |y(t)|^2 \, dx \, dt + C s \lambda \int_\Gamma_0 \int_0^T e^{2s \varphi_\lambda(\cdot, t)} (|\nabla y \cdot n|^2 + c^4 |\nabla y \cdot n|^2) \, d\sigma \, dt,
\]
for all \(s \geq s_0\) and for all \(y \in L^2(0, T; H^1_0(\Omega))\) satisfying \(L_y := y_{tt} + \alpha y_{tt} - c^2 \Delta y - b \Delta y_t \in L^2(\Omega \times (0, T))\), \(y(\cdot, 0) = y_t(\cdot, 0) = 0\) in \(\Omega\), and \(y_t(\cdot, 0) \in L^2(\Omega)\).

Let us mention that, in order to obtain estimate \((2.11)\), we do not follow the classical procedure of decomposing the differential operator \(L_y\) of the MGT system. Instead of that, we use in an appropriate way the well-known Carleman estimate for the wave operator, from which we are able to obtain \((2.11)\).
thanks to the fact that we are asking that the initial conditions $y(\cdot, 0)$ and $y_t(\cdot, 0)$ are null. For instance, this result is not enough to obtain controllability, but this is coherent with the fact the MGT equation has poor control properties: in [22] is proved that the interior null controllability of this system is not true, and then, the boundary null controllability is also false. A similar idea was considered in [28], where a Carleman estimate for the Laplace operator was used to prove the unique continuation property for a linearized Benjamin–Bona–Mahony equation.

The Bukhgeim–Klibanov method and Carleman estimates have been widely used for obtaining stability of coefficients with one-measurement observations. Concerning inverse problems for wave equations with boundary observations, in [25] is studied the problem of recovering a source term of the equation, [27] deals with the problem of recovering a coefficient of the zero-order term, and [8] concerns the recovering of the main coefficient. In addition, we can mention the works [14, 29] related to coefficient inverse problems for hyperbolic equations. We refer to [9] for an account of classic and recent results concerning the use of Carleman estimates on the study of inverse problems for hyperbolic equations.

3. Auxiliary results

In this section, we state and prove some auxiliary results concerning estimates for the Laplacian of a solution of (1.1) and a hidden regularity estimate for the solution of the MGT equation.

3.1. Bound of Laplacian of the solutions. From now, throughout the article, we define

$$\gamma(x) := \alpha(x) - \frac{c^2}{b}. \quad (3.1)$$

Let us note that $\alpha \in A_M$ if and only if $0 \leq \gamma \leq M$ in $\Omega$. We also define the energy

$$E_c(y) := \frac{b}{2} \|\nabla y\|^2_{L^2(\Omega)} + \frac{1}{2} \|y_t\|^2_{L^2(\Omega)}. \quad (3.2)$$

In order to prove our main results, some technical estimations are necessary. One of them is the following:

**Lemma 3.1.** Let $b > 0$ and $M > 0$ such that $\alpha \in A_M$. Then there exists $C > 0$ such that the total energy

$$E(t) := E_c(u_t(t)) + E_c(u(t)), \quad (3.3)$$

satisfies

$$E(t) \leq C \left( E(0) + \|f\|^2_{L^2(0,T;L^2(\Omega))} \right), \quad t \in [0,T],$$

for every $(u_0, u_1, u_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(0,T; L^2(\Omega))$, where $u$ be the unique solution of (1.1).

**Proof.** Without loss of generality, we assume that $b = 1$. Then, the equation

$$u_{ttt} + \alpha(x)u_{tt} - c^2 \Delta u - b \Delta u_t = f$$

can be write as follows (recall the definition of $\gamma$ in (3.1))

$$Lu := L_0u_t + c^2 L_0u + \gamma(x)u_{tt} = f, \quad (3.4)$$

where $L_0$ is the wave operator given by

$$L_0 := \partial_t^2 - \Delta. \quad (3.5)$$

Let us multiply the equation (3.4) by $u_{tt}(t) + c^2 u_t(t) \in L^2(\Omega)$ and after integrating on $\Omega$, we deduce that
\[
\frac{d}{dt} E_c(u_t + c^2 u) + \int_{\Omega} \gamma u_{tt}(u_t + c^2 u_t) = \int_{\Omega} f(u_t + c^2 u_t),
\]
thus we have
\[
\frac{d}{dt} E_c(u_t + c^2 u) + \frac{c^2}{2} \frac{d}{dt} \|\gamma^{1/2} u_t\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|f\|_{L^2(\Omega)}^2 + E_c(u_t + c^2 u).
\]
And using Gronwall’s inequality, there exists a constant \( C > 0 \), such that
\[
E_c(u_t + c^2 u) + \frac{c^2}{2} \|\gamma^{1/2} u_t\|_{L^2(\Omega)}^2 \leq C(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0)), \forall t \in [0, T].
\]
On the other side, a direct computation give us
\[
E_c(u_t + c^2 u) = E_c(u_t) + c^4 E_c(u) + c^2 \frac{d}{dt} E_c(u),
\]
and replacing (3.7) in (3.6), we have
\[
c^2 \frac{d}{dt} E_c(u) \leq C(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0)).
\]
Hence, integrating we obtain that, there exists a constant \( C > 0 \), such that
\[
E_c(u_t) \leq C(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0)), \forall t \in [0, T].
\]
Finally, if we take \( \varepsilon < 1 \), we observe that
\[
c^2 \frac{d}{dt} E_c(u) = c^2 \int_{\Omega} (u_t u_{tt} + \nabla u \cdot \nabla u_{tt}) \geq -\varepsilon^2 E_c(u_t) - \frac{c^4}{\varepsilon^2} E_c(u),
\]
replacing (3.9) in (3.6) and using (3.8), we obtain that, there exists a constant \( C > 0 \), such that,
\[
E_c(u_t) \leq C(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0)), \forall t \in [0, T],
\]
which together with (3.8), we can conclude the proof.

Lemma 3.2. Let \( b = 1 \) and \( M > 0 \) such that \( \alpha \in A_M \). Let \( (u, u_t, u_{tt}) \) be the unique solution of (1.1) with data \( (u_0, u_1, u_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \) and \( f \in L^2(0,T;L^2(\Omega)) \). Then, the term \( \Delta u(t) \) can be bounded as follows
\[
\|\Delta u(t)\|_{L^2(\Omega)}^2 \leq C \left( \|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0) + \|\Delta u_0\|_{L^2(\Omega)}^2 \right), \forall t \in [0, T].
\]
Proof. Since the term \( u_{tt}(t), \Delta u(t) \in L^2(\Omega) \), let us multiply the equation (3.4) by \( L_0 u \) and after integrating on \( \Omega \), we deduce that
\[
\frac{d}{dt} \|L_0 u(t)\|_{L^2(\Omega)}^2 + 2c^2 \|L_0 u(t)\|_{L^2(\Omega)}^2 = 2\langle f(t) - \gamma u_{tt}(t), L_0 u(t) \rangle_{L^2(\Omega)}.
\]
By standard argument, from (3.11) we immediately obtain
\[
\frac{d}{dt} \|L_0 u(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{c^2} \|f(t) - \gamma u_{tt}(t)\|_{L^2(\Omega)}^2.
\]
Integrating (3.12) from 0 to \( t > 0 \), we obtain that
\[
\|L_0 u(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{c^2} \int_0^t \|f(\tau) - \gamma u_{tt}(\tau)\|_{L^2(\Omega)}^2 d\tau.
\]
Then, we have
\[
\| L_0u(t) \|_{L^2(\Omega)}^2 - \| L_0u(t) \|_{L^2(\Omega)}^2 \bigg|_{t=0} \leq \frac{2}{c^2} \int_0^t \| f(\tau) \|_{L^2(\Omega)}^2 d\tau + \frac{2}{c^2} \int_0^t \| \gamma u_{tt}(\tau) \|_{L^2(\Omega)}^2 d\tau,
\]
and then using Theorem 3.1 we obtain the desired estimate.

3.2. Hidden regularity. We can observe that the inverse problem considered in this paper needs that the normal derivative of the solution can be defined on the boundary. It is well known that the wave equation satisfies certain extra regularity called *hidden regularity* [14], it is natural to expect an analogous result for the Moore–Gibson–Thompson equation, due its hyperbolic nature [17]. In the following result, using the multiplier method, we obtain a hidden regularity for the solutions of this equation.

**Proposition 3.3.** The unique solution \((u, u_t, u_{tt}) \in C([0, T]; (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega))\) of (1.1) satisfies
\[
\frac{\partial u}{\partial n} \in H^1(0, T; L^2(\Gamma)).
\]
Moreover, the normal derivative satisfies
\[
\left\| \frac{\partial u}{\partial n} \right\|_{H^1(0, T; L^2(\Gamma))}^2 \leq C \left( \| u_0 \|_{H^2(\Omega) \cap H^1_0(\Omega)}^2 + \| u_1 \|_{H^1_0(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^2 + \| f \|_{L^2(0, T; L^2(\Omega))}^2 \right). \tag{3.14}
\]
Consequently, the mapping
\[
(f, u_0, u_1, u_2) \mapsto \frac{\partial u}{\partial n}
\]
is linear continuous from \(L^2(0, T; L^2(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega))\) into \(H^1(0, T; L^2(\Gamma))\).

**Proof.** We use the multiplier method for the proof. Let \(m \in W^{1, \infty}(\Omega; \mathbb{R}^N)\) and let us multiply \(L_0u\) by \(m\nabla u\) and \(L_0(u_1)\) by \(m\nabla u_t\). Using the summation convention for repeated index, we obtain, respectively

\[
\int_0^T \int_\Omega L_0(u_1) m \nabla u_t dx dt = \frac{1}{2} \int_0^T \int_\Omega \nabla (m) |u_{tt}|^2 dx dt + \int_\Omega u_{tt} m \nabla u_t |_{t=0}^T dx \\
+ \int_0^T \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial m}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt - \frac{1}{2} \int_0^T \int_\Omega \nabla u_t |_{t=0}^T dx - \frac{1}{2} \int_0^T \int_{\partial \Omega} |\nabla u_t \cdot n|^2 (m \cdot n) d\sigma dt. \tag{3.15}
\]

and

\[
\int_0^T \int_\Omega L_0 u m \nabla u dx dt = \frac{1}{2} \int_0^T \int_\Omega \nabla (m) |u|^2 dx dt + \int_\Omega u_t m \nabla u |_{t=0}^T dx \\
+ \int_0^T \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial m}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt - \frac{1}{2} \int_0^T \int_\Omega \nabla u |_{t=0}^T dx - \frac{1}{2} \int_0^T \int_{\partial \Omega} |\nabla u \cdot n|^2 (m \cdot n) d\sigma dt. \tag{3.16}
\]

Now, taking the multiplier \(m\) as a lifting of the outward unit normal \(n\), so that \(m \cdot n = 1\), on \(\Gamma\) and using that \((u, u_t, u_{tt}) \in C([0, T]; (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega))\) we obtain...
\[
\frac{1}{2} \int_0^T \int_{\partial \Omega} |\nabla u \cdot n|^2 d\sigma dt + \frac{1}{2} \int_0^T \int_{\partial \Omega} |\nabla u_t \cdot n|^2 d\sigma dt \\
= - \int_0^T \int_{\Omega} (f - c^2 L_0 u - \gamma u_t) m \nabla u_t dx dt + \frac{1}{2} \int_0^T \int_{\partial \Omega} \text{div}(m)|u_t|^2 dx dt \\
+ \int_{\Omega} u_{tt} m \nabla u_t dx + \int_0^T \int_{\Omega} \frac{\partial u_t}{\partial x_i} \frac{\partial m_j}{\partial x_i} \frac{\partial u_t}{\partial x_j} dx dt - \frac{1}{2} \int_0^T \int_{\partial \Omega} \text{div}(m)|\nabla u_t|^2 d\sigma dt \\
- \int_0^T \int_{\Omega} m L_0 u \nabla u dx dt + \frac{1}{2} \int_0^T \int_{\Omega} \text{div}(m)|u_t|^2 dx dt + \int_{\Omega} u_{tt} m \nabla u_t^0 dx \\
+ \int_0^T \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial m_j}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt - \frac{1}{2} \int_0^T \int_{\partial \Omega} \text{div}(m)|u_t|^2 d\sigma dt \\
\leq C \left( \|f\|_{L^2(0,T;L^2(\Omega))}^2 + E(0)^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 \right). \tag{3.17}
\]

From (3.17), using the continuous dependence of the solution with respect to the data, we obtain the desired estimate (3.14) and the proof is finished.

\[\square\]

4. PROOF OF MAIN RESULTS

In this section we prove our main results, that is, Theorem 2.1 and Theorem 2.6. First, we obtain the Carleman estimate given in Theorem 2.6 and then we apply this inequality to solve our inverse problem.

We use the following notation for the weighted energy of the wave operator \(L_0\)

\[W(y) := s\lambda \int_0^T \int_{\Omega} e^{2s\varphi_\lambda} \varphi_\lambda |y|^2 + |\nabla y|^2 dx dt + s^3 \lambda^3 \int_0^T \int_{\Omega} e^{2s\varphi_\lambda} \varphi_\lambda^3 |y|^2 dx dt, \tag{4.1}\]

with \(\varphi_\lambda\) is given by (2.8). Also, we recall the operator \(L\) defined in Section 3

\[Ly := L_0 y_t + c^2 L_0 y + \gamma y_t.\]

**Proof of Theorem 2.6** Let \(y \in L^2(0, T; H^1_0(\Omega))\) satisfying \(Ly = f \in L^2(\Omega \times (0,T))\), \(y(\cdot, 0) = y_t(\cdot, 0) = 0\) in \(\Omega\), and \(y_{tt}(\cdot, 0) = y_{t}(\cdot, 0) \in L^2(\Omega)\). Then, by [16] Theorem 2.10 then \((y, y_t, y_{tt}) \in C([0,T];(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega))\) and satisfies the boundary value problem

\[
\begin{cases}
  L_0 y_t + c^2 L_0 y + \gamma y_t = f, & \Omega \times (0,T) \\
  y = 0, & \Gamma \times (0,T). \\
  y(\cdot, 0) = 0, y_t(\cdot, 0) = 0, y_{tt}(\cdot, 0) = y_2, & \Omega.
\end{cases} \tag{4.2}
\]

For a given function \(F\) defined in \([0,T]\), we will denote by \(\hat{F}\) its even extension, and by \(\tilde{F}\) its odd extension to \([-T, T]\).

Then \(w = \tilde{y}\) satisfies

\[
\begin{cases}
  L_0 w_t + c^2 L_0 w + \gamma w_t = \tilde{f}, & \Omega \times (-T,T) \\
  w = 0, & \Gamma \times (-T,T). \\
  w(\cdot, 0) = 0, w_t(\cdot, 0) = 0, w_{tt}(\cdot, 0) = y_2, & \Omega.
\end{cases} \tag{4.3}
\]

We denote by \(P\) the operator

\[P := \partial_t L_0 + c^2 L_0 + \gamma \partial_t^2,\]
and by \( \| \cdot \|_s \) the weighted norm
\[
\| w \|_s^2 := \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |w|^2 \, dx \, dt,
\]
where \( \varphi_\lambda \) is given by (2.8). Then,
\[
\| Pw - \hat{\gamma} w_t \|_s^2 = \| L_0 w_t \|_s^2 + c^4 \| L_0 w \|_s^2 + \int_{-T}^T \int_\Omega c^2 e^{s\varphi_\lambda} \partial_t |L_0 w|^2 \, dx \, dt,
\]
and, subsequently
\[
\int_{-T}^T \int_\Omega \hat{\gamma}^2 e^{s\varphi_\lambda} \partial_t |L_0 w|^2 \, dx \, dt \geq -2c^2 \int_{-T}^T \int_\Omega \hat{\gamma}^2 e^{s\varphi_\lambda} |\partial_t w|^2 \, dx \, dt - \int_{-T}^T \int_\Omega c^2 e^{s\varphi_\lambda} \partial_t |L_0 w|^2 \, dx \, dt
\]
\[
= \int_{-T}^T \int_\Omega \hat{\gamma}^2 e^{s\varphi_\lambda} |\partial_t w|^2 \, dx \, dt + \int_{-T}^T \int_\Omega c^2 e^{s\varphi_\lambda} \partial_t |L_0 w|^2 \, dx \, dt.
\]
Also, from the definition of the weight function, we have
\[
\left\{ \begin{array}{ll}
\partial_t \varphi_\lambda < 0, & t \in (0, T), \\
\partial_t \varphi_\lambda > 0, & t \in (-T, 0),
\end{array} \right.
\]
and then
\[
\int_{-T}^T \int_\Omega \hat{\gamma}^2 e^{s\varphi_\lambda} \partial_t |L_0 w|^2 \, dx \, dt \geq -2c^2 \int_{-T}^T \int_\Omega \hat{\gamma}^2 e^{s\varphi_\lambda} |\partial_t w|^2 \, dx \, dt.
\]
From (4.4) and (4.5), using that \( w(\cdot, 0) = 0 \), we deduce that
\[
\| L_0 w_t \|_s^2 + c^4 \| L_0 w \|_s^2 - 2c^2 \int_\Omega y_2(x)^2 e^{s\varphi_\lambda(-,0)} \, dx \leq \| Pw \|_s^2 + \| \hat{\gamma} w_t \|_s^2.
\]
Hence, taking into account that \( \phi(x,t) \leq \phi(x,0) \) for all \( x \in \Omega \), and Lemma (3.1), we get
\[
\int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |Pw|^2 \, dx \, dt \leq C \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |f|^2 \, dx \, dt + C \int_\Omega e^{2s\varphi_\lambda(-,0)} |y_2|^2 \, dx \, dt,
\]
which together with (4.6) gives
\[
\| L_0 w_t \|_s^2 + c^4 \| L_0 w \|_s^2 \leq C \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |f|^2 \, dx \, dt + C \int_\Omega e^{2s\varphi_\lambda(-,0)} |y_2|^2 \, dx + \| \hat{\gamma} w_t \|_s^2.
\]
Since \( \hat{\gamma} \in L^\infty(\Omega \times (-T,T)) \), from (4.8) we obtain that \( L_0 w \) and \( L_0 w_t \) belongs to \( L^2(\Omega \times (-T,T)) \). Therefore, using the hidden regularity for the wave equation, we have that \( \frac{\partial w}{\partial n} \in H^1(-T,T;L^2(\Gamma_0)) \). Then, we can apply the Carleman estimates given by Theorem 2.10 in [4] for the wave equation to each term \( L_0 w \) and \( L_0 w_t \). Namely, we have
\[
s\lambda \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |w_t|^2 + |\nabla w|^2 \, dx \, dt + s^3 \lambda^3 \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |\partial_t w|^2 \, dx \, dt
\]
\[
\leq C \int_{-T}^T \int_\Omega e^{2s\varphi_\lambda} |L_0 w|^2 \, dx \, dt + C s \lambda \int_{-T}^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left| \frac{\partial w}{\partial n} \right|^2 \, d\sigma \, dt,
\]
where we use the fact that \( w_t(\cdot, 0) = 0 \), and

\[
\sqrt{s} \int_{\Omega} e^{2s\varphi_\lambda(\cdot, 0)}|y_2|^2 \, dx + s\lambda \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda}(|w_t|^2 + |\nabla w_t|^2) \, dx \, dt \\
+ s^3 \lambda^3 \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda} |w_t|^2 \, dx dt \leq C \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda} |L_0 w_t|^2 \, dx dt \\
+ C s \lambda \int_{-T}^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left| \frac{\partial w_t}{\partial n} \right|^2 \, dx dt. \tag{4.10}
\]

Thus, from (4.8), (4.9) and (4.10) we obtain

\[
\sqrt{s} \int_{\Omega} e^{2s\varphi_\lambda(\cdot, 0)}|y_2|^2 \, dx + c^4 W(y) + W(y) \leq C \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda(\cdot, 0)}|f|^2 \, dx \, dt \\
+ \|\tilde{w}_t\|_2^2 + C \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda} \left( \frac{\partial y_t}{\partial n}^2 + c^4 \left| \frac{\partial y}{\partial n} \right|^2 \right) \, dx dt. \tag{4.11}
\]

Then, there exists \( s_0 > 0 \) and \( \lambda \) such that for every \( s \geq s_0 \) we absorb the second and third term in the right hand side of (4.11) which implies

\[
\sqrt{s} \int_{\Omega} e^{2s\varphi_\lambda(\cdot, 0)}|y_2|^2 \, dx + c^4 W(y) + W(y) \leq C \int_{-T}^T \int_{\Omega} e^{2s\varphi_\lambda(\cdot, 0)}|f|^2 \, dx \, dt \\
+ s\lambda C \int_{0}^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{\partial y_t}{\partial n}^2 + c^4 \left| \frac{\partial y}{\partial n} \right|^2 \right) \, dx dt.
\]

Finally, without loss of generality, we can take \( M_0 > 0 \) and \( C > 1 \) in definition (2.9) such that \( \phi(x, 0) \leq C\phi(x, t) \) for all \( x \in \Omega \) and \( t \in [0, T] \). Then we have \( \varphi_\lambda(x, 0) \leq C_1 \varphi_\lambda(x, t) \) for some \( C_1 = C_1(\lambda) \) independent of \( (x, t) \in \Omega \times [0, T] \), from where we conclude the desired estimate (2.11).

\[ \square \]

With the previous Carleman inequality, we can prove the main result of this article.

**Proof of Theorem 2.1.** Using the notation settled in the previous section (see (3.1) and (3.3)), we write the MGT equation in the following way.

\[
\begin{align*}
\begin{cases}
L_0 u_t + c^2 L_0 u + \gamma u_t = f, & \Omega \times (0, T) \\
u = h, & \Gamma \times (0, T) \\
u(\cdot, 0) = u_0, & u_t(\cdot, 0) = u_1, \quad u_{tt}(\cdot, 0) = u_2, & \Omega.
\end{cases}
\end{align*}
\tag{4.12}
\]

Hence, we will prove a stability estimate for coefficient \( \gamma = \gamma(x) \) in equation (4.12).

Let us denote by \( u^k \) the solution of equation (4.12) with coefficient \( \gamma_k \), for \( k = 1, 2 \), which existence is guaranteed by Theorem 2.10 in [16]. Hence \( z := u^1 - u^2 \) solves the following system.

\[
\begin{align*}
\begin{cases}
L_0 z_t + c^2 L_0 z + \gamma_1(x) z_{tt} = (\gamma_2 - \gamma_1) R(x, t), & \Omega \times (0, T) \\
z = 0, & \Gamma \times (0, T) \\
z(\cdot, 0) = z_t(\cdot, 0) = 0, & \Omega.
\end{cases}
\end{align*}
\tag{4.13}
\]

where \( R = \partial_x^2 u^2 \). Then \( y := \partial_t z \) satisfies

\[
\begin{align*}
\begin{cases}
L_0 y_t + c^2 L_0 y + \gamma_1(x) y_{tt} = (\gamma_2 - \gamma_1) \partial_t R(x, t), & \Omega \times (0, T) \\
y = 0, & \Gamma \times (0, T) \\
y(\cdot, 0) = y_t(\cdot, 0) = 0, & y_{tt}(\cdot, 0) = (\gamma_2 - \gamma_1) R(x, 0), & \Omega.
\end{cases}
\end{align*}
\tag{4.14}
\]
Moreover, from Theorem 3.3 the normal derivative $\frac{\partial u}{\partial n}$ belongs to $H^1(0, T; L^2(\Gamma))$ and satisfy
\[
\left\| \frac{\partial y}{\partial n} \right\|^2_{H^1(0, T; L^2(\Gamma))} \leq C\|y_2 - y_1\|^2_{L^2(\Omega)} + \|\partial_t R\|_{L^2(0, T; L^\infty(\Omega))}.
\]
This last estimate gives that $\frac{\partial n}{\partial z}$ $\in H^2(0, T; L^2(\Gamma_0))$ and proves the second inequality of (2.6).

Next, we apply Theorem 2.6 to $y$. From system (4.14) we have
\[
\int_0^T \int_{\Omega} e^{2s\varphi_\lambda} |Ly|^2 \, dx \, dt \leq C(\|\gamma_1\|_{L^\infty(\Omega)}, \|\partial_t R\|_{L^2(0, T; L^\infty(\Omega))}) \int_0^T \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 \, dx.
\]
Thus, from (2.11)
\[
\sqrt{s} \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 |R(x, 0)|^2 \, dx \leq C \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 \, dx
\]
\[
\quad + Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{|\partial y|}{\partial n} \right)^2 + c^4 \left( \frac{|\partial y|}{\partial n} \right)^2 ) \, d\sigma dt,
\]
which implies, using that $|R(x, 0)| = |u_2| \geq \eta > 0 \text{ a.e. in } \Omega$,
\[
\eta^2 \sqrt{s} \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 \, dx \leq C \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 \, dx
\]
\[
\quad + Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{|\partial y|}{\partial n} \right)^2 + c^4 \left( \frac{|\partial y|}{\partial n} \right)^2 ) \, d\sigma dt.
\]
Therefore, taking $s$ large enough we absorb the first term in the right hand side and we have
\[
\eta^2 \int_{\Omega} e^{2s\varphi_\lambda} |\gamma_2 - \gamma_1|^2 \, dx \leq C\sqrt{s}\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{|\partial y|}{\partial n} \right)^2 + c^4 \left( \frac{|\partial y|}{\partial n} \right)^2 ) \, d\sigma dt,
\]
which is the first inequality of (2.6) and the proof is finished.

5. RECONSTRUCTION OF THE COEFFICIENT

In this section we shall propose an reconstruction algorithm for the unknown parameter $\gamma$, from measurements of the normal derivative of the solution $u(\gamma)$ of the MGT equation (4.12). This algorithm is an extension of the work of Baudouin, Buhan and Ervedoza [4], in which they propose a reconstruction algorithm for the potential of the wave equation.

By Theorem 2.1 we known that the knowledge of $\frac{\partial u}{\partial n}$ on $\Gamma_0 \times (0, T)$ is enough to identify the parameter $\gamma$. Then $\alpha \in A_M$ is equivalent to ask that $\gamma$ belongs to
\[
\mathcal{B}_M := \{ \gamma \in L^\infty(\Omega), \ 0 \leq \gamma(x) \leq M, \ \forall x \in \overline{\Omega} \}.
\]
Let $\gamma \in \mathcal{B}_M$. Let $g \in L^2(\Omega \times (0, T))$ and $\mu \in H^1(0, T; L^2(\Gamma_0))$. Given $\varphi_{\lambda}$ defined in (2.8) with $\lambda > 0$ given by Theorem 2.6 we define the functional

$$J[\mu, g](y) = \frac{1}{2s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |Ly - g|^2 dx dt + \frac{1}{2s} \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} \left( \frac{\partial y}{\partial n} - \mu \right)^2 d\sigma dt, \quad (5.2)$$

defined in the space

$$\mathcal{V} = \{ y \in L^2(0, T; H^1_0(\Omega)) \text{ with } Ly \in L^2(\Omega \times (0, T)), y(\cdot, 0) = y_0(\cdot, 0) = 0, \text{ and } y_0(\cdot, 0) \in L^2(\Omega) \}, \quad (5.3)$$

with the family of semi–norms

$$\|y\|_{\mathcal{V},s}^2 := \frac{1}{s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |Ly|^2 dx dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} \left( \frac{\partial y}{\partial n} \right)^2 d\sigma dt, \quad (5.4)$$

A few remarks about this semi–norms (for more details see [4] Section 4):

**Remark 5.1.**

(a) Since the weighted functions $e^{s\varphi_{\lambda}}$ are bounded from below and from above by a positive constants depending on $s$, the semi–norms (5.4) are equivalent to

$$\|y\|_{\mathcal{V}}^2 := \int_0^T \int_{\Omega} |Ly|^2 dx dt + \int_0^T \int_{\Gamma_0} \left( \frac{\partial y}{\partial n} \right)^2 d\sigma dt,$$

in the sense that there exists a constant $C = C(s)$, such that for all $y \in \mathcal{V}$

$$\frac{1}{C} \|y\|_{\mathcal{V}}^2 \leq \|y\|_{\mathcal{V},s}^2 \leq C \|y\|_{\mathcal{V}}^2.$$  

(b) By Theorem 2.6 there exists $s_0 > 0$ such that for every $s \geq s_0$ the semi–norm (5.4) is actually a norm. Hence, from 1. we have that $\| \cdot \|_{\mathcal{V},s}$ is a norm for all $s > 0$. In the rest of the paper, we will omit the subscript $s$ in the notation.

The first result concerning the reconstruction of $\gamma$, is to guarantee that the functional $J[\mu, g]$ reaches the minimum. Moreover, we have the following uniqueness result.

**Theorem 5.2.** Assume the same hypotheses of Theorem 2.6 and assume that $g \in L^2(\Omega \times (0, T))$ and $\mu \in H^1(0, T; L^2(\Gamma_0))$. Then, for all $s > 0$ and $\gamma \in \mathcal{B}_M$, the functional $J[\mu, g]$ defined by (5.2) is continuous, strictly convex and coercive on $\mathcal{V}$. Besides, the minimizer $y^*$ satisfies

$$\|y^*\|_{\mathcal{V}}^2 \leq \frac{4}{s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |g|^2 dx dt + 4 \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} (|\mu|^2 + |\mu_1|^2) d\sigma dt.$$

**Proof.** The continuity and convexity is immediately. Let us see the coercivity.

$$J[\mu, g](y) = \frac{1}{2s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |Ly|^2 dx dt + \frac{1}{2s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |g|^2 dx dt - \frac{1}{s} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} gLy dx dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} \left( \frac{\partial y}{\partial n} \right)^2 d\sigma dt - \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} \left( \frac{\partial y}{\partial n} + \frac{\partial y_0}{\partial n} \right) \mu^2 d\sigma dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} (|\mu|^2 + |\mu_1|^2) d\sigma dt.$$
Using the fact that $2ab \leq 2a^2 + \frac{b^2}{2}$, we deduce

$$J[\mu, g](y) \geq \frac{1}{4s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |Ly|^2 \, dx \, dt - \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |g|^2 \, dx \, dt$$

$$+ \frac{1}{4} \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{\partial y}{\partial n} \right)^2 + \left( \frac{\partial y_t}{\partial n} \right)^2 \, d\sigma \, dt - \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} (|\mu|^2 + |\mu_t|^2) \, d\sigma \, dt$$

$$= \frac{1}{4} \|y^*\|^2_\mathcal{V} - \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |g|^2 \, dx \, dt - \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} (|\mu|^2 + |\mu_t|^2) \, d\sigma \, dt.$$

Therefore, the functional $J[\mu, g]$ admits a unique minimizer $y^*$ in $\mathcal{V}$.

Now, let us prove the estimates on the minimizer. First, we develop the functional $J[\mu, g](y^*)$:

$$J[\mu, g](y^*) = \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |Ly^*|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \left( \frac{\partial y^*}{\partial n} \right)^2 + \left( \frac{\partial y_t^*}{\partial n} \right)^2 \right) \, d\sigma \, dt$$

$$+ \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |g|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} (|\mu|^2 + |\mu_t|^2) \, d\sigma \, dt$$

$$- \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} g Ly^* \, dx \, dt - \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{\partial y^*}{\partial n} + \mu_t \frac{\partial y_t^*}{\partial n} \right) \, d\sigma \, dt.$$

Next, since $y^*$ is the minimizer, we have that $J[\mu, g](y^*) \leq J[\mu, g](0)$, which implies in particular

$$\frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |Ly^*|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \left( \frac{\partial y^*}{\partial n} \right)^2 + \left( \frac{\partial y_t^*}{\partial n} \right)^2 \right) \, d\sigma \, dt$$

$$\leq \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} g Ly^* \, dx \, dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( \frac{\partial y^*}{\partial n} + \mu_t \frac{\partial y_t^*}{\partial n} \right) \, d\sigma \, dt.$$

Therefore, using that $2ab \leq 2a^2 + \frac{b^2}{2}$ and the definition of the norm $\| \cdot \|_\mathcal{V}$, we deduce

$$\frac{1}{4} \|y^*\|^2_\mathcal{V} \leq \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi_\lambda} |g|^2 \, dx \, dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} (|\mu|^2 + |\mu_t|^2) \, d\sigma \, dt.$$

□

Secondly, the following Theorem gives a relationship between the unique minimizer of $J[\mu, g]$ and $g$. This is, together with the Theorem 2.6 an essential result for the proof of convergence of our algorithm of reconstruction.

**Theorem 5.3.** Assume the same hypotheses of Theorem 2.6 and assume that $\mu \in H^1(0, T; L^2(\Gamma_0))$ and $g^1, g^2 \in L^2(\Omega \times (0, T))$. Let $y^{i\star}$ be the unique minimizer of the functional $J[\mu, g^i]$, for $i = 1, 2$. Then, there exists $s_0 > 0$ and a constant $C > 0$ such that for all $s \geq s_0$

$$\sqrt{s} \int_\Omega e^{2s\varphi_\lambda} |y^{i\star}|^2 \, dx \leq C \int_0^T \int_\Omega e^{2s\varphi_\lambda} |g^1 - g^2|^2 \, dx \, dt. \quad (5.5)$$
Proof. Since $y^{*,i}$ is the unique minimizer of $J[\mu, g^i]$, for $i = 1, 2$, we have that for all $y \in V$

$$\frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi\lambda} (Ly^{*,1} - g^1) Ly\,dx\,dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi\lambda} \left[ \left( \frac{\partial y^{*,1}}{\partial n} - \mu \right) \frac{\partial y}{\partial n} + \left( \frac{\partial y^{*,1}}{\partial n} - \mu_t \right) \frac{\partial y_t}{\partial n} \right] \,d\sigma dt = 0, \quad (5.6)$$

and

$$\frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi\lambda} (Ly^{*,2} - g^2) Ly\,dx\,dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi\lambda} \left[ \left( \frac{\partial y^{*,2}}{\partial n} - \mu \right) \frac{\partial y}{\partial n} + \left( \frac{\partial y^{*,2}}{\partial n} - \mu_t \right) \frac{\partial y_t}{\partial n} \right] \,d\sigma dt = 0. \quad (5.7)$$

Subtracting (5.6) and (5.7), for $y = y^{*,1} - y^{*,2}$, we deduce that

$$\frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi\lambda} |Ly|^2 \,dx\,dt + \int_0^T \int_{\Gamma_0} e^{2s\varphi\lambda} \left( \left| \frac{\partial y}{\partial n} \right|^2 + \left| \frac{\partial y_t}{\partial n} \right|^2 \right) \,d\sigma dt = \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi\lambda} (g^1 - g^2) Ly\,dx\,dt, \quad (5.8)$$

Then, applying again $2ab \leq 2a^2 + b^2$ we obtain

$$\frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi\lambda} |Ly|^2 \,dx\,dt + \frac{s}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi\lambda} \left( \left| \frac{\partial y}{\partial n} \right|^2 + \left| \frac{\partial y_t}{\partial n} \right|^2 \right) \,d\sigma dt \leq 2 \int_0^T \int_\Omega e^{2s\varphi\lambda} |g^1 - g^2|^2 \,dx\,dt, \quad (5.8)$$

Finally, by the estimate (2.11) of Theorem 2.6 we obtain the desired result.

Finally, we present our algorithm and the convergence result of this.

Algorithm:

(a) **Initialization:** $\gamma^0 = 0$.

(b) **Iteration:** From $k$ to $k + 1$

Step 1 - Given $\gamma^k$ we consider $\mu^k = \partial_t \left( \frac{\partial u(\gamma^k)}{\partial n} - \frac{\partial u(\gamma)}{\partial n} \right)$ and

$$\mu^k_t = \partial_t \left( \frac{\partial u(\gamma^k)}{\partial n} - \frac{\partial u(\gamma)}{\partial n} \right)$$

on $\Gamma_0 \times (0, T)$, where $u(\gamma^k)$ and $u(\gamma)$ are the solution of the problems

$$\begin{cases}
L_0 u_t + c^2 L_0 u + \gamma^k(x) u_{tt} = f, & \Omega \times (0, T) \\
u = g, & \Gamma \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega
\end{cases} \quad (5.9)$$
We consider proof.

Theorem 5.4. Assume the same hypotheses of Theorem 2.6, and the following assumption of admissible trajectories \( y \in \mathcal{V} \):

\[
J[\mu^k, 0](y) = \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi \lambda} |L_0 y_t + c^2 L_0 y + \gamma(x) y_{tt}|^2 \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi \lambda} \left( \left| \frac{\partial y}{\partial n} - \mu_k \right|^2 + \left| \frac{\partial y}{\partial n} - \mu_{tt} \right|^2 \right) \, d\sigma \, dt
\]

Step 2 - Minimize the functional \( J[\mu^k, 0] \) on the admissible trajectories \( y \in \mathcal{V} \):

\[
J[\mu^k, 0](y) = \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi \lambda} |L_0 y_t + c^2 L_0 y + \gamma^k(x) y_{tt}|^2 \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi \lambda} \left( \left| \frac{\partial y}{\partial n} - \mu^k \right|^2 + \left| \frac{\partial y}{\partial n} - \mu_{tt}^k \right|^2 \right) \, d\sigma \, dt
\]

Step 3 - Let \( y^{*, k} \) the minimizer of \( J[\mu^k, 0] \) and

\[
\gamma^{k+1} = \gamma^k + \frac{y^{*, k} \mid_{t=0}}{u_2},
\]

Step 4 - Finally, consider \( \gamma^{k+1} = T(\gamma^{k+1}) \), where

\[
T(\gamma) = \begin{cases} 
M & \text{if } \gamma(\cdot) > M \\
\gamma & \text{if } 0 \leq \gamma(\cdot) \leq M \\
0 & \text{if } \gamma(\cdot) < 0.
\end{cases}
\]

Therefore, under the previous Theorem 5.3 we can prove the convergence of this algorithm:

**Theorem 5.4.** Assume the same hypotheses of Theorem 2.6 and the following assumption of \( u(\gamma) \):

\[
u(\gamma) \in H^3(0, T; \mathcal{L}^\infty(\Omega)) \text{ and } |u_2| \geq \eta > 0.
\]

Then, there exists a constant \( C > 0 \) and \( s_0 > 0 \) such that for all \( s \geq s_0 \) and \( k \in \mathbb{N} \)

\[
\int_\Omega e^{2s\varphi \lambda(\cdot, 0)} (\gamma^{k+1} - \gamma)^2 \, dx \leq \frac{C}{\sqrt{s}} \int_\Omega e^{2s\varphi \lambda(\cdot, 0)} (\gamma^k - \gamma)^2 \, dx.
\]

**Proof.** We consider \( y^k = \partial_t (u(\gamma^k) - u(\gamma)) \), which is the solution of

\[
\begin{cases} 
L_0 y_t^k + c^2 L_0 y^k + \gamma(x) y_{tt}^k = (\gamma - \gamma^k) \partial_t R(x, t), & \Omega \times (0, T) \\
y^k = 0, & \Gamma \times (0, T) \\
y^k(\cdot, 0) = 0, y_t^k(\cdot, 0) = 0, y_{tt}^k(\cdot, 0) = (\gamma - \gamma^k) R(x, 0), & \Omega
\end{cases}
\]

where \( R(x, t) = \partial_t^2 u(\gamma) \). Thus,

\[
\mu^k = \frac{\partial y^k}{\partial n}, \quad \mu_{tt}^k = \frac{\partial y_{tt}^k}{\partial n}.
\]

We observe that \( y^k \) belongs to \( \mathcal{V} \). Therefore, by (5.10), the solution \( y^k \) of (5.15) satisfy the Euler–Lagrange equations associated with the functional \( J[\mu^k, y^k] \), where \( g^k = (\gamma - \gamma^k) \partial_t R(x, t) \). Since \( J[\mu^k, g^k] \) admits a unique minimizer, \( y^k \) corresponds to minimum of \( J[\mu^k, g^k] \).

Let \( y^{*, k} \) be the minimizer of \( J[\mu^k, 0] \). From Theorem 5.3 we obtain that

\[
\sqrt{s} \int_\Omega e^{2s\varphi \lambda(\cdot, 0)} |y^{*, k} \mid_{t=0} - y_{tt}^k(\cdot, 0)|^2 \, dx \leq C \int_0^T \int_\Omega e^{2s\varphi \lambda(\cdot, 0)} (\gamma - \gamma^k) \partial_t R(x, t)^2 \, dx \, dt.
\]
This implies that, using that $\|u_2\| \geq \eta > 0$

$$
\eta^2 \sqrt{s} \int_{\Omega} e^{2s\varphi_s}(-0)(\gamma^{k+1} - \gamma^k)^2 \, dx \leq C \int_0^T \int_{\Omega} e^{2s\varphi_s} \left| (\gamma - \gamma^k) \partial_t R(x, t) \right|^2 \, dx \, dt.
$$

(5.18)

Since the function $T$ defined in (5.12) is Lipschitz continuous and satisfy $T(\gamma) = \gamma$, we obtain

$$
|\gamma^{k+1} - \gamma| \geq |T(\gamma^{k+1}) - T(\gamma)| = |\gamma^{k+1} - \gamma|.
$$

(5.19)

On the other hand, since $\phi(\cdot, t)$ is decreasing in $t \in (0, T)$ and $\partial_t R(x, t) \in L^2(0, T; L^\infty(\Omega))$, we conclude

$$
\int_{\Omega} e^{2s\varphi_s}(-0)(\gamma^{k+1} - \gamma^k)^2 \, dx \leq C \sqrt{s} \frac{\|\partial_t R(x, t)\|_{L^2(0, T; L^\infty(\Omega))}}{\eta^2} \int_{\Omega} e^{2s\varphi_s}(-0)(\gamma - \gamma^k)^2 \, dx \, dt.
$$

□

Let us finish this section with the following observation.

Remark 5.5. We can observe that this algorithm, from a theoretical point of view, is based on the minimization of a convex and coercive functional. Therefore, we can expect that numerical simulations can be done using, for instance, CasADi open-source tool for nonlinear optimization and algorithmic differentiation [1]. However, some drawbacks appears in its numerical simulations. This can be seen in

From (5.11) and (5.15)

$$
y^{k, \lambda}_t(x, 0) = (\gamma^{k+1} - \gamma^k)u_2, \quad y^{k, \lambda}_t(x, 0) = (\gamma - \gamma^k)u_2.
$$

This implies that, using that $|u_2| \geq \eta > 0$

$$
\eta^2 \sqrt{s} \int_{\Omega} e^{2s\varphi_s}(-0)(\gamma^{k+1} - \gamma^k)^2 \, dx \leq C \int_0^T \int_{\Omega} e^{2s\varphi_s} \left| (\gamma - \gamma^k) \partial_t R(x, t) \right|^2 \, dx \, dt.
$$

(5.18)

Since the function $T$ defined in (5.12) is Lipschitz continuous and satisfy $T(\gamma) = \gamma$, we obtain

$$
|\gamma^{k+1} - \gamma| \geq |T(\gamma^{k+1}) - T(\gamma)| = |\gamma^{k+1} - \gamma|.
$$

(5.19)

On the other hand, since $\phi(\cdot, t)$ is decreasing in $t \in (0, T)$ and $\partial_t R(x, t) \in L^2(0, T; L^\infty(\Omega))$, we conclude

$$
\int_{\Omega} e^{2s\varphi_s}(-0)(\gamma^{k+1} - \gamma^k)^2 \, dx \leq C \sqrt{s} \frac{\|\partial_t R(x, t)\|_{L^2(0, T; L^\infty(\Omega))}}{\eta^2} \int_{\Omega} e^{2s\varphi_s}(-0)(\gamma - \gamma^k)^2 \, dx \, dt.
$$

□

Both parameters $\lambda$ and $s$ are chosen large enough, in order to use the Carleman estimate given in Theorem

This implies a immediately problem from a numerical point of view. For example, if we consider

$s = \lambda = 3$, $\Omega = (0, 1)$, $x_0 = 0$, $T = 1$ and $\beta = 1$, the following ratio

$$
\frac{\max_{\Omega \times (0, T)} e^{2s\varphi_s}}{\min_{\Omega \times (0, T)} e^{2s\varphi_s}}
$$

is of order of $10^{340}$ (see for instance [5]).

It seems reasonable modify the algorithm presented here in order to obtain a numerical implementation, to validate at least with an example, the coefficient inverse problem studied in this article. In this direction, the modified algorithm is part of our forthcoming work.

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