Fractional operators as traces of operator-valued curves

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**Abstract.** We relate non integer powers $L^s$, $s > 0$ of a given (unbounded) positive self-adjoint operator $L$ in a real separable Hilbert space $\mathcal{H}$ with a certain differential operator of order $2\lceil s \rceil$, acting on even curves $\mathbb{R} \to \mathcal{H}$. This extends the results by Caffarelli–Silvestre and Stinga–Torrea regarding the characterization of fractional powers of differential operators via an extension problem.

**Keywords:** Higher order fractional operators; Operator-valued functions.

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1 Introduction

The study of the fractional powers of differential operators via their relations with generalized harmonic extensions and corresponding Dirichlet-to-Neumann operators began more than fifty years ago [15] and became popular thanks to the celebrated work [4] of Caffarelli and Silvestre, which stimulated a fruitful line of research. The idea of relating the operators $(-\Delta)^s$, $s \in (0, 1)$, acting on $\mathbb{R}^n$ and $-\text{div}(y^{1-2s}\nabla)$ acting on $\mathbb{R}^n \times \mathbb{R}_+$,

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has been adapted to cover much more general situations. The first contribution in this
direction is due to Stinga and Torrea [20]; important generalizations were given in [1, 11].

The case of higher order powers of \((-\Delta)^s\) has been investigated firstly in [7] via con-
formal geometry techniques. We also cite [6, 9, 12, 17, 19], the more recent papers [5, 8]
and references there-in.

Before describing our results, let us notice that any extension \(w = u(\cdot, y)\) of a given
\(u = u(\cdot)\) can be related with the curve \(y \mapsto w(\cdot, y)\) taking values in a suitable function
space. In the present paper we use this interpretation to handle any non-integer power
\(s > 0\) of a linear operator \(L\) in quite a general framework.

Let \(\mathcal{H}\) be a separable real Hilbert space with scalar product \((\cdot, \cdot)_\mathcal{H}\) and norm \(||\cdot||_\mathcal{H}\).
Let
\[
\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}, \quad \mathcal{D}(\mathcal{L}) \subseteq \mathcal{H}
\]
be a given unbounded, self-adjoint operator. In order to simplify the exposition, we
first assume that \(\mathcal{L}\) is positive definite and has discrete spectrum (some generalizations
are given in Section 5). We organize the spectrum of \(\mathcal{L}\) in a nondecreasing sequence
of eigenvalues \(\lambda_j\)\(j \geq 1\), counting with their multiplicities, and denote by \(\varphi_j \in \mathcal{D}(\mathcal{L})\) a
complete orthonormal system of corresponding eigenvectors.

Given \(s \in \mathbb{R}\), the \(s\)-th power of \(\mathcal{L}\) in the sense of spectral theory is the operator
\[
\mathcal{L}^s u = \sum_{j=1}^{\infty} \lambda_j^s u_j \varphi_j, \quad \text{where} \quad u_j = (u, \varphi_j)_\mathcal{H}, \quad (1.1)
\]
so that \(\mathcal{L}^0\) is the identity in \(\mathcal{H}\). If \(s > 0\), the natural domain of the quadratic form
\[
u \mapsto (\mathcal{L}^s u, u)_\mathcal{H} = \sum_{j=1}^{\infty} \lambda_j^s u_j^2
\]
is denoted by \(\mathcal{H}_L^s\). Clearly \(\mathcal{H}_L^s\) coincides with the domain of \(\mathcal{L}^{\frac{s}{2}}\); it is a Hilbert space with
scalar product and norm given by
\[
(u,v)_{\mathcal{H}_L^s} = (\mathcal{L}^{\frac{s}{2}} u, \mathcal{L}^{\frac{s}{2}} v)_{\mathcal{H}}, \quad ||u||_{\mathcal{H}_L^s} = ||\mathcal{L}^{\frac{s}{2}} u||_{\mathcal{H}}. \quad (1.2)
\]
We identify the dual space \((\mathcal{H}_L^s)\)' with \(\mathcal{H}_L^{-s} = \{\mathcal{L}^s u \mid u \in \mathcal{H}_L^s\}\) via the identity
\[
(L^s u, v) = (L^{\frac{s}{2}} u, L^{\frac{s}{2}} v)_{\mathcal{H}} \quad \text{for any} \ u, v \in \mathcal{H}_L^s.
\]
Notice that $L^s$ is an isometry $H^s_L \to H^{-s}_L$ with inverse $L^{-s}$.

In this paper we relate the operator $L^s : H^s_L \to H^{-s}_L$ for $s > 0$ non-integer to certain linear operator acting on even curves $\mathbb{R} \to H^s_L$ (this simplifies the treatment in case of higher powers $s > 1$, compare with [8]).

Let $b \in (-1, 1)$. Denote by $L^{2,b} (\mathbb{R} \to \mathcal{H})$ the Hilbert space of curves $U : \mathbb{R} \to \mathcal{H}$ such that $\|U(y)\|^2_{\mathcal{H}}$ is integrable on $\mathbb{R}$ with respect to the measure $|y|^b dy$. Further, $L^{2,b}_{\text{e}} (\mathbb{R} \to \mathcal{H})$ stands for the subspace of even curves.

For $U \in L^{2,b}_{\text{e}} (\mathbb{R} \to \mathcal{H})$ we consider the (unbounded) operators

$$\mathbb{D}_b U = -\partial^2_{yy} U - b y^{-1} \partial y U = -|y|^{-b} \partial_y (|y|^b \partial_y U), \quad \mathbb{L}_b U = \mathbb{D}_b U + \mathcal{L}_b U.$$  \hspace{1cm} (1.3)

Denoting by $U_j(y) = (U_j(y), \varphi)_\mathcal{H}$ the coordinates of $U(y)$, we have

$$\mathbb{L}_b U = \sum_{j=1}^{\infty} ((\mathbb{D}_b + \lambda_j) U_j) \varphi_j,$$

and the corresponding quadratic form reads

$$(\mathbb{L}_b U, U)_{L^{2,b}} = \int_{-\infty}^{+\infty} |y|^b (\| \partial_y U(y) \|^2_{\mathcal{H}} + \| \mathcal{L}_b^2 (U(y)) \|^2_{\mathcal{H}}) dy = \sum_{j=1}^{\infty} \int_{-\infty}^{+\infty} |y|^b (|\partial_y U_j|^2 + \lambda_j |U_j|^2) dy.$$

In Section 4 we study in detail the natural domain $H^{k,b}_{L,e} (\mathbb{R} \to \mathcal{H}) \subset L^{2,b}_{\text{e}} (\mathbb{R} \to \mathcal{H})$, $k \in \mathbb{N}$, of the quadratic form $U \mapsto (\mathbb{L}_b U, U)_{L^{2,b}}$. Lemma 4.2 provides explicit expressions for its Hilbertian scalar product and related norm, which are denoted by $(\cdot, \cdot)_{H^{k,b}_{L,e}}, \| \cdot \|^2_{H^{k,b}_{L,e}}$, respectively, and shows that the Dirac-type trace function $\delta_0(V) = V(0)$ is continuous from $H^{k,b}_{L,e} (\mathbb{R} \to \mathcal{H})$ into $\mathcal{H}^{k-\frac{1}{2}}_L$.

Our main results involve the linear transform

$$\mathcal{P}_s [u] (y) = \frac{2^{1-s}}{\Gamma(s)} \sum_{j=1}^{\infty} (\sqrt{\lambda_j} |y|)^s K_s (\sqrt{\lambda_j} |y|) u_j \varphi_j$$ \hspace{1cm} (1.4)
for \( u = \sum u_j \varphi_j \in \mathcal{H} \) and \( y \in \mathbb{R} \), where \( K_s \) is the modified Bessel function of the second kind (the Macdonald function; compare with [20], where \( s \in (0, 1) \) is assumed).

Due to the regularity and decaying properties of the Bessel functions, in Lemma A.1 of Appendix A, we prove that for any \( u \in \mathcal{H} \), \( \mathcal{P}_s[u] \) is an even curve in \( \mathcal{H} \); in addition \( \mathcal{P}_s[u] \in C^\infty(\mathbb{R}_+ \to \mathcal{H}_s^\sigma) \) for every \( \sigma > 0 \).

To state our main result we introduce the floor and ceiling notation

\[
[s] := \text{integer part of } s; \quad \lceil s \rceil := [s] + 1.
\]

**Theorem 1.1** Let \( s > 0 \) be non-integer. We put

\[
b := 1 - 2(s - [s]) \in (-1, 1).
\]

For any \( u \in \mathcal{H}_s^\sigma \) the following facts hold.

i) \[
\| \mathcal{P}_s[u] \|_{\mathcal{H}_s^\sigma}^2 = 2d_s \| u \|_{\mathcal{H}_s}^2 \quad \text{where} \quad d_s = 2^b \Gamma \left( \frac{1 + b}{2} \right) \frac{[s]!}{\Gamma(s)}.
\]

That is, up to a constant, the transform \( \mathcal{P}_s \) is an isometry \( \mathcal{H}_s^\sigma \to \mathcal{H}_s^{\lceil s \rceil} \).

ii) \( \mathcal{P}_s[u] \) achieves

\[
\min_{u \in \mathcal{H}_s^{\lceil s \rceil} \text{(R \to H)}} \| U \|_{\mathcal{H}_s^{\lceil s \rceil}}^2 = 2d_s \| u \|_{\mathcal{H}_s^\sigma}^2; \quad (1.6)
\]

iii) \( (\mathcal{P}_s[u], V)_{\mathcal{H}_s^{\lceil s \rceil}} = 2d_s \langle \mathcal{L}_s u, V(0) \rangle \) for any \( V \in \mathcal{H}_s^{\lceil s \rceil} \text{(R \to H)} \);

iv) \( \mathcal{P}_s[u] \) solves the differential equation

\[
\mathcal{L}_b^{\lceil s \rceil} \mathcal{P}_s[u] = 0 \quad \text{in} \quad \mathbb{R}_+
\]

and satisfies

\[
\lim_{y \to 0^+} \mathcal{P}_s[u](0) = u \quad \text{in} \quad \mathcal{H}_s^\sigma, \quad \lim_{y \to 0^+} y^b \partial_y \left( \mathcal{L}_b^{\lceil s \rceil} \mathcal{P}_s[u] \right)(y) = -d_s \mathcal{L}_s u \quad \text{in} \quad \mathcal{H}_s^{-s}. \quad (1.8)
\]
Additional information on the regularity of $P_s[u]$ and on its behavior at \( \{y = 0\} \) is given in Appendix A, see in particular Theorems A.6 and A.9. Corollary A.2 improves the convergence in [20, Theorem 1.1], where \( s \in (0, 1) \) is assumed; in Subsection A.2 we point out some isometric properties of the operator $P_s$ in the spirit of [18].

We can also consider negative, non-integer orders.

Let \( s > 0 \). If \( \zeta \in \mathcal{H}_L^{-s} \) then \( L^{-s} \zeta \in \mathcal{H}_L^s \), so that for any \( y \in \mathbb{R} \) we can compute

\[
P_s[L^{-s}\zeta](y) = \frac{2^{1-s}}{\Gamma(s)} \sum_{j=1}^{\infty} \lambda_j^{-s} \zeta_j (\sqrt{\lambda_j}|y|)^s K_s(\sqrt{\lambda_j}|y|) \varphi_j.
\]

The next result is in fact a corollary of Theorem 1.1.

**Theorem 1.2** Let \( s > 0, b \in (-1, 1) \) be as in Theorem 1.1. For any \( \zeta \in \mathcal{H}_L^{-s} \) the following facts hold.

i) \[
\|P_s[\zeta]\|^2_{\mathcal{H}_L^{[s],b}} = 2 d_s \|\zeta\|^2_{\mathcal{H}_L^{-s}} \quad \text{where} \quad P_s := (P_s \circ L^{-s}).
\] That is, up to a constant, the transform $P_s$ is an isometry \( \mathcal{H}_L^{-s} \to \mathcal{H}_L^{[s],b} (\mathbb{R} \to \mathcal{H}) \);

ii) \( P_s[\zeta] \) achieves

\[
\min_{U \in \mathcal{H}_L^{[s],b} (\mathbb{R} \to \mathcal{H})} \left( \|U\|^2_{\mathcal{H}_L^{[s],b}} - 4 d_s \langle \zeta, U(0) \rangle \right) = -2 d_s \|\zeta\|^2_{\mathcal{H}_L^{-s}}.
\]

iii) \( (P_s[\zeta], V)_{\mathcal{H}_L^{[s],b}} = 2 d_s \langle \zeta, V(0) \rangle \) for any \( V \in \mathcal{H}_L^{[s],b} (\mathbb{R} \to \mathcal{H}) \);

iv) \( P_s[\zeta] \) solves the differential equation

\[
L_b^{[s]} P_s[\zeta] = 0 \quad \text{in} \quad \mathbb{R}_+
\]

and satisfies

\[
\lim_{y \to 0^+} y^b \partial_y (L_b^{[s]} P_s[\zeta])(y) = -d_s \zeta \quad \text{in} \quad \mathcal{H}_L^{-s}, \quad \lim_{y \to 0^+} P_s[\zeta](y) = L^{-s} \zeta \quad \text{in} \quad \mathcal{H}_L^s.
\]
The paper is organized as follows. We start by introducing and studying, in Section 2, some Sobolev-type spaces $H_{e}^{k,b}(\mathbb{R})$ depending on the integer $k \geq 1$ and on the parameter $b \in (-1,1)$. In Section 3 we investigate the properties of the functions
\[ \psi_s(y) = c_s |y|^b K_s(|y|), \quad c_s = \frac{2^{1-s}}{\Gamma(s)}, \] (1.11)
which are involved in the definition of the operator $u \mapsto \mathcal{P}_s[u]$. The main result here is Theorem 3.3, which constitutes the basic tool in the proof of Theorem 1.1.

Section 4 contains the description of the Hilbert space $H^{k,b}_{e,L}(\mathbb{R} \to \mathcal{H})$ of even curves in $\mathcal{H}$ mentioned above, and the proofs of Theorems 1.1 and 1.2.

Generalizations and examples are given in Section 5.

As already mentioned, the Appendix contains several results about the operator $\mathcal{P}_s$.

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**Notation.** Let $X$ be a Hilbert space with scalar product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$. For any $b \in (-1,1)$ and any open interval $I \subseteq \mathbb{R}$, the space

\[ L^{2,b}(I \to X) := L^2(I \to X; |y|^b dy) \]
is endowed with the Hilbertian scalar product

\[ (U, V)_{L^{2,b}} = \int_{-\infty}^{+\infty} |y|^b(U(y), V(y))_X dy \quad U, V \in L^{2,b}(I \to X) \]
and corresponding norm $\|\cdot\|_{L^{2,b}}$ (notice that we avoid the longer notation $\|\cdot\|_{L^{2,b}(I \to X)}$).

Let $k \geq 0$ be an integer. We denote by $C^k(I \to X)$ the space of curves $I \to X$ which are continuously differentiable up to the order $k$. If $U \in C^k(I \to X)$, then $\partial^\ell_y U$ is the derivative of order $\ell = 0, \ldots, k$ (however, we will often write $\partial^2_{yy}$ instead of $\partial^2_y$). Further, $C^\infty(I \to X) = \bigcap_{k \geq 0} C^k(I \to X)$.

Accordingly with a commonly used notation, curves in $C^{k,\sigma}(I \to X) \subset C^k(I \to X)$ have H"older continuous derivatives of order $k$. For our purposes, it is convenient to put

\[ \widetilde{C}^\alpha(I \to X) = \begin{cases} \mathcal{C}^{[\alpha], \alpha-[\alpha]}(I \to X) & \text{if } \alpha > 0 \text{ is not an integer} \\ \mathcal{C}^{[\alpha]-1,1}(I \to X) & \text{if } \alpha \geq 1 \text{ is an integer}. \end{cases} \] (1.12)
Also, for $U \in \tilde{C}^\alpha (I \to X)$ we put
\[
\|U\|_{\tilde{C}^\alpha} = \left\{ \begin{array}{ll}
\sup_{y_1, y_2 \in \mathbb{R}} \frac{\|\partial_y^\alpha U(y_1) - \partial_y^\alpha U(y_2)\|}{|y_1 - y_2|^{\alpha - \lfloor \alpha \rfloor}} & \text{if } \alpha \notin \mathbb{N}, \\
\sup_{y_1, y_2 \in \mathbb{R}} \frac{\|\partial_y^{\alpha-1} U(y_1) - \partial_y^{\alpha-1} U(y_2)\|}{|y_1 - y_2|} & \text{if } \alpha \in \mathbb{N}.
\end{array} \right.
\]

Notice that $\tilde{C}^\alpha (I \to X) \subset C^{\lfloor \alpha \rfloor} (I \to X)$ if and only if $\alpha$ is not an integer.

Let $k \in \mathbb{N} \cup \{\infty\}$. The spaces of even curves in $L^{2b}(\mathbb{R} \to X), C^k(\mathbb{R} \to X)$ are denoted by $L^{2b}_e(\mathbb{R} \to X), C^k_e(\mathbb{R} \to X)$, respectively, and $C^k_{c,e}(\mathbb{R} \to X)$ is the space of compactly supported functions in $C^k_e(\mathbb{R} \to X)$.

We simply write $L^{2b}_e(\mathbb{R}), C^k_e(\mathbb{R}), C^\infty_{c,e}(\mathbb{R})$ instead of $L^{2b}(\mathbb{R} \to \mathbb{R}), C^k(\mathbb{R} \to \mathbb{R}), C^\infty_{c,e}(\mathbb{R} \to \mathbb{R})$.

## 2 Spaces of real valued functions

In this section, for any parameter $b \in (-1, 1)$ and any integer $k \geq 0$ we introduce the Sobolev-type space $H^{k;b}_e(\mathbb{R})$, which is related to the differential operators $(\mathbb{D}_b + \lambda)^k$, $\lambda > 0$.

The choice of working with even functions has been inspired by [8]. This strategy is needed in case $b \neq 0$ to overcome some technical difficulties produced by the singularity of the operator $\mathbb{D}_b$ in (1.3) at $y = 0$.

In fact, as noticed in [8], if $\psi \in C^2_e(\mathbb{R})$, then $y^{-1} \partial_y^3 \psi(y) = \partial_y^2 \psi(0) + o(1)$ as $y \to 0$, which implies $\mathbb{D}_b \psi \in C^0_e(\mathbb{R})$ More generally,

\[(\mathbb{D}_b + \lambda)^m \psi \in C^{-2m}_e(\mathbb{R}) \quad \text{for any integer } m \leq k/2 \text{ and any } \psi \in C^k_e(\mathbb{R}). \quad (2.1)\]

Our definition of $H^{k;b}_e(\mathbb{R})$ is based on induction procedure, starting from the lower order cases $k = 1, 2$.

**First order** For $\lambda > 0$, we endow the weighted Hilbert space
\[
H^{1,b}_e(\mathbb{R}) := H^1(\mathbb{R}; |y|^b dy) = \{ \psi \in L^{2b}(\mathbb{R}) \mid \partial_y \psi \in L^{2b}(\mathbb{R}) \}.
\]
with the scalar product
\[(\psi, \eta)_{\lambda,H^{1,b}} = \int_{-\infty}^{+\infty} |y|^b (\partial_y \psi \partial_y \eta + \lambda \psi \eta) \, dy\]
and the corresponding norm \(\|\psi\|_{\lambda,H^{1,b}}\). If \(\lambda = 1\) we drop it and simply write \((\psi, \eta)_{H^{1,b}}\) and \(\|\psi\|_{H^{1,b}}\). Clearly, the norms \(\|\cdot\|_{\lambda,H^{1,b}}\) are equivalent for all \(\lambda > 0\) and moreover
\[
\|\psi(\cdot)\sqrt{\lambda}\|_{\lambda,H^{1,b}}^2 = \lambda^{1 - \frac{2b}{1+b}} \|\psi(\cdot)\|_{H_{b}^{1,b}}^2. \tag{2.2}
\]

Lemma 2.1

i) \(C_c^\infty(\mathbb{R})\) is dense in \(H^{1,b}(\mathbb{R})\);

ii) \(H^{1,b}(\mathbb{R}) \subset H_{\text{loc}}^1(\mathbb{R})\) if \(b \in (-1,0]\) and \(H^{1,b}(\mathbb{R}) \subset W_{\text{loc}}^{1,p}(\mathbb{R})\) for any \(p \in [1, \frac{2}{1+b})\) if \(b \in (0,1)\);

iii) \(H^{1,b}(\mathbb{R}) \subset C_{\text{loc}}^{0,\frac{1-b}{2}}(\mathbb{R})\);

iv) There exists \(m_b > 0\) such that \(\|\psi\|_{H^{1,b}}^2 \geq m_b |\psi(0)|^2\) for any \(\psi \in H^{1,b}(\mathbb{R})\).

Proof. For i) see [14]. The first part of ii) is trivial; to prove the second one use Hölder’s inequality.

If \(b \leq 0\) then ii) implies iii) immediately. Assume \(b \in (0,1)\) and take \(\psi \in C_c^\infty(\mathbb{R})\). Since
\[
\psi(y_2) - \psi(y_1) = \int_{y_1}^{y_2} |t|^\frac{b}{2} (|t|^{\frac{b}{2}} \partial_t \psi(t)) \, dt
\]
for any \(y_1, y_2 \in \mathbb{R}\), then Hölder’s inequality and the density result in i) imply that
\[
|\psi(y_2) - \psi(y_1)|^2 \leq \frac{1}{1-b} \|\partial_y \psi\|_{L^{2,b}}^2 |y_2 - y_1|^{1-b} \leq \frac{2^{1-b}}{1-b} \|\partial_y \psi\|_{L^{2,b}}^2 |y_2 - y_1|^{1-b} \tag{2.3}
\]
for any \(\psi \in H^{1,b}(\mathbb{R})\), \(y_1, y_2 \in \mathbb{R}\). Since \(C_c^\infty(\mathbb{R})\) is dense in \(H^{1,b}(\mathbb{R})\) and since \(\psi\) was arbitrarily chosen in \(H^{1,b}(\mathbb{R})\), the inclusion in iii) easily follows.
Lastly, given \( \psi \in H^{1;b}(\mathbb{R}) \) we use (2.3) to get the existence of a constant \( c > 0 \) depending only on \( b \) such that
\[
c|y|^b|\psi(0)|^2 \leq |y|\|\partial_y \psi\|_{L^2}^2 + |y|^b|\psi(y)|^2
\]
for any \( y \in \mathbb{R} \). Then \( iii \) follows via integration over \((0,1)\). \( \square \)

**Remark 2.2** It follows from Theorem 3.3 in Section 3 that the best constant in \( iv \) is
\[
m_b = 2^{1+b}\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1-b}{2}\right)^{-1},
\]
and it is achieved by the function \( \psi_s \), see (1.11), for \( s = \frac{1-b}{2} \).

We will be mainly concerned with \( H^{1;b}_e(\mathbb{R}) \), the subspace of even functions in \( H^{1;b}(\mathbb{R}) \). For future convenience, we notice that the proof of Lemma 2.1 gives
\[
|\psi(y_2) - \psi(y_1)|^2 \leq \frac{1}{1-b}||\partial_y \psi||_{L^2}^2 |y_2|^{1-b} - |y_1|^{1-b}
\]
for any \( \psi \in H^{1;b}_e(\mathbb{R}) \), \( y_1, y_2 \in \mathbb{R} \).

**Second order** If \( \psi \in H^{1;b}_e(\mathbb{R}) \) then \( |y|^b\partial_y \psi \in L^{2;-b}(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R}) \). We put
\[
H^{2;b}_e(\mathbb{R}) = \{ \psi \in H^{1;b}_e(\mathbb{R}) \mid |y|^b\partial_y \psi \in H^{1;-b}(\mathbb{R}) \}.
\]
Let \( \psi \in C^2_c(\mathbb{R}) \). Then \( \partial_y(|y|^b\partial_y \psi) = -|y|^b\mathbb{D}_b \psi \), which implies \( \psi \in H^{2;b}_e(\mathbb{R}) \) by (2.1). We extend the pointwise defined operator \( \mathbb{D}_b \) to \( H^{2;b}_e(\mathbb{R}) \) by putting
\[
\mathbb{D}_b \psi := -|y|^{-b}\partial_y(|y|^b\partial_y \psi) \quad \text{for} \quad \psi \in H^{2;b}_e(\mathbb{R}),
\]
so that \( \mathbb{D}_b : H^{2;b}_e(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \).

**Lemma 2.3** Let \( \psi \in H^{2;b}_e(\mathbb{R}) \). Then
\[
(\mathbb{D}_b \psi, \eta)_{L^2} = (\partial_y \psi, \partial_y \eta)_{L^2} \quad \text{for any} \quad \eta \in H^{1;b}_e(\mathbb{R}); \quad (2.5)
\]
\[
(\mathbb{D}_b \psi, \eta)_{L^2} = (\psi, \mathbb{D}_b \eta)_{L^2} \quad \text{for any} \quad \eta \in H^{2;b}_e(\mathbb{R}). \quad (2.6)
\]
Proof. Let $\eta \in C^\infty_c(\mathbb{R})$. We can use integration by parts to compute
\[\int_{-\infty}^{+\infty} |y|^b (\mathbb{D}_b \psi) \eta \, dy = - \int_{-\infty}^{+\infty} \partial_y (|y|^b \partial_y \psi) \eta \, dy = \int_{-\infty}^{+\infty} |y|^b \partial_y \psi \partial_y \eta \, dy.\]
Thus $i)$ follows, thanks to the density result in Lemma 2.1. Clearly $ii)$ is an immediate consequence of $i)$. \hfill \Box

It remains to introduce a Hilbertian structure on $H^{2,b}_e(\mathbb{R})$. Given $\lambda > 0$, we put
\[ (\psi, \eta)_{\lambda, H^{2,b}_e} = ((\mathbb{D}_b + \lambda) \psi, (\mathbb{D}_b + \lambda) \eta)_{L^2,b}, \quad \|\psi\|_{\lambda, H^{2,b}_e} = \|(\mathbb{D}_b + \lambda) \psi\|_{L^2,b}. \]
If $\lambda = 1$ we drop it and simply write $(\psi, \eta)_{H^{2,b}_e}$ and $\|\psi\|_{H^{2,b}_e}$. Notice that
\[ (\mathbb{D}_b + \lambda) \psi(\cdot \sqrt{\lambda}) = \lambda \left[ (\mathbb{D}_b + 1) \psi \right](\cdot \sqrt{\lambda}), \quad (2.7) \]
which implies
\[ \|\psi(\cdot \sqrt{\lambda})\|^2_{\lambda, H^{2,b}_e} = \lambda^{-\frac{1+b}{2}} \|\psi(\cdot)\|^2_{H^{2,b}_e} \quad \text{for any} \ \psi \in H^{2,b}_e(\mathbb{R}). \quad (2.8) \]

Lemma 2.4 Let $\lambda > 0, \psi \in H^{2,b}_e(\mathbb{R})$. Then
\[ \|\psi\|^2_{\lambda, H^{2,b}_e} \geq \lambda \|\psi\|^2_{\lambda, H^{1,b}}. \]
Therefore, $H^{2,b}_e(\mathbb{R})$ is a Hilbert space, and is continuously embedded in $H^{1,b}_e(\mathbb{R})$.

Proof. Thanks to (2.8) we can assume that $\lambda = 1$. By Lemma 2.3 with $\eta = \psi$ we have
\[ (\mathbb{D}_b \psi, \psi)_{L^2,b} = \|\partial_y \psi\|^2_{L^2,b}. \]
Thus
\[ \int_{-\infty}^{+\infty} |y|^b (\mathbb{D}_b + 1) \psi|^2 \, dy = \int_{-\infty}^{+\infty} |y|^b |\mathbb{D}_b \psi|^2 \, dy + 2 \int_{-\infty}^{+\infty} |y|^b (\mathbb{D}_b \psi) \psi \, dy + \int_{-\infty}^{+\infty} |y|^b |\psi|^2 \, dy \]
\[ \geq 2 \int_{-\infty}^{+\infty} |y|^b |\partial_y \psi|^2 \, dy + \int_{-\infty}^{+\infty} |y|^b |\psi|^2 \, dy \]
which implies $\|\psi\|^2_{H^{2,b}_e} \geq \|\psi\|^2_{H^{1,b}}$. The conclusion of the proof is standard. \hfill \Box
Higher order  If $k > 2$ and $\lambda > 0$ we use induction to define

$$H^k_{\lambda}(\mathbb{R}) = \left\{ \psi \in H^{k-1}_{\lambda}(\mathbb{R}) \mid \nabla \psi \in H^{k-2}_{\lambda}(\mathbb{R}) \right\}$$

$$(\psi, \eta)_{H^k_{\lambda}} = ((\nabla + \lambda)\psi, (\nabla + \lambda)\eta)_{H^k_{\lambda}}.$$

As usual, if $\lambda = 1$ we drop it and simply write $(\psi, \eta)_{H^k_{\lambda}}$ and $\| \psi \|_{H^k_{\lambda}}$.

Notice that $C^k_{-1,1}(\mathbb{R}) \subset H^k_{\lambda}(\mathbb{R})$ by (2.1). In the next lemma we collect the main properties of the spaces $H^k_{\lambda}(\mathbb{R})$ for $k \geq 1$. In particular it implies that $\| \cdot \|_{\lambda,H^k_{\lambda}}$, for different $\lambda$’s, define the same Hilbertian structure on $H^k_{\lambda}(\mathbb{R})$. We omit its easy proof, which is based on previous results and induction.

**Lemma 2.5** Let $k \geq 1$, $b \in (-1, 1)$, $\psi \in H^k_{\lambda}(\mathbb{R})$ and $\lambda > 0$. The following facts hold.

i) $\| \psi \|^2_{\lambda,H^k_{\lambda}} \leq \| \partial_y (\nabla + \lambda)^{\frac{k-1}{2}} \psi \|^2_{L^2_{\lambda}} + \lambda \left\| (\nabla + \lambda)^{\frac{k-1}{2}} \right\|^2_{L^2_{\lambda}}$ if $k$ is odd

ii) $\| \psi \|^2_{\lambda,H^k_{\lambda}} = \lambda^{k-\frac{1+4b}{2}} \| \psi(\cdot) \|^2_{H^k_{\lambda}}$;

iii) $(\nabla + \lambda)^m \psi \in H^k_{\lambda}(\mathbb{R})$ for any positive integer $m < k/2$;

iv) $\| \psi \|^2_{\lambda,H^k_{\lambda}} \geq \lambda^{k-j} \| \psi \|^2_{\lambda,H^k_{\lambda}} \geq \lambda^k \| \psi \|^2_{L^2_{\lambda}}$ for any $j = 1, \ldots, k$;

v) $\| \psi \|^2_{\lambda,H^k_{\lambda}} \geq m_b \lambda^{k-\frac{1+4b}{2}} |\psi(0)|^2$, where $m_b$ is the constant in Lemma 2.1.

We now establish some integration by parts formulae. It suffices to take $\lambda = 1$.

**Lemma 2.6** Let $k \geq 2$, $\psi \in H^2_{\lambda}(\mathbb{R})$, $\eta \in H^k_{\lambda}(\mathbb{R})$. Then

$$(\psi, \eta)_{H^k_{\lambda}} = ((\nabla + 1)^{k-1}\psi, (\nabla + 1)^k\eta)_{L^2_{\lambda}}.$$

**Proof.** Notice that $H^2_{\lambda}(\mathbb{R}) \subset H^k_{\lambda}(\mathbb{R})$.

If $k = 2$, the equality in the lemma holds by definition.
If $k = 2m \geq 4$ is even, we use (2.6) with $((\mathcal{D}_b + 1)^m \psi, (\mathcal{D}_b + 1)^m \eta)_{L^2; b} = ((\mathcal{D}_b + 1)^{m+1} \psi, (\mathcal{D}_b + 1)^{m+1} \eta)_{L^2; b}$.

If $m = 2$ we are done. Otherwise, repeat the same procedure $m - 1$ times to get

$$(\psi, \eta)_{H^m; b} = ((\mathcal{D}_b + 1)^{2m-1} \psi, (\mathcal{D}_b + 1)^{2m-1} \eta)_{L^2; b},$$

which concludes the proof in the even case.

If $k = 2m + 1 \geq 3$ is odd we apply (2.5) with $((\mathcal{D}_b + 1)^m \psi, (\mathcal{D}_b + 1)^m \eta)_{L^2; b} = ((\mathcal{D}_b + 1)^{m+1} \psi, (\mathcal{D}_b + 1)^{m+1} \eta)_{L^2; b}$.

It follows that

$$(\psi, \eta)_{H^m; b} = ((\mathcal{D}_b + 1)^{m+1} \psi, (\mathcal{D}_b + 1)^{m+1} \eta)_{L^2; b} = (\mathcal{D}_b + 1)^{m+1} \psi, (\mathcal{D}_b + 1)^{m+1} \eta)_{L^2; b}.$$

To conclude the proof, use (2.9) with $\psi$ replaced by $(\mathcal{D}_b + 1)\psi$.

**Remark 2.7** It is well known that smooth, compactly supported functions are dense in $H^k(\mathbb{R})$ for any $k > 0$. Recall that $C^\infty_c(\mathbb{R})$ is dense in $H^1; b(\mathbb{R})$ for any $b \in (-1, 1)$ by [14]. If would be of interest to prove the density of $C^\infty_c(\mathbb{R})$ in $H^k; b(\mathbb{R})$ in case $b \neq 0$, $k > 1$.

### 3 Bessel functions and related issues

The basic properties of the Bessel function $K_\alpha$ can be found for instance [13, Sections 8.4, 8.5]. For any $\alpha \in \mathbb{R}$ the standard modified Bessel function of the second kind $K_\alpha = K_{-\alpha}$ solves

$$\partial^2_{yy} K_\alpha(y) + y^{-1} \partial_y K_\alpha(y) - (1 + \alpha^2 y^{-2}) K_\alpha(y) = 0 \quad \text{on } \mathbb{R}_+,$$

and decays exponentially as $y \to +\infty$. If $\alpha \neq 0$ then

$$K_\alpha(y) = 2^{\alpha-1} \Gamma(|\alpha|) y^{-|\alpha|} + o(y^{-|\alpha|}) \quad \text{as } y \to 0^+.$$
Bessel functions of different orders are related by the formulae
\[ \partial_y (y^\alpha K_\alpha (y)) = -y^\alpha K_{\alpha - 1}(y), \quad K_\alpha (y) - K_{\alpha - 2}(y) = 2(\alpha - 1)y^{-1}K_{\alpha - 1}(y). \]

Next, for \( s > 0 \) and \( \lambda > 0 \) we put
\[ \psi_{s,\lambda} (y) := \psi_s (\sqrt{\lambda} y) = c_s (\sqrt{\lambda} |y|)^s K_s (\sqrt{\lambda} |y|), \quad (3.1) \]
see (1.11). Notice that
\[ \psi_{s,\lambda} \in C^0 (\mathbb{R}), \quad \psi_{s,\lambda} (0) = 1, \quad \psi_{s,\lambda} \in C^\infty (\mathbb{R}_+), \]
and \( \psi_{s,\lambda} \) decays exponentially at infinity together with its derivatives of any order. Further, (2.7) readily implies
\[ (\mathbb{D}_b + \lambda)^m \psi_{s,\lambda} (y) = \lambda^m \left[ (\mathbb{D}_b + 1)^m \psi_s \right] (\sqrt{\lambda} y) \quad (3.2) \]
for any \( y \neq 0 \) and any integer \( m \geq 1 \).

**Lemma 3.1** Let \( s > 0 \) be non-integer and put \( b = 1 - 2(s - \lfloor s \rfloor) \). Then \( \psi_s \) solves the following differential equations on \( \mathbb{R}_+ \):

\( i \) \quad \[ \partial_y \psi_s = \begin{cases} -d_s y^{2s-1} \psi_{1-s}(y) & \text{if } 0 < s < 1, \\ -\frac{1}{2(s-1)} y \psi_{s-1}(y) & \text{if } s > 1; \end{cases} \]

\( ii \) \quad \[ -\partial_y^2 \psi_s (y) + \psi_s (y) = \begin{cases} d_s (2s-1)y^{2(s-1)} \psi_{1-s}(y) & \text{if } 0 < s < 1, \\ \frac{2s-1}{2(s-1)} \psi_{s-1} & \text{if } s > 1; \end{cases} \]

\( iii \) \quad \( (\mathbb{D}_b + 1)^[s] \psi_s = 0; \)

\( iv \) \quad If \( s > 1 \) then for any \( m = 1, \ldots, [s] \)
\[ (\mathbb{D}_b + 1)^m \psi_s = \frac{d_s}{d_{s-m}} \psi_{s-m} = \frac{|s|!}{(s - m)!} \frac{\Gamma(s - m)}{\Gamma(s)} \psi_{s-m}. \quad (3.3) \]
**Proof.** Let \( s \in (0, 1) \). By the properties of the Bessel functions we get
\[
\partial_y \psi_s(y) = -c_s y^s K_{1-s}(y) = -c_s y^{2s-1} - K_{1-s}(y)) = -d_s y^{2s-1} \psi_{1-s}(y).
\]
This gives the first equality in \( i) \). Now we notice that we can compute \( \partial_y \psi_{1-s} \) via the first equality in \( i) \), where \( s \) is replaced by \( 1 - s \). The proofs of \( ii), iii) \) readily follow. This completes the proof in this case.

Now let \( s > 1 \). We compute
\[
\partial_y \psi_s(y) = c_s \partial_y (y^s K_s(y)) = -c_s y^{s-1} K_{s-1}(y) = -\frac{c_s}{c_{s-1}} \psi_{s-1}(y)
\]
which gives the second equality in \( i) \). Also, we get
\[
\partial_y^2 \psi_s(y) = -c_s \partial_y (y^s K_{s-1}(y)) = -c_s y^{s-2} (y) + y^{-1} K_{s-1}(y))
\]
by the recurrence formula for \( K_s \). Hence
\[
\partial_y^2 \psi_s(y) = \frac{c_s}{c_{s-1}} (1 - 2s) \psi_{s-1} + \psi_s(y),
\]
which gives \( ii) \) for \( s > 1 \). To prove \( iv) \) we notice that this last equality implies
\[
(\mathbb{D}_b + 1) \psi_s = -\partial_y^2 \psi_s - (1 - 2s + 2[s]) \partial_y \psi_s + \psi_s = \frac{|s|}{s-1} \psi_{s-1} = \frac{d_s}{d_{s-1}} \psi_{s-1}.
\]
Thus (3.3) holds for \( m = 1 \). To conclude the proof of \( iv) \) repeat the same argument a finite number of times.

It remains to prove \( iii) \) in this case. We use \( iv) \) with \( m = [s] \) and then \( iii) \) with \( s \) replaced by \( s - [s] \in (0, 1) \) to get
\[
(\mathbb{D}_b + 1)^[s] \psi_s = \frac{d_s}{d_{1-[s]}} (\mathbb{D}_b + 1) \psi_{s-[s]} = 0.
\]
The lemma is completely proved. \( \square \)
Remark 3.2 Since $K_s > 0$ on $\mathbb{R}_+$, from i) in Lemma 3.1 it readily follows that the positive function $\psi_s$ achieves its maximum at the origin.

The next theorem contains our main result on the functions $\psi_s$ (recall our non-standard definition of H"older spaces in (1.12)).

**Theorem 3.3** Let $s > 0$ be non-integer, put $b = 1 - 2(s - \lfloor s \rfloor)$ and let $\lambda > 0$. Then

\[
\psi_{s,\lambda} \in H_0^{[s]:b}(\mathbb{R}) \cap \tilde{C}^{2s}(\mathbb{R}) ;
\]

\[
\lim_{y \to 0^+} y^b \partial y \left( (D_b + \lambda)^{[s]} \psi_{s,\lambda} \right) = -d_s \lambda^s
\]

where $d_s$ is the constant in (1.5). Moreover $\psi_{s,\lambda}$ satisfies

\[
(\psi_{s,\lambda}, \eta)_{\lambda,H_0^{[s]:b}} = 2d_s \lambda^s \eta(0) \quad \text{for any} \quad \eta \in H_0^{[s]:b}(\mathbb{R}).
\]

Finally, $\psi_{s,\lambda}$ admits the following variational characterization,

\[
\|\psi_{s,\lambda}\|_{\lambda,H_0^{[s]:b}}^2 = \inf_{\eta \in H_0^{[s]:b}(\mathbb{R})} \|\eta\|_{\lambda,H_0^{[s]:b}}^2 = 2d_s \lambda^s.
\]

**Proof.** Thanks to (3.2), we assume that $\lambda = 1$. We divide the proof in two steps.

**Step 1.** Let $\lfloor s \rfloor = 0$. Then $b = 1 - 2s$ and

\[
\partial y \psi_s(y) = -d_s y^{-b} \psi_{1-s}(y) = -d_s y^{-b} + o(y^{-b}) \quad \text{as} \quad y \to 0^+,
\]

which proves (3.5). Since in addition $\psi_s$ decays exponentially at infinity, from (3.8) we first infer that $\psi_s \in H_0^{1:1-2s}(\mathbb{R})$.

To prove that $\psi_s \in \tilde{C}^{2s}(\mathbb{R})$ we fix two points $y_1, y_2 \in \mathbb{R}$. By the symmetry of $\psi_s$, we can assume that $y_1, y_2 \geq 0$.

Let $0 < 2s \leq 1$. For $y > 0$ we have $|\partial y \psi_s(y)| = d_s y^{2s-1} \psi_{1-s}(y) \leq d_s y^{2s-1}$. Thus $\psi_s \in \tilde{C}^{2s}(\mathbb{R})$ follows from

\[
|\psi_s(y_1) - \psi_s(y_2)| \leq d_s \left| \int_{y_1}^{y_2} y^{2s-1} dy \right| = d_s \left| y_1^{2s} - y_2^{2s} \right| \leq d_s \frac{2s}{2s} |y_1 - y_2|^{2s}.
\]
If $1 < 2s < 2$ we use $ii)$ in Lemma 3.1 to estimate
\[ |\partial_{yy}^2 \psi_s(y)| = |\psi_s(y) - d_s(2s - 1)y^{2(s-1)}\psi_{1-s}(y)| \leq 1 + cy^{2(s-1)} \]
for $y > 0$. Using integration as before, we plainly get
\[ |\partial_y \psi_s(y_1) - \partial_y \psi_s(y_2)| \leq |y_1 - y_2| + c|y_1 - y_2|^{2s-1}. \]
Since $\partial_y \psi_s$ decays exponentially at infinity, we infer that there exists a constant $c > 0$ depending only on $s$, such that
\[ |\partial_y \psi_s(y_1) - \partial_y \psi_s(y_2)| \leq c|y_1 - y_2|^{2s-1} \]
which, in turns concludes the proof of (3.4).

Next, by $iii)$ in Lemma 3.1 we have that
\[ \partial_y (y^b \partial_y \psi_s) = y^b \psi_s \quad \text{on } \mathbb{R}_+. \tag{3.9} \]
We test (3.9) with an arbitrary $\eta \in C^\infty_{c,0}(\mathbb{R})$. Taking (3.8) into account we obtain
\[ \int_0^\infty y^b \psi_s \eta \, dy = \int_0^\infty \partial_y (y^b \partial_y \psi_s) \eta \, dy = d_s \eta(0) - \int_0^\infty y^b \partial_y \psi_s \partial_y \eta \, dy. \]
By the evenness of $\psi_s$ and $\eta$, this implies that $(\psi_s, \eta)_{H^{1,b}_e} = 2d_s \eta(0)$. Thus (3.6) holds in case $\lfloor s \rfloor = 0$, thanks to the density result in Lemma 2.1.

From (3.6) it follows that $(\psi_s, \eta - \psi_s)_{H^{1,b}_e} = 0$ for any $\eta \in H^{1,b}_e(\mathbb{R})$ such that $\eta(0) = 1$. Thus, $\psi_s$ is the minimal distance projection of 0 on the hyperplane $\{\eta(0) = 1\} \subset H^{1,b}_e(\mathbb{R})$, that is, $\psi_s$ is the unique solution to the minimization problem in (3.7). This completes the proof in the case $s \in (0, 1)$.

**Step 2:** Let $\lfloor s \rfloor \geq 1$. Thanks to $i)$ in Lemma 3.1 we see that
\[ \partial_y \psi_s(y) = -\frac{1}{2(s-1)} y \psi_{s-1}(y) = -\frac{1}{2(s-1)} y + o(y) \quad \text{as } y \to 0^+, \]
hence $\psi_s \in C^2(\mathbb{R})$. Next, as in case $2s \in (1, 2)$ we use $ii)$ in Lemma 3.1 to infer that $\partial_{yy}^2 \psi_s$ has the same regularity as $\psi_{s-1}$. If $s \in (1, 2)$ we obtain $\psi_s \in \tilde{C}^{2s}(\mathbb{R})$ by Step 1; if $s > 2$
one can use a bootstrap argument to prove that $\psi_s \in \tilde{C}^{2s}(\mathbb{R})$. By the decaying of $\psi_s$ at infinity we also infer that

$$\psi_s \in H_e^{[s]}(\mathbb{R}) \subset H_e^{[s]}(\mathbb{R}),$$

which concludes the proof of (3.4).

To prove (3.5) it suffices to notice that (3.3) and Step 1 give

$$\lim_{y \to 0^+} y^b \partial_y ((D_b + 1)^{[s]} \psi_s) = \frac{d_s}{d_{s-[s]}} \lim_{y \to 0^+} y^b \partial_y \psi_{s-[s]} = -d_s.$$

We now prove (3.6). Take any $\eta \in H_e^{[s]}(\mathbb{R})$. We apply Lemma 2.6 with $k = \lceil s \rceil$ and $\psi = \psi_s$ to obtain

$$(\psi_s, \eta)_{H_e^{[s]}(\mathbb{R})} = (\langle (D_b + 1)^{[s]} \psi_s, (D_b + 1)\eta \rangle_{L^2(\mathbb{R}^n)}).$$

Therefore, (3.3), (2.5) and Step 1 with $s$ replaced by $s - \lfloor s \rfloor \in (0, 1)$ give

$$(\psi_s, \eta)_{H_e^{[s]}(\mathbb{R})} = \frac{d_s}{d_{s-[s]}} (\psi_{s-[s]}, (D_b + 1)\eta)_{L^2(\mathbb{R}^n)} = \frac{d_s}{d_{s-[s]}} (\psi_{s-[s]}, \eta)_{H_e^{[s]}(\mathbb{R})} = 2d_s \eta(0),$$

and (3.6) follows. For (3.7) argue as in Step 1. □

**Remark 3.4** The recurrence formulae (3.3) plainly imply the identities

$$(D_b + 1)^m \psi_s(0) = \frac{d_s}{d_{s-m}}, \quad m = 1, \ldots, \lceil s \rceil$$

$$\lim_{y \to 0^+} y^{-1} \partial_y ((D_b + 1)^m \psi_s) = -\frac{d_s}{d_{s-m}} \frac{1}{2(s-m-1)}, \quad m = 0, \ldots, \lceil s \rceil - 1$$

We conclude this section with a corollary of Theorem 3.3, which might be of independent interest.

**Corollary 3.5** Let $\lfloor s \rfloor$ be even. Then the following virial-type formulae hold:

$$\int_{-\infty}^{+\infty} |y|^b ((D_b + 1)^{[s]} \psi_s)|^2 = \frac{\lfloor s \rfloor}{|s|} \cdot 2d_s,$$

$$\int_{-\infty}^{+\infty} |y|^b \partial_y ((D_b + 1)^{[s]} \psi_s)|^2 = \frac{|s| - s}{|s|} \cdot 2d_s.$$
Proof. We use (3.7) with $s$ replaced by $s + 1$ and then (3.3) to get

$$2d_{s+1} = \|\psi_{s+1}\|_{H^{2+s+1}}^2 = \int_{-\infty}^{+\infty} |y|^b |(\mathbb{D}_b + 1)^{s+1}\psi_{s+1}|^2 \, dy$$

$$= \frac{d_{s+1}^2}{d_s^2} \int_{-\infty}^{+\infty} |y|^b |(\mathbb{D}_b + 1)^{s}\psi_s|^2 \, dy,$$

and the first equality follows. For the second one, recall that $2d_s = \|\psi_s\|_{H^{2+s+1}}^2$. □

4 Spaces of curves in $\mathcal{H}$; proof of the main results

We start this section by studying the (unbounded) operators $\mathbb{L}^k_b U = (\mathbb{D}_b + \mathcal{L})^k U$ on $L^2_{e,b}(\mathbb{R} \to \mathcal{H})$, for any $b \in (-1, 1)$ and any integer $k \geq 0$.

Any function $U \in L^2_{e,b}(\mathbb{R} \to \mathcal{H})$ can be decomposed as follows,

$$U(y) = \sum_{j=1}^{\infty} U_j(y) \varphi_j,$$

where $U_j = (U, \varphi_j)_{\mathcal{H}} \in L^2_{e,b}(\mathbb{R})$ for any $j \geq 1$, and

$$\|U\|_{L^2_{e,b}}^2 = \sum_{j=1}^{\infty} \int_{-\infty}^{+\infty} |y|^b |U_j|^2 \, dy = \sum_{j=1}^{\infty} \|U_j\|_{L^2_{e,b}}^2.$$

Recall that

$$\mathbb{L}_b U = (\mathbb{D}_b + \mathcal{L}) U = -\partial_{yy}^2 U - by^{-1}\partial_y U + \mathcal{L} U , \quad \mathcal{L} \varphi_j = \lambda_j \varphi_j$$

and that we are assuming $\lambda_j \geq \lambda_1 > 0$. Thus, at least formally we have

$$\mathbb{L}^k_b U = \sum_{j=1}^{\infty} [(\mathbb{D}_b + \lambda_j)^k U_j] \varphi_j.$$
We define
\[ H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) = \left\{ U \in L_{e}^{2;b}(\mathbb{R} \to \mathcal{H}) \mid U_j = (U, \varphi_j)_\mathcal{H} \in H_{e}^{k;b}(\mathbb{R}) \text{ and } \|U\|_{H_{L,e}^{k;b}} < \infty \right\}, \]
where
\[ \|U\|_{H_{L,e}^{k;b}}^2 := \sum_{j=1}^{\infty} \|U_j\|_{\lambda_j,H_{L,e}^{k;b}}^2 \]
(we recall that \(\|\cdot\|_{\lambda_j,H_{L,e}^{k;b}}\) are equivalent norms in the space \(H_{e}^{k;b}(\mathbb{R})\), see Section 2). Thanks to Lemma 2.5, it is easily checked that \(H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H})\) is a Hilbert space with scalar product
\[ (U, V)_{H_{L,e}^{k;b}} = \sum_{j=1}^{\infty} (U_j, V_j)_{\lambda_j,H_{e}^{k;b}}. \]

For future convenience we provide another definition of the space \(H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H})\). Consider the standard weighted Sobolev space
\[ H^{1;b}(\mathbb{R} \to \mathcal{H}) := H^1(\mathbb{R} \to \mathcal{H}; |y|^b dy) = \left\{ U \in L^{2;b}(\mathbb{R} \to \mathcal{H}) \mid \partial_y U \in L^{2;b}(\mathbb{R} \to \mathcal{H}) \right\}, \]
and denote by \(H_{e}^{1;b}(\mathbb{R} \to \mathcal{H})\) the space of even curves in \(H^{1;b}(\mathbb{R} \to \mathcal{H})\). Then we let
\[ H_{e}^{2;b}(\mathbb{R} \to \mathcal{H}) = \left\{ U \in H_{e}^{1;b}(\mathbb{R} \to \mathcal{H}) \mid |y|^b \partial_y U \in H^{1;-b}(\mathbb{R} \to \mathcal{H}) \right\} \]
so that
\[ \mathbb{D}_b U := -|y|^{-b} \partial_y (|y|^b \partial_y U) \in L_{e}^{2;b}(\mathbb{R} \to \mathcal{H}) \quad \text{for any } U \in H_{e}^{2;b}(\mathbb{R} \to \mathcal{H}). \]

Finally, for \(k \geq 3\) we use induction to define
\[ H_{e}^{k;b}(\mathbb{R} \to \mathcal{H}) = \left\{ U \in H_{e}^{k-1;b}(\mathbb{R} \to \mathcal{H}) \mid \mathbb{D}_b U \in H_{e}^{k-2;b}(\mathbb{R} \to \mathcal{H}) \right\}. \]

The proof of the next lemma is simple but boring, and we omit it.

**Lemma 4.1** Let \(k \geq 1\) be an integer, \(b \in (-1, 1)\). Then
\[ H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) = H_{e}^{k;b}(\mathbb{R} \to \mathcal{H}) \cap L^{2;b}(\mathbb{R} \to \mathcal{H}_{L}). \]
The next lemma will be useful for the proof of our main results.

Lemma 4.2  

i) If \( U \in H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) \) then the following facts hold,

\[
\|U\|_{H_{L,e}^{k;b}}^2 = \begin{cases} 
\|\partial_y (\frac{b}{L} U)\|_{L^2}^2 + \|\mathcal{L}^{\frac{k}{2}}(\frac{b}{L} U)\|_{L^2}^2 & \text{if } k \text{ is odd} \\
\|\frac{b}{L} U\|_{L^2}^2 & \text{if } k \text{ is even}
\end{cases}
\]

\[
\|U\|_{H_{L,e}^{k;b}}^2 \geq \lambda_k^{-j}\|U\|_{H_{L,e}^{j;b}}^2 \geq \lambda_k^j\|U\|_{L^2}^2 \quad \text{for any } j = 1, \ldots, k;
\]

(4.1)

ii) the Dirac delta-type function

\[
\delta_0 : H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) \to \mathcal{H}_{L}^{k-\frac{1+b}{2}}, \quad \delta_0(V) = V(0)
\]

is well defined and continuous.

Proof. To prove i) use Lemma 2.5. Next, let \( U = \sum_{j=1}^{\infty} U_j \varphi_j \) be any curve in \( H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{L}) \). Thanks to v) in Lemma 2.5 we can estimate

\[
\|U\|_{H_{L,e}^{k;b}}^2 = \sum_{j=1}^{\infty} \|U_j\|_{H_{L,e}^{j;b}}^2 \geq m_b \sum_{j=1}^{\infty} \lambda_j^{-\frac{1+b}{2}} |U_j(0)|^2 = m_b \|U(0)\|_{H_{L}^{1+b}}^2,
\]

which concludes the proof. \( \square \)

Remark 4.3 It turns out that \( H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) \subset C_{\text{loc}}^{0,\frac{1-b}{2}}(\mathbb{R} \to \mathcal{H}) \). For the proof, take \( U = \sum_{j=1}^{\infty} U_j \varphi_j \in H_{L,e}^{1;b}(\mathbb{R} \to \mathcal{H}) \) and \( y_1, y_2 \in \mathbb{R} \). We use (2.4) with \( \psi = U_j \in H_{e}^{k;b}(\mathbb{R}) \) to estimate

\[
\|U(y_2) - U(y_1)\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |U_j(y_2) - U_j(y_1)|^2 \leq \frac{1}{1-b} \|U\|_{H_{L,e}^{1;b}}^2 |y_2|^{1-b} - |y_1|^{1-b}.
\]

Since \( H_{L,e}^{k;b}(\mathbb{R} \to \mathcal{H}) \) is continuously embedded in \( H_{L,e}^{1;b}(\mathbb{R} \to \mathcal{H}) \) by (4.1), the claim follows.
Proof of Theorem 1.1 Recall that \( b = 1 - 2(s - \lfloor s \rfloor) \in (-1, 1) \). For \( u = \sum_j u_j \varphi_j \in \mathcal{H} \), we use the notation introduced in (3.1) to rewrite (1.4) as

\[
P_s[u](y) = \sum_{j=1}^{\infty} \psi_{s,\lambda_j}(y) u_j \varphi_j.
\] (4.2)

We fix any \( u \in \mathcal{H}^s_{L} \). Theorem 3.3 gives \( \psi_{s,\lambda_j} \in H_{\leq}^{[s]:b}(\mathbb{R}) \) and \( \|\psi_{s,\lambda_j}\|_{L^2_{[s]:b}}^2 = 2 d_s \lambda_j^s \) by (3.7). Thus

\[
\|P_s[u]\|^2_{H_{\leq}^{[s]:b}} = \sum_{j=1}^{\infty} u_j^2 \|\psi_{s,\lambda_j}\|^2_{L^2_{[s]:b}} = 2 d_s \sum_{j=1}^{\infty} \lambda_j^s u_j^2 = 2 d_s \|\mathcal{L}^s u\|^2_{\mathcal{H}} = 2 d_s \|u\|^2_{H_{\leq}^{[s]:b}}
\]

and (1.5) is proved.

Next, take any \( V \in H_{L,\leq}^{[s]:b}(\mathbb{R} \rightarrow \mathcal{H}) \) and put \( V_j(y) = (V(y), \varphi_j)_{\mathcal{H}} \). We have

\[
(P_s[u],V)_{H_{\leq}^{[s]:b}} = \sum_{j=1}^{\infty} u_j (\psi_{s,\lambda_j}, V_j)_{L^2_{[s]:b}} = 2 d_s \sum_{j=1}^{\infty} \lambda_j^s u_j V_j(0) = 2 d_s \langle \mathcal{L}^s u, V(0) \rangle
\]

by (3.6), which proves \( iii) \).

Evidently \( iii) \) implies that \( P_s[u] \) is a weak solution to (1.7). Since \( P_s[u] \) is smooth on \( \mathbb{R}_+ \) by Lemma A.1, we see that in fact \( P_s[u] \) solves (1.7) pointwise. The first equality in (1.8) is satisfied by \( iii) \) in Lemma A.1.

To conclude the proof of (1.8), we first compute

\[
\|L_{[s]:b}\|_{P_s[u]}(y) = \sum_{j=1}^{\infty} \lambda_j^s ((D_{[s]:b} + 1)^{[s]} \psi_s)(\sqrt{\lambda_j} y) u_j \varphi_j.
\]

Now we use two items in Lemma 3.1, namely, \( iv) \) (with \( m = [s] \)) and then \( i) \) (with \( s - [s] \) instead of \( s \)). This gives

\[
y^b(\partial_y \|_{[s]:b} P_s[u])(y) = \frac{d_s}{d_s-[s]} \sum_{j=1}^{\infty} \lambda_j^s y^b (\partial_y \psi_{s-[s]})(\sqrt{\lambda_j} y) u_j \varphi_j
\]

\[
= -d_s \sum_{j=1}^{\infty} \psi_{s-[s]} (\sqrt{\lambda_j} y) \lambda_j^s u_j \varphi_j = -d_s P_{s-[s]}[\mathcal{L}^s u](y).
\] (4.3)
The second limit in (1.8) follows from $iii$) in Lemma A.1, and $iv$) is proved.

It remains to prove $ii$). Let $V \in H_{\mathcal{L},\mathcal{E}}^{[s],b}(\mathbb{R} \to \mathcal{H})$ be such that $V(0) = u$. Then $V_j(0) = u_j$ for any $j \geq 1$. Thus (3.7) gives

$$u_j^2\|\psi_{s,\lambda_j}\|_{\lambda_j,H_{\mathcal{E}}^{[s],b}}^2 \leq \|V_j\|_{\lambda_j,H_{\mathcal{E}}^{[s],b}}^2$$

for any $j \geq 1$. Thus

$$\|P_s[u]\|_{H_{\mathcal{L},\mathcal{E}}^{[s],b}}^2 = \sum_{j=1}^{\infty} u_j^2\|\psi_{s,\lambda_j}\|_{\lambda_j,H_{\mathcal{E}}^{[s],b}}^2 \leq \sum_{j=1}^{\infty} \|V_j\|_{\lambda_j,H_{\mathcal{E}}^{[s],b}}^2 = \|V\|_{H_{\mathcal{L},\mathcal{E}}^{[s],b}}^2,$$

and $ii$) follows. The theorem is completely proved. \hfill \Box

**Proof of Theorem 1.2** Recall that $P_s[u] : \mathcal{H}_s \to H_{\mathcal{L},\mathcal{E}}^{[s],b}(\mathbb{R} \to \mathcal{H})$ is, up to the constant $2d_s$, an isometry by item $i$) in Theorem 1.1, and that $\mathcal{L}_\mathcal{E}^{-s} : \mathcal{H}_\mathcal{E}^{-s} \to \mathcal{H}_\mathcal{E}^s$ is an isometry. Thus for any $\zeta \in \mathcal{H}_\mathcal{E}^{-s}$ we have that

$$\|P^{-s}[\zeta]\|_{H_{\mathcal{L},\mathcal{E}}^{[s],b}}^2 = 2d_s\|\mathcal{L}_\mathcal{E}^{-s}\zeta\|_{\mathcal{H}_\mathcal{E}^s}^2 = 2d_s\|\zeta\|_{\mathcal{H}_\mathcal{E}^{-s}},$$

and (1.9) is proved. The conclusions in $iii), iv$) are immediate consequences of Theorem 1.1 (with $u := \mathcal{L}_\mathcal{E}^{-s}\zeta$).

Finally, notice that the strictly convex minimization problem in (1.10) has a unique solution $\hat{U} \in H_{\mathcal{L},\mathcal{E}}^{[s],b}(\mathbb{R} \to \mathcal{H})$, and that $\hat{U}$ satisfies

$$\langle \hat{U}, V \rangle_{H_{\mathcal{L},\mathcal{E}}^{[s],b}} = 2d_s\langle \zeta, V(0) \rangle = 2d_s\langle \mathcal{L}_\mathcal{E}^s u, V(0) \rangle$$

for any $V \in H_{\mathcal{L},\mathcal{E}}^{[s],b}(\mathbb{R} \to \mathcal{H})$. Thus $\hat{U} = P_s[u] = P^{-s}[\zeta]$ by $iii$) in Theorem 1.1. \hfill \Box

### 5 Generalizations and examples

In this section we provide some possible generalizations of our main result. They are based on Theorem 3.3.
5.1 Nonnegative operators

Assume that \( \mathcal{L} \) is self-adjoint, with a discrete spectrum, nonnegative and with a nontrivial kernel. Trivially, for any \( s > 0 \) we have \( \text{ker} \mathcal{L}^s = \text{ker} \mathcal{L} \), hence

\[
\mathcal{L}^s u = \mathcal{L}^s(u - \Pi u),
\]

where \( \Pi : \mathcal{H} \to \ker \mathcal{L} \) is the orthogonal projection on \( \ker \mathcal{L} \). The domain of the quadratic form \( u \mapsto (\mathcal{L}^s u, u)_{\mathcal{H}} \) is

\[
H^s_\mathcal{L} = \ker \mathcal{L} \oplus H^s_{\mathcal{L}, 1}, \quad \mathcal{L}_1 := \mathcal{L}_{|_{(\ker \mathcal{L})^\perp}} : (\ker \mathcal{L})^\perp \to (\ker \mathcal{L})^\perp.
\]

Notice that \( \mathcal{L}_1 \) is self-adjoint, with a discrete spectrum and positive. Thus Theorem 1.1 provides a full characterization of \( \mathcal{L}^s_1 \) and of the corresponding quadratic form on \( \mathcal{H}^s_{\mathcal{L}, 1} \). This gives, in turn, corresponding results for \( \mathcal{L}^s \) and for its quadratic form on \( \mathcal{H}^s_\mathcal{L} \).

In particular, the operator \( u \mapsto \mathcal{P}_s[u] \) in (1.4) is the identity on \( \ker \mathcal{L} \) and

\[
\mathcal{P}_s[u](y) = \Pi[u] + \mathcal{P}^\perp_s[u - \Pi u](y), \quad (5.1)
\]

where \( \mathcal{P}^\perp_s \) is the isometry given by Theorem 1.1 for the operator \( \mathcal{L}_1 \). Since \( \mathcal{P}_s[u] \) differs from \( \mathcal{P}^\perp_s[u - \Pi u] \) by a constant curve, then \( \mathcal{P}_s[u], \mathcal{P}^\perp_s[u] \) enjoy the same regularity properties in the Appendix.

5.2 Non-discrete spectrum

Let \( \mathcal{L} \) be a nonnegative, self-adjoint operator in the Hilbert space \( \mathcal{H} \). Then there exists a unique projector-valued spectral measure \( E \) on \( \mathbb{R} \) supported on the spectrum \( \sigma(\mathcal{L}) \subset [0, \infty) \), such that

\[
\mathcal{L} = \int_{[\Lambda, \infty)} \lambda dE(\lambda),
\]

where \( \Lambda \geq 0 \) is the bottom of \( \sigma(\mathcal{L}) \) (see e.g., [3, Ch. 6]).

For \( s > 0 \), the \( s \)-power of \( \mathcal{L} \) is formally defined via

\[
\mathcal{L}^s = \int_{[\Lambda, \infty)} \lambda^s dE(\lambda).
\]
We denote by $H^s_L$ the domain of the corresponding quadratic form, which is a Hilbert space with norm $\| \cdot \|_{H^s_L}^2 = \| L^s \cdot \|_H^2 + \| \cdot \|_H^2$.

Let us first assume that $L$ be positive definite, i.e. $\Lambda > 0$. Then $\| L^s \cdot \|_H^2$ is an equivalent norm in $H^s_L$.

For $s > 0$ non-integer and $u \in H^s_L$ we consider the curve

$$P_s[u](y) = \int_{[\Lambda, \infty)} \psi_s(\sqrt{\lambda} y) dE(\lambda) u,$$

where $\psi_s$ is the function in (1.11). As in the discrete case, we have that $P_s$ maps any $u \in H$ into an even curve in $H^s_L$; in addition $P_s[u] \in C^\infty(\mathbb{R}_+ \to H^s_L)$ for every $u \in H$, $\sigma > 0$.

Further, for $b \in (-1, 1)$ we introduce the following (unbounded) operators acting on even curves $U \in L^2_b(\mathbb{R} \to H),$

$$I_b U = \int_{[\Lambda, \infty)} (\mathbb{D}_b + \lambda) dE(\lambda) U, \quad D_b U = -\partial^2_{yy} U - by^{-1} \partial_y U,$$

compare with (1.3).

For any integer $k \geq 1$ we introduce the space

$$H^{k,b}_{L,\infty}(\mathbb{R} \to H) = \left\{ U \in L^2_b(\mathbb{R} \to H) \mid \| U \|_{H^{k,b}_{L,\infty}} < \infty \right\}.$$

Here $\| \cdot \|_{H^{k,b}_{L,\infty}}$ is defined similarly as we did in the discrete case. More precisely, if $k$ is even then

$$\| U \|^2_{H^{k,b}_{L,\infty}} := \int_{\mathbb{R}} |y|^b \left[ \int_{[\Lambda, \infty)} d(E(\lambda)V(y, \lambda), V(y, \lambda)) \right] dy,$$

where $V(y, \lambda) = (\mathbb{D}_b + \lambda)^{k/2} U(y)$. If $k$ is odd then

$$\| U \|^2_{H^{k,b}_{L,\infty}} := \int_{\mathbb{R}} |y|^b \left[ \int_{[\Lambda, \infty)} d(E(\lambda)\partial_y V(y, \lambda), \partial_y V(y, \lambda)) + \int_{[\Lambda, \infty)} \lambda d(E(\lambda)V(y, \lambda), V(y, \lambda)) \right] dy,$$

where $V(y, \lambda) = (\mathbb{D}_b + \lambda)^{k/2} U(y)$.

With the above definitions, Theorem 1.1 holds true, and its proof can be carried out with no essential modifications.
If Λ = 0 is an eigenvalue of \( L \) one can use a decomposition similar to (5.1) and the above remarks in the present subsection for the restriction of \( L \) to \( \ker L^\perp \).

A more complicated case is when \( 0 \in \sigma(L) \) is not an eigenvalue but a point of continuous spectrum. Clearly \( \|L^2 \cdot \|_{\mathcal{H}} \) cannot bound \( \| \cdot \|_{\mathcal{H}} \) and therefore it is only a seminorm in \( \mathcal{H}_L^s \). Denote by \( \hat{\mathcal{H}}_L^s \) the completion of \( \mathcal{H}_L^s \) with respect to \( \|L^2 \cdot \|_{\mathcal{H}} \).

To avoid additional difficulties, we assume that \( \|L^2 \cdot \|_{\mathcal{H}} \) is a norm in \( \hat{\mathcal{H}}_L^s \). In this case one can define a suitable space of curves, and prove a result similar to Theorem 1.1.

### 5.3 Examples

The approach proposed in the present paper can be used, for instance, to recover non-integer powers of a large class of differential operators.

The case of the Dirichlet Laplacian in a bounded, smooth domain \( \Omega \subset \mathbb{R}^n \) is included in Theorem 1.1. Any curve \( y \mapsto U(y) \in L^2(\Omega) = \mathcal{H} \) is identified with the function \( (x,y) \mapsto U(y)(x), \Omega \times \mathbb{R} \to \mathbb{R} \), so that \( L^{2,\mathbb{R}}(\mathbb{R} \to L^2(\Omega)) \equiv L^2(\Omega \times \mathbb{R}; |y|^b dx dy) \), and

\[
\|U\|_{L^{2,\mathbb{R}}(\mathbb{R} \to L^2(\Omega))}^2 = \int_{-\infty}^{+\infty} |y|^b \|U(y)\|_{L^2(\Omega)}^2 dy = \iint_{\Omega \times \mathbb{R}} |y|^b |U(x,y)|^2 \, dx dy.
\]

Further, \( L^{2,\mathbb{R}}_e(\mathbb{R} \to L^2(\Omega)) \) is identified with the space of functions in \( L^2(\Omega \times \mathbb{R}; |y|^b dx dy) \) which are even in the \( y \)-variable, that is denoted by \( L^2_e(\Omega \times \mathbb{R}; |y|^b dx dy) \).

We choose \( L = -\Delta_D \), the standard Laplace operator with domain \( H^1_0(\Omega) \cap H^2(\Omega) \). Its eigenvalues \( \lambda_j \) and corresponding eigenfunctions \( \varphi_j \) solve the Dirichlet problem

\[
\begin{cases}
-\Delta \varphi_j = \lambda_j \varphi_j & \text{in } \Omega \\
\varphi_j = 0 & \text{on } \partial \Omega,
\end{cases}
\]

The natural domain \( \mathcal{H}^s_{-\Delta_D}(\Omega) \) of the quadratic form \( u \mapsto ((-\Delta)^s u, u)_{L^2} \) can be described by the results in [21, Section 1], see also [16, Lemma 3]:

\[
\mathcal{H}^s_{-\Delta_D}(\Omega) = \left\{ u \in H^s(\Omega) \mid (-\Delta)^m u|_{\partial \Omega} = 0 \text{ if } m \in \mathbb{N}_0, \ 2m < s - \frac{1}{2} \right\}
\]
(recall that functions in $H^s(\Omega)$ have a trace on $\partial \Omega$ if and only if $s > \frac{1}{2}$).

We see that

$$L_b U = -\Delta U - b y^{-1} \partial_y U = -|y|^{-b} \text{div}(|y|^b \nabla U),$$

(5.3)

where $-\Delta$ is the Dirichlet Laplacian in $\Omega \times \mathbb{R}$.

For $s$ non-integer, Theorem 1.1 relates the nonlocal operator $(-\Delta^D)_s$, with the local operator $L_{\lceil s \rceil} b \Delta_{\lceil s \rceil}$ acting on the space $H^{\lceil s \rceil}_b(\Omega \times \mathbb{R})$. For instance, with obvious notation, we have

$$H^{1,b}_{-\Delta^D, e}(\Omega \times \mathbb{R}) = \left\{ U \in H^1_e(\Omega \times \mathbb{R}; |y|^b dx dy) \mid U(\cdot, y) \in H^1_0(\Omega) \text{ for } y \neq 0 \right\},$$

$$||U||^2_{H^{1,b}_{-\Delta^D, e}} = \iint_{\Omega \times \mathbb{R}} |y|^b |\nabla U|^2 dx dy;$$

$$H^{2,b}_{-\Delta^D, e}(\Omega \times \mathbb{R}) = \left\{ U \in H^{1,b}_{-\Delta^D, e}(\Omega \times \mathbb{R}) \mid |y|^b \nabla U \in H^1(\Omega \times \mathbb{R}; |y|^b dx dy) \right\},$$

$$||U||^2_{H^{2,b}_{-\Delta^D, e}} = \iint_{\Omega \times \mathbb{R}} |y|^b ||L_b U||^2 dx dy = \iint_{\Omega \times \mathbb{R}} |y|^{-b} |\text{div}(|y|^b \nabla U)|^2 dx dy.$$

The Neumann Laplacian in $\Omega$ fits in the situation described in Subsection 5.1. Now we choose $\mathcal{L} = -\Delta_N$. It is an unbounded operator on $L^2(\Omega)$ with eigenvalues $\lambda_j \geq 0$ and eigenfunctions $\varphi_j$ solving

$$\begin{cases}
-\Delta \varphi_j = \lambda_j \varphi_j & \text{in } \Omega \\
\partial_\nu \varphi_j = 0 & \text{on } \partial \Omega,
\end{cases}$$

$$\int_\Omega \varphi_j \varphi_h \, dx = \delta_{jh}.$$

The natural domain $\mathcal{H}_s^{\Delta_N}(\Omega)$ of the quadratic form $u \mapsto (-\Delta_N)^s u, u)_{L^2}$ is

$$\mathcal{H}_s^{\Delta_N}(\Omega) = \left\{ u \in H^s(\Omega) \mid \partial_\nu (-\Delta)^m u \big|_{\partial \Omega} = 0 \text{ if } m \in \mathbb{N}_0, 2m < s - \frac{3}{2} \right\},$$

see [21, Section 1].

In this case, the operator $L_b$ is pointwise defined as in (5.3). For $s \notin \mathbb{N}$, the nonlocal operator $(-\Delta_N)^s$ is related to $L_{[s]} b$, acting on a different domain

$$H_{-\Delta_N, e}^{[s], b}(\mathbb{R} \to L^2(\Omega)) \equiv H_{-\Delta_N, e}^{[s], b}(\Omega \times \mathbb{R}).$$

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The approach described in Subsection 5.1 covers this example, as $\lambda_1 = 0$.

Lastly, if $n > 2s$ then the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^n$ fits into the general approach in Subsection 5.2. In this case, thanks to Hardy inequality the space $\mathcal{H}^s_{-\Delta}$ can be identified with the standard homogeneous Sobolev space $\mathcal{D}^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n; |x|^{-2s}dx)$. The resulting space of curves can be identified with the space $\mathcal{D}^{[s];b}_{e}(\mathbb{R}^{n+1})$ in [8].

\section{On the transforms $\mathcal{P}_s$}

Here we assume that $s > 0$ is non-integer and study the transform $\mathcal{P}_s[\cdot]$, see (4.2). We start by noticing that formulae (1.1) and (1.2) hold and that $\mathcal{H}^s_{\mathcal{L}}$ is the domain of the quadratic form of $\mathcal{L}^s$, for negative orders $s$ as well.

\begin{lemma} \label{lem:transforms}
Let $s > 0$, $\sigma \in \mathbb{R}$.

i) For any $u \in \mathcal{H}$, we have $\mathcal{P}_s[u] \in \mathcal{C}^\infty(\mathbb{R} \to \mathcal{H}^\sigma_{\mathcal{L}})$, and $\|\partial_y^k \mathcal{P}_s[u](y)\|_{\mathcal{H}^\sigma_{\mathcal{L}}}$ decays exponentially as $y \to \infty$, for any order $k \geq 0$;

ii) The linear operator $u \mapsto \mathcal{P}_s[u](y)$ is nonexpansive in $\mathcal{H}^\sigma_{\mathcal{L}}$, that is,

$$\|\mathcal{P}_s[u](y)\|_{\mathcal{H}^\sigma_{\mathcal{L}}} \leq \|u\|_{\mathcal{H}^\sigma_{\mathcal{L}}} \quad \text{for any } y \in \mathbb{R}; \quad (A.1)$$

iii) If $u \in \mathcal{H}^\sigma_{\mathcal{L}}$ then $\mathcal{P}_s[u] \in \mathcal{C}^0(\mathbb{R} \to \mathcal{H}^\sigma_{\mathcal{L}})$ and $\mathcal{P}_s[u](0) = u$;

iv) The operator $u \mapsto \mathcal{P}_s[u](y)$ commutes with the fractional powers of $\mathcal{L}$, that is,

$$\mathcal{P}_s[\mathcal{L}^\sigma u](y) = \mathcal{L}^\sigma(\mathcal{P}_s[u](y)). \quad (A.2)$$

\end{lemma}

\textbf{Proof.} By the properties of the Bessel functions, for any integer $k \geq 0$ and any $\delta > 0$ we have $|\partial_y^k \psi_\delta(y)| \leq c(\delta)e^{-y}$ for $y > \sqrt{\lambda_1 \delta}$, where the constant $c(\delta)$ depends on $\delta, s$ and $k$ but not on $y$. Thus, for $y > \delta$ we have

$$\lambda_j^{k+\sigma}|(\partial_y^k \psi_\delta)(\sqrt{\lambda_j y})|^2 \leq c(\delta)^2 \lambda_j^{k+\sigma} e^{-2\sqrt{\lambda_j y}} \leq C(\delta) e^{-\sqrt{\lambda_j y}}$$
because $\lambda_j \geq \lambda_1 > 0$, where the new constant $C(\delta)$ depends only on $\delta, s, \sigma$ and $k$. It readily follows that

$$
\|\partial_y^k \mathcal{P}_s[u](y)\|_{H^\sigma_L}^2 = \sum_{j=1}^\infty \lambda_j^{s+\sigma} u_j^2 (\psi_{s,\lambda_j}^s(y))^2 \leq C(\delta) \|u\|_{H^\sigma_L}^2 e^{-\sqrt{\lambda_1}y}
$$

for any $u \in \mathcal{H}$, provided that $y > \delta$, and i) is proved.

Now we take $u \in H^\sigma_L$. By Remark 3.2, we have $0 < \psi_{s,\lambda_j}(y) \leq \psi_{s,\lambda_j}(0) = 1$. Thus

$$
\|\mathcal{P}_s[u](y)\|_{H^\sigma_L}^2 = \sum_{j=1}^\infty \lambda_j^s u_j^2 (\psi_{s,\lambda_j}^s(y))^2 \leq \sum_{j=1}^\infty \lambda_j^s u_j^2 = \|u\|_{H^\sigma_L}^2,
$$

which proves ii). Further, we have

$$
\|u - \mathcal{P}_s[u](y)\|_{H^\sigma_L}^2 = \sum_{j=1}^\infty \lambda_j^s u_j^2 (\psi_{s,\lambda_j}^s(0) - \psi_{s,\lambda_j}(y))^2 \leq \sum_{j=1}^\infty \lambda_j^s u_j^2. \quad (A.3)
$$

The first series in (A.3) is dominated by a convergent number series and converges to zero termwise as $y \to 0$. We infer that $\|u - \mathcal{P}_s[u](y)\|_{H^\sigma_L}^2 \to 0$ as $y \to 0$, which implies $\mathcal{P}_s[u] \in C^0(\mathbb{R} \to H^\sigma_L)$, and iii) is proved.

Since the equality in iv) is trivial, the proof is complete. $\square$

Thanks to Lemma A.1 and (4.3), we can improve the convergences in (1.8) as follows.

**Corollary A.2** Let $s > 0$ be non-integer, $b = 1 - 2(s - \lfloor s \rfloor)$. Assume that $u \in H^\sigma_L$ for some $\sigma \in \mathbb{R}$. Then $\mathcal{P}_s[u]$ solves the differential equation (1.7) and satisfies the boundary conditions

$$
\begin{align*}
\lim_{y \to 0^+} \mathcal{P}_s[u](0) &= u \quad \text{in} \quad H^\sigma_L \\
\lim_{y \to 0^+} y^b \partial_y (L_b^s \mathcal{P}_s[u])(y) &= -d_s L^s u \quad \text{in} \quad H^{\sigma-2s}_L.
\end{align*}
$$

**Remark A.3** For any integer $k \geq 0$ we have

$$
\psi_{k+\frac{1}{2}}(y) = \frac{1}{(k+1)!} |y|^{k+\frac{1}{2}} e^{-|y|}, \quad \mathcal{P}_{k+\frac{1}{2}}[u](y) = \frac{1}{(k+1)!} y^k \mathcal{P}_2[L_2^k u](y).
$$
A.1 Derivatives

The regularity of the curve $P_s[u]$ given in Lemma A.1 improves as $s$ increases. We start by proving a technical result which involves the Beta function

$$B(\tau, t) = \int_0^1 x^{\tau-1}(1-x)^{t-1} \, dx = \frac{\Gamma(t)\Gamma(\tau)}{\Gamma(t+\tau)}.$$ 

The coefficients in the next lemma are computed by taking inspiration from [8, Section 4].

Lemma A.4 Let $\sigma \in \mathbb{R}, u \in \mathcal{H}_l^\sigma, y > 0$.

i) If $s \in (0, 1)$ then $\partial_y P_s[u](y) = -d_s y^{2s-1} P_{1-s}[L^s u](y)$;

ii) If $s > 1$ then for any $m = 1, \ldots, \lfloor s \rfloor$ it holds that

$$\partial^2 y^m P_s[u](y) = \frac{1}{B(s, \frac{1}{2})} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell B(s - \ell, \frac{1}{2}) \cdot P_{s-\ell}[L^m u](y)$$

(A.4)

$$\partial^2 y^{m-1} P_s[u](y) = y \cdot \frac{1}{B(s, \frac{1}{2})} \sum_{\ell=1}^{m} \binom{m-1}{\ell-1} (-1)^\ell B(s - \ell, \frac{3}{2}) \cdot P_{s-\ell}[L^m u](y).$$

(A.5)

Proof. If $s \in (0, 1)$ then $\partial_y \psi_{s,\lambda}(y) = -d_s y^{2s-1} \lambda^s \psi_{1-s,\lambda}(y)$ by Lemma 3.1. Thus

$$\partial_y P_s[u](y) = -d_s y^{2s-1} \cdot \sum_{j=1}^{\infty} \partial^2 y^{m} \psi_{s-\ell,\lambda_j}(y) \lambda_j^s u_j \varphi_j,$$

and the identity in i) follows.

To handle the case $s > 1$ we put $\gamma_{s,\ell} = \frac{B(s - \ell, \frac{1}{2})}{B(s, \frac{1}{2})} = \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} \cdot \frac{\Gamma(s)}{\Gamma(s + \frac{1}{2} - \ell)}$. Using

ii) in Lemma 3.1 and induction one gets

$$\partial^2 y^m \psi_s(y) = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell \gamma_{s,\ell} \psi_{s-\ell}(y).$$

(A.6)
for any integer $m = 1, \ldots, \lfloor s \rfloor$. Since $\partial_y^{2m} \psi_{s, \lambda}(y) = \lambda^m (\partial_y^{2m} \psi)(\sqrt{\lambda} y)$, we infer that

$$
\partial_y^{2m} \mathcal{P}_s[u](y) = \sum_{j=1}^{\infty} \partial_y^{2m} \psi_{s, \lambda_j}(y) u_j \varphi_j = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell \gamma_{s, \ell} \sum_{j=1}^{\infty} \psi_{s-\ell, \lambda_j}(y) \lambda_j^m u_j \varphi_j,
$$

which proves (A.4).

Arguing as for $i)$ we obtain

$$
\partial_y \mathcal{P}_s[u](y) = -\frac{y}{2(s-1)} \mathcal{P}_{s-1}[\mathcal{L}u](y),
$$

i.e. (A.5) holds if $m = 1$. If $m > 1$ we use (A.6) for $m - 1$ and then $i)$ in Lemma 3.1 to compute

$$
\partial_y^{2m-1} \psi_s(y) = \partial_y \partial_y^{2(m-1)} \psi_s = \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (-1)^\ell \gamma_{s, \ell} \partial_y \psi_{s-\ell}(y)
$$

$$
= y \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (-1)^{\ell+1} \frac{\gamma_{s, \ell}}{2(s-\ell-1)} \psi_{s-\ell-1}(y)
$$

$$
= y \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^\ell \frac{\ell \gamma_{s, \ell}}{2m(s+\frac{1}{2}-\ell)} \psi_{s-\ell}(y).
$$

Then (A.5) follows by arguing as in the "even" case. □

**Theorem A.5** Let $2s \geq 1$, $\sigma \in \mathbb{R}$ and let $k$ be an integer, with $1 \leq k \leq \lfloor 2s \rfloor$.

i) Let $u \in \mathcal{H}_\sigma^\sigma$. Then

$$
\|\partial_y^k \mathcal{P}_s[u](y)\|_{\mathcal{H}_\sigma^{\sigma-k}} \leq c_k \|u\|_{\mathcal{H}_\sigma^\sigma} \quad \text{for any } y > 0,
$$

where the constant $c_k$ depends only on $s$ and $k$. Thus, for any $y > 0$ the linear operator $u \mapsto \partial_y^k \mathcal{P}_s[u](y)$ is continuous $\mathcal{H}_\sigma^\sigma \to \mathcal{H}_\sigma^{\sigma-k}$.

ii) If in addition$^1$ $k < 2s$ then $\partial_y^k \mathcal{P}_s[u] \in C^0(\mathbb{R} \to \mathcal{H}_\sigma^{\sigma-k})$ for any $u \in \mathcal{H}_\sigma^\sigma$.

$^1$this is a restriction only if $s$ is a half integer
Proof. It is convenient to define

\[ M_{\alpha, \beta} = \max_{y \geq 0} y^{2\beta} \psi_\alpha(y)^2, \quad \alpha, \beta > 0. \]

If \( \frac{1}{2} \leq s < 1 \), then \( i) \) in Lemma A.4 gives

\[ \| \partial_y P_s[u](y) \|^2_{H^\sigma_{\ell-1}} = d_s^2 \| y^{2s-1} P_{1-s}([L^s u](y)) \|^2_{H^\sigma_{\ell-1}}. \]

The conclusion in \( i) \) follows, because

\[ \| y^{2s-1} P_{1-s}([L^s u](y)) \|^2_{H^\sigma_{\ell-1}} = \sum_{j=1}^{\infty} \lambda_j^{2s} u_j^2 \psi_{1-s}(\sqrt{\lambda_j y})^2 \]

\[ \leq \sum_{j=1}^{\infty} \lambda_j^{2s} u_j^2 \psi_{1-s}(\sqrt{\lambda_j y})^2 \leq M_{1-s, 2s-1} \| u \|^2_{H^\sigma_{\ell-1}}. \] (A.8)

If \( 2s > 1 \), then the series in (A.8) are dominated by a convergent number series and converge to zero termwise as \( y \to 0 \). We infer that \( \partial_y P_s[u](y) \to 0 \) in \( H^\sigma_{\ell-1} \) as \( y \to 0 \), which proves \( ii) \) in this case.

Next, let \( s > 1 \). We first face the case when \( k \leq 2 \lfloor s \rfloor \) is even. Take integers \( \ell, m \) with \( 0 \leq \ell \leq m \leq \lfloor s \rfloor \). By Lemma A.1 we have

\[ \| P_{s-\ell}[L^m u](y) \|^2_{H^\sigma_{\ell-2m}} \leq \| L^m u \|^2_{H^\sigma_{\ell-2m}} = \| u \|^2_{H^\sigma_{\ell-2m}} \], \( P_{s-\ell}[L^m u] \in C^0(\mathbb{R} \to H^\sigma_{\ell-2m}). \)

Taking also (A.4) into account, we see that the conclusions hold in this case.

Let now \( k \leq 2 \lfloor s \rfloor - 1 \) be odd. For \( 1 \leq \ell \leq m \) we estimate

\[ \| y P_{s-\ell}[L^m u](y) \|^2_{H^\sigma_{\ell-2m+1}} = \| y L^m (P_{s-\ell}[u](y)) \|^2_{H^\sigma_{\ell-2m+1}} = \| y P_{s-\ell}[u](y) \|^2_{H^\sigma_{\ell+1}} \]

\[ = \sum_{j=1}^{\infty} \lambda_j^{2s} u_j^2 (\sqrt{\lambda_j y})^2 \psi_{s-\ell}(\sqrt{\lambda_j y})^2 \leq M_{s-\ell, 1} \| u \|^2_{H^\sigma_{\ell+1}}. \] (A.9)

In view of (A.5), we see that (A.7) holds also in this case. By repeating the argument we used for \( \frac{1}{2} \leq s < 1 \) one plainly conclude the proof also in this case.

It remains to discuss the case \( \lfloor s \rfloor + \frac{1}{2} \leq s < \lfloor s \rfloor \) and \( k = 2 \lfloor s \rfloor + 1 = \lfloor 2s \rfloor \). We differentiate formula (A.4) for \( m = \lfloor s \rfloor \). To compute \( \partial_y P_{s-\ell}[L^s u](y) \), we use (A.5) for
\( \ell = 1, \ldots, \lfloor s \rfloor - 1 \) and \( i \) in Lemma A.4 for the last \( \ell \). It gives

\[
\partial_y^{2 \lfloor s \rfloor + 1} P_s[u](y) = - \sum_{\ell=1}^{\lfloor s \rfloor} a_{s,\ell} \cdot (yP_{s-\ell}[L^{\lfloor s \rfloor}u](y)) \\
- a_s \cdot (y^{2(\lfloor s \rfloor - 1)} P_{s-\lfloor s \rfloor} [L^s u](y))
\]

where the coefficients \( a_{s,\ell}, a_s \in \mathbb{R} \) depend only on \( s \) and \( \ell \). One can easily adapt the arguments we used for (A.9), (A.8). In this way one proves 

\[ i \] if \( \lfloor s \rfloor + \frac{1}{2} \leq s < \lceil s \rceil \), and

\[ ii \] if \( \lfloor s \rfloor + \frac{1}{2} < s < \lceil s \rceil \).

\( \square \)

**Theorem A.6** Let \( s > 1, \sigma \in \mathbb{R}, u \in \mathcal{H}_L^\sigma \). Then, for any \( k = 1, \ldots, \lfloor s \rfloor \) we have

\[
P_s[u](y) = \frac{1}{\Gamma(s)} \sum_{m=1}^{k} \Gamma(s-m) \frac{2^{2m} m!}{2m!} \cdot L^m u \cdot y^{2m} + o(y^{2k}) \quad \text{as } y \to 0
\]

with convergence in \( \mathcal{H}_L^\sigma - 2k \).

**Proof.** Take an integer \( k = 1, \ldots, \lfloor s \rfloor \). By \( ii \) in Theorem A.5 we have that \( P_s[u] \in C^k(\mathbb{R} \to \mathcal{H}_L^\sigma - 2k) \). Further, for any \( m = 1, \ldots, k \), Lemma A.4 gives

\[
\partial_y^{2m} P_s[u](0) = \frac{1}{B(s, \frac{1}{2})} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell B(s-\ell, \frac{1}{2}) \cdot L^m u
\]

and \( \partial_y^{2m-1} P_s[u](0) = 0 \). Then (A.11) follows via Taylor expansion formula, thanks to Lemma A.7 below. \( \square \)

**Lemma A.7** Let \( m \leq \lfloor s \rfloor \) be a positive integer. Then

\[
\kappa_{s,m} := \frac{1}{B(s, \frac{1}{2})} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell B(s-\ell, \frac{1}{2}) = (-1)^m \frac{\Gamma(s-m)}{\Gamma(s)} \frac{1}{2^{2m} m!} (2m)!.
\]

**Proof.** We compute

\[
\sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell B(s-\ell, \frac{1}{2}) = \int_{0}^{1} x^{-\frac{1}{2}} (1-x)^{s-m-1} \left( \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell (1-x)^{m-\ell} \right) dx
\]

\[
= (-1)^m \int_{0}^{1} x^{m-\frac{1}{2}} (1-x)^{s-m-1} dx = (-1)^m B(s-m, m + \frac{1}{2}).
\]

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Recalling the Legendre duplication formula, we infer that
\[
\kappa_{s,m} = (-1)^{m} \frac{B(s - m, m + \frac{1}{2})}{B(s, \frac{1}{2})} = (-1)^{m} \frac{\Gamma(s - m) \Gamma(m + \frac{1}{2})}{\Gamma(s) \sqrt{\pi}} = (-1)^{m} \frac{\Gamma(s - m) \Gamma(2m)}{\Gamma(s) \Gamma(m) \Gamma(m + \frac{1}{2})} \sqrt{\pi} = (-1)^{m} \frac{\Gamma(s - m) \Gamma(s)}{2^{2m} m!} (2m)! ,
\]
which completes the proof. \(\square\)

**Corollary A.8** Let \(s > 1, u \in \mathcal{H}_C^s\). Then for any integer \(m = 1, \ldots, \lfloor s \rfloor\) we have that
\[
\mathbb{L}_b^m P_s[u](y) = \frac{d_s}{d_{s-m}} P_{s-m}[\mathcal{L}^m u](y), \quad y \in \mathbb{R}.
\]

\[
\lim_{y \to 0} y^{-1} \partial_y^{2m-1} P_s[u] = \lim_{y \to 0} \partial_y^{2m} P_s[u] = \kappa_{s,m} \mathcal{L}^m u
\]
where \(\kappa_{s,m}\) is the constant in Lemma A.7. The limits are taken in the \(\mathcal{H}_C^{s-2m}\) topology.

**Proof.** The first equality follows from formulae (3.2) and (3.3):
\[
\mathbb{L}_b^m P_s[u](y) = \sum_{j=1}^{\infty} u_j (\mathbb{D}_b + \lambda_j)^m \psi_{s,\lambda_j}(y) u_j \varphi_j = \sum_{j=1}^{\infty} \lambda_j^m (\mathbb{D}_b + 1)^m \psi_s(\sqrt{\lambda_j} y) u_j \varphi_j
\]
\[
= \frac{d}{d_{s-m}} \sum_{j=1}^{\infty} \psi_{s-m,\lambda_j}(y) \lambda_j^m u_j \varphi_j.
\]
To conclude the proof, use ii) in Lemma A.4 and then iii) in Lemma A.1. \(\square\)

Our last result in this section involves the Hölder-type spaces \(\tilde{C}^\alpha\) in (1.12).

**Theorem A.9** Let \(s > 0\) non-integer, \(\sigma \in \mathbb{R}, u \in \mathcal{H}_C^s, \alpha \in (0, 2s]\). Then
\[
P_s[u] \in \tilde{C}^\alpha(\mathbb{R} \to \mathcal{H}_C^{s-\alpha}), \quad \|P_s[u]\|_{\tilde{C}^\alpha} \leq c\|u\|_{\mathcal{H}_C^s}.
\]  
(A.12)

**Proof.** Thanks to ii) in Theorem A.5, we only have to investigate the Hölderianity of \(\partial_y^{\lceil \alpha \rceil} P_s[u]\) if \(\alpha > \lceil \alpha \rceil\), and the Lipschitz properties of \(\partial_y^{\alpha-1} P_s[u]\) if \(\alpha\) is integer.
Theorem 3.3 already gives $\psi_s \in \tilde{C}^{2s}(\mathbb{R})$. Since $\psi_s$ decays exponentially at infinity together with its derivatives of any order, we infer that $\psi_s \in \tilde{C}^{\alpha}(\mathbb{R})$ for any $\alpha \in (0, 2s]$. Since trivially $\partial_y^k \psi_{s, \lambda}(y) = \lambda^{\frac{k}{2}} (\partial_y^k \psi_s)(\sqrt{\lambda} y)$ for any integer $k$ and any $\lambda > 0$, then $\|\psi_{s, \lambda}\|_{\tilde{C}^{\alpha}} = \lambda^{\frac{\alpha}{2}} \|\psi_s\|_{\tilde{C}^{\alpha}}$ for any $\alpha \in (0, 2s]$.

Take arbitrary points $y_1, y_2 \in \mathbb{R}$. Without loss of generality, we can assume that $y_1, y_2 \geq 0$. If $\alpha$ is not an integer, then

$$\|\partial_y^\lfloor \alpha \rfloor \mathcal{P}_s[u](y_1) - \partial_y^\lfloor \alpha \rfloor \mathcal{P}_s[u](y_2)\|^2_{\mathcal{H}_s^\sigma - \alpha} \leq \|\psi_s\|_{\tilde{C}^{\alpha}}^2 \sum_{j=1}^{\infty} \lambda_j^{\sigma - \alpha} u_j^2 |y_1 - y_2|^{2(\alpha - \lfloor \alpha \rfloor)}.$$

If $\alpha$ is integer, with a similar computation we get

$$\|\partial_y^{\alpha - 1} \mathcal{P}_s[u](y_1) - \partial_y^{\alpha - 1} \mathcal{P}_s[u](y_2)\|^2_{\mathcal{H}_s^\sigma - \alpha} \leq c \sum_{j=1}^{\infty} \lambda_j^{\sigma} u_j^2 |y_1 - y_2|^2 = c \|u\|^2_{\mathcal{H}_s^\sigma} |y_1 - y_2|^2.$$

In both cases, this concludes the proof. \(\square\)

### A.2 Isometric properties

From Theorem 1.1 we already know that the linear transform $u \mapsto \mathcal{P}_s[u]$ is, up to a constant, an isometry $\mathcal{H}_s^\sigma \rightarrow H^{[s], \sigma}_{b, \mathcal{H}}(\mathbb{R} \rightarrow \mathcal{H})$ for $b := 1 - 2(s - \lfloor s \rfloor)$. In this section we point out more isometric properties of $\mathcal{P}_s$. We stress the fact that $s > 0$ might be an integer number.

**Theorem A.10** Let $s > 0$, $b \in (-1, 1)$ and $\sigma \in \mathbb{R}$. Up to a constant (not depending on $\sigma$), the operator $\mathcal{P}_s$ is an isometry $\mathcal{H}_s^\sigma \rightarrow L^{2b}_{e, e}(\mathbb{R} \rightarrow \mathcal{H})$. More precisely,

$$\|\mathcal{P}_s[u]\|_{L^{2b}_{e, e}(\mathbb{R} \rightarrow \mathcal{H}_s^\sigma)} = \|\psi_s\|_{L^{2b}(\mathbb{R})} \|u\|_{\mathcal{H}_s^\sigma} \text{ for any } u \in \mathcal{H}_s^\sigma. \quad (A.13)$$
Proof. For \( u \in \mathcal{H}_L^s \) we compute
\[
\int_{-\infty}^{+\infty} |y|^b \|P_s[u](y)\|_{\mathcal{H}_L^{s+\frac{b}{2}}}^2 dy = \sum_{j=1}^{+\infty} \lambda_j^{s+\frac{b}{2}} u_j^2 \int_{-\infty}^{+\infty} |y|^b |\psi_s(\sqrt{\lambda_j} y)|^2 dy
\]
\[
= \left( \int_{-\infty}^{+\infty} |y|^b |\psi_s(y)|^2 dy \right) \sum_{j=1}^{+\infty} \lambda_j^{s} u_j^2 = \left( \int_{-\infty}^{+\infty} |y|^b |\psi_s(y)|^2 dy \right) \|u\|_{\mathcal{H}_L^s}^2,
\]
and the Lemma is proved. \( \square \)

Let \( \alpha > 0 \). We recall the definition of the Sobolev–Slobodetskii spaces and corresponding seminorms
\[
\mathcal{H}^\alpha(\mathbb{R}) = \left\{ \psi \in L^2(\mathbb{R}) \mid \|\psi\|_{\mathcal{H}^\alpha} < \infty \right\}
\]
\[
\|\psi\|_{\mathcal{H}^\alpha}^2 = \int_{-\infty}^{+\infty} |(-\partial^2_{yy})^\frac{\alpha}{2} \psi(y)|^2 dy = \int_{-\infty}^{+\infty} |\xi|^{2\alpha} |\hat{\psi}(\xi)|^2 d\xi,
\]
where \( \hat{\psi} \) stands for the unitary Fourier transform of \( \psi \in L^2(\mathbb{R}) \), namely,
\[
\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} \psi(y) dy.
\]

We first compute the Fourier transform of the function \( \psi_s \) in (1.11).

**Proposition A.11** Let \( s > 0 \) (possibly integer). Then
\[
\hat{\psi}_s(\xi) = \frac{\sqrt{2}\Gamma(s + \frac{1}{2})}{\Gamma(s)} (1 + \xi^2)^{-\frac{s}{4} + \frac{1}{2}}.
\]
In particular, \( \psi_s \in \mathcal{H}^\alpha(\mathbb{R}) \) if and only if \( \alpha < 2s + \frac{1}{2} \), and in this case
\[
\|\psi_s\|_{\mathcal{H}^\alpha}^2 = \frac{\Gamma(s + \frac{1}{2})^2}{s\Gamma(2s)\Gamma(s)^2} \Gamma(\alpha + \frac{1}{2}) \Gamma(2s - \alpha + \frac{1}{2}). \tag{A.14}
\]

**Proof.** It is well known, see for instance [8, Lemma 4.2] for a simple proof, that
\[
(1 + |\cdot|^2)^{-\frac{s+1}{2}}(y) = \frac{\Gamma(s)}{\sqrt{2\Gamma(s + \frac{1}{2})}} \psi_s(y).
\]
To conclude, use the symmetry of \( \psi_s \) and make direct computations. \( \square \)
For $\alpha > 0$ we introduce a Sobolev-type space of curves $\mathbb{R} \to \mathcal{H}$ and corresponding seminorm as follows:

$$H^\alpha(\mathbb{R} \to \mathcal{H}) = \{ U \in L^2(\mathbb{R} \to \mathcal{H}) \mid \|U\|_{H^\alpha}^2 := \sum_{j=1}^{+\infty} \int_{-\infty}^{+\infty} |\xi|^{2\alpha} |\widehat{U}_j(\xi)|^2 d\xi < \infty \},$$

where the Fourier transform of a function $U = \sum_j U_j \varphi_j \in L^2(\mathbb{R} \to \mathcal{H})$ is defined via the Fourier transform of its coordinates, that is,

$$\widehat{U}(\xi) = \sum_{j=1}^{\infty} \widehat{U}_j(\xi) \varphi_j.$$

It is evident that $H^\alpha(\mathbb{R} \to \mathcal{H})$ is a Hilbert space with norm

$$\|U\|_{H^\alpha}^2 = \|U\|_{L^2}^2 + \sum_{j=1}^{+\infty} \int_{-\infty}^{+\infty} (|\xi|^{2\alpha} + 1) |\widehat{U}_j(\xi)|^2 d\xi.$$

**Theorem A.12** Let $s > 0, \sigma \in \mathbb{R}$, $\alpha \in (-\frac{1}{2}, 2s)$. Then $\mathcal{P}_s$ is a continuous transform $\mathcal{P}_s : \mathcal{H}_s^\sigma \to H^{\alpha+\frac{1}{2}}(\mathbb{R} \to \mathcal{H}_s^{\sigma-\alpha})$. Moreover,

$$\|\mathcal{P}_s[u]\|_{H^{\alpha+\frac{1}{2}}(\mathbb{R} \to \mathcal{H}_s^{\sigma-\alpha})}^2 = \frac{\Gamma(s + \frac{1}{2})^2 \Gamma(\alpha + 1) \Gamma(2s - \alpha)}{\Gamma(s)^2 s \Gamma(2s)} \|u\|_{\mathcal{H}_s^\sigma}^2. \quad (A.15)$$

**Proof.** Thanks to (A.13) we already know that

$$\|\mathcal{P}_s[u]\|_{L^2(\mathbb{R} \to \mathcal{H}_s^{\sigma-\alpha})} \leq \|\psi_s\|_{L^2(\mathbb{R})} \|u\|_{\mathcal{H}_s^{\sigma-\alpha-\frac{1}{2}}} \leq \lambda_1^{-\alpha-\frac{1}{2}} \|\psi_s\|_{L^2(\mathbb{R})} \|u\|_{\mathcal{H}_s^\sigma}^2$$

for any $u \in \mathcal{H}_s^\sigma$, which gives the continuity of $\mathcal{P}_s : \mathcal{H}_s^\sigma \to L^2(\mathbb{R} \to \mathcal{H}_s^{\sigma-\alpha})$, as $\lambda_1 > 0$.

Next, take $u = \sum_j u_j \varphi_j \in \mathcal{H}_s^\sigma$. By the rescaling properties of the Fourier transform we have

$$\widehat{\mathcal{P}_s[u]}(\xi) = \sum_{j=1}^{\infty} \psi_{s, \lambda_j}(\xi) u_j \varphi_j = \sum_{j=1}^{\infty} \lambda_j^{-\frac{1}{2}} \psi_{s}(\lambda_j^{-\frac{1}{2}} \xi) u_j \varphi_j.$$

This readily gives
\[ \| \mathcal{P}_s[u] \|_{H^{s+\frac{1}{2}}(\mathbb{R} \to H^{s-a})}^2 = \sum_{j=1}^{\infty} \lambda_j^{s-a-1} u_j^2 \int_{-\infty}^{+\infty} |\xi|^{2a+1} \left| \hat{\psi}_s(\lambda_j^{-\frac{1}{2}} \xi) \right|^2 d\xi \]
\[ = \left( \int_{-\infty}^{+\infty} |\xi|^{2a+1} \left| \hat{\psi}_s(\xi) \right|^2 d\xi \right) \sum_{j=1}^{\infty} \lambda_j^s u_j^2, \]

which proves (A.15). This concludes the proof by Proposition A.11 and Lemma A.1. \[ \square \]

We conclude by stating the next immediate consequence of Theorems A.10 and A.12, which is related to some results in [18].

**Corollary A.13** Let \( s > 0 \). For any \( u \in \mathcal{H}_L^s \) it holds that

\[ \| \mathcal{P}_s[u] \|_{L^2(\mathbb{R} \to \mathcal{H}_L^{s+\frac{1}{2}})}^2 = \frac{\sqrt{\pi} \Gamma\left(2s + \frac{1}{2}\right) \Gamma\left(s + \frac{1}{2}\right)^2}{s \Gamma(2s) \Gamma(s)^2} \| u \|_{\mathcal{H}_L^s}^2 \]
\[ \| \mathcal{P}_s[u] \|_{H^{s+\frac{1}{2}}(\mathbb{R} \to \mathcal{H})}^2 = \frac{\Gamma\left(s + \frac{1}{2}\right)^2}{\Gamma(2s)} \| u \|_{\mathcal{H}_L^s}^2. \]

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