Fast Track Communications

Creation and revival of ring dark solitons in an annular Bose–Einstein condensate

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Abstract

We propose a protocol for the simultaneous controlled creation of multiple concentric ring dark solitons in a toroidally trapped flat Bose–Einstein condensate. The decay of these solitons into a vortex–antivortex necklace shows revivals of the soliton structure, but eventually becomes an example of quantum turbulence.

Keywords: ring dark soliton, ring trap, quantum turbulence, snake instability

The nonlinear nature of a Bose–Einstein condensate (BEC) makes it possible to support dark solitons [1–4], which display a remarkable form-stability in one dimension, e.g. they propagate without dispersion [5]. This stability is possible because the nonlinearity of the system has an opposite and cancelling effect to dispersion, but in higher dimensions, in general, the stability can be lost when the dark solitons decay into vortex–antivortex pairs via the snake instability [6–10].

Dark solitons can be experimentally prepared by creating a phase step in the condensate and letting the density respond by generating the dark soliton (phase imprinting) [11–13], but it is also possible to consider the complementary process of imposing a density profile and letting the phase respond by producing a step (density imprinting). In [14, 15], density imprinting is achieved by letting two BECs collide and produce an interference pattern, which then evolves into dark (or grey) solitons.

In this communication, we consider the challenging case of ring dark solitons (RDSs) [16, 17], which have not yet been observed with cold atoms. Compared to planar solitons with tanh-shaped amplitude, analytic descriptions for RDSs are not easy to obtain [18]. Instead of considering the interference of two condensates [19], we propose a density imprinting protocol involving a time-dependent double-well potential to make a single condensate produce fringes, which are let to evolve into RDSs. We show how the fringes could arise from self-interference, which underlines the long-range coherence of a BEC, and is an example of nonlinear matter-wave interference [20]. In [21], RDSs were defined by nodes of numerically found radial solutions, resembling the regular Bessel functions in the noninteracting limit. This approach is similar to our case, where the RDSs also evolve from nodes in the wavefunction.

Our idea for the protocol arises from the adiabatic passage by light-induced potentials (APLIP) method for molecular bond extension [22–24]. The method relies on three states with spatially dependent potentials, which are coupled by laser pulses that are applied in counterintuitive order. It is similar to the widely used method of stimulated Raman adiabatic passage [25], as used in [26, 27], for example, but with important differences: e.g. it can be reduced to a single-state process [28] in two dimensions. The success of APLIP in the noninteracting case makes it plausible to extend similar ideas also to the weakly interacting BEC.

Focussing on the case of cylindrical symmetry as expected for RDSs, we present an analytical expression for the time-dependent double-well potential that produces fringes, which are let to evolve into RDSs. We show how the fringes could arise from self-interference, which underlines the long-range coherence of a BEC, and is an example of nonlinear matter-wave interference [20]. In [21], RDSs were defined by nodes of numerically found radial solutions, resembling the regular Bessel functions in the noninteracting limit. This approach is similar to our case, where the RDSs also evolve from nodes in the wavefunction.

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Focussing on the case of cylindrical symmetry as expected for RDSs, we present an analytical expression for the time-dependent double-well potential that produces from one to four concentric RDSs. We note that similar potentials would work for the creation of planar solitons as well [28]. The stability of the RDS is discussed, and we show that the...
original RDS(s) decay into vortex–antivortex necklaces when the snake instability is induced. Remarkably, even in the case of a many-RDS system, we observe revivals of the RDSs before the ultimate (and spontaneous) onset of quantum turbulence. In that respect, our protocol could also serve as a method to produce a turbulent state. We examine the presence of turbulence by two signatures: power-law tails in the superfluid velocity statistics and the Kolmogorov scaling law of the kinetic energy.

To begin, let us consider a scalar order parameter, \( \psi \), representing the macroscopic wavefunction of a BEC trapped in a potential given by \( V_{\text{trap}} \), which is a solution to the Gross–Pitaevskii equation:

\[
iv\psi = -\nabla^2 \psi + V_{\text{trap}} \psi + C_{2D} |\psi|^2 \psi.
\]

(1)

Here we have assumed a two-dimensional condensate whereby the \( z \)-direction is tightly trapped to the corresponding harmonic oscillator ground state (\( \omega_z \gg \omega_x = \omega_y \approx \omega_{\text{osc}} \)) and has been projected onto the \( xy \)-plane. Then \( C_{2D} = 4 \sqrt{\pi} Na/\omega_{\text{osc}}^2 \), where \( N, a \) and \( \omega_{\text{osc}} \) are the number of atoms in the cloud, the \( s \)-wave scattering length of the atoms, and the characteristic trap length in the \( z \)-direction respectively. We have obtained dimensionless quantities by measuring time, length and energy in terms of \( \omega^{-1}, a_{\text{osc}} \equiv a_{\text{osc}} = \sqrt{\hbar/(2m\omega_z)} \) and \( h \omega_z \) respectively, where \( \omega_z \) is the angular frequency of the trap in the \( z \)-direction. This basis is equivalent to setting \( \omega_z = \hbar = 2m = 1 \).

By choosing \( V_{\text{trap}} \) appropriately, it is possible to control the quantum dynamics of the condensate. The main result of this communication is a simple analytic time dependence for \( V_{\text{trap}} \) that generates a single or multiple concentric RDS(s) in an annular trap. We will not perform an exhaustive mapping of the large parameter space, but the process can be extended to an even higher number of solitons.

To generate multiple concentric RDSs, \( V_{\text{trap}} \) takes the form of a combination of two simple harmonic potentials with time-dependent energy minima and radii, i.e. we set \( V_{\text{trap}} = \tilde{V} \), where

\[
\tilde{V}(r, t) = \begin{cases} 
\tilde{V}_1(r, t) = \frac{\omega_z^2}{4} (r - r_i(t))^2 + \Delta_1(t) & (r \leq r_e) \\
\tilde{V}_2(r, t) = \frac{\omega_z^2}{4} (r - r_o(t))^2 + \Delta_2(t) & (r > r_e),
\end{cases}
\]

(2)

where (\( i = 1, 2 \)):

\[
\begin{align*}
 r_i(t) &= \alpha_i - \beta_i \text{sech} \left[ \gamma \left( t - \frac{t_1 + t_2}{2} \right) \right], \\
\Delta_i(t) &= -\Omega e^{-\left[ \frac{\gamma^2 (t - \frac{t_1 + t_2}{2})^2}{\gamma^2 \tau^2} \right]} + \Delta_i,
\end{align*}
\]

(3) \hspace{1cm} (4)

and where \( r_e \) is the radius of the point of intersection of \( \tilde{V}_{1,2} \). As the initial state we choose the ground state of the potential \( \tilde{V}_1 \) at \( t = 0 \); for the parameters we use \( \omega = 20.0, \Delta_1 = 0.0, t_1 = 15.0, t_2 = 10.0, \alpha_1 = 1.5, \alpha_2 = 4.5, \beta_1 = -\beta_2 = 1.0, \) and \( \gamma = 0.30 \). Throughout this communication we use \( C_{2D} = 50.0 \) and \( \Omega = 10 \) unless stated otherwise. The remaining parameter \( \Delta_2 \) determines the number of RDSs produced, and it is separately specified for each case.

The time evolution given by equations (1) and (2) is shown in figures 1 and 2. In figure 1, we produce a single RDS, which remains stable throughout the process because we use a sufficiently low value of \( C_{2D} \) to suppress the decay into vortex–antivortex pairs via the snake instability [10]. As also shown...
in [10], breaking the rotation-invariance of the potentials can induce the snake instability. With various choices for \(\Delta_2\), we obtain between one and four RDSs (see figure 2).

It is worth mentioning that we could equally well produce the soliton structure so that it occupies the inner potential well at the end of the process [23, 28]. The population of the inner well is approximatively controlled by the parameter \(\Omega\) alone, while the population of (together with the number of RDSs in) the outer well is approximatively determined by \(\Delta_2\) alone. This property is the reason we have chosen \(\Omega = 10\) for definiteness, as we want the inner well to be empty after the process. To find the initial state self-consistently, we have performed a relaxation by propagating the GPE (1) in imaginary time until the density has converged to a steady state. This process is not done at the iterations of the time evolution.

To qualitatively see why \(\Delta_2\) controls the outer well, we note that the process in general relies on the adiabatic passage of the initial state with possible diabatic jumps, but the visible changes are due to the adiabatic state itself changing (in the diabatic basis). Our explanation for the mechanism is that once the condensate has tunneled into the outer well, it undergoes reflection and self-interference; subsequently, the RDSs evolve from the fringes. This process can be modelled either by the method of images [29] or simply by considering the de Broglie wavelength \(\lambda = h/mv\), where \(v\) is the relative centre of mass speed between the (point-like) condensate and its reflection. Both ways give the approximate fringe spacing of \(\Delta r \approx \pi/k\), where \(k\) is the momentum imparted by the outer well by being lower in energy compared to the inner well. For \(k = \sqrt{|\Delta 2|^2 + \Delta 2^2} = \pm n\), where \(n \in \mathbb{N}^+\), we therefore obtain that the total number of fringes and therefore RDSs in the outer well is \(4n/\pi \approx 1.27n\), which rounding down to the nearest integer (for small \(n\)) is simply \(n\), agreeing with the choices of \(\Delta_2\) in figure 2. It should be noted that the validity of the assumption \(k = \sqrt{|\Delta 2|^2 + \Delta 2^2}\) depends strongly on the value of \(\Omega\) as well, and in general the overall outcome is a function of both of the parameters \(\Omega\) and \(\Delta_2\) for a given \(C_{2D}\).

The time evolution is quite sensitive to the form of the potential around \(r_n\), with the height of the barrier between the two wells playing a crucial role, but the sharp cusp arising from mathematical convenience in the potential (2) is not a physically necessary feature. It is only important that this barrier be gradually lowered, preventing full sudden expansion of the outer well condensate onto the outer well to ensure smooth control. The process is robust against small variations of the parameters and for small nonlinearities (\(C_{2D} \lesssim 50\)), beyond which there can be high loss of adiabaticity (as we do not enforce a self-consistent relaxation at every time step), and even total loss of the ring soliton(s) via the snake instability. Also, the well minima must be separated enough to prevent inter-well oscillations, but not too much so to have reasonable tunnelling rates. For higher nonlinearities, one can use a Feshbach resonance [30] to tune \(C_{2D}\) to be small enough for the process described here to work without the possible snake instability, and then ramp it back up after the population transfer.

We note that with the recent advances in painting arbitrary dynamic potentials for BECs [31, 32], experimental realization of relevant potentials is feasible. Using the experimental parameters of [31] and \(^{87}\text{Rb}\), \(a_{\text{osc}} = 0.8 \, \mu\text{m}, C_{2D} \sim 200 - 2000\), and the duration of the passage e.g. in figure 1 corresponds to 40 ms. The toroidal BEC of [32] was calculated to have an actual radial thickness corresponding to the experimental conditions of less than 1 \(\mu\text{m}\), which is comparable to our case, and adiabatic passage of the toroidal BEC was demonstrated on the time scale of 200 ms. Furthermore, in [31], the improved spatial resolution of the experimental protocol was reported to be \(\sim 1.5 \mu\text{m}\), which was experimentally demonstrated to be small enough to create Josephson junctions on a toroidal BEC with significant and tunable tunnelling rates. We also note that in our protocol, the minimum of the inter-well separation is of the order of the spatial resolution of the experimental protocol, and that the sharp cusp at \(r = r_n\) is not physically relevant. The potential (2) is a model, and quite similar but smoother and experimentally easier potentials with the desired effect also exist. For example, we can model the barrier between the two wells by a Gaussian, e.g. (see figure 3(a))

\[
V_{\text{trap}}(r,t) = \frac{25}{4}(r - 4)^2 + \alpha(t) e^{-5(r-4)^2} - 10(r - 4) - 15,
\]

where \(\alpha(t) = 100H(-t) + H(t)H(6 - t)(100 - 75t/6 + H(12 - t)H(t - 6)(25 + 75(t - 6)/6) + 100H(t - 12),\) where \(H\) is the Heaviside step function. The potential 5 creates three RDSs in the outer well (see figure 3(b),(c)). Therefore, it should be feasible to print the barrier between the two harmonic wells using similar methods as in the case of building the Josephson tunnelling junctions, and obtain significant tunnelling into the outer well. We have numerically confirmed that the protocol in equation (2) works without loss of fidelity when \(C_{2D} = 500\) in radial coordinates where the RDS is stable [10], but to avoid the possible snake instability due to roughness of the potential, prior down-ramping of \(C_{2D}\) might be necessary.

Higher \(C_{2D}\), in general, will induce the snake instability of the RDS. To study the decay, we set nonadiabatically \(C_{2D} = 400\) at \(t = 25\) on an \(x, y\) grid. Interestingly, the RDSs (irrespective of their number) are revived after the first snake instability, after which the regime of quantum turbulence begins. This behaviour is similar to an experimentally observed oscillating soliton–vortex ring [15], but here the soliton is much longer giving rise to more vortex–antivortex pairs. Furthermore, we observe revivals for many-RDS systems as well. We conjecture that the revival mechanism is possible when the vortex necklace state is nearby in energy and the number of atoms to the original dark soliton state, but the details of this analysis will be reported later [29].

To gain insight on the decay and revival, we consider the normalized equal-time first-order correlation function (see, e.g. [33])

\[
g^{(1)}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{\langle \psi^\dagger(\mathbf{r}_1)\psi(\mathbf{r}_2)\langle\psi(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\rangle}}
\]

where \(\mathbf{r}_1 = (r, \theta)\) and \(\mathbf{r}_2 = (r, \theta)\), and for which \(0 \leq |\langle \psi^{(1)}(\mathbf{r}_1, \mathbf{r}_2)\rangle| \leq 1\). Let us further define a dimensionless quantity, g, by

\[
g = \frac{1}{N_r} \sum_{r = r_{\text{min}} - \epsilon}^{r_{\text{max}}} \langle |\langle \psi^{(1)}(\mathbf{r}_1, \mathbf{r}_2)\rangle| \rangle_{\theta}.
\]
where we take $r$ equally spaced with $N_r = 10$ steps over the range of $2\epsilon$ (chosen to cover the ring), and where $\langle \rangle_\theta$ denotes the average over $\theta$. Figure 4 shows the dependence of $g$ on time after $t = 25$. Local (the limit of the correlation function at infinity is not necessarily affected) phase coherence is gradually lost, but the vortex–antivortex necklace(s) recombine(s) back to produce RDS(s). As is evident in figure 4, a higher number of RDSs slows down the loss of coherence and delays the onset of the decay and subsequent revival, but the effect seems to saturate above 3RDS. Eventually, the (local) decoherence is enough for the onset of quantum turbulence, preventing orderly dynamics needed by further revivals of the RDS.

Quantum turbulence, a complex spaghetti of tangled vortex lines, can be characterized by power-law tails in the superfluid velocity probability distribution $P(v)$ (see [34, 35] and references therein), making a difference to classical turbulence with Gaussian distributions. This result has been
a Gaussian distribution, characteristic of classical turbulence. We calculate the velocity distributions \( v \propto \nabla S \), where \( S \) is the phase (see figure 5). We observe the emergence of power-law tails in the velocity statistics as the modified coherence function \( g \) becomes smaller, which signifies the onset of quantum turbulence, in accordance with the previous literature.

Furthermore, we calculate the kinetic energy density \( \epsilon(k) \), defined by \( E_{\text{kin}} = \int_0^\infty dk \, \epsilon(k) \), where \( E_{\text{kin}} \) is the total kinetic energy and \( k \) is the wavenumber. Using the Madelung transformation \( \psi = |\psi|e^{i\phi} \), we get

\[
E_{\text{kin}} = \int d\mathbf{r} |\nabla \psi(\mathbf{r})|^2 = \int d\mathbf{r} \left( \frac{|\psi|^2 \mathbf{v} - \mathbf{v} |\psi|^2}{4} + (\nabla |\psi|^2)^2 \right),
\]

where only the first term corresponds to particle currents. Dropping the second term and taking the Fourier transformation, we obtain

\[
\epsilon(k) = \frac{1}{4(2\pi)^2} \int_0^{2\pi} d\theta k \int_{\mathbb{R}^2} d\mathbf{r} e^{i\mathbf{r} \cdot \mathbf{k}} |\psi(\mathbf{r})|^2 |\mathbf{v}(\mathbf{r})|^2,
\]

where \( \mathbf{k} = k_x \mathbf{i} + k_y \mathbf{j} \) and \( \theta = \tan^{-1}(k_y/k_x) \). The Kolmogorov signature for turbulence [37] states that \( \epsilon(k) \sim k^{-5/3} \) over the inertial range of \( k \) (e.g. between the healing length and the Thomas–Fermi radius in a trapped condensate [38]). Strictly speaking, we use the ‘incompressible’ (i.e. \( \nabla \cdot (|\psi|^2 \mathbf{v}) = 0 \)) part of the kinetic energy \( \epsilon^{(k)}(k) \), which is related to the particle currents that arise from the vortices alone (e.g. no sound waves). The result of evaluating equation (9) at various times is shown in figure 6, and we can see that the spectrum agrees qualitatively with the Kolmogorov scaling in the inertial region, which is another signature for the presence of turbulence after the snake instability.

In summary, we have proposed a protocol for the creation of multiple concentric RDSs in a BEC. The RDSs evolve experimentally shown by using solid hydrogen tracers (much smaller than the average vortex separation) in superfluid helium [36], giving \( P(u_i) \propto v_i^{-3} \), where \( i \) refers to the velocity component.

We calculate the velocity distributions \( v \propto \nabla S \), where \( S \) is the phase (see figure 5). We observe the emergence of power-law tails in the velocity statistics as the modified coherence function \( g \) becomes smaller, which signifies the onset of quantum turbulence, in accordance with the previous literature.

Figure 5. Superfluid velocity statistics at three different times for the 1RDS case, showing the emergence of power-law tails characteristic of quantum turbulence. At around \( t = 27 \), the vortex–antivortex necklace recombines to reproduce the RDSs, which then later decays again. The solid line shows \( P(u_i) \propto v_i^{-3} \), and the dashed line shows a Gaussian distribution, characteristic of classical turbulence.

Figure 6. The incompressible energy spectrum \( \epsilon^{(k)}(k) \) at different times for the 1RDS case. The solid line is the Kolmogorov scaling law \( \sim k^{-5/3} \), and the dashed lines give the inertial range for \( k \), \( k_{\text{lower}} = 2\pi/R \), where \( R \) is the size of the condensate (left dashed line) and \( k_{\text{upper}} = 2\pi/\xi \), where \( \xi \) is the healing length (right dashed line). Note the logarithmic scale on both axes.

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References

[1] Kivshar Y S and Luther-Davies B 1998 Dark optical solitons: physics and applications Phys. Rep. 298 81–197
[2] Perlick C and Smith H 2008 Bose–Einstein Condensation in Dilute Gases 2nd edn (Cambridge: Cambridge University Press)
[3] Kevrekidis P G, Frantzeskakis D J and Carretero-González R (ed) 2008 Emergent Nonlinear Phenomena in Bose–Einstein Condensates (Berlin: Springer)
[4] Frantzeskakis D J 2010 Dark solitons in atomic Bose–Einstein condensates: from theory to experiments J. Phys. A: Math. Theor. 43 213001
[5] Drazin P G and Johnson R S 1989 Solitons: An Introduction 2nd edn (Cambridge: Cambridge University Press)
[6] Muryshkev A E, van Linden van den Heuvell H B and Shlyapnikov G V 1999 Stability of standing matter waves in a trap Phys. Rev. A 60 R2665–8
[7] Martikainen J-P, Suominen K-A, Santos L, Schulte T and Sanpera A 2001 Generation and evolution of vortex–antivortex pairs in Bose–Einstein condensates Phys. Rev. A 64 063602
