The Dehn invariants of the Bricard octahedra

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Abstract

We prove that the Dehn invariants of any Bricard octahedron remain constant during the flex and that the Strong Bellows Conjecture holds true for the Steffen flexible polyhedron.

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Key words: flexible polyhedron, Dehn invariant, scissors congruent, Napier’s analogies, spherical trigonometry.

1 Introduction

A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., if every face remains congruent to itself during the flex.

For the first time flexible sphere-homeomorphic polyhedra in Euclidean 3-space were constructed by R. Connelly in 1976 [5]. Since that time, many properties of flexible polyhedra were discovered, for example: the total mean curvature [1] and the oriented volume [15], [7], [16], [18], are known to be constant during the flex. Nevertheless, many interesting problems remain open. One of them is the so called Strong Bellows Conjecture which was posed in [6] and reads as follows: If an embedded polyhedron (i.e., self-intersection free polyhedron) $P_0$ is obtained from an embedded polyhedron $P_0$ by a continuous flex then $P_1$ and $P_0$ are scissors congruent, i.e., $P_1$ can be partitioned in a finite set of polyhedra $\{Q_j\}$, $j = 1, \ldots, n$, with the following property: for every $j = 1, \ldots, n$ there is an isometry $F_j : \mathbb{R}^3 \to \mathbb{R}^3$ such that the set $\{F_j(Q_j)\}$, is a partition of $P_0$.

Let us recall the following well-known

THEOREM 1. Given two embedded polyhedra $P_0$ and $P_1$ in $\mathbb{R}^3$ the following conditions are equivalent:

1. $P_0$ and $P_1$ are scissors congruent;
2. $\text{Vol} P_0 = \text{Vol} P_1$ and $\text{D}_f P_0 = \text{D}_f P_1$ for every $\mathbb{Q}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = 0$. Here $\text{Vol} P$ stands for the volume of $P$ and $\text{D}_f P = \sum |\ell| f(\alpha_\ell)$ stands for its Dehn invariant; $\alpha_\ell$ is the internal dihedral angle of $P$ at the edge $\ell$; $|\ell|$ is the length of $\ell$.

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The implication (1)⇒(2) was proved by M. Dehn [8]; the implication (2)⇒(1) was proved independently by J.P. Sydler [21] and B. Jessen [10]. We refer the reader to [3] for more detail.

Since we know that volume is constant during the flex [15], [7], [16], [18], Theorem 1 reduces the Strong Bellows Conjecture to the problem whether the Dehn invariants are constant. Note that the latter problem makes sense also for polyhedra with self-intersections. All we need is the notion of the dihedral angle for polyhedra with self-intersections.

**DEFINITION.** A dihedral angle at the edge ℓ of an oriented (not necessarily embedded) polyhedron P is a multi-valued real-analytic function α∗ = α∗ 0 + 2πm, m ∈ ℤ. Here 0 < α∗ 0 < 2π stands just for one of the values (or branches) of this multi-valued function and can be calculated as follows. Let x be an internal point of the edge ℓ, let Q1 and Q2 be the two faces of P that are adjacent to ℓ, let \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be unit normal vectors to the faces Q1 and Q2 respectively which set the orientation of P. Let \( B \) be a closed ball centered at x of a so small radius that \( B \) (a) contains no vertices of P, (b) has no common points with any edge of P other than ℓ, and (c) has no common points with any face of P other than Q1 or Q2. First suppose that \( \mathbf{n}_1 \neq \mathbf{n}_2 \). Rotate the semi-circle \( B \cap Q_1 \) around its diameter \( B \cap ℓ \) in the direction of the vector \( \mathbf{n}_1 \) until this semi-circle coincides with the semicircle \( B \cap Q_2 \) for the first time. During the process of rotation the points of the semi-circle \( B \cap Q_1 \) fill in a sector of \( B \). Denote this sector by S and put by definition \( \alpha^0 = \text{Vol } S/\text{Vol } B \). If \( \mathbf{n}_1 = \mathbf{n}_2 \) we put by definition \( \alpha^0 = 2\pi \).

Given an edge ℓ of a flexible polyhedron P(t) we choose an arbitrary univalent branch \( \alpha(t) \) of the multi-valued function \( \alpha^* = \alpha^* 0 + 2\pi m \) provided that \( \alpha(t) \) is continuous in t and use it in the calculations below.

The main result of this paper is that the Dehn invariants of any Bricard octahedron remain constant during the flex. Using this result we prove also that the Strong Bellows Conjecture holds true for the Steffen flexible polyhedron.

Our description of the Bricard octahedra of types 1–3 and of the Steffen polyhedron is very brief; it is aimed mainly at fixing notations and recalling the properties needed for our study. We refer the reader for details to [4], [11], [12], [14], and [20].

### 2 Bricard octahedra of type 1

Any Bricard octahedron of type 1 in \( \mathbb{R}^3 \) can be constructed in the following way. Consider a disk-homeomorphic piece-wise linear surface S in \( \mathbb{R}^3 \) composed of four triangles \( A_1 B_1 C_1, B_1 A_2 C_1, A_2 B_2 C_1, \) and \( B_2 A_1 C_1 \) such that \( |A_1 B_1| = |A_2 B_2| \) and \( |B_1 A_2| = |B_2 A_1| \) (see Fig. 1). It is known that such a spatial quadrilateral \( A_1 B_1 A_2 B_2 \) is symmetric with respect to a line L passing through the middle points of its diagonals \( A_1 A_2 \) and \( B_1 B_2 \) [11]. Glue together S and its symmetric image with respect to L (see Fig. 2). Denote by \( C_2 \) the symmetric image of \( C_1 \) under the symmetry with respect to L. The resulting polyhedral surface with self-intersections is flexible (because S is flexible) and is combinatorially equivalent to the surface of the regular octahedron. This is
known as the Bricard octahedron of type 1. Each of the spatial quadrilaterals $A_1B_1A_2B_2$, $A_1C_1A_2C_2$ or $B_1C_1B_2C_2$ is called its equator.

THEOREM 2. For every equator $E$ of any Bricard octahedron of type 1 and every $\mathbb{Q}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = 0$ yields

$$D_f(E) \overset{\text{def}}{=} \sum_{\ell \in E} |\ell| f(\alpha_\ell(t)) = 0.$$ 

Proof. Split $D_f(E)$ into 2 groups of the form $|\ell| f(\alpha_\ell(t)) + |\ell^*| f(\alpha_{\ell^*}(t))$, where $\ell$ and $\ell^*$ are opposite edges of the octahedron (i.e., such edges that the pair of the corresponding edges of the regular octahedron is symmetric with respect to the center of symmetry). Obviously, $|\ell| = |\ell^*|$. Moreover, $\alpha_\ell(t) + \alpha_{\ell^*}(t) = 2\pi m$ for some $m \in \mathbb{Z}$, since the dihedral angles attached to $\ell$ and $\ell^*$ are symmetric to each other with respect to the line $L$. Hence, $|\ell| f(\alpha_\ell(t)) + |\ell^*| f(\alpha_{\ell^*}(t)) = |\ell| f(2\pi m) = 0$ and, thus, $D_f(E) = 0$ for all $t$. \hfill $\square$

THEOREM 3. Any Dehn invariant of any Bricard octahedron of type 1 is constant during the flex; moreover, it equals zero.

Proof follows immediately from Theorem 2. \hfill $\square$

3 Bricard octahedra of type 2

Any Bricard octahedron of type 2 in $\mathbb{R}^3$ can be constructed in the following way. Consider a disk-homeomorphic piece-wise linear surface $R$ in $\mathbb{R}^3$ composed of four triangles $A_1B_1C_1$, $A_1B_2C_1$, $A_2B_2C_1$, and $B_2A_1C_1$ such that $|A_1B_2| = |B_2A_2|$ and $|A_1B_2| = |B_2A_1|$ (see Fig. 3). It is known that such a spatial quadrilateral $A_1B_1A_2B_2$ is symmetric with respect to a plane $P$ which dissects the dihedral angle between the half-planes $A_1B_1B_2$ and $A_2B_1B_2$. Glue together $R$ and its symmetric image with respect to $P$. The resulting polyhedral surface with self-intersections is flexible (because $R$ is flexible) and is combinatorially equivalent to the regular octahedron (see Fig. 4). This is known as the Bricard octahedron of type 2. Each of the spatial quadrilaterals $A_1B_1A_2B_2$, $A_1C_1A_2C_2$ or $B_1C_1B_2C_2$ is called its equator.
THEOREM 4. For every equator \( E \) of any Bricard octahedron of type 2 and every \( \mathbb{Q} \)-linear function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(\pi) = 0 \) yields \( D_f(E) = 0 \) for all \( t \).
Proof is similar to the proof of Theorem 2: symmetry of the Bricard octahedron of type 2 with respect to the plane \( P \) implies that symmetric edges have equal lengths and the sum of the dihedral angles attached is a multiple of \( \pi \). □

THEOREM 5. Any Dehn invariant of any Bricard octahedron of type 2 is constant during the flex; moreover, it equals zero.
Proof follows immediately from Theorem 4. □

4 Bricard octahedra of type 3

Any Bricard octahedron \( \mathcal{O} \) of type 3 in \( \mathbb{R}^3 \) can be constructed in the following way. Let \( K_C \) and \( K_B \) be two different circles in \( \mathbb{R}^2 \) with a common center. Let \( A_1B_1A_2B_2 \) be a convex quadrilateral with the sides tangent to \( K_C \) as it is shown in Fig. 5 (in fact it is not prohibited that three points, say \( A_1, B_1, \) and \( A_2 \), lie on a straight line; in this case \( B_1 \) is a tangent point). Let \( A_1C_1A_2C_2 \) be a quadrilateral with self-intersections such that every straight line containing a side of \( A_1C_1A_2C_2 \) is tangent to \( K_B \) as it is shown in Fig. 6. A Bricard octahedron \( \mathcal{O} \) of type 3 in a flat position is composed of the vertices \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) and of the edges of the quadrilaterals \( A_1B_1A_2B_2, A_1C_1A_2C_2, \) and \( B_1C_1B_2C_2 \) (see Fig. 7). The faces of \( \mathcal{O} \) are defined as the triangles \( \triangle A_iB_jC_k \) for all choices of the indices \( i, j, k \in \{1, 2\} \). It is known that the quadrilateral
Figure 5: Construction of a Bricard octahedron $\mathcal{O}$ of type 3. Step 1

$B_1C_1B_2C_2$ is circumscribed about a circle $K_A$ which shares the center with the circles $K_B$ and $K_C$ (see Fig. 7).

**Theorem 6.** For every equator $E$ of any Bricard octahedron $\mathcal{O}$ of type 3 and every $\mathbb{Q}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = 0$ yields $\mathcal{D}_f(E) = 0$ for all $t$.

**Proof.** Let $E = A_1B_1A_2B_2$. Let $\mathcal{B}$ be a closed ball centered at $A_1$ of a so small radius $r$ that $\mathcal{B}$ contains no vertices of octahedron $\mathcal{O}$ other than $A_1$. The intersection of $\mathcal{B}$ and $\mathcal{O}$ is a quadrilateral. Denote it by $Q(t)$ and denote its vertices as follows: $\tilde{B}_1 = A_1B_1 \cap \partial \mathcal{B}$, $\tilde{B}_2 = A_1B_2 \cap \partial \mathcal{B}$, $\tilde{C}_1 = A_1C_1 \cap \partial \mathcal{B}$, and $\tilde{C}_2 = A_1C_2 \cap \partial \mathcal{B}$. The length of a side of $Q(t)$ equals the corresponding angle of a face of $\mathcal{O}$ multiplied by the radius $r$ of $\mathcal{B}$, e.g., the length of the side $\tilde{B}_2\tilde{C}_1$ is equal to $\angle B_2A_1C_1$ multiplied by $r$. Thus, the length of any side of $Q(t)$ remains constant during the flex. On the other hand, it follows from Fig. 7 that $\angle B_2A_1C_1 = \angle B_1A_1C_2$ and $\angle B_2A_1C_2 = \angle B_1A_1C_1$. Hence, the opposite sides of $Q(t)$ have equal lengths.

If $\mathcal{O}$ is in the flat position shown in Fig. 7 then all the vertices of the spherical quadrilateral $Q(t)$ are located on a single great circle (see Fig. 8a). If $\mathcal{O}$ is in a non-flat position close to a flat position shown in Fig. 7 then, generally speaking, there are two possibilities for $Q(t)$: either $Q(t)$ is convex (see Fig. 8b) or $Q(t)$ has self-intersections (see Fig. 8c). However, $Q(t)$ can not be convex because in this case $\mathcal{O}$ is embedded (at least when $\mathcal{O}$ is close enough to the flat position shown in Fig. 7) while it is known that no embedded octahedron is flexible [13]. Taking into account that opposite sides of $Q(t)$ have equal lengths we conclude
Figure 6: Construction of a Bricard octahedron $O$ of type 3. Step 2

Figure 7: Construction of a Bricard octahedron $O$ of type 3. Step 3
that the sum of any two opposite angles of \( Q(t) \) equals \( 2\pi m \) for some \( m \in \mathbb{Z} \).

Hence, the sum of any two opposite dihedral angles of the solid angle of \( O \) with the vertex \( A_1 \) equals \( 2\pi m \) for some \( m \in \mathbb{Z} \). According to our agreement made in the Introduction, this means, for example, that if \( \alpha_{A_1 B_1}(t) \) and \( \alpha_{A_1 B_2}(t) \) are any branches of the dihedral angles of \( O \) at the edges \( A_1 B_1 \) and \( A_1 B_2 \), respectively, then there is \( n \in \mathbb{Z} \) such that for all \( t \)

\[
\alpha_{A_1 B_1}(t) + \alpha_{A_1 B_2}(t) = 2\pi n. \tag{1}
\]

Strictly speaking, the above arguments prove (1) only for the positions of \( O \) close enough to the flat position shown in Fig. 7. But, in fact, (1) holds true for all positions of \( O \) obtained from the flat position shown in Fig. 7 by a continuous flex. The reason is that we may assume that the coordinates of the vertices of \( O \) are analytic functions of the flexing parameter \( t \), see [9] for more detail. Then any branch of the dihedral angle is an analytic function of \( t \). As soon as we know that (1) holds true for all \( t \) corresponding to any position of \( O \) close enough to the flat position shown in Fig. 7, we conclude that (1) holds true for all \( t \) corresponding to any position of \( O \) obtained from the flat position shown in Fig. 7 by a continuous flex. But the configuration space \( (\text{i.e., the space of all positions}) \) of any Bricard octahedron of type 3 is known to be homeomorphic to a circle. Thus the relation between dihedral angles established for \( O \) close enough to the flat position shown in Fig. 7 holds true for all positions of \( O \).

Similar arguments show that, for any solid angle of \( O \) with the vertex \( B_1 \), \( A_2 \), or \( B_2 \), the sum of any two opposite dihedral angles equals \( 2\pi m \) for some \( m \in \mathbb{Z} \).

By definition put \( A_i B_j \cap K_C = c_{ij}; \ i, j = 1, 2 \). In other words, denote by \( c_{ij} \) the point where the segment \( A_i B_j \) touches the circle \( K_C \). Then

\[
D_f(E) = (|A_1 c_{12}|f(\alpha_{A_1 B_2}) + |A_1 c_{11}|f(\alpha_{A_1 B_1}))
+ (|B_1 c_{11}|f(\alpha_{A_1 B_1}) + |B_1 c_{21}|f(\alpha_{A_2 B_1}))
+ (|A_2 c_{21}|f(\alpha_{A_2 B_1}) + |A_2 c_{22}|f(\alpha_{A_2 B_2}))
+ (|B_2 c_{22}|f(\alpha_{A_2 B_2}) + |B_2 c_{12}|f(\alpha_{A_1 B_2}))
= |A_1 c_{12}|f(\alpha_{A_1 B_2} + \alpha_{A_1 B_1}) + |B_1 c_{11}|f(\alpha_{A_1 B_1} + \alpha_{A_2 B_1})
+ |A_2 c_{21}|f(\alpha_{A_2 B_2} + \alpha_{A_2 B_1}) + |B_2 c_{22}|f(\alpha_{A_2 B_2} + \alpha_{A_1 B_2}) = 0.
\]

Here we use the fact that the segments of the two lines passing through a point and tangent to a circle have equal lengths (e.g., \( |A_1 c_{12}| = |A_1 c_{11}| \)) and that the sum of the opposite angles of \( Q(t) \) equals \( 2\pi m \) for some \( m \in \mathbb{Z} \) (e.g., (2)).

Exactly the same arguments, we have used above to prove the formula \( D_f(E) = 0 \) for the equator \( E = A_1 B_1 A_2 B_2 \), can be used to prove it for the equator \( E = B_1 C_1 B_2 C_2 \). This part of the proof is left to the reader. On the contrary, in order to prove the formula \( D_f(E) = 0 \) for the equator \( E = A_1 C_2 A_2 C_1 \), we have to use additional arguments that are given below.

Let \( E = A_1 C_2 A_2 C_1 \). We need some relations for the dihedral angles at the edges of the equator \( E \). We already know that if \( \alpha = \alpha(t) \) is one of the univalent
Figure 8: Spherical quadrilateral $Q(t)$: (a) in a flat position; (b) as a convex polygon (impossible); (c) as a polygon a with self-intersection.

Branches of the angle $\alpha_{A_1B_2}(t)$ (i.e., $\alpha_{A_1B_2}(t) = \alpha(t)$) then $\alpha_{A_2B_2}(t) = 2\pi k - \alpha$, $\alpha_{A_2B_1}(t) = \alpha + 2\pi m$, and $\alpha_{B_2A_1}(t) = 2\pi n - \alpha$ for some $k, m, n \in \mathbb{Z}$. Similarly, we know that if $\beta = \alpha_{A_1C_1}(t)$ and $\gamma = \alpha_{C_1A_2}(t)$ then $\alpha_{A_1C_2}(t) + \beta = 2\pi p$ and $\alpha_{A_2C_1}(t) + \gamma = 2\pi q$ for some $p, q \in \mathbb{Z}$. Let’s prove that, for some $s \in \mathbb{Z}$, yields $\beta(t) + \gamma(t) = 2\pi s$.

Recall one of the four Napier’s analogies (known also as the Napier’s rules) from the spherical trigonometry [19]: Let a spherical triangle have sides $a$, $b$, and $c$ with $A$, $B$, and $C$ the corresponding opposite angles. Then

$$
\frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}} = \frac{\tan \frac{A - B}{2}}{\cot \frac{C}{2}}.
$$

(3)

Applying (3) to the spherical triangle $\tilde{C}_1 \tilde{B}_2 \tilde{C}_2$ shown in Fig. 8c we get

$$
\frac{\sin \frac{\angle B_2A_1C_2 - \angle B_2A_1C_1}{2}}{\sin \frac{\angle B_2A_1C_2 + \angle B_2A_1C_1}{2}} = \frac{\tan \frac{\beta}{2}}{\cot \frac{\alpha}{2}},
$$

(4)

where $r$ stands for the radius of the ball $B$ and $\angle XYZ$ stands for the value of the ’plane’ angle with the vertex $Y$ of the triangular face $XYZ$ of the Bricard octahedron $O$ shown in Fig. 7.

Similarly, apply (3) to a spherical triangle obtained by intersecting the solid angle of the octahedron $O$ with the vertex $A_2$ and the ball $B^*$ of the radius $r$ centered at $A_2$ (we assume that $B^*$ contains no vertices of $O$ other than $A_2$ and the balls $B$, $B^*$ have equal radii). More precisely, put $\tilde{B}_1 = A_2B_1 \cap \partial B^*$, $\tilde{C}_1 = A_2C_1 \cap \partial B^*$, and $\tilde{C}_2 = A_2C_2 \cap \partial B^*$. Applying (3) to the spherical triangle $\tilde{B}_1 \tilde{C}_1 \tilde{C}_2$ (see Fig. 9) we get

$$
\frac{\sin \frac{\angle B_1A_2C_2 - \angle B_1A_2C_1}{2}}{\sin \frac{\angle B_1A_2C_2 + \angle B_1A_2C_1}{2}} = \frac{\tan \left( \frac{\pi q - \gamma}{2} \right)}{\cot \left( \frac{\alpha}{2} + \pi m \right)}.
$$

(5)
the point where the line passing through the segment $A_1B_2$ implying that $K \in \Delta$ where Steffen (see [2] or [17]). Each of them belongs to the class $F_R$. Connelly (see [5] or [17]), by P. Deligne and N. Kuiper (see [17]), and by K.

There are several examples of embedded flexible polyhedra in $\Delta$. Now we can make the main computation for the equator $E = A_1C_2A_2C_1$:

\[
D_f(E) = (|A_1b_{11}|f(\alpha_{A_1C_1}) + |A_1b_{12}|f(\alpha_{A_1C_2}))
- (|C_1b_{11}|f(\alpha_{A_1C_1}) - |C_1b_{21}|f(\alpha_{A_2C_1}))
+ (|A_2b_{21}|f(\alpha_{A_2C_1}) + |A_2b_{22}|f(\alpha_{A_2C_2}))
- (|C_2b_{22}|f(\alpha_{A_2C_2}) - |C_2b_{12}|f(\alpha_{A_1C_2}))
= |A_1b_{11}|f(\alpha_{A_1C_1} + \alpha_{A_1C_2}) + |C_1b_{21}|f(\alpha_{A_2C_1} - \alpha_{A_1C_1})
+ |A_2b_{22}|f(\alpha_{A_2C_1} + \alpha_{A_2C_2}) + |C_2b_{12}|f(\alpha_{A_1C_2} - \alpha_{A_2C_2})
= |A_1b_{11}|f(2\pi p) + |C_1b_{21}|f(2\pi q - 2\pi s)
+ |A_2b_{22}|f(2\pi q) + |C_2b_{12}|f(2\pi p - 2\pi s) = 0.
\]

THEOREM 7. Any Dehn invariant of any Bricard octahedron of type 3 is constant during the flex; moreover, it equals zero.

Proof follows immediately from Theorem 6. □

REMARK. Bricard octahedra of type 3 are unexpectedly symmetric: we have seen above that there are linear relations between edge lengths, plane and dihedral angles, and trigonometric relations between dihedral angles. Let us now mention one more relation that did not appear above: $\angle A_1C_1B_2 + \angle A_2C_1B_1 = \angle A_1C_1B_1 + \angle A_2C_1B_2 = \pi$ (see Fig. 7). A proof is left to the reader.

5 Steffen polyhedron

There are several examples of embedded flexible polyhedra in $\mathbb{R}^3$ proposed by R. Connelly (see [5] or [17]), by P. Deligne and N. Kuiper (see [17]), and by K. Steffen (see [2] or [17]). Each of them belongs to the class $\mathcal{F}_n$ for some $n \geq 0$, where $\mathcal{F}_n$ is defined as follows:

Figure 9: Spherical quadrilateral $B_1^*C_1^*B_2^*C_2^*$

Using Fig. 6 we easily conclude that $\angle B_2A_1C_2 = \angle B_1A_2C_2$ and $\angle B_2A_1C_1 = \angle B_1A_2C_1$. Hence, the left-hand sides of (4) and (5) are equal to each other implying that $\beta + \gamma = 2\pi s$ for some $s \in \mathbb{Z}$. Thus, the formula (2) is proved.
Figure 10: Constructing the Steffen polyhedron

(i) $\mathcal{F}_0$ consists of all convex polyhedra in $\mathbb{R}^3$ and all Bricard octahedra;
(ii) a polyhedron $P$ belongs to $\mathcal{F}_n$, $n \geq 1$, if and only if one of the following holds true: (ii$_1$) $P$ is obtained from $P_1, P_2 \in \mathcal{F}_k$, $0 \leq k \leq n - 1$, by gluing them together along congruent faces $Q_1 \subset P_1$ and $Q_2 \subset P_2$; (ii$_2$) $P$ is obtained from $P_1 \in \mathcal{F}_{n-1}$ by gluing together two its faces $Q_1, Q_2 \subset P_1$ provided that they coincide in $\mathbb{R}^3$; (ii$_3$) $P$ is obtained from $P_1 \in \mathcal{F}_{n-1}$ by a subdivision of its faces.

For example, the Steffen polyhedron (which has only 9 vertices and 14 faces and, hypothetically, is an embedded flexible polyhedron with the least possible number of vertices) can be constructed from a tetrahedron $T$ and two copies $O$ and $O^\dagger$ of the Bricard octahedron of type 1 in the following way (see Fig. 10).

The tetrahedron $T = DEFL$ has the following edge lengths: $|DE| = |EF| = |FL| = |LD| = 12$ and $|DF| = 17$. It does not change its spatial shape during the flex of the Steffen polyhedron. The edge $EL$ is not shown in Fig. 10 because it does not appear in the Steffen polyhedron.

The Bricard octahedron of type 1 $O = A_1A_2B_1B_2C_1C_2$ has the following edge lengths: $|A_1C_1| = |C_2B_2| = |A_2C_2| = |C_2B_1| = 12$, $|B_1C_1| = |C_1A_2| = |A_1C_2| = |C_2B_2| = 10$, $|A_1B_1| = |A_2B_2| = 5$, and $|A_1B_2| = |A_2B_1| = 11$. The edge $A_1B_2$ is not shown in Fig. 10 because it does not appear in the Steffen polyhedron.

The Bricard octahedron of type 1 $O^\dagger = A_1^\dagger A_2^\dagger B_1^\dagger B_2^\dagger C_1^\dagger C_2^\dagger$ is obtained from $O$ by an orientation-preserving isometry of $\mathbb{R}^3$. The edge $A_1^\dagger B_2^\dagger$ is not shown in Fig. 10 because it does not appear in the Steffen polyhedron.

Glue $T$ and $O$ along the triangles $\triangle DEL$ and $\triangle C_1B_2A_1$ (more precisely, identify the points $D$ and $C_1$, $E$ and $B_2$, $L$ and $A_1$). The resulting polyhedron
\( S_1 \) belongs to the class \( \mathcal{F}_1 \), is flexible but is not embedded.

Glue \( \mathcal{F}_1 \) and \( \mathcal{O} \) along the triangles \( \triangle EFL \) and \( \triangle A_1^1C_1^1B_2^1 \) (more precisely, identify the points \( E \) and \( A_1^1 \), \( F \) and \( C_1^1 \), \( L \) and \( B_2^1 \)). The resulting polyhedron \( S_2 \) belongs to the class \( \mathcal{F}_2 \), is flexible but is not embedded.

Note that, during the flex of \( S_2 \), the both vertices \( C_2 \) and \( C_1^† \) move along the circle that lies in the plane perpendicular to the segment \( EL \) and is centered at the middle point of \( EL \). Hence, for every position of \( C_2 \) (originated from the flex of \( 0^† \)) we can bend \( 0^† \) in such a way that \( C_1^† \) coincide with \( C_2 \). It means that even when we glued the triangles \( \triangle LEC_2 \) and \( \triangle LEC_2^† \) the resulting polyhedron is flexible. It belongs to the class \( \mathcal{F}_3 \) and is known as the Steffen flexible polyhedron, see \([2]\) or \([14]\).

**THEOREM 8.** For every \( n \geq 0 \) any flexible embedded polyhedron \( P \in \mathcal{F}_n \) satisfies the Strong Bellows Conjecture, i.e., every embedded polyhedron \( P' \) obtained from \( P \) by a continuous flex is scissors congruent to \( P \).

**Proof.** According to Theorem 1 it suffice to prove that \( \text{Vol} \, P' = \text{Vol} \, P \) and

\[
D_f P' = D_f P \quad (6)
\]

for every \( \mathbb{Q} \)-linear function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(\pi) = 0 \). The equality \( \text{Vol} \, P' = \text{Vol} \, P \) follows directly from the fact that the oriented volume is constant during the flex \([15]\), \([7]\), \([16]\), \([18]\). The equality (6) can be proven by induction on \( n \). In fact, Theorems 3, 5, and 7 yields (6) for \( P \in \mathcal{F}_0 \). Now suppose that (6) holds true for all polyhedra of the classes \( \mathcal{F}_k \), \( 0 \leq k \leq n - 1 \). If \( P \) is constructed according to (ii) from \( P_1, P_2 \in \mathcal{F}_k \), \( 0 \leq k \leq n - 1 \), then \( D_f P = \pm D_f P_1 \pm D_f P_2 \) (depending on the orientation of \( P, P_1, \) and \( P_2 \)) and, thus, is constant during the flex. Similarly, if \( P \) is constructed according to (ii) or (iii) from \( P_1 \in \mathcal{F}_{n-1} \), then \( D_f P = \pm D_f P_1 \) and, thus, is again constant during the flex.

**COROLLARY.** If \( P \) is a Steffen flexible polyhedron and \( P' \) is obtained from \( P \) by a continuous flex then \( P \) and \( P' \) are scissors congruent.

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