Some Features of Geometric Nonlinear Damping on Isolation Performance

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Abstract. The addition of nonlinear characteristics to improve the dynamic performance of isolation systems and vibration absorbers has been extensively investigated in the past decades. Both nonlinear stiffness and nonlinear damping have been largely analysed. A common way in which this can be achieved in practice, is by arranging linear elements (springs and viscous dampers) in a specific geometrical configuration. This paper focuses on the fundamental effect of geometrical nonlinear damping in a vibration isolation system, and briefly revises two of the classical implementation mechanisms. They can achieve a better performance than linear systems, by assuring lower damping force at lower displacements, and higher damping force at higher displacements. However, strong nonlinear effects can manifest around resonances, causing non-negligible higher-order harmonics to appear, and making the approximate frequency response obtained by the harmonic balance, useless. This paper proposes a new strategy to limit such undesired dynamic effects. For this aim, a coupling is introduced on purpose in the damping mechanism, which allows tailoring the damping force for the specific need.

1. Introduction

The benefits of introducing nonlinear elements to improve the isolation performance of vibrating devices have been of particular attention in the past decades [1]. Jazar et al. [2] assumed a quadratic function of displacement for the stiffness and damping of an engine mount for modelling and performance investigation. Lang et al. [3] studied the transmissibility of a vibration isolator with cubic viscous damping characteristic, highlighting how the resonant region was modified by the damping and the non-resonant one remained unaffected. Sapsis et al. [4] analysed three different applications of an essentially nonlinear energy sink and showed how these attachments could drastically increase the effective damping properties. Tang and Brennan [5] studied the vibration transmissibility characteristics of a single-degree-of-freedom passive vibration isolator comparing the performance of linear viscous dampers, cubic viscous dampers and geometrical nonlinear viscous dampers, for low and high amplitudes of vibration. Elliot et al. [6] reviewed the diverse sources of nonlinear damping and analysed some examples subject to sinusoidal and random excitations. Geometrical nonlinear stiffness and damping systems have been implemented using a scissor-like structure in [7], and an investigating on the effect of nonlinear cubic viscous damping in a vibration isolator with a magnetic spring has been investigated in [8, 9]. Two of the main mechanisms adopted to achieve geometric
nonlinear effects exploit linear elements swinging as in [5], or linear elements driven by swinging rigid rods, as in [9]. In most of the cases and applications, a predominately harmonic response is obtained when a harmonic excitation is considered. However, frequency components other than the fundamental one may affect the system response and its actual performance [10].

This paper presents a strategy on how to reduce such undesired effects by introducing a coupling between the mass motion and the mechanism to achieve geometrical nonlinear damping.

2. Nonlinear damping mechanisms

A schematic model of the vibration isolator considered in this work is illustrated in figure 1, where it is used to reduce the transmitted force to its base. It consists of a mass \( m \), subject to a harmonic excitation force of amplitude \( F \) and frequency \( \omega \), suspended through a linear spring of stiffness \( k \) and a damper, whose damping force could generally be a function \( f_d(x,\dot{x}) \) of displacement \( x \) and velocity \( \dot{x} \). The transmitted force to ground is denoted by \( f_t \).

![Figure 1. Simple model of a vibration isolator subject to force excitation.](image)

The equation of motion of the system is

\[
m\ddot{x} + f_d(x,\dot{x}) + kx = F\cos(\omega t)
\]

where \( t \) is the time, and the transmitted force is \( f_t = f_d(x,\dot{x}) + kx \).

2.1. Linear damping

For a linear system, where \( f_d(x,\dot{x}) = c_i\dot{x} \), the non-dimensional form of equation (1) can be written as

\[
\ddot{x} + \sigma_i\dot{x} + \ddot{x} = \hat{F}\cos(\Omega\tau + \phi)
\]

where \( \sigma_i \) is twice the damping ratio, i.e. \( \sigma_i = c_i/\sqrt{km} \); \( \Omega = \omega/\omega_n \), where \( \omega_n = \sqrt{k/m} \); the non-dimensional displacement is normalized to some characteristic dimension of the system, \( a \), i.e. \( \ddot{x} = x/a ; \hat{F} = F/ka \), \( \tau = t\omega_n \) is the non-dimensional time, primes denote derivation respect to \( \tau \), and \( \phi \) is the phase angle relative to the displacement response.

The force transmissibility for such a linear system is [11]

\[
T = \sqrt{\frac{1 + \sigma_i^2\Omega^2}{(1 - \Omega^2)^2 + \sigma_i^2\Omega^2}}
\]

When \( \Omega = 1 \), the transmissibility is \( \sqrt{1 + \sigma_i^2/\sigma_i} \), which reduces to \( 1/\sigma_i \) for relatively low values of damping, such that \( \sigma_i \ll 1 \).
At higher frequencies, when $\Omega \gg 1$, the transmissibility is $\frac{1}{\Omega^2}$, which reduces to $1/\Omega^2$ for relatively low values of damping (i.e. $\sigma_i/\Omega \ll 1$), and to $\sigma_i/\Omega$ otherwise.

It can be then clearly seen that the performance of a linear isolator should be selected as a well-known trade-off [11] between good performance at high frequencies, where a light damping would be more beneficial, and the performance at resonance, where a higher damping would be more effective.

2.2. Nonlinear damping

A way to cope with the desirable requirement of having both a high damping effect for large amplitudes of vibration, and a light damping effect for small amplitudes of vibration, is to exploit some nonlinear behaviour. What is specifically of interest in this work is the case where the nonlinear effect is achieved by some geometrical arrangement of linear damping elements.

Two main configurations have been predominately investigated in the past, and they are illustrated schematically in figure 2(a) and (b).

In figure 2(a), the linear damper inclines when the oscillator mass moves, while in figure 2(b) the linear damper is horizontal and it is driven by a rigid rod, which inclines as the oscillator mass moves.

The expression for the non-dimensional damping force for the two configurations in figure 2(a) and (b) may be found, for instance, in [5,9] respectively

$$\hat{f}_d (\hat{x}, \hat{x}') = \sigma_{nl} \frac{\hat{x}^2}{1 \pm \hat{x}^2} \hat{x}'$$  \hspace{1cm} (4a,b)

where $\sigma_{nl} = c_n/\sqrt{km}$. It is clear that equation 4(b) provides larger damping force for high displacements than that in equation (4a). Such damping force per unit velocity and per unit damping coefficient is plotted in figure 3 with a solid thicker line for the system in figure 2(a), and with a solid thinner line for the system in figure 2(b).

Expanding equations (4a,b) in Taylor series leads to

$$\hat{f}_d (\hat{x}, \hat{x}') = \sigma_{nl} \left(\hat{x}^2 \pm \hat{x}^4 \mp \hat{x}^6 \mp \hat{x}^8 \mp \hat{x}^{10} \mp \ldots\right) \hat{x}'$$  \hspace{1cm} (5a,b)
Figure 3. Damping force per unit velocity and per unit damping coefficient for the system in figure 2(a) (thick solid line) and figure 2(b) (thin solid line). Asymptotic trends are denoted with dotted lines. The lower order approximation for small displacements is denoted by the dashed line.

so that, to the lower order approximation, both the systems in figure 2 behave the same. Such lower order approximation is depicted in figure 3 as a dashed line, and holds for small deflections. The two thin dotted lines in figure 3 represent the asymptotic trends for larger deflections. In particular, the system in figure 2(a) will behave as a rigid connection as approaching the value of \( \hat{x} = \pm 1 \), while the system in figure 2(b) will tend to a constant value.

For relatively small deflections, so that the lower order approximation holds, both system configurations assure a very small damping force for relatively small deflections (useful for high frequency isolation), and very large damping force for relatively large deflections (potentially useful at resonance).

3. Force transmissibility

The Taylor series expansion given in equations (5a,b) can be used to derive the frequency response of the system in the assumption of predominately harmonic response. If it is assumed that \( \hat{x} = \hat{X} \cos(\Omega \tau) \), and applying the harmonic balance method and the equivalent viscous damping [6], the following amplitude-frequency equation can be obtained

\[
(1 - \Omega^2)^2 \hat{X}^2 + \Omega^2 \hat{X}^2 \sigma_{al}^2 \left( \frac{1}{4} \hat{X}^2 + \frac{1}{8} \hat{X}^4 + \frac{5}{64} \hat{X}^6 + \frac{7}{128} \hat{X}^8 + \frac{21}{512} \hat{X}^{10} + \ldots \right)^2 = \hat{F}^2
\]  

which is quadratic in \( \Omega^2 \) and can be solved to plot the frequency response of the displacement amplitude for different values of the nonlinear damping and excitation amplitude. The force transmissibility can then be found as

\[
T = \frac{\hat{X}}{\hat{F}} \sqrt{1 + \Omega^2 \sigma_{al}^2 \left( \frac{1}{4} \hat{X}^2 + \frac{1}{8} \hat{X}^4 + \frac{5}{64} \hat{X}^6 + \frac{7}{128} \hat{X}^8 + \frac{21}{512} \hat{X}^{10} + \ldots \right)^2}
\]  

As detailed in [5], by considering the approximate force transmissibility, the performance of the nonlinear damper is better than that of the linear damper when the force amplitude is such that \( \hat{F} > 2 \sigma_{al} \sqrt{ \sigma_{al} } \). Furthermore, the differences between equation (7a) and (7b) starts to become relevant for even much higher levels, so that the two systems in figure 2(a) and (b) performs similarly in practical situations.

However, at higher amplitudes of vibrations, the system response is affected by higher order harmonics, so that the approximate transmissibility in equations 7(a,b) underestimates the maximum effective peak. For \( \hat{F} = 0.6 \) and \( \sigma_{al} = 20 \), figure 4(a) shows the approximate force transmissibility as a function of frequency, figure 4(b) shows the steady-state response of the actual transmissibility as a function of time at \( \Omega = 1 \), and figure 4(c) shows the corresponding normalized Fourier coefficients.
Figure 4. (a) Approximate force transmissibility as a function of frequency, (b) actual transmissibility as a function of time, and (c) corresponding normalised Fourier coefficients, for the system in figure 2(a) (thick line) and in figure 2(b) (thin line). The parameters used are $\hat{F} = 0.6$ and $\sigma_{nl} = 20$. In (a), the circle and square denote the amplitude of the response at the excitation frequency for the system in figure 2(a) and (b), respectively.

It can be seen that the two damping mechanisms in figure 2 perform qualitatively the same, although the mechanism in figure 2(a) is slightly less damped. One can also note the significant presence of higher order harmonics, which are visually limited to six in figure 4(c). These cause the distortion of the time history in figure (b). Furthermore, it is seen that the maximum amplitude in the transmissibly in figure 2(b) is about 2.7, while the approximate value in the frequency response in figure 4(a) is about 1.3, well below the actual value.

4. Effect of variable geometry

With the aim to investigate how to limit the presence of higher harmonics in the transmissibility for higher displacement amplitudes, the effect of changing the position of the damper support to ground is investigated. The proposed mechanism is illustrated in figure 5, and is obtained from the configuration in figure 2(a), where the support to ground of the linear damper is assumed to be varied according to a function $y$ of the mass displacement $x$.

Such damping mechanism is used to synthesise a proper coupling between the mass motion and the support motion, in order to achieve a specified nonlinear damping force.
Figure 5. Damping mechanism obtained as an alternative configuration to those in figure 2(a) and (b). The support to ground of the damper is varied according to a function $y$ of the mass displacement $x$.

In a first attempt to tailor the damping force to limit higher harmonics content in the transmissibility for high vibration amplitudes, the one selected is plotted in figure 6(a) per unit velocity and for $\sigma_0 = 20$, and compared to the approximate ones given in figure 3. A close-up is shown in figure 6(b), where a 10th order polynomial approximation is also reported as a dotted line.

Figure 6. (a) Selected damping force per unit velocity for the damping mechanism in figure 5 for $\sigma_0 = 20$ (solid red line), superimposed to those in figure 3. (b) Close-up with polynomial approximation (dotted line).

The reason behind the damping force trend shown in figure 6 is essentially to shift the asymptotic behaviour shown in figure 3 for the thick line, to a more usable displacement range. It can then be seen that the damping force trend in figure 6(b) is very low for small displacement and reaches an asymptotic constant value at about $\hat{x} = 0.3$.

The damping force per unit velocity of the mechanism in figure 5 is given by

$$\hat{f}_d(\hat{x},\hat{x}') = \sigma_0 \left( \dot{\hat{x}} + \hat{y} \frac{\partial \hat{y}}{\partial \hat{x}} \right) \frac{\hat{x}}{\hat{x}^2 + \hat{y}^2}$$

which is a first order differential equation in $\hat{y} = \hat{y}(\hat{x})$, where $\hat{y} = y/a$. Equation (8) is solved numerically after the damping force is approximated by the 10th order polynomial expression plotted in figure 6(b) as a dotted line. The result is illustrated in figure 7 as a solid line. As a reference, the position of the corresponding pin connection of the rigid rod in figure 2(b) is also shown as a dashed line.

From figure 7, it can be noted that the change in position of the damper support is relatively only slightly larger than the position it would achieve if a rigid rod would be used instead.
Figure 7. Relation between the damper support displacement, $\hat{y}$, and the mass displacement, $\hat{x}$, (solid line) to achieve the damping force trend illustrated in figure 6(b) as dotted line. The dashed line indicates the same relation for the rigid rod in the mechanism in figure 2(b).

5. Frequency response and numerical simulation

Provided that the left hand side of equation (8) is written as a polynomial expression of even powers of $\hat{x}$, as $\dot{f}(\hat{x},\hat{x}')/\hat{x}' = c_2\hat{x}^2 + c_4\hat{x}^4 + c_6\hat{x}^6 + c_8\hat{x}^8 + c_{10}\hat{x}^{10} + \ldots$ the equivalent viscous damping can be adopted, and the frequency response in terms of force transmissibility can be computed as in equation (7), which now reads as

$$T = \frac{\hat{X}}{\hat{F}} \sqrt{1 + \Omega^2 \sigma_{nl}^2 \left( \frac{1}{4} c_2 \hat{X}^2 + \frac{1}{8} c_4 \hat{X}^4 + \frac{5}{64} c_6 \hat{X}^6 + c_8 \frac{7}{128} \hat{X}^8 + c_{10} \frac{21}{512} \hat{X}^{10} + \ldots \right)}$$

Equation (9) is plotted in figure 8(a) for the same excitation amplitude and nonlinear damping as in figure 4(a).

Figure 8. (a) Approximate force transmissibility as a function of frequency, (b) actual transmissibility as a function of time, and (c) corresponding normalized Fourier coefficients, for the system in figure 5. The parameters used are $\hat{F} = 0.6$ and $\sigma_{nl} = 20$. In (a), the circle denotes the amplitude of the response at the excitation frequency.
Also, a numerical simulation is performed to investigate the dynamic performance of the proposed methodology. The calculated actual transmissibility as a function of time, and the corresponding normalized Fourier coefficients, are thus reported in figure 8(b) and (c), respectively.

It can be seen that the amplitude of the higher order components in the transmissibility is largely reduced – the third component is now less than 20% of the amplitude of the first harmonic, whereas in figure 4 it was about 70%. As a consequence, the estimate given by the frequency response in figure 8(a) is much closer to the maximum amplitude in the response shown in figure 8(b).

6. Conclusions
The paper has presented a preliminary investigation on a novel strategy to achieve tailored geometrical nonlinear damping in a vibration isolator. Classical nonlinear mechanisms are characterised by a very low (quasi-zero) damping at very low deflections, but a relatively high damping at higher deflection, so that around resonance strong nonlinear effects manifest. These may be characterised by the presence of higher order frequency components, which are not negligible and largely affect the maximum amplitude of the force transmitted to ground. As a result, the approximate closed-form solution for the frequency response sensibly underestimates the actual performance. Also, higher order frequency components can be dangerous as they can introduce new frequencies and excite internal resonances. The strategy proposed in this paper investigates the introduction of a mathematical coupling between the oscillator mass movement and the internal geometry of the damping mechanism. Such relation can be used to achieve a desired damping force characteristic. In a preliminary attempt to show the feasibility of the technique, a specific damping force is assumed and it is shown how this can largely reduce the undesired dynamic effects. On-going work is to investigate the benefits of nonlinear strategies in adaptive control, ways in which the mathematical connection can be mechanically implemented, as well as strategies for optimal design.

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