On the cohomologies of the de Rham complex over weighted isotropic and anisotropic Hölder spaces

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ABSTRACT
We consider the de Rham complex over scales of weighted isotropic and anisotropic Hölder spaces with prescribed asymptotic behavior at infinity. Starting from theorems on the solvability of the system of operator equations generated by the de Rham differential $d$ and the operator $d^*$ formally adjoint to it, a description of the cohomology groups of the de Rham complex over these scales was obtained. It was also proved that in the isotropic case the cohomology space is finite-dimensional, and in the anisotropic case the general form of an element from the cohomology space is presented.

1. Introduction

It is well known that de Rham cohomologies provide a lot of information on important invariants of a differentiable manifold, see for instance, [1]. However advances in the theory of differential equations and, more generally, in the theory of complexes of differential operators, lead us to a slightly different concept: differential complexes over function spaces on the corresponding manifold (Hilbert spaces, Banach spaces, Fréchet spaces, etc.), see, for instance, [2–4]. As applied to the de Rham complex, this concept can be traced in the theory of cohomologies over Lebesgue and Sobolev spaces, see, for instance, [5,6]. On this way, the classical de Rham cohomologies (over the Fréchet spaces of smooth functions) and cohomologies over Banach spaces may essentially differ.

This article is a logical continuation of papers [7,8], in which we discussed solvability conditions for the operator equations generated by the differentials of the de Rham complex over isotropic weighted Hölder spaces on the Euclidean space $\mathbb{R}^n$ and the induced de Rham differentials over anisotropic weighted Hölder spaces on the strip $\mathbb{R}^n \times [0, T]$ from the Euclidean space $\mathbb{R}^{n+1}$, where the variable $t \in [0, T]$, is considered as a parameter included in the coefficients of differential forms. One of the motivations for studying the cohomology groups of the induced complex is the connection between the de Rham complex and various models of hydrodynamics, see, for example, [9,10].

More precisely, using the method of integral representations, the technique of working in weighted spaces elaborated in articles [11,12], and the results obtained in [7,8], it is proved in the paper that the cohomology groups of the de Rham complex over isotropic
weighted Hölder spaces is finite-dimensional, and their dimension is indicated in dependence on the weight index of the space and the degree of the forms, and one of the possible bases in the cohomology space is indicated. It is pertinent to note that there is no countable basis in the induced cohomology space over non-separable weighted Hölder spaces.

2. Isotropic case

As usual, we denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geq 1$. Let us first consider the classical de Rham complex over the domain $\mathcal{X}$ in $\mathbb{R}^n$:

$$0 \rightarrow \Omega^0(\mathcal{X}) \xrightarrow{d^0} \Omega^1(\mathcal{X}) \xrightarrow{d^1} \Omega^2(\mathcal{X}) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \Omega^n(\mathcal{X}) \rightarrow 0;$$

where $d^k$ are de Rham differentials given by exterior derivatives, $\Omega^0(\mathcal{X})$ is the space of infinitely differentiable functions on $\mathcal{X}$, and $\Omega^k(\mathcal{X})$ is the space of exterior differential $k$-forms whose coefficients are smooth functions on $\mathcal{X}$.

As is well known, the cohomology groups of this complex are trivial for $0 < q \leq n$ if $\mathcal{X}$ is a convex (or, more generally, star-shape) domain, see [1]. However, when considering the de Rham complex over other function spaces, the cohomology spaces may differ significantly from the classical ones, see, for example, [5,6].

In the papers [7,8] the following systems of operator equations were considered in weighted Hölder spaces over $\mathbb{R}^n$:

$$\begin{cases} du = f, \\ d^* u = g, \end{cases}$$

where $d$, as above, denotes the de Rham differential, and $d^*$ is formally adjoint to it, and the weights regulate the decreasing order of the considered differential forms at the point at infinity. It turned out that as the order of decrease of the desired differential form increases, sufficiently restrictive conditions for the solvability of this system of operator equations appear. This led us to the question of describing the cohomology groups of the de Rham complex on the chosen spaces.

Let’s describe the situation in more detail. Namely, we put

$$w(x) = \sqrt{1 + |x|^2}, \quad w(x, y) = \max \{w(x), w(y)\}, \quad x, y \in \mathbb{R}^n.$$  

For $s \in \mathbb{Z}_+$ and $\delta \in \mathbb{R}$ we denote by $C^{s,0}_\delta$ the space of $s$ times continuously differentiable functions in $\mathbb{R}^n$ with finite norm

$$\|u\|_{C^{s,0}_\delta} = \sum_{|\alpha| \leq s} \sup_{x \in \mathbb{R}^n} w^{\delta + |\alpha|}(x) |\partial^\alpha u(x)|.$$  

Let $\mathcal{X}$ be some subset in $\mathbb{R}^n$. Then let $U \subset \mathbb{R}^n$ be a non-empty bounded neighborhood of zero in the topology $\mathbb{R}^n$, if $\mathcal{X}$ is not bounded, or an empty set otherwise. For $0 < \lambda \leq 1$
we set
\[ \langle u \rangle_{\lambda, \delta, \overline{X}} = \sup_{x, y \in \overline{X}, x \neq y} \left( u(x) - u(y) \right) \frac{|u(x) - u(y)|}{|x - y|^\lambda}. \]

Let \( C^0_{\delta} \) consists of all continuous functions on \( \mathbb{R}^n \) with a finite norm
\[ ||u||_{C^0_{\delta}} = ||u||_{C^0(\mathbb{R}^n)} + ||u||_{C^0_{\delta}}, \]
where \( || \cdot ||_{C^0(\mathbb{R}^n)} = || \cdot ||_{C^0(\overline{U})} \) is the norm of the usual Hölder space \( C^0_{\delta}(\overline{U}) \) on the compact \( \overline{U} \). Finally, for \( s \in \mathbb{Z}_+ \), let \( C^s_{\delta} \) denotes the space of all \( s \) times continuously differentiable functions on \( \mathbb{R}^n \) with a finite norm
\[ ||u||_{C^s_{\delta}} = \sum_{|\alpha| \leq s} ||\partial^\alpha u||_{C^0_{\delta + |\alpha|}}. \]

We proceed to the construction of the de Rham complex over the selected Banach spaces. To this end, for the differential operator \( A \), acting on differential forms over \( \mathbb{R}^n \), we denote by \( C^s_{\delta, A} \cap S_A \) the space of differential forms \( u \in C^s_{\delta, A} \), satisfying \( Au = 0 \) in the sense of distributions in \( \mathbb{R}^n \). Obviously, this is a closed subspace in \( C^s_{\delta, A} \), which means it is a Banach space with the induced norm.

Below we will consider the following complex of Banach spaces
\[ 0 \rightarrow C^s_{\delta, A} \rightarrow C^{s-1, A} \rightarrow \cdots \rightarrow C^1_{\delta, A} \rightarrow C^0_{\delta, A} \rightarrow 0, \quad (1) \]
where \( d^q \) are de Rham differentials given by external derivatives, and \( A^q \) are bundles of exterior differential forms of degree 0 \( \leq q \leq n \) over \( \mathbb{R}^n \).

The first goal of this paper is to describe the cohomology groups of the complex (1), therefore, let the set \( Z^s_{\delta, A} \cap S_A \) denotes the space of cocycles and consists of the forms contained in intersection \( C^s_{\delta, A} \cap S_A \), the set \( B^s_{\delta, A} \) denotes the space of coboundaries and consists of forms \( f \in C^s_{\delta, A} \), for which exists \( u \in C^s_{\delta, A} \), satisfying the equation \( f = du \), and the set \( H^s_{\delta, A} \) denotes the cohomology space, that is, equal to set \( Z^s_{\delta, A}/B^s_{\delta, A} \).

In order to pass to the description of cohomology spaces, it is necessary to cite some previous results, namely: in the paper [7] continuous linear operators
\[ (d^q, (d^q - 1)^*): C^{s+1}_{\delta, A} \rightarrow C^s_{\delta, A} \cap S_{d^q} \oplus C^s_{\delta, A} \cap S_{d^{q-2}}, \quad (2) \]
\[ (d^q, (d^q - 1)^*): C^{s+1}_{\delta, A} \rightarrow C^s_{\delta, A} \cap S_{d^q} \oplus C^s_{\delta, A} \cap S_{d^{q-2}}, \quad (3) \]
were considered, where \( C^s_{\delta, A} \) is the closure of the space \( \mathcal{D}(\mathbb{R}^n) \) of infinitely differentiable functions with compact supports in \( \mathbb{R}^n \) in the space \( C^s_{\delta, A} \). A more detailed description of this space was given in [7, Theorem 2]. Taking as \( \mathcal{H}_{s, m, A} \) the space of all differential forms of degree \( q \), whose coefficients are harmonic polynomials of degree \( \leq m \), the following theorem was formulated and proved.

**Theorem 2.1:** Let \( n \geq 2, s \in \mathbb{Z}_+, 0 < \lambda < 1 \). If \( \delta > 0 \) and \( \delta + 1 - n \notin \mathbb{Z}_+ \), then the operators (2) and (3) are Fredholm. Moreover,
• (2) and (3) are isomorphisms if $0 < \delta < n - 1$;
• (2) and (3) are injections with a closed image if $n - 1 + m < \delta < n + m$ for $m \in \mathbb{Z}_+$; more precisely, the image of the operator (2) consists of all pairs $f \in C^{s,\lambda}_{\delta+1,A^{q+1}} \cap S_{d^{q+1}}$, $g \in C^{s,\lambda}_{\delta+1,A^{q-1}} \cap S_{(d^{q-2})^*}$, satisfying
\[
(f, d^i h)_{A^{q+1}}(\mathbb{R}^n) + (g, (d^{i-1})^* h)_{A^{q-1}}(\mathbb{R}^n) = 0 \quad \text{for all } h \in H_{\leq m+1,A^n},
\]
and the image of the operator (3) consists of all pairs $f \in C^{s,\lambda}_{\delta+1,A^{q+1}} \cap S_{d^{q+1}}$, $g \in C^{s,\lambda}_{\delta+1,A^{q-1}} \cap S_{(d^{q-2})^*}$, satisfying (4).

The main tools in the proof of the theorem were the following defined integral operators. Namely, let
\[
(E_q u)(x) = \int_{\mathbb{R}^n} u(y) \wedge e_q(x,y)
\]
for a suitable $q$-form $u$, where
\[
e_q(x,y) = \sum_{|I|=q} e(x-y)(\star dy_I)dx_I, \quad e(x) = \begin{cases} \frac{1}{\ln |x|}, & \text{for } n = 2, \\ \frac{\pi}{|x|^{2-n}}, & \text{for } n \geq 3, \\ \frac{\ln |x|}{\sigma_n 2 - n}, & \text{for } n \geq 3, \end{cases}
\]
that is, $e$ is a standard fundamental solution of convolution type for the Laplace operator in $\mathbb{R}^n$, and $e_q$ is its analog for action on exterior differential forms (here $\sigma_n$ is the area of the unit sphere in $\mathbb{R}^n$). Now, for $f \in C^{s,\lambda}_{\delta+1,A^{q+1}}$, $g \in C^{s,\lambda}_{\delta+1,A^{q-1}}$, we define
\[
(\Phi_q f)(x) = \int_{\mathbb{R}^n} f(y) \wedge \phi_q(x,y), \quad (\hat{\Phi}_q g)(x) = \int_{\mathbb{R}^n} g(y) \wedge \hat{\phi}_q(x,y),
\]
where
\[
\phi_q(x,y) = (d^{n-q-1})_y^* e_q(x,y), \quad \hat{\phi}_q(x,y) = d^n_y e_q(x,y), \quad n \geq 2.
\]
Next, we use the following decomposition of the fundamental solution of the Laplace operator in homogeneous harmonic polynomials $\{h^{(i)}_k\}$, forming an orthonormal basis in $L^2(\partial B_1)$ on the unit sphere $\partial B_1$ in $\mathbb{R}^n$:
\[
e(x-y) = e(x-0) - \sum_{k=1}^{\infty} \sum_{j=1}^{J(k)} \frac{h^{(j)}_k(x)h^{(j)}_k(y)}{(n+2k-2)|x|^{n+2k-2}},
\]
here $n \geq 2$, $k$ is the degree of homogeneity of the polynomial, $j$ is the number of a homogeneous polynomial of degree $k$ in the basis, and the series converges uniformly together with all derivatives on compact sets of their cone $\{|x| > |y|\}$ in $\mathbb{R}^{2n}$ (see for example [12,13]). We put
\[
\phi_{m,q}(x,y) = \phi_q(x,y) + \sum_{|I|=q} \sum_{k=1}^{m+1} \sum_{j=1}^{J(k)} \frac{h^{(j)}_k(x)(d^{n-q-1})_y^* (h^{(j)}_k(y)(\star dy_I))}{(n+2k-2)|x|^{n+2k-2}},
\]
where $\phi_q(x,y)$ is the analog for action on exterior differential forms (here $\sigma_n$ is the area of the unit sphere in $\mathbb{R}^n$). Now, for $f \in C^{s,\lambda}_{\delta+1,A^{q+1}}$, $g \in C^{s,\lambda}_{\delta+1,A^{q-1}}$, we define
\[
(\Phi_q f)(x) = \int_{\mathbb{R}^n} f(y) \wedge \phi_q(x,y), \quad (\hat{\Phi}_q g)(x) = \int_{\mathbb{R}^n} g(y) \wedge \hat{\phi}_q(x,y),
\]
where
\[
\phi_q(x,y) = (d^{n-q-1})_y^* e_q(x,y), \quad \hat{\phi}_q(x,y) = d^n_y e_q(x,y), \quad n \geq 2.
\]
\[ \Phi_{m,q}(x,y) = \hat{\Phi}(x,y) + \sum_{|\ell| = q} \sum_{k=1}^{m+1} \sum_{j=1}^{l(k)} h^{(j)}_k(x) \left( d^{n-q} \right)_y \left( h^{(j)}_k(y) \ast dy_l \right) dx_l, \]

where \( m \in \mathbb{Z}_+ \) and \( \vartheta(x) \) is some smooth function such that \( \vartheta(x) = |x| \) for \( |x| \geq 2 \) and \( 0 < 1/\vartheta(x) \leq 1 \).

Then we define the potential \( \Phi_{m,q} \) as follows

\[ \Phi_{m,q}f(x) = \int_{\mathbb{R}^n} f(y) \wedge \Phi_{m,q}(x,y), \quad \hat{\Phi}_{m,q}g(x) = \int_{\mathbb{R}^n} g(y) \wedge \hat{\Phi}_{m,q}(x,y). \]

The main property of these potentials is the following.

**Lemma 2.2:** If \( \delta > 0, \lambda \in (0, 1) \), then the potentials \( \Phi_qf, \hat{\Phi}_qg \), given by formula (5), satisfy, in the sense of distributions on \( \mathbb{R}^n \), equalities

\[ (d^\delta \Phi_qf) = f, \quad (d^{\delta-1} \ast (\Phi_qf)) = 0, \quad (d^{\delta-1} \ast (\hat{\Phi}_qg)) = g, \quad (d^\delta)(\hat{\Phi}_qg) = 0 \]

for all \( f \in C^{0,\lambda}_{\delta+1,\mathbb{R}^{q+1}} \cap S^1_{\delta+1,\mathbb{R}^{q+1}}, \ g \in C^{0,\lambda}_{\delta+1,\mathbb{R}^{q-1}} \cap S^1_{\delta+1,\mathbb{R}^{q-1}}. \) Moreover, if \( 0 < \delta < n - 1 \), then the potentials (5) induce the bounded linear operators

\[ \Phi_q : C^{0,\lambda}_{\delta+1,\mathbb{R}^{q+1}} \to C^{1,\lambda}_{\delta,\mathbb{R}^{q}}, \quad \hat{\Phi}_q : C^{0,\lambda}_{\delta+1,\mathbb{R}^{q-1}} \to C^{1,\lambda}_{\delta,\mathbb{R}^{q}}. \]

Let us extract from Theorem 2.1 and Lemma 2.2 a rather expected consequence; we will use to describe the cohomology spaces of complex (1).

**Corollary 2.3:** Let \( n \geq 2, \ s \in \mathbb{Z}_+, \ 0 < \lambda < 1, \ \delta > n/2 \) and \( f \in Z^{\delta,\lambda}_{\delta+1,\mathbb{R}^{q+1}}. \) If the form \( u \in \mathcal{C}^{s,\lambda}_{\delta,\mathbb{R}^{q}} \) is a solution to the equation \( du = f \), then there is a form \( v \in \mathcal{C}^{s,\lambda}_{\delta,\mathbb{R}^{q}} \), satisfying

\[ \begin{align*}
  dv &= f, \\
  d^s v &= 0.
\end{align*} \]

**Proof:** Let the form \( u \) satisfy the conditions of the corollary. Then (see [7, Corollary 1]) the following decomposition takes place:

\[ u = d^s \hat{\Phi} u + d\Phi u. \]

Applying the operator \( d \) to identity (7) and taking into account that \( d \circ d = 0 \), we see that \( v = d^s \hat{\Phi} u \) is also a solution to the equation \( du = f \). And since \( d^s \circ d^s = 0 \), then \( v \) is a solution to the system (6). Since the operator \( d^s \hat{\Phi} \) is continuous in \( \mathcal{C}^{s+1,\lambda}_{\delta,\mathbb{R}^{q}} \) (also, by Corollary 1), it follows that \( v \in \mathcal{C}^{s+1,\lambda}_{\delta,\mathbb{R}^{q}} \). \( \square \)

Finally, we can go directly to the description of the cohomology groups of complex (1). It follows from Theorem 2.1 that they are trivial for \( 0 < \delta < n - 1 \). Therefore, below we will consider the case when \( n + m - 1 < \delta < n + m \).

**Theorem 2.4:** Let \( n \geq 2, \ s \in \mathbb{Z}_+, \ 0 < \lambda < 1 \) and \( n + m - 1 < \delta < n + m \). Then the cohomology groups of complex (1) are finite-dimensional and isomorphic to the image of the operator \( d(\Phi - \hat{\Phi}_m) \), acting from \( \mathbb{Z}^{\delta,\lambda}_{\delta+1,\mathbb{R}^{q+1}} \) to \( \mathbb{Z}^{\delta,\lambda}_{\delta+1,\mathbb{R}^{q+1}} \).
**Proof:** First, we prove that $d(\Phi - \Phi_m)$ is a continuous operator. Since $d$ acts continuously from $C^{s+1,\lambda}_{\delta,A^q}$ to $C^{s,\lambda}_{\delta+1,A^q+1}$, and $\Phi_m$ acts continuously from $C^{s,\lambda}_{\delta+1,A^q+1}$ to $C^{s+1,\lambda}_{\delta,A^q}$ (see [7]), we have

$$
\| d(\Phi_m f) \|_{C^{s,\lambda}_{\delta+1,A^q+1}} \leq C_1 \| \Phi_m f \|_{C^{s+1,\lambda}_{\delta,A^q}} \leq C_1 C_2 \| f \|_{C^{s,\lambda}_{\delta+1,A^q+1}}.
$$

It follows from Theorem 2.1 that the operator $\Phi$ for $n + m - 1 < \delta < n + m$ does not act continuously from $C^{s,\lambda}_{\delta+1,A^q+1}$. But since the operator $\Phi$ under the conditions of the theorem being proved is considered on $Z^{s,\lambda}_{\delta+1,A^q+1}$, then $d\Phi f = f$. Thus, we get

$$
\| d(\Phi f) - d(\Phi_m f) \|_{C^{s,\lambda}_{\delta+1,A^q+1}} \leq \| d(\Phi) \|_{C^{s,\lambda}_{\delta+1,A^q+1}} + \| d(\Phi_m) \|_{C^{s,\lambda}_{\delta+1,A^q+1}} \leq (1 + C_1 C_2) \| f \|_{C^{s,\lambda}_{\delta+1,A^q+1}}.
$$

The next step in the proof is to establish the equality of spaces $C^{s,\lambda}_{\delta+1,A^q+1} \cap \mathcal{S}_d(\Phi - \Phi_m)$ and $B^{s,\lambda}_{\delta+1,A^q+1}$. Suppose

$$
d(\Phi - \Phi_m) f = 0. \quad (8)
$$

As we noted above, $d\Phi f = f$. Then it follows from Equation (8) that $f$ can be represented as $f = d\Phi_m f$. Since the operator $\Phi_m$ acts continuously (see [7, Lemma 5]) from $C^{s,\lambda}_{\delta+1,A^q+1}$ to $C^{s,\lambda}_{\delta,A^q}$, then $f \in B^{s,\lambda}_{\delta+1,A^q+1}$.

Suppose now that $f \in B^{s,\lambda}_{\delta+1,A^q+1}$. Then there is a form $u \in C^{s+1,\lambda}_{\delta,A^q}$ satisfying the condition $f = du$. Corollary 2.3 implies that in this case there also exists $v \in C^{s+1,\lambda}_{\delta,A^q}$, satisfying system (6). Then it follows from Theorem 2.1 that the form $f$ is orthogonal to any element of the set $\mathcal{H}^{s,\lambda}_{\leq m+1}$, and hence $d(\Phi - \Phi_m)f = 0$.

The last part of the proof consists in constructing the desired isomorphism. Consider a map $\omega$ that assigns to each class $[g]$ in $H^{s,\lambda}_{\delta+1,A^q+1}$ an element in the image of an operator $d(\Phi - \Phi_m)$, acting from $Z^{s,\lambda}_{\delta+1,A^q+1}$ to $Z^{s,\lambda}_{\delta+1,A^q+1}$. Let this map acts according to rule

$$
\omega ([g]) = d(\Phi - \Phi_m) g,
$$

where $g$ is some representative of the class $[g]$.

Let us show that $\omega$ does not depend on the choice of a representative of the class $[g]$. Let $g_1, g_2 \in [g]$ and $g_1 \neq g_2$, then $g_2$ can be represented in the form $g_2 = g_1 + du$, where $du$ is an element of the space $B^{s,\lambda}_{\delta+1,A^q+1}$. We obtain

$$
d(\Phi - \Phi_m) g_2 = d(\Phi - \Phi_m) (g_1 + du) = d(\Phi - \Phi_m) g_1 + d(\Phi - \Phi_m) (du).
$$

But $du$ is contained in the set $B^{s,\lambda}_{\delta+1,A^q+1}$, which means that it is also contained in $C^{s,\lambda}_{\delta+1,A^q+1} \cap \mathcal{S}_d(\Phi - \Phi_m)$. Thus

$$
d(\Phi - \Phi_m) g_2 = d(\Phi - \Phi_m) g_1.
$$

Next, we turn to the proof of the injectivity of the map $\omega$. Let $g_1 \in [g_1], g_2 \in [g_2]$ and $[g_1] \neq [g_2]$. Suppose $\omega ([g_1]) = \omega ([g_2])$. We obtain

$$
\omega ([g_1]) = d(\Phi - \Phi_m) g_1 = g_1 + d\Phi_m g_1,
$$

$$
\omega ([g_2]) = d(\Phi - \Phi_m) g_2 = g_2 + d\Phi_m g_2.
$$

Since $\omega ([g_1]) = \omega ([g_2])$, we have $d(\Phi - \Phi_m) g_1 = d(\Phi - \Phi_m) g_2$. Then it follows from the definition of the map $\omega$ that $g_1 = g_2$. Thus, $\omega$ is injective.
\[ \omega ([g_2]) = d(\Phi - \Phi_m)g_2 = g_2 + d\Phi_mg_2. \]

Then, if \( \omega ([g_1]) = \omega ([g_2]) \) then \( g_1 + d\Phi_mg_1 = g_2 + d\Phi_mg_2 \), and therefore \( g_1 \) and \( g_2 \) belong to the same class, which contradicts the condition.

Finally, it remains to prove that the mapping is surjectivity. If a form \( f \) is an element of the image of an operator \( d(\Phi - \Phi_m) \), then there exists a form \( g \in Z^s_{\delta+1, A^q+1} \) such that \( f = d(\Phi - \Phi_m)g \) and hence there is also a class \([g]\) containing this form. Thus, the mapping \( \omega \) is the isomorphism we were looking for.

Remark 2.1: Note that the operator \((\Phi - \Phi_m)\) has the form

\[
(\Phi - \Phi_m)f(x) = \int_{\mathbb{R}^n} f(y) \wedge \sum_{|j|=q} \sum_{k=1}^{m+1} \sum_{j=1}^J h_k^{(j)}(x)(d^{n-q-1}y_j y_j(y)(\star dy_1))dx_1 \quad \text{for } x \in (0,T), \quad f \in C^0_{\delta,T}. \tag{9}
\]

Thus, all elements of the image of the operator \((\Phi - \Phi_m)\) are represented as finite linear combinations of basis vectors of the space \( \mathcal{H} \leq m, \Lambda q \) of \( q \)-forms with coefficients in the space \( \mathcal{H} \leq m \), divided by functions \( \vartheta^{n+2k-2} \). Applying operator \( d \) to the right-hand side of (9), we see that the dimension of the image of operator \( d(\Phi - \Phi_m) \) does not exceed the dimension of the image of \((\Phi - \Phi_m)\).

3. Anisotropic case

Similarly to the usual anisotropic Hölder spaces (see, for example, [14] or [15]), anisotropic weighted Hölder spaces on a cylindrical domain \( \mathcal{X} \times (0,T) \) were introduced in paper [10] (see also [16] as well as [17] for the case when the base \( \mathcal{X} \) is a Riemannian manifold with a conical singularity or [18] for polyhedral domains). The specificity of these spaces is that, although the dilation principle with respect to the element smoothness index is fulfilled, it is violated with respect to the weight index. This is done intentionally in order to guarantee, under certain conditions, the continuity of both parabolic and elliptic potentials on the scale of these spaces, see [10, § 3, b § 4].

Consider a cylinder \( \mathbb{R}^n \times (0,T) \), where \( T > 0 \). On this set, we construct a space \( C^{0,0,0}_{\delta,T} \) of continuous functions with a weight only in the variables \((x_1, \ldots, x_n)\) and with a finite norm

\[ \|u\|_{C^{0,0,0}_{\delta,T}} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_{C^{0,0}_{\delta}}. \]

For \( 0 < \lambda \leq 1 \) and \( 0 < \mu \leq 1 \) we define weighted Hölder spaces \( C^{0,0,\lambda,\mu}_{\delta,T} \), for which the norm is finite

\[ \|u\|_{C^{0,0,\lambda,\mu}_{\delta,T}} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_{C^{0,0}_{\delta}} + \langle u \rangle_{\lambda,\mu,\delta,\mathbb{R}^n,T}, \]

where

\[
\langle u \rangle_{\lambda,\mu,\delta,\mathbb{R}^n,T} = \begin{cases} 
\sup_{t \in [0,T]} \langle u(\cdot, t) \rangle_{\lambda,\delta,\mathbb{R}^n}, & \mu = 0, \\
\sup_{t \notin \tau, \tau \in [0,T]} \|u(\cdot, t) - u(\cdot, \tau)\|_{C^{0,0}_{\delta}} / |t - \tau|^\mu, & \mu \in (0, 1].
\end{cases}
\]
Further, let \( s \in \mathbb{Z}_+ \). We denote by \( C^{2s,\lambda,\mu}_{\delta,T} \) the spaces of \( s \) times continuously differentiable functions over \( \mathbb{R}^n \times [0, T] \), having a finite norm

\[
\| u \|_{C^{2s,\lambda,\mu}_{\delta,T}} = \sum_{|\alpha|+2j \leq 2s} \| \partial^\alpha_x \partial_t^j u \|_{C^{0,\lambda,\mu}_{\delta+|\alpha|,T}}.
\]

Finally, for \( k \in \mathbb{Z} \) we define spaces \( C^{2s+k,\lambda,\mu}_{\delta,T} \) with additional smoothness in the variables \((x_1, \ldots, x_n)\) with a finite norm

\[
\| u \|_{C^{2s+k,\lambda,\mu}_{\delta,T}} = \sum_{|\beta| \leq k} \| \partial_x^\beta u \|_{C^{0,\lambda,\mu}_{\delta+|\beta|,T}}.
\]

### 3.1. Case \( \mu = 0 \)

Now let \( A^q(t) \) denote the induced bundle of exterior differential forms of degree \( q \) over the half-space \( \mathbb{R}^{n+1}_{t \geq 0} = \mathbb{R}^n \times [0, +\infty) \) with coordinates \((x, t)\), that is, its sections are exterior differential forms on \( \mathbb{R}^n \), whose coefficients depend on the parameter \( t \in [0, +\infty) \):

\[
U = \sum_{|I|=q} U_I(x, t) dx_I, \quad I = (i_1, \ldots, i_q), \quad 1 \leq i_j \leq n, \quad 0 \leq q \leq n,
\]

where, as usual \( dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_q} \), and the symbol \( \wedge \) denotes the exterior product of differential forms.

Consider the following complex of Banach spaces:

\[
0 \rightarrow C^{2s+k,\lambda,0}_{\delta,A^0} \xrightarrow{d_0} C^{2s+k-1,\lambda,0}_{\delta+1,A^1} \xrightarrow{d_1} \cdots \xrightarrow{d_{k-2}} C^{2s+1,\lambda,0}_{\delta+(k-1),A^{k-1}} \xrightarrow{d_{k-1}} C^{2s,\lambda,0}_{\delta+k,A^k} \cap S d^k \xrightarrow{d_k} 0,
\]

where

\[
(d^q U)(x, t) = \sum_{j=1}^n \sum_{|I|=q} \frac{\partial U_I(x, t)}{\partial x_j} dx_I \wedge dx_I.
\]

In the article [8] Theorem 2.1 was extended to bounded operators

\[
\begin{align*}
(d^q, (d^q)^{-1}) & : C^{2s+k,\lambda,0}_{\delta,T,A^q} \rightarrow C^{2s+k,\lambda,0}_{\delta+1,T,A^{q+1}} \cap S_d \oplus C^{2s+k,\lambda,0}_{\delta+1,T,A^{q+1}} \cap S_d^*, \quad k \in \mathbb{Z}_+, \quad (11) \\
(d^q, (d^q)^{-1}) & : C^{2s+k,\lambda,0}_{\delta,T,A^q} \rightarrow C^{2s+k,\lambda,0}_{\delta+1,T,A^{q+1}} \cap S_d \oplus C^{2s+k,\lambda,0}_{\delta+1,T,A^{q+1}} \cap S_d^*, \quad k \in \mathbb{Z}_+, \quad (12)
\end{align*}
\]

where \( C^{2s+k,\lambda,0}_{\delta,T} \) denotes the closure of the set \( C^\infty([0, T], D(\mathbb{R}^n)) \) in the space \( C^{2s+k,\lambda,0}_{\delta,T} \), and \( C^\infty([0, T], D(\mathbb{R}^n)) \), in turn, is the space of infinitely differentiable mappings \( U : [0, T] \rightarrow D(\mathbb{R}^n) \). Thus, we have

**Corollary 3.1:** Let \( n \geq 2, \ s \in \mathbb{Z}_+, \ 0 < \lambda < 1 \). If \( \delta > 0 \) and \( \delta + 1 - n \notin \mathbb{Z}_+ \), then the operators (11) and (12) are normal solvable. Moreover,

- (11) and (12) are isomorphisms if \( 0 < \delta < n - 1 \);
• (11) and (12) are injections with a closed image if \( n - 1 + m < \delta < n + m \) for \( m \in \mathbb{Z}_+ \); more precisely, the image of the operator (11) consists of all pairs \( f \in C^{2s+k,s,0}_{\delta+1,T,A_{q+1}} \cap S_d, \ g \in C^{2s+k,s,0}_{\delta+1,T,A_{q-1}} \cap S_d, \) satisfying

\[
(f(\cdot, t), dh)_{L^2_{A^{q+1}}(\mathbb{R}^n)} + (g(\cdot, t), d^*h)_{L^2_{A^{q-1}}(\mathbb{R}^n)} = 0
\]  

for all \( t \in [0, T] \) and \( h \in H^2_{\leq m+1} \), and the image of the operator (12) consists of all pairs \( f \in C^{2s+k,s,0}_{\delta+1,T,A_{q+1}} \cap S_d, \ g \in C^{2s+k,s,0}_{\delta+1,T,A_{q-1}} \cap S_d, \) satisfying (13).

As in the case of isotropic spaces, we define cocycles, coboundaries, and cohomology as follows. Let \( Z^{2s+k,s,0}_{\delta,A_q} \) denotes the space of cocycles and contain forms belonging to the set \( C^{2s+k,s,0}_{\delta,A_q} \cap S_d \), and \( B^{2s+k,s,0}_{\delta,A_q} \) denotes the space of coboundaries and contains elements \( f \in C^{2s+k,s,0}_{\delta,A_q} \), for which there exists a form \( u \in C^{2s+k+1,s,0}_{\delta-1,A_{q-1}} \) such that the equation \( f = du \) is valid. Finally, let \( H^{2s+k,s,0}_{\delta,A_q} \) denotes the cohomology space of complex (10) and contains forms lying in \( Z^{2s+k,s,0}_{\delta,A_q} / B^{2s+k,s,0}_{\delta,A_q} \).

We now get the following statement.

**Corollary 3.2:** Let \( n \geq 2, s, k \in \mathbb{Z}_+ \), \( 0 < \lambda < 1, \delta > n/2 \) and \( f \in Z^{2s+k,s,0}_{\delta+1,A_{q+1}} \). If the form \( u \in C^{2s+k+1,s,0}_{\delta,A_q} \) is a solution to the equation \( du = f \), then there exists a form \( v \in C^{2s+k+1,s,0}_{\delta,A_q} \), satisfying the system of operator equations

\[
\begin{align*}
  dv &= f, \\
  d^*v &= 0.
\end{align*}
\]

**Proof:** Carrying out the proof as in Corollary 2.3 and relying on Corollary 3.4 of [8], we obtain the assertion of the corollary. 

Finally, we proceed to describe the cohomology spaces of complex (10).

**Theorem 3.3:** Let \( n \geq 2, s \in \mathbb{Z}_+ \), \( 0 < \lambda < 1 \) and \( n + m - 1 < \delta < n + m \). Then the cohomology groups of complex (10) are isomorphic to the image of the operator \( d(\Phi - \Phi_m) \), acting from \( Z^{2s+k,s,0}_{\delta+1,A_{q+1}} \) to \( Z^{2s+k+1,s,0}_{\delta+1,A_{q+1}} \).

**Proof:** The proof is carried out similarly to the proof of Theorem 2.4, relying on Corollary 3.1 instead of Theorem 2.1. 

Note that the operator \((\Phi - \Phi_m)\) has the form

\[
(\Phi - \Phi_m)f(x, t) = \int_{\mathbb{R}^n} f(y, t) \wedge \sum_{|I|=q} \sum_{k=1}^{m+1} \sum_{j=1}^{J(k)} \frac{h^{(j)}_k(x)(d^{n-q-1})^y(h^{(j)}_k(y)(\ast dy_I))}{(n+2k-2)(n^{2k-2}(x))}.
\]
Thus, the elements of the image of the operator $d(\Phi - \Phi_m)$ acting from $Z_{\delta + 1, A^q+1}^{2s+k,s,\lambda,0}$ to $Z_{\delta + 1, A^q+1}^{2s+k,s,\lambda,0}$ are represented as follows:

$$d \left( \sum_{|l|=q} \sum_{k=1}^{m+1} \sum_{j=1}^{f(k)} a_l(t) h_{l}^{(j)}(x) \, dx_l \right),$$

where $a_l(t)$ are functions of a variable $t$ of class $C^0([0, T])$.

### 3.2. Case $\mu = \lambda/2$

A direct generalization of Corollary 3.1 to the scale of spaces $C_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ is impossible, since the operators $\Phi$, $\hat{\Phi}$ do not act continuously on it. Therefore, we need to change slightly the definitions of the spaces. To this end, we denote by $\Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ the completion of the space $C_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ with respect to the norm

$$\|u\|_{\Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}} = \|u\|_{C_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}} + \|du\|_{C_{\delta, T, A^q+1}^{2s+k,s,\lambda,\frac{\lambda}{2}}} + \|d^*u\|_{C_{\delta, T, A^q-1}^{2s+k,s,\lambda,\frac{\lambda}{2}}}.\tag{14}$$

Now we consider an operator of the following form

$$(d, d^*) : \Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}} \to \Gamma_{\delta+1, T, A^{q+1}}^{2s+k,s,\lambda,\frac{\lambda}{2}} \cap S_d \oplus \Gamma_{\delta+1, T, A^{q-1}}^{2s+k,s,\lambda,\frac{\lambda}{2}} \cap S_d^*$$

and formulate for it an analog of Theorem 2.1 and Corollary 3.1.

**Theorem 3.4:** Let $n \geq 2$, $s \in \mathbb{Z}_+$, $0 < \lambda < 1$. If $\delta > 0$ and $\delta + 1 - n \notin \mathbb{Z}_+$, then the operator (14) is normal solvable. Moreover,

- (14) is an isomorphism if $0 < \delta < n - 1$;
- (14) is an injection with a closed image if $n - 1 + m < \delta < n + m$ for $m \in \mathbb{Z}_+$; more precisely, the image of the operator (14) is the set of forms $(f, g) \in \Gamma_{\delta+1, T, A^{q+1}}^{2s+k,s,\lambda,\frac{\lambda}{2}} \oplus \Gamma_{\delta+1, T, A^{q-1}}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ for which conditions

$$df(x, t) = 0,$$

$$d^*g(x, t) = 0,$$

$$(f(\cdot, t), dh)_{L^2} + (g(\cdot, t), d^*h)_{L^2} = 0,$$

are satisfied, and where (16) is true for any $h$ from $H_{\leq m+1}$.

**Proof:** It is easy to check that for $0 < \delta < n - 1$ the operator (14) acts continuously from $\Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ to $\Gamma_{\delta+1, T, A^{q+1}}^{2s+k,s,\lambda,\frac{\lambda}{2}} \cap S_d$. Let us prove that the operators $\Phi$ and $\hat{\Phi}$, acting from $\Gamma_{\delta+1, T, A^{q+1}}^{2s+k,s,\lambda,\frac{\lambda}{2}} \cap S_d$ to $\Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ and from $\Gamma_{\delta, T, A^q-1}^{2s+k,s,\lambda,\frac{\lambda}{2}} \cap S_d$ to $\Gamma_{\delta, T, A^q}^{2s+k,s,\lambda,\frac{\lambda}{2}}$ respectively, are
also continuous. To do this, consider the norm
\[
\| \Phi f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}} \leq \| f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}} + \| d^* \Phi f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}}
\]
Since \( d\Phi f = f \), the second term is \( \| f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}} \). And since \( df = 0 \), then \( d^* \Phi f = 0 \) by Lemma 2.2. Let us estimate the first term using embedding theorems, as well as the continuity of the operator \( \Phi \) from \( C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}} \) to \( C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}} \) and from \( C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}} \) to \( C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}} \). Indeed,

\[
\| \Phi f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}} \leq \| f \|_{C^{2s+k,\lambda,\frac{1}{2}}_{\delta+1,T,A^q^{+1}}}
\]

The corresponding estimate showing the boundedness of the operator \( \hat{\Phi} \), is carried out in a similar way.

Thus, the first part of the theorem immediately follows from the continuity of the operators \( \Phi \), \( \hat{\Phi} \) and \( (d, d^*) \), as well as the completeness of the spaces \( \Gamma_{\delta+1,T,A^q^{+1}}^{2s+k,\lambda,\frac{1}{2}} \) and \( \Gamma_{\delta+1,T,A^q^{+1}}^{2s+k,\lambda,\frac{1}{2}} \).

We divide the rest of the proof into two statements.

**Lemma 3.5:** Let \( s \in \mathbb{Z}_{\geq 0}, k \geq 1, 0 < \lambda < 1 \) and \( n - 1 + m < \delta < n + m \). Then the set of pairs of forms \( (f, g) \), lying in the space \( \Gamma_{\delta+1,T,A^q^{+1} \oplus A^{q'-1}}^{2s+k,\lambda,\frac{1}{2}} \), for which conditions (15) and (16), are valid is closed.

**Proof:** Consider a sequence \( \{f_j, g_j\}_{j=1}^{\infty} \), converging to \( (f, g) \) in the space \( \Gamma_{\delta+1,T,A^q^{+1} \oplus A^{q'-1}}^{2s+k,\lambda,\frac{1}{2}} \), such that conditions (15) and (16) are satisfied for all \( (f_j, g_j) \).

Since \( \{f_j, g_j\}_{j=1}^{\infty} \) converges to a pair \( (f, g) \) in the space \( \Gamma_{\delta+1,T,A^q^{+1} \oplus A^{q'-1}}^{2s+k,\lambda,\frac{1}{2}} \), then \( \{df_j, d^* g_j\}_{j=1}^{\infty} \) converges to \( (df, d^* g) \) in \( C_{\delta+2,T,A^{q+2} \oplus A^{q'-2}}^{2s+k,\lambda,\frac{1}{2}} \). But for any \( j \in \mathbb{N} \) the element \( (df_j, d^* g_j) \) is equal to 0. This means that sequence \( \{df_j, d^* g_j\}_{j=1}^{\infty} \) converges to 0 in the space \( C_{\delta+2,T,A^{q+2} \oplus A^{q'-2}}^{2s+k,\lambda,\frac{1}{2}} \). Since the limit is unique, we can conclude that condition (15) is satisfied.

Let us check the fulfillment of condition (16) for the pair \( (f, g) \):

\[
| (f, dh)_{L_2} + (g, d^* h)_{L_2} | \\
\leq | (f - f_j, dh)_{L_2} | + | (f, dh)_{L_2} | + | (g - g_j, d^* h)_{L_2} | + | (g, d^* h)_{L_2} | \\
= | (f - f_j, dh)_{L_2} | + | (g - g_j, d^* h)_{L_2} |.
\]

(18)
Next, using (18) and the Cauchy inequality we obtain

\[
| (f - f_j, dh)_{L^2} | + | (g - g_j, d^* h)_{L^2} |
\leq \| f - f_j \|_{L^2} \| dh \|_{L^2} + \| g - g_j \|_{L^2} \| d^* h \|_{L^2}
\leq C_1 \| f - f_j \|_{C^2+\lambda_2,T,A^{q-1}} + C_2 \| g - g_j \|_{C^2+\lambda_2,T,A^q}.
\]

and since the norm of \( L^2 \) is weaker than the norm of \( C^2+\lambda_2,T,A^{q+1} \), we see that the right hand side of (19) does not exceed

\[
C_1 \| f - f_j \|_{C^2+\lambda_2,T,A^{q+1}} + C_2 \| g - g_j \|_{C^2+\lambda_2,T,A^{q-1}}.
\]

Thus, we obtain inequality

\[
| (f, dh)_{L^2} + (g, d^* h)_{L^2} | \leq C_1 \| f - f_j \|_{C^2+\lambda_2,T,A^{q+1}} + C_2 \| g - g_j \|_{C^2+\lambda_2,T,A^{q-1}}.
\]

Letting \( j \) in (20) to infinity, we get

\[
(f, dh)_{L^2} + (g, d^* h)_{L^2} = 0.
\]

This means that the subspace of \( \Gamma_{\delta+1,T,A^q+1}^{2s+k,s,\lambda,\frac{1}{2}} \), consisting of the forms satisfying conditions (15) and (16), is closed. \( \square \)

It remains to prove that all elements of the image of operator (14) satisfy conditions (15) and (16).

**Lemma 3.6:** Let \( s \in \mathbb{Z}_{\geq 0}, k \geq 1, 0 < \lambda < 1 \) and \( n - 1 + m < \delta < n + m \). Then the image of the operator (14) is the set of forms \((f,g) \in \Gamma_{\delta+1,T,A^{q+1}}^{2s+k,s,\lambda,\frac{1}{2}}, A^q-1\), for which conditions (15) and (16) are valid.

**Proof:** Let \((f,g) \in \Gamma_{\delta+1,T,A^{q+1}}^{2s+k,s,\lambda,\frac{1}{2}} \subset C^{2s+k,s,\lambda,0}_{\delta,T,A^q+1} \subset C^{2s+k,s,\lambda,0}_{\delta,T,A^q+1} \subset A^q-1\), then, according to Corollary 3.1, the differential form

\[
u = \Phi f + \hat{\Phi} g,
\]

belongs to the space \( C^{2s+k,1,s,\lambda,0}_{\delta,T,A^q} \), and conditions (15) and (16) are fulfilled for it.

Let us show that the form \( u \) is contained in the space \( \Gamma_{\delta,T,A^q}^{2s+k,s,\lambda,\frac{1}{2}} \). Indeed,

\[
\| u \|_{\Gamma_{\delta,T,A^q}^{2s+k,s,\lambda,\frac{1}{2}}} = \| u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,T,A^q}} + \| du \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q+1}} + \| d^* u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^{q-1}}}.
\]

Since \( df = 0 \) and \( d^* g = 0 \), it is known from Lemma 2.2 that \( d\Phi g = 0 \) and \( d^* \Phi f = 0 \). Thus, the second and third terms in the last formula have the form

\[
\| du \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q+1}} + \| d^* u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^{q-1}}}.
\]
Using the representation \( u = \Phi f + \hat{\Phi} g \), in the norm \( \| u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q}} \), carrying out an estimate similar to (17), we obtain that

\[
\| u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q}} \leq \| f \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q+1}} + \| g \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,T,A^q-1}}.
\]

This means that \( u \) is contained in the space \( \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,T,A^q} \). And from (21) we see that \( u \) belongs to the preimage of the forms \( (f, g) \), for which conditions (15) and (16) are valid. \( \blacksquare \)

Thus, Lemmas 3.5 and 3.6 provide a proof of the second part of the theorem. \( \blacksquare \)

As in the previous sections, we define cocycles, coboundaries, and cohomology on spaces \( \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \). Let the set of coboundaries \( Z^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \) consist of all forms \( f \in \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \), for which \( df = 0 \), and the set of cocycles \( B^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \) contains \( f \in \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \) such that there exists \( u \in \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta-1,A^{q-1}} \) satisfying the equation \( du = f \). As usual, cohomology groups are defined as \( H^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} = Z^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} / B^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \). Now we can consider the following complex of Banach spaces:

\[
0 \to \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \xrightarrow{d_0} \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,A^q} \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+(n-1),A^{q-1}} \xrightarrow{d_{n-1}} \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+n,A^q} \xrightarrow{d_n} 0 \quad (22)
\]

**Corollary 3.7:** Let \( n \geq 2, s, k \in \mathbb{Z}_+, 0 < \lambda < 1 \) and \( f \in Z^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,A^{q+1}} \). If the form \( u \in \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \) is a solution to the equation \( du = f \), then there exists a form \( v \in \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \), satisfying the system of operator equations

\[
\begin{aligned}
\begin{cases}
v = f, \\
d^* v = 0.
\end{cases}
\end{aligned}
\]

**Proof:** Since the space \( C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \) is embedded continuously into the space \( C^{2s+k,s,\lambda,0}_{\delta,A^q} \), the form \( u \) can be represented in the form

\[
u = d\Phi u + d^* \hat{\Phi} u. \quad (23)
\]

Consider the norm of an operator \( d^* \hat{\Phi} \), acting on \( \Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \):

\[
\| d^* \hat{\Phi} u \|_{\Gamma^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q}} = \| d^* \hat{\Phi} u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q}} + \| d\Phi u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,A^{q+1}}} + \| d^* d^* \hat{\Phi} u \|_{C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta+1,A^{q+1}}}
\]

Here the last term is equal to zero, because \( d^* \circ d^* = 0 \). Using the embedding theorems and the continuity of the operator \( d^* \hat{\Phi} \) on the spaces \( C^{2s+k,s,\lambda,0}_{\delta,T,A^q} \) and \( C^{2s+k,s,\lambda,\frac{1}{2}}_{\delta,A^q} \), we obtain an
estimate for the first term:

\[ \| d^* \hat{\Phi} f \|_{C^{2s+k,0}_{\delta,1,\Lambda q+1}} \leq \| f \|_{C^{2s+k,0}_{\delta,1,\Lambda q+1}} + \sum_{j=1}^{s} \sup_{t, \tau \in [0,T]} \frac{\| \partial^j (f(\cdot, t) - f(\cdot, \tau)) \|_{C^{2s+k,0}_{\delta,1,\Lambda q+1}}}{|t - \tau|^{\frac{k}{2}}} = \| f \|_{C^{2s+k,0}_{\delta,1,\Lambda q+1}}. \]

To estimate the second term, we apply the operator \( d \) to Equation (23) and obtain

\[ \| dd^* \hat{\Phi} u \|_{\Gamma^{2s+k,0}_{\delta,1,\Lambda q+1}} = \| f \|_{\Gamma^{2s+k,0}_{\delta,1,\Lambda q+1}}. \]

Thus, \( v = d^* \hat{\Phi} u \) belongs to the space \( \Gamma^{2s+k,0}_{\delta,1,\Lambda q} \) and, similarly to Corollaries 3.2 and 2.3, is the desired form.

We can now describe the cohomology groups of complex (22).

**Theorem 3.8:** Let \( n \geq 2, s \in \mathbb{Z}_+, 0 < \lambda < 1 \) and \( n + m - 1 < \delta < n + m \). Then the cohomology groups of complex (22) are isomorphic to the image of the operator \( d(\Phi - \Phi_m) \) acting from \( Z^{2s+k,0}_{\delta,1,\Lambda q+1} \) to \( Z^{2s+k,0}_{\delta,1,\Lambda q+1} \).

**Proof:** The proof is carried out similarly to the proof of Theorem 2.4, relying on Corollary 3.7 instead of Theorem 2.1.

Also, as in the previous case, the elements of the image of the operator \( d(\Phi - \Phi_m) \) acting from \( Z^{2s+k,0}_{\delta,1,\Lambda q+1} \) to \( Z^{2s+k,0}_{\delta,1,\Lambda q+1} \) are represented as follows

\[ d \left( \sum_{|l|=q} \sum_{k=1}^{m+1} \sum_{j=1}^f a_l(t) h_k^{(j)}(x) \frac{dx_l}{(n + 2k - 2) \delta^{n+2k-2}(x)} \right), \]

where \( a_l(t) \) are functions of the variable \( t \) of the class \( C^{\frac{k}{2}}([0, T]) \).

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