A thermodynamical model for paleomagnetism in Earth’s crust

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Abstract
A thermodynamically consistent model for soft deformable viscoelastic magnets is formulated in actual space (Eulerian) coordinates. The possibility of a ferro-paramagnetic-type (or ferri-antiferromagnetic) transition exploiting the Landau phase-transition theory as well as mechanical melting or solidification is considered, being motivated and applicable to paleomagnetism (involving both thermo- and isothermal and viscous remanent magnetization) in rocks in Earth’s crust and to rock-magma transition. The temperature-dependent Jeffreys rheology in the deviatoric part combined with the Kelvin-Voigt rheology in the spherical (volumetric) part is used. The energy balance and the entropy imbalance behind the model are demonstrated, and its analysis is performed by time discretization, proving existence of weak solutions.

Keywords
Thermo-viscoelasticity, creep, Euler description, ferro-paramagnetic transition, melting/solidification, objective time derivatives, weak solutions.

1. Introduction
Magnetism in our planet Earth has two main mechanisms. The mains source of geomagnetic field is in the fluidic part of the Earth core, called the outer core, composed primarily from electrically conductive hot Iron with Nickel flowing in high speed of the order 10 km/year and inducing magnetic field via magnetodynamo effect. The second mechanism is in the very upper part of the silicate mantle, called the crust, which is rather cold and exhibits para-to-ferro (or antiferro-to-ferrimagnetic) phase transition under the geomagnetic field generated primarily by magnetic dynamo in the fluidic Iron-Nickel outer core, considered time-space dependent but given in this paper. These two very different magnetic phenomena are primarily related, respectively, with the names of Hannes O. G. Alfvén and Louis E. F. Néel, both Nobel prize winners in 1970. For completeness, let us mention that other contributions are due to the (relatively weak) magnetodynamo effects in moving salty oceans or in hot silicate mantle and due to the magnetic field from the Sun which interacts mainly with the magnetosphere around the Earth and does not substantially influence Earth’s interior.

In this paper, we will focus on the second phenomenon of magnetism in rocks containing magnetic materials, called paleomagnetism. Magnetism in some rocks forms a vital part of rock physics and mechanics (cf. [1–5]). Magnetism in the oceanic or the continental Earth crust is an important phenomenon which gives information about history of geomagnetic field and about various mechanical processes, e.g., behind folding and faulting of rocks or formation mountains or even or even movement of continents in the far history of the Earth or even in some other terrestrial-type planets/moons as particularly Mars and “our” Moon. As in all magnetic materials,
the magnetism is heavily dependent on temperature and, above certain “critical” temperature disappears because spontaneous magnetism in ferromagnetic materials undergoes the transition to a non-magnetic paramagnetic variant. Ferromagnetism is caused by a synchronized (parallel) orientation of spins of atoms in crystal lattices while paramagnetism is characterized by random orientation. In the case of ferro-to-para magnetic transition, the mentioned critical temperature is called the Curie temperature.

Actually, in rocks, orientation of magnetic moments in neighbouring atoms is opposite. Their magnitude can be equal or one of them can dominate. Then, we speak about antiferro- or ferri-magnetism, respectively. The spontaneous magnetization in antiferromagnets is zero, similarly as in paramagnets. Like in ferro/para-magnetism, it depends on temperature and, instead of the Curie temperature, we speak about the Néel temperature.

There are several mechanisms behind natural remanent magnetism in cold rocks. The most typical is related with the antiferro-to-ferri-magnetic phase transition (phenomenologically quite similar to para-to-ferromagnetic phase transition) during cooling of initially hot rock containing magnetic minerals in the geomagnetic field. This is referred to as a thermoremanent magnetization (TRM) (cf. Figure 1).

Another magnetization in cold rock with magnetic minerals might be by strong magnetic fields typically due to lightning strikes, called an isothermal remanent magnetization (IRM). Beside these two, there is also a viscous remanent magnetization (VRM) which may occur when rocks are exposed by a modern-day geomagnetic field which is stronger than geomagnetic field but anyhow not so strong to lead to an immediate (re)magnetization. Actually, there are some other processes and mechanisms occasionally relevant in paleomagnetism, as chemical or depositional (detrital) remanent magnetization but we will not address them in our model. All these mechanisms belong to paleomagnetism (cf. in particular [1, 6] and historical references [7, 8]).

The scope of this paper is the following: In section 2, we briefly present relevant physics and formulate the governing system of partial differential equations. Then, in section 3, we justify the model as far as its desired energy balances and the entropy imbalance. Finally, in section 4, the analysis of an initial-boundary-value problem for the system from section 2 is performed by time discretization, proving existence of suitably defined weak solutions.

2. The thermo-magneto-mechanical model

Before writing the system of partial differential equations describing the model, let us first articulate the main concepts and motivation we want to employ, and related attributes of the model.

The main attribute of the model will be its formulation in Eulerian coordinates. This reflects the reality that, within geological time scales of instantaneous evolution of Earth, there is no reference configuration. It contrasts to engineering work-pieces where a reference configuration may refer to the shape in which they have
be manufactured and the Lagrangian description which uses such a reference configuration is well motivated even for largely strained elastic solids.

Rocks considered in a limited (and not much large) space-time region of the crust can be considered not much compressible, and in particular, the mass density does not vary substantially and can be (and often is) considered constant. This simplifies the model and its analysis substantially. Often, the models are simplified even more by imposing incompressibility. Yet, incompressible models do not facilitate propagation of longitudinal seismic waves, which substantially limit geophysical applications, although in some situation (in particular in paleomagnetism), even quasistatic models which neglects all dynamical effects have reasonable applications. Yet, forgetting inertial effects seems to bring analytical difficulties by lacking an immediate control on acceleration. Therefore, we will use a dynamical and not fully incompressible model but we will adopt a compromise concept of only slightly compressible (so-called semi-compressible) materials [9] with mass densities constant in space and time (cf. Remark 6 for a fully compressible model).

Various gradient theories are applicable and allow for inventing various dispersion of velocities of elastic waves into the model and facilitate its analysis (cf. [9]). The particular enhancement by dissipative gradient terms exploits the general ideas of multipolar (also called non-simple) media by Green and Rivlin [10] adopted for fluids by Nečas at al. [11–14] and Fried and Gurtin [15]. More specifically, we will use non-linear second-grade non-simple fluids.

Essentially, rocks will then be considered as viscoelastic fluids but very high viscosity (typically of the order $10^{22} \pm 2$ Pa s) unless being melted to magma when the viscosity drops down substantially (typically being of the order $10^{4} \pm 3$ Pa s) (cf. Figure 2). This rock-magma transition will be covered by our model, too. Of course, the magma is hot and surely much above the Curie temperature, thus nonmagnetic.

For TRM of rocks, we need a ferro-para-magnetic transition that is formulated and analyzed in Podio-Guidugli et al. [16] by using the Landau [17] idea. Possibly, this may be quite equally interpreted as ferro-antiferro-magnetic transition, too.

Further important attribute is a proper choice of rates (time derivatives) of intensive variables in the Eulerian setting. The adjective “intensive” refers to variables or properties which do not depend on the system size or the amount of material in the system, as e.g., velocity or temperature, in contrast to extensive variables as, e.g., momentum or entropy. Intensive scalar variables are transported by the convective (also called material) time derivative defined, for a scalar variable $\alpha$ and velocity field $v$, as

$$\dot{\alpha} = \frac{\partial \alpha}{\partial t} + v \cdot \nabla \alpha.$$  \hspace{1cm} (1)

It is applicable componentwise also for a velocity vector itself for which it reads as $\dot{v} = \frac{\partial}{\partial t} v + (v \cdot \nabla) v$, i.e., the last term means componentwise $[(v \cdot \nabla) v]_i = \sum_{j=1}^{d} v_j \frac{\partial}{\partial x_j} v_i$. This convective derivative is however not objective for general vector- or tensor-valued intensive variables (as, e.g., magnetization or stress) which have to be transported by other time derivatives, however. Objectivity here means that the time derivatives do not depend on the frame of reference. For such rates, it is reasonable to require, beside objectivity, also being the so-called corotational. The simplest corotational form is the Zaremba–Jaumann time derivative [18, 19] justified
for stress rates by M. Biot [20, p. 494] (cf. also [21]), defined as:

$$\dot{A} = \frac{\partial A}{\partial t} + (v \cdot \nabla) A - WA + AW$$

with “skw” denoting the skew-symmetric part, i.e., (change globally) $\text{skw}(\nabla v) = \frac{1}{2}(\nabla v - (\nabla v)^\top)$. In isotropic materials, it is relevant also for strain rates (cf. [22]). The important attribute of the corotational derivative is that it commutes with the transposition and with convective derivatives for for traces and pressure-like tensors:

$$(A^\top)^* = (A^*)^\top, \quad \text{tr} \dot{A} = (\text{tr} A)^*, \quad \text{and} \quad (a I)^* = \hat{a} I.$$ (3)

These properties are important to keep properties of symmetric or deviatoric preserved during evolution governed by such derivatives as noticed, e.g., in Thielmann et al. [23], and to gain the expected convective derivative for transition from solid to fluid models during melting of rocks to magmas as noticed in Roubíček [22]. Consistently with the Zaremba–Jaumann corotational derivative used for stress or strain tensors in the model, it is natural to use it also for vectors (cf., e.g., [24, Sect. 5.5]), defined as:

$$\dot{m} = \frac{\partial m}{\partial t} + (v \cdot \nabla)m - Wm$$

where again $W = \text{skw}(\nabla v)$. (4)

Actually, it was used for magnetization in [25, Sect. 2.5].

We neglect thermal expansion (and the related buoyancy effects – cf. Remark 4) and also we neglect variation of mass density not only by thermal expansion but also by pressure. Anyhow, we admit the medium being slightly compressible. This is a reasonable compromise for solid or fluid geophysical materials if one does not want to exclude pressure (longitudinal) waves like it would happen in fully incompressible models. However, allowing for a slight compressibility but taking fixed mass density is relevant if pressure variations are negligible in comparison with elastic bulk modulus (which is usually quite large of the order of GPa’s). It leads either to allowing for a slight compressibility but taking fixed mass density is relevant if pressure variations are negligible in comparison with elastic bulk modulus (which is usually quite large of the order of GPa’s). It leads either to (slight) violation of Galilean invariancy (cf. Remark 6). Here, we will adopt the latter option. Fully convective variants of such a compromise fluidic models are sometimes called quasi-compressible (cf. [9]). In the former energy-violating variant, they are sometimes used in physical modelling (cf. e.g. [27–29]).

We consider a fixed bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $3$ and with a Lipschitz boundary $\Gamma$ and a time interval $I = [0, T]$. The basic variables and data for this preliminary model are summarized in the following table:

| Symbol | Description |
|--------|-------------|
| $v$    | velocity (in m/s) |
| $E$    | the elastic-strain tensor (symmetric) |
| $\Pi$  | the inelastic-strain tensor (deviatoric) |
| $R$    | the inelastic-strain rate (in s$^{-1}$) |
| $m$    | magnetization vector (in A/m) |
| $r$    | magnetization rate (in Am$^{-1}$s$^{-1}$) |
| $b$    | magnetic induction (in T) |
| $h_{\text{driv}}$ | driving magnetic field (in A/m) |
| $h_{\text{dem}}$ | intensity of demagnetizing field (in A/m) |
| $u$    | potential of demagnetizing field (in A) |
| $\theta$ | temperature (in K) |
| $w$    | enthalpy (in Pa = J/m$^3$) |
| $\zeta$ | magnetic dissipation potential (in Pa/s = W/m$^3$) |
| $\mu_0$ | vacuum permeability ($\approx 1.257 \times 10^{-6}$ H/m) |
| $\eta$ | entropy (in Pa/K) |
| $\psi$ | Helmholtz free energy (in Pa) |
| $c(v)$ | small strain rate (in s$^{-1}$) |
| $\Delta$ | elastic stress (symmetric - in Pa) |
| $\Delta_v$ | viscous stress (symmetric - in Pa) |
| $\Delta_c$ | capillarity/couple stress (in Pa) |
| $\delta$ | hyperstress (in Pa m) |
| $\rho$ | mass density (in kg/m$^3$) |
| $g$    | gravity acceleration (in ms$^{-2}$) |
| $M$    | Maxwell viscosity modulus (in Pa s) |
| $\nu_1$ | Stokes viscosity modulus (in Pa s) |
| $K$    | heat conductivity (in Wm$^{-1}$K$^{-1}$) |
| $j_{\text{ext}}$ | external heat flux (in W/m$^2$) |
| $h_{\text{geo}}$ | intensity of geomagnetic field |

The basic ingredient of the model is a specific free energy $\psi$. The simplest form in terms of the elastic strain $E$ with a rather weak (linearized) coupling of mechanical and magnetic effects is $\psi(E, m, \theta) = \phi_0(E, m) + \theta \varphi_1(m) + \mu_0 h_{\text{dem}} m - \phi(\theta)$ or, for the purpose of the analysis below, rather as

$$\psi(E, m, \theta) = \varphi(E, m) + \theta \omega(m) + \mu_0 h_{\text{dem}} m - \phi(\theta),$$ (5)
where \( h_{\text{dem}} \) is the intensity of demagnetizing field satisfying the equations:

\[
\nabla \times h_{\text{dem}} = 0 \quad \text{and} \quad \nabla \cdot (h_{\text{dem}} + \chi_{\Omega} \mathbf{m}) = 0 \quad \text{on} \quad \mathbb{R}^d,
\]

which is the quasistatic rest of the Maxwell electromagnetic system in electrically non-conductive magnetic media. The second equation is the Gauss law for magnetism \( \nabla \cdot \mathbf{b} = 0 \) with the magnetic induction \( \mathbf{b} = \mu_0 (h_{\text{dem}} + \chi_{\Omega} \mathbf{m}) \) with \( \mu_0 \) the vacuum permeability. Here, \( \chi_{\Omega} \) is the characteristic function of the domain \( \Omega \), i.e. \( \chi_{\Omega}(x) = 1 \) for \( x \in \Omega \) while \( = 0 \) otherwise. The static system (6) is thus considered on the whole Universe \( \mathbb{R}^d \) at each time instant. The first equation in (6) implies existence of a (scalar) magnetic potential \( u \) such that \( h_{\text{dem}} = -\nabla u \). Thus equation (6) can be written as a single Poisson equation:

\[
\Delta u = \nabla \cdot (\chi_{\Omega} \mathbf{m}) \quad \text{on} \quad \mathbb{R}^d
\]

to be understood in the sense of distributions.

Let us now write the system of six partial differential equations (one of them being an inclusion) for \( v, \mathbf{E}, \Pi, \mathbf{m}, u, \) and \( \theta \). More specifically, it is composed from a momentum equation, the Green–Naghdi’s additive decomposition of the total strain written in terms of rates, a flow rule for the inelastic strain \( \Pi \), a flow rule for the magnetization \( \mathbf{m} \), the Poisson equation for the demagnetizing-field potential \( u \), and the heat-transfer equation for temperature \( \theta \). Namely,

\[
\varphi \dot{v} = \nabla \cdot (S_e + S_v + S_c - \mu_0 (h \cdot \mathbf{m}) I - \nabla \phi) + \mu_0 (\nabla h)^\top m - \frac{\partial}{\partial t} (\nabla v) \cdot \varphi + \varphi g,
\]

where \( S_e = \psi'_{\mathbf{E}}(\mathbf{E}, \mathbf{m}) - \psi(\mathbf{E}, \mathbf{m}, \theta) \), \( S_v = v_1 e(v) \), \( h = h_{\text{geo}} + h_{\text{dem}} \),

\[
\text{and} \quad S_c = \frac{\kappa \mu_0}{\gamma} \nabla m \otimes \nabla m - \frac{1}{2} |\nabla m|^2 I \quad - \frac{\mu_0}{\gamma} \text{skw}(h_{\text{dev}} \otimes m)
\]

couple stress due to magnetic dipoles

\[
\text{and} \quad h_{\text{dem}} = -\nabla u, \quad \dot{\phi} = v_2 |\nabla v|^p - 2 \nabla v \cdot \nabla u,
\]

\[
\dot{\mathbf{E}} + \dot{\Pi} = e(v),
\]

\[
M(\theta)\dot{\Pi} + \text{dev} S_e = \kappa \Delta \dot{\Pi},
\]

\[
\partial_m \xi(\theta; \mathbf{m}) \ni \dot{h}_{\text{dev}} \quad \text{with} \quad \dot{h}_{\text{dev}} = h - \frac{\psi(\mathbf{E}, \mathbf{m}, \theta)}{\mu_0} + \kappa \Delta m,
\]

\[
\Delta u = \nabla \cdot (\chi_{\Omega} \mathbf{m}) \quad \text{on} \quad \mathbb{R}^d,
\]

\[
\frac{\partial}{\partial t} + \nabla (w - \kappa (\theta) \nabla \nabla) = \xi(\theta; e(v), \dot{\Pi}, \dot{\mathbf{m}}) + \theta \omega' \mathbf{m} \cdot \dot{\mathbf{m}} + \left( \theta \omega(\mathbf{m}) + \phi(\theta) \right) \nabla v
\]

with \( w = \gamma(\theta) \) and \( \xi(\theta; e, \dot{\Pi}, \dot{\mathbf{m}}) = \partial_m e + \phi + \nabla e + M(\theta) \dot{\Pi}^2 \]

\[
\quad + \kappa |\nabla \mathbf{m}|^2 + \mu_0 \partial_m \xi(\theta; \dot{\mathbf{m}} \cdot \mathbf{m}) \cdot \mathbf{m}.
\]

The equations (8a-d,f) are considered on the domain \( \Omega \), in contrast to (8e) which is considered on the whole \( \mathbb{R}^d \). Of course, the corotational derivatives \( \dot{E} \) and \( \dot{\Pi} \) are from equation (2) and \( \dot{\mathbf{m}} \) is from equation (4). Note that, in view of equation (5), we could equivalently write \( S_e = \psi'_{\mathbf{E}}(\mathbf{E}, \mathbf{m}, \theta) - \psi(\mathbf{E}, \mathbf{m}, \theta) \). where all the equations/inclusion except equation (8e) is considered on the domain \( \Omega \). For the capillarity-like term and the skew-symmetric stress, see also [25, 30]. The magnetization flow rule (8d) with the corotational derivative \( \dot{\mathbf{m}} \) (see also [25, 31]) where it is articulated that the magnetization is “frozen” in the deforming medium if \( \dot{\mathbf{m}} = 0 \) which then means that the magnetization is transported and rotates at the same local rate as the deforming medium; this is the situation below the blocking temperature \( \theta_b \) and when the total driving field \( h_{\text{dev}} \) has small magnitude.

The first term in equation (5) allows for involving also magnetostrictive effects in general. Its simplest (and isotropic) form without magnetostrictive effects (which can surely be neglected in paleomagnetism in rocks) is:

\[
\varphi(\mathbf{E}, \mathbf{m}) = \frac{d}{2} K_e |\text{sph} E|^2 + G_e |\text{dev} E|^2 + b_0 |\mathbf{m}|^4 - a_0 \theta_c |\mathbf{m}|^2 \quad \text{and} \quad \omega(\mathbf{m}) = a_0 |\mathbf{m}|^2
\]
where \( K_E \) is the bulk elastic modulus, \( G_E \) is the shear elastic modulus, and with some \( a_0 > 0 \) and \( b_0 \) and with \( \theta_c \) denoting the Curie temperature. Neglecting the demagnetizing field, i.e. for \( h_{\text{dem}} = 0 \), the minimum of such \( \psi(E, \cdot, \theta) \) with respect to \( m \) is attained at the orbit \( |m| = m_s \) with the saturation magnetization:

\[
m_s = m_s(\theta) = \begin{cases} \sqrt{a_0(\theta_c-\theta)/b_0} & \text{for } \theta < \theta_c, \\ 0 & \text{for } \theta \geq \theta_c; \end{cases}
\]  
(10)

note that \( \psi(E, \cdot, \theta) \) from equation (9) becomes convex with the single minimizer \( m = 0 \) for \( \theta \) above the Curie temperature \( \theta_c \) (see Figure 2).

The further ingredient for building the model is the (pseudo)potential of dissipative forces as a convex functional of rates. In our case, we will choose it temperature dependent as:

\[
(v, \dot{\Pi}, \dot{m}) \mapsto \frac{1}{2} |e(v)|^2 + \frac{\nu_1}{p} |\nabla e(v)|^p + \frac{M(\theta)}{2} |\dot{\Pi}|^2 + \frac{\kappa}{2} |\nabla \Pi|^2 + \zeta(\theta; \dot{m}).
\]  
(11)

For notational simplicity, we consider the same Stokes-type (hyper)viscosity in the deviatoric and the spherical parts by considering single parameters \( \nu_1 \) and \( \nu_2 \). The simplest dissipation potential \( \zeta = \zeta(\theta; \dot{m}) \) which gives the typical hysteretic response in the \( m/h \)-diagrammes is non-differentiable at the magnetization rate \( \dot{m} = 0 \):

\[
\zeta(\theta; \dot{m}) = h_c(\theta)|\dot{m}| \quad \text{with} \quad h_c(\theta) = \begin{cases} \text{high} & \text{for } \theta \text{ below } \theta_b, \\ \text{low (or zero)} & \text{for } \theta \text{ above } \theta_b, \end{cases}
\]  
(12)

For temperature below blocking temperature \( \theta_b \), the subdifferential of \( \zeta(\theta; \cdot) \) is then multivalued at \( \dot{m} = 0 \), as depicted in Figure 3(a). Together with equation (9), this simplest scenario equation (9) gives the hysteretic loops as in Figure 3(b).

The slope of the hysteretic loops in Figure 3(b) is due to demagnetizing effects and depends on the shape of the magnets; actually, the loops typically are rather curved than piecewise affine (cf., e.g., [32]). Actually, the models (9)–(12) cover also the IRM when the intensity of the external magnetic field \( h_{\text{geo}} \) is sufficiently large.

To cover also the VRM, one should modify the dissipation potential (12) in a small neighbourhood of 0. Conceptually, one can consider:

\[
\zeta(\theta; \dot{m}) = h_c(\theta)|\dot{m}| + \epsilon|\dot{m}|^p + \begin{cases} \tau_c(\theta)|\dot{m}|^2 & \text{for } |\dot{m}| \leq m_r, \\ \tau_c(\theta)m_r^2 & \text{for } |\dot{m}| > m_r, \end{cases}
\]  
(13)
with \( h_c \) from (12) and with some presumably large \( \tau_c(\theta) \) and presumably very small \( \epsilon > 0 \) and small \( m_r > 0 \); the physical unit of the magnetization rate \( m_r \) is \( \text{Am}^{-1}\text{s}^{-1} \) while \( \tau_c \) is some time constant (in the dimension seconds) similarly as \( \epsilon \) if the exponent \( r \) would equal 2. The potential \( \zeta(\theta; \cdot) \) is strictly convex, uniformly with respect to \( \theta \), which will simplify the analysis. A subdifferential of such a potential is illustrated in Figure 4.

An important modelling assumption that the elastic-strain tensor \( \mathbf{E} \) is kept symmetric during the evolution, which is granted by initial conditions on both \( \mathbf{E} \) and \( \Pi \) and by the choice of the objective time derivative as corotational (cf. the first property in (3)). Then, also the elastic-stress tensor \( \mathbf{\phi}' \mathbf{E} \) is symmetric. Moreover, we will use the second property in equation (3) and the initial condition on \( \Pi \) to keep \( \Pi \) not only symmetric but also trace-free, i.e., deviatoric.

Remark 1 (Jeffreys and Kelvin-Voigt rheologies). To reveal the rheological model behind equation (8a–c), assuming for a moment that \( \kappa = 0 \) and \( \nu_2 = 0 \), we can eliminate the internal variables \( \mathbf{E} \) and \( \Pi \) and obtain a relation between the total stress \( \mathbf{S} \) and total strain rate \( \mathbf{e}(v) \). Denoting by \( \mathbf{C} \), \( \mathbf{D} \), and \( \mathbf{M} \), the tensors of elasticity modulus, the Stokes-viscosity modulus, and the Maxwellian viscosity modulus, respectively, we can write

\[
\mathbf{S}_e = \mathbf{C} \mathbf{E} = \mathbf{D} \Pi \quad \text{and} \quad \mathbf{S}_v = \mathbf{D} \mathbf{e}(v).
\]

Using also equation (8b), by some algebraic manipulation, we obtain:

\[
\mathbf{C}^{-1} \dot{\mathbf{S}} + \mathbf{M}^{-1} \mathbf{S} = \mathbf{C}^{-1} \mathbf{D} \mathbf{e}(v) + (\mathbf{I} + \mathbf{M}^{-1} \mathbf{D}) \mathbf{e}(v),
\]

at small strains (see [33, Sect. 6.6]). This is a relation for the Jeffreys rheology written in Eulerian coordinates. For vanishing \( \mathbf{D} \), equation (14) degenerates to:

\[
\mathbf{C}^{-1} \dot{\mathbf{S}} + \mathbf{M}^{-1} \mathbf{S} = \mathbf{e}(v),
\]

which is the relation for the Maxwell rheology. For \( \mathbf{M} \) very large “going to infinity”, i.e., for \( \mathbf{M}^{-1} \) vanishing, equation (14) degenerates to:

\[
\dot{\mathbf{S}} = \mathbf{D} \mathbf{e}(v) + \mathbf{C} \mathbf{e}(v),
\]

which is the relation for the Kelvin–Voigt rheology. In terms of acceleration \( \mathbf{\dot{v}} \), one can also express \( \mathbf{e}(v) \) occurring in equations (14) and (16) as \( \mathbf{e}(\mathbf{\dot{v}}) = \text{sym}(\nabla \mathbf{v} \nabla \mathbf{v}) + \text{skw}(\nabla \mathbf{v} \mathbf{\top} \nabla \mathbf{v}) \). In equations (9) and (11), the above moduli can be written componentwise as \( [\mathbf{C}]_{ijkl} = K_{ik} \delta_{ij} \delta_{kl} + G_{ik} d_{ijkl} \) with \( d_{ijkl} = \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} - 2 \delta_{ij} \delta_{kl} / d \) with \( \delta \) being the Kronecker symbol and with \( K_{ik} \) and \( G_{ik} \) from equation (9). \( [\mathbf{D}]_{ijkl} = \nu_1 d_{ijkl} \), and \( \mathbf{M} = M(\theta) d_{ijkl} \). This means that our model combines temperature-dependent Jeffreys rheology in the deviatoric part with the Kelvin–Voigt rheology in the spherical (volumetric) part.
Remark 2 (Elimination of $\Pi$). In fact, $\Pi$ itself occurs in equation (8) only as its corotation rate $\dot{\Pi}$. Thus, introducing the new variable $\mathbf{R} := \dot{\mathbf{H}}$, one can write equation (8) in terms of $\mathbf{R}$ without $\Pi$. In particular, the parabolic equation (8c) turns into the elliptic equation $M(\theta)\mathbf{R} + \text{dev} \mathbf{S}_e = \kappa \Delta \mathbf{R}$ and one does not need any initial condition for $\Pi$. Knowing a solution $(v, \mathbf{K}, \mathbf{m}, u, \theta)$ of such system and prescribing an initial condition for $\Pi$, one can then reconstruct the inelastic strain $\Pi$ by solving the non-homogeneous transport equation $\dot{\Pi} = \mathbf{R} + \text{skw}(\nabla v)\Pi - \Pi \text{skw}(\nabla v)$.

3. Energetics and thermodynamics of the model

To reveal the energetics behind the system, we must specify some boundary conditions for the particular equations (8a,c–f), say:

$$v \cdot \mathbf{n} = 0, \nabla e(v) : (n \otimes n) = 0, \tag{17a}$$

$$[(\mathbf{S}_e + \mathbf{S}_v + \mathbf{S}_c - \mu_0 (h \cdot \mathbf{m}) \mathbb{I} - \text{div} \mathbf{J}) \mathbf{n} + \text{div}(\mathbf{J} \mathbf{m})]_\Gamma = 0, \tag{17b}$$

$$(n \cdot \nabla)\Pi = 0, \quad (n \cdot \nabla)\mathbf{m} = 0, \quad \lim_{|x| \to \infty} u(x) = 0, \quad \text{and} \tag{17c}$$

$$n \cdot \mathbf{K}(\theta) \nabla \theta = j_{\text{ext}}, \tag{17d}$$

where $\text{div} \mathbf{J} = \text{tr}(\nabla \mathbf{J})$ with $\text{tr}(\cdot)$ being the trace of a $(d-1) \times (d-1)$-matrix, denotes the $(d-1)$-dimensional surface divergence and $\nabla v = \nabla v - \frac{\text{div} v}{\text{div} n} n$ being the surface gradient of $v$. The first condition in equation (17a) allows for considering a fixed domain $\Omega$, otherwise the situation would be extremely complicated because, beside other analytical complications, evolving domains might exhibit self-penetration. The condition in equation (17b) employs the notation $[ \cdot ]_\Gamma$ for the tangential component of a vector. The last condition involves an external heat flux $j_{\text{ext}}$ considered prescribed.

To see the magneto-mechanical energy balance behind the system (8a–c) is seen by testing the equation (8a) by $v$, equation (8c) by $\dot{\mathbf{H}}$, equation (8d) by $\dot{\mathbf{m}}$, and equation (8e) by $\mu_0 \frac{\partial}{\partial t} u$. In this section, we perform the calculus only formally, i.e. assuming that the system (8) with the boundary conditions (17) has a solution which is sufficiently smooth.

Executing the first mentioned test of equation (8a) by $v$, the inertial terms integrated over $\Omega$ gives, using the Green formula and the boundary condition $v \cdot \mathbf{n} = 0$, that:

$$\int_\Omega \varrho \left( \dot{v} + \frac{1}{2} v \text{div} v \right) \cdot v \, dx = \int_\Omega \varrho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v + \frac{1}{2} v \text{div} v \right) \cdot v \, dx$$

$$= \frac{d}{dt} \int_\Omega \frac{\varrho}{2} |v|^2 \, dx + \int_\Omega \frac{\varrho}{2} |v|^2 \text{div} v - \varrho v \cdot \text{div}(v \otimes v) - |v|^2 \left( v \cdot \nabla \varrho \right) \, dx + \int_{\Gamma} \frac{\varrho}{2} |v|^2 (v \cdot \mathbf{n}) \, dS$$

$$= \frac{d}{dt} \int_\Omega \frac{\varrho}{2} |v|^2 \, dx + \int_\Omega \varrho \left( \frac{1}{2} |v|^2 \text{div} v - v \cdot \text{div}(v \otimes v) \right) \, dx = \frac{d}{dt} \int_\Omega \frac{\varrho}{2} |v|^2 \, dx,$$  

where we used also that $\int_\Omega \varrho v \cdot \text{div}(v \otimes v) \, dx = \int_\Omega |v|^2 \text{div} v + (v \otimes v) \nabla v \, dx \neq 0$, $\int_{\Gamma} |v|^2 (v \cdot \mathbf{n}) \, dS \neq 0$, $\int_\Omega |v|^2 \text{div} v - v \cdot \text{div}(v \otimes v) \, dx \neq 0$, and the Green formula with the boundary condition $v \cdot \mathbf{n} = 0$. For the stress terms in equation (8a), we use simply the Green theorem. The peculiarity is behind the second-grade hyperstress for which we should use twice the Green formula over $\Omega$ and once a surface Green formula over the boundary $\Gamma$ (cf. [34, Sect. 2.4.4]) for details. The other stress terms need only once the Green theorem, which leads to $\int_\Omega \mathbf{S}_e : \mathbf{e}(v) \, dx$, $\int_\Omega \mathbf{S}_c : \mathbf{e}(v) \, dx$, and $\int_\Omega \mathbf{S}_e : \nabla v \, dx$. The first term $\mathbf{S}_c : \mathbf{e}(v)$ is balanced with terms resulting from equations (19) and (24) below, while the second term $\mathbf{S}_c : \mathbf{e}(v)$ contributes to the dissipation rate $\xi$ in equation (8f). The last term $\mathbf{S}_c : \nabla v$ is balanced with the particular terms arising in equations (22) and (25).
The flow rule (8c) for $\Pi$ is to be multiplied by $\tilde{\Pi}$. From the term $\text{dev} S_\varepsilon$, we obtain:

$$\text{dev} S_\varepsilon \cdot \tilde{\Pi} = (S_\varepsilon - \text{sph} S_\varepsilon) : \tilde{\Pi} = S_\varepsilon \cdot \tilde{\Pi} = S_\varepsilon \cdot \left( e(v) - \dot{E} \right)$$

$$= S_\varepsilon \cdot e(v) - S_\varepsilon \cdot \left( \frac{\partial E}{\partial t} + (v \cdot \nabla)E \right) - S_\varepsilon \cdot \left( E \text{skw}(\nabla v) - \text{skw}(\nabla v)E \right)$$

$$= S_\varepsilon \cdot e(v) - S_\varepsilon \cdot \left( \frac{\partial E}{\partial t} + (v \cdot \nabla)E \right) + \text{skw}(S_\varepsilon E^\top - S_\varepsilon E) : \nabla v$$

$$= 0 \text{ as } E \text{ and } S_\varepsilon \text{ are symmetric.}$$

The second equality in equation (19) has used the orthogonality of the deviatoric and the spherical parts of $S_\varepsilon$ and the mentioned attribute that the inelastic strain $\Pi$ is deviatoric during the whole evolution. The third equality has used equation (8b). The penultimate equality has used the matrix algebra $A: (BC) = (B^\top A) : C = (AC^\top) : B$ while for the last equality in equation (19) we used $S_\varepsilon = \psi_\varepsilon(E, m) - \psi(E, m, \theta)$ together with the mentioned symmetry of both $E$ and $S_\varepsilon$ so that the skew-symmetric part vanishes.

The test of equation (8d) by $\mu_0 \hat{m}$ is quite technical. The term $\kappa \Delta m - \mu_0 (v \cdot \nabla)m$ is to be handled using the Green’s formula twice. Namely,

$$\int_\Omega \kappa \Delta m - \mu_0 v \cdot \nabla)m \, dx = \int_\Gamma \kappa (n \cdot \nabla)m \cdot \mu_0 (v \cdot \nabla)m \, dS$$

$$- \int_\Omega \mu_0 \kappa \nabla^2 m : (v \otimes \nabla m) + \mu_0 \kappa (\nabla m \otimes \nabla m) : e(v) \, dx$$

$$= \int_\Gamma \mu_0 \kappa \left( (n \cdot \nabla)m \cdot (v \cdot \nabla)m - \frac{1}{2} |\nabla m|^2 v \cdot n \right) \, dS$$

$$+ \int_\Omega \frac{\mu_0 \kappa}{2} |\nabla m|^2 \nabla v - \mu_0 \kappa (\nabla m \otimes \nabla m) : e(v) \, dx,$$

where the boundary integral vanishes due to the boundary conditions $(n \cdot \nabla)m = 0$ and $v \cdot n = 0$. In the last equality, we see the terms which (counting the factor $\mu_0 \kappa$) are balanced by the Korteweg-like stress in equation (8a). Moreover, we use the Green theorem also for the driving magnetic field $h = h_\text{geo} + h_\text{dem}$:

$$\int_\Omega h \cdot (v \cdot \nabla)m \, dx = \int_\Gamma (h \cdot m) v \cdot n \, dS - \int_\Omega (\nabla h)^\top m \cdot v + (h \cdot m) \, div \, v \, dx,$$

which (counting the factor $\mu_0$) is balanced with the force $\mu_0 \nabla h^\top m$ and the pressure $\mu_0 h \cdot m$ in equation (8a). This is used to handle the terms in $\mu_0 \text{h}_{\text{dev}} \cdot \hat{m}$ by the calculus:

$$\int_\Omega \mu_0 \text{h}_{\text{dev}} \cdot \hat{m} \, dx = \int_\Omega \left( \mu_0 \kappa \Delta m - \psi'_m(E, m, \theta) + \mu_0 (h_\text{geo} + h_\text{dem}) \right) \cdot \hat{m} - \mu_0 \text{h}_{\text{dev}} \cdot \text{skw}(\nabla v)m \, dx$$

$$= \frac{d}{dt} \int_\Omega \mu_0 h_\text{geo} \cdot m - \frac{\mu_0 \kappa}{2} |\nabla m|^2 \, dx$$

$$+ \int_\Omega \mu_0 \kappa \hat{m} \cdot (n \cdot \nabla)m - \frac{\mu_0 \kappa}{2} |\nabla m|^2 v \cdot n + \mu_0 (h_\text{dev} \cdot m) \cdot \hat{m} \, dS$$

$$+ \int_\Omega \left( \frac{\mu_0 \kappa}{2} |\nabla m|^2 \nabla v - \mu_0 \kappa (\nabla m \otimes \nabla m) : e(v) - \psi'_m(E, m) \cdot \frac{\partial m}{\partial t} \right.$$}

$$\left. + (\mu_0 h_\text{dem} \cdot \theta \omega'(m)) \cdot \hat{m} - \mu_0 \frac{\partial h_\text{geo}}{\partial t} m - \psi'_m(E, m) \cdot (v \cdot \nabla)m + \mu_0 (\nabla h)^\top m \cdot v - \mu_0 \text{h}_{\text{dev}} \cdot m \cdot v - \mu_0 \text{skw}(h_\text{dev} \otimes m) \cdot \nabla v \right) \, dx.$$

Here, we see the skew-symmetric contribution $\text{skw}(h_\text{dev} \otimes m)$ which is balanced by the skew-symmetric couple-like stress in equation (8a), as well as the pressure-like term $h \cdot m$ and the force $(\nabla h)^\top m$ in equation
The term $\theta\omega'(m)\dot{\mathbf{n}}$ will occur in equation (26) and will be balanced in the total energy with the adiabatic heat source/sink in equation (8f).

The partial time derivative terms in equations (19) and (22) can be merged by the calculus as:

$$
\varphi'_E(E, m)\frac{\partial E}{\partial t} + \varphi'_m(E, m)\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \varphi(E, m).
$$

The convective terms in equations (19) and (22) can be merged for the calculus and by the Green formula:

$$
\int_\Omega \varphi'_E(E, m) : (\mathbf{v} \cdot \nabla) \mathbf{E} + \varphi'_m(E, m) \cdot (\mathbf{v} \cdot \nabla) \mathbf{m} \, d\mathbf{x} = \int_\Omega \nabla \varphi(E, m) \cdot \mathbf{v} \cdot \mathbf{n} \, dS - \int_\Omega \varphi(E, m) \text{div} \mathbf{v} \, d\mathbf{x}.
$$

Here, we can see the term $\varphi(E, m)$ which are balanced by the pressure contribution in the elastic stress $S_E$.

For equation (8e) tested by $\mu_0 \frac{\partial}{\partial t} \mathbf{u}$, we use the calculus, including the Green theorem for the convective term, to obtain:

$$
0 = \int_{\mathbb{R}^d} \mu_0 \text{div} (\chi_\Omega \mathbf{m} - \mathbf{v} \mathbf{u}) \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{x} = \int_{\mathbb{R}^d} \mu_0 (\mathbf{v} \mathbf{u} - \chi_\Omega \mathbf{m}) \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{x} = \frac{d}{dt} \int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mu_0 \mathbf{m} \cdot \frac{\partial \mathbf{h}_{\text{dem}}}{\partial t} \, d\mathbf{x} = \frac{d}{dt} \left( \int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mu_0 \mathbf{m} \cdot \mathbf{h}_{\text{dem}} \, d\mathbf{x} \right) - \int_\Omega \mu_0 \mathbf{m} \cdot \mathbf{h}_{\text{dem}} \, d\mathbf{x}
$$

$$
= \frac{d}{dt} \left( \int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mu_0 \mathbf{m} \cdot \mathbf{h}_{\text{dem}} \, d\mathbf{x} \right) + \int_\Omega \mu_0 \left( \text{skw} (\nabla \mathbf{v} - \dot{\mathbf{m}} - (\mathbf{v} \cdot \nabla) \mathbf{m}) \right) \cdot \mathbf{h}_{\text{dem}} \, d\mathbf{x} = \frac{d}{dt} \left( \int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mu_0 \mathbf{m} \cdot \mathbf{h}_{\text{dem}} \, d\mathbf{x} \right) - \int_\Gamma \mu_0 (\mathbf{h}_{\text{dem}} \cdot \mathbf{n}) \left. \mathbf{v} \cdot \mathbf{n} \right|_{\mathbf{n} = 0} \, dS + \int_\Omega \mu_0 \text{skw} (\mathbf{h}_{\text{dem}} \otimes \mathbf{m}) \cdot \nabla \mathbf{v} - \mu_0 \dot{\mathbf{m}} \cdot \mathbf{h}_{\text{dem}}
$$

$$
+ \mu_0 \mathbf{v} \cdot (\nabla \mathbf{h}_{\text{dem}}) \mathbf{m} + \mu_0 (\mathbf{h}_{\text{dem}} \mathbf{m}) \text{div} \mathbf{v} \right) \, d\mathbf{x}.
$$

The last integral reveals how the force $\mu_0 (\nabla \mathbf{h}_{\text{dem}}) \mathbf{m}$ and the stress contribution $\mu_0 \text{skw} (\mathbf{h}_{\text{dem}} \otimes \mathbf{m}) - \mu_0 (\mathbf{h}_{\text{dem}} \mathbf{m}) \mathbb{I}$ arises energetically in the momentum equation. The term $\mu_0 \dot{\mathbf{m}} \cdot \mathbf{h}_{\text{dem}}$ is canceled with the term $\mu_0 \nabla \mathbf{u} \cdot \dot{\mathbf{m}}$ arising from equation (8d) tested by $\mu_0 \dot{\mathbf{m}}$.

To summarize the above calculations, we state the following.

**Proposition 1** (Magneto-mechanical energy balance). Any smooth solution of the systems (8a–e) with the boundary conditions (17) satisfies the identity:

$$
\frac{d}{dt} \int_\Omega \left[ \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 + \varphi(E, m) + \frac{\kappa \mu_0}{2} |\nabla \mathbf{m}|^2 + \mu_0 \mathbf{h}_{\text{geo}} \cdot \mathbf{m} \right] \, dx + \int_\Omega \frac{\mu_0}{2} |\nabla \mathbf{u}|^2 \, dx
$$

$$
+ \int_\Omega \left[ \xi(\theta; \mathbf{e}(\mathbf{v}), \mathbf{\Pi}, \dot{\mathbf{m}}) \mathbf{v} \cdot \mathbf{n} \right. \mathbf{dx} = \int_\Omega \left( \frac{\partial}{\partial t} \mathbf{h}_{\text{geo}} \cdot \mathbf{m} - \theta \omega'(m) \mathbf{n} - (\theta \omega(m) + \phi(\theta)) \right) \mathbf{v} \cdot \mathbf{n} \, d\mathbf{x}.
$$

Adding equation (8f) tested by 1 and using also the boundary condition in equation (17d), the adiabatic effects are absorbed in the internal heat energy $w$ and we obtain the following.
Proposition 2 (Total energy balance). Any smooth solution of the system (8) with the boundary conditions (17) satisfies the identity:

\[
\frac{d}{dt} \left( \int_{\Omega} \frac{\rho}{2} |v|^2 + \varphi(E, m) + \frac{\kappa}{2} |\nabla m|^2 + \mu_0 h_{\text{geo}} \cdot m + w \, dx \right) + \int_{\Omega} \frac{\mu_0}{2} |\nabla u|^2 \, dx = \int_{\Omega} \varphi \cdot v + \frac{\partial h_{\text{geo}}}{\partial t} \cdot m \, dx + \int_{\Gamma} j_{\text{ext}} \cdot dS. \tag{27}
\]

The thermodynamical context of this model relies on an additive splitting of the specific free energy \( \psi \) into the purely mechanical part and the thermal part \( \phi \) (cf. equation (9)). An important attribute of the model, beside keeping the energetics (26)–(27), is the entropy imbalance, i.e. the Clausius–Duhem inequality. The specific entropy is then \( \eta = -\psi_{,m}(m, \theta) = \phi(\theta) - \omega(m) \) with \( \omega \) being an extensive variable (in J K\(^{-1}\) m\(^{-3}\)) and transported by the entropy equation:

\[
\frac{\partial \eta}{\partial t} + \text{div}(\nu \eta) = \frac{\xi}{\theta} - \text{div} j
\]

(28)

with \( \xi \) denoting the heat production rate from equation (8f) and \( j \) the heat flux (here governed by the Fourier law \( j = -K \nabla \theta \)). Substituting \( \eta = \phi(\theta) - \omega(m) \) into equation (28), we obtain the heat-transfer equation:

\[
c(\theta) \dot{\theta} = \xi - \text{div} j + \theta \omega'(m) \cdot \dot{m} - \eta \text{div} v \text{ with the heat capacity } c(\theta) = \theta \phi''(\theta),
\]

(29)

note that temperature (in K) is an intensive variable and is transported by the material derivative. There are the adiabatic heat source/sink terms \( \theta \omega'(m) \cdot \dot{m} \) and \( \eta \text{div} v = \theta(\phi'(\theta) - \omega(m)) \text{div} v \) on the right-hand side due to both the magnetic phase transition and the compressibility of the continuum. Note that only the convective time derivative \( \dot{m} \) but not the rotation swk(\( \nabla v \)m influences the adiabatic magnetic heat.

Furthermore, the internal energy is given by the Gibbs relation \( e = \psi + \theta \eta \), and splits here into the purely magneto-mechanical part and the purely thermal part \( w = \theta \phi'(\theta) - \phi(\theta) =: \gamma(\theta) \), namely,

\[
e = \frac{\varphi(E, m) + \theta \omega(m) - \phi(\theta)}{\psi(E, m, \theta)} + \frac{\theta \phi'(\theta) - \theta \omega(m)}{\theta \omega(m, \theta)} = \varphi(E, m) + w. \tag{30}
\]

Since \( \gamma'(\theta) = \theta \phi''(\theta) \), the heat equation (29) can be written as:

\[
\dot{w} = \xi - \text{div} j + \theta \omega'(m) \cdot \dot{m} + \theta(\omega(m) - \phi(\theta)) \text{div} v.
\]

(31)

Alternatively, we can write (31) as:

\[
\frac{\partial w}{\partial t} + \text{div}(w v) = \xi - \text{div} j + \theta \omega'(m) \cdot \dot{m} + (\theta \omega(m) + \phi(\theta)) \text{div} v,
\]

(32)

which reflects the property that the thermal internal energy \( w \) in J m\(^{-3}\) is again an extensive variable and is transported like equation (28). This reveals the structure of equation (8f). In particular, from equation (28), we can see at least formally by the usual calculus, relying on positivity of temperature.

Proposition 3 (Clausius-Duhem entropy inequality). Assuming \( \xi \geq 0 \) and \( K \geq 0 \), any smooth solution of the systems (8a–e) with positive temperature \( \theta \) with the boundary conditions (17) satisfies the identity:

\[
\frac{d}{dt} \int_{\Omega} \eta \, dx \geq \int_{\Gamma} (v \eta - j / \theta) \cdot n \, dS = -\int_{\Gamma} j_{\text{ext}} \cdot dS.
\]

The mentioned positivity of temperature can be ensured by non-negativity of the boundary heat flux \( j_{\text{ext}} \) and positivity of the initial temperature together with at least linear decay of \( \phi(\cdot) \) to zero for \( \theta \to 0^+ \).
Remark 3 (Oberbeck–Bousinesq buoyancy enhancement). An important phenomenon is that hot or even molten rocks (in particular magma) are lighter than solid rocks, which rise a tendency of magma floating up in the gravitation field \( g \). This can be included in a simplified way by the Oberbeck–Bousinesq buoyancy model, usually used for incompressible media which, anyhow, exhibit a slight thermal expansibility. This gives rise to an extra force by replacing \( g(1−b(\theta)) \) in equation (8a) with some \( b(\cdot) \) continuous. Then, equation (8f) expands by the adiabatic heat source/sink term \( b(\theta)\nu g \) and the analysis in section 4 below can easily be enhanced like (even more easily) for the term \( (\theta\omega(m)+\phi(\theta))\text{div}\;v \) in equation (8f).

Remark 4 (More general free energies). The free energy (5) has a specific feature that \( \psi''_{\theta\theta} \) is independent of \( \omega \). This leads to a specific “transition” temperature (Curie or Néel) at which the spontaneous (saturation) magnetization \( m_s \) sharply falls to zero (see Figure 2). A “fuzzy” transition temperature which would correspond more realistically to a mixture of various magnetic minerals in rock would need a general non-linear function instead of \( \theta \mapsto \theta\omega(m) \) in equation (5). This would however make the specific heat \( -\theta\psi''_{\theta\theta} \) dependent also on \( m \) and the analysis more complicated.

Remark 5 (Compressible variant with varying mass density). In situations where mass density \( \rho \) varies in space and time, the system (8) is to be complete by the continuity equation \( \partial_t\theta=−\rho\text{div}\;v \) while the force \( -\rho(\text{div}\;v)\nu/2 \) in equation (8a) is to be omitted. The model (8) itself used the simplification based on the assumption of a constant mass density \( \rho \) (cf. the calculus (18)), and neglects its variations during volumetric deformation (which is typically indeed small in liquids and in solids, too, in contrast to gases). Keeping the energy balance without the continuity equation then needs a compensation by the “structural” force \( -\rho(\text{div}\;v)\nu/2 \), invented by R. Temam [26] rather for numerical purposes. This extra force is presumably very small (as \( \text{div}\;v \) is typically small) and slightly violates Galilean invariancy, as pointed out in Tomassetti [35], which is the price for simplifying the model and its analysis. Here, in fact, the analysis would however work for the full model because the concept of non-linear non-simple material ensures \( \nabla v \) bounded in space, although a lot of very sophisticated technicalities and non-trivial arguments would still be needed.

Remark 6 (Convective transport of magnetization vectors). Based on the concept of micropolar fluids [36], an isothermal model of transport of magnetization vector by the time derivative \( \dot{\omega} = \omega \times \dot{m} \) uses an angular velocity \( \omega \) (cf. [37–41]). It is possible to be approximated as \( \text{curl}\;v/2 \). Our choice Zaremba–Jaumann objective transport (4) actually used \( W = \text{skw}(\nabla v) \) instead of \( \frac{1}{2}(\text{curl}\;v) \times \cdot \). A completely different approach can use rotation of magnetization vector described in Lagrangian (reference) configuration [42] or also [33, Sect. 4.5.4], which would easily allow for varying mass density but brings a lot of other technicalities, and is not well fitted with fluids and thus with solid–fluid transition.

4. The analysis by a time-discrete approximation

We will consider an initial-value problem for the boundary-value problem equations (8)–(17) considered with \( R := \vec{H} \) as in Remark 2 by imposing the initial conditions:

\[
\begin{align*}
 v|_{t=0} &= v_0, & E|_{t=0} &= E_0, & m|_{t=0} &= m_0, & w|_{t=0} &= w_0.
\end{align*}
\]

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely, \( L^p(\Omega;\mathbb{R}^n) \) for Lebesgue measurable functions \( \Omega \to \mathbb{R}^n \) whose Euclidean norm is integrable with \( p \)-power, and \( W^{k,p}(\Omega;\mathbb{R}^n) \) for functions from \( L^p(\Omega;\mathbb{R}^n) \) whose all derivative up to the order \( k \) have their Euclidean norm integrable with \( p \)-power. We also write briefly \( H^k = W^{k,2} \). The notation \( p^* \) will denote the exponent from the embedding \( W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \), i.e., \( p^* = dp/(d-p) \) for \( p < d \) while \( p^* = +\infty \) for \( p > d \). Moreover, for a Banach space \( X \) and for \( I = [0,T] \), we will use the notation \( L^p(I;X) \) for the Bochner space of Bochner measurable functions \( I \to X \) whose norm is in \( L^p(I) \) while \( W^{1,p}(I;X) \) denotes for functions \( I \to X \) whose distributional derivative is in \( L^p(I;X) \). Also, \( C_w(I;X) \) will denote the Banach space of weakly* continuous functions \( I \to X \), and \( C_{ww}(I;X) \) of weakly* continuous if \( X \) has a predual, i.e., there is \( X^* \) such that \( X = (X^*)^* \) where \((\cdot)^*\) denotes the dual space. Occasionally, we will use \( L^p_{ww}(I;X) \) the space of weakly* measurable mappings \( I \to X \); recall that \( L^p_{ww}(I;X) = L^p(I;X) \) if \( X \) is separable reflexive. Eventually, \( C^1(\cdot) \) will stand for the space of continuously differentiable functions.
The philosophy of tuning the assumptions is quite peculiar and worth articulating the main points:

- $p > d$ so that the velocity field gradient $\nabla v$ will be surely in $L^p(I; L^\infty(\Omega; \mathbb{R}^{d\times d}))$ and the transport of $E$ will be qualitatively controlled by initial conditions,
- due to the estimation of the convective term $v \cdot \nabla \theta$ in equation (57) below, we need the heat capacity to be bounded, so that the temperature gradient $\nabla \theta$ will have only the basic expected regularity (cf. equation (36f) below), and
- the exponent $r$ controlling the coercivity of $\zeta(\theta; \cdot)$ and thus integrability of $\dot{m}$ is large enough to make the term $\theta \omega'(m) \dot{m}$ integrable when covering the standard Landau ansatz (9).

More specifically, we will assume, with some $C \in \mathbb{R}$ and some $\epsilon > 0$ arbitrarily small, that:

\[ \varphi \in C^1(\mathbb{R}^{d\times d} \times \mathbb{R}^d), \quad \varphi(E, m) = \tilde{\varphi}(E, m) + \check{\varphi}(m) \quad \text{with} \quad \tilde{\varphi} \in C^1(\mathbb{R}^{d\times d} \times \mathbb{R}^d) \quad \text{convex}, \]

\[ \forall (E, m) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d : \quad \varphi(E, m) \leq C(1 + |E|^2 + |m|^{2^* - \epsilon}), \]

\[ |\varphi'_E(E, m)| \leq C(1 + |E|^{2^* - 1 - \epsilon} + |m|^{2^* - 1 - \epsilon}), \]

\[ |\varphi'_m(E, m)| \leq C(1 + |E|^{2^{r'/2}} + |m|^{2^{r'/2}}), \]

\[ |\check{\varphi}'(m)| \leq C(1 + |m|^{2^{r'}}), \quad (34a) \]

\[ \varphi \text{ is coercive:} \quad \varphi(E, m) \geq \epsilon |E|^2 + \epsilon |m|^2 - C, \quad (34b) \]

\[ \omega \in C^1(\mathbb{R}^d), \quad \forall m, \in \mathbb{R}^d : \quad |\omega(m)| \leq C(1 + |m|^2) \quad \text{and} \quad |\omega'(m)| \leq C(1 + |m|), \quad (34c) \]

\[ \phi \in C^1(\mathbb{R}), \quad \phi(0) = 0, \quad \forall \theta \in \mathbb{R}^+ : \quad |\phi(\theta)| \leq C(1 + \theta^{1+\epsilon}), \quad (34d) \]

\[ \gamma : \theta \mapsto \phi(\theta) - \theta \phi'(\theta) \quad \text{and} \quad \gamma^{-1} \quad \text{have at most linear growth}, \]

\[ M : \mathbb{R} \rightarrow \mathbb{R} \quad \text{and} \quad K : \mathbb{R} \rightarrow \mathbb{R} \quad \text{continuous, bounded, inf } M(\cdot) > 0 \quad \text{and} \quad \text{inf } K(\cdot) > 0, \quad (34e) \]

\[ \zeta : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \quad \text{continuous,} \]

\[ \zeta(\theta; \cdot) : \mathbb{R}^d \rightarrow \mathbb{R} \quad \text{strictly convex and smooth on } \mathbb{R}^d \setminus \{0\}, \]

\[ \forall (r, \theta) \in \mathbb{R}^d \times \mathbb{R}^+ : \quad \epsilon |r|^\gamma \leq \zeta(\theta; r) \leq (1 + |r|^\gamma) / \epsilon, \quad (34f) \]

\[ r \in \begin{cases} (30/7, 6] & \text{if } d = 3, \\ (2, +\infty) & \text{if } d = 2, \end{cases} \quad p > r, \quad \theta, \kappa, \nu, v_1, v_2, \mu_0 > 0, \quad (34g) \]

\[ v_0 \in L^2(\Omega; \mathbb{R}^d), \quad E_0 \in H^1(\Omega; \mathbb{R}^{d\times d}), \quad m_0 \in H^1(\Omega; \mathbb{R}^d), \quad \theta_0 \in L^2(\Omega), \quad \theta_0 \geq 0, \quad (34h) \]

\[ g \in L^1(I; L^2(\Omega; \mathbb{R}^d)), \quad h_{\text{geo}} \in L^1(I; H^1(\Omega; \mathbb{R}^d)), \quad j_{\text{ext}} \in L^1(I \times \Gamma), \quad j_{\text{ext}} \geq 0. \quad (34i) \]

Let us note that we do not assume to cover $\tilde{\varphi}$ and $\omega$ convex in order to include the Landau magnetic phase-transition model (9), using particularly $\tilde{\varphi}(m) = -a_0 \theta_c |m|^2$, and the split $\varphi(E, m) = \tilde{\varphi}(E, m) + \check{\varphi}(m)$ is related with the time discretization used in the proof itself. For this we need to make a regularization of the magnetization flow rule when discretized in time (cf. the magnetic flow rule (40d) and then also the (40a, f) below). Note also that equation (34b) admits a canonical choice $\phi(\theta) = c_\gamma \theta(1 - \ln \theta)$, which gives $\gamma(\theta) = c_\gamma \theta$ with a constant heat capacity $c_\gamma$.

Since $\nabla^2 u$ has a sense as a distribution only, for the weak formulation (37a) of the momentum equation (8a) below, we rather use the Green formula:

\[ \int_{\Omega} \tilde{v}(\nabla^2 u)m \, dx = \int_{\Gamma} (\tilde{v} - n)(\nabla u \cdot m) \, dx - \int_{\Omega} (\nabla u \cdot m) \text{div} \tilde{v} + (\tilde{v} \otimes \nabla u) : \nabla m \, dx. \quad (35) \]
Definition 1 (Weak solutions). A six-tuple \((v, E, R, m, u, \theta)\) with
\[
v \in C_w(I; L^2(\Omega; \mathbb{R}^d)) \cap L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))
\]
with \(\frac{\partial}{\partial t} v \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)^*) + L^1(I; L^2(\Omega; \mathbb{R}^d))\),
\[
E \in C_w(I; H^1(\Omega; \mathbb{R}^{d\times d}) \cap H^1(I, L^2(\Omega; \mathbb{R}^{d\times d})),
\]
\[
R \in L^2(I; H^2(\Omega; \mathbb{R}^{d\times d}))
\]
\[
m \in C_w(I; H^1(\Omega; \mathbb{R}^d)) \text{ with } \hat{m} \in L'(I \times \Omega; \mathbb{R}^d) \text{ and } \Delta m \in L'(I \times \Omega; \mathbb{R}^d),
\]
\[
u \in C_w(I; H^1(\mathbb{R}^d)) \text{ with } \frac{\partial}{\partial t} u \big|_{I \times \Omega} \in L^2(I; H^1(\Omega)),
\]
\[
\theta \in C_w(I; L^1(\Omega)) \cap L^q(I; W^{1,q}(\Omega)) \text{ with } 1 \leq q < \frac{d+2}{d+1}, \text{ and } \theta \geq 0 \text{ a.e. on } I \times \Omega,
\]
will be called a weak solution to the boundary-value problem (8)–(17) with the initial conditions (33) if \(S_h = \varphi'_h(E, m) - \psi(E, m, \theta) \Pi \in L^\infty(I; L^1(\Omega; \mathbb{R}^{d\times d})), \varphi'_m(E, m) + \theta \omega'(m) \in L^2(I \times \Omega; \mathbb{R}^d), n \cdot v = 0 \text{ on } I \times \Gamma, w = \gamma(\theta) \in C_w(I; L^1(\Omega)) \) with \(\frac{\partial}{\partial t} w \in L^1(I; H^{d+1}(\Omega)^*)\), and if:
\[
\int_0^T \int_\Omega \left( \varphi(v \cdot \nabla v + \frac{1}{2} (\nabla \cdot \nu)) + (S_e + v_1 e(v)) : \nabla \nabla \nu + S_c \cdot \nabla \nu \right)
\]
\[
\quad + v_2 \nabla(\nabla e(v)) : \nabla \nabla \nu + \mu_0 (\nabla h_{\text{geo}})^\top m \cdot \nabla \nu - \mu_0 \nabla u \cdot m \nabla \nu
\]
\[
- \mu_0(\nabla \nabla \nu : m) - \mu_0 \nabla v \cdot \frac{\partial \nabla \nu}{\partial t} \right) \, dx \, dt
\]
\[
\quad + \int_0^T \int_\Omega \varphi \nu \tilde{v}(0) \, dx \, dt + \int_0^T \int_\Omega \varphi \nu \tilde{v} \, dx \, dt
\]
\[
(37a)
\]
with \(S_c \in L^1(I \times \Omega; \mathbb{R}^{d\times d})\) from equation (8a) for all \(\nu \in C^1(I \times \Omega; \mathbb{R}^d) \cap L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))\) with \(n \cdot \nu = 0\) on \(I \times \Gamma\) and \(\nu(T) = 0\), and if:
\[
\int_0^T \int_\Omega \left( M(w) R + \text{dev } S \right) \tilde{p} + \kappa \nabla R : \nabla \tilde{p} \, dx \, dt = 0
\]
\[
(37b)
\]
holds for all \(\tilde{p} \in H^1(I \times \Omega; \mathbb{R}^{d\times d})\), and if:
\[
\int_0^T \int_\Omega \left( \xi(\theta; \tilde{r}) + \left( h_{\text{geo}} - \frac{\varphi'_m(E, m) + \theta \omega'(m)}{\mu_0} - \nabla \nu \right) \cdot (\tilde{r} - m) \right)
\]
\[
\quad + \kappa \nabla m : \nabla \tilde{r} + \kappa \Delta m \cdot \left( (v \cdot \nabla) m - \text{skw}(\nabla v) m \right) \right) \, dx \, dt
\]
\[
+ \int_\Omega \int_0^T |\nabla m_0|^2 \, dx \geq \int_\Omega \int_0^T |\nabla \tilde{p}(T)|^2 \, dx + \int_0^T \int_\Omega \xi(\theta; \tilde{m}) \, dx \, dt
\]
\[
(37c)
\]
holds for all \(\tilde{r} \in L^2(I; H^1(\Omega; \mathbb{R}^d))\), and at all time instants:
\[
\int_{\mathbb{R}^d} \nabla u \cdot \nabla \tilde{u} \, dx = \int_{\Omega} m \cdot \nabla \tilde{u} \, dx
\]
\[
(37d)
\]
holds for all \(\tilde{u} \in H^1(\mathbb{R}^d)\), and
\[
\int_0^T \int_\Omega \left( (K(\theta) \nabla \theta - \nu \nu m) \cdot \nabla \tilde{w} - \left( \xi(\theta; e(v), R, \tilde{m}) + \theta \omega'(m) \cdot \tilde{m} + (\theta \omega(m) + \phi(\theta)) \nu \right) \nabla v \right) \tilde{w}
\]
\[
- \nu \frac{\partial \tilde{w}}{\partial t} \right) \, dx \, dt = \int_\Omega \gamma(\theta_0) \tilde{w}(0) \, dx + \int_0^T \int_\Gamma j \tilde{w} \, dS \, dt
\]
\[
(37e)
\]
with \(\xi(\theta; e(v), R, \tilde{m})\) from equation (8f) for any \(\tilde{w} \in W^{1,\infty}(I \times \Omega)\) with \(\tilde{w}(T) = 0\), and eventually also \(\hat{E} = e(v) - R\) holds a.e. on \(I \times \Omega\) and \(\Pi(0) = \Pi_0\) and \(E(0) = E_0\) a.e. on \(\Omega\).
Proposition 4 (Existence of weak solutions). Let equation (34) be valid. Then the initial-boundary-value problem for the system (8) with the boundary and initial conditions (17) and (33) has a weak solution \((v, E, R, m, u, \theta)\) according Definition 1 and every such solution also satisfies the mechanical-energy balance (26) and conserves the total energy in the sense (27).

Proof. For lucidity, we divide the proof into six steps.

Step 1: approximation by time discretization. As we need testing by convective (but not mere partial) time derivatives, using of the Galerkin method would be very technical (if not just impossible). Therefore, we use the Rothe method, i.e., the fully implicit time discretization with an equidistant partition of the time interval \(I\) with the time step \(\tau > 0\). This approximation is also rather technical because the stored energy \(\varphi(E, \cdot) + \theta\omega(\cdot)\) is intentionally non-convex for \(\theta < \theta_C\), and its test by the time difference of \(m\) is to be estimated “on the right-hand side”. For this reason, we denote the possibly non-convex part of the stored energy by:

\[
\varpi(m, \theta) := \varphi(m) + \theta\omega(m)
\]

for this proof. In addition to time discretization, we thus use also a modification of \(\varpi\) by some \(\varpi_t(m, \theta) = \varphi_t(m) + \theta\omega_t(m)\) with \(\varphi_t, \omega_t \in W^{1,\infty}(\mathbb{R}^d)\), specifically we put:

\[
\varphi_t(m) := \frac{\varphi(m)}{1+\varepsilon |m|^2} \quad \text{and} \quad \omega_t(m) := \frac{\omega(m)}{1+\varepsilon |m|^2}, \quad \text{so that} \quad [\varpi_t]_0(m, \theta) := \omega_t(m).
\]

Consequently, we also put \(\psi_t(\varphi_t(m, \theta)) = \varphi_t(m) + \varphi_t(m) + \theta\omega_t(m) + \mu_0 h_{\text{dem}} - \phi(\theta)\).

We denote by \(v^{k, t}, E^{k, t}, S^{k, t}, \ldots\) the approximate values of \(v, E, S, \ldots\) at time \(k\tau\) with \(k = 1, 2, \ldots, T/\tau\). We introduce a shorthand notation for the bi-linear operators:

\[
b_{\tau}(v, m) = (v \cdot \nabla)m - \text{skw}(\nabla v)m \quad \text{and} \quad B_{\tau}(v, E) = (v \cdot \nabla)E - \text{skw}(\nabla v)E + E\text{skw}(\nabla v).
\]

We devise a fully coupled time discretization; here, let us remark that usual efficient decoupled (staggered) schemes would be problematic in this case because the convective corotation terms and the adiabatic terms bond intimately the particular equations and their decoupling would complicate the a-priori estimates. However, we easily can use delayed temperature in some dissipation terms (cf. equation (40c, d, f) below). Beside the mentioned regularization of \(\omega_t\), we use a bit lower dissipation heat source in equation (40f) by using the factor \(1 - \varepsilon\). We will then use the following recursive regularized time-discrete scheme written in the classical form as:

\[
\varphi_t\left(\frac{v^{k+1, t} - v^{k, t}}{\tau} + (v^{k, t} \cdot \nabla)v^{k, t}\right) = \text{div}\left(S^{k, t} + S^{k, t} + S^{k, t} - \mu_0((h^{k, t} - m^{k, t})\Pi - \text{div} S^{k, t})\right) - \frac{\mu_0}{2}(\text{div} v^{k, t})v^{k, t} + \mu_0(\nabla h^{k, t})^T m^{k, t} + \varpi^{k, t}
\]

with:

\[
S^{k, t} = \varphi_t(E^{k, t}, m^{k, t}) - E^{k, t}, \quad S^{k, t} = \psi_t(E^{k, t}, m^{k, t}, \theta^{k, t}), \quad h^{k, t} = h^{k, t} \geq 0 - \nabla u^{k, t},
\]

\[
S^{k, t} = \frac{\psi_t(E^{k, t}, m^{k, t}) - \psi_t(E^{k, t}, m^{k, t})}{\mu_0} \frac{\psi_t(E^{k, t}, m^{k, t}) - \psi_t(E^{k, t}, m^{k, t})}{\mu_0} - h^{k, t} + \mu_0 h_{\text{shock}} - e(v^{k, t}), \quad (\text{div} v^{k, t})^T m^{k, t} - \frac{\mu_0}{2} |(\text{div} m^{k, t})^2| - \mu_0 \text{skw}((h^{k, t} - m^{k, t}))^T m^{k, t},
\]

\[
E^{k, t} = \frac{E^{k, t} - E^{k-1, t}}{\tau} + B_{\tau}(v^{k, t}, E^{k, t}) = e(v^{k, t}) - R^{k, t}, \quad M_{\tau}(\theta^{k-1}) R^{k, t},
\]

\[
\Delta v^{k, t} = \text{div}(\chi^{\Omega}(m^{k, t})) \quad \text{on} \quad \mathbb{R}^d,
\]

\[
\Delta h^{k, t} = \psi_t(E^{k, t}, m^{k, t}) + \frac{\psi_t(m^{k, t}, \theta^{k, t})}{\mu_0} - \nabla u^{k, t} + \kappa \Delta v^{k, t},
\]

\[
\Delta u^{k, t} = \psi_t(E^{k, t}, m^{k, t})
\]
\[ \frac{w_{k+1} - w_k}{\tau} + \text{div} (v_{k+1} v_k - K(a_{k+1}) \nabla \theta_{k+1}) = (1 - \varepsilon) \left( M(a_{k+1}) |R_{k+1}|^2 + v_1 |e(v_k)|^2 \right) + v_2 |\nabla e(v_k)|^p + \partial_\varepsilon (\theta_{k+1}; r_t^k) \cdot r_t^k + \varepsilon |\nabla R_{k+1}|^2 \]

\[ + \theta_{k+1} \omega_{k+1} (m_{k+1}) \left( \frac{m_{k+1} - m_k}{\tau} + (v_{k+1} \nabla) m_{k+1} \right) \]

\[ + \left( \partial_{\Theta_{k+1}} \omega_{\Theta_{k+1}} (m_{k+1}) + \phi(\Theta_{k+1}) \right) \text{div} v_{k} \quad \text{with } w_{k+1} = \gamma(\Theta_{k+1}). \]  

(40f)

We complete the system (40a–d,f) by the corresponding boundary conditions, i.e.

\[ \left[ \begin{array}{c}
\left( S_{\varepsilon_{k+1}} + S_{\varepsilon_{k+1}^d} + S_{\varepsilon_{k+1}^d} - \mu_0 (h_{k+1} \cdot m_{k}) \| - \text{div} S_{\varepsilon_{k+1}} \right) n + \text{div}(S_{\varepsilon_{k+1}} n) = 0, \\
v_{k+1} \cdot n = 0, \nabla e(v_{k+1}) (n \otimes n) = 0, (n \cdot \nabla) R_{k+1} = 0, (n \cdot \nabla) m_{k+1} = 0, \text{ and} \\
\n \n \end{array} \right] \]

(41a)

while (40e) completes with the condition \( \lim_{\varepsilon \to \infty} u_{k+1}(\varepsilon) = 0 \) for all \( k = 1, ..., T/\tau \). Here, we used \( g_{k+1} = \int_{(k-1)\tau}^{k\tau} g(t) \, dt \) and similarly also for \( h_{\text{geo}, k} \) and \( f_{\text{ext}, k} \). This system of boundary-value problems is to be solved recursively for \( k = 1, 2, ..., T/\tau \), starting with the initial conditions for \( k = 1 \):

\[ v_0^0 = v_0, \quad E_0^0 = E_0, \quad m_0^0 = m_0, \quad \text{and} \quad w_0^0 = w_0. \]

(42)

The existence of a weak solution \((v_k^0, E_k^0, R_k^0, m_k^0, u_k^0, w_k^0) \in W^{2, p}(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d \times d}) \times \mathbb{R}^{d \times d} \times H^1(\Omega; \mathbb{R}^{d \times d}) \times H^1(\Omega; \mathbb{R}^d) \times H^1(\mathbb{R}^d) \times W^{1, 1}(\Omega) \) of the coupled quasi-linear boundary-value problem (40)–(41) can thus be seen by a combination of the quasi-linear technique for equation (40a) involving the quasi-linear term \( \text{div}^2 (v_2 |\nabla e(v_{k+1})|^p - \nabla e(v_{k+1})) \), with the usual semi-linear technique for equation (40c–d), with the \( L^1 \)-technique for the heat equation (40f), and with the set-valued inclusion (in fact a variational inequality) (40d) provided \( v_{k+1}^1, E_{k+1}^1, m_{k+1}^1, \) and \( w_{k+1}^1 \) are known from the previous time step. Actually, \( w_{k+1}^1 \in W^{1, q}(\Omega) \) for any \( q < d \). Let us note that the system (40) is indeed fully coupled due to the convective derivatives and due to the adiabatic effects (cf. the term \( \phi(\Theta_{k+1}) \) in equation (40a)), and it seems not possible to devise some decoupled (staggered) discrete scheme which would allow for some reasonable estimation strategy of the recursive scheme. The mentioned coercivity is a particular consequence of the a-priori estimates derived below. Thus, also \( R_k^0 \in H^1(\Omega; \mathbb{R}^{d \times d}) \) is obtained. Let us note that, due to the convective terms, this system does not have any potential so that the non-constructive Brower fixed-point arguments combined with the Galerkin approximation. Also strict monotonicity of the main parts of equation (40a–d) so that the approximated right-hand side of the semi-linear equation (40f) converges strongly in \( L^1(\Omega) \). In general, one cannot expect any uniqueness of this solution.

Moreover, \( w_{k+1}^1 \geq 0 \) a.e. on \( \Omega \) for at least one weak solution. To show it, we test equation (40f) by the negative part of \( \partial_{\Theta_{k+1}} \). Here, we exploit \( \phi(0) = 0 \) so that the adiabatic terms vanish for \( \theta \to 0+ \), and thus prove that \( \theta_{k+1} \geq 0 \). Using also \( \gamma(0, +\infty) \leq 0 \), then also \( w_{k+1}^1 = \gamma(\Theta_{k+1}) \geq 0 \).

Using the values \((\varepsilon_{k+1}^0)^{T/\tau})_{k=0}^\infty \), we define the piecewise constant and the piecewise affine interpolants, respectively, as:

\[ \bar{\varepsilon}_{k+1}(t) := \varepsilon_{k+1}^0, \quad \bar{\varepsilon}_{k+1}(t) := \varepsilon_{k+1}^{k-1}, \quad \text{and} \quad \bar{\varepsilon}_{k+1}(t) := \left( \frac{t}{\tau} - k + 1 \right) \varepsilon_k + \left( k - \frac{t}{\tau} \right) \varepsilon_k^{k-1}. \]

(43)

for \((k-1)\tau < t \leq k \tau \) with \( k = 0, 1, ..., T/\tau \). Analogously, we define also \( E_{k+1}, \bar{\varepsilon}_{k+1}, \bar{\varepsilon}_{k+1}, \bar{\varepsilon}_{k+1}^1, \bar{\varepsilon}_{k+1}^2, \bar{\varepsilon}_{k+1}^1, \bar{\varepsilon}_{k+1}^2, \) etc. Thus, equation (40) holding a.e. on \( \Omega \) for \( k = 1, ..., T/\tau \) can be written “compactly” as:

\[ \begin{aligned}
\phi \left( \frac{\partial v_{k+1}}{\partial t} + (\bar{\varepsilon}_{k+1} \nabla) \bar{\varepsilon}_{k+1} \right) &= \text{div} (\bar{S}_{k+1} + \bar{S}_{k+1} + \bar{S}_{k+1} + \mu_0 (h_{k} \cdot \bar{m}_{k}) \| - \text{div} \bar{S}_{k+1}) \\
&\quad - \frac{\partial}{2} \left( \text{div} \bar{\varepsilon}_{k+1} \right) \sigma_{k+1} + \mu_0 (\bar{h}_{k} \cdot \bar{m}_{k}) + \phi(\Theta_{k+1})
\end{aligned} \]

with \( \bar{S}_{k+1} = \phi'_{\bar{E}}(\bar{E}_{k+1}, \bar{m}_{k+1}), \quad \bar{\varepsilon}_{k+1} = \bar{h}_{\text{geo}, k} - \nabla \bar{u}_{k+1} \).
\[ \mathbf{S}_{v,tt} = v_1 e(\mathbf{v}_{tt}), \quad \mathbf{S}_{tt} = v_2 |\nabla e(\mathbf{v}_{tt})|^2 \nabla e(\mathbf{v}_{tt}), \quad \text{and} \]
\[ \mathbf{S}_{c,tt} = \kappa \mu_0 \nabla \mathbf{m}_{ct} \otimes \nabla \mathbf{m}_{ct} - \frac{\kappa \mu_0}{2} |\nabla \mathbf{m}_{ct}|^2 \mathbb{I} - \mu_0 \text{skw}(\mathbf{h}_{dv,rt} \otimes \mathbf{m}_{ct}), \quad (44a) \]
\[ \frac{\partial \mathbf{E}_{rt}}{\partial t} + B_2(\mathbf{r}_{rt}, \mathbf{E}_{rt}) = e(\mathbf{v}_{rt}) - \mathbf{R}_{rt}, \quad (44b) \]
\[ M(\mathbf{w}_{rt}) \mathbf{R}_{rt} + \text{dev} \mathbf{S}_{c,rt} = \kappa \Delta \mathbf{R}_{rt}, \quad (44c) \]
\[ \partial_t \zeta(\mathbf{r}_{rt}; \mathbf{r}_{rt}) = \mathbf{h}_{dv,rt} \quad \text{with} \quad \mathbf{r}_{rt} = \frac{\partial \mathbf{m}_{rt}}{\partial t} + b_{m}(\mathbf{v}_{rt}, \mathbf{m}_{rt}) \quad \text{and} \]
\[ \mathbf{h}_{dv,rt} = \mathbf{h}_{geo,\omega} - \frac{\bar{\phi}'(\mathbf{E}_{rt}, \mathbf{m}_{rt}) + [\sigma e]'(\mathbf{m}_{rt}, \theta_{rt})}{\mu_0} - \nabla \mathbf{v}_{rt} + \kappa \Delta \mathbf{m}_{rt}, \quad (44d) \]
\[ \frac{\partial \mathbf{w}_{rt}}{\partial t} + \text{div}(\mathbf{w}_{rt} \mathbf{v}_{rt} - K(\mathbf{w}_{rt}) \nabla \mathbf{v}_{rt}) = (1 - \varepsilon) \left( M(\mathbf{w}_{rt}) \mathbf{R}_{rt} \right)^2 + v_1 |e(\mathbf{v}_{rt})|^2 
+ v_2 |\nabla e(\mathbf{v}_{rt})|^2 + \partial_t \zeta(\mathbf{r}_{rt}; \mathbf{r}_{rt}) |\nabla \mathbf{R}_{rt}|^2 \right) + \mathbf{v}_{rt} \omega(\mathbf{m}_{rt}) \left( \frac{\partial \mathbf{m}_{rt}}{\partial t} + \mathbf{v}_{rt} \cdot \nabla \mathbf{m}_{rt} \right) 
+ \left( \mathbf{v}_{rt} \omega(\mathbf{m}_{rt}) + \phi(\mathbf{v}_{rt}) \right) \text{div} \mathbf{v}_{rt} \quad \text{with} \quad \mathbf{w}_{rt} = \gamma(\mathbf{v}_{rt}) \quad (44e) \]
holding on \( I \times \Omega \) either a.e. or in a weak sense involving also the boundary conditions (41) which are to be written analogously in terms of the above introduced interpolants, and:
\[ \Delta \mathbf{v}_{rt} = \text{div}(\chi_\omega \mathbf{m}_{rt}) \quad (44f) \]
holding on \( I \times \mathbb{R}^d \) in the weak sense together with the condition \( \lim_{|x| \to \infty} \mathbf{v}_{rt}(t, x) = 0 \) for a.a. \( t \in I \).

**Step 2: \( \alpha \)-priori estimates.** The \( \alpha \)-priori estimation is based on the energy test for the mechanical part combined with the heat problem. This means here the test of equation (40a) by \( \mathbf{v}^k_{rt} \), while using also equation (40b) tested by \( \mathbf{S}^k_{rt} = \phi'_e \left( \mathbf{E}^k_{rt}, \mathbf{m}^k_{rt} \right) \), then the test (40c) by \( \mathbf{R}^k_{rt} \), the inclusion (40d) by \( \mu_0 \mathbf{R}^k_{rt} \), (40e) by \( (u^k_{rt} - u^k_{rt-1})/\tau \), and (40f) by 1. We thus obtain an energy-like inequality for the time-discrete approximation corresponding to equation (27), but in contrast to equation (27), we treat the non-convex terms \( \omega_\varepsilon \) and \( \phi_\varepsilon \) “on the right-hand side”.

More specifically, the terms related to inertia in equation (40a) use the calculus:
\[ \left( \frac{\mathbf{v}^k_{rt} - \mathbf{v}^{k-1}_{rt}}{\tau} + \phi(\mathbf{v}^k_{rt} \cdot \nabla \mathbf{v}^k_{rt} + \frac{\theta}{2} (\text{div} \mathbf{v}^k_{rt}) \mathbf{v}^k_{rt}) \right) \cdot \mathbf{v}^k_{rt} = \frac{\theta}{2} \left( \frac{|\mathbf{v}^k_{rt}|^2 - |\mathbf{v}^{k-1}_{rt}|^2}{\tau} \right) \]
\[ + \phi(\mathbf{v}^k_{rt} \cdot \nabla \mathbf{v}^k_{rt} + \frac{\theta}{2} (\text{div} \mathbf{v}^k_{rt}) |\mathbf{v}^k_{rt}|^2 + \tau \frac{\theta}{2} \left( \frac{|\mathbf{v}^k_{rt} - \mathbf{v}^{k-1}_{rt}|^2}{\tau} \right). \quad (45) \]

This holds pointwise and, when integrated over \( \Omega \), we further use also:
\[ \int_\Omega \phi(\mathbf{v}^k_{rt} \cdot \nabla \mathbf{v}^k_{rt}) \mathbf{v}^k_{rt} \cdot \mathbf{v}^k_{rt} \, dx = - \int_\Omega \frac{\theta}{2} |\mathbf{v}^k_{rt}|^2 (\text{div} \mathbf{v}^k_{rt}) \, dx + \int_\Gamma \frac{\theta}{2} |\mathbf{v}^k_{rt}|^2 (\mathbf{v}^k_{rt} \cdot \mathbf{n}) \, dS \]
\[ = - \int_\Omega \left( \frac{\theta}{2} |\mathbf{v}^k_{rt}|^2 \right) : e(\mathbf{v}^k_{rt}) \, dx + \int_\Gamma \frac{\theta}{2} |\mathbf{v}^k_{rt}|^2 (\mathbf{v}^k_{rt} \cdot \mathbf{n}) \, dS. \quad (46) \]

The last term in equation (45) is non-negative and can simply be forgotten, which will give the inequality:
\[ \int_\Omega \phi(\mathbf{v}^k_{rt} \cdot \nabla \mathbf{v}^k_{rt} + \frac{\theta}{2} (\text{div} \mathbf{v}^k_{rt}) \mathbf{v}^k_{rt}) \cdot \mathbf{v}^k_{rt} \, dx \]
\[ \geq \int_\Omega \frac{\theta}{2} \left( \frac{|\mathbf{v}^k_{rt}|^2 - |\mathbf{v}^{k-1}_{rt}|^2}{\tau} \right) + \phi(\mathbf{v}^k_{rt} \cdot \nabla \mathbf{v}^k_{rt} + \frac{\theta}{2} (\text{div} \mathbf{v}^k_{rt}) |\mathbf{v}^k_{rt}|^2) \, dx \]
\[ = \int_\Omega \frac{\theta}{2} \left( \frac{|\mathbf{v}^k_{rt}|^2 - |\mathbf{v}^{k-1}_{rt}|^2}{\tau} \right) \, dx + \int_\Gamma \frac{\theta}{2} |\mathbf{v}^k_{rt}|^2 (\mathbf{v}^k_{rt} \cdot \mathbf{n}) \, dS. \quad (47) \]
The last term vanishes due to the boundary condition (41a). The further term in equation (40a) uses the calculus:

\[
\int_{\Omega} (\text{div} \, S_{E,\tau}^k) \cdot v_{E,\tau}^k \, dx = \int_{\Gamma} S_{E,\tau}^k \cdot (v_{E,\tau}^k \otimes n) \, dS - \int_{\Omega} S_{E,\tau}^k \cdot \epsilon(v_{E,\tau}^k) \, dx \\
\overset{(40b)}{=} \int_{\Gamma} S_{E,\tau}^k \cdot (v_{E,\tau}^k \otimes n) \, dS - \int_{\Omega} S_{E,\tau}^k \cdot \left( \frac{E_{E,\tau}^k - E_{E,\tau}^{k-1}}{\tau} + B_{\tau}(v_{E,\tau}^k, E_{E,\tau}^k) - R_{E,\tau}^k \right) \, dx
\]

\[
\overset{(40b,c)}{=} \int_{\Gamma} S_{E,\tau}^k \cdot (v_{E,\tau}^k \otimes n) \, dS + \int_{\Omega} \left( M(\theta_{\tau,\tau}^{-1}) |R_{E,\tau}^k|^2 + \varepsilon^2 |\nabla R_{E,\tau}^k|^2 + \left( \psi_m(E_{E,\tau}^k, m_{E,\tau}^k, \theta_{E,\tau}) \right) \right)
\]

\[\frac{\mu \bar{\phi}}{\mu \bar{\phi}} (E_{E,\tau}^k, m_{E,\tau}^k); \left( \frac{E_{E,\tau}^k - E_{E,\tau}^{k-1}}{\tau} + B_{\tau}(v_{E,\tau}^k, E_{E,\tau}^k) \right) \, dx, \quad (48)
\]

where we used also equation (40c) tested by $R_{E,\tau}^k$.

Furthermore, equation (40d) is to be tested by $\mu_0 R_{E,\tau}^k$ and use an analogue of the calculus (20)–(22) by exploiting the algebra:

\[
h_{E,\tau}^k \cdot m_{E,\tau}^k - \frac{m_{E,\tau}^{k-1}}{\tau} = h_{E,\tau}^k \cdot m_{E,\tau}^k - \frac{h_{E,\tau}^{k-1}}{\tau} - \frac{h_{E,\tau}^k - h_{E,\tau}^{k-1}}{\tau} m_{E,\tau}^k - \frac{m_{E,\tau}^{k-1}}{\tau} m_{E,\tau}^k
\]

and convexity of the functional $m \mapsto \int_{\Omega} \frac{1}{2} k |\nabla m|^2 \, dx$. Thus we will get the inequality as a discrete analogue of equation (22), i.e.

\[
\int_{\Omega} \mu_0 h_{E,\tau}^k \cdot v_{E,\tau}^k \, dx \leq \int_{\Omega} \mu_0 h_{E,\tau}^k \cdot m_{E,\tau}^k - \frac{m_{E,\tau}^{k-1}}{\tau} - \mu_0 k \frac{|\nabla m_{E,\tau}^k|^2 - |\nabla m_{E,\tau}^{k-1}|^2}{2\tau} \, dx
\]

\[+ \int_{\Omega} \left( \mu_0 k \frac{|\nabla m_{E,\tau}^k|^2 \text{div} v_{E,\tau}^k - \mu_0 k (\nabla m_{E,\tau}^k \otimes \nabla m_{E,\tau}^k); \epsilon(v_{E,\tau}^k) - \bar{\psi}_m(E_{E,\tau}^k, m_{E,\tau}^k); m_{E,\tau}^k - m_{E,\tau}^{k-1}}{\tau} \right)
\]

\[\times \left( \frac{m_{E,\tau}^k - m_{E,\tau}^{k-1}}{\tau} + (v_{E,\tau}^k \cdot \nabla) m_{E,\tau}^k \right) - \mu_0 (h_{E,\tau}^k \cdot m_{E,\tau}^k) \text{div} v_{E,\tau}^k - \mu_0 (\nabla h_{E,\tau}^k)^T m_{E,\tau}^k \cdot v_{E,\tau}^k - \mu_0 \text{skw}(h_{E,\tau}^k \otimes m_{E,\tau}^k) ; \nabla v_{E,\tau}^k \right) \, dx. \quad (50)
\]

For this test, we also exploit that $\partial_r \zeta(\theta; \tau)$ is uniquely defined even though the subdifferential $\partial_r \zeta(\theta; \cdot)$ is admitted to be set-valued at the magnetization rate $r = 0$.

Furthermore, we test equation (40e) by $\mu_0 (u_{E,\tau}^k - u_{E,\tau}^{k-1})/\tau$. By the analogue of the calculus (25) and exploiting convexity of the functional $u \mapsto \int_{\Omega} \frac{1}{2} \mu_0 |\nabla u|^2 \, dx$, we obtain the inequality:

\[
\int_{\Omega} \mu_0 \frac{|\nabla u_{E,\tau}^k|^2}{\tau} - \frac{|\nabla u_{E,\tau}^{k-1}|^2}{\tau} \, dx + \int_{\Omega} \left( \mu_0 \left( \frac{m_{E,\tau}^k - m_{E,\tau}^{k-1}}{\tau} - \frac{h_{E,\tau}^k \cdot h_{E,\tau}^{k-1}}{\tau} - \frac{h_{E,\tau}^k \cdot h_{E,\tau}^{k-1}}{\tau} \right) \right)
\]

\[\times \left( \frac{m_{E,\tau}^k - m_{E,\tau}^{k-1}}{\tau} + \mu_0 \text{skw}(h_{E,\tau}^k \otimes m_{E,\tau}^k); \nabla v_{E,\tau}^k \right) + \mu_0 \text{skw}(h_{E,\tau}^k \otimes m_{E,\tau}^k) \text{div} v_{E,\tau}^k \right) \, dx \leq 0. \quad (51)
\]

The discrete analogue of equation (23) exploits the assumption of convexity of $\bar{\psi}$ and that $\bar{\psi}_E = \bar{\psi}_E'$ and results to the inequality

\[
\bar{\psi}_E'(E_{E,\tau}^k, m_{E,\tau}^k); \frac{E_{E,\tau}^k - E_{E,\tau}^{k-1}}{\tau} + \bar{\psi}_m(E_{E,\tau}^k, m_{E,\tau}^k); \frac{m_{E,\tau}^k - m_{E,\tau}^{k-1}}{\tau} \geq \frac{\bar{\psi}(E_{E,\tau}^k, m_{E,\tau}^k) - \bar{\psi}(E_{E,\tau}^{k-1}, m_{E,\tau}^{k-1})}{\tau}. \quad (52)
\]
Summing the above estimates for $k = 1, \ldots, l$ and the convexity of $\bar{\varphi}$, we see the discrete analogue of equation (26) as an inequality:

$$
\int_{\Omega} \frac{Q}{2} |v_{\varepsilon}^{l}|^2 + \bar{\varphi}(E_{\varepsilon}^{l}, m_{\varepsilon}^{l}) + \frac{\kappa \mu_{0}}{2} |\nabla m_{\varepsilon}^{l}|^2 \, dx + \int_{\mathbb{R}^d} \frac{\mu_{0}}{2} |\nabla u_{\varepsilon}^{l}|^2 \, dx
$$

$$
+ \tau \sum_{k=1}^{l} \int_{\Omega} M(\tau^{k-1})|R_{\varepsilon}^{k}|^2 + v_{1} |e(v_{\varepsilon}^{k})|^2 + v_{2} |\nabla e(v_{\varepsilon}^{k})|^2 + \frac{\partial_{\varepsilon} (\dot{\theta}_{\varepsilon}^{k-1}; \bar{r}_{\varepsilon}^{k}) \cdot \bar{r}_{\varepsilon}^{k} + \tau |\nabla R_{\varepsilon}^{k}|^2 \, dx
$$

$$
\leq \int_{\Omega} \mu_{0} h_{\varepsilon}^{l} \mathbf{m}_{\varepsilon}^{l} + \frac{\mu_{0}}{2} |v_{0}|^2 + \bar{\varphi}(E_{0}, m_{0}) + \frac{\kappa \mu_{0}}{2} |\nabla m_{0}|^2 - \mu_{0} h_{\varepsilon}^{0} \cdot m_{0} \, dx
$$

$$
+ \int_{\mathbb{R}^d} \frac{\mu_{0}}{2} |\nabla u_{0}|^2 \, dx + \tau \sum_{k=1}^{l} \int_{\Omega} \left( \frac{h_{\varepsilon}^{k-1} - h_{\varepsilon}^{0}}{\tau} \mathbf{m}_{\varepsilon}^{k-1} - \varphi_{\varepsilon}(m_{\varepsilon}^{k}) \frac{m_{\varepsilon}^{k} - m_{\varepsilon}^{k-1}}{\tau} \right)
$$

$$
+ \left( \theta_{\varepsilon}^{k} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k}) \frac{m_{\varepsilon}^{k} - m_{\varepsilon}^{k-1}}{\tau} \right) = \left( \partial_{\varepsilon} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k}) + \varphi_{\varepsilon}^{k} \right) \text{div} v_{\varepsilon}^{k} \right) \, dx, \quad (53)
$$

where $u_{0} \in H^{1}(\mathbb{R}^d)$ denotes the weak solution to equation (7) with $\mathbf{m} = m_{0}$. For $h_{\varepsilon}^{0}$ we consider an (arbitrary) extension of $h_{\varepsilon}^{0}$ from $I = [0, T]$ to $[-\tau, T]$ by continuity so that $h_{\varepsilon}^{0} \rightarrow h_{\varepsilon}^{0}(0)$ for $\tau \rightarrow 0$ in $H^{1}(\Omega; \mathbb{R}^d)$. Adding equation (40f) tested by 1 summed for $k = 1, \ldots, l$, we see cancellation of the adiabatic terms $\partial_{\varepsilon} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k})$, ($m_{\varepsilon}^{k} - m_{\varepsilon}^{k-1}$)/$\tau$ + ($v_{\varepsilon}^{k} \cdot \nabla$) $m_{\varepsilon}^{k}$, and ($\partial_{\varepsilon} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k}) + \varphi_{\varepsilon}^{k}$)div $v_{\varepsilon}^{k}$ and an “$\varepsilon$-cancelation” of the dissipative terms. We thus obtain an analogue of the total energy balance (27) except that an $\varepsilon$-part of the dissipation still remains:

$$
\int_{\Omega} \frac{Q}{2} |v_{\varepsilon}^{l}|^2 + \bar{\varphi}(E_{\varepsilon}^{l}, m_{\varepsilon}^{l}) + \frac{\kappa \mu_{0}}{2} |\nabla m_{\varepsilon}^{l}|^2 + w_{\varepsilon}^{l} \, dx + \int_{\mathbb{R}^d} \frac{\mu_{0}}{2} |\nabla u_{\varepsilon}^{l}|^2 \, dx
$$

$$
+ \varepsilon \tau \sum_{k=1}^{l} \int_{\Omega} M(\tau^{k-1})|R_{\varepsilon}^{k}|^2 + v_{1} |e(v_{\varepsilon}^{k})|^2 + v_{2} |\nabla e(v_{\varepsilon}^{k})|^2 + \frac{\partial_{\varepsilon} (\dot{\theta}_{\varepsilon}^{k-1}; \bar{r}_{\varepsilon}^{k}) \cdot \bar{r}_{\varepsilon}^{k} + \tau |\nabla R_{\varepsilon}^{k}|^2 \, dx
$$

$$
\leq \int_{\Omega} \mu_{0} h_{\varepsilon}^{l} \mathbf{m}_{\varepsilon}^{l} + \frac{\mu_{0}}{2} |v_{0}|^2 + \bar{\varphi}(E_{0}, m_{0}) + \frac{\kappa \mu_{0}}{2} |\nabla m_{0}|^2 - \mu_{0} h_{\varepsilon}^{0} \cdot m_{0} \, dx
$$

$$
+ \int_{\mathbb{R}^d} \frac{\mu_{0}}{2} |\nabla u_{0}|^2 \, dx + \tau \sum_{k=1}^{l} \int_{\Omega} \left( \frac{h_{\varepsilon}^{k-1} - h_{\varepsilon}^{0}}{\tau} \mathbf{m}_{\varepsilon}^{k-1} - \varphi_{\varepsilon}(m_{\varepsilon}^{k}) \frac{m_{\varepsilon}^{k} - m_{\varepsilon}^{k-1}}{\tau} \right)
$$

$$
+ \left( \theta_{\varepsilon}^{k} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k}) \frac{m_{\varepsilon}^{k} - m_{\varepsilon}^{k-1}}{\tau} \right) = \left( \partial_{\varepsilon} \omega_{\varepsilon}^{k} (m_{\varepsilon}^{k}) + \varphi_{\varepsilon}^{k} \right) \text{div} v_{\varepsilon}^{k} \right) \, dx, \quad (54)
$$

The regularization (38) makes $\varphi_{\varepsilon}^{k} (m_{\varepsilon}^{k})$ bounded for fixed $\varepsilon > 0$ while the time difference of $\mathbf{m}$ can be estimated if written as:

$$
\frac{m_{\varepsilon}^{l} - m_{\varepsilon}^{l-1}}{\tau} = \bar{r}_{\varepsilon}^{k} - (v_{\varepsilon}^{k} \cdot \nabla) m_{\varepsilon}^{k} + \text{skw}(\nabla v_{\varepsilon}^{k}) m_{\varepsilon}^{k}. \quad (55)
$$

Then, by the discrete Gronwall inequality and by the coercivity of $\varphi$, we obtain the a-priori estimates:

$$
\|v_{\varepsilon}^{l}\|_{L^{\infty}(t; L^{2}(\Omega; \mathbb{R}^{d})) \cap L^{p}(t; W^{2,p}(\Omega; \mathbb{R}^{d}))} \leq C_{\varepsilon}, \quad (56a)
$$

$$
\|E_{\varepsilon}^{l}\|_{L^{\infty}(t; L^{2}(\Omega; \mathbb{R}^{d}))} \leq C_{\varepsilon}, \quad (56b)
$$

$$
\|m_{\varepsilon}^{l}\|_{L^{\infty}(t; H^{1}(\Omega; \mathbb{R}^{d}))} \leq C_{\varepsilon}, \quad (56c)
$$

$$
\|u_{\varepsilon}^{l}\|_{L^{\infty}(t; H^{1}(\mathbb{R}^{d}))} \leq C_{\varepsilon}, \quad (56d)
$$

$$
\|R_{\varepsilon}^{l}\|_{L^{2}(t; H^{1}(\Omega; \mathbb{R}^{d}))} \leq C_{\varepsilon}, \quad (56e)
$$

$$
\|W_{\varepsilon}^{l}\|_{L^{\infty}(t; L^{1}(\Omega))} \leq C_{\varepsilon} \quad \text{and} \quad \|\theta_{\varepsilon}^{l}\|_{L^{\infty}(t; L^{1}(\Omega))} \leq C_{\varepsilon}. \quad (56g)
$$
The later estimate in equation (56f) follows from \( \frac{\partial}{\partial t} \mathbf{m}_{\varepsilon} + (\mathbf{v}_{\varepsilon \cdot \nabla}) \mathbf{m}_{\varepsilon} = \mathbf{r}_{\varepsilon} + \text{skw}(\nabla \mathbf{v}_{\varepsilon \cdot \nabla}) \mathbf{m}_{\varepsilon} \) since \( \text{skw}(\nabla \mathbf{v}_{\varepsilon \cdot \nabla}) \mathbf{m}_{\varepsilon} \in L^p(I; L^2(\Omega'; \mathbb{R}^d)) \subset L^r(I; \mathbb{R}^d) \) because \( r \leq \min(p, 2^*) \) is assumed. Moreover, from equation (55) and the estimates (56a,c,f), we also have:

\[
\left\| \frac{\partial \mathbf{m}_{\varepsilon}}{\partial t} \right\|_{L^1(I; \mathbb{R}^d)} \leq C_\varepsilon. \tag{56h}
\]

Having now estimated the dissipation heat source on the right-hand side of the discrete heat equation (44e), as the next step we can use the \( L^1 \)-technique to estimate of temperature gradient developed by Boccardo and Gallouët [43] exploiting sophisticatedly the Gagliardo–Nirenberg inequality (cf. [33, Prop. 8.2.1]). The essence is to test the heat-transfer equation (44e) by a smoothened Heaviside function, say \( h(\theta) = (1 - (1 + \varepsilon)^{-1}) \) for \( \varepsilon > 0 \), as suggested in Feireisl et al. [44]. The modification in comparison with the usual “heat operator” in the form \( \frac{\partial}{\partial t} \gamma(\theta) - \text{div}(K(\theta) \nabla \theta) \) and with an \( L^1 \)-right-hand side consists in that \( \tilde{w} \) contains one more term, namely, \( \text{div} (v \gamma(\theta)) \). In the discrete form, this additional convective term is \( \text{div}(\mathbf{v}_{\varepsilon \cdot \gamma}(\tilde{\theta}_{\varepsilon \cdot t})) \) and, when tested by \( h(\tilde{\theta}_{\varepsilon \cdot t}) \), we can estimate it “on the right-hand side” as:

\[
-\int_{\Omega} \text{div}(\mathbf{v}_{\varepsilon \cdot \gamma}(\tilde{\theta}_{\varepsilon \cdot t})) h(\tilde{\theta}_{\varepsilon \cdot t}) \, dx = \int_{\Omega} \tilde{\mathbf{v}}_{\varepsilon \cdot \gamma}(\tilde{\theta}_{\varepsilon \cdot t}) h'(\tilde{\theta}_{\varepsilon \cdot t}) \nabla \tilde{\theta}_{\varepsilon \cdot t} \, dx \\
\leq \int_{\Omega} \frac{1}{\varepsilon} |\tilde{\mathbf{v}}_{\varepsilon \cdot \gamma}|^2 \gamma(\tilde{\theta}_{\varepsilon \cdot t})^2 h'(\tilde{\theta}_{\varepsilon \cdot t}) \, dx + \varepsilon \int_{\Omega} h'(\tilde{\theta}_{\varepsilon \cdot t}) |\nabla \tilde{\theta}_{\varepsilon \cdot t}|^2 \, dx. \tag{57}
\]

As \( \gamma(\theta) = \theta(\theta) \) while \( h'(\theta) = \theta'(1/\theta) \), we have \[ \gamma^2 h'(\theta) = \theta'(1/\theta) \] so that \( \gamma(\tilde{\mathbf{v}}_{\varepsilon \cdot \gamma})^2 h'(\tilde{\theta}_{\varepsilon \cdot t}) \) is bounded in \( L^\infty(I; L^1(\Omega')) \) while \( |\varphi_{\varepsilon \cdot \gamma}|^2 \) is surely bounded in \( L^\infty(I; L^1(\Omega')) \), so that the integrand in the penultimate integral in (57) is bounded in \( L^2(I; L^1(\Omega')) \). For the last integral, this is exactly fitted with the estimation in the \( L^1 \)-theory, and for \( \varepsilon > 0 \) sufficiently small can be absorbed in the respective estimation (cf. [33, Formula (8.2.17)]). Using also the already obtained estimate (56g), we have a “prefabricated” estimate \( \varepsilon \|\varphi_{\varepsilon \cdot \gamma}\|_{L^\infty(I; L^1(\Omega'))} + \varepsilon \|\nabla \varphi_{\varepsilon \cdot \gamma}\|_{L^2(I; L^2(\Omega'))} \leq \frac{1}{\epsilon} + \|\text{rhs}_{\varepsilon t}\|_{L^1(I; \Omega)} \) for some \( \delta_{\epsilon} > 0 \) at disposal with “rhs\( \varepsilon t \)” abbreviating the right-hand side of equation (44e) and with \( \sigma < 2/d + 2/d \) and \( \sigma < (d+2)/(d+1) \) (cf. [33, Prop. 8.2.1]). We add this estimate to the discrete mechanical energy balance (53) for \( I = \tau \) multiplied by the factor 2. Assuming (without loss of generality) \( \tilde{w} \geq 0 \) and then forgetting the first (non-negative) integral in equation (53), we thus obtain the estimate:

\[
\int_{0}^{T} \int_{\Omega} \left( M(\tilde{\theta}_{\varepsilon \cdot t}) |\tilde{\mathbf{R}}_{\varepsilon \cdot t}|^2 + v_1 (\mathbf{e}(\tilde{\mathbf{v}}_{\varepsilon \cdot t}))^2 + v_2 |\nabla \mathbf{e}(\tilde{\mathbf{v}}_{\varepsilon \cdot t})|^2 \right) + \frac{\partial}{\partial t} \varepsilon \left( \tilde{\mathbf{R}}_{\varepsilon \cdot t} \cdot \tilde{\mathbf{r}}_{\varepsilon \cdot t} + \varepsilon |\nabla \tilde{\mathbf{R}}_{\varepsilon \cdot t}|^2 + \varepsilon |\tilde{\mathbf{r}}_{\varepsilon \cdot t}|^2 + \varepsilon |\nabla \tilde{\theta}_{\varepsilon \cdot t}|^2 \right) \, dx \, dt \\
\leq \int_{\Omega} \int_{0}^{T} 2\mu_0 \mathbf{h}_{\varepsilon \cdot \gamma, \tau}(T) \cdot \mathbf{m}_{\varepsilon \cdot t}(T) + \varphi_{\varepsilon \cdot \gamma} |\mathbf{u}_{\varepsilon \cdot \gamma}|^2 + 2\varphi_{\varepsilon \cdot \gamma}(\mathbf{E}_0, \mathbf{m}_0) + \kappa \mu_0 |\nabla \mathbf{m}_0|^2 - 2\mu_0 \mathbf{h}_{\varepsilon \cdot \gamma, \tau} \cdot \mathbf{m}_0 \, dx \, dt \\
+ \int_{\Omega} \int_{0}^{T} \mu_0 |\nabla \mathbf{u}_{\varepsilon \cdot \gamma}|^2 + 2 \int_{0}^{T} \int_{\Omega} \mathbf{e}(\mathbf{v}_{\varepsilon \cdot \gamma}) \cdot \mathbf{m}_{\varepsilon \cdot t} - \tilde{\mathbf{v}}_{\varepsilon \cdot t}(\tilde{\mathbf{m}}_{\varepsilon \cdot t}) - \frac{\partial}{\partial t} \mathbf{m}_{\varepsilon \cdot t} \right) + \tilde{\mathbf{v}}_{\varepsilon \cdot t}(\tilde{\mathbf{m}}_{\varepsilon \cdot t}+\phi(\tilde{\theta}_{\varepsilon \cdot t})) \text{div} \tilde{\mathbf{v}}_{\varepsilon \cdot t} \, dx \, dt \tag{58}
\]

The term \( \tilde{\mathbf{v}}_{\varepsilon \cdot t}(\tilde{\mathbf{m}}_{\varepsilon \cdot t}) \cdot \frac{\partial}{\partial t} \mathbf{m}_{\varepsilon \cdot t} \) can be estimated by using equation (56h) and the growth assumption (34a). The penultimate term in equation (58) can be estimated by means of the growth condition (34c) of \( \omega' \) and the latter estimate in equation (56f), we have by the Hölder and the Young inequalities the estimate:

\[
\int_{0}^{T} \int_{\Omega} \left( \tilde{\mathbf{v}}_{\varepsilon \cdot t} \omega'_{\varepsilon \cdot t}(\tilde{\mathbf{m}}_{\varepsilon \cdot t}) \right) \left| \frac{\partial \mathbf{m}_{\varepsilon \cdot t}}{\partial t} + (\mathbf{v}_{\varepsilon \cdot \gamma}) \mathbf{m}_{\varepsilon \cdot t} \right| \, dx \, dt \\
\leq \frac{\delta_{\epsilon}}{3} \|\tilde{\mathbf{v}}_{\varepsilon \cdot t}\|_{L^1(I; \Omega')} + C \left( 1 + \|\mathbf{m}_{\varepsilon \cdot t}\|_{L^2(I; \Omega')} \right) \left\| \frac{\partial \mathbf{m}_{\varepsilon \cdot t}}{\partial t} + (\mathbf{v}_{\varepsilon \cdot \gamma}) \mathbf{m}_{\varepsilon \cdot t} \right\|_{L^2(I; \Omega')} \tag{59}
\]
estimate is even uniform with respect to \( \varepsilon > 0 \). The last term in equation (58) can be estimated by using the growth assumption (34c,d) on \( \omega \) and on \( \phi \) as:

\[
\int_0^T \int_\Omega \left| (\widetilde{\theta}_{\varepsilon \tau} \omega_\varepsilon (\mathbf{m}_{\varepsilon \tau}) + \phi(\mathbf{m}_{\varepsilon \tau})) \right| \, dx \, dt \\
\leq \frac{\gamma}{3} \, \| \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)} + C \left( 1 + \| \mathbf{m}_{\varepsilon \tau} \|_{L^\infty(I \times \Omega)} \right) \| \nabla \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)},
\]

which holds for \( 1/p + 1/d \leq 1 \) (which is satisfied if \( p > d \)) and for \( 1/s + 1/2^s \leq 1 \) (which is satisfied if \( s = 2^s \)).

Merging equations (59)–(60) with equation (58), we obtain the estimates:

\[
\| \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)} \leq C_{\varepsilon, \sigma}, \quad \text{with} \quad 1 \leq s < 1 + \frac{2}{d}, \quad 1 \leq \sigma < \frac{d + 2}{d + 1}.
\]

The \( L^p \) estimate (61b) of \( \nabla \mathbf{m}_{\varepsilon \tau} \) can be read from (61a) due to \( \nabla \mathbf{m}_{\varepsilon \tau} = \nabla \mathbf{m}_{\varepsilon \tau} \nabla \mathbf{m}_{\varepsilon \tau} \). The \( L^p \) estimate of \( \mathbf{m}_{\varepsilon \tau} \) is to be read by the Gagliardo–Nirenberg interpolation of the first estimate in equation (61b) with equation (56g).

In addition, by comparison from equation (44d), we obtain also:

\[
\| \Delta \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)} \leq C_{\varepsilon}.
\]

More in detail, using also the assumption (34f) and the bound (56f), we have \( \{ \partial_\varepsilon \xi (\mathbf{m}_{\varepsilon \tau}, \mathbf{m}_{\varepsilon \tau}) \}_{\varepsilon > 0} \) bounded in \( L^p(I \times \Omega; \mathbb{R}^d) \). Also \( \nabla \mathbf{m}_{\varepsilon \tau} \) bounded in \( L^p(I \times \Omega; \mathbb{R}^d) \) due to equation (34a); here we used the growth assumption (34c) on \( \omega \) together with the estimates (56c) and (61a), so that:

\[
\int_0^T \int_\Omega \left| \partial_{x_t} \omega_\varepsilon (\mathbf{m}_{\varepsilon \tau}) \right|^r \, dx \, dt \leq C \int_0^T \int_\Omega \left| \partial_{x_t} \mathbf{m}_{\varepsilon \tau} \right|^r \, dx \, dt \\
\leq C \left( 1 + \| \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)} \mathbf{m}_{\varepsilon \tau} \|_{L^p(I \times \Omega)} \right)
\]

provided \( d/(d + 2) + 1/2^s < 1/r^* \), which gives the restriction \( r > \frac{2^d - 2^{d+2}}{2^d + 2^d - 2} \), i.e., \( r > 30/7 \) for \( d = 3 \) or \( r > 2 \) for \( d = 2 \) as assumed in equation (34g).

An important attribute of the model is that the convective transport of variables via the velocity field \( \mathbf{v} \in L^1_{\text{reg}}(\Omega; \mathbb{R}^d) \) or, in the discrete variant by \( \mathbf{v}_\tau \), bounded in \( L^1(I; \mathbb{R}^d) \), qualitatively well copies regularity properties of the initial conditions. We use this phenomenon particularly for the Zaremba–Jaumann time difference of the elastic strain \( E = [E_{ij}] \). Let us consider a general tensor-valued source \( \mathbf{F} = [F_{ij}] \), i.e., in the difference variant:

\[
\frac{E_{ij}^{k} - E_{ij}^{k-1}}{\tau} + B_{ij}(v_i^k, E_{ij}^k) = F_{ij}^k.
\]

For \( \sigma > 1 \), we use the following calculus exploiting the Green formula with the boundary condition \( \mathbf{v} \cdot \mathbf{n} = 0 \):

\[
\int_\Omega (\mathbf{v} \cdot \nabla z) |z|^{\sigma - 2} z \, dx = \int_\Omega (1 - \sigma) |z|^{\sigma - 2} (\mathbf{v} \cdot \nabla z) - (\nabla \mathbf{v}) |z| \, dx \\
+ \int_\Gamma |z|^{\sigma} (\mathbf{v} \cdot \mathbf{n}) \, dS = - \frac{1}{\sigma} \int_\Omega (\text{div} \mathbf{v}) |z|^{\sigma} \, dx.
\]

We test equation (63) by \( |E_{ij}^{k}|^{\sigma - 2} E_{ij}^{k} \), which gives:

\[
\frac{1}{\sigma} \int_\Omega \left| E_{ij}^{k} \right|^{\sigma} + \left| E_{ij}^{k-1} \right|^{\sigma} \, dx \leq \int_\Omega \left| \text{div} \mathbf{v}^k \right| \left| E_{ij}^{k} \right|^{\sigma} + 2 [\text{skw} \nabla \mathbf{v}^k] \left| E_{ij}^{k} \right|^{\sigma} + |E_{ij}^{k}|^{\sigma - 2} E_{ij}^{k} \cdot F_{ij}^k \, dx,
\]
where we used equation (64) for each component \( z = E_y \). From equation (65), by the Young and the discrete Gronwall inequalities, we obtain the estimate:

\[
\| E^k \|_{L^q(I; \mathbb{R}^{d\times d})} \leq C e^{1 + 2\varepsilon \max_{i=1,...,k} \| \nabla v^k_i \|_{L^\infty(\Omega; \mathbb{R}^{d\times d})}} \left( \| E^0 \|_{L^q(I; \mathbb{R}^{d\times d})} + \tau \sum_{i=1}^k \| F^i \|_{L^q(I; \mathbb{R}^{d\times d})} \right)
\]  

(66)

for some \( C \) and for \( \tau \leq 1/(2 + 4\sigma) \) \( \max_{i=1,...,k} \| \nabla v^k_i \|_{L^\infty(\Omega; \mathbb{R}^{d\times d})} + 2\sigma \).

Moreover, we can also test (63) by the \( q \)-Laplacian \( -\text{div}(|\nabla E^k|^q \nabla E^k) \). More specifically, we can apply the \( \nabla \)-operator to equation (63) and test it by \( |\nabla E^k|^q \nabla E^k \). Instead of equation (65), this gives:

\[
\int_\Omega \nabla(v \cdot \nabla z) |\nabla z|^{q-2} \nabla z \, dx = \int_\Omega |\nabla z|^{q-2} \nabla v : (\nabla z \otimes \nabla z) + (q-1)|\nabla z|^{q-2} \nabla (v \cdot \nabla z) + (q-1)(\text{div} v) |\nabla z|^q \, dx
\]

\[+ \int \nabla |\nabla z|^q (v \cdot n) \, dS = \int_\Omega |\nabla z|^{q-2} \nabla v : (\nabla z \otimes \nabla z) - \frac{1}{q} (\text{div} v) |\nabla z|^q \, dx, \]

(67)

In the tensorial situation (63), we use it again for \( z = E_y \) and then we use also:

\[
\nabla \left( \text{skew}(v^k)E^k - E^k \text{skew}(v^k) \right) |\nabla E^k|^q \nabla E^k
\]

\[\leq 2 |\nabla v^k| |\nabla E^k|^q + 2 |\nabla^2 v^k| |\nabla E^k|^q |\nabla E^k|^{q-1}. \]

(68)

Instead of equation (65), this gives:

\[
\frac{1}{q} \int_{\Omega} |\nabla E^k|^q - |\nabla E^{k-1}|^q \, dx \leq \int_{\Omega} \left( \frac{\text{div} v^k}{q} |\nabla E^k|^q + \left( 2 + \frac{1}{q} \right) |\nabla v^k| |\nabla E^k|^q
\]

\[+ |\nabla E^k|^q - |\nabla E^{k-1}|^q \right) \, dx \leq C + C \left( 1 + \| v^k \|_{L^\infty(\Omega)} \| E^k \|_{L^q(\Omega; \mathbb{R}^{d\times d})} \right) \left( \| E^k \|_{L^q(\Omega; \mathbb{R}^{d\times d})} + |\nabla E^k|^p \right)
\]

\[+ \| \nabla F^k \|_{L^q(\Omega; \mathbb{R}^{d\times d})} + \| \nabla E^k \|_{L^q(I; \mathbb{R}^{d\times d})} \right) \cdot (69)

for some \( C \) sufficiently large, provided \( 1/p + 1/\sigma + 1/q' \leq 1 \) with \( p \) from equation (56a). By a discrete Gronwall inequality like equation (66), provided \( \tau \) is sufficiently small, we obtain:

\[
\| \nabla E^k \|_{L^q(I; \mathbb{R}^{d\times d})} \leq C \left( \| E^0 \|_{L^q(I; \mathbb{R}^{d\times d})} + \tau \sum_{i=1}^k \| F^i \|_{L^q(I; \mathbb{R}^{d\times d})} \right).
\]

(70)

with some \( C \) depending on \( \max_{i=1,...,k} \| \nabla v^k_i \|_{L^\infty(\Omega; \mathbb{R}^{d\times d})} \) and on \( \tau \sum_{i=1}^k \| \nabla^2 v^k_i \|_{L^p(I)} + \| E^k \|_{L^q(I)}. \)

Considering \( v^k = v^k_t \) and \( F^k = 2(\nabla v^k) - R^k \) and using the already obtained estimates (4a, e) and the initial condition \( E_0 \in H^1(\Omega; \mathbb{R}^{d\times d}) \), we use the calculus (65)–(66) with \( \sigma = 2^* \) and \( q = 2 \) for equation (40b). Thus we obtain:

\[
\| \nabla E^k \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^{d\times d}))} \leq C.
\]

(71)

**Step 3: convergence in the mechanical part for \( \varepsilon \to 0 \) with \( \varepsilon > 0 \) fixed.** By the Banach selection principle, we obtain a subsequence converging weakly* with respect to the topologies indicated in equations (56) and (61) to some limit \( (v_\varepsilon, E_\varepsilon, R_\varepsilon, m_\varepsilon, u_\varepsilon, \theta_\varepsilon) \).
Moreover, we use the Aubin–Lions compact-embedding theorem generalized for functions with measure time derivatives as in [34, Cor. 7.9] to show the strong convergence:

\[
\begin{align*}
\nabla \vec{\omega}_{\epsilon,t} &\to \nabla \omega_t & &\text{in } L^2(I;L^\infty(\Omega;\mathbb{R}^{d\times d})), \\
\vec{E}_{\epsilon,t} &\to E_\epsilon & &\text{in } L^{1/\epsilon}(I;L^{2^\epsilon}(\Omega;\mathbb{R}^{d\times d})), \\
\vec{m}_{\epsilon,t} &\to m_\epsilon & &\text{in } L^{1/\epsilon}(I;L^{2^{-\epsilon}}(\Omega;\mathbb{R}^{d})), \\
\varphi(\vec{E}_{\epsilon,t},\vec{m}_{\epsilon,t}) &\to \varphi(E_\epsilon,m_\epsilon) & &\text{in } L^{1/\epsilon}(I;L^1(\Omega)), \\
\vec{S}_{\epsilon,t} &\to \vec{S}_{\epsilon} = \varphi'_\epsilon(E_\epsilon,m_\epsilon) & &\text{in } L^{1/\epsilon}(I;L^{2^\epsilon}(\Omega;\mathbb{R}^{d\times d})), \\
\varphi'_m(\vec{E}_{\epsilon,t},\vec{m}_{\epsilon,t}) &\to \varphi'_m(E_\epsilon,m_\epsilon) & &\text{in } L^{1/\epsilon}(I;L^{\sigma}(\Omega;\mathbb{R}^d)), \\
\omega_\epsilon(\vec{m}_{\epsilon,t}) &\to \omega_\epsilon(m_\epsilon) & &\text{in } L^{1/\epsilon}(I;L^{2/\epsilon}(\Omega)), \\
\omega'_\epsilon(\vec{m}_{\epsilon,t}) &\to \omega'_\epsilon(m_\epsilon) & &\text{in } L^{1/\epsilon}(I;L^{2^{-\epsilon}}(\Omega;\mathbb{R}^d)), \\
\end{align*}
\]

for any \(0 < \epsilon \leq 2^* - 1\); here for equation (72d–f) we used the growth assumptions (34a,c). For (72a–c), we need some information about time derivatives to be able to use the mentioned Aubin–Lions theorem generalized for piece-wise constant functions in time. For equation (72a), a boundedness of the sequence \(\{\frac{\partial}{\partial t}E_{\epsilon,t}\}_{t>0}\) in \(L^1(I;W^{2,p}(\Omega;\mathbb{R}^d))\) can be seen by comparison from equation (44a). For the convergence equation (72b), we can see some information by comparison from equation (44b), from which we can see that \(\{\frac{\partial}{\partial t}m_{\epsilon,t}\}_{t>0}\) is bounded in \(L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}))\). Eventually, for equation (72c), an estimate of \(\frac{\partial}{\partial t}m_{\epsilon,t}\) is directly at disposal in the a-priori estimate equation (56h). The strong convergence in equation (72d–f) is then simply by continuity of the Nemytskii operator induced by \(\varphi, \varphi'_E\), and \(\varphi'_m\) even without having any information about the time derivative.

As \(\nabla m\) occurs non-linearly in the capillarity stress and also in the weak formulation (37c) multiplied by \(\Delta m\), we need to prove also a strong convergence \(\nabla \vec{m}_{\epsilon,t} \to \nabla m_\epsilon\). To prove it, we take a sequence \(\{\vec{m}_{\epsilon,t}\}_{t>0}\) piecewise constant in time with respect to the partition with the time step \(\tau\) and converging strongly towards \(m_\epsilon\) for \(\tau \to 0\). Then, using the variational inequality arising from the inclusion (44d) tested by \(\vec{m}_{\epsilon,t} - \vec{m}_t\), we can see that, written in terms of the interpolants, that it holds:

\[
\begin{align*}
\int_0^T \int_{\Omega} \kappa |\nabla (\vec{m}_{\epsilon,t} - \vec{m}_t)|^2 \, dx dt &= \int_0^T \int_{\Omega} \left( \varphi'_m(\vec{E}_{\epsilon,t},\vec{m}_{\epsilon,t}) + \vec{S}_{\epsilon,t} \omega'_\epsilon(\vec{m}_{\epsilon,t}) - \partial_\epsilon \xi(\theta_{\epsilon,t};\vec{r}_{\epsilon,t}) + \vec{S}_{\epsilon} \omega_\epsilon(\vec{m}_\epsilon) - \kappa \nabla \vec{m}_{\epsilon,t} : \nabla (\vec{m}_{\epsilon,t} - \vec{m}_t) \right) \, dx dt \\
&\quad + \int_0^T \int_{\Omega} \kappa |\nabla (\vec{m}_{\epsilon,t} - \vec{m}_t)|^2 \, dx dt \\
&\to 0.
\end{align*}
\]

In fact, here we mean a suitable selection from the set \(\partial_\epsilon \xi(\theta_{\epsilon,t};\vec{r}_{\epsilon,t})\). Here we again used equation (34f) with equation (56f) so that the set \(\partial_\epsilon \xi(\theta_{\epsilon,t};\vec{r}_{\epsilon,t})\) is bounded (uniformly with respect to \(\tau\)) in \(L^\sigma(I \times \Omega;\mathbb{R}^d)\). Moreover, we used growth assumptions (34a,c,f) and the a-priori estimates (56c,d), (61), and (71), so that \(\varphi'_m(\vec{E}_{\epsilon,t},\vec{m}_{\epsilon,t})\) and \(\vec{S}_{\epsilon,t} \omega'_\epsilon(\vec{m}_{\epsilon,t})\) are bounded in \(L^\sigma(I \times \Omega;\mathbb{R}^d)\) while \(\vec{m}_{\epsilon,t} - \vec{m}_t \to 0\) strongly surely in \(L^{2^\epsilon} (I \times \Omega;\mathbb{R}^d) \subset L^\sigma (I \times \Omega;\mathbb{R}^d)\) (cf. equation (72c)). In equation (73), we used also that both \(\nabla \vec{m}_{\epsilon,t}\) is bounded in \(L^2(I \times \Omega;\mathbb{R}^{d\times d}) \subset L^\sigma (I \times \Omega;\mathbb{R}^d)\); here, we used \(1/2^* + 1/\sigma' < 1\) which is granted if \(r < 2^*\) (cf. the assumption (34g)). We thus obtained the desired strong convergence in \(L^2(I \times \Omega;\mathbb{R}^{d\times d})\). By interpolation with the estimate (56c), we obtain even:

\[
\nabla \vec{m}_{\epsilon,t} \to \nabla m_\epsilon \quad \text{strongly in } L^{1/\epsilon}(I;L^2(\Omega;\mathbb{R}^{d\times d})).
\]

As \(M(\cdot), \xi(\cdot;\vec{m})\), and \(\phi\) depend non-linearly on the temperature \(\theta\), we need a strong convergence of \(\vec{\theta}_{\epsilon,t}\) and \(\theta_{\epsilon,t}\):

\[
\vec{\theta}_{\epsilon,t} \to \theta_t \quad \text{and} \quad \theta_{\epsilon,t} \to \theta_t \quad \text{strongly in } L^\epsilon(I \times \Omega)
\]
with $s$ from equation (61b). This follows from the estimates on the gradient (61b), when using the (generalized) Aubin–Lions compact embedding theorem interpolated also with the a-priori estimates (56g). Then we obtain also:

$$
\begin{align*}
\vec{\theta}_{\varepsilon, t} \omega_{\varepsilon}(\vec{m}_{\varepsilon}) & \to \theta_{\varepsilon} \omega_{\varepsilon}(m) \quad \text{in} \ L^s(I; L^{2s/(2s+2)}(\Omega; \mathbb{R}^d)), \\
\vec{\theta}_{\varepsilon} \omega_{\varepsilon}(\vec{m}_{\varepsilon}) & \to \theta_{\varepsilon} \omega_{\varepsilon}(m) \quad \text{in} \ L^{2s/(2s+1)}(I \times \Omega; \mathbb{R}^d), \\
\phi(\vec{\theta}_{\varepsilon}) & \to \phi(\theta) \quad \text{in} \ L^r(I \times \Omega; \mathbb{R}^d)
\end{align*}
$$

(76a)

(76b)

(76c)

with $s$ from equation (61). Note that, for $d = 3, s < 5/3$ and $2^* = 6$ so that the exponent in equation (76a) is below $15/14$ but still bigger than $1$.

Using also equations (72d), (74), (75), and (76), we can see that the capillarity/couple stress converges as:

$$
\vec{S}_{\varepsilon, t} \to \vec{S}_{\varepsilon} = \kappa \mu_0 \left( \nabla m \otimes \nabla m - \frac{1}{2} |\nabla m|^2 \right) - \mu_0 \text{skw}(h_{\varepsilon, \varepsilon} \otimes \nabla m)
$$

weakly in $L^{\min(s, r')}(I; L^1(\Omega; \mathbb{R}^{d \times d}))$. (77)

The term which makes this convergence only weak is $\text{skw}(\Delta \vec{m}_{\varepsilon} \otimes \vec{m}_{\varepsilon})$ contained in $\vec{S}_{\varepsilon, t}$ from equation (44a) with $h_{\varepsilon, \varepsilon}$ from equation (44d); note that $\Delta \vec{m}_{\varepsilon}$ is bounded (and converges weakly) in $L^{s'}(I \times \Omega; \mathbb{R}^d)$ while $\vec{m}_{\varepsilon}$ converges strongly in $L^{1/2}(I; L^{2^*}(\Omega; \mathbb{R}^d))$ and that $1/s' + 1/2^* \leq 1$ because $2^* \geq r$ is assumed. Furthermore, the term $\mu_0 \kappa (\nabla \vec{m}_{\varepsilon} \otimes \nabla \vec{m}_{\varepsilon} - |\nabla \vec{m}_{\varepsilon}|^2/2)$ converges strongly in $L^{1/4}(I; L^1(\Omega; \mathbb{R}^{d \times d}))$ (cf. equation (74)), and is bounded in $L^{\infty}(I; L^1(\Omega; \mathbb{R}^{d \times d}))$. The weak convergence is again due to the term $\Delta \vec{m}_{\varepsilon} \vec{m}_{\varepsilon}$ contained in $h_{\varepsilon, \varepsilon} \vec{m}_{\varepsilon}$.

By the uniform monotonicity of the operators $\frac{\partial}{\partial t}$ and $\text{div}(\text{vol}(\varepsilon \varepsilon))$, we obtain the strong convergence:

$$
\begin{align*}
\vec{\theta}_{\varepsilon, t} & \to \varepsilon \varepsilon \quad \text{in} \ L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \quad \text{and} \\
v_{\varepsilon, t}(T) & \to \varepsilon \varepsilon(T) \quad \text{in} \ L^2(\Omega; \mathbb{R}^d).
\end{align*}
$$

(78a)

(78b)

More in detail, we use the discrete momentum equation (44a) tested by $\vec{\theta}_{\varepsilon, t} - \vec{\theta}_{\varepsilon}$ with some $\vec{\theta}_{\varepsilon}$ piecewise constant on the time-partition of the time step $\tau$ and converging strongly to $\varepsilon \varepsilon$ in $L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \cap C_w(I; L^2(\Omega; \mathbb{R}^d))$. After integration over time and using also equation (35) to avoid $\partial \vec{u}$, we obtain:

$$
\begin{align*}
\frac{\partial}{\partial t} \left[ \frac{\varepsilon \varepsilon(T) - \vec{\theta}_{\varepsilon}(T)}{2} + \varepsilon \varepsilon(T) \right] & \leq \int_0^T \left[ \varepsilon \varepsilon(T) + \varepsilon \varepsilon(T) \right] dt \\
& \leq \int_0^T \left[ \varepsilon \varepsilon(T) + \varepsilon \varepsilon(T) \right] dt \\
& \leq \int_0^T \left( \varepsilon \varepsilon(T) + \varepsilon \varepsilon(T) \right) dt
\end{align*}
$$

(79)

Using equations (72a,c) and (77) and also the a-priori estimates equation (56), we can see that the convergence to 0 in equation (79), and we thus we obtain equation (78). Using also $\nabla \vec{\theta}_{\varepsilon} \to \nabla \varepsilon \varepsilon$ weakly* in $L^\infty(I; L^2(\Omega; \mathbb{R}^d))$, the convergence in the discrete momentum equation (44a) is then easy.

The limit passage in the variational inequality for $\vec{m}_{\varepsilon}$ which is behind the inclusion (44d) (cf. equation (37c)), exploits the strong convergences (72f) and (76b).

Exploiting that $\nabla \vec{\theta}_{\varepsilon} \to \nabla \varepsilon \varepsilon$ strongly in $L^w_w(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$, we can see that $\vec{\theta}_{\varepsilon}$ from equation (68) converges to $\vec{\theta}_{\varepsilon} = \vec{m}_{\varepsilon}$ weakly in $L^s(I \times \Omega; \mathbb{R}^d)$. 

Step 4: convergence of the dissipation rate and of the heat-transfer equation for $\tau \to 0$. Here, we use the already proved convergence in the mechanical part together with the mechanical-energy conservation. Using equation (53) for $l = T/\tau$, we have the chain of estimates:

$$
\int_0^T M(\theta_\tau) |R_\tau|^2 + v_1 |e(v_\tau)|^2 + v_2 |D e(v_\tau)|^2 + \varrho \xi (\theta_\tau; r_\tau) \cdot r_\tau + \varkappa |\nabla R_\tau|^2 \, dx \, dt \\
\leq \liminf_{\tau \to 0} \int_0^T M(\theta_\tau) |\bar{R}_{\tau}|^2 + v_1 |e(\bar{e}_\tau)|^2 + v_2 |D e(\bar{e}_\tau)|^2 + \varrho \xi (\theta_\tau; r_\tau) \cdot r_\tau + \varkappa |\nabla \bar{R}_{\tau}|^2 \, dx \, dt
$$

Using equation (53), we have:

$$
\leq \int_\Omega \frac{\varrho}{2} |v_\tau|^2 + \varphi(E_\tau, m_\tau) + \frac{\kappa \mu_0}{2} |\nabla m_\tau|^2 + \frac{\mu_0}{2} |\nabla u_\tau(0)|^2 - \mu_0 h_{geo, \tau}(0) \cdot m_0 \, dx \\
- \liminf_{\tau \to 0} \int_\Omega \frac{\varrho}{2} |v_\tau(T)|^2 + \varphi(E_\tau(T), m_\tau(T)) + \frac{\kappa \mu_0}{2} |\nabla m_\tau(T)|^2 + \frac{\mu_0}{2} |\nabla u_\tau(T)|^2 \, dx
$$

The first inequality is due to the weak lower semicontinuity. The last equality in equation (80) is just the magneto-mechanical energy balance (26) written for the $\varepsilon$-solution.

This magneto-mechanical energy balance follows from the tests as used for the (formal) calculations (18)–(25) written for the $\varepsilon$-solution. The validity of this balance is however not automatic and the rigorous proof needs to have granted that testing the particular magneto-mechanical equations by $v_\tau$, $S_{str, \tau}$, $R_\tau$, $\mu_0 r_\tau = \mu_0 \dot{m}_\tau$, and $\mu_0 \frac{\partial}{\partial t} u_\tau$ is indeed legitimate. Here, in particular it is important that $S_{str, \tau} \nabla v_\tau \in L^1(I \times \Omega)$ because $S_{str, \tau} \in L^\infty(I; L^1(\Omega; \mathbb{R}^{d \times d}))$ and $\nabla v_\tau \in L^\infty_s(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$. Also,

$$
\Delta m_\tau = \frac{\partial}{\partial t} \xi(\theta; \dot{m}_\tau) + \frac{\varrho \xi (E_\tau, m_\tau) + \varphi_m(E_\tau, m_\tau)}{\mu_0} - h_{geo} - \nabla u_\tau
$$

holds pointwise a.e. in the sense of $L^\sigma(I \times \Omega; \mathbb{R}^d)$, we can legitimately test it by $\dot{m}_\tau = r_\tau + \text{skw}(\nabla v_\tau) m_\tau \in L^\sigma(I \times \Omega)$ provided $r \leq \min(p, 2^*)$ as indeed assumed (cf. equation (34g)). Since $\Delta m(\theta; \cdot)$ is single-valued except 0 (cf. equation (34f)), $\frac{\partial}{\partial t} \xi(\theta; \dot{m}_\tau) \cdot \dot{m}_\tau \in L^1(I \times \Omega)$ is single-valued. To make the test (25) legitimate for the $\varepsilon$-solution, we need $\nabla \frac{\partial}{\partial t} u_\tau \mid_{\times \Omega} \in L^1(I; L^2(\Omega; \mathbb{R}^d))$. By differentiating equation (8e) in time, we have $\Delta \frac{\partial}{\partial t} u_\tau = \text{div}(\chi_{\Omega} \frac{\partial}{\partial t} m_\tau)$. Realizing that $\frac{\partial}{\partial t} m_\tau = r_\tau - (v_\tau \cdot \nabla) m_\tau + \text{skw}(\nabla v_\tau) m_\tau$ belongs surely to $L^2(I \times \Omega; \mathbb{R}^d)$, we obtain even $\nabla \frac{\partial}{\partial t} u_\tau \mid_{\times \Omega} \in L^2(I \times \Omega; \mathbb{R}^d)$.

Altogether, this reveals that there are actually equalities in equation (80). Since the dissipation rate is uniformly convex in terms of rates on the uniformly convex $L^2$- or $L^p$-spaces, these rates converge not only weakly but even strongly in these $L^2$-spaces. Thus, the dissipation rate itself converges strongly in $L^2(I \times \Omega)$.

The limit passage in the resting semi-linear terms in equation (44e) is then easy.

Step 5: a-priori estimates uniform in $\varepsilon > 0$. The arguments can only slightly modify the strategy of Step 2 and the calculus from section 3.

We exploit the magneto-mechanical energy balance as the equality used already in equation (80) yetconsidered on the interval $[0, l]$ and add to it the heat-transfer equation written for the $\varepsilon$-solution integrated over
The non-convexity of the stored energy for $\theta < \theta_c$ now makes no problem and testing by $\frac{\partial}{\partial t} m_\varepsilon$ can be performed as in the time-continuous variant in equation (22). Due to the factor $1 - \varepsilon$ in equation (44e), we obtain the total energy balance (27) for $\varepsilon$-solution with “$\varepsilon$-part” of the dissipative heat source “on the left-hand side” like in equation (54) which is now to be neglected for obtaining uniform estimates with respect to $\varepsilon > 0$. Thus, we obtain the inequality:

$$\int_0^t \int_\Omega \left[ \frac{\mathcal{Q}}{2} |v_\varepsilon(t)|^2 + \varphi(E_\varepsilon(t), m_\varepsilon(t)) + \frac{\kappa \mu_0}{2} |\nabla m_\varepsilon(t)|^2 + \frac{\mu_0}{2} |\nabla u_\varepsilon(t)|^2 + \mu_0 h_{\text{geo}}(t) \cdot m_\varepsilon(t) + w_\varepsilon(t) \right] \mathrm{d}x \mathrm{d}t$$

$$\leq \int_0^t \left( \int_\Omega \mathcal{Q} \mathbf{g} \cdot v_\varepsilon + \frac{\partial h_{\text{geo}}}{\partial t} m_\varepsilon \mathrm{d}x + \int_{\Gamma} j_{\text{ext}} \mathrm{d}S \right) \mathrm{d}t$$

$$+ \int_0^t \frac{\mathcal{Q}}{2} |v_0|^2 + \varphi(E_0, m_0) + \frac{\kappa \mu_0}{2} |\nabla m_0|^2 + \frac{\mu_0}{2} |\nabla u_0|^2 + \mu_0 h_{\text{geo}}(0) \cdot m_0 + w_0 \mathrm{d}x \, .$$

(81)

Using the Young and the Gronwall inequalities, we obtain the a-priori estimates uniform with respect to $\varepsilon > 0$:

$$\| v_\varepsilon \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(82a)

$$\| E_\varepsilon \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d \times \mathbb{R}^d))} \leq C \, ,$$

(82b)

$$\| m_\varepsilon \|_{L^\infty(I; H^1(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(82c)

$$\| u_\varepsilon \|_{L^\infty(I; H^1(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(82d)

$$\| w_\varepsilon \|_{L^\infty(I; L^2(\Omega))} \leq C \quad \text{and} \quad \| \theta_\varepsilon \|_{L^\infty(I; L^2(\Omega))} \leq C \, .$$

(82e)

Then, we employ the magneto-mechanical energy balance itself as the equality used already in equation (80) for the $\varepsilon$-solution. Now, in the time-continuous situation, we can use also the identity $r_\varepsilon = \dot{m}_\varepsilon$. Thus, as in equation (58), we arrive to:

$$\int_0^t \int_\Omega M(\theta_\varepsilon) \nabla \dot{\mathbf{H}}_\varepsilon \cdot \nabla \varepsilon + v_1 \varepsilon v_2 \nabla \varepsilon + \partial_\varepsilon \xi (\dot{\theta}_\varepsilon, \dot{m}_\varepsilon) \cdot \dot{m}_\varepsilon$$

$$+ \mathcal{K} \nabla \dot{\mathbf{H}}_\varepsilon \cdot \nabla \varepsilon + \delta_\varepsilon |\dot{\theta}_\varepsilon|^2 + \delta_\varepsilon |\nabla \dot{\theta}_\varepsilon|^2 \mathrm{d}x \mathrm{d}t$$

$$= \int_\Omega \frac{\mathcal{Q}}{2} |v_0|^2 + \varphi(E_0, m_0) + \frac{\kappa \mu_0}{2} |\nabla m_0|^2 + \frac{\mu_0}{2} |\nabla u_0|^2 - \mu_0 h_{\text{geo}}(0) \cdot m_0 + \mu_0 h_{\text{geo}}(T) \cdot m_\varepsilon(T) \mathrm{d}x$$

$$+ \int_0^t \int_\Omega 2 \mathcal{Q} \mathbf{g} \cdot v_\varepsilon + 2 \frac{\partial h_{\text{geo}}}{\partial t} m_\varepsilon + \mathcal{K} |\dot{\theta}_\varepsilon \omega_\varepsilon(\theta_\varepsilon) | \mathrm{d}x \mathrm{d}t \, .$$

(83)

Exploiting equation (82e) with the estimation as equations (59)–(60), we can treat the adiabatic terms on the right-hand side of equation (83). We thus obtain the estimates:

$$\| v_\varepsilon \|_{L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(84a)

$$\| \dot{\mathbf{H}}_\varepsilon \|_{L^p(I; L^2(\Omega; \mathbb{R}^d \times \mathbb{R}^d))} \leq C \, ,$$

(84b)

$$\| \dot{\theta}_\varepsilon \|_{L^p(I; H^1(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(84c)

$$\| \dot{E}_\varepsilon \|_{L^p(I; H^1(\Omega; \mathbb{R}^d))} \leq C \, ,$$

(84d)

$$\| \dot{\theta}_\varepsilon \omega_\varepsilon(\theta_\varepsilon) \|_{L^p(I; H^1(\Omega; \mathbb{R}^d))} \leq C_s \sigma \, ,$$

(84e)

The estimates (56h), (62), and (71) for $\varepsilon$-solution uniform with respect to $\varepsilon > 0$ can be obtained analogously as above.

**Step 6:** convergence for $\varepsilon \to 0$. This final convergence towards a weak solution due to Definition 1 copies the arguments in the Step 4 above. By the Banach selection principle, we obtain a subsequence converging weakly* with respect to the topologies indicated in estimates in Step 5 to some limit $(v, E, \Pi, R, m, u, \theta)$. The only small difference is that equations (72g,h) and (76a,b) now involves also the convergence $\omega_\varepsilon \rightarrow \omega, \omega'_\varepsilon \rightarrow \omega', \omega_\varepsilon \rightarrow \omega, \text{ and } [\omega'_\varepsilon]_m \rightarrow \omega'_m$. 

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Due to the publisher SAGE requirement, “the author declares that there is no conflict of interest”. Yet, the author claims that he has no jurisdiction education and that he is not completely sure whether there are really no conflicts of interests existing on the planet Earth.

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