Stieltjes Bochner spaces and applications to the study of parabolic equations

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Abstract

This work is devoted to the mathematical analysis of Stieltjes Bochner spaces and their applications to the resolution of a parabolic equation with Stieltjes time derivative. This novel formulation allows us to study parabolic equations that present impulses at certain times or lapses where the system does not evolve at all and presents an elliptic behavior. We prove several theoretical results related to existence of solution, and propose a full algorithm for its computation, illustrated with some realistic numerical examples related to population dynamics.

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1. Introduction

The main goal of this work is to analyze the existence of solution of the partial differential equation

\[
\begin{align*}
    u_g' - \nabla \cdot (k_1 \nabla u) + k_2 u &= f, \quad \text{in } [(0,T) \setminus C_g] \times \Omega, \\
    u &= 0, \quad \text{on } (0,T) \times \partial \Omega, \\
    u(0,x) &= u_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(1)

where \( \Omega \subset \mathbb{R}^3 \) is a domain with a smooth enough boundary \( \partial \Omega \) and \( u_g' \) is the Stieltjes derivative in some Banach space \( V \) with respect to a left-continuous nondecreasing function \( g : \mathbb{R} \to \mathbb{R} \). This is, given a function \( u : [0,T] \to V \), we define for each \( t \in (0,T) \setminus C_g \), \( u_g'(t) \) as the following limit in \( V \) in the case it exists:

\[
    u_g'(t) := \begin{cases} 
        \lim_{s \to t} \frac{u(s) - u(t)}{g(s) - g(t)}, & \text{if } t \notin D_g, \\
        \frac{u(t^+) - u(t)}{g(t^+) - g(t)}, & \text{if } t \in D_g,
    \end{cases}
\]

(2)

where

\[
    D_g = \{ s \in \mathbb{R} : g(s^+) - g(s) > 0 \}
\]

(3)

and

\[
    C_g = \{ s \in \mathbb{R} : g \text{ is constant on } (s - \varepsilon, s + \varepsilon) \text{ for some } \varepsilon \in \mathbb{R}^+ \}.
\]

(4)

The study of this type of derivatives and its application to the field of ODEs appears in [1,2,3]. We use the notation established in previous works. We further assume that \( k_1 > 0 \) is a positive constant, \( k_2 \geq 0, \ u_0 \in L^2(\Omega) \) and \( f \in L^2_0([0,T], L^2(\Omega)) \), with \(([0,T], \mathcal{M}_g, \mu_g)\) a suitable measure space associated to \( g \) [1]. It is important to mention that if \( u : A \subset \mathbb{R} \to H \) is \( g \)-continuous for every \( t_0 \in A \) in the sense of:

\[
    \forall \varepsilon > 0 \exists \delta > 0 : [t \in A, \ |g(t) - g(t_0)| < \delta] \Rightarrow \| u(t) - u(t_0) \|_H < \varepsilon,
\]

(5)

then \( f \) is constant in the same intervals as \( g \) [2, Proposition 3.2]. Moreover, continuity in the previous sense does not imply continuity in the classical sense,
but if $g$ is continuous at $t_0 \in [0,T]$, then so is $f$. Taking into account that $g$ es left-continuous, we observe that the spaces of bounded $g$-continuous functions $\mathcal{BC}_g([0,T], L^2(\Omega))$ and $\mathcal{BC}_g([0,T), L^2(\Omega))$ are basically the same since any function in $\mathcal{BC}_g([0,T], L^2(\Omega))$ must be continuous at $T$.

Observe that the Stieltjes derivative is not defined at the points of $C_g$. The connected components of $C_g$ correspond to lapses when our system does not evolve at all and presents an elliptic behavior. The set $D_g$ of discontinuities of $g$ correspond with times when sudden changes occur and which are usually introduced in the form of impulses. Finally, in the remaining set of times $[0,T] \setminus (C_g \cup D_g)$ the system presents a parabolic behavior and the different slopes of the derivator $g$ (see [3]) correspond to different influences of the corresponding times, namely, the bigger the slope of $g$ the more important the corresponding times are for the process. In a certain sense, system (1) can be considered as a degenerate parabolic system.

The main difficulty in the mathematical analysis of system (1) lies in the fact that we cannot consider the distributional derivative in time for defining the concept of solution. Thence, we will define the solution in terms of its integral representation and prove new Lebesgue-type differentiation results in order to recover the Stieltjes derivative $g$-almost everywhere in $[0,T]$. Results proven in the appendix of [5] suggest that it might be possible to define the concept of $g$-distributional derivative, thus proving the relationship between the $g$-absolute continuous functions and the $W^{1,1}_g$-type spaces. It is important to mention that in the case where $g(t) = t$, we recover the standard derivative, so all of the results that we will prove extend the classical theory.

In this work we will establish the basis of the mathematical analysis for system (1) as well as a first numerical approximation of its solution. In order to organize the contents of the paper, we will divide the work in the following sections: In Section 2 we will introduce the Stieltjes-Bochner spaces in which we will define the concept of solution. We will also prove new Lebesgue-differentiation–type results for the Stieltjes derivatives and some continuous injections. In Section 3 we will define the concept of solution of problem (1). In Section 4 we
will prove an existence result for system (1) that generalizes some aspects of the classical theory of parabolic partial differential equations. Finally, in Section 5, we will present a realistic example and we will propose a numerical scheme. In this example we will have a parabolic-elliptical behavior, showing the advantage of considering derivatives of the Stieltjes type.

2. Stieltjes Bochner spaces

We start by defining the spaces in which to look for the solution of the problem and its fundamental properties. In order to achieve this, and for convenience of the reader, we start by reviewing some concepts related to Bochner spaces \([5,8]\). Let us consider the measure space \((\mathbb{R}, \mathcal{M}, \mu)\) induced by \(g\) and \(V\), a real Banach space.

Definition 2.1 (g-measurable functions). Given \(f : \mathbb{R} \to V\) we say:

- \(f\) is a simple \(g\)-measurable function if there exits a finite set \(\{x_k\}_{k=1}^n \subset V\) such that \(A_k = f^{-1}(\{x_k\}) \in \mathcal{M}\), with \(\mu_g(A_k) < \infty\) and \(f = \sum_{k=1}^n x_k \chi_{A_k}\).
  
  In this case, we its integral as
  \[
  \int f(s) \, d\mu_g(s) = \sum_{k=1}^n x_k \mu_g(A_k) \in V.  \tag{6}
  \]

- \(f\) is a strongly \(g\)-measurable function (or simply \(g\)-measurable function) if there exists a sequence \(\{f_n\}_{n\in \mathbb{N}}\) of simple \(g\)-measurable functions such that \(f_n(s) \to f(s)\) in \(V\) for \(g\)-a.e. \(s \in \mathbb{R}\).

- \(f\) is a weakly \(g\)-measurable function if \(s \in \mathbb{R} \to v(f(s)) \in \mathbb{R}\) is \(g\)-measurable for every \(v \in V'\).

Pettis’ Theorem (cf. [9] §V.4, Theorem 1) establishes that a function \(f : \mathbb{R} \to V\) is strongly \(g\)-measurable if and only if it is weakly \(g\)-measurable and \(g\)-almost separably-valued. Therefore if we consider a separable Banach space \(V\) both concepts are equivalent.

Now we define the concept of a \(g\)-integrable \(V\)-valued function.
Definition 2.2 (g-integrable V-valued function). A g-measurable function \( f : \mathbb{R} \to V \) is said to be a g-integrable V-valued function if there exists a sequence of simple g-measurable functions \( \{ \varphi_n \}_{n \in \mathbb{N}} \) such that \( \varphi_n(t) \to f \) in \( V \) for g-a.e. \( t \in \mathbb{R} \) and
\[
\lim_{n \to \infty} \int \| \varphi_n(s) - f(s) \|_V \, d\mu_g(s) = 0.
\] (7)

The integral of \( f \) in \( B \in \mathcal{M}_g \) is defined as
\[
\int_B f(s) \, d\mu_g(s) = \lim_{n \to \infty} \int \chi_B(s) \varphi_n(s) \, d\mu_g(s) \in V.
\] (8)

Bochner’s Theorem (cf. [6, §V.5, Theorem 1]) allows us to characterize the g-integrable V-valued functions in terms of the g-integrability of its norm, that is a g-measurable function \( f : \mathbb{R} \to V \) is g-integrable if and only if
\[
\int \| f(s) \|_V \, d\mu_g(s) < \infty,
\] (9)
and, in such a case,
\[
\left\| \int_B f(s) \, d\mu_g(s) \right\|_V \leq \int_B \| f(s) \|_V \, d\mu_g(s),
\] (10)
for every \( B \in \mathcal{M}_g \).

Furthermore, we have the following lemma that we will allow us to establish the concept of solution for our problem. From now on, given \( v \in V \) and \( w \in V' \) we will write \( \langle w, v \rangle := w(v) \).

Lemma 2.3 ([6 §V.5, Corollary 2]). Let \( W \) be a Banach space. \( T : V \to W \) a bounded linear operator. Then, if \( f : \mathbb{R} \to V \) is g-integrable, we have that \( T \circ f : \mathbb{R} \to W \) is g-integrable and
\[
\int_B (T \circ f)(s) \, d\mu_g(s) = T \left( \int_B f(s) \, d\mu_g(s) \right), \forall B \in \mathcal{M}_g.
\] (11)
In particular, for \( f : \mathbb{R} \to V' \) and \( v \in V \),
\[
\int_B \langle f(s), v \rangle \, d\mu_g(s) = \left\langle \int_B f(s) \, d\mu_g(s), v \right\rangle.
\] (12)

Definition 2.4 (L^p_g spaces). With the usual equivalence relation functions which are equal g-a.e., we define, for \( 1 \leq p < \infty \), the space \( L^p_g([0,T], V) \) as the
set of \( g \)-measurable functions \( f : [0, T] \rightarrow V \) such that

\[
\int_{[0, T]} \|f(s)\|^p_V \, d\mu_g(s) < \infty.
\]  

(13)

Analogously, we define the space \( L^\infty_g([0, T], V) \) of those functions which are essentially bounded.

**Remark 2.5.** We have that the set \( L^p_g([0, T], V) \) with \( 1 \leq p \leq \infty \) is a Banach space with the norm

\[
\|f\|_{L^p_g([0, T], V)} = \begin{cases} 
\left[ \int_{[0, T]} \|f(s)\|_V^p \, d\mu_g(s) \right]^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\sup_{t \in [0, T]} \|f(t)\|_V, & p = \infty,
\end{cases}
\]  

(14)

–see [8, Theorem 8.15].

From now on, let \( V \) be a real reflexive separable Banach space and let \( H \) be a Hilbert space such that \( V \) continuously and densely embedded in \( H \). Identifying \( H \) with its dual \( H' \) we have that \( V \subset H \equiv H' \subset V' \).

Now we will adapt [6, Theorem 2, p. 134] to our setting (see Theorem 2.9) to guarantee that an indefinite Bochner \( g \)-integral is \( g \)-differentiable. This result will be fundamental in order to recover the existence of \( g \)-derivative \( g \)-almost everywhere for the solutions of problem (1). In order to check this we present some previous definitions and results.

**Theorem 2.6** ([1, Theorem 2.4]). Assume that \( f : [0, T] \rightarrow \mathbb{R} \) is integrable on \([0, T]\) with respect to \( \mu_g \) and consider its indefinite Lebesgue-Stieltjes integral

\[
F(t) = \int_{[0,t]} f(s) \, d\mu_g(s) \quad \text{for all } t \in [0, T].
\]  

(15)

Then there is a \( g \)-measurable set \( N \subset [0, T] \) such that \( \mu_g(N) = 0 \) and

\[
F'_g(t) = f(t) \quad \text{for all } t \in [0, T] \setminus N.
\]  

(16)

**Definition 2.7.** Let \( X, Y \) be vector spaces. An operator \( L : X \rightarrow Y \) is said to be of **finite rank** if \( L(X) \) is contained in a finite dimensional vector subspace of \( Y \).
Observe that any simple $g$-measurable function is of finite rank. The extension of the previous theorem to finite rank functions is straightforward, so we have the following theorem.

**Theorem 2.8** (Generalized Lebesgue’s differentiation Theorem for finite rank functions). Let $f : [0, T] \to V$ is a Bochner $g$-integrable finite rank function and consider its indefinite Lebesgue-Stieltjes integral

$$F(t) = \int_{[0, t]} f(s) \, d\mu_g(s) \in V \quad \text{for all } t \in [0, T]. \quad (17)$$

Then there is a $g$-measurable set $N \subset [0, T]$ such that $\mu_g(N) = 0$ and

$$F'_g(t) = f(t) \in V \quad \text{for all } t \in [0, T] \setminus N. \quad (18)$$

**Theorem 2.9** (Generalized Lebesgue’s differentiation Theorem). Let $T \in \mathbb{R}^+$ and $f : [0, T] \to V$ be a Bochner $g$-integrable function and consider its indefinite Lebesgue-Stieltjes integral

$$F : t \in [0, T] \to \int_{[0, t]} f(s) \, d\mu_g(s) \in V. \quad (19)$$

Then there exists a $g$-measurable set $N \subset [0, T]$ such that $\mu_g(N) = 0$ and

$$F'_g(t) = f(t), \; \forall t \in [0, T] \setminus N. \quad (20)$$

**Proof.** Let us consider the sequence of simple $g$-measurable functions $\{f_n\}_{n \in \mathbb{N}}$ such that

- $\|f_n(s)\|_V \leq \|f(s)\|_V \left(1 + \frac{1}{n}\right)$,
- $\lim_{n \to \infty} f_n(s) = f(s)$, $g$-a.e. $s \in [0, T]$.

Let $t \in [0, T] \setminus (C_g \cup D_g)$. For every $s \in [0, T]$, $s \neq t$, we have that $g(s) \neq g(t)$ and we can consider, assuming that $s > t$,

$$\frac{F(s) - F(t)}{g(s) - g(t)} - f(t) = \frac{1}{g(s) - g(t)} \int_{[t, s]} [f(\xi) - f(t)] \, d\mu_g(\xi)$$

$$= \frac{1}{g(s) - g(t)} \int_{[t, s]} [f(\xi) - f_n(\xi) + f_n(\xi) - f_n(t)] \, d\mu_g(\xi) + f_n(t) - f(t).$$
Thus,
\[ \left\| \frac{F(s) - F(t)}{g(s) - g(t)} - f(t) \right\|_V \leq \frac{1}{g(s) - g(t)} \int_{[s,t]} \|f(\xi) - f_n(\xi)\|_V \, d\mu_g(\xi) \]
\[ + \left\| \frac{1}{g(s) - g(t)} \int_{[t,s]} [f_n(\xi) - f_n(t)] \, d\mu_g(\xi) \right\|_V + \|f_n(t) - f(t)\|_V. \]

Let us define
\[ \bar{u}_n : t \in [0, T] \rightarrow \bar{u}_n(t) = \int_{[0,t]} \|f(\xi) - f_n(\xi)\|_V \, d\mu_g(\xi). \] (21)

It is clear that \( \bar{u}_n \in L^1_g([0,T]) \). Hence, we can use Theorem 2.6 to conclude that
\[ \lim_{s \to t^+} \frac{\bar{u}_n(s) - \bar{u}_n(t)}{g(s) - g(t)} = \|f(t) - f_n(t)\|_V. \] (22)

Since \( f_n \) is finite rank we can use Theorem 2.8 so, in the topology of \( V \),
\[ \lim_{s \to t^+} \frac{1}{g(s) - g(t)} \int_{[t,s]} [f_n(\xi) - f_n(t)] \, d\mu_g(\xi) = 0. \] (23)

Finally,
\[ \lim_{s \to t^+} \left\| \frac{F(s) - F(t)}{g(s) - g(t)} - f(t) \right\|_V \leq 2 \|f(t) - f_n(t)\|_V. \] (24)

Hence, taking \( n \to \infty \), we obtain the desired result. The case \( s < t \) is analogous.

We denote by \( C_g([0, T], V) \) the set of \( g \)-continuous functions on interval \([0, T]\) in the sense of (2), and by \( BC_g([0, T], V) \) the subset of bounded \( g \)-continuous functions on \([0, T]\). We have that the space \( BC_g([0, T], V) \) equipped with the supremum norm
\[ ||h||_0 = \sup_{t \in [0,T]} ||h(t)||_V, \forall h \in C_g([0, T], V), \] (25)

is a Banach space. The proof is analogous to one given in [2, Theorem 3.4].

Given \( 1 \leq p, q \leq \infty \), we define
\[ W^{1,p,q}_g([0, T], V, V') := \left\{ u \in L^p_g([0, T], V) : \exists \bar{u} \in L^q_g([0, T], V'), u(t) = u(0) + \int_{[0,t]} \bar{u}(s) \, d\mu_g(s) \in V', t \in [0, T] \right\}. \] (26)
Proof. Let 
\[ v \in L^q_g([0, T], V', V') \] 
and \( \tilde{u} \in L^q_g([0, T], V') \) be such functions. Then, for every \( v \in V' \),
\[
\int_{[0,t]} [v(\tilde{u}_2(s)) - v(\tilde{u}_1(s))] \, d\mu_g(s) = 0. \tag{27}
\]

Now, using Theorem \ref{thm:w1pq} and differentiating on both sides, \( v(\tilde{u}_2(s)) - v(\tilde{u}_1(s)) = 0 \) for \( g \)-a.e. \( t \in [0, T] \) and every \( v \in V' \), so \( \tilde{u}_2(t) = \tilde{u}_2(t) \) \( g \)-a.e. Furthermore, by Theorem \ref{thm:w1pq} \( \tilde{u} = u'_g \) \( g \)-a.e.

If we endow the space \( W^{1,p,q}_g([0, T], V, V') \) with the norm
\[
\|u\|_{W^{1,p,q}_g([0, T], V, V')} = \|u\|_{L^p_g([0, T], V)} + \|u'_g\|_{L^q_g([0, T], V')}, \tag{28}
\]

it is clear that \( \left(W^{1,p,q}_g([0, T], V, V'), \| \cdot \|_{W^{1,p,q}_g([0, T], V, V')} \right) \) is a normed vector space.

**Remark 2.10.** Observe that given \( u \in W^{1,p,q}_g([0, T], V, V') \), \( \tilde{u} \in L^q_g([0, T], V') \) is unique up to a set of \( g \)-measure zero. To see this assume there are two, \( \tilde{u}_1 \) and \( \tilde{u}_2 \), such functions. Then, for every \( v \in V' \),
\[
\int_{[0,t]} [v(\tilde{u}_2(s)) - v(\tilde{u}_1(s))] \, d\mu_g(s) = 0.
\]

**Lemma 2.11.** Given \( 1 \leq p, q \leq \infty \) we get the following continuous inclusion
\[
\left(W^{1,p,q}_g([0, T], V, V'), \| \cdot \|_{W^{1,p,q}_g([0, T], V, V')} \right) \hookrightarrow (BC_g([0, T], V'), \| \cdot \|_0). \tag{29}
\]

**Proof.** Let \( u \in \tilde{W}^{1,p,q}_g([0, T], V, V') \) and define
\[
v : t \in [0, T] \to v(t) = \int_{[0,t]} \tilde{u}(s) \, d\mu_g(s). \tag{30}
\]

We have that \( v \in \mathcal{AC}_g([0, T], V') \). Moreover,
\[
\|v(t)\|_{V'} \leq \int_{[0,t]} \|\tilde{u}(s)\|_{V'} \, d\mu_g(s)
= \|\tilde{u}\|_{L^q_g([0, T], V')} \leq \mu_g([0, T])^{1-\frac{1}{q}} \|\tilde{u}\|_{L^q_g([0, T], V')}, \tag{31}
\]

where \( \mu_g([0, T])^{1-\frac{1}{q}} \) is the embedding constant of \( L^q_g([0, T], V') \) into \( L^q_g([0, T], V') \) – cf. \cite{9} Theorem 13.17. Thus,
\[
\|v\|_{L^p_g([0, T], V')} = \left( \int_{[0,T]} \|\tilde{u}(s)\|_{V'}^p \, d\mu_g(s) \right)^{\frac{1}{p}}
\leq \left( \int_{[0,T]} (\mu_g([0, T]))^{1-\frac{1}{q}} \|\tilde{u}\|_{L^q_p([0, T], V')}^p \, d\mu_g(s) \right)^{\frac{1}{p}}
= \mu([0, T])^{1+\frac{1}{q} - \frac{1}{p}} \|\tilde{u}\|_{L^q_p([0, T], V')}, \tag{32}
\]
Now, \( v = u + c \) with \( c = u(0) \in V' \) and, if \( \tilde{N} \) is the embedding constant of \( V \) in \( V' \), using (32),

\[
\|c\|_{V'} = \mu_g([0, T])^{-\frac{1}{p}} \left[ \int_{[0, T]} \|c\|_{V'}^p \, d\mu_g(s) \right]^{\frac{1}{p}} \\
\leq \mu_g([0, T])^{-\frac{1}{p}} \|v - u\|_{L^p_\alpha([0, T], V')} \\
\leq \mu_g([0, T])^{-\frac{1}{p}} \|u\|_{L^p_\alpha([0, T], V')} + \|c\|_{V'} \\
\leq \tilde{N} \mu_g([0, T])^{-\frac{1}{p}} \|u\|_{L^p_\alpha([0, T], V')} + \mu_g([0, T])^{-\frac{1}{p}} \|u\|_{L^p_\alpha([0, T], V')}.
\]

Finally,

\[
\|u\|_0 = \sup_{t \in [0, T]} \left\| \int_{[0, t]} \tilde{u}(s) \, d\mu_g(s) - c \right\|_{V'} \\
\leq \int_{[0, T]} \|\tilde{u}(s)\|_{V'} \, d\mu_g(s) + \|c\|_{V'} \\
\leq 2 \mu_g([0, T])^{-\frac{1}{p}} \|\tilde{u}\|_{L^p_\alpha([0, T], V')} + \tilde{N} \mu_g([0, T])^{-\frac{1}{p}} \|u\|_{L^p_\alpha([0, T], V')}.
\]

Thus,

\[
\|u\|_0 \leq \max \left\{ 2 \mu_g([0, T])^{-\frac{1}{p}}, \tilde{N} \mu_g([0, T])^{-\frac{1}{p}} \right\} \|u\|_{\tilde{W}^{1, p, q}_\alpha([0, T], V, V')}.
\]

\( \Box \)

**Corollary 2.12.** The space \( (\tilde{W}^{1, p, q}_g([0, T], V, V'), \| \cdot \|_{\tilde{W}^{1, p, q}_\alpha([0, T], V, V')} ) \) is a Banach space.

**Proof.** We first prove that \( \tilde{W}^{1, p, q}_g([0, T], V, V') \) is a Banach space. Consider a Cauchy sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( \tilde{W}^{1, p, q}_g([0, T], V, V') \). In particular, the sequences \( \{u_n\}_{n \in \mathbb{N}} \) and \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \) are Cauchy sequences in \( L^p_\alpha([0, T], V) \) and \( L^q_\beta([0, T], V') \). Furthermore, thanks to Lemma 2.11, they will also be so in \( \mathcal{B} \mathcal{C}_g([0, T], V') \).

Since the previous spaces are complete, there will exist \( u \in L^p_\alpha([0, T], V) \) and \( \tilde{u} \in L^q_\beta([0, T], V') \) such that \( u_n \to u \) in \( L^p_\alpha([0, T], V) \), \( \tilde{u}_n \to \tilde{u} \) strongly in \( L^q_\beta([0, T], V') \), \( u_n(t) \to u(t) \) strongly in \( V' \) for every \( t \in [0, T] \). Thus, we can take the following expression to the limit for every \( v \in V' \) and every \( t \in [0, T] \),

\[
\langle u_n(t), v \rangle_{V', V} = \langle u_n(0), v \rangle_{V', V} + \int_{[0, t]} \langle \tilde{u}_n(s), v \rangle_{V', V} \, d\mu_g(s),
\]

(36)
and we get, for every \( v \in V' \) and every \( t \in [0, T] \),

\[
\langle u(t), v \rangle_{V', V} = \langle u(0), v \rangle_{V', V} + \int_{[0,t]} \langle \bar{u}(s), v \rangle_{V', V} \, d\mu_g(s). \tag{37}
\]

Since

\[
\langle u_n(t), v \rangle_{V', V} - \langle u(t), v \rangle_{V', V} \leq \|u_n(t) - t(t)\|_V \|v\|_V, \tag{38}
\]

we have that, for every \( v \in V \) and every \( t \in [0, T] \),

\[
\int_{[0,t]} \left( \langle \bar{u}_n(s), v \rangle_{V', V} - \langle \bar{u}(s), v \rangle_{V', V} \right) \, d\mu_g(s)
\]

\[
\leq \int_{[0,t]} \|\bar{u}_n(s) - \bar{u}(s)\|_{V'} \|v\|_V \, d\mu_g(s) \tag{39}
\]

\[
\leq \|v\| \int_{[0,T]} \|\bar{u}_n(s) - \bar{u}(s)\|_{V'} \, d\mu_g(s).
\]

Therefore \( \{u_n\}_{n \in \mathbb{N}} \to u \) in \( \tilde{W}^{1,p,q}_{g}([0,T], V, V') \).

Now we are going to prove that the space \( \tilde{W}^{1,p,q}_{g}([0,T], V, V') \) is also reflexive. In order to achieve that, we need some results that we are going to review for the convenience of the reader.

**Definition 2.13 ([10]).** A pair \((X, Y)\) of Banach spaces \(X\) and \(Y\) is called a compatible couple if there is some Hausdorff topological vector space in which each of \(X\) and \(Y\) is continuously embedded. Let \(((X, \|\cdot\|_X), (Y, \|\cdot\|_Y))\) be a compatible couple, then \(X \cap Y\) with the norm \(\|x\| = \max\{\|x\|_X, \|x\|_Y\}\) and \(X + Y := \{x + y : x \in X, y \in Y\}\) with the norm

\[
\|z\|_{X+Y} = \inf_{\substack{x \in X \\ y \in Y \\ x + y = z}} (\|x\|_X + \|y\|_Y) \tag{40}
\]

are Banach spaces. The cartesian product \(X \times Y\) with the norm \(\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y\) is such that \(X + Y \cong (X \times Y)/L\) where \(L := \{(z, -z) \in X \cap Y\}\). A compatible couple \((X, Y)\) with the property that \(X \cap Y\) is dense in \(X\) and in \(Y\) is called a conjugate couple.

**Lemma 2.14 ([11, Theorem 3.1, p. 15]).** If \((X, Y)\) is a conjugate couple, then (\(X \cap Y\))' is isometric to \(X' + Y'\) and \((X + Y)'\) is isometric to \(X' \cap Y'\).
Let us define the space

\[
\tilde{L}^q_g([0, T], V') := \left\{ u : [0, T] \to V' : \exists \tilde{u} \in L^q_g([0, T], V'), \quad u(t) = u(0) + \int_{[0,t]} \tilde{u}(s) \, d\mu_g(s), \forall t \in [0, T] \right\}.
\] (41)

It is clear that \( \tilde{L}^q_g([0, T], V') \subset BC_g([0, T], V') \) so, \( \forall u \in \tilde{L}^q_g([0, T], V') \), \( u(0) \) has sense in \( V' \). We also have \( u'_g(t) = \tilde{u}(t) \in V' \), \( g \)-a.e. \( t \in [0, T] \).

**Lemma 2.15.** \( \tilde{L}^q_g([0, T], V') \) is a Banach space with the norm

\[
\|u\|_{\tilde{L}^q_g([0, T], V')} := \|u(0)\|_{V'} + \|\tilde{u}\|_{L^q_g([0, T], V')}.
\] (42)

**Proof.** First, \( \| \cdot \|_{\tilde{L}^q_g([0, T], V')} \) is a norm. It is clearly subadditive and absolutely homogeneous. It is left to check that it is positive definite. If \( \|u\|_{\tilde{L}^q_g([0, T], V')} = 0 \) then \( \|u(0)\|_{V'} = 0 \) and \( \|\tilde{u}\|_{L^q_g([0, T], V')} = 0 \). Since they both are norms, \( u(0) = 0 \) and \( \tilde{u} = 0 \). By definition of \( u \),

\[
u(t) = u(0) + \int_{[0,t]} \tilde{u}(s) \, d\mu_g(s) = 0.
\] (43)

Now, take a Cauchy sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( \tilde{L}^q_g([0, T], V') \). Then

\[
u_n(t) = u_n(0) + \int_{[0,t]} \tilde{u}_n(s) \, d\mu_g(s).
\] (44)

Thus, \( \{u_n(0)\}_{n \in \mathbb{N}} \) and \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \) are Cauchy sequences and, since both \( V' \) and \( L^q_g([0, T], V') \) are Banach spaces, they converge to \( x \) and \( v \) respectively. Now, define

\[
u(t) = x + \int_{[0,t]} v(s) \, d\mu_g(s).
\] (45)

Clearly \( u \in \tilde{L}^q_g([0, T], V') \) and \( \{u_n\}_{n \in \mathbb{N}} \to u \) in \( \tilde{L}^q_g([0, T], V') \). Hence, we have that \( \tilde{L}^q_g([0, T], V') \) is a Banach space. \( \square \)

**Lemma 2.16.** \( \tilde{W}^{1,p,q}_g([0, T], V, V') \) and \( L^p_g([0, T], V) \cap \tilde{L}^q_g([0, T], V') \) are isomorphic.

**Proof.** To see this remember that

\[
\|u\|_{\tilde{W}^{1,p,q}_g((0, T], V, V')} = \|u\|_{L^p_g([0, T], V)} + \|u'_g\|_{L^q_g([0, T], V')},
\] (46)
and

$$
\|u(0)\|_{V'} \leq \|u\|_0 \leq C\|u\|_{\tilde{W}^{\gamma}^p([0,T],V,V')},
$$

(47)

where \(C := \max \left\{ 2\mu_g([0,T])^{1-\frac{1}{2}}, N\mu_g([0,T])^{-\frac{1}{2}} \right\} \). Hence,

$$
\|u\|_{L^\gamma_g([0,T],V) \cap \tilde{W}^{\gamma}_p([0,T],V')}
= \max \{\|u\|_{L^\gamma_g([0,T],V)} , \|u\|_{\tilde{W}^\gamma_p([0,T],V')} \}
= \max \{\|u\|_{L^\gamma_g([0,T],V)} , \|u(0)\|_{V'} + \|u'_g\|_{L^\gamma_g([0,T],V')} \}
\leq \max \{\|u\|_{\tilde{W}^\gamma_p; \gamma([0,T],V,V')} , \|u\|_{L^\gamma_g; \gamma([0,T],V,V')} + C\|u\|_{\tilde{W}^\gamma_p; \gamma([0,T],V,V')} \}
= (C + 1)\|u\|_{\tilde{W}^\gamma_p; \gamma([0,T],V,V')}.
$$

On the other hand,

$$
\|u\|_{\tilde{W}^\gamma_p; \gamma([0,T],V,V')}
= \|u\|_{L^\gamma_g([0,T],V)} + \|u'_g\|_{L^\gamma_g([0,T],V')}
\leq 2\max \{\|u\|_{L^\gamma_g([0,T],V)} , \|u'_g\|_{L^\gamma_g([0,T],V')} \}
\leq 2\max \{\|u\|_{L^\gamma_g([0,T],V)} , \|u(0)\|_{V'} + \|u'_g\|_{L^\gamma_g([0,T],V')} \}
= 2\|u\|_{L^\gamma_g([0,T],V) \cap \tilde{W}^{\gamma}_p([0,T],V')}.
$$

\(\square\)

**Lemma 2.17.** \(\bar{L}^\gamma_g([0,T],V')\) is reflexive.

**Proof.** Take the map

$$
\varphi : V' \times L^\gamma_g([0,T],V') \to \bar{L}^\gamma_g([0,T],V')
(x,v) \to \varphi(x,v) = u,
$$

(50)

where, \(\forall t \in [0,T],\)

$$
u(t) = x + \int_{[0,t]} u(s) \, d\mu_g(s) \in V.
$$

(51)

Clearly, \(\varphi\) is an isometric isomorphism with inverse

$$
\varphi^{-1} : \bar{L}^\gamma_g([0,T],V') \to V' \times L^\gamma_g([0,T],V')
\quad u \to \varphi^{-1}(u) = (u(0),u'_g).
$$

(52)

\(\varphi\) induces the isomorphism \(\varphi^* : \bar{L}^\gamma_g([0,T],V')' \to (V' \times L^\gamma_g([0,T],V'))'\). We know that \((X \times Y)' = X' \times Y'\) with the norm \(\|(f,g)\|_{X' \times Y'} = \max\{\|f\|_{X'}, \|g\|_{Y'}\}\) – and
vice-versa, see [11, p. 14]. Hence, thanks to Riesz representation theorem (8
Theorem 8.17), we have that \( \tilde{L}_q^g([0, T], V') \) is isomorphic to \( V \times L_q^g([0, T], V) \),
where \( p^* \in [1, \infty] \) such that \( p + p^* = pp^* \), with the norm
\[
\|(f, g)\|_{V \times L_q^g([0, T], V')} = \max\{\|f\|_{V'}, \|g\|_{L_q^g([0, T], V')}\}.
\] (53)
Taking the dual again, we obtain that \( \tilde{L}_q^g([0, T], V') \) is reflexive. \( \Box \)

**Lemma 2.18.** \((L_p^g([0, T], V), \tilde{L}_q^g([0, T], V'))\) is a conjugate couple.

*Proof.* Observe that we have the continuous inclusion
\[
L_p^g([0, T], V) \cap \tilde{L}_q^g([0, T], V') \simeq \tilde{W}_g^{1,p,q}([0, T], V, V') \hookrightarrow \mathcal{BC}_g([0, T], V').
\] (54)
Therefore, \((L_p^g([0, T], V), \tilde{L}_q^g([0, T], V'))\) is a compatible couple when embedded in \( L_g^p([0, T], V') \). Since we have the dense embeddings
\[
\tilde{W}_g^{1,p,q}([0, T], V, V') \hookrightarrow L_g^p([0, T], V)
\] (55)
and
\[
\tilde{W}_g^{1,p,q}([0, T], V, V') \hookrightarrow \tilde{L}_g^q([0, T], V'),
\] (56)
\((L_g^p([0, T], V), \tilde{L}_g^q([0, T], V'))\) is a conjugate couple. \( \Box \)

**Corollary 2.19.** \( \tilde{W}_g^{1,p,q}([0, T], V, V') \) is reflexive.

*Proof.* Using Lemma 2.14 and the fact that \((L_g^p([0, T], V), \tilde{L}_g^q([0, T], V'))\) is a conjugate couple,
\[
\tilde{W}_g^{1,p,q}([0, T], V, V')' = L_g^p([0, T], V)' + \tilde{L}_g^q([0, T], V').
\] (57)
Thus,
\[
\tilde{W}_g^{1,p,q}([0, T], V, V')'' = L_g^p([0, T], V)'' \cap \tilde{L}_g^q([0, T], V)''
\] (58)
and so, \( \tilde{W}_g^{1,p,q}([0, T], V, V') \) is reflexive. \( \Box \)
3. The concept of solution

In this section we will establish the concept of solution of system (1). In order to properly motivate this concept, we will proceed by analogy with the classic case. So, we consider the following system:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nabla \cdot (k_1 \nabla u) + k_2 u = f, & \text{in } (0, T) \times \Omega, \\
u = 0, & \text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x), & \text{in } \Omega,
\end{cases}
\]  

with \( f \in L^2([0, T], L^2(\Omega)) \), \( u_0 \in L^2(\Omega) \). If we denote by

\[
W^{1,p,q}([0, T], V, V') = \left\{ u \in L^p([0, T], V) : \frac{du}{dt} \in L^q([0, T], V') \right\},
\]

where \( \frac{du}{dt} \) is distributional derivative of \( u \). We have that there exists an unique element \( u \in W^{1,2,2}([0, T], H^1_0(\Omega), H^{-1}(\Omega)) \cap C([0, T], L^2(\Omega)) \) (where \( H^1_0(\Omega) = \{ u \in W^{1,2}(\Omega) : u|_{\partial \Omega} = 0 \} \) and \( H^{-1}(\Omega) = H^1_0(\Omega)' \)) such that \( u(0) = u_0 \) and, for every \( v \in H^1_0(\Omega) \), \( u \) satisfies the following variational formulation in \( D'(0, T) \):

\[
\frac{d}{dt} (u(t), v)_{L^2(\Omega)} + k_1 \int_{\Omega} \nabla u(t) \cdot \nabla v \, dx + k_2 \int_{\Omega} u(t)v \, dx = \int_{\Omega} f(t)v \, dx,
\]

Thus, the distributional derivative is such that

\[
\frac{du}{dt} = f + \nabla \cdot (k_1 \nabla u) - k_2 u \in L^2([0, T], H^{-1}(\Omega)),
\]

and then, by [5, Proposition A.6], we can identify \( u \) with an element of the space

\[
\tilde{W}^{1,2,2}([0, T], H^1_0(\Omega), H^{-1}(\Omega)) = \left\{ u \in L^2([0, T], H^1_0(\Omega)) : \exists \tilde{u} \in L^2([0, T], H^{-1}(\Omega)), \right. \\
\left. u(t) = u(0) + \int_{[0,t]} \tilde{u}(s) \, ds \in H^{-1}(\Omega), \forall t \in [0, T] \right\}
\]

with \( \tilde{u} = \frac{du}{dt} \) almost everywhere in \([0, T]\). So, we have that the spaces

\[
\tilde{W}^{1,2,2}([0, T], H^1_0(\Omega), H^{-1}(\Omega)) \quad \text{and} \quad W^{1,2,2}([0, T], H^1_0(\Omega), H^{-1}(\Omega))
\]

are essentially the same and, for every \( t \in [0, T] \), \( v \in H^1_0(\Omega) \) we have

\[
\langle u(t), v \rangle = \langle u_0, v \rangle + \int_{[0,t]} \int_{\Omega} f(s) v - k_1 \nabla u(s) \cdot \nabla v - k_2 u(s) v \, dx \, ds,
\]
Moreover, thanks to the Lebesgue Differentiation Theorem [7, Theorem 1.6], there exists
\[
\lim_{h \to 0} \frac{u(t + h) - u(t)}{h} = f(t) + \nabla \cdot (k_1 \nabla u(t)) - k_2 u(t) \in H^{-1}(\Omega),
\]
for a.e. \( t \in [0, T] \). That is, there exists the classical derivative in time, \( u'(t) \in H^{-1}(\Omega) \), almost everywhere in \([0, T]\) and it satisfies
\[
u'(t) = f(t) + \nabla \cdot (k_1 \nabla u(t)) - k_2 u(t) \in H^{-1}(\Omega),
\]
for a.e. \( t \in [0, T] \). (67)

In our case, we cannot define the space \( W^{1,2,2}_{g}([0, T], H^1_0(\Omega), H^{-1}(\Omega)) \) because we don’t have a \( g \)-distributional derivative. Still, we can use the generalized Lebesgue’s Differentiation Theorem (Theorem 2.9) and define the solution of system (1) in the following way.

**Definition 3.1 (Solutions of system (1)).** Given \( u_0 \in L^2(\Omega) \) and \( f \in L^2_g([0, T], L^2(\Omega)) \), we say that
\[
u \in W^{1,2,2}_{g}([0, T], H^1_0(\Omega), H^{-1}(\Omega)) \cap BC_g([0, T], L^2(\Omega))
\]
is a solution of equation (1) if, for every \( v \in V \) and every \( t \in [0, T] \),
\[
(u(t), v) = (u_0, v)_H + \int_{[0,t]} (f(s), v) \, d\mu_g(s)
- k_1 \int_{[0,t]} (\nabla u(s), \nabla v) \, d\mu_g(s) - k_2 \int_{[0,t]} (u(s), v) \, d\mu_g(s).
\]
(69)

In the following corollary, a direct consequence of Theorem 2.9 we will see that we can recover the \( g \) derivative of the solution \( g \)-almost everywhere in time.

**Corollary 3.2.** If \( u \) is a solution of equation (1) then there exists a \( g \)-measurable set, \( N \subset [0, T] \), with \( \mu_g(N) = 0 \) such that
\[
u'_g(t) = f(t) + \nabla \cdot (k_1 \nabla u(t)) - k_2 u(t) \in H^{-1}(\Omega), \ \forall t \in [0, T] \setminus N.
\]
(70)

**4. An existence result**

In this section we will study the existence and uniqueness of solution of the equation (1) where \( \Omega \subset \mathbb{R}^3 \) is a domain with a sufficiently regular boundary
We take $V = H^1_0(\Omega)$ and $H = L^2(\Omega)$ in the functional framework of the previous section and we use the classical diagonalization method –see [12]– in order to prove existence of solution. The fundamental goal is to recover those results known for the case $D_g = \emptyset$.

Let us establish some notation. Let $\{w_k\}_{k \in \mathbb{N}}$ be an eigenvector basis of $H^1_0(\Omega)$, orthonormal with respect to $L^2(\Omega)$, related to the following spectral problem

\[
\begin{align*}
  w_n &\in H^1_0(\Omega), \quad \forall n \in \mathbb{N}, \\
  k_1 \int_{\Omega} \nabla w_n \cdot \nabla v \, dx + k_2 \int_{\Omega} w_n v \, dx &= \lambda_n \int_{\Omega} w_n v \, dx, \quad \forall v \in H^1_0(\Omega), \quad \forall n \in \mathbb{N},
\end{align*}
\]

where $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ and such that $\{w_n/\sqrt{\lambda_n}\}_{n \in \mathbb{N}}$ is a basis $H^1_0(\Omega)$ for the scalar product of $H^1_0(\Omega)$,

\[
(u, v) = \int_{\Omega} (k_1 \nabla u \cdot \nabla v + k_2 uv) \, dx. \tag{72}
\]

Let $\Delta g(t) = g(t^+) - g(t)$ for any function $g$. For our next result we recall the following.

**Theorem 4.1** ([2, Lemma 6.4]). Let $x_0 \in \mathbb{R}$, $h, d \in L^1([a, b]),$ with $d(t)\Delta g(t) \neq 1$ for every $t \in [a, b] \cap D_g$ and $\sum_{t \in [a, b] \cap D_g} |\ln|1 - d(t)\Delta g(t)|| < \infty$. Then

\[
\begin{align*}
  x'_g(t) + d(t)x(t) &= h(t) \quad \text{for } g\text{-a.e. in } [a, b), \\
  x(a) &= x_0
\end{align*}
\]

has a unique solution $x \in AC_g([a, b])$.

**Theorem 4.2** (Existence of solution of the system [1]). Let $u_0 \in L^2(\Omega)$, $f \in L^2_\delta([0, T], L^2(\Omega))$ and $g$ a nondecreasing function, continuous in a neighborhood of $t = 0$ and left-continuous in $(0, T]$. Assume that, for every $n \in \mathbb{N},$

(H1) $\lambda_n \Delta g(t) \neq 1, \quad \forall t \in [0, T] \cap D_g,$ $\sum_{t \in [a, b] \cap D_g} |\ln|1 - \lambda_n \Delta g(t)|| < \infty,$

(H2) $e^{-2\lambda_n \mu_g([0, t] \cap D_g)} \prod_{u \in [0, t] \cap D_g} |1 - \lambda_n \Delta g(u)|^2 \leq C_1, \quad \forall t \in [0, T],$
\[(H3) \int_{[0,T]} \lambda_n e^{-2 \lambda_n \mu_g ([0,t) \cap \mathbb{D}_g)} \prod_{u \in [0,t) \cap \mathbb{D}_g} |1 - \lambda_n \Delta g(u)|^2 \, d \mu_g(t) \leq C_1,\]

\[(H4) \int_{[0,t)} e^{-2 \lambda_n \mu_g ([s,t) \cap \mathbb{D}_g)} \prod_{u \in [s,t) \cap \mathbb{D}_g} |1 - \lambda_n \Delta g(u)|^2 \frac{1}{|1 - \lambda_n \Delta g(s)|^2} \, d \mu_g(s) \leq C_2, \forall t \in [0,T],\]

\[(H5) \int_{[0,T]} \int_{[0,t)} \lambda_n e^{-2 \lambda_n \mu_g ([s,t) \cap \mathbb{D}_g)} \prod_{u \in [s,t) \cap \mathbb{D}_g} |1 - \lambda_n \Delta g(u)|^2 \frac{1}{|1 - \lambda_n \Delta g(s)|^2} \, d \mu_g(s) \, d \mu_g(t) \leq C_2,\]

where \(C_1, C_2 \in \mathbb{R}^+\) are constants. Then there exists
\[u \in W^{1,2,2}_g([0,T], H^{1}_0(\Omega), H^{-1}(\Omega)) \cap \mathcal{BC}_g([0,T], L^2(\Omega)),\] (74)

unique solution of the equation \([1]\) in the sense of the formulation \([69]\), such that satisfies the following bounds with respect to the data
\[
\|u\|_{L^{\infty}_g([0,T], L^2(\Omega))} + \|u\|_{L^2_g([0,T], H^{1}_0(\Omega))} + \|u'_g\|_{L^2([0,T], H^{-1}(\Omega))} \\
\leq \tilde{C}_1 \|u_0\|_{L^2(\Omega)} + \tilde{C}_2 \|f\|_{L^2([0,T], L^2(\Omega))},
\] (75)

where \(\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}^+\) are constants.

Proof. In order to make a clearer proof, we will divide it into five parts. In the first part we will approximate problem \([69]\) using the functions of the spectral basis. Then, in Part 2, we will obtain bounds for the solutions associated to the discrete problem. In the third part, we will analyze how to take the limit and recover a solution of the continuous problem. Later, in Part 4, we will analyze the continuity with respect to the data. In the last part we will prove the uniqueness of solution.

* Part 1, spectral basis approximation.

Given \(k \in \mathbb{N}\), let us write
\[u^k_0 = (u_0, w_k), \quad f^k(t) = (f(t), w_k),\] (76)

for every \(k = 1, \ldots, n\). We have that
\[u_0 = \sum_{k=1}^{\infty} u^k_0 w_k, \quad f(t) = \sum_{k=1}^{\infty} f^k(t) w_k.\] (77)
We look for a solution of the form

$$u(t) = \sum_{k=1}^{\infty} \xi^k(t) w_k,$$  \hspace{1cm} (78)

which, substituting formally in (69),

$$(u(t), w_k) = (u_0, w_k) + \int_{[0,t]} (f(s), w_k) \, d\mu_g(s)
- k_1 \int_{[0,t]} \int_{\Omega} \nabla u(s) \cdot \nabla w_k \, dx \, d\mu_g(s)
- k_2 \int_{[0,t]} \int_{\Omega} (u(s), w_k) \, dx \, d\mu_g(s), \hspace{0.5cm} \forall t \in [0,T], \hspace{0.5cm} k \in \mathbb{N}, \hspace{1cm} (79)$$

it will satisfy, for every $k \in \mathbb{N}$, the approximated problem

$$\xi^k(t) = u^k_0 + \int_{[0,t]} [f^k(s) - \lambda_k \xi^k(s)] \, d\mu_g(s), \hspace{0.5cm} \forall t \in [0,T]. \hspace{1cm} (80)$$

Thanks to the Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral (see [2, Theorem 5.1]) that the previous problem is equivalent to

$$\begin{cases}
(\xi^k)'(t) + \lambda_k \xi^k(t) = f^k(t), & \text{for } g\text{-almost all } t \in [0,T], \\
\xi^k(0) = u^k_0, & k \in \mathbb{N}.
\end{cases} \hspace{1cm} (81)$$

Now, by Theorem 4.1 if the compatibility conditions (H1) are satisfied, there exists a unique solution $\xi^k \in AC_g([0,T]) \cap BC_g([0,T]), \hspace{0.5cm} k = 1, \ldots, n$, which, furthermore, we can compute explicitly taking into account the exponential function in [2, Lemma 6.4], this is,

$$\xi^k(t) = \hat{e}^k(t)^{-1} u^k_0 + \hat{e}^k(t)^{-1} \int_{[0,t]} \hat{e}^k(s) \tilde{f}^k(s) \, d\mu_g(s), \hspace{1cm} (82)$$

with, for every $k = 1, \ldots, n$,

$$\hat{e}^k(t) = \begin{cases}
\hat{d}^k(t), & 0 \leq t \leq t_1, \\
(-1)^j \hat{d}^k(t) \Delta g(t), & t_j < t \leq t_{j+1}, \hspace{0.5cm} k = 1, \ldots, N_k,
\end{cases} \hspace{1cm} (83)$$

with $t_{N_k+1} = T$ and $\hat{d}^k$ given by

$$\hat{d}^k(t) = \begin{cases}
\hat{d}^k(t), & t \in [0,T] \setminus D_g, \\
\frac{\ln |1 + \hat{d}^k(t) \Delta g(t)|}{\Delta g(t)}, & t \in [0,T] \cap D_g,
\end{cases} \hspace{1cm} (84)$$
where
\[ \tilde{d}^k(t) = \frac{\lambda_k}{1 - \lambda_k \Delta g(t)}, \quad \tilde{f}^k(t) = \frac{f^k(t)}{1 - \lambda_k \Delta g(t)}, \]
are functions in \( L^1_g([0,T]) \), as it was pointed out in the proof of [2, Proposition 6.8]. Finally, the points \( \{t_1, \ldots, t_{N_k}\} \), with \( k = 1, \ldots, n \), are those in
\[ T^{-\tilde{d}}_k = \{ t \in [0,T] \cap D_g : 1 + \tilde{d}^k(t) \Delta g(t) < 0 \} . \]
This is a finite set because
\[ \sum_{t \in T^{-\tilde{d}}_k} 1 < \sum_{t \in T^{-\tilde{d}}_k} |\tilde{d}^k(t)\Delta g(t)| \leq \|\tilde{d}^k\|_{L^1_g([0,T])} < \infty . \]
Observe that, given \( t \in [0,T] \cap D_g \), we have that
\[ \hat{d}^k(t) = \begin{cases} \lambda_k, & t \in [0,T] \setminus D_g, \\ -\frac{\ln|1 - \lambda_k \Delta g(t)|}{\Delta g(t)}, & t \in [0,T] \cap D_g. \end{cases} \]
Thence, for a given \( t \in [0,T] \cap D_g \), there exists an index \( k \) from which \( \ln|1 - \lambda_k \Delta g(t)| \) is a strictly positive number.

**Part 2: obtaining of bounds related to the solution of the discrete problem.**

In what follows we will obtain a series of bounds of the solution of (82) of the approximated problem (81). Taking the absolute value on (82) we have that
\[ |\xi^k(t)| \leq \frac{|u^k_0|}{|\tilde{c}^k(t)|} + \int_{[0,t]} \frac{|\tilde{c}^k(s)|}{|\tilde{c}^k(t)|} |\tilde{f}^k(s)| \, d\mu_g(s), \]
Thence, taking into account Hölder’s inequality and the parallelogram law,
\[ |\xi^k(t)|^2 \leq 2 \frac{|u^k_0|^2}{|\tilde{c}^k(t)|^2} + 2 \int_{[0,t]} \frac{|\tilde{c}^k(s)|^2}{|\tilde{c}^k(t)|^2 |1 - \lambda_k \Delta g(s)|^2} \, d\mu_g(s) \int_{[0,t]} |\tilde{f}^k(s)|^2 \, d\mu_g(s). \]
Now,
\[
\frac{1}{|\hat{c}^k(t)|^2} = \exp \left( -2 \int_{[0,t] \cap D_g} \hat{c}^k(u) \, d\mu_g(u) \right) \exp \left( -2 \int_{[0,t] \cap D_g} \hat{c}^k(u) \, d\mu_g(u) \right) \\
= e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \exp \left( 2 \sum_{u \in [0,t] \cap D_g} \ln |1 - \lambda_k \Delta g(u)| \right) \\
= e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2.
\]
(92)

Analogously, using (92),
\[
\int_{[0,t]} \frac{|\hat{c}^k(s)|^2}{|\hat{c}^k(t)|^2} \left[ 1 - \lambda_k \Delta g(s) \right]^2 \, d\mu_g(s) = e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \, d\mu_g(s) \\
= e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \frac{1}{|1 - \lambda_k \Delta g(s)|^2} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \, d\mu_g(s).
\]
(93)

Thus, from (91) and using (H2) and (H4) we have that, for every \( k \in \mathbb{N} \) and every \( t \in [0,T] \),
\[
|\xi^k(t)|^2 \leq 2 \left( e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \right) |u_0^k|^2 \\
+ 2 \left( \int_{[0,t]} e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \, d\mu_g(s) \right) \cdot \int_{[0,t]} |f^k(s)|^2 \, d\mu_g(s) \leq 2C_1 |u_0^k|^2 + 2C_2 \|f^k\|^2_{L^2([0,T],\mathbb{R})}.
\]
(94)

On the other hand, from (91) and using (H3) and (H5) we have that
\[
\int_{[0,T]} \lambda_k |\xi^k(t)|^2 \, d\mu_g(t) \\
\leq 2 \left( \int_{[0,T]} \lambda_k e^{-2\lambda_k \mu_g([0,t] \setminus D_g)} \prod_{u \in [0,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \, d\mu_g(t) \right) |u_0^k|^2 \\
+ 2 \int_{[0,T]} \lambda_k e^{-2\lambda_k \mu_g([s,t] \setminus D_g)} \prod_{u \in [s,t] \cap D_g} |1 - \lambda_k \Delta g(u)|^2 \, d\mu_g(s) \, d\mu_g(t) \cdot \|f^k\|^2_{L^2([0,T],\mathbb{R})} \leq 2C_1 |u_0^k|^2 + 2C_2 \|f^k\|^2_{L^2([0,T],\mathbb{R})}.
\]
(95)
From (94) we deduce that, given \( n, p \in \mathbb{N} \),
\[
\sum_{k=n}^{n+p} |\xi^k(t)|^2 \leq 2C_1 \sum_{k=n}^{n+p} |u_0^k|^2 + 2C_2 \int_{[0,T]} \sum_{k=n}^{n+p} |f^k(s)|^2 \, d\mu_g(s), \quad \forall t \in [0,T] \tag{96}
\]
and, from (95),
\[
\sum_{k=n}^{n+p} \int_{[0,T]} \lambda_k |\xi^k(t)|^2 \, d\mu_g(t) \leq 2C_1 \sum_{k=n}^{n+p} |u_0^k|^2 + 2C_2 \int_{[0,T]} \sum_{k=n}^{n+p} |f^k(s)|^2 \, d\mu_g(s). \tag{97}
\]

- Part 3: taking to the limit the discrete problem.

Given \( n \in \mathbb{N} \), we write \( u_n(t) := \sum_{k=1}^{n} \xi^k(t)w_k \), since \( \xi^k \in AC_g([0,T]) \cap BC_g([0,T]) \), we have that \( u_n \in BC_g([0,T], L^2(\Omega)) \). Thanks to the bound in (96) we observe that \( \{u_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( BC_g([0,T], L^2(\Omega)) \). Indeed, on one hand,
\[
\sup_{t \in [0,T]} \|u_n(t)\|_{L^2(\Omega)} = \sup_{t \in [0,T]} \left[ \sum_{k=1}^{n} |\xi^k(t)|^2 \right]^{1/2}, \tag{98}
\]
so, taking into account (96) and the subadditivity of the square root,
\[
\sup_{t \in [0,T]} \|u_{n+p}(t) - u_n(t)\|_{L^2(\Omega)} = \sup_{t \in [0,T]} \left[ \sum_{k=n}^{n+p} |\xi^k(t)|^2 \right]^{1/2} \\
\leq \sqrt{2C_1} \left[ \sum_{k=n}^{n+p} |u_0^k|^2 \right]^{1/2} + \sqrt{2C_2} \left[ \int_{[0,T]} \sum_{k=n}^{n+p} |f^k(s)|^2 \, d\mu_g(s) \right]^{1/2}, \tag{99}
\]
from where we deduce the Cauchy character of the series at the left hand side of the equality. In particular, since \( BC_g([0,T], L^2(\Omega)) \) is a Banach space, the sequence will be convergent to an element \( u \in BC_g([0,T], L^2(\Omega)) \). Furthermore, \( u(0) = u_0 \). To see this, observe that, since \( g \) is continuous at 0 and \( u_n \in BC_g([0,T], L^2(\Omega)) \), \( u_n \) are continuous at 0, so \( u_n(0) = u_n(0^+) \).

From equation (92) and the continuity of \( g \) at 0 we have that
\[
\|u_n(0^+) - u_0\|_{L^2(\Omega)}^2 = \sum_{k=1}^{n} |\xi^k(0^+) - u_0^k|^2 + \sum_{k=n+1}^{\infty} |u_0^k|^2 \\
= \sum_{k=1}^{\infty} \left| \left( \hat{c}^k(0^+)^{-1} - 1 \right) u_0^k + \hat{c}^k(0)^{-1} \int_{(0,0^+]} \hat{c}^k(s)\hat{f}^k(s) \, d\mu_g(s) \right|^2 + \sum_{k=n+1}^{\infty} |u_0^k|^2 \tag{100}
\]
+ \sum_{k=n+1}^{\infty} |u_0^k|^2 = \sum_{k=n+1}^{\infty} |u_0^k|^2 \to 0.

Hence, \( \{u_n(0)\}_{n \in \mathbb{N}} \to u_0 \) and so \( u(0) = u_0 \). On the other hand, if we take into account that
\[
\|u_n\|_{L^2([0,T], H^1_0(\Omega))} = \left[ \int_{[0,T]} \sum_{k=1}^{n} \lambda_k |\xi^k(t)|^2 \, d\mu_g(t) \right]^{1/2},
\]
we have that, thanks to (97),
\[
\|u_{n+p} - u_n\|_{L^2([0,T], H^1_0(\Omega))} = \left[ \int_{[0,T]} \sum_{k=n}^{n+p} \lambda_k |\xi^k(t)|^2 \, d\mu_g(t) \right]^{1/2} \leq \sqrt{2C_1} \left[ \sum_{k=n}^{n+p} |u_k^h|^2 \right]^{1/2} + \sqrt{2C_2} \left[ \int_{[0,T]} \sum_{k=n}^{n+p} |f^k(s)|^2 \, d\mu_g(s) \right]^{1/2}.
\]
Thus, \( \{u_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the Banach space \( L^2_{\lambda}([0,T], H^1_0(\Omega)) \) and so \( \{u_n\}_{n \in \mathbb{N}} \to u \) in \( L^2_{\lambda}([0,T], H^1_0(\Omega)) \). Finally, given \( n \in \mathbb{N} \) and \( 1 \leq k \leq n \), (99) can be written for every \( t \in [0,T] \) and \( k = 1, \ldots, n \),
\[
(u_n(t), w_k) = (u_{0,n}, w_k) + \int_{[0,t]} (f_n(s), w_k) \, d\mu_g(s) - k_1 \int_{[0,t]} \int_{\Omega} \nabla u_n(s) \cdot \nabla w_k \, dx \, d\mu_g(s) - \delta \int_{[0,t]} \int_{\Omega} u_n(s) w_k \, dx \, d\mu_g(s).
\]
Let us fix an element \( k < n \) and take \( n \to \infty \). We have that, for every \( t \in [0,T] \),
\[
\lim_{n \to \infty} \|u_n(t) - u(t), w_k\| \leq \lim_{n \to \infty} \|u_n(t) - u(t)\|_{L^2(\Omega)} = 0,
\]
\[
\lim_{n \to \infty} \int_{[0,t]} (u_n(s) - u(s), w_k) \, d\mu_g(s) \leq \lim_{n \to \infty} \int_{[0,t]} \|u_n(s) - u(s)\|_{H^1_0(\Omega)} \|w_k\|_{H^1_0(\Omega)} \, d\mu_g(s) = 0,
\]
\[
\lim_{n \to \infty} \int_{[0,t]} (f_n(s) - f(s), w_k) \, d\mu_g(s) \leq \lim_{n \to \infty} \int_{[0,t]} \|f_n(s) - f(s)\|_{L^2(\Omega)} \, d\mu_g(s) = 0,
\]
Furthermore, \( \lim_{n \to \infty} (u_n(0), w_k) = (u_0, w_k) \). Thus, since we can choose \( k \) arbitrarily, we deduce, by the density of the system of vectors \( \{w_k\}_{k \in \mathbb{N}} \) in \( H^1_0(\Omega) \),
that
\[
(u(t), w) = (u_0, w) + \int_{[0,t]} (f(s), w) \, d\mu_g(s) - k_1 \int_{[0,t]} \int_{\Omega} \nabla u(s) \cdot \nabla w \, dx \, d\mu_g(s)
- \delta \int_{[0,t]} \int_{\Omega} u(s) w \, dx \, d\mu_g(s), \quad \forall t \in [0, T], \quad \forall w \in H^1_0(\Omega).
\] (105)

**Part 4: bounding with respect to the data.**

On one hand, we have by (94) and (95) that, for every \( t \in [0, T] \),
\[
\|u_n(t)\|_{L^2(\Omega)}^2 + \int_{[0,T]} \|u_n(s)\|_{H^1_0(\Omega)}^2 \, d\mu_g(s)
\leq 4C_1 \|u_0\|_{L^2(\Omega)}^2 + 4C_2 \|f\|_{L^2([0,T],L^2(\Omega))}^2,
\] (106)
so, taking \( n \to \infty \),
\[
\|u\|_{L^\infty([0,T],L^2(\Omega))} + \|u\|_{L^2([0,T],H^1_0(\Omega))} \leq \hat{C}_1 \|u_0\|_{L^2(\Omega)} + \hat{C}_2 \|f\|_{L^2([0,T],L^2(\Omega))}.
\] (107)

Recovering the \( g \)-time derivative from (105),
\[
u'_g = f - k_1 \Delta u - k_2 u \in L^2_g([0,T],H^{-1}(\Omega)),
\] (108)
and using the bounds in (107), we obtain, redefining the constants if necessary, that
\[
\|u\|_{L^\infty([0,T],L^2(\Omega))} + \|u\|_{L^2([0,T],H^1_0(\Omega))} \leq \hat{C}_1 \|u_0\|_{L^2(\Omega)} + \hat{C}_2 \|f\|_{L^2([0,T],L^2(\Omega))}.
\] (109)

**Part 5: uniqueness of solution.** Suppose that there exists another solution of system (1) \( \hat{u} \in W^{1,2,2}_g([0,T], H^1_0(\Omega), H^{-1}(\Omega)) \) in the sense of Definition 3.1. Then,
\[
\hat{u}(t) = \sum_{k=1}^{\infty} \hat\xi^k(t) w_k, \quad \forall t \in [0, T],
\] (110)
where
\[
\hat\xi^k(t) = \int_{\Omega} \hat{u}(t) w_k \, dx
\] (111)
and the convergence of series occurring in (110) is considered in \( L^2(\Omega) \) for all \( t \in [0, T] \) and in \( H^1_0(\Omega) \) for \( g \)-almost all \( t \in [0, T] \). To see this, observe that since
\( \tilde{u}(t) \in L^2(\Omega) \), for all \( t \in [0, T] \), and \( \{w_k\}_{k \in \mathbb{N}} \) is an orthonormal basis of \( L^2(\Omega) \), we have that

\[
\tilde{u}(t) = \sum_{k=1}^{\infty} \left( \int_{\Omega} \tilde{u}(t) \, w_k \, dx \right) w_k, \quad (112)
\]

where the convergence is in \( L^2(\Omega) \). Now, \( \{w_k/\sqrt{\lambda_k}\}_{k=1}^{\infty} \) is a orthonormal basis of \( H^1_0(\Omega) \) associated to scalar product in (72), so, since \( \tilde{u}(t) \in H^1_0(\Omega) \) for \( g \)-almost all \( t \in [0, T] \), we have that

\[
\tilde{u}(t) = \sum_{k=1}^{\infty} \left( \int_{\Omega} \tilde{u}(t) \, w_k \, dx \right) w_k = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (\tilde{u}(t), w_k) w_k
\]

\[
= \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k} \left( \int_{\Omega} \tilde{u}(t) \, w_k \, dx \right) w_k,
\]

where the convergence is in \( H^1_0(\Omega) \). Now, given elements \( t < s \) in \([0, T]\) (analogous for the case \( s < t \)) and \( k \in \mathbb{N} \), we have that

\[
\tilde{\xi}^k(s) - \tilde{\xi}^k(t) = \int_{\Omega} \frac{\tilde{u}(s) - \tilde{u}(t)}{g(s) - g(t)} \, w_k \, dx = \left\langle \frac{\tilde{u}(s) - \tilde{u}(t)}{g(s) - g(t)}, w_k \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)},
\]

thus,

\[
\lim_{s \to t^+} \left| \frac{\tilde{\xi}^k(s) - \tilde{\xi}^k(t)}{g(s) - g(t)} - \left\langle \tilde{u}'_g(t), w_k \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \\
\leq \lim_{s \to t^+} \left\| \frac{\tilde{u}(s) - \tilde{u}(t)}{g(s) - g(t)} - \tilde{u}'_g(t) \right\|_{H^{-1}(\Omega)} \|w_k\|_{H^1_0(\Omega)} = 0. \quad (115)
\]

Where the convergence is a consequence of \( \tilde{u} \in W^{1,2,2}_g([0, T], H^1_0(\Omega), H^{-1}(\Omega)) \). From the previous expression we deduce that

\[
(\tilde{\xi}^k)'_g(t) = \left\langle \tilde{u}'_g(t), w_k \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \text{ for } g\text{-a.e. } t \in [0, T]. \quad (116)
\]

Therefore, \( \tilde{\xi}^k \in \tilde{W}^{1,2}_g(0,T) \cap BC_g([0,T]) \). Now, assume \( \tilde{u} \) is a solution of system \([1]\). Therefore, for \( g\text{-a.e. } t \in [0, T] \) and every \( k \in \mathbb{N} \),

\[
(\tilde{\xi}^k)'_g(t) = \left\langle \tilde{u}'_g(t), w_k \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}
\]

\[
= \left\langle f(t) + k_1 \nabla \cdot \nabla \tilde{u}(t) - k_2 \tilde{u}(t), w_k \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}
\]

\[
= f^k(t) - (\tilde{u}(t), w_k) = f^k(t) - \lambda_k \tilde{\xi}^k(t). \quad (117)
\]

25
From the previous expression we have that \( \hat{\xi}^k \) satisfies the following equation:

\[
\begin{cases}
(\hat{\xi}_g^k)'(t) + \lambda_i \hat{\xi}_g^k(t) = f^k(t), \quad \text{for } g\text{-almost all } t \in [0, T], \\
\hat{\xi}_g^k(0) = u^k_0,
\end{cases}
\]

which is the same equation that satisfies \( \xi^k \) for \( k \in \mathbb{N} \), in \( (81) \). Hence, by the uniqueness of solution of previous system, we have that \( \xi^k(t) = \hat{\xi}^k(t), t \in [0, T], \forall k \in \mathbb{N} \), and then, \( u \) and \( \hat{u} \) are essentially the same element.

**Remark 4.3.** We must point out that the solution we have obtained in \( (80) \) is not, in general, continuous at the points of \( [0, T] \cap D_g \). Indeed, given \( t \in [0, T] \cap D_g \), we have that

\[
\xi^k(t^+) = \xi^k(t)(1 - \lambda_k \Delta_g(t)) + f^k(t)\Delta_g(t). \tag{119}
\]

Let us consider now sufficient conditions in order to guarantee the fulfillment of the existence hypotheses (H1)–(H5). As we can see from the proof, such conditions are necessary in order to establish some bounds of the partial sums in some spaces. These appear naturally while establishing the bounds that concern the initial condition and source term.

**Corollary 4.4** (Sufficient conditions). Let \( u_0 \in L^2(\Omega) \), \( f \in L^2_g([0, T], L^2(\Omega)) \) and \( g \) a nondecreasing function, continuous in a neighborhood of \( t = 0 \) and left-continuous in \( (0, T] \). A sufficient condition for (H2)–(H5) to hold is

\[
\sum_{u \in [s, t) \cap D_g} \frac{\ln |1 - \lambda_k \Delta g(u)|}{\lambda_k} < \mu_g([s, t) \setminus D_g), \forall 0 \leq s < t \leq T, \forall k \in \mathbb{N}. \tag{120}
\]

**Proof.** On one hand,

\[
e^{-2\lambda_k \mu_g([0, t) \setminus D_g)} \prod_{u \in [0, t) \cap D_g} |1 - \lambda_k \Delta g(u)|^2
\]

\[
= \begin{cases}
e^{-2\lambda_k \mu_g([0, t))}, & t \in [0, t_0), \\
\exp \left( -2\lambda_k \left[ \mu_g([0, t) \setminus D_g) - \sum_{u \in [0, t) \cap D_g} \frac{\ln |1 - \lambda_k \Delta g(u)|}{\lambda_k} \right] \right), & t \in (t_0, T],
\end{cases}
\]

\( \square \)
for any \( t_0 \in [0, T] \) such that \( g \) is continuous in \([0, t_0)\). Therefore we obtain the bounds in (H2) and (H3). Let us check now what happens with conditions (H4) and (H5). Given \( t \in [0, T] \) and \( s \in [0, t) \), we have that
\[
e^{-2\lambda_k \mu_g([s, t) \setminus D_g)} \prod_{u \in [s, t) \cap D_g} \frac{|1 - \lambda_k \Delta g(u)|^2}{|1 - \lambda_k \Delta g(s)|^2} \leq \exp \left(-2\lambda_k \left[ \mu_g([s, t) \setminus D_g) - \sum_{u \in [s, t) \cap D_g} \frac{\ln |1 - \lambda_k \Delta g(u)|}{\lambda_k} \right] \right) .
\]
In particular, we have that
\[
e^{-2\lambda_k \mu_g([s, t) \setminus D_g)} \prod_{u \in [s, t) \cap D_g} \frac{|1 - \lambda_k \Delta g(u)|^2}{|1 - \lambda_k \Delta g(s)|^2} \leq \exp \left(-2\lambda_k \left[ \mu_g([s, t) \setminus D_g) - \sum_{u \in [s, t) \cap D_g} \frac{\ln |1 - \lambda_k \Delta g(u)|}{\lambda_k} \right] \right) , \quad t \in [0, t_0),
\]
and
\[
e^{-2\lambda_k \mu_g([s, t) \setminus D_g)} \prod_{u \in [s, t) \cap D_g} \frac{|1 - \lambda_k \Delta g(u)|^2}{|1 - \lambda_k \Delta g(s)|^2} \leq \exp \left(-2\lambda_k \left[ \mu_g([s, t) \setminus D_g) + \sum_{u \in [t_0, t) \cap D_g} \frac{\ln |1 - \lambda_k \Delta g(u)|}{\lambda_k} \right] \right), \quad t \in [t_0, T),
\]
from where obtain estimations (H4) and (H5).

**Remark 4.5.** We can easily extend the results above to the case of Neumann homogeneous boundary conditions:
\[
\begin{aligned}
&u'(g) - \nabla \cdot (k_1 \nabla u) + k_2 u = f(t, x), \quad \text{in } ([0, T) \setminus C_g) \times \Omega, \\
k_1 \nabla u \cdot n = 0, &\quad \text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x), &\quad \text{in } \Omega.
\end{aligned}
\]
In this case, we obtain a solution in the space \( \tilde{W}^{1,2,2}_g([0, T], H^1(\Omega), H^1(\Omega)^c) \cap BC_g([0, T], L^2(\Omega)) \).

**Remark 4.6.** Observe that in the case of \( g(t) = t \), hypothesis (H1)-(H5) are trivially satisfied and we recover the classical results for the parabolic partial dif-
So, in a certain sense, the theory that we have developed generalizes the classical theory for this type of partial differential equations.

5. Applications to population dynamics

In this section we present a possible application of the theory that we have developed in the previous section to a silkworm population model based on the example presented in [3, Section 5]. In our case, we will consider that we have a diffusion term that allows us to study the spatial distribution of the silkworm in an island (for instance, Gran Canaria). Thus, consider the following equation:

\[
\begin{align*}
&u'(t) - \nabla \cdot (\eta \nabla u) = f(t, u(t), u), \quad \text{in } [(0, T) \setminus C_g] \times \Omega, \\
&\eta \nabla u \cdot \mathbf{n} = 0, \quad \text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(125)

where \( \Omega \subset \mathbb{R}^2 \), \( \eta > 0 \), \( u_0 \in L^2(\Omega) \), \( f : [0, T] \setminus C_g \times \mathbb{R} \times L^1_{\text{loc}}(\mathbb{R}) \to \mathbb{R} \) is such that

\[
f(t, x, \varphi) = \begin{cases} 
-cx, & \text{if } t \in (5k, 5k + 4), k = 0, 1, 2, \ldots \\
-x, & \text{if } t = 5k + 4, k = 0, 1, 2, \ldots \\
\lambda \int_{t-5}^{t-1} \varphi(s) \, ds, & \text{if } t = 5(k + 1), k = 0, 1, 2, \ldots
\end{cases}
\]

(126)

with \( c > 0 \) and \( \lambda > 0 \) and \( g : t \in [0, \infty) \to \mathbb{R} \) defined as

\[
g(t) = \begin{cases} 
\frac{1}{2} \sqrt{4t - t^2}, & \text{if } 0 \leq t \leq 2, \\
1, & \text{if } 2 < t \leq 3, \\
2 - \sqrt{6t - t^2 - 8}, & \text{if } 3 < t \leq 4, \\
3, & \text{if } 4 < t \leq 5, \\
4 + g(t - 5), & \text{if } 5 > t.
\end{cases}
\]

(127)

A detailed description of the relationship between the previous functions and the life cycle of silkworms can be found in [3]. If we integrate (125) in the whole domain \( \Omega \) and denote by \( \overline{u}(t) = \int_\Omega u(t) \, dx \), we have

\[
\int_\Omega u'_g(t) \, dx - \int_\Omega \nabla \cdot (\eta \nabla u) \, dx = \int_\Omega f(t, u(t), u) \, dx,
\]

(128)
where we have assumed that we can interchange the integral with the Stieltjes derivative, we recover the 0-space-dimensional model studied in \[3\]:

\[
\begin{aligned}
\dot{\overline{u}}_g(t) &= f(t, \overline{u}(t), \overline{u}), \quad \text{in } (0, T) \setminus C_0, \\
u(0) &= \overline{u}_0.
\end{aligned}
\] (129)

The mathematical analysis of equation (125) can be done utilizing the same techniques that we have used for the general model (1). That is, we can consider a spectral basis of $H^1(\Omega)$, solve the corresponding problem associated to each eigenvalue and finally pass to the limit. We leave the details to the reader and we will focus on the numerical approximation of the model.

We consider a polygonal approximation of Gran Canaria island (the domain $\Omega$), and the following triangulation $\{\tau_h^k\}_{k=1}^{nt}$ of the domain:

![Image of Gran Canaria island](image1)

(a) Image of Gran Canaria island generated by Google Earth.

![Triangulation](image2)

(b) Triangulation generated with FreeFem++ from the image of the left (88018 triangles).

Figure 1: Computational domain.

Associated to the previous triangulation, we consider the following finite element space:

\[
V_h = \{u \in C(\Omega) : u|_{\tau_h^k} \in \mathbb{P}_1(\tau_h^k), \forall k = 1, \ldots, nt \} \subset H^1(\Omega).
\] (130)

Now, let $nv = \text{dim}(V_h)$ (number of vertices) and $\{\varphi_{h,j}^k\}_{j=1}^{nv}$ a basis of $V_h$ such that

\[
\varphi_{h,j}^k(e_k) = \delta_{k,j}, \quad \forall k, j \in \{1, \ldots, nv\},
\] (131)
where \( \{e_k\}_{k=1}^{nv} \) are the vertices of the mesh. For \( k, j = 1, \ldots, nv \) we write

\[
[R_h] \in \mathcal{M}_{nv \times nv}(\mathbb{R}) : [R_h]_{k,j} = \eta \int_{\Omega} \nabla \varphi_k \cdot \nabla \varphi_j \, dx + \int_{\Omega} \varphi_k \varphi_j \, dx,
\]

\[
[M_h] \in \mathcal{M}_{nv \times nv}(\mathbb{R}) : [M_h]_{k,j} = \int_{\Omega} \varphi_k \varphi_j \, dx.
\]

We will approximate the solution of system (125) by

\[
u_h(x,t) = \sum_{k=1}^{nv} \xi^k_h(t) \psi^k_h(x), \tag{133}
\]

where, for \( k = 1, \ldots, nv \), \( \xi^k_h(t) \in AC_g([0,T]) \) is the solution of

\[
\begin{cases}
(\xi^k_h)'_g(t) + \lambda^k_h \xi^k_h(t) = f(t, \xi^k_h(t), \xi^k_h), & g\text{-a.e. } (0,T) \setminus C_g, \\
\xi^k_h(0) = (u_0, \psi^k_h)_{L^2(\Omega)},
\end{cases}
\]

(134)

where

\[
\psi^k_h(x) = \sum_{j=1}^{nv} v^k_j \varphi_j(x) \tag{135}
\]

with \( 0 < \lambda^1_h \leq \lambda^2_h \leq \cdots \leq \lambda^{nv}_h \) and \( \{v^k\}_{k=1}^{nv} \subset \mathbb{R}^{nv} \) such that

\[
[R_h] v^k_h = \lambda^k_h [M_h] v^k_h, \quad \forall k = 1, \ldots, nv. \tag{136}
\]

We have (see [14] Theorem 6.4-1]) that \( \{\psi^k_h\}_{k=1}^{nv} \) is a basis of \( V_h \) orthonormal in \( L^2(\Omega) \) and \( (\psi^k_h, \varphi^j) = \lambda^k_h \langle \psi^k_h, \varphi^j \rangle \), \( \forall k, j = 1, \ldots, nv \), where, given \( u, v \in H^1(\Omega) \),

\[
(u,v) = \eta \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} u v \, dx. \tag{137}
\]

Finally, using the same arguments as in [3], we have the following expression for the exact solution of (134):

\[
\xi^k_h(t) = \begin{cases}
(u_0, \psi^k_h) e^{-(\lambda^k_h+c-1)g(t)}, & \text{if } 0 \leq t \leq 4, \\
\lambda \int_{5(k-1)}^{5k-1} \xi^k_h(s) \, ds \cdot \exp \left(- (\lambda^k_h+c-1)[g(t) - g(5k^+)] \right), & \text{if } 5k < t \leq 5k + 4, k \in \mathbb{N}, \\
0, & \text{in other case.}
\end{cases}
\]

(138)
In order to implement all the previous approximations we have used the software FreeFem++. We present some results that we have obtaining using the first 150 eigenfunctions and the same data as in [3], with

\[ u_0(x, y) = \frac{x_0}{\int_\Omega (r^2 + s^2) \, dr \, ds} (x^2 + y^2), \]  

(139)

and \( T = 15 \). First, in Figure 2 we can see a comparison between the solution of the 0-dimensional model and the 2-dimensional model. We observe that the evolution of the spatial mean of the 2-d model solution coincides with the 0-d model solution, which was expected in view of the fact that equation (129) has to verify the spatial mean of the 2-d solution.

Figure 2: Comparison between the solution of the 0-d model and the 2-d model.

Secondly, in Figures 3a, 3b and 3c we can observe, respectively, the initial condition and the solution in the first (\( t = 5 \)) and second impulse (\( t = 10 \)).
Figure 3: Evolution of the model from the initial condition to time $t = 10$.

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33