PERMUTATION STATISTICS AND MULTIPLE PATTERN AVOIDANCE

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Abstract. For a set of permutation patterns \( \Pi \), let \( F_{n}^{st}(\Pi; q) \) be the st-polynomial of permutations avoiding all patterns in \( \Pi \). Suppose \( 312 \in \Pi \). For a class of permutation statistics which includes inversion and descent statistics, we give a formula that expresses \( F_{n}^{st}(\Pi; q) \) in terms of these st-polynomials where we take some subblocks of the patterns in \( \Pi \). Using this formula, we can construct many examples of nontrivial st-Wilf equivalences. In particular, this disproves a conjecture by Dokos, Dwyer, Johnson, Sagan, and Selsor that all inv-Wilf equivalences are trivial.

1. Introduction

Let \( S_{n} \) be the set of permutations of \([n] := \{1, 2, ..., n\} \) and let \( S = \bigcup_{n \geq 0} S_{n} \), where \( S_{0} \) contains only one element \( \epsilon \) - the empty permutation. For permutation \( \sigma, \pi \in S \) we say that the permutation \( \sigma \) contains \( \pi \) if there is a subsequence of \( \sigma \) having the same relative order as \( \pi \). In particular, every permutation contains \( \epsilon \), and every permutation except \( \epsilon \) contains \( 1 \in S_{1} \). For consistency, we will use the letter \( \sigma \) as a permutation and \( \pi \) as a pattern. We say that \( \sigma \) avoids \( \pi \) (or \( \sigma \) is \( \pi \)-avoiding) if \( \sigma \) does not contain \( \pi \). For example, the permutation 46127538 contains 3142 since it contains the subsequence while the permutation 46123578 avoids 3142. We denote by \( S_{n}(\pi) \), where \( \pi \in S \), the set of permutations \( \sigma \in S_{n} \) avoiding \( \pi \). More generally we denote by \( S_{n}(\Pi) \), where \( \Pi \subseteq S \), the set of permutations avoiding each pattern \( \pi \in \Pi \) simultaneously, i.e. \( S_{n}(\Pi) = \bigcap_{\pi \in \Pi} S_{n}(\pi) \). Two sets of patterns \( \Pi \) and \( \Pi' \) are called Wilf equivalent, written \( \Pi \equiv \Pi' \), if \( |S_{n}(\Pi)| = |S_{n}(\Pi')| \) for all integers \( n \geq 0 \).

Now we define the \( q \)-analogue of pattern avoidance using permutation statistics. A permutation statistic (or sometimes just statistic) is a function \( st : S \to \mathbb{N} \), where \( \mathbb{N} \) is the set of nonnegative integers. Given a permutation statistic \( st \), we define the st-polynomial of \( \Pi \)-avoiding permutations to be

\[
F_{n}^{st}(\Pi) = F_{n}^{st}(\Pi; q) := \sum_{\sigma \in S_{n}(\Pi)} q^{st(\sigma)}.
\]

We may drop the \( q \) if it is clear from the context. The set of patterns \( \Pi \) and \( \Pi' \) are said to be st-Wilf equivalent if \( F_{n}^{st}(\Pi; q) = F_{n}^{st}(\Pi'; q) \) for all \( n \geq 0 \).

The study of \( q \)-analogue of pattern avoidance using permutation statistics and the st-Wilf equivalences began 2002, as initiated by Robertson, Saracino, and Zeilberger \[6\], with the emphasis on the number of fixed points. Elizalde subsequently refined results of Robertson et al. by considering the excedance statistic \[2\] and later extended the study to cases of multiple patterns \[3\]. A bijective proof was later given by Elizalde and Pak \[4\]. Dokos et al. \[1\] studied pattern avoidance on the the inversion and major statistics, as remarked by Savage and Sagan in their study of Mahonian pairs \[7\].
In this paper, we study multiple pattern avoidance on a class of permutation statistics which includes the inversion and descent statistics. The inversion number of $\sigma \in S_n$ is
\[
\text{inv}(\sigma) = \#\{(i,j) \in [n]^2 : i < j \text{ and } \sigma(i) > \sigma(j)\}.
\]
The descent number of $\sigma \in S_n$ is
\[
\text{des}(\sigma) = \#\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}.
\]
For example $\text{inv}(3142) = \#\{(1,2), (1,3), (3,4)\} = 3$ and $\text{des}(3142) = \#\{1,3\} = 2$.

In [1], Dokos et al. conjectured that there are only essentially trivial inv-Wilf equivalences, obtained by rotations and reflections of permutation matrices. Let us describe these more precisely. The notations used below are mostly taken from [1].

For $\sigma \in S_n$, we represent it geometrically using the squares $(1,\sigma(1)), (2,\sigma(2)), \ldots, (n,\sigma(n))$ of the $n$-by-$n$ grid, which is coordinated according to the $xy$-plane. This will be referred as the permutation matrix of $\sigma$. The diagram to the left in the Figure 1 is the permutation matrix of 46127538. In the diagram to the right, the red squares correspond to its subsequence 4173, which is an occurrence of the pattern 3142.

By representing each $\sigma \in S$ using a permutation matrix, we have an action of the dihedral group of square $D_4$ on $S$ by the corresponding action on the permutation matrices. We denote the elements of $D_4$ by
\[
D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_0, r_1, r_{-1}, r_{\infty}\},
\]
where $R_\theta$ is the counter-clockwise rotation by $\theta$ degrees and $r_m$ is the reflection in a line of slope $m$. We will sometimes write $\Pi^f$ for $r_{-1}(\Pi)$. Note that $R_0, R_{180}, r_{-1},$ and $r_1$ preserve the inversion statistic while the others reverse it, i.e.
\[
\text{inv}(f(\sigma)) = \begin{cases} 
\text{inv}(\sigma) & \text{if } f \in \{R_0, R_{180}, r_{-1}, r_1\}, \\
\binom{n}{2} - \text{inv}(\sigma) & \text{if } f \in \{R_{90}, R_{270}, r_0, r_{\infty}\}.
\end{cases}
\]
It follows that $\Pi$ and $f(\Pi)$ are inv-Wilf equivalent for all $\Pi \in S$ and $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. We call these equivalences trivial. With these notations, the conjecture by Dokos et al. can be stated as the following.
Conjecture 1.1. ([1], conj. 2.4) \(\Pi\) and \(\Pi'\) are inv-Wilf equivalent iff \(\Pi = f(\Pi')\) for some \(f \in \{R_0, R_{180}, r_{-1}, r_1\}\).

Given permutations \(\pi = a_1a_2...a_k \in S_k\) and \(\sigma_1, ..., \sigma_k \in \mathcal{S}\), the inflation \(\pi[\sigma_1, ..., \sigma_k]\) of \(\pi\) by the \(\sigma_i\) is the permutation whose permutation matrix is obtained by putting the permutation matrices of \(\sigma_i\) in the relative order of \(\pi\); for instance, \(213[123,1,21]=234165\) as illustrated in Figure 2.

For convenience, we write \(\pi_\ast := 21[\pi, 1]\).
In other words, \(\pi_\ast\) is the permutation whose permutation matrix is obtained by adding a box to the lower right corner of the permutation matrix of \(\pi\).

The next proposition is one of the main results of this paper, which disproves the conjecture above. This is a special case of the corollary of the theorem 2.4 in the next section.

Proposition 1.2. Let \(\pi_1, ..., \pi_r, \pi'_1, ..., \pi'_r\) be permutations such that \(\{312, \pi_i\} \equiv \{312, \pi'_i\}\) for all \(i\). Set \(\pi = \iota_r[\pi_1, ..., \pi_r]\) and \(\pi' = \iota_r[\pi'_1, ..., \pi'_r]\). Then \(\{312, \pi\}\) and \(\{312, \pi'\}\) are also inv-Wilf equivalent, i.e. \(F_n^{\text{inv}}(312, \pi) = F_n^{\text{inv}}(312, \pi')\) for all \(n\).

In particular, if we set each \(\pi'_i\) to be either \(\pi_i\) or \(\pi_i^{\text{inv}}\), then the conditions \(\{312, \pi_i\} \equiv \{312, \pi'_i\}\) are satisfied. By this construction \(\Pi'\) is generally not of the form \(f(\Pi)\) for any \(f \in \{R_0, R_{180}, r_{-1}, r_1\}\). For example, the pair \(\Pi = \{312, 32415\}\) and \(\Pi' = \{312, 24315\}\) is a smallest example of a nontrivial inv-Wilf equivalent constructed this way.

## 2. Avoiding two patterns

In this section, we give a recursive formula for the polynomial \(F_n^{\text{st}}(\Pi)\) when \(\Pi\) consists of 312 and another permutation \(\pi\). Then we will present its corollary, which gives a construction of nontrivial st-Wilf equivalences. The idea in the proof of the main theorem is similar to those in [5].

Suppose \(\sigma \in S_{n+1}(312)\) with \(\sigma(k + 1) = 1\). Then, for every pair of indexes \((i, j)\) with \(i < k + 1 < j\), we must have \(\sigma(i) < \sigma(j)\); otherwise \(\sigma(i)\sigma(k + 1)\sigma(j)\) forms a pattern 312 in \(\sigma\). So \(\sigma\) can be written as \(\sigma = 213[\sigma_1, 1, \sigma_2]\) with \(\sigma_1 \in S_k\) and \(\sigma_2 \in S_{n-k}\). In the rest of the paper, we will always consider \(\sigma\) in its inflation form.

For the rest of the paper, we assume that the permutation statistic \(\text{st} : S_n \to \mathbb{N}\) satisfies

\[
\text{st}(\sigma) = f(k, n-k) + \text{st}(\sigma_1) + \text{st}(\sigma_2)
\]
for some function $f : \mathbb{N}^2 \to \mathbb{N}$ that is independent of the statistic $st$. Some examples of such statistics are the inversion number, the descent number, and the number of occurrences of the consecutive pattern $213$:

\[ 213(\sigma) = \#\{ i \in [n-2] : \sigma(i+1) < \sigma(i) < \sigma(i+2) \}. \]

For these mentioned statistics, we have

\[
\begin{align*}
\text{inv}(\sigma) &= k + \text{inv}(\sigma_1) + \text{inv}(\sigma_2), \\
\text{des}(\sigma) &= 1 - \delta_{0,k} + \text{des}(\sigma_1) + \text{des}(\sigma_2), \\
213(\sigma) &= (1 - \delta_{0,k})(1 - \delta_{k,n}) + 213(\sigma_1) + 213(\sigma_2).
\end{align*}
\]

For a pattern $\pi$, it will be more beneficial to consider $\pi$ in its block decomposition as stated in the following proposition.

**Proposition 2.1.** Every $312$-avoiding permutation $\pi \in \mathfrak{S}_n(312)$ can be written uniquely as

\[ \pi = \iota_r[\pi_{1*}, ..., \pi_{r*}] \]

where $r \geq 0$ and $\pi_i \in \mathfrak{S}(312)$. Here $\iota_r$ denotes the identity element $12...r$ of $\mathfrak{S}_r$.

**Proof.** The uniqueness part is trivial. The proof of existence of $\pi_1, ..., \pi_r$ is by induction on $n$. If $n = 0$, there is nothing to proof. Suppose the result holds for $n$. Suppose that $\pi(k+1) = 1$. Then $\pi = 213[\pi_1, 1, \pi'] = 12[\pi_{1*}, \pi']$ where $\pi_1 \in \mathfrak{S}_k(312)$ and $\pi' \in \mathfrak{S}_{n-k}(312)$. Applying the inductive hypothesis on $\pi'$, we are done. \qed

Suppose that $\pi \in \mathfrak{S}_n(312)$ has the block decomposition $\pi = \iota_r[\pi_{1*}, ..., \pi_{r*}]$. For $1 \leq i \leq r$, we define $\pi(i)$ and $\overline{\pi}(i)$ as

\[
\pi(i) = \begin{cases} 
\pi_1 & \text{if } i = 1, \\
\iota_r[\pi_{1*}, ..., \pi_{i*}] & \text{otherwise,}
\end{cases}
\]

and

\[
\overline{\pi}(i) = \iota_{r-i+1}[\pi_{i*}, ..., \pi_{r*}].
\]

Let $\Pi = \{312, \pi\}$. If $\pi$ contains the pattern $312$, then every permutation avoiding $312$ will automatically avoid $\pi$, which means $F_n^{\text{inv}}(\Pi) = F_n^{\text{inv}}(312)$. So for the rest of this paper we will assume that every pattern besides $312$ in a set of patterns $\Pi$ avoids $312$. We will need the following lemma which gives a recursive condition for a permutation $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ to avoid $\pi$, in terms of $\sigma_1, \sigma_2$, and the blocks $\pi_{i*}$ of $\pi$.

**Lemma 2.2.** Let $\sigma = 213[\sigma_1, 1, \sigma_2], \pi = \iota_r[\pi_{1*}, ..., \pi_{r*}] \in \mathfrak{S}(312)$. Then $\sigma$ avoids $\pi$ if and only if the condition

\[(C_i) : \sigma_1 \text{ avoids } \overline{\pi}(i) \text{ and } \sigma_2 \text{ avoids } \pi(i).\]

hold for some $i \in [r]$

**Proof.** First, suppose that $\sigma$ contains $\pi$. Let $j$ be the largest number for which $\sigma_1$ contains $\pi(j)$. Then $\sigma_2$ must contain $\pi(j+1)$. So $\sigma_1$ contains $\pi(i)$ for all $i \leq j$, and $\sigma_2$ contains $\pi(i)$ for all $i > j$. Thus none of the $C_i$ holds.

On the other hand, suppose that there is a permutation $\sigma \in \mathfrak{S}(312)$ that avoids $\pi$ but does not satisfy any $C_i$. This means, for every $i$, either $\sigma_1$ contains $\overline{\pi}(i)$ or $\sigma_2$ contains $\pi(i)$. Let
Let \( r \) be the chain of \( r \) elements \( 0 < 1 < \ldots < r - 1 \). Let \( L_r \) be the poset obtained by taking the elements of \( r \times r \) of rank 0 to \( r - 1 \), i.e. the elements of \( L_r \) are the lattice points \((a, b)\) where \( a, b \geq 0 \) and \( a + b < r \). For instance, \( L_5 \) is the poset shown in Figure 3. We denote the minimal element \((0, 0)\) in \( L_r \) by \( \hat{0} \). Let \( \hat{L}_r \) be the poset \( L_r \) with the unique maximum element \( \hat{1} \) adjoined. Note that for every element \( a \in \hat{L}_r \) the up-set \( U(a) := \{ x \in \hat{L}_r : x \geq a \} \) of \( a \) is isomorphic to \( \hat{L}_{r-l(a)} \) where \( l(a) \) is the rank of \( a \) in \( L_r \). So to understand the Möbius function \( \mu_{\hat{L}_r} \) on these \( \hat{L}_r \), it suffices to know the value of \( \mu_{\hat{L}_r}(\hat{0}, \hat{1}) \) for every \( r \), which is given by the following lemma. The proof is omitted since it is by a straightforward calculation.

**Lemma 2.3.** We have

\[
\mu_{\hat{L}_r}(\hat{0}, \hat{1}) = \begin{cases} (-1)^r, & \text{if } r = 1, 2, \\ 0, & \text{otherwise.} \end{cases}
\]

We now state the main theorem of this section.

**Theorem 2.4.** Let \( \Pi = \{312, \pi\} \). Suppose that the statistic \( st : \mathcal{S} \rightarrow \mathbb{N} \) satisfies the condition \((\dagger)\). Then \( F_{n+1}^{st}(\Pi; q) \) satisfies

\[
F_{n+1}^{st}(\Pi; q) = \sum_{k=0}^{n} q^{f(k, n-k)} \left[ \sum_{i=1}^{r} F_k^{st}(312, \pi(i)) \cdot F_{n-k}^{st}(312, \pi(i)) - \sum_{i=1}^{r-1} F_k^{st}(312, \pi(i)) \cdot F_{n-k}^{st}(312, \pi(i+1)) \right],
\]

(*) for all \( n \geq 0 \), where \( F_0^{st}(\Pi; q) = 0 \) if \( \pi = \epsilon \), and 1 otherwise.
Proof. For \( k \in \{0, 1, \ldots, n\} \), we write \( \mathcal{S}_{n+1}^k(\Sigma) \) where \( \Sigma \subset \mathcal{S} \) to denote the set of permutations \( \sigma \in \mathcal{S}_{n+1}(\Sigma) \) such that \( \sigma(k+1) = 1 \). In particular, \[
\mathcal{S}_{n+1}^k(312) = \{ \sigma = 213[\sigma_1, 1, \sigma_2] : \sigma_1 \in \mathcal{S}_{k}^1(312) \text{ and } \sigma_2 \in \mathcal{S}_{n-k}(312) \}.
\]

Fix \( k \), and let \( A_i(i \in [r]) \) be the set of permutations in \( \mathcal{S}_{n+1}^k(312) \) satisfying the condition \( C_i \). So \( \mathcal{S}_{n+1}(\Pi) = A_1 \cup A_2 \cup \cdots \cup A_n =: A \). Observe that if \( i_1 < \ldots < i_k \) then \( A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = A_{i_1} \cap A_{i_k} =: A_{i_1,i_k} \).

This is since satisfying the conditions \( C_{i_1}, \ldots, C_{i_k} \) is equivalent to satisfying the conditions \( C_{i_1} \) and \( C_{i_k} \).

Let \( P \) be the intersection poset of \( A_1, \ldots, A_n \), which consists of the unique maximal element \( A \), the \( A_i \), and \( A_{i,j} \) for \( 1 \leq i < j \leq r \). So \( P \) is isomorphic to the set \( \hat{L}_r \). Thus the Möbius function \( \mu_P(T, A) \) for \( T \in P \) is given by \[
\mu_P(T, A) = \begin{cases} 1 & \text{if } T = A \text{ or } A_{i,i+1} \text{ for some } i, \\ -1 & \text{if } T = A_i \text{ for some } i, \\ 0 & \text{otherwise}. \end{cases}
\]

For \( T \in P \), we define \( g : P \to \mathbb{C}(x : x \in A) \) by \[
g(T) = \sum_{x \in T} x.
\]

So the Möbius inversion formula ([8], section 3.7) implies that \[
g(A) = -\sum_{T < A} \mu_P(T, A)g(T)
= \sum_{i=1}^{r} g(A_i) - \sum_{i=1}^{r-1} g(A_i \cap A_{i+1}).
\]

By mapping \( \sigma \mapsto q^{st(\sigma)} \) for all \( \sigma \in A \), \( g(A) \) is sent to \( F_{n+1,1}^{st}(\Pi; q) := \sum_{\sigma \in \mathcal{S}_{n+1}(\Pi)} q^{st(\sigma)} \).

Hence, \[
F_{n+1,1}^{st}(\Pi; q) = \sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{st(\sigma)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{st(\sigma)}
= q^{f(k,n-k)} \left[ \sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{st(\sigma)+st(\sigma_2)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{st(\sigma_1)+st(\sigma_2)} \right],
\]

where the second equality is obtained from the condition (†).

Note that \( \sigma \in A_i \) iff \( \sigma_1 \) avoids \( \overline{\pi}(i) \) and \( \sigma_2 \) avoids \( \overline{\pi}(i) \), and \( \sigma \in A_i \cap A_{i+1} \) iff \( \sigma_1 \) avoids \( \overline{\pi}(i) \) and \( \sigma_2 \) avoids \( \overline{\pi}(i+1) \). Thus \[
\sum_{\sigma \in A_i} q^{st(\sigma_1)+st(\sigma_2)} = F_k^{st}(312, \overline{\pi}(i)) \cdot F_{n-k}^{st}(312, \overline{\pi}(i))
\]
and
\[
\sum_{\sigma \in A_n \cap A_{n+1}} q^{st(\sigma_1) + st(\sigma_2)} = F_{k}^{st}(312, \pi(i)) \cdot F_{n-k}^{st}(312, \pi(i + 1)).
\]

Therefore
\[
F_{n+1,k}^{st}(\Pi; q) = q^{f(k,n-k)} \left[ \sum_{i=1}^{r} F_{k}^{st}(312, \pi(i)) \cdot F_{n-k}^{st}(312, \pi(i)) \right.
\]
\[
- \sum_{i=1}^{r-1} F_{k}^{st}(312, \pi(i)) \cdot F_{n-k}^{st}(312, \pi(i + 1)) \right].
\]

Summing the equation above from \(k = 0\) to \(n\), we get the stated result. \(\square\)

**Example 2.5.** (A \(q\)-analogue of odd Fibonacci numbers) It is well-known that the permutations avoiding 312 and 1432 are counted by the Fibonacci numbers \(F_{2n+1}\) assuming \(F_1 = F_2 = 1\) (see [9] for example). Let \(A_n = F_{2n-1}\). It can be shown that the \(A_n\) satisfy
\[
A_{n+1} = A_n + \sum_{k=0}^{n-1} 2^{n-k-1} A_k.
\]

Theorem 2.4 gives \(q\)-analogues of this relation. Here, we will consider the inversion statistic \(inv\).

Let \(\pi = 1432 = 12[\epsilon_3, 21_2]\) and \(\Pi = \{312, \pi\}\). Since \(\pi(1) = \epsilon\) and \(F_{n}^{inv}(312, \epsilon) = 0\) for all \(n\), theorem 2.4 implies
\[
F_{n+1}^{inv}(\Pi) = \sum_{k=0}^{n} F_{k}^{inv}(\Pi) F_{n-k}^{inv}(312, 321)
\]
\[
= q^n F_{n}^{inv}(\Pi) + \sum_{k=0}^{n-1} q^k(1 + q)^{n-k-1} F_{k}^{inv}(\Pi),
\]
where the last equality is by [1], Proposition 4.2.

**Corollary 2.6.** Let \(\pi_1, ..., \pi_r, \pi'_1, ..., \pi'_r\) be permutations such that \(\{312, \pi_i\} \equiv \{312, \pi_i'\}\) for all \(i\). Set \(\pi = t_r[\pi_1, ..., \pi_r]\) and \(\pi' = t_r[\pi'_1, ..., \pi'_r]\). Then \(\{312, \pi\}\) and \(\{312, \pi'\}\) are also st-Wilf equivalent, i.e. \(F_{n}^{st}(312, \pi) = F_{n}^{st}(312, \pi')\) for all \(n\).

**Proof.** The proof is by induction over \(n\). If \(n = 0\), then the statement trivially holds. Now suppose the statement holds up to \(n\). Then for \(0 \leq k \leq n\) and \(1 \leq i \leq r\), we have
\[
F_{k}^{st}(312, \pi(i)) = F_{k}^{st}(312, \pi'(i)) \quad \text{and} \quad F_{n-k}^{st}(312, \pi(i)) = F_{n-k}^{st}(312, \pi'(i)).
\]
Hence \(F_{n+1}^{st}(312, \pi) = F_{n+1}^{st}(312, \pi')\) by comparing the terms on the right hand side of (*). \(\square\)

As mentioned at the end of section one, for the inversion statistic we can choose take each \(\pi'_i\) to be either \(\pi_i\) or \(\pi'_i\). Indeed, this construction works for every statistic \(st\) satisfying (†) and that \(st(\sigma) = st(\sigma')\) for all \(\sigma \in S(312)\). Besides the inversion statistic, the descent statistic also possesses this property. To justify this fact, we write \(\sigma = 213[\sigma_1, 1, \sigma_2] \in S(312)\) where \(\sigma_1, \sigma_2 \in S\). Observe that \(\sigma' = 132[\sigma_2, \sigma'_1, 1]\) and
\[
\text{des}(\sigma') = \text{des}(\sigma'_2) + \text{des}(\sigma'_1) + (1 - \delta_{0,k})
\]
where \( k = |\sigma_1'| = |\sigma_1| \). The proof then proceeds by induction on \( n = |\sigma| \). Note that, however, it is not true in general that the matrix transposition preserves the descent number.

## 3. Generalization

In this section, we will generalize the theorem \([2.4]\) to the case when \( \Pi \) consists of 312 and other patterns. Then we give a generalized version of the corollary \([2.6]\).

**Lemma 3.1.** Let \( L \) be the poset \( L_{r_1} \times \cdots \times L_{r_m} \) and \( \hat{L} \) the poset \( L \) adjoined by the maximal element \( \hat{1} \). Let \( \mu = \mu_{\hat{L}} \) be the Möbius function on \( \hat{L} \). Then \( \mu(0, \hat{1}) = 0 \) unless each \( r_i \in \{1, 2\} \), in which case \( \mu(0, \hat{1}) = (-1)^{|S|+1} \), where \( |S| = \{i : r_i = 2\} \).

**Proof.** Let \( a = (a_1, \ldots, a_m) \in L \). Then \( \mu(0, a) = \prod_{i=1}^{m} \mu_i(0, a_i) \), where \( \mu_i \) is the Möbius function on \( L_{r_i} \). So

\[
\mu(0, \hat{1}) = -\sum_{a \in L} \mu(0, a) = -\prod_{i=1}^{m} \left( \sum_{a_i \in L_{r_i}} \mu_i(0, a_i) \right).
\]

Note that for \( r \geq 3 \) \( \mu_{L_{r}}(0, a) = 0 \) unless \( a \in \{(0, 0), (1, 0), (0, 1), (1, 1)\} \), in which cases \( \mu_{L_{r}}(0, a) = 1, -1, -1, 1 \), respectively. So \( \sum_{a \in L_{r}} \mu_{L_{r}}(0, a_i) = 1 \) if \( r = 1 \) and \(-1 \) if \( r = 2 \). So if \( r_i \geq 3 \) for some \( i \), then \( \mu(0, \hat{1}) = 0 \). If each \( r_i \in \{1, 2\} \), then each index \( i \) for which \( r_i = 2 \) contributes a \(-1\) to the product on the right hand side of the previous equation. Thus \( \mu(0, \hat{1}) = (-1)^{|S|+1} \).

The following theorem is a generalization of \([2.4]\). For convenience, we introduce the following notations. Let \( \Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\} \) where \( \pi^{(i)} = \iota_{r_j}([\pi^{(1)}_{i}]_{s}, \ldots, [\pi^{(j)}_{i}]_{s}) \). For \( I = (i_1, \ldots, i_m) \), we define

\[
\Pi_I = \{312, \pi^{(1)}(i_1), \ldots, \pi^{(m)}(i_m)\}
\]

and

\[
\Pi_I = \{312, \pi^{(1)}(i_1), \ldots, \pi^{(m)}(i_m)\}.
\]

**Theorem 3.2.** Let \( \Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\} \) where \( \pi^{(i)} = \iota_{r_j}([\pi^{(1)}_{i}]_{s}, \ldots, [\pi^{(j)}_{i}]_{s}) \). Then \( F_{0}^{st}(\Pi) = 0 \) if some \( \pi_i = \epsilon \) and \( 1 \) otherwise, and for \( n \geq 1 \)

\[
F_{n+1}^{st}(\Pi; g) = \sum_{k=0}^{n} q^{f(k,n-k)} \left[ \sum_{S \subseteq [m]} (-1)^{|S|} \sum_{I=(i_1, \ldots, i_m): 1 \leq i_j \leq r_j - \delta_j} F_{k}^{st}(\Pi_I) \cdot F_{n-k}^{st}(\Pi_{I+\delta}) \right].
\]

Here \( \delta_j = 1 \) if \( j \in S \) and \( 0 \) if \( j \notin S \).

**Proof.** Recall that by \([2.2]\) \( \sigma = 213[\sigma_1, 1, \sigma_2] \in \mathcal{S}(312) \) avoids \( \pi^{(j)} \) iff \( \sigma \) satisfies the condition \((C_i^j)\): \( \sigma_1 \) avoids \( \pi^{(j)}(i) \) and \( \sigma_2 \) avoids \( \pi^{(j)}(i) \) for some \( i \in [r_j] \). So \( \sigma \in \mathcal{S}(312) \) belongs to \( \mathcal{S}(\Pi) \) if for all \( j \), there is an \( i \in [r_j] \) for which \( \sigma \) satisfies \((C_i^j)\). Fix \( k \) and let \( \mathcal{S}_{n+1}^k(312) \) be as in the proof of \([2.4]\). Let \( A_{i}^j \) be the set of permutations in \( \mathcal{S}_{n+1}^k(312) \) avoiding \( \pi^{(j)} \) and satisfying the condition \((C_i^j)\). For \( I = (i_1, \ldots, i_m) \in [r_1] \times [r_2] \times \cdots \times [r_m] \), we define

\[
A_I = A_{i_1,i_2,\ldots,i_m} := A_{i_1}^1 \cap A_{i_2}^2 \cap A_{i_m}^m.
\]
So $\mathcal{S}_{n+1}^k(\Pi)$ is the union

$$\mathcal{S}_{n+1}^k(\Pi) = \bigcup_{i_1, \ldots, i_m} A_{i_1, i_2, \ldots, i_m},$$

where the union is taken over all $m$-tuples $I = (i_1, \ldots, i_m)$ in $[r_1] \times [r_2] \times \cdots \times [r_m]$. Let $\hat{P}_j$ be the intersection poset of the $A_j^1, \ldots, A_j^r$, and let $P_j$ be the poset $\hat{P}_j \setminus \{\hat{1}\}$, where $\hat{1} = A_j^1 \cup \cdots \cup A_j^r$ is the unique maximum element of $\hat{P}_j$. Recall that $P_j$ is isomorphic to $L_{r_j}$. Let $P$ be the intersection poset of the $A_j$. The elements of $P$ are the unique maximal element $A = \mathcal{S}_{n+1}^k(\Pi)$ and

$$T = T^1 \cap T^2 \cap \cdots \cap T^m,$$

where each $T^j$ is an element of $P_j$. Thus $P$ is isomorphic to $L_{r_1} \times \cdots \times L_{r_m}$ with the unique maximum element $\hat{1}$ adjoined. For $S \subseteq [n]$, we say that $T \in P$ has type $S$ if $T^j = A_j^i$ for some $i$ when $j \notin S$ and $T^j = A_j^i \cap A_j^{i+1}$ for some $i$ when $j \in S$. Using the previous lemma, we see that the Möbius function on $P$ for $T = T^1 \cap T^2 \cap \cdots \cap T^m \neq \hat{1}$ is

$$\mu(T, A) = \begin{cases} (-1)^{|S|+1}, & \text{if } T \text{ has type } S, \\ 0, & \text{otherwise.} \end{cases}$$

For $T \in P$, we define $g : P \to \mathbb{C}(x : x \in A)$ by $g(T) = \sum_{x \in T} x$, so that

$$g(A) = -\sum_{T < A} \mu(T, A)g(T)$$

$$= \sum_{S \subseteq [n]} (-1)^{|S|} \sum_{T \text{ has type } S} g(T)$$

by the Möbius inversion formula. Now, by definition of type $S$, we have

$$\sum_{T \text{ has type } S} g(T) = \sum_{i_1, \ldots, i_m : 1 \leq i_j \leq r_j - \delta_j} g\left(\bigcap_{j \notin S} A_j^{i_j} \cap \bigcap_{j \in S} (A_j^{i_j} \cap A_j^{i_j+1})\right),$$

where $\delta_j = 1$ if $j \in S$ and $0$ if $j \notin S$. Recall that $\sigma \in A_j^{i_j}$ iff $\sigma_1$ avoids $\pi(j)(i_j)$ and $\sigma_2$ avoids $\pi(j)(i_j)$, and $\sigma \in A_j^{i_j} \cap A_j^{i_j+1}$ iff $\sigma_1$ avoids $\pi(j)(i_j)$ and $\sigma_2$ avoids $\pi(j)(i_j+1)$. Therefore, by mapping $\sigma \mapsto q^{\text{st}(\sigma)}$, we have

$$g\left(\bigcap_{j \notin S} A_j^{i_j} \cap \bigcap_{j \in S} (A_j^{i_j} \cap A_j^{i_j+1})\right) \mapsto q^{f(k, n-k)} F_k^{\text{st}}(312, \pi(1)(i_1), \ldots, \pi(m)(i_m))$$

$$\cdot F_{n-k}^{\text{st}}(312, \pi(1)(i_1 + \delta_1), \ldots, \pi(m)(i_m + \delta_m)).$$

Thus

$$F_{n+1, k}^{\text{st}}(\Pi; q) = q^{f(k, n-k)} \left[ \sum_{S \subseteq [m]} (-1)^{|S|} \sum_{i_1, \ldots, i_m : 1 \leq i_j \leq r_j - \delta_j} F_k^{\text{st}}(\Pi_{I}) \cdot F_{n-k}^{\text{st}}(\Pi_{I + \delta}) \right].$$

The theorem then follows by summing $F_{n+1, k}^{\text{st}}(\Pi; q)$ over $k$ from 1 to $n$. \qed
Example 3.3. Let $\Pi = \{312, \pi^{(1)}, \pi^{(2)}\}$ where $\pi^{(1)} = 2314 = 12[1_2, \epsilon_4]$ and $\pi^{(2)} = 2143 = 12[1_4, 1_2]$. We want to compute $a_n = F_n^{\pi}(\Pi)$ by using the theorem 3.2. There are four possibilities of $S \subseteq \{1, 2\}$, and for each possibility the following table shows the appearing terms.

| $S = \emptyset$ | $F_k^{\pi}(312, 12, 1) \cdot F_{n-k}^{\pi}(\Pi)$ | $= \delta_{0,k} \cdot a_{n-k}$
|-----------------|-----------------------------------------------|-------------------------------
|                 | $F_k^{\pi}(312, 2314, 1) \cdot F_{n-k}^{\pi}(312, 1, 2143)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$
|                 | $F_k^{\pi}(312, 12, 2143) \cdot F_{n-k}^{\pi}(312, 2314, 21)$ | $= 1$
|                 | $F_k^{\pi}(\Pi) \cdot F_{n-k}^{\pi}(312, 1, 21)$ | $= \delta_{0,n-k} \cdot a_k$
| $S = \{1\}$    | $F_k^{\pi}(312, 12, 1) \cdot F_{n-k}^{\pi}(312, 1, 2143)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$
|                 | $F_k^{\pi}(312, 12, 2143) \cdot F_{n-k}^{\pi}(312, 1, 21)$ | $= \delta_{0,n-k}$
| $S = \{2\}$    | $F_k^{\pi}(312, 12, 1) \cdot F_{n-k}^{\pi}(312, 2314, 21)$ | $= \delta_{0,k}$
|                 | $F_k^{\pi}(312, 2314, 1) \cdot F_{n-k}^{\pi}(312, 1, 21)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$
| $S = \{1, 2\}$ | $F_k^{\pi}(312, 21, 1) \cdot F_{n-k}^{\pi}(312, 1, 21)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$

Here $\delta$ is the Kronecker delta function. Thus

$$a_{n+1} = \sum_{q=0}^{n} q^k \left[ \delta_{0,k} a_{n-k} + \delta_{0,n-k} a_k + 1 - \delta_{0,k} - \delta_{0,n-k} \right]$$

$$= (1 + q^n) a_n + \frac{1 - q^{n+1}}{1 - q} - (1 + q^n)$$

$$= (1 + q^n) a_n + q \left( \frac{1 - q^{n+1}}{1 - q} \right).$$

In particular, by setting $q = 1$ we get $a_{n+1} = 2a_n + n - 1$ with $a_0 = a_1 = 1$. Thus

$$|\mathfrak{S}_n(312, 2314, 2143)| = 2^n - n.$$

The following corollary of the theorem 3.2 is a generalization of the corollary 2.6. It can be proved using a similar argument to that of 2.6 so we will omit the proof.

Corollary 3.4. Let $\pi^{(j)}_i, r^{(j)}_i, 1 \leq j \leq m, 1 \leq i \leq r_m$, be permutations such that

$$\{312, \pi^{(1)}_i, \ldots, \pi^{(m)}_i \} \equiv \{312, \pi^{(1)}_{i_1}, \ldots, \pi^{(m)}_{i_m} \}$$

for all $m$-tuples $I = (i_1, \ldots, i_m) \in [r_1] \times \ldots \times [r_m]$. Set $\pi^{(j)}_I = \tau_I[\pi^{(j)}_1, \ldots, \pi^{(j)}_{r_m}]$ and $r^{(j)}_I = \tau_I[\pi^{(j)}_1, \ldots, \pi^{(j)}_{r_m}]$. Then $\Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ and $\Pi' = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ are st-Wilf equivalent.
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