Magnetic Catalysis of Dynamical Symmetry Breaking and Aharonov-Bohm Effect

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The phenomenon of the magnetic catalysis of dynamical symmetry breaking is based on the dimensional reduction $D \rightarrow D - 2$ in the dynamics of fermion pairing in a magnetic field. We discuss similarities between this phenomenon and the Aharonov-Bohm effect. This leads to the interpretation of the dynamics of the (1+1)-dimensional Gross-Neveu model with a non-integer number of fermion colors as a quantum field theoretical analogue of the Aharonov-Bohm dynamics.

I. INTRODUCTION.

It has been recently shown [1–4] that a constant magnetic field in 2+1 and 3+1 dimensions is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions (the magnetic catalysis of dynamical symmetry breaking).

The effect has been extensively studied in different models [5–8], confirming the universality of this phenomenon. There might be interesting applications of this effect in condensed matter physics [11,12] and cosmology [3,4].

As it has been recently pointed out [14], the effect should play the important role in the phenomenon of the spontaneous generation of a magnetic field in the vacuum of 2+1 dimensional QED with the Chern-Simons term [15].

The essence of the effect is the dimensional reduction $D \rightarrow D - 2$ (i.e. $2 + 1 \rightarrow 0 + 1$ and $3 + 1 \rightarrow 1 + 1$) in the infrared dynamics of the fermion pairing in a magnetic field [1–4]. The physical reason of this reduction is the fact that the motion of charged particles is restricted in those directions that are perpendicular to the magnetic field. This is in turn connected with the point that, at weak coupling between fermions, the fermion pairing, leading to the chiral condensate, is mostly provided by fermions from the lowest Landau level (LLL) whose dynamics are $(D - 2)$-dimensional.

The most explicit description of this dimensional reduction was done by Elias et al. [6]. In that paper, it was shown that in the “continuum” limit, when both the strength of the magnetic field and the ultraviolet cutoff go to infinity, the (weakly coupling) (3 + 1)-dimensional NJL models with $N_c$ colors are reduced to a continuum set of independent (1+1)–dimensional Gross-Neveu (GN) models [16], labeled by coordinates $x_\perp$ in the plane perpendicular to the magnetic field $B$. The number of colors in the GN models is $\tilde{N}_c = (b\pi/2)N_c$, where $b = |eB|/\Lambda^2$ in the “continuum” limit (here $\Lambda$ is the ultraviolet cutoff). The factor $b\pi/2$ is proportional to a (local) magnetic flux attached to each point in the $x_\perp$-plane (see Sec.3 below). Actually, $b\pi/2$ is equal to $|e\Phi|/2\pi$, where $\Phi$ is the local magnetic flux, and therefore

$$\tilde{N}_c = N_c(|e\Phi|/2\pi).$$

If we asserted that $\tilde{N}_c$ is a (non-negative) integer, we would be led to the quantization condition for $N_c(|e\Phi|/2\pi)$ coinciding with the quantization condition for a magnetic flux, $q(\Phi/2\pi) = n$, at which the Aharonov-Bohm scattering of particles with the electric charge $q = eN_c$ disappears [5].

We will argue that this intriguing similarity reflects a deep connection between the phenomenon of the dimensional reduction in a magnetic field and the Aharonov-Bohm effect. This in turn yields the interpretation of the dynamics in the GN model with a non-integer number of colors $\tilde{N}_c$ as a quantum field theoretical analogue of the Aharonov-Bohm dynamics with a non-integer $q(\Phi/2\pi)$.

1The fact that an external magnetic field enhances a fermion dynamical mass was known from studying the NJL model in 3+1 and 2+1 dimensions [9,10].

2As it has been shown in Refs. [7,8], at weak coupling constants of quantum dynamics (such as gauge and Yukawa interactions) the magnetic catalysis is irrelevant for the phase transitions in the early Universe. However, it may become relevant if some coupling constants are strong, as technicolor interactions during the electroweak phase transition (compare with Ref. [13]).
But first, following Ref. [3], we shall discuss the connection between the (1+1)-dimensional GN model and the 
(3+1)-dimensional NJL model in a magnetic field.

II. EFFECTIVE ACTION IN THE GROSS–NEVEU MODEL.

In this section, for completeness, we shall derive the effective action for the GN model. The Lagrangian density of
the GN model is:
\[ \mathcal{L}_{GN} = \frac{1}{2} \left[ \tilde{\Psi}, (i\gamma^\mu \partial_\mu) \Psi \right] + \frac{\tilde{G}}{2} \left[ (\tilde{\Psi} \Psi)^2 + (\tilde{\Psi} i\gamma^5 \Psi)^2 \right] \]  
where \( \mu = 0, 1 \) and the fermion field carries an additional “color” index \( \tilde{a} = 1, 2, \ldots, \tilde{N}_c \) (for simplicity, we consider the case of the chiral \( U_L(1) \times U_R(1) \) symmetry). The theory is equivalent to the theory with the Lagrangian density
\[ \mathcal{L}'_{GN} = \frac{1}{2} \left[ \tilde{\Psi}, (i\gamma^\mu \partial_\mu) \Psi \right] - \tilde{G} (\sigma + i\gamma^5 \pi) \Psi - \frac{1}{2\tilde{G}} (\sigma^2 + \pi^2) . \] 
The Euler–Lagrange equations for the auxiliary fields \( \sigma \) and \( \pi \) take the form of constraints:
\[ \sigma = -\tilde{G} \tilde{\Psi} \Psi, \quad \pi = -\tilde{G} \tilde{\Psi} i\gamma^5 \Psi, \] 
and the Lagrangian density (3) reproduces Eq. (2) upon application of the constraints (4). The effective action for the
composite fields \( \sigma \) and \( \pi \) can be obtained by integrating over fermions in the path integral. It is given by the standard
relation:
\[ \Gamma_{GN}(\sigma, \pi) = \tilde{\Gamma}_{GN}(\sigma, \pi) = \frac{1}{2\tilde{G}} \int d^2 x (\sigma^2 + \pi^2) , \] 
\[ \tilde{\Gamma}_{GN}(\sigma, \pi) = -i Tr \ln [i \gamma^\mu \partial_\mu - (\sigma + i\gamma^5 \pi)] . \] 
The low energy quantum dynamics are described by the path integral (with the integrand \( \exp(i\Gamma_{GN}) \)) over the
composite fields \( \sigma \) and \( \pi \). As \( \tilde{N}_c \rightarrow \infty \), the path integral is dominated by the stationary points of the action:
\[ \delta \Gamma_{GN}/\delta \sigma = \delta \Gamma_{GN}/\delta \pi = 0 . \] 
We will analyze the dynamics by using the expansion of the action \( \Gamma_{GN} \) in powers of derivatives of the composite fields.

We begin the calculation of \( \Gamma_{GN} \) by calculating the effective potential \( V_{GN} \). Since \( V_{GN} \) depends only on the
\( U_L(1) \times U_R(1) \)-invariant \( \rho^2 = \sigma^2 + \pi^2 \), it is sufficient to consider a configuration with \( \pi = 0 \) and \( \sigma \) independent of \( x \). Then we find from Eqs. (3) and (4):
\[ V_{GN}(\rho) = \frac{\rho^2}{2\tilde{G}} - \tilde{N}_c \int \frac{d^2 k}{(2\pi)^2} \ln \left( \frac{k^2 + \rho^2}{k^2} \right) = \frac{\rho^2}{2\tilde{G}} - \frac{\tilde{N}_c \rho^2}{4\pi} \left[ \ln \frac{\Lambda^2}{\rho^2} + 1 \right] , \] 
where the integration is done in Euclidean region (\( \Lambda \) is an ultraviolet cutoff). As is known, in the GN model the
equation of motion \( dV_{GN}/d\rho = 0 \) has a nontrivial solution \( \rho = \bar{\sigma} \equiv m_{dyn} \) for any value of the coupling constant \( \tilde{G} \). Then the potential \( V_{GN} \) can be rewritten as
\[ V_{GN}(\rho) = \frac{\tilde{N}_c \rho^2}{4\pi} \left[ \ln \frac{\rho^2}{m_{dyn}^2} - 1 \right] , \] 
where
\[ m_{dyn}^2 = \Lambda^2 \exp \left( -\frac{2\pi}{\tilde{N}_c \tilde{G}} \right) . \] 
Due to the Mermin-Wagner-Coleman (MWC) theorem [13], there cannot be spontaneous breakdown of continuous
symmetries at \( D = 1 + 1 \). The parameter \( m_{dyn} \) is an order parameter of chiral symmetry breaking only in leading
order in \( 1/\tilde{N}_c \) (this reflects the point that the MWC theorem is not applicable to systems with \( \tilde{N}_c \rightarrow \infty \))}. In
the exact GN solution, spontaneous chiral symmetry breaking is washed out by interactions (strong fluctuations) of
would-be NG bosons π (i.e. after integration over π and σ in the path integral). The exact solution in this model corresponds to the realization of the Berezinsky–Kosterlitz–Thouless (BKT) phase: though chiral symmetry is not broken in this phase, the parameter $m_{dyn}$ still defines the fermion mass, and the would-be NG boson π transforms into a BKT gapless excitation [19].

Let us now turn to calculating the kinetic term in $\Gamma_{GN}$. The chiral $U_L(1) \times U_R(1)$ symmetry implies that the general form of the kinetic term is

$$L^{(k)}_{\text{GN}} = \frac{f_{1\mu}^{\nu}}{2} (\partial_\mu \rho_\nu \partial_\nu \rho_\mu) + \frac{f_{2\mu}^{\nu}}{\rho^2} (\rho_\mu \partial_\nu \rho_\mu)(\rho_\nu \partial_\nu \rho_\nu)$$

where $\rho = (\sigma, \pi)$ and $f_{1\mu}^{\nu}$, $f_{2\mu}^{\nu}$ are functions of $\rho^2$. To find the functions $f_{1\mu}^{\nu}$ and $f_{2\mu}^{\nu}$, one can use different methods. We utilize the same method as in Ref. [4] (see Appendix A in that paper). The result is:

$$f_{1\mu}^{\nu}(\rho^2) = -\frac{i}{2} \int \frac{d^2k}{(2\pi)^2} \text{tr} \left[ S(k) i \gamma_5 \frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} \right],$$

$$f_{2\mu}^{\nu}(\rho^2) = -\frac{i}{4} \int \frac{d^2k}{(2\pi)^2} \text{tr} \left[ S(k) \frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} - S(k) i \gamma_5 \frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} i \gamma_5 \right],$$

with $S(k) = i(k^\mu \gamma_\mu + \rho)/(k^2 - \rho^2)$. The explicit form of these functions is:

$$f_{1\mu}^{\nu} = \frac{g^{\mu\nu}}{4\pi \rho^2}, \quad f_{2\mu}^{\nu} = -\frac{g^{\mu\nu}}{12\pi \rho^2}$$

III. THE INTERPLAY BETWEEN THE GN MODEL AND THE NJL MODEL IN A MAGNETIC FIELD

In this section, we compare the effective actions in the GN model and in the NJL model in a magnetic field, and we establish a rather interesting connection between these two models.

The analog of the Lagrangian density [3] in the NJL model in a magnetic field is

$$L' = \frac{1}{2} \left[ \bar{\Psi} \left( i \gamma^\mu D_\mu - A^\mu_{\text{ext}} \right) \right] - \bar{\Psi} \left( \sigma + i \gamma^5 \pi \right) \Psi - \frac{1}{2G} (\sigma^2 + \pi^2)$$

where $D_\mu = \partial_\mu - ieA^\mu_{\text{ext}}$, $A^\mu_{\text{ext}} = B x_1 \delta^3_\mu$ (the magnetic field is in $+x_1$ direction).

In leading order in $1/N_c$, the effective action in the NJL model in a magnetic field is derived in Refs. [2,4]. The effective potential and the kinetic term are ($\rho = (\sigma, \pi)$):

$$V(\rho) = \frac{\rho^2}{2G} + \frac{N_c}{8\pi^2} \left[ \frac{\Lambda^4}{2} + \frac{1}{3!} \ln(\Lambda l)^2 + \frac{1 - \gamma - \ln 2}{3!} + \frac{\rho^4}{2} \ln(\Lambda l)^2 \right]$$

$$+ \frac{\rho^4}{2} (1 - \gamma - \ln 2) + \frac{\rho^2}{2} \ln \left( \frac{\rho l}{2} \right)^2 - \frac{4}{l^2} \zeta(1, \frac{(\rho l)^2}{2} + 1) + O \left( \frac{1}{\Lambda} \right),$$

$$L^{(k)} = \frac{f_{1\mu}^{\nu}}{2} (\partial_\mu \rho_\nu \partial_\nu \rho_\mu) + \frac{f_{2\mu}^{\nu}}{\rho^2} (\rho_\mu \partial_\nu \rho_\mu)(\rho_\nu \partial_\nu \rho_\nu)$$

with $f_{1\mu}^{\nu}$ and $f_{2\mu}^{\nu}$ being diagonal tensors:

$$f_{100}^{00} = -f_{11}^{11} = \frac{N_c}{8\pi^2} \left[ \ln \left( \frac{\Lambda l}{2} \right)^2 - \psi \left( \frac{(\rho l)^2}{2} + 1 \right) + \frac{1}{(\rho l)^2} - \gamma + \frac{1}{3} \right],$$

$$f_{12}^{33} = \frac{N_c}{8\pi^2} \left[ \ln \left( \frac{\Lambda l}{\rho^2} \right) - \gamma + \frac{1}{3} \right],$$

$$f_{100}^{00} = -f_{11}^{11} = \frac{N_c}{24\pi^2} \left[ \psi \left( 2, \frac{(\rho l)^2}{2} + 1 \right) + \frac{1}{(\rho l)^2} \right],$$

$$f_{12}^{33} = \frac{N_c}{8\pi^2} \left[ (\rho l)^2 - \psi \left( \frac{(\rho l)^2}{2} + 1 \right) - 2(\rho l)^2 \ln \Gamma \left( \frac{(\rho l)^2}{2} \right) \right]$$

$$- (\rho l)^2 \ln \left( \frac{(\rho l)^2}{2} - (\rho l)^2 + 1 \right).$$
Here $G$ is the NJL coupling constant, $N_c$ is the number of colors, $\zeta(\nu, x)$ is the generalized Riemann zeta function, $\zeta'(\nu, x) = \partial \zeta(\nu, x)/\partial \nu$, $\gamma \approx 0.577$ is the Euler constant, $\psi(x) = d(\ln \Gamma(x))/dx$, and $l \equiv |eB|^{-1/2}$ is the magnetic length. The gap equation $dV/d\rho = 0$ is\[^{[1]}\]

$$\rho \Lambda^2 \left( \frac{1}{g} - 1 \right) = -\rho^3 \ln \left( \frac{\Lambda l^2}{2} \right) + \gamma \rho^3 + \frac{\rho}{l^2} \ln \left( \frac{\rho l^2}{4\pi} \right) + \frac{2\rho}{l^2} \ln \left( \frac{\rho l^2}{2} \right) + O\left( \frac{1}{\Lambda} \right), \quad (18)$$

where the dimensionless coupling constant $g = N_c \Lambda^2 / 4\pi^2$. In the derivation of this equation, we used the relations $[2]$

$$\frac{\partial}{\partial x} \zeta(\nu, x) = -\nu \zeta(\nu + 1, x), \quad (19)$$

$$\frac{\partial}{\partial \nu} \zeta(\nu, x) \bigg|_{\nu = 0} = \ln \Gamma(x) - \frac{1}{2} \ln 2\pi, \quad \zeta(0, x) = \frac{1}{2} - x. \quad (20)$$

As $B \to 0$ ($l \to \infty$), we recover the known gap equation in the NJL model (for a review see Ref. $[21]$):

$$\rho \Lambda^2 \left( \frac{1}{g} - 1 \right) = -\rho^3 \ln \frac{\Lambda^2}{\rho l^2}. \quad (21)$$

This equation admits a nontrivial solution only if $g$ is supercritical, $g > g_c = 1$ (as Eq. $(14)$ implies, a solution to the gap equation, $\rho = \bar{\sigma}$, coincides with the fermion dynamical mass, $\bar{\sigma} = m_{dyn}$). As was shown in Refs. $[2,4]$, at $B \neq 0$, a non–trivial solution exists for all $g > 0$.

Let us consider the case of small subcritical $g$, $g \ll g_c = 1$, in detail. A solution is seen to exist for this case if $\rho l$ is small. Specifically, for $g \ll 1$, the left–hand side of Eq. $(18)$ is positive. Since the first term of the right–hand side in this equation is negative, we conclude that a non–trivial solution to this equation may exist only for

$$\rho^2 \ln(\Lambda l^2) \ll \frac{1}{l^2} \ln \frac{1}{(\rho l)^2} \quad (22)$$

($\Gamma(\rho l^2/2) \approx 2/(\rho l)^2$). We then find the solution:

$$m_{dyn}^2 \equiv \bar{\sigma}^2 = \frac{|eB|}{\pi} \exp \left( -\frac{4\pi^2(1-g)}{|eB| N_c G} \right) = \frac{|eB|}{\pi} \exp \left( -\frac{(1-g)\Lambda^2}{g|eB|} \right). \quad (23)$$

Actually, since Eq. $(23)$ implies that condition $(22)$ is violated only if $(1-g) < |eB|/\Lambda^2$, the expression $(23)$ is valid for all $g$ outside that (scaling) region near the critical value $g_c = 1$. Note that in the scaling region ($g \to g_c - 0$) the expression for $m_{dyn}^2$ is different $[4]$:

$$m_{dyn}^2 \simeq |eB| \ln \left( \frac{\ln \Lambda^2 l^2/\pi}{\ln \Lambda^2 l^2} \right). \quad (24)$$

Let us compare relation $(23)$ with relation $(3)$ for the dynamical mass in the GN model. The similarity between them is evident: $|eB|$ and $|eB| G$ in Eq. $(23)$ play the role of an ultraviolet cutoff and the dimensionless coupling constant $\tilde{G}$ in Eq. $(3)$. Let us discuss this connection and show that it is intimately connected with the dimensional reduction $3 + 1 \to 1 + 1$ in the dynamics of the fermion pairing in a magnetic field.

Eq. $(3)$ implies that the GN model is asymptotically free, with the bare coupling constant $\tilde{G} = 2\pi/\tilde{N}_c \ln(\Lambda^2 / m_{dyn}^2) \to 0$ as $\Lambda \to \infty$. Let us now consider the following limit in the NJL model in a magnetic field: $|eB| \to \infty$, $|eB|/\Lambda^2 = b \ll 1$. Then relation $(23)$, which can be rewritten as

$$m_{dyn}^2 = \frac{b\Lambda^2}{\pi} \exp \left( -\frac{(1-g)}{bg} \right), \quad (25)$$

implies that the behavior of the bare coupling constant $g$ must be

\[^{3}\text{We consider the case of a large ultraviolet cutoff: } \Lambda^2 \gg \bar{\sigma}^2, |eB|, \text{ where } \bar{\sigma} \text{ is a minimum of the potential } V.\]
\[ g \simeq \frac{1}{b \ln(b \Lambda^2/\pi m_{\text{dyn}}^2)} \to 0, \]  

in order to get a finite value for \( m_{\text{dyn}}^2 \) in this limit. Thus in this “continuum” limit, we recover the same behavior for the coupling \( g \) in the NJL model as for the coupling constant \( \tilde{G} \) in the GN model.

Let us now compare the effective potentials in these two models. At first glance, the expressions (3) and (13) for the effective potentials in these models look very different: the character of ultraviolet divergences in 1+1 and 3+1 dimensional theories is essentially different. However, using Eqs.(19) and (20), the expression (13) can be rewritten, for small \( \rho_l \), as

\[
V(\rho) = V(0) + \frac{N_c |eB|}{8\pi^2} \rho^2 \left[ \ln \left( \frac{\rho^2}{m_{\text{dyn}}^2} \right) - 1 + O((\rho l)^2) \right].
\]  

Then, expressing the coupling constant \( g \) through \( m_{\text{dyn}} \) from Eq.(24), we find that

\[
V(\rho) = V(0) + \frac{N_c |eB|}{8\pi^2} \rho^2 \left[ \ln \left( \frac{\rho^2}{m_{\text{dyn}}^2} \right) - 1 + O((\rho l)^2) \right].
\]  

Here we used the fact that, because of Eq.(24), the ratio \((\rho l)^2\) is small near the minimum \( \rho = m_{\text{dyn}} \).

The expressions (8) and (28) for the potentials in these two models look now quite similar. There is however an important difference: the character of ultraviolet divergences in 1+1 and 3+1 dimensional theories is essentially different. However, using Eqs.(19) and (20), the expression (13) can be rewritten, for small \( \rho_l \), as

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V(\rho) = V(0) + \frac{N_c |eB|}{8\pi^2} \rho^2 \left[ \ln \left( \frac{\rho^2}{m_{\text{dyn}}^2} \right) - 1 + O((\rho l)^2) \right].
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\]  

Here we used the fact that, because of Eq.(24), the ratio \((\rho l)^2\) is small near the minimum \( \rho = m_{\text{dyn}} \).

Because of the expression (23) for \( m_{\text{dyn}} \) at small \( g \), the term \( 1/(\rho l)^2 \) dominates in the functions \( f_1^{00} = -f_1^{11} \) and \( f_2^{00} = -f_2^{11} \), around the minimum \( \rho = m_{\text{dyn}} \):

\[
f_1^{00} = -f_1^{11} \simeq \frac{N_c}{8\pi^2} \frac{1}{\rho^2}, \quad f_2^{00} = -f_2^{11} \simeq -\frac{N_c}{24\pi^2} \frac{1}{\rho^2}. \]  

Up to the additional factor \( |eB|/2\pi \), these functions coincide with those in (13) in the GN model. On the other hand, the functions \( f_1^{22} = f_1^{33} \) and \( f_2^{22} = f_2^{33} \), connected with derivatives with respect to the transverse coordinates, are strongly supressed, as compared to the functions (23), and the ratios of the functions \( f_1^{22} = f_1^{33} \) and \( f_2^{22} = f_2^{33} \) to those in (23) go rapidly (as \( m_{\text{dyn}}^2/\Lambda^2 \)) to zero as \( |eB| \to \infty \).

As a result, the coordinates \( x_2 \) and \( x_3 \) become redundant variables in this limit: there are no transitions of \( f_1^{00} = -f_1^{11} \) and \( f_2^{00} = -f_2^{11} \), around the minimum \( \rho = m_{\text{dyn}} \):

\[
f_1^{00} = -f_1^{11} \simeq \frac{N_c}{8\pi^2} \frac{1}{\rho^2}, \quad f_2^{00} = -f_2^{11} \simeq -\frac{N_c}{24\pi^2} \frac{1}{\rho^2}. \]

As a result, the coordinates \( x_2 \) and \( x_3 \) become redundant variables in this limit: there are no transitions of \( f_1^{00} = -f_1^{11} \) and \( f_2^{00} = -f_2^{11} \), around the minimum \( \rho = m_{\text{dyn}} \):

\[
\Gamma_{NJL}(\sigma, \pi) = \int dx_2 dx_3 \int dx_4 dx_1 L^{(\text{eff})}_{NJL}(\sigma(x), \pi(x)) \simeq \frac{1}{2\pi |eB| a^4} \sum_{i,j=-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \tilde{L}^{(\text{eff})}_{NJL}(\sigma_{ij}(n, m), \pi_{ij}(n, m))
\]

where \( \sigma_{ij}(n, m) = \sigma(x) \), \( \pi_{ij}(n, m) = \pi(x) \), with \( x_2 = ia \), \( x_3 = ja \), \( x_4 = ix_0 = na \), \( x_1 = ma \), and here the factor \( |eB|/2\pi \) was explicitly factorized from \( L^{(\text{eff})}_{NJL} \). Now, taking into account Eqs.(8),(13) and Eqs.(23), (28), we find that

\[
\Gamma_{NJL} = \frac{1}{2\pi |eB| a^4} \sum_{i,j=-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \tilde{L}^{(\text{eff})}_{NJL}(\sigma_{ij}(n, m), \pi_{ij}(n, m)) \to \sum_{x_2, x_3} \int dx_4 dx_1 L^{(\text{eff})}_{GN}(\pi_{x_2 x_3}(x_\parallel), \pi_{x_2 x_3}(x_\parallel))
\]
in the “continuum” limit with \( (|eB|/2\pi)a^2 = \pi|eB|/2\Lambda^2 = b\pi/2 \) (here \( \Lambda = \pi/\alpha \) is the ultraviolet cutoff on the lattice\). The lagrangian density \( L^{(\text{eff})}_{\text{GN}} \) corresponds to the GN model with the number of colors \( N_c = (\pi/2C)N_c \). Note that the symbol \( \sum_{x_2x_3} \) here is somewhat formal and it just implies that the GN model occurs at each point in the \( x_2x_3 \)-plane.

The physical meaning of this reduction of the NJL model in a magnetic field is rather clear. At weak coupling, the fermion pairing in a magnetic field takes place essentially for fermions in the LLL with the momentum \( |x| \) zero. Then, because of the degeneracy in the LLL \( [22] \), there are \( (|e\Phi|/2\pi)N_c \), where \( \Phi \) is the (local) magnetic flux across a plaquette on the lattice. It leads to changing the number of colors, \( N_c \to \tilde{N}_c = N_c(|e\Phi|/2\pi) \), in the GN model. Note that since \( \tilde{N}_c \) appears analytically in the path integral of the theory, one can give a non-perturbative meaning to the theory with non-integral \( \tilde{N}_c \).

A few comments are in order. Since these GN models are independent, the parameters of the chiral \( U_L(1) \times U_R(1) \) transformations can depend on \( x_\perp \). In other words, here the chiral group is \( \prod_{x_\perp} U_L^{(\pi)}(1) \times U_R^{(\pi)}(1) \). As a result, there are an infinite number of gapless modes \( \pi_{x_2x_3}(x_\parallel) \) in the “continuum” limit.

Since there is no spontaneous breakdown of continuous symmetries at \( D = 1+1 \), the fields \( \pi_{x_2x_3}(x_\parallel) \) do not describe NG bosons (though they do describe gapless BKT excitations) \( [19] \). Since the magnetic field depends on \( \Lambda \) in the “continuum” limit, it can be considered as an additional parameter (“coupling constant”) in the renormalization group. The ratio \( b = |eB|/\Lambda^2 \) is arbitrary here. From the point of view of the renormalization group, this can be interpreted as the presence of a line of ultraviolet fixed points for the dimensionless coupling \( b \). The values of \( b \) on the line define the local magnetic flux and, therefore, the number of colors \( \tilde{N}_c \) in the corresponding GN models.

We emphasize that the reduction of the NJL model, described above, takes place only as \( |\epsilon| \) tends to zero. At finite values of the magnetic field, the dynamics in the NJL and GN models are different: while there is spontaneous chiral symmetry breaking in the NJL model, the BKT phase is realized in the GN model \( [18] \). The connection between these two sets of dynamics is similar to that between the dynamics of 2-dimensional and \( (2+\epsilon) \)-dimensional GN models.

Also, we emphasize that this discussion pertains only to the NJL model with a weak coupling constant, when relation \( (23) \) is valid. In the case of the NJL model with a near–critical \( \epsilon \), the situation is different: when \( \epsilon \to \epsilon_c \to 0 \), \( (24) \) is valid. The difference between these two dynamical regimes reflects the fact that, while at weak coupling the LLL dominates, at near–critical \( \epsilon \) all Landau levels are relevant \( [4] \).

### IV. MAGNETIC CATALYSIS AND THE AHARONOV-BOHM EFFECT.

The relation between \( \tilde{N}_c \) and \( N_c \) obtained in the previous section is:

\[
\tilde{N}_c = N_c(|e\Phi|/2\pi),
\]

(32)

If we asserted that \( \tilde{N}_c \) is a (non-negative) integer, we would be led to the quantization condition for \( N_c(|e\Phi|/2\pi) \) coinciding with the quantization condition for a magnetic flux, \( q(\Phi/2\pi) = n \), at which the Aharonov-Bohm (AB) effect is characterized by the following features:

1. When \( q(\Phi/2\pi) \) is not integer, there is a nontrivial scattering of particles with the charge \( q \) in a line-like (or point-like, in 2+1 dimensions) solenoid field. The effect looks as a non-local one: particles being outside of the solenoid somehow feel the magnetic field inside it.

The effect is intimately connected with the boundary conditions for the wave functions of particles on the solenoid surface.

\[ \text{Of course, the ultraviolet cutoff on the lattice is different from the cutoff in the proper-time regularization used above. However, since the constant } b = |eB|/\Lambda^2 \text{ is anyway arbitrary here, we use the same notation for the cutoff on the lattice as in the proper-time regularization.} \]
2. When \( q(\Phi/2\pi) \) is integer, the magnetic flux in a line-like solenoid is not observable.

Let us now turn to the dimensional reduction in the NJL model in a magnetic field. The number of colors \( \hat{N}_c \) in the corresponding GN model is given by expression (22). Its physical meaning is clear: it is just the number of degrees of freedom connected with one plaquette on the lattice described in the previous section. Now, has this number to be integer? The answer is “no”: the plaquette is just a part of the lattice, and there is no such an additional constraint in the system.

When this number is non-integer, the dynamics of the plaquette is not completely factorized from the dynamics of the rest of the lattice even in the “continuum” limit considered in the previous section. In other words, those degrees of freedom are not confined inside one plaquette (that has to be reflected in a nontrivial boundary condition for the fields on the plaquette boundary). As a result, even in the “continuum” limit the dynamics of a plaquette is non-local in this case: indeed, it is described by a (1+1) dimensional GN model with a non-integer \( \hat{N}_c \), which certainly is not a local field theory.

These features are similar to the features of the AB effect discussed in item 1 above.

On the other hand, for integer \( \hat{N}_c \) the dynamics connected with one plaquette does factorize from the dynamics of the rest of the lattice in the “continuum” limit, and therefore it is described by a local GN model. Like in the case of the magnetic flux satisfying the AB condition (see item 2), the flux connected with a plaquette is not observable (for a (1+1)-dimensional observer) in this case.

Therefore we are led to the interpretation of the dynamics of the GN model with a non-integer \( \hat{N}_c \) as a quantum field theoretical analogue of the AB dynamics with a non-integer \( q(\Phi/2\pi) \). The AB dynamics occurs from the dynamics in a solenoid field when the cross-section of the solenoid goes to zero. Similarly, the (1+1)-dimensional GN model, with both integer and non-integer \( \hat{N}_c \), occurs from the (3+1)-dimensional NJL model in a magnetic field in the “continuum” limit, when the radius of the LLL shrinks to zero.

It would be interesting to study Green’s functions in the GN model with a non-integer \( \hat{N}_c \).

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