INEQUALITIES FOR THE DERIVATIVES OF THE RADON TRANSFORM ON CONVEX BODIES

BY

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ABSTRACT

It was proved in [22] that the sup-norm of the Radon transform of an arbitrary probability density on an origin-symmetric convex body of volume 1 is bounded from below by a positive constant depending only on the dimension. In this note we extend this result to the derivatives of the Radon transform. We also prove a comparison theorem for these derivatives.

1. A slicing inequality for functions

Let $K$ be an origin-symmetric convex body of volume 1 in $\mathbb{R}^n$, and let $f$ be any non-negative measurable function on $K$ with $\int_K f = 1$. Does there exist a constant $c_n$ depending only on $n$ so that for any such $K$ and $f$ there exists a direction $\xi \in S^{n-1}$ with $\int_{K \cap \xi^\perp} f \geq c_n$? Here $\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$ is the central hyperplane perpendicular to $\xi$, and integration is with respect to Lebesgue measure on $\xi^\perp$. It was proved in [22] that, in spite of the generality of the question, the answer to this question is positive, and one can take $c_n \sim \frac{1}{\sqrt{n}}$. In [5] this result was extended to non-symmetric bodies $K$. Moreover, it was shown in [13] that this estimate is optimal up to a logarithmic term, and the logarithmic term was removed in [14], so, finally, $c_n \sim \frac{1}{\sqrt{n}}$. We write $a \sim b$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1a \leq b \leq c_2a$.

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Note that the same question for volume, where \( f \equiv 1 \), is the matter of the slicing problem of Bourgain \([1, 2]\). Bourgain \([3]\) proved that \( \frac{1}{c_n} \leq O(n^{1/4} \log n) \). Klartag \([12]\) removed the logarithmic term from Bourgain’s estimate. In a recent breakthrough result, Chen \([6]\) proved that \( \frac{1}{c_n} \leq O(n^\epsilon) \) for every \( \epsilon > 0 \), as the dimension goes to infinity.

The constant \( c_n \) does not depend on the dimension for several classes of bodies \( K \). For example, it was proved in \([23]\) that if \( K \) belongs to the class of unconditional convex bodies, the constant \( c_n = \frac{1}{2e} \) works for all functions \( f \) and all dimensions \( n \). The same happens for intersection bodies \([21]\), and for the unit balls of subspaces of \( L_p, p > 2 \), where the constant is of the order \( p^{-1/2} \) \([25]\).

Denote by 
\[
R_f(\xi, t) = \int_{K \cap \{x \in \mathbb{R}^n : (x, \xi) = t\}} f(x) \, dx, \quad \xi \in S^{n-1}, t \in \mathbb{R}
\]
the Radon transform of \( f \). The result described above means that the sup-norm of the Radon transform of a probability density on an origin-symmetric convex body in \( \mathbb{R}^n \) is bounded from below by a positive constant depending only on the dimension \( n \).

In this note we prove a similar estimate for the derivatives of the Radon transform. Let us define the fractional derivatives. Let \( m \in \mathbb{N} \cup \{0\} \) and suppose that \( h \) is an even continuous function on \( \mathbb{R} \) that is \( m \) times continuously differentiable in some neighborhood of zero. For \( q \in \mathbb{C}, -1 < \Re(q) < m, q \neq 0, 1, \ldots, m-1 \), the fractional derivative of the order \( q \) of the function \( h \) at zero is defined as the action of the distribution \( t_+^{1-q}/\Gamma(-q) \) on the function \( h \), as follows:

\[
h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left( h(t) - h(0) - \cdots - h^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} \right) \, dt
\]

\[
+ \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t) \, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!(k-q)}.
\]

It can be seen that for a fixed \( q \) the definition does not depend on the choice of \( m > \Re(q) \), as long as \( h \) is \( m \) times continuously differentiable. Note that without dividing by \( \Gamma(-q) \) the expression for the fractional derivative represents an analytic function in the domain \( \{ q \in \mathbb{C} : \Re(q) > -1 \} \) not including integers, and has simple poles at integers. The function \( \Gamma(-q) \) is analytic in the same domain and also has simple poles at non-negative integers, so after the division we get an analytic function in the whole domain \( \{ q \in \mathbb{C} : m > \Re(q) > -1 \} \),
which also defines fractional derivatives of integer orders. Moreover, computing
the limit as $q \to k$, where $k$ is a non-negative integer, we see that the fractional
derivatives of integer orders coincide with usual derivatives up to a sign (when
we compute the limit the first two summands in the right-hand side of (1)
converge to zero, since $\Gamma(-q) \to \infty$, and the limit in the third summand can be
computed using the property $\Gamma(x + 1) = x\Gamma(x)$ of the $\Gamma$-function):

$$h^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} h(t)\big|_{t=0}.$$  

The sign does not matter, because $h$ is an even function, and its derivatives of
odd orders at the origin are equal to zero. Also, in the case where $h$ is even, for
$m - 2 < \Re q < m$ the expression (1) becomes

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} \left( h(t) - \sum_{j=0}^{(m-2)/2} \frac{t^{2j}}{(2j)!} h^{(2j)}(0) \right) dt.$$

We also note that if $-1 < q < 0$ then

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} h(t) \, dt.$$

A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line
passing through the origin crosses the boundary of $K$ at exactly two points, the
origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. If $x \in S^{n-1}$, then $\|x\|_K^{-1} = r_K(x)$ is the radius
of $K$ in the direction of $x$. A star body $K$ is **origin-symmetric** if $K = -K$.
A star body $K$ is called **convex** if for any $x, y \in K$ and every $0 < \lambda < 1$,
$\lambda x + (1 - \lambda)y \in K$.

We say that a star body $K$ in $\mathbb{R}^n$ is **infinitely smooth** if the restriction to the
unit sphere of the Minkowski functional of $K$ belongs to the space $C^\infty(S^{n-1})$
of infinitely differentiable functions on the sphere. For an origin-symmetric
infinitely smooth convex body $K$ in $\mathbb{R}^n$, an infinitely differentiable function $f$
on $K$, fixed $\xi \in S^{n-1}$, and $q \in \mathbb{C}$, $\Re q > -1$, we denote the fractional derivative
of the order $q$ at zero of the function $t \to Rf(\xi, t)$, $t \in \mathbb{R}$, by

$$(Rf(\xi, t))^{(q)}_t(0) = \left( \int_{K \cap \{x : (x, \xi) = t\}} f(x) \, dx \right)^{(q)}_t(0).$$
The estimate that we prove is as follows.

**Theorem 1:** There exists an absolute constant \( c > 0 \) so that for any infinitely smooth origin-symmetric convex body \( K \) of volume 1 in \( \mathbb{R}^n \), any even infinitely smooth probability density \( f \) on \( K \), and any \( q \in \mathbb{R}, \ 0 \leq q \leq n - 2 \), which is not an odd integer, there exists a direction \( \xi \in S^{n-1} \) so that

\[
\left( \frac{c(q+1)}{\sqrt{n \log^3 \left( \frac{n}{q+1} \right)} } \right)^{q+1} \leq \frac{1}{\cos \left( \frac{\pi q}{2} \right)} (Rf(\xi,t))^{(q)}(0).
\]

If \( q = 2k, \ k \in \mathbb{N} \cup \{0\} \), is an even integer, then \( (Rf(\xi,t))^{(2k)}(0) \) is the usual derivative and

\[
\left( \frac{c(2k+1)}{\sqrt{n \log^3 \left( \frac{n}{2k+1} \right)} } \right)^{2k+1} \leq (-1)^k (Rf(\xi,t))^{(2k)}(0).
\]

If \( q = 2k - 1, \ k \in \mathbb{N} \), is an odd integer, then in the right-hand side of (5) we have \( 0 \), and computing the limit as \( q \to 2k - 1 \) we get that there exists a direction \( \xi \in S^{n-1} \) so that

\[
\left( \frac{2kc}{\sqrt{n \log^3 \left( \frac{n}{2k} \right)} } \right)^{2k} \leq (-1)^k (2k-1)! \int_0^\infty t^{-2k} \left( Rf(\xi,t) - \sum_{j=0}^{k-1} \frac{t^{2j}}{(2j)!} (Rf(\xi,t))^{(2j)}(0) \right) dt.
\]

We deduce Theorem 1 from a more general result. Note that in the case where \( q = 0 \) the following theorem was proved in [23].

**Theorem 2:** Suppose \( K \) is an infinitely smooth origin-symmetric convex body in \( \mathbb{R}^n \), \( f \) is a non-negative even infinitely smooth function on \( K \) and \(-1 < q < n-1\) is not an odd integer. Then

\[
\int_K f(x) dx \leq \frac{n}{(n-q-1) 2^{q+1} \pi^{q+1} n^{q+1}} |K|^{q+1} (d_{ovr}(K, L_{n-q}^{-1}))^{q+1} \times \max_{\xi \in S^{n-1}} \frac{1}{\cos \left( \frac{\pi q}{2} \right)} (Rf(\xi,t))^{(q)}(0).
\]
Here $|K|$ stands for the volume of $K$. By $L^n_{-1-q}$ we denote the class of star bodies $D$ in $\mathbb{R}^n$ for which the space $(\mathbb{R}^n, \| \cdot \|_D)$ embeds in $L_{-1-q}$, i.e., the function $\| \cdot \|^{-1}_D$ represents a positive definite distribution; see Section 3 for details.

If $\mathcal{A}$ is a class of compact sets in $\mathbb{R}^n$, the outer volume ratio distance from $K$ to $\mathcal{A}$ is defined by

$$d_{ovr}(K, \mathcal{A}) = \inf \left\{ \left( \frac{|D|}{|K|} \right)^{1/n} : K \subset D, \ D \in \mathcal{A} \right\}.$$  

Let $0 \leq q \leq n - 2$ in Theorem 2. Since $n/(n - q - 1) < e^{q+1}$ and by the Stirling formula, if $|K| = 1$ and $\int_K f = 1$, the estimate (6) turns into

$$\max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0) \geq \left( \frac{c(q + 1)}{d_{ovr}(K, L^n_{-1-q})} \right) \frac{q+1}{2},$$

where $c$ is an absolute constant. This means that the lower estimate for the derivatives of the Radon transform is completely controlled by the distance $d_{ovr}(K, L^n_{-1-q})$. Indeed, if this distance is equal to 1 or is bounded by an absolute constant, then, for every probability density $f$ and every $K$ with volume 1, the right-hand side of (7) depends only on $q$:

$$\frac{1}{\cos(\frac{\pi q}{2})} Rf(\xi,t)_t^{(q)}(0) \geq (c(q + 1)) \frac{q+1}{2}.$$  

The distance $d_{ovr}(K, L^n_{-1-q})$ is known to be bounded by an absolute constant in the following cases.

**Proposition 1:** Let $q \in \mathbb{R}$ be any number from the interval $(-1, n - 1)$.

(i) If $K$ is the unit ball of an $n$-dimensional subspace of $L_p$, $0 < p \leq 2$, then $K \in L^n_{-1-q}$, so

$$d_{ovr}(K, L^n_{-1-q}) = 1.$$  

(ii) If $K$ is an unconditional convex body in $\mathbb{R}^n$, i.e., for every vector $(x_1, \ldots, x_n) \in K$ the vectors $(\pm x_1, \ldots, \pm x_n) \in K$ for all choices of signs, then

$$d_{ovr}(K, L^n_{-1-q}) \leq c.$$  

(iii) If $K$ is the unit ball of an $n$-dimensional subspace of $L_p$, $p > 2$, then

$$d_{ovr}(K, L^n_{-1-q}) \leq c \sqrt{p},$$

where $c$ is an absolute constant.
Proof. (i) Proved in [17] and [15, Theorem 6.17].
(ii) It follows from (i) that the \( \ell_1^n \)-ball
\[
B_1^n = \{ x \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq 1 \}
\]
belongs to \( L_{-1-q}^n \) for every \( q \in (-1, n-1) \). Also, the classes \( L_{-1-q}^n \) are invariant with respect to linear transformations of \( \mathbb{R}^n \), which follows from the connection between the Fourier transform and linear transformations, so \( T(B_1^n) \in L_{-1-q}^n \) for every linear operator \( T \) on \( \mathbb{R}^n \), \( \det(T) \neq 0 \).

By a result of Lozanovskii [30] (see the proof in [36, Corollary 3.4]), there exists a linear operator \( T \) on \( \mathbb{R}^n \) so that \( T(B_1^n) \subset K \subseteq nT(B_1^n) \), where \( B_1^n \) is the cube with sidelength 2 in \( \mathbb{R}^n \). Let \( D = nT(B_1^n) \in L_{-1-q}^n \). Since \( |B_1^n| = 2^n/n! \), we have \( |D|^{1/n} \leq 2e|\det T|^{1/n} \). On the other hand, \( |T(B_1^n)| = 2^n|\det T| \), and \( T(B_1^n) \subset K \), so \( |D|^{1/n} \leq 2e|K|^{1/n} \).

(iii) It follows from (i) with \( p = 2 \) that all \( n \)-dimensional origin-symmetric ellipsoids belong to the class \( L_{-1-q}^n \) for every \( q \in (-1, n-1) \). This can also be shown directly using formula (14). Now the result follows from [33] (see also [25]), where it was proved that the outer volume ratio distance from the unit ball of a subspace of \( L_p \), \( p > 2 \) to the class of ellipsoids is bounded by \( c\sqrt{p} \), where \( c \) is an absolute constant.

Let us show how to get the result of Theorem 1 from the estimate of Theorem 2. In general, one cannot expect to get an estimate for the distance \( d_{\text{over}}(K, L_{-1-q}^n) \) independent of the dimension \( n \). In fact, it was shown in [13, 14] that in the case \( q = 0 \) this distance can be of the order \( \sqrt{n} \). Since the classes \( L_{-1-q}^n \) contain ellipsoids, one can use John’s theorem [10] to prove that \( d_{\text{over}}(K, L_{-1-q}^n) \leq \sqrt{n} \) for every origin-symmetric convex body \( K \) in \( \mathbb{R}^n \) and every \( q \in (-1, n-1) \). However, the estimate of Theorem 1 is better for large values of \( q \).

To prove this better estimate, we need to introduce another class of bodies. For \( p > 0 \), the radial \( p \)-sum of star bodies \( K \) and \( L \) in \( \mathbb{R}^n \) is defined as a new star body \( K_{\tilde{+}} p L \) whose radius in every direction \( \xi \in S^{n-1} \) is given by
\[
r_{K_{\tilde{+}} p L}(\xi) = r_K^n(\xi) + r_L^n(\xi), \quad \forall \xi \in S^{n-1}.
\]
The radial metric in the class of origin-symmetric star bodies is defined by
\[
\rho(K, L) = \sup_{\xi \in S^{n-1}} |r_K(\xi) - r_L(\xi)|.
\]
Definition 1: Let $0 < p < n$. We define the class of generalized $p$-intersection bodies $\mathcal{BP}_p^n$ in $\mathbb{R}^n$ as the closure in the radial metric of radial $p$-sums of finite collections of origin-symmetric ellipsoids in $\mathbb{R}^n$.

Note that when $p = k$ is an integer, we get the class of generalized $k$-intersection bodies introduced by Zhang [39].

The following estimate was proved in [27, Th. 1.1.] for integers $p$, but the proof remains exactly the same for non-integers. Also note that a mistake in the proof in [27] was corrected in [23, Section 5].

Proposition 2 ([27]): For every $p \in [1, n - 1]$ and every origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$d_{ovr}(K, \mathcal{BP}_p^n) \leq C \sqrt{\frac{n \log^3 (ne_p)}{p}},$$

where $C$ is an absolute constant.

It was proved in [20] (see also [34, 28]) that for any integer $k$, $1 \leq k < n$ every generalized $k$-intersection body belongs to the class $L_{-k}^n$. We need an extension of this fact to non-integers, as follows.

Proposition 3: For every $0 < p < n$, we have

$$\mathcal{BP}_p^n \subset L_{-p}^n.$$  

Proof. We need to prove that for any star body $K \in \mathcal{BP}_p^n$, the function $\| \cdot \|^{-p}_K$ represents a positive definite distribution in $\mathbb{R}^n$. As mentioned in the proof of Proposition 1, the powers of the Euclidean norm $|x|^{-p}_{2}$, $0 < p < n$ represent positive definite distributions in $\mathbb{R}^n$; see formula (14). Because of the connection between the Fourier transform of distributions and linear transformations, for any origin-symmetric ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$, the function $\| \cdot \|^{-p}_\mathcal{E}$ represents a positive definite distribution. Note that for any unit vector $x \in S^{n-1}$ and any star body $K$,

$$r_K(x) = \|x\|^{-1}_K.$$

Therefore, radial $p$-sums of ellipsoids in $\mathbb{R}^n$ belong to the class $L_{-p}^n$. The fact that positive definiteness is preserved under limits in the radial metric follows from [15, Lemma 3.11], which proves the result.  \[\blacksquare\]
Deduction of Theorem 1 from Theorem 2. By Propositions 2 and 3, for any $0 \leq q \leq n - 2$,

$$d_{ovr}(K, L_{n-1-q}^n) \leq d_{ovr}(K, BP_{q+1}^n) \leq C \sqrt{\frac{n \log^3\left(\frac{ne}{q+1}\right)}{q+1}}.$$  

Now the first estimate (5) of Theorem 1 follows from Theorem 2, in the form of (7), combined with (8).

In order to prove the case of even integers in Theorem 1, put $q = 2k$ in (5). In the case of odd integers, we use the expression for the fractional derivative (3) to find the limit in (5) as $q \to 2k - 1$:

$$\lim_{q \to 2k-1} \frac{(Rf(\xi, t))^{(q)}(0)}{\cos\left(\frac{\pi q}{2}\right)} = \lim_{q \to 2k-1} \frac{1}{\Gamma(-q) \cos\left(\frac{\pi q}{2}\right)}$$

$$\times \int_0^{\infty} t^{-2k}(Rf(\xi, t) - \sum_{j=0}^{k-1} \frac{t^{2j}}{(2j)!}(Rf(\xi, t))^{(2j)}(0))dt.$$ 

Now use $\Gamma(x + 1) = x\Gamma(x)$ to compute

$$\lim_{q \to 2k-1} \Gamma(-q) \cos\left(\frac{\pi q}{2}\right)$$

$$= \lim_{q \to 2k-1} \frac{\Gamma(-q + 2k)}{(-q)(1 - q) \cdots (2k - 1 - q)} \sin\left(\frac{(q - 2k + 1)\pi}{2}\right) (-1)^k$$

$$= \frac{\pi}{2(2k - 1)!} (-1)^k. \quad \blacksquare$$

The proof of Theorem 2 is presented in Section 4.

We conclude this section by showing the place of the classes $L_{n-1-q}^n$ in the general theory of convex bodies. These classes are generalizations of the concept of an intersection body introduced by Lutwak in [31]. Intersection bodies are an important component of Lutwak’s dual Brunn–Minkowski theory, and they played the crucial role in the solution of the Busemann–Petty problem; see Section 2.

Definition 2 ([31]): For star bodies $D, L$ in $\mathbb{R}^n$, we say that $D$ is the intersection body of $L$ if

$$r_D(\xi) = |L \cap \xi^\perp|, \quad \forall \xi \in S^{n-1}.$$ 

Taking the closure in the radial metric of the class of intersection bodies of star bodies, we define the class of intersection bodies.
A generalization of the concept of an intersection body was introduced in [20].

**Definition 3:** For an integer $k$, $1 \leq k < n$ and star bodies $D, L$ in $\mathbb{R}^n$, we say that $D$ is the $k$-intersection body of $L$ if

$$|D \cap H^\perp| = |L \cap H|, \quad \forall H \in Gr_{n-k}.$$

Taking the closure in the radial metric of the class of $k$-intersection bodies of star bodies, we define the class of $k$-intersection bodies.

It was proved in [17] for $k = 1$, and in [20] for $k > 1$, that an origin-symmetric star body $K$ in $\mathbb{R}^n$ is a $k$-intersection body if and only if the function $\| \cdot \|_K^{-k}$ represents a positive definite Schwartz distribution in $\mathbb{R}^n$. This result is related to embeddings in $L_p$-spaces. By $L_p$, $p > 0$ we mean the $L_p$-space of functions on $[0, 1]$ with Lebesgue measure. It was shown in [16] that an $n$-dimensional normed space embeds isometrically in $L_p$, where $p > 0$ and $p$ is not an even integer, if and only if the Fourier transform in the sense of Schwartz distributions of the function $\Gamma((-p/2)) \cdot \| \cdot \|^p$ is a non-negative distribution outside of the origin in $\mathbb{R}^n$. The concept of embedding of finite dimensional normed spaces in $L_p$ with negative $p$ was introduced in [19, 20], as an analytic extension of embedding into $L_p$ with $p > 0$.

**Definition 4:** For $0 < p < n$, we say that star body $D$ belongs to the class $L_{-p}$, or, in other words, the space $(\mathbb{R}^n, \| \cdot \|_D)$ embeds in $L_{-p}$, if the function $\| \cdot \|_D^{-p}$ represents a positive definite Schwartz distribution on $\mathbb{R}^n$.

A connection between $k$-intersection bodies and embedding in $L_p$ with negative $p$ was found in [20].

**Proposition 4 ([20]):** Let $1 \leq k < n$. An origin-symmetric star body $D$ in $\mathbb{R}^n$ is a $k$-intersection body if and only if $D \in L_{-k}$, or, equivalently, the space $(\mathbb{R}^n, \| \cdot \|_D)$ embeds in $L_{-k}$.

The classes $L_{-p}$ were studied by a number of authors. The advantage of Proposition 4 is that one can take any result about the usual $L_p$-spaces, extend it to $L_{-p}$, and immediately get a geometric application to intersection bodies. Let us mention one of the results. If $n - 3 \leq p < n$, the class $L_{-p}$ contains all origin-symmetric convex bodies in $\mathbb{R}^n$; see [15, Corollary 4.9]. This result was proved as an extension of the fact that every two-dimensional normed space embeds in $L_1$. The result implies that every origin-symmetric convex body in $\mathbb{R}^4$...
is an intersection body, which provides an affirmative answer to the Busemann–Petty problem in the critical 4-dimensional case. More results about embeddings in $L_{-p}$ can be found in [11, 27, 24, 33, 34, 29, 38], [15, Chapter 6] and [28].

2. A comparison theorem for the derivatives of the Radon transform

Our next result is related to the Busemann–Petty problem [4] which is the following question. Let $K, L$ be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of $K$ is smaller than the same for $L$, i.e.,

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|$$

for every $\xi \in S^{n-1}$. Does it necessarily follow that the $n$-dimensional volume of $K$ is smaller than the volume of $L$, i.e., $|K| \leq |L|$? The answer is affirmative if the dimension $n \leq 4$, and it is negative when $n \geq 5$; see [7, 15] for the solution and its history.

It was proved in [18] (see also [15, Theorem 5.12]) that the answer to the Busemann–Petty problem becomes affirmative if one compares the derivatives of the parallel section function of high enough orders. Namely, denote by

$$A_{K,\xi}(t) = R(\chi_K)(\xi, t) = |K \cap \{x \in \mathbb{R}^n : (x, \xi) = t\}|, \quad t \in \mathbb{R}$$

the parallel section function of $K$ in the direction $\xi$. If $K, L$ are infinitely smooth origin-symmetric convex bodies in $\mathbb{R}^n$, $n \geq 4$, $q \in [n-4, n-1)$ is not an odd integer, and for every $\xi \in S^{n-1}$ the fractional derivatives of the order $q$ of the parallel section functions at zero satisfy

$$\frac{1}{\cos(\pi q/2)} A_{K,\xi}^{(q)}(0) \leq \frac{1}{\cos(\pi q/2)} A_{L,\xi}^{(q)}(0),$$

then $|K| \leq |L|$. For $-1 < q < n-4$ this is no longer true.

Another generalization of the Busemann–Petty problem, known as the isomorphic Busemann–Petty problem, asks whether the inequality for volumes holds up to an absolute constant. Does there exist an absolute constant $C$ so that for any dimension $n$ and any origin-symmetric convex bodies $K, L$ in $\mathbb{R}^n$ satisfying $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|$ for all $\xi \in S^{n-1}$, we have $|K| \leq C|L|$? As shown in [35], this question is equivalent to the slicing problem of Bourgain mentioned in Section 1.
Zvavitch [40] considered an extension of the Busemann–Petty problem to general Radon transforms, as follows. Suppose that $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $f$ is an even continuous strictly positive function on $\mathbb{R}^n$. Suppose that $R(f|_K)(\xi,t)(0) \leq R(f|_L)(\xi,t)(0)$ for every $\xi \in S^{n-1}$, where $f|_K$ is the restriction of $f$ to $K$. Does it necessarily follow that $|K| \leq |L|$?

The answer is exactly the same as in the case of volume. Isomorphic versions of this result were proved in [29, 26].

In this note we generalize these results to general Radon transforms as follows. In the case $q = 0$ this result was proved in [26].

**Theorem 3:** Let $K, L$ be infinitely smooth origin-symmetric convex bodies in $\mathbb{R}^n$, $f, g$ non-negative infinitely differentiable functions on $K$ and $L$, respectively, $\|g\|_\infty = g(0) = 1$, and $q \in (-1, n - 1)$ is not an odd integer. If for every $\xi \in S^{n-1}$

$$\frac{1}{\cos(\frac{\pi q}{2})}(Rf(\xi,t))(q)(0) \leq \frac{1}{\cos(\frac{\pi q}{2})}(Rg(\xi,t))(q)(0),$$

then

$$\int_K f(x)dx \leq \frac{n}{n - q - 1}(\text{dvol}(K, L^n_{-q-1}))^{q+1} \left( \int_L g(x)dx \right)^{\frac{n-q-1}{n}} |K|^{\frac{2q+1}{n}}.$$

**3. The main tools**

We often use integration in polar coordinates $x = r\theta$, $x \in \mathbb{R}^n$, $r \geq 0$, $\theta \in S^{n-1}$; see [32, Ch.6, Th. 5.2]. If $f$ is an integrable function on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_{S^{n-1}} \left( \int_0^\infty r^{n-1} f(r\theta)dr \right) d\theta. \tag{9}$$

If $K$ is a star body in $\mathbb{R}^n$, putting $f(x) = \chi_K(x)$, the indicator function of $K$, we get a formula for volume:

$$|K| = \int_{\mathbb{R}^n} \chi_K(x)dx = \int_{S^{n-1}} \left( \int_0^{\|\theta\|^{-1}_K} r^{n-1}dr \right) d\theta = \frac{1}{n} \int_{S^{n-1}} \|\theta\|^{-n}_K d\theta. \tag{10}$$

Our main tool is the Fourier transform of distributions; see [15, 28] for a comprehensive introduction to the Fourier approach in convex geometry. The Fourier transform of a distribution $f$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \phi \rangle$ for every test
function $\phi$ from the Schwartz space $S$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. For any even distribution $f$, we have

$$(\hat{f})^\wedge = (2\pi)^nf;$$

see [37, Th. 7.7] for the inversion formula for the Fourier transform.

If $K$ is a star body and $0 < p < n$, then $\|\cdot\|_K^p$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution acting on test functions by integration. Suppose that $K$ is infinitely smooth, i.e., $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [15, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^p$ is an extension of some function $g \in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n + p$ on $\mathbb{R}^n$. When we write $(\|\cdot\|_K^p)^\wedge(\xi)$, we mean $g(\xi)$, $\xi \in S^{n-1}$. If $K, L$ are infinitely smooth star bodies, the following spherical version of Parseval’s formula was proved in [18] (see [15, Lemma 3.22]): for any $p \in (-n, 0)$

$$(11) \int_{S^{n-1}} (\|\cdot\|_K^p)^\wedge(\xi)(\|\cdot\|_L^{n+p})^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^p \|x\|_L^{n+p} \, dx.$$  

A distribution is called **positive definite** if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi$.

We need a lemma that can be found in [15, Lemma 3.14]. We include the proof for completeness.

**Lemma 1:** Let $0 < q < 1$. For every even test function $\phi$ and every fixed vector $\theta \in S^{n-1}$

$$\int_{\mathbb{R}^n} |(\theta, \xi)|^{-1-q} \phi(\xi) d\xi = \frac{2\Gamma(-q) \cos(\pi q/2)}{\pi} \int_0^\infty t^q \hat{\phi}(t\theta) dt.$$  

**Proof.** A well-known connection between the Fourier and Radon transforms is that, for any test function $\phi$, the function $t \to \hat{\phi}(t\theta)$ is the Fourier transform of the function

$$z \to \int_{(\theta, \xi) = z} \phi(\xi) d\xi;$$

see, for example, [15, Lemma 2.11]. Using the Fubini theorem and the formula for the Fourier transform of $|z|^{-1-q}$ (see [15, Lemma 2.23])

$$(|z|^{-1-q})^\wedge(t) = 2\Gamma(-q) \cos(\pi q/2)|t|^q,$$
we get
\[ \int_{\mathbb{R}^n} |(\theta, \xi)|^{-1-q} \phi(\xi) d\xi = \int_{\mathbb{R}} |z|^{-1-q} \left( \int_{(\theta, \xi) = z} \phi(\xi) d\xi \right) dz \]
\[ = \left\langle |z|^{-1-q} \int_{(\theta, \xi) = z} \phi(\xi) d\xi \right\rangle \]
\[ = \frac{1}{2\pi} (2\Gamma(-q) \cos(\pi q/2) |t|^q \hat{\phi}(t\theta)) \]
\[ = \frac{\Gamma(-q) \cos(\pi q/2)}{\pi} \int_{\mathbb{R}} |t|^q \hat{\phi}(t\theta) dt. \]

Finally, recall that \( \phi \) is an even function. 

Our next lemma generalizes Theorem 1 from [8] (see also [15, Th. 3.18]).

**Lemma 2:** Let \( K \) be an infinitely smooth origin-symmetric convex body in \( \mathbb{R}^n \), let \( f \) be an even infinitely smooth function on \( K \), and let \( q \in (-1, n-1) \). Then for every fixed \( \xi \in S^{n-1} \)
\[ (Rf(\xi, t))_t^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi} \left( |x|^{-n+q+1} \left( \int_0^{|x|^2 K^{-n-q-2} f(r \frac{x}{|x|^2}) dr \right) \right) (\xi). \]

**Proof.** Let \(-1 < q < 0\). Then, using the definitions of the Radon transform and the fractional derivative (4), the Fubini theorem and integration in polar coordinates (9) with \( x = r\theta \), we get
\[ (Rf(\xi, t))_t^{(q)}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |t|^{-1-q} \left( \int_{K \cap \{x: (x, \xi) = t\}} f(x) dx \right) dt \]
\[ = \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |(x, \xi)|^{-1-q} f(x) \chi_K(x) dx \]
\[ = \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |(\theta, \xi)|^{-1-q} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta) dr \right) d\theta. \]

Consider the latter as a homogeneous of degree \(-1 - q\) function of \( \xi \in \mathbb{R}^n \setminus \{0\} \), apply it to an even test function \( \phi \) and use Lemma 1:
\[ \langle (Rf(\xi, t))_t^{(q)}(0), \phi \rangle \]
\[ = \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} \phi(\xi) \left( \int_{S^{n-1}} |(\theta, \xi)|^{-1-q} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta) dr \right) d\theta \right) d\xi \]
\[ = \frac{\cos(q\pi/2)}{\pi} \int_{S^{n-1}} \left( \int_0^{\infty} t^q \hat{\phi}(t\theta) dt \right) \left( \int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta) dr \right) d\theta. \]
On the other hand, if we apply the function in the right-hand side of (12) to the test function \( \phi \) we get

\[
\left\langle \frac{\cos(\pi q/2)}{\pi} \left| x \right|^{-n+q+1}_2 \left( \int_0^{\|x\|_K} r^{n-q-2} f \left( \frac{rx}{|x|_2} \right) dr \right) \right\rangle^\wedge_x (\xi, \phi(\xi))
\]

\[
= \frac{\cos(\pi q/2)}{\pi} \int_{\mathbb{R}^n} \left| x \right|^{-n+q+1}_2 \left( \int_0^{\|x\|_K} r^{n-q-2} f \left( \frac{rx}{|x|_2} \right) dr \right) \hat{\phi}(x) dx
\]

\[
= \frac{\cos(\pi q/2)}{\pi} \int_{S^{n-1}} \left( \int_0^{\infty} t^q \hat{\phi}(t\theta) dt \right) \left( \int_0^{\|\theta\|_K} r^{n-q-2} f(r\theta) dr \right) d\theta,
\]

where in the last step we use integration in polar coordinates (9) with \( x = t\theta \). Comparing the computations, we see that, for any even test function \( \phi \),

\[
\left\langle (Rf(\xi, t))^{(q)}(0), \phi \right\rangle
\]

\[
= \left\langle \frac{\cos(\pi q/2)}{\pi} \left| x \right|^{-n+q+1}_2 \left( \int_0^{\|x\|_K} r^{n-q-2} f \left( \frac{rx}{|x|_2} \right) dr \right) \right\rangle^\wedge_x (\xi, \phi(\xi)).
\]

Since both distributions are even, this proves the lemma for \(-1 < q < 0\). By an argument similar to that in Lemma 2.22 from [15], one can see that both sides of (13) are analytic functions of \( q \) in the domain \(-1 < \Re q < n - 1\). By analytic extension, (13) holds for all \(-1 < q < n - 1\), which completes the proof. 

Let us compute the fractional derivatives of the Radon transform in the case where \( f \equiv 1 \) and \( K = B^n_2 \), the unit Euclidean ball.

**Corollary 1:** For \(-1 < q < n - 1\) and every \( \xi \in S^{n-1} \)

\[
(R(\chi_{B^n_2})(\xi, t))^{(q)}(0) = \frac{2^{q+1}\pi^{\frac{n+2}{2}}\Gamma(\frac{q+1}{2})\cos\left(\frac{\pi q}{2}\right)}{(n-q-1)\Gamma(\frac{n-q-1}{2})}.
\]

**Proof.** First, we use Lemma 2 with \( f \equiv 1 \) and \( K = B^n_2 \):

\[
(R(\chi_{B^n_2})(\xi, t))^{(q)}(0) = \frac{\cos\left(\frac{\pi q}{2}\right)}{\pi(n-q-1)} \left| x \right|^{-n+q+1}_2 \wedge (\xi).
\]

Next, we apply the formula for the Fourier transform of powers of the Euclidean norm (see [9]):

\[
\left| x \right|^{-q+1}_2 \wedge (\xi) = \frac{2^{q+1}\pi^{\frac{n}{2}}\Gamma(\frac{q+1}{2})}{\Gamma(\frac{n-q-1}{2})} |\xi|_2^{-1-q}, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\]
Note that if we replace the unit Euclidean ball by the Euclidean ball of volume 1, the constant in Corollary 1 has to be divided by $|B^n_2|^{\frac{1-q}{2}}$, and it matches the estimate of Theorem 2 for bodies $K$ of volume 1; see inequality (7). We leave this computation for the interested reader.

4. Proofs of the main results

Proof of Theorem 3. The assumption of the theorem is that, for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\frac{\pi q}{2})}(Rf(\xi,t))^{(q)}(0) \leq \frac{1}{\cos(\frac{\pi q}{2})}(Rg(\xi,t))^{(q)}(0).$$

By Lemma 2, for every $\xi \in S^{n-1}$, this assumption is equivalent to

$$\left(|x|_2^{-n+q+1} \left( \int_0^{\frac{|x|_2}{r \xi_2} \int k \int_0^{\frac{|x|_2}{r \xi_2}} r^{n-q-2} f(r \xi_2 dr) \right)^x (\xi) \right)^\wedge \leq \left(|x|_2^{-n+q+1} \left( \int_0^{\frac{|x|_2}{r \xi_2} \int k \int_0^{\frac{|x|_2}{r \xi_2}} r^{n-q-2} g(r \xi_2 dr) \right)^x (\xi) \right)^\wedge.$$

(15)

Let $\delta > 0$, and let $D \in L^n_{1-q}$ be such that $K \subset D$ and

$$|D|^{1/n} \leq (1 + \delta)\text{d_{ovr}(K, L^n_{1-q})}|K|^{1/n}.$$

(16)

By approximation, we can assume that $D$ is infinitely smooth; see [15, Lemma 4.10]. Then $(\|x\|_D^{-1-q})^\wedge$ is a non-negative function on the sphere. Multiplying both sides of (15) by $(\|x\|_D^{-1-q})^\wedge (\xi)$, integrating over the sphere and using Parseval’s formula on the sphere (11) we get

$$\int_{S^{n-1}} \|\theta\|_D^{-1-q} \left( \int_0^{\|\theta\|_D} r^{n-q-2} f(r \theta dr) \right)^x \leq \int_{S^{n-1}} \|\theta\|_D^{-1-q} \left( \int_0^{\|\theta\|_D} r^{n-q-2} g(r \theta dr) \right)^x,$$

or, using the formula for integration in polar coordinates (9) with $x = r \theta$,

$$\int_K \|x\|_D^{-1-q} f(x) dx \leq \int_L \|x\|_D^{-1-q} g(x) dx.$$

(17)

Since $K \subset D$, we have $\|x\|_D \leq 1$ for every $x \in K$, and thus

$$\int_K \|x\|_D^{-1-q} f(x) dx \geq \int_K f(x) dx.$$

(18)
On the other hand, by the Lemma from section 2.1 in Milman–Pajor [35, p. 76],
\[
\left( \int_{L} \frac{g(x)dx}{\|x\|^{1-q}Dx} \right)^{1/(n-q-1)} \leq \left( \frac{\int_{D} g(x)dx}{\int_{D} dx} \right)^{1/n}.
\]
By (9) and (10),
\[
\int_{D} \|x\|^{-1-q}Dx = \int_{\mathbb{R}^{n}} \|x\|^{-1-q} \chi_{D}(x)dx
\]
\[
= \int_{S^{n-1}} \|\theta\|^{-1-q} \left( \int_{0}^{\|\theta\|_{D}^{-1}} r^{n-q-2} dr \right) d\theta
\]
\[
= \frac{1}{n-q-1} \int_{S^{n-1}} \|\theta\|^{-n} d\theta = \int \frac{n}{n-q-1} |D|.
\]
Now we can rewrite (19) as
\[
\int_{L} \|x\|^{-1-q}g(x)dx \leq \frac{n}{n-q-1} \left( \int_{L} g(x)dx \right)^{\frac{n-q-1}{n}} |D|^\frac{q+1}{n}.
\]
Combining estimates (17), (18) and (20) with the definition of $D$, (16), and sending $\delta$ to zero, we get
\[
\int_{K} f(x)dx \leq \frac{n}{n-q-1} \left( \int_{L} g(x)dx \right)^{\frac{n-q-1}{n}} (d_{\text{ovt}}(K, L_{n-1-q}))^{q+1} |K|^\frac{q+1}{n},
\]
which is the conclusion of the theorem. \hfill \blacksquare

Proof of Theorem 2. Consider a number $\varepsilon > 0$ such that, for every $\xi \in S^{n-1},$
\[
\frac{1}{\cos(\pi q/2)} (Rf(\xi, t))^{(q)}(0) \leq \frac{\varepsilon}{\cos(\pi q/2)} (R(\chi_{B_{2}^{n}})(\xi, t))^{(q)}(0).
\]
By Lemma 2, for every $\xi \in S^{n-1},$
\[
(\frac{|x|}{2})^{-n+q+1} \left( \int_{0}^{\frac{|x|}{2}} r^{n-2} f(r \frac{x}{|x|}) dr \right)^{\frac{1}{q}}(\xi) \leq \frac{\varepsilon}{n-q-1} |x|^{-n+q+1}(\xi).
\]
Let $\delta > 0$, and let $D \in L_{n-1-q}$ be such that $K \subset D$ and
\[
|D|^{1/n} \leq (1 + \delta) d_{\text{ovt}}(K, L_{n-1-q}) |K|^\frac{1}{n}.
\]
By approximation, we can assume that $D$ is infinitely smooth. Then $(\|x\|^{-1-q})^$ is a non-negative function on the sphere. Multiplying both sides of the latter
inequality by $\left(\|x\|_D^{1-q}\right)^{(\xi)}$, integrating over the sphere and using Parseval’s formula on the sphere we get (like in the proof of Theorem 3)

$$\int_K \|x\|_D^{-1-q} f(x) dx \leq \frac{\varepsilon}{n - q - 1} \int_{S^{n-1}} \|x\|_D^{-1-q} dx.$$  

Since $K \subset D$, we have

$$\int_K \|x\|_D^{-1-q} f(x) dx \geq \int_K \|x\|_D^{-1-q} f(x) dx \geq \int_K f(x) dx.$$  

On the other hand, by Hölder’s inequality with the exponents $\frac{n}{q+1}$ and $\frac{n}{n-q-1}$, and by (10),

$$\int_{S^{n-1}} \|x\|_D^{-1-q} dx \leq |S^{n-1}| \frac{n^{-q-1}}{n} n^{\frac{q+1}{n}} |D|^{\frac{q+1}{n}},$$  

where (see, for example, [15, Corollary 2.20])

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$  

Combining (22), (23) and (24) we get

$$\int_K f(x) dx \leq \frac{\varepsilon |S^{n-1}| \frac{n^{-q-1}}{n} n^{\frac{q+1}{n}}}{n - q - 1} |D|^{\frac{q+1}{n}}.$$  

Now replace $D$ by $K$ using (21):

$$\int_K f(x) dx \leq \frac{\varepsilon |S^{n-1}| \frac{n^{-q-1}}{n} n^{\frac{q+1}{n}}}{n - q - 1} (1 + \delta)^{q+1} (d_{ovr}(K, L_{n-1}^{n-1}))^{q+1} |K|^{\frac{q+1}{n}},$$  

and put

$$\varepsilon = \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)} \left(\frac{Rf(\xi, t)}{(Rf(\xi, t))^{(q)}(0)}\right).$$  

Replace the denominator in the expression for $\varepsilon$ using Corollary 1, and substitute the formula for $|S^{n-1}|$:

$$\int_K f(x) dx \leq c(n, q) (1 + \delta)^{q+1} (d_{ovr}(K, L_{n-1}^{n-1}))^{q+1} |K|^{\frac{q+1}{n}}$$

$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)} (Rf(\xi, t))^{(q)}(0),$$  

where

$$c(n, q) = \frac{\pi \Gamma\left(\frac{n-q-1}{2}\right) n^{\frac{q+1}{2}} 2^{\frac{n-q-1}{2}} \frac{n}{\pi}^{\frac{n-q-1}{2}} 2^{q+1} \pi^{\frac{n}{2}} \Gamma\left(\frac{q+1}{2}\right) (\Gamma\left(\frac{n}{2}\right))^{-\frac{n-q-1}{n}}.$$  

Now use $\Gamma(x + 1) = x\Gamma(x)$ and the inequality
\[
\frac{\Gamma\left(\frac{n-q-1}{2} + 1\right)}{(\Gamma\left(\frac{n}{2} + 1\right))^\frac{n-q-1}{n}} \leq 1,
\]
which follows from the log-convexity of the $\Gamma$-function (see [15, Lemma 2.14]). We get
\[
c(n, q) = \frac{n \Gamma\left(\frac{n-q-1}{2} + 1\right)}{2q\pi^\frac{n-q}{2}\Gamma\left(\frac{n+1}{2}\right)}(n - q - 1)(\Gamma\left(\frac{n}{2} + 1\right))^\frac{n-q-1}{n}
\]
\[
\leq \frac{n}{2q\pi^\frac{n-q}{2}\Gamma\left(\frac{n+1}{2}\right)}(n - q - 1),
\]
and (25) implies
\[
\int_K f(x)dx \leq \frac{n}{2q\pi^\frac{n-q}{2}\Gamma\left(\frac{n+1}{2}\right)}(n - q - 1)(1 + \delta)^{q+1}(d_{ovr}(K, L^{n}_{n-1-q}))^{q+1}|K|^{\frac{q+1}{n}}
\]
\[
\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)}(Rf(\xi, t))^{(q)}(0).
\]
Finally, send $\delta$ to zero to get the result. 

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