BOUNDREDNESS OF SINGULAR INTEGRALS ON THE FLAG HARDY SPACES ON THE HEISENBERG GROUP

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ABSTRACT. We prove that the classical one-parameter convolution singular integrals on the Heisenberg group are bounded on multiparameter flag Hardy spaces, which satisfy \( \cdots \), dilation between the one-parameter anisotropic dilation and the product dilation on \( \mathbb{C}^n \times \mathbb{R} \) implicitly.

1. Introduction and statement of main results

The purpose of this note is to show that the classical one-parameter convolution singular integrals on the Heisenberg group are bounded on multiparameter flag Hardy spaces. Recall that the Heisenberg \( \mathbb{H}^n \) is the Lie group with underlying manifold \( \mathbb{C}^n \times \mathbb{R} = \{ [z,t] : z \in \mathbb{C}^n, t \in \mathbb{R} \} \) and multiplication law

\[
[z,t] \circ [z', t'] = [z_1, \cdots, z_n, t] \circ [z'_1, \cdots, z'_n, t'] := [z_1 + z'_1, \cdots, z_n + z'_n, t + t' + 2\text{Im}(\sum_{j=1}^{n} z_j z'_j)].
\]

The identity of \( \mathbb{H}^n \) is the origin and the inverse is given by \( [z,t]^{-1} = [-z,-t] \). Hereafter we agree to identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) and to use the following notation to denote the points of \( \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1} : g = [z,t] \equiv [x,y,t] = [x_1, \cdots, x_n, y_1, \cdots, y_n, t] \) with \( z = [z_1, \cdots, z_n] \) and \( z_j = x_j + iy_j \) and \( x_j, y_j, t \in \mathbb{R} \) for \( j = 1, \cdots, n \). Then, the composition law \( \circ \) can be explicitly written as

\[
g \circ g' = [x,y,t] \circ [x',y',t'] = [x + x', y + y', t + t' + 2(y,x') - 2(x,y')],
\]

where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^n \).

Consider the dilations

\[
\delta_r : \mathbb{H}^n \to \mathbb{H}^n, \quad \delta_r(g) = [r z, r^2 t].
\]

A trivial computation shows that \( \delta_r \) is an automorphism of \( \mathbb{H}^n \) for every \( r > 0 \). Define a “norm” function \( \rho \) on \( \mathbb{H}^n \) by

\[
\rho(g) = \rho([z,t]) := \max\{|z|, |t|^{1/2}\}.
\]

It is easy to see that \( \rho(g^{-1}) = \rho(-g) = \rho(g), \rho(\delta_r(g)) = r \rho(g), \rho(g) = 0 \) if and only if \( g = 0 \), and \( \rho(g \circ g') \leq \gamma (\rho(g) + \rho(g')) \), where \( \gamma > 1 \) is a constant.

The Haar measure on \( \mathbb{H}^n \) is known to just coincide with the Lebesgue measure on \( \mathbb{R}^{2n+1} \). For any measurable set \( E \subset \mathbb{H}^n \), we denote by \( |E| \) its (Harr) measure. The vector fields

\[
T := \frac{\partial}{\partial t}, \quad X_j := \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j := \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n,
\]

form a natural basis for the Lie algebra of left-invariant vector fields on \( \mathbb{H}^n \). For convenience we set \( X_{n+j} := Y_j \) for \( j = 1, 2, \cdots, n \), and set \( X_{2n+1} := T \). Denote by \( \tilde{X}_j \), \( j = 1, \cdots, 2n+1 \), the right-invariant vector field which coincides with \( X_j \) at the origin. Let \( \mathbb{N} \) be the set of

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all non-negative integers. For any multi-index $I = (i_1, \ldots, i_{2n+1}) \in \mathbb{N}^{2n+1}$, we set $X^I := X_1^{i_1}X_2^{i_2} \cdots X_{2n+1}^{i_{2n+1}}$ and $\tilde{X}^I := \tilde{X}_1^{i_1}\tilde{X}_2^{i_2} \cdots \tilde{X}_{2n+1}^{i_{2n+1}}$. It is well known that (9)

$$X^I(f_1 * f_2) = f_1 * (X^I f_2), \quad \tilde{X}^I(f_1 * f_2) = (\tilde{X}^I f_1) * f_2, \quad (X^I f_1) * f_2 = f_1 * (\tilde{X}^I f_2),$$

and

$$X^I \tilde{f} = (-1)^{|I|} \tilde{X}^I f,$$

where $\tilde{f}$ is given by $\tilde{f}(g) := f(g^{-1})$. We further set

$$|I| := i_1 + \cdots + i_{2n+1} \quad \text{and} \quad d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}.$$ Then $|I|$ is said to be the order of the differential operators $X^I$ and $\tilde{X}^I$, while $d(I)$ is said to be the homogeneous degree of $X^I$ and $\tilde{X}^I$.

**Definition 1.1** (17). A function $\phi$ is called a normalized bump function on $\mathbb{H}^n$ if $\phi$ is supported in the unit ball $\{g = [z, t] \in \mathbb{H}^n : \rho(g) \leq 1\}$ and

$$|\partial^I_{z, t}\phi(z, t)| \leq 1$$

uniformly for all multi-indices $I \in \mathbb{N}^{2n+1}$ with $|I| \leq N$, for some fixed positive integer $N$.

**Remark 1.2.** The condition (1.1) is equivalent (modulo a constant) to the following one:

$$|X^I \phi(g)| \leq 1$$

for all multi-indices $I$ with $|I| \leq N$. Indeed, this follows from the following homogeneous property of the “norm” $\rho$ and the fact that

$$X^I f(g) = \sum_{|J| \leq |I|, \; d(J) \geq d(I)} P_{IJ}(g)(\partial^J_{z, t}f)(g)$$

and

$$(\partial^I_{z, t}f)(g) = \sum_{|J| \leq |I|, \; d(J) \geq d(I)} Q_{IJ}(g)(X^J f)(g),$$

where $P_{IJ}, Q_{IJ}$ are polynomials of homogeneous degree $d(J) - d(I)$ (see [9]).

We assume that $K$ is a distribution on $\mathbb{H}^n$ that agrees with a function $K(g), \; g = [z, t] \neq [0, 0]$, and satisfies the following regularity conditions:

$$|K(g)| \leq C\rho(g)^{-2n-2}, \quad |\nabla_z K(g)| \leq C\rho(g)^{-2n-3}, \quad |\partial_{\partial t} K(g)| \leq C\rho(g)^{-2n-4},$$

and the cancellation condition

$$|K(\phi^r)| \leq C$$

for all normalized bump function $\phi$ and for all $r > 0$, where $\phi^r(g) = \phi(\delta_r(g))$. It is well known that the classical one-parameter convolution singular integral $T$ defined by $T(f) = f * K$ is bounded on $L^p$, $1 < p < \infty$, and on the classical Hardy spaces on the Heisenberg group $H^p(\mathbb{H}^n)$ for $p \in (p_0, 1]$. See [9] and [17] for more details and proofs.

Müller, Ricci and Stein ([13], [14]) proved that Marcinkiewicz multipliers are $L^p$ bounded for $1 < p < \infty$ on the Heisenberg group $\mathbb{H}^n$. This is surprising since these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is no two parameter group of automorphic dilations on $\mathbb{H}^n$. Moreover, they show that Marcinkiewicz multiplier can be characterized by convolution operator with the form $f * K$ where, however, $K$ is a flag kernel. At the endpoint estimates, it is natural to expect that Hardy space and BMO bounds are available. However, the lack of automorphic dilations underlies the failure of such multipliers to be in general bounded on the classical Hardy space $H^1$ and also precludes a pure product Hardy space theory on the Heisenberg group. This was the original motivation in [11] (see also [12]) to develop a theory of flag Hardy spaces $H^p_{\text{flag}}$ on the Heisenberg group, $0 < p \leq 1$, that is in a sense ‘intermediate’ between the classical
Hardy spaces $H^p(\mathbb{H}^n)$ and the product Hardy spaces $H^p_{\text{product}}(\mathbb{C}^n \times \mathbb{R})$ (A. Chang and R. Fefferman [11, 12, 13, 14, 15]). They show that singular integrals with flag kernels, which include the aforementioned Marcinkiewicz multipliers, are bounded on $H^p_{\text{flag}}$, as well as from $H^p_{\text{flag}}$ to $L^p$, for $0 < p \leq 1$. Moreover, they construct a singular integral with a flag kernel on the Heisenberg group, which is not bounded on the classical Hardy spaces $H^1(\mathbb{H}^n)$. Since, as pointed out in [11, 12], the flag Hardy space $H^p_{\text{flag}}(\mathbb{H}^n)$ is contained in the classical Hardy space $H^p(\mathbb{H}^n)$, this counterexample implies that $H^1_{\text{flag}}(\mathbb{H}^n) \nsubseteq H^1(\mathbb{H}^n)$.

A natural question arises: Is it possible that the classical one-parameter singular integrals on the Heisenberg group are bounded on flag Hardy spaces $H^p_{\text{flag}}(\mathbb{H}^n)$?

Note that the classical singular integrals on the Heisenberg group satisfy the one-parameter anisotropic dilation as mentioned above. However, the flag Hardy spaces do not satisfy such a dilation, but satisfy ‘intermediate’ dilation between the one-parameter anisotropic dilation and the product dilation on $\mathbb{C}^n \times \mathbb{R}$ implicitly. We would like to point out that Nagel, Ricci and Stein [15] introduced a class of singular integrals with flag kernels on the Euclidian space. They also pointed that singular integrals with flag kernels on the Euclidean space belong to product singular integrals, see Remark 2.1.7 and Theorem 2.1.11 in [15], where the characterizations in terms of the corresponding multipliers between the flag and product singular integrals are given. See also [16] for singular integrals with flag kernels on homogeneous groups. Recently, in [18] it was proved that the classical Calderon-Zygmund convolution operators on the Euclidean space are bounded on the product Hardy spaces.

In this note we address this deficiency by showing that the classical one-parameter convolution singular integrals on $\mathbb{H}^n$ are bounded for flag Hardy spaces on $\mathbb{H}^n$.

Before stating the main results in this note, we begin with recalling the Calderón’s reproducing formula, Littlewood–Paley square function and the flag Hardy space $H^p_{\text{flag}}(\mathbb{H}^n)$. Let $\psi_1(1) \in C_\infty(\mathbb{H}^n)$ and all arbitrarily large moments vanish and such that the following Calderón reproducing formula holds:

$$f = \int_0^\infty (\psi_1(1))^{\vee} \ast \psi_1(1) \ast f \frac{ds}{s}, \quad f \in L^2(\mathbb{H}^n),$$

where $\ast$ is Heisenberg convolution, $(\psi_1(1))^{\vee}(\zeta) = \psi_1(1)(\zeta^{-1})$ and $\psi_1(1)(z,u) = s^{-2n-2} \psi_1(1)(\frac{z}{s}, \frac{u}{s})$ for $s > 0$. See Corollary 1 of [10] for the existence of the function $\psi_1(1)$.

Let $\psi_2(1) \in \mathcal{S}(\mathbb{R})$ satisfying

$$\int_0^\infty |\widehat{\psi_2(1)}(t\eta)|^2 \frac{dt}{t} = 1$$

for all $\eta \in \mathbb{R} \setminus \{0\}$. Assume along with the following moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta \psi_1(1)(z,u) dz du = 0, \quad |\alpha| + 2\beta \leq M,$n

$$\int_{\mathbb{R}} v^\gamma \psi_2(1)(v) dv = 0, \quad \gamma \geq 0.$$

Here the positive integer $M$ may be taken arbitrarily large. Thus, we have

$$f(z,u) = \int_0^\infty \int_0^\infty (\psi_{s,t})^{\vee} \ast \psi_{s,t} \ast f(z,u) \frac{ds \, dt}{s \, t},$$

where $f \in L^2(\mathbb{H}^n)$, $\widehat{\psi_{s,t}}(\zeta) = \psi_{s,t}(\zeta^{-1})$ for every $\zeta \in \mathbb{H}^n$, and the series converges in the $L^2(\mathbb{H}^n)$ norm. Following [14], a Littlewood–Paley component function $\psi$ is defined on $\mathbb{H}^n \simeq$
\(C^n \times \mathbb{R}\) by the partial convolution \(*_2\) in the second variable only:
\[
\psi(z, u) = \psi^{(1)} *_2 \psi^{(2)}(z, u) = \int_{\mathbb{R}} \psi^{(1)}(z, u - v) \psi^{(2)}(v) dv,
\]
where \(z, u \in C^n \times \mathbb{R}\), and the function \(\psi_{s,t}(z, u)\) is given by
\[
\psi_{s,t}(z, u) = \psi^{(1)}_{s} *_2 \psi^{(2)}_{t}(z, u) = \int_{\mathbb{R}} \psi^{(1)}_{s}(z, u - v) \psi^{(2)}_{t}(v) dv.
\]

We now set
\[
\psi'_Q = \psi^{(1)}_j \text{ if } Q \in Q(j),
\]
\[
\psi'_R = \psi_{j,k} = \psi^{(1)}_j *_2 \psi^{(2)}_{k} \text{ if } R \in R(j, k),
\]
where \(Q \in Q(j)\) are cubes and \(R \in R(j, k)\) with \(k < j\) are rectangles, and
\[
Q = \bigcup_{j \in \mathbb{Z}} Q(j),
\]
and the collection of all strictly vertical dyadic rectangles as
\[
R_{vert} = \bigcup_{j > k} R(j, k).
\]
The wavelet Calderón reproducing formula is then given by the following (Theorem 3 in [11])
\[(1.6) \quad f(z, u) = \sum_{Q \in Q} f_Q \Psi_Q(z, u) + \sum_{R \in R_{vert}} f_R \Psi_R(z, u), \quad f \in M_{flag}^{M'+\delta}(\mathbb{H}^{n}),\]
where
\[
f_Q \equiv c_a |Q| \psi_{j,k} * f(z_Q, u_Q), \quad \text{for } Q \in Q(j) \text{ and } k \geq j,
\]
\[
f_R \equiv c_a |R| \psi_{j,k} * f(z_R, u_R), \quad \text{for } R \in R(j, k) \text{ and } k < j,
\]
the functions \(\Psi_Q\) and \(\Psi_R\) are in \(M_{flag}^{M'+\delta}(\mathbb{H}^{n})\) satisfying \(\|\Psi_Q\|_{M_{flag}^{M'+\delta}(\mathbb{H}^{n})} \lesssim \|\psi'_Q\|_{M_{flag}^{M'+\delta}(\mathbb{H}^{n})}\)
and \(\|\Psi_R\|_{M_{flag}^{M'+\delta}(\mathbb{H}^{n})} \lesssim \|\psi'_R\|_{M_{flag}^{M'+\delta}(\mathbb{H}^{n})}\), and the convergence of the series holds in both \(L^p(\mathbb{H}^{n})\) and the Banach space \(M_{flag}^{M'+\delta}(\mathbb{H}^{n})\).

Based on the above reproducing formula, the wavelet Littlewood–Paley square function is defined by
\[
S_{flag}(f)(z, u) := \left\{ \sum_{Q \in Q} |\psi'_Q * f(z_Q, u_Q)|^2 \chi_Q(z, u) + \sum_{R \in R_{vert}} |\psi'_R * f(z_R, u_R)|^2 \chi_R(z, u) \right\}^\frac{1}{2},
\]
where \((z_Q, u_Q)\) is any fixed point in the cube \(Q\); and \((z_R, u_R)\) is any fixed point in the rectangle \(R\).

We now recall the precise definition of the flag Hardy spaces.

**Definition 1.3** ([11] [12]). Let \(0 < p < \infty\). Then for \(M\) sufficiently large depending on \(n\) and \(p\) we define the flag Hardy space \(H_{flag}^p(\mathbb{H}^{n})\) on the Heisenberg group by
\[
H_{flag}^p(\mathbb{H}^{n}) := \left\{ f \in M_{flag}^{M'+\delta}(\mathbb{H}^{n}): S_{flag}(f) \in L^p(\mathbb{H}^{n}) \right\},
\]
and for \(f \in H_{flag}^p(\mathbb{H}^{n})\) we set
\[(1.7) \quad \|f\|_{H_{flag}^p} := \|S_{flag}(f)\|_p.\]
See [11] [12] for more details about structures of dyadic cubes and strictly vertical rectangles, test function space $M_{flag}^{M+\delta}(\mathbb{H}^n)$ and its dual $M_{flag}^{M+\delta}(\mathbb{H}^n)'$.

The main results in this note are the following

**Theorem 1.4.** Suppose that $K$ is a distribution kernel on $\mathbb{H}^n$ satisfying the regularity conditions [1.3] and the cancellation condition [1.4]. Then the operator $T$ defined by $T(f) := f*K$ is bounded on $H^p_{flag}(\mathbb{H}^n)$ for $\frac{4n}{4n+1} < p \leq 1$.

We remark that the lower bound $\frac{4n}{4n+1}$ for $p$ in Theorem 1.4 can be getting smaller if the regularity and cancellation conditions on $K$ are required to be getting higher. We leave these details to the reader.

As a consequence of Theorem 1.4 and the duality of $H^1_{flag}(\mathbb{H}^n)$ with $BMO_{flag}(\mathbb{H}^n)$ as given in [11] [12], we obtain

**Corollary 1.5.** Suppose that $K$ is a distribution kernel on $\mathbb{H}^n$ as given in Theorem 1.4. Then the operator $T$ defined by $T(f) := f*K$ is bounded on $BMO_{flag}(\mathbb{H}^n)$.

The main idea to show our results is to apply the discrete Calderón reproducing formula, almost orthogonal estimates associated with the flag structure and the Fefferman–Stein vector valued maximal function.

**Notations:** Throughout this paper, $\mathbb{N}$ will denote the set of all nonnegative integers. For any function $f$ on $\mathbb{H}^n$, we define $\bar{f}(g) = f(g^{-1})$ and $f^\vee(g) = \bar{f}(g) = \bar{f}(g^{-1})$, $g \in \mathbb{H}^n$. If $h$ is a fixed point on $\mathbb{H}^n$, we define the function $f_h$ by $f_h(g) := f(h \circ g)$, $g \in \mathbb{H}^n$. Finally, if $f$ is a function or distribution on $\mathbb{H}^n$ and $r > 0$, we set $D_r f(g) = r^{2n+2} \bar{f}(\delta_r(g))$.

**2. Proof of Theorem 1.4**

Note that it was proved in [11] [12] that $L^2(\mathbb{H}^n) \cap H^p_{flag}(\mathbb{H}^n)$ is dense in $H^p_{flag}(\mathbb{H}^n)$. To show Theorem 1.4 by the Definition 1.3 of the flag Hardy space, it suffices to prove that there exists a constant $C$ such that for every $f \in L^2(\mathbb{H}^n) \cap H^p_{flag}(\mathbb{H}^n)$,

$$
\left\| \left\{ \sum_{Q \in Q} \left| \psi_Q \ast T(f)(z_Q, u_Q) \right|^2 \chi_Q(z, u) \right\} \right\|_p \leq C \| f \|_{H^p_{flag}(\mathbb{H}^n)}
$$

and

$$
\left\| \left\{ \sum_{\mathcal{R} \in \mathcal{R}_{pert}} \left| \psi_{\mathcal{R}} \ast T(f)(z_{\mathcal{R}}, u_{\mathcal{R}}) \right|^2 \chi_{\mathcal{R}}(z, u) \right\} \right\|_p \leq C \| f \|_{H^p_{flag}(\mathbb{H}^n)}.
$$

To achieve the estimates in (2.1) and (2.2), we need the almost orthogonality estimates and a new version of discrete Calderón-type reproducing formula. We first give the almost orthogonality estimate as follows.

**Lemma 2.1.** Suppose that $\varphi, \phi$ are functions on $\mathbb{H}^n$ satisfying that for all $g \in \mathbb{H}^n$,

$$
\int_{\mathbb{H}^n} \varphi(g) dg = 0, \int_{\mathbb{H}^n} \phi(g) dg = 0,
$$

$$
|\varphi(g)|, |\phi(g)| \leq C \frac{1}{(1 + \rho(g))^{2n+3}},
$$

$$
|\nabla_{z} \varphi(g)|, |\nabla_{z} \phi(g)| \leq C \frac{1}{(1 + \rho(g))^{2n+4}}, \text{ and}
$$

$$
\left| \frac{\partial}{\partial t} \varphi(g) \right|, \left| \frac{\partial}{\partial t} \phi(g) \right| \leq C \frac{1}{(1 + \rho(g))^{2n+5}}.
$$
Then for any $\varepsilon \in (0,1)$, there is a constant $C > 0$ such that for all $j,j' \in \mathbb{Z}$,

$$|\varphi_j * \phi_{j'}(g)| \lesssim 2^{-|j-j'|\varepsilon} \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + \rho(g))^{2n+3}},$$

where $\varphi_j := (D_2 \varphi)(g) = 2^{j(2n+2)}\varphi(\delta_{2^j}(g))$.

The proof of Lemma 2.1 is routine and we omit the details of the proof.

**Lemma 2.2.** Suppose $K$ is a classical Calderón–Zygmund kernel and $\psi^{(1)}$ is a smooth function on $\mathbb{R}^n$ with support in $B(0,1/100\gamma b)$ (where $\gamma > 1$ is the constant in the quasi-triangle inequality for the “norm”) and $b > 1$ is the constant in the stratified mean value theorem [9], and $\int_{\mathbb{R}^n} \psi^{(1)}(g)dg = 0$. Then for any $\varepsilon \in (0,1)$, there is a constant $C > 0$ such that for any $0 < \varepsilon < 1$ and all $j,j' \in \mathbb{Z}$,

$$|\psi^{(1)}_j * K * \psi^{(1)}_{j'}(g)| \lesssim 2^{-|j-j'|\varepsilon} \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + \rho(g))^{2n+3}},$$

where $\psi^{(1)}_j(g) := (D_2 \psi^{(1)})(g) = 2^{j(2n+2)}\psi(\delta_{2^j}(g))$.

**Proof.** We first recall that there is a constant $C_i$ independent of $j$ such that

$$|(D_{2^{-j}} K) * \psi^{(1)}(g)| \leq C \frac{1}{(1 + \rho(g))^{2n+3}}.$$  

See [17] for the detail of the proof. Note that we also have

$$|\psi^{(1)}_j * (D_{2^{-j}} K)(g)| \lesssim \frac{1}{(1 + \rho(g))^{2n+3}}.$$  

Indeed, this follows from (2.4), the observation $\psi^{(1)}_j * (D_{2^{-j}} K)(g) = (D_{2^{-j}} K) * \tilde{\psi}^{(1)}(g^{-1})$, and the fact that $\tilde{K}$ satisfies the same size, smoothness, and cancellation conditions to $K$.

Now we can derive (2.3) from (2.4) and (2.5). To see this, we write

$$\psi^{(1)}_j * K * \psi^{(1)}_{j'} = (D_2 \psi^{(1)}_j) * K * (D_{2^{j'}} \psi^{(1)}_{j'})$$

$$= \begin{cases} (D_2 (\psi^{(1)}_j * (D_{2^{-j}} K)) * \psi^{(1)}_{j'}) & \text{if } j \geq j', \\ (D_2 \psi^{(1)}_j) * D_{2^{j'}} [(D_{2^{-j}} K) * \psi^{(1)}_{j'}] & \text{if } j < j'. \end{cases}$$

Thus by Lemma 2.1 we obtain

$$|\psi^{(1)}_j * K * \psi^{(1)}_{j'}(g)| \lesssim \begin{cases} 2^{-|j-j'|\varepsilon} \frac{2^{-j'}}{(2^{-j} + \rho(g))^{2n+3}} & \text{if } j \geq j', \\ 2^{-|j-j'|\varepsilon} \frac{2^{-j}}{(2^{-j'} + \rho(g))^{2n+3}} & \text{if } j < j'. \end{cases}$$

for any $\varepsilon \in (0,1)$. The proof of Lemma 2.2 is concluded. \qed

The key estimate is the following

**Lemma 2.3.** Let $\psi^{(1)}$ be as in Lemma 2.2 and let $\psi^{(2)} \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \psi^{(2)}(u)du = 0$. Set $\psi^{(1)}_j(g) := 2^{j(2n+2)}\psi^{(1)}(\delta_{2^j}(g))$, $\psi^{(2)}_k(u) := 2^k \psi^{(2)}(2^ku)$, and $\psi_{j,k}(g) = \psi_{j,k}(z,u) := [\psi^{(1)}_j(z,\cdot) * \psi^{(2)}_k(u)](t) = \int_{\mathbb{R}} \psi^{(1)}_j(z,t-u)\psi^{(2)}_k(u)du$. Then, for $\varepsilon \in (0,1)$,

$$|\psi_{j,k} * K * \psi^{(1)}_{j',k'}(z,t)|$$

...
We write

\[ |\psi_{j,k}^* K \ast \psi_{j',k'}^* (z,t)| \lesssim 2^{-|k-k'|} (2^{-\langle j,j'\rangle} + |z|^2 + |t-u|^{(n+1)+\frac{1}{2}})^{2-(k\wedge k')/4} \]

and

\[ |\psi_{j,k}^* K \ast \psi_{j',k'}^* (z,t)| \lesssim 2^{-|k-k'|} (2^{-\langle j,j'\rangle}/2 + |t-u|^{(n+1)+\frac{1}{2}} + \sqrt{|t|})^{2+(k\wedge k')/4} \]

if \(2(j \wedge j') \leq k \wedge k'\).}

**Case 1:** If \(2(j \wedge j) \geq k \wedge k'\) and \(|t| \geq 2^{-\langle k\wedge k'\rangle}\), write

\[
\int_{\mathbb{R}} \frac{2^{-\langle j,j'\rangle}}{(2^{-\langle j,j'\rangle}/2 + |t-u|^{(n+1)+\frac{1}{2}} + |t|)^{2-(k\wedge k')/4}} du = I + II.
\]

It is easy to see that

\[
|I| \lesssim \frac{2^{-\langle j,j'\rangle}}{(2^{-\langle j,j'\rangle} + |z|^2 + |t-u|^{(n+1)+\frac{1}{2}})^{2-(k\wedge k')/4}} \lesssim \frac{2^{-\langle j,j'\rangle}/2}{(2^{-\langle j,j'\rangle}/2 + |t|^{1+\frac{1}{2}})^{2-(k\wedge k')/4}} \lesssim \frac{2^{-\langle j,j'\rangle}/2}{(2^{-\langle j,j'\rangle}/2 + |z|^{2n+\frac{1}{2}})^{2-(k\wedge k')/4}}.
\]

Next, we estimate

\[
|II| \lesssim \frac{2^{-\langle k\wedge k'\rangle}}{(2^{-\langle k\wedge k'\rangle} + |t|^{2})^{2}} \int_{\mathbb{R}} \frac{2^{-\langle j,j'\rangle}}{(2^{-\langle j,j'\rangle}/2 + |z|^2 + |t-u|^{(n+1)+\frac{1}{2}})^{2-(k\wedge k')/4}} du \lesssim \frac{2^{-\langle k\wedge k'\rangle}}{(2^{-\langle k\wedge k'\rangle} + |t|^{2})^{2}} \int_{\mathbb{R}} \frac{2^{-\langle j,j'\rangle}/2}{(2^{-\langle j,j'\rangle}/2 + |t-u|^{(n+1)+\frac{1}{2}} + |t|)^{1+\frac{1}{4}}} du \lesssim \frac{2^{-\langle j,j'\rangle}/2}{(2^{-\langle j,j'\rangle} + |z|^{2n+\frac{1}{2}})^{2-(k\wedge k')/4}}.
\]
Case 2: If \(2(j \land j') \geq k \land k'\) and \(|t| \leq 2^{-(k \land k')}\), then

\[
\int_{\mathbb{R}} \frac{2^{-j \land j'}}{2^{-(k \land k')}} \frac{2^{-(k \land k')}}{2} \frac{2^{-(k \land k')}}{2} (2^{-(k \land k')} + |u|)^{2} \, du \\
\lesssim \frac{1}{2^{-(k \land k')}} \int_{\mathbb{R}} \frac{2^{-2(j \land j')}}{2^{-j \land j'}} \frac{2^{-j \land j'}}{2} (2^{-(k \land k')} + |t-u|)^{(n+1)+\frac{1}{2}} (2^{-(k \land k')} + |u|)^{2} \, du \\
\lesssim \frac{1}{2^{-(k \land k')}} (2^{-(k \land k')} + |t|)^{1+\frac{1}{2}} (2^{-(j \land j')} + |z|)^{2n+\frac{1}{2}}.
\]

Case 3: We now consider the case \(2(j \land j') \leq k \land k'\) and \(|t| \leq 2^{-(j \land j')}\). Then

\[
\int_{\mathbb{R}} \frac{2^{-(j \land j')}}{2^{-(k \land k')}} \frac{2^{-(j \land j')}}{2} (2^{-(j \land j')} + |u|)^{2} \, du \\
\lesssim \frac{1}{2^{-(j \land j')}} (2^{-(j \land j')} + |z|)^{2(1+\frac{1}{2}} + (2^{-(j \land j')} + |t|)^{1+\frac{1}{2}}
\]

\[
= \frac{2^{-(j \land j')}}{2^{-(j \land j')}} (2^{-(j \land j')} + |z|)^{2n+\frac{1}{2}} (2^{-(j \land j')} + \sqrt{|t|})^{2+\frac{1}{2}}.
\]

Case 4: If \(2(j \land j') \leq k \land k'\) and \(|t| \geq 2^{-(j \land j')}\), write

\[
\int_{\mathbb{R}} \frac{2^{-(j \land j')}}{2^{-(k \land k')}} \frac{2^{-(j \land j')}}{2} (2^{-(j \land j')} + |u|)^{2} \, du \\
= \int_{|u| \leq \frac{1}{2}|t|, \text{ or } |u| \geq 2|t|} \int_{\frac{1}{2}|t| \leq |u| \leq 2|t|} = I + II.
\]

It is easy to see that

\[
|I| \lesssim \frac{2^{-(j \land j')}}{2^{-(2(j \land j'))} + |z|^{2} + |t|)^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|)^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|)^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|)^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|)^{(n+1)+\frac{1}{2}}}
\]

To estimate \(II\), we have

\[
|II| \lesssim \frac{2^{-(k \land k')}}{2^{-(2(k \land k')} + |t|^{2}} \int_{\mathbb{R}} \frac{2^{-(j \land j')}}{2^{-(2(j \land j'))} + |z|^{2} + |t|^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(k \land k')}}{2^{-(2(k \land k')} + |t|^{2}} \int_{\mathbb{R}} \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|^{(n+1)+\frac{1}{2}}}
\]

\[
\lesssim \frac{2^{-(k \land k')}}{2^{-(2(k \land k')} + |t|^{2}} \int_{\mathbb{R}} \frac{2^{-(j \land j')/2}}{2^{-(2(j \land j'))/2} + |z|^{2} + |t|^{(n+1)+\frac{1}{2}}}
\]

This finishes the proof. \(\Box\)
Now we prove the following new version of discrete Calderon’s reproducing formula.

**Theorem 2.4.** Suppose $0 < p < 1$. For any given $f \in L^2(\mathbb{H}^n) \cap H^p_{\text{flag}}(\mathbb{H}^n)$, there exists $h \in L^2(\mathbb{H}^n) \cap H^p_{\text{flag}}(\mathbb{H}^n)$ such that, for a sufficiently large integer $N \in \mathbb{N}$,

\[
(2.6) \quad f(z,u) = \sum_{j,k \in \mathbb{Z}} \sum_{\ell(I) = 2^{-j-N}, \ell(J) = 2^{-j-N}} |R| \psi_{j,k}(z,u) (z_1, u_j)^{-1}(\psi_{j,k} * h)(z_1, u_j),
\]

where the series converges in $L^2(\mathbb{H}^n)$ and $z_1, u_j$ are any fixed points in $I, J$, respectively. Moreover,

\[
(2.7) \quad \|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)} \approx \|h\|_{H^p_{\text{flag}}(\mathbb{H}^n)}, \quad \|f\|_{L^2(\mathbb{H}^n)} \approx \|h\|_2.
\]

**Proof.** Following [11] (see also [12]) and beginning with the Calderon reproducing formula in [13], we discretize the one-parameter structure of the Heisenberg group for $T_{\alpha}$, which holds for $f \in L^2(\mathbb{H}^n)$ and converges in $L^2(\mathbb{H}^n)$, for any given $\alpha > 0$, as follows:

\[
f(z,u) = \int_0^\infty \int_0^\infty \tilde{\psi}_{s,t} * f(z,u) \, ds \, dt,
\]

where

\[
\psi_{j,k} = \tilde{\psi}_{2^{-\alpha j}, 2^{-2\alpha k}},
\]

\[
c_\alpha = \int_{2^{-\alpha(j+1)}}^{2^{-\alpha j}} \int_{2^{-2\alpha(k+1)}}^{2^{-2\alpha k}} \frac{dt \, ds}{t \, s} = \ln \frac{2^{-\alpha j}}{2^{-\alpha(j+1)}} \ln \frac{2^{-2\alpha k}}{2^{-2\alpha(k+1)}} = 2(\alpha \ln 2)^2.
\]

We further discretize the terms $T_{\alpha}^{(1)} f(z,u)$ and $T_{\alpha}^{(2)} f(z,u)$ in different ways, exploiting the one-parameter structure of the Heisenberg group for $T_{\alpha}^{(1)}$, and exploiting the implicit product structure for $T_{\alpha}^{(2)}$. More precisely,

\[
T_{\alpha}^{(1)} f(z,u) = \sum_{j,k} \sum_{Q \in \mathcal{Q}(j)} f_Q \psi_Q(z,u) + R_{\alpha,N}^{(1)} f(z,u),
\]

\[
T_{\alpha}^{(2)} f(z,u) = \sum_{j,k} \sum_{R \in \mathcal{R}(j,k)} f_R \psi_R(z,u) + R_{\alpha,N}^{(2)} f(z,u),
\]

where

\[
f_Q = c_\alpha |Q| \psi_{j,k} * f(z_0, u_0), \quad \text{for } Q \in \mathcal{Q}(j) \text{ and } k \geq j,
\]

\[
f_R = c_\alpha |R| \psi_{j,k} * f(z_0, u_0), \quad \text{for } R \in \mathcal{R}(j,k) \text{ and } k < j,
\]

\[
\psi_Q(z,u) = \frac{1}{|Q|} \int_Q \tilde{\psi}_{j,k}((z,u) \circ (z', u')^{-1}) \, dz' \, du', \quad \text{for } Q \in \mathcal{Q}(j) \text{ and } k \geq j,
\]

\[
\psi_R(z,u) = \frac{1}{|R|} \int_R \tilde{\psi}_{j,k}((z,u) \circ (z', u')^{-1}) \, dz' \, du', \quad \text{for } R \in \mathcal{R}(j,k) \text{ and } k < j.
\]
where we denote \( \|TA\| \) the right-hand side of the equality (2.8) as \( \|T\|_{L^p(H^n)^*} \leq C \|T\|_{L^p(H^n)} \). Altogether we have

\[
R^{(1)}_{\alpha,N} f (z, u) = c_\alpha \sum_{j \leq k} \sum_{Q \in Q(j)} \int_Q \tilde{\psi}_{j,k} \left( (z, u) \circ (z', u')^{-1} \right) \times \left[ \psi_{j,k} * f (z', u') - \psi_{j,k} * f (z_{Q, u_Q}) \right] dz' du',
\]

\[
R^{(2)}_{\alpha,N} f (z, u) = c_\alpha \sum_{j > k} \sum_{R \in R(j,k)} \int_R \tilde{\psi}_{j,k} \left( (z, u) \circ (z', u')^{-1} \right) \times \left[ \psi_{j,k} * f (z', u') - \psi_{j,k} * f (z_{R, u_R}) \right] dz' du'.
\]

Recall that we denote by \( Q = \bigcup_{j \in \mathbb{Z}} Q(j) \) the collection of all dyadic cubes, and by \( R_{vert} = \bigcup_{j > k} R(j,k) \) the collection of all strictly vertical dyadic rectangles. Finally, we can rewrite the right-hand side of the equality (2.5) as

\[
f(z, u) = \left( \sum_{Q \in Q} f_Q \psi_Q (z, u) + \sum_{R \in R_{vert}} f_R \psi_R (z, u) \right) + \left\{ R_{\alpha} f (z, u) + R^{(1)}_{\alpha,N} f (z, u) + R^{(2)}_{\alpha,N} f (z, u) \right\}
\]

where the series converge in the norm of \( L^2(\mathbb{H}^n) \).

It was proved in [11, 12] that

\[
\|R_{\alpha} f\|_{L^p(H^n)} + \|R^{(1)}_{\alpha,N} f\|_{L^p(H^n)} + \|R^{(2)}_{\alpha,N} f\|_{L^p(H^n)} \leq C 2^{-N} \|f\|_{L^p(H^n)}
\]

for all \( f \in L^p(\mathbb{H}^N) \), \( 1 < p < \infty \),

\[
\|R_{\alpha} f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)} + \|R^{(1)}_{\alpha,N} f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)} + \|R^{(2)}_{\alpha,N} f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)} \leq C 2^{-N} \|f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)}
\]

for all \( f \in \mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n) \).

Thus, we have

\[
\|\left\{ R_{\alpha} + R^{(1)}_{\alpha,N} + R^{(2)}_{\alpha,N} \right\} (f)\|_{L^2(\mathbb{H}^n)} \leq C 2^{-N} \|f\|_{L^2(\mathbb{H}^n)}.
\]

Next we claim that

\[
\left\| \left\{ R_{\alpha} + R^{(1)}_{\alpha,N} + R^{(2)}_{\alpha,N} \right\} (f)\right\|_{H^p_{\text{flag}}(\mathbb{H}^n)} \leq C 2^{-N} \|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)}.
\]

Indeed, the above claim follows from the following general result:

**Proposition 2.5.** If \( T \) is a bounded operator on \( L^2(\mathbb{H}^n) \) and molecular space \( \mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n) \), then \( T \) is bounded on \( H^p_{\text{flag}} \). Moreover,

\[
\|T(f)\|_{H^p_{\text{flag}}} \leq C \left( \|T\|_{2,2} + \|T\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}} \right) \|f\|_{H^p_{\text{flag}}},
\]

where we denote \( \|T\|_{2,2} \) for the operator norm of \( T \) on \( L^2(\mathbb{H}^n) \) and \( \|T\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}} \) for the operator norm on the molecular space \( \mathcal{M}^{M'+\delta}_{\text{flag}} \).
Lemma 6 in [12], we have

To estimate the term $A$

Since $T_{\tilde{\psi}}$, we have

We have

Thus,

Thus, by (1.6), it follows that

Proposition 2.5 follows from the discrete Calderon’s reproducing formula (1.6) (Theorem 3 in [11]) and the almost orthogonality estimates (Lemma 6 in [11]). We only give an outline of the proof.

Suppose $f \in L^2(\mathbb{H}^n) \cap H^p_{flag}(\mathbb{H}^n)$. By (1.6), it follows that

Thus,

To estimate the term $A_1$, note that

We have

Since $T$ is bounded on the molecular space $\mathcal{M}^{M' + \delta}_{flag}(\mathbb{H}^n)$, we obtain that $T_{\tilde{\psi}'}$ satisfies the same conditions as $\tilde{\psi}'$, and does with an extra constant $\|T\|_{\mathcal{M}^{M' + \delta}_{flag}(\mathbb{H}^n)}$. Thus, by Lemma 6 in [12], we have

Thus, we obtain that
Proof of Theorem 1.4.

Thus, choosing
\[ A \]
Similarly we can estimate the terms
\[ A \]
set
\[ h \in T \]
and
\[ f \]
for
\[ f \]
and
\[ f \]
see Theorem 4 in [12], we obtain that
\[ A_1 \leq C \left( \|T\|_{L^2 + |T|} \|M^{p+\delta}_{\text{flag}} \|_{L^p} \|f\|_{L^p} \|H^p_{\text{flag}}(\mathbb{H}^n) \right). \]

Thus, choosing \( N \) large enough implies that \( T_N \) is invertible and \( T_N^{-1} \) is bounded on \( H^p_{\text{flag}}(\mathbb{H}^n) \). Set \( h = T_{a,k} \).

Now by Proposition 2.5 we obtain that the claim \( 2.10 \) holds, which implies that
\[ \|R_N(f)\|_{H^p_{\text{flag}}(\mathbb{H}^n)} \leq C 2^{-N} \|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)}. \]

We now return to Theorem 1.4.

Proof of Theorem 1.4. We first verify \( 2.2 \). To this end, applying the discrete version of the reproducing formula \( 2.6 \) for \( f \) in the term \( \psi_{R}^j * T(f)(z_R, u_R) \) given in \( 2.2 \) implies that
\[ \psi_{R}^j * T(f)(z_R, u_R) = \psi_{R}^j * K \]
\[ * \left( \sum_{j',k' \in \mathbb{Z}} \sum_{R'=I',J', \ell(I')=2^{-j'-N}, \ell(J')=2^{-j'-N}} \|R'\|_{\psi_{J',k'}^j((x,y) \circ (x_I', y_J')^{-1})(\psi_{J',k'}^j \cdot h)(x_I', y_J')} (z_R, u_R) \right) \]
\[ = \sum_{j',k' \in \mathbb{Z}} \sum_{R'=I',J', \ell(I')=2^{-j'-N}, \ell(J')=2^{-j'-N}} \|R'\|_{\psi_{R}^j * K * \psi_{J',k'}^j((z_R, u_R) \circ (x_I', y_J')^{-1})(\psi_{J',k'}^j \cdot h)(x_I', y_J')} (z_R, u_R) \]

Then, by Lemma 2.3 to the term \( \psi_{R}^j * K * \psi_{J',k'}^j((z_R, u_R) \circ (x_I', y_J')^{-1}) \) in the right-hand side of the last equality above, we obtain that
\[ \|\psi_{R}^j * T(f)(z_R, u_R)\| \leq \sum_{j',k' \in \mathbb{Z}} \sum_{R'=I',J', \ell(I')=2^{-j'-N}, \ell(J')=2^{-j'-N}} |R'| \frac{2^{-(j \wedge j')/2}}{(2^{-(j \wedge j')} + |z_R - x_I'|)^{2n+\frac{3}{2}}} \]
Applying H"older's inequality and Fefferman-Stein vector valued maximal inequality and summing over 

\[ \sum_{j', k' \in \mathbb{Z}} 2^{-|j-j'|2-|k-k'|} \left( \sum_{R' \in I' \times J', \ell(J') = 2^{-j'-N}, \ell(J) = 2^{-j-N} + 2^k - N} |R'| \left( 2^{-2k'}/4 + |z_R - x_R| \right)^{2n+1} \right)^{\frac{1}{4}} \left( 2^{-2k'/2} \left( 2^{-2(j \wedge j')/2} + \sqrt{|u_R - y_J|} \right)^{2n+1} \right) \]

and

\[ |\psi_{j'} \ast T(f)(z_R, u_R)| \leq C \sum_{j', k' \in \mathbb{Z}} 2^{-|j-j'|2-|k-k'|} \left( \sum_{R' \in I' \times J', \ell(J') = 2^{-j'-N}, \ell(J) = 2^{-j-N} + 2^k - N} |R'| \left( 2^{-2k'}/4 + |z_R - x_R| \right)^{2n+1} \right)^{\frac{1}{4}} \left( 2^{-2k'/2} \left( 2^{-2(j \wedge j')/2} + \sqrt{|u_R - y_J|} \right)^{2n+1} \right) \]

Using Lemma 7 in [11] [12], for \( \frac{4n}{4n+1} < r < p \) and any \((z_R^*, u_R^*) \in R\), we get that

\[ |\psi_{j'} \ast T(f)(z_R, u_R)| \leq C \sum_{j', k' \in \mathbb{Z}} 2^{-|j-j'|2-|k-k'|} \left( \sum_{R' \in I' \times J', \ell(J') = 2^{-j'-N}, \ell(J) = 2^{-j-N} + 2^k - N} |R'| \left( 2^{-2k'}/4 + |z_R - x_R| \right)^{2n+1} \right)^{\frac{1}{4}} \left( 2^{-2k'/2} \left( 2^{-2(j \wedge j')/2} + \sqrt{|u_R - y_J|} \right)^{2n+1} \right) \]

where \( M \) is the Hardy-Littlewood maximal function and \( M_s \) is the strong maximal function on \( \mathbb{R}^n \), respectively.

Applying Hölder's inequality and Fefferman-Stein vector valued maximal inequality and summing over \( R \in R_{\text{vert}} \) yield

\[ \left\{ \sum_{R \in R_{\text{vert}}} |\psi_{j'} \ast T(f)(z_R, u_R)|^2 \chi_R (z, u) \right\}^{\frac{1}{2}} \]

\[ \leq C \left\{ \sum_{R \in R_{\text{vert}}} \left( \sum_{j', k' \in \mathbb{Z}} 2^{-|j-j'|2-|k-k'|} \left( \sum_{R' \in I' \times J', \ell(J') = 2^{-j'-N}, \ell(J) = 2^{-j-N} + 2^k - N} |R'| \left( 2^{-2k'}/4 + |z_R - x_R| \right)^{2n+1} \right)^{\frac{1}{4}} \left( 2^{-2k'/2} \left( 2^{-2(j \wedge j')/2} + \sqrt{|u_R - y_J|} \right)^{2n+1} \right) \right\}^{\frac{1}{2}} \]
\[
\leq C \left\| \sum_{j', k' \in \mathbb{Z}} \sum_{\ell(I')=2^{-j'}-N, \ell(J')=2^{-j'}-N+2^{-k'}-N} \left( \psi_{j', k'} \ast h \right)(x_{I'}, y_{J'})^{2} \chi_{I'} \chi_{J'} \right\|_{p}
\]

\[
\leq C \| h \|_{H^{p}(\mathbb{R}^{n})}
\]

\[
\leq C \| f \|_{H^{p}(\mathbb{R}^{n})}.
\]

The proof for (2.1) is similar and easier. The proof of Theorem 1.4 is concluded. □

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