Covariant quantization of the Maxwell field in de Sitter space from SO$_0(2,4)$-invariance

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Abstract. We present a SO$_0(2,4)$-invariant quantization of the free electromagnetic field in de Sitter space. Precisely, we quantize the Maxwell (“massless spin one”) de Sitter field in a conformally invariant gauge. This result is obtained thanks to a canonical quantization scheme of the Gupta-Bleuler type and to a geometrical formalism in which the Minkowski, de Sitter and anti-de Sitter spaces are realized as intersections of the five dimensional null cone of $\mathbb{R}^6$ and a moving hyperplane. We obtain a new and simple de Sitter invariant two-point function.

1. Introduction and methodology

The aim of the work presented in [1] (to which the Reader is referred for details) is to provide a quantization of the free Maxwell field in de Sitter space in which the SO$_0(2,4)$-invariance is preserved. To this end we quantize the one-form field $A_\mu^H$ which fulfills the de Sitter Maxwell equations together with a conformal gauge condition:

\[
\begin{align*}
\Box H A_\mu^H - \nabla_\mu \nabla A^H + 3H^2 A_\mu^H &= 0 \\
(\Box H + 2H^2) \nabla A^H &= 0,
\end{align*}
\]

where $12H^2 = R$, $R$ being the Ricci scalar and $\Box H$ the usual Laplace-Beltrami operator. As a result, we obtain the following de Sitter-covariant two-point function

\[
D_{\mu\nu}(p,p') = \frac{H^2}{8\pi^2} \left( \frac{1}{Z-1} g_{\mu\nu} - n_\mu n_\nu \right),
\]

where $g_{\mu\nu}(p,p')$ is the parallel propagator, $Z$, $n_\mu$, $n_\nu$ being related to the geodesic distance between the two points $p$ and $p'$ of the de Sitter space (see Appendix C).

The problem posed by such a quantization is twofold. First, we have to set up a geometrical framework in which the SO$_0(2,4)$-invariance is easily implemented. Then, we have to deal, in that context, with the known problem of quantizing an Abelian gauge field, that is to say to keep manifest the invariance in the quantized field.

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To answer the first part of the problem we use the formalism already described in [2, 3], which are summarized in Sec. 2. Essentially, we consider the null cone in $\mathbb{R}^6$ which is the natural geometric object related to the $SO_0(2,4)$-invariance. Then, we realize the de Sitter and Minkowski spaces as the intersections of the light cone and a moving hyperplane whose position is parameterized by the curvature $H$.

The second side of the problem is well known in the case of the usual covariant canonical quantization of the electromagnetic field in Lorenz gauge in Minkowski space. In short, it is not possible to build a quantum field operator which satisfies at the same time both the Maxwell equations and the Lorenz condition. A solution to this problem is to quantize a field which, in place of the Maxwell equations, satisfies another set of equations whose set of solutions contains as an invariant subset the solutions of the Maxwell equations in the Lorenz gauge. The quantum states which correspond to this subset are then specified by a condition which corresponds at the quantum level to the gauge condition. In that case the additional solutions correspond to the longitudinal and scalar photons and the quantum condition allows to select the physical transverse photons. This method is essentially the Gupta-Bleuler quantization [4, 5]. On a technical side, that method is implemented by adding a gauge-fixing term in the Lagrangian which leads to the equation mentioned above whose set of solutions is larger than that of the initial problem. In our conformal framework, that scheme is implemented in a different way which is an extension of the method used in [6]. There is no gauge-fixing term in the Lagrangian, instead, the Maxwell equations and the covariant gauge condition are obtained from a larger set of equations which in addition to the Maxwell field contains two more auxiliary fields. They encode both the gauge condition and transversality constraints. This is described in Sec. 3 and 4.

For convenience, our notations and conventions are collected in Appendix A, explicit formulas are given for reference in Appendix B and the definitions of quantities related to the geodesic distance which appears in the Wightman two-point function are given in Appendix C.

2. The geometrical framework in brief
Since our goal is to preserve the $SO_0(2,4)$-invariance the natural geometric object, invariant under the action of that group, is the five dimensional null cone $C$ of $\mathbb{R}^6$

$$\mathcal{C} = \{ y \in \mathbb{R}^6 : (y^0)^2 - y^2 - (y^4)^2 + (y^5)^2 = 0 \}.$$  \hspace{1cm} (3)

The de Sitter and Minkowski spaces are obtained as the intersection $X_H$ of the five dimensional cone $\mathcal{C}$ of and a moving hyperplane $P_H$ whose equation reads $f_H(y) = 2$ where

$$f_H(y) = (1 + H^2)y^5 + (1 - H^2)y^4,$$  \hspace{1cm} (4)

where $H$ is a non-negative real number which identifies with the curvature of $X_H$.

As already shown in [2], the manifold $X_H$ is invariant under a subgroup of $SO(2,4)$ isomorphic to the de Sitter group $SO_0(1,4)$ for $H > 0$ and to the Poincaré group for $H = 0$. As a consequence, the manifold $X_H$ is identified with the de Sitter space for $H > 0$ and Minkowski space for $H = 0$. This realization allows to make a continuous transition between spaces by a continuous change of the value of the parameter $H$. Note that, as a consequence of this property, the zero-curvature limit of all local objects depending on $H$ in the de Sitter space (for instance, group generators, modes, two-point functions,...) is obtained by setting $H = 0$ in the concerned expression.

The anti-de Sitter space is also obtained by this construction. In this work we are interested in quantization and thus we only consider globally hyperbolic spaces. At a classical level our results apply to the anti-de Sitter space as well. The expressions relevant for this space can be obtained directly from the substitution $H^2 \rightarrow -H^2$.  

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Figure 1. The main construction: to each point of $X_H = P_H \cap C$ corresponds a unique half-line $\Delta$, that is a point of $C'$.

In this construction each point of a manifold $X_H$ is associated to a half-line issued from the origin of $\mathbb{R}^6$ (see Fig.1), or, in other words, is associated to a point of the set $C'$ of half-lines, which is the cone up to the dilations. The most important consequence of this fact is that each manifold $X_H$ can be realized as a subset of $C'$. Thus, all the manifolds $X_H$ (i.e., for any value of $H$) are realized on the same underlying set $C'$.

From the point of view of group theory note that thanks to the linearity of the action of $\text{SO}_0(2,4)$, there is a natural action of this group on $C'$. Another consequence of the relation between the points of $X_H$ and the half-lines is that the functions on $X_H$ and the homogeneous functions on the cone $C$ are in one-to-one correspondence. Precisely, if $y$ is a point of the cone and $y^H$ the intersection of $X_H$ with the half-line linking $y$ to the origin of $\mathbb{R}^6$, then one can verify using the equations defining $C$ and $P_H$ that

$$y^H = 2y/f_H(y).$$

Thus, to any $\tilde{F}$ homogeneous function of degree $r$ defined on $C$ corresponds a function $F^H$ on $X_H$ defined through

$$F^H(x) = \left(\frac{2}{f_H(y)}\right)^r \tilde{F}(y),$$

where $y \in C$ and $x = y^H$ is the corresponding point of $X_H$.

Finally, the metric $g_{\mu\nu}^H$ on $X_H$ is inherited from $\eta_{\alpha\beta}$ on $\mathbb{R}^6$. For $H = 0$ this metric is the Minkowskian metric, while for $H > 0$ it is a de Sitterian metric hereafter denoted $g_{\mu\nu}$. It can be verified, for instance through a straightforward calculation in some coordinate system, that these metrics are related through a (local) Weyl rescaling. In the system $\{x^\mu\}$ (see (9) below) one has

$$g_{\mu\nu} = (K^H)^2 \eta_{\mu\nu}, K^H = \frac{1}{1 - \frac{H^2}{F^2} \eta_{\mu\nu} x^\mu x^\nu}$$

$K^H$ being the Weyl factor and $\eta_{\mu\nu}$ the usual Minkowski metric.
3. The Maxwell field in a conformal gauge

We use the Dirac’s six-cone formalism [7, 8] as a starting point: we consider the SO\(_{(2,4)}\)-invariant homogeneous wave equation

\[ \tilde{\eta}^{\gamma \beta} \tilde{\partial}_\gamma \tilde{\partial}_\beta \tilde{a}_\alpha = 0, \quad (8) \]

where \( \tilde{a} = \tilde{a}_\alpha dy^\alpha \) is a one-form field in \( \mathbb{R}^5 \). As shown in [7, 8], the one-forms \( \tilde{a} \) must be homogeneous of degree \(-1\) in order to consider the field and the equation on the cone \( C \).

In addition, we make use of auxiliary fields (see [6] for instance). Basically, they implement two constraints: they set the SO\(_{(2,4)}\) homogeneous of degree \(-1\) in order to consider the field and the equation on the cone \( C \).

Now, it is apparent that for \( A^H = 0 \) the first line of (12) are precisely the Maxwell equations. Then, eliminating \( A^c \) from the remaining two equations and using the relation

\[ \nabla_\mu (K^H)^2 = (K^H)^2 W_\mu, \]

one obtains

\[ \{ \begin{align*} &\Box H A^H_\mu - \nabla_\mu \nabla A^H - 3H^2 A^H_\mu = -\frac{1}{2} \nabla_\mu (\Box H + 2H^2) A^H_\mu \\ &\nabla - W)A^H + (K^H)^{-2} A^H_\mu = \frac{1}{2} (\Box H + 2H^2) A^H_\mu \\ &\nabla - W)\nabla A^H_\mu = -K^H H^2 (\Box H + 2H^2) A^H_\mu \end{align*} \]

where \( \nabla A^H \) is the divergence of \( A^H \), and \( W \) is the one-form \( W := d \ln (K^H)^2 \) of components \( W_\mu = \nabla_\mu \ln (K^H)^2 \).

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one obtains

\[ \{ \begin{align*} &\Box H A^H_\mu - \nabla_\mu \nabla A^H - 3H^2 A^H_\mu = 0 \\ &\nabla^\mu (W_\mu + W_\nu) (\nabla_\nu - W_\nu) A^H_\nu = 0, \end{align*} \]

where \( \nabla \) means that each component \( a^H_\alpha \) must be considered as a scalar.
Note that, in the above system each equation is invariant under both the SO$_0(2,4)$
transformations and the Weyl rescaling. One can show (see [1]) that this system is equivalent to
\begin{equation}
\begin{cases}
\square_H A^\mu_H - \nabla_\mu \nabla A^\mu_H + 3H^2 A^\mu_H = 0 \\
(\square_H + 2H^2) \nabla A^\mu_H = 0.
\end{cases}
\end{equation}

In the second line we recognize the Eastwood-Singer gauge [9] specialized to our constant
curvature space $X_H$. The gauge condition of the second line is invariant under both the SO$_0(2,4)$
group and Weyl transformations only on the space of solutions of the Maxwell equations.

Two comments are in order before closing this section. First, it is apparent that the condition
$A^\mu_+ = 0$ characterize the subset of solutions of the system (12) which verify both Maxwell’s
equations and the conformal gauge condition. This condition is invariant under the SO$_0(2,4)$
transformation since it can be shown that $\tilde{A}_+ = y^\alpha \tilde{a}_\alpha$ (see [1] again). Finally, the system (12)
which is the the system we will quantize is in fact the equation (11) in which $a^\mu$ appears as a
conformal scalar field.

4. Quantization

Let us consider a tensor field $F$ satisfying some linear equations: $\mathcal{E}F = 0$ on $X_H$. When $F$ is
not an abelian gauge field, one can obtain a covariant and causal quantized field $\hat{F}$ satisfying
the equations $\mathcal{E}\hat{F} = 0$ by applying the following procedure:

(i) Select a Hilbert space $\mathcal{K}$ of solutions of the classical equation equipped with a scalar product
$\langle \cdot, \cdot \rangle$ and carrying a unitary representation of the symmetry group.

(ii) Obtain a causal reproducing kernel $\mathcal{W}$ for $\mathcal{K}$, the Wightman two-point function. More
precisely, $\mathcal{W}$ is a bitensor such that, for each $x \in X_H$, $\mathcal{W}(x, \cdot) : x' \to \mathcal{W}(x, x')$ is, up to a
smearing function on the variable $x$, an element of $\mathcal{K}$ satisfying
\begin{equation}
\langle \mathcal{W}(x, \cdot), \psi \rangle = \psi(x),
\end{equation}
for any $\psi \in \mathcal{K}$, and such that $\mathcal{W}(x, x') = \mathcal{W}(x', x)$ as soon as $x$ and $x'$ are causally separated.

(iii) Define the quantum field $\hat{F}$ through
\begin{equation}
\hat{F}(x) = a \mathcal{W}(x, \cdot) + a^\dagger \mathcal{W}(x, \cdot),
\end{equation}
where $a$ and $a^\dagger$ are the usual creator and annihilator of the Fock space built onto $\mathcal{K}$.

In practice, one determines a family of modes $\{\phi_k\}$ solutions of $\mathcal{E}F = 0$ and finds a scalar
product such that $\{\phi_k\}$ is an orthonormal Hilbert basis for $\mathcal{K}$. Then, the two-point function
reads
\begin{equation}
\mathcal{W}(x, x') = \sum_k \phi_k(x) \otimes \phi_k(x').
\end{equation}

From this expression, using (16) and the anti-linearity and linearity of $a$ and $a^\dagger$ respectively, one
obtains the familiar expression of the quantum field:
\begin{equation}
\hat{F}(x) = \sum_k \phi_k(x) b_k + \phi^*_k(x) b_k^\dagger,
\end{equation}
where $b_k : = a(\phi_k)$ and $b_k^\dagger : = a^\dagger(\phi_k)$ are the annihilators and creators of the modes $\phi_k$. The
Hilbert space of quantum states $|\psi\rangle$, is then built as usual through the action of the $b_k^\dagger$ on the
vacuum state of the theory.

As already mentioned in the introduction, in gauge context, due to the presence of pure
gauge solutions, such a two-point reproducing kernel does not exist. The reason for that is
the following: the pure gauge solutions (the gradients of complex functions in the case of the Maxwell equations) are known to be orthogonal to any modes including themselves, so replacing $\psi$ by a pure gauge modes in (15) should make vanishing the left hand side and not the right hand side. In the Gupta-Bleuler scheme, one overcomes this problem by considering an enlarged space $H \supset K$ containing some elements not orthogonal to the pure gauges. This space, which is generally a Krein space, is that is a space equipped with an indefinite inner product \cite{10, 11}, is defined through another equations $E' F = 0$ also invariant under the group. The elements of $K$, called in this context the physical solutions, satisfy, in addition to the new field equation, a constraint $GF = 0$. This classical condition, which allows to characterize the classical physical solutions, translates into a quantum condition which allows in turn to characterize the subspace of physical states. For instance, in the usual Gupta-Bleuler quantization of the Maxwell field in Minkowski space: the equation defining the enlarged space $H$ is $\Box A = 0$, the space $H$ contains modes with negative norm (those corresponding to the scalar photons); the subspace of physical states, for which $H \supset K$, is defined through another equations $E' F = 0$ also invariant under the group. The elements of $K$, called in this context the physical solutions, satisfy, in addition to the new field equation, a constraint $GF = 0$. This classical condition, which allows to characterize the classical physical solutions, translates into a quantum condition which allows in turn to characterize the subspace of physical states. For instance, in the usual Gupta-Bleuler quantization of the Maxwell field in Minkowski space: the equation defining the enlarged space $H$ is $\Box A = 0$, the space $H$ contains modes with negative norm (those corresponding to the scalar photons); the subspace of physical solutions (corresponding to the transverse photons) is characterized by the constraint $\Box A = 0$, the Lorenz gauge condition, it’s quantum counterpart reads $\partial A^{(+)} | \psi \rangle = 0$, where $\hat{A}^{(+)}$ is the annihilator part of $\hat{A}$ and $| \psi \rangle$ a physical state. The new quantum field $\hat{F}$ is covariant and causal. It is obtained through

$$\hat{F}(x) = \sum_k \zeta_k \left( \phi_k(x) b_k + \phi_k^*(x) b_k^\dagger \right),$$

where $\zeta_k = \pm 1$. One can show (see \cite{1} again) that it satisfy:

$$\langle \psi_1 | E \hat{F} | \psi_2 \rangle = 0,$$

as soon as $| \psi_1 \rangle, | \psi_2 \rangle$ are physical states.

We now apply this quantization process to the Maxwell de Sitter field in conformal gauge (14). As for the non conformal case, it can be shown that the pure gauge solutions $(A^\mu_H = \nabla_H f, \text{with } (\Box_H + 2H^2) \Box_H f = 0, f \text{ being an arbitrary function})$ are orthogonal to all the solutions including themselves. As a consequence, the space of solutions of (14) is degenerate and the canonical quantization process fails. Following the Gupta-Bleuler method, we consider the system (12), instead of (14). Now, as we have already pointed out at the end of Sec. 3, the system (12) is nothing but the equation (11) that is, basically, the equation for the conformal scalar field. It follows that the practical quantization scheme described above can be realized directly from the knowledge of the conformal scalar field (see \cite{2} for details):

- The scalar product on the space of solutions of $(\Box_H + 2H^2) a^H = 0$ is defined through

$$\langle a^H, b^H \rangle := -\tilde{\eta}^{\alpha\beta} \langle a^H_\alpha, b^H_\beta \rangle_s$$

where $\langle . \rangle_s$ is (with a slightly different notation from that used in \cite{2}) the Klein-Gordon scalar product on the space of solutions of the conformal scalar equation on $X_H$.

- The modes on $X_H$ reads

$$a_{LM(\gamma)}^H(x) = \epsilon_{(\gamma)} \Phi_{LM}^H(x),$$

where the one-forms $\epsilon_{(\gamma)}$ are defined through $\epsilon_{(\gamma)\beta} = -\tilde{\eta}_{\gamma\delta}$ and $\Phi_{LM}^H(x)$ are the modes solutions of the scalar equation $(\Box_H + 2H^2) \Phi^H = 0$ (see \cite{2} for details). These solutions are normalized with respect to (21) precisely

$$\langle a_{LM(\gamma)}, a_{LM'(\delta)}^H \rangle = -\tilde{\eta}^{\gamma\delta} \delta_{LM} \delta_{LM'}.\tag{23}$$

- The Wightman two-point function is obtained through the formula

$$W^H = \sum_{LM\gamma} \zeta_\gamma \epsilon_{(\gamma)} (\Phi_{LM}^H)^* \otimes \epsilon_{(\gamma)} \Phi_{LM}^H,$$
with \( \zeta = -\tilde{\eta}_{\gamma\gamma} \), applied to the above modes. A straightforward calculation using the results of [2] gives

\[
W_{\alpha\beta}^H(x, x') = -\tilde{\eta}_{\alpha\beta} D_{\alpha\beta}^H(x, x'),
\]

where \( D_{\alpha\beta}^H(x, x') \) is the usual two-point function for the conformal scalar field in de Sitter space (see Appendix B).

- The quantum field \( \hat{a}^\mu \) reads

\[
\hat{a}^\mu(x) = \sum_{LM\gamma} a_{LM(\gamma)}^\mu(x) b_{LM(\gamma)} + a_{LM(\gamma)}^{\mu*}(x) b_{LM(\gamma)}^\dagger,
\]

where \( b_{LM(\gamma)} \) and \( b_{LM(\gamma)}^\dagger \) being the annihilators and creators of the mode \( a_{LM(\gamma)}^\mu \).

- The physical states are created from physical solutions:

\[
|\alpha^\mu\rangle \text{ is a physical state iff } |\alpha^\mu\rangle = a^{\dagger}(\alpha^\mu) |0\rangle_H, \quad \text{and } A_+^{\mu} [a^{\mu}] := L^{-1} a^{\mu} = 0.
\]

The quantum field \( \hat{A}^\mu \) and the corresponding two-point function are obtained from the above relations thanks to the linear relations \( A^\mu = L a^\mu \). In particular, The Maxwell de Sitter two-point function defined as usual through

\[
D_{\mu\nu}^H(x, x') = H^2 \langle 0 | \hat{A}_{\mu}^H(x) \hat{A}_{\nu}^H(x') | 0 \rangle_H,
\]

is found to be (see [1] for details)

\[
D_{\mu\nu}^H(x, x') = \frac{H^2}{8\pi^2} \left( \frac{1}{Z_\varepsilon - 1} g_{\mu\nu} - n_\mu n_\nu \right),
\]

where \( Z_\varepsilon := Z - i\varepsilon(x^0 - x'^0) \), \( \varepsilon \) being a small positive real number, and the geometrical objects \( Z, g_{\mu\nu}, n_\mu \) and \( n'_\nu \) are explicitly defined in Appendix C.

Note finally that this two-point function possesses no other singular point than \( Z = 1 \) and has clearly the Hadamard behavior (thus our vacuum is the Euclidean one).

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Appendix A. Conventions and notations

Here are the conventions:

\[
\begin{align*}
\alpha, \beta, \gamma, \delta, \ldots & = 0, \ldots, 5, \\
\mu, \nu, \rho, \sigma, \kappa, \ldots & = 0, \ldots, 3, \\
i, j, k, l, \ldots & = 1, \ldots, 3.
\end{align*}
\]

The indices and superscripts \( I, J \) stand for the set \( \{c, \mu, +\} \), for instance \( \{A_I\} = \{A_c, A_\mu, A_+\} \). The space \( \mathbb{R}^6 \), is provided with the natural orthogonal coordinates \( \{y^\alpha\} \) and the metric \( \tilde{\eta}_{\alpha\beta} = \text{diag}(+, -, -, -, -, +) \). whose coefficients are denoted \( \tilde{\eta}_{\alpha\beta} \):

\[
\tilde{\eta}_{00} = 1 = -\tilde{\eta}_{ii} = -\tilde{\eta}_{44}.
\]

For convenience we set \( \eta_{\mu\nu} := \tilde{\eta}_{\mu\nu} \). Partial derivatives with respect to the variables \( \{y^\alpha\} \) of \( \mathbb{R}^6 \) are denoted by \( \tilde{\partial}_\alpha \).

Various spaces and maps are used throughout this paper. Except otherwise stated, quantities related to \( \mathbb{R}^6 \) and its null cone \( C \) are labeled with a tilde, those defined on \( X_H \) (see Sec. 2 hereafter) are denoted with a super or subscript \( H \) except when \( H \) takes the null value (Minkowski space) in which case the super or subscript 0 is omitted. The quantum operator associated with a classical quantity \( Q \) is denoted with a hat: \( \hat{Q} \).
Appendix B. Explicit formulas

The linear relations between the $A^\mu$ and the $a^\mu$ reads

$$
\begin{align*}
A^H_c &= (K^H)^2 \left\{ a^H_5 (1 - H^2) - a^H_4 (1 + H^2) - H^2 a^H_\mu x^\mu \right\} \\
A^H_\mu &= \frac{(K^H)^2}{2} \left\{ \left( a^H_4 (1 + H^2) - a^H_5 (1 - H^2) \right) \eta_{\mu\nu} x^\nu + H^2 a^H_\sigma x^\sigma \eta_{\mu\nu} x^\nu + \frac{1}{2} K^H a^H_\mu \right\} \\
A^H_+ &= K^H \left\{ a^H_5 \left( 1 - \frac{1}{4} \eta_{\mu\nu} x^\mu x^\nu \right) + a^H_4 \left( 1 + \frac{1}{4} \eta_{\mu\nu} x^\mu x^\nu \right) + a^H_\mu x^\mu \right\},
\end{align*}
$$

from which the matrix $L$ can be read. These relations are obtained by first expressing the basis \{dy\} in the left hand side of (10) in terms of the basis \{dx\} and identifying both sides and then applying the correspondence (6).

The above system can be inverted in

$$
\begin{align*}
a^H_5 &= \frac{1}{2K^H} \left\{ A_c^H \left( 1 - \frac{1}{4} \eta_{\mu\nu} x^\mu x^\nu \right) - A_\sigma^H x^\sigma + A_\mu^H (1 + H^2) \right\} \\
a^H_4 &= \frac{1}{2K^H} \left\{ A_c^H \left( -1 - \frac{1}{4} \eta_{\mu\nu} x^\mu x^\nu \right) - A_\sigma^H x^\sigma + A_\mu^H (1 - H^2) \right\} \\
a^H_\mu &= \frac{1}{2K^H} \left\{ A_c^H \eta_{\mu\nu} x^\nu + 2A_\mu^H \right\}.
\end{align*}
$$

The Wightman two-point function for the conformal scalar field in the de Sitter space reads

$$
D_+^H(x, x') = -\frac{H^2}{8\pi^2} \frac{1}{(Z(x, x') - 1 - i\epsilon \text{sign}(t-t'))},
$$

in which $\epsilon$ is a small positive real number and $Z(x, x')$ is defined in Appendix C.

Appendix C. Intrinsic quantities for bitensors in de Sitter space

Following Allen and Jacobson [12] (where the reader is referred for proofs and details), any maximally symmetric bitensor (that is, invariant under the isometry group of a maximally symmetric manifold, here the de Sitter space) can be decomposed in a unique way as sum of products of fundamental objects. They are: the metric at points $p$ of the manifold and three quantities related to the length $\mu(p, p')$, of the geodesic from $p$ to $p'$ ($\mu$ being imaginary when the geodesic is spacelike), namely:

\begin{align*}
    n_\mu(p, p') = \nabla_\mu \mu(p, p') &\text{ is the unit tangent vector to the geodesic at the point } p, \\
n_\nu(p, p') = \nabla_\nu \mu(p, p') &\text{ is the unit tangent vector to the geodesic at the point } p', \\
g_{\mu\nu}(p, p') = \frac{1}{C} \nabla_\mu n_\nu(p, p') - n_\mu(p, p') n_\nu(p, p') &\text{ is the parallel propagator along the geodesic,}
\end{align*}

where we use the usual convention that a primed (resp. not primed) index refers to a primed (resp. not primed) point. The factor $C$ will be given in what follows.

In order to define the standard variable $Z$, let us introduce the five-dimensional “ambient” Minkowski space with metric $\overline{\eta} = \text{diag}(+, -, -, -, -)$. We will use small roman letters $a, b, c, ...$ to denote indices running from 0 to 4. The de Sitter space can be viewed as the sub-manifold defined by the equation

\begin{equation}
\overline{\eta}_{ab} X^a X^b = -H^{-2},
\end{equation}

where $\{X^a\}$ denotes ambient space cartesian coordinates. A point $p$ on the de Sitter space is associated to the vector $X(p)$ of coordinates $X^a(p)$. The ambient coordinates are related to the coordinates $\{x^\mu\}$ through

\begin{align*}
    X^\mu &= K^\mu x^\mu, \\
    X^4 &= \frac{1}{H} (2K^\mu - 1).
\end{align*}
The function \( Z = Z(p, p') \) is then defined through
\[
Z := -H^2 \bar{\eta}_{ab} X^a X^b, \tag{C.3}
\]
where \( X = X(p) \) and \( X' = X(p') \). The geodesic distance \( \mu(p, p') \) is related to \( Z \) by
\[
Z = \cosh (H \mu), \quad Z \geq -1. \tag{C.4}
\]
The case \( Z < -1 \) corresponds to the situation where \( p' \) is lying in the interior of the light cone of the antipodal of \( p \) and, in this case, there is no geodesic connecting \( p \) and \( p' \). Nevertheless, \( Z \) is always defined and one can define \( \mu(p, p') \) through an analytic continuation (see [12] again). As a function of \( Z \) the factor \( C \) reads
\[
C = \frac{-H}{\sqrt{(Z^2 - 1)}}. \tag{C.5}
\]

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