HOMOTOPY BASES AND FINITE DERIVATION TYPE
FOR SUBGROUPS OF MONOIDS

R. GRAY
School of Mathematics & Statistics, University of St Andrews,
St Andrews KY16 9SS, Scotland.
robertg@mcs.st-andrews.ac.uk

A. MALHEIRO
Centro de Álgebra, da Universidade de Lisboa,
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal.
amheiro@cii.fc.ul.pt

Abstract. Given a monoid defined by a presentation, and a homotopy base
for the derivation graph associated to the presentation, and given an arbitrary
subgroup of the monoid, we give a homotopy base (and presentation) for the
subgroup. If the monoid has finite derivation type (FDT), and if under the action
of the monoid on its subsets by right multiplication the strong orbit of the
subgroup is finite, then we obtain a finite homotopy base for the subgroup, and
hence the subgroup has FDT. As an application we prove that a regular monoid
with finitely many left and right ideals has FDT if and only if all of its
maximal subgroups have FDT. We use this to show that a regular monoid
with finitely many left and right ideals satisfies the homological finiteness
condition $FP_3$ if all of its maximal subgroups satisfy the condition $FP_3$.

2000 Mathematics Subject Classification: 20M50 (primary), 20M05, 68Q42 (secondary)

1. Introduction

Geometric methods play an important role in combinatorial group theory (see [15]).
More recently, in [27] Squier (and in independent work Pride [22], and Kilibrarda [12]) has
developed a homotopy theory for monoids. Given a monoid presentation, Squier constructs
a graph, called a derivation graph, and then defines certain equivalence relations called
homotopy relations on the set of paths in the graph. A set of closed paths that generates
the full homotopy relation is called a homotopy base. A monoid defined by a finite presentation
is said to have finite derivation type (FDT) if there is a finite homotopy base for the
presentation. Squier showed that FDT is a property of the monoid, in the sense that it is
independent of the choice of (finite) presentation.

The homotopical finiteness condition FDT was originally introduced as a tool for the
study of finite string rewriting systems. It was shown in [27] that if a monoid is defined
by a finite complete rewriting system then that monoid has FDT. In [13] Kobayashi
showed that every one-relator monoid has FDT; it is still an open question whether every
one-relator monoid admits a finite complete rewriting system. More background on the
relationship with finite complete rewriting systems may be found in [21].

Key words and phrases. Finitely presented groups and monoids, Finiteness conditions, Homotopy bases,
Finite derivation type.

Supported by an EPSRC Postdoctoral Fellowship EP/E043194/1.

This work was developed within the projects POCTI-ISFL-1-143 and PTDC/MAT/69514/2006 of
CAUL, financed by FCT and FEDER.
There are also important connections to the homology theory of monoids. If a monoid has FDT then it must satisfy the homological finiteness condition FP$_3$, and for groups FDT and FP$_3$ are equivalent; see [4], [22], [14] and [5]. In [23] it was shown that for monoids FDT and the homological finiteness condition FHT (in the sense of [31]) are not equivalent. In addition to this, Squier’s homotopy theory has been applied in [9] to the study of diagram groups, which are precisely fundamental groups of Squier complexes of monoid presentations.

One important area in the study of finiteness conditions is the consideration of their closure properties. Given a finiteness condition how does the property holding in a monoid relate to it holding in the substructures of that monoid (and vice versa)? For groups when passing to finite index subgroups, or finite index extensions, the properties of being finitely generated, finitely presented, having FP$_n$, or having FDT are all preserved; see [10], [2]. Results about the behaviour of FDT for monoids, when passing to submonoids and extensions appear in [29], [17], [22], [20], [30] and [19].

In this paper we investigate the relationship between the property FDT holding in a monoid, and the property holding in the subgroups of that monoid. Results of this type have been obtained for the properties of being finitely generated and presented (see [25]), and for being residually finite (see [6]). Specifically, given a monoid $S$ defined by a presentation $\mathcal{P}_S$, given an associated homotopy base $X$, and given a subgroup $G$ of $S$, we show how to obtain a homotopy base $Y$ for a presentation $\mathcal{P}_G$ of $G$ (this will be done in Section 3). Moreover, when the presentation $\mathcal{P}_S$ and the homotopy base $X$ are both finite, and $G$ has only finitely many cosets in $S$ (where a coset of $G$ is a set $Gs$, with $s \in S$, such that there exists $s' \in S$ with $Gss' = G$) then the presentation $\mathcal{P}_G$ and homotopy base $Y$ will both be finite. A subgroup $G$ of a monoid $S$ with finitely many cosets, in the above sense, is said to have finite translational index. This is equivalent to saying that $G$ acts on the left of its $R$-class with finite quotient. This notion of index was first considered in [25] where it was shown that the properties of being finitely generated and presented are both inherited by subgroups with finite translational index (see also [28] for a quick topological proof of this result, in the special case of inverse semigroups). Note that in [25] the author uses the term index rather than translational index for the number of (right) cosets (in the above sense) of a subgroup of a monoid, but here we have opted for translational index to distinguish it from the various other notions of index for that have appeared in the literature; see [1], [20] and [7] for example.

This gives our first main result.

**Theorem 1.** Let $S$ be a monoid and let $G$ be a subgroup of $S$. If $S$ has finite derivation type, and $G$ has finite translational index in $S$, then $G$ has finite derivation type.

Theorem 1 proves one direction of our second main result.

**Theorem 2.** Let $S$ be a regular monoid with finitely many left and right ideals. Then $S$ has finite derivation type if and only if every maximal subgroup of $S$ has finite derivation type.

Combining Theorem 2 with results of Cremanns and Otto [4], [5] we obtain the following.

**Corollary 1.** Let $S$ be a regular monoid with finitely many left and right ideals. If every maximal subgroup of $S$ satisfies the homological finiteness condition FP$_3$, then so does $S$.

**Proof.** In [4] (see also [22] and [12]) it was shown that if a monoid $S$ has finite derivation type then it also satisfies the homological finiteness condition FP$_3$. Moreover, in [5] it was shown that for groups having finite derivation type is equivalent to being of type FP$_3$. Thus if every maximal subgroup of $S$ is of type FP$_3$ then they all have finite derivation type, which by Theorem 2 implies that $S$ has finite derivation type, which in turn implies that $S$ is of type FP$_3$. □
These results answer several open problems posed in [25, Open Problem 4.5]. It is still open whether the converse of Corollary [1] holds.

The paper is structured as follows. In Section 2 we give the preliminaries required for the rest of the paper, in particular we define what it means for a monoid to have finite derivation type. Then in Section 3 we prove our first main result which gives a homotopy base for an arbitrary subgroup of a monoid, and we prove Theorem [1]. Finite derivation type for regular monoids and their maximal subgroups is the topic of Section 4, and this is the place where our second main result, Theorem [2] is proved.

2. Preliminaries: homotopy relations and the homotopical finiteness property finite derivation type

The background and results stated in Section 1 above have all been given in terms of monoids. In fact, all of the concepts that we intend to work with in the paper may be defined for semigroups and we shall find it convenient throughout to work in this slightly more general context.

Presentations and rewriting systems. Let $A$ be a non-empty set, that we call an alphabet. We use $A^{*}$ to denote the free monoid over $A$, and $A^{+}$ to denote the free semigroup. For words $u, v \in A^{*}$ we write $u \equiv v$ to mean that $u$ and $v$ are identically equal as words. We write $|w|$ to denote the total number of letters in $w$, which we call the length of the word $w$. A rewriting system over $A$ is a subset $R \subseteq A^{+} \times A^{+}$. An element of $R$ is called a rule, and we often write $r_{+1} = r_{-1}$ for $(r_{+1}, r_{-1}) \in R$. For $u, v \in A^{+}$ we write $u \to_{R} v$ if $u \equiv w_{1}r_{+1}w_{2}$, and $v \equiv w_{1}r_{-1}w_{2}$ where $(r_{+1}, r_{-1}) \in R$ and $w_{1}, w_{2} \in A^{*}$. The reflexive symmetric transitive closure $\to_{R}^{*}$ of $\to_{R}$ is precisely the congruence on $A^{+}$ generated by $R$. The ordered pair $(A, R)$ is called a semigroup presentation with generators $A$ and set of defining relations $R$. If $S$ is a semigroup that is isomorphic to $A^{+} / \to_{R}^{*}$ we say that $S$ is the semigroup defined by the presentation $(A, R)$. A semigroup is said to be finitely presented if it may be defined by a presentation with finitely many generators and a finite number of defining relations. For a word $w \in A^{*}$ we use $[w]_{R}^{*}$ to denote the $\to_{R}^{*}$-class of $w$. So $[w]_{R}^{*}$ is the element of $S = A^{+} / \to_{R}^{*}$ that the word $w$ represents. For any subset $T$ of the semigroup $S$ we use $\mathcal{L}(A, T)$ to denote the set of all words in $A^{+}$ representing elements of $T$, i.e.

$$\mathcal{L}(A, T) = \{ w \in A^{+} : [w]_{R}^{*} \in T \}.$$ 

Derivation graphs, homotopy bases, and finite derivation type. Let $P = (A, R)$ be a semigroup presentation. The derivation graph associated with $P$ is an infinite graph $\Gamma = \Gamma(P) = (V, E, \iota, \tau,^{-1})$ with vertex set $V = A^{+}$, and edge set $E$ consisting of the following collection of 4-tuples:

$$\{(w_{1}, r, \epsilon, w_{2}) : \ w_{1}, w_{2} \in A^{*}, r = (r_{+1}, r_{-1}) \in R, \text{ and } \epsilon \in \{+1, -1\}\}.$$ 

The functions $\iota, \tau : E \to V$ associate with each edge $E = (w_{1}, r, \epsilon, w_{2})$ (with $r = (r_{+1}, r_{-1}) \in R$) its initial and terminal vertices $\iota(E) = w_{1}r_{+1}w_{2}$ and $\tau(E) = w_{1}r_{-1}w_{2}$, respectively. The mapping $^{-1} : E \to E$ associates with each edge $E = (w_{1}, r, \epsilon, w_{2})$ an inverse edge $E^{-1} = (w_{1}, r, -\epsilon, w_{2})$.

A path is a sequence of edges $P = E_{1} \circ E_{2} \circ \ldots \circ E_{n}$ where $\tau E_{i} = \iota E_{i+1}$ for $i = 1, \ldots, n-1$. Here $P$ is a path from $\iota E_{1}$ to $\tau E_{n}$ and we extend the mappings $\iota$ and $\tau$ to paths by defining $\iota P \equiv \iota E_{1}$ and $\tau P \equiv \tau E_{n}$. The inverse of a path $P = E_{1} \circ E_{2} \circ \ldots \circ E_{n}$ is the path $P^{-1} = E_{n}^{-1} \circ E_{n-1}^{-1} \circ \ldots \circ E_{1}^{-1}$, which is a path from $\tau P$ to $\iota P$. A closed path is a path $P$ satisfying $\iota P = \tau P$. For two paths $P$ and $Q$ with $\tau P \equiv \iota Q$ the composition $P \circ Q$ is defined.

To unclutter notation slightly we shall usually omit the symbol $\circ$ when composing paths, writing simply $PQ$ in place of $P \circ Q$.

We denote the set of paths in $\Gamma$ by $P(\Gamma)$, where for each vertex $w \in V$ we include a path $1_{w}$ with no edges, called the empty path at $w$. We call a path $P$ positive if it is either
empty or it contains only edges of the form \((w_1, r, +1, w_2)\). We use \(P_+(\Gamma)\) to denote the set of all positive paths in \(\Gamma\). Dually we have the notion of negative path, and \(P_-(\Gamma)\) denotes the set of all negative paths. The free monoid \(A^*\) acts on both sides of the set of edges \(E\) of \(\Gamma\) by
\[
x \cdot E \cdot y = (xw_1, r, \epsilon, w_2y)
\]
where \(E = (w_1, r, \epsilon, w_2)\) and \(x, y \in A^*\). This extends naturally to a two-sided action of \(A^*\) on \(P(\Gamma)\) where for a path \(P = E_1 \circ E_2 \circ \ldots \circ E_n\) we define
\[
x \cdot P \cdot y = (x \cdot E_1 \cdot y) \circ (x \cdot E_2 \cdot y) \circ \ldots \circ (x \cdot E_n \cdot y).
\]
If \(P\) and \(Q\) are paths such that \(\iota P \equiv \iota Q\) and \(\tau P \equiv \tau Q\) then we say that \(P\) and \(Q\) are parallel, and write \(P \parallel Q\). We use \(\parallel \subseteq P(\Gamma) \times P(\Gamma)\) to denote the set of all parallel paths.

An equivalence relation \(\sim\) on \(P(\Gamma)\) is called a homotopy relation if it is contained in \(\parallel\) and satisfies the following conditions:

(H1) If \(E_1\) and \(E_2\) are edges of \(\Gamma\), then
\[
(E_1 \cdot \iota E_2)(\tau E_1 \cdot E_2) \sim (\iota E_1 \cdot E_2)(E_1 \cdot \tau E_2).
\]
(H2) For any \(P, Q \in P(\Gamma)\) and \(x, y \in A^*
\[
P \sim Q \implies x \cdot P \cdot y \sim x \cdot Q \cdot y.
\]
(H3) For any \(P, Q, R, S \in P(\Gamma)\) with \(\tau R \equiv \iota P \equiv \iota Q\) and \(\tau S \equiv \tau P \equiv \tau Q\)
\[
P \sim Q \implies R \parallel S \sim R \parallel Q \parallel S.
\]
(H4) If \(P \in P(\Gamma)\) then \(PP^{-1} \sim 1_{\iota P}\), where \(1_{\iota P}\) denotes the empty path at the vertex \(\iota P\).

It is a straightforward exercise to check that the collection of all homotopy relations is closed under arbitrary intersection, and that \(\parallel\) itself is a homotopy relation. Therefore, for any subset \(C\) of \(\parallel\) there is a unique smallest homotopy relation \(\sim_C\) on \(P(\Gamma)\) containing \(C\). We call this the homotopy relation generated by \(C\). A subset \(C\) of \(\parallel\) that generates \(\parallel\) is called a homotopy base for \(\Gamma\).

**Definition 1.** We say that the presentation \(P = \langle A | R \rangle\) has finite derivation type (written FDT for short) if there is a finite homotopy base for \(\Gamma = \Gamma(P)\). A finitely presented semigroup \(S\) has finite derivation type if some (and hence any by [27, Theorem 4.3] and [17, Theorem 3]) finite presentation for \(S\) has finite derivation type.

Clearly, a set \(B\) of parallel paths is a homotopy base if and only if the set \(\{(P \circ Q)^{-1}, 1_{\iota P}\} : (P, Q) \in B\}\) is. Hence we say that a set \(C\) of closed paths is a homotopy base if \(\{(P, 1_{\iota P}) : P \in C\}\) is a homotopy base. So a homotopy base \(X\) for \(\Gamma = \Gamma(P)\) may be given either as a subset of \(\parallel\), so that \(X\) is a set of parallel paths, or may be given as a set of closed paths. Both of these ways of expressing homotopy bases will be used in this article. In addition, sometimes we shall refer to \(X\) as a homotopy base of a presentation \(P\) (rather than of the graph \(\Gamma(P)\)) and also, when it is clear from context, as a homotopy base of the semigroup.

The definition of FDT given above was first introduced for monoids by Squier in [27]. An equivalent geometric definition of FDT, stated in terms of fundamental groups of 2-complexes, may be found in [22]. Here, following [17], we have opten to work with semigroups, semigroup presentations and the corresponding definition of FDT. If one chose instead to work with monoid presentations, then the definitions given above would all have to be modified by replacing every occurrence of \(A^+\) by \(A^*\). Fortunately, when considering the properties of being finitely generated, presented or having FDT for monoids, it does not matter whether one chooses to work with semigroup presentations (and the corresponding definition of FDT) or with monoid presentations (and corresponding definition of FDT). Indeed, in [27, Theorem 4.3] it was shown that for monoids the property of having FDT is independent of the choice of finite presentation for the monoid. The semigroup presentation analogue of this result was given in [17, Theorem 3]. It is a very easy exercise to check that for a monoid, being finitely presented as a monoid is equivalent to being...
finitely presented as a semigroup. The analogue of this result for FDT is stated below; the proof is standard and so is omitted, full details may be found in [18, Section 2.4].

**Lemma 1.** Let $M$ be a finitely presented monoid. Let $P$ be a finite semigroup presentation defining $M$, and let $Q$ be a finite monoid presentation defining $M$. Then $P$ has finite derivation type if and only if $Q$ has finite derivation type.

Therefore when considering FDT for monoids, it makes no difference whether we work with semigroup presentations or with monoid presentations. Throughout the paper we shall work only with semigroup presentations.

### 3. A homotopy base for a subgroup of a semigroup

In this section we present our first main result: Given a homotopy base for a semigroup we find a homotopy base for an arbitrary subgroup of that semigroup. When the semigroup has finite derivation type, and the subgroup has finite translational index, the obtained homotopy base for the subgroup will be finite, thus proving Theorem 1.

Let $S$ be a semigroup and let $G$ be a subgroup of $S$. As mentioned in the introduction, the right cosets of $G$ are the elements of the strong orbit of $G$ under the action of $S$ on the set $\{Gs : s \in S\}$. So, $Gs$ is a coset if and only if there exists $s' \in S$ with $Gss' = G$. By taking $s = s' = e$ where $e$ is the identity of $G$, which is an idempotent of $S$, we see that $G$ is a right coset of $G$. We call the number of right cosets the (right) translational index of $G$ in $S$. We say that $G$ has finite right translational index if $G$ has finitely many right cosets. There is also an obvious dual notion of left coset, and left translational index. Throughout we shall work with right cosets, and by the translational index we shall always mean the right translational index. This definition of index for a subgroup of a semigroup was first considered in [25]. These ideas have a natural interpretation in terms of Green’s relations; see the end of this section for more details on this.

Let $S$ be defined by the semigroup presentation $P = \langle A | R \rangle$. The first thing that we need is a presentation for the group $G$. A method for finding such a presentation, given in terms of $P$ and the action of $S$ on the cosets of $G$, was the main result of [25]. We now give details of that result, which will be stated as Theorem 3 before going on to consider the problem of homotopy bases.

Let $C = \{C_i : i \in I\}$ be the collection of (right) cosets of $G$ in $S$. Now $S$ acts on the set $C \cup C_0$ in a natural way by right multiplication where $C_i s = C_0$ if and only if $C_i s \notin C$. This is the same as the action of $S$ on the index set $I \cup \{0\}$ given by $C_i s = C_i s$. We extend this notation to $A^+$ writing $iw \in I \cup \{0\}$ to mean the element of $I \cup \{0\}$ obtained by acting on $i$ by the element of $S$ that the word $w \in A^+$ represents.

By convention we let $1 \in I$ and set $C_1 = G$. For all $i \in I \setminus \{1\}$ let $r_i, r'_i$ be fixed words in $A^+$, representing the elements $s_{r_i}$ and $s_{r'_i}$ of $S$ respectively, and chosen so that we have $Gs_{r_i} = C_i$ and $gs_{r_i}, s_{r'_i} = g$ for all $g \in G$. (To show that such elements exist is an easy exercise; see [25, Proposition 2.4].) For notational convenience we set $r_1 = r'_1 = 1 \in A^+$ and also define $i 1 = i$ for all $i \in I$ (where $1$ denotes the empty word). Also let $e \in A^*$ be a fixed word representing the identity of the group $G$. Next define a new alphabet

$$B = \{[i, a] : i \in I, a \in A, ia \neq 0\},$$

and define a mapping

$$\phi : \{(i, w) : i \in I, w \in A^*, iw \neq 0\} \to B^*$$

inductively by

$$\phi(i, 1) = 1,$$

$$\phi(i, aw) = [i, a]\phi(ia, w) \quad (i \in I, \ a \in A, \ w \in A^*, \ iaw \neq 0).$$

This easily extends to

$$\phi(i, w_1 w_2) = \phi(i, w_1) \phi(iw_1, w_2) \quad (i \in I, \ w_1, w_2 \in A^*, \ iw_1 w_2 \neq 0).$$

(1)
For \( w \in A^* \) such that \( 1w \neq 0 \), abusing notation slightly, we write \( \phi(w) = \phi(1,w) \) and we note that for any two words \( w_3, w_4 \in \mathcal{L}(A,G) \) we have
\[
\phi(1, w_3 w_4) = \phi(1, w_3) \phi(1w_3, w_4) = \phi(1, w_3) \phi(1, w_4)
\]
so that
\[
\phi(w_3 w_4) = \phi(w_3) \phi(w_4) \quad \text{(for all } w_3, w_4 \in \mathcal{L}(A,G)).\tag{2}
\]
Note that even though \( \phi \) is defined for the pairs \((i, 1) \) \( (i \in I) \), and its image includes the empty word, we have \( \phi(i, w) \in B^+ \) if and only if \( w \in A^+ \).

With this notation the main theorem of \([25]\) states:

**Theorem 3.** \([25, \text{Theorem 2.9}]\) If \( S \) is defined by a presentation \( \langle A|R \rangle \) then, with the above notation, the presentation with generators \( B \) and set of relations \( U \):
\[
\phi(i, u) = \phi(i, v) \quad (i \in I, \, (u = v) \in R, \, iu \neq 0),
\]
\[
[i, a] = \phi(ear_i a) \quad (i \in I, \, a \in A, \, ia \neq 0)
\]
defines \( G \) as a semigroup.

Note that when \( A, R \) and \( I \) are finite, Theorem 3 gives a finite presentation for \( G \).

Given a homotopy base \( X \) for a presentation \( \mathcal{P} = \langle A|R \rangle \) of \( S \) we shall now show how to obtain a homotopy base \( Y \) for the presentation \( \mathcal{Q} = \langle B|U \rangle \) of \( G \) given in Theorem 3. When \( A, R, I \) and \( X \) are all finite, our homotopy base \( Y \) (along with \( B \) and \( Q \) will be finite.

The derivation graph \( \Gamma(\mathcal{P}) \) has vertices \( A^+ \) while the graph \( \Gamma(\mathcal{Q}) \) has vertices \( B^+ \). So \( \phi \) gives a mapping from the vertices of \( \Gamma(\mathcal{P}) \) to the vertices of \( \Gamma(\mathcal{Q}) \). Next we let \( \psi \) be the homomorphism extending
\[
\psi : B^* \to A^*, \quad [i, a] \mapsto er_i a.
\]
Note that in particular the empty word is mapped to the empty word under \( \psi \), which we write as \( \psi(1) = 1 \). It follows from the definition that for \( u, v \in B^+ \), if \( u \) and \( v \) represent the same element of \( G \) then \( \psi(u) \) and \( \psi(v) \) represent that same element of \( S \). Now by restricting to \( B^+ \), \( \psi \) gives a mapping from the vertices of \( \Gamma(\mathcal{Q}) \) to the vertices of \( \Gamma(\mathcal{P}) \).

In order to define a homotopy base for \( \langle B|U \rangle \) we shall carry out the following steps.

(I) Extend \( \phi \) and \( \psi \) to mappings between paths of \( \Gamma(\mathcal{P}) \) and paths of \( \Gamma(\mathcal{Q}) \).

(II) For each \( w \in B^+ \) define a path \( \Lambda_w \) in \( \Gamma(\mathcal{Q}) \) from \( w \) to \( \phi(\psi(w)) \in B^+ \).

In fact, \( \phi \) will actually be defined on certain pairs \( (i, P) \) where \( i \in I \) and \( P \) is a path from \( \Gamma(\mathcal{P}) \). Once these definitions have been made, we shall use them in Theorem 4 below to define a homotopy base for the presentation \( Q \) of \( G \).

**Extending the definitions of \( \phi \) and \( \psi \) to mappings between paths.** Now we shall extend \( \phi \) to a mapping, also denoted \( \phi \), with
\[
\phi : \{(i, P) : i \in I, \, P \in \mathcal{P}(\Gamma(\mathcal{P})), \, i(P) \neq 0 \} \to \mathcal{P}(\Gamma(\mathcal{Q}))
\]
where for an edge \( E = (w_1, r, e, w_2) \) in \( \Gamma(\mathcal{P}) \) such that \( i(E) \neq 0 \) we define
\[
\phi(i, E) = (\phi(i, w_1), \phi(1w_1, r+1) = \phi(iw_1, r-1), e, \phi(iw_1 r+1, w_2))
\]
which is an edge of \( \Gamma(\mathcal{Q}) \), and then for a path \( P = E_1 E_2 \ldots E_k \) in \( \Gamma(\mathcal{P}) \) we define
\[
\phi(i, P) = \phi(i, E_1) \phi(i, E_2) \ldots \phi(i, E_k).
\]
Also, for any word \( w \in A^+ \) and \( i \in I \) such that \( iw \neq 0 \) we define \( \phi(i, 1w) = 1 \phi(i, w) \). From the definition, and using (1), it is easy to see that \( \psi(i, P) = \phi(i, 1P) \), \( \tau \phi(i, P) = \phi(i, tP) \), and \( \phi(i, P)^{-1} = \phi(i, P^{-1}) \). As above we write \( \phi(P) \) as shorthand for \( \phi(1, P) \), for any paths \( P \) such that \( 1(P) \neq 0 \).

Immediately from this definition, combining with (1) we obtain the identity
\[
\phi(i, w_1 \cdot P \cdot w_2) = \phi(i, w_1) \cdot \phi(iw_1, P) \cdot \phi(iw_1 P, w_2) \quad \text{(3)}
\]
for any path $P$ of $\Gamma(P)$ and $w_1, w_2 \in A^*$, $i \in I$ with $iw_1Pw_2 \neq 0$.

Next we extend $\psi : B^* \rightarrow A^*$ to a mapping, also denoted $\psi$, where

$$\psi : P(\Gamma(Q)) \rightarrow P(\Gamma(P)).$$

To do this, firstly for any word $w \in B^+$ we define $\psi(1_w) = 1_{\psi(w)}$. Next, for each $u \in U$ let $E_u = (1, u, +1, 1)$, which is an edge in $\Gamma(Q)$. Then for every $u \in U$ let $\psi(E_u)$ be a fixed path in $\Gamma(P)$ from $\psi(u+1)$ to $\psi(u-1)$. Such a path exists since $\psi(u+1)$ and $\psi(u-1)$ represent the same element of $S$. Then for an arbitrary edge $E = (w_1, u, \epsilon, w_2)$ of $\Gamma(Q)$ we define

$$\psi(E) = \psi(w_1) \cdot \psi(E_u) \cdot \psi(w_2)$$

which is a path from $\psi(E)$ to $\psi(\tau E)$ since by definition $\psi : B^* \rightarrow A^*$ is a homomorphism. Then for any path $P = E_1E_2 \ldots E_k$ in $\Gamma(Q)$ we define

$$\psi(P) = \psi(E_1)\psi(E_2) \ldots \psi(E_k).$$

**Definition of the paths $\Lambda_w$.** For each $w \in B^+$ we shall define a path $\Lambda_w$ in $\Gamma(Q)$ from $w \in B^+$ to $\phi(\psi(w))$. We define $\Lambda_w$ by induction on the length of the word $w$. If $|w| = 1$ then we have $w \equiv [i, a]$ for some $i \in I$, $a \in A$ such that $ia \neq 0$, and we set

$$\Lambda_w = (1, [i, a] = \phi(er_iar_i)_{ia}, +1, 1)$$

which is an edge of $\Gamma(Q)$. If $|w| > 1$ then write $w \equiv bw'$ where $b \in B$ and $w' \in B^+$, say $b = [i, a] \in B$. Then we define inductively

$$\Lambda_w = (\Lambda_b \cdot w')(\phi(er_iar_i)_{ia} \cdot \Lambda_w').$$

**Lemma 2.** For all $w \in B^+$, $\Lambda_w$ is a path in $\Gamma(Q)$ with initial vertex $w$ and terminal vertex $\phi(\psi(w))$.

**Proof.** The result holds trivially for words of length 1. Now let $w \in B^+$ with $|w| \geq 2$ and assume inductively that the result holds for all words of strictly smaller length than $w$. Write $w \equiv bw'$ where $b = [i, a] \in B$. By definition

$$\Lambda_w = (\Lambda_b \cdot w')(\phi(er_iar_i)_{ia} \cdot \Lambda_w') = (\Lambda_b \cdot w')(\phi(\psi(b)) \cdot \Lambda_w').$$

By induction $\Lambda'_w$ is a path in $\Gamma(Q)$ from $w'$ to $\phi(\psi(w'))$. Now $\Lambda_b \cdot w'$ has initial vertex $bw'$ and terminal vertex $\phi(\psi(b)w')$, while $\phi(\psi(b)) \cdot \Lambda_w'$ has initial vertex $\phi(\psi(b)w')$ and terminal vertex $\phi(\psi(b)\phi(\psi(w')))$. Since $\phi(\psi(b))$ and $\phi(\psi(w'))$ both belong to $L(A, G)$ it follows from [2] and the fact $\psi$ is a homomorphism that

$$\phi(\psi(b)\phi(\psi(w'))) \equiv \phi(\psi(b)\phi(\psi(w'))) \equiv \phi(\psi(w)),$$

so $\Lambda_w$ is a path from $bw' \equiv w$ to $\phi(\psi(w))$.

We are now in a position to define the homotopy base for $G$.

**Theorem 4.** Let $X$ be a homotopy base for a presentation $\langle A|R \rangle$ defining a semigroup $S$, where $X$ is a set of closed paths. Then, with the above notation, the set of closed paths $K \cup W$ is a homotopy base for the presentation $\langle B|U \rangle$ of the group $G$ where

$$K = \{ \phi(j, P) : P \in X, j \in I, j(P) \neq 0 \}$$

and

$$W = \{ E_r \Lambda_{r-1} \phi(\psi(E_r))^{-1} \Lambda_{r+1}^{-1} : r \in U, i \in I \},$$

with $E_r = (1, r, +1, 1)$.

Note that once proved, this result has Theorem [1] as a corollary, since if $X$, $A$, $R$ and $I$ are all finite then $B$, $U$ and $K \cup W$ will be finite.

**Proving Theorem 4.** The rest of this section will be devoted to proving Theorem 4. We begin by writing an infinite homotopy base $Z$ for $\Gamma(Q)$ and then show that $K \cup W$
generates all the closed paths of \( Z \) and hence is itself a homotopy base. Let \( Z \) denote the following infinite set of closed paths of the graph \( \Gamma(Q) \)
\[
\mathbb{E} \Lambda_{\mathbb{E}} (\phi(\psi(E)))^{-1} \Lambda_{\mathbb{E}}^{-1}, \text{ for } \mathbb{E} \text{ in } \Gamma(Q) 
\] (4)
and \( \phi([E_1, E_2]), \text{ for } E_1, E_2 \text{ in } \Gamma(P) \) such that \( iE_1iE_2 \in L(A,G) \),
\[
\phi((E_1 \cdot E_2)(\tau E_1 \cdot E_2)(E_1 \cdot \tau E_2)^{-1}(E_1 \cdot E_2)^{-1}) 
\] (5)
where \([E_1, E_2]\) is the path \( (E_1 \cdot iE_2)(\tau E_1 \cdot E_2)(E_1 \cdot \tau E_2)^{-1}(iE_1 \cdot E_2)^{-1} \)
\[
\phi(w_1 \cdot P \cdot w_2), \text{ for } P \in X \text{ and } w_1, w_2 \in A^* \text{ such that } w_1(iP)w_2 \in L(A,G). 
\] (6)

Lemma 3. The set \( Z \) is a homotopy base for \( \Gamma(Q) \).

Proof. This actually follows from a more general result proved in [10]. Since in this case it is not difficult, and for the sake of completeness, we offer a sketch of the proof here. For full details we refer the reader to [10].

Let \( C \) be an arbitrary closed path in \( \Gamma(Q) \). We must prove that \( C \sim_Z 1_{iC} \).

Let \( D = \psi(C) \), noting that \( D \) is a closed path in \( \Gamma(P) \) and that each vertex of \( D \) belongs to \( L(A,G) \). First we consider the closed path \( \phi(\psi(C)) = \phi(D) \) in \( \Gamma(Q) \). Since \( X \) is a homotopy base for the presentation \( P \) it follows that in \( \Gamma(P) \) we have \( D \sim_X 1_{iD} \). This means (see [13, Lemma 2.1]) that there is a finite sequence of applications of (H1)–(H4) that can be used to transform \( D \) into \( 1_{iD} \). Using (5) each application of (H1) can be turned into a corresponding application in \( \Gamma(Q) \), and using (6) and the fact that the vertices of \( D \) belong to \( L(A,G) \), each application of (H2) can be turned into a corresponding application in \( \Gamma(Q) \). The relations (H3) may be dealt with since by definition \( \phi(\mathbb{P}Q\mathbb{R}) = \phi(\mathbb{P})\phi(Q)\phi(\mathbb{R}) \), and the relations (H4) are handled using \( \phi(\mathbb{P}^{-1}) = \phi(\mathbb{P})^{-1} \). We conclude that \( \phi(\psi(C)) = \phi(D) \) is a closed path in \( \Gamma(Q) \) with \( \phi(D) \sim_Z 1_{i\phi(D)} \).

Now we return our attention to the original closed path \( C \). Using the paths (4), applied to each edge of the closed path \( C \) in turn, we see that \( C \sim_Z 1_{iC} \phi(\psi(C)) \Lambda_{\mathbb{E}}^{-1} \) where \( iC \equiv \tau C \) since \( C \) is a closed path. We now have
\[
C \sim_Z 1_{iC} \phi(\psi(C)) \Lambda_{\mathbb{E}}^{-1} \sim_Z 1_{iC} \phi(\psi(C)) \Lambda_{\mathbb{E}}^{-1} = 1_{iC} \Lambda_{\mathbb{E}} \Lambda_{\mathbb{E}}^{-1} \sim_Z 1_{iC} 
\]
as required. \( \square \)

It follows that to prove Theorem 4 it is sufficient to show the following.

Lemma 4. The homotopy relation generated by \( K \cup W \) contains \( Z \), and hence \( K \cup W \) is a homotopy base for \( \Gamma(Q) \).

The proof of Lemma 4 will follow from the lemmas below. We consider each of the sets of closed paths (4), (5), and (6) in turn.

Generating the paths (5).

Lemma 5. Let \( E_1 \) and \( E_2 \) be edges from \( \Gamma(P) \) where \( iE_1iE_2 \in L(A,G) \). Then \( \phi([E_1, E_2]) \sim 1_v \) in \( \Gamma(Q) \), where \( v \) denotes the vertex \( \psi([E_1, E_2]) \) of the graph \( \Gamma(Q) \).

Proof. First observe that by definition \( \phi(1\tau E_1, E_2) \) is a path from \( \phi(1\tau E_1, iE_2) \) to \( \phi(1\tau E_1, \tau E_2) \), and we have the equality \( \phi(1\tau E_1, E_2) \equiv \phi(1\tau E_1, iE_2) \) since \( iE_1 \) and \( \tau E_1 \) both represent the same element of \( S \). Also the paths \( \phi(1\tau E_1, E_2) \) and \( \phi(1\tau E_1, iE_2) \) are identically equal, since \( 1\tau E_1 = 1iE_1 \in I \). Now we have
\[
\phi(E_1 \cdot E_2)\phi(\tau E_1 \cdot E_2) = (\phi(E_1) \cdot (1\tau E_1, iE_2)) (\phi(\tau E_1) \cdot (1\tau E_1, E_2)) \quad \text{(by (3))}
\]
\[
\sim (\phi(E_1) \cdot (1\tau E_1, iE_2)) (\phi(E_2) \cdot (1\tau E_1, E_2)) \quad \text{(applying (H1))}
\]
\[
= (\phi(E_1) \cdot (1\tau E_1, iE_2)) (\phi(E_2) \cdot (1\tau E_1, E_2)) \quad \text{(since } 1\tau E_1 = 1iE_1) 
\]
\[
= (\phi(E_1) \cdot E_2)\phi(E_1 \cdot \tau E_2) \quad \text{(by (3))}.
\]
Along with the definition of \( \phi \) on paths and the fact that \( \phi(E)^{-1} = \phi(E^{-1}) \) this completes the proof of the lemma. \( \square \)
Generating the paths \([86x123]\).

**Lemma 6.** For any \(\mathbb{P} \in X\), and \(w_1, w_2 \in A^*\) such that \(w_1(i\mathbb{P})w_2 \in \mathcal{L}(A, G)\) we have \(\phi(w_1 \cdot \mathbb{P} \cdot w_2) \sim_{K} 1_v\), where \(v\) denotes the vertex \(\phi(w_1 \cdot \mathbb{P} \cdot w_2)\) of the graph \(\Gamma(Q)\).

**Proof.** Since \(\phi(1w_1, \mathbb{P}) \in K\), using \([44x107]\) we obtain

\[
\phi(w_1 \cdot \mathbb{P} \cdot w_2) = \phi(1, w_1) \cdot \phi(1w_1, \mathbb{P}) \cdot \phi(1w_1, \mathbb{P}) \sim_{K} 1_v.
\]

\[\square\]

Generating the paths \([44x107]\). For this family of paths we must first prove two straightforward lemmas.

**Lemma 7.** For any edge \(E = (w_1, r, \epsilon, w_2)\) of \(\Gamma(Q)\), with \(E_r = (1, r, \epsilon, 1)\) we have

\[
\phi(\psi(E)) = \phi(\psi(w_1)) \cdot \phi(\psi(E_r)^c) \cdot \phi(\psi(w_2)).
\]

**Proof.** It follows from the definition of \(\psi(E)\) and \([44x107]\) that

\[
\phi(\psi(E)) = \phi(\psi(w_1)) \cdot \phi(\psi(E_r)^c) \cdot \phi(\psi(w_2))
= \phi(1, \psi(w_1)) \cdot \phi(\psi(w_1), \psi(E_r)^c) \cdot \phi(\psi(w_1)\psi(E_r)^c, \psi(w_2))
= \phi(\psi(w_1)) \cdot \phi(\psi(E_r)^c) \cdot \phi(\psi(w_2))
\]

since \(\psi(w_1), \psi(w_1)\psi(E_r)^c \in \mathcal{L}(A, G)\) implies that \(1\psi(w_1) = 1\psi(w_1)\psi(E_r)^c = 1.\)

**Lemma 8.** For all \(w_1, w_2, w_3 \in B^+\) we have

(i) \(\Lambda_{w_1w_2} = (\Lambda_{w_1} \cdot w_2)(\phi(\psi(w_1)) \cdot \Lambda_{w_2})\)

(ii) \(\Lambda_{w_1w_2w_3} = (\Lambda_{w_1} \cdot w_2w_3)(\phi(\psi(w_1)) \cdot \Lambda_{w_2w_3})(\phi(\psi(w_1))\phi(\psi(w_2)) \cdot \Lambda_{w_3}).\)

**Proof.** Part (i) follows straight from the definition of \(\Lambda_{w}\), and part (ii) is proved by applying part (i) twice.

Now we can finish the proof.

**Lemma 9.** For any edge \(E\) of \(\Gamma(Q)\) we have

\(\Lambda_{E} \phi(\psi(E)) \sim_{W} E \Lambda_{E}.\)

**Proof.** Let \(E = (w_1, r, \epsilon, w_2)\) be an arbitrary edge of \(\Gamma(Q)\). First suppose that \(w_1\) and \(w_2\) are both non-empty. Then we have

\[
\Lambda_{E} \phi(\psi(E)) = (\Lambda_{w_1} \cdot r \cdot w_2) (\phi(\psi(w_1)) \cdot \Lambda_{r} \cdot w_2) (\phi(\psi(w_1)) \phi(\psi(r)) \cdot \Lambda_{w_2})
\phi(\psi(w_1)) \cdot \phi(\psi(E_r)^c) \cdot \phi(\psi(w_2))
\]

(by Lemmas \([7x107]\) and \([8x107]\) )

\[
\sim (\Lambda_{w_1} \cdot r \cdot w_2) (\phi(\psi(w_1)) \cdot \Lambda_{r} \cdot w_2) (\phi(\psi(w_1)) \cdot \phi(\psi(E_r)^c) \cdot w_2)
\phi(\psi(w_1)) \phi(\psi(r)) \cdot \Lambda_{w_2}
\]

(applying (H1))

\[
\sim_W (\Lambda_{w_1} \cdot r \cdot w_2) (\phi(\psi(w_1)) \cdot E_r \cdot w_2) (\phi(\psi(w_1)) \cdot \Lambda_{r} \cdot w_2)
\phi(\psi(w_1)) \phi(\psi(r)) \cdot \Lambda_{w_2}
\]

(since \(\Lambda_{r} \phi(\psi(E_r))^c \sim_W E_r \Lambda_{r}\))

\[
\sim (w_1 \cdot E_r \cdot w_2) (\Lambda_{w_1} \cdot r \cdot w_2) (\phi(\psi(w_1)) \cdot \Lambda_{r} \cdot w_2)
\phi(\psi(w_1)) \phi(\psi(r)) \cdot \Lambda_{w_2}
\]

(applying (H1))

\[
= \mathbb{E} \Lambda_{r} \mathbb{E}
\]

(by Lemma \([8x107]\) and the definition of \(\mathbb{E}\)).
If $w_1$ is empty, then this sequence of deductions holds if we replace every expression of the form $\Lambda w_1 \cdot u$ ($u \in B^+$) by the empty path $1_u$ at $u$. Similarly, if $w_2$ is empty then the deductions above are valid if we replace each occurrence of $u \cdot \Lambda w_2$ ($u \in B^+$) by $1_u$.

This completes the proof of Lemma 4 and hence also of Theorem 4. Moreover, since when all of $X$, $A$, $R$ and $I$ are finite it implies that all of $B$, $U$, and $K \cup W$ are finite, we obtain our first main result, Theorem 1, as a corollary.

**Corollary 2.** Let $S$ be a semigroup and let $G$ be a subgroup of $S$. If $S$ has finite derivation type, and $G$ has finite translational index in $S$, then $G$ has finite derivation type.

### 3.1. Cosets and Green’s relations

The notion of coset being used in this article is best thought of when viewed from the point of view of Green’s relations on the semigroup.

Green’s relations are the equivalence relations $R$, $L$, $D$, $H$ and $J$ defined on a semigroup $S$ by

$$
xRy \iff xS^1 = yS^1, \quad xLy \iff S^1x = S^1y, \quad xJy \iff S^1xS^1 = S^1yS^1
$$

$$
D = R \circ L = R \circ L, \quad H = R \cap L.
$$

Here $S^1$ denotes the semigroup $S$ with an identity element $1$ adjoined. Since their introduction in [8], Green’s relations have played a fundamental role in the development of the structure theory of semigroups. For more background on Green’s relations and their importance in semigroup theory we refer the reader to [10]. In particular, the maximal subgroups of a semigroup $S$ are precisely the $H$-classes of $S$ that contain idempotents. Hence Theorem 1 has the following corollary relating the property of FDT in a semigroup with the property in its maximal subgroups.

**Corollary 3.** Let $S$ be a semigroup and let $H$ be a maximal subgroup of $S$. If $S$ has finite derivation type, and the $R$-class of $H$ contains only finitely many $H$-classes, then $H$ has finite derivation type.

**Proof.** The subgroup $H$ has finite translational index since its right cosets are precisely the $H$-classes of $S$ contained in the $R$-class of $H$. 

### 4. Regular semigroups and finite derivation type

In this section we turn our attention to regular semigroups. For a full account of regular semigroups and the other standard semigroup theoretic concepts mentioned in this section, such as completely 0-simple semigroups, we refer the reader to [10].

A semigroup $S$ is regular if for all $x \in S$ there exists $y \in S$ such that $xyx = x$. This is equivalent to saying that every $R$-class and every $L$-class of $S$ contains an idempotent, and hence contains a maximal subgroup. In this sense we may think of a regular semigroup as having “lots” of maximal subgroups and as a consequence it is often the case that the behaviour of such semigroups is closely linked to the behaviour of their maximal subgroups. Examples of this kind of result include [25] where finite generation and presentability are considered, and [3] which is concerned with residual finiteness. Here we consider the property FDT, and the relationship between FDT holding in a regular semigroup, and FDT holding in the maximal subgroups of the semigroup.

As a special case of Corollary 3 we see that if a regular semigroup $S$ has finitely many left and right ideals (which is equivalent to having finitely many $R$- and $L$-classes) then FDT holding in $S$ implies FDT holding in all of the maximal subgroups of $S$. The main aim of this section is to prove that the converse of this result is also true. We shall prove the following.

**Theorem 5.** Let $S$ be a regular semigroup with finitely many left and right ideals. If every maximal subgroup of $S$ has FDT then $S$ has FDT.
Along with Corollary 2 this proves the second main result, Theorem 2 of our paper.

Our approach to the proof of Theorem 5 requires us to introduce a little more theory. A semigroup is said to be completely (0-)simple if it is (0-)simple and has (0-)minimal left and right ideals. Recall that a completely 0-simple semigroup $S$ is isomorphic to a 0-Rees matrix semigroup $M^0[G, I, \Lambda, P]$, where $G$ is a group isomorphic to any (and hence all) maximal subgroups of $S$, $I$ is a set indexed by the set of all 0-minimal right ideals of $S$, $\Lambda$ is a set indexed by the set of all 0-minimal left ideals of $S$, and $P = (p_{\lambda i})$ is a regular $\Lambda \times I$ matrix with entries from $G \cup \{0\}$. Multiplication in $M^0[G, I, \Lambda, P] = (I \times G \times \Lambda) \cup \{0\}$ is given by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

$$0(i, g, \lambda) = (i, g, \lambda)0 = 00 = 0.$$

There is an analogous construction for completely simple semigroups, given by taking Rees matrix semigroups $M[G, I, \Lambda, P] = I \times G \times \Lambda$ over groups, where $P$ is a matrix with entries from $G$, and multiplication is given by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$.

Given semigroups $S, T$ and $U$, we say that $S$ is an ideal extension of $T$ by $U$ if $T$ is isomorphic to an ideal of $S$ and the Rees quotient $S/T$ is isomorphic to $U$.

In order to prove Theorem 5 it will be sufficient to establish the following.

**Theorem 6.** Let $S$ be an ideal extension of $T$ by a completely 0-simple semigroup $U = M^0[G; I, \Lambda; P]$ with $I$ and $\Lambda$ both finite. If $T$ has FDT and $G$ has FDT then $S$ has FDT.

Before embarking on a proof of of Theorem 6 we now show how Theorem 5 may be deduced from it.

**Proof of Theorem 5.** We begin with an easy observation. Let $N$ be a finitely presented semigroup, and let $N^0$ denote $N$ with a zero element 0 adjoined. We claim that if $N^0$ has FDT then $N$ has FDT. Indeed, let $P_1 = \langle A | R \rangle$ be a finite presentation for $N$. Then $P_0 = \langle A, 0 | R, a0 = 0a = 00 = 0 \rangle$ is a finite presentation for $N^0$. If $X$ is a finite homotopy base for $P_0$ then the subset $X' \subseteq X$ defined by $X' = \{ p \in X : i_p \in \mathbb{A}^+ \}$ is clearly a finite homotopy base for the presentation $P_1$ of $N$. (This is really just a special case of the semigroup presentation analogue of the result proved in [24] stating that FDT is inherited by submonoids with ideal complement.)

Now we use Theorem 6 to prove Theorem 5. From the assumption that $S$ has finitely many left and right ideals it follows (see for example [3, Chapter 6]) that in $S$ we have $J = \mathcal{D}$. The proof of Theorem 5 now goes by induction on the number of $(\mathcal{J} = \mathcal{D})$-classes of $S$. When $S$ has just one $\mathcal{J}$-class it follows, since $S$ has finitely many $\mathcal{R}$- and $\mathcal{L}$-classes, that $S$ is isomorphic to a completely simple semigroup $M[G; I, \Lambda, P]$ with $I$ and $\Lambda$ finite, and where $G$ has FDT by assumption. By Theorem 6 this implies that $S^0$ has FDT, since it is an ideal extension of the trivial semigroup $T = \{0\}$ by the completely 0-simple semigroup $S^0$ where $G$ has FDT. Now by the observation given in the previous paragraph, since $S^0$ has FDT it follows that $S$ has FDT.

Now suppose that $S$ has at least two $\mathcal{J}$-classes. Let $J_M$ be a maximal $\mathcal{J}$-class (in the natural ordering of $\mathcal{J}$-classes $J_x \leq J_y$ if $xS^1 \subseteq yS^1$). Then $T = S \setminus J_M$ is an ideal of $S$ where $T$ is regular and, since $T$ is a union of $(\mathcal{J} = \mathcal{D})$-classes of $S$ each of which is regular and contains only finitely many $\mathcal{R}$- and $\mathcal{L}$-classes, it follows that $T$ has strictly fewer $(\mathcal{J} = \mathcal{D})$-classes than $S$. So $T$ is a regular semigroup with finitely many left and right ideals, and every maximal subgroup of $T$ is a maximal subgroup of $S$ and thus has FDT by assumption. Hence by induction $T$ has FDT. But now $S$ is an ideal extension of $T$ by the Rees quotient $S/T \cong M^0[G; I, \Lambda; P]$ where $I$ and $\Lambda$ are finite (since $S$ has finitely many left and right ideals), and $G$ has FDT. Applying Theorem 6 we conclude that $S$ has FDT.

The rest of this section will be devoted to the proof of Theorem 6.
Proof of Theorem 6. Let $S$ be an ideal extension of a semigroup $T$ by the completely 0-simple semigroup $U = M^0[G; I, \Lambda; P]$. So the semigroup $S$ decomposes as the disjoint union $S = T \cup (I \times G \times \Lambda)$. Without loss of generality we may suppose that $1 \in I, 1 \in \Lambda$ and that $p_{11} \neq 0$ so that $\{1\} \times G \times \{1\}$ is a group $H$-class. Let $\mathcal{P}_T = \langle Z(Q) \rangle$ be a semigroup presentation for $T$, and let $\mathcal{P}_G = \langle A|R \rangle$ be a semigroup presentation for the group $G$. Our first task is to write down a presentation for $S$ with respect to which we shall define our homotopy base. We obtain a presentation for $S$ by applying the results [11, Theorem 6.2] and [25, Proposition 4.4] in the following way.

We start by fixing some notation. Let $e \in A^+$ is a fixed word representing the identity element of $G$. Let $B = \{b_i : i \in I \setminus \{1\}\}$ and $C = \{c_\lambda : \lambda \in \Lambda \setminus \{1\}\}$. Ultimately in the presentation for $S$ we give below, $A \cup B \cup C$ will be a subset of the generators, where $b_i$ represents the element $(i,1,1)$, $c_\lambda$ represents the element $(1,1,\lambda)$, and $a \in A$ represents the element $(1,ga,1)$ where $ga \in G$ is the element of $G \cong \langle A|R \rangle$ represented by $a \in A$. Given this, for every word $u \in (A \cup B \cup C)^+$ representing the zero of $U = M^0[G; I, \Lambda; P]$ fix a word $\rho(u) \in Z^+$ such that the relation $u = \rho(u)$ holds in $S$. Similarly for every pair of letters $x \in A \cup B \cup C$ and $z \in Z$ fix words $\sigma(z, x), \tau(x, z) \in Z^+$ such that the relations $zx = \sigma(z, x)$ and $xz = \tau(x, z)$ hold in $S$.

Now, from [11, Theorem 6.2] we can use the presentation $\langle A|R \rangle$ and the matrix $P$ to obtain a presentation for $U = M^0[G; I, \Lambda; P]$. In [11, Theorem 6.2] this is given as a presentation of a semigroup with zero which we must convert into a genuine semigroup presentation by adding a generator $0$ and relations $0x = x0 = 00 = 0$ for all other generators. As well as this, we also add the following additional (redundant) relations:

\[
\begin{align*}
abla_i = ap_{11} & (a \in A, i \in I), \quad c_\lambda a = p_{11}a \ (a \in A, \lambda \in \Lambda), \\
c_\lambda c_\mu = p_{11}c_\mu & (\mu, \lambda \in \Lambda), \quad b_i b_j = b_i p_{11}j \ (i, j \in I).
\end{align*}
\]

where the symbols $p_{11}$ appearing in the relations are really fixed words from $A^+$ representing the elements $p_{11}$ of $G$. These relations are all easily seen to be consequences of the relations appearing in the presentation of [11, Theorem 6.2] and hence the resulting presentation also defines $U = M^0[G; I, \Lambda; P]$. Then taking this presentation for $U = M^0[G; I, \Lambda; P]$ together with the presentation $\langle Z(Q) \rangle$ of $T$, and applying [25, Proposition 4.4] we obtain the presentation for $S$ given below.

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.1)

\[
\begin{align*}
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.2)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.3)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.4)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.5)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.6)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.7)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.8)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.9)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.10)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.11)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.12)

\[
\begin{align*}
\mathcal{P}_S = \langle A, B, C, Z \mid R, Q, \rangle, \\
b_i e = b_i, \quad ec_\lambda = c_\lambda, \\
xx = \sigma(z, x), \quad xx = \tau(x, z) & (x \in A \cup B \cup C, z \in Z), \\
\end{align*}
\]

(4.13)

We group the relations of this presentation together into the following sets

\[
\begin{align*}
R_e = (4.2), \quad R_0 = (4.3) \\
R_U = (4.4) \cup (4.5) \cup (4.6) \cup (4.7) \cup (4.8) \\
R_T = (4.9) \cup (4.10) \cup (4.11) \cup (4.12) \cup (4.13),
\end{align*}
\]

so that

\[
\mathcal{P}_S = \langle A, B, C, Z \mid R, R_e, R_0, R_U, R_T, Q, \rangle.
\]
Let \( \Gamma_G = \Gamma(\mathcal{P}_G) \), \( \Gamma_T = \Gamma(\mathcal{P}_T) \) and \( \Gamma_S = \Gamma(\mathcal{P}_S) \). In the natural way we view \( \Gamma_G \) and \( \Gamma_T \) as subgraphs of \( \Gamma_S \). Let \( \Gamma \) be the subgraph of \( \Gamma_S \) with the same vertex set, but which only contains those edges \((w_1, r, \epsilon, w_2)\) with \( r \in R_0 \cup R_U \cup R_T \). Recall that we use \( \mathcal{P}_A(\Gamma) \) to denote the set of all positive paths in \( \Gamma \) (i.e. those paths that are either empty or have the property that \( \epsilon = +1 \) for every edge of the path). Let \( \Gamma_U \) (respectively \( \Gamma_0 \)) denote the subgraph of \( \Gamma_S \) with the same vertex set as \( \Gamma_S \), but which contains only those edges of the form \((w_1, r, \epsilon, w_2)\) where \( r \in R_U \) (respectively \( r \in R_0 \)).

Before we begin the process of constructing a homotopy base for \( S \), we must prove some lemmas about the presentation \( \mathcal{P}_S \). We write \( B^1A^*C^1 \) to denote the set of all non-empty words from the set \( \{bwc : b \in B \cup \{1\}, c \in C \cup \{1\} \land w \in A^*\} \). Likewise we use the notation \( B^1A^* \) and \( A^*C^1 \). We shall refer to the set of words \( B^1A^*C^1 \cup Z^+ \) as the set of quasi-normal forms for the presentation \( \mathcal{P}_S \).

**Lemma 10.**

(i) The word \( b_iwc_\lambda \) with \( i \in I \setminus \{1\}, \lambda \in \Lambda \setminus \{1\} \) and \( w \in A^* \) represents the element \((i,g,\lambda)\) of \( S \), where \( g \in G \) is the element represented by \( w \) if \( w \in A^+ \), or \( g = 1 \) if \( w \) is the empty word.

(ii) The word \( b_iw \) with \( i \in I \setminus \{1\} \) and \( w \in A^* \) represents the element \((i,g,1)\) of \( S \), where \( g \in G \) is the element represented by \( w \) if \( w \in A^+ \), or \( g = 1 \) if \( w \) is the empty word.

(iii) The word \( wc_\lambda \) with \( \lambda \in \Lambda \setminus \{1\} \) and \( w \in A^* \) represents the element \((1,g,\lambda)\) of \( S \), where \( g \in G \) is the element represented by \( w \) if \( w \in A^+ \), or \( g = 1 \) if \( w \) is the empty word.

(iv) The word \( w \in A^+ \) represents the element \((1,g,1)\) where \( g \in G \) is the element represented by the word \( w \).

**Proof.** This follows from the construction of the presentation \( \mathcal{P}_S \), together with the proof of [11, Theorem 6.2].

Given \( b \in B^1 \) and \( c \in C^1 \) we write \( b \cdot P(\Gamma_G) \cdot c \) to denote the set of all paths \( b \cdot \mathcal{P} \cdot c \) where \( \mathcal{P} \in P(\Gamma_G) \), and we write \( B^1 \cdot \mathcal{P}(\Gamma_G) \cdot C^1 = \bigcup_{b \in B^1, c \in C^1} b \cdot P(\Gamma_G) \cdot c \). Since the vertices of the graph \( \Gamma_G \) are the (non-empty) words \( A^+ \) it follows that the set of vertices appearing in paths from \( B^1 \cdot \mathcal{P}(\Gamma_G) \cdot C^1 \) is \( B^1A^*C^1 \). So there are words in \( B^1A^*C^1 \) that are not vertices of paths in \( B^1 \cdot \mathcal{P}(\Gamma_G) \cdot C^1 \), namely the words \( B \cup C \cup BC \). We shall often be interested in paths with vertices from \( B^1A^*C^1 \) and edges of the form \((w_1, r, \epsilon, w_2)\) where \( r \in R \cup R_\epsilon \). So, given \( b \in B^1 \) and \( c \in C^1 \) we write \( b \cdot \mathcal{P}(\Gamma_G) \cdot c \) to denote the set of all paths whose vertices all belong to \( bA^* \cup \{bc\} \) (where \( \{bc\} = \{b\} \) when \( c = 1 \), and \( \{bc\} = \{c\} \) when \( b = 1 \)) and whose edges are all either empty or of the form \((w_1, r, \epsilon, w_2)\) where \( r \in R \cup R_\epsilon \). Clearly \( b \cdot \mathcal{P}(\Gamma_G) \cdot c \subseteq \bigcup_{b \in B^1} b \cdot \mathcal{P}(\Gamma_G) \cdot c \) for all \( b \in B^1, c \in C^1 \). We also write \( B^1 \cdot \mathcal{P}(\Gamma_G) \cdot C^1 = \bigcup_{b \in B^1, c \in C^1} b \cdot \mathcal{P}(\Gamma_G) \cdot c \) noting that the set of vertices arising in paths of \( B^1 \cdot \mathcal{P}(\Gamma_G) \cdot C^1 \) is \( B^1A^*C^1 \).

The next lemma shows how using only positive relations from the set \( R_0 \cup R_U \cup R_T \) we can transform an arbitrary word into a word in quasi-normal form.

**Lemma 11.** Let \( w \in (A \cup B \cup C \cup Z)^+ \) be arbitrary.

(i) If \( w \) represents an element of \( S \setminus T \) then there is a path \( \mathcal{P} \in \mathcal{P}_+(\Gamma) \) from \( w \) to some \( w' \in B^1A^*C^1 \).

(ii) If \( w \) represents an element of \( T \) then there is a path \( \mathcal{P} \in \mathcal{P}_+(\Gamma) \) from \( w \) to some \( w' \in Z^+ \). Moreover, if \( w \) contains a letter from \( Z \) then \( \mathcal{P} \) may be chosen with \( \mathcal{P} \in \mathcal{P}_+(\Gamma_0) \).

**Proof.** We prove (i) and (ii) simultaneously. The proof is by induction on the total number \( |w|_{B \cup C} \) of letters in the word that come from the set \( B \cup C \). If \( |w|_{B \cup C} = 0 \) then either \( w \)
contains a letter from $Z$, in which case there is an obvious path in $P_\alpha(T_0)$ from $w$ to a word $w' \in Z^+$, or $w \in A^+$, so $w$ represents an element of $G$ and is already written in the required form, and we are done by setting $w' \equiv w$ and $P$ to be the empty path.

Now suppose that $|w|_{B_0C} > 0$. If $w \in B^1A^*C^1 \cup Z^+$ then we are done by setting $w' \equiv w$, $P$ to be the empty path, and using the fact that the words in $B^1A^*C^1$ all represent elements of $S \setminus T$. Now suppose that $w \not\in B^1A^*C^1 \cup Z^+$. Then by inspection of the relations in the presentation $P_S$ it follows there exists a positive edge $\mathbb{E} \in P_+(\Gamma)$ with $v\mathbb{E} \equiv w$. Now the result follows by induction since by considering the relations $R_0 \cup R_U \cup R_T$ we see that with $v \equiv \tau\mathbb{E}$ we have $|v|_{B_0C} < |w|_{B_0C}$.

The last part of (ii) is an obvious consequence of the definition of $\Gamma_0$.

The following technical lemma gives some information about what happens if we try to rewrite a word into quasi-normal form by first splitting the word into three parts, and then rewriting each of the parts in turn.

**Lemma 12.** Let $w_1, w_3 \in B^1A^*C^1 \cup \{1\}$ and let $w_2 \in (A \cup B \cup C \cup Z)^+$ be arbitrary. Write $w_1 \equiv w_1'\alpha$ and $w_3 \equiv \beta w_3'$ where $\alpha, \beta \in A \cup B \cup C \cup \{1\}$, and where $\alpha$ (resp. $\beta$) is empty only if $w_1$ (resp. $w_3$) is empty. If $\alpha\omega_2\beta$ represents an element in $S \setminus T$ and $u = \alpha\omega_2\beta$ holds in $S$ with $u \in B^1A^*C^1$, then

$$w_1'uw_3' \in B^1A^*C^1.$$

In particular, $w_1w_2w_3 = w_1'uw_3'$ represents an element of $S \setminus T$.

**Proof.** There are several cases to consider.

**Case 1.** $w_1'$ and $w_3'$ are both empty.

In this case $w_1'uw_3' \equiv u \in B^1A^*C^1$ by assumption.

**Case 2.** Neither $w_1'$ nor $w_3'$ is empty.

In this case it follows that $\alpha \in C \cup A$ and $\beta \in B \cup A$, since $w_1, w_3 \in B^1A^*C^1$ where $w_1 \equiv w_1'\alpha$ and $w_3 \equiv \beta w_3'$. Now consider the element of the semigroup $S$ represented by the word $\alpha\omega_2\beta$. By assumption $\alpha\omega_2\beta$ represents an element in $S \setminus T$. Moreover since $\alpha \in C \cup A$ it follows from Lemma 10 that the element that $\alpha$ represents is a triple $(i, g, \lambda) \in I \times G \times \Lambda$ with $i = 1$. Similarly the letter $\beta$ represents a triple $(i, g, \lambda)$ with $\lambda = 1$. Therefore, since by assumption $\alpha\omega_2\beta$ represents an element of $S \setminus T$ it follows from the definition of multiplication in a 0-Rees matrix semigroup that it represents a triple of the form $(1, g, 1)$ for some $g \in G$. So by Lemma 10 since $u \in B^1A^*C^1$ and $u$ also represents the element $(1, g, 1)$ we conclude that in fact $u \in A^+$. In conclusion we have $w_1' \in B^1A^*$, $u \in A^*$ and $w_3' \in A^*C^1$, and hence $w_1'uw_3' \in B^1A^*C^1$, as required.

**Case 3.** $w_1'$ is non-empty and $w_3'$ is empty (and the dual of this case).

We argue in a similar way to the previous case. Since $w_1'$ is non-empty we have $\alpha \in A \cup C$, and since $w_3'$ is empty it follows that $|w_3'| \leq 1$. Now consider the word $\alpha\omega_2\beta$. Since $\alpha \in A \cup C$ it follows that $\alpha$ represents an element of the form $(1, g, \lambda)$ and hence, by the definition of multiplication in a 0-Rees matrix semigroup, since $\alpha\omega_2\beta$ represents an element of $S \setminus T$, it must represent an element $(1, h, \mu)$ for some $h \in G$, $\mu \in \Lambda$. Therefore since $u \in B^1A^*C^1$ with $u = \alpha\omega_2\beta$ in $S$, it follows from Lemma 10 that $u \in A^*C^1$. Since $w_1'$ being non-empty implies that $\alpha$ is non-empty we see that $w_1' \in B^1A^*$. Now since $w_3'$ is empty we have $w_1'uw_3' \equiv w_1'u \in B^1A^*C^1$ as required.

The fact that $w_1w_2w_3$ represents an element of $S \setminus T$ follows from Lemma 10.

**A homotopy base for $P_S$.** Now we begin the process of building a homotopy base for $P_S$. Given a homotopy base $X_G$ for the presentation $P_G = \langle R \rangle$, and a homotopy base $X_T$ for the presentation $P_T = \langle Z | Q \rangle$, we shall construct a homotopy base $X = X_1 \cup X_1' \cup X_2 \cup X_3 \cup \cup X_\alpha \cup X_G \cup X_T$ for the presentation $P_S$ where $X_1, X_1', X_2, X_3$ and
The parallel paths $X$ and $X'$. For every pair $β_1, β_2 ∈ Z^+$ such that $β_1 = β_2$ in $T$, let $P_{β_1, β_2}$ be a fixed path in $Γ_T$ from $β_1$ to $β_2$, where these paths are chosen in such a way that $P_{β_2, β_1} = P_{−1}^{-1}$, for all $β_1, β_2 ∈ Z^+$. Let $X_1$ be the set of parallel paths of $Γ_S$ of the form:

\[(x, l = r, +1, 1), (1, xz = τ(x, z), +1, l')\]

\[(P_{τ(x, z)'}, τ(x, z')r')(1, xz' = τ(x, z'), −1, r')\]

where $x ∈ A ∪ B ∪ C$, $(l, r) ∈ Q$, and $l ≡ zl', r ≡ z'r'$ where $z, z' ∈ Z$. Let $X'_1$ be the set of parallel paths of $Γ_S$ of the form:

\[(1, l = r, +1, x), (l', xx = σ(x, x), +1, 1)\]

\[(P_{τ(x, x)'}, τ(x, x)r')(r', xz = σ(x', x), −1, 1)\]

where $x ∈ A ∪ B ∪ C$, $(l, r) ∈ Q$, and $l ≡ zl', r ≡ z'r'$ where $z, z' ∈ Z$. Note that $X_1$ and $X'_1$ are both finite when $Q, A, B,$ and $C$ are all finite.

**Lemma 13.** Let $P$ be a non-empty path in $Γ_T$ and let $u, v ∈ (A ∪ B ∪ C ∪ Z)^*$ such that $uv$ is non-empty. Then there exist paths $Q_1, Q_2 ∈ P_+(Γ_0)$, words $u', v' ∈ (A ∪ B ∪ C ∪ Z)^*$, and a path $P' ∈ Γ_T$ such that

\[u · P · v ∼ X_1 ∪ X'_1 Q_1(u' · P' · v')Q_2^{-1}\]

and with $|u'v'| < |uv|$. 

**Proof.** Let $P = E_1 ∪ E_2 ∪ ... ∪ E_n$ be a non-empty path in $Γ_T$ and let $u, v ∈ (A ∪ B ∪ C ∪ Z)^*$ such that $uv$ is non-empty. Suppose that $u$ is non-empty. The case that $v$ is non-empty is dealt with using a dual argument and the set of parallel paths $X'_1$. Decompose $u ≡ u'x$ where $x ∈ A ∪ B ∪ C ∪ Z$ and $u'$ is a, possibly empty, word over the same alphabet.

If $x ∈ Z$ then the path $(x · P)$ also belongs to $Γ_T$ and the result follows by setting $Q_1, Q_2$ to be empty, $P' := (x · P)$, $u' = u'$ and $v' = v$.

Now suppose $x ∉ Z$. Let $z_i$ be the first letter of $iE_i$ for $i = 1, 2, ..., n$, and let $z_{n+1}$ be the first letter of $τE_n$, noting that each $z_i$ belongs to $Z$. For each $i = 1, ..., n + 1$ define

\[F_i = (u', xz_i = τ(x, z_i), +1, w_i)\]

where the $w_i$ are obtained by the identities $iE_i ≡ z_iw_i$ (for $i = 1, ..., n$) and $τE_n ≡ z_{n+1}w_{n+1}$. Note that we can have $xz_i ≠ xz_{i+1}$ in $S$ for some (or even all) values of $i$.

Now for each $i ∈ \{1, ..., n\}$ we shall define a path $S_i$ of $Γ_T$. For each $i ∈ \{1, ..., n\}$ let $E_i = (γ_i, l_i = r_i, ε_i, δ_i)$. The definition of $S_i$ varies depending on whether or not $γ_i$ is empty.

Suppose first that $γ_i$ is non-empty, so that $γ_i ≡ z_iγ'_i$ and $z_{i+1} ≡ z_i$ which implies $τ(x, z_i) ≡ τ(x, z_{i+1})$. In this case we define

\[S_i = (τ(x, z_i)γ'_i, l_i = r_i, ε_i, δ_i)\]

which is an edge of $Γ_T$. Applying (H1) we observe that

\[(u · E_i · v)F_{i+1} ∼ F_i(u'τ(x, z_i)γ'_i, l_i = r_i, ε_i, δ_i)\]

Rearranging this expression gives

\[u · E_i · v ∼ F_i(u' · S_i · v)F_{i+1}^{-1}.\]
On the other hand, if \( \gamma_i \) is empty, so that \( l_i \equiv z_i l'_i \) and \( r_i \equiv z_{i+1} r'_i \), we define
\[
S_i = P_{r(x,z_i)l'_i} \cdot P_{r(x,z_{i+1})r'_i} \cdot \delta_i,
\]
which is a path in \( \Gamma_T \). By the definition of \( X_1 \) we have
\[
\begin{align*}
    u \cdot E_i \cdot v &= (u' x \gamma_i, l'_i = r_i, \epsilon_i, \delta_i v)
    = u' \cdot (x, l'_i = r_i, \epsilon_i, 1) \cdot \delta_i v \\
    \sim_{X_1} u' \cdot (1, x z_i = \tau(x, z_i), +1, l'_i) \cdot \delta_i v
    = u' \cdot P_{r(x,z_i)l'_i} \cdot \delta_i v \\
    &\sim_{X_1} u' \cdot (1, x z_{i+1} = \tau(x, z_{i+1}), -1, r'_i) \cdot \delta_i v \\
    &= F_i (u' \cdot S_i \cdot v) F_i^{-1}.
\end{align*}
\]
Combining these observations completes the proof of the lemma since
\[
\begin{align*}
    u \cdot P \cdot v &= (u' \cdot E_1 \cdot v) (u \cdot E_2 \cdot v) \ldots (u \cdot E_n \cdot v) \\
    \sim_{X_1} F_1 (u' \cdot S_1 \cdot v) F_2^{-1} F_2 (u' \cdot S_2 \cdot v) F_3^{-1} \ldots F_n^{-1} (u' \cdot S_n \cdot v) F_n^{-1} \\
    \sim F_1 (u' \cdot S_1 S_2 \ldots S_n \cdot v) F_n^{-1} \\
    &= Q_1 (u' \cdot P' \cdot v') Q_2
\end{align*}
\]
where \( Q_1 = F_1, Q_2 = F_n^{-1}, v' \equiv v \) and \( P' = S_1 S_2 \ldots S_n \) is a path in \( \Gamma_T \).

**Corollary 4.** Let \( P \) be a path in \( \Gamma_T \) and let \( u, v \in (A \cup B \cup C \cup Z)^* \). Then there exist paths \( P_1 \) and \( P_2 \) in \( P_+(\Gamma_T) \) and a path \( Q \) in \( P(\Gamma_T) \) such that
\[
u \sim_{X_1 \cup X_1} P_1 Q P_2^{-1}.
\]

**Proof.** If \( P \) is empty then the result holds trivially by applying Lemma \( \text{[1]} \)(ii). Otherwise the result follows by repeated application of Lemma \( \text{[13]} \). \( \Box \)

**The parallel paths \( X_2 \).** For every edge \( E \) in \( \Gamma_S \) by Lemmas \( \text{[1]} \) and \( \text{[10]} \) we can fix paths \( P_{ie} \), \( P_{re} \) in \( P_+(\Gamma) \) and \( Q_E \) such that
\[
Q_E \in \begin{cases} P(\Gamma_T) \cdot C^1 & \text{if } e \text{ represents an element of } \Gamma_T, \\ B^1 \cdot P(\Gamma_G) \cdot C^1 & \text{if } e \text{ represents an element of } S \setminus T \end{cases}
\]
and where \( iP_{ie} \equiv \tau E, rP_{re} \equiv \tau E, \tau P_{re} \equiv \tau Q_E \) and \( \tau P_{ie} \equiv \tau Q_E \). Let \( X_2 \) be the set of parallel paths
\[
\{ (E, P_E Q_E P_{\tau E}) : E = (\alpha, r, \epsilon, \beta) \in P(\Gamma_S), \alpha, \beta \in A \cup B \cup C \cup Z \cup \{ 1 \} \text{ & } \epsilon \in \{ +1, -1 \} \}.
\]
Note that \( X_2 \) is finite when \( \Gamma_S \) is a finite presentation.

**Lemma 14.** Let \( E \) be an edge in \( \Gamma_S \).

(i) If \( \tau E \) represents an element of \( S \setminus T \) then there exist paths \( P_3, P_4 \in P_+ (\Gamma) \) and \( Q \in B^1 \cdot P(\Gamma_G) \cdot C^1 \) such that \( E \sim_{X_2} P_3 Q P_4^{-1} \).

(ii) If \( \tau E \) represents an element of \( T \) then there exist paths \( P_3, P_4 \in P_+ (\Gamma) \) and \( Q \in P(\Gamma_T) \) such that \( E \sim_{X_2} P_3 Q P_4^{-1} \).

**Proof.** (i) Let \( E = (u, r, \epsilon, v) \) be an arbitrary edge of the graph \( \Gamma_S \). Since \( \tau E \) represents an element of \( S \setminus T \), and since \( T \) is an ideal, it follows that every subword of the word \( \tau E \) also represents an element of \( S \setminus T \). It now follows by Lemma \( \text{[1]}(i) \) there is a path \( P_1 \in P_+(\Gamma) \) from \( u \) to some \( u' \in B^1 A^* C^1 \), and a path \( P_2 \in P_+(\Gamma) \) from \( v \) to some \( v' \in B^1 A^* C^1 \). Appling \( \text{[1]} \) gives
\[
E \sim (P_1 \cdot r \cdot v)(u' r_\epsilon \cdot P_2)(u', r, \epsilon, v')(P^{-1} \cdot r_\epsilon v')(w r_\epsilon \cdot P_2^{-1}).
\]
Now consider the edge \( E' = (u', r, \epsilon, v') \). Write \( u' \equiv u' \alpha \) and \( v' \equiv v \beta \) where \( \alpha, \beta \in A \cup B \cup C \cup \{ 1 \} \), and where \( \alpha \) (resp. \( \beta \)) is empty only if \( u' \) (resp. \( v' \)) is empty. Putting \( F = (\alpha, r, \epsilon, \beta) \), from the definition of \( X_2 \) we have
\[
E' = u'' \cdot F \cdot v'' \sim_{X_2} u'' \cdot P_{ie} Q_E P_{\tau E}^{-1} \cdot v''.
\]
Next by applying Lemma 12 since $\iota Q_E, \tau Q_F \in B^1 \cdot A^* \cdot C^1$ we deduce that $u''v Q_E v'', u''\tau Q_F v'' \in B^1 A^* C^1$ and by the definitions of $Q_E$ and of $B^1 \cdot P(\Gamma_G) \cdot C^1$ this implies

$$u'' \cdot Q_E \cdot v'' \in B^1 \cdot P(\Gamma_G) \cdot C^1.$$ 

This completes the proof of part (i) since we have $E \sim_X \tau_3 Q_3 P_3^{-1}$ where $P_3 = (P_1 \cdot r \cdot v) (u' r \cdot P_2) (u'' \cdot P_2 \cdot P_1)$, $P_4 = (u' \cdot P_2 \cdot (P_1 \cdot r \cdot v') (u'' \cdot P_2')$, and $Q = u'' \cdot Q_E \cdot v'' \in B^1 \cdot P(\Gamma_G) \cdot C^1$.

(ii) We proceed in much the same way as in part (i) except that here we need also to apply Corollary 4. Let $E = (u_\gamma, r, \iota, \tau)$ be an arbitrary edge of the graph $\Gamma_S$. It now follows by Lemma 11 that there is a path $P_1 \in P_+(\Gamma)$ from $u$ to some $u' \in B^1 A^* C^1 \cup Z^+$ (depending on whether $u$ represents an element of $T$ or of $S \setminus T$), and a path $P_2 \in P_+(\Gamma)$ from $v$ to some $v' \in B^1 A^* C^1 \cup Z^+$. Applying (H1) gives

$$E \sim (P_1 \cdot r \cdot v) (u' r \cdot P_2) (u', r, \iota, v') (P_2 \cdot r \cdot v') (u' \cdot P_2^{-1}) (u \cdot P_2^{-1}).$$

Now consider the edge $E' = (u', r, \iota, v')$. Write $u' \equiv u'' \alpha$ and $v' \equiv \beta v''$ where $\alpha, \beta \in Z \cup A \cup B \cup C \cup \{1\}$, and where $\alpha$ (resp. $\beta$) is empty only if $u'$ (resp. $v'$) is empty. It now follows from Lemma 12 that the word $\alpha \beta \gamma \alpha$ (and hence also the word $\alpha \beta \gamma$) represents an element of $T$. Indeed, if $\alpha \beta \gamma$ represented an element of $S \setminus T$ then by Lemma 12 it would follow that $ur_{\iota, \alpha} \beta \gamma$ represents an element of $S \setminus T$, contrary to the assumption that $ur_{\iota, \alpha} \beta \gamma$ represents an element of $T$.

Now putting $F = (\alpha, r, \iota, \tau)$ from definition of $X_2$ we have

$$E' \sim_{X_2} u'' \cdot P_{r E}^{-1} \cdot P_{Q F}^{-1} \cdot v''.$$ 

Since both $\alpha \beta \gamma$ and $\alpha \beta \gamma$ represent an element of $T$ it follows that $\iota Q_E, \tau Q_F \in Z^+$ and hence by definition $Q_E \in P(\Gamma_T)$. Now by Corollary 4 applied to $u'' \cdot Q_E \cdot v''$ there are paths $P_3, P_4 \in P_+(\Gamma)$ and $Q \in P(\Gamma_T)$ such that

$$u'' \cdot Q_E \cdot v'' \sim_X P_3 Q_3 P_3^{-1}.$$

This completes the proof of part (ii) since we have $E \sim_X P_3 Q_3 P_3^{-1}$ where $P_3 = (P_1 \cdot r \cdot v) (u' r \cdot P_2) (u'' \cdot P_2 \cdot P_1)$, $P_4 = (u' \cdot P_2 \cdot (P_1 \cdot r \cdot v') (u'' \cdot P_2')$, and $Q \in P(\Gamma_T)$. \hfill \square

The parallel paths $X_3$. For any pair of positive edges $E_1, E_2$ from $P_+(\Gamma)$ with $\iota E_1 \equiv \iota E_2$, by Lemma 11 and Lemma 10 we can fix paths $P_{\tau E_1}, P_{\tau E_2} \in P_+(\Gamma)$ and

$$Q_{E_1, E_2} \in \begin{cases} \{B^1 \cdot P(\Gamma_G) \cdot C^1\} & \text{if } \iota E_1 \text{ represents an element of } S \setminus T \\ P(\Gamma_T) & \text{if } \iota E_1 \text{ represents an element of } T \end{cases}$$

such that $\tau E_1 \equiv \tau P_{\tau E_1}, \tau E_2 \equiv \tau P_{\tau E_2}, \tau P_{\tau E_1} \equiv \tau Q_{E_1, E_2}$ and $\tau P_{\tau E_2} \equiv \tau Q_{E_1, E_2}$. We then let $X_3$ be the following set of parallel paths:

$$\{(E_1 P_{\tau E_1}, E_2 P_{\tau E_2}) : E_1, E_2 \in P_+(\Gamma), \iota E_1 \equiv \iota E_2 \& |\iota E_1| \leq 2|\iota| + 3\}.$$ 

Recall that $e$ is the word from $A^+$ representing the identity of the group $G$ that appears in the presentation $P_S$. The restriction $|\iota E_1| \leq 2|\iota| + 3$ comes from consideration of lengths of words appearing on left hand sides of the relations in $R_0 \cup R_U \cup R_T$.

If $P_S$ is a finite presentation then $X_3$ is finite. To see this, first observe that for every vertex $w$ of $\Gamma_S$ the number of edges $E$ of $\Gamma_S$ with $\iota E \equiv w$ is finite, as $w$ has finite length and there are only finitely many relations that can be applied to $w$. So since in the definition of $X_3$, $\iota E_1$ and $\iota E_2$ can be only one of finitely many words, since the generating set in the presentation $P_S$ is finite, it follows that there are only finitely many possibilities for $E_1$ and $E_2$. Hence $X_3$ is finite.

For the following proof the notion of the applications of relations to a word “overlapping” is important. For two edges $E_1, E_2 \in P_+(\Gamma)$ with $\iota E_1 \equiv \iota E_2$ we say that $E_1$ and $E_2$ do not overlap if we can write $E_1 \equiv ur_{\iota, v} w, w, w \in (A \cup B \cup C \cup Z)^*$ and

$$E_1, E_2 \} = \{(u, r_{\iota} = r_{\iota} + 1, + 1, vs_{\iota} + w), (ur_{\iota} v, s_{\iota} = s_{\iota} + s_{\iota} + 1, w)\}. $$
Otherwise we say that \( E_1 \) and \( E_2 \) overlap.

**Lemma 15.** Let \( E_1, E_2 \in P_+ (\Gamma) \) with \( iE_1 \equiv iE_2 \).

(i) If \( iE_1 \) represents an element of \( S \setminus T \) then there are paths \( P, P' \in P_+ (\Gamma) \) and \( Q \in B^1 \cdot P(\Gamma_G) \cdot C^1 \) such that \( E_1 \cdot PQ \sim_X E_2 \cdot P' \).

(ii) If \( iE_1 \) represents an element of \( T \) then there are paths \( P, P' \in P_+ (\Gamma) \) and \( Q \in P(\Gamma_T) \) such that \( E_1 \cdot PQ \sim_X E_2 \cdot P' \).

**Proof.** (i) If \( E_1 = E_2 \) then the result holds trivially by Lemma 11, so suppose that \( E_1 \neq E_2 \).

Next suppose that \( E_1 \) and \( E_2 \) do not overlap. So without loss of generality we have \( iE_1 \equiv iE_2 \equiv ur + vs_{+1} w, \ E_1 = (u, r_{+1} = r_{-1}, +1, vs_{+1} w), \) and \( E_2 = (ur + v, s_{+1} = s_{-1}, +1, w) \).

Then define \( F_1 = (ur_{-1} v, s_{+1} = s_{-1}, +1, w) \) and \( F_2 = (u, r_{+1} = r_{-1}, +1, vs_{-1} w) \). By Lemma 11 we can fix a path \( \bar{P} \in P^+ (\Gamma) \) from \( ur_{-1} vs_{-1} w \) to a word \( x \in B^1 A^* C^1 \). Now set \( Q \) to be the empty path \( 1_x \), and set \( P = F_1 P_1 \) and \( P' = F_2 P_1 \). Applying (H1) we have \( E_1 F_1 \sim E_2 F_2 \) and hence \( E_1 PQ \sim E_2 P' \), as required.

Now suppose that \( E_1 \) and \( E_2 \) do overlap. It follows that there exist \( F_3, F_4 \in P_+ (\Gamma) \) and words \( w_1, w_2 \in (A \cup B \cup C \cup Z)^* \) such that \( iF_3 = iF_4, E_1 = w_1 \cdot F_3 \cdot w_2, E_2 = w_1 \cdot F_4 \cdot w_2 \) and \( |F_3| = |F_4| \leq 2 |e| + 1 \). This inequality comes from the fact that \( E_1 \) and \( E_2 \) overlap, and that the lengths of the words appearing as left hand sides of the relations in \( R_0 \cup R_U \cup R_T \) are bounded above by \( |e| + 1 \).

As in the proof of Lemma 12(i), applying Lemma 11(i) there is a path \( P_1 \in P_+ (\Gamma) \) from \( w_1 \) to some word \( w'_1 \in B^1 A^* C^1 \), and a path \( P_2 \in P_+ (\Gamma) \) from \( w_2 \) to some word \( w'_2 \in B^1 A^* C^1 \). Write \( w'_1 \equiv w''_1 \alpha \) and \( w'_2 \equiv w''_2 \beta \) where \( \alpha, \beta \in A \cup B \cup C \cup \{ \epsilon \} \), and where \( \alpha \) (resp. \( \beta \)) is empty only if \( w''_1 \) (resp. \( w''_2 \)) is empty. Let \( E'_1 = w''_1 \cdot (\alpha \cdot F_3 \cdot \beta) \cdot w''_2, \quad E'_2 = w''_2 \cdot (\alpha \cdot F_4 \cdot \beta) \cdot w''_2 \).

Applying (H1) we obtain

\[
E_1(P_1 \cdot \tau F_3 w_2)(w'_1 \tau F_3 \cdot P_2) \sim (P_1 \cdot iF_3 w_2)(w'_1 iF_3 \cdot P_2)E'_1
\]

and

\[
E_2(P_1 \cdot \tau F_4 w_2)(w'_1 \tau F_4 \cdot P_2) \sim (P_1 \cdot iF_4 w_2)(w'_1 iF_4 \cdot P_2)E'_2
\]


\[
= (P_1 \cdot iF_3 w_2)(w'_1 iF_3 \cdot P_2)E'_2
\]

since \( iF_3 \equiv iF_4 \). Then by definition of \( X_3 \), and since \( |i(\alpha \cdot F_3 \cdot \beta)| = |iF_3| \leq 2 |e| + 3 \) we have

\[
((\alpha \cdot F_3 \cdot \beta)P_{\tau(\alpha \cdot F_3 \cdot \beta)}Q_{\alpha \cdot F_3 \cdot \beta}P_{\tau(\alpha \cdot F_3 \cdot \beta)}) \in X_3.
\]

Hence by acting on the left by \( w''_1 \) and on the right by \( w''_2 \) we obtain

\[
E'_1(w''_1 \cdot P_{\tau(\alpha \cdot F_3 \cdot \beta)}Q_{\alpha \cdot F_3 \cdot \beta}P_{\tau(\alpha \cdot F_3 \cdot \beta)} \cdot w''_2)
\]

\[
\sim_X E'_2(w''_2 \cdot P_{\tau(\alpha \cdot F_4 \cdot \beta)}Q_{\alpha \cdot F_4 \cdot \beta}P_{\tau(\alpha \cdot F_4 \cdot \beta)} \cdot w''_2).
\]

To complete the proof of part (i) we now set

\[
P = (P_1 \cdot \tau F_3 w_2)(w'_1 \tau F_3 \cdot P_2)(w''_1 \cdot P_{\tau(\alpha \cdot F_3 \cdot \beta)} \cdot w''_2),
\]

which is a path in \( P_+ (\Gamma) \), define

\[
P' = (P_1 \cdot \tau F_4 w_2)(w'_1 \tau F_4 \cdot P_2)(w''_1 \cdot P_{\tau(\alpha \cdot F_4 \cdot \beta)} \cdot w''_2),
\]

which is a path in \( P_+ (\Gamma) \), and set \( Q = w''_1 \cdot Q_{\alpha \cdot F_3 \cdot \beta}P_{\tau(\alpha \cdot F_3 \cdot \beta)} \cdot w''_2 \). As in the proof of Lemma 14 to see that \( Q \in B^1 \cdot P(\Gamma_G) \cdot C^1 \) we apply Lemma 14 to \( w''_1 (\alpha \tau F_3 \beta)w''_2 \) and \( w''_1 (\alpha \tau F_4 \beta)w''_2 \), deducing that \( iQ, \tau Q \in B^1 A^* C^1 \), which along with the definition of \( Q_{\alpha \cdot F_3 \cdot \beta}P_{\tau(\alpha \cdot F_3 \cdot \beta)} \cdot w''_2 \) implies that \( Q \in B^1 \cdot P(\Gamma_G) \cdot C^1 \).

(ii) We proceed in much the same way as in part (i) except that here we need also to apply Corollary 4. The cases when \( E_1 = E_2 \), or when \( E_1 \) and \( E_2 \) do not overlap, are dealt with in exactly the same way as in part (i).
Now suppose that $E_1$ and $E_2$ do overlap. As in part (i) it follows that there exist $P_3, P_4 \in P_+(\Gamma)$ and words $w_1, w_2 \in (A \cup B \cup C \cup Z)^*$ such that $\iota P_3 \equiv \iota P_4$, $E_1 \equiv w_1 \cdot P_3 \cdot w_2$, $E_2 \equiv w_1 \cdot P_4 \cdot w_2$ and $|\iota P_3| = |\iota P_4| = 2|e| + 1$.

As in the proof of Lemma 11(ii), applying Lemma 11 there is a path $P \in P_+(\Gamma)$ from $w_1$ to some word $w'_1 \in B^1 A^* C^1 \cup Z^+$, and a path $P_2 \in P_+(\Gamma)$ from $w_2$ to some $w'_2 \in B^1 A^* C^1 \cup Z^+$. Write $w'_1 \equiv w''_1 \alpha$ and $w'_2 \equiv w''_2 \beta$ where $\alpha, \beta \in Z \cup A \cup B \cup C \cup \{1\}$, and where $\alpha$ (resp. $\beta$) is empty only if $w'_1$ (resp. $w'_2$) is empty. Let $E'_1$ and $E'_2$ be defined in the same way as in part (i). Then in exactly the same way as in part (i) we can prove that the equations (7), (8) and (9) all hold. Here we set $Q' = w''_1 \cdot P_{\alpha \cdot P_4 \cdot \beta} \cdot w''_2$.

Next, in contrast to part (i), we claim that here $Q'$ also represents an element of $S \setminus T$, a contradiction. Similarly we see that $Q''$ also represents an element of $T$, and this completes the proof that $Q_{\alpha \cdot P_4 \cdot \beta} \in P(\Gamma)$. Corollary 3 there exist paths $P_3, P_4 \in P_+(\Gamma)$ and $Q \in P(\Gamma T)$ such that $Q' \sim_{X} P_3 Q P_4^1$. This completes the proof of the lemma, with $Q$ defined as above, and by setting $P = (P_1 \cdot \tau P_3 w_2) (w'_1 \tau P_3 \cdot P_2) (w''_1 \cdot P_{\tau(\alpha \cdot P_4 \cdot \beta)} \cdot w''_2) P_3$ and $P' = (P_1 \cdot \tau P_4 w_2) (w'_1 \tau P_4 \cdot P_2) (w''_1 \cdot P_{\tau(\alpha \cdot P_4 \cdot \beta)} \cdot w''_2) P_4$, which are both paths in $P_+(\Gamma)$.

The next thing we want to do is to extend Lemma 15 to a statement about paths. To do this we need to introduce a function on words that, roughly speaking, gives a measure of how close that word is to being in quasi-normal form $B^1 A^* C^1 \cup Z^+$.

As in the proof of Lemma 11 given a word $w \in W^+$ and a subset $V \subseteq W$ we write $|w|_V$ to denote the total number of letters of $w$ that belong to $V$. Now we define $F : (A \cup B \cup C \cup Z)^+ \rightarrow \mathbb{N} \times \mathbb{N}$ $F(w) = (|w|_{B \cup C}, |w|_A)$. Note that $F(w)$ does not depend in any way on the number of letters from $Z$ in $w$. Now define the following total order $< on \mathbb{N} \times \mathbb{N}$ by $(n_1, m_1) < (n_2, m_2) \Leftrightarrow \begin{cases} n_1 < n_2 \mbox{ or } \\ n_1 = n_2 \mbox{ and } m_1 < m_2. \end{cases}$ Clearly with this ordering $(\mathbb{N} \times \mathbb{N}, <)$ is well-founded (since it is the lexicographic product of two well-founded orders) with a unique minimal element $(0, 0)$, which allows us to apply well-founded induction (as we shall do below in the proof of Lemma 17).

Also define addition on pairs of natural numbers in the usual way where $(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$. The idea behind the definition of $F$ is that given a vertex of $\Gamma$ if there are no edges from $P_+(\Gamma)$ coming out of it, it means that that the vertex belongs either to $B^1 A^* C^1 \cup Z^+$, and as the number $F(w)$ decreases, the closer we are to having our word in quasi-normal form.

Lemma 16. The function $F$ has the following properties.

(i) $F(w_1 w_2) = F(w_1) + F(w_2)$ for all words $w_1, w_2$;
(ii) $F(w) = (0, 0)$ if and only if $w \in Z^+$; 
(iii) $F(\iota E) < F(\iota E)$ for all $E \in P_+(\Gamma)$. 


Proof. Parts (i) and (ii) follows straight from the definitions. Part (iii) is proved by using part (i) and checking that (iii) holds for the edges $E = (1, r, +1, 1)$ where $r \in R_0 \cup R_U \cup R_T$. \hfill \Box

Lemma 17. Let $P_1, P_2 \in P_+(\Gamma)$ with $\iota P_1 \equiv \iota P_2$.

(i) If $\iota P_1$ represents an element of $S \setminus T$ then there are paths $P, P' \in P_+(\Gamma)$ and $Q \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P_1 P Q \sim_X P_2 P'$.

(ii) If $\iota P_1$ represents an element of $T$ then there are paths $P, P' \in P_+(\Gamma)$ and $Q \in P(\Gamma_T)$ such that $P_1 P Q \sim_X E_2 P'$.

Proof. For $m, n \in \mathbb{N}$, let $R(m, n)$ denote the following statement:

$R(m, n)$: Lemma 17 holds for all paths $P_1$ and $P_2$ such that $F(\iota P_1) = (m, n)$.

We shall prove $R(m, n)$ for all $m, n \in \mathbb{N}$ by well-founded induction. First note that $R(0, 0)$ holds trivially. Now we shall prove

For any $m, n \in \mathbb{N}$, if $R(x, y)$ holds for all $(x, y) < (m, n)$ then $R(m, n)$ holds.

First consider part (i). Let $P_1, P_2 \in P_+(\Gamma)$ with $\iota P_1 \equiv \iota P_2$ and where $\iota P_1$ represents an element of $S \setminus T$. If $\iota P_1$ belongs to $B^1 \cdot A^+ \cdot C^1$ then the result holds trivially, so suppose otherwise. Also, if one of $P_1$ or $P_2$ is the empty path then the result is easily seen to be true by applying Lemma 14(i), so suppose not. So we can write $P_1 = E_1 P_1'$ and $P_2 = E_2 P_2'$ where $E_i$ are edges in $P_+(\Gamma)$ and $P_1'$ and $P_2'$ both belong to $P_+(\Gamma)$. By Lemma 15(i) there are paths $P_3, P_4 \in P_+(\Gamma)$ and $Q' \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $E_1 P_3 Q' \sim_X E_2 P_4$.

Now $P_1', P_3 \in P_+(\Gamma)$, $\iota P_1' \equiv \iota P_3$ and by Lemma 14(ii) we have $F(\iota P_1') < F(\iota P_1)$. So by the inductive hypothesis there exist paths $P_1', Q_1 \in P_+(\Gamma)$ and $Q_1 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P_1' P_1 Q_1 \sim_X P_3$. Likewise there exist paths $P_2', Q_2 \in P_+(\Gamma)$ and $Q_2 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P_2' P_2 Q_2 \sim_X P_2'$. Combining these observations together, we have found $P_1', Q_1, Q_2 \in P_+(\Gamma)$, and $Q_1 Q_2 \equiv P_1 P_2^1$. As required.

The proof of part (ii) is done in exactly the same way as the proof of part (i) but by applying Lemma 14(iii).

The following corollary is now immediate.

Corollary 5. Let $P_1, P_2 \in P_+(\Gamma)$ with $\iota P_1 \equiv \iota P_2$.

(i) If $\iota P_1$ represents an element of $S \setminus T$ and $\tau P_1, \tau P_2 \in B^1 \cdot A^+ \cdot C^1$ then there is a path $Q \in B^1 \cdot P(\Gamma_G) \cdot C$ such that $P_1 \sim_X P_2 Q$.

(ii) If $\iota P_1$ represents an element of $T$ and $\tau P_1, \tau P_2 \in Z^+$ then there is a path $Q \in P(\Gamma_T)$ such that $P_1 \sim_X P_2 Q$.

Lemma 18. Let $P$ be any path in $\Gamma_S$.

(i) If $\iota P$ represents an element of $S \setminus T$ then there exist paths $P_1', P_m \in P_+(\Gamma)$ and $Q \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P \sim_X P_1' P_m Q_{m-1}$.

(ii) If $\iota P$ represents an element of $T$ then there exist paths $P_1', P_m \in P_+(\Gamma)$ and $Q \in P(\Gamma_T)$ such that $P \sim_X P_1' Q_{m-1}$.

Proof. (i) If $P$ is empty then the result holds trivially by Lemma 11, so suppose otherwise. Let $P = E_1 E_2 \ldots E_m$, where each $E_i$ is an edge of $\Gamma_S$. By Lemma 14(i) there are paths $P_i, P_i' \in P_+(\Gamma)$ and $Q_i \in B^1 \cdot P(\Gamma_G) \cdot C^1$, such that $\tau P_i, \tau P_i' \in B^1 \cdot A^+ \cdot C^1$ and $E_i \sim_X P_i' Q_i P_i^{-1}$ for $i = 1, \ldots, m$. Hence

\[ P \sim_X P_1' Q_1 P_1^{-1} P_2' Q_2 P_2^{-1} \ldots P_m' Q_m P_m^{-1}. \]
Since $\iota P_i \equiv \iota P'_i$ and $\tau P_i, \tau P'_i \in B^1 \cdot A^* C^1$, by Corollary 5(i) there is a path $Q' \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P'_i \sim_X P_i Q'_i$ so that $P_i^{-1} P'_i \sim_X Q_i$. Thus

$$P \sim_X P_i Q_1 Q'_1 Q_2 Q'_2 \ldots Q_{m-1} Q_m' \sim P_m^{-1},$$

as required.

The proof of part (ii) is done in exactly the same way as part (i), except that we apply Lemma 14(ii) and Corollary 5(ii). \hfill $\square$

**The parallel paths $X_e$.** Let $X_e$ denote the following set of parallel paths of $\Gamma_S$

$$\{(b_i, ec_\lambda = c_\lambda + 1, 1), (1, b_i e = b_i, +1, c_\lambda ) : i \in I \setminus \{1\}, \lambda \in \Lambda \setminus \{1\}\}.$$

Note that $X_e$ is finite when both $I$ and $\Lambda$ are finite.

**Lemma 19.** Let $Q$ be a path in $B^1 \cdot P(\Gamma_G) \cdot C^1$.

(i) If neither $\iota Q$ nor $\tau Q$ belongs to $B \cup C \cup BC$ then there exists a path $Q'$ in $B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $Q \sim_X Q'$.

(ii) If both $\iota Q$ and $\tau Q$ belong to $B \cup C \cup BC$ then it follows that $\iota Q \equiv \tau Q$, and there exists a path $Q'$ in $B^1 \cdot P(\Gamma_G) \cdot C^1$ and edges $E_1, E_2$ of $B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $Q \sim_X E_1 Q' E_2$ and $E_1 \sim_X E_2^{-1}$.

**Proof.** (i) Suppose that neither $\iota Q$ nor $\tau Q$ belongs to $B \cup C \cup BC$. If $Q$ is empty or is already a path in $B^1 \cdot P(\Gamma_G) \cdot C^1$ then the result is trivial. Otherwise there exists an edge $E$ of the path $Q$ with $\tau E \in B \cup C \cup BC$. By the construction of $S$ it follows that in the path $Q$ the edge $E$ must be immediately followed by its inverse $E^{-1}$. On the other hand, if $\iota E \in BC$ then the edge $E$ equals either $E_i = (b_i, ec_\lambda, +1, 1)$ or $E_\lambda = (1, b_i e = b_i, +1, c_\lambda)$ for some $i \in I \setminus \{1\}$, $\lambda \in \Lambda \setminus \{1\}$. By construction of the relations in the presentation $P_S$ it follows that in the path $Q$ the edge $E$ must be immediately followed by its inverse $E^{-1}$. On the other hand, if $\tau E \in BC$ then the edge $E$ equals either $E_i = (b_i, ec_\lambda, +1, 1)$ or $E_\lambda = (1, b_i e = b_i, +1, c_\lambda)$, for some $i \in I \setminus \{1\}$, $\lambda \in \Lambda \setminus \{1\}$. In this case $\tau E = b_i c_\lambda$ and in the path $Q$ the edge $E$ must be immediately followed by $E_{c_i}^{-1}$ or $E_\lambda^{-1}$. Since $(E_{c_i}, E_\lambda) \in X_e$ it follows that in any of these situations we can remove $E$ and the edge that follows it from the path $Q$ to obtain a shorter path that is $\sim_X$ related to the original. Repeating this process, after a finite number of steps we obtain the desired parallel path $Q'$ from $B^1 \cdot P(\Gamma_G) \cdot C^1$ with $Q \sim_X Q'$.

(ii) Suppose that $\iota Q, \tau Q \in B \cup C \cup BC$. The fact that $\iota Q \equiv \tau Q$ follows from Lemma 11. If $Q$ is empty then the result holds trivially by setting $E_1$ to be any edge with initial vertex $\iota Q$ and terminal vertex in $BA^+$ or $CA^+$ or $BA^+ C$ depending on the value of $\iota Q$ and letting $E_2 = E_1^{-1}$. Now suppose that $Q$ is not empty, and let $E$ be its first edge. Now $Q$ is a closed path, and $\iota Q$ belongs either to $B, C$ or $BC$. In the first two cases there is only one edge in $B^1 \cdot P(\Gamma_G) \cdot C^1$ with initial vertex $\iota E_1$ and hence since $Q$ is closed, the last edge of $Q$ must be $E_1^{-1}$. Then the result follows by applying part (i).

On the other hand, if $\iota E_1 \in BC$, then the last edge $E_2$ of $Q$ satisfies $\tau E_1 \equiv \iota E_2 \equiv b_i c_\lambda$, for some $i \in I$, $\lambda \in \Lambda$, and either $E_4 = E_2^{-1}$ or $(E_1, E_2) \in X_e$. Now the result follows by applying part (i). \hfill $\square$

We are now in a position to complete the proof of Theorem 3. With the definitions given above we define $X = X_1 \cup X'_1 \cup X_2 \cup X_3 \cup X_e \cup X_G \cup X_T$ where $X_G$ is a homotopy base for the presentation $<A,R>$ of $G$, and $X_T$ is a homotopy base for the presentation $<Z,Q>$ of $T$. Here $X_G$ is a set of parallel paths in the subgraph $\Gamma_G$ of $\Gamma_S$, and $X_T$ is a set of parallel paths in the subgraph $\Gamma_T$ of $\Gamma_S$.

The proof of Theorem 3 is then wrapped up with the following lemma.

**Lemma 20.** The set $X$ is a homotopy base for the presentation $P_S$ of $S$.

**Proof.** Let $P_1, P_2 \in P(\Gamma_S)$ with $(P_1, P_2) \in \parallel$. We must prove that $P_1 \sim_X P_2$. There are two main cases depending on whether or not $\iota P_1$ represents an element of $T$. 

First suppose that $iP_1$ represents an element of $S \setminus T$. By Lemma 18 there are paths $P_1', P_2' \in P_+(\Gamma)$, $P_1'', P_2'' \in P_-(\Gamma)$, and $Q_1, Q_2 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P_i \sim_X P_i'Q_iP_i''$ for $i = 1, 2$. By Corollary 5 there are paths $Q_3, Q_4 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $P_1'Q_3 \sim_X P_2', \quad \text{and} \quad P_2'' \sim_X Q_4P_2''$.  

**Claim.** $Q_1 \sim_X Q_3Q_2Q_4$.  

**Proof of Claim.** First we show that for any pair of parallel paths $(Q, Q')$ with $Q, Q'$ both belonging to $B^1 \cdot P(\Gamma_G) \cdot C^1$, we have $Q \sim_{X_G} Q'$. Indeed, in this situation by Lemma 10 there exist $b \in B^1$ and $c \in C^1$ with $Q, Q' \in b \cdot \Gamma_G \cdot c$. Hence we have $Q = b \cdot R \cdot c$ for some $R \in \Gamma_G$, and $Q' = b \cdot R' \cdot c$ for some $R' \in \Gamma_G$. Since $X_G$ is a homotopy base for $\Gamma_G$ it follows that $R \sim_{X_G} R'$ which, by acting on the left by $b$ and the right by $c$, implies that $Q \sim_{X_G} Q'$.  

Returning to the proof of the claim, suppose that neither $iQ_1$ nor $\tau Q_1$ belongs to $B \cup C \cup BC$. Then by Lemma 19(i) it follows that there exist paths $Q_5, Q_6 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $Q_1 \sim_X Q_5$ and $Q_5Q_2Q_4 \sim_X Q_6$. By the previous paragraph, since $Q_5, Q_6 \in B^1 \cdot P(\Gamma_G) \cdot C^1$ are parallel paths we obtain $Q_5 \sim_X Q_6$ and hence $Q_1 \sim_X Q_5 \sim_X Q_6 \sim_X Q_3Q_2Q_4$.  

Now consider the case that one of $iQ_1$ or $\tau Q_1$ belongs to $B \cup C \cup BC$. Suppose the former, the latter case is dealt with in much the same way. Now let $Q'' = Q_1Q_4^{-1}Q_2^{-1}Q_3^{-1}$, so $Q''$ is a closed path with $iQ'' \equiv \tau Q'' \in B \cup C \cup BC$. By Lemma 19(ii) there exists a path $Q''_1$ in $B^1 \cdot P(\Gamma_G) \cdot C^1$ and edges $E_1, E_2$ of $B^1 \cdot P(\Gamma_G) \cdot C^1$ such that $Q'' \sim_X E_1Q''_1E_2$ and $E_1 \sim_X E_2^{-1}$. Since $(1, Q'', Q'')$ is a pair of parallel paths both from $B^1 \cdot P(\Gamma_G) \cdot C^1$ it follows from the observation above that $Q'' \sim_{X_G} 1_{Q''}$. We conclude  

\begin{align*}
Q_1 & \sim_{X} (Q_1Q_3Q_2^{-1}Q_4)Q_5 \sim_{X} Q_1Q''_1Q_5 \sim_{X} E_1Q''_1E_2Q_5 \sim_{X} E_1E_2Q_5 \sim_{X} E_2Q_5.
\end{align*}

This covers all possible cases and completes the proof of the claim. \hfill \Box

Using the claim we conclude  

\begin{align*}
P_1 & \sim_X P_1'Q_1P_1'' \sim_X P_1'Q_3Q_2Q_4P_1'' \sim_X P_2'Q_2P_2'' \sim_X P_2',
\end{align*}

completing the proof in the case that $iP_1$ represents an element of $S \setminus T$. If $iP_1$ does represent an element of $T$ then the proof is more straightforward. We follow the same steps as above (using part (ii) each time a Lemma is applied) except that $Q_i \in P(\Gamma_T)$ for $i = 1, 2, 3, 4$, and we make use of the fact that $X_T$ is a homotopy base for $\Gamma_T$ to prove $Q_1 \sim_{X_T} Q_3Q_2Q_4$. \hfill \Box

To complete the proof of Theorem 6 suppose that $G$ and $T$ have finite derivation type. Let $P_G$ and $P_T$ be finite presentations for $G$ and $T$ respectively. Then $P_S$ defined above is a finite presentation for $S$. Let $X_G$ and $X_T$ be finite homotopy bases for $P_G$ and $P_T$ respectively. Then we have seen above that $X = X_1 \cup X_1' \cup X_2 \cup X_3 \cup X_e \cup X_G \cup X_T$ is a finite homotopy base for $P_S$. Hence $S$ has finite derivation type.

**References**

1. G. M. Bergman. The index of a group in a semigroup. *Pacific J. Math.*, 36:55–62, 1971.
2. K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.
3. A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. II*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1967.
4. R. Cremanns and F. Otto. Finite derivation type implies the homological finiteness condition $FP_3$. *J. Symbolic Comput.*, 18(2):91–112, 1994.
[5] R. Cremanns and F. Otto. For groups the property of having finite derivation type is equivalent to the homological finiteness condition \( \text{FP}_3 \). *J. Symbolic Comput.*, 22(2):155–177, 1996.

[6] É. A. Golubov. Finitely approximable regular semigroups. *Mat. Zametki*, 17:423–432, 1975.

[7] R. Gray and N. Ruškuc. Green index and finiteness conditions for semigroups. *J. Algebra*, 320(8):3145–3164, 2008.

[8] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951.

[9] V. Guba and M. Sapir. Diagram groups. *Mem. Amer. Math. Soc.*, 130(620):viii+117, 1997.

[10] J. M. Howie. *Fundamentals of semigroup theory*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1995. L.M.S. Monographs, No. 7.

[11] J. M. Howie and N. Ruškuc. Constructions and presentations for monoids. *Comm. Algebra*, 22(15):6209–6224, 1994.

[12] V. Kilibarda. On the algebra of semigroup diagrams. *Internat. J. Algebra Comput.*, 7(3):313–338, 1997.

[13] Y. Kobayashi. Finite homotopy bases of one-relator monoids. *J. Algebra*, 229(2):547–569, 2000.

[14] Y. Lafont. A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier). *J. Pure Appl. Algebra*, 98(3):229–244, 1995.

[15] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[16] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial group theory*. Dover Publications Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[17] A. Malheiro. Finite derivation type for Rees matrix semigroups. *Theoret. Comput. Sci.*, 355(3):274–290, 2006.

[18] A. Malheiro. *Finiteness Conditions of Semigroup Presentations*. PhD Thesis. Universidade de Lisboa, 2006.

[19] A. Malheiro. On trivializers and subsemigroups. In *Semigroups and formal languages*, pages 188–204. World Sci. Publ., Hackensack, NJ, 2007.

[20] A. Malheiro. Finite derivation type for large ideals. *Semigroup Forum*, (to appear (2008)).

[21] F. Otto and Y. Kobayashi. Properties of monoids that are presented by finite convergent string-rewriting systems—a survey. In *Advances in algorithms, languages, and complexity*, pages 225–266. Kluwer Acad. Publ., Dordrecht, 1997.

[22] S. J. Pride. Low-dimensional homotopy theory for monoids. *Internat. J. Algebra Comput.*, 5(6):631–649, 1995.

[23] S. J. Pride and F. Otto. For rewriting systems the topological finiteness conditions FDT and FHT are not equivalent. *J. London Math. Soc. (2)*, 69(2):363–382, 2004.

[24] S. J. Pride and J. Wang. Rewriting systems, finiteness conditions, and associated functions. In *Algorithmic problems in groups and semigroups (Lincoln, NE, 1998)*, Trends Math., pages 195–216. Birkhäuser Boston, Boston, MA, 2000.

[25] N. Ruškuc. Presentations for subgroups of monoids. *J. Algebra*, 220(1):365–380, 1999.

[26] N. Ruškuc and R. M. Thomas. Syntactic and Rees indices of subsemigroups. *J. Algebra*, 205(2):435–450, 1998.

[27] C. C. Squier, F. Otto, and Y. Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[28] B. Steinberg. A topological approach to inverse and regular semigroups. *Pacific J. Math.*, 208(2):367–396, 2003.

[29] J. Wang. Finite complete rewriting systems and finite derivation type for small extensions of monoids. *J. Algebra*, 204(2):493–503, 1998.
[30] J. Wang. Finite derivation type for semigroups and congruences. *Semigroup Forum*, 75(2):388–392, 2007.

[31] X. Wang and S. J. Pride. Second order Dehn functions of groups and monoids. *Internat. J. Algebra Comput.*, 10(4):425–456, 2000.