Invariant bilinear forms under the operator group of order $p^3$ with odd prime $p$

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Abstract

In this paper we formulate the number of $n$ degree representations of a group of order $p^3$ with $p$ an odd prime and the dimensions of corresponding spaces of invariant bilinear forms over a field $\mathbb{F}$ which contains a primitive $p^3$ root of unity. We explicitly discuss about the existence of a non-degenerate invariant bilinear form.

Keywords: Bilinear forms, Representation theory, Vector spaces, Direct sums, Semi direct product.

2010 MSC: 15A63, 11E04, 06B15, 15A03

1. Introduction

Representation theory enables the study of a group as operators on certain vector spaces and as an orthogonal group with respect to a corresponding bilinear form. Also since last several decades the search of non-degenerate invariant bilinear forms has remained of great interest. Such type of studies acquire an important place in quantum mechanics and other branches of physical sciences.

Let $G$ be a group and $V$ a vector space over a field $\mathbb{F}$, then we have following.

Definition 1.1. A homomorphism $\rho : G \to GL(V)$ is called a representation of the group $G$. $V$ is also called a representing space of $G$. The dimension of $V$ over $\mathbb{F}$ is called degree of the representation $\rho$.

Definition 1.2. A class function is a map $f : G \to \mathbb{F}$ so that $f(g) = f(h)$ if $g$ is a conjugate of $h$ in $G$.

Definition 1.3. A bilinear form on a finite dimensional vector space $V(\mathbb{F})$ is said to be invariant under the representation $\rho$ of a finite group $G$ if

$$B(\rho(g)x, \rho(g)y) = B(x, y), \quad \forall \ g \in G \text{ and } x, y \in V(\mathbb{F}).$$

Let $\Xi$ denotes the space of bilinear forms on the vector space $V(\mathbb{F})$ over $\mathbb{F}$ and $C_{\mathbb{F}}(G)$ the set of all class functions on $G$.

Definition 1.4. The set of invariant bilinear forms under the representation $\rho$ is given by

$$\Xi_G = \{B \in \Xi \mid B(\rho(g)x, \rho(g)y) = B(x, y), \quad \forall \ g \in G \text{ and } x, y \in V\}.$$
is a vector space over \( \mathbb{F} \) with dimension \( r \), where \( r \) is the number of conjugacy classes of \( G \). By the Frobenius theorem (see pp 319, Theorem (5.9)) there are \( r \) irreducible representations \( \rho_i \) (say), \( 1 \leq i \leq r \) of \( G \) and \( \chi_i \) (say) the corresponding characters of \( \rho_i \). Also by Maschke’s theorem (see pp 316, corollary (4.9)) every \( n \) degree representation of \( G \) can be written as a direct sum of copies of irreducible representations. For \( \rho = \bigoplus_{i=1}^{r} \rho_i \) an \( n \) degree representation of \( G \), the coefficient of \( \rho_i \) is \( k_i \), \( 1 \leq i \leq r \), so that \( \sum_{i=1}^{r} d_i k_i = n \), and \( \sum_{i=1}^{r} d_i^2 = |G| \), where \( d_i \) is the degree of \( \rho_i \) and \( d_i | | G | \) with \( d_j \geq d_i \) when \( j > i \). It is already well understood in the literature that the invariant space \( \Xi_G \) under \( \rho \) can be expressed by the set \( \Xi'_G = \{ X \in M_n(\mathbb{F}) | C^{g}_{\rho(g)} X C^{t}_{\rho(g)} = X, \forall g \in G \} \) with respect to an ordered basis \( e \) of \( V(\mathbb{F}) \), where \( M_n(\mathbb{F}) \) is the set of square matrices of order \( n \) with entries from \( \mathbb{F} \) and \( C_{\rho(g)} \) is the matrix representation of the linear transformation \( \rho(g) \) with respect to \( e \).

Here we consider \( G \) to be a group of order \( p^3 \) with \( p \) an odd prime, \( \mathbb{F} \) a field with \( \text{char}(\mathbb{F}) \neq p \), which consists of a primitive \( p^3 \)th root of unity and \( (\rho, V) \) an \( n \) degree representation of \( G \) over \( \mathbb{F} \). Then the corresponding set \( \Xi_G \) of invariant bilinear forms on \( V \) under \( \rho \), forms a subspace of \( \Xi \). In this paper our investigation is about the following questions.

**Question.** How many \( n \) degree representations (upto isomorphism) of \( G \) can be there? What is the dimension of \( \Xi_G \) for every \( n \) degree representation? What are the necessary and sufficient conditions for the existence of a non-degenerate invariant bilinear form.

The primary focus is on the existence of a non-degenerate invariant bilinear forms. Over complex numbers it has been seen with positive findings, as an evidence we present here one.

It is well known that every maximal (proper) subgroup of \( G \) has index \( p \) and is normal (As finite \( p \) groups are nilpotent and any proper subgroup of a nilpotent group is properly contained in its normaliser). Thus there are epimorphisms from \( G \) to the cyclic group \( C \) of order \( p \).

Fix a generator \( c \) of \( C \) and \( 1 \neq \zeta \) a primitive \( p \)th-root of unity. Let \( U \) and \( V \) be the one-dimensional representations of \( C \) on which \( c \) acts respectively by \( \zeta \) and \( \bar{\zeta} \). We claim that \( U \oplus V \) admits a \( C \)-invariant non-degenerate bilinear form. Via some epimorphism to \( C \) one can pullback these representations and the forms to \( G \).

To prove the claim let us fix the vectors \( 0 \neq u \in U \) and \( 0 \neq v \in V \). Using these we define a bilinear form \( B \) on \( U \oplus V \) as follows: \( B(u, u) = 0 = B(v, v), B(u, v) = 1 = B(v, u) \) so that \( B(\lambda u + \mu v, \lambda' u + \mu' v) = \lambda \mu' + \lambda' \mu \).

Now we may easily check the \( C \) invariance as follows: \( B(c(\lambda u + \mu v), c(\lambda' u + \mu' v)) = B(\zeta (\lambda u + \mu v), \bar{\zeta}(\lambda' u + \mu' v)) = \zeta \bar{\zeta} \lambda \mu' + \lambda' \mu = B(\lambda u + \mu v, \lambda' u + \mu' v) \).

The questions in concern have been studied in the literature in several distinct contexts. Gongopadhyay and Kulkarni studied the existence of \( T \)-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Kulkarni and Tanti formulated the dimension of space of \( T \)-invariant bilinear forms. Malto and Tanti formulated the dimensions of invariant spaces and explicitly discussed about the existence of the non-degenerate invariant bilinear forms under \( n \) degree representations of a group of order \( 8 \). Sergeichuk studied systems of forms and linear mappings by associating with them self-adjoint representations of a category with involution. Frobenius proved that every endomorphism of a finite dimensional vector space \( V \) is self-adjoint for at least one non-degenerate symmetric bilinear form on \( V \). Later, Stenzel determined when an endomorphism could be skew-selfadjoint for a non-degenerate quadratic form, or self-adjoint or skew-self adjoint for a symplectic form.
on complex vector spaces. However his results were later generalized to an arbitrary field $\mathbb{F}$. Pazzis [12] tackled the case of the automorphisms of a finite dimensional vector space that are orthogonal (resp. symplectic) for at least one non-degenerate quadratic form (resp. symplectic form) over an arbitrary field of characteristics 2.

In this paper we investigate about the counting of $n$ degree representations of a group of order $p^3$ with $p \geq 3$ a prime, over a field $\mathbb{F}$ which consists of a primitive $p^3$th root of unity, dimensions of their corresponding spaces of invariant bilinear forms and establish a characterization criteria for existence of a non-degenerate invariant bilinear form. Our investigations are stated in the following three main theorems.

**Theorem 1.1.** The number of $n$ degree representations (upto isomorphism) of a group $G$ of order $p^3$, with $p$ an odd prime is $(n+p^3-1)/p^3-1$ when $G$ is abelian and $\sum_{\mu=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \left( \frac{\mu+p-2}{p-2} \right) \left( \frac{n-\mu p+p^2-1}{p^2-1} \right)$ otherwise.

**Theorem 1.2.** The space $\Xi_G$ of invariant bilinear forms of a group $G$ of order $p^3$ ($p$ an odd prime), under an $n$ degree representation $(\rho, \mathcal{V}(\mathbb{F}))$ is isomorphic to the direct sum of the subspaces $\mathcal{W}_{(i,j)}$, $(i,j) \in A_G$ of $\mathbb{M}_n(\mathbb{F})$, i.e.,

$$\Xi_G = \bigoplus_{(i,j) \in A_G} \mathcal{W}_{(i,j)},$$

where $A_G = \{(i,j) \mid \rho_i$ and $\rho_j$ are dual to each other and for every $(i,j) \in A_G$, $\mathcal{W}_{(i,j)} = \{ X \in \mathbb{M}_n(\mathbb{F}) \mid X_{d_i k_i \times d_j k_j} = C_{k_i,\rho_i(g)}^d X_{d_i k_i \times d_j k_j} C_{k_j,\rho_j(g)}, \forall g \in G \text{ and rest blocks are zeros} \}$. Also for $(i,j) \in A_G$, the dimension of $\mathcal{W}_{(i,j)} = k_i k_j$.

**Theorem 1.3.** If $G$ is group of order $p^3$, with $p$ an odd prime, then an $n$ degree representation of $G$ consists of a non-degenerate invariant bilinear form if and only if every irreducible representation and its dual have the same multiplicity.

Thus we are able to give the answers to the questions in concern. Here for $n \in \mathbb{N}$ Theorem [1.1] counts all $n$ degree representations, Theorem [1.2] computes the dimension of the space of invariant bilinear forms and Theorem [1.3] characterises those $n$ degree representations, each of which admits a non-degenerate invariant bilinear form for a group of order $p^3$ with $p$ an odd prime over the field $\mathbb{F}$.

**Remark 1.1.** Thus we get the necessary and sufficient condition for the existence of a non-degenerate invariant bilinear form under an $n$ degree representation.

2. Preliminaries

The classification of groups of order $p^3$, with $p$ an odd prime has been well understood in the literature. Due to the structure theorem of finite abelian groups, there are only three abelian groups (upto isomorphism) of this order viz, $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Amongst non-abelian groups of this order Hisenberg group $\mathbb{H}$ is well known and named after a German theoretical physicist Werner Heisenberg. In this group every non identity element is of order $p$. The elements of this group are usually seen in the form of $3 \times 3$ upper triangular matrices whose diagonal entries consist of 1 and other three entries are chosen from the finite field $\mathbb{Z}_p$. If at all there exists any other non-abelian group of this order then it must have a non identity element of order $p^2$. Let us consider the $2 \times 2$ upper triangular matrices with $a_{11} = 1 + pm, (m \in \mathbb{Z}_p)$, $a_{12} = a \in \mathbb{Z}_{p^2}$ and $a_{22} = 1$. Here the element
with entries $a_{11} = a_{12} = a_{22} = 1$ has order $p^2$ making it non-isomorphic to the Heisenberg group. We denote this group by $G_p$. Thus upto isomorphism there are five groups of order $p^3$ with an odd prime $p$. For an abelian group of order $p^3$, there are $p^3$ number of irreducible representations each having degree 1 and for non-abelian cases, the number of trivial conjugacy classes is $|Z(G)| = p$. To find a non-trivial conjugacy class we refer to the theory of group action and class equation

$$|G| = |Z(G)| + \sum_{g \notin Z(G) \text{ varying over distinct conjugacy classes}} |C_g|$$

with $|C_g| = \frac{|G|}{|C(g)|}$, where $C_g$ and $C(g)$ are the conjugacy class and the centralizer respectively of $g$ in $G$. If $g \notin Z(G)$ then $|C(g)| = p^2$. Therefore there are $p^2 - 1$ non trivial conjugacy classes of order $p$. Thus total number of conjugacy classes for a non-abelian group is $r = p^2 - 1 + p$, which is same as the number of irreducible representations with degree $d_i$ since $d_i||G|$ and $\sum_{i=1}^{r} d_i^2 = |G|$, therefore $d_i = 1$ or $p$. Thus there are $p^2$ representations of degree 1 and $p-1$ representations of degree $p$ for a non-abelian group. We here formulate every irreducible representation $\rho_i$ in such a way that the entries of $C_{\rho_i(g)}$ are either 0 or $p^3$th primitive roots of unity.

**Definition 2.1.** The character of $\rho$ is a function $\chi : G \rightarrow K$, $\chi(g) = \text{tr}(\rho(g))$ and is also called character of the group $G$.

**Theorem 2.1.** (Maschke’s Theorem): If char($F$) does not divide $|G|$, then every representation of $G$ is a direct sum of irreducible representations.

*Proof.* See p -316, corollary (4.9) \[1\].

**Theorem 2.2.** Two representations $(\rho, \mathbb{V}(F))$ and $(\rho', \mathbb{V}(F))$ of $G$ are isomorphic iff their character values are same i.e, $\chi(g) = \chi'(g)$ for all $g \in G$.

*Proof.* See p -319, corollary (5.13) \[1\].

3. Irreducible representation (irrep.) of group of order $p^3$ with an odd prime $p$.

In this section $G$ is a group of order $p^3$ with $p$ an odd prime, $(\rho_i, \mathbb{W}_{d_i}(F))$ stands for an irreducible representation $\rho_i$ of degree $d_i$ of $G$ over a field $F$ with char($F$) $\neq p$, which consists of $\omega \in F$, a primitive $p^3$th root of unity.

Let $\sigma_s = \rho_{p^s+s}$, $1 \leq s \leq p-1$ denote the irreducible representations of degree $p$ when $G$ is non-abelian. Since $\sigma_s$ is a homomorphism from $G$ to $GL(\mathbb{W}_p) \cong GL(p, F)$, by the fundamental theorem of homomorphism $\frac{G}{\text{Ker}(\sigma_s)} \cong \sigma_s(G)$ and the possible value of $|\text{Ker}(\sigma_s)|$ is 1 or $p$. If $|\text{Ker}(\sigma_s)| = p$ then $(\chi_s, \chi_s) \geq 1$, therefore $(\chi_s, \chi_s) = 1$ only when $g \notin \text{Ker}(\sigma_s)$, so we have $\chi_s(g) = 0$. Also we have trivial character $\chi_1(g) = 1, \forall g \in G$ and $(\chi_1, \chi_1) = \frac{p^2}{p^2} \neq 0$, which fails the orthonormality property of the irreducible characters. Thus $|\text{Ker}(\sigma_s)| = 1$ and hence $\sigma_s(G) \cong \frac{G}{\{e_G\}}$ which is isomorphic to a non-abelian group of order $p^3$. Now $\sigma_s(G)$ has subgroups $H$ and $K$ of orders $p$ and $p^2$ respectively [see p-132, \[4\], exercise-29]. Since $K$ is a maximal subgroup of $G$ so $K$ must be a normal subgroup and there exists a subgroup $H_p$ of order $p$ which is not normal (not contained in $Z(G)$)[see p-188, \[4\], Theorem 1], thus we have, $\sigma_s(G) = KH_p$ and every element of $\sigma_s(G)$ can be expressed uniquely in the form of $kh$ for some $k \in K$ and $h \in H_p$ (this uniqueness follows from the condition $K \cap H_p = \{\sigma_s(e_G)\}$), therefore...
$\sigma_s(\text{Heis}(\mathbb{Z}_p)) \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ and $\sigma_s(G_p) \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$. As $|Z(G)| = p$, we can choose a subgroup of order $p$ from $GL(p, \mathbb{F})$ and say that it is the Image of $Z(G)$ (subset of a normal subgroup of order $p^2$ of $G$) denoted by $\text{Im}(Z(G)) = \{\omega^{sp^2}I_p | 1 \leq s \leq p\}$ under the irreducible representation $\sigma_s$ (since center elements commute and are scalar matrices). Thus $\text{Im}(Z(G))(\in \sigma_s(K)) \cong \mathbb{Z}_p$, each non-identity element of $Z(G)$ have $p - 1$ choices in $\mathbb{Z}_p$ under $\sigma_s$ and rest $p^3 - p$ elements of $G$ map to rest $p^3 - p$ elements of $\sigma_s(G)$. We decide all $p$ degree representations by the elements of $Z(G)$ and elements of $G - Z(G)$ by mapping to the set $\sigma_s(G) - \text{Im}Z(G) \subseteq GL(p, \mathbb{F})$, which consists of those elements whose trace is zero and order of every element is either $p$ or $p^2$. We will depict all irreducible representations of $G$ in the next subsections.

### 3.1. Heisenberg group.

$$G = \text{Heis}(\mathbb{Z}_p) = \left\{ (\alpha, \beta, \gamma) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{Z}_p \right\}.$$ Another presentation of Heisenberg group is ([4], pp 179)

$$G = \langle x, a \mid x^p = a^p = 1, x(a^{-1}xa)x^{-1} = a^{-1}xa, a(xax^{-1})a^{-1} = xax^{-1} \rangle.$$ For the Heisenberg group, center $Z(G) = \langle xax^{-1}a^{-1} \rangle = \langle (0, 1, 0) \rangle$. Each of the irreducible representations $\sigma_s$, $1 \leq s \leq p - 1$ of degree $p$ maps the center of $G$ to the center of $GL(p, \mathbb{F})$, i.e., if $z \in Z(G)$ then $\sigma_s(z)$ is a scalar matrix say $c_1I_p$, $c_s \in \mathbb{F}$ and since order of $z$ is $p$ so $c_1^p = 1$. Hence $c_s \in \{1, \omega^s, \omega^{2s}, \ldots, \omega^{(p-1)s}\}$. Thus each of the $p - 1$ irreducible representations of degree $p$ maps $Z(G)$ into the $Z(GL(p, \mathbb{F}))$ even maps to the $Z(\sigma_s(G))$, it has been recorded in the following table.

| Table 1: All $p$ degree irreducible representations of $\text{Heis}(\mathbb{Z}_p)$ |
|---|
| All irrep. $\rightarrow$ | $\sigma_{2\eta - 1}$ | $\sigma_{2\eta}$ |
| Running variable | $1 \leq \eta \leq \frac{p - 1}{2}$, |
| $g \in Z(G)$ | $\epsilon^n I_p$ | $\epsilon^{(p - n)} I_p$ |
| $g \notin Z(G)$ | See Note 3.1 | See Note 3.1 |
| Dual irrep. | $\sigma_{2\eta - 1}^* = \sigma_{2\eta}$ |
| $\#$ irrep. | $p - 1$ |

**Note 3.1.** For the $p$ degree representations $\sigma_s$ of Heisenberg group $G$, if $xax^{-1}a^{-1} \in Z(G)$ then $\sigma_s(xax^{-1}a^{-1}) = \rho_{p^2 + s}(xax^{-1}a^{-1}) = \epsilon^m I_p$, for some $m$, $1 \leq m < p$ and the elements of $G - Z(G)$ get mapped bijectively to the following set

$$\sigma_s(G) - \sigma_s(Z(G)) = \{A_u \in GL(p, \mathbb{F}) \mid \text{Tr}(A_u) = 0 \text{ and } A_u^p = I_p \text{ for } 1 \leq u \leq p^3 - p\}.$$
Thus the $p$ degree irreducible representations $\sigma_s$ can be expressed as below

$$\sigma_{2\eta-1}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^\eta \quad \text{and} \quad \sigma_{2\eta-1}(a) = \begin{bmatrix} e^\eta & 0 & 0 & 0 & \cdots & 0 \\ 0 & e^{\eta+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{\eta+2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{\eta+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e^{\eta+p-1} \end{bmatrix},$$

$$\sigma_{2\eta}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{-\eta} \quad \text{and} \quad \sigma_{2\eta}(a) = \begin{bmatrix} e^{p+1-\eta} & 0 & 0 & 0 & \cdots & 0 \\ 0 & e^{p+2-\eta} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{p+3-\eta} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{p+4-\eta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e^{p-\eta} \end{bmatrix}.$$

Non-abelian groups of order $p^3$ have $p^2$ representations of degree 1. Let $\sigma_{(s-1,t-1)}$, $1 \leq s, t \leq p$, denote representations of degree 1. Here we present all 1 degree representations for the Heisenberg group.

Table 2: All irreducible representations of degree 1 for the Heisenberg group.

| All irrep. $\rightarrow$ | $\rho_{(t-1)p+2s-2}$ | $\rho_{(t-1)p+2s-1}$ | $\rho_{(t-1)p+2s}$ | $\rho_{(t-1)p+2s}$ |
|--------------------------|------------------------|------------------------|------------------------|------------------------|
| Running Variable $\rightarrow$ | $\sigma_{(s-1,t-1)}$ | $\sigma_{(p-s+1,p-t+1)}$ | $\sigma_{(s-1,t-1)}$ | $\sigma_{(p-s+1,p-t+1)}$ |
| 1 $\beta$               | $e^p = 1$              | 1                      | 1                      | 1                      |
| 0 1 0                    |                         |                        |                        |                        |
| 0 0 1                    |                         |                        |                        |                        |
| 1 $\alpha$               | $e^{(a(s-1))}$          | $e^{(p-s+1)}$          | $e^{(a(s-1)+\gamma(t-1))}$ | $e^{(a(p-s+1)+\gamma(p-t+1))}$ |
| 0 1 $\gamma$             |                         |                        |                        |                        |
| 0 0 1                    |                         |                        |                        |                        |
| Dual irrep.              | $\rho_{2s-2} = \rho_{2s-1} \& \rho_0 = \rho_1$ | $\rho_{t-1)p+2s-1} = \rho_{(t-1)p+2s}$ |                         |                        |
| # irrep.                 | p                       | p^2 - p                |                        |                        |

Since $1 \leq s, t \leq p$, and $\alpha, \gamma \in \mathbb{Z}_p$, so $e^{(a(s-1)+\gamma(t-1))}$ is generated by $e$, thus the representation $\sigma_{(s-1,t-1)}$ maps an element $g \in G$ to $e^m$, for some $m$, $1 \leq m \leq p$.

3.2. The non-abelian group $G_p$

$$G_p = \left\{ (p, \delta) = \begin{bmatrix} 1 + p & \gamma \\ 0 & 1 \end{bmatrix} \mid \gamma \in \mathbb{Z}_p, \delta \in \mathbb{Z}_{p^2} \right\}.$$  

Another presentation of this group is ([4], pp 180)
\[ G_p = \langle x, y \mid x^p = y^{p^2} = 1, xy = y^{p+1}x \rangle. \]

The center of \( G_p \) is \( Z(G_p) = \left\langle y^p = \begin{bmatrix} 1 + p & 0 \\ 0 & 1 \end{bmatrix} \right\rangle. \)

**Note 3.2.** For the \( p \) degree representations \( \sigma_s \) of group \( G_p \), if \( y^p \in Z(G_p) \) then \( \sigma_{2\eta-1}(y^p) = \rho_{p^2+2\eta-1}(y^p) = \omega^{\eta p^2}I_p \) and its dual \( \sigma_{2\eta}(y^p) = \rho_{p^2+2\eta}(y^p) = \omega^{(p-\eta)p^2}I_p \), \( 1 \leq \eta \leq \frac{p-1}{2} \). Also the elements of \( G_p - Z(G_p) \) map bijectively to the set

\[ \sigma_s(G_p) - \sigma_s(Z(G_p)) = \{ A_u \in GL(p, \mathbb{F}) \mid \text{Tr}(A_u) = 0 \text{ and } A_u^p = I_p \text{ or } A_u^{p^2} = I_p \text{ for } 1 \leq u \leq p^3 - p \}. \]

We have recorded the \( p \) ordered irreducible representations of \( G_p \) in the following table.

| Table 3: All irreducible representations of degree \( p \) for the group \( G_p \). |
|---|---|---|
| All irrep. \( \rightarrow \) | \( \sigma_{2\eta-1} \) | \( \sigma_{2\eta} \) |
| Running variable \( \rightarrow \) | \( 1 \leq \eta \leq \frac{p-1}{2} \), \( g \in Z(G_p) \) | \( \omega^{\eta p^2}I_p \) | \( \omega^{(p-\eta)p^2}I_p \) |
| \( g \notin Z(G_p) \) | See Note 3.2 | See Note 3.2 |
| Dual irrep. | \( \sigma_{2\eta-1} = \sigma_{2\eta} \) | \( p - 1 \) |

Thus the \( p \) degree irreducible representations \( \sigma_s \) is defined as below

\[
\sigma_{2\eta-1}(x) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}^\eta \\
\text{and } \sigma_{2\eta-1}(y) = \begin{bmatrix}
\omega^{\eta p^2} & 0 & \cdots & 0 \\
0 & \omega^{p^2+\eta p} & 0 & \cdots & 0 \\
0 & 0 & \omega^{2p^2+\eta p} & \cdots & 0 \\
0 & 0 & 0 & \cdots & \omega^{(p-1)p^2+\eta p}
\end{bmatrix},
\]

\[
\sigma_{2\eta}(x) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}^{-\eta} \\
\text{and } \sigma_{2\eta}(y) = \begin{bmatrix}
\omega^{p^2-\eta p} & 0 & \cdots & 0 \\
0 & \omega^{2p^2-\eta p} & 0 & \cdots & 0 \\
0 & 0 & \omega^{3p^2-\eta p} & \cdots & 0 \\
0 & 0 & 0 & \cdots & \omega^{p^2-\eta p}
\end{bmatrix}.
\]

There are \( p^2 \) representations \( \sigma_{(s-t, t-1)} \), \( 1 \leq s, t \leq p \) of degree 1. As \( g \in G_p - Z(G_p) \), we have \( g = (p\gamma, \delta) = \begin{bmatrix} 1 + p\gamma & \delta \\ 0 & 1 \end{bmatrix}, \delta \in Z_{p^2}^*, |G_p - Z(G_p)| = (p^2 - 1)p. \) In the table 4 we have recorded all 1 degree representations of \( G_p \).
3.3. The cyclic group $\mathbb{Z}/p^3\mathbb{Z}$.

Here $\omega^{p^2(\gamma(s-1)+\delta'(t-1))}$ is generated by $\omega^{p^2}$, thus the representation $\sigma_{(s-1,t-1)}$ maps an element $g \in G_p$ to $\omega^{mp^2}$, for some $m$, $1 \leq m \leq p$.

**Note 3.3.** As a finite abelian group is finitely generated, here for the abelian groups $\mathbb{Z}/p^3\mathbb{Z}$, $\mathbb{Z}/p^2\times \mathbb{Z}/p$ and $\mathbb{Z}/p\times \mathbb{Z}/p\times \mathbb{Z}/p$, there exist the finite generating subsets $\{a\}$, $\{b,c\}$ and $\{d,e,f\}$ respectively. For example we may take $a = 1+p^3\mathbb{Z}$, $b = (1+p^2\mathbb{Z}, 0 + p\mathbb{Z})$, $c = (0 + p^2\mathbb{Z}, 1 + p\mathbb{Z})$, $d = (1 + p\mathbb{Z}, 0 + p\mathbb{Z}, 0 + p\mathbb{Z})$, $e = (0 + p\mathbb{Z}, 1 + p\mathbb{Z}, 0 + p\mathbb{Z})$ and $f = (0 + p\mathbb{Z}, 0 + p\mathbb{Z}, 1 + p\mathbb{Z})$. Further order of an element $g$ under $\rho_i, 1 \leq i \leq r$ is $|\rho_i(g)|$.

| $\rho_{(t-1)p+2s-1}$ | $\rho_{(t-1)p+2s-1}$ | $\rho_{(t-1)p+2s}$ | $\rho_{(t-1)p+2s}$ |
|-----------------------|-----------------------|-----------------------|-----------------------|
| $\sigma_{(s-1,t-1)}$ | $\sigma_{(s-1,t-1)}$ | $\sigma_{(s-1,t-1)}$ | $\sigma_{(s-1,t-1)}$ |

| Running Variable $t$ | $1 \leq s \leq \frac{p^2+1}{2}$ | $1 \leq s \leq p$, $2 \leq t \leq \frac{p^2+1}{2}$ |
|---------------------|-------------------------------|-------------------------------|
| $y^{p\gamma} = \begin{bmatrix} 1 + p\gamma + 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ | $\omega^{p^3} = 1$ | 1 |
| $t \neq 0$ | $\omega^{p^2\gamma(s-1)} = \omega^{p^2\gamma(p-s+1)}$ | $\omega^{p^2(\gamma(s-1)+\delta'(t-1))}$ | $\omega^{p^2(\gamma(p-s+1)+\delta'(p-t+1))}$ |
| Dual irrep. $\rho_{2s-2}^* = \rho_{2s-1} \& \rho_0 = \rho_1$ | $\rho_{(t-1)p+2s-1}^* = \rho_{(t-1)p+2s}$ | $\rho_{(t-1)p+2s-1}^* = \rho_{(t-1)p+2s}$ |
| # irrep. $p$ | $p^2 - p$ | $p^2 - p$ |

Table 4: All irreducible representations of degree 1 for the group $G_p$.

Here $G_p = \langle a, b | a^{p^3} = b^p = 1, ab = ba \rangle$.

3.4. The group $\mathbb{Z}/p^2\times \mathbb{Z}/p$.

Here $\mathbb{Z}/p^2\times \mathbb{Z}/p = \langle a, b | a^{p^3} = b^p = 1, ab = ba \rangle$. 

Table 5: All irreducible representations of the group $G_p$.

| All irrep. $\rightarrow$ | $\rho_i$ | $\rho_{p^{i-1}+2t}$ | $\rho_{p^{i-1}+2t+1}$ |
|------------------------|---------|------------------|------------------|
| Running variable $t$ | $i = 1$ | $i \in \{x \mid gcd(p^3, xp^{3-s}) = p^{3-s}, 1 \leq x \leq \frac{p(p^{3-p^{3-s}}-2)}{2(p-1)} + 1\}$ | $t = i$’s place in above set |
| $|\rho_i(a)|$ | 1 | $p^s$, $s = 1, 2, 3$ |
| $a$ | $\omega^{p^3} = 1$ | $\omega^{ip^{3-s}}$, | $\omega^{p^{3-3s}}$ |
| Dual irrep. self | $\rho_{p^{i-1}+2t}^{*} = \rho_{p^{i-1}+2t+1}$ |
| # irrep. $1$ | $p^s - p^{s-1}$ |

Table 6: All irreducible representations of the group $G_p$.

$$T_{\mathbb{Z}/p^3\mathbb{Z}} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 & \rho_8 \\ a & \omega^{8} & \omega^{4} & \omega^{2} & \omega^{6} & \omega & \omega^{7} & \omega^{3} & \omega^{5} \end{bmatrix}.$$
Table 7: All irreducible representations of group $\mathbb{Z}_p^2 \times \mathbb{Z}_p$. 

| Runing variable $\rightarrow$ | $\sigma_{(s,t)}$ | $\sigma_{(p^2,t)}$ | $\sigma_{(p^2,p-t)}$ | $\sigma_{(s,p)}$ | $\sigma_{(p-s,p)}$ | $\sigma_{(s,t)}$ | $\sigma_{(p-s,p-t)}$ | $\sigma_{(s,t)}$ | $\sigma_{(p^2-s,p-t)}$ | $\sigma_{(s,p)}$ | $\sigma_{(p-s,p)}$ |
|-----------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $s$                         | $t \in \{1, \ldots, p-1\}$ | & $1 \leq s \leq \frac{p-1}{2}$ | $s \in \{1, \ldots, p-1\}$ | & $1 \leq s \leq \frac{p-1}{2}$ | $1 \leq s \leq \frac{p-1}{2}$ | $t \in \{1, \ldots, p-1\}$ | & $1 \leq t \leq \frac{p-1}{2}$ | $s \in \{1, \ldots, p^2-p\}$ | & $1 \leq s \leq \frac{p^2-p}{2}$ |
| $t$                         | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ | $1 \leq t \leq \frac{p-1}{2}$ |
| $p$                         | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             | $p$             |
| $\rho_1(a)$                 | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               |
| $\rho_2(a)$                 | $\omega^{sp^2}$ | $\omega^{(p-s)p^2}$ | $\omega^{sp^2}$ | $\omega^{(p-s)p^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ |
| $\rho_3(a)$                 | $\omega^{p^2}$  | $\omega^{(p-t)p^2}$ | $\omega^{p^2}$  | $\omega^{(p-t)p^2}$ | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  |
| $\rho_4(a)$                 | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             |
| $\rho_1(b)$                 | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               |
| $\rho_2(b)$                 | $\omega^{sp^2}$ | $\omega^{(p-s)p^2}$ | $\omega^{sp^2}$ | $\omega^{(p-s)p^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ | $\omega^{sp^2}$ |
| $\rho_3(b)$                 | $\omega^{p^2}$  | $\omega^{(p-t)p^2}$ | $\omega^{p^2}$  | $\omega^{(p-t)p^2}$ | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  | $\omega^{p^2}$  |
| $\rho_4(b)$                 | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             | $1$             |
| Dual irrep. $\sigma_{(s,t)}$ | $\sigma_{(p^2,t)} = \sigma_{(p^2,p-t)}$ | $\sigma_{(s,p)} = \sigma_{(p-s,p)}$ | $\sigma_{(s,t)} = \sigma_{(p-s,p-t)}$ | $\sigma_{(s,t)} = \sigma_{(p^2-s,p-t)}$ | $\sigma_{(s,t)} = \sigma_{(p-s,p)}$ | $\sigma_{(s,p)} = \sigma_{(p-s,p)}$ |
| # irrep. $\rho_1(a)$        | 1               | $p-1$           | $p-1$           | $(p-1)(p-1)$    | $(p^2-p)(p-1)$  | $p^2-p$         | $p^2-p$         | $p^2-p$         | $p^2-p$         | $p^2-p$         | $p^2-p$         |

Table 8: All irreducible representations of group $\mathbb{Z}_4 \times \mathbb{Z}_2$. 

$T_{\mathbb{Z}_4 \times \mathbb{Z}_2} = \begin{array}{cccccccc}
\rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 & \rho_8 \\
\begin{array}{cccccccc}
a & \omega^8 & \omega^4 & \omega^4 & \omega^2 & \omega^2 & \omega^8 & \\
b & \omega^8 & \omega^4 & \omega^4 & \omega^2 & \omega^2 & \omega^8 & \\
\end{array}
\end{array}$

3.5. The group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. 

Here $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p = \langle a, b, c | a^p = b^p = c^p = 1, ab = ba, ac = ca, bc = cb \rangle$. The corresponding tables are given by
\[ T_{Z_p \times Z_p \times Z_p} = \]

| 3-tuples → | All place are same | Exactly two place are same | All place are distinct |
|------------|--------------------|----------------------------|-----------------------|
| \((s, t, m)\) | \((s, s, s)\) | \((s, s, m), (s, m, s), (m, s, s), m \neq s\) | \((s, t, m), s \neq t \neq m \neq s\) |
| All irrep. → | \(\sigma_{(s, s, s)}\) | \(\sigma_{(p-s,p-s,p-s)}\) | \(\sigma_{(s, t, m)}\) |
| \(\sigma_{(p-s,p-s,p-s)}\) | \(\frac{p+1}{2} \leq s \leq p\) | \(s = p & 1 \leq m \leq \frac{p-1}{2}\) or \(1 \leq s \leq \frac{p-1}{2} & m \in \{1, 2, \cdots p\}\) | \(1 \leq s, t, m \leq p,\) |

| Dual irrep. | \(\sigma^*_{(s, s, s)} = \sigma_{(p-s,p-s,p-s)}\) | \(\sigma^*_{(s, s, m)} = \sigma_{(p-s,p-s,p-m)}\) | \(\sigma^*_{(s, t, m)} = \sigma_{(p-s,p-t,p-m)}\) |
| Distinct | \# irrep. | \(p\) | \(3p(p-1)\) | \(p(p-1)(p-2)\) |

**Table 9: All irreducible representations of \(Z_p \times Z_p \times Z_p\)**

\[ T_{Z_2 \times Z_2 \times Z_2} = \]

\[
\begin{array}{cccccccc}
|\rho_1| & |\rho_2| & |\rho_3| & |\rho_4| & |\rho_5| & |\rho_6| & |\rho_7| & |\rho_8| \\
\hline
| a | & | \omega^8 | & | \omega^4 | & | \omega^8 | & | \omega^4 | & | \omega^4 | & | \omega^4 | \\
| b | & | \omega^8 | & | \omega^4 | & | \omega^4 | & | \omega^8 | & | \omega^4 | & | \omega^8 | \\
| c | & | \omega^8 | & | \omega^4 | & | \omega^8 | & | \omega^4 | & | \omega^8 | & | \omega^4 | \\
\end{array}
\]

**Note 3.4.** An irreducible representation \(\rho_{2i}\) is seated together (preceded or succeeded) to its dual \(\rho_{2i+1}\) in \(\rho\).

Now
\[
\rho = k_1 \rho_1 \oplus k_2 \rho_2 \oplus \cdots \cdots \oplus k_r \rho_r,
\]
where for every \(1 \leq i \leq r\), \(k_i \rho_i\) stands for the direct sum of \(k_i\) copies of the irreducible representation \(\rho_i\).

Let \(\chi\) be the corresponding character of the representation \(\rho\), then
\[
\chi = k_1 \chi_1 + k_2 \chi_2 + \cdots + k_r \chi_r,
\]
where \(\chi_i\) is the character of \(\rho_i\), \(\forall 1 \leq i \leq r\). Dimension of the character \(\chi\) is being calculated at the identity element of the group, i.e,
\[
dim(\rho) = \chi(1) = tr(\rho(1)).
\]
\[
\Rightarrow d_1 k_1 + d_2 k_2 + \cdots + d_r k_r = n.
\]

**Note 3.5.** Equation \(\boxed{2}\) holds in more general situation, which helps us to find all possible distinct \(r\)-tuples \((k_1, k_2, \ldots, k_r)\), which correspond to the distinct \(n\) degree representations (up to isomorphism) of a given finite group.
Theorem 3.1. Let $G$ be a group of order $p^3$ with $p$ a prime. If $\sigma$ is an irreducible representation of $G$ of degree $p$, then $\sigma$ is a faithful representation.

Proof. For non-abelian groups it is clear from the tables 1 and 3 in the subsections 3.1 and 3.2 in this paper if $p$ is an odd prime, whereas for $p = 2$ it follows from the tables $T_{D_4}$ and $T_{Q_8}$ in the article [11]. For an abelian group there is no irreducible representation of degree $p$. 

4. Existence of non-degenerate invariant forms.

An element in the space of invariant bilinear forms under representation of a finite group is either non-degenerate or degenerate. All the elements of the space are degenerate when $k_{2i} \neq k_{2i+1}$, such a space is called a degenerate invariant space which has also been discussed in [11] for the groups of order 8. How many such representations exist out of total representations, is a matter of investigation. Some of the spaces contain both non-degenerate and degenerate invariant bilinear forms under a particular representation. In this section we compute the number of such representations of the group $G$ of order $p^3$, with $p$ an odd prime.

Remark 4.1. The space $\Xi_G$ of invariant bilinear forms under an $n$ degree representation $\rho$ contains only those $X \in M_n(\mathbb{F})$ whose $(i, j)^{th}$ block is a $0$ sub-matrix of order $d_i k_i \times d_j k_j$ when $(i, j) \notin A_G = \{(i, j) \mid \rho_i$ and $\rho_j$ are dual to each other$\}$ whereas for $(i, j) \in A_G$ with $d_i = d_j = 1$, the block matrix $X_{d_i k_i \times d_j k_j}^{ij}$ is given by

$$X_{d_i k_i \times d_j k_j}^{ij} = X_{k_i \times k_j}^{ij} = \begin{bmatrix}
  x_{1,1}^{ij} & x_{1,2}^{ij} & \cdots & x_{1,k_j}^{ij} \\
  x_{2,1}^{ij} & x_{2,2}^{ij} & \cdots & x_{2,k_j}^{ij} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{k_i,1}^{ij} & x_{k_i,2}^{ij} & \cdots & x_{k_i,k_j}^{ij}
\end{bmatrix}.$$

And for $(i, j) \in A_{G_p}$ with $d_i = d_j = p$ it is,

$$X_{d_i k_i \times d_j k_j}^{ij} = X_{p k_i \times p k_j}^{ij} = \begin{bmatrix}
  x_{1,p}^{ij} L & x_{1,2p}^{ij} L & \cdots & x_{1,pk_j}^{ij} L \\
  x_{p+1,p}^{ij} L & x_{p+1,2p}^{ij} L & \cdots & x_{p+1,pk_j}^{ij} L \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{(k_i-1)p+1,p}^{ij} L & x_{(k_i-1)p+1,2p}^{ij} L & \cdots & x_{(k_i-1)p+1,pk_j}^{ij} L
\end{bmatrix},$$

where $L = \begin{bmatrix}
  0 & \cdots & 0 & 1 \\
  0 & \cdots & 1 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  1 & \cdots & 0 & 0
\end{bmatrix}_{p \times p}$.

Lemma 4.1. If $X \in \Xi_G$, and $k_{2i} \neq k_{2i+1}$, then $X$ must be singular.

Proof. With reference to the above remark and Note 3.3 for every $X \in \Xi_G$, we have $X = [X_{d_i k_i \times d_j k_j}^{ij}]_{(i, j) \in A_G}$.

I.e.
on the dual of $V$ linear form if and only if every irreducible representation and its dual have same multiplicity in the representation.

Suppose assume that the representation is faithful. Being a non-trivial finite non-trivial central element in $G$, for $k_{2t} \neq k_{2t+1}$ and since $d_{2t} = d_{2t+1}$, so the number of rows and columns of $X_{d,k,k}^{ij}$ is different hence either rows (or columns) is linearly dependent. Thus the result follows.

In the next lemma we characterise the representations of $G$ each of which admits a non-degenerate invariant bilinear form. To prove the next lemma we will choose only those $X \in M_n(\mathbb{F})$ whose $(i, j)^{th}$ block is zero for $(i, j) \notin A_G$, whereas for $(i, j) \in A_G$ with $k_i = k_j$ and the block matrices $X_{d,k,k}^{ij}$, is non-singular.

**Lemma 4.2.** For $n \in \mathbb{Z}^+$, an $n$-degree representation of $G$ has a non-degenerate invariant bilinear form if and only if $k_{2i} = k_{2i+1}, 1 \leq i \leq \frac{r-1}{2}$.

**Proof.** From equation (2) we have $d_1 k_1 + d_2 k_2 + \ldots + d_r k_r = n$ and choose $X \in M_n(\mathbb{F})$ such that

$$X = \begin{bmatrix} X_{d_1 k_1}^{11} & 0 & \cdots & 0 \\ 0 & X_{d_2 k_2}^{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{d_r k_r}^{r-1} \\ \end{bmatrix}$$

with $X_{d,k,k}^{ij} = C_{k_1 p_i(g)}^t X_{d,k,k}^{ij} C_{k_1 p_i(g)}$, for $(i, j) \in A_G$. If $k_{2i} \neq k_{2i+1}$ and since $d_{2i} = d_{2i+1}$, so the number of rows and columns of $X_{d,k,k}^{ij}$ is different hence either rows (or columns) is linearly dependent. Thus the result follows.

Note that the Lemmas 4.1 and 4.2 can be covered in a more general situation by stating as 'no non-trivial irreducible representation of a finite $p$-group can be self dual'. For if $L$ a finite $p$-group and $V$ a non-trivial irreducible representation of $L$, replacing $L$ by its image in the matrix group (the general linear group), we may assume that the representation is faithful. Being a non-trivial finite $p$-group, $L$ has non-trivial centre. Let $g$ be a non-trivial central element in $L$. The action of $g$ on $V$ is by multiplication by a root of unity $\zeta \neq \pm 1$. Its action on the dual of $V$ is by multiplication by $\overline{\zeta}(= \zeta^{-1})$. Since $\zeta \neq \overline{\zeta}$, it follows that $V$ is not self-dual.

**Corollary 4.1.** For $n \in \mathbb{Z}^+$, an $n$ degree representation of a group of order $p^3$ has a non-degenerate invariant bilinear form if and only if every irreducible representation and its dual have same multiplicity in the representation.
Lemma 4.3. Let $G$ be a group of order $p^3$ and $n \in \mathbb{N}$. Then the number of $n$ degree representations of $G$ each of which admits a non-degenerate invariant bilinear form is
\[
\sum_{\ell=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \left\lfloor \frac{\ell + \frac{p^3-3}{2}}{2} \right\rfloor \sum_{s=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} \left( s + \frac{p^2-3}{2} \right) \]
when $G$ is non-abelian whereas it is
\[
\sum_{\ell=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \left( \ell + \frac{p^3-3}{2} \right) \]
when $G$ is abelian.

Proof. Let $\rho = \bigoplus_{i=1}^{r} k_i \rho_i$ be an $n$ degree representation of $G$ which admits a non-degenerate bilinear form. So, we have $k_{2i} = k_{2i+1}$, $1 \leq i \leq \frac{p-1}{2}$. In the non-abelian case, $G$ is either $G_p$ or $Heis(\mathbb{Z}_p)$ and for each of these two, we have $r = p^2 + p - 1, d_i = 1$ for $1 \leq i \leq p^2$ and $d_i = p$ for $p^2 + 1 \leq i \leq p^2 + p - 1$. Now from equation (2), we have
\[
k_{\ell} + 2(k_2 + k_4 + \cdots + k_{p^2-1}) + 2p(k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2}) = n
\]

\[
\Rightarrow k_{\ell} + 2(k_2 + k_4 + \cdots + k_{p^2-1}) = n - 2p(k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2})
\]

\[
\Rightarrow k_{\ell} + 2(k_2 + k_4 + \cdots + k_{p^2-1}) = n - 2p\ell.
\]

To solve the above equation we have $\left( k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2} \right) = \ell$, $0 \leq \ell \leq \left\lfloor \frac{n}{2p} \right\rfloor$, i.e, we have $\left\lfloor \frac{n}{2p} \right\rfloor + 1$ equations. The $\ell$th equation is
\[
k_{p^2+1} + k_{p^2+3} + \cdots + k_{p^2+p-2} = \ell.
\]

The number of distinct solutions to above equations is $\binom{\ell + \frac{p^2-1}{2}}{\frac{p-1}{2} - 1}$, $0 \leq \ell \leq \left\lfloor \frac{n}{2p} \right\rfloor$.

Further from equation 3 we have
\[
k_1 = n - 2p\ell - 2(k_2 + k_4 + \cdots + k_{p^2-1})
\]

\[
\Rightarrow k_1 = n - 2p\ell - 2\lambda,
\]

where $k_2 + k_4 + \cdots + k_{p^2-1} = \lambda$, $0 \leq \lambda \leq \left\lfloor \frac{n-2p\ell}{2} \right\rfloor$, i.e, we have $\left\lfloor \frac{n-2p\ell}{2} \right\rfloor + 1$ equations and the number of solutions for every $\lambda$ to the equation is $\binom{\lambda + \frac{p^2-1}{2}}{\frac{p-1}{2} - 1}$. 

Remark 4.2. If $F$ is algebraically closed, it has infinitely many non-zero elements, hence if there is one non-degenerate invariant bilinear form in the space $\Xi_G$, it has infinitely many.
Thus the number of all distinct $p^2 + p - 1$ tuples $(k_1, k_2, k_3, \ldots, k_{p^2+p-2}, k_{p^2+p-1})$ with $k_{2i} = k_{2i+1}$ is

$$\sum_{\ell=0}^{\left\lfloor \frac{p^2}{2} \right\rfloor} \left( \ell + \frac{p^2 - 1}{2} - 1 \right) \sum_{\lambda=0}^{\left\lfloor \frac{n-2p\ell}{2} \right\rfloor} \left( \lambda + \frac{p^2 - 1}{2} - 1 \right).$$

Now in the abelian case $G$ is either of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p^2 \times \mathbb{Z}_p$ and $\mathbb{Z}_p^3$, for each of which $r = p^3$ and $d_i = 1$ for $1 \leq i \leq p^3$. Now from (2), we have

$$k_1 + 2(k_2 + k_4 + \ldots + k_{p^3-1}) = n.$$ 

Thus the number of all distinct $p^3$-tuples $(k_1, k_2, k_3, \ldots, k_{p^3})$ with $k_{2i} = k_{2i+1}$ is $\sum_{\ell=0}^{\left\lfloor \frac{p^3}{2} \right\rfloor} \left( \ell + \frac{p^3 - 1}{2} - 1 \right)$. 

Thus from equation (2) and Theorem 2.2 the number of $n$ degree representations (upto isomorphism) of a group $G$ consisting non-degenerate invariant bilinear form is $\sum_{\ell=0}^{\left\lfloor \frac{p^3}{2} \right\rfloor} \left( \ell + \frac{p^3 - 1}{2} - 1 \right)$ for non-abelian groups and $\sum_{\ell=0}^{\left\lfloor \frac{p^3}{2} \right\rfloor} \left( \ell + \frac{p^3 - 1}{2} - 1 \right)$ for abelian groups of order $p^3$, with $p$ an odd prime.

\[\square\]

**Definition 4.1.** The space $\Xi_G$ of invariant bilinear forms is called degenerate if it’s all elements are degenerate.

We will discuss about the degenerate invariant space in the later section.

5. Dimensions of spaces of invariant bilinear forms under the representations of groups of order $p^3$ with prime $p > 2$.

The space of invariant bilinear forms under an $n$ degree representation is generated by finitely many vectors, so its dimension is finite along with its symmetric and the skew-symmetric subspace. In this section we formulate the dimension of the space of invariant bilinear forms under a representation of a group of order $p^3$, with $p$ an odd prime.

**Theorem 5.1.** If $\Xi_G$ is the space of invariant bilinear forms under an $n$ degree representation $\rho = \bigoplus_{i=1}^{r} k_i \rho_i$ of a group $G$ of order $p^3$, then $\dim(\Xi_G) = \sum_{(i,j) \in A_G} k_i k_j$.

**Proof.** For every $X \in \Xi_G$, we have

$$X = \begin{bmatrix} X_{d_1 k_1 \times d_1 k_1}^{11} & 0 & \ldots & 0 \\ 0 & X_{d_2 k_2 \times d_2 k_2}^{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X_{d_r k_r \times d_r k_r}^{rr} \end{bmatrix}.$$
with \( X_{d,k_i \times d,k_j}^{ij} = C_{k_i \rho_i(g)}^i X_{d,k_i \times d,k_j}^{ij} C_{k_j \rho_j(g)}^j \), for \((i,j) \in A_G\) and to generate each of these blocks of \( X \) it needs \( k_i k_j \) vectors from \( \Xi_G' \). Thus the result follows.

**Corollary 5.1.** The space of invariant symmetric bilinear forms under an \( n \) degree representation \( \rho = \oplus_{i=1}^r k_i \rho_i \) of a group \( G \) of order \( p^3 \) has dimension \( \frac{k_1(k_1+1)}{2} + \sum_{(i,j) \in A_G} \frac{k_i k_j}{2} \).

**Proof.** Follows from the proof of theorem 5.1.

**Corollary 5.2.** The space of invariant skew-symmetric bilinear forms under an \( n \) degree representation \( \rho = \oplus_{i=1}^r k_i \rho_i \) of a group \( G \) of order \( p^3 \) has dimension \( \frac{k_1(k_1-1)}{2} + \sum_{(i,j) \in A_G} \frac{k_i k_j}{2} \).

**Proof.** Follows from the proof of theorem 5.1.

6. Main results

Here we present the proofs of the main theorems stated in the Introduction section.

**Proof of Theorem 1.1** Since \( G \) is the group of order \( p^3 \), with an odd prime \( p \) and degree of the representation \( \rho \) is \( n \), if \( G \) is either \( G_p \) or \( Heis(Z_p) \), we have \( r = p^2 + p - 1 \), \( d_i = 1 \) for \( 1 \leq i \leq p^2 \) and \( d_i = p \) for \( p^2 + 1 \leq i \leq p^2 + p - 1 \). Now from equation (2), we have

\[
k_1 + k_2 + \ldots + k_p^2 + pk_{p^2+1} + \ldots + pk_{p^2+p-1} = n
\]

Or,

\[
k_1 + k_2 + \ldots + k_p^2 = n - p(k_{p^2+1} + \ldots + k_{p^2+p-1}).
\]

Or,

\[
k_1 + k_2 + \ldots + k_p^2 = n - p\mu,
\]

where \( \mu = k_{p^2+1} + \ldots + k_{p^2+p-1}, 0 \leq \mu \leq \lfloor p^2 \rfloor \). i.e, we have \( \lfloor p^2 \rfloor + 1 \) equations placed in the chronological order and the \( \mu^{th} \) equation is given by

\[
k_{p^2+1} + k_{p^2+2} + \ldots + k_{p^2+p-1} = \mu.
\]

The number of distinct solutions to equation \( 9 \) is \( \binom{n+p-2}{p-2}, 0 \leq \mu \leq \lfloor p^2 \rfloor \).

Thus the number of all distinct \( p^2 + p - 1 \) tuples \((k_1, k_2, \ldots, k_{p^2+p-1})\) is \( \sum_{\mu=0}^{\lfloor p^2 \rfloor} \binom{n+p-2}{p-2}(n-\mu p + p^2 - 1)\).

On the otherhand if \( G \) is either of \( Z_p \times Z_p \times Z_p, Z_{p^2} \times Z_p \) and \( Z_{p^3} \) then \( r = p^3 \) and \( d_i = 1 \) for \( 1 \leq i \leq p^3 \). Now from (2), we have

\[
k_1 + k_2 + \ldots + k_p^2 + \ldots + k_{p^3} = n.
\]

Thus the number of all distinct \( p^3 \)-tuples \((k_1, k_2, k_3, \ldots, k_{p^3})\) is \( \binom{n+p^3-1}{p^3-1} \).

Thus from equation (2) and Theorem 2.2 the number of \( n \) degree representations (upto isomorphism) of a group \( G \) of order \( p^3 \) is \( \sum_{\mu=0}^{\lfloor p^2 \rfloor} \binom{n+p-2}{p-2}(n-\mu p + p^2 - 1)\), when \( G \) is non-abelian, whereas it is \( \binom{n+p^3-1}{p^3-1} \), when \( G \) is abelian. 

\(\square\)
6.1. Degenerate invariant spaces

From Lemma 4.1 if $k_{2i} \neq k_{2i+1}$ then all the elements of the space are degenerate. Thus by the Theorem 4.1 and Lemma 7.1, the number of $n$ degree representations whose corresponding invariant spaces of bilinear forms contain only degenerate invariant bilinear forms are

$$\sum_{p=0}^{\lfloor \frac{n}{p} \rfloor} \left( \binom{n-p-2}{p-2} \binom{n-p^2-1}{p^2-1} \right) - \sum_{\ell=0}^{\lfloor \frac{n}{p^2} \rfloor} \left( \binom{\frac{n-p^2}{2}}{\frac{p^2-1}{2}} \right) \left( \binom{\frac{n-2p\ell}{p^2}}{\frac{p^2-1}{2}} \right).$$

In the non-abelian case and it is $(n+p^2-1) - \sum_{\ell=0}^{\lfloor \frac{n}{p^2} \rfloor} \left( \binom{\frac{n-p^2-3}{2}}{\frac{p^2-1}{2}} \right)$ in the abelian case.

**Proof of Theorem 7.1**

Let $A_G = \{(i, j) | \rho_i and \rho_j are dual to each other\}$ and for every $(i, j) \in A_G$, $W_{(i,j)} = \{ X \in M_n(F) | X^{ij}_{d_i, k_i \times d_j, k_j} = C_k^{i, \rho_i(g)} X^{ij}_{d_i, k_i \times d_j, k_j} C_k^{i, \rho_i(g)}, \forall g \in G and rest blocks are zeros\}$. Then for $(i, j) \in A_G$, $W_{(i,j)}$ is a subspace of $M_n(F)$.

Let $X$ be an element of $\Xi'_G$, then

$$C_{\rho(g)} X C_{\rho(g)} = X and X = [X^{ij}_{d_i, k_i \times d_j, k_j}]_{(i,j) \in A_G}.$$

Existence:

Let $X \in \Xi'_G$ then for every $(i, j) \in A_G$, there exists at least one $X_{(i,j)} \in W_{(i,j)}$, such that $\sum_{(i,j) \in A_G} X_{(i,j)} = X$.

Uniqueness:

For every $(i, j) \in A_G$, suppose there are $Y_{(i,j)}$ and $X_{(i,j)} \in W_{(i,j)}$, such that $\sum_{(i,j) \in A_G} Y_{(i,j)} = X = \sum_{(i,j) \in A_G} X_{(i,j)}$, then $\sum_{(i,j) \in A_G} X_{(i,j)} = \sum_{(i,j) \in A_G} Y_{(i,j)}$, i.e., $Y_{(i',j')} - X_{(i',j')} = \sum_{(i,j) \in A_G} (X_{(i,j)} - Y_{(i,j)}).$ Therefore $Y_{(i',j')} - X_{(i',j')} = 0 \implies Y_{(i',j')} = X_{(i',j')} for all (i', j') \in A_G.$

Thus we have

$$\Xi'_G = \oplus_{(i,j) \in A_G} W_{(i,j)} and dim(\Xi'_G) = \sum_{(i,j) \in A_G} dim(W_{(i,j)}). \tag{7}$$

Now as for $(i, j) \in A_G$, $W_{(i,j)} = \{ X \in M_n(F) | (i, j)^{th} block 'X^{ij}' is a sub - matrix of order $d_i k_i \times d_j k_j$ satisfying $X^{ij} = C_k^{i, \rho_i(g)} X^{ij} C_k^{i, \rho_i(g)}, \forall g \in G and rest blocks are zeros\}$. Now by the remark 4.1 we see that for $(i, j) \in A_G$, the sub-matrices $X^{ij}$ in $W_{(i,j)}$ have $k_i k_j$ free variables & $W_{(i,j)} \cong M_{k_i \times k_j}(F)$. Thus $\Xi'_G \cong \oplus_{(i,j) \in A_G} M_{k_i \times k_j}(F)$ and $dim(W_{(i,j)}) = k_i k_j$.

Thus substituting these in equation (7) we get the dimension of $\Xi'_G$.

**Proof of Theorem 7.3** Follows immediately from Lemmas 4.1 and 7.2. \hfill \Box

7. Representation over a field of characteristic $p$.

**Remark 7.1.** If characteristic of the field $\mathbb{F}$ is $p$ then a group $G$ of order $p^3$ has only trivial irreducible representation. Therefore for every $g \in G$, we have $\rho(g) = n \rho_1(g)$. 

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where $\rho_1$ is the trivial representation of group $G$. So the representation $\rho$ is a trivial representation of degree $n$, i.e.,

$$\rho(g) = I_n, \text{ for all } g \in G.$$ 

**Note 7.1.** As $X = I_n'X I_n, \forall X \in \mathbb{M}_n(F)$, therefore every $X \in \mathbb{M}_n(F)$ gives an invariant bilinear form in the case when characteristic of the field is $p$. This is summarised in the following result.

**Proposition 7.1.** The space of invariant bilinear forms under an $n$ degree representation of a group $G$ of order $p^3$ with $\text{char}(F) = p$ an odd prime is isomorphic to $\mathbb{M}_n(F)$ and contains a non-degenerate invariant bilinear form.

Thus here we have completely characterised the representations of a group of order $p^3$ each of which admits a non-degenerate invariant bilinear form over a field of characteristic different from $p$ consisting of a primitive $p^3$th root of unity. Authors hope to evaluate these results for a group of higher order in future.

**Acknowledgement** The first author would like to thank UGC, India for providing the research fellowship and to the Central University of Jharkhand, India for facilitating this research work. The second author would like to express his gratitude towards Babasaheb Bhimrao Ambedkar University, Lucknow, India where he got affiliated while finalizing this paper.

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