Operator space entanglement entropy in transverse Ising chain

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The efficiency of time dependent density matrix renormalization group methods is intrinsically connected with the rate of entanglement growth. We introduce a new measure of entanglement in the space of operators and show, for transverse Ising spin 1/2 chain, that the simulation of observables, contrary to simulation of typical pure quantum states, is efficient for initial local operators. For initial operators with a finite index in Majorana representation, the operator space entanglement entropy saturates with time to a level which is calculated analytically, while for initial operators with infinite index the growth of operator space entanglement entropy is shown to be logarithmic.

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I. INTRODUCTION

The entanglement is an intrinsic property of composite quantum systems and represents a cornerstone in quantum information theory [1]. It is important to understand the role of quantum entanglement in classical manipulation of quantum objects, and to quantify the degree of entanglement. Although the question of quantification is not clearly resolved, the quantum information theory offers several measures [2, 3] of entanglement. Quantum information theory also gave a new birth or fresh interpretation of a class of methods for numerical simulation of many-body quantum systems which, due to the exponential growth of Hilbert space, cannot be manipulated using exact diagonalization. The methods originally known as density matrix renormalization group (DMRG) [4] deploy the fact that many degrees of freedom are redundant in quantum state description; the system is therefore adequately described by taking into account maximally entangled components only. Thus, sufficiently slow growth of the entanglement is of crucial importance. DMRG enjoyed remarkable success in determining the ground state properties of large one-dimensional quantum models, for which the degree of entanglement scales at most logarithmically with size [5, 6, 7, 8, 9, 10]; however its time-dependent version (tDMRG) [11, 12] is often plagued by abundance of entanglement with time evolution [13]. For efficient classical simulation of many-body quantum dynamics using tDMRG it is required that the computational costs grow polynomially in time meaning that the degree of entanglement of any quantum object which can be represented as an element of a scalable tensor product Hilbert space (either a pure state, or a mixed state/operator, etc) must grow no faster than logarithmically. It was recently shown that this is generically not the case for quantum chaotic Ising spin chain in a tilted magnetic field where the entanglement entropies grow linearly and hence the computation costs increase exponentially in time [13].

In this paper we shall consider the model of quantum Ising chain in transverse magnetic field which is integrable and an explicit analytical solution exists. Calabrese and Cardy [14] have shown numerically that the growth of entanglement entropy is linear for evolution of pure initial states which are ground states of quenched hamiltonians; see also Ref. [15]. However, from efficiency of tDMRG for time evolution of local operators [13] one may conclude that entanglement entropy computed in the space of operators grows only logarithmically. Here we address this problem theoretically using the idea [16] of re-formulating the Heisenberg evolution in an algebra of operators in terms of a Schrödinger evolution generated by a different - adjoint hamiltonian acting on the Hilbert space of operator algebra. We show that operator space entanglement entropy saturates in time for initial local operators of finite index (precise definitions follow) and explicitly compute the saturation values in the critical case. Further, for initial local operators of infinite index we give accurate numerical evidence and a theoretical hint that the growth is logarithmic (in thermodynamic limit) with prefactor 1/6 in critical, or 1/3 in non-critical case.

We note that, to best of our knowledge, the entanglement in operator space is a new concept which has not yet been considered theoretically, and it is clearly not equivalent to the conventional concept of entanglement of density operators as discussed in sect [16]. Yet, it is the one which we expect to be more closely related to computational complexity of time-evolution in infinite interacting quantum systems.

II. FERMION REPRESENTATION OF DYNAMICS IN OPERATOR SPACE

The dynamics of a transverse Ising chain of length 2L is described in terms of canonical Pauli matrices $\sigma_j^x$ for sites $j \in \mathbb{Z}_{2L} \equiv \{ -L + 1, \ldots, 0, 1, \ldots, L \}$ and the hamiltonian

$$H = \sum_{j=-L+1}^{L-1} \sigma_j^x \sigma_{j+1}^x + \hbar \sum_{j=-L+1}^{L} \sigma_j^z$$

(1)

with open boundary conditions, which can be diagonalized by means of Jordan-Wigner transformation and introduction of Majorana fermion operators [16, 17], $X_n = (\prod_{j<n} \sigma_j^x) \sigma_n^x$ and $Y_n = (\prod_{j<n} \sigma_j^x) \sigma_n^z$ fulfilling the...
anti-commutation relations \(\{X_i, X_j\} = \{Y_i, Y_j\} = 2\delta_{ij}\) and \(\{X_i, Y_j\} = 0\). Heisenberg equations of motion \(dA/dt = [H, A]\) for Majorana operators can be written:

\[
\dot{X}_n = 2(Y_{n-1} - hY_n), \quad \dot{Y}_n = -2(X_{n+1} - hX_n). \tag{2}
\]

An operator corresponding to an arbitrary physical observable can be written as a superposition of products of Majorana operators \(X_i, Y_j\), namely \(P_{n,n'} = X_{n-L+1}^{s_{n-L+1}} \cdots X^{s_{n-L}}_{n-L} Y^{s_{n}}_{n}\), with powers \(n_j, n'_j \in \{0, 1\}\). A set of \(4L^2\) operators \(\{|P_{n,n'}\rangle\rangle\) spans an orthonormal basis of a Hilbert space, namely the matrix algebra \(A = C^{2L \times 2L}\), with an inner product \(\langle A|B\rangle = 2^{-2L} trA^\dagger B\).

\(A\) can be conveniently interpreted as a Fock space of \(2L\) adjoint fermions (we shall call them a-fermions) with pseudo-spin (distinguishing between Majorana \(X_j\) and \(Y_j\) operator). An arbitrary operator \(A\) is then a-fermion state \(|A\rangle = \sum_{n,n'} a_{n,n'}|P_{n,n'}\rangle\rangle\). A-fermi operators over \(A\), \(c_{n,j}, \hat{c}_{n,j}\), can be introduced by

\[
\hat{c}_{n,j}|P_{n,n'}\rangle = n_j|X_j P_{n,n'}\rangle, \quad \hat{c}_{n,j}|P_{n,n'}\rangle = n'_j|Y_j P_{n,n'}\rangle, \tag{3}
\]

satisfying canonical anti-commutation relations.

The index of an operator \(P_{n,n'}\) is defined as \(I_{n,n'} = \sum_j (n_j + n'_j)\) and for index-one operators eq. (2) rewrites:

\[
\frac{d}{dt}\langle X_n\rangle = 2(c_{n-1}^\dagger - h\hat{c}_{n}^\dagger)\hat{c}_{n'}\langle X_n\rangle, \tag{5}
\]

\[
\frac{d}{dt}\langle Y_n\rangle = -2(c_{n+1}^\dagger - h\hat{c}_{n}^\dagger)\hat{c}_{n}\langle Y_n\rangle, \tag{6}
\]

which can be interpreted as a Schrödinger equation \((d/dt)|A\rangle = -i\hat{H}|A\rangle\) for the adjoint hamiltonian

\[
\hat{H} = 2i \sum_{n,n' \in Z_{2L}} \left[c_{n-1}^\dagger (\hat{c}_{n'} - \hat{c}_{n'}^\dagger) + h(\hat{c}_{n'}\hat{c}_{n} - \hat{c}_{n'}^\dagger\hat{c}_{n}^\dagger)\right]. \tag{7}
\]

Since the adjoint time-evolution is a homomorphism the Schrödinger equation extends to arbitrary element of the operator algebra \(|A\rangle\in A\). Note that the number of a-fermions, \(\hat{N} = \sum_{n,A} \hat{c}_{n,s}^\dagger \hat{c}_{n,s}\) is conserved, unlike the number of ordinary Majorana fermions.

III. OPERATOR SPACE ENTANGLEMENT ENTROPY

It is clear that classical simulability of quantum states is quantified by the entanglement entropy of half-half (or worst case) bipartition of the lattice. However, for simulability of quantum observables (or density operators of mixed states), the decisive quantity is an analog of entanglement entropy defined for an arbitrary element of operator algebra \(A \ni |A\rangle = \sum_{n,n'} a_{n,n'}|P_{n,n'}\rangle\rangle\), with the adjoint reduced density matrix

\[
R_{(n,L+1,n',L+1)}(n_0,...,n_L,m_0,...,m_L) = \sum_{n_1,n'_1,...,n_L,n'_L} a_{(n_1,...,n_L,n_1,...,n_L)} a_{(n_1,...,n_L,n_1,...,n_L)}^\dagger \tag{8}
\]

namely

\[
S = -trR \ln R. \tag{9}
\]

For a spin 1/2 chain it is perhaps more natural to use a set of \(4L\) Pauli operators \(Q_{s_{n-L+1}...s_{n-L}} = \sigma_{s_{n-L+1}}^z \cdots \sigma_{s_{n-L}}^z\), where \(s_j \in \{0, x, y, z\}\), \(\sigma^0 = 1\), as a physical basis of operator algebra \(A\), and define bi-partition and entanglement entropy with respect to \(Q_s\). However, it is easy to show that the result is identical to \((\ref{7})\) since the transformation between the bases \(\{P_{n,n'}\}\) and \(\{Q_s\}\) is a simple permutation of multindices \((n,n')\leftrightarrow s\) (with multiplications by \(\pm 1\)), and even though it is non-local it maps first \(L\) a-fermions to only first \(L\) spins and vice versa.

Let us now try to compute time-dependence of operator space entanglement entropy \(S(t)\) for some simple initial operators \(A\). For convenience, we introduce staggered a-fermi operators \(\tilde{w}_j, j \in Z_{2L} = \{-2L + 1, \ldots, 0, 1, \ldots, 2L\}\), such that \(\tilde{w}_{2n-1} = \tilde{c}_{n_{n}}\) and \(\tilde{w}_{2n} = \tilde{c}_{n_{n}}\). Any operator acting solely in a space of first \(L\) a-fermions (or first \(L\) spins) can be expressed in terms of \(2L\) anti-comuting operators \(\tilde{w}_j\) with \(j \in Z_{2L} = \{-2L + 1, \ldots, -1, 0\}\). We follow Ref. [10] and express \(2L^2\) eigenvalues of adjoint reduced density matrix \(R\), as \(\rho_{mn} = \prod_j (n_j \gamma_j + (1-n_j)(1-\gamma_j))\), \(n_j \in \{0, 1\}\) where \(\gamma_j\) are eigenvalues of time-dependent \(2L \times 2L\) correlation matrix

\[
\Gamma_{mn}(t) = \langle A|\hat{\tilde{w}}_m^\dagger(t)\hat{\tilde{w}}_n(t)|A\rangle, \quad m, n \in Z_{2L} \tag{10}
\]

Then, the entanglement entropy \((\ref{7})\) simply reads

\[
S(t) = \sum_j e(\gamma_j), \quad e(x) = -x \ln x - (1-x) \ln(1-x). \tag{11}
\]

This procedure (see [14] for details) results in an efficient numerical method which essentially only requires diagonalization of \(2L\) dimensional matrix \(\Gamma\) for the solution of a quantum problem on \(2L\) dimensional Hilbert space.

The time-dependent a-fermi operators \(\tilde{w}_m(t)\) are obtained from linear Heisenberg type equations \(\dot{\tilde{w}}_m = -i\tilde{w}_m, H]\), namely, \(\tilde{w}_{2j} = 2(\tilde{w}_{2j+1} - hw_{2j-1})\) and \(\tilde{w}_{2j-1} = 2(-\tilde{w}_{2j-2} + hw_{2j})\). The solution of such Heisenberg equations, written as \(\tilde{w}_m = -i\sum_n G_{mn}\tilde{w}_n\), is obtained by diagonalizing a \(2L \times 2L\) matrix \(G = V \cdot \Lambda \cdot V^\dagger\) which yields

\[
\tilde{w}_m(t) = \sum_n \left(\sum_k V_{mk}e^{-it\Lambda_{kk}}V_{nk}^\dagger\right)\tilde{w}_n. \tag{12}
\]

However, in the ‘critical case’ \(h = 1\), the time-evolution of \(\tilde{w}_m(t)\) can be solved exactly and some analytical solutions can be given. Namely in such a case the sets of
Heisenberg eqs. are identical and are solved via discrete sine-transform with \( V_{mn} = i^m \sqrt{\frac{2}{4L+1}} \sin \left( \frac{m(2L)k\pi}{4L+1} \right) \).

\[
\hat{w}_m(t) = \sum_n \left[ \sum_{k=0}^{4L} V_{mn} e^{i4\cos(\frac{k\pi}{2L+1})t} V_{nk}^* \right] \hat{w}_n.
\] (13)

In the following we shall be interested in the results in the thermodynamic limit (TL), \( L \to \infty \). The infinite sum over \( k \) in (13) is transformed onto an integral which yields \( \hat{w}_m(t) = \int_{\mathbb{Z}} \Phi_{nm}(4t) \hat{w}_n \) in terms of Bessel functions \( \Phi_{nm}(x) \equiv J_{m-k}(x) \). The correlation matrix elements are therefore (using \( \hat{n}_b = \hat{w}_b^\dagger \hat{w}_b \))

\[
\Gamma_{mn}(t) = \sum_{b \in \mathbb{Z}} \Phi_{bm}(4t) \Phi_{bn}(4t) \langle A | \hat{n}_b | A \rangle_{t^4}.
\] (14)

We also assume that the initial operator \( A \) is local, i.e. a product of finite number of Pauli matrices \( \sigma_\alpha^g \). This implies that: (i) either \( A \) has a finite index, i.e. \( \langle \hat{n}_b \rangle_0 \equiv \langle A | \hat{n}_b | A \rangle = 0 \), for \( |b| > b_0 \) for some \( b_0 \in \mathbb{Z} \), or (ii) \( A \) has an infinite index and \( \langle \hat{n}_b \rangle = 1 \) for \( b < -b_0 \) and \( \langle \hat{n}_b \rangle = 0 \) for \( b > b_0 \) (such as e.g. \( A = \sigma_\alpha^g \)). Then as \( L \to \infty \), an arbitrary large fixed finite piece of correlation matrix can be asymptotically written as

\[
\Gamma_{mn}(t) = \sum_{b \in \mathbb{Z}} J_{m-k}(4t) J_{m-n}(4t) \langle \hat{n}_b \rangle.
\] (15)

Note that \( \Gamma_{mn} \) has effectively finite rank \( \sim x = 4t \), namely \( \Gamma_{m,n} \sim \delta_{mn} \), for \( -m,-n > x \). For brevity we shall be omitting the argument of Bessel functions always equal to \( x = 4t \).

### A. Initial operators of finite index

First, we focus on the case (i) of finite index initial operators \( A \). It was conjectured in (13) that in such cases the entanglement entropy in thermodynamic limit saturates in time. Using the a-fermion algebra we are now able to calculate the exact saturation value since RHS in (15) is a finite sum. Consider \( \Gamma_{mn} \) as a real matrix over \( \mathbb{R}^\infty \) with canonical basis \( \{ |m \rangle, m \in \mathbb{Z}^- \} \) and write a set of non-orthogonal vectors \( \{ |\psi_\alpha \rangle \} \), namely \( \langle m | \psi_\alpha \rangle = (-1)^\alpha J_{m-\alpha}(4t) \langle \psi_\alpha | m \rangle \). Let us write initial operator of finite index \( K \) as \( A = O_{j_1} \cdots O_{j_K} \) where \( O_{2j-1} \equiv X_j \) and \( O_{2j} \equiv Y_j \). Then we have \( \Gamma_{mn} = J_{j_1-m} J_{j_1-n} + \cdots + J_{j_K-m} J_{j_K-n} \), or

\[
\Gamma_{mn}(t) = \langle |m \rangle |\psi_\alpha \rangle \langle \psi_\beta | n \rangle + \cdots + \langle |m \rangle |\psi_{j_K} \rangle \langle \psi_{j_K} | n \rangle.
\] (16)

This means that the rank of \( \Gamma_{mn} \) is bounded by \( K \), in fact it is \( K \), and its non-trivial eigenspaces are spanned by \( \{ |\psi_{j_k} \rangle \}, 1 \leq k \leq K \). Let \( \{ |\phi_k \rangle \}, 1 \leq k \leq K \) be an orthonormalized set obtained from \( \{ |\psi_{j_k} \rangle \}, 1 \leq k \leq K \} \) by a standard Gramm-Schmidt procedure, for which the only input is the set of scalar products \( \langle \psi_\alpha | \psi_\beta \rangle = \sum_{k=1}^{K} \langle \phi_k | \psi_\alpha \rangle \langle \psi_\beta | \phi_k \rangle \). (17)

Thus, eq. (17) is our main result for the case of finite index initial operators. For illustration, let us calculate the asymptotic value \( S(t \to \infty) \) for the simplest two cases: (a) \( A = O_j \), e.g. \( A = \cdots \sigma_\alpha^g \sigma_\beta^g \sigma_\gamma^g \sigma_\delta^g \), and (b) \( A = O_j O_{j+1} \), e.g. \( A = \sigma_\alpha^g \). In case (a), \( K = 1 \), the result is \( \gamma_1 = \langle \psi_j | \psi_j \rangle \) with \( \gamma_1 | t = \infty = 1 \). Then we have \( \Gamma_{mn} = J_{j_1-m} J_{j_1-n} \).

### B. Initial operators of infinite index

Second, we consider the case (ii) of infinite index initial operator \( A \). In thermodynamic limit, local spin operators such as \( \sigma_\alpha^g \) are products of infinite number of Majorana operators \( X_\alpha, Y_\alpha \), in particular \( \sigma_\alpha^g \equiv | X_\alpha Y_\alpha \rangle \langle X_\alpha Y_\alpha | \) and previous discussion does not apply. As conjectured in (13) time complexity for such initial operators only grows polynomially in time which corresponds to logarithmic growth of the entanglement

**FIG. 1:** Entanglement entropy for finite-index operators \( |X_1 \rangle \) (black) and \( |X_1 Y_1 \rangle \) (gray) compared to theoretical saturation value for \( t \to \infty \) and \( h = 1 \) (thick lines). Three different values of magnetic field are considered: \( h = 1 \) (solid curve), \( h = 0.5 \) (dotted), \( h = 2 \) (dashed), \( h = 3 \) (dash-dotted). We set \( 2L \) such that no finite size effect were noticable.

The sum of eq. (11) entering eq. (11) is thus given by the eigenvalues of the following \( K \times K \) matrix

\[
\hat{F}_{kl} = \langle \phi_k | \psi_j \rangle \langle \psi_j | \phi_l \rangle + \cdots + \langle \phi_k | \psi_{j_K} \rangle \langle \psi_{j_K} | \phi_l \rangle.
\] (17)
It should be noted that correlation matrix can be factorized $\Gamma'_{mn} = \sum_{l=0}^{\infty} \Psi_{ml} \Psi_{ln}$ as a square of a matrix $\Psi_{mn} = (-1)^{m+n} J_{m-n}$. Note that $\Psi_{mn}$ is a real symmetric infinite block Toeplitz matrix

$$\Psi = \begin{pmatrix} \Pi_0 & \Pi_1 & \cdots \\ \Pi_{-1} & \Pi_0 & \ddots \\ \vdots & \ddots & \ddots \\ \Pi_{-l} & \cdots & \cdots & \Pi_0 & \Pi_{-l+1} & \cdots \\ -J_{2l} & -J_{2l-1} & \cdots & \cdots & \cdots & \cdots \\ -J_{2l+1} & -J_{2l} & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}.$$  \tag{20}

Following Ref. [10] we express the time-dependent entanglement entropy \( S(t) \) in terms of a formula involving Block Toeplitz determinant

$$S = \frac{1}{2\pi i} \int_{\Xi} e(\lambda^2) \left[ \frac{d}{d\lambda} \ln \det(\lambda \mathbb{1} - \Psi) \right] d\lambda \tag{21}$$

where $\Xi$ is a closed curve in complex plane enclosing unit disk, avoiding point 1 and interval $[-1,0]$. Note that eigenvalues of infinite dimensional matrix $\Psi$ come in pairs $\lambda, -\lambda$ with accumulation points $\pm 1$. For any $\epsilon > 0$ there is only a finite number, $N_\epsilon(t) \sim t$, of eigenvalues of $\Psi$ which are not in $\epsilon$ vicinity of $\pm 1$. However, we find numerically that most of these eigenvalues cluster around 0, and only $\sim \ln t$ of them lie outside $\epsilon$ vicinity of 0 which are the only eigenvalues contributing to entanglement entropy result \([13]\).

At present state of the theory of block Toeplitz determinants - in connection to the theory of integrable Fredholm operators and the Riemann-Hilbert problem \([19]\) - the formula \((21)\) can be explicitly evaluated (see e.g. Ref. \([20]\)) provided the matrix symbol $\Phi(z) = \lambda \mathbb{1} - \sum_{k \in \mathbb{Z}} \Pi_k z^k$, admits explicit Wiener-Hopf factorizatons $\Phi(z) = U^+(z) U^-(z) = V^+(z) V^-(z)$ where the matrix functions $U^\pm(z), V^\pm(z)$ are analytic inside(+)/outside(-) the unit circle. Even though the matrix symbol has an appealing explicit form

$$\Phi(z) = \begin{pmatrix} \lambda - f f^* + g g^* & f g^* - g f^* \\ z g f^* - f g^* & \lambda + f f^* - g g^* \end{pmatrix} \tag{22}$$

where $f = f(z), \bar{f} = f(z^{-1}), g = g(z), \bar{g} = g(z^{-1})$ and $f(z) \equiv \cosh(2\sqrt{\epsilon} z)$ and $g(z) \equiv \sinh(2\sqrt{\epsilon} z)/\sqrt{\epsilon}$ are entire analytic functions, its Wiener-Hopf factorization is at present unknown and poses a challenging problem.

**IV. CONCLUSIONS**

We have studied complexity of time evolution of initial local operators under dynamics given by the transverse Ising chain. Such complexity can be characterized in terms of entanglement entropy of operators treated as elements of a product Hilbert space corresponding to a bi-partition of a chain and is directly related to time efficiency of simulation methods such as tDMRG. Note that operator space entanglement entropy, of say a density operator, is not simply related to a traditional notion of entanglement of the corresponding mixed state. For example, consider a macroscopic convex combination of $2^L$ product states. This corresponds to a non-entangled
mixed state but has a macroscopic operator space entanglement entropy $\sim L$ and hence it is difficult to simulate classically. Thus it seems that the traditional concept of state entanglement is not sufficient to characterize classical complexity of quantum operators. In this paper we have shown, in parts analytically and numerically, that operator space entanglement entropy of transverse Ising model grows at most logarithmically for initial operators which are local products of Pauli matrices. This result has to be contrasted with a a linear growth of entanglement entropy for time evolution of pure states [14]. Explanation of deeper physical reasons for this dramatic effect is needed.

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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1 (2007).
[3] J. Eisert and M. B. Plenio, J. Mod. Opt. 46, 3496 (1999).
[4] S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
[5] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003); J. I. Latorre, E. Rico, and G. Vidal, Quant. Inf. Comp. 4, 48 (2004).
[6] A. Osterloh et al., Nature (London) 416, 608 (2002).
[7] T. J. Osborne and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[8] J. P. Keating and F. Mezzadri, Commun. Math. Phys. 252, 543 (2004).
[9] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B424, 44 (1994).
[10] B.-Q. Jin and V. E. Korepin, J. Stat. Phys. 116, 79 (2004); quant-ph/0304108
[11] G. Vidal, Phys. Rev. Lett. 91, 147902 (2003).
[12] S. R. White and A. E. Feiguin, Phys. Rev. Lett. 93, 076401 (2004).
[13] T. Prosen and M. Žnidarič, Phys. Rev. E 75, 015202(R) (2007).
[14] P. Calabrese and J. Cardy, J. Stat. Mech. Theor. Exp. P04010 (2005).
[15] G. De Chiara et al., J. Stat. Mech. Theor. Exp. P03001 (2006).
[16] T. Prosen, Phys. Rev. E 60, 1658 (1999); Prog. Theor. Phys. Suppl. 139, 191 (2000).
[17] U. Brandt and K. Jacoby, Z. Physik B25, 181 (1976); Z. Physik B26, 245 (1977).
[18] P. Calabrese and J. Cardy, J. Stat. Math. Theor. Exp. P06002 (2004).
[19] P. Deift, Amer. Math. Soc. Transl. 189, 69 (1999).
[20] A. R. Its, B.-Q. Jin, and V. E. Korepin, J. Phys. A 38, 2975 (2005); quant-ph/0606178