Real algebraic curves of bidegree $(5,5)$ on the quadric ellipsoid

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Abstract

We complete the topological classification of real algebraic non-singular curves of bidegree $(5,5)$ on the quadric ellipsoid. We show in particular that previously known restrictions form a complete system for this bidegree. Therefore, the main part of the paper concerns the construction of real algebraic curves. Our strategy is first to reduce to the construction of curves in the second Hirzebruch surface by degenerating the quadric ellipsoid to the quadratic cone. Next, we combine different classical construction methods on toric surfaces, such as Dessin d’enfants and and Viro’s patchworking method.

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1 Introduction

The study of real algebraic curves on the quadric ellipsoid is related to the classical Hilbert’s 16-th problem whose first part is about the classification of the oval arrangements of real algebraic non-singular plane curves. Let $X = \mathbb{C}P^1 \times \mathbb{C}P^1$, it admits an anti-holomorphic involution

$$\sigma : X \longrightarrow X, \quad (x, y) \longrightarrow (\overline{y}, \overline{x}),$$

where $x = [x_0 : x_1], y = [y_0 : y_1]$ are in $\mathbb{C}P^1$ and $\overline{x} = [\overline{x}_0 : \overline{x}_1], \overline{y} = [\overline{y}_0 : \overline{y}_1]$ are respectively the images of $x, y$ via the standard complex conjugation on $\mathbb{C}P^1$. The real part of $X$ is denoted by $RX := \text{fix}(\sigma)$ and is homeomorphic to $S^2$. It is well known that $X$ is isomorphic to the quadric ellipsoid in $\mathbb{C}P^3$. A real algebraic non-singular curve $A$ on $X$ is defined by a bi-homogeneous polynomial of bidegree $(d, d)$

$$P(x_0, x_1, y_0, y_1) = \sum_{i,j=1}^{d,d} a_{i,j} x_i^i x_0^{d-i} y_j^j y_0^{d-j}$$

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where \( d \) is a positive integer and the coefficients satisfy \( a_{i,j} = \overline{a_{j,i}} \). Let \( \mathbb{R}A \) be the set of real points of \( A \). The connected components of \( \mathbb{R}A \) are called ovals. We are interested in the classification of the oval arrangements of real algebraic non-singular curves on \( X \). The rigid isotopy classification of real algebraic non-singular curves of bidegree \( (d,d) \), for \( d < 5 \) is known (see [Zvo01], [NS05a], [NS07], [NS05b], [DZ99], [Nik85] and for a general idea see Section 4.9 in [DK00]). Therefore, we are interested in bidegree \((5,5)\): in this paper, we give the classification, up to homeomorphism, of the topological types of the pair \((\mathbb{R}X, \mathbb{R}A)\) where \( A \) is a real algebraic non-singular curve of bidegree \((5,5)\) on \( X \).

Due to the Harnack-Klein inequality, for a compact real algebraic non-singular curve \( A \), the number \( l \) of its connected components is bounded by \( g + 1 \), where \( g \) is the genus of \( A \). When \( l = g + 1 \), we talk about \( M \)-curves or maximal curves. Otherwise, for \( 0 \leq l \leq g \) , we speak about \((M-i)\)-curves with \( i = g + 1 - l \). Thanks to the adjunction formula, for a real algebraic non-singular curve of bidegree \((d,d)\) on \( X \) we have \( g = (d-1)^2 \).

Given a collection \( \sqcup_{i=1}^{l} B_i \) of \( l \) disjoint circles embedded in \( S^2 \), we encode the topological pair \((S^2 \setminus \sqcup_{i=1}^{l} B_i)\) as follows. Let \( p \) be any point in \( S^2 \setminus \sqcup_{i=1}^{l} B_i \). Each oval \( B_i \) bounds two non-homeomorphic parts in \( S^2 \setminus \{p\} \). Let us call the interior the part homeomorphic to a disc and the exterior the other one. For each pair of ovals, if one is in the interior of the other we speak about injective pair, otherwise of non-injective one. We shall adopt the following notation to encode a given topological pair \((S^2 \setminus \{p\}, \sqcup_{i=1}^{l} B_i)\). An empty union of ovals is denoted by \( \emptyset \). We say that a union of \( l \) ovals realizes \( l \) if there are no injective pairs. Therefore, the symbol \( <S> \) denotes the disjoint union of a collection of ovals realizing \( S \), and an oval forming an injective pair with each oval of the collection. Finally, the disjoint union of any two collections of ovals, realizing in \( S^2 \setminus \{p\} \) respectively \( S', S'' \), is denoted by \( S' \sqcup S'' \) if none of the ovals of one collection forms an injective pair with the ovals of the other one.

We say that the pair \((S^2 \setminus \{p\}, \sqcup_{i=1}^{l} B_i)\) realizes \( S \) if there exists a point \( p \in S^2 \setminus \sqcup_{i=1}^{l} B_i \) such that \((S^2 \setminus \{p\}, \sqcup_{i=1}^{l} B_i)\) realizes \( S \). Moreover, let \( A \) be a real algebraic curve of bidegree \((d,d)\) on \( X \), we say that \( A \) has real scheme \( S \) if the pair \((\mathbb{R}X, \mathbb{R}A)\) realizes \( S \). Let us give the main result of this paper about the classification of \( M \)-curves.

**Theorem 1.1** (Maximal curves). Let \( A \) be a real algebraic non-singular \( M \)-curve of bidegree \((5,5)\) on \( X \). Then the pair \((\mathbb{R}X, \mathbb{R}A)\) has to realize one of the following real schemes:

\[ \alpha \sqcup < \beta > \sqcup < \gamma >, \ \alpha \equiv 1 \pmod{4}, \ \text{with} \ \alpha + \beta + \gamma = 15. \]

Moreover, all such real schemes are realizable by real algebraic non-singular \( M \)-curves of bidegree \((5,5)\).

Let \( A \) be a real algebraic non-singular curve, if \( CA \setminus \mathbb{R}A \) is connected we say that \( A \) is of type II, otherwise of type I. If \( A \) is of type I the number of its real connected components has the same parity of \( g(A) + 1 \). The maximal curves are of type I (see for example Proposition [Vir 2.6.B]).

**Theorem 1.2** ((\(M-1\))-curves). Let \( A \) be a real algebraic non-singular \((M-1)\)-curve of bidegree \((5,5)\) on \( X \). Then the pair \((\mathbb{R}X, \mathbb{R}A)\) has to realize one of the following real schemes:

\[ \alpha \sqcup < \beta > \sqcup < \gamma >, \ \alpha \equiv 0 \text{ or } 1 \pmod{4}, \ \text{with} \ \alpha + \beta + \gamma = 14. \]

Moreover, all such real schemes are realizable by real algebraic non-singular \((M-1)\)-curves of bidegree \((5,5)\).
Theorem 1.3 ((M−2)-curves). Let $A$ be a real algebraic non-singular (M−2)-curve of bidegree (5, 5) on $X$. If it is of type I, then the pair $(\mathbb{R}X, \mathbb{R}A)$ has to realize one of the following real schemes:

\begin{enumerate}
  \item $\alpha \sqcup <\beta> \sqcup <\gamma>$, $\alpha \equiv 0 \pmod{2}$, with $\alpha + \beta + \gamma = 13$;
\end{enumerate}

if it is of type II, then one of the following ones:

\begin{enumerate}
  \item $\alpha \sqcup <\beta> \sqcup <\gamma>$, $\alpha \not\equiv 2 \pmod{4}$, with $\alpha + \beta + \gamma = 13$.
\end{enumerate}

Moreover, all the real schemes in (1) and (2) are realizable by real algebraic non-singular (M−2)-curves of bidegree (5, 5) respectively of type I and II.

Theorem 1.4 (Type I and II curves). Let $A$ be a real algebraic non-singular (M−i)-curve of bidegree (5, 5) on $X$, where $3 \leq i \leq 17$. If it is of type II, then the pair $(\mathbb{R}X, \mathbb{R}A)$ has to realize one of the following real schemes:

\begin{enumerate}
  \item $0, 1$.
  \item $\alpha \sqcup <\beta> \sqcup <\gamma>$, with $\alpha + \beta + \gamma = 17 - (i - 2)$;
\end{enumerate}

if it is of type I, then $i = 4, 6, 8, 10, 12$ and the pair $(\mathbb{R}X, \mathbb{R}A)$ has to realize one of the following real schemes:

\begin{enumerate}
  \item $<<<<1>>>>$, when $\alpha + \beta + \gamma = 5, 9$,
  \item with $\alpha \equiv 1 \pmod{2}$ when $\alpha + \beta + \gamma = 7, 11$.
\end{enumerate}

Moreover, the real schemes in (1), (2) and (3), (4) are realizable by real algebraic non-singular curves of bidegree (5, 5) respectively of type II and I.

Remark 1.5. It is easy to see that the classification up to homeomorphism of the pairs $(\mathbb{R}X, \mathbb{R}A)$ in Theorems 1.1, 1.2, 1.3, 1.4 is equivalent to the classification up to isotopy.

Previous results concerning the restrictions part of the classification are given in Proposition 2.1, Theorem 2.3 and Proposition 2.4. Such results give a system of restrictions for real schemes of algebraic curves of bidegree (5, 5) on $X$, the content of this paper is the proof that such system of restrictions is complete. Therefore, the main part of the paper concerns the construction part of the classification. In [Mik94] it is claimed to have constructed some real algebraic M-curves which realize almost all real schemes of Theorem 1.1. Here, in Sections 4 we exploit different constructions and realize all of them.

The real schemes which are not prohibited by restrictions in Section 2 are those listed in Theorems 1.1, 1.2, 1.3 and 1.4. In Section 3 we introduce the tools for the construction part: in particular, in Section 3.2 we explain how constructing a real algebraic curve on the second Hirzebruch surface $\Sigma_2$ imply constructing a real algebraic curve on the quadric ellipsoid in $\mathbb{C}P^3$. Then, on $\Sigma_2$ we exploit different construction techniques: in Section 4.1 we realize some intermediate constructions using the method introduced in [Ore03] by Orevkov via dessins d’enfants; in Section 4.2 using the Viro’s patchworking method and a variant of it developed by Shustin (see [Vir84], [Vir99], [Vir06] and [Shu99], [Shu02], [Shu06]) we finally construct all remaining real algebraic curves of bidegree (5, 5) on $X$ which realize all real schemes listed in Theorems 1.1, 1.2, 1.3 and 1.4.

1.1 Acknowledgments

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2 Restrictions

2.1 Restriction on the depth of nests

A collection $B$ of $h$ disjoint embedded circles in $S^2$ is called a nest of depth $h$ if any connected component of $S^2 \setminus B$ is either a disc or an annulus. Two nests are said to be disjoint if each of them lies on one of the discs bounded by the other.

**Proposition 2.1.** [DK00, Proposition 4.9.2] Let $A$ be a real algebraic non-singular curve of bidegree $(d,d)$ on $X$. Then the total number of ovals in any collection of three pairwise disjoint nests of $\mathbb{R}A$ does not exceed $d$.

Proposition 2.1 implies in particular that the maximal depth for a nest of such a curve $A$ is $d$. Furthermore, it is well known that if $A$ is of type I and has $d$ oval, it has a nest of maximal depth $d$ (see Proposition 2.4).

The Harnack-Klein inequality combined with Proposition 2.1 immediately implies the following corollary.

**Corollary 2.2.** Let $A$ be a real algebraic non-singular curve of bidegree $(5,5)$ on $X$. Then the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the following real schemes:

- $0, 1$.
- $\alpha \sqcup <\beta > \sqcup <\gamma >$, for $0 \leq \alpha + \beta + \gamma \leq 15$.
- $<<<<1>>>>$.

2.2 Congruences and complex orientation formula on the ellipsoid

As an application of Guillou-Marin congruences [GM77] (which are a generalization of Rokhlin’s congruences), G. Mikhalkin proved the following theorem.

**Theorem 2.3.** [Mik94, Theorem 1] Let $A$ be a real algebraic non-singular curve of bidegree $(d,d)$ on $X$, with $d$ odd. Let $B$ be a disjoint union of connected components of $\mathbb{R}X \setminus \mathbb{R}A$ such that $\mathbb{R}A$ bounds $B$.

- If $A$ is a $M$-curve, then
  \[ \chi(B) \equiv \frac{d^2 + 1}{2} \pmod{8}. \]

- If $A$ is a $(M - 1)$-curve, then
  \[ \chi(B) \equiv \frac{d^2 + 1}{2} \pm 1 \pmod{8}. \]

- If $A$ is a $(M - 2)$-curve and
  \[ \chi(B) \equiv \frac{d^2 - 7}{2} \pmod{8}, \]
  then $A$ is of type I.

- If $A$ is of type I, then
  \[ \chi(B) \equiv 1 \pmod{4}. \]
Given a real algebraic non-singular curve \( A \) of type I on \( X \), the two halves of \( CA \setminus RA \) induce two opposite orientations on \( RA \) called complex orientations of the curve. Fix any of these two complex orientations on \( RA \). Since any pair of ovals of \( RA \) bounds an annulus in \( RX \), we distinguish two types of pairs: denote by \( \Pi_- \) (respectively \( \Pi_+ \)) the number of pairs of ovals realizing the same (resp. different) first homology class of the corresponding annulus. As an application of the generalizations of Rohklin’s formula of complex orientations, V. I. Zvonilov in [Zvo83] gave a complex orientation formula for type I real algebraic non-singular curves on \( X \). This formula depends on the choice of an auxiliary point in \( RX \setminus RA \). Afterwards, S. Y. Orekov in [Ore07] reformulated it with no dependence on the choice of an auxiliary point.

\[ \text{Proposition 2.4.} \quad [\text{Zvo83}], [\text{Ore07}] \quad \text{Proposition 1.2}] \quad \text{Let } A \text{ be a real algebraic type I curve of bidegree } (d, d) \text{ on } X \text{. Denoting by } l \text{ the number of connected components of } RA, \text{ one has the following complex orientations formula:} \]

\[ 2(\Pi_+ - \Pi_-) = l - d^2 \quad (1) \]

\[ \text{Corollary 2.5.} \quad [\text{Ore07}] \quad \text{Proposition 1.3}] \quad \text{Let } A \text{ be a real algebraic type I curve of bidegree } (d, d) \text{ on } X \text{. Then } RA \text{ has at least } d \text{ connected components. Furthermore, in the case where } RA \text{ has } d \text{ connected components, it consists of a nest of maximal depth } d \text{.} \]

Corollary 2.2 and Theorem 2.3 give a complete system of restrictions for real schemes of algebraic curves of bidegree \((5, 5)\) on \(X\). Moreover, Theorem 2.3 and Proposition 2.4 allow us to give even finer restrictions on which real schemes, listed in Corollary 2.2, may be realized by type I (resp. type II) real non-singular algebraic curves of bidegree \((5, 5)\) on \(X\). Therefore, given a real algebraic non-singular curve \(A\) of bidegree \((5, 5)\) on \(X\), the pair \((RX, RA)\) has to realize one of the real schemes listed in Theorems 1.1, 1.2, 1.3, 1.4.

In the next sections we pass to the construction part of the classification.

## 3 Constructions tools

### 3.1 Hirzebruch surfaces

A Hirzebruch surface is a compact complex surface which admits a holomorphic fibration over \(CP^1\) with fiber \(CP^1\). Every Hirzebruch surface is biholomorphic to exactly one of the surfaces \(\Sigma_n = \mathbb{P}(O_{CP^1}(n) \oplus \mathbb{C})\) for \(n \geq 0\). The surface \(\Sigma_n\) admits a natural fibration

\[ \pi_n : \Sigma_n \rightarrow CP^1 \]

with fiber \(CP^1 =: F_n\). Denote by \(B_n\) (resp. \(E_n\)) the section \(\mathbb{P}(O_{CP^1}(n) \oplus \{0\})\) (resp. \(\mathbb{P}((0) \oplus \mathbb{C})\)). The self-intersection of \(B_n\) (resp. \(E_n\) and \(F_n\)) is \(n\) (resp. \(-n\) and \(0\)). When \(n \geq 1\), the exceptional divisor \(E_n\) determines uniquely the Hirzebruch surface since it is the only irreducible and reduced algebraic curve in \(\Sigma_n\) with negative self-intersection.

For example \(\Sigma_0 = CP^1 \times CP^1\). The Hirzebruch surface \(\Sigma_1\) is the complex projective plane blown-up at a point, and \(\Sigma_2\) is the quadratic cone with equation \(Q_0 : X^2 + Y^2 - Z^2 = 0\) blown-up at the node in \(CP^3\). The fibration of \(\Sigma_2\) (resp. of \(\Sigma_1\)) is the extension of the projection from the blown-up point to a hyperplane section (resp. to a line) which does not pass through the blown-up point.

The group \(H_2(\Sigma_n, \mathbb{Z})\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}\) and is generated by the classes \([B_n]\) and \([F_n]\). An algebraic curve \(C\) in \(\Sigma_n\) is said to be of bidegree \((a, b)\) if it realizes the homology class \(a[B_n] + b[F_n]\) in \(H_2(\Sigma_n, \mathbb{Z})\). Note that \([E_n] = [B_n] - n[F_n]\) in \(H_2(\Sigma_n, \mathbb{Z})\). An algebraic curve of bidegree \((3, 0)\) on \(\Sigma_n\) is called a trigonal curve.
We can obtain $\Sigma_{n+1}$ from $\Sigma_n$ via a birational transformation $\beta^n_{\pi_n} : \Sigma_n \dashrightarrow \Sigma_{n+1}$ which is the composition of a blow-up at a point $p \in E_n \subset \Sigma_n$ and a blow-down of the strict transform of the fiber $\pi^{-1}_{n}(p)$.

The surface $\Sigma_n$ is also the projective toric surface which corresponds to the polygon of vertices $(0,0), (0,1), (1,1), (n+1,0)$ (see Fig. 1a), the number labeling an edge corresponds to its integer length. The Newton polygon of an algebraic curve $C$ of bidegree $(a,b)$ on $\Sigma_n$, lies inside the trapeze with vertices $(0,0), (0,a), (b,a), (an+b,0)$ as in Fig. 1b. The surface $\Sigma_n$ is canonically endowed by a real structure induced by the standard complex conjugation in $(\mathbb{C}^*)^2$. For this real structure the real part of $\Sigma_n$, denoted by $\mathbb{R}\Sigma_n$, is a torus if $n$ is even and a Klein bottle if $n$ is odd. We will depict $\mathbb{R}\Sigma_n$ as a quadrangle whose opposite sides are identified in a suitable way. Moreover, the horizontal sides will represent $\mathbb{R}E_n$.

![Polygon defining $\Sigma_n$](image1)

![Newton polygon of a curve of bidegree $(a,b)$ on $\Sigma_n$](image2)

Figure 1:

The restriction of $\pi_n$ to $\mathbb{R}\Sigma_n$ defines a $S^1$-bundle over $S^1$ that we denote by $\mathcal{L}$. We are interested in the isotopy types with respect to $\mathcal{L}$ of real algebraic curves in $\mathbb{R}\Sigma_n$.

**Definition 3.1.**

- Two arrangements of circles and points immersed in $\mathbb{R}\Sigma_n$ are $\mathcal{L}$-isotopic if there exists an isotopy of $\mathbb{R}\Sigma_n$ which brings one arrangement to the other, each line of $\mathcal{L}$ to another line of $\mathcal{L}$ and whose restriction to $\mathbb{R}E_n$ is an isotopy of $\mathbb{R}E_n$.

- An arrangement of circles and points immersed in $\mathbb{R}\Sigma_n$ up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_n$ is called an $\mathcal{L}$-scheme.

- A $\mathcal{L}$-scheme is realizable by a real algebraic curve of bidegree $(a,b)$ in $\Sigma_n$ if there exists such a curve whose real part is $\mathcal{L}$-isotopic to the arrangement of circles and points in $\mathbb{R}\Sigma_n$.

- A trigonal $\mathcal{L}$-scheme is a $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ which intersects each fiber in 1 or 3 real points counted with multiplicities and which does not intersect $\mathbb{R}E_n$.

- A trigonal $\mathcal{L}$-scheme $\eta$ in $\mathbb{R}\Sigma_n$ is hyperbolic if it intersects each fiber in 3 real points counted with multiplicities.

### 3.2 The quadric ellipsoid and the second Hirzebruch surface

We explain in this section how to construct a real algebraic curve of bidegree $(d,d)$ on the quadric ellipsoid in $\mathbb{C}P^3$ (see Section 1) starting from a real algebraic curve of bidegree $(d,0)$ in $\Sigma_2$, endowed with the real canonical structure (see Section 3.1). The advantage of working with the real toric surface $\Sigma_2$ is that we can use Viro's
patchworking method to construct real algebraic non-singular curves (see for example [Vir06]).

Let \( C \) be a real algebraic curve of degree \((d, 0)\) in \( \Sigma_2 \). Let \( \eta \) be the real scheme realized by \( \mathbb{R}C \) in \( \mathbb{R}\Sigma_2 \) (a torus). Now, cut \( \mathbb{R}\Sigma_2 \) along \( \mathbb{R}E_2 \) (Fig. 2, a)) and glue two discs \( D_1, D_2 \) as depicted in Fig. 2, b). By this construction we obtain a 2-sphere \( S^2 \). Moreover, from the arrangement of the triplet \((\mathbb{R}\Sigma_2, \eta, \mathbb{R}E_2)\) we obtain an arrangement \( B \) of embedded circles in \( S^2 \). As example, look at Fig 3 where we obtain the real scheme \( 1\sqcup <1> \) in \( S^2 \).

\[
\begin{align*}
&\mathbb{R}\Sigma_2 & \longrightarrow & D_1 \quad D_2 \\
&\mathbb{R}E_2 & \longrightarrow & S^2 \\
\end{align*}
\]

Figure 2: From a torus to a 2-sphere

\[
\begin{align*}
&\mathbb{R}\Sigma_2 & \longrightarrow & S^2 \\
\end{align*}
\]

Figure 3: Example: from an arrangement of embedded circles in \( \mathbb{R}\Sigma_2 \) to an arrangement in \( S^2 \)

**Proposition 3.2.** Let \( C \) be a real algebraic non-singular curve of bidegree \((d, 0)\) in \( \Sigma_2 \). Let \( B \) be the real scheme on the sphere \( S^2 \) obtained from the pair \((\mathbb{R}\Sigma_2, \mathbb{R}C)\) by the construction above. Then, \( B \) is realizable by a real algebraic curve of bidegree \((d, d)\) on the quadric ellipsoid in \( \mathbb{C}P^3 \).

\[
\begin{align*}
&\mathbb{R}E_2 & \longrightarrow & \mathbb{R}Q_0 \quad \mathbb{R}Q_\varepsilon \\
&\mathbb{R}\Sigma_2 & \longrightarrow & q \\
\end{align*}
\]

Figure 4:

**Proof.** Let \([X : Y : Z : W]\) be the homogeneous coordinates in the complex projective space. Let \( Q_0 \) be the quadratic cone with equation \( X^2 + Y^2 - Z^2 = 0 \) in \( \mathbb{C}P^3 \). Recall that we obtain \( \Sigma_2 \) blowing-up \( Q_0 \) at the point \( q = [0 : 0 : 0 : 1] \). The image of \( C \)
via the blow-up is a real algebraic curve $\tilde{C}$ of degree $2d$ in $\mathbb{CP}^3$. The curve $\tilde{C}$ is the intersection of a real algebraic non-singular surface $S_d$ of degree $d$, not passing through the node of $Q_0$, and $Q_0$. Observe that we can perturb $Q_0$ to the quadric ellipsoid $Q_\varepsilon$ of equation $X^2 + Y^2 - Z^2 = -\varepsilon W^2$, where $\varepsilon > 0$ (see Fig. 4). Since a real algebraic curve of bidegree $(d, d)$ on $Q_\varepsilon$ is the intersection of the quadric ellipsoid and a surface of degree $d$, the intersection of $S_d$ and $Q_\varepsilon$ is a real algebraic curve $A$ of bidegree $(d, d)$. Moreover, the pair $(\mathbb{R}Q_\varepsilon, RA)$ realizes $B$. 

3.3 Dessin d’enfants

Orevkov in [Ore03] has formulated the existence of real algebraic trigonal curves realizing a given trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ in terms of the existence of a real rational graph on $\mathbb{CP}^1$. Later on, Degtyarev, Itenberg and Zvonilov in [DIZ14] have given a general way to determine if such real algebraic curves are of type I or II.

In Section 4.1, we will exploit such construction techniques in order to construct real algebraic trigonal curves in rational geometrically ruled surfaces. Therefore, we present here some results of [Ore03] and [DIZ14].

Definition 3.3. Let $n$ be a fixed positive integer. We say that a graph $\Gamma$ is a real trigonal graph of degree $n$ if

- it is a finite oriented connected graph embedded in $\mathbb{CP}^1$, invariant under the standard complex conjugation of $\mathbb{CP}^1$;
- it is decorated with the additional following structure:
  - every edge of $\Gamma$ is colored solid, bold or dotted;
  - every vertex of $\Gamma$ is $\bullet$, $\circ$, $\times$ (said essential vertices) or monochrome

and satisfying the following conditions:

1. any vertex is incident to an even number of edges; moreover, any $\circ$-vertex (resp. $\bullet$-vertex) to a multiple of 4 (resp. 6) number of edges;
2. for each type of essential vertices, the total sum of edges incident to the vertices of a same type is $12n$;
3. the orientations of the edges of $\Gamma$ form an orientation of $\partial(\mathbb{CP}^1 \setminus \Gamma)$ which is compatible with an orientation of $\mathbb{CP}^1 \setminus \Gamma$ (see Fig. 4);
4. all edges incident to a monochrome vertex have the same color;
5. $\times$-vertices are incident to incoming solid edges and outgoing dotted edges;
6. $\circ$-vertices are incident to incoming dotted edges and outgoing bold edges;
7. $\bullet$-vertices are incident to incoming bold edges and outgoing solid edges.

Definition 3.4. Let $\eta$ be a trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$. Let us consider the restriction of the projection $\pi_n$ (see Section 3.1) to $\mathbb{R}\Sigma_n$. Thanks to $\pi_{n|\Sigma_n}$ we can encode $\eta$ into a colored oriented graph $\Gamma$ on $\mathbb{RP}^1 \subset \mathbb{CP}^1$ in the following way (in Fig. 5 the dashed lines denote fibers of $\pi_{n|\Sigma_n}$):

1. To each fiber of $\pi_{n|\Sigma_n}$ intersecting $\eta$ in two points we associate a $\times$-vertex on $\mathbb{RP}^1$.
2. Let $F_1, F_2$ be two fibers of $\pi_{n|\Sigma_n}$ intersecting $\eta$ in two points such that $\eta$, up to $\mathcal{L}$-isotopy, is locally as depicted in Fig. 5 (b) or (c). Let $F_3$ be another fiber between $F_1, F_2$. Then, we associate to $F_3$ a $\circ$-vertex on $\mathbb{RP}^1$. Moreover, if between $F_1$ and $F_2$ each other fiber intersects $\eta$ in only one real point (as in Fig. 5 (c)), then we associate to a fiber between $F_1$ and $F_3$ (resp. $F_3$ and $F_2$) a $\bullet$-vertex on $\mathbb{RP}^1$. Between $\bullet$ and $\circ$-vertices we put bold edges.
3. Except for the fibers of $\pi_{\nu|_{\Sigma_{\nu}}}$ to which we associate essential vertices and bold edges, to a fiber which intersects $\eta$ in three distinct real points (resp. only one real point) we associate dotted (resp. solid) edges on $\mathbb{R}P^1$.

4. The orientations of the edges incident to a vertex are in an alternating order. In particular, the orientations of the edges incident to an essential vertex are respectively as described in 5., 6., 7. of Definition 3.3.

The graph $\Gamma$ associated to $\eta$ is called real graph.

We say that $\overline{\Gamma}$ is completable in degree $n$ if there exists a complete real trigonal graph $\Gamma'$ of degree $n$ such that $\Gamma' \cap \mathbb{R}P^1 = \overline{\Gamma}$.

![Figure 5: Local topology of trigonal $\mathcal{L}$-schemes and their corresponding real graphs.](image)

Theorem 3.5 ([Ore03]). A trigonal $\mathcal{L}$-scheme on $\Sigma_{\nu}$ is realizable by a real algebraic trigonal curve if and only if its real graph is completable in degree $n$.

Given a real graph $\overline{\Gamma}$, we depict only the completion to a real trigonal graph $\Gamma$ on a hemisphere of $\mathbb{C}P^1$ since $\Gamma'$ is symmetric with respect to the standard complex conjugation. Moreover, we can omit orientations in figures representing real trigonal graphs because each vertex is adjacent to an even number of edges oriented in an alternating order as, for example, depicted in Fig. 6 and such orientations are compatible with each others.

Theorem 3.5 is improved in [DIZ14] in order to check if a given trigonal $\mathcal{L}$-scheme is realizable by a real trigonal curve of type I. We say that a real algebraic singular curve is of type I (resp. of type II) if its normalization is of type I (resp. of type II).

![Figure 6: Colored vertices of a real trigonal graph.](image)

Definition 3.6. Let $\Gamma$ be a real trigonal graph of degree $n$. We say that $\Gamma$ is of type I if we can label each connected component of $\mathbb{C}P^1 \setminus \Gamma$, with the numbers 1, 2 or 3, such that:

- neighboring connected components of a $\bullet$-vertex, or a $\circ$-vertex of $\Gamma$, are labeled as depicted in one of the pictures in Fig. 7(a);
- neighboring connected components of a $\times$-vertex, which does not belong to $\Gamma \cap \mathbb{R}P^1$, are labeled as depicted in Fig. 7(b);
- neighboring connected components of $\times$-vertices, belonging to $\Gamma \cap \mathbb{R}P^1$, are labeled as depicted in in Fig. 7(c).
Otherwise, we say that $\Gamma$ is of type II.

The original statement in [DIZ14] of the following theorem treats only the case of non-singular real trigonal curve, but it is possible to extend it to real nodal trigonal curves.

**Theorem 3.7** ([DIZ14]). A non-hyperbolic trigonal $L$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve of type I (resp. of type II) if and only if its real graph has a completion in degree $n$ which is of type I (resp. type II).

**Remark 3.8.** For each non-hyperbolic trigonal $L$-scheme on $\mathbb{R}\Sigma_n$ realizable by an irreducible real trigonal curve of type I, each completion in degree $n$ of its real graph has a unique type I labeling (see [DIZ14]). So, later on, each time we have to assign a labeling to a real trigonal graph of type I, we could label only one component.

**Remark 3.9.** If a hyperbolic trigonal $L$-scheme in $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve $C$ in $\Sigma_n$, then the curve $C$ is of type I because the projection $\pi_n : \Sigma_n \to \mathbb{C}P^1$ (see Section 3.1) gives a totally real morphism on $\mathbb{C}P^1$.

### 3.3.1 Gluing real trigonal graphs

We call cubic trigonal graph of type I (resp. type II) a real trigonal graph of degree 1 and type I (resp. type II). The graph in Fig. 8 (a) is a cubic trigonal graph of type I, it has a unique type I labeling and associated trigonal $L$-scheme on $\mathbb{R}\Sigma_1$ as depicted in Fig. 8 (d). While the graphs depicted in Fig. 8 (b), (c) are of type II and have associated trigonal $L$-schemes on $\mathbb{R}\Sigma_1$ as depicted respectively in Fig. 8 (e), (f).
Let $\Gamma_1$ (resp. $\Gamma_2$) be a real trigonal graph. Denote by $D_1$ (resp. $D_2$) the disk on which one of the two symmetric halves of $\Gamma_1$ (resp. $\Gamma_2$) lies. Consider the disjoint union $\Gamma_1 \sqcup \Gamma_2 \subset D_1 \sqcup D_2$. Let $I_i \subset D_i, i = 1, 2$, be a segment in $\mathbb{R}P^1$ whose endpoints are not vertices of $\Gamma_i$ and such that $I_i$ contains a single $\circ$-vertex or a monochrome dashed vertex $\circ$. Let $\phi : I_1 \to I_2$ be an isomorphism preserving orientation, i.e. $\Gamma_1 \cap I_1 \to \Gamma_2 \cap I_2$ is an isomorphism preserving the types of vertices and edges, and preserving orientation. Consider the quotient $D_1 \sqcup \phi D_2 = D_1 \sqcup D_2 / (x \sim \phi(x))$ and $\Gamma_\phi \subset D_1 \sqcup \phi D_2$ the quotient of the image of $\Gamma_1 \sqcup \Gamma_2$. We call such operation gluing. The gluing of real trigonal graphs is still a real trigonal graph (see [DIK08, Section 5.6] for details).

Gluing type I real trigonal graphs, which are glued to each other along vertices whose neighboring connected components have the same labels, we get a type I real trigonal graph. As example, look at the gluing of two cubic trigonal graphs of type I in Fig. 9a (resp. b)) and 10a (resp. c)). The obtained graphs are real trigonal graphs of degree 2 and type I. The respective associated trigonal L-schemes are depicted in Fig. 10b, d).

4 Construction

4.1 Trigonal construction

In this section we give some intermediate constructions of real algebraic curves that we will need later on.
Figure 11: The union of a trigonal $\mathcal{L}$-scheme and two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$: $\eta_{a,b,c,c'}$.

**Proposition 4.1.** Let $\eta_{a,b,c,c'}$ be, up to isotopy of $\mathbb{R}\Sigma_5$, the union of a trigonal $\mathcal{L}$-scheme with two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$ as depicted in Fig. 11, where letters $a, b, c, c'$ denote numbers of ovals. Let $h, j, t$ be non-negative integer numbers. Then, there exist real algebraic trigonal curves in $\Sigma_5$ realizing the real schemes $\eta_{a,b,c,c'}$ for all $a, b, c, c'$ such that $0 \leq c + c' \leq h, 0 \leq a \leq j, 0 \leq b \leq t$, where $h, j, t$ are the following:

1. $j + h + t = 12$ with
   - $h = 1$ and $j \in \{0, 1, 4, 7, 10, 11\}$,
   - $h = 5$ and $j \in \{0, 1, 2, 3, 4, 5, 6, 7\}$,
   - $h = 9$ and $j \in \{0, 1, 2, 3\}$.

2. $j + h + t = 10$ with
   - $h = 0$ and $j \in \{4, 6, 8\}$,
   - $h = 2$ and $j \in \{0, 1, 2, 4, 5, 6, 8\}$,
   - $h = 4$ and $j \in \{0, 1, 2, 4, 5, 6\}$,
   - $h = 6$ and $j \in \{0, 1, 2, 4\}$,
   - $h = 8$ and $j \in \{0, 1, 2\}$.

3. $j + h + t = 8$ with
   - $h = 1$ and $j = 3$,
   - $h = 3$ and $j \in \{1, 2, 5\}$.

In particular, such real trigonal curves are of type I for $c + c' = h, a = j, b = t$. Moreover, there exist real trigonal curves of type I in $\Sigma_5$ realizing $\eta_{a,b,c,c'}$ for $(a, b, c, c') = (1, 5, 0, 0)$ and $(3, 3, 0, 0)$.

Figure 12: Real trigonal graph of degree 5 and type I.
Proof. Thanks to Theorems 3.5, 3.7 if the real graphs associated to \( \eta_{a,b,c,c'} \) are completable in degree 5 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing \( \eta_{a,b,c,c'} \).

Opportunely gluing 5 cubic trigonal graphs (see Section 3.3.1) we obtain type I (resp. type II) real trigonal graphs of degree 5, which complete the real graph associated to \( \eta_{a,b,c,c'} \) where \( a, b, c, c' \) are such that \( c + c' = h, a = j, b = t \) (resp. \( 0 \leq c + c' < h, 0 \leq a < j, 0 \leq b < t \)), for \( h, j, t \) as listed in 1. - 3. . Finally, a type I completion of the real graph associated to \( \eta_{a,b,c,c'} \) for \((a, b, c, c') = (3, 3, 0, 0) \) (resp. \((a, b, c, c') = (1, 5, 0, 0)\)), is pictured in Fig. 12(a) (resp. b) (see Remark 3.8). □

![Figure 13: a) The union of a trigonal L-scheme and a fiber of L on \( \mathbb{R} \Sigma_6 \). b) A nodal trigonal L-scheme on \( \mathbb{R} \Sigma_6 \). \( \tilde{\eta}_{a,b,c,c'} \).](image1)

![Figure 14: a) Real trigonal graph of degree 1 and type I: \( \xi \). b) Local type I labeling.](image2)

**Proposition 4.2.** Let \( \tilde{\eta}_{a,b,c,c'} \) be, up to isotopy of \( \mathbb{R} \Sigma_6 \), a trigonal L-scheme on \( \mathbb{R} \Sigma_6 \) as depicted in Fig. 13(b), where letters \( a, b, c, c' \) denote numbers of ovals. Then, there exist real algebraic trigonal curves in \( \Sigma_6 \) realizing the real schemes \( \tilde{\eta}_{a,b,c,c'} \) for all \( a, b, c, c' \) as listed in Proposition 4.1.

**Proof.** Thanks to Theorems 3.5, 3.7 if the real graphs associated to \( \tilde{\eta}_{a,b,c,c'} \) are completable in degree 6 to a real trigonal graph of type I (resp. type II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing \( \tilde{\eta}_{a,b,c,c'} \).

For \( a, b, c, c' \) as listed in Proposition 4.1 the existence of real trigonal graphs of degree 6 and type I (resp. type II) completing the real graph associated to \( \tilde{\eta}_{a,b,c,c'} \), is equivalent to the existence of those of type I (resp. type II) associated to the L-scheme depicted in Fig. 13(a) (see Ore[13]).

Let \( \xi \) be the cubic trigonal graph of type I pictured in Fig. 14(a). Take any real trigonal graph \( \Gamma \) of degree 5 constructed in the proof of Proposition 4.1 realizing a trigonal L-scheme \( \eta_{a,b,c,c'} \). In a neighborhood of \( \Gamma \cap \mathbb{R}P^1 \), let us denote by \( \delta \) the sub-graph of \( \Gamma \) which is as depicted in Fig. 14(b) and whose associated L-scheme is the part of \( \eta_{a,b,c,c'} \) for which passes one fixed fiber of \( L \) (see Fig. 11). Glue \( \Gamma \) along the \( \alpha \)-vertex of \( \delta \) to the \( \alpha \)-vertex, with same labeling if \( \Gamma \) is of type I, of the cubic trigonal graph \( \xi \). The gluing is a real trigonal graph of degree 6 which completes the real graph associated to the union of a trigonal L-scheme with one fiber of \( L \) as depicted in Fig. 13(a) on \( \mathbb{R} \Sigma_6 \) for all \( a, b, c, c' \) as in Proposition 4.1. Besides, the gluing is of type I (resp. type II) for all \( a, b, c, c' \) for which \( \Gamma \) is of type I (resp. type II). □

**Proposition 4.3.** Let \( \eta_{i,d,e,h,g} \) be, up to isotopy of \( \mathbb{R} \Sigma_6 \), the trigonal L-scheme on \( \mathbb{R} \Sigma_6 \) depicted in Fig. 13(b), where letters \( i, d, e, h, g_j \), for \( j = 1, 2, 3, 4 \), denote numbers.
of ovals. Let \( g \) be \( \sum_{j=1}^{4} g_j \) and let \( s, k \) be non-negative integer numbers. Then, there exist real algebraic trigonal curves in \( \Sigma_6 \) realizing the real schemes \( \eta_{i,d,e,h,g} \) for all\( i, d, e, h, g \) such that \( 0 \leq g \leq s, 0 \leq i + d + e + h \leq k \), where \( s, k \) are the following:

1. \( s + k = 12 \) with \( s \in \{6, 10\} \),
2. \( s + k = 10 \) with \( s \in \{5, 9\} \),
3. \( s + k = 8 \) with \( s \in \{0, 4, 6, 8\} \).

In particular, such real algebraic trigonal curves are of type I for \( g = s, i + d + e + h = k \). Moreover, there exist real algebraic trigonal curves of type I in \( \Sigma_6 \) realizing \( \eta_{i,d,e,h,g} \) for

4. \( i + d + e + h + g = 8 \) with \( g = 0 \);
5. \( i + d + e + h + g = 6 \) with \( g \in \{1, 3, 5\} \),
6. \( i + d + e + h + g = 4 \) with \( g \in \{2, 4\} \).

**Proof.** Thanks to Theorems 3.5 and 3.7 if the real graphs associated to \( \eta_{i,d,e,h,g} \) are completable in degree 6 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing \( \eta_{i,d,e,h,g} \). Let \( \tilde{\eta}_{i,d,e,h,g} \) be, up to isotopy of \( \mathbb{R} \Sigma_6 \), the union of a trigonal \( \mathcal{L} \)-scheme with four fibers of \( \mathcal{L} \) on \( \mathbb{R} \Sigma_6 \) as depicted in Fig. 15(a). Remark that, for \( i, d, e, h, g \) as listed in 1.−6., the existence of real trigonal graphs of degree 6 and type I (resp. type II) completing the real graph associated to \( \eta_{i,d,e,h,g} \) is equivalent to the existence of those of type I (resp. type II) associated to \( \tilde{\eta}_{i,d,e,h,g} \) (see [Ore07]).

Opportunely gluing 6 cubic trigonal graphs, we obtain real trigonal graphs of degree 6, which complete the real graph associated to \( \tilde{\eta}_{i,d,e,h,g} \) where \( i, d, e, h, g \) are such that \( 0 \leq g \leq s, 0 \leq i + d + e + h \leq k \), for \( s, k \) as listed in 1.−3. Type I completions of the real graph associated to \( \tilde{\eta}_{i,d,e,h,g} \), for values listed in 4.−6., are pictured in Fig. 16.

**Proposition 4.4.** There exist real algebraic trigonal curves of type I in \( \Sigma_6 \) realizing the trigonal \( \mathcal{L} \)-schemes respectively depicted in Fig. 17(a) and b). Moreover, there exists a real algebraic trigonal curve in \( \Sigma_6 \) realizing the hyperbolic (see Remark 3.9) \( \mathcal{L} \)-scheme depicted in Fig. 17(c).

**Proof.** Thanks to Theorems 3.5 and 3.7 if the real graphs associated to the real schemes in the statement are completable in degree 6 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing them.

Respective completions in degree 6 of the real graphs associated to the \( \mathcal{L} \)-schemes in Fig. 17(a), b) and c) are pictured in Fig. 17(d), e) and f). Furthermore, the trigonal graphs depicted in Fig. 17(d), e) are of type I.
Figure 16: Real trigonal graph of degree 6 and type I.

Figure 17: Trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_6$ and the completion of their real graphs in degree 6.

Figure 18: $\mathcal{L}$-schemes $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ on $\mathbb{R}\Sigma_2$. 

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Proposition 4.5. Let $\eta_1$ (resp. $\eta_2, \eta_3, \eta_4$) be, up to isotopy of $\mathbb{R} \Sigma_2$, a trigonal $\mathcal{L}$-scheme on $\mathbb{R} \Sigma_2$ as depicted in Fig. 18 (resp. $\mathcal{L}$) (resp. $\mathcal{L}$, $\mathcal{L}$, $\mathcal{L}$), where $a, b, c, c'$ (resp. $i, d, e, h, g$, resp. $a', b'$) denote numbers of ovals. Then, such $\mathcal{L}$-scheme is realizable by a real algebraic non-singular curve $C_1$ (resp. $C_2, C_3, C_4$) of bidegree $(3, 4)$ on $\Sigma_2$ for $a, b, c, c'$ as listed in Proposition 4.1 (resp. $i, d, e, h, g$ as listed in Proposition 4.3, resp. $(a', b') = (3, 2)$ and $(9, 0)$).

Proof. Let us denote by $\tilde{C}_1$ (resp. $\tilde{C}_2, \tilde{C}_3$ and $\tilde{C}_4$) any real algebraic trigonal curves in $\Sigma_6$ constructed in Propositions 4.2 (resp. Proposition 4.3, Proposition 4.4). For each curve $\tilde{C}_j$, let us consider, as defined in Section 3.1, the birational transformation
algebraic non-singular curves of bidegree $\circ$ (resp. $\star$). All the real schemes in the following list are realizable by real Proposition 4.8.

Notation 4.7.

all real schemes listed in Theorem 1.1 but the real schemes $\sqcup < 4 > \sqcup < 10 >$, $\sqcup < 7 > \sqcup < 7 >$;

2. all real schemes listed in Theorem 1.2 but the real schemes $< 4 > \sqcup < 10 >$, $< 7 > \sqcup < 7 >$;

4.2 Final constructions and patchworking

In this Section we end the proof of Theorems 1.1, 1.2, 1.3, 1.4. We will need the Viro’s patchworking method in Section 4.2 to construct real algebraic curves of bidegree $(5, 0)$ on $\Sigma_2$ (see [Vir84], [Vir89], [Vir06]), in Fig. 23 we depict the charts of real algebraic non-singular curves of bidgree $(3, 4)$ in $\Sigma_2$ constructed in Proposition 4.5. Moreover, performing a coordinates transformation to a curve $C_4$ with chart as depicted in Fig. 23, we obtain a type I real algebraic curve $C_5$ of bidegree $(3, 4)$ in $\Sigma_2$ with chart (resp. real $L$-scheme) as depicted in Fig. 23 (resp. Fig. 18 e), where $a', b'$ still denote numbers of ovals and $(a', b') = (3, 2)$ or $(9, 0)$.

4.2 Final constructions and patchworking

In this Section we end the proof of Theorems 1.1, 1.2, 1.3, 1.4. We will need the Viro’s patchworking method in Section 4.2 to construct real algebraic curves of bidegree $(5, 0)$ on $\Sigma_2$ (see [Vir84], [Vir89], [Vir06]), in Fig. 23 a), b), c), d) we depict the charts of real algebraic non-singular curves of bidgree $(3, 4)$ in $\Sigma_2$ constructed in Proposition 4.5. Moreover, performing a coordinates transformation to a curve $C_4$ with chart as depicted in Fig. 23 d), we obtain a type I real algebraic curve $C_5$ of bidegree $(3, 4)$ in $\Sigma_2$ with chart (resp. real $L$-scheme) as depicted in Fig. 23 (resp. Fig. 18 e), where $a', b'$ still denote numbers of ovals and $(a', b') = (3, 2)$ or $(9, 0)$.

Remark 4.6. In the proofs of Propositions 4.8, 4.9, 4.10 we construct by patchworking some real algebraic non-singular curves of bidegree $(5, 0)$ on $\Sigma_2$. This, because of the perturbation explained in Proposition 3.2 immediately implies the existence of real algebraic non-singular curve of bidegree $(5, 5)$ on the quadric ellipsoid (see Section 1).

Notation 4.7. In Propositions 4.8, 4.9, 4.10 the real schemes marked with the symbol $^*$ (resp. $^\ast$) are realized by a real algebraic curve of type I (resp. type II).

Proposition 4.8. All the real schemes in the following list are realizable by real algebraic non-singular curves of bidegree $(5, 5)$ on the quadric ellipsoid:

1. all real schemes listed in Theorem 1.1 but the real schemes $\sqcup < 4 > \sqcup < 10 >$, $\sqcup < 7 > \sqcup < 7 >$;

2. all real schemes listed in Theorem 1.2 but the real schemes $< 4 > \sqcup < 10 >$, $< 7 > \sqcup < 7 >$;
Proposition 4.10.

The real schemes listed in Theorem 1.3 but the real scheme \(4 > 1 < 9 >\);

4 all real schemes listed in Theorem 1.4 but the real schemes \(13^\circ, 1 < 1 < 4 < 9 >, 1 < 3 > 4 < 7 >, < 1 > 4 < 7 >, 1, 0\).

![Diagrams of D1 and D2](image)

Figure 24: Charts and arrangements with respect of the coordinate axes \(\{y = 0\}\) of real algebraic curves of bidegree (2, 0) on \(\Sigma_2\).

Proof. Let us consider charts of real algebraic curves on \(\Sigma_2\) with \(\{y = 0\}\) as coordinates axis. Due to the small perturbations method (see for example Propositions [Vir, 1.5.A, 7.2.C]), for any fixed real four points on \(\{y = 0\}\) there exist real algebraic curves \(D_1, D_2\) of bidegree (2, 0) on \(\Sigma_2\) whose charts are respectively as depicted in Fig. 24(a), (b) and which intersect the coordinates axis \(\{y = 0\}\) in the fixed four points. Thanks to Viro's patchworking method and Remark 4.6 we realize the real schemes listed in 1. – 4. glueing the polynomials and the charts of a real algebraic curve \(C_i, i = 1, 2, 3, 4, 5\), constructed in Proposition 4.5 and at the end of Section 4.1 and of a real algebraic curve \(D_j, j = 1, 2\) (in Fig. 25 it is depicted the patchworking of charts of the \(C_i\) of type I with \(D_1\)).

Proposition 4.9. The real schemes \(1 < 4 > 1 < 10 >, < 4 > 1 < 10 >, 1 < 4 < 9 >\), \(13^\circ, 1 < 1 < 9 >\), \(1 < 3 > 7 >\), \(1 < 4 < 7 >\) are realizable by real algebraic non-singular curves of bidegree (5, 5) on the quadric ellipsoid.

Proof. Let us consider charts of real algebraic curves on \(\Sigma_3\) with \(\{y = 0\}\) as coordinates axis. For any fixed real ten points on \(\{y = 0\}\) there exist a real algebraic curve \(\tilde{C}\) of bidegree (2, 0) on \(\Sigma_3\) whose charts are as depicted in Fig. 26 and which intersects the coordinates axis \(\{y = 0\}\) in the fixed ten points. For any fixed connected component \(\mathcal{O}\) of \(\mathbb{R}\tilde{C}\) we can pick four real points, \(p_1, p_2, p_3, p_4\) on it as depicted in Fig. 26(a) (resp. b)). Then, consider the birational transformation, as defined in Section 3.1 \(\beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_3, \tilde{C}) \to (\Sigma_1, C')\) where we call \(p_i\) also the image of \(p_i\) via \(\beta_{p_i}^{-1}\beta_{p_j}^{-1}\), \(i, j = 1, 2, 3\). Choose the coordinates axes in \(\mathbb{R}\Sigma_1\) such that \(\mathbb{R}C'\) has an arrangement as depicted in Fig. 27(d), (resp. c), where \(t, s\) denote numbers of ovals and \(t + s = 3\) (the coordinates axes are pictured in red). The charts \(C'\) are as depicted in Fig. 28(c) (resp. b)).

In [OS16], Shustin and Orevkov have constructed some real algebraic curves of bidegree (4, 0) and type I (resp. type II) in \(\Sigma_2\) whose charts, up to a coordinates change, and arrangements with respect to the coordinates axis \(\{x = 0\}\) are as depicted in Fig. 28(e) for \((\alpha, \beta, \gamma) = (5, 0, 2), (8, 1, 0)\) and \((4, 1, 0)\) (resp. \((7, 0, 1)\)), where \(\alpha, \beta, \gamma\) denote numbers of ovals. Thanks to Viro's patchworking method and Remark 4.6 we realize the real schemes \(< 4 > 1 < 9 >\), \(13^\circ, 1 < 4 < 1 < 10 >, 1 < 1 > 9 >\), \(1 < 3 > 7 >\) and \(< 4 > 1 < 10 >\) glueing the polynomials and the charts of these latter algebraic curves and of the curves \(C'\) (the patchworking of their charts is depicted in Fig. 29).

Proposition 4.10. The real schemes \(1 < 4 > 1 < 7 >, < 7 > 1 < 7 >\), \(1 < 4 > 1 < 7 >, 1, 0\) are realizable by real algebraic non-singular curves of bidegree (5, 5) on the quadric ellipsoid \(Q\).
Proof. First of all, in order to realize the real scheme $0$ (resp. $1$), we just smooth the union of five hyperplane sections in $\mathbb{CP}^3$ not intersecting $RQ$ (resp. whose just one intersects $RQ$). For the realization of the real schemes $\langle 1 \rangle \cup \langle 4 \rangle \circ \langle 1 \rangle$ (resp. $\langle 1 \rangle \cup \langle 7 \rangle \circ \langle 1 \rangle \cup \langle 7 \rangle$) we will give some intermediate constructions, then we will apply Viro’s (resp. Shustin's) patchworking method.

For any fixed five distinct real points on the infinity line of the projective plane, we can construct real algebraic plane quintics passing through such fixed points, obtained as evolving of the singularity $N_{16}^2$ (see Proposition [Vir, 7.3.D]). In particular, there exist real algebraic non-singular quintics $R_1, R_2, R_3$ whose charts are respectively

Figure 25: Charts of curves of bidegree $(5,0)$ in $\Sigma_2$ obtained patchworking the charts of the real algebraic curves $C_i$ from Proposition 4.3 and $D_1$.
as depicted in Fig. 30 (a), (b), (c) for \( n = 4, (p, q) = (6, 0), (0, 6) \), where \( n, p, q \) denote numbers of ovals. Moreover, let

\[ P_h(x, y, z) = \sum_{i+j \leq 5} a_{i,j} x^i y^j z^{5-i-j} \]

be the polynomial of a quintic \( R_h, h = 2, 3 \), passing through five fixed points on the infinity line. Then, applying to the polynomials \( P_h \) the transformation \( T: P_h(x, y, z) \mapsto P_h(xz, yz, x^2) \) we construct real algebraic plane curves \( \tilde{R}_h \) whose polynomials are

\[ \tilde{P}_h(x, y, z) = \sum_{i+j \leq 5} a_{i,j} x^{10-i-j} y^j z^{i+j} \]
and whose charts are as depicted in Fig. 30 (d), (e) respectively for \( n = 4 \), \( (p, q) = (0, 6), (6, 0) \), where \( n, p, q \) still denote numbers of ovals. Later on, we will use Viro’s patchworking theorem, so remark that \( P_h \) and \( \tilde{P}_h \) have monomials with same coefficient \( a_{i,j} \) for \( i + j = 5 \).

In order to realize the real scheme \(< 1 > \sqcup < 4 >^0 \) (resp. \( 1 \sqcup < 7 > \sqcup < 7 > \)), we apply Viro’s patchworking method gluing the polynomials and the charts of the real plane quintic \( R_1 \) (resp. \( R_2 \) with \( (p, q) = (0, 6) \)) with the real algebraic curve \( \tilde{R}_3 \) (resp. \( \tilde{R}_2 \) with \( (p, q) = (6, 0) \)).

Finally, the construction of a real algebraic curve of bidegree \((5, 5)\) on the quadric ellipsoid realizing the real scheme \(< 7 > \sqcup < 7 > \) requires a variant of the patchworking theorem presented in \cite{Shu02}, \cite{Shu06} and \cite{Shu98} by Shustin. For any four fixed real points on the infinity line in the projective plane, there exists a real algebraic plane quintic passing through three such points and tangent to the infinity line at the remaining fixed point and whose chart is as depicted in Fig. 31 (a), where \( (p, q) = (6, 0) \) and \( (0, 6) \) denote numbers of ovals. Moreover, there exists a real algebraic plane curve with chart as depicted in Fig. 31 (b), constructed from the quintic via the transformation \( T \) above. Gluing the charts of those two real algebraic plane curves respectively with \( (p, q) = (6, 0) \) and \( (p, q) = (0, 6) \) (see Fig. 31 (c)) we do not obtain a chart of a polynomial. But, the variant of the patchworking method developed by Shustin (see \cite{Shu06}) allows us to replace a neighborhood of the two tangency points with a deformation pattern (see \cite{Shu06}, Sections 3.5 and 5 and Fig. 32 (b)) whose chart is as depicted in Fig. 32 (a), in order to obtain a chart, as depicted in Fig. 31 (d), of a real algebraic curve of bidegree \((5, 0)\) in \( \Sigma_2 \) (see Remark 4.6).

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a) Chart of real algebraic plane quintic.  

b) Chart of real algebraic plane curve.

c)  

d) Chart of real algebraic non-singular curve of bidegree (5,0) on $\Sigma_2$.

Figure 31:

a) Deformation pattern.  

b) Chart of real algebraic non-singular curve of bidegree (5,0) on $\Sigma_2$.

Figure 32:

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