On the sectional category of certain maps

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Abstract

We give a simple characterisation of the sectional category of rational maps admitting a homotopy retraction which generalises the Félix-Halperin theorem for rational LS category. As a particular case, we prove a conjecture of Jessup-Murillo-Parent concerning rational topological complexity and generalise it to Rudyak’s higher topological complexity. We also give a characterisation of Doeraene-El Haouari’s relative category of such maps and thus of Iwase-Sakai’s monoidal topological complexity for rational spaces.

Introduction

Sectional category is an invariant of the homotopy type of maps introduced by Schwarz in [29]. If \( f : X \to Y \) is a continuous map, its sectional category is the smallest \( m \) for which there are \( m + 1 \) local homotopy sections for \( f \) whose sources form an open cover of \( Y \). Its most studied particular case is the well known Lusternik-Schnirelmann category of a space \( X \) introduced in [26] as a lower bound for the number of critical points on any smooth map defined on \( X \). Namely, the Lusternik-Schnirelmann category of a pointed space \( X \), cat\((X)\), is the sectional category of the base point inclusion map, \(* \leftrightarrow X\).

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Throughout this work we will consider all spaces to be simply connected of finite type and use standard rational homotopy techniques which are explained in the excellent text, [14], by Y. Félix, S. Halperin and J.-C. Thomas.

Lusternik-Schnirelmann category can be characterized in a more categorical way through the Whitehead or Ganea characterizations. The advantage of these approaches is that they can be used to obtain models of LS category in other categories through functors. A remarkable result on this direction is the Félix-Halperin characterisation of LS category of certain spaces in terms of their Sullivan models. Explicitly, if $X$ is a simply connected space of finite type modelled by $(\Lambda V, d)$ and $X_0$ its rationalisation (see [14], [31]) then $\text{cat}(X_0)$ is the smallest $m$ for which the cdga projection

$$(\Lambda V, d) \rightarrow \left( \frac{\Lambda V}{\Lambda^{>m} V}, d \right)$$

admits a homotopy retraction.

In his famous paper, [10], M. Farber introduced the concept of topological complexity of a space $X$, $\text{TC}(X)$, which can be seen as the sectional category of the diagonal map $\Delta: X \rightarrow X \times X$. This invariant is used to estimate the motion planning complexity of a mechanical system and has also applications to other fields of mathematics, see [11], for instance. As a direct generalisation of this invariant, Rudyak introduced in [28] the concept of higher topological $n$-complexity of a space, $\text{TC}_n(X)$, as the sectional category of the $n$-diagonal map $\Delta_n: X \rightarrow X^n$. Several explicit computations of topological complexity of rational spaces have been done in [25], [24], [18] and [2].

Inspired on the above characterisation of Félix and Halperin, Jessup, Murillo and Parent, in [24], conjectured that $\text{TC}(X_0)$ is the smallest $m$ such that the projection

$$(\Lambda V \otimes \Lambda V, d) \rightarrow \left( \frac{\Lambda V \otimes \Lambda V}{K^{m+1}}, d \right)$$

admits a homotopy retraction, where $K$ denotes the kernel of the multiplication morphism $\mu_2: \Lambda V \otimes \Lambda V \rightarrow \Lambda V$. 

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In [21] and [22], Iwase and Sakai introduce the monoidal topological complexity of a space $X$, $TC^M(X)$, as a way to measure the smart motion planning complexity of a mechanical system. They conjecture that $TC(X) = TC^M(X)$ and prove that they differ by, at most, one. Later on, in [8], Doeraene and El Haouari introduced the concept of relative category of a map $f$, $relcat(f)$. In [9] they conjecture that, if $f$ admits a homotopy retraction, then $secat(f) = relcat(f)$ and prove that they also differ at most by one. In [3] the authors show that the Iwase-Sakai conjecture is a particular case of the Doeraene-El Haouari conjecture and give several partial results supporting these conjectures although a general answer is yet to be given.

In [16], Ganea asks whether, $cat(X \times S^n) = cat(X) + cat(S^n)$. This is known as the Genea conjecture. The proof of this conjecture for rational spaces is a combination of two independent results. Firstly, that of B. Jessup, [23] saying that the conjecture holds for a weaker version of LS category called the module LS category: $mcat(X \times S^n) = mcat(X) + mcat(S^n)$. And secondly, Hess’ theorem, [19], that says that LS category of rational spaces equals module category: $mcat(X) = cat(X_0)$. Later on, N. Iwase gave an example in [20] of a space $X$ for which $cat(X \times S^n) < cat(X) + cat(S^n)$.

If $\varphi : A \to B$ is a surjective cdga morphism, denote $sc(\varphi)$ the smallest $m$ such that the projection

$$\rho_m : A \to \frac{A}{(\ker \varphi)^{m+1}}$$

admits a homotopy retraction. If $f$ is a continuous map, define $sc(f)$ as the smallest $sc(\varphi)$ with $\varphi$ a surjective cdga model for $f$. It is shown in [2] that $secat(f_0) \leq sc(f)$. The main result of this work reads

**Theorem 1.** If $f$ admits a homotopy retraction, then $secat(f_0) = sc(f)$.

This theorem reduces to the Félix-Halperin theorem for rational LS category in [13] and proves the Murillo-Jessup-Parent conjecture in [24]. It also proves the analogous result for Rudyak’s higher topological complexity:

**Theorem 2.** Let $(\Lambda V, d)$ be the Sullivan model of a space $X$, then $TC_n(X_0)$ is the smallest $m$ for which the projection

$$(\Lambda V, d) \otimes^n \to \left(\frac{(\Lambda V)^\otimes_n}{K^{m+1}}, d\right)$$
admits a homotopy retraction, being $K$ the kernel of the $n$-multiplication morphism $\mu_n: (\Lambda V, d)^\otimes n \rightarrow (\Lambda V, d)$.

This theorem combined with [24, Theorem 1.6] yields a proof of the Ganea conjecture for module topological complexity. Explicitly, if $X$ is a space, then $mTC(X \times S^n) = mTC(X) + mTC(S^n)$, this is an analogous result of Jessup’s theorem.

Our main result is also used to give an algebraic description for rational relative category, which should help clarify the Doeraene-El Haouari and the Iwase-Sakai conjectures in the rational homotopy theory context.

The ideas on this paper come from some parts of [14] based in [5]. In Section 1 we describe homotopy in the fibrewise pointed category $\text{cdga}(B)$ and deduce a lifting lemma. Section 2 defines relative nilpotency and proves a key lemma saying that relative nilpotency can be controlled when modelling homotopy cofibres in $\text{Top}(B)$. In Section 3 we give a model for the $m$-Ganea map with relative nilpotency $m$. The main result is proven in Section 4, while applications to higher topological complexity are exposed in Section 5 and to the Doeraene-El Haouari conjecture in Section 6. We finally end this paper with a small remark on a general Hess’ theorem.

1 Fibrewise pointed cdga’s

In this section we develop some technical tools that will be needed later on. Let $\mathcal{C}$ be a J-category in the sense of Doeraene ([6],[7]) and fix an object $B$ of $\mathcal{C}$. One can then consider the fibrewise pointed category over $B$, denoted $\mathcal{C}(B)$, ([11, Pg 85]) whose objects are factorisations of $\text{Id}_B$,

$$
\begin{array}{ccc}
B & \xrightarrow{s_X} & X & \xrightarrow{p_X} & X,
\end{array}
$$

1 Fibrewise pointed cdga’s

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and morphism are commutative diagrams

These morphism are said to be fibrations (\(\to\)), cofibrations (\(\hookrightarrow\)) or weak equivalences (\(\sim\)) if the underlying morphism \(f\) is such in \(\mathcal{C}\). These definitions make \(\mathcal{C}(B)\) inherit the structure of \(\mathcal{C}\). We would like to point out that the structure considered here is not the same as that considered in [17].

Now let \(B\) be a topological space. Recall that, in \(\text{Top}(B)\), two maps \(f, g\),

are fibrewise pointed homotopic when there exists another map

with \(I_B(X) := X \times I/\sim\), being \((s_X(b), t) \sim (s_X(b), 0)\), \(s(b) := (s_X(b), 0)\) and \(p(x, t) := p_X(x)\), verifying \(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\).

If \(s_X\) is a cofibration, \(I_B(X)\) is constructed as the following homotopy
Fix now a cdga $B$ and consider the model for the interval $\Lambda(t, dt)$ with $|t| = 0$. Analogously, if we have an object $C$ in the category cdga$(B)$, then the following pullback diagram gives a cylinder object for $C$,

Denote $K = \ker p_C$, then, as a vector space, $I_B(C) = B \oplus (K \otimes \Lambda(t, dt))$, with multiplication given by multiplication in $C$, $B \cdot K \subset K$. So if we want a morphism $\Phi: A \to C \otimes \Lambda(t, dt)$ to determine a morphism $A \to I_B(C)$, we must impose that $\text{Im } \Phi \subset C \oplus (\ker(p_C) \otimes \Lambda^+(t, dt))$.

This remark motivates the following construction:

**Definition 3.** Two cdga$(B)$ morphisms, $\varphi_0$, $\varphi_1$, 

\begin{align*}
B & \xrightarrow{s_B} X \times I \\
& \xrightarrow{\text{Id}} X \times I \\
& \xrightarrow{\text{Id}} B.
\end{align*}
are said to be fibrewise pointed homotopic if there exists a diagram

\[
\begin{array}{ccc}
B & \xrightarrow{s_B} & C \\
\downarrow{s_A} & & \downarrow{s_Y} \\
A & \xrightarrow{\Phi} & C \otimes \Lambda(t,dt),
\end{array}
\]

with \( \text{Im} \Phi \subset C \oplus (\ker(p_C) \otimes \Lambda^+(t,dt)) \) and \( \epsilon_i \circ \Phi = \varphi_i \), where \( \epsilon_i(t) := i \), \( i = 0,1 \).

Following the proof of [14, Prop. 12.7], one can verify that pointed fibrewise homotopy is an equivalence relation if \( s_A \) is a cofibration. In this case, denote \([A,C]_B\) the set of pointed fibrewise homotopy classes.

There is a family of objects in \( \text{cdga}(B) \) that will be very important, those are relative Sullivan models \( (B \otimes \Lambda V,D) \) with a projection \( p: (B \otimes \Lambda V,D) \to B \) that is the identity on \( B \). These turn out to be the fibrant-cofibrant objects of \( \text{cdga}(B) \).

The following lemma is a consequence of Quillen’s model category theory, [27]. We include its proof for the convenience of the reader.

**Lemma 4.** Suppose \( \theta: A \to C \) is a quasi-isomorphism in \( \text{cdga}(B) \) and \( (B \otimes \Lambda V,D) \) a fibrant-cofibrant object of \( \text{cdga}(B) \), then composition with \( \theta \) induces a bijection

\[
\theta_\#: [B \otimes \Lambda V,A]_B \to [B \otimes \Lambda V,C]_B.
\]

**Proof.** We will first prove the result in the case that \( \theta \) is surjective. Let \( \varphi: B \otimes \Lambda V \to C \) be a morphism of \( \text{cdga}(B) \). We then have the commutative diagram
By the relative lifting lemma for surjective quasi-isomorphisms, there is a morphism \( \varphi' : B \otimes \Lambda V \to A \) such that \( \theta \circ \varphi' = \varphi \) and \( \varphi' \circ s = s_A \). Then the morphism \( \varphi' \) is clearly a map in \( \text{cdga}(B) \), which proves the surjectivity of \( \theta_\# \).

Let us now prove the injectivity of \( \theta_\# \). Consider two morphisms \( \varphi_0, \varphi_1 : B \otimes \Lambda V \to A \) such that \([\theta \circ \varphi_0] = [\theta \circ \varphi_1]\) and choose \( \Phi : B \otimes \Lambda V \to C \otimes \Lambda(t, dt) \), a fibrewise pointed homotopy between \( \theta \circ \varphi_0 \) and \( \theta \circ \varphi_1 \). Consider the diagram

\[
\begin{array}{ccc}
A \otimes \Lambda(t, dt) & \xrightarrow{\sim} & \widetilde{T} \\
\|
\end{array}
\]

and construct the commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\mathbf{\Phi}} & A \otimes \Lambda(t, dt) \\
\|
\end{array}
\]

with \( \epsilon_i \circ \widetilde{\Phi} = \varphi_i \). Since \( p_A = p_C \circ \theta \), we have that \( \text{Im} \, \widetilde{\Phi} \subset A \oplus (\ker(p_A) \otimes \Lambda^+(t, dt)) \) and then \([\varphi_0] = [\varphi_1]\).

We now prove the general case. Factor \( \theta \) as

\[
\begin{array}{ccc}
A & \xrightarrow{\theta} & C \\
\|
\end{array}
\]

where \( A \otimes \Lambda(C, dC) \) can be regarded as an object of \( \text{cdga}(B) \) by taking \( s = i \circ s_A \) and \( p = p_C \circ \pi \). Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}_A} & A \\
\|
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes \Lambda(C, dC) & \xrightarrow{p_C \circ \pi} & B \\
\|
\end{array}
\]

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then there is a morphism $\lambda$ in $\text{cdga}(B)$ such that $\lambda \circ \iota = \text{Id}_A$. This tells us that $\iota_\#$ is a bijection, and since $\pi_\#$ is also bijective then $\theta_\#$ is a bijection. □

The usefulness of this lemma, for our purpose, relies on the fact that the morphism given by it will commute with the projection morphisms.

2 Relative nilpotency and homotopy pullbacks

Given an object $A$ of $\text{cdga}(B)$, we define its relative nilpotency as

$$\text{nil}_B A := \text{nil} \ker p_A.$$ 

Next lemma gives an explicit surjective replacement for a cdga morphism that will be needed afterwards.

**Lemma 5.** Any cdga morphism $\psi: B \rightarrow E$ admits a factorisation

$$\begin{array}{ccc}
B & \xrightarrow{\psi} & E \\
\alpha \simeq & \downarrow h & \\
B \oplus (E \otimes \Lambda^+ (t, d)) & \rightarrow & \\
\end{array}$$

where $\alpha$ is just the inclusion in the first summand, $h(b) = \psi(b)$, $h(e) = e$, $h(t) = 1$, $h(dt) = 0$ and $be = \psi(b)e$.

**Proof.** It follows from the fact that $E \otimes \Lambda^+ (t, dt)$ is an acyclic ideal of $B \oplus (E \otimes \Lambda^+ (t, dt))$. □

The following lemma is crucial, it tells us that we can control relative nilpotency while doing homotopy pullbacks.

**Lemma 6.** Suppose we have a diagram

$$\begin{array}{ccc}
B & \xrightarrow{s_C} & C \\
\downarrow \text{Id}_B & \xrightarrow{i} & \downarrow \text{Id}_B \\
B & \xrightarrow{\beta} & (C \otimes \Lambda V, D) \\
\downarrow \psi & \xrightarrow{\text{Id}_B} & \downarrow \text{Id}_B \\
B & \xrightarrow{\gamma} & B \\
\end{array}$$

where
with $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$ and $\psi(V) = 0$. If $\text{nil}_B C = m$ then there exists an object $D$ in $\text{cdga}(B)$ weakly equivalent to the homotopy pullback of $i$ and $\beta$ with $\text{nil}_B D = m + 1$.

Proof. By previous lemma, one can take a factorisation of $\beta$ as

\[
\begin{array}{c}
\xymatrix{
B \ar[r]^{\beta} & C \otimes \Lambda V \\
\downarrow_{\alpha} \ar@{=}[r] & \downarrow_{h} \\
S,
}
\end{array}
\]

where

\[ S := B \oplus (C \otimes V \otimes \Lambda^+ \{t, dt\}). \]

Observe that, in $S$,

\[ b \cdot (c \otimes v \otimes \xi) = (\beta(b)c) \otimes v \otimes \xi. \]

We then have the commutative diagram

\[
\begin{array}{c}
\xymatrix{
B \ar[rr]^{s_C} & & C \\
\downarrow_{\text{Id}_B} \ar@/_/[uur]^{s_M} & & \downarrow_{p_C} \\
M \ar[rr]^{\eta} & & C \\
S \ar[rr]^{h} \ar[uur]_{\approx} \ar[urr]_{\approx} & & (C \otimes \Lambda V, D) \\
\downarrow_{\text{Id}_B} & & \downarrow_{\beta} \\
B & & B
}
\end{array}
\]

which, shows $M$ as the homotopy pullback of $\beta$ and $i$. Moreover, taking $p_M := p_C \circ \eta$ exposes $M$ as an object of $\text{cdga}(B)$. To finish the proof, we will construct an object $D$ of $\text{cdga}(B)$ with $\text{nil}_B D = m + 1$ weakly equivalent to $M$.

Write $K_\epsilon$ the kernel of $\epsilon: \Lambda^+(t, dt) \to \mathbb{Q}$, $t \mapsto 1$, then there is a $\text{cdga}$ isomorphism having as source a sub $\text{cdga}$ of $S$,

\[ \eta: B \oplus (C \otimes \Lambda^+(t, dt)) \oplus (C \otimes \Lambda^+ V \otimes K_\epsilon) \to M, \]
where $\eta(b) = (b, \beta(b))$, $\eta(c \otimes \xi) = (c \otimes 1 \otimes \xi, c\epsilon(\xi))$ and $\eta(c \otimes v \otimes \omega) = (c \otimes v \otimes \omega, 0)$, for $b \in B$, $c \in C$, $\xi \in \Lambda^+(t, dt)$, $\omega \in K_e$, and $v \in V$. Then, through this isomorphism, the projection $p_M: M \to B$ is such that $p_M(b) = b$, $p_M(c \otimes \xi) = p_C(c)\epsilon(\xi)$ and $p_M(c \otimes v \otimes \xi) = 0$ and the section $s_M: B \to M$ verifies $s_M(b) = b$.

Since $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$,
\[ N := B \oplus ((\ker p_C) \otimes \Lambda^+(t, dt)) \oplus (C \otimes \Lambda^+ V \otimes K_e) \]
is a sub-$\text{cdga}$ of $M$. The subcomplexes $C \otimes \Lambda^+(t, dt)$ and $(\ker p_C) \otimes \Lambda^+(t, dt)$ are acyclic. Since the respective quotients of $M$ and $N$ by these subcomplexes are isomorphic, the inclusion $\iota: N \hookrightarrow M$ is a quasi-isomorphism. Observe that these sub-complexes are not ideals and that we can see $N$ as an object of $\text{cdga}(B)$ by setting $s_N(b) := b$ and $p_N = p_M \circ \iota$. Therefore $p_N$ vanishes on $(\ker p_C) \otimes \Lambda^+(t, dt) \oplus C \otimes \Lambda^+ V \otimes K_e$.

Now, write $J$ the acyclic ideal of $\Lambda(t, dt)$ generated by $t^2 - t$ and $d(t^2 - t)$ and $I := ((\ker p_C) \otimes J) \oplus (C \otimes \Lambda^+ V \otimes J)$. Clearly, $I$ is an acyclic ideal of $N$, hence the projection $\pi: N \to \frac{N}{I}$ is a quasi-isomorphism. Since $p_N$ vanishes on $I$, we have a commutative diagram
\[ \begin{array}{ccc}
B & \xrightarrow{s_D} & M \\
\downarrow{s_N} & & \downarrow{s_M} \\
N & \xrightarrow{\pi} & \frac{N}{I} \\
\downarrow{p_D} & & \downarrow{p_M} \\
B, & & \\
\end{array} \]
where $s_D := \pi \circ s_N$ and $p_D$ is induced by $p_N$, which exposes $D := \frac{N}{I}$ as an object in $\text{cdga}(B)$ weakly equivalent to $M$.

Let us now prove that $\text{nil}_B D = m + 1$. Observe that
\[ D = B \oplus (\ker p_C) \oplus ((\ker p_C) \otimes dt) \oplus (C \otimes \Lambda^+ V \otimes dt) \]
where an element $x \in \ker p_C$ corresponds to $xt$. Therefore its differential is given by $d(x) + (-1)^{|x|} x dt$. Also,
\[ \ker p_D = (\ker p_C) \oplus ((\ker p_C) \otimes dt) \oplus (C \otimes \Lambda^+ V \otimes dt), \]
and thus a maximal length product in ker $p_D$ is given by $z(1 \otimes v \otimes dt)$ where $z$ is a maximal length product in ker $p_C$. This proves that $\text{nil}_B D = m + 1$. 

### 3 Relative nilpotency and Ganea models

Let $f: X \to Y$ be a continuous map. Recall ([12], [1], [3]) that one can fit the $m$-Ganea morphism for $f$, $G_m(f)$, into a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
P^m(f) & \xrightarrow{G_m(f)} & Y,
\end{array}$$

and that $\text{secat}(f) \leq m$ if and only if $G_m(f)$ admits a homotopy section. Also, if $\varphi: A \to B$ is a surjective model for $f$, then Diagram 3 can be modelled by the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\kappa_m} & C_m \\
\varphi \downarrow && \downarrow p_m \\
B, & & 
\end{array}$$

where $\kappa_m$ models $G_m(f)$ and can be constructed inductively by taking the homotopy pullback of the induced maps by the homotopy pushout of $\varphi$ and any model, $g: A \to D$, of $G_{m-1}(f)$ as in

$$\begin{array}{ccc}
A & \xrightarrow{g} & D \\
\varphi \downarrow && \downarrow p_b \\
B & \xrightarrow{\cong} & A \otimes \Lambda V & \xrightarrow{\cong} & D \otimes \Lambda V,
\end{array}$$

where the large square is a pushout. By [2], $\text{secat}(f_0) \leq m$ if and only if $k_m$ admits a homotopy retraction.

We now give the key model for the $m$-Ganea map $G_m(f)$:
Proposition 7. Let \( f \) be a map and \( \varphi : A \rightarrow B \) be a cdga model for \( f \) admitting a section \( s : B \hookrightarrow A \) which is a cofibration (\( A \) is fibrant-cofibrant in cdga\((B)\)). Then there is a model \( \lambda_m \) for \( G_m(f) \) which is a morphism in cdga\((B)\),

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda_m} & C_m \\
\downarrow s & & \downarrow p_m \\
B & \xleftarrow{\varphi} & \end{array}
\]

with \( \text{nil}_B C_m = m \).

Proof. We will proceed by induction. For \( m = 0 \) this is clearly the case since \( p_0 = \text{Id}_B \). Suppose such diagram exists for the \((m - 1)\)-th Ganea map \( G_{m-1}(f) \).

Since \( \varphi \) is surjective, one can take a relative Sullivan model for \( \varphi \),

\[
\theta : (A \otimes \Lambda V, D) \xrightarrow{\sim} B,
\]

such that \( D(V) \subset \ker(\varphi) \oplus (A \otimes \Lambda^+ V) \) and \( \theta(V) = 0 \). Now take the homotopy pushout

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda_{m-1}} & C_{m-1} \\
\downarrow j & & \downarrow p_{m-1} \\
A \otimes \Lambda V & \xrightarrow{\lambda_{m-1} \otimes \text{Id}} & C_{m-1} \otimes \Lambda V \\
\downarrow \theta & & \downarrow \leftarrow \\
B & \xrightarrow{\text{Id}} & \end{array}
\]

and factor \( \lambda_{n-1} \otimes \text{Id} \) as \( q \circ w \) with \( q : E \rightarrow C_{m-1} \otimes \Lambda V \) a surjective cdga morphism and \( w : A \otimes \Lambda V \xrightarrow{\sim} E \) a weak equivalence. Then pullbacks
universal property

\[
\begin{array}{c}
A \\
\downarrow g \\
\downarrow \lambda_{m-1} \\
\downarrow T \\
\downarrow E \\
\downarrow C_{m-1} \\
\downarrow i \\
\downarrow C_{m-1} \otimes \Lambda V \\
\downarrow \theta \\
\downarrow B \\
\end{array}
\]

This gives a model for \(G_m(f)\) which can be seen as a morphism in \(\text{cdga}(B)\) by taking \(p_T = p_{m-1} \circ q \) and \(s_T = g \circ s\).

Now, define \(\beta : B \rightarrow C_{m-1} \otimes \Lambda V\) as \(\beta := i \circ s_{m-1}\) and consider the commutative diagram

\[
\begin{array}{c}
B \\
\downarrow j \circ s \simeq \\
\downarrow C_{m-1} \otimes \Lambda V \\
\downarrow A \otimes \Lambda V \\
\end{array}
\]

where \(j \circ s\) is a quasi-isomorphism, because \(\theta \circ (j \circ s) = \text{Id}_B\) and \(\theta\) is a quasi-isomorphism. We have then deduced a factorisation of \(\beta\) as quasi-isomorphism followed by a fibration \(\beta = q \circ (w \circ j \circ s)\). On the other hand, consider also the factorisation of \(\beta = h \circ \alpha\) as in the proof of Lemma 6. Diagram 4. Applying [7, Lemma 1.8] to previous factorisations and commutative square, in \(\text{cdga}(B)\),

\[
\begin{array}{c}
B \\
\downarrow \text{Id}_B \quad s_{m-1} \\
\downarrow C_{m-1} \\
\end{array}
\]

\[
\begin{array}{c}
B \\
\downarrow \beta \quad C_{m-1} \otimes \Lambda V \\
\end{array}
\]
gives a diagram commutative diagram

\[
\begin{array}{c}
\xymatrix{ & M \ar[dr]^{s_M} & \\
B \ar[ur]^{s_T} \ar[rr]_{r} & & T. }
\end{array}
\]

Now, applying Lemma 6 to the following diagram

\[
\begin{array}{c}
\xymatrix{ B \ar[r]^{s_{m-1}} & C_{m-1} \\
B \ar[u]^{\text{Id}_B} \ar[r]^\beta & C_{m-1} \otimes \Lambda V \ar[u]^{i} \\
B \ar[r]_{\text{Id}_B} \ar[u] & B, \ar[u]^{p_{m-1}} }
\end{array}
\]

we get an object $C_m$ of $\text{cdga}(B)$, with $\text{nil}_B C_m = m$, which is weakly equivalent $M$. Observe that we cannot use pullback’s universal property to get a model of $G_m(f)$ because, in general, $\beta \circ \varphi$ does not coincide with $i \circ \lambda_{m-1}$.

Considering previous diagrams we get diagram in $\text{cdga}(B)$:

\[
\begin{array}{c}
\xymatrix{ A \ar[d]_{g} & \\
T \ar@{=}[r] & M \ar@{=}[r] & C_m. }
\end{array}
\]

Since $A$ is a fibrant-cofibrant object of $\text{cdga}(B)$, we can apply Lemma 4 to get a model for $G_m(f)$ in $\text{cdga}(B)$, $\lambda_m: A \to C_m$, with $\text{nil}_B C_m = m$. 

4 Rational sectional category and the main result

Recall from [2] that if $\varphi: A \to B$ is a surjective cdga morphism then we can consider the projection

$$
\rho_m: A \to \frac{A}{(\ker \varphi)^{m+1}}
$$
and define:

- $\text{msc}(\varphi)$ the smallest $m$ such that $\rho_m$ admits a homotopy retraction as $A$-module,

- $\text{Hsc}(\varphi)$ the smallest $m$ such that $H(\rho_m)$ is injective.

- If $f$ is a map, $\text{msc}(f)$ ($\text{Hsc}(f)$) the smallest $\text{msc}(\varphi)$ ($\text{Hsc}(\varphi)$) with $\varphi$ surjective model for $f$.

We can now easily prove our main result:

**Theorem 8.** Let $\varphi: A \to B$ a cdga morphism with section $s: B \hookrightarrow A$ which is a cofibration. Then we have

(i) $\text{secat}(\varphi) = \text{sc}(\varphi)$,

(ii) $\text{msecat}(\varphi) = \text{msc}(\varphi)$,

(iii) $\text{Hsecat}(\varphi) = \text{Hsc}(\varphi)$.

**Proof.** Proposition 7 gives a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\kappa} & C \\
\downarrow{\varphi} & & \downarrow{p} \\
B, & & \\
\end{array}
\]

where $\kappa$ is a model for the $m$-th Ganea morphism of $\varphi$, $\text{nil ker } p = m$ and $K := \ker \varphi$. Then $\kappa(K^{m+1}) = 0$ and we get a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\rho_m} & C \\
\downarrow{A} & & \downarrow{C} \\
K^{m+1} & & \\
\end{array}
\]

If $\kappa$ admits a cdga (respectively $A$-module) homotopy retraction, one can built a cdga (respectively $A$-module) homotopy retraction for $\rho_m$ by standard techniques. The third assertion can by deduced by applying homology to the previous diagram. The opposite inequalities hold for any surjective morphism $\varphi$ as seen in [2, Proposition 12].
Observe that [2, Example 10] shows that the hypothesis $s$ is a cofibration is necessary.

**Proof of Theorem 1.** There exists a model $\varphi$ for $f$ admitting a strict section $s$. Now factor $s$ as $s = \theta \circ s'$ with $\theta$ a quasi-isomorphism and $s'$ a cofibration. Then $\varphi \circ \theta$ is a surjective model for $f$ with a section $s'$ which is a cofibration, thus by Theorem 8,

$$\text{secat}(f_0) = \text{secat}(\varphi \circ \theta) = \text{sc}(\varphi \circ \theta),$$

and the proof follows from the inequality

$$\text{secat}(f_0) \leq \text{sc}(f) \leq \text{sc}(\varphi \circ \theta).$$

\[\square\]

Obviously, in such conditions we also have $\text{msecat}(f_0) = m\text{sc}(f)$ and $\text{Hsecat}(f_0) = H\text{sc}(f)$.

5 Application to rational topological complexity

Our main theorem applied to the higher topological complexity is a bit more general than the Murillo-Jessup-Parent analogues. Namely, if $A$ is any cdga model for a space $X$, then the $n$-diagonal map $\Delta: X \to X^n$ is modelled by the map $\varphi := (\text{Id}_A, \eta, \cdots, \eta): A \otimes (\Lambda V)^{\otimes n-1} \to A$ where $\eta: \Lambda V \to A$ is a Sullivan model for $A$. Observe that the cofibration $s: A \to A \otimes (\Lambda V)^{\otimes n-1}$ is a section for $\varphi$. Applying our main result, we obtain

**Theorem 9.** Let $X$ be a topological space, then $\text{TC}_n(X_0)$ is the smallest $m$ such that the projection

$$A \otimes (\Lambda V)^{\otimes n-1} \to \frac{A \otimes (\Lambda V)^{\otimes n-1}}{K^{m+1}}$$

admits a homotopy retraction, where $K := \ker \varphi$.

Observe that $K$ is generated by elements $\eta(v) - v_i$, with $v \in V$ and $i = 1, \ldots, m - 1$ where $\eta(v)$ denotes $\eta(v) \otimes 1 \otimes \cdots \otimes 1$ and $v_i$ denotes $v$ included in the $i$-th factor of $(\Lambda V)^{\otimes m-1}$. 
Remark 10. Observe also that previous theorem remains true for $mTC_n$ and $HTC_n$ asking previous projection to respectively have a homotopy retraction as $A \otimes (\Lambda V)^{\otimes n-1}$-module and to be injective in homology.

If we take $A = \Lambda V$, we deduce Theorem 2.

We now use this result to compute the topological complexity of the space $X$ in [15, Example 6.5].

Example 11. Recall that the minimal model for $X$ is given by $(\Lambda V,d)$ with $V^{\leq 8} = \langle a_3, b_3, x_5 \rangle$, $dx = ab$ and that $H^*(X, \mathbb{Q}) = \langle 1, [a], [b], [ax], [bx] \rangle$. Consider the cycle

$$\omega = (a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)(x \otimes 1 - 1 \otimes x) \in K^3.$$

We have that $\omega$ represents a non-zero class of $\Lambda V \otimes \Lambda V$, this means that $HTC(X) \geq 3$. On the other hand, if we take the model for $X$, $A = \Lambda a,b,x$, we have that $\text{nil ker}(\mu_2: A \otimes A \to A) = 3$. This proves that $TC(X_0) = 3$.

Previous theorem combined with [24, Theorem 1.6] gives the Ganea conjecture for $mTC$.

Theorem 12. If $X$ is a space then

$$mTC(X \times S^n) = mTC(X) + mTC(S^n).$$

6 Application to the D-EH conjecture

Consider Diagram 3 and recall, [8], that $\text{relcat}(f)$ is the smallest $m$ such that $G_m(f)$ admits a homotopy section $s$ verifying $s \circ f \simeq \iota$. Recall also the Doeraene-El Haouari conjecture from [9],

$$\text{if } f \text{ admits homotopy retraction then } \text{secat}(f) = \text{relcat}(f).$$

If a map $f: X \to Y$ admits a homotopy retraction, we can take, as in the proof of Theorem 1, a cdga model for $f$, $\varphi: A \to B$ admitting a strict
section which is a cofibration. Consider the diagram

\[
\begin{array}{c}
A \otimes \Lambda Z_m \\
\downarrow i_m \searrow \theta_m \\
\uparrow \rho_m \\
A \\
\downarrow \varphi \\
B,
\end{array}
\]

where \( i_m \) is a relative Sullivan model for \( \rho_m \) and \( K := \ker \varphi \).

**Theorem 13.** With previous notation, \( \text{relcat}(f_0) \) is the smallest \( m \) such that \( i_m \) admits a retraction \( r \) verifying \( \varphi \circ r \simeq \varphi \circ \theta_m \) rel \( A \).

**Proof.** Consider the commutative diagram in the proof of Theorem 8

\[
\begin{array}{c}
A \\
\downarrow \rho_m \\
A \\
\downarrow \varphi \\
B,
\end{array}
\]

with \( \lambda_m \) a model for \( G_m(f) \) with nil \( \ker p_m = m \) and the lower triangle being a model for Diagram 3. Taking \( j_m \) a relative model of \( \lambda_m \) and applying Lemma 4 we get a diagram

\[
\begin{array}{c}
(A \otimes \Lambda Z_m, D) \\
\downarrow i_m \\
A \\
\downarrow \varphi \\
B,
\end{array}
\]

\[
\begin{array}{c}
(A \otimes \Lambda W_m, D) \\
\downarrow p_m \\
\uparrow j_m \\
(A \otimes \Lambda Z_m, D)
\end{array}
\]

with \( p_m \circ w = \varphi \circ \theta_m \). If \( j_m \) admits a retraction \( r' \) such that \( \varphi \circ r' \simeq p_m \) rel \( A \) then \( i_m \) admits a retraction \( r := r' \circ w \) such that \( \varphi \circ r = \varphi \circ r' \circ w \simeq p_m \circ \omega = \varphi \circ \theta_m \) rel \( A \).

\( \square \)
7 A note on Hess’ theorem

In [30], D. Stanley gives an example of a map \( f \) for which \( \text{msecat}(f) < \text{secat}(f_0) \). This example tells us that a general Hess’ Theorem for sectional category in impossible. However, the example given does not verify the hypothesis of our main theorem. On the other hand, the proof of Hess’ theorem uses strongly the fact that \( \text{cat}(X_0) = \text{sc}(\ast \hookrightarrow X) \). These remarks lead us to a

Conjecture. If \( f \) admits a homotopy retraction, then

\[
\text{msecat}(f) = \text{secat}(f_0).
\]

A proof of this conjecture would yield important consequences.

- Combining it with the main result of [4], we would get that \( \text{secat}(f_0) = \text{Hsecat}(f) \) if the base of \( f \) is a Poincaré duality complex. In particular, we would have \( \text{TC}_n(X_0) = \text{HTC}_n(X_0) \) for Poincaré duality complexes.

- If combined with Theorem 12 it would give a proof of the Ganea conjecture for rational topological complexity.

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