SAMPLING AND GALERKIN RECONSTRUCTION IN REPRODUCING KERNEL SPACES

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Abstract. In this paper, we introduce a fidelity measure depending on a given sampling scheme and we propose a Galerkin method in Banach space setting for signal reconstruction. We show that the proposed Galerkin method provides a quasi-optimal approximation, and the corresponding Galerkin equations could be solved by an iterative approximation-projection algorithm in a reproducing kernel subspace of $L^p$. Also we present detailed analysis and numerical simulations of the Galerkin method for reconstructing signals with finite rate of innovation.

1. Introduction

Digital processing of signals $f$ may start from sampling on a discrete set $\Gamma$,

\begin{equation}
    f \mapsto (f(\gamma_n))_{\gamma_n \in \Gamma}
\end{equation}

[5, 32, 42, 43]. The celebrated Whittaker-Shannon-Kotelnikov sampling theorem states that a bandlimited signal can be recovered from its samples taken at a rate greater than twice the bandwidth [32, 45]. In the last two decades, that paradigm for bandlimited signals has been extended to represent signals in a shift-invariant space [5, 7, 42], signals with finite rate of innovation [13, 28, 31, 34, 36, 43], and signals in a reproducing kernel space [11, 19, 24, 29, 30].

A fundamental problem in sampling theory is how to obtain a good approximation of the signal $f$ when only the noisy sampling data $(f(\gamma_n) + \epsilon(\gamma_n))_{\gamma_n \in \Gamma}$ is available [3, 5, 34, 42]. The above problem is well studied and many algorithms, such as the frame algorithm and the approximation-projection algorithm, have been proposed [4, 12, 14, 17, 29, 34, 38]. In this paper, we introduce a Galerkin method for signal

2010 Mathematics Subject Classification. 94A20, 46E22, 65J22.

Key words and phrases. sampling, Galerkin reconstruction, oblique projection, reproducing kernel space, finite rate of innovation, iterative approximation-projection algorithm.
reconstruction and we propose a fast and stable algorithm to solve the corresponding Galerkin equations.

A conventional way to reconstruct signals $f$ in a linear space $V$ from their sampling data is to solve a minimization problem

$$Rf := \arg\min_{h \in V} \| h - f \|,$$

where the fidelity measure $\| h - f \|$ depends only on the sampling data of $h - f$ on $\Gamma$. Typical examples of fidelity measures in the bandlimited setting are weighted sampling energy $\sum_{\gamma_n \in \Gamma} w_n |f(\gamma_n) - h(\gamma_n)|^2$ and weighted pre-reconstruction energy $\| \sum_{\gamma_n \in \Gamma} w_n (f(\gamma_n) - h(\gamma_n)) \text{sinc}(-\gamma_n) \|_2$, where $w_n$ are positive weights appropriately selected.

The fidelity of perceptual signals, such as acoustic and visual signals, might not be well measured by some weighted square errors [10, 44]. Alternatives of fidelity measures are weighted sampling error $\left( \sum_{\gamma_n \in \Gamma} w_n |f(\gamma_n) - h(\gamma_n)|^p \right)^{\frac{1}{p}}$ and weighted pre-reconstruction error $\| \sum_{\gamma_n \in \Gamma} w_n (f(\gamma_n) - h(\gamma_n)) K(\cdot, \gamma_n) \|_p$, $1 \leq p < \infty$, for signals in a reproducing kernel subspace of $L^p := L^p(\mathbb{R}^d)$ with kernel $K$. In this paper, we introduce a general fidelity measurement associated with a linear operator $S$ on a Banach space $V$, that depends on the sampling scheme (1.1). Then the minimization problem (1.2) becomes

$$Rf := \arg\min_{h \in V} \| Sh - Sf \|_V.$$ 

The operator $S$ in the above minimization problem can be selected as

$$Sf := \sum_{\gamma_n \in \Gamma} w_n f(\gamma_n) \text{sinc}(\cdot - \gamma_n)$$

for the bandlimited setting, and

$$Sf := \sum_{\gamma_n \in \Gamma} w_n f(\gamma_n) K(\cdot, \gamma_n)$$

for the reproducing kernel space setting.

The nonlinear minimization problem (1.3) does not give a tractable signal reconstruction. Observe that

$$\| Sh - Sf \|_V = \sup_{\| g \|_{V^*} = 1, g \in V^*} |\langle Sh - Sf, g \rangle|,$$

where $\langle \cdot, \cdot \rangle$ is the standard dual product between elements in $V$ and its dual $V^*$. So we propose the following linear approach

$$\langle Sh, g \rangle = \langle Sf, g \rangle \quad \text{for all } g \in \tilde{U},$$

$$\sum_{\gamma_n \in \Gamma} w_n f(\gamma_n) \text{sinc}(-\gamma_n)$$

for the bandlimited setting, and

$$\sum_{\gamma_n \in \Gamma} w_n f(\gamma_n) K(\cdot, \gamma_n)$$

for the reproducing kernel space setting.
where $\tilde{U} \subset V^*$ is a (finite-dimensional) trial space. Clearly, the solution of the Galerkin equations (1.4) with $\tilde{U} = V^*$ is also a solution of the minimization problem (1.3).

The approach (1.4) may not recover all signals in $V$, especially when the sampling scheme (1.1) is taken only on a finite duration. To apply the approach (1.4) for signal reconstruction, a realistic additional requirement is that the consistence

$$h = f$$

hold only for signals $f$ in some subspaces $U$ of $V$, where $Rf := h$ in (1.4) is the reconstructed signal. Then natural questions are whether the Galerkin reconstruction (1.4) and (1.5) is numerically stable

$$\|Rf\|_V \leq C\|f\|_V$$

and quasi-optimal

$$\|Rf - f\|_V \leq C \inf_{h \in U\{f \in V, \ |h - f\|_V \},}$$

where $C$ is a positive constant. In Theorem 2.3, we establish numerical stability and quasi-optimality of the Galerkin reconstruction (1.4) and (1.5) when the linear operator $S$ is admissible, which is a frame-like requirement on the sampling scheme (1.1).

The next topic of this paper is how to solve the Galerkin reconstruction (1.4) and (1.5) for signals in a reproducing kernel space (RKS). Particularly, we are interested in RKSs of the form

$$V_{K,p} := \left\{ T_0 f : f \in L^p \right\} = \left\{ f \in L^p : T_0 f = f \right\}, \quad 1 \leq p \leq \infty,$$

where $T_0$ is an idempotent integral operator with kernel $K$.

$$T_0 f(x) := \int_{\mathbb{R}^d} K(x,y)f(y)dy, \quad f \in L^p.$$

The RKS of the form (1.6) has rich geometric structure, lots of flexibility and technical suitability for sampling. It has been used for modeling bandlimited signals, wavelet (spline) signals, and signals with finite rate of innovation [5, 29, 30, 36, 42].

For the sampling scheme (1.1) on $V_{K,p}$, take a disjoint covering

$$\{ I_n \subset B(\gamma_n, \delta) : \gamma_n \in \Gamma \}$$

of $B(\Gamma, \delta) := \cup_{\gamma_n \in \Gamma} B(\gamma_n, \delta) = \cup_{\gamma_n \in \Gamma} \{ x : |x - \gamma_n| \leq \delta \}$, and define

$$S_{\Gamma,\delta} f(x) := \sum_{\gamma_n \in \Gamma} |I_n| f(\gamma_n)K(x,\gamma_n), \quad f \in V_{K,p},$$

where $\delta > 0$. The operator $S_{\Gamma,\delta}$ depends only on the sampling scheme (1.1). We call it a pre-reconstruction operator, as $S_{\Gamma,\delta} f(x)$ is a good
approximation to $f(x)$ when $\delta$ is sufficiently small and $x \in B(\Gamma, \delta)$ is away from the complement of $B(\Gamma, \delta)$, see Figure 1. Due to the

Figure 1. Plotted on the left is a bandlimited signal $f_0 = \sum \alpha_i \text{sinc}(\cdot - i)$ with $\alpha_i \in [-1, 1]$ randomly selected. On the right is the difference between $f_0$ and its pre-reconstruction $h_0 = S_{\Gamma, \delta} f_0$, where $\delta = 1$ and $\Gamma := \{ \gamma_k, k = 1, 2, \ldots, 80 \}$ is a nonuniform sampling set with $\gamma_1 = -40$ and $\gamma_k - \gamma_{k-1} \in [0.9, 1.1], 2 \leq k \leq 80$, being randomly selected. In this figure, the maximal amplitude $\max_{-38 \leq t \leq 38} |f_0(t)|$ of the signal $f_0$ is 1.7498, while the maximal pre-reconstruction error $\max_{-38 \leq t \leq 38} |h_0(t) - f_0(t)|$ on $[-38, 38] \subset B(\Gamma, 1)$ is 0.6708.

above approximation property of the pre-reconstruction operator $S_{\Gamma, \delta}$, we propose the following iterative approximation-projection algorithm

\begin{equation}
  g_0 \in U \quad \text{and} \quad g_{m+1} = g_m - P_{U,\tilde{U}} S_{\Gamma, \delta} g_m + g_0, \quad m \geq 0,
\end{equation}

to solve the Galerkin reconstruction (1.4) and (1.5) for $S_{\Gamma, \delta}$, where $P_{U,\tilde{U}}$ is an oblique projection for the trial-test space pair $(U, \tilde{U})$. The above algorithm is shown in Theorem 4.2 to have exponential convergence, cf. [4, 6, 16, 29, 40].

This paper is organized as follows. In Section 2, we introduce the concept of admissible operators in a Banach space setting. We show that (sub-)Galerkin reconstruction for an admissible operator provides a quasi-optimal approximation (Theorem 2.3), and such (sub-)Galerkin reconstruction exists whenever the trial and test spaces are finite-dimensional (Theorem 2.4, Corollaries 2.5 and 2.6). In Section 3, we discuss admissibility of the pre-reconstruction operator $S_{\Gamma, \delta}$ in (1.8) (Theorem 3.1). In Section 4, we propose to apply the iterative approximation-projection algorithm (1.9) for the Galerkin reconstruction (1.4) and (1.5) (Theorem 4.2 and Theorem 4.3). Many signals with finite rate of innovation live in some reproducing kernel spaces of the form (1.6). In Section 5, we provide detailed analysis for pre-reconstruction
operators, and we obtain matrix formulation of Galerkin reconstructions for signals with finite rate of innovation. In Section 6, we present some numerical simulations to demonstrate our Galerkin method. In the last section, we include all proofs.

2. **Galerkin reconstruction in Banach spaces**

In this section, we consider numerical stability and quasi-optimality of a (sub-)Galerkin reconstruction in a Banach space setting. First we introduce admissibility of operators for a trial-test space pair.

**Definition 2.1.** Let $(U, V, B)$ be a triple of Banach spaces with $U \subset V \subset B$, and let $\tilde{U} \subset B^*$. We say that a bounded linear operator $S : V \to V$ is admissible for the trial-test space pair $(U, \tilde{U})$ if there exist positive constants $D_1$ and $D_2$ such that

\[
\sup_{g \in \tilde{U}, \|g\| \leq 1} |\langle Sf, g \rangle| \geq D_1 \|f\| \quad \text{for all } f \in U,
\]

and

\[
\sup_{g \in \tilde{U}, \|g\| \leq 1} |\langle Sf, g \rangle| \leq D_2 \|f\| \quad \text{for all } f \in V.
\]

The above admissibility concept in a Hilbert space setting is a frame-like requirement, which was introduced in [2, Definition 3.2]. In our model for sampling, $S$ is the pre-reconstruction operator $S_{\Gamma, \delta}$ in (1.8), and the triple of Banach spaces contains the reconstruction space $U$, the reproducing kernel space $V_{K,p}$ and the space $L^p$.

Next we introduce a general notion of Galerkin reconstructions.

**Definition 2.2.** Let $S : V \to V$ be a bounded linear operator, and $(U, \tilde{U})$ be a trial-test space pair as in Definition 2.1. We say that a linear operator $R : V \to U$ is a Galerkin reconstruction for $S$ if

\[
Rh = h, \quad h \in U
\]

and

\[
\langle SRf, g \rangle = \langle Sf, g \rangle, \quad f \in V \text{ and } g \in \tilde{U};
\]

and a sub-Galerkin reconstruction for $S$ if (2.3) holds and

\[
\sup_{g \in \tilde{U}, \|g\| \leq 1} \|\langle SRf, g \rangle\| \leq D_3 \sup_{g \in \tilde{U}, \|g\| \leq 1} \|\langle Sf, g \rangle\|, \quad f \in V,
\]

for some $D_3 > 0$.

In the following theorem, we establish numerical stability and quasi-optimality of (sub-)Galerkin reconstructions associated with admissible operators.
Theorem 2.3. Let $V, U, \tilde{U}$ be as in Definition 2.1, and $S$ be admissible for the pair $(U, \tilde{U})$ with bounds $D_1$ and $D_2$. If $R : V \to U$ is a sub-Galerkin reconstruction for $S$ with bound $D_3$, then

(i) $R$ is numerically stable,
$$\|Rf\| \leq \frac{D_2 D_3}{D_1} \|f\|, \ f \in V; \text{ and}$$

(ii) $R$ is quasi-optimal,
$$\|Rf - f\| \leq \frac{D_1 + D_2 D_3}{D_1} \inf_{h \in \tilde{U}} \|f - h\|, \ f \in V.$$

By Theorem 2.3, the existence of a quasi-optimal approximation follows by finding a sub-Galerkin reconstruction. Now we show that such a sub-Galerkin reconstruction always exists when $U$ and $\tilde{U}$ are finite-dimensional.

Theorem 2.4. Let $V, U, \tilde{U}$ be as in Definition 2.1, and $S$ be admissible for the pair $(U, \tilde{U})$. If $U$ and $\tilde{U}$ are finite-dimensional, then there is a sub-Galerkin reconstruction for $S$.

For the case that $U$ and $\tilde{U}$ have the same dimension, we have

Corollary 2.5. Let $V, U, \tilde{U}$ be as in Definition 2.1, and $S$ be admissible for the pair $(U, \tilde{U})$. If dimensions of $U$ and $\tilde{U}$ are the same, then for $f \in V$, the unique solution of Galerkin equations

$$\langle SRf, g \rangle = \langle Sf, g \rangle, \ g \in \tilde{U},$$

defines a Galerkin reconstruction for $S$.

In a Hilbert space setting, we can establish the following result for least squares solutions.

Corollary 2.6. Let $V$ be a Hilbert space, $U$ and $\tilde{U}$ be linear subspaces of $V$, and let $S$ be admissible for the pair $(U, \tilde{U})$. If $U$ and $\tilde{U}$ are finite-dimensional, then the least squares solution of Galerkin equations (2.6),

$$Rf := \arg\min_{h \in U} \sup_{g \in \tilde{U}, \|g\| \leq 1} |\langle S(h - f), g \rangle|, \ f \in V,$$

defines a sub-Galerkin reconstruction for $S$ with bound $D_3 \leq 1$.

We remark that the above conclusion on least squares solutions with $\tilde{U} = U$ has been established by Adcock, Gataric and Hansen for non-uniform sampling [1, 2].
3. Admissible Pre-reconstruction Operator in Reproducing Kernel Spaces

To consider sampling and reconstruction in $V_{K,p}$, we always assume that the kernel $K$ of the space $V_{K,p}$ in (1.6) satisfies

\[(3.1) \quad \|K\|_W := \max \left\{ \sup_{x \in \mathbb{R}^d} \|K(x, \cdot)\|_1, \sup_{y \in \mathbb{R}^d} \|K(\cdot, y)\|_1 \right\} < \infty\]

and

\[(3.2) \quad \lim_{\delta \to 0} \|\omega_\delta(K)\|_W = 0,\]

where

$$\omega_\delta(K)(x, y) := \sup_{|x'|, |y'| \leq \delta} |K(x + x', y + y') - K(x, y)|.$$

Under the above hypothesis, the integral operator $T_0$ in (1.7) is a bounded operator on $L^p$,

$$\|T_0f\|_p \leq \|K\|_W \|f\|_p, \quad f \in L^p.$$ 

More importantly, its range space $V_{K,p}$ is a reproducing kernel space [29]. The model space (5.1) for FRI signals to live in is a reproducing kernel space of the form (1.6) with kernel $K$ satisfying (3.1) and (3.2), see Theorem A.1 in the appendix.

In this section, we discuss admissibility of the pre-reconstruction operator $S_{\Gamma,\delta}$ in (1.8). To do so, we introduce the residue $E(U, F)$ of signals in a linear space $U \subset L^p$ outside a measurable set $F$,

$$E(U, F) := \sup_{0 \neq f \in U} \frac{\|f\|_{L^p(\mathbb{R}^d \setminus F)}}{\|f\|_p},$$

where $\|\cdot\|_{L^p(E)}$ is the $p$-norm on a measurable set $E$. The reader may refer to [1, 25, 26] for some applications of residues of bandlimited signals.

**Theorem 3.1.** Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in (1.6) and (1.8) respectively. Assume that $U \subset V_{K,p}$ and $\tilde{U} \subset L^{p/(p-1)}$. If

\[(3.3) \quad \sup_{g \in \tilde{U}, \|g\|_{p/(p-1)} \leq 1} |\langle f, g \rangle| \geq D_4 \|f\|_p, \quad f \in U\]

for some constant $D_4$ satisfying

\[(3.4) \quad r_0 := D_4^{-1}(E(U, B(\Gamma, \delta)))\|K\|_W + \|\omega_\delta(K)\|_W (1 + \|K\|_W + \|\omega_\delta(K)\|_W) < 1,\]

then $S_{\Gamma,\delta}$ is admissible for the pair $(U, \tilde{U})$. 

Given a sampling set $\Gamma$, we say that the sampling scheme (1.1) has \textit{weighted $\ell^p$-stability} on $U$ if there exist positive constants $C_1, C_2$ and $\delta$ such that

$$C_1 \| f \|_p \leq \left( \sum_{\gamma_n \in \Gamma} |I_n| |f(\gamma_n)|^p \right)^{1/p} \leq C_2 \| f \|_p, \quad f \in U,$$

if $1 \leq p < \infty$, and

$$C_1 \| f \|_\infty \leq \sup_{\gamma_n \in \Gamma} |f(\gamma_n)| \leq C_2 \| f \|_\infty, \quad f \in U,$$

if $p = \infty$, where $\{I_n \subset B(\gamma_n, \delta), \gamma_n \in \Gamma\}$ is a disjoint covering of the $\delta$-neighborhood $B(\Gamma, \delta)$ of the sampling set $\Gamma$. Weighted stability of a sampling scheme is an important concept for the robustness and uniqueness of signal reconstructions, see [5, 6, 9, 15, 29, 34, 39, 40, 42] and references there.

In the next theorem, we show that weighted stability of the sampling scheme (1.1) follows from admissibility of the pre-reconstruction operator in (1.8).

\textbf{Theorem 3.2.} Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in (1.6) and (1.8) respectively. Assume that $U \subset V_{K,p}$ and $\tilde{U} \subset L^{p/(p-1)}$. If $S_{\Gamma,\delta}$ is admissible for the pair $(U, \tilde{U})$, then the sampling scheme (1.1) on $\Gamma$ has weighted $\ell^p$-stability on $U$.

By the regularity assumption (3.2) on the reproducing kernel $K$, the second requirement (3.4) in Theorem 3.1 is satisfied if $\delta$ is sufficiently small and $B(\Gamma, \delta)$ is the whole Euclidean space $\mathbb{R}^d$. For the case that $B(\Gamma, \delta)$ contains an open domain $F_0$ but not necessarily the whole space $\mathbb{R}^d$, we obtain the following samplability result from Theorems 3.1 and 3.2.

\textbf{Corollary 3.3.} Let $U \subset V_{K,p}$ and $D_4$ be as in Theorem 3.1. Assume that $F_0$ is an open domain satisfying $E(U, F_0)\|K\|_W < D_4$. If $\Gamma$ is a sampling set with $B(\Gamma, \delta) \supset F_0$ for some sufficiently small $\delta > 0$, then signals in $U$ are uniquely determined by their samples taken on $\Gamma$.

The samplability of various signals is well-studied, see, e.g., [2, 16, 23] for band-limited signals, [5, 42] for signals in a shift-invariant space, [34, 36] for signals with finite rate of innovation, and [24, 29] for signals in a reproducing kernel space.
4. GALERKIN RECONSTRUCTION AND ITERATIVE APPROXIMATION-PROJECTION ALGORITHM

In this section, we apply the iterative approximation-projection algorithm (1.9) to define a unique Galerkin reconstruction associated with the pre-reconstruction operator $S_{\Gamma,\delta}$.

To define the iterative approximation-projection algorithm (1.9), we recall the oblique projection for a pair $(U, \tilde{U})$ of Banach spaces.

**Definition 4.1.** Given $U \subset V_{K,p}$ and $\tilde{U} \subset L^p/(p-1)$, a bounded operator $P_{U,\tilde{U}} : V_{K,p} \to U$ is said to be an oblique projection for the pair $(U, \tilde{U})$ if

\begin{align}
P_{U,\tilde{U}}h &= h, \quad h \in U, \quad (4.1) \\
\langle P_{U,\tilde{U}}f, g \rangle &= \langle f, g \rangle, \quad f \in V_{K,p}, g \in \tilde{U}. \quad (4.2)
\end{align}

In Hilbert space setting, an oblique projection $P_{U,\tilde{U}}$ exists when cosine of the subspace angle between $U$ and $\tilde{U}^\perp$ is positive [3, 9, 15, 41]. Following the argument used in Theorem 2.4, we can show that if $U$ and $\tilde{U}$ have the same dimension and satisfy the first requirement (3.3) of Theorem 3.1, then there is an oblique projection $P_{U,\tilde{U}}$ for the pair $(U, \tilde{U})$.

In the next theorem, we prove that the iterative approximation-projection algorithm (1.9) associated with the oblique projection $P_{U,\tilde{U}}$ has exponential convergence, cf. Remark 6.1.

**Theorem 4.2.** Let $V_{K,p}$, $S_{\Gamma,\delta}$ and $r_0 \in (0, 1)$ be as in (1.6), (1.8) and (3.4) respectively. Assume that $U \subset V_{K,p}$ and $\tilde{U} \subset L^p/(p-1)$ satisfy (3.3) and (3.4), and an oblique projection $P_{U,\tilde{U}}$ associated with the pair $(U, \tilde{U})$ exists. Then for any $g_0 \in U$, the sequence $g_m, m \geq 0$, in the iterative algorithm (1.9) converges to some $g_\infty \in U$,

\begin{align}
\|g_m - g_\infty\|_p &\leq \frac{r_0^{m+1}}{1 - r_0} \|g_0\|_p, \quad m \geq 0. \quad (4.3)
\end{align}

Moreover, if $g_0 = P_{U,\tilde{U}}S_{\Gamma,\delta}h + \tilde{g}$ for some $h, \tilde{g} \in U$, then

\begin{align}
\|g_\infty - h\|_p &\leq \frac{\|\tilde{g}\|_p}{1 - r_0}. \quad (4.4)
\end{align}

The algorithm (1.9) has been widely used to reconstruct various signals. The reader may refer to [16, 40] for band-limited signals, [4, 6] for signals in a shift-invariant space, and [29] for signals in a reproducing kernel space.
Applying exponential convergence of the iterative approximation-projection algorithm (1.9), we can define a unique Galerkin reconstruction.

**Theorem 4.3.** Let $V_{K,p}, S_{Γ,δ}, U,  \bar{U}$ and $P_{U,\bar{U}}$ be as in Theorem 4.2. Then Galerkin equations

\begin{equation}
\langle S_{Γ,δ} h, g \rangle = \langle S_{Γ,δ} f, g \rangle, \quad g \in  \bar{U},
\end{equation}

have a unique solution $h \in U$ for $f \in V_{K,p}$. Moreover, the mapping $f \to h$ defines a Galerkin reconstruction.

We finish this section with a remark on the iterative approximation-projection algorithm (1.9).

**Remark 4.4.** Given $δ > 0$, a sampling set $Γ$ and probability measures $μ_n$ supported on $I_n$, we define

\[
\tilde{S}_{Γ,δ} f(x) = \sum_{γ_n \in Γ} |I_n| f(γ_n) \int_{I_n} K(x, y) dμ_n(y), \quad f \in V_{K,p},
\]

where $\{I_n \subset B(γ, δ), \ γ_n \in Γ\}$ is a disjoint covering of $B(Γ, δ)$. The operator $\tilde{S}_{Γ,δ}$ just defined becomes the pre-reconstruction operator $S_{Γ,δ}$ in (1.8) when $μ_n$ are point measures supported on $γ_n$, and the pre-reconstruction operator

\[
S_{Γ,δ} f(x) = \sum_{ω_n \in Γ} f(γ_n) \int_{I_n} K(x, y) dy, \quad f \in V_{K,p}
\]

when $μ_n$ are normalized Lebesgue measure supported on $I_n$. Following the argument used in Theorem 3.1 and Theorem 4.2, we can show that the approximation-projection algorithm (1.9) with $S_{Γ,δ}$ replaced by $\tilde{S}_{Γ,δ}$ has exponential convergence if

\[
D^{-1}_1(E(U,B(Γ,δ))\|K\|_W + \|ω_{2δ}(K)\|_W(1 + \|K\|_W + \|ω_{2δ}(K)\|_W)) < 1,
\]

cf., the second requirement (3.4) in Theorem 3.1.

5. Sampling signals with finite rate of innovation

A signal with finite rate of innovation (FRI) has finitely many degrees of freedom per unit of time [13, 28, 31, 34, 36, 43]. Define the Wiener amalgam space by

\[
\mathcal{W}^1 := \{φ, \|φ\|_{\mathcal{W}^1} := \sum_{k ∈ \mathbb{Z}} \sup_{0 ≤ x ≤ 1} |φ(x + k)| < ∞\}.
\]
It is observed in [36] that lots of FRI signals live in a space of the form
\[ V_2(\Phi) := \left\{ \sum_{i \in \mathbb{Z}} c_i \phi_i(\cdot - i), \sum_{i \in \mathbb{Z}} |c_i|^2 < \infty \right\}, \]
where the generator \( \Phi := (\phi_i)_{i \in \mathbb{Z}} \) satisfies
\[ \|\Phi\|_{W^1} := \sup_{i \in \mathbb{Z}} |\phi_i|_{W^1} < \infty \quad \text{and} \quad \lim_{\delta \to 0} \sup_{i \in \mathbb{Z}} \omega_\delta(\phi_i) = 0. \]

For \( \Phi := (\phi_i)_{i \in \mathbb{Z}} \) and \( \tilde{\Phi} := (\tilde{\phi}_j)_{j \in \mathbb{Z}} \) satisfying (5.2), define their correlation matrix by
\[ A_{\Phi, \tilde{\Phi}} := \left( \langle \phi_i(\cdot - i), \tilde{\phi}_j(\cdot - j) \rangle \right)_{i,j \in \mathbb{Z}}. \]
In this section, we always assume that \( A_{\Phi, \tilde{\Phi}} \) has bounded inverse on \( \ell^2 \).

Write \( A_{\Phi, \tilde{\Phi}}^{-1} = (b_{ij})_{i,j \in \mathbb{Z}} \). Applying Wiener’s lemma for Baskakov-Gohberg-Sjöstrand class, one may verify that the space \( V_2(\Phi) \) for FRI signals to live in is the range space \( V_{K_{\Phi, \tilde{\Phi}}, 2} \) of an idempotent integral operator with kernel
\[ K_{\Phi, \tilde{\Phi}}(x, y) := \sum_{i,j \in \mathbb{Z}} \phi_i(x - i) b_{ij} \tilde{\phi}_j(y - j) \]
satisfying (3.1) and (3.2), see Theorem A.1 in the appendix.

Given a sampling set \( \Gamma = \{\gamma_n\}_{n=1}^N \) ordered as \( \gamma_1 < \gamma_2 < \cdots < \gamma_N \), define
\[ S_{\Phi, \tilde{\Phi}, \Gamma} f(x) := \sum_{n=1}^N \frac{\gamma_{n+1} - \gamma_{n-1}}{2} f(\gamma_n) K_{\Phi, \tilde{\Phi}}(x, \gamma_n), \quad f \in V_2(\Phi), \]
where \( \gamma_0 = \gamma_1 \) and \( \gamma_{N+1} = \gamma_N \). In the next theorem, we establish the equivalence between admissibility of the operator \( S_{\Phi, \tilde{\Phi}, \Gamma} \) and its corresponding Galerkin reconstruction in a finite-dimensional space, cf. Corollary 2.5, and Theorems 3.1 and 4.3.

**Theorem 5.1.** For \( L \geq 1 \), define
\[ V_{2,L}(\Phi) := \left\{ \sum_{i=-L}^L c_i \phi_i(\cdot - i), \sum_{i=-L}^L |c_i|^2 < \infty \right\} \]
and
\[ V_{2,L}(\tilde{\Phi}) := \left\{ \sum_{i=-L}^L d_i \tilde{\phi}_i(\cdot - i), \sum_{i=-L}^L |d_i|^2 < \infty \right\}. \]
Assume that \( \Phi, \tilde{\Phi} \) satisfy (5.2), and the correlation matrix \( A_{\Phi, \tilde{\Phi}} \) in (5.3) has bounded inverse on \( \ell^2 \). Then the following statements are equivalent:
(i) The $L \times L$ matrix

$A_{\Phi,\tilde{\Phi},\Gamma} := \left( \sum_{n=1}^{N} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_i(\gamma_n - i) \tilde{\phi}_j(\gamma_n - j) \right)_{-L \leq i,j \leq L}$

is nonsingular.

(ii) $S_{\Phi,\tilde{\Phi},\Gamma}$ is admissible for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde{\Phi}))$.

(iii) For any $f \in V_{2,L}(\Phi)$, Galerkin equations

$\langle S_{\Phi,\tilde{\Phi},\Gamma} h, g \rangle = \langle S_{\Phi,\tilde{\Phi},\Gamma} f, g \rangle, ~ g \in V_{2,L}(\tilde{\Phi})$

have a unique solution $h$ in $V_{2,L}(\Phi)$.

(iv) For any $g \in V_{2}(\tilde{\Phi})$, dual Galerkin equations

$\langle S_{\Phi,\tilde{\Phi},\Gamma} f, \tilde{h} \rangle = \langle S_{\Phi,\tilde{\Phi},\Gamma} f, g \rangle, ~ f \in V_{2,L}(\Phi)$

have a unique solution $\tilde{h}$ in $V_{2,L}(\tilde{\Phi})$.

To solve the Galerkin equations (5.9) by the iterative approximation-projection algorithm (1.9), we need an oblique projection for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde{\Phi}))$.

**Theorem 5.2.** Let $L \geq 1$, and let $\Phi$ and $\tilde{\Phi}$ satisfy (5.2). Assume that the correlation matrix $A_{\Phi,\tilde{\Phi}}$ in (5.3) has bounded inverse on $\ell^2$. Then the principal submatrix

$A_{\Phi,\tilde{\Phi},L} := \left( \langle \phi_i(\cdot - i), \tilde{\phi}_j(\cdot - j) \rangle \right)_{-L \leq i,j \leq L}$

of the correlation matrix $A_{\Phi,\tilde{\Phi}}$ is nonsingular if and only if there exists a unique oblique projection for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde{\Phi}))$. Moreover, the oblique projection could be defined by

$P_{\Phi,\tilde{\Phi},L} f := \sum_{-L \leq i,j \leq L} \langle f, \tilde{\phi}_i(\cdot - i) \rangle \tilde{b}_{ij} \phi_j(\cdot - j), ~ f \in V_{2}(\Phi),$

where $(A_{\Phi,\tilde{\Phi},L})^{-1} = (\tilde{b}_{ij})_{-L \leq i,j \leq L}$.

We conclude this section by examining exponential convergence of an iterative algorithm for the recovery of signals with finite rate of innovation. Replacing $P_{\Gamma,\tilde{\Gamma}}$ and $S_{\Gamma,\delta}$ in the iterative algorithm (1.9) by $P_{\Phi,\tilde{\Phi},L}$ and $S_{\Phi,\tilde{\Phi},\Gamma}$ respectively, it becomes

$g_{m+1} = g_m - \sum_{n=1}^{N} \sum_{i,j=-L}^{L} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} g_m(\gamma_n) \tilde{\phi}_i(\gamma_n - i) \tilde{b}_{ij} \phi_j(\cdot - j) + g_0, ~ m \geq 0,$
with \( g_0 \in V_{2,L}(\Phi) \). The above iterative algorithm has exponential convergence when

\[
\| A_{\Phi,\tilde{\Phi},\Gamma}(A_{\Phi,\tilde{\Phi},L})^{-1} - I \| < 1.
\]

**Theorem 5.3.** Let \( \Phi \) and \( \tilde{\Phi} \) satisfy (5.2). Assume that \( A_{\Phi,\tilde{\Phi},L} \) is non-singular. If (5.13) holds, then the iterative algorithm (5.12) has exponential convergence. Moreover, it recovers the original signal \( h \in V_{2,L}(\Phi) \) when

\[
g_0 = \sum_{n=1}^{N} \sum_{i,j=-L}^{L} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} h(\gamma_n) \tilde{\phi}_i(\gamma_n - i) b_{ij} \phi_j(\cdot - j).
\]

### 6. Numerical Simulation

In this section, we present several examples to illustrate our Galerkin reconstruction of signals with finite rate of innovation.

Let \( \Theta := \{ \theta_i \} \) be either \( \Theta_O := \{ 0 \} \) (the identical zero set), or \( \Theta_I \) with \( \theta_i \) being randomly selected in \([-0.2, 0.2]\). Set

\[
\Phi_0 = \{ \phi_0(\cdot - \theta_i) \}_{i \in \mathbb{Z}},
\]

where the generating function \( \phi_0 \) is either (i) the sinc function \( \text{sinc}(t) := \frac{\sin(\pi t)}{\pi t} \), or (ii) the Gaussian function \( \text{gauss}(t) := \exp(-3t^2/2) \), or (iii) the cubic \( B \)-spline \( \text{spline}(t) \), see Figure 2 for examples of signals in \( V_2(\Phi_0) \).

In our numerical simulations, reconstructed signals live in the space

\[
V_{2,L}(\Phi_0) = \left\{ \sum_{i=-L}^{L} c_i \phi_0(t - i - \theta_i) : \sum_{i=-L}^{L} |c_i|^2 < \infty \right\}, \quad L \geq 1
\]

and sampling schemes are

- Nonuniform sampling on \( \Gamma_N := \{ \gamma_k, |k| \leq L+2 \} \), where \( \gamma_{-L-3} = -L - 2 \) and \( \gamma_k - \gamma_{k-1} \in [0.9, 1.1], |k| \leq L + 2 \), are randomly selected.
- Jittered sampling on \( \Gamma_J := \{ \gamma_k := k + \delta_k, |k| \leq L + 2 \} \), where \( \delta_k \in [-0.1, 0.1] \) are randomly selected.
- Adaptive sampling on \( \Gamma_C := \{ \gamma_k \in [-L-2, L+2] \} \) of a bounded signal \( x \in V_2(\Phi) \) via crossing time encoding machine (C-TEM), where \( x(t) \neq \|x\|_{\infty} \sin(\pi t) \) for all \( t \in [-L-2, L+2] \) except \( t = \gamma_k \) for some \( k \), see Figure 3 [18, 21, 27].

To reconstruct signals via our Galerkin method, take

\[
\tilde{\Phi}_0 = \{ \tilde{\phi}_0 \} \quad \text{with} \quad \tilde{\phi}_0 = \chi_{[-1/2,1/2]}.
\]
Figure 2. Plotted above are bandlimited signals $x(\text{sinc}, 0) = \sum_i \alpha_i \text{sinc}(t - i)$ with $(1 + |i|)\alpha_i \in [-1, 1]$ randomly selected (left), and $x(\text{sinc}, 1) = \sum_i \beta_i \text{sinc}(t - i)$ with $\beta_i = (1 + |i|)^{-1}\cos(\pi i/8)$ (right). Shown below are signals $x(\text{sinc}, 2) = \sum_i \alpha_i \text{sinc}(t - i - \theta_i)$ (left) and $x(\text{sinc}, 3) = \sum_i \beta_i \text{sinc}(t - i - \theta_i)$ with $\theta_i \in [-0.2, 0.2]$ randomly selected (right).

Then the equation (5.9) to determine the Galerkin reconstruction

$$G_{\Phi_0, \tilde{\Phi}_0, \Gamma} f := \sum_{i=\pm L}^L c_i \phi_0(\cdot - i - \theta_i) \in V_{2,L}(\Phi_0)$$

can be reformulated as follows:

$$\sum_{i=\pm L}^L \left( \sum_{n=1}^N \frac{\gamma_n + 1 - \gamma_{n-1}}{2} \phi_0(\gamma_n - i - \theta_i) \tilde{\phi}_0(\gamma_n - j) \right) c_i$$

$$= \sum_{n=1}^N \frac{\gamma_{n+1} - \gamma_{n-1}}{2} f(\gamma_n) \tilde{\phi}_0(\gamma_n - j), -L \leq j \leq L,$$

(6.1)

where $f \in V_2(\Phi_0)$ and $\Gamma := \{\gamma_n\}_{n=1}^N$ is either the nonuniform sampling set $\Gamma_N$, or the jittered sampling set $\Gamma_J$, or the adaptive C-TEM sampling set $\Gamma_C$. Considering the bandlimited signal $x(\text{sinc}, 0)$ described in
Figure 3. Plotted above is the signal $x(sinc, 0)$ in Figure 2 and the crossing signal $\|x(sinc, 0)\|_\infty \sin \pi t$ on $[-L - 2, L + 2]$, while plotted below is the sampling data of $x(sinc, 0)$ on the sampling set $\Gamma_C \subset [-L - 2, L + 2]$, where $L = 30$.

In Figure 5, we illustrate their best approximation in $V_{2, L}(\Phi_0)$ and solutions of the Galerkin system (6.1) with $f$ replaced by $x(\phi_0, l)$, $0 \leq l \leq 3$, respectively. We observe that given a signal in $V_2(\Phi_0)$, its Galerkin reconstruction in $V_{2, L}(\Phi_0)$ could almost match its best approximation in $V_{2, L}(\Phi_0)$, except near the boundary of the sampling interval. The boundary effect is viewable especially when $\phi_0$ has slow decay at infinity.

Given signals $x(\phi_0, l), 0 \leq l \leq 3$, let $y_L(\phi_0, l)$ be their best approximators in $V_{2, L}(\Phi_0)$, and denote by

$$e(\phi_0, l) = \|x(\phi_0, l) - y_L(\phi_0, l)\|$$

their best approximation error in $V_{2, L}(\Phi_0)$. For $\Gamma = \Gamma_N$ or $\Gamma_J$ or $\Gamma_C$, set

$$\epsilon_\Gamma(\phi_0, l) = \|z_L(\Gamma, \phi_0, l) - y_L(\phi_0, l)\|,$$
Figure 4. Plotted on the top left is the difference between the signal \( x(\text{sinc}, 0) \) in Figure 2 and its pre-reconstructed signal \( S_{\Phi_0, \Phi_0, \Gamma_N} x(\text{sinc}, 0) \), while on the top right is the difference between \( x(\text{sinc}, 0) \) and its Galerkin reconstruction \( G_{\Phi_0, \Phi_0, \Gamma_N} x(\text{sinc}, 0) \). Shown in the middle are differences \( x(\text{sinc}, 0) - S_{\Phi_0, \Phi_0, \Gamma_J} x(\text{sinc}, 0) \) (left) and \( x(\text{sinc}, 0) - G_{\Phi_0, \Phi_0, \Gamma_J} x(\text{sinc}, 0) \) (right) associated with jittered sampling. Figures in the bottom row are differences \( x(\text{sinc}, 0) - S_{\Phi_0, \Phi_0, \Gamma_C} x(\text{sinc}, 0) \) (left) and \( x(\text{sinc}, 0) - G_{\Phi_0, \Phi_0, \Gamma_C} x(\text{sinc}, 0) \) (right) associated with adaptive C-TEM sampling.

where \( z_L(\Gamma, \phi_0, l) \) is obtained from solving Galerkin system (6.1) with \( f \) replaced by \( x(\phi_0, l) \). For signals \( x(\phi_0, l), 0 \leq l \leq 3 \), and sampling sets \( \Gamma = \Gamma_N, \Gamma_J \) and \( \Gamma_C \), Galerkin reconstruction (6.1) provides quasi-optimal approximation in \( V_{2,L}(\Phi_0) \), and the quasi-optimal constant in
Figure 5. Plotted are differences between best approximations of signals $x(\phi_0, 0)$ in $V_{2.30}(\Phi_0)$ and their Galerkin reconstructions associated with operators $S_{\Phi_0, \tilde{\phi}_0, \Gamma}$, where on the above, $\phi_0 = \text{sinc}$, $\Gamma = \Gamma_N$ (left) and $\Gamma = \Gamma_J$ (right), while on the bottom $\Gamma = \Gamma_N$, $\phi_0 = \text{gauss}$ (left) and $\phi_0 = \text{spline}$ (right).

Theorem 2.3 is well behaved,

$$\frac{\| z_L(\Gamma, \phi_0, l) - x(\phi_0, l) \|}{\| y_L(\phi_0, l) - x(\phi_0, l) \|} \leq 1 + \frac{\epsilon_{\Gamma}(\phi_0, l)}{e(\phi_0, l)} \leq \frac{3}{2},$$

see Table 1 for numerical results with abbreviated notations.

Numerical stability of Galerkin reconstruction (6.1) could be reflected by the condition number $\text{cond}_{\Gamma, \Theta}(\phi_0)$ of the square matrix

$$A_{\Phi_0, \tilde{\phi}_0, \Gamma} = \left( \sum_{n=1}^{N} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_0(\gamma_n - i - \theta_i) \tilde{\phi}_0(\gamma_n - j) \right)_{-L \leq i, j \leq L}.$$

Some numerical results of condition numbers $\text{cond}_{\Gamma, \Theta}(\phi_0)$ with $\Gamma = \Gamma_N$ or $\Gamma_J$, and $\Theta = \Theta_O$ or $\Theta_I$, are presented in Table 2 with abbreviated notations. For the robust (sub-)Galerkin reconstruction, the generating
Table 1. Quasi-optimality of Galerkin reconstructions for bandlimited/Gauss/spline signals

|   | 10  | 15  | 20  | 25  | 30  |
|---|-----|-----|-----|-----|-----|
| $e(sinc,0)$ | 0.2176 | 0.1711 | 0.1388 | 0.1166 | 0.1024 |
| $\epsilon_N(sinc,0)$ | 0.0795 | 0.0668 | 0.0197 | 0.0201 | 0.0294 |
| $\epsilon_J(sinc,0)$ | 0.0770 | 0.0668 | 0.0201 | 0.0214 | 0.0290 |
| $\epsilon_C(sinc,0)$ | 0.0789 | 0.0715 | 0.0239 | 0.0263 | 0.0325 |
| $e(sinc,1)$ | 0.2600 | 0.2124 | 0.1816 | 0.1457 | 0.1303 |
| $\epsilon_N(sinc,1)$ | 0.0344 | 0.0809 | 0.0370 | 0.0294 | 0.0431 |
| $\epsilon_J(sinc,1)$ | 0.0353 | 0.0806 | 0.0372 | 0.0301 | 0.0433 |
| $\epsilon_C(sinc,1)$ | 0.0363 | 0.0831 | 0.0379 | 0.0319 | 0.0442 |
| $e(sinc,2)$ | 0.2095 | 0.1703 | 0.1365 | 0.1167 | 0.1007 |
| $\epsilon_N(sinc,2)$ | 0.0619 | 0.0618 | 0.0256 | 0.0163 | 0.0281 |
| $\epsilon_J(sinc,2)$ | 0.0596 | 0.0618 | 0.0260 | 0.0177 | 0.0275 |
| $\epsilon_C(sinc,2)$ | 0.0608 | 0.0664 | 0.0284 | 0.0226 | 0.0308 |
| $e(sinc,3)$ | 0.2655 | 0.2180 | 0.1863 | 0.1477 | 0.1322 |
| $\epsilon_N(sinc,3)$ | 0.0461 | 0.0810 | 0.0374 | 0.0258 | 0.0406 |
| $\epsilon_J(sinc,3)$ | 0.0446 | 0.0809 | 0.0375 | 0.0265 | 0.0401 |
| $\epsilon_C(sinc,3)$ | 0.0474 | 0.0837 | 0.0392 | 0.0298 | 0.0418 |
| $e(gauss,0)$ | 0.2055 | 0.1682 | 0.1398 | 0.1250 | 0.1086 |
| $\epsilon_N(gauss,0)$ | 0.0437 | 0.0515 | 0.0270 | 0.0158 | 0.0093 |
| $\epsilon_J(gauss,0)$ | 0.0439 | 0.0523 | 0.0259 | 0.0160 | 0.0096 |
| $\epsilon_C(gauss,0)$ | 0.0433 | 0.0527 | 0.0270 | 0.0181 | 0.0108 |
| $e(spline,0)$ | 0.1482 | 0.1325 | 0.1110 | 0.0924 | 0.0664 |
| $\epsilon_N(spline,0)$ | 0.0405 | 0.0298 | 0.0204 | 0.0266 | 0.0176 |
| $\epsilon_J(spline,0)$ | 0.0403 | 0.0299 | 0.0204 | 0.0281 | 0.0184 |
| $\epsilon_C(spline,0)$ | 0.0407 | 0.0292 | 0.0209 | 0.0279 | 0.0181 |

function $\tilde{\phi}_0$ of the test space $V_{2,L}(\Phi_0)$ should be so chosen that the corresponding matrice $A_{\Phi_0,\tilde{\phi}_0,\Gamma}$ is well-conditioned, cf. Theorem 2.3.

We conclude this section with two more remarks.

**Remark 6.1.** The iterative approximation-projection algorithm (5.12) could have better performance on solving Galerkin equations (6.1), especially while matrices $A_{\Phi_0,\tilde{\phi}_0,\Gamma}$ have large condition number, which is the case when the sampling set $\Gamma$ and/or the shifting set $\Theta$ are not chosen appropriately.

**Remark 6.2.** For the admissibility of the pre-reconstruction operator $S_{\Gamma,\delta}$, the test space $\tilde{U}$ must have its dimension larger than or equal to the one of the reconstruction space $U$. For $U = V_{2,L}(\Phi_0)$ and $\tilde{U} =$
Table 2. Stability of Galerkin reconstructions for nonuniform/jittered sampling

|       | L 10 | L 15 | L 20 | L 25 | L 30 |
|-------|------|------|------|------|------|
| cond<sub>N,O</sub>(sinc) | 1.2059 | 1.2367 | 1.3458 | 1.4273 | 1.2904 |
| cond<sub>N,I</sub>(sinc) | 1.9190 | 1.8946 | 1.9828 | 2.0635 | 2.0421 |
| cond<sub>N,O</sub>(gauss) | 3.0162 | 2.7000 | 2.7908 | 3.3314 | 2.8362 |
| cond<sub>N,I</sub>(gauss) | 3.2850 | 3.1447 | 3.1421 | 4.0283 | 3.4391 |
| cond<sub>N,O</sub>(spline) | 3.7677 | 3.7534 | 3.0534 | 3.1400 | 4.1708 |
| cond<sub>N,I</sub>(spline) | 4.4768 | 5.2417 | 3.3507 | 3.5354 | 5.0292 |
| cond<sub>J,O</sub>(sinc) | 1.3737 | 1.4164 | 1.4105 | 1.4149 | 1.3763 |
| cond<sub>J,I</sub>(sinc) | 1.9723 | 1.9351 | 2.3328 | 2.2037 | 2.1744 |
| cond<sub>J,O</sub>(gauss) | 2.7066 | 2.7074 | 2.6936 | 2.6957 | 2.7190 |
| cond<sub>J,I</sub>(gauss) | 3.0847 | 3.1591 | 3.0696 | 3.0197 | 3.0878 |
| cond<sub>J,O</sub>(spline) | 3.1052 | 3.2109 | 3.2218 | 3.3257 | 3.2331 |
| cond<sub>J,I</sub>(spline) | 3.5570 | 3.7388 | 3.7140 | 3.9172 | 4.1830 |

V<sub>2,L</sub>(Φ<sub>0</sub>) with ˜L ≥ L, least square solutions of the linear system (6.1) with −L ≤ j ≤ L replaced by −ˆL ≤ j ≤ ˆL defines a sub-Galerkin reconstruction ∑<sup>ˆL</sup><sub>i=−L</sub> c<sub>i</sub>φ<sub>0</sub>(· − i − θ<sub>i</sub>) ∈ V<sub>2,ˆL</sub>(Φ<sub>0</sub>) by Corollary 2.6, where f ∈ V<sub>2</sub>(Φ<sub>0</sub>) and Γ := Γ<sub>N</sub>, Γ<sub>J</sub>, Γ<sub>C</sub>. Our numerical simulations show that the above sub-Galerkin reconstructions for different ˜L ≥ L have comparable approximation errors.

7. Proofs

In this section, we include proofs of Theorems 2.3, 2.4, 3.1, 3.2, 4.2, 4.3, 5.1, 5.2 and 5.3.

7.1. Proof of Theorem 2.3. (i) For f ∈ V, we obtain from (2.1), (2.2) and (2.5) that

\[ D_1 \|Rf\| \leq \sup_{g \in \tilde{U}, \|g\| \leq 1} |\langle Sf, g \rangle| \leq D_3 \sup_{g \in \tilde{U}, \|g\| \leq 1} |\langle Sf, g \rangle| \leq D_2 D_3 \|f\|. \]

This proves numerical stability of the reconstruction operator R.

(ii) For f ∈ V and h ∈ U,

\[ \|f - Rf\| \leq \|f - h\| + \|h - Rf\| = \|f - h\| + \|R(f - h)\| \leq \frac{D_1 + D_2 D_3}{D_1} \|f - h\|, \]

where we have used the facts that R is a sub-Galerkin reconstruction and has numerical stability. Then quasi-optimality of the reconstruction operator R holds by taking infimum over h ∈ U.
7.2. Proof of Theorem 2.4. Let \( \{f_i\}_{i=1}^m \) and \( \{g_i\}_{i=1}^n \) be bases of \( U \) and \( \tilde{U} \) respectively. From the admissibility of \( S \) it follows immediately that \( (\langle Sf_i, g_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \) has full rank \( m \). Without loss of generality, we assume that \( M := ((Sf_i, g_j))_{1 \leq i,j \leq m} \) is nonsingular. Let \( \tilde{U}_* \) be the space spanned by \( \{g_j\}_{j=1}^n \). By the non-singularity of the matrix \( M \), there is a positive constant \( C_0 \) such that

\[
C_0 \|h\| \leq \sup_{g \in \tilde{U}_*, \|g\| \leq 1} \left| \langle Sh, g \rangle \right|, \quad h \in U.
\]

(7.1)

Write \( M^{-1} = (b_{ij})_{1 \leq i,j \leq m} \), and define linear operator \( R \) by

\[
Rf := \sum_{i,j=1}^m \langle Sf, g_i \rangle b_{ij} f_j, \quad f \in V.
\]

One may easily verify that \( R \) satisfies (2.3), and \( Rf \) solves the Galerkin equations

\[
\langle SRf, g \rangle = \langle Sf, g \rangle, \quad g \in \tilde{U}_*
\]

for any \( f \in V \). Therefore

\[
\sup_{g \in \tilde{U}_*, \|g\| \leq 1} \left| \langle SRf, g \rangle \right| \leq D_2 \|Rf\| \leq \frac{D_2}{C_0} \sup_{g \in \tilde{U}_*, \|g\| \leq 1} \left| \langle SRf, g \rangle \right|
\]

\[
= \frac{D_2}{C_0} \sup_{g \in \tilde{U}_*, \|g\| \leq 1} \left| \langle Sf, g \rangle \right|
\]

\[
\leq \frac{D_2}{C_0} \sup_{g \in \tilde{U}, \|g\| \leq 1} \left| \langle Sf, g \rangle \right|, \quad f \in V,
\]

by (7.1), (7.2) and the admissibility of \( S \).

7.3. Proof of Theorem 3.1. To prove Theorem 3.1, we need the following lemma.

Lemma 7.1. Let \( V_{K,p} \) and \( S_{\Gamma, \delta} \) be as in (1.6) and (1.8) respectively. Then

\[
\|S_{\Gamma, \delta}f\|_p \leq (\|K\|_W + \|\omega_\delta(K)\|_W) (1 + \|\omega_\delta(K)\|_W) \|f\|_p, \quad f \in V_{K,p}.
\]
Proof. Let \( \{ I_n \} \) be the disjoint covering of \( B(\Gamma, \delta) \) in (1.8). For \( f \in V_{K,p} \), write

\[
S_{\Gamma,\delta} f(x) = \sum_n \int_{I_n} \int_{\mathbb{R}^d} K(x, \gamma_n) K(\gamma_n, z) f(z) dz dy
\]

\[
= \sum_n \int_{I_n} \int_{\mathbb{R}^d} \left\{ K(x, y) K(y, z) + (K(x, \gamma_n) - K(x, y)) \times K(y, z) + K(x, \gamma_n)(K(\gamma_n, z) - K(y, z)) \right\} f(z) dz dy
\]

\[
\text{(7.3)} \quad =: I + II + III + IV.
\]

Observe that

\[
\|I\|_p = \left\| \int_{B(\Gamma, \delta)} K(\cdot, y) f(y) dy \right\|_p \leq \|K\|_{W} \|f\|_p,
\]

\[
\|II\|_p \leq \left\| \int_{\mathbb{R}^d} \omega_\delta(K)(\cdot, y) |f(y)| dy \right\|_p \leq \|\omega_\delta(K)\|_{W} \|f\|_p,
\]

\[
\|III\|_p \leq \left\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(\cdot, y)| \omega_\delta(K)(y, z) |f(z)| dz dy \right\|_p
\]

\[
\leq \|K\|_{W} \|\omega_\delta(K)\|_{W} \|f\|_p,
\]

and

\[
\|IV\|_p \leq \left\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \omega_\delta(K)(\cdot, y) \omega_\delta(K)(y, z) |f(z)| dz dy \right\|_p
\]

\[
\leq \|\omega_\delta(K)\|_{W}^2 \|f\|_p.
\]

Combining the above four estimates with (7.3) completes the proof. \( \square \)

We finish this subsection with proof of Theorem 3.1.

**Proof of Theorem 3.1.** The upper bound estimate (2.2) for the operator \( S_{\Gamma,\delta} \) follows immediately from Lemma 7.1.

Define

\[
T_0^* g(x) := \int_{\mathbb{R}^d} K(y, x) g(y) dy, \quad g \in L^{p/(p-1)}.
\]
For \( f \in U \) and \( g \in \tilde{U} \subset L^{p/(p-1)} \) with \( \|g\|_{p/(p-1)} \leq 1 \), we obtain
\[
|\langle S_{\Gamma,\delta} f, g \rangle - \langle f, g \rangle| \leq \left| \int_{\mathbb{R}^d \setminus B(\Gamma, \delta)} f(x)T_{\delta}^* g(x)dx \right| \\
+ \left| \sum_n \int_{I_n} f(\gamma_n)(T_{\delta}^* g)(\gamma_n) - f(x)(T_{\delta}^* g)(x)dx \right| \\
\leq \|K\|_W \|f\|_{L^p(\mathbb{R}^d \setminus B(\Gamma, \delta))} + \|\omega_{\delta}(K)\|_W (1 + \|K\|_W + \|\omega_{\delta}(K)\|_W) \|f\|_p,
\]
(7.4)
where \( \{I_n\} \) is the disjoint covering of \( B(\Gamma, \delta) \) in (1.8). This together with (3.3) and (3.4) proves the lower bound estimate (2.1) for the operator \( S_{\Gamma,\delta} \).

7.4. Proof of Theorem 3.2. Take \( f \in \mathbf{V} \). Following the argument used in Lemma 7.1, we obtain
\[
\left( \|K\|_W + \|\omega_{\delta}(K)\|_W \right)^{-1} \|S_{\Gamma,\delta} f\|_p \leq \left( \sum_n |I_n| |f(\omega_n)|^p \right)^{1/p} \leq \left( 1 + \|K\|_W + \|\omega_{\delta}(K)\|_W \right) \|f\|_p
\]
for \( 1 \leq p < \infty \), and
\[
\left( \|K\|_W + \|\omega_{\delta}(K)\|_W \right)^{-1} \|S_{\Gamma,\delta} f\|_\infty \leq \sup_n |f(\omega_n)| \leq \|f\|_\infty
\]
for \( p = \infty \). The above two estimates together with admissibility of the operator \( S_{\Gamma,\delta} \) complete the proof.

7.5. Proof of Theorem 4.2. Combining (3.3), (4.2) and (7.4), we obtain
\[
\left( \|K\|_W + \|\omega_{\delta}(K)\|_W \right)^{-1} \|S_{\Gamma,\delta} f\|_p \leq \left( \sum_n |I_n| |f(\omega_n)|^p \right)^{1/p} \leq \left( 1 + \|K\|_W + \|\omega_{\delta}(K)\|_W \right) \|f\|_p
\]
for \( 1 \leq p < \infty \), and
\[
\left( \|K\|_W + \|\omega_{\delta}(K)\|_W \right)^{-1} \|S_{\Gamma,\delta} f\|_\infty \leq \sup_n |f(\omega_n)| \leq \|f\|_\infty
\]
for \( p = \infty \). The above two estimates together with admissibility of the operator \( S_{\Gamma,\delta} \) complete the proof.

Observe from (1.9) that
\[
g_{m+1} - g_m = (I - P_{\tilde{U}} S_{\Gamma,\delta})(g_m - g_{m-1}), \quad m \geq 1.
\]
This together with (7.5) proves (4.3).

Now we prove (4.4). Taking limit in (1.9) leads to the following consistence condition
\[
P_{\tilde{U}} S_{\Gamma,\delta} g_\infty = g_0.
\]
(7.6)
Replacing \( g_0 \) in (7.6) by \( P_{\tilde{U}} S_{\Gamma,\delta} h + \tilde{g} \) gives
\[
P_{\tilde{U}} S_{\Gamma,\delta}(g_\infty - h) = \tilde{g}.
\]
This together with (7.5) completes the proof.

7.6. Proof of Theorem 4.3. Take $f \in V_{K,p}$, set $g_0 = P_{U,\tilde{U}} S_{\Gamma,\delta} f$, and let $g_\infty \in U$ be the limit of $g_m, m \geq 0$, in the iterative algorithm (1.9). The existence of such a limit follows from Theorem 4.2. Taking limit in (1.9) leads to

\begin{equation}
P_{U,\tilde{U}} S_{\Gamma,\delta} f = P_{U,\tilde{U}} S_{\Gamma,\delta} g_\infty.
\end{equation}

Then for any $g \in \tilde{U}$,

\begin{equation}
\langle S_{\Gamma,\delta} g_\infty, g \rangle = \langle P_{U,\tilde{U}} S_{\Gamma,\delta} g_\infty, g \rangle = \langle P_{U,\tilde{U}} S_{\Gamma,\delta} f, g \rangle = \langle S_{\Gamma,\delta} f, g \rangle
\end{equation}

by (4.2) and (7.7). This proves that $g_\infty$ is a solution of Galerkin equations (4.5).

Next, we show that $g_\infty$ is the unique solution of Galerkin equations (4.5). Let $h \in U$ be another solution. Then

\begin{equation}
\langle P_{U,\tilde{U}} S_{\Gamma,\delta} (h - g_\infty), g \rangle = \langle S_{\Gamma,\delta} (h - g_\infty), g \rangle = 0.
\end{equation}

This together with (3.3) implies that

\begin{equation}
P_{U,\tilde{U}} S_{\Gamma,\delta} (h - g_\infty) = 0.
\end{equation}

Recall from (7.5) that $P_{U,\tilde{U}} S_{\Gamma,\delta}$ is invertible on $U$. Then $h = g_\infty$ and the uniqueness follows.

Observe that any $f \in U$ satisfies Galerkin equations (4.5). This together with (7.8) proves that the unique solution of Galerkin equations (4.5) defines a Galerkin reconstruction.

7.7. Proof of Theorem 5.1. For $h = \sum_{i=-L}^{L} c_i \phi_i(\cdot - i) \in V_{2,L}(\Phi)$ and $g = \sum_{j=-L}^{L} d_j \tilde{\phi}_j(\cdot - j) \in V_{2,L}(\tilde{\Phi})$, we obtain

\begin{equation}
\begin{aligned}
\langle S_{\Phi,\tilde{\Phi},L} h, g \rangle &= \sum_{i,j=-L}^{L} \left( \sum_{n=1}^{N} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_i(\gamma_n - i) \langle K_{\Phi,\tilde{\Phi}}(t, \gamma_n), \tilde{\phi}_j(t - j) \rangle \right) c_i d_j \\
&= \sum_{i,j=-L}^{L} \left( \sum_{n=1}^{N} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_i(\gamma_n - i) \tilde{\phi}_j(\gamma_n - j) \right) c_i d_j \\
(7.9) &= c^T A_{\Phi,\tilde{\Phi},L} d,
\end{aligned}
\end{equation}

where $c = (c_i)_{-L \leq i \leq L}$ and $d = (d_j)_{-L \leq j \leq L}$. By the invertibility assumption on $A_{\Phi,\tilde{\Phi}}$, $\{\phi_i(\cdot - i), -L \leq i \leq L\}$ and $\{\tilde{\phi}_i(\cdot - i), -L \leq i \leq L\}$ are Riesz bases of $V_{2,L}(\Phi)$ and $V_{2,L}(\tilde{\Phi})$ respectively. This together with (7.9) proves the desired equivalent statements.
7.8. **Proof of Theorem 5.2.** The sufficiency is obvious. Now we prove the necessity. Suppose, to the contrary, that $A_{\Phi, \tilde{\Phi}, L}$ in (5.10) is singular. Take a nonzero vector $e = (e_i)_{-L \leq i \leq L}$ in the null space $N((A_{\Phi, \tilde{\Phi}, L})^T)$ and a nonzero linear functional $\mathcal{J}$ on $V_2(\Phi)$ such that $\mathcal{J}(h) = 0$ for all $h \in V_2(\Phi)$. Define

$$Q(f) := \mathcal{J}(f) \sum_{-L \leq i \leq L} e_i \phi_i(\cdot - i), \quad f \in V_2(\Phi).$$

Then $Q$ is a nonzero linear operator from $V_2(\Phi)$ to $V_2(\Phi, \tilde{\Phi})$.

$$Qh = 0, \quad h \in V_2(\Phi),$$

and

$$\langle Qf, g \rangle = \mathcal{J}(f) \sum_{-L \leq i,j \leq L} e_i \langle \phi_i(\cdot - i), \tilde{\phi}_j(\cdot - j) \rangle d_j = 0,$$

where $g = \sum_{-L \leq j \leq L} d_j \tilde{\phi}_j(\cdot - j) \in V_2(\Phi)$. This contradicts the uniqueness of oblique projections.

7.9. **Proof of Theorem 5.3.** Write $g_m = \sum_{-L \leq i \leq L} c_m(i) \phi_i(\cdot - i)$ and set $c_m = (c_m(i))_{-L \leq i \leq L}$. Then we can reformulate the iterative algorithm (5.12) as

$$c_{m+1}^T = c_m^T - c_m^T A_{\Phi, \tilde{\Phi}, L}(A_{\Phi, \tilde{\Phi}, L})^{-1} + c_0^T, \quad m \geq 0.$$

This together with (5.13) proves the desired conclusions.

**Appendix A. Reproducing kernel spaces for FRI signals to live in**

In this appendix, we show that the space $V_2(\Phi)$ for FRI signals to live in is, in fact, the range space of some idempotent integral operator.

**Theorem A.1.** Let $\Phi$ and $\tilde{\Phi}$ satisfy (5.2), and the correlation matrix $A_{\Phi, \tilde{\Phi}}$ in (5.3) have bounded inverse on $\ell^2$. Then

$$V_2(\Phi) = V_{K_{\Phi, \tilde{\Phi}}},$$

for the kernel $K_{\Phi, \tilde{\Phi}}$ in (5.4), which satisfies (3.1) and (3.2).

Let $\mathcal{C}_1$ contain all infinite matrices $A := (a_{ij})_{i,j \in \mathbb{Z}}$ with

$$\|A\|_{\mathcal{C}_1} : = \sum_{k \in \mathbb{Z}} \left( \sup_{i-j=k} |a_{ij}| \right) < \infty.$$

To prove Theorem A.1, we recall Wiener’s lemma for the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$, see [8, 20, 22, 33, 35, 37] and references therein.
Lemma A.2. If \( A \in \mathcal{C}_1 \) has bounded inverse on \( \ell^2 \), then its inverse \( A^{-1} \) belongs to \( \mathcal{C}_1 \) too.

Proof of Theorem A.1. By direct calculation, we have
\[
\|A_{\Phi, \tilde{\Phi}}\|_{\mathcal{C}_1} \leq \|\Phi\|_{W^1} \|\tilde{\Phi}\|_{W^1}.
\]
Thus the inverse of the correlation matrix \( A_{\Phi, \tilde{\Phi}} \) belongs to the Baskakov-Gohberg-Sjöstrand class by Lemma A.2. One may then verify immediately that the kernel \( K_{\Phi, \tilde{\Phi}} \) in (5.4) satisfies all requirements of the theorem. \( \square \)

Acknowledgement The authors thank professor Ben Adcock for his comments and suggestions. The project is partially supported by the National Natural Science Foundation of China (Nos. 11201094 and 11161014), Guangxi Natural Science Foundation (2014GXNSF-BA118012), Program for Innovative Research Team of Guilin University of Electronic Technology (Differential Equation and Dynamic System, Computer Software), Guangxi Key Laboratory of Cryptography and Information Security, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, and the National Science Foundation (DMS-1109063 and DMS-1412413).

References
[1] B. Adcock, M. Gataric and A. C. Hansen, On stable reconstruction from nonuniform Fourier measurements, *SIAM J. Imaging Sci.*, 7(2014), 1690–1723.
[2] B. Adcock, M. Gataric and A. C. Hansen, Weighted frames of exponentials and stable recovery of multidimensional functions from nonuniform Fourier samples, *Appl. Comput. Harmon. Anal.*, in Press, doi:10.1016/j.acha.2015.09.006
[3] B. Adcock, A. C. Hansen and C. Poon, Beyond consistent reconstructions: optimality and sharp bounds for generalized sampling, and application to the uniform resampling problem, *SIAM J. Math. Anal.*, 45(2013), 3114–3131.
[4] A. Aldroubi and H. Feichtinger, Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the \( L_p \) theory. *Proc. Amer. Math. Soc.*, 126(1998), 2677–2686.
[5] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, 43(2001), 585–620.
[6] A. Aldroubi, Q. Sun and W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, *Constr. Approx.*, 20(2004), 173–189.
[7] A. Aldroubi, Q. Sun and W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.*, 11(2005), 215–244.
A. G. Baskakov, Wiener’s theorem and asymptotic estimates for elements of inverse matrices, *Funktsional Anal i Prilozhen*, 24(1990), 64–65; translation in *Funct. Anal. Appl.*, 24(1990), 222–224.

P. Berger and K. Gröchenig, Sampling and reconstruction in different subspaces by using oblique projections, arXiv: 1312.1717

C. Cheng, Y. Jiang and Q. Sun, Spatially distributed sampling and reconstruction, arXiv: 1511.08541

J. G. Christensen, Sampling in reproducing kernel Banach spaces in Lie group, *J. Approx. Theory*, 164(2012), 179–203.

O. Christensen and T. Strohmer, The finite section method and problems in frame theory, *J. Approx. Theory*, 133(2005), 221–237.

P. L. Dragotti, M. Vetterli and T. Blu, Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Stranks-Fix, *IEEE Trans. Signal Process.*, 55(2007), 1741–1757.

T. G. Dvorkind, Y. C. Eldar and E. Matusiak, Nonlinear and nonideal sampling: theory and methods, *IEEE Trans. Signal Process.*, 56(2008), 5874–5890.

Y. C. Eldar and T. Werther, General framework for consistent sampling in Hilbert spaces, *Int. J. Wavelets Multiresolution Inf. Process.*, 3(2005), 347–359.

H. G. Feichtinger and K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, *SIAM J. Math. Anal.*, 231(1992), 244–261.

H. G. Feichtinger, K. Gröchenig and T. Strohmer, Efficient numerical methods in non-uniform sampling theory, *Numer. Math.*, 69(1995), 423–440.

H. G. Feichtinger, J. C. Principe, J. L. Romero, A. A. Singh, and G. A. Alexander, Approximate reconstruction of bandlimited functions for the integrate and fire sampler, *Adv. Comput. Math.*, 36(2012), 67–78.

A. G. Garcia and A. Portal, Sampling in reproducing kernel Banach spaces, *Mediterr. J. Math.*, 103(2013), 1401–1417.

I. Gohberg, M. A. Kaashoek and H. J. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equation Oper. Theory*, 12(1989), 343–382.

D. Gontier and M. Vetterli, Sampling based on timing: time encoding machines on shift-invariant subspaces, *Appl. Comput. Harmon. Anal.*, 36(2014), 63–78.

K. Gröchenig, Wiener’s lemma: theme and variations, an introduction to spectral invariance and its applications, In: *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, editor by P. Massopust and B. Forster, Birkhäuser, Boston, 2010.

K. Gröchenig, Reconstructing algorithms in irregular sampling, *Math. Comput.*, 59(1992), 181–194.

D. Han, M. Z. Nashed and Q. Sun, Sampling expansions in reproducing kernel Hilbert and Banach spaces, *Numer. Funct. Anal. Optim.*, 30(2009), 971–987.

J. A. Hogan and J. D. Lakey, *Duration and Bandwidth Limiting: Prolate Functions, Sampling, and Applications*, Birkhäuser, 2012.
[26] P. Jaming, A. Karoui, R. Kerman and S. Spektor, Approximation of almost time and band limited functions I: Hermite expansion, arXiv: 1407.1293
[27] A. A. Lazar and L. T. Toth, Perfect recovery and sensitivity analysis of time encoded bandlimited signals, IEEE Trans. Circuits System, 51(2004), 2060–2073.
[28] M. Mishali, Y. C. Eldar and A. J. Elron, Xampling: signal acquisition and processing in union of subspaces, IEEE Trans. Signal Process., 59(2011), 4719–4734.
[29] M. Z. Nashed and Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(R^d)$, J. Funct. Anal., 258(2010), 2422–2452.
[30] M. Z. Nashed and G. G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces, Math. Control Signals Systems, 4(1991), 363–390.
[31] H. Pan, T. Blu and P. L. Dragotti, Sampling curves with finite rate of innovation, IEEE Trans. Signal Process., 62(2014), 458–471.
[32] C. E. Shannon, Communication in the presence of noise, Proc. IRE, 37(1949), 10–21.
[33] J. Sjöstrand, Wiener type algebra of pseudodifferential operators, Cent. Math., Ecole Polytechnique, Palaiseau France, Seminaire 1994, 1995, December 1994.
[34] Q. Sun, Non-uniform average sampling and reconstruction for signals with finite rate of innovations, SIAM J. Math. Anal., 38(2006), 1389–1422.
[35] Q. Sun, Wiener’s lemma for infinite matrices, Trans. Amer. Math. Soc., 359(2007), 3099–3123.
[36] Q. Sun, Frames in spaces with finite rate of innovation, Adv. Comput. Math., 28(2008), 301–329.
[37] Q. Sun, Wiener’s lemma for infinite matrices II, Constr. Approx., 34(2011), 209–235.
[38] Q. Sun, Localized nonlinear functional equations and two sampling problems in signal processing, Adv. Comput. Math., 40(2014), 415–458.
[39] Q. Sun and J. Xian, Rate of innovation for (non-)periodic signals and optimal lower stability bound for filtering, J. Fourier Anal. Appl., 20(2014), 119–134.
[40] W. Sun and X. Zhou, Reconstruction of bandlimited signals from local averages, IEEE Trans. Inf. Theory, 48(2002), 2955–2963.
[41] W.-S. Tang, Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces, Proc. Amer. Math. Soc., 128(1999), 463–473.
[42] M. Unser, Sampling – 50 years after Shannon, Proc. IEEE, 88(2000), 569–587.
[43] M. Vetterli, P. Marziliano and T. Blu, Sampling signals with finite rate of innovation, IEEE Trans. Signal Process., 50(2002), 1417–1428.
[44] Z. Wang and A. C. Bovik, Mean squared error: love it or leave it?- A new look at signal fidelity measures, IEEE Signal Process. Mag., 98(2009), 98C117.
[45] J. M. Whittaker, Interpolating Function Theory, Cambridge University Press, London, 1935.

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