FILTRATIONS OF FREE GROUPS ARISING FROM THE LOWER CENTRAL SERIES

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To the memory of O.V. Melnikov

Abstract. We make a systematic study of filtrations of a free group $F$ defined as products of powers of the lower central series of $F$. Under some assumptions on the exponents, we characterize these filtrations in terms of the group algebra, the Magnus algebra of non-commutative power series, and linear representations by upper-triangular unipotent matrices. These characterizations generalize classical results of Grün, Magnus, Witt, and Zassenhaus from the 1930’s, as well as later results on the lower $p$-central filtration and the $p$-Zassenhaus filtrations. We derive alternative recursive definitions of such filtrations, extending results of Lazard. Finally, we relate these filtrations to Massey products in group cohomology.

1. Introduction

Given a group $G$, we denote its lower central filtration by $G^{(n,0)}$, $n = 1, 2, \ldots$. Thus $G^{(1,0)} = G$ and $G^{(n+1,0)} = [G^{(n,0)}, G]$ for $n \geq 1$. Let $F = F_X$ be the free group on a finite basis $X$. Classical results from the 1930’s give the following three alternative descriptions of $F^{(n,0)}$.

(I)° Power series description (Magnus [Mag37 p. 111]): Let $\mathbb{Z} \langle \langle X \rangle \rangle$ be the ring of all formal power series over the set $X$ of non-commuting variables and with coefficients in $\mathbb{Z}$. Let $\mathfrak{d}_\mathbb{Z}$ be the ideal in $\mathbb{Z} \langle \langle X \rangle \rangle$ generated by $X$, and define the Magnus homomorphism $\mu_\mathbb{Z}: F \to \mathbb{Z} \langle \langle X \rangle \rangle^\times$ by $\mu_\mathbb{Z}(x) = 1 + x$ for $x \in X$. Then $F^{(n,0)} = \mu_\mathbb{Z}^{-1}(1 + \mathfrak{d}_\mathbb{Z}^n)$.

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(II)

Group algebra description ([Mag37] and Witt [Wit37]): $F^{(n,0)} = F \cap (1 + c^n_Z)$, where $c_Z$ is the augmentation ideal in the group algebra $\mathbb{Z}[F]$.

(III)

Description by unipotent matrices (Gr"un [Gr"u36]; see also Magnus [Mag35]): $F^{(n,0)}$ is the intersection of all kernels of homomorphisms $F \to \mathbb{U}_n(\mathbb{Z})$, where $\mathbb{U}_n(\mathbb{Z})$ is the group of all upper-triangular unipotent $n \times n$ matrices over $\mathbb{Z}$ (this is somewhat implicit in Gr"un’s paper, and we refer to R"ohl [R"oh85] for a more modern exposition, where this is shown for lower-triangular matrices).

For a detailed history of these results see [CM82]. These descriptions of $F^{(n,0)}$ have natural analogs in the context of free profinite groups, where one considers free profinite groups, complete group rings, and continuous homomorphisms.

Next, for a prime number $p$, one defines the lower $p$-central filtration $G^{(n,p)}$, $n = 1, 2, \ldots$, of a group $G$ by

$$G^{(n,p)} = \prod_{i=1}^{n} (G^{(i,0)})^{p^{n-i}}.$$ 

Alternatively, it has the following inductive definition [NSW08 Prop. 3.8.6]:

$$G^{(1,p)} = G, \quad G^{(n+1,p)} = (G^{(n,p)})^p \langle [G^{(n,p)}, G] \rangle \text{ for } n \geq 1.$$ 

For a free group $F = F_X$ this filtration has the following analogs of (I)- (III):

(I) As shown by Skopin [Sko50] (for $p \neq 2$) and Koch [Koc60],

$$F^{(n,p)} = \mu_Z^{-1} (1 + (p\mathbb{Z}\langle X \rangle + \mathfrak{d}_Z)^n).$$

(II) By a result of Koch [Koc02 Th. 7.14] (who however works in a pro-$p$ context),

$$F^{(n,p)} = F \cap (1 + (p\mathbb{Z}[F] + c_Z^n)).$$

(III) $F^{(n,p)}$ is the intersection of all kernels of homomorphisms $F \to \mathbb{U}_{d+1}(\mathbb{Z}/p^{n-d}\mathbb{Z})$, where $1 \leq d \leq n - 1$ (Mináč and Tân [MT15, §2]); Alternatively, $F^{(n,p)}$ is the intersection of all kernels of homomorphisms $F \to G(n, p)$, where $G(n, p)$ is the group of all upper-triangular unipotent $n \times n$-matrices $(c_{ij})$ over $\mathbb{Z}/p^n\mathbb{Z}$ such that $c_{ij} \in p^{j-i} \mathbb{Z}/p^n\mathbb{Z}$ for every $i \leq j$ [Efr14b].

A third well-studied filtration of a group $G$ of a similar nature is the $p$-Zassenhaus filtration $G_{(n,p)}$, $n = 1, 2, \ldots$, where $p$ is a prime number (see
Here \[ G(n,p) = \prod_{i=1}^{n} (G(i,0))^p^{[\log_p(n/i)]} = \prod_{ip' \geq n} (G(i,0))^{p'} \].

By a result of Lazard, it has the following alternative inductive definition (see [Laz65, p. 209, (3.14.5)], [DDMS99, Th. 11.2]):

\[ G(1,p) = G, \quad G(n,p) = (G([n/p],p))^p \prod_{i+j=n} [G(i,p), G(j,p)] \text{ for } n \geq 2. \]

For this filtration one has the following analogs of (I)\(^0\)–(III)\(^0\):

(I)\(_p\) In the pro-

p case \(F(n,p) = \mu_{F_p}^{-1}(1 + \mathfrak{d}_{F_p}^{e(n,p)})\), where \(\mu_{F_p}: F \to F_p\langle\langle X\rangle\rangle\) and \(\mathfrak{d}_{F_p}\) are defined similarly to (I)\(^0\) (see Morishita [Mor12, §8.3]), Vogel [Vog05, Th. 2.19(ii)], or [Efr14a, Prop. 6.2]).

(II)\(_p\) One has \(G(n,p) = G \cap (1 + c_{F_p}^{e(n,p)})\), where \(c_{F_p}\) denotes the augmentation ideal in the group \(F_p\)-algebra \(F_p[G]\) of \(G\). This was proved by Jennings and Brauer [Jen41, Th. 5.5] for a finite \(p\)-group \(G\), and was extended to arbitrary groups by Lazard [Laz54, Cor. 6.10] (see also Quillen [Qui68, Th. 2.4]).

(III)\(_p\) \(F(n,p)\) is the intersection of all kernels of homomorphisms \(\varphi: F \to \mathbb{U}_{d+1}(\mathbb{Z}/e(n,d)\mathbb{Z})\), with \(1 \leq d \leq n - 1\). This generalizes (III)\(^0\), (III)\(_p\), (III)\(_p\).
Finally, in §6–§8 we study inductive definitions of the above subgroups \( \prod_{i=1}^{n} (G^{(i,0)})^{e(n,i)} \), as we had for the motivating examples \( G^{(n,0)} \), \( G^{(n,p)} \), \( G_{(n,p)} \). After developing some general machinery in §6, we generalize in §7 the inductive definition of the lower central series \( G^{(n,0)} \), and lower \( p \)-central series \( G^{(n,p)} \), to the following situation: for a sequence \( A = (a_i)_{i=1}^{\infty} \) of non-negative integers we define the \( A \)-filtration \( G^{(n,A)} \), \( n = 1, 2, \ldots \), of \( G \) by \( G^{(1,A)} = G \) and \( G^{(n+1,A)} = (G^{(n,A)})^{a_n} [G^{(n,A)}, G] \) for \( n \geq 1 \). We then prove in Theorem 7.4 that

\[
G^{(n,A)} = \prod_{i=1}^{n} (G^{(i,0)})^{e(n,i)},
\]

where \( e(n,i) \) is the gcd of all products of \( n - i \) elements from \( a_1, \ldots, a_{n-1} \) (and \( e(n,n) = 1 \)). Note that taking \( a_i = 0 \) or \( a_i = p \) for every \( i \), recovers the inductive definitions of \( G^{(n,0)} \), \( G^{(n,p)} \), respectively.

In §8 we use the general machinery of §6 to extend Lazard’s description of the \( p \)-Zassenhaus filtration, by showing that for a \( p \)-power \( q \) one has

\[
G^{(n,q)} = \prod_{i=1}^{n} (G^{(i,0)})^{q^{\lceil \log_2(n/i) \rceil}},
\]

where the \( q \)-Zassenhaus filtration \( G^{(n,q)} \) is defined inductively by \( G^{(1,q)} = 1 \) and \( G^{(n,q)} = (G^{([n/p],q)})^q \prod_{i+j=n} [G^{(i,q)}, G^{(j,q)}] \) (Theorem 8.3).

Finally, in §9 we investigate these filtrations from a cohomological viewpoint, and relate them to Massey products in \( H^2 \). As a new contribution to the classical theory of Magnus, Witt and Zassenhaus, we apply our techniques to compute the subgroup of \( H^2(F/F^{(n,0)}, \mathbb{Z}) \) generated by the \( n \)-fold Massey products corresponding to words in the alphabet \( X \). Specifically, we show that it is a free \( \mathbb{Z} \)-module whose rank is the necklace function of \( n \) and \( |X| \); see Corollary 9.4 for details.

While throughout the paper we work in the context of discrete groups, many of our main results have quite straightforward profinite analogs. The study of such filtrations in the profinite context has become increasingly interesting in recent years in infinite Galois theory – see for instance [BT12], [CEM12], [EM11], [EM16], [MT15], [Mor04], [Top12], [Vog05].

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2. Graded Lie algebras

We fix a set $X$. For a commutative unital ring $R$ let $R\langle X \rangle$ be the free associative $R$-algebra on $X$, i.e., the algebra of non-commuting polynomials in elements of $X$ and with coefficients in $R$. It is graded by total degree, and we denote by $R\langle X \rangle^{(n)}$ its homogenous component of degree $n$. Thus $R\langle X \rangle \cong \bigoplus_{n=0}^{\infty} R\langle X \rangle^{(n)}$ as additive groups. We also consider the ring $R\langle \langle X \rangle \rangle = \prod_{n=0}^{\infty} R\langle X \rangle^{(n)}$. When $X$ is finite, $R\langle \langle X \rangle \rangle$ is the ring of formal power series over the set $X$ of non-commuting variables and coefficients in $R$. Given $\alpha \in R\langle X \rangle$ or $\alpha \in R\langle \langle X \rangle \rangle$, we denote the homogenous component of $\alpha$ of degree $n$ by $\alpha^{(n)}$.

Let $\mathfrak{d}_R = \langle X \rangle$ be the ideal in $R\langle \langle X \rangle \rangle$ generated by $X$. We observe:

**Lemma 2.1.** If $\alpha \in \mathfrak{d}_R$, then $\sum_{k=0}^{\infty} \alpha^k$ is a well-defined element of $1 + \mathfrak{d}_R$, and is the unique two-sided inverse of $1 - \alpha$. Consequently, if $\mathfrak{a} \subseteq \mathfrak{d}_R$ is an ideal of $R\langle \langle X \rangle \rangle$, then $1 + \mathfrak{a}$ is a subgroup of $R\langle \langle X \rangle \rangle^\times$.

As with any associative algebra, we consider $R\langle X \rangle$ and $R\langle \langle X \rangle \rangle$ also as Lie $R$-algebras with the Lie brackets $[\alpha, \beta] = \alpha\beta - \beta\alpha$. This makes $R\langle X \rangle$ a graded Lie $R$-algebra.

Let $L_{X,R}$ be the free Lie $R$-algebra on $X$. It has a natural grading [Ser92, p. 19], and we denote the $n$-th homogenous component of $L_{X,R}$ by $L_{X,R}^{(n)}$. The universal property of $L_{X,R}$ gives rise to a natural graded Lie algebra homomorphism $\phi_R: L_{X,R} \to R\langle X \rangle$ which is the identity on $X$. The algebra $R\langle X \rangle$ may be identified with the universal algebra of the Lie algebra $L_{X,R}$ via this map, and $L_{X,R}$ is a free $R$-module [Ser92, Part I, Ch. IV, Th. 4.2(1)(3)].

Now let $F = F_X$ be the free group on the basis $X$, and $F^{(n,0)}$, $n = 1, 2, \ldots$, its lower central filtration of $F$ (see the Introduction). The quotients $F^{(n,0)}/F^{(n+1,0)}$ are commutative. Therefore we have a graded $\mathbb{Z}$-module $\text{gr}(F) = \bigoplus_{n=1}^{\infty} F^{(n,0)}/F^{(n+1,0)}$. Moreover, the commutator map induces on $\text{gr}(F)$ a structure of a graded Lie $\mathbb{Z}$-algebra. The identity map on $X$ induces a graded Lie algebra homomorphism $L_{X,\mathbb{Z}} \to \text{gr}(F)$. It is surjective in degree 1, and by induction, in all degrees (in fact, it is an isomorphism [Ser92, Part I, Ch. IV, Th. 6.1], but we will not need this fact).

We write $R[F]$ for the group $R$-algebra of $F = F_X$. Let $\mathfrak{c}_R$ be its augmentation ideal, i.e., the ideal generated by all elements of the form $g - 1$ with $g \in F$. We identify $F$ as a subset of $R[F]$.
Lemma 2.2. Let \( c \) be an ideal in \( R[F] \). Then \( F \cap (1 + c) \) is a subgroup of \( F \).

**Proof.** The identity \( \alpha^{-1} = 1 - \alpha^{-1}(\alpha - 1) \) shows that \( F \cap (1 + c) \) is closed under inversion. The rest is immediate. \( \square \)

For every \( x \in X \) the element \( 1 + x \) of \( R\langle\langle X\rangle\rangle \) is invertible, by Lemma 2.1. Let

\[
\mu_R: F \to 1 + \mathfrak{d}_R \leq R\langle\langle X\rangle\rangle^\times, \quad x \mapsto 1 + x \quad \text{for} \quad x \in X,
\]

be the Magnus homomorphism. It extends canonically to an \( R \)-algebra homomorphism \( \mu_R: R[F] \to R\langle\langle X\rangle\rangle \). We note that

\[
(2.1) \quad \mu_R(c_R) \subseteq d_R.
\]

There are inclusions

\[
(2.2) \quad F^{(n,0)} \subseteq F \cap (1 + c_R^n) \subseteq \mu_R^{-1}(1 + \mathfrak{d}_R^n).
\]

For \( R = \mathbb{Z} \), the classical results of Magnus and Witt ([Mag35], [Mag37], [Wit37]) show that these are equalities:

\[
(2.3) \quad F^{(n,0)} = F \cap (1 + c_R^n) = \mu_R^{-1}(1 + \mathfrak{d}_R^n).
\]

For every \( n \geq 1 \) we have a group homomorphism \( \mu_R^{(n)}: F^{(n,0)}/F^{(n+1,0)} \to R\langle\langle X\rangle\rangle^{(n)} = R\langle X\rangle^{(n)} \) given by \( \mu_R^{(n)}(gF^{(n+1,0)}) = \mu_R(g)^{(n)} \) for \( g \in F^{(n,0)} \). For \( g \in F^{(n,0)} \) and \( h \in F^{(m,0)} \) we have as in [Ser92, p. 25, (\*)], \( [g,h] \in F^{(n+m,0)} \) and

\[
\mu_R([g,h])^{(n+m)} = \mu_R(g)^{(n)}\mu_R(h)^{(m)} - \mu_R(h)^{(m)}\mu_R(g)^{(h)}.
\]

Therefore the group homomorphisms \( \mu_R^{(n)} \) combine to a graded Lie \( R \)-algebra homomorphism

\[
\text{gr} \mu_R = \bigoplus_{n=1}^{\infty} \mu_R^{(n)}: \text{gr}(F) \to R\langle X\rangle.
\]

The triangle of graded Lie \( \mathbb{Z} \)-algebra homomorphisms

\[
(2.4) \quad L_{X,\mathbb{Z}} \xrightarrow{\phi_\mathbb{Z}} \mathbb{Z}\langle X\rangle \xrightarrow{\text{gr} \mu_\mathbb{Z}} \text{gr}(F)
\]

commutes on every \( x \in X \), and hence on all of \( L_{X,\mathbb{Z}} \).

Let \( \Delta_R: R\langle X\rangle \to R\langle X\rangle \otimes_R R\langle X\rangle \) be the \( R \)-algebra homomorphism induced by the diagonal embedding \( \lambda \mapsto (\lambda, \lambda) \) for \( \lambda \in L_{X,R} \) (compare
When $R$ is torsion-free as a $\mathbb{Z}$-module, \cite[Part I, Ch. III, Th. 5.4]{Ser92} gives an exact sequence of $R$-modules
\begin{equation}
L_{X,R} \xrightarrow{\phi_R} R\langle X \rangle \to R\langle X \rangle \otimes_R R\langle X \rangle,
\end{equation}
where the right homomorphism is given by $\lambda \mapsto \Delta_R(\lambda) - \lambda \otimes 1 + 1 \otimes \lambda$.

**Lemma 2.3.** For $R = \mathbb{Z}$, the cokernel of $\text{gr} \mu_\mathbb{Z} : \text{gr}(F) \to \mathbb{Z}\langle X \rangle$ is torsion-free as a $\mathbb{Z}$-module.

**Proof.** By (2.4), this cokernel coincides with the cokernel $\mathbb{Z}\langle X \rangle / \phi_\mathbb{Z}(L_{X,Z})$ of $\phi_\mathbb{Z}$. By (2.5), the latter cokernel is a submodule of the torsion-free $\mathbb{Z}$-module $\mathbb{Z}\langle X \rangle \otimes_\mathbb{Z} \mathbb{Z}\langle X \rangle$. Hence it is also torsion-free. \hfill $\square$

**Corollary 2.4.** Suppose that either $R = \mathbb{Z}$ or $R$ is a field. Then the cokernel of $\mu_R^{(n)} : F^{(n,0)}/F^{(n+1,0)} \to R\langle X \rangle^{(n)}$ is a torsion-free $R$-module.

**Proof.** The case $R = \mathbb{Z}$ follows from Lemma 2.3. The case where $R$ is a field follows from the general structure theory of $R$-linear spaces. \hfill $\square$

### 3. Multiplicatively descending maps

**Definition 3.1.** We say that a map
e : \{(n, i) \in \mathbb{Z}^2 \mid 1 \leq i \leq n\} \to \mathbb{Z}_{\geq 0}

is multiplicatively descending if it satisfies the following conditions:

(i) $e(n, n) = 1$ for every $n$;

(ii) $e(n, i) \in e(n, i + 1)\mathbb{Z}$ for every $1 \leq i < n$.

**Example 3.2.** The trivial multiplicatively descending map is defined by $e(n, i) = 0$ for $1 \leq i < n$, and $e(n, n) = 1$.

**Example 3.3.** Let $(a_i)_{i=1}^\infty$ be a sequence of non-negative integers. We define a multiplicatively descending map by setting for $1 \leq i < n$,

$$e(n, i) = \gcd\left\{\prod_{j \in J} a_j \mid J \subseteq \{1, 2, \ldots, n - 1\}, \ |J| = n - i \right\},$$

and setting $e(n, n) = 1$.

**Example 3.4.** In the previous example take the constant sequence $(a)_{i=1}^\infty$, where $a$ is a non-negative integer. We obtain that $e(n, i) = a^{n-i}$ is a multiplicatively descending map. For $a = 0$ (and $e(n, n) = 1$) this recovers the trivial multiplicatively descending map.
Example 3.5. Let $t$ be a positive integer and $p$ a prime number. For integers $1 \leq i \leq n$, let $j(n, i) = \lceil \log_p \left( \frac{n}{i} \right) \rceil$. Then $e(n, i) = p^{j(n, i)}$ is a multiplicatively descending map. Indeed, (i) is immediate, and condition (ii) follows from $j(n, i) \geq j(n, i + 1)$.

Definition 3.6. We call a multiplicatively descending map $e$ binomial if for every positive integers $n, i, l$ such that $l \leq e(n, i)$ and $il \leq n$ one has $(\binom{e(n, i)}{l}) \in e(n, il) \mathbb{Z}$.

For example, the trivial multiplicatively descending map is clearly binomial. A useful way to verify this property in more involved situations is given by the next lemma. For a prime number $p$ let $v_p$ be the $p$-adic valuation on $\mathbb{Z}$.

Lemma 3.7. Let $e$ be a multiplicatively descending map satisfying the following condition:

(iii) For every positive integers $n, i, r$ and a prime number $p$ such that $ip^r \leq n$, if $v_p(e(n, i)) \geq r$, then $v_p(e(n, i)) - r \geq v_p(e(n, ip^r))$.

Then $e$ is binomial.

Proof. Let $n, i, l$ be as in Definition 3.6. It suffices to show that for every prime number $p$,

$$v_p \left( \binom{e(n, i)}{l} \right) \geq v_p(e(n, il)).$$

To this end let $r = v_p(l)$ and $s = v_p(e(n, i))$.

When $s < r$ we have $ip^s < ip^r \leq il \leq n$, so by (iii) (with $r$ replaced by $s$), $v_p(e(n, ip^s)) = 0$. As $ip^r \leq il \leq n$, conditions (iii), and (ii), respectively, therefore give

$$v_p \left( \binom{e(n, i)}{l} \right) \geq s - r \geq v_p(e(n, ip^r)) \geq v_p(e(n, il))$$

in this case as well. \qed

Example 3.8. The map of Example 3.3 is binomial. Indeed, take a sequence $(a_i)_{i=1}^{n}$ of non-negative integers, a prime number $p$, and $1 \leq i \leq n - 1$. We first show that if $v_p(e(n, i)) \geq 1$, then $v_p(e(n, i+1)) < v_p(e(n, i))$. Indeed, there exists a subset $J$ of $\{1, 2, \ldots, n - 1\}$ of size $n - i$ such
that \( v_p(e(n, i)) = v_p(\prod_{j \in J} a_j) \). Further, there exists \( j_0 \in J \) such that \( v_p(a_{j_0}) \geq 1 \). Then
\[
v_p(e(n, i)) > v_p(\prod_{j \in J \setminus \{j_0\}} a_j) \geq v_p(e(n, i + 1)).
\]
Iterating this, we obtain that if \( v_p(e(n, i)) \geq r \geq 0 \) and \( i + r \leq n \), then
\[
v_p(e(n, i)) - r \geq v_p(e(n, i + r)).
\]
By (ii), this implies that \( v_p(e(n, i)) - r \geq v_p(e(n, ip^r)) \) whenever \( ip^r \leq n \). This proves (iii). We now apply Lemma 3.7.

**Example 3.9.** The map of Example 3.5 is binomial. Indeed, consider a prime number \( p \) and integers \( t, n, i, r \geq 1 \). Suppose that \( v_p(e(n, i)) \geq r \). Therefore
\[
v_p(e(n, i)) - r = t\left[\log_p \frac{n}{i}\right] - r \geq t\left[\log_p \frac{n}{ip^r}\right] = v_p(e(n, ip^r)).
\]
Thus condition (iii) holds, and we use again Lemma 3.7.

### 4. Powers of the Lower Central Series

As before, let \( X \) be a fixed set, let \( R \) be a unital commutative ring, and let \( F = F_X \) be the free group on basis \( X \). For a multiplicatively descending map \( e \), we consider the ideals
\[
\mathfrak{c}^{(e, n)}_R = \sum_{i=1}^{n} e(n,i)\mathfrak{c}^i_R, \quad \mathfrak{d}^{(e, n)}_R = \sum_{i=1}^{n} e(n,i)\mathfrak{d}^i_R
\]
of \( R[F], \ R\langle\langle X\rangle\rangle \), respectively. We record the following alternative description of \( \mathfrak{d}^{(e, n)}_R \):

**Lemma 4.1.** One has \( \mathfrak{d}^{(e, n)}_R = \bigcap_{d=1}^{n} (e(n,d)\mathfrak{d}_R + \mathfrak{d}^{d+1}_R) \).

*Proof.* We denote the ideal \( \bigcap_{d=1}^{n} (e(n,d)\mathfrak{d}_R + \mathfrak{d}^{d+1}_R) \) by \( J \).

Let \( 1 \leq i, d \leq n \). If \( 1 \leq i \leq d \), then \( e(n,i) \in e(n,d)\mathbb{Z} \), so \( e(n,i)\mathfrak{d}^i_R \subseteq e(n,d)\mathfrak{d}_R \). If \( d < i \leq n \), then \( e(n,i)\mathfrak{d}^i_R \subseteq \mathfrak{d}^{d+1}_R \). Thus in both cases, \( e(n,i)\mathfrak{d}^i_R \subseteq e(n,d)\mathfrak{d}_R + \mathfrak{d}^{d+1}_R \). Since \( i, d \) were arbitrary, \( \mathfrak{d}^{(e, n)}_R \subseteq J \).

Conversely, we show by induction on \( 1 \leq m \leq n \) that
\[
J \subseteq \sum_{i=1}^{m-1} e(n,i)\mathfrak{d}^i_R + \mathfrak{d}^m_R.
\]
For \( m = 1 \) this is trivial. Assume that the induction hypothesis holds for \( 1 \leq m \leq n - 1 \). We note that
\[
\mathfrak{d}^m_R \cap J \subseteq \mathfrak{d}^m_R \cap (e(n,m)\mathfrak{d}_R + \mathfrak{d}^{m+1}_R) = e(n,m)\mathfrak{d}^m_R + \mathfrak{d}^{m+1}_R.
\]
As \( \sum_{i=1}^{m-1} e(n, i) \mathcal{D}_R^i \subseteq \mathcal{D}_R^{(e, n)} \subseteq J \) \( \square \) implies that
\[
J \subseteq \sum_{i=1}^{m-1} e(n, i) \mathcal{D}_R^i + (\mathcal{D}_R \cap J) \subseteq \sum_{i=1}^{m-1} e(n, i) \mathcal{D}_R^i + e(n, m) \mathcal{D}_R^m + \mathcal{D}_R^{m+1}
\]
\[
= \sum_{i=1}^{m} e(n, i) \mathcal{D}_R^i + \mathcal{D}_R^{m+1},
\]
completing the induction.

For \( m = n \) we obtain \( J \subseteq \sum_{i=1}^{n} e(n, i) \mathcal{D}_R^i = \mathcal{D}_R^{(e, n)} \), whence \( \mathcal{D}_R^{(e, n)} = J \). \( \square \)

**Corollary 4.2.** One has \( \mathcal{D}_R^{(e, n)} \cap \mathcal{D}_Z^m \subseteq e(n, m) \mathcal{D}_Z^m + \mathcal{D}_Z^{m+1} \).

**Proof.** By Lemma 4.1,
\[
\mathcal{D}_R^{(e, n)} \cap \mathcal{D}_Z^m \subseteq (e(n, m) \mathcal{D}_Z + \mathcal{D}_Z^{m+1}) \cap \mathcal{D}_Z^m
\]
\[
= (e(n, m) \mathcal{D}_Z \cap \mathcal{D}_Z^m) + \mathcal{D}_Z^{m+1} = e(n, m) \mathcal{D}_Z^m + \mathcal{D}_Z^{m+1}.
\]
\( \square \)

**Theorem 4.3.** Let \( e \) be a binomial multiplicatively descending map. Then
\[
\prod_{i=1}^{n} (F^{(i, 0)})^{e(n, i)} \subseteq F \cap (1 + c_R^{(e, n)}) \subseteq \mu^{-1}_R(1 + \mathcal{D}_R^{(e, n)}).
\]
Moreover, when \( R = \mathbb{Z} \), these are equalities.

**Proof.** The right inclusion follows from (2.1).

For the left inclusion, let \( 1 \leq i \leq n \) and let \( g \in F^{(i, 0)} \). By (2.2), \( g = 1 + \alpha \) with \( \alpha \in c_R^i \). We have
\[
g^{e(n, i)} = 1 + \sum_{l=1}^{e(n, i)} \binom{e(n, i)}{l} \alpha^l.
\]
Let \( 1 \leq l \leq e(n, i) \). Since \( e \) is binomial, if \( il \leq n \), then
\[
\binom{e(n, i)}{l} \alpha^l \in e(n, il) c_R^l \subseteq c_R^{(e, n)}.
\]
If \( il \geq n \), then
\[
\binom{e(n, i)}{l} \alpha^l \in c_R^il \subseteq c_R^n = e(n, n) c_R^n \subseteq c_R^{(e, n)}
\]
in this case as well. Therefore \( g^{e(n, i)} \in 1 + c_R^{(e, n)} \). It remains to recall that \( F \cap (1 + c_R^{(e, n)}) \) is a subgroup of \( F \) (Lemma 2.2).
Finally, we prove that when $R = \mathbb{Z}$, the set 

$$
\Gamma = \mu^{-1}_\mathbb{Z}(1 + \mathfrak{d}_\mathbb{Z}^{(e,n)}) \setminus \prod_{i=1}^{n}(F^{(i,0)})^{\langle n,i \rangle}
$$

is empty. Indeed, assume that $\Gamma \neq \emptyset$. Since $e(n,n) = 1$ we have $\Gamma \cap F^{(n,0)} = \emptyset$. Therefore for every $g \in \Gamma$ there is an integer $m(g)$ such that $1 \leq m(g) < n$ and $g \in F^{(m(g),0)} \setminus F^{(m(g)+1,0)}$. We may choose $g \in \Gamma$ with $m := m(g)$ maximal. The definition of $\Gamma$ implies that $\mu_\mathbb{Z}(g) \in 1 + \mathfrak{d}_\mathbb{Z}^{(e,n)}$, and $g \in F^{(m,0)}$ implies that $\mu_\mathbb{Z}(g) \in 1 + \mathfrak{d}_\mathbb{Z}^m$, by (2.3). It therefore follows from Corollary 4.2 that $\mu_\mathbb{Z}(g)^{(m)} = e(n,m)\lambda \neq 0$ for some $\lambda \in \mathbb{Z}\langle X \rangle^{(m)}$. By Corollary 2.4 the cokernel of $\mu_\mathbb{Z}^{(m)}$ is a torsion-free $\mathbb{Z}$-module. Hence there exists $h \in F^{(m,0)}$ with $\mu_\mathbb{Z}(h)^{(m)} = \lambda$. We obtain that $gh^{-e(n,m)} \in F^{(m,0)}$ and $\mu_\mathbb{Z}^{(m)}(gh^{-e(n,m)}) = 0$, so by (2.3), $gh^{-e(n,m)} \in F^{(m+1,0)}$.

On the other hand, by the first part of the theorem, $h^{-e(n,m)} \in \mu^{-1}_\mathbb{Z}(1 + \mathfrak{d}_\mathbb{Z}^{(e,n)})$. Since $1 + \mathfrak{d}_\mathbb{Z}^{(e,n)}$ is multiplicatively closed (Lemma 2.1), $gh^{-e(n,m)} \in \mu^{-1}_\mathbb{Z}(1 + \mathfrak{d}_\mathbb{Z}^{(e,n)})$. Further, $h^{-e(n,m)} \in \prod_{i=1}^{n}(F^{(i,0)})^{\langle n,i \rangle}$, so we have $gh^{-e(n,m)} \not\in \prod_{i=1}^{n}(F^{(i,0)})^{\langle n,i \rangle}$. Therefore $gh^{-e(n,m)} \not\in \Gamma$, contrary to the maximality of $m$.

Example 4.4. Let $R = \mathbb{Z}$ and let $e(n,i)$ be the trivial multiplicatively descending map (Example 3.2). Then Theorem 4.3 contains the classical results (2.2, 2.3) of Magnus and Witt as special cases (note however that (2.3) was used in the proof of Theorem 4.3).

Example 4.5. Let $R = \mathbb{Z}$, let $a$ be a non-negative integer, let $e(n,i) = a^{-i}$ as in Example 3.4, and recall that it is binomial (Example 3.8). We obtain from Theorem 4.3 that 

$$
\prod_{i=1}^{n}(F^{(i,0)})^{a^{-i}} = F \cap (1 + \sum_{i=1}^{n}a^{-i}c_Z^i) = \mu^{-1}_\mathbb{Z}(1 + \sum_{i=1}^{n}a^{-i}c_Z^i).
$$

The sequence $\prod_{i=1}^{n}(F^{(i,0)})^{a^{-i}}$, $n = 1, 2, \ldots$, is a generalization of the descending $p$-lower sequence of $F$ (when one takes $a = p$). In this sense, Theorem 4.3 for this choice of $e$ generalizes the results of Skopin and Koch ([Sko50, Koc60, Koc02, Th. 7.14]) discussed in the Introduction (the latter result is however in a pro-$p$ context).

Example 4.6. Let $R = \mathbb{Z}$, and take a prime number $p$ and an integer $t \geq 1$. Consider the multiplicatively descending map $e$ of Example 3.5 and
recall that it is binomial (Example 3.3). We obtain from Theorem 4.3 that
\[
\prod_{i=1}^{n}(F^{(i,0)})^{p^{[\log p \frac{n}{i}]}} = F \cap \left(1 + \sum_{i=1}^{n} p^{[\log p \frac{n}{i}]} c_{i}^{\mathbb{Z}}\right) = \mu_{\mathbb{Z}}^{-1}\left(1 + \sum_{i=1}^{n} p^{[\log p \frac{n}{i}]} d_{i}^{\mathbb{Z}}\right).
\]
As we will later show in §8 for \( t = 1 \) the product \( \prod_{i=1}^{n}(F^{(i,0)})^{p^{[\log p \frac{n}{i}]}} \) is the \( n \)th term \( F_{(n,p)} \) of the \( p \)-Zassenhaus filtration of \( F \).

**Remark 4.7.** The proof of Theorem 4.3 does not use condition (ii) of Definition 3.1. However, given a map \( e : \{(n, i) \mid 1 \leq i \leq n\} \to \mathbb{Z}_{\geq 0} \) satisfying only \( e(n, n) = 1 \), we may define a map \( e' \) by \( e'(n, i) = \gcd_{1 \leq j \leq i} e(n, j) \). Then \( e' \) is a multiplicatively descending map. Clearly, \( e(n, i) \in e'(n, i)\mathbb{Z} \) for every \( i \leq n \). Furthermore, \( e'(n, i) \) is a linear combination of \( e(n, 1), \ldots, e(n, i) \) with integral coefficients. Therefore
\[
\prod_{i=1}^{n}(F^{(i,0)})^{e(n,i)} = \prod_{i=1}^{n}(F^{(i,0)})^{e'(n,i)}, \quad c_{R}^{(e,n)} = c_{R}^{(e',n)}, \quad d_{R}^{(e,n)} = d_{R}^{(e',n)}.
\]
Hence we lose nothing by assuming (ii) here.

## 5. Intersections of Kernels

Let \( R \) be again a unital commutative ring and \( n \geq 2 \) an integer. Let \( \mathbb{U}_{n}(R) \) be the group of \( n \times n \) upper-triangular unipotent matrices over \( R \). For an integer \( t \geq 0 \) let \( T_{n,t}(R) \) be the set of \( n \times n \) matrices \( (a_{ij}) \) over \( R \) with \( a_{ij} = 0 \) for every \( 1 \leq i, j \leq n \) such that \( j - i < t \). Thus \( T_{n,0}(R) = \{0\} \) for \( n \leq t \). We view \( T_{n,0}(R) \) as an \( R \)-algebra, filtered by the powers of the ideal \( T_{n,1}(R) \). Note that \( T_{n,t}(R)T_{n,t'}(R) \subseteq T_{n,t+t'}(R) \), and in particular \( T_{n,1}(R)^{n} = \{0\} \). We write \( I_{n} \) for the identity matrix of order \( n \times n \).

As before, let \( F = F_{X} \) be the free group on a set \( X \) of generators. Let \( \varphi : F \to \mathbb{U}_{n}(R) \) be a group homomorphism. We filter \( R\langle\langle X \rangle\rangle \) by the powers of \( \partial_{R} \). Its universal property then gives rise to a unique \( R \)-algebra homomorphism \( \hat{\varphi} : R\langle\langle X \rangle\rangle \to T_{n,0}(R) \) which is compatible with the filtrations, and such that \( \hat{\varphi}(x) = \varphi(x) - I_{n} \in T_{n,1}(R) \) for \( x \in X \). Thus \( \varphi(\partial_{R}) \subseteq T_{n,1}(R) \).

For \( 1 \leq i < j \leq n \) let \( \varphi_{ij} : F \to R \) be the composition of \( \varphi \) with the projection on the \( (i, j) \)-entry of \( \mathbb{U}_{n}(R) \).

Also let \( X^{*} \) be the free monoid on \( X \), i.e., the set of finite words in the alphabet \( X \). We denote the length of a word \( w \in X^{*} \) by \( |w| \). Given \( g \in F \) we write
\[
\mu_{R}(g) = \sum_{w \in X^{*}} \mu_{R,w}(g)w,
\]
where $\mu_{R,w}$ is the coefficient of the Magnus homomorphism $\mu_R : F \to R(\langle X \rangle)^\times$ at $w$ (see [2]). The fact that $\mu_R$ is a group homomorphism implies, as in [Efr14a, Lemma 7.5], the following lemma:

**Lemma 5.1.** Let $w = (x_1 \cdots x_{n-1}) \in X^*$ be a word of length $n - 1$. The map $\varphi_{R,w} : F \to \mathbb{U}_n(R)$, defined by $(\varphi_{R,w})_{ij} = \mu_{R,(x_1,\ldots,x_{j-1})}$ for $1 \leq i < j \leq n$, is a group homomorphism.

**Proposition 5.2.** For every positive integer $n$ we have

$$\mu_R^{-1}(1 + \mathfrak{d}_R^n) = \bigcap_{w \in X^*} \{ \ker(\varphi) \mid \varphi \in \text{Hom}(F, \mathbb{U}_n(R)) \} = \bigcap_{w \in X^*} \ker(\varphi_{R,w}).$$

**Proof.** Take $\varphi \in \text{Hom}(F, \mathbb{U}_n(R))$ and let $\phi : R(\langle X \rangle) \to T_{n,0}(R)$ be an $R$-algebra homomorphism as above. Then $\varphi(\mathfrak{d}_R^n) \subseteq T_{n,1}(R)^n = \{0\}$. Since $\varphi = \phi \circ \mu_R$ on $F$, this implies that $\mu_R^{-1}(1 + \mathfrak{d}_R^n) \subseteq \ker(\varphi)$.

The middle intersection is trivially contained in the right intersection.

Finally, take $g$ in the right intersection. Every word $u \in X^*$ of length $k$, with $1 \leq k \leq n - 1$, can be extended to a word $w \in X^*$ of length $n - 1$ with prefix $u$. Then $\mu_{R,u}(g) = (\varphi_{R,w}(g))_{1,k+1} = 0$. We conclude that $g \in \mu_R^{-1}(1 + \mathfrak{d}_R^n)$. \( \square \)

We can now add to Theorem 4.3 the following characterization of $\mu_R^{-1}(1 + \mathfrak{d}_R^{(e,n)})$ in terms of linear representations by upper-triangular unipotent matrices.

**Theorem 5.3.** Let $e$ be multiplicatively descending map. Then

$$\mu_R^{-1}(1 + \mathfrak{d}_R^{(e,n)}) = \bigcap_{d=1}^{n-1} \bigcap_{w \in X^*} \ker(\varphi_{R/e(n,d)R,w}).$$

**Proof.** By Lemma 4.1, $\mathfrak{d}_R^{(e,n)} = \bigcap_{d=1}^{n} (e(n,d)\mathfrak{d}_R + \mathfrak{d}_R^{d+1})$. Therefore

$$\mu_R^{-1}(1 + \mathfrak{d}_R^{(e,n)}) = \bigcap_{d=1}^{n} \mu_R^{-1}(1 + \mathfrak{d}_R^{d+1})^{-1} \mathfrak{d}_R^{e(n,d)R}(1 + \mathfrak{d}_R^{d+1})^{-1}.$$  

The Theorem now follows from Proposition 5.2. \( \square \)

**Example 5.4.** Let $e(n,i)$ be the trivial multiplicatively descending map (Example 3.2). Then, by Proposition 5.2

$$\mu_R^{-1}(1 + \mathfrak{d}_R^{(e,n)}) = \mu_R^{-1}(1 + \mathfrak{d}_R^n) = \bigcap_{w \in X^*} \ker(\varphi_{R,w}).$$
In particular, when $R = \mathbb{Z}$, the group $\mu_{\mathbb{Z}}^{-1}(1 + \mathfrak{d}_{\mathbb{Z}}^{(e,n)})$ is the $n$-th term $F^{(n,0)}$ of the lower central series of $F$ (see (2.3)). Then (5.1) is essentially due to Grün [Grü36] (see also [Röh85]; note that Grün works with lower-triangular unipotent matrices).

**Example 5.5.** Let $a$ be a non-negative integer and consider the multiplicatively descending map $e(n,i) = a^{n-i}$ (Example 3.4). We obtain that

$$\mu_{\mathbb{Z}}^{-1}(1 + \mathfrak{d}_{\mathbb{Z}}^{(e,n)}) = \bigcap_{d=1}^{n-1} \left\{ \text{Ker}(\varphi) \mid \varphi \in \text{Hom}(F, \mathbb{U}_{d+1}(\mathbb{Z}/a^{n-d}\mathbb{Z})) \right\}.$$ 

When $a = p$ is a prime number, $\mu_{\mathbb{Z}}^{-1}(1 + \mathfrak{d}_{\mathbb{Z}}^{(e,n)})$ is the $n$-th term $F^{(n,p)}$ in the lower $p$-central filtration of $F$, by the results of Skopin and Koch (see Example 4.5). Thus, in this special case, the first equality in Theorem 5.3 recovers the characterization of $F^{(n,p)}$ due to Mináč and Tân [MT15, Th. 2.7(c)] mentioned in (III)$^p$ of the Introduction (an equivalent characterization was independently proved in [Efr14b]).

**Example 5.6.** Let $R = \mathbb{Z}$, let $t$ be a positive integer, and let $p$ be a prime number. Consider the multiplicatively descending map $e(n,i) = p^{j(n,i)}$ of Example 3.5, where $j(n,i) = \lceil \log_p \left( \frac{n}{i} \right) \rceil$. Then $\mu_{\mathbb{Z}}^{-1}(1 + \mathfrak{d}_{\mathbb{Z}}^{(e,n)})$ is the $n$-th term $F^{(n,p)}$ in the $p$-Zassenhaus filtration (see Example 4.6). We obtain from Theorem 5.3 that

$$F^{(n,p)} = \bigcap_{d=1}^{n-1} \left\{ \text{Ker}(\varphi) \mid \varphi \in \text{Hom}(F, \mathbb{U}_{d+1}(\mathbb{Z}/p^{j(n,d)}\mathbb{Z})) \right\}.$$ 

**Example 5.7.** Let $R = F_p$ for a prime number $p$ and let $e(n,i)$ be the trivial multiplicatively descending map. Then $\mu_{F_p}^{-1}(1 + \mathfrak{d}_{F_p}^{(e,n)}) = \mu_{F_p}^{-1}(1 + \mathfrak{d}_{F_p}^{n})$ is again the $n$-th term $F^{(n,p)}$ of the $p$-Zassenhaus filtration of $F$ (see (1)$^p$ in the Introduction). Hence Proposition 5.2 gives

$$F^{(n,p)} = \bigcap \{ \text{Ker}(\varphi) \mid \varphi \in \text{Hom}(F, \mathbb{U}_n(F_p)) \}.$$ 

A profinite version of (5.2) was proved in [Efr14a, Th. A] as a special case of a more general result on Massey products in profinite cohomology. More direct alternative proofs were later given in [Efr14b, Ex. 6.4] (also in the discrete setting) and [MT15, Th. 2.7].

We conclude this section with some terminology and facts that will be needed in §9. Let $Z_n(R)$ be the subgroup of $\mathbb{U}_n(R)$ consisting of all unipotent matrices which are zero except for the main diagonal and (possibly)
entry \((1,n)\). It lies in the center of \(\mathbb{U}_n(R)\), so we may define
\[
\bar{\mathbb{U}}_n(R) = \mathbb{U}_n(R)/\mathbb{Z}_n(R).
\]
We may view the elements of \(\bar{\mathbb{U}}_n(R)\) as upper-triangular unipotent \(n \times n\)-matrices without the \((1,n)\)-entry.

Given a word \(w = (x_1 \cdots x_n)\), let \(\bar{\varphi}_w: F \to \bar{\mathbb{U}}_{n+1}(R)\) be the homomorphism induced by \(\varphi_w\). The following proposition complements Proposition 5.2.

**Proposition 5.8.** For \(n \geq 1\) we have \(\mu_{R}^{-1}(1 + \vartheta^n_R) = \bigcap_{w \in X^* \mid |w| = n} \text{Ker}(\bar{\varphi}_w)\).

**Proof.** For \(x_1, x_2, \ldots, x_{n-1}, x_n \in X\) one has inclusions
\[
\text{Ker}(\varphi(x_1 \cdots x_{n-1})) \supseteq \text{Ker}(\varphi(x_1 \cdots x_n)) = \text{Ker}(\varphi(x_1 \cdots x_{n-1})) \cap \text{Ker}(\varphi(x_2 \cdots x_n)).
\]
It follows that
\[
\bigcap_{w \in X^* \mid |w| = n-1} \text{Ker}(\varphi_w) = \bigcap_{w \in X^* \mid |w| = n} \text{Ker}(\bar{\varphi}_w).
\]
Now apply Proposition 5.2. \(\square\)

It follows from Proposition 5.8 that, for every \(w \in X^*\) of length \(n \geq 1\), the restriction of \(\varphi_w\) to \(\mu_{R}^{-1}(1 + \vartheta^n_R)\) is into \(\mathbb{Z}_{n+1}(R)\). We therefore obtain:

**Corollary 5.9.** The map \(\mu_{R}^{-1}(1 + \vartheta^n_R) \to R, g \mapsto \varphi_w(g)_{1,n+1}\), is a group homomorphism.

6. Inductive definitions of filtrations

We now turn to study inductive definitions of the general filtrations discussed in \(\S\) similarly to what we had for the lower \(p\)-central series and \(p\)-Zassenhaus filtrations. First we develop in the current section a general machinery for such inductive constructions. In \(\S\) and \(\S\) we apply it to obtain the above known examples, and extend them to new cases.

Suppose that \(T \subseteq \mathbb{Z}_{\geq 1}\), and let \(f: \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 0}, g: \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 1}\) be maps such that \(g(n) < n\) for every \(n \geq 2\). For a group \(G\) we define inductively subgroups \(G(n), n = 1, 2, \ldots,\) by
\[
G(1) = G, \quad G(n) = G^{f(n)}_{(g(n))} \prod_{(s,t) \in T, s+t=n} [G(s), G(t)].
\]
The subgroups \(G(n)\) are characteristic, whence normal in \(G\). Note that the sequence \(G(n), n = 1, 2, \ldots,\) need not be decreasing.

Let \(e(n,i)\) be a multiplicatively descending map, and assume that:
(1) For every \((s, t) \in T, 1 \leq i \leq s\) and \(1 \leq j \leq t\) one has \(e(s, i)e(t, j) \in e(s + t, i + j)\mathbb{Z}\).

(2) For every \(n \geq 2\) and \(1 \leq j_1, \ldots, j_t \leq g(n)\) such that \(1 \leq l \leq f(n)\) and \(j_1 + \cdots + j_t \leq n\) one has
\[
\left(\frac{f(n)}{l}\right)e(g(n), j_1) \cdots e(g(n), j_t) \in e(n, j_1 + \cdots + j_t)\mathbb{Z}.
\]

**Remark 6.1.** Condition (1) implies that \(\varnothing Z^{(e,s)}\varnothing Z^{(e,t)} \subseteq \varnothing Z^{(e,s+t)}\) for \((s, t) \in T\). Condition (2) implies that \(\left(\frac{f(n)}{l}\right)\alpha^l \in \varnothing Z^{(e,n)}\) for every \(n \geq 2\), \(1 \leq l \leq f(n)\), and \(\alpha \in \varnothing Z^{(e,g(n))}\).

**Theorem 6.2.** Let \(F = F_X\) be a free group. Then \(F(n) \subseteq \mu_Z^{-1}(1 + \varnothing Z^{(e,n)})\) for every \(n\).

**Proof.** We argue by induction on \(n\). For \(n = 1\) we have \(\varnothing Z^{(e,1)} = \varnothing Z\), so \(F(1) = F = \mu_Z^{-1}(1 + \varnothing Z^{(e,1)})\).

Suppose that \(n > 1\). For every \(\alpha \in \varnothing Z^{(e,g(n))} = \sum_{j=1}^{g(n)} e(g(n), j)\varnothing Z\) and \(1 \leq l \leq f(n)\), assumption (2) implies that \(\left(\frac{f(n)}{l}\right)\alpha^l \in \varnothing Z^{(e,n)}\) (see Remark 6.1). Therefore \((1 + \alpha)^{f(n)} = \sum_{l=0}^{f(n)} \left(\frac{f(n)}{l}\right)\alpha^l \in 1 + \varnothing Z^{(e,n)}\). By the induction hypothesis, this shows that \(F^{f(n)}_{g(n)} \subseteq \mu_Z^{-1}(1 + \varnothing Z^{(e,n)})\).

Next take \((s, t) \in T\) with \(s + t = n\) and consider \(\alpha \in \varnothing Z^{(e,s)}\) and \(\beta \in \varnothing Z^{(e,t)}\). By (1), \(\varnothing Z^{(e,s)}\varnothing Z^{(e,t)} \subseteq \varnothing Z^{(e,n)}\) (see Remark 6.1). As \(1 + \alpha, 1 + \beta \in \mathbb{Z}\langle\langle X\rangle\rangle^{(e,s)}\), we obtain that
\[
[1 + \alpha, 1 + \beta] = 1 + (1 + \alpha)^{-1}(1 + \beta)^{-1}((1 + \alpha)(1 + \beta) - (1 + \beta)(1 + \alpha))
\]
\[
= 1 + (1 + \alpha)^{-1}(1 + \beta)^{-1}(\alpha \beta - \beta \alpha) \subseteq 1 + \varnothing Z^{(e,n)}.
\]
From this and the induction hypothesis we conclude that \([F(s), F(t)] \subseteq \mu^{-1}(1 + \varnothing Z^{(e,n)})\). Consequently, \(F(n) \subseteq \mu^{-1}(1 + \varnothing Z^{(e,n)})\), completing the induction. \(\square\)

7. The \(A\)-filtration

Let \(A = (a_i)_{i=1}^{\infty}\) be a sequence of non-negative integers. Let \(e\) be the multiplicatively descending map of Example 3.3. Thus for \(1 \leq i < n\) we set
\[
e(n, i) = \gcd\left\{\prod_{j \in J} a_j \mid J \subseteq \{1, 2, \ldots, n - 1\}, \ |J| = n - i\right\},
\]
and by convention, \(e(n, n) = 1\). In this section we give an inductive definition for the subgroups \(\prod_{i=1}^{n} (G^{(i,0)})^{e(n,i)}\) of a group \(G\). Namely, we show
that they coincide with the $A$-filtration $G^{(n,A)}$, $n = 1, 2, \ldots$, of $G$ (Theorem 7.4). Recall that we defined $G^{(1,A)} = G$, and 

$$G^{(n,A)} = (G^{(n-1,A)})^{n-1}[G^{(n-1,A)}, G],$$

for every $n > 1$. For this we use the general framework of \[\mathbb{G}\] with

$$f(n) = a_{n-1}, \quad g(n) = n - 1, \quad T = \mathbb{Z}_{\geq 1} \times \{1\}.$$

**Lemma 7.1.** Conditions (1) and (2) of \[\mathbb{G}\] hold in this setup.

**Proof.** For condition (1) observe that $e(s, i)e(1, 1) = e(s, i) \in e(s+1, i+1)\mathbb{Z}$ for every $1 \leq i \leq s$.

For condition (2), take $1 \leq j_1, \ldots, j_l \leq n - 1$ with $1 \leq l \leq a_{n-1}$ and $j_1 + \cdots + j_l \leq n$. We need to show that

$$\left(\begin{array}{c}
a_{n-1} \\
l
\end{array}\right) \cdot e(n - 1, j_1) \cdot \cdots \cdot e(n - 1, j_l) \in e(n, j_1 + \cdots + j_l)\mathbb{Z}.$$

When $l = 1$ we have in fact $e(n - 1, j_1) \in e(n, j_1)\mathbb{Z}$. When $2 \leq l \leq a_{n-1}$ we have $j_1 + 1 \leq j_1 + \cdots + j_l$, so

$$e(n - 1, j_1) \cdot \cdots \cdot e(n - 1, j_l) \in e(n - 1, j_1)\mathbb{Z} \subseteq e(n, j_1 + 1)\mathbb{Z} \subseteq e(n, j_1 + \cdots + j_l)\mathbb{Z}.$$

Next we recall some basic facts about commutators of subgroups.

**Lemma 7.2.** Let $H$ be a normal subgroup of $G$ and let $a, b$ be non-negative integers. Then:

(a) $[H^a, G] = [H, G]^a \pmod{[H, G], G]}.$

(b) $(H^a[H, G])^b[H^a[H, G], G] = (H^b[H, G])^a[H^b[H, G], G].$

**Proof.** (a) For $h \in H$ and $g \in G$ we have

$$[h^a, g] = h^{-a}(g^{-1}hg)^a = h^{-a}(h[h, g])^a \equiv [h, g]^a \pmod{[H, G], G]}.$$

(b) Since both subgroups contain the normal subgroup $[[H, G], G]$, we prove the equality modulo $[[H, G], G]$. By (a),

$$(H^a[H, G])^b \equiv H^{ab}[H, G]^b \equiv H^{ab}[H^b, G] \pmod{[[H, G], G]}.$$ 

Also, for $h \in H$ and $k \in [H, G]$ we have

$$[h^a k, g] = k^{-1}h^a [g, k, g] \in k^{-1}[H^a, G]k[[H, G], G] = [H^a, G][[H, G], G],$$

since $[H^a, G]$ is normal in $G$. Hence

$$[[H^a, G], G] \equiv [H^a, G] \pmod{[[H, G], G]}.$$ 

Therefore the left hand side of (b) is $H^{ab}[H^b, G][H^a, G][[H, G], G]$. By symmetry, this is also the right hand side of (b). \qed
Lemma 7.3. Let $\sigma \in S_{n-1}$ be a permutation, and $A_\sigma$ the sequence obtained from $A$ by applying $\sigma$ to the first $n-1$ elements. Then $G^{(n,A)} = G^{(n,A_\sigma)}$.

Proof. Every permutation is a composition of transpositions, so we can assume that $\sigma$ transposes $k-1$ and $k$, where $2 \leq k < n$. We apply Lemma 7.2(b) with $H = G^{(k-1,A)} = G^{(k-1,A_\sigma)}$, $a = a_{k-1}$, and $b = a_k$, to obtain that
\[
G^{(k+1,A)} = (G^{(k,A)})^b [G^{(k,A)}, G] = (H^a[H,G])^b [H^a[H,G], G] = (H^b[H,G])^a [H^b[H,G], G] = (G^{(k,A_\sigma)})^a [G^{(k,A_\sigma)}, G] = G^{(k+1,A_\sigma)}.
\]
Hence also $G^{(n,A)} = G^{(n,A_\sigma)}$. \qed

We now obtain the main result of this section:

Theorem 7.4. Let $A = (a_i)_{i=1}^\infty$ be a sequence of non-negative integers and $G$ a group. Let $e(n, i)$ be as in Example 5.5. Then for every $n \geq 1$ we have
\[
G^{(n,A)} = \prod_{i=1}^n (G^{(i,0)})^{e(n,i)}.
\]

Proof. Let $T$ be a subset of $\{1, 2, \ldots, n-1\}$ of size $n - i$. There exists $\sigma \in S_{n-1}$ such that $T = \{\sigma(i), \ldots, \sigma(n-1)\}$. Let $A_\sigma$ be again the sequence obtained from $A$ by applying $\sigma$ to the first $n-1$ elements. By the definition of the filtrations, for every $1 \leq i \leq n$ we have $G^{(i,0)} \subseteq G^{(i,A_\sigma)}$. Therefore $(G^{(i,0)})_{i \in T} \subseteq G^{(i,A_\sigma)}$, and by Lemma 7.3 $G^{(n,A_\sigma)} = G^{(n,A)}$. Further, $e(n, i)$ is a linear combination of the products $\prod_{j \in T} a_j$ with integral coefficients. We conclude that $(G^{(i,0)})^{e(n,i)} \subseteq G^{(n,A)}$, whence $\prod_{i=1}^n (G^{(i,0)})^{e(n,i)} \subseteq G^{(n,A)}$.

For the opposite inclusion let $F = F_X$ be a free group on $X$. By Lemma 7.1 and Theorem 5.2 $F^{(n,A)} \subseteq \mu Z^{-1} (1 + \Omega^{(e,n)} Z)$. Since $e$ is binomial (Example 3.8), Theorem 1.3 shows that $\mu Z^{-1} (1 + \Omega^{(e,n)} Z) = \prod_{i=1}^n (F^{(i,0)})^{e(n,i)}$, so $F^{(n,A)} \subseteq \prod_{i=1}^n (F^{(i,0)})^{e(n,i)}$. Since every group is an epimorphic image of a free group, this completes the proof. \qed

In particular, for a non-negative integer $a$, we define the $a$-lower central series $G^{(n,a)}$, $n = 1, 2, \ldots$, of $G$ by
\[
G^{(1,a)} = G, \quad G^{(n,a)} = (G^{(n-1,a)})^a [G^{(n-1,a)}, G]
\]
for $n \geq 2$. Note that when $a = 0$ (resp., $a = p$ is prime) this is just the lower central (resp., $p$-central) series, and therefore there is no conflict of notation. We obtain:

Corollary 7.5. For every integer $n \geq 1$ we have $G^{(n,a)} = \prod_{i=1}^n (G^{(i,0)})^{a^{n-i}}$.

When $a = q$ is a prime power, this is contained in [NSW08, Prop. 3.8.6].
8. The $q$-Zassenhaus filtration

In this section, let $p$ be a prime number, $t$ a positive integer, and $q = p^t$ a $p$-power. Let $e(n,i)$ be the multiplicatively descending map of Example 3.5. Thus for integers $1 \leq i \leq n$ we set $j(n,i) = \lceil \log_p \left( \frac{n}{i} \right) \rceil$ and $e(n,i) = q^{j(n,i)}$.

We give an inductive construction of the subgroups $\prod_{i=1}^{n} (G^{i,0})^{e(n,i)}$ of a group $G$ in this case. More specifically, we show that they coincide with the $q$-Zassenhaus filtration $G_{(n,q)}$, $n = 1, 2, \ldots$, defined inductively by

$$G_{(1,q)} = G, \quad G_{(n,q)} = G_{([n/p],q)} \prod_{s+t=n} [G_{(s,q)}, G_{(t,q)}].$$

**Lemma 8.1.** $G_{(n-1,q)} \geq G_{(n,q)}$ for every $n \geq 2$.

**Proof.** We may assume inductively that $G = G_{(1,q)} \geq \cdots \geq G_{(n-1,q)}$. We show that all the factors in the definition of $G_{(n,q)}$ are contained in $G_{(n-1,q)}$.

We first note that $\lceil (n-1)/p \rceil \leq \lceil n/p \rceil \leq n-1$. By the induction hypothesis, $G_{((n-1)/p),q} \geq G_{([n/p],q)}$, whence $G_{(n-1,q)} \geq G_{([n/p],q)}^q$.

Next, consider $s, t \geq 1$ with $s + t = n$. If, say $s \geq 2$, then

$$G_{(n-1,q)} \geq [G_{(s-1,q)}, G_{(t,q)}] \geq [G_{(s,q)}, G_{(t,q)}],$$

and similarly when $t \geq 2$. Finally, when $s = t = 1$ and $n = 2$ the latter inclusion is trivial. \hfill \Box

We consider the general framework of $\mathcal{G}$ with

$$f(n) = q, \quad g(n) = \lceil n/p \rceil, \quad T = \mathbb{Z}_{\geq 1}^2.$$

**Lemma 8.2.** Conditions (1) and (2) of $\mathcal{G}$ hold in this setup.

**Proof.** For (1), take integers $1 \leq i \leq s$ and $1 \leq j \leq t$. As $st/ij \geq (s+t)/(i+j)$ we have

$$\left\lfloor \log_p \left( \frac{s}{i} \right) \right\rfloor + \left\lfloor \log_p \left( \frac{t}{j} \right) \right\rfloor \geq \left\lfloor \log_p \left( \frac{st}{ij} \right) \right\rfloor \geq \left\lfloor \log_p \left( \frac{s+t}{i+j} \right) \right\rfloor,$$

whence $e(s,i)e(t,j) \in e(s+t,i+j)\mathbb{Z}$.

For (2), let $1 \leq j_1, \ldots, j_l \leq g(n)$ where $1 \leq l \leq q$ and $j_1 + \cdots + j_l \leq n$.

When $p \leq l$ we have $lg(n) \geq n$, so (1) implies that $e(g(n), j_1) \cdots e(g(n), j_l) \in e(lg(n), j_1 + \cdots + j_l)\mathbb{Z} \subseteq e(n, j_1 + \cdots + j_l)\mathbb{Z}$.

When $1 \leq l < p$ the equality $q^{(l-1)} = l\binom{q}{l}$ shows that $q$ divides $\binom{q}{l}$.

Further, $n/(j_1 + \cdots + j_l) \leq n/j_1 \leq pg(n)/j_1$, so

$$\left\lfloor \log_p \frac{n}{j_1 + \cdots + j_l} \right\rfloor \leq \left\lfloor \log_p \frac{g(n)}{j_1} \right\rfloor \leq \left\lfloor \log_p \frac{g(n)}{j_1} \right\rfloor + \cdots + \left\lfloor \log_p \frac{g(n)}{j_l} \right\rfloor.$$
Therefore \( q e(g(n), j_1) \cdots e(g(n), j_l) \in e(n, j_1 + \cdots + j_l) \mathbb{Z} \), and we get (2) in this case as well. \( \square \)

We now prove the main result of this section. It extends Lazard’s result ([Laz65], p. 209, (3.14.5)], [DDMS99, Th. 11.2]) mentioned in the Introduction, which is the case \( t = 1, q = p \) of our theorem. Note that our method of proof is different from Lazard’s.

**Theorem 8.3.** Let \( G \) be a group. Then \( G_{(n,q)} = \prod_{i=1}^n (G^{(i,0)})^{e(n,i)} \).

**Proof.** For every \( 1 \leq i \leq n \) we have by definition \( G^{(i,0)} \subseteq G_{(i,q)} \) and \( (G_{(i,q)})^q \subseteq G_{(ip,q)}. \) For \( j = j(n,i) \) we have \( ip^j \geq n \). Using these observations and Lemma 8.1, we obtain

\[
(G^{(i,0)})^{q^j} \subseteq (G_{(i,q)})^{q^j} \subseteq G_{(ip^j,q)} \subseteq G_{(n,q)}.
\]

This shows that \( \prod_{i=1}^n (G^{(i,0)})^{e(n,i)} \subseteq G_{(n,q)}. \)

For the opposite inclusion we may assume that \( G = F_X \) is a free group. By Lemma 8.2 and Theorem 6.2 \( (F_X)_{(n,q)} \subseteq \mu^{-1}_F(1 + \partial^{(e,n)}_R) \), and by Example 6.3 the latter group is \( \prod_{i=1}^n (F^{(i,0)})^{e(n,i)} \). \( \square \)

**9. Massey products**

We now use the machinery of [3] to investigate the cohomology of the quotients \( F/\mu^{-1}_R(1 + \partial_R^n) \) for a free group \( F \). We show that the quotients of consecutive subgroups \( \mu^{-1}_R(1 + \partial_R^n) \) are dual to a certain subgroup of \( H^2(F/\mu^{-1}_R(1 + \partial_R^n), R) \) generated by \( n \)-fold Massey products (Theorem 9.1). The discussion is partly inspired by [Efr14a].

Throughout this section we fix a commutative unital ring \( R \). Let \( G \) be a group acting from the left on \( R \), considered as an abelian group. We write \( C^i(G, R) \) for the group of (inhomogeneous) \( i \)-cochains on \( G \) with values in \( R \) and \( \partial : C^i(G, R) \to C^{i+1}(G, R) \) for the differential. Let \( H^i(G, R) \) be the \( i \)th cohomology group. We refer, e.g., to [Wei94, Ch. 6] for the basic notions in group cohomology.

Let \( F = F_X \) be as before the free group on basis \( X \). We let it act trivially on \( R \). One has \( H^2(F, R) = 0 \) ([Wei94], Cor. 6.2.7]. Let \( N \) be a normal subgroup of \( F \) contained in \([F,F]\). Then every homomorphism \( F \to R \) factors via \( F/N \), i.e., the inflation map \( H^1(F/N, R) \to H^1(F, R) \) is surjective. The 5-term sequence in group cohomology ([Wei94], 6.8.3] therefore shows that the transgression map \( \text{trg} : H^1(N, R)^F \to H^2(F/N, R) \) (which is the \( \partial_2^{-1} \)-differential in the Hochschild–Serre spectral sequence) is an isomorphism. We may therefore define a bilinear map

\[
(\cdot,\cdot) : N \times H^2(F/N, R) \to R, \quad (g, \alpha)' = (\text{trg}^{-1}(\alpha))(g).
\]
It has a trivial right kernel (see [EM16 (3.3)]).

Next we recall a few notions and facts about Massey products in group cohomology; see e.g. [Kra66], [Dwy75], [Fen83] for more details. Let \( n \geq 2 \).

An array

\[
M = \{ c_{ij} \mid 1 \leq i < j \leq n + 1, \ (i, j) \neq (1, n + 1) \}
\]

in \( C^1(G, R) \) is called a defining system if \( \partial c_{ij} = -\sum_{k=i+1}^{j-1} c_{ik} \cup c_{kj} \) for every \( i, j \) as above. In particular, \( \partial c_{i,i+1} = 0 \) for every \( 1 \leq i \leq n \). One can show that \( \sum_{k=2}^{n} c_{1k} \cup c_{k,n+1} \) is a 2-cocycle (see [Fen83, p. 233], [Kra66, p. 432]; note that various sources have different sign conventions). Its cohomology class \( \text{Val}(M) \) in \( H^2(G, M) \) is the value of the defining system \( M \). Given \( \chi_1, \ldots, \chi_n \in H^1(G, M) \), the \( n \)-fold Massey product \( \langle \chi_1, \ldots, \chi_n \rangle \) is the subset of \( H^2(G, M) \) consisting of all values of defining systems \( M \) as above such that \( c_{i,i+1} \) is a representative of \( \chi_i \), \( i = 1, 2, \ldots, n \). For example, when \( n = 2 \) the 2-fold Massey product \( \langle \chi_1, \chi_2 \rangle \) consists only of the cup product \( \chi_1 \cup \chi_2 \).

By Dwyer [Dwy75, Th. 2.4], when \( G \) acts trivially on \( R \) there is a bijective correspondence between defining systems \( M = (c_{ij}) \) and group homomorphisms \( \varphi : G \to \widehat{\mathbb{U}}_{n+1}(R) \) (see [5], given by \( \varphi(g)_{ij} = (-1)^{j-i}c_{ij}(g) \) for \( 1 \leq i < j \leq n + 1 \) with \( (i, j) \neq (1, n + 1) \) (the other entries being obvious).

For the rest of this section we abbreviate

\[
F_{(n,R)} = \mu_R^{-1}(1 + \mathfrak{d}_R^n).
\]

Given a word \( w = (x_1 \cdots x_n) \in X^* \) of length \( n \), let \( \varphi_w = \varphi_{R,w} : F \to \mathbb{U}_{n+1}(R) \) and \( \varphi_w : F \to \widehat{\mathbb{U}}_{n+1}(R) \) be as in [5]. By Proposition 5.8, \( \varphi_w \) factors via a homomorphism \( \varphi'_w : F/F_{(n,R)} \to \widehat{\mathbb{U}}_{n+1}(R) \). It gives rise as above to a defining system \( M = (\tilde{c}_{ij}) \) in \( C^1(F/F_{(n,R)}, R) \). We set \( \psi_w = \text{Val}(M) \). Thus \( \psi_w \in \langle [c_{12}], \ldots, [c_{n,n+1}] \rangle \subset H^2(F/F_{(n,R)}, R) \), where \( [c_{ij}] \) denote the cohomology class of \( c_{ij} \) in \( H^1(F/F_{(n,R)}, R) \). For \( 1 \leq i < j \leq n + 1 \) let \( c_{ij} \in C^1(F, R) \) be the projection of \( \varphi_w \) on the \((i, j)\)-entry.

**Lemma 9.1.** \( \psi_w = \text{trg}[c_{1,n+1}|F_{(n,R)}] \).

**Proof.** Since \( \varphi_w \) is a group homomorphism, for every \( g_1, g_2 \in F \) we have \( c_{1,n+1}(g_1 g_2) = \sum_{k=1}^{n+1} c_{1k}(g_1)c_{k,n+1}(g_2) \), whence

\[
(\partial c_{1,n+1})(g_1, g_2) = -\sum_{k=2}^{n} c_{1k}(g_1)c_{k,n+1}(g_2)
\]

\[
= -\sum_{k=2}^{n} \tilde{c}_{1k}(g_1 F_{(n,R)})\tilde{c}_{k,n+1}(g_2 F_{(n,R)})
\]

where \( \tilde{c}_{ij} \) are the \( (i, j) \)-entries of \( c_{ij} \).
The explicit definition of the transgression map, as in [NSW08, Prop. 1.6.5], therefore implies that
\[ \psi_w = - \left[ \sum_{k=2}^{n} c_{1k} \cup \bar{c}_{k,n+1} \right] = \text{trg}[c_{1,n+1}|_{F(n,R)}]. \]

We now define a homomorphism
\[ \Psi_{(n,R)} : \bigoplus_{w \in \mathfrak{X}^*} R \to H^2(F/F(n,R), R), \quad (r_w)_w \mapsto \sum_w r_w \psi_w. \]

In view of Corollary 5.9 and Proposition 5.2, there is a bilinear map (9.2)
\[ (\cdot, \cdot)'': \quad F(n,R)/F(n+1,R) \times \bigoplus_{w \in \mathfrak{X}^*} R \to R, \quad (\bar{g}, (r_w)_w)'' = \sum_{|w|=n} r_w \varphi_w(g)_{1,n+1} \]
with a trivial left kernel. Combining this with (9.1) (for \( N = F(n,R) \)), we obtain a diagram of bilinear maps of \( \mathbb{Z} \)-modules

\[ (9.3) \quad F(n,R)/F(n+1,R) \times \bigoplus_{w \in \mathfrak{X}^*} R \overset{(\cdot, \cdot)''}{\longrightarrow} R \]
\[ H^2(F/F(n,R), R) \overset{(\cdot, \cdot)'}{\longrightarrow} R. \]

This diagram commutes, in the sense that for \( g \in F(n,R) \) and for \( (r_w)_w \in \bigoplus_{|w|=n} R \) one has, by Lemma 9.1
\[ (g, \Psi_{(n,R)}((r_w)_w))' = (g, \sum_w r_w \psi_w)' = \sum_w r_w(g, \text{trg}[c_{1,n+1}|_{F(n,R)}])' \]
\[ = \sum_w r_w c_{1,n+1}(g) = \sum_w r_w \varphi_w(g)_{1,n+1} = (\bar{g}, (r_w)_w)''. \]

We denote
\[ H^2(F/F(n,R), R)_{n-\text{Massey}} = \text{Im}(\Psi_{(n,R)}). \]

**Theorem 9.2.** For every integer \( n \geq 2 \) there is a canonical non-degenerate bilinear map
\[ F(n,R)/F(n+1,R) \times H^2(F/F(n,R), R)_{n-\text{Massey}} \to R. \]

**Proof.** This follows formally from the commutativity of (9.3), and the triviality of the left- (resp., right-) kernel of the upper (resp., lower) bilinear map (see [Efr14a, Lemma 2.2]). \( \Box \)

**Example 9.3.** Take \( R = \mathbb{Z} \). We recall that, by Magnus’ theorem, \( F(n,\mathbb{Z}) \) is the \( n \)-th term \( F^{(n,0)} \) in the lower central sequence. Theorem 9.2 therefore gives a canonical non-degenerate bilinear map
\[ (9.4) \quad F^{(n,0)}/F^{(n+1,0)} \times H^2(F/F^{(n,0)}, \mathbb{Z})_{n-\text{Massey}} \to \mathbb{Z}. \]
By the classical results of Magnus and Witt [Ser92, Part I, Ch. IV, Cor. 6.2], $F^{(n,0)}/F^{(n+1,0)}$ is a free $\mathbb{Z}$-module of rank $l_{|X|}(n)$; here $l_m(n)$ is the \textit{necklace function}, defined for a positive integer $m$ by

$$l_m(n) = \frac{1}{n} \sum_{d|n} \mu(d) m^{n/d},$$

where $\mu$ is the Möbius function, and with the convention $l_\infty(n) = \infty$. We deduce:

\textbf{Corollary 9.4.} For $n \geq 2$, the $\mathbb{Z}$-module $H^2(F/F^{(n,0)}, \mathbb{Z})_{n-\text{Massey}}$ is free of rank $l_{|X|}(n)$.

For $n = 2$ the quotient $\bar{F} = F/F^{(2,0)}$ is the free abelian group on basis $X$. Then Corollary 9.4 recovers the well-known fact that the image of the cup product map $H^2(\bar{F}, \mathbb{Z}) \otimes \mathbb{Z} H^2(\bar{F}, \mathbb{Z}) \to H^2(\bar{F}, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $l_{|X|}(2) = \left(\frac{|X|}{2}\right)$ (if $X$ is infinite). Indeed, the 2-fold Massey product is the cup product, and the cohomology ring $H^*(\bar{F}, \mathbb{Z})$ is the exterior algebra over $H^1(\bar{F}, \mathbb{Z}) \cong \bigoplus_X \mathbb{Z}$.

\textbf{Example 9.5.} Let $R = \mathbb{F}_p$ with $p$ prime. Recall that, $F_{(n,p)} = \mu_p^{-1}(1+\mathfrak{o}_{\mathbb{F}_p}^n)$ is the $n$th term $F_{(n,p)}$ in the $p$-Zassenhaus filtration of $F$ (see (I) of the Introduction). We obtain for $n \geq 2$ a non-degenerate bilinear map

$$F_{(n,p)}/F_{(n+p)} \times H^2(F/F_{(n,p)}, \mathbb{F}_p)_{n-\text{Massey}} \to \mathbb{F}_p.$$  

A profinite variant of this result was proved in [Efr14a, Th. B].

\textbf{REFERENCES}

[BT12] F. Bogomolov and Y. Tschinkel, \textit{Introduction to birational anabelian geometry}, In: “Current Developments in Algebraic Geometry” (L. Caporaso et al, ed.), MSRI Publications, vol. 59, Cambridge Univ. Press, 2012, pp. 17–63.

[CM82] B. Chandler and W. Magnus, \textit{The History of Combinatorial Group Theory: a case study of the history of ideas}, Studies in the History of Mathematics and Physical Sciences, vol. 9, Springer-Verlag, 1982.

[CEM12] S. K. Chebolu, I. Efrat, and J. Mináč, \textit{Quotients of absolute Galois groups which determine the entire Galois cohomology}, Math. Ann. 352 (2012), 205–221.

[DDMS99] J. D. Dixon, M. P. F. Du Sautoy, A. Mann, and D. Segal, \textit{Analytic Pro-p Groups}, Cambridge Univ. Press, Cambridge, 1999.

[Dwy75] W. G. Dwyer, \textit{Homology, Massey products and maps between groups}, J. Pure Appl. Algebra 6 (1975), 177–190.

[Efr14a] I. Efrat, \textit{The Zassenhaus filtration, Massey products, and representations of profinite groups}, Adv. Math. 263 (2014), 389–411.
I. Efrat, *Filtrations of free groups as intersections*, Arch. Math. (Basel) 103 (2014), 411–420.

I. Efrat and J. Mináč, *On the descending central sequence of absolute Galois groups*, Amer. J. Math. 133 (2011), 1503–1532.

I. Efrat and J. Mináč, *Galois groups and cohomological functors*, Trans. Amer. Math. Soc. (2016), to appear, available at arXiv:1103.1508.

R. A. Fenn, *Techniques of Geometric Topology*, London Math. Soc. Lect. Notes Series, vol. 57, Cambridge Univ. Press, Cambridge, 1983.

O. Grün, *Über eine Faktorgruppe freier Gruppen I*, Deutsche Math. 1 (1936), 772–782.

S. A. Jennings, *The structure of the group ring of a p-group over a modular field*, Tran. Amer. Math. Soc. 50 (1941), 169–187.

H. Koch, *Über die Faktorgruppen einer absteigenden Zentralreihe*, Math. Nach. 22 (1960), 159–161.

H. Koch, *Galois Theory of p-Extensions*, Springer, 2002.

D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. 124 (1966), 431–449.

M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954), 101–190.

M. Lazard, *Groupes analytiques p-adiques*, Inst. Hautes Études Sci., Publ. Math. (1965), 389–603.

W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. 111 (1935), 259–280.

W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. reine angew. Math. 177 (1937), 105–115.

J. Mináč and N. D. Tàn, *The kernel unipotent conjecture and the vanishing of Massey products for odd rigid fields*, Adv. Math. 273 (2015), 242–270. with an appendix by I. Efrat, J. Mináč, and N.D. Tàn.

M. Morishita, *Milnor invariants and Massey products for prime numbers*, Compos. Math. 140 (2004), 69–83.

M. Morishita, *Knots and Primes: An Introduction to Arithmetic Topology*, Universitext, Springer, London, 2012.

J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, Springer, 2008.

D. G. Quillen, *On the associated graded ring of a group ring*, J. Algebra 10 (1968), 411–418.

F. Röhl, *Review and some critical comments on a paper of Grün concerning the dimension subgroup conjecture*, Bol. Soc. Braz. Mat. 16 (1985), 11–27.

J.-P. Serre, *Lie Algebras and Lie Groups*, Springer, 1992.

A. I. Skopin, *The factor groups of an upper central series of free groups*, Doklady Akad. Nauk SSSR (N.S.) 74 (1950), 425–428 (Russian).

A. Topaz, *Commuting-liftable subgroups of Galois groups II*, J. reine angew. Math., posted on 2012, DOI 10.1515/crelle-2014-0115, (to appear in print), available at arXiv:1208.0583v5.
[Vog05] D. Vogel, *On the Galois group of 2-extensions with restricted ramification*, J. reine angew. Math. **581** (2005), 117–150.

[Wei94] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wit37] E. Witt, *Treu Darstellungen Liescher Ringe*, J. reine angew. Math. **177** (1937), 152–160.

[Zas39] H. Zassenhaus, *Ein Verfahren, jeder endlichen p-Gruppe einen Lie-Ring mit Charakteristik p zuzuordnen*, Abh. Math. Sem. Univ. Hambg. **13** (1939), 200–206.

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