Ferromagnetism of electron gas

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Abstract

In current work, we investigate the density and temperature dependence of polarization parameter; using the relativistic formalism for the electron-electron interaction within the fermi liquid model. we calculate the spin dependent scattering matric elements in relativistic region, and then obtain the non-relativistic behavior to study the magnetic properties of an electron gas. By varying the polarization parameter, we minimized the free energy and then obtain the polarization of the system as a function of density and temperature. At zero temperature the exact results for polarization and magnetic susceptibility have obtained. It has been shown that for a given temperature (density) there is critical density (temperature) that the ferromagnetic phase can appears in electron gas. Our results show that at nonzero temperatures and in very low and very hight densities the ferromagnetism phase can not be exist.

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I. INTRODUCTION

The properties of electron gas are investigated in several works \[1, 7\]. One of the important cases, is the ferromagnetism of electron gas \[7\]. The spontaneous magnetization may appear in different densities for different temperatures and the polarization of the system is a function of density and temperature. By assuming the spin-spin interaction inside the system we can study the magnetic property of the system. Several models study the possibility of existence of ferromagnetism phase inside the solids by assuming the lattice system \[7\]. For gas systems by using the statistical methods for an imperfect fermi gas we can show that the ferromagnetism phase can be exist \[7\]. In current work we examine the possibility of existence of ferromagnetism phase inside the fermi gas by one photon exchange interaction between two electron and Fermi liquid model to calculate the energy density of the system. Because the spin dependence of the Landau Fermi liquid interaction function is due to the relativistic effects (spin-spin and spin-orbit interaction) and the exchange interaction \[7\], we calculate the direct and exchange diagrams contributions (with defined spin directions) to thermodynamic quantities, in relativistic formalism. After this we obtain the non-relativistic limit, because in the ultra-relativistic region the exchange interaction becomes repulsive and can not cause to ferromagnetic phase inside system. By varying the free energy with respected to the \( p \) (polarization parameter) at various density and temperature we can find the minimum of the free energy for given density and temperature. At end, we obtained the magnetic susceptibility of system at zero temperature, and our results shoe that the \( \chi \) has a divergence. This means that the phase transition is second order inside the system.

II. EXCHANGE AND DIRECT CONTRIBUTIONS IN ENERGY DENSITY

To obtain the exchange and direct diagrams contributions we write the Lorentz invariant matric elements for two electron scattering in order \( g^2 \). It can be shown that the direct diagram portion in energy density is 

\[
E_d \sim -\rho V^2, \tag{1}
\]

the \( \frac{E_d}{N} \sim V^{-\frac{1}{3}} \) becomes zero in the thermodynamic limit. So we can ignore the direct interactions and only calculate the exchange interactions contribution in the thermodynamic limit. We define the up and down spin states in the rest frame of each electron, with the
vector $\vec{\xi} = (0,0,\pm1)$. Where the up and down spin states corresponded to the $\pm$ signs. Electrons with momentum $p$ and spin $s$ are described by spinors $u(p,s)$, where the $s^\mu$ is a Lorentz vector. In the rest frame of particles the $s^\mu$ reduced to the unit spatial vector $\vec{\xi} = (0,0,\pm1)$. The components of $s^\mu$ in the frame which the particle moves with momentum $\vec{p}$ are obtained by Lorentz transformation [2]:

$$s^\mu = (\frac{\vec{p}.\vec{\xi}}{m}, \vec{\xi} + \frac{\vec{p}.\vec{\xi}}{m(E+m)}\vec{p}).$$

(2)

Where $E = \sqrt{m^2 + p^2}$. It can be checked that the Eq(2) satisfied the normalization and orthogonality conditions, which in the rest frame are evidence. State with definite polarization obtained by applying the projection operator $\Sigma(s) = \frac{1}{2}(1 + \gamma_5 \delta)$. The polarization density matrix can be written as [2]:

$$\rho(p,s) = \frac{1}{2m}(q' + m)\Sigma(s).$$

(3)

For exchange interaction we can write the Lorentz invariant matrix element as follow [2]:

$$\mathcal{M}^{\xi}_{k.q',q} = g^2 tr(\gamma_\mu \rho(k,\xi)\gamma^\mu \rho(q,\xi')) \frac{1}{(k-q)^2}.$$  

(4)

It must mentioned that if we use above relation in Landau theory, then the stability of the fermi liquid do not satisfied. If we add a term like $\delta^2$ in the dominator of the Eq.(4), then the stability can satisfied. The $\delta^2$ can interpreted as higher order corrections to the gauge boson propagator. For simplicity we ignore the $\delta^2$ and set it equal to zero. After performing traces the result becomes [2]:

$$\mathcal{M}^{\xi}_{k.q,q'} = \frac{1}{2m^2} [2m^2 - k.q - (k,\xi)(q,\xi')]$$

$$m^2 \xi.\xi' + \frac{1}{(\varepsilon_k + m)(\varepsilon_q + m)}$$

$$\times \{m(\varepsilon_k + m)(\xi.q)(\xi'.q) +$$

$$m(\varepsilon_q + m)(\xi.k)(\xi'.k) +$$

$$(k.q)(\xi.k)(\xi'.q)\} \frac{1}{(k-q)^2}.$$  

(5)

The Landau Fermi liquid interaction function related to Lorentz invariant matrix element via:

$$f_{k,q,q'} = \frac{m^2}{E_k E_q} \mathcal{M}^{\xi}_{k.q,q'}.$$  

(6)
If $\xi = \xi'$ (parallel spins) we have the spin non-flip interaction and if $\xi = -\xi'$ (anti parallel spins) we have flip interaction. So the exchange energy density for flip and non-flip interactions can be written:

$$
\varepsilon_{ex}^{\text{flip}} = \int \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} n(k^+) n(q^-) f_{k,q}^{\text{flip}}
$$

$$
\varepsilon_{ex}^{\text{non-flip}} = \frac{1}{2} \sum_{i=\pm} \int \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \times n(k^i) n(q^j) f_{k,q}^{\text{non-flip}}
$$

In above equations $\pm$ correspond to

$$
n_+ = n_q(1 + p)/2
$$

$$
n_- = n_q(1 - p)/2
$$

Where the $n_{\pm}$ and $p$ are the density of spin up and spin down quarks and polarization parameter, respectively. $n(k^i)$ are the Fermi distribution functions. We can calculate integrals over angles using following relations [3]:

$$
\int \frac{d\Omega}{1 + k, \hat{r}} = \frac{2\pi}{k} \ln\left(\frac{1 + k}{1 - k}\right)
$$

$$
\int d\Omega(\hat{r}, \hat{a})(\hat{r}, \hat{b}) = \frac{4\pi}{3} \hat{a} \cdot \hat{b}
$$

$$
\int d\Omega \frac{\hat{a}, \hat{r}}{1 + \hat{k}, \hat{r}} = \frac{4\pi}{k^2} \hat{a} \cdot \hat{k} [1 - \frac{1}{2k} \ln\left(\frac{1 + k}{1 - k}\right)]
$$

$$
\int d\Omega (\hat{a}, \hat{r})(\hat{b}, \hat{r}) = \frac{2\pi}{a^3} \ln\left(\frac{1 + k}{1 - k}\right) \hat{a} \cdot \hat{b}
$$

After integrating over angles we perform the integration on the momentums by numerical methods. The numerical integration arguments are as follow:

$$
\varepsilon_{ex} = \frac{1}{(2\pi)^6} \int k^2 q^2 dk dq A(k, q) B(k, q) \left(\sum_{i=1}^{6} T_i\right)
$$
with:

\[ A(k, q) = \frac{m^2}{\varepsilon_k \varepsilon_q} \frac{2g^2}{18m^2kq} \]
\[ B(k, q) = \frac{m^2 - \varepsilon_k \varepsilon_q}{kq} \]
\[ B_1(k, q) = 2m^2 + m^2 \zeta \tilde{\zeta} - \varepsilon_k \varepsilon_q \]
\[ Z_1(k, q) = \frac{m(\varepsilon_k + m)}{(\varepsilon_k + m)(\varepsilon_q + m)} \]
\[ Z_2(k, q) = \frac{m(\varepsilon_q + m)}{(\varepsilon_k + m)(\varepsilon_q + m)} \]
\[ Z_3(k, q) = \frac{1}{(\varepsilon_k + m)(\varepsilon_q + m)} \]

\[ T_1 = 8\pi^2 B_1(k, q) \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \]
\[ T_2 = 16\pi^2 kq \left( 1 - \frac{B(k, q)}{2} \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \right) \]
\[ T_3 = -\frac{16\pi^2}{3} kq \left( 1 - \frac{B(k, q)}{2} \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \right) \tilde{\zeta} \tilde{\zeta}' \]
\[ T_4 = \frac{8\pi^2}{3} Z_1(k, q) q^2 \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \tilde{\zeta} \tilde{\zeta}' \]
\[ T_5 = \frac{8\pi^2}{3} Z_2(k, q) k^2 \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \tilde{\zeta} \tilde{\zeta}' \]
\[ T_6 = Z_3(k, q) \frac{8\pi^2}{3} B(k, q)^2 k^2 q^2 \ln \left( \frac{B(k, q) + 1}{B(k, q) - 1} \right) \tilde{\zeta} \tilde{\zeta}' \]

Integration results for \( p = 0 \) at zero temperature are the same as unpolarized exchange energy results that obtained before \[4, 5\].

\[ \varepsilon_{\text{ex unpol}}^\text{unpol} = -\frac{\alpha}{4\pi^2} \{ k_F^4 \}
- \frac{3}{2} \left[ E_F k_F - m_q^2 \ln \left( \frac{E_F + k_F}{m_q} \right) \right]^2 , \]

where \( E_F = \sqrt{m_q^2 + k_F^2} \) is the Fermi energy \[4, 6\] and the \( \alpha = g^2/4\pi \). According to fermi liquid theory the interaction function on the Fermi surface has the form \[7\]:

\[ \left( \frac{k_F}{\pi^2 v_f} \right)^{-2} f_{k \xi, q \xi} = F(\theta) + \tilde{s} \tilde{s} G(\theta) . \]

In Eq. \(14\) \( \theta \) is the angle between two electron momentum on the Fermi surface. Comparing Eq. \(14\) with Eq. \(10\) and Eq. \(11\) one can see that on the Fermi surface only the scalar product
of two electron spin operators appears \[7\]. In non-relativistic region we can use the \(k_f \ll m\) approximation and then we have:

\[
\mathcal{M}_{\text{non-rel}} = \frac{-g^2}{2} \frac{1 + \xi \xi'}{|k - q|^2}
\]  

(15)

One can see that if we have \(\xi = -\xi'\) then the Lorentz invariant matrix elements vanishes, this means that in non-relativistic region the spin flip contribution in interaction energy density vanishes. The spin non-flip exchange and kinetic energy density in non-relativistic case at zero temperature is:

\[
\varepsilon_{\text{ex}}^{\text{non-rel}}(T = 0) = \frac{-\alpha}{8\pi^3} k_f^4 ((1 + p)^{\frac{4}{3}} + (1 - p)^{\frac{4}{3}})
\]

\[
\varepsilon_{\text{kin}}^{\text{non-rel}}(T = 0) = \frac{k_f^5}{20m\pi^2} ((1 + p)^{\frac{5}{3}} + (1 - p)^{\frac{5}{3}})
\]

(16)

### III. EQUATION OF STATE AT LOW TEMPERATURE

In the low temperature the kinetic and exchange energy densities, and entropy density of the system become:

\[
\varepsilon_{\text{kin}}^{\text{non-rel}}(T) = \varepsilon_{\text{kin}}^0 [1 + \frac{5\pi^2}{12} \left(\frac{2m^*T}{k_f^2}\right)^2]
\]

\[
\varepsilon_{\text{ex}}^{\text{non-rel}}(T) = \varepsilon_{\text{ex}}^0 [1 - \frac{\pi^2}{6} \left(\frac{2m^*T}{k_f^2}\right)^2]
\]

\[
S = s^+ + s^- = \frac{\pi^2 nm^*T}{2k_f^2} [(1 + p)^{\frac{1}{3}} + (1 - p)^{\frac{1}{3}}]
\]

(17)

where the \(m^*, \varepsilon_{\text{kin}}^0\) and \(\varepsilon_{\text{ex}}^0\) are the effective mass of electrons, non-relativistic kinetic and exchange energies at zero temperature, respectively. Using the results of equation (17) one can obtain the following result for free energy density:

\[
F = \varepsilon_{\text{ex}} + \varepsilon_{\text{kin}} - TS
\]

\[
= \varepsilon_{\text{ex}}^0 [1 - \frac{\pi^2}{6} \left(\frac{2m^*T}{k_f^2}\right)^2]
\]

\[
+ \varepsilon_{\text{kin}}^0 [1 + \frac{5\pi^2}{12} \left(\frac{2m^*T}{k_f^2}\right)^2]
\]

\[
- \frac{\pi^2 nm^*T}{2k_f^2} [(1 + p)^{\frac{1}{3}} + (1 - p)^{\frac{1}{3}}]
\]

(18)
According to the thermodynamics fundamental relations, the free energy must be minimum at the equilibrium state, so we must have:

\[ dF|_{T,n} = 0 \]  

(19)

At fixed density and temperature we can write:

\[ \frac{\partial F}{\partial m^*} dm^* + \frac{\partial F}{\partial p} dp = 0 \]  

(20)

Because the effective mass and polarization parameter are the independent variables for free energy, then we must have:

\[ \frac{\partial F}{\partial m^*} = 0 \]
\[ \frac{\partial F}{\partial p} = 0 \]  

(21)

Solving the above equations simultaneously, we can obtain the density and temperature dependence of \( m^* \) and \( p \). The derivatives of free energy with respected to \( p \) and \( m^* \) can be written as follow:

\[
\frac{\partial F}{\partial m^*} = \left( \frac{T}{k_f^2} \right)^2 m^* \left[ \varepsilon_0^0 \left( \frac{10\pi^2}{3} \right) - \varepsilon_0^0 \frac{4\pi^2}{3} \right]
\]
\[ -\left( \frac{T}{k_f^2} \right)^2 \varepsilon_{\text{kin}}^0 \frac{\pi^2 n}{2} \left[ (1 + p)^{\frac{1}{3}} + (1 - p)^{\frac{1}{3}} \right] \]  

(22)

\[
\frac{\partial F}{\partial p} = \left[ 1 + \frac{5\pi^2}{12} \left( \frac{2m^* T^2}{k_f^2} \right) \right] \frac{\partial \varepsilon_{\text{kin}}^0}{\partial p}
+ \left[ 1 - \frac{\pi^2}{6} \left( \frac{2m^* T}{k_f^2} \right)^2 \right] \frac{\partial \varepsilon_{\text{ex}}^0}{\partial p}
- \frac{\pi^2 nm^*}{6} \left( \frac{T}{k_f} \right)^2 \left[ (1 + p)^{\frac{7}{2}} - (1 - p)^{\frac{7}{2}} \right] \]  

(23)

The derivatives of kinetic and exchange energies with respected to \( p \) are given bellow:

\[
\frac{\partial}{\partial p} \varepsilon_{\text{ex}}^0 = -\frac{4}{3} \frac{\alpha}{8\pi^\frac{3}{2}} k_f^4 [(1 + p)^{\frac{7}{2}} - (1 - p)^{\frac{7}{2}}] \]  

(24)

\[
\frac{\partial}{\partial p} \varepsilon_{\text{kin}}^0 = \frac{5}{3} \frac{k_f^5}{20\pi^2 n} [(1 + p)^{\frac{7}{2}} - (1 - p)^{\frac{7}{2}}] \]  

(25)

By solving the equation (22), one can find \( m^* \) as a function of \( p \).

\[
m^* = \frac{4k_f^2 n}{3} \frac{(1 + p)^{\frac{7}{2}} + (1 - p)^{\frac{7}{2}}}{5 \varepsilon_{\text{kin}}^0 - 2 \varepsilon_{\text{ex}}^0} \]  

(26)
Inserting this function in equation (23), we can obtain the density and temperature dependence of polarization parameter. At zero temperature the equation (23) becomes simpler and one can obtain the below result:

\[
\frac{\partial}{\partial p} \varepsilon_{ex}^0 + \frac{\partial}{\partial p} \varepsilon_{kin}^0 = 0
\] (27)

The resultant equation from Eq.(27) is:

\[
\frac{k_f}{2m} \left[ (1 + p)^{\frac{2}{3}} - (1 - p)^{\frac{2}{3}} \right] = \frac{\alpha}{\pi} \left[ (1 + p)^{\frac{1}{3}} - (1 - p)^{\frac{1}{3}} \right]
\] (28)

This result is very look like to the well known results of the spontaneous magnetization of an imperfect Fermi gas [8]. But in our calculus the interaction part of the hamiltonian is not independent of the spins alignment, and this dependence change the right side of the equation (28).

IV. RESULTS AND DISCUSSION

![FIG. 1: The polarization parameter of system at zero temperature. The horizontal axis shows the \( \frac{2\alpha m}{\pi k_f} \). For \( \alpha m = \pi k_f \) the polarization of the system vanishes. The value of critical density for phase transition is proportional to the coupling constant. If we ignore the interaction inside system the ferromagnetism can not appeared and if we use the effective and great coupling constant then the critical density increases.](image)

At zero temperature we can solve the Eq.(23). The result is shown in figure(1), there is a specific density (depended on coupling constant), that for densities higher than it, the ferromagnetism phase will appeared. This figure also shows that if density becomes small the ferromagnetism phase can not exist inside the system. At non-zero temperature we can
solve the Eq.(22) and find \( m^* \) as a function of \( p, T \) and density. Minimizing the free energy with respected to \( p \), at constant temperature, yields to figures 2 and 3. The fig [2] shows the density dependence of the polarization parameter at various temperatures. Increasing the temperature leads to disappearing the ferromagnetic phase. At low densities, also the ferromagnetism disappeared. Comparing the fermi energy of the system with respected to the thermal energy of the system can help us to find the reason. For states that their fermi energy is smaller than thermal energy, the fermi distribution function reduced to the Maxwell distribution and the difference between different spin states ignorable. Figure [3] display the temperature dependence of the polarization parameter at different densities. As we expected, by increasing the temperature, the polarization of system reduced and suddenly vanishes. To obtain the magnetic susceptibility of the system we can use the following relation [9]:

\[
\chi^{-1} = \frac{1}{\rho^2 \mu_B} \frac{d^2 F}{dp^2} \bigg|_{p=0} 
\]

(29)

At zero temperature the \( p = 0 \) equivalent to \( k_f = \frac{m \alpha}{\pi} \). The magnetic susceptibility of
FIG. 3: The polarization parameter as a function of temperature at different densities. By increasing temperature, the polarization of system reduced.

the system is plotted as a function of fermi momentum in figure 4. It has a divergence at $k_f = \frac{mA}{\pi}$. This divergence is the sign of second order phase transition inside the system.

FIG. 4: The magnetic susceptibility of system at zero temperature as a function of fermi momentum of system. As $k_f \rightarrow \frac{mA}{\pi}$ the $\chi$ tends to infinity and diverges. This means that the phase transition is second order phase transition.
V. SUMMARY

The possibility of existence of the ferromagnetism phase inside the electron gas investigated. The equations have written at low temperature limit. However we can write the equations in general form. we used the one photon exchange interaction at relativistic region and then calculate the non-relativistic limit of interaction. The spin dependence of the hamiltonian appears automatically. According the results, the ferromagnetism phase can appeared at low temperatures. The coupling constant has important rule in critical density and temperature for phase transition. The magnetic susceptibility at zero temperature calculated. The magnetic susceptibility becomes infinite at $k_f \rightarrow \frac{m \pi}{a}$, and this means that the phase transition is second order. We also can obtain the critical exponents of the system by expanding the magnetization and magnetic susceptibility of the system near the critical point.

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