A Feynman-Kac formula for magnetic monopoles

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May 20, 2020

Abstract

We consider the quantum mechanics of a charged particle in the presence of Dirac’s magnetic monopole. Wave functions are sections of a complex line bundle and the magnetic potential is a connection on the bundle. We establish a Feynman-Kac formula expressing solutions of the imaginary time Schrödinger equation as stochastic integrals.

1 Introduction

Consider a single charged particle (an electron) in the presence of a magnetic potential one-form $A = \sum_k A_k dx^k$ with magnetic field 2-form $B = dA$. The quantum mechanical Hamiltonian for this particle is an operator $H$ on $L^2(\mathbb{R}^3)$ which has the form

$$H = -\frac{1}{2} \sum_k (\partial_k - iA_k(x))^2$$

(1)

The Feynman-Kac formula expresses the semi-group $e^{-tH}$ in terms of stochastic integrals. If $X_t$ is Brownian motion in $\mathbb{R}^3$ starting at $x$, then the equation is for $f \in L^2(\mathbb{R}^3)$

$$(e^{-tH}f)(x) = E^x \left( \exp \left( -i \int_0^t \sum_k A_k(X_t) \circ dX_t^k \right) f(X_t) \right)$$

(2)

Here $\int_0^t \sum_k A_k(X_t) \circ dX_t^k$ is the Stratonovich stochastic integral. This has been rather thoroughly studied, often in the presence of an additional scalar potential. See for example Hinz [4] for references.

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In this paper we establish a version of this for Dirac’s magnetic monopole. The situation is more difficult since a magnetic monopole creates a magnetic field for which there is no smooth magnetic potential. Instead one has to formulate the problem on a Hilbert space of sections of a certain $U(1)$ line bundle over $M = \mathbb{R}^3 - \{0\}$, and replace the potential by a connection on this bundle.

The problem can be regarded as a special case of the problem of establishing a Feynman-Kac formula for a Hamiltonian on sections of a general vector bundle on a manifold. It this generality results have been obtained by Norris [6] and Güneysu [2]. However Norris assumes a compact manifold and Güneysu assumes the manifold is geodesically complete. Neither assumption is true for the monopole so it needs the special treatment. On the other hand the analysis is somewhat easier in that the manifold is an open subset of $\mathbb{R}^3$ and the bundle has the abelian structure group $U(1)$.

In section 2 we define the bundle and the connection. In section 3 we introduce various eigenfunction expansions and use them to define the Hamiltonian as a self-adjoint operator. In section 4 we review some facts about stochastic integrals. In section 5 we define stochastic parallel translation. Finally in section 6 we prove the Feynman-Kac formula.

2 The monopole bundle

The $U(1)$ bundle $E$ is a manifold together with a smooth map $\pi : E \to M$ such that the fibers $E_x = \pi^{-1}x$ are vector spaces isomorphic to $\mathbb{C}$. We take an covering of $M = \mathbb{R}^3 - \{0\}$ by two open set $U_{\pm}$. In spherical coordinates they are defined by

$$U_+ = \left\{ x \in M : 0 \leq \theta < \frac{\pi}{2} + \alpha \right\}$$
$$U_- = \left\{ x \in M : \frac{1}{2}\pi - \alpha < \theta \leq \pi \right\}$$

(3)

Here $0 < \alpha < \frac{1}{2}\pi$ is a fixed angle. If $\delta = \sin \alpha$ then $0 < \delta < 1$ and $\cos(\frac{\pi}{2} \pm \alpha) = \mp \delta$.

Then in Cartesian coordinates

$$U_+ = \left\{ x \in M : 1 \geq \frac{x_3}{|x|} > -\delta \right\}$$
$$U_- = \left\{ x \in M : \delta > \frac{x_3}{|x|} \geq -1 \right\}$$

(4)

$E$ is defined so in each region there is a trivialization which is a diffeomorphism

$$h_\pm : \pi^{-1}(U_\pm) \to U_\pm \times \mathbb{C}$$

(5)

such that for $x \in U_\pm$ the map $h_\pm : E_x \to \{x\} \times \mathbb{C}$ is a linear isomorphism. These maps are related in $U_+ \cap U_-$ by the transition functions for fixed integer $n$

$$h_+ h_-^{-1} = e^{2in\phi}$$

(6)
which means that if \( v \in E_x \) and \( h_{\pm}v = (x, v_{\pm}) \) then \( v_+ = e^{i2n\phi(x)}v_- \). Concretely \( E \) can be constructed as equivalence classes in \( M \times \mathbb{C} \) with \( (x, v_+) \sim (x, v_-) \) if \( x \in U_+ \cap U_- \)
and \( v_+ = e^{i2n\phi(x)}v_- \). There is an inner product on \( E_x \) defined unambiguously by

\[<v, w> = \frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2} \]

Together with \( n \), \( \phi \) is designed to compensate the gauge transformation (7).

A connection can be defined by a one-form \( A^\pm \) on \( U_\pm \) which in \( U_+ \cap U_- \) are related by the gauge transformation

\[ A^+ = A^- + 2nd\phi \]

They are defined by

\[ A^\pm = -n(cos \theta \mp 1)d\phi = n\left(\frac{x_3}{|x|} \mp 1\right)\frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2} \]

These each yield the magnetic field \( B \) of a monopole of strength \( n \in \mathbb{Z}, n \neq 0 \):

\[ dA^\pm = n \sin \theta \, d\theta \wedge d\phi = \frac{n}{r^2} dr \equiv B \]

Neither magnetic potential \( A^\pm \) is singular, but there is no one-form \( A \) defined on all of \( M \) such that \( dA = B \), which is why we need the vector bundle. The transition functions were designed to compensate the gauge transformation (7).

Let us check that \( A^+ \) on \( U_+ \) is not singular and get an estimate on its size. It has a possible singularity on the \( x_3 \) axis from the factor \( (x_1^2 + x_2^2)^{-1} \). However we can write

\[ \frac{x_3}{|x|} - 1 = \frac{1}{|x|}(x_3 - |x|) = -\frac{1}{|x|}x_3 \]

and hence

\[ A^+ = \frac{-n}{|x|(x_3 + |x|)}(x_2 dx_1 - x_1 dx_2) \]

Since \( x_3 + |x| \) is bounded away from zero this is smooth on \( U_+ \). Indeed \( x_3 \geq -\delta|x| \) on \( U_+ \), hence \( x_3 + |x| \geq (1 - \delta)|x| \), and hence \( x_3(x_3 + |x|)^{-1} \leq (1 - \delta)^{-1} \). Thus for all \( x \in U_+ \) we have

\[ |A^+_k(x)| \leq |n|(1 - \delta)^{-1}|x|^{-1} \]

Parallel translation on a curve \( x: [0, t] \rightarrow M \) is a linear isomorphism \( \Pi_t : E_{x_0} \rightarrow E_{x_t} \) defined as follows. Let \( v \in E_{x_0} \) and suppose the curve is entirely contained in some \( U_{\pm} \).

If \( h_{\pm}v = (x_0, v_{\pm}) \) then \( \Pi_t v \) is defined by

\[ h_{\pm}(\Pi_t v) = (x_t, (\Pi_t v)_{\pm}) = (x_t, \Pi_t^\pm v_{\pm}) \]

where \( \Pi_t^\pm \) is multiplication by (summation convention)

\[ \Pi_t^\pm = \exp \left( i \int_0^t A^\pm_k(x_s)dx_s \right) \]

If the curve is entirely contained in both \( U_{\pm} \) then (7) implies \( \Pi_t^+ = e^{2in\phi(x_t)}\Pi_t^- e^{-2in\phi(x_0)}. \)

Together with \( v_+ = e^{2in\phi(x_0)}v_- \) this implies that \( \Pi_t^+ v_+ = e^{2in\phi(x_t)}\Pi_t^- v_- \) and hence the
The definition of $\Pi_t$ in (13) is independent of the choice $U_\pm$. Finally parallel translation for a curve not contained in a single trivialization can be defined by patching together pieces which stay within one trivialization.

Now we can define a covariant derivative on sections of $E$. A section of $E$ is a map $f : M \to E$ such that $\pi(f(x)) = x$ which says $f(x) \in E_x$. The set of all smooth sections is denoted $\Gamma(E)$. For $x \in M$ and $e_k$ the standard basis for $\mathbb{R}^3$ consider parallel transport along the curve $x_t = x + te_k$. For $f \in \Gamma(E)$ define $\nabla_k f \in \Gamma(E)$ as the limit $\lim_{t \to 0} t^{-1} \left( \Pi_{-t}^{-1} f(x_t) - f(x) \right)$

\begin{equation}
(\nabla_k f)(x) = \lim_{t \to 0} t^{-1} \left( \Pi_{-t}^{-1} f(x_t) - f(x) \right)
\end{equation}

If $x \in U_\pm$ and $h_\pm f(x) = (x, f_\pm(x))$ this is computed as

\begin{equation}
(\nabla_k f)_\pm(x) = \left( (\partial_k - iA_k^\pm) f_\pm \right)(x)
\end{equation}

3 The Hamiltonian

3.1 definitions

The Hamiltonian for our problem is initially defined on smooth sections $\psi \in \Gamma(E)$ by

\begin{equation}
H \psi = \frac{-1}{2} \left( \sum_k \nabla_k \nabla_k \right) \psi
\end{equation}

We want to define it as a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(E, dx)$ consisting of all measurable sections $\psi$ such that $\|\psi\|^2 = \int_M |\psi(x)|^2 dx < \infty$.

We use the strategy of Wu-Yang [10] and the author [1]. First change to spherical coordinates. For any function $\Psi$ on $M = \mathbb{R}^3 - \{0\}$ we define $\hat{\Psi}$ on $\mathbb{R}^+ \times S^2$ by

\begin{equation}
\hat{\Psi}(r, \theta, \phi) = \Psi(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)
\end{equation}

If $\Psi$ is a section of $\pi: E \to M$, then $\hat{\Psi}$ is a section of a vector bundle $\pi: E' \to \mathbb{R}^+ \times S^2$. With $U'_\pm = U_\pm \cap S^2$ the bundle $E'$ has trivializations on $\mathbb{R}^+ \times U'_\pm$ still with transition functions $h_+ h_-^{-1} = e^{2in\phi}$. For any $\Psi \in \mathcal{C}^2(E)$ we have $\hat{\Psi} \in \mathcal{C}^2(\hat{E})$ and $(H \Psi)^\wedge = \hat{H} \hat{\Psi}$ where

\begin{equation}
\hat{H} f = \frac{1}{2} \left( - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} + \frac{1}{r^2}(L^2 - n^2) \right)
\end{equation}

Here $L^2 = L_1^2 + L_2^2 + L_3^2$ and $L_i$ are angular momentum operators. In the trivialization on $\mathbb{R}^+ \times U'_\pm$ we have

\begin{align}
L_1^\pm &= i \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) - n \cos \phi \left( \frac{1 + \cos \theta}{\sin \theta} \right) \\
L_2^\pm &= i \left( - \cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) - n \sin \phi \left( \frac{1 + \cos \theta}{\sin \theta} \right) \\
L_3^\pm &= - i \frac{\partial}{\partial \phi} \mp n
\end{align}

\text{\textsuperscript{4}}
The map $\Psi \rightarrow \hat{\Psi}$ is unitary from $\mathcal{H} = L^2(E, dx)$ to $\hat{\mathcal{H}} = L^2(E', r^2 dr d\Omega)$ where $d\Omega = \sin \theta d\theta d\phi$ is the Haar measure on $S^2$. In fact since the the transition functions only depend on the angular variable $\phi$ we can make the identification

$$\hat{\mathcal{H}} = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(\tilde{E}, d\Omega)$$

where $\tilde{E}$ is a vector bundle $\pi : \tilde{E} \rightarrow S^2$ with trivializations on $U^\prime_\pm$ which still satisfy $h_+ h_-^{-1} = e^{2i n \phi}$.

The joint spectrum of the commuting operators $L^2, L^3$ has been studied by Wu-Yang [10]. They find a complete set of eigenfunctions $Y_{n,\ell,m}(\theta, \phi)$ which are sections of $L^2(\tilde{E}, d\Omega)$. They take values $\ell \geq |n|$ and $|m| \leq \ell$ and satisfy

$$L^2 Y_{n,\ell,m} = \ell(\ell + 1) Y_{n,\ell,m} \quad L^3 Y_{n,\ell,m} = m Y_{n,\ell,m}$$

The explicit expression for $Y_{n,\ell,m}$ is given in section 3.3. Let $K_{n,\ell}$ be the eigenspace spanned by $Y_{n,\ell,m}$ with $|m| \leq \ell$. Then the Hilbert space can be identified with

$$\hat{\mathcal{H}} = \bigoplus_{\ell = |n|}^{\infty} L^2(\mathbb{R}^+, r^2 dr) \otimes K_{n,\ell}$$

The Hamiltonian is now

$$\hat{H} = \bigoplus_{\ell = |n|}^{\infty} h_{\ell} \otimes I$$

where on smooth functions

$$h_{\ell} = \frac{1}{2} \left( - \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell + 1) - n^2}{r^2} \right)$$

### 3.2 radial eigenfunctions

To further study the radial Hamiltonian $h_{\ell}$ we make a continuum eigenfunction expansion. Continuum eigenfunctions for $h_{\ell}$ are

$$(kr)^{-\frac{3}{2}} J_\mu(kr) \quad \mu^2 = \ell(\ell + 1) - n^2 + \frac{1}{4}$$

where $J_\mu$ is the Bessel function of order $\mu > 0$ regular at the origin. We have

$$h_{\ell} \left( (kr)^{-\frac{3}{2}} J_\mu(kr) \right) = \frac{1}{2} k^2 \left( (kr)^{-\frac{3}{2}} J_\mu(kr) \right)$$

For future reference we note that since $\ell \geq |n| \geq 1$ we have

$$\mu \geq \frac{1}{2} \sqrt{5} \geq 1.12$$

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The eigenfunction expansion is the Fourier-Bessel transform and we recall some relevant facts [9, 1]. The transform is defined by the formula

\[ \psi^\#(k) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr) \psi(r) r^2 dr \]  

(29)

at first for say \( \psi \in C_0^\infty(\mathbb{R}^+) \). It satisfies

\[ \int_0^\infty |\psi^\#(k)|^2 k^2 dk = \int_0^\infty |\psi(r)|^2 r^2 dr \]  

(30)

and extends to a unitary operator from \( L^2(\mathbb{R}^+, r^2 dr) \) to \( L^2(\mathbb{R}^+, k^2 dk) \). The inverse is given by exactly the same formula

\[ \psi(r) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr) \psi^\#(k) k^2 dk \]  

(31)

Now define the dense domain

\[ \mathcal{D}_0 = \{ \psi \in L^2(\mathbb{R}^+, r^2 dr) : \psi^\# \in C_0^\infty(\mathbb{R}^+) \} \]  

(32)

To analyze this domain we use the fact that \( x^{-\frac{1}{2}} J_\mu(x) \) is a bounded smooth function with the asymptotics

\[ x^{-\frac{1}{2}} J_\mu(x) = \begin{cases} 
\mathcal{O}(|x|^{\mu - \frac{1}{2}}) & x \to 0 \\
\mathcal{O}(|x|^{-1}) & x \to \infty 
\end{cases} \]  

(33)

\textbf{Lemma 1.} \( \mathcal{D}_0 \subset C^2(\mathbb{R}^+) \). If \( \psi \in \mathcal{D}_0 \) we have the asymptotics for \( m = 0, 1, 2 \)

\[ \psi^{(m)}(r) = \begin{cases} 
\mathcal{O}(r^{\mu - \frac{1}{2} - m}) & r \to 0 \\
\mathcal{O}(r^{-1-m}) & r \to \infty 
\end{cases} \]  

(34)

Furthermore for \( \psi \in \mathcal{D}_0 \) we have \( \psi, \psi', \psi'' \in L^2(\mathbb{R}^+, r^2 dr) \).

\textbf{Remark.} In fact \( \mathcal{D}_0 \subset C^\infty(\mathbb{R}^+) \), and if we worked harder using Bessel function identities as in [1], we could show the long distance asymptotics are \( \mathcal{O}(r^{-N}) \) for any \( N \).

\textbf{Proof.} \( \psi^\# \in C_0^\infty(\mathbb{R}^+) \) means that \( k \) is bounded away from zero and infinity. So the asymptotics for \( \psi \) follow from (33). We know \( \psi \in L^2(\mathbb{R}^+, r^2 dr) \).

For the derivative differentiate (31) under the integral sign and then integrate by parts to obtain

\[ \psi'(r) = \int_0^\infty d/dr \left((kr)^{-\frac{1}{2}} J_\mu(kr)\right) \psi^\#(k) k^2 dk \]

\[ = \int_0^\infty -\frac{k}{r} \frac{d}{dk} \left((kr)^{-\frac{1}{2}} J_\mu(kr)\right) \psi^\#(k) k^2 dk \]  

(35)

\[ = \frac{-1}{r} \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr) \left(3\psi^\#(k) + k(\psi^\#)'(k)\right) k^2 dk \]
So $\psi'(r)$ is $r^{-1}$ times a function in $D_0$ and the asymptotics follow. Further $\psi' \in L^2(\mathbb{R}^+, r^2 dr)$ since for $r \geq 1$ it is bounded by a function in $D_0 \subset L^2$, and for $r \leq 1$ we have

$$\int_{r \leq 1} |\psi'(r)|^2 r^2 dr < O(1) \int_{r \leq 1} (r^{\mu-2})^2 r^2 dr < O(1) \int_{r \leq 1} r^{2\mu-1} dr < \infty$$

Similarly $\psi''(r)$ is $r^{-2}$ times a function in $D_0$ and the asymptotics follow. For $\psi'' \in L^2$ the relevant integral is $\int_{r \leq 1} r^{2\mu-3} dr < \infty$. (Higher derivatives would not be in $L^2$). This completes the proof.

**Lemma 2.**

1. For $\psi \in D_0$ we have $h_\ell \psi \in D_0$ with $(h_\ell \psi)^\#(k) = \frac{1}{2} k^2 \psi^\#(k)$

2. $h_\ell$ is self-adjoint on $D(h_\ell) = \{ \psi \in L^2(\mathbb{R}^+, r^2 dr) : \frac{1}{2} k^2 \psi^\# \in L^2(\mathbb{R}^+, k^2 dk) \}$ and essentially self-adjoint on $D_0$.

3. For $\psi \in L^2(\mathbb{R}^+, r^2 dr)$ we have $(e^{-h_\ell t} \psi)^\#(k) = e^{-\frac{1}{4} k^2 t} \psi^\#(k)$. Hence if $\psi \in D_0$ then $e^{-h_\ell t} \in D_0$

**Proof.**

1. In (31) differentiate under the integral sign and use (27)

2. multiplication by $\frac{1}{2} k^2$ is self-adjoint on on $f \in L^2 : \frac{1}{2} k^2 f \in L^2$ and essentially self-adjoint on $C^\infty(\mathbb{R}^+)$. The stated result is the unitary transform of these facts.

3. The unitary transform $\psi \rightarrow \psi^\#$ provides the spectral resolution $h_\ell$ and hence the definition of $e^{-h_\ell t}$.

### 3.3 Angular eigenfunctions

We consider the monopole harmonics $\mathcal{Y}_{n,\ell,m}$ as defined by Wu-Yang [10]. They are given in the trivializations on $U'_\pm = U_\pm \cap S^2$ by

$$\mathcal{Y}_{n,\ell,m}^\pm(\xi, \phi) = \text{const}(1-\xi)^{\frac{1}{2} \alpha} (1+\xi)^{\frac{1}{2} \beta} P_{\ell+m}^{\alpha,\beta}(\xi) e^{i(m \pm n) \phi} \quad \xi = \cos \theta$$

where $\alpha = -n - m$, $\beta = n - m$ and $P_{\ell+m}^{\alpha,\beta}$ are Jacobi polynomials given by

$$P_{\ell+m}^{\alpha,\beta}(\xi) = \text{const}(1-\xi)^{-\alpha} (1+\xi)^{-\beta} d_{\ell+m}^{\alpha+\ell+m}(1-\xi)^{\alpha+\ell+m}(1+\xi)^{\beta+\ell+m}$$

These are smooth functions of $\theta, \phi$. But $\theta, \phi$ are not a smooth coordinate patch around the poles $\xi = \pm 1$. We need to express $\mathcal{Y}_{n,\ell,m}^\pm$ as a smooth function on all of $U'_\pm$. This is accomplished when we pass to Cartesian coordinates as follows
Lemma 3. There exist functions $\mathcal{Y}_{n,\ell,m}^\pm$ defined on a neighborhood of $U_\pm$ in $\mathbb{R}^3$ which are bounded and smooth with bounded derivatives and satisfy

1. $\mathcal{Y}_{n,\ell,m}^\pm(x/|x|)$ in spherical coordinates is $\mathcal{Y}_{n,\ell,m}^\pm(\xi, \phi)$
2. $\mathcal{Y}_{n,\ell,m}^\pm(x/|x|)$ as a function on $U_\pm$ in $M = \mathbb{R}^3 - \{0\}$ is in $C^2(U_\pm)$ and there is a constant $c$ such that

$$|\partial_i(\mathcal{Y}_{n,\ell,m}^\pm(x/|x|))| \leq c|x|^{-1} \quad \text{and} \quad |\partial_i\partial_j(\mathcal{Y}_{n,\ell,m}^\pm(x/|x|))| \leq c|x|^{-2} \quad (39)$$

Proof. (1.) Consider $\mathcal{Y}_{n,\ell,m}^+$. Since $\beta = \alpha + 2n$ we have

$$\left(1 - \xi \right)^{\frac{1}{2}}(1 + \xi)^{\frac{1}{2}} = (1 - \xi^2)^{\frac{1}{2}}(1 + \xi)^n = (\sin \theta)^\alpha(1 + \cos \theta)^n \quad (40)$$

and we also have $e^{i(m+n)\phi} = e^{-i\alpha\phi} = (\cos \phi - i \sin \phi)^\alpha$. Thus for the product

$$\left(1 - \xi \right)^{\frac{1}{2}}(1 + \xi)^{\frac{1}{2}}e^{i(m+n)\phi} = (\sin \theta \cos \phi - i \sin \theta \sin \phi)^\alpha(1 + \cos \theta)^n$$

$$= \left(\frac{x_1 - ix_2}{|x|}\right)^\alpha \left(1 + \frac{x_3}{|x|}\right)^n \quad (41)$$

If $\alpha$ is positive the first factor is bounded, smooth, etc. Even if $n$ is negative, the same holds for the second factor on $U_+$ where $x_3 \geq -\delta|x|$. This gives the result since $P_{\ell+m}(\xi) = P_{\ell+m}(x_3/|x|)$ is a polynomial.

If $\alpha$ is negative we combine (37), (38) and use $\alpha + \ell + m = \ell - n$ and $\beta + \ell + m = \ell + n$ to write

$$\mathcal{Y}_{n,\ell,m}^\pm(\xi, \phi) = \text{const}(1 - \xi)^{-\frac{1}{2}}(1 + \xi)^{-\frac{1}{2}} \frac{d^{\ell+m}}{d\xi^{\ell+m}}(1 - \xi)^{\ell-n}(1 + \xi)^{\ell+n}e^{i(m+n)\phi} \quad (42)$$

Now we use $e^{i(m+n)\phi} = e^{-i\alpha\phi} = (\cos \phi + i \sin \phi)^{-\alpha}$ to write

$$\left(1 - \xi \right)^{-\frac{1}{2}}(1 + \xi)^{-\frac{1}{2}}e^{i(m+n)\phi} = (\sin \theta \cos \phi + i \sin \theta \sin \phi)^{-\alpha}(1 + \cos \theta)^{-n}$$

$$= \left(\frac{x_1 + ix_2}{|x|}\right)^{-\alpha} \left(1 + \frac{x_3}{|x|}\right)^{-n} \quad (43)$$

which is bounded, smooth, etc. Since $\ell + m$, $\ell - n$, $\ell + n$ are all non-negative the derivatives of $(1 - \xi)^{\ell-n}(1 + \xi)^{\ell+n}$ just give a polynomial in $\xi = x_3/|x|$ and so this factor satisfies the hypotheses as well.

The analysis on $U_-$ is similar.

(2.) For the derivatives we compute for example

$$\partial_k(\mathcal{Y}_{n,\ell,m}^\pm(x/|x|)) = \sum_j \partial_j \mathcal{Y}_{n,\ell,m}^\pm(x/|x|) \left|\frac{|x|^2\delta_{jk} - x_jx_k}{|x|^3}\right| \quad (44)$$
which is bounded by $c|x|^{-1}$. The second derivative is similar. This completes the proof.

Remark. The construction shows that monopole harmonics may not be polynomials in $x/|x|$, unlike the usual spherical harmonics.

3.4 self-adjointness

For self-adjointness we start in spherical coordinates. We introduce the dense domain in $\hat{\mathcal{H}} = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(\tilde{E}, d\Omega)$:

$$\hat{D} = \text{finite sums of } \psi \otimes Y_{n,\ell,m} \quad \psi \in \mathcal{D}_0 \quad \ell \geq |n|, |m| \leq \ell$$

(45)

Lemma 4.

1. $\hat{H} = \bigoplus_\ell (h_\ell \otimes I)$ maps $\hat{D}$ to itself.

2. $\hat{H}$ is essentially self-adjoint on $\hat{D}$.

3. The semi-group $e^{-\hat{H}t}$ defined with the self-adjoint closure satisfies

$$e^{-\hat{H}t} = \bigoplus_{\ell=|n|}^{\infty} (e^{-h_\ell t} \otimes I)$$

(46)

and maps $\hat{D}$ to itself.

Proof.

1. This is clear since $h_\ell$ preserves $\mathcal{D}_0$.

2. $h_\ell$ is essentially self-adjoint on $\mathcal{D}_0$ by lemma 2. It follows that it follow $h_\ell \otimes I$ is essentially self-adjoint in $L^2(\mathbb{R}^+) \otimes \mathcal{K}_{n,\ell}$ on the domain of finite sums of vectors $\psi \otimes Y_{n,\ell,m}$ with $\psi \in \mathcal{D}_0$ and for fixed $n, \ell$ and $|m| \leq \ell$. Then $\hat{H} = \bigoplus_\ell (h_\ell \otimes I)$ is essentially self-adjoint in $\hat{\mathcal{H}}$ on $\hat{D}$.

3. Both sides of (46) are continuous semi-groups. Taking derivatives we see that the generators agree on $\mathcal{D}$ by part 1. Since this is a domain of essential self-adjointness the generators are the same and hence the identity. The domain $\mathcal{D}$ is preserved since $e^{-h_\ell t}$ preserves $\mathcal{D}_0$.

We want to translate this to a statement about $H$ back in Cartesian coordinates on the Hilbert space $\mathcal{H} = L^2(E, dx)$. The domain $\mathcal{D}$ becomes

$$\mathcal{D} = \text{finite sums of } \psi(|x|)Y_{n,\ell,m}(x/|x|) \quad \psi \in \mathcal{D}_0 \quad \ell \geq |n|, |m| \leq \ell$$

(47)
Lemma 5. \( \mathcal{D} \subset C^2(E) \) and for \( \Psi \in \mathcal{D} \) we have the asymptotics

\[
|\Psi(x)| = \begin{cases} 
O(|x|^{\frac{1}{2}(\sqrt{5}-1)}) & x \to 0 \\
O(|x|^{-1}) & x \to \infty
\end{cases}
\]

\[
|\nabla_k \Psi(x)| = \begin{cases} 
O(|x|^{\frac{1}{2}(\sqrt{5}-3)}) & x \to 0 \\
O(|x|^{-2}) & x \to \infty
\end{cases}
\]

\[
|\nabla_j \nabla_k \Psi(x)| = \begin{cases} 
O(|x|^{\frac{1}{2}(\sqrt{5}-5)}) & x \to 0 \\
O(|x|^{-3}) & x \to \infty
\end{cases}
\]

Furthermore \( \Psi, \nabla_k \Psi, \nabla_j \nabla_k \Psi \) are all in \( \mathcal{H} = L^2(E, dx) \)

Proof. It suffices to consider \( \Psi(x) = \psi(|x|) \mathcal{Y}_{n,\ell,m}(x/|x|) \) with \( \psi \in \mathcal{D}_0 \). Then \( \psi(|x|) \) has the stated asymptotics by lemma 1 and \( \mu \geq \sqrt{5}/2 \). Also \( \mathcal{Y}_{n,\ell,m}(x/|x|) \) is bounded by lemma 3. Hence the result. Note that \( \Psi \) is bounded.

For the derivatives it suffices to prove the bounds separately in \( U_\pm \), say \( U_+ \). In the trivialization on \( U_+ \) we have

\[
\left( \nabla_k (\psi \otimes \mathcal{Y}_{n,\ell,m}) \right)_+ (x) = \left( \partial_k - iA_k^+(x) \right) \psi(|x|) \mathcal{Y}_{n,\ell,m}^+(x|x|^{-1})
\]

\[
= \frac{x_k}{|x|} \psi'(|x|) \mathcal{Y}_{n,\ell,m}^+(x|x|^{-1}) + \psi(|x|) \partial_k \left( \mathcal{Y}_{n,\ell,m}^+(x|x|^{-1}) \right) - iA_k^+(x) \psi(|x|) \mathcal{Y}_{n,\ell,m}^+(x|x|^{-1})
\]

(49)

The first term is bounded except for the \( \psi'(|x|) \) which has the stated asymptotics by lemma 1. For the second term combine the asymptotics for \( \psi(|x|) \) with the bound \( |\partial_k (\mathcal{Y}_{n,\ell,m}^+(x|x|^{-1}))| \leq O(|x|^{-1}) \) from lemma 3 to get the stated bounds. For the last term \( |A_k^+| \leq c|x|^{-1} \) which combined with the asymptotics for \( \psi(|x|) \) gives the bound.

The statement that \( \nabla_k \Psi \) is in \( L^2(E, dx) \) follows from the asymptotics.

The second derivatives are handled in the same way. This completes the proof.

In Cartesian coordinates lemma 4 becomes:

Lemma 6. For \( \Psi \in \mathcal{D} \subset L^2(E, dx) \), \( H = -\frac{1}{2} \sum_k \nabla_k^2 \) satisfies \( (H \Psi)^\wedge = \hat{H} \hat{\Psi} \) and

1. \( H \) maps \( \mathcal{D} \) to itself.

2. \( H \) is essentially self-adjoint on \( \mathcal{D} \).

3. \( e^{-Ht} \) defined with the self-adjoint closure maps \( \mathcal{D} \) to itself.

Proof. Since \( \mathcal{D} \subset C^2(E) \) by lemma 3 the statement \( (H \Psi)^\wedge = \hat{H} \hat{\Psi} \) is just a repeat of section 3.1. The rest is a translation of the results for \( \hat{H} \) in lemma 4 by the unitary change of variables operator.
4 Stochastic integrals

We recall some definitions of stochastic integrals. General references are [3], [7], [8]. First let $X_t$ be Brownian motion in $\mathbb{R}$ starting at $X_0 = x$. It is a Gaussian process with mean $x$ and covariance $\text{Cov}(X_t, X_s) = \min(s, t)$. Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $\{X_s\}_{0 \leq s \leq t}$ and let $Y_t$ be a real valued process which is non-anticipating in the sense that $Y_t$ is $\mathcal{F}_t$ measurable.

The Ito integral is defined by

$$Z_t = \int_0^t Y_s dX_s = \lim_{n \to \infty} \sum_{k=0}^{[2^n t]-1} \frac{1}{2} \left[ Y \left( \frac{k}{2n} \right) + Y \left( \frac{k+1}{2n} \right) \right] \left[ X \left( \frac{k+1}{2n} \right) - X \left( \frac{k}{2n} \right) \right]$$

(50)

The limit exists in $L^2$ and satisfies

$$E^x(|Z_t|^2) = \int_0^t E^x(|Y_s|^2) \, ds$$

(51)

provided the right side is finite. $Z_t$ has expectation zero (and is a martingale). The equation $Z_t = \int_0^t Y_s dX_s$ is also written as the stochastic differential equation

$$dZ_t = Y_t dX_t$$

(52)

Similarly we define the Stratonovich integral by

$$Z_t = \int_0^t Y_s \circ dX_s = \lim_{n \to \infty} \sum_{k=0}^{[2^n t]-1} \frac{1}{2} \left[ Y \left( \frac{k}{2n} \right) + Y \left( \frac{k+1}{2n} \right) \right] \left[ X \left( \frac{k+1}{2n} \right) - X \left( \frac{k}{2n} \right) \right]$$

(53)

which we write as

$$dZ_t = Y_t \circ dX_t$$

(54)

The Stratonovich integral obeys many of the usual rules of calculus. In particular if $Z_t = f(X_t)$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s$$

(55)

which we write as the chain rule

$$d(f(X_t)) = f'(X_t) \circ dX_t$$

(56)

The difference between the two integrals is the quadratic integral

$$\int_0^t Y_s \circ dX_s - \int_0^t Y_s dX_s$$

$$= \lim_{n \to \infty} \sum_{k=0}^{[2^n t]-1} \frac{1}{2} \left[ Y \left( \frac{k+1}{2n} \right) - Y \left( \frac{k}{2n} \right) \right] \left[ X \left( \frac{k+1}{2n} \right) - X \left( \frac{k}{2n} \right) \right]$$

(57)

$$\equiv \frac{1}{2} \int_0^t \langle dY_s, dX_s \rangle ds$$
which we write as
\[ Y_t \circ dX_t = Y_t dX_t + \frac{1}{2} \langle dY_t, dX_t \rangle dt \] (58)

The quadratic term vanishes if \( Y_t \) has finite variation. Sometimes the quadratic term can be simplified. For example if \( Y_t = \alpha(X_t) \) for a continuously differentiable function \( \alpha \) then the quadratic term is \( \frac{1}{2} \langle d\alpha(X_t), dX_t \rangle \) and so
\[ \int_0^t \alpha(X_s) \circ dX_s = \int_0^t \alpha(X_t) dX_s = \frac{1}{2} \int_0^t \alpha'(X_s) ds \] (59)

Now let \( X_t = (X^1_t, X^2_t, X^3_t) \) be Brownian motion in \( \mathbb{R}^3 \) starting at \( X_0 = x \) consisting of three independent Brownian motions. So \( X_t \) is a Gaussian process with mean \( x \in \mathbb{R}^3 \) and covariance \( \text{Cov}(X^i_t X^j_s) = \delta_{ij} \min(s,t) \). If the start point \( X_0 = x \neq 0 \) then \( X_t \neq 0 \) with probability one. Thus we can also regard \( X_t \) as a process in \( M = \mathbb{R}^3 - \{0\} \) which we do hereafter.

The Ito integral of a non-anticipating \( Y_t \) taking values in \( \mathbb{R}^3 \) is now (summation convention)
\[ Z_t = \int_0^t (Y_s)_k dX_s^k \] (60)
and it exists if
\[ \int_0^t E^x(|Y_s|^2) \, ds < \infty \] (61)
Similarly there is a Stratonovich integral
\[ \int_0^t (Y_s)_k \circ dX_s^k \] (62)
and they differ by a quadratic integral.

The following result covers the cases of interest to us.

**Lemma 7.** Let \( f \) be a continuous vector valued function on \( M = \mathbb{R}^3 - 0 \) satisfying

1. \( |f(x)| \) is bounded for \( |x| \geq 1 \)
2. \( |f(x)| \leq \mathcal{O}(|x|^{-\alpha}) \) for some \( \alpha < \frac{3}{2} \) and \( |x| \leq 1 \). (So \( \int_{|x|\leq1} |f(x)|^2 dx < \infty \).)

Then for Brownian motion starting at \( x \in M \) and \( t > 0 \)
\[ \int_0^t E^x(|f(X_s)|^2) \, ds < \infty \] (63)

Hence condition (61) is satisfied and the stochastic integral \( \int_0^t f_k(X_s) dX_s^k \) exists.
Proof. For $0 < s < t$

$$E^x(|f(X_s)|^2) = (2\pi s)^{-\frac{3}{2}} \int |f(y)|^2 e^{-|y|y^2/2s} dy$$  \hspace{1cm} (64)$$

For $|y| \geq 1$, $|f(y)|^2$ is bounded by a constant, the integral gives one and so the result is finite. For $|y| \leq 1$ we use $|f(y)|^2 \leq \mathcal{O}(|y|^{-2\alpha})$, then enlarge the integral to all space, and have a bound which is a constant times

$$(2\pi s)^{-\frac{3}{2}} \int |y|^{-2\alpha} e^{-|x-y|^2/2s} dy = (2\pi s)^{-\frac{3}{2}} \int |x-y|^{-2\alpha} e^{-|y|^2/2s} dy$$  \hspace{1cm} (65)$$

We break the integral into two regions. The first is $|x-y| \geq |x|/2$, and in this case the integral is bounded by $4|x|^{-2\alpha}$. The second is $|x-y| \leq |x|/2$. In this case we have $|y| \geq |x|/2$ which implies $e^{-|y|^2/2s}(2\pi s)^{\frac{3}{2}} \leq \mathcal{O}(|x|^{-3})$. Then the integral over the second region is bounded by

$$\mathcal{O}(|x|^{-3}) \int_{|x-y| \leq |x|/2} |x-y|^{-2\alpha} dy = \mathcal{O}(|x|^{-3}) \int_0^{|x|/2} r^{2-2\alpha} dr = \mathcal{O}(|x|^{-2\alpha})$$  \hspace{1cm} (66)$$

Hence everything is finite and the proof is complete.

This result still holds if $f_k$ takes values in a normed vector space (for example $E_x$) rather than just $\mathbb{R}$.

5 Stochastic parallel translation

For parallel translation along a Brownian path $X_t$ we would like to define a stochastic parallel translation operator $\Pi_t = \Pi(t,0) : E_{X_0} \to E_{X_t}$. This should involve replacing functions $\exp(i \int_0^t A^\pm_k(x_s)dx_s)$ in section 2 by stochastic $\exp(i \int_0^t A^\pm_k(X_s) \circ dX_s^k)$ with Stratonovich integrals. However we have to be mindful of the fact that the Brownian path may visit each of $U_\pm$ multiple times. Our treatment roughly follows Norris [6].

For Brownian paths $X_t$, one can find stopping times $\tau_0 = 0$, $\tau_1$, $\tau_2$, \ldots such that $\tau_k \to \infty$ as $k \to \infty$ and such that for $t \in [\tau_k, \tau_k+1]$ the $X_t$ is entirely contained in at least one of $U_\pm$. For details see for example [3], p 426.

Fix the specification of stopping times and suppose we have defined $\Pi(t,0)$ for $t \in [0, \tau_k]$. We extend the definition to $t \in [0, \tau_{k+1}]$ defining

$$\Pi(t,0) = \Pi(t, \tau_k)\Pi(\tau_k,0)$$  \hspace{1cm} (67)$$

If $X_t \in U_\pm$ for $t \in [\tau_k, \tau_{k+1}]$ and $v \in E_{X_{\tau_k}}$ then $\Pi(t, \tau_k)v \in E_{X_t}$ is defined by $(\Pi(t, \tau_k)v)_\pm = \Pi^\pm(t, \tau_k)v_\pm$ where

$$\Pi^\pm(t, \tau_k) = \exp \left( i \int_{\tau_k}^t A^{\pm}_k(X_s) \circ dX^k_s \right)$$  \hspace{1cm} (68)$$
So our precise definition of $\Pi_t : E_{X_t} \to E_{X_t}$ is

$$\Pi_t = \Pi(t,0) = \sum_{k=0}^{\infty} 1(\tau_k < t \leq \tau_{k+1})\Pi(t, \tau_k)\Pi(\tau_k,0)$$ (69)

The sum is actually finite due to the $\tau_k \to \infty$ condition.

**Lemma 8.** $\Pi(t,0)$ is well-defined since

1. If $X_t \in U_\pm$ for $t \in [\tau_k, \tau_{k+1}]$ the integral $\int_{\tau_k}^t A_\pm^k(X_s) \circ dX_s^k$ exists.
2. If $X_t \in U_+ \cap U_-$ for $t \in [\tau_k, \tau_{k+1}]$ then $\Pi^\pm(t, \tau_k)$ give the same $\Pi(t, \tau_k)$.
3. The definition is independent of the stopping times.

**Proof.** For the integral suppose $X_t \in U_+$ for $t \in [\tau_k, \tau_{k+1}]$. The quadratic term in this case can be identified as the divergence div$A$ and we can write the Stratonovich integral as an Ito integral by

$$\int_{\tau_k}^t A_\pm^k(X_s) \circ dX_s^k = \int_{\tau_k}^t A_\pm^k(X_s)dX_s^k + \frac{1}{2} \int_{\tau_k}^t (\text{div} A^\pm)(X_s)ds$$ (70)

However div$A^+ = 0$ as one can see from (8). Therefore the Stratonovich and Ito integrals coincide and we work with the Ito integral. We have

$$\int_{\tau_k}^t A_\pm^k(X_s)dX_s^k = \int_{0}^{t} \tilde{A}_\pm^k(X_s)dX_s^k$$ (71)

where

$$\tilde{A}_\pm^k(X_s) = A_\pm^k(X_s)1(s \geq \tau_k)$$ (72)

From (12) we have $|A^\pm(x)| \leq C|x|^{-1}$ for all $x \in M$ so that $|\tilde{A}^\pm(X_s)| \leq C|X_s|^{-1}$. The existence of the integral then follows from

$$\int_{0}^{t} E^s\left(|\tilde{A}^\pm(X_s)|^2\right)ds \leq C \int_{0}^{t} E^s\left(|X_s|^{-2}\right)ds < \infty$$ (73)

where the second inequality follows from lemma 7 with $\alpha = 1$. The first point is established.

For the second point the independence of $\pm$ follows just as in the deterministic case thanks to the identity

$$\int_{\tau_k}^t (\partial_k \phi)(X_s) \circ dX_s^k = \phi(X_t) - \phi(X_{\tau_k})$$ (74)
Finally we show that the definition is independent of the choice of stopping times \( \tau_k \). Suppose we add an intermediate stopping time \( \tau_k < \sigma_k < \tau_{k+1} \). Then for \( t \in [\sigma_k, \tau_{k+1}] \) the definition is the same as before. For \( t \in [\sigma_k, \tau_{k+1}] \) the definition is now

\[
\Pi(t, 0) = \Pi(t, \sigma_k)\Pi(\sigma_k, 0) = \Pi(t, \sigma_k)\Pi(\sigma_k, \tau_k)P(\tau_k, 0)
\]

and we want to compare this with the original \( \Pi(t, 0) = \Pi(t, \tau_k)P(\tau_k, 0) \). Thus we need to show for \( t \in [\sigma_k, \tau_{k+1}] \)

\[
\Pi(t, \sigma_k)\Pi(\sigma_k, \tau_k) = \Pi(t, \tau_k)
\]

But for \( t \in [\tau_k, \tau_{k+1}] \) we are in some \( U_\pm \) say \( U_+ \), and so the identity follows by

\[
\Pi^+(t, \sigma_k)\Pi^+(\sigma_k, \tau_k) = \exp \left( i \int_{\sigma_k}^t A^+_k(X_s) \circ dX^k_s \right) \exp \left( i \int_{\tau_k}^{\sigma_k} A^+_k(X_s) \circ dX^k_s \right) = \Pi^+(t, \tau_k)
\]

This result implies that if one choice of stopping times is a refinement of another then they give the same result. Finally any two choices of stopping times agree with their common refinement and hence give the same result. This completes the proof.

**Lemma 9.** Let \( \Psi \in \Gamma(E) \) with bounded covariant derivatives. Then \( \Pi_t^{-1}\Psi(X_t) \) is a non-anticipating stochastic process taking values in \( E_{X_0} = E_x \) and

1. The Stratonovich differential satisfies

\[
d\left( \Pi_t^{-1}\Psi(X_t) \right) = \Pi_t^{-1}(\nabla_k \Psi)(X_t) \circ dX^k_t
\]

2. The Ito differential satisfies

\[
d\left( \Pi_t^{-1}\Psi(X_t) \right) = \Pi_t^{-1}(\nabla_k \Psi)(X_t)dX^k_t - \Pi_t^{-1}(H\Psi)(X_t)
\]

**Proof.** (1.) In general if \( s < t \) we define \( \Pi(s, t) = \Pi(t, s)^{-1} \) Take stopping times \( 0 < \tau_1 < \tau_2 < \cdots \to \infty \) as before. For \( t \in [\tau_k, \tau_{k+1}] \)

\[
\Pi_t^{-1} = \Pi(t, 0)^{-1} = \Pi(\tau_k, 0)^{-1}\Pi(t, \tau_k)^{-1} = \Pi(0, \tau_k)\Pi(\tau_k, t)
\]

Then our precise definition is

\[
\Pi_t^{-1}\Psi(X_t) = \sum_{k=0}^{\infty} 1(\tau_k < t \leq \tau_{k+1})\Pi(0, \tau_k)\Pi(\tau_k, t)\Psi(X_t)
\]
Then we have for the Stratonovich differential

\[
\mathcal{D}^{\Pi_t^{-1}\Psi(X_t)} = \sum_{k=0}^{\infty} 1(\tau_k < t \leq \tau_{k+1}) \Pi(0, \tau_k) \mathcal{D}^{\Pi(\tau_k, t)\Psi(X_t)}
\]  

(82)

Let us check this. It means

\[
\Pi_t^{-1}\Psi(X_t) - \Psi(x) = \sum_{k=0}^{\infty} \Pi(0, \tau_k) \int_0^t 1(\tau_k < s \leq \tau_{k+1}) \mathcal{D}^{\Pi(\tau_k, s)\Psi(X_s)}
\]  

(83)

Choose \( n \) so \( t \in (\tau_n, \tau_{n+1}] \). Then right side of (83) is

\[
\sum_{k=0}^{n-1} \Pi(0, \tau_k) \int_{\tau_k}^{\tau_{k+1}} \mathcal{D}^{\Pi(\tau_k, s)\Psi(X_s)} + \Pi(0, \tau_n) \int_{\tau_n}^t \mathcal{D}^{\Pi(\tau_n, s)\Psi(X_s)}
\]

\[
\sum_{k=0}^{n-1} \Pi(0, \tau_k) \Pi(\tau_k, \tau_{k+1}) \Psi(X_{\tau_{k+1}}) - \Psi(X_{\tau_k})
\]

\[
+ \Pi(0, \tau_n) \Pi(\tau_n, t)\Psi(X_t) - \Psi(X_{\tau_n})
\]

\[
= \sum_{k=0}^{n-1} \left( \Pi(0, \tau_{k+1}) \Psi(X_{\tau_{k+1}}) - \Pi(0, \tau_k) \Psi(X_{\tau_k}) \right) + \left( \Pi(0, t)\Psi(X_t) - \Pi(0, \tau_n)\Psi(X_{\tau_n}) \right)
\]

\[
= \Pi_t^{-1}\Psi(X_t) - \Psi(x)
\]

(84)

which is the same as the left side of (83). Thus (82) is confirmed.

Now it suffices to show that restricted to the event \( \{ \tau_k < t \leq \tau_{k+1} \} \)

\[
\mathcal{D}^{\Pi(\tau_k, t)\Psi(X_t)} = \Pi(\tau_k, t)(\nabla_j \Psi)(X_t) \circ dX_t^j
\]

(85)

which is an identity in \( E_{X_t^{(k)}} \). This is sufficient since if we make this substitution in (82) we get \( \Pi_t^{-1}(\nabla_k \Psi)(X_t) \circ dX_t^k \).

By construction we have \( X_t \) in at least one of \( U_{\pm} \) for \( \tau_k < t \leq \tau_{k+1} \). Suppose it is \( U_+ \). In the trivialization on \( U_+ \) the claim is that

\[
\mathcal{D}^{\Pi^+(\tau_k, t)\Psi^+(X_t)} = \Pi^+(\tau_k, t)((\partial_j - iA_j^+)\Psi_+)(X_t) \circ dX_t^j
\]

(86)

Now

\[
\Pi^+(\tau_k, t) = \exp \left( -i \int_{\tau_k}^t A_j^+(X_s) \circ dX_s^j \right)
\]

(87)

The integral here is a semimartingale (in fact a martingale since it coincides with the Ito integral) and so \( \Pi^+(\tau_k, t) \) is a semimartingale. For semimartingales \( Z_t \) we have a
chain rule $df(Z_t) = f'(Z_t) dZ_t \[5\]$. Therefore

$$d \Pi^+(\tau_k, t) = \Pi^+(\tau_k, t) d \left( -i \int_{\tau_k}^t A^+_j(X_s) \circ dX_j^s \right)$$

$$= \Pi^+(\tau_k, t) \left( -i A^+_j(X_t) \right) \circ dX_j^t$$

Furthermore by the three dimensional version of \[56\]

$$d \Psi_+(X_t) = \partial_j \Psi_+(X_t) \circ dX_j^t$$

Combining the last two with the product rule (for semimartingales) gives \[56\] and completes the proof of part (1.)

(2.) We change from a Stratonovich differential to an Ito differential as in \[58\]

$$\Pi_t^{-1}(\nabla_k \Psi)(X_t) \circ dX_t^k = \Pi_t^{-1}(\nabla_k \Psi)(X_t) dX_t^k + \frac{1}{2} \left\langle d \left( \Pi_t^{-1}(\nabla_k \Psi)(X_t) \right), dX_t^k \right\rangle$$

In the quadratic term here we can exchange Stratonovich differentials and Ito differentials since the difference is the differential of a function with finite variation. Then by \[78\] again it is

$$\frac{1}{2} \left\langle d \left( \Pi_t^{-1}(\nabla_k \Psi)(X_t) \right), dX_t^k \right\rangle = \frac{1}{2} \left\langle \Pi_t^{-1}(\nabla_j \nabla_k \Psi)(X_t) \circ dX_j^k, dX_t^k \right\rangle$$

$$= \frac{1}{2} \left\langle \Pi_t^{-1}(\nabla_j \nabla_k \Psi)(X_t) dX_j^k, dX_t^k \right\rangle$$

$$= \frac{1}{2} \Pi_t^{-1}(\nabla_k \nabla_k \Psi)(X_t) dt = -\Pi_t^{-1}(H \Psi)(X_t) dt$$

Here we used $<dX_j^k, dX_t^k> = \delta_{ij} dt$. This completes the proof.

This result would be adequate if for example we took $\Psi \in \mathcal{C}_0^\infty(E)$. But we want $\Psi \in \mathcal{D}$ which means we have to relax the condition that $\Psi$ have bounded covariant derivatives and allow growth as $x \to 0$.

**Lemma 10.** The results of lemma \[9\] still hold if $\Psi \in \mathcal{D}$.

**Proof.** The expression \[79\] for the Ito differential means that

$$\Pi_t^{-1} \Psi(X_t) = \Psi(x) + \int_0^t \Pi_s^{-1}(\nabla_k \Psi)(X_s) dX_s^k - \int_0^t \Pi_s^{-1}(H \Psi)(X_s) ds$$

We need to check that these integrals exist for $\Psi \in \mathcal{D}$. By lemma \[5\] we have $(\nabla_k \Psi)(t, x)$ is bounded for $|x| \geq 1$ and is $O(|x|^{\frac{1}{2}(\sqrt{7} - 3)})$ for $|x| \leq 1$. Hence the same holds for $\Pi_t^{-1}(\nabla_k \Psi)(t, x)$. The asymptotics as $|x| \to 0$ is less severe than $O(|x|^{-\frac{4}{7}})$ so the existence of the first integral follows by lemma \[7\]. For the second integral $H \Psi$ is again in $\mathcal{D}$, hence it is bounded and the integral exists.
6 The Feynman-Kac formula

We now prove the Feynman-Kac formula, roughly following the strategy of Norris [6].

**Theorem 1.** Let $\Psi \in \mathcal{H} = L^2(E, dx)$, let $X_t$ be Brownian motion with $X_0 = x$, and let $\Pi_t$ be the stochastic parallel translation operator. Then

$$ (e^{-Ht}\Psi)(x) = E^x \left( \Pi_t^{-1}\Psi(X_t) \right) $$

(93)

**Proof.** It suffices to prove the result pointwise for $\Psi$ in our dense domain $\mathcal{D}$ of smooth sections. This is true since both sides of the equation define bounded operators on $L^2(E)$. This is true for the left side since $H$ is a positive operator. To see it is true for the right side note that $|\Psi| \in L^2(M) = L^2(\mathbb{R}^3)$ so we can estimate for $\Psi \in \mathcal{D}$

$$ \int |E^x(\Pi_t^{-1}\Psi(X_t))|^2 dx \leq \int \left( E^x|\Psi(X_t)| \right)^2 dx $$

(94)

Hence the right side extends to a bounded operator on all $L^2(E, dx)$.

Now with $\Psi \in \mathcal{D}$ define $\chi(t, x)$ for fixed $T$ and $t \leq T$ by

$$ \chi(t, x) = (e^{-H(T-t)}\Psi)(x) $$

(95)

Then $\chi(t, \cdot)$ is still in the domain $\mathcal{D}$ by lemma 6.

Since $\chi$ is a function of $t$ as well as $x$ the equation (78) becomes

$$ d\left( \Pi_t^{-1}\chi(t, X_t) \right) = \Pi_t^{-1}\frac{\partial \chi}{\partial t}(t, X_t) + \Pi_t^{-1}(\nabla_k \chi)(t, X_t) \circ dX_t^k $$

(96)

Changing to the Ito differential by lemma 10 gives

$$ d\left( \Pi_t^{-1}\chi(t, X_t) \right) = \Pi_t^{-1}\frac{\partial \chi}{\partial t}(t, X_t) + \Pi_t^{-1}(\nabla_k \chi)(t, X_t) dX_t^k - \Pi_t^{-1}(H\chi)(t, X_t) dt $$

(97)

But $\partial \chi/\partial t - H\chi = 0$ so simplifies this simplifies to

$$ d\left( \Pi_t^{-1}\chi(t, X_t) \right) = \Pi_t^{-1}(\nabla_k \chi)(t, X_t) dX_t^k $$

(98)

which means that

$$ \Pi_t^{-1}\chi(t, X_t) = \chi(0, x) + \int_0^t \Pi_s^{-1}(\nabla_k \chi)(s, X_s) dX_s^k $$

(99)

The Ito integral in (99) has expectation zero. Taking expectations in this equation gives $E^x(\Pi_t^{-1}\chi(t, X_t)) = \chi(0, x)$. At $t = T$ this says

$$ E^x \left( \Pi_T^{-1}\Psi(X_T) \right) = (e^{-HT}\Psi)(x) $$

(100)
Since $T$ is arbitrary this is our result.

**Remark.** This proof depended on finding a nice dense domain invariant under $e^{-Ht}$. If we added a radial scalar potential and wanted to consider $e^{-(H+V)t}$, more work on continuum eigenfunctions for the radial Hamiltonian would be needed to identify such a domain in this case.

**References**

[1] J. Dimock, Scattering on the Dirac magnetic monopole, ArXiv 2001.10327, (2020).

[2] Güneysu, B., The Feynman-Kac formula for Schrödinger operators on vector bundles over complete manifolds, Journal of Geometry and Physics 60, 1997-2010 (2010).

[3] Hackenbroch, W., Thalmaier, A., Stochastische Analysis, B.G. Teubner, Stuttgart (1994).

[4] Hinz, M., Magnetic energies and Feynman-Kac formulas for symmetric Markov processes (2015).

[5] Ikeda, N., Watanabe, S, Stochastic differential equations and diffusion processes, North Holland, Amsterdam, (1981).

[6] Norris, J.R., A complete differential formalism for stochastic calculus on manifolds, Seminar de Probabilites (Strasbourg), Springer, 189-209 (1992).

[7] Øksendal, B., Stochastic Differential Equations, Springer, Berlin-Heidelberg-New York, (2003).

[8] Simon, B., Functional Integration and Quantum Physics, Academic Press, New York, (1979).

[9] E.C. Titchmarsh, *Theory of Fourier Integrals*, Oxford University Press, London (1948).

[10] T.T. Wu, C.N. Yang, Dirac monopole without strings: monopole harmonics, *Nuclear Physics* B107, (1976), 365-380.