Many Body Theory for Quartets, Trions, and Pairs in Low Density Multi-Component Fermi-Systems *

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A selfconsistent many body approach for the description of gases with quartets, trions, and pairs is presented. Applications to 3D Fermi systems at low density are discussed.

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1. Introduction

Cluster formation in strongly interacting Fermi-systems is one of the most interesting subjects in many body physics. So far, in condensed matter, only the study of formation of two body clusters has been studied in a wide perspective in the context of pairing, that is superfluidity and superconductivity (see e.g. in the context of cold atoms). However, in nuclear physics, due to the existence of four different fermions, all attracting one another, the scenario of cluster formation is much richer. Of course, there also exist Cooper pairs of neutrons and protons, and to a lesser degree proton-neutron pairs, but in addition there exist tritons, helions (trions) and α-particles (quartets), to cite only the lightest ones. For instance, the formation and condensation of α-particles is presently in the focus of our studies. However, adding to a gas of α-particles two neutrons per α-particle, at some higher excitation energy, this system may be transformed into a gas of tritons. For example, one may imagine that $^{12}\text{Be}$ is composed out of four tritons at sufficiently high energies. Also nucleons themselves are trions when seen as bound states formed out of three quarks. Then

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the study of the transition from a Fermi gas of quarks to a new Fermi gas of trions, i.e. nucleons is interesting.

Recently in cold atoms physics the trapping of three different fermions has been achieved by two experimental groups and the formation of a gas of trions seems possible [2]. May be in the future, one will even capture four different fermionic atoms and study quartet condensation. Theoretical work on trions and quartets already has appeared in quite a number (see articles in [3] and references therein) and it seems that the field will rapidly expand in the future. The theoretical works mostly have been performed in one dimensional (1D) models. In this work, we will try to develop a many body approach for the treatment of a gas of clusters of fermions valid in 3D.

2. Alpha-particle Condensation

The condensation of $\alpha$-particles can be treated in analogy to the pairing case. To this end one has to consider the equation for the four fermion order parameter $\Psi_{1234}$ (summation convention is used) [4]:

$$\Psi_{1234} = \frac{1 - n_1 - n_2}{4\mu - \epsilon_{1234}} v_{121/2} \Psi_{1/2:34} + \text{permutations}$$

where $\mu$ is the chemical potential and $\epsilon_{1234} = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$ where the $\epsilon_i$’s are the single particle energies. Besides the matrix elements of the interaction $v_{121/2}$, the most important ingredients are the single particle occupation numbers $n_i = \langle c_i^+ c_i \rangle$. They, in principle, have to be calculated from a Dyson equation with a mass operator containing the quartet condensate. Equation (1) then constitutes the quartet "gap-equation" [3].

A lowest order expression for the single particle mass operator entering the Dyson equation can be obtained in the following way where the four particle $T$-matrix $T_4$, in the one pole approximation corresponding to (1), is convoluted with three uncorrelated hole lines:

$$\Sigma(1, z_\nu) = \sum_{234, \Omega_4} T_4(1234, 1234; \Omega_4) G^0_3(234, \Omega_4 - z_\nu)$$

where we used summation over the Matsubara frequency and the uncorrelated three body Matsubara Green’s function is given by

$$G^0_3(234, z_\nu) = \frac{(1 - f_2)(1 - f_3)(1 - f_4) + f_2 f_3 f_4}{z_\nu - \epsilon_2 - \epsilon_3 - \epsilon_4}$$

However, in order to simplify in a first step, we linearize and replace, at finite temperature, the $n_i$ by the Fermi-Dirac occupation numbers $n_i \to f_i = [1 + e^{(\epsilon_i - \mu)/T}]^{-1}$. Still (1) is very difficult to solve, since it is a four body problem rendered more complicated by the presence of the Pauli blocking factors. However, for clusters involving more than two fermions, and certainly for strongly bound
Many Body Theory for Quartets, Trions, and Pairs

quartets like the \( \alpha \)-particle a projected product ansatz \(^5\) for \( \Psi_{1234} \) is known to be a good approximation:

\[
\Psi_{1234} \rightarrow \delta(K - k_1 - k_2 - k_3 - k_4)\phi_0(k_1)\phi_0(k_2)\phi_0(k_3)\phi_0(k_4)\chi^{ST} \quad (4)
\]

where \( K \) is the total c.o.m. momentum and \( \phi_0(k) \) is a mean field \( 0S \)-wave function. Spin and isospin are taken care of by \( \chi^{ST} \) with \( S = T = 0 \) for the \( \alpha \)-particle. We will not consider it further. Inserting (4) into (1) (with \( K = 0 \), for condensation), one straightforwardly obtains a Hartree-Fock (HF)-type of equation for \( \phi_0 \) which can be solved for various critical temperatures \( T = T_c \), while \( \mu \) is determined from the particle number condition \( N/V = 4 \int \frac{d^3p}{(2\pi)^3} \int (\epsilon_p), \) see elsewhere \(^6\) for the detailed calculation. For simplicity we only take a spin-isospin averaged separable interaction

\[
v_{121'2'} = \lambda w(k_1 - k_2)w(k_1' - k_2')\delta(k_1 + k_2 - k_1' - k_2') \quad \text{with the form factor} \quad w(k) = e^{-k^2/k_0^2}.
\]

The two open parameters \( \lambda = -991 \text{ MeV fm}^3 \) and \( k_0 = 1.43 \text{ fm}^{-1} \) are adjusted that energy (-28 MeV) and radius (1.71 fm) of \( \alpha \)-particle come right. In Fig. 1 we show the result for \( T_c \) as a function of \( \mu \) and the density \( n \) where the free gas Equation of State (EOS) has been used to relate \( n \) and \( \mu \). It is very remarkable that the obtained results for \( T_c^\alpha \) well agree with a direct solution of (1) using a realistic NN force.\(^7\) These results for \( T_c^\alpha \) are by about 25 percent higher than the ones of our earlier publication.\(^4\) We, however, checked that the underlying radius of the \( \alpha \)-particle in that work is with 2 fm larger than the experimental value and that \( T_c^\alpha \) decreases with increasing radius of \( \alpha \). Furthermore a different variational wave function was used in \(^4\).

In Fig. 1 we also show the critical temperature for deuteron condensation. In this case we take \( \lambda = -1305 \text{ MeV fm}^3 \) and \( k_0 = 1.46 \text{ fm}^{-1} \) to get experimental energy (-2.2 MeV) and radius (1.95 fm) of the deuteron. It is seen that at higher densities deuteron condensation wins over the one of \( \alpha \)-particles. This stems from the fact that Fermi-Dirac distributions in the four body case, see \(^1\), can never become step-like, as in the two body case at zero temperature, since the pairs in an \( \alpha \)-particle are always in motion. As a consequence, \( \alpha \)-condensation only exists as a BEC phase and the weak coupling regime is absent.

The mean field ansatz \(^1\), of course, simplifies the problem of quartet condensation enormously. In the future it, therefore, should be possible to solve the quartet gap-equation \(^2\) with \(^1\) fully self-consistently. At zero temperature this will correspond (approximately) to the minimization of the energy with a four body coherent state

\[
|\Psi_\alpha\rangle \sim e^{\frac{i}{\hbar} \sum_{1234} \phi(1234) c_1^+ c_2^+ c_3^+ c_4^+ |\text{vac}\rangle} \quad (5)
\]

The relation between \( \Psi_{1234} \) and \( \phi(1234) \) is not trivial but can be established in an analogous way to BCS theory.

Condensation phenomena can have precursor signs in finite systems, even small. We know that, e.g., from nuclear pairing. Evidence for \( \alpha \)-particle condensation exists
in light self conjugate nuclei. For example the so-called Hoyle state at 7.65 MeV in $^{12}$C turns out to be a product state of three $\alpha$-particles in identical 0$^+_S$ orbit to over 70 percent \cite{8}. Indications from theoretical results have been put forward recently that the 6-th 0$^+_+$ state at 15.1 MeV in $^{16}$O is a four $\alpha$-particle condensate state \cite{9}. If this prediction is further verified experimentally and theoretically, then $\alpha$-particle condensation very likely is a general phenomenon in nuclear systems \cite{8}. Of course, for the description of the condensation of a very small number of $\alpha$-particles a number conserving variant of (5) must be employed. Also the c.o.m. motion is not described by plane waves as in (4) for homogeneous systems but the c.o.m. orbitals obey themselves a mean field equation. The corresponding ansatz, therefore, is

$$\langle \mathbf{r}_1, \mathbf{r}_2, ... | \Phi_\alpha \rangle \sim A [ \phi(1234) \phi(5678) ... ]$$

(6)

with $A$ the antisymmetriser and $\phi(1234) = \Phi_0(\mathbf{R}_{1234}) \varphi_0(\mathbf{r}_{ij})$ where $\mathbf{R}_{1234} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)/4$ is the c.o.m. coordinate and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ stands for all the possible combinations of relative coordinates of the quartet. Of course, at the end the dependence on the total c.o.m. coordinate of the system $X_G = [\mathbf{R}_{1234} + \mathbf{R}_{5678} + ...]/n_\alpha$, with $n_\alpha$ the number of quartets has to be eliminated, that is one has to project on good total c.o.m. momentum zero. In the case of a Gaussian ansatz for $\Phi_0(\mathbf{R})$ and $\varphi_0(\mathbf{r}_{ij})$, this is easy to write down and we obtain

$$\langle \mathbf{r}_1, \mathbf{r}_2, ... | \Phi_\alpha \rangle \sim A [ \Phi_0^{c.o.m.} \varphi_0^{intrinsic} ]$$

(7)

with

$$\Phi_0^{c.o.m.} = e^{-\mathbf{R}_{1234}^2/B^2} e^{-\mathbf{R}_{5678}^2/B^2} ...$$

(8)

$$\varphi_0^{intrinsic} = e^{-s_{1234}^2/b^2} e^{-s_{5678}^2/b^2} ...$$

(9)

where $\mathbf{R}_{1234}^G = \mathbf{R}_{1234} - X_G$, and $s_{1234}^2 = (\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_1 - \mathbf{r}_3)^2 + ...$.

Fig. 1. Transition temperature of alpha and deuteron as function of chemical potential (left) and density (right).
We see that the $\alpha$-condensate wave function consists in the anti summarized product of two interdependent mean field parts, one concerning the intrinsic part of each $\alpha$-particle and one concerning the c.o.m. motion of the $\alpha$-particles. Each part is a mean field wave function with the c.o.m. part eliminated. The wave function is, therefore, totally translational invariant. This type of wave function is known in the literature as THSR wave function. It depends on two parameters $B$ and $b$ which in were taken as Hill-Wheeler coordinates to quantize the corresponding energy surface $E_0(B, b)$. For $B \gg b$ one can neglect the antisymmetrizer in and, as we see from , the wave function becomes a pure product state of $\alpha$-particles, i.e. a condensate. On the other hand for $B = b$ is equivalent to a pure Slater determinant. The ansatz gives excellent results for all observables of the Hoyle state with the Pauli principle only acting mildly. It also predicts similar $\alpha$-condensate states for $^{16}$O, $^{20}$Ne, ... close to the corresponding threshold energies for $\alpha$-particle break up. For the case where in nuclei many $\alpha$-particles existed (e.g. $^{40}$Ca with 10 $\alpha$-particles), one also must employ the coherent state formulation, since the explicit antisymmetrization becomes very difficult.

3. Trions
The case of a gas of trions has not been considered very much in nuclear physics. He has an excited state at $\sim 12$ MeV which can be interpreted as mainly consisting out of two tritons. One can speculate that, e.g. in $^{12}$Be there exists a state around 30 MeV excitation energy with four tritons. Since tritons are fermions, the corresponding wave functions of the c.o.m. motion will develop nodes as in the usual shell model with $0S, 0P$, etc. orbitals. The THSR wave function for a few tritons may have the form

$$|nt\rangle \sim A[\phi(123)\phi(456)\ldots]$$

with

$$\phi(123) = e^{-R_{123}^2/B^2} \bar{\varphi}(r_{ij})$$

with definitions analogous to the ones of the quartet case. As already mentioned, the antisymmetrizer will make out of the product of c.o.m. Gaussians a Slater determinant, whereas the intrinsic wave functions of the tritons may stay essentially unaltered.

In the nuclear case, a gas of tritons will contain only a very small number of such clusters. Considering nucleons as trions formed by three quarks, the number of trions can be considerable, that is the number of nucleons in a nucleus or in a Heavy Ion Collision. In the case of cold atoms where, as mentioned, recently trapping of three different fermions has been achieved, the number of trions may go in the
In such cases the trion wave function $|nt\rangle$ of above cannot be handled directly and we must elaborate some approximate scheme. To put a trion creation operator in the exponent like for the quartet is not possible, unless one introduces Grassman algebra, since we deal with fermionic clusters. One way to proceed is via the well known Equation of Motion (EOM) method, in analogy to Self Consistent Random Phase Approximation for two particles\cite{11}. In this case one first defines in the present case a three particle creation operator

$$Q^+_t = \sum_{123} \chi^+_t 123$$

with $C^+_{123} = c^+_1 c^+_2 c^+_3 / [(1 - n_1)(1 - n_2)(1 - n_3) + n_1 n_2 n_3]^{1/2}$ the product of three fermionic single particle creators. The equation for the amplitudes $\chi$ can be obtained by minimizing the following energy weighted sum rule:

$$E_t = \frac{\langle [Q_t, [H, Q^+_t]]_+ \rangle}{\langle [Q_t, Q^+_t]_+ \rangle}$$

This leads to a non linear hermitian secular equation for the $\chi$’s. The nonlinearity stems from the fact that in the double commutator in (13) appear three-body correlation functions of the type $\langle c^+_1 c^+_2 c^+_3 c c c \rangle$. The relation between $Q^+_t$ and $C^+_{123}$ can be inverted and the three body correlation function be expressed by the $\chi$’s with the help of the relation $Q_t|0\rangle = 0$. However, two body and one body correlation functions appear as well. Though approximate formulas expressing those lower rank correlation functions by the $\chi$-amplitudes may be derived, thus obtaining a fully self consistent system of equations, the procedure seems extremely heavy. A strong simplification could consist in factorizing all correlation functions into an antisymmetrized product of single particle correlation functions (single particle density matrices) and express them via $\chi$-amplitudes in a similar way as we discussed above for the alpha particle condensation. This procedure is generally known as the so-called renormalized RPA, see, e.g.,\cite{11}.

On the other hand, if in the gas the trions stay more or less compact, one may try a similar mean field ansatz as for the $\alpha$-particles, inspite of the fact that a mean field description of only three fermions may not be as good as the one for a quartet. However, in the case of three colors, the three fermions can occupy the lowest $0S$ level of the mean field and with a projection on good total momentum the description may still be quite reasonable. For instance, this may be a quite attractive approach for the constituent quark model where the three quarks in a nucleon are bound by a harmonic oscillator field. Let us, therefore, make a THSR ansatz\cite{8} also for the trion wave function for the case of homogeneous infinite matter

$$\chi^+_t 123 \rightarrow \epsilon^{KR}_{123} \delta(K - k_1 - k_2 - k_3) \phi(k_1) \phi(k_2) \phi(k_3) \zeta_{123}$$

(14)
where $\zeta_{123}$ takes care of spin, isospin, and color indices. With (13) we can insert this trial wave function into (12) and vary with respect to $\varphi(k)$. Since the trions are fermions, we also must take care of the fact that the trions should carry different c.o.m. momenta $K$ and, therefore, in (12) a summation over the c.o.m momenta must be contained. The case of trion formation has been studied theoretically in solving exactly 1D-model Hamiltonians with three colors (3). It would be interesting to compare the results of above approach to these exact solutions.

In the case of nucleons, one pair of quarks is forming a strongly correlated di-quark state. It seems reasonable to approximate the di-quark by a boson. Then the transition of a gas of quarks and di-quarks can be considered as a mixture of fermions and bosons. The nucleons then constitute a bound state between a fermion and a boson. Interesting physics has recently been found in treating the fermion-boson scattering in the background of a gas of bosons and fermions (the one pair approximation). Most interestingly the fermion-boson pair seems to form a stable pair even for infinitesimal attraction, quite similar to what happens for the formation of Cooper pairs in a two component Fermi gas. However, the boson-fermion pairs have, of course, Fermi statistics, thus forming a new Fermi gas of composites (12).

4. Pairs

In the pairing case, the focus is on the BEC-BCS transition. Researches of BEC-BCS crossover have been done not only for the condensed matter (13) but also for the nuclear matter (14). The application of the Nozières Schmitt-Rink approach (15) is fairly standard by now, taking care of the purely bosonic (strong coupling) limit of the pairs and their c.o.m. motion in the Bose-Einstein distribution. However, in the strong coupling limit, the pairs behave as (composite) particles and create their own mean field. This effect is not taken into account in the work of (15), as discussed by the authors themselves. In order to include this effect, we can derive two coupled mean field equations, one fermionic, the other bosonic (fermion pairs). This is achieved by the minimization of the following generalized sum rules

$$\varepsilon_k = \frac{\langle q_k | H, q_k^+ | \rangle}{\langle q_k, q_k^+ \rangle}$$

(15)

with $q_k^+ = u_k c_k^+ - v_k c_k^-$ being the standard Bogoliubov transformation of fermions. The corresponding bosonic equation is obtained from

$$E_{K,\nu} = \frac{\langle Q_{K,\nu} | H, Q_{K,\nu}^+ | \rangle}{\langle Q_{K,\nu}, Q_{K,\nu}^+ \rangle}$$

(16)

where

$$Q_{K,\nu}^+ = \sum_{k_1, k_2} [X_{K,\nu}^{k_1, k_2} q_{k_1}^+, q_{k_2}^+ - Y_{K,\nu}^{k_1, k_2} q_{k_1} q_{k_2}]/(1 - n_{k_1} - n_{k_2})^{1/2}$$

(17)

represents a Bogoliubov transformation for fermion pairs.
A momentum conserving delta function between $K$ and $k_1,k_2$ is implicit and spin and isospin indices have been suppressed. The expectation values are with the correlated ground state defined (approximately) by $Q|0\rangle = 0$.

Equations (15)-(17) can be generalized to finite temperature in introducing corresponding Gorkov equations. The fully selfconsistent set of equations for amplitudes $u,v,X,Y$ can be worked out (see 16 and references therein) and their solutions give very promising results in non-trivial model cases 16. They constitute a fully selfconsistent extension of the Nozières Schmitt-Rink approach 15.

5. Conclusions

Selfconsistent many body approaches for gas phases of quartets, trions, and pairs in multicomponent Fermi-systems have been established. They are applicable to realistic 3D systems accounting for strong cluster phenomena.

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