On duality for nonconvex minimization problems within the framework of abstract convexity

Ewa M. Bednarczuk\textsuperscript{a} and Monika Sygab\textsuperscript{b}

\textsuperscript{a}Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland; \textsuperscript{b}Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

\textbf{ABSTRACT}

By applying the perturbation function approach, we propose the Lagrangian and conjugate duals for minimization problems of the sum of two, generally nonconvex, functions. The main tools are the \( \Phi \)-convexity theory and minimax theorems for \( \Phi \)-convex functions. We provide conditions ensuring zero duality gap and introduce \( \Phi \)-Karush–Kuhn–Tucker conditions that characterize solutions to primal and dual problems. We also discuss the relationship between the dual problems introduced in the present investigation and some conjugate-type duals existing in the literature.

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\section{1. Introduction}

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be real vector spaces, \( \mathcal{X} + \mathcal{Y} \subset \mathcal{Y} \). We consider the minimization problem of the form

\[
\text{Min}_{x \in \mathcal{X}} f(x) + g(x),
\]

where \( f : \mathcal{X} \to (-\infty, +\infty], g : \mathcal{Y} \to (-\infty, +\infty] \).

Our standing assumptions and notations are as follows.

(a) \( \Phi \) and \( \Psi \) are classes (closed under addition of real constants) of real-valued functions defined on \( \mathcal{X} \) and \( \mathcal{Y} \) respectively, called elementary functions, of simple structure, e.g. affine, quadratic, step functions. In the sequel, we put additional requirements (of algebraic character) on sets of elementary functions when needed, e.g. in some constructions and theorems we assume that \( \Phi \) is symmetric (\( \Phi = -\Phi \)) and/or \( \Phi \) is additive (\( \Phi + \Phi \subset \Phi \)).

(b) A function \( f : \mathcal{X} \to (-\infty, +\infty] \) is proper, i.e. the domain of \( f \) is nonempty, i.e.

\[
\text{dom}(f) := \{ x \in \mathcal{X} : f(x) < +\infty \} \neq \emptyset.
\]

and \( g : \mathcal{Y} \to (-\infty, +\infty] \) is proper and \( \text{dom}(f) \cap \text{dom}(g) \neq \emptyset \).
(c) When $\mathcal{X}$ is a Hilbert space, the inner product is denoted by $\langle \cdot | \cdot \rangle$ and the associated norm is $\| \cdot \|$. 

Main tool of the present investigation is the abstract convexity theory called $\Phi$-convexity. The origins of the $\Phi$-convexity theory go back to the investigations of Ky Fan [1], Moreau [2] and Rubinov and Kutateladze [3]. Applications in optimization were investigated, e.g. by Balder [4], Dolecki and Kurcyusz [5], Pallaschke & Rolewicz [6], Rubinov [7], in mass transport by Rüschendorf [8]. 

The underlying idea of $\Phi$-convexity also called convexity without linearity is to replace the classical bi-linear coupling functions used in the convex analysis by general (possibly nonlinear) coupling functions. $\Phi$-convexity provides a framework for the analysis of important classes of nonconvex problems. In the case of bi-linear coupling, this framework allows for the retrieval, and sometimes refinement, of classical results of convex analysis.

$\Phi$-convexity provides global tools for investigating nonconvex objects and offers a framework for investigating global optimization problems, see, e.g. the monographs by Alexander Rubinov [7] and by Diethard Pallaschke & Stefan Rolewicz [6]. Basic concepts of $\Phi$-convexity theory from a historical perspective, its rôle in global optimization and duality theory have been recently discussed in several presentations during the online WOMBAT 2020 meeting (https://wombat.mocao.org/wombat-2020/recordings/).

An important class of $\Phi$-convex functions are $\Phi_{\text{lsc}}$-convex functions defined on a Hilbert space with elementary functions defined by (12). In a series of papers [9–11], Stefan Rolewicz investigated particular subclass of $\Phi_{\text{lsc}}$-convex functions called paraconvex functions (weakly convex, semiconvex) functions.

Formulae for $\Phi$-subdifferentials, $\Phi$-conjugates, $\Phi$-inffimal convolution for the sum of two functions have been studied by Jeyakumar, Rubinov & Wu [12] who also provided generalizations of the results obtained for convex problems by Burachik & Jeyakumar in [13]. The results of [12] have been generalized by Bui, Burachik, Kruger & Yost in [14] to the sum of any finite number of functions $f_i$, $i = 1, \ldots, m$. In [14], the dual problem $(ICD)$ is formulated on the basis of the $\Phi$-inffimal convolution of $\Phi$-convex conjugates of functions $f_i$ in the classes of elementary functions $\Phi$ for which $0 \in \Phi$ and $\Phi + \Phi \subset \Phi$.

In the present paper, we construct a $\Phi$-conjugate dual $(CD)$, where $0 \in \Phi$, for the problem of minimizing the sum of two proper functions which is based on the perturbation function $p(\cdot, \cdot)$. We calculate the $c$-conjugate $p_c^*(\cdot, \cdot)$ with respect to a suitably chosen coupling function $c$ and we define the dual $(CD)$ as the problem of maximizing the function $-p_c^*(0, \cdot)$. This approach coincides with the approach to conjugate duality in the convex case, see, e.g. [15,16].

We also introduce the $\Phi$-Lagrangian function $L(\cdot, \cdot)$ for which the $\Phi$-Lagrangian dual (LD) is equivalent to the $\Phi$-conjugate dual problem $(CD)$.

The question of conditions ensuring zero duality gap is approached via minimax theorems obtained by Syga in [17] and [18] for classes of elementary
functions $\Phi$ which are convex sets. In Theorem 3.1 and Theorem 4.1 of [17], the so-called intersection property is proved to be a necessary and sufficient condition for the minimax equality to hold. These results allow us to prove that the intersection property for the $\Phi$-Lagrange function is necessary and sufficient for zero duality gap both in the $\Phi$-Lagrangian and $\Phi$-conjugate dualities.

We investigate the relationships between $\Phi$-infimal convolution dual ($ICD$) and $\Phi$-conjugate dual problem ($CD$) zero duality gap conditions obtained in [14]. The contribution of the paper is as follows.

- Construction of the $\Phi$-conjugate dual ($CD$) via the conjugate $p^*_c(\cdot, \cdot)$ of the perturbation function $p$ with respect to the coupling function $c$ (formula (cpl)) (Section 3 and Section 4).
- Construction of the $\Phi$-Lagrangian dual equivalent to the $\Phi$-conjugate dual (Section 5).
- Derivation of conditions for zero duality gap in the form of the so-called intersection property and discussion of its relationship to respective conditions for $\Phi$-inf convolution-based zero duality gap proved in [14] (Section 6).
- Definition of $\Phi$-KKT conditions and characterization of solutions to problems ($P$) and ($CD$) (Section 7).

2. Preliminaries

2.1. Abstract convexity

Let $\mathcal{X}$ be a set and let $\Phi$ be a set of elementary real-valued functions $\varphi : \mathcal{X} \rightarrow \mathbb{R}$ and $f : \mathcal{X} \rightarrow (-\infty, +\infty]$. The set

$$ \text{supp}_\Phi(f) := \{ \varphi \in \Phi : \varphi \leq f \} $$

is called the support of $f$ with respect to $\Phi$, where, for any $g, h : \mathcal{X} \rightarrow (-\infty, +\infty]$, $g \leq h$ $\Leftrightarrow$ $g(x) \leq h(x) \ \forall \ x \in \mathcal{X}$. We will use the notation $\text{supp}(f)$ whenever the class $\Phi$ is clear from the context. Elements of class $\Phi$ are called elementary functions.

Definition 2.1 ([5–7]): A function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ is called $\Phi$-convex on $\mathcal{X}$ if

$$ f(x) = \sup\{\varphi(x) : \varphi \in \text{supp}(f)\} \ \forall \ x \in \mathcal{X}. $$

If the set $\mathcal{X}$ is clear from the context, we simply say that $f$ is $\Phi$-convex.

2.2. $\Phi$-subgradients

Definition 2.2 (see, e.g. formula 1.1.1 of [6] and Definition 1.7 of [7]): An element $\varphi \in \Phi$ is called a $\Phi$-subgradient of a function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ at
\( \bar{x} \in \text{dom} f \), if the following inequality holds:

\[
  f(x) - f(\bar{x}) \geq \varphi(x) - \varphi(\bar{x}) \quad \forall x \in \mathcal{X}.
\]  

(1)

The set of all \( \Phi \)-subgradients of \( f \) at \( \bar{x} \) is denoted as \( \partial_\Phi f(\bar{x}) \). For characterizations of subgradients, see Proposition 1.2, Proposition 1.3 of [7] and formula 1.1.7 max of [6].

**Definition 2.3** (see, e.g. Definition 7.8 of [7], formula 1.1.1 \( \varepsilon \) of [6]): An element \( \varphi \in \Phi \) is called a \( \Phi \)-\( \varepsilon \)-subgradient of a function \( f : \mathcal{X} \rightarrow (-\infty, +\infty] \) at \( \bar{x} \in \text{dom} f \), if the following inequality holds:

\[
  f(x) - f(\bar{x}) \geq \varphi(x) - \varphi(\bar{x}) - \varepsilon \quad \forall x \in \mathcal{X}.
\]  

(2)

The set of all \( \Phi \)-\( \varepsilon \)-subgradients of \( f \) at \( \bar{x} \) is denoted as \( \partial_\Phi^\varepsilon f(\bar{x}) \). If \( f \) is \( \Phi \)-convex on \( \mathcal{X} \), then, for every \( \varepsilon > 0 \), the domain of the \( \varepsilon \)-subdifferential mapping \( \text{dom} \partial_\Phi^\varepsilon f = \{ x \in \mathcal{X} \mid \partial_\Phi^\varepsilon f(x) \neq \emptyset \} \) coincides with \( \text{dom} f \).

### 2.3. \( \Phi \)-conjugation

Let \( f : \mathcal{X} \rightarrow (-\infty, +\infty] \). The function \( f_\Phi^* : \Phi \rightarrow (-\infty, +\infty] \),

\[
  f_\Phi^*(\varphi) := \sup_{x \in \mathcal{X}} (\varphi(x) - f(x))
\]  

(3)

is called the \( \Phi \)-conjugate of \( f \) (see, e.g. [2]). The function \( f_\Phi^* \) is convex whenever \( \Phi \) is convex. For the characterization of the epigraph of \( f_\Phi^* \), see, e.g. Proposition 7.8 of [7].

Accordingly, the \( \Phi \)-bi-conjugate of \( f \) is defined as

\[
  f_{\Phi^*}^*(x) := \sup_{\varphi \in \Phi} (\varphi(x) - f_\Phi^*(\varphi)).
\]

The following relationships between \( \varepsilon \)-\( \Phi \)-subgradients, conjugate and biconjugate functions hold.

**Theorem 2.4**: Let \( f : \mathcal{X} \rightarrow (-\infty, +\infty] \).

(i) Fenchel–Moreau inequality: For every \( x \in \text{dom} (f) \) and every \( \varphi \in \Phi \)

\[
  f(x) + f_\Phi^*(\varphi) \geq \varphi(x).
\]  

(4)

(ii) For every \( x \in \text{dom} (f) \) and every \( \varepsilon > 0 \)

\[
  \varphi \in \partial_\Phi^\varepsilon f(x) \iff f(x) + f_\Phi^*(\varphi) \leq \varphi(x) + \varepsilon.
\]  

(5)

(iii) For every \( x \in \text{dom} (f) \), \( f_{\Phi^*}^*(x) \leq f(x) \).
(iv) For every $x \in \text{dom}(f)$

$$\forall \varepsilon > 0 \; \partial_{\Phi}^\varepsilon f(x) \neq \emptyset \Rightarrow f(x) = f_{\Phi}^{**}(x).$$

(v) $f$ is $\Phi$-convex on $\mathcal{X}$ $\iff$ $f(x) = f_{\Phi}^{**}(x) \; \forall \; x \in \mathcal{X}$.

**Proof:** (i) Follows directly from the definition of the $\Phi$-conjugate function, see, e.g. Proposition 1 [19] and Proposition 1.2.2 of [6].

(ii) Follows from the definition of the conjugate and the $\varepsilon$-$\Phi$-subgradient, see Proposition 1 of [19], Proposition 1.2.4 of [6], Proposition 7.10 of [7].

(iii) Follows directly from Fenchel–Moreau inequality (i).

(iv) Let $\varepsilon > 0$ and $\phi \in \partial_{\Phi}^\varepsilon f(x)$. By (ii) and the definition of the $\Phi$-bi-conjugate

$$f(x) \leq \phi(x) - f_{\Phi}^*(\phi) + \varepsilon \leq f_{\Phi}^{**} + \varepsilon.$$

Since $\varepsilon$ is arbitrary, $f(x) \leq f_{\Phi}^{**}(x)$. By (iii), the conclusion follows.

(v) For the proof see Theorem 1.2.6 of [6] and Theorem 7.1 of [7], $\blacksquare$

As noted in Proposition 1.2.3 of [6], the space $\mathcal{X}$ induces on $\Phi$ the family of functions $x : \Phi \to \mathbb{R}$ defined as $x(\phi) = \phi(x)$. This family of functions is also denoted by $\mathcal{X}$. Hence, for any $\bar{x} \in \mathcal{X}$, $\bar{\phi} \in \Phi$, we write $\bar{x} \in \partial_{\mathcal{X}} f_{\Phi}^*(\bar{\phi})$ if

$$f_{\Phi}^*(\phi) - f_{\Phi}^*(\bar{\phi}) \geq \phi(\bar{x}) - \bar{\phi}(\bar{x}) \; \text{for all} \; \phi \in \Phi.$$  

The following proposition holds (see also Proposition 1 of [19]).

**Proposition 2.5:** Let $f : \mathcal{X} \to (-\infty, +\infty]$ be a $\Phi$-convex function. Let $\bar{\phi} \in \Phi$ and $\bar{x} \in \mathcal{X}$. The following conditions are equivalent:

(i) $f(\bar{x}) + f_{\Phi}^*(\bar{\phi}) = \bar{\phi}(x)$.

(ii) $\bar{\phi} \in \partial_{\Phi} f(\bar{x})$.

(iii) $\bar{x} \in \partial_{\mathcal{X}} f^*(\bar{\phi})$.

**Proof:** The equivalence between (i) and (ii) was proved in [7], Proposition 7.7 and [6], Proposition 1.2.4. We show the equivalence between (i) and (iii).

Assume that (i) holds. Then, by $\Phi$-convexity of $f$, and Theorem 2.4(v), we get

$$f_{\Phi}^{**}(\bar{x}) + f_{\Phi}^*(\bar{\phi}) = \bar{\phi}(x),$$

which is equivalent to

$$\phi(x) - f_{\Phi}^*(\phi) + f_{\Phi}^*(\bar{\phi}) \leq \bar{\phi}(x) \; \forall \; \phi \in \Phi,$$

i.e. $\bar{\phi} \in \partial_{\mathcal{X}} f_{\Phi}^*(\bar{x})$. 

Assume that \((iii)\) holds. We have the following inequality:
\[
\bar{\varphi}(\bar{x}) \geq \varphi(\bar{x}) - f^*_\Phi(\varphi) + f^*_\Phi(\bar{\varphi}) \quad \forall \ \varphi \in \Phi.
\]
Taking the supremum over \(\varphi \in \Phi\), we obtain
\[
\bar{\varphi}(\bar{x}) \geq \sup_{\varphi \in \Phi} \{\varphi(\bar{x}) - f^*_\Phi(\varphi)\} + f^*_\Phi(\bar{\varphi}) \quad \forall \ \varphi \in \Phi,
\]
which is equivalent to
\[
\bar{\varphi}(\bar{x}) \geq f(\bar{x}) + f^*_\Phi(\varphi) \quad \forall \ \varphi \in \Phi.
\]
This, together with the Fenchel–Moreau inequality, gives \((i)\).

3. Perturbation function and its conjugate

Let \(\mathcal{X}\) and \(\mathcal{Y}\) be real linear vector spaces, \(\mathcal{X} + \mathcal{Y} \subset \mathcal{Y}\). Let \(f : \mathcal{X} \to (-\infty, +\infty]\), \(g : \mathcal{Y} \to (-\infty, +\infty]\). The perturbation function \(p : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]\) related to problem \((P)\) is defined as
\[
p(x, y) := f(x) + g(x + y). \quad (8)
\]
Clearly, \(p(x, 0) = f(x) + g(x)\).

In this section, we investigate the conjugate \(p^*\) to the perturbation function \(p\). The obtained formulae, will be used in Section 4 to define the conjugate dual to problem \((P)\).

Let \(\Phi\) and \(\Psi\) be two classes of elementary functions defined on \(\mathcal{X}\) and \(\mathcal{Y}\), respectively. To define the conjugate \(p^*\), we introduce the coupling function \(c\) on the Cartesian product \(\Phi \times \Psi\), \(c : (\Phi \times \Psi) \times (\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}\) as follows:
\[
c((\varphi, \psi), (x, y)) := \varphi(x) + \psi(x + y) - \psi(x). \quad (cpl)
\]
In other words, elementary functions \(c = (\varphi, \psi) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}\) defined on the product \((\Phi \times \Psi)\) are of the form
\[
(\varphi, \psi)(x, y) := \varphi(x) + \psi(x + y) - \psi(x). \quad (c)
\]
By \((\Phi \times \Psi)^c\) we denote the Cartesian product \(\Phi \times \Psi\) equipped with the coupling \((cpl)\). For other coupling functions defined on the Cartesian product \(\Phi \times \Psi\), see [19].

Clearly, if the class \(\Psi\) consists of all affine functions, i.e. \(\psi(y) = \langle w, y \rangle + d\), \(w \in \mathcal{Y}^\circ\) and \(\mathcal{Y}^\circ\) is the algebraic dual of \(\mathcal{Y}\), \(d \in \mathbb{R}\), then
\[
c((\varphi, \psi), (x, y)) := \varphi(x) + \psi_0(y),
\]
where \(\psi_0(y) = \langle w, y \rangle\), which is the standard bi-linear coupling, see, e.g. [15,16].
The conjugate $p_c^* : (\Phi \times \Psi)^c \rightarrow (-\infty, +\infty]$ with respect to the coupling (cpfl), i.e. with respect to the set of elementary functions $(\Phi \times \Psi)^c$, is given as

$$p_c^*(\varphi, \psi) := \sup_{x \in \mathcal{X}, y \in \mathcal{Y}} \{\varphi(x) + \psi(x + y) - \psi(x) - p(x, y)\}.$$  \hspace{1cm} (9)

Clearly, the conjugate $p_c^*$ depends on the choice of the coupling between the Cartesian products $\mathcal{X} \times \mathcal{Y}$ and $\Phi \times \Psi$ (for other definitions of nonlinear couplings, see, e.g. [19]). In the sequel, we simplify the notation and put $p^* := p_c^*$

Keeping in mind our application, from now on we assume that $\mathcal{X} = \mathcal{Y}$. We get

$$p^*(\varphi, \psi) = \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \{\varphi(x) + \psi(x + y) - \psi(x) - f(x) - g(x + y)\}
= \sup_{x \in \mathcal{X}} \varphi(x) - f(x) - \psi(x) + \sup_{y \in \mathcal{X}} \{\psi(x + y) - g(x + y)\}.$$  \hspace{1cm} (10)

By putting $z := x + y$, we obtain

$$p^*(\varphi, \psi) = \sup_{x \in \mathcal{X}} \varphi(x) - f(x) - \psi(x) + \sup_{z \in \mathcal{X}} \{\psi(z) - g(z)\}
= \sup_{x \in \mathcal{X}} \varphi(x) - f(x) - \psi(x) + g^*_{\Phi}(\psi).$$

(P1) When $\Psi = \Phi, 0 \in \Phi$, and $\mathcal{X} = \mathcal{Y}$, by (10), for any $\varphi \in \Phi$

$$p^*(0, \varphi) = \sup_{x \in \mathcal{X}} -f(x) - \varphi(x) + g^*_{\Phi}(\varphi)
= *f_{\Phi}(\varphi) + g^*_{\Phi}(\varphi),$$

where $*f_{\Phi}(\varphi) := \sup_{x \in \mathcal{X}} -f(x) - \varphi(x)$. Clearly, $*f_{\Phi}(\varphi) = f^*_1(-\varphi)$ whenever $-\varphi \in \Phi$. When $\Phi$ is a convex set, then $p^*(0, \cdot) : \Phi \rightarrow (-\infty, +\infty]$ is convex.

(P2) When $\mathcal{X} = \mathcal{Y}$ is a Hilbert space and $\Phi = \Phi_{lsc} = \Psi$, where

$$\Phi_{lsc} := \{\varphi : \mathcal{X} \rightarrow \mathbb{R} \mid \varphi(x) := -a\|x\|^2 + \langle v, x \rangle + c, a \geq 0, v \in \mathcal{X}, c \in \mathbb{R}\},$$

by (10), for any $\varphi \in \Phi$,

$$p^*(0, \varphi) = \sup_{x \in \mathcal{X}} -f(x) + a\|x\|^2 - \langle v, x \rangle - c + g^*_{\Phi_{lsc}}(\varphi)
= \sup_{x \in \mathcal{X}} -f(x) + a\|x\|^2 - \langle v, x \rangle + g^*_{\Phi_{lsc}}(\varphi_1)
= \tilde{f}_{\Phi_{lsc}}^*(0, -v) + g^*_{\Phi_{lsc}}(a, v) = p^*(0, \varphi_1)$$

where $\varphi_1(x) := -a\|x\|^2 + \langle v, x \rangle$ and $\tilde{f}(x) := \tilde{f}_{\varphi_1}(x) = f(x) - a\|x\|^2$ and we identify functions from the class $\Phi_{lsc}$ of the form $\varphi_1$ with pairs $(a, v), a \in \mathbb{R}_+, v \in \mathcal{X}$. By (13), the domain of $p^*(0, \cdot)$ can be restricted to elementary functions of the form $\varphi_1$ (with $c = 0$). As previously, $*f_{\Phi_{lsc}}(\varphi_1) := \sup_{x \in \mathcal{X}} -\varphi_1(x) - f(x)$. Clearly, $*f(\varphi_1) = *f(a, v) = \tilde{f}^*(0, -v)$. By Proposition 6.3 of [7] $\tilde{f}$ is $\Phi_{lsc}$-convex whenever $f$ is. Moreover, $p^*(0, \cdot)$ is a convex function on $\Phi_{lsc}$. 


(P3) When $\mathcal{X} = \mathcal{Y}$ and $\Phi = \Psi$, $\Phi$ is symmetric, i.e. $\Phi = -\Phi$ with $(-\varphi)(x) := -\varphi(x)$ we have

$$ p^*(0, \varphi) = -f^*_\Phi(-\varphi) - g^*_\Phi(\varphi). \quad (14) $$

Consider now $\Phi = \Phi_{\text{conv}}$, where $\mathcal{X}$ is a Banach space with the topological dual $\mathcal{X}^*$,

$$ \Phi_{\text{conv}} := \{ \varphi : \mathcal{X} \to \mathbb{R} | \varphi(x) = (v, x) + c, \ v \in \mathcal{X}^*, \ c \in \mathbb{R} \}. \quad (15) $$

By (11),

$$ p^*(0, \varphi) = \sup_{x \in \mathcal{X}} -f(x) - (v, x) + c + c + g^*(v) $$

$$ = f^*(-v) + g^*(v) = p^*(0, \varphi_0), \quad (16) $$

where $g^*(v) := \sup_{y \in \mathcal{X}} (v, y) - g(y)$ is the conjugate to $g$ in the sense of convex analysis, $\varphi_0(y) := (v, y)$. By (16), we can restrict the domain of $p^*(0, \cdot)$ to linear functionals.

(P4) When $\Psi \subset \Phi$, $\Psi$ is symmetric, i.e. $\Psi = -\Psi$ with $(-\psi)(x) := -\psi(x)$ we have

$$ p^*(0, \psi) = -f^*_\Psi(-\psi) - g^*_\Psi(\psi). \quad (17) $$

### 4. The $\Phi$-conjugate dual

Recall that, we assume that $\mathcal{X} = \mathcal{Y}$. Following the classical (convex) approach (see, e.g. Boţ [15] and Bonnans, Shapiro [16]), we introduce the conjugate dual to $(P)$ by the formula

$$ \max_{\psi \in \Psi} - p^*_c (0, \psi). \quad \text{(GCD)} $$

(D1) When $\Psi = \Phi$, $0 \in \Phi$, and the coupling $c$ is given by (cpl), by (11), the $\Phi$-conjugate dual (GCD) takes the form

$$ \max_{\varphi \in \Phi} - *f_\Phi(\varphi) - g^*_\Phi(\varphi). \quad \text{(CD)} $$

where $*f_\Phi(\varphi) = \sup_{x \in \mathcal{X}} -f(x) - \varphi(x)$. For dual problems resulting from other coupling functions $c$, see, e.g. [19].

(D2) When $\Psi = \Phi$, $0 \in \Phi$ and the set $\Phi$ is symmetric, i.e. $\Phi = -\Phi$, problem (CD) takes the form

$$ \max_{\varphi \in \Phi} - f^*_\Phi(-\varphi) - g^*_\Phi(\varphi). \quad \text{(CD}_{\text{sym}}) $$

(c.f. Corollary 5.2 of [12].)

(D3) When $\Psi = \Phi$, $0 \in \Phi$, and $\Phi + \Phi \subset \Phi$ and $\Phi = -\Phi$ the problem (CD$_{\text{sym}}$) becomes the $\Phi$-infimal convolution dual (ICD) as introduced in
\[ \text{Max}_{\varphi_1, \varphi_2 \in \Phi, \varphi_1 + \varphi_2 = 0} - f_\Phi^*(\varphi_2) - g_\Phi^*(\varphi_1). \]  

(1CD)

In general, when \( \Psi = \Phi \) and \( \Phi \) is not symmetric we have

\[ \text{val}(CD) \geq \text{val}(CD^{sym}). \]  

(18)

**Example 4.1:**

1. Let \( \mathcal{X} \) be a Hilbert space and \( \Psi = \Phi, \Phi = \Phi_{lsc} \). The \( \Phi_{lsc} \)-conjugate dual (CD) takes the form

\[ \text{Max}_{(a,w) \in \Phi_{lsc}} - \tilde{f}_p(a,w)^* (0, -w) - g_{\Phi_{lsc}}^*(a,w), \]  

\[ (CD_{lsc}) \]

where functions of the form \( \varphi_1(x) := -a\|x\|^2 + \langle w, x \rangle \) are identified with pairs \((a, w), w \in \mathcal{X}, a \in \mathbb{R}^+ \) and \( \tilde{f}_p(x) := f_p(a,w) = f(x) - a\|x\|^2 \) (according to (13) we can neglect constants). Since \( \Phi_{lsc} \neq -\Phi_{lsc} \), the \( \Phi_{lsc} \)-conjugate dual does not coincide, in general, with \( \Phi_{lsc} \)-infimal convolution dual (1CD), see Example 6.6.

2. When \( \mathcal{X} \) is a Banach space, and \( \Psi = \Phi, \Phi = \mathcal{X}^* \), the \( \mathcal{X}^* \)-conjugate dual (CD) becomes the classical Fenchel dual

\[ \text{Max}_{v \in \mathcal{X}^*} - f^*(-v) - g^*(v). \]  

(FD)

(FD) coincides with (1CD) and the \( \mathcal{X}^* \)-infimal convolution dual (1CD).

1. When \( \mathcal{X} \) is a Banach spaces \( \Psi = X^* \) and \( \Psi \subset \Phi \), the conjugate dual (GCD) takes the form analogous to the classical Fenchel dual (FD).

### 4.1. Weak (conjugate) duality

Let \( \mathcal{X} \) be a real linear space.

By (11), for every \( x \in \mathcal{X} \) and \( \varphi \in \Phi \)

\[ p(x, 0) + p^*(0, \varphi) = p(x, 0) + \varphi^*(\varphi) + g_\Phi^*(\varphi), \]

\[ \geq f(x) + g(x) - f(x) - \varphi(x) + \varphi(x) - g(x) \]

\[ = 0 \]

In consequence,

\[ p(x, 0) \geq -p^*(0, \varphi) \quad x \in \mathcal{X}, \varphi \in \Phi \]  

(19)

which yields the weak duality

\[ \text{val}(P) := \inf_{x \in \mathcal{X}} p(x, 0) = \inf_{x \in \mathcal{X}} f(x) + g(x) \geq \sup_{\varphi \in \Phi} -p^*(0, \varphi) =: \text{val}(CD). \]  

(20)

The problem of zero duality gap will be addressed in Section 6.
5. Lagrangian dual

In this section, we assume that $\mathcal{X}$ is a nonempty set and we introduce the $\Phi$-Lagrangian function (L) with the $\Phi$-Lagrangian dual equivalent to the $\Phi$-conjugate dual (CD). Let $f, g : \mathcal{X} \to (-\infty, +\infty]$.

For problem (P), we consider the $\Phi$-Lagrangian $L : \mathcal{X} \times \Phi \to \bar{\mathbb{R}}$ defined as

$$L(x, \varphi) := f(x) + \varphi(x) - g^*_\Phi(\varphi)$$

with the $\Phi$-Lagrangian primal

$$\inf_{x \in \mathcal{X}} \sup_{\varphi \in \Phi} L(x, \varphi)$$

and the $\Phi$-Lagrangian dual

$$\sup_{\varphi \in \Phi} \inf_{x \in \mathcal{X}} L(x, \varphi).$$

Then

$$\sup_{\varphi \in \Phi} L(x, \varphi) = f(x) + \sup_{\varphi \in \Phi} \varphi(x) - g^*_\Phi(\varphi) = f(x) + g^{**}(x).$$

(21)

**Proposition 5.1:** If $g : \mathcal{X} \to (-\infty, +\infty]$ is $\Phi$-convex on $\mathcal{X}$, the $\Phi$-Lagrangian primal (LP) is equivalent to (P), i.e.

$$\inf_{x \in \mathcal{X}} f(x) + g(x) = \inf_{x \in \mathcal{X}} \sup_{\varphi \in \Phi} L(x, \varphi),$$

(22)

**Proof:** Follows from Theorem 2.4 (v). ■

On the other hand, by (L),

$$\inf_{x \in \mathcal{X}} L(x, \varphi) = \inf_{x \in \mathcal{X}} f(x) + \varphi(x) - g^*_\Phi(\varphi) = -\sup_{x \in \mathcal{X}} -f(x) - \varphi(x) - g^*_\Phi(\varphi).$$

By using the notation $^*f_\Phi(\varphi) = \sup_{x \in \mathcal{X}} -\varphi(x) - f(x),$

$$\sup_{\varphi \in \Phi} \inf_{x \in \mathcal{X}} L(x, \varphi) = \sup_{\varphi \in \Phi} -^*f_\Phi(\varphi) - g^*_\Phi(\varphi)$$

(23)

which shows that the $\Phi$-conjugate dual (CD) is equivalent to the $\Phi$-Lagrangian dual (LD) with the Lagrangian defined by (L).
Example 5.2: Let $\mathcal{X}$ be a Hilbert space. For $\varphi \in \Phi_{lsc}$, $\varphi(x) := -a\|x\|^2 + \langle w, x \rangle$, $a \geq 0$, $w \in \mathcal{X}$, we have

$$\inf_{x \in \mathcal{X}} \mathcal{L}(x, \varphi) = \inf_{x \in \mathcal{X}} f(x) + \varphi(x) - g^*_\Phi_{lsc}(\varphi) = -\sup_{x \in \mathcal{X}} -f(x) - \varphi(x) - g^*_\Phi_{lsc}(\varphi)$$

$$= -\sup_{x \in \mathcal{X}} -f(x) + a\|x\|^2 - \langle w, x \rangle - g^*_\Phi_{lsc}(\varphi)$$

$$= -\sup_{x \in \mathcal{X}} -\tilde{f}(x) - \langle w, x \rangle - g^*_\Phi_{lsc}(\varphi)$$

$$= -\tilde{f}^*_\Phi_{lsc}(0, -w) - g^*_\Phi_{lsc}(a, w),$$

where $\varphi$ is identified with the pair $(a, w)$, and, for a given $\varphi, \tilde{f} := f(\cdot) - a\| \cdot \|^2$.

$$\sup_{\varphi \in \Phi} \inf_{x \in \mathcal{X}} \mathcal{L}(x, \varphi) = \sup_{\varphi \in \Phi_{lsc}} -\tilde{f}^*_\Phi_{lsc}(0, -w) - g^*_\Phi_{lsc}(a, w), \quad (24)$$

and the $\Phi_{lsc}$-conjugate dual $(CD_{lsc})$ coincides with the $\Phi_{lsc}$-Lagrangian dual (LD).

Example 5.3: Let $\mathcal{X}$ be a Banach space. Let $\Phi := \{ \varphi : \mathcal{X} \to \mathbb{R} \mid \varphi(x) := \langle v, x \rangle + c, v \in \mathcal{X}^*, c \in \mathbb{R}\}$ and let $g(\cdot := \text{ind}_A(\cdot)$ be the indicator function of a set $A$,

$$A := \{ x \in \mathcal{X} \mid x \in K \}$$

where $K$ is a cone in $\mathcal{X}$. For the problem $(P)$ with $f : \mathcal{X} \to (-\infty, +\infty]$,

$$\mathcal{L}(x, \varphi) = f(x) + \varphi(x) - (\text{ind}_A)^*(\varphi)$$

$$= f(x) + \langle v, x \rangle + c - \sup_{x \in \mathcal{X}} \{ \langle v, x \rangle + c - \text{ind}_K(x) \}$$

$$= f(x) + \langle v, x \rangle - \sup_{x \in \mathcal{X}} \{ \langle v, x \rangle - \text{ind}_K(x) \}$$

$$= f(x) + \langle v, x \rangle - \sup_{x \in K} \langle v, x \rangle$$

Since

$$\sup_{x \in K} \langle v, x \rangle = \begin{cases} 0 & v \in K^c \\ +\infty & v \notin K^c \end{cases}$$

where $K^c := \{ v \in \mathcal{X} \mid \langle v, x \rangle \leq 0 \ \forall x \in K \}$ is the polar cone to $K$, we get

$$\mathcal{L}(x, \varphi) = \begin{cases} f(x) + \langle v, x \rangle & v \in K^c \\ -\infty & v \notin K^c \end{cases}.$$
6. Zero duality gap for $\Phi$-conjugate duality

In view of Proposition 5.1, and formula (23), the question of zero duality gap for $\Phi$-conjugate and $\Phi$-Lagrangian dualities can be investigated simultaneously, by seeking conditions ensuring minimax equality for $\Phi$-Lagrangian.

We begin this section by discussing zero duality gap for problems (LP), (LD) from the point of view of minimax theorems. The characterization of zero duality gap for problems (LP), (LD) is expressed with the help of the so-called intersection property, which is used in general minimax theorems formulated within the framework of $\Phi$-convexity as it is done in [18] for the case, where the elementary functions may admit infinite values. For convenience of the reader, we provide the outline of the proof based on Lemma 6.2. The intersection property together with the condition $\inf_{x \in X} f(x) + g(x) = \inf_{x \in X} f(x) + g_\Phi^*(x)$ immediately gives the zero duality gap condition for the pair of dual problems (P) and (CD).

**Theorem 6.1:** Let $X$ be a nonempty set. Let $\Phi$ be a convex set of elementary functions $\varphi : X \to \mathbb{R}$ and $f, g : X \to (-\infty, +\infty]$ and the $\Phi$-Lagrangian is given by (L).

The following are equivalent:

(i) for every $\alpha < \inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi)$ there exist $\varphi_1, \varphi_2 \in \Phi$ and $\bar{\varphi}_1 \in \text{supp} L(\cdot, \varphi_1)$ and $\bar{\varphi}_2 \in \text{supp} L(\cdot, \varphi_2)$ such that functions $\bar{\varphi}_1$ and $\bar{\varphi}_2$ have the intersection property on $X$ at the level $\alpha$, i.e. for all $t \in [0, 1]$

$$[t\bar{\varphi}_1 + (1 - t)\bar{\varphi}_2 < \alpha] \cap [\bar{\varphi}_1 < \alpha] = \emptyset \quad [t\bar{\varphi}_1 + (1 - t)\bar{\varphi}_2 < \alpha]$$

$$\cap [\bar{\varphi}_2 < \alpha] = \emptyset,$$

where $[\bar{\varphi} < \alpha] := \{x \in X : \bar{\varphi}(x) < \alpha\}$.

(ii)

$$\inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi) = \sup_{\varphi \in \Phi} \inf_{x \in X} L(x, \varphi).$$

The proof of Theorem 6.1 is based on the following lemma (c.f. [18], Lemma 4.1).

**Lemma 6.2:** Let $X$ be a nonempty set, $\alpha \in \mathbb{R}$, and let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be any two functions. The functions $\varphi_1$ and $\varphi_2$ have the intersection property on $X$ at the level $\alpha$ if and only if $\exists t_0 \in [0, 1]$ such that

$$t_0 \varphi_1 + (1 - t_0) \varphi_2 \geq \alpha \quad \forall x \in X.$$  

Lemma 4.1 proved in [18] refers to a more general situation, where $\varphi_1, \varphi_2 : X \to [-\infty, +\infty]$ and reduces to Lemma 6.2 whenever $\varphi_1, \varphi_2 : X \to \mathbb{R}$. 

Proof: Let \( \alpha < \inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi) \). By (i), there exist \( \varphi_1, \varphi_2 \in \Phi \) and \( \tilde{\varphi}_1 \in \text{supp}L(\cdot, \varphi_1) \) and \( \tilde{\varphi}_2 \in \text{supp}L(\cdot, \varphi_2) \) such that \( \tilde{\varphi}_1 \in \text{supp}L(\cdot, \varphi_1) \) and \( \tilde{\varphi}_2 \in \text{supp}L(\cdot, \varphi_2) \) have the intersection property on \( X \) at the level \( \alpha \). By Lemma 6.2 and (26), there exists \( t \in [0, 1] \) such that
\[
t \tilde{\varphi}_1 + (1 - t) \tilde{\varphi}_2 \geq \alpha \quad \forall \ x \in X. \tag{27}
\]
By the definition of the support set and the inequality (27), we get
\[
tL(x, \varphi_1) + (1 - t)L(x, \varphi_2) \geq \alpha \quad \forall \ x \in X. \tag{28}
\]
By the concavity of \( L \) as a function of \( \varphi \), we have
\[
L(x, \varphi_0) \geq \alpha \quad \forall \ x \in X, \tag{29}
\]
where \( \varphi_0 = t \varphi_1 + (1 - t) \varphi_2 \) and, by convexity of \( \Phi \), \( \varphi_0 \in \Phi \).
From this we deduce the following inequality:
\[
\sup_{\varphi \in \Phi} \inf_{x \in X} L(x, \varphi) \geq \alpha \quad \forall \ x \in X. \tag{30}
\]
By the fact that the inequality (30) holds for every \( \alpha < \inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi) \) we get the desired conclusion.

The second implication follows directly from Theorem 2.1 of [18]. \( \blacksquare \)

Remark 1: Let us note that in some classes of functions (e.g. \( \Phi_{lsc} \) and \( \Phi_{conv} \), see Proposition 2 and Proposition 4 of [20]) the intersection property at the level \( \alpha \) is equivalent to the condition
\[
[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset. \tag{31}
\]

Theorem 6.1 allows us to formulate the following zero duality gap conditions for \( \Phi \)-conjugate dual (CD).

**Theorem 6.3:** Let \( X \) be a nonempty set. Let \( \Phi \) be a convex set of elementary functions \( \varphi : X \to \mathbb{R}, f, g : X \to (-\infty, +\infty] \) and the \( \Phi \)-Lagrangian is given by (L). Assume that
\[
\inf_{x \in X} f(x) + g(x) = \inf_{x \in X} f(x) + g^*_{\Phi}(x). \tag{32}
\]
The following are equivalent:

(i) for every \( \alpha < \inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi) \) there exist \( \varphi_1, \varphi_2 \in \Phi \) and \( \tilde{\varphi}_1 \in \text{supp}L(\cdot, \varphi_1) \) and \( \tilde{\varphi}_2 \in \text{supp}L(\cdot, \varphi_2) \) such that functions \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_1 \) have the intersection property at the level \( \alpha \), i.e. for all \( t \in [0, 1] \)
\[
[t \tilde{\varphi}_1 + (1 - t) \tilde{\varphi}_2 < \alpha] \cap [\tilde{\varphi}_1 < \alpha] = \emptyset \quad \text{or} \quad [t \tilde{\varphi}_1 + (1 - t) \tilde{\varphi}_2 < \alpha] \cap [\tilde{\varphi}_2 < \alpha] = \emptyset. \tag{33}
\]
(ii) \[
\inf_{x \in X} \{ f(x) + g(x) \} = \sup_{\varphi \in \Phi} -f_{\varphi}(\varphi) - g_{\varphi}^*(\varphi),
\]

where \( f_{\varphi}(\varphi) = \sup_{x \in X} -\varphi(x) - f(x) \).

**Proof:** Follows directly from Theorem 6.1 and formula (21). ☐

By Proposition 5.1 if \( g \) is \( \Phi \)-convex on \( X \), then (32) holds. The following conditions for zero duality gap for problems (P) and (ICD) were proved in [14].

**Theorem 6.4 ([14], Theorem 3.5):** Let \( X \) be a nonempty set. Let \( f, g : X \to (-\infty, +\infty] \) and \( 0 \in \Phi \) and \( \Phi + \Phi \subset \Phi \). The following properties are equivalent:

(i) \[
0 \in \bigcap_{\epsilon > 0} (\partial_{\Phi}^\epsilon f + \partial_{\Phi}^\epsilon g)(X),
\]

(ii) \[
\inf_{x \in X} (f(x) + g(x)) = \sup_{\varphi_1, \varphi_2 \in \Phi, \varphi_1 + \varphi_2 = 0} (-f_{\varphi_1}^*(\varphi_1) - g_{\varphi_2}^*(\varphi_2)) = \text{val}(ICD) < +\infty.
\]

**Remark 2:** The following inequalities hold:

\[
\text{val}(P) \overset{(1)}{\geq} \text{val}(LP) \overset{(2)}{\geq} \text{val}(LD) \overset{(3)}{=} \text{val}(CD) \overset{(4)}{\geq} \text{val}(ICD),
\]

where (1) holds by Theorem 2.4 (iii), (2) holds by general minimax inequality, (3) holds by (23) and (4) holds by (18).

In particular, condition (34) of Theorem 6.4 implies that \( \text{val}(P) = \text{val}(LP) \), i.e.

\[
\inf_{x \in X} f(x) + g(x) = \inf_{x \in X} f(x) + g_{**}(x).
\]

**Theorem 6.5:** Let \( X \) be a nonempty set, \( 0 \in \Phi \), and \( \kappa := \inf_{x \in X} \sup_{\varphi \in \Phi} \mathcal{L}(x, \varphi) \)

\(< +\infty \).

Consider the following conditions:

1. **condition (34):**

\[
0 \in \bigcap_{\epsilon > 0} (\partial_{\Phi}^\epsilon f + \partial_{\Phi}^\epsilon g)(X),
\]

2. **condition (33):** for every \( \alpha < \inf_{x \in X} \sup_{\varphi \in \Phi} \mathcal{L}(x, \varphi) \) there exist \( \bar{\varphi}, \tilde{\varphi} \in \Phi \), \( \varphi_1 \in \supp \mathcal{L}(., \bar{\varphi}) \) and \( \varphi_2 \in \supp \mathcal{L}(., \tilde{\varphi}) \) such that \( \varphi_1 \) and \( \varphi_2 \) have the intersection property at the level \( \alpha \).
Then

(1) If $\Phi$ is convex, $\Phi = -\Phi$, and

$$\inf_{x \in X} f(x) + g(x) = \inf_{x \in X} f(x) + g^*_{\Phi}(x) = \inf_{x \in X} \sup_{\varphi \in \Phi} L(x, \varphi) = \kappa.$$ \hspace{1cm} (36)

then (2) implies (1).

(2) If $\Phi + \Phi \subset \Phi$, then (1) implies (2).

**Proof:** (2) $\Rightarrow$ (1). Let $\varepsilon > 0$. By Lemma 6.2, there exists $t_0 \in [0, 1]$ such that $t_0 \psi_1 + (1 - t_0) \psi_2 \geq \alpha := \kappa - \varepsilon$ for all $x \in X$. Hence,

$$L(x, t_0 \psi_1 + (1 - t_0) \psi_2) \geq t_0 L(x, \psi_1) + (1 - t_0) L(x, \psi_2) \geq t_0 \psi_1 + (1 - t_0) \psi_2$$

for all $x \in X$ and

$$L(x, \psi_0) \geq \kappa - \varepsilon = (\varepsilon + \kappa) - 2\varepsilon \quad \forall x \in X,$$ \hspace{1cm} (37)

where $\psi_0 := t_0 \psi_1 + (1 - t_0) \psi_2 \in \Phi$ (in view of the convexity of $\Phi$). By Assumption (36), and Theorem 2.4, (iii), there exists $\tilde{x} \in X$ satisfying

$$\varepsilon + \kappa > f(\tilde{x}) + g(\tilde{x}) \geq \sup_{\varphi \in \Phi} L(\tilde{x}, \varphi) = f(\tilde{x}) + g^*_{\Phi}(\tilde{x}).$$ \hspace{1cm} (38)

Moreover, $\varepsilon + f(\tilde{x}) + g^*_{\Phi}(\tilde{x}) \geq \varepsilon + \kappa > f(\tilde{x}) + g(\tilde{x})$, hence $g^*_{\Phi}(\tilde{x}) \geq g(\tilde{x}) - \varepsilon$ and

$$\varepsilon + \kappa > f(\tilde{x}) + g(\tilde{x}) \geq \sup_{\varphi \in \Phi} L(\tilde{x}, \varphi) = f(\tilde{x}) + g^*_{\Phi}(\tilde{x}) \geq f(\tilde{x}) + g(\tilde{x}) - \varepsilon.$$ \hspace{1cm} (39)

By (37), (38), (39) for all $x \in X$,

(a). $L(x, \psi_0) \geq \sup_{\varphi \in \Phi} L(\tilde{x}, \varphi) - 2\varepsilon \geq L(\tilde{x}, \psi_0) - 3\varepsilon$

(b). $L(x, \psi_0) \geq f(\tilde{x}) + g(\tilde{x}) - 3\varepsilon$.

In particular, by (b),

(b'). $L(\tilde{x}, \psi_0) \geq f(\tilde{x}) + g(\tilde{x}) - 3\varepsilon$.

Since $\Phi = -\Phi$, hence $-\psi_0 \in \Phi$ and by (a),

$$f(x) + \psi_0(x) - g^*(\psi_0) = L(x, \psi_0) \geq f(\tilde{x}) + \psi_0(\tilde{x}) - g^*(\psi_0) - 3\varepsilon \quad \forall x \in X.$$ \hspace{1cm} (40)

which shows that $-\psi_0 \in \partial^{3\varepsilon} f(\tilde{x})$. By (b'),

$$L(\tilde{x}, \psi_0) \geq f(\tilde{x}) + g(\tilde{x}) - 3\varepsilon \quad \text{i.e.} \quad f(\tilde{x}) + \psi_0(\tilde{x}) - g^*(\psi_0) \geq f(\tilde{x}) + g(\tilde{x}) - 3\varepsilon$$

which gives $\psi_0(\tilde{x}) - g^*(\psi_0) \geq g(\tilde{x}) - 3\varepsilon$.

By Theorem 2.4 (ii), the latter is equivalent to $\psi_0 \in \partial^{3\varepsilon} g(\tilde{x})$. 

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This means that

\[ 0 \in \bigcap_{\varepsilon > 0} (\partial^\varepsilon f + \partial^\varepsilon g)(\mathcal{X}) \]

i.e. (1) holds.

(1) ⇒ (2). Let \( \alpha < \inf_{x \in \mathcal{X}} \sup_{\varphi \in \Phi} \mathcal{L}(x, \varphi) \) and take any \( \beta \) such that \( \alpha < \beta < \inf_{x \in \mathcal{X}} \sup_{\varphi \in \Phi} \mathcal{L}(x, \varphi) \). Let \( \varepsilon = \beta - \alpha > 0 \). By assumption, there exist \( \bar{x} \in \mathcal{X} \) and \( \tilde{\varphi} \in \partial^\varepsilon f(\bar{x}) \) and \( \tilde{\varphi} \in \partial^\varepsilon g(\bar{x}) \) such that

\[ \tilde{\varphi} + \bar{\varphi} = 0. \] (41)

Since \( \bar{\varphi} \in \partial^\varepsilon f(\bar{x}) \) the following inequality holds for all \( x \in \mathcal{X} \):

\[ f(x) - f(\bar{x}) \geq \bar{\varphi}(x) - \bar{\varphi}(\bar{x}) - \varepsilon, \]

which is equivalent to

\[ f(x) + (-\bar{\varphi}(x)) - g^*_\Phi(-\bar{\varphi}) - (f(\bar{x}) + (-\bar{\varphi}(\bar{x})) - g^*_\Phi(-\bar{\varphi})) \geq 0 - \varepsilon \quad \forall \ x \in \mathcal{X}. \]

Equivalently,

\[ \mathcal{L}(x, \bar{\varphi}) - \mathcal{L}(\bar{x}, \bar{\varphi}) \geq -\varepsilon. \] (42)

By Equality (41) we have \( \bar{\varphi} = -\bar{\varphi} \) and Inequality (42) takes the form

\[ \mathcal{L}(x, \bar{\varphi}) \geq \mathcal{L}(\bar{x}, \bar{\varphi}) - \varepsilon \quad \forall \ x \in \mathcal{X}. \] (43)

Since \( \bar{\varphi} \in \partial^\varepsilon g(\bar{x}) \), the inequality \( g^*_\Phi(\bar{\varphi}) + g(\bar{x}) \leq \bar{\varphi}(\bar{x}) + \varepsilon \) holds (see Theorem 2.4 ii). We have

\[ \mathcal{L}(\bar{x}, \bar{\varphi}) = f(\bar{x}) + \bar{\varphi}(\bar{x}) - g^*_\Phi(\bar{\varphi}) \geq f(\bar{x}) + g(\bar{x}) - \varepsilon \geq \inf_{x \in \mathcal{X}} \{f(x) + g(x)\} - \varepsilon \]

The above inequality together with (43) and (36) give

\[ \mathcal{L}(x, \bar{\varphi}) \geq \inf_{x \in \mathcal{X}} \{f(x) + g(x)\} - \varepsilon = \inf_{x \in \mathcal{X}} \sup_{\varphi \in \Phi} \mathcal{L}(x, \varphi) - \varepsilon \quad \forall \ x \in \mathcal{X}. \] (44)

By the inequality (44), we have

\[ \mathcal{L}(x, \bar{\varphi}) \geq \beta - \varepsilon = \beta - \beta + \alpha = \alpha. \]

Let \( \varphi_1 \equiv \alpha \) then \( \varphi_1 \in \text{supp} \mathcal{L}(\cdot, \bar{\varphi}) \) and \( [\varphi_1 < \alpha] = \emptyset \). Let \( \varphi_2 \in \text{supp} \mathcal{L}(\cdot, \bar{\varphi}) \), then \( \varphi_1, \varphi_2 \) have the intersection property at the level \( \alpha \). \[ \blacksquare \]

**Remark 3:** (1) Let \( \mathcal{X} \) be a topological vector space equipped with closed convex pointed cone \( S \) which induces the ordering relation: \( x \leq y \iff y - x \in S \). The family of functions \( L := \{\ell_y : \mathcal{X} \to \mathbb{R} \mid y \in \mathcal{X}\} \) is defined as

\[ \ell_y(x) = \max\{\lambda \geq 0 \mid \lambda y \leq x\}. \]

It was shown in [21] (see also [22]) that \( 0 \in L, L + L \subset L, \) and \( L = -L \) and a function \( f : \mathcal{X} \to [0, +\infty] \) is increasing positive homogeneous (IPH) if
and only if $f$ is $L$-convex. $L$-conjugate dual (CD) coincides with $L$-infimal convolution dual (ICD).

(2) By Theorem 6.5, if $\Phi$ is convex, $0 \in \Phi$, $\Phi + \Phi \subset \Phi$ and $\Phi = -\Phi$,

condition (34) $\iff$ condition (33) + equality (36).

(3) If $0 \in \Phi$, $\Phi + \Phi \subset \Phi$ and $\Phi = -\Phi$, then (CD) is equivalent to (ICD) (see Corollary 6.7) and Theorem 6.4 is stronger than Theorem 6.3 since the convexity of $\Phi$ is not required in Theorem 6.4.

(4) If $0 \in \Phi$, $\Phi + \Phi \subset \Phi$ and $\Phi \neq -\Phi$, (e.g. $\Phi = \Phi_{lsc}$) the $\Phi$-infimal convolution dual (ICD) is defined but is not equivalent to (CD) and it may happen that zero duality gap holds for (CD) but not for (ICD) (see Example 6.6).

**Example 6.6:** Let $\Phi = \Phi_{lsc}$. Let $g(x) = -x^2$ and $f(x) = 2x^2$. It is easy to see that $\inf_{x \in X} (f(x) + g(x)) = 0$. For every $\varepsilon > 0$, the elements of the set $\partial_{\varepsilon} g$ are of the form $(a, b) \in \mathbb{R} \times \mathbb{R}$ with $a \geq 1$ and some $b \in \mathbb{R}$, this means that

$$0 \notin \bigcap_{\varepsilon > 0} (\partial_{\varepsilon} f + \partial_{\varepsilon} g)(X).$$

On the other hand,

$$L(x, 1, 0) = 2x^2 - x^2 - \sup_{x \in X} \{-x^2 + x^2\} = x^2.$$

This means that $\hat{\varphi} \equiv 0$ belongs to the set $\text{supp} L(x, 1, 0)$. We have

$$L(x, 3, 0) = 2x^2 - 3x^2 - \sup_{x \in X} \{-3x^2 + x^2\} = -x^2,$$

and the set $\text{supp} L(x, 3, 0) \neq \emptyset$. Hence the functions $\hat{\varphi}$ and any other $\varphi \in \text{supp} L(x, 3, 0)$ have the intersection property at every level $\alpha < 0$.

For any $a \geq 0$, $b \in \mathbb{R}$ and $\tilde{f} = (2 - a)x^2$ we have

$$g^*(a, b) = \sup_{x \in \mathbb{R}} \{bx - ax^2 + x^2\} = \begin{cases} +\infty & \text{for } 0 \leq a < 1 \\ +\infty & \text{for } a = 1 \ b \neq 0 \\ 0 & \text{for } a = 1 \ b = 0, \\ \frac{b^2}{4(a-1)} & \text{for } a > 1 \end{cases}$$

$$\tilde{f}^*(0, -b) = \sup_{x \in \mathbb{R}} \{-bx + ax^2 - 2x^2\} = \begin{cases} +\infty & \text{for } a = 2, \ b = 0 \\ +\infty & \text{for } a = 2, \ b \neq 0 \end{cases}$$

$$f^*(0, b) = \sup_{x \in \mathbb{R}} \{bx - 2x^2\} = \frac{b^2}{8}.$$ 

In consequence,

$$\text{val} (ICD) = \sup_{b \in \mathbb{R}} -f^*(0, -b) - g^*(0, b) = \sup_{b \in \mathbb{R}} -\frac{b^2}{8} - \infty = -\infty$$
and
\[
\text{val}(CD) = \sup_{(a,b) \in \mathbb{R}^+ \times \mathbb{R}} -\tilde{f}^*(0, -b) - g^*(a, b) \\
= \max\{0, \sup_{1 < a < 2 b \in \mathbb{R}} b^2 \frac{1}{4(a - 2)} - \frac{b^2}{4(a - 1)}\} \\
= \max\{0, \sup_{1 < a < 2 b \in \mathbb{R}} b^2 \frac{1}{4(a - 1)(a - 2)}\} \\
= 0
\]

**Corollary 6.7:** Let \( \mathcal{X} \) be a real vector space. Let \( g : \mathcal{X} \to (-\infty, +\infty] \) be \( \Phi \)-convex. If \( \Phi + \Phi \subset \Phi \), \( 0 \in \Phi \), \( \Phi = -\Phi \) and \( \Phi \) is a convex set, then the intersection property (25) is equivalent to (34). Consequently, (ICD) is equivalent to (CD_{sym}), i.e.
\[
\text{val}(ICD) = \sup_{\varphi \in \Phi} -f^*_\Phi(-\varphi) - g^*_\Phi(\varphi) = \text{val}(CD_{sym}). \tag{45}
\]

**Proof:** Follows directly from Theorem 6.5.

\[\blacksquare\]

7. \( \Phi \)-Karush–Kuhn–Tucker conditions

In this section, we provide a characterization of solutions to (P) and (CD)/(CD_{sym}) in terms of the \( \Phi \)-Karush–Kuhn–Tucker conditions.

Let \( \mathcal{X} \) be a real vector space. Consider problem (P)
\[
\min_{x \in \mathcal{X}} f(x) + g(x). \tag{P}
\]

where \( f, g : \mathcal{X} \to (-\infty, +\infty] \) are \( \Phi \)-convex.

The existence of solutions to the dual problem (CD_{sym}) was investigated in [12] and the following result was proved.

**Proposition 7.1 ([12], Corollary 5.2):** Let \( \Phi \) be an additive and symmetric set of elementary functions, i.e. \( -\varphi \in \Phi \) if \( \varphi \in \Phi \). Assume that \( f \) and \( g \) are \( \Phi \)-convex. If the mapping \( \text{supp}(\cdot, \Phi) \) is additive in \( f \), \( g \), then there exists \( \varphi^* \in \Phi \), such that
\[
\inf_{x \in \mathcal{X}} \{f(x) + g(x)\} = -f^*_\Phi(\varphi^*) - g^*_\Phi(-\varphi^*) = \sup_{\varphi \in \Phi} \{-f^*_\Phi(\varphi) - g^*_\Phi(-\varphi)\}
\]

**Definition 7.2:** Let \( \Phi \) be symmetric i.e. \( \Phi = -\Phi \). We say that \( x^* \in \mathcal{X} \) and \( \varphi^* \in \Phi \) satisfy the \( \Phi \)-Karush–Kuhn–Tucker conditions (KKT) for the pair of dual problems (P) and (CD_{sym}) if
\[
-\varphi^* \in \partial f(x^*), \quad x^* \in \partial g^*_\Phi(\varphi^*). \tag{KKT}
\]
Theorem 7.3: Let $\mathcal{X}$ be a vector space and $\Phi$ be a symmetric set of elementary functions. Let $f, g : \mathcal{X} \to (-\infty, +\infty]$ be $\Phi$-convex functions. Let $x^* \in \mathcal{X}$ and $\varphi^* \in \Phi$.

The following conditions are equivalent.

(i) $x^*$ and $\varphi^*$ are solutions to $(P)$ and $(CD_{\text{sym}})$, respectively, i.e.

$$
\inf_{x \in \mathcal{X}} (f(x) + g(x)) = f(x^*) + g(x^*) = \sup_{\varphi \in \Phi} (-f^*_\Phi(-\varphi) - g^*_\Phi(\varphi))
$$

$$
= -f^*_\Phi(-\varphi^*) - g^*_\Phi(\varphi^*) \quad (46)
$$

(ii) $x^*$ and $\varphi^*$ satisfy the $\Phi$-KKT conditions, i.e.

$$
- \varphi^* \in \partial_{\Phi} f(x^*), \quad x^* \in \partial_{\mathcal{X}} g^*_{\Phi}(\varphi^*). \quad (47)
$$

Proof: Assume that $(46)$ holds, i.e.

$$
f(x^*) + g(x^*) = -f^*_\Phi(-\varphi^*) - g^*_\Phi(\varphi^*). \quad (48)
$$

By Theorem 2.4 and the $\Phi$-convexity of $g$, $(46)$ is equivalent to

$$
f(x^*) + g^*_{\Phi}(x^*) = -f^*_\Phi(-\varphi^*) - g^*_\Phi(\varphi^*).
$$

By the definition of $g^*_{\Phi}(x^*)$, $f(x^*) + \varphi^*(x^*) - g^*_\Phi(\varphi^*) \leq -f^*_\Phi(-\varphi^*) - g^*_\Phi(\varphi^*)$, i.e.

$$
f(x^*) + f^*_\Phi(-\varphi^*) \leq -\varphi^*(x^*).
$$

This, together with the Fenchel–Moreau inequality yields to $f(x^*) + f^*_\Phi(-\varphi^*) = -\varphi^*(x^*)$ i.e. $-\varphi^* \in \partial_{\Phi} f(x^*)$.

Analogously, by replacing in $(48)$ function $f$ with $f^*_{\Phi}(x^*)$ we obtain $x^* \in \partial_{\mathcal{X}} g^*_{\Phi}(\varphi^*)$.

Assume now that the conditions $(47)$ hold. By Proposition 2.5,

$$
- \varphi^*(x^*) = f^*_\Phi(-\varphi^*) + f(x^*) \quad (49)
$$

and

$$
\varphi^*(x^*) = g^*_\Phi(\varphi^*) + g(x^*). \quad (50)
$$

Hence,

$$
f(x^*) + g(x^*) = -f^*_\Phi(-\varphi^*) - g^*_\Phi(\varphi^*).
$$

From $(50)$, we get $\varphi^* \in \partial_{\Phi} g(x^*)$, which means that $0 \in \partial_{\Phi} f(x^*) + \partial_{\Phi} g(x^*) \subset \partial_{\Phi} (f + g)(x^*)$. From Equality $(49)$, we have $x^* \in \partial_{\mathcal{X}} f^*_{\Phi}(-\varphi^*)$, this, together with
the assumption that \( x^* \in \partial \chi g_\Phi^*(\varphi^*) \), yields to
\[
f_\Phi^*(-\varphi) + g_\Phi^*(\varphi) - f_\Phi^*(-\varphi^*) - g_\Phi^*(\varphi^*) \geq -\varphi(x^*) + \varphi^*(x^*) + \varphi(x^*)
- \varphi^*(x^*) \quad \forall \varphi \in \Phi.
\]
Equivalently, \(-f_\Phi^*(-\varphi) - g_\Phi^*(\varphi) \leq -f_\Phi^*(-\varphi^*) - g_\Phi^*(\varphi^*) \) for all \( \varphi \in \Phi \) which means that \(-f_\Phi^*(\varphi^*) - g_\Phi^*(\varphi^*) = \sup_{\varphi \in \Phi} (-f_\Phi^*(-\varphi) - g_\Phi^*(\varphi)) \). This completes the proof. 

### 7.1. KKT for \( \Phi = \Phi_{lsc} \)

Let \( \mathcal{X} \) be a Hilbert space. In the present section, we prove a variant of Theorem 7.3 with \( f \) and \( g \) which are \( \Phi_{lsc} \)-convex, where
\[
\Phi_{lsc} = \{ \varphi : \mathcal{X} \to \mathbb{R} \mid \varphi(x) := -a\|x\|^2 + \langle v, x \rangle + c, \ a \geq 0, \ v \in \mathcal{X}, \ c \in \mathbb{R} \}.
\]  
(51)

The set \( \Phi_{lsc} \) is nonsymmetric and forms a nonpointed cone with the lineality space \( L = \mathcal{X} \). By Proposition 6.3 of [7], the class of \( \Phi_{lsc} \)-convex functions defined on Hilbert space \( \mathcal{X} \) coincides with the class of all lower semicontinuous functions minorized by a function \( \varphi \in \Phi_{lsc} \). Clearly, the class \( \Phi_{lsc} \) is additive and the sum \( f + g \) of any \( \Phi_{lsc} \)-convex functions \( f \) and \( g \) is a \( \Phi_{lsc} \) function.

Recall that the \( \Phi_{lsc} \)-conjugate dual to problem (P) with \( \Phi_{lsc} \)-convex functions \( f \) and \( g \) has the form (\( CD_{lsc} \))
\[
\text{Max}_{(a,w) \in \Phi_{lsc}} (-\tilde{f}(a,w)) - g_{\Phi_{lsc}}^*(a,w), \quad (CD_{lsc})
\]
where functions of the form \( \psi_1(x) := -a\|x\|^2 + \langle w, x \rangle \) are identified with pairs \((a, w), \ a \geq 0, \ w \in \mathcal{X} \) and \( \tilde{f}_\psi_1(x) = \tilde{f}_\psi_1(x) := f(x) - a\|x\|^2 \).

**Theorem 7.4:** Let \( \mathcal{X} \) be a Hilbert space. Let \( f, g : \mathcal{X} \to (-\infty, +\infty] \) be \( \Phi_{lsc} \)-convex functions.

In order that \( x^* \in \mathcal{X} \) and \( \varphi^* = (a^*, w^*) \in \Phi_{lsc} \) be such that
\[
\inf_{x \in \mathcal{X}} (f(x) + g(x)) = f(x^*) + g(x^*)
= \sup_{(a,w) \in \Phi_{lsc}} (-\tilde{f}(a,w))_{\Phi_{lsc}}^*(0,-w) - g_{\Phi_{lsc}}^*(0,w))
= -\tilde{f}^*(a^*,w^*)_{\Phi_{lsc}}(0,-w^*) - g_{\Phi_{lsc}}^*(a^*,w^*)
\]  
(52)
i.e. \( x^* \in \mathcal{X} \) solves (P) and \( \varphi^* = (a^*, w^*) \in \Phi_{lsc} \) solves (\( CD_{lsc} \)) it is necessary and sufficient that \( x^* \in \mathcal{X} \) and \( \varphi^* \in \Phi_{lsc} \) satisfy the \( \Phi \)-Karush–Kuhn–Tucker conditions (KKT),
\[
(0,-w^*) \in \partial_{lsc} \tilde{f}(x^*), \quad x^* \in \partial \chi g_{\Phi_{lsc}}^*(\varphi^*)
\]  
(53)
where \( \partial_{lsc} \) denotes the \( \Phi_{lsc} \)-subgradient.
By Theorem 2.4 and the inequality, \( x \) gives
\[
\tilde{f}(x^*) = f(x^*) - a^* \|x^*\|^2 \quad \text{and} \quad \varphi(x^*) = -a^* \|x^*\|^2 + \langle w^*, x^* \rangle = g_{lsc}^*(\varphi^*) + g(x^*). 
\]

Consequently,
\[
\tilde{f}(x^*) = f(x^*) - g(x^*) = -g_{lsc}^*(\varphi^* - g(x^*))
\]
and, in view of (19), we obtain (52).

Assume now that (52) holds with \( \varphi^*(x) = -a^* \|x\|^2 + \langle w^*, x \rangle \), i.e.
\[
f(x^*) + g(x^*) = -(\tilde{f}(w^*, x^*))_{lsc}(0, -w^*) - g_{lsc}^*(\varphi^*),
\]
where
\[
(\tilde{f}(w^*, x^*))_{lsc}(0, -w^*) = \sup_{x \in \mathcal{X}} (-\langle w^*, x \rangle - (f(x) + a^* \|x\|^2)
\]
and, as previously, \( \tilde{f}(\cdot) = \tilde{f}(a, w) = f(\cdot) - a \cdot \| \cdot \|^2 \) for any \( \varphi \in \Phi \).

By Theorem 2.4 and the \( \Phi \)-convexity of \( g \), (55) is equivalent to
\[
f(x^*) + g_{lsc}^*(x^*) = -(\tilde{f}(w^*, x^*))_{lsc}(0, -w^*) - g_{lsc}^*(\varphi^*).
\]

By definition of \( g_{lsc}^* \),
\[
f(x^*) + g_{lsc}^*(x^*) = \sup_{x \in \mathcal{X}} (-\langle w^*, x \rangle - (f(x) + a^* \|x\|^2)
\]
and, as previously, \( \tilde{f}(\cdot) = \tilde{f}(w^*, x^*) = f(\cdot) - a \cdot \| \cdot \|^2 \) for any \( \varphi \in \Phi \).

Hence, \( \tilde{f}(x^*) + (\tilde{f}(w^*, x^*))_{lsc}(0, -w^*) \leq -\langle w^*, x^* \rangle \) and, by Fenchel–Moreau inequality, \( \tilde{f}(a^*, x^*) = (\tilde{f}(w^*, x^*))_{lsc}(0, -w^*) = -\langle w^*, x^* \rangle \) i.e. \( (0, -w^*) \in \partial_{lsc} \tilde{f}(a^*, x^*) \).

Analogously, by (55), \( \tilde{f}(a^*, x^*) = a^* \|x^*\|^2 + g(x^*) = -\langle \tilde{f}(a^*, x^*) \rangle_{lsc}(0, -w^*) - g_{lsc}^*(\varphi^*), \) and since \( \tilde{f} \) is \( \Phi \)-convex, \( \tilde{f}(a, w) = (\tilde{f}(a, w))_{lsc}(0, -w^*) \) and
\[
- (\tilde{f}(a^*, x^*))_{lsc}(0, -w^*) = -\langle w^*, x^* \rangle + a^* \|x^*\|^2 + g(x^*) \leq - (\tilde{f}(a^*, x^*))_{lsc}(0, -w^*) - g_{lsc}^*(\varphi^*),
\]
i.e. \( g(x^*) + g_{lsc}^*(\varphi^*) \leq \varphi^*(x^*) \) which, together with Fenchel–Moreau inequality, gives \( x^* \in \partial_{\mathcal{X}} g_{lsc}^*(\varphi^*) \).

The example below illustrates Theorem 7.4.
Example 7.5: Let $\mathcal{X} = \mathbb{R}$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by the following formulas:

$$f(x) = \begin{cases} 2(x - 1)^2 & \text{for } x \geq 0 \\ 2(x + 1)^2 & \text{for } x < 0 \end{cases}, \quad g(x) = -x^2.$$ 

It is easy to see that $f$ and $g$ are nonconvex but $\Phi_{lsc}$-convex, where the set $\Phi_{lsc}$ is defined by (12). Functions $\varphi \in \Phi_{lsc}$ such that $c = 0$ will be identified with pairs $(a, v)$, $a, v \in \mathbb{R}$.

Consider the problem

$$\inf_{x \in \mathbb{R}} \{f(x) + g(x)\}.$$ 

Let $\varphi^*(x) = -x^2$, which we identify with the pair $(1, 0)$, it is easy to see that

$$f(x) - f(2) \geq x^2 - (2)^2$$ 

i.e. $(-1, 0) \in \partial_{lsc} f(2)$. Now we show that $2 \in \partial_{\mathcal{X} lsc} \varphi^*(1, 0)$, i.e. that the KKT conditions hold and this means that $\varphi^*(x) = -x^2$ and $x^* = 2$ are the solutions of the dual and primal problems, respectively. By simple calculations, we get

$$g^*(a, b) = \sup_{x \in \mathbb{R}} \{-ax^2 + bx + x^2\} = \begin{cases} +\infty & \text{for } a \leq 1, b \neq 0 \\ 0 & \text{for } a = 1, b = 0 \\ \frac{-b^2}{4(1-a)} & \text{for } a > 1, b \neq 0 \end{cases},$$

and it is easy to see that $2 \in \partial_{\mathcal{X} lsc} \varphi^*(1, 0)$ and $(0, 0) \in \partial_{lsc} \tilde{f}(2)$, where $\tilde{f}(a, v) = f - a\|\cdot\|^2$.

8. Conclusions

In conclusion, Theorem 6.3 provides sufficient and necessary conditions for zero duality gap for primal (P), $\Phi$-Lagrangian (LP), $\Phi$-Lagrangian dual (LD) and $\Phi$-conjugate dual (CD) problems, for a suitably defined $\Phi$-Lagrangian function when $0 \in \Phi$, and $\Phi$ is a convex set. Theorem 6.3, together with Theorem 7.3 and Theorem 7.4 reveal the importance of properties of the elementary functions $\Phi$ in general duality theory.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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