Differentially Private Stochastic Gradient Descent with Low-Noise

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Abstract

Modern machine learning algorithms aim to extract fine-grained information from data to provide accurate predictions, which often conflicts with the goal of privacy protection. This paper addresses the practical and theoretical importance of developing privacy-preserving machine learning algorithms that ensure good performance while preserving privacy. In this paper, we focus on the privacy and utility (measured by excess risk bounds) performances of differentially private stochastic gradient descent (SGD) algorithms in the setting of stochastic convex optimization. Specifically, we examine the pointwise problem in the low-noise setting for which we derive sharper excess risk bounds for the differentially private SGD algorithm. In the pairwise learning setting, we propose a simple differentially private SGD algorithm based on gradient perturbation. Furthermore, we develop novel utility bounds for the proposed algorithm, proving that it achieves optimal excess risk rates even for non-smooth losses. Notably, we establish fast learning rates for privacy-preserving pairwise learning under the low-noise condition, which is the first of its kind.

Keywords: Stochastic Gradient Descent, Differential Privacy, Generalization, Low-Noise

1 Introduction

Stochastic gradient descent (SGD) iteratively updates model parameters using the gradient information over a small batch of random examples, which reduces the computation cost and makes it amenable to solving large-scale problems. Due to its low computational overhead and easy implementation, it has become the workhorse algorithm for training many machine learning models \[11, 18, 29, 30, 32, 36, 37, 48, 51\].

On the other important front, we have witnessed a significant risk of privacy leakage by sharing gradient information of machine learning models because the gradient often embeds knowledge about the training data. For instance, \[53\] provides paradigms for breaching privacy and reconstructing training examples from publicly shared gradients and \[40\] shows that the membership of a data record can be inferred from a binary classifier trained on gradients. As SGD is widely deployed in machine learning models, it is crucial to develop private SGD algorithms to mitigate the privacy leakage posted by gradients.

In this paper, we are interested in differentially private SGD (DP-SGD) for both pointwise and pairwise learning problems. Differential privacy (DP) \[12\] is a de facto concept for designing private algorithms, which defines a rigorous attack model independent of background knowledge and gives a quantitative representation of the degree of privacy leakage. There is a considerable amount of work \[2, 3, 5, 14, 42, 44, 45, 47, 48\] on analyzing the utility guarantee (i.e., statistical generalization performance) of DP-SGD algorithms. In particular, \[2, 5, 14, 44, 45\] have shown that private SGD algorithms can achieve the optimal excess population risk bound \(O\left(\frac{1}{\sqrt{n}} + \frac{1}{\Delta^2} \sqrt{d \log(1/\delta)}\right)\)
for solving convex problems in different settings. Here, $n$ is the size of the training dataset, $d$ is the dimension, and $(\epsilon, \delta)$ are privacy parameters. One nature question then arises: can DP-SGD algorithms achieve faster utility rates beyond $O\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)}\right)$?

We provide an affirmative answer to the above question under a low-noise condition (also referred as a realizability condition in the literature)\cite{38, 41, 39, 27, 35}, which assumes that there exists a model within the considered hypothesis space perfectly fits the underlying data distribution. Under this condition, we conduct a comprehensive study of DP-SGD for both pointwise and pairwise learning as well as both smooth and non-smooth losses, which is able to provide faster utility bounds in terms of the excess population risk. Our main contributions are listed as follows:

- Firstly, we are concerned with the standard pointwise learning problems where the loss function $f(\cdot; z)$ on a single datum $z = (x, y)$. For this case, we show that DP-SGD with gradient perturbation algorithm can achieve the rate $O\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)}\right)$ for both strongly smooth and $\alpha$-Hölder smooth losses, which match the results in the recently work \cite{14}. Under a low-noise condition, we remove the term $O\left(\frac{1}{\sqrt{n}}\right)$ and achieve the excess risk bound of the order $O\left(\frac{1}{n}\sqrt{d\log(1/\delta)}\right)$ for strongly smooth losses. Further, a better excess risk rate $O\left(n^{-\frac{1+\alpha}{1+\alpha}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)}\right)$ is established for $\alpha$-Hölder smooth losses.

- Secondly, we study the pairwise learning setting where the loss $f(\cdot; z, z')$ involves a pair of examples $(z, z')$. In this learning setting, we propose a simple differentially private SGD algorithm for pairwise learning with utility guarantees. Specifically, for strongly smooth losses, our algorithm only requires gradient complexity $O(n)$ to achieve the optimal excess risk rate, while \cite{17} and \cite{48} require $O(n^3 \log(1/\delta))$ and $O(\log(1/\delta))$, respectively. We also show that this rate can be achieved even if the loss is non-smooth. Further, for both strongly smooth and non-smooth pairwise losses, we establish faster excess risk bounds under a low-noise condition. To the best of our knowledge, this is the first utility analysis which provides the excess risk bounds better than $O\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)}\right)$ for privacy-preserving pairwise learning.
Table 2: Comparison of different $(\epsilon, \delta)$-DP algorithms for pairwise learning. We report the results for Gradient descent with output perturbation (Output GD), Localized Gradient descent (Localized GD) and SGD with gradient perturbation (Gradient SGD).

| Work | Method            | Lipschitz | Smooth | Low-noise | Gradient complexity                                                                 | Utility          |
|------|-------------------|-----------|--------|-----------|-------------------------------------------------------------------------------------|------------------|
|      | Output GD         | ✓         | ✓      | ×         | $O(n^2)$                                                                            | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
|      | Localized GD      | ✓         | ✓      | ×         | $O(n^3 \log(1/\delta))$                                                            | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
|      | Localized SGD     | ✓         | ✓      | ×         | $O(n \log(1/\delta))$                                                              | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(\frac{1}{\delta})})$ |
|      | Localized SGD     | ✓         | ×      | ×         | $O(n^2 \log(1/\delta))$                                                            | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
| Ours | Gradient SGD      | ✓         | ✓      | ×         | $O(n)$                                                                             | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
|      | Gradient SGD      | ✓         | ✓      | ✓         | $O(n)$                                                                             | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
|      | Gradient SGD      | ✓         | α-Hölder | ×         | $O(n^{\frac{2-\alpha}{1+\alpha}} + n)$                                            | $O(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |
|      | Gradient SGD      | ✓         | α-Hölder | ✓         | $O(n^{\frac{1-\alpha}{1+\alpha}})$                                               | $O(n^{\frac{1-\alpha}{1+\alpha}} + \frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)})$ |

1.1 Related Work

In this subsection, we review the relevant work on DP-SGD which are close to our work. We discuss them in the pointwise and pairwise learning settings, respectively.

For pointwise learning, [8] established the excess population risk bounds in the order of $O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)}\right)$ for $(\epsilon, \delta)$-differentially private stochastic convex optimization algorithms for both strongly smooth and non-smooth losses, which match the lower bound given in [5]. However, their algorithms have a large gradient complexity (measured by the total number of computing the gradient). Specifically, their analysis establishes gradient complexity $O(n^{1.5} \sqrt{\epsilon} + (n \epsilon)^{2.5} (\log(1/\delta))^{-1})$ and $O(n^{1.5} \sqrt{\epsilon} + (n)^{6/5} \epsilon^{1/5} (\log(1/\delta))^{-2})$ for strongly smooth and non-smooth losses, respectively. [14] proposed a private phased SGD algorithm for strongly smooth losses, which can achieve the optimal excess risk rate with a linear gradient complexity $O(n)$. The work [2] developed a DP-SGD algorithm with gradient perturbation which improved the gradient complexity to $O(n^2)$ for non-smooth losses. However, they didn’t obtain the fast rates in the low-noise case which is the main focus of our paper. For clarity, we list in Table 1 the comparison of our work against the existing methods in terms of utility (excess risk) bounds, assumptions on loss function and the gradient complexity of DP-SGD in the pointwise learning setting.

For pairwise learning, [21] studied private gradient descent (GD) with output perturbation and proved that the proposed algorithm can achieve the excess risk rate $O\left(\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)}\right)$ for Lipschitz and strongly smooth losses. [47] proposed a private localized GD algorithm, which can achieve the optimal excess risk rate with gradient complexity $O(n^{3} \log(1/\delta))$ for Lipschitz and strongly smooth losses. The work [48] developed a DP-SGD algorithm with an iterative localization technique and derived the (nearly) optimal excess risk bounds for strongly smooth and non-smooth losses with gradient complexity $O(n \log(1/\delta))$ and $O(n^2 \log(1/\delta))$, respectively. In this work, we are interested in DP-SGD for both strongly smooth and α-Hölder smooth losses as well as the low-noise case. Table 2 summarizes the comparison of our work against the existing methods in terms of the utility (excess risk) bounds, assumptions on losses and the gradient of DP-SGD in the pairwise learning setting.

Organization of the paper. The remaining parts of the paper are organized as follows. In Section 2 we
present the formulations of pointwise and pairwise learning together with basic concepts of differential privacy. In Sections 3, we introduce the DP-SGD algorithms in the settings of pointwise learning and pairwise learning and present our main results. The main proofs are given in Section 4. Section 5 concludes the paper.

2 Learning Setting and Preliminaries

Let $\rho$ be a probability measure defined on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subset \mathbb{R}^d$ is an input space and $\mathcal{Y} \subset \mathbb{R}$ is an output space. In the standard framework of statistical learning theory [7, 43], one considers the problem of learning from a training dataset $S = \{z_i\}_{i=1}^n$, where $z_i$ is independently drawn from $\rho$. In the subsequent subsections, we describe the settings of pointwise and pairwise learning, the definition of differential privacy, and illustrate the goal of utility analysis.

2.1 Pointwise and Pairwise Learning

In the task of pointwise learning such as classification and regression, we aim to learn a model $\mathbf{w} \in \mathcal{W} \subset \mathbb{R}^d$ from training data $S$ and measure the quality of $\mathbf{w}$ using a pointwise loss function $f(\mathbf{w}; z)$ on a single datum $z = (x, y)$. The expected population risk for pointwise learning is given by $F(\mathbf{w}) = \mathbb{E}_{z \sim \rho}[f(\mathbf{w}; z)]$. The corresponding empirical risk minimization (ERM) problem based on training dataset $S$ is defined by

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ F_S(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^n f(\mathbf{w}; z_i) \right\}. \quad (1)$$

In contrast to pointwise learning, the performance of a model $\mathbf{w}$ for pairwise learning is measured on a pair of examples $(z, z')$ by a loss function $f(\mathbf{w}; z, z')$ [46, 48, 25, 26]. Many machine learning problems can be formulated as learning with pairwise loss functions including AUC maximization [10, 15, 33, 49, 52], metric learning [6, 8, 22], a minimum error entropy principle [20], and ranking [1, 9]. We use $\bar{F}(\mathbf{w})$ to denote the population risk, i.e., $\bar{F}(\mathbf{w}) = \mathbb{E}_{z, z' \sim \rho}[f(\mathbf{w}; z, z')]$. Let $\mathbf{w}^* = \arg \min_{\mathbf{w} \in \mathcal{W}} \bar{F}(\mathbf{w})$ be the best model, and let $[n] := \{1, \ldots, n\}$. The ERM problem on training data $S$ is given by

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ \bar{F}_S(\mathbf{w}) = \frac{1}{n(n-1)} \sum_{i,j \in [n], i \neq j} f(\mathbf{w}; z_i, z_j) \right\}. \quad (2)$$

2.2 Definition and Property of Differential Privacy

As a privacy-preserving technology with a rigorous mathematical guarantee, DP has been widely used in several areas [16, 17, 28, 35]. Its definition is stated formally as follows.

**Definition 1** (Differential Privacy (DP) [12]). We say a randomized algorithm $\mathcal{A}$ satisfies $(\epsilon, \delta)$-DP if, for any two neighboring datasets $S$ and $S'$ differing at one data point and any event $E$ in the output space of $\mathcal{A}$, there holds

$$\Pr(\mathcal{A}(S) \in E) \leq e^\epsilon \Pr(\mathcal{A}(S') \in E) + \delta.$$

In particular, we call it satisfies $\epsilon$-DP if $\delta = 0$.

To show a randomized algorithm satisfies DP, we need the following concept called $\ell_2$-sensitivity. Let $\| \cdot \|_2$ denote the Euclidean norm.

**Definition 2.** The $\ell_2$-sensitivity of a function (mechanism) $\mathcal{M} : \mathcal{Z}^n \rightarrow \mathcal{W}$ is defined as $\Delta = \sup_{S, S'} \| \mathcal{M}(S) - \mathcal{M}(S') \|_2$, where $S$ and $S'$ are neighboring datasets differing at one data point.
A basic mechanism to achieve $(\epsilon, \delta)$-DP is called Gaussian mechanism, which is shown as follows.

**Lemma 1** ([13]). *Given a function $\mathcal{M} : \mathbb{Z}^n \rightarrow \mathcal{W}$ with the $l_2$-sensitivity $\Delta$ and a dataset $S \subset \mathbb{Z}^n$, and assume that $\sigma \geq \sqrt{\frac{2\log(1.25/\delta)}{\epsilon}}$. The following Gaussian mechanism yields $(\epsilon, \delta)$-DP:*

$$G(S, \sigma) := \mathcal{M}(S) + b, \quad b \sim \mathcal{N}(0, \sigma^2 I_d),$$

where $I_d$ is the identity matrix in $\mathbb{R}^{d \times d}$.

We are interested in DP-SGD with strongly smooth and $\alpha$-Hölder smooth losses, respectively.

**Definition 3.** We say a function $w \rightarrow f(w)$ is $L$-strongly smooth with $L > 0$ if, for any $w, w' \in \mathcal{W}$, there holds $f(w) \leq f(w') + \langle \partial f(w'), w - w' \rangle + \frac{L}{2}\|w - w'\|^2_2$, where $\partial f(\cdot)$ denotes a (sub)gradient of $f$. We say a function $w \rightarrow f(w)$ is $\alpha$-Hölder smooth with $\alpha \in [0, 1)$ and parameter $L$ if for any $w, w' \in \mathcal{W}$, there holds $\|\partial f(w) - \partial f(w')\|_2 \leq L\|w - w'\|_2^\alpha$.

The smoothness parameter $\alpha \in [0, 1)$ characterizes the smoothness of the function $f$. Specifically, if $\alpha = 0$, then $f$ is Lipschitz continuous as considered in Definition [4] below. This definition instantiates many non-smooth loss functions including the hinge loss $\max \{0, 1 - yw^\top x\}$ for $q$-norm soft margin SVM and the $q$-norm loss $|y - w^\top x|^q$ in regression with $q \in [1, 2]$.

### 2.3 Target of Utility Analysis

We move on to describing the target of utility analysis of a randomized algorithm $A$ to solve the ERM problems (1) or (2). For simplicity, we elaborate this by taking pointwise learning as example and the same procedure can apply to the case of pairwise learning.

To this end, let $A(S)$ denote the output of $A$ based on the training dataset $S$ for pointwise learning. The utility of the output of a randomized algorithm is measured by the *excess population risk* $F(A(S)) - F(w^*)$, where $w^* = \arg\min_{w \in \mathcal{W}} F(w)$ is the one with the best prediction performance over $\mathcal{W}$. To examine the excess population risk, we use the following error decomposition:

$$E_{S,A}[F(A(S)) - F(w^*)] = E_{S,A}[F(A(S)) - F_S(A(S))] + E_{S,A}[F_S(A(S)) - F_S(w^*)],$$

(3)

where $E_{S,A}[\cdot]$ denotes the expectation w.r.t. both the randomness of $S$ and the internal randomness of $A$. The first term $E_{S,A}[F(A(S)) - F_S(A(S))]$ is called the generalization error, which measures the discrepancy between the expected risk and the empirical one. It can be handled by the stability analysis [2, 7, 19, 24, 27]. The second term is called the optimization error. We will use tools in optimization theory to control this term.

Throughout the paper, we assume the loss function $f$ is convex and Lipschitz continuous with respect to (w.r.t.) the first argument.

**Definition 4.** We say a function $w \rightarrow f(w)$ is convex if, for any $w, w' \in \mathcal{W}$, there holds $f(w) \geq f(w') + \langle \partial f(w'), w - w' \rangle$. We say a function $w \rightarrow f(w)$ is $G$-Lipschitz continuous with $G > 0$ if, for any $w, w' \in \mathcal{W}$, there holds $|f(w) - f(w')| \leq G\|w - w'\|_2$.

### 3 Main Results

We present our main results in this section. First, we propose the differentially private SGD algorithm for pointwise learning, and systematically study the privacy and utility guarantees of the proposed algorithm. Then, we turn to pairwise learning problems. We present a simple differentially private SGD algorithm for pairwise learning and provide its privacy and utility guarantees.
Algorithm 1 DP-SGD for pointwise learning

1: **Inputs:** Data $S = \{z_i \in Z : i = 1, \ldots, n\}$, loss function $f(w;z)$ with Lipschitz parameter $G$, the convex set $W \subseteq \mathbb{R}^d$, step size $\{\eta_t\}$, privacy parameters $\epsilon, \delta$, and constant $\beta$.
2: **Set:** $w_1 = 0$
3: for $t = 1$ to $T$ do
4:     Sample $i_t \sim \text{Unif}[\{n\}]$
5:         $w_{t+1} = \text{Proj}_W (w_t - \eta_t (\partial f(w_t;z_{i_t}) + b_t))$, where $b_t \sim \mathcal{N}(0, \sigma^2 I_d)$ with $\sigma^2 = \frac{14G^2T}{\beta n \epsilon} \left( \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)$
6:     end for
7: **return:** $w_{\text{priv}} = \frac{1}{T} \sum_{t=1}^{T} w_t$

3.1 DP-SGD for Pointwise Learning

In this subsection, we are interested in differentially private SGD for pointwise learning. To achieve $(\epsilon, \delta)$-differential privacy, we resort to the gradient perturbation mechanism, i.e., adding Gaussian noise to the stochastic gradient. The detailed algorithm is described in Algorithm 1. In particular, in each iteration $t$, the algorithm randomly selects a sample $z_{i_t}$, according to the uniformly distribution over $[n]$, and then updates the model parameter $w_{t+1}$ based on the noising gradient $\partial f(w_t;z_{i_t}) + b_t$ with $b_t \sim \mathcal{N}(0, \sigma^2 I_d)$. After $T$ iterations, Algorithm 1 outputs the private average model $w_{\text{priv}} = \frac{1}{T} \sum_{t=1}^{T} w_t$, whose privacy guarantee is established in the following algorithm.

**Theorem 2 (Privacy guarantee).** Suppose that the loss function $f$ is convex and $G$-Lipschitz. Then Algorithm 1 with some $\beta \in (0,1)$ satisfies $(\epsilon, \delta)$-DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{\alpha^2} \log \left( \frac{n}{\lambda (1 + \frac{\alpha^2}{G^2})} \right)$ with $\lambda = \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1$.

**Remark 1.** In Algorithm 1 the variance $\sigma^2$ of the Gaussian noise $b_t$ depends on a constant $\beta \in (0,1)$, which should satisfy the conditions $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{\alpha^2} \log \left( \frac{n}{\lambda (1 + \frac{\alpha^2}{G^2})} \right)$. [44] studied DP-SGD with gradient perturbation for $\alpha$-Hölder smooth losses and gave a sufficient condition for the existence of $\beta$ under a specific parameter setting. Specifically, they proved that if $n > 18$, $T = n$ and $\delta = 1/n^2$, then there exists at least one $\beta \in (0,1)$ such that DP-SGD satisfies $(\epsilon, \delta)$-DP when $\epsilon \geq 7(n^{2} - 1) + 4 \log(n)n + 7/(2n(n^{2} - 1))$. Indeed, our algorithm can be seen as a special case of their algorithm with $\alpha = 0$. Hence, we can also show the existence of $\beta$ under the same setting.

Now, we establish the utility guarantee for strongly smooth losses. Part (a) in the following theorem provides the optimal utility bound for a general setting, i.e., the “pessimistic” case $F(w^*) > 0$. Part (b) of Theorem 3 focuses on the low-noise setting, i.e., the optimistic case $F(w^*) = 0$, where the best possible model $w^*$ can achieve zero error. This setting is particularly intriguing in the context of deep learning, where models may possess more parameters than training examples.

**Theorem 3 (Utility guarantee for smooth losses).** Suppose $f$ is nonnegative, convex, $G$-Lipschitz and $L$-smooth. Let $w_{\text{priv}}$ be the output by Algorithm 1 with $T$ iterations. Then the following statements hold true.

(a) If we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then

$$E_{S,A}[F(w_{\text{priv}}) - F(w^*)] = O \left( \frac{1}{n} + \frac{d \log(1/\delta)}{n \epsilon} \right).$$

(b) If $F(w^*) = 0$, we choose $\eta_t = \frac{c \epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then

$$E_{S,A}[F(w_{\text{priv}}) - F(w^*)] = O \left( \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right).$$

**Remark 2.** [44] established the optimal rate for DP-SGD algorithm and improved the gradient complexity to $O(n)$ when the loss is strongly smooth and the parameter space is bounded. Our bound (part (a) in Theorem 3)
can achieve the optimal rate with gradient complexity \( O(n) \) when the loss is strongly smooth and Lipschitz continuous. Compared with [44], we need a further Lipschitz continuous assumption. However, this assumption can be removed when we assume the parameter domain is bounded in our setting. Indeed, the smoothness of \( f \) implies that the upper bound of the gradient can be controlled by the diameter of parameter domain \( R \), i.e., \( \|\partial f(w;z)\|_2 \leq \sup_z \|\partial f(0;z)\|_2 + L\|w\|_2 \leq \sup_z \|\partial f(0;z)\|_2 + LR \), where \( L \) is the smoothness parameter. Hence, our result can achieve the optimal rate under the same assumptions as [44]. In the optimistic case with \( F(w^*) = 0 \), Part (b) in Theorem 4 removes the term \( \mathcal{O}\left(\frac{1}{n} \sqrt{d \log(1/\delta)}\right) \) and further improves the excess population risk rate to \( \mathcal{O}\left(\frac{1}{n} \sqrt{d \log(1/\delta)}\right) \) with gradient complexity \( O(n) \) for strongly smooth losses under a low-noise condition. A very recent work [23] provided the excess population risk rate \( \mathcal{O}\left(\frac{1}{n} \sqrt{d \log(1/\delta)}\right) \) for the private gradient descent algorithm, while they focused on the non-convex setting and assumed Polyak-Lojasiewicz condition holds.

Now, we turn to the more general case, i.e., the loss function is \( \alpha \)-Hölder smooth with \( \alpha \in [0,1) \). The following theorem presents the excess population risk bound for \( \alpha \)-Hölder smooth losses.

**Theorem 4 (Utility guarantee for non-smooth losses).** Suppose \( f \) is nonnegative, convex, \( G \)-Lipschitz and \( \alpha \)-Hölder smooth with parameter \( L \) and \( \alpha \in [0,1) \). Let \( w_{\text{priv}} \) be the output of Algorithm 7 with \( T \) iterations. Then the following statements hold true.

(a) If \( \alpha \geq 1/2 \), we choose \( \eta_t = c \min\left\{ \frac{1}{\sqrt{T}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \) and \( T \approx n \). If \( \alpha < 1/2 \), we choose \( \eta_t = c \min\left\{ n^{\frac{2(1-\alpha)}{2(1-\alpha)}} \frac{\epsilon}{\sqrt{d \log(1/\delta)}}, \frac{\epsilon}{T^{1/2} \sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \) and \( T \approx n^{\frac{2}{1-\alpha}} \).

Then

\[
\mathbb{E}_{S,A}[F(w_{\text{priv}})] - F(w^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{ne} \right).
\]

(b) If \( F(w^*) = 0 \), we choose \( \eta_t = c \min\left\{ n^{\frac{2\alpha}{2(1-\alpha)}}, \frac{\epsilon}{T^{1/2} \sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \) and \( T \approx n^{\frac{2}{1-\alpha}} \).

Then

\[
\mathbb{E}_{S,A}[F(w_{\text{priv}})] - F(w^*) = \mathcal{O}\left(\frac{1}{n^{1-\alpha}} + \frac{\sqrt{d \log(1/\delta)}}{ne} \right).
\]

**Remark 3.** [44] studied DP-SGD with gradient perturbation for \( \alpha \)-Hölder smooth losses and showed that the algorithm can achieve the optimal rate \( \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{ne} \sqrt{d \log(1/\delta)}\right) \) with gradient complexity \( O(n^{\frac{2}{1-\alpha}} + n) \). Our result (Part (a) in Theorem 4) matches their bounds with the same gradient complexity. As discussed in Remark 2, although we need a further Lipschitz condition, we can also recover their result under the same setting when the parameter domain is bounded. Analogous to the smooth case, Part (b) in Theorem 4 derives the excess population risk bound better than \( \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{ne} \sqrt{d \log(1/\delta)}\right) \). To the best of our knowledge, this is the first excess population risk bound of the order \( \mathcal{O}(n^{-\frac{2}{1-\alpha}} + \frac{1}{ne} \sqrt{d \log(1/\delta)}) \) for private SGD with non-smooth losses.

### 3.2 DP-SGD for Pairwise Learning

In this subsection, we first present the differentially private SGD algorithm for pairwise learning, and then establish its privacy and utility guarantees. The proposed algorithm is described in Algorithm 2. In particular, in iteration \( t \), the algorithm draws a pair \( \{(i_t, j_t)\} \) from the uniform distribution over all pairs \( \{(i, j) : i, j \in [n], i \neq j\} \). Then the parameter is updated by the noised gradient \( \partial f(w_i, z_{i_t}, z_{j_t}) + b_t \) with \( b_t \sim \mathcal{N}(0, \sigma^2 I_d) \). The following theorem establishes the privacy guarantee for Algorithm 2.

**Theorem 5 (Privacy guarantee).** Suppose that the loss function \( f \) is convex and \( G \)-Lipschitz. Then Algorithm 2 with some \( \beta \in (0,1) \) satisfies \( (\epsilon, \delta) \)-DP if \( \sigma^2 \geq 2.68G^2 \) and \( \lambda = \frac{1}{2} \log\left(\frac{n}{2\lambda(1+\frac{\epsilon}{\epsilon(1-\beta)\delta})}\right) \) with \( \lambda \geq \frac{\log(1/\delta)}{1-\beta} + 1 \).
Algorithm 2 DP-SGD for pairwise learning (DP-SGD-pairwise)

1: **Inputs:** Data $S = \{z_i \in \mathbb{Z} : i = 1, \ldots, n\}$, loss function $f(w; z, z')$ with Lipschitz parameter $G$, the convex set $W \subseteq \mathbb{R}^d$, step size $\{\eta_t\}$, privacy parameters $\epsilon, \delta$, and constant $\beta$.

2: **Set:** $w_1 = 0$

3: for $t = 1$ to $T$ do

4: Sample $(i_t, j_t)$ uniformly over all pairs $(i, j) : i, j \in [n], i \neq j$

5: $w_{t+1} = \text{Proj}_{W}(w_t - \eta_t(\partial f(w_t; z_{i_t}, z_{j_t}) + b_i))$, where $b_i \sim \mathcal{N}(0, \sigma^2 I_d)$ with $\sigma^2 = \frac{56G^2T}{\beta n^2 \epsilon} \left( \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)$

6: end for

7: return: $w_{\text{priv}} = \frac{1}{T} \sum_{t=1}^{T} w_t$

By combining the stability results and the optimization error bounds (Lemmas 19 and 20 below) together, we establish the following utility guarantees for Algorithm 2 for strongly smooth and non-smooth losses, respectively.

**Theorem 6** (Utility guarantee for smooth losses). Suppose $f$ is nonnegative, convex, $G$-Lipschitz and $L$-smooth. Let $\{w_t\}$ be produced by Algorithm 2 with $T$ iterations. Then the following statements hold true.

**(a)** If we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then

$\mathbb{E}_{S,A}[\bar{F}(w_{\text{priv}})] - \bar{F}(w^*) = O\left( \frac{1}{\sqrt{n}} + \frac{d \log(1/\delta)}{n \epsilon} \right)$.

**(b)** If $\bar{F}(w^*) = 0$, we choose $\eta_t = \frac{c \epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then

$\mathbb{E}_{S,A}[\bar{F}(w_{\text{priv}})] - \bar{F}(w^*) = O\left( \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right)$.

**Remark 4.** We now compare our results with the related work for pairwise learning. Under the strongly smooth and Lipschitz continuous assumptions, [21] proposed the gradient descent with output perturbation algorithm to achieve DP and provided the excess population risk bound in the order of $O\left( \frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right)$ with gradient complexity $O(n^2)$. [17] improved the excess population risk rate to $O\left( \frac{1}{\sqrt{n}} + \frac{1}{n^\epsilon} \sqrt{d \log(1/\delta)} \right)$ by proposing a localized gradient descent algorithm with a large gradient complexity $O(n^3 \log(1/\delta))$. [48] presented a simple localized DP-SGD algorithm which can achieve the optimal excess risk rate $O\left( \frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right)$ up to a $\log(1/\delta)$ term. Their algorithm needs the gradient complexity $O(n \log(1/\delta))$. Our result (Part (a) in Theorem 6) shows that our algorithm can achieve the optimal excess risk rate $O\left( \frac{1}{\sqrt{n}} + \frac{1}{n^\epsilon} \sqrt{d \log(1/\delta)} \right)$ only with the gradient complexity $O(n)$ for strongly smooth losses, which significantly reduces the computational complexity of the algorithm. Under a low-noise condition, Part (b) removes the term $O\left( \frac{1}{n^\epsilon} \right)$ and derives the excess population risk bound of the order $O\left( \frac{1}{n^\epsilon} \sqrt{d \log(1/\delta)} \right)$, which only need the gradient complexity in the order of $O(n)$. To the best of our knowledge, this is the first excess population risk bound in the order of $O\left( \frac{1}{n^\epsilon} \sqrt{d \log(1/\delta)} \right)$ for privacy-preserving pairwise learning.

The following theorem establishes the utility bounds for Algorithm 2 when the loss is non-smooth.

**Theorem 7** (Utility guarantee for non-smooth losses). Suppose $f$ is nonnegative, convex, $G$-Lipschitz and $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$. Let $\{w_t\}$ be produced by Algorithm 2 with $T$ iterations. Then the following statements hold true.

**(a)** If $\alpha \geq 1/2$, we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$. If $\alpha < 1/2$, we choose $\eta_t = c \min \left\{ n^{\frac{3\alpha-1}{2(1-\alpha)}}, \frac{1}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$, and $T \asymp n^{\frac{2}{1-\alpha}}$. 

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Lemma 10 (From RDP to $\epsilon, \delta$-DP). If a randomized mechanism $A$ satisfies $(\lambda, \rho)$-RDP, then $A$ satisfies $(\rho + \log(1/\delta)/(\lambda - 1), \delta)$-DP for all $\delta \in (0, 1)$.

4 Proofs of Main Results

Before presenting the detailed proof, we first introduce some definitions and useful lemmas. To establish tighter privacy analysis of DP-SGD, we introduce the definition of Rényi differential privacy (RDP) which provides tighter composition and amplification results for iterative algorithms.

Definition 5 (RDP [48]). For $\lambda > 1$, $\rho > 0$, a randomized mechanism $A$ satisfies $(\lambda, \rho)$-RDP, if, for all neighboring datasets $S$ and $S'$, we have

$$D_\lambda(A(S) \parallel A(S')) := \frac{1}{\lambda - 1} \log \int \left( \frac{P_{A(S)}(\theta)}{P_{A(S')}^{A(S')}(\theta)} \right)^\lambda dP_{A(S')}^{A(S')}(\theta) \leq \rho,$$

where $P_{A(S)}(\theta)$ and $P^{A(S')}(\theta)$ are the density of $A(S)$ and $A(S')$, respectively.

The following lemma shows the privacy amplification of RDP by uniform subsampling, which is fundamental to establish privacy guarantees of noisy SGD algorithms.

Lemma 8 (59). Consider a function $M : \mathcal{Z}^n \to \mathcal{W}$ with the $\ell_2$-sensitivity $\Delta$, and a dataset $S \subset \mathcal{Z}^n$. The Gaussian mechanism $G(S, \sigma) = M(S) + b$, where $b \sim N(0, \sigma^2 I)$, applied to a subset of samples that are drawn uniformly without replacement with subsampling rate $p$ satisfies $(\lambda, 3.5p^2\lambda \Delta^2/\sigma^2)$-RDP if $\sigma^2 \geq 0.67\Delta^2$ and $\lambda - 1 \leq 3.5p^2\lambda \Delta^2/\sigma^2$.

We say a sequence of mechanisms $(A_1, \ldots, A_k)$ are chosen adaptively if $A_i$ can be chosen based on the outputs of the previous mechanisms $A_1(S), \ldots, A_{i-1}(S)$ for any $i \in [k]$.

Lemma 9 (Adaptive Composition of RDP [59]). If a mechanism $A$ consists of a sequence of adaptive mechanisms $(A_1, \ldots, A_k)$ with $A_i$ satisfying $(\lambda_i, \rho_i)$-RDP, $i \in [k]$, then $A$ satisfies $(\lambda, \sum_{i=1}^k \rho_i)$-RDP.

The relationship between RDP and $(\epsilon, \delta)$-DP is given as follows.

Lemma 10 (From RDP to $(\epsilon, \delta)$-DP [59]). If a randomized mechanism $A$ satisfies $(\lambda, \rho)$-RDP, then $A$ satisfies $(\rho + \log(1/\delta)/(\lambda - 1), \delta)$-DP for all $\delta \in (0, 1)$.
A fundamental property of DP called post-processing property is introduced as follows. It implies that a differentially private output can be arbitrarily transformed by using some data-independent functions.

**Lemma 11** (Post-processing property). Let \( \mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{W}_1 \) satisfy \((\lambda, \rho)-\text{DP}\) and \( f : \mathcal{W}_1 \rightarrow \mathcal{W}_2 \) be an arbitrary function. Then \( f \circ \mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{W}_2 \) satisfies \((\lambda, \rho)-\text{DP}\).

Let \( M = \sup_{z \in \mathcal{Z}} f(0; z) \). Define

\[
\alpha_{n,1} = \begin{cases} 
(1 + 1/\alpha)^{\frac{\sigma^2}{\beta^2}} L^\frac{1}{\alpha}, & \text{if } \alpha > 0, \\
M + L, & \text{if } \alpha = 0.
\end{cases}
\]

(4)

Our analysis requires to use a self-bounding property for strongly smooth and \(\alpha\)-Hölder smooth losses, which means that gradients can be controlled by function values.

**Lemma 12** (Self-bounding property). Suppose \( f \) is nonnegative. If \( f \) is \(L\)-strongly smooth, then there holds \( \|\partial f(w; z)\|_2 \leq \sqrt{L} f(w; z) \) for any \( w \in \mathbb{R}^d, z \in \mathcal{Z} \). If \( f \) is \(\alpha\)-Hölder smooth with \( L > 0 \) and \( \alpha \in [0, 1) \), then for \( c_{n,1} \) defined in (4) we have \( \|\partial f(w; z\|_2 \leq c_{n,1} f(w; z) \) for any \( w \in \mathbb{R}^d, z \in \mathcal{Z} \).

We will use the following concept of on-average argument stability to study the generalization error.

**Definition 6** (On-average argument stability). Let \( S = \{z_1, \ldots, z_n\} \) and \( S' = \{z'_1, \ldots, z'_n\} \) be drawn independently from \( \rho \). For any \( i \in [n] \), denote \( S^{(i)} = \{z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_n\} \) as the set from \( S \) by replacing the \( i \)-th element with \( z'_i \). We say an algorithm \( \mathcal{A} \) is on-average argument \( \epsilon \)-stable if

\[
\mathbb{E}_{S, S', \mathcal{A}} \left[ \frac{1}{n} \sum_{i=1}^{n} \|\mathcal{A}(S) - \mathcal{A}(S^{(i)})\|_2^2 \right] \leq \epsilon.
\]

### 4.1 Proofs for Pointwise Learning

We first give the proof of the privacy guarantee for Algorithm\( \text{ priv} \). Specifically, according to the Lipschitz continuity of \( f \), we can show that the \( \ell_2 \)-sensitivity of \( \mathcal{M}_t = \partial f(w_t; z_t) \) is \( 2G \). Then by Lemma 1 and the post-processing property, we know that \( w_{t+1} = \left( \frac{\log(1/\delta)}{1-\beta^2} + \frac{1}{\beta \epsilon} \right) - \text{RDP} \) for any \( t = 1, \ldots, T \). Further, we use the adaptive composition theorem (Lemma 9) and the connection between RDP and DP (Lemma 10) to show that \( w_{\text{priv}} \) satisfies \((\epsilon, \delta)\)-DP. The detailed proof is shown as follows.

**Proof of Theorem 3**. For each iteration \( t \), let \( \mathcal{A}_t = \mathcal{M}_t + b_t \), where \( \mathcal{M}_t = \partial f(w_t; z_t) \). For any \( w_t \in \mathcal{W} \) and any \( z_t, z'_t \in \mathcal{Z} \), the Lipschitz continuity of \( f \) implies

\[
\|\partial f(w_t; z_t) - \partial f(w; z'_t)\|_2 \leq \|\partial f(w_t; z_t)\|_2 + \|\partial f(w_t; z'_t)\|_2 \leq 2G.
\]

From the definition of sensitivity (see Definition 2), we know the \( \ell_2 \)-sensitivity of \( \mathcal{M}_t \) is bounded by \( 2G \). Note that

\[
\sigma^2 = \frac{14G^2T}{\beta n^2} \left( \frac{\log(1/\delta)}{1-\beta^2} + 1 \right).
\]

According to Lemma 8 with \( p = 1/n \), we know \( \mathcal{A}_t \) is \((\lambda, \frac{\lambda \sigma^2}{\beta n^2} + 1)\)-RDP as long as \( \sigma^2 \geq 2.68G^2 \) and \( \lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left( \frac{n}{\lambda (1+\frac{\sigma^2}{6G^2})} \right) \) hold.

Let \( \lambda = \frac{\log(1/\delta)}{1-\beta^2} + 1 \), then we get \( \mathcal{A}_t \) is \((\lambda, \frac{\lambda \sigma^2}{\beta n^2} + 1, \frac{\beta \epsilon}{T})\)-RDP. Further, Lemma 11 implies that \( w_{t+1} \) is \((\lambda, \frac{\lambda \sigma^2}{\beta n^2} + 1, \frac{\beta \epsilon}{T})\)-RDP for any \( t = 1, \ldots, T \). According to the adaptive composition theorem of RDP (see Lemma 9), we know Algorithm\( \text{ priv} \) is \((\lambda, \frac{\lambda \sigma^2}{\beta n^2} + 1, \beta \epsilon)\)-RDP. Finally, the relationship between RDP and DP (Lemma 10) implies that Algorithm\( \text{ priv} \) is \((\epsilon, \delta)\)-DP if \( \sigma^2 \geq 2.68G^2 \) and \( \lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left( \frac{n}{\lambda (1+\frac{\sigma^2}{6G^2})} \right) \) hold. The proof is completed. \( \square \)
To study the utility guarantee of Algorithm 1, we need to estimate the generalization error $\mathbb{E}_{S,A}[F(w_{\text{priv}}) - F_S(w_{\text{priv}})]$ and the optimization error $\mathbb{E}_{S,A}[F_S(w_{\text{priv}}) - F(w^*)]$, respectively. We will use on-average argument stability to study the generalization error, which measures the sensitivity of the output model of an algorithm. The relationship between generalization error and on-average argument stability is established in the following lemma [27].

**Lemma 13 (Generalization via on-average stability).** Let $\mathcal{A}$ be on-average $\nu$-stable. Let $\gamma > 0$.

(a) If $f$ is nonnegative and $L$-smooth, then
\[
\mathbb{E}_{S,A}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))] \leq \frac{L}{\gamma^2} \mathbb{E}_{S,A}[F_S(\mathcal{A}(S))] + \frac{(L + \gamma)\nu}{2}.
\]

(b) If $f$ is nonnegative, convex and $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$, then
\[
\mathbb{E}_{S,A}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))] \leq \frac{c_{\alpha,1}^2}{2\gamma} \mathbb{E}_{S,A}[F_S^{1/n}(\mathcal{A}(S))] + \frac{\nu^2}{2}.
\]

Since the noise added to the gradient in each iteration is the same for the neighboring datasets, then the noise addition does not impact the stability analysis. Therefore, the on-average argument stability of non-private SGD equals that of private SGD. We can use the following lemma directly to give the stability bounds of Algorithm 1 for both strongly smooth and non-smooth losses [27].

**Lemma 14 (On-average stability bounds).** Suppose $f$ is nonnegative and convex. Let $S, S'$ and $S^{(i)}$ be as Definition 2. Let $\{w_t\}$ and $\{w^{(i)}_t\}$ be produced by Algorithm 1 based on $S$ and $S^{(i)}$, respectively.

(a) If $f$ is $L$-smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then
\[
\mathbb{E}_{S,S',A}[\frac{1}{n} \sum_{i=1}^n \|w_{t+1} - w^{(i)}_{t+1}\|^2_2] \leq \frac{8e(1 + t/n)L}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,A}[F_S(w_j)].
\]

(b) If $f$ is $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$, then
\[
\mathbb{E}_{S,S',A}[\frac{1}{n} \sum_{i=1}^n \|w_{t+1} - w^{(i)}_{t+1}\|^2_2] \leq c_{\alpha,3}^2 \sum_{j=1}^t \eta_j^{2/(1+\alpha)} + \frac{4\eta_t^2 c_{\alpha,1}^2(1 + t/n)}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,A}[F_S^{1/n}(w_j)],
\]

where $c_{\alpha,3} = \sqrt{\frac{1-\alpha}{1+\alpha}(2-\alpha)L^{1+\alpha}}$.

The following theorem presents generalization bounds of DP-SGD for both smooth and non-smooth losses, which directly follows from Lemma 13 and Lemma 14.

**Theorem 15 (Generalization bounds).** Suppose $f$ is nonnegative and convex. Let $\mathcal{W} = \mathbb{R}^d$ and let $\mathcal{A}$ be Algorithm 1 with $T$ iterations. Let $\gamma > 0$.

(a) If $f$ is $L$-smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then
\[
\mathbb{E}_{S,A}[F(w_{\text{priv}}) - F_S(w_{\text{priv}})] \leq \frac{L}{\gamma^2} \mathbb{E}_{S,A}[F_S(w_{\text{priv}})] + \frac{4e(1 + \gamma)(1 + t/n)L}{n} \sum_{t=1}^T \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)].
\]

(b) If $f$ is $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$, then
\[
\mathbb{E}_{S,A}[F(w_{\text{priv}}) - F_S(w_{\text{priv}})] \leq \frac{c_{\alpha,1}^2}{2\gamma} \mathbb{E}_{S,A}[F_S^{1/n}(w_{\text{priv}})] + \frac{1}{2} \left( c_{\alpha,3}^2 \sum_{t=1}^T \eta_t^{1/(1+\alpha)} + \frac{4e\eta_t^2 c_{\alpha,1}^2(1 + t/n)}{n} \sum_{t=1}^T \eta_t^2 \mathbb{E}_{S,A}[F_S^{1/n}(w_t)] \right).
\]
In the following theorem, we use techniques in optimization theory to control the optimization error in expectation. Recall \( w^* = \arg \min_{w \in \mathcal{W}} F(w) \). Let
\[
c_{\alpha,2} = \begin{cases} 
\frac{1-\alpha}{1+\alpha} \left( 2 \alpha/(1+\alpha) \right)^{2\alpha} c_{\alpha,1}^{2+2\alpha}, & \text{if } \alpha > 0 \\
\frac{1}{2} c_{\alpha,1}^2, & \text{if } \alpha = 0.
\end{cases}
\]

(5)

**Theorem 16 (Optimization error).** Suppose \( f \) is non-negative and convex. Let \( \{w_t\} \) be produced by Algorithm 7. Assume the step size \( \eta_t \) is non-increasing.

(a) If \( f \) is \( L \)-smooth, then
\[
\sum_{j=1}^{t} \eta_j \mathbb{E}_A [F_S(w_j) - F_S(w^*)] \leq \left( \frac{1}{2} + 3L\eta_1 \right) \|w^*\|_2^2 + 3L \sum_{j=1}^{t} (3\eta_j^2 \sigma^2 d + 2\eta_j^2 F_S(w^*)) + \sum_{j=1}^{t} 3\eta_j^2 \sigma^2 d.
\]

(b) If \( f \) is \( \alpha \)-Hölder smooth with parameter \( L \) and \( \alpha \in [0, 1) \),
\[
\sum_{j=1}^{t} \eta_j \mathbb{E}_A [F_S(w_j) - F_S(w^*)] \leq \frac{1}{2} \|w^*\|_2^2 + \frac{3}{4} c_{\alpha,1}^2 \left( \sum_{j=1}^{t} \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[ 2\eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (6\eta_j^3 \sigma^2 d + 4\eta_j^2 F_S(w^*) + 3c_{\alpha,2} \eta_j^2) \right]^{\frac{2\alpha}{1+\alpha}} + \sum_{j=1}^{t} 3\eta_j^2 \sigma^2 d.
\]

**Proof.** Note the projection operator \( \text{Proj} \) is non-expansive. Then for any \( \alpha \in [0, 1) \), we have
\[
\|w_{t+1} - w^*\|_2^2 = \|w_t - \eta_t (\partial f(w_t; z_i) + b_t) - w^*\|_2^2 \\
= \|w_t - w^*\|_2^2 + \eta_t^2 \|\partial f(w_t; z_i) + b_t\|_2^2 + 2\eta_t \langle w^* - w_t, \partial f(w_t; z_i) + b_t \rangle \\
\leq \|w_t - w^*\|_2^2 + \frac{3}{2} \eta_t^2 \|\partial f(w_t; z_i)\|_2^2 + 3\eta_t^2 \|b_t\|_2^2 + 2\eta_t \langle w^* - w_t, \partial f(w_t; z_i) + b_t \rangle \\
\leq \|w_t - w^*\|_2^2 + \frac{3}{2} c_{\alpha,1} \eta_t^2 f_{\alpha+1}(w_t; z_i) + 3\eta_t^2 \|b_t\|_2^2 + 2\eta_t \langle f(w^*; z_i) - f(w_t; z_i) \rangle + 2\eta_t \langle w^* - w_t, b_t \rangle,
\]
where in the second inequality we used \((a+b)^2 \leq (1+p)a^2 + (1+1/p)b^2\) with \( p = 1/2 \), and the last inequality is due to the self-bounding property (Lemma 12) and the convexity of \( f \).

Rearranging the above inequality, we get
\[
2\eta_t [f(w_t; z_i) - f(w^*; z_i)] \\
\leq \|w_t - w^*\|_2^2 + \frac{3}{2} c_{\alpha,1} \eta_t^2 f_{\alpha+1}(w_t; z_i) + 3\eta_t^2 \|b_t\|_2^2 + 2\eta_t \langle w^* - w_t, b_t \rangle.
\]
Taking a summation over \( j \) and noting \( w_1 = 0 \), we know
\[
2 \sum_{j=1}^{t} \eta_j [f(w_j; z_i) - f(w^*; z_i)] \\
\leq \|w^*\|_2^2 + \frac{3}{2} c_{\alpha,1} \sum_{j=1}^{t} \eta_j^2 f_{\alpha+1}(w_j; z_i) + \sum_{j=1}^{t} (3\eta_j^2 \|b_j\|_2^2 + 2\eta_j \langle w^* - w_j, b_j \rangle).
\]

Note that \( w_j \) is independent of \( i_j \), we can take an expectation w.r.t. \( A \) and get
\[
\sum_{j=1}^{t} \eta_j \mathbb{E}_A [F_S(w_j) - F_S(w^*)] = \sum_{j=1}^{t} \eta_j \mathbb{E}_A [f(w_j; z_i) - f(w^*; z_i)] \\
\leq \frac{1}{2} \|w^*\|_2^2 + \frac{3}{2} c_{\alpha,1} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_A [f_{\alpha+1}(w_j; z_i)] + \sum_{j=1}^{t} 3\eta_j^2 \sigma^2 d,
\]

(7)
where we used $\mathbb{E}_\mathcal{A}[\|b_j\|^2_2] = \sigma^2 d$ and $\mathbb{E}_\mathcal{A}[\langle w^* - w_j, b_j \rangle] = 0$ since $b_j$ is a Gaussian vector with mean 0 and variance $\sigma^2$, and $w^* - w_j$ is independent of $b_j$.

To control the right hand side of (7), we have to estimate $\sum_{j=1}^t \eta_j^2 \mathbb{E}_\mathcal{A}[f^{\alpha \eta_j}(w_j; z_i)]$. By Young’s inequality $ab \leq p^{-1}|a|^p + q^{-1}|b|^q$ with $a, b \in \mathbb{R}$ and $p^{-1} + q^{-1} = 1$, for any $t \in [T]$, we have

$$
\eta_t c_{\alpha,1}^{2} f^{\alpha \eta_t}(w_t; z_{i_t}) = \left( \frac{1 + \alpha}{2\alpha} f(w_t; z_{i_t}) \right)^{\frac{2\alpha}{1 + \alpha}} \left( \frac{2\alpha}{1 + \alpha} \right)^{\frac{1}{\alpha} \eta_t} c_{\alpha,1}^{2} \eta_t 
\leq \left( \frac{2\alpha}{1 + \alpha} \right)^{\frac{1}{\alpha} \eta_t} c_{\alpha,1}^{2} \eta_t 
= f(w_t; z_{i_t}) + c_{\alpha,2} \eta_t^{\frac{1}{\alpha} \eta_t}.
$$

Putting the above inequality back into (6) yields

$$
\|w_t+1 - w^*\|^2_2 \leq \|w_t - w^*\|^2_2 + 3\eta_t^2 \|b_t\|^2_2 + 2\eta_t f(w^*; z_{i_t}) - \frac{1}{2} \eta_t f(w_t; z_{i_t}) + \frac{3}{2} \eta_t c_{\alpha,2} \eta_t^{\frac{1}{\alpha}} + 2\eta_t (w^* - w_t, b_t).
$$

Rearranging the above inequality and multiplying both sides by $\eta_t$, we get

$$
\eta_t f(w_t; z_{i_t}) \leq 2\eta_t (\|w_t - w^*\|^2_2 - \|w_{t+1} - w^*\|^2_2) + 6\eta_t^2 \|b_t\|^2_2 + 4\eta_t f(w^*; z_{i_t}) + 3c_{\alpha,2} \eta_t^{\frac{3}{\alpha}} + 4\eta_t^2 (w^* - w_t, b_t)
\leq 2\eta_t (\|w_t - w^*\|^2_2 - \eta_t \|w_{t+1} - w^*\|^2_2) + 6\eta_t^2 \|b_t\|^2_2 + 4\eta_t f(w^*; z_{i_t}) + 3c_{\alpha,2} \eta_t^{\frac{3}{\alpha}} + 4\eta_t^2 (w^* - w_t, b_t),
$$

where we assume $\eta_t \geq \eta_{t+1}$ for all $t \in [T-1]$.

Taking a summation over $j$ and noting $w_1 = 0$, we know

$$
\sum_{j=1}^t \eta_j^2 f(w_j; z_{i_j}) \leq \sum_{j=1}^t 2\eta_j (\|w_j - w^*\|^2_2 + \sum_{j=1}^t (6\eta_j^2 \|b_j\|^2_2 + 4\eta_j f(w^*; z_{i_j}) + 3c_{\alpha,2} \eta_j^{\frac{3}{\alpha}} + 4\eta_j^2 (w^* - w_j, b_j))).
$$

Note $x \mapsto x^{\frac{2\alpha}{1 + \alpha}}$ is concave. Then Jensen’s inequality implies

$$
\sum_{j=1}^t \eta_j^2 f^{\alpha \eta_j}(w_j; z_{i_j}) \leq \left( \sum_{j=1}^t \eta_j^2 \right)^{\frac{2\alpha}{1 + \alpha}} \left( \sum_{j=1}^t \eta_j^2 f(w_j; z_{i_j}) \right)^{\frac{2\alpha}{1 + \alpha}} = \left( \sum_{j=1}^t \eta_j^2 \right)^{\frac{2\alpha}{1 + \alpha}} \left[ \sum_{j=1}^t \eta_j^2 f(w_j; z_{i_j}) \right]^{\frac{2\alpha}{1 + \alpha}}.
$$

Plugging the above inequality back into (7), we have

$$
\sum_{j=1}^t \eta_j \mathbb{E}_\mathcal{A}[F_S(w_j) - F_S(w^*)] \leq \frac{1}{2} \|w^*\|^2_2 + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d + \frac{3}{4} \eta_j c_{\alpha,1}^{2} \mathbb{E}_\mathcal{A} \left[ \sum_{j=1}^t 2\eta_j (\|w_j - w^*\|^2_2 + \sum_{j=1}^t (6\eta_j^2 \|b_j\|^2_2 + 4\eta_j f(w^*; z_{i_j}) + 3c_{\alpha,2} \eta_j^{\frac{3}{\alpha}} + 4\eta_j^2 (w^* - w_j, b_j))) \right]^{\frac{2\alpha}{1 + \alpha}} + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d,
$$

where the last inequality used Jensen’s inequality for concave mapping and $\mathbb{E}_\mathcal{A}[\langle w^* - w_j, b_j \rangle] = 0$. Part (b) is proved. From the definition we know that $\alpha$-Hölder smoothness with $\alpha = 1$ corresponds to the strongly smoothness of $f$. Hence, Part (a) in the theorem directly follows by setting $\alpha = 1$ in the above inequality. \qed
Now, we can establish the proofs of the excess population risk bounds of DP-SGD for pointwise learning by combining Theorem 15 and Theorem 16 together. First, we give the proof for the strongly smooth case (i.e., Theorem 15).

**Proof of Theorem 15.** Putting stability bounds for smooth losses (Part (a) in Lemma 14) back into Part (a) of Lemma 13 we get

\[
E_{S,A}[F(w_{i+1})] \leq (1 + \frac{L}{\gamma}) E_{S,A}[F_S(w_i)] + \frac{4\epsilon(L + \gamma)(1 + t/n)L}{n} \sum_{j=1}^{t} \eta_j^2 E_{S,A}[F_S(w_j)].
\]

Note that \(w_j\) is independent of \(b_j\) and \(i_j\). Eq. 8 implies

\[
\sum_{j=1}^{t} \eta_j^2 E_{S,A}[F_S(w_j)] = \sum_{j=1}^{t} \eta_j^2 E_{S,A}[f(w_j; z_i)]
\]

\[
\leq 2\eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (6\eta_j^2 E_{A}[\|b_j\|_2^2] + 4\eta_j^2 E_{S,A}[f(w^*; z_i)] + 4\eta_j^2 E_{S,A}[(w^* - w_j; b_j)])
\]

\[
\leq 2\eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (6\eta_j^2 \sigma^2 d + 4\eta_j^2 F(w^*))
\]

where we used \(E_A[\|b_j\|_2^2] = \sigma^2 d, E_{S,A}[f(w^*; z_i)] = F(w^*)\) and \(E_{S,A}[(w^* - w_j; b_j)] = 0\).

Combining the above two inequalities together, we get

\[
E_{S,A}[F(w_{i+1})] \leq (1 + \frac{L}{\gamma}) E_{S,A}[F_S(w_i)] + \frac{8\epsilon(L + \gamma)(1 + t/n)L}{n} \left[ \eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (3\eta_j^2 \sigma^2 d + 2\eta_j^2 F(w^*)) \right].
\]

Multiplying both sides by \(\eta_{i+1}\) followed with a summation gives

\[
\sum_{t=1}^{T} \eta_t E_{S,A}[F(w_t)] \leq \left(1 + \frac{L}{\gamma}\right) T \sum_{t=1}^{T} \eta_t E_{S,A}[F_S(w_t)] + \frac{8\epsilon(L + \gamma)(1 + T/n)L}{n} \sum_{t=1}^{T} \eta_t \left[ \eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (3\eta_j^2 \sigma^2 d + 2\eta_j^2 F(w^*)) \right].
\] (10)

Part (a) in Theorem 16 implies

\[
\sum_{t=1}^{T} \eta_t E_{S,A}[F_S(w_t)] \leq \sum_{t=1}^{T} \eta_t F_S(w^*) + \left(\frac{1}{2} + 3L\eta_t\right) \|w^*\|_2^2 + 3 \sum_{t=1}^{T} (3L\eta_t + 1) \eta_t^2 \sigma^2 d + 4 \sum_{t=1}^{T} \eta_t^2 F_S(w^*).
\]

Plugging the above inequality back into (10) and noting \(E_S[F_S(w^*)] = F(w^*)\), we get

\[
\sum_{t=1}^{T} \eta_t E_{S,A}[F(w_t)]
\]

\[
\leq \left(1 + \frac{L}{\gamma}\right) \left( \sum_{t=1}^{T} \eta_t F(w^*) + \left(\frac{1}{2} + 3L\eta_t\right) \|w^*\|_2^2 + 3 \sum_{j=1}^{t} (3L\eta_j + 1) \eta_j^2 \sigma^2 d + 4 \sum_{j=1}^{t} \eta_j^2 F(w^*) \right)
\]

\[
+ \frac{8\epsilon(L + \gamma)(1 + T/n)L}{n} \sum_{t=1}^{T} \eta_t \left[ \eta_1 \|w^*\|_2^2 + \sum_{j=1}^{t} (3\eta_j^2 \sigma^2 d + 2\eta_j^2 F(w^*)) \right].
\]
Let \( \eta = \eta \leq \min\{2/L, 1\} \) and assume \( T \geq n \). Note \( w_{\text{priv}} = \frac{1}{T} \sum_{t=1}^{T} w_t \). Then according to Jensen’s inequality, there holds
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) \|w^*\|_2^2 + \left( \frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) F(w^*)
\]
\[
+ \left( 1 + \frac{T \eta}{n^2} \right) \frac{T \log(1/\delta)}{n^2}.
\]

Recalling that \( \sigma^2 = \frac{14C^2 d}{n^2} \left( \frac{\log(1/\delta)}{1 - 2\gamma T} \right) + 1 \), we further have
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) \|w^*\|_2^2 + \left( \frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) F(w^*)
\]
\[
+ \left( 1 + \frac{T \eta}{n^2} \right) \left( \frac{T \log(1/\delta)}{n^2} \right) \frac{T \log(1/\delta)}{n^2}.
\]

(a) If we set \( T \approx n \) and \( \gamma = \sqrt{n} \), then Eq. (11) implies
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) \|w^*\|_2^2 + \left( \frac{1}{\sqrt{n}} + \frac{T \eta}{n^2} \right) F(w^*)
\]
\[
+ \left( 1 + \frac{T \eta}{n^2} \right) \left( \frac{T \log(1/\delta)}{n^2} \right) \frac{T \log(1/\delta)}{n^2}.
\]

Further let \( \eta = \frac{c}{\max\{\sqrt{n}, \frac{\sqrt{d \log(1/\delta)}}{\epsilon}\}} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \), then there holds
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right),
\]
where we assume \( \sqrt{d \log(1/\delta)} = O(nc) \) (otherwise the bound will not converge).

(b) Consider the low noise case, i.e., \( F(w^*) = 0 \). Let \( \gamma = 1 \) and \( T \approx n \), then
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{1}{n \eta} \right) \|w^*\|_2^2 + \left( \frac{1}{n \eta} \right) \frac{\sqrt{d \log(1/\delta)}}{n \epsilon}.
\]

Let \( \eta = \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \), then
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathcal{O}
\left(\frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right).
\]

The proof of the theorem is completed. \( \square \)

Finally, we provide the proof of utility guarantee for Algorithm 1 when the loss is non-smooth.

**Proof of Theorem 1** Note \( \mathbb{E}[F_S(w^*)] = F(w^*) \) and \( w_{\text{priv}} = \frac{1}{T} \sum_{t=1}^{T} w_t \). By Jensen’s inequality we know
\[
\mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)] = \mathbb{E}_{S, A}[F(w_{\text{priv}}) - F(w^*)]
\]
\[
= \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S, A}[F(w_t) - F(w^*)]
\]
\[
= \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S, A}[F(w_t) - F(w^*)] + \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S, A}[F_S(w_t) - F(w^*)].
\] (12)
We first estimate the term \( (\sum_{i=1}^{T} \eta_t)^{-1} \sum_{i=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t) - F_S(w_t)] \). Putting part (b) in Lemma 13 back into part (b) of Lemma 13, we get
\[
\mathbb{E}_{S,A}[F(w_{t+1}) - F_S(w_{t+1})] \\
\leq \frac{c_{\alpha,3}^2}{2\gamma} \mathbb{E}_{S,A}[F(x^{\text{w}}_{t+1})] + \frac{c_{\alpha,3}^2}{2} \sum_{j=1}^{t} \eta_j + \frac{2c_{\alpha,1} \gamma (1 + t/n)}{n} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}[F^{\text{w}}_{S}(w_j)]
\]

Let \( \delta_j = \max \{ \mathbb{E}_{S,A}[F(w_j)] - \mathbb{E}_{S,A}[F_S(w_j)], 0 \} \). Due to the concavity of \( x \mapsto x^{\frac{\gamma}{1-t}} \), there holds
\[
\mathbb{E}_{S,A}[F(x^{\text{w}}_{t+1})] \leq (\mathbb{E}_{S,A}[F(w_{t+1})] - \mathbb{E}_{S,A}[F_S(w_{t+1})] + \mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} \\
\leq \delta_{t+1} + (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}}.
\]

Combining the above two inequalities together yields
\[
\delta_{t+1} \leq \frac{c_{\alpha,1}^2}{2\gamma} (\delta_{t+1} + (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}}) + \frac{c_{\alpha,3}^2}{2} \sum_{j=1}^{t} \eta_j + \frac{2c_{\alpha,1} \gamma (1 + t/n)}{n} \sum_{j=1}^{t} \eta_j^2 (\mathbb{E}_{S,A}[F(w_j)])^{\frac{\gamma}{1-t}}.
\]

Solving the above inequality of \( \delta_{t+1} \) we get
\[
\delta_{t+1} = O \left( \left( \gamma^{\frac{1}{1-t}} + \gamma^{-1} (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} \right)^{\frac{1}{\gamma}} + \gamma \sum_{j=1}^{t} \eta_j^{\frac{\gamma}{1-t}} + \gamma (n^{-1} + T n^{-2}) \sum_{j=1}^{t} \eta_j^2 (\mathbb{E}_{S,A}[F(w_j)])^{\frac{\gamma}{1-t}} \right).
\]

Assuming \( T \geq n \), from the definition of \( \delta_{t+1} \) we have
\[
\left( \sum_{i=1}^{T} \eta_t \right)^{-1} \sum_{i=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t) - F_S(w_t)] \\
= O \left( \gamma^{\frac{1}{1-t}} + \gamma \sum_{i=1}^{T} \eta_t^{\frac{\gamma}{1-t}} + \gamma^{-1} \sum_{i=1}^{T} \eta_t (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} + \gamma T n^{-2} \sum_{i=1}^{T} \eta_t^2 (\mathbb{E}_{S_A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} \right).
\]

If we set \( \eta_t = \eta \), then there holds
\[
\left( \sum_{i=1}^{T} \eta \right)^{-1} \sum_{i=1}^{T} \eta \mathbb{E}_{S,A}[F(w_t) - F_S(w_t)] \\
= O \left( \left( \gamma^{\frac{1}{1-t}} + \gamma T \eta^{\frac{\gamma}{1-t}} + (\gamma T \eta)^{-1} \sum_{i=1}^{T} \eta (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} + \gamma T n^{-2} \sum_{i=1}^{T} \eta^2 (\mathbb{E}_{S_A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} \right) \right)
\]

Since \( \mathbb{E}_{A}[w^* - w_t, b_t] = 0 \), Eq. 8 with \( \eta_t = \eta \) implies
\[
\sum_{i=1}^{T} \eta^2 (\mathbb{E}_{S,A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} \leq \sum_{i=1}^{T} \eta^2 \left( \sum_{i=1}^{T} \eta^2 \mathbb{E}_{S,A}[F(w_{t+1})] \right)^{\frac{\gamma}{1-t}} = \left( \sum_{i=1}^{T} \eta^2 \right)^{\frac{1}{1-t}} \left( \sum_{i=1}^{T} \eta^2 \mathbb{E}_{S,A}[F(w_{t+1})] \right)^{\frac{\gamma}{1-t}} \\
\leq (T \eta^2)^{\frac{1}{1-t}} \left( 2\eta ||w^*||_2^2 + 6T \eta^3 \sigma^2 d + 4T \eta^2 F(w^*) + 3c_{\alpha,2} \eta^2 \right)^{\frac{\gamma}{1-t}} \\
= O \left( (T \eta^2)^{\frac{1}{1-t}} \left( \eta + T \eta^3 \sigma^2 d + T \eta^2 F(w^*) + T \eta^{\frac{\gamma}{1-t}} \right) \right).
\]

Dividing both sides by \( \eta \), we get
\[
\sum_{i=1}^{T} \eta (\mathbb{E}_{S_A}[F_S(w_{t+1})])^{\frac{\gamma}{1-t}} = O \left( (T \eta^2)^{\frac{1}{1-t}} \left( \eta + T \eta^3 \sigma^2 d + T \eta^2 F(w^*) + T \eta^{\frac{\gamma}{1-t}} \right) \right).
Now, plugging the above two inequalities back into (13), we have

\[
\begin{align*}
(\sum_{t=1}^{T} \eta_t)^{-1} & \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t) - F_{S}(w_t)] \\
& = O \left( \gamma^{\frac{1}{1+\alpha}} + \gamma T \eta^{\frac{2}{1+\alpha}} + (\gamma T)^{-1} T \frac{\eta}{1+\alpha} \left( \eta + T \eta^3 \sigma^2 d + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right) \right)^{\frac{1}{1+\alpha}} \\
& \quad + \gamma T n^{-2} (T \eta^3) \left( \eta + T \eta^3 \sigma^2 d + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right)^{\frac{2}{1+\alpha}} \\
& = O \left( \gamma^{\frac{1}{1+\alpha}} + \gamma T \eta^{\frac{2}{1+\alpha}} + \left[ \gamma^{-1} T \frac{\eta}{1+\alpha} \right] + \eta n^{-2} T \eta^2 \eta^{\frac{2-2\alpha}{1+\alpha}} \right) \left( \eta + T \eta^3 \sigma^2 d + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right)^{\frac{2}{1+\alpha}} \\
& \quad + \gamma T n^{-2} \eta^{\frac{2-2\alpha}{1+\alpha}} + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right)^{\frac{2}{1+\alpha}}. 
\end{align*}
\]

(14)

Part (b) in Theorem 16 with \( \eta_t = \eta \) implies

\[
\begin{align*}
\sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t) - F(w^*)] &= \sum_{t=1}^{T} \eta \mathbb{E}_{S,A}[F_S(w_t) - F(w^*)] \\
& = O \left( \frac{1}{T \eta} + T \frac{\eta}{1+\alpha} \left( \eta + T \eta^3 \sigma^2 d + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right)^{\frac{2}{1+\alpha}} + \eta^2 d \right). 
\end{align*}
\]

(15)

Plugging (14) and (16) back into (12) yields

\[
\mathbb{E}_{S,A}[F(w_{priv})] - F(w^*) \\
= O \left( \gamma^{-1} T \frac{\eta}{1+\alpha} + \gamma T n^{-2} T \frac{\eta}{1+\alpha} + T \frac{\eta}{1+\alpha} \right) \left( \eta + T \eta^3 \sigma^2 d + \eta^2 F(w^*) + T \eta^{\frac{3-\alpha}{1+\alpha}} \right)^{\frac{2}{1+\alpha}} \\
+ \gamma T n^{-2} + \gamma T \eta^{\frac{2}{1+\alpha}} + \frac{1}{T \eta} + \eta^2 d \right). 
\]

(16)

Now, we can prove part (a) by choosing suitable \( \gamma, \eta \) and \( T \). Let \( \gamma = \sqrt{n} \) and \( \eta = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \). Recall that \( \sigma^2 d = O \left( \frac{T d \log(1/\delta)}{n \epsilon} \right) \). Note we assume \( T \eta \geq 1 \). Then

\[
\eta + T \eta^3 \sigma^2 d + T \eta^2 + T \eta^{\frac{3-\alpha}{1+\alpha}} = O \left( \frac{T^2 \eta^3 \frac{d \log(1/\delta)}{n \epsilon^2}}{n \epsilon^2} + T \eta^2 \right) = O \left( T^2 n^{-2} \eta + T \eta^3 \right).
\]

Combing the above equation with Eq.(16), we get

\[
\mathbb{E}_{S,A}[F(w_{priv})] - F(w^*) = O \left( \frac{T \eta^3 \frac{d \log(1/\delta)}{n \epsilon^2}}{n \epsilon} + T \eta^2 \right) \\
+ n^{\frac{1+\alpha}{2(\alpha-1)}} \frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon}. 
\]

If we further choose \( T \propto n \), then for any \( \alpha \in [1/2, 1) \) there holds

\[
\mathbb{E}_{S,A}[F(w_{priv})] - F(w^*) = O \left( \frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon} \right) 
\]

For the case \( \alpha \in [0, 1/2) \), let \( \gamma = \sqrt{n} \) and \( \eta = c \min \left\{ n^{\frac{3(\alpha-1)}{2(1+\alpha)}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \).

Similar to the discussion of Part (a), this choice of \( \eta \) implies

\[
\eta + T \eta^3 \sigma^2 d + T \eta^2 + T \eta^{\frac{3-\alpha}{1+\alpha}} = O \left( \frac{T^2 \eta^3 \frac{d \log(1/\delta)}{n \epsilon^2}}{n \epsilon^2} + T \eta^2 \right) = O \left( \frac{T^2 \eta^2 \sqrt{d \log(1/\delta)}}{n(n \epsilon)} + T \eta^2 \right).
\]
Further setting $T \sim n^{2/\alpha}$, then combining the above equation with Eq. (10) implies

$$
E_{S,A}[F(w_{\text{priv}})] - F(w^*) = O\left(\frac{\sqrt{\log(1/\delta)}}{n\epsilon} + \frac{d\log(1/\delta)}{n^{\alpha}} + \frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right),
$$

where the last equality used $\alpha < 1/2$. The proof of part (a) is completed.

Finally, we consider the low noise case, i.e., $F(w^*) = 0$. Let $\eta = c \min\left\{n^{\alpha/2 + \epsilon/\alpha} \frac{\epsilon}{d\log(1/\delta)} \right\} \leq \min\{2/L, 1\}$. Then (10) implies

$$
E_{S,A}[F(w_{\text{priv}})] - F(w^*) = O\left(\gamma^{-1}T^2\eta^{1/2} + \gamma n^{-2}T^3\eta^{1/4} + T^4\eta^{1/8}\right)\left(\eta + T\eta^3\sigma^2d + T\eta^2\right)^{\frac{3}{2\alpha}}
$$

$$
+ \gamma \frac{\sigma^2}{n} + T\eta \frac{\sigma^2}{n} + \frac{1}{T\eta} + \eta \sigma^2 d.
$$

Note for any $\alpha \in [0, 1)$, there holds

$$
\eta + T\eta^3\sigma^2d + T\eta^{\frac{3}{2\alpha}} = O\left(\eta\left(1 + \frac{T^2\eta^2\log(1/\delta)}{n^2\epsilon^2}\right)\right) = O(\eta),
$$

where we used $T^2\eta^2 = O(n^2\epsilon^2/(d\log(1/\delta)))$. Further, if we choose $\gamma = n^{1/2\alpha}$ and $T \sim n^{2/\alpha}$, there holds

$$
E_{S,A}[F(w_{\text{priv}})] - F(w^*) = O\left(\frac{1}{n^{1/2\alpha}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right),
$$

which completes the proof.

\[\square\]

### 4.2 Proofs for Pairwise Learning

We now turn to the analysis of DP-SGD for pairwise learning algorithm (i.e. Algorithm 2) and provide the proofs for Theorems 6 and 7.

We start with the proof of Theorem 6. Specifically, we first prove that each iteration $t$ of the algorithm satisfies RDP by applying Lemma 8 with sampling rate $2/p$. Then according to Lemma 9 and Lemma 10, we can show that the proposed algorithm satisfies $(\epsilon, \delta)$-DP. The detailed proof is shown as follows.

**Proof of Theorem 6.** For each $t \in [T]$, we consider the mechanism $A_t = M_t + b_t$, where $M_t = \partial f(w_t; z_i, z_j)$. Similar to before, we can show that the $l_2$-sensitivity of $M_t$ is $2G$ by using Lipschitz continuity of $f$. Notice that

$$
\sigma^2 = \frac{56G^2T}{\beta n^2\epsilon}\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1\right).
$$

Note that $z_i$ and $z_j$ are drawn uniformly without replacement from the training set $S$. Then according to Lemma 8 with $p = 2/n$, we know $A_t$ satisfies $\left(\lambda, \frac{\lambda \beta \epsilon}{T(\log(1/\delta)/(1-\beta)\epsilon) + 1}\right)$-RDP as long as $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2}\log\left(\frac{n}{2\lambda(1+\frac{\beta}{2\epsilon})}\right)$ hold. Now, let $\lambda = \frac{\log(1/\delta)/(1-\beta)\epsilon + 1}{2\epsilon}. Then we get $A_t$ satisfies $\left(\frac{\log(1/\delta)/(1-\beta)\epsilon + 1, \beta \epsilon}{2\epsilon}\right)$-RDP. According to Lemma 11 and Lemma 9 we can show that Algorithm 2 is $(\epsilon, \delta)$-DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2}\log\left(\frac{n}{\lambda(1+\frac{\beta}{2\epsilon})}\right)$ hold. The proof is completed. \[\square\]
To establish the generalization analysis of Algorithm 2, we first introduce the connection between stability and generalization error in the following lemma.

**Lemma 17** (Generalization via stability for pairwise learning). Let $A$ be on-average $\nu$-argument stable. Let $\gamma > 0$.

(a) If $f$ is nonnegative and $L$-smooth, then
\[
\mathbb{E}_{S,A}[\tilde{F}(A(S)) - \tilde{F}_S(A(S))] \leq \frac{L}{\gamma} \mathbb{E}_{S,A}[\tilde{F}_S(A(S))] + 2(L + \gamma)\nu.
\]

(b) If $f$ is nonnegative, convex and $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$, then
\[
\mathbb{E}_{S,A}[\tilde{F}(A(S)) - \tilde{F}_S(A(S))] \leq \frac{c_{\alpha,1}^2}{2\gamma} \mathbb{E}_{S,A}[\tilde{F}^{2\alpha}_{\gamma}(A(S))] + 2\nu.
\]

**Proof.** Part (a) was established in [20]. We only consider Part (b). Recall that $S = \{z_1, \ldots, z_n\}$ and $S' = \{z'_1, \ldots, z'_n\}$ are drawn independently from $\rho$. For any $i \in [n]$, denote $S^{(i)} = \{z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_n\}$. Further, let $S^{(i,j)} = \{z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_{j-1}, z'_j, z_{j+1}, \ldots, z_n\}$.

According to the symmetry between $z_i, z_j$ and $z'_i, z'_j$, we have
\[
\mathbb{E}_{S,S'}[\tilde{F}(A(S)) - \tilde{F}_S(A(S))]
= \frac{1}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'}[\tilde{F}(A(S^{(i,j)})) - \tilde{F}_S(A(S))]
= \frac{1}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'}[f(A(S^{(i,j)}; z_i, z_j)) - f(A(S); z_i, z_j)]
\leq \frac{1}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'}[\langle \partial f(A(S^{(i,j)}; z_i, z_j)), A(S^{(i,j)}) - A(S) \rangle],
\tag{17}
\]

where in the second equality we used $\mathbb{E}_{z_i, z_j}[f(A(S^{(i,j)}; z_i, z_j))] = \tilde{F}(A(S^{(i,j)}))$ since $z_i, z_j$ are independent of $A(S^{(i,j)})$, and in the last inequality we used the convexity of $f$.

By the Schwartz’s inequality and self-bounding property (Lemma [12]) we know
\[
\langle \partial f(A(S^{(i,j)}; z_i, z_j)), A(S^{(i,j)}) - A(S) \rangle
\leq \frac{1}{2\gamma} \|\partial f(A(S^{(i,j)}; z_i, z_j))\|_2^2 + \frac{\gamma}{2} \|A(S^{(i,j)}) - A(S)\|_2^2
\leq \frac{c_{\alpha,1}^2}{2\gamma} f^{2\alpha}_{\gamma}(A(S^{(i,j)}; z_i, z_j)) + \gamma \|A(S^{(i,j)}) - A(S^{(i)})\|_2^2 + \gamma \|A(S^{(i)}) - A(S)\|_2^2.
\]

Plugging the above inequality back into Eq. (17) we get
\[
\mathbb{E}_{S,S'}[\tilde{F}(A(S)) - \tilde{F}_S(A(S))]
\leq \frac{1}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'} \left[ \frac{c_{\alpha,1}^2}{2\gamma} f^{2\alpha}_{\gamma}(A(S^{(i,j)}; z_i, z_j)) + \gamma \|A(S^{(i,j)}) - A(S^{(i)})\|_2^2 + \gamma \|A(S^{(i)}) - A(S)\|_2^2 \right]
= \frac{c_{\alpha,1}^2}{2\gamma n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'}[f^{2\alpha}_{\gamma}(A(S^{(i,j)}; z_i, z_j)) + \frac{2\gamma}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S'}[\|A(S^{(i)}) - A(S)\|_2^2]]
= \mathbb{E}_{S,S'}[\|A(S^{(i)}) - A(S)\|_2^2] = \mathbb{E}_{S,S'}[\|A(S^{(j)}) - A(S)\|_2^2].
\]

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Since $x \mapsto x^{2\alpha}$ is concave and $z_i, z_j$ are independent of $\mathcal{A}(S^{(i,j)})$, we know

$$
\mathbb{E}_{S,S',A}[f^{2\alpha}(\mathcal{A}(S^{(i,j)}); z_i, z_j)] \leq \mathbb{E}_{S,S',A}[\langle \mathbb{E}_{z_i,z_j}[f(\mathcal{A}(S^{(i,j)}); z_i, z_j)] \rangle^{2\alpha}]
$$

Combining the above two inequalities together implies

$$
\mathbb{E}_{S,S',A}[\tilde{F}^{2\alpha}(\mathcal{A}(S)) - F^{2\alpha}(\mathcal{A}(S))]
$$

Our stability analysis for $\alpha$-Hölder smooth losses requires the following lemma, which shows the approximately non-expansive behavior of the gradient mapping $w \mapsto w - \eta \partial f(w; z, z')$.

**Lemma 18** (27). Assume for all $z, z' \in Z$, the map $w \mapsto f(w; z, z')$ is convex, and $w \mapsto \partial f(w; z, z')$ is $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$. Then for all $w, w'$ and $\eta > 0$ we have

$$
\|w - \eta \partial f(w; z, z') - w' + \eta \partial f(w'; z, z')\|_2^2 \leq \|w - w'\|_2^2 + c_{\alpha, \eta}^2 \eta^{2\alpha}.
$$

As discussed in Section 4.1, adding noise to gradient will not impact stability results. Hence, we only need to address the on-average stability bounds of non-private SGD for pairwise learning.

**Lemma 19** (Stability bounds). Suppose $f$ is nonnegative and convex. Let $S, S'$ and $S^{(i)}$ be constructed as Definition 6. Let $\{w_t\}$ and $\{w_t^{(i)}\}$ be produced by Algorithm 2 based on $S$ and $S^{(i)}$, respectively.

(a) If $f$ is $L$-smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then

$$
\mathbb{E}_{S,S',A}\left[\frac{1}{n} \sum_{i=1}^{n} \|w_{t+1} - w_{t+1}^{(i)}\|_2^2\right] \leq \frac{16L(1 + 2t/n)e}{n} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}[F_S(w_j)].
$$

(b) If $f$ is $\alpha$-Hölder smooth with parameter $L$ and $\alpha \in [0, 1)$, then

$$
\mathbb{E}_{S,S',A}\left[\frac{1}{n} \sum_{i=1}^{n} \|w_{t+1} - w_{t+1}^{(i)}\|_2^2\right] \leq \frac{8c_{\alpha,1}^2 (1 + 2t/n)}{n} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}\left[F_{S}^{\frac{2\alpha}{1 + \alpha}}(w_j)\right] + c_{\alpha,3}^2 \sum_{j=1}^{t} \eta_j^{\frac{2\alpha}{1 + \alpha}},
$$

where $c_{\alpha,3} = \sqrt{\frac{c_{1 - \alpha, \alpha} (2 - \alpha) L}{1 - \alpha}}$.

**Proof.** The proof of part (a) can be found in [29]. We only give the proof of part (b). For any $i \in [n]$, let $S, S^{(i)}$ and $S'$ be constructed as Definition 6. For any $S$ and $i \in [n]$, we consider the following three cases.

**Case 1.** If $i_t \neq i$ and $j_t \neq i$, it then follows from the update rule of $w_{t+1}$ and Lemma 18 that

$$
\|w_{t+1} - w_{t+1}^{(i)}\|_2^2 \leq \|w_t - \eta \partial f(w_t; z_{i_t}, z_{j_t}) - w_t^{(i)} + \eta \partial f(w_t^{(i)}; z_{i_t}, z_{j_t})\|_2^2
\leq \|w_t - w_t^{(i)}\|_2^2 + c_{\alpha,3}^2 \eta_t^{\frac{2\alpha}{1 + \alpha}}.
$$
Case 2. If \(i_t = i\), it then follows from the update rule and the standard inequality \((a+b)^2 \leq (1+p)a^2 + (1+1/p)b^2\) that

\[
\|w_{t+1} - w_{t+1}^{(i)}\|^2 \leq (1+p)\|w_t - w_t^{(i)}\|^2 + (1+1/p)\eta_i^2 (\|\partial f(w_t; z_i, z_j)\|^2 - \partial f(w_t^{(i)}; z_i, z_j)) + (1+p)\|w_t - w_t^{(i)}\|^2 + 2(1+1/p)\eta_i^2 (f^{2\alpha}(w_t; z_i, z_j) + f^{2\alpha}(w_t^{(i)}; z_i, z_j))
\]

Further, taking an expectation over both sides yields

\[
\Pr \text{ Case 2.}
\]

Case 3. If \(j_t = i\), similar to Case 2, we have

\[
\|w_{t+1} - w_{t+1}^{(i)}\|^2 \leq (1+p)\|w_t - w_t^{(i)}\|^2 + 2c_{\alpha,1}^2(1+1/p)\eta_i^2 (f^{2\alpha}(w_t; z_i, z_j) + f^{2\alpha}(w_t^{(i)}; z_i, z_j))
\]

Note \(\Pr(i_t \neq i) = \frac{(n-1)(n-2)}{n(n-1)}\) and \(\Pr(i_t = i) = \frac{1}{n(n-1)}\) for any \(j \neq i\). We can combine the above three cases together and get

\[
\mathbb{E}_{i_t,j_t}[\|w_{t+1} - w_{t+1}^{(i)}\|^2] \leq \frac{(n-1)(n-2)}{n(n-1)} \left(\|w_t - w_t^{(i)}\|^2 + c_{\alpha,3}^2\eta_t^{2}\right)
\]

Taking an average over \(i\) we have

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{i_t,j_t}[\|w_{t+1} - w_{t+1}^{(i)}\|^2] \leq \left(1 + \frac{2p}{n}\right)\frac{1}{n} \sum_{i=1}^{n} \|w_t - w_t^{(i)}\|^2 + c_{\alpha,3}^2\eta_t^{2} + \frac{2(1+1/p)c_{\alpha,1}^2\eta_i^2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \in [n]; j \neq i} [f^{\alpha}(w_t; z_i, z_j) + f^{\alpha}(w_t^{(i)}; z_i, z_j) + f^{\alpha}(w_t^{(i)}; z_i, z_j') + f^{\alpha}(w_t^{(i)}; z_i, z_j')].
\]

Further, taking an expectation over both sides yields

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S,S',A}[\|w_{t+1} - w_{t+1}^{(i)}\|^2] \leq \left(1 + \frac{2p}{n}\right)\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S,S',A}[\|w_t - w_t^{(i)}\|^2] + \frac{2(1+1/p)c_{\alpha,1}^2\eta_i^2}{n^2(n-1)} \sum_{i=1}^{n} \sum_{j \in [n]; j \neq i} [f^{\alpha}(w_t; z_i, z_j) + f^{\alpha}(w_t^{(i)}; z_i, z_j) + f^{\alpha}(w_t^{(i)}; z_i, z_j') + f^{\alpha}(w_t^{(i)}; z_i, z_j')]
\]

Due to the symmetry between \(z_i\) and \(z_i'\) we know

\[
\mathbb{E}_{S,A} \left[ \sum_{j \in [n]; j \neq i} [f^{\alpha}(w_t; z_i, z_j) + f^{\alpha}(w_t; z_i, z_j')] \right] = \mathbb{E}_{S,S',A} \left[ \sum_{j \in [n]; j \neq i} [f^{\alpha}(w_t^{(i)}; z_i, z_j) + f^{\alpha}(w_t^{(i)}; z_i, z_j')] \right].
\]
Further, according to Jensen’s inequality and \(w\), where in the last equality we used 
\[ \sum_{i=1}^{n} \sum_{j \in [n]: j \neq i} f_{\frac{2\alpha}{t} + a}(w_t; z_i, z_j) = \sum_{i=1}^{n} \sum_{j \in [n]: j \neq i} f_{\frac{2\alpha}{t}}(w_t; z_i, z_j). \]

Further, according to Jensen’s inequality and \(w_1 = w_1^t\), we know
\[ \frac{1}{n} \sum_{i=1}^{n} E_{S, S', A}[\|w_{t+1} - w_t^{(i)}\|^2_2] \leq \left(1 + \frac{2p}{n}\right) \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in [n]: j \neq i} f_{\frac{2\alpha}{t} + a}(w_t; z_i, z_j) + c_{\alpha, 3} \eta t^{2\gamma - \frac{2\beta}{t}} \]
\[ + \frac{8(1 + 1/p)c_{\alpha, 1}^2 \eta^2}{n} E_{S, A}[\frac{1}{n(n - 1)} \sum_{i=1}^{n} \sum_{j \in [n]: j \neq i} f_{\frac{2\alpha}{t}}(w_t; z_i, z_j)] \]

Now, we can apply the above inequality recursively and get
\[ \frac{1}{n} \sum_{i=1}^{n} E_{S, S', A}[\|w_{t+1} - w_t^{(i)}\|^2_2] \leq \frac{8(1 + 1/p)c_{\alpha, 1}^2 \eta^2}{n} \sum_{j=1}^{t} \left(1 + \frac{2p}{n}\right)^{t-j} \eta^2 t^{\frac{2\gamma}{t} - \frac{2\beta}{t}} E_{S, A}[f_{\frac{2\alpha}{t}}(w_j)] \]
\[ + c_{\alpha, 3}^2 \sum_{j=1}^{t} \left(1 + \frac{2p}{n}\right)^{t+1-j} \eta^2 t^{\frac{2\gamma}{t} - \frac{2\beta}{t}}. \]

Finally, we can set \(p = \frac{2t}{\sqrt{t}}\) and use \((1 + 1/t)^t \leq e\) to get
\[ \frac{1}{n} \sum_{i=1}^{n} E_{S, S', A}[\|w_{t+1} - w_t^{(i)}\|^2_2] \leq \frac{8e(1 + 2t/n)c_{\alpha, 1}^2 \eta^2}{n} \sum_{j=1}^{t} \eta^2 E_{S, A}[f_{\frac{2\alpha}{t}}(w_j)] + c_{\alpha, 3}^2 \sum_{j=1}^{t} \eta^2 t^{2\gamma - \frac{2\beta}{t}}, \]
which completes the proof.

To prove Theorem 6, we introduce the following lemma on optimization error. As discussed in [26], the optimization error analysis of DP-SGD (Algorithm 2) for pairwise learning is the same as that for pointwise learning (Algorithm 1). Here, \(\alpha = 1\) corresponds to the strongly smooth case due to the definition of \(\alpha\)-Hölder smoothness.

**Lemma 20.** Suppose \(f\) is nonnegative, convex and \(\alpha\)-Hölder smooth with parameter \(L\) and \(\alpha \in [0, 1]\). Let \(\{w_t\}\) be produced by Algorithm 2 with \(\eta_t = \eta\). Then
\[ \sum_{j=1}^{t} \eta_j E_{S, A} [F_S(w_j) - F_S(w^*)] \]
\[ \leq \frac{1}{2} \|w^*\|^2_2 + \frac{3}{4} c_{\alpha, 1}^2 \left(\sum_{j=1}^{t} \eta_j^2 \right)^{1 + \frac{2\gamma}{\eta}} \left[2\eta_1 \|w^*\|^2_2 + \sum_{j=1}^{t} (6n_j^3 \sigma^2 + 4n_j^2 F_S(w^*) + 3c_{\alpha, 2} n_j^{\frac{3-\gamma}{\eta}}) \right]^{1 + \frac{2\gamma}{\eta}} + \sum_{j=1}^{t} 3n_j^2 \sigma^2 d \]
and
\[ \sum_{j=1}^{t} \eta_j^2 E_{S, A} [F_S(w_t)] \leq 2\eta_t \|w^*\|^2_2 + \sum_{j=1}^{t} (6n_j^3 \|b_j\|^2_2 + 4n_j^2 F(w^*) + 3c_{\alpha, 2} n_j^{\frac{3-\gamma}{\eta}}). \]
Now, we are ready to prove the utility guarantees of Algorithm 2 for strongly smooth and non-smooth cases. We first present the proof for strongly smooth case (i.e., Theorem 6).

**Proof of Theorem 6.** Similar to the proof of Theorem 3, combining Lemma 19, Lemma 20 and part (a) in Lemma 17, together we have

\[
E_{S,A}[\bar{F}(w_{t+1})] \leq \left(1 + \frac{L}{\gamma}\right) E_{S,A}[\bar{F}_S(w_t)] + \frac{32\epsilon(L + \gamma)(1 + 2T/n)L}{n} \left[ 2\eta_t \|w^*\|_2^2 + \sum_{j=1}^{t} \left( 6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(w^*) \right) \right].
\]

Multiplying both sides by \( \eta_{t+1} \) and taking a summation gives

\[
\sum_{t=1}^{T} \eta_t E_{S,A}[\bar{F}(w_t)] \leq \left(1 + \frac{L}{\gamma}\right) \sum_{t=1}^{T} \eta_t E_{S,A}[\bar{F}_S(w_t)] + \frac{32\epsilon(L + \gamma)(1 + 2T/n)L}{n} \sum_{t=1}^{T} \eta_t \left[ 2\eta_t \|w^*\|_2^2 + \sum_{j=1}^{t} \left( 6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(w^*) \right) \right] + \sum_{t=1}^{T} 3\eta_t^2 \sigma^2 d.
\]

Combining the above two inequalities together yields

\[
\sum_{t=1}^{T} \eta_t E_{S,A}[\bar{F}(w_t)] \leq \left(1 + \frac{L}{\gamma}\right) \sum_{t=1}^{T} \eta_t \bar{F}(w^*) + \left(\frac{1}{2} + 3L\eta_t\right) \|w^*\|_2^2 + 3 \sum_{j=1}^{t} (3\eta_j^3 \sigma^2 d + 2\eta_j^2 \bar{F}_S(w^*)) + \sum_{j=1}^{t} \eta_j^2 \bar{F}(w^*)
\]

\[
+ \frac{32\epsilon(L + \gamma)(1 + 2T/n)L}{n} \sum_{t=1}^{T} \eta_t \left[ 2\eta_t \|w^*\|_2^2 + \sum_{j=1}^{t} \left( 6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(w^*) \right) \right].
\]

Let \( \eta_t = \eta \leq \min\{2/L, 1\} \) and assume \( T \geq n \). Recall that \( \sigma^2 d = \mathcal{O}\left(\frac{Td \log(1/\delta)}{n^2 \epsilon^2}\right) \). According to Jensen’s inequality, there holds

\[
E_{S,A}[\bar{F}(w_{\text{priv}}) - \bar{F}(w^*)] = \mathcal{O}\left(\left(\frac{1}{\gamma} + \frac{T^2 \eta^2 (1 + \gamma)}{n^2} + \left(\frac{1}{\gamma} + 1\right)\eta \bar{F}(w^*) + \left(\frac{1 + \gamma^{-1}}{T \eta} + \frac{(1 + \gamma) T \eta}{n^2}\right)\|w^*\|_2^2 + \left(\left(1 + \frac{1}{\gamma}\right)\eta + \frac{T^2 \eta^3 (1 + \gamma)}{n^2}\right) T d \log(1/\delta)\right) n^2 \epsilon^2 \right).
\]

Now, we give the proof of part (a). We can set \( T \asymp n \), \( \gamma = \sqrt{n} \) and \( \eta_t = c/ \max\left\{ \sqrt{n}, \frac{\sqrt{d \log(1/\delta)}}{c} \right\} \leq \min\{2/L, 1\} \) for some constant \( c > 0 \). Then from Eq. (18) we obtain

\[
E_{S,A}[\bar{F}(w_{\text{priv}}) - \bar{F}(w^*)] = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \epsilon}\right),
\]

where we also assume \( \sqrt{d \log(1/\delta)} = \mathcal{O}(n \epsilon) \).
(b) We now consider the low-noise case $F(w^*) = 0$. By setting $\gamma \geq 1$, $T = n$ and $\eta_t = \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$, we get

$$E_{S,A}[\bar{F}(w_{\text{priv}}) - \bar{F}(w^*)] = O\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right),$$

which completes the proof. \hfill \Box

Finally, we give the proof for Theorem 17.

Proof of Theorem 17. The proof is similar to that of Theorem 4. Specifically, we can plug part (b) in Lemma 19 back into part (b) in Lemma 17 to get that

$$\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta E_{S,A}[\bar{F}(w_t) - \bar{F}_S(w_t)]$$

$$= O\left(\frac{\gamma^{1+\c}}{\eta T} + \gamma T \eta^{\frac{3\gamma}{1+\c}} + \gamma T \eta \sum_{t=1}^{T} \eta E_{S,A}[\bar{F}_S(w_{t+1})] \right)^{\frac{1}{1+\c}} + \gamma T n^{-2} \sum_{t=1}^{T} \eta^2 E_{S,A}[\bar{F}_S(w_t)]^{\frac{1}{1+\c}}. \tag{19}$$

Further, combining Eq. (19) and Lemma 20 together we can obtain

$$\left(\sum_{t=1}^{T} \eta\right)^{-1} \sum_{t=1}^{T} \eta E_{S,A}[\bar{F}_S(w_t) - \bar{F}w^*]$$

$$= O\left(\frac{1}{T \eta} + T \eta^{\frac{1+\c}{1+\c}} \left(\eta + T \eta^3 \sigma^2 n + T \eta^2 \bar{F}(w^*) + T \eta \sigma^2 d^{\frac{3}{n-\alpha}}\right)\right)^{\frac{1}{1+\c}} + \eta \sigma^2 d. \tag{20}$$

Plugging Eq. (19) and Eq. (20) back into Eq. (12) we have

$$E_{S,A}[\bar{F}(w_{\text{priv}}) - \bar{F}(w^*)] = O\left((1 + \gamma^{-1}) T \eta + \frac{(1 + \gamma) T \eta}{n^2} \|w^*\|_2^2 + \left(\gamma^{-1} + \frac{T \eta^2 (1 + \gamma)}{n^2} + (\gamma^{-1} + 1) \eta\right) \bar{F}(w^*)\right.$$

$$\left. + \left((1 + \gamma^{-1}) T \eta + \frac{T \eta^4 (1 + \gamma)}{n^2}\right) T \frac{d \log(1/\delta)}{n^2 \epsilon^2}\right). \tag{21}$$

The rest of the proof is similar to Theorem 4. We omit it for simplicity. \hfill \Box

5 Conclusion

In this paper, we conducted a systematic analysis of DP-SGD with gradient perturbation for both pointwise and pairwise learning problems. For pointwise learning, we introduced a low-noise condition and derived sharper excess population risk bounds. Specifically, we achieved bounds in the order of $O\left(\frac{1}{n \epsilon} \sqrt{d \log(1/\delta)}\right)$ and $O\left(n^{-\frac{1}{1+\c}} + \frac{1}{n \epsilon} \sqrt{d \log(1/\delta)}\right)$ for strongly smooth and $\alpha$-Hölder smooth losses, respectively.

Regarding pairwise learning, we presented a computationally efficient DP-SGD algorithm with utility guarantees. Our analysis demonstrated that our algorithm achieves the optimal excess risk bounds of the order $O\left(\frac{1}{n \epsilon} \sqrt{d \log(1/\delta)}\right)$ and $O\left(n^{-\frac{1}{1+\c}} + \frac{1}{n \epsilon} \sqrt{d \log(1/\delta)}\right)$ for both strongly smooth and $\alpha$-Hölder smooth losses. Furthermore, we established faster excess risk bounds for both strongly smooth and $\alpha$-Hölder smooth losses under a low-noise condition. Notably, our work represents the first utility analysis for privacy-preserving pairwise learning that provides excess risk rates tighter than $O\left(\frac{1}{\sqrt{n}} + \frac{1}{n \epsilon} \sqrt{d \log(1/\delta)}\right)$.
There are several open questions that remain for further study. Firstly, it would be interesting to explore whether our analysis of DP-SGD with uniform sampling can be extended to DP-SGD with Markov sampling, which poses a more challenging task. Secondly, an unexplored area for us is to investigate the utility analysis of DP-SGD with a neural network structure. Addressing these questions would contribute to a deeper understanding of privacy-preserving machine learning algorithms.

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