Abstract. In this article, we have studied the difference-difference Lotka-Volterra equations in $p$-adic number space and its $p$-adic valuation version. We pointed out that the structure of the space given by taking the ultra-discrete limit is the same as that of the $p$-adic valuation space.

Introduction

In soliton theory, difference-difference equations, whose domain space-time are given by integers, and the ultra-discrete difference-difference equations, whose all, domain and range are given by integers, are currently studied [HT, TS, TTMS].

On the other hand, recently number theory and physics might be considered as a missing link of each other. For example, a set of geodesics in a compact Riemannian surface with genus $g \geq 2$ are investigated in the framework of chaos because any geodesics, or orbits, part from each other due to its negative curvature [V,Su] (whereas the Jacobi varieties of the Riemannian surfaces are completely classified by a soliton equation [M]). By quantization of the orbits, there appears quantum chaos and, as it is very mysterious, its partition function has very resemble structure of zeta functions in number theory [V,Su]. (Level statistics in quantum chaos is also connected with the integrable system [So].) Using the resemblance of zeta functions, Connes proposed a kind of unification of number theory and quantum statistical physics in order to solve the Riemannian conjecture of $\zeta$-function [BC,Co].

Further on the discrimination problem of integrability of Hamiltonian system, there appears Galois theory in the category of differential equation [MR, Y], which plays the same role in the category of the number theory.

Thus in order to know what is the integrability or quantization, it is not surprising that there appears integer theory in physics. In fact, there are many other studies pointing out that the $p$-adic number theory and non-archimedean valuation theory are closely related to statistical and quantum physics [RTV, BF, VVZ], even though $p$-adic space has a metric which differs from euclidean sense. These correspondences might imply that there is a deep hidden symmetry behind physics and number theory and give a novel step to mathematical physics.

Thus I believe that it is very important to interpret such recent development of soliton theory using $p$-adic number theory and non-archimedean valuation theory.

In this article, we will mainly deal with the Lotka-Volterra equation as a typical difference-difference soliton equation. We will show that 1) even in $p$-adic space of the number theory, the $p$-adic difference-difference Lotka-Volterra equation has mathematical meanings and has solutions. Next we will define the $p$-adic valuation version of the equation. Then we will show that 2) a quantity obtained by the ultra-discrete limit in the soliton theory should be regarded as the non-archimedean valuation as $p$-adic space has the $p$-adic valuation [C,VVZ].
In this article we will start from a preliminary of $p$-adic number theory. Next we will review the recent development of difference-difference and the ultra-discrete soliton theory [TTMS]. We will deal with the difference-difference and ultra-discrete difference-difference Lotka-Volterra equations. After we formally construct a $p$-adic difference-difference Lotka-Volterra equation, we will investigate its existence and explicit forms of its solutions. Then we will show that even in the $p$-adic space the difference-difference Lotka-Volterra equation has mathematical meanings and has soliton solutions. Next by computations of $p$-adic valuation, we will show resemblance between the $p$-adic valuation of the $p$-adic difference-difference Lotka-Volterra equation and ultra-discrete difference-difference Lotka-Volterra equation. Then it will be shown that the ultra discrete limit has the same structure as the $p$-adic valuation. Finally we will comment upon physical and mathematical meanings of the ultra-discrete limit.

**Preliminary: $p$-adic Space**

Let us consider $p$-adic field $\mathbf{Q}_p$ for a prime number $p$ [BF, C, I, RTV, VVZ]. For a rational number $u \in \mathbf{Q}$ which are given by $u = \frac{v}{w} p^m$ ($v$ and $w$ are coprime to the prime number $p$ and $m$ is an integer), we will define a symbol $\left[ \frac{u}{p} \right] = p^m$. Here we will define the $p$-adic valuation of $u$ as a map from $\mathbf{Q}$ to a set of integers $\mathbf{Z}$,

$$\text{ord}_p(u) := \log_p \left[ \frac{u}{p} \right], \quad \text{for } u \neq 0, \quad \text{and } \text{ord}_p(u) := \infty, \quad \text{for } u = 0.$$  

This valuation has following properties ($I_p$):

$I_p$: For $u, v \in \mathbf{Q}$,

1. $\text{ord}_p(uv) = \text{ord}_p(u) + \text{ord}_p(v)$.
2. $\text{ord}_p(u + v) \geq \min(\text{ord}_p(u), \text{ord}_p(v))$.
   If $\text{ord}_p(u) \neq \text{ord}_p(v)$, $\text{ord}_p(u + v) = \min(\text{ord}_p(u), \text{ord}_p(v))$.

This property ($I_p$-1) means that $\text{ord}_p$ is a homomorphism from the multiplicative group $\mathbf{Q}^\times$ of $\mathbf{Q}$ to the additive group $\mathbf{Z}$. The $p$-adic metric is given by $|v|_p = p^{-\text{ord}_p(v)}$. It is obvious that it is a metric because it has the properties ($II_p$):

$II_p$: For $u, v \in \mathbf{Q}$,

1. if $|v|_p = 0$, $v=0$.
2. $|v|_p \geq 0$.
3. $|vu|_p = |v|_p |u|_p$.
4. $|u + v|_p \leq \max(|u|_p, |v|_p) \leq |u|_p + |v|_p$.

The $p$-adic field $\mathbf{Q}_p$ is given as a completion of $\mathbf{Q}$ with respect to this metric so that properties ($I_p$) and ($II_p$) survive for $\mathbf{Q}_p$.

Further, we note that $\mathbf{Z}_p$, integer part of $\mathbf{Q}_p$, is a "localized ring" and has only prime ideals $\{0\}$ and $p\mathbf{Z}_p$. As the properties of $p$-adic metric, the convergence condition of series $\sum_m x_m$ is identified with the vanishing condition of sequence $|x_m|_p \to 0$ for $m \to \infty$ due to the relationship,

$$ \left| \sum_m x_m \right|_p = \max |x_m|_p. \quad (1) $$
Let us define $|u|_\infty$ as a natural metric or absolute value over real field $\mathbb{R}$, $|u|_\infty := |u|$, and $\mathbb{R}$ is regarded as $\infty$ point of prime numbers; we will denote $\mathbb{R}$ as $\mathbb{Q}_\infty$. Then we have a relation for any non-zero $u \in \mathbb{Q}$,

$$\prod_{p \in \mathfrak{A}} |u|_p = 1,$$

where $\mathfrak{A}$ is a set of prime numbers and $\infty$. This is an adelic property of $p$-adic metric.

**Difference-Difference Lotka-Volterra Equation**

Here we will review the difference-difference Lotka-Volterra equation [HT, TTMZ]. First, we will consider the Korteweg-de Vries (KdV) equation,

$$\partial_t u + 6u\partial_s u + \partial^3_s u = 0,$$

(2)

where $\partial_t := \partial/\partial t$ and $\partial_s := \partial/\partial s$ and $u = u(s, t)$ whose domain $(t, s)$ is $\mathbb{R}^2$. This differential equation was found by Korteweg and de Vries. However this differential equation and related (KdV) hierarchy were also discovered by Baker about one hundred years ago [B]. He studied the hyperelliptic functions and essentially discovered KdV hierarchy as differential equations generating periodic and algebraic functions over a hyperelliptic curve. He defined hyperelliptic $\sigma$ function as the best tuning theta function there and hyperelliptic $\wp$ functions as meromorphic functions over the hyperelliptic curve; e.g., they are connected as $\wp = \partial^2_s \log \sigma$. To evaluate the $\wp$-functions, he also used Pfaffians a bilinear operator, and bilinear equations, which, latter two, are recently called Hitora bilinear operator and bilinear equations [B]. These $\sigma$ and $\wp$ functions are closely related to the problems in number theory [O].

Along the line of arguments of [TTMS], we will deal with the difference-difference Lotka-Volterra as a difference-difference analogue of (2) here and next ultra-discrete difference-difference Lotka-Volterra.

The difference-difference Lotka-Volterra equation is given as [HT],

$$\frac{c_m^{m+1}}{c_m^m} = \frac{1 + \delta c_{m-1}^m}{1 + \delta c_{m+1}^m},$$

(3)

According to the arguments in [DJM1-3, H, HT, TTMZ], (3) is related to the bilinear difference-difference equation,

$$\tau_{n+1}^{m+1} \tau_{n}^{m} - (1 + \delta) \tau_{n+1}^{m} \tau_{n}^{m} + \delta \tau_{n-1}^{m} \tau_{n+2}^{m} = 0,$$

(4)

where

$$c_m^m = \frac{\tau_{n+1}^{m+1}}{\tau_{n}^{m+1}}.$$  

(5)

For example, the two-soliton solution is expressed as [H],

$$\tau_{n}^{m} = 1 + c^{n(m,n)} + e^{n_2(m,n)} + A c^{n(m,n) + n_2(m,n)},$$

(6)

where $k_a$, $\omega$, $n_a^0$ ($a = 1, 2$) are real numbers satisfied with,

$$\eta_{a}(m,n) = k_a n - \omega_{a} m + n^0_a,$$

$$e^{\omega_a} = \frac{1 + \delta(e^{k_a} + 1)}{1 + \delta(e^{-k_a} + 1)},$$

$$A = \frac{\sinh^2((k_1 - k_2)/2)}{\sinh^2((k_1 + k_2)/2)}.$$  

(7)
Similarly, we have more general solutions [H, DJM1-3, TTMS].

**Ultra-Discrete Space**

Next we will introduce the ultra-discrete limit following [TTMS]. In order to make our argument easy, we will change its notation but its essential definition does not differ.

Let $\mathcal{A}_\beta$ be a subset of non-negative functions over $(m, n, \beta) \in \mathbb{Z}^2 \times (\mathbb{R}_{>0})$ where $\mathbb{R}_{>0}$ is a set of positive real numbers. Here we regard that $\beta$ is a parameter of the function over $\mathbb{Z}^2$.

Let us define a map $\text{ord}_\beta : \mathcal{A}_\beta \rightarrow \mathbb{R}$. We set $\text{ord}_\beta(0) = \infty$ for zero and for $u \in \mathcal{A}_\beta$,

$$\text{ord}_\beta(u) := - \lim_{\beta \rightarrow +\infty} \frac{1}{\beta} \log(u). \tag{8}$$

$\mathcal{A}_\beta$ is characterized so that $\text{ord}_\beta(u)$ has a meaning value.

Typically it behaves like,

$$\text{ord}_\beta(e^{-\beta A} + e^{-\beta B} + \cdots) = \min(A, B, \cdots).$$

We note that this map $\text{ord}_\beta$ has the properties (I$_\beta$);

$I_\beta$.

For $u, v \in \mathcal{A}_\beta$,

1. $\text{ord}_\beta(uv) = \text{ord}_\beta(u) + \text{ord}_\beta(v)$.
2. $\text{ord}_\beta(u + v) = \min(\text{ord}_\beta(u), \text{ord}_\beta(v))$.

We note that this is a non-archimedean valuation because for $A > B$, there does not exist a finite integer $n$ such that $\text{ord}_\beta(e^{-\beta A}) < \text{ord}_\beta(ne^{-\beta B})$ [I,C, VVZ]. First we will note that this valuation is resemble to the property $I_p$ of $p$-adic valuation. We will call it ultra-valuation.

By introducing new variables $f^m_n := -\text{ord}_\beta(c^m_n)$ and $d := -\text{ord}_\beta(\delta)$ [T], we have an ultra-valuation version of the difference-difference Lotka-Volterra equation (3),

$$f^{m+1}_n - f^m_n = \text{ord}_\beta(1 + \delta_p c^{m}_{n-1}) - \text{ord}_\beta(1 + \delta_p c^m_{n-1}) \tag{9}$$

or

$$f^{m+1}_n - f^m_n = \max(0, f^m_{n-1} + d) - \max(0, f^{m+1}_{n-1} + d). \tag{9'}$$

This is an ultra-discrete difference-difference Lotka-Volterra equation, which is also integrable; its integrability was proved in [TTMS]. Of course, when $f$'s are given by quantities of integers times $d$ respectively, we can normalize it as $d = 1$ by dividing $d$.

**$p$-adic Difference-Difference Lotka-Volterra Equation and Its Valuation Version**

Next we will show that even in the $p$-adic space, difference-difference Lotka-Volterra equation has mathematical meaning and has solutions.

First we will formally introduce the $p$-adic difference-difference Lotka-Volterra equation for a $p$-adic series $\{c^m_n \in \mathbb{Q}_p\}$ ($p \neq 2$),

$$\frac{c^{m+1}_n}{c^m_n} = \frac{1 + \delta_p c^m_{n-1}}{1 + \delta_p c^{m+1}_{n+1}}, \tag{10}$$
where $\delta_p \in p\mathbb{Z}_p$. Noting that from (1), $p\mathbb{Z}_p$ is domain of exponential function and $1 + p\mathbb{Z}_p$ is domain of logarithm function [VVZ]. Further addition of elements of $p\mathbb{Z}_p$ belongs to $p\mathbb{Z}_p$ because $p\mathbb{Z}_p$ is an ideal. If $k_a$ and $\eta^0_a (a = 1, 2)$ belong to $p\mathbb{Z}_p$, two-soliton solution (6) and the conditions (7) replacing $\delta$ with $\delta_p$ are well-defined in $p$-adic space. In fact, in the equation,

$$\omega_a = \log \frac{1 + \delta_p(e^{k_a} + 1)}{1 + \delta_p(e^{-k_a} + 1)}$$

$(1 + \delta_p(e^{k_a} + 1))/(1 + \delta_p(e^{-k_a} + 1))$ can be expanded in $p$-adic space and belongs to $1 + p\mathbb{Z}_p$ and $\omega_a$ has a value in $\mathbb{Q}_p$. Since region of logarithm function for $1 + p\mathbb{Z}_p$ is $p\mathbb{Z}_p$, $\omega_p$ is an element of $p\mathbb{Z}_p$. Due to the properties of ideal, $\eta_a(m,n) := k_a n - \omega_a m + \eta^0_a$ is in a domain of exponential function in $p$-adic space. Further $p$-adic version of $A$ in (7) also can be computed. Hence $p$-adic version $\tau$ in (6) and $c$ has a finite value in $p$-adic space. In other words, one- and two-soliton solutions exist in (10).

For the case of $p = 2$, since $4\mathbb{Z}_2$ is domain of exponential function, $\delta_2$, $k_a$ and $\eta^0_a (a = 1, 2)$ belong to $4\mathbb{Z}_2$. Further though $k_a$ must also be satisfied with $k_1 \pm k_2 \in 8\mathbb{Z}_2$, we can argue it in similar way.

Further similarly we can construct other soliton solutions for $p$-adic equation (10) following the procedure in [H, DJM1-3, TTMS].

Hence it is very surprising that (10) does not has only formal meaning but also mathematical meanings in the $p$-adic space. In other words, they has $n$-soliton solutions. We emphasize that the fact that even in $p$-adic space, the difference-difference equation has a soliton solution stands upon a very subtle balance and is absolutely different from trivial phenomena.

Due to the properties of $p$-adic space, there are positive integers $n$ and $m$ such that these solutions module $p^n$ are satisfied with (10) module $p^n$. For example, $p \gg 1$ and $n = 3$ case: $\omega_a \equiv 2\delta_p k_a - 4\delta^2_p$ module $p^3$ and $A \equiv \frac{k_1 - k_2}{k_1 + k_2} \left(1 + \frac{1}{24}(k_1^2 + k_2^2)\right) \text{ modulo } p^3$. Since $\exp(p) \equiv 1 + p + \frac{1}{2}p^2$ modulo $p^3$, $\tau$'s and $c$'s are determined in $p^3$.

As the $p$-adic difference-difference Lotka-Volterra equation is well-defined, we will consider the $p$-adic valuation of the equation. By letting $f^{m}_n := -\text{ord}_p (c^m_n)$ and $d_p := -\text{ord}_p (\delta_p)$, we have

$$f^{m+1}_n - f^n_m = \text{ord}_p (1 + \delta_p c^m_{n-1}) - \text{ord}_p (1 + \delta_p c^m_{n-1}).$$

When we assume that $f^n_m \neq -d_p$, (11) becomes

$$f^{m+1}_n - f^n_m = \max(0, f^n_{m-1} + d_p) - \max(0, f^{m+1}_{n+1} + d_p).$$

It is also surprising that (12) has the same form as the ultra-discrete difference-difference Lotka-Volterra equation (9) at all.

In other words, we conclude that the structure of ultra-discrete limit has the same as that in $p$-adic analysis.

**Ultra-Discrete Metric From Point of View of Valuation Theory**

As we saw the similarity between ultra valuation and $p$-adic valuation, we will construct the ultra metric following the definition of $p$-adic metric.

Since soliton theory is defined over the field whose characteristic is zero, we might regard it as theory of $\mathbb{Q}_\infty$. However it should be also noted that since the ultra-valuation is a natural non-archimedean valuation of real valued functions, another real valued metric is naturally defined, which seem to slightly differ from the ordinary metric $|x|_\infty \equiv |x|$.
By introducing a real number $\bar{\beta} \gg 1$, it is defined as

$$|x|_\beta := (e^{-\bar{\beta}})^{\ord_\beta(x)} ,$$

which is a kind of exponential valuation [I]. We call this ultra-metric. It has properties (II$_\beta$);

\[ \text{II}_\beta. \]

For $u, v \in \mathcal{A}_\beta$, ($|v|_\beta$ depends upon $\bar{\beta}$)

1. if $|v|_\beta = 0$, $v=0$.
2. $|v|_\beta \geq 0$.
3. $|vu|_\beta = |v|_\beta |u|_\beta$.
4. $|u + v|_\beta \leq |u|_\beta + |v|_\beta$.

Since the ultra-discrete and the $p$-adic valuation are given by for $u \in \mathcal{A}_\beta$ and $v \in \mathbb{Q}_p$, ($u \neq 0$, $v \neq 0$),

$$\ord_\beta(u) = \lim_{\beta \to +\infty} \log_{e^{-\beta}}(u), \quad \text{and} \quad \ord_p(v) = \log_p([v]_p),$$

$e^{-\beta}$ plays the same role of $p$. However it should be noted that since this ultra valuation is defined in $\mathbb{R}$, $|x|_\beta$ is defined above rather than $(e^{-\bar{\beta}})^{-\ord_\beta(x)}$ whereas $|x|_p = p^{-\ord_p x}$. As we have assumed that $x \in \mathcal{A}_\beta$ has a value at $\beta \to \infty$,

$$|x|_{\beta \to \infty} \sim \exp(-\bar{\beta}(-(\log x)/\beta))|_{\beta \to \infty} = |x|^{\bar{\beta}/\beta}|_{\beta \to \infty},$$

and thus, as it is not guaranteed, it may be regarded that $|x|_\beta \sim |x|$, in heart, by synchronizing $\bar{\beta}$ and $\beta$. It implies that the ultra-metric $|x|_\beta$ might be also considered as the natural metric at $\mathbb{Q}_\infty$.

In this metric, the convergence condition of series is also equivalent with the vanishing condition of sequence and we have the relation,

$$\left| \sum_m x_m \right|_{\beta \to \infty} = e^{-\bar{\beta}\min(\ord_\beta(x_m))} . \quad (13)$$

We should note that this metric appears in the low temperature treatment of the statistical physics and in the semi-classical treatment of path integral [D, FH]. For the low temperature limit $\bar{\beta} \sim \beta = 1/T, T \to 0$ or the classical limit of deformation parameter $\bar{\beta} \sim \beta = 1/\hbar, \hbar \to 0$, only the minimal point survives and contributes to zero temperature or classical phenomena. Thus the ultra-discrete limit is sometimes called "quantization", as a terminology of discretization in digital picture, in the literature in the soliton theory but it should be regarded as low temperature limit of statistical mechanical phenomena or classical limit of quantum phenomena. The reason why the domain of $\mathcal{A}_\beta$ is non-negative might be related to the positiveness of the probability.

Then there arises a question why the ultra-discrete limit is related to integer valued solutions for a soliton solution. Function form of finite type solution of (1) including soliton solution is completely determined at the infinity point of the spectral parameters $k = \infty$ [DJM1, K]. The soliton solution is given by exponential function whose power is polynomial of $(k, s, t)$ owing to algebraic properties of soliton solutions. Since polynomial of integer valued $(k, s, t)$ is also integer, ultra-discrete is associated with integer valued solutions.

Further it is known that some of properties in the $q$ analysis can be regarded as those in $p$-adic analysis by setting $q = 1/p$ [VVZ]. We have correspondence among $p$, $q$ and $e^\beta$ as

$$e^{-\beta} \leftrightarrow p (\beta \sim \infty), \quad p \leftrightarrow 1/q, \quad q \leftrightarrow e^\beta (\beta \sim 0).$$

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From quantum mechanical point of view, it must be emphasized that the classical regime appears as a non-archimedean valuation, which is an algebraic manipulation, for quantum mechanical values. In this analogy, we might regard that $\mathbb{Z}$ is in a classical regime whereas $\mathbb{Q}_p$’s ($p \in \mathbb{N}$) are of quantum world in number theory. In fact, Bost and Connes applied the method in quantum statistical field theory (or methods in type III factor of non-commutative ring) to the problem of number theory and proposed a strategy to Riemannian hypothesis of $\zeta$-function [BC,Co].

Even though the non-degenerated hyperelliptic functions might not be related to the discrete Lotka-Volterra equation (3), the discrete Lotka-Volterra equation are related to invariant theory as their conserved quantities [HT]. Thus this $p$-adic approach might give another aspect on relations between soliton and number theory besides [IMO, O, P]. Further as mentioned in the introduction, $p$-adic analysis is closely connected with quantum and statistical physics [BC, Co, BF, RTV, VVZ]. In fact, due to their translation symmetry, the Green functions (or the correlation functions) in the quantum and statistical physics usually have the properties of Toeplitz matrix and their product are given by a convolution. A difference-difference soliton equation, in general, can be regarded as an identity of Toeplitz determinant [So]. On the other hand, the Heck algebra in number theory is defined using a convolution in the adelic space. Thus it is natural to expect that they should be written in single framework. Accordingly we hope that the correspondence between $p$-adic and ultra-discrete structures might have an effect on these and other studies on physics.

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References

[B] H.F. Baker, On a system of differential equations leading to periodic functions, Acta Math. 27 (1903), 135-156.
[BC] J.-B. Bost and A. Connes, Hecke Algebras, Type III Factors and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory, Selecta Math. New Series 1 (1995), 411-457.
[BF] L. Brekke and P. G. O. Freund, p-adic numbers in physics, Phys. Report 233 (1993), 1-66.
[BV] N.L. Blaza and A. Voros, Chos on the Pseudosphere, Phys. Report 143, 109-240.
[C] J.W.S. Cassels, Lectures on Elliptic Curves, Cambridge Univ. Press, Cambridge, 1991.
[Co] A. Connes, Formule de trace en géométrie non-commutative et hypothèse de Riemann, C. R. Acad. Sci. Paris 323 (1996), 1231-1236.
[D] P. A. M. Dirac, The Principles of Quantum Mechanics, forth edition, Oxford, Oxford, 1958.
[DJM1] E. Date, M. Jimbo and T. Miwa, Methods for Generating Discrete Soliton Equations I, J. Phys. Soc. Jpn. 51 (1982), 4116-4124.
[DJM2] , Methods for Generating Discrete Soliton Equations II, J. Phys. Soc. Jpn. 51 (1982), 4125-4131.
[DJM3] , Methods for Generating Discrete Soliton Equations III, J. Phys. Soc. Jpn. 52 (1983), 388-393.
[DH] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integral, McGraw-Hill, Auckland, 1965.
[H] R. Hirota, Nonlinear Partial Difference Equations. I. A Difference Analogue of the Koreweg-de Vries Equation, J. Phys. Soc. Jpn. 43 (1977), 1424-1433.
[HT] R. Hirota and S. Tsujimoto, Conserved Quantities of a Class of Nonlinear Difference-Difference Equations, J. Phys. Soc. Jpn. 64 (1995), 3125-3127.
[I] K. Iwasawa, Algebraic Function Theory, Iwanami, Tokyo, 1952. (japanese)
[IMO] S. Ishiwata, S. Matsutani and Y. Ônishi, Localized State of Hard Core Chain and Cyclotomic Polynomial: Hard Core Limit of Diatomic Toda Lattice, Phys. Lett. A 231 (1997), 208-216.
[K] I. M. Krichever, Methods of Algebraic Geometry in the Theory of Non-linear Equations, Russian Math. Surveys 32 (1977), 185-213.
[M] M. Mulase, Cohomological Structure in Soliton Equations and Jacobian Varieties, J. Diff. Geom. (1984), 403-430.
[MR] J.J. Morales-Ruiz and C. Simó, Picard-Vessiot theory and Ziglin’s theorem, J. Diff. Eq. 107 (1994), 140-162.
[O] Y. Ônishi, Complex multiplication formulae for curves of genus three, Tokyo J. Math. 21 (1998), 381-431.
[P] M. Pigli, Adelic Integrable Systems, J. Math. Phys. 36 (1995), 6829-6845.
[RTV] R. Rammal, G. Toulouse and M. A. Virasoro, Ultrametricity for physicists, Rev. Mod. Phys. 58 (1986), 765-788.
[So] K. Sogo, Time-Dependent Orthogonal Polynomials and Theory of Soliton - Applications to Matrix Model, Vertex Model and Level Statistics, J. Phys. Soc. Jpn 62 (1993), 1887-1894.
[Su] K. Sunada, Laplacian to Kihongun (Laplacian and fundamental group), Kinokuniya, Tokyo, 1985. (Japanese)
[T] D. Takahashi, Ultra-discrete Toda Lattice Equation -A Grandchild of Toda-, International Symposium, Advances in soliton theory and its applications: The 30th anniversary of the Toda lattice (1996), 36-37.
[TS] D. Takahashi and J. Satsuma, A Soliton Cellular Automaton, J. Phys. Soc. Jpn. 59 (1990), 3514-3519.
[TTMS] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure, Phys.Rev.Lett. 76 (1996), 3427-3250.
[VVZ] V.S.Vladimirov, I.V.Volovich and E.I.Zelenov, P-adic Analysis and Mathematical Physics, World Scientific, Singapore, 1994.