A STATISTICAL THEORY FOR THE ANALYSIS OF UNCERTAIN SYSTEMS

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Abstract. This paper addresses the issues of conservativeness and computational complexity of probabilistic robustness analysis. We solve both issues by defining a new sampling strategy and robustness measure. The new measure is shown to be much less conservative than the existing one. The new sampling strategy enables the definition of efficient hierarchical sample reuse algorithms that reduce significantly the computational complexity and make it independent of the dimension of the uncertainty space. Moreover, we show that there exists a one to one correspondence between the new and the existing robustness measures and provide a computationally simple algorithm to derive one from the other.

1. Introduction

Robustness analysis is used to predict if a system will perform satisfactorily in the presence of uncertainties. It is generally accepted as an essential step in the design of high-performance control systems. In practice, the analysis has to be very efficient because it has to use models as realistic as possible and, usually, it takes many cycles of analysis-design to come up with a satisfactory controller. The outcome of the robustness analysis should allow the designer not only to evaluate the robust performance of a controller, but also to compare various controllers in order to obtain the best control strategy. Needless to say, unnecessary conservativeness prevents a realistic analysis.

Aimed at overcoming the computational complexity and conservatism of the classical deterministic worst-case approach, there are growing interests in developing probabilistic methods and randomized algorithms (see, [1]-[6], [11]-[15] and the references therein). Specially, a probabilistic robustness measure, referred to as the confidence degradation function or robustness function is proposed in [3]. Such robustness measure has been demonstrated to be much superior than the classical deterministic robustness margin in terms of conservatism, computational complexity and generality of application.

The computation of the robustness function using Monte Carlo simulations requires uniform sampling from bounding sets in the uncertainty space, which can reach high dimensions very quickly; for example if the uncertainty is modelled by a $5 \times 5$ complex-valued matrix then the dimension of the uncertainty space is 50. We will show here that such sampling suffers from what we term surface effect and may introduce undue conservativeness in the evaluation of system robustness. We address this conservativeness with a new sampling technique and a new probabilistic robustness measure that is significantly less conservative. Moreover, with a suitable computing structure it can be evaluated for arbitrarily dense gridding of uncertainty radius with a computational complexity that is very low and is independent of the dimension of uncertainty.

We shall use the following notation throughout this paper. The uncertainty is denoted as boldface $\Delta$ and its realization is denoted as $\Delta$. The probability density function of $\Delta$ is denoted as $f_\Delta$. We measure the
size of uncertainty by a function $||.||$ which has the scalable property that $||\rho \Delta|| = \rho ||\Delta||$ for any uncertainty instance $\Delta$ and any $\rho > 0$. Obviously, the most frequently used $H_{\infty}$ or $l_p$ norm of uncertainty possesses such scalable property. The uncertainty bounding set of radius $r$ is denoted as $\mathcal{B}_r = \{ \Delta : ||\Delta|| \leq r \}$. We use $\partial \mathcal{B}_r$ to denote $\{ \Delta : ||\Delta|| = r \}$. Specially, $\mathcal{B}$ denotes $\{ \Delta : ||\Delta|| \leq 1 \}$ and $\partial \mathcal{B}$ denotes $\{ \Delta : ||\Delta|| = 1 \}$.

For a subset $S_r$ of $\partial \mathcal{B}_r$, its “area” is defined as

$$\text{area}(S_r) = \lim_{\varepsilon_1 \downarrow 0, \varepsilon_2 \uparrow 0} \frac{\int_{\varepsilon_1 \leq r \leq r + \varepsilon_2, \Delta \in S_r} dq}{\varepsilon_1 + \varepsilon_2}$$

where “$\int$” denotes the multivariate Lebesgue integration and the down arrow “$\downarrow$” means “decreases to”.

The indicator function $\mathbb{I}(.)$ means that $\mathbb{I}(\Delta) = 1$ if the robustness requirement is guaranteed for $\Delta$ and $\mathbb{I}(\Delta) = 0$ otherwise. The probability of an event is denoted as $\text{Pr}\{ . \}$. The conditional probability is denoted as $\text{Pr}\{ . \mid . \}$. The set of complex number is denoted as $\mathbb{C}$. The set of real matrices of size $m \times p$ is denoted as $\mathbb{R}^{m \times p}$. The set of complex matrices of size $m \times p$ is denoted as $\mathbb{C}^{m \times p}$. The real and complex parts of a number is denoted as $\Re(.)$ and $\Im(.)$ respectively. The largest and the second largest singular values of a matrix are denoted as $\sigma_1(.)$ and $\sigma_2(.)$ respectively. The ceiling function is denoted as $[.]$ and the floor function is denoted as $\lfloor . \rfloor$.

1.1. The Surface Effect of Uniform Sampling. In order to illustrate the surface effect, consider a uniform sampling extracting samples from the uncertainty set $\mathcal{B}_r$. Let $E_\rho$ denote the event that a sample chosen uniformly from $\mathcal{B}_r$ lies outside the bounding set $\mathcal{B}_\rho$ of radius $\rho < r$. Under the assumption of uniform distribution it is easy to see that such event will have the probability $\text{Pr}\{ E_\rho \} = 1 - \left( \frac{\rho}{r} \right)^d$ where $d$ is the dimension of uncertainty. As $d$ increases this probability approaches one for all $\rho < r$. For example when $\frac{\rho}{r} = 0.9$ and $d = 50$ then $\text{Pr}\{ E_\rho \} = 0.9948$. Hence out of 1000 samples extracted uniformly from the bounding set of radius $r$ one would expect that about 995 will be outside the bounding set with radius $\rho = 0.9r$. If the uncertainty is well modeled one can reasonably assume that large uncertainties are less likely than small ones and we are faced with the fact that the uniform sampling selects cases that are not indicative of the actual situation but present a very unfavorable picture. In Section 2 we discuss in detail the modeling of uncertainties and show that uniform sampling can give a very conservative evaluation of system robustness. In Section 3 we introduce a new sampling technique and a new robustness measure which overcomes the conservativeness issue. Section 4 establishes a one to one mapping between our measure and the existing one and considers other capabilities of the new robustness function. The detail algorithms are presented in Section 5. Section 6 addresses the issue of computational complexity for the evaluation of robustness function. In particular we show that by using a special type of hierarchial data structure it is possible to design computational algorithms that have a complexity that is independent of the dimension of the uncertainty. The proofs of theorems are given in the Appendix.

2. Modeling Uncertainty

In this section, we shall discuss the characteristics of uncertainty from the perspective of modelling practices.

Consider an uncertain system shown in Figure 1. In control engineering, one usually takes into account all possible directional information about the uncertainty by introducing weighting matrices and absorbing it into the generalized plant $P$. Therefore, it is reasonable to assume that the uncertainty $\Delta$ is radially symmetrical in distribution in the sense that, for any $r > 0$ and any $S_r \subseteq \{ \Delta : ||\Delta|| = r \}$,

$$\text{Pr}\{ \Delta \in S_r \mid ||\Delta|| = r \} = \frac{\text{area}(S_r)}{\text{area}(\partial \mathcal{B})}$$
if \( f_{||\Delta||}(.) \) is continuous at \( r \), where the conditional probability in the left hand side is defined as

\[
\lim_{\epsilon_1, \epsilon_2 \downarrow 0} \frac{\Pr\{\Delta \in \{\Delta \in S_\rho : r - \epsilon_1 \leq \rho \leq r + \epsilon_2\}\}}{\Pr\{r - \epsilon_1 \leq ||\Delta|| \leq r + \epsilon_2\}}.
\]

On the other hand, one usually attempts to make the magnitude of modelling error, measured by \( ||\Delta|| \), as small as possible. Due to the effort to minimize \( ||\Delta|| \) in modelling, it is reasonable to assume that small modelling error is more likely than large modelling error. This gives rise to the rationale of treating \( ||\Delta|| \) as a random variable such that its density, \( f_{||\Delta||}(r) = \frac{d\Pr\{||\Delta|| \leq r\}}{dr} \), is non-increasing with respect to \( r \). In the sequel, we shall use \( \mathcal{F} \) to denote the family of radially symmetrical and non-increasing density function \( f_{||\Delta||} \).

It should be noted that a wider class of probability density functions, denoted by \( \mathcal{G} \), has been proposed in [3] to model uncertainty. Such family \( \mathcal{G} \) consists of radially symmetrical density function \( f_{\Delta} \) that is non-increasing in the sense that \( f_{\Delta}(\Delta_1) \leq f_{\Delta}(\Delta_2) \) if \( ||\Delta_1|| \geq ||\Delta_2|| \). It can be shown that \( \mathcal{G} \) is a superset of \( \mathcal{F} \), i.e., \( \mathcal{G} \supseteq \mathcal{F} \) (see Lemma 2 in Appendix A).

### 2.1. Existing Robustness Function

The existing robustness function, proposed in [3], is given by

\[
\overline{\mathbb{P}}(r) \overset{\text{def}}{=} \inf_{\rho \in [0, r]} \mathbb{P}(ho)
\]

with

\[
\mathbb{P}(r) = \Pr\{I(\Delta^u) = 1\}
\]

where \( \Delta^u \) is uniformly distributed over \( \mathcal{B}_r \). It has been shown in [3] that \( \overline{\mathbb{P}}(r) \) is a lower bound of the probability of guaranteeing the robustness requirement if the density of uncertainty belongs to \( \mathcal{G} \) and the uncertainty is bounded in \( \mathcal{B}_r \).

An attracting feature of the existing robustness function is that it relies on very mild assumptions about uncertainty. However, as can be seen from Theorem 6.1 (in page 856) of [3], the associated computational complexity can be very high for large uncertainty dimension. Another issue of the existing measure is that it can be very conservative from the perspective of modelling practices. For illustration of this point, we consider a conceptual example as follows.

Suppose it is known that the norm of uncertainty \( ||\Delta|| \) cannot exceed \( \gamma \). Without loss of generality, assume \( \gamma = 1 \). That is, all instances of \( \Delta \) are included in the bounding set \( \mathcal{B} = \{\Delta : ||\Delta|| < 1\} \). We partition \( \mathcal{B} \) as \( m \) layers \( \mathcal{S}_\ell = \{\Delta : r_{\ell-1} \leq ||\Delta|| < r_\ell\}, \ell = 1, 2, \cdots, m \) by radii \( r_\ell = \frac{\gamma}{m}, \ell = 0, 1, \cdots, m \). From the consideration of modelling practices, it is reasonable to assume that the density of uncertainty \( \Delta \) belongs to \( \mathcal{F} \). Hence, for sufficiently large \( m \), we have \( \Pr\{\Delta \in \mathcal{S}_\ell\} \geq \Pr\{\Delta \in \mathcal{S}_{\ell+1}\}, \ell = 1, 2, \cdots, m - 1 \). In reality, it is not impossible that not only the outer layers are “bad” and some inner layer is also “bad”. Such scenario is described as follows:

The robustness requirement is violated for \( \Delta \in \mathcal{S}_i \) and for \( \Delta \in \mathcal{S}_\ell, \ell = j, j + 1, \cdots, m \) where \( i \) and \( j \) are integers such that \( 2 \leq i + 1 < j < m \). See Figure 2 for an illustration. Let \( d \) be the dimension of uncertainty.
Figure 2. Conceptual Example (The robustness requirement is violated for red layers and is satisfied for green layers. Existing robustness measure tends to completely ignore uncertainty instances in the inner layers as $d$ increases. Based on the existing robustness measure, a very thin bad layer may lead to an unrealistic judgement that the system has very poor robustness. However, the instances in the inner layers are more probably to occur in reality. Hence, they should have at least equal impact on the evaluation of system robustness as compared to the instances in the outer layer.)

space. By direct computation, we obtain the existing robustness function as $P(r) = \inf_{\rho \in [0, r]} P(\rho)$ where

$$P(\rho) = \begin{cases} 1, & \text{for } \rho < r_{i-1}; \\
\frac{(i-1)^d}{(m_\rho)^d}, & \text{for } r_{i-1} \leq \rho < r_i; \\
\frac{(i-1)^d}{(m_\rho)^d} + \frac{(i-1)^d}{(m_\rho)^d}, & \text{for } r_i \leq \rho < r_{j-1}; \\
\frac{(i-1)^d}{(m_\rho)^d} + \frac{(i-1)^d}{(m_\rho)^d}, & \text{for } r_{j-1} \leq \rho < 1. 
\end{cases}$$

Clearly, $\lim_{d \to \infty} P(r) = 1$ for $r < r_{i-1}$ and $\lim_{d \to \infty} P(r) = 0$ for $r_{i-1} \leq r < 1$. This indicates that the existing robustness function tends to be a discontinuous function as $d$ increases. An undesirable feature of existing measure resulted from such discontinuity is that a very small variation in the knowledge of the uncertainty bound, $\gamma$, may lead to an opposite evaluation of the system robustness.

For practical systems, large uncertainty instance is less probably while the robustness requirement is more likely to be violated for larger uncertainty instance. Consequently, unduly conservatism may be introduced if the uncertainty instances near the surface of uncertainty bounding sets assume a dominant role. This is indeed the case for the existing probabilistic robustness measure. This can be illustrated as follows. Suppose $Pr(||\Delta|| < \gamma) = 1$. For the existing measure, the corresponding density of $||\Delta||$ of the sampling distribution that determines $P(\gamma)$ is often times close to $f_{||\Delta||}(r) = d \left(\frac{r}{\rho^*}\right)^{d-1}$ where $\rho^* = \max\{\rho : P(\rho) = P(\gamma)\}$. For $\rho \approx \rho^*$, the probability that a sample falls into $\{\Delta : \rho < ||\Delta|| \leq \rho^*\}$ is $1 - \left(\frac{\rho}{\rho^*}\right)^d$ which is very close to 1 when the dimension $d$ is high. This shows that the uncertainty instances near the surface of $B_{\rho^*}$ are dominating in the evaluation of system robustness.

3. New Sampling Technique and Robustness Function

We have shown before that uniform sampling in high dimensional sets suffers from a surface effect. In the following we introduce a new sampling technique that offsets such effect and we use the modified sampling technique to define the new robustness measure.

3.0.1. A New Sampling Technique. To offset the surface effect for uncertainties with radial symmetry we define two independent random variables. One, $U$ is uniformly distributed in the surface of the unit bounding set, $\{\Delta : ||\Delta|| = 1\}$, in the uncertainty space. The second random variable is $R$ which is a scalar
variable uniformly distributed over \([0, r]\). Clearly, for a given value of the scalar random variable \(R\), the uncertainties lay on the surface of a ball and since \(R\) is scalar the surface effect is reduced.

3.0.2. A New Robustness Function. Now that have established the sampling technique to be used, we define the robustness measure for the radius \(r\) as

\[
P(r) = \inf_{\rho \in (0, r]} P(\rho) \quad \text{with} \quad P(r) = \Pr\{I(UR) = 1\}
\]

where \(U\) is a sample from \(U\) and \(R\) a sample from \(R\). The probabilistic implication of such robustness measure can be seen from the following theorem.

**Theorem 1.** For any robustness requirement,

\[
\inf_{\Delta \in \mathcal{F}} \Pr\{I(\Delta) = 1 \mid ||\Delta|| \leq \gamma\} = P(\gamma) \geq P(\gamma).
\]

See Appendix A for a proof. The intuition behind Theorem 1 is that, in the worst-case, the uncertainty instances in the inner layers should assume equal importance as that of uncertainty instances in the outer layers in the evaluation of system robustness. It should be noted that the density \(f_{||\Delta||}(\cdot)\) can be unbounded and has infinitely many and arbitrarily distributed discontinuities. An example of unbounded density is

\[
f_{||\Delta||}(\rho) = k - 1 \rho^k, \quad k > 1.
\]

Now we revisit the conceptual example discussed in Section 2.1. Our robustness function is

\[
P(r) = \inf_{\rho \in [0, r]} P(\rho)
\]

where

\[
\begin{align*}
P(\rho) &= \begin{cases} 
1, & \text{for } \rho < r_{i-1}; \\
i-1 \rho^{i-1}, & \text{for } r_{i-1} \leq \rho < r_i; \\
m^{i-1} \rho^{i-1}, & \text{for } r_i \leq \rho < r_{j-1}; \\
j-2 \rho^{j-2}, & \text{for } r_{j-1} \leq \rho < 1.
\end{cases}
\end{align*}
\]

As can be seen from Figure 3, our robustness measure is significantly less conservative than the existing one.

4. Mapping of Robustness Functions

In this section, we shall demonstrate that there exists a fundamental relationship between our robustness measure and the existing probabilistic robustness measure. This relationship can be exploited, for example, to reduce the computational complexity of existing probabilistic robustness measure.

4.1. Integral Transforms. The following theorem shows that there exists an integral transform between our proposed robustness function and existing robustness function.

**Theorem 2.** Define \(\phi(r) = \Pr\{I(rU) = 1\}\) where \(U\) is a random variable uniformly distributed over \(\{\Delta : ||\Delta|| = 1\}\). Suppose that the distribution of uncertainty \(\Delta\) is radially symmetrical and that both \(f_{||\Delta||}(\cdot)\) and \(\phi(\cdot)\) are piece-wise continuous. Then, for any \(r > 0\),

\[
P(r) = \frac{P(r)}{n} + \frac{n-1}{n} \int_{0}^{1} P(rp) \, dp,
\]

\[
P(r) = n \, P(r) - n(n-1) \int_{0}^{1} P(rp) \, \rho^{n-1} \, dp
\]

where \(n\) is the dimension of uncertainty space.

See Appendix B for a proof. Theorem 2 shows that once one of \(P(\cdot)\) and \(P(\cdot)\) is available from Monte Carlo simulation, the other can be obtained without simulation.
I recursively because of the relationship numerical problem for computing the product $n$

Clearly, the major computation is on the integration over $n$

For a transform to be useful, we shall develop efficient method for its computation. The efficiency can be achieved by recursive computation. We first discuss the computation of transform from $\mathcal{P}(\cdot)$ to $\mathbb{P}(\cdot)$.

It can be seen that the expression of $\mathbb{P}(\cdot)$ in terms of $\mathcal{P}(\cdot)$ is not amenable for recursive computation. By a change of variable, we rewrite the second equation of Theorem 2 as $\mathbb{P}(r) = n \mathcal{P}(r) - \frac{n(n-1)}{(r+h)^n} \int_0^r \mathcal{P}(\rho) \rho^{n-1} d\rho$. Clearly, the major computation is on the integration $I(r) = \int_0^r \mathcal{P}(\rho) \rho^{n-1} d\rho$, which can be computed recursively because of the relationship $I(r+h) = I(r) + \int_r^{r+h} \mathcal{P}(\rho) \rho^{n-1} d\rho$. Unfortunately, there will be a numerical problem for computing the product $\frac{n(n-1)}{(r+h)^n} \times I(r)$ in the situation that $n$ is large and $r < 1$. For example, $\frac{n(n-1)}{(r+h)^n}$ can be a huge number and cause intolerable numerical error when $n = 36$ and $r = 0.5$. To overcome this problem, we derive the following recursive relationship

$$\mathbb{P}(r+h) = n \mathbb{P}(r+h) - \left( \frac{r}{r+h} \right)^n [n\mathcal{P}(r) - \mathbb{P}(r)] - \frac{n(n-1)}{(r+h)^n} \int_r^{r+h} \mathcal{P}(\rho) \rho^{n-1} d\rho.$$

Since $\mathcal{P}(\cdot)$ can be approximated by a simple function, we can decompose $\frac{n(n-1)}{(r+h)^n} \int_r^{r+h} \mathcal{P}(\rho) \rho^{n-1} d\rho$ as a summation of integrations of the form $\frac{n(n-1)}{(r+h)^n} \int_{r}^{\alpha} \mathcal{P}(\rho) \rho^{n-1} d\rho$ with $\mathcal{P}(\rho) = c$, $\forall \rho \in [\alpha, \beta]$. Clearly, we have the explicit formula $\frac{n(n-1)}{(r+h)^n} \int_{\alpha}^{\beta} \mathcal{P}(\rho) \rho^{n-1} d\rho = (n-1)c \left( \frac{\alpha}{\beta} \right)^{n-1} \left[ \left( \frac{\alpha}{\beta} \right)^n - 1 \right]$.

In a similar manner, $\mathcal{P}(\cdot)$ can be computed recursively by relationship

$$\mathcal{P}(r+h) = \frac{\mathbb{P}(r+h)}{n} + \frac{r}{r+h} \left[ \mathcal{P}(r) - \frac{\mathbb{P}(r)}{n} \right] + \frac{n-1}{n} \int_r^{r+h} \mathcal{P}(\rho) d\rho.$$

5. Computational Algorithms and Hierarchical Sample Reuse

In this section we shall discuss the evaluation of $\mathcal{P}(\cdot)$ for uncertainty radius $[\frac{r_1}{m}, a]$ with sample size $N$ and $m$ grid points $\frac{r_1}{m} = r_1 < \cdots < r_m = a$. First, we shall introduce basic subroutines. Second, we present
The basic idea of our algorithms is as follows. Let \( U^k, k = 1, \ldots, N \) be \( N \) i.i.d. samples uniformly generated from \( \{ \Delta : ||\Delta|| = 1 \} \). For \( i = 1, \ldots, m \), we can estimate \( \mathcal{S}(r_i) \) as \( \sum_{k=1}^{N} \frac{I(\Delta^k,i)}{N} \) with \( \Delta^k,i = U^k R^k,i \) where \( R^k,i \) is uniformly distributed over \([0, r_i]\) and is independent of \( U^k \) for \( k = 1, \ldots, N \). It should be noted that \( R^k,i, i = 1, \ldots, m \) are not necessarily mutually independent to ensure that \( \Delta^k,i, k = 1, \ldots, N \) are i.i.d samples. Due to the uniform distribution of \( R^k,i \), sample reuse techniques can be employed to save a substantial amount of computation for the generation of \( R^k,i \), \( \Delta^k,i \) and the evaluation of \( I(\Delta^k,i) \) in the following manner. Let \( k \) be fixed. Let \( R \) be a sample uniformly generated from interval \( [0, r_p] \). Then, for any index \( j \) such that \( r_j \in [R, r_p] \), we can use \( R \) as \( R^k,j \), \( U^k R \) as \( \Delta^k,j \), and \( I(U^k R) \) as \( I(\Delta^k,j) \). It can be shown that the minimum index \( j \) can be computed by explicit formula \( (5.1) \) as

\[
j = \begin{cases} 
1 + \max \left( 0, \left\lfloor \frac{(m - 1) \ln a}{\ln \lambda} \right\rfloor \right) & \text{for uniform gridding;} \\
1 + \max \left( 0, \left\lfloor \frac{m - 1}{m \ln \lambda} \left(1 + \frac{\ln R}{\ln a}\right)\right\rfloor \right) & \text{for geometric gridding}
\end{cases}
\]

where “uniform gridding” means that \( \frac{r_i}{r_{i-1}} \) is the same for \( i = 2, \ldots, m \) and “geometric gridding” means that \( \frac{r_i}{r_{i-1}} \) is the same for \( i = 2, \ldots, m \).

For a specific \( k \), the sample \( U^k \) is referred to as a directional sample and the simulation with sample reuse techniques to obtain \( I(\Delta^k,i), i = 1, \ldots, m \) is referred to as “Radial Sampling”. Clearly, \( I(\Delta^k,i), i = 1, \ldots, m \) can be expressed as a matrix \( D \) of 3 columns and random number of rows such that its \( i \)-th row \([D_{i1}, D_{i2}, D_{i3}]\) means that

\[
I(\Delta^k,i) = \begin{cases} 
1 & \text{if } D_{i3} = 1; \\
0 & \text{if } D_{i3} = 0
\end{cases}
\]

for \( D_{i1} \leq j \leq D_{i2} \). The algorithm of “Radial Sampling” is formally described in Section 5.1.

The process of obtaining the summation \( \sum_{k=1}^{N} I(\Delta^k,i), i = 1, \ldots, m \) is accomplished by the subroutine “Merging”, which is described in Section 5.2.

5.1. Radial Sampling. For a directional sample \( U \), the goal of radial sampling is to create a matrix \( D \). The input of the subroutine “Radial Sampling” is \( U, \lambda, a, m \) and the corresponding output is \( D = RS(U, \lambda, a, m) \). The algorithm is presented as follows.

- Let \( p \leftarrow m \) and do the following.
  - Generate a sample \( R \) uniformly from \([0, r_p]\).
  - Let \( \Delta \leftarrow UR \) and evaluate \( I(\Delta) \).
  - Determine the smallest index \( j \) such that \( r_j \geq R \) by \( (5.1) \).
  - Let \( D \leftarrow [j, p, I(\Delta)] \) and \( s \leftarrow I(\Delta) \).
  - Let \( p \leftarrow j - 1 \).
- While \( p > 0 \) do the following.
  - Generate a sample \( R \) uniformly from \([0, r_p]\).
  - Let \( \Delta \leftarrow UR \) and evaluate \( I(\Delta) \).
  - Determine the smallest index \( j \) such that \( r_j \geq R \) by \( (5.1) \).
  - If \( I(\Delta) \neq s \), add \([j, p, I(\Delta)]\) to \( D \) as the first row and let \( s \leftarrow I(\Delta) \). Otherwise, update the first element of the first row of \( D \) as \( j \).
  - Let \( p \leftarrow j - 1 \).
- Return \( D \) as the outcome of radial sampling.
5.2. **Merging.** The operation of merging involves two matrices $D$ and $H$. Matrix $D$ defines a segmented function $f_D(\cdot)$ over domain $\{1, \cdots, m\}$ in the sense that, for the $j$-th row of $D$, $f_D(i) = D_{j1}$ for any $i$ such that $D_{j1} \leq i \leq D_{j2}$. Similarly, matrix $H$ defines a segmented function $f_H(\cdot)$ over domain $\{1, \cdots, m\}$ in the sense that, for the $j$-th row of $H$, $f_H(i) = H_{j1}$ for any $i$ such that $H_{j1} \leq i \leq H_{j2}$. For input matrices $D$ and $H$, the merging operation finds $M = \text{Merge}(D, H)$ such that

\[ f_M(i) = f_D(i) + f_H(i), \quad i = 1, \cdots, m \]

where $f_M(\cdot)$ is a segmented function $f_M(\cdot)$ over domain $\{1, \cdots, m\}$ in the sense that, for the $j$-th row of $M$, $f_M(i) = M_{j1}$ for any $i$ such that $M_{j1} \leq i \leq M_{j2}$.

5.3. **Sequential Sample Reuse Algorithm (SSRA).** The sequential algorithm derives its name from the sequential nature of the data merging process. The input variable is $N, \lambda, a, m$ and the output is a matrix $H$ of random number of rows and 3 columns. The main algorithm is presented as follows.

- Let $k \leftarrow 1$ and do the following.
  - Generate a directional sample $U$.
  - Perform radial sampling and let $D \leftarrow \text{RS}(U, \lambda, a, m)$.
  - Let $H \leftarrow D$.
- While $k < N$ do the following.
  - Generate a directional sample $U$.
  - Perform radial sampling and let $D \leftarrow \text{RS}(U, \lambda, a, m)$.
  - Perform merging and let $H \leftarrow \text{Merge}(D, H)$.
  - Let $k \leftarrow k + 1$.
- Return $H$.

Once we have $H$ from the execution of SSRA, we can estimate $\mathcal{P}(r_i)$ as $\sum_{k=1}^{N} \frac{H_{k,i}}{N} = \frac{f_H(i)}{N}$, $i = 1, \cdots, m$.

5.4. **Hierarchy Sample Reuse Algorithm (HSRA).** A major problem with the sequential algorithm is that the computational effort devoted to merging becomes an enormous burden as the sample size $N$ becomes large.

The merging time for $N = 1000, 5000, 10000$ and $50000$ are respectively 4, 120, 722 and 92119 seconds, which is obtained by simulation on a PC of 1024M RAM and 3.2G CPU. As can be seen from Figure 4, the merging time required for $N = 10^5, 10^6$ and $N = 5 \times 10^6$ is predicted respectively as, 12 days, 366 years, and $9 \times 10^5$ years, by fitting the simulation data into a quadratic function (in log scale) based on regression techniques. For a better understanding of the complexity issue, a theoretical analysis of the computational complexity of data merging is as follows.

From the merging process, it can be seen that the computational complexity of merging two matrices can be quantified by the sum of the numbers of the rows of the two input matrices. Thus, it suffices to study how the number of rows is growing when matrices $D^k = \text{RS}(U^k, \lambda, a, m), k = 1, \cdots, N$ are sequentially merged.

Note that the average numbers of rows for all $D^k$ are identical. Let this average be $L$. To merge $D^1$ with $D^2$, the required computation is $2L$. The computation to merge the outcome with $D^3$ is $3L$. The
A statistical theory for the analysis of uncertain systems

Figure 4. Merging Time

Figure 5. Illustration of Successive Binary Merging with \( N = 8 \).

Computation for all steps of merging forms a series, \( 2L, 3L, \ldots, NL \), of constant increment \( L \). Hence, the total number of computation is \( \frac{L(N+2)(N-1)}{2} \). This can be a huge number because \( N \) is usually large.

To overcome the difficulty of sequential algorithm, we propose a merging method of hierarchy structure. We first introduce a subroutine called \textit{successive binary merging} for \( N = 2^p \) data matrices as follows.

Divide these \( N \) matrices \( D_1, \ldots, D_N \) into \( \frac{N}{2} \) groups so that each group has two matrices. After merging each group, we have \( \frac{N}{4} \) matrices. Repeating the operations of dividing and merging, we obtain a matrix in the final stage. This process can be associated with a binary tree as illustrated by Figure 5.

For the general case that \( N \) is not a power of 2, we decompose \( N \) as a summation of numbers which are powers of 2. For example, for \( N = 1000 \), we have \( N = 512 + 256 + 128 + 64 + 32 + 8 \). Such
decomposition corresponds to the decimal-to-binary conversion. In general, for $N = \sum_{\ell=1}^{\tau} N_\ell$ with $N_\ell = 2^{p_\ell}$ and $N_1 < N_2 < \cdots < N_\tau$, the merging can be performed as follows.

1. Let $\ell \leftarrow 1$. Applying successive binary merging to $N_1$ to create data matrix $M_1$. Let $H \leftarrow M_1$.
2. While $\ell < \tau$ do the following.
   - Applying successive binary merging to $N_\ell$ to create data matrix $M_\ell$.
   - Let $H \leftarrow \text{Merge}(H, M_\ell)$.
   - Let $\ell \leftarrow \ell + 1$.

The merging for $N = 1000$ is shown by Figure 6.

The complexity of such hierarchy can be analyzed as follows. For successive binary merging with $N = 2^p$, the computation is $p \times N \log_2 N$. For $N = \sum_{\ell=1}^{\tau} N_\ell$, the computation is bounded by $L \sum_{\ell=1}^{\tau} N_\ell \log_2(N_\ell) + L \sum_{\ell=1}^{\tau} (\tau-\ell+1)N_\ell - LN_1$. Therefore, the computation is reduced from the sequential algorithm by a factor of $\Upsilon = \frac{(N+1)(N-1)}{2 \log_2(N)}$. Specially, for $N = 2^p$, we have $\Upsilon = \frac{(N+1)(N-1)}{2 \log_2(N)} > \frac{N}{2 \log_2(N)}$, which is usually a very large number.

6. Computational Complexity

In this section, we discuss the computational complexity for the evaluation of $\mathcal{P}(.)$ over uncertainty radius interval $([\frac{a}{N}, a])$. For practical designs, the robustness requirement is guaranteed for the nominal model. Hence, $\mathcal{P}(\rho) = 1$ for small $\rho$, and we have $\inf_{\rho \in [\frac{a}{N}, a]} \mathcal{P}(\rho) = \inf_{\rho \in [\frac{a}{N}, a]} \mathcal{P}(\rho)$ for a sufficiently large $\lambda$. A direct Monte Carlo simulation method is to partition the interval $([\frac{a}{N}, a])$ by $m$ grid points $\frac{a}{N} = r_1 < \cdots < r_m = a$ and estimate $\mathcal{P}(r_i)$ by $N$ i.i.d. Monte Carlo simulations. The estimate of $\inf_{\rho \in [\frac{a}{N}, a]} \mathcal{P}(\rho)$ is obtained by taking the minimum of the results for the $m$ grid points. Such direct method requires $mN$ simulations. As $m$ gets large, the computing time and the memory complexity becomes a challenging problem. Fortunately, by employing our hierarchy sample reuse algorithms, the computational complexity is absolutely bounded and very low for arbitrarily dense gridding and arbitrarily large dimension of uncertainty.
For quantifying the computational complexity, we define the equivalent number of grid points, \( m_{eq} \) as the ratio
\[
m_{eq} = \frac{\text{Average total number of simulations}}{N}.
\]
We shall interpolate the value of \( \mathcal{P}(r) \) for \( r \in [r_i, r_{i+1}] \) as
\[
\mathcal{P}^*(r) = \frac{(r - r_i) \mathcal{P}(r_{i+1}) + (r_{i+1} - r) \mathcal{P}(r_i)}{r_{i+1} - r_i}.
\]

For a uniform gridding, we have

**Theorem 3.** Let \( \epsilon \in (0, 1) \) and \( m = 2 + \left[ \frac{2(\lambda - 1)}{\epsilon} \right] \). Let \( r_i = \frac{a}{N} + \frac{(i-1)(a-N)}{m-1} \) for \( i = 1, \cdots, m \). Then,
\[
|\mathcal{P}(r) - \mathcal{P}^*(r)| < \epsilon, \quad \forall r \in [r_i, r_{i+1}]
\]
for \( i = 1, \cdots, m - 1 \). Moreover, \( m_{eq}(\epsilon) = m - \sum_{i=1}^{m-1} \left( 1 - \frac{1}{N(i+1)} \right) \) < 1 + ln \( \lambda \) for any \( \epsilon \in (0, 1) \).

See Appendix C for a proof. For a geometric gridding, we have

**Theorem 4.** Let \( \epsilon \in (0, 1) \) and \( m = 2 + \left[ \frac{\ln N}{\ln(1+\lambda)} \right] \). Let \( r_i = a \left( \frac{1}{\lambda} \right)^{\frac{i}{m-1}} \) for \( i = 1, \cdots, m \). Then,
\[
|\mathcal{P}(r) - \mathcal{P}^*(r)| < \epsilon, \quad \forall r \in [r_i, r_{i+1}]
\]
for \( i = 1, \cdots, m - 1 \). Moreover, \( m_{eq}(\epsilon) = 1 + (m - 1) \left[ 1 - \left( \frac{1}{\lambda} \right)^{\frac{m-1}{m-1}} \right] < 1 + \ln \lambda \) for any \( \epsilon \in (0, 1) \).

See Appendix C for a proof. For completeness, we note that, for arbitrarily large \( m \), the memory complexity is also absolutely bounded and independent of uncertainty dimension.

To compare the computational complexity of our probabilistic measure with that of [3], we recall Theorem 6.1 of [3], which states that if
\[
m \geq 1 + \frac{2(\lambda - 1)d}{\epsilon}
\]
then \( |\mathcal{P}(r) - \mathcal{P}(r_i)| < \epsilon \) \( \forall r \in [r_i, r_{i+1}] \) for \( i = 1, \cdots, m - 1 \). This bound shows that, for fixed error \( \epsilon \), the complexity is polynomial. From another perspective, it also shows that the number of grid points and computational complexity tend to infinity as the tolerance tends to zero. The computational complexity can be reduced by the sample reuse techniques of [3]. It is recently shown in [7] that the equivalent number of grid points is bounded by \( 1 + d \ln \lambda \) (see Appendix C for a proof). In applications, \( d \) can be very large. For example, the dimension \( d \) is \( 2n^2 \) for a complex block of size \( n \times n \). Since the complexity of computing \( \mathcal{P}(.) \) is independent of dimension \( d \), the integral transform can be applied to obtain \( \mathcal{P}(.) \) from \( \mathcal{P}(.) \) and thus significantly reduced the computational complexity.

### 7. Examples

In this section, we shall demonstrate the power of our techniques by examples. By the definition of the indicator function \( \mathbb{I}(.) \), for \( N \) i.i.d. samples \( \Delta_1, \cdots, \Delta_N \) generated from \( B_r \),
\[
\mathbb{I}(\Delta_i) = \begin{cases} 
1 & \text{if the robustness requirement is satisfied for } \Delta_i; \\
0 & \text{otherwise.}
\end{cases}
\]
Specially, for the robustness stability problem in the $M - \Delta$ setup with $M(s) = C(sI - A)^{-1}B$,

$$\|\Delta_i\| = \begin{cases} 1 & \text{if } A + B\Delta_iC \text{ is stable;} \\ 0 & \text{otherwise.} \end{cases}$$

Of course, the $N$ samples are obtained by the HSRA. A minimum variance unbiased estimator of $\mathcal{P}(r)$ is taken as $\hat{\mathcal{P}}(r) = \sum_{i=1}^{N} \|\Delta_i\|$. Since $\|\Delta_i\|$, $i = 1, \cdots, N$ are i.i.d. Bernoulli random variables with a success probability $\mathcal{P}(r)$, the Chernoff bound [8] asserts that, for any $\varepsilon$, $\delta \in (0, 1)$, $\Pr\left\{ \left| \hat{\mathcal{P}}(r) - \mathcal{P}(r) \right| < \varepsilon \right\} > 1 - \delta$ if the sample size $N > \frac{\ln \frac{1}{\delta}}{\varepsilon^2}$.

In all examples, we first apply our previous method in [9] to obtain an estimate of the probabilistic robustness margin with a risk probability $\alpha = 0.05$ (Roughly speaking, we are only interested in the curve of robustness function above $1 - \alpha = 0.95$). Then, we evaluate the robustness function $\mathcal{P}(r)$ for $r \in [\frac{1}{2}, a]$ by our hierarchy sample reuse algorithms. The existing robustness measure is computed from our measure by the integral transform. The algorithms are implemented in MATLAB and all programs are executed on a PC of 1024M RAM and 3.2G CPU.

We first consider the case that the uncertainty is of a single block. A typical robustness problem is to determine the robustness margin which is specified as the maximum size of uncertainty under the condition that all poles of the closed-loop system are restricted in a certain domain $\mathcal{C}_g$. For single blocked uncertainty, there exists formulas for computation of the robustness margin in a $M - \Delta$ setup with $M(s) = C(sI - A)^{-1}B$ (see, e.g., [10] for illustration). For complex uncertainty, the robustness margin is

$$r_C = \inf\{\mathcal{P}(\Delta) : \Delta \in \mathbb{C}^{m \times p} \text{ and all eigenvalues of } A + B\Delta C \text{ are in } \mathcal{C}_g\} = \frac{1}{\sup_{s \in \partial\mathcal{C}_g} \mathcal{P}(C(sI - A)^{-1}B)}$$

where $\partial\mathcal{C}_g$ denotes the boundary of domain $\mathcal{C}_g$. This formula was essentially obtained by Doyle and Stein [9]. For real uncertainty, the robustness margin is

$$r_R = \inf\{\mathcal{P}(\Delta) : \Delta \in \mathbb{R}^{m \times p} \text{ and all eigenvalues of } A + B\Delta C \text{ are in } \mathcal{C}_g\}$$

$$= \frac{1}{\sup_{s \in \partial\mathcal{C}_g} \inf_{\gamma \in (0, 1]} \sigma_2 \left( \begin{bmatrix} \Re(M) & -\gamma \Im(M) \\ \gamma^{-1} \Im(M) & \Re(M) \end{bmatrix} \right)}$$

where the function to be minimized is a unimodal function on $(0, 1]$. This formula was established by Qiu and his coworkers [13].

To compare the power of our randomized algorithms with that of these formulas, we revisit two examples of [13]. In example 2 of [13], the domain $\mathcal{C}_g$ is defined as $\mathcal{C}_g = \{s \in \mathbb{C} : \Re(s) < 0\}$. The data of matrices $A$, $B$, $C$ can be found in page 889 and is thus omitted here. The robustness margins for the complex and real uncertainty are obtained, respectively, as $r_C = 0.3914$ and $r_R = 0.5141$. The robustness functions are shown in Figures 7 and 8 for the cases of complex and real uncertainty respectively. It can be seen that our randomized algorithms can provide useful information for the system robustness beyond the deterministic robustness margin. Specially, the deterministic robustness margin can be estimated from both types of robustness functions. Moreover, it can be seen that our robustness measure is significantly less conservative than the existing robustness measure.

In example 3 of [13], the domain $\mathcal{C}_g$ is defined as $\mathcal{C}_g = \{s \in \mathbb{C} : |s| < 1\}$ and the data of matrices $A$, $B$, $C$ are given in page 889. The robustness margins for the complex and real uncertainty are obtained as $r_C = 0.7472$ and $r_R = 1.0374$ respectively. The robustness functions are shown in Figures 9 and 10 for the cases of complex and real uncertainty respectively.
Figure 7. Robustness Functions (Sample Size $N = 26482$). The vertical line marks the deterministic robustness margin.

Figure 8. Robustness Functions (Sample Size $N = 26482$). The vertical line marks the deterministic robustness margin.
Figure 9. Robustness Functions (Sample Size $N = 26482$). The vertical line marks the deterministic robustness margin.

Figure 10. Robustness Functions (Sample Size $N = 26482$). The vertical line marks the deterministic robustness margin.
We now consider the stability margin problem where the uncertainty consists of multiple blocks. A particularly important special case is that the uncertainty is real parameters. When the number of uncertainty blocks is more than one, the formulas of [9] and [13] are not applicable and the branch and techniques are needed. We explore the application of our HSRA for the stability margin problem studied in [10] by a deterministic approach. The system considered in [10] is represented by Figure 11. The compensator is $C(s) = \frac{s+2}{s+10}$ and the plant is $P(s) = \frac{800(1+0.1\delta_1)}{(s+4+0.2\delta_2)(s+6+0.3\delta_3)}$ with parametric uncertainty $\Delta = [\delta_1, \delta_2, \delta_3]$.

![Figure 11. Uncertain System](image1)

The deterministic robustness margin is found to be 3.44 by a branch and bound technique (see, page 163 of [10]). The robustness functions are shown in Figure 12, which provides more insight for the system robustness than the deterministic robustness margin.

![Figure 12. Robustness Functions (Sample size $N = 119,830.$)](image2)

We now consider the robustness problem involving time-domain specifications for the same system shown by Figure 11. The robustness requirement is that the rise time and settling time should be no more than 0.25 and 3.5 seconds respectively and the overshoot should be no more than 70% under the condition that the closed-loop system is stable. It is well-known that this type of problems are, in general, intractable by
the deterministic approach. However, our HSRA can readily provided insightful solution. The robustness functions are shown in Figure 13.

![Figure 13. Robustness Functions (Sample size $N = 26482$.)](image)

Now we present more extensive numerical experiments for testing the efficiency of our hierarchy sample reuse algorithms. We consider the robust stability of a system of transfer function $H(s) = C(sI - A)^{-1}B + D$ with uncertain matrix $A = -10 I_{k \times k} + \sum_{\ell=1}^d q_{\ell} \sqrt{\ell} W$ where $I_{k \times k}$ is a $k$ by $k$ identity matrix, $d = k^2$ is the dimension of uncertainty and $W$ is a matrix with all elements equal to 1. This is a special case of multiple blocks of real uncertainty. Although this may not be a realistic system, it can be representative for realistic systems in the respect of computational complexity.

When the size of matrix $A$ increases from 2 to 10, the dimension of uncertainty increases from 4 to 100. The robustness functions for the case that $A$ is of size $10 \times 10$ is shown in Figure 14. The computing time is shown in Figure 15 for various problem sizes. The sample size is chosen by the Chernoff bound $N = \left\lceil \frac{m \ln \frac{1}{\varepsilon}}{\varepsilon^2} \right\rceil$ as 738, 26482, 119830, 3800452 corresponding to $\varepsilon = \delta = 0.05, 0.01, 0.005, 0.001$ respectively.

Traditionally, it is widely believed that the classical deterministic robustness analysis are usually more efficient than randomized algorithms. However, as can be seen from Figure 16, our numerical experiments indicates that, if one is willing to accept our probabilistic robustness measure, the robustness analysis via hierarchy sample reuse algorithms can be generally far more efficient.

8. Conclusion

In this paper, we develop a new statistical approach for robustness analysis which requires an extremely low complexity that is independent of the dimension of uncertainty space. Our proposed robustness measure is less conservative as compared to the existing probabilistic robustness measure. The fundamental connection between our measure and the existing one is also established.
Figure 14. Robustness Functions (Dimension $d = 100$. Sample size $N = 119,830$.)

Figure 15. Simulation Time
The following Lemma 1 is due to [3].

**Lemma 1.** For any robustness requirement, \( \inf_{f_\Delta \in \mathcal{G}} \Pr \{ \| \Delta \| = 1 \mid \| \Delta \| \leq \gamma \} = \mathbb{P}(\gamma). \)

**Lemma 2.** \( \mathcal{G} \) is a superset of \( \mathcal{F} \), i.e., \( \mathcal{G} \supseteq \mathcal{F} \).

**Proof.** Let \( f_\Delta \in \mathcal{F} \). We need to show \( f_\Delta \in \mathcal{G} \). Let \( 0 < r_1 < r_2 \) be two numbers such that, for any \( \Delta_1, \Delta_2 \) satisfying \( \| \Delta_1 \| = r_1, \| \Delta_2 \| = r_2 \), both \( f_\Delta(\Delta_1) \) and \( f_\Delta(\Delta_2) \) exist. By the radial symmetry of the distribution of \( \Delta \), we can write \( f_\Delta(\Delta_i) \) as \( g(r_i) \) for \( i = 1, 2 \). Clearly, the existence implies that \( g(.) \) is continuous at \( r = r_i, \ i = 1, 2 \). Let \( c = \int_{\overline{\mathcal{B}}}(\rho) \, d\rho \). By the radial symmetry of the distribution of \( \Delta \) and the scaling property of the function \( \| . \| \), we have \( f_\Delta(\Delta_i)(r_i) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{r_1-\varepsilon}^{r_1+\varepsilon} g(\rho) \, ncp^{n-1} \, d\rho \) for \( i = 1, 2 \), where \( n \) is the dimension of \( \Delta \). Hence, \( f_\Delta(\Delta_i)(r_i) \) is continuous at \( r = r_i, \ i = 1, 2 \). Recall that \( f_\Delta \in \mathcal{F} \), we have \( f_\Delta(\Delta_i)(r_1) \geq f_\Delta(\Delta_i)(r_2) \). On the other hand, by the radial symmetry of the distribution of \( \Delta \) and the scaling property of the function \( \| . \| \), we have \( g(r_i) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\overline{\mathcal{B}}}(\rho) \, d\rho \) for \( i = 1, 2 \). By the continuity of
For any $S \subseteq \partial B$, \( \text{area}(S_r) = r^{n-1} \text{area}(S) \) where \( S_r = \{ r \Delta : \Delta \in S \} \) and \( n \) is the dimension of \( B \).

Proof. By the scalable property of \( ||\cdot|| \),

\[
\left\{ \frac{\rho \Delta}{r} : r - \varepsilon_1 < \rho < r + \varepsilon_2 , \Delta \in S_r \right\} = \left\{ \rho \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \Delta \in S \right\} = \left\{ \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \frac{\Delta}{\rho} \in S \right\}.
\]

Hence, by invoking the definition (1.1), \( \text{area}(S_r) = \lim_{\varepsilon_1 \downarrow 0 \varepsilon_2 \downarrow 0} \frac{\int_{\varepsilon_1 \varepsilon_2} \{ \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \frac{\Delta}{\rho} \in S \} \, dq}{(\varepsilon_1 + \varepsilon_2) r} \). Making a change of variable \( q = r q' \) yields

\[
\text{area}(S_r) = \lim_{\varepsilon_1 \downarrow 0 \varepsilon_2 \downarrow 0} \frac{\int_{\varepsilon_1 \varepsilon_2} \{ \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \frac{\Delta}{\rho} \in S \} \, dq}{(\varepsilon_1 + \varepsilon_2) r} = \lim_{\varepsilon_1 \downarrow 0 \varepsilon_2 \downarrow 0} \frac{\int_{\varepsilon_1 \varepsilon_2} \{ \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \frac{\Delta}{\rho} \in S \} \, dq'}{\varepsilon_1 + \varepsilon_2} = \lim_{\varepsilon_1 \downarrow 0 \varepsilon_2 \downarrow 0} \frac{\int_{\varepsilon_1 \varepsilon_2} \{ \Delta : r - \varepsilon_1 < \rho < r + \varepsilon_2, \frac{\Delta}{\rho} \in S \} \, dq'}{\varepsilon_1 + \varepsilon_2} = \text{area}(S).
\]

\( \square \)

Lemma 4. Suppose the distribution of \( \Delta \) is radially symmetrical. Let \( S \) be a subset of \( \partial B = \{ \Delta : ||\Delta|| = 1 \} \). Then, \( \Pr \left\{ \frac{\Delta}{||\Delta||} \in S \mid ||\Delta|| = \rho \right\} = \frac{\text{area}(S)}{\text{area}(\partial B)} \) for any \( \rho > 0 \) such that \( f_{||\Delta||}(\rho) \) is continuous.

Proof. By the definition of the conditional probability,

\[
\Pr \left\{ \frac{\Delta}{||\Delta||} \in S \mid ||\Delta|| = \rho \right\} = \lim_{\varepsilon_1 \downarrow 0 \varepsilon_2 \downarrow 0} \frac{\Pr \{ \Delta : \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \} \in S \}}{\Pr \{ \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \}}.
\]

We claim that \( \left\{ \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \right\} = \{ \Delta : \Delta : \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq \rho' \leq \rho + \varepsilon_2 \right\} \) where \( S_{\rho, \varepsilon_1, \varepsilon_2} = \{ \Delta : \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq \rho' \leq \rho + \varepsilon_2 \} \). To show this claim, it suffices to show that \( \left\{ \Delta : \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \right\} = S_{\rho, \varepsilon_1, \varepsilon_2} \). Let \( \frac{\Delta}{||\Delta||} \in S_{\rho, \varepsilon_1, \varepsilon_2} \). By definition, there exists \( \rho' \in [\rho - \varepsilon_1, \rho + \varepsilon_2] \) such that \( \frac{\Delta}{||\Delta||} \in S \). Therefore, by the scalable property of the function \( ||\cdot|| \), we have \( ||\Delta|| = ||\rho' \frac{\Delta}{||\Delta||}|| = \rho' ||\frac{\Delta}{||\Delta||}|| = \rho' \in [\rho - \varepsilon_1, \rho + \varepsilon_2] \) and \( \frac{\Delta}{||\Delta||} = \frac{\rho'}{\rho} \frac{\Delta}{\rho} \in S \). This implies that \( \Delta \in \{ \Delta : \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 < ||\Delta|| \leq \rho + \varepsilon_2 \} \).

Now let \( \Delta \in \{ \Delta : \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \} \) and \( \rho' \equiv ||\Delta|| \). By definition, \( \rho - \varepsilon_1 \leq \rho' \leq \rho + \varepsilon_2, \frac{\rho'}{\rho} \in S \). Hence, \( \Delta \in S_{\rho, \varepsilon_1, \varepsilon_2} \). The claim is thus proved and we have

\[
\Pr \left\{ \frac{\Delta}{||\Delta||} \in S, \rho - \varepsilon_1 \leq ||\Delta|| \leq \rho + \varepsilon_2 \right\} = \Pr \{ \Delta \in S_{\rho, \varepsilon_1, \varepsilon_2} \}.
\]

Let \( S_{\rho'} = \{ \rho' \Delta : \Delta \in S \} \). Then, \( S_{\rho'} \subseteq \partial B_{\rho'} \) and \( S_{\rho, \varepsilon_1, \varepsilon_2} = \{ \Delta : \Delta \in S_{\rho'}, \rho - \varepsilon_1 < \rho' \leq \rho + \varepsilon_2 \} \). By the notion of the radially symmetrical distribution of \( \Delta \) and the property of the area function shown in
Lemma 4. we have \( \Pr\{ \Delta \in S_{\rho^*} \mid \|\Delta\| = \rho^* \} = \frac{\text{area}(S_{\rho^*})}{\text{area}(\partial B_{\rho^*})} = \frac{\rho^{m-1} \text{area}(S)}{\rho^{m-1} \text{area}(\partial B)} = \frac{\text{area}(S)}{\text{area}(\partial B)}. \) On the other hand, by the definition of the conditional probability,

\[
\Pr\{ \Delta \in S_{\rho^*} \mid \|\Delta\| = \rho^* \} = \lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon_2 \downarrow 0} \frac{\Pr\{ \Delta \in S_{\rho^* \varepsilon_1 \varepsilon_2} \}}{\Pr\{ \rho - \varepsilon_1 \leq \|\Delta\| \leq \rho + \varepsilon_2 \}}.
\]

It follows that \( \Pr\{ \frac{\Delta}{\|\Delta\|} \in S \mid \|\Delta\| = \rho \} = \lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon_2 \downarrow 0} \frac{\Pr\{ \Delta \in S_{\rho^* \varepsilon_1 \varepsilon_2} \}}{\Pr\{ \rho - \varepsilon_1 \leq \|\Delta\| \leq \rho + \varepsilon_2 \}} = \frac{\text{area}(S)}{\text{area}(\partial B)}. \)

\( \square \)

Lemma 5. Suppose \( f_{\|\Delta\|}(\cdot) \) is continuous in \((a, b)\). Then, \( \Pr\{ \frac{\Delta}{\|\Delta\|} \in S \mid a < \|\Delta\| < b \} = \frac{\text{area}(S)}{\text{area}(\partial B)}. \)

Proof. Let \( \eta > 0 \) and \( \delta \in (0, \frac{b-a}{2}) \). For notational simplicity, let \( c = \frac{\text{area}(S)}{\text{area}(\partial B)}. \) By Lemma 4 for any \( \rho \in [a + \delta, b - \delta] \), we can find \( \varepsilon = \varepsilon(\rho) \) such that \( \Pr\{ \frac{\Delta}{\|\Delta\|} \in S \mid \rho - \varepsilon_1 \leq \|\Delta\| \leq \rho + \varepsilon_2 \} - c < \eta \) for any positive \( \varepsilon_1, \varepsilon_2 \) less than \( \varepsilon(\rho) \). Hence, the union of the open intervals \( \cup_{\rho \in [a + \delta, b - \delta]} \{ \rho - \varepsilon(\rho), \rho + \varepsilon(\rho) \} \) will cover interval \([a + \delta, b - \delta]\). By the finite coverage theorem, we can choose finite number of \( \rho_i \) from \([a + \delta, b - \delta]\) such that \( \cup_{i=1}^{k} (\rho_i - \varepsilon(\rho_i), \rho_i + \varepsilon(\rho_i)) \) covers interval \([a + \delta, b - \delta]\) and that none of \((\rho_i - \varepsilon(\rho_i), \rho_i + \varepsilon(\rho_i))\) is nested in another. By using the mid-points of the intersections of every two consecutive intervals as dividing points, we can partition \([a + \delta, b - \delta]\) as \( k \) intervals \([a_i, b_i]\) such that \( \Pr\{ \frac{\Delta}{\|\Delta\|} \in S \mid a_i \leq \|\Delta\| \leq b_i \} - c \eta \Pr\{ a_i \leq \|\Delta\| \leq b_i \} < \eta \) for \( i = 1, \ldots, k \). Therefore,

\[
\Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a_i \leq \|\Delta\| \leq b_i \} - c \eta \Pr\{ a_i \leq \|\Delta\| \leq b_i \} < \eta \sum_{i=1}^{k} \Pr\{ a_i \leq \|\Delta\| \leq b_i \}.
\]

That is,

\[
\Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a + \delta \leq \|\Delta\| \leq b - \delta \} - c \Pr\{ a + \delta \leq \|\Delta\| \leq b - \delta \} < \eta \Pr\{ a + \delta \leq \|\Delta\| \leq b - \delta \}.
\]

As a result, \( \Pr\{ \frac{\Delta}{\|\Delta\|} \in S \mid a + \delta \leq \|\Delta\| \leq b - \delta \} - c \) \( < \eta \). Since \( \eta \) can be arbitrarily small, we have

\[
\Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a + \delta \leq \|\Delta\| \leq b - \delta \} = c \Pr\{ a + \delta \leq \|\Delta\| \leq b - \delta \}.
\]

By the assumption that \( f_{\|\Delta\|}(\cdot) \) is piece-wise continuous, we have \( \Pr\{ \rho \leq \|\Delta\| \leq \rho + \delta \} \rightarrow 0 \) as \( \delta \downarrow 0 \) for all \( \rho \geq 0 \). Hence,

\[
\lim_{\delta \downarrow 0} \Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a + \delta \leq \|\Delta\| \leq b - \delta \} - \Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a < \|\Delta\| < b \} = 0,
\]

and so \( \lim_{\delta \downarrow 0} \Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a + \delta \leq \|\Delta\| \leq b - \delta \} = \Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a < \|\Delta\| < b \} \). Similarly,

\[
\lim_{\delta \downarrow 0} \Pr\{ a + \delta \leq \|\Delta\| \leq b - \delta \} - \Pr\{ a < \|\Delta\| < b \} = 0,
\]

and so \( \lim_{\delta \downarrow 0} \Pr\{ a + \delta \leq \|\Delta\| \leq b - \delta \} = \Pr\{ a < \|\Delta\| < b \} \). It follows that

\[
\Pr\{ \frac{\Delta}{\|\Delta\|} \in S, a < \|\Delta\| < b \} = \Pr\{ a < \|\Delta\| < b \}.
\]

This completes the proof. \( \square \)
Lemma 6. Suppose that the distribution of $\Delta$ is radially symmetrical and that $f_{||\Delta||}(.)$ is piece-wise continuous over $(0, \infty)$. Then, $\frac{\Delta}{||\Delta||}$ is independent with $||\Delta||$. Moreover, $\frac{\Delta}{||\Delta||}$ is uniformly distributed over $\{\Delta : ||\Delta|| = 1\}$.

Proof. Since $f_{||\Delta||}(.)$ is piece-wise continuous over $(0, \infty)$, we can represent $(0, \infty)$ as a union of open intervals $(a, b)$ where $f_{||\Delta||}(.)$ is continuous and the set of discrete values $\rho$, $j = 1, 2, \cdots$ for which $f_{||\Delta||}(.)$ is discontinuous. We can enumerate the intervals and the discrete values such that $b_i - a_i$ is non-increasing with respect to $i$ and that $\rho_j - \rho_{j-1}$ is non-increasing with respect to $j$. Then, $\Pr\left\{\frac{\Delta}{||\Delta||} \in S, ||\Delta|| = \rho_j\right\} = 0$, $j = 1, 2, \cdots$ and, by Lemma 6

$$\Pr\left\{\frac{\Delta}{||\Delta||} \in S\right\} = \sum_i \Pr\left\{\frac{\Delta}{||\Delta||} \in S, a_i < ||\Delta|| < b_i\right\} + \sum_j \Pr\left\{\frac{\Delta}{||\Delta||} \in S, ||\Delta|| = \rho_j\right\} = \frac{\text{area}(S)}{\text{area}(\partial B)} \left[\sum_i \Pr\{a_i < ||\Delta|| < b_i\} + \sum_j \Pr\{||\Delta|| = \rho_j\}\right] = \frac{\text{area}(S)}{\text{area}(\partial B)} \cdot \frac{\text{area}(\partial B)}{\text{area}(\partial B)} = \frac{\text{area}(S)}{\text{area}(\partial B)} \cdot \frac{\text{area}(\partial B)}{\text{area}(\partial B)}.$$

Therefore, invoking Lemma 6, we have $\Pr\left\{\frac{\Delta}{||\Delta||} \in S, ||\Delta|| = \rho\right\} = \Pr\left\{\frac{\Delta}{||\Delta||} \in S\right\}$ for any $\rho$ such that $f_{||\Delta||}(.)$ is continuous. This implies the independence between $\frac{\Delta}{||\Delta||}$ and $||\Delta||$. Moreover, since the argument holds for any $S \subseteq \{\Delta : ||\Delta|| = 1\}$, we have that $\frac{\Delta}{||\Delta||}$ is uniformly distributed over $\{\Delta : ||\Delta|| = 1\}$. The proof is thus completed.

Lemma 7. Suppose that $\phi(.)$ is continuous over $(a, b)$ and that the distribution of uncertainty $\Delta$ is radially symmetrical and continuous over $(a, b)$. Then $\Pr[I(\Delta) = 1, a < ||\Delta|| < b] = \int_a^b \phi(r) f_R(r) dr$.

Proof. Define $U = \frac{\Delta}{||\Delta||}$, $R = ||\Delta||$ and $f_R(\rho) = \frac{d\Pr[R \leq \rho]}{d\rho}$. By Lemma 6, we have that $U$ and $R$ are independent and that $U$ is uniform over $\partial B$. Hence, the probability density function of $UR$ is $\frac{1}{\text{area}(\partial B)} \times f_R(r)$ and, by the Fubini’s Theorem,

$$\Pr[I(\Delta) = 1, a < ||\Delta|| < b] = \int_a^b \int_{u: \phi(u) = 1} \frac{1}{\text{area}(\partial B)} f_R(r) du dr = \int_a^b \int_{u: \phi(u) = 1} \frac{1}{\text{area}(\partial B)} du f_R(r) dr = \int_a^b \phi(r) f_R(r) dr$$

where the last equality follows from the definition of $\phi(.)$.

Lemma 8. Suppose that $\phi(.)$ is piece-wise continuous and that $f_{||\Delta||}(.)$ is piece-wise continuous and non-increasing. Then, $\Pr[I(\Delta) = 1, ||\Delta|| \leq \gamma] = \int_0^\gamma \phi(\rho) f_{||\Delta||}(\rho) d\rho$. 

\[ \text{A STATISTICAL THEORY FOR THE ANALYSIS OF UNCERTAIN SYSTEMS} \]
Lemma 11. $\inf_{\frac{a}{\rho} \leq \gamma} \mathcal{P}(\rho) = \inf_{0 < \rho \leq \gamma} \mathcal{P}(\rho)$ where $\mathbb{Q}$ denotes the set of all rational numbers.
Proof. Let $a = \inf_{0 < \rho \leq \gamma} \mathcal{P}(\rho)$ and $b = \inf_{0 < \rho \leq \gamma} \mathcal{P}(\rho)$. Clearly, $a \geq b \geq 0$. Suppose $a > b$. Then, there exists a real number $\rho^* \in (0, \gamma]$ such that $\mathcal{P}(\rho^*) < \frac{a + b}{2}$. By the dense property of the rational numbers, for any $\delta \in (0, \rho^*)$, there exists a number $\theta$ such that $\frac{\delta}{2} < \theta \in \mathbb{Q}$ and that $|\theta - \rho^*| < \delta$. Thus, by Lemma 10, $|\mathcal{P}(\theta) - \mathcal{P}(\rho^*)| \leq \frac{2\delta}{\rho^* - \theta}$, leading to $\mathcal{P}(\theta) \leq \mathcal{P}(\rho^*) + \frac{2\delta}{\rho^* - \theta} < \frac{a + b}{2} + \frac{2\delta}{\rho^* - \theta}$. Since $\delta$ can be arbitrarily small, we have $\mathcal{P}(\theta) \leq \frac{a + b}{2}$. Hence, $a \leq \frac{a + b}{2}$, i.e., $a \leq b$, contradicting to $a > b$. This shows that $a > b$ is not true. Therefore, $a = b$.

We are now in the position to prove Theorem 1. For every $f_\Delta \in \mathcal{F}$, define $f_{||\Delta||}(\rho, \gamma) = \frac{1}{\Pr(\|\Delta\| \leq \gamma)} \frac{d \Pr(\|\Delta\| \leq \rho)}{d \rho}$. Then, $f_{||\Delta||}(\rho, \gamma) = \frac{1}{\Pr(\|\Delta\| \leq \gamma)} \frac{d \Pr(\|\Delta\| \leq \rho)}{d \rho} = \frac{f_{\|\Delta||}(\rho)}{\Pr(\|\Delta\| \leq \gamma)}$, and the set of all such functions constitute a family of conditional density functions, denoted by $\mathcal{F}_\gamma$. Clearly, every conditional density $f_{||\Delta||}(\rho, \gamma)$ in $\mathcal{F}_\gamma$ is non-increasing with respect to $\rho$. For every positive integer $k$, we use $\mathcal{F}_{\gamma,k}$ to denote the set of conditional density functions of the form: $f_{||\Delta||}(\rho, \gamma) = \sum_{i=1}^{k} \xi_i I_{\{r_{i-1}, r_i\}}(\rho)$, $\forall \rho \in (0, \gamma]$ where $r_i = \frac{i-\gamma}{k}$, $i = 0, 1, \ldots, k$,

$$I_{\{r_{i-1}, r_i\}}(x) = \begin{cases} 1 & \text{if } x \in (r_{i-1}, r_i); \\ 0 & \text{otherwise} \end{cases}$$

and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k \geq 0$ with $\frac{1}{k} \sum_{i=1}^{k} \xi_i = 1$. By Lemma 8,

$$\Pr[I(\Delta) = 1 | \|\Delta\| \leq \gamma] = \frac{\Pr[I(\Delta) = 1, \|\Delta\| \leq \gamma]}{\Pr(\|\Delta\| \leq \gamma)} = \frac{\int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho}{\Pr(\|\Delta\| \leq \gamma)} = \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho.$$

Therefore,

$$\inf_{f_\Delta \in \mathcal{F}} \Pr[I(\Delta) = 1 | \|\Delta\| \leq \gamma] = \inf_{f_{||\Delta||}(\rho, \gamma) \in \mathcal{F}_\gamma} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho.$$ 

Since $\phi(\rho) I_{\{r_{i-1}, r_i\}}(\rho)$ is bounded and piece-wise continuous over $(0, \gamma]$, it is Riemann integrable. It follows that, for a conditional density $f_{||\Delta||}(\rho, \gamma)$ in the family $\mathcal{F}_{\gamma,k}$,

$$\int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho = \int_{0}^{\gamma} \phi(\rho) \left[\sum_{i=1}^{k} \xi_i I_{\{r_{i-1}, r_i\}}(\rho)\right] d\rho = \sum_{i=1}^{k} \int_{0}^{\gamma} \phi(\rho) I_{\{r_{i-1}, r_i\}}(\rho) d\rho \xi_i = \sum_{i=1}^{k} a_i \xi_i$$

where $a_i = \int_{0}^{\gamma} \phi(\rho) I_{\{r_{i-1}, r_i\}}(\rho) d\rho$ for $i = 1, \ldots, k$. Since $a_i$ is independent of $(\xi_1, \ldots, \xi_k)$ for $i = 1, \ldots, k$, we have that $\sum_{i=1}^{k} a_i \xi_i$ is a linear function of $\xi_i$, $i = 1, \ldots, k$ for any given $k > 0$. Therefore, the infimum $\inf_{f_{||\Delta||}(\rho, \gamma) \in \mathcal{F}_{\gamma,k}} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho$ equals to the minimum of $\sum_{i=1}^{k} a_i \xi_i$ subject to the constraint that $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k \geq 0$ and $\frac{1}{k} \sum_{i=1}^{k} \xi_i = 1$. Note that the minimum of a linear program over a bounded set is achieved at the extreme points. By Lemma 2.2 of [2], for every extreme point of the convex set

$$\{(\xi_1, \ldots, \xi_k) : \xi_1 \geq \xi_2 \geq \cdots \geq \xi_k \geq 0, \frac{1}{k} \sum_{i=1}^{k} \xi_i = 1\},$$

we can find an integer $\ell$ such that $\xi_i = \frac{i}{k}$ for $i = 1, \ldots, \ell$ and $\xi_i = 0$ for $i = \ell + 1, \ldots, k$. For such extreme point associated with $\ell$, we have $\sum_{i=1}^{k} a_i \xi_i = \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho = \int_{0}^{\gamma} \phi(\rho) \frac{1}{k} \gamma d\rho = \mathcal{P}\left(\frac{\ell}{k}\right), \gamma$, where the last equality follows from Lemma 8. Therefore,

$$\inf_{f_{||\Delta||}(\rho, \gamma) \in \mathcal{F}_{\gamma,k}} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho = \min \left\{ \mathcal{P}\left(\frac{\ell}{k}\right) : 0 \leq \ell \leq k \right\}.$$ 

It follows that

$$\inf_{f_{||\Delta||}(\rho, \gamma) \in \cup_{k=1}^{\infty} \mathcal{F}_{\gamma,k}} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho = \inf_{k = 1}^{\infty} \left\{ \mathcal{P}\left(\frac{\ell}{k}\right) : 0 \leq \ell \leq k \right\} = \inf \left\{ \mathcal{P}(\rho) : 0 < \rho \leq \gamma, \frac{\rho}{\gamma} \in \mathbb{Q} \right\}.$$
It can be shown that
\[ \inf_{f_{||\Delta||} \in \bigcup_{k=1}^{\infty} F_{\gamma,k}} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho = \inf_{f_{||\Delta||} \in \mathcal{F}} \int_{0}^{\gamma} \phi(\rho) f_{||\Delta||}(\rho, \gamma) d\rho. \]

Hence, by (A.2),
\[ \inf_{f_{\Delta} \in \mathcal{F}} \Pr\{I(\Delta) = 1 | ||\Delta|| \leq \gamma\} = \inf_{\rho \in \mathcal{Q}} \mathcal{P}(\rho), \]
where the last equality follows from Lemma 11. Finally, by Lemma 1 and Lemma 2, we have
\[ \mathcal{P}(\gamma) = \inf_{f_{\Delta} \in \mathcal{F}} \Pr\{I(\Delta) = 1 | ||\Delta|| \leq \gamma\} \geq \inf_{f_{\Delta} \in \mathcal{G}} \Pr\{I(\Delta) = 1 | ||\Delta|| \leq \gamma\} = \mathcal{P}(\gamma). \]

The proof is thus completed.

**Appendix B. Proof of Theorem 2**

We shall first define some terminologies that will be used in the proof.

**Definition 1.** A value of the uncertainty radius is said to be a discontinuity if \( \phi(\cdot) \) is discontinuous for that value.

**Definition 2.** An open interval \((a, b)\) is said to be a continuous interval if \( \phi(r) \) is continuous for any \( r \in (a, b) \).

**Definition 3.** A discontinuity, \( p \), is said to be a cluster point if, for any \( \epsilon > 0 \), there exists another discontinuity, \( q \), such that \( |p - q| < \epsilon \).

The proof of the transform formulas is largely focused on the investigation of discontinuities, cluster points and continuous intervals. By the assumption that \( \phi(\cdot) \) is piece-wise continuous, we can see that the distributions of discontinuities and cluster points can be arbitrary. For example, it is possible that there are infinitely many discontinuities distributed over \((0, r)\) as \( \frac{1}{(i+1)(j+1)} \) where \( i = 1, \ldots, \infty \) and \( j = 1, \ldots, \infty \). In this example, there are infinitely many cluster points \( \frac{1}{i+1} \), \( i = 1, \ldots, \infty \).

Despite the complexity of the distributions of discontinuities and cluster points, it suffices to prove the transform formulas for the following four cases:

**Case (1):** There are a finite number of discontinuities.

**Case (2):** There are infinitely many discontinuities such that \( r = 0 \) is the unique cluster point.

**Case (3):** There are infinitely many discontinuities such that there is a cluster point at \( r = 0 \) and that there is at least one more cluster point at \( r > 0 \).

**Case (4):** There are infinitely many discontinuities such that there is no cluster point at \( r = 0 \).

Before addressing each case in details, we need to establish some preliminary results.

The following lemma is on the enumeration and classification of continuous intervals.

**Lemma 12.** For any \( \epsilon > 0 \), the set of all continuous intervals defined by the end points \( q, r \) or discontinuities of interval \((q, r)\) can be divided into two classes such that i) the first class, denoted by \( \mathcal{F}_\epsilon \), has a finite number of intervals; ii) the second class, denoted by \( \mathcal{F} \), has infinitely many intervals and the total length is less than \( \epsilon \).

Proof. Such classification can be performed as follows. Let \( k = 1 \) and \( c_k = \frac{1}{2^k} \). Find all intervals with length greater than \( c_k \). Rank these intervals by the lengths and include it in set \( \mathcal{A} \). Include the remaining intervals in set \( \mathcal{B} \). Increment \( k \) and update \( c_k = \frac{1}{2^k} \). From \( \mathcal{B} \) find all intervals with length greater than \( c_k \). Add these intervals to set \( \mathcal{A} \) and rank all intervals by the lengths. Eliminate those intervals from set \( \mathcal{B} \).
Repeating these steps for infinitely many values of \( k \) leads to a sequence of intervals of decreasing lengths. Let \((a_i, b_i), i = 1, 2, \ldots \) denote this sequence. Let \( L_i = b_i - a_i \). Then, \( \sum_{i=1}^{\infty} L_i = r - q \) and \( L_i \) is decreasing with respect to \( i \). Thus, by Cauchy’s theorem, there must be an integer \( K \) such that \( \sum_{i=K}^{\infty} L_i < \varepsilon \). This implies that we have the desired two classes. The first class \( \mathcal{I}_\varepsilon \) consists of intervals \((a_i, b_i), i = 1, \ldots, K-1 \) and the second class \( \mathcal{I}_\varepsilon \) consists of intervals \((a_i, b_i), i = K, \ldots, \infty \).

**Lemma 13.** For any \( r > 0 \), \( \mathbb{P}(r) = \frac{n}{r^n} \int_0^r \phi(\rho) \rho^{n-1} \, d\rho \) where \( n \) is the dimension of uncertainty space.

Proof. Since \( \Delta^u \) is uniformly distributed over \( B \), we can derive the density function of \(|\Delta^u|\) as \( f_{|\Delta^u|}(\rho) = \frac{n\rho^{n-1}}{r^n} \). By definition, \( \mathbb{P}(r) = \operatorname{Pr}\{|\Delta^u| = 1\} = \operatorname{Pr}\{|\Delta^u| = 1, \ |\Delta^u| \leq r\} \). By Lemma 8

\[
\mathbb{P}(r) = \int_0^r \phi(\rho) f_{|\Delta^u|}(\rho) \, d\rho = \int_0^r \phi(\rho) \frac{n\rho^{n-1}}{r^n} \, d\rho = \frac{n}{r^n} \int_0^r \phi(\rho) \rho^{n-1} \, d\rho.
\]

The following two lemmas establish connections between \( \phi(.) \), \( \mathbb{P}(.) \) and \( \mathcal{P}(.) \).

**Lemma 14.** For any continuous interval \((a, b)\) with \( 0 < a < b \),

\[
\int_a^b \phi(\rho) \, d\rho = \frac{b\mathbb{P}(b) - a\mathbb{P}(a)}{n} + \frac{n-1}{n} \int_a^b \mathbb{P}(\rho) \, d\rho.
\]

Proof. By Lemma 13 we have \( \mathbb{P}(r) = \frac{n}{r^n} \int_0^r \phi(\rho) \rho^{n-1} \, d\rho \). Since \( \phi(\rho) \) is continuous over \((a, b)\), we have that \( \mathbb{P}(r) \) is differentiable with respect to \( r \) and that \( \phi(\rho) = \frac{d\mathbb{P}(\rho)}{\rho^n} \) for any \( \rho \in (a, b) \). Consequently,

\[
\int_a^b \phi(\rho) \, d\rho = \int_a^b \frac{d\mathbb{P}(\rho)}{\rho^n} \, d\rho = \int_a^b \frac{1}{n\rho^{n-1}} \, d[\rho^n \mathbb{P}(\rho)] = \lim_{\epsilon \to 0} \frac{(b - \epsilon) \mathbb{P}(b - \epsilon) - (a + \epsilon) \mathbb{P}(a + \epsilon)}{n} + \frac{n-1}{n} \int_a^b \mathbb{P}(\rho) \, d\rho
\]

(B.1)

\[
= \frac{b\mathbb{P}(b) - a\mathbb{P}(a)}{n} + \frac{n-1}{n} \int_a^b \mathbb{P}(\rho) \, d\rho
\]

(B.2)

where we have used the technique of integration by part in (B.1) and the fact that \( \mathbb{P}(\rho) \) is continuous for any \( \rho > 0 \) in (B.2).

**Lemma 15.** For any continuous interval \((a, b)\) with \( 0 < a < b \),

\[
\int_a^b \phi(\rho) \rho^{n-1} \, d\rho = [b^n \mathcal{P}(b) - a^n \mathcal{P}(a)] - (n-1) \int_a^b \mathcal{P}(\rho) \rho^{n-1} \, d\rho.
\]

Proof. By Lemma 8 we have \( \mathcal{P}(\rho) = \frac{1}{n} \int_0^\rho \phi(\rho) \, d\rho \). Since \( \phi(\rho) \) is continuous over \((a, b)\), we have that \( \mathcal{P}(\rho) \) is differentiable with respect to \( \rho \) and that \( \phi(\rho) = \frac{d\mathcal{P}(\rho)}{d\rho} \) for any \( \rho \in (a, b) \). Hence,

\[
\int_0^\rho \rho^{n-1} \phi(\rho) \, d\rho = \int_a^b \rho^{n-1} \, d[\rho^n \mathcal{P}(\rho)] = \lim_{\epsilon \to 0} [(b - \epsilon)^n \mathcal{P}(b - \epsilon) - (a + \epsilon)^n \mathcal{P}(a + \epsilon)] - \int_a^b \rho \mathcal{P}(\rho) \rho^n \rho^{n-2} \, d\rho = [b^n \mathcal{P}(b) - a^n \mathcal{P}(a)] - (n-1) \int_a^b \mathcal{P}(\rho) \rho^{n-1} \, d\rho
\]
where we have used the technique of integration by part and the fact that $\mathcal{P}(\rho)$ is continuous for any $\rho > 0$.

\[\square\]

**Lemma 16.** Let $q \leq a < b \leq r$. Then, $|b\mathcal{P}(b) - a\mathcal{P}(a)| \leq \left(\frac{nq}{q} + 1\right)(b-a)$.

**Proof.** Note that, for $q \leq a < b \leq r$, we have

\[
|b\mathcal{P}(b) - a\mathcal{P}(a)| = |b\mathcal{P}(b) - b\mathcal{P}(a) + b\mathcal{P}(a) - a\mathcal{P}(a)| \\
\leq b|\mathcal{P}(b) - \mathcal{P}(a)| + (b-a)\mathcal{P}(a) \\
\leq \frac{bn(b-a)}{a} + (b-a) \leq \left(\frac{nr}{q} + 1\right)(b-a)
\]

where we have used the bound $|\mathcal{P}(b) - \mathcal{P}(a)| \leq \frac{n(b-a)}{a}$, which was derived in the proof of Theorem 6.1 in page 856 of [3]. \[\square\]

**Lemma 17.** Let $q \leq a < b \leq r$. Then, $|b^n \mathcal{P}(b) - a^n \mathcal{P}(a)| < \left(\frac{2r^n}{q} + nr^{n-1}\right)(b-a)$.

**Proof.** Note that, by Lemma 10, $|\mathcal{P}(b) - \mathcal{P}(a)| \leq \frac{2(b-a)}{a}$, we have

\[
|b^n \mathcal{P}(b) - a^n \mathcal{P}(a)| = |b^n \mathcal{P}(b) - b^n \mathcal{P}(a) + b^n \mathcal{P}(a) - a^n \mathcal{P}(a)| \\
\leq b^n|\mathcal{P}(b) - \mathcal{P}(a)| + (b^n - a^n)\mathcal{P}(a) \\
\leq \frac{2b^n(b-a)}{a} + (b^n - a^n) \\
< \frac{2b^n(b-a)}{a} + nb^{n-1}(b-a) \\
= \left(\frac{2b^n}{a} + nb^{n-1}\right)(b-a) \\
\leq \left(\frac{2r^n}{q} + nr^{n-1}\right)(b-a)
\]

where we have used the inequality $b^n - a^n < nb^{n-1}(b-a)$ which can be shown by using Taylor’s expansion formula $b^n = a^n + n\xi^{n-1}(b-a) < a^n + nb^{n-1}(b-a)$ with some $\xi \in (a,b)$.

\[\square\]

We are now in the position to prove the transform formulas for each cases.
Case (1): Let \( 0 = p_0 < p_1 < \cdots < p_k < p_{k+1} = r \) where \( p_1, \ldots, p_k \) are \( k \geq 0 \) discontinuities. By Lemma 14 we have

\[
\int_0^r \phi(\rho) d\rho = \lim_{\epsilon \downarrow 0} \int_{\epsilon}^r \phi(\rho) d\rho = \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{p_1} \phi(\rho) d\rho + \sum_{i=1}^{k} \int_{p_i}^{p_{i+1}} \phi(\rho) d\rho = \lim_{\epsilon \downarrow 0} \left[ \frac{p_1 P(p_1) - \epsilon P(\epsilon)}{n} + \frac{n-1}{n} \int_{\epsilon}^{p_1} P(\rho) d\rho \right] + \sum_{i=1}^{k} \left[ \frac{p_{i+1} P(p_{i+1}) - p_i P(p_i)}{n} + \frac{n-1}{n} \int_{p_i}^{p_{i+1}} P(\rho) d\rho \right] = \lim_{\epsilon \downarrow 0} \left[ -\frac{\epsilon P(\epsilon)}{n} + \frac{n-1}{n} \int_{\epsilon}^{p_1} P(\rho) d\rho + \frac{r P(r)}{n} + \frac{n-1}{n} \int_{p_1}^{r} P(\rho) d\rho. \right]
\]

Since \( 0 \leq P(\rho) \leq 1, \forall \rho > 0 \), we have \( \lim_{\epsilon \downarrow 0} (\epsilon P(\epsilon) = 0 \) and \( \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{p_1} P(\rho) d\rho = \int_{\epsilon}^{p_1} P(\rho) d\rho \). It follows that \( \int_{0}^{r} \phi(\rho) d\rho = \frac{P(r)}{n} + \frac{n-1}{n} \int_{0}^{r} P(\rho) d\rho \) and that \( P(r) = \frac{1}{r} \int_{0}^{r} \phi(\rho) d\rho = \frac{P(r)}{n} + \frac{n-1}{n} \int_{0}^{r} P(\rho) d\rho \).

By Lemma 15 and similar techniques, we can show the expression for \( P(r) \) in this case.

Case (2): In this case, the discontinuities can be represented as a monotone decreasing sequence \( \{p_i\}_{i=1}^{\infty} \) such that \( r = p_0 > p_1 > p_2 > \cdots > p_k > \cdots \) and \( \lim_{k \to \infty} p_k = 0 \). By Lemma 14 we have

\[
\int_0^r \phi(\rho) d\rho = \lim_{k \to \infty} \sum_{i=1}^{k} \int_{p_i}^{p_{i+1}} \phi(\rho) d\rho = \lim_{k \to \infty} \sum_{i=1}^{k} \left[ \frac{p_{i-1} P(p_{i-1}) - p_i P(p_i)}{n} + \frac{n-1}{n} \int_{p_i}^{p_{i-1}} P(\rho) d\rho \right] = \lim_{k \to \infty} \left[ \frac{r P(r) - p_k P(p_k)}{n} + \frac{n-1}{n} \int_{p_k}^{r} P(\rho) d\rho \right].
\]

Since \( 0 \leq P(\rho) \leq 1, \forall \rho > 0 \) and \( \lim_{k \to \infty} p_k = 0 \), we have \( \lim_{k \to \infty} p_k P(p_k) = 0 \) and \( \lim_{k \to \infty} \int_{p_k}^{r} P(\rho) d\rho = \int_{p_k}^{r} P(\rho) d\rho \). It follows that \( \int_{0}^{r} \phi(\rho) d\rho = \frac{r P(r)}{n} + \frac{n-1}{n} \int_{0}^{p_1} P(\rho) d\rho \) and \( P(r) = \frac{1}{r} \int_{0}^{r} \phi(\rho) d\rho = \frac{P(r)}{n} + \frac{n-1}{n} \int_{0}^{r} P(\rho) d\rho \).

By Lemma 15 and similar techniques, we can show the expression for \( P(r) \) in this case.

Case (3): In this case, let \( r_\epsilon \) be the smallest positive cluster point. Let \( q = \frac{r}{r_\epsilon} \). We can write \( \int_{0}^{r} \phi(\rho) d\rho = \int_{0}^{q \phi(\rho) d\rho + \int_{q}^{r} \phi(\rho) d\rho \). Applying the result of Case (2), we have \( \int_{0}^{q} \phi(\rho) d\rho = \frac{q P(q)}{n} + \frac{n-1}{n} \int_{0}^{q} P(\rho) d\rho \). We consider \( \int_{q}^{r} \phi(\rho) d\rho \). For any \( \epsilon > 0 \), by Lemma 12 we can write

\[
\int_{q}^{r} \phi(\rho) d\rho = \sum_{(a,b) \in \mathcal{F}_\epsilon} \int_{(a,b)} \phi(\rho) d\rho + \sum_{(a,b) \in \mathcal{F}_\epsilon} \phi(\rho) d\rho
\]

where \( \int_{(a,b)} \) means the integration over interval \( (a, b) \) and \( \sum_{(a,b) \in \mathcal{F}_\epsilon} \) means the summation over all intervals of \( \mathcal{F}_\epsilon \). The notion of \( \sum_{(a,b) \in \mathcal{F}_\epsilon} \) is similar.

To evaluate \( \sum_{(a,b) \in \mathcal{F}_\epsilon} \int_{(a,b)} \phi(\rho) d\rho \), we arrange the intervals in \( \mathcal{F}_\epsilon \) as \( (a_i, b_i), i = 1, \ldots, k \) such that \( a_1 = q, b_i < a_{i+1}, i = 1, \ldots, k-1 \) (Here \( k \) is the total number of intervals). Note that, by
Therefore, by (B.3), (B.4), (B.7) and (B.8),
\[
\sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(\rho) d\rho = \sum_{i=1}^{k} \left[ b_i \mathbb{P}(b_i) - a_i \mathbb{P}(a_i) + \frac{n-1}{n} \int_{a_i}^{b_i} \mathbb{P}(\rho) d\rho \right]
\]
\[
= \frac{r \mathbb{P}(r) - q \mathbb{P}(q) - n-1}{n} \int_{a}^{b} \mathbb{P}(\rho) d\rho + \sum_{i=1}^{k-1} \left[ \frac{a_{i+1} \mathbb{P}(a_{i+1}) - b_i \mathbb{P}(b_i)}{n} + \frac{n-1}{n} \int_{b_i}^{a_{i+1}} \mathbb{P}(\rho) d\rho \right].
\]

By Lemma 16, we have $|a_{i+1} \mathbb{P}(a_{i+1}) - b_i \mathbb{P}(b_i)| < \left( \frac{nr}{q} + 1 \right) (a_{i+1} - b_i)$, $i = 1, \ldots, k-1$ and
\[
\sum_{i=1}^{k-1} \left| \frac{a_{i+1} \mathbb{P}(a_{i+1}) - b_i \mathbb{P}(b_i)}{n} \right| < \sum_{i=1}^{k-1} \left( \frac{nr}{q} + 1 \right) (a_{i+1} - b_i) = \left( \frac{nr}{q} + 1 \right) \sum_{i=1}^{k-1} (a_{i+1} - b_i)
\]
\[
= \left( \frac{nr}{q} + 1 + \frac{n-1}{n} \right) \varepsilon.
\]

By (B.3), (B.4), and (B.6),
\[
\left| \sum_{i=1}^{k-1} \left[ \frac{a_{i+1} \mathbb{P}(a_{i+1}) - b_i \mathbb{P}(b_i)}{n} + \frac{n-1}{n} \int_{b_i}^{a_{i+1}} \mathbb{P}(\rho) d\rho \right] \right| < \left( \frac{nr}{q} + 1 \right) \varepsilon + \frac{n-1}{n} \varepsilon
\]
\[
= \left( \frac{nr}{q} + 1 + \frac{n-1}{n} \right) \varepsilon.
\]

Now we bound $\sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(\rho) d\rho$. By Lemmas 14 and 16
\[
\sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(\rho) d\rho = \sum_{(a,b) \in \mathcal{F}} \left[ \frac{b \mathbb{P}(b) - a \mathbb{P}(a)}{n} + \frac{n-1}{n} \int_{a}^{b} \mathbb{P}(\rho) d\rho \right]
\]
\[
< \sum_{(a,b) \in \mathcal{F}} \left[ \left( \frac{nr}{q} + 1 \right) (b-a) + \frac{n-1}{n} (b-a) \right]
\]
\[
= \left( \frac{nr}{q} + 1 + \frac{n-1}{n} \right) \sum_{(a,b) \in \mathcal{F}} (b-a)
\]
\[
= \left( \frac{nr}{q} + 1 + \frac{n-1}{n} \right) \varepsilon.
\]

Therefore, by (B.3), (B.4), (B.7) and (B.8),
\[
\left| \int_{a}^{b} \phi(\rho) d\rho - \frac{r \mathbb{P}(r) - q \mathbb{P}(q) - n-1}{n} \int_{a}^{b} \mathbb{P}(\rho) d\rho \right|
\]
\[
\leq \left| \sum_{i=1}^{k-1} \left[ \frac{a_{i+1} \mathbb{P}(a_{i+1}) - b_i \mathbb{P}(b_i)}{n} + \frac{n-1}{n} \int_{b_i}^{a_{i+1}} \mathbb{P}(\rho) d\rho \right] \right| + \sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(\rho) d\rho
\]
\[
< 2 \left( \frac{nr}{q} + 1 + \frac{n-1}{n} \right) \varepsilon.
\]
Since the above argument holds for arbitrarily small \( \varepsilon > 0 \), it must be true that \( \int_r^a \phi(\rho) \, d\rho = \frac{r^n \Phi(r) - q^n \Phi(q)}{n} + \frac{n-1}{n} \int_q^r \Phi(\rho) \, d\rho \). It follows that

\[
\int_0^r \phi(\rho) \, d\rho = \int_0^q \phi(\rho) \, d\rho + \int_q^r \phi(\rho) \, d\rho
\]

\[
= \frac{q^n \Phi(q)}{n} + \frac{n-1}{n} \int_0^q \Phi(\rho) \, d\rho + \frac{r^n \Phi(r) - q^n \Phi(q)}{n} + \frac{n-1}{n} \int_q^r \Phi(\rho) \, d\rho
\]

\[
= \frac{r^n \Phi(r)}{n} + \frac{n-1}{n} \int_0^r \Phi(\rho) \, d\rho,
\]

leading to the formula for \( \mathcal{P}(r) \).

To show the formula for \( \mathcal{P}(r) \), recall that \( r^n \Phi(r) = n \int_0^r \phi(\rho) \rho^{n-1} \, d\rho \). We write

\[
\int_0^r \phi(\rho) \rho^{n-1} \, d\rho = \int_0^q \phi(\rho) \rho^{n-1} \, d\rho + \int_q^r \phi(\rho) \rho^{n-1} \, d\rho.
\]

By Lemma \ref{lemma12} we can write

\[
\int_q^r \phi(\rho) \rho^{n-1} \, d\rho = \sum_{(a,b) \in I_r} \int_{(a,b)} \phi(\rho) \rho^{n-1} \, d\rho + \sum_{(a,b) \in I_r} \int_{(a,b)} \phi(\rho) \rho^{n-1} \, d\rho.
\]

To evaluate \( \sum_{(a,b) \in I_r} \int_{(a,b)} \phi(\rho) \rho^{n-1} \, d\rho \), we arrange the intervals in \( I_r \) as \( (a_i, b_i), \ i = 1, \cdots, k \) such that \( a_1 = q, b_1 < a_{i+1}, i = 1, \cdots, k-1 \) (Here \( k \) is the total number of intervals). Note that, by Lemma \ref{lemma15}

\[
\sum_{(a,b) \in I_r} \int_{(a,b)} \phi(\rho) \rho^{n-1} \, d\rho = \sum_{i=1}^k \left[ b_i^n \mathcal{P}(b_i) - a_i^n \mathcal{P}(a_i) - (n-1) \int_{a_i}^{b_i} \phi(\rho) \rho^{n-1} \, d\rho \right]
\]

\[
= r^n \mathcal{P}(r) - q^n \mathcal{P}(q) - (n-1) \int_q^r \mathcal{P}(\rho) \rho^{n-1} \, d\rho
\]

\[
- \sum_{i=1}^{k-1} \left[ a_{i+1}^n \mathcal{P}(a_{i+1}) - b_i^n \mathcal{P}(b_i) - (n-1) \int_{b_i}^{a_{i+1}} \mathcal{P}(\rho) \rho^{n-1} \, d\rho \right].
\]

By Lemma \ref{lemma17} we have \( |a_{i+1}^n \mathcal{P}(a_{i+1}) - b_i^n \mathcal{P}(b_i)| < \left( \frac{2r^n}{q} + nr^{n-1} \right) (a_{i+1} - b_i) \). Hence,

\[
\left| \sum_{i=1}^{k-1} \left[ a_{i+1}^n \mathcal{P}(a_{i+1}) - b_i^n \mathcal{P}(b_i) \right] \right| < \left( \frac{2r^n}{q} + nr^{n-1} \right) \sum_{i=1}^{k-1} (a_{i+1} - b_i)
\]

\[
= \left( \frac{2r^n}{q} + nr^{n-1} \right) \varepsilon.
\]

On the other hand, observing that \( \int_a^b \mathcal{P}(\rho) \rho^{n-1} \, d\rho < r^{n-1}(b-a) \), we have

\[
\sum_{i=1}^{k-1} \int_{b_i}^{a_{i+1}} \mathcal{P}(\rho) \rho^{n-1} \, d\rho < r^{n-1} \sum_{i=1}^{k-1} (a_{i+1} - b_i) = r^n \varepsilon.
\]
By (B.11), (B.12) and (B.13),

\[
\left| \sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(r)\rho^{n-1}d\rho - \left[ r^n \mathcal{P}(r) - q^n \mathcal{P}(q) - (n-1) \int_q \mathcal{P}(\rho)\rho^{n-1}d\rho \right] \right| < \left( \frac{2r^n}{q} + nr^{n-1} \right) \varepsilon - (n-1)r^{n-1}\varepsilon
\]

\[
= \left( \frac{2r^n}{q} + r^{n-1} \right) \varepsilon.
\]

(B.14)

Now we bound \( \sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(r)\rho^{n-1}d\rho \). By Lemmas 15 and 17,

\[
\sum_{(a,b) \in \mathcal{F}} \int_{(a,b)} \phi(r)\rho^{n-1}d\rho = \sum_{(a,b) \in \mathcal{F}} \left[ b^n \mathcal{P}(b) - a^n \mathcal{P}(a) - (n-1) \int_a^b \mathcal{P}(\rho)\rho^{n-1}d\rho \right]
\]

\[
< \sum_{(a,b) \in \mathcal{F}} \left[ \left( \frac{2r^n}{q} + nr^{n-1} \right) (b-a) - (n-1)r^{n-1}(b-a) \right]
\]

\[
= \left( \frac{2r^n}{q} + r^{n-1} \right) \sum_{(a,b) \in \mathcal{F}} (b-a)
\]

\[
= \left( \frac{2r^n}{q} + r^{n-1} \right) \varepsilon.
\]

(B.15)

Therefore, by (B.10), (B.14) and (B.15),

\[
\left| \int_q^r \phi(r)\rho^{n-1}d\rho - \left[ r^n \mathcal{P}(r) - q^n \mathcal{P}(q) - (n-1) \int_q^r \mathcal{P}(\rho)\rho^{n-1}d\rho \right] \right| < \left( \frac{2r^n}{q} + r^{n-1} \right) \varepsilon + \left( \frac{2r^n}{q} + r^{n-1} \right) \varepsilon = 2 \left( \frac{2r^n}{q} + r^{n-1} \right) \varepsilon.
\]

Since the argument applies to arbitrarily small \( \varepsilon > 0 \), it must be true that \( \int_q^r \phi(r)\rho^{n-1}d\rho = \left[ r^n \mathcal{P}(r) - q^n \mathcal{P}(q) - (n-1) \int_q^r \mathcal{P}(\rho)\rho^{n-1}d\rho \right] \). Therefore,

\[
\int_0^r \phi(r)\rho^{n-1}d\rho = \int_0^q \phi(r)\rho^{n-1}d\rho + \int_q^r \phi(r)\rho^{n-1}d\rho
\]

\[
= q^n \mathcal{P}(q) - (n-1) \int_0^q \mathcal{P}(\rho)\rho^{n-1}d\rho + r^n \mathcal{P}(r) - q^n \mathcal{P}(q) - (n-1) \int_q^r \mathcal{P}(\rho)\rho^{n-1}d\rho
\]

\[
= r^n \mathcal{P}(r) - (n-1) \int_0^r \mathcal{P}(\rho)\rho^{n-1}d\rho,
\]

from which we find the formula for \( \mathcal{P}(r) \).

**Case (4):** In this case, let \( r_* \) be the smallest positive cluster point. Let \( q = \frac{r_*}{2} \). We can write

\[
\int_0^r \phi(r)d\rho = \int_0^q \phi(r)d\rho + \int_q^r \phi(r)d\rho.
\]

Applying the result of Case (1), we have \( \int_0^q \phi(r)d\rho = \frac{q^n \mathcal{P}(q)}{n} + \frac{n+1}{n} \int_q^r \mathcal{P}(\rho)d\rho \). By a method similar to that of Case (3), we have \( \int_q^r \phi(r)d\rho = \frac{r^n \mathcal{P}(r) - q^n \mathcal{P}(q)}{n} + \frac{n+1}{n} \int_q^r \mathcal{P}(\rho)d\rho \). Combining the two integrals gives the formula for \( \mathcal{P}(r) \). The proof for the formula of \( \mathcal{P}(r) \) is similar.
Appendix C. Proofs of Theorem 3 and 4

For completeness of argument, we need to quote a general complexity result established in [7] as Theorem 5 at below. This theorem concerns the sampling complexity of the Sample Reuse Algorithm proposed in page 1963 of [5].

**Theorem 5.** Let $d$ be the dimension of uncertainty parameter space. Then, for arbitrary gridding scheme, the equivalent number of grid points based on the Sample Reuse Algorithm [5] is strictly bounded from above by $1 + d \ln \lambda$, i.e., $m_{eq} < 1 + d \ln \lambda$.

Proof. We first establish the following inequality \( (C.1) \) that will be used to prove Theorem 5.

\[
\frac{1}{x} + \ln x > 1, \quad \forall x > 1.
\]

To prove \( (C.1) \) let $f(x) = \frac{1}{x} + \ln x$. Then $f(1) = 1$ and $\frac{df(x)}{dx} = \frac{x-1}{x^2} > 0$, $\forall x > 1$. It follows that $f(x) > 1$, $\forall x > 1$.

Now we are in the position to prove Theorem 5. Observing that $\left( \frac{r_{m+1}}{r_1} \right)^d = \prod_{i=1}^{m-1} \left( \frac{r_{i+1}}{r_i} \right)^d$, we have

\[
\ln \left( \frac{r_{m+1}}{r_1} \right)^d = \sum_{i=1}^{m-1} \ln \left( \frac{r_{i+1}}{r_i} \right)^d.
\]

Therefore,

\[
\sum_{i=1}^{m-1} \left( \frac{r_i}{r_{i+1}} \right)^d + \ln \left( \frac{r_m}{r_1} \right)^d = \sum_{i=1}^{m-1} \left( \frac{1}{\frac{r_i}{r_{i+1}}} \right)^d + \ln \left( \frac{r_{i+1}}{r_i} \right)^d.
\]

Since $\left( \frac{r_{i+1}}{r_i} \right)^d > 1$, $i = 1, \ldots, m-1$, it follows from \( (C.1) \) that $\frac{1}{\frac{r_i}{r_{i+1}}} + \ln \left( \frac{r_{i+1}}{r_i} \right)^d > 1$ for $i = 1, \ldots, m-1$.

Hence, $\sum_{i=1}^{m-1} \left( \frac{r_i}{r_{i+1}} \right)^d + \ln \left( \frac{r_m}{r_1} \right)^d > m-1$, or equivalently, $m - \sum_{i=1}^{m-1} \left( \frac{r_i}{r_{i+1}} \right)^d < 1 + \ln \left( \frac{r_m}{r_1} \right)^d = 1 + d \ln \lambda$.

Finally, by Theorem 1 of [5] and the definition of $m_{eq}$, we have $m_{eq} = m - \sum_{i=1}^{m-1} \left( \frac{r_i}{r_{i+1}} \right)^d < 1 + d \ln \lambda$. □

C.1. Proof of Theorem 3

By Lemma 5 \( |\mathcal{P}(r) - \mathcal{P}^*(r)| \leq \frac{2}{\lambda} \left( \frac{r_{i+1} - r_i}{r_i} \right) \), $\forall r \in [r_i, r_{i+1}]$. Thus, it suffices to show $\frac{2}{\lambda} \left( \frac{r_{i+1} - r_i}{r_i} \right) < \epsilon$, i.e.,

\[
\frac{r_{i+1}}{r_i} < 1 + \frac{\epsilon}{2}.
\]

By the definition of uniform gridding, for $i = 1, \ldots, m-1$,

\[
\frac{r_{i+1}}{r_i} = \frac{a - \frac{(m-i-1)(\lambda-1)}{\lambda} a}{a - \frac{(m-1)(\lambda-1)}{\lambda} a} = 1 + \frac{\lambda - 1}{m - 1 + (\lambda - 1)(i - 1)} \leq 1 + \frac{\lambda - 1}{m - 1}.
\]

By virtue of \( (C.2) \), to guarantee that the gridding error is less than $\epsilon$, it suffices to ensure $1 + \frac{\lambda - 1}{m - 1} < 1 + \frac{\epsilon}{2}$, i.e., $m > 1 + \frac{2(\lambda - 1)}{\epsilon}$. Hence, it suffices to have $m \geq 2 + \left\lfloor \frac{2(\lambda - 1)}{\epsilon} \right\rfloor$. It can be verified that $\frac{r_i}{r_{i+1}} = 1 - \frac{1}{\lambda}$ for $i = 1, \ldots, m-1$.

Let $n^k$ be the total number of simulations on the direction associated with directional sample $U^k$, $k = 1, \ldots, N$. Applying Theorem 1 of [7] and Theorem 5 in this paper to a sample reuse process conditioned upon a direction with grid points $r_1, \ldots, r_m$ and sample size $N = 1$, we have $E[n^k | U^k] = m - \sum_{i=1}^{m-1} \frac{r_i}{r_{i+1}} < 1 + d \ln \lambda$ and consequently $E[n^k] = E[E[n^k | U^k]] = m - \sum_{i=1}^{m-1} \frac{r_i}{r_{i+1}} < 1 + d \ln \lambda$ for $k = 1, \ldots, N$. Finally, the proof is completed by invoking the definition of equivalent number of grid points.
C.2. **Proof of Theorem 4.** By the definition of uniform gridding, we have $r_{i+1}/r_i = \lambda^{1/m}$. Hence, by (C.2), it suffices to show $\lambda^{1/m} < 1 + \frac{\epsilon}{2}$, which can be reduced to $m > 1 + \frac{\ln \lambda}{m(1+\frac{\epsilon}{2})}$. This inequality is equivalent to $m \geq 2 + \left\lfloor \frac{\ln \lambda}{\ln(1+\frac{\epsilon}{2})} \right\rfloor$. By letting $n_k$ be the total number of simulations on the direction associated with directional sample $U^k$, $k = 1, \cdots, N$ and applying Theorem 1 of [5] and Theorem 5 in this paper to a sample reuse process conditioned upon a direction with grid points $r_1, \cdots, r_m$ and sample size $N = 1$, we have $\mathbb{E}[n^k | U^k] = m - (m - 1) \left( \frac{1}{\lambda} \right)^{1/m} < 1 + d \ln \lambda$ and consequently $\mathbb{E}[n^k] = \mathbb{E}[\mathbb{E}[n^k | U^k]] = m - (m - 1) \left( \frac{1}{\lambda} \right)^{1/m} < 1 + d \ln \lambda$ for $k = 1, \cdots, N$. The proof is completed by using the definition of equivalent number of grid points.