TAILORING A PAIR OF PANTS: THE PHASE TROPICAL VERSION

I. Zharkov
Kansas State University
138, Cardwell Hall, Manhattan 66506, USA
zharkov@ksu.edu

We show that the phase tropical pair-of-pants \( P^\circ \subset (\mathbb{C}^*)^n \) is (ambient) isotopic to the complex pair-of-pants \( P^o \subset (\mathbb{C}^*)^n \). The existence of an isotopy between the complex and ober-tropical pairs-of-pants was recently established by the author jointly with H. Ruddat. Thereby all the three (complex, phase tropical, and ober-tropical) versions are isotopic.

Bibliography: 10 titles. Illustrations: 8 figures.

1 Introduction

The \((n-1)\)-dimensional (complex) pair-of-pants \( P^o \subset (\mathbb{C}^*)^n \) is the main building block for many problems in complex and symplectic geometries. Its projection under the \( \log |z| \) map is called the amoeba, and its projection under the argument map is called the coamoeba. The phase tropical pair-of-pants \( P^\circ \subset (\mathbb{C}^*)^n \) is the fibration over the tropical hyperplane with fibers over tropical strata given by the corresponding coamoebas. It is natural to consider closed spaces, so we compactify \((\mathbb{C}^*)^n \) to \( \Delta \times \mathbb{T}^n \), where \( \Delta \) is the standard \( n \)-simplex and \( \mathbb{T}^n \) is the \( n \)-torus. The main result of the paper is Theorem 3.1 which states that the closures \( \mathcal{P} \) and \( P \) of the two versions of the pair-of-pants in \( \Delta \times \mathbb{T}^n \) are (ambient) isotopic (in the PL category).

Instead of trying to build an isotopy explicitly, we build regular cell decompositions of both pairs and show that they are homeomorphic. The cell structures respect the natural stratification of \( \Delta \times \mathbb{T}^n \), so the homeomorphisms will glue well at the boundary. Thus, with a tiny bit of effort the isotopy can be extended to any general affine hypersurface by using the pair-of-pants decomposition of [1] and [2].

The proof follows the same path as for the ober-tropical case in [3]. Namely, the main ingredient is to show that, in the CW-decomposition, the pair \(( (\mathbb{C}^*)^n, \mathcal{P} ) \) restricted to each cell is the standard ball pair. This involves two statements: local flatness (Lemma 3.1) and the complement homotopic to a circle (Lemma 3.2).

The main application of the isotopy is to address the following question in mirror symmetry. Given an integral affine manifold with singularities, we want to build a topological SYZ fibration [4] with discriminant in codimension 2 (rather than codimension one) (cf. [5] for the quintic...
3-fold case and an announcement [6]). To compare with the ober-tropical version, the phase tropical one has the advantage that no unwiggling is required when gluing local models. On the other hand, the singular fibers are not equi-dimensional. For example, in dimension 3, the negative vertex fiber looks like $S^1$-fibration over $T^2$, where the circle collapses to a point over two triangles (the coamoeba), rather than over its skeleton (the $\Theta$-graph in the ober-tropical case).

2 CW-Structure of Complex and Phase Tropical Pairs of Pants

2.1. Notation. We set $\widehat{n} := \{0, \ldots, n\}$. We think of $(\mathbb{C}^*)^n \cong (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ as the product $\Delta^\circ \times T^n$, where $\Delta^\circ$ is the interior of the $n$-simplex

$$\Delta := \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum x_i = 1\right\}$$

and $T^n := (\mathbb{R}/2\pi \mathbb{Z})^{n+1}/(\mathbb{R}/2\pi \mathbb{Z})$ is the $n$-torus with homogeneous coordinates $[\theta_0, \ldots, \theta_n]$. It is more natural to work with closed spaces, so we compactify $(\mathbb{C}^*)^n$ to $\Delta \times T^n$ and all subspaces in $(\mathbb{C}^*)^n$ by taking their closures in $\Delta \times T^n$. We denote by bsd $\Delta$ the first barycentric subdivision of $\Delta$. We also consider the dualizing subdivision dsd $\Delta$ which is a coarsening of bsd $\Delta$ by combining all simplices in bsd $\Delta$ from a single interval $[I, J]$ together, i.e., a cell $\Delta_{IJ}$ in dsd $\Delta$ has the form

$$\Delta_{IJ} := \text{Conv} \{\widehat{\Delta}_K : I \subseteq K \subseteq J\},$$

where $\widehat{\Delta}_K$ stands for the barycenter of $\Delta_K$. The hypersimplex $\Delta^n(2) \subset \Delta^n$ is obtained from the ordinary simplex by cutting the corners halfway, i.e.,

$$\Delta^n(2) := \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 2\pi \text{ and } 0 \leq x_i \leq \pi\right\}.$$

We use $2\pi = 1$ for the amoeba and $2\pi = 6.28\ldots$ for the coamoeba.

2.2. Two stratifications of $\Delta \times T^n$. The stratification $\{\Delta_J\}$ of the simplex $\Delta$ is given by its face lattice, namely, by the nonempty subsets $J \subseteq \widehat{n}$ with dim $\Delta_J = |J| - 1$. The most refined decomposition of $\Delta$ we will ever need is dsd $\Delta$ whose faces $\Delta_{IJ}$ are the pairs $I \subseteq J \subseteq \widehat{n}$. The face lattice is given by the inclusion on the $J$’s and the reverse inclusion on the $I$’s.

On the torus side, the set of hyperplanes $\theta_i = \theta_j$, $i, j \in \widehat{n}$, stratifies $T^n$ by cyclic orderings of the points $\theta_0, \ldots, \theta_n$ on the circle. The strata $T_\sigma$ are labeled by cyclic partitions $\sigma = \langle I_1, \ldots, I_k \rangle$ of the set $\widehat{n}$, i.e., $\widehat{n} = I_1 \sqcup \cdots \sqcup I_k$ and the sets $I_1, \ldots, I_k$, called the parts of $\sigma$, are cyclically ordered. The elements within each part $I_i$ are not ordered. If all parts are 1-element sets, then we call this partition maximal and write $\sigma = \langle i_0, \ldots, i_n \rangle$. A cyclic partition $\sigma = \langle I_1, \ldots, I_k \rangle$ can be depicted by marking $|\sigma| := k$ points on a circle, called vertices of $\sigma$, and labeling the arcs between them by parts in $\sigma$ in the counterclockwise order. Any coarsening of $\sigma$ is specified by a subset of vertices of $\sigma$.

Each $T_\sigma \subseteq T^n$ can be thought of as the interior of the simplex

$$\Delta_\sigma := \left\{(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k : \alpha_i \geq 0, \sum \alpha_i = 2\pi\right\}.$$  

(2.1)
Here, the coordinates $\alpha_i$ play the role of differences between the consecutive in the order $\sigma$ original arguments $\theta_i$’s. More precisely, $T_\sigma$ is embedded into $T^n$ via $\theta_i = \theta_j$ for $i, j \in I_s$ and $\theta_i + \alpha_s = \theta_j$ for $i \in I_s, j \in I_{s+1}$, where we assume the periodic indexing $I_{s+k} = I_s$.

The closure $\overline{T_\sigma}$ of each $T_\sigma$ in $T^n$ lifts to the cover of $T^n$ as a simplex $\Delta_\sigma$ in the cube $[0, 2\pi]^n$. The covering map $\Delta_\sigma \to \overline{T_\sigma}$ is one-to-one away from the vertices of $\Delta_\sigma$ which are all mapped to $\{0\}$, the single vertex of $\overline{T_\sigma}$. We can then distinguish the preimages of $\{0\}$ in $\Delta_\sigma$ by specifying a vertex from the set of vertices of $\sigma$.

There is a (twice)$\nu^n$ more refined alcove decomposition $\mathcal{A}$ of $T^n$ by the hyperplanes

$$\theta_i - \theta_j \in \pi \mathbb{Z} \quad \text{for all pairs } i, j \in \widehat{n}. \quad (2.2)$$

One can parametrize the alcoves $\mathcal{A}_\nu$ by nonempty nets of chords $\nu$ as follows (cf. [7] for details). Draw a circle in the plane, which we assume to be (counterclockwise) oriented. A chord is an interval (possibly an infinitesimal tangent) in the circle connecting two points, to be vertices of the partition $[\nu]$. We say that a nonempty collection $\nu$ of chords in the disk with at most $n+1$ vertices is a net if any two chords intersect (possibly at a vertex). Then we label the arcs between the vertices by disjoint subsets $I_s$ such that $\pi = I_1 \sqcup \cdots \sqcup I_k$, they form a cyclic partition $[\nu] = \langle I_1, \ldots, I_k \rangle$.

A net of chords $\nu$ defines an alcove $\mathcal{A}_\nu$ by defining the relations among the arguments $\theta_i$ as follows.

1. If two elements $i, j \in \widehat{n}$ are separated by no chords in $\tau$, i.e., $i, j$ belong to the same $I_s$ in $\sigma(\tau)$, then $\theta_i = \theta_j$.

2. If two elements $i, j \in \widehat{n}$ are separated by all chords in $\nu$, then $\theta_i - \theta_j = \pi \mod 2\pi$.

3. If two elements $i, j \in \widehat{n}$ are separated by some, but not all chords in $\nu$, then all nonseparating chords define the same counterclockwise order (otherwise, they would not intersect), say, $i$ comes before $j$, then $\theta_j - \theta_i \in [0, \pi] \mod 2\pi$.

The alcove decomposition of any simplex $\Delta_\sigma$ is denoted by $\mathcal{A} \Delta_\sigma$.

Finally, we combine the stratification of $\Delta$ by $\Delta_J$ with the stratification of $T^n$ by either $T_\sigma$ or $\mathcal{A}_\nu$ to get the product stratification of $\Delta \times T^n$. We illustrate cells $\Delta_J \times T_\sigma$ and $\Delta_J \times \mathcal{A}_\nu$ in $\Delta \times T^n$ by marking the arcs between vertices of $\sigma$, drawing the net $\nu$, and underlining the elements in $J \subseteq \pi$ (cf. Figure 1).

![Figure 1](image)

**Figure 1.** Example: $J = \{0, 3, 5\}$, $\sigma = \langle 1, 2, 3, \{45\}, 0 \rangle$ and two nets of chords $\nu_{1,2}$ with $[\nu_{1,2}] = \sigma$. The bold points on the circle are vertices. Tangent chord on the right is depicted as an outside loop.

**2.3. A cell decomposition of the complex pair-of-pants.** The $(n - 1)$-dimensional pair-of-pants $P^\circ$ is the complement of $n + 1$ generic hyperplanes in $\mathbb{CP}^{n-1}$. By an appropriate
We define the compactified pair-of-pants $P$ to be the closure of $P^o$ in $\Delta \times \mathbb{T}^n$ via the map

$$\mu_1 \times \mu_2 : (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \to \Delta \times \mathbb{T}^n, \quad [z_0, \ldots, z_n] \mapsto \left( \frac{|z_0|}{\sum |z_i|}, \ldots, \frac{|z_n|}{\sum |z_i|}; \arg z_0, \ldots, \arg z_n \right).$$

$P$ is a manifold with boundary, and it can be thought of as a real oriented blow-up of $\mathbb{CP}^{n-1}$ along its intersections with the coordinate hyperplanes in $\mathbb{CP}^n$.

The image $\mu_1(P) \subset \Delta$ is called the (compactified) amoeba of the hypersurface $P$ (cf. [8]). One can easily identify $\mu_1(P)$ with the hypersimplex $\Delta(2)$ since the only restrictions on lengths of $z_i$ are given by the triangle inequalities. Namely, if we normalize the perimeter to be 1, then $\mu_1(P)$ is cut out by the inequalities $|z_i| \leq 1/2$.

The image $\mu_2(P) \subset \mathbb{T}^n$ is called the coamoeba of the hypersurface $P$. Its restriction to the stratum $\hat{T}_\sigma$ is denoted by $\mathcal{C}_\sigma$. Since the covering map $\Delta_\sigma \to \hat{T}_\sigma$ is bijective away from the vertices of $\Delta_\sigma$ and $\mathcal{C}_\sigma$ misses $\{0\} \in \mathbb{T}^n$, we may view $\mathcal{C}_\sigma$ as a subset of the simplex $\Delta_\sigma$. It is interesting that the coamoeba cell $\mathcal{C}_\sigma \subset \Delta_\sigma$ is cut out by the inequalities $\alpha_s \leq \pi$, $\sum \alpha_s = 2\pi$, i.e., $\mathcal{C}_\sigma$ is also the hypersimplex $\Delta_\sigma(2) \subset \Delta_\sigma$.

We also consider partial coamoebas $\mathcal{C}'I \subset \mathbb{T}^n$ for $I \subset \mathbb{N}$ defined as the closure of the image $\mu_2(P^I) \subset \mathbb{T}^n$, where $P^I \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ is a hypersurface given by $\sum_{i \in I} z_i = 0$. Any partial coamoeba is the product of a lower-dimensional coamoeba with a complementary-dimensional torus.

The product stratification $(J, \sigma)$ of $\Delta \times \mathbb{T}^n$ induces the subdivision of $P$ which was shown in [7] to be a regular CW-complex. Let us briefly describe the face lattice $\{P_{\sigma,J}\}$ of this complex. We say that $\sigma$ divides $J$ if $J$ contains elements in at least two parts of $\sigma = (I_1, \ldots, I_k)$. A face $P_{\sigma,J}$ is nonempty if and only if $\sigma$ divides $J$. Then the dimension of $P_{\sigma,J}$ is $|\sigma| + |J| - 4$. This, in particular, means that $|\sigma| \geq 2$ and $|J| \geq 2$. $P_{\sigma',J'}$ is a face of $P_{\sigma,J}$ if $J' \subseteq J$ and $\sigma'$ is a coarsening of $\sigma$.

Again, since the covering map $\Delta_\sigma \to \hat{T}_\sigma$ is bijective away from $\{0\} \in \mathbb{T}^n$ and $P_{\sigma,J}$ does not have any points lying over $\{0\}$, we may view it sitting in the product of simplices

$$P_{\sigma,J} \subseteq \Delta_J \times \Delta_\sigma.$$

It was shown in [3] that $(\Delta_J \times \Delta_\sigma, P_{\sigma,J})$ is homeomorphic to the standard ball pair for every $(J, \sigma)$. Our main goal is to prove the same result for the phase tropical case.

### 2.4. Phase tropical pair-of-pants as a CW-complex.

Consider the spine $H$ of the amoeba $\mu_1(P)$ (also known as the tropical hyperplane) which is a polyhedral subcomplex of $\text{dsd} \Delta$ defined by

$$H = \{(x_0, \ldots, x_n) \in \Delta : x_i = x_j \geq x_k \text{ for some } i \neq j \text{ and all } k \neq i, j\}.$$ 

Its faces $H_{I,J}$ are cubes of dimension $|J| - |I|$ parameterized by pairs of subsets $I \subseteq J \subseteq \mathbb{N}$ with $|I| \geq 2$. Namely, $H_{I,J}$ is defined by $x_i = x_{i'} \geq x_k$ for all $i, i' \in I$, $k \notin I$ and $x_j = 0$, $j \notin J$. 

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The phase tropical pair-of-pants \( \mathcal{P} \subset \Delta \times \mathbb{T}^n \) is the union

\[
\mathcal{P} := \bigcup_{I \subseteq J} (H_{IJ} \times C^I).
\]

Similar to the complex pair-of-pants \( P \), the stratification \( \{ \Delta_J \times \mathbb{T}^n \} \) of \( \Delta \times \mathbb{T}^n \) induces a stratification \( \{ \mathcal{P}_{\sigma, J} \} \) of \( \mathcal{P} \). It was shown in \([7]\) that \( \{ \mathcal{P}_{\sigma, J} \} \) forms a regular CW-complex isomorphic to \( \{ P_{\sigma, J} \} \) which proves that \( P \) and \( \mathcal{P} \) are homeomorphic.

Again, since each \( \mathcal{P}_{\sigma, J} \) does not touch the vertices of \( \Delta_\sigma \), we view it as sitting inside the product of two simplices

\[
\mathcal{P}_{\sigma, J} \subset \Delta_J \times \Delta_\sigma.
\]

Our main goal is to show that this is the standard ball pair.

Let us briefly recall the polyhedral structure of \( \mathcal{P}_{\sigma, J} \) in this complex (cf. \([7]\) for details). For a subset \( I \subseteq \hat{n} \) we say a chord in \( \nu \) divides \( I \) if \( I \) does not lie on one side of it. Then \( \mathcal{P}_{\sigma, J} \) is the union of products \( H_{IK} \times \mathcal{A}_\nu \) such that \( I \subseteq K \subseteq J \), \( [\nu] \) is a coarsening of \( \sigma \), and each chord in \( \nu \) divides \( I \).

![Figure 2. Face \( H_{IJ} \times \mathcal{A}_\nu \) in \( \mathcal{P}_{\sigma, \hat{n}} \) for \( \sigma = \{0, 1, 2, 3, 45\} \) and \( I = \{0, 3\} \) in bold.](image)

The face lattice is \( (I, K, \nu) \preceq (I', K', \nu') \) if \( I \supseteq I' \), \( K \subseteq K' \) and \( \nu \subseteq \nu' \). The dimension of the \((I, K, \nu)\)-stratum is given by

\[
\dim H_{IK} + \dim \mathcal{A}_\nu = (|K| - |I|) + (|\nu| - 1).
\]  

(2.3)

The final remark we would like to make before going into the proof section is that the alcove subdivision is not the coarsest polyhedral structure one can put on \( \mathcal{P} \), but it is the coarsest one which refines the \( \sigma \)-stratification of the \( \mathbb{T}^n \)-factor.

3 Isotopy

As shown in \([7]\), \( \mathcal{P} \) is a topological manifold homeomorphic to the complex pairs-of-pants.

We prove the relative (much stronger) version of this homeomorphism, which is the main result of the paper.

**Theorem 3.1.** The two spaces \( P \) and \( \mathcal{P} \) are (ambient) isotopic in \( \Delta \times \mathbb{T}^n \). An isotopy can be chosen such that it respects the stratification \( \{ \Delta_J \times \mathbb{T}^n \} \).

The key ingredient of the proof of Theorem 3.1 is a compatible collection of homeomorphisms between the cell pairs \( (\Delta_J \times \Delta_\sigma, P_{\sigma, J}) \) and \( (\Delta_J \times \Delta_\sigma, \mathcal{P}_{\sigma, J}) \). More precisely, we show that both are unknotted ball pairs.
3.1. Unknotted ball pairs. Here, we collect some basic results from PL topology which we will need to prove the isotopy. A ball pair \((B^4, B^m)\) is proper if \(\partial B^m = B^m \cap \partial B^4\). The standard ball pair \(B^q,m\) is the pair of cubes \([-1,1]^q \times \{0,0\} \times [-1,1]^m\). A ball pair is locally flat if any point has a neighborhood homeomorphic to a neighborhood of a point in the standard pair. We say that a ball pair is unknotted if it is homeomorphic to the standard ball pair. We will be mainly concerned with ball pairs of codimension 2.

**Proposition 3.1** (cf., for example, [9, Chapters 4 and 7]). In the PL category, the following assertions hold.

1. For \(q \neq 4\) a locally flat ball pair \((B^4, B^{q-2})\) is unknotted if and only if \(B^4 \setminus B^{q-2}\) has the homotopy type of a circle.
2. A proper ball pair \((B^4, B^2)\) is unknotted if it is the cone over the locally flat sphere pair \((S^3, S^1)\) with \(S^3 \setminus S^1\) homotopic to a circle.
3. A homeomorphism between the boundaries of unknotted balls extends to their interiors. Moreover, one can choose the extension to agree with any given extension on the subball.

3.2. Proof of the main theorem. The main building block of the proof of Theorem 3.1 is a homeomorphism of the pairs \((\Delta_J \times \Delta_{\sigma}, P_{\sigma,J})\) and \((\Delta_J \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J})\).

**Proposition 3.2** ([3, Propositions 11 and 12]). The ball pair \((\Delta_J \times \Delta_{\sigma}, P_{\sigma,J})\) is unknotted.

**Proof.** We give an idea of the proof here, the details can be found in [3]. We rely on Proposition 3.1 (1). Local flatness follows from the fact that \(P\) is a submanifold in \((\mathbb{C}^*)^n\). Then the complement \(\Delta_J \times \Delta_{\sigma} \setminus P_{\sigma,J}\) is homotopic to the subcomplex \(L_{\sigma,J} \subset \Delta_J \times \Delta_{\sigma}\) (cf. the proof of Lemma 3.2), which, in turn, collapses to a circle. The 4-dimensional case is done by hands in the proof of Proposition 3.3 below.

We prove the same result for the phase-tropical ball pair \((\Delta_J \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J})\). Looking at the polyhedral structure of \(\mathcal{P}_{\sigma,J}\), one can easily observe that \((\Delta_J \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J})\) is a proper ball pair.

**Lemma 3.1.** The pair \((\Delta_J \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J})\) is locally flat.

**Proof.** Since \(\mathcal{P}_{\sigma,J}\) is a polyhedral subcomplex of dsd \(\Delta_J \times \mathcal{A} \Delta_{\sigma}\), it is enough to consider the local fan at a vertex of \(\mathcal{P}_{\sigma,J}\). Let \(v\) be a vertex of \(\mathcal{C}_{\sigma}\) which corresponds to a 2-partition \(\langle I_- , I_+\rangle\), a coarsening of \(\sigma\), and let \(K\) be a subset of \(J\) with nonempty \(K \cap I_\pm\). To avoid cumbersome notation, we assume that \(K = J\) and \(\sigma\) is maximal. Let \(J_\pm := J \cap I_\pm\). Any nonmaximal case is the product of a lower-dimensional maximal one with the simplicial cone on \(J \setminus K\).

The local fan \(\mathcal{P}_{\sigma,J}^v\) of \(\mathcal{P}_{\sigma,J}\) at the vertex \(\widehat{\Delta}_J \times v\) projects to the \(\Delta_J\) factor as a locally flat codimension 1 fan \(H_{v,J} \cong \mathbb{R}^{[J]^{d-2}}\) isomorphic to the product of the normal fans to simplices \(\Delta_{J_\pm}\). Thus, the problem is reduced to a codimension 1 ball pair \((H_{v,J} \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J}^v)\) which is related to the Schönflies conjecture. We avoid the inductive dependence on dimension 4 and show the local flatness of \((H_{v,J} \times \Delta_{\sigma}, \mathcal{P}_{\sigma,J}^v)\) explicitly.

We draw the circle with arcs labelled by the parts in \(\sigma\) such that \(J_-\) is on the bottom and \(J_+\) is on the top from the horizontal chord \(v\) and order the parts in \(\sigma\) from right to left (cf. Figure 3). The codimension 1 subball \(\mathcal{P}_{\sigma,J}^v\) breaks the ambient ball \(H_{v,J} \times \Delta_{\sigma}\) into two
connected components, positive and negative. A cell $H_I \times \mathcal{A}_\nu$ does not belong to $\mathcal{P}_{\sigma,J}^v$ if there are chords in $\nu$ not dividing $I$. Since all chords intersect, $I$ lies on the same side for all non-dividing chords in the above right-to-left order. If $I$ lies on the left, then we call $H_I \times \mathcal{A}_\nu$ positive; otherwise, we call it negative. For example, in Figure 3 with $\nu$ including the 4 chords, the cell $H_{03} \times \mathcal{A}_\nu$ is positive and the cell $H_{02} \times \mathcal{A}_\nu$ is negative (only one chord is non-dividing in both cases).

Let $w$ be a vector parallel to the edge of the simplex $\Delta_\nu$ containing $v$. Here, we have to make a choice of the direction of $w$, which is equivalent to choosing which part is positive in the 2-partition $\langle I_-, I_+ \rangle$. Then we choose a vector $\lambda$ in the product of the normal fans to simplices $\Delta_{J \pm}$ which as a linear functional defines our right-to-left order on vertices of each simplex $\Delta_{J \pm}$. Then we claim that $w + \lambda$ defines the desired product structure on $(H_{v,J} \times \Delta_\sigma, \mathcal{P}_{\sigma,J}^v)$. Geometrically, it means that $w + \lambda$ “pokes” through $\mathcal{P}_{\sigma,J}^v$ from the negative side to the positive side of $H_{v,J} \times \Delta_\sigma$ (cf. Figure 4).

Indeed, we just need to look at the facets of $\mathcal{P}_{\sigma,J}^v$, which are of two types.

Type 1: $|I| = 2$ and $\nu$ is 1 chord short of being maximal.

Type 2: $|I| = 3$ and $\nu$ is maximal.

We depict the two types and the poking by $w + \lambda$ from negative to positive cobounding facets of $H_{v,J} \times \Delta_\sigma$ in Figure 5. A type 1 facet is given by the hyperplane $\theta_i - \theta_j = \pi$, where $\{i,j\} = I$. The vector $\lambda$ is parallel to it, and the vector $w$ agrees with its negative/positive coorientation. A type 2 facet is given by the hyperplane $x_i = x_j$, where $I = \{i,j,k\}$ and $i,j$ are in the same part $I_-$ or $I_+$. Here, the vector $w$ is parallel and $\lambda$ agrees with the coorientation. The right/left and negative/positive choices are made such that the projection from $H_{v,J} \times \Delta_\sigma$ along $w + \lambda$...
defines a PL homeomorphism from $\mathcal{P}_{\sigma,J}^v$ to its quotient image $\mathbb{R}|J|−2 \times (\mathcal{C}_{\sigma}^v/w)$, thus giving a product structure $H_{v,J} \times \Delta_\sigma \cong \mathcal{P}_{\sigma,J}^v \times \mathbb{R}(w + \lambda)$.

Figure 5. Type 1: $\lambda$ is parallel to the facet, $w$ is poking through.

Figure 6. Type 2: $w$ is parallel to the facet, $\lambda$ is poking through.

Lemma 3.2. The complement $\Delta_J \times \Delta_\sigma \setminus \mathcal{P}_{\sigma,J}$ is homotopic to a circle.

Proof. Let $L_{\sigma,J}$ be the subcomplex of $\Delta_J \times \Delta_\sigma$ consisting of pairs $(\sigma', K) \preceq (\sigma, J)$ such that $\sigma'$ does not divide $K$, i.e., $L_{\sigma,J}$ consists of noninterlacing pairs $(K, V)$ (i.e., the entire $K$ lies between two vertices), where $K \subseteq J$ and $V$ is a subset of the vertices of $\sigma$. Lemma 9 in [3] asserts that $L_{\sigma,J}$ collapses to a circle. Next, we show that the complement of the open star neighborhood of $P_{\sigma,J}$ in $dsd \Delta_J \times \Delta_\sigma$ is a refinement of $L_{\sigma,J}$.

Indeed, a face $H_{IK} \times \mathcal{A}_\nu$ belongs to the open star neighborhood of $P_{\sigma,J}$ in $dsd \Delta_J \times \Delta_\sigma$ if and only if it has a vertex in $P_{\sigma,J}$. A vertex of $H_{IK} \times \mathcal{A}_\nu$ is given by a subset $I'$ between $I$ and $K$ and a single chord in $\nu$. Decreasing $K$ does not do anything in terms of changing divisibility of $I$ by $\nu$. Increasing $I$, on the other hand, does. Thus, $H_{IK} \times \mathcal{A}_\nu$ has no vertex in $P_{\sigma,J}$ if and only if no chord in $\nu$ divides $K$. Since all chords are intersecting, this can happen only if $K$ belongs to a single part of $[\nu]$. Conversely, if $K$ is in a single part of $[\nu]$, then, clearly, no chord divides it.

Now, notice that the union of all faces $H_{IK}$ for a fixed $K$ is just the dualizing subdivision of the face $\Delta_K \subseteq \Delta_J$. Specifying a net $\nu$ with a fixed set of vertices $V$ gives the alcove decomposition of the corresponding face $\Delta_V$ of $\Delta_\sigma$.

Finally, we show that $\Delta_J \times \Delta_\sigma \setminus \mathcal{P}_{\sigma,J}$ is homotopic to the complement of the open star neighborhood of $\mathcal{P}_{\sigma,J}$ in $dsd \Delta_J \times \Delta_\sigma$. Consider a face $F := H_{IK} \times \mathcal{A}_\nu$. Let $M := F \cap \mathcal{P}$, which we can assume to be a proper subcomplex of $F$. We show that $M$ is collapsible. Then its regular neighborhood $N(M)$ in $F$ is a ball (cf., for example, [9, Theorem 3.26]) and then its (closed) complement collapses to $\partial F \setminus N(M)$. Thus, inductively on dimension, we can collapse all faces in $\Delta_J \times \Delta_\sigma \setminus \mathcal{P}_{\sigma,J}$ which belong to the open star neighborhood of $\mathcal{P}_{\sigma,J}$.

To see that $M$ is collapsible, we note that which alcove face $\mathcal{A}_{I'} \subseteq \mathcal{A}_\nu$ sits over a cubical face $H_{IK'} \subseteq H_{IK}$ depends only on $I'$. Thus, the lattice of faces with a fixed $\mathcal{A}_{I'}$ is Boolean on
the $K'$-index and thus inductively collapses to just the vertex $H_{KK}$. At last, we collapse the remaining alcove $\mathcal{A}_{\nu_0}$ which sits over the vertex $H_{KK}$, where $\nu_0 \subseteq \nu$ contains all chords from $\nu$ dividing $K$.

**Proposition 3.3.** The ball pair $(\Delta_J \times \Delta_\sigma, \mathcal{P}_{\sigma,J})$ is unknotted.

**Proof.** The case $\dim \Delta_J \times \Delta_\sigma \neq 4$ follows from Lemmas 3.1 and 3.2 and Proposition 3.1 (1). The cases $\Delta_J^1 \times \Delta_\sigma^2$ and $\Delta_J^2 \times \Delta_\sigma^1$ are trivial. Thus, it only remains to show the case $\Delta_J^2 \times \Delta_\sigma^2$. It may be possible to extend the explicit isotopy from [10] to an ambient one in this low-dimensional case. However, it is a lot easier to compare $\mathcal{P}_{\sigma,J}$ to its ober-tropical analog $\mathcal{P}_{\sigma,J}$, which we now describe.

We consider the trivalent skeleton $S \subset \text{dsd } \Delta_\sigma$ of the coamoeba triangle similar to the spine $H$ of the amoeba. Then we define $\mathcal{P}_{\sigma,J}$ to be the union of 6 squares (cf. Figure 7):

$$\mathcal{P}_{\sigma,J} = \bigcup_{ij \neq kl} H_{ij} \times S_{kl} \subset \Delta_J \times \Delta_\sigma.$$  

![Figure 7](image1.png)

**Figure 7.** Two skeleta and ober-tropical cell $\mathcal{P}_{\sigma,J} \subset \Delta_J^2 \times \Delta_\sigma^2$.  

The ober-tropical pair $(\Delta_J^2 \times \Delta_\sigma^2, \mathcal{P}_{\sigma,J})$ was shown in [3] to be unknotted, which, in this dimension, is almost trivial. Just look at how the boundary circle $\partial \mathcal{P}_{\sigma,J}$ sits inside the boundary 3-sphere $\partial(\Delta_J^2 \times \Delta_\sigma^2)$. Namely, note that its complement collapses to a circle. Then note that the ball pair $(\Delta_J^2 \times \Delta_\sigma^2, \mathcal{P}_{\sigma,J})$ is the cone over its boundary.

![Figure 8](image2.png)

**Figure 8.** The cellular “roofing” move from $\mathcal{P}_{\sigma,J}$ to $\mathcal{P}_{\sigma,J}$ over the leg $H_{02}$.  

Finally, to see an isotopy from $\mathcal{P}_{\sigma,J}$ to $\mathcal{P}_{\sigma,J}$, we refine the central triangle in $\Delta_\sigma$ by its skeleton $S$ and perform three “raising the roof” elementary cellular moves relative boundary. The moves are in the triangular prisms over the legs of $H$. Figure 8 shows one of the three, the back-facing triangle is on the boundary. Any elementary move can be extended to an ambient isotopy (cf., for example, [9, Proposition 4.15]). This finishes the proof. □

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Proof of Theorem 3.1. Propositions 3.2 and 3.3 provide homeomorphisms of ball pairs $(\Delta_J \times \Delta_\sigma, P_{\sigma,J})$ and $(\Delta_J \times \Delta_\sigma, \mathcal{P}_{\sigma,J})$ which respect the stratifications. By the Alexander trick, the homeomorphisms are homotopic to the identity on all cells. An inductive application of Proposition 3.1 (3) on strata gives an isotopy which respects the stratification.

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