QUANTUM VARIANCE ON QUATERNION ALGEBRAS, II

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Abstract. A method for determining quantum variance asymptotics on compact quotients attached to non-split quaternion algebras is developed in general and applied to “microlocal lifts” in the non-archimedean setting. The results obtained are in the spirit of recent work of Sarnak–Zhao.

The arguments involve a careful analytic study of the theta correspondence, the interplay between additive and multiplicative harmonic analysis on quaternion algebras, the equidistribution of translates of elementary theta functions, and the Rallis inner product formula.

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1. Introduction

1.1. Overview. The quantum variance problem (see e.g. [19, §1], [5], [31, §15.6], [21, §4.1.3], [24, 13, 11, 14, 15, 32]) concerns sums of the shape

$$\sum_{\varphi \in F} \langle \varphi, \Psi_1 \varphi \rangle \langle \Psi_2 \varphi, \varphi \rangle.$$  (1.1)

Here $\Psi_1, \Psi_2$ are fixed mean zero functions on the unit cotangent bundle of a Riemannian manifold $M$ with ergodic geodesic flow, $F$ traverses a sequence of families of microlocal lifts of Laplace eigenfunctions with eigenvalues in $[0, T^2]$, and $T \to \infty$. The problem is to determine the leading order asymptotic behavior of (1.1). The
difficulty of the problem may be appreciated by comparing the expected magnitude $\sim T$ for (1.1) for typical $\Psi_1 = \Psi_2$ with the best known general upper bound $O(T^{\dim(M)/\log T})$ (see e.g. [19, §1] and references for details).

Although a mathematically rigorous solution to the problem seems hopeless on general $M$, Sarnak–Zhao [24] (following Luo–Sarnak [14] and Zhao [32]) managed to solve it completely on $M = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ for $\mathcal{F}$ consisting of Hecke eigenfunctions. It is natural to seek analogous results on other arithmetic quotients, such as the compact quotients attached to orders in quaternion division algebras. The method of Luo, Sarnak and Zhao demonstrates the tremendous power of parabolic Fourier expansions, such as the $q$-expansions $\sum a_n q^n$ enjoyed by classical holomorphic modular forms on $\text{SL}_2(\mathbb{Z})$ at the cusp $\infty$, to establish results that are inaccessible by means of semiclassical analysis or trace formulas alone. Conversely, their technique is fundamentally limited to split quotients, such $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ and its congruence covers, on which such expansions are available.

In this article, we develop systematically a method for studying quantum variance on non-split arithmetic quotients arising from non-split quaternion algebras, in contrast to the split matrix algebra $M_2(\mathbb{Q})$ underlying the quotient $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ considered by Luo, Sarnak and Zhao. Our main result (Theorem 1, stated in §1.8) concerns families of “microlocal lifts” on non-split $p$-adic arithmetic quotients attached to quaternion algebras. We focus on $p$-adic quotients to simplify the analysis; this passage does not affect the fundamental difficulty of the problem, and is orthogonal to the primary novelty that the quotient is non-split. Our core estimates (Theorem 2, stated in §4) are developed in some generality.

1.2. Trace formulas and linear statistics. Let $X := \Gamma \backslash G$ be the quotient by an arithmetic lattice of the points of a semisimple $\mathbb{Q}$-group over a local field, such as the real numbers, and let $\mathcal{F}$ be a “large” collection of eigenfunctions $\varphi : X \to \mathbb{C}$. It is natural to ask for the asymptotic statistics, as $\mathcal{F} \to \infty$ in some sense, of the random measure on $X$ sending a test function $\Psi$ to $\langle \varphi, \Psi \varphi \rangle$, where $\varphi \in \mathcal{F}$ is sampled randomly with respect to (say) the normalized counting measure.

The linear statistics of this random measure are captured by the mean $\Psi \mapsto \mathbb{E}_{\varphi \in \mathcal{F}} \langle \varphi, \Psi \varphi \rangle$. When $\mathcal{F}$ admits a nice harmonic-analytic description, it can be (at least approximately) picked off by a convolution kernel $f \in C_c^\infty(G)$. The mean can then be studied using trace formula techniques: by integrating the pretrace formula

$$\sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \approx \sum_{\varphi \in \mathcal{F}} \overline{\varphi(x)} \varphi(y) \quad (x, y \in \Gamma \backslash G) \quad (1.2)$$

over the diagonal against $\Psi$, one obtains an identity

$$\sum_{\varphi \in \mathcal{F}} \langle \varphi, \Psi \varphi \rangle \approx \int_{x \in \Gamma \backslash G} \Psi(x) \sum_{\gamma \in \Gamma} f(x^{-1} \gamma x) \quad (1.3)$$

whose RHS may be studied by methods for bounding orbital integrals much as in the “Weyl’s law” case $\Psi \equiv 1$. (For an example of such arguments, see §9.2.)

Higher-order statistics such as the $n$-point correlations

$$\langle \Psi_1, \cdots, \Psi_n \rangle \mapsto \mathbb{E}_{\varphi \in \mathcal{F}} \langle \varphi, \Psi_1 \varphi \rangle \cdots \langle \varphi, \Psi_n \varphi \rangle$$

are more mysterious. The quantum variance problem concerns the quadratic statistics (1.1) about which trace formulas alone say little.
1.3. Hecke multiplicativity and variance statistics. Until this work and its prequel, the only known asymptotic formulas for higher-order statistics in this setting of §1.2 were those of Luo–Sarnak–Zhao concerning $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$. The point of departure for their method is that when the eigenfunctions $\varphi$ admit Fourier expansions with coefficients $\lambda(n)$ enjoying a “doubling identity” of the shape

$$\lambda(m)\lambda(n) = \sum \lambda(\cdots),$$

one can try to reduce variance statistics to linear ones and apply trace formulas such as (1.2). This method does not apply when such expansions are not available.

1.4. Theta functions and variance statistics. When the space $X$ arises from a quaternion algebra $B$ (over $\mathbb{Q}$, say), the Eichler/Shimizu theta correspondence provides an analogue of the doubling identity (1.4) that suggests a natural strategy for studying quantum variance. We pursue this strategy here. Let $\mathcal{F}$ be a family of eigenfunctions on $X$. Oversimplifying for now, Shimizu’s theorem (see [28, II.1]) says that one can find

- a space $X'$ (a congruence cover of $\text{PGL}_2(\mathbb{Z}) \backslash \text{PGL}_2(\mathbb{R})$),
- a function of three variables $\Theta : X \times X \times X' \to \mathbb{C}$ (a theta kernel), and
- for each $\varphi \in \mathcal{F}$, a function $\Phi_\varphi : X' \to \mathbb{C}$ (an Eichler/Jacquet–Langlands lift)

with the property that

$$\overline{\varphi(x)\varphi(y)} = \int_z \Phi_\varphi(z) \Theta(x, y; z) \quad \text{for all } \varphi \in \mathcal{F} \text{ and } x, y \in X. \quad (1.5)$$

By integrating the diagonal case $x = y$ of (1.5) against $\Psi$, one obtains

$$\langle \varphi, \Psi \varphi \rangle = \int_z \Phi_\varphi(z) \int_x \Psi(x) \Theta(x, x; z). \quad (1.6)$$

If the functions $\Phi_\varphi$ are orthogonal to one another and the family $\mathcal{F}$ is sufficiently “complete,” then a cavalier application of Parseval’s formula to (1.6) suggests that

$$\sum_{\varphi \in \mathcal{F}} \left( \int_x |\Phi_\varphi|^2 \right)^{-1} \langle \varphi, \Psi_1 \varphi \rangle \langle \Psi_2 \varphi, \varphi \rangle = \int_x \left( \int_y |\Psi_1(x)\Theta(x, x; z)| \int_y |\Psi_2(y)\Theta(y, y; z)| \right). \quad (1.7)$$

The LHS (1.7) may be understood as a reasonable proxy for the quantum variance (1.1) of $\mathcal{F}$ provided that the weights $\int_x |\Phi_\varphi|^2$ are sufficiently uniform in $\varphi$. One aim of this article is to develop robust techniques for determining the asymptotics of the RHS of (1.7), which is not a priori any simpler to analyze than the LHS. A second aim is to apply the resulting machinery to an interesting family of automorphic forms.

We have oversimplified by neglecting that the theta kernel $\Theta$ produced by Shimizu’s theorem may (and generally does) depend upon the automorphic form $\varphi$. For the above argument to make sense, we need to choose one $\Theta$ that works for every element of the family $\mathcal{F}$. It is natural instead to interpret (1.7) as defining a (weighted) family $\mathcal{F}$ in terms of $\Theta$. A third aim of this article is then to clarify in general how to invert the association $\Theta \mapsto \mathcal{F}$.
1.5. Microlocal lifts in the non-archimedean setting. Let $k$ be a non-archimedean local field of characteristic zero. Denote by $\mathfrak{o}$ its maximal order, $q$ its maximal ideal, and $q := \#\mathfrak{o}/q$. For example, one can take $(k, \mathfrak{o}, q, q) := (\mathbb{Q}_p, \mathbb{Z}_p, p\mathbb{Z}_p, p)$ for some rational prime number $p$.

Fix a totally real number field $F$ having $k$ as its completion. Fix a discrete cocompact subgroup $\Gamma$ of $\text{PGL}_2(k)$ arising from a maximal order in a (non-split) totally definite quaternion algebra over $F$ (see §4.5.2 for details). Set

$$\mathbf{X} := \Gamma \backslash \text{PGL}_2(k).$$

To simplify the present exposition, we assume that $F$ has odd narrow class number, so that $\mathbf{X}$ comes with a natural family of commuting Hecke operators (see §4.5.3), which also commute with the right translation action by $\text{PGL}_2(k)$. Fix any $\text{PGL}_2(k)$-invariant measure on $\mathbf{X}$; denote by $\text{vol}(\mathbf{X})$ the total volume and define $L^2(\mathbf{X})$ and $\langle , \rangle$ by integrating.

**Definition 1.** By an eigenfunction, we shall mean a nonzero function $\varphi : \mathbf{X} \to \mathbb{C}$ that is smooth (i.e., right-invariant under some open subgroup), is an eigenfunction under every Hecke operator, and generates an irreducible representation of $\text{PGL}_2(k)$ under right translation.

Some of the Hecke operators are involutions; an eigenfunction will be called even if it has eigenvalue $+1$, rather than $-1$, under each such involution.

**Definition 2.** Let $N$ be a large integer and let $\omega : \mathfrak{o}^\times \to \mathbb{C}^\times$ be a unitary character of conductor $N$, thus $\omega$ is trivial on $1+q^N$ but not on $1+q^{N-1}$. An eigenfunction $\varphi$ will be called a microlocal lift of orientation $\omega$ if it satisfies $\varphi(xg) = \omega(a^2/\det(g))\varphi(x)$ for all $x \in \mathbf{X}$ and all

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathfrak{o}) \cap \begin{pmatrix} \mathfrak{o} & \frac{1}{2}q^{\lfloor N/2 \rfloor} \\ 0 & \mathfrak{o} \end{pmatrix}. \quad (1.8)$$

There is a finite set $\mathcal{F}_\omega$ consisting of unit vector microlocal lifts of orientation $\omega$ so that $\{\text{microlocal lifts of orientation } \omega \} = \bigsqcup_{\varphi \in \mathcal{F}_\omega} \mathbb{C}^\times \varphi$. (see §9 or [18]). Let $\mathcal{F}_N$ denote the union of $\mathcal{F}_\omega$ over all $\omega$ of conductor $N$.

**Lemma.** There is an explicit $c > 0$ so that for $N$ large enough,

$$|\mathcal{F}_N| = cq^{2N}. \quad (1.9)$$

**Proof.** By a trace formula computation (see §9.2).

**Remark.** From a representation-theoretic or microlocal-analytic perspective, $\mathcal{F}_N$ is analogous to the family of microlocal lifts of Hecke–Maass eigenforms on a compact arithmetic hyperbolic surface with Casimir eigenvalues $1/4+t^2$, $t \in [T, 2T]$, $T \approx q^N$. For example, their limit measures enjoy diagonal invariance for local reasons. We refer to [18, Thm 25, Rmk 26] for a discussion of the sense in which the microlocal lifts considered here are actually “lifts.”

1.6. Context: worst case behavior and mean statistics. It was shown recently in [18] that microlocal lifts on $\mathbf{X}$ satisfy an analogue of the arithmetic quantum unique ergodicity theorem [26, 12]: for each continuous $\Psi : \Gamma \backslash \text{PGL}_2(k) \to \mathbb{C}$,

$$\lim_{N \to \infty} \max_{\varphi \in \mathcal{F}_N} |\langle \varphi, \Psi \varphi \rangle - \langle \Psi, 1 \rangle/\langle 1, 1 \rangle| = 0. \quad (1.10)$$

The Lindelöf hypothesis predicts more precisely that

$$\langle \varphi, \Psi \varphi \rangle = \langle \Psi, 1 \rangle/\langle 1, 1 \rangle + O(q^{-N/2+o(N)}) \text{ for } \Psi \text{ fixed and } \varphi \in \mathcal{F}_N, \quad (1.11)$$
and it is generally believed that this prediction is essentially optimal.

**Lemma.** For each continuous \( \Psi : \Gamma \setminus \mathrm{PGL}_2(k) \to \mathbb{C} \),

\[
\lim_{N \to \infty} |F_N|^{-1} \sum_{\varphi \in F_N} \langle \varphi, \Psi \varphi \rangle = \langle \Psi, 1 \rangle / (1, 1).
\]  

(1.12)

**Proof.** This follows by averaging (1.10); alternatively, one can argue as in §1.2 using the pretrace formula (see §9.2). \( \square \)

**1.7. Definition of variance sums.** Consider for each large natural number \( N \) the random measure on \( X \) sending a continuous function \( \Psi : X \to \mathbb{C} \) to \( \langle \varphi, \Psi \varphi \rangle \), where \( \varphi \in F_N \) is sampled uniformly at random. The lemma of §1.6 says that this random measure has expectation tending as \( N \to \infty \) to the invariant probability measure on \( X \). The most natural definition of the variance of that random measure would then be the bilinear form on smooth mean zero functions \( \Psi_1, \Psi_2 : X \to \mathbb{C} \) given by

\[
|F_N|^{-1} \sum_{\varphi \in F_N} \langle \varphi, \Psi_1 \varphi \rangle \langle \Psi_2 \varphi, \varphi \rangle.
\]

For technical reasons related to our method (which the reader may infer already from (1.7)), we consider instead the slightly modified (and unnormalized) sums

\[
V_N(\Psi_1, \Psi_2) := \sum_{\varphi \in F_N} \iota_\varphi \langle \varphi, \Psi_1 \varphi \rangle \langle \varphi \Psi_2, \varphi \rangle,
\]  

(1.13)

where \( \iota_\varphi := L^{(S)}(\varphi, 1) \) is a standard “harmonic weight” (see §3.3.3, §9.3). Such weights are positive and mild in that their size is \( \iota_\varphi = q^{o(N)} \) (see (3.9)) and their mean essentially constant. Although there are well-developed techniques for removing them as in [24], we retain them here for simplicity.

It follows from the compactness of \( X \) that every smooth function on \( X \) is a finite linear combination of eigenfunctions, so in studying (1.13), we may assume by linearity that \( \Psi_1, \Psi_2 \) are eigenfunctions. It is natural to assume further that

- \( \Psi_1, \Psi_2 \) are **even**, as otherwise \( V_N(\Psi_1, \Psi_2) = 0 \) by parity considerations, and that
- \( \Psi_1, \Psi_2 \) are **strongly of mean zero** in that they are orthogonal to the finite collection of one-dimensional subrepresentations of \( L^2(X) \), as otherwise \( V_N(\Psi_1, \Psi_2) = 0 \) by basic properties of the families \( F_N \) (see [18, Lem 51]).

**1.8. Statement of main result.** By (1.9) and (1.11), one expects \( V_N(\Psi_1, \Psi_2) \) to have magnitude at most \( O(q^N) \). It should be possible to confirm this expected upper bound using the Cauchy–Schwarz inequality, the triple product formula, and well-developed techniques for averaging families of \( L \)-functions as in [24], but the true asymptotics of \( V_N(\Psi_1, \Psi_2) \) are much subtler: in the off-diagonal case that \( \Psi_1, \Psi_2 \) generate distinct irreducible representations \( \pi_1 \neq \pi_2 \subseteq L^2(X) \), one expects the **signs** of the quantities \( \langle \varphi, \Psi_1 \varphi \rangle \) and \( \langle \varphi, \Psi_2 \varphi \rangle \) to vary independently, suggesting additional cancellation in (1.13). The primary novelty in the following result is that we detect such cancellation in the off-diagonal case; a secondary novelty is that in the diagonal case, we determine the main term.

**Theorem 1.** For even eigenfunctions \( \Psi_1, \Psi_2 \) strongly of mean zero, the limit

\[
\lim_{N \to \infty} q^{-N} V_N(\Psi_1, \Psi_2)
\]  

(1.14)
exists. The limit is zero unless $\Psi_1, \Psi_2$ generate the same irreducible representation $\pi \subseteq L^2(X)$. In that case, it is equal to

$$c_0 L(S)(\pi, \frac{1}{2}) \int_{h \in N(H)} \left( \int_{x \in X} \Psi_1(xh) \bar{\Psi_2(x)} \right),$$

(1.15)

where (see §9 for details)

• $c_0$ is an explicit positive constant,

• $S$ denotes the set of “bad places,”

• $L(S)(\pi, \frac{1}{2})$ denotes the central $L$-value without Euler factors in $S$, and

• $N(H)$ denotes the normalizer of the diagonal torus $H$ in $\text{PGL}_2(k)$.

Remark 1. The integral (1.15) converges as written (see §2.1.5, §2.1.6).

Remark 2. As in [24], the identity (1.15) admits an intriguing semiclassical interpretation whereby the arithmetical values $L(S)(\pi, \frac{1}{2})$ quantify the deviation of the asymptotic quantum variance from the (symmetrized) classical variance of the diagonal flow.

Remark 3. We establish the stronger assertion that $q^{-N} V_N(\Psi_1, \Psi_2)$ differs from its limit by $O(Nq^{-N})$ (see §9). We expect that our method is capable of refining that error term to $O(q^{-N})$ and proving that such an estimate is essentially best possible (see [19, §6.5]), but we do not pursue such refinements here.

Remark 4. We note as in [15] that Theorem 1 confirms the Lindelöf prediction (1.11) on average and implies that it is essentially optimal if it is true.

Remark 5. The off-diagonal case is that in which the method of Luo–Sarnak–Zhao requires a cusp. It is conceivable that one could establish the diagonal case of Theorem 1 by averaging the triple product formula as in [24]. Our method does not use the triple product formula.

1.9. Comparison with the prequel. We highlight some differences between the results and aims of [19] and those of this article.

1. In [19], the observables $\Psi_1, \Psi_2 : X \to \mathbb{C}$ were restricted to be right-invariant by the maximal compact subgroup $K := \text{PGL}_2(o)$. In this article, we study arbitrary fixed observables on the full “phase space” $X$ rather than on the “configuration space” $X/K$. The simplifying restriction of the prequel allowed us to get by with some ad hoc computations in places where a more systematic approach is required here.

This jump in complexity is analogous to that from [15, 32] to [24], but the manner in which the new complexity is addressed differs completely (in a stronger sense than that the methods themselves differ completely): In [24], phase space observables were treated by an inductive technique involving weight isotypic vectors, while the methods developed here apply directly to general phase space observables.

2. In the prequel, we considered families $F'_N$ of balanced newvectors. Here, we consider families $F_N$ of microlocal lifts. The methods developed in Part 1 of this article apply robustly to a large class of families; given those methods, neither $F_N$ nor $F'_N$ is much more difficult to analyze than the other. We focused in [19] on newvector families because of their familiarity. We focus here on the families $F_N$ because of their strong analogy with those in the
motivating work of Luo–Sarnak–Zhao and because the formulas (1.15) for
the main term for \( \mathcal{F}_N \) are more aesthetically appealing than those for \( \mathcal{F}'_N \).

(3) The aim of [19] was to introduce the method by application to the simplest
non-trivial non-split case of the quantum variance problem and in the most
elementary language possible.\(^1\) The aim here is instead to develop the
technique as clearly as possible in its natural generality. We hope the two
articles serve complementary purposes.

(4) It may be worth recording that for the reasons indicated in the previous
three points, the two articles have essentially no logical overlap.

1.10. Discussion of method. The general strategy of §1.4 applies in the setting
of Theorem 1: modulo a preliminary technical partition of the family \( \mathcal{F} = \mathcal{F}_N \), we
construct \( f = f_N \) for which (1.2) holds and then \( \Theta \) for which (1.7) holds. (For a
precise definition of \( f \), see the end of §5.) The integral \( \int \Psi_i(z) \Theta(x, x; z) \) does not
define a theta lift of \( \Psi_i \) in the traditional sense, but instead decomposes as a sum
of products \( \theta_i(z) h_i(z) \), where \( \theta_i \) is a variant of the Jacobi theta function and \( h_i \) is
a theta lift of \( \Psi_i \). The RHS of (1.7) then decomposes as a sum of inner products
\[
\langle \theta_1 h_1, \theta_2 h_2 \rangle
\]
Suppose we can approximate each such inner product by
\[
\langle \theta_1, \theta_2 \rangle \langle h_1, h_2 \rangle.
\]
The Rallis inner product formula [6, 7] for theta lifts applies to (1.17); summing it
up, we obtain
\[
\sum_{\phi \in \mathcal{F}} (| \Phi_\phi |^2)^{-1} \langle \phi, \Psi_1 \phi \rangle \langle \Psi_2 \phi, \phi \rangle \approx (\ast) \int_{h \in \mathbb{G}} I_f(h) \left( \int_{x \in X} \Psi_1(xh) \Psi_2(x) \right),
\]
where:

- \( \approx \) means up to the error incurred by replacing each term (1.16) with (1.17);
- \( (\ast) \) means “modify by a central \( L \)-value as in Theorem 1;” and
- \( I_f(h) := \int_{g \in \mathbb{G}} \mathfrak{S} f(h^{-1} gh) \mathfrak{S} f(g) \), where \( \mathfrak{S} f(g) := (f(g) + f(g - \text{tr}(g)))/2.\)

To complete the proof of Theorem 1, it suffices now to show that

(i) \( I_f : \mathbb{G} \to \mathbb{C} \) tends to the “delta distribution” on the normalizer \( N(H) \) as
the parameter \( N \) tends to \( \infty \), and that
(ii) the error hidden by \( \approx \) is negligible.

Problem (i) is purely local. Problem (ii) involves both local and global difficulties;
a critical global input to its solution was developed in [20].

\(^1\)In the prequel, the base field is unimportantly restricted to be \( \mathbb{Q}_2 \) (rather than \( \mathbb{Q}_p \) for an odd
prime \( p \)) in order to reduce the set of bad primes from \( \{2, p\} \) to \( \{2\} \), noting that 2 is always bad
when studying half-integral weight forms.
To indicate in more detail how this works, assume for simplicity that \( k \) arises as a completion of \( \mathbb{Q} \). The proof may then be summarized by the sequence

\[
q^{-N} V_N(\Psi_1, \Psi_2) = \langle \theta(z) h_1(q^{2N} z), \theta(z) h_2(q^{2N} z) \rangle
\]
\[
= \langle \sqrt{|\theta|^2(z)}, \overline{h_1 h_2(q^{2N} z)} \rangle
\]
\[
= \langle \sqrt{|\theta|^2}, 1 \rangle(1, \overline{h_1 h_2}) + O(Nq^{-N})
\]
\[
= c_0 L^{(S)}(\pi, 1/2) \int_{h \in N(H)} (\int_{x \in \mathcal{X}} \Psi_1(xh) \overline{\Psi_2(x)}) + O(Nq^{-N}),
\]

where \( \theta \) is essentially the weight 1/2 Jacobi theta function, \( z \) denotes an integration parameter in the upper half-plane, \( h_1, h_2 \) are fixed weight 3/2 theta lifts of \( \Psi_1, \Psi_2 \) to some congruence cover of \( \Gamma_0(4) \backslash \mathbb{H} \), and inner products are taken on congruence covers of \( \Gamma_0(4) \backslash \mathbb{H} \) with respect to the natural probability measures. The first step (1.19) was discussed in §1.4 in high level terms. The obvious second step (1.20) forms the cornerstone of the argument; see [23] for related discussion. The third step (1.21) is a variant of the equidistribution of Hecke operators; we have discussed it extensively in [19, 20]. The fourth step (1.22) is an explication of the Rallis inner product formula followed by the asymptotic analysis of the integral \( I_f(h) \) discussed above.

1.11. The shape of the key inner product. The precise shape of the RHS of (1.19) (namely, the “separation” of \( \theta(z) \) and \( h_i(q^{2N} z) \) by the dilation \( z \to q^{2N} z \)) is evidently crucial to the success of the method, so we sketch how it arises. Let \( \phi = \phi_N : M_2(k) \to \mathbb{C} \) denote the Schwartz–Bruhat function related to the kernel \( f = f_N \) by

\[
\phi(x) := \begin{cases} 0 & \text{if } x \notin \text{GL}_2(k) \\ 1 \times (\det(x)) f(pr(x)) & \text{if } x \in \text{GL}_2(k), \end{cases}
\]

where \( pr : \text{GL}_2(k) \to \text{PGL}_2(k) \) denotes the canonical projection. Let \( \mathcal{F} \) denote the Fourier transform on \( M_2(k) \). (We hope the dual use of this symbol for Fourier transforms and families introduces no confusion.) Then

\[
\mathcal{F} \phi \left( \begin{pmatrix} d + a & b \\ c & d - a \end{pmatrix} \right) \approx \begin{cases} 1 & |d| = O(1); |b|, |c| = O(q^{N/2}); |a| \asymp q^N \\ 0 & \text{otherwise} \end{cases}
\]

The detailed statement and proof of the computation (1.24) may be found in §6. The key features are the shape of the support and the uniform smoothness under simultaneous dilation of the parameters \( a, b, c \). Ignoring the less important variable \( d \), one can think of \( \mathcal{F} \phi \) as capturing roughly the Fourier transform of the pullback of \( f \) to the Lie algebra of \( \text{PGL}_2(k) \). We note in passing that the subgroup \( N(H) \) arises eventually from (1.24) as the “normalizer of the limiting support” of \( \mathcal{F} \phi \).

Consider now the Hecke twisted pretrace formula

\[
\sum_{\gamma \in M_n} f(x^{-1} \gamma y) = \sum_{\varphi \in \mathcal{F}_N} \sqrt{\lambda_\varphi(n)} \overline{\varphi(x)} \varphi(y),
\]

where \( M_n \) is as in the classical definition of the Hecke operator \( T_n \) and \( \sqrt{\lambda_\varphi(n)} \) denotes the Hecke eigenvalue. Let \( R \subseteq B \) denote the maximal order underlying the
construction of $\Gamma$. Taking $x = y =: g$ in (1.25) and summing against $e(nz)$ over positive integers $n$ having only “good” prime divisors gives

$$\sum_{\gamma \in R} \phi(g^{-1}\gamma g)e(\det(\gamma)z) = \sum_{\varphi \in F_N} |\varphi|^2(g)\Phi_{\varphi}(z)$$

where $e(z) := e^{2\pi iz}$ and $\Phi_{\varphi}(z) := \sqrt{\pi} \lambda_{\varphi}(n)e(nz)$ denotes an Eichler/Jacquet−Langlands lift of $\varphi$. Suppose henceforth that $k = \mathbb{Q}_p$. By the inversion formula for theta functions, we obtain

$$\varphi \in F_N \quad \sum_{\varphi \in F_N} |\varphi|^2(g)\Phi_{\varphi}(-1/z) \approx \sum_{\gamma \in R[1/p]} F\phi(g^{-1}\gamma g)e(\det(\gamma)z)$$

$$\approx \sum_{m, \beta} F\phi(m + g^{-1}\beta g)e(m^2 z)e(\det(\beta)z),$$

where the sum is over $m \in \mathbb{Z}[1/p]$ and $\beta \in R[1/p]^0$. (Here $\approx$ means “up to unimportant inaccuracies.”) By integrating against $\Psi(g)$, we obtain

$$\sum_{\varphi \in F_N} \Phi_{\varphi}(-1/z)(\varphi, \Psi) = \sum_{m, \gamma} c(m^2 z)e(\det(\beta)z) \int_{g \in \Gamma \setminus G} \Psi(g) F\phi(m + g^{-1}\beta g).$$

The multiplicity one theorem on $X$ implies that for $\varphi, \varphi' \in F_N$,

$$\langle \Phi_{\varphi}, \Phi_{\varphi'} \rangle \approx \iota_{\varphi} \text{ if } \varphi = \varphi', \quad = 0 \text{ otherwise.}$$

By (1.24) and the assumption that $\Psi$ is fixed, one deduces that

$$\int_{g \in \Gamma \setminus G} \Psi(g) F\phi(m + g^{-1}\beta g) \approx 1_{\mathbb{Z}_p}(m)\phi''_{\Psi}(p^N \beta)$$

where $\phi''_{\Psi}$ is independent of $N$. (A “toy version” of such reasoning: if a function on $\mathbb{R}^3$ is smooth with respect to polar coordinates and supported away from the origin, then it is smooth.) By Parseval applied to (1.26), (1.27) and (1.28), one arrives at an identity of the form (1.19).

1.12. Organization of this paper. The main result is Theorem 1, stated somewhat informally above and more precisely (and generally) as Theorem 3 in the final section §9 of this paper, which also contains the proof.

To present the proof as clearly as possible, we separate the difficulties concerning general families from those specific to the $F_N$ considered above. The former may be found in Part I, the latter in Part II. The two parts are weakly coupled.

The main result of Part I is Theorem 2, stated in §4. Its proof involves local (§2) and global (§3) preliminaries. Its conclusion reduces the quantum variance problem to local problems of three sorts, which are treated in Part II for the families $F_N$:

- (§5) The comparatively mild difficulties associated with producing a convolution kernel $f$ that picks off a given family $F$, as in (1.2).
- (§7) Those associated with determining the main term (i.e., asymptotically evaluating distributions such as the $I_f$ considered in §1.10).
- (§8) Those concerned with bounding error terms (i.e., differences between inner products as in (1.16) and (1.17)). An important point is that Theorem 2 reduces this global problem to a local one.
For the latter two problems, the key input is a careful analysis (§6) of the “Fourier transform” of the convolution kernel $f$.

The reader might first study carefully §1.13, §4.1, §4.2, §4.3, §4.5.1, §5.4.3, §7.1, §8.1, §9.1 and §9.3, skipping or skimming any sections before or between; we have made some effort to keep those sections essentially self-contained.

1.13. General notation. For a quaternion algebra $B$ over a field or adele ring $A$, we denote by $\iota : B \to B$ the main involution, by $\text{nr} : B \to A$ the reduced norm $\text{nr}(x) := xx^\prime$, by $\text{tr} : B \to A$ the reduced trace $\text{tr}(x) := x + x^\prime$, and by

$$B^0 := \{ x \in B : \text{tr}(x) = 0 \}$$

the subspace of traceless quaternions. We employ the notations

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad t(y) := \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \quad n'(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

and

$$\text{Ad}(g)x := gxg^{-1}, \quad \text{Ad}(g)f(x) := f(g^{-1}xg)$$

and

$$\mathcal{G}f(x) := \frac{f(x) + f(x - \text{tr}(x))}{2}$$

whenever they make sense. For example, this is the case if $g$ belongs to the unit group $B^\times$ of a quaternion algebra $B$ over $A$ as above and $x$ belongs to (resp. $f$ is a function on) one of the sets $B^\times/A^\times, B, B^0$. (As we explain below, one may interpret $\mathcal{G}$ as “projection onto $O_1$-invariants.”)

We define the right regular representation $\rho_{\text{reg}}(g)f(x) := f(xg)$ whenever it makes sense.

Given a local (resp. global) field $F$ and a nontrivial unitary character $\psi$ of $F$ (resp. of $\mathbb{A}/F$) and an element $a \in F^\times$, we denote by $\psi^a$ the nontrivial unitary character with the same domain as $\psi$ given by $\psi^a(x) := \psi(ax)$.

For a finite-dimensional vector space $V$ over a local field or adele ring, we denote by $\mathcal{S}(V)$ the space of Schwartz–Bruhat functions $\phi : V \to \mathbb{C}$, topologized as usual (see e.g. [30, §11]).

Let $G$ be a group over an adele ring or a finite product of local fields. We let $C_c^\infty(G)$ denote the space of smooth compactly supported functions; as usual, smooth means infinitely differentiable (resp. locally constant) with respect to the archimedean (resp. non-archimedean) variables. Assume that we have equipped $G$ with a Haar measure. Let $\pi$ be a smooth representation of a group that contains $G$. Let $f \in C_c^\infty(G)$. We then define the operator $\pi(f) \in \text{End}(\pi)$ by $\pi(f)v := \int_{g \in G} f(g)\pi(g)v$.

The use of Vinogradov notation is standard: $A = O(B)$, $A \ll B$ and $B \gg A$ each signify that $|A| \leq c|B|$ for some “constant” $c$, with dependencies indicated by subscripts; $A \asymp B$ signifies that $A \ll B \ll A$.

We write $1_E$ for the characteristic function of a subset $E$ of some set $X$. For an assertion $A$, we set $1_A := 1$ if $A$ is true and $1_A := 0$ if $A$ is false.

We set $\mathbb{C}^{(1)} := \{ z \in \mathbb{C}^\times : |z| = 1 \}$.

We adopt the convention that main results (Theorems 1, 2, 3 in §1.8, §4.3, §9.3, respectively) are called theorems, the most important intermediary results original to this article (Propositions 1, 2, 3, 4, 5, 6 in §3.5.6, §4.4.8, §5.4.3, §6.3, §7.1, §8.1) are called propositions, and everything else (including deep cited work) is called a lemma.
Part 1. Quantum variance and theta functions

2. Local preliminaries

The purpose of this section is collect local definitions, notation and identities for later use. The notation introduced here should be self-descriptive with the exception of that for the similitude Weil representation $\Omega$ defined in §2.2.5.

Let $k$ be a local field of characteristic $\neq 2$, thus $k$ is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ or (for $p \neq 2$) of $\mathbb{F}_p(t)$. The assumption on the characteristic is relevant only for sections discussing the Weil representation.

Let $\psi : k \to \mathbb{C}^{(1)}$ be a nontrivial unitary character of $k$, and let $B$ be a quaternion algebra over $k$. Set $G := B^\times /k^\times$. When $k$ is non-archimedean, let $R \subset B$ be a maximal order.

2.1. Generalities.

2.1.1. The number field. We denote by $|\cdot| := |\cdot|_k: k \to \mathbb{R}_{\geq 0}$ the normalized absolute value, so that $d(cx) = |c| \, dx$ for $c \in k$ and any Haar measure $dx$ on $k$.

Let $\zeta_k(s)$ denote the local zeta function, thus $\zeta_k(s) = \pi^{-s/2}\Gamma(s/2), 2(2\pi)^{-s}\Gamma(s)$ or $(1 - q^{-s})^{-1}$ as $k = \mathbb{R}, \mathbb{C}$, or a non-archimedean local field with residue field of cardinality $q$.

Recall that $B$ is split if it is isomorphic to the algebra $M_2(k)$ of $2 \times 2$ matrices. Otherwise, $B$ is called non-split or ramified; in that case, it is the unique (up to isomorphism) quaternion division algebra over $k$, and the group $G$ is compact.

When $k$ is non-archimedean, we denote by $\sigma$ the maximal order, by $q$ the maximal ideal, by $\omega := \#\sigma/q$ the cardinality of the residue field, by $\infty \in q = \omega o$ a uniformizer (thus $|\omega| = q^{-1}$), and by $\Delta_\psi$ the absolute conductor of $\psi : k \to \mathbb{C}^{(1)}$, thus $\Delta_\psi = q^d$ if $\psi$ is trivial on $q^{-d}$ but not on $q^{-d-1}$. Recall that $\psi$ is unramified if $\Delta_\psi = 1$.

2.1.2. Measures. For $X \in \{k, B^0, B\}$, define the perfect pairing $\langle \cdot, \cdot \rangle : X \otimes X \to k$ by $\langle x, y \rangle := xy$ if $X = k$ and by $\langle x, y \rangle := \text{tr}(x'y)$ if $X = B^0, B$. Equip $X$ with the Haar measure $dx$ for which the Fourier transform $\mathcal{F} : \mathcal{S}(X) \to \mathcal{S}(X)$ defined by $\mathcal{F} f(\xi) := \int_{x \in X} f(x)\psi(\langle x, \xi \rangle) \, dx$ satisfies the inversion formula

$$\mathcal{F} \mathcal{F} f(x) = f(-x).$$

(2.1)

Equip $k^\times$ with the Haar measure $\int_{k^\times} f := \int_{x \in k^\times} f(x) \, dx_{|z|}$.

The quotient $k^\times /k^{\times 2}$ is finite. We equip it with the Haar measure compatible with the squaring map, so that for $f \in C_c(k^\times),$

$$\int_{x \in k^\times} f(x) \frac{dx}{|x|} = \int_{y \in k^{\times 2}/k^{\times 2}} \int_{z \in k^\times} f(yz^2) \frac{dz}{|z|}.$$ (2.2)

For $f : k^\times /k^{\times 2} \to \mathbb{C}$, one has explicitly $\int_{x \in k^\times /k^{\times 2}} f(x) = \frac{|k|}{2} \sum_{x \in k^\times /k^{\times 2}} f(x)$.

Equip $G$ with the Haar measure $dg$ for which the integral formula

$$\int_{x \in B} f(x) \, dx = \int_{g \in G} \left( \int_{x \in k^\times} |n(x)|^2 f(xg) \frac{dz}{|z|} \right) \, dg$$ (2.3)

holds for $f \in C_c(B)$. When $B = M_2(k)$ is split, so that $G = \text{PGL}_2(k)$, a direct calculation with differential forms gives for $f \in C_c(G)$ that

$$\int_G f = \int_{x_1, x_2 \in k^\times} \int_{y \in k^\times} f(n'(x_1)n(x_2)a(y)) \, dx_1 \, dx_2 \frac{dy}{|y|}.$$ (2.4)
2.1.3. Volume formulas. Assume (for all but the final assertion of §2.1.3) that \( k \) is non-archimedean. Write \( \text{vol}(E \subseteq X) \) to denote the volume of \( E \) with respect to the measure that we have defined on \( X \). Let \( J \subseteq G \) denote the image of \( R^\times \); if \( B \) is split, then \( J \) is a maximal compact subgroup of \( G \), otherwise it has index 2 in the compact group \( G \). Abbreviate \( \text{vol}(\mathfrak{o}) := \text{vol}(\mathfrak{o} \subseteq k) \), \( \text{vol}(\mathfrak{o}^\times) := \text{vol}(\mathfrak{o}^\times \subseteq k^\times) \), \( \text{vol}(J) := \text{vol}(J \subseteq G) \), \( \text{vol}(R) := \text{vol}(R \subseteq B) \) and \( \Delta := \Delta_\psi \). Let \( \Delta_B \) denote the reduced discriminant, thus \( \Delta_B^\times = 1 \) or \( q \) according as \( B \) splits or ramifies. Set \( \zeta_B(s) := \zeta_k(2s)\zeta_k(2s - 1) \) if \( B \) splits and \( \zeta_B(s) := \zeta_k(2s) \) otherwise.

Lemma.

(i) \( \text{vol}(\mathfrak{o}) = \Delta^{-1/2}, \text{vol}(\mathfrak{o}^\times) = \zeta_k(1)^{-1}\Delta^{-1/2} \).

(ii) \( \text{vol}(R) = \Delta_B^{-1}\Delta^{-4/2}, \text{vol}(J) = \zeta_k(1)\zeta_B(1)\Delta_B^{-1}\Delta^{-3/2} \).

(iii) If \( B \) is split, then

\[
\frac{\text{vol}(R)}{\text{vol}(J)\Delta^{-1/2}} = \zeta_k(2).
\]

(iv) If \( k \) is real, \( B \) is non-split and \( \psi(x) = e^{2\pi ix} \), then \( \text{vol}(G) = 4\pi^2 \).

Proof. For (i)—(iii), we may reduce by dimensional analysis to the case \( \Delta = 1 \). The required formulas then follow from (2.2) applied to \( f = 1_\mathfrak{o} \) or \( f = 1_R \) and by (2.3) applied to \( f = 1, \psi \) (see [27, Lem 2.4.3] for details). For (iv), set \( f(x) := e^{-2\pi \text{nr}(x)} \). Apply (2.1) to see that \( \int_G f = 1 \). Apply (2.3) and the substitution \( z \mapsto z/(2\pi \text{nr}(g)) \) to deduce that \( (2\pi)^2 = \text{vol}(G) \int_{z \in \mathbb{R}^\times} |z|^{4e^{-2|z|^2}} \frac{dz}{|z|^2} = \text{vol}(G) \). \( \Box \)

2.1.4. Cartan decomposition. Suppose \( B = M_2(k) \), so that \( G = \text{PGL}_2(k) \). Let \( K \subseteq G \) be the standard maximal compact subgroup. Then \( G = \cup_{g \in k^\times, |g| \leq 1} Ka(y)K \). When \( k \) is non-archimedean, one has for \( f \in \mathcal{C}_c(K\backslash G/K) \) the integral formula

\[
\int_G f = \text{vol}(K) \sum_{m \geq 0} q^m (1 + 1_{m > 0}q^{-1}) f(a(z^m)). \tag{2.5}
\]

2.1.5. The \( \Xi \)-function. Given a maximal compact subgroup \( K \subseteq G \), let \( \Xi : G \to \mathbb{R}_{\geq 0} \) denote the Harish-Chandra function relative to \( K \):

- If \( B \) is non-split, then \( \Xi \equiv 1 \).
- If \( B \) is split, then \( \Xi(g) = \langle gv, v \rangle \) where \( v \) is a \( K \)-invariant unit vector in the unitary induction of the trivial character of a Borel subgroup of \( G \) (see [4]).

The following properties of \( \Xi \) are relevant for us:

(1) It satisfies \( \Xi(1) = 1 \), and is bi-\( K \)-invariant.

(2) If \( B \) is split, then under any identification \( G = \text{PGL}_2(k) \), one has \( \Xi(a(y)) \propto \log(t)/t^{1/2} \) with \( t := |y| + |y|^{-1} \).

(3) Let \( \pi \) be an irreducible unitary representation of \( G \). If \( B \) is split, assume that \( \dim(\pi) > 1 \). Then there exists \( \delta > 0 \) so that for \( v_1, v_2 \in \pi \), one has \( \langle gv_1, v_2 \rangle \ll_{v_1, v_2} \Xi(g)^\delta \) for all \( g \in G \). (See for instance [16, §2.5.1] for a more precise assertion).

2.1.6. Convergence lemmas. The estimates collected here are standard.

Lemma 1. Either let \( X \) be one of the spaces \( B^0, B \) and take \( \phi_1, \phi_2 \in S(X) \), or let \( X = G \) and take \( \phi_1, \phi_2 \in \mathcal{C}_c^\infty(G) \). For \( g \in G \), one then has

\[
\langle \text{Ad}(g)\phi_1, \phi_2 \rangle_{L^2(X)} \ll_{\phi_1, \phi_2} \Xi(g)^2.
\]
Proof. We treat the case $X = B^0$; the other cases are similar. The estimate is trivial unless $B$ is split, so assume that $B = M_2(k)$. By the Cartan decomposition, we may assume that $g = a(y)$ with $|y| \leq 1$. It suffices then (ignoring logarithmic factors) to show that
\[
\int_{a,b,c \in k} \overline{\phi_1(a \cdot y^{-1} b)} \phi_2(a \cdot b) \ll_{\phi_1, \phi_2} |y|.
\]
For this, we substitute $b \mapstoyb$ and appeal to the rapid decay of $\phi_1, \phi_2$. \hfill $\Box$

Lemma 2. Let $\delta > 0$.
(1) The integral $\int_{g \in G} \Xi^{2+\delta}(g)$ converges.
(2) Let $E$ be a separable quadratic subalgebra of $B$. Let $H \leq G$ denote the image of $E^\times$. Equip $H$ with some Haar measure. Then the integral $\int_{h \in H} \Xi^3(h)$ converges.

Proof. We may assume that $B = M_2(k)$ is split and that $E$ is the split diagonal torus, as otherwise the groups involved are compact. The convergence of the second integral then follows from that of $\int_{g \in k^\times} \min(|y|, |y|^{-1})^\delta$. For the first integral, we integrate using the Cartan decomposition; the convergence follows similarly. \hfill $\Box$

2.1.7. Conventions. By a representation of a $k$-group, we shall always mean

- a smooth representation, if $k$ is non-archimedean, and otherwise
- the space of smooth vectors in a unitary representation.

2.2. Weil representations.

2.2.1. Quadratic spaces. Let $V$ be a quadratic space over $k$, thus $V$ is a finite-dimensional $k$-vector space equipped with a non-degenerate quadratic form $q_V : V \times V \to k$. We denote by $b_V : V \otimes V \to k$ the associated non-degenerate bilinear form given by $b_V(x, y) := q_V(x + y) - q_V(x) - q_V(y)$.

Recall that $O(V) := \{g \in GL(V) : q_V(gx) = q_V(x) \text{ for all } x \in V\}$, $GO(V) := \{g \in GL(V) : \text{there exists } \lambda \in k^\times \text{ so that } q_V(gx) = \lambda q_V(x) \text{ for all } x \in V\}$, and $SO(V) := SL(V) \cap O(V)$. The group $GO(V)$ contains the subgroup $k^\times$ of scalar operators, and we set $PGO(V) := GO(V)/k^\times$.

Let $\mu_V$ denote the measure on $V$ that is $(\psi, b_V)$-self dual, i.e., that for which $\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V)$ defined by $\mathcal{F}\phi(\xi) := \int_{x \in V} \phi(x) \psi(b_V(x, \xi)) d\mu_V(x)$ satisfies $\mathcal{F}\mathcal{F}\phi(x) = \phi(-x)$.

The following examples of quadratic spaces are relevant for us:

(1) $V = B$, $q_V = nr$, so that $b_V(x, \xi) = tr(x^t \xi) = \langle x, \xi \rangle$.
(2) $V = B^0$, $q_V$ the restriction of $nr$. The natural map $Ad : G \to SO(B^0)$ is an isomorphism.
(3) $V = k$, regarded as a subspace of $B$, and $q_V$ the restriction of $nr$, thus $q_V(x) = x^2$ and $b_V(x, y) = 2xy$ for $x \in V$. In this case, we denote the orthogonal group by $O_1(k) := O(V) \cong \{\pm 1\}$.

For $V = B, B_0$, the measure $d\mu_V(x)$ coincides with $dx$ as defined in §2.1.2.

2.2.2. Metaplectic group. Let $Mp_2(k)$ denote the metaplectic double cover of $SL_2(k)$. It is convenient to identify $Mp_2(k)$ with $SL_2(k) \times \mu_2$, where $\mu_2 := \{\pm 1\}$, with the group law given by $(s_1, \zeta_1)(s_2, \zeta_2) = (s_1s_2, \zeta_1\zeta_2c(s_1, s_2))$ for a cocycle $c : SL_2(k) \times SL_2(k) \to \{\pm 1\}$ as in [9, p.19] or [20, §4.4]. Thus $\mu_2$ is a central subgroup of $Mp_2(k)$, and one has a short exact sequence $1 \to \mu_2 \to Mp_2(k) \xrightarrow{pr} SL_2(k) \to 1$. 
2.2.3. Weil representation. For a quadratic space $V$, one has the Weil representation [30] on the Schwartz–Bruhat space $S(V)$:

$$\rho_{\text{Weil}}^{\psi, V} : \text{Mp}_2(k) \times \text{O}(V) \to \text{GL}(S(V))$$.

This representation is continuous [30, §39] for the standard topologies on all spaces involved and extends to a unitary representation on $L^2(V) := L^2(V, \mu_V)$.

For the remainder of §2.2.3, abbreviate $\rho := \rho_{\text{Weil}}$. For $s \in \text{Mp}_2(k)$ or $g \in \text{O}(V)$ we abbreviate $\rho(s) := \rho(s, 1)$ and $\rho(g) := \rho(1, g)$; one then has $\rho(s)\rho(g) = \rho(g)\rho(s)$.

Elements $\zeta$ of the central subgroup $\mu_2$ of $\text{Mp}_2(k)$ act by the scalar operators $\rho(\zeta) = (-1)^{\dim(V)}$, so $\rho$ factors through $\text{SL}_2(k)$ if and only if $\dim(V)$ is even.

There is a quartic character $\chi_{\psi, V} : k^x \to \mathbb{C}^{(1)}$ and an eighth root of unity $\gamma_{\psi, V} \in \mathbb{C}^{(1)}$ so that, abbreviating $n(b) := (n(b), 1), t(a) := (t(a), 1), w := \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $1 \in \text{Mp}_2(k)$ for $b \in k, a \in k^x$, one has for $\phi \in S(V), x \in V$ that

$$\rho(n(b))\phi(x) = \psi(bq_{\psi}(x))\phi(x),$$
$$\rho(t(a))\phi(x) = \chi_{\psi, V}(a)|a|^{\dim(V)/2}\phi(ax),$$
$$\rho(w)\phi(x) = \gamma_{\psi, V} \mathcal{F}\phi(x).$$

If $V = M_2(k)$, then $\chi_{\psi, V}$ is trivial and $\gamma_{\psi, V} = 1$.

Elements $g$ of the orthogonal group $\text{O}(V)$ act by $\rho(g)\phi(v) := \phi(g^{-1}v)$. Suppose that $V = B^0$, so that $\text{Ad} : G \cong \text{SO}(B^0)$. For $g \in G$ and $\phi \in S(B^0)$, the function $\text{Ad}(g)\phi$ as defined in §1.13 agrees with $\rho(\text{Ad}(g))\phi$: both send $x \in B^0$ to $\phi(g^{-1}xg)$.

2.2.4. Factorization. Let $V$ be a quadratic space that admits an orthogonal decomposition $V = V' \oplus V''$ as a sum of two quadratic spaces. (The relevant example is when $V = B, V' = k, V'' = B^0$.)

Recall the dense inclusion $S(V') \otimes S(V'') \to S(V)$ obtained by identifying $\phi' \otimes \phi'' \in S(V') \otimes S(V'')$ with the function $V' \oplus V'' \ni \alpha' + \alpha'' \mapsto \phi'(\alpha')\phi''(\alpha'')$.

Given continuous linear operators $T, T', T''$ on $S(V), S(V'), S(V'')$, respectively, write $T = T' \otimes T''$ to denote that $T(\phi' \otimes \phi'') = T'\phi' \otimes T''\phi''$ for all $\phi' \in S(V'), \phi'' \in S(V'')$. In this sense, one has $\rho_{\text{Weil}}(s) = \rho_{\text{Weil}}^{V', \psi}(s) \otimes \rho_{\text{Weil}}^{V'', \psi}(s)$ for all $s \in \text{Mp}_2(k)$.

We denote by $1 \otimes \rho_{\text{Weil}}^{V'', \psi}(s)$ the operator on $S(V)$ sending $\phi' \otimes \phi''$ to $\phi' \otimes \rho_{\text{Weil}}^{V'', \psi}(s)\phi''$.

2.2.5. Extension to similitudes. The following definitions were inspired by [28, I.3]. Let $\Omega$ denote the space of functions $\phi : k^x \times B \to \mathbb{C}$ satisfying the conditions:

- For each $t \in k^x$, the function $\phi[t] : B \to \mathbb{C}$ given by $\phi[t](x) := \phi(t, x)$ belongs to the Schwartz–Bruhat space $S(B)$.
- One has $\phi(z^2t, x) = \phi(t, x)$ for all $t, z \in k^x, x \in B$.

Let $\rho_{\text{Weil}} : \text{PGL}_2(k) \times \text{PGO}(B) \to \text{GL}(\Omega)$ denote the representation characterized by the identities: for $s \in \text{SL}_2(k), y \in k^x, g \in \text{GO}(B)$,

$$(\rho_{\text{Weil}}(s)\phi)[t] = \rho_{\text{Weil}}^{s, B}(s)\phi[t],$$
$$(\rho_{\text{Weil}}(a(g))\phi)[t] = |g|\phi[ty],$$
$$(\rho_{\text{Weil}}(g)\phi)(t, x) = \phi(\lambda(g)t, g^{-1}x)$$

where $\lambda : \text{GO}(B) \to k^x$ denotes the similitude factor.
Remark. More generally, if \( V \) is any even-dimensional quadratic space, then the representation \( \rho_{\text{Weil}} \) factors through \( \text{SL}_2(k) \times \text{O}(V) \). One can induce it to a representation of \( \text{GL}_2(k) \times \text{GO}(V) \) on \( \mathcal{S}(k^* \times V) \), whose isomorphism class is independent of \( \psi \). By taking coinvariants for the action by the center, one arrives at a representation of \( \text{PGL}_2(k) \times \text{PGO}(V) \). In the relevant case that \( V = B \), the representation obtained in that way is realized by \( \Omega \). Our global discussion concerns the restriction of \( \Omega \) to \( \text{SL}_2(k) \times \text{O}_1(k) \times \text{O}(B^0) \), which embeds as the “even subspace” of \( \oplus_{t \in k^*/k \times 2} \rho_{\text{Weil}}^e \circledast \rho_{\text{Weil}}^{0} \).

Equip \( \Omega \) with the invariant hermitian norm \( \| \cdot \|_\Omega \) given by
\[
\| \phi \|^2_{\Omega} := \int_{t \in k^*/k \times 2} |t|^2 \int_{x \in B} |\phi|^2(t, x),
\]
(2.6)
or equivalently (by (2.3), (2.2)),
\[
\| \phi \|^2_{\Omega} = \int_{g \in G} |nr(g)|^2 \int_{t \in k^*} |t|^2 |\phi(t, g)|^2 \frac{dt}{|t|} dg.
\]
(2.7)

Define \( \mathfrak{S} : \Omega \to \Omega \) and \( \text{Ad}(g) : \Omega \to \Omega \) \((g \in G)\) by applying the general definition (1.13) to the second coordinate, so that for \( \phi \in \Omega \) and \((t, x) \in k^* \times B\), one has \((\mathfrak{S}\phi)[t] = \mathfrak{S}(\phi[t]), \mathfrak{S}\phi(t, x) = (\phi(t, x) + \phi(t, \text{tr}(x) - x))/2\), \(\text{Ad}(g)\phi = \rho_{\text{Weil}}(\text{Ad}(g))\phi\), \((\text{Ad}(g)\phi)[t] = \text{Ad}(g)(\phi[t]), \text{Ad}(g)\phi(t, x) = \phi(t, g^{-1}xg)\).

2.2.6. The distinguished element. Suppose temporarily that \( k \) is non-archimedean and that \( B \cong M_2(k) \) is split; similar considerations apply to non-split \( B \), but we do not need them. The distinguished element \( \phi^0 \in \Omega \) (with respect to the chosen maximal order \( R \subset B \)) is then defined by
\[
\phi^0(t, x) := \int_{z \in k^*} \frac{1_R(zx)1_{\sigma^e \times (z^{-2}t)} \frac{dz}{|z|^2}}{\int_{z \in k^*} 1_{\sigma^e}(z) \frac{dz}{|z|^2}}.
\]
(2.8)

Note that \( \phi^0 \) takes values in \( \{0, 1\} \). One verifies directly that \( \phi^0 \) is \( \text{PGL}_2(\sigma) \times K' \)-invariant, where \( K' \leq \text{PGO}(B) \) denotes the image of the \( \text{O}(B) \)-stabilizer of \( R \). Moreover, \( \mathfrak{S}\phi^0 = \phi^0 \).

Lemma 1. \( \| \phi^0 \|^2_{\Omega} = \text{vol}(R) \).

Proof. Since \( \phi \) takes values in \( \{0, 1\} \), one has \( \| \phi^0 \|^2_{\Omega} = \int_{t \in k^*/k \times 2} |t|^2 \int_{x \in B} \phi^0(t, x) \). By expanding the definition of \( \phi^0 \) and using that \( \int_{x \in B} 1_R(zx) = |z|^{-4} \text{vol}(R) \) and that \( |t|^2 |z|^{-4} 1_{\sigma^e \times (z^{-2}t)} = 1_{\sigma^e \times (z^{-2}t)} \), our task reduces to showing that \( \int_{t \in k^*/k \times 2} \frac{dz}{|z|^2} = \int_{z \in k^*} 1_{\sigma^e}(z^{-2}t) \frac{dz}{|z|^2} \), as follows from (2.2). \( \square \)

Let \( K \leq G \) denote the image of \( R^* \). We may then fix an identification \( B = M_2(k) \) under which \( G = \text{PGL}_2(k), R = M_2(\sigma), K = \text{PGL}_2(\sigma) \).

Lemma 2. Let \( \phi_1, \phi_2 \in \mathbb{C}\phi^0 \). Let \( g \in K \sigma(\varpi^m)K \) for some \( m \in \mathbb{Z}_{>0} \) (see §2.1.4). Then \( (\text{Ad}(g)\phi_1, \phi_2)\Omega = q^{-m}(\phi_1, \phi_2)\Omega \).

Proof. We expand the definitions and use that \( \text{vol}(gRg^{-1} \cap R) = q^{-m} \text{vol}(R) \). \( \square \)
2.3. Generic representations of PGL₂. We refer to [3, §4.4, §4.6] for details on and proofs of the facts collected here. Let \( \pi \) be an irreducible representation of \( \text{PGL}_2(k) \). Recall that \( \pi \) is generic if it admits a Whittaker model \( \mathcal{W}(\pi, \psi) \), consisting of \( W : \text{PGL}_2(k) \to \mathbb{C} \) satisfying \( W(n(x)g) = \psi(x)W(g) \). It then admits a Kirillov model \( \mathcal{K}(\pi, \psi) \), consisting of \( W : k^\times \to \mathbb{C} \) of the form \( W(y) := W'(a(y)) \) for some \( W' \in \mathcal{W}(\pi, \psi) \). The vector space \( \mathcal{K}(\pi, \psi) \) is independent of \( \psi \) and contains \( C^\infty_c(k^\times) \). Recall that \( \pi \) is unramified if the space \( \pi^{\text{PGL}_2(\mathfrak{o})} \) of \( \text{PGL}_2(\mathfrak{o}) \)-invariant vectors in \( \pi \) is nonzero, and that in that case, \( \dim(\pi^{\text{PGL}_2(\mathfrak{o})}) = 1 \).

Suppose for the remainder of §2.3 that \( \pi \) is generic and unramified. Let \( \psi^0 \) be an unramified unitary character of \( k \). There is then a unique \( \text{PGL}_2(\mathfrak{o}) \)-invariant vector \( \psi^0_k(1) \) in the Kirillov model \( \mathcal{K}(\pi, \psi^0) \) of \( \pi \) for which \( \psi^0_k(1) = 1 \). There is a unique unordered pair \( \{\alpha, \beta\} \) of complex numbers, the Satake parameters of \( \pi \), so that for \( y \in k^\times \) with \( |y| = q^{-n} \),

\[
\psi^0_k(y) = |y|^{1/2} \sum_{i,j \in \mathbb{Z}_{2\alpha}^{1+i+j=n}} \alpha^i \beta^j = 1_{\alpha^x}(y)|y|^{1/2} q^{n+1} - \beta^{n+1} \alpha - \beta. \tag{2.9}
\]

One has in general \( \alpha \beta = 1 \); if moreover \( \pi \) is unitary, then either \( |\alpha| = |\beta| = 1 \) or \( \alpha, \beta \in (-q^{1/2}, q^{1/2}) \subseteq \mathbb{R} \). The adjoint \( L \)-factor is defined for \( s \in \mathbb{C} \) by

\[
L(\text{ad} \pi, s) := (1 - \alpha^{-1} q^{-s})^{-1}(1 - q^{-s})^{-1}(1 - \alpha^{-1} \beta q^{-s})^{-1}.
\]

Lemma. If \( \pi \) is unitary and \( \text{Re}(s) \geq 0 \), then \( L(\text{ad} \pi, s) \) is finite, and one has the identity

\[
\int_{y \in k^\times} \psi^0_k(y)^2 |y|^s \frac{dy}{|y|} = \frac{L(\text{ad} \pi, 1 + s)}{\zeta_k(2 + 2s)} \Delta_q^{1/2} \zeta_k(1 + s) \tag{2.10}
\]

in which the LHS converges absolutely.

Proof. This is a standard calculation.\(^2\)

2.4. Representations of \( G \). Let \( \pi \) be an irreducible representation of \( G \). Define a compact open subgroup \( J \subseteq G \) in the following two cases:

- if \( k \) is non-archimedean, take for \( J \subseteq G \) the image of the unit group \( R^\times \) of the chosen maximal order \( R \subseteq B \);
- if \( k \) is real and \( B \) is non-split, set \( J := G \).

In either case, set \( \text{vol}(J) := \int_{y \in G} 1_J(y) \) and \( e_J := \text{vol}(J)^{-1} 1_J \in C^\infty_c(G) \).

2.4.1. Hecke kernels and theta kernels. Assume that \( k \) is non-archimedean. For \( y \in k^\times \), the normalized Hecke kernel \( T_y \) is defined to be the element with the property that \( |y|^{-1} \text{vol}(J) T_y \) is the characteristic function of the image in \( G \) in the set \( \{b \in R : \text{nr}(b) = |y|\} \) of \( B^\times \). For example, if \( y \in \mathfrak{o}^\times \), then \( T_y = e_J \).

Lemma. Let \( y \in k^\times \), \( g \in G \). Choose \( \tilde{g} \in B^\times \) with image \( g \). Then

\[
\rho_{\text{weil}}(a(y)) \phi^0(\text{nr}(\tilde{g})^{-1}, \tilde{g}) = |y| \phi^0(\text{nr}(\tilde{g})^{-1}, \tilde{g}) = \text{vol}(J) T_y(g)
\]

where \( \phi^0 \in \Omega \) is the distinguished element (§2.2.6).

\(^2\)Since we lack a convenient reference sharing our measure normalizations, we record the proof. The LHS of (2.10) expands to \( \text{vol}(\mathfrak{o}^\times) \sum_{n \geq 0} |a_n|^2 x^n \), where \( a_n := (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) \) and \( x := q^{-(1+s)n} \). One has \( \text{vol}(\mathfrak{o}^\times) = \zeta_k(1)^{-1} \Delta_q^{1/2} \). The identity \( \sum a_n x^n = (1 - \alpha x)^{-1}(1 - \beta x)^{-1} \) and [3, Lem.1.6.1] imply that \( \sum_{n \geq 0} |a_n|^2 x^n = (1 - x^2)(1 - \alpha \beta x)^{-1}(1 - \beta x)^{-1}(1 - \beta \alpha x)^{-1}(1 - \beta x)^{-1}(1 - \alpha x)^{-1} \). Invoking the consequence \( \{\alpha, \alpha \beta, \beta \alpha, \beta \} = \{1, \alpha, \beta \} \) of the assumed unitarity of \( \pi \), we obtain \( \sum_{n \geq 0} |a_n|^2 x^n = \zeta_k(2 + 2s)^{-1} L(\text{ad} \pi, 1 + s) \zeta_k(1 + s) \), as required.
Proof. We must verify that
\[ |y|^{-1} \text{vol}(J) T_y(g) = \frac{\int_{z \in k^\times} 1_R(z \tilde{y}) 1_{\sigma^\times}(y \text{nr}(z \tilde{y})^{-1}) \frac{dz}{|z|}}{\int_{z \in k^\times} 1_{\sigma^\times}(z) \frac{dz}{|z|}}. \tag{2.11} \]
Let \( g \in G \). The RHS of (2.11) is independent of \( \tilde{y} \), and both sides take values in \( \{0, 1\} \). The RHS of (2.11) is nonzero iff its integrand is nonzero for some \( z \in k^\times \), i.e., iff for some \( z \in k^\times \) the element \( b := z \tilde{y} \) lies in \( R \) and \( |\text{nr}(b)| = |y| \), i.e., iff the LHS of (2.11) is nonzero.

\[ \square \]

2.4.2. Hecke functionals and standard L-factors. Continue to assume that \( k \) is non-archimedean. Recall that \( \pi \) is unramified if the space \( \pi^J \) of \( J \)-invariant vectors in \( \pi \) is nonzero; it is known then that \( \dim(\pi^J) = 1 \).

Suppose for remainder of §2.4.2 that \( \pi \) is unramified. There is then a unique functional \( \lambda_{\pi} : C^\infty_c(J \setminus G/J) \to \mathbb{C} \) so that \( \pi(T)v = \lambda_{\pi}(T)v \) for all \( T \in C^\infty_c(J \setminus G/J), \) \( v \in \pi^J \). We may evaluate this functional on the elements \( T_y \) attached to \( y \in k^\times \):

- If \( B \) is split, then there is a unique unordered pair \( \{\alpha, \beta\} \) of complex numbers (the Satake parameters) satisfying \( \alpha \beta = 1 \) so that with \( |y| = q^{-n} \),
  \[ \lambda_{\pi}(T_y) = |y|^{1/2} \sum_{i,j \in \mathbb{Z}, i+j=n} \alpha^i \beta^j = 1_{\sigma^\times}(y)|y|^{1/2} \frac{q^{n+1} - \beta^{n+1}}{\alpha - \beta} \]  \tag{2.12}

(see e.g. [3, §4.6]).

- If \( B \) is non-split, then there is a unique unramified quadratic character \( \eta \) of \( k^\times \) so that \( \lambda_{\pi}(T_y) = |y|\eta(y) \).

The standard L-factor is then the meromorphic function defined for \( s \in \mathbb{C} \) by

\[ L(\pi, s) := \begin{cases} (1 - \alpha q^{-s})^{-1}(1 - \beta q^{-s})^{-1} & \text{if } B \text{ is split}, \\ (1 - \eta(\omega)q^{-s-1})^{-1} & \text{if } B \text{ is non-split}. \end{cases} \]

2.4.3. The local Jacquet–Langlands correspondence. The Jacquet–Langlands lift \( \pi_{\text{JL}} \) of \( \pi \) is an irreducible representation of \( \text{PGL}_2(k) \) attached to \( \pi \). The following properties of the association \( \pi \mapsto \pi_{\text{JL}} \) are relevant for us:

- \( \pi_{\text{JL}} \) is generic if (and only if) either
  - \( B \) is non-split, or
  - \( B \) is split and \( \dim(\pi) > 1 \).

- If \( B \) is split, then \( \pi_{\text{JL}} \) corresponds to \( \pi \) under the isomorphism \( G \cong \text{PGL}_2(k) \). In particular, if \( \pi \) is unramified, then so is \( \pi_{\text{JL}} \), and Satake parameters (see §2.3, §2.4.2) are preserved.

Assume now that that \( k \) is non-archimedean, that \( B \) is split, that \( \pi \) is unramified, and that \( \dim(\pi) > 1 \). Then \( \pi_{\text{JL}} \) is generic and unramified. Let \( W^0_{\pi} : k^\times \to \mathbb{C} \) denote the function attached to \( \pi_{\text{JL}} \) in §2.3. By (2.9) and (2.12),
\[ W^0_{\pi}(y) = \lambda_{\pi}(T_y). \]  \tag{2.13}

2.4.4. Local integrals. Assume first that \( k \) is non-archimedean, that \( \pi \) is unramified, and that \( \pi \) is unitary. Retain the notation of §2.4.2.

Lemma 1. Suppose that \( B \) is split and that \( \dim(\pi) > 1 \).

- (i) \( \alpha, \beta < q^{-1/2} \). In particular, \( L(\pi, \frac{1}{2}) \) is finite.
(ii) Let \( \phi_1, \phi_2 \) belong to the line \( \mathbb{C} \phi^0 \) spanned by the distinguished element \( \phi^0 \in \Omega \). Let \( v_1, v_2 \in \pi^\prime \). Then the identity
\[
\int_{g \in G} \langle \text{Ad}(g)\phi_1, \phi_2 \rangle \langle gv_1, v_2 \rangle = \frac{L(\pi, \frac{1}{2})}{\zeta_k(2)} \text{vol}(J) \langle \phi_1, \phi_2 \rangle \langle v_1, v_2 \rangle \tag{2.14}
\]
holds, with the LHS converging absolutely.

**Proof.** For (i), see [3, Thm 4.6.7]. For (ii), the convergence follows from §2.1.6. Let \( \{ \alpha, \beta \} \) denote the Satake parameters of \( \pi \) and set \( t_1 := \alpha q^{-1/2}, t_2 := \beta q^{-1/2} \), so that \( L(\pi, \frac{1}{2})^{-1} = (1 - t_1)(1 - t_2) \). The Macdonald formula [3, Thm 4.6.6] says that
\[
\langle gv_1, v_2 \rangle = (u_1 t_1^m + u_2 t_2^n) \langle v_1, v_2 \rangle, \quad \text{where}
\]
\[
u_1 := \frac{1}{1 + q^{-1}} \frac{1 - q^{-1}\beta/\alpha}{1 - \beta/\alpha}, \quad \nu_2 := \frac{1}{1 + q^{-1}} \frac{1 - q^{-1}\alpha/\beta}{1 - \alpha/\beta}.
\]
By the Cartan decomposition and Lemma 2, we obtain
\[
\int_{g \in G} \langle \text{Ad}(g)\phi_1, \phi_2 \rangle \langle gv_1, v_2 \rangle = \text{vol}(J) \langle \phi_1, \phi_2 \rangle \langle v_1, v_2 \rangle \Sigma,
\]
where \( \Sigma := \sum_{i=1,2} \sum_{m \geq 0} (1 + m > 0 q^{-1}) t_i^m \). We compute that \( \sum_{i=1,2} u_i (1 + q^{-1} t_i)(1 - t_i)^{-1} = L(\pi, \frac{1}{2})\Sigma' \) with \( \Sigma' := \sum_{i=1,2} u_i (1 + q^{-1} t_i)(1 - t_i)^{-1}, \{i, i'\} = \{1, 2\} \). Direct calculation gives \( \Sigma' = \zeta_k(2)^{-1} \), as required. \( \square \)

Suppose now that \( B \) is non-split, so that \( \pi \) is the one-dimensional representation corresponding to the character \( G \ni g \to \eta(nr(g)) \in \{ \pm 1 \} \), as in §2.4.2. Let \( v_1, v_2 \in \pi \). Recalling that \( [G : J] = 2 \), we have
\[
\int_{g \in G} \langle \text{Ad}(g)e_J, e_J \rangle L^2(G) \langle gv_1, v_2 \rangle = \langle v_1, v_2 \rangle \cdot \begin{cases} 0 & \text{if } \eta \text{ is nontrivial}, \\ 2 & \text{if } \eta \text{ is trivial}. \end{cases} \tag{2.15}
\]

Suppose finally that \( k \cong \mathbb{R} \), that \( B \) is non-split, and that \( \pi \) is trivial. For \( v_1, v_2 \in \pi \), one then has
\[
\int_{g \in G} \langle \text{Ad}(g)e_J, e_J \rangle L^2(G) \langle gv_1, v_2 \rangle = \langle v_1, v_2 \rangle. \tag{2.16}
\]

3. Global preliminaries

In this section we collect those preliminaries for the proof of Theorem 2 whose discussion makes sense independently of that proof.

Let \( F \) be a number field with adele ring \( \mathbb{A} \), let \( B \) be a quaternion algebra over \( F \), and let \( \psi \) be a nontrivial unitary character of \( \mathbb{A}/F \).

3.1. Generalities.

3.1.1. **Notation.** We denote by \( \mathcal{O}_F \) or simply \( \mathcal{O} \) the ring of integers in \( F \). We denote by \( p \) a place of \( F \), finite or infinite. A subscripted \( p \) denotes completion; for example, \( \mathcal{O}_p \) denotes the ring of integers of \( F_p \) if \( p \) is finite. For a finite set of places \( S \), a subscripted \( S \) denotes a product taken over \( S \), such as in \( F_S := \prod_{p \in S} F_p, B_S := \prod_{p \in S} B_p \).

The character \( \psi \) factors as \( \psi(x) = \prod \psi_p(x_p) \), where \( \psi_p \) is a nontrivial unitary character of \( F_p \).

For a place \( p \), let \( \zeta_p := \zeta_{F_p} \) denote the local Euler factor. Let \( \xi_F(s) := \prod \zeta_p(s) \) denote the Dedekind zeta function (absolutely convergent for \( \text{Re}(s) > 1 \)) and
\[ \xi_p^*(1) := \text{res}_{s \to 1} \xi_F(s) \] its residue. For a finite set \( S \) of places that contains the infinite places, let \( \zeta^{(S)}(s) := \prod_{p \in S} \zeta_p(s) \) denote the partial Dedekind zeta function.

### 3.1.2. Groups

For an algebraic \( F \)-group \( G \) we write \( G := G(F), G_p := G(F_p) \), \( G_\mathbf{\hat{A}} := G(\mathbf{\hat{A}}), \) \( G_S := G(F_S) = \prod_{p \in S} G_p, \) \( |G| := G/G_\mathbf{\hat{A}}. \) This notation applies notably to the \( F \)-group \( \text{PB}^\times \) given by \( \text{PB}^\times(A) := (B \otimes_F A)^\times/A^\times \) and also to the \( F \)-groups \( \text{PGL}_2, \text{SL}_2. \) We similarly abbreviate \( [\text{Mp}_2] := \text{SL}_2(F) \backslash \text{Mp}_2(\mathbf{\hat{A}}) \) (see §3.4.1).

### 3.1.3. Measures

When \( G \) is semisimple, we equip \( G_\mathbf{\hat{A}} \) and \( |G| \) with Tamagawa measures. Then \( \text{vol}([\text{SL}_2]) = 1 \) and \( \text{vol}([\text{PGL}_2]) = \text{vol}([\text{PB}^\times]) = 2. \) Denote by \( \langle , \rangle_G \) the corresponding inner product on \( L^2(|G|) \); we omit the subscripted \( G \) if it is clear by context.

For each place \( p \), the character \( \psi_p \) induces (via the recipe of §2.1.2) a Haar measure on \( F_p, B_p, F_p^2/F_p^\times, \text{PB}_p^\times; \) we equip \( \mathbf{\hat{A}}, B_\mathbf{\hat{A}}, \mathbf{\hat{A}}^\times/\mathbf{\hat{A}}^\times \) and \( \text{PB}_\mathbf{\hat{A}}^\times \) with the corresponding restricted product measures. (This defines the Tamagawa measure on \( \text{PB}_\mathbf{\hat{A}}^\times. \) The quotient measures on \( \mathbf{\hat{A}}/F \) and \( B_\mathbf{\hat{A}}/B \) are then probability measures. We likewise equip finite products such as \( F_S \) or \( \text{PB}_S^\times \) with product measures.

We equip \( \mathbf{\hat{A}}^\times \) with the regularized product of the measures constructed in §2.1.2: for a factorizable function \( f = \prod f_p \in C_c^\infty(\mathbf{\hat{A}}^\times) \) for which \( f_p = 1 \) for almost all finite primes \( p \), we set

\[
\int_{y \in \mathbf{\hat{A}}^\times} f(y) \frac{dy}{|y|} := \frac{1}{\xi_p^*(1)} \prod_p \zeta_p(1) \int_{y \in F_p^\times} f_p(y) \frac{dy}{|y|}.
\]

We thereby obtain a quotient Haar \( \frac{dy}{|y|} \) on \( \mathbf{\hat{A}}^\times/F^\times \) whose pushforward under \( |.| : \mathbf{\hat{A}}^\times/F^\times \to \mathbb{R}_+^\times \) is the standard Haar measure \( \frac{dt}{t} \) on \( \mathbb{R}_+^\times \), where \( dt \) denotes Lebesgue measure.

The quotient measure on the discrete group \( F^\times/F^\times \) compatible with the squaring map is half the counting measure, i.e., for finitely-supported \( f : F^\times \to \mathbb{C}, \) one has \( \sum_{x \in F^\times} f(x) = \frac{1}{2} \sum_{y \in F^\times/F^\times} (\sum_{z \in F^\times} f(yz^2)). \) On \( \mathbf{\hat{A}}^\times/F^\times \mathbf{\hat{A}}^\times, \) we take the quotient measure induced by the exact sequence \( 1 \to F^\times/F^\times \to \mathbf{\hat{A}}^\times/\mathbf{\hat{A}}^\times \to \mathbf{\hat{A}}^\times/F^\times \mathbf{\hat{A}}^\times \to 1, \) where \( F^\times/F^\times \) is equipped with half the counting measure. Thus for \( f \in C_c(\mathbf{\hat{A}}^\times/\mathbf{\hat{A}}^\times), \)

\[
\int_{y \in \mathbf{\hat{A}}^\times/F^\times \mathbf{\hat{A}}^\times} \frac{1}{2} \sum_{a \in F^\times/F^\times} f(ay) = \int_{\mathbf{\hat{A}}^\times/F^\times} f. \tag{3.1}
\]

By decomposing the Haar on \( \mathbf{\hat{A}}^\times \) in two ways, one finds for \( f \in C_c(\mathbf{\hat{A}}^\times/F^\times) \) that \( \int_{\mathbf{\hat{A}}^\times/F^\times} f = \int_{x \in \mathbf{\hat{A}}^\times/F^\times} \int_{y \in \mathbf{\hat{A}}^\times/F^\times} f(xy^2); \) moreover, \( \text{vol} (\mathbf{\hat{A}}^\times/F^\times \mathbf{\hat{A}}^\times) = 2 \) \( \text{vol}([\text{PGL}_2]) = 2. \) Finally, for \( f \in C_c^\infty([\text{PGL}_2]), \)

\[
\int_{[\text{PGL}_2]} f = \int_{y \in \mathbf{\hat{A}}^\times/F^\times \mathbf{\hat{A}}^\times} \int_{x \in [\text{SL}_2]} f(xa(y)). \tag{3.2}
\]

### 3.1.4. The \( \Xi \)-function

Fix a maximal compact subgroup \( K = \prod K_p \leq \text{PB}^\times_\mathbf{\hat{A}}. \) Let \( \Xi : \text{PB}^\times_\mathbf{\hat{A}} \to \mathbb{C} \) be the product \( \Xi(g) := \prod \Xi_p(g_p) \) of the functions \( \Xi_p \) on \( \text{PB}_p^\times \) attached in §2.1.5 to the factors \( K_p. \)
3.1.5. Conventions. A cusp form is a smooth vector in the Hilbert space $L^2_{\text{cusp}}([G])$ of square-integrable cuspidal functions. A cuspidal automorphic representation $\pi$ of $G_\mathbb{A}$ is the space of smooth vectors in an irreducible subrepresentation of $L^2_{\text{cusp}}([G])$.

3.2. Automorphic forms on $\text{PGL}_2$.

3.2.1. Fourier expansions. Let $\varphi : [\text{PGL}_2] \to \mathbb{C}$ be a smooth function. It admits the Fourier expansion $\varphi(n(a)y)) = c_\varphi(y) + \sum_{\tau \in F^\times} \psi(\tau x) W_\varphi(\tau y)$, where $c_\varphi(y) := \int_{x \in \mathbb{A}/F} \varphi(n(a)y))$ denotes the constant term and

$$W_\varphi(y) := \int_{x \in \mathbb{A}/F} \psi(-x) \varphi(n(a)y))$$

denotes the (diagonal restriction of) the Whittaker function. The standard Borel subgroup of $\text{PGL}_2(\mathbb{A})$ has dense image in $[\text{PGL}_2]$, so $\varphi$ is determined by the values $\varphi(n(a)y))$ for $x \in \mathbb{A}, y \in \mathbb{A}^\times$. Recall that $\varphi$ is cuspidal if $c_\varphi = 0$; in that case, $\varphi$ is determined by $W_\varphi$.

3.2.2. Kirillov model. Let $\pi \subseteq L^2([\text{PGL}_2])$ be a cuspidal automorphic representation; it is the (smooth completion of) a restricted tensor product $\otimes_p \pi_p$, where $\pi_p$ is a generic for every $\mathfrak{p}$ and unramified for almost all finite $\mathfrak{p}$. Let $K(\pi, \psi) := \{W_\varphi : \varphi \in \pi\}$. The natural map $\pi \to K(\pi, \psi)$ is a linear isomorphism under which the pure tensors in $\pi$ correspond to the factorizable functions $V(y) = \prod W_\varphi(y)$, where $W_\varphi$ belongs to the local Kirillov model $K(\pi_p, \psi_\mathfrak{p})$ and satisfies $W_\varphi = W_\varphi^{\mathfrak{p}}$ (see §2.4.3) for almost all finite $\mathfrak{p}$.

Lemma. Let $S$ be a finite set of places of $F$ that contains all infinite places as well as any places $\mathfrak{p}$ at which $\pi$ ramifies. Let $\varphi \in \pi$. The integral $I(s) := \int_{y \in \mathbb{A}^\times} |W_\varphi(y)|^2 |y|^s \frac{dy}{|y|}$ converges absolutely for complex numbers $s$ with positive real part, extends to a meromorphic function on the complex plane, and satisfies

$$2 \text{res}_{s=0} I(s) = \|\varphi\|^2. \tag{3.3}$$

Proof. See for instance [16, Lem 2.2.3].

3.2.3. Adjoint $L$-function. For $\pi, S$ as in the lemma of §3.2.2, the partial adjoint $L$-function is defined for $\text{Re}(s) > 1$ by the absolutely-convergent Euler product $L^{(S)}(\text{ad} \pi, s) := \prod_{\mathfrak{p} \notin S} L(\text{ad} \pi_\mathfrak{p}, s)$; it continues meromorphically to the complex plane, and is holomorphic for (at least) $\text{Re}(s) \geq 1$ (see [8]).

3.3. Automorphic forms on $\text{PB}^\times$.

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3The cited reference normalizes measures differently than we do. For the convenience of the reader, we sketch here the proof that (3.3) is normalized correctly, taking for granted the meromorphicity of $I(s)$. By unfolding, $I(s) = \int_{y \in \mathbb{A}^\times/F} \int_{x \in \mathbb{A}/F} |\varphi(n(x)a(y))|^2 |y|^s \frac{dy}{|y|}$. Because the pushforward under $|.| : \mathbb{A}^\times/F^\times \to \mathbb{R}_+^\times$ of $\frac{dy}{|y|}$ is the standard Haar measure $d\lambda$ on $\mathbb{R}_+^\times$ (see §3.1.3), it follows that $\text{res}_{s=1} I(s) = \lim_{t \to 0} \mathbb{E}_{y \mid |y| = t} \int_{x \in \mathbb{A}/F} |\varphi(n(x)a(y))|^2$ where $\mathbb{E}$ denotes an average with respect to the probability measure invariant by the norm one idele class group. By the equidistribution of the horocycle flow and the normalization $\text{vol}(\mathbb{A}/F) = 1$, that limit is the integral of $|\varphi|^2$ with respect to the probability Haar on $[\text{PGL}_2]$. By the consequence $\text{vol}([\text{PGL}_2]) = 2$ of our measure normalizations, we obtain (3.3).
3.3.1. Jacquet–Langlands lifts. Let $\pi = \otimes_p \pi_p \subseteq L^2([B\pi^\times])$ be a cuspidal automorphic representation with $\dim(\pi) > 1$. By [25, Prop 4],
\[
\dim(\pi_p) > 1 \text{ for any prime } p \text{ at which } B \text{ splits.}
\]
The Jacquet–Langlands lift $\pi_{JL} = \otimes_p \pi_{JL,p} \subseteq L^2([PGL_2])$ is the unique cuspidal automorphic automorphic representation for which $\pi_{JL,p} = (\pi_p)_{JL}$ for each place $p$. If $p$ is a finite prime at which $B$ splits and for which $\pi_p$ is unramified, then $(\pi_{JL})_p$ is unramified. The association $\pi \mapsto \pi_{JL}$ is injective.

3.3.2. The pretrace formula. Assume that $B$ is non-split, so that $[B\pi^\times]$ is compact. Fix a maximal compact subgroup $K$ of $B\pi^\times$. The pretrace formula asserts that for $f \in C_c^\infty(B\pi^\times)$ and $x \in B\pi^\times$,
\[
\sum_\pi \sum_\varphi \overline{\varphi(x)} \pi(f) \varphi(x) = \sum_{\gamma \in PB^\times} f(x^{-1} \gamma x),
\]
where $\pi$ traverses the irreducible subrepresentations of $L^2([B\pi^\times])$ and $\varphi$ traverses an orthonormal basis $B(\pi)$ of $\pi$ consisting of $K$-isotypic vectors. Only finitely many summands on the RHS of (3.5) are nonzero, while the condition on $B(\pi)$ implies that the LHS of (3.5) converges absolutely, or indeed, rapidly: Let $C(\pi) := \prod_p C((\pi_p)_{JL}) \in \mathbb{R}_{\geq 1}$ denote the analytic conductor of $\pi$ (see e.g. [16, §3.1.8, §4.1.4]). Then for each $A \geq 0$, one has
\[
\sum_\pi C(\pi)^A \sum_\varphi |\overline{\varphi(x)} \pi(f) \varphi(x)| < \infty.
\]
Note also that there exists $A_0 > 3$ so that (see e.g. [16, (2.15)]).
\[
\sum_\pi C(\pi)^{-A_0} < \infty
\]

Let $S$ be a set of places containing the infinite ones. Let $R \subseteq B$ be a maximal order. For each $p \not\in S$, let $J_p \leq B\pi^\times$ denote the image of $R_p^\times$, as in §2.4, and set $J := \prod_{p \not\in S} J_p$. Suppose that $f = f_S \otimes (\otimes_{p \in S} T_{y_p})$ for some $f_S \in C_c^\infty(B\pi^\times)$ and $y \in \pi_{JL}$, where $T_{y_p}$ is the Hecke kernel as defined in §2.4.1 relative to $J_p$. The formula (3.5) then specializes to
\[
\sum_{\pi} \sum_\varphi \overline{\varphi(x)} \pi(f_S) \varphi(x) \prod_{p \not\in S} \lambda_{y_p}(T_{y_p}) = \sum_{\gamma \in PB^\times} f_S(x^{-1}_S \gamma x_S) \prod_{p \not\in S} T_{y_p}(x_p^{-1} \gamma x_p),
\]
where $\pi \subseteq L^2([B\pi^\times])$ now traverses the subrepresentations that are unramified outside $S$ (i.e., that contain a nonzero $J$-fixed vector) and $\varphi$ traverses an orthonormal basis of $K$-isotypic vectors for the $J$-fixed subspace $\pi^J$ of $\pi$.

3.3.3. $L$-functions. Let $\pi \subseteq L^2([B\pi^\times])$ be a cuspidal automorphic representation with $\dim(\pi) > 1$. Let $S$ be a finite set of places containing all infinite places as well as any places at which either $B$ or $\pi$ ramifies.

The partial standard $L$-function is defined for $\text{Re}(s) > 1$ by the absolutely-convergent Euler product $L^{(S)}(\pi, s) := \prod_{p \not\in S} L(\pi_p, s)$; it continues meromorphically to the complex plane, and is holomorphic for (at least) $\text{Re}(s) \geq 1/2$ (see e.g. [3, §3.5]).

---

For instance, this follows (in overkill fashion) from the proof of [17, Thm 9.1].
Set $L(S)(\text{ad } \pi, s) := L(S)(\text{ad } \pi_{\text{II}}, s)$ (see §3.2.3). By [10] (cf. [1, §2.9]), one has

$$C(\pi)^{-\varepsilon} \ll_{\varepsilon} L(S)(\text{ad } \pi, 1) \ll_{\varepsilon} C(\pi)^{\varepsilon} \text{ for each } \varepsilon > 0.$$  

(3.9)

3.4. Theta functions.

3.4.1. Metaplectic group. Let $\text{Mp}_2(\mathbb{A})$ denote the metaplectic double cover of $\text{SL}_2(\mathbb{A})$; it fits into a short exact sequence $1 \to \mu_2 \to \text{Mp}_2(\mathbb{A}) \xrightarrow{\text{pr}} \text{SL}_2(\mathbb{A})$. We may identify it with $\text{SL}_2(\mathbb{A}) \times \mu_2$ with the group law given by $(s_1, \zeta_1)(s_2, \zeta_2) = (s_1s_2, \zeta_1\zeta_2c(s_1, s_2))$, where $c$ is the product of the cocycles from §3.4.1. We identify $\text{SL}_2(F)$ with its image under the unique splitting $\text{SL}_2(F) \hookrightarrow \text{Mp}_2(\mathbb{A})$.

We may similarly define $\text{Mp}_2(F_S)$ as a double cover of $\text{SL}_2(F_S)$ for any collection $S$ of places of $F$.

3.4.2. Quadratic spaces. One defines quadratic spaces $V$ over $F$ as in §2.2.1. The relevant examples are still $V = B, B^0, F$. We equip $V_{\mathbb{A}}$ with the $(\psi, b_V)$-self dual measure $\mu_V$. That measure is the product of the measures $\mu_B$ on the local spaces $V_{\mathfrak{p}}$ attached to $\psi_{\mathfrak{p}}$, and is independent of $\psi$: it assigns volume one to a fundamental domain for $V_{\mathfrak{A}}/V$.

3.4.3. Weil representation. For a quadratic space $V$ over $F$, the Schwartz–Bruhat space $S(V_{\mathbb{A}})$ factors as the (completed) restricted tensor product $S(V_{\mathbb{A}}) = \otimes S(V_{\mathfrak{p}})$. The Weil representation $\rho_{\text{Weil}} : \text{Mp}_2(\mathbb{A}) \times O(V_{\mathbb{A}}) \to \text{GL}(S(V_{\mathbb{A}}))$ is given by $\rho_{\text{Weil}} = \otimes \rho_{\text{Weil}}^V$. We identify $\text{SL}_2(F)$ with its image under the unique splitting $\text{SL}_2(F) \hookrightarrow \text{Mp}_2(\mathbb{A})$.

We may similarly define a Weil representation $\rho_{\text{Weil}}^V : \text{Mp}_2(F_S) \times O(V_S) \to \text{GL}(S(V_S))$ for a finite set $S$ of places of $F$.

3.4.4. Theta kernels. Let $V$ be a quadratic space over $F$. For $\phi \in S(V_{\mathbb{A}})$, $s \in \text{Mp}_2(\mathbb{A})$ and $g \in O(V_{\mathbb{A}})$, set $\theta_{\psi}(\phi)(s, g) := \sum_{x \in V} \rho_{\text{Weil}}^V(s, g)\phi(x)$. The sum converges absolutely and defines a smooth function $\theta_{\psi}(\phi) : [\text{Mp}_2] \times [O(V)] \to \mathbb{C}$. We employ notation such as $\theta_{\psi}(\phi; s, g) := \theta_{\psi}(\phi)(s, g)$. Observe that

$$\theta_{\psi}(\phi; ss', gg') = \theta_{\psi}(\rho_{\text{Weil}}^V(s', g')\phi; s, g)$$

(3.10)

3.4.5. Elementary theta functions. Let $V = F$, regarded as a quadratic subspace of $B$ as in §2.2.1. In that case, we abbreviate $O_1(F) := O(V) \cong \{ \pm 1 \}$. For $\phi \in S(V_{\mathbb{A}}) = S(\mathbb{A})$, we denote also by $\theta_{\psi}(\phi)$ the elementary theta function on $[\text{Mp}_2]$ obtained by restricting to the first factor the theta kernel defined in §3.4.4, thus $\theta_{\psi}(\phi)(s) := \theta_{\psi}(\phi)(s, 1) = \sum_{x \in F} \rho_{\text{Weil}}^F(s)\phi(x)$. By (3.10),

$$\rho_{\text{reg}}(s)\theta_{\psi}(\phi) = \theta_{\psi}(\rho_{\text{Weil}}^F(s)\phi) \text{ for } s \in \text{Mp}_2(\mathbb{A}).$$

(3.11)

The $O_1(F)$-invariance of the theta kernel says that for $\phi \in S(\mathbb{A})$,

$$\theta_{\psi}(\phi) = \theta_{\psi}(\phi_{-}) \text{ with } \phi_{-}(x) := \phi(-x).$$

(3.12)

3.4.6. Ternary theta lifts. Suppose $V = B^0$. Given $\Psi : [\text{PB}^\times] \to \mathbb{C}$ and $\phi \in S(B^0_{\mathbb{A}})$ and $s \in \text{Mp}_2(\mathbb{A})$, set $\theta_{\psi}(\phi, \Psi; s) := \int_{g \in [\text{PB}^\times]} \Psi(g)\theta_{\psi}(\phi; s, \text{Ad}(g))$ where $\text{Ad} : \text{PB}^\times \xrightarrow{\cong} \text{SO}(B^0_{\mathbb{A}})$ is the isomorphism induced by the notation of §1.13. If $\Psi$ is a cusp form, then the integral converges absolutely and defines a cusp form $\theta_{\psi}(\phi, \Psi) : [\text{Mp}_2] \to \mathbb{C}$. By (3.10),

$$\rho_{\text{reg}}(s)\theta_{\psi}(\phi, \Psi) = \theta_{\psi}(\rho_{\text{Weil}}^F(s)\phi, \Psi) \text{ for } s \in \text{Mp}_2(\mathbb{A}),$$

(3.13)
\[ \theta_\psi(\text{Ad}(g)\phi, \rho_{\text{reg}}(g)\Psi) = \theta_\psi(\phi, \Psi) \text{ for } g \in \text{PB}^\times_k. \] (3.14)

### 3.4.7. Factorization

If the quadratic space \( V \) decomposes as the direct sum \( V' \oplus V'' \) of quadratic subspaces, then the factorization of the Weil representation (§2.2.4) implies the factorization of theta functions: for \( g = g' \times g'' \in O(V'_k) \times O(V''_k) \leq O(V'_k) \) and \( \phi = \phi' \otimes \phi'' \in \mathcal{S}(V'_k) \) with \( \phi' \in \mathcal{S}(V'_k), \phi'' \in \mathcal{S}(V''_k) \) (see §2.2.4),

\[ \theta_\psi(\phi_1, t_1, t_2) = \theta_\psi(\phi'_1, t'_1, t'_2) \theta_\psi(\phi''_1, t''_1, t''_2). \] (3.15)

With \( \phi'_1 \) as in (3.12) and notation as in §1.13, one has

\[
\text{Ad}(g)\phi = \phi' \otimes \text{Ad}(g)\phi'' \quad (3.16)
\]

\[
\mathcal{S}\phi = \phi'_1 \otimes \phi'', \quad (3.17)
\]

### 3.5. Equidistribution of products of pairs of elementary theta functions

The purpose of this section is to recall and apply some results from [20]. Let \( \tau_1, \tau_2 \in F^\times \). Throughout this section we regard \( \psi, \tau_1, \tau_2, F, B \) as fixed: implied constants may depend upon them without explicit mention. We assume also (for technical convenience) that \( B \) is non-split.

#### 3.5.1. Some asymptotic notation

Given a topological vector space \( S \), we adopt the convention (similar to “big O notation”) of denoting by \( C(\phi) \) any quantity depending continuously upon \( \phi \in \mathcal{S} \): the continuity is assumed uniform in all auxiliary parameters except those explicitly labelled “fixed.” The space \( S \) itself is always regarded as fixed, of course. This convention applies in particular to Schwartz–Bruhat spaces of finite-dimensional vector spaces over local fields, over finite products of local fields, or over adele rings.

Similarly to the “\( \epsilon \)-convention” of analytic number theory, we allow the precise meaning of \( C(\phi) \) to change from one occurrence to the next. When we specifically wish to distinguish between several such quantities, we use the notation \( C'(\phi), C''(\phi), \) and so on.

For example, let \( V \) be a vector space over \( F \) (always assumed finite-dimensional). Let \( \hat{F} \) denote the ring of finite adeles, so that \( \hat{A} = F_\infty \times \hat{F} \) with \( F_\infty := \prod_{\ell \mid \infty} F\ell \). Similarly, write \( V_\hat{A} = V_\infty \times \hat{V} \). The Schwartz–Bruhat space \( \mathcal{S}(V_\hat{A}) \) factors as the algebraic tensor product \( \mathcal{S}(V_\infty) \otimes \mathcal{S}(\hat{V}) \). Suppose given some quantities \( a(\phi; t_1, t_2) \) and \( b(\phi; t_1, t_2) \) depending upon \( \phi \in \mathcal{S}(V_\hat{A}) \) and some auxiliary parameters \( t_1, t_2 \). The notation

\[
|a(\phi; t_1, t_2)| \leq b(\phi; t_1, t_2)C(\phi) \quad (3.18)
\]

means that for each \( t_2 \) and each \( \phi_2 \in \mathcal{S}(\hat{V}) \) there is a finite collection \( P \) of polynomials on \( V_\infty \) and a finite collection \( D \) of translation-invariant differential operators on \( V_\infty \) (thus \( D \) consists of linear combinations of monomials \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \) with respect to some coordinates \( x_j : V_\infty \to \mathbb{R} \)) so that for all \( \phi_\infty \in \mathcal{S}(V_\infty) \) and all \( t_1, t_2 \),

\[
|a(\phi_\infty \otimes \phi_2; t_1, t_2)| \leq b(\phi_\infty \otimes \phi_2; t_1, t_2) \sum_{P \in P} \sum_{D \in D} \|PD\phi_\infty\|_{L^\infty(V_\infty)}. \]

One could just as well write “the functional \( \mathcal{S}(V_\hat{A}) \ni \phi \mapsto a(\phi; t_1, t_2)/b(\phi; t_1, t_2) \) is defined and continuous, uniformly in \( t_1 \)” but that would be stilted.

**Remark.** It would be “better” to work not with the notation \( C \) but instead with a system of adelic Sobolev norms on \( \mathcal{S}(V_\hat{A}) \), such as those attached by the recipe of [16, §2] and [20, §4.6, §5.3] to the basic Weil representation of the metaplectic group.
of the symplectic space $V^*_A \oplus V_A$. The advantage of doing so would be to obtain polynomial dependence in Theorem 1 upon the levels of $\Psi_1, \Psi_2$. The disadvantages would be to lengthen the article and introduce technical overhead irrelevant to our primary aims.

3.5.2. Simple estimates for lattice sums.

Lemma 1. Let $n \in \mathbb{Z}_{\geq 0}$. Let $A \geq 0$ and $t_0 > 0$ be fixed. For each $\phi \in S(\mathbb{R}^n)$ and $t > t_0$, one has $\sum_{v \in \mathbb{Z}^n - \{0\}} |\phi(tv)| \ll |t|^{-A}C(\phi)$.

Proof. The LHS of the required estimate is bounded by

\[ C|t|^{-A} \sup_{x \in \mathbb{R}^n} |x|^{n+1-A} |\phi(x)| \]

with $C := |t_0|^{-(n+1)} \sum_{v \in \mathbb{Z}^n - \{0\}} |v|^{-(n+1+A)} < \infty$. \hfill \Box

Lemma 2. Let $V$ be a vector space over $F$. Let $A \geq 0$ and $t_0 > 0$ be fixed. For $\phi \in S(V_A)$ and $y \in \mathbb{A}^\times$ with $|y| > t_0$, one has $\sum_{v \in V - \{0\}} |\phi(yv)| \ll |y|^{-A}C(\phi)$.

Proof. Observe first that the action of $\mathbb{A}^\times$ on $S(V_A)$ by dilation is continuous. Let $A^{(1)} := \{ y \in \mathbb{A}^\times : |y| = 1 \}$ denote the subgroup of norm one ideles. By the compactness of $A^{(1)}/F^\times$ and the previous observation, it suffices to consider the case that $y_F = 1$ for all finite $p$ and $y_p = t$ for all infinite $p$, where $t \in \mathbb{R}_+^\times$ satisfies $t > t_1 := t_0^{1/[F:Q]}$. Each $\phi_f \in S(\mathbb{V})$ is bounded and satisfies $\text{supp}(\phi_f) \cap V \subseteq L$ for some lattice $L \subseteq V$, so it suffices to show for each fixed $A \geq 0$ and fixed lattice $L \subseteq V$ that for all $\phi \in S(V_\infty)$ and $t > t_1$, one has $\sum_{v \in L - \{0\}} |\phi(tv)| = O(|t|^{-A}C(\phi))$.

By choosing a $\mathbb{Z}$-basis of $L$, we reduce to Lemma 1. \hfill \Box

3.5.3. Simple estimates for theta functions. Recall that the Iwasawa decomposition asserts that each $s \in \text{SL}_2(\mathbb{A})$ may be written in the form $s = n(x)t(y)k$, where $x \in \mathbb{A}, y \in \mathbb{A}^\times$ and $k$ belongs to the standard maximal compact subgroup of $\text{SL}_2(\mathbb{A})$. The decomposition is not unique, but the quantities $x$ and $|y|$ depend only upon $s$.

Recall that the height function $ht : [\text{Mp}_2] \to \mathbb{R}_{>0}$ factors through $ht : [\text{SL}_2] \to \mathbb{R}_{>0}$ where it is given for $g \in [\text{SL}_2]$ by $ht(g) := \max_{\gamma \in G} \text{ht}_A(\gamma g)$, where $ht_A : \text{SL}_2(\mathbb{A}) \to \mathbb{R}_{>0}$ is defined with respect to the Iwasawa decomposition $s = n(x)t(y)k$ by $ht_A(s) := |y|^{-1/2}$. One has $\int_{[\text{SL}_2]} \text{ht}^{1-c} < \infty$ for $c > 0$. Reduction theory says that the image of $ht$ is bounded from below by some $c > 0$ depending only upon $F$.

Recall that the nontrivial unitary character $\psi$ of $\mathbb{A}/F$ is regarded as fixed.

Lemma 1. Let $A \geq 0$ be fixed. Let $\Psi \in L^1([\text{PB}_A^\times])$ with $\langle \Psi, 1 \rangle = 0$. Let $\phi \in S(\text{B}_A^0)$. For $s \in [\text{Mp}_2(\mathbb{A})$, one has $\theta_\phi(\psi, \Psi, s) \ll \text{ht}(s)^{-A}C(\phi)\|\Psi\|_{L^1}$.

Proof. Since $\Psi$ has mean zero,

$\theta_\phi(\psi, \Psi, s) = \int_{g \in [\text{PB}_A^\times]} \Psi(g) \sum_{v \in V - \{0\}} \rho_\text{Weil}(s, \text{Ad}(g)) \phi(v)$.

By the Iwasawa decomposition and reduction theory, we may assume that $s = n(x)t(y)k$ with $|y| \gg 1$. Since $B$ is non-split, we may fix a compact subset $U$ of $\text{PB}_A^\times$ containing a fundamental domain for $[\text{PB}_A^\times]$. Then

$|\theta_\phi(\psi, \Psi, s)| \ll \|\Psi\|_{L^1}|y|^{3/2} \sum_{g \in U} \sum_{v \in B^0 - \{0\}} |\rho_\text{Weil}(k, \text{Ad}(g)) \phi(yv)|$. 
Since the Weil representation is continuous ([30, §39]), we may reduce to the case $k = 1$ and $g = 1$, in which the required estimate follows from Lemma 2 of §3.5.2. □

Lemma 2. For $\phi \in \mathcal{S}(\mathbb{A})$ and $s \in \text{Mp}_2(\mathbb{A})$, one has $\theta_{\psi}(\phi; s) \ll \text{ht}(s)^{1/4}\mathcal{C}(\phi)$.

Proof. We argue as in the proof of Lemma 1, but take into account the contribution from $0 \in F$ to the definition of $\theta_{\psi}(\phi)$. □

3.5.4. Main estimate: the case of pure tensors.

Lemma. Let $\phi_1', \phi_2' \in \mathcal{S}(\mathbb{A})$ and $\phi_1'', \phi_2'' \in \mathcal{S}(B^0_\mathbb{A})$. Let $\Psi_1, \Psi_2 : [PB^\times] \to \mathbb{C}$ be integrable functions of mean zero. Let $s$ be fixed. Abbreviate $\theta_i := \theta_{\phi_i'}(\phi_i')$ and $h_i := \theta_{\phi_i''}(\phi_i'', \Psi_i)$. Then for all $s \in \text{Mp}_2(\mathbb{A})$,

$$\langle \theta_1 \cdot \rho_{\text{reg}}(s)h_1, \theta_2 \cdot \rho_{\text{reg}}(s)h_2 \rangle = \langle \theta_1, \theta_2 \rangle \langle h_1, h_2 \rangle + O(\Xi(s) \prod_{i=1,2} \mathcal{C}(\phi_i')\mathcal{C}(\phi_i''))\|\Psi_i\|_{L^1}.$$  

Proof. The main result of [20] gives an estimate nearly of the required shape, but instead with the error term $\Xi(s)\mathcal{S}(\phi_i')\mathcal{S}(\phi_i'')\mathcal{S}^X(h_1h_2)$, where $\mathcal{S}, \mathcal{S}^X$ are adelic Sobolev norms whose relevant properties we recall shortly. By the cuspidality of $h_1, h_2$ and axioms (S3b) and (S4e) of [16], we may replace the expression $\mathcal{S}(h_1h_2)$ first with $\mathcal{S}^X(h_1h_2)$ and then with $\mathcal{S}(h_1)\mathcal{S}(h_2)$. Our task thereby reduces to showing for $i = 1, 2$ that $\mathcal{S}(\phi_i') \ll \mathcal{C}(\phi_i')$ and $\mathcal{S}(h_i) \ll \mathcal{C}(\phi_i'')\|\Psi_i\|_{L^1}$.

To that end, we must recall something about the norms $\mathcal{S}$ (see [16, §2] and [20, §4.6, §5.3] for details). They have the form $\mathcal{S}(v) = \|\Delta^d v\|$, where $d \in \mathbb{Z}_{\geq 0}$ is fixed but large enough, and $\Delta$ acts linearly on the space of smooth vectors in any unitary representation $\pi$ of $\text{Mp}_2(\mathbb{A})$. If $\pi$ factors as $\pi_{\text{fin}} \otimes \pi_{\text{inf}}$, then likewise $\Delta = \Delta_{\text{inf}} \otimes \Delta_{\text{fin}}$.

If $\pi$ is the Weil representation on the Schwartz–Bruhat space $\mathcal{S}(V_\mathbb{A}) = \mathcal{S}(V_\mathbb{C}) \otimes \mathcal{S}(V)$ of a quadratic space $V$ over $F$, then $\Delta_{\text{fin}}$ is a finite order differential operator with polynomial coefficients, hence is continuous for the Schwartz topology; since any linear operator on $\mathcal{S}(V)$ is continuous, it follows that $\Delta$ defines a continuous operator on $\mathcal{S}(V_\mathbb{A})$ for the Schwartz–Bruhat topology.

The continuity of $\Delta$ implies that $\mathcal{S}(\phi_i') = \|\Delta^d \phi_i'\| \ll \mathcal{C}(\phi_i')$, giving one of the two required estimates. The operator $\Delta$ is moreover natural in that for an equivariant morphism $f : \pi \to \pi'$ of $\text{Mp}_2(\mathbb{A})$-representations, one has $\Delta \circ f = f \circ \Delta$. Thus

$$\mathcal{S}(h_i) = \|\Delta^d h_i\| = \|\theta_{\phi_i''}(\Delta^d \phi_i'', \Psi_i)\|.$$  

By Lemma 1 of §3.5.3, we have $\mathcal{S}(h_i) \ll \mathcal{C}(\Delta^d \phi_i'')\|\Psi_i\|_{L^1}$. The continuity of $\Delta$ gives $\mathcal{C}(\Delta^d \phi_i'') \ll \mathcal{C}'(\phi_i'')$, and the required estimate follows. □

3.5.5. Factorization. Let $V', V''$ be vector spaces over $F$ and $V := V' \oplus V''$.

Lemma. Let $\ell : \mathcal{S}(V'_\mathbb{A}) \otimes \mathcal{S}(V''_\mathbb{A}) \to \mathbb{C}$ be an algebraic linear functional on the algebraic tensor product of Schwartz–Bruhat spaces satisfying an estimate of the form

$$\ell(\phi' \otimes \phi'') \ll \mathcal{C}'(\phi')\mathcal{C}''(\phi'').$$  

Then $\ell$ extends to a continuous functional $\ell : \mathcal{S}(V_\mathbb{A}) \to \mathbb{C}$ satisfying

$$\ell(\phi) \ll C(\phi)$$  

for all $\phi \in \mathcal{S}(V_\mathbb{A})$, where $C$ depends only upon $C'$ and $C''$. 
Proof. This is essentially the Schwartz kernel theorem, as extended by Bruhat [2, §5]. For concreteness, we sketch a proof. There exists \( \nu = \prod \nu_\ell \in C^\infty_c(V'_A) \) such that \( \sum_{\lambda \in V} \nu(\lambda + x)^2 = 1 \) for all \( x \in V'_A \); thus \( \nu \) is a \( \text{“square-root of a partition of unity”} \). Let \( m \in \mathbb{A}^\times \) be large enough that the map \( \text{supp}(\nu) \to V'_A/mV' \) is injective.

Set \( \Lambda_1 := V' \) and \( \Lambda_2 := m^{-1}V' \). Fix non-degenerate bilinear forms \( V'_A \otimes V'_A \to \mathbb{A} \) and \( V''_A \otimes V''_A \to \mathbb{A} \); denote them by \( (x, y) \mapsto x \cdot y \in \mathbb{A} \). By Fourier inversion on the compact group \( V''_A/mV' \), there exists \( c > 0 \) so that for all \( f \in C^\infty_c(V''_A) \) and \( x \in V'_A \),

\[
\nu(x)^2 f(x) = c \nu(x) \sum_{\lambda_2 \in \Lambda_2} \psi(\lambda_2 \cdot x) \int_{y \in V'_A} \nu(y) f(y) \psi(-\lambda_2 \cdot y). \tag{3.19}
\]

For \( t' \in V'_A, t'' \in V''_A \), we apply (3.19) to \( f(x) := \phi((t'' - \lambda_1) + x) \) and set \( x := t' + \lambda_1 \), giving

\[
\phi(t' + t'') = \sum_{\lambda_2 \in \Lambda_2} \nu(t' + \lambda_1)^2 \phi((t'' - \lambda_1) + (t' + \lambda_1)) \tag{3.20}
\]

\[
= \sum_{\lambda_2 \in \Lambda_2} \phi'_{\lambda_1, \lambda_2}(t') \phi''_{\lambda_1, \lambda_2}(t''), \tag{3.21}
\]

where

\[
\phi'_{\lambda_1, \lambda_2}(t') := c \nu(t' + \lambda_1) \psi(\lambda_2 \cdot (t' + \lambda_1)), \tag{3.22}
\]

\[
\phi''_{\lambda_1, \lambda_2}(t'') = \int_{y \in \mathbb{A}} \nu(y) \phi(t'' - \lambda_1 + y) \psi(-\lambda_2 \cdot y). \tag{3.23}
\]

“Integration by parts” gives readily that

\[
\sum |\ell(\phi'_{\lambda_1, \lambda_2} \otimes \phi''_{\lambda_1, \lambda_2})| \ll \sum |C'(\phi'_{\lambda_1, \lambda_2})C''(\phi''_{\lambda_1, \lambda_2})| \ll C(\phi) \tag{3.24}
\]

for suitable \( C(\phi) \). It follows that \( \ell \) admits the required extension and satisfies the required estimate.

3.5.6. Main estimate: the general case. Temporarily denote by \( \mathcal{A}_0 \) denote the space of integrable functions \( \Psi : \mathbb{P}B^+ \to \mathbb{C} \) of mean zero. Let \( \mathcal{E}_{\tau_1, \tau_2} : \mathcal{S}(B_\mathbb{A}) \otimes \mathcal{S}(B_\mathbb{A}) \otimes \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathbb{C} \) denote the sesquilinear form given for \( \phi_i = \phi'_i \otimes \phi''_i \in \mathcal{S}(B_\mathbb{A}) \) with \( \phi'_i, \phi''_i \in \mathcal{S}(B_\mathbb{A}) \) by

\[
\mathcal{E}_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2) := \langle \theta_1 h_1, \theta_2 h_2 \rangle - \langle \theta_1, \theta_2 \rangle \langle h_1, h_2 \rangle,
\]

where we abbreviate \( \theta_i := \theta_{\tau_i}(\phi'_i) \) and \( h_i := \theta_{\tau_i}(\phi''_i, \Psi_i) \). (The definition makes sense: \textit{a priori} estimates as in §3.5.3 and the density of \( \mathcal{S}(B_\mathbb{A}) \otimes \mathcal{S}(B^0_\mathbb{A}) \) in \( \mathcal{S}(B_\mathbb{A}) \) allow us to extend \( \mathcal{E}_{\tau_1, \tau_2} \) continuously from its initial domain.)

Proposition 1. For \( \phi_1, \phi_2 \in \mathcal{S}(B_\mathbb{A}), \Psi_1, \Psi_2 \in \mathcal{A}_0 \) and \( s \in \mathbb{M}_p(\mathbb{A}) \), one has with \( \rho_0(s) := \rho^\psi_{\mathcal{E}_{\tau_1, \tau_2}}(s) \) the estimate

\[
\mathcal{E}_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2) \ll \Xi(s) \prod_{j=1,2} C(\phi_j) ||\Psi_j||L^1. \tag{3.25}
\]

The implied constant and the uniformity of the continuity of \( C(\phi_j) \) depend at most upon \( \psi, \tau_1, \tau_2, F, B \). The operator \( 1 \otimes \rho^{\tau_1}(s) \) is defined as in §2.2.4.

Proof. The lemma of §3.5.5 reduces the general case of Proposition 1 to the special case in which \( \phi_i = \phi'_i \otimes \phi''_i \) for \( i = 1, 2 \), which follows from the lemma of §3.5.4 upon recalling from (3.13) that \( \theta_{\psi_i} \) intertwines \( \rho_0 \) with \( \rho_{\text{reg}} \). \qed
3.5.7. **Invariance properties.** We record these for later use.

**Lemma.** For \( g_1, g_2 \in \text{PB}_A^\times \) and \( s \in \text{Mp}_2(A) \), one has

\[
\mathcal{E}_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2) = \mathcal{E}_{\tau_1, \tau_2}(\mathcal{G}\phi_1, \phi_2, \Psi_1, \Psi_2)
\]

\[
= \mathcal{E}_{\tau_1, \tau_2}(\text{Ad}(g_1)\phi_1, \text{Ad}(g_2)\phi_2, \rho_{\text{reg}}(g_1)\Psi_1, \rho_{\text{reg}}(g_2)\Psi_2)
\]

\[
= \mathcal{E}_{\tau_1, \tau_2}(\rho_{\text{Weil}}^\tau(s)\phi_1, \rho_{\text{Weil}}^\tau(s)\phi_2, \Psi_1, \Psi_2).
\]

**Proof.** The first two identities follow from (3.17) and (3.12), the remaining from (3.11), (3.13), (3.14), (3.16), (3.17) and the translation invariance of the Petersson inner product. \( \square \)

### 3.6. Similitude theta functions.

#### 3.6.1. **Weil representation.** For each place \( p \) of \( F \), let \( \Omega_p \) denote the representation of \( \text{PGL}_2(F_p) \times \text{GO}(B_p) \) attached as in §2.2.5 to the tuple \((F_p, B_p, \psi_p)\). Let \( \Omega \) denote the restricted tensor product of the spaces \( \Omega_p \) with respect to the distinguished elements, which we denote now by \( \phi_0^p \in \Omega_p \). We may and shall identify \( \Omega \) with the space of functions \( \phi : \mathbb{A}^\times \times B_\mathbb{A} \rightarrow \mathbb{C} \) such that

- For each \( t \in \mathbb{A}^\times \), the function \( \phi(t) : B_\mathbb{A} \rightarrow \mathbb{C} \) given by \( \phi(t)(x) := \phi(t, x) \) belongs to the Schwartz–Bruhat space \( S(B_\mathbb{A}) \);
- \( \phi(z^2t, x) = \phi(t, zx) \) for all \( z, t \in \mathbb{A}^\times, x \in B_\mathbb{A} \).
- There is a compact subset \( C \) of \( \mathbb{A}^\times / \mathbb{A}^\times_2 \) such that \( \phi(t) = 0 \) for all \( t \notin C \) (i.e., for all \( t \in \mathbb{A}^\times \) whose image in \( \mathbb{A}^\times / \mathbb{A}^\times_2 \) lies outside \( C \));
- There is an open subgroup \( U \) of \( \mathbb{A}^\times / \mathbb{A}^\times_2 \) such that \( \phi(tu) = \phi(t) \) for all \( t \in \mathbb{A}^\times, u \in U \).

We equip \( \Omega \) with the invariant hermitian norm \( \| \cdot \| \) obtained by tensoring those on the factors \( \Omega_p \), thus

\[
\|\phi\|^2 := \int_{t \in \mathbb{A}^\times / \mathbb{A}^\times_2} |t|^2 \int_{x \in B_\mathbb{A}} |\phi|^2(t, x). \tag{3.26}
\]

The group \( \text{PGL}_2(A) \times \text{GO}(B_\mathbb{A}) \) acts on \( \Omega \) by the representation \( \rho_{\text{Weil}} \) obtained as the restricted tensor product of those defined in §2.2.5. We define \( \mathcal{G} : \Omega \rightarrow \Omega \) and (for \( g \in \text{PB}_A^\times \)) \( \text{Ad}(g) : \Omega \rightarrow \Omega \) as in §2.2.5. Note that \( \mathcal{G} \) does not preserve pure tensors: for \( \phi = \otimes \phi_p \in \Omega \),

\[
\mathcal{G}\phi(t, x) = (\phi(t, x) + \phi(t, x - nr(x)))/2,
\]

\[
\otimes \mathcal{G} \phi_p(x, t) = \prod \left( \phi(t_p, x_p) + \phi(t_p, x_p - nr(x_p)) \right)/2,
\]

and these are not in general the same. They do coincide if \( \#\{ p : \mathcal{G}\phi_p \neq \phi_p \} \leq 1 \).

#### 3.6.2. **Theta functions.** For \( \phi \in \Omega, s \in \text{PGL}_2(A), g \in \text{GO}(B_\mathbb{A}) \), set

\[
\Theta(\phi; s, g) := \frac{1}{2} \sum_{\tau \in F^\times / F^\times 2} \sum_{x \in B} \rho_{\text{Weil}}(s, g) \phi(\tau, x). \tag{3.27}
\]

The sum is well-defined, converges absolutely and defines a smooth function \( \Theta(\phi) \) on \( [\text{PGL}_2] \times [\text{GO}(B)] \). For a cusp form \( \Psi : \text{[PB}_A^\times] \rightarrow \mathbb{C} \) and \( s \in \text{PGL}_2(A) \), set

\[
\Theta(\phi, \Psi; s) := \int_{g \in \text{[PB}_A^\times]} \Psi(g) \Theta(\phi; s, \text{Ad}(g)).
\]
The integral (together with similar integrals below) converges absolutely and defines a \( \Theta(\phi, \Psi) : [\text{PGL}_2] \to \mathbb{C} \).

**Remark.** \( \Theta(\phi, \Psi) \) is not a theta lift in the traditional sense: the integral in its definition is with respect to the orthogonal group of \( B^0 \) rather than that of \( B \).

**3.6.3. Fourier expansion.** Let \( \phi \in \Omega \), and let \( \Psi : [\text{PB}^x] \to \mathbb{C} \) be a cusp form.

**Lemma.** For \( x \in \mathbb{A}, y \in \mathbb{A}^\times \), one has

\[
\Theta(\phi, \Psi; n(x)a(y)) = \sum_{\tau \in F^x} \psi(\tau x)W(\Theta(\phi, \Psi), \tau y)
\]

where \( W(\Theta(\phi, \Psi), y) := \int_{g \in [\text{PB}^x]} \Psi(g) \sum_{\gamma \in \text{PB}^x} |y|\phi(y \nr(\gamma)^{-1}, g^{-1}\gamma g) \).

**Proof.** By direct unfolding as in [25], one has for \( g \in \text{PB}^x_A \) that

\[
\Theta(\phi, n(x)a(y); \Ad(g)) = \frac{1}{2} \sum_{\tau \in F^x/F^x} |y|\phi(0, \tau y) + \sum_{\tau \in F^x} \psi(\tau x)W(\Theta(\phi), \Ad(g), \tau y),
\]

where \( W(\Theta(\phi), \Ad(g), y) := \sum_{\gamma \in \text{PB}^x} |y|\phi(y \nr(\gamma)^{-1}, g^{-1}\gamma g) \). We conclude by integrating against \( \Psi \). \( \square \)

**3.6.4. Restriction to SL\(_2\).** Let \( \phi, \Psi \) be as above.

**Lemma.** Suppose that for each \( y \in \mathbb{A}^\times \), one has \( \phi[y] = \phi'[y] \otimes \phi''[y] \) for some \( \phi'[y] \in \mathcal{S}(\mathbb{A}), \phi''[y] \in \mathcal{S}(B^0_A) \). Then for \( y \in \mathbb{A}^\times \) and \( s \in \text{SL}_2(\mathbb{A}) \), one has

\[
\Theta(\phi, \Psi; sa(y)) = \frac{1}{2} \sum_{\tau \in F^x/F^x} |y|\theta_{\phi'}(\phi'[\tau y]; s)\theta_{\phi''}(\phi''[\tau y], \Psi; s).
\]

**Proof.** We derive first using (3.27) that for \( g \in \text{O}(B_A) \),

\[
\Theta(\phi; sa(y), g) = \frac{1}{2} \sum_{\tau \in F^x/F^x} |y|\theta_{\phi'}(\phi[ay]; s, g),
\]

hence by (3.15) that for \( g \in \text{O}(B^0_A) \),

\[
\Theta(\phi; sa(y), g) = \frac{1}{2} \sum_{\tau \in F^x/F^x} |y|\theta_{\phi'}(\phi'[\tau y]; s)\theta_{\phi''}(\phi''[\tau y]; s, g).
\]

We integrate against \( \Psi \) to conclude. \( \square \)

**3.6.5. Unfolding the inner product.** Let \( \phi_1, \phi_2 \in \Omega \). Let \( \Psi_1, \Psi_2 : [\text{PB}^x] \to \mathbb{C} \) be cusp forms.

**Lemma.** Suppose that for each \( y \in \mathbb{A}^\times \), one has \( \phi_1[y] = \phi'_1[y] \otimes \phi''_1[y] \) for some \( \phi'_1[y] \in \mathcal{S}(\mathbb{A}), \phi''_1[y] \in \mathcal{S}(B^0_A) \). Abbreviate \( \theta_i := \theta_{\phi'_i}, (\phi'_i[\tau y], \Psi), h_i := \theta_{\phi'_i}(\phi''_i[\tau y], \Psi_i) \). Then the identity

\[
\langle \Theta(\phi_1, \Psi_1), \Theta(\phi_2, \Psi_2) \rangle_{\text{PGL}_2} = \int_{y \in \mathbb{A}^\times/F^x \mathbb{A}^\times} |y|^2 \frac{1}{2^2} \sum_{\tau_1, \tau_2 \in F^x/F^x} \langle \theta_1 h_1, \theta_2 h_2 \rangle_{\text{SL}_2}
\]

holds, with both sides converging absolutely. \( \square \)
Proof. The LHS is an inner product of cusp forms, hence convergent. On the RHS, we may replace the $y$-integral by a finite sum, since the domain $\mathbb{A}^\times / F^\times \mathbb{A}^{x,2}$ is compact and the integrand is invariant under an open subgroup. For individual $y$, the sum over $\tau_1, \tau_2$ has only finitely many nonzero summands, each of which consists of an inner product whose convergence is clear (see §3.5.3). The expansion (3.28) implies for $y \in \mathbb{A}^\times, s \in \text{SL}_2(\mathbb{A})$ that $\Theta(\phi_1, \Psi_i, a(y)s) = \frac{1}{2} \sum_{\tau, \in F^\times / F^{x,2}} |y| \theta(s) h_i(s)$, so the required identity follows from the formula (3.2) relating integrals over $[\text{PGL}_2]$ and $[\text{SL}_2]$. □

Remark. In this paper, we consider several expressions shaped like the RHS of (3.29). On a first (or perhaps second reading), one should focus on the contributions from $y = \tau_1 = \tau_2 = 1$; under some class number and unit group restrictions, these turns out to be the relevant ones for the proof of Theorem 1. (We considered imposing such restrictions for the sake of presentation, but found that doing so obfuscated rather than simplified.)

3.7. Inner product formulas.

3.7.1. Elementary theta functions. We recall part of [20, Thm 2].

Lemma. Suppose $\phi_1, \phi_2 \in \mathcal{S}(A)$ satisfy $\phi_1(x) = \phi_1(-x), \phi_2(x) = \phi_2(-x)$. Let $\tau_1, \tau_2 \in F^\times$. Set $\theta_1 := \theta_{\psi, \tau_1}(\phi_1), \theta_2 := \theta_{\psi, \tau_2}(\phi_2)$. Then $\langle \theta_1, \theta_2 \rangle_{\text{SL}_2} = 0$ unless $\tau_1 = \tau_2$, in which case $\langle \theta_1, \theta_2 \rangle_{\text{SL}_2} = 2 \langle \phi_1, \phi_2 \rangle_{L^2(\mathbb{A})}$.

3.7.2. Ternary theta lifts. As in [19, §12.3], we explicate Gan–Takeda [7, Thm 6.6] (compare with [22, Prop 2.8 (i)]).

Lemma. Let $\pi_1, \pi_2 \subset L^2([\text{PB}^\times])$ be cuspidal automorphic representations that are not one-dimensional. Let $\Psi_1 \in \pi_1, \Psi_2 \in \pi_2$ and $\phi_1, \phi_2 \in \mathcal{S}(B^0_\mathbb{A})$. Let $\tau \in F^\times$. Set $h_i := \theta_{\psi, \tau}(\phi_i, \Psi_i)$ for $i = 1, 2$.

1. If $\pi_1 \neq \pi_2$, then $\langle h_1, h_2 \rangle_{\text{SL}_2} = 0$.

2. Suppose $\pi_1 = \pi_2 = \pi$. Let $S$ be a finite set of places of $F$ that containing all archimedean places, as well as any finite places at which $B$ ramifies, and that is sufficiently large in terms of $\Psi_i, \phi_i$. Then $\langle h_1, h_2 \rangle_{\text{SL}_2}$ equals

$$\frac{L^{(S)}(\pi, \frac{1}{2})}{\zeta^{(S)}(2)} \prod_{p \notin S} \text{vol}(K_p) \int_{g \in \text{PB}^S_{\pi}} \langle \text{Ad}(g) \phi_1, \phi_2 \rangle_{L^2(B^0_\mathbb{A})} \langle \pi(g) \Psi_1, \Psi_2 \rangle_{\text{PB}^\times}$$

with $L^{(S)}(\pi, \frac{1}{2})$ as in §3.3.3.

3.7.3. Induction to $\Omega$. We now combine the previous two lemmas and sum them up. Temporarily denote by $\mathcal{A}_0$ the space of cusp forms $\Psi : [\text{PB}^\times] \to \mathbb{C}$ that are orthogonal to all one-dimensional representations. For $\tau_1, \tau_2 \in F^\times$, let $m : \Omega \otimes \Omega \otimes \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathbb{C}$ denote the sesquilinear form given for $\phi_1, \phi_2 \in \Omega$ admitting factorizations $\phi_i[y] = \phi'_i[y] \otimes \phi''_i[y]$ by

$$m(\phi_1, \phi_2, \Psi_1, \Psi_2) := \int_{g \in \mathcal{A}_0 \times [\text{PB}^\times]_2} |y|^2 \frac{1}{2} \sum_{\tau_1, \tau_2 \in F^\times / F^{x,2}} \langle \theta_1, \theta_2 \rangle_{\langle h_1, h_2 \rangle},$$

where we abbreviate $\theta_1 := \theta_{\psi, \tau_1}(\phi'_1[\tau_1 y])$ and $h_1 := \theta_{\psi, \tau_1}(\phi''_1[\tau_1 y], \Psi_i)$. The relevance of $m$ may be inferred from §3.6.5.

The definition makes sense: as in the proof of §3.6.5, the $y$-integral is really a finite sum, and the sum over $\tau_1, \tau_2$ has only finitely many nonzero summands.
Each summand defines a sesquilinear form on $S(A) \otimes S(B_\mathbb{R}) \otimes A_0$ that extends continuously to $S(B_\mathbb{R}) \otimes A_0$ by the \textit{a priori} estimates of §3.5.3.

**Lemma.** Let $\pi_1, \pi_2 \subseteq L^2([PB^\times])$ be cuspidal automorphic representations that are not one-dimensional. Let $\Psi_1 \in \pi_1, \Psi_2 \in \pi_2$. Let $\phi_1, \phi_2 \in \Omega$.

1. If $\pi_1 \neq \pi_2$ then $m(\phi_1, \phi_2, \Psi_1, \Psi_2) = 0$.
2. Suppose $\pi_1 = \pi_2 =: \pi$. Let $S$ be a large enough finite set of places. Then $m(\phi_1, \phi_2, \Psi_1, \Psi_2)$ equals

$$\frac{L(S)(\pi, \frac{1}{2})}{\zeta_{PB}^S(2)} \left( \prod_{p \notin S} \text{vol}(K_p) \right) \int_{g \in PB_\mathbb{R}} \langle \text{Ad}(g)\mathcal{S}\phi_1, \mathcal{S}\phi_2 \rangle_{\Omega}(\pi(g)\Psi_1, \Psi_2)_{\Omega}.$$

**Proof.** It suffices to consider the case that $\phi_1, \phi_2$ admit factorizations as in the definition of $m$. By (3.12) and (3.17), we may assume that $\mathcal{S}\phi_1 = \phi_i$, or equivalently, that $\phi_2(t, x) = \phi_i(t, -x)$. By the lemmas of §3.7.1 and §3.7.2, we have $\langle \theta_1, \theta_2 \rangle = 0$ unless $\tau_1 = \tau_2$ and then $\langle h_1, h_2 \rangle = 0$ unless $\pi_1 = \pi_2$; in that case, the formulas from those lemmas and the identities

$$\langle \phi_1[y\tau], \phi_2[y\tau] \rangle_{L^2(B_\mathbb{R})} = \langle \text{Ad}(g)\phi_1[y\tau], \phi_2[y\tau] \rangle_{L^2(B_\mathbb{R})}$$

and (see (3.1), (3.26))

$$\int_{y \in A \times F^\times A^\times 2} |y|^2 \sum_{\tau \in F^\times F^\times 2} \langle \text{Ad}(g)\phi_1[y\tau], \phi_2[y\tau] \rangle_{L^2(B_\mathbb{R})}$$

combine to give the required conclusion. \hfill \square

**4. Estimates for general quantum variance sums**

In this section we introduce general families of quantum variance sums, propose a candidate for their leading asymptotics, and state a general “estimate” comparing the two.

**4.1. Notation.** Let $F$ be a number field with adele ring $A$. Fix a nontrivial unitary character $\psi$ of $A/F$. Let $B$ be a non-split quaternion algebra over $F$. Fix a maximal order $R \subseteq B$ and a finite set $S$ of places of $F$, containing all archimedean places as well as any finite places at which $B$ ramifies. Retain the (unsurprising) notation of §3.1.

Since $B$ is non-split, the quotient $[PB^\times] = PB^\times \setminus PB_\mathbb{A}^\times$ is compact, and so $L^2([PB^\times])$ is completely reducible. Let $A^\psi$ denote the set of irreducible subrepresentations of the Hilbert space $L^2([PB^\times])$. For each $\pi^b \in A^\psi$, let $\pi \leq \pi^b$ denote the subspace of smooth vectors. Set $A := \{ \pi : \pi^b \in A^\psi \}$. Let $A$ denote the algebraic direct sum $\oplus_{\pi \in A} \pi$, regarded as a pre-unitary representation of the group $PB_\mathbb{A}^\times$.

We introduce the following additional notation:

- $K = \prod K_p$: a maximal compact subgroup of $PB_\mathbb{A}^\times$. For $p \notin S$, we assume that $K_p \leq PB_\mathbb{p}^\times$ is the image of $R_\mathbb{p}^\times$.  

\[ A_0 \leq A: \text{the orthogonal complement of the one-dimensional subrepresentations. (We had earlier, in §3.5 and §3.7.3, used the same symbol to denote some larger spaces than what we call here } A_0. \text{ This abuse of notation should introduce no confusion.)}\]

- \[ A_0 := \{ \pi \in A : \pi \subseteq A_0 \} = \{ \pi \in A : \text{dim}(\pi) > 1 \}, \text{ so that } A_0 = \oplus_{\pi \in A_0} \pi. \]
- \[ A^S := \{ \varphi \in A : \rho_{\text{reg}}(k)\varphi = \varphi \text{ for all } k \in K_p, p \notin S \}, A^S_0 := A_0 \cap A^S: \text{ the } \text{“unramified outside } S^\prime \text{” subspaces of } A, A_0. \]
- \[ A^S_{\varnothing} := \{ \pi \in A : \pi \cap A^S_{\varnothing} = \{ 0 \}, A^S_0 := A^S \cap A_0: \text{ the subsets consisting of those } \pi \text{ that are unramified outside } S. \]
- \[ B(V), \text{ for } V \text{ a } K\text{-invariant subspace of } A: \text{ an orthonormal basis for the closure of } V \text{ that consists of } K\text{-isotypic elements of } V. \]

Fix Haar measures on \( PB^\times_S \) and \([PB^\times]^\times \); we do not require any compatibility between them. Because \( B \) is non-split, each \( \pi \in A_0 \) is cuspidal. Let \( L(S)(\pi, s), L(S)(\text{ad } \pi, s) \) be as in §3.3.3.

### 4.2. Key definitions.

#### 4.2.1. The basic distributions.

For \( \pi \in A^S \), define \( \omega_\pi : C^\infty_c(PB^\times_S) \otimes A^S \rightarrow \mathbb{C} \) by \( \omega_\pi(f, \Psi) := \sum_{\varphi \in \mathcal{B}(\pi \cap A^S_{\varnothing})} \langle \varphi, \Psi, \pi(f)\varphi \rangle. \) The definition is independent of the choice of orthonormal basis.

**Example.** If \( \pi(f) = 0 \), then \( \omega_\pi(f, \Psi) = 0. \) If \( \pi(f) \) is the orthogonal projector onto a one-dimensional subspace \( \mathbb{C}\varphi \) of \( \pi \) with unit basis vector \( \varphi \), then \( \omega_\pi(f, \Psi) = \langle \varphi, \Psi, \varphi \rangle. \)

#### 4.2.2. Quantum variance sums.

For \( f \in C^\infty_c(PB^\times_S) \), define the sesquilinear form \( V_f : A^S_0 \otimes A^S_0 \rightarrow \mathbb{C} \) by
\[
V_f(\Psi_1, \Psi_2) := \sum_{\pi \in A^S_0} L(S)(\text{ad } \pi, 1)\omega_\pi(f, \Psi_1)\overline{\omega_\pi(f, \Psi_2)}.
\]

#### 4.2.3. Proposed limiting variance.

For \( f \in C^\infty_c(PB^\times_S) \), define the sesquilinear form \( M_f : A^S_0 \otimes A^S_0 \rightarrow \mathbb{C} \) by requiring for \( \Psi_1 \in \pi_1 \in A^S_0, \Psi_2 \in \pi_2 \in A^S_0 \) that \( M_f(\Psi_1, \Psi_2) := 0 \) unless \( \pi_1 = \pi_2 := \pi \), in which case
\[
M_f(\Psi_1, \Psi_2) := c_3 L(S)(\pi, \frac{1}{2}) \int_{g \in PB^\times_S} \langle \text{Ad}(g)\mathcal{S}f, \mathcal{S}f \rangle_{L^2(PB^\times_S)} \langle \pi(g)\Psi_1, \Psi_2 \rangle_{PB^\times_S}.
\]

where
\[
c_3 := c_{PB^\times}(2) \text{ vol}(PB^\times)^{-1}.
\]

The integral converges absolutely (see §2.1.6).

#### 4.2.4. Thickening \( PB^\times_S \) inside \( B \).

Fix, once and for all, a nonzero element \( W_S \in C^\infty_c(F^\times_S) \). For \( \tau \in F^\times \), define the linear map \( \nabla^\tau : C^\infty_c(PB^\times_S) \rightarrow S(B_S) \) by
\[
\nabla^\tau f(x) := \frac{W_S(\tau \text{nr}(x))}{|\text{nr}(x)|} _{B^\times_S} 1_{B^\times_S}(x)f(\text{pr}(x)),
\]
where \( \text{pr} : B^\times_S \rightarrow PB^\times_S \) denotes the natural projection.
4.3. Statement of main result. The statement involves the metaplectic group (§3.4.1) and the Weil representation (§3.4.3). For \( s \in \text{Mp}_2(F_S) \), we abbreviate \( \rho^\tau(s) := \rho^\psi_{\text{Weil}}(s) \) and \( \rho_0^\tau(s) := \rho^\psi_{\text{Weil}}(s) \); these operators act respectively on \( S(B_S) \) and \( S(B_0^S) \). The operators \( 1 \otimes \rho_0^\tau(s) \) on \( S(B_S) \) are defined using the decomposition \( B_S = F_S \oplus B_0^S \), as in §2.2.4.

**Theorem 2.** There is a finite subset \( X \) of \( F^\times \) and a finite collection \( (\varepsilon_{\tau_1, \tau_2})_{\tau_1, \tau_2 \in X} \) of sesquilinear forms \( \varepsilon_{\tau_1, \tau_2} : S(B_S) \otimes S(B_S) \otimes A_0^S \otimes A_0^S \to \mathbb{C} \), depending only upon \( F, \psi, S \) and \( W_S \), with the following properties:

1. **Relevance.** For \( f \in C^\infty_c(PB_S^\times) \), one has the following identity of sesquilinear forms on \( A_0^S \):
   \[
   \mathcal{V}_f = \mathcal{M}_f + \sum_{\tau_1, \tau_2 \in X} \varepsilon_{\tau_1, \tau_2}(\otimes^{\tau_1} f, \otimes^{\tau_2} f, \cdot, \cdot).
   \]  

2. **\( O_1(F) \)-invariance.**
   \[
   \varepsilon_{\tau_1, \tau_2}(\otimes \phi_1, \phi_2, \Psi_1, \Psi_2) = \varepsilon_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2),
   \]
   \[
   \varepsilon_{\tau_1, \tau_2}(\phi_1, \otimes \phi_2, \Psi_1, \Psi_2) = \varepsilon_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2).
   \]

3. **\( \text{SO}(B_0^S) \)-invariance.** For \( g_1, g_2 \in PB_S^\times \),
   \[
   \varepsilon_{\tau_1, \tau_2}(\text{Ad}(g_1)\phi_1, \text{Ad}(g_2)\phi_2, \rho_{\text{reg}}(g_1)\Psi_1, \rho_{\text{reg}}(g_2)\Psi_2)
   \]
   \[= \varepsilon_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2).\]

4. **Metaplectic invariance.** For \( s \in \text{Mp}_2(F_S) \),
   \[
   \varepsilon_{\tau_1, \tau_2}(\rho^\tau(s)\phi_1, \rho^\tau(s)\phi_2, \Psi_1, \Psi_2) = \varepsilon_{\tau_1, \tau_2}(\phi_1, \phi_2, \Psi_1, \Psi_2).
   \]

5. **Main estimate.** For \( s \in \text{Mp}_2(F_S) \),
   \[
   \varepsilon_{\tau_1, \tau_2}((1 \otimes \rho_0^\tau(s))\phi_1, (1 \otimes \rho_0^\tau(s))\phi_2, \Psi_1, \Psi_2)
   \]
   \[\ll \Xi(s) \prod_{i=1,2} C(\phi_i) \|\Psi\|_{L^1},\]
   where \( \Xi \) denotes the Harish–Chandra function (§3.1.4) and \( C(\phi_i) \) denotes a quantity that varies continuously with \( \phi_i \) (see §3.5.1). The implied constants and the uniformity in the continuity of \( C(\cdot) \) depend at most upon \( F, \psi, S, W_S \).

6. **Construction.** \( \varepsilon_{\tau_1, \tau_2} \) factors explicitly through the theta correspondence in the sense of §4.4.9 and the remark of §8.5.

**Remark 1.** For the application to Theorem 1, the crucial assertions are the relevance, the \( \text{SO}(B_0^S) \)-invariance, and the main estimate. The \( O_1(F) \)-invariance and metaplectic invariance are employed to simplify the presentation of the proof. The construction is not applied in this paper, but may be useful for further refinements and extensions.

**Remark 2.** One purpose of Part II is to give evidence that Theorem 2 is useful.

**Remark 3.** The formulation of Theorem 2 is independent of the choice of measures on \( \text{PB}_S^\times \) and on \( [\text{PB}^\times] \).
Remark 4. Theorem 2 minus the “main estimate” is like a trace formula: $\mathcal{V}_f$ is a sum over automorphic forms, $\mathcal{M}_f$ is like the “identity” contribution, and the $\varepsilon_{\tau_1,\tau_2}$ are the “interesting” contributions which one would like in practice to show have negligible size. One difference is that $\mathcal{V}_f$ has a quadrilinear (rather than bilinear) dependence upon the automorphic forms $\varphi$.

Remark 5. Theorem 2 likely extends to the split case $B = M_2(F)$ after incorporating contributions from the continuous spectrum into the definitions of $\S 4.2$ and replacing $\|\Psi_t\|_{L^1}$ with $\|\operatorname{ht}^A \Psi_t\|_{L^1}$ for some fixed large enough $A > 0$.

4.4. Proof of Theorem 2.

4.4.1. Measures. With a view to applications, we have formulated Theorem 2 in a measure-independent fashion. For the proof, it is convenient to take on $[PB^\times]$ the Tamagawa measure, so that

$$c_3 = \frac{1}{2} \lambda_p(S)(2),$$

(4.3)

and to fix measures on $\text{PB}_k, \text{PB}_p$ and hence on $\text{PB}_S = \prod_{p \in S} \text{PB}_p^\times$ as in $\S 3.1.3$.

4.4.2. The $\heartsuit$ operator: local. Suppose temporarily (for $\S 4.4.2$ only) that $k$ is a local field, $\psi$ is a nontrivial unitary character of $k$, $B$ is a quaternion algebra over $k$, $G := B^\times/k^\times$, and $W \in C_c^\infty(k^\times)$. Recall from $\S 2.2.5$ the definition of $\Omega$. We define a linear map $\heartsuit : C_c^\infty(G) \to \Omega$ by

$$\heartsuit f(t, x) := \frac{W(t \operatorname{nr}(x))}{|t \operatorname{nr}(x)|} 1_{B^\times}(x) f(x).$$

(4.4)

(By abuse of notation, we write $f(x)$ for the value taken by $f$ at the image of $x$ under the natural projection $B^\times \to G$.)

By inspecting the definitions, one has the identities of maps $C_c^\infty(G) \to \Omega$

$$\mathcal{S} \heartsuit = \heartsuit \mathcal{S}, \quad \operatorname{Ad}(g) \heartsuit = \heartsuit \operatorname{Ad}(g) \quad (\text{for } g \in G).$$

By inspecting the definitions, one has for $y \in k^\times, b \in B^\times$ that

$$\rho_{\text{well}}(a(y)) \heartsuit f(y \operatorname{nr}(b)^{-1}, b) = |y| \heartsuit f(y \operatorname{nr}(b)^{-1}, b) = \mathcal{M}(y) f(b).$$

(4.5)

By the formula (2.7) for $\|\cdot\|_{\Omega}$, one obtains

$$\|\heartsuit f\|_{\Omega} = \|f\|_{L^2(G)} \|\mathcal{M}\|_{L^2(k^\times, |x|^{-1} dx)}. $$

(4.6)

4.4.3. The $\heartsuit$ operator: global. We revert to the global setting of $\S 4$. Let $\pi \in A^S_\text{st}$. Recall from $\S 3.6.1$ the definition of $\Omega$. We define a linear map $\heartsuit : C_c^\infty(\text{PB}^\times) \to \Omega$ by

$$\heartsuit f(t, x) := \frac{W_S(t_S \operatorname{nr}(x_S))}{|t_S \operatorname{nr}(x_S)|} 1_{B^\times_S}(x_S) f(x_S) \prod_{p \notin S} \operatorname{vol}(K_p)^{-1} \phi^0_p(t_p, x_p),$$

where $\phi^0_p \in \Omega_p$ is defined with respect to $R_p$ (see $\S 2.2.6$). This definition and that of $\S 4.2.4$ are obviously similar; we record their precise relationship below in $\S 4.4.9$.

By (4.5) and the lemma of $\S 2.4.1$, one has for $y \in A^\times, b \in B^\times_k$ that

$$|y| \heartsuit f(y \operatorname{nr}(b)^{-1}, b) = W_S(y_S) f(b_S) \prod_{p \notin S} T_{y_p}(b_p),$$

(4.7)
with $T_{\psi}$ as in §2.4.1. By combining (4.6) with Lemma 1 of §2.2.6, one obtains
\[
\|\varpi f\|_{K}^{2} = \|f\|_{L^{2}(PB^{\infty}_{S})}^{2} \int_{t \in F_{S}^{\times}} |m|^{2}(t) \frac{dt}{|t|} \prod_{p \in S} \frac{\vol(R_{p})}{\vol(K_{p})^{2}}. \quad (4.8)
\]

**Lemma.** Let $\pi \in A_{0}^{S}$. Let $\Psi_{1}, \Psi_{2} \in \pi$ be $\prod_{p \in S} K_{p}$-invariant vectors. For $f \in C_{c}^{\infty}(PB^{\infty}_{S})$, the quantity $m(\varpi f, \varpi f, \Psi_{1}, \Psi_{2})$ (see §3.7.3) equals
\[
c_{2}L^{(S)}(\pi, \frac{1}{2}) \int_{y \in PB^{\infty}_{S}} \langle \text{Ad}(g)\mathcal{G}f, \mathcal{G}f \rangle_{L^{2}(PB^{\infty}_{S})} \langle \pi(g)\Psi_{1}, \Psi_{2} \rangle_{PB^{\times}},
\]
where
\[
c_{2} := \frac{1}{\zeta_{F}^{(S)}(2)} \prod_{p \in S} \frac{\vol(R_{p})}{\vol(K_{p})} \int_{y \in F_{S}^{\times}} |W_{S}|^{2}(y) \frac{dy}{|y|}. \quad (4.9)
\]

**Proof.** By the lemma of §3.7.3, the polarization of (4.8) and the commutativity $\varpi \text{Ad}(g) = \text{Ad}(g)\varpi$, the required identity holds if we replace $S$ with some possibly larger finite set of places $S' \supseteq S$. To deduce the identity as written, we apply (2.14) (using (3.4) to verify its hypotheses).

4.4.4. A specific Eichler/Jacquet–Langlands lift. For $\pi \in A_{0}^{S}$, let $\Phi_{\pi} \in \pi_{JL}$ denote the element of the Jacquet–Langlands lift of $\pi$ having the Fourier expansion $\Phi_{\pi}(n(x)a(y)) = \sum_{\tau \in F_{S}^{\times}} \psi(\tau x) W_{\pi}(\tau y)$, where the Whittaker function $W_{\pi} : \mathbb{A}^{\infty} \to \mathbb{C}$ is given by $W_{\pi}(y) := W_{S}(y_{S}) \prod_{p \in S} W_{\pi_{p}}(y_{p})$ (see §3.2.1, §3.2.2).

**Lemma.** One has $\|\Phi_{\pi}\|^{2} = c_{3}L^{(S)}(\text{ad} \pi, 1)$, where
\[
c_{1} := \frac{1}{\zeta_{F}^{(S)}(2)} \prod_{p \in S} \frac{\Delta_{\psi_{p}}^{-1/2}}{\vol(K_{p})} \int_{y \in F_{S}^{\times}} |W_{S}|^{2}(y) \frac{dy}{|y|}. \quad (4.10)
\]
If $\pi, \pi' \in A_{0}^{S}$ are distinct, then $\langle \Phi_{\pi}, \Phi_{\pi'} \rangle = 0$.

**Proof.** The conclusion in the case $\pi \neq \pi'$ is the multiplicity one theorem for $PB^{\times}$ combined with the injectivity of $\pi \mapsto \pi_{JL}$. The formula (4.10) is a consequence of the lemma of §3.2.2 and the corresponding local calculation (§2.3).

4.4.5. The normalizing scalar. Recall from (4.10), (4.9) and (4.3) the scalars $c_{1}, c_{2}, c_{3}$. By the local volume formulas of §§2.1.3, 2.2.6,
\[
c_{1}^{-1} c_{2} = c_{3}. \quad (4.11)
\]

4.4.6. Application of the pretrace formula. Recall the theta functions $\Theta(\phi, \Psi)$ attached in §3.6 to each $\phi \in \Omega$, $\Psi \in A_{0}$. Let $f \in C_{c}^{\infty}(PB^{\infty}_{S})$, $\Psi \in A_{0}^{S}$.

**Lemma 1.** $\sum_{\pi \in A_{0}^{S}} |\omega_{\pi}(f, \Psi)||\Phi_{\pi}|_{L^{p}(\text{PGL})_{2}} < \infty$ for $p = 2, \infty$.

**Proof.** By (3.6) and (3.7), it suffices to show that $\|\Phi_{\pi}\|_{L^{p}(\text{PGL})_{2}} \ll C(\pi)^{O(1)}$. The case $p = 2$ follows from the lemma of §4.4.4 and (3.9). The case $p = \infty$ reduces to the case $p = 2$ by axioms (S2a) and (S3b) of [16, §2.4], wherein the quantities $S_{d}(\Phi_{\pi})$ may be estimated using [16, §3.2.5]. A direct proof of this convergence also follows by a rearrangement of the arguments given below.

**Lemma 2.** $\Theta(\varpi f, \Psi) = \sum_{\pi \in A_{0}^{S}} \omega_{\pi}(f, \Psi) \Phi_{\pi}$. 

Proof. Set $\Phi_1 := \Theta(\nabla f, \Psi)$ and $\Phi_2 := \sum_{\pi \in A_0} \omega_\pi(f, \Psi) \Phi_\pi$; we must show that $\Phi_1 = \Phi_2$. Since $\Phi_1, \Phi_2$ are cuspidal, it will suffice to demonstrate the equality of their Whittaker functions $W_1, W_2 : \mathbb{A}^\times \to \mathbb{C}$ as defined in §3.2.1. By the lemma of §3.6.3 and (4.7), we have

$$W_1(y) = \sum_{\gamma \in \mathbb{P}B^\times} \int_{g \in [\mathbb{P}B^\times]} \Psi(g) W_S(y s) f(g S^{-1} \gamma g s) \prod_{p \notin S} T_{y_p}(g_p^{-1} \gamma g_p).$$

The definition of $\Phi_\pi$ implies (using Lemma 1 to justify the interchange of summation with the Fourier integral over the compact group $\mathbb{A}/F$) that

$$W_2(y) = \sum_{\pi \in A_0} \omega_\pi(f, \Psi) W_S(y s) \prod_{p \notin S} W_{\pi_p}(y_p).$$

Since $\Psi \in A_0^S$, we have $\omega_\pi(f, \Psi) = 0$ for all $\pi \in A^S$ with $\pi \notin A_0^S$, so it suffices to establish for all $y \in \mathbb{A}^\times$, $g \in \mathbb{P}B^\times$ the pointwise identity

$$\sum_{\gamma \in \mathbb{P}B^\times} f(g S^{-1} \gamma g s) \prod_{p \notin S} T_{y_p}(g_p^{-1} \gamma g_p) = \sum_{\pi \in A^S} \sum_{\pi \in A^S} \Theta(g) \pi(f) \varphi(g) \prod_{p \notin S} W_{\pi_p}(y_p),$$

which follows from the pretrace formula (§3.3.2) and the identity $W_{\pi_p}(y_p) = \lambda_{\pi_p}(T_{y_p})$ (see (2.13)). □

Remark. Lemma 2 and its proof are in the spirit of arguments of Shimizu [25, §4], but we were unable to relate them precisely (e.g., by deducing one from the other).

4.4.7. Some sesquilinear forms. Define $\mathcal{V}, \mathcal{M}, \mathcal{E} : \Omega \otimes \Omega \otimes A_0 \otimes A_0 \to \mathbb{C}$ by requiring that for $\phi_1, \phi_2 \in \Omega$ satisfying $\phi_1[y] = \phi_1'[y] \otimes \phi_1''[y]$ with $\phi_1'[y] \in S(A)$ and $\phi_1''[y] \in \mathcal{S}(B_\mathbb{A})$ for all $y \in \mathbb{A}^\times$, one has with the abbreviations $\theta_i := \theta_{\psi_i}(\phi_i'[\tau_i y])$ and $h_i := \theta_{\psi_i}(\phi_i''[\tau_i y], \Psi_i)$ that

$$\mathcal{V}(\phi_1, \phi_2, \Psi_1, \Psi_2) := c_1^{-1} \int_{y \in \mathbb{A}^\times / F \times \mathbb{A}^\times} |y|^2 \frac{1}{22} \sum_{\tau_1, \tau_2 \in F \times F \times \mathbb{A}^\times} \langle \theta_1 h_1, \theta_2 h_2 \rangle,$$

$$\mathcal{M}(\phi_1, \phi_2, \Psi_1, \Psi_2) := c_1^{-1} \int_{y \in \mathbb{A}^\times / F \times \mathbb{A}^\times} |y|^2 \frac{1}{22} \sum_{\tau_1, \tau_2 \in F \times F \times \mathbb{A}^\times} \langle \theta_1, \theta_2 \rangle \langle h_1, h_2 \rangle$$

and $\mathcal{E} := \mathcal{V} - \mathcal{M}$, or equivalently,

$$\mathcal{E}(\phi_1, \phi_2, \cdot, \cdot) := c_1^{-1} \int_{y \in \mathbb{A}^\times / F \times \mathbb{A}^\times} |y|^2 \frac{1}{22} \sum_{\tau_1, \tau_2 \in F \times F \times \mathbb{A}^\times} \mathcal{E}_{\tau_1, \tau_2}(\phi_1[\tau_1 y], \phi_2[\tau_2 y], \cdot, \cdot),$$

where $\mathcal{E}_{\tau_1, \tau_2} : S(B_\mathbb{A}) \otimes S(B_\mathbb{A}) \otimes A_0 \otimes A_0 \to \mathbb{C}$ is as in §3.5.6. The definitions makes sense for the same reasons as in §3.7.3. The identity

$$\mathcal{V}(\phi_1, \phi_2, \Psi_1, \Psi_2) = c_1^{-1} \langle \Theta(\phi_1, \Psi_1), \Theta(\phi_2, \Psi_2) \rangle. \tag{4.13}$$

follows from §3.6.5 when $\phi$ is a pure tensor, hence in general by linearity.

4.4.8. The main identities.

**Proposition 2.** Let $f \in C_c^\infty(\mathbb{P}B_\mathbb{A})$ and $\Psi_1, \Psi_2 \in A_0^S$. Then

$$\mathcal{V}(\nabla f, \nabla f, \Psi_1, \Psi_2) = \mathcal{V}_f(\Psi_1, \Psi_2), \tag{4.14}$$

$$\mathcal{M}(\nabla f, \nabla f, \Psi_1, \Psi_2) = \mathcal{M}_f(\Psi_1, \Psi_2), \tag{4.15}$$

$$\mathcal{V}_f(\Psi_1, \Psi_2) = \mathcal{M}_f(\Psi_1, \Psi_2) + \mathcal{E}(\nabla f, \nabla f, \Psi_1, \Psi_2). \tag{4.16}$$
Proof. (4.14): By (4.13), Lemma 2 of §4.4.6, and the lemma of §4.4.4,

\[ V(\varnothing f, \varnothing f, \Psi_1, \Psi_2) = c_1^{-1}(\Theta(\varnothing f, \Psi_1), \Theta(\varnothing f, \Psi_2)) \]

\[ = c_1^{-1} \sum_{\pi_1, \pi_2 \in A_0} \langle \omega_{\pi_1}(f, \Psi_1), \omega_{\pi_2}(f, \Psi_2) \rangle \]

\[ = \sum_{\pi \in A_0} L^S(\text{ad} \, \pi, 1) \omega_{\pi}(f, \Psi_1), \omega_{\pi}(f, \Psi_2) \]

\[ = V_1(f, \Psi_1, \Psi_2). \]

(4.15): by the lemma of §4.4.3 and (4.11).

(4.16): by (4.14), (4.15) and the definition of \( E \).

\[ \square \]

4.4.9. Completion of the proof. We now apply Proposition 1 (see §3.5.6) and Proposition 2 to prove Theorem 2. This final part of the argument is of a purely technical nature and involves no major new ideas. Indeed, its main purpose is to recast the content of those propositions in terms of \( S(B_S) \) rather than the less “user-friendly” space \( \Omega \).

By weak approximation, we may choose a compact fundamental domain \( Y \subset A^x / A^{x^2} \) for \( A^x / F^x \), with the property that \( y_p = 1 \) for all \( y \in Y \) and \( p \in S \). Choose a finite set \( X \subset F^x \) of representatives for the finite set

\[ \{ \tau \in F^x / F^{x^2} : \text{there exists } y \in Y \text{ so that } y_p \tau \in F_p^x \Omega_p^x \text{ for all } p \notin S \}. \] (4.17)

For \( y \in Y, \tau \in X \), let \( \Diamond^{\tau y} : S(B_S) \hookrightarrow S(B_\Lambda) \) denote the map \( \Diamond^{\tau y} \Phi := \Phi \otimes ((\otimes_p S) \phi_p[\tau y]) \), where \( \phi_p \in \Omega_p \) denotes as usual the distinguished element. For \( f \in C^\infty_c(\text{PB}_S^x) \), one then has \( \Diamond^{\tau y} \Diamond f = \Diamond f[y] \). Observe that for \( p \notin S \) and \( t \in F_p^x \), one has \( \phi_p[t] = 0 \) unless \( t \in F_p^{x^2} \Omega_p^x \). It follows that for \( \tau \in F^x \) and \( y \in Y \), one has \( \Diamond f[y] = 0 \) unless \( \tau \) belongs to the set (4.17), hence that

\[ E(\varnothing f, \varnothing f, \cdot, \cdot) = c_1^{-1} \int_{y \in Y} |y|^2 \sum_{\tau_1, \tau_2 \in X} E_{\tau_1, \tau_2}(\Diamond^{\tau_1 y} \Diamond^{\tau_1 f}, \Diamond^{\tau_2 y} \Diamond^{\tau_2 f}, \cdot, \cdot). \] (4.18)

Define \( \varepsilon_{\tau_1, \tau_2} : S(B_S) \otimes S(B_S) \otimes A_0^x \otimes A_0^x \rightarrow \mathbb{C} \) by

\[ \varepsilon_{\tau_1, \tau_2}(\Phi_1, \Phi_2, \Psi_1, \Psi_2) := c_1^{-1} \int_{y \in Y} |y|^2 E_{\tau_1, \tau_2}(\Diamond^{\tau_1 y} \Phi_1, \Diamond^{\tau_2 y} \Phi_2, \Psi_1, \Psi_2). \] (4.19)

We verify the assertions made in Theorem 2:

1. The “relevance” follows from (4.18), (4.19) and Proposition 2.
2. Since \( S \phi_p^0 = \phi_p^0 \), one has \( S \Diamond^{\tau y} = \Diamond^{\tau y} S \). For \( g \in \text{PB}_S^x \) and \( s \in \text{Mp}_2(F_S) \), one has \( \text{Ad}(g) \Diamond^{\tau y} = \Diamond^{\tau y} \text{Ad}(g) \) and \( \rho^y(s) \Diamond^{\tau y} = \Diamond^{\tau y} \rho^y(s) \). Thus the “\( O_1(F) \)-invariance,” “\( SO(B_S^x) \)-invariance” and “metaplectic invariance” follow from §3.5.7.
3. The “main estimate” is the content of Proposition 1.

4.5. Classicalization. We begin discussing how to relate the setting of Theorem 2 to that of Theorem 1. We complete this discussion in §9.
4.5.1. **Specialization to a single place.** We specialize the definitions of §4.2 to the case that ramification is concentrated at a single place \( q \) of \( F \), finite or infinite. This is the case required for the proof of Theorem 1.

Assume that \( S \) is the set of places \( p \) for which either

- \( p \) is infinite,
- \( p \) is a finite place at which \( B \) ramifies, or
- \( p = q \).

Assume that for each \( p \notin S - \{q\} \), the completion \( B_q \) is non-split, or equivalently, that \( \mathbb{P}B_q^\times \) is compact. There are the following possibilities:

1. \( q \) is real, in which case \( F \) is totally real and \( B \) ramifies at every infinite place other than \( q \).
2. \( q \) is complex, in which case \( F \) is real and \( B \) ramifies at every infinite place other than \( q \).
3. \( q \) is finite, in which case \( F \) is totally real and \( B \) is totally definite.

For each place \( p \), define the compact open subgroup \( J_p \subseteq \mathbb{P}B_q^\times \) as in §2.4 by taking for \( J_p \) the image of \( R_p^\times \) if \( p \) is finite and taking \( J_p := \mathbb{P}B_p^\times \) if \( p \) is infinite. Set \( J := \prod_{p \notin S} J_p \). In addition to the notation of §4.1, we now introduce a superscripted \( J \), as in \( A^J, A^J_0, \pi^J \) to denote the \( J \)-fixed subspace. We denote by \( A^J_+ \subseteq A^J \) the “even” subspaces consisting of \( \varphi \) that are \( \mathbb{P}B_p^\times \)-invariant for all \( p \in S - \{q\} \). Thus, for instance, \( A^J_0 \subseteq A^J_0 \subseteq A^J_0 \subseteq A \). We denote by \( A_0, A^J_0, A^J_0, A^J_0 \) the set of all \( \pi \in A \) having nonzero intersection with the space having the corresponding scripted notation.

Set \( G := \mathbb{P}B_q^\times \), and let \( f \in C_c^\infty(G) \). For \( p \in S - \{q\} \), set \( e_{J_p} := \text{vol}(J_p)^{-1}J_p \in C_c^\infty(\mathbb{P}B_q^\times) \). Define \( \tilde{f} \in C_c^\infty(\mathbb{P}B_q^\times) \) by the formula

\[
\tilde{f}(g) := f(g_q) \prod_{p \in S - \{q\}} e_{J_p}(g_p).
\]

For \( \pi \in A^J_0 \) and \( \Psi \in A^J_0 \), set \( \omega_\pi(f, \Psi) := \sum_{\varphi \in B(\pi \cap A^J)} \langle \varphi, \Psi : \pi(f) \varphi \rangle \). Then \( \omega_\pi(\tilde{f}, \Psi) = \omega_\pi(f, \Psi) \) (see §4.2.1). Since \( \dim(\pi_p) = 1 \) for all \( p \in S - \{q\} \), one has for \( \Psi \in \pi^J_0 \) that \( \omega_\pi(f, \Psi) = 0 \) unless \( \pi^J \subseteq A^J_0 \).

Let \( V_f, M_f : A^J_0 \otimes A^J_0 \to \mathbb{C} \) denote the sesquilinear forms obtained by restricting the forms \( V_f, M_f : A^J_0 \otimes A^J_0 \to \mathbb{C} \). Then

\[
V_f(\Psi_1, \Psi_2) = \sum_{\pi \in A^J_0} L(S)(\text{ad} \pi, 1) \omega_\pi(f, \Psi_1) \bar{\omega}_\pi(f, \Psi_2).
\] (4.20)

By the observation that \( \mathcal{S}e_{J_p} = e_{J_p} \) for \( p \in S - \{q\} \) and the local calculations (2.15) and (2.16), we see for \( \Psi_1 \in \pi_1 \in A^J_0 \) and \( \Psi_2 \in \pi_2 \in A^J_0 \) that \( M_f(\Psi_1, \Psi_2) = 0 \) unless \( \pi_1 = \pi_2 =: \pi \), in which case

\[
M_f(\Psi_1, \Psi_2) = c_4 L(S)(\pi, \frac{1}{2}) \int_{g \in G} \langle \text{Ad}(g) \mathcal{S}f, \mathcal{S}f \rangle L^2(G)(\pi(g) \Psi_1, \Psi_2)_{\mathbb{P}B^\times} \] (4.21)

where

\[
c_4 := 2^t \zeta_p^{(S)}(2) \text{vol}(\mathbb{P}B^\times))^{-1}.
\] (4.22)

with \( t \) the number of finite primes \( p \in S - \{q\} \).
4.5.2. **Strong approximation.** Retaining the notation of §4.5.1, we record here how the quotient $[\mathbb{P}^1_B]/J$ unadèles under some assumptions. Recall that $G := \mathbb{P}^1_B$. Let $\Gamma \leq G$ denote the image of $\mathbb{P}^1_B \cap J$ under the inclusion $\mathbb{P}^1_B \to G$. Then $\Gamma$ is a discrete cocompact subgroup of $G$, and the natural map

$$\Gamma \backslash G \to [\mathbb{P}^1_B]/J$$

is injective.

**Lemma.** Suppose that $F$ has odd narrow class number and either that

1. $B_q$ is split, or that
2. $q$ is infinite and $B$ has class number one.

Then $\iota$ is bijective.

**Proof.** The class number assumption on $F$ implies that

$$F_+^\times F_\infty^\times F_q^\times \prod_{p < \infty} \mathcal{O}_p^\times \mathbb{A}^\times_2 = \mathbb{A}^\times := F_\infty^\times \mathbb{A}^\times_f$$

where $F_+^\times$ denotes the connected component of the unit group of $F_\infty := \prod_{p \mid \infty} F_p$, $\mathbb{A}^\times_f := \prod_{p < \infty} F_p^\times$ denotes the group of finite ideles, and $F_+^\times := F^\times \cap F_\infty^\times$. Set $J'_f := \mathbb{P}^1_B \cap J$; it is open in $\mathbb{P}^1_B$. Under the reduced norm map $\operatorname{nr} : \mathbb{P}^1_B \to \mathbb{A}^\times_2$, we have $\operatorname{nr}(\mathbb{P}^1_B) = F_\infty^\times \mathbb{A}^\times_2 / \mathbb{A}^\times_2$ and $\operatorname{nr}(J'_f) = F_\infty^\times F_p^\times \prod_{p < \infty} \mathcal{O}_p^\times \mathbb{A}^\times_2 / \mathbb{A}^\times_2$, from our assumption (4.24), it follows that

$$\operatorname{nr}(\mathbb{P}^1_B) \cap \operatorname{nr}(J'_f) = \operatorname{nr}(\mathbb{P}^1_B).$$

If $B_q$ splits, then $\mathbb{P}^1_B$ is non-compact, so the strong approximation theorem and (4.25) imply that $\mathbb{P}^1_B \cap J_f = \mathbb{P}^1_B$, hence that $\iota$ is surjective. In the remaining case, the surjectivity of $\iota$ holds by the definition of “the class number of $B$.”

**Remark.** Suppose the conclusion of the lemma holds and that $q$ is finite. It is then natural to ask for the volume of $\Gamma \backslash G$ with respect to a Haar measure $\nu$ on $\Gamma \backslash G$ obtained as the quotient of some given Haar measure $\mu$ on $G$. Using that $\operatorname{vol}(\mathbb{P}^1_B) = 2$ with respect to Tamagawa measure, one can show (as in §9.2 below) that

$$\nu(\Gamma \backslash G) = \frac{\zeta_F(2) \Delta_B \Delta_F^{3/2}}{(4\pi^2)^{g-1} \prod_{p \in \text{ram}_{J_f}(B)} \zeta_p(1)} \mu(K_q),$$

where $K_q \leq G$ denotes a maximal compact subgroup, $\Delta_F$ the absolute discriminant, $\Delta_B$ the absolute reduced discriminant, and $\text{ram}_{J_f}(B)$ the set of finite places at which $B$ ramifies.

4.5.3. **Hecke operators.** For each finite place $p \neq q$, let $\mathcal{H}_p$ denote the Hecke algebra consisting of compactly-supported bi-$J_p$-invariant distributions on $\mathbb{P}^1_B$. It acts on $\mathcal{A}_1$. It admits a standard generator $T_p$ given by convolution against $\{w\}^{-1} T_\infty \in C_c(\mathbb{P}^1_B \setminus \mathbb{P}^1_B^+ / J_p^\times)$, where $w$ is a generator of the maximal ideal in $\mathcal{O}_p$ and $T_\infty$ is the normalized Hecke kernel defined in §2.4.1. If $B$ ramifies at $p$, then $T_p$ is an involution; otherwise, it has degree $|p| + 1$, where $|p|$ denotes the absolute norm.

**Lemma.** The following are equivalent for $\varphi \in \mathcal{A}_1$.

1. $\varphi$ generates an irreducible representation of $G$ and is an eigenfunction of $T_p$ for each finite $p \neq q$.
2. $\varphi$ generates an irreducible representation $\pi$ of $\mathbb{P}^1_B^\times$. 
Assume that these conditions hold. Then $\pi \in A^J$, and the following are equivalent:

1. $T_p \varphi = \varphi$ for each finite prime $p \neq q$ at which $B$ ramifies.
2. $\pi \in A^J_+$.

In view of the lemma, the following definition faithfully extends and makes precise Definition 1 of §1.5:

Definition. An eigenfunction is a nonzero element $\varphi \in A^J$ that belongs to some $\pi \in A^J$; it is even if $\pi \in A^J_+$ and strongly of mean zero if $\pi \in A^J_0$ (and both if $\pi \in A^J_{0+}$).
Part 2. Application to microlocal lifts

Our aim is now to prove Theorem 1 by application of Theorem 2. We retain the general notation of §1.13. The following additional notation is in effect for §5–§8:

- $k$: a non-archimedean local field of characteristic $\neq 2$.
- $o, q, |.| = |.|_k$, ord = ord$_k$: as in §2.1.1.
- $B := M_2(k)$: the algebra of $2 \times 2$ matrices, so that $nr = \det : B \to k$.
- $G := B^\times /k^\times = PGL_2(k)$.

 Equip $G$ with any Haar measure; we choose it more precisely when we must.

5. Preliminaries

5.1. Conductors. Let $c(\omega)$ denote the (log-)conductor of a character $\omega : o^\times \to \mathbb{C}^\times$, that is, the smallest nonnegative integer $n$ for which $\omega$ has trivial restriction to $o^\times \cap (1 + q^n)$. Note that $c(\omega) = 0$ precisely when $\omega = 1$. Let $c(\chi) := c(\chi|_{o^\times})$ denote the conductor of a character $\chi : k^\times \to \mathbb{C}^\times$.

5.2. Principle series representations. For a character $\chi : k^\times \to \mathbb{C}^\times$, we denote by $\chi \oplus \chi^{-1}$ the corresponding induced representation; it consists of smooth functions $v : G \to \mathbb{C}$ satisfying $v(n(x)a(y)g) = |y|^{1/2}\chi(y)v(g)$ for all $x, y, g \in k, k^\times, G$, with the group $G$ acting by right translation. If $c(\chi^2) \neq 0$, then $\chi \oplus \chi^{-1}$ is irreducible and generic.

5.3. Microlocal lifts. We recall the specialization to trivial central characters of the main definition from [18] and record some basic properties.

5.3.1. Definition. For each integer $N > ord_k(2)$, fix a decomposition $N = ord_k(2) = N_1 + N_2$ into nonnegative integers $N_1, N_2$ that tend to $\infty$ with $N$. For $N$ large enough, we assume for the sake of consistency with §1.5 that $N_1 = \lfloor N/2 \rfloor - ord_k(2), N_2 = \lfloor N/2 \rfloor$.

Let $A$ be a representation of $G$. We say that a vector $\varphi \in A$ is a microlocal lift if

(i) $\varphi \neq 0$,
(ii) $\varphi$ generates an irreducible subrepresentation $\pi \subseteq A$, and
(iii) there is a character $\omega$ of $o^\times$ whose square $\omega^2$ is nontrivial so that with $N := c(\omega)$ and for all

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(o) \cap \begin{pmatrix} 0 & q^{N_1} \\ q^{N_2} & 0 \end{pmatrix},$$

one has $g\varphi = \omega(a^2/\det(g))\varphi$.

We call $\omega$ the orientation of $\varphi$; it is determined by $\varphi$.

Remark 1. We refer again to [18, Thm 25, Rmk 26] for a discussion of the sense in which the microlocal lifts considered here are actually “lifts.”

Remark 2. The quantity denoted $N$ in [18] has been renamed here to $N - ord_k(2)$ in order to make the formulas for the variance statistics hold more uniformly when $k$ extends $\mathbb{Q}_2$. Amusingly, the convention chosen here is less natural from the perspective of linear statistics (see §9.2).
5.3.2. Classification. Let \( \pi \) be an irreducible representation of \( G \). By [18, Lem 22], we have:

**Lemma.**

1. Suppose that \( \pi \) is isomorphic to \( \chi \boxplus \chi^{-1} \) for some character \( \chi \) of \( k^\times \) for which \( c(\chi^2) \neq 0 \). Then the set of microlocal lifts in \( \pi \) is a disjoint union
\[
\mathbb{C}^\times \varphi_+ \bigsqcup \mathbb{C}^\times \varphi_-, \quad \text{where} \quad \varphi_+ \quad \text{and} \quad \varphi_- \quad \text{are microlocal lifts of orientations} \quad \omega := \chi|_{k^\times} \quad \text{and} \quad \omega^{-1}, \quad \text{respectively.}
\]

2. Otherwise, \( \pi \) contains no microlocal lifts.

**Corollary.** Let \( X \) be a set consisting of characters \( \omega \) of \( \mathfrak{o}^\times \) for which \( \omega^2 \neq 1 \). Assume that \( \omega \in X \implies \omega^{-1} \notin X \). The set of microlocal lifts in \( \pi \) with orientation in \( X \) is then either empty or of the form \( \mathbb{C}^\times \varphi \) for some \( 0 \neq \varphi \in \pi \).

5.3.3. Projectors. Let \( \omega \) be a character of \( \mathfrak{o}^\times \) for which \( \omega^2 \) is nontrivial. Set \( N := c(\omega) \). Let \( \mathfrak{3} \) denote the subgroup of \( \text{GL}_2(\mathfrak{o}) \) appearing on the RHS of (5.1). Let \( \mathfrak{3} \leq G \) denote the image of \( \mathfrak{3} \) under the projection \( \text{GL}_2(\mathfrak{k}) \to G \). By matrix multiplication and the inequalities \( \min(N_1, N_2) \geq 0 \), \( \max(N_1, N_2) \geq 1 \), one has for \( x_1, x_2 \in k \) and \( y \in k^\times \) that
\[
n'(x_2)n(x_1)a(y) \in \mathfrak{3} \iff x_1 \in q^{N_1}, x_2 \in q^{N_2}, y \in \mathfrak{o}^\times \quad (5.2)
\]

Define \( f_{\omega} \in C_c^\infty(G) \) as follows, for \( g \in G \):

- If \( g \notin \mathfrak{3} \), then \( f_{\omega}(g) := 0 \).
- If \( g \in \mathfrak{3} \) is image of \( \tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{3} \), then \( f_{\omega}(g) := \text{vol}(\mathfrak{3})^{-1}\omega(\text{nr}(\tilde{g})/a^2) \).

The definition is independent of the choice of \( \tilde{g} \).

**Lemma.** Let \( \pi \) be an irreducible unitary representation of \( G \).

1. If \( \pi \) contains a microlocal lift \( \varphi \) of orientation \( \omega \), then \( \pi(f_{\omega}) \) is the rank one orthogonal projector onto the subspace \( \mathbb{C}\varphi \) of \( \pi \).

2. Otherwise, \( \pi(f_{\omega}) = 0 \).

**Proof.** \( f_{\omega} \) is supported on \( \mathfrak{3} \). By direct calculation, the restriction of \( \text{vol}(\mathfrak{3})f_{\omega} \) to \( \mathfrak{3} \) defines a unitary character \( \xi \) of \( \mathfrak{3} \) with the property that condition (iii) in §5.3.1 reads “\( g\varphi = \xi^{-1}(g)\varphi \) for all \( g \in \mathfrak{3} \).” The conclusion follows from elementary Fourier analysis on the compact group \( \mathfrak{3} \).

5.4. The convolution kernel of interest.

5.4.1. Taylor expansion of the logarithm. Let \( N, N_0 \) be positive integers. Assume that \( N_0 \) is large enough in terms of \( \text{ord}_k(2) \) and that \( N \) is large enough in terms of \( N_0 \) that
\[
N > N_0 > \text{ord}_k(2), \quad (5.3)
\]
\[
N \geq 2N_0 + \text{ord}_k(2). \quad (5.4)
\]
In applications, we take \( N_0 \) large enough but fixed, and let \( N \to \infty \).

Set \( \mathfrak{o}_0^\times := 1 + q^{N-N_0} \), regarded as a subgroup of \( \mathfrak{o}^\times \). Then
\[
[\mathfrak{o}^\times : \mathfrak{o}_0^\times] = q^{N-N_0}\zeta_k(1)^{-1}. \quad (5.5)
\]

Regard \( 1 + q^N \) as a subgroup of \( \mathfrak{o}_0^\times \). Fix a uniformizer \( \varpi \), i.e., a generator of the \( \mathfrak{o} \)-ideal \( \varpi \).
Lemma. The map
\[ \iota : \mathfrak{o}_0^\times/(1 + \mathfrak{q}^N) \to \mathfrak{q}^{-N_0}/\mathfrak{o} \]
\[ \iota(u) := \varpi^N(u - 1) \]
is an isomorphism of groups.

Proof. By inspection, \( \iota \) is well-defined and has a well-defined inverse \( \iota^{-1} \), hence is bijective. For \( x, y \in \mathfrak{q}^{-N_0}/\mathfrak{o} \), one has \( \iota^{-1}(x)\iota^{-1}(y) = \iota^{-1}(x+y) + \varepsilon \) with \( \varepsilon := \varpi^{2N}xy \).
By (5.4), one has \( \varepsilon \in \mathfrak{q}^{2N-2N_0} \subseteq \mathfrak{q}^N \), so \( \iota^{-1} \) is a homomorphism. \( \square \)

5.4.2. Partitioning the characters of given conductor. Motivated by the corollary of §5.3.2, we record here a partition of the characters of \( \mathfrak{o}^\times \) having given conductor into nice subsets that do not contain the inverses of any characters that they contain. Recall that a unitary character \( \sigma : k \to \mathbb{C}^{(1)} \) is unramified if it is trivial on \( \mathfrak{o} \) but not on \( \mathfrak{q}^{-1} \). Let \( \Sigma \) denote the set of equivalence classes of unramified unitary characters \( \sigma \) of \( k \), with two such characters declared equivalent if they have the same restriction to \( \mathfrak{q}^{-N_0} \). For any \( \sigma \in \Sigma \), the map
\[ \mathfrak{o}^\times/(1 + \mathfrak{q}^{N_0}) \to \Sigma \]
\[ \xi \mapsto (\text{the class of the character } [x \mapsto \sigma(\xi x)]) \]
is bijective, so \( \Sigma \) is a finite set of cardinality
\[ |\Sigma| = \zeta_K(1)^{-1}q^{N_0}. \] (5.6)

For each \( \sigma \in \Sigma \), let \( \omega_\sigma : \mathfrak{o}_0^\times \to \mathbb{C}^\times \) denote the function defined by \( \omega_\sigma(u) := \sigma(\iota(u)) \). Both \( \omega_\sigma \) and the association \( \Sigma \ni \sigma \mapsto \omega_\sigma \) are then well-defined. By the lemma of §5.4.1, we see that
- \( \omega_\sigma \) is a character of \( \mathfrak{o}_0^\times \) of conductor \( N \) (in the sense that it has trivial restriction to \( 1 + \mathfrak{q}^N \) but not to \( 1 + \mathfrak{q}^{N-1} \), and that
- each character of \( \mathfrak{o}_0^\times \) of conductor \( N \) is of the form \( \omega_\sigma \) for some unique \( \sigma \in \Sigma \).

Let \( \mathcal{X}_N \) denote the set of characters \( \omega \) of \( \mathfrak{o}^\times \) for which \( c(\omega) = N \). Since each such \( \omega \) restricts to a character of \( \mathfrak{o}_0^\times \), we have a partition
\[ \mathcal{X}_N = \bigsqcup_{\sigma \in \Sigma} \mathcal{X}_N^\sigma \] (5.7)
where \( \mathcal{X}_N^\sigma := \{ \omega \in \mathcal{X}_N : \omega|_{\mathfrak{o}_0^\times} = \omega_\sigma \} \).

Lemma. \( \omega \in \mathcal{X}_N^\sigma \implies \omega^{-1} \notin \mathcal{X}_N^\sigma \).

Proof. If \( \omega, \omega^{-1} \in \mathcal{X}_N^\sigma \) then \( \omega_\sigma = \omega|_{\mathfrak{o}_0^\times} = \omega_\sigma^{-1} \), hence \( \omega_\sigma^2 = 1 \), hence \( \sigma(2x) = 1 \) for all \( x \in \mathfrak{q}^{-N_0} \), contrary to our assumptions that \( \sigma \) is unramified and \( N_0 > \text{ord}_k(2) \). \( \square \)

5.4.3. Definition. By the element \( f \in C_c^\infty(G) \) attached to \( (N, \sigma) \) we shall mean the function
\[ f := \sum_{\omega \in \mathcal{X}_N^\sigma} f_\omega, \]
where \( f_\omega \) is as in §5.3.3. The element \( f \) depends also upon \( N_0 \); we regard that element as fixed in applications, and so omit its dependence from the terminology.

Proposition 3. Let \( f \in C_c^\infty(G) \) be attached to \( (N, \sigma) \). Let \( \pi \) be an irreducible unitary representation of \( G \).
(1) If \( \pi \) contains a microlocal lift \( \varphi \) of orientation \( \omega \) for some \( \omega \in X_N^\times \), then \( \pi(f) \) is the rank one orthogonal projector onto the subspace \( \mathbb{C} \varphi \) of \( \pi \).

(2) Otherwise, \( \pi(f) = 0 \).

\textbf{Proof.} By combining the lemmas of §5.3.2, §5.3.3, §5.4.2. \hfill \Box

\section{5. \( \nabla^\tau \) operator.}

For \( \tau \in k^\times \), define \( \nabla^\tau : C_c^\infty(G) \to \mathcal{S}(B) \) by the formula \( \nabla^\tau f(g) := 1_{a^\times}(\tau \text{nr}(g))f(g) \). (This is a variant of the definition of §4.2.4, adapted to the local setting.)

\section{6. Fourier transforms of convolution kernels}

We study here the “Fourier transform” of the \( f \in C_c^\infty(G) \) attached to \( (N, \sigma) \) (see §5.4.3).

This section may be safely skipped on a first reading: the results stated here are used in the proofs of the main results of §7 and §8, but the details of those proofs are not needed to understand the overall structure of the proof given in §9 of the main result of the article.

Throughout this section, we fix \( \sigma \in \Sigma \) and choose an arbitrary unramified unitary character \( \psi : k \to \mathbb{C}^{(1)} \) belonging to the class \( \sigma \).

\subsection{6.1. Measures.}

We use \( \psi \) to define measures on \( k, B, G \) as in §2.1.2. The measures so obtained on \( k \) and \( B \) assign volume one to maximal compact subrings. The measure on \( G \) satisfies the integral formulas (2.3), (2.4).

\textbf{Lemma.} \( \text{vol}(\mathfrak{J}) = |2|_k^{-1} q^{-N} \zeta_k(1)^{-1} \).

\textbf{Proof.} This follows from the description of \( \mathfrak{J} \) given by (5.2), the integral formula (2.4), the local volume formulas (§2.1.3), and the consequence \( q^{-N_1 - N_2} = |2|_k^{-1} q^{-N} \) of the definitions of \( N_1, N_2 \). \hfill \Box

\subsection{6.2. Coordinates.}

On \( B \), we employ the coordinates

\[ x = \begin{pmatrix} d - a/2 & b \\ c & d + a/2 \end{pmatrix}, \quad \xi = \begin{pmatrix} \delta/2 + \alpha \\ \delta/2 - \alpha \end{pmatrix}. \tag{6.1} \]

By a simple change of variables, one verifies that the Haar measure on \( B \) is given in either set of coordinates (6.1) by integrating over the coordinate variables with respect to the chosen Haar measure on \( k \). The main involution \( \iota : B \to B \) is given by \( (a, b, c, d) \mapsto (-a, -b, -c, d), (\alpha, \beta, \gamma, \delta) \mapsto (-\alpha, -\beta, -\gamma, \delta) \) and the Fourier transform \( \mathcal{F} : \mathcal{S}(B) \to \mathcal{S}(B) \), defined as in §2.1.2 or §2.2.1 by \( \mathcal{F} \phi(\xi) = \int_{x \in B} \phi(x)\psi((x, \xi)) \, dx \) with \( (x, \xi) := \text{tr}(x^t \xi) \), by

\[ \mathcal{F} \phi(\xi) = \int_{a,b,c,d \in k} \phi(x)\psi(\alpha a + \delta d - \beta c - \gamma b). \tag{6.2} \]

The general notation of §1.13 gives us for each space \( V \in \{ C_c^\infty(G), \mathcal{S}(B) \} \) maps \( \nabla^\tau : V \to V \) and \( \text{Ad}(g) : V \to V \) for \( g \in G \). These maps commute with each other and with \( \mathcal{F}, \nabla^\tau \) in every conceivable sense: \( \mathcal{S} \nabla^\tau = \nabla^\tau \mathcal{S}, \mathcal{S} \text{Ad}(g) = \text{Ad}(g)\mathcal{S}, \nabla^\tau \text{Ad}(g) = \text{Ad}(g) \nabla^\tau, \mathcal{F} \text{Ad}(g) = \text{Ad}(g)\mathcal{F}, \mathcal{F} \mathcal{S} = \mathcal{S} \mathcal{F} \).
6.3. Statement of result. Asymptotic notation here refers to the \( N \to \infty \) limit. Implied constants depend at most upon the field \( k \) and the integer \( N_0 \).

**Proposition 4.** Let \( f \in C_{\infty}^{\infty}(G) \) be attached to \((N, \sigma)\) (see §5.4.3). Set
\[
\phi := F\mathfrak{S}\nabla^1 f = \mathfrak{S}F\nabla^1 f = F\nabla^1 \mathfrak{S}f \in \mathcal{S}(B).
\]

Define \( \Phi : k^4 \to \mathbb{C} \) in terms of the coordinates (6.1) by \( \Phi(\alpha, \beta, \gamma, \delta) := \phi(\xi) \). One then has \( \Phi(\alpha, \beta, \gamma, \delta) \neq 0 \) only if
\[
|\alpha| \asymp q^N, \quad |\beta|, |\gamma| = o(q^N), \quad |\delta| = O(1).
\]

One has
\[
\Phi(ua, u\beta, u\gamma, \delta) = \Phi(\alpha, \beta, \gamma, \delta) \quad \text{for all} \quad u \in 1 + q^{N_0}.
\]

The function \( I(\alpha, \delta) := \int_{\beta, \gamma \in k} \Phi(\alpha, \beta, \gamma, \delta) \) satisfies
\[
I(\alpha, \delta) = I(-\alpha, \delta)
\]
and
\[
q^{-N}I(\varpi^{-N}\alpha, \delta) \text{ is independent of } N.
\]

Moreover,
\[
\int_{\alpha, \delta \in k} |2\alpha|^{-2} |I(\alpha, \delta)|^2 = Cq^{N-N_0}
\]
with \( C := (2\zeta_k(1))^{-1} \).

**Remark.** We shall subsequently refer only to the properties laid out in the statement of Proposition 4, but it may be instructive to record that if \( q \) is odd, one can establish (extending the proof of Proposition 4) the explicit formula
\[
\Phi(\alpha, \beta, \gamma, \delta) = \frac{q^{-N_0}}{\zeta_k(1)}1_{\varpi^{-N}\sigma^\times}(\alpha)1_{q^{-N_0}}(\beta)1_{q^{-N_0}}(\gamma)1_{q^{-N_0}}(\delta)\co(\delta/\varpi^N\alpha),
\]
where \( \co(t) := (\psi(t) + \psi(-t))/2 \). Otherwise, \( k \) is a finite extension of \( \mathbb{Q}_2 \), and a similar but more complicated formula holds.

6.4. Proofs. The purpose of this section (which may be safely skipped on a first reading) is to prove Proposition 4.

**Lemma.** Let \( f \) be attached to \((N, \sigma)\). Let \( x \in B \). In the coordinates (6.1),
\[
\nabla^1 f(x) = C_01_{\varpi^\sigma}(d)1_{q^N-N_0}(a)1_{q^N}(b)1_{q^N}(c)\psi\left(\frac{ad - bc}{\varpi^Ndx}\right)
\]
where \( C_0 := q^{2N-N_0}|2|_k \).

**Proof.** By (5.5) and the lemma of §6.1,
\[
C_0 = [\sigma^\times : \mathfrak{o}_0^\times] \vol(\mathfrak{J})^{-1}.
\]

(6.10)

By Fourier analysis on \( \mathfrak{o}^\times \), we have for \( u \in \mathfrak{o}^\times \) the expansion
\[
[\sigma^\times : \mathfrak{o}_0^\times]1_{\varpi^\sigma}(u)\omega_\sigma(u) = \sum_{\omega \in \mathfrak{K}_0^\sigma} \omega(u).
\]

(6.11)

Applying (6.10) and (6.11) to the definition of \( f \) gives \( \nabla^1 f(x) = C_0\kappa\psi(\zeta) \), where
\[
\kappa := 1_{q^N}(b)1_{q^N}(c)1_{\varpi^\sigma}(d - a/2)1_{\varpi^\sigma}(d + a/2)1_{q^N-N_0}\left(\frac{nr(g)}{(d - a/2)^2} - 1\right),
\]
\[
\zeta := \varpi^{-N}\left(\frac{nr(x)}{(d - a/2)^2} - 1\right).
\]
By our assumption (5.3) on the largeness of \( N \) relative to \( N_0 \) and identifies such as
\[
\frac{nr(x)}{(d-a/2)^2} - 1 = \frac{a}{d-a/2} - \frac{bc}{(d-a/2)^2},
\]
we verify directly that
\[
\kappa = 1_{q^N} (d) 1_{q^{N-N_0}}(a) 1_{q^{N_1}}(b) 1_{q^{N_2}}(c).
\]
Similarly, we verify using (5.4) that for \( a, b, c, d \) in the support of the RHS of (6.13), the congruences
\[
\frac{a}{d-a/2} \equiv \frac{ad}{d^2} \pmod{q^N}, \quad \frac{bc}{(d-a/2)^2} \equiv \frac{bc}{d^2} \pmod{q^N}
\]
hold. It follows from these and (6.12) that
\[
\kappa \psi(\zeta) = \kappa \psi\left(\frac{ad-bc}{2\varpi N d^2}\right).
\]
This completes the proof of (6.9). \( \square \)

**Corollary.** With notation as in the lemma, the quantity \( f(x) \) depends only upon the congruence classes
\[
a \pmod{q^N}, \quad b \pmod{2q^{N_1}}, \quad c \pmod{2q^{N_2}}, \quad d \pmod{q^{N_0}}.
\]

**Proof.** Immediate; it may help to recall that \( N = \text{ord}_k(2) + N_1 + N_2 \). \( \square \)

We now prove Proposition 4. By the formula for \( \Diamond f \) given in the lemma and the explication (6.2) of the Fourier transform, we have
\[
\Phi(\alpha, \beta, \gamma, \delta) = \int_{a,b,c,d \in k} F(a, b, c, d) \psi(\alpha a + \delta d - \beta c - \gamma b),
\]
where
\[
F(a, b, c, d) := C_0 1_{q^N} (d) 1_{q^{N-N_0}}(a) 1_{q^{N_1}}(b) 1_{q^{N_2}}(c) \co\left(\frac{ad-bc}{\varpi N d^2}\right),
\]
with \( \co(x) := (\psi(x) + \psi(-x))/2 \). The smoothness properties of \( f \) (hence of \( F \)), as enunciated in the corollary, imply corresponding decay properties of its Fourier transform, namely that \( \Phi(\alpha, \beta, \gamma, \delta) \neq 0 \) only if \( \alpha \in q^{-N}, \beta, \gamma, \delta \in 2^{-1} q^{-N_1}, \gamma \in 2^{-1} q^{-N_2}, \delta \in q^{-N_0} \). We thereby obtain all assertions in (6.4) except for the lower bound on \( |\alpha| \). To establish the latter, we compute for \( d \in \varpi^2 \) that
\[
\int_{a \in k} 1_{q^{N-N_0}}(a) \co\left(\frac{ad}{\varpi N d^2}\right) \psi(\alpha a) = q^{N-N_0} \rho(\alpha, d),
\]
where
\[
\rho(\alpha, d) := 1_{q^{N_0-N}}(\alpha + \frac{1}{\varpi N d}) + 1_{q^{N_0-N}}(\alpha - \frac{1}{\varpi N d}).
\]
One has \( \rho(\alpha, d) \neq 0 \) only if \( \alpha \in \pm \frac{1}{\varpi N d} + q^{N_0-N} \equiv \frac{1}{\varpi N d} \varpi^\times \), in which case \( |\alpha| \asymp q^N \).

The invariance (6.5) is equivalent to the identity \( F(ua, ub, uc, d) = F(a, b, c, d) \) for \( u \in 1 + q^{N_0} \), which follows from (6.14) and the congruence \( uad - u^2bc \equiv ad - bc \pmod{q^N} \) for \( (a, b, c, d) \in \supp(F) \).

We now verify (6.6). By Fourier inversion,
\[
I(\alpha, \delta) = \int_{a, d \in k} F(a, 0, 0, d) \psi(\alpha a + \delta d).
\]
Thus (6.14) reduces to \( F(a, 0, 0, d) = F(-a, 0, 0, d) \), follows from (6.14).
To establish (6.7), we recall the definition of $C_0$ from the lemma and apply to (6.14) and (6.16) the change of variables $a \rightarrow w^{-N}a$, giving

$$I(w^{-N}\alpha, \delta) = C'q^N \int_{\alpha, d \in k} 1_{\alpha \times}(d)1_{q^{-N_0}}(a)\co(a/d)\psi(\alpha a + \delta d)$$

for some unimportant scalar $C' = |2|q^{-N_0}$ that does not depend upon $N$.

We turn finally to (6.8). Denote temporarily by $I$ its LHS. By (6.16) and Parseval,

$$I = \int_{\alpha, d \in k} |2\alpha|^{-2} \int_{a \in k} F(a, 0, 0, d)\psi(\alpha a)^2. \quad (6.17)$$

We compute with the help of (6.15) that

$$\int_{a \in k} F(a, 0, 0, d)\psi(\alpha a) = q^{N_0-N}C_0 1_{\alpha \times}(d)\rho(\alpha, d).$$

Substituting this and the identity $q^{N_0-N}C_0 = q^N|2|_k$ into (6.17) gives

$$I = q^{2N} \int_{\alpha, d \in k} |\alpha|^{-2} 1_{\alpha \times}(d)|\rho(\alpha, d)|^2.$$

We substitute $\alpha \rightarrow w^{-N}\alpha$; since $|w^{-N}|^{-1} = q^{-N}$, we obtain

$$I = q^{N} \int_{\alpha, d \in k} |\alpha|^{-2} 1_{\alpha \times}(d)\left|1_{q^{N_0}}(\alpha + 1/d) + 1_{q^{N_0}}(\alpha - 1/d)\right|^2.$$

By the consequence $2 \notin q^{N_0}$ of the assumption (5.3), we have

$$1_{\alpha \times}(d)1_{q^{N_0}}(\alpha + 1/d)1_{q^{N_0}}(\alpha - 1/d) = 0,$$

whence

$$I = q^{N} \int_{\alpha, d \in k} |\alpha|^{-2} 1_{\alpha \times}(d)\frac{1_{q^{N_0}}(\alpha + 1/d) + 1_{q^{N_0}}(\alpha - 1/d)}{2^2} = \frac{2^{N-N_0}(1 - \frac{1}{q})}{2^2} = \frac{q^{N-N_0}}{2\zeta_k(1)}.$$

We thereby arrive at (6.8) with the constant

$$C = q^{N_0-N} \cdot \frac{q^{N-N_0}}{2\zeta_k(1)} = \frac{1}{2\zeta_k(1)},$$

as required.

7. Estimates for the main term

7.1. Statement of result. Let $E$ denote the diagonal subalgebra of $B$. Let $H \subseteq G$ denote the image of $E^\times$. Thus $H = \{a(y) : y \in k^\times\}$ is the diagonal split torus. Let $N(H)$ denote the normalizer in $G$ of $H$. Let

$$W := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

denote the Weyl group $W \cong N(H)/H$. Equip $H$ and $N(H)$ the measures

$$\int_H f := \int_{y \in k^\times} f(y) \frac{dy}{|y|}, \quad \int_{N(H)} f := \sum_{w \in W} \int_{h \in H} f(wh),$$

where $dy$ denotes (as in §6.1) the Haar measure on $k$ assigning volume one to $w$. 


Proposition 5. Let \( \Psi : G \to \mathbb{C} \) be a function with the following properties:

1. There is an open subgroup \( U \) of \( G \) so that
   \[ \Psi(u_1gu_2) = \Psi(g) \text{ for all } u_1, u_2 \in U, \ g \in G. \]  
   (7.1)

2. One has (see \( \S 2.1.5 \) concerning \( \Xi \))
   \[ \Psi(g) \ll \Xi(g)^\delta \text{ for some } \delta > 0. \]  
   (7.2)

Let \( N \) be a positive integer taken sufficiently large in terms of \( \Psi \). Let \( f \in C_c^\infty(G) \) be attached to \( (N, \sigma) \) for some \( \sigma \in \Sigma \) (see \( \S 5.4.3 \)). Then
\[
\int_{g \in G} \langle \text{Ad}(g)Sf, Sf \rangle_{L^2(G)} \Psi(g) = q^{N-N_0} \frac{1}{2} \int_{N(H)} \Psi. \quad (7.3)
\]

Remark 1. The condition (7.2) implies the absolute convergence of both sides of (7.3) (see \( \S 2.1.6 \)).

Remark 2. Let \( \pi \) be an irreducible unitary representation of \( G \) with \( \dim(\pi) > 1 \), and let \( v_1, v_2 \in \pi \). The hypotheses of Proposition 5 are then satisfied by \( \Psi(g) := \langle gv_1, v_2 \rangle \) (see \( \S 2.1.5 \)).

Remark 3. The LHS of (7.3) is independent of the choice of Haar measure on \( G \) (noting that the definition of \( f \) involves the factor \( \text{vol}(\mathfrak{H})^{-1} \)).

The proof of Proposition 5 occupies the remainder of \( \S 7 \).

7.2. Reduction to matrix calculus. We fix measures on \( G, B, k \) and define \( \mathcal{F} \) as in \( \S 6.1 \).

7.2.1. Application of Parseval. Set \( \phi := \mathcal{F} \mathbb{S} \mathcal{V} f \in \mathcal{S}(B) \). By Parseval, (2.3) and the volume formulas of \( \S 2.1.3 \), we have for \( g \in G \) that
\[
\langle \text{Ad}(g)Sf, Sf \rangle_{L^2(G)} = \zeta_k(1) \langle \text{Ad}(g)\phi, \phi \rangle_{L^2(B)}. \quad (7.4)
\]

Remark. We may now informally explain (7.3) as follows: The support properties (6.4) say that \( \phi \) is concentrated on the subspace of diagonal matrices, whose normalizer is \( N(H) \). Thus \( \langle \text{Ad}(g)\phi, \phi \rangle \) should be small unless \( g \) is close to \( N(H) \). One has (morally) \( \text{Ad}(h)\phi \approx \phi \) for elements \( h \in H \) of size \( O(1) \). The identity (6.6) says (morally) that \( \text{Ad}(w)\phi \approx \phi \) for all \( w \in W \). Thus the distribution \( g \mapsto \langle \text{Ad}(g)\phi, \phi \rangle \) should conceivably approximate some multiple of the Haar measure on \( N(H) \).

7.2.2. Principal congruence subgroups. For a positive integer \( m \), we let \( K[m] \) denote the \( m \)th principal congruence subgroup of \( G \); we define this to mean the image in \( G \) of the depth \( m \) principal congruence subgroup
\[
\left( \begin{array}{cc} 1 + q^m & q^n \\ q^m & 1 + q^m \end{array} \right) \subseteq \text{GL}_2(\mathfrak{O}).
\]

One has a diffeomorphism
\[
q^m \times q^m \times (1 + q^m) \xrightarrow{\cong} K[m] \quad (x_1, x_2, y) \mapsto n(x_2)n(x_1)a(y). \quad (7.5)
\]
7.2.3. Properties of $\phi$. Let $m$ be any positive integer for which $m \geq N_0$. Proposition 4 implies that
\[ \phi\left(\begin{array}{cc} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{array}\right) = \phi\left(\begin{array}{cc} \delta/2 + u\alpha & u\beta \\ u\gamma & \delta/2 - u\alpha \end{array}\right) \text{ for all } u \in 1 + q^m \tag{7.6} \]
and for $N$ large enough also that
\[ \text{supp}(\phi) \subseteq E(m), \tag{7.7} \]
where $E(m)$ denotes the following set of “near-diagonal” matrices:
\[ E(m) := \left\{ \left(\begin{array}{cc} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{array}\right) : \delta \in k, \alpha \in k; \beta, \gamma \in 2aq^m \right\}. \tag{7.8} \]
Recall also from (6.6) that
\[ \int_{\beta,\gamma \in k} \phi\left(\begin{array}{cc} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{array}\right) = \int_{\beta,\gamma \in k} \phi\left(\begin{array}{cc} \delta/2 - \alpha & \beta \\ \gamma & \delta/2 + \alpha \end{array}\right). \tag{7.9} \]

7.2.4. The key computation. Proposition 5 follows immediately from (7.4), the normalizing computation (6.8) (giving the factor $(2\zeta_k(1))^{-1}$) and the following:

Lemma. Let $\Psi : G \to \mathbb{C}$ and $\phi \in S(B)$ be arbitrary. Suppose there exists an open subgroup $U$ of $G$ and a positive integer $m$ so that
\[ K[m] \leq U. \tag{7.10} \]
and so that (7.1), (7.2), (7.6), (7.7) and (7.9) hold. Then
\[ \int_{g \in G} \langle \text{Ad}(g)\phi, \phi \rangle \Psi(g) = \left(\int_{N(H)} \Psi \int_{\alpha,\beta \in k} \frac{1}{2a^2} \left|\int_{\beta,\gamma \in k} \phi\left(\begin{array}{cc} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{array}\right)\right|^2 \right)^2. \]
The proof involves Hensel’s lemma, the Weyl integral formula, and related arguments; we complete it in §7.3.3.

Remark. The lemma and its proof generalize readily to non-split quaternion algebras $B$ and/or non-split separable quadratic subalgebras $E \subseteq B$. We focus on the relevant split diagonal case for the sake of concreteness.

7.3. Some matrix calculus. One purpose of this section is to prove the lemma of §7.2.4. Another is to develop preliminaries for the proof of Proposition 6, below.

7.3.1. Notation. We introduce on $B^0$ the coordinates
\[ [\alpha, \beta, \gamma] := \left(\begin{array}{c} \alpha \\ \beta \\ -\alpha \end{array}\right). \]
The Weyl group $W \cong N(H)/H$ has a natural right action on $G/H$, denoted by juxtaposition. Let $(G/H)[m]$ denote the image of $K[m]$ under the quotient map $G \to G/H$; by (7.5), the map $q^m \times q^m \ni (x_1, x_2) \mapsto n'(x_1)n(x_2)H \in (G/H)[m]$ is a bijection. Let $E^0 := E \cap B^0 = \{[\alpha, 0, 0] : \alpha \in k\}$ denote the subspace of traceless diagonal matrices. Set $E^0(m) := \{[\alpha, \beta, \gamma] : \alpha \in k; \beta, \gamma \in 2aq^m \} \subseteq B^0$. The sets $E^0(m) - \{0\}$ form a shrinking system of open neighborhoods of $E^0 - \{0\}$. Their images $\mathbb{P}E^0(m)$ in the projective plane $\mathbb{P}B^0$ form a fundamental system of open neighborhoods of the point $\mathbb{P}E^0$. 
7.3.2. Measures. We fix measures on $G, B, B^0, k$ as in §6.1. One then has
\[ \int_{B^0} f = \int_{\alpha, \beta, \gamma \in k} f([\alpha, \beta, \gamma]), \]
\[ \int_B f = \int_{\alpha, \beta, \gamma, \delta \in k} f\left(\begin{pmatrix} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{pmatrix}\right) = \int_{\delta \in k} \left(\int_{B^0} f_{\delta}\right), \tag{7.11} \]
where for $f \in C_c(B)$ we define $f_{\delta} \in C_c(B^0)$ by $f_{\delta}(\xi) := f(\delta/2 + \xi)$.

We use the notation $E$ to denote an average over a compact group or orbit thereof with respect to the evident invariant probability measure. Averages over $K[m]$ may be computed by the formula (cf. (7.5))
\[ E_{g \in K[m]} f(g) = \mathbb{E}_{x, y \in q^m} f(n'(x)n(y)a(z)). \tag{7.12} \]

Equip $G/H$ with the quotient Haar; by (2.4),
\[ \int_{g \in G/H} f(g) = \int_{x_1, x_2 \in k} f(n'(x_2)n(x_1)), \tag{7.13} \]
\[ \int_{x \in (G/H)[m]} f(x) := \int_{x \in G/H} 1_{(G/H)[m]}(x)f(x) = \int_{x_1, x_2 \in q^m} f(n'(x_1)n(x_2)), \tag{7.14} \]
\[ E_{x \in (G/H)[m]} f(x) = \mathbb{E}_{x_1, x_2 \in q^m} f(n'(x_1)n(x_2)). \tag{7.15} \]

7.3.3. Basic observations. By inspection,
\[ \text{Ad}(N(H))E^0 = E^0, \quad \text{Ad}(W)E^0(m) = E^0(m). \tag{7.16} \]

By direct calculations such as
\[ \text{Ad}(n'(x)n(y)a(z)[1, 0, 0]) = [1 + 2xy, -2y, 2x(1 + 2y)]. \tag{7.17} \]
one verifies also that
\[ \text{Ad}(K[m])E^0(m) = E^0(m). \tag{7.18} \]

7.3.4. Applications of Hensel’s lemma.

Lemma 1. Let $m, n \in \mathbb{Z}_{\geq 1}$. Set $M := (q^m)^{\oplus n}$. Let $f_1, \ldots, f_n \in \mathfrak{o}[X_1, \ldots, X_n]$ be polynomials in the variables $X_1, \ldots, X_n$ with coefficients in $\mathfrak{o}$ satisfying $f_1(0) = \cdots = f_n(0) = 0$. Let $f : M \to k^{\oplus n}$ denote the function given by $x \mapsto (f_1(x), \ldots, f_n(x))$. Set $J := \det(\partial f_i/\partial x_j) \in \mathfrak{o}[X_1, \ldots, X_n]$. Assume that $J(x) \in \mathfrak{o}^\times$ for all $x \in M$. Then $f$ induces a diffeomorphism $f : M \to M$.

Proof. One argues as in the proof of Hensel’s lemma that $f$ is bijective. The conclusion then follows from the inverse function theorem. \qed

Lemma 2. Let $\alpha_0 \in k - \{0\}$. Then the map
\[ (1 + q^m) \times (G/H)[m] \to [(1 + q^m)\alpha_0, 2q^m\alpha_0, 2q^m\alpha_0] \]
\[ (\lambda, x) \mapsto \lambda \text{Ad}(x)\alpha_0, 0, 0 \]
is a well-defined diffeomorphism whose Jacobian has constant valuation.
Proof. We may assume that $\alpha_0 = 1$. For $x_1, x_2, x_3 \in q^m$, define $y_1, y_2, y_3 \in k$ by $(1 + x_1) \text{Ad}(n'(x_2)n(x_3))[1,0,0] = [1 + y_1, 2y_2, 2y_3]$. By a calculation similar to (7.17),

$$
y_1 = x_1 + 2x_2x_3 + 2x_1x_2x_3,$$
$$y_2 = -x_3(1 + x_1),$$
$$y_3 = x_2(1 + x_1)(1 + 2x_3).$$

Since $m \geq 1$, one sees that $y_1, y_2, y_3 \in q^m$ and that the Jacobian of the map $(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$ belongs to $\sigma^x$ at every point of its domain. The conclusion follows now from Lemma 1.

Corollary. For each $\xi = [\alpha, \beta, \gamma] \in E^0(m)$ there exists $\alpha_0 \in (1 + q^m)$ and $g \in K[m]$ so that $\xi = \text{Ad}(g)[\alpha_0, 0, 0]$.

Lemma 3. Let $f \in C_c(B^0)$ and $\xi_0 = [\alpha_0, \beta_0, \gamma_0] \in E^0(m) - \{0\}$. Then

$$E_{\lambda \in 1 + q^m} f(\lambda \text{Ad}(g)\xi_0) = E_{\alpha \in (1 + q^m)\alpha_0} f([\alpha, \beta, \gamma]).$$

Proof. By the corollary, we may write $\xi_0 = \text{Ad}(g_0)[\alpha_0', 0, 0]$ with $\alpha_0' \in (1 + q^m)\alpha_0$ and $g_0 \in K[m]$. By the change of variables $g \mapsto gg_0^{-1}$, we reduce to proving the required identity in the special case $\beta_0 = \gamma_0 = 0$. In that case, we replace the average over $g \in K[m]$ with an average over $x \in (G/H)[m]$ and appeal to Lemma 2 and the change of variables formula. 

Lemma 4. Suppose $f \in C_c(B^0)$ is supported on $E^0(m)$ and satisfies $f(\lambda x) = f(x)$ for all $x \in B^0$ and $\lambda \in 1 + q^m$. Let $\xi_0 \in B^0 - \{0\}$. Then

$$E_{g \in K[m]} f(\text{Ad}(g)\xi_0) = 1_{E^0(m)}(\xi_0)E_{\beta, \gamma \in 2q^m, \alpha_0} f([\alpha_0, \beta, \gamma])$$

$$= 1_{\alpha_0 \neq 0} e_{2q^m, \alpha_0}(\beta_0) e_{q^m, 0}(\gamma_0) \int_{\beta, \gamma \in k} f([\alpha_0, \beta, \gamma, -\alpha_0])$$

(7.19)

where $e_{\alpha} := \text{vol}(a)^{-1} 1_a$ denotes the normalized characteristic function of a compact open subgroup $a$ of $k$.

Proof. If $\xi_0 \notin E^0(m)$, then the vanishing of the LHS follows from (7.18), so suppose that $\xi_0 \in E^0(m)$. The $(1 + q^m)$-invariance of $f$ allows us to rewrite the LHS of (7.19) as $E_{\lambda \in 1 + q^m, g \in K[m]} f(\lambda \text{Ad}(g)\xi_0)$. We apply Lemma 3 to the latter, invoking again the $(1 + q^m)$-invariance of $f$ to simplify the conclusion.

Lemma 5. Let $f$ be as in Lemma 4 and $\alpha_0 \in k - \{0\}$. Then

$$\int_{x \in (G/H)[m]} f(\text{Ad}(x)[\alpha_0, 0, 0]) = |2\alpha_0|^{-2} \int_{\beta, \gamma \in k} f([\alpha_0, \beta, \gamma]).$$

(7.20)

Proof. Using the integral formulas (7.12) and (7.14), the LHS of (7.20) may be rewritten $\text{vol}(q^m)^2 E_{g \in K[m]} f(\text{Ad}(g)[\alpha_0, 0, 0])$. The conclusion follows from Lemma 4 upon noting that $\text{vol}(q^m)^2 e_{2q^m, 0}(0)^2 = |2\alpha_0|^{-2}$. 

7.3.5. Expanding near the singular points.

Lemma 1. For $x \in G/H$, the following are equivalent:

(i) There exists $w \in W$ so that $xw \in (G/H)[m]$.
(ii) There exists $\tau \in E^0 - \{0\}$ so that $\text{Ad}(x)\tau \in E^0(m)$.
(iii) For all $\tau \in E^0$, one has $\text{Ad}(x)\tau \in E^0(m)$.

Proof. (i) implies (ii): immediate from (7.16) and (7.18). (ii) implies (iii): immediate from the definitions and the fact that $\dim(E^0) = 1$. (iii) implies (i): By the corollary to Lemma 2 of §7.3.4, we have $\text{Ad}(x)[1,0,0] \in \text{Ad}(g)E^0$ for some $g \in K[m]$; then $\text{Ad}(g^{-1}x)E^0 = E^0$, hence $g^{-1}x \in N(H) = WH$, hence $x \in K[m]WH$. □

Lemma 2.

(i) Suppose $f \in C_c(G/H)$ is supported on $(G/H)[m]W$. Then

$$\int_{G/H} f = \sum_{w \in W} \int_{x \in (G/H)[m]} f(xw).$$

(ii) Suppose $f_1 \in C_c(B^0)$ is supported on $E^0(m)$ and $f_2 \in C_c(G/H)$ is arbitrary. Let $\tau \in E^0$. Then

$$\int_{x \in G/H} f_1(\text{Ad}(x)\tau)f_2(x) = \sum_{w \in W} \int_{x \in (G/H)[m]} f_1(\text{Ad}(xw)\tau)f_2(xw).$$

Proof. (i) The nontrivial Weyl element $w \in W$ acts on $x = n'(x_1)n(x_2)H \in (G/H)[m]$ by the formula (for $x_2 \neq 0$) $xw = n'(x_1+1/x_2)n(-x_2)H$, hence $(G/H)[m]\cap (G/H)[m]w = \emptyset$. The conclusion follows.

(ii) By Lemma 1, the function $f(x) := f_1(\text{Ad}(x)\tau)f_2(x)$ is supported on $(G/H)[m]$, so we may apply (i). □

7.3.6. A variant of the Weyl integral formula.

Lemma. Let $f \in C_c(B^0)$ be supported on $\cup_{g \in gG}E^0g^{-1}$. Then

$$\int_{\alpha,\beta,\gamma \in k} f([\alpha,\beta,\gamma]) = \frac{1}{|W|} \int_{\alpha \in k} [2\alpha]^{-2} \int_{x \in G/H} f(\text{Ad}(x)[\alpha,0,0]).$$

We omit the proof. In the special case that $\text{supp}(f) \subseteq (G/H)[m]W$, the conclusion follows from Lemma 5 of §7.3.4 and Lemma 2 of §7.3.5; this special case suffices for our purposes and serves also to check the normalization.

7.3.7. Proof of the lemma of §7.2.4. Let $\phi \in S(B)$. By (7.11),

$$\langle \text{Ad}(g)\phi, \phi \rangle_{L^2(B)} = \int_{\beta \in k} \langle \text{Ad}(g)\phi_\delta, \phi_\delta \rangle_{L^2(B^0)},$$

where $\phi_\delta \in S(B^0)$ is given by $\phi_\delta(\xi) := \phi(\delta/2 + \xi)$ for $\xi \in B^0$. The proof of the lemma of §7.2.4 thereby reduces to that of the following:

Lemma. Let $\Psi : G \rightarrow \mathbb{C}$ and $\phi \in S(B^0)$ be arbitrary. Define $I : E^0 \rightarrow \mathbb{C}$ by $I([\alpha,0,0]) := \int_{\beta,\gamma \in k} \phi([\alpha,\beta,\gamma])$. Suppose there exists an open subgroup $U$ of $\text{PGL}_2(k)$ and a positive integer $m$ so that

$$K[m] \leq U$$

(7.21)
\[ \phi(\lambda x) = \phi(x) \text{ for all } x \in B^0 \text{ and } \lambda \in 1 + q^m. \] (7.22)

\[ \text{supp}(\phi) \subseteq E^0(m) \] (7.23)

\[ I([\alpha, 0, 0]) = I([-\alpha, 0, 0]), \] (7.24)

and so that the decay and smoothness assumptions (7.2), (7.1) concerning \( \Psi \) hold. Then

\[ \int_{g \in G} \Psi(g) (\text{Ad}(g) \phi, \phi) = \left( \int_{\tau \in E^0} \Psi \right) \int_{\alpha \in k} |2\alpha|^{-2} |I([\alpha, 0, 0])|^2. \] (7.25)

**Proof.** Note first that the RHS of (7.25) converge absolutely, thanks to (7.2), Lemma 2 of §2.1.6, and the compactness of the support of \( \phi \). We thereby reduce to establishing the claimed identity in the special case that \( \Psi \) is the characteristic function of some \( U \times U \)-orbit; in particular, we may assume that \( \Psi \) is compactly-supported. For this reason, we may neglect convergence issues in the arguments to follow.

Equip \( E^0 \) with the measure \( \int_{E^0} f := \int_{\alpha \in k} f([\alpha, 0, 0]) \) and define \( D : E^0 \to \mathbb{R}_{>0} \) by \( D([\alpha, 0, 0]) := |2\alpha|^2 \), so that the RHS of (7.25) reads

\[ \sum_{w \in W} \int_{h \in H} \Psi(wh) \int_{\tau \in E^0} D(\tau)^{-1} |I(\tau)|^2. \] (7.26)

We now successively transform the LHS of (7.25). By expanding the definitions and applying the integral formula of §7.3.6, we obtain

\[ \frac{1}{|W|} \int_{g \in G} \Psi(g) \int_{\tau \in E^0} D(\tau) \int_{x \in G/H} \overline{\phi(\text{Ad}(g^{-1}x)\tau)} \phi(\text{Ad}(x)\tau). \]

We execute the (cosmetic) change of variables \( x \mapsto gx \), swap orders of integration, and apply the (crucial) substitution \( g \mapsto gx^{-1} \) to arrive at

\[ \frac{1}{|W|} \int_{\tau \in E^0} D(\tau) \int_{x \in G/H} \overline{\phi(\text{Ad}(x)\tau)} \int_{g \in G} \Psi(gx^{-1}) \phi(\text{Ad}(g)\tau). \]

By factoring \( g = yh \) with \( y \in G/H, h \in H \), we obtain

\[ \frac{1}{|W|} \int_{\tau \in E^0} D(\tau) \int_{x, y \in G/H} \overline{\phi(\text{Ad}(x)\tau)} \phi(\text{Ad}(y)\tau) \int_{h \in H} \Psi(yhx^{-1}). \]

We apply Lemma 2 of §7.3.5 to the \( x, y \) integrals, giving

\[ \frac{1}{|W|} \sum_{w_1, w_2 \in W} \int_{\tau \in E^0} \int_{x, y \in (G/H)[m]} D(\tau) \overline{\phi(\text{Ad}(xw_1)\tau)} \phi(\text{Ad}(yw_2)\tau) I'(x, y) \]

where \( I'(x, y) := \int_{h \in H} \Psi(yw_2hw_1^{-1}x^{-1}) \). For each \( \tau \in E^0 \setminus \{0\} \), one has by our assumption (7.21) that \( I'(x, y) = \int_{h \in H} \Psi(w_2hw_1^{-1}) = \int_{h \in H} \Psi(w_2w_1^{-1}h) \), by Lemma 5 of §7.3.4 that \( \int_{x \in (G/H)[m]} \phi(\text{Ad}(x)\tau) = D(\tau)^{-1} I(\tau), \) and by our assumption (7.24) that \( I(\text{Ad}(w)\tau) = I(\tau) \) for all \( w \in W \). We obtain

\[ \frac{1}{|W|} \sum_{w_1, w_2 \in W} \int_{h \in H} \Psi(w_2w_1^{-1}h) \int_{\tau \in E^0} D(\tau)^{-1} |I(\tau)|^2, \]

which simplifies to (7.26). \[ \square \]
8. Estimates for the error terms

8.1. Statement of result. Recall the definitions of the Harish–Chandra function $\Xi$ (§2.1.5) and the Weil representation (§2.2). Let $\psi : k \to \mathbb{C}^{(1)}$ be an unramified unitary character. For $\tau \in k^\times$, set

$$\rho^\tau := \rho_{\text{Weil}}^\psi, \quad \rho_0^\tau := \rho_{\text{Weil}}^{\psi, B^0}.$$ 

Let $\tau_1, \tau_2 \in k^\times$. The relevance of the following definition may be inferred from the statement of Theorem 2.

**Definition.** Let $\ell : S(B) \otimes S(B) \to \mathbb{C}$ be a sesquilinear form. We say that $\ell$ is good (relative to $\psi, \tau_1, \tau_2$) if:

1. There is an open subgroup $U$ of $G$ so that for $g_1, g_2 \in U$, $\phi_1, \phi_2 \in S(B)$ and $s \in \text{Mp}_2(k)$, one has

$$\ell(\phi_1, \phi_2) = \ell(\mathcal{S}\phi_1, \phi_2) = \ell(\phi_1, \mathcal{S}\phi_2) = \ell(\text{Ad}(g_1)\phi_1, \text{Ad}(g_2)\phi_2)$$

$$= \ell(\rho^{\tau_1}(s)\phi_1, \rho^{\tau_2}(s)\phi_2).$$

2. For all $\phi_1, \phi_2 \in S(B)$ there exists $C \geq 0$ so that for all $s \in \text{Mp}_2(k)$,

$$|\ell((1 \otimes \rho_0^{\tau_1}(s))\phi_1, (1 \otimes \rho_0^{\tau_2}(s))\phi_2)| \leq C\Xi(s).$$

**Proposition 6.** For each good sesquilinear form $\ell : S(B) \otimes S(B) \to \mathbb{C}$ and $N_0 > \text{ord}_k(2)$ there exists $C \geq 0$ so that for large positive integers $N$ and all $\sigma \in \Sigma$, the element $f \in C^\infty_c(G)$ attached to $(N, \sigma)$ (see §5.4.3) satisfies

$$|\ell(\nabla^{\tau_1} f, \nabla^{\tau_2} f)| \leq CN.$$  

The proof of Proposition 6 occupies §8.2–§8.5.

8.2. Preliminary reduction. If Proposition 6 holds for $(\psi, \tau_1, \tau_2)$, then it holds formally also for $(\psi^{1/u}, \tau_1 u, \tau_2 u)$ for each $u \in \sigma^\times$. We may and shall thereby reduce (for notational convenience) to the case that $\psi \in \sigma$ (see §5.4.2).

8.3. Smoothing with respect to the adjoint action. Let $m$ be a positive integer. Assume the following:

$$m \geq N_0,$$  

$$N \text{ is large enough in terms of } m.$$  

Let $U \leq G$ denote the $m$th principal congruence subgroup (see §7.2.2). Let $f \in C^\infty_c(G)$ be attached to $(N, \sigma)$. We study the effect of smoothing $f$ under the adjoint action of $U$.

**Lemma.** Set $\phi := \mathcal{F}\mathcal{S} \nabla^{1} f \in S(B)$, as in §6.3. Define $\phi^U \in S(B)$ by

$$\phi^U(x) := \mathbb{E}_{g \in U} \phi(\text{Ad}(g)x).$$

Then

$$\phi^U\left(\begin{pmatrix} \delta/2 + \alpha & \beta \\ \gamma & \delta/2 - \alpha \end{pmatrix}\right) = I(\alpha, \delta)e_{2\alpha q^m}(\beta)e_{2\alpha q^m}(\gamma).$$

In particular,

$$q^{-N}\phi^U\left(\begin{pmatrix} \delta/2 + \varpi^{-N}\alpha & \varpi^{-N}\beta \\ \varpi^{-N}\gamma & \delta/2 - \varpi^{-N}\alpha \end{pmatrix}\right) \text{ is independent of } N.$$  

Proof. Proposition 4 implies that for $N$ large enough,
\[
supp(\phi) \subseteq E(m) \tag{8.11}
\]
with $E(m)$ as in (7.8), and also that $\phi$ satisfies the smoothness property (7.6) noted above. To deduce (8.9) from (7.6) and (8.11) is a calculus problem; to solve it, we apply Lemma 4 of §7.3.4 to the functions $\phi_\delta \in S(B^0)$ attached to $\phi \in S(B)$ and $\delta \in k$ by $\phi_\delta(\xi) := \phi(\delta/2 + \xi)$ for $\xi \in B^0$. The final assertion (8.10) follows from (6.7).

\[\square\]

8.4. Metaplectic interpretation. Retain the notation and setting of §8.3. We now interpret (8.10) in terms of the Weil representation. Recall the double cover $pr: \text{Mp}_2(k) \rightarrow \text{SL}_2(k)$ and the elements $n(b), t(a), w \in \text{Mp}_2(k)$ as in §2.2.3. To reduce clutter in the formulas to follow, we introduce some notation and terminology:

Definition. Let $a_1, a_2$ be quantities depending implicitly upon the large positive integer $N$ and a field element $\tau \in k^\times$ (as well as the field $k$ containing $\tau$, of course). Thus $a_1 = a_1(\tau, N)$. Write $a_1 \approx a_2$ to denote that $a_1 = \gamma |\tau|^{c/4} b$, where

- $\gamma \in \mathbb{C}(1)$ is an eighth root of unity that may depend upon $\tau$ and $N$, and
- $c \in \mathbb{Z}$ depends neither upon $\tau$ nor upon $N$.

We say that a quantity $a$ is essentially independent of $N$ if $a \approx b$, where $b$ is independent of $N$.

Lemma. Let $\tau \in k^\times$. If $\tau \notin \sigma^\times k^{\times 2}$, then $\text{Def}^\tau f = 0$. Otherwise there exists $\phi_0 \in S(B)$ that is essentially independent of $N$ so that
\[
\text{Ad}(e_U) \rho^\tau(w) \text{Def}^\tau f = q^{N/2}(1 \otimes \rho_0^\tau(t(w))\phi_0). \tag{8.12}
\]

Proof. The first assertion is immediate from the definitions of $\text{Def}^\tau$ and $f$. Abbreviate $f' := \text{Def} f$. Since $\text{Def}^\tau f' = \text{Def}^\tau f$, it suffices now to show for $\tau \in \sigma^\times k^{\times 2}$ that the required conclusion holds in the inverted form
\[
q^{-N/2}(1 \otimes \rho_0^\tau(t(w))) \text{Ad}(e_U) \rho^\tau(w) \text{Def}^\tau f' \text{ is essentially independent of } N. \tag{8.13}
\]
The case $\tau = 1$ of (8.13) follows from the lemma of §8.3 and the formulas describing of Weil representation. In general, we see by inspecting the definitions that $\text{Def}^\tau f' = \text{Def}^\tau f \approx \rho^\tau(t(\nu))\text{Def}^\tau f$ and that $\rho^\tau(w) \approx \rho^\tau(t(\nu))\rho^\tau(t(\nu))^\tau f'$, hence that
\[
\rho^\tau(w)\text{Def}^\tau f' \approx \rho^\tau(t(\nu))\rho^\tau(t(\nu))^\tau f' \approx \rho^\tau(1)\text{Def}^\tau f'. \tag{8.14}
\]
Moreover, $\rho_0^\tau(t(w)) \approx \rho_0^\tau(t(w^2))$. Thus the identity (8.13) for general $\tau \in \sigma^\times k^{\times 2}$ reduces to the $\tau = 1$ case already established.

\[\square\]

8.5. Completion of the proof. We now prove Proposition 6. For $i = 1, 2$, set $\phi := \phi^\tau_i(w)^\tau f$. By (8.1) and (8.3), we have
\[
\ell(\text{Def}_\tau^1 f, \text{Def}_\tau^2 f) = \ell(\phi_1, \phi_2). \tag{8.15}
\]
By averaging (8.2) over $g_1, g_2 \in U$, we have
\[
\ell(\phi_1, \phi_2) = \ell(\phi_1^U, \phi_2^U), \tag{8.16}
\]
with notation as in §8.3. By shrinking $U \leq G$ as necessary, we may assume that $U$ is the $m$th principal congruence subgroup for some $m \geq N_0$; since $\ell$ and $N_0$ are independent of $N$, we may assume also that $N$ is sufficiently large in terms of $m$. By the lemma of §8.4, we may suppose that $\tau_1, \tau_2 \in \sigma^\times k^{\times 2}$, in which case
\[
\phi_i^U \approx q^{N/2}(1 \otimes \rho_0^\tau(t(w))\phi_0),
\]
where $\phi_0 \in S(B)$ is essentially independent of $N$ (see §8.4). Our task thereby reduces to establishing the estimate
\[ \ell((1 \otimes \rho^0_\varphi(t(\mathbb{W}^N))))\phi_0, (1 \otimes \rho^0_\varphi(t(\mathbb{W}^N)))\phi_0) \ll Nq^{-N}, \]
which follows finally from the condition (8.4) in the definition of “good.”

**Remark.** The sesquilinear forms $\ell$ to which we apply Proposition 6 below may be assumed to have an additional property (beyond being “good”). That property is not directly relevant for the immediate purposes of this paper, but may be useful in future work, so we briefly record it: There are irreducible unitary representations $\pi_1, \pi_2$ of $G$ of dimension $> 1$ (arising as local components of cuspidal automorphic representations) so that $\ell$ factors as $\ell = \ell \circ ((1 \otimes \theta_1) \otimes (1 \otimes \theta_2))$, where

1. $\sigma_i$ is the local $\psi^\gamma$-theta lift of $\pi_i$ as in [29], i.e., the irreducible representation of $\text{Mp}_2(k)$ for which one has $\text{Hom}_G(\rho^0_\varphi, \pi_i) = \text{Hom}_C(\sigma_i, C)$,

2. $\ell : S(k) \otimes S(k) \otimes \sigma_1 \otimes \sigma_2 \to C$ is a sesquilinear form invariant by the diagonal action of $\text{Mp}_2(k)$,

3. $\theta_i : S(B^0) \to \sigma_i$ is a basis element for the one-dimensional space of $G$-equivariant maps $\rho^0_\varphi \to \sigma_i$, and

4. $1 \otimes \theta_1 : S(B) = S(k) \otimes S(B^0) \to S(k) \otimes \sigma_i$ and $(1 \otimes \theta_1) \otimes (1 \otimes \theta_2) : S(B) \otimes S(B) \to S(k) \otimes S(k) \otimes \sigma_1 \otimes \sigma_2$ are the evident maps.

The $\phi = \nabla f$ of interest in this paper concentrate on semisimple elements. If one instead considers $\phi$ supported close to the nilcone (as arise naturally when studying classical newvectors), then one may exploit bounds towards temperedness of the $\sigma_i$ and the above factorization of $\ell$ to produce estimates sharper than those that follow from $\ell$ being good.

### 8.6. Bounds for partial orbital integrals.

The contents of this short miscellaneous section are used below to deduce the assertions made in §1 concerning the family cardinality and mean statistics; they are directly related neither to the rest of §8 nor to the main new ideas of this paper.

The parameter $N_0$ as in §5.4.3 is regarded here as fixed once and for all. Recall that a regular semisimple element $\gamma \in B^\times$ is one which is diagonalizable with distinct eigenvalues over an algebraic closure $\mathcal{K}$ of $k$, or equivalently, for which $\text{tr}(\gamma)^2 \neq 4\text{nr}(\gamma)$.

**Lemma.** Let $\gamma \in B^\times$ be regular semisimple. Let $U_2$ be a compact open subgroup of $G$ that is small enough in terms of $\gamma$. Let $N$ be large enough in terms of $(\gamma, U_2)$. Let $f \in C^\infty_c(G)$ be attached to $(N, \sigma)$ for some $\sigma \in \Sigma$. Then $\int_{u \in U_2} f(u^{-1} \gamma u) = 0$.

**Proof.** Let $M_1$ denote the semidirect product $k \rtimes k^\times$. Set $M := M_1 \times G$. The group $M$ consists of triples $(x, y, z) \in k \times k^\times \times G$ with the group law
\[ (x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + y_1 x_2, y_1 y_2, z_1 z_2). \]
For $(x, y, z) \in M$ and $b \in B$, set $(x, y, z) \cdot b := z(by + x)z^{-1}$. This formula defines an action of $M$ on $B$.

Set $\tau := \text{nr}(\gamma)^{-1}$ and $\phi := \nabla^\tau f \in S(B)$. For $g \in G$, one then has $\tau \text{nr}(g^{-1} \gamma g) = 1$ and thus $\phi(g^{-1} \gamma g) = f(g^{-1} \gamma g)$. By the lemma of §6.4 (or directly from the definitions), there is an open subgroup $U_1 \leq M_1$, depending only upon $\tau$, so that $\phi(u_1 \cdot b) = \phi(b)$ for all $u_1 \in U_1, b \in B$. Setting $U := U_1 \times U_2 \leq M$, our task reduces to showing that
\[ \mathbb{E}_{u \in U} \phi(u^{-1} \gamma u) = 0 \] (8.18)
for $N$ large enough in terms of $(\gamma, U)$. To that end, observe that the orbit map $M \to B$ given by $m \mapsto m \cdot \gamma$ is submersive at the identity: it suffices to check this claim over $\mathcal{F}$, where it follows by direct calculation in the diagonal case. By Hensel’s lemma as in §7.3.4, together with the assumption that $U \leq M$ is small enough in terms of $\gamma$, we see that

1. the orbit $O := U \cdot \gamma \subseteq B$ is open,
2. the orbit map $U \to O$ is a diffeomorphism, and
3. the pushforward of the probability Haar on $U$ under the orbit map is the probability measure on $O$ induced by the Haar on $B$.

Our goal (8.18) is thus equivalent to showing that $\int_O \phi = \langle \phi, 1_O \rangle_{L^2(B)} = 0$. Since the Fourier transform of $1_O$ has compact support, we reduce by Parseval to showing

$$\int_O \phi = \langle \phi, 1_O \rangle = 0.$$  

To that end, observe that the orbit map $\mathcal{F}$ is submersive at the identity: it suffices to check this claim over $\mathcal{F}$, where it follows by direct calculation in the diagonal case. By Hensel’s lemma as in §7.3.4, together with the assumption that $U \leq M$ is small enough in terms of $\gamma$, we see that

$$\int_O \phi = \langle \phi, 1_O \rangle = 0.$$  

9. Deduction of the main theorem

9.1. Setting. We adopt here the notation and setting of §4.5.1, but assume now that $\mathfrak{q}$ is a finite place and that $B_\mathfrak{q}$ is split. Our existing assumptions imply then that $F$ is totally real and that $B$ is totally definite.

Set $k := F_\mathfrak{q}$, fix an identification $PB_\mathfrak{q} = PGL_2(k)$, and adopt the notation $\sigma, \mathfrak{q}, \mathfrak{q}$ from §2.1.1. Let $N$ be a large positive integer, and let $\omega : \sigma^\times \to C^\times$ be a character of conductor $N$. Let $\mathcal{F}_\omega$ denote the set of nonzero vectors $\varphi \in A_0^J$ for which

- there exists $\pi \in A_0^J$ so that $\varphi \in \pi$, and
- $\varphi$ is a microlocal lift of orientation $\omega$ in the sense of §5.3.

For $\varphi \in A_0^J$, the set $\pi \cap \mathcal{F}_\omega$ is either empty or of the form $C^\times \varphi_1 \cup C^\times \varphi_2$ for some $\varphi_1, \varphi_2 \in \pi^J$ (see §5.3.2), hence $\mathcal{F}_\omega$ is a union of scaling classes $C^\times \varphi$. Choose a set $\mathcal{F}_\omega$ consisting of one unit vector from each such scaling class, so that $\mathcal{F}_\omega = \bigcup_{\varphi \in \mathcal{F}_\omega} C^\times \varphi$. The discussion of §1.5 and §1.7 applies, giving us a sequence of sets $\mathcal{F}_N := \bigcup_{\omega \in \Sigma_N} \mathcal{F}_\omega \subseteq A_0^J$ indexed by large positive integers $N$.

Define $\Sigma$ and $\mathcal{X}_N^\sigma$ as in §5.4.2 with respect to some fixed but large enough natural number $N$. The partition $\mathcal{X}_N = \bigsqcup_{\varphi \in \Sigma_N} \mathcal{X}_N^{\varphi}$ of the group of characters of $\sigma^\times$ of conductor $N$ induces a partition $\mathcal{F}_N = \bigsqcup_{\varphi \in \Sigma_N} \mathcal{F}_N^{\varphi}$ of the family of microlocal lifts. We emphasize that $|\Sigma| = O(1)$.

9.2. Mean statistics. We have included this section to complete the discussion of §1; it has nothing to do with the main new ideas of this paper.

Since smooth functions on $X$ are uniformly dense in the space of continuous functions, the lemmas of §1.5 and §1.6 are consequences of the following:

Lemma. Fix $\Psi \in A$. Assume that $N$ is large enough in terms of $\Psi$. Then

$$\sum_{\varphi \in \mathcal{F}_N} \langle \varphi, \Psi \varphi \rangle = cq^{2N} \frac{\langle 1, \Psi \rangle}{\langle 1, 1 \rangle}$$

where with $\Delta_F, \Delta_B$ the absolute (reduced) discriminants and $\text{ram}_f(B)$ the set of finite places at which $B$ ramifies,

$$c := 2 \frac{\zeta(2) \Delta_B^{3/2}}{(4\pi^2)^{\delta_Q} \prod_{p \in \text{ram}_f(B)} \zeta_p(1) \zeta_k(1) \zeta_k(2)}.$$
Example. Suppose that $F = \mathbb{Q}$, that $B$ is ramified precisely at $\{\infty, D\}$ for some prime $D \in \mathbb{Z}_{\geq 1}$, and that $q$ corresponds to some prime $q \in \mathbb{Z}_{\geq 1}$. By taking $\Psi = 1$ in the lemma and evaluating $c$, we obtain that for $N$ large enough,

$$|F_N| = q^{2N} \frac{D-1}{12} (1 - \frac{1}{q})(1 - \frac{1}{q^2}) \int 1/2 \quad q = 2 \text{ is odd.}$$

Proof of the lemma. It will suffice to show for each $\sigma \in \Sigma$ that

$$|\Sigma| \sum_{\phi \in F_N} \langle \phi, \Psi \sigma \rangle \quad (9.2)$$

is eventually equal to the RHS of $(9.1)$. For the remainder of the proof, set $G := \mathbb{PB}^\infty$. Let $f = \prod_p f_p \in C_c^\infty(G_\kappa)$ be given by $f_p := \text{vol}(J_p)^{-1} J_p$ for $p \neq q$ (see §4.5.1) and by taking for $f_q \in C_c^\infty(G_q) = C_c^\infty(\text{PGL}_2(k))$ the element attached to $(N, \sigma)$ as in §5.4.3. By Proposition 3, the example of §4.2.1, and the pretrace formula (§3.3.2), we have

$$\sum_{\phi \in F_N} \langle \phi, \Psi \sigma \rangle = \int_{g \in [G]} \Psi(g) \sum_{\gamma \in G} f(g^{-1} \gamma g). \quad (9.3)$$

As in the proof of the trace formula, the RHS of $(9.3)$ may be factored as $\sum_{\gamma} I(\gamma)$, where $\gamma$ traverses a set of representatives for the $G$-conjugacy classes in $G$ and $I(\gamma) := \int_{h \in [G]} \int_{g \in G_{\gamma}} \Psi(hg)f(g^{-1} \gamma g)$; here $G_\gamma$ denotes the centralizer. The function $f$ is supported in a fixed (i.e., independent of $N$) compact subset of $G_\kappa$, hence $I(\gamma) = 0$ for $\gamma$ outside some fixed finite collection of representatives.

Let $\mu$ denote the Tamagawa measure on $\mathbb{PB}^\infty$. Recall the subgroup $J := G_q$ arising in the definition of $f$. Set $j := J \times \prod_{p \neq q} J_p$. Since $|\Sigma| \cdot |\mathcal{X}_N| = |\mathcal{X}_N|$, one has

$$|\Sigma| I(1) = \langle \Psi, 1 \rangle |\Sigma| f(1) = \langle \Psi, 1 \rangle |\mathcal{X}_N| |\mu(j')^{-1}. \quad (9.4)$$

To compute the latter volume, it is convenient to factor $\mu = \prod \mu_p$ into the local measures $\mu_p$ as defined in §2.1.2 relative to the standard nontrivial unitary character $\psi = \prod_x \psi_p$ of $\mathbb{A}/F$, i.e., that for which $\psi_p(x) = e^{2\pi i x}$ for infinite places $p$. For a finite place $p$, $\Delta_{F_p} = \Delta_{\psi_p}$ is then the absolute discriminant of $F_p$. We record some consequences of the volume formulas of §2.1.3 and §6.1:

1. If $p$ is finite and $B_p$ splits, then $\mu_p(J_p) = \zeta_p(2)^{-1} \Delta_{B_p}^{-1} \Delta_{F_p}^{-3/2}$.
2. If $p$ is finite and $B_p$ ramifies, then $\mu_p(J_p) = \zeta_p(1) \zeta_p(2)^{-1} \Delta_{B_p}^{-1} \Delta_{F_p}^{-3/2}$.
3. If $p$ is infinite, then $\mu_p(J_p) = \mu_p(G_p) = 4\pi^2$.
4. $\mu_q(j)/\mu_q(J_q) = |2|^{-1} q^{-N} \zeta_k(1)^{-1} \zeta_k(2)$.

Therefore

$$\mu(j')^{-1} = \frac{\zeta_F(2) \Delta B \Delta_F^{3/2}}{(4\pi^2)^{|F:Q|} \prod_{p \text{ ram}} s_p(1) \zeta_k(2)^2} \cdot \frac{|2| k q^N \zeta_k(1)}{\zeta_k(2)}. \quad (9.5)$$

Since $|\mathcal{X}_N| = q^N/\zeta_k(1)^2$ and $(1, 1) = \mu([\mathbb{PB}^\infty]) = 2$, we obtain

$$|\mathcal{X}_N| |\mu(j')^{-1} = cq^{2N}/2 = cq^{2N}/(1, 1). \quad (9.6)$$

By $(9.4)$ and $(9.6)$, the contribution from $I(1)$ to $(9.2)$ gives the required RHS of $(9.1)$.

To complete the proof, it suffices now to show for each fixed $1 \neq \gamma \in G$ that $I(\gamma) = 0$ for $N$ large enough. Let $U$ be a small enough but fixed compact open
subgroup of $G_\mathfrak{q}$ under which $\Psi$ is invariant. By a change of variables in the definition of $I(\gamma)$, our task reduces to showing for all $g \in G_\mathfrak{h}$ that
\[
\int_{u \in U} f(u^{-1}g^{-1}gu) = 0 \text{ for } N \text{ large enough.} \quad (9.7)
\]
Fix a compact set $E$ (independent of $N$) containing the support of $f$, and let $j : G_\mathfrak{h} \to G_\mathfrak{h}$ denote the orbit map $j(g) := g^{-1}g$. Since $B$ is non-split, the group element $\gamma$ is regular semisimple (see §8.6). The map $G_\gamma \backslash G_\mathfrak{h} \to G_\mathfrak{h}$ induced by $\gamma$ is thus proper, and so the set $j^{-1}(E)$ meets only a fixed finite collection of double cosets $G_\gamma \backslash g U$. For this reason, it suffices to establish (9.7) for each fixed $g \in G_\mathfrak{h}$. The conclusion follows in that case from the lemma of §8.6.

\section{Variance statistics.}

\subsection{The sums.}

Define the sesquilinear forms $V_N : A_{i,0+}^d \otimes A_{i,0+}^d \to \mathbb{C}$ by
\[
V_N(\Psi_1, \Psi_2) = \sum_{\varphi \in \mathcal{F}_N} L^{(S)}(\text{ad } \varphi, 1) \langle \varphi, \Psi_1 \rangle \langle \Psi_2, \varphi \rangle.
\]
We have written $L^{(S)}(\text{ad } \varphi, 1) := L^{(S)}(\text{ad } \pi, 1)$ for $\varphi \in \pi \in A_{i,0}^d$.

\subsection{The leading constant.}

Set $X := [PB^\times]/J$. Equip it with the quotient measure induced by any Haar measure on $[PB^\times]$; the formulation of our results will not depend upon this choice. Set
\[
c_0 := 2^{\#\text{ram}(B)} \zeta_F^{(S)}(2) \frac{\text{vol}(X)}{2 \zeta(1)}. \quad (9.8)
\]

\textbf{Example.} In the setting of the example of §9.2, suppose that we identify $X$ with $\Gamma \backslash \text{PGL}_2(\mathbb{Q}_p)$ as in §4.5.2 and equip $X$ with the quotient measure induced by the Haar on $\text{PGL}_2(\mathbb{Q}_p)$ assigning volume one to $\text{PGL}_2(\mathbb{Z}_p)$. Then $\text{vol}(X) = (D - 1)/12$ (see the remark of §4.5.2). After some calculation, we obtain
\[
c_0 = 2\pi q \frac{2(1 + D^{-1})(1 - q^{-1})}{2 \zeta(1)}(1 - q^{-2} - D)
\]

\subsection{The proposed limit.}

Let $V_\infty : A_{i,0+}^d \otimes A_{i,0+}^d \to \mathbb{C}$ denote the sesquilinear form given for $\Psi_i \in \pi_i \in A_{i,0+}^d$ ($i = 1, 2$) by $V_\infty(\Psi_1, \Psi_2) := 0$ unless $\pi_1 = \pi_2 =: \pi$, in which case
\[
V_\infty(\Psi_1, \Psi_2) := c_0 L^{(S)}(\pi, \frac{1}{2}) \int_{h \in N(H)} \langle \pi(h)\Psi_1, \Psi_2 \rangle L^2(X),
\]
where the measure on $N(H)$ is as in §7.

\subsection{Main result.}

In view of the discussion of §4.5.2 and §4.5.3, the following result makes precise (and mildly generalizes) Theorem 1:

\textbf{Theorem 3.} Let $\Psi_1, \Psi_2 \in A_{i,0+}^d$ be fixed (i.e., independent of $N$). Then
\[
q^{-N}V_N(\Psi_1, \Psi_2) = V_\infty(\Psi_1, \Psi_2) + O(N q^{-N}). \quad (9.9)
\]

\textbf{Proof.} The proof divides into five steps:

\begin{enumerate}
\item The partition $\mathcal{F}_N = \bigsqcup_{\sigma \in \Sigma} \mathcal{F}_N^\sigma$ of the family induces a decomposition $V_N = \sum_{\sigma \in \Sigma} V_N^\sigma$ of the variance. Since $|\Sigma| \asymp 1$, it will suffice to show that
\[
q^{-N}V_N^\sigma(\Psi_1, \Psi_2) = |\Sigma|^{-1}V_\infty(\Psi_1, \Psi_2) + O(N q^{-N}). \quad (9.10)
\]
\end{enumerate}
(2) Let \( f \in C_\infty C(PB_q^\infty) \) be attached to \((N, \sigma)\) (see §5.4.3). Recall from §4.5.1 the definitions of \( V_f, M_f \). By Proposition 3 and the example of §4.2.1, we have \( V_N^g = V_f \).

(3) By (4.22) and (5.6), we have \( c_4 q^{N-N_0} \frac{1}{2} = q^N |\Sigma|^{-1} c_0 \). Feeding this calculation into Proposition 5 gives
\[
M_f (\Psi_1, \Psi_2) = q^N |\Sigma|^{-1} V_\infty (\Psi_1, \Psi_2)
\]
for \( N \) large enough in terms of \( \Psi_1, \Psi_2 \). We reduce to showing that
\[
V_f (\Psi_1, \Psi_2) - M_f (\Psi_1, \Psi_2) \ll N.
\]

(4) Fix a nontrivial unitary character \( \psi \) of \( \mathbb{A}/F \) whose component \( \psi_q \) is unramified. Fix a nonzero element \( W_S = \prod_{p \mid S} W_p \in C_\infty (F_S^\times) \) for which \( W_q := 1_q \). We apply Theorem 2 with respect to \( \psi \) and \( W_S \); our task thereby reduces to showing for fixed \( \tau_1, \tau_2 \in F^\times \) that
\[
\epsilon_{\tau_1, \tau_2} (\nabla^{\tau_1} f, \nabla^{\tau_2} f, \Psi_1, \Psi_2) \ll N,
\]
where \( f \in C_\infty C(PB_q^\infty) \) is as in §4.5.1 and \( \nabla^{\tau} \) is as in §4.2.4.

(5) One has \( \nabla^{\tau_1} f = (\nabla^{\tau_1} f) \otimes \phi^{\tau_1} \) where \( \nabla^{\tau} f \in S(B_q) \) is as in §5.5 and \( \phi^{\tau} = \otimes_{p \mid S{-}\{q\}} \phi^{\tau}_p \) with \( \phi^{\tau}_p \in S(B_p) \) independent of \( N \). The LHS of (9.13) is thus equal to \( \ell (\nabla^{\tau_1} f, \nabla^{\tau_2} f) \), where \( \ell : S(B_q) \otimes S(B_q) \to \mathbb{C} \) denotes the sesquilinear form given by
\[
\ell (\Phi_1, \Phi_2) := \epsilon_{\tau_1, \tau_2} (\Phi_1 \otimes \phi^{\tau_1}, \Phi_2 \otimes \phi^{\tau_2}, \Psi_1, \Psi_2).
\]
Observe that \( \ell \) is independent of \( N \). Theorem 2 says that \( \ell \) is good in the sense of §8.1 with respect to \( (\psi_q, \tau_1, \tau_2) \) (taking for \( U \leq PB_q^\times \) any open subgroup under which \( \Psi_1, \Psi_2 \) are invariant). Proposition 6 implies finally that \( \ell (\nabla f, \nabla f) \ll N \), giving (9.13), as required.

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