Restrained Whole Domination in Graphs

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Abstract. In this paper, we introduce a new definition of domination number in graphs called restrained whole domination number and denoted by \( \gamma_{rw} \). Some bounds and results for restrained whole domination number are established. Moreover, this number is computed to the complement of certain graphs. Finally, the effective the deletion or addition an edge or the deletion of a vertex of the graph has this number is discussed.

1. Introduction

We consider a graph \( G = (V, E) \) for simplicity \( G \) as finite, simple and undirected. In recent years many fields of graph theory in mathematics are appears to solve many problems in as labeled graph [1],[2] and [3], topological graph [10], fuzzy graph [16], and others Claude Berge in 1962 introduced concept of domination in graphs [5], after this time many parameters of domination are discussed as [4], [8], [11] and [12]. The set \( D \) in a graph \( G \) is a restrained dominating if each vertex in \( V - D \) is adjacent to at least one vertex in the set \( D \) and in the set \( V - D \). The minimum cardinality of all these sets is called the restrained domination number of \( G \) denoted by \( \gamma_r(G) \). The concept of restrained domination is studied and investigated in [6], [7], [14] and [15]. Ibrahim introduced the parameter whole domination number in his thesis [9]. For more details for the concepts used in this work, the reader can be found in [7] and [13].

Definition 1.1.[9] Let a graph \( G \) be a simple and connected, a proper subset \( D \subset V \) is called whole dominating set (WDS), if every vertex in the set \( V - D \) is adjacent to all vertices in the set \( D \). The set \( D \) is called minimal WDS (MWDS) if it has no proper WDS. The whole domination number denoted by \( \gamma_{w}(G) \) for simplicity \( \gamma_{w} \) is the minimum cardinality of a MWDS. The MWDS has \( \gamma_{w} \) is called \( \gamma_{w} - \) set.

Observation 1.2.[9]

1) \( \gamma_{wh}(P_n) = 1 \) if \( 2 \leq n \leq 3 \), otherwise \( P_n \) has no WDS.
2) \( \gamma_{wh}(C_n) = 1, if \, n = 3 \), \( otherwise \, C_n \) has no WDS.
3) \( \gamma_{wh}(K_n) = \gamma_{wh}(W_n) = 1 \).
4) \( \gamma_{wh}(K_{m,n}) = \min\{m,n\} \).
Remark 1.3.\[9\]
If $G$ is a disconnected graph, then $G$ has no WDS.

2. Restrained Whole Domination Number
Throughout this section a new definition of dominating set is initiated which is called an upper whole dominating set with study some of properties.

Definition 2.1. A WDS $D \subset V$ is Restrained whole dominating set (RWDS), if each vertex in $V - D$ is adjacent to at least one vertex in the set $D$ and in the set $V - D$.

Definition 2.2. The restrained whole domination number of $G$ denoted by $\gamma_{rwh}(G)$, is the smallest cardinality of a SWDS of $G$.

Example 2.3. In the following Figure:

\[
D = \{ v_1, v_2, v_3 \} \text{ is minimum whole dominating set of } G, \text{ thus, } \gamma_{wh}(G) = 3 \\
D_1 = \{ v_4, v_6, v_7 \} \text{ is minimum restrained whole dominating set of } G, \gamma_{rwh}(G) = 3
\]

Observation 2.4. If $G$ is a graph and $D$ is a RWDS with minimum cardinality.
1) If $G$ has an isolated vertex, then $G$ has no RWDS.
2) $|V - D|$ has no an isolated vertex.
3) $deg(v) \geq \gamma_{rwh}(G) + 1, \forall v \in V - D$.
4) $|V - D| \geq 2$.
5) $\gamma(G) \leq \gamma_{wh}(G) \leq \gamma_{rwh}(G)$.

Theorem 2.5.
1) $\gamma_{rwh}(K_n) = 1$, for any $n, \; n \geq 3$.
2) $\gamma_{rwh}(C_n) = 1$ if $n = 3$, otherwise $C_n$ has no RWDS.
3) $\gamma_{rwh}(W_n) = 1$, for any $n$.
4) $\gamma_{rwh}(P_n)$ has no RWDS in graph.
5) $\gamma_{rwh}(K_{m,n})$ has no RWDS in graph.

Proof.
1) Let $D = \{ v_i \}, i = 1, 2, ..., n$, it is clear that the set $D$ is minimum RWDS and each vertex in $V - D$ is adjacent to the other vertex in $V - D$, since $n \geq 3$. Thus, the result is obtained.
2) If $n = 3$, then $C_3 = K_3$, and $\gamma_{rwh}(C_3) = 1$ by (1). If $n \neq 3$, then the $\Delta(G) = 2$, so the maximum cardinality of neighborhood for each vertex (say $v$) in this graph is two say $\{ w_1, w_2 \}$. Now, if the set d
contains the vertex v, then the set V−d contain \{w_1, w_2\}. Since n ≠ 3, so there is at least one other vertex say w_3, if this vertex belong to the set V − D, then the vertex v must adjacent to it according to the definition 2.1, and this is a contradiction with \(\Delta(G) = 2\). In other hand, if w_3 belong to the set D, then again this vertex must adjacent to the two vertices w_1 and w_2 and this a contradiction with property of cycle since the vertex v is adjacent to these vertices. Thus, every cycle of order four or more has no RWDS.

3) Let D = \{ν\}, where ν is the center of the wheel graph, it is clear that the set D is minimum RWDS and each vertex in V − D is adjacent to the other vertex in V − D, since n ≥ 3. Thus, the result is obtained.

4) It is clear that \(P_n\) has WDS only when n = 2, 3 by observation 1.2 (1), but in each cases the set V − D contains an isolated vertex. Thus, the graph has no RWDS, according to observation 2.6 (1).

5) By observation 1.2 (iv) the set of WDS contains all vertices of partite of minimum cardinality. Thus, the set V − D contains an isolated vertex, therefore the graph has no RWDS, according to observation 2.4 (1).

Note 2.6. The star graph \(S_n \equiv K_{1,n−1}\) has no RWDS.

Proposition 2.7. If a graph G has WDS, then G has restrained whole dominating set if and only if G[V−D] has no isolated vertex.

**Proof.** It is straightforward from observation 1.2. □

Corollary 2.8. If G is a tree, then G has no RWDS.

Proof. Since G has no cycle, then all vertices in V − D is not adjacent. So, V − D are independent and G has no RWDS. □

Proposition 2.9. Let a graph G has RWDS. If D has no isolated in G[D], then V−D has RWDS in G.

Proof. Let D be a minimum RWDS, the each vertex is adjacent to all vertices in \(〈V−D〉\). In other hand, each vertex in V − D is also adjacent to all vertices in D. Furthermore, \(〈D〉\) has no an isolated vertex, thus \(V−D\) has RWDS in G. □

3. Complement the certain graphs

**Theorem 3.1.** If \(G \equiv \overline{P_n}\) then \(y_{rwh}(\overline{P_n}) = 2\), \(∀n > 5\) otherwise \(\overline{P_n}\) has no RWDS.

**Proof.** There are four cases depend on n as follows.

Case 1. If \(n \leq 3\), then \(\overline{P_n}\) has an isolated vertex. Therefore, \(\overline{P_n}\) has no RWDS, by Observation 2.4(1).

Case 2. If \(n = 4\), then \(\overline{P_4} \equiv P_4\) so again it has no RWDS, by Observation 2.4(1).

Case 3. If \(n = 5\), then let \(\{v_1, v_2, v_3, v_4, v_5\}\) the set of vertices of the graph \(\overline{P_5}\). It is clear that the set \(\{v_1, v_3\}\) is the only whole dominating set in \(\overline{P_5}\), but this is not restrained, since the vertex \(v_3\) is an isolated in \(V−D\). Therefore, \(\overline{P_5}\) has no RWDS according to Observation 2.5(1).

Case 4. If \(n ≥ 6\), the set \(\{v_1, v_n\}\) is a minimum WDS and the set \(V−D\) has no an isolated vertex. Thus, \(y_{rwh}(\overline{P_n}) = 2\).

From all cases above, the result is obtained. □

**Theorem 3.2.** If \(G \equiv \overline{C_n}\), then \(y_{rwh}(\overline{C_n}) = 2\), \(∀n > 5\) otherwise \(\overline{C_n}\) has no RWDS.

**Proof.** There are three cases depend on n as follows.

Case 1. If \(n ≤ 4\), then \(\overline{C_n}\) is disconnected graph. Therefore, \(\overline{C_n}\) has no WDS according to Remark 1.3. Thus, \(\overline{C_n}\) has no restrained whole.
Case 2. If \( n = 5 \), then \( C_5 \equiv C_5 \) so again it has no RWDS, according to Theorem 2.6.

Case 3. If \( n \geq 6 \), the set \( \{ v_i, v_j \}; d(v_i, v_j) \geq 2 \) is a minimum RWDS and the set \( V - D \) has no an isolated vertex. Thus, \( \gamma_{rwh}(P_n) = 2 \).

From all cases above, the result is obtained. \( \square \)

Proposition 3.3. The graphs \( \overline{W_n}, K_n, \overline{S_n} \), and \( \overline{K_{n,m}} \) have no RWDS.

Proof. All these graphs are disconnected. Thus, according to Theorem 2.5, these graphs have no RWDS. \( \square \)

4-Effective the deletion or addition

Theorem 4.1. Let \( G \) be a graph has RWDS \( D \) with \( \gamma_{rwh} - \) set and \( \gamma_{rwh}(G) > 1 \), then if \( v \in D \) then \( \gamma_{rwh}(G - v) < \gamma_{rwh}(G) \) otherwise all probability are possible.

Proof. Let \( D \) be a \( \gamma_{rwh} \) - set of a graph \( G \). Then there are two cases as follows:

Case 1: If a vertex \( v \), where \( v \in D \) is deleting, then \( \gamma_{rwh}(G - v) < \gamma_{rwh}(G) \).

Case 2: If we delete a vertex \( v \), where \( v \in V - D \), then three cases are obtained as follows:

1) If \( v \in D' \) such that \( D' \) is RWDS\( \gamma_{rwh} \) - set, then

\[
\gamma_{wh}(G - v) < \gamma_{wh}(G).
\]

2) If \( |D| < |V - D| \) and \( (V - D) - v \) has RWDS, then \( \gamma_{rwh}(G - v) = \gamma_{rwh}(G) \).

3) If \( (V - D) - v \) has an isolated vertex, then \( D \) is not RWDS. Thus, there are two subcases as follows.

Subcase 1. If there is other RWDS in \( G - v \) say \( D' \), then \( \gamma_{wh}(G - v) > \gamma_{wh}(G) \).

Subcase 2. If there is no other RWDS in \( G - v \), then \( G - v \) has no RWDS.

From all cases above, the result is obtained.

Theorem 4.2. Let \( G \) be a graph has restrained whole domination number, then \( G \) has no RWDSor \( \gamma_{rwh}(G - e) \geq \gamma_{rwh}(G) \).

Proof. There are several cases depending on the location of the edge \((e = uv)\) as follows.

Case 1. If \( e \) belongs to the induced subgraph \( \langle D \rangle \), then the deleting \( e \) not influence to the restrained whole domination.

Case 2. If \( e \) belongs to the induced subgraph \( \langle V - D \rangle \), then there are two subcases as follows.

I) At least one vertex \( u \) or \( v \) becomes an isolated in \( \langle V - D \rangle \) after deleting the edge \( e \), then \( G \) has no restrained whole domination.

II) If \( u \) and \( v \) are not isolated vertices in \( \langle V - D \rangle \) after deleting the edge \( e \), then the deleting \( e \) not influence to the restrained whole domination.

Case 3. If \( u \in D \) and \( v \in V - D \), then there are two subcases as follows.

I) If all vertices in the induced subgraph \( \langle V - D \rangle \) are adjacent to a vertex \( v \), then \( v \) will belongs to the set \( D \) after deleting the edge \( e \). Therefore, \( \gamma_{rwh}(G - e) > \gamma_{rwh}(G) \).

II) If all vertices in the induced subgraph \( \langle V - D \rangle \) are not adjacent to a vertex \( v \), then \( G - e \) has no RWDS.

From all cases above, the result is obtained. \( \square \)

Theorem 4.3. If \( G \) has a \( \gamma_{rwh} \), then \( \gamma_{rwh}(G + e) \leq \gamma_{rwh}(G) \), where \( e \in \bar{G} \).

Proof.

Let \( D' \) be a minimum RWDS that means \( |D'| = \gamma_{rwh} \). By adding an edge \( e \) to a graph \( G \), two different cases are gotten as follows.

Case 1. If \( e \) joins two vertices from \( V - D \), then the whole domination is not influenced by this addition. Thus, \( \gamma_{rwh}(G + e) = \gamma_{rwh}(G) \).
Case 2. If \( e \) joins two vertices in \( D \), then if at least one vertex which is incident on this edge becomes adjacent to all vertices in \( D \), then \( \gamma_{rwh}(G + e) = 1 \). Thus, \( \gamma_{rwh}(G + e) \leq \gamma_{rwh}(G) \).

We cannot add an edge \( e \) to a graph \( G \) if one of its vertices belongs to the set \( D \) and the other belongs to the set \( V - D \), since \( e \notin G \) in this case according to definition of whole dominating set. Therefore, we get the result.

\[ \square \]

Theorem 4.4. If \( G \) has a \( \gamma_{rwh} \), then \( \gamma_{rwh}(G \setminus e) \leq \gamma_{rwh}(G) \).

**Proof.** Let \( D' \) be a \( \gamma_{rwh} \) set of a graph \( G \). By contracting an edge \( e \) of graph \( G \), three different cases are obtained as follows.

Case 1. If we contract the edge \( e \) which is incident on two vertices of \( V - D \), then there are two cases as follows.
(i) If \(|D| = |V - D|\), then \( V - D \) becomes the minimum RWDS. Thus, \( \gamma_{rwh}(G \setminus e) < \gamma_{rwh}(G) \).
(ii) If \(|D| < |V - D|\), then the minimum RWDS is not influenced by this contraction. Thus, \( \gamma_{rwh}(G \setminus e) = \gamma_{rwh}(G) \).

Case 2. If contracting an edge \( e \) incident on two vertices from \( D \), then it is clear that the set \( D \) is decreasing by one vertex and will still minimum RWDS. Thus, \( \gamma_{rwh}(G \setminus e) < \gamma_{rwh}(G) \).

Case 3. If \( e \) is incident on two vertices one of them from \( V - D \) and the other from \( D \), then the new vertex obtained by this contraction is adjacent to all vertices in \( G \), hence \( \gamma_{rwh} = 1 \).

Therefore, from all cases above, \( \gamma_{rwh}(G \setminus e) \leq \gamma_{rwh}(G) \). \[ \square \]

5. **Conclusion**

Throughout the results which are obtained in this work, new properties and boundaries of a new parameter are been getting. Also, for certain graphs as a cycle, complete, path, wheel, and complete bipartite are been determined of restrained whole domination. The complement of graphs especially the certain graphs are mentioned above are been determined of this number. Finally, the effective of deleting or adding an edge or deleting a vertex of the restrained domination number is discussed.

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