Quantum Preconditioners and Hydrological Linear Systems

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Abstract

Modeling hydrological fracture networks is a hallmark challenge in computational earth sciences. Accurately predicting critical features of fracture systems, e.g., percolation, can require solving large linear systems far beyond current or future high performance capabilities. Quantum computers can theoretically bypass the memory and speed constraints faced by classical approaches, however several technical issues must first be addressed. Chief amongst these difficulties is that such systems are often ill-conditioned, i.e. small changes in the system can produce large changes in the solution, which can slow down the performance of linear solving algorithms. We test several existing quantum techniques to improve the condition number, but find they are insufficient. We then introduce the inverse Laplacian preconditioner, which significantly reduces the condition number of the system and readily admits a quantum implementation. These results are a critical first step in developing a quantum solver for fracture systems, both advancing the state of hydrological modeling and providing a novel real-world application for quantum linear systems algorithms.

1 Introduction

Quantum algorithms for solving linear systems offer a potential for exponential speed-up over classical algorithms. However, accessing the full quantum advantage of these algorithms requires several advancements. Many of these algorithms can only be implemented on future error-corrected hardware [7], while others are designed for today’s noisy intermediate-sized quantum devices [1] but feature a less pronounced speed-up. Beyond the current physical limitations, formulating real-world problems to take advantage of this speed-up remains an outstanding problem. This is in part due to limitations of the quantum algorithms, as well as advances in classical techniques. On the quantum side, the existing algorithms are largely restricted to sparse systems with low condition number. Meanwhile, quantum speed-ups have been ruled out for many applications because large systems of equations can often be replaced by small systems
of equations easily solved on classical computers while retaining the desired accuracy [14]. Existing applications for quantum linear systems (QLS) algorithms have therefore been largely synthetic or specific examples, carefully chosen to avoid these constraints but also lacking in real-world application [8].

In this work we argue that simulating fluid flow through fracture systems, one of the most challenging problems in geophysics, is a promising application for quantum linear systems algorithms. Fracture systems are generally sparse and cannot be coarse-grained, however they tend to have large condition number. Classical methods to reduce the condition number, known as preconditioners, have been successfully developed and tailored specifically for fracture systems, the most prominent example being multigrid preconditioners [6]. However, they cannot simply be ported directly onto quantum computers due to the drastically different underlying technologies and algorithmic design constraints. Therefore, in order to use quantum computers to solve large linear systems for fracture networks of relevance and outpace classical techniques, an effective quantum preconditioning algorithm for fracture networks must be developed. A good preconditioner is generally tied to the type of linear system being solved and exploits some underlying mathematical or physical structure in the problem. Meanwhile, one must also find an efficient way of implementing the preconditioner on a quantum computer. These two constraints – tailored to a specific application while also having an efficient quantum implementation – are likely to play a prominent role in future attempts to solve large, interesting linear systems on quantum computers.

We show that existing techniques to reduce the condition number either do
Figure 2: Fracture systems feature critical behavior, such as percolation, which is only apparent when taking all length scales into account. The red/blue/green colors indicate connected components in the fracture network – the blue connected component in the right image results in percolation.

not have quantum implementations or are not well-suited for fracture systems. However, we find that the inverse Laplacian is both effective on fracture systems and can be efficiently implemented on a quantum computer. See Fig. 1 for a summary of our results. Furthermore, we show that realistic fracture systems can be solved by a quantum computer with a polynomial advantage over classical approaches. Finally, we discuss algorithmic bottlenecks which need to be resolved to unlock the full potential of QLS for fracture systems.

2 Results

2.1 Fracture Networks and Coarse-Graining

In a complex fracture network, fractures of many scales – from kilometers to centimeters – intersect. Critically, small fractures cannot generally be neglected because these can transform the network topology radically, e.g. pushing a system over a percolation threshold, see Fig. 2. Small fractures may also collectively contribute a large surface area to the network providing a critical connection between fractures and the underlying rock. When modeling flow in these networks, it is therefore critical to include the full range of fracture scales, which has led to the development of advanced meshing techniques [10] and high-performance simulators [13]. However, these approaches do not provide a viable path to modeling the full range of scales. Even state-of-the-art high-performance computers and cutting-edge methods can only model large fracture networks with fracture lengths varying over three orders of magnitude [9]. Modeling real-world fracture networks to a high degree of accuracy requires meshes far beyond current or future classical capabilities—e.g., a 1km domain with a 1cm resolution would require $10^{15}$ degrees of freedom. Larger systems of equations would be needed for larger domains or more finely resolved meshes.

Numerical models of subsurface flow in a fracture network are based on a
discretized version of
\[ \nabla \cdot (k \nabla h) = f \]  
(1)

where \( k \) is the permeability, \( f \) is the fluid flux, and \( h \) is the pressure. The pressure is key for applications of subsurface flow such as waste fluid disposal and hydraulic fracturing. It is also critical for transport applications, since pressure gradients drive the transport. In fractured systems, \( k \) is highly heterogeneous with a sharp increase when moving from the rock matrix (where \( k \) is small) to a fracture (where \( k \) is large).

2.2 Test of Existing Quantum Preconditioners

Recently developed QLS algorithms provide a novel path to modeling the full complexity of fracture networks. Since a quantum computer with \( n \) qubits can represent a \( 2^n \) dimensional vector, vast systems of equations can be solved with a small number of qubits. That is, for the 1km domain with a 1cm resolution problem described above, a quantum computer would require \( O(\log(10^{15}) \approx 50) \) qubits, whereas a classical computer would require \( O(10^{15}) \) classical bits. This reduction in memory requirements gives the quantum approach an advantage for huge systems of equations and does not require meshes to be split across nodes, as is needed for classical approaches. Furthermore, the computational complexity of quantum linear systems algorithms can in some cases be exponentially better than the best classical counterparts.

The original QLS algorithm introduced by Harrow, Hassidim, and Lloyd (HHL) \(^7\) solves a sparse \( N \)-variable system of equations \( Ax = b \) with a runtime of \( O(\log(N)\kappa(A)^2) \), where \( \kappa(A) := \|A\|\|A^{-1}\| \) is the condition number of \( A \). The condition number describes how sensitive the system is to small perturbations, or equivalently, how much it can stretch or shrink an arbitrary vector. The best classical algorithm, the Conjugate Gradient method, runs in \( O(N\sqrt{\kappa(A)}) \) on sparse matrices, so the quantum algorithm provides an exponential speed-up when \( \kappa(A) \) is small.\(^2\)

The HHL algorithm is just one example of a burgeoning class of quantum linear system solvers for both near- and far-term quantum computers. A feature common to all of these quantum approaches is a scaling with \( \kappa \) that is linear at best, as compared to the \( O(\sqrt{\kappa}) \) scaling of the best classical approaches. A common technique in classical analysis is to further reduce \( \kappa \) by using a preconditioner. This technique relies on finding a matrix \( M \) such that \( \kappa(MA) \ll \kappa(A) \). One then finds the \( x \) satisfying \( MAx = Mb \). The matrix \( M \) is generally dependent on the specific matrix \( A \), and different preconditioning approaches have been developed for different contexts.

Despite the significant interest and activity in QLS algorithms, relatively little work has been done to develop application-specific preconditioning algorithms. There are currently three general purpose quantum preconditioning

\(^1\)In this work we use \( \|A\| \) with no subscripts to refer to the 2-norm (or operator norm), i.e. \( \|A\| = \|A\|_2 \), and explicitly use \( \|A\|_F \) to refer to the Frobenius norm.

\(^2\)For notational clarity, we often use ‘n’ alone to signify \( \kappa(A) \).
algorithms in the literature: the circulant method [17], the sparse approximate inverse method [2], and the fast-inverse method [20]. These algorithms are described in detail in Sec. 4, here we give only the salient points.

The circulant method is a one-size-fits-all approach, that is, the only input is a matrix $A$ and the output is a preconditioner $M$. With SPAI one gives a sparsity pattern for the preconditioner $M$, and several techniques have been developed for determining good sparsity patterns for fracture systems [11]. The fast-inverse method is designed for systems of the form $A + B$, where $A^{-1}$ can be easily calculated. One then uses $A^{-1}$ as the preconditioner. As discussed in Sec. 4, fracture systems can be decomposed into $\Delta + A_F$, where the Laplacian $\Delta$ describes the system in the absence of fractures, and $A_F$ is the contribution of the fractures. Because the singular value decomposition of the Laplacian is known [3], $\Delta^{-1}$ can be efficiently calculated and used as a preconditioner.

We applied these three techniques to a variety of 2D fracture systems. The simplest system we studied involved two fractures intersecting in a ‘+’ configuration, and we then studied fractal-style recursion of the ‘+’ system to generate more complicated fracture networks, see Fig. 3. The relative permeability of the fractures as compared to the underlying rock is a critical parameter in the analysis of fracture systems, and we studied five different types:

1. “Simple, Low” is the ‘+’ system with fracture 10% more permeable as the underlying rock.
2. “Simple, High” is the ‘+’ system with fractures $10^4$ times more permeable than the underlying rock.
3 & 4. “Fractal, Low/High” are the same as above, but with the fractal system.
5. “Fractal, Var.” is the fractal system where the fractures have permeability contrast that scales down as the fractures get smaller, i.e. largest fractures have contrast $10^4$ and the smallest fractures have contrast 1.1.

See Sec. 4 for a more thorough description of the fracture systems.
Figure 4: Efficacy of the preconditioners under analysis. The inverse Laplacian \( \Delta^{-1} \) gives the best scaling in all cases, while the circulant and SPAI preconditioners reduce \( \kappa(A) \) but do not significantly improve the scaling in \( N \). There are fewer data points for the circulant and SPAI preconditioners due to computational constraints.

In Fig. 4 we show the efficacy of the circulant, SPAI, and inverse Laplacian, along with the scaling of the non-preconditioned matrices \( A \) for the systems described above. We find that the circulant and SPAI methods are poor choices for the fracture systems under consideration. While these two preconditioners consistently reduce the condition number of the system, they do not improve how the condition number scales in \( N \), which is necessary to unlock the full potential of QLS algorithms for large problems. The inverse Laplacian preconditioner, however, does improve the scaling of the condition number, often by a considerable amount. In the cases with low permeability contrast, the condition number of the system \( \Delta^{-1} A \) is very low, scaling as \( \leq O(N^{0.05}) \). The high permeability contrast systems do not fair as well, with the \( \kappa \) of the preconditioned fractal system scaling as \( O(N^{0.6}) \). The fractal system with variable permeability, which is the most realistic of the systems under consideration, has a preconditioned \( \kappa \) which asymptotically scales as approximately \( O(N^{0.55}) \).
2.3 Algorithmic Scaling with Inverse Laplacian Preconditioner

Identifying a preconditioner $M$ which reduces the condition number of a system $A$ is generally not sufficient to guarantee good performance of a QLS algorithm to solve the preconditioned system $MAx = Mb$. This is because one must find an efficient way to calculate the product $MA$ in such a way as to make it readily accessible on a quantum computer. Generic matrix multiplication is an $O(N^2)$ operation on a quantum computer [16] and would remove any benefit from the reduced condition number. Each of the three methods previously discussed get around this limitation through clever techniques: the circulant method calculates the product with the quantum Fourier Transform algorithm, the SPAI method exploits sparseness of $M$ and $A$, and the fast-inverse method assumes efficient block-encodings of $M$ and $A$.

Our results in Sec. 2.2 show that the inverse Laplacian is a good preconditioner, however it remains to examine the potential algorithmic runtime of incorporating this into a preconditioner into a QLS algorithm. As described in Sec. 4 because the Laplacian $\Delta$ is normal and has a known eigenvalue decomposition, one can efficiently prepare a block encoding of $\Delta^{-1}$. The techniques of [20] then give a method for solving the preconditioned system $\Delta^{-1}$, with runtime bounded below by

$$O(||\Delta^{-1}|| \cdot ||A-\Delta|| \cdot ||A^{-1}\Delta|| \cdot \log(||A^{-1}\Delta||)/\epsilon).$$

In Fig. 5 we show the scaling of these components for the fractal system with variable contrast. In Table 1 we summarize the scaling of each component as well as the overall scaling (modulo $\log(N)$) compared with the scaling for Conjugate Gradient on the same systems.

Using the fast inverse QLS algorithm with the inverse Laplacian preconditioner, we can achieve a polynomial improvement over the best generic classical scaling for all fracture systems considered here. This approach utilizes block encodings of $\Delta^{-1}$ and $A-\Delta$ to calculate the product $\Delta^{-1}A$. However, since $||\Delta^{-1}||$ scales linear in $N$, the block encoding takes at least this long. Future algorithms for QLS, specifically tailored to fracture systems, could be developed which calculates $\Delta^{-1}A$ even more efficiently by exploiting the sparseness of $A$.

Table 1: Scaling of the various components entering the overall scaling of the fast-inverse QLS algorithm. The Overall scaling reported is modulo $\log(N)$.

|                        | $\kappa(A)$ | $\|\Delta^{-1}\|$ | $\|A-\Delta\|$ | $\|A^{-1}\Delta\|$ | Overall | Classical |
|------------------------|-------------|---------------------|----------------|---------------------|---------|-----------|
| Simple, Low Contrast   | 1           | 1.01                | 0              | 0.03                | 1.03    | 1.5       |
| Simple, High Contrast  | 1.02        | 1.01                | 0              | 0.26                | 1.26    | 1.51      |
| Fractal, Low Contrast  | 1           | 1.01                | 0              | 0.11                | 1.11    | 1.5       |
| Fractal, High Contrast | 1.09        | 1.01                | 0              | 0.34                | 1.34    | 1.54      |
| Fractal, Var. Contrast | 1.05        | 1.01                | -0.02          | 0.31                | 1.3     | 1.52      |
3 Discussion

We have shown that two previously introduced quantum preconditioners, the circulant and SPAI methods, do not improve the scaling in $N$ of $\kappa(A)$ and therefore will not help gain a quantum advantage for these fracture systems. However, the inverse Laplacian is an effective preconditioner for fracture systems and readily admits a quantum implementation. In particular, it can be implemented via the fast-inverse QLS algorithm, and the overall scaling of this solver scales better than the best generic classical algorithm.

In comparing against classical techniques, we have so far not addressed the fact that for PDE-based systems on a uniform mesh, such as those considered here, more specialized methods can be used. Geometric multigrid methods, which exploit the structured mesh, can solve systems of equations in $O(N)$ or $O(N \log N)$ [18]. This is comparable to the performance of the fast-inverse QLS method, thus negating the quantum advantage.

It is therefore necessary to further develop the quantum approaches with an even greater eye towards the specific physics and mathematical structures of the fracture systems at hand. An obvious area of improvement is a more efficient quantum means of implementing the preconditioned system $\Delta^{-1}A$. In the most realistic case we studied, the fractal system with variable contrast, $\kappa(\Delta^{-1}A)$ asymptotes to roughly $O(N^{0.55})$. Therefore in principle the scaling of solving just the preconditioned system, e.g. with recent adiabatic QLS algorithms [19], would be $O(N^{0.55} \log N)$, a significant improvement over the geometric multigrid methods.

Figure 5: Scaling of the components of the fast-inverse QLS algorithm for problems of increasing size for a fractal fracture network with high permeability contrast.
Alternatively, while the inverse Laplacian opens the door for polynomial speed-up over classical, an even better preconditioner is needed for exponential speed-up. For example, while a direct quantum port of the classical multigrid methods is not plausible, some of the ideas may be used to construct an analogous approach that can be implemented on a quantum computer. Reducing the condition number scaling to $O(\log N)$ in the quantum context would then be possible.

This study shows that fracture networks are a challenging real-world problem with a potential for serious advancements from quantum computation. The large linear systems necessary to accurately model flow behavior are sparse yet cannot be coarse-grained. The condition number $\kappa$ of the systems tends to scale linearly with $N$, however the inverse Laplacian preconditioner, which readily admits a quantum implementation, can improve this scaling considerably, and it is likely even further advancements can be made in $\kappa$. Future work will be devoted to incorporating our application-specific preconditioning techniques into a full quantum linear solver, ideally targeting an implementation on NISQ devices.

4 Methods

4.1 Fracture Networks

We discretize Eq. 1 using a two-point flux finite volume method, which is one of the standard numerical schemes used in subsurface flow solver code bases including: FEHM [21], TTOUGH2 [15], and PFLOTRAN [12]. The two-point flux finite volume method ensures mass conservation, which is a highly desirable property for these numerical solvers. This results in a coefficient matrix which is sparse, symmetric, and positive definite. We treat the permeability of the rock matrix as being constant. The permeability within the fractures is treated in two different ways. In four of the cases we study, the permeability is constant in the fractures. In the fifth case, the permeability of smaller fractures is smaller than the permeability of larger fractures. In this case, the permeability is defined by a power law relating the length of the fracture to the permeability, which is commonly used in practice [9],

$$k \propto \sqrt{r}. \quad (3)$$

4.2 Circulant Preconditioner

In [17] they introduce an efficient quantum implementation of a circulant preconditioner $C$ based on the quantum Fourier transform $F$. An $n \times n$ matrix $C$ is circulant if $C_{ij} = C_{(i-j) \mod n}$. The use of circulant preconditioners in classical applications is motivated by the fact that, for a given circulant matrix $C$ and an arbitrary matrix $A$, $CA$ and $C^{-1}A$ can be computed in $O(n \log n)$ steps using the fast Fourier transform. Circulant preconditioners are particularly useful in solving Toeplitz systems [5].
For an arbitrary matrix $A$, one can construct the circulant preconditioner via
\[ C(A) = F^\dagger \text{diag}(FAF^\dagger)F, \]
where $F_{jk} = \frac{1}{\sqrt{n}} \omega^{jk}$ with $\omega = e^{-2\pi i/n}$. $C^{-1}(A)$ is then used the preconditioner. $F$ can be efficiently implemented via the quantum Fourier transform, and the middle term simplifies to
\[ \text{diag}(FAF^\dagger)_k = \frac{1}{n} \sum_{p,q} \omega^{(p-q)k}A_{p,q}. \]

An algorithm for efficiently preparing the state in Eq. 5 is given in [17]. This approach works for arbitrary dense non-Hermitian matrices, however there is no upper bound on $\kappa(CA)$, and in practice for random dense matrices $\kappa(CA) = O(\kappa(A))$.

4.3 Sparse Approximate Inverse

The Sparse Approximate Inverse (SPAI) approach for solving a system $Ax = b$ attempts to find a matrix $M$ such that $MA \approx I$, where $M$ has a (user-defined) sparsity pattern. For example, if one gives a sparsity pattern involving $n$ non-zero rows and $d$ non-zero elements per row, then $M$ is given by solving $n \times d$ independent least squares problems. The trick with this approach is determining which sparsity pattern to choose for $M$.

In [2] they show that the preconditioned system
\[ MAx = Mb \]
can be solved via a slightly modified version of the HHL algorithm. The overall scaling for actually solving Eq. 6 with error $\epsilon$ is
\[ \tilde{O}(d^7 \kappa(MA) \log(N)/\epsilon^2). \]

In Sec. 2.2 we adopt the relatively standard approach of using the sparsity pattern of $A$ for $M$. One can also try other methods, e.g. [11]. These can significantly reduce the condition number, but again do not improve the scaling in $N$ for the fracture systems studied here. Finally, $d$ does not need to be a constant. As long as $\kappa(MA) = O(1)$, the sparsity pattern can scale with $d \propto N^{\leq 1/7}$ in order to at least recover some quantum advantage. However, for the systems and sparsity patterns considered here, a small increase in $d$ has a corresponding small decrease in $\kappa(MA)$. For example, when applying the technique described in [11] to the “simple” systems, i.e. fractal depth 1, increasing the density of $M$ by a factor of five only decreases $\kappa(MA)$ by a factor of two. It is therefore difficult to imagine a system size or sparsity pattern where such an incremental increase in $d$ could produce sufficiently large reductions in $\kappa(MA)$ as to make the procedure worthwhile.
4.4 Fast Inverse

In [20] they give an algorithm for solving linear systems $Ax = b$ where $A$ can be decomposed as

$$ A = A_{\text{big}} + A_{\text{small}}, \quad (8) $$

where $\|A_{\text{big}}\| \gg \|A_{\text{small}}\|$. They then give a QLS algorithm which uses $A_{\text{big}}^{-1}$ as a preconditioner and solves the system $(I + A_{\text{big}}^{-1}A_{\text{small}})x = A_{\text{big}}^{-1}b$ with scaling bounded by $\|A_{\text{small}}\|$, $\|A_{\text{big}}^{-1}\|$, and $\|A^{-1}\|$.

This technique is dependent on efficient block-encodings of $A_{\text{big}}^{-1}$ and $A_{\text{small}}$. An $(\alpha, m, \epsilon)$-block-encoding of the matrix $A$ is given by the unitary $U_A$:

$$ U_A = \begin{bmatrix} A/\alpha & * \\ * & * \end{bmatrix} \quad (9) $$

where $*$ denotes arbitrary matrix blocks, $\alpha$ is a rescaling constant such that $\|U_A\| = 1$, and the error $\epsilon$ is bounded by $\|A - \alpha(\langle 0^m \rangle \otimes I_n)U_A(\langle |O^m \rangle \otimes I_n)\| \leq \epsilon$. Since the magnitude of $\alpha$ plays a critical role in the scaling of this algorithm, we note that $\|U_A\| = 1$ implies that $\alpha \geq \|A\|$.

In order to use $A_{\text{big}}^{-1}$ as the preconditioner, $A_{\text{big}}$ must be fast-invertible. A matrix $M$ is fast-invertible matrix if one can efficiently prepare a $(\Theta(1), m, \epsilon)$-block-encoding $U'_M$ of $M^{-1}$. This requires access to an oracle for $M^{-1}$, and the number of queries to this oracle in preparing $U'_M$ must be independent of $\kappa(M)$. For example, if $M$ is normal, and the eigenvalue decomposition $M = VDV^\dagger$ gives a $V$ that can be efficiently implemented in a quantum circuit and the elements of $D$ can be accessed through an oracle, then $M$ is fast-invertible.

The fast-inverse QLS algorithm takes as inputs an $(\alpha_s, m_s, 0)$-block-encoding $U_s$ of $A_{\text{small}}$, and an $(\alpha'_b, m'_b, 0)$-block-encoding $U'_b$ of $A_{\text{big}}^{-1}$ implemented via fast-inversion. They then use a modified version of the quantum singular value transformation [4] to construct a block encoding of $(A_{\text{big}} + A_{\text{small}})^{-1}$ with error $\epsilon$ in

$$ O \left( \frac{\alpha'_b \alpha_s}{\bar{\sigma}_\text{min}} \log \left( \frac{\alpha'_b}{\bar{\sigma}_\text{min} \epsilon} \right) \right) \quad (10) $$

applications of $U_s, U'_b$ along with their inverses, controlled versions, and other primitive gates. Here $\bar{\sigma}_\text{min}$ is a lower bound for the smallest singular value of $I + A_{\text{big}}^{-1}A_{\text{small}}$, i.e. the preconditioned system.

This approach has the benefit of providing an upper bound on the condition number of the preconditioned matrix, with the downside of needing a decomposition of $A$ that matches a lengthy list of requirements. For fracture problems, we have the natural decomposition of

$$ A = \Delta + A_F, \quad (11) $$

where the Laplacian $\Delta$ describes the flow in the absence of fractures, and $A_F$ denotes the fracture matrix. Fortunately the discretized Laplacian is fast-invertible, and so we may use it as the preconditioner. However, we have no guarantee that $\|\Delta\| \gg \|A_F\|$, which is required to get good scaling. Still, we
can numerically test the scaling of the algorithm to see how it performs in the absence of performance guarantees.

The parameters contributing to the performance of this algorithm, Eq. 10, are the block-encoding parameters $\alpha'_b$ and $\alpha_s$, along with a lower bound on the smallest singular value of the preconditioned system $MA$, $\tilde{\sigma}_{\text{min}}$. In order to assess the potential usefulness of this algorithm for our application, we will explore the most optimistic values for these parameters. Due to minor technical details, we rescale the entire system by $\|\Delta^{-1}\|$, which gives $A_{\text{big}} = \Delta \|\Delta^{-1}\|$ and $\alpha'_b$ (the block-encoding parameter for $A^{-1}_{\text{big}}$) = $\Theta(1)$. We also have $A_{\text{small}} = (A - \Delta)\|\Delta^{-1}\|$, so $\alpha_s \geq \|A - \Delta\|\|\Delta^{-1}\|$, and $1/\tilde{\sigma}_{\text{min}} \geq \|A^{-1}\Delta\|$. Therefore the overall scaling for the fast-inverse QLS algorithm is bounded below by

$$O\left(\frac{\alpha'_b \alpha_s}{\tilde{\sigma}_{\text{min}}} \log \left(\frac{\alpha'_b}{\tilde{\sigma}_{\text{min}} \epsilon}\right)\right) \geq O \left(\|\Delta^{-1}\|\|A - \Delta\|\|A^{-1}\Delta\| \log \left(\|A^{-1}\Delta\|/\epsilon\right)\right).$$

(12)

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