Frobenius functors and Gorenstein projective precovers

Jiangsheng Hu, Huanhuan Li, Jiafeng Lü and Dongdong Zhang*

Abstract

We establish relations between Gorenstein projective precovers linked by Frobenius functors. This is motivated by an open problem that how to find general classes of rings for which modules have Gorenstein projective precovers. It is shown that if \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a separable Frobenius functor between abelian categories with enough projective objects, then every object in \( \mathcal{C} \) has a Gorenstein projective precover provided that every object in \( \mathcal{D} \) has a Gorenstein projective precover. This result is applied to Frobenius extensions and excellent extensions as well as to the category of unbounded complexes over a ring \( R \).

1. Introduction

Gorenstein homological algebra is the relative version of homological algebra that replaces the classical projective (injective, flat) objects with the Gorenstein projective (Gorenstein injective, Gorenstein flat) ones. A basic problem in Gorenstein homological algebra is to try to get Gorenstein analogues of results in the classical homological algebra. Perhaps the fundamental problem is to find general classes of rings for which modules have Gorenstein projective precovers. So far the existence of Gorenstein projective precovers (of left modules) is known over a left coherent ring for which the projective dimension of any flat right module is finite (see \([8]\)). Examples of such rings include but are not limited to: Gorenstein rings, commutative noetherian rings of finite Krull dimension, as well as two sided noetherian rings \( R \) such that the injective dimension of \( R \) (as a right \( R \)-module) is finite. But for arbitrary rings this is still an open question. Work on this problem can be seen in \([1, 2, 6, 8, 9, 14]\) for instance.

Recall that a pair of functors \( (F, G) \) is said to be a Frobenius pair \([3]\) if \( G \) is at the same time a left and a right adjoint of \( F \). That is a standard name which we use instead of Morita’s original “strongly adjoint pairs” in \([16]\). The functors \( F \) and \( G \) are known as Frobenius functors. A prominent example of Frobenius functors is provided by Frobenius extensions. Recall that an extension \( S \subseteq R \) of rings is called a Frobenius extension if \( R \) is finitely generated and projective as a left \( S \)-module and \( R \cong \text{Hom}_S(\mathcal{S}R, S) \) as an \( R \)-\( S \)-bimodule. The invariant properties of rings under Frobenius functors have been studied by many authors, see \([4, 12, 18, 19, 21, 22, 24, 25]\) for instance.

*Corresponding author.

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In this paper, we shall establish relations between Gorenstein projective precovers (in abelian categories with enough projective objects) linked by Frobenius functors, including Frobenius extensions of rings.

To state our main result more precisely, let us first introduce some definitions.

Assume that \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) are covariant functors between abelian categories. Recall from [11] that if \((F, G)\) is an adjoint pair, then the unit \( \eta : 1_{\mathcal{C}} \to GF \) and the counit \( \varepsilon : FG \to 1_{\mathcal{D}} \) of the adjunction satisfy the identities \( \varepsilon_{F(X)}F(\eta_X) = 1_{F(X)} \) and \( G(\varepsilon_Y)\eta_G(Y) = 1_{G(Y)} \) for all \( X \in \mathcal{C} \) and \( Y \in \mathcal{D} \). By Lemma 2.2, for any adjoint pair \((F, G)\), one has \( F \) is separable if and only if \( \eta : 1_{\mathcal{C}} \to GF \) is a split monomorphism and \( G \) is separable if and only if \( \varepsilon : FG \to 1_{\mathcal{D}} \) is a split epimorphism. Moreover, if \((F, G)\) and \((G, F)\) are adjoint pairs, then we say that \( F \) and \( G \) are Frobenius functors and \((F, G)\) is a Frobenius pair by [3].

Let \( \mathcal{A} \) be an abelian category with enough projective objects. Recall that an object \( M \) in \( \mathcal{A} \) is called Gorenstein projective [7, 23] if there exists an exact complexes of projective objects in \( \mathcal{A} \):

\[
\mathbf{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots ,
\]

with \( M \cong \ker(P^0 \to P^1) \) such that \( \text{Hom}_{\mathcal{A}}(\mathbf{P}, Q) \) is exact for any projective object \( Q \). In what follows, we denote by \( \mathcal{GP}(\mathcal{A}) \) the subcategory of \( \mathcal{A} \) consisting of Gorenstein projective objects.

Let \( \mathcal{X} \) be a class of objects in an abelian category \( \mathcal{A} \). A homomorphism \( \varphi : X \to M \) with \( X \in \mathcal{X} \) is called an \( \mathcal{X} \)-precover of \( M \) [5] if for any homomorphism \( f : X' \to M \) with \( X' \in \mathcal{X} \), there is a homomorphism \( g : X' \to X \) such that \( \varphi g = f \). The class \( \mathcal{X} \) is called precovering in \( \mathcal{A} \) if every object in \( \mathcal{A} \) has an \( \mathcal{X} \)-precover.

Now, our main result can be stated as follows.

**Theorem 1.1.** Let \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) are covariant functors between abelian categories with enough projective objects, and let \((F, G)\) be a Frobenius pair.

1. Assume that \( G \) is a separable Frobenius functor. If \( \mathcal{GP}(\mathcal{C}) \) is precovering in \( \mathcal{C} \), then \( \mathcal{GP}(\mathcal{D}) \) is precovering in \( \mathcal{D} \).
2. Assume that \( F \) is a separable Frobenius functor. If \( \mathcal{GP}(\mathcal{D}) \) is precovering in \( \mathcal{D} \), then \( \mathcal{GP}(\mathcal{C}) \) is precovering in \( \mathcal{C} \).

We will apply Theorem 1.1 to Frobenius extensions and excellent extensions. As a result, we produce some examples of rings such that the class of Gorenstein projective modules is precovering over them (see Example 2.10).

Let \( R \) be a ring and \( A = R[x] / (x^2) \) the quotient of the polynomial ring, where \( x \) is a variable which is supposed to commute with all the elements of \( R \). By [10, Section 3.2], \( A \) may be viewed as a graded ring with a copy of \( R \) (generated by 1) in degree 0 and a copy of \( R \) (generated by \( x \)) in degree \(-1\), and 0 otherwise. One can check that the category graded left \( A \)-modules is isomorphic to the category of unbounded chain complexes of left \( R \)-modules, where the differential \( d \) corresponds to multiplication by \( x \). As a consequence of Theorem 1.1, we have the following corollary, which parallels [6, Proposition 11].

**Corollary 1.2.** Let \( R \) be a ring. If every left \( R \)-module has a Gorenstein projective precover, then every complex of left \( R \)-modules has a Gorenstein projective precover.
The proof of the above results will be carried out in the next section.

2. Proofs of the results

We begin this section with the following definition.

**Definition 2.1.** [17] A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is said to be **separable** if for all objects $M, N$ in $\mathcal{C}$ there are maps $\varphi : \text{Hom}_\mathcal{D}(F(M), F(N)) \to \text{Hom}_\mathcal{C}(M, N)$, satisfying the following conditions:

1. For $\theta \in \text{Hom}_\mathcal{C}(M, N)$ we have $\varphi(F(\theta)) = \theta$.
2. Given $M', N' \in \mathcal{C}$, $\alpha \in \text{Hom}_\mathcal{C}(M, M')$, $\beta \in \text{Hom}_\mathcal{C}(N, N')$, $f \in \text{Hom}_\mathcal{D}(F(M), F(N))$, $g \in \text{Hom}_\mathcal{D}(F(M'), F(N'))$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
F(M) & \xrightarrow{f} & F(N) \\
\downarrow{F(\alpha)} & & \downarrow{F(\beta)} \\
F(M') & \xrightarrow{g} & F(N'),
\end{array}
$$

then the following diagram is also commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi(f)} & N \\
\downarrow{\alpha} & & \downarrow{\beta} \\
M' & \xrightarrow{\varphi(g)} & N'.
\end{array}
$$

The following lemma collects some results on adjoint functors (see [15, Theorem 1, p.89] and [20, Theorem 1.2]):

**Lemma 2.2.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between abelian categories, and let $(F, G)$ be an adjoint pair.

1. $F$ is faithful if and only if $\eta_X : X \to GF(X)$ is a monomorphism for any $X \in \mathcal{C}$.
2. $G$ is faithful if and only if $\varepsilon_Y : FG(Y) \to Y$ is an epimorphism for any $Y \in \mathcal{D}$.
3. $F$ is separable if and only if $\eta : 1_{\mathcal{C}} \to GF$ is a split monomorphism, i.e. there exists a natural transformation $\psi : GF \to 1_{\mathcal{C}}$ such that $\psi \eta = 1$.
4. $G$ is separable if and only if $\varepsilon : FG \to 1_{\mathcal{D}}$ is a split epimorphism, i.e. there exists a natural transformation $\xi : 1_{\mathcal{D}} \to FG$ such that $\varepsilon \xi = 1$.

The following fact collects some results on Frobenius functors (see [3, Section 2]):

**Fact 2.3.** Let $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and $(F, G)$ a Frobenius pair.

1. $F$ and $G$ are exact functors.
2. If $\mathcal{C}$ and $\mathcal{D}$ have projective objects, then both $F$ and $G$ preserve projective objects.
3. $\text{Ext}_\mathcal{C}^i(X, G(Y)) \cong \text{Ext}_\mathcal{D}^i(F(X), Y)$ and $\text{Ext}_\mathcal{C}^i(G(Y), X) \cong \text{Ext}_\mathcal{D}^i(Y, F(X))$ for all $i \geq 0$, $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

In what follows, we always assume $(F, G)$ is a Frobenius pair, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are covariant functors between abelian categories with enough projective objects.
Proposition 2.4. Let $X$ be an object in $\mathcal{C}$ and $Y$ an object in $\mathcal{D}$.

(1) If $X \in \mathcal{GP}(\mathcal{C})$, then $F(X) \in \mathcal{GP}(\mathcal{D})$. The converse is true if $F$ is faithful.

(2) If $Y \in \mathcal{GP}(\mathcal{D})$, then $G(Y) \in \mathcal{GP}(\mathcal{C})$. The converse is true if $G$ is faithful.

Proof. We only need to prove (1), and the proof of (2) is similar. The proof is model that of [12, Theorem 3.3]. Let $X$ be a Gorenstein projective object in $\mathcal{C}$. Then there exists an exact complex of projective objects in $\mathcal{C}$:

$$\mathbf{P} : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

with $X \cong \ker(P^0 \rightarrow P^1)$ such that $\text{Hom}_\mathcal{C}(\mathbf{P}, Q)$ is exact for every projective object $Q$ in $\mathcal{C}$. Applying the functor $F$ to the exact sequence $\mathbf{P}$, we have the following exact sequence of projective objects in $\mathcal{D}$:

$$F(\mathbf{P}) : \cdots \rightarrow F(P_1) \rightarrow F(P_0) \rightarrow F(P^0) \rightarrow F(P^1) \rightarrow \cdots.$$

Let $Q$ be a projective object in $\mathcal{D}$. It follows that $G(Q)$ is projective in $\mathcal{C}$. Hence the complex $\text{Hom}_\mathcal{C}(\mathbf{P}, G(Q))$ is exact, and the complex $\text{Hom}_\mathcal{D}(F(\mathbf{P}), Q)$ is exact by adjoint isomorphism. So $F(X)$ is Gorenstein projective in $\mathcal{D}$.

Conversely, we assume that $F(X)$ is a Gorenstein projective in $\mathcal{D}$ and $F$ is faithful. Let $P$ be a projective object in $\mathcal{C}$. Note that $\text{Ext}^i_\mathcal{C}(X, GF(P)) \cong \text{Ext}^i_\mathcal{D}(F(X), F(P))$ for all $i \geqslant 1$. Note that $F(P)$ is projective in $\mathcal{D}$ by Proposition 2.4. Then $\text{Ext}^i_\mathcal{D}(F(X), F(P)) = 0$. By Lemma 2.2, the counit $GF(P) \rightarrow P$ is an epimorphism and then $P$ is a direct summand of $GF(P)$. Hence $\text{Ext}^i_\mathcal{C}(X, P) = 0$ for all $i \geqslant 1$. It suffices to construct the right part of the complete projective resolution of $X$. It is easy to check that $GF(X)$ is a Gorenstein projective object in $\mathcal{C}$ by a similar proof above, there exists an exact sequence $0 \rightarrow GF(X) \rightarrow P^0 \rightarrow L^1 \rightarrow 0$ in $\mathcal{C}$ with $P^0$ projective and $L^1$ Gorenstein projective. Note that there exists an exact sequence $0 \rightarrow X \xrightarrow{\eta_X} GF(X) \rightarrow K \rightarrow 0$ in $\mathcal{C}$ by Lemma 2.2. Consider the following pushout diagram:
Applying the functor $F$ to the commutative diagram above, we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(X) & \rightarrow & F(GF(X)) & \rightarrow & F(K) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(P^0) & \rightarrow & F(H^1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(L^1) & \rightarrow & F(L^1) & & & & & & & \\
0 & & 0 & & & & & & & \\
\end{array}
\]

Note that $F(\eta_X)$ is a split monomorphism, thus $F(K)$ is Gorenstein projective because the class of Gorenstein projective objects is closed under direct summands by [23, Proposition 4.11], and so is $F(H^1)$ because the class of Gorenstein projective objects is closed under extension by [23, Corollary 4.5]. Hence we have an exact sequence $0 \rightarrow X \rightarrow P^0 \rightarrow H^1 \rightarrow 0$ in $\mathcal{C}$ where $P^0$ projective in $\mathcal{C}$ and $F(H^1)$ is Gorenstein projective in $\mathcal{D}$. By the forgoing proof, we can get that $\text{Ext}^i_{\mathcal{C}}(H^1, Q) = 0$ for all $i \geq 1$ and any projective object $Q$. Proceed in this manner, we have an exact sequence $0 \rightarrow X \rightarrow P^0 \rightarrow \cdots \rightarrow P^1 \rightarrow \cdots$ in $\mathcal{C}$ with each $P^i$ projective, which is $\text{Hom}_{\mathcal{C}}(-, Q)$-exact for all projective objects $Q$. So $X$ is Gorenstein projective. □

Let $X$ be an object in $\mathcal{C}$. The Gorenstein projective dimension, $\text{Gpd}(X)$, of $X$ is defined by declaring that $\text{Gpd}(X) \leq n$ if, and only if there is an exact sequence $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow X \rightarrow 0$ with all $G_i$ Gorenstein projective.

**Corollary 2.5.** If $F$ is faithful, then $\text{Gpd}(X) = \text{Gpd}(F(X))$ for any $X \in \mathcal{C}$.

**Proof.** Let $X$ be an object in $\mathcal{C}$. It is easy to check that $\text{Gpd}(F(X)) \leq \text{Gpd}(X)$, now we will show $\text{Gpd}(X) \leq \text{Gpd}(F(X))$. If $\text{Gpd}(F(X)) = \infty$, the equality is trivial. Now assume that $\text{Gpd}(F(X)) = m < \infty$. Consider the following exact sequence in $\mathcal{C}$:

\[
0 \rightarrow K \rightarrow G_{m-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow X \rightarrow 0,
\]

where $G_i$ is Gorenstein projective in $\mathcal{C}$ for $0 \leq i \leq m - 1$. By Proposition 2.4, we have the following exact sequence in $\mathcal{D}$:

\[
0 \rightarrow F(K) \rightarrow F(G_{m-1}) \rightarrow \cdots \rightarrow F(G_1) \rightarrow F(G_0) \rightarrow F(X) \rightarrow 0,
\]

where $F(G_i)$ is Gorenstein projective for $0 \leq i \leq m - 1$. It follows from $\text{Gpd}(F(X)) = m$ that $F(K)$ is a Gorenstein projective. Thus $K$ is Gorenstein projective in $\mathcal{C}$ by Proposition 2.4, and hence $\text{Gpd}(X) \leq m$. So $\text{Gpd}(X) = \text{Gpd}(F(X))$. This completes the proof. □

**Lemma 2.6.** Let $M$ be an object in $\mathcal{C}$ and $N$ an object in $\mathcal{D}$. 
(1) If $f : X \to M$ is a Gorenstein projective precover of $M$, then $F(f) : F(X) \to F(M)$ is a Gorenstein projective precover of $F(M)$.

(2) If $g : Y \to N$ is a Gorenstein projective precover of $N$, then $G(g) : G(Y) \to G(N)$ is a Gorenstein projective precover of $G(N)$.

**Proof.** We only prove (1), and the proof of (2) is similar. Assume that $f : X \to M$ is a Gorenstein projective precover of $M$. Let $L$ be an object in $\mathcal{GP}(D)$ and $h : L \to F(M)$ a morphism in $D$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_D(L,F(X)) & \xrightarrow{\text{Hom}_D(L,F(f))} & \text{Hom}_D(L,F(M)) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_C(G(L),X) & \xrightarrow{\text{Hom}_C(G(L),f)} & \text{Hom}_C(G(L),M).
\end{array}
$$

Since $L \in \mathcal{GP}(D)$, $G(L) \in \mathcal{GP}(C)$ by Proposition 2.4. Note that $f : X \to M$ is a Gorenstein projective precover. Then $\text{Hom}_C(G(L),f) : \text{Hom}_C(G(L),X) \to \text{Hom}_C(G(L),M)$ is epic. Hence $\text{Hom}_D(L,F(f)) : \text{Hom}_D(L,F(X)) \to \text{Hom}_D(L,F(M))$ is epic. This completes the proof. □

**Lemma 2.7.**

(1) If $\mathcal{GP}(C)$ is precovering in $C$, then $F(\mathcal{GP}(C))$ is precovering in $D$.

(2) If $\mathcal{GP}(D)$ is precovering in $D$, then $G(\mathcal{GP}(D))$ is precovering in $C$.

**Proof.** We will prove (1), and the proof of (2) is similar. Let $N$ be an object in $D$. Since $\mathcal{GP}(C)$ is precovering in $C$ by hypothesis, there exists a Gorenstein projective precover $f : X \to G(N)$ of $G(N)$. Set $\gamma := \varepsilon_N F(f) : F(X) \to N$, where $\varepsilon_N : FG(N) \to N$ is the counit of the adjoint pair $(F,G)$. Next we claim that $\gamma : F(X) \to N$ is a $F(\mathcal{GP}(C))$-precover. Let $L$ be an object in $\mathcal{GP}(C)$ and $h : F(L) \to N$ a morphism in $D$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_D(F(L),F(X)) & \xrightarrow{\text{Hom}_D(F(L),F(f))} & \text{Hom}_D(F(L),FG(N)) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_C(L,GF(X)) & \xrightarrow{\text{Hom}_C(L,GF(f))} & \text{Hom}_C(L,GFG(N)) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_C(L,G(N)) & \xrightarrow{\text{Hom}_C(L,G(\varepsilon_N))} & \text{Hom}_C(L,G(N)).
\end{array}
$$

Note that $G(\varepsilon_N) \circ \eta_{G(N)} = 1_{G(N)}$. Then $\text{Hom}_C(L,G(\varepsilon_N)) : \text{Hom}_C(L,GFG(N)) \to \text{Hom}_C(L,G(N))$ is epic. Hence $\text{Hom}_D(F(L),\varepsilon_N) : \text{Hom}_C(L,GF(N)) \to \text{Hom}_C(F(L),N)$ is epic. Since $f$ is a Gorenstein projective precover of $G(N)$, so is $GF(f) : GF(X) \to GFG(N)$ by Lemma 2.6. Thus $\text{Hom}_C(L,GF(f)) : \text{Hom}_C(L,GF(X)) \to \text{Hom}_C(L,GFG(N))$ is epic, and hence $\text{Hom}_D(F(L),F(f)) : \text{Hom}_D(F(L),F(X)) \to \text{Hom}_D(F(L),FG(N))$ is epic. So $\text{Hom}_D(F(L),\gamma) = \text{Hom}_D(F(L),\varepsilon_N) \circ \text{Hom}_D(F(L),F(f))$ is epic. This completes the proof. □

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We only prove (1), and the proof of (2) is similar. Let $N$ be an object in $D$. Then there exists a $F(\mathcal{GP}(C))$-precover $\alpha : F(X) \to N$ with $X \in \mathcal{GP}(C)$ by Lemma 2.7. We will show $\alpha : F(X) \to N$ is a $\mathcal{GP}(D)$-precover of $N$. Firstly, we have $F(X) \in \mathcal{GP}(D)$ by Proposition 2.4. Now let $g : L \to N$ be a morphism with $L \in \mathcal{GP}(D)$. Since $G$ is a separable...
functor, there exists a morphism \( h : L \to FG(L) \) such that \( \varepsilon_L h = 1_L \). Since \( FG(L) \in F(GP(C)) \) by Proposition 2.4, there exists a morphism \( \beta : FG(L) \to F(X) \) such that \( \alpha \beta = g \varepsilon_L \) by Lemma 2.7. So \( \alpha(\beta h) = (\alpha \beta) h = (g \varepsilon_L) h = g(\varepsilon_L h) = g \). This completes the proof. \( \square \)

Recall from [17, Proposition 1.3] that a Frobenius extension \( S \subseteq R \) is separable if and only if \( F := \mathcal{S}R \otimes_R - : R\mathcal{M} \to \mathcal{S}\mathcal{M} \) is a separable functor, where \( R\mathcal{M} \) is the class of left \( R \)-modules and \( \mathcal{S}\mathcal{M} \) is the class of left \( S \)-modules. As a consequence of Theorem 1.1, we have the following corollary.

**Corollary 2.8.** Assume that \( S \subseteq R \) is a separable Frobenius extension. If \( GP(S) \) is precovering, then \( GP(R) \) is precovering.

It follows from [13, Lemma 4.7] that an excellent extension \( S \subseteq R \) with \( S \) a commutative ring is a separable Frobenius extension. By Corollary 2.8, we have the following corollary.

**Corollary 2.9.** Assume that \( S \subseteq R \) is an excellent extension with \( S \) a commutative ring. If \( GP(S) \) is precovering, then \( GP(R) \) is precovering.

As a consequence of Corollary 2.9 and [13, Example 2.2], we have the following example.

**Example 2.10.** (1) Let \( S \) be a ring and \( G \) a finite group. If \( |G|^{-1} \in S \), then the skew group ring \( R = S \ast G \) is an excellent extension of \( S \) by [13, Example 2.2]. It follows from Corollary 2.9 that \( GP(R) \) is precovering whenever \( S \) is a commutative ring such that \( GP(S) \) is precovering.

(2) Let \( S \) be a finite-dimensional commutative algebra over a field \( K \), and let \( S' \) be a finite separable field extension of \( K \). Then \( R = S \otimes_K S' \) is an excellent extension of \( S \). Hence \( GP(R) \) is precovering by Corollary 2.9.

We end this paper by giving the proof of Corollary 1.2 as follows.

**Proof of Corollary 1.2.** Let \( \mathcal{C} \) be the category of graded left \( A \)-modules and \( \mathcal{D} \) the category of left \( A \)-modules. Assume that \( F : \mathcal{C} \to \mathcal{D} \) is the forgetful functor. By the note above Corollary 4.4 in [3], we get that \( F \) is a separable Frobenius functor. Note that every left \( R \)-module has a Gorenstein projective precover by hypothesis. It is not hard to check that every left \( A \)-module has a Gorenstein projective precover by noting that \( A \) is a graded ring with a copy of \( R \) in degree 0 and a copy of \( R \) in degree \(-1\), and 0 otherwise. So every complex of left \( R \)-modules has a Gorenstein projective precover by Theorem 1.1. \( \square \)

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Jiangsheng Hu
School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China
E-mail: jiangshenghu@jsut.edu.cn

Huanhuan Li
School of Mathematics and Statistics, Xidian University, Xi’an 710071, China
lihuanhuan0416@163.com

Jiafeng Lü
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
jiafenglv@zjnu.edu.cn

Dongdong Zhang
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
E-mail: zdd@zjnu.cn