A risk measurement approach from risk-averse stochastic optimization of score functions

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Abstract

We propose a risk measurement approach for a risk-averse stochastic problem. We provide results that guarantee that our problem has a solution. We characterize and explore the properties of the argmin as a risk measure and the minimum as a deviation measure. We provide a connection between linear regression models and our framework. Based on this conception, we consider conditional risk and provide a connection between the minimum deviation portfolio and linear regression. Moreover, we also link the optimal replication hedging to our framework.

Keywords: Risk management; Uncertainty modeling; risk measures; deviation measures; robust stochastic programming.

1 Introduction

The theoretical discussion of risk measures gained prominence since the seminal work of Artzner et al. (1999), who developed the class of coherent risk measures. From there, other proprieties and classes of risk measures were proposed, including the convex (Föllmer and Schied, 2002; Frittelli and Gianin, 2002), spectral (Acerbi, 2002), and generalized deviation measures (Rockafellar et al., 2006). From that, an entire stream of literature has proposed and discussed distinct features for risk measures, including axiom sets, dual representations, and mathematical properties. For detailed reviews, we recommended the books of Pflug and Römisch (2007), Delbaen (2012), Rüschendorf (2013), and Föllmer and Schied (2016) and the studies of Föllmer and Knispel (2013) and Föllmer and Weber (2015).

Recently, the discussion of statistical properties that a risk measure must respect has also gained space in the literature that discusses characteristics for risk measures. A prominent statistical property

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is elicitability. This property is very useful for risk management because it enables comparing competing forecast models using the scoring rule. Examples of elicitable functionals are quantiles and expectiles, which makes Value at Risk (VaR) and Expectile Value at Risk (EVaR) elicitable risk measures. We recommended Gneiting (2011), Bellini and Bignozzi (2015), Ziegel (2016), Kou and Peng (2016), Fissler and Ziegel (2016), Fissler and Ziegel (2021), and the references therein for more details. A functional $T$ on a vector space of random variables as $X$ is elicitable if exists a scoring function $S : \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$
T(X) = - \arg \min_{y \in \mathbb{R}} E [S(X, y)] .
$$

We present more details in Definition 1 below, and Section 3 describe some examples of $S$.

Inspired on the elicitability reasoning, even without keeping its technical definition, we consider a robust/risk-averse counterpart to the optimization problem as

$$
\inf_{y \in \mathbb{R}} \sup_{Q' \in \mathcal{Q}'} E_Q [S(X, y)] ,
$$

where $\mathcal{Q}'$ is a suitable set of probability measures, which may represent beliefs or scenarios. Regarding the ambiguity set, one can choose $\mathcal{Q}' \subseteq \mathcal{P}$ in an ad hoc sense according to some a priori established risk aversion parameter. Another possibility is to consider those probability measures representing beliefs absolutely continuous inside some distance from a nominal measure $\mathcal{P}$, as in Shapiro (2017). We consider $\mathcal{Q}'$ linked to dual representations of coherent risk measures (sub-linear expectations as in Sun and Ji (2017)). We consider risk measures coherent in the sense of Artzner et al. (1999) because these maps have a dual representation as the supremum of expectations over a closed (in total variation norm) convex set of probability measures. Thus, with coherent risk measures $\rho$ replacing the expectation, we define our risk measurement approach as a risk-averse stochastic problem as

$$
\inf_{y \in \mathbb{R}} \rho (- S(X, y)) .
$$

A possible, but not limited to, interesting direct application of this kind of risk measurement process could seek to minimize capital determination errors to reduce the costs linked to it. As pointed out in Righi et al. (2020), from the regulatory point of view, risk underestimation, and consequently capital determination underestimation, is the main concern. In this case, capital charges are desirable to avoid costs from unexpected and uncovered losses. However, from the perspective of institutions, it is also desirable to reduce the regret costs arising from risk overestimation because the latter reduces profitability.

We provide results that guarantee that our risk measurement approach has a solution. We characterize the argmin as a risk measure per se, and the minimum as a deviation measure in the sense of Rockafellar et al. (2006). We also explore the main proprieties of both functionals. Our proposal is inspired by the study of Righi et al. (2020). The authors propose a risk measurement procedure that represents the capital determination for a financial position that minimizes the expected value of the sum
between costs from risk overestimation and underestimation and considers a supremum of probability measures to the expectation. However, they explore a single score instead general ones. A similar idea has been pursued in Mao and Cai (2018), where the expectation has been replaced by functionals arising from rank-dependent expected utility and cumulative prospect theory. In this way, both studies of can be thought of as special cases in our framework.

The paper of Rockafellar and Uryasev (2013) also relates to risk and deviation measures linked by a common optimization problem, and Bellini et al. (2014) study generalized quantiles as risk measures by minimizing asymmetric loss functions. Unlike we do in this current approach, both mentioned studies do not consider the supremum of probability measures to the expectation. Thus, our approach can be considered robust since it is not sensitive to choosing a specific probability measure representing a particular belief about the world. In this sense, our approach is in concordance with the stream of Shapiro (2017), Bellini et al. (2021), Righi (2018), Righi et al. (2020), for instance. In line with our study, Embrechts et al. (2021) introduce the notions of Bayes pairs and Bayes risk measures as the counterpart of elicitable risk measures as the minimum of the scores. Nonetheless, their minimum scores are also risk measures instead of deviations.

We also make a connection between our framework and linear regression analysis. The most common functional form of regression analysis is linear regression, widely known through the method of ordinary least squares that minimizes the sum of squared differences. Other forms of regression use slightly different scores to estimate parameters, such as the quantile regression, see Koenker and Bassett (1978), Koenker (2005), expectile regression, see Newey and Powell (1987), and extremile regression, see Daouia et al. (2019) and Daouia et al. (2021), for instance. The link between linear regression models and our risk measurement approach allows us to have conditional versions of both risk and deviation. We explore the proprieties of conditional risk and prove that the minimizer is unique. Discussions of conditional versions of risk are not new in the literature. However, the focus has been on score functions related mainly to quantile regression, i.e., VaR regressions. Guillen et al. (2021) point that this approach is extremely useful for identifying covariates that influence the worst-case outcomes. We extend this discussion to different score functions. Wu et al. (2023) explores, as a counterpart to the generalized quantiles studied in Bellini et al. (2014), conditional generalized quantiles. They, contrary to us, do not consider a robust optimization approach.

The concept of deviation is present in finance since Markowitz (1952) with the standard deviation. Such concept is axiomatized and generalized for convex functionals in Rockafellar et al. (2006), Pflug (2006) and Grechuk et al. (2009). The problem of minimizing the deviation of a portfolio and its implications are explored in Rockafellar et al. (2007). Recently, Righi and Ceretta (2016) and Righi and Borenstein (2018) consider both risk and deviation measures. Furthermore, representing the portfolio choice problem in terms of an estimation problem of a linear regression model is well known. Britten-Jones (1999) proposes a regression approach for the tangency portfolio, and Kempf and Memmel (2006) as well as Fan et al. (2012) show that the plug-in estimator for the GMVP (global minimum variance portfolio)
weights can also be obtained by means of linear regression. More recently, Li (2015) provides a regression representation of the mean-variance portfolio. The approach in Frey and Pohlmeier (2016) differs from the regression representation mentioned above by avoiding the choice for a n-th asset \(X_n\) as a dependent variable. We provide a similar connection between minimum deviation problems with linear regression under the same score that generates the deviation. Our results guarantee that our minimum deviation portfolio optimization problem has a solution.

In a complete market model, any derivative is attainable and thus admits a perfect hedge. The cost of replication equals the price of the derivative, which is the expected discounted claim payoff under the unique equivalent martingale measure (Huang and Guo, 2013). However, completeness is only an idealization of a financial market. Relaxing the idealized assumption leads to incomplete market models, where financial products bear an intrinsic risk that cannot be hedged away completely, see Carr et al. (2001), and Balter and Pelsser (2020) for details. For hedging procedures proposed in the literature for expected utility maximization in the form of minimization of a score/loss function, typically quadratic or quantile one, see Bessler et al. (2016), Halkos and Tsirivis (2019) and Barigou et al. (2022), for instance.

We then provide a direct connection between optimal hedging strategies with linear regression under the same score that the hedge is taken. We explore results that guarantee that our problem has a solution.

Regarding structure, the remainder of this paper divides in the following contents: Section 2 describes definitions and results concerning the existence of a solution to our risk measurement approach problem, and explores the properties of our risk and deviation measures. Section 3 exposes, in more detail, examples of possible choices for \(\rho\) and \(S\). Section 4 connects our approach to linear regression models, allowing conditional risk and its properties, besides solving minimum deviation portfolio optimization and optimal replication hedging problems.

## 2 Proposed approach

Consider the real-valued random result \(X\) of any asset \((X \geq 0\) is a gain, \(X < 0\) is a loss) that is defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). All equalities and inequalities are considered almost surely in \(\mathbb{P}\). We define \(X^+ = \max(X,0)\), \(X^- = \max(-X,0)\), and \(1_A\) as the indicator function for an event \(A\). Let \(L^p := L^p(\Omega, \mathcal{F}, \mathbb{Q})\) the space of (equivalent classes of) random variables such that \(\|X\|_p^p = E|X|^p < \infty\) for \(p \in [1, \infty)\) and \(\|X\|_\infty = \text{ess sup}|X| < \infty\) for \(p = \infty\), where \(E\) is the expectation. When not explicit, it means that definitions and claims are valid for any fixed \(L^p\), \(p \in [1, \infty]\). We have that \(L^p_{\mathbb{Q}}\) is its cone of non-negative elements. We denote by \(X_n \to X\) convergence in the \(L^p\) norm, while \(\lim_{n \to \infty} X_n = X\) means \(\mathbb{P}\)-a.s. convergence.

We let \(\mathbb{Q}\) denote the set composed of probability measures \(\mathbb{Q}\) defined on \((\Omega, \mathcal{F})\) that are absolutely continuous with respect to \(\mathbb{P}\), with Radon-Nikodym derivative \(\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\frac{1}{q}\) for \(\frac{1}{p} + \frac{1}{q} = 1\), with the convention \(q = \infty\) when \(p = 1\) and \(q = 1\) when \(p = \infty\). Moreover, \(E_\mathbb{Q}[X] = \int_\Omega X d\mathbb{Q}\), \(F_{X,\mathbb{Q}}(x) = \mathbb{Q}(X \leq x)\) and \(F_{X,\mathbb{Q}}^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_{X,\mathbb{Q}}(x) \geq \alpha\}\) are, respectively, the expected value, the distribution function and
the (left) quantile of $X$ under $Q$. We drop the subscript when it is regarding $P$.

We now formally define the framework we need to build our proposed approach.

**Definition 1.** A map $S : \mathbb{R}^2 \to \mathbb{R}_+$ is called scoring function if the map $\omega \to S(X(\omega), Y(\omega))$ belongs to $L^1$ for any $X, Y \in L^p$, and satisfy the following properties for any $x, y \in \mathbb{R}$:

(i) $S(x, y) \geq 0$ and $S(x, y) = 0$ if and only if $x = y$.

(ii) There is a function $f_S : \mathbb{R} \to \mathbb{R}$ such that $S(x, y) = f_S(x - y)$.

(iii) $y \to S(x, y)$ is convex and continuous.

A function $T : L^p \to \mathbb{R}$ is elicitable if exists a scoring function $S$ such that

$$
T(X) = - \arg \min_{y \in \mathbb{R}} E[S(X, y)], \forall X \in L^p.
$$

**Remark 1.** As a consequence of properties (i) and (iii) we have that $y \to S(x, y)$ is non-decreasing for $y > x$ and non-increasing for $y < x$. Moreover, some more generality can be obtained. In fact, for most of the paper one could relax continuity of $y \to S(x, y)$ to only lower semi-continuity. Further, one can drop the demand for existence of a $f_S$ at the cost of dropping the Translation Invariance/Insensitivity (see below). It is straightforward to verify that there is the preservation of such properties if and only if there exists such a real $f_S$.

**Remark 2.** We would like to highlight that the assumption on the scoring function $S$ implies some properties in the function $f_S$. In particular, we have that $f_S(x) = 0$ if and only if $x = 0$, $f_S(x)$ is convex, continuous, non-decreasing for $x < 0$ and non-increasing for $x > 0$. This also implies that the map $x \mapsto S(x, y)$ has the same properties as $y \mapsto S(x, y)$. Furthermore, note that when the necessary derivatives exist, we have that

$$
\frac{\partial S(x, y)}{\partial x} = \frac{\partial f_S(x - y)}{\partial x} = f_S'(x - y) \quad \text{and} \quad \frac{\partial S(x, y)}{\partial y} = \frac{\partial f_S(x - y)}{\partial y} = -f_S'(x - y).
$$

A robust counterpart to this optimization problem, even without keeping the technical definition of elicitation, involves sets of probability measures obtained from coherent risk measures. Thus, we expose some definitions and results from the risk measures literature we use alongside the paper. We choose to consider only finite maps since it is the kind that fits our proposed approach.

**Definition 2.** A functional $\rho : L^p \to \mathbb{R}$ is a risk measure. Its acceptance set is defined as $A_\rho = \{X \in L^p : \rho(X) \leq 0\}$. $\rho$ may possess the following properties:

(i) **Monotonicity:** if $X \leq Y$, then $\rho(X) \geq \rho(Y)$, $\forall X, Y \in L^p$.

(ii) **Translation Invariance:** $\rho(X + c) = \rho(X) - c$, $\forall X \in L^p$, $\forall c \in \mathbb{R}$.

(iii) **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$, $\forall X, Y \in L^p$, $\forall \lambda \in [0, 1]$.

(iv) **Positive Homogeneity:** $\rho(\lambda X) = \lambda \rho(X)$, $\forall X \in L^p$, $\forall \lambda \geq 0$. 
We have that $\rho$ is called monetary if it fulfills (i) and (ii), convex if it is monetary and respects (iii), and coherent if it is convex and fulfills (iv).

**Theorem 1** (Theorems 2.11 and 3.1 of Kaina and Rüschendorf (2009)). A map $\rho : L^p \to \mathbb{R}$, $p \in [1, \infty)$, is a coherent risk measure if and only if it can be represented as:

$$\rho(X) = \max_{Q \in Q_p} E_Q[-X], \forall X \in L^p,$$

(2)

where $Q_p \subseteq Q$ is non-empty, closed, and convex set called the dual set of $\rho$. Moreover, $\rho$ is lower semi-continuous in the $L^p$ norm and continuous in the bounded $\mathbb{P}$-a.s. convergence (Lebesgue continuous).

Thus, we can have risk measures replacing the expectation under an appropriate choice for the dual set. Hence, we define our risk measurement approach as a risk-averse stochastic problem as

$$\inf_{y \in \mathbb{R}} \rho(-S(X, y)).$$

In order to guarantee risk-averseness of the problem in the sense of worst values for the objective function, we assume that $\mathbb{P} \in Q_\rho$, which implies $\rho(X) \geq E[-X]$ for any $X \in L^p$. This property is known as Loadedness in the literature. This is always the case when $\rho$ is law invariant in the sense that $F_X = F_Y$ implies $\rho(X) = \rho(Y)$, see Follmer and Schied (2016) for details.

We now formally define the functionals from our proposed risk measurement approach. The negative sign is to keep the pattern for losses. Moreover, in Propositions 2 and 3 we prove that our both functionals are in fact finite and, thus, well defined.

**Definition 3.** Let $\rho : L^1 \to \mathbb{R}$ be a coherent risk measure and $S : \mathbb{R}^2 \to \mathbb{R}$ a score function. The risk and deviation measures they generate are functionals $R, D : L^p \to \mathbb{R}$, $p \in [1, \infty)$, respectively, defined as

$$R(X) := R_{\rho,S}(X) = -\min \left\{ \arg \min_{y \in \mathbb{R}} \rho(-S(X, y)) \right\},$$

(3)

and

$$D(X) := D_{\rho,S}(X) = \min_{y \in \mathbb{R}} \rho(-S(X, y)).$$

(4)

Furthermore, for any $X \in L^p$ define the set of minimizers as $B_X := \arg \min_{y \in \mathbb{R}} \rho(-S(X, y))$.

**Remark 3.** In this definition, we consider $L^1$ as the domain for $\rho$ to be a more general and easy notation. Nonetheless, any $L^r$, $r \in [1, \infty)$ could be considered by adjusting the definition of score $S$ to fulfills $S(X, Y) \in L^r$ for any $X, Y \in L^p$. All results in this paper are directly adaptable to the $\rho : L^r \to \mathbb{R}$ if that would be the case. We just do not consider $L^\infty$ for the domain of $\rho$ since we want the supremum in its dual representation to be attained. In $L^\infty$, this would be the case under further continuity properties. See Follmer and Schied (2016) for details. We do not want to consider parsimony. Similar reasoning applies to both $R$ and $D$. 

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Remark 4. The task of assessing the performance of financial investments is central, with indexes such as the Sharpe ratio used to assess the trade-off between risk and return. In the last decade, performance has been analyzed through acceptability indexes since the seminal paper of Cherny and Madan (2009), which is extended in Gianin and Sgarra (2013). These authors discuss the properties such functionals must fulfill. Under our framework, we can have a reward to deviation ratio for acceptability as a map \( RD: L^p \to [0, \infty] \) defined as

\[
RD(X) = \begin{cases} 
-\frac{R(Y)}{D(Y)} & \text{if } R(X) < 0 \text{ and } D(X) > 0, \\
\infty & \text{if } R(X) \leq 0 \text{ and } D(X) = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Other adaptations and properties of this structure are discussed in and Righi (2021).

We now expose a formal result that guarantees our minimization problems have a solution.

Proposition 1. Let \( X \in L^p \) and \( B_X \) defined as in Definition 3. Then:

(i) \( B_X \) is a closed interval.

(ii) \( y \in B_X \) if and only if \( y \) satisfies the first order condition given by

\[
E_{Q^*} \left[ \frac{\partial^- S(X, x)}{\partial x} (y) \right] \leq 0 \leq E_{Q^*} \left[ \frac{\partial^+ S(X, x)}{\partial x} (y) \right],
\]

where \( Q^* = \arg \max_{Q \in Q} \rho(\rho(-S(X, y))) = E_{Q^*}[S(X, y)]. \)

(iii) if \( y \to \rho(-S(X, y)) \) is, for any \( X \in L^p \), differentiable with strictly increasing derivative, then \( B_X \) is a singleton.

Proof. For (i), fix \( X \in L^p \) and let \( f : \mathbb{R} \to \mathbb{R} \) be defined as \( f(x) := \rho(-S(X, x)) \). Clearly, \( f \) is finite, convex, and, hence, a continuous function. Note that \( f \) is proper and level bounded. Thus, \( \inf_{x \in \mathbb{R}} f(x) \) is finite and the set \( \arg \min_{x \in \mathbb{R}} f(x) \) is non-empty and compact. Moreover, since \( f \) is convex, \( B_X \) is an interval.

Regarding (ii), since \( f \) is convex, we have that \( y \in \mathbb{R} \) is a minimizer if and only if

\[
0 \in \left[ \frac{\partial^- f(x)}{\partial x} (y), \frac{\partial^+ f(x)}{\partial x} (y) \right].
\]

Dominated convergence yields

\[
\frac{\partial^- f(x)}{\partial x} (y) = E_{Q^*} \left[ \frac{\partial^- S(X, x)}{\partial x} (y) \right] \quad \text{and} \quad \frac{\partial^+ f(x)}{\partial x} (y) = E_{Q^*} \left[ \frac{\partial^+ S(X, x)}{\partial x} (y) \right],
\]

where \( Q^* \in Q \) such that \( \rho(-S(X, y)) = E_{Q^*}[S(X, y)]. \).
For (iii) the f.o.c. becomes

\[ E_Q^* \left[ \frac{\partial S(X, x)}{\partial x}(y) \right] = \frac{\partial \rho(-S(X, x))}{\partial x}(X, y) = 0. \]

Then, the strictly increasing behavior assures the minimizer is unique.

We now explore the main properties of our risk and deviation measures.

**Proposition 2.** Let \( R \) and \( B_X \) be as in Definition 3. Then:

(i) \( R \) is monetary and \( A_R = \{ X \in L^p : y \notin B_X \forall \ y < 0 \} \).

(ii) if \( y \to \rho(-S(X, y)) \) is, for any \( X \in L^p \), differentiable with strictly increasing derivative and \( \frac{\partial f_S(x)}{\partial x} \) is convex, then \( R \) fulfills Convexity. In this case, \( R \) is a lower semi-continuous in the \( L^p \) norm and continuous in the bounded \( \mathcal{P} \)-a.s. convergence (Lebesgue continuous).

(iii) if \( f_S \) is positive homogeneous, then \( R \) fulfills Positive Homogeneity.

(iv) \( R(X) \in [\text{ess inf } X, \text{ess sup } X] \).

**Proof.** Regarding (i), Translation Invariance is straightforward since \( S(x, y) = f_S(x-y) \) with \( f_S(0) = 0 \).

For Monotonicity, let \( g, h : L^p \times \mathbb{R} \to \mathbb{R} \) be as

\[ g(X, x) = \frac{\partial^- \rho(-S(X, y))}{\partial y}(X, x), \quad \text{and} \quad h(X, x) = \frac{\partial^+ \rho(-S(X, y))}{\partial y}(X, x). \]

Since \( x \mapsto \rho(-S(X, x)) \) is a convex real function for all \( X \in L^p \), the left and right derivatives above are well defined. Furthermore, \( g \) and \( h \) are non-decreasing in the second argument. Additionally, note that

\[ g(X, x) = E_Q^* \left[ \frac{\partial^- S(X, y)}{\partial y}(X, x) \right]. \]

Hence, we have that \( X(\omega) \mapsto S(X(\omega), x) \) is also a convex real function. Therefore, \( X \mapsto g(X, x) \) is non-increasing. Similarly for \( h \). Now, let \( X, Y \in L^p \) with \( X \leq Y \). Then \( h(Y, x) \leq h(X, x) \) for any \( x \in \mathbb{R} \).

Furthermore, as \( x \mapsto g(X, x) \) is non-decreasing and \( g \leq h \), the condition \( g(X, x) \leq 0 \) in the following is non-binding in the sense that

\[ \inf \{ x \in \mathbb{R} : g(X, x) \leq 0, h(X, x) \geq 0 \} = \inf \{ x \in \mathbb{R} : h(X, x) \geq 0 \}. \]

Then, we get from the first order condition of Proposition 1 that

\[ -R(X) = \min B_X = \inf \{ x \in \mathbb{R} : h(X, x) \geq 0 \}, \]
\[ -R(Y) = \min B_Y = \inf \{ x \in \mathbb{R} : h(Y, x) \geq 0 \}. \]
Note that such expressions are well defined because, from Proposition 1, the argmin set is a closed interval.
We then must have \(-R(X) \leq -R(Y)\) since \(\{x \in \mathbb{R} : h(Y, x) \geq 0\} \subseteq \{x \in \mathbb{R} : h(X, x) \geq 0\}\). By multiplying both sides by \(-1\) we get the claim. Moreover, we then have that

\[
A_R = \{X \in L^p : \min B_X \geq 0\} = \{X \in L^p : y \in B_X \Rightarrow y \geq 0\} = \{X \in L^p : y \notin B_X, \forall y < 0\}.
\]

Concerning (ii), let \(f'_S\) be convex. The f.o.c. becomes

\[
E_{\mathbb{Q}^*}[f'_S(X - x)] = \frac{\partial p(-f_S(X - x))}{\partial x}(X - x) = 0.
\]
Let then \(g: L^p \times \mathbb{R} \rightarrow \mathbb{R}\) be as

\[
g(X, x) = \frac{\partial p(-f_S(X - y))}{\partial y}(x),
\]
which is convex in its domain and non-increasing in \(x\) for any \(X \in L^p\). Let \(\lambda \in [0, 1]\) and \(X, Y \in L^p\). Then we have

\[
g(\lambda X + (1 - \lambda)Y, \lambda R(X) + (1 - \lambda)R(Y)) \leq \lambda g(X, R(X)) + (1 - \lambda)g(Y, R(Y)) = 0.
\]

Furthermore, \(0 = g(\lambda X + (1 - \lambda)Y, R(\lambda X + (1 - \lambda)Y))\). This yields

\[
g(\lambda X + (1 - \lambda)Y, \lambda R(X) + (1 - \lambda)R(Y)) \leq g(\lambda X + (1 - \lambda)Y, R(\lambda X + (1 - \lambda)Y)).
\]

Thus, due to its non-increasing behavior in the second argument, we obtain \(R(\lambda X + (1 - \lambda)Y) \leq \lambda R(X) + (1 - \lambda)R(Y))\). In this case, \(R\) is a convex risk measure. The continuity properties are then directly obtained from Theorem 1, in fact, from Theorems 2.11, and 3.1 of Kaina and Rüschendorf (2009).

Regarding (iii), the result follows immediately since for any \(X \in L^p\), any \(y \in \mathbb{R}\) and \(\lambda \geq 0\)

\[
R(\lambda X) = -\min \left\{ \arg \min_{y \in \mathbb{R}} \rho \left(-f_S(\lambda X - y)\right) \right\}
\]

\[
= -\min \left\{ \arg \min_{y \in \mathbb{R}} \lambda \rho \left(-f_S(\lambda X - y)\right) \right\}
\]

\[
= -\lambda \min \left\{ \arg \min_{y \in \mathbb{R}} \rho \left(-f_S(\lambda X - y)\right) \right\} = \lambda R(X).
\]

For (iv), for \(X \in L^\infty\) note that \(f_S(X - y) \geq f_S(X - \text{ess inf } X)\) for \(y < \text{ess inf } X\), and \(f_S(X - y) \geq f_S(X - \text{ess sup } X)\) for \(y \geq \text{ess sup } X\). Additionally, when \(\text{ess sup } X = \infty\) or \(\text{ess inf } X = -\infty\), in other words, when \(X \in L^p \setminus L^\infty\), the condition becomes trivial as \(R(X)\) is finite. Thus, the argmin must be in the interval.

\[
\square
\]

Remark 5. Under the conditions of the items (ii) and (iii), we have by Theorem 1 the following dual
representation:

\[ R(X) = \max_{Q \in \mathcal{Q}_R} \mathbb{E}_Q[-X], \ \forall \ X \in L^p, \]

where

\[ \mathcal{Q}_R = \{ Q \in \mathcal{Q} : \mathbb{E}_Q[-X] \leq R(X), \ \forall \ X \in L^p \} \]

\[ = \{ Q \in \mathcal{Q} : \mathbb{E}_Q[X] \geq \min B_X, \ \forall \ X \in L^p \} \]

\[ = \bigcap_{X \in L^p} \{ Q \in \mathcal{Q} : \mathbb{E}_Q[X] \geq \min B_X \}. \]

We now characterize the minimum \( D_{\rho,S} \) as a deviation measure. In this sense, we first formally define deviation measures.

**Definition 4.** A functional \( D : L^p \to \mathbb{R}_+ \) is a deviation measure. It may fulfill the following properties:

(i) Non-Negativity: \( D(X) = 0 \) for \( X \in \mathbb{R} \) and \( D(X) > 0 \) for \( X \in L^p \setminus \mathbb{R} \);

(ii) Translation Insensitivity: \( D(X + c) = D(X) \), \( \forall X \in L^p \), \( \forall c \in \mathbb{R} \);

(iii) Convexity: \( D(\lambda X + (1 - \lambda)Y) \leq \lambda D(X) + (1 - \lambda)D(Y) \), \( \forall X,Y \in L^p \), \( \forall \lambda \in [0,1] \);

(iv) Positive Homogeneity: \( D(\lambda X) = \lambda D(X) \), \( \forall X \in L^p \), \( \forall \lambda \geq 0 \);

A deviation measure \( D \) is called convex if it fulfills (i), (ii), and (iii); generalized (also called coherent) if it is convex and fulfills (iv).

**Proposition 3.** Let \( D \) be defined as in Definition 3. Then it has the following properties:

(i) \( D \) is a convex deviation. Moreover, \( D(X) = \sup_{Q \in \mathcal{Q}_p} \min_{y \in \mathbb{R}} \mathbb{E}_Q[S(X,y)] \), \( \forall X \in L^p \).

(ii) if \( f_S \) is positive homogeneous, then \( D \) fulfills Positive Homogeneity.

(iii) if \( f_S \) is sub-additive and \( f_S(X) \leq \|X\|_p \) \( \forall X \in L^p \), then \( D \) lower semi-continuous in the \( L^p \) norm.

(iv) \( D \) is continuous in the bounded \( \mathbb{P}-a.s. \) convergence (Lebesgue continuous).

(v) if \( \rho_1 \geq \rho_2 \) \( (S_1 \geq S_2) \), then \( D_{\rho_1,S} \geq D_{\rho_2,S} \) \( (D_{\rho,S_1} \geq D_{\rho,S_2}) \).

**Proof.** Regarding (i), Translation insensitivity is direct from \( D(X) = \rho(-f_S(X + R(X))) \). For Non-negativity, since \( R(c) = -c, \ \forall c \in \mathbb{R} \), we have that \( D(c) = 0 \). If \( X \) is not a constant, with abuse of notation, we have that \( \mathbb{P}(X \neq -R(X)) > 0 \). We then get that \( \mathbb{P}(S(X,R(X)) > 0) > 0 \), which, together to \( S(X,R(X)) \geq 0 \) guarantees that \( \mathbb{E}[S(X,R(X))] > 0 \). Hence, \( D(X) = \rho(-S(X,R(X))) \geq \mathbb{E}[S(X,R(X))] > 0 \). For convexity, remember that \( f_S \) is convex. Then, consider any pair \( X,Y \in L^p \) and
any \( \lambda \in [0, 1] \). We then obtain that

\[
D(\lambda X + (1 - \lambda)Y) = \min_{\lambda x_1 + (1 - \lambda)x_2 \in \mathbb{R}} \rho(-f_S(\lambda X - \lambda R(X) + (1 - \lambda)Y - (1 - \lambda)R(Y))) \\
\leq \min_{x_1, x_2 \in \mathbb{R}} \{ \lambda \rho(-f_S(X - R(X))) + (1 - \lambda)\rho(-f_S(Y - R(Y))) \} \\
= \lambda \min_{x_1 \in \mathbb{R}} \rho(-f_S(X - R(X))) + (1 - \lambda) \min_{x_2 \in \mathbb{R}} \rho(-f_S(Y - R(Y))) \\
= \lambda D(X) + (1 - \lambda) D(Y).
\]

The representation result follows from the Sion’s minimax theorem, see Sion (1958), because the map \((x, Q) \rightarrow E_Q[S(X, x)]\) has the needed continuity and quasi-convex properties, \(Q_p\) is convex. The optimization over \(x \in \mathbb{R}\) can be done in the compact interval \(B_X\).

Positive Homogeneity in (ii) is straightforwardly obtained when \(f_S\) is positive homogeneous.

Regarding (iii), let \(X_n \rightarrow X\). Since \(D(X) \leq \rho(-S(X, y))\), \(\forall y \in \mathbb{R}\), we thus get that \(D(X) \leq \liminf_{n \rightarrow \infty} \rho(-f_S(X + R(X_n)))\). We then have that

\[
D(X) \leq \liminf_{n \rightarrow \infty} \rho(-f_S(X + R(X_n))) \\
= \liminf_{n \rightarrow \infty} \rho(-f_S(X + X_n - X_n + R(X_n))) \\
\leq \liminf_{n \rightarrow \infty} \{ \rho(-f_S(X - X_n)) + \rho(-f_S(X_n + R(X_n))) \} \\
\leq \liminf_{n \rightarrow \infty} \{ \|X - X_n\|_p + \rho(-f_S(X_n + R(X_n))) \} \\
= \liminf_{n \rightarrow \infty} \|X - X_n\|_p + \liminf_{n \rightarrow \infty} \rho(-f_S(X_n + R(X_n))) = \liminf_{n \rightarrow \infty} D(X_n).
\]

For (iv), for any \(x \in \mathbb{R}\) and any \(Q \in \mathcal{Q}_p\) by Dominated convergence we have that \(\lim_{n \rightarrow \infty} X_n = X\), \(\{X_n\} \subset L^\infty\) bounded implies, for any bounded sequence \(\{x_n\} \subset \mathbb{R}\) such that \(\lim_{n \rightarrow \infty} x_n = x\), in

\[
E_Q[S(X, x)] = \lim_{n \rightarrow \infty} E_Q[S(X_n, x_n)].
\]

Let \(\{x^{*}_{n, Q}\}\) be a sequence where each member is from the argmin set, i.e., \(x^{*}_{Q, n} \in \arg\min_{y \in \mathbb{R}} E_Q[S(X_n, y)]\). Since \(\{X_n\}\) is bounded, \(\{x^{*}_{n, Q}\}\) also is bounded for any \(Q \in \mathcal{Q}_p\). By the Bolzano-Weierstrass Theorem, we have, by taking a subsequence if needed, that \(x^{*}_{Q, n} = \lim_{n \rightarrow \infty} x^{*}_{Q, n}\) is well defined and finite. We then get that

\[
D(X) \leq \sup_{Q \in \mathcal{Q}_p} E_Q[S(X_n, x^{*}_{Q, n})] \\
= \sup_{Q \in \mathcal{Q}_p} \lim_{n \rightarrow \infty} E_Q[S(X_n, x^{*}_{Q, n})] \\
\leq \liminf_{n \rightarrow \infty} \sup_{Q \in \mathcal{Q}_p} E_Q[S(X_n, x^{*}_{Q, n})] = \liminf_{n \rightarrow \infty} D(X_n).
\]

Furthermore, if \(\{X_n\}\) is bounded, then also is \(\{S(X_n, x)\}\) for any \(x \in \mathbb{R}\) since we have \(S(X_n, x) \leq \max\{S(M, x), S(-M, x)\}\), where \(M\) is the uniform bound. By continuity of \(S\) and Lebesgue continuity of
\[ D(X) = \min_{x \in \mathbb{R}} \lim_{n \to \infty} \rho(-S(X_n, x)) \geq \limsup_{n \to \infty} \min_{x \in \mathbb{R}} \rho(-S(X_n, x)) = \limsup_{n \to \infty} D(X_n). \]

Hence \( D(X) = \lim_{n \to \infty} D(X_n) \).

Finally, (v) is trivial from the Monotonicity of \( \rho, \rho_1 \) and \( \rho_2 \).

**Remark 6.** Under the conditions of item (iii), we have by Theorem 1 of Rockafellar et al. (2006) and The Main Theorem of Pflug (2006), the following dual representation:

\[ D(X) = \sup_{Z \in \mathcal{Z}_Q^\rho} E[Z], \forall X \in L^p, \]

where

\[ \mathcal{Z}_Q^\rho = \{ Z \in L^q : E[Z] = 0, E[XZ] \leq D(X), \forall X \in L^p \} \]

\[ = \{ Z \in L^q : E[Z] = 0, \exists Q \in \mathcal{Q}_\rho \text{ s.t. } E[XZ] \leq E_Q(S(X, x)) \forall x \in \mathbb{R}, \forall X \in L^p \} \]

\[ = \text{clconv}( \bigcup_{Q \in \mathcal{Q}_\rho} \{ Z \in L^q : E[Z] = 0, E[XZ] \leq E_Q[S(X, x)] \forall x \in \mathbb{R}, \forall X \in L^p \} ), \]

where \( \text{clconv} \) means the closed convex hull.

**Remark 7.** Recently, Castagnoli et al. (2021) proposed the class of star-shaped risk measures, which are characterized by the star-shaped property of the generated acceptance set. The reasoning for star-shapedness as a sensible axiomatic requirement is that if a position is acceptable, any scaled reduction is also possible. This class is in the literature in Liebrich (2021), Moresco and Righi (2022), Herdegen and Khan (2022) and Righi (2021). This property for some functional is defined as \( T(\lambda X) \leq \lambda T(X) \) for \( \lambda \in [0, 1] \).

This property is implied by convexity under \( T(0) \leq 0 \). In our framework, \( R \) fulfills this property when we replace convexity of \( \frac{\partial f_S(x)}{\partial x} \) by star-shapedness and \( \frac{\partial f_S(x)}{\partial x}(0) \leq 0 \) in item (ii) of Proposition 2. For \( D \), this property is automatically obtained since it is a convex deviation measure.

### 3 Examples

In this section, we present a description of possible, but not all, choices for \( \rho \) and \( S \). The examples described for both quantities can be considered in the practical use of the proposed approach.

#### 3.1 Risk measures

In this subsection, we present examples of functionals that can be considered possible choices for \( \rho \). However, it is noteworthy that the choices of \( \rho \) are not limited to the risk measures presented.

**Example 1.** (Expected Loss). Expected Loss (EL) is the most parsimonious coherent risk measure, and
it indicates the expected value (mean) of a loss. Thus, EL is a functional $EL : L^1 \to \mathbb{R}$ defined as

$$EL(X) = E[-X].$$

For this measure, the dual set is a singleton $Q_{EL} = \{\mathbb{P}\}$, that is, it only considers the basic belief. Henceforth, we will omit the subscript $E$ in $R_{E,S}$ and $D_{E,S}$ whenever the risk measure is the expected loss.

**Example 2.** (Mean plus Semi-Deviation). The Mean plus Semi-Deviation (MSD) is a functional $MSD : L^2 \to \mathbb{R}$ defined by

$$MSD^\beta(X) = -E[X] + \beta \sqrt{E[(X - E[X])^2]},$$

where $\beta \in [0, 1]$. MSD penalizes the EL by the semi-deviation. The proportion of deviation that has to be considered is given by $\beta$. This measure is studied in detail by Ogryczak and Ruszczyński (1999) and Fischer (2003), and it is a well known law invariant coherent risk measure, which belongs to loss-deviation measures discussed by Righi (2019). The advantages of MSD are its simplicity and financial meaning. The dual set of this measure can be represented by

$$Q_{MSD^\beta} = \left\{ Q \in Q : \frac{dQ}{d\mathbb{P}} = 1 + \beta(V - E[V]), V \geq 0, E[|V|^2] = 1 \right\}.$$

Despite not being finite for any $X \in L^1$, it is readily useful in our approach if we consider $L^2$ as the domain for $\rho$.

**Example 3.** (Expected Shortfall). Expected Shortfall (ES) is the canonical example of a coherent risk measure, being the basis of many representation theorems in this field. Nowadays, it is recommended, together with Value at Risk (VaR), by the Basel Committee as a functional basis for quantifying market risk. The ES is a functional $ES : L^1 \to \mathbb{R}$ defined as

$$ES^\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR^s(X)ds,$$

where $\alpha \in (0, 1)$ is the significance level, and $VaR^\alpha(X) = -F_X^{-1}(\alpha)$, i.e., the maximum expected loss for a given period and significance level. Its acceptance set is $A_{VaR^\alpha} = \{X \in L^1 : \mathbb{P}(X < 0) \leq \alpha\}$. ES represents the expected value of a loss, given it is beyond the $\alpha$-quantile of interest, i.e., $VaR$. We have

$$A_{ES^\alpha} = \left\{ X \in L^1 : \int_0^\alpha VaR^s(X)ds \leq 0 \right\}.$$

For ES, the dual set is

$$Q_{ES^\alpha} = \left\{ Q \in Q : \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\alpha} \right\}.$$

**Example 4.** (Expectile Value at Risk). Expectile Value at Risk (EVaR) links to the concept of an
expectile, which is a generalization of the quantile function used for VaR estimation. EVaR is a functional

\[ EVaR^\alpha(X) = -\arg\min_{x \in \mathbb{R}} E[\alpha((X-x)^+] + (1-\alpha)((X-x)^-)]^2]. \]

In accordance with Bellini et al. (2014), the EVaR is a law invariant coherent risk measure for \( \alpha \leq 0.5 \). In addition, this measure is the only example of elicitable coherent risk measure beyond EL. Bellini and Di Bernardino (2017) points out that according to EVaR, the position is acceptable when the ratio between the expected value of the gain and the loss is sufficiently high. In this case, we have

\[ A_{EVaR^\alpha} = \left\{ X \in L^1 : \frac{E[X^+]}{E[X^-]} \geq \frac{1-\alpha}{\alpha} \right\}. \]

The dual set of EVaR can be given by

\[ Q_{EVaR^\alpha} = \left\{ Q \in Q : \exists a > 0, a \leq \frac{dQ}{d\mathbb{P}} \leq a \frac{1-\alpha}{\alpha} \right\}. \]

Example 5. (Maximum Loss). Maximum Loss (ML) is the most extreme coherent risk measure. It is a functional \( ML : L^\infty \to \mathbb{R} \) defined as

\[ ML(X) = -\essinf X. \]

ML leads to more protective situations since \( ML(Y) \geq \rho(Y) \) for any coherent risk measure \( \rho \). For this measure, the dual set is given by \( Q_{ML} = Q \), i.e., all beliefs are considered. This measure does not directly fit into our framework since the supremum in its dual representation is not necessarily attained because it does not has finiteness assured in any \( L^p, p \in [1, \infty) \).

3.2 Score functions

In this subsection, we present some examples of \( S \). We describe possible but not limited choices for \( S \). We also commented on some functions that do not fit our approach to avoid leaving out important scores.

Example 6 (Squared Error). The squared error is one of the most common score functions. It is tied to the standard tools of least-squares regression. It is well known that its minimum is the variance, and its minimizer is the expectation. In our setup, we have that:

\[ S_{EL}(x, y) = (x - y)^2, \quad f_{S_{EL}}(x) = x^2; \]

\[ R_{E,S_{EL}}(X) = -\min\{\arg\min_{y \in \mathbb{R}} E[(X-y)^2]\} = E[-X]; \]

\[ D_{E,S_{EL}}(X) = \min_{y \in \mathbb{R}} E[(X-y)^2] = \sigma^2(X). \]

It is clear that \( S_{EL} \) is a scoring function when \( L^p \subseteq L^2 \). Furthermore, \( y \mapsto S_{EL}(x, y) \) is, for any \( x \in \mathbb{R}, \)

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differentiable with strictly increasing derivative and \( \frac{\partial f_{SEL}(x)}{\partial x} \) is convex. However, \( S_{EL} \) is not positive homogeneous, while the resulting \( R_{E,SEL} \) has this property, the deviation \( D_{E,SEL} \) does not. This also yields that \( R_{\rho,SEL} \) and \( D_{\rho,SEL} \) are convex risk/deviation measures for any coherent \( \rho \).

**Example 7 (Value at Risk).** Value at Risk is a well known monetary risk measure in academia and industry. This measure is elicitable. As seen in example 3, VaR is defined as the lower quantile. Its connection to quantile regression is self-evident.

\[
\begin{align*}
S_{VaR}(x, y) &= \alpha (x - y) + (1 - \alpha) (x - y)^-, \\
R_{VaR}(X) &= \min \{ \arg \min_{y \in \mathbb{R}} E[\alpha (X - y)^+ + (1 - \alpha) (X - y)^+] \} = VaR^\alpha(X), \\
D_{VaR}(X) &= \min_{y \in \mathbb{R}} \{ E[\alpha (X - y)^+ + (1 - \alpha) (X - y)^-] \} = ESD^\alpha(X),
\end{align*}
\]

where \( ESD^\alpha \) is the expected shortfall deviation, a generalized deviation measure based on the expected shortfall, defined as:

\[ ESD^\alpha(X) = ES^\alpha(X - E[X]). \]

\( S_{VaR} \) is a scoring function when \( L^p \subseteq L^1 \). Furthermore, \( y \mapsto S_{VaR}(x, y) \) is not, for any \( x \in \mathbb{R} \) differentiable with strictly increasing derivative for all \( y \), in particular, it is not differentiable at \( y = x \). This violates the assumption in item (iii) of proposition 1, which would yield that the set \( B_X \) is a singleton; indeed, it is well known that quantiles are intervals. It also prevents us from applying an item (ii) of Proposition 2 which would yield convexity, and again, it is well known that VaR is not convex. However, to guarantee that \( D_{VaR} \) is convex and positive homogeneous, it is enough for \( f_{VaR} \) to be convex and positive homogeneous, which is the case. For different choices of \( \rho \), our results guarantee that \( R_{\rho, VaR} \) is a positive homogeneous monetary risk measure; and \( D_{\rho, VaR} \) is a generalized deviation measure.

\[
\begin{align*}
R_{\rho,VaR}(X) &= \min \{ \arg \min_{y \in \mathbb{R}} \rho(-\alpha (X - y)^+ - (1 - \alpha)(X - y)^-) \}, \\
D_{\rho, VaR}(X) &= \min_{y \in \mathbb{R}} \{ \rho(-\alpha (X - y)^+ - (1 - \alpha)(X - y)^-) \}.
\end{align*}
\]

**Example 8 (Absolute Error).** The Median is a special case of the VaR with \( \alpha = 0.5 \). In such a case, the VaR scoring function degenerates to the Absolute Error. The Median and the Absolute Error are widely used in evaluating point forecasts. See Gneiting (2011).

\[
\begin{align*}
AE(x, y) &= |x - y|, \\
f_{AE}(x) &= |x|, \\
R_{AE}(X) &= VaR^{\frac{1}{2}}(X) = \text{median}(X), \\
D_{AE}(X) &= ESD^{\frac{1}{2}}(X).
\end{align*}
\]

Of course, the same characteristics and issues of the VaR scoring function are carried over the Absolute Error. Keeping the Absolute Error as the scoring function, we can change the risk measure inside the
argmin from the expected loss to the Maximum Loss, defined as $ML(X) = \text{ess sup}(-X)$. The ML is only well defined in $L^\infty$ and thus not directly fitted into our approach. Nonetheless, in this case, we could define

$$R_{ML,AE}(X) = -\min\{\arg\min_{y\in\mathbb{R}} \text{ess sup} |X - y|\} = \frac{1}{2} (\text{ess sup} X + \text{ess inf} X),$$

$$D_{ML,AE}(X) = -\min\{\arg\min_{y\in\mathbb{R}} \text{ess sup} |X - y|\} = \frac{1}{2} (\text{ess sup} X - \text{ess inf} X).$$

We have that $R_{ML,AE}(X)$ is the center of the range of $X$. It is also worth noting that $R_{ML,AE} = R_{ML,SE}$. Additionally, $D_{ML,AE}(X)$ is the full range of $X$ and is a generalized deviation measure.

**Example 9** (Minimum and Maximum Loss, a non-example). The Minimum Loss (MinL) is a very forgiven coherent risk measure, it is defined as $MinL(X) = \text{ess inf} -X = -\text{ess sup} X$, it can be seen as a VaR with $\alpha = 1$, note that in this case, while the left quantile is well defined (for $X \in L^\infty$) the right quantile assumes $\infty$.

$$S_{MinL}(x, y) = (x - y)^+, f_{S_{MinL}}(x) = x^+,$$

$$R_{S_{MinL}}(X) = MinL(X),$$

$$D_{S_{MinL}}(X) = 0.$$  

This example is a clear warning to the importance of using a scoring function and not any seemingly fine function, note that $D_{S_{MinL}}$ is identically 0 that is because $S_{MinL}$ is not a scoring function, it does not fulfill requirement (i) of Definition 1 as $S_{MinL}(x, y) = 0$ for all $y \geq x$. It is relevant to highlight that under $S_{MinL}$, $B_X = [\sup X, \infty)$. If we were to allow our scoring function to assume $\infty$ and not be continuous, we could use the following:

$$S_{MinL}(x, y) = \begin{cases} y - x & \text{if } x \leq y, \\ \infty & \text{if } x > y, \end{cases}$$

$$f_{S_{MinL}}(x) = \begin{cases} -x & \text{if } x \leq 0, \\ \infty & \text{if } x > 0, \end{cases}$$

$$R_{S_{MinL}}(X) = MinL(X),$$

$$D_{S_{MinL}}(X) = \sup X - E[X].$$
We can similarly obtain the Maximum Loss.

$$S_{ML}(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ \infty & \text{if } x < y. \end{cases}$$

$$f_{SML}(x) = \begin{cases} x & \text{if } x \geq 0, \\ \infty & \text{if } x < 0. \end{cases}$$

$$R_{SML}(X) = ML(X),$$

$$D_{SML}(X) = E[X] - \inf X.$$
aversion of the user through the exponential utility function. It is a prime example of a convex risk measure that is not coherent. This measure is the map $ENT^\gamma : L^1 \rightarrow \mathbb{R}$ defined as

$$ENT^\gamma (X) = \frac{1}{\gamma} \log E[e^{-\gamma X}]$$

for a risk aversion parameter $\gamma > 0$. It is associated with the linear-exponential loss function (LINEX). The intuition behind this loss is that it is an asymmetric approximation to the usual quadratic loss function. This is a popular loss function in econometrics.

$$S_\gamma (x, y) = e^{\gamma (y-x)} - \gamma (y-x) - 1,$$
$$f_{S_\gamma} (x) = e^{-\gamma x} + \gamma x - 1,$$
$$R_{S_\gamma} (X) = ENT^\gamma (X),$$
$$D_{S_\gamma} (X) = ENT^\gamma (X - E[X]).$$

$S_\gamma$ is a scoring function such that $y \mapsto S_\gamma (x, y)$ is, for any $x \in \mathbb{R}$, differentiable with strictly increasing derivative and $\frac{\partial f_{S_\gamma} (x)}{\partial x}$ is convex. Hence, for any coherent $\rho$, $R_{\rho, S_\gamma}$ is a convex risk measure and $D_{\rho, S_\gamma}$ is a convex deviation measure. For more details on LINEX score functions, see Zellner (1986).

**Example 12** (Expectile and Variantile). Expectile Value at Risk (see example 4) and Variantile are directly defined as an argmin and a minimum, respectively, for a given scoring function. EVaR has a score function analog quadratic form of the VaR score function (see example 7). The EVaR arises as a solution for asymmetric least squares. Taking $\alpha = 0.5$, we recover traditional least squares, and the EVaR coincides with the Expected Loss (see example 1).

$$S_{EVaR} (x, y) = \alpha [(x - y)^+]^2 + (1 - \alpha) [(x - y)^-]^2,$$
$$f_{S_{EVaR}} (x) = \alpha [(x)^+]^2 + (1 - \alpha) [(x)^-]^2.$$

The EVaR is defined as $R_{S_{EVaR}}$, and the Variantile is defined as $D_{S_{EVaR}}$; the first is a coherent risk measure while the second is a generalized deviation measure.

**Example 13.** We now present a score which is a generalization of the Cauchy/Lorentzian, Geman-McClure, Welsch/Leclerc, generalized Charbonnier, Charbonnier/pseudo-Huber/L1-L2, and L2 loss func-
tions. This score function was proposed by Barron (2019), and it is defined as

\[
f_{S_\alpha}(x) = \begin{cases} 
\frac{x^2}{2} & \text{if } \alpha = 2, \\
\log \left( \frac{x^2}{2} + 1 \right) & \text{if } \alpha = 0, \\
1 - \exp \left( -\frac{x^2}{2} \right) & \text{if } \alpha = -\infty, \\
\frac{\lvert \alpha - 2 \rvert}{\alpha} \left( \left( \frac{x^2}{\lvert \alpha - 2 \rvert} + 1 \right)^{\frac{\alpha}{2}} - 1 \right) & \text{otherwise.}
\end{cases}
\]

\[S_\alpha(x, y) = f_{S_\alpha}(x - y)\].

The map \( \alpha \mapsto S_\alpha(x, y) \) is continuous in the parameter \( \alpha \), which is a shape parameter that controls the robustness of the loss. When \( \alpha = 2 \) this loss resembles the \( S_{EL} \) loss, when \( \alpha = 1 \) it is a pseudo Huber loss, see Huber (1992). \( \alpha = 0 \) yields the Cauchy or Lorentzian loss. \( S_{-2} \) is the Geman-McClure loss. Lastly, letting \( \alpha = -\infty \) yields the Welch loss (see Dennis Jr and Welsch (1978)). It is convex, hence, a scoring function in our framework for \( \alpha \geq 1 \). We highlight that \( S_\alpha \) is a symmetric loss for all \( \alpha \).

**Example 14** (Absolute percentage error and relative error). Absolute percentage error (APE) and relative error (RE) are also widely used score functions to assess point forecasts. Both scores are defined in the following way

\[
APE(x, y) = \frac{|x - y|}{|y|}, \\
RE(x, y) = \frac{|x - y|}{|x|}.
\]

However, neither fits our approach as there is no real function \( f: \mathbb{R} \to \mathbb{R} \) such that \( APE(x, y) = f(x - y) \) or \( RE(x, y) = f(x - y) \).

**Example 15** (Location of minimum variance squared distance). Landsman and Shushi (2022) proposed the Location of minimum variance squared distance (LVS) to measure multivariate risk. LVS is defined as

\[LVS(\mathbf{X}) = \arg \min_{y \in \mathbb{R}^n} Var(||\mathbf{X} - y||^2),\]

when \( \mathbf{X} \) is a vector of \( n \) random variables, \( ||.|| \) denotes the Euclidean distance, and \( Var \) is the variance. For \( n = 1 \), LVS becomes similar to our approach, i.e.,

\[LVS(X) = \arg \min_{y \in \mathbb{R}} Var((X - y)^2) = \arg \min_{y \in \mathbb{R}} \sigma((X - y)^2),\]

where \( \sigma \) is the standard deviation. This is in a similar spirit to our approach. However, it uses a generalized deviation measure (variance) instead of a coherent risk measure. Therefore, it cannot be used in our approach.
Example 16 (Co-elicitability). Unfortunately, the ES (see example 3) is not elicitable, that is, there is no score function such that the ES is its minimizer. However, as shown by Fissler and Ziegel (2016) the function $T(X) \mapsto (VaR^\alpha(X), ES^\alpha(X)) \in \mathbb{R}^2$ is. In this case, the score function has its domain in $\mathbb{R}^3$. Considering the family of scoring functions for ES proposed by Fissler and Ziegel (2016), Gerlach et al. (2016) suggest using the following score function to assess ES point forecasts

$$S_{ES^\alpha}(x, y, z) = y(1_{x < y} - \alpha) - x1_{x < y} + e^z \left(z - y + \frac{1}{\alpha} (y - x)\right) - e^z + 1 - \log(1 - \alpha),$$

$$(VaR^\alpha(X), ES^\alpha(X)) = \arg\min_{y \in \mathbb{R}^2} E[S_{ES^\alpha}(X, y)].$$

On the same topic, the Range Value at Risk (RVaR) proposed by Cont et al. (2010), is defined as

$$RVaR^{\alpha, \beta}(X) = -\frac{1}{\beta - \alpha} \int_\alpha^{\beta} F_X^{-1}(s) ds, \quad 0 \leq \alpha \leq \beta \leq 1.$$ 

Similar to ES, this measure is not elicitable but, $T(X) \mapsto (VaR^\alpha, VaR^\beta, RVaR^{\alpha, \beta})$ is, as shown by Fissler and Ziegel (2021), under the following score

$$S_{RVaR^{\alpha, \beta}}(x, y, z, w) = y(1_{x < y} - \alpha) - x1_{x < y} + z(1_{x < z} - \beta) - x1_{x < z} - \log(\cosh((\alpha - \beta)w)) - \log(1 - \alpha) + (\beta - \alpha) \tanh((\beta - \alpha)w) \left[w + \frac{1}{\beta - \alpha} (S_{VaR^\beta}(x, z) - S_{VaR^\alpha}(x, y))\right] + 1.$$ 

Clearly, the scores above do not fit our framework, as we limit ourselves to scores with domain in $\mathbb{R}^2$. Nonetheless, it is possible to extend the framework for the domain of $S$ be $\mathbb{R}^{k+1}$, where $k$ is the degree of co-elicitability in the sense of Fissler and Ziegel (2016), with the minimization taken on $\mathbb{R}^k$. In this setup, one can define $R$ as the coordinate of interest from the argmin vector. For instance, $ES$ implies $k = 2$ and the second coordinate, while for $RVaR$ one gets $k = 3$.

Example 17 (Cost minimization). Righi et al. (2020) brings forward a similar approach. They propose a robust risk measurement approach that minimizes the expectation of overestimation ($G$) and underestimation ($L$) costs. Their loss function intrinsically depends on the exogenous random variables $G$ and $L$. The study also proposes a deviation measure that is strikingly similar to ours. In fact, if $G$ and $L$ were to be taken as constants, it is equal to $D_{\rho, S_{VaR^\alpha}}$; under some mild conditions and proper choice of dual set, see their Proposition 3. For suitable $G$, $L$ and dual set $\mathcal{Q}'$, their score function, positive homogeneous monetary risk measure, and generalized deviation measure are defined as

$$S_{GL}(x, y) = (x - y)^+ G + (x - y)^- L,$$

$$GL(X) = \sup_{Q \in \mathcal{Q}'} \left\{- \min \left\{ \arg\min_{x \in \mathbb{R}} E_Q[(X - x)^+ G + (X - x)^- L] \right\}\right\},$$

$$GLD(X) = \sup_{Q \in \mathcal{Q}'} \left\{ \min_{x \in \mathbb{R}} E_Q[(X - x)^+ G + (X - x)^- L] \right\} = \min_{x \in \mathbb{R}} \rho((X - x)^+ G + (X - x)^- L).$$
$S_{GL}$ is a scoring function in our framework if $G \in (0, 1)$ and $L = 1 - G$. In this case, $S_{GL} = S_{VaR}$.

4 Conditional risk, linear regression and optimal portfolio weights

In this section, we define the conditional version of risk and deviation measures, which can be obtained from the connection between regression models and our framework. We discuss the properties that conditional risk respects. Moreover, we formalize a connection between minimum deviation portfolio optimization and regression model considering the score that results from the deviation measure. We also provide a solution for the optimal replication hedging problem based on conditional risk measures obtained by the connection between our approach and linear regression.

We now provide a link between linear regression and our framework.

**Definition 5.** Let $X \in (L^p)^n$ and condition (iii) in Proposition 1 holds. The conditional risk is a map $R_{\rho,S} : L^p \to L^p$ given by

$$R_{\rho,S}(Y|X) = -\left(\mu^* + \sum_{i=1}^{n} \beta^*_i X_i\right),$$

where $(\mu^*, \beta^*_1, \ldots, \beta^*_n) = \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho\left(-S\left(Y, \mu + \sum_{i=1}^{n} \beta_i X_i\right)\right).$ \hfill (9)

**Remark 8.** In the case $Q = \{\mathbb{P}\}$ and $S = S_{VaR^*}$ or $S = S_{EVaR^*}$, we obtain, respectively, the conditional $\alpha$ quantile and $\alpha$ expectile obtained from quantile regression and expectile regression. In particular, with $S = S_{EVaR^0.5} = S_{EL}$, one recovers the usual ordinary least squares linear regression. We also define a conditional version of the deviation $D_{\rho,S} : L^p \to L^p$ as

$$D_{\rho,S}(Y|X) = \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho\left(-S\left(Y, \mu + \sum_{i=1}^{n} \beta_i X_i\right)\right) = \rho\left(-S\left(Y, -R_{\rho,S}(Y|X)\right)\right).$$

Since it is not directly used in the context we consider, we do not fully explore it due to parsimony.

We now explore properties of the conditional risk similarly as in Propositions 1 and 2.

**Proposition 4.** Let $\beta(Y, X)$ be the argmin in equation (9) for $Y \in L^p$, $X = (X_1, \ldots, X_n) \in (L^p)^n$. We have the following:

(i) $R_{\rho,S}(Y|X)$ is well defined.

(ii) $\mu^* = -R(Y - \sum_{i=1}^{n} \beta^*_i X_i)$.

(iii) if $Y \leq Z$, then $R(Y|X) \geq R(Z|X)$ for any $Y, Z \in L^p$, and $R(Y + C|X) = R(Y|X) - C$ for any $C \in \mathbb{R}$.

(iv) $\beta(Y + CX, X) = \beta(Y, X) + C$ for any $C \in \mathbb{R}^n$. Hence, $R_{\rho,S}(Y + CX|X) = R_{\rho,S}(Y|X) - CX$. 

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(v) if \( \frac{\partial f_S(x)}{\partial x} \) is convex, then \( R(\lambda Y + (1 - \lambda)Z|X) \leq \lambda R(Y|X) + (1 - \lambda)R(Z|X) \) for any \( Y, Z \in L^p \) and any \( \lambda \in [0, 1] \).

(vi) if \( f_S \) is positive homogeneous, then \( \beta(\lambda Y, X) = \lambda \beta(Y, X) \) for any \( \lambda \geq 0 \). Hence, \( R_{\rho,S}(\lambda Y|X) = \lambda R_{\rho,S}(Y|X) \).

(vii) \( \beta(Y, XA) = A^{-1}\beta(Y, X) \) for any \( n \times n \) non-singular matrix \( A \).

(viii) \( R_{\rho,S}(Y|X) = - (\mu^* + \sum_{i=1}^{n} \beta_i^* X_i) \) if and only if \( Y = \mu^* + \sum_{i=1}^{n} \beta_i^* X_i + \epsilon \), where \( R_{\rho,S}(\epsilon|X) = 0 \).

**Proof.** For (i), we have that \( - (\mu + \sum_{i=1}^{n} \beta_i X_i) \in L^p \) for any \( (\mu, \beta_1, \ldots, \beta_n) \). Further, defining \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) as \( f(\mu, \beta_1, \ldots, \beta_n) = \rho (-S(Y, \mu + \sum_{i=1}^{n} \beta_i X_i)) \), the deduction to prove that the argmin is a singleton is similar to the one in Proposition 1, but adapted to \( \mathbb{R}^{n+1} \).

Regarding (ii), note that for any \( \beta \in \mathbb{R}^n \) we have that \[
\arg\min_{\mu \in \mathbb{R}} \rho \left( - f_S \left( Y - \sum_{i=1}^{n} \beta_i (X_i) - \mu \right) \right) = - R_{\rho,S} \left( Y - \sum_{i=1}^{n} \beta_i (X_i) \right). \]

Thus, we obtain that
\[
\begin{align*}
\arg\min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( - f_S \left( Y - \sum_{i=1}^{n} \beta_i (X_i) - \mu \right) \right) \\
= \arg\min_{\beta \in \mathbb{R}^n} \rho \left( - f_S \left( Y - \sum_{i=1}^{n} \beta_i (X_i) + R_{\rho,S} \left( Y - \sum_{i=1}^{n} \beta_i (X_i) \right) \right) \right) \\
= \rho \left( - f_S \left( Y - \sum_{i=1}^{n} \beta_i^* (X_i) + R_{\rho,S} \left( Y - \sum_{i=1}^{n} \beta_i^* (X_i) \right) \right) \right). \\
\end{align*}
\]

Hence, \( \mu^* = - R(Y - \sum_{i=1}^{n} \beta_i^* X_i) \).

Concerning (iii), let \( Y \leq Z \). By Proposition 2, we have \( p - a.s. \) that \( R(Y|X)(\omega) = R \left( Y - \mu^* - \sum_{i=1}^{n} \beta_i^* X_i(\omega) \right) \geq R \left( Z - \mu^* - \sum_{i=1}^{n} \beta_i^* X_i(\omega) \right) = R(Z|X)(\omega) \).

Hence, \( R(Y|X) \geq R(Z|X) \). Further, for any \( C \in \mathbb{R} \) we have that
\[
\begin{align*}
\arg\min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( - f_S \left( Y + C - \mu - \sum_{i=1}^{n} \beta_i X_i \right) \right) \\
= \arg\min_{\mu - C \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( - f_S \left( Y - \mu + \sum_{i=1}^{n} \beta_i X_i \right) \right) \\
= \arg\min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( - f_S \left( Y - \mu + \sum_{i=1}^{n} \beta_i X_i \right) \right) + (C, 0). \\
\end{align*}
\]

Then, \( R(Y + C|X) = R(Y|X) - C \).
Regarding (iv), for any $C \in \mathbb{R}^n$, we have that
\[
\arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( -f_S \left( Y + \sum_{i=1}^n C_i X_i - \mu - \sum_{i=1}^n \beta_i X_i \right) \right) \\
= \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( -f_S \left( Y - \mu + \sum_{i=1}^n (\beta_i - C_i) X_i \right) \right) \\
= \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( -f_S \left( Y - \mu + \sum_{i=1}^n \beta_i X_i \right) \right) + (0, C)
\]
Thus, $R_{\rho, S}(Y + CX|X) = -(\mu^* + \sum_{i=1}^n (\beta_i^* - C_i) X_i) = R_{\rho, S}(Y|X) - CX$.

Concerning (v), the claim follows similarly to that in Proposition 2 by considering the f.o.c.
\[
\frac{\partial \rho(-f_S(Y - \sum_{i=1}^n \beta_i X_i - R(Y - \sum_{i=1}^n \beta_i X_i)))}{\partial \beta_i} = 0, \; i = 1, \ldots, n.
\]
Let then $g: L^p \times (L^p)^n \to \mathbb{R}^n$ be as
\[
g(Y, X) = \frac{\partial \rho(-f_S(Y - \sum_{i=1}^n \beta_i X_i - R(Y - \sum_{i=1}^n \beta_i X_i)))}{\partial \beta_i}(\beta), \; i = 1, \ldots, n
\]
which is convex in its domain and non-increasing in $X$ for any $Y \in L^p$. Let $\lambda \in [0, 1]$ and $Y, Z \in L^p$. Then we have
\[
g(\lambda Y + (1 - \lambda) Z, \lambda R(Y|X) + (1 - \lambda) R(Z|X)) \\
\leq \lambda g(Y, R(Y|X)) + (1 - \lambda) g(Z, R(Z|X)) = 0 \\
= g(\lambda Y + (1 - \lambda) Z, R(\lambda Y + (1 - \lambda) Z|X)).
\]
Thus, we obtain $R(\lambda Y + (1 - \lambda) Z|X) \leq \lambda R(Y|X) + (1 - \lambda) R(Z|X)$.

For (vi), if $\lambda = 0$ the result is trivial. Further, we have for any $\lambda > 0$ that
\[
\arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( -f_S \left( \lambda Y - \mu - \sum_{i=1}^n \beta_i X_i \right) \right) \\
= \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \lambda \rho \left( -f_S \left( \frac{Y - \mu + \sum_{i=1}^n \beta_i X_i}{\lambda} \right) \right) \\
= \lambda \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left( -f_S \left( \frac{Y - \mu - \sum_{i=1}^n \beta_i X_i}{\lambda} \right) \right).
\]
We thus get that $R(\lambda Y|X) = \lambda \left( -\mu^* - \sum_{i=1}^n \beta_i^* X_i \right) = \lambda R(Y|X)$.  

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Concerning (vii), we have for any $n \times n$ non-singular matrix $A$ that

$$
\arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho(-f_S(Y - \mu - \beta X A)) \nonumber \\
= \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho(-f_S(Y - \mu - (\beta A) X)) 
$$

$$
= (1, A^{-1}) \arg \min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^n} \rho \left(-f_S \left(Y - \mu + \sum_{i=1}^n \beta_i X_i \right) \right). 
$$

For (viii), if $Y = \mu^* + \sum_{i=1}^n \beta_i^* X_i + \epsilon$ with $R_{\rho,S}(\epsilon|X) = 0$, then it is direct that

$$
0 = R_{\rho,S}(\epsilon|X) = R_{\rho,S} \left(Y - \left(\mu^* + \sum_{i=1}^n \beta_i^* X_i \right) \right|X) = R_{\rho,S}(Y|X) + \mu^* + \sum_{i=1}^n \beta_i^* X_i.
$$

Thus, the if part of the claim follows. For the converse, let $\epsilon = Y - (\mu^* + \sum_{i=1}^n \beta_i^* X_i)$. Then $Y = \mu^* + \sum_{i=1}^n \beta_i^* X_i + \epsilon$ and

$$
R_{\rho,S}(\epsilon|X) = R_{\rho,S} \left(Y - \left(\mu^* + \sum_{i=1}^n \beta_i^* X_i \right) \right|X) = R_{\rho,S}(Y|X) + \mu^* + \sum_{i=1}^n \beta_i^* X_i = 0. \tag{10}
$$

\[ \square \]

Remark 9. Let $\sigma(X) \subseteq F$ be the sub-sigma-algebra generated by $X$. It is straightforward to verify that the previous Proposition 4 implies that $R(Y + C|X) = R(Y|X) - C$ for any $C \in L^\rho(\Omega, \sigma(X), \mathbb{P})$. Furthermore, if if $f_S$ is positive homogeneous, then $R_{\rho,S}(\lambda Y|X) = \lambda R_{\rho,S}(Y|X)$ or any $\lambda \in L^\rho_+(\Omega, \sigma(X), \mathbb{P})$. Thus, we indeed have that $R_{\rho,S}$ is in fact a conditional risk measure in the sense of Ruszczyński and Shapiro (2006).

Remark 10. Based on such framework, one can have metrics in our setup that are similar to the usual coefficient of determination $R^2$ as

$$
CD_{\rho,S}(Y, X) = 1 - \frac{\rho(S(Y, -R(Y|X)))}{\rho(S(Y, -R(Y)))},
$$

where $Y \in L^p$, $X = (X_1, \ldots, X_n) \in (L^p)^n$. Such quantity can be used to summarize the association of $Y$ and $X$. Furthermore, it is also possible to study inference properties of estimated parameters $\beta(Y, X)$ as well as hypothesis tests such as counterparts to the usual $t$ and $F$ tests for OLS approaches. Such topics are outside our current scope and left for future research.

We now formalize the minimum deviation and replication hedging problems to our framework and state a result for our setup that guarantees the existence of a solution and how to obtain it.

**Definition 6.** Let $X = (X_1, \ldots, X_n) \in (L^p)^n$. The minimum deviation portfolio optimization problem for $X$, $P(X)$, is defined as

$$
\min_{w \in \mathbb{R}^n} \quad D_{\rho,S} \left( \frac{\sum_{i=1}^n w_i X_i}{\sum_{i=1}^n w_i} \right) \tag{10}
$$
Proposition 5. We have \( w^* = (w_1^*, \ldots, w_n^*) \in \arg \min P(X) \) if and only if \( R_{\rho,S}(Y|(Y-X_1, \ldots, Y-X_n)) = -(\mu^* + \sum_{i=1}^{n} w_i^*(Y-X_i)) \), where \( Y = \frac{1}{n} \sum_{i=1}^{n} X_i \), \( w_i^* = w_i^* \) and \( \mu^* = R_{\rho,S}(\sum_{i=1}^{n} w_i^*(Y-X_i)) \).

Proof. We have by Definition that

\[
R_{\rho,S}(Y|(Y-X_1, \ldots, Y-X_n)) = -(\mu^* + \sum_{i=1}^{n} w_i^*(Y-X_i))
\]

\[\iff (\mu^*, w^*) = \arg \min_{\mu \in \mathbb{R}, w \in \mathbb{R}^n} \rho \left( -S \left( Y, \mu + \sum_{i=1}^{n} w_i(Y-X_i) \right) \right).\]

Note that \( w' \in \mathbb{R}^n \iff w_i^* = w_i^* \left( 1 - \sum_{i=1}^{n} w_i^* \right) \in \mathbb{R} \) and \( \sum_{i=1}^{n} w_i^* = 1 \). The equivalence then follows by:

\[
\begin{align*}
&\min_{\mu \in \mathbb{R}, w \in \mathbb{R}^n} \rho \left( -S \left( Y, \mu + \sum_{i=1}^{n} w_i(Y-X_i) \right) \right) \\
= &\min_{w \in \mathbb{R}^n} \rho \left( -f_S \left( \left( 1 - \sum_{i=1}^{n} w_i \right) Y + \sum_{i=1}^{n} w_i X_i - \mu \right) \right) \\
= &\min_{\sum_{i=1}^{n} w_i = 1} \rho \left( -f_S \left( \sum_{i=1}^{n} \left( \frac{1}{n} \left( 1 - \sum_{i=1}^{n} w_i \right) + w_i \right) X_i + R_{\rho,S} \left( \sum_{i=1}^{n} \left( \frac{1}{n} \left( 1 - \sum_{i=1}^{n} w_i \right) + w_i \right) X_i \right) \right) \right) \\
= &\min_{\sum_{i=1}^{n} w_i = 1} D_{\rho,S} \left( \sum_{i=1}^{n} w_i X_i \right).
\end{align*}
\]

\( \square \)

Definition 7. Let \( Y \in L^p \) be given and \( X = (X_1, \ldots, X_n) \in (L^p)^n \). The optimal replication hedging problem for \( X \), \( H(X) \), is defined as

\[
\min_{w \in \mathbb{R}^n} \rho \left( -S \left( Y, \mu + \sum_{i=1}^{n} w_i X_i \right) \right) \quad (11)
\]

Proposition 6. \( w^* = (w_1^*, \ldots, w_n^*) \in \arg \min H(X) \) if and only if \( R(Y|X) = -(\mu^* + \sum_{i=1}^{n} w_i^* X_i) \), where \( \mu^* = -R(Y - \sum_{i=1}^{n} w_i^* X_i) \).

Proof. We have by Definition that

\[
R_{\rho,S}(Y|X_1, \ldots, X_n) = -(\mu^* + \sum_{i=1}^{n} w_i^* X_i)
\]

\[\iff (\mu^*, w^*) = \arg \min_{\mu \in \mathbb{R}, w \in \mathbb{R}^n} \rho \left( -S \left( Y, \mu + \sum_{i=1}^{n} w_i X_i \right) \right).\]
Further, notice that
\[
\arg\min_{w \in \mathbb{R}^n} \rho \left( -S \left( Y, \sum_{i=1}^{n} w_i X_i \right) \right) = \arg\min_{w \in \mathbb{R}^n} \rho \left( -S \left( Y, k + \sum_{i=1}^{n} w_i X_i \right) \right), \quad \forall k \in \mathbb{R}.
\]
We also have that
\[
\min_{\mu \in \mathbb{R}, w \in \mathbb{R}^n} \rho \left( -S \left( Y, \sum_{i=1}^{n} w_i X_i \right) \right) = \min_{w \in \mathbb{R}^n} \rho \left( -f_S \left( \sum_{i=1}^{n} w_i X_i + R \left( Y - \sum_{i=1}^{n} w_i X_i \right) \right) \right).
\]
From these facts, we get the equivalence between both \( w^* = (w_1^*, \ldots, w_n^*) \in \arg\min H(X) \) and \( R(Y \mid X) = - (\mu^* + \sum_{i=1}^{n} w_i^* X_i) \).

\[
\square
\]

References

Acerbi, C., 2002. Spectral measures of risk: A coherent representation of subjective risk aversion. Journal of Banking & Finance 26, 1505 – 1518.

Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203–228.

Balter, A.G., Pelsser, A., 2020. Pricing and hedging in incomplete markets with model uncertainty. European Journal of Operational Research 282, 911–925.

Barigou, K., Bignozzi, V., Tsanakas, A., 2022. Insurance valuation: A two-step generalised regression approach. ASTIN Bulletin: The Journal of the IAA 52, 211–245.

Barron, J.T., 2019. A general and adaptive robust loss function, in: Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition, pp. 4331–4339.

Bellini, F., Bignozzi, V., 2015. On elicitable risk measures. Quantitative Finance 15, 725–733.

Bellini, F., Di Bernardino, E., 2017. Risk management with expectiles. The European Journal of Finance 23, 487–506.

Bellini, F., Klar, B., Müller, A., Gianin, E.R., 2014. Generalized quantiles as risk measures. Insurance: Mathematics and Economics 54, 41 – 48.

Bellini, F., Laeven, R.J.A., Gianin, E.R., 2021. Dynamic robust Orlicz premia and Haezendonck–Goovaerts risk measures. European Journal of Operational Research 291, 438–446.

Bessler, W., Leonhardt, A., Wolff, D., 2016. Analyzing hedging strategies for fixed income portfolios: A bayesian approach for model selection. International Review of Financial Analysis 46, 239–256.

Britten-Jones, M., 1999. The sampling error in estimates of mean-variance efficient portfolio weights. The Journal of Finance 54, 655–671.
Carr, P., Geman, H., Madan, D.B., 2001. Pricing and hedging in incomplete markets. Journal of financial economics 62, 131–167.

Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C., Wang, R., 2021. Star-shaped risk measures. URL: https://arxiv.org/abs/2103.15790, doi:10.48550/ARXIV.2103.15790.

Cherny, A., Madan, D., 2009. New measures for performance evaluation. Review of Financial Studies 22, 2371–2406.

Cont, R., Deguest, R., Scandolo, G., 2010. Robustness and sensitivity analysis of risk measurement procedures. Quantitative finance 10, 593–606.

Daouia, A., Gijbels, I., Stupfler, G., 2019. Extremiles: A new perspective on asymmetric least squares. Journal of the American Statistical Association 114, 1366–1381.

Daouia, A., Gijbels, I., Stupfler, G., 2021. Extremile regression. Journal of the American Statistical Association, 1–8.

Delbaen, F., 2012. Monetary utility functions. Osaka University Press.

Dennis Jr, J.E., Welsch, R.E., 1978. Techniques for nonlinear least squares and robust regression. Communications in Statistics-simulation and Computation 7, 345–359.

Embrechts, P., Mao, T., Wang, Q., Wang, R., 2021. Bayes risk, elicitability, and the Expected Shortfall. Mathematical Finance 31, 1190–1217.

Fan, J., Zhang, J., Yu, K., 2012. Vast portfolio selection with gross-exposure constraints. Journal of the American Statistical Association 107, 592–606.

Fischer, T., 2003. Risk capital allocation by coherent risk measures based on one-sided moments. Insurance: Mathematics and Economics 32, 135–146.

Fissler, T., Ziegel, J.F., 2016. Higher order elicitability and Osband’s principle. The Annals of Statistics 44, 1680–1707.

Fissler, T., Ziegel, J.F., 2021. On the elicitability of range value at risk. Statistics & Risk Modeling 38, 25–46.

Föllmer, H., Knispel, T., 2013. Convex risk measures: Basic facts, law-invariance and beyond, asymptotics for large portfolios, in: MacLean, L., Ziemba, W. (Eds.), Handbook of the Fundamentals of Financial Decision Making. World Scientific, pp. 507–554.

Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance and stochastics 6, 429–447.
Follmer, H., Schied, A., 2016. Stochastic finance: an introduction in discrete time. Walter de Gruyter GmbH.

Föllmer, H., Weber, S., 2015. The axiomatic approach to risk measures for capital determination. Annual Review of Financial Economics 7, 301–337.

Frey, C., Pohlmeier, W., 2016. Bayesian shrinkage of portfolio weights. Available at SSRN 2730475.

Frittelli, M., Gianin, E.R., 2002. Putting order in risk measures. Journal of Banking & Finance 26, 1473–1486.

Gerlach, R., Walpole, D., Wang, C., 2017. Semi-parametric Bayesian tail risk forecasting incorporating realized measures of volatility. Quantitative Finance 17, 199–215.

Gianin, E.R., Sgarra, C., 2013. Acceptability indexes via 'g-expectations': An application to liquidity risk. Mathematics and Financial Economics 7, 457–475.

Gneiting, T., 2011. Making and evaluating point forecasts. Journal of the American Statistical Association 106, 746–762.

Grechuk, B., Molyboha, A., Zabarankin, M., 2009. Maximum Entropy Principle with General Deviation Measures. Mathematics of Operations Research 34, 445–467.

Guillen, M., Bermúdez, L., Pitarque, A., 2021. Joint generalized quantile and conditional tail expectation regression for insurance risk analysis. Insurance: Mathematics and Economics 99, 1–8.

Halkos, G.E., Tsirivis, A.S., 2019. Energy commodities: A review of optimal hedging strategies. Energies 12, 3979.

Herdegen, M., Khan, N., 2022. Sensitivity to large losses and \( \rho \)-arbitrage for convex risk measures. URL: https://arxiv.org/abs/2202.07610, doi:10.48550/ARXIV.2202.07610.

Huang, S.F., Guo, M., 2013. An optimal multi-step quadratic risk-adjusted hedging strategy. Journal of the Korean Statistical Society 42, 37–49.

Huber, P.J., 1992. Robust estimation of a location parameter, in: Breakthroughs in statistics. Springer, pp. 492–518.

Kaina, M., Rüschendorf, L., 2009. On convex risk measures on lp-spaces. Mathematical Methods of Operations Research 69, 475–495.

Kempf, A., Memmel, C., 2006. Estimating the global minimum variance portfolio. Schmalenbach Business Review 58, 332–348.

Koenker, R., 2005. Quantile regression. Cambridge University Press New York.
Koenker, R., Bassett, G., 1978. Regression quantiles. Econometrica 46, 33–50.

Kou, S., Peng, X., 2016. On the measurement of economic tail risk. Operations Research 64, 1056–1072.

Landsman, Z., Shushi, T., 2022. The location of a minimum variance squared distance functional. Insurance: Mathematics and Economics 105, 64–78.

Li, J., 2015. Sparse and Stable Portfolio Selection With Parameter Uncertainty. Journal of Business & Economic Statistics 33, 381–392. doi:10.1080/07350015.2014.954.

Liebrich, F.B., 2021. Risk sharing under heterogeneous beliefs without convexity. URL: https://arxiv.org/abs/2108.05791, doi:10.48550/ARXIV.2108.05791.

Mao, T., Cai, J., 2018. Risk measures based on behavioural economics theory. Finance and Stochastics 22, 367–393.

Markowitz, H., 1952. Portfolio selection. The Journal of Finance 7, 77–91.

Moresco, M.R., Righi, M.B., 2022. On the link between monetary and star-shaped risk measures. Statistics & Probability Letters, 109345.

Newey, W.K., Powell, J.L., 1987. Asymmetric least squares estimation and testing. Econometrica: Journal of the Econometric Society, 819–847.

Ogryczak, W., Ruszczyński, A., 1999. From stochastic dominance to mean-risk models: Semideviations as risk measures. European Journal of Operational Research 116, 33–50.

Pflug, G., Römisch, W., 2007. Modeling, Measuring and Managing Risk. 1 ed., World Scientific.

Pflug, G.C., 2006. Subdifferential representations of risk measures. Mathematical Programming 108, 339–354.

Righi, M.B., 2018. A theory for combinations of risk measures. URL: https://arxiv.org/abs/1807.01977, doi:10.48550/ARXIV.1807.01977.

Righi, M.B., 2019. A composition between risk and deviation measures. Annals of Operations Research 282, 299–313.

Righi, M.B., 2021. Star-shaped acceptability indexes. URL: https://arxiv.org/abs/2110.08630, doi:10.48550/ARXIV.2110.08630.

Righi, M.B., Borenstein, D., 2018. A simulation comparison of risk measures for portfolio optimization. Finance Research Letters 24, 105–112.

Righi, M.B., Ceretta, P.S., 2016. Shortfall deviation risk: an alternative for risk measurement. Journal of Risk 19, 81–116.
Righi, M.B., Müller, F.M., Moresco, M.R., 2020. On a robust risk measurement approach for capital
determination errors minimization. Insurance: Mathematics and Economics 95, 199–211.

Rockafellar, R., Uryasev, S., 2013. The fundamental risk quadrangle in risk management, optimization
and statistical estimation. Surveys in Operations Research and Management Science 18, 33–53.

Rockafellar, R.T., Uryasev, S., Zabarankin, M., 2006. Generalized deviations in risk analysis. Finance
and Stochastics 10, 51–74.

Rockafellar, R.T., Uryasev, S., Zabarankin, M., 2007. Equilibrium with investors using a diversity of
deviation measures. Journal of Banking & Finance 31, 3251–3268.

Rüschendorf, L., 2013. Mathematical Risk Analysis. Springer.

Ruszczyński, A., Shapiro, A., 2006. Optimization of Risk Measures. Springer. pp. 119–157.

Shapiro, A., 2017. Distributionally robust stochastic programming. SIAM Journal on Optimization 27,
2258–2275.

Sion, M., 1958. On general minimax theorems. Pacific Journal of Mathematics 8, 171–176.

Sun, C., Ji, S., 2017. The least squares estimator of random variables under sublinear expectations.
Journal of Mathematical Analysis and Applications 451, 906 – 923.

Wu, Q., Yang, F., Zhang, P., 2023. Conditional generalized quantiles based on expected utility model and
equivalent characterization of properties. URL: https://arxiv.org/abs/2301.12420.

Zellner, A., 1986. Bayesian estimation and prediction using asymmetric loss functions. Journal of the
American Statistical Association 81, 446–451.

Ziegel, J.F., 2016. Coherence and elicitability. Mathematical Finance 26, 901–918.