GRADIENT CONTINUITY ESTIMATES FOR THE NORMALIZED $p$–POISSON EQUATION

AGNID BANERJEE AND ISIDRO H. MUNIVE

Abstract. In this paper, we obtain gradient continuity estimates for viscosity solutions of
\[ \Delta^N_p u = f \] in terms of the scaling critical $L(n,1)$ norm of $f$, where $\Delta^N_p$ is the normalized $p$–Laplacian operator defined in (1.2) below. Our main result, Theorem 2.2, corresponds
to the borderline gradient continuity estimate in terms of the modified Riesz potential $\tilde{I}^q_f$.
Moreover, for $f \in L^m$ with $m > n$, we also obtain $C^{1,\alpha}$ estimates, see Theorem 2.3 below.
This improves one of the regularity results in [3], where a $C^{1,\alpha}$ estimate was established
depending on the $L^m$ norm of $f$ under the additional restriction that $p > 2$ and $m > \max(2, n, \frac{p}{2})$ (see Theorem 1.2 in [3]). We also mention that differently from the approach
in [3], which uses methods from divergence form theory and nonlinear potential theory in
the proof of Theorem 1.2, our method is more non-variational in nature, and it is based on
separation of phases inspired by the ideas in [36]. Moreover, for $f$ continuous, our approach
also gives a somewhat different proof of the $C^{1,\alpha}$ regularity result, Theorem 1.1, in [3].

1. Introduction

The aim of this paper is to obtain pointwise gradient continuity estimates for viscosity solutions of
\[ \Delta^N_p u = f \] in terms of the scaling critical $L(n,1)$—norm of $f$. Here, $\Delta^N_p$ denotes the normalized $p$–Laplace
operator given by
\[ \Delta^N_p u = \left( \delta_{ij} + (p-2) \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij}. \]

The fundamental role of these borderline, or end-point regularity, estimates in the theory
of elliptic and parabolic partial differential equations is well known. In order to put our result

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in the correct historical perspective, we note that in 1981, E. Stein in his visionary work [34] showed the following.

**Theorem 1.1.** Let $L(n, 1)$ denote the standard Lorentz space, then the following implication holds:
\[
\nabla v \in L(n, 1) \implies v \text{ is continuous.}
\]

The Lorentz space $L(n, 1)$ appearing in Theorem 1.1 consists of those measurable functions $g$ satisfying the condition
\[
\int_0^\infty |\{x : g(x) > t\}|^{1/n} dt < \infty.
\]

Theorem 1.1 can be regarded as the limiting case of Sobolev-Morrey embedding that asserts
\[
\nabla v \in L^{n+\varepsilon} \implies v \in C^{0, \frac{\varepsilon}{n+\varepsilon}}.
\]

Note that indeed $L^{n+\varepsilon} \subset L(n, 1) \subset L^n$ for any $\varepsilon > 0$, with all the inclusions being strict.

Now Theorem 1.1 coupled with the standard Calderon-Zygmund theory has the following interesting consequence.

**Theorem 1.2.** $\Delta u \in L(n, 1) \implies \nabla u$ is continuous.

The analogue of Theorem 1.2 for general nonlinear, and possibly degenerate elliptic and parabolic equations, has become accessible not so long ago through a rather sophisticated and powerful nonlinear potential theory (see for instance [15, 26, 27] and the references therein).

The first breakthrough in this direction came up in the work of Kuusi and Mingione in [25], where they showed that the analogue of Theorem 1.2 holds for operators modelled after the $p$-Laplacian. Such a result was subsequently generalized to $p$-Laplacian-type systems by the same authors in [28].

Since then, there has been several generalizations of Theorem 1.2 to operators with various kinds of nonlinearities. In the context of fully nonlinear elliptic equations, the analogue of Theorem 1.2 was established by Daskalopoulos-Kuusi-Mingione in [14]. More precisely, they showed the following (see Theorem 1.1 in [14]).

**Theorem 1.3.** Let $u$ be a $W^{2, q}$ viscosity solution of
\[
F(x, \nabla^2 u) = f \text{ in } B_1,
\]
where $F$ is uniformly elliptic fully nonlinear operator and $f \in L(n, 1)$. Then, there exists $\theta \in (0, 1)$, depending only on $n$ and the ellipticity constants of $F$, such that if $F(\cdot)$ has $\theta$-BMO coefficients, then $\nabla u$ is continuous in the interior of $B_1$. Moreover, the following estimates hold for some $\alpha = \alpha(n, \lambda, \Lambda)$ and $\delta \in (0, 1)$,
\[
\begin{aligned}
|\nabla u(x_0)| &\leq C \left( \frac{1}{B_r(x_0)} \int_{B_r(x_0)} |\nabla u|^p \right)^{1/p} \\
|\nabla u(x_1) - \nabla u(x_2)| &\leq C \left( \left| \nabla u \right|_{L^n(B_{3r})} |x_1 - x_2|^{\alpha(1-\delta)} + \sup_{x \in \{x_1, x_2\}} \tilde{I}_q f(x, 4|x_1 - x_2|^\delta) \right),
\end{aligned}
\]
whenever $x_1, x_2 \in B_r$. Here $\tilde{I}_q f(x_0, r)$ is the “modified Riesz potential” defined by
\[
\tilde{I}_q f(x_0, r) = \int_0^r \left( \frac{1}{B_s(x_0)} \int_{B_s(x_0)} |f|^q \right)^{1/q} ds,
\]
and $C = C(n, p, r, \lambda, \Lambda)$.

Before proceeding further, we make the following important remark.
Remark 1. The reader should note that from the Hardy-Littlewood rearrangement inequality (see for instance [14]) we have that

\begin{equation}
\int_0^r \left( \int_{B_s} |f|^q \right)^{1/q} ds \leq \frac{C}{|B_1|} \int_0^{|B_1|} \left[ f^{**}(\rho) \rho^2 \right]^{\frac{1}{q}} \frac{d\rho}{\rho},
\end{equation}

where \( f^{**} \) is defined as

\[ f^{**}(\rho) = \frac{1}{\rho} \int_0^\rho f^*(t) dt, \]

with \( f^* \) being the radial non-increasing rearrangement of \( f \). Now, when \( f \in L(n, 1) \), we have from an equivalent characterization of Lorentz spaces that

\begin{equation}
\int_0^\infty \left[ f^{**}(\rho) \rho^2 \right]^{\frac{1}{q}} \frac{d\rho}{\rho} < \infty, \quad \text{for } q < n.
\end{equation}

Therefore, it follows from the inequalities in (1.6) and (1.7) that when \( f \in L(n, 1) \) and \( q < n \), \( \mathbb{E}^{\rho}_{(x_0, r)}(\epsilon) \rightarrow 0 \) as \( r \rightarrow 0 \). Consequently, the gradient continuity follows from the estimates in (1.4) above.

We also refer to the recent work [2] of one of us and Adimurthi where an analogous regularity result has been obtained under Dirichlet boundary conditions when the domain is \( C^{1,\text{Dini}} \). The result was established using Caffarelli style compactness arguments as in [9].

In this paper we establish a similar estimate as in (1.4) above when the fully nonlinear operator \( F \) gets replaced by the normalized \( p \)-Laplacian operator \( \Delta_p^N \). In order to provide the reader with the right viewpoint concerning our approach, we note that getting \( C^1 \)-regularity result in general amounts to show that the graph of \( u \) can be touched by an affine function so that the error is of order \( o(r) \) in a ball of radius \( r \) for every \( r \) small enough. The proof of this is based on iterative argument where one ensures improvement of flatness at every successive scale by comparing to a solution of a limiting equation with more regularity. At each step, via rescaling, it reduces to show that if \( < p_0, x > + u \) solves (1.1) in \( B_1 \), then the oscillation of \( u \) is strictly smaller in a smaller ball up to a linear function. This is accomplished via compactness arguments which crucially relies on apriori estimates. Such estimates in the context of \( \Delta_p^N \) come from the Krylov-Safonov theory because the equation (1.1) lends itself to a uniformly elliptic structure.

Now, for a \( u \) that solves (1.1), we have that \( u - < p_0, x > \) is a solution of the following perturbed equation

\begin{equation}
\left( \delta_{ij} + (p - 2) \frac{(u_i + (p_0) i)(u_j + (p_0) j)}{|\nabla u + p_0|^2} \right) u_{ij} = f.
\end{equation}

Therefore, in order to obtain improvement of flatness at each scale after a rescaling, it is imperative to get uniform \( C^1 \)-type estimates independent of \( |p_0| \) for the limiting equations corresponding to the case \( f \equiv 0 \). This is precisely done in [3] by an adaptation of the Ishii-Lions approach as in [20], where the authors obtained uniform Lipschitz estimates for solutions to (1.8) for large \( |p_0|/s \) when \( f = 0 \). In this paper, we follow an approach which is different from that in [3]. Our proofs of Theorem 2.2 and Theorem 2.3 are based instead on separation of the degenerate and the non-degenerate phase, and do not rely on the uniform Lipschitz estimates for equations of the type (1.8) for large \( |p_0|/s \). This is inspired by ideas in [36], where an alternate proof of \( C^{1,\alpha} \)-regularity for the \( p \)-Laplacian was given. Moreover, in the case of continuous \( f \), our method also provides a different proof of the \( C^{1,\alpha} \)-regularity result.
for (1.1) established in [3] (see also [4] for \( p \geq 2 \)). We believe that this alternate viewpoint would definitely be of independent interest.

Finally, we mention that over the last decade, there has been a growing attention on equations of the type (1.1) because of their connections to tug-of-war games with noise. This aspect was first studied in [33]. In recent times, the parabolic normalized \( p \)-Laplacian, as well as its degenerate and singular variants, have been studied in various contexts in a number of papers, see [1, 22, 13, 5, 6, 7, 18, 32, 19, 23, 21, 31]. Such equations have also found applications in image processing (see for instance [13]).

The paper is organized as follows. In Section 2 we introduce some basic notations, list some preliminary results, and then state our main theorems. In Section 3 we first establish approximation lemmas that play a crucial role in the separation of phases in the iterative argument in the proof of our main results. We then subsequently establish our main results Theorem 2.2 and Theorem 2.3. In closing, we would like to mention that it remains to be seen whether one can obtain similar borderline estimates for more general equations of the type

\[
|\nabla u|^\gamma \left( \delta_{ij} + (p - 2) \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij} = f,
\]

with appropriate restrictions on the parameter \( \gamma \). This seems to be an interesting open question to which we would like to come back in a future study.

2. Notations, Preliminaries and statement of the main results

We denote points in \( \mathbb{R}^n \) by \( x, y, x_1, x_2 \) etc. We let \( |x| \) be the norm of \( x \), and \( |A| \) will denote the Lebesgue measure of \( A \subset \mathbb{R}^n \). Let \( B_r(x) = \{ x : |x| < r \} \). When \( x = 0 \), we will occasionally denote such a set by \( B_r \). By \( \partial B_r(x) \), we will denote the boundary of the set \( B_r(x) \). We will also denote by \( S(n) \) the space of \( n \times n \) symmetric matrices. In our ensuing discussion, at times we will be using the notation \( \int_A h dx \) to indicate the integral average of a function \( h \) over a set \( A \).

We now fix an exponent \( q \in (n - n_0, n) \), where \( n_0 \) (denoted by \( \varepsilon \) in [16]) is a small universal constant as obtained in [16], such that the Krylov-Safanov type Hölder estimate holds for functions which belong to extremal Pucci class \( S(\lambda, \Lambda, f) \) in the \( W^{2,q} \) viscosity sense. Here

\[
\lambda = \min(1, p - 1), \quad \Lambda = \max(1, p - 1),
\]

\( f \in L^q \), and \( S(\lambda, \Lambda, f) \) is the set of all functions \( u \) which solves in the \( W^{2,q} \) viscosity sense (we refer to [10] for the precise notion of \( W^{2,q} \) viscosity solutions)

\[
(2.2) \quad \mathcal{P}^-_{\lambda, \Lambda}(\nabla^2 u) \leq f \leq \mathcal{P}^+_{\lambda, \Lambda}(\nabla^2 u).
\]

The operators \( \mathcal{P}^-_{\lambda, \Lambda} \) and \( \mathcal{P}^+_{\lambda, \Lambda} \) are the minimal and maximal Pucci operators, respectively, defined in the following way

\[
(2.3) \quad \begin{cases}
\mathcal{P}^-_{\lambda, \Lambda}(M) = \inf_{\{A \in S(n) : \lambda \leq A \leq \Lambda\}} \text{trace } (AM), \\
\mathcal{P}^+_{\lambda, \Lambda}(M) = \sup_{\{A \in S(n) : \lambda \leq A \leq \Lambda\}} \text{trace } (AM).
\end{cases}
\]

We now turn our attention to the relevant notion of solution to (1.1). For \( p \in \mathbb{R}^n - \{0\} \) and \( X = [m_{ij}] \in S(n) \), following [8], we define

\[
F(p, X) = \left( \delta_{ij} + (p - 2) \frac{p_i p_j}{|p|^2} \right) m_{ij}.
\]

Then as in [11], the lower semicontinuous relaxation \( F_* \) is defined as follows
\begin{equation}
F_*(q, X) = \begin{cases} 
F(q, X) & \text{if } q \neq 0, \\
\inf_{a \in \mathbb{R}^n \setminus \{0\}} F(a, X) & \text{if } q = 0,
\end{cases}
\end{equation}

while the upper semicontinuous relaxation \( F^* \) is defined as

\begin{equation}
F^*(q, X) = \begin{cases} 
F(q, X) & \text{if } q \neq 0, \\
\sup_{a \in \mathbb{R}^n \setminus \{0\}} F(a, X) & \text{if } q = 0.
\end{cases}
\end{equation}

**Definition 2.1.** We say that \( u \) is a \( W^{2,q} \) viscosity sub-solution of (1.1) in a domain \( \Omega \subset \mathbb{R}^n \) if given \( \phi \in W^{2,q} \) such that \( u - \phi \) has a local maximum at \( x_0 \in \Omega \), then one has

\begin{equation}
\limsup_{x \to x_0} F^*(\nabla \phi(x), \nabla^2 \phi(x)) - f(x) \geq 0.
\end{equation}

In an analogous way, the notion of viscosity supersolution of (1.1) is defined using \( F_* \) instead of \( F^* \), and where \( \liminf \) gets replaced by \( \limsup \) in the equation (2.6) above. Finally, we say that \( u \) is a \( W^{2,q} \) viscosity solution to (1.1) if it is both a subsolution and a supersolution. It is easy to deduce that if \( u \) is a \( W^{2,q} \) viscosity solution to (1.1), then \( u \) belongs to the Pucci class \( \mathcal{S}(\lambda, \Lambda, f) \) in the \( W^{2,q} \) viscosity sense where \( \lambda, \Lambda \) are as in (2.1). Hence, \( u \) satisfies universal Hölder estimates as in [16], which depend on \( n, p \) and \( ||u||_{L^\infty} \).

### 2.1. Statement of the main results

We now state our first main result. This result corresponds to the regularity estimate in the borderline case, i.e., gradient continuity estimates with dependence on the \( L(n,1) \) norm of \( f \).

**Theorem 2.2.** For a given \( p > 1 \), let \( u \) be a \( W^{2,q} \) viscosity solution of (1.1) in \( B_1 \) where \( f \in L(n,1) \). Then \( \nabla u \) is continuous inside of \( B_1 \). Moreover, the following borderline estimates hold

\begin{equation}
|\nabla u(x_0)| \leq C(\tilde{l}_q(x_0, 1/2) + ||u||_{L^\infty(B_1)}) \quad \text{for } x_0 \in B_1/2, \\
|\nabla u(x_1) - \nabla u(x_2)| \leq C(n,p)
\left( ||u||_{L^\infty(B_{1/4})} + \sup_{x \in \{x_1, x_2\}} \tilde{l}_q(x, 1) |x_1 - x_2|^\alpha/4 + \sup_{x \in \{x_1, x_2\}} \tilde{l}_q(x, 4|x_1 - x_2|^{1/4}) \right),
\end{equation}

whenever \( x_1, x_2 \in B_{1/2} \), and where \( \alpha = \alpha(n,p) \).

In the case \( f \in L^m(\mathbb{R}^n) \) with \( m > n \), we obtain the following regularity result that improves Theorem 1.2 in [3].

**Theorem 2.3.** For \( p > 1 \) and \( m > n \), let \( u \) be a \( W^{2,m} \) viscosity solution of (1.1) in \( B_1 \), where \( f \in L^m \). Then, \( \nabla u \in C^{\alpha_0}(B_{1/2}) \) for some \( \alpha_0 = \alpha_0(n,p,m) \). Moreover, we have that the following estimate holds,

\[ ||u||_{C^{1,\alpha_0}(B_{1/2})} \leq C(n,p,||f||_{L^m},||u||_{L^\infty(B_1)}). \]

### 3. Proof of the main results

#### 3.1. Proof of Theorem 2.2

We now fix a universal parameter which plays a crucial role in our compactness arguments. Let \( \beta > 0 \) be the optimal Hölder exponent such that any arbitrary solution \( u \) of

\[ \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{is in } C^{1,\beta}_{loc}. \]
The fact that $\beta > 0$ follows from the regularity results in [12], [29] and [35]. We then fix some $\alpha > 0$ such that

$$
\alpha < \beta.
$$

We now state our first relevant approximation lemma which plays a very crucial role in the separation of phases. This is analogous to Lemma 2.3 in [36].

**Lemma 3.1.** Let $u$ be a $W^{2,q}$ viscosity solution of

$$
\left( \delta_{ij} + (p-2) \frac{(\delta u_i + A_i)(\delta u_j + A_j)}{\delta \nabla u + A^2} \right) u_{ij} = f \quad \text{in } B_1,
$$

with $|u| \leq 1$, $u(0) = 0$ and $|A| \geq 1$. Given $\tau > 0$, there exists $\delta_0 = \delta_0(\tau) > 0$ such that if

$$\delta_0 \left( \frac{1}{|B_{3/4}|} \int_{B_{3/4}} |f|^q \right)^{1/q} \leq \delta_0,$$

then for some $w \in C^2(B_{1/2})$ with universal $C^2$ bounds depending only on $n, p$ and independent of $|A|$, we have that

$$
\begin{aligned}
&w(0) = 0 \\
&||w-u||_{L^\infty(B_{1/2})} \leq \tau
\end{aligned}
$$

**Proof.** We argue by contradiction. If not, then there exists $\tau_0 > 0$ and a sequence of pairs $\{u_k, f_k\}$ that solves (3.2) corresponding to $\{\delta_k, A_k\}$ with $\delta_k \to 0, f_k \to 0$ in $L^q(B_{3/4})$ as $k \to \infty$ and such that $u_k'$s are not $\tau_0$ close to any such $w$. We note that the equation satisfied by $u_k$ can be rewritten as

$$
\left( \delta_{ij} + (p-2) \frac{(\delta_k u_{ij} + (A_k)_{ij})(\delta_k (u_k)_j + (A_k)_j)}{\delta_k \nabla u_k + A_k^2} \right)(u_k)_{ij} = f_k,
$$

where $\delta_k = \frac{\delta_k}{|A_k|}$ and $A_k = \frac{A_k}{|A_k|}$. Since $|A_k| \geq 1$, we have $\delta_k \to 0$ as $k \to \infty$.

From the Krylov-Safonov-type estimates as in [16], we now observe that $u_k$'s are uniformly Hölder continuous in $B_{3/4}$. Therefore, up to a subsequence, by Arzela-Ascoli we may assume that $u_k \to u_0$ uniformly on $B_{3/4}$ and, moreover, we can also assume that $A_k \to A_0$ by possibly passing to another subsequence) such that $|A_0| = 1$.

We now make the following claim.

**Claim:** $u_0$ solves

$$
\left( \delta_{ij} + (p-2)(A_0)_{ij}(A_0)_j \right)(u_0)_{ij} = 0.
$$

By standard theory, it suffices to check that $u_0$ is a $C^{2-}$ viscosity solution to the above limiting equation. We note that the stability result in Theorem 3.8 in [10] can not be directly applied here, because of the singular dependence of the operator in the “gradient” variable. We, however, show that the proof of Theorem 3.8 can still be adapted in this situation. Let $\phi$ be a $C^2$ function such that the graph of $\phi$ strictly touches the graph of $u_0$ from above at $x_0 \in B_{1/2}$. We show that at $x_0$,

$$
\left( \delta_{ij} + (p-2)(A_0)_{ij}(A_0)_j \right) \phi_{ij} \geq 0.
$$
Suppose that is not the case. Then, there exists \( \varepsilon, \eta, r > 0 \) small enough such that

\[
\begin{aligned}
\delta_{ij} + (p-2)(A_0)_{ij}(A_0)_{ij} \phi_{ij} &\leq -\varepsilon \text{ in } B_r(x_0), \\
\phi - u &\geq \eta \text{ on } \partial B_r(x_0).
\end{aligned}
\] (3.7)

We now show that for every \( k \), there exists perturbed test functions \( \phi + \phi_k \) with \( \phi_k \in W^{2,q} \) such that

\[
F_k^*(\nabla(\phi + \phi_k), \nabla^2(\phi + \phi_k)) \leq f_k - \varepsilon \text{ in } B_r(x_0),
\] (3.8)

where \( F_k^* \) is the upper semicontinuous relaxation of the operator in (3.4). Moreover, we can also ensure that \( (\phi + \phi_k) - u_k \) has a minimum in \( B_r(x_0) \) for large enough \( k' \)'s. This would then contradict the viscosity formulation for \( u_k \) for such \( k' \)'s and hence (3.6) would follow.

Therefore, under the assumption that (3.7) holds, we now show the validity of (3.8). We first observe that from (3.7), the following differential inequality holds,

\[
F_k^*(\nabla(\phi + \phi_k), \nabla^2(\phi + \phi_k)) \leq \mathcal{P}_{\lambda,\Lambda}^+(\nabla^2\phi_k) + C_0|\bar{A}_k - A_0| + C_0\bar{\delta}_k|\nabla\phi_k| + C_0\tilde{\delta}_k|\nabla\phi| - \varepsilon,
\] (3.9)

where \( C_0 = C_0(||\nabla^2\phi||, p, n) \) and \( \lambda, \Lambda \) are as in (2.1). This inequality above follows by adding and subtracting \( \delta_{ij} + (p-2)(A_0)_{ij}(A_0)_{ij} \phi_{ij} \), by using (3.7), and then by splitting the considerations depending on whether

\[
|A_0 - (\bar{A}_k + \tilde{\delta}_k(\nabla\phi + \nabla\phi_k))| < 1/2 \text{ or } > 1/2.
\]

We now let \( \phi_k \) be a strong solution to the following boundary value problem

\[
\begin{aligned}
\mathcal{P}_{\lambda,\Lambda}^+(\nabla^2\phi_k) + C_0|\bar{A}_k - A_0| + C_0\bar{\delta}_k|\nabla\phi_k| + C_0\tilde{\delta}_k|\nabla\phi| &= f_k \text{ in } B_r(x_0), \\
\phi_k &= 0 \text{ on } \partial B_r(x_0).
\end{aligned}
\] (3.10)

The existence of such strong \( W^{2,q} \) solutions is guaranteed by Corollary 3.10 in [10]. Therefore, with such \( \phi_k \), we have that (3.8) holds.

We now observe that since \( f_k \to 0 \) in \( L^q \) and also \( \bar{\delta}_k, |\bar{A}_k - A_0| \to 0 \), from the generalized maximum principle, as in [10], we have that

\[
||\phi_k||_{L^\infty(B_r)} \to 0 \text{ as } k \to \infty.
\]

Now, since \( \phi - u \) has a strict minimum at \( x_0 \), it follows for large \( k' \)'s that \( (\phi + \phi_k) - u_k \) has a minimum in the inside of \( B_r(x_0) \) (since \( \phi_k \equiv 0 \) on \( \partial B_r(x_0) \) and \( \phi - u > \eta \) on \( \partial B_r(x_0) \)). From this, as we mentioned before, (3.6) follows.

Then, by an analogous argument we would have that the opposite inequality holds in (3.6), when instead the graph of \( \phi \) touches the graph of \( u \) from below at \( x_0 \) and consequently it follows that \( u_0 \) solves (3.5). Moreover, since \( |u_0| \leq 1 \), we have from the classical theory that \( u_0 \) is smooth with universal \( C^2 \) bounds in \( B_{1/2} \). This would then be a contradiction for large enough \( k' \)'s since \( u_k \to u_0 \) uniformly. This finishes the proof of the lemma.

\( \square \)

As a consequence of Lemma 3.1, we have the following result on the affine approximation of \( u \) at 0, provided there is a sufficiently large non-degenerate slope at a certain scale. As the reader will see, such is ensured by the fast geometric convergence of the approximations.
Lemma 3.2. Let $u$ be a viscosity solution of
\[
\left( \delta_{ij} + (p-2) \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij} = f \quad \text{in } B_1,
\]
with $u(0) = 0$. Then, there exists a universal $\delta_0 > 0$ such that if for some $A \in \mathbb{R}^n$, satisfying $M \geq |A| \geq 2$, we have
\[
||u - < A, x >||_{L^\infty(B_1)} \leq \delta_0,
\]
and also
\[
\int_0^1 \left( \int_{B_s} |f|^q \right)^{1/q} ds \leq \delta_0^2,
\]
then there exists an affine function $L_0$ such that
\[
\begin{cases}
1 \leq |\nabla L_0| \leq M + 1 \\
|u(x) - L_0(x)| \leq C|x| K(|x|)
\end{cases}
\]
Here $K(r) \doteq r^{\alpha/2} + \int_0^{r^{1/2}} (\int_{B_s} |f|^q)^{1/q} ds$ and $\alpha$ is the universal parameter as in (3.1). Moreover $\delta_0$ can be chosen independent of $M$. In view of Remark 1, we note that for $f \in L(1,1)$, we have that $K(r) \to 0$ as $r \to 0$.

Proof. We will show that for every $k = 0, 1, 2, \ldots$, there exist linear functions $\bar{L}_k x \doteq < A_k, x >$ such that
\[
\begin{cases}
||u - \bar{L}_k||_{L^\infty(B_{rk})} \leq r^k \omega(r^k), \\
|A_k - A_{k-1}| \leq C \omega(r^{k-1}),
\end{cases}
\]
for some $r < 1$ universal, independent of $\delta_0$. Here we let for a given $k$,
\[
\omega(r^k) = \frac{1}{\delta_0} \sum_{i=0}^k r^i \omega_1 \left( \frac{3}{4} r^{k-i} \right),
\]
with $\omega_1$ defined in the following way
\[
\omega_1(t) = \max \left( t \left( \int_{B_{2t}} |f|^q \right)^{1/q}, \delta_0 \frac{4}{3} t \right).
\]
We note that $\delta_0$ is to be fixed later. We also let $A_0 \doteq A$. Now, suppose $A_k$ exists upto some $k$ with the bounds as in (3.12). Then, we observe that
\[
|A_k| \geq |A_0| - (|A_1 - A_0| + \ldots + |A_k - A_{k-1}|)
\]
\[
> 2 - C \sum \omega(r^i) > 2 - \frac{C}{\delta_0} \sum \omega_1 \left( \frac{3}{4} r^i \right) \quad \text{(using the Cauchy product formula)}
\]
\[
\geq 2 - C_1 \delta_0 > 1 \quad \text{(if $\delta_0$ is small enough)}.
\]
In the last inequality above we also used the fact that
\[
\sum \omega_1 \left( \frac{3}{4} r^i \right) \leq C \left( \delta_0^2 \sum r^i + \frac{3}{4} \left( \int_{B_{2r^i}} |f|^q \right)^{1/q} \right)
\]
\[
\leq C \left( \delta_0^2 + \int_0^1 \left( \int_{B_{2s}} |f|^q \right)^{1/q} ds \right) \leq C_2 \delta_0^2.
\]
Note that the last inequality in (3.15) is a consequence of the following estimate
\[
\sum \frac{3r^i}{4} \left( \int_{B_{\frac{4}{3}r^i}} |f|^q \right)^{1/q} \leq C \int_0^1 \left( \int_{B_s} |f|^q \right)^{1/q} ds,
\]
which in turn follows by breaking the integral in the above expression into integrals over dyadic subintervals of the type $[\frac{3}{4}r^i, \frac{3}{4}r^{i-1}]$.

Thus the estimate in (3.14) ensures that the non-degeneracy condition in Lemma 3.1 holds for every $k$. We prove the claim in (3.12) by induction. From the hypothesis of the lemma, the case when $k = 0$ is easily verified with $A_0 = A$ with our choice of $\omega$. Let us now assume that the claim as in (3.12) holds up to some $k$. We then consider
\[
v = \frac{u - \tilde{L}_k(r^k x)}{r^k \omega(r^k)},
\]
which solves
\[
(3.16) \quad (\delta_{ij} + (p - 2)(\omega(r^k)v_i + (A_k)_i)(\omega(r^k)v_j + (A_k)_j)/|\omega(r^k)\nabla v + A_k|^2) v_{ij} = \frac{r^k}{\omega(r^k)} f(r^k x).
\]
Now, by a change of variable formula and the definition of $\omega$ it follows that, with $f_k(x) = \frac{r^k}{\omega(r^k)} f(r^k x)$, we have
\[
(3.17) \quad \left( \frac{1}{|B_{3/4}|} \int_{B_{3/4}} |f_k|^q \right)^{1/q} = \frac{r^k}{\omega(r^k)} \left( \frac{1}{|B_{3r^k/4}|} \int_{B_{3r^k/4}} |f|^q dy \right)^{1/q} \leq \frac{r^k}{\omega(\frac{3r^k}{4})} \left( \frac{1}{|B_{3r^k/4}|} \int_{B_{3r^k/4}} |f|^q dy \right)^{1/q} \leq \frac{1}{3\delta_0} \left( \frac{1}{|B_{3r^k/4}|} \int_{B_{3r^k/4}} |f|^q dy \right)^{1/q} \leq 4\delta_0.
\]
Moreover,
\[
\omega(r^k) \leq \sum \omega(r^i) \leq C_0 \delta_0.
\]
Therefore, $v$ satisfies an equation for which the conditions in Lemma 3.1 are satisfied. Consequently for a given $\tau > 0$, we can find $\delta_0 > 0$ such that for some $w$ with universal $C^2$ bounds we have that $||w - v||_{L^\infty(B_{1/2})} \leq \tau$. Now since $w$ has uniform $C^2$ bounds and $w(0) = 0$, there exists a universal $C > 0$ such that
\[
|w - Lx| \leq C|x|^2,
\]
where $L$ is the linear approximation for $w$ at 0. We then choose $r$ small enough such that
\[
Cr^2 = \frac{r^{1+\alpha}}{2},
\]
where $\alpha$ is as in (3.1). Subsequently, we let $\tau = \frac{r^{1+\alpha}}{2}$ which decides the choice of $\delta_0$. Then, by an application of triangle inequality we have,

$$||v - L||_{L^\infty(B_1)} \leq r^{1+\alpha}.$$ 

Consequently by scaling back to $u$ we obtain

$$||u - \tilde{L}_{k+1}||_{L^\infty(B_{r^{k+1}})} \leq r^{k+1}r^\alpha \omega(r^k) \leq r^{k+1}\omega(r^{k+1}),$$

where $\tilde{L}_{k+1}(x) = \tilde{L}_k + r^k \omega(r^k)L\left(\frac{x}{r^k}\right)$. Note that in the last inequality in (3.18) we also used the following $\alpha$–decreasing property of $\omega$

$$r^\alpha \omega(r^k) \leq \omega(r^{k+1}),$$

which is easily seen from the expression of $\omega$ as in (3.13) (see also the proof of Lemma 4.7 in [2]). This verifies the induction step. The conclusion now follows by a standard analysis argument as in the proof of Lemma 4.9 in [2].

□

The next result is an improvement of flatness result that allows to handle the case when the affine approximation have small slopes at a “kth step”. This corresponds to the degenerate alternative in the iterative argument in the proof of the main result Theorem 2.2.

**Lemma 3.3.** Let $u$ be a solution to

$$\left(\delta_{ij} + (p-2)\frac{u_iu_j}{|\nabla u|^2}\right)u_{ij} = f \quad \text{in } B_1,$$

with $|u| \leq 3$ and $u(0) = 0$. There exists a universal $\varepsilon_0 > 0$ such that if

$$\int_0^1 \left(\int_{B_s} |f|^q\right)^{1/q} ds \leq \varepsilon_0,$$

then there exists an affine function $L$, with universal bounds, and a universal $\eta \in (0,1)$ such that

$$||u - L||_{L^\infty(B_{\eta})} \leq \delta_0 \eta^{1+\alpha}.$$ 

Here $\delta_0$ is as in Lemma 3.2 above. Without loss of generality we may take $\varepsilon_0 < \delta_0^2$. 

**Proof.** First note that (3.21) implies that

$$||f||_{L^q(B_{s/4})} < C\varepsilon_0.$$ 

We first show that given $\kappa > 0$, there exists $\varepsilon_0 > 0$ such that if $u$ solves (3.20) and $f$ satisfies the bound in (3.21), then there exists a $p$–harmonic function $w$ such that

$$||w - u||_{L^\infty(B_{1/2})} \leq \kappa.$$ 

Assume that (3.22) actually holds. It then follows from the $C^{1,\beta}$ regularity results for $p$-harmonic functions in [12], [29] and [35] that there exists an affine function $L$ such that

$$|w(x) - L(x)| \leq C|x|^{1+\beta}.$$ 

We now choose $\eta > 0$ such that

$$C\eta^{1+\beta} = \frac{\delta_0}{2}\eta^{1+\alpha} \quad (\text{This crucially uses } \alpha < \beta).$$

Subsequently, we choose $\kappa = \frac{\delta_0}{2}\eta^{1+\alpha}$, and this decides the choice of $\varepsilon_0$. The conclusion of the lemma now follows by an application of the triangle inequality.
We are now going to prove (3.22). Then there exists \( \kappa_0 > 0 \) and a sequence of pairs \( \{u_k, f_k\} \) which solves (3.20) with \( f_k \) satisfying (3.21) (with \( \varepsilon = \frac{1}{k} \)) such that \( u_k \) is not \( \kappa_0 \) close to any such \( w \). Then from uniform Krylov-Safanov type Hölder estimates as in [16] and Arzela-Ascoli, it follows that \( u_k \to u_0 \) uniformly in \( B_{1/2} \) up to a subsequence. We aim the following claim.

**Claim:** \( u_0 \) is \( p \)-harmonic.

Once the claim is established, this would then be a contradiction for large enough \( k \)'s and thus (3.22) would follow.

The proof is similar to that of the **Claim** in Lemma 3.1. As before, we note that the stability result in Theorem 3.8 in [10] cannot be directly applied because the operator \( \Delta^N_p \) does not satisfy the structural assumptions in [10] because of singular dependence in the “gradient” variable. We first observe that it follows from [24] that in order to show that \( u_0 \) is \( p \)-harmonic, it suffices to show that \( u_0 \) satisfies the viscosity formulation at points where the gradient of the test function does not vanish.

Let \( \phi \) be a \( C^2 \) test function which strictly touches the graph of \( u \) from above at some point \( x_0 \in B_{1/2} \) such that \( \nabla \phi(x_0) \neq 0 \). We claim that

\[
\Delta^N_p \phi(x_0) \geq 0. \tag{3.23}
\]

Suppose such is not the case. Then there exists \( \varepsilon, r, \delta > 0 \) small enough such that

\[
\begin{cases}
\Delta^N_p \phi(x) \leq -\varepsilon & \text{for } x \in B_r(x_0), \\
\phi - u > \delta & \text{on } \partial B_r(x_0).
\end{cases} \tag{3.24}
\]

Moreover, we can also assume that in \( B_r(x_0) \), we have that

\[
|\nabla \phi| \geq \kappa > 0. \tag{3.25}
\]

We now show that for every \( k \), there exists perturbed test functions \( \phi + \phi_k \) with \( \phi_k \in W^{2,q} \) such that

\[
F^*(\nabla(\phi + \phi_k), \nabla^2(\phi + \phi_k)) \leq f_k - \varepsilon \quad \text{in } B_r(x_0), \quad \text{with } F^* \text{ as in } (2.5). \tag{3.26}
\]

Moreover, we can also ensure that \( (\phi + \phi_k) - u_k \) has a minimum in \( B_r(x_0) \) for large enough \( k \)'s. This would then contradict the viscosity formulation for \( u_k \), and hence (3.23) would follow. In an entirely analogous way, we will have that if a \( C^2 \) test function strictly touches \( u \) from below at \( x_0 \) then

\[
\Delta^N_p \phi(x_0) \leq 0,
\]

and consequently we can assert from the results in [24] that \( u_0 \) is \( p \)-harmonic.

Hence under the assumption that (3.24) is valid, we now turn our attention to establish (3.26). We first observe that because of (3.24), (3.25), the following inequality holds,

\[
F^*(\nabla(\phi + \phi_k), \nabla^2(\phi + \phi_k)) \leq \mathcal{P}^+_{\lambda, \Lambda}(\nabla^2 \phi_k) + C(\kappa, ||\nabla^2 \phi||)||\nabla \phi_k|| - \varepsilon, \tag{3.27}
\]

with \( \lambda, \Lambda \) as in (2.1). Here \( \mathcal{P}^+_{\lambda, \Lambda} \) is the maximal Pucci operator defined as in (2.3). This inequality again follows by adding and subtracting \( \Delta^N_p \phi \), by using (3.24) and then by splitting considerations depending on whether

\[
|\nabla \phi_k| < \kappa/2 \text{ or } > \kappa/2.
\]

At this point, given \( k \), we look for \( \phi_k \) which is a strong solution to

\[
\begin{cases}
\mathcal{P}^+_{\lambda, \Lambda}(\nabla^2 \phi_k) + C(\kappa, ||\nabla^2 \phi||)||\nabla \phi_k|| = f_k \quad \text{in } B_r(x_0), \\
\phi_k = 0 \quad \text{on } \partial B_r(x_0).
\end{cases} \tag{3.28}
\]
The existence of such strong solutions is again guaranteed by Corollary 3.10 in [10]. Moreover since \( f_k \to 0 \) in \( L^q \), therefore from the generalized maximum principle we have that

\[
||\phi_k||_{L^\infty(B_r)} \to 0 \text{ as } k \to \infty.
\]

Now since \( \phi - u \) has a strict minimum at \( x_0 \), it follows that for large \( k \)'s that \( (\phi + \phi_k) - u_k \) would have a minimum in the inside of \( B_r(x_0) \) (since \( \phi_k \equiv 0 \) on \( \partial B_r(x_0) \) and \( \phi - u > \delta \) on \( \partial B_r(x_0) \)). However because of (3.27) and (3.28) we also have that (3.26) holds which violates the viscosity formulation for \( u_k \)'s for large enough \( k \)'s. Thus in view of our discussion above, we can assert that \( u_0 \) is \( p \)-harmonic and this concludes the proof.

With this Lemma 3.2 and Lemma 3.3 in hand, we now proceed with the proof of our main result.

**Proof of Theorem 2.2.** We will show that there exists an affine function \( \tilde{L} \) such that

\[
(3.29) \quad |u(x) - \tilde{L}(x)| \leq C|x|K_0(4|x|),
\]

where \( K_0(|x|) \) is defined as

\[
K_0(|x|) = \left( \int_0^1 \left( \int_{B_s} |f|^q \right)^{1/q} ds + 1 \right) |x|^\alpha/4 + C_0(\alpha) \int_0^{|x|^{1/4}} \left( \int_{B_s} |f|^q \right)^{1/q} ds
\]

, and some universal \( C \).

Likewise a similar affine approximation holds at all points in \( B_{1/2} \) and consequently the estimates in (2.7) follow by a standard real analysis argument.

We may assume that \( u(0) = 0 \). Now with \( \eta, \varepsilon_0 \) as in lemma 3.3 and \( \delta_0 \) as in lemma 3.2, assume the following hypothesis for a given \( i \in \mathbb{N} \),

\[
[H] \quad \begin{cases}
\text{There exists affine function } L_i(x) = \langle B_i, x \rangle \text{ such that } ||u - L_i||_{L^\infty(B_{\eta^i})} \leq \delta_0 \eta^i \omega(\eta^i) \\
\text{and } |B_i| \leq 2\omega(\eta^i).
\end{cases}
\]

Here \( \omega \) is defined instead as

\[
(3.30) \quad \omega(\eta^k) = \frac{1}{\varepsilon_0} \sum_{i=0}^k \eta^i \omega_1(\eta^{k-i}),
\]

where we let \( \omega_1 \) to be

\[
\omega_1(t) = \max \left( \int_0^t \left( \int_{B_s} |f|^q \right)^{1/q} ds, t \right).
\]

By multiplying \( u \) with a suitable constant we can assume that the Statement \([H]\) holds when \( i = 0 \) with \( L_0 = 0 \). Let \( k \) be the first integer such that the Statement \([H]\) breaks. Then there are two possibilities.

**Case 1:** Suppose \( k = \infty \). Then let given \( x \), let \( i \in \mathbb{N} \) be such that \( |x| \sim \eta^i \). Then from the inequalities in \([H]\) and triangle inequality, it follows that

\[
(3.31) \quad |u(x)| \leq |u(x) - \langle B_i, x \rangle| + |\langle B_i, x \rangle| \leq C_1 \eta^i \omega(\eta^i) \leq C_1 |x| \omega(2|x|) \leq C|x|K_0(4|x|),
\]
and thus (3.29) follows with $\tilde{L} = 0$. The last inequality in (3.31) is seen as follows:

\begin{align}
\omega(\eta^{i'}) &= \frac{1}{\varepsilon_0} \sum_{j=0}^{i} \eta^{i' \alpha} \omega(\eta^{j'}) \\
&\leq C \omega_1(\eta^{i'/2}) \sum_{j=0}^{i/2} \eta^{j \alpha} + C \omega_1(1) \sum_{j=i/2}^{i} \eta^{j \alpha} \quad \text{(here we use $\omega_1$ is increasing)} \\
&\leq C \left( \int_0^1 \left( \int_{B_s} |f|^q \right)^{1/q} \, ds + 1 \right) \eta^{i'/2} + C_0(\alpha) \int_0^{\eta^{i'/2}} \left( \int_{B_s} |f|^q \right)^{1/q} \, ds \\
&\leq C \tilde{K}_0(4|x|) \quad \text{(using $|x| \sim \eta^i$)}.
\end{align}

**Case 2:** Suppose instead that $k < \infty$. Then we have that the Statement $[H]$ is satisfied upto $k - 1$. Now let

$$v(x) = \frac{u(\eta^{-1} x)}{\eta^{-1} \omega(\eta^{-1})},$$

which solves

$$\left( \delta_{ij} + (p-2) \frac{v_i v_j}{|v|^2} \right) v_{ij} = \frac{\eta^{-1} f(\eta^{-1} x)}{\omega(\eta^{-1})}.$$

Moreover, from the estimates in (3.1) for $i = k - 1$ it follows that $|v| \leq 2 + \delta_0 \leq 3$. Also by change of variable, we have that for $f_k(x) = \frac{\eta^{-1} f(\eta^{-1} x)}{\omega(\eta^{-1})}$, the following holds,

\begin{align}
\int_0^1 \left( \int_{B_s} |f_k|^q \right)^{1/q} \, ds \\
&\leq \varepsilon_0 \frac{\eta^{-1} \int_0^1 \left( \int_{B_s} |f(\eta^{-1} x)|^q \, dx \right)^{1/q} \, ds}{\int_0^{\eta^{-1}} \left( \int_{B_s} |f|^q \, dx \right)^{1/q} \, ds} \\
&= \varepsilon_0 \frac{\int_0^{\eta^{-1}} \left( \int_{B_s} |f|^q \, dx \right)^{1/q} \, ds}{\int_0^{\eta^{-1}} \left( \int_{B_s} |f|^q \, dx \right)^{1/q} \, ds} \quad \text{(by change of variable)} \\
&= \varepsilon_0.
\end{align}

Here we have used also that

$$\omega(\eta^{-1}) \geq \frac{1}{\varepsilon_0} \int_0^{\eta^{-1}} \left( \int_{B_s} |f|^q \, dx \right)^{1/q}.$$

Hence, $v$ solves an equation of the type (1.1) such that the hypothesis in Lemma 3.3 is satisfied. Therefore, by applying Lemma 3.3, we obtain that there exists an affine function $Lx = \langle \tilde{A}, x \rangle$ such that

$$||v - L||_{L^\infty(B_{\eta})} \leq \delta_0 \eta^{1+\alpha}.$$

Scaling back to $u$, we obtain with $L_k x = \langle B_k, x \rangle$, where $B_k = \omega(\eta^{-1}) \tilde{A} x$, that

\begin{align}
||u - L_k||_{L^\infty(B_{\eta})} \leq \delta_0 \eta^k \eta^\alpha \omega(\eta^{-1}) \leq \delta_0 \eta^k \omega(\eta^k),
\end{align}

where in the last inequality, we used the $\alpha-$decreasing property of $\omega$ (as in (3.19)). This property is easily seen from the expression of $\omega$ in (3.30). However, since the Statement $[H]$
does not hold for \( i = k \), we must necessarily have
\[
(3.35) \quad |B_k| \geq 2\omega(\eta^k). 
\]
We now let
\[
\tilde{v} = \frac{u(\eta^k x)}{\eta^k \omega(\eta^k)}. 
\]
Then, we observe that \( \tilde{v} \) solves
\[
\left( \delta_{ij} + (p - 2) \frac{\tilde{v}_i \tilde{v}_j}{|\nabla \tilde{v}|^2} \right) \tilde{v}_{ij} = \frac{\eta^k f(\eta^k x)}{\omega(\eta^k)}. 
\]
Moreover, from (3.34) we have, with
\[
(3.36) \quad A = \frac{\omega(\eta^{k-1}) \tilde{A}}{\omega(\eta^k)}, 
\]
that the following inequality holds
\[
(3.37) \quad \|\tilde{v} - A, x\|_{L^\infty(B_1)} \leq \delta_0. 
\]
Moreover, using that \( |\tilde{A}| \leq C \), where \( C \) is universal, and the \( \alpha \)-decreasing property of \( \omega \), we obtain
\[
(3.38) \quad |A| = \frac{|\tilde{A}| \eta^\alpha \omega(\eta^{k-1})}{\eta^\alpha \omega(\eta^k)} \leq \frac{C}{\eta^\alpha}. 
\]
Also (3.35) implies
\[
|A| \geq 2. 
\]
Now again by change of variables it is seen that \( \tilde{f}_k \), defined by
\[
(3.39) \quad \tilde{f}_k(x) = \frac{\eta^k f(\eta^k x)}{\omega(\eta^k)}, 
\]
satisfies the estimate as in (3.33). Now using the fact that \( \varepsilon_0 < \delta_0^2 \), we find that \( \tilde{v} \) satisfies the conditions in Lemma 3.2. Hence, there exists an affine function \( L_0 x \doteqdot < A_0, x > \), with universal bounds depending on \( \eta \) (more specifically on \( \frac{C}{\eta^\alpha} \)), such that
\[
(3.40) \quad |\tilde{v}(x) - L_0(x)| \leq C|x|K_{\tilde{f}_k}(\eta^{k}/|x|), \quad |x| < 1, 
\]
where \( K_{\tilde{f}_k}(|x|) = |x|^{\alpha/2} + \int_0^{|x|^{1/2}} (f_{\tilde{f}_k})^{1/q} ds \) with \( \tilde{f}_k \) as in (3.39). Then, by scaling back to \( u \), letting \( \eta^k x \) as our new \( x \), we obtain for \( |x| \leq \eta^k \) that the following holds by change of variables,
\[
(3.41) \quad |u(x) - \omega(\eta^k) < A_0, x > | \leq C|x| \left( \omega(\eta^k)|y|^{\alpha/2} + \int_0^{\eta^k |y|^{1/2}} (\int_{B_s} |f|^q)^{1/q} ds \right) \quad (y = \eta^{-k}x) 
\]
\[
\leq C|x| \left( \omega(\eta^{k/2})|y|^{\alpha/2} + \int_0^{\eta^{k/2} |y|^{1/2}} (\int_{B_s} |f|^q)^{1/q} ds \right) \quad \text{(using} \ \eta^k \leq \eta^{k/2} \ \text{and} \ \omega(\eta^k) \leq \omega(\eta^{k/2}) \text{)} 
\]
\[
= C|x| \left( \omega(\eta^{k/2})|y|^{\alpha/2} + \int_0^{|x|^{1/2}} (\int_{B_s} |f|^q)^{1/q} ds \right). 
\]
Now, let $j$ be the smallest integer such that $|y| \leq \eta^j$. Then, we have that
\begin{equation}
\omega(\eta^{k/2})|y|^{\alpha/2} \leq \omega(\eta^{k/2})\eta^{j\alpha/2}
\end{equation}
\begin{equation}
= \frac{1}{\varepsilon_0} \sum_{i=j/2}^{k+j/2} \eta^{\alpha_1} \omega_1(\eta^{k+j/2-i}) \leq \omega(\eta^{k/2})
\end{equation}
\begin{equation}
\leq C \left[ \left( \int_0^1 \left( \int_{B_s} |f|^q \right)^{1/q} \right) \left| x \right|^{\alpha/4} + \int_0^{|x|/4} \left( \int_{B_s} |f|^q \right)^{1/q} ds \right]
\end{equation}
\begin{equation}
\leq CK_0(4|x|) \quad \text{(using } y = \eta^{-k} x), \end{equation}
where the last inequality in (3.42) follows from a computation as in (3.32). This implies that (3.29) holds with $L x \doteqdot <\omega(\eta^k)A_0, x >$, when $|x| \leq \eta^k$.

Now when $|x| \geq \eta^k$, one can show that
\begin{equation}
|u(x)| \leq C|x|\omega(2|x|) \leq C|x|K_0(4|x|).
\end{equation}
This follows from the fact that with $L_i x \doteqdot < B_i, x >$ we have for $i = 0, \ldots, k - 1$,
\begin{equation}
||u-L_i||_{L^\infty(B_{\eta^i})} \leq \delta_0 \eta^i \omega(\eta^i)
\end{equation}
and
\begin{equation}
|B_i| \leq 2\omega(\eta^i)
\end{equation}
because (3.1) holds upto $k - 1$. And moreover for $i = k$, we again have
\begin{equation}
||u-L_k||_{L^\infty(B_{\eta^k})} \leq \delta_0 \eta^k \omega(\eta^k).
\end{equation}
In this case, instead the following bound holds
\begin{equation}
|B_k| \leq C\omega(\eta^{k-1}) \leq \frac{C\omega(\eta^k)}{\eta^s} \quad \text{using } \alpha-\text{decreasing property of } \omega
\end{equation}
Using such estimates, it is easy to see that (3.43) holds. Now note that with $\tilde{L} x \doteqdot < \tilde{B}, x >$, with $\tilde{B} \doteqdot \omega(\eta^k)A_0$, we also have the following bound
\begin{equation}
|\tilde{B}| \leq C\omega(\eta^k).
\end{equation}
Therefore, it follows from (3.43) and the estimate (3.44) above that
\begin{equation}
|u(x) - \tilde{L}(x)| \leq C|x|K_0(4|x|)
\end{equation}
also holds when $|x| \geq \eta^k$, for a possibly different $C$. Hence the estimate in (3.29) follows with $\tilde{L} x \doteqdot < \tilde{B}, x >$ and this finishes the proof of the theorem. \qed

3.2. Proof of Theorem 2.3. In this subsection, we assume that $u$ is a $W^{2,m}$ viscosity solution to
\begin{equation}
\left( \delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla u|^2} \right) u_{ij} = f,
\end{equation}
where $f \in L^m$ for some $m > n$. We now state and prove the counterparts of the approximation lemmas in this situation. The analogue of Lemma 3.1 is as follows.
Lemma 3.4. Let $u$ be a $W^{2,m}$ viscosity solution to

$$
(\delta_{ij} + (p - 2)\frac{\delta u_i + A_j}{|\delta \nabla u + A|^2}) u_{ij} = f \quad \text{in } B_1,
$$

with $|u| \leq 1$ and $|A| \geq 1$. Given $\tau > 0$, there exists $\delta_0 = \delta_0(\tau) > 0$ such that if

$$
\delta_i \left( \frac{1}{|B_{3/4}|} \int_{B_{3/4}} |f|^m \right)^{1/m} \leq \delta_0,
$$

then $||w - u||_{L^\infty(B_{1/2})} \leq \tau$ for some $w \in C^2(B_{1/2})$ satisfying $w(0) = 0$ with universal $C^2$ bounds depending only on $n, p$ and independent of $|A|$.

Proof. The proof is identical to that of Lemma 3.1 and so we omit the details. □

We now state the counterpart of Lemma 3.2.

**Lemma 3.5.** Let $u$ be a viscosity solution to

$$
(\delta_{ij} + (p - 2)\frac{u_i u_j}{|\nabla u|^2}) u_{ij} = f
$$

in $B_1$ with $u(0) = 0$. Then there exists a universal $\delta_0 > 0$, such that if for some $A \in \mathbb{R}^n$ satisfying $M \geq |A| \geq 2$ we have

$$
||u - <A, x>||_{L^\infty(B_1)} \leq \delta_0,
$$

and also

$$
||f||_{L^m(B_1)} \leq \delta_0^2,
$$

then there exists an affine function $L_0$ such that

$$
\begin{cases}
1 \leq |\nabla L_0| \leq M + 1 \\
|u(x) - L_0(x)| \leq C|x|^{1+\alpha_0}
\end{cases}
$$

where $\alpha_0 < \min(\alpha, 1 - n/m)$. Moreover, $\delta_0$ can be chosen independent of $M$.

Proof. As in the proof of Lemma 3.2, we show that for every $k \in \mathbb{N}$, there exists affine functions $\tilde{L}_k x = <A_k, x>$ such that

$$
\begin{cases}
||u - \tilde{L}_k x||_{L^\infty(B_{1/2})} \leq \delta_0 r^{k(1+\alpha_0)}, \\
|A_k - A_{k+1}| \leq C\delta_0 r^{k\alpha_0},
\end{cases}
$$

for some $r < 1$ universal independent of $\delta_0$. The conclusion of the lemma then follows from (3.49) in a standard way. We first observe that (3.49) holds for $k = 0$ with $A_0 = A$. Moreover the non-degeneracy condition as in (3.14) is easily verified in this situation provided $\delta_0$ is small enough. Now assume (3.49) holds up to some $k$. We then define

$$
v = \frac{u - \tilde{L}_k(r^k x)}{\delta_0 r^{k(1+\alpha_0)}}.
$$

Then $v$ solves in $B_1$

$$
(\delta_{ij} + (p - 2)\frac{\delta r^{k\alpha_0} v_i + (A_k)_i)(\delta r^{k\alpha_0} v_j + (A_k)_j)}{|\delta r^{k\alpha_0} \nabla v + A_k|^2}) v_{ij} = f_k,
$$
where \( f_k \) is defined as
\[
f_k(x) = r^{k(1-\alpha_0)} f(r^k x) / \delta_0.
\]
Now by change of variable it is seen that
\[
\|f_k\|_{L^m(B_1)} = r^{k(1-\alpha_0) \frac{1}{\delta_0}} \|f\|_{L^m(B_r)} \leq \delta_0.
\]
Note that over here, we crucially used the hypothesis of the lemma i.e,
\[
\|f\|_{L^m(B_1)} \leq \delta_0^2,
\]
and the fact that \( \alpha_0 < 1 - n/m \). Therefore, \( \nu \) satisfies the hypothesis of Lemma 3.4 and at this point we can repeat the arguments in the proof of Lemma 3.2 to conclude that there exists \( \tilde{L}_{k+1}(x) = \tilde{L}_k(x) + \delta_0 r^{k(1+\alpha_0)} L \left( \frac{x}{r} \right) \), where \( L \) has universal bounds such that (3.49) holds for \( k+1 \). This verifies the induction step and the conclusion of the lemma thus follows. □

We also have the following lemma which is the analogue of Lemma 3.3.

**Lemma 3.6.** Let \( u \) be a solution of
\[
\left( \delta_{ij} + (p-2) \frac{u_i u_j}{|u|^2} \right) u_{ij} = f \quad \text{in } B_1,
\]
with \( |u| \leq 3 \) and \( u(0) = 0 \). There exists a universal \( \varepsilon_0 > 0 \) such that if
\[
\|f\|_{L^m(B_1)} \leq \varepsilon_0,
\]
then there exists an affine function \( L \) with universal bounds and a universal \( \eta \in (0,1) \) such that
\[
\|u - L\|_{L^\infty(B_\eta)} \leq \delta_0 \eta^{1+\alpha_0},
\]
where \( \delta_0 \) is as in Lemma 3.5 above. Without loss of generality we may take \( \varepsilon_0 < \delta_0^2 \).

**Proof.** The proof is again identical to that of Lemma 3.3 and thus we skip the details. □

With Lemmas 3.4–3.6 in hand, we now proceed with the proof of Theorem 2.3.

**Proof of Theorem 2.3.** It suffices to show that at 0, there exists an affine function \( \tilde{L} \) with universal bounds such that
\[
|u(x) - \tilde{L}(x)| \leq C|x|^{1+\alpha_0}.
\]
We also assume that \( u(0) = 0 \). Now with \( \eta, \varepsilon_0 \) as in Lemma 3.6 and \( \delta_0 \) as in Lemma 3.5, assume the following hypothesis for a given \( i \in \mathbb{N} \),
\[
[H_1] \begin{cases}
\text{There exists affine function } L_i(x) \prec B_i, x > \text{ such that } \|u - L_i\|_{L^\infty(B_{\eta^i})} \leq \frac{\delta_0}{\varepsilon_0} \eta^{i(1+\alpha_0)} \\
\text{and } |B_i| \leq \frac{\varepsilon_0}{\delta_0} \eta^{i\alpha_0}.
\end{cases}
\]
By multiplying \( u \) with a suitable constant, we may assume that the hypothesis holds for \( i = 0 \) with \( L_0 = 0 \). We can also assume that
\[
\|f\|_{L^m(B_1)} \leq 1.
\]
Let \( k \) be the smallest integer such that (3.53) fails. Then as in the proof of Theorem 2.2, there are two possibilities.

**Case 1:** Suppose \( k = \infty \). Then in this case, (3.52) is seen to hold with \( \tilde{L} = 0 \).
Case 2: Suppose instead that $k < \infty$. Then we have that the hypothesis is satisfied up to $k - 1$. As before, we let

$$v(x) = \varepsilon_0 \frac{u(\eta^{k-1}x)}{\eta^{|k-1|}(1+\alpha_0)},$$

which solves in $B_1$

$$\left( \delta_{ij} + (p-2) \frac{v_i v_j}{|\nabla v|^2} \right) v_{ij} = f_k,$$

where

$$f_k(x) = \varepsilon_0 \eta^{(k-1)(1-\alpha_0)} f(\eta^{k-1}x).$$

Then by change of variable and (3.54), it is again seen that $\|f_k\|_{L^\infty} \leq \varepsilon_0$. Moreover, from (3.53) and triangle inequality it follows that $|v| \leq 2 + \delta_0 \leq 3$. Thus the hypothesis of Lemma 3.6 is satisfied and consequently there exists an affine function $Lx = \langle A, x \rangle$ such that

$$\|v - Lx\|_{L^\infty(B_1)} \leq \delta_0 \eta^{1+\alpha_0}.$$

By scaling back to $u$, we obtain with $L_k x = B_k x$, with $B_k = \eta^{(k-1)\alpha_0} \varepsilon_0 \langle \tilde{A}, x \rangle$, that the following holds,

$$\|u - L_k\|_{L^\infty(B_{\eta k})} \leq \frac{\delta_0}{\varepsilon_0} \eta^{k(1+\alpha_0)}.$$

However since Statement [H1] fails, we must necessarily have

$$|B_k| \geq \frac{2}{\varepsilon_0} \eta^{k\alpha_0}.$$

If we now let

$$\tilde{v}(x) = \varepsilon_0 \frac{u(\eta^k x)}{\eta^{|k|}(1+\alpha_0)},$$

then, as in the proof of Theorem 2.2, it can be easily checked that $\tilde{v}$ solves an equation of the type (1.1) such that the hypothesis of Lemma 3.5 is verified. Hence there exists an affine function $L_0 x = \langle A_0, x \rangle$, with universal bounds depending on $\eta$, such that

$$|\tilde{v} - L_0 x| \leq C |x|^{1+\alpha_0}.$$

By scaling back to $u$ we obtain that, with $\tilde{L}(x) = \frac{\eta^{k\alpha_0}}{\varepsilon_0} < A_0, x >$, the following estimate holds for $|x| \leq \eta^k$,

$$(3.55) \quad |u(x) - \tilde{L}(x)| \leq C |x|^{1+\alpha_0}.$$

The rest of the argument is again the same as in the proof of Theorem 2.2, which allows us to conclude that the estimate (3.55) holds also when $|x| \geq \eta^k$. This finishes the proof of the theorem.

\[\square\]

References

[1] A. Attouchi, *Local regularity for quasi-linear parabolic equations in non-divergence form*, arXiv:1809.03241.
[2] K. Adimurthi & A. Banerjee, *Borderline regularity for fully nonlinear equations in Dini domains*, arXiv:1806.07652.
[3] A. Attouchi, M. Parviainen & E. Ruosteenoja, *C^{1,\alpha} regularity for the normalized p-Poisson problem*, J. Math. Pures Appl. (9) 108 (2017), no. 4, 553-591.
[4] I. Birindelli & F. Demengel, *Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators*, J. Differential Equations 249(5) (2010) 1089-1110.
[5] A. Banerjee & N. Garofalo, *On the Dirichlet boundary value problem for the normalized p-Laplacian evolution*, Commun. Pure Appl. Anal. 14 (2015), 1–21.
Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations, Indiana Univ. Math. J. 62 (2013), 699–736.

Modica type gradient estimates for an inhomogeneous variant of the normalized p-Laplacian evolution, (English summary) Nonlinear Anal. 121 (2015), 458–468.

A. Banerjee & B. Kawohl, overdetermined problems for the normalized p-Laplacian, (English summary) Proc. Amer. Math. Soc. Ser. B 5 (2018), 18–24.

L. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math., 130(1) (1989), pp. 189–213.

L. Caffarelli, M. Crandall, M. Kocan & A. Swiech, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure. Appl. Math. 49 (1996), 365–397.

Crandall, M. H. Ishii, H. & Lions, P.-L., User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., 27 (1992), 1–67.

E. Di Benedetto, C1+α local regularity of solutions to degenerate elliptic equations, Nonlinear Anal. 7 (1983) 827-850.

K. Does, An evolution equation involving the normalized p-Laplacian, Commun. Pure Appl. Anal. 10 (2011), 361–396.

P. Daskalopoulos, T. Kuusi, and G. Mingione, Borderline estimates for fully nonlinear elliptic equations, Comm. Partial Differential Equations 39 (2014), no. 3, 574–590.

Frank Duzaar and Giuseppe Mingione, Gradient estimates via non-linear potentials, Amer. J. Math. 133 (2011), no. 4, 1093–1149.

L. Escauriaza, W2,n a priori estimates for solutions to fully nonlinear equations, (English summary) Indiana Univ. Math. J. 42 (1993), no. 2, 413-423.

P. Fok, Some Maximum Principles and Continuity Estimates for Fully Nonlinear Equations of Second Order, PhD Thesis. Santa Barbara: Univ. California 1996.

F. Hoeg & P Lindqvist, Regularity of Solutions of the Parabolic Normalized p-Laplace Equation, arXiv:1802.04568

C. Imbert, T. Jin & L. Silvestre, Hölder gradient estimates for a class of singular or degenerate parabolic equations, arXiv:1609.01123.

H. Ishii & P. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations, 83 (1990), pp. 26–78.

T. Jin & L. Silvestre, Hölder gradient estimates for parabolic homogeneous p-Laplacian equations, J. Math. Pures Appl. (9) 108 (2017), 63–87.

Juutinen, P. & Kawohl, B., On the evolution governed by the infinity Laplacian, Math. Ann. 335 (2006), 819–851.

Juutinen, P., Decay estimates in the supremum norm for the solutions to a nonlinear evolution equation., Proc. Roy. Soc. Edinburgh Sect. A, 144 (2014), 557–566.

, P. Juutinen, P. Lindqvist & J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation, SIAM J. Math. Anal. 33 (2001), no. 3, 699-717.

T. Kuusi & G. Mingione Linear potentials in nonlinear potential theory, Arch. Ration. Mech. Anal. 207 (2013), no. 1, 215–246. MR 3004772

, Universal potential estimates, J. Funct. Anal. 262 (2012), no. 10, 4205–4269.

, Guide to nonlinear potential estimates, Bull. Math. Sci. 4 (2014), no. 1, 1–82.

A nonlinear Stein theorem, Calc. Var. Partial Differential Equations 51 (2014), no. 1-2, 45-86.

J. Lewis, Regularity of derivatives of solutions to certain degenerate elliptic equations, Indiana Univ. Math. J. 32 (1983) 849-858.

Lu, G. & Wang, P., A uniqueness theorem for elliptic equations. Lect. Notes Semin. Interdisciplinare di Matematica (Potenza) 7, (2008) 207–222.

Manfredi, M. Parviainen & J. Rossi, An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games, SIAM J. Math. Anal., 42(5) (2010), 2058-2081.

M. Parviainen & E. Ruosteenoja, Local regularity for time-dependent tug-of-war games with varying probabilities, J. Differential Equations 261 (2016), 1357–1398.

Y. Peres and S. Sheffield. Tug-of-war with noise: a game-theoretic view of the p-Laplacian, Duke Math. J. 145(1) (2008), 91-120.

E. M. Stein, Editor’s note: the differentiability of functions in R^n, Ann. of Math. (2) 113 (1981), no. 2, 383-385.
[35] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1) (1984) 126-150.
[36] L. Wang, *Compactness methods for certain degenerate elliptic equations*, J. Differential Equations **107** (1994), no. 2, 341-350.
[37] Niki Winter, *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, Z. Anal. Anwend. **28** (2009), no. 2, 129–164.

Tata Institute of Fundamental Research, Centre For Applicable Mathematics, Bangalore-560065, India

*E-mail address*, Agnid Banerjee: agnidban@gmail.com

Instituto de Matemáticas, México

*E-mail address*, Isidro Munive: imunivel@gmail.com