ASSOUAD-NAGATA DIMENSION OF LOCALLY FINITE GROUPS AND ASYMPTOTIC CONES

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Abstract. In this paper we study two problems concerning Assouad-Nagata dimension:
(1) Is there a metric space of positive Assouad-Nagata dimension such that all of its asymptotic cones are of Assouad-Nagata dimension zero? (Question 4.5 of [11])
(2) Suppose $G$ is a locally finite group with a proper left invariant metric $d_G$. If $\dim_{AN}(G, d_G) > 0$, is $\dim_{AN}(G, d_G)$ infinite? (Problem 5.3 of [6])

The first question is answered positively. We provide examples of metric spaces of positive (even infinite) Assouad-Nagata dimension such that all of its asymptotic cones are ultrametric. The metric spaces can be groups with proper left invariant metrics.

The second question has a negative solution. We show that for each $n$ there exists a locally finite group of Assouad-Nagata dimension $n$. As a consequence this solves for non finitely generated countable groups the question about the existence of metric spaces of finite asymptotic dimension whose asymptotic Assouad-Nagata dimension is larger but finite.

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1. INTRODUCTION

Nagata dimension (also called Assouad-Nagata dimension) was introduced by Assouad in [1] influenced from the papers of Nagata. This notion and its asymptotic version (asymptotic Assouad-Nagata dimension) has been studied in recent years as a geometric invariant, see for example [4], [8], [9] and specially [14]. In those papers
many properties of the asymptotic dimension (see [2] for a good survey about it) were generalized to asymptotic Assouad-Nagata dimension.

One interesting problem is about the relationship between the Assouad-Nagata dimension of a metric space and the topological dimension of its asymptotic cones. In [11] the following improvement of a result of [8] was obtained:

**Theorem 1.1.** [Dydak, Higes [11]] \( \dim(\text{Cone}(X, c, d)) \leq \dim_{\text{AN}}(\text{Cone}(X, c, d)) \leq \text{asdim}_{\text{AN}}(X, d_X) \) for any metric space \((X, d_X)\).

The main idea of the proof of this theorem is that the \( n \)-dimensional Nagata property of a metric space is inherited by its asymptotic cones. Recall that a metric space has the 0-dimensional Nagata property if and only if it is ultrametric. In section 3 we show that the converse is not true by giving a class of non-ultrametric spaces such that all of its asymptotic cones are ultrametric. Such construction will also answer the first main question.

Another problem that remains open is about the size of the difference between the asymptotic dimension and the asymptotic Assouad-Nagata dimension in a discrete group. P. Nowak in [15] found for each \( n \geq 1 \) a finitely generated group of asymptotic dimension \( n \) but infinite Assouad-Nagata dimension. If \( n \geq 2 \) such group can be finitely presented. As a final problem of his paper, Nowak asked about the existence of discrete groups such that the asymptotic dimension differs from asymptotic Assouad-Nagata dimension but both of them are finite. The paper of Nowak was complemented by one of Brodskiy, Dydak and Lang [6] who related the growth of a finitely generated group \( G \) with the Assouad-Nagata dimension of the wreath product \( H \wr G \) with \( H \neq 1 \) a finite group. As a consequence of their results many examples of countable locally finite groups with infinite Assouad-Nagata dimension can be found. Countable locally finite groups satisfy many remarkable geometric properties: they are the unique countable groups of asymptotic dimension zero [17]. Each metric space of asymptotic dimension zero and bounded geometry can be embedded coarsely in any infinite countable locally finite group. And they are the unique countable groups that admits a proper left invariant ultrametric [3].

The main target of section 4 is to study asymptotic Assouad-Nagata dimension of locally finite groups and countable groups. In particular we find for each \( n \) a locally finite group \((G, d_G)\) with \( d_G \) a proper left invariant metric that satisfies \( \text{asdim}_{\text{AN}}(G, d_G) = n \). In [5] a method was shown to build a locally finite group of infinite Assouad-Nagata dimension. Our construction seems quite similar to that one but it was not clear how to define the group and the metric in such a way its asymptotic Assouad-Nagata dimension was finite and positive. This problem was asked explicitly by two of the authors of [5] and Lang in [9]. We also show in Corollary 4.11 that for each \( n \geq 0 \) and \( k \geq 0 \) there is a countable group \( G \) and a proper left invariant metric \( d_G \) such that \( G \) is of asymptotic dimension \( n \) but \( \text{asdim}_{\text{AN}}(G, d_G) = n + k \). This solves Nowak’s problem for countable groups and answers negatively the second main problem of this paper. As far as we know the problem for finitely generated groups and finitely presented groups remains open.

2. **Asymptotic cones and cubes**

Let \( s \) be a positive real number. An \( s \)-scale chain (or \( s \)-path) between two points \( x \) and \( y \) of a metric space \((X, d_X)\) is defined as a finite sequence points \( \{x = x_0, x_1, \ldots, x_m = y\} \) such that \( d_X(x_i, x_{i+1}) < s \) for every \( i = 0, \ldots, m - 1 \). A
A metric space \((X, d_X)\) is said to be of asymptotic dimension at most \(n\) (notation asdim\(_n\)(\(X, d\)) \(\leq n\)) if there is an increasing function \(D_X : \mathbb{R}_+ \to \mathbb{R}_+\) such that for all \(s > 0\) there is a cover \(\mathcal{U} = \{U_0, ..., U_n\}\) so that the \(s\)-scale connected components of each \(U_i\) are \(D_X(s)\)-bounded i.e. the diameter of such components is bounded by \(D_X(s)\).

The function \(D_X\) is called an \(n\)-dimensional control function for \(X\). Depending on the type of \(D_X\), one can define the following two invariants:

A metric space \((X, d_X)\) is said to be of Assouad-Nagata dimension at most \(n\) (notation dim\(_{AN}\)(\(X, d\)) \(\leq n\)) if it has an \(n\)-dimensional control function \(D_X\) of the form \(D_X(s) = C \cdot s + k\) with \(C > 0\) and \(k \in \mathbb{R}\) two fixed constants.

A metric space \((X, d_X)\) is said to be of asymptotic Assouad-Nagata dimension at most \(n\) (notation asdim\(_{AN}\)(\(X, d\)) \(\leq n\)) if it has an \(n\)-dimensional control function \(D_X\) of the form \(D_X(s) = C \cdot s\) with \(C > 0\) some fixed constant.

One important fact about the asymptotic dimension is that it is invariant under coarse equivalences. Given a map \(f : (X, d_X) \to (Y, d_Y)\) between two metrics spaces it is said to be a coarse embedding if there exists two increasing functions \(\rho_+ : \mathbb{R}_+ \to \mathbb{R}_+\) and \(\rho_- : \mathbb{R}_+ \to \mathbb{R}_+\) with \(\lim_{x \to \infty} \rho_-(x) = \infty\) such that:

\[
\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))
\]

for every \(x, y \in X\).

The functions \(\rho_+\) and \(\rho_-\) are usually called contraction and dilatation functions of \(f\) respectively.

Now a coarse equivalence between two metrics spaces \((X, d_X)\) and \((Y, d_Y)\) is defined as a coarse embedding \(f : (X, d_X) \to (Y, d_Y)\) for which there exists a constant \(K > 0\) such that \(d_Y(y, f(X)) \leq K\) for every \(y \in Y\). If there exists a coarse equivalence between \(X\) and \(Y\) both spaces are said to be coarsely equivalent.

Next result relates \(n\)-dimensional control functions and coarse embeddings.

**Proposition 2.2.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces and let \(f : (X, d_X) \to (Y, d_Y)\) be a coarse embedding with \(\rho_+\) and \(\rho_-\) the contraction and dilatation functions of \(f\) respectively. If \(D_Y^n\) is an \(n\)-dimensional control function of \((Y, d_Y)\) then the function \(D_X^n\) defined by \(D_X^n = \rho_-^{-1} \circ D_Y^n \circ \rho_+\) is an \(n\)-dimensional control function of \((X, d_X)\).

**Proof.** Fix \(s > 0\) a positive real number. As \(D_Y^n\) is an \(n\)-dimensional control function there exists a cover \(\mathcal{U} = \{U_0, ..., U_n\}\) in \(Y\) so that the \(\rho_+(s)\)-scale connected components of each \(U_i\) are \(D_Y^n(\rho_+(s))\)-bounded. Take the cover \(\mathcal{V} = \{V_0, ..., V_n\}\) in \(X\) defined as \(V_i = f^{-1}(U_i)\). Notice that if two points \(x, y \in X\) satisfies \(d_X(x, y) < s\) then \(d_Y(f(x), f(y)) < \rho_+(s)\). Hence given an \(s\)-scale chain \(\{x_0, x_1, ..., x_m\}\) in \(X\) we get that \(\{f(x_0), f(x_1), ..., f(x_m)\}\) is an \(\rho_+(s)\)-scale chain. Therefore \(d_Y(f(x_0), f(x_m)) \leq D_Y^n(\rho_+(s))\) what implies \(d(x_0, x_m) \leq D_X^n(\rho_+(s))\). The proof is complete.

The following easy corollary will be useful in the next section:

**Corollary 2.3.** If \((X, d)\) is a metric space and asdim\(_{AN}\)(\(X, \log(1 + d)\)) \(\leq n\) then there is a polynomial \(n\)-dimensional control function of \((X, d)\).
Proof. Suppose \( \text{asdim}_{AN}(X, \log(1 + d)) \leq n \). This implies there exists a linear \( n \)-dimensional control function \( D^n_X(s) = C \cdot s + b \) of \( (X, \log(1 + d)) \) with \( C \) and \( b \) two fixed constants. Suppose without lost of generality that \( C \in \mathbb{N} \). Now the identity 
\[(X, d|_X) \to (X, \log(1 + d)) \]

is clearly a coarse equivalence with \( \rho_{+}(d) = \log(1 + d) = \rho_{-}(d) \). By proposition 2.2 we will get that \( Q^n_X = \rho_{-}^{-1} \circ D^n_X \circ \rho_{+} \) is an \( n \)-dimensional control function of \( (X, d) \). That means \( Q^n_X(s) = (10^b \cdot (1 + s)^C - 1 \). Therefore there is a polynomial dimensional control function of \((X, d)\).

It is clear that the asymptotic dimension of a metric space is less than or equal the asymptotic Assouad-Nagata dimension and this is greater or equal than the Assouad-Nagata dimension. In [4] it was shown that for a discrete space the asymptotic Assouad-Nagata dimension and the Assouad-Nagata dimension are equal.

Our target in this section is to find some sufficient conditions that give lower bounds for the Assouad-Nagata dimension. Next definition plays an important role for such purpose.

**Definition 2.4.** We define an \emph{\( n \)-dimensional dilated cube} in a metric space \((X, d_X)\) as a dilatation function \( f : \{0, 1, \ldots, k\}^n \to X \), that means there exists a constant \( C \geq 1 \) (dilatation constant) such that \( C \cdot \|x - y\|_1 = d_X(f(x), f(y)) \) for every \( x, y \in \{0, 1, \ldots, k\}^n \).

**Remark 2.5.** \( n \)-dimensional dilated cubes are particular cases of the \( n \)-dimensional \( s \)-cubes introduced by Brodskiy, Dydak and Lang in [6]. Recall that an \( n \)-dimensional \( s \)-cube in a metric space \((X, d_X)\) is defined as a function \( f : \{0, 1, \ldots, k\}^n \to X \) with the property that \( d(f(x), f(x + e_i)) < s \) for all \( x \in \{0, 1, \ldots, k\}^n \) such that \( x + e_i \in \{0, 1, \ldots, k\}^n \) with \( e_i \) belonging to the standard basis of \( \mathbb{R}^n \).

In the whole paper we will take in \( \mathbb{R}^n \) the \( l_1 \)-metric instead of the euclidean metric.

Now we will relate the existence of some sequences of \( n \)-dimensional dilated cubes in a space with the existence of cubes of the form \([0, s]^n \subset \mathbb{R}^n\) in its asymptotic cones. Let \((X, d_X)\) be a metric space. Given a non-principal ultrafilter \( \omega \) of \( \mathbb{N} \) and a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of points of \( X \), the \( \omega \)-limit of \( \{x_n\}_{n \in \mathbb{N}} \) (notation: \( \lim_{\omega} x_n \)) is defined as the element \( y \) of \( X \) such that for every neighborhood \( U \) of \( y \) the set \( F_U = \{n|x_n \in U\} \) belongs to \( \omega \). Analogously if for every ball \( B(x, r) \) of radius \( r \) the set \( F_{B(x, r)} = \{n|x_n \in X \setminus B(x, r)\} \) belongs to \( \omega \) then it is said that the \( \omega \)-limit of \( \{x_n\}_{n \in \mathbb{N}} \) is infinity and the sequence is an \( \omega \)-divergent sequence. It can be proved easily that the \( \omega \)-limit always exists in a compact space.

Assume \( \omega \) is a non principal ultrafilter of \( \mathbb{N} \). Let \( d = \{d_n\}_{n \in \mathbb{N}} \) be an \( \omega \)-divergent sequence of positive real numbers and let \( c = \{c_n\}_{n \in \mathbb{N}} \) be any sequence of elements of \( X \). Now we can construct the \emph{asymptotic cone} (notation: \( \text{Cone}_{\omega}(X, c, d) \)) of \( X \) as follows:

Firstly define the set of all sequences \( \{x_n\}_{n \in \mathbb{N}} \) of elements of \( X \) such that \( \lim_{\omega} \frac{d_X(x_n, c_n)}{d_n} \) is bounded. In such set take the pseudo metric given by:

\[
D(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) = \lim_{\omega} \frac{d_X(x_n, y_n)}{d_n}.
\]

By identifying sequences whose distances is 0 we get the metric space \( \text{Cone}_{\omega}(X, c, d) \).

Asymptotic cones were originally introduced by Gromov in [12]. There has been a lot of research relating properties of groups with topological properties of its
asymptotic cones. For example a finitely generated group is virtually nilpotent if and only if all its asymptotic cones are locally compact \([13]\) or a group is hyperbolic if and only if all of its asymptotic cones are \(\mathbb{R}\)-trees \([12]\) and \([10]\).

**Proposition 2.6.** Let \((X, d_X)\) be a metric space and let \(\{f_m\}_{m=1}^\infty\) be a sequence of \(n\)-dimensional dilated cubes, \(f_m : \{0, 1, \ldots, k_m\}^n \to X\) such that \(\lim_{\omega} k_m = \infty\) for some non principal ultrafilter \(\omega\) of \(\mathbb{N}\). If \(\{d_m\}_{m=1}^\infty\) is an \(\omega\)-divergent sequence of positive real numbers that satisfies

\[
\lim_{\omega} \frac{C_m \cdot k_m}{d_m} = s
\]

with \(\{C_m\}_{m=1}^\infty\) the sequence of dilatation constants and \(0 \leq s < \infty\), then \([0, s]^n \subset \text{Cone}_{\omega}(X, c, d)\) where \(c = \{f_m(0)\}_{m=1}^\infty\).

**Proof.** Let us prove firstly the case \(n = 1\). For each \(t \in [0, s]\) let \(A_m\) be the subset of elements of \(\{0, 1, \ldots, k_m\}\) such that, for every \(x \in A_m\), \(\frac{d_m}{C_m} x\) is closest to \(t\). It means \(C_m \cdot x\) is closest to \(d_m \cdot t\). Take now the sequence \(\{r_m^t\}_{m=1}^\infty\) where \(r_m^t\) is the minimum element of \(A_m\).

Define the map \(g : [0, s] \to \text{Cone}_{\omega}(X, c, d)\) by \(g(t) = x^t\) if the sequence \(\{f_m(r_m^t)\}_{m=1}^\infty\) is in the class \(x^t\). As:

\[
\lim_{\omega} \frac{d(f_m(0), f_m(r_m^t))}{d_m} = \lim_{\omega} \frac{C_m \cdot r_m^t}{d_m} \leq \lim_{\omega} \frac{C_m \cdot k_m}{d_m} = s
\]

the map is well defined. Let us prove that it is in fact an isometry. From the definition of \(r_m^t\) we get that if \(t_1 < t_2\) then \(r_m^t_1 \leq r_m^t_2\) which implies \(\lim_{\omega} \frac{d(f_m(r_m^t_1), f_m(r_m^t_2))}{d_m} = \lim_{\omega} \frac{C_m(r_m^t_2 - r_m^t_1)}{d_m}\). So the unique thing we need to show is that \(\lim_{\omega} \frac{C_m \cdot r_m^t}{d_m} = t\) for every \(t\). Firstly notice that as \(\lim_{\omega} \frac{C_m \cdot k_m}{d_m} = s\) and \(s \geq t\) then there exists an \(F \in \omega\) such that \(\frac{C_m \cdot k_m}{d_m} \geq t\) for every \(m \in F\). We have also \(\lim_{\omega} \frac{C_m}{d_m} = 0\) as \(\lim_{\omega} k_m = \infty\) but \(\lim_{\omega} \frac{C_m}{d_m} = s < \infty\). This implies that given \(\epsilon > 0\) there exists \(G_\epsilon \in \omega\) such that \(\frac{C_m}{d_m} < \epsilon\) for every \(m \in G_\epsilon\). Therefore if \(m \in F \cap G_\epsilon\) we have \(|C_m \cdot r_m^t - d_m \cdot t| < C_m\) and then \(|C_m \cdot r_m^t - t| < \frac{C_m}{d_m} \leq \epsilon\).

Now let us do the general case. Let \((s_1, \ldots, s_n) \in [0, s]^n\). By the previous case we get that for every \(j = 1, \ldots, n\) there exists a sequence \(\{r_m^n\}_{m \in \mathbb{N}}\) with \(r_m^n \in \{0, 1, \ldots, k_m\}\) such that \(\lim_{\omega} \frac{C_m \cdot r_m^n}{d_m} = s_j\). In a similar way as before we construct a map \(g : [0, s]^n \to \text{Cone}_{\omega}(X, c, d)\) by defining \(g(s_1, \ldots, s_m)\) as the class that contains the sequence \(\{f_m(r_m^s, \ldots, r_m^n)\}_{m=1}^\infty\). To finish the proof it will be enough to check that for every \(s, t \in [0, s]^n\) with \(s = (s_1, \ldots, s_n)\) and \(t = (t_1, \ldots, t_n)\), the following equality holds:

\[
\lim_{\omega} \frac{d_X(f_m(r_m^{s_1}, \ldots, r_m^n), f_m(r_m^{t_1}, \ldots, r_m^n))}{d_m} = \sum_{i=1}^n |s_i - t_i|
\]

As \(f_m\) is a dilatation of constant \(C_m\) we can write:

\[
\lim_{\omega} \frac{d_X(f_m(r_m^{s_1}, \ldots, r_m^n), f_m(r_m^{t_1}, \ldots, r_m^n))}{d_m} = \sum_{i=1}^n \lim_{\omega} \frac{C_m \cdot |r_m^{s_i} - r_m^{t_i}|}{d_m}
\]
And again by the case \( n = 1 \) we can deduce that the last term satisfies the equality:

\[
\sum_{i=1}^{n} \lim_{\omega} \frac{C_m \cdot |r_{m}^i - r_{m}^j|}{d_m} = \sum_{i=1}^{n} |s_i - t_i|
\]

Combining theorem 1.1 with proposition 2.6 we can get the following lower bound of Assouad-Nagata dimension for certain spaces. Such estimation will be very useful to prove the main results of section 4.

**Corollary 2.7.** If a metric space \((X, d_X)\) contains a sequence \(\{f_m\}_{m=1}^{\infty}\) of \(n\)-dimensional dilated cubes \(f_m : \{0, 1, \ldots, k_m\}^n \to X\) with \(\lim_{m \to \infty} k_m = \infty\) then

\[
asdim_{AN}(X, d_X) \geq n
\]

**Proof.** Let \(\{C_m\}_{m=1}^{\infty}\) be the sequence of dilatation constants. Define the divergent sequence \(d = \{d_m\}_{m=1}^{\infty}\) as \(d_m = C_m \cdot k_m\). Then for any non principal ultrafilter \(\omega\) of \(\mathbb{N}\) the hypothesis of proposition 2.6 are satisfied with \(s = 1\) so we get \([0, 1]^n \subset \text{Cone}_\omega(X, c, d)\). Applying 1.1 we obtain immediately:

\[
n \leq \dim(\text{Cone}_\omega(X, c, d)) \leq \dim_{AN}(\text{Cone}_\omega(X, c, d)) \leq \asdim_{AN}(X, d_X)
\]

3. **Ultrametric Asymptotic Cones**

The main aim of this section is to find metric spaces of positive asymptotic Assouad-Nagata dimension such that all of its asymptotic cones are ultrametric spaces. In particular we are interested when the space is a group with a proper left invariant metric. A metric \(d_G\) defined in a group \(G\) is said to be a proper left invariant metric if it satisfies the following conditions:

1. \(d_G(g_1 \cdot g_2, g_1 \cdot g_3) = d_G(g_2, g_3)\) for every \(g_1, g_2, g_3 \in G\).
2. For every \(K > 0\) the number of elements \(g\) of \(G\) such that \(d(1, g)_G \leq K\) is finite.

We will say that a metric \(d_X\) defined in a set \(X\) is an asymptotic ultrametric if there exists a constant \(k \geq 0\) such that:

\[
d_X'(x, y) \leq \max\{d_X'(x, z), d_X'(y, z)\} + k \text{ for every } x, y, z \in X
\]

If \(k = 0\) the space is said to be ultrametric.

Next theorem is a combination of two different results of [4] and [5].

**Theorem 3.1.** For every metric space \((X, d_X)\) of finite asymptotic dimension there exists a metric \(d'_X\) coarsely equivalent to \(d_X\) such that:

1. \(\asdim_{AN}(X, d'_X) = \asdim(X, d_X)\).
2. \(d'_X\) is an asymptotic ultrametric.

Moreover if \((X, d_X)\) is a countable group with \(d_X\) a proper left invariant metric we can take \(d'_X\) as a proper left invariant metric.

**Proof.** The first part of the theorem follows directly from the proof of [4], Theorem 5.1.

The second part can be got by an easy modification of the proof of [5], proposition 6.6.
Next result appeared in Gromov [12], page] but without proof. We provide a proof here.

**Proposition 3.2.** Let \((X, d_X)\) be a metric space that satisfies:

\[
    d_X(x, y) \leq (1 + \epsilon) \cdot \max\{d_X(x, z), d_X(y, z)\} + k \quad \text{for every } x, y, z \in X
\]

where \(\epsilon\) is some function in \(d = d_X(x, y)\) which goes to 0 when \(d\) goes to \(\infty\) and \(k > 0\) some fixed constant. Then every asymptotic cone of \((X, d_X)\) is an ultrametric space.

**Proof.** Suppose \((X, d_X)\) is an asymptotic ultrametric space with constant \(k > 0\). Let \(x, y\) and \(z\) be three points of \(\text{Cone}_\omega(X, c, d)\) and let \(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset X\) be three sequences in the classes \(x, y\) and \(z\) respectively. Without loss of generality assume \(D(x, y) \geq D(x, z)\) and \(D(x, y) \geq D(y, z)\). It will be enough to check that \(D(x, y) \leq \max\{D(x, z), D(y, z)\}\).

First notice that \(\lim_{\omega} \epsilon(d_X(x_n, y_n)) = 0\). If not there exists an \(M > 0\) such that the set \(H = \{n | \epsilon(d_X(x_n, y_n)) \geq M\}\) is in \(\omega\). But if we assume the non trivial case \(D(x, y) \neq 0\) then the set \(G = \{d_X(x_n, y_n) | n \in H\}\) is bounded. Taking now a divergent subsequence \(\{d_X(x_n, y_n)\}_{n=1}^\infty \subset G\) we get a contradiction as \(\lim_{n \to \infty} \epsilon(d_X(x_n, y_n)) = 0\) but \(d_X(x_n, y_n) \geq M\) for every \(i \in \mathbb{N}\).

Take now \(F\) the subset of natural numbers defined as:

\[
    F = \{n | d_X(x_n, z_n) \geq d_X(y_n, z_n)\}
\]

As \(\omega\) is an ultrafilter then \(F \subset \omega\) or \(\mathbb{N} \setminus F \subset \omega\). Let us assume the first case. In this first case we will obtain that \(D(x, z) \geq D(y, z)\) and by the asymptotic ultrametric property we deduce that:

\[
    \frac{d_X(x_n, y_n)}{d_n} \leq \frac{(1 + \epsilon(d_X(x_n, y_n))) \cdot d_X(x_n, z_n)}{d_n} + k \quad \text{for every } n \in F.
\]

Taking limits in the previous inequality and applying the fact \(\lim_{\omega} \epsilon(d_X(x_n, y_n)) = 0\) we obtain \(D(x, y) \leq D(x, z) = \max\{D(x, z), D(y, z)\}\). Finally let us do the case \(\mathbb{N} \setminus F \subset \omega\). This implies the set \(\mathbb{N} \setminus F = \{n | d_X(y_n, z_n) > d_X(x_n, z_n)\}\) belongs to the ultrafilter what yields \(D(y, z) \geq D(x, z)\) applying now the same reasoning we get:

\[
    \frac{d_X(x_n, y_n)}{d_n} \leq \frac{(1 + \epsilon(d_X(x_n, y_n))) \cdot d_X(y_n, z_n)}{d_n} + k \quad \text{for every } n \in \mathbb{N} \setminus F.
\]

Taking again limits we obtain \(D(x, y) \leq D(y, z) = \max\{D(x, z), D(y, z)\}\) what finish the proof.

**Corollary 3.3.** If \((X, d_X)\) is a metric space with \(d_X\) an asymptotic ultrametric then \(\text{Cone}_\omega(X, c, d)\) is an ultrametric space for every \(c, d\) and \(\omega\).

Recall that ultrametric spaces are of Assouad-Nagata dimension zero. So an immediate consequence of theorem 3.1 and proposition 3.3 is the following theorem that solves the first target of this paper:

**Theorem 3.4.** For every metric space \((X, d_X)\) with finite asymptotic dimension there exists a coarsely equivalent metric \(d'_X\) such that asdim\(_{\text{AN}}(X, d'_X) = \text{asdim}(X, d_X)\) and every asymptotic cone of \((X, d'_X)\) is an ultrametric space.

Moreover if \((X, d_X)\) is a countable group with \(d_X\) a proper left invariant metric then we can take \(d'_X\) a proper left invariant metric.
We could ask now about the existence of a metric space of infinite asymptotic Assouad-Nagata dimension such that all its asymptotic cones are of Assouad-Nagata dimension zero, in particular ultrametric spaces. The rest of the section is devoted to show an example of this type. Such example will be a finitely generated group with some proper left invariant metric. We will need the following result of \[6\].

**Theorem 3.5.** (Brodskiy, Dydak and Lang [6]) Suppose $H$ and $G$ are finitely generated and $K$ is the kernel of the projection of $H \wr G \to G$ equipped with the metric induced from $H \wr G$. If $\gamma$ is the growth function of $G$ and $D_{n-1}^K$ is an $n-1$ dimensional control function of $K$, then the integer part of $\gamma(r)$ is at most $D_{n-1}^K(3 \cdot n \cdot r)$.

**Theorem 3.6.** Let $G$ be a finitely generated group of exponential growth and let $H$ be a finite group. Suppose $d$ is a word metric of $H \wr G$ then:

1. All the asymptotic cones of $(H \wr G, \log(1 + d))$ are ultrametric.
2. $\text{asdim}_{AN}(H \wr G, \log(1 + d)) = \infty$.

**Proof.** The first assertion is a consequence of proposition 3.2. In fact it is not hard to check that if the metric $D(x, y)$ of $X$ is of the form $D(x, y) = \log(1 + d(x, y))$ where $d(x, y)$ is another distance, then $\epsilon(d) = \frac{\log(2)}{\log(\frac{d}{2} + 1)}$ satisfies the conditions of the cited proposition.

Second assertion is a consequence of corollary 2.3 and theorem 3.5. The proof is by contradiction. Suppose $\text{asdim}_{AN}(H \wr G, \log(1 + d)) \leq n$. By corollary 2.3 we get that there is a polynomial $n$-dimensional control function of $(H \wr G, d)$ and then there is a polynomial $n$-dimensional control function of the kernel $K \subset H \wr G$ with the restricted metric. But as the growth of $G$ is exponential by theorem 3.5 any $n$-dimensional control function of $(K, d|_K)$ must be at least exponential, a contradiction. \qed

**Remark 3.7.** As the metric spaces $(H \wr G, d)$ and $(H \wr G, \log(1 + d))$ are coarsely equivalent then both have the same asymptotic dimension. Hence we can also assume that the group $(H \wr G, \log(1 + d))$ is of finite asymptotic dimension. Let us show an example of this fact.

**Example 3.8.** Let $G = F_2$ be the free group of two generators and $H = \mathbb{Z}_2$. For any word metric $d$ of $\mathbb{Z}_2 \wr F_2$ we have $\text{asdim}(\mathbb{Z}_2 \wr F_2, d) = 1$ and such group satisfies the conditions of the theorem.

It is clear that the kernel $K$ of the projection $H \wr G \to G$ is a locally finite group when $H$ is finite. From the proof of 3.6 we get easily:

**Corollary 3.9.** There exists a locally finite group $K$ with a proper left invariant metric $d_K$ such that $\text{asdim}_{AN}(K, d_K) = \infty$ and all of its asymptotic cones are ultrametric.

**Remark 3.10.** Notice that if $G$ is a finitely generated group and $d_G$ is any word metric then every asymptotic cone $\text{Cone}_\omega(G, c, d)$ is geodesic what implies $\text{dim}_{AN}(\text{Cone}_\omega(G, c, d)) \geq 1$. Therefore the metrics $d_G'$ of the theorems of this section can not be quasi-isometric to any word metric of $G$. 
4. Locally finite groups and positive Assouad Nagata dimension

In this section we will study the Assouad-Nagata dimension of locally finite groups. Firstly let us check that all locally finite groups admits a proper left invariant metric with positive Assouad-Nagata dimension.

Given a countable group $G$ we want to build a proper left invariant metric on it. Associated to each proper left invariant metric there exist a proper norm. A map $\| \cdot \| : G \rightarrow \mathbb{R}_+$ is called to be a proper norm if it satisfies the following conditions:

1. $\|g\|_G = 0$ if and only if $g$ is the neutral element of $G$.
2. $\|g\|_G = \|g^{-1}\|_G$ for every $g \in G$.
3. $\|g \cdot h\|_G \leq \|g\|_G + \|h\|_G$ for every $g, h \in G$.
4. For every $K > 0$ the number of elements of $G$ such that $\|g\|_G \leq K$ is finite.

So if we build a proper norm $\| \cdot \|_G$ in $G$ then the map $d_G(g, h) = \|g^{-1} \cdot h\|_G$ defines a proper left invariant metric. Conversely for every proper left invariant metric $d_G$ the map $\|g\|_G = d_G(1, g)$ defines a proper norm.

One method of obtaining a norm in a countable group was described by Smith in [17](see also [16]). Let $S$ be a symmetric system of generators(possibly infinite) of a countable group $G$ and let $w : L \rightarrow \mathbb{R}_+$ be a function(weight function) that satisfies:

1. $w(s) = 0$ if and only if $s = 1_G$
2. $w(s) = w(s^{-1})$.

Then the function $\| \cdot \|_w : G \rightarrow \mathbb{R}_+$ defined by:

$$\|g\|_w = \min\{\sum_{i=1}^{n} w(s_i) | x = \Pi_{i=1}^{n} s_i, s_i \in S\}$$

is a norm. Moreover if $w$ satisfies also that $w^{-1}[0, N]$ is finite for every $N$ then $\| \cdot \|_w$ is a proper norm.

Notice that if we define $w(g) = 1$ for all the elements $g \in S$ of a finite generating system $S \subset G$ (a finitely generated group) we will obtain the usual word metric.

This method for finite groups has the following nice(and obvious) extension property:

**Lemma 4.1.** Let $G$ be a countable group and let $(G_1, d_{G_1})$ be a finite subgroup of $G$ with $d_{G_1}$ a proper left invariant metric. Let $S \subset G$ be a symmetric subset of $G$ such that:

1. $S \cap G_1 = \emptyset$.
2. $G$ is generated by $G_1 \cup S$.

If $\| \cdot \|_w : G \rightarrow \mathbb{R}_+$ is a norm defined by a weight function $w : G_1 \cup S \cup S^{-1} \rightarrow \mathbb{R}_+$ that satisfies:

1. $w(g) = \|g\|_{G_1}$ if $g \in G_1$
2. $w(g) \geq \text{diam}(G_1)$ if $g \in S \cup S^{-1}$.

then for every $g \in G_1$ $\|g\|_w = \|g\|_{G_1}$.

A group is said to be locally finite if all of its finitely generated subgroups are finite.

In a locally finite group $G$ we can take a filtration $\mathcal{L}$ of finite subgroups $\mathcal{L} = \{\{1\} = G_0 \subset G_1 \subset G_2 \ldots\}$ of $G$. Lemma 4.1 applied successively to $\mathcal{L}$ allows us to build a sequence of norms $\| \cdot \|_i : G_i \rightarrow \mathbb{R}_+$ and a norm $\| \cdot \|_G : G \rightarrow \mathbb{R}_+$ such that
the restriction of $\| \cdot \|$ to $G_i$ coincides with $\| \cdot \|_{G_i}$ and each $\| \cdot \|_i$ is an extension of $\| \cdot \|_{i-1}$. This idea was used in [3] to prove that each locally finite group is coarsely equivalent to a direct sum of cyclic groups.

Let $L = \{ \{0\} = G_0 \subset G_1 \subset \ldots \}$ be a filtration of a countable group $G$. We will say that the sequence $\{g_n\}_{n=1}^\infty$ of elements of $G$ is a system of generators of $L$ if for every $i \geq 1$ there exists an $n_i$ such that $G_i = \langle g_1, \ldots, g_{n_i} \rangle$. If it is also satisfied that $n_i = i$ and $G_{i-1} \neq G_i$ for every $i \geq 1$ then we will say that the filtration $L$ is a one-step ascending chain. In one-step ascending chains we can estimate easily the cardinality of $G_i$.

**Lemma 4.2.** Let $G$ be a locally finite group and let $L = \{ \{1\} = G_0 < G_1 < \ldots \}$ be a one-step ascending chain then $|G_i| \geq 2^i$ for every $i \in \mathbb{N}$.

**Proof.** Let $\{g_i\}_{i \in \mathbb{N}}$ be a sequence of generators of $L$. For $i = 1$ the result is obvious. Let us assume the result is true for some $i \geq 1$. Take $2^i$ different elements $\{x_n\}_{n=1}^{2^i}$ of $G_i$. The subset $X$ of $G_{i+1}$ defined as $X = \{x_n\}_{n=1}^{2^i} \cup \{x_n \cdot g_{i+1}\}_{n=1}^{2^i}$ contains $2^{i+1}$ different elements as $g_{i+1} \notin G_i$. □

**Theorem 4.3.** Let $G$ be a locally finite group. For every increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to \infty} f(x) = \infty$ there exists a proper left invariant metric $d_G$ in $G$ such that $f$ is not a 0-dimensional control function.

**Proof.** The proof is by contradiction. Let $L = \{ G_0 < G_1 < \ldots \}$ be a one-step ascending chain of $G$ with $\{g_i\}_{i=1}^\infty$ a sequence of generators of $L$. Assume there exists some function $f$ as in the hypothesis that is a 0-dimensional control function such that for every proper left invariant metric $d_G$. In such case we claim:

There exists a $K > 0$ such that for every $i > 0$ every element $g \in G_i$ has length less than or equal K as a word of $\{g_1, \ldots, g_i\}$.

Suppose that the claim were false. In such case we will show there is a proper left invariant metric $d_G$ such that $f$ is not a 0-dimensional control function. Fix $J_1 = 1$ and take $h_1 \in G_{i_1}$ an element such that its minimum length in $G_{i_1}$ is greater than $f(1)$. Define the norms in the system of generators $\{g_j\}_{j=1}^\infty$ as $\|g_j\|_{G_{i_1}} = 1$ with $\epsilon = \pm 1$. By viewing such norms as weights we have a proper left invariant metric in $G_{i_1}$. Notice that $\|h_1\|_{G_{i_1}} > f(J_1)$ and $h_1$ is in the same 1-connected component as the origin. Now suppose defined a proper left invariant metric in $G_{i_1}$ that verifies the following property:

There exists an $h_r \in G_{i_r}$ with $\|h_r\|_{G_{i_r}} > f(J_r)$, $J_r \geq \text{diam}(G_{i_{r-1}})$ and $h_r$ is in the same $J_r$-connected component as the origin.

Let $J_{r+1} > \text{diam}(G_{i_r})$ and let $h_{r+1}$ be an element of the group $G_{i_{r+1}}$ with its minimum length greater than $f(J_{r+1})$. Apply $\{1, -1\}$ to $G_{i_r} \cup S$ with $S = \{g_{i_{r+1}}, \ldots, g_{i_{r+1}}\}$ and a weight function $u$ that satisfies $u(g) = J_{r+1}$ for every generator $g \in S \cup S^{-1}$. Then we get a norm in $G_{i_{r+1}}$ that is an extension of $\| \cdot \|_{G_{i_r}}$ with the same property as above. Repeating this procedure we can get a proper norm $\| \cdot \|_G$ defined in $G$ so that for every $J_r$ there is an element $h_r$ with $\|h_r\|_G > f(J_r)$ and $h_r$ is in the same $J_r$ connected component as the origin. We deduce that $f$ cannot be a 0-dimensional control function of $d_G$ so the claim must be true.
Now applying an easy combinatorial argument we can estimate the cardinality of every subgroup $G_i$:

$$|G_i| \leq \sum_{j=0}^{K} (2 \cdot i)^j \leq (K + 1) \cdot 2^K \cdot i^K$$

This contradicts lemma 4.2 for $i$ sufficiently large. 

**Corollary 4.4.** A countable locally finite group $G$ is finite if and only if $asdim_{AN}(G, d_G) = 0$ for every proper left invariant metric $d_G$.

**Proof.** It is clear that every finite group satisfies $asdim_{AN}(G, d_G) = 0$ for every metric $d_G$.

For the converse firstly notice that if $(X, d_X)$ is a discrete metric space and $f$ is not a 0-dimensional control function of $(X, d_X)$ then $g$ is not a 0-dimensional control function of $(X, d_X)$ for every function $g$ such that $g \leq f$ asymptotically that means there exists an $x_0$ such that $g(x) \leq f(x)$ for every $x \geq x_0$. Therefore by previous theorem if $G$ is non finite we can take a metric $d_G$ such that the function $f(x) = x^2$ is not a 0-dimensional control function of $(G, d_G)$. Then any linear function cannot be a 0-dimensional control function of $(G, d_G)$.

The remainder of this section will be focused on finding locally finite groups with non zero but still positive Assouad-Nagata dimension. Such groups will be of the form $G = \bigoplus_{i=0}^{\infty} G_i$ with $\{G_i\}_{i \in \mathbb{N}}$ some sequence of finite groups and $G_0 = \{0\}$. Our first step consists in defining a nice proper left invariant metric in $G$ using a sequence of proper left invariant metrics $\{d_{G_i}\}_{i \in \mathbb{N}}$ of $\{G_i\}$. Suppose $d_{G_i}(x, y) \geq 1$ for every two different elements of $G_i$. To build such metric we take a sequence of positive numbers $\{s_i\}_{i \in \mathbb{N}}$ such that $s_1 \geq 1$ and $s_i \geq s_{i-1} \cdot \text{diam}(G_{i-1}) + 1$ and we define the map $k : G \rightarrow \mathbb{N} \cup \{0\}$ by $k(g) = \max\{i | \pi_i(g) \neq 1_{G_i}\}$. The functions $\pi_i : G \rightarrow G_i$ are here the canonical projections. In this situation we can construct a proper norm as following:

**Lemma 4.5.** Let $G = \bigoplus_{i=0}^{\infty} G_i$ with $G_0 = \{0\}$ and $\{G_i, d_{G_i}\}_{i \in \mathbb{N}}$ be a sequence of finite groups with proper left invariant metrics $d_{G_i}$. Let $\{s_i\}_{i \in \mathbb{N}}$ be the sequence of finite numbers as above. Then the map $\| \cdot \| : G \rightarrow \mathbb{R}_+$ defined by $\|g\|_G = s_{k} \cdot \|\pi_{k}(g)\|_{G_i}$ satisfies:

1. $\| \cdot \|_G$ is a proper norm in $G$.
2. If $g \in G_i$ then $\|j_{i}(g)\|_G = s_i \|g\|_G$, with $j_i : G_i \rightarrow G$ the canonical inclusion.

**Proof.** (2) is obvious by definition of $\| \cdot \|_G$ so let us prove (1). We will check firstly that $\| \cdot \|_G$ defines a norm in $G$. We have to prove that $\| \cdot \|_G$ satisfies the three first conditions of a norm. We are using the convention $k(g) = 0$ if and only if $\pi_i(g) = 1_{G_i}$ for every $i$. This implies $g = 1_G$. Hence the first condition can be easily derived from the fact that each $\| \cdot \|_{G_i}$ is a (proper) norm. The second condition is trivial. For the third one let $g, h \in G$ and assume without loss of generality that $k(g) \geq k(h)$. As $\pi_i(g \cdot h) = \pi_i(g)$ for every $i > k(h)$ the case $k(g) > k(h)$ is obvious. Consider now that $k(g) = k(h)$. In such case we have $\pi_i(g \cdot h) = 1_{G_i}$ for every $i > k(g)$. There are two possibilities:

1. $\pi_{k}(g \cdot h) \neq 1_{G_i}$: Then as each $\| \cdot \|_{G_i}$ is a norm we get:

$$\|g \cdot h\|_G = s_{k} \cdot \|\pi_{k}(g) \cdot \pi_{k}(h)\|_{G_{k}} \leq s_{k} \cdot \left( \|\pi_{k}(g)\|_{G_{k}} + \|\pi_{k}(h)\|_{G_{k}} \right) = \|g\|_G + \|h\|_G$$
(2) \( \pi_{k(G)}(g \cdot h) = 1_{G_{k(G)}} \). It implies trivially \( \|g \cdot h\|_G < s_{k(G)} \leq \|g\|_G + \|h\|_G \).

Finally to prove it is a proper norm given \( K > 0 \) we take an \( s_i \) so that \( K < s_i \). Hence the number of elements \( g \in G \) with norm less than or equal \( K \) will be bounded by \( \Pi_{j=1}^{s} |G_j| \) with \( |G_j| \) the cardinality of \( G_j \).

The proper left invariant metric \( d_G \) associated to this norm will be called the quasi-ultrametric generated by \( \{d_{G_i}\}_{i \in \mathbb{N}} \). The reason of this name is shown in next lemma.

**Lemma 4.6.** Let \( G = \bigoplus_{i=0}^{\infty} G_i \) be the group defined above with \( d_G \) the proper left invariant metric of the previous lemma. Then for every \( g_1, g_2, g_3 \in G \) such that \( k(g_i) \neq k(g_j) \) with \( i, j = 1, 2, 3 \) we have:

\[
d_G(g_1, g_2) \leq \max\{d_G(g_2, g_3), d_G(g_1, g_3)\}
\]

**Proof.** If \( k(g_1) > k(g_2) \) and \( k(g_1) > k(g_3) \) then we will have \( \pi_{k(g_1)}(g_1^{-1} \cdot g_i) = \pi_{k(g_1)}(g_1^{-1}) \) with \( j = 2, 3 \), it will imply that \( d_G(g_1, g_j) = \|g_1^{-1} \cdot g_j\|_G = \|g_1^{-1}\|_G \).

Hence \( d_G(g_1, g_2) = d_G(g_1, g_3) \). Now suppose \( k(g_2) > k(g_1) \) and \( k(g_2) > k(g_3) \). Applying the same reasoning we get: \( d_G(g_1, g_2) = d_G(g_2, g_3) \). Finally if \( k(g_3) > k(g_1) \) and \( k(g_3) > k(g_2) \) we obtain by an analogous reasoning that \( d_G(g_1, g_3) < s_{k(g_i)} \leq d_G(g_3, g_2) \).

We can estimate the Assouad-Nagata dimension of \( G \) from the Assouad-Nagata dimension of each \( G_i \).

**Lemma 4.7.** Let \( G = \bigoplus_{i=0}^{\infty} G_i \) where \( G_0 = \{0\} \) and \( \{G_i, d_{G_i}\}_{i \in \mathbb{N}} \) is a sequence of finite groups with \( d_{G_i} \) a proper left invariant metric. Let \( d_G \) be the quasi-ultrametric generated by \( \{d_{G_i}\}_{i \in \mathbb{N}} \). If there is a constant \( C \geq 1 \) such that for every \( s \in (1, \text{diam}(G_i)) \) there exists a cover \( U = \{U_0, \ldots, U_n\} \) of \( (G_i, d_{G_i}) \) so that the \( s \)-scale connected components of each \( U_j \) are \( C \cdot s \)-bounded then \( \text{asdim}_{AN}(G, d_G) \leq n \).

**Proof.** Let \( s \in (s_i, s_{i+1}] \) and let \( U = \{U_0, \ldots, U_n\} \) be a cover of \( (G_i, d_{G_i}) \) such that the \( \frac{s}{s_i} \)-scale connected components of each \( U_j \) are \( C \cdot \frac{s}{s_i} \)-bounded. If \( s \in (s_i, \text{diam}(G_i), s_i \cdot \text{diam}(G_i) + 1] \) we take as \( U_j = G_i \) for every \( j = 0, \ldots, n \). Define the cover \( V = \{V_0, \ldots, V_n\} \) of \( (G, d_G) \) by the property \( g \in V_j \) if and only if \( \pi_i(g) \in U_j \). Let us prove that the \( s \)-scale connected components of \( V_j \) are \( C \cdot s \)-bounded. Given an \( s \)-scale chain \( x_1, x_2, \ldots, x_m \) of \( V_j \), define the associated chain \( y_1, \ldots, y_m \) by \( y_r = x_1^{-1} \cdot x_r \). By proving \( \|y_m\|_G \leq C \cdot s \) we will complete the result.

Firstly notice that \( y_1 = 1_G \). Now suppose it is true for some \( r < m \) that \( \pi_j(y_r) = 1_{G_j} \) for every \( j > i \) then as a consequence of the fact \( d_G(y_r, y_{r+1}) < s \leq s_{i+1} \) we will get \( \pi_j(y_{r+1}) = 1_{G_j} \) for every \( j > i \). It shows that \( \pi_j(y_r) = 1_{G_j} \) for every \( j > i \) and each \( r = 1, \ldots, m \). By construction of the metric \( d_G \) we obtain \( d_{G_i}(\pi_i(y_r), \pi_i(y_{r+1})) < \frac{s}{s_i} \) and then \( \pi_i(y_1), \ldots, \pi_i(y_m) \) is an \( \frac{s}{s_i} \)-scale chain of \( \pi_i(x_1^{-1}) \cdot U_j \). Hence it will be \( C \cdot \frac{s}{s_i} \)-bounded. Combining this and the fact \( \pi_i(y_1) = 1_G \), we finally get that \( \|\pi_i(y_m)\|_G \leq C \cdot \frac{s}{s_i} \), what implies \( \|y_m\|_G \leq C \cdot s \).

So in order to get the main theorem of this section we have to find nice finite groups with proper left invariant metrics that satisfy previous lemma and guarantee that \( (G, d_G) \) has non zero Assouad-Nagata dimension. This is the aim of next lemma.
Lemma 4.8. Let \( n \) be a fixed natural number. There exists a constant \( C_n \geq 1 \) such that for each \( s \in \mathbb{R}_+ \), every finite group \( (\mathbb{Z}^n_k, d^k) \) with \( k > 1 \) and \( d^k \) the canonical word metric of \( \mathbb{Z}^n_k \) has a cover \( U = \{U_0, ..., U_n\} \) where the \( s \)-scale connected components of each \( U_i \) are \( C_n \cdot s \)-bounded.

Proof. Define \( r \) as the integer part of \( \frac{k}{2} \). In this proof we will see the groups \( \mathbb{Z}_k \) as:

\[
\mathbb{Z}_k = \{-r, -r + 1, ..., -1, 0, 1, ..., r - 1, r\}.
\]

As a trivial consequence of the results of [13] it can be showed that \( \dim_{AN} \mathbb{Z}^n \leq n \) so let \( D(s) = C'_n \cdot s \) be an \( n \)-dimensional control function of \( \mathbb{Z}^n \) with \( C'_n \geq 1 \). Fix \( s \) a positive number and take a cover \( V = \{V_0, ..., V_n\} \) of \( \mathbb{Z}^n \) such that the \( s \)-scale connected components of each \( V_i \) are \( C'_n \cdot s \)-bounded. Define \( U' = \{U'_0, ..., U'_n\} \) a cover in \( I^n_r = \{0, 1, ..., r\} \) by the rule \( U'_i = V_i \cap I^n_r \). Notice that the restriction of \( d^k \) to \( I^n_r \) coincides with the \( l_1 \)-metric of \( I^n_r \). Let \( \pi_i : \mathbb{Z}^n_k \rightarrow \mathbb{Z}_k \) be the canonical projection over the ith coordinate. For each subset \( \lambda \) of \( \{1, 2, ..., n\} \) take the automorphism \( p_{\lambda} : Z^n_k \rightarrow \mathbb{Z}^n_k \) given by \( p_{\lambda}(x) = \epsilon(\lambda, i) \cdot \pi_i(x) \) with \( \epsilon(\lambda, i) = 1 \) or \( \epsilon(\lambda, i) = -1 \) depending on \( i \in \lambda \) or \( i \notin \lambda \). Define the cover \( U = \{U_0, ..., U_n\} \) of \( \mathbb{Z}^n_k \) by the rule \( x \in U_i \) if and only if there exists a \( \lambda \) such that \( p_{\lambda}(x) \in U_i \). Let us estimate the diameter of the \( s \)-scale connected components of \( U_i \). Firstly we claim that if \( L \) and \( M \) are two different \( s \)-scale connected components of \( U_i \) then \( d(p_{\lambda_1}(L), p_{\lambda_2}(M)) \geq s \) for every \( \lambda_1 \) and \( \lambda_2 \) subsets of \( \{1, 2, ..., n\} \). Suppose on the contrary that there exists \( x \in L \) and \( y \in M \) such that \( d(p_{\lambda_1}(x), p_{\lambda_2}(y)) < s \). But if \( x = (x_1, ..., x_n) \) and \( y = (y_1, ..., y_n) \) then:

\[
d(x, y) = |x_1 - y_1| + ... + |x_n - y_n| \leq \sum_{i=1}^{n} d_k(\epsilon(\lambda_1, i) \cdot x_i, \epsilon(\lambda_2, i) \cdot y_i) < s
\]

A contradiction with the fact that \( L \) and \( M \) are different \( s \)-scale connected components.

From this we deduce that if \( L' \) is an \( s \)-scale connected component of \( U_i \) then there exists an \( s \)-scale connected component \( L \) of \( U'_i \) and some subsets \( \lambda_1, ..., \lambda_m \) of \( \{1, ..., n\} \) so that \( L' = \bigcup_{j=1}^{m} p_{\lambda_j}(L) \). Each \( p_{\lambda_j}(L) \) is \( s \)-scale connected and \( C'_n \cdot s \)-bounded as the maps \( p_{\lambda_j} \) are isometries. It is clear that \( m \leq 2^n \). Hence we get:

\[
diam(L') = diam\left( \bigcup_{j=1}^{m} p_{\lambda_j}(L) \right) \leq \sum_{i=1}^{m} (C'_n \cdot s + s) \leq 2^n \cdot (C'_n + 1) \cdot s
\]

Therefore the cover \( U = \{U_0, ..., U_n\} \) satisfies the requirements of the lemma with \( C_n = 2^n \cdot (C'_n + 1) \).

Theorem 4.9. For each \( n \in \mathbb{N} \cup \{\infty\} \) there exists a locally finite group \( G^n \) with a proper left invariant metric \( d_n \) such that \( asdim_{AN}(G^n, d_n) = n \).

Proof. Case \( n < \infty \). Suppose \( n \) fixed. Let \( \{k_i\}_{i \in \mathbb{N}} \) be an increasing sequence of natural numbers with \( k_i > 1 \) and let \( \{r_i\}_{i \in \mathbb{N}} \) be the sequence given by the integer part of \( \frac{k_i}{2} \). Take the finite groups \( (G_i^n = \mathbb{Z}^n_{k_i}, d_{G_i^n}) \) with \( d_{G_i^n} \) the canonical word metric and let \( (G, d_G) \) be the group \( G = \bigoplus_{i=1}^{\infty} G_i^n \) with \( d_G \) the quasi-ultrametric generated by \( \{d_{G_i^n}\} \). It is clear that if we embed the subsets \( \{0, 1, ..., r_i\}^n \) of each \( G_i^n \) in \( G \) we get a sequence of \( n \)-dimensional dilated cubes of increasing size so applying [27] we obtain \( asdim_{AN}(G, d_G) \geq n \). On the other hand by lemmas 4.7 and 4.8 we conclude \( asdim_{AN}(G, d_G) \leq n \).
Case $n = \infty$. Just take the group $G = \bigoplus_{i=1}^{\infty} G_i$ with $G_i$ the groups defined in the previous case and construct the quasi-ultrametric $d_G$ generated by $\{d_{G_i}\}_{i \in \mathbb{N}}$. Applying an analogous reasoning as above we get that $\text{asdim}_{\text{AN}}(G, d_G) \geq n$ for each $n$.

Remark 4.10. It was asked in [5] about the possibility of defining the asymptotic Assouad-Nagata dimension of arbitrary groups $G$ as the supremum of the asymptotic Assouad-Nagata dimensions of its finitely generated subgroups $H$. From the previous results we can deduce that this approach does not work well. Even if we assume that the asymptotic Assouad-Nagata dimension of $G$ is finite.

Corollary 4.11. For every $n, k$ with $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$ there exists a countable abelian group $(G^{(n,k)}, d_{(n,k)})$ with $d_{(n,k)}$ a proper left invariant metric such that $\text{asdim}(G^{(n,k)}, d_{(n,k)}) = n$ but $\text{asdim}_{\text{AN}}((G^{(n,k)}, d_{(n,k)})) = n + k$.

Proof. Fix $n$ and $k$ as in the hypothesis and take the group $(G^k, d_k)$ as in theorem 4.9. Define the group $G^{(n,k)} = \mathbb{Z}^n \oplus G^k$ with the proper left invariant metric $d_{(n,k)}$ given by $d_{(n,k)}((x_1, y_1), (x_2, y_2)) = \|x_1 - x_2\|_1 + d_k(y_1, y_2)$. Multiplying an $n$-dimensional dilated cube of $\mathbb{Z}^n$ with a $k$-dimensional dilated cube of $G^k$ we will get an $n + k$-dimensional dilated cube in $G^{(n,k)}$. Applying 2.7 we deduce $\text{asdim}_{\text{AN}}((G^{(n,k)}, d_{(n,k)})) \geq n + k$. The other inequalities follow easily from the by the subadditivity of the asymptotic dimension and the Assouad-Nagata dimension with respect to the cartesian product (see for example [5]) and the well known fact $\text{asdim}(\mathbb{Z}^n, d_1) = n$.

Problem 4.12. Does any countable group $G$ of finite asymptotic dimension satisfy the following condition: There exists a proper left invariant metric $d_G$ such that $\text{asdim}(G, d_G) < \text{asdim}_{\text{AN}}(G, d_G) < \infty$?

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