On brane solutions in M(atrix) theory

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ABSTRACT

In this paper we consider brane solutions of the form $G/H$ in M(atrix) theory, showing the emergence of world volume coordinates for the cases where $G = SU(n)$. We examine a particular solution with a world volume geometry of the form $\mathbb{CP}^2 \times S^1$ in some detail and show how a smooth manifold structure emerges in the large $N$ limit. In this limit the solution becomes static; it is not supersymmetric but is part of a supersymmetric set of configurations. Supersymmetry in small locally flat regions can be obtained, but this is not globally defined. A general group theoretic analysis of the previously known spherical brane solutions is also given.

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1. Introduction

The matrix theory proposed by BFSS [1, 2] as a version of M-theory has been rather intensively investigated over the last year or so. It is by now clear that it does capture many of the expected features of M-theory such as the 11-dimensional supergravity regime and the existence of extended objects of appropriate dimensions, although there are still some issues to be resolved regarding recently found discrepancies [3,4]. Smooth extended objects pertain to the large $N$-limit as well as nonperturbative regimes of the theory and as such, the study of these objects, particularly of curved brane solutions, is of interest. The emergence of the two-brane or the standard membrane was analyzed many years ago in the paper of de Wit, Hoppe and Nicolai [5]. More recently, spherically symmetric membranes have been obtained [6,7]. As regards the five-brane, which is the other extended object of interest, there has been no satisfactory construction or understanding of the transverse brane where all five spatial dimensions are a subset of the nine manifest dimensions of the matrix theory. The longitudinal five-branes, called $L^5$-branes, which have four manifest dimensions and one along the compactified direction (either the 11-th dimension or the lightlike circle) have been obtained. These include flat branes [8,9] and stacks of $S^4 \times S^1$-branes [10]. In this paper we analyze the construction of brane configurations. We obtain a group theoretical rephrasing of the known solutions and some new solutions such as an $L^5$-brane of $\mathbb{C}P^2 \times S^1$-geometry. For this latter case it is possible to have a single smooth static brane configuration, rather than stacks of them, in the large $N$-limit. Static solutions, as opposed to merely static configurations, are possible if $N \to \infty$ with $R$ fixed, where $R$ is the radius of the lightlike circle or the 11-th dimension.

In our analysis, the condition for finite potential energy in the large $N$-limit can be easily obtained in a group theoretic way. The static solution we obtain has finite energy as $N \to \infty$. In general, the geometry of curved brane solutions must clearly be constrained by the requirement that they be solutions in M-theory. The fact that one can obtain finite static energy only for a small class of configurations is presumably related to a M(atrix) theory version of this condition.
The plan of this paper is as follows: In section two we explain our ansatz and its group theoretical structure. In this section we also show how the known spherical branes have a simple description in the framework of our analysis. Section three is devoted to the discussion of one particular solution, namely, a five-brane with the world volume geometry of $\mathbb{CP}^2 \times S^1$. In this section we also indicate the way to introduce local coordinates in the world volume of the brane and discuss the emergence of the smooth manifold structure in the large $N$ limit. The immersion of the $\mathbb{CP}^2$ in $\mathbb{R}^9$ is also discussed. In section four we discuss the response of the $\mathbb{CP}^2 \times S^1$ solution to supersymmetry transformations. Our solution is not supersymmetric, but is part of a set of degenerate configurations which are related by supersymmetry. We argue that it also admits some unbroken $N = 4$ supersymmetry in small locally flat neighborhoods; this notion is however not globally defined and hence the implications of this symmetry are unclear. In section five we discuss some other solutions and give a different descriptions of the spherical membrane. The paper concludes with a brief discussion.

2. The ansatz

The matrix theory Lagrangian can be written as

$$\mathcal{L} = Tr \left[ \frac{\dot{X}_I^2}{2R} + \frac{R}{4} [X_I, X_J]^2 + \theta^T \dot{\theta} + iR\theta^T \Gamma_I [X_I, \theta] \right] \tag{1}$$

where $I, J = 1, \ldots, 9$ and $\theta$ is a 16-component spinor of $O(9)$ and $\Gamma_I$ are the appropriate gamma matrices. $X_I$ are hermitian $(N \times N)$-matrices; they are elements of the Lie algebra of $U(N)$ in the fundamental representation. The theory is defined by this Lagrangian supplemented by the Gauss law constraint

$$[X_I, \dot{X}_I] - [\theta, \theta^T] \approx 0 \tag{2}$$

In the following we shall be concerned with bosonic solutions and the $\theta$'s will be set to zero. The relevant equations of motion are thus

$$\frac{1}{R} \dddot{X}_I - R [X_J, [X_I, X_J]] = 0 \tag{3}$$
We shall look for solutions which carry some amount of symmetry. In this case, simple ansätze can be formulated in terms of a group coset space $G/H$ where $H \subset G \subset U(N)$. The ansätze we consider will have spacetime symmetries, a spacetime transformation being compensated by an $H$-transformation. In the large $N$-limit, the matrices $X_I$ will go over to continuous brane-like solutions with the geometry of $G/H$.

Let $t^A, A = 1, \ldots, \text{dim}G$ denote a basis of the Lie algebra of $G$. We split this set of generators into two groups, $t^\alpha, \alpha = 1, \ldots, \text{dim}H$ which form the Lie algebra of $H$ and $t_i, i = 1, \ldots, (\text{dim}G - \text{dim}H)$ which form the complementary set. Our ansatz will be to take $X_I$’s to lie along the entire algebra of $G$ or to be linear combinations of the $t_i$’s. In the latter case, in order to satisfy the equation of motion (3), we shall then need the double commutator $[X_J, [X_I, X_J]]$ to be combinations of the $t_i$’s themselves. This is guaranteed if $[t_i, t_j] \subset H$, since the $t_i$’s themselves transform as representations of $H$. In this case we have

$$[t^\alpha, t^\beta] = if^{\alpha\beta\gamma}t^\gamma, \quad [t^\alpha, t_i] = if_{ij}^\alpha t_j \quad (4a)$$

$$[t_i, t_j] = ic_{ij}^\alpha t^\alpha \quad (4b)$$

and $G/H$ is a symmetric space. If $H = 1$ the $t_i$’s will belong to the full algebra $G$. In this case, $c_{ij}^\alpha$ of Eq.(4b) will be the structure constants of $G$. In such a case, even though the ansatz involves the full algebra $G$, it can satisfy further algebraic constraints which have only a smaller invariance group $H$. The solution will then again reduce to the $G/H$-type. We shall see an example of this in the next section. These are the cases we analyze.

Since $X_I$ are elements of the Lie algebra of $U(N)$, having chosen a $G$ and an $H$, we must consider the embedding of $G$ in $U(N)$. This is done as follows. We consider a value of $N$ which corresponds to the dimension of a unitary irreducible representation (UIR) of $G$. The embedding is then specified by identifying the fundamental $N$-dimensional representation of $U(N)$ with the $N$-dimensional UIR of $G$. Eventually we would want to consider the limit $N \to \infty$ as well. Thus we need to consider an infinite sequence of UIR’s of $G$. Different choices of such sequences are possible, presumably corresponding to different ways of defining the $N \to \infty$ limit. A simple convenient choice is to take the
symmetric rank \( s \)-tensors of \( G \), of dimension, say, \( d(s) \). Thus we choose \( N = d(s) \), defining the large \( N \)-limit by \( s \rightarrow \infty \).

The ansatz we take is of the form

\[
X_i = r(t) \frac{t_i}{N^a}
\]  

(5)

for a subset \( i = 1, \ldots, p \) of the nine \( X \)'s. \( t_i \) are generators in \( G/H \), in the symmetric rank \( s \)-tensor representation of \( G \). The eigenvalues fo any of the \( t_i \)'s in the \( s \)-tensor representation will range from \( cs \) to \(-cs \), where \( c \) is a constant. The eigenvalues thus become dense with a finite range of the variation of the \( X_i \)'s as \( s \rightarrow \infty \) if \( N^a \sim s \). In this case, as \( s \rightarrow \infty \), the \( X_i \)'s will tend to a smooth brane-like configuration. Our choice of the index \( a \) will be fixed by this requirement, viz., \( N^a \sim s \). Notice that this ansatz is consistent with the Gauss law (2).

The ansatz (5) has a symmetry of the form

\[
R_{ij} \ U X_j U^{-1} = X_i
\]  

(6)

where \( R_{ij} \) is a spatial rotation of the \( X_i \)'s and \( U \) is an \( H \)-transformation for the \( G/H \) case or more generally it can be in \( U(N) \). \( R_{ij} \) is determined by the choice of the \( X_i \)'s involved in (5). Further, the ansatz (5) is to be interpreted as being given in a specific gauge. A \( U(N) \)-transformation, common to all the \( X_i \)'s, is a gauge transformation and does not bring in new degrees of freedom. We may alternatively say that the meaning of Eq.(6) is that \( X_j \) is invariant under rotations \( R_{ij} \) upto a gauge transformation.

For the ansatz (5), the Lagrangian simplifies to

\[
\mathcal{L} = A_s \left[ \frac{\dot{r}^2}{2RN^{2a}} - \frac{c_{ij\alpha}c_{ij\alpha}}{4N^{4a}} R r^4 \right]
\]  

(7)

where \( A_s \) is defined * by \( \text{Tr}(t_A t_B) = A_s \delta_{AB} \) and the matrices and trace are in the \( s \)-tensor of \( G \). From its definition, \( A_s = d(s)c_2(s)/\text{dim}G \), where \( c_2(s) \) is the quadratic Casimir of

* We normalize the generators of \( G \) such that in the fundamental representation \( \text{Tr}(t_A t_B) = \frac{1}{2} \delta_{AB} \).

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the $s$-tensor representation which goes like $s^2$ for large $s$. Thus, with $N^a \sim s$, the kinetic term in (7) will always go like $N/R$ for large $s$.

We now turn to some specific cases. Consider first $G = SU(2)$. In this case, $d(s) = s+1$, $A_s = s(s+1)(s+2)/12$. The smooth brane limit thus requires $a = 1$ or $N^a \sim N \sim s$. The kinetic energy term in (7) goes like $N/R$, while the potential term goes like $R/N$. Thus both these terms would have a finite limit if we take $N \to \infty$, $R \to \infty$, keeping $(N/R)$ fixed. In fact, this particular property holds only for $G = SU(2)$ or products thereof, such as $G = O(4)$. For this reason some of the branes which are realized as cosets of products of the $SU(2)$ group can perhaps be regarded as being transverse as their energy will not depend on $R$ in the large $R$ limit. We can use this case to obtain a slightly different description of the spherical membrane of [6,7] as well as a ”squashed” $S^2$ or $\mathbb{CP}^1$. The round $\mathbb{CP}^1$ corresponds to the case where the three generators of $SU(2)$ lie along three of the nine $X'_i$s.

As another example, consider $G = O(6) \sim SU(4)$. In this case, $d(s) = (s+1)(s+2)(s+3)/6$, $A_s = (1/240)(s+4)!/(s−1)!$ and we need $a = \frac{1}{3}$. The kinetic energy term goes like $N/R$ while the potential energy goes like $sR$. This corresponds to the longitudinal five-brane with $S^4$-geometry discussed in [10]. $SU(4)$ has an $O(5)$ subgroup under which the 15-dimensional adjoint representation of $SU(4)$ splits into the adjoint of $O(5)$ and the 5-dimensional vector representation. The coset generators, corresponding to the 5-dimensional representation, may be represented by the $(4 \times 4)$-gamma matrices, $\gamma_i$, $i = 1, ..., 5$. The ansatz (5) thus takes the form $X_i = r\gamma_i/N^{\frac{3}{4}}$. It is shown in [9] that the sum of the squares of the $X_i$’s is proportional to the identity, thus giving effectively a four-dimensional brane. This is interpreted as $s$ copies of a longitudinal five-brane, one of the directions being along the compactified 11-th or lightcone coordinate, of extent $R$, an interpretation consistent with the potential energy $\sim sR$.

An interesting special case which we shall analyze in some detail corresponds to $G = SU(3)$. In this case, $d(s) = (s+1)(s+2)/2 \sim s^2$ and $A_s = (1/48)(s+3)!/(s−1)!$. We choose $a = \frac{1}{2}$. The kinetic energy again goes like $N/R$ for large $N$ while the potential energy goes
like $R$. The potential energy is independent of $s$ and thus, for $r$ independent of $t$, we have a single static smooth five-brane wrapped around the compactified dimension in the $s \to \infty$ limit. In other words, the tension defined by the static energy per unit volume remains finite as $s \to \infty$. An appropriate choice of $H$ in this case is $H = U(2) \sim SU(2) \times U(1)$. The emergent world volume geometry is then $\mathbb{CP}^2 \times S^1$, whose immersion in $\mathbb{R}^9$ can be complicated and will depend on the details of the ansatz. As we shall see in the next section the coset embedding will produce a singular surface in a 9-dimensional Euclidean space, while when the eight generators of $SU(3)$ are set parallel to eight of nine $X_I$’s we shall obtain the standard $CP^2$ embedded in an $S^7$ contained in $R^9$.

Notice also that since the effective mass for the degree of freedom corresponding to $r$ goes like $N/R$, oscillations in $r$ are suppressed as $N \to \infty$ with $R$ fixed; in this limit this configuration becomes a static solution. We can fix $r$ to any value and time-evolution does not change this.

3. World volume coordinates and analysis of $\mathbb{CP}^2 \times S^1$-brane

We shall now analyze the $\mathbb{CP}^2 \times S^1$ solution in some detail. The key to interpreting this as a five-brane is the emergence of the world volume coordinates in the large $N$-limit. In this limit, the matrices $X_i$ become truly infinite dimensional and one has something like a semiclassical limit, the configurations being smooth and matrix commutators becoming Poisson brackets. The emergence of the world volume coordinates can be seen in a suitable parametrization of the generators $t_i$. (Notice that the energy of a static configuration, given by the potential energy in Eq.(7), goes like $R r^4$, which is naively suggestive of a five-dimensional extended object. However, this behaviour of the potential energy is identical for all configurations, i.e., for any kind of extended object, and one cannot identify the dimensionality, much less any geometry, from this consideration.)

Since $SU(3)/U(2)$ is $\mathbb{CP}^2$, we start with a parametrization of the $SU(3)$ generators in terms of homogeneous coordinates on $\mathbb{CP}^2$. Introduce three complex numbers $\pi_\alpha$, $\alpha = 1, 2, 3$. The identification $\pi_\alpha \sim \lambda \pi_\alpha$, $\lambda \in \mathbb{C} - \{0\}$ gives $\mathbb{CP}^2$. As homogeneous coordinates, valid for a coordinate patch around $\pi_3 \neq 0$, we can use $\xi_i = \pi_i/\pi_3$, $i = 1, 2$. The generators
belonging to SU(3) − U(2) are given by

\[ t_i = \pi_3 \frac{\partial}{\partial \pi_i}, \quad t_i^\dagger = \pi_i \frac{\partial}{\partial \pi_3} \] (8)

We are interested in the s-tensor representation which has the form \[ \pi_{\alpha_1} \cdots \pi_{\alpha_s} = \pi_3^s f(\xi). \]

Simplifying the above expressions, we find for these states,

\[ t_i = \frac{\partial}{\partial \xi_i}, \quad t_i^\dagger = \xi_i \left( s - \xi_k \frac{\partial}{\partial \xi_k} \right) \]

\[ h_{ij} \equiv [t_i, t_j^\dagger] = \left( s - \xi_k \frac{\partial}{\partial \xi_k} \right) \delta_{ij} - \xi_j \frac{\partial}{\partial \xi_i} \] (9)

The states of the s-tensor representation are of the form \( f = 1, \xi_1, \ldots, (\xi_1 \cdots \xi_s) \). The scalar product for these states is given by

\[ \langle f | g \rangle = \frac{(s+1)(s+2)}{2} \int \frac{2 d^4 \xi}{\pi^2 (1 + \xi \cdot \xi)^3} \frac{1}{(1 + \xi \cdot \xi)^s} \bar{f} g \] (10)

where \( \bar{\xi} \cdot \xi = \xi_k \xi_k \). \( t_i, t_i^\dagger \) are adjoints with this inner product. Notice also that finiteness of norm restricts the powers of \( \xi_i \) to be less than or equal to \( s \), giving \( N = d(s) = \frac{1}{2}(s+1)(s+2) \) states in all. Also, in matrix elements of these operators we can use

\[ t_i = (s + 3) \frac{\bar{\xi}_i}{(1 + \xi \cdot \xi)}, \quad t_i^\dagger = (s + 3) \frac{\xi_i}{(1 + \xi \cdot \xi)} \]

\[ h_{ij} = (s + 3) \frac{(\delta_{ij} - \bar{\xi}_i \xi_j)}{(1 + \xi \cdot \xi)} \] (11)

To see the emergence of the continuous world volume coordinates from the matrix model, let us look, for example, at the states near the state \( f = 1 \sim \pi_3^s \). On such states we have \( [\sqrt{N} t_i, t_j^\dagger] \approx \frac{s}{N} \delta_{ij} \); we can thus use semiclassical simplifications as \( s \to \infty, s/N \to 0 \). *

In this case, any operator involving products of SU(3) generators, \( t_i, t_i^\dagger, h_{ij} \), etc., may be replaced by the \( c \)-number versions as in Eq.(11). The trace of an operator \( \mathcal{O} \) is given by

\[ \text{Tr} \mathcal{O} = \frac{(s+1)(s+2)}{2} \int \frac{2 d^4 \xi}{\pi^2 (1 + \xi \cdot \xi)^3} \mathcal{O}(\bar{\xi}, \xi) \] (12)

* This is similar to the large spin approximation in Holstein-Primakoff formulation of magnetic systems in condensed matter physics.
Again, as $s \to \infty$, the matrix commutator of any two operators $F, G$ goes over to a Poisson bracket of the corresponding functions given by

$$[F, G] \to \{F, G\} = (\Omega^{-1})^{\bar{a}a}(\partial_{\bar{a}}F\partial_a G - \partial_a F\partial_{\bar{a}} G)$$

$$(\Omega^{-1})^{\bar{a}a} = \frac{(1 + \bar{\xi} \cdot \xi)}{s + 3} (\delta^{\bar{a}a} + \bar{\xi}^a \xi^a)$$  \hspace{1cm} (13)$$

$(\Omega^{-1})^{\bar{a}a}$ is the inverse of the Kähler form of $\mathbb{C}P^2$.

The ansatz for the five-brane may now be stated as follows. The simplest case to consider is the following. We define the complex combinations $Z_i = \frac{X_i + iX_{i+2}}{\sqrt{2}}, \ i = 1, 2$. The symmetric ansatz is then given by

$$Z_i = r(t) \frac{t_i}{\sqrt{N}}, \quad i = 1, 2$$
$$X_i = 0, \quad i = 5, ..., 9$$  \hspace{1cm} (14)$$

In the large $s$-limit, $Z_i \approx r \frac{s}{\sqrt{N}} \frac{\bar{\xi}_i}{1 + \xi_\bar{\xi}}$, realizing a continuous map from $\mathbb{C}P^2$ to the space $\mathbb{R}^9$. This map is not one-to-one; the region $\bar{\xi} \cdot \xi < 1$ and the region $\bar{\xi} \cdot \xi > 1$ are mapped into the same spatial region $|Z| < r\sqrt{2}$, corresponding to a somewhat squashed $\mathbb{C}P^2$.

The standard $\mathbb{C}P^2$ is obtained by considering an $\mathbb{R}^8$-subspace of $\mathbb{R}^9$, whose coordinates can be identified with the $SU(3)$ generators as in Eq.(5), i.e., $X_A = rt_A/\sqrt{N}, \ A = 1, 2, ..., 8, \ X_9 = 0$. Specifically in the parametrization of the $SU(3)$ generators in terms of the $\xi$’s as in Eq.(11), this ansatz is given by (recall that in the large $N$ limit $s \approx \sqrt{2N}$)

$$X_i = \frac{r}{\sqrt{2}} \frac{\bar{\xi}_i \bar{\sigma}^i \xi}{1 + \xi_\bar{\xi}}$$
$$X_4 + iX_5 = \frac{r}{\sqrt{2}} \frac{2\xi_1}{1 + \xi_\bar{\xi}}$$
$$X_6 + iX_7 = \frac{r}{\sqrt{2}} \frac{2\xi_2}{1 + \xi_\bar{\xi}}$$
$$X_8 = \frac{r}{\sqrt{6}} \frac{2 - \xi_\bar{\xi}}{1 + \xi_\bar{\xi}}$$
$$X_9 = 0$$  \hspace{1cm} (15a)$$

where $\sigma^i, \ i = 1, 2, 3$ are Pauli matrices. Notice that the singularity of the squashed $\mathbb{C}P^2$ is removed by this ansatz since the regions $\bar{\xi} \cdot \xi < 1$ and $\bar{\xi} \cdot \xi > 1$ are mapped to different
regions of the nine-dimensional space. In fact it is easy to see by a direct inspection that the map is actually one-to-one. Furthermore it is not hard to verify that \( \sum_{A=1}^{8} X_A X_A = \frac{2r^2}{3} \).

Thus our surface is embedded in an \( S^7 \). In fact we can use the above relations and express \( X^1, X^2 \) and \( X^3 \) in terms of \( X^4, ..., X^8 \) as

\[
X^a = \frac{3}{\sqrt{2}} \frac{\bar{\zeta} \sigma^a \zeta}{r + \sqrt{6} X_8}
\]

where \( \zeta \) is a two-component vector defined by \( \zeta^1 = \frac{1}{\sqrt{2}} (X_4 + iX_5) \) and \( \zeta^2 = \frac{1}{\sqrt{2}} (X_6 + iX_7) \).

To see how the smooth \( \mathbb{CP}^2 \) emerges from this construction, define the \((3 \times 3)\) hermitian traceless matrix \( H = \sum_{A=1}^{8} X_A \lambda_A \), where \( \lambda_A \) are the eight hermitian generators of \( SU(3) \) in the fundamental representation and the \( X' \)'s are parametrized as above. By a somewhat lengthy but straightforward calculation we can show that \( H \) satisfies the following constraint

\[
H^2 = \frac{r}{3\sqrt{2}} H + \frac{r^2}{9}
\]

The manifest \( SU(3) \) covariance of this equation indicates that our four-manifold is indeed \( SU(3) \) invariant. In fact \( SU(3) \) acts transitively on the surface defined (16), because any \( H \) satisfying the constraint (16) can be obtained from a diagonal one given by \( H_{\text{diag.}} = (r/3\sqrt{2}) \text{ diag } (-1, -1, 2) \) by the action of an \( SU(3) \)-transformation. Furthermore \( H_{\text{diag.}} \) is invariant under a \( SU(2) \times U(1) \) subgroup of \( SU(3) \). It can thus depend only on four real parameters. These properties are sufficient to identify the manifold covered by the two complex \( \xi \)'s as the standard smooth \( \mathbb{CP}^2 \) embedded in a one-to-one way in an \( S^7 \) subspace of \( \mathbb{R}^9 \).

The coset structure is clearer directly in terms of the ansatz for (14), which is why we started with this squashed \( \mathbb{CP}^2 \). The smooth configuration (15) may be regarded as a relaxation of (14) along some of the \( \mathbb{R}^9 \)-directions.

A comment regarding the large \( s \) limit is in order. The identification of the generators \( t_i \) with the functions of \( \bar{\xi}, \xi \) as in Eq.(11) can be done for finite \( N \) for matrix elements of single powers of \( t_i \). Furthermore the isomorphism between the \( SU(3) \) Lie algebra and the algebra defined by the Poisson brackets is valid for any \( s \). However, this isomorphism
extends to the enveloping algebras, or general functions of the generators, only in the large $s$ limit. Therefore it is only for large $s$ that the Lie bracket operation on any function of the generators of $SU(3)$ will go over to the Poisson brackets of the corresponding classical functions. Also, the states are limited to the $s$-th power of the $\xi$’s by the condition of the finiteness of the norm and one can get arbitrary world volume deformations of the five-brane (which involve arbitrarily high powers of $\xi$’s) only for $s \to \infty$. Thus it is only in this limit that we can expect a continuous five-brane which is an immersion of $\mathbb{C}P^2$ in spacetime.

The Lagrangian for ansätze (14,15) becomes

$$L = \frac{(s+3)!}{(s-1)!} \left[ \frac{\dot{r}^2}{12NR} - \frac{Rr^4}{8N^2} \right]$$

$$\approx \left[ \frac{N \dot{r}^2}{R \ 3} - \frac{Rr^4}{2} \right] (1 + \mathcal{O}(\frac{1}{s}))$$

(17)

In terms of the world volume coordinates, we can also write

$$L \approx \int \frac{2 \ d^4\xi}{\pi^2(1 + \xi \cdot \xi)^3} \left[ \frac{s^4 \dot{r}^2}{2NR} \left( \frac{\tilde{\xi} \cdot \xi}{(1 + \xi \cdot \xi)^2} \right) - Rr^4 \frac{s^4}{4N^2} \frac{2 - 2\tilde{\xi} \cdot \xi + (\tilde{\xi} \cdot \xi)^2}{(1 + \xi \cdot \xi)^2} \right]$$

(18a)

$$\approx \int \frac{2 \ d^4\xi}{\pi^2(1 + \xi \cdot \xi)^3} \left[ \frac{N \dot{r}^2}{R \ 3} - \frac{Rr^4}{2} \right]$$

(18b)

Expression (18a) applies to ansatz (14), expression (18b) to ansatz (15). The energy densities are uniformly distributed over the world volume for (15), but not for (14).

The equation of motion for $r$ becomes

$$\frac{N}{R} \ddot{f} + 6f^3 = 0$$

(19)

where $r = f/\sqrt{R}$. The effective mass for the degree of freedom corresponding to $r$ is $\frac{2}{3}(N/R)$. Thus in the limit $N \to \infty$ with $R$ fixed, any solution with finite energy would have to have a constant $r$ or $f$. In this limit, we thus get a five-brane which is a static solution of the matrix theory. Alternatively, if we consider $N, R \to \infty$ with $(N/R)$ fixed, $f$ can have a finite value. However, in this limit, the physical dimension of the brane as given by $r$ would vanish.
Explicit solutions to Eq.(19) may be written in terms of the sine-llemniscate function as

\[ f = A \sin \text{lemn} \left( \sqrt{\frac{3R}{N}} A(t - t_0) \right) \]  

(20)

The supersymmetry algebra calculation identifies the brane charge as [7]

\[ Q = \frac{R}{4!} \text{Tr}(X_i X_j X_k X_l) \epsilon^{ijkl} \]

\[ = \frac{R}{12} \text{Tr} \left( [Z_1, \bar{Z}_1][Z_2, \bar{Z}_2] - [Z_1, \bar{Z}_2][Z_2, \bar{Z}_1] - [Z_1, Z_2][\bar{Z}_1, \bar{Z}_2] \right) \]  

(21)

For the rest of this section, we shall consider the squashed CP^2 of Eq.(14); we have written out Q for this case in Eq.(21). In the large s-limit we find

\[ Q \approx \frac{R r^4 s^4}{24 N^2} \int \frac{2 \, d^4 \xi}{\pi^2(1 + \xi \cdot \xi)^3} \frac{1 - \bar{\xi} \cdot \xi}{(1 + \xi \cdot \bar{\xi})^2} \]  

(22)

We see that there is nonuniformly distributed charge density on the world volume of the brane. Of course, if the integral is evaluated, the result will be zero, in consistency with the evaluation of the trace in Eq.(21). This trace will also vanish for the L5-brane discussed in [10]. This is a reflection of the fact that there is no good definition of Q for finite dimensional matrices; it is only as N → ∞ that a charge can be expected. In reference [10], a nonzero charge was obtained essentially by restricting the trace to a subset of states. A similar definition of Q will give a nonzero result for our solution as well. For example, if in the evaluation of the trace we consider only the contributions coming from the states very near f = 1 ∼ πs, which corresponds to the approximation \((1 - \bar{\xi} \cdot \xi)/(1 + \bar{\xi} \cdot \xi)^2 \approx 1\) in the integrand of (22), we obtain

\[ Q \approx \frac{R r^4}{6}. \]

A similar result will hold for the regular CP^2-solution of Eq.(15). In this case, we need to choose a subset of four out of the eight X_A’s for use in the formula (21). For ξ’s near zero, the local coordinates correspond to X_4, X_5, X_6, X_7. If this set is used, the result is the same as in Eq.(22), except for an additional factor of \(1/4\) since \(\zeta_i \sim (r/2\sqrt{N}) t_i\).

As regards spacetime properties of the solution, the energy we have evaluated is the lightcone component \(T^{+-}\). Other components can be evaluated, following the general formula of [11], and they will act as a source for gravitons, again along the general lines of
The current $\mathcal{T}^{IJK}$ which is the source for the antisymmetric tensor field of eleven-dimensional supergravity is another quantity of interest. This vanishes for the spherically symmetric configurations considered in [10], since there are no invariant $O(9)$-tensors of the appropriate rank and symmetry. However, for the $\mathbb{CP}^2$ geometry, there is the Kähler form and the possibility that $\mathcal{T}^{-ij}$ can be proportional to the Kähler form has to be checked explicitly. The kinetic terms of $\mathcal{T}^{-ij}$ which depend on $\dot{r}$ are easily seen to vanish for the solution (15), essentially because of the symmetric nature of the ansatz. For solution (14), we find, by direct evaluation,

$$
\begin{align*}
\mathcal{T}^{-11} &= \mathcal{T}^{-22} = -\frac{i}{60} \frac{s^5}{N^2} \dot{r}^2 + ... \\
\mathcal{T}^{-12} &= \mathcal{T}^{-21} = 0
\end{align*}
$$

Naively, this diverges as $s \to \infty$. However, as we have noticed before, in this limit, the solution becomes static, $\dot{r} = 0$ and hence this vanishes. This holds for other components of $\mathcal{T}^{IJK}$ as well. The nonkinetic terms in $\mathcal{T}^{IJK}$ are of the form $R^2 r^6 (\frac{s^5}{N^2})$ and also vanish as $s \to \infty$. Thus the source for the antisymmetric tensor field is zero in the $s$ (or $N$) $\to \infty$ limit.

In the alternative limit with $N/R$ fixed, if we write $r = f/\sqrt{R}$, $f$ has a finite limit and $\mathcal{T}^{-ij}$ of Eq.(23) vanishes as $R \to \infty$, eliminating possible radiation of the antisymmetric tensor field. (This holds also for the nonkinetic terms of $\mathcal{T}^{IJK}$ which we have not displayed.) If the limit is taken in this way, the physical dimension $r$ shrinks to zero. Presumably the interpretation of this limit is that the squashed $\mathbb{CP}^2$-brane can collapse by radiation of the antisymmetric tensor field as well as by radiation of gravitons.

4. Supersymmetry and the $\mathbb{CP}^2$ solution

The Lagrangian of the matrix theory is invariant under supersymmetry transformations. The supersymmetry variation of $\theta$ is given by

$$
\delta \theta \equiv K \epsilon + \rho = \frac{1}{2} \left[ \dot{X}_I \Gamma_I + [X_I, X_J] \Gamma_{IJ} \right] \epsilon + \rho
$$

where $\epsilon$ and $\rho$ are 16-component spinors of $O(9)$. 

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In this section, we shall examine the question of supersymmetry of the solutions. Within the class of solutions considered in this paper, the potential energy, for a fixed $R$, is finite as $s \to \infty$ only for (14) and (15). Therefore we shall examine only these cases.

Since $\delta \theta$ has $s$-dependent terms, the question of supersymmetry is best understood by considering fermionic collective coordinates. These are introduced by using the supersymmetry variation (24) with the parameters $\epsilon, \rho$ taken to be time-dependent. Upon substitution in the Lagrangian, the term $\text{Tr}[\theta^T \dot{\theta}]$ generates the symplectic structure for $(\epsilon, \rho)$. We can then construct the supersymmetry generators for fluctuations around our solution. If the starting configuration is supersymmetric, there will be zero modes in the symplectic form so constructed and we will have only a smaller number of fermionic parameters appearing in $\text{Tr}[\theta^T \dot{\theta}]$. Now, in the large $s$-limit, we have $\text{Tr}[\theta^T \dot{\theta}] \sim s^2 [\epsilon^T K^T K \dot{\epsilon} + \rho^T \dot{\rho}]$ which goes to zero as $s \to \infty$ if $\delta \theta \sim s^{-1-\eta}, \eta > 0$. Thus if $\delta \theta$ vanishes faster than $1/s$, we can conclude that the starting bosonic configuration is supersymmetric.

We now turn to the specific solutions, taking up the squashed $\mathbb{C}P^2$ first. The finiteness of the kinetic energy in the large $N$ limit requires that the leading term in $r$ must be a constant, which is how we obtained a static solution. The equation of motion then shows that $\dot{r}$ must go like $\frac{1}{N} \sim \frac{1}{s^2}$. In other words, we can write $r = r_0 + \frac{1}{N} r_1 + ...$. The $\dot{X}_I$-term of $\delta \theta$ thus vanishes to the order required. The vanishing of $\delta \theta$ (or the BPS-like condition) then becomes, to leading order,

$$-\frac{8r_0^2}{N} (\lambda_a L_a + \sqrt{3} \lambda_8 R_1) \epsilon + \rho = 0 \quad (25)$$

where the set $\lambda_a, \lambda_8, a = 1, 2, 3$ generate an $SU(2) \times U(1)$ subgroup of $SU(3)$ while the operators $L_a, R_1$ generate an $SU(2)_L \times U(1)_R$ subgroup of $O(4) \sim SU(2)_L \times SU(2)_R$. In terms of $(16 \times 16)$ $\Gamma$-matrices they are given by

$$L_1 = \frac{i}{4} (\Gamma_1 \Gamma_3 + \Gamma_4 \Gamma_2)$$

$$L_2 = \frac{i}{4} (\Gamma_1 \Gamma_2 + \Gamma_3 \Gamma_4)$$

$$L_3 = \frac{i}{4} (\Gamma_2 \Gamma_3 + \Gamma_1 \Gamma_4)$$

$$R_1 = \frac{i}{4} (\Gamma_1 \Gamma_3 - \Gamma_4 \Gamma_2) \quad (23)$$
Notice that the $\epsilon$-term is of order $1/s$ since $\lambda_a, \lambda_8$ have eigenvalues of order $s$ and in the large $s$ limit $N \approx \frac{1}{2}s^2$; the $\rho$-term is of order one. Thus condition (25) is required for supersymmetry as explained above. Further, in our problem the $O(9)$ group is broken to $O(4) \times O(5)$. With respect to this breaking, the 16-component spinor of $O(9)$ decomposes according to $16 = ((1,2), 4) + ((2,1), 4)$, where 4 denotes the spinor of $O(5)$. In terms of the $SU(2)_L \times U(1)_R$ subgroup generated by the $L_a$ and $R_1$ we have $(1,2) = 1_1 + 1_{-1}$ and $(2,1) = 2_0$ where the subscripts denote the $U(1)$ charges. Clearly we have no singlets under $SU(2)_L \times U(1)_R$; $SU(2)_L$-singlets necessarily carry $U(1)$ charges. Therefore the operator $(\lambda_a L_a + \sqrt{3} \lambda_8 R_1)$ can neither annihilate $\epsilon$ nor can it be a multiple of the unit operator in the $SU(3)$ space. Hence a nontrivial $\epsilon$-supersymmetry cannot be compensated by a $\rho$-transformation. Thus we have no supersymmetry.

The supersymmetry variation produces a $\theta$ of the form $(\lambda_a L_a + \sqrt{3} \lambda_8 R_1) \epsilon$, where we set $\rho = 0$ for the moment. The contribution of this $\theta$ to the Hamiltonian via the term $\text{Tr}(\theta^T [X_i, \theta])$ is zero due to the orthogonality of the $SU(N)$ generators. In other words, the configurations $(X_i, 0)$ and $(X_i, \delta \theta)$ have the same energy. This gives a supersymmetric set of degenerate configurations, or supermultiplets upon quantization. The starting bosonic configuration is not supersymmetric but is part of a set of degenerate configurations related by supersymmetry.

Consider now the solution (15). In this case also, a $\rho$-transformation cannot compensate for an $\epsilon$-transformation and the condition for supersymmetry becomes $f_{IJK} \Gamma_I \Gamma_J \epsilon = 0$, where $f_{IJK}$ are the structure constants of $SU(3)$. $L_K = f_{IJK} \Gamma_I \Gamma_J$ obey the $SU(3)$ commutation rules and indeed this defines an $SU(3)$ subgroup of $O(8)$. The spinors of $O(8)$ do not contain singlets under this $SU(3)$ and hence there is again no supersymmetry.

Although we do not have supersymmetry in the strict sense it is perhaps interesting to note that we do have unbroken supersymmetries in what has been called locally flat regions in reference [6]. Again, considering the squashed $\mathbb{C}P^2$ for simplicity, this can be seen by choosing $\epsilon$ and $\rho$ to be of the $(1,2)$ type in the notation introduced above. We shall choose them to be $1_1$ of the $SU(2) \times U(1)$ group. Similar arguments will apply to
1−1 component. The BPS condition then reduces to

$$-\frac{8r_0^2}{N} \sqrt{3} \lambda_8 \epsilon + \rho = 0$$

(27)

To apply the notion of local flatness of [6] to our problem we consider those regions of the diagonal matrix \( \lambda_8 \) where the matrix elements are close to their maximum values, which are of the order of \( s \). In these regions \((1/s)\lambda_8\) will behave like \(1 + c/s\), where \( c = -\frac{3}{2}(k+l) \).

In general \( k \) and \( l \), which indicate the number of the indices equal to 1 and 2 in a symmetric rank \( s \) tensor of \( SU(3) \), can range between 1 and \( s \) such that \( k + l \leq s \). The locally flat region is defined by the condition that \( k + l \) is much smaller than \( s \). In such a region the BPS equation (27) will become

$$\frac{8r_0^2}{s} (1 + \frac{c}{s}) \epsilon + \rho = 0$$

(28a)

When \( c \) becomes close to \( s \) we should end up in a different locally flat region. It is clear from (28) that with an appropriate choice of \( \rho \) it is possible to compensate the supersymmetry generated by \( \epsilon \).

A more geometrical way of expressing this observation is to substitute the semi-classical large \( s \) expression for \( \lambda_8 \) in (27). We then obtain the following local expression

$$\frac{4r_0^2}{s} \left[ \frac{2 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} \right] \epsilon + \rho = 0$$

(28b)

Equation (28) corresponds to the region \( \xi = 0 \). Other values of \( \xi \) will correspond to different submatrices of \( \lambda_8 \) where it can be written as the unit matrix plus small corrections of the order of \( 1/s \).

From this discussion we see that in the large \( N \) limit we have a kind of local supersymmetry, which cannot be extended globally. It is of course well known that \( \mathbb{CP}^2 \) does not admit global spin structure. In matrix theory this is a consequence of the fact that the matrix \((1/s)\lambda_8\) cannot be globally written as the sum of the unit matrix and a matrix with elements of the order of \( 1/s \). A global extension of this result can only be obtained if the supersymmetry parameters also transform nontrivially under the gauge group.
5. Other Solutions

So far we have presented a class of solutions of the form $G/H$ for the equations of motion of M(atrix) theory. We shall now comment on how smooth manifolds of various dimensions smaller than 11 can emerge from Matrix theory in the large $N$ limit.

As mentioned in section 2, the condition for the solvability of the equations of motion for our class of anz"atze is that $[X_J, [X_I, X_J]]$ should belong to the same set as the $X_I$. The case of $\mathbb{CP}^2$, squashed or otherwise, given in section 3 can clearly be generalized to $\mathbb{CP}^3$, $\mathbb{CP}^4$ and other coset spaces. For the $\mathbb{CP}^n$ case, the exponent $a$ in (5) is given by $a = \frac{1}{n}$ and the potential energy goes like $s^{n-2}Rr^4$. Thus it is only for $n = 2$ that the potential energy becomes independent of $s$.

The required condition on the double commutators is satisfied if the $X_I$ span the entire Lie algebra of $G$ as well. The equations of motion for all these cases are generically of the form

$$\left(\frac{N}{R}\right)^{2a} \ddot{f} + c_2(\text{adj})f^3 = 0$$

(29)

where $c_2(\text{adj})$ represents the quadratic Casimir of $G$ in the adjoint representation and $r = fR^{1-a}$. Since there are only nine $X_I$, if $G$ is not a product group, its dimension for this type of solutions cannot exceed 9.

First consider the case that the generators are parametrized in terms of $\mathbb{CP}^n$ coordinates, as in Eqs.(11), eventhough the $X_i$ span the whole Lie algebra of $G$. * The solutions in this case amount to different immersions of the $\mathbb{CP}^n$ solutions in $\mathbb{R}^9$. The case of $G = SU(2)$ reproduces the spherical membrane. In this case $a = 1$, and, as noted before, both kinetic and potential energies have well-defined limits as $N, R \to \infty$ with their ratio fixed.

* For all the $\mathbb{CP}^n$ cases the coset embedding will produce a squashed manifold of the type we discussed in the previous section for the case of $n = 2$. 

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The semiclassical form of the solution in this case is

\[ Z = \frac{X_1 + iX_2}{\sqrt{2}} = r(t) \frac{(s + 2)}{(s + 1)} \frac{\bar{\xi}}{(1 + \xi \cdot \bar{\xi})} \approx r(t) \frac{\bar{\xi}}{(1 + \xi \cdot \bar{\xi})} \]

\[ \zeta = X_3 = \frac{1}{2}r(t) \frac{(s + 2)}{(s + 1)} \frac{1 - \bar{\xi} \cdot \xi}{(1 + \xi \cdot \bar{\xi})} \approx \frac{1}{2}r(t) \frac{1 - \bar{\xi} \cdot \xi}{(1 + \xi \cdot \bar{\xi})} \]

Clearly we have a two-sphere defined by

\[ ZZ + \zeta^2 \approx \frac{1}{4}r(t)^2 \]  

(31)

The radius of the sphere remains finite as \( s \to \infty \). Even though the ansatz has the full \( SU(2) \)-symmetry, there is a further algebraic constraint, viz., Eq. (31), and this reduces the space of free parameters to \( SU(2)/U(1) \).

We have already considered the case of \( G = SU(3) \), which leads to the ansatz (15) obeying the algebraic conditions (16). Also, as mentioned in section 2, the case of \( S^4 \)-geometry involves the coset \( O(6)/O(5) \), with a further algebraic condition which reduces the dimension to four.

There are many other interesting cases which can be considered along these lines; for example, for \( SU(2) \times SU(2) \), we can set six of the \( X_i \)'s proportional to the generators and the energies depend only on \( N/R \) as \( N, R \to \infty \). This can be embedded in \( U(N) \) in a block-diagonal way by choosing the representation \( (N_1, 1) + (1, N_2) \) with \( N = N_1 + N_2 \). Presumably this can give two copies of the two-brane in some involved geometrical arrangement in \( R^9 \).

The world volume geometry, it is clear for the above solutions, is tied to the representation (11) of the generators. In order to have different world geometries, we need to consider different parametrizations of the group generators, involving more world volume coordinates, for example. Within our ansatz (5), we do not expect more possibilities. For example, for \( SU(2) \), consider the parametrization of the generators

\[ t_i = \epsilon_{ijk} y_j \frac{\partial}{\partial y_k} \]  

(32)

which would naively suggest a three-dimensional world geometry. However, it is consistent, within this parametrization, to set \( y_i y_i = 1 \), so that effectively, this reverts to the previous
case, viz., \( \mathbb{CP}^1 \) or \( SU(2)/U(1) \). Likewise, for \( SU(3) \), we can consider a parametrization in terms of eight real \( y_i \)'s. The quadratic and cubic invariants of \( SU(3) \) allow a reduction to six world coordinates, with a local geometry of the form \( \frac{SU(3)}{U(1) \times U(1)} \); in other words, we can take \( H = U(1) \times U(1) \). It may be possible to find such a solution, but it would be necessary to go beyond our ansatz (5).

6. Conclusion

In this paper we constructed solutions to the matrix theory equations of motion which, in the large \( N \) limit, approach a continuum manifold of the form \( G/H \). We gave simple group theoretical description of the known spherical solutions and showed how other solutions of similar type can be obtained. We then examined a particular case, namely \( G = SU(3) \), in some detail and showed how in the large \( N \) limit we can endow the solution with complex local coordinates and complex functions which can be expressed in terms of these coordinates. We gave an explicit map from the enveloping algebra of \( SU(3) \) to the algebra of functions on \( \mathbb{CP}^2 \). Under this correspondence the Lie bracket is mapped to the Poisson bracket on the space of functions. This solution is not supersymmetric, but is part of a set of degenerate configurations which are related by supersymmetry. Further, in the large \( N \) and finite \( R \) limit in which the solution becomes static, we argued that there is some residual supersymmetry in small locally flat neighborhoods, where we used the matrix theory version of local flatness along the lines of reference [6]. However, there is no global extension of this result and hence the meaning or significance of this symmetry is not clear. Finally, we also showed how other solutions of similar variety or even a slightly more general type, in which the homogenous space \( G/H \) is not necessarily symmetric, can be generated. The equations of motion of the M(atrix) theory seem to admit many solutions of all dimensionalities between one and nine. It is still, however, an open question to find solutions to these equations which have a better resemblance to the BPS-states of M-theory.

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