A new look at the integral methods for solving heat and mass transfer problems

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Abstract. A new concept of construction of integral relations for approximate solving problems on heat and mass transfer is proposed. This concept is based on the introduction of local functions for the heat flow or the temperature that are determined directly from the differential equation of heat conduction with boundary conditions at the temperature disturbance front. This made it possible to obtain a number of new efficient integral relations, mainly an improved integral relation for the temperature momentum and integrals of the square-law heat flow and the square-law temperature function. New schemes for optimization of the exponent \( n \) in the parabolic description of the temperature field with the use of the error norms \( E_1 \) and \( L_1^E \) introduced for the first time are proposed. In comparison with the Langford norm \( E_1 \) (the scheme of T. Myers), the effectiveness of determining optimum parabolic solutions has been substantially increased. On the basis of the integral relations proposed in combination with the new schemes for minimizing an error, optimum solutions of simple parabolic form have been obtained with an error of 1.23\% and an integral error \( E_i = 0.0301 \). The solutions obtained are much better in approximation representation and error than the solutions obtained by known integral methods.

1. Introduction
In the analytical heat-conduction theory, methods using the notion of the temperature-disturbance front \( \delta(t) \) are used widely [1–5]. Analysis of known works shows that a fundamental problem is the adequate and full description of a nonstationary temperature field with the use of a simple parabolic profile with a known exponent \( n \). It should be noted that almost all methods using the temperature-disturbance front notion and a parabolic temperature profile are problematic and cannot be used in many important cases. Therefore, in the present work, some new conceptions of construction of determining integral relations on the basis of principles differing radically from the classical ones are proposed for the purpose of further development of the theory of approximate description of potential (temperature) fields with the use of a simple parabolic representation. The definition of the temperature function is associated, as a rule, with the prescription of one of the following boundary conditions on the surface of a body: the Dirichlet condition, the Neuman condition, or the heat exchange by the Newton law. Of special importance is the Dirichlet condition that makes it possible to realize the possibilities of any integral method. In this connection, when new approaches to the obtaining of approximate solutions are considered, of crucial importance is the test problem in which the surface of a semi-bounded space has a constant temperature.
2. Mathematical formulation of the problem

The mathematical formulation of the problem is as follows:

\[
\frac{\partial \bar{T}}{\partial \bar{t}} = \kappa \frac{\partial^2 \bar{T}}{\partial \bar{x}^2}, \quad 0 < \bar{x} < \infty, \quad \bar{t} > 0,
\]

\[
\bar{T}(0, \bar{t}) = T_0, \quad \bar{T}(\infty, \bar{t}) = 0.
\]

Here, \( \bar{T} \) is the temperature of a body, \( \bar{x} \) is the coordinate, \( \bar{t} \) is the time, \( \kappa \) is the thermal diffusivity of the body, and \( T_0 \) is its initial temperature. We introduce the dimensionless parameters

\[
T = (\bar{T} - T_0) / (T_{\text{ref}} - T_0), \quad x = \bar{x} / \tau, \quad t = \bar{t} / \tau, \quad \tau = L^2 / \kappa, \quad \text{and} \quad h = (T_0 - T_{\text{ref}}) / (T_{\text{ref}} - T_0),
\]

where \( \tau \) is the time scale and \( L \) is the characteristic length. Putting \( T_{\text{ref}} = T_s \), instead of (1)–(2) we obtain the problem formulation

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,
\]

\[
T(0, t) = 1, \quad T(x, 0) = 0, \quad T(\infty, t) = 0.
\]

Its exact solution is

\[
T = \text{erfc}(x / 2 \sqrt{t}).
\]

To characterize approximate solutions, we introduce the parameters

\[
\varepsilon = |\Delta T| / T(0, t)100\%, \quad \text{and} \quad E_i = \int_0^h |T - T_0| \, dx,
\]

where \( E_i \) is the error, \( \Delta T \) is the local deviation of an approximate solution from the exact one, and \( \varepsilon \) is the relative error.

3. Contemporary integral methods based on the temperature-disturbance front notion

We consider contemporary methods based on integration of Eq. (3) over the region \( \Omega = [0, \delta(t)] \) with the conditions \( T(\delta, t) = 0 \) and \( \partial T(\delta, t) / \partial x = 0 \) at the temperature-disturbance front.

3.1. Heat-balance integral method (HBIM)

A solution to the equations (3)–(4) is represented as a rule in the form of a parabola [1, 2]:

\[
T = \left(1 - \frac{x}{\delta}\right)^n.
\]

Integration of (3) over the region \( \Omega = [0, \delta] \) gives a classical heat-balance integral [1]:

\[
\frac{d}{dt} \int_0^\delta T \, dx = -\frac{\partial T(0, t)}{\partial x}.
\]

Substituting profile (5) into (6) we obtain the ordinary differential equation \( \delta \delta' = n(n+1) \) having the solution \( \delta = \sqrt{2n(n+1)} t \). Analysis of the temperature profiles (5) at \( n = 2 \) and \( n = 3 \) presented in [3] points to their low accuracy. In the case of the most satisfactory solution \( (n = 2) \), we have \( E_i = 0.0576 \) and \( \varepsilon = 3.29\% \).

3.2. Refined integral method (RIM)

The RIM [4, 5] is oriented to the surface temperature, to which corresponds the \( x \) momentum

\[
\frac{d}{dt} \int_0^\delta T \, x \, dx = T(0, t) = 1.
\]
Substitution of profile (5) into (7) gives the differential equation \( 2\delta \delta' = 2 + 3n + n^2 \), whose solution is \( \delta = \sqrt{(n+1)(n+2)}t \) [4]. The temperature profiles constructed at \( n = 2 \) by the RIM and the HBIM are identical, and the solution obtained at \( n = 3 \) \( (E_4 = 0.0359, \varepsilon = 2.30\%) \) is better than the solution obtained at \( n = 2 \) by the HBIM \( (E_4 = 0.0576, \varepsilon = 3.29\%) \).

### 3.3. Combined integral method (CIM)

The combination of the HBIM with the RIM gives the CIM [5]. The common use of the HBIM and RIM made it possible to equate the obtained temperature fronts: \( (2n+1)(n+1)(n+2)^{1/2} = [(n+1)(n+2)t]^{1/2} \). Herefrom it follows that \( n = 2 \) [5].

### 3.4. T. G. Myers scheme

To optimize the exponent \( n \), the Langford norm \( E_L \) [2] and its minimization are used [4]:

\[
\psi = \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}, \quad E_L = \int_0^{\frac{\delta}{\partial t}} \psi^2 \, dx = \int_0^{\frac{\delta}{\partial t}} \left( \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right)^2 \, dx > 0 \rightarrow \text{min}. \quad (8)
\]

Substitution of (5) into (8) at \( \delta = \sqrt{2(n+1)t} \) (HBIM) gives \( n = 2.2335 \) [4]. Using the relation \( \delta = \sqrt{(n+1)(n+2)}t \) (RIM), from (8) we find \( n = 2.2187 \) [4]. It follows from the graphs for the deviation \( |\Delta T| \), presented in [4], that the best approximate solution is obtained using the RIM \( (E_4 = 0.0436, \varepsilon = 2.48\%) \). However, as seen, Myers scheme does not improve the solution obtained by the RIM \( (n = 3) \) but makes it worse.

### 4. New schemes of minimizing an error

We will start from the solution obtained by the RIM. Instead of the norm \( E_L \) we write the condition

\[
E_1^L = \int_0^{\frac{\delta}{\partial t}} \left( \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right) \, dx = 0 \rightarrow \text{min}. \quad (9)
\]

From (9) we obtain the expression \( (t = 1) \)

\[
E_1^L = \frac{n}{2} \int_0^{\frac{\delta}{\partial t}} \left( \frac{2 - 2n + x\sqrt{(n+1)(n+2) - x^2}}{\sqrt{(n+1)(n+2) - x^2}} \right)^n \left( \frac{1 - x}{\sqrt{(n+1)(n+2)}} \right) \, dx > 0 \rightarrow \text{min}. \quad (10)
\]

Numerical integration of (10) allows one to find the minimum of \( E_1^L \) at \( n = 2.4841 \). In this case, the approximation error decreases, as compared to \( n = 2.2187 \) [4]: \( E_4 = 0.0349, \varepsilon = 1.81\% \).

On the other hand, the knowledge of the exact solution allows one to write the condition

\[
E_i = \int_0^{\frac{\delta}{\partial t}} \left| T - T_0 \right| \, dx \rightarrow \text{min}. \quad (11)
\]

For the RIM, condition (11) takes the form \( (t = 1) \)

\[
E_i = \int_0^{\frac{\delta}{\partial t}} \left| \left( 1 - \left[ (n+1)(n+2) \right]^{1/2} x \right)^n - \text{erfc}(x/2) \right| \, dx \rightarrow \text{min}. \quad (12)
\]

The dependence \( E_i(n) \) has a minimum \( (E_i = 0.0332) \) at \( n = 2.670 \). In this case, the approximation error is much smaller, as compared to the known solution [4] obtained based on minimization of the norm \( E_i \): \( \varepsilon = 1.50\%, E_i = 0.0332 \).
5. Integral relation for the $T$ momentum (TMIR) and its new representations

In [1], the possibility of using the $T$ momentum equivalent to the integral

$$
\int_{\delta}^{b} \left( \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right) T \, dx = 0
$$

(13)

was demonstrated. Substitution of (5) into (13) gives the differential equation

$$
2n(2n^2 - n - 1)\delta' + (2n - 1)\delta^2 \delta'' = 0,
$$

whose solution is

$$
\delta = 2\sqrt{n(n-1)(2n+1)/(2n-1)}.
$$

5.1. T. F. Zien method

Relation (13) in combination with the neat-balance integral (6) was used by T.F. Zien [6] when constructing solutions with the use of the exponent $T = \exp(-x/\delta)$. Using (5), we equate the expression for $\delta(t)$, corresponding to the TMIR and HBIM:

$$
2\left[n(n-1)(2n+1)(2n-1)^{-1}t\right]^{1/2} = \left[2n(n+1)t\right]^{1/2}.
$$

It follows herefrom that $n=1.7808$. The value $n=1.7808$ is very close to the value $n$ calculated by the scheme of W. Braga and M. Mantelli ($n=1.75$).

5.2. New representation of the TMIR

Expanding (13) and using the heat-balance integral (6), we obtain the relation

$$
\frac{d}{dt} \int_{\delta}^{b} \left( 1 - \frac{T}{2} \right) dx = \int_{\delta}^{b} \left( \frac{\partial T}{\partial x} \right)^2 dx.
$$

(14)

Substitution of (5) into (14) gives the equation

$$
2n(1 + 3n + 2n^2)(1 + 6n^2)\delta' = 0,
$$

whose solution is

$$
\delta = 2n\left[\frac{(n+1)(2n+1)}{(2n-1)(3n+1)}t\right]^{1/2}.
$$

(15)

Common use of (15) and (7) (RIM) allows equating the expressions for the front $\delta(t)$:

$$
2n\left[(n+1)(2n+1)(2n-1)^{-1}(3n+1)^{-1}t\right]^{1/2} = [(n+1)(n+2)t]^{1/2}.
$$

From here it follows that $n=2.8508$. Profile (24) at $n=2.8508$ has much better characteristics ($E_1 = 0.0340$, $\varepsilon = 1.94\%$), as compared to the profiles obtained using the CIM ($n=2$) and RIM ($n=2.219$).

Now we will use the norm $E^1_1$ and its minimization by the scheme

$$
E^1_1 = \int_{\delta}^{b} \left( 1 - \frac{6n^2 - n - 1}{\sqrt{1 + 3n + 2n^2}} \frac{x}{2n} \right)^n \left[ 1 - \frac{x^2}{2} + n \left( \frac{1 + 3n + 2n^2}{6n^2 - n - 1} x - 1 \right) \right] \left[ x - 2n \sqrt{\frac{1 + 3n + 2n^2}{6n^2 - n - 1}} x \right] \, dx.
$$

(16)

The dependence $E^1_1(n)$ has a minimum ($E^1_1 = 0.1954$) at $n=2.4155$. The graph for $|\Delta T|$ (Figure 1) and the accuracy parameters ($E_1 = 0.0310$, $\varepsilon = 1.37\%$) point to the higher efficiency of the norm $E^1_1$ compared to $E_L$ ($E_1 = 0.0343$, $\varepsilon = 1.65\%$).

5.3. Refined TMIR (RTMIR)

Integration of the differential equation (3) over the region $\Omega = [x, \delta]$ gives

$$
\int_{\delta}^{b} \left( \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right) dx = 0 \Rightarrow \frac{d}{dt} \left[ T \left|_{\delta}^{b} \right. \right] = 0 \Rightarrow \frac{\partial T}{\partial x} = \frac{d}{dt} \left[ T \left|_{\delta}^{b} \right. \right]
$$

(17)
Eliminating $\partial T/ \partial x$ from (14) and substituting (17), we obtain the integral relation

$$E_1 \int_0^\delta \left( \frac{\partial T}{\partial x} \right)^2 dx = \int_0^\delta \left( \frac{d}{dt} \int_0^\delta T \, dx \right)^2 dx.$$  \hspace{1cm} (18)

Minimizing the error by scheme (11) for the norm $E_1$ we obtain $n = 2.4989$. The solution obtained has the accuracy properties $\epsilon = 1.33\%$ and $E_1 = 0.03007$. This points to the fact that the RTMIR is a more “powerful” relation compared to the TMIR ($\epsilon = 1.32\%$, $E_1 = 0.0308$).

Use of scheme (9) based on $E_1^L$ gives $n = 2.409$, to which corresponds the solution with $\epsilon = 1.24\%$ and $E_1 = 0.0302$. As seen, for the RTMIR, the norm $E_1^L$ is more “powerful” compared to $E_1$. Hence, when more “adequate” integral relations are used, the error norms oriented to the differential equation of heat conduction are more efficient.

Common use of the RTMIR and the TMIR with equating their fronts $\delta(t)$ gives the equation $\left[(n + 1)(2n + 3)(3n + 1)(5n + 3)^r \right]^{1/2} = 2n \left[(n + 1)(2n + 1)(2n-1)^r (3n + 1)^r \right]^{1/2}$, from which it follows that $n = 2.3479 \pm 0.6142$ with the modulus $|n| = 2.427$. The solution obtained defines the temperature profile more exactly compared to the above-considered approximations ($E_1 = 0.03017$, $\epsilon = 1.23\%$).

6. Integral of the square-law heat flow (SLHFI)

The left sides of relations (14) and (18) are identical, which allows one to equate their right sides:

$$\int_0^\delta \left( \frac{\partial T}{\partial x} \right)^2 dx = \int_0^\delta \left( \frac{d}{dt} \int_0^\delta T \, dx \right)^2 dx \Rightarrow \int_0^\delta \left( \frac{\partial T}{\partial x} \right)^2 dx = \int_0^\delta \left( \frac{d}{dt} \int_0^\delta T \, dx \right)^2 dx.$$ \hspace{1cm} (19)

Relation (19) represents the integral of the square-law heat flow ($Q = \partial T/ \partial x$), which is equivalent to the expression $\int_0^\delta Q^2 dx = \int_0^\delta \hat{Q}^2 dx$, where $\hat{Q} = \frac{d}{dt} \int_0^\delta T \, dx$ is an equivalent of the heat flow obtained from the differential heat-conduction equation (3).

From (5) and (19) we obtain the equation $n(n + 1) \left(4n^2 + 8n + 3 \right)^{1/2} = \delta \delta' \left(10n^2 + n - 3 \right)^{1/2}$, whose solution is

$$\delta = \left[ \frac{(2n + 1)(2n + 3)}{(2n - 1)(5n + 3)} \right]^{1/4} \left(2n(n + 1)t\right)^{1/2}. \hspace{1cm} (20)$$

Use of the Myers scheme (8) with the norm $E_1$ gives $n = 2.1957$ with the accuracy parameters $\epsilon = 1.61\%$ and $E_1 = 0.0337$.

Using scheme (9) with the norm $E_1^L$ for optimization, we arrive at the condition

$$E_1^L = \int_0^\delta \frac{\left( \frac{n}{\delta} \delta' \left(1 + \frac{x}{\delta} \right) - \frac{n - 1}{\delta} \left(1 - \frac{x}{\delta} \right)^{n-2} \right)^2}{\left| \delta' \left(1 + \frac{x}{\delta} \right) - \frac{n - 1}{\delta} \left(1 - \frac{x}{\delta} \right)^{n-2} \right|} \rightarrow \min. \hspace{1cm} (21)$$

Substitution of expression (20) for $\delta(t)$ and its derivative $\delta'$ into (21) gives $n = 2.4120$ with a good solution: $\epsilon = 1.31\%$, $E_1 = 0.0306$.

Using scheme (11) with the norm $E_1$, we obtain $n = 2.4919$. The accuracy parameters $\epsilon = 1.28\%$ and $E_1 = 0.0304$ are close to those for the solution by scheme (9) with the norm $E_1^L$. This points to the fact that the solution obtained is almost optimum.
7. Integral of the square-law temperature function (SLTFI)

By analogy with the SLHFI we write the integral relation
\[
\hat{T}^2 - T^2 = \frac{d}{dt} \int_0^\infty \hat{T}^2 \, dt.
\]
Then we arrive at the integral relation
\[
\int_0^\infty \hat{T}^2 \, dt = \int_0^\infty \left( \frac{d}{dt} \int_0^\infty \hat{T}^2 \, dt \right)^2 \, dt.
\]  
(22)

From (22) and (5) we obtain the equation
\[
\delta = \frac{(2n+3)(2n+5)}{(2n+1)(13n+20)} \left(2(n+1)(n+2)\right)^{\frac{1}{12}}.
\]  
(23)

Since the SLHFI is similar in essence to the RIM (only the temperature function is considered), we can obtain a fairly good approximate solution on the basis of common use of the corresponding integral relations. Equating the right sides of the expressions for the front \( \delta(t) \), we obtain the equation
\[
\left[ \frac{(2n+3)(2n+5)}{(2n+1)(13n+20)} \right]^{\frac{1}{12}} \left[2(n+1)(n+2)\right]^{\frac{1}{12}} = \left[ (n+1)(n+2) \right]^{\frac{1}{12}}.
\]  
(24)

From here it follows that \( n = 2.6242 \). Analysis of this solution (\( \varepsilon = 1.38\%, \ E_1 = 0.0333 \)) and other solutions obtained using the RIM shows that solution (24) is the best solution for the RIM (Figure 2).

![Figure 1](image-url)  
**Figure 1.** Deviation \(|\Delta T|\) for the RTMIR and the optimization schemes: HBIM (dash-dot line), RIM (dashed line); \( E_1 \) (dotted line); \( E_1^L \) (thin line); \( E_1 \) (heavy line).

![Figure 2](image-url)  
**Figure 2.** Temperature profiles constructed for the half-space on the basis of the exact solution (solid lines) and the approximate solution by the \{SLHFI–RIM\} according to (24) at \( n = 2.6242 \) (dashed line).

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