A Set-Theoretic Decision Procedure for Quantifier-Free, Decidable Languages Extended with Restricted Quantifiers

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1 Introduction

Restricted quantifiers (RQ) are formulas of the following forms:

\begin{align*}
\forall x \in A & : \phi(x) \\
\exists x \in A & : \phi(x)
\end{align*}
\begin{align*}
(1) & \\
(2) &
\end{align*}

where \( A \) is a set called \textit{quantification domain}. The first form is called \textit{restricted universal quantifier} (RUQ), while the second is called \textit{restricted existential quantifier} (REQ). The semantics of such formulas is, respectively:

\begin{align*}
\forall x(x \in A \Rightarrow \phi(x)) & \\
\exists x(x \in A \land \phi(x))
\end{align*}
\begin{align*}
(3) & \\
(4) &
\end{align*}

RQ are present in formal notations such as B [47], TLA+ [36] and Z [48] making it important to be able to automatically reason about RQ. In fact, RQ allow to...
express important program or system properties. For example, one may need to express that some property, \( \phi \), holds for all the users (\( U \)) of a system. Then, it can be expressed by means of a RUQ:

\[
I \triangleq \forall u \in U : \phi(u)
\]

Later, one might need to prove that \( I \) is a state invariant of that system by discharging proofs of the form:

\[
I \land T \Rightarrow I'
\]  

(5)

where \( T \) is a state transition and \( I' \) is the result of substituting every state variable \( v \) in \( I \) by \( v' \)—i.e., the next-state variable. In this scenario it would be important if many (or all) of those proofs can be performed automatically.

In a recent article [18] we have presented a decision procedure for a language based on extensional and intensional sets called \( \mathcal{L}_{\text{RTS}}(X) \), where \( X \) is a first-order, decidable theory. \( \mathcal{L}_{\text{RTS}} \) can express RQ where the inner formula does not contain other RQ. Then, \( \mathcal{L}_{\text{RTS}} \) can automatically discharge a proof such as (5). However, in general, \( \mathcal{L}_{\text{RTS}} \) does not allow for nested RQ. For example, if \( \psi \) is a formula depending on a user and a process, the following:

\[
I_2 \triangleq \forall u \in U : (\forall p \in P : \psi(u, p))
\]  

(6)

where \( P \) is the set of processes of the system, is a \( \mathcal{L}_{\text{RTS}} \) formula only if \( U \) is not part of \( \psi \). Consequently, the decision procedure defined for \( \mathcal{L}_{\text{RTS}} \) is unable to automatically reason about all formulas such as (5) where \( I \) is substituted by \( I_2 \). Therefore, finding a decision procedure for formulas of the form (5) but involving predicates such as \( I_2 \) would be a valuable contribution to the formal verification community.

In this paper we depart from \( \mathcal{L}_{\text{RTS}}(X) \) to define a new language, \( \mathcal{L}_{\text{RQ}}(X) \), admitting finitely nested RQ at any level, when \( L_X \) is a quantifier-free, first-order, decidable language\(^1\). Then, departing from the decision procedure defined for \( \mathcal{L}_{\text{RTS}} \) we define a decision procedure for \( \mathcal{L}_{\text{RQ}} \). In particular, we provide a precise condition defining the class of decidable formulas where RUQ and REQ can be arbitrarily nested. The implementation of these results as part of the \( \{\text{log}\} \) (‘setlog’) tool [46] is also briefly discussed. More space is committed to show that the implementation works in practice by providing several examples of non trivial properties and verification conditions, drawn from real-world case studies, that \( \{\text{log}\} \) is able to deal with.

The paper is structured as follows. Section 2 introduces \( \mathcal{L}_{\text{RQ}} \) by first giving an informal presentation (2.1) and then its formal syntax (2.2) and semantics (2.3). The solver for \( \mathcal{L}_{\text{RQ}} \) is presented in Section 3. In Section 4 soundness and completeness (4.1) and termination (4.2) of the solver are proved. Some extensions to \( \mathcal{L}_{\text{RQ}} \), helping to avoid the introduction of existential variables, are introduced in Section 5. The implementation of \( \mathcal{L}_{\text{RQ}} \) and its solver as part of \( \{\text{log}\} \) is shown in Section 6. In that section we also comment on three case studies carried out with \( \{\text{log}\} \) involving RQ. Our results are discussed and compared with similar works in Section 7. Section 8 gives our conclusions.

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\(^1\) Although in \( \mathcal{L}_{\text{RTS}}(X) \), \( L_X \) can be a quantified language it make little sense to extend such a language with RQ. Hence, in this paper we focus on quantifier-free languages that need to be extended to support at least a restricted form of quantification.
2 Formal Syntax and Semantics

This section describes the syntax and semantics of the set-theoretic language of Restricted Quantifiers, $\mathcal{L}_{RQ}$. In other words, $\mathcal{L}_{RQ}$ builds RQ from fundamental concepts drawn from set theory. A gentle, informal introduction is provided in Section 2.1, followed by the formal presentation of the language.

$\mathcal{L}_{RQ}$ is a first-order predicate language with terms of sort set and terms designating ur-elements\(^2\). The latter are provided by an external first-order theory $\mathcal{X}$ (i.e., $\mathcal{L}_{RQ}$ is parametric with respect to $\mathcal{X}$). $\mathcal{X}$ must include: a class $\Phi_\mathcal{X}$ of admissible $\mathcal{X}$-formulas based on a set of function symbols $F_\mathcal{X}$ and a set of predicate symbols $\Pi_\mathcal{X}$ (providing at least equality); an interpretation structure $I_\mathcal{X}$ with domain $D_\mathcal{X}$ and interpretation function $(\cdot)^{I_\mathcal{X}}$; and a decision procedure $\text{SAT}_\mathcal{X}$ for $\mathcal{X}$-formulas. $\mathcal{L}_{RQ}(\mathcal{X})$ denotes the instance of $\mathcal{L}_{RQ}$ based on theory $\mathcal{X}$.

$\mathcal{L}_{RQ}$ provides special set constructors, and a handful of basic predicate symbols endowed with a pre-designated set-theoretic meaning. Set constructors are used to construct both restricted intensional sets (RIS) and extensional sets. Set elements are the objects provided by $\mathcal{X}$, which are manipulated through the primitive operators that $\mathcal{X}$ offers. Hence, $\mathcal{L}_{RQ}$ sets represent untyped unbounded finite hybrid sets, i.e., unbounded finite sets whose elements are of arbitrary sorts. $\mathcal{L}_{RQ}$ formulas are built in the usual way by using conjunction and disjunction of atomic formulas.

2.1 $\mathcal{L}_{RQ}$ in a Nutshell

$\mathcal{L}_{RQ}$ provides three kinds of set terms: $\emptyset$, the empty set; $\{x \cup A\}$, called extensional set whose interpretation is $\{x\} \cup A$; and $\{c : D \mid \phi\}$, called restricted intensional set (RIS) whose interpretation is $\{c : c \in D \land \phi(c)\}$, where $D$ is called domain and $\phi$ is called filter. At the same time, $\mathcal{L}_{RQ}$ is a parametric language w.r.t. the language of some theory $\mathcal{X}$. The elements of sets are $\mathcal{X}$ elements and the filters of RIS can be either $\mathcal{X}$ formulas or a very specific kind of $\mathcal{L}_{RQ}$ formulas. $\mathcal{X}$ is expected to be a decidable theory providing at least equality. For example, if $\mathcal{X}$ is the theory of linear integer arithmetic (LIA) then $\mathcal{L}_{RQ}(\mathcal{X})$ will allow to reason about formulas combining RQ over integer formulas.

In $\mathcal{L}_{RQ}$ formulas are conjunctions and disjunctions of $\mathcal{L}_{RQ}$ and $\mathcal{X}$ constraints. In turn, $\mathcal{L}_{RQ}$ provides the set equality ($=$), membership ($\in$) and subset ($\subseteq$) relations, as constraints.

Example 1 If $\mathcal{X}$ is the theory of LIA then the following is a $\mathcal{L}_{RQ}(\mathcal{X})$ formula:

$$\text{min} \in \{y \cup S\} \land \{y \cup S\} \subseteq \{x : \{y \cup S\} \mid \text{min} \leq x\}$$

where $\text{min} \leq x$ is a $\mathcal{X}$ constraint. \(\square\)

$\mathcal{L}_{RQ}$ allows the definition of RUQ in set-theoretic terms by exploiting the following identity:

$$\forall x \in A : \phi(x) \Leftrightarrow A \subseteq \{x : x \in A \land \phi(x)\}$$

\(^2\) Ur-elements (also known as atoms or individuals) are objects which contain no elements but are distinct from the empty set.
In this way, in $\mathcal{L}_{RQ}$ we can define a constraint for RUQ as follows:

$$\text{foreach}(x \in A, \phi) \equiv A \subseteq \{x : A \mid \phi\}$$

(8)

Then, the formula of Example 1 can be written more compactly as follows:

$$\min \in \{y \cup S\} \land \text{foreach}(x \in \{y \cup S\}, \min \leq x)$$

Likewise, REQ can also be defined as constraints:

$$\exists (x \in A, \phi) \equiv n \in A \land \phi(n)$$

where $n$ is a new variable.

Furthermore, in $\mathcal{L}_{RQ}$, the filter of a RIS can be a conjunction of $X$, foreach and exists constraints. This is an important difference w.r.t. $\mathcal{L}_{RIS}$ [18] because there, RIS filters can only be $X$ formulas. The possibility of including foreach and exists constraints in RIS filters allows for the definition of nested RQ (what is not possible in $\mathcal{L}_{RIS}$). For example:

$$\forall x \in X : (\forall y \in Y : \phi(x, y))) \Rightarrow X \subseteq \{x : x \in X \land (\forall y \in Y : \phi(x, y))\}$$

$$\Leftrightarrow X \subseteq \{x : x \in X \land (Y \subseteq \{y : y \in Y : \phi(x, y))\})\}$$

The latter being equivalent to the following $\mathcal{L}_{RQ}$ formula:

$$\text{foreach}(x \in X, \text{foreach}(y \in Y, \phi(x, y)))$$

which can be further simplified by introducing some syntactic sugar:

$$\text{foreach}([x \in X, y \in Y], \phi(x, y))$$

In the next two subsections a formal presentation of $\mathcal{L}_{RQ}$ is made and in later sections its decidability is analyzed.

2.2 Syntax

The $\mathcal{L}_{RQ}$ syntax is defined primarily by giving the signature upon which terms and formulas of the language are built.

**Definition 1 (Signature)** The signature $\Sigma_{RQ}$ of $\mathcal{L}_{RQ}$ is a triple $\langle F, \Pi, V \rangle$ where:

- $F$ is the set of function symbols, partitioned as $F = F_S \cup F_X$, where $F_S$ contains $\emptyset$, $\{\cup\}$ and $\{\cdot : \cdot | \cdot\}$, while $F_X$ contains the function symbols provided by the theory $X$ (at least, a constant and the binary function symbol $(\cdot, \cdot)$).
- $\Pi$ is the set of primitive predicate symbols, partitioned as $\Pi = \Pi_S \cup \Pi_T \cup \Pi_X$ where $\Pi_S \equiv \{=, \in, \subseteq\}$ and $\Pi_T \equiv \{\text{set}, \text{is}\}$, while $\Pi_X$ contains the predicate symbols provided by the theory $X$ (at least $=_X$).
- $V$ is a denumerable set of variables, partitioned as $V = V_S \cup V_X$.  

$\square$
Intuitively, $\emptyset$, $\{\cdot \cup \cdot\}$ and $\{\cdot \mid \cdot\}$ are interpreted as outlined at the beginning of Section 2.1. $\equiv_X$ is interpreted as the identity in $D_X$, while $(\cdot, \cdot)$ will be used to represent ordered pairs.

Sorts of function and predicate symbols are specified as follows: if $f$ (resp., $\pi$) is a function (resp., a predicate) symbol of arity $n$, then its sort is an $n+1$-tuple $(s_1, \ldots, s_{n+1})$ (resp., an $n$-tuple $(s_1, \ldots, s_n)$) of non-empty subsets of the set $\{\text{Set}, X\}$ of sorts. This notion is denoted by $f : (s_1, \ldots, s_{n+1})$ (resp., by $\pi : (s_1, \ldots, s_n)$).

Specifically, the sorts of the elements of $F$ and $V$ are the following.

Definition 2 (Sorts of function symbols and variables) The sorts of the symbols in $F$ are as follows:

- $\emptyset : \langle \{\text{Set}\} \rangle$
- $\{\cdot \cup \cdot\} : \langle \{X\}, \{\text{Set}\}, \{\text{Set}\} \rangle$
- $\{\cdot \mid \cdot\} : \langle \{X\}, \{\text{Set}\}, \{\Phi_{RQ}\}, \{\text{Set}\} \rangle$
- $f : \langle (X)_{n}, \ldots, (X), \{X\} \rangle$, if $f \in F_X$ is of arity $n \geq 0$.

where $\Phi_{RQ}$ represents the set of $RQ$-formulas defined in Definition 6. The sorts of variables are as follows:

- $v : \langle \{\text{Set}\} \rangle$, if $v \in V_S$
- $v : \langle \{X\} \rangle$, if $v \in V_X$

Definition 3 (Sorts of predicate symbols) The sorts of the predicate symbols in $\Pi$ are as follows:

- $\equiv_S : \langle \{\text{Set}\}, \{\text{Set}\} \rangle$
- $\equiv_X : \langle \{X\}, \{X\} \rangle$
- $\in : \langle \{X\}, \{\text{Set}\} \rangle$
- $\subseteq : \langle \{\text{Set}\}, \{\text{Set}\} \rangle$
- $\text{set, is}_X : \langle \{\text{Set}, X\} \rangle$

Whenever it is clear from context we will write $=$ instead of $=_{S}$ or $=_{X}$.

Definition 4 (RQ-terms) Let $T^0_{RQ}$ be the set of terms generated by the following grammar:

- $T^0_{RQ} ::= \text{Elem} \mid \text{Set}$
- $\text{Elem} ::= T_X \mid V_X$
- $\text{Ctrl} ::= V_X \mid (\text{'}Ctrl \text{', 'Ctrl'})$
- $\text{Ext} ::= \emptyset \mid V_S \mid \{'\text{Elem'}\cup\text{'}\text{Ext}\'}$
- $\text{Ris} ::= \{'\text{Ctrl'} : \text{'Ext'} \mid \Phi_{RQ} \text{'\'}\}$
- $\text{Set} ::= \text{Ris} \mid \text{Ext}$

where $T_X$ represents the set of non-variable $X$-terms; $\Phi_{RQ}$ is the set of $RQ$-formulas defined in Definition 6; and variables occurring in a Ctrl-term must all be distinct from each other.

The set of $RQ$-terms, denoted by $T_{RQ}$, is the maximal subset of $T^0_{RQ}$ complying with the sorts as given in Definition 2.
If \( t \) is a term \( f(t_1, \ldots, t_n), f \in \mathcal{F}, n \geq 0 \), and \( \langle s_1, \ldots, s_{n+1} \rangle \) is the sort of \( f \), then we say that \( t \) is of sort \( \langle s_{n+1} \rangle \). The sort of any \( \mathcal{RQ} \)-term \( t \) is always \( \langle \{ \text{Set} \} \rangle \) or \( \langle \{ X \} \rangle \).

For the sake of simplicity, we simply say that \( t \) is of sort \( \text{Set} \) or \( X \), respectively. In particular, we say that a \( \mathcal{RQ} \)-term of sort \( \text{Set} \) is a set term, that set terms of the form \( \{ t_1 \sqcup t_2 \} \) are extensional set terms, and that terms of the form \( \{ t_1 : t_2 \mid \phi \} \) are RIS terms. The first argument of an extensional set term is called element part and the second is called set part. In turn, the first argument of a RIS term is called control term, the second is the domain and the third one is the filter.

As can be seen in Definition 4, control terms can be either variables or nested ordered pairs. The utility of the latter will be precisely motivated and discussed in Section 5. Note that the domain of a RIS term can be the empty set, a set variable \( X \), or \( \{ t \} \), while \( \exists \{ t \} \) is always \( \{ t \} \).

Hereafter, we will use the following notation for extensional set terms: \( \{ t_1, t_2, \ldots, t_n \mid t \}, n \geq 1 \), is a shorthand for \( \{ t_1 \sqcup \{ t_2 \sqcup \cdots \{ t_n \sqcup t \} \} \} \), while \( \{ t_1, t_2, \ldots, t_n \} \) is a shorthand for \( \{ t_1, t_2, \ldots, t_n \sqcup \emptyset \} \).

**Definition 5 (\( \mathcal{RQ} \)-constraints)** If \( \pi \in \Pi \) is a predicate symbol of sort \( \langle s_1, \ldots, s_n \rangle \), and for each \( i = 1, \ldots, n \), \( t_i \) is a \( \mathcal{RQ} \)-term of sort \( \langle s'_i \rangle \) with \( s'_i \subseteq s_i \), then:

1. If \( \pi \) is \( \subseteq \), then \( \pi(t_1, t_2) \) is a \( \mathcal{RQ} \)-constraint if \( t_2 \equiv \{ \text{Ctrl} : t_1 \mid \Phi_{\mathcal{RQ}} \} \), where \( \text{Ctrl} \) and \( \Phi_{\mathcal{RQ}} \) are as in Definition 4.
2. If \( \pi \) is \( \in \), then \( \pi(t_1, t_2) \) is a \( \mathcal{RQ} \)-constraint if \( t_2 \) is an \( \text{Ext} \) term as in Definition 4.
3. If \( \pi \) is \( =_S \), then \( \pi(t_1, t_2) \) is a \( \mathcal{RQ} \)-constraint if \( t_1 \) and \( t_2 \) are \( \text{Ext} \) terms as in Definition 4.
4. If \( \pi \) is any other element of \( \Pi \), then \( \pi(t_1, \ldots, t_n) \) a \( \mathcal{RQ} \)-constraint.

The set of \( \mathcal{RQ} \)-constraints is denoted by \( \mathcal{C}_{\mathcal{RQ}} \).

**The set of \( \mathcal{RQ} \)-constraints based on symbols in \( \Pi_S \) will be called set constraints.** Note that the conditions on \( \subseteq \)-constraints forces them to be RUQ as in (7).

Finally, we define the set of \( \mathcal{RQ} \)-formulas as follows.

**Definition 6 (\( \mathcal{RQ} \)-formulas)** The set of \( \mathcal{RQ} \)-formulas, denoted by \( \Phi_{\mathcal{RQ}} \), is given by the following grammar:

\[
\Phi_{\mathcal{RQ}} ::= \text{true} \mid \text{false} \mid \Phi_\mathcal{X} \mid \mathcal{C}_{\mathcal{RQ}} \mid \Phi_{\mathcal{RQ}} \land \Phi_{\mathcal{RQ}} \mid \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}}
\]

where \( \Phi_\mathcal{X} \) and \( \mathcal{C}_{\mathcal{RQ}} \) represent any element belonging to the class of \( \mathcal{X} \)-formulas and \( \mathcal{RQ} \)-constraints, respectively.

As can be seen, \( \mathcal{L}_{\mathcal{RQ}} \) is based solely on fundamental concepts of set theory.

**Remark 1 (Notation)** We will use the following naming conventions, unless stated differently: \( A, B, C, D \) stand for terms of sort \( \text{Set} \); \( a, b, c, d, x, y, z \) stand for terms of sort \( \mathcal{X} \); and \( t, u, v \) stand for terms of any of the two sorts. A symbol such as \( A \) states that \( A \in \mathcal{V} \). Finally, \( n, n_i \) stand for new variables of sort \( \mathcal{X} \); and \( N, N_i \) for new variables of sort \( \text{Set} \); no dot above them will be used.

**Remark 2 (\( \mathcal{L}_{\mathcal{RQ}} \) vs. \( \mathcal{L}_{\mathcal{RIS}} \)** As we have pointed out in Section 1, \( \mathcal{L}_{\mathcal{RQ}} \) departs from \( \mathcal{L}_{\mathcal{RIS}} \) [18]. \( \mathcal{L}_{\mathcal{RQ}} \) is a sublanguage of \( \mathcal{L}_{\mathcal{RIS}} \) except for one modification which extends \( \mathcal{L}_{\mathcal{RIS}} \). Indeed, \( \mathcal{L}_{\mathcal{RIS}} \) admits the same function and predicate symbols
than $\mathcal{L}_{RQ}$, plus some other or more complex versions of them. For example, in $\mathcal{L}_{RIS}$ RIS terms have a more complex structure and union, intersection, etc. of RIS terms are available.

However, in $\mathcal{L}_{RIS}$ filters can only be $\mathcal{L}_X$ formulas. In $\mathcal{L}_{RQ}$ filters can be nested RQ ending in an $\mathcal{L}_X$ formula. This extension is crucial to extend the expressiveness of the language (cf. formula (6)). The restriction on filters to $\mathcal{L}_X$ formulas in $\mathcal{L}_{RIS}$ is key to define a decision procedure for it. If this restriction is lifted, termination of the decision algorithm is compromised. As we will shown in Section 4, there are subclasses of formulas admitting nested RQ that do not compromise termination.$\blacksquare$

2.3 Semantics

Sorts and symbols in $\Sigma_{RQ}$ are interpreted according to the interpretation structure $\mathcal{R} = (\mathcal{D}, (\cdot)^{\mathcal{R}})$, where $\mathcal{D}$ and $(\cdot)^{\mathcal{R}}$ are defined as follows.

Definition 7 (Interpretation domain) The interpretation domain $\mathcal{D}$ is partitioned as $\mathcal{D} = \mathcal{D}_{\text{Set}} \cup \mathcal{D}_X$ where:

- $\mathcal{D}_{\text{Set}}$ is the set of all hereditarily finite hybrid sets built from elements in $\mathcal{D}$.
  Hereditarily finite sets are those sets that admit (hereditarily finite) sets as their elements, that is sets of sets.
- $\mathcal{D}_X$ is a collection of other objects.

Definition 8 (Interpretation function) The interpretation function $(\cdot)^{\mathcal{R}}$ is defined as follows:

- Each sort $S \in \{\text{Set}, X\}$ is mapped to the domain $\mathcal{D}_S$.
- For each sort $S$, each variable $x$ of sort $S$ is mapped to an element $x^{\mathcal{R}}$ in $\mathcal{D}_S$.
- The constant and function symbols in $\mathcal{F}_S$ are mapped to elements in $\mathcal{D}_S$ as follows:
  - $\emptyset$ is interpreted as the empty set, namely $\emptyset^{\mathcal{R}} = \emptyset$.
  - $\{x \cup A\}$ is interpreted as the set $\{x^{\mathcal{R}} \cup A^{\mathcal{R}}\}$.
  - Let $\vec{x}$ be a vector of variables occurring in $c$ and $\vec{v}$ a vector of other variables, then the set $\{c(\vec{x}) : X | \phi(\vec{x}, \vec{v})\}$ is interpreted as the set:
    $$\{y : \exists \vec{x}(c(\vec{x}) \in X \land \phi(\vec{x}, \vec{v}))\}$$
  Note that in RIS terms, $\vec{x}$ are “local” variables whose scope is the RIS itself, while $\vec{v}$ are “non-local” variables whose scope is the formula where the RIS is participating in.
- The predicate symbols in $\Pi$ are interpreted as follows:
  - $A =_S B$ is interpreted as $A^{\mathcal{R}} = B^{\mathcal{R}}$, where $=$ is the identity relation in $\mathcal{D}_{\text{Set}}$
  - $x =_X y$ is interpreted as $x^{\mathcal{R}} = y^{\mathcal{R}}$, where $=$ is the identity relation in $\mathcal{D}_X$.
  - $x \in A$ is interpreted as $x^{\mathcal{R}} \in A^{\mathcal{R}}$.
  - $A \subseteq B$ is interpreted as $A^{\mathcal{R}} \subseteq B^{\mathcal{R}}$.
  - $\text{is}X(t)$ is interpreted as $t^{\mathcal{R}} \in \mathcal{D}_X$.
  - $\text{set}(t)$ is interpreted as $t^{\mathcal{R}} \in \mathcal{D}_{\text{Set}}$. $\blacksquare$
The interpretation structure \( \mathcal{R} \) is used to map each \( \mathcal{R}_Q \)-formula \( \Phi \) to a truth value \( \Phi^\mathcal{R} = \{ \text{true}, \text{false} \} \) in the following way: set constraints (resp., \( \mathcal{X} \) constraints) are evaluated by \( (\cdot)^\mathcal{R} \) according to the meaning of the corresponding predicates in set theory (resp., in theory \( \mathcal{X} \)) as defined above; \( \mathcal{R}_Q \)-formulas are evaluated by \( (\cdot)^\mathcal{R} \) according to the rules of propositional logic. A \( \mathcal{L}_{\mathcal{R}_Q} \)-formula \( \Phi \) is satisfiable if and only if there exists an assignment \( \sigma \) of values from \( D \) to the free variables of \( \Phi \), respecting the sorts of the variables, such that \( \Phi[\sigma] \) is true in \( \mathcal{R} \), i.e., \( \mathcal{R} \models \Phi[\sigma] \). In this case, we say that \( \sigma \) is a successful valuation (or, simply, a solution) of \( \Phi \).

3 A Solver for \( \mathcal{L}_{\mathcal{R}_Q} \)

In this section we present a constraint solver for \( \mathcal{L}_{\mathcal{R}_Q} \), called \( \text{SAT}_{\mathcal{R}_Q} \). The solver provides a collection of rewrite rules for rewriting \( \mathcal{L}_{\mathcal{R}_Q} \) formulas that are proved to be a decision procedure for some subclasses of \( \mathcal{L}_{\mathcal{R}_Q} \) formulas (see Section 4). As already observed, however, checking the satisfiability of \( \mathcal{R}_Q \)-formulas depends on the existence of a decision procedure for \( \mathcal{X} \)-formulas (i.e., formulas over \( \mathcal{L}_\mathcal{X} \)).

3.1 The Solver

\( \text{SAT}_{\mathcal{R}_Q} \) is a rewriting system whose global organization is shown in Algorithm 1, where \( \text{STEP} \) is the core of the algorithm.

\( \text{sort\_infer} \) is used to automatically add \( \Pi_T \)-constraints to the input formula \( \Phi \) to force arguments of \( \mathcal{R}_Q \)-constraints in \( \Phi \) to be of the proper sorts (see Remark 3 below). \( \text{sort\_infer} \) is called twice in Algorithm 1: first, at the beginning of the algorithm, and second, within procedure \( \text{STEP} \) for the constraints that are generated during constraint processing. \( \text{sort\_check} \) checks \( \Pi_T \)-constraints occurring in \( \Phi \): if they are satisfiable, then \( \Phi \) is returned unchanged; otherwise, \( \Phi \) is rewritten to false.

**Algorithm 1** The \( \text{SAT}_{\mathcal{R}_Q} \) solver. \( \Phi \) is the input formula.

```plaintext
procedure \text{STEP}(\Phi)
    for all \( \pi \in \Pi_S \cup \Pi_T : \Phi \leftarrow \text{rw}_\pi(\Phi) \); 
    \( \Phi \leftarrow \text{sort\_check}(\text{sort\_infer}(\Phi)) \)
    return \( \Phi \)

procedure \text{rw}_\pi(\Phi)
    if \( \Phi = \cdots \land \text{false} \land \cdots \) then
        return \( \text{false} \)
    else
        repeat
            let \( \phi \) be \( c_1 \land \cdots \land c_m \)
            select a \( \pi \)-constraint \( c_i \) in \( \phi \)
            apply any applicable rule to \( c_i \)
        until no rule applies to any \( \pi \)-constraint
    return \( \Phi \)

procedure \text{SAT}\_\mathcal{R}_Q(\Phi)
    \( \Phi \leftarrow \text{sort\_infer}(\Phi) \)
    repeat
        \( \Phi' \leftarrow \Phi \)
        \( \Phi \leftarrow \text{STEP}(\Phi) \)
        until \( \Phi = \Phi' \)
    if \( \Phi_S \land \Phi_X \)
        \( \Phi \leftarrow \Phi_S \land \text{SAT}_\mathcal{X}(\Phi_X) \)
    return \( \Phi \)
```

\( \text{STEP} \) applies specialized rewriting procedures to the current formula \( \Phi \) and returns either false or the modified formula. Each rewriting procedure applies a
few non-deterministic rewrite rules which reduce the syntactic complexity of \( R\mathcal{Q} \)-
constraints of one kind. Procedure \( \pi \) in Algorithm 1 represents the rewriting
procedure for \((\Pi_S \cup \Pi_T)\)-constraints. The execution of \( \text{STEP} \) is iterated until a
fixpoint is reached—i.e., the formula cannot be simplified any further. \( \text{STEP} \) re-
turns \text{false} whenever (at least) one of the procedures in it rewrites \( \Phi \) to \text{false}. In
this case, a fixpoint is immediately detected, since \( \text{STEP}(\text{false}) \) returns \text{false}.

\( SAT_X \) is the constraint solver for \( X \)-formulas. The formula \( \Phi \) can be seen,
without loss of generality, as \( \Phi_S \land \Phi_X \), where \( \Phi_S \) is a pure \( R\mathcal{Q} \)-formula (basically,
a \( R\mathcal{Q} \)-formula with with no \( X \)-formula in it—see Definition 12) and \( \Phi_X \) is an \( X \)-
formula. \( SAT_X \) is applied only to the \( \Phi_X \) conjunct of \( \Phi \). Note that, conversely,
\( \text{STEP} \) rewrites only \( R\mathcal{Q} \)-constraints, while it leaves all other atoms unchanged.
Nonetheless, as the rewrite rules show, \( SAT_{R\mathcal{Q}} \) generates \( X \)-formulas that are
conjoined to \( \Phi_X \) so they are later solved by \( SAT_X \).

As we will show in Section 4, when all the non-deterministic computations of
\( SAT_{R\mathcal{Q}}(\Phi) \) return \text{false}, then we can conclude that \( \Phi \) is unsatisfiable; otherwise,
we can conclude that \( \Phi \) is satisfiable and each solution of the formulas returned
by \( SAT_{R\mathcal{Q}} \) is a solution of \( \Phi \), and vice versa.

\textbf{Remark 3} \( L_{R\mathcal{Q}} \) does not provide variable declarations. The sort of variables are
enforced by adding suitable \textit{sort constraints} to the formula to be processed. Sort
constraints are automatically added by the solver. Specifically, a constraint \( \text{set}(y) \) (resp., \( \text{isX}(y) \)) is added for each variable \( y \) which is required to be of sort \( \text{set} \) (resp.,
\( X \)). For example, given \( B = \{ y \uplus A \} \), \( \text{sort}\_\text{infer} \) conjoins the sort constraints \( \text{set}(B) \),
\( \text{isX}(y) \) and \( \text{set}(A) \). If the set of function and predicate symbols of \( L_{R\mathcal{Q}} \) and \( L_X \)
are disjoint, there is a unique sort constraint for each variable in the formula. \( \square \)

\subsection*{3.2 Rewrite Rules}

The rewrite rules used by \( SAT_{R\mathcal{Q}} \) are defined as follows.

\textbf{Definition 9 (Rewrite rules)} If \( \pi \) is a symbol in \( \Pi_S \cup \Pi_T \) and \( p \) is a \( R\mathcal{Q} \)-
constraint based on \( \pi \), then a \textit{rewrite rule for \( \pi \)-constraints} is a rule of the form
\[ p \rightarrow \Phi_1 \lor \cdots \lor \Phi_n, \]
where \( \Phi_i, \ i \geq 1, \) are \( R\mathcal{Q} \)-formulas. Each atom matching
\( p \) is non-deterministically rewritten to one of the \( \Phi_i \). Variables appearing in the
right-hand side but not in the left-hand side are assumed to be fresh variables,
implicitly existentially quantified over each \( \Phi_i \). Conjunction has higher precedence
than disjunction. \( \square \)

A \textit{rewriting procedure} for \( \pi \)-constraints consists of the collection of all the rewrite
rules for \( \pi \)-constraints. For each rewriting procedure, \( \text{STEP} \) selects rules in the order
they are listed in Figure 1. The first rule whose left-hand side matches the input
\( \pi \)-constraint is used to rewrite it.

Rules whose right-hand side is \textsc{irreducible} indicate that the constraint at the
left-hand side is not rewritten and will remain as it is all the way to the final
answer returned by Algorithm 1. In Figure 1, we have made explicit equality in
\( L_X \) by means of \( =_X \). All other instances of \( = \) correspond to equality in \( L_{R\mathcal{Q}} \) (i.e.,
set equality).

As shown in Figure 1, there are rewriting procedures for \( \leq \)-constraints (\textsc{Subset}), \( \epsilon \)-
constraints (\textsc{Membership}) and \( = \)-constraints (\textsc{Equality}). The \textsc{Membership}
and
Subset
\[
\emptyset \subseteq \{ x : A \mid \phi(x) \} \rightarrow true \tag{9}
\]
\[
\{ a \sqcup A \} \subseteq \{ x : \{ a \sqcup A \} \mid \phi(x) \} \rightarrow \phi(a) \land A \subseteq \{ x : A \mid \phi(x) \} \tag{10}
\]
\[
\dot{A} \subseteq \{ x : \dot{A} \mid \phi(x) \} \rightarrow \text{irreducible} \tag{11}
\]

Membership
\[
a \in \emptyset \rightarrow false \tag{12}
\]
\[
a \in \{ b \sqcup A \} \rightarrow a =_X b \lor a \in A \tag{13}
\]
\[
a \in \dot{A} \rightarrow \dot{A} = \{ a \sqcup N \} \tag{14}
\]

Equality
\[
\emptyset = \emptyset \rightarrow true \tag{15}
\]
\[
\dot{A} = \dot{A} \rightarrow true \tag{16}
\]
\[
B = \dot{A} \rightarrow \dot{A} = B, \text{ if } B \notin V \tag{17}
\]
\[
\dot{A} = B \rightarrow \dot{A} = B, \text{ and substitute } A \text{ by } B \text{ in the rest of the formula} \tag{18}
\]
\[
\{ a \sqcup A \} = \emptyset \rightarrow false \tag{19}
\]
\[
\emptyset = \{ a \sqcup A \} \rightarrow false \tag{20}
\]
\[
\{ a \sqcup A \} = \{ b \sqcup B \} \rightarrow
a =_X b \land A = B
\lor a =_X b \land A = \{ a \sqcup A \} \lor A = \{ b \sqcup B \} \lor B = \{ a \sqcup A \}
\tag{21}
\]
\[
\dot{A} = B \rightarrow \text{irreducible, if } \dot{A} \text{ does not occur elsewhere in the formula} \tag{22}
\]

Fig. 1 Rewrite rules for $\mathcal{R}_Q$-constraints

Equality rules deal only with extensional sets due to the restrictions given in Definition 5. Observe that all other constraints generated by the rules of Figure 1 are $X$-constraints which are dealt with by SAT$_X$. All the rules in the figure are borrowed from the L$_{RIS}$ solver [18]. This is important because it simplifies the proof of some important properties of SAT$_{\mathcal{R}_Q}$ (Section 4).

As can be seen, most of the rules are straightforward. Rule (21) is the main rule of set unification [29]. Set unification is pervasive in other logics developed by the authors [16,18]. This rule states when two non-empty, non-variable sets are equal by non-deterministically and recursively computing four cases. These cases implement the Absorption and Commutativity on the left properties of set theory [28]. As an example, by applying rule (21) to $\{1\} = \{1,1\}$ we get: $(1 = 1 \land \emptyset = \{1\}) \lor (1 = 1 \land \emptyset = \{1\}) \lor (\emptyset = \{1\} \land \{1\} = \{1\}) \lor (\emptyset = \{1\} \land \{1\} = \{1\})$, which turns out to be true (due to the second disjunct).

Rules (9)-(11) process RUQ by implementing (7). Rule (10) iterates over all the elements of the domain of the RIS until it becomes the empty set or a variable. In each iteration one of the elements of the domain is proved to verify the filter (if not, the rule fails), and a new iteration is fired. If the domain becomes a
variable the constraint is not processed any more. Note that a constraint such as 
\( A \subseteq \{ x : A | \phi(x) \} \) is trivially satisfied by substituting \( A \) by the empty set.

As \( L_{RQ} \), \( SAT_{RQ} \) is based solely on fundamental concepts of set theory.

**Remark 4** Observe that when \( \subseteq \) are rewritten only the following are generated:
- \( \phi \in \Phi_X \)
- \( A \subseteq \{ x : A | \phi(x) \} \)

### 3.3 Irreducible Constraints

When no rewrite rule is applicable to the current \( RQ \)-formula \( \Phi \) and \( \Phi \) is not \textit{false},
the main loop of \( SAT_{RQ} \) terminates returning \( \Phi \) as its result. This formula can be
seen, without loss of generality, as \( \Phi_S \wedge \Phi_X \), where \( \Phi_X \) contains all (and only) \( X \)
constraints and \( \Phi_S \) contains all other constraints occurring in \( \Phi \).

The following definition precisely characterizes the form of atomic constraints
in \( \Phi_S \).

**Definition 10 (Irreducible formula)** Let \( \Phi \) and \( \phi \) be \( RQ \)-formulas, \( A \in \forall_S \), \( x \) a
control term (thus it is a term of sort \( X \)) and \( t \) a term of sort \( Set \). A \( RQ \)-constraint
\( p \) occurring in \( \Phi \) is \textit{irreducible} if it has one of the following forms:
1. \( A = t \), and neither \( t \) nor \( \Phi \setminus \{ A = t \} \) contain \( A \)
2. \( A \subseteq \{ x : A | \phi(x) \} \)

A \( RQ \)-formula \( \Phi \) is irreducible if it is \textit{true} or if all of its \( RQ \)-constraints are irre-
ducible.

\( \Phi_S \), as returned by \( SAT_{RQ} \)'s main loop, is an irreducible formula. This fact can
be checked by inspecting the rewrite rules presented in Figure 1. This inspection
is straightforward as there are no rules rewriting irreducible constraints and all
non-irreducible form constraints are rewritten by some rule.

It is important to observe that the atomic constraints occurring in \( \Phi_S \) are
indeed quite simple. In particular, all non-variable set terms occurring in the in-
put formula have been removed, except those occurring as right-hand side of \( = \)
constraints.

### 4 Decidability of \( L_{RQ} \) Formulas

In this section we analyze the soundness, completeness and termination properties
of \( SAT_{RQ} \) for different subclasses of \( RQ \)-formulas.

As we have explained in Remark 2, \( L_{RQ} \) is a sublanguage of \( L_{RIS} \) except for
the fact that \( L_{RQ} \) admits \( RQ \) in filters. We also pointed out that accepting \( RQ \) in
filters poses termination problems; soundness and completeness are not affected.
Actually, as noted in Section 3, all the rules of Figure 1 are rules borrowed from the
\( L_{RIS} \) solver. Hence, we will briefly analyze soundness and completeness of
\( SAT_{RQ} \) and will spend more time analyzing its termination.

As \( RUQ \) and \( REQ \) play a central role in this work, we provide some syntactic
sugar for them.
Definition 11 (Restricted Quantifiers) Given a control term \( x \), an extensional set term \( A \) and a formula \( \phi \), a restricted universal quantifier (RUQ), noted \( \forall x \in A, \phi \), is defined as:

\[
\forall x \in A, \phi \equiv A \subseteq \{ x : A | \phi(x) \}
\]  

(23)

Under the same terms, a restricted existential quantifier (REQ), noted \( \exists x \in A, \phi \), is defined as:

\[
\exists x \in A, \phi \equiv \exists n \in A \land \phi(n)
\]  

(24)

where all variables occurring in \( n \) are fresh variables not occurring elsewhere in the formula of which the REQ is a part.

In a RQ: \( x \) is called control term or quantified variable, \( A \) is called domain and \( \phi \) is called filter (following the vocabulary of RIS terms). Note that in both RUQ and REQ, \( \phi \) must depend on \( x \).

Definition 12 (Subclasses of \( \Phi_{\mathcal{RQ}} \)) The following are the subclasses of \( \Phi_{\mathcal{RQ}} \)-formulas for which decidability will be analyzed:

- \( \Phi_{\text{nrq}} \) is the subclass of \( \Phi_{\mathcal{RQ}} \) whose elements are nested RQ. The subclasses of formulas to be analyzed will be subclasses of \( \Phi_{\text{nrq}} \).

\[
\Phi_X^X := \forall \text{ (} Ctrl \in \text{ Ext, } \Phi_X \text{)}
\]

\[
\Phi_X^X := \exists \text{ (} Ctrl \in \text{ Ext, } \Phi_X \text{)}
\]

\[
\Phi_{\text{mix}} := \Phi_X^X \lor \Phi_X^X \lor \exists \text{ (} Ctrl \in \text{ Ext, } \Phi_{\text{mix}} \text{)} \lor \exists \text{ (} Ctrl \in \text{ Ext, } \Phi_{\text{mix}} \text{)}
\]

\[
\Phi_{\text{nrq}} := \text{true} \lor \text{false} \lor \Phi_X \lor \Phi_{\text{mix}} \lor \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}}
\]

- \( \Phi_{\forall} \) is the subclass of \( \Phi_{\text{nrq}} \) whose elements are built from \( \Phi_X \) and nested RUQ.

\[
\Phi_{\forall} := \Phi_X \lor \Phi_{\forall} \lor \Phi_{\forall} \lor \Phi_{\forall} \lor \Phi_{\forall} \lor \Phi_{\mathcal{RQ}}
\]

- \( \Phi_{\exists} \) is the subclass of \( \Phi_{\text{nrq}} \) whose elements are built from \( \Phi_X \) and nested REQ.

\[
\Phi_{\exists} := \Phi_X \lor \Phi_{\exists} \lor \Phi_{\exists} \lor \Phi_{\exists} \lor \Phi_{\exists} \lor \Phi_{\mathcal{RQ}}
\]

- \( \Phi_{\exists \forall} \) is the subclass of \( \Phi_{\text{nrq}} \) whose elements are built from \( \Phi_X \) and nested RQ where all REQ are before all RUQ (if any).

\[
\Phi_{\exists \forall} := \Phi_X \lor \exists \text{ (} Ctrl \in \text{ Ext, } \text{Filter} \text{)} \lor \Phi_{\exists \forall} \lor \Phi_{\exists \forall} \lor \Phi_{\exists \forall}
\]

Filter := \Phi_{\exists \forall} \lor \Phi_{\exists \forall}

- \( \Phi_{\mathcal{RQ}} \) is the subclass of \( \Phi_{\text{nrq}} \) whose elements are pure \( \mathcal{RQ} \)-formulas.

\[
\Phi_{\mathcal{RQ}} := \text{true} \lor \text{false} \lor \Phi_{\text{mix}} \lor \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}} \lor \Phi_{\mathcal{RQ}}
\]

Similar definitions can be given for pure \( \Phi_{\forall}, \Phi_{\exists} \) and \( \Phi_{\exists \forall} \) formulas.
Remark 5 (Notation) and, vice versa, every solution of one of these formulas is a solution for the other. The proof rests on a series of lemmas each showing that the set of solutions of one of these formulas is equisatisfiable to the set of solutions of the other.

As can be seen, formulas in $\Phi_{\text{nrq}}$ are conjunctions and disjunctions of $\forall\exists$-formulas and nested $\forall\exists$; the filter of the innermost $\forall\exists$ is an $\forall\exists$-formula. Note that not every $\forall\exists$-constraint can be part of a formula in $\Phi_{\text{nrq}}$; the idea is to restrict them to be $\forall\exists$. Hence, basically, we analyze the decidability of $\forall\exists$-formulas strictly encoding $\forall\exists$. However, note that when $\text{SAT}_{\forall\exists}$ processes a $\Phi_{\text{nrq}}$-formula it may generate a formula outside $\Phi_{\text{nrq}}$. For example, $\exists x A \phi(x)$ is rewritten into $\exists n \in A \land \phi(n)$ which then is rewritten into $A = \{n \cup N\} \land \phi(n)$, which is not a $\Phi_{\text{nrq}}$-formula due to the presence of $A = \{n \cup N\}$.

Remark 5 (Notation) From now on, we will write $\forall x A \phi(x)$ as a shorthand for $\forall x A \forall y B \phi(x, y)$ and $\exists x A \exists y B \phi(x, y)$ as a shorthand for $\exists x A \exists y B \phi(x, y)$. Besides, $\forall x A \phi(x)$ denotes $\forall x A \forall y B \phi(x, y)$ and $\exists x A \phi(x)$ denotes $\exists x A \exists y B \phi(x, y)$. Hence, it is easy to see that: in pure $\Phi^0_\forall$ formulas there are only constraints of the form $\forall x A \phi(x)$ for some $\forall\exists$-formula $\phi$; in pure $\Phi^0_\exists$ formulas there are only constraints of the form $\exists x A \phi(x)$ for some $\forall\exists$-formula $\phi$; and in pure $\Phi^0_{\forall\exists}$ formulas there are only constraints of the form $\exists x A \forall y B \phi(x, y)$ for some $\forall\exists$-formula $\phi$ ($0 < n, 0 \leq m$).

4.1 Soundness and Completeness

The following theorem ensures that, after termination, the rewriting process implemented by $\text{SAT}_{\forall\exists}$ preserves the set of solutions of the input formula.

**Theorem 1 (Equisatisfiability)** Let $\Phi$ be a $\Phi_{\text{nrq}}$-formula and $\Phi^0, \Phi^1, \ldots, \Phi^n$ be the collection of $\forall\exists$-formulas returned by $\text{SAT}_{\forall\exists}(\Phi)$. Then $\Phi^0 \lor \Phi^1 \lor \cdots \lor \Phi^n$ is equisatisfiable to $\Phi$, that is, every possible solution of $\Phi$ is a solution of one of the $\Phi^i$'s and, vice versa, every solution of one of these formulas is a solution for $\Phi$.

Proof The proof rests on a series of lemmas each showing that the set of solutions of left and right-hand sides of each rewrite rule are the same. Given that the rewrite rules of Figure 1 are those used to define the solver for $\mathcal{L}_{\text{RIS}}$, then the lemmas proved for $\mathcal{L}_{\text{RIS}}$ still apply [18, Appendix C.4]. The only concern with those lemmas might be the fact that they were proved under the assumption that RIS filters do not admit $\forall\exists$. However, it is trivial to see that all the MEMBERSHIP and EQUALLY rules and rules (9) and (11) are unaffected by the fact that filters admit $\forall\exists$. For the remaining rule, i.e. (10), we reproduce in Appendix A the proof made for $\mathcal{L}_{\text{RIS}}$ so readers can check that it do not depend on any limitation over RIS filters.

---

3 More precisely, each solution of $\Phi^i$ expanded to the variables occurring in $\Phi^i$ but not in $\Phi$, so as to account for the possible fresh variables introduced into $\Phi^i$. 

---
Theorem 2 (Satisfiability of the output formula) Any $\mathcal{RQ}$-formula different from $\text{false}$ returned by $\text{SAT}_{\mathcal{RQ}}$ is satisfiable w.r.t. the underlying interpretation structure $\mathcal{R}$.

Proof As we have explained, each disjunct of the formula returned by $\text{SAT}_{\mathcal{RQ}}$ can be written as $\Phi_S \land \Phi_X$, where $\Phi_S$ is a pure $\mathcal{RQ}$-formula and $\Phi_X$ is an $\mathcal{X}$-formula.

Since $\text{SAT}_{\mathcal{X}}$ is called on $\Phi_X$ we know that it is satisfiable (under the assumption that $\text{SAT}_{\mathcal{RQ}}$ has not returned $\text{false}$).

Now we prove that $\Phi_S$ is satisfiable, too. We know that $\Phi_S$ is an irreducible formula (Definition 10). Then, we have to prove that an irreducible formula is always satisfiable. Given that an irreducible formula is a conjunction of irreducible constraints, we have to prove that all these constraints can be simultaneously satisfied. Constraints of the form $\dot{A} = t$ are satisfied by binding $\dot{A}$ to $t$ (recall from Definition 10 that $\dot{A}$ does not occur in the rest of an irreducible formula); constraints of the form $\dot{A} \subseteq \{x : \phi(x)\}$ are satisfied by substituting the domain of the RIS by the empty set. Hence, there is always a solution for an irreducible $\mathcal{RQ}$-formula.

Finally, we prove that $\Phi_S \land \Phi_X$ can be satisfied. Indeed, observe that the solution for $\Phi_S$ do not bind variables of sort $\mathcal{X}$ and that $\Phi_X$ do not contain variables of sort $\mathcal{Set}$. So the values of the solution for $\Phi_X$ do no conflict with the values of the solution for $\Phi_S$. $\Box$

The following example shows how Theorem 2 works in practice.

Example 3 Consider the following nested RUQ where $\phi$ is an $\mathcal{X}$-formula:

\[
\text{foreach}(\{x \in \{a \cup \dot{A}\}, y \in \{b \cup \dot{B}\}, \phi(x, y)\})
\]

$\text{SAT}_{\mathcal{RQ}}$ applies (23) and rule (10) twice yielding:

\[
\phi(a, b) \land \text{foreach}(y \in \dot{B}, \phi(a, y)) \land \text{foreach}(\{x \in \dot{A}, y \in \{b \cup \dot{B}\}, \phi(x, y)\})
\]

Now, it calls $\text{SAT}_{\mathcal{X}}(\phi(a, b))$ because both $\text{foreach}$ constraints are irreducible. Thus, determining the satisfiability of (25) is reduced to determining the satisfiability of $\phi(a, b)$ because satisfiability of the two $\text{foreach}$ constraints is guaranteed by Theorem 2 (with $\dot{A} \leftarrow \emptyset$, $\dot{B} \leftarrow \emptyset$). $\Box$

Thanks to Theorems 1 and 2 we can conclude that, given a $\Phi_{\text{nrq}}$-formula $\Phi$, then $\Phi$ is satisfiable with respect to the intended interpretation structure $\mathcal{R}$ if and only if there is a non-deterministic choice in $\text{SAT}_{\mathcal{RQ}}(\Phi)$ that returns a $\mathcal{RQ}$-formula different from $\text{false}$. Conversely, if all the non-deterministic computations of $\text{SAT}_{\mathcal{RQ}}(\Phi)$ terminate with $\text{false}$, then $\Phi$ is surely unsatisfiable. Note that these theorems have been proved for any $\Phi_{\text{nrq}}$-formula.

4.2 Termination

The problem is that termination of $\text{SAT}_{\mathcal{RQ}}$ cannot be proved for every $\Phi_{\text{nrq}}$-formula, as shown by the following example.
Example 4 The following nested RQ where \( \phi \) is an \( X \)-formula, is rewritten as indicated.

\[
\begin{align*}
\text{foreach}(x \in \{a \cup A\}, \exists y \in \{b \cup A\}, \phi(x, y)) & \quad \text{[by rule (10)]} \\
\rightarrow \exists y \in \{b \cup A\}, \phi(a, y) & \\
\land \text{foreach}(x \in A, \exists y \in \{b \cup A\} : \phi(x, y)) & \quad \text{[by Def. 11, (24)]} \\
\rightarrow n \in \{b \cup A\} \land \phi(a, n) \land \text{foreach}(x \in A, \exists y \in \{b \cup A\} : \phi(x, y)) &
\end{align*}
\]

Now there are two cases from \( n \in \{b \cup A\} \): \( n = b \) and \( n \in A \) (rule (13)). Let us see the second one:

\[
\begin{align*}
n \in A \land \phi(a, n) \land \text{foreach}(x \in A, \exists y \in \{b \cup A\} : \phi(x, y)) & \quad \text{[by rule (14)]} \\
\rightarrow A = \{n \cup N\} \land \phi(a, n) & \\
\land \text{foreach}(x \in A, \exists y \in \{b \cup A\} : \phi(x, y)) & \quad \text{[by rule (18)]} \\
\rightarrow A = \{n \cup N\} \land \phi(a, n) \land \text{foreach}(x \in \{n \cup N\}, \exists y \in \{b, n \cup N\} : \phi(x, y)) &
\end{align*}
\]

It is clear that the last \text{foreach} is structurally equal to the initial formula. Without more information about \( \phi \) this could potentially cause an infinite loop making \( SAT_{RQ} \) not to terminate. \( \square \)

Before presenting the theorems stating termination on different subclasses of \( \Phi_{nrq} \)-formulas, consider the following analysis. Let \( \phi \) be a \( \Phi_{nrq} \)-formula. If \( \phi = \phi_1 \lor \phi_2 \), then we prove \( SAT_{RQ} \) terminates on \( \phi_1 \) and then on \( \phi_2 \). Hence, as concerns termination, we can consider \( \phi \) to be a conjunction of \( \Phi_{nrq} \)-formulas. In this case \( \phi \) can be written as \( \phi_S \land \phi_X \) where \( \phi_S \) is a pure \( \Phi_{nrq} \)-formula and \( \phi_X \) is an \( X \)-formula.

We need to prove termination of \( SAT_{RQ} \) on \( \phi_S \), as termination of \( SAT_X \) on \( \phi_X \) is guaranteed by the assumption that \( SAT_X \) is a decision procedure for \( L_X \). Now, if \( \phi_S \) is a disjunction, we prove termination for each disjunct. Hence, in the following theorems we prove termination of \( SAT_{RQ} \) on conjunctions of pure \( RQ \)-constraints belonging to different subclasses of formulas.

**Theorem 3 (Termination on \( \Phi_{q} \) formulas)** The \( SAT_{RQ} \) procedure can be implemented as to ensure termination for every conjunction of pure \( \Phi_{q} \) constraints.

**Proof** Recall that the only constraint in pure \( \Phi_{q} \) formulas is of the form \( \text{foreach}(\bar{x}_n \in \bar{A}_n, \phi), 0 < n \) and \( \phi \in \Phi_X \). First we will prove that \( SAT_{RQ} \) terminates on these constraints. The proof is by induction on \( n \).

- **Base case.** Let \( \phi \) be a \( L_X \) formula. We will show that \( SAT_{RQ} \) terminates on the following RUQ:

\[
\text{foreach}(x \in A, \phi(x)) \tag{26}
\]

- \( A = \emptyset \), this case is trivial as rule (9) terminates immediately.
- \( A \in V_S \), this case is trivial as rule (11) terminates immediately.
- \( A = \{b \cup B\} \), rule (10) is applied to (26) yielding:

\[
\phi(b) \land \text{foreach}(x \in B, \phi(x))
\]

The recursive call to \text{foreach} is made with a domain strictly smaller than \( A \). This is so because the call is made with \( B \) and because \( \phi(b) \) cannot bind a
value to $B$ since $\phi$ is $X$-formula and $B$ is of sort $\text{Set}$ (the only way of binding a value to $B$ is by means of $t \in B$ or $B = t$, for some term $t$, which are not generated during $\Phi_\exists$ processing, Remark 4). Then, $\text{SAT}_RQ$ will terminate when the ‘end’ of $B$ is reached (i.e., when a variable or the empty set is found).

- Induction hypothesis. $\text{SAT}_RQ$ terminates on every constraint of the form $\text{foreach}(\vec{x}_k \in \vec{A}_k, \phi)$ with $k \leq n$, for any $X$-formula $\phi$.
- Induction step. Let $\phi$ be any $X$-formula. We will prove that $\text{SAT}_RQ$ terminates on the following constraint:

$$\text{foreach}(x \in A, \text{foreach}(\vec{x}_n \in \vec{A}_n, \phi(x, \vec{x}))) \quad (27)$$

- $A = \emptyset$, this case is trivial as rule (9) terminates immediately.
- $A \in V_S$, this case is trivial as rule (11) terminates immediately.
- $A = \{b \cup B\}$, rule (10) is applied to (27) yielding:

$$\text{foreach}(\vec{x}_n \in \vec{A}_n, \phi(b, \vec{x})) \land \text{foreach}(x \in B, \text{foreach}(\vec{x}_n \in \vec{A}_n, \phi(x, \vec{x})))$$

$\text{SAT}_RQ$ terminates on the first conjunct by the induction hypothesis. Besides, the recursive call in the second conjunct is made with a domain strictly smaller than $A$. This is so because the call is made with $B$ and because the first conjunct cannot bind a value to $B$ since it is a RUQ or an $X$-formula and $B$ is of sort $\text{Set}$. Then, $\text{SAT}_RQ$ will terminate when the ‘end’ of $B$ is reached (i.e., when a variable or the empty set are found).

Observe that termination depends solely on the size of the domain of the RUQ. If $\phi_S$ is a conjunction of RUQ, then termination of $\text{SAT}_RQ$ on each RUQ implies termination of $\text{SAT}_RQ$ for the whole formula. Indeed, when a given RUQ is processed it can only generate a shorter RUQ or a $L_X$ formula. In either case, nothing is generated that can bind a value to a domain. Then, the domain of a RUQ in $\phi_S$ is not affected by the processing of the other RUQ in the formula.

Before proving termination of $\text{SAT}_RQ$ on $\Phi_\exists$ formulas we need the following lemma.

**Lemma 1** $\text{SAT}_RQ$ terminates on any $RQ$-formula without $RQ$.

**Proof** Let $\phi$ be a $RQ$-formula without $RQ$. Write $\phi$ as $\phi_S \land \phi_X$. Hence, $\phi_S$ is comprised solely of membership and equality constraints. Then, only the Membership and Equality rules of Figure 1 will be used by $\text{SAT}_RQ$. These rules have been proved to constitute a terminating rewriting system elsewhere [28, Theorem 10.10].

**Theorem 4 (Termination on $\Phi_\exists$ formulas)** The $\text{SAT}_RQ$ procedure can be implemented as to ensure termination for every conjunction of pure $\Phi_\exists$ constraints.

**Proof** Let $\phi$ be an $X$-formula. Consider the following rewriting:

$$\begin{align*}
\exists x \in A, y \in B, \phi(x, y) & \quad \text{[by Def. 11, (24)]} \\
\rightarrow n_1 \in A & \land \exists y \in B, \phi(n_1, y) \quad \text{[by Def. 11, (24)]} \\
\rightarrow n_1 \in A & \land n_2 \in B \land \phi(n_1, n_2)
\end{align*}$$
Then, all REQ are quickly eliminated from the formula. This can be easily generalized to \(\exists(x_n \in A_n, \phi)\) for any \(\lambda\)-formula \(\phi\) and any \(0 < n\). The resulting non-\(\lambda\) subformula is a \(\mathcal{RQ}\)-formula without RQ. Hence, by Lemma 1, \(\text{SAT}_{\mathcal{RQ}}\) terminates on that formula. Given that conjunctions of REQ are rewritten into conjunctions of formulas such as the last one above, \(\text{SAT}_{\mathcal{RQ}}\) terminates on every conjunction of pure \(\Phi_3\) formulas.

**Theorem 5 (Termination on \(\Phi_{3\lambda}\) formulas)** The \(\text{SAT}_{\mathcal{RQ}}\) procedure can be implemented as to ensure termination for every conjunction of pure \(\Phi_{3\lambda}\) constraints.

**Proof** Recall that the only constraints in pure \(\Phi_{3\lambda}\) formulas are of the form:

\[
\exists(x_k \in \bar{A}_k, \forall(x_m \in \bar{B}_m, \phi)) \tag{28}
\]

for some \(\lambda\)-formula \(\phi\), \(0 < k\) and \(0 \leq m\) (if \(m = 0\), then \(\phi\) is the innermost filter).

First we prove termination on such a constraint. By using the same reasoning of Theorem 4, (28) is rewritten into:

\[
n_1 \in A_1 \land \cdots \land n_k \in A_k \land \forall(x_m \in \bar{B}_m, \phi) \tag{29}
\]

Then, \(\text{SAT}_{\mathcal{RQ}}\) process \(n_1 \in A_1 \land \cdots \land n_k \in A_k\). By Lemma 1, \(\text{SAT}_{\mathcal{RQ}}\) terminates on that conjunction. The processing of this conjunction either terminates in false, and so \(\text{SAT}_{\mathcal{RQ}}\) stops, or it yields a conjunction of the form:

\[
(v \land i=1 \land X_i = t_i) \land (w \land i=1 \land Y_i = u_i) \tag{30}
\]

where \(X_i \in V_S\), \(Y_i \in V_X\), \(t_i\) are set terms and \(u_i\) are \(\lambda\) terms.

If some \(B_j\) in \(\bar{B}\) is either \(X_i\) or \(\{\cdots \cup X_i\}\), then \(X_i\) is substituted by \(t_i\). This rewrites \(\forall(x_m \in \bar{B}_m, \phi)\) into \(\forall(x_m \in \bar{B}_m', \phi)\). Then we have the following formula:

\[
(v \land i=1 \land X_i = t_i) \land (w \land i=1 \land Y_i = u_i) \land \forall(x_m \in \bar{B}_m', \phi) \tag{31}
\]

By Theorem 3 \(\text{SAT}_{\mathcal{RQ}}\) terminates on \(\forall(x_m \in \bar{B}_m', \phi)\). While processing the RUQ, \(\text{SAT}_{\mathcal{RQ}}\) can only generate either RUQ or \(\lambda\)-formulas (Remark 4). Then, the main loop of \(\text{SAT}_{\mathcal{RQ}}\) terminates. It only remains to call \(\text{SAT}_{\lambda}\) on the \(\lambda\)-subformula, which will terminate under the assumption that it is a decision procedure. Hence, \(\text{SAT}_{\mathcal{RQ}}\) terminates on (28).

Now we prove that \(\text{SAT}_{\mathcal{RQ}}\) terminates on a conjunction of constraints such as (28). We can think that \(\text{SAT}_{\mathcal{RQ}}\) will process each such constraint by going through formulas (29)-(31). Processing each final RUQ can only generate either RUQ or \(\lambda\)-formulas. Then, the main loop of \(\text{SAT}_{\mathcal{RQ}}\) terminates.

Now we consider a more general subclass of \(\Phi_{\text{ rq}}\)-formulas which, however, must obey a restriction concerning the domains of REQ that go after RUQ in mixed RQ (some times called alternating quantifiers [31]).

We say that an RQ has a variable domain if its domain is either a variable or an extensional set whose set part is a variable. Moreover, we say that the variable...
of the domain is the domain itself (if it is a variable) or its set part. For example, in:

\[
\text{foreach}(x \in \{h \cup A\}, \exists y \in \hat{B}, \phi(x, y))
\]

\(\{h \cup A\}\) is a variable domain whose variable is \(A\), and \(\hat{B}\) is variable domain whose variable is \(B\).

The class of formulas we are about to define will avoid formulas such as the one in Example 4. The problem with that formula is that there is an \(\exists\) constraint after a \text{foreach} constraint with the same domain variable \((A)\). In this situation when the \text{foreach} constraint picks an element \((a)\) of its domain the \(\exists\) constraint hypothesizes the existence of a new element \((n)\) in \(A\) as to satisfy \(\phi\). As now \(n \in A\), the \text{foreach} constraint must pick \(n\) making the \(\exists\) constraint to hypothesize the existence of another new element in \(A\). This behavior may produce an infinite rewriting loop. In a sense, the \(\exists\) constraint feeds back the \text{foreach} constraint with new elements if they have the same domain variable. This problem can be generalized to conjunctions of RQ.

**Example 5** The following formula:

\[
\text{foreach}(x \in \{h \cup A\}, \exists y \in \hat{B}, \phi(x, y)) \\
\land \text{foreach}(z \in B, \exists w \in \{b \cup A\}, \psi(z, w))
\]

may produce an infinite loop even though the \text{foreach} and \(\exists\) constraints sharing the same domain variable \((A)\) are in different RUQ. Still, in a sense, the \(\exists\) constraint with domain variable \(A\) is after the \text{foreach} constraint with the same domain variable: from the \(A\) in the \text{foreach} we go to the \(B\) in the \(\exists\) constraint, from this we go to the \(B\) in the \text{foreach} which leads us to the \(A\) in the \(\exists\).

Therefore, the mathematics we are going to define are meant to characterize formulas such as those in Examples 4 and 5.

Let \(\phi_1 \land \cdots \land \phi_n\) be a conjunction of nested RQ. Each RQ in a nested RQ is indexed by its position in the chain. For instance, in (32) the \text{foreach} constraint has index 1 while the \(\exists\) constraint has index 2. For each \(\phi_i\) build the function, called domain function of \(\phi_i\), whose ordered pairs are of the form \((i, j), (Q^i_j, D^i_j)\) where:

- A pair with first component \((i, j)\) is in the domain function of \(\phi_i\) iff the RQ with index \(j\) in \(\phi_i\) has a variable domain.
- \(D^i_j\) is the domain variable of the RQ with index \(j\) in \(\phi_i\).
- \(Q^i_j\) is \(\forall\) if the \(j\) RQ is a \text{foreach} constraint and is \(\exists\) if the RQ is an \(\exists\) constraint, in \(\phi_i\).

Hence, the domain function of the formula of Example 5 is:

\[\{(1, 1), (\forall, A), (1, 2), (\exists, B), ((2, 1), (\forall, B)), ((2, 2), (\exists, A))\}\]

From the domain functions build a directed graph, called domain graph, whose nodes are the ordered pairs of the domain functions. The edges are built as follows:

1. If \((i, j), (\forall, D^i_j)\) and \((i, k), (\exists, D^k_j)\), with \(j < k\), are in a domain function, then \((i, j), (\forall, D^i_j) \rightarrow ((i, k), (\exists, D^k_j))\) is an edge of the domain graph.
2. If \(((i,j), (\exists, D))\) and \(((b,a), (\forall, D))\) are in domain functions with \(i \neq b\), then \(((i,j), (\exists, D)) \rightarrow ((b,a), (\forall, D))\) is an edge of the domain graph.

Hence, the domain graph of the formula of Example 5 is:

\[
((1,1), (\forall, A)) \rightarrow ((1,2), (\exists, B)), ((2,1), (\forall, B)) \rightarrow ((2,2), (\exists, A)), ((1,2), (\exists, B)) \rightarrow ((2,1), (\forall, B)), ((2,2), (\exists, A)) \rightarrow ((1,1), (\forall, A))
\]

Consider the domain graph of a conjunction of pure RQ-constraints. A path in the graph such as:

\[
((i_1,j_{i1}), (\forall, D)) \rightarrow ((i_1,j_{i2}), (\exists, D_1)) \rightarrow ((i_2,j_{i2}), (\forall, D_1)) \rightarrow ((i_2,j_{22}), (\exists, D_2)) \rightarrow \cdots \cdots \cdots \cdots \rightarrow ((i_n,j_{n1}), (\forall, D_{n-1})) \rightarrow ((i_n,j_{n2}), (\exists, D))
\]

where \(i_a \neq i_b\) if \(a \neq b\) for all \(a, b \in [1,n]\), is called a \(\exists\) loop. Note that the first and last domain variables in a \(\exists\) loop are the same (D).

We say that a conjunction of RQ is free of \(\exists\) loops if there are no \(\exists\) loop in its domain graph. \(\Phi_{\exists}\) is the set of conjunctions of RQ free of \(\exists\) loops. Note that \(\Phi_{\exists}\) includes all the \(\Phi_{n_{\forall}}\)-formulas where there are no variable domains.

**Theorem 6 (Termination on \(\Phi_{\exists}\) formulas)** The \(\text{SAT}_{RQ}\) procedure can be implemented as to ensure termination for every formula in \(\Phi_{\exists}\).

**Proof** First we prove that \(\text{SAT}_{RQ}\) terminates on an atomic formula, \(\psi\). It starts by removing from \(\psi\) all the leading \(\exists\) constraints (if any) and then proceeding as in Theorem 5. Hence, we get a formula such as (31):

\[
(\bigwedge_{i=1}^{v} X_i = t_i) \land (\bigwedge_{i=1}^{u} Y_i = u_i) \land \text{foreach}(\vec{y}_m \in \vec{B}_m', \Phi)
\]

but where \(\Phi\) is a mixed RQ (eventually ending in a \(\forall\)-formula) whose domains might have been changed during the substitution step (see proof of Theorem 5).

Now, \(\text{SAT}_{RQ}\) processes the \(\text{foreach}\) constraint as in Theorem 3. Here, though, we cannot easily conclude that RUQ processing cannot bind a value to \(\vec{B}'\) because after the leading RUQ there might be some REQ. The problem with REQ is that they generate constraints of the form \(n \in A\), for some domain \(A\). Then, if the variable of \(A\) happens to be a variable in \(\vec{B}'\) (or in some RUQ in \(\Phi\)) we will have an infinite loop as in Examples 4 and 5. However, since \(\psi\) belongs to \(\Phi_{\exists}\), we know that there is no domain variable shared between a RUQ and a REQ ahead of it because \(\psi\) is free from \(\exists\) loops. Hence, we can arrive at the same conclusion of Theorem 3 meaning that \(\text{SAT}_{RQ}\) terminates on \(\psi\).

Now we prove that \(\text{SAT}_{RQ}\) terminates on a conjunction of constraints such as \(\psi\). Again, all the leading \(\exists\) constraints (if any) are removed from each \(\psi_i\) thus generating a formula such as (31) but with a conjunction of \(\text{foreach}\) constraints:

\[
(\bigwedge_{i=1}^{v} X_i = t_i) \land (\bigwedge_{i=1}^{u} Y_i = u_i) \land \left(\bigwedge_{i=1}^{q} \text{foreach}_i(\vec{y}_m \in \vec{B}'_m, \Phi)\right)
\]

\(^4\text{foreach}_i\) means that all its elements are renamed accordingly: \(\vec{y}_{i_{m_i}}, \vec{B}_{i_{m_i}}, \Phi_i\).
with \(0 \leq v, w, q\).

As above, \(\text{SAT}_{RQ}\) processes all the \(\text{foreach}_i\) (as in Theorem 3) and, again, the problem are the \(n \in A\) constraints (\(A\) a variable domain) that might be generated by the possible REQ present in each \(\Phi_i\). Differently from the base case, here a \(n \in A\) constraint generated when \(\text{foreach}_a\) is processed might affect a domain variable of a \(\text{foreach}\) constraint present in \(\text{foreach}_b\) with \(a \neq b\), as shown in Example 5. However, since the formula belongs to \(\Phi_{\forall\exists}\) we know is free from \(\forall\exists\) loops. This includes loops starting with a \(\text{foreach}\) domain variable in \(\text{foreach}_a\) and ending with the same domain variable in an \(\exists\) constraint present in \(\text{foreach}_b\). Then, no \(n \in A\) constraint can affect a \(\text{foreach}\) domain variable. Therefore, \(\text{SAT}_{RQ}\) terminates. \(\Box\)

We close this section with the following two observations.

Remark 6 \(\text{SAT}_{RQ}\) terminates for some formulas in \(\Phi_{\forall\exists} \setminus \Phi_{\forall\exists}\). For example, the following is a slight variation of the formula of Example 4:

\[
\text{foreach}(x \in \hat{A}, \exists y \in \{b \sqcup \hat{A}\}, \phi(x, y)))
\]

for which \(\text{SAT}_{RQ}\) trivially terminates because the formula is irreducible given that the domain is a variable (rule (11) applies). We could have tighten the definition of \(\Phi_{\forall\exists}\) as to include this kind of formulas. However, we consider that these formulas do not constitute a proper subclass as termination depends on whether or not some RUQ remain irreducible throughout the constraint solving procedure. \(\Box\)

Remark 7 \(\mathcal{L}_{RQ}\) has been designed by imposing some restrictions on its fundamental elements, namely: \(V_S \cap V_X = \emptyset\) and \(\Pi_S \cap \Pi_X = \emptyset\). These restrictions are used to prove Theorems 2-6. However, they can be relaxed to some extent as to accept a wider class of theories as the parameter for \(\mathcal{L}_{RQ}\).

It is possible to accept an \(\mathcal{L}_X\) such that \(V_S \cap V_X \neq \emptyset\) and \(\Pi_S \cap \Pi_X \neq \emptyset\) provided the solutions returned by \(\text{SAT}_X\) are compatible with the irreducible form of Definition 3.3. That is, if \(\text{SAT}_X\) returns, as part of its solutions, a conjunction of constraints including set variables, this conjunction must be satisfiable by substituting all set variables by the empty set. This would be enough as to prove Theorem 2.

Along the same lines, if \(\Pi_X\) contains \(\in\) then termination of \(\text{SAT}_{RQ}\) might be compromised as \(\text{SAT}_X\) might generate \(\in\)-constraints where the right term is the domain of a \(\text{foreach}\) (much as when an \(\exists\) is after a \(\text{foreach}\)). This can be generalized to any predicate symbol in \(\Pi_X\) that can bind values to set terms. In this case Theorems 3-6 can be proved if \(\text{foreach}\) domains are not affected by the constraints generated by \(\text{SAT}_X\).

As we will shown in Section 6 there are expressive \(\mathcal{L}_X\) such that \(V_S \cap V_X \neq \emptyset\) and \(\Pi_S \cap \Pi_X \neq \emptyset\) for which \(\text{SAT}_{RQ}\) is still an effective solver. \(\Box\)

5 Avoiding Existential Variables Inside RQ

All the considerations made in this section concerning RUQ apply equally to REQ. The concepts introduced here are adapted from those developed by us for RIS [18, Section 6].
Assume $R$ is a set of ordered pairs. We can try to write a formula stating that $R$ is the identity relation:

$$\text{foreach}(x \in R, x = (e, e))$$

(33)

where $e$ is intended to be a variable existentially quantified inside the RUQ. As defined in Section 2, $L_{RQ}$ does not allow to introduce these variables and so (33) would not be one of its formulas. Besides, it is not clear what a quantification domain could be for $e$. However, the following is an $RQ$-formula stating the same property:

$$\text{foreach}((x, y) \in R, x = y)$$

Note that we use an ordered pair as the control term (see Definition 4). Precisely, allowing (nested) ordered pairs as control terms makes it possible to avoid many existential variables inside RUQ. As binary relations are a fundamental concept in Computer Science [35,6], the introduction of ordered pairs as control terms is sensible as it enables to quantify over binary relations, without introducing existential variables.

Even if we allow (unrestricted) existential variables inside RQ, it is important to avoid them because the negation of such a RQ would not be a $L_{RQ}$ formula. Indeed, if there are existential variables inside the RQ the negation will introduce a universally quantified formula, which is not a $L_{RQ}$ formula. For instance, assuming $R$ is a set of numeric ordered pairs, the following is a predicate stating that the sum of any of its elements is greater than $z$:

$$\text{foreach}((x, y) \in R, \text{sum}(x, y, n) \land z < n)$$

(34)

where $\text{sum}(x, y, n)$ is interpreted as $n = x + y$, and $n$ is intended to be a variable existentially quantified inside the RUQ. Here, control terms do not help and, again, we do not have a quantification domain for $n$. Then, the negation of this formula would inevitably introduce a universal quantification. Furthermore, it would negate $\text{sum}(x, y, n)$ for all $n$ which would mean that there is no result for $x + y$.

Hence, the language is extended by introducing a $\text{foreach}$ constraint of arity 4:

$$\text{foreach}(x \in A, [e_1, \ldots, e_n], \phi(x, e_1, \ldots, e_n), \psi(x, e_1, \ldots, e_n))$$

(35)

where $e_1, \ldots, e_n$ are variables implicitly existentially quantified inside the RUQ and $\psi$ is a conjunction of so-called functional predicates. A predicate $p$ of arity $n + 1$ ($0 < n$) is a functional predicate iff for each $x_1, \ldots, x_n$ there exists exactly one $y$ such that $p(x_1, \ldots, x_n, y)$ holds; $y$ is called the result of $p$. For instance, $\text{sum}$ is a functional predicate. In an extended RUQ, $e_1, \ldots, e_n$ must be the results of the functional predicates in $\psi$.

The semantics of (35) is:

$$\forall x(x \in A \Rightarrow (\exists e_1, \ldots, e_n(\psi(x, e_1, \ldots, e_n) \land \phi(x, e_1, \ldots, e_n))))$$

whereas its negation is:

$$\exists x(x \in A \land (\exists e_1, \ldots, e_n(\psi(x, e_1, \ldots, e_n) \land \neg \phi(x, e_1, \ldots, e_n))))$$

given the functional character of $\psi$ [18, Section 6.2]. By means of functional predicates the introduction of existential variables inside RUQ is harmless while the expressiveness of the language is widened.
Example 6  Formula (34) should be written by means of an extended RUQ:

\[
\text{foreach}( (x, y) \in R, [n], z < n, \text{sum}(x, y, n))
\]

Note that \( n \) is the result of \( \text{sum} \). The negation of the above formula is:

\[
\text{exists}( (x, y) \in R, [n], z \geq n, \text{sum}(x, y, n))
\]

which is consistent with the intended meaning of the original formula. ⊓ ⊔

6 \( \mathcal{LRQ} \) in Practice

\( \mathcal{LRQ} \) and \( \text{SAT}_{\mathcal{LRQ}} \) have been implemented as part of the \{log\} (‘setlog’) tool [46]. \{log\} is a constraint logic programming (CLP) language implemented in Prolog. It also works as a satisfiability solver (and thus as an automated theorem prover) for a few theories rooted in the theory of finite sets. \{log\} and the theories underlying it have been thoroughly described elsewhere [28, 25, 16, 18, 22, 23, 21]. Empirical evidence of the practical capabilities of \{log\} has been provided as well [24, 17, 20, 15, 19].

Theory \( X \) in \{log\} is the combination between the theories known as \( \mathcal{LIA} \) and \( \mathcal{BR} \). \( \mathcal{LIA} \) stands for linear integer arithmetic and implements a decision procedure for systems of linear equations and disequations over the integer numbers. \( \mathcal{BR} \) stands for binary relations and implements a decision procedure for an expressive fragment of finite set relation algebra (RA) [16]. In \( \mathcal{BR} \), binary relations are sets of ordered pairs and all the RA operators are available as constraints, namely: union \( (C = A \cup B \to \text{un}(A,B,C)) \), intersection \( (C = A \cap B \to \text{inters}(A,B,C)) \), identity relation over a set \( (\text{id}(A) = R \to \text{id}(A,R)) \), converse of a binary relation \( (R^- = S \to \text{inv}(R,S)) \) and composition \( (T = R \circ S \to \text{comp}(R,S,T)) \). These operators can be combined in \( \mathcal{BR} \)-formulas to define many other operators such as: domain \( (\text{dom } R = A \to \text{dom}(R,A)) \) and range \( (\text{ran } R = A \to \text{ran}(R,A)) \) of a binary relation, a predicate constraining a binary relation to be a function \( (\text{pfun}(R)) \), function application \( (F(X) = Y \to \text{applyTo}(F,X,Y)) \), etc.

\( \mathcal{LIA} \) satisfies all the restrictions discussed in Remark 7, but \( \mathcal{BR} \) does not. However, the solutions returned by \{log\} when solving \( \mathcal{BR} \)-formulas are compatible with the irreducible form of Definition 10 [16, Definition 15 and Theorem 3]. That is, irreducible formulas in \( \mathcal{BR} \) are satisfied by substituting all the set and relational variables by the empty set. However, when a \( \mathcal{BR} \)-formula is processed, RQ domains may be affected. As discussed in Remark 7 this may compromise termination; \( \mathcal{LRQ}(\mathcal{LIA} + \mathcal{BR}) \) decidability is discussed more deeply in Section 7. Nevertheless, as the following case studies show, \{log\} is still an effective and efficient tool to automatically reason about \( \mathcal{LRQ}(\mathcal{LIA} + \mathcal{BR}) \) formulas. Termination do not seem to be an issue for many classes of practical problems expressible in \( \mathcal{LRQ}(\mathcal{LIA} + \mathcal{BR}) \).

\{log\}’s concrete syntax is a slight variation of the syntax used in this paper: \{_\cup_\} is \{_/_/\}; \{_\_\} is \{_/_\}; \in is \text{in}; \land is \&; \( A \subseteq B \) is \text{subset}(A,B); variables begin with a capital letter.

The following simple example shows \{log\} syntax and how to use it to prove invariance lemmas.
Example 7 Let $\text{Usr}$ and $\text{Admin}$ be the sets of users and administrators of some system. Let us say that the security policy requires these sets to be disjoint. We can express that in \{\text{log}\} as follows:

$$\text{inv(Usr, Admin)} : - \text{foreach([U in Usr, A in Admin], U neq A)}.$$ 

We can model the operation adding user $X$ to $\text{Usr}$ yielding $\text{Usr}_-$ as the new set:

$$\text{addUsr(Usr, Admin, X, Usr_, Admin_) : - Usr_ = \{X / Usr\} & Admin_ = Admin}.$$ 

We would like to know if $\text{addUsr}$ preserves $\text{inv}$, so we run the following query:

$$\text{neg(\text{inv(Usr, Admin)} \& \text{addUsr(Usr, Admin, X, Usr_, Admin_) implies inv(Usr_, Admin_)})}.$$ 

As $\text{addUsr}$ fails to preserve $\text{inv}$, \{\text{log}\} provides a counterexample ($N$ new variable): $\text{Admin} = \{X / N\}$, $\text{Usr}_- = \{X / \text{Usr}\}$, $\text{Admin}_- = \{X / N\}$.

So we can fix $\text{addUsr}$ by adding a pre-condition:

$$\text{addUsr(Usr, Admin, X, Usr_, Admin_) : - X \in \text{Usr} & Usr_ = \{X / Usr\} & Admin_ = Admin}.$$ 

Now the answer to the query is no meaning that the formula is unsatisfiable.

The next three subsections present real-world case studies where \{\text{log}\} is used as a CLP language and as an automated verifier. The focus is on how RQ are used.

6.1 The Landing Gear System

In the fourth edition of the ABZ Conference held in Toulouse (France) in 2014, Boniol and Wiels proposed a real-life, industrial-strength case study, known as the Landing Gear System (LGS) [9]. Mammar and Laleau [39] developed an Event-B [1] specification of the LGS, formally verified using Rodin [2], ProB [37] and AnimB\(^7\). Basically, we encoded in \{\text{log}\} the Event-B specification and used \{\text{log}\} to automatically discharge all the proof obligations generated by Rodin. This work is thoroughly described elsewhere [19].

This is the simplest model in terms of RQ as it does not require nested RQ. A typical use of RQ in the LGS is the following:

$$\text{ta_inv5(Positions, DULDC)} : -$$

\begin{align*}
\text{pfun(DULDC) \&}
\text{dompf(DULDC, Positions) \&}
\text{foreach([X, Y] in DULDC, 0 =< Y).}
\end{align*}

\(^5\) Other encodings are possible; we deliberately choose to use a RUQ.
\(^6\) Given that \{\text{log}\} is a satisfiability solver we call it on the negation of the lemma waiting for a no (i.e., false) answer.
\(^7\) [\text{http://www.animb.org}]
\(^8\) \{\text{log}\} code of the LGS: [\text{http://www.ciplset.unipr.it/SETLOG/APPLICATIONS/lgc.zip}].
\(^9\) Some variable names are changed to save some space.
That is, ta_inv5 defines a state invariant corresponding to the Event-B machine named TimedAspects. In mathematical notation the invariant states \( DULDC \in Positions \rightarrow N \). As in \{log\} we cannot express \( N \) we use a RUQ to ascertain that the second component of each element in \( DULDC \) is non-negative. Then, invariance lemmas such as:

\[
\text{neg}(di_inv1(Positions,Dcp) \land \text{ta_inv1}(CT) \land \text{ta_inv5}(Positions,DULDC) \land \text{ta_make_DoorClosed}(...,Dcp,...,CT,...,DULDC,...,Dcp_,...,DULDC_)) \implies \text{ta_inv5}(Positions,DULDC_)).
\]

are automatically discharged by \{log\}. \( di_inv1(Positions,Dcp) \) and \( \text{ta_inv1}(CT) \) are other invariants that are needed as hypothesis and \( \text{ta_make_DoorClosed} \) is one of the state transitions of the LGS model (ellipses stand for variables). Hence, \( \text{ta_make_DoorClosed} \) changes the value of \( DULDC \) during the state transition and so we need to check that \( \text{ta_inv5} \) is still valid in the new state. \{log\} discharges all the 465 proof obligations in less than 5 minutes.

6.2 The Bell-LaPadula Security Model

Around 1973 D.E. Bell and L. LaPadula published the first formal model of a secure operating system [3,4]. Today this model is known as the Bell-LaPadula model, abbreviated as BLP. BLP is described as a state machine by means of first-order logic and set theory. The model also formalizes two state invariants known as security condition and \( *\)-property. We encoded BLP and its properties in \{log\} and used it to automatically discharge all the invariance lemmas. This work is presented with detail elsewhere [17].

The following is the \{log\} encoding of the \( *\)-property:

\[
\text{starProp(State)} : - \\
\text{State = } [[[br,Br],[bw,Bw],[fo,Fo],[fs,Fs],[m,M]] \land \text{foreach([S1,O1] in Br, [S2,02] in Bw), [Sco1,Sco2]}, \\
\text{S1 = S2 implies dominates(Sco1,Sco2),} \\
\text{applyTo(Fo,O1,Sco1) \& applyTo(Fo,O2,Sco2)).}
\]

As can be seen it requires the use of the extended version of nested RUQ (Section 5). That is, it declares two existential variables inside the RUQ (Sco1 and Sco2) and uses the functional predicate section (applyTo(Fo,O1,Sco1) \& ...). Extended RUQ are also used in the state transitions, for instance:

\[
\text{getRead(State,S,O,State_)} : - \\
\text{State = } [[[br,Br],[bw,Bw],[fo,Fo],[fs,Fs],[m,M]] \land \text{[O,[S,read]] in M \& [S,0] min Br \&} \\
\text{applyTo(Fo,0,Sco) \& applyTo(Fs,S,Scs) \&} \\
\text{dominates(Sco,Scs) \&} \\
\text{foreach([S1,O1] in Bw,[Sco1],} \\
\text{S1 = S implies dominates(Sco1,Scoi),applyTo(Fo,Oi,Scoi)) \&} \\
\text{Br_ = } {[[S,0]/Br]} \& \\
\text{State_ = } [[[br,Br_],[bw,Bw],[fo,Fo],[fs,Fs],[m,M]]}.
\]

\( {log} \) code of BLP: http://www.clpset.unipr.it/SETLOG/APPLICATIONS/blp2.zip.
getRead grants read permission to subject S on object O in which case changes the value of variable Br. Then, the following invariance lemma must be proved:

$$\neg(\starProp(State) \land \text{getRead}(State, S, O, State_) \implies \starProp(State_)).$$

Due to an optimization introduced since our first experiments with BLP, now $\{\log\}$ proves all the 60 invariance lemmas in less than 2 seconds instead of the 11.5 seconds reported previously [17].

6.3 Android’s Permission System

In a series of articles a group of Uruguayan and Argentinian researchers and students developed a certified Coq model and implementation of Android’s permission system [8, 7, 38, 26]. They model the system as a state machine, then propose a number of properties and use Coq to verify them against the model. Properties are classified in two classes: valid state properties and security properties. The first class ensures the state machine preserves some well-formedness properties of the state variables, while the second ensures Android behaves as expected in some security-related scenarios. As with the previous case studies, we translated the Coq model into $\{\log\}$ and used it to automatically prove properties. This is the most challenging case study we have developed so far. It takes $\{\log\}$ to its limits concerning reasonable computing times to discharge proof obligations. It also uses the most complex nested RQ we have used so far, as the following one which formalizes one of the valid state properties.

$$\text{notDupPerm}(DP) :-$$

$$\forall [A1, SP1] \in DP, [A2, SP2] \in DP,$$

$$\forall [P1 \in SP1, P2 \in SP2], [IP1, IP2],$$

$$\text{idP}(P1, IP1) \land \text{idP}(P2, IP2).$$

Note that there is one nested RUQ whose filter is an extended nested RUQ. Furthermore, in the innermost RUQ SP1 and SP2 are domains whereas they are part of the outermost control term. That is, first $[A1, SP1]$ quantifies over DP and then P1 quantifies over SP1.

The following is another state consistency property fitting in the $\Phi_{\forall\exists}$ subclass.

$$\text{permsDom}(PR, Apps, SS) :-$$

$$\forall [A, P] \in PR,$$

$$A \in Apps \lor \exists (SI \in SS, [IA], IA = A, \text{idSI}(SI, IA)).$$

As can be seen, the domain of the RUQ does not appear inside the REQ, thus making the formula free of $\forall\exists$ loops. The REQ uses a functional predicate.

$\{\log\}$ automatically discharges 801 proof obligations in around 22 minutes.

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11 $\{\log\}$ code of Android 10’s permission system: [http://www.clpset.unipr.it/SETLOG/APPLICATIONS/android.zip](http://www.clpset.unipr.it/SETLOG/APPLICATIONS/android.zip)

12 Actually, $\{\log\}$ is unable to prove only three of the properties proposed in the Coq model.

13 In the following formulas some simplifications are introduced to save some space.
7 Discussion and Related Work

The problem of deciding the satisfiability of quantified formulas is obviously undecidable. Hence, we can only hope to find expressive fragments that are decidable. A possible path for this is to restrict the form of the quantifiers and another is to allow only certain quantifier-free formulas. A class of quantified formulas that has been studied for many years is that of restricted quantifiers. However, the full fragment of restricted quantifiers as well as some of its sub-fragments are undecidable (e.g. [10, 43, 31]). Hence, further restrictions must be imposed. One of such restrictions is to deal with formulas where RUQ are after REQ. This fragment has been proved to be decidable in different contexts [10, 42, 31]. We started by working with quantifier-free formulas that do not affect quantification domains (Definition 1, Remark 7). We have not seen other works taking this path. Then, we relaxed that restriction loosing termination but nonetheless gaining expressiveness. This combination seems to be useful in practice (Section 6).

We identify Computable Set Theory (CST) [11, 14] as the main and closest source of works related to the one presented in this paper. However, there are works outside CST dealing with similar problems, specially in the realm of STM solvers. We start with the latter.

The STM solving community deals with unrestricted quantifiers. The usual practical technique employed in SMT solvers to deal with quantified formulas is heuristics-based quantifier instantiation [32, 30, 40, 33]. In particular, Simplify’s E-matching algorithm [27] is used by some of these tools. Heuristic instantiation manages to solve problems of software verification. However, it suffers from some shortcomings as stated by Ge and de Moura [34]. For this reason, Ge and de Moura propose some decidable fragments of first order logic modulo theories. The proposed decision procedures can solve complex quantified array properties. The authors show how to construct models for satisfiable quantified formulas in these fragments.

In a more recent work, Feldman et al. [31] study the problem of discharging inductive invariants with quantifier alternation using SMT solvers. They depart from formulas belonging to the Effectively Propositional logic (EPR), also known as the Bernays-Schönfinkel-Ramsey class. In this logic, formulas are of the form $\exists^*\forall^*(\delta)$, where $\delta$ is a quantifier-free formula over some first-order vocabulary. This logic has been proved to be decidable and useful in automatically discharging verification conditions of software involving linked-lists, distributed protocols, etc. Feldman and his colleagues then go to extend EPR with formulas of the form $\forall^*\exists^*(\delta)$. The first conclusion they get is that this fragment is undecidable. However, a second conclusion is that some techniques can be put to work as to solve many interesting problems in that fragment. The main technique is instantiations that are bounded in the depth of terms. However, bounded instantiations guarantee termination a-priori even when the invariant is not correct. In these cases the algorithm returns an approximated counterexample. The invariants approached by Feldman et al. are of the same form of most of the proof obligations present in our case studies.

As can be seen, the SMT solving community approaches the problem of finding decision procedures for quantified fragments of logic languages in a quite different way as we do. They do not use RQ nor a theory of sets. RQ have an interesting property: $\forall x \in A : \phi$, with $A$ a variable, is satisfied with $A = \emptyset$. If this is combined
with a set constructor such as \{·⊔·\}, it is possible to iterate over the elements of the quantification domain until the ‘end’ is reached: if it is ∅, then the quantifier can be eliminated; if it is a variable, then the iteration can be stopped because we know that we have a good candidate solution for the quantification domain. Quantification domains are crucial to find out a decision procedure for formulas where REQ are after RUQ. As we have shown in Theorem 6, REQ with the same domain variable than a preceding RUQ, in general, generate infinite feedback loops. These loops can be easily detected by following the flow of hypothesized elements through quantification domains. We believe all this is harder to see when the language admits general quantifiers and is not based on a theory of sets. At the same time, RQ do not pose a threat on expressiveness when it comes to software verification. Finally, concerning counterexample generation, within the decidable fragments presented in this paper, \{log\} is not only always able to generate a counterexample of any given satisfiable formula but it (interactively) generates a finite representation of all its solutions (Theorem 1).

Our work is closer to CST. CST has been looking for decidability results on quantified fragments of set theory since at least forty years ago. In many cases, CST is interested in proving decidability results (in the form of satisfiability tests) but not so much in providing efficient algorithms or in implementing them in some software tool. Breban et al. [10] present a semi-decision algorithm for a wide class of quantified formulas where the quantifier-free theory is decidable. In this work, quantifiers are RQ but no quantified variable can be a quantified domain of a deeper RQ (see (†) below). For some theories the algorithm becomes complete. In general, the quantifier-free theories are sub-languages of set theory. In particular they consider a language based on \{=, ∈\}. The resulting quantified language allows to express many set-theoretic operators (e.g., union). From that article, several researchers of the CST community have found a number of (un)decidability results about different fragments of quantified languages of set theory [45, 43, 41, 5, 42].

As can be seen, the decidability results represented by Theorems 1-5 have already been proved. On the contrary, we believe the result of Theorem 6 is new. Besides, as far as we understand, all of our results are new in terms of the algorithm we use and in particular the set of rewrite rules we use, not to mention the fact that we put these results to work in a software tool that is able to solve real-world problems (Section 6).

More recently, Cantone and Longo [12, 13] worked on the language \∀π0,2, part of Cantone’s long work on CST. \∀π0,2 helps to analyze the decidability and expressiveness of \LQRQ(\LIA + \BR).14 \∀π0,2 is a two-sorted quantified fragment of set theory allowing the following literals: \forall x \in y, (x,y) \in R, x = y and R = S, where x and y are set variables and R and S are variables ranging over binary relations. Note that in \∀π0,2 sets are pure meaning that their elements are sets where the empty set is the base element (semantics of \∀π0,2 is given in terms of the von Neumann standard cumulative hierarchy of sets). Formulas in \∀π0,2 are Boolean combinations of expressions of the following two forms: \forall x_1 \in z_1 : \ldots \forall x_h \in z_h : \forall(x_{h+1}, y_{h+1}) \in R_{h+1} : \ldots \forall(x_n, y_n) \in R_n : \delta, and the same expression where \forall is replaced by \exists. In these expressions: \delta is a propositional combination of \∀π0,2-literals; x_i, y_i, z_i are set variables; R_i are binary relation variables; and (†) no \forall x_i or \exists y_i can also occur as

14 The same could be achieved by using as a reference the work on CST by Breban et al. [10]. We opted by Cantone’s because is newer. Nonetheless, Breban’s is also duly referenced.
a $z_1$ (i.e., no quantified variable can occur also as a domain variable in the same quantifier prefix). Note that in $\forall_{0.2}^\pi$ RUQ and REQ cannot be mixed in the same expression.

$\forall_{0.2}^\pi$ is a decidable language which allows to express all the operators of RA with the exception of composition. Indeed, $\forall_{0.2}^\pi$ only allows to express $R \circ S \subseteq T$ but the other inclusion cannot be written. The impossibility to express the other inclusion comes from the fact that RUQ and REQ cannot be mixed in the same expression, which is tantamount to preserve decidability of $\forall_{0.2}^\pi$. In effect, $T \subseteq R \circ S$ is equivalent to:

$$\forall (x, z) \in T : (\exists (x_1, y_1) \in R : (\exists (y_2, z_1) \in S : x_1 = x \land y_1 = y_2 \land z_1 = z))$$

which is not a $\forall_{0.2}^\pi$ formula (as it mixes RUQ and REQ).

Now we analyze the decidability and expressiveness of $L_{\forall \exists}(L_{\forall \exists} + BR)$ in terms of $\forall_{0.2}^\pi$:

1. $\forall_{0.2}^\pi$ sets are not necessarily finite; $L_{\forall \exists}(L_{\forall \exists} + BR)$ sets are finite. However, since we are interested in software verification this is not a real restriction.
2. $\forall_{0.2}^\pi$ is not a parametric language as $L_{\forall \exists}$, although other works on CST provide parametric languages in the line of $\forall_{0.2}^\pi$ [10]. Parametrization of $L_{\forall \exists}$ enables hybrid sets.
3. $\forall_{0.2}^\pi$ sets are pure, while $L_{\forall \exists}(L_{\forall \exists} + BR)$ sets are hybrid. Pure sets allow to encode ordered pairs, natural numbers, etc. However, these encodings tend to reduce the efficiency of solvers. Working directly with hybrid sets facilitates the integration with efficient solvers for other theories, such as $L_{\forall \exists}$. For instance, the formula in Example 1 encodes the minimum of a set which would require a complex $\forall_{0.2}^\pi$ formula.
4. $L_{\forall \exists}(L_{\forall \exists} + BR)$ extends the decidability result of $\forall_{0.2}^\pi$. On one hand, $\forall_{0.2}^\pi$ almost expresses RA so if in $L_{\forall \exists}(L_{\forall \exists} + BR)$ composition is used as in $\forall_{0.2}^\pi$, the former is a fragment of the latter in what concerns to RA. On the other hand, $L_{\forall \exists}(L_{\forall \exists} + BR)$ allows to fully express composition by a suitable encoding of formula (36):

$$\forall x, z \in T : (\exists (x_1, y_1) \in R : (\exists (y_2, z_1) \in S : x_1 = x \land y_1 = y_2 \land z_1 = z))$$

As can be seen, this formula is free of $\forall \exists$ loops as long as the variable of $T$ is different from the variables of $R$ and $S$. Hence, Cantone and Longo go to far in restricting $\forall_{0.2}^\pi$ as the real problem with composition comes with formulas such as $R \subseteq R \circ S$ or $S \subseteq R \circ S$. This is aligned with our results concerning the decidability of $L_{BR}$ [16, Section 5.3, Definition 16]. Finding larger decidable fragments of RA is important as it is a fundamental theory in Computer Science due to its expressiveness [35, last paragraph Section 1]. Besides, as shown in Section 6.3 with formula $\text{permDom}$, allowing $\Phi_{0.2}$ formulas is useful in practice.

5. $L_{\forall \exists}(L_{\forall \exists} + BR)$ allows quantified variables to occur as domains in the same quantifier prefix (cf. (i) above). Recall, for instance, formula $\text{notDupPerm}$ in Section 6.3. These formulas are ruled out from $\forall_{0.2}^\pi$ because they compromise completeness, not soundness [10, Sections 2 and 4]. The problem is that there are formulas not adhering to (i) that are satisfied only by infinite sets when $\in$ is part of the quantifier-free theory [44]. So if we allow these formulas in $\log$ its answers are correct because the tool is still sound, although it will not
terminate for those formulas that are satisfied only by infinite sets. As shown in Section 6.3, allowing formulas not adhering to (1) is useful in practical cases.

6. $\forall^n\pi_0\exists^n\pi_2$ is proved to be decidable by encoding each of its formulas as a $\forall^n\pi_0$-formula. In turn, $\forall^n\pi_0$ is shown to be decidable by means of the notions of skeletal representation and of its realization [13]. The authors do not provide an algorithm with an obvious operative semantics for the decidability problem of $\forall^n\pi_2$ formulas. Conversely, by adapting our results on RIS, we provide a simple and concrete solver for $L_{RQ}$ (i.e., $SAT_{RQ}$) with CLP properties easily implementable as part of \{log\}. In turn, we provide empirical evidence of its practical capabilities. The algorithms presented by Breban [10] are closer to $SAT_{RQ}$.

8 Final Remarks

We have presented a decision procedure for quantifier-free, decidable languages extended with restricted quantifiers. The decision procedure is based on a small collection of rewrite rules for primitive set-theoretic operators ($\subseteq$, $\in$, $=$). Although all but one of the decidability results underlying our decision procedure are not new, as far as we understand, the decision procedure and its rewrite rules are novel. The new decidability result concerns quantified formulas where a restricted existential quantifier comes after a restricted universal quantifier. The result is based on building a graph linking the universally quantified domains with those existentially quantified. Then, a path analysis is performed to find out whether or not there is a path from a universally quantified domain to the same domain but making part of an existential quantifier. Finally, the implementation of the decision procedure as part of a software tool (\{log\}) and its successful application to real-world, industrial-strength case studies as an automated software verifier provide empirical evidence of the usefulness of the approach.

Our strongest decidability results are possible by imposing some restrictions on the quantifier-free language—namely, that it does not include terms denoting sets. Although non-trivial languages fulfill these restrictions (e.g., linear integer arithmetic), the greatest expressiveness is reached when some of these restrictions are lifted at the expense of termination. Hence, as a future work we plan to study what quantifier-free languages preserve termination even though they support sets to some extent. In particular, the long and fruitful work on CST should help us in finding those languages.

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A Proofs

In the following, a set of the form \{P(x) : F(x)\} (where pattern and filter are separated by a colon (:) instead of a bar (|)), and the pattern is before the colon) is a shorthand for \{y : \exists x(P(x) = y \land F(x))\}. That is, the set is written in the classic notation for intensional sets used in mathematics.

**Proposition 1**
\[ \forall d,D : \{ x : (d \cup D) \mid \phi \circ u\} = \{u(d) : \phi(d)\} \cup \{u(x) : x \in D \land \phi(x)\} \]

*Proof* Taking any \(d\) and \(D\) we have:
\[\begin{align*}
\{x : (d \cup D) \mid \phi \circ u\} &= \{u(x) : x \in (d \cup D) \land \phi(x)\} \\
&= \{u(x) : (x = d \lor x \in D) \land \phi(x)\} \\
&= \{u(x) : (x = d \land \phi(x)) \lor (x \in D \land \phi(x))\} \\
&= \{u(x) : x \in (d \land \phi(x)) \cup (x \in D \land \phi(x))\} \\
&= \{u(d) : \phi(d)\} \cup \{u(x) : x \in D \land \phi(x)\}
\end{align*}\]

\(\square\)

**Lemma 2 (Equivalence of rule (10))**
\[ \forall t,A : t \notin A \Rightarrow \{t \cup A\} \cup \{x : \{t \cup A\} \mid \phi\} = \{x : \{t \cup A\} \mid \phi\} \]

*Proof* First note that
\[t \notin A \Rightarrow \{t\} \parallel A \land \{t\} \parallel \{A \mid \phi\} \]
and
\[\{x : \{t\} \mid \phi\} \subseteq \{t\} \]
and
\[\{x : \{t\} \mid \phi\} = \{t\} \Leftrightarrow \phi(t)\]

\[\begin{align*}
\{t \cup A\} \cup \{x : \{t \cup A\} \mid \phi\} &= \{x : \{t \cup A\} \mid \phi\} \\
\Leftrightarrow \{t\} \cup A \cup \{x : \{t\} \mid \phi\} \cup \{x : A \mid \phi\} &= \{x : \{t\} \mid \phi\} \cup \{x : A \mid \phi\} \quad \text{[by Prop. 1; semantics \(\cup\)]} \\
\Leftrightarrow \{t\} \cup A \cup \{x : A \mid \phi\} &= \{x : \{t\} \mid \phi\} \cup \{x : A \mid \phi\} \quad \text{[by (38); \{t\} in left-hand side]} \\
\Leftrightarrow \phi(t) \cup \{t\} \cup A \cup \{x : A \mid \phi\} &= \{t\} \cup \{x : A \mid \phi\} \quad \text{[by (39)]} \\
\Leftrightarrow \phi(t) \land A \cup \{x : A \mid \phi\} &= \{x : A \mid \phi\} \quad \text{[by (37); basic property of \(\parallel\) and \(\cup\)]}
\end{align*}\]