THE MULTIVARIATE COVERING LEMMA
AND ITS CONVERSE

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ABSTRACT. The multivariate covering lemma states that given a collection of
$k$ codebooks, each of sufficiently large cardinality and independently generated
according to one of the marginals of a joint distribution, one can with prob-
ability arbitrarily close to one choose one codeword from each codebook such
that the resulting $k$-tuple of codewords is jointly typical with respect to the
joint distribution. Prior proofs of the multivariate covering lemma primarily
employ strong typicality. We give a proof of this lemma for weakly typical sets.
This allows achievability proofs that rely on the covering lemma to go through
for continuous (e.g., Gaussian) channels without the need for quantization.
The covering lemma and its converse are widely used in information theory,
including in rate-distortion theory and in achievability results for multi-user
channels.

1. INTRODUCTION

The covering lemma and its extensions play a crucial role in achievability results
in network information theory. Covering lemmas are useful for enabling network
nodes to transmit codewords that "look like" they are generated from a dependent
distribution, whereas in reality, they are carefully selected from sufficiently large
codebooks that are independently generated. This allows nodes to obtain the ben-
efits of both independent and dependent codewords: like independent codewords,
such codewords can be decoded in different locations; like dependent codewords
they have the potential to achieve rates higher than those achieved by independent
codewords. This benefit, however, comes at a cost in rate. Thus the strategy is
useful when the benefit transmitting dependent codewords exceeds its cost.

In the context of the covering lemma, the concept of "looking like" dependent
codewords is captured by the notion of being jointly typical with respect to a
dependent distribution. As there are various ways to define the typical set (here
we specifically focus on weakly typical [2] and strongly typical sets [3]), one may
ask whether a specific version of the covering lemma holds for a given definition
of the typical set. The weakly typical set has two advantages over the strongly typical
set. First, it is easily defined for continuous (e.g., Gaussian) distributions. Second,
the weakly typical set has a simple one-shot counterpart, which allows proofs using
the weakly typical set to be written in the one-shot framework in a simple manner.
On the other hand, some results hold for the strongly typical set that do not hold
for the weakly typical set. Thus it is helpful to review the covering lemma and its
extensions and see for which definition of the typical set each result is currently
known to hold.

The simplest case of the covering lemma is the situation where given a random
vector and an independently generated codebook, a node looks for a codeword in
the codebook that is jointly typical (with respect to a dependent distribution) with the given random vector. The result obtained in this case, simply referred to as the “covering lemma”, appears in the achievability proof of the rate distortion theorem using weakly typical sets [2]. The second case, called the “mutual covering lemma,” treats the case where given two independently generated codebooks, a node looks for a jointly typical pair of codewords, where each codeword is from one of the codebooks. This result is used in Marton’s inner bound for the two-user broadcast channel and is proved for strongly typical sets [4,7]. Recently, by extending the proof of [2], the authors of [6,8] prove a one-shot version of the mutual covering lemma. This proof can be used to show the validity of the mutual covering lemma for weakly typical sets in the asymptotic setting. The proof in [6,8], however, requires stronger independence assumptions on the codebooks than the proof using strongly typical sets in [3,4]. Finally, the “multivariate covering lemma” is the extension of the mutual covering lemma to \( k \) independently generated codebooks, and can be used to obtain an inner bound on the broadcast channel with \( k \) users [3]. As stated in [3], one can show this result holds for strongly typical sets by extending the proof of the mutual covering lemma [4].

In this work, using the general strategy of El Gamal and Van der Meulen [4] and some ideas regarding weakly typical sets from Koetter, Effros, and Médard [5], we give a proof of the multivariate covering lemma for weakly typical sets. We also provide a converse, a special case of which is usually referred to as the packing lemma [3]. We remark that while similar to the argument in [4], we use Chebyshev’s inequality for the direct result (Section 1), it is also possible to use the Cauchy-Schwarz inequality (see Appendix A), which leads to a more accurate upper bound.

2. Problem Statement

For every positive integer \( n \), define the set \([n] = \{1, \ldots, n\}\). Let \( k \) be a positive integer and

\[
p(u_0, u_1, \ldots, u_k, u_{k+1})
\]

be a probability distribution on the set

\[
\prod_{j=0}^{k+1} U_j.
\]

For every nonempty \( S \subseteq [k] \) define

\[
U_S = \prod_{j \in S} U_j.
\]

For every \( j \in [k] \), let \( M_j \) be a nonnegative integer. For every nonempty \( S \subseteq [k] \), define the set \( M_S \) as

\[
M_S = \prod_{j \in S} [M_j],
\]

and let \( M = M_{[k]} \). For every \( m = (m_1, \ldots, m_k) \in M \), let the random vector

\[
(U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1})
\]

have distribution

\[
p(u_0) \prod_{j=1}^{k+1} p(u_j \mid u_0),
\]
where \( p(u_0) \) and each \( p(u_j | u_0) \) are the conditional marginals of \( p(u_0, \ldots, u_{k+1}) \). In addition, let \( \mathcal{F} \) be an arbitrary subset of \( \mathcal{U}_0 \times \mathcal{U}_{k+1} \). We want to find upper and lower bounds on the probability

\[
P \left\{ \forall \mathbf{m} \in \mathcal{M} : (U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1}) \notin \mathcal{F} \right\}.
\]

We derive the lower bound (Section 3) using the union bound, which does not depend on the statistical dependencies of the vectors

\[
(U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1})
\]

for different values of \( \mathbf{m} \). For the upper bound (Section 4), which leads to the multivariate covering lemma, we require a stronger assumption, which we next describe.

Let \( \mathbf{m} = (m_j)_{j \in [k]} \) and \( \mathbf{m}' = (m'_j)_{j \in [k]} \) be in \( \mathcal{M} \). Define the set \( S_{\mathbf{m}, \mathbf{m}'} \) as

\[
S_{\mathbf{m}, \mathbf{m}'} = \{ j \in [k] : m_j = m'_j \}.
\]

When \( \mathbf{m} \) and \( \mathbf{m}' \) are clear from context, we denote \( S_{\mathbf{m}, \mathbf{m}'} \) with \( S \). In the proof of the upper bound we require

\[
P \left\{ \forall j \in [k] : U_j(m_j) = u_j \text{ and } U_j(m'_j) = u'_j \mid U_0 = u_0, U_{k+1} = u_{k+1} \right\}
\]

\[
= \prod_{j=1}^k p(u_j | u_0) \times \prod_{j \in S} p(u'_j | u_0),
\]

for all \( u_0 \) and all \((u_j)_j \) and \((u'_j)_j \) such that if \( j \in S \), then \( u_j = u'_j \) (Assumption I). Note that if there exists a \( j \in S \) where \( u_j \neq u'_j \) then the probability on the left hand side equals zero.

In the corresponding asymptotic problem (Section 5), we apply our bounds to

\[
P \left\{ \forall \mathbf{m} : (U_0^n, U_1^n(m_1), \ldots, U_k^n(m_k), U_{k+1}^n) \notin A_{\delta}^{(n)} \right\},
\]

where for every \( \mathbf{m} \),

\[
(U_0^n, U_1^n(m_1), \ldots, U_k^n(m_k), U_{k+1}^n)
\]

is simply \( n \) i.i.d. copies of the original random vector

\[
(U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1}),
\]

(Assumption II) and \( A_{\delta}^{(n)} \) is the weakly typical set for the distribution \( p(u_0, u_1, \ldots, u_k, u_{k+1}) \). Our main result follows.

**Theorem 1** (Multivariate Covering Lemma). Suppose Assumptions (I) and (II) hold for the joint distribution of

\[
U_0^n, \{U_1^n(m_1), \ldots, U_k^n(m_k)\}_{\mathbf{m}}, U_{k+1}^n.
\]

For the direct part, suppose for all \( j \in [k], M_j \geq e^{nR_j} \). If for all nonempty \( S \subseteq [k] \),

\[
\sum_{j \in S} R_j > \sum_{j \in S} H(U_j | U_0) - H(U_S | U_0, U_{k+1}) + (8k - 2|S| + 10)\delta,
\]

then

\[
\lim_{n \to \infty} P \left\{ \exists \mathbf{m} : (U_0^n, U_1^n(m_1), \ldots, U_k^n(m_k), U_{k+1}^n) \in A_{\delta}^{(n)} \right\} = 1.
\]
For the converse, assume for all \( j \in [k] \), \( M_j \leq e^{nR_j} \). If Equation (3) holds, then
\[
\sum_{j \in S} R_j \geq \sum_{j \in S} H(U_j | U_0) - H(U_S | U_0, U_{k+1}) - 2(|S| + 1)\delta,
\]
for all nonempty \( S \subseteq [k] \).

In the direct part of Theorem 1, we can weaken the lower bound on \( \sum_{j \in S} R_j \) when \( S = [k] \). Specifically, we can replace Equation (1) with
\[
\sum_{j=1}^{k} R_j > \sum_{j=1}^{k} H(U_j | U_0) - H(U_{[k]} | U_0, U_{k+1}) + 2(k + 1)\delta.
\]
for \( S = [k] \).

3. THE LOWER BOUND

For every \( S \subseteq [k] \), define \( F_S \) as the projection of \( F \) on \( U_0 \times U_S \times U_{k+1} \). Then for every \((u_0, u_S, u_{k+1}) \in F_S\), let \( F(u_0, u_S, u_{k+1}) \) be the set of all \( u_{S'} \) such that \((u_0, u_{[k]}, u_{k+1}) \in F\). In addition, for every nonempty \( S \subseteq [k] \), let \( \alpha_S \) and \( \beta_S \) be constants such that
\[
\alpha_S \leq \log \frac{p(u_S | u_0, u_{k+1})}{\prod_{j \in S} p(u_j | u_0)}
\]
for all \((u_0, u_S, u_{k+1}) \in F_S\) and
\[
\beta_S \leq \log \frac{p(u_S | u_0, u_{S'}, u_{k+1})}{\prod_{j \in S} p(u_j | u_0)}
\]
for all \((u_0, u_S, u_{S'}, u_{k+1}) \in F\). Furthermore, let the constant \( \gamma \) satisfy
\[
\gamma \geq \log \frac{p(u_{[k]} | u_0, u_{k+1})}{\prod_{j \in [k]} p(u_j | u_0)}
\]
for all \((u_0, u_{[k]}, u_{k+1}) \in F\).

For every \( m = (m_1, \ldots, m_k) \in M \), define the random variable \( Z_m \) as
\[
Z_m = 1 \left\{ (U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1}) \in F \right\}
\]
and set
\[
Z = \sum_{m \in M} Z_m.
\]

Our aim is to find a lower bound for \( P\{Z = 0\} \). Note that for every nonempty \( S \subseteq [k] \),
\[
P\{\exists m: Z_m = 1\} = P\left\{\exists m: (U_0, U_1(m_1), \ldots, U_k(m_k), U_{k+1}) \in F \right\}
\]
\[
\leq P\left\{\exists m: (U_0, (U_j(m_j))_{j \in S}, U_{k+1}) \in F_S \right\}
\]
\[
\leq |M_S| \sum_{F_S} p(u_0, u_{k+1}) \prod_{j \in S} p(u_j | u_0)
\]
\[
\leq |M_S| e^{-\alpha_S} \sum_{F_S} p(u_0, u_S, u_{k+1})
\]
\[
\leq |M_S| e^{-\alpha_S}.
\]
Thus
\[ P\{Z = 0\} = 1 - P\{\exists m : Z_m = 1\} \]
\[ \geq 1 - \min_{|S| \neq 0} |\mathcal{M}_S| e^{-\alpha_S}. \]  
(3)

4. The Upper Bound

In deriving our upper bound on \( P\{Z = 0\} \), we apply conditioning and Chebyshev’s inequality. Thus, the factor
\[
\left( \frac{1}{P\{F(u_0, u_{k+1})\}} \right)^2
\]
appears, where
\[
P\{F(u_0, u_{k+1})\} = P\{U_{[k]} \in F(u_0, u_{k+1})| U_0 = u_0, U_{k+1} = u_{k+1}\}
= \sum_{u_{[k]} \in F(u_0, u_{k+1})} p(u_{[k]}|u_0, u_{k+1})
\]
and \( F(u_0, u_{k+1}) \) (Section 3) is simply the set of all \( u_{[k]} \)’s that satisfy \( (u_0, u_{[k]}, u_{k+1}) \in F \). Thus to get a reasonably accurate upper bound, we require \( P\{F(u_0, u_{k+1})\} \) to be large. However, as we cannot guarantee this for all \( (u_0, u_{k+1}) \), we partition the \( (u_0, u_{k+1}) \) pairs into “good” and “bad” sets, corresponding to large and small values of \( P\{F(u_0, u_{k+1})\} \), respectively. The probability of the good set is large when \( P\{(U_0, U_{[k]}, U_{k+1}) \in F\} \) is sufficiently large. To see this, fix \( \epsilon > 0 \) and following Appendix III of [5], define the set \( \mathcal{G} \subseteq U_0 \times U_{k+1} \) as
\[
\mathcal{G} = \{ (u_0, u_{k+1}) : P\{F(u_0, u_{k+1})\} \geq 1 - \epsilon \},
\]
Note that \( \mathcal{G} \) is the set of all good \( (u_0, u_{k+1}) \) pairs as defined above. We have
\[
P\{(U_0, U_{[k]}, U_{k+1}) \in F\} = \sum_{u_0, u_{k+1}} \sum_{u_{[k]} \in F\{u_0, u_{k+1}\}} p(u_0, u_{k+1})p(u_{[k]}|u_0, u_{k+1})
= \sum_{u_0, u_{k+1}} p(u_0, u_{k+1})P\{F(u_0, u_{k+1})\}
\leq (1 - \epsilon)P\{(U_0, U_{k+1}) \notin \mathcal{G}\} + P\{(U_0, U_{k+1}) \in \mathcal{G}\}
= 1 - \epsilon P\{(U_0, U_{k+1}) \notin \mathcal{G}\}.
\]
Thus
\[
P\{(U_0, U_{k+1}) \notin \mathcal{G}\} \leq \frac{1}{\epsilon} P\{(U_0, U_{[k]}, U_{k+1}) \notin F\}. \]  
(4)

Our aim is to find an upper bound for \( P\{Z = 0\} \). To do this, we write
\[
P\{Z = 0\} = \sum_{u_0, u_{k+1}} p(u_0, u_{k+1})P\{Z = 0| u_0, u_{k+1}\}
\leq \frac{1}{\epsilon} P\{(U_0, U_{[k]}, U_{k+1}) \notin F\} + \sum_{(u_0, u_{k+1}) \in \mathcal{G}} p(u_0, u_{k+1})P\{Z = 0| u_0, u_{k+1}\},
\]
where the inequality follows from Equation (4). Therefore, to find an upper bound on \( P\{Z = 0\} \), it suffices to find an upper bound on \( P\{Z = 0| U_0 = u_0, U_{k+1} = u_{k+1}\} \) for all \( (u_0, u_{k+1}) \in \mathcal{G} \). Fix \( (u_0, u_{k+1}) \in \mathcal{G} \). We use Chebyshev’s inequality to find an upper bound on \( P\{Z = 0| U_0 = u_0, U_{k+1} = u_{k+1}\} \). Thus we need to
calculate $E[Z|U_0 = u_0, U_{k+1} = u_{k+1}]$ and $E[Z^2|U_0 = u_0, U_{k+1} = u_{k+1}]$. For a given $m$, from the definition of $\gamma$ (Section 3) it follows

$$E[Z_m|u_0, u_{k+1}] = P\left\{ (U_1(m_1), \ldots, U_k(m_k)) \in F(u_0, u_{k+1}) | u_0, u_{k+1} \right\}$$

$$= \sum_{F(u_0, u_{k+1})} p(u_1|u_0) \ldots p(u_k|u_0)$$

$$\geq \sum_{F(u_0, u_{k+1})} e^{-\gamma} p(u_k|u_0, u_{k+1})$$

$$= e^{-\gamma} P\{F(u_0, u_{k+1})\} \geq (1 - \epsilon)e^{-\gamma}.$$  

where the last inequality follows from the fact that $(u_0, u_{k+1}) \in G$. Thus, by linearity of expectation,

$$E[Z|U_0 = u_0, U_{k+1} = u_{k+1}] \geq |M|e^{-\gamma}(1 - \epsilon).$$  

(6)

Next, we find an upper bound on $E[Z^2|U_0 = u_0, U_{k+1} = u_{k+1}]$. We have

$$Z^2 = \sum_m Z_m^2 + \sum_{m \neq m'} Z_m Z_{m'} = Z + \sum_{m \neq m'} Z_m Z_{m'},$$

since $Z_m^2 = Z_m$ and $Z = \sum_m Z_m$. Thus

$$E[Z^2|u_0, u_{k+1}] = E[Z|u_0, u_{k+1}] + E\left[ \sum_{m \neq m'} Z_m Z_{m'} | u_0, u_{k+1} \right]$$

For any pair of distinct $m$ and $m'$ with nonempty $S = S_{m,m'}$, we have

$$E[Z_m Z_{m'}|u_0, u_{k+1}]$$

$$= \sum_{F_S(u_0, u_{k+1}) \in S} \prod_{u_{S \subseteq F(u_0, u_{k+1}) \cup j \in S^c}} p(u_j|u_0)^2$$

$$\leq e^{-\alpha_S - 2\beta_{S^c}} \sum_{F_S(u_0, u_{k+1})} p(u_{S}|u_0, u_{k+1}) \left( \sum_{u_{S^c} \in F(u_0, u_{k+1})} p(u_{S^c}|u_0, u_{S, u_{k+1}})^2 \right)$$

$$\leq e^{-\alpha_S - 2\beta_{S^c}},$$

where $F_S(u_0, u_{k+1})$ is the set of all $u_S$ that satisfy $(u_0, u_S, u_{k+1}) \in F_S$. On the other hand, if $S = S_{m,m'}$ is empty, then $Z_m$ and $Z_{m'}$ are independent given $(U_0, U_{k+1}) = (u_0, u_{k+1})$, and

$$E[Z_m Z_{m'}|u_0, u_{k+1}] = (E[Z_m|u_0, u_{k+1}])^2.$$  

Thus (assume $|M_S| = 1$)

$$E[Z^2|u_0, u_{k+1}] = E[Z|u_0, u_{k+1}] + \sum_{S \subseteq \{k\}} |M_S| \prod_{j \in S^c} (|M_j| - |M_j|) E[Z_m Z_{m'}|u_0, u_{k+1}]$$

$$\leq E[Z|u_0, u_{k+1}] + (E[Z|u_0, u_{k+1}])^2 + \sum_{\emptyset \subseteq S \subseteq \{k\}} |M_S| |M_{S^c}|^2 e^{-\alpha_s - 2\beta_{S^c}}.$$  

(7)
where the notation $\emptyset \subset S \subset [k]$ means that $S$ is a nonempty proper subset of $[k]$. We have

$$P\{Z = 0|u_0, u_{k+1}\} \leq P\left\{|Z - \mathbb{E}[Z|u_0, u_{k+1}]| \geq \mathbb{E}[Z|u_0, u_{k+1}]\right\}$$

(a) \quad \frac{\text{Var}(Z|u_0, u_{k+1})}{(\mathbb{E}[Z|u_0, u_{k+1}])^2} = \frac{\mathbb{E}[Z^2|u_0, u_{k+1}]}{(\mathbb{E}[Z|u_0, u_{k+1}])^2} - 1

(b) \quad \frac{1}{1 - \epsilon}|M|^{-1}e^\gamma + \frac{1}{(1 - \epsilon)^2} \sum_{\emptyset \subset S \subset [k]} |M_S|^{-1}e^{-\alpha_S - 2\beta_S + 2\gamma},

where (a) follows from Chebyshev’s inequality and (b) follows from Equations (6) and (7). Now using Equation (5), we get

$$P\{Z = 0\} \leq \frac{1}{\epsilon}P\{\mathcal{F}^c\} + \frac{1}{1 - \epsilon}|M|^{-1}e^\gamma + \frac{1}{(1 - \epsilon)^2} \sum_{\emptyset \subset S \subset [k]} |M_S|^{-1}e^{-\alpha_S - 2\beta_S + 2\gamma},$$

(8)

5. THE ASYMPTOTIC RESULT

In this section, using our lower and upper bounds, we prove Theorem 1. We first prove the direct part using our upper bound from Section 4. Set $\mathcal{F} = A^{(n)}_S$ and for every $j \in [k]$, choose an integer $M_j \geq \epsilon^n R_j$. Choose a sequence $\{\epsilon_n\}_n$ such that

$$\lim_{n \to \infty} \frac{1}{\epsilon_n} P\{(A^{(n)}_S)^c\} = 0.$$

This is simple to do, since $P\{(A^{(n)}_S)^c\}$ decays exponentially in $n$ (see Appendix B). Fix a nonempty $S \subseteq [k]$. Notice that if $(U^n_0, (U^n_j)_{j \in S}, U^n_{k+1}) \in \mathcal{F}_S$, then

$$\left| \log \frac{p(u^n_S|u^n_0, u^n_{k+1})}{\prod_{j \in S} p(u^n_j|u^n_0)} - n \left( \sum_{j \in S} H(U_j|U_0) - H(U_S|U_0, U_{k+1}) \right) \right| \leq 2n(|S| + 1)\delta.$$

Thus we may choose

$$\alpha_S = n \left( \sum_{j \in S} H(U_j|U_0) - H(U_S|U_0, U_{k+1}) - 2(|S| + 1)\delta \right)$$

and

$$\gamma = n \left( \sum_{j=1}^k H(U_j|U_0) - H(U_{k+1}|U_0, U_{k+1}) + 2(k + 1)\delta \right).$$

Similarly, for every nonempty $S \subseteq [k]$, we choose $\beta_S$ as

$$\beta_S = n \left( \sum_{j \in S} H(U_j|U_0) - H(U_S|U_0, U_{S^c}, U_{k+1}) - 2(|S| + 1)\delta \right),$$

since for every $(U^n_0, (U^n_j)_{j \in S}, (U^n_j)_{j \in S^c}) \in \mathcal{F}$,

$$\left| \log \frac{p(u^n_S|u^n_0, u^n_{S^c}, u^n_{k+1})}{\prod_{j \in S} p(u^n_j|u^n_0)} - n \left( \sum_{j \in S} H(U_j|U_0) - H(U_S|U_0, U_{S^c}, U_{k+1}) \right) \right| \leq 2n(|S| + 1)\delta.$$
From our upper bound, Equation (8), it now follows that if for all nonempty \( S \subseteq [k] \),

\[
\sum_{j \in S} R_j > \frac{1}{n}(2\gamma - \alpha_S - 2\beta_S),
\]

\[
= 2 \sum_{j=1}^k H(U_j | U_0) - 2H(U_0 | U_0, U_{k+1}) - \sum_{j \in S} H(U_j | U_0) + H(U_S | U_0, U_{k+1})
\]

\[
- 2 \sum_{j \in S^c} H(U_j | U_0) + 2H(U_{S^c} | U_0, U_{S}, U_{k+1}) + (8k - 2|S| + 10)\delta
\]

\[
= \sum_{j \in S} H(U_j | U_0) - H(U_S | U_0, U_{k+1}) + (8k - 2|S| + 10)\delta,
\]

and for \( S = [k] \),

\[
\sum_{j=1}^k R_j > \frac{1}{n}\gamma = \sum_{j=1}^k H(U_j | U_0) - H(U_0 | U_0, U_{k+1}) - 2(k + 1)\delta,
\]

then

\[
\lim_{n \to \infty} P \left\{ \exists m : (U_0^n, U_1^n(m_1), \ldots, U_k^n(m_k), U_{k+1}^n) \in A^{(n)}_S \right\} = 1. \quad (9)
\]

Next we prove the converse. Suppose for each \( j \in [k] \), \( M_j \leq e^{nR_j} \) and Equation (9) holds. Then from our lower bound, Equation (3), it follows

\[
\sum_{j \in S} R_j \geq \frac{1}{n}\alpha_S = \sum_{j \in S} H(U_j | U_0) - H(U_S | U_0, U_{k+1}) - 2(|S| + 1)\delta,
\]

for all nonempty \( S \subseteq [k] \).

**APPENDIX A. CAUCHY-SCHWARZ INEQUALITY**

Let \( Z \) be any random variable that is nonnegative with probability one and has positive first and second moments. Then

\[
Z = Z \mathbf{1}\{Z > 0\}
\]

almost surely. Thus

\[
E[Z] = E[Z \mathbf{1}\{Z > 0\}]
\]

\[
\leq \sqrt{E[Z^2]} \times P\{Z > 0\},
\]

where the inequality follows from Cauchy-Schwarz. Hence

\[
P\{Z > 0\} \geq \frac{(E[Z])^2}{E[Z^2]}
\]

and

\[
P\{Z = 0\} \leq 1 - \frac{(E[Z])^2}{E[Z^2]}.
\]

On the other hand, using Chebyshev’s inequality we get

\[
P\{Z = 0\} = P\left\{ |Z - E[Z]| \geq E[Z] \right\}
\]

\[
\leq \frac{\text{Var}(Z)}{(E[Z])^2} = \frac{E[Z^2]}{(E[Z])^2} - 1.
\]
Now note that the bound resulting from Cauchy-Schwarz is stronger, since for any $t > 0$,

$$1 - t \leq \frac{1}{t} - 1.$$  

**APPENDIX B. LARGE DEVIATIONS**

The moment generating function of a random variable $X$ is defined as

$$M(t) = \mathbb{E}[e^{tX}]$$

for all real $t$ for which the expectation on the right hand side is finite. If $M$ is defined on a neighborhood of 0, say $(-t_0, t_0)$ for some $t_0 > 0$, then it has a Taylor series expansion with a positive radius of convergence [1, pp. 278-280]. In particular,

$$\frac{d}{dt} M(t) \bigg|_{t=0} = \mathbb{E}[X].$$

We want to find an upper bound for $\mathbb{P}\{X \geq a\}$ for some real number $a$. Choose $t > 0$. Using Markov’s inequality, we get

$$\mathbb{P}\{X \geq a\} = \mathbb{P}\{tX \geq ta\} = \mathbb{P}\{e^{tX} \geq e^{ta}\} \leq e^{-ta} \mathbb{E}[e^{tX}] = e^{t \log M(t) - ta}.$$  

Since $t > 0$ was arbitrary, we get

$$\mathbb{P}\{X \geq a\} \leq e^{\inf_{t>0} (\log M(t) - ta)}.$$  

Define the function $f$ as

$$f(t) = \log M(t) - ta.$$  

Then $f(0) = 0$ and $f'(0) = \mathbb{E}[X] - a$. Thus if $a > \mathbb{E}[X]$,

$$\inf_{t>0} (\log M(t) - ta) < 0.$$  

(10)

If we apply the same inequality to the random variable

$$\frac{1}{n} \sum_{i=1}^{n} X_i,$$

where the $X_i$’s are i.i.d. copies of $X$, we get

$$\mathbb{P}\left\{ \sum_{i=1}^{n} X_i \geq na \right\} \leq e^{n \inf_{t>0} (\log M(t) - ta)}.$$  

(11)

Now consider a random vector $(U_1, \ldots, U_k)$ with distribution $p(u_1, \ldots, u_k)$. For every nonempty $S \subseteq [k]$, let $U_S$ denote the random vector $(U_j)_{j \in S}$. Let $(U_1^n, \ldots, U_k^n)$ be $n$ i.i.d. copies of $(U_1, \ldots, U_k)$. By applying inequality (11) to the random variables $\left\{ \log \frac{1}{p(U_{S_i})} \right\}_{i=1}^{n}$ and setting $a = H(U_S) + \epsilon$ for some $\epsilon > 0$, we get

$$\mathbb{P}\left\{ \sum_{i=1}^{n} \log \frac{1}{p(U_{S_i})} \geq n(H(U_S) + \epsilon) \right\} \leq e^{-n I_S(\epsilon)},$$  

(12)

where $I_S(\epsilon)$ is given by

$$I_S(\epsilon) = \inf_{t>0} \left\{ t(H(U_S) + \epsilon) - \log \mathbb{E}[p(U_S)^{-t}] \right\}.$$
By the union bound we get
\[
P\left\{ (U_1^n, \ldots, U_k^n) \notin A_c^n(U_1, \ldots, U_k) \right\} \leq 2 \sum_{\emptyset \subseteq S \subseteq [k]} e^{-nI_S(\epsilon)} \\
\leq 2(2^k - 1)e^{-n\min_S I_S(\epsilon)} \\
\leq e^{-nI(\epsilon)},
\]
where
\[
I(\epsilon) = \min_{S \subseteq [k]} I_S(\epsilon) + o\left(\frac{1}{n}\right).
\]
Finally, note that by Equation (10), each \( I_S(\epsilon) \) is positive, thus so is \( I(\epsilon) \).

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