Noncommutativity in interpolating string: A study of gauge symmetries in noncommutative framework

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Abstract

A new Lagrangian description that interpolates between the Nambu–Goto and Polyakov version of interacting strings is given. Certain essential modifications in the Poisson bracket structure of this interpolating theory generates noncommutativity among the string coordinates for both free and interacting strings. The noncommutativity is shown to be a direct consequence of the nontrivial boundary conditions. A thorough analysis of the gauge symmetry is presented taking into account the new modified constraint algebra, which follows from the noncommutative structures and finally a smooth correspondence between gauge symmetry and reparametrisation is established.

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1 Introduction

For the last few decades string theory has been regarded as the most promising step towards the fundamental theory uniting all the basic interactions at the Planck scale [1]. The dynamics of a bosonic string is described either by Nambu–Goto (NG) or Polyakov action. Both these actions, though very well-known in the literature, poses certain degrees of difficulty. NG formalism is inconvenient for path integral quantisation whereas Polyakov action involves many redundant degrees of freedom. However, yet another formulation, interpolating between these two versions of string action, has also been put forward in the literature [2]. This interpolating Lagrangian is a better description of the theory in the sense that it neither objects to quantization nor has as many redundancies as in the Polyakov version. Further, it gives a perfect platform to study the gauge symmetries vis-à-vis reparametrisation symmetries of the various free string actions by a constrained Hamiltonian approach [3-4].

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Noncommutative (NC) theories, on the other hand, have been one of the central areas of research recently [5]. In this context the study of open strings propagating in the presence of a background Neveu–Schwarz two-form field $B_{\mu\nu}$ has become important because it exhibits a manifest NC structure among the space-time coordinates of the D-branes [6]. Several approaches have been taken to obtain such results, for example a Dirac approach [7] is employed with the string boundary conditions (BC(s)) imposed as second class constraints in [9, 10]. However, in a series of recent papers [2, 11, 12], it has been shown explicitly that noncommutativity can be obtained in a more transparent way by modifying the canonical Poisson bracket (PB) structure, so that it is compatible with the BC(s). This is similar in spirit to the treatment of Hanson, Regge and Teitelboim [13], where modified PB(s) were obtained for the free NG string. In a very recent paper [8], it has also been obtained using a conformal field theoretic approach.

In the present paper, acknowledging the above facts, we derive a master action for interacting bosonic strings, interpolating between the NG and Polyakov formalism. Modification of the basic PB structure compatible with BC(s) followed by the emergence of the noncommutativity is shown in this formalism (in case of both free and interacting strings) following the approach in [2, 11, 12]. Our results go over smoothly to the Polyakov version once proper identifications are made. Interestingly, we observe that a gauge fixing is necessary to give an exact NC solution between the string coordinates. Further, this gauge fixing condition restrict us to a reduced phase space of the interpolating theory which in turn minimizes the gauge redundancy of the theory by identifying a particular combination of the constraints (that occurs in the full gauge independent theory) leading to a new involutive constraint algebra which is markedly different from that given in [2]. With the above results at our disposal, we go over to the study of gauge symmetry in the NC framework. Owing to the new constraint algebra we find surprising changes in the structure constants of the theory. Finally, we compute the gauge variations of the fields and show the underlying unity of diffeomorphism with the gauge symmetry in the NC framework.

The organisation of the paper is as follows. In the next section we briefly review interacting string in NG formalism. This fixes the notations. Here we also extend the domain of definition of the fields and give the closure relations of Virasoro algebra. In section 3 we formulate the interpolating Lagrangian for interacting string, derive their corresponding boundary conditions and full set of constraints. Their identification with the usual NG and Polyakov version is demonstrated. Section 4 elaborates on the modification of the basic PB(s) of the interpolating theory (for both free and interacting case) to make the canonical structure compatible with the BC(s). In this section, one finds the emergence of noncommutative behavior among the coordinates. To give an explicit NC solution one needs to put a restriction on the phase space of the interpolating theory. In section 5 we systematically analyse the reduced gauge symmetry of bosonic string in light of the modified canonical setup. Finally we conclude in section 6.

2 Interacting String in Nambu-Goto Formalism

In this section, we analyse the NG formulation of the interacting bosonic string. As we shall see in the next section, this is essential in the construction of the Interpolating Lagrangian for the interacting string. The NG action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field $B_{\mu\nu}$ reads:

$$S_{NG} = \int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \left[L_0 + eB_{\mu\nu}\dot{X}^\mu X'^\nu\right]$$ (1)

As far as the study of gauge symmetry is concerned, we consider only free strings in the remainder of the paper.

1
where $\mathcal{L}_0$ is the free NG Lagrangian density given by:

$$\mathcal{L}_0 = - \left[ (\dot{X} \cdot X')^2 - X'^2 X''^2 \right]^{\frac{1}{2}}. \quad (2)$$

The string tension is kept implicit for convenience. The Euler-Lagrange (EL) equations and BC obtained by varying the action read:

$$\dot{\Pi}^\mu + K'^\mu = 0$$
$$K^\mu|_{\sigma=0,\pi} = 0 \quad (3)$$

where,

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \mathcal{L}^{-1}_0 \left( -X'^2 \dot{X}^\mu + (\dot{X} \cdot X') X'^\mu \right) + eB_{\mu\nu} \dot{X}^\nu$$
$$K^\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} = \mathcal{L}^{-1}_0 \left( -\dot{X}^2 X'^\mu + (\dot{X} \cdot X') \dot{X}^\mu \right) - eB_{\mu\nu} \dot{X}^\nu. \quad (4)$$

Note that $\Pi^\mu$ is the canonically conjugate momentum to $X^\mu$. The nontrivial PB(s) of the theory are given by:

$$\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = \eta^{\mu\nu} \delta(\sigma - \sigma'). \quad (5)$$

The primary constraints of the theory are:

$$\Omega_1 = \Pi^\mu X'^\mu \approx 0$$
$$\Omega_2 = (\Pi^\mu - eB_{\mu\nu} \dot{X}^\nu)^2 + X'^2 \approx 0. \quad (6)$$

From the above PB structure (5), it is easy to generate a first class (involutive) algebra:

$$\{\Omega_1(\sigma), \Omega_1(\sigma')\} = [\Omega_1(\sigma) + \Omega_1(\sigma')] \partial_{\sigma} \delta(\sigma - \sigma')$$
$$\{\Omega_1(\sigma), \Omega_2(\sigma')\} = [\Omega_2(\sigma) + \Omega_2(\sigma')] \partial_{\sigma} \delta(\sigma - \sigma')$$
$$\{\Omega_2(\sigma), \Omega_2(\sigma')\} = 4 [\Omega_1(\sigma) + \Omega_1(\sigma')] \partial_{\sigma} \delta(\sigma - \sigma'). \quad (7)$$

Now as happens for a reparametrisation invariant theory, the canonical Hamiltonian density defined by a Legendre transform vanishes

$$\mathcal{H}_c = \Pi^\mu \dot{X}^\mu - \mathcal{L} = 0. \quad (8)$$

This can be easily seen by substituting (4) in (8). The total Hamiltonian density is thus given by a linear combination of the first class constraints (6):

$$\mathcal{H}_T = -\rho \Omega_1 - \frac{\lambda}{2} \Omega_2 \quad (9)$$

where $\rho$ and $\lambda$ are Lagrange multipliers. It is easy to check that time preserving the primary constraints yields no new secondary constraints. Hence the total set of constraints of the interacting NG theory is given by the first-class system (6).

Now we enlarge the domain of definition of the bosonic field $X^\mu$ from $[0, \pi]$ to $[-\pi, \pi]$ by defining

$$X^\mu(\tau, -\sigma) = X^\mu(\tau, \sigma) ; \quad B_{\mu\nu} \rightarrow -B_{\mu\nu} \quad \text{under} \quad \sigma \rightarrow -\sigma. \quad (10)$$

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2 Here $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$ and $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$.

3 This is done in order to write down the generators of $\tau$ and $\sigma$ reparametrisation in a compact form.
The second condition implies that $B_{\mu \nu}$, albeit a constant, transforms as a pseudo scalar under $\sigma \rightarrow -\sigma$ in the extended interval. This ensures that the interaction term $e B_{\mu \nu} \dot{X}^\mu X'^\nu$ in (11) remains invariant under $\sigma \rightarrow -\sigma$ like the free NG Lagrangian density $L_0$ (2). Consistent with this, we have

$$\Pi^\mu(\tau, -\sigma) = \Pi^\mu(\tau, \sigma), \quad X'^\mu(\tau, -\sigma) = -X'^\mu(\tau, \sigma). \quad (11)$$

Now, from (6), (10) we note that the constraints $\Omega_1(\sigma) \approx 0$ and $\Omega_2(\sigma) \approx 0$ are odd and even respectively under $\sigma \rightarrow -\sigma$. Now demanding the total Hamiltonian density $H_T$ (9) also remains invariant under $\sigma \rightarrow -\sigma$, one finds that $\rho$ and $\lambda$ must be odd and even respectively under $\sigma \rightarrow -\sigma$.

We may then write the generator of all $\tau$ and $\sigma$ reparametrisation as the functional [13]:

$$L[f] = \frac{1}{2} \int_{-\pi}^{\pi} d\sigma \{ f_+(\sigma) \Omega_2(\sigma) + 2 f_-(\sigma) \Omega_1(\sigma) \}, \quad (12)$$

where, $f_\pm(\sigma) = \frac{1}{2} (f(\sigma) \pm f(-\sigma))$ are by construction even and odd function and $f(\sigma)$ is an arbitrary differentiable function defined in the extended interval $[-\pi, \pi]$. The above expression can be simplified to:

$$L[f] = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma f(\sigma) \left[ \Pi_\mu(\sigma) + X'_\mu(\sigma) - e B_{\mu \nu} X'^\nu(\sigma) \right]^2. \quad (13)$$

It is now easy to verify (using (7)) that the above functional (13) generates the following Virasoro algebra:

$$\{ L[f(\sigma)], L[g(\sigma)] \} = L[f(\sigma) g'(\sigma) - f'(\sigma) g(\sigma)]. \quad (14)$$

Defining

$$L_m = L[e^{-im\sigma}], \quad (15)$$

one can write down an equivalent form of the Virasoro algebra

$$\{ L_m, L_n \} = i(m - n)L_{m+n}. \quad (16)$$

Note that we do not have a central extension here, as the analysis is entirely classical.

3 Interpolating Lagrangian, boundary conditions and constraint structure of the Interacting String

In the previous section we have reviewed the salient features of the interacting NG string. We now pass on to the construction of the interpolating action of the interacting string. To achieve this end, we write down the Lagrangian of the interacting NG action in the first-order form:

$$L_I = \Pi_\mu \dot{X}^\mu - H_T. \quad (17)$$

Substituting (9) in (17), $L_I$ becomes

$$L_I = \Pi_\mu \dot{X}^\mu + \rho \Pi_\mu X'^\mu + \frac{\lambda}{2} \left[ (\Pi^2 + X'^2) - 2 e B_{\mu \nu} \Pi^\mu X'^\nu + e^2 B_{\mu \nu} B_\rho X'^\nu X'^\rho \right]. \quad (18)$$

\(^4\text{The construction of the interpolating action for the free string has been discussed in [2].}\)
The advantage of working with the interpolating action is that it naturally leads to either the NG or the Polyakov formulations of the string. In the Lagrangian \( (18) \), \( \lambda \) and \( \rho \) originally introduced as Lagrange multipliers, will be treated as independent fields, which behave as scalar and pseudo-scalar fields respectively in the extended world-sheet, as was discussed in the previous section. We will eliminate \( \Pi_\mu \) from \( (18) \) as it is an auxiliary field. The EL equation for \( \Pi_\mu \) is:

\[
\dot{X}^\mu + \rho X'^\mu + \lambda \Pi^\mu - e\lambda B^{\mu\nu} X'_\nu = 0.
\]

Substituting \( \Pi_\mu \) from \( (19) \) back in \( (18) \) yields:

\[
L_I = -\frac{1}{2\lambda} \left[ \dot{X}^2 + 2\rho(\dot{X} \cdot X') + (\rho^2 - \lambda^2)X'^2 - 2\lambda eB_{\mu\nu}\dot{X}^\mu X'^\nu \right].
\]

This is the form of the interpolating Lagrangian of the interacting string.

The reproduction of the NG action \( (11) \) from the interpolating action of the interacting string is trivial and can be done by eliminating \( \rho \) and \( \lambda \) from their respective EL equations of motion following from \( (20) \),

\[
\rho = -\frac{\dot{X} \cdot X'}{X'^2},
\]

\[
\lambda^2 = \frac{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}{X'^2 X'^2}.
\]

If, on the other hand, we identify \( \rho \) and \( \lambda \) with the following contravariant components of the world-sheet metric,

\[
g^{ab} = (\frac{-\dot{g}}{\lambda}) \left[ \frac{1}{2} \rho \frac{\rho}{\lambda} \frac{(\rho^2 - \lambda^2)}{\lambda} \right]
\]

then the above Lagrangian \( (20) \) reduces to the Polyakov form,

\[
L_P = -\frac{1}{2} \left( \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu - e\epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad ; \quad (a, b = \tau, \sigma).
\]

In this sense, therefore, the Lagrangian in \( (20) \) is referred to as an interpolating Lagrangian\(^5\). We can now, likewise construct the interpolating BC from the interpolating Lagrangian \( (20) \),

\[
K^\mu = \left[ \frac{\partial L_I}{\partial X_\mu} \right]_{\sigma=0,\pi} = \left( \frac{\rho}{\lambda} \dot{X}^\mu + \frac{\rho^2 - \lambda^2}{\lambda} X'^\mu + eB_{\mu\nu} \dot{X}^\nu \right)_{\sigma=0,\pi} = 0.
\]

The fact that this can be easily interpreted as interpolating BC, can be easily seen by using the expressions \( (21) \) for \( \rho \) and \( \lambda \) in \( (24) \) to yield:

\[
\left[ L_0^{-1} \left( -\dot{X}^2 X'^\mu + \dot{X} X'^\mu \right) - eB^{\mu\nu} \dot{X}^\nu \right]_{\sigma=0,\pi} = 0.
\]

This is the BC of the interacting NG string \( (4) \).

On the other hand, we can identify \( \rho \) and \( \lambda \) with the metric components as in \( (22) \) to recast \( (24) \) as:

\[
\left[ g^{1a} \partial_a X^\mu (\sigma) + \frac{1}{\sqrt{-g}} eB_{\mu\nu} \partial_0 X^\nu (\sigma) \right]_{\sigma=0,\pi} = 0.
\]

\(^5\)It should be noted that the interpolating action has only two additional degrees of freedom, \( \lambda \) and \( \rho \), which does not fully match the three degrees of freedom of the worldsheet metric of the Polyakov action. However, due to Weyl invariance of the Polyakov action, only two of the three different metric coefficients \( g_{ab} \) are really independent. This Weyl invariance is special to the Polyakov string, the higher branes do not share it.
which is easily identifiable with Polyakov form of BC [2] following from the action (23).

Using phase space variables $X^\mu$ and $\Pi_\mu$, (24) can be rewritten as

$$K^\mu = \left[(\rho \Pi^\mu + \lambda X'^\mu) + eB^\nu_{\rho} (\Pi'^\nu - eB^\rho_\nu X'^\nu)\right]_{\sigma=0,\pi} = 0.$$  (27)  

Hence it is possible to interpret either of (24) or (27) as an interpolating BC.

Now we come to the discussion of the constraint structure of the interpolating interacting string. Note that the independent fields in (20) are $X^\mu$, $\rho$ and $\lambda$. The corresponding momenta denoted by $\Pi_\mu$, $\pi_\rho$ and $\pi_\lambda$, are given as:

$$\Pi_\mu = \frac{1}{\lambda}(\dot{X}_\mu + \rho X'_\mu) + eB_{\mu\nu}X'^\nu$$

$$\pi_\rho = 0$$

$$\pi_\lambda = 0.$$  (28)  

In addition to the PB(s) similar to (5), we now have:

$$\{\rho (\tau, \sigma), \pi_\rho (\tau, \sigma')\} = \delta (\sigma - \sigma')$$

$$\{\lambda (\tau, \sigma), \pi_\lambda (\tau, \sigma')\} = \delta (\sigma - \sigma').$$  (29)  

The canonical Hamiltonian following from (20) reads:

$$H_c = -\rho \Pi_\mu X'^\mu - \frac{\lambda}{2} \left\{(\Pi_\mu - eB_{\mu\nu}X'^\nu)^2 + X'^2\right\}$$  (30)  

which reproduces the total Hamiltonian (9) of the NG action. From the definition of the canonical momenta we can easily identify the primary constraints:

$$\Omega_3 = \pi_\rho \approx 0$$

$$\Omega_4 = \pi_\lambda \approx 0.$$  (31)  

The conservation of the above primary constraints leads to the secondary constraints $\Omega_1$ and $\Omega_2$ of (6). The primary constraints of the NG action appear as secondary constraints in this formalism\textsuperscript{6}. The system of constraints for the Interpolating Lagrangian thus comprises of the set (31) and (6). The PB(s) of the constraints of (31) vanish within themselves. Also the PB of these with (6) vanish.

4 Modified brackets for Interpolating String

4.1 Free Interpolating String:

Let us consider boundary condition for free interpolating string which can be obtained by setting $B_{\mu\nu} = 0$ in (27):

$$K^\mu = \left[(\rho \Pi^\mu + \lambda X'^\mu)\right]_{\sigma=0,\pi} = 0.$$  (32)  

It is now easy to note that the above BC is not compatible with the basic PB (5). To incorporate this, an appropriate modification in the PB is in order. In [13, 2, 11, 12], the equal time brackets were given in terms of certain combinations ($\Delta_+ (\sigma, \sigma')$) of periodic delta function:\textsuperscript{7}

$$\{X^\mu (\tau, \sigma), \Pi_\rho (\tau, \sigma')\} = \delta_\mu^\nu \Delta_+ (\sigma, \sigma')$$  (33)  

\textsuperscript{6}No more secondary constraints are obtained.

\textsuperscript{7}The form of the periodic delta function is given by $\delta_P(x - y) = \delta_P(x - y + 2\pi) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} e^{in(x - y)}$ and is related to the usual Dirac $\delta$-function as $\delta_P(x - y) = \sum_{n \in \mathbb{Z}} \delta(x - y + 2\pi n)$.
where,

\[
\Delta_+ (\sigma, \sigma') = \delta P (\sigma - \sigma') + \delta P (\sigma + \sigma') = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \cos(n \sigma') \cos(n \sigma)
\]

\[
\Delta_- (\sigma, \sigma') = \delta P (\sigma - \sigma') - \delta P (\sigma + \sigma') = \frac{1}{\pi} \sum_{n \neq 0} \sin(n \sigma') \sin(n \sigma)
\] (34)

rather than an ordinary delta function to ensure compatibility with Neumann BC

\[\partial_\sigma X^\mu (\sigma) |_{\sigma = 0, \pi} = 0 , \] (35)

in the bosonic sector. Observe that the other brackets

\[
\{X^\mu (\sigma), X^{\nu'} (\sigma')\} = 0
\]

\[\{\Pi^\mu (\sigma), \Pi^{\nu'} (\sigma')\} = 0
\] (36) (37)

are consistent with the Neumann boundary condition (35).

Now a simple inspection shows that the BC (32) is also compatible with (33) and (37), but not with (29) and (36). Hence the brackets (29) and (36) should be altered suitably.

Now, since \(\rho\) and \(\lambda\) are odd and even functions of \(\sigma\) respectively, we propose:

\[
\{\rho (\tau, \sigma), \pi_\rho (\tau, \sigma')\} = \Delta_-(\sigma, \sigma')
\]

\[\{\lambda (\tau, \sigma), \pi_\lambda (\tau, \sigma')\} = \Delta_+(\sigma, \sigma'). \] (38)

and also make the following ansatz for the bracket among the coordinates (36):

\[
\{X^\mu (\tau, \sigma), X^{\nu'} (\tau, \sigma')\} = C^\mu\nu (\sigma, \sigma') ; \text{ where } C^\mu\nu (\sigma, \sigma') = - C^{\nu\mu} (\sigma', \sigma). \] (39)

One can easily check that the brackets (38) are indeed compatible with the BC (32). Now imposing the BC (32) on the above equation (39), we obtain the following condition:

\[\partial_\sigma C^\mu\nu (\sigma, \sigma') |_{\sigma = 0, \pi} = \frac{\rho}{\lambda} g^\mu\nu \Delta_+ (\sigma, \sigma') |_{\sigma = 0, \pi}. \] (40)

Now to find a solution for \(C^\mu\nu (\sigma, \sigma')\), we choose \(\partial_\sigma \left( \frac{\rho}{\lambda} \right) = 0 \) (41)

which gives a solution of \(C^\mu\nu (\sigma, \sigma')\) as:

\[
C^\mu\nu (\sigma, \sigma') = \eta^\mu\nu \left[ \kappa(\sigma) \Theta(\sigma, \sigma') - \kappa(\sigma') \Theta(\sigma', \sigma) \right]
\] (42)

where the generalised step function \(\Theta(\sigma, \sigma')\) satisfies,

\[\partial_\sigma \Theta(\sigma, \sigma') = \Delta_+(\sigma, \sigma'). \] (43)

Here, \(\kappa(\sigma) = \frac{\rho}{\lambda}(\sigma)\) is a pseudo-scalar. The \(\sigma\) in the parenthesis has been included deliberately to remind the reader that it transforms as a pseudo-scalar under \(\sigma \rightarrow -\sigma\) and should not be read as a functional dependence. The pseudo-scalar property of \(\kappa(\sigma)\) is necessary for \(C^\mu\nu (\sigma, \sigma')\)

Note that there is no inconsistancy in (35) as \(\partial_\sigma \Delta_+ (\sigma, \sigma') |_{\sigma = 0, \pi} = 0.\)

The condition (41) reduces to a restricted class of metric for Polyakov formalism that satisfy \(\partial_\sigma g_{01} = 0\). Such conditions also follow from a standard treatment of the light-cone gauge [1].
to be an even function of \(\sigma\) as \(X(\sigma)\) is also an even function of \(\sigma\) in the extended interval \([-\pi, \pi]\) of the string \((10)\). An explicit form of \(\Theta(\sigma, \sigma')\) is given by \([13]\):

\[
\Theta(\sigma, \sigma') = \frac{\sigma}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \frac{1}{n} \sin(n\sigma)\cos(n\sigma')
\]  

(44)

having the properties,

\[
\Theta(\sigma, \sigma') = 1 \quad \text{for} \quad \sigma > \sigma'
\]

and

\[
\Theta(\sigma, \sigma') = 0 \quad \text{for} \quad \sigma < \sigma'.
\]

(45)

Using the above relations, the simplified structure of \((42)\) reads,

\[
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0 \quad \text{for} \quad \sigma = \sigma'
\]

\[
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = \kappa(\sigma) \eta^{\mu\nu} \quad \text{for} \quad \sigma > \sigma'
\]

\[
= -\kappa(\sigma') \eta^{\mu\nu} \quad \text{for} \quad \sigma < \sigma'.
\]

(46)

We therefore propose the brackets \((33)\) and \((46)\) as the basic PB(s) of the theory and using these one can easily obtain the following involutive algebra between the constraints:

\[
\{\Omega_1(\sigma), \Omega_1(\sigma')\} = \Omega_1(\sigma') \partial_\sigma \Delta_+ (\sigma, \sigma') + \Omega_1(\sigma) \partial_\sigma \Delta_- (\sigma, \sigma')
\]

\[
\{\Omega_1(\sigma), \Omega_2(\sigma')\} = \left(\Omega_2(\sigma) + \Omega_2(\sigma')\right) \partial_\sigma \Delta_+ (\sigma, \sigma')
\]

\[
\{\Omega_2(\sigma), \Omega_2(\sigma')\} = 4 \left(\Omega_1(\sigma) \partial_\sigma \Delta_+ (\sigma, \sigma') + \Omega_1(\sigma') \partial_\sigma \Delta_- (\sigma, \sigma')\right).
\]

(47)

Note that a crucial intermediate step in the above derivation is to use the relation

\[
\{X^\mu(\sigma), X^\nu(\sigma')\} = 0
\]

(48)

which follows from the basic bracket \((46)\) \([2]\).  

We now compute the algebra between the Virasoro functionals using the modified constraint algebra \((47)\),

\[
\{L[f(\sigma)], L[g(\sigma)]\} = L[f(\sigma)g'(\sigma) - f'(\sigma)g(\sigma)].
\]

(49)

Interestingly, the Virasoro algebra has the same form as that of \((14)\) at the classical level. Consequently, the alternative forms of Virasoro algebra \((16)\) is also reproduced here.

It is now interesting to observe that the condition \((41)\) (which is necessary for giving an exact NC solution \((12)\)) reduces the gauge redundancy of the interpolating formalism as \(\rho\) and \(\lambda\) are no more independent. Consequently, one should look for only a particular combination of the constraints \((6)\) which gives a involutive algebra.

To this end we go back to \((20)\) and study the effect of \((41)\) on the free interpolating Lagrangian. Earlier it contained two additional fields \(\rho\) and \(\lambda\). However the interpolating Lagrangian depends only on one of these fields \(\lambda\) (say) once the condition \((41)\) is imposed and one gets the following reduced form of the Lagrangian:

\[
L_{\text{red}} = -\frac{1}{2\lambda} \dot{X}^2 - \kappa(\sigma) \dot{X} \cdot X'.
\]

(50)

Owing to the condition \((41)\), the free canonical Hamiltonian reduces to:

\[
H_c = -\kappa(\sigma) \lambda \Pi \cdot X' - \frac{\lambda}{2} \left\{\Pi^2 + X'^2\right\}
\]

(51)

\footnote{Note that there were some errors in \([2]\) and the correct constraint algebra was given in \([12]\).}
having only one primary constraint,
\[ \pi_\lambda \approx 0. \] (52)
Conserving (52) with the canonical Hamiltonian (51) we get the secondary constraint
\[ \Omega(\sigma) = \frac{1}{2} \left[ \Pi^2 + X'^2 + 2\kappa (\sigma) \Pi \cdot X' \right] \approx 0 \] (53)
which generates the first class algebra (in NC framework):
\[ \{ \Omega(\sigma), \Omega(\sigma') \} = 2 \left[ \kappa (\sigma) \Omega(\sigma) \partial _\sigma \Delta_+ (\sigma, \sigma') - \kappa (\sigma') \Omega(\sigma') \partial _{\sigma'} \Delta_+ (\sigma, \sigma') \right] \] (54)
We shall study the consequences of the above algebra (47) and (54) in Section 5 where we make an exhaustive analysis of gauge symmetry.

4.2 Interacting Interpolating String:

The Interpolating action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field \( B_{\mu \nu} \) is given by,
\[ S_I = \int d\tau d\sigma \left\{ -\frac{1}{2\lambda} \left[ X^2 + 2\rho (\dot{X} \cdot X') + (\rho^2 - \lambda^2) X'^2 - \lambda e^{ab} B_{\mu \rho} \partial _\sigma X^\mu \partial _b X^\nu \right] \right\} \] (55)
where \( e^{01} = -e^{10} = +1 \). The constraint structure has already been discussed in the section 3.

The boundary condition (BC) (27) can be written in a completely covariant form as:
\[ [M_{\nu}^\mu (\partial _\sigma X^\nu) + N^\mu \Pi _\nu] |_{\sigma=0,\pi} = 0 \] (56)
where,
\[ M_{\nu}^\mu = \left( \lambda \delta _\mu ^\nu - e^2 B_{\mu \rho} B_{\nu \rho} \right) \]
\[ N^\mu = (\rho \eta ^{\mu \nu} + eB^{\mu \nu}) \] (57)
This nontrivial BC leads to a modification in the original (naive) PBs (5). The BC (56) can be recast as:
\[ \left( \partial _\sigma X^\mu + \Pi _\rho \left( NM^{-1} \right)^{\rho \mu} \right) |_{\sigma=0,\pi} = 0. \] (58)
The \( \{ X^\mu(\sigma), \Pi ^\nu(\sigma') \}_{PB} \) is the same as that of the free string (33). We therefore make similar ansatz like (39) and using the BC (58), we get:
\[ \partial _\sigma C_{\mu \nu}(\sigma, \sigma') |_{\sigma=0,\pi} = (NM^{-1})_{\nu \mu} \Delta_+ (\sigma, \sigma') |_{\sigma=0,\pi} \] (59)
As in the free case, we restrict to the class defined by \( \partial _\sigma (NM^{-1})_{\nu \mu} = 0 \) which reduces to a restricted class of metric for Polyakov formalism. This reproduces the corresponding equation in interacting Polyakov string theory (see [2], in particular Eq 52). We therefore, obtain the following solution:
\[ C_{\mu \nu}(\sigma, \sigma') = \frac{1}{2} (NM^{-1})_{(\nu \mu)} (\sigma) \Theta (\sigma, \sigma') - \frac{1}{2} (NM^{-1})_{(\nu \mu)} (\sigma') \Theta (\sigma', \sigma) + \frac{1}{2} (NM^{-1})_{[\nu \mu]} (\sigma) [\Theta (\sigma, \sigma') - 1] + \frac{1}{2} (NM^{-1})_{[\nu \mu]} (\sigma') \Theta (\sigma', \sigma) \] (60)
with \( (NM^{-1})_{(\nu \mu)} \) the symmetric and \( (NM^{-1})_{[\nu \mu]} \) the antisymmetric part of \( (NM^{-1})_{\nu \mu} \).
5 Gauge symmetry

In this section we will discuss the gauge symmetries of the different actions and investigate their correspondence with the reparametrisation invariances. This has been done earlier for the free string case \[3\], however the canonical symplectic structure for the open string were not compatible with the general BC(s) of the theory. Now we shall investigate the gauge symmetry with the new modified PB structures (discussed in the earlier sections) which correctly takes into account the BC(s) of the theory. Importantly, the modified PB structure reveals a NC behavior among the string coordinates \[39 \[42\]. As we have seen in the previous section, an explicit account of noncommutativity requires a gauge fixing \[41\], thereby reducing the gauge redundancy of the interpolating picture. Note that the new generator of gauge transformation in the reduced phase space is \[53\]. For simplicity the following analysis of the gauge symmetry is done for the case of the free strings.

Our discussion will be centered on the reduced interpolating Lagrangian \[50\] as it provides an easy access to the analysis of gauge symmetry. The constraint structure of the reduced interpolating Lagrangian has already been discussed in the section 4.1. All the constraints are first class and therefore generate gauge transformations on \[L_{\text{red}}\] but the number of independent gauge parameters is equal to the number of independent primary first class constraints, i.e. one. In the following analysis we will apply a systematic procedure of abstracting the most general local symmetry transformations of the Lagrangian. A brief review of the procedure of \[15\] will thus be appropriate.

Consider a theory with first class constraints only. The set of constraints \(\Omega_a\) is assumed to be classified as

\[\{\Omega_a\} = \{\Omega_{a_1}; \Omega_{a_2}\}\]

where \(a_1\) belong to the set of primary and \(a_2\) to the set of secondary constraints. The total Hamiltonian is

\[H_T = H_c + \sum \lambda^{a_1} \Omega_{a_1}\]

where \(H_c\) is the canonical Hamiltonian and \(\lambda^{a_1}\) are Lagrange multipliers enforcing the primary constraints. The most general expression for the generator of gauge transformations is obtained according to the Dirac conjecture as

\[G = \Sigma \epsilon^a \Omega_a\]

where \(\epsilon^a\) are the gauge parameters, only \(a_1\) of which are independent. By demanding the commutation of an arbitrary gauge variation with the total time derivative, (i.e. \(\frac{d}{dt} (\delta q) = \delta \left(\frac{d}{dt} q\right)\)) we arrive at the following equations \[15 \[16\]

\[\delta \lambda^{a_1} = \frac{d \epsilon^{a_1}}{dt} - \epsilon^a \left(V^{a_1}_{a} + \lambda^{b} C^{a_1}_{b} a\right)\]

\[0 = \frac{d \epsilon^{a_2}}{dt} - \epsilon^a \left(V^{a_2}_{a} + \lambda^{b} C^{a_2}_{b} a\right)\]

Here the coefficients \(V^{a_1}_{a}\) and \(C^{a_1}_{b}\) are the structure functions of the involutive algebra, defined as

\[\{H_c, \Omega_a\} = V^{b}_{a} \Omega_b\]

\[\{\Omega_a, \Omega_b\} = C^{c}_{ab} \Omega_c\]

Solving \[65\] it is possible to choose \(a_1\) independent gauge parameters from the set \(\epsilon^a\) and express \(G\) of \[63\] entirely in terms of them. The other set \[64\] gives the gauge variations of the Lagrange multipliers \[11\].

\[11\] It can be shown that these equations are not independent conditions but appear as internal consistency conditions. In fact the conditions \[64\] follow from \[63\] \[15\].
We begin the analysis with the interpolating Lagrangian \((20)\). It contains additional fields \(\rho\) and \(\lambda\). We shall calculate the gauge variation of these extra fields and explicitly show that they are connected to the reparametrization by a mapping between the gauge parameters and the diffeomorphism parameters. These maps will be obtained later in this section by demanding the consistency of the variations \(\delta X^\mu\) due to gauge transformation and reparametrization.

The full constraint structure of the theory comprises of the constraints \((31)\) along with \((6)\). We could proceed from these and construct the generator of gauge transformations. The generator of the gauge transformations of \((20)\) is obtained by including the whole set of first class constraints \(\Omega_i\) given by \((31)\) and \((6)\) as

\[
G = \int d\sigma \alpha_i \Omega_i \tag{67}
\]

where only two of the \(\alpha_i\)'s are the independent gauge parameters. Using \((65)\) the dependent gauge parameters could be eliminated. After finding the gauge generator in terms of the independent gauge parameters, the variations of the fields \(X^\mu\), \(\rho\) and \(\lambda\) can be worked out. But the number of independent gauge parameters are same in both N–G \((1)\) and interpolating \((20)\) version. So the gauge generator \([12]\) is the same for both the cases, namely:

\[
G = \int d\sigma (\alpha_1 \Omega_1 + \alpha_2 \Omega_2) \tag{68}
\]

Also, looking at the intermediate first order form \((18)\) it appears that the fields \(X^\mu\) were already there in the N–G action \((1)\). The other two fields of the interpolating Lagrangian are \(\rho\) and \(\lambda\) which are nothing but the Lagrange multipliers enforcing the first class constraints \((6)\) of the N–G theory. Hence their gauge variation can be worked out from \((64)\). We prefer to take this alternative route. For convenience we relabel \(\rho\) and \(\lambda\) by \(\lambda_1\) and \(\lambda_2\)

\[
\lambda_1 = \rho \quad \text{and} \quad \lambda_2 = \frac{\lambda}{2} \tag{69}
\]

and their variations are obtained from \((64)\)

\[
\delta \lambda_i (\sigma) = -\dot{\lambda}_i - \int d\sigma' d\sigma'' C_{kj} \delta_i (\sigma', \sigma'') \frac{\lambda_k}{\lambda_j} (\sigma') \alpha_j (\sigma'') \tag{70}
\]

where \(C_{kj} (\sigma', \sigma'', \sigma)\) are given by

\[
\{\Omega_\alpha (\sigma), \Omega_\beta (\sigma')\} = \int d\sigma'' C_{\alpha \beta} (\sigma, \sigma', \sigma'') \Omega_\gamma (\sigma'') \tag{71}
\]

Observe that the structure function \(V_{ab}^b\) does not appear in \((70)\) since \(H_c = 0\) for the NG theory. The nontrivial structure functions \(C_{\alpha \beta \gamma} (\sigma, \sigma', \sigma'')\) are obtained from the constraint algebra \((47)\) as:

\[
\begin{align*}
C_{11} (\sigma, \sigma', \sigma'') & = (\partial_\sigma \Delta_+ (\sigma, \sigma')) \Delta_- (\sigma', \sigma'') + (\partial_\sigma \Delta_- (\sigma, \sigma')) \Delta_- (\sigma, \sigma'') \\
C_{22} (\sigma, \sigma', \sigma'') & = 4 (\partial_\sigma \Delta_+ (\sigma, \sigma')) \Delta_- (\sigma, \sigma'') + 4 (\partial_\sigma \Delta_- (\sigma, \sigma')) \Delta_- (\sigma', \sigma'') \\
C_{12} (\sigma, \sigma', \sigma'') & = \partial_\sigma \Delta_+ (\sigma, \sigma') \Delta_+ (\sigma, \sigma'') + \Delta_+ (\sigma, \sigma'') \\
C_{21} (\sigma, \sigma', \sigma'') & = \partial_\sigma \Delta_- (\sigma, \sigma') \Delta_+ (\sigma, \sigma'') + \Delta_+ (\sigma', \sigma'')
\end{align*} \tag{72}
\]

all other \(C_{\alpha \beta \gamma}\)'s are zero. Note that these structure functions are potentially different from those appearing in \([3,4]\) in the sense that here periodic delta functions are introduced to make

\[\text{Note that the gauge parameters } \alpha_1 \text{ and } \alpha_2 \text{ are odd and even respectively under } \sigma \to -\sigma.\]

12
the basic brackets compatible with the nontrivial BC. Using the expressions of the structure functions (72) in equation (70) we can easily derive:

\[
\begin{aligned}
\delta \lambda_1 &= -\dot{\alpha}_1 + (\alpha_1 \partial_1 \lambda_1 - \lambda_1 \partial_1 \alpha_1) + 4 (\alpha_2 \partial_1 \lambda_2 - \lambda_2 \partial_1 \alpha_2) \\
\delta \lambda_2 &= -\dot{\alpha}_2 + (\alpha_2 \partial_1 \lambda_1 - \lambda_1 \partial_1 \alpha_2) + (\alpha_1 \partial_1 \lambda_2 - \lambda_2 \partial_1 \alpha_1)
\end{aligned}
\]  

(73)

From the correspondence (69), we get the variations of \(\rho\) and \(\lambda\) as:

\[
\begin{aligned}
\delta \rho &= -\dot{\alpha}_1 + (\alpha_1 \partial_1 \rho - \rho \partial_1 \alpha_1) + 2 (\alpha_2 \partial_1 \lambda - \lambda \partial_1 \alpha_2) \\
\delta \lambda &= -2\dot{\alpha}_2 + 2 (\alpha_2 \partial_1 \rho - \rho \partial_1 \alpha_2) + (\alpha_1 \partial_1 \lambda - \lambda \partial_1 \alpha_1)
\end{aligned}
\]

(74)

In the above we have found out the full set of symmetry transformations of the fields in the interpolating Lagrangian (20). These symmetry transformations (74) were earlier given in [17] for the free string case. But the results were found there by inspection. In our approach [3, 4] the appropriate transformations are obtained systematically by a general method applicable to a whole class of string actions.

Now choosing

\[\alpha_1(\sigma) = 2\kappa(\sigma)\alpha_2(\sigma)\]  

(75)

one can write (68) as:

\[G = \int d\sigma \alpha_1(\sigma) \left[ \Pi^2 + X'^2 + 2\kappa(\sigma) \Pi \cdot X' \right]\]  

(76)

which is nothing but the generator of gauge transformation in the reduced interpolating framework (54).

The nontrivial structure functions \(C(\sigma, \sigma', \sigma'')\) obtained from (54) and (74) are:

\[C(\sigma, \sigma', \sigma'') = 2 \left[ 1_{\sigma} \partial_{\sigma} (\Delta_{+}(\sigma, \sigma')) \Delta_{+}(\sigma, \sigma'') - 1_{\sigma} \partial_{\sigma'} (\Delta_{+}(\sigma, \sigma')) \Delta_{+}(\sigma', \sigma'') \right].\]  

(77)

Substituting the structure functions in equation (70) yields the variation in \(\lambda\) to be:

\[\delta \lambda = -\dot{\alpha} + 2 1_{\sigma} (\alpha \partial_{\sigma} \lambda - \lambda \partial_{\sigma} \alpha).\]  

(78)

which can also be obtained by substituting (75) in (74). We are still to investigate to what extent the exact correspondence between gauge symmetry and reparametrisation holds in our modified NC framework. This can be done very easily if we stick to the method discussed in [3, 4].

To work out the mapping between the gauge parameters and the diffeomorphism parameters we now take up the Polyakov action (23). Here the only dynamic fields are \(X^\mu\). The transformations of \(X^\mu\) under (68) can be worked out resulting in the following:

\[\delta X_\mu(\sigma) = \{X_\mu(\sigma), G\} = \left( \alpha_1 X'_\mu(\sigma) + 2\alpha_2 \Pi_\mu(\sigma) \right)\]  

(79)

We can substitute for \(\Pi_\mu\) to obtain:

\[\delta X_\mu = \left( \alpha_1 - 2\alpha_2 \sqrt{-g} g^{01} \right) X'_\mu - 2\sqrt{-g} g^{00} \alpha_2 \dot{X}_\mu\]  

(80)

This is the gauge variation of \(X^\mu\) in terms of \(X'_\mu\) and \(\dot{X}_\mu\) where the coefficients appear as arbitrary functions of \(\sigma\) and \(\tau\). So we can identify them with the arbitrary parameters \(\Lambda_1\) and \(\Lambda_0\) characterising the infinitesimal reparametrisation [14]:

\[
\begin{aligned}
\tau' &= \tau - \Lambda_0 \\
\sigma' &= \sigma - \Lambda_1 \\
\delta X^\mu &= \Lambda^a \partial_a X^\mu = \Lambda_0 \dot{X}^\mu + \Lambda_1 X'^\mu
\end{aligned}
\]

(81)

\[\text{For easy comparison identify } \alpha_1 = \eta \text{ and } 2\alpha_0 = \epsilon\]
and that of $g_{ab}$ as:

$$\delta g_{ab} = D_a \Lambda_b + D_b \Lambda_a$$  \hspace{1cm} (82)

where

$$D_a \Lambda_b = \partial_a \Lambda_b - \Gamma_{abc} \Lambda_c$$  \hspace{1cm} (83)

$\Gamma_{ab}^c$ being the usual Christoffel symbols \[14\]. The infinitesimal parameters $\Lambda^a$ characterizes reparametrisation.

Comparing (80) and (81), we get the map connecting the gauge parameters with the diffeomorphism parameters:

$$\Lambda_0 = -2 \sqrt{-g} g^{00} \alpha_2$$
$$\Lambda_1 = \left( \alpha_1 - 2 \alpha_2 \sqrt{-g} g^{01} \right)$$  \hspace{1cm} (84)

Using the definitions (21), this map can be cast in a better shape:

$$\Lambda_1 = \left( \alpha_1 - 2 \frac{\alpha_2 \rho}{\lambda} \right)$$
$$\Lambda_0 = -2 \frac{\alpha_2 \lambda}{\lambda}$$  \hspace{1cm} (85)

All that remains now is to get the variation of $\rho$ and $\lambda$ induced by the reparametrisation (82). The identification (22) and (82) reproduces (74) as the variations of $\rho$ and $\lambda$. This establishes complete equivalence of the gauge transformations with the diffeomorphisms of the string.

Once again in the reduced case the condition (75) leads to the following map:

$$\Lambda^0 = -\frac{1}{\lambda} \alpha$$ \hspace{1cm} ; \hspace{1cm} \Lambda^1 = 0$$  \hspace{1cm} (86)

This along with (82) reproduces (78) as the variation of $\lambda$. The mapping (86) thus establishes complete equivalence in the reduced case.

6 Discussion

In this paper, we have developed a new action formalism for interacting bosonic string and demonstrated that it interpolates between the NG and Polyakov form of interacting bosonic actions. This is similar to the interpolating action formalism for free string proposed in [2]. We have also modified the basic PBs in order to establish consistency of the BC with the basic PBs. We stress that contrary to standard approaches, BC(s) are not treated as primary constraints of the theory. Our approach is similar in spirit with the previous treatment of string theory \[13, 2, 12\]. The NC structures derived in our paper go over smoothly to the Polyakov version once suitable identifications are made. However, to give explicit forms of the NC structures suitable gauge fixing needs to be done. We then set out to study the status of gauge symmetries vis-à-vis reparametrisation in this NC set up and establish the connection between gauge symmetry and diffeomorphism transformations. Finally, we feel that it would be interesting to investigate whether non-critical strings can be discussed using the interpolating action in a path-integral framework.

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References

[1] J. Polchinski, *String Theory*, Vol. I, Cambridge University Press, 1998.

[2] R. Banerjee, B. Chakraborty and S. Ghosh, Phys. Lett. B 537 (2002), 340, [hep-th/0203199]

[3] R. Banerjee, P. Mukherjee, A. Saha, Phys. Rev. D 70 026006, 2004, [hep-th/0403065]

[4] R. Banerjee, P. Mukherjee, A. Saha, Phys. Rev. D 72 066015, 2005, [hep-th/0501030]

[5] for a brief review see R. J. Szabo, Phys. Repts. 278 (2003) 207 and the references therein.

[6] N. Seiberg, E. Witten, JHEP 9909 (1999) 032.

[7] P. A. M. Dirac, *Lectures on Quantum Mechanics*, (Yeshiva University Press, New York, 1964).

[8] B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra, Phys. Rev. D 74 105011, 2006, [hep-th/0608065]

[9] F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari, JHEP 9902 (1999) 016; W. T. Kim, J. J. Oh, Mod.Phys. Lett. A 15 (2000) 1597; C. -S. Chu, P. M. Ho, Nucl. Phys. B 568, (2000) 447.

[10] N. R. F. Braga, C. F. L. Godinho, [hep-th/ 0110297] Phys. Rev. D 65 (2002) 085030.

[11] R. Banerjee, B. Chakraborty and K. Kumar, Nucl. Phys. B 668, 179 (2003), [hep-th/0306122]

[12] B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra, F. G. Scholtz, Phys. Letts. B 625 (2005), 302, [hep-th/0508156]

[13] A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian System*, Roma, Accademia Nazionale Dei Lincei, (1976).

[14] See for example J.W.van Holten, *Aspects of BRST Quantisation*, [hep-th/0201124]

[15] R. Banerjee, H. J. Rothe and K. D. Rothe, Phys. Lett. B 479 (2000) 429- 438, [hep - th/9907217]; R. Banerjee, H. J. Rothe and K. D. Rothe, Phys. Lett.B 463 (1999) 248- 251, [hep - th/9906072].

[16] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B 332 (1990) 169.

[17] M.Kaku, *Introduction to Superstring Theory*, Springer-Verlag, 1988.