COMBINATORICS OF THE $K$-THEORY OF AFFINE
GRASSMANNIANS

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Abstract. We introduce a family of tableaux that simultaneously generalizes
the tableaux used to characterize Grothendieck polynomials and $k$-Schur func-
tions. We prove that the polynomials drawn from these tableaux are the affine
Grothendieck polynomials and $k$-$K$-Schur functions—Schubert representatives
for the $K$-theory of affine Grassmannians and their dual in the nil Hecke ring.
We prove a number of combinatorial properties including Pieri rules.

1. INTRODUCTION AND BACKGROUND

The Schur functions form a fundamental basis for the symmetric function space
$\Lambda$. Many problems in geometry and representation theory have been solved using
the combinatorics behind Schur functions. Natural combinatorics associated to the
more general families of Grothendieck polynomials and $k$-Schur functions has sim-
ilarly led to an understanding of geometric and representation theoretic questions.
Here we explore the underlying combinatorics of two families of affine Grothendieck
polynomials.

1.1. Schur functions. To begin, the Schur role in geometry is nicely illustrated by
the problem of calculating intersections of Schubert varieties. The cohomology ring
of the Grassmannian $Gr_{\ell n}$ has a basis of Schubert classes $\sigma_\lambda$, indexed by partitions
$\lambda \in P^{\ell \times n}$ that fit inside an $\ell \times (n-\ell)$ rectangle. The structure constants of $H^*(Gr_{\ell n})$
in this basis:

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \in P^{\ell \times n}} c^\nu_{\lambda \mu} \sigma_\nu,$$

are the number of points in the intersection of the Schubert varieties $X_\lambda \cap X_\mu \cap X_\nu$.

There is an isomorphism, $H^*(Gr_{\ell n}) \cong \Lambda_{(e_n, \ldots, e_1)}$, where the Schur func-
tion $s_\lambda$ maps to the Schubert class $\sigma_\lambda$ when $\lambda \in P^{\ell \times n}$ and to zero otherwise. Thus,
the structure constants $c^\nu_{\lambda \mu}$ of (1) are none other than Littlewood-Richardson coeffi-
cients in the expansion

$$s_\nu \cdot s_\mu = \sum_{\lambda} c^\lambda_{\nu \mu} s_\lambda,$$

and the Schur functions are representatives of the Schubert classes.

The beauty of this identification is that the geometry can be studied in parallel
to Schur theory, where elegant solutions are given by way of combinatorics. For

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example, the Schur functions can be defined explicitly as the weight generating function of semi-standard tableaux:

$$s_\lambda = \sum_{\text{semi-standard } T \text{ with } \text{shape}(T) = \lambda} x^{\text{weight}(T)}.$$  \hspace{1cm} (3)

The ring structure can be determined by the Pieri rule,

$$s_\ell s_\mu = \sum_{\lambda = \mu + \ell \text{-strip}} s_\lambda,$$  \hspace{1cm} (4)

which is a simple matter of adding \( \ell \) boxes to the diagram of \( \mu \), and the Littlewood-Richardson coefficients can be characterized in terms of skew yamanouchi tableaux. Other properties of Schur functions (and thus the Schubert classes) also amount to simple combinatorial operations such as

$$\omega s_\lambda = s_{\lambda'},$$  \hspace{1cm} (5)

where \( \omega \) is the algebra automorphism defined by sending \( e_\ell \) to \( h_\ell \).

1.2. Grothendieck polynomials. Lasoux and Schützenberger introduced the Grothendieck polynomials in [LS82] as representatives for the \( K \)-theory classes determined by structure sheaves of Schubert varieties. Grothendieck polynomials are connected to combinatorics, representation theory, and algebraic geometry in a way that mimics ties between these theories and Schur functions (e.g. [Dem74, KK90, Las90, FK97]). This study leads to a generalization of Schubert calculus and combinatorics is again at the forefront. For example, the stable Grothendieck polynomials \( G_\lambda \) are inhomogeneous symmetric polynomials whose lowest homogeneous component is a Schur function. They are characterized by Buch as the weight generating function:

$$G_\lambda = \sum_{T \text{ set-valued with } \text{shape}(T) = \lambda} (-1)^{|\lambda| - |\text{weight}(T)|} x^T,$$  \hspace{1cm} (6)

where set-valued tableaux contain the semi-standard tableaux as a subset. The Pieri rules are in terms of binomial numbers [Len00] and there is a natural generalization for yamanouchi tableaux [Buc12] that gives a combinatorial rule for the structure constants.

Contrary to Schur functions, Grothendieck polynomials are not self-dual with respect to the Hall-inner product \( \langle \cdot, \cdot \rangle \). This gives rise to the family of polynomials \( g_\lambda \), dual to \( G_\mu \), whose top homogeneous component is a Schur function. Although less well-explored, the theory of these dual Grothendieck polynomials is equally as interesting.

1.3. \( k \)-Schur functions. There is a generalization of Schur functions along other lines that arose circuitously in a study of Macdonald polynomials [LLM03]. Pursuant work led to a new basis for \( \mathbb{Z}[h_1, \ldots, h_k] \) that satisfies properties analogous to (1) – (3) in an affine setting. In particular, the \( k \)-Schur functions \( s^{(k)}_\lambda \) were introduced in [LM07] and defined by inverting the system:

$$h_\lambda = \sum_{\mu} K^{(k)}_{\mu \lambda} s^{(k)}_\mu,$$  \hspace{1cm} (7)

where \( K^{(k)}_{\mu \lambda} \) enumerate a family of tableaux in bijection with elements of the type-\( A \) affine Weyl group called \( k \)-tableaux [LM05].
Geometrically, it was proven that the $k$-Schur functions are fundamental to the structure of the quantum and affine (co)homology of Grassmannians analogous to the Schur role in the usual cohomology. Quantum cohomology originated in string theory and symplectic geometry and is connected through the work of Kontsevich and Manin to the Gromov-Witten invariants. It was shown in [LM08] that certain Gromov-Witten invariants and calculation in the quantum cohomology of the Grassmannian can be reduced to computing the product of $k$-Schur functions. It was then shown in [Lam08] that $k$-Schur functions are the Schubert basis for homology of the affine Grassmannian. Again, combinatorics behind $k$-Schur functions is key to their study, as well as to the geometry. Properties such as their Pieri rule are proven in [LM07].

A second affine analog for Schur functions was introduced in [LM08]. These dual $k$-Schur functions (or affine Schur functions) can be defined by $(s^{(k)}_\lambda, \Theta^{(k)}_{\mu}) = \delta_{\mu \lambda}$. These also have significance in the geometry of the affine Grassmannian and are studied by way of combinatorial identities such as their weight generating function

$$\Theta^{(k)}_\lambda = \sum_{T \text{ k-tableau \ shape(T)=c(\lambda)}} x^{\text{weight}(T)} \tag{8}$$

and their Pieri rule (see [LLMS09a]).

1.4. **Affine Grothendieck polynomials.** The extension of ideas in $k$-Schur theory to an inhomogeneous setting underlies our investigation of affine combinatorics in the $K$-theoretic framework. We present a family of affine set-valued tableaux (or affine s-v tableaux) that simultaneously generalizes those used to characterize Grothendieck polynomials and $k$-Schur functions. We produce a bijection between affine s-v tableaux and certain elements that arise from the affine nil Hecke algebra. From this, we prove that the polynomials drawn from these tableaux:

$$G^{(k)}_\lambda = \sum_{T \text{ affine s-v tableau \ shape(T)=c(\lambda)}} (-1)^{|\lambda|+\text{weight}(T)} x^{\text{weight}(T)} \tag{9}$$

are affine stable Grothendieck polynomials introduced in [Lam06]. We also study their dual with respect to the Hall-inner product, the $k$-$K$-Schur functions $g^{(k)}_\lambda$.

We prove that the affine s-v tableaux associated to integer $k > 0$ contain $k$-tableaux as a subset and reduce to set-valued tableaux when $k$ is large. As a consequence, affine Grothendieck polynomials and $k$-$K$-Schur functions reduce to Grothendieck polynomials and their dual in a limiting case. Moreover, the term of lowest degree in $G^{(k)}_\lambda$ is the dual $k$-Schur function $\Theta^{(k)}_\lambda$ and the highest term of $g^{(k)}_\lambda$ is the $k$-Schur function $s^{(k)}_\lambda$.

We also give a number of combinatorial properties for the $k$-$K$-Schur functions such as Pieri rules. In particular, for $k$-bounded partition $\lambda$ and $r \leq k$,

$$g^{(k)}_\lambda g^{(k)}_\mu = \sum_{(\mu, \rho) \in \mathcal{H}^{k}_{\lambda,r}} (-1)^{r+|\lambda|-|\mu|} g^{(k)}_{\rho} \tag{10}$$

$$g^{(k)}_\lambda g^{(k)}_\mu = \sum_{(\mu, \rho) \in \mathcal{E}^{k}_{\lambda,r}} (-1)^{r+|\lambda|-|\mu|} g^{(k)}_{\rho} \tag{11}$$

where the elements of $\mathcal{H}^{k}_{\lambda,r}$ and $\mathcal{E}^{k}_{\lambda,r}$ are obtained by way of an affine set-valued notion of horizontal and vertical strips, respectively. In addition to extending [9]...
and (1) to the affine $K$-theoretic setting, we find that $k$-$K$-Schur functions satisfy a natural analog to (5). The image of $g^{(k)}_\lambda$ under an involution $\Omega$ on $\Lambda$ is simply another $k$-$K$-Schur function:

$$\Omega g^{(k)}_\lambda = g^{(k)}_{\lambda^{\omega_k}},$$

where $\lambda^{\omega_k}$ is a certain unique “$k$-conjugate” partition associated to $\lambda$.

Our results establish that the $k$-$K$-Schur functions and affine Grothendieck polynomials are the affine $K$-theoretic Schur functions in a combinatorial sense. Lam, Schilling, and Shimozono show in [LSS] that these polynomials satisfy an analog geometrically along the lines of (1) and (2). We thus have a combinatorial framework within which to study this geometry as has been so fruitful in classical Schur function theory.

1.5. Related work. This study grew out of an FRG problem solving session in Viña del Mar, Chile (2008) during which Thomas Lam posed the problem of exploring polynomials to play the Schur role in an affine $K$-theoretic setting. Our results, establishing that the affine Grothendieck polynomials and $k$-$K$-Schur functions are the appropriate candidate, are obtained purely combinatorially. Recent work of Lam, Schilling, and Shimozono [LSS] carries out a similar investigation from the geometric viewpoint and they prove that $G^{(k)}_\lambda$ and $g^{(k)}_\lambda$ are Schubert representatives for $K$-theory classes of the affine Grassmannian and their dual in the nil Hecke ring, respectively.

Another direction explores ties between the theories of Grothendieck and Macdonald polynomials. $k$-Schur functions are conjectured to be the $t=1$ specialization of atoms, a family of polynomials that arose in the study of Macdonald polynomials [LLM03]. In a forthcoming paper with Jason Bandlow [BM09], we conjecture that the Macdonald polynomials can be expanded positively (up to degree-alternating sign) in terms of the $\{g_\lambda\}_\lambda$ basis. We prove such an expansion in the Hall-Littlewood case using a statistic on set-valued tableaux that naturally generalizes the Lascoux-Schützenberger charge [LS78]. In addition, we explore a one parameter family of affine Grothendieck polynomials by $t$-generalizing the methods of [LM07] where $k$-Schur functions were defined so they could be connected to geometry.

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2. Definitions

2.1. Partitions. A partition is an integer sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)$ whose degree is $|\lambda| = \lambda_1 + \cdots + \lambda_m$ and whose length $\ell(\lambda)$ is $m$. Each partition $\lambda$ has an associated Ferrers shape with $\lambda_i$ lattice squares in the $i$th row, from the bottom to top. Given a partition $\lambda$, its conjugate $\lambda'$ is the shape obtained by reflecting $\lambda$ about the diagonal. A partition $\lambda$ is $k$-bounded if $\lambda_i \leq k$ and $\mathcal{P}_k$ denotes the set of all such partitions. Any lattice square in the shape is called a cell, where the cell $(i,j)$ is in the $i$th row and $j$th column of the shape. We say that $\lambda \subseteq \mu$ when $\lambda_i \leq \mu_i$ for all $i$. When $\rho \subseteq \gamma$, the skew shape $\gamma/\rho$ is the set theoretic difference $\gamma - \rho$. Dominance order $\lambda \succeq \mu$ on partitions is defined by $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i$. 


Given a partition $\gamma$, a $\gamma$-removable corner is a cell $(i,j) \in \gamma$ with $(i,j+1),(i+1,j) \not\in \gamma$ and a $\gamma$-addable corner is a square $(i,j) \not\in \gamma$ with $(i,j-1),(i-1,j) \in \gamma$. A cell $(i,j) \in \gamma$ where $(i+1,j+1) \not\in \gamma$ is called extremal. In particular, removable corners are extremal. In the skew shape $(5,3,3,2)/(1,1)$ below, all addable corners are labeled by $a$, extremals labeled by $e$, and removable corners are framed.

The hook-length of a cell $c$ in a partition $\gamma$ is the number of cells above and to the right of $c$, including $c$ itself. $h(\gamma)$ is the hook-length of $c = (0,0)$. A $p$-core is a partition that does not contain any cells with hook-length $p$. Let $C_p$ denote the collection of $p$-cores. The $p$-residue of square $(i,j)$ is $j - i \mod p$. That is, the integer in this square when squares are periodically labeled with $0,1,\ldots,p-1$, and zeros lie on the main diagonal. The $5$-residues of the $5$-core $(6,4,3,1,1,1)$ are

Hereafter we work with a fixed integer $k > 0$ and all cores/residues are $k+1$-cores/$k+1$-residues. For convenience, we refer to a corner of residue $i$ as an $i$-corner and a cell $c$ of residue $i$ is denoted by $c(i)$. The set of residues used to label cells of the shape $\gamma/\beta$ is denoted by $\text{Res}(\gamma/\beta)$.

Several basic properties of cores will be used throughout. For example, Property 15 of [LM05] is particularly useful in our study:

**Property 1.** Given cell $c(i)$ in row $r$ of a $p$-core $\gamma$,

- if $c$ lies at the end of its row then all extremals of $p$-residue $i$ in a row higher than $r$ lie at the end of their row
- if $c$ lies at the top of its column then all extremals of $p$-residue $i$ in a row lower than $r$ lie at the top of their column.

Note then that a core never has both an addable and a removable corner of the same residue. Further, given a core $\gamma$ and any addable $i$-corner, the shape obtained by adding all $i$-corners to $\gamma$ is also core. A bijection $p$ from $k+1$-cores to $k$-bounded partitions was defined in [LM05] by the map

$$p(\gamma) = \lambda,$$

where $\lambda$ is obtained by deleting all hooks larger than $k$ from $\gamma$ and reading the rows of the resulting skew shape. We denote the inverse by $c = p^{-1}$. Note that $|\lambda|$ is the number of $k$-bounded hooks in $c(\lambda)$.

**Remark 2.** (Proposition 22 in [LM05]) If $\gamma$ is obtained by adding all addable corners of some residue to a core $\beta$, then $|p(\gamma)| = |p(\beta)| + 1$.

Our study also requires the use of compositions where the length $\ell(\alpha)$ denotes the number of parts in $\alpha$. We work often with $k$-bounded compositions; $\alpha$ where $\alpha_i \leq k$. 
2.2. Symmetric functions. Let $\Lambda$ denote the ring of symmetric functions, generated by the elementary symmetric functions $e_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$, or equivalently by the complete functions $h_r = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}$. The Hall inner product on $\Lambda$ is defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda \mu},$$

where $m_\mu$ is a monomial symmetric function. Complete details on symmetric functions can be found in e.g. [Mac95, Sta99, Las03].

Schur functions $s_\lambda$ are the orthonormal basis with respect to the Hall inner product. They can be combinatorially defined using (semi-standard) tableaux – the filling of a Ferrers shape with integers that strictly increase in columns and are not decreasing in rows. The weight of tableau $T$ is the composition $w(T) = \alpha$ where $\alpha_i$ is the multiplicity of $i$ in $T$. Schur functions are the weight generating functions:

$$s_\lambda = \sum_{\text{shape}(T) = \lambda} x^{w(T)}.$$ (14)

Tableaux can equivalently be viewed as a sequence of shapes differing by horizontal strips where a horizontal $r$-strip is a skew shape with $r$ cells and whose columns have at most one cell. To be precise, a semi-standard tableau of weight $\alpha$ and shape $\lambda$ is a sequence

$$\emptyset \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(\ell(\alpha))} = \lambda,$$

where $\lambda^{(x)}/\lambda^{(x-1)}$ is an $\alpha$-strip, for $x = 1, \ldots, \ell(\alpha)$.

2.3. Set-valued tableaux. The weight generating function for Grothendieck polynomials is given in [Buc02] using set-valued tableaux – fillings of a Ferrers shape with sets of integers where a set $X$ below (west of) $Y$ satisfies $\max X \leq \min Y$. The weight is again the composition determined by the multiplicities of each letter. For any partition $\lambda$, the symmetric Grothendieck polynomial is

$$G_\lambda = \sum_{\text{set valued } T} (-1)^{|w(T)|+|\lambda|} x^{w(T)}.$$ (15)

A set-valued tableau where every cell contains a set of cardinality one is a semi-standard tableau of the same shape $\lambda$ and weight $\alpha$. This occurs iff $|\alpha| = |\lambda|$ and otherwise $|\alpha| > |\lambda|$. If $K_{\lambda \alpha}$ enumerates the set-valued tableaux of shape $\lambda$ and weight $\alpha$, then

$$G_\lambda = \sum_{\substack{|\mu| \geq |\lambda|}} (-1)^{|\lambda|+|\mu|} K_{\lambda \mu} m_\mu = s_\lambda + \text{terms of higher degree}.$$ (16)

Dual Grothendieck polynomials $g_\lambda$ (e.g. [Len00, SZ, LP]) can be defined by

$$\langle g_\lambda, G_\mu \rangle = \delta_{\lambda \mu}.$$ (17)

Duality and (17) imply that

$$h_\mu = \sum_{|\lambda| \leq |\mu|} (-1)^{|\lambda|+|\mu|} K_{\lambda \mu} g_\lambda.$$ (18)
by the triangularity of $h_\mu$ in terms of Schur functions.

2.4. $k$-tableaux. Let $\gamma$ be a $k+1$-core and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of $[p(\gamma)]$. A “$k$-tableau” of shape $\gamma$ and weight $\alpha$ is a semi-standard filling of $\gamma$ with integers $1, 2, \ldots, r$ such that the collection of cells filled with letter $i$ are labeled by exactly $\alpha_i$ distinct $k+1$-residues.

**Example 3.** The $3$-tableaux of weight $(1, 3, 1, 2, 1, 1)$ and shape $(8, 5, 2, 1)$ are:

![Example 3](image)

**Remark 4.** When $k \geq h(\gamma)$, a $k$-tableau $T$ of shape $\gamma$ and weight $\alpha$ is a semi-standard tableau of weight $\mu$ since no two diagonals of $T$ can have the same residue.

The symmetric family of dual $k$-Schur functions was introduced in [LM08] and defined to be the weight generating function of $k$-tableaux: for any $k$-bounded partition $\lambda$,

$$
\mathcal{S}_\lambda^{(k)} = \sum_{T: \text{shape}(T) = \epsilon(\lambda)} x^{w(T)}. 
$$

(20)

It is shown in [LM05] that the number $K^{(k)}_{\mu\lambda}$ of $k$-tableaux of shape $\epsilon(\mu)$ and weight $\alpha$ satisfies the property

$$
K^{(k)}_{\mu\lambda} = 0 \text{ when } \mu \not\sqsubset \lambda \text{ and } K^{(k)}_{\mu\mu} = 1, 
$$

(21)

for any $\lambda, \mu \in \mathcal{P}^k$. Therefore, the monomial expansion has the form

$$
\mathcal{S}_\lambda^{(k)} = m\lambda + \sum_{\mu \in \mathcal{P}^k} K^{(k)}_{\lambda\mu} m\mu, 
$$

(22)

revealing that $\{\mathcal{S}_\lambda^{(k)}\}_{\lambda \in \mathcal{P}^k}$ forms a basis for

$$
\Lambda/\mathcal{T}^k \text{ where } \mathcal{T}^k = \langle m_\lambda : \lambda_1 > k \rangle.
$$

This space is natural paired with

$$
\Lambda^{(k)} = \mathbb{Z}[h_1, h_2, \ldots, h_k].
$$

Since $\langle h_i : i > k \rangle$ is dual to $\mathcal{T}^k$ with respect to the Hall inner product, $\Lambda^{(k)}$ is dual to $\Lambda/\mathcal{T}^k$. The basis for $\Lambda^{(k)}$ that is dual to $\{\mathcal{S}_\lambda^{(k)}\}_{\lambda \in \mathcal{P}^k}$ is made up of the $k$-Schur functions $s_\lambda^{(k)}$. Since the matrix $[\{K^{(k)}\}_{\lambda,\mu \in \mathcal{P}^k}]$ is invertible, the system:

$$
h_\lambda = s_\lambda^{(k)} + \sum_{\mu: \mu \not\sqsubset \lambda} K^{(k)}_{\mu\lambda} s_\mu^{(k)} \text{ for all } \lambda_1 \leq k 
$$

(23)

can be taken as the definition of $k$-Schur functions [LM07].
3. Affine set-valued tableaux

3.1. **Definition.** In this section, we introduce and derive properties for a family of tableaux that generalizes both $k$-tableaux and set-valued tableaux. In subsequent sections, from these tableaux we will extract an inhomogeneous generalization of (dual) $k$-Schur functions and an affine analog of (dual) Grothendieck polynomials.

Let $T_{\leq x}$ denote the subtableau obtained by deleting all letters larger than $x$ from $T$. For example,

$$T = \begin{array}{c}
7 \\
2, 5 \\
1 \\
2, 3, 4, 6
\end{array} 
\quad T_{\leq 4} = \begin{array}{c}
2 \\
1 \\
2, 3, 4
\end{array}$$

**Definition 5.** A standard affine set-valued tableau $T$ of degree $n$ is a set-valued filling such that, for each $1 \leq x \leq n$, $\text{shape}(T_{\leq x})$ is a core and there is an $x$ exactly in all removable corners of $T_{\leq x}$ with the same residue.

**Example 6.** With $k = 2$, the standard affine s-v tableaux of degree 5 with shape $c(2, 1, 1) = (3, 1, 1)$ are

$$\begin{array}{c}
\{3, 4\} \\
\{2\} \\
\{1, 2, 3, 4, 5\}
\end{array} 
\quad \begin{array}{c}
\{3, 5\} \\
\{2\} \\
\{1, 2, 3, 4, 5\}
\end{array} 
\quad \begin{array}{c}
\{5\} \\
\{4\} \\
\{1, 2, 3, 4, 5\}
\end{array} 
\quad \begin{array}{c}
\{4\} \\
\{3\} \\
\{1, 2, 3, 4, 5\}
\end{array} 
\quad \begin{array}{c}
\{4\} \\
\{3\} \\
\{1, 2, 3, 4, 5\}
\end{array}$$

To define the semi-standard case of affine s-v tableaux, we impose certain conditions on certain reading words of standard affine s-v tableaux. Recall that the reading word of a semi-standard tableau is made of the letter $s$ from top to bottom and left to right. Semi-standard tableaux of weight $\alpha$ are in bijection with standard tableaux having increasing reading words in the alphabets

$$A_{\alpha,x} = [1 + \Sigma^{x-1} \alpha, \Sigma^x \alpha] \quad \text{where} \quad \Sigma^x \alpha = \sum_{i \leq x} \alpha_i,$$

for $x = 1, \ldots, \ell(\alpha)$. This arises through $\alpha$-standardization defined on $T$ iteratively as follows: start with $r = |\alpha|$ and relabel the rightmost $\ell(\alpha)$ by $r$. Let $r = r - 1$ and repeat. When there are no $\ell(\alpha)$ remaining, perform the relabeling of $\ell(\alpha) - 1$. $\alpha$-standardization applies to set-valued tableaux as well, where the reading word is defined as usual and letters in the same cell are read in decreasing order.

It is in this spirit that we have defined our affine $K$-theoretic generalization of tableaux. Since letters in standard affine s-v tableaux can occur with multiplicity, we must consider the **lowest reading word** in $A$, obtained by reading the lowest occurrence of the letters in $A$ from top to bottom and left to right. Again, letters
in the same cell are read in decreasing order. In Example 6 the lowest reading words in \{1, \ldots, 5\} are 21435, 52134, 52134, 32145, 32145, 51324, 51243, 41253.

**Definition 7.** For any \( k \)-bounded composition \( \alpha \), an affine s-v tableau of weight \( \alpha \) is a standard affine s-v tableau of degree \( |\alpha| \) where, for each \( 1 \leq x \leq \ell(\alpha) \),

1. the lowest reading word in \( A_{\alpha,x} \) is increasing
2. the letters of \( A_{\alpha,x} \) occupy \( \alpha_x \) distinct residues
3. the letters of \( A_{\alpha,x} \) form a horizontal strip.

Let \( T^k(\lambda) \) denote the affine s-v tableaux of shape \( \epsilon(\lambda) \) and weight \( \alpha \) and \( T^k(\lambda) \) be those of shape \( \epsilon(\lambda) \) and any weight. Note the definition extends simply to skew affine s-v tableaux by deleting letters \( 1, \ldots, a \) from an affine s-v tableau.

**Example 8.** The set \( T^2_{(2,1,1,1)}(2,1,1) \) of affine s-v tableaux with weight \( (2,1,1,1) \) contains the affine s-v tableaux in \( \{25\} \) of Example 6.

3.2. Retrieving set-valued and \( k \)-tableaux. To justify that affine s-v tableaux are in fact an affine \( K \)-theoretic version of tableaux, we connect them to \( k \)-tableaux and set-valued tableaux.

**Proposition 9.** For any \( k \)-bounded partition \( \lambda \) where \( h(\lambda) \leq k \), the affine s-v tableaux of shape \( \lambda \) are simply the set-valued tableaux of shape \( \lambda \).

**Proof.** Given \( h(\lambda) \leq k \), \( \epsilon(\lambda) = \lambda \) and no two diagonals of \( \lambda \) have the same residue. Thus, an affine s-v tableau of weight \( \alpha \) is a set-valued tableau with no repeated letters. The substitution of all letters in \( A_{\alpha,x} \) by \( x \) gives a set-valued tableau of weight \( \alpha \) since conditions on affine s-v tableaux imply these letters form a horizontal strip and occupy \( \alpha_x \) distinct cells. On the other hand, the \( \alpha \)-standardization of a set-valued tableau of weight \( \alpha \) ensures that the reading word of the \( \alpha_x \) letters in \( A_{\alpha,x} \) is increasing and forms a horizontal strip.

To make the connection with \( k \)-tableaux, we need several basic properties of affine s-v tableaux. We say that \( x \) is lonely when a letter \( x \) occurs in cell without another letter.

**Property 10.** Given an affine s-v tableau \( T \), \( \text{shape}(T) / \text{shape}(T_{\leq x}) \neq \emptyset \) if and only if every \( x \) is lonely in \( T_{\leq x} \), for any letter \( x \in T \).

**Proof.** If \( \text{shape}(T_{\leq x}) = \text{shape}(T_{\leq x-1}) \) then no \( x \) can be lonely since \( T_{\leq x-1} \) is obtained by deleting the letter \( x \) from \( T_{\leq x} \). On the other hand, we will show that if \( T_{\leq x} \) has some \( x \) that is not lonely then no \( x \) are lonely implying the shapes must be equal. Suppose \( T_{\leq x} \) has a cell of some residue \( i \) containing \( x \) by itself and one containing \( x \) with another letter. Then \( \text{shape}(T_{\leq x-1}) \) has both a removable and an addable \( i \)-corner. This contradicts that a core never contains an addable and removable \( i \)-corner.

**Property 11.** Given an affine s-v tableau \( T \), if \( \gamma(x) = \text{shape}(T_{\leq x}) \) then

\[
|p(\gamma(x))| = |p(\gamma(x-1))| + 1 \quad \text{for any} \quad \gamma(x) \neq \gamma(x-1).
\]

**Proof.** Given \( T \) is an affine s-v tableau, there is an \( x \) in all removable \( i \)-corners of the core \( \gamma(x) \). If \( \gamma(x) \neq \gamma(x-1) \) then these \( x \) are all lonely by Property 10. Therefore, \( \gamma(x) \) is \( \gamma(x-1) \) plus addable \( i \)-corners and the result follows from Remark 2.

\[\square\]
From Property [11] an affine s-v tableau \( T \) of weight \( \alpha \) satisfies \(|p(\text{shape}(T_{k,x}))| = |p(\text{shape}(T_{k,x-1}))| + \{0, 1\}\) for all \( x = 1, 2, \ldots, |\alpha| \). Consequently:

**Corollary 12.** If there is an affine s-v tableau of weight \( \alpha \) and shape \( c(\lambda) \), then \(|\alpha| \geq |\lambda|\).

We are now prepared to prove that the family of affine s-v tableaux includes \( k \)-tableaux.

**Proposition 13.** The set of affine s-v tableaux \( T^k_\alpha(\lambda) \) when \(|\alpha| = |\lambda|\) is the set of \( k \)-tableaux with weight \( \alpha \) and shape \( c(\lambda) \).

**Proof.** Consider an affine s-v tableau \( T \) of weight \( \alpha \) and shape \( \gamma = c(\lambda) \) where \( n = |\alpha| = |\lambda| \). For \( x = 1, \ldots, n \), let \( \lambda^{(x)} = p(\text{shape}(T_{k,x})) \). Since \(|\lambda^{(x)}| = |\lambda^{(x-1)}| + \{0, 1\}\) by Property [11] and \(|\lambda^{(n)}| = n\), \(|\lambda^{(x)}| = |\lambda^{(x-1)}| + 1\) for all \( x \). Thus, \( \lambda^{(x-1)} \neq \lambda^{(x)} \) which implies that all \( x \) are lonely in \( T_{k,x} \) by Property [11] for all \( x \). Therefore no cell of \( T \) has a set of cardinality more than one. The conditions on affine s-v tableau imply that replacing all letters in \( A_{\alpha,x} \) by \( x \) gives a semi-standard filling where \( x \) occupies \( \alpha_x \) distinct residues and thus, \( T \) is a \( k \)-tableau.

On the other hand, given a \( k \)-tableau \( T \) of weight \( \alpha \), we can \( \alpha \)-standardize \( T \) iteratively from \( r = |\alpha| \) as follows: relabel every \( x(i) \in T \) by \( r \), where \( x(i) \) is the rightmost letter of \( T \) that is not larger than \( r \) and \( i \) is its residue. Let \( r = r - 1 \). It was shown in [LM05] that the resulting tableau \( U \) is a standard \( k \)-tableau and by construction, the lowest reading word in \( A_{\alpha,x} \) is clearly increasing. Therefore, since the letter \( x \) occupies \( \alpha_x \) distinct residues and forms a horizontal strip in \( T \), \( U \) meets the conditions of an affine s-v tableau of weight \( \alpha \).

4. Alternate characterizations

4.1. Affine Weyl group characterization. In the theory of \( k \)-Schur functions, the discovery that \( k \)-tableaux are reduced words for grassmannian permutations in the affine symmetric group was the spring board to understanding the \( k \)-Schur role in geometry. Here, we investigate a similar interpretation for affine s-v tableaux.

To start, we consider the standard case and recall results in the \( k \)-tableaux case.

Let \( \tilde{S}_n \) denote the affine Weyl group of \( A_{n-1} \), generated by \( \langle s_0, s_1, \ldots, s_{n-1} \rangle \) and satisfying the relations

\[
\begin{align*}
    s_i^2 &= 1 \quad \text{for all } i \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \quad \text{for all } i \\
    s_is_j &= s_js_i \quad \text{if } |i - j| > 1,
\end{align*}
\]

(27)

where indices are taken modulo \( n \) (hereafter we will always be working mod \( n \)). A word \( i_1i_2\cdots i_m \) in the alphabet \( \{0, 1, \ldots, n - 1\} \) corresponds to the permutation \( w \in \tilde{S}_n \) if \( w = s_{i_1} \ldots s_{i_m} \). The length \( \ell(w) \) of \( w \) is defined to be the length of its shortest word. Any word of this length is said to be reduced and we denote the set of all reduced words for \( w \) by \( R(w) \). The set \( \tilde{S}_n^0 \) of grassmannian elements are the minimal length coset representatives of \( \tilde{S}_n/S_n \), where \( S_n \) is the finite symmetric group. In fact, \( w \in \tilde{S}_n^0 \) iff every reduced word for \( w \) ends in 0.

Consider operators on set-valued tableaux defined for \( i = 0, \ldots, k \) by

\[
s_{i,x} : \tilde{T} \rightarrow T,
\]
where $T$ is obtained by adding an $x$ to all addable or all removable $i$-corners of $\hat{T}$. It turns out \cite{LM05} that the set of reduced words for a fixed $w \in S_{k+1}^0$ is in bijection with the set of $k$-tableaux of some shape $c(\lambda)$. In particular, each reduced word $i_\ell i_\ell \cdots i_1$ for $w \in S_{k+1}^0$ is sent to the standard $k$-tableau $s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1} \emptyset$ on $\ell$ letters.

It is natural to work with operators defined on shapes, for $i = 0, \ldots, k$, by

$$s_i : \gamma \to \gamma + \text{ its addable } i\text{-corners}.$$  

These can be viewed as an affine analog of operators introduced in \cite{Buc02} and where the set-up follows that of \cite{FG98}.

**Remark 14.** A number of useful properties are satisfied by the $s_i$ operators.

1. If $\beta$ is a core with an addable $i$-corner, then $s_i(\beta)$ is a core and $|p(s_i(\beta))| = |p(\beta)| + 1$ by Remark 14(2).

2. The bijection between $\hat{S}_{k+1}^0$ and $\mathcal{C}^k$ defined by the map

$$t : w \to c(\lambda) = s_{i_k} \cdots s_{i_1} \emptyset,$$

for any $i_k \cdots i_1 \in R(w)$, has the property that $|\lambda| = \ell(w)$ since a word of length $\ell$ corresponds to a $k$-tableaux on $\ell$ letters.

3. If $i_\ell i_{\ell-1} \cdots i_1$ is a reduced word for an affine grassmannian permutation, then $s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1} \emptyset$ has an addable $i_j$-corner for all $j < \ell$ by (1) and (2).

4. When acting on cores, we have the relations

$$s_i^2 = s_i \text{ for all } i, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ when } |i - j| > 1.$$

Denote the affine grassmannian permutation associated to $\lambda \in P^k$ by

$$w_\lambda = t^{-1}(c(\lambda)).$$

**Remark 15.** A quick way to construct $w_\lambda$ from $\lambda$ is to take the residues of $\lambda$ read from **right** to **left** and to **top** to **bottom** \cite{LM05}. For example, $w_{(2,1,1)} \in \hat{S}_3$ is

$$\begin{bmatrix}
3 & 3 \\
2 & 2 \\
1 & 1 \\
\end{bmatrix} \rightarrow 1210 \quad (28)$$

The relations in Remark 14(4) arise in the nil Hecke algebra. We will discuss this further in \S 5 but for now are interested in studying the equivalence classes of words under these relations. To be precise, we consider the set $\mathcal{W}(w_\lambda)$ of all words whose reduced expression is in $R(w_\lambda)$ under the relations

$$u_i^2 = u_i \quad \text{for all } i,$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad \text{for all } i,$$

$$u_i u_j = u_j u_i \quad \text{when } |i - j| > 1 \quad (29)$$

Note that $R(w_\lambda) = \{ u \in \mathcal{W}(w_\lambda) : \ell(u) = \ell(w_\lambda) \}$.

**Remark 16.** For any $i_\ell \cdots i_1 \in \mathcal{W}(w_\lambda)$, $s_{i_\ell} \cdots s_{i_1} \emptyset$ has a removable or an addable $i_m$-corner for all $m \leq \ell$. In particular, $i_\ell \cdots i_1 \in \mathcal{W}(w_\lambda)$ if and only if $i_1 = 0$. This follow by iterating the following argument: if $i_{\ell+1} i_\ell \cdots i_1 \in \mathcal{W}(w_\mu)$ then there is some $\ell$ where $j_\ell \cdots j_1 \cdots j_1 \sim i_\ell i_{\ell-1} \cdots i_1$, for $\ell = \ell(w_\mu)$. Then $s_{j_\ell} \cdots s_{j_1} \emptyset$ has an addable $j_m$-corner by Remark 14(3) implying $s_{j_\ell} \cdots s_{j_1} s_{j_\ell} \cdots s_{j_1} \emptyset$ has an addable or removable $j_m$-corner. Therefore, $s_{i_{\ell+1}} \cdots s_{i_1} \emptyset$ has an addable or a removable $i_m$-corner for any $m \leq \ell + 1$ by Remark 14(4).
Just as the $k$-tableaux represent reduced words for affine Grassmannian permutations, we find that standard affine s-v tableaux of fixed shape $\gamma(\lambda)$ are none other than the words whose reduced expression is in $\mathcal{R}(w_\lambda)$. 

**Proposition 17.** For $\lambda \in \mathcal{P}^k$, there is a bijection $s : \mathcal{W}(w_\lambda) \rightarrow T_{1m}^k(\lambda)$ defined by $s(i_m \cdots i_1) = s(i_{m,m} \cdots s_{i_{m-1,m-1}} \cdots s_{i_1,1})$. 

**Proof.** Given $i_m \cdots i_1 \in \mathcal{W}(w_\lambda)$, we claim that $T = s(i_m \cdots i_1)$ is a standard affine s-v tableau of degree $m$. Since $T_{\leq 1} = [1]$, assume by induction that $T_{\leq m-1}$ is a standard affine s-v tableau of core shape $\gamma$. Remark 16 implies $\gamma$ has either an addable or a removable $i_m$-corner and thus $T = T_{\leq m}$ is obtained by putting an $m$ in all such corners. Therefore the shape of $T$ is either $\gamma$ or $\gamma$ plus its addable $i_m$-corners; in both cases it is a core. Further, $T$ has shape $\gamma(\lambda)$ by Remark 14.

For any $T \in T_{1m}^k(\lambda)$, $T = s(j_m,j_{m-1},\cdots,j_1)$ where $j_a$ denotes the residue of letter $a$ in $T$. Thus, $\gamma(\lambda) = s(i_m \cdots s_{i_1,1})$ and to prove $s$ is onto, we need $j_m \cdots j_1 \in \mathcal{W}(w_\lambda)$. Consider a reduced word $j_m' \cdots j_1'$ equivalent to $j_m \cdots j_1$. Remark 14 implies $\gamma(\lambda) = s(j_m' \cdots j_1')$ and therefore $j_m' \cdots j_1' \in \mathcal{R}(w_\lambda)$.

To see that $s$ is 1-1, consider $s(i_m \cdots i_1) = s(i_{m,m} \cdots i_1)$ for $i_m \cdots i_1, i_m, \cdots, i_1 \in \mathcal{W}(w_\lambda)$. By definition, $s_{i_{m,m}}T$ has an $x$ in the addable or removable $i$-corners of $T$. Therefore, if there is some (minimal) $j$ where $i_j \neq i_j$ clearly $s_{i_{j,j}}T \neq s_{i_{j,j}}T$. 

4.2. **Horizontal strip characterization.** In classical and $k$-tableaux theory, characterizing tableaux as a sequence of shapes satisfying horizontality conditions has many applications. Most notably, the Pieri rule for affine and usual Schur functions is readily apparent in this context. Moreover, given such an interpretation, the tie between the affine Weyl group and $k$-tableaux in the semi-standard case can be made. With this in mind, we set-out to coin affine s-v tableaux in terms of certain horizontal strips.

We start with the characterization of set-valued and $k$-tableaux as a sequence of shapes satisfying horizontality conditions.

A **set-valued** $r$-strip $(\gamma/\beta, \rho)$ is such that

1. $\gamma/\rho$ is a horizontal $r$-strip
2. $\beta/\rho$ is a set of $r - |\gamma/\beta|$ $\beta$-removable corners

The set-valued tableaux of weight $\alpha$ and shape $\lambda$ are in bijection with the set of sequences having the following form:

$$(0,0) \subseteq (\lambda^{(1)}, \rho^{(1)}) \subseteq (\lambda^{(2)}, \rho^{(2)}) \subseteq \cdots \subseteq (\lambda^{(\ell(\alpha))}, \rho^{(\ell(\alpha))})$$

where $(\lambda^{(x)}, \rho^{(x)})$ is a set-valued $\alpha_x$-strip, for $x = 1, \ldots, \ell(\alpha)$.

For $0 \leq r \leq k$, an **affine** $r$-strip $\gamma/\beta$ is a horizontal strip where

1. $\alpha_1$: $\gamma$ and $\beta$ are cores
2. $\alpha_2$: $|\gamma(\beta)| - |\beta(\alpha)| = r$
3. $\alpha_3$: $\gamma/\beta$ occupies $r$ distinct residues.

It can be deduced from results in [LM05] that $k$-tableaux of weight $\alpha$ and shape $\gamma$ are in bijection with chains of the form:

$$0 \subseteq \gamma^{(1)} \subseteq \gamma^{(2)} \subseteq \cdots \subseteq \gamma^{(\ell(\alpha))} = \gamma,$$

where $\gamma^{(x)}/\gamma^{(x-1)}$ is an affine $\alpha_x$-strip, for $x = 1, \ldots, \ell(\alpha)$.
We impose additional conditions on these strips in order to view affine s-v tableaux in a similar way. Given cores $\beta \subseteq \gamma$, a cell of $\beta$ that lies below a cell of $\gamma$ is $\gamma$-blocked.

**Definition 18.** For $0 \leq r \leq k$, an “affine set-valued” $r$-strip $(\gamma/\beta, \rho)$ is such that
- **asv1:** $\gamma/\rho$ is a horizontal strip
- **asv2:** $\gamma/\beta$ is an affine horizontal $r - m$-strip, where $m = |\text{Res}(\beta/\rho)|$.
- **asv3:** $\beta/\rho$ is a subset of $\beta$-removable corners such that if $c(i) \in \beta/\rho$ then all $\beta$-removable $i$-corners that are not $\gamma$-blocked lie in $\beta/\rho$.

**Remark 19.** A priori, if $(\gamma/\beta, \rho)$ is an affine s-v $r$-strip then $\gamma/\beta$ is an affine strip. In fact, $\gamma/\beta$ is an affine $r$-strip when $\beta = \rho$. When $k = \infty$, an affine s-v strip is simply a set-valued strip since residues can occur at most once in a horizontal strip.

To prove that affine s-v tableaux can be characterized in terms of affine s-v strips, we need a number of properties about these and affine strips.

**Proposition 20.** If $\gamma/\beta$ is an affine $r$-strip and $i$ is the residue of its rightmost cell, then $\gamma/\beta$ is an affine $r - 1$-strip where $\gamma$ is $\gamma$ minus its removable $i$-corners.

*Proof.* Since $|p(\gamma)| = |p(\beta)| - 1$ by Remark 2, it suffices to prove that $\beta \subseteq \gamma$. To this end, we will prove that the $\gamma$-removable $i$-corners are exactly the cells of residue $i$ in $\gamma/\beta$, where $i$ is the residue of the rightmost cell in $\gamma/\beta$. Since $\gamma/\beta$ is a horizontal strip whose rightmost cell has residue $i$ then all cells of residue $i$ in $\gamma/\beta$ are $\gamma$-removable by Property 1 since they are all extremal in $\gamma$. It thus remains to show that any $\gamma$-removable $i$-corner is in $\gamma/\beta$.

Suppose by contradiction there is a $\gamma$-removable $c(i)$ that is also $\beta$-removable (choose the lowest). Note that it lies higher than any $a(i) \in \gamma/\beta$ since the $z(i+1)$ beneath $a$ is at the top of its column in $\beta$ implying that all extremals of residue $i + 1$ lower than $a$ are at the top of their column in $\beta$ by Property 1. Then, the cell left-adj to any $a$ must be in $\gamma/\beta$ since otherwise it would end its row in $\beta$ whereas the extremal left-adj to $c(i)$ does not (contradicting Property 1). Therefore, the column with $c$ has more $k$-bounded hooks in $\beta$ than in $\gamma$. Let $a_1, \ldots, a_r$ denote the lowest cells of residues $i_1, \ldots, i_r$, respectively, in $\gamma/\beta$. Each of these cells lies in a column with one more $k$-bounded hook in $\gamma$ than in $\beta$ by Property 1. In fact, these are the only columns where $\gamma$ has more $k$-bounded hooks than $\beta$. Since $|p(\gamma)| - |p(\beta)| = r$ it must be that no column of $\beta$ has more $k$-bounded hooks than in $\gamma$ and we have our contradiction. \qed

**Property 21.** Let $\gamma/\beta$ be an affine $r$-strip for some $r \leq k$. If $\beta$ has a removable $i$-corner that is not $\gamma$-blocked then $i \notin \text{Res}(\gamma/\beta)$.

*Proof.* Consider cell $c(i)$, in some row $r_c$, that is $\beta$-removable and at the top of its column in $\gamma$. Suppose by contradiction that there is a $y(i) \in \gamma/\beta$ and let $r_y$ be the lowest row containing such a cell. With $\gamma = \gamma^{(0)}$, let $\gamma^{(j)}$ be the core obtained by deleting all $i_j$-corners of $\gamma^{(j-1)}$, where $i_j$ is the residue of the rightmost element in $\gamma^{(j-1)}/\beta$. Proposition 20 implies that $\gamma^{(j)}/\beta$ is an affine strip.

Note that $y(i) \in \gamma/\beta$ implies that there is some $t$ where $i_t = i$. Since $y$ is the rightmost element of $\gamma^{(t-1)}/\beta$ and all cells in the same row and to the right of a $\beta$-removable cell are in $\gamma/\beta$, $c(i)$ lies at the end of its row in $\gamma^{(t-1)}$ (and thus in $\gamma^{(t)}$) if $r_c < r_y$. In this case, the cell below $c(i)$ is extremal in $\gamma^{(t)}$. However, the cell below $y(i)$ lies at the top of its column in $\gamma^{(t)}$ violating Property 1. Therefore,
Since $\gamma/\rho$ is horizontal for any affine s-v strip $(\gamma/\beta, \rho)$, no cell in $\beta/\rho$ is $\gamma$-blocked. We therefore deduce that:

**Corollary 22.** For any affine set-valued r-strip $(\gamma/\beta, \rho)$, $\text{Res}(\gamma/\beta) \cap \text{Res}(\beta/\rho) = \emptyset$.

In particular, since $|\text{Res}(\gamma/\beta)| = r - m$ by the definition of affine strip, we have:

**Corollary 23.** For any affine set-valued r-strip $(\gamma/\beta, \rho)$, $|\text{Res}(\gamma/\rho)| = r$.

**Property 24.** If $(\gamma/\beta, \rho)$ is an affine s-v strip and $i$ is the residue of the rightmost cell in $\gamma/\rho$ then the cells of residue $i$ in $\gamma/\rho$ are exactly the $\gamma$-removable $i$-corners.

**Proof.** If the rightmost cell $c(i)$ of $\gamma/\rho$ is in $\gamma/\beta$, Property 20 implies that the $\gamma$-removable $i$-corners are exactly the cells of $\gamma/\beta$ with residue $i$, which are exactly the cells of $\gamma/\rho$ with residue $i$ by Corollary 22. If $c(i) \in \beta/\rho$, note that all cells of residue $i$ in $\gamma/\rho$ are $\gamma$-removable corners since the lowest cell of residue $i$ in $\gamma/\rho$ is at the end of its row implying all $i$-extremals are at the end of their row in $\gamma$ by Property 1. If there is a $\gamma$-removable corner $\bar{c}(i) \notin \gamma/\rho$, then $\bar{c} \in \rho \subseteq \beta \subseteq \gamma$ implies $\bar{c} \notin \beta/\rho$ and is a $\beta$-removable $i$-corner. Therefore $\bar{c}$ is $\beta$-blocked by definition of affine s-v strip, contradicting that $\bar{c}$ is $\gamma$-removable.

**Proposition 25.** Consider the affine s-v strip $(\gamma/\beta, \rho)$ whose rightmost cell in $\gamma/\rho$ has residue $i$. If $i \in \text{Res}(\gamma/\beta)$ then $(\hat{\gamma}/\beta, \rho)$ is an affine s-v strip where $\hat{\gamma}$ is $\gamma$ minus its $i$-corners. Otherwise, $(\gamma/\beta, \rho)$ is an affine s-v strip where $\hat{\rho}$ is $\rho$ plus its $i$-corners.

**Proof.** Let $c(i)$ be the rightmost cell of $\gamma/\rho$. If $c(i) \in \beta/\rho$ then $i \notin \text{Res}(\gamma/\beta)$ by Corollary 22 and there are no $\gamma$-addable $i$-corners since $c(i)$ is $\gamma$-removable by Property 23 and $\gamma$ is a core. Therefore, all addable $i$-corners of $\rho$ are in $\beta/\rho$ implying $\hat{\rho} \subseteq \beta$ and thus that $(\gamma/\beta, \hat{\rho})$ is an affine s-v strip.

If $c(i) \in \gamma/\beta$ then Proposition 20 implies (asv1) and (asv2). Now suppose there are $\beta$-removable corners $c_1(j) \in \beta/\rho$ and $c_2(j) \notin \beta/\rho$. Since $(\gamma/\beta, \rho)$ is an affine s-v strip, $c_2(j)$ must be $\gamma$-blocked. If, by contradiction, $c_2$ is not $\gamma$-blocked then $j = i + 1$. $c(i)$ lies at the end of its row in $\gamma$ implying all higher $i$-extremals are at the end of their row in $\gamma$ by Property 1. However, by horizontality of $\gamma/\rho$, the cell left-adj to $c_1$ is an $i$-extremal.

Now we are equipped to rephrase the definition of affine s-v tableaux. For $\lambda \in \mathcal{P}_k$, consider the set of pairs obtained by adding affine s-v strips to $c(\lambda)$,

$$\mathcal{H}^k_{\lambda, \tau} = \{ (\mu, \rho) : (c(\mu)/c(\lambda), \rho) = \text{affine set-valued r-strip} \}.$$ 

**Theorem 26.** For any $\lambda \in \mathcal{P}_k$ and $k$-bounded composition $\alpha$, there is a bijection between $\mathcal{T}_k^k(\lambda)$ and

$$\left\{ (\emptyset, \emptyset) \subset (\gamma^{(1)}, \rho^{(1)}) \subseteq \cdots \subseteq (\gamma^{(t(\alpha))}, \rho^{(t(\alpha))}) : (p(\gamma^{(x)}), \rho^{(x)}) \in \mathcal{H}^k_{p(\gamma^{(x-1)}), \alpha_{x}} \right\},$$

where $\gamma^{(0)} = \emptyset$ and $\gamma^{(t(\alpha))} = c(\lambda)$. 
Proof. \((\Leftarrow):\) For \(x = 1, \ldots, \ell(\alpha)\), we will construct \(T\) by filling the cells of \(\gamma(x)/\rho(x)\) iteratively as follows: start with \(N = \Sigma^x\alpha\). Let \(i\) denote the residue of the rightmost cell in \(\gamma(x)/\rho(x)\) that contains no letter larger than \(N\). Put letter \(N\) in all cells of \(\gamma(x)/\rho(x)\) with residue \(i\). Let \(N = N - 1\) and repeat.

Since \(\gamma(0) = \emptyset\), \(\gamma(1)\) is the row shape \((\alpha_1)\) and thus \(T_{\leq \alpha_1} = \emptyset\) and is an affine \(s\)-v tableau of weight \((\alpha_1)\). By induction, assume \(T_{\leq \Sigma^x-1\alpha}\) is an affine \(s\)-v tableau of weight \((\alpha_1, \ldots, \alpha_{x-1})\). Note that Conditions (1) and (3) on affine \(s\)-v tableaux are met by construction since \(\gamma(x)/\rho(x)\) is a horizontal strip. Further, Corollary 23 implies Condition (2). It thus remains to show that \(T_{\leq \Sigma^x \alpha - j}\) is a set-valued tableau with \(\Sigma^x\alpha - j\) in all removable corners of some residue and its shape \(\tau(j)\) is a core, for \(j = 0, \ldots, \alpha_x - 1\).

Let \(\beta = \gamma^1(x)\) and \(\eta(0) = \rho(x)\). Note that \(\tau(0) = \gamma(x)\). By construction, there is an \(N = \Sigma^x\alpha\) in all cells of \(\tau(0)/\eta(0)\) with residue \(i_0\), where \(i_0\) is the residue of the rightmost cell \(c\) in \(\tau(0)/\eta(0)\). Property 24 implies there is a \(N\) in exactly the \(\tau(0)\)-removable \(i_0\)-corners.

Note that \(\tau(1)\) is \(\tau(0)\) minus cells containing a lonely \(N\). If \(c \in \tau(0)/\beta\) this is \(\tau(0)\) minus its \(i_0\)-corners and otherwise \(\tau(1) = \tau(0)\). In the later case, set \(\eta(1) = \eta(0)\) plus its \(i_0\)-corners and otherwise let \(\eta(1) = \eta(0)\). Proposition 25 thus implies that \((\tau(1)/\beta, \eta(1))\) is an affine \(s\)-v strip and in particular, \(\tau(1)\) is a core. By construction, there is an \(N - 1\) in all cells of \(\tau(1)/\eta(1)\) with residue \(i_1\), where \(i_1\) is the residue of the rightmost cell \(c_1 \in \tau(1)/\eta(1)\). Thus, there is an \(N - 1\) in exactly the \(\tau(1)\)-removable \(i_1\)-corners by Property 24. Iterating this argument proves the claim.

\((\Rightarrow)\) Given \(U \in T_\alpha^\Sigma(\lambda)\), let \(\gamma(x) = \text{shape}(U_{\leq \Sigma^x\alpha})\) and \(\rho(x)\) be the shape obtained by deleting any cell containing an element of \(A_{\alpha,x}\) from \(U_{\leq \Sigma^x\alpha}\), for \(x = 1, \ldots, \ell(\alpha)\). We claim that each \((\gamma(x)/\gamma^1(x-1), \rho(x))\) is an affine set-valued \(\alpha_x\)-strip.

Let \(T = U_{\leq \Sigma^x\alpha}\) and note that
\[
\begin{align*}
\gamma(x)/\rho(x) &\text{ are the cells in } T \text{ containing a letter in } A_{\alpha,x} \\
\gamma(x-1)/\rho(x) &\text{ are the cells in } T \text{ containing a letter in } A_{\alpha,x} \text{ and a letter weakly smaller than } \Sigma^{x-1}\alpha \\
\gamma(x)/\gamma^1(x-1) &\text{ are the cells in } T \text{ containing only letters larger than } \Sigma^{x-1}\alpha.
\end{align*}
\]

The definition of affine \(s\)-v tableaux implies that \(\gamma(x)/\rho(x)\) is a horizontal strip and that \(\gamma(x)\) and \(\gamma(x-1)\) are cores.

To show that \(\gamma(x)/\gamma^1(x-1)\) is an affine \(\alpha_x - |Res(\gamma(x-1)/\rho(x))|\)-strip, we note it is a horizontal strip since \(\rho(x) \geq \gamma(x-1)\). We claim that no letter lies in both \(\gamma(x)/\gamma^1(x-1)\) and \(\gamma(x-1)/\rho(x)\) implying (af3) since the \(\alpha_x\) letters of \(A_{\alpha,x}\) each occupy a distinct residue and lie in \(\gamma(x)/\rho(x)\). To this end, suppose \(y = a + \Sigma^{x-1}\alpha \in \gamma(x)/\gamma^1(x-1)\). It must be lonely in \(T_{\leq \Sigma^x}\) since \(\gamma(x)/\gamma^1(x-1)\) has no letters smaller than \(\Sigma^{x-1}\alpha\) and no two letters of \(A_{\alpha,x}\) share a cell. The proof of Property 10 then implies that all \(y\) are lonely in \(T_{\leq \Sigma^x}\) and thus all \(y \in \gamma(x)/\gamma^1(x-1)\). To verify (af2), let
\[
\tau(a) = \text{shape}(T_{\leq a + \Sigma^{x-1}\alpha}),
\]
for \(a = 1, \ldots, \alpha_x\). Since any \(a + \Sigma^{x-1}\alpha \in \gamma(x)/\gamma^1(x-1)\) is lonely, \(\tau(a) \neq \tau(a-1)\) for these \(a\) and \(|\tau(a)| = |\tau(a-1)| + 1\) by Property 11. Otherwise, \(|\tau(a)| = |\tau(a-1)|\) by definition of \(\gamma(x-1)/\rho(x)\).

To prove Condition (asv3), note that all cells of \(\gamma(x-1)/\rho(x)\) are removable corners of \(\gamma(x-1)\) since these cells contain both a letter larger and weakly smaller than \(\Sigma^{x-1}\alpha\) and rows/columns are non-decreasing. We thus must show that if \(c\) and \(\tilde{c}\)}
are removable $i$-corners of $\gamma^{(x-1)}$ where $c \in \gamma^{(x-1)}/\rho(x)$ and $\bar{c} \not\in \gamma^{(x-1)}/\rho(x)$, then $\bar{c}$ lies below a cell in $\gamma^{(x)}$.

Let $y = a + \sum x - 1 \alpha$ denote the letter of $A_{a,x}$ in cell $c$ of $T$. Then $c$ is a removable $i$-corner of the core $\tau^{(a)}$ and Definition 7 implies there must be a $y$ in all $\tau^{(a)}$-removable $i$-corners. Since $\bar{c}$ contains no letter of $A_{a,x}$, it is not $\tau^{(a)}$-removable. Therefore there is a letter $x$ in $A_{a,x}$ for some $x \leq y$ above or right-adj to $\bar{c}$. In fact, $x < y$ since $y$ occupies only cells of residue $i$. Assuming the later case, $c$ must lie weakly lower than $c$ since Property 4 implies all $i$-extremals in $\tau^{(a)}$ that are higher than $c$ must lie at the end of their row (and all removable corners of $\gamma^{(x)}$ are extremal in $\gamma^{(x)}$ by horizontality). If we choose $c$ to be the cell containing the lowest $y$, there is a letter of $A_{a,x}$ that is smaller and weakly lower than this $y$ contradicting that the lowest reading word of an affine s-v tableau is increasing. □

5. Affine Grothendieck polynomials

Affine stable Grothendieck polynomials were introduced in [Lam06] in terms of the nil Hecke algebra. Recall that the nil Hecke algebra $K$ for the type-A affine Weyl group is generated over $Z$ by $A_0, A_1, \ldots, A_k$ and relations

$A_i^2 = -A_i$ for all $i$, $A_iA_j = A_jA_i$ if $|i - j| > 2$, $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$

where the indices are taken modulo $k + 1$ [KK86]. The algebra $K$ is a free $Z$-module with basis $\{A_w : w \in \tilde{S}_{k+1}\}$ where $A_w = A_{i_1} \cdots A_{i_{\ell}}$ for any reduced word $i_1 \cdots i_{\ell}$ of $w$. In this basis, the multiplication is given by

$A_iA_u = \begin{cases} A_{si,u} & \text{if } \ell(si,u) > \ell(u) \\ -A_u & \text{if } \ell(si,u) < \ell(u) \end{cases}$

The definition of affine Grothendieck polynomials requires elements defined by cyclically decreasing permutations. To be precise, let $i_1 \cdots i_{\ell}$ be a sequence of numbers where each $i_i \in [0, k]$, $i_1 \cdots i_\ell$ is cyclically decreasing if no number is repeated and $j$ precedes $j - 1$ (taken modulo $k + 1$) when both $j, j - 1 \in \{i_1, \ldots, i_{\ell}\}$. If $i_1 \cdots i_{\ell}$ is cyclically decreasing then we say the permutation $w = s_{i_1} \cdots s_{i_{\ell}}$ is cyclically decreasing. Note that $w$ is reduced and depends only on the set $\{i_1, \ldots, i_{\ell}\}$ of indices involved. This given, consider

$h_i = \sum_{w \in \tilde{S}_{k+1} \text{ cyclically decreasing}} A_w$.

Then, for any $w \in \tilde{S}_{k+1}$, the affine stable Grothendieck polynomial is defined by

$G_w^{(k)}(x_1, x_2, \ldots) = \sum_{\alpha} \langle h_{\alpha_i} h_{\alpha_{i-1}} \cdots h_{\alpha_1}, A_w \rangle x^\alpha.$  \hspace{1cm} (31)

We can explicitly describe the coefficients in this expression using certain factorizations of permutations. Define an $\alpha$-factorization of $w$ to be a decomposition of the form $w = w^{(\alpha)} \cdots w^{(1)}$ where $w^{(i)}$ is a cyclically decreasing permutation of length $\alpha_i$. From this viewpoint, the coefficient of $A_w$ in

$h_{\alpha_i} \cdots h_{\alpha_1} = \sum_{\ell(w^{(i)}) = \alpha_i} A_{w^{(i)}} \cdots \sum_{\ell(w^{(1)}) = \alpha_1} A_{w^{(1)}}$
is the signed enumeration of \( \alpha \)-factorizations of \( w \). Therefore,
\[
G_{w}^{(k)}(x_1, x_2, \ldots) = \sum_{\alpha} (-1)^{|\alpha| - \ell(w)} \sum_{w = \alpha - \text{factorization}} x_{\alpha}.
\]  

We give a bijection between affine s-v tableaux of shape \( \lambda \) and weight \( \alpha \) and \( \alpha \)-factorizations of \( w_\lambda \). From this, the stable affine Grothendieck polynomials indexed by grassmannian permutations are none other than generating functions for affine s-v tableaux. One advantage of such an identification is that properties of the tableaux given in prior sections immediately reveal basic facts about affine Grothendieck polynomials.

**Lemma 27.** Given a cyclically decreasing word \( i_r \cdots i_1 \), let \( T = s_{i_r-1} \cdots s_{i_1-1}(\beta) \) for any core \( \beta \). If the ones in \( T \) occupy \( r \) distinct residues, then \( (\gamma/\beta, \rho) \) is an affine s-v strip for \( \gamma = \text{shape}(T) \) and \( \rho \) the shape obtained by deleting all ones from \( T \).

**Proof.** Since \( s_{i_r-1} \) is never applied after \( s_{i_1} \) by the definition of cyclically decreasing, \( \gamma/\rho \) is horizontal. Let \( \beta^{(0)} = \beta \) and set \( \beta^{(x)} = s_{i_x}(\beta^{(x-1)}) \) for \( x = 1, \ldots, r \). An element of \( \beta/\rho \) arises only when \( s_{i_{x+1}} \) is applied to \( \beta^{(x)} \) and there is \( \beta^{(x)} \)-removable \( i_{x+1} \)-corner. In this case, all \( i_{x+1} \)-corners of \( \beta^{(x)} \) are \( \beta \)-removable since \( \beta^{(x)}/\beta \) has no \( i_{x+1} \)-residue. Further, any cell \( c \) right-adj to a \( \beta \)-removable \( i_{x+1} \)-corner \( c \) has residue \( i_{x+1} + 1 \) and thus is not in \( \beta^{(x)} \) by definition of cyclically decreasing. Therefore \( c \) is either \( \beta \)-removable (and in \( \beta/\rho \)) or it is \( \gamma \)-blocked.

It thus remains to prove that \( \gamma/\beta \) is an affine \( r - m \) strip, where \( m = |\text{Res}(\beta/\rho)| \). Since the ones in \( T \) occupy \( r \) distinct residues, \( \text{Res}(\gamma/\rho) = \{i_1, \ldots, i_r\} \) and \( \beta^{(x-1)} \) has a removable or an addable \( i_{x-1} \)-corner. Since \( \{i_1, \ldots, i_r\} \) are distinct and a core never has both an addable and removable corner of the same residue, \( \text{Res}(\beta^{(x)}/\beta) \cap \text{Res}(\beta/\rho) = \emptyset \). Therefore, by Remark 14(1), \( |\mathbf{p}(\gamma)| = |\mathbf{p}(\beta)| + r - m \).

**Theorem 28.** For \( \lambda \in \mathcal{P}^k \), there is a bijection between \( T^k_\alpha(\lambda) \) and the set of \( \alpha \)-factorizations for \( w_\lambda \).

**Proof.** (\( \Rightarrow \)) Consider an \( \alpha \)-factorization \( w_\lambda = w^\ell w^{\ell-1} \cdots w^1 \) and let \( i_1, \ldots, i_{\ell} \) be a cyclically decreasing word for \( w^\ell \), for each \( x = 1, \ldots, \ell = \ell(\alpha) \). From this, iteratively construct \( U^{(x)} = s_{i_x \cdot \cdot \cdot i_{x-1} x}(U^{(x-1)}) \). Let \( \gamma^{(x)} = \text{shape}(U^{(x)}) \) and let \( \rho^{(x)} \) be the shape of \( U^{(x)} \) minus its cells containing the letter \( x \). By Theorem 26 it suffices to show that \( \gamma^{(x)}/\rho^{(x)} \) is an affine s-v \( \alpha_x \)-strip.

Since \( w_\lambda \) is grassmannian and \( w^1 = s_{a_1-1} \cdots s_{a_0} \) is cyclically decreasing, \( w^1 = s_{a_1-1} \cdots s_{a_0} \). Thus the result holds for \( x = 1 \) since \( \gamma^{(1)} = s_{a_1-1} \cdots s_{a_0} \) is horizontal and \( \rho^{(1)} = \emptyset \). By induction, \( \gamma^{(\ell-1)} \) is a core and by Remark 16 the letter \( \ell \) in \( U^{(\ell)} \) occupies \( \alpha_\ell \) distinct residues. The result then follows by applying Lemma 27 since \( \gamma^{(\ell)} = s_{i_{\ell+1} \cdot \cdot \cdot i_{\ell+1}} \gamma^{(\ell-1)} \).

(\( \Rightarrow \)); Given \( T \in T^k_\alpha(\lambda) \), let \( w^\ell = s_{j_x \cdot \cdot \cdot i_{x+1}} \) where \( j_x \) denotes the residue of letter \( a \) in \( T \), for \( x = 1, \ldots, \ell(\alpha) \). Proposition 17 implies that \( w_\lambda = w^\ell \cdot \cdot \cdot w^1 \) and it remains to show that \( w^\ell \) is cyclically decreasing.

The letters of \( A_{\alpha, x} \) occupy residues \( S = \{j_{x+1}, \ldots, j_{x+1}\} \). The definition of affine s-v tableau implies residues in \( S \) are distinct, the lowest reading word of \( A_{\alpha, x} \) is increasing, and \( \gamma/\rho \) is horizontal, where \( \gamma = \text{shape}(T_{\leq \Sigma^\alpha}) \) and \( \rho \) is the shape of \( T_{\leq \Sigma^\alpha} \) minus cells containing an element of \( A_{\alpha, x} \). Suppose \( i, i-1 \in S \) and \( i-1 \) precedes \( i \) in \( j_{x+1} \cdot \cdot \cdot j_{x+1} \). Then there are letters \( t_1(i) \) and \( t_2(i-1) \) in \( A_{\alpha, x} \).
where \( t_1 < t_2 \). Since the lowest reading word is increasing, the lowest \( t_2 \) occurs to the right of \( t_1 \) and they do not lie in the same row since \( \alpha_x \leq k \). Further, there is no element of \( A_{\alpha,x} \) right-adj to \( t_2(i - 1) \) since this could only be \( t_1(i) \). Therefore, all extremals of residue \( i - 1 \) lie at the end of their row by Property 1. However, \( t_1(i) \) is right-adj to an extremal of residue \( i - 1 \) by the horizontality of \( \gamma/\rho \). □

Corollary 29. For any \( \lambda \in \mathcal{P}^k \),

\[
G^{(k)}_\lambda = \sum_{T \in \mathcal{T}_k(\lambda)} (-1)^{|\lambda|+|w(T)|} x^{w(T)},
\]

(33)

where \( G^{(k)}_\lambda = G^{(k)}_{w_\lambda} \).

This interpretation for the \( G^{(k)}_\lambda \) allows us to establish a number of properties using our results on affine s-v tableaux. To start, Mark Shimozono conjectured that affine Grothendieck polynomials for the grassmannian reduce to Grothendieck polynomials in limiting cases of \( k \). In fact, we find precisely that

Property 30. If \( h(\lambda) \leq k \) then \( G^{(k)}_\lambda = G_\lambda \).

Proof. Proposition 9 tells us that the elements of \( \mathcal{T}_k(\lambda) \) are set-valued tableaux of shape \( \lambda \) when \( h(\lambda) \leq k \). The result thus follows from Corollary 29 and the definition for Grothendieck polynomials. □

It was shown in [Lam06] that \( G^{(k)}_w \) of (31) are symmetric functions. Thus, letting

\[
K^{(k)}_{\lambda\alpha} = |\mathcal{T}_{\alpha}(\lambda)|
\]

enumerate the affine s-v tableaux, we deduce a symmetry of this affine K-theoretic refinement of the Kostka numbers from Corollary 29.

Corollary 31. \(^1\) Given any \( \lambda \in \mathcal{P}^k \) and \( k \)-bounded composition \( \alpha \),

\[
K^{(k)}_{\lambda\alpha} = K^{(k)}_{\lambda\beta}
\]

for any rearrangement \( \beta \) of \( \alpha \).

The affine Grothendieck polynomial can then be written as, for \( \lambda \in \mathcal{P}^k \),

\[
G^{(k)}_\lambda = \sum_{\mu \in \mathcal{P}^k} (-1)^{|\lambda|+|\mu|} K^{(k)}_{\lambda\mu} m_\mu.
\]

(34)

Our earlier result showing affine s-v tableaux are simply \( k \)-tableaux in certain cases also enables us to refine this expansion and connect affine Grothendieck polynomials to dual \( k \)-Schur functions.

Property 32. For any \( k \)-bounded partitions \( \lambda \) and \( \mu \),

\[
K^{(k)}_{\lambda\mu} = \begin{cases} 
1 & \text{when } \mu = \lambda \\
0 & \text{when } |\mu| = |\lambda| \text{ and } \lambda \nleq \mu \\
0 & \text{when } |\mu| < |\lambda|
\end{cases}
\]

(35)

\(^1\) A direct combinatorial proof of this symmetry will appear in [BM09] using an involution from the set of \( \alpha \)-factorizations of \( w_\lambda \) to the set of \( \alpha \)-factorizations, where \( \alpha \) is obtained by transposing two adjacent components of \( \alpha \). The involution generalizes the Lascoux-Schützenberger symmetric group action on words.
Proof. Consider an affine $s$-v tableau $T$ of shape $c(\lambda)$ and weight $\mu$. We have that $|\lambda| \geq |\mu|$ by Corollary 12. Further, Proposition 13 implies that $T$ is a $k$-tableau when $|\mu| = |\lambda|$, in which case the desired relation follows from (21). □

We thus have the unitriangularity relation:

$$G^{(k)}_\lambda = m_\lambda + \sum_{\mu \in \mathcal{P}^k \mu \subset \lambda} K^{(k)}_{\lambda \mu} m_\mu + \sum_{\mu \in \mathcal{P}^k |\mu| > |\lambda|} (-1)^{|\lambda|+|\mu|} K^{(k)}_{\lambda \mu} m_\mu,$$

and it follows immediately that these affine Grothendieck polynomials are a basis.

**Property 33.** $\{G^{(k)}_\lambda\}_{\lambda \in \mathcal{P}^k}$ is a basis for $\Lambda/I^k$.

In analogy to (16), we also deduce that the dual $k$-Schur expansion of an affine Grothendieck polynomial has integer coefficients and the polynomial made up of the lowest homogeneous degree terms is precisely a dual $k$-Schur function.

**Property 34.** For any $k$-bounded partition $\lambda$,

$$G^{(k)}_\lambda = \mathcal{S}^{(k)}_\lambda + \sum_{\mu \in \mathcal{P}^k \mu \subset \lambda} a_{\lambda \mu}^{(k)} \mathcal{S}^{(k)}_\mu \quad \text{for} \quad a_{\lambda \mu}^{(k)} \in \mathbb{Z}. \quad (37)$$

Proof. The bottom degree terms of expression (36) matches the monomial expansion (22) for the dual $k$-Schur function $\mathcal{S}^{(k)}_\lambda$ since $K^{(k)}_{\lambda \mu} = K^{(k)}_{\lambda \mu}$ when $|\mu| = |\lambda|$ by Proposition 13. The higher degree terms involve $m_\mu \in \Lambda/I^k$ and can thus be expanded into the $\{\mathcal{S}^{(k)}_\lambda\}$-basis. The coefficient remain integral by the unitriangularity of expansion (22). □

On one hand, as their name suggests, the affine Grothendieck polynomials can be viewed as an affine analog of the Grothendieck polynomials. At a fundamental level, because the expansion coefficients in (16) are in fact positive (up to a degree-alternating sign), Thomas Lam conjectured the same about the coefficients $a_{\lambda \mu}^{(k)}$. In the same vein, it was proven in Len00 that the coefficients in

$$s_\lambda = \sum_{\mu} f_{\lambda \mu} G_\mu$$

have a simple combinatorial interpretation as the number of certain restricted skew tableaux. Evidence suggests that the affine analog of this identity is also positive.

**Conjecture 35.** For any $k$-bounded partition $\lambda$, the coefficients $f_{\lambda \mu}^{(k)}$ in

$$\mathcal{S}^{(k)}_\lambda = \sum_{\mu \in \mathcal{P}^k \mu \subset \lambda} f_{\lambda \mu}^{(k)} G_\mu$$

are non-negative integers.

On the other hand, instead viewing affine Grothendieck polynomials as the $K$-theoretic analog of dual $k$-Schur functions suggests that these polynomials satisfy even more refined combinatorial properties. For example, it is proven in the forthcoming paper LLMS09 that the coefficients in

$$\mathcal{S}^{(k+1)}_\lambda = \sum_{\mu \in \mathcal{P}^k} a_{\lambda \mu}^{k+1,k} \mathcal{S}^{(k)}_\mu \mod I^k$$

are non-negative integers.
are non-negative integers. Since a dual $k$-Schur function reduces to a Schur function for large $k$, this expression can be iterated to imply the positivity of coefficients in

$$s_\lambda = \sum_{\mu \in \mathcal{P}_k} a_{\lambda,\mu}^{(k)} \mathcal{S}_\mu^{(k)} \mod I^k,$$

for any $k > 0$. Naturally following suite in our setting leads to the $K$-theoretic version of these ideas.

**Conjecture 36.** For any $k + 1$-bounded partition $\lambda$, the coefficients $d^{k+1,k}_{\lambda,\mu}$ in

$$G^{(k+1)}_\lambda = \sum_{\mu \in \mathcal{P}_k \mid |\mu| \geq |\lambda|} (-1)^{|\lambda|+|\mu|} d^{k+1,k}_{\lambda,\mu} G^{(k)}_\mu \mod I^k$$

are non-negative integers. By Property 34 this implies

$$G_\lambda = \sum_{\mu \in \mathcal{P}_k \mid |\lambda| \leq |\mu|} (-1)^{|\lambda|+|\mu|} d^{k}_{\lambda,\mu} G^{(k)}_\mu \text{ where } d^k_{\lambda,\mu} \in \mathbb{N}.$$

Note that (43) is the analog of (41) where the Schur function $s_\lambda$ is considered to be $g_\lambda^{(\infty)}$ and the dual $k$-Schur functions are then all replaced by their $K$-theoretic counterparts, the affine Grothendieck polynomials. Alternatively, if we do not interpret $s_\lambda$ as a dual $\infty$-Schur function, we can derive a conjecture about the affine Grothendieck expansion of a Schur function.

**Conjecture 37.** For any $k$-bounded partition $\lambda$, the coefficients $d^k_{\lambda,\mu}$ in

$$s_\lambda = \sum_{\mu \in \mathcal{P}_k \mid |\mu| \geq |\lambda|} d^k_{\lambda,\mu} G^{(k)}_\mu \mod I^k$$

are non-negative integers.

### 6. $k$-K-Schur functions

Thomas Lam conjectured in an FRG wiki post, and at the 2008 FRG problem solving session, that there is a basis $g^{(k)}_\lambda$ of $\Lambda^{(k)}$ such that

1. $(g^{(k)}_\lambda, G^{(k)}_\mu) = \delta_{\lambda,\mu}$
2. as $k \to \infty$, $g^{(k)}_\lambda$ reduces to the dual Grothendieck polynomial $g_\lambda$
3. The top homogeneous component of $g^{(k)}_\lambda$ is the k-Schur function $s^{(k)}_\lambda$
4. $g^{(k)}_\lambda$ can be expanded positively in terms of k-Schur functions.

It was with this in mind that we began the study of a second family of polynomials called $k$-K-Schur functions. Our point of departure is similar to what was done to define dual Grothendieck and $k$-Schur functions, but now exploiting the invertibility of the matrix $||K^{(k)}||_{|\lambda|,\mu \in \mathcal{P}_k}$ given by Property 32.

**Definition 38.** $k$-K-Schur functions $g^{(k)}_\lambda$ are defined by the system of equations,

$$h_\lambda = \sum_{\mu \in \mathcal{P}_k \mid |\mu| \leq |\lambda|} (-1)^{|\lambda|+|\mu|} K^{(k)}_{\mu,\lambda} g^{(k)}_\mu \text{ for all } \lambda \in \mathcal{P}_k.$$
In particular, we let \( \| K^{(k)} \| \) denote the inverse of \( \| K^{(k)} \| \) and invert (45). The conditions on this matrix imposed by Property 32 imply

\[
g^{(k)}_\lambda = h_\lambda + \sum_{\mu \in P^k_{\kappa \lambda}} K^{(k)}_{\mu \lambda} h_\mu + \sum_{\mu \in P^k_{\kappa \lambda}, |\mu| < |\lambda|} (-1)^{|\mu|+|\lambda|} \bar{K}^{(k)}_{\mu \lambda} h_\mu. \tag{46}
\]

From this, we extract a number of properties including proofs of Conjectures (1–3).

**Property 39.** The set \( \{ g^{(k)}_\lambda \}_{\lambda, \kappa \leq k} \) forms a basis for \( \Lambda^{(k)} \).

**Property 40.** For all \( \lambda, \mu \in \mathcal{P}^k \),

\[
\langle g^{(k)}_\lambda, G^{(k)}_\mu \rangle = \delta_{\lambda \mu}.
\]

*Proof.* From Equation (46) for \( k - K \)-Schur functions, we have that

\[
\langle G^{(k)}_\mu, g^{(k)}_\lambda \rangle = \sum_{|\alpha| \leq |\beta|} (-1)^{|\alpha|+|\beta|} K^{(k)}_{\mu \beta} m_\beta, \sum_{|\alpha| \leq |\lambda|} (-1)^{|\lambda|+|\alpha|} \bar{K}^{(k)}_{\alpha \lambda} h_\alpha,
\]

implying

\[
\langle G^{(k)}_\mu, g^{(k)}_\lambda \rangle = (-1)^{|\lambda|+|\mu|} \sum_\beta K^{(k)}_{\mu \beta} \bar{K}^{(k)}_{\beta \lambda} = \delta_{\lambda \mu}
\]

by the duality of \( \{ m_\beta \} \) and \( \{ h_\alpha \} \).

We have seen that the term of lowest degree in the affine Grothendieck polynomial is a dual \( k \)-Schur function. The affine analog of (13) is that the highest degree term of a \( k - K \)-Schur function is a \( k \)-Schur function. We also find that their expansion coefficients in terms of \( k \)-Schur functions and the dual Grothendieck polynomials are integers.

**Property 41.** For any \( k \)-bounded partition \( \lambda \),

\[
g^{(k)}_\lambda = s^{(k)}_\lambda + \sum_{\mu \in \mathcal{P}^k_{\kappa \lambda}, |\mu| < |\lambda|} d^{(k)}_{\lambda \mu} s^{(k)}_\mu \text{ for } d^{(k)}_{\lambda \mu} \in \mathbb{Z}. \tag{47}
\]

*Proof.* Proposition 13 implies that \( K^{(k)}_{\mu \lambda} = K^{(k)}_{\mu \lambda} \) for all \( |\lambda| = |\mu| \). Making this replacement in Equation (46) gives that

\[
g^{(k)}_\lambda = s^{(k)}_\lambda + \sum_{\mu \in \mathcal{P}^k_{\kappa \lambda}, |\mu| < |\lambda|} (-1)^{|\mu|+|\lambda|} \bar{K}^{(k)}_{\mu \lambda} h_\mu.
\]

We can then expand \( h_\mu \) in terms of \( k \)-Schur functions using (23) and since \( \bar{K}^{(k)}_{\mu \lambda} \in \mathbb{Z} \) by unitriangularity, we find that the expansion coefficients of (47) are integers.

The long-standing conjecture that \( k \)-Schur functions are Schur positive,

\[
s^{(k)}_\lambda = \sum_{\mu} b^{k}_{\lambda, \mu} s^{(k)}_\mu \text{ where } b^{k}_{\lambda, \mu} \in \mathbb{N}, \tag{48}
\]

has recently been proven in [AB09] and in [LLMS09b]. The path to proving this result that is taken in [LLMS09b] proves the more refined property that

\[
s^{(k)}_\lambda = \sum_{\mu \in \mathcal{P}^{k+1}} b^{k, k+1}_{\lambda, \mu} s^{(k+1)}_\mu \text{ where } b^{k, k+1}_{\lambda, \mu} \in \mathbb{N}. \tag{49}
\]

The Schur positivity follows from this because a \( k \)-Schur function reduces to a Schur function for large \( k \).
We conjecture that the theory of $k$-$K$-Schur functions follows a similar path. To be precise, the homogeneous symmetric functions that arise in (46) can be integrally expanded in terms of the dual Grothendieck polynomials by (17).

**Property 42.** For any $k$-bounded partition $\lambda$,
\[
g^{(k)}_{\lambda} = g_{\lambda} + \sum_{\mu \triangleright \lambda} b_{\lambda \mu} g_{\mu} + \sum_{|\mu| < |\lambda|} b_{\lambda \mu} g_{\mu} \quad \text{for } b_{\lambda \mu} \in \mathbb{Z}.
\] (50)

Similarly, the homogeneous symmetric functions that arise in (46) can be instead be expanded integrally in terms of the $k+1$-$K$-Schur functions using Definition 38.

**Property 43.** For any $\lambda \in \mathcal{P}^k$,
\[
g^{(k)}_{\lambda} = g^{(k+1)}_{\lambda} + \sum_{\mu \triangleright \lambda} b^k_{\lambda \mu} g^{(k+1)}_{\mu} + \sum_{|\mu| < |\lambda|} b^k_{\lambda \mu} g^{(k+1)}_{\mu} \quad \text{for } b^k_{\lambda \mu} \in \mathbb{Z}.
\] (51)

**Conjecture 44.** For all $\lambda, \mu \in \mathcal{P}^k$, the integer coefficient $(-1)^{|\lambda|+|\mu|} b^k_{\lambda \mu}$ is non-negative.

Then, when $k$ is large, a $k$-$K$-Schur function reduces simply to a dual Grothendieck polynomial.

**Property 45.** If $|\lambda| \leq k$, then $g^{(k)}_{\lambda} = g_{\lambda}$.

**Proof.** Proposition 9 implies that $K^{(k)}_{\lambda \mu} = K_{\lambda \mu}$ when $h(\lambda) \leq k$. In particular, if $|\lambda| \leq k$ then all partitions $\mu$ where $|\mu| \leq |\lambda|$ have $k$-bounded hook-length. □

From this, iterating Property 43 will eventually lead to the positive (up to alternating sign) expansion coefficients in (50) of a $k$-$K$-Schur function in terms of dual Grothendieck polynomials.

**Conjecture 46.** For all $\lambda, \mu \in \mathcal{P}^k$, the integer coefficient $(-1)^{|\lambda|+|\mu|} b^k_{\lambda \mu}$ is non-negative.

**Property 47.** For any partition $\lambda$ with $h(\lambda) \leq k$, we have that
\[
g^{(k)}_{\lambda} = g_{\lambda} + \text{lower degree terms}.
\] (52)

Note: we conjecture that all the lower degree terms cancel.

**Proof.** When $k \geq h(\lambda)$, it was shown in LM07 that $s^{(k)}_{\lambda} = s_{\lambda}$. Thus, the $k$-Schur expansion of Property 41 reduces in this case to
\[
g^{(k)}_{\lambda} = s_{\lambda} + \text{lower degree terms}.
\] (53)

The result follows from (18) expressing $g_{\lambda}$ as $s_{\lambda}$ plus lower degree terms. □

7. Pieri rules

In addition to the basic properties of $k$-$K$-Schur functions extracted from the definition, we have also determined explicit Pieri rules for these polynomials.
7.1. Row Pieri Rule.

**Theorem 48.** For any $k$-bounded partition $\lambda$ and $r \leq k$,

$$g^{(k)}_{r} g^{(k)}_{\lambda} = \sum_{(\mu, \rho) \in \mathcal{H}^{k}_{\lambda, r}} (-1)^{|\lambda| + r - |\mu|} g^{(k)}_{\mu}, \quad (54)$$

where $\mathcal{H}^{k}_{\lambda, r} = \{ (\mu, \rho) : (c(\mu)/c(\lambda), \rho) = \text{affine set-valued } r\text{-strip} \}$.

**Example 49.**

$$g^{(3)}_{2} g^{(3)}_{3, 2, 1} = g^{(3)}_{3, 2, 1} + g^{(3)}_{3, 3, 1, 1} - g^{(3)}_{3, 2, 1, 1} - 2g^{(3)}_{3, 2, 2} + g^{(3)}_{3, 3, 1}$$

**Proof.** Note that $g^{(k)}_{k} = h^{(k)}_{\ell}$. Since the $k$-$K$-Schur functions form a basis of $\Lambda^{(k)}$, there is an expansion

$$h^{(k)}_{\ell} g^{(k)}_{\nu} = \sum_{\mu} c^{(k)}_{\mu \nu} g^{(k)}_{\mu} \quad (55)$$

for some coefficients $c^{(k)}_{\mu \nu}$. To determine the $c^{(k)}_{\mu \nu}$, we examine $h^{(k)}_{\ell} h^{(k)}_{\lambda}$. Using the $k$-$K$-Schur expansion (45) for $h^{(k)}_{\lambda}$, we find that

$$h^{(k)}_{\ell} h^{(k)}_{\lambda} = \sum_{\nu} K^{(k)}_{\nu \lambda} h^{(k)}_{\nu} = \sum_{\mu} K^{(k)}_{\nu \lambda} \sum_{\nu} c^{(k)}_{\mu \nu} g^{(k)}_{\mu} \quad (56)$$

On the other hand, we can use (45) to expand $h^{(k)}_{\ell} h^{(k)}_{\lambda} = h^{(k)}_{\tau}$, where $\tau$ is the partition rearrangement of $(\ell, \lambda)$:

$$h^{(k)}_{\ell} h^{(k)}_{\lambda} = h^{(k)}_{\tau} = \sum_{\mu} K^{(k)}_{\mu \lambda} h^{(k)}_{\mu} = \sum_{\mu} K^{(k)}_{\mu \lambda} \sum_{\nu} c^{(k)}_{\mu \nu} g^{(k)}_{\mu} \quad (57)$$

where the last equality holds by Corollary 31. Then, since Theorem 26 implies

$$K^{(k)}_{\mu \lambda} = \sum_{\nu} K^{(k)}_{\nu \lambda} c^{(k)}_{\mu \nu} \quad (58)$$

we have

$$h^{(k)}_{\ell} h^{(k)}_{\lambda} = \sum_{\nu} \sum_{\mu \in \mathcal{H}^{k}_{\nu, \ell}} K^{(k)}_{\nu \lambda} g^{(k)}_{\mu} \quad (59)$$

Equating the coefficient of $g^{(k)}_{\mu}$ in the right side of this expression to that of (56) to get the system:

$$\sum_{\nu \in \mathcal{H}^{k}_{\nu, \ell}} K^{(k)}_{\nu \lambda} c^{(k)}_{\mu \nu} = \sum_{\nu} K^{(k)}_{\nu \lambda} c^{(k)}_{\mu \nu} \quad (60)$$

We thus find our desired solution

$$c^{(k)}_{\mu \nu} = \begin{cases} 1 & \text{if } \mu \in \mathcal{H}^{k}_{\nu, \ell} \\ 0 & \text{otherwise} \end{cases} \quad (61)$$

It is unique since any other solution satisfies

$$\sum_{\nu} K^{(k)}_{\nu \lambda} (\bar{c}^{(k)}_{\mu \nu} - c^{(k)}_{\mu \nu}) = 0 \quad (62)$$

and the invertibility of $K^{(k)}_{\nu \lambda}$ implies $\bar{c}^{(k)}_{\mu \nu} = c^{(k)}_{\mu \nu}$. □

The Pieri rule can equivalently be phrased in the notation of affine permutations. In particular, Theorem 28 identifies affine s-v strips with cyclically decreasing words and we know (e.g. Remark 15) that that $w_{(r)} = s_{r-1} \cdots s_{0}$. 


Corollary 50. For any $w \in \tilde{S}_k^{n+1}$ and $r \leq k$,
\[
g^{(k)}_{w, r-1, \ldots, 0} g^{(k)}_w = \sum_{u, v \text{ cyclically decreasing}} (-1)^{\ell(u) + r - \ell(v)} g^{(k)}_v.
\] (63)

Note that the highest degree terms in the rhs of (63) are simply the terms given by the Pieri rule [LM07] for $k$-Schur functions:
\[h_{\ell} g^{(k)}_v = h_{\ell} s^{(k)}_v + \text{lower terms},\]
obtained by adding affine $\ell$-strips to $c(\nu)$.

7.2. Column Pieri Rule. There is also a combinatorial rule to compute the $k$-$K$-Schur function expansion of $g^{(k)}_v$ in terms of vertical strips rather than horizontal. The dual Grothendieck polynomial indexed by a column is
\[g_{\ell t} = \sum_{j=1}^{\ell} \binom{\ell - 1}{j - 1} e_j,
\] (64)
and $g^{(k)}_{\ell t} = g_{\ell t}$ when $\ell \leq k$ by Property 45. To determine the associated Pieri rule, we start with a “K-theoretic” version of Newton’s formula (e.g. [Mac95]):
\[\sum_{r=0}^{\ell} (-1)^r h_{\ell - r} e_r = 0.
\] (65)

Proposition 51. For any integer $\ell \geq 0$,
\[\sum_{r=0}^{\ell} \sum_{j=0}^{r} (-1)^j e_r \binom{r - 2}{j} g \ell = 0.
\] (66)

Proof. By expression (64) for $g_{\ell t}$, this follows from the identity
\[\sum_{r=0}^{\ell} \sum_{j=0}^{r} \sum_{t=1}^{r-j} (-1)^j e_r \binom{r - 2}{j} \binom{r-j-1}{t-1} h_{\ell - r} e_t = 0.
\] (67)

Equivalently,
\[\sum_{r=0}^{\ell} \sum_{t=1}^{r-j} (-1)^j e_r \binom{r - 2}{j} \binom{r-j-1}{t-1} h_{\ell - r} e_t = 0.
\] (68)

In fact, the orthogonality identity implies that
\[\sum_{j=0}^{r-t} (-1)^j \binom{r - 2}{j} \binom{r-j-2}{t-1} + (-1)^{r-t} \delta_{r-2,t-2}
\] (69)
and thus the l.h.s. of (68) reduces to
\[\sum_{r=0}^{\ell} \sum_{t=1}^{r} (-1)^r \left( (-1)^{r+t-1} \delta_{r-2,t-1} + (-1)^{r-t} \delta_{r-2,t-2} \right) h_{\ell - r} e_t
\] (70)
\[= \sum_{r=0}^{\ell} (-1)^r h_{\ell - r} e_{r-1} + \sum_{r=0}^{\ell} (-1)^r h_{\ell - r} e_r
\] (71)
which vanishes by Newton’s identity. \qed
**Theorem 52.** For any $k$-bounded partition $\lambda$ and integer $r \leq k$,
\[
g^{(k)}_1 g^{(k)}_\lambda = \sum_{(\mu, \rho) \in E^{(k)}_{k,r}} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_{\mu},
\]
where $(\mu, \rho) \in E^{(k)}_{k,r}$ if $(\mu^\omega, \rho') \in \mathcal{H}^{(k)}_{\lambda^\omega, k-r}$.

**Example 53.**
\[
g_{1,1} g_{3,2,1} = (3)^3 (3)^3 + (3)^3 - (3)^3 - (3)^3 = (3)^3.
\]

**Proof.** Since $g_1 = h_1$, Theorem 45 implies the case when $r = 1$ and we assume by induction that the action of $g_1$ for all $s < r$ is given by (73). To prove our assertion for multiplication by $g_1$, since Proposition 51 can be rewritten as
\[
\sum_{s=0}^{r-1} \sum_{j=0}^r (-1)^s \binom{s+j-2}{j} g_{r-s-j} g_{1^s} + (-1)^r g_{1^r} = 0,
\]
it suffices to show
\[
\sum_{s=0}^{r-1} \sum_{j=0}^r (-1)^s \binom{s+j-2}{j} g_{r-s-j} g_{1^s}^{(k)} + \sum_{(\mu^\omega, \rho') \in \mathcal{H}^{(k)}_{\lambda^\omega, k-r}} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_{\mu} = 0.
\]

We claim that the coefficient of $g^{(k)}_{\nu}$ in the left side of this expression is zero for any $\nu \in \mathcal{P}^k$.

By induction, for $s < r$, the coefficient of $g^{(k)}_{\nu}$ in $g_{r-s-j} g_{1^s}^{(k)}$ is $(-1)^{|\nu|+|\lambda|-|\nu|-r+j}$ times the number of $vh$-fillings with weight $(s, r - s - j)$ defined by:

(i) letter $x$ lies in cells of $c(\mu)/\rho$ where $(\mu^\omega, \rho') \in \mathcal{H}^{k}_{\lambda^\omega, s}$

(ii) letter $y$ lies in cells of $c(\nu)/\tau$ where $(\nu, \tau) \in \mathcal{H}^{k}_{\mu, r-s-j}$

Denote the set of such fillings by $\mathcal{V} \mathcal{H}^{k}_{s, r-s-j}(\nu, \lambda)$. This given, for fixed $\nu, \lambda \in \mathcal{P}^k$, (75) follows by proving the identity
\[
\sum_{s=0}^{r} \sum_{j=0}^{r-s} (-1)^{r-s-j} \binom{s+j-2}{j} |\mathcal{V} \mathcal{H}^{k}_{s, r-s-j}(\nu, \lambda)| = 0.
\]

We take a combinatorial approach. First we rewrite this expression as a single sum by introducing ordered multisets of signed $vh$-fillings to account for the binomial numbers and then describe a sign-reversing involution to achieve cancellation. The desired involution will act by permuting certain “free” entries of $T \in \mathcal{V} \mathcal{H}^{k}_{(a,b)}(\nu, \lambda)$, defined:

- $\{x(i), y(i)\}$ is free if every $x(i)$ and $y(i)$ in $T$ share a cell
- $\{x(i)\}$ is free if every $x(i) \in T$ occurs alone and at the top of its column
- $\{y(i)\}$ is free if every $y(i) \in T$ occurs alone and is not right-adj to $x$ or $y$.

Let $a = \binom{s+j-2}{j}$, $b = \binom{s+j-3}{j}$, and $c = \binom{s+j-3}{j-1}$. For $T \in \mathcal{V} \mathcal{H}^{k}_{s, r-s-j}(\nu, \lambda)$, let $S_T$ be the multiset containing $|a|$ copies of $(\text{sign}(a), T)$ if the lowest free entry is not $\{x\}$. Otherwise, let $S_T$ be the ordered multiset with $|b|$ copies of $(\text{sign}(b), T)$ followed by $|c|$ copies of $(\text{sign}(c), T)$. Eq. (76) is then
\[
\sum_{s=0}^{r} \sum_{j=0}^{r-s} (-1)^{r-s-j} \sum_{T \in \mathcal{V} \mathcal{H}^{k}_{s, r-s-j}(\nu, \lambda)} \sum_{(\sigma, T) \in S_T} \sigma = 0.
\]
If $T^k(v, \lambda)$ is the union of multisets $S_T$, for all $T \in \mathcal{VH}_s^{k-r-j}(\nu, \lambda)$ where $j, s \geq 0$ and $0 \leq j + s \leq r$, then the above expression reduces to

$$\sum_{(\sigma, T) \in T^k(v, \lambda)} (-1)^{\text{weight}_v(T)} \times \sigma = 0,$$

(78)

where $\text{weight}_y(T)$ is the number of residues occupied by $y$'s. Our result will follow from Property 58 which gives an involution $m$ on $T^k(v, \lambda)$ where $m(\sigma, T) = (\sigma, T)$ with the property that $\text{weight}_y(T) = \text{weight}_y(\hat{T}) \pm 1$. \hfill \Box

**Definition 54.** Define the map

$$m : T^k(v, \lambda) \to T^k(v, \lambda)$$

on $(\sigma, T)$ in position $p$ of $S_T$ as follows: let $s, j, m, i$ be integers where $\text{weight}(T) = (s, r - s - j)$ and the lowest free entry of $T$ has residue $i$ and lies in row $m$. Then $m(\sigma, T) = (\sigma, \hat{T})$ is in position $\hat{p}$ where

1. if row $m$ contains a free $\{y(i)\}$, then $\hat{p} = p$ and $\hat{T}$ is obtained by replacing all $\{y(i)\}$ in $T$ by $\{x(i)\}$
2. if row $m$ contains a free $\{x(i), y(i)\}$, then $\hat{p} = |(s+j-2)| + p$ and $\hat{T}$ is obtained by replacing all $\{x(i), y(i)\}$ in $T$ by $\{x(i)\}$
3. if row $m$ contains a free $\{x(i)\}$, then
   a. if $p \leq |(s+j-3)|$ then $\hat{p} = p$ and $\hat{T}$ is obtained by replacing all $\{x(i)\}$ in $T$ by $\{y(i)\}$
   b. otherwise $\hat{p} = p - |(s+j-3)|$ and $\hat{T}$ is obtained by replacing all $\{x(i)\}$ in $T$ by $\{x(i), y(i)\}$

Let us emphasize that a vh-filling $T \in \mathcal{VH}_{(a,b)}^k(\nu, \lambda)$ is constructed as follows: take the transpose of the tableau obtained by putting $x$'s in an affine $s$-$v$ $a$-strip added to $c(\lambda)'$. To the resulting tableau, add an affine $s$-$v$ $b$-strip filled with y's.

**Lemma 55.** Given a vh-filling $T$, if there is an $\{x(i), y(i)\} \in T$ that is not free then there must be an $x(i-1)$ or a $y(i-1)$ in $T$.

**Proof.** Consider $T \in \mathcal{VH}_{(a,b)}^k(\nu, \lambda)$ with a cell $c(i)$ containing a non-free $\{x, y\}$. Given $x$ lies in cells of $c(\mu)/\rho$ where $\mu \in \mathcal{H}_{(a,b)}^k(\nu, \lambda)$, we have that $\beta = c(\mu)$ is the shape obtained by deleting all lonely $y$'s from $T$. Thus $c$ is $\beta$-removable since no $x$ lies above a $y$ and the $x$'s form a vertical strip.

If $c'(i)$ contains a $\{y\}$ then $c' \notin \beta$ is above a cell in $\beta$ since the $y$'s form a horizontal strip. Therefore the cell left-adj to $c'$ (of residue $i-1$) contains a $y$ since $\beta$ cannot have an addable and removable $i$-corner. On the other hand, if $c'(i)$ contains an $\{x\}$, assume $x(i-1) \notin T$. Then $c'$ is at the top of its column in $\beta$ and is thus $\beta$-removable. Further, $c / \beta / \rho$ where $\rho$ is the shape obtained by deleting from $T$ any cell containing a $y$. Thus, all non-blocked $\beta$-removable $i$-corners are in $\beta/\rho$ implying that $c'$ is blocked (by a $y(i-1)$). \hfill \Box

**Property 56.** $m$ is well-defined

**Proof.** Consider $(\sigma, T) \in T^k(v, \lambda)$. By definition of free, no row of $T$ contains more than one free entry since $x$'s form a vertical strip in vh-fillings. It thus suffices to show that $T$ contains a free $\{x\}$, a free $\{y\}$, or a free $\{x, y\}$. Suppose no entries are free. $T$ contains an arbitrary letter $x(i)$ or $y(i)$ and thus an entry $\{x(i)\}, \{y(i)\} \text{ or } \{x(i), y(i)\}$. If $\{x(i)\}$ or $\{y(i)\}$ is in $T$, then it is not free implies there is an
Therefore, there is an \( x(i - 1) \) or a \( y(i - 1) \) in \( T \). On the other hand, if there is an \( \{x(i), y(i)\} \in T \), then Lemma 55 implies there is an \( x(i - 1) \) or a \( y(i - 1) \) in \( T \). Thus, this, the same argument implies there must be an \( x(i - 2) \) or a \( y(i - 2) \) in \( T \). By iteration, \( T \) contains the letters \( z(i), z(i - 1), z(i - 2), \ldots, z(i + 2), z(i + 1) \), where each \( z(t) \) is \( x(t) \) or \( y(t) \). This contradicts that \( T \) has weight \( (s, r - j - s) \) for \( r - j \leq k \).

**Lemma 57.** Given \( T \in T^k_r(\nu, \lambda) \)

1. if \( \{x(i)\} \) is the lowest free entry in \( T \) then \( y(i) \notin T \)
2. if \( \{y(i)\} \) is the lowest free entry in \( T \) then \( x(i) \notin T \)

**Proof.** Given \( T \in V H^k_{(a, b)}(\nu, \lambda) \), \( x \) lies in cells of \( c(\mu)/\rho \) where \( (\mu^a, \rho^b) \in H^k_{(\nu, \lambda)} \) and letter \( y \) lies in cells of \( c(\nu)/\tau \) where \( (\nu, \tau) \in H^k_{\rho, b} \).

1. Any free \( \{x(i)\} \) is a removable corner of \( c(\mu) \) since it lies at the top of its column and the \( x \)'s form a vertical strip. Therefore, by Property 51 there can be no \( y(i) \) in the affine strip \( c(\nu)/c(\mu) \). Suppose \( y(i) \in c(\mu) \). Then there is a \( y \) in all removable \( i \)-corners of \( c(\mu) \) that are not \( c(\nu) \)-blocked. Since the free \( \{x(i)\} \) is not blocked, it shares a cell with \( y(i) \), violating the definition of free.

2. Given the lowest free entry is a \( \{y(i)\} \) in cell \( c_y \) of row \( m \), suppose there is an \( x(i) \in T \) in cell \( c_x \) of row \( m_x \). Since \( \{y(i)\} \) is free, \( c_x \) contains a lonely \( \{x(i)\} \). In \( c(\mu) \), cell \( c_y \) lies above an \( i + 1 \)-extremal by verticality of \( x \)'s and the cell beneath \( c_y \) is at the top of its column by horizontality of \( y \)'s. Therefore, by Property 41 \( m_x > m \). Let \( \beta \) be the shape obtained by deleting from \( T \) all lonely \( y \)'s and all lonely \( x(j) \)'s for any residue \( j \) that lies higher than the highest \( x(i) \) and is occupied by an \( x \). Proposition 51 implies that \( \beta \) is a core and \( c_x(i) \) lies at the top of its column. However, the \( \{y\} \) in cell \( c_y \) is free and thus is right-adj to a cell in \( \beta \) that lies above an \( i \)-extremal. We reach a contradiction by Property 41.

**Property 58.** The map \( m \) is an involution on \( T^k_r(\nu, \lambda) \) and for \( m(\sigma, T) = (\sigma, T) \), \( \text{weight}_g(T) = \text{weight}_g(T) \pm 1 \)

**Proof.** Given \( (\sigma, T) \in T^k_r(\nu, \lambda) \), define \( p, r, s \) so that \( (\sigma, T) \) is in position \( p \) of \( S_T \) and \( \text{weight}(T) = (s, r - j - s) \). We will show that \( m(\sigma, T) = (\sigma, T_1) \in T^k_r(\nu, \lambda) \), \( \text{weight}_g(T_1) = r - j - s \pm 1 \), and \( m^2 = 1 \). Let \( m \) denote the lowest row with a free entry in \( T \) and set \( a = (s + j - 2), b = (s - 3), c = (s - 3). \)

Consider the case when row \( m \) of \( T \) has a free \( \{y(i)\} \). Then \( p \leq |S_T| = |a| \) and \( \sigma = \text{sign}(a) \). In this case, \( T_1 \) is obtained by replacing each \( \{y(i)\} \) in \( T \) by \( \{x(i)\} \). Any \( y(i) \in T \) is lonely by definition of free and \( x(i) \notin T \) by Lemma 57. Therefore, any \( x(i) \) in \( T_1 \) is lonely and lies at the top of its column by horizontality of \( y \)'s. From this, the lowest free entry in \( T_1 \) is the \( \{x(i)\} \) in row \( m \) and the weight of \( T_1 \) is \( (s + 1, r - j - s - 1) \). Thus \( S_{T_1} \) has \( \text{sign}(a), T_1 \) in its first \( |a| \) positions by definition of \( S_T \). In particular, there is a \( (\sigma, T_1) \) in position \( p = p \leq |a| \) of \( S_T \). Moreover, since the lowest free in \( T_1 \) is an \( \{x(i)\} \) and \( y(i) \notin T_1 \), \( m \) acts on \( (\sigma, T_1) \) by replacing each \( \{x(i)\} \) by \( \{y(i)\} \) and the \( (\sigma, T) \) in position \( p \) is recovered.

In the case that row \( m \) of \( T \) contains a free \( \{x(i), y(i)\} \), we again have \( \sigma = \text{sign}(a) \) and \( p \leq |a| \). \( T_1 \) is obtained by replacing each \( \{x(i), y(i)\} \) in \( T \) by \( \{x(i)\} \). The definition of free implies there are no lonely \( x(i) \) or \( y(i) \) in \( T \) and therefore \( T_1 \) has weight \( (s, r - j - s - 1) \). Further, the lowest free entry in \( T_1 \) is an \( \{x(i)\} \) in row \( m \) since each \( \{x(i), y(i)\} \) in \( T \) lies at the top of its column by horizontality of \( y \)'s and
is sent to a lonely $x(i)$ in $T_1$. This given, there are $|\binom{s+j-2}{j+1}| + |\mathcal{S}_T|$ elements of $\mathcal{S}_T$ of which the last $|\mathcal{S}_T|$ entries are $(\text{sign}(a), T_1)$. Thus, $p \leq |\mathcal{S}_T|$ implies $(\text{sign}(a), T_1)$ is in position $\hat{p} = p + |\binom{s+j-2}{j+1}|$. Further, $m$ acts by replacing each $\{x(i)\}$ in $T_1$ by $\{x(i), y(i)\}$ and we have $m^2 = id$.

The last case is when there is a free $\{x(i)\}$ in row $m$. There are two scenarios depending on $p$. When $p \leq |b|$, $\sigma = \text{sign}(b)$ and $T_1$ is obtained by replacing $\{x(i)\}$ with $\{y(i)\}$ in $T$. Since $y(i) \notin T$ by Lemma 57, the weight of $T_1$ is $(s-1, r-j-s+1)$. Further, the lowest free in $T_1$ is $\{y(i)\}$ in row $m$ since there is at most one $x$ in each row of $T$ implies that no $x$ or $y$ is left-adj to $\{y(i)\}$ in $T_1$. Thus, the $|b|$ entries of $\mathcal{S}_T$ are $(\text{sign}(b), T_1)$. Therefore, there is a $(\sigma, T_1)$ in position $\hat{p} = p$ of $\mathcal{S}_T$. When $m$ acts on $(\sigma, T_1), each \{y(i)\}$ in $T_1$ is replaced by $\{x(i)\}$ and we recover $\{\sigma, T\}$ in position $p$.

Otherwise, $|b| + 1 \leq p \leq |b| + |c|$ and $\sigma = \text{sign}(c)$. $T_1$ is obtained by replacing $\{x(i)\}$ by $\{x(i), y(i)\}$. Since $y(i) \notin T$ by Lemma 57, the weight of $T_1$ is $(s, r-j-s+1)$ and every $x(i)$ lies with $y(i)$ and vice versa. Thus the lowest free entry is an $\{x(i), y(i)\}$ in row $m$ implying that $\mathcal{S}_T$ is $|c|$ copies of $(\text{sign}(c), T_1)$. Therefore, there is a $(\sigma, T_1)$ in position $p - |b| \leq |c|$ of $\mathcal{S}_T$. When $m$ acts on $(\sigma, T_1), each \{x(i), y(i)\}$ in $T_1$ is replaced by $\{x(i)\}$ and we recover $(\sigma, T)$ in position $p$. □

### 8. Conjugating affine Grothendieck polynomials

An important property in the theory of Schur functions and $k$-Schur functions involves the algebra automorphism defined on $\Lambda$ by $\omega : h_\ell \rightarrow e_\ell$. Not only does $\omega$ send $s_\lambda$ to the single Schur function $s_{\lambda^k}$, but it was proven in [LM07] that

$$\omega \left(s_\lambda^{(k)}\right) = s_{\lambda^k}^{(k)}, \quad (79)$$

where $\lambda^{(k)} = p(\epsilon(\lambda))'$.

In our study, we consider the algebra endomorphism defined on $\Lambda$ by

$$\Omega h_\ell = \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j \quad (80)$$

to be an inhomogeneous analog of $\omega$. Note: the transformation $e_\ell \rightarrow \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j$ has been studied [BGT11] and is needed to relate the cohomology ring to the Grothendieck ring. In fact, the polynomials $\sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j$ are connected to the study of classes of a Schubert subvariety of the Grassmannian in these rings [Las74].

**Remark 59.** A manipulator proof that $\Omega$ is an involution on $\Lambda^{(k)}$, supplied by Adriano Garsia, shows that

$$\sum_{\ell \geq 1} \binom{u}{u-1}^\ell \Omega h_\ell = \sum_{\ell \geq 1} \binom{u}{u-1}^\ell \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j \quad = \sum_{j \geq 1} e_j \sum_{\ell \geq j} \binom{\ell-1}{j-1} \left(\frac{u}{u-1}\right)^{\ell-j} \left(\frac{u}{u-1}\right)^j \quad = \sum_{j \geq 1} e_j \left(\frac{u}{u-1}\right)^j \left(1 - \frac{u}{u-1}\right)^j \quad = \sum_{j \geq 1} (-1)^j e_j u^j$$
implies by Newton’s formula (65) that
\[
\left( \sum_{\ell \geq 0} u^\ell h_\ell \right) \left( \sum_{\ell \geq 0} \left( \frac{u}{u-1} \right)^\ell \Omega h_\ell \right) = 1.
\]

The result follows by substituting \( u = \frac{u}{u-1} \) into this expression and applying \( \Omega \):
\[
\left( \sum_{\ell \geq 0} \left( \frac{u}{u-1} \right)^\ell \Omega h_\ell \right) \left( \sum_{\ell \geq 0} u^\ell \Omega^2 h_\ell \right) = 1.
\]

Remark 60. By Jacobi-Trudi we have
\[
\Omega(s_\lambda) = s_\lambda' + \text{lower degree terms}.
\]

This involution acts beautifully on the \( k \)-Schur functions just as \( \omega \) acts on a Schur function (or more generally, a \( k \)-Schur function).

**Theorem 61.** For any \( k \)-bounded partition \( \lambda \),
\[
\Omega g_{\lambda}^{(k)} = g_{\lambda^\omega k}^{(k)}. \tag{81}
\]

**Proof.** Let \( F_\lambda = \Omega \left( g_{\lambda^\omega k}^{(k)} \right) \). Since \( h_\ell \Omega \left( g_{\lambda}^{(k)} \right) = \Omega \left( \sum_{j=1}^\ell \left( \begin{array}{c} j-1 \\ \ell-1 \end{array} \right) e_j g_{\lambda}^{(k)} \right) \), we can apply the column Pieri rule (Theorem 52) to obtain
\[
h_\ell F_\lambda = \Omega \left( g_{(1^\ell)} g_{\lambda^\omega k}^{(k)} \right) = \sum_{(\mu,\rho) \in E_{\lambda^\omega k},\ell} (-1)^{|\mu|-|\lambda|-\ell} \Omega g_{\mu}^{(k)} = \sum_{(\mu,\rho) \in E_{\lambda^\omega k},\ell} (-1)^{|\mu|-|\lambda|-\ell} F_{\mu^\omega k}
\]
\[
= \sum_{(\mu^\omega k, \rho) \in E_{\lambda^\omega k},\ell} (-1)^{|\mu|-|\lambda|-\ell} F_{\mu} = \sum_{(\mu, \rho') \in H_{\lambda,\ell}} (-1)^{|\mu|-|\lambda|-\ell} F_{\mu}. \tag{82}
\]

By Theorem 26, the iteration of this expression from \( F_0 = \Omega g_0 = 1 \) matches the iteration of the row Pieri rule (54) from \( g_0 = 1 \). Thus, \( F_\mu \) satisfies
\[
h_\lambda = \sum_{\mu \in P_{\lambda}^{(k)}} F_{\mu} \tag{83}
\]
implying that \( F_\mu = g_{\mu}^{(k)} \) by Definition 38 of the \( k \)-\( K \)-Schur functions. \( \square \)

The result can also be translated into the language of affine permutations since \( c(\lambda') = c(\lambda^\omega k) \).

**Corollary 62.** For any \( w \in \tilde{S}^{+1}_{k+1} \),
\[
\Omega g_{w}^{(k)} = g_{w'}^{(k)}
\]
where \( w' \) is obtained by replacing \( s_i \) in \( w \) with \( s_{k+1-i} \).

9. Computability

The notion of affine \( s \)-v strips and Theorem 26 give an efficient recursive algorithm to compute \( k \)-\( K \)-Schur functions and affine Grothendieck polynomials. This enabled us to check all conjectures extensively.
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