Guarded Traced Categories

Sergey Goncharov and Lutz Schröder
Friedrich-Alexander-Universität Erlangen-Nürnberg

FoSSaCS 2018, 16-19 April 2018, Thessaloniki, Greece
Guarded Traced (Symmetric Monoidal) Categories

Sergey Goncharov and Lutz Schröder
Friedrich-Alexander-Universität Erlangen-Nürnberg

FoSSaCS 2018, 16-19 April 2018, Thessaloniki, Greece
Introduction

- Recursion / iteration
  - order-theoretic / unguarded
  - process-theoretic / guarded

- Generic categorical models:
  - Total:
    - Axiomatic/synthetic domain theory (Hyland, Fiore, Taylor et al.)
    - let-ccc’s with fixpoint objects (Crole/Pitts, Simpson)
    - Traced monoidal categories (Joyal/Street/Verity, Hasegawa)
    - Elgot monads/theories (Bloom/Esik, Adámek, Milius et al.)
  - Partial:
    - Completely iterative monads/theories (Bloom/Esik, Adámek, Milius et al.)
    - later-modality (Nakano, Appel, Melliès, Benton, Birkedal et al.)
    - Partial traced categories (Heghverdi, Scott, Malherbe, Selinger)
    - Functorial dagger (Milius, Litak)

- Here: Unifying framework for guarded and unguarded feedback in monoidal categories
Guarded Fixpoints: Overview

- Guarded traced categories
- Guarded iteration
- Guarded complete Elgot monads
  - Completely iterative monads
    - Process algebra examples
  - Complete Elgot monads
    - Domain-enriched examples
- 8-dimensional Hilbert spaces
Guarded Fixpoints: Overview

FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration

guarded traced categories

guarded iteration

guarded complete Elgot monads

completely iterative monads

\(\dagger\)-congruent

retraction

complete Elgot monads

domain-enriched examples

process algebra examples
FoSSaCS17: G., Schröder, Rauch, Piró, Unifying Guarded and Unguarded Iteration

- Guarded traced categories
- Guarded iteration
- Guarded complete Elgot monads
- Completely iterative monads
- Process algebra examples

FoSSaCS18: G., Schröder, GTC

- Completely iterative monads
- Retraction
- Convergent retraction
- Complete Elgot monads
- Domain-enriched examples

- Infinite-dimensional Hilbert spaces
- Topos of trees, total Conway recursion, complete metric spaces

- Elgot monads
Motivating Example: Process Algebra

In process algebra, we solve tail-recursive process definitions, like

\[ x = a \cdot x + \tau \cdot x + y \]

More abstractly, we involve a monad \( T_\Sigma X = \nu \gamma. T(X + \Sigma \gamma) \) of infinite process trees and axiomatize guardedness of \( f : X \rightarrow T_\Sigma Y \) in a coproduct summand \( \sigma : Y' \leftrightarrow Y \) as follows (in Klesili):

\[
\begin{align*}
\text{(vac)} & \quad f : X \rightarrow Z \\
\text{inl } f : X \rightarrow Y & \quad \frac{f : X \rightarrow Z}{\text{inl } f : X \rightarrow Y + Z}
\end{align*}
\]

\[
\begin{align*}
\text{(cmp)} & \quad f : X \rightarrow Y + Z \\
[\text{g, h}] \circ f : X & \quad \frac{g : Y \rightarrow \sigma V \quad h : Z \rightarrow V}{[\text{g, h}] \circ f : X \rightarrow \sigma V}
\end{align*}
\]

\[
\begin{align*}
\text{(par)} & \quad [f, g] : X + Y \rightarrow \sigma Z \\
[f, g] & \quad \frac{f : X \rightarrow \sigma Z \quad f : Y \rightarrow \sigma Z}{[f, g] : X + Y \rightarrow \sigma Z}
\end{align*}
\]
Guarded iteration is a (partial) operation

\[ f : X \to Y + X \]
\[ f^\dagger : X \to Y \]

with \( f \) guarded in \( X \)

Dualization should yield guarded recursion:

\[ f : X \times Y \to X \]
\[ f^\dagger : Y \to X \]

Can we make sense of this intuition? : 

Diagram: 

- \( X \) and \( Y \) are connected to a box labeled \( a \). 
- \( X \) is also connected to a cloud representing the partial function, indicating guarding. 
- \( Y \) is connected to the output, showing the recursion direction.
**Guarded iteration** is a (partial) operation

\[
f : X \rightarrow Y + X
\]

\[
f^\dagger : X \rightarrow Y
\]

with \( f \) guarded in \( X \)

Dualization should yield **guarded recursion**:

\[
f : X \times Y \rightarrow X
\]

\[
f^\dagger : Y \rightarrow X
\]

Can we make sense of this intuition?

**Pivotal Idea**: Keep the notion of guardedness independent of fixpoint calculations
Going Monoidal
(We only consider symmetric monoidal categories, think of $\otimes = +, \times$)

Identity $id$:

Composition $g \circ f$:

Tensor $g \otimes f$:

Symmetry:
Going Monoidal: Additional Structure

Trace $tr(f : U \otimes A \to B \otimes U)$\(^1\):

$$A \to B$$

Compact closure:

- unit $\eta : I \to A \otimes A^*$ and
- counit $\epsilon : A^* \otimes A \to I$

where $(-)^*$ is a contravariant involutive endofunctor

In compact closed categories, trace is definable and unique, for:

---

\(^1\)The twist of input wires is nonstandard, but bear with me
Iteration and recursion are typically viewed as corner cases:

- With $\otimes = +$, we obtain $\left(f : A \to B + A\right)^! = tr(f \circ \nabla)$:

- With $\otimes = \times$, we obtain $\left(f : A \times B \to A\right)^! = tr(\Delta \circ f)$:

Corresponding converse definitions can also be produced. So, traces and (Conway) fixpoints are equivalent in the requisite cases!
Guarded Categories
A monoidal category is **guarded** if it is equipped with distinguished families \( \text{Hom}^\bullet(A \otimes B, C \otimes D) \subseteq \text{Hom}(A \otimes B, C \otimes D) \), drawn as follows:

![](image)

where

- \( A \) is unguarded input
- \( B \) is guarded input
- \( C \) is unguarded output
- \( D \) is guarded output

The idea is to prevent feedback on \((A, D)\). Hence we introduce axioms:
The Axioms
Some (Easy) Observations

- There is a greatest notion of guardedness,
  \[ \text{Hom}^\ast(A \otimes B, C \otimes D) = \text{Hom}(A \otimes B, C \otimes D) \]

- There is a least (vacuous) notion of guardedness,

- Axioms are stable under 180°-rotations, hence \( C \) is guarded iff \( C^{\text{op}} \) is guarded, i.e. we maintain duality of recursion and iteration
A particularly common case is **ideal guardedness**

A **guarded ideal** is a family $\text{Hom}^\searrow(X, Y) \subseteq \text{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides.

The general form of a partially guarded morphism over a guarded ideal is

$$\begin{align*}
p & \xrightarrow{f_1} g_1 h_1 \xrightarrow{\ldots} g_n h_n \xrightarrow{\ldots} q
\end{align*}$$
A particularly common case is **ideal guardedness**

A **guarded ideal** is a family $\text{Hom}^\bullet(X, Y) \subseteq \text{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides.

The general form of a partially guarded morphism over a guarded ideal is

\[
\begin{array}{c}
p \rightarrow f_1 \rightarrow g_1 \rightarrow h_1 \\
\vdots & & \vdots \\
p \rightarrow f_n \rightarrow g_n \rightarrow h_n \rightarrow q
\end{array}
\]

In the (co-)Cartesian case this simplifies greatly, generating standard notions, e.g. $f : X \rightarrow_{2} Y + Z$ iff

\[
X \xrightarrow{h} Y + W \xrightarrow{[\text{inl}, g]} Y + Z
\]

with some $g \in \text{Hom}^\bullet(W, Y + Z)$ and $h : X \rightarrow Y + W$
Guarded Traces
A guarded category is guarded traced if it is equipped with a trace:

satisfying a collection of axioms adapted from the standard case.

Guarded Conway iteration/recursion operators are obtained analogously to the standard case.
For guarded categories we have coherence of structural and geometric notions: a term is in $\text{Hom}^\bullet(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from $A$ to $D$ runs through some atomic box via an unguarded input and a guarded output.

This is no longer true for guarded traces:

Geometrically, this is OK but there is no structured way to derive it!
For guarded categories we have coherence of structural and geometric notions: a term is in $\text{Hom}^\bullet(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from $A$ to $D$ runs through some atomic box via an unguarded input and a guarded output.

This is no longer true for guarded traces:

![Diagram](image)

Geometrically, this is OK but there is no structured way to derive it!

**But:** This discrepancy does not arise in the ideal case

**Conjecture:** The same is true for the (co-)Cartesian case
Consider the category **CMS** of inhabited complete metric spaces and non-expansive maps.

Let $f : X \times Y \rightarrow Z$ be guarded in $Y$ if for all $x \in X$, $f(x, -)$ is contractive.

This makes **CMS** into a guarded traced monoidal category (fixpoints calculated via Banach’s fixpoint theorem) but not ideally guarded, because a contraction factor depends on $x$ and may not be chosen uniformly.
A standard way to do recursion with monads is in the category $\mathbf{C}^\mathbb{T}$ with $\mathbb{T}$-algebras as objects and $\mathbf{C}$-morphisms of carriers as morphisms.

**Example:** $\mathbf{C} =$ point-free dcpo’s and continuous functions; $\mathbb{T} =$ lifting monad $X \mapsto X_{\perp}$

Alternatively, following [Milius and Litak, 2013], we consider guarded recursion operators on $\mathbf{C}$ where $\mathbf{C}$ is ideally guarded over $\text{Hom}^\uparrow(X, Y) = \{f \circ \eta \mid f : TX \to Y\}$

**Example:** with $\mathbf{C}$ and $\mathbb{T}$ as above, we allow only recursion on $X$ of $f : X_{\perp} \times Y \to X$
We consider the following axioms:

- **Dinaturality:**

  \[
  \begin{array}{c}
  B \\
  f \quad g \\
  A \\
  \end{array}
  \quad = \quad
  \begin{array}{c}
  B \\
  g \quad f \\
  A \\
  \end{array}
  \]

- **Squaring** (is not a property of Conway recursion but a property of Conway uniform recursion):

  \[
  \begin{array}{c}
  B \\
  f \quad f \\
  A \\
  \end{array}
  \quad = \quad
  \begin{array}{c}
  B \\
  f \\
  A \\
  \end{array}
  \]

**Theorem:** There is a bijective correspondence between guarded squarable dinatural operators on \( C \) and unguarded squarable dinatural on \( C^T \).
Guarded Traces in Hilbert Spaces
Finite-Dimensional Hilbert Spaces

Recall the multiplicative compact closed category of relations \((\text{Rel}, \times, 1)\).

Relations can be thought of as Boolean matrices, with transposition \((-)^*\) and (unparamerized) trace being the trace of the square matrices

\[
\text{tr} \begin{pmatrix}
  b_{11} & \cdots & b_{1n} \\
  \cdots & \cdots & \cdots \\
  b_{n1} & \cdots & b_{nn}
\end{pmatrix} = \sum_i b_{ii}
\]

Analogously, linear operators on finite-dimensional Hilbert spaces can be represented as matrices over a field – we stick to the field of reals.

Thus, Hilbert spaces are compact closed with tensors

\[(f \otimes g)(x \otimes y) = f(x) \otimes g(y),\]  
\(R\) as tensor unit, \(X^* = X\) on objects, \(f^*\) as the unique \textbf{adjoint operator} \(\langle f(x), y \rangle = \langle x, f^*(y) \rangle\) and unit/counit induced by \textbf{inner products}.
Infinite-Dimensional Hilbert Spaces

More generally, Hilbert spaces are vector spaces with inner products, complete as a **normed spaces** under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$

**Category Hilb:**

- Objects are Hilbert spaces
- Morphisms are **bounded linear operators**, i.e. $\|f(x)\| \leq c \cdot \|x\|$ for a fixed $c$ and every $x$
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^* = f^* \otimes g^*$, $f^{**} = f$, $id^* = id$, $(f \circ g)^* = g^* \circ f^*$
Infinite-Dimensional Hilbert Spaces

More generally, Hilbert spaces are vector spaces with inner products, complete as a normed spaces under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$

Category $\text{Hilb}$:

- Objects are Hilbert spaces
- Morphisms are bounded linear operators, i.e. $\|f(x)\| \leq c \cdot \|x\|$ for a fixed $c$ and every $x$
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^* = f^* \otimes g^*$, $f^{**} = f$, $\text{id}^* = \text{id}$, $(f \circ g)^* = g^* \circ f^*$

But there is no (total) trace, because the trace formula $\text{tr}(f) = \sum_i \langle f(e_i), e_i \rangle$ may diverge ($\{e_i\}_i$ is any orthonormal basis)!

E.g. it diverges with $f = \text{id} : X \to X$ with infinite-dimensional $X$
Nontrivial Trace from Vacuous Guardedness

[Abramsky, Blute, and Panangaden, 1999] already give a construction of unparameterized partial traces in \( \text{Hilb} \) via nuclear ideals. We generalize and reconcile it with our approach by equipping \( \text{Hilb} \) with the vacuous guardedness structure:

What is nontrivial though is that this is independent of the decomposition into \( g \) and \( h \)!

Morphisms \( f : X \to X \) from the induced guarded ideal are precisely those for which \( tr(f) = \sum_i \langle f(e_i), e_i \rangle \) absolutely converges for any choice of an orthonormal basis \( (e_i)_i \); the sum is then independent of the basis.
The Diversity of Further Avenues

- Deepen the theory of guarded traced categories: coherence, expressiveness, completeness (w.r.t. Hilbert spaces?), Int-construction
- Elaborate the relationship between guarded ideals and nuclear ideals ⇒ further examples
- Yet more examples: hybrid iteration, neural networks, ...
- Metalanguages for guarded iteration and recursion
Questions?
References

S. Abramsky, R. Blute, and P. Panangaden. Nuclear and trace ideals in tensored *-categories. *J. Pure Appl. Algebra*, 143:3–47, 1999.

Stefan Milius and Tadeusz Litak. Guard your daggers and traces: On the equational properties of guarded (co-)recursion. In *Fixed Points in Computer Science, FICS 2013*, volume 126 of *EPTCS*, pages 72–86, 2013.