Well-posed Cauchy formulation for Einstein-æther theory

Olivier Sarbach\textsuperscript{1,\textcopyright}, Enrico Barausse\textsuperscript{2,3,4,5,\textcopyright} and Jorge A Preciado-López\textsuperscript{6,\textcopyright}

\textsuperscript{1} Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, Mexico
\textsuperscript{2} Institut d’Astrophysique de Paris, CNRS & Sorbonne Universités, UMR 7095, 98 bis bd Arago, 75014 Paris, France
\textsuperscript{3} SISSA, Via Bonomea 265, 34136 Trieste, Italy
\textsuperscript{4} INFN Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy
\textsuperscript{5} IFPU—Institute for Fundamental Physics of the Universe, Via Beirut 2, 34014 Trieste, Italy
\textsuperscript{6} Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON, N2L 2Y5, Canada

E-mail: sarbach@ifm.umich.mx, barausse@sissa.it and jpreciado@perimeterinstitute.ca

Received 14 February 2019, revised 31 May 2019
Accepted for publication 1 July 2019
Published 23 July 2019

Abstract
We study the well-posedness of the initial value (Cauchy) problem of vacuum Einstein-æther theory. The latter is a Lorentz-violating gravitational theory consisting of general relativity with a dynamical timelike ‘æther’ vector field, which selects a ‘preferred time’ direction at each spacetime event. The Einstein-æther action is quadratic in the æther, and thus yields second order field equations for the metric and the æther. However, the well-posedness of the Cauchy problem is not easy to prove away from the simple case of perturbations over flat space. This is particularly problematic because well-posedness is a necessary requirement to ensure stability of numerical evolutions of the initial value problem. Here, we employ a first-order formulation of Einstein-æther theory in terms of projections on a tetrad frame. We show that under suitable conditions on the coupling constants of the theory, the resulting evolution equations can be cast into strongly or even symmetric hyperbolic form, and therefore they define a well-posed Cauchy problem.

Keywords: Einstein-æther theory, strongly hyperbolic systems, well-posed Cauchy formulations
1. Introduction

Lorentz symmetry has long been one of the cornerstones of theoretical physics, and has been tested to high precision in a variety of experiments. The standard model of particle physics is obviously Lorentz invariant, but parametrized formalisms such as the standard model extension \cite{1–3} have been introduced and used to place very strong constraints on Lorentz violations (LVs) in matter \cite{4–7} and in the matter-gravity sector \cite{8}. As for the purely gravitational sector, tests of LVs have historically been less compelling, partly because of the absence of a parametrized formalism applicable to strong gravitational fields. Indeed, theory independent tests of Lorentz symmetry can be performed in the solar system \cite{9} and in binary/isolated pulsars \cite{10–13} at first post-Newtonian (PN) order\textsuperscript{7} in the conservative dynamics, and more recently with the propagation of gravitational waves (by parameterizing their dispersion relation \cite{15}), but it is very difficult to extend these tests to the highly dynamical, relativistic and strong-field regime relevant e.g. to sources for Advanced LIGO and Virgo. To understand the effect of LVs in these systems, it is much more fruitful to consider specific gravitational theories explicitly breaking Lorentz symmetry.

General relativity (GR) is of course Lorentz (and diffeomorphism) invariant, but the gravitational theory can be made Lorentz violating by introducing additional Lorentz violating gravitational degrees of freedom besides the spin-2 polarizations of GR. Focusing, for concreteness, on boost violations rather than on general LVs, these extra graviton polarizations can be for instance a vector field $u^\mu$—constrained to be timelike at the level of the action, so as to represent a ‘preferred time’ direction at each spacetime event—or a scalar field—again constrained to have a timelike gradient $u^\mu$ at the level of the action, so as to represent a ‘preferred foliation’ of the spacetime. By writing the most generic gravitational action (to quadratic order in the gradient of $u^\mu$) for these vector or scalar degrees of freedom, one obtains respectively Einstein-æther theory \cite{16} and khronometric gravity \cite{17–19}. These are indeed the most generic boost violating extensions of GR at low energies, and have been extensively used as a framework to place bounds on LVs in purely gravitational experiments, e.g. from solar system tests \cite{20–23}; from the coincident gravitational and electromagnetic detection of GW170817 and GRB 170817A \cite{15, 24}; from binary and isolated pulsars \cite{25–27}; and from isolated \cite{28–30} and binary black holes \cite{31}. Similarly, one can in principle extend GR by breaking invariance under spatial diffeomorphisms (i.e. spatial rotations), see e.g. \cite{32, 33}. In this paper, however, we will focus on boost-violating theories, and more specifically on Einstein-æther theory.

It should be stressed that boost-violating theories are not only interesting from a phenomenological point of view (as strawmen to test LVs in gravity), but also from a more fundamental perspective. Indeed, by violating boost symmetry one can introduce anisotropies (Lifschitz scaling) between the time and spatial coordinates, which result in a better ultraviolet (UV) behavior of the theory. Indeed, khronometric theory turns out to coincide with the low-energy limit of Hořava gravity \cite{18}, which, unlike GR, is power counting \cite{18} and also perturbatively renormalizable \cite{34}, thanks exactly to the presence of the aforementioned anisotropic scaling between space and time. While it is yet unclear whether Hořava gravity can be a fully viable theory of quantum gravity when matter is included—mainly because one needs to suppress (e.g. via renormalization group running \cite{35–37}\textsuperscript{8}, supersymmetry \cite{39} or a large energy scale

\textsuperscript{7} The PN formalism \cite{14} is an expansion of the dynamics in powers of $v/c$, $v$ being the characteristic velocity of the system. Terms of order $(v/c)^{2n}$ relative to the leading one are referred to as of ‘$n$PN’ order.

\textsuperscript{8} Note however that \cite{38} finds that the renormalization group flow may not approach GR at low energies.
the percolation of large LVs from gravity to the matter sector, the improved UV behavior makes boost-violating gravitational theories particularly attractive.

A conspicuous practical problem with studying gravitational wave emission from systems of two compact objects (neutron stars or black holes) in theories extending GR is (in general) the absence of results on the stability of the initial value (Cauchy) problem. In GR, the Cauchy problem can be put in a ‘well-posed’ form, i.e. such that for given initial data there exists a unique time evolution which depends continuously on the initial data (see e.g. [41, 42] for reviews). This property is clearly crucial to integrate systems of compact objects on a computer (currently the only way we have to study rigorously their merger), as it prevents numerical errors to grow unbounded. Very few results for the well-posedness of the Cauchy problem exist beyond GR, with the exception of scalar tensor theories of the Fierz–Jordan–Brans–Dicke type [43–45] (see e.g. [46, 47]). Attempts have been made to enforce stability of the initial value problem via reduction of order techniques [48] or by ‘smoothing’ higher derivatives (e.g. using techniques from relativistic hydrodynamic) [49, 50], but it is unclear how these procedures impact the results of numerical simulations.

From this point of view, boost-violating gravity is a perfect case study to assess the stability (or lack thereof) in theories of gravity extending GR. As mentioned, both khronometric and Einstein-æther present additional graviton polarizations besides those of GR, namely a spin-0 graviton in the former [22] and a spin-0 graviton and two spin-1 polarizations in the latter [51]. In both theories, these extra gravitons satisfy wave equations in flat space, for generic values of the theory parameters, but it is unclear if similar results apply on curved backgrounds. Nevertheless, the flat space results provide hope that the Cauchy problem may be well posed on generic backgrounds.

In the following we will focus on Einstein-æther theory, whose dynamics is richer and more complicated than khronometric theory because of the presence of spin-1 degrees of freedom. In section 2 we briefly review the action and field equations of the theory, as well as the experimental constraints on the coupling constants. In section 3 we present a toy problem highlighting the idea that inspired us to use a frame formulation of Einstein-æther theory, which we present in section 4. We proceed to show in section 5 that under suitable conditions on the coupling constants, the field equations in this frame formulation of the theory are strongly hyperbolic. Next, in section 6 we derive a three-parameter subfamily of this formulation which satisfies the stronger requirement of symmetric hyperbolicity and thus yields a well-posed Cauchy problem. We briefly summarize our conclusions in section 7 and include technical details in appendices.

Throughout this paper, we will use the ($-++$) metric signature and set the speed of light $c = 1$. We will also introduce the following combinations of the coupling constants $c_i$ (with $i = 1, 2, 3, 4$) of Einstein-æther theory: $c_{ij} \equiv c_i + c_j$ and $c_{ijk} \equiv c_i + c_j + c_k$.

2. Einstein-æther theory: action and field equations

Einstein-æther theory breaks boost invariance explicitly in the gravitational sector by introducing a preferred time ‘direction’ at each spacetime event, via a timelike ‘æther’ vector field $u^\alpha$ with unit norm $g_{\alpha\beta} u^\alpha u^\beta = -1$. The most generic covariant modification to the GR action that is quadratic in the æther is then given by [16]

$$S = \frac{1}{16\pi G_{æ}} \int \left[ R - M^{\alpha\beta}_{\mu\nu} (\nabla_{\alpha} u^\mu)(\nabla_{\beta} u^\nu) + \lambda (g_{\alpha\beta} u^\alpha u^\beta + 1) \right] \sqrt{-g} d^4x + S_m \left[ \psi, g_{\alpha\beta} \right],$$  (1)
\[ M^{\alpha\beta}_{\mu\nu} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + c_3 \delta^{\alpha}_{\nu} \delta^{\beta}_{\mu} - c_4 u^\alpha u^\beta g_{\mu\nu}, \]  
\[ (2) \]

where \( c_1, c_2, c_3, c_4 \) denote four dimensionless coupling constants, \( R \) is the Ricci scalar, \( \lambda \) is a Lagrange multiplier that enforces the unit norm constraint on the æther field, and \( G_\text{B} \) is the bare gravitational constant, related to the value \( G_N \) measured in the solar system by
\[ \frac{G_\text{B}}{G_N} = \frac{1 - c_2/2}{1 - c_14/2} \]
\[ (52) \]

The matter fields, collectively denoted by \( \psi \) and appearing in the matter action \( S_m \), are then supposed to couple only to the metric at tree level, so as to avoid the appearance of unwanted ‘fifth forces’ and to pass existing particle physics tests of Lorentz invariance (see the discussion in the Introduction). In the following, however, we will focus on the character of the Cauchy problem in vacuum, and we will therefore set \( S_m = 0 \).

Varying the vacuum action with respect to \( g^{\alpha\beta}, u^\beta \) and \( \lambda \) yields the field equations
\[ G^{\alpha\beta} = T^{\text{æ}}_{\alpha\beta}, \]
\[ (3) \]

\[ \nabla_\alpha I^{\alpha}_{\beta} + c_4 a_\alpha \nabla_\beta u^\alpha = -\lambda u_\beta, \]
\[ (4) \]

\[ u^\alpha u_\alpha = -1, \]
\[ (5) \]

where \( G_{\alpha\beta} \) denotes the Einstein tensor, \( T^{\text{æ}}_{\alpha\beta} \) is the Æther stress-energy tensor explicitly given by
\[ T^{\text{æ}}_{\alpha\beta} = \nabla_\mu \left( J^{\mu}_{\alpha} u_\beta \right) - J^{\alpha}_{\mu} u_\beta + J^{(\alpha \beta)} u^\mu \]
\[ + c_1 \left[ (\nabla_\alpha u^\mu)(\nabla_\beta u_\mu) - (\nabla^\mu u_\alpha)(\nabla_\mu u_\beta) \right] - \frac{1}{2} g_{\alpha\beta} \nabla_\mu u^\nu + c_4 a_\alpha a_\beta + \lambda u_\alpha u_\beta, \]
\[ (6) \]

and where we have introduced
\[ J^{\alpha}_{\mu} \equiv M^{\alpha\beta}_{\mu\nu} \nabla_\beta u^\nu, \]
\[ (7) \]

\[ a_\alpha \equiv u^\mu \nabla_\mu u_\alpha. \]
\[ (8) \]

Note that the Lagrange multiplier can be eliminated from the system by projecting equation (4) on the space orthogonal to the æther, or equivalently by contracting the same equation with \( u^\beta \) (thus solving for \( \lambda \)).

Expanding the field equations over flat space reveals that the theory has additional degrees of freedom compared to GR. Indeed, the theory presents two spin-2 polarizations, as in GR, but also one spin-0 and two spin-1 polarizations. The (squared) propagation speeds on flat space are respectively given by
\[ s^2_2 = \frac{1}{1 - c_13}, \]
\[ (9) \]

\[ s^2_1 = \frac{2c_1 - c_1^2 + c_3^2}{2c_14(1 - c_13)}, \]
\[ (10) \]

\[ s^2_0 = \frac{c_122(2 - c_14)}{c_14(1 - c_13)(2 + c_13 + 3c_2)}. \]
\[ (11) \]

Stability at the classical and quantum levels requires \( s^2_i, s^2_i, \) and \( s^2_0 \) to be positive [51, 53]. Moreover, ultrahigh energy cosmic ray observations require \( s^2_i \geq 1 - O(10^{-15}) \) (with \( i = 0, 1, 2 \)), to prevent cosmic rays from losing energy into gravitational modes via a Cherenkov-like cascade.
More recently, the coincident gravitational and electromagnetic detection of GW170817 and GRB 170817A has constrained $-3 \times 10^{-15} < s_2 - 1 < 7 \times 10^{-16}$ [15].

Additional bounds come from the requirement that the theory should agree with solar system experiments. At 1PN order and for weakly gravitating sources such as those encountered in the solar system, the theory only deviates from GR through the preferred frame parameters $\alpha_1$ and $\alpha_2$ appearing in the parametrized PN expansion [9]. Those parameters are related to the coupling constants by [20]

$$\alpha_1 = -\frac{8(c_1^2 + c_4 c_4)}{2c_1 - c_1^2 + c_3^2},$$

$$\alpha_2 = \frac{\alpha_1}{2} - \frac{(c_1 + 2c_3 - c_4)(2c_1 + 3c_2 + c_3 + c_4)}{c_{123}(2 - c_{14})}.$$  \hspace{1cm} (12)

Solar systems tests require $|\alpha_1| \lesssim 10^{-4}$ and $|\alpha_2| \lesssim 10^{-7}$ [9]. Saturating these constraints (i.e. assuming in particular that $|\alpha_1| \lesssim 10^{-4}$ but not $|\alpha_1| \ll 10^{-4}$), together with the aforementioned constraints on the propagation speeds, yields $c_1 \approx -c_3 + O(10^{-15})$, $c_4 \approx c_3 + O(10^{-4})$, and $c_2 \approx (c_4 - c_3)[1 + O(10^{-3})]$. Therefore, to within an accuracy of $10^{-4}$ or better in the coupling parameters, Einstein-ether theory possesses a one-dimensional viable parameter space, i.e. one has $c_1 + c_3 \approx 0$, $c_4 - c_3 \approx c_2 \approx 0$, while $c_1 - c_3$ is essentially unconstrained. This latter combination corresponds to the coefficient multiplying the ether vorticity in the action, as shown explicitly in [55]. In particular, it can be easily shown that one can send $|c_1 - c_3| \rightarrow \infty$ or $|c_1 - c_3| \rightarrow 0$, while passing all aforementioned experimental bounds.

Note that gravitational wave generation (e.g. in binary pulsars/black holes) is not expected to further constrain the theory at low PN orders, at least in the limits $|c_1 - c_3| \rightarrow \infty$ and $|c_1 - c_3| \rightarrow 0$. Indeed, in both limits gravitational wave emission should approach the GR predictions. This happens, in the latter case, because all the coupling parameters $c_i$ go to zero (to within $O(10^{-4})$ or better), hence the action of the theory approaches that of GR. In the former case, instead, one can show [55, 56] that the theory’s solutions converge to those of a krnonometric theory with coupling parameters of $O(10^{-4})$ or smaller, hence deviations from GR in the solutions should be small. We leave the task of deriving detailed predictions for gravitational wave emission (especially away from these two limits) to future work, as for the present one it is sufficient to show that the theory still has a viable parameter space$^9$.

We note in passing that another viable portion of the parameter space may be obtained by requiring that $|\alpha_1|$ be much smaller than its bound, so that the bound on $\alpha_2$ is also satisfied automatically (since $\alpha_2 \propto \alpha_1$ if $c_1 + c_3 = 0$, as required by GW170817). Indeed, much of the literature about Einstein-ether theory prior to GW170817 set $\alpha_1$ and $\alpha_2$ exactly to zero. Such a choice, combined with the bound from GW170817, yields $c_4 = c_3$ and thus a two-dimensional parameter space $(c_2, c_1 - c_3)$. However, both the spin-0 and spin-1 propagation speeds diverge in this limit. Of course, requiring $0 \neq |c_3 - c_4| \gtrsim 10^{-7}$ provides in principle a viable two-dimensional parameter space $(c_2, c_1 - c_3)$ (with the only further requirement that $|c_2| \lesssim 0.1$ to pass Big Bang Nucleosynthesis bounds [52]) and large but finite speeds. These large speeds also make the spin-0 and spin-1 fields non-dynamical and therefore suppress the

$^9$Note that constraints from binary pulsars were derived in [26] in a different portion of the parameter space than the one currently favored after the GW170817 detection. Indeed, neutron star sensitivities, which are a crucial ingredient to constrain gravitational theories with pulsar data, where computed under the assumption that $\alpha_1 = \alpha_2 = 0$. That choice, together with the requirement that the graviton propagation speeds be $\gtrsim 1$ to avoid vacuum Cherenkov radiation, excluded the parameter space ’line’ with $c_1 + c_3 \approx 0$, $c_4 - c_3 \approx c_2 \approx 0$ and variable $c_1 - c_3$ discussed above.
deviations away from GR in gravitational wave data (e.g. binary pulsars) \[26, 31\]. While such a choice is less generic than that of saturating the bound on $|\alpha_1|$, it is in principle a possibility.

Having considered all presently known experimental constraints on the theory, here we will focus on a new class of theoretical bounds that have never been considered thus far. In more detail, in this work we aim to investigate whether the system given by equations (3)–(5) can provide a well-posed Cauchy problem. Indeed, while perturbations over flat space do produce a strongly hyperbolic system (because they can be recast as wave equations for the spin-2, spin-1 and spin-0 modes), it is unclear if the same result can be obtained for the full system. The main difficulty consists in the second derivatives of the metric fields $g_{\alpha\beta}$, which appear in the effective stress-energy tensor $T^{\alpha\beta}$ through the second covariant derivatives of $u^\alpha$.

3. Toy model example

To gain some insight into the well-posedness of the initial value problem in Einstein-ether theory, we start with a simpler toy theory with a somewhat similar structure. To this purpose we consider a $U(1)$ gauge field $A_\mu$ on flat spacetime which is coupled to a complex massless scalar field $\Phi$ of charge $q \neq 0$ subject to the constraint $|\Phi| = 1$. The corresponding action is

$$S_{\text{toy}} = \int \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (D^\mu \Phi)^* (D_\mu \Phi) - \lambda \left( |\Phi|^2 - 1 \right) \right] d^4x, \tag{14}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor, $D_\mu = \partial_\mu + iq A_\mu$ is the covariant derivative operator, and $\lambda$ is a real Lagrange multiplier. The equations of motion are

$$\partial_\nu F^{\mu\nu} = i \frac{q}{2} \left[ \Phi^* D^\mu \Phi - \Phi (D^\mu \Phi)^* \right], \tag{15}$$

$$D^\mu D_\mu \Phi = \lambda \Phi, \tag{16}$$

$$|\Phi| = 1. \tag{17}$$

Both the action $S_{\text{toy}}$ and equations (15)–(17) are invariant with respect to local gauge transformations

$$\Phi \mapsto e^{-iq \Lambda} \Phi, \quad A_\mu \mapsto A_\mu + \partial_\mu \Lambda, \tag{18}$$

for some arbitrary function $\Lambda$. With an appropriate choice of $\Lambda$ we can always arrange that $\Phi$ is real and positive, in which case the constraint (17) yields $\Phi = 1$. In this gauge, equation (16) simplifies to

$$iq \partial^\mu A_\mu - q^2 A^\mu A_\mu = \lambda. \tag{19}$$

The real part of this equation fixes the Lagrange multiplier $\lambda = -q^2 A^\mu A_\mu$. More interestingly, the imaginary part of this equation yields the Lorenz gauge condition\[10\]

$$\partial^\mu A_\mu = 0. \tag{19}$$

Using this result in equation (15) yields the Proca-like equation

$$-\partial_\mu \partial^\nu A_\nu + q^2 A^\mu = 0 \tag{20}$$

for the gauge field.

\[10\] This gauge condition is due to Ludvig Lorenz (1829–1891), while the symmetry is due to Hendrik Lorentz (1853–1928). Note the different spelling.
Therefore, instead of enforcing the Lorenz gauge by hand (as is usually done in electromagnetism to obtain a wave equation for the gauge potential), this condition emerges as a consequence of the field equation for the scalar field $\Phi$ and the $U(1)$-gauge adapted to $\Phi$ (such that $\Phi = 1$). Remarkably, this gauge leads naturally to a hyperbolic equation for $A_\mu$. Alternatively, taking into account the Lorenz gauge condition (19), we can also cast equation (20) in first-order form:

\[
\partial^\mu A_\mu = 0, \tag{21}
\]
\[
\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu \nu}, \tag{22}
\]
\[
\partial_\nu F^{\mu \nu} = -q^2 A^\mu, \tag{23}
\]
\[
\partial_{[\alpha} F_{\mu \nu]} = 0, \tag{24}
\]
which yields a symmetric hyperbolic system for $(A_\mu, F_{\mu \nu})$.

This toy model suggests that we should consider a gauge-formulation of GR, in which the role of the scalar field $\Phi$ is replaced by the æther field $u^\alpha$. A gauge-like formulation of GR is provided by the frame formalism.

4. First-order reformulation of Einstein-æther theory in frame variables

Motivated by the example discussed in the previous section, we switch to a frame formulation of Einstein-æther theory, in which the spin-2 gravitational field (i.e. the metric) is described by an orthonormal frame $\{e_0, e_1, e_2, e_3\}$, such that

\[
g(e_\alpha, e_\beta) = g_{\mu \nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha \beta},
\]
with $(\eta_{\alpha \beta}) = \text{diag}(-1, 1, 1, 1)$. In the following, the Greek indices $\alpha, \beta, \gamma, \delta$ from the beginning of the alphabet denote frame indices, while mid-alphabet letters $\mu, \nu, \ldots$ denote coordinate indices. The frame indices are raised and lowered with the symbol $\eta_{\alpha \beta}$. The coordinate components of the metric can be reconstructed from the frame fields in the following way:

\[
g^{\mu \nu} = \eta^{\alpha \beta} e_\alpha^\mu e_\beta^\nu.
\]

The Ricci rotation coefficients with respect to the Levi-Civita connection $\nabla$ of $g$, defined such that they are antisymmetric in the first two indices, are

\[
\Gamma_{\alpha \beta \gamma} := g(e_\alpha, \nabla_{\gamma} e_\beta) = -\Gamma_{\beta \alpha \gamma},
\]
or $\nabla_{\gamma} e_\beta = \Gamma_{\alpha \beta \gamma} e_\alpha$.

The transformations that are analogous to (18) consist of local Lorentz transformations,

\[
e_\alpha \mapsto (\Lambda^{-1})^\beta_\alpha e_\beta, \quad A_\mu \mapsto \Lambda A_\mu \Lambda^{-1} - (\partial_\mu \Lambda) \Lambda^{-1}, \tag{25}
\]
where $\Lambda(x)$ is a Lorentz matrix at each point $x$ of the spacetime manifold (and varying smoothly with $x$), and $A_\mu$ is the matrix-valued connection 1-form whose components are the matrices $(A_\mu)_{\alpha \beta} := \Gamma_{\alpha \beta \mu}$. Henceforth, we fix part of this freedom by aligning the timelike leg $e_0$ of the frame with the æther field $u$. As a result of the unit-norm constraint (5), this implies that

\[
e_0 = u.
\]

In this gauge, one has

\[
J^\alpha_{\beta} = c_1 \Gamma^\alpha_{\beta 0} + c_2 \delta^\alpha_{\beta} \Gamma^\gamma_{0 \gamma} + c_3 \Gamma^\alpha_{0 \beta} - c_4 \delta^0_\alpha \delta^\gamma_0 \Gamma^\gamma_{\beta 0}.
\]
and the spatial frame components of equation (4) yield
\[ \nabla_\alpha J^\alpha_b + c_4 a_b \Gamma^\alpha_{0b} = 0, \quad b = 1, 2, 3, \] (26)
while the zeroth component fixes the Lagrange multiplier:
\[ \lambda = \nabla_\alpha J^\alpha_0 + c_4 a_0. \] (27)

For the following, it is convenient to use the dyadic formalism of [57], in which the 24 independent Ricci rotation coefficients are decomposed into two 3-vectors
\[ a_b := \Gamma^b_{00} , \quad \omega_b := -\frac{1}{2} \varepsilon_{b}^{cd} \Gamma^d_{eab}, \] (28)
and two \(3 \times 3\) matrices
\[ K_{ab} := \Gamma^c_{0ab} , \quad N_{ab} := \frac{1}{2} \varepsilon_{b}^{cd} \Gamma^d_{ead}, \] (29)
where from now on \(a, b, c, d = 1, 2, 3\) refer to spatial tetrad indices. \(a_b\) is the acceleration of the observers with four-velocity \(e_0\) (and therefore coincides with the æther acceleration, since \(e_0 = u\)), and \(\omega_b\) is the angular velocity of the spacelike triad relative to a Fermi-propagated frame along such observers. If \(e_0\) is hypersurface-orthogonal, then \(K_{ab} = K_{ba}\) is symmetric and describes the second fundamental form of the hypersurfaces orthogonal to \(e_0\), while \(N_{ab}\) encodes the induced connection of this surface. Below, we will not necessarily assume that \(e_0 = u\) is hypersurface-orthogonal (since that is not generically the case in Einstein-æther theory), so in this case \(K_{ab}\) is not necessarily symmetric. Based on this decomposition, one obtains quite naturally a symmetric hyperbolic formulation of Einstein’s vacuum equations in general relativity, see [58] and references therein. We will now show that such a formulation can also be obtained in the Einstein-æther theory case.

In terms of the variables \(a_b\) and \(K_{ab}\) defined above, the \(3 + 1\) split of \(J^\alpha_\beta\) yields
\[ J^\alpha_0 = c_2 K^\alpha , \quad J^b_0 = -c_4 a_b , \quad J^\alpha_0 = c_3 a^\alpha , \]
\[ J^\alpha_b = c_1 K^\alpha_b + c_3 K^\alpha_b + c_2 \delta^\alpha_b K , \]
where \(K = K^c_{c}\) denotes the trace of \(K_{ab}\). Up to lower-order terms in the derivatives, equations (26) and (27) respectively yield
\[ c_1 \partial_\alpha a_b = c_1 D_\alpha K^\alpha_b + c_3 D_\alpha K^\alpha_b + c_2 D_\alpha K + l.o., \] (30)
and
\[ \lambda = c_2 D_0 K + c_3 D^b_0 a_b + l.o., \] (31)
where \(D_\alpha = e^\mu_\alpha \partial_\mu\) are the directional derivatives along the tetrad fields. As an instructive example, consider the case where all the \(c_i\’s\) vanish except \(c_1\). Equation (30) then reduces to
\[ D_\alpha a_b = D_\alpha K^\alpha_b + l.o., \]
which coincides (up to lower-order terms in the derivatives) to what is obtained from the Lorenz gauge condition \(D_\gamma \Gamma_{\alpha \beta \gamma} + \Gamma_{\alpha \beta \gamma} \Gamma_{\delta \gamma \delta} = 0\), see [58]. Therefore, at least in this example, the æther equation of motion selects the Lorenz gauge, i.e. it plays the role of the scalar field \(\Phi\) in the toy model of section 3.

In terms of the variables \(K_{ab}, N_{ab}, a_b\) and \(\omega_b\), the modified Einstein equation (3) can be obtained from the corresponding equations in [58, 59]:
\[ D_b K_{ab} - \varepsilon_a^{\alpha d} D_d N_{ab} = (D_a + a_b) u_b - \varepsilon_a^{\alpha d} N_{ab} u_d + 2 \varepsilon_a^{\alpha d} K_{p_d} \omega_d + \omega_b \varepsilon_a^{\alpha d} K_{cd} + NN_{ab} \]

\[ + \frac{1}{2} \varepsilon_a^{\alpha d} \varepsilon_b^{\gamma e} (K_{ab} K_{\gamma} - N_{ab} N_{\gamma}) - K_{e}^{\alpha} K_{eb} - N_{e}^{\alpha} N_{eb} + T_{ab}^{\varepsilon} - \frac{1}{2} \delta_{ab} \left( [T_{e}^{\varepsilon}]_{c} + T_{00}^{\varepsilon} \right). \]

(32)

\[ D_b N_{ab} + \varepsilon_a^{\alpha d} D_d K_{ab} = -(D_a + a_b) \omega_b + \varepsilon_a^{\alpha d} (K_{ab} a_d + N_{ab} \omega_d) + \varepsilon_a^{\alpha d} N_{ab} \omega_d + a_b \varepsilon_a^{\alpha d} N_{cd} \]

\[ - N K_{ab} + N_b^e \varepsilon_a^{\alpha e} K_{ab} + N_a^e \varepsilon_b^{\gamma e} N_{ab} K_{\gamma} + \varepsilon_a^{\alpha d} T_{ab}^{\varepsilon}, \]

(33)

where \( N \) is the trace of \( N_{ab} \). As explained previously, the variables \( \omega_b \) appearing on the right-hand sides of these equations are related to the rotational freedom in the choice of the triad fields \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \), and in this work we will assume that they are \textit{a priori} given functions. Up to lower-order terms, explicit expressions for \( T_{\alpha \beta}^{\varepsilon} \) are obtained from equation (6),

\[ T_{00}^{\varepsilon} = c_{14} D_b a_b + \text{l.o.s.}, \]

(34)

\[ T_{0b}^{\varepsilon} = c_{13} D_b K_{(ab)} + c_2 D_b K + \text{l.o.s.}, \]

(35)

\[ T_{ab}^{\varepsilon} = c_{13} D_b K_{(ab)} + c_2 \delta_{ab} D_b K + \text{l.o.s.}, \]

(36)

with the notation \( K_{(ab)} := (K_{ab} + K_{ba})/2 \). Here, we have also used equation (26) in order to simplify the expression for \( T_{00}^{\varepsilon} \).

Formally, the system (30), (32) and (33) would seem a closed evolution system for the variables \( (K_{ab}, N_{ab}, a_b) \); however, one needs to remember that the operators \( D_b \) and \( D_a \) are directional derivatives along the tetrad fields \( \mathbf{e}_0 \) and \( \mathbf{e}_a \), respectively. Therefore, in order to close the system, equations determining the components of the tetrad fields have to be provided. Following [58], we assume a given foliation of the spacetime manifold in spacelike hypersurfaces with adapted coordinates \((t, x^i)\) and we decompose the tetrad fields as

\[ \mathbf{e}_0 = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \beta^i \frac{\partial}{\partial x^i} \right), \quad \mathbf{e}_b = A_b \mathbf{e}_0 + B^k_b \frac{\partial}{\partial x^i}. \]

(37)

with \( \alpha \) and \( \beta^i \) denoting the time- and spatial components of the aether field \( \mathbf{u} = \mathbf{e}_0 \), and where the fields \( A_b \) and \( B^k_b \) refer to the components of \( \mathbf{e}_b \) with respect to the non-orthogonal basis of tangent vectors \( \mathbf{e}_b, \partial / \partial x^i \). We assume that \( \alpha \) and \( \beta^i \) are specified by some appropriate gauge conditions, not modifying the principal symbol of the system (for example, they could be \textit{a priori} specified functions). In contrast to this, the fields \( A_b \) and \( B^k_b \) are determined by a set of advection equations along the aether field \( \mathbf{u} \), given by

\[ D_b A_b = a_b - B^k_b \frac{\partial}{\partial x^k} \log \alpha - (K_{b}^d + \varepsilon_b^{\gamma d} \omega_d) A_d, \]

(38)

\[ D_b B^k_b := \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \beta^i \right) B^k_b = -(K_{b}^d + \varepsilon_b^{\gamma d} \omega_d) B^k_d, \]

(39)

which follow from the property that the connection is torsion-free [58]. The system (30), (32), (33), (38) and (39), provides a closed evolution system for the variables \( (A_b, B^k_b, K_{ab}, N_{ab}, a_b) \), whose hyperbolicity will be analyzed in the next section. Finally, it should be mentioned that this evolution system is subject to several constraint equations. First, there are the Hamiltonian and momentum constraint equations.

\textit{Note that in general} \( \alpha \) and \( \beta^i \) \textit{are different than the definitions of lapse and shift in the standard} \( 3 + 1 \) \textit{decomposition, since} \( \mathbf{e}_0 \) \textit{does not necessarily coincide with the normal vector to the} \( t = \text{const} \) \textit{slices.}
\[ 2D^n a^a = -2\omega_a^\alpha \Omega_a + N^{ab} N_{ab} + \frac{1}{2} K^{ab} K_{ba} - \frac{1}{2} N^{ab} N_{ba} - \frac{1}{2} (K^2 + N^2) + T_0^{0\alpha}. \]  

(40)

\[ D_b K_a^b - D_a K = -2\varepsilon_a^{bc} a_b \Omega_c - \varepsilon_a^{bc} K_b^d N_{dc} + 2n^b K_{ab} + T_0^{0\alpha}. \]  

(41)

where we have introduced the shorthand notation \( n_a := \varepsilon_a^{bc} N_{be} / 2 \) and \( \Omega_a := \varepsilon_a^{bc} K_{be} / 2 \). Next, there are analogues constraints on the fields \( \Omega_a \) and \( N_{ab} \), which read

\[ D^n \Omega_a = (\alpha^b + 2n^b) \Omega_b, \]  

(42)

\[ D_b N_a^b - D_a N = 2\varepsilon_a^{bc} \omega_b \Omega_c - \varepsilon_a^{bc} N_b^d N_{dc} - 2\Omega^b K_{ab}. \]  

(43)

Finally, as a further consequence of zero torsion, one obtains the following constraints on the fields \( A_b \) and \( B_b \) (see [58]):

\[ \varepsilon^{cab} B_a^d \frac{\partial A_b}{\partial N^c} = (N^{de} - \delta^{de} N) A_d + \varepsilon^{cab} A_a (K_b^d + \varepsilon_b^{ed} \omega_e - a_b) A_d + 2\Omega^c, \]  

(44)

\[ \varepsilon^{cab} B_a^d \frac{\partial B_b}{\partial N^c} = (N^{de} - \delta^{de} N) B_d^b + \varepsilon^{cab} A_a (K_b^d + \varepsilon_b^{ed} \omega_e) B_d^b. \]  

(45)

Note that unless the æther field \( u \) is hypersurface orthogonal, such that \( 2\Omega^c = \varepsilon^{abc} K_{ab} = 0 \), it is not possible to choose the \( t = \text{const} \) hypersurface to lie perpendicular to \( u \); hence, in general one has \( A_b \neq 0 \). Consequently, equations (40)–(45) should be called ‘quasi-constraints’ as in [58], because although they contain only directional derivatives along the spatial tetrad legs \( e_a \), the latter contain partial time derivatives, see equation (37). However, these equations can be converted into genuine constraints without time derivatives of the fields by substituting each appearance of \( D_b \) by \( A_b D_0 + B_b \partial_0 \), and then using the evolution equations in order to eliminate all terms involving \( D_0 \). We will refer to the final equations obtained in this way as the ‘constraints’.

Although we do not propose a detailed procedure for solving the initial constraints in this article, we nevertheless make the following observations, which should be relevant for this problem. Another way of turning the ‘quasi-constraints’ (40) and (41) into bona-fide constraints (only depending on initial data for the time evolution) consists of noting that due to the diffeomorphism invariance of the action (1), there exists a generalized Bianchi identity [29, 60–64]

\[ \nabla_\mu (2E^{\mu\nu} - u^\mu A^\nu) = A_\mu \nabla^\nu u^\mu, \]  

(46)

where

\[ E_{\alpha\beta} \equiv -\frac{1}{2} (G_{\alpha\beta} - T_{\alpha\beta}), \]  

(47)

\[ A_\mu \equiv \nabla_\alpha J_\mu^\alpha + c_4 a_\alpha \nabla_\mu u^\alpha + \lambda u_\mu = (\nabla_\alpha J_\mu^\alpha + c_4 a_\alpha \nabla_\nu u^\alpha) (g_{\mu\nu} + u_\mu u_\nu), \]  

(48)

(i.e. \( E_{\alpha\beta} = 0 \) and \( A_\mu = 0 \) are respectively the Einstein and æther equations). Expanding the identity (46) in terms of partial derivatives with respect to a coordinate chart \((t, x^i)\) and the corresponding Christoffel symbols, it is easy to show that the four combinations

\[ C_\nu \equiv 2E^{\nu\nu} - u^\nu A_\nu \]  

(49)

depend on one less partial time derivative than the Einstein and æther equations [29]. Since those depend on partial time derivatives of \( N_{ab} \), \( K_{ab} \) and \( \omega_a \), and they do not contain partial time
derivatives of $\alpha$, $\beta^i$, $A_b$, $B^b$, and $\omega_b$, it then follows that the combinations $C^\nu$ can only depend on $N_{ab}$, $K_{ab}$ and $\alpha_b$ (but not on their partial time derivatives, nor on $\alpha$, $\beta^i$ and $\omega_b$)\(^{12}\). Indeed, it can also be verified explicitly that the combinations $T^\nu_{\mu\nu} - u^\nu \mathbf{A}^\nu$ only depend on $\nabla_{\nu} u_{\mu3}$ and on first-order spatial derivatives of $J_{\alpha\beta}$, which in turn depend algebraically on $a_b$ and $K_{ab}$ (while the Einstein part $G^\nu_{\nu}$ only depends on spatial derivatives of the first and second fundamental forms ($\gamma_{ij}$, $K_{ij}$)). Furthermore, it is possible to express $K_{ab}$ in terms of $a_b$, $K_{ij}$, as well as the spatial components $u_i$ of the aether field and their first-order spatial derivatives.

Consequently, the equations $C^\nu = 0$ yield four constraint equations for the data ($\gamma_{ij}$, $K_{ij}$, $u_i$, $a_i$), which should be solved on the initial slice $t = 0$ (the tetrad is not required for this, one can solve these constraints in terms of the local spatial coordinates $x^i$). Once these have been solved, one specifies lapse and shift, which allows one to determine completely the components of the 4-metric and aether field at the initial time $t = 0$. Next, one chooses (e.g. by a Gram–Schmidt method) a tetrad whose timelike leg $e_0$ coincides with the aether field, at each point of the initial hypersurface. Finally, one specifies, by a gauge choice, $\omega_b$ (see equation (28)), which determines the time derivative of the triad at the initial slice. $N_{ab}$ is then defined by equation (29). The constraints given by equations (42)–(45) will then be satisfied by construction.

5. Strong hyperbolicity

In this section we analyze under what conditions on the coupling constants $c_i$ the first-order evolution system (30), (32), (33), (38) and (39) is strongly hyperbolic, such that the associated Cauchy problem is well-posed, at least locally in time. The analysis is performed in several steps, with each reducing the system to one with a smaller number of variables. Roughly speaking, the idea is the following. If we discard all lower-order (undifferentiated) terms in the equations, the evolution equations for the tetrad fields become trivial:

\[ D_0 A_b = 0, \quad D_0 B^b = 0, \]

while the principal part of the remaining evolution equations (30), (32) and (33) gives

\[ D_0 K_{ab} = + \varepsilon_a^{cd} D_c N_{db} + D_a \alpha_b + c_{13} D_0 K_{(ab)} - \frac{1}{2} c_{12} \delta_{ab} D_0 u - \frac{1}{2} c_{14} \delta_{ab} D_a \omega, \]

\[ (50) \]

\[ D_0 N_{ab} = - \varepsilon_a^{cd} D_c K_{db} - D_a \omega_b + c_{13} \varepsilon_{ab}^c D^c K_{cd} + c_2 \varepsilon_{ab}^c D^c K, \]

\[ (51) \]

\[ c_{14} D_0 \alpha_b = c_1 D^a K_{ab} + c_3 D^a K_{ba} + c_2 D_a K, \]

\[ (52) \]

where we have kept, for the moment, the derivatives of the fields $\omega_b$ in the equations, for future generalizations to more general gauge choices. Note that equations (50)–(52) are exact in the limit where the fields represent linear perturbations of Minkowski spacetime with a constant aether field; however in general they are only exact up to lower order terms in the derivatives. Next, we note that by applying the derivative operator $D_0$ on both sides of equation (50), commuting $D_0$ with $D_a$ and using equations (51) and (52) in order to eliminate $D_0 N_{ab}$ and $D_0 \alpha_b$, one obtains (up to lower-order terms) a second-order equation for $K_{ab}$, which reads

\[ \text{Note instead that } C^\nu \text{ may depend on } A_b, B^b, \text{ since when applying the argument above, their time derivatives can be eliminated via the advection equations (38) and (39).} \]

\[ 12 \]
\[ D_0^2 K_{ab} = D^\alpha D_\alpha K_{ab} + \left( \frac{c_1}{c_{14}} - 1 \right) D_a D^\alpha K_{ab} + \frac{c_3}{c_{14}} D_\beta D_\lambda K_{ab} + c_2 \left( \frac{1}{c_{14}} - 1 \right) D_a D_b K \]

\[- c_{13} D_b D^\gamma K_{(ac)} + c_{13} D_0^2 K_{(ab)} + \frac{1}{2} \delta_{ab} \left( c_{13} D^\gamma D^\delta K_{cd} + c_2 D^\delta D_a K - c_{123} D_0^2 K \right) - \varepsilon_{a^c d^e} D_a D_b \omega_{b} \cdot \frac{1}{3} \]

In the following, we will derive sufficient conditions on the principal symbol of this second order system that guarantee that the original, first-order system (30), (32), (33), (38) and (39) is strongly hyperbolic. Besides considerably simplifying the analysis (ending up with a system for the 9 components of \( K_{ab} \) instead of a first-order system with 33 independent variables), this method will also be useful to compute the characteristic fields of the system and to determine whether they are physical fields (i.e. lying in the kernel of the principal symbol associated with both the constraints and gauge-transformations), constraint-violating fields (i.e. lying in the kernel of the principal symbol associated with gauge transformations but outside the one associated with the constraints) or gauge fields according to the classification of [65].

Before analyzing the hyperbolicity of the system, it is important to remark that equations (30), (32) and (33) are not in the standard form with only time derivatives of the fields on one side and only spatial derivatives on the other side. The reason is two-fold: on one hand, there are terms involving the \( D^0 \) operator on the right-hand side of equation (32); on the other hand, the directional derivative operators \( D_a \) contain partial time derivatives, see equation (37). The first issue is easily dealt with by introducing the linear algebraic operator \( L \) defined by

\[ L K_{ab} := K_{ab} - c_{13} K_{(ab)} + \frac{1}{2} c_{123} \delta_{ab} K, \]

which is seen to be invertible as long as \( 1 - c_{13} \neq 0 \) and \( 2(1 + c_2) + c_{123} \neq 0 \)\(^{13}\). Assuming that these conditions are satisfied, equation (32) can be rewritten in the equivalent form

\[ D_0 K_{ab} = L^{-1} \left( \varepsilon_{a^c d^e} D_a N_{db} + D_\alpha a b - \frac{1}{2} c_{14} \delta_{ab} D_\gamma a_c \right), \]

where no \( D_0 \) derivatives appear on the right-hand side. To deal with the second issue, in the following we will resort to an elegant, fully covariant definition of hyperbolicity discussed in [66, 67], which does not rely on any particular foliation of spacetime and thus is well-adapted to our formulation in terms of directional derivatives.

5.1. Reduction to the first-order system for the connection variables \( K_{ab}, N_{ab} \) and \( a_b \)

From now on let us assume that \( c_{13} \neq 1, c_{14} \neq 0 \) and \( 2(1 + c_2) + c_{123} \neq 0 \), such that the operator \( L \) defined above is invertible and such that the aether equation (52) is non-degenerate. In this case, the full system (30), (32), (33), (38) and (39) of evolution equations can be written schematically in the form

\[ D_0 U = A^0 D_\alpha U + F(U), \]

with the 33-component column vector defined by \( U = (A_b, B^b_{ab}, K_{ab}, N_{ab}, a_b)^T \), where the \( 33 \times 33 \) matrices \( A^1, A^2, \) and \( A^3 \) can be read off the principal part of the equations, and where \( F(U) \) is a nonlinear function of \( U \) that represents the lower-order terms. By defining \( A^0 := -I \), with \( I \) the identity matrix, we can trivially rewrite this system in the form

\(^{13}\) Note that \( L \) can be thought of as a matrix acting on the vector \( (K_{ab}) \). These conditions then follow from the requirement that the determinant of that matrix be non-zero.
\[ A^\alpha D_\alpha U + \mathcal{F}(U) = 0. \]  

(56)

The principal symbol of this equation is defined by \( \mathcal{A}(k) := A^\alpha k_\alpha \), for any co-vector \( k_\alpha \). Let us now recall the following definitions from [66, 67], which will allow us to define hyperbolicity even though our system contains directional (as opposed to partial) derivatives:

**Definition 1.** A first-order system of the form (56) with an \( m \)-component state vector \( U \) is called \( C \)-strongly hyperbolic if there exists a co-vector field \( n_\alpha \), and, for each \( k_\alpha \), an \( m \times m \) matrix \( h(k) \) depending smoothly on \( k_\alpha \), such that

(i) \( h(k)A(k) \) is symmetric for all \( k_\alpha \),
(ii) \( h(k)A(n) \) is symmetric and positive definite for all \( k_\alpha \).

If the matrix \( h(k) \) can be chosen to be independent of \( k \), the first-order system (56) is called \( C \)-symmetric hyperbolic.

Since this definition might not be familiar to the reader, we show the relation with the usual definitions of strong and symmetric hyperbolicity for quasi-linear partial differential equations in appendix A. For the applications in this work, it is important to stress that \( C \)-symmetric hyperbolicity implies (local in time) well-posedness of the nonlinear Cauchy problem, while a similar result in the strongly hyperbolic case seems to require some additional smoothness conditions (see the discussion at the end of appendix A for details); as far as we are aware of \( C \)-strong hyperbolicity as defined above only guarantees the well-posedness of the frozen coefficient problem.

In order to apply definition 1 to our system, we partition the state vector \( U \) in the form \( U = (E, V) \), with \( E = (A_b, B_b) \) the components of the tetrad fields and \( V = (K_ab, N_ab, a_b) \) the connection fields. With respect to this decomposition, the principal symbol has the following block structure:

\[ \mathcal{A}(k) = -k_0 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P(k) \end{pmatrix}, \]

with \( P(k) \) a symbol depending linearly on \( k := (k_\alpha) = (k_1, k_2, k_3) \in \mathbb{R}^3 \) that can be inferred from the system (50)–(52). We make the following ansatz for the family of matrices \( h(k) \) in definition 1:

\[ h(k) := \begin{pmatrix} I & 0 \\ 0 & H(k) \end{pmatrix}, \]

with \( H(k) = H(k)^T \) a family of symmetric, positive definite \( 21 \times 21 \) matrices to be determined, depending smoothly on \( k \). Then, taking \( n_\alpha := u_\alpha \) to be the co-vector field associated with the \( \alpha \)th field (such that \( A(u) = I \) one finds \( h(k)A(u) = h(k) \) which, by definition, satisfies condition (ii)). Furthermore,

\[ h(k)A(k) = -k_0 \begin{pmatrix} I & 0 \\ 0 & H(k) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & H(k)P(k) \end{pmatrix} \]

which is symmetric provided \( H(k) \) satisfies

\[ H(k)P(k) = P(k)^T H(k) \]  

(57)

for all \( k \in \mathbb{R}^3 \). (In fact, since \( P(k) \) depends linearly on \( k \), it is sufficient to assume that \( k \in S^2 \) has unit norm, provided that \( H(k) \) is chosen to depend on the direction of \( k \), but not on its
norm.) Condition (57) means that the evolution system (50)–(52) for the connection variables is strongly hyperbolic:

**Definition 2.** The first-order system (50)–(52) is called strongly hyperbolic if there exists a family of symmetric, positive-definite matrices \(H(k) = H(k)^T > 0\) depending smoothly on \(k \in S^2\) such that equation (57) is satisfied for all \(k \in S^2\). If the symmetrizer \(H = H(k)\) can be chosen to be independent of \(k\), then the system is called symmetric hyperbolic.

The condition (57) implies that \(P(k)\) is diagonalizable (i.e. having a complete set of eigenvectors with only real eigenvalues\(^{14}\)). Conversely, if \(P(k)\) has only real eigenvalues and a complete set of eigenvectors that are arranged in the columns of a matrix \(S(k)\), then \(H(k) = [S(k)^{-1}]^T S(k)^{-1}\) satisfies equation (57). Therefore, up to the smoothness requirement on \(H(k)\), strong hyperbolicity is equivalent to the principal symbol \(P(k)\) being diagonalizable with only real eigenvalues.

Summarizing what we have achieved so far, we have shown that the full system of evolution equations (30), (32), (33), (38) and (39) is C-strongly (C-symmetric) hyperbolic if the system (50)–(52) of the connection variables is strongly (symmetric) hyperbolic. In the next step, we will show that the 21 \(\times\) 21 symbol \(P(k)\) can be further reduced to a 9 \(\times\) 9 symbol.

### 5.2. Reduction to a second-order system for \(K_{ab}\)

Further partitioning the state vector \(V = (V_1, V_2)\) into the components \(V_1 = (K_{ab})\) and \(V_2 = (N_{ab}, a_{ab})\), the symbol \(P(k)\) associated with the system (50)–(52) has the particular block structure

\[
P(k) = \begin{pmatrix} 0 & Q(k) \\ R(k) & 0 \end{pmatrix},
\]

with

\[
Q(k)V_2 = L^{-1}\left(\varepsilon_{a}^{cd}k_{c}N_{db} + k_{a}d_{b} - \frac{1}{2}c_{14}\delta_{ab}k^{c}a_{c}\right),
\]

\[
R(k)V_1 = \begin{pmatrix} -\varepsilon_{a}^{cd}k_{c}K_{db} + c_{13}\varepsilon_{abc}k_{d}K^{(cd)} + c_{2}\varepsilon_{abc}k^{(c}K) \\ \frac{c_{1}}{c_{14}}k^{c}K_{ab} + \frac{c_{1}}{c_{14}}k^{c}K_{ab} + \frac{c_{1}}{c_{14}}k_{b}K \end{pmatrix},
\]

where we recall that the linear algebraic operator \(L\) has been defined in equation (54).

The analysis of the symbol \(P(k)\) is greatly simplified by exploiting its special block structure (58), following ideas described in [68, 69]. If \(\lambda\) is an eigenvalue of \(P(k)\), then there exists \(V = (V_1, V_2) \neq (0, 0)\) such that

\[
\lambda \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 & Q(k) \\ R(k) & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.
\]

There are two possible type of solutions: either \(\lambda = 0\) in which case \(V_1\) has to lie in the kernel of \(R(k)\) and \(V_2\) in the kernel of \(Q(k)\), or \(\lambda \neq 0\), in which case \(V_1\) must be an eigenvector of \(M(k) := Q(k)R(k)\) with eigenvalue \(\lambda^2\). Note that \(M(k)\) is precisely the symbol associated

---

\(^{14}\)This can be seen by noting that equation (57) implies that \(P := P(k)\) is symmetric with respect to the scalar product \((V_1, V_2) := V_1^T H(k) V_2\), and thus by the spectral theorem it is diagonalizable with real eigenvalues. Equivalently, equation (57) implies that \(H^{1/3}PH^{-1/3}\) (where \(H^{1/3}\) is well defined because \(H := H(k)\) is positive definite) is symmetric. If \(\lambda\) is an eigenvector for \(H^{1/3}PH^{-1/3}\), then \(H^{1/2}\) is an eigenvector for \(P\) with the same eigenvalue. It then trivially follows that \(P\) has a complete set of eigenvectors and is therefore diagonalizable with real eigenvalues.
with the second-order wave-like equation for $K_{ab}$, equation (53). Indeed, that equation can be written (up to lower order terms) as $D_0^2 V_1 = M^{ab} D_0 D_b V_1$, with $V_1$ the state vector $V_1 = (K_{ab})$, and $M^{ab}$ is the matrix associated to $M(k)$, i.e. $M(k) = M_{ab} k_a k_b$. The following lemma, whose proof is given in the appendix B, gives sufficient conditions on the second-order symbol $M(k)$ for the first-order system to be strongly hyperbolic.

**Lemma 1.**

(a) Suppose the symbol $M(k) = Q(k) R(k)$ is diagonalizable and that all its eigenvalues $0 < \mu_1 < \mu_2 < \ldots < \mu_4$ are strictly positive. Then, $P(k)$ is diagonalizable with real eigenvalues of the form $0, \pm \sqrt{\mu_j}$, $j = 1, 2, \ldots, 4$. (Hence, the system is strongly hyperbolic provided a smooth symmetrizer $H(k)$ can be constructed.)

(b) If there exists a family of symmetric, positive-definite matrices $H_1(k)$ depending smoothly on $k \in S^2$ such that

$$H_1(k) M(k) = M(k)^T H_1(k)$$

is symmetric and positive definite for all $k \in S^2$, then $P(k)$ is strongly hyperbolic in the sense of definition 2.

The significance of the lemma relies in the fact that it provides a sufficient condition on the symbol associated with the second-order system for the original system to be strongly hyperbolic. This is clearly a huge simplification, since the former system is $21 \times 21$. Explicitly, the symbol $M(k)$ is given by

$$LM(k) K_{ab} = L K_{ab} + \left( \frac{c_1}{c_{14}} - 1 \right) k_b k^d K_{db} + \frac{c_1}{c_{14}} k_a k^d K_{bc} + c_2 \left( \frac{1}{c_{14}} - 1 \right) k_b k_e K_{ab} + c_{13} \left[ K_{(ab)} - k_a k^d K_{(db)} - \frac{1}{2} k_b (K - k^d K_{cd}) \right].$$

(61)

A more explicit form for the symbol $M(k)$ can be obtained by multiplying both sides of the equations to the left with the inverse $L^{-1}$ of $L$; however, it turns out to be simpler to perform this operation after the next step, in which we decompose $K_{ab}$ into scalar, vector and tensor contributions.

### 5.3. Decomposition into scalar, vector and tensor blocks

In order to determine under which conditions on the constants $c_i$’s the hypothesis on the second-order symbol (61) in lemma 1 are satisfied, it is convenient to decompose $K_{ab}$ into its components parallel and perpendicular to the unit vector $k$. Introducing the operator $\gamma_a^b = \delta_a^b - k_a k^b$, which projects on the plane orthogonal to $k$, this decomposition reads

$$K_{ab} = k_a k_b K_{kk} + k_a K_{kb} + K_{ak} k_b + K_{ab} + \frac{1}{2} \gamma_{ab} (K - K_{kk}),$$

(62)

with the quantities $K_{kk} := k^d k^e K_{cd}$ and $K := \delta_{ab} K_{cd}$ constituting the scalar block, the vectors orthogonal to $k$ defined by $K_{kb} := k^d K_{cd} \gamma_d^b$ and $K_{ab} := \gamma_{ab} k_a k^d$ constituting the vector block, and the transverse, trace-less part $K_{ab} := \gamma_a^c \gamma_b^d (K_{cd} - \frac{1}{2} \gamma_{cd} \gamma_{ef})$. The latter can be further decomposed into symmetric and anti-symmetric parts, the symmetric part $\hat{K}_{ab}$ describing the tensor block and the antisymmetric part being dual to the pseudo-scalar $k_a \gamma_{ab} K_{bc}$. With respect to this decomposition, the eigenvalue problem $\lambda^2 K_{ab} = M(k) K_{ab}$ decouples, and one
can analyze the conditions of Lemma 1 separately in each block, which further simplifies the problem. Using equations (54) and (61), the results obtained in each block are the following:

1. **Tensor and pseudo-scalar blocks**

   In this case one obtains the set of equations
   \[ \lambda^2 \hat{K}_{(ab)} = \lambda^2 \hat{K}_{(ab)}, \quad \lambda^2 \hat{K}_{[ab]} = \hat{K}_{[ab]}, \] (63)

   for the symmetric and anti-symmetric parts of \( \hat{K}_{ab} \), respectively, where \( \lambda^2 := (1 - c_{13})^{-1} \).
   
   The system (63) is already in diagonal form and its eigenvalues are equal to \( \lambda^2 \) and 1, which are positive and real provided \( c_{13} < 1 \).

2. **Vector block**

   In this case one obtains the coupled system
   \[ \lambda^2 \begin{pmatrix} K_{ib} \\ K_{bk} \end{pmatrix} = \begin{pmatrix} I \& a_V b_V^T \\ b_V^T \& I \end{pmatrix} \begin{pmatrix} K_{ib} \\ K_{bk} \end{pmatrix}, \] (64)

   with \( a_V \) a column vector and \( b_V^T \) a row vector given by
   \[ a_V = \frac{1}{2(1 - c_{13})} \begin{pmatrix} 2 - c_{13} \\ c_{13} \end{pmatrix}, \quad b_V^T = \begin{pmatrix} c_{13} \\ \frac{c_{13} + c_1}{c_{14}} - 1, \frac{c_{13} + c_2}{c_{14}} \end{pmatrix}. \]

   In equation (64), it is understood that the first component of \( b_V^T \) acts on the three components of \( K_{ib} \) and its second component on the three components of \( K_{bk} \), and likewise for \( a_V \).

3. **Scalar block**

   In this case one obtains a coupled system, which can be written in a form similar to that of the vector case:
   \[ \lambda^2 \begin{pmatrix} K_{kk} \end{pmatrix} = \begin{pmatrix} I \& a_S b_S^T \\ b_S^T \& I \end{pmatrix} \begin{pmatrix} K_{kk} \end{pmatrix}, \] (65)

   with
   \[ a_S = \frac{1}{2(1 - c_{13})} \begin{pmatrix} 2 - c_{13} \\ c_{13} \end{pmatrix}, \quad b_S^T = \begin{pmatrix} c_{13} \\ \frac{c_{13} + c_1}{c_{14}} - 1, \frac{c_{13} + c_2}{c_{14}} \end{pmatrix}. \]

   In order to determine under what conditions the vector and scalar blocks are diagonalizable, one can exploit the particular structure of the matrices in each case and use the following simple lemma, whose proof is included in appendix B for completeness:

   **Lemma 2.** Let \( a, b \in \mathbb{R}^n \) be two non-vanishing constant column vectors in \( \mathbb{R}^n \), and consider the matrix
   \[ M := I + ab^T. \]

   Then, \( M \) is diagonalizable if and only if \( a \) is not orthogonal to \( b \). In this case, its eigenvalues are \( 1 + a^T b \) (with multiplicity 1 and eigenvector \( a \)) and 1 (with multiplicity \( n - 1 \) and eigenvectors orthogonal to \( b \)). Furthermore, the symmetric matrices
   \[ H := \kappa_0 (a^2 I - aa^T) + \kappa_1 bb^T, \]
with positive constants $\kappa_0 > 0$ and $\kappa_1 > 0$, constitute a family of symmetrizers for $\mathcal{M}$, i.e. they are symmetric, positive-definite and satisfy $\mathcal{H} \mathcal{M} = \mathcal{M}^T \mathcal{H}$.

Applying this lemma to the systems obtained in the tensor, vector and scalar blocks, we conclude that the symbol $\mathcal{M}(\mathbf{k})$ is diagonalizable with strictly positive eigenvalues if and only if

$$\lambda_\gamma^2 := \frac{1}{1 - c_{13}} > 0,$$

$$\lambda_V^2 := 1 + b_V^T a_V = \frac{(2 - c_1)c_1 + c_1^2}{2c_{14}(1 - c_{13})} > 0,$$  \hfill (66)

$$\lambda_S^2 := 1 + b_S^T a_S = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_{13})(2 + 2c_2 + c_{123})} > 0,$$  \hfill (67)

and $\lambda_\gamma^2 \neq 1 \neq \lambda_S^2$. If these conditions are satisfied, one can use the previous Lemma again to construct a symmetrizer $H_1(\mathbf{k})$ for the second-order symbol $\mathcal{M}(\mathbf{k})$, defined by

$$\mathbf{(K, K) :} \mathcal{M}(\mathbf{k}) := \mathcal{K}^{ab} H_1(\mathbf{k}) K_{ab}$$

$$= \mathcal{K}^{ab} K_{ab} + \left( \begin{array}{c} \mathcal{K}^{kb} \mathcal{K}_{kb} \\ \mathcal{K}^{ab} \end{array} \right) \left( \begin{array}{c} \mathcal{K}^{an} \mathcal{K}_{an} \\ \mathcal{K}^{bn} \mathcal{K}_{bn} \end{array} \right) \left( \mathcal{K}^{bc} \mathcal{K}_{bc} \right)$$

$$+ \left( \begin{array}{c} \mathcal{K}^{ab} \mathcal{K}_{ab} \\ \mathcal{K}^{bc} \mathcal{K}_{bc} \end{array} \right) \left( \begin{array}{c} \mathcal{K}^{an} \mathcal{K}_{an} \\ \mathcal{K}^{bn} \mathcal{K}_{bn} \end{array} \right) \left( \mathcal{K}^{bc} \mathcal{K}_{bc} \right).$$

By construction, the matrix $H_1(\mathbf{k})$ defined in this way is symmetric, positive-definite and satisfies the condition (b) of Lemma 1. Furthermore, $H_1(\mathbf{k})$ depends smoothly on the vector $\mathbf{k}$, since the projections of $K_{ab}$ onto its pieces parallel and orthogonal to $\mathbf{k}$ are smooth (in particular, note that the projection operator $\gamma_a^b = \delta_a^b - k_a k^b$ depends smoothly on $\mathbf{k}$) and since the components of the vectors $a_V, b_V, a_S, b_S$ only depend on the constants $c_i$. Therefore, it follows from Lemma 1 that the first-order system with symbol $P(\mathbf{k})$ is strongly hyperbolic, provided the conditions (66)–(68) and $\lambda_\gamma^2 \neq 1 \neq \lambda_S^2$ are fulfilled. This is the main result of this paper, whose implications will be further discussed in the conclusion section. Before doing so, however, it is instructive to determine the eigenvalue-eigenvector pairs associated with the symbol $P(\mathbf{k})$ since (at least in the regime of small amplitude, high-frequency perturbations) they describe the propagation speeds and modes of the system.

### 5.4. Characteristic speed and fields and their physical interpretation

Before proceeding, we briefly note that the quantities $\lambda_\gamma^2$, $\lambda_V^2$ and $\lambda_S^2$ defined in (66)–(68) coincide precisely with the squared propagation speed $s_2^2$, $s_1^2$ and $s_0^2$, respectively, given in equations (9)–(11). This is expected, because, as already discussed, the speeds $s_i$ ($i = 0, 1, 2$) regulate the propagation of the physical spin-$i$ modes of the theory, on flat backgrounds.

From Lemma 2, the eigenfields of the second-order symbol $\mathcal{M}(\mathbf{k})$ corresponding to the eigenvalues $\lambda_\gamma^2$, $\lambda_V^2$ and $\lambda_S^2$ are given by

$$\lambda_\gamma^2 = s_2^2 \quad \text{(two modes)} : K^{(T,1)}_{ab} = c_a e_b - f_a f_b, \quad K^{(T,2)}_{ab} = 2e_i f_b,$$  \hfill (69)

$$\lambda_V^2 = s_1^2 \quad \text{(two modes)} : K^{(V,1)}_{ab} = (2 - c_{13})k_a e_b + c_{13} e_a k_b, \quad K^{(V,2)}_{ab} = (2 - c_{13})k_a f_b + c_{13} f_a k_b,$$  \hfill (70)
with \( e_b f_b \) two mutually orthogonal unit vectors such that \( \{ k, e, f \} \) forms an oriented orthonormal basis of \( \mathbb{R}^3 \), i.e. \( e_{abc} = 6 k_a e_b f_c \). The corresponding eigenfields of the first-order symbol \( P(k) \) can be constructed using the method described in the proof (given in appendix B) of lemma 1(a), which yields the eigenvalue-eigenvector pairs

\[
\pm \lambda_T \text{ (four modes)}: V^{(T1)}_{\pm} = (K^{(T1)}_{ab}, \pm \lambda_T^{-1} R(k) V^{(T1)}_{ab}) = (K^{(T1)}_{ab}, \pm \lambda_T^{-1} K^{(T2)}_{ab}, 0),
\]

\[
\pm \lambda_V \text{ (four modes)}: V^{(V1)}_{\pm} = (K^{(V1)}_{ab}, \pm \lambda_V^{-1} R(k) V^{(V1)}_{ab}) = \left( K^{(V1)}_{ab}, \pm \frac{c_{13} \lambda_V}{\lambda_V c_{14}} k_{ab} f_b, \pm \left( \frac{2 - c_1 c_1 + c_3^2}{\lambda_V c_{14}} \right) e_b \right),
\]

\[
\pm \lambda_S \text{ (two modes)}: V^{(S)} = (K^{(S)}_{ab}, \pm \lambda_S^{-1} R(k) V^{(S)}_{ab}) = (K^{(S)}_{ab}, \pm \lambda_S^{-1} e_{ab} f_b, \pm \lambda_S^{-1} c_{14} f_b),
\]

where we recall the notation \( V = (K_{ab}, N_{ab}, a_b) \) and that the symbol \( R(k) \) is defined in equation (60).

The remaining eigenfields of the second-order symbol \( M(k) \) have eigenvalues 1. According to Lemma 2, they are explicitly given by \( 2 e_a f_b \); by two non-trivial linear combinations of \( K_{ab} \) and \( K_{bh} \) orthogonal to the vector \( b_v \); and by a non-trivial linear combination of \( K_{kh} \) and \( K \) orthogonal to \( b_v \). As we show now, these fields correspond to constraint-violating modes. In order to do so, we consider the symbol associated with the momentum constraint (41):

\[
k^b \left[ K_{ab} - c_{13} K_{(ab)} - (1 + c_2) \delta_{ab} K \right] = 0.
\]

In the scalar and vector blocks, respectively, this yields

\[
(1 - c_{13}) K_{kh} - (1 + c_2) K = 0, \quad c_{13} K_{ab} - (2 - c_{13}) K_{bh} = 0,
\]

while there are no restrictions in the tensor and pseudo-scalar blocks. We see that these equations are precisely satisfied for the eigenvectors proportional to \( a_k \) and \( a_V \), respectively. Therefore, the five eigenvectors defined in equations (69)–(71), representing the physical modes, lie in the kernel of the symbol associated with the momentum constraint, as expected. In contrast to this, the vector and scalar modes propagating with speed 1 are orthogonal to \( b_v \) and \( b_S \), respectively, and hence they cannot be parallel to \( a_k \) or \( a_V \). Consequently, the vector and scalar modes propagating with speed 1 are constraint-violating modes.

Next, we analyze the eigenfield \( 2 e_a f_b \) in the pseudo-scalar block, which also propagates with speed 1 but lies in the kernel of the symbol associated with the momentum constraint. The corresponding eigenfields of the first-order symbol \( P(k) \) are

\[
\left( 2 e_a f_b, \pm R(k) e_a f_b \right) = \left( 2 e_a f_b, \pm (e_a e_b + f_a f_b), 0 \right),
\]

which do not lie in the kernel associated with the constraint equation (43):

\[
N_{bh} = 0, \quad N_{kh} - N = 0,
\]

whereas the physical modes defined in equations (72)–(74) do. Therefore, all the modes (with the exception of the modes \( V^{(T1)}_{\pm} \) and \( V^{(T2)}_{\pm} \) when \( c_{13} = 0 \) propagating with speed 1 are
constraint-violating. For completeness, one may also consider the symbol associated with the Hamiltonian constraint (40), which is
\[ \varepsilon^{abc} k_a N_{bc} - c_{14} k_b a_b = 2 \epsilon [w f^b] N_{ab} - c_{14} a_k = 0, \]
and is automatically satisfied by the physical modes defined in equations (72)–(74).

The remaining 15 eigenvectors of the first-order system propagate with zero speed. They correspond to the 12 tetrad fields \( A_b \) and \( B^b_k \) and to the 3 independent vectors lying in the kernel of the symbol \( Q(k) \), as discussed in lemma 1(a). The kernel of \( Q(k) \) is easily shown to be of the form
\[ V = (K_{ab}, N_{ab}, a_b) = (0, k_w b, 0), \]
with \( w_b \) an arbitrary vector, and they correspond to gauge-modes because they can be eliminated by an appropriate choice of the angular velocity \( \omega_a \), see equation (51). This concludes our discussion of the characteristic speeds and fields of the system.

6. Symmetric hyperbolic formulations

In this section, we show that by taking suitable combinations of the evolution and (quasi-)constraint equations, it is possible to recast the evolution equations into symmetric (instead of merely strongly) hyperbolic form. To simplify the analysis, in the following we focus on the three-parameter space
\[ c_1 = \Delta s_1^2, \quad c_2 = \Delta \Gamma, \quad c_{13} = 0, \quad c_{14} = \Delta, \quad (75) \]
with \( \Delta, s_1 \) and \( \Gamma \) real parameters satisfying \( s_1 \neq 0 \) and \( \Delta \Gamma > -2/3 \) (so that the evolution equations and symmetrizer below are well-defined). In terms of the new parametrization, the squared propagation speeds (66)–(68) are given by
\[ \lambda_T^2 = 1, \quad \lambda_F^2 = s_1^2, \quad \lambda_S^2 = (2 - \Delta) \Gamma / 2 + 3 \Delta \Gamma, \quad (76) \]
and the post-Newtonian parameters (12) and (13) reduce to
\[ \alpha_1 = -4 \Delta, \quad \alpha_2 = \Delta \left( 1 - (2 - 2\Delta) \Gamma / (2 - \Delta) \Gamma \right). \quad (77) \]

Hence, the parameter choices (75) are compatible with the observational constraints obtained in section 2 by saturating the bound on \( \alpha_1 \), provided that \( |\Delta| \lesssim 0.25 \times 10^{-4} \) and \( |\Gamma - 1| \lesssim 8 \times 10^{-3} \) (and \( s_1 \geq 1 \) and \( \Gamma - 1 \geq 2\Delta / (1 - 2\Delta) \) to ensure \( s_1^2 \geq 1 \) and thus satisfy the Cherenkov bound). Note that the bounds from gravitational wave generation, though yet to be worked out in detail in this region of the parameter space, should be satisfied. Indeed, if \( s_1 \sim 1 \), then \( |c_1 - c_3| \sim \mathcal{O}(10^{-4}) \), while if \( s_1 \gg 1 \) and \( \Delta s_1^2 \gg 1 \), then \( |c_1 - c_3| \gg 1 \), and the theory should reproduce the GR predictions for gravitational wave generation in both limits as discussed in section 2. Note that our parametrization can also cover the two-dimensional parameter space \((c_2, c_1 - c_3)\) obtained in section 2 by setting \( 0 \neq |c_3 - c_4| \lesssim 10^{-7} \). Here, that corresponds to taking \( 0 \neq |\Delta| \lesssim 10^{-7} \).

To obtain a symmetric hyperbolic system, we use the momentum quasi-constraint equation (41) to eliminate the divergence term \( D^a K_{ab} \) in the evolution equation for \( a_b \). Furthermore, for the sake of gaining flexibility to achieve a symmetric symbol, we use the Hamiltonian quasi-constraint equation (40) to modify the right-hand side of the evolution equation for \( K_{ab} \). After these operations, the principal part of the evolution equations (50)–(52) becomes
\[ D_{\alpha}K_{\beta} = +\varepsilon_{\alpha}^{\quad \epsilon d}D_{\epsilon}N_{\beta d} + D_{\epsilon\alpha b} - \frac{\Delta}{2 + 3\Delta\Gamma}\delta_{\alpha b}\left[(\Gamma - \sigma)\varepsilon^{\epsilon d}\epsilon_{d}N_{\epsilon d} + (1 + \Gamma + \Delta\sigma)D_{\epsilon}a_{\epsilon}\right], \quad (78) \]

\[ D_{\alpha}N_{\beta \alpha} = -\varepsilon_{\alpha}^{\quad \epsilon d}D_{\epsilon}K_{\beta d} + \Delta\Gamma\varepsilon_{\alpha\beta}^{\epsilon d}D_{\epsilon}K, \quad (79) \]

\[ D_{\alpha\beta} = s_{1}^{2}\mathbf{D}^{\alpha\beta}K_{\alpha\beta} + \left[\Gamma - s_{1}^{2}\left(1 + \Delta\Gamma\right)\right]D_{\alpha}K, \quad \text{(80)} \]

with the new real parameter \(\sigma\) being the coefficient determining the considered linear combination between the evolution equation for \(K_{\alpha\beta}\) and the Hamiltonian quasi-constraint. For the particular choice \(\Delta = 0\) and \(\Gamma = s_{1}^{2} > 0\), the system (78)–(80) simplifies considerably, and it is a simple task to verify that it is symmetric hyperbolic with respect to the \((k\text{-independent})\) symmetrizer \(H\) defined by

\[ V^{T}HV := K_{\alpha\beta}^{ab}K_{\alpha\beta} + N_{\alpha\beta}N_{\alpha\beta} + \frac{1}{s_{1}^{2}}d^{\alpha}a_{\alpha}, \quad V = (K_{\alpha\beta}, N_{\alpha\beta}, a_{\alpha})^{T}, \quad \text{(81)} \]

that is, its principal symbol \(P(k)\) satisfies \(V^{T}HP(k)V_{2} = V^{T}HP(k)V_{1}\) for all state vectors \(V_{1}\) and \(V_{2}\) and all \(k \in S^{2}\). More generally, one can show that equation (81) provides a symmetrizer for the system (78)–(80), provided that the following relations hold:

\[ \Gamma = -\frac{\Gamma - \sigma}{2 + 3\Delta\Gamma}, \quad \frac{\Gamma}{s_{1}} = (1 + \Delta\Gamma) = -\Delta\frac{1 + \Gamma + \Delta\sigma}{2 + 3\Delta\Gamma}. \]

The first one can always be satisfied by defining

\[ \sigma := 3\Gamma(1 + \Delta\Gamma), \quad \text{(82)} \]

while the second reduces the dimensionality of the parameter space \((s_{1}, \Delta, \Gamma)\) from three to two; e.g. the second relation is satisfied if we define \(s_{1}^{2}\) by

\[ s_{1}^{2} := \Gamma\frac{2 + 3\Delta\Gamma}{(\Gamma + \Delta\Gamma + (1 - \Gamma)(1 + 3\Delta\Gamma + 3\Delta^{2}\Gamma^{2})}. \quad \text{(83)} \]

We conclude that the system (78)–(80) is symmetric hyperbolic for any choice for \(\Delta\) and \(\Gamma\) such that the right-hand side of equation (83) is strictly positive, provided that \(\sigma\) and \(s_{1}^{2}\) are defined by equations (82) and (83), respectively. This shows that there is a (at least) two-dimensional parameter space of Einstein-ether theory whose evolution equations can be cast into symmetric hyperbolic form. This parameter space contains the values \((\Delta, \Gamma)\) for which \(0 \leq \Delta \leq 1\) and \(\Gamma > 0\), and hence it also contains the open region in the \((\Delta, \Gamma)\)-plane defined by \(|\Delta| \lesssim 0.25 \times 10^{-4}\) and \(|\Gamma - 1| \lesssim 8 \times 10^{-3}\), which is compatible with the solar system constraints when saturating the \(a_{1}\) bound. Note indeed that for small \(|\Delta|\) and \(|\Gamma - 1|\), equation (83) gives \(s_{1}^{2} = 1 + (\Gamma - 1) + O[(\Gamma - 1)^{2}, \Delta^{2}]\), which is enough to satisfy the experimental bounds discussed above (from Cherenkov radiation and gravitational wave emission) if \(\Gamma > 1\). Note however that the condition equation (83) does not allow us to prove symmetric hyperbolicity in the case in which \(s_{1}\) diverges (which is a necessary condition to cover the limit \(|c_{1} - c_{3}| \rightarrow \infty\), also allowed by the experimental bounds when one saturates the \(a_{1}\) constraint).

By adding a term proportional to \(K^{2}\) to the right-hand side of equation (81), one can obtain an even larger class of symmetric hyperbolic systems for an open set of parameters \((s_{1}, \Delta, \Gamma)\) in \(\mathbb{R}^{3}\), characterized by the requirement that

\[ \frac{(1 + 3s_{0}^{2})^{2}}{2s_{1}^{2}}\Gamma^{2} + (1 - 3s_{0}^{2})\Gamma^{2} - 4s_{0}^{2}\Gamma - s_{0}^{2}(1 - 3s_{0}^{2}) > 0, \quad s_{0}^{2} = \frac{2 - \Delta\Gamma}{2 + 3\Delta\Gamma}. \quad \text{(84)} \]
When \((\Delta, \Gamma) \approx (0, 1)\) one has \(s_0^2 \approx 1\), and this set restricts the value of \(s_1^2\) to be approximately smaller than \(3/2\), so one can still not access the regime where \(\Delta s_1^2\) is very large, which is in principle also compatible with the experimental bounds, as discussed above.

As for the two-dimensional parameter space \((c_2, c_1 - c_3)\) obtained in section 2 by setting \(0 \neq |c_3 - c_1| \lesssim 10^{-7}\), that is fully included in our analysis since \(|\Delta|\) can be made arbitrarily small while still satisfying equation (83). Note also that unless \(c_2\) and \(c_3\) are also \(O(\Delta)\) (which can be avoided simply by choosing \(\Gamma \sim c_2/\Delta \sim O(0.1)/\Delta\)), the spin-0 and spin-1 squared propagation speeds automatically become \(\sim \Gamma \sim O(0.1)/\Delta \sim 10^6\). As already mentioned in section 2, this renders the scalar and vector polarizations close to non-dynamical, and therefore more likely to pass binary pulsar bounds.

7. Conclusions

We have analyzed the well-posedness of the Cauchy (initial value) problem in Einstein-ether theory. In the standard metric formulation of the theory it is far from clear that the evolution equations can be cast in hyperbolic form, since they are second order in both the metric fields and the ether vector field. This is worrisome as well-posedness is a fundamental ingredient for the predictive power of the theory, meaning that for a given set of initial data there exists (at least locally in time) a unique solution that depends continuously on the data. In particular, well-posedness is a necessary requirement to ensure stability and convergence of numerical initial value evolutions. Although well-posedness is relatively easy to prove for linear perturbations on flat spacetime, since the linear equations reduce to a system of decoupled wave equations [51] for the spin-0, spin-1 and spin-2 modes of the theory, a generalization to the full system of field equations was so far lacking. We have succeeded in showing that a first-order reformulation of Einstein-ether theory in terms of projections onto a tetrad frame quite naturally leads to strongly hyperbolic evolution equations, as long as the (squared) propagation speeds \(s_0^2, s_1^2\) and \(s_2^2\) are strictly positive and finite with \(s_0^2 \neq 1 \neq s_1^2\). The covariant notion of strong hyperbolicity employed in this article ensures (modulo technical smoothness requirements, see appendix A) the local in time well-posedness of the vacuum initial value (Cauchy) problem in Einstein-ether theory in this region of the coupling constants’ parameter space. Furthermore, by suitably modifying the evolution equations by means of the constraints, we have obtained a three-parameter family of formulations for which \(s_2^2 = 1\) and \(s_0^2\) and \(s_1^2\) lie close enough to 1, which is symmetric (rather than merely strongly) hyperbolic, and in this case (local in time) well-posedness of the Cauchy problem follows without any additional assumptions.

At least some of the conditions on the propagation speeds are easy to understand on physical grounds. Strict positiveness of \(s_0^2, s_1^2\) and \(s_2^2\) is needed to ensure absence of ghosts and gradient instabilities (see [51, 53] and discussion in section 2). At the level of the evolution equations, a negative value of \(s_0^2, s_1^2\) or \(s_2^2\) would imply catastrophic, unbounded frequency-dependent instabilities of the solutions. Similarly, if any of the propagation speeds diverges, the field equations cannot be strongly hyperbolic (not even in flat space), as the degree of freedom whose speed diverges becomes non-dynamical and satisfies an elliptic equation in flat space [51]. Therefore, the system can at best be elliptic-hyperbolic. It is instead more difficult to intuitively make sense of the requirement that the spin-0 and spin-1 speeds should be different than one. If equal to one, it follows from our analysis in section 5 that the evolution equations are only weakly hyperbolic, a property that (at the nonlinear level) usually leads to frequency-dependent instabilities [42, 70]. Nevertheless, it seems very likely that these special cases still yield a well-posed Cauchy problem if the constraints are taken into account (indeed
the restrictions \( x_0^2 \neq 1 \neq x_1^2 \) go away in our symmetric hyperbolic subfamilies for which the constraints have been used). This could be analyzed in a systematic and elegant way based on the recent method introduced in [71]. We will leave this problem aside for future work. Nevertheless, the experimental constraints on the coupling constants (discussed in section 2) show that while the spin-2 speed needs to be very close to unity (to within \( O(10^{-15}) \)), the spin-0 and spin-1 speeds are typically different from one. Therefore, our proof of the well-posedness of the Cauchy problem applies to the whole viable parameter space of the coupling constants.

Acknowledgments

We would like to warmly thank Luis Lehner for providing insightful comments about this work. We also thank Ted Jacobson, Thomas Sotiriou and Diego Blas for useful discussions about Lorentz violating gravity, and Oscar Reula for fruitful discussions about covariant definitions of strong hyperbolicity and their implications for the well-posedness of the Cauchy problem. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant agreement No. GRAMS-815673; project title ‘GRavity from Astrophysical to Microscopic Scales’); from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant agreement No. 690904; from the CONACyT Network Project No. 294625 ‘Agujeros Negros y Ondas Gravitatorias’; and from a CIC grant to Universidad Michoacana de San Nicolás de Hidalgo. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade.

Appendix A. Equivalent definitions of strong hyperbolicity

Consider a first-order quasi-linear system of the form of equation (56),

\[
\mathcal{A}^\alpha D_\alpha U + \mathcal{F}(U) = 0,
\]

(A.1)

with corresponding principal symbol \( \mathcal{A}(k) := \mathcal{A}^\alpha k_\alpha \). Suppose that this system is C-strongly hyperbolic in the sense of definition 1. In this appendix we reproduce parts of the arguments in [67], which show that this definition is equivalent to the usual definition of strong hyperbolicity. To this purpose, let \( T^\alpha \) denote any vector field such that \( n_\alpha T^\alpha = 1 \), and let \( h_\alpha^\beta := \delta_\alpha^\beta - n_\alpha T^\beta \) denote the projection operator along \( n_\beta \) onto the subspace of co-vectors that annihilate \( T^\alpha \) (i.e. \( h_\alpha^\beta n_\beta = 0 \) and \( T^\beta h_\alpha^\beta = 0 \)). Inserting \( \delta_\alpha^\beta = n_\alpha T^\beta + h_\alpha^\beta \) into equation (A.1), this equation can be rewritten as

\[
T_\beta D^\beta U = -\mathcal{A}(n)^{-1} \left[ \mathcal{A}^\alpha h_\alpha^\beta D_\beta U + \mathcal{F}(U) \right] =: \tilde{\mathcal{A}}_\beta D_\beta U + \tilde{\mathcal{F}}(U),
\]

(A.2)

where we have used the fact that, because of the second condition of definition 1, the matrix \( \mathcal{A}(n) \) is invertible. The symbol of the operator on the right-hand side of equation (A.2) is defined by

\[
\tilde{\mathcal{A}}(k) := \tilde{\mathcal{A}}^\alpha k_\alpha = -\mathcal{A}(n)^{-1} \mathcal{A}^\alpha k_\alpha,
\]

(A.3)
for all co-vectors \( k_\alpha \) satisfying \( k_\alpha T^\alpha = 0 \), and as a consequence of definition 1, it possesses the smooth symmetrizer

\[
\tilde{H}(k) := h(k)A(n) = \tilde{H}(k)^T > 0, \quad k_\alpha T^\alpha = 0.
\]

To establish the relation with the usual definition of strong hyperbolicity, suppose first that \( n_\alpha \) is hypersurface-orthogonal, such that (at least locally) \( n_\alpha = -ND_\alpha t \), for some functions \( N \) and \( t \). Introduce local coordinates \( x^1, x^2, x^3 \) on the \( t = \text{const} \) hypersurfaces which are transported along the vector field \( T^\alpha \), such that \( T^3 D_\beta = \partial / \partial t \). Since \( h_\alpha^\beta D_\beta t = -N^{-1} h_\alpha^\beta n_\beta = 0 \), the differential operator \( \tilde{A}^\alpha D_\beta \) is tangent to the \( t = \text{const} \) hypersurfaces, and equation (A.2) reduces to a strongly hyperbolic quasi-linear partial differential equation with associated principal symbol \( \tilde{A}(n) \). In this case, local in time well-posedness of the Cauchy problems follows from standard theorems, see for instance [72].

If \( n_\alpha \) is not hypersurface-orthogonal we take a point \( p \in M \) on the manifold and approximate \( n_\alpha \) by a different co-vector field \( \tilde{n}_\alpha \) which is hypersurface-orthogonal in an open neighbourhood \( U \subset M \) of \( p \) such that \( \tilde{n}_\alpha|_p = n_\alpha|_p \). According to Proposition 1 in [67], the system (A.1) is also C-hyperbolic with respect to \( \tilde{n}_\alpha \) provided \( U \) is chosen small enough, and hence the arguments above show that the system (A.2) with \( n_\alpha \) replaced by \( \tilde{n}_\alpha \) is again strongly hyperbolic.

A relevant question is whether or not the new symmetrizer \( \tilde{H}(k) \) (with \( n_\alpha \) replaced by \( \tilde{n}_\alpha \)) is still smooth in \( k \), which is required to apply the standard theorems in the variable-coefficient or the quasi-linear cases. Unfortunately, we do not know of any general results which guarantee this property, and hence in the case in which the co-vector field \( n_\alpha \) is not hypersurface-orthogonal one can a priori only guarantee well-posedness of the frozen coefficient problems. However, there is an important special case in which local in time well-posedness for the quasi-linear problem does follow, namely when the system is symmetric hyperbolic, in which case the symmetrizers \( h(k) \) and \( H(k) \) are independent of \( k \).

## Appendix B. Proofs of lemmas 1 and 2

In this appendix we provide the proofs for the two technical lemmas used in section 5.

**Proof of lemma 1.** For convenience, we rewrite the vector \( (V_1, V_2) \) as \( (v, w) \) and denote by \( V \) and \( W \) the vector spaces of \( v \) and \( w \) live in, such that \( \dim V := n \leq m := \dim W \) and \( R(k) : V \to W, Q(k) : W \to V \). In order to prove (a), we first note that the hypothesis implies that \( Q(k) \) and \( R(k) \) have full rank, since otherwise \( M(k) \) would not be invertible and would have zero eigenvalues. Next, let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of eigenvectors of \( M(k) \) corresponding eigenvalues \( \lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2 \). Then, the \( 2n \) vectors

\[
(e_j, \pm \lambda_j^{-1} R(k) e_j), \quad j = 1, 2, \ldots, n
\]

are linearly independent eigenvectors of \( P(k) \) with nonzero eigenvalues \( \pm \lambda_j \). Furthermore, since \( Q(k) \) has full rank,

\[
\dim \ker Q(k) = m - n,
\]

which provides the remaining \( m - n \) linearly independent eigenvectors of \( P(k) \), which have zero eigenvalues.
To prove (b) we first note that the hypothesis implies that $M(k)$ is diagonalizable with strictly positive eigenvalues, so that the statement in (a) holds. It remains to show that a smooth symmetrizer $H(k)$ can be constructed for $P(k)$. We make the ansatz

$$H(k) = \begin{pmatrix} H_1(k) & 0 \\ 0 & H_2(k) \end{pmatrix}$$

with $H_2(k)$ a symmetric, positive-definite $m \times m$ matrix to be determined. The condition for $H(k)P(k)$ to be symmetric is equivalent to $H_1(k)Q(k) = R(k)^T H_2(k)$. Instead of $H(k)$ we may equip the vector spaces $V$ and $W$ with the scalar products

$$(v, v')_1 := u^T H_1(k) u', \quad v, v' \in V,$$

$$(w, w')_2 := w^T H_2(k) w', \quad w, w' \in W,$$

and the condition for $H(k)P(k)$ to be symmetric can be rewritten as

$$(v, Q(k)w)_1 = (R(k)v, w)_2, \quad v \in V, w \in W,$$  \hspace{1cm} (B.4)

that is, $R(k)$ is the adjoint of $Q(k)$. In order to define $H_2(k)$ or, equivalently, $(v, w)_2$, we denote by $W_0 := \ker Q(k)$ the kernel of $Q(k)$ (which has dimension $m - n$) and by $W_1 \subset W$ the image of $R(k)$ (which has dimension $n$). These subspaces are transversal to each other, since $w \in W_0 \cap W_1$ implies that $Q(k)w = 0$ and $w = R(k)\nu$ for some $\nu \in V$, which in turn implies that $M(k)\nu = Q(k) R(k) \nu = 0$ and hence $\nu = 0$ and $w = 0$. Therefore, we can decompose

$$W = W_0 \oplus W_1,$$

and correspondingly, each $w \in W$ can be written uniquely in the form $w = w_0 + w_1$ with $w_0 \in W_0$ and $w_1 \in W_1$. Let $R(k) : V \rightarrow W_1$ denote the restriction of $R(k)$ to its image, and $R(k)^{-1} : W_1 \rightarrow V$ its inverse. Then, we define

$$(w, w')_2 := w_0^T w_0^T + (R(k)^{-1} w_1)^T H_1(k) M(k) R(k)^{-1} w'_1$$

for $w, w' \in W$, which is clearly symmetric, positive-definite, and smooth in $k$.\footnote{The smoothness property can be established by noticing that the orthogonal projector $\pi_1(k) : W \rightarrow W$ onto $W_1$ is given by $\pi_1(k) = R(k) R(k)^T R(k)^{-1} R(k)^T$, which is smooth in $k$, such that $R(k)^{-1} w_1 = [R(k)^T R(k)]^{-1} R(k)^T w$ for all $w \in W$. Likewise, the orthogonal projector $\pi_2(k)$ onto $W_0$ is given by $\pi_2(k) = 1 - Q(k)^T [Q(k) Q(k)^T]^{-1} Q(k)$, which is smooth in $k$.} We now show that this scalar product is such that the relation (B.4) is satisfied. For this, let $v \in V$ and $w = w_0 + w_1 \in W$. Then, $R(k) v \in W_1$, and hence $R(k) R(k)^{-1} R(k) v = v$, and also $R(k) R(k)^{-1} w_1 = w_1 = w - w_0$. Therefore,

$$R(k) v, w) = v^T H_1(k) M(k) R(k)^{-1} w_1$$

$$= (v, M(k) R(k)^{-1} w_1)_1$$

$$= (v, Q(k) w - Q(k) w_0)_1 = (v, Q(k) w)_1,$$

$$\pi_1(k) = R(k) R(k)^T R(k)^{-1} R(k)^T,$$

which is smooth in $k$, such that $R(k)^{-1} w_1 = [R(k)^T R(k)]^{-1} R(k)^T w$ for all $w \in W$. Likewise, the orthogonal projector $\pi_2(k)$ onto $W_0$ is given by $\pi_2(k) = 1 - Q(k)^T [Q(k) Q(k)^T]^{-1} Q(k)$, which is smooth in $k$.}
which concludes the proof of statement (b).

**Proof of lemma 2.** If \( a \) and \( b \) are orthogonal to each other, one can introduce an orthonormal basis \( e_1, e_2, \ldots, e_n \) of \( \mathbb{R}^n \) such that \( e_1 \) and \( e_2 \) are parallel to \( a \) and \( b \), respectively. In this basis,
\[
\mathcal{M} e_1 = e_1, \quad \mathcal{M} e_2 = |a| |b| e_1,
\]
and \( \mathcal{M} e_j = e_j \) for \( j = 3, 4, \ldots, n \). Therefore, \( \mathcal{M} \) has a non-trivial Jordan block and is not diagonalizable.

From now on, suppose \( a \) and \( b \) are not orthogonal to each other. Let \( e_1 \) be a unit vector parallel to \( a \) and let \( e_2, \ldots, e_n \) be a basis of \( \ker(b^T) \). Since \( b^T a \neq 0 \), the vectors \( e_1, e_2, \ldots, e_n \) form a basis of \( \mathbb{R}^n \) with respect to which
\[
\mathcal{M} e_1 = (1 + b^T a) e_1, \quad \mathcal{M} e_j = e_j, \quad j = 2, 3, \ldots, n,
\]
which shows that \( \mathcal{M} \) is diagonalizable with eigenvalues \( 1 + a^T b \) and 1.

Finally, we note that a given vector \( \bar{v} \in \mathbb{R}^n \) satisfies \( (\bar{v}, \mathcal{H}\bar{v}) = 0 \) only if \( \bar{v} \) is proportional to \( a \) and orthogonal to \( b \) at the same time. Since \( a^T b \neq 0 \) this is only possible if \( \bar{v} = 0 \), which shows that \( \mathcal{H} = a^T b \) is positive definite. Further, a simple calculation reveals that
\[
\mathcal{H} \mathcal{M} = \kappa_0 (|a|^2 I - aa^T) + \kappa_1 (1 + a^T b) bb^T,
\]
which is clearly symmetric. □

**ORCID iDs**

Olivier Sarbach ‡ https://orcid.org/0000-0002-0753-4449

Enrico Barausse ‡ https://orcid.org/0000-0001-6499-6263

Jorge A Preciado-López ‡ https://orcid.org/0000-0002-4146-0113

**References**

[1] Colladay D and Kostelecky V A 1998 Phys. Rev. D **58** 116002

[2] Kostelecky V A 1998 Physics of Mass. Proc., 26th Int. Conf., Orbis Scientiae (Miami Beach, USA, December 12–15 1997) pp 89–94

[3] Kostelecky V A 1999 Beyond the Desert: Accelerator, Non-Accelerator and Space Approaches into the Next Millennium. Proc., 2nd Int. Conf. on Particle Physics Beyond the Standard Model (Ringberg Castle, Tegernsee, Germany, June 6–12 1999) pp 151–63

[4] Kostelecky V A 2004 Phys. Rev. D **69** 105009

[5] Kostelecky V A and Russell N 2011 Rev. Mod. Phys. **83** 11

[6] Mattingly D 2005 Living Rev. Relativ. **8** 5

[7] Jacobson T, Liberati S and Mattingly D 2006 Ann. Phys., NY **321** 150

[8] Kostelecky A V and Tasson J D 2011 Phys. Rev. D **83** 016013

[9] Will C M 2014 Living Rev. Relativ. **17** 4

[10] Nordvedt K 1987 Astrophys. J. **320** 871

[11] Damour T and Esposito-Farese G 1992 Phys. Rev. D **46** 4128

[12] Shao L, Caballero R N, Kramer M, Wex N, Champion D J and Jessner A 2013 Class. Quantum Grav. **30** 165019

[13] Shao L and Wex N 2012 Class. Quantum Grav. **29** 215018

[14] Blanchet L 2014 Living Rev. Relativ. **17** 2
[15] Abbott B P et al (Virgo, Fermi-GBM, INTEGRAL, LIGO Scientific) 2017 Astrophys. J. 848 L13
[16] Jacobson T and Mattingly D 2001 Phys. Rev. D 64 024028
[17] Blas D, Pujolas O and Sibiryakov S 2010 Phys. Rev. Lett. 104 181302
[18] Hořava P 2009 Phys. Rev. D 79 084008
[19] Jacobson T 2010 Phys. Rev. D 81 101502
[20] Foster B Z and Jacobson T 2006 Phys. Rev. D 73 064015
[21] Jacobson T 2007 Proc., Workshop on from Quantum to Emergent Gravity: Theory and Phenomenology (QG-Ph) (Trieste, Italy, June 11–15 2007) PoS QG-Ph 020
[22] Blas D and Sanctuary H 2011 Phys. Rev. D 84 064004
[23] Bonetti M and Barausse E 2015 Phys. Rev. D 91 084053
[24] Emir Gumrukcuoglu A, Saravani M and Sotiriou T P 2018 Phys. Rev. D 97 024032
[25] Foster B Z 2007 Phys. Rev. D 76 084033
[26] Yagi K, Blas D, Barausse E and Yunes N 2014 Phys. Rev. D 89 084067
[27] Yagi K, Blas D, Barausse E and Yunes N 2014 Phys. Rev. D 90 096901 (erratum)
[28] Eling C and Jacobson T 2006 Class. Quantum Grav. 23 5643
[29] Eling C and Jacobson T 2010 Class. Quantum Grav. 27 049802 (erratum)
[30] Blas D and Sibiryakov S 2011 Phys. Rev. D 84 124043
[31] Ramos O and Barausse E 2019 Phys. Rev. D 99 024034
[32] Dubovsky S L 2004 J. High Energy Phys. JHEP10(2004)076
[33] Blas D, Comelli D, Nesti F and Pilo L 2009 Phys. Rev. D 80 044025
[34] Barausse E, Sotiriou T P and Vega I 2016 Phys. Rev. D 93 064022
[35] Chadha S and Nielsen H B 1983 Nucl. Phys. B 217 125
[36] Bednía G, Pujolas O and Sibiryakov S 2013 J. High Energy Phys. JHEP11(2013)064
[37] Barausse E, Blas D, Herrero-Valea M, Sibiryakov S M and Steinwachs C F 2016 Phys. Rev. D 93 064022
[38] Knorr B 2018 Phys. Lett. B 792 142–8
[39] Groot Nibbelink S and Pospelov M 2005 Phys. Rev. Lett. 94 081601
[40] Pospelov M and Shang Y 2012 Phys. Rev. D 85 105001
[41] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[42] Sarbach O and Tiguio M 2012 Living Rev. Relativ. 15 9
[43] Fierz M 1956 Helv. Phys. Acta 29 128
[44] Jordan P 1959 Z. Phys. 157 112
[45] Brans C and Dicke R H 1961 Phys. Rev. 124 925
[46] Salgado M 2006 Class. Quantum Grav. 23 4719
[47] Barausse E, Palenzuela C, Ponce M and Lehner L 2013 Phys. Rev. D 87 081506
[48] Okounkova M, Stein L C, Scheel M A and Hemberger D A 2017 Phys. Rev. D 96 044020
[49] Cayuso J, Ortiz N and Lehner L 2017 Phys. Rev. D 96 084043
[50] Allwright G and Lehner L 2018 Class. Quantum Grav. 36 084001
[51] Jacobson T and Mattingly D 2004 Phys. Rev. D 70 024003
[52] Carroll S M and Lim E A 2004 Phys. Rev. D 70 124025
[53] Garfinkle D and Jacobson T 2011 Phys. Rev. Lett. 107 191102
[54] Eichhorn J W, Moore G D and Stoica H 2005 J. High Energy Phys. JHEP08(2005)066
[55] Jacobson T 2014 Phys. Rev. D 89 081501
[56] Barausse E, Sotiriou T P and Vega I 2016 Phys. Rev. D 93 044044
[57] Estabrook F and Wahlquist H 1964 J. Math. Phys. 5 1629
[58] Buchman L and Bardeen J 2003 Phys. Rev. D 67 084017
[59] Buchman L and Bardeen J 2005 Phys. Rev. D 72 044003 (erratum)
[60] Barausse E and Sotiriou T P 2013 Class. Quantum Grav. 30 244010
[61] Jacobson T 2011 Class. Quantum Grav. 28 245011
[62] Seifert M D 2007 Phys. Rev. D 76 064002
[63] Seifert M D and Wald R M 2007 Phys. Rev. D 75 084029
[64] Bergmann P G 1949 Phys. Rev. 75 680
[65] Calabrese G, Pullin J, Reula O, Sarbach O and Tiguio M 2003 Commun. Math. Phys. 240 377
[66] Geroch R 1996 General Relativity: Proc. ed G S Hall and J R Pulham (Bristol: IOP Publishing) p 19
[67] Reula O 2004 Differ. Equ. 01 251
[68] Kreiss H-O and Ortiz O 2002 The Conformal Structure of Space-Time (Lecture Notes in Physics vol 604) ed J Frauendiener and H Friedrich (Berlin: Springer) pp 359–70
[69] Sarbach O and Tiglio M 2002 Phys. Rev. D 66 064023
[70] Kreiss H and Lorenz J 1989 Initial-Boundary Value Problems and the Navier-Stokes Equations (San Diego, CA: Academic)
[71] Abalos F and Reula O 2018 (arXiv:1811.05558)
[72] Taylor M 1999 Partial Differential Equations III, Nonlinear Equations (New York: Springer)