SQUARE FUNCTION ESTIMATES FOR CONICAL REGIONS

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Abstract. We prove square function estimates for certain conical regions. Specifically, let \( \{ \Delta_j \} \) be regions of the unit sphere \( S^{n-1} \) and let \( S_j f \) be the smooth Fourier restriction of \( f \) to the conical region \( \{ \xi \in \mathbb{R}^n : \xi / |\xi| \in \Delta_j \} \). We are interested in the following estimate

\[
\left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|_p \lesssim \varepsilon \delta^{-\varepsilon} \|f\|_p.
\]

The first result is: when \( \{ \Delta_j \} \) is a set of disjoint \( \delta \)-balls, then the estimate holds for \( p = 4 \). The second result is: in \( \mathbb{R}^3 \), when \( \{ \Delta_j \} \) is a set of disjoint \( \delta \times \delta^{1/2} \)-rectangles contained in the band \( S^2 \cap N_\delta(\{ \xi_1^2 + \xi_2^2 = \xi_3^2 \}) \) and \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^3 : \xi / |\xi| \in S^2 \cap N_\delta(\{ \xi_1^2 + \xi_2^2 = \xi_3^2 \}) \} \), then the estimate holds for \( p = 8 \). The two estimates are sharp.

1. Introduction

The whole Littlewood-Paley theory concerns orthogonality properties of the Fourier transform, and square function gives a way to express and quantify orthogonality of the Fourier transform on \( L^p \) space. In particular, one seeks estimates of the form

\[
\left\| \left( \sum_{\sigma \in \Sigma} |S_\sigma f|^2 \right)^{1/2} \right\|_p \leq C_{n,p,\Sigma} \|f\|_p,
\]

where \( \Sigma = \{ \sigma \} \) is a collection of geometric objects in \( \mathbb{R}^n \) and \( S_\sigma f \) is the Fourier restriction of \( f \) to \( \sigma \). Two different types of the operator are frequently studied: smooth operator and sharp operator. If one studies the smooth operator, then one uses the definition \( S_\sigma f = (1_\ast \hat{f})^\vee \sigma \), where \( 1_\ast \) is a smooth bump function at \( \sigma \). Similarly, one uses the definition \( S_\sigma f = (1_\sigma \hat{f})^\vee \) where \( 1_\sigma \) is the indicator function of \( \sigma \) to study the sharp operator. Usually, the smooth version is easier than the sharp version.

Let us discuss some well-known square function estimates of the form (1.1). The classic Littlewood-Paley theory justifies (1.1) for \( 1 < p < \infty \) when \( \Sigma \) is the collection of dyadic annuli and \( \{ S_\sigma \}_{\sigma \in \Sigma} \) are sharp (or smooth) Fourier projection operators associated to the annuli; the Rubio de Francia’s square function estimate justifies (1.1) for \( 2 \leq p < \infty \) when \( \Sigma \) is a collection of disjoint rectangles whose edges are parallel to the coordinate axes and \( \{ S_\sigma \}_{\sigma \in \Sigma} \) are sharp (or smooth) projections (see [RdF85], [Jon85] and [Lac07]). If one uses the sharp operator and seeks for an square function estimate, then the shape of each \( \sigma \in \Sigma \) is quite limited. Indeed, due to Fefferman’s ball-multiplier example [Fef71], we see that (1.1) makes sense only when \( \sigma \) is “flat” in some sense. In the case of sharp operator, we replace each \( \sigma \in \Sigma \) by a rectangular box of the same size to make sure the boundaries of \( \sigma \) are flat. For this specific choice of \( \Sigma \), the estimate (1.1) is related to the maximal Nikodym or Kakeya conjecture (see for instance [C97], [Bon91]), which is one of
Theorem 1.1. A well-known result for square function in $\mathbb{R}^2$ proved by Córdoba \cite{Cor82} says that for some absolute constant $C$ the following estimate holds:

$$ (1.2) \quad \left\| \left( \sum_{j=1}^{N} |S_j f|^2 \right)^{1/2} \right\|_4 \lesssim (\log N)^C \| f \|_4. $$

Here $S_j f$ is the Fourier restriction to the conical regions determined by

$$ (1.3) \quad \Delta_j = \{ \omega \in S^1 : 2\pi j/N \leq \arg(\omega) \leq 2\pi(j+1)/N \}, $$

namely, $S_j f(\xi) = 1_{\Delta_j}(\xi/|\xi|) \hat{f}(\xi)$.

Our first result is a generalization of Córdoba’s estimate \cite{Cor82} to higher dimensions. In Córdoba’s square function estimate, the regions are chosen as $\Sigma = \{ \sigma_j \}$ where $\sigma_j = \{ \xi : \xi/|\xi| \in \Delta_j \}$ is a sector of angle $N^{-1}$. A natural way to generalize it to higher dimensions is by choosing $\{ \Delta_j \}$ to be a set of disjoint $\delta$-balls in $\mathbb{S}^{n-1}$. We first discuss the smooth version. Let $\{ \Delta_j \}$ be a set of disjoint $\delta$-balls and let $\{ 1_{\Delta_j}^* \}$ be the corresponding smooth bump functions. More precisely, $1_{\Delta_j}^*$ is supported in $\Delta_j$ and $1$ on $\frac{1}{2} \Delta_j \cup \frac{1}{2} \Delta_j$ is the ball that has the same center as $\Delta_j$ but half the radius, and it satisfies the decay condition $| \nabla^k 1_{\Delta_j}^*(\omega) | \lesssim_k \delta^{-k}$.

Theorem 1.1. Let $\{ \Delta_j \}$ be a set of disjoint $\delta$-balls in $\mathbb{S}^{n-1}$ and $\{ 1_{\Delta_j}^* \}$ be the corresponding smooth bump functions. Let $T_j f = (1_{\Delta_j}^*(\cdot/|\cdot|) \hat{f}((\cdot))')^\vee$ be the smooth Fourier projection of $f$ to the conical region $\{ \xi \in \mathbb{R}^n : \xi/|\xi| \in \Delta_j \}$. Then for some constant $C$ depending only on the dimension, we have

$$ (1.4) \quad \left\| \left( \sum_{j} |T_j f|^2 \right)^{1/2} \right\|_4 \lesssim (\log \delta^{-1})^{1/2} \| f \|_4. $$

Remark 1.2. The exponent $p = 4$ is sharp in the sense that for $p > 4$, the factor $(\log \delta^{-1})^{1/2}$ in (1.4) should be replaced by some factor like $\delta^{-p}$, which depends exponentially on $\delta$. Also, a logarithmic loss like $(\log \delta^{-1})^{1/4}$ is inevitable due to the existence of Besicovitch set (see for instance \cite{ADPH20} Section 8.6).

Next, let us discuss the sharp-projection version of Theorem 1.1. We still hope each $\Delta_j$ is roughly a $\delta$-ball in $\mathbb{S}^{n-1}$. While due to Fefferman’s ball-multiplier example, the boundaries of $\sigma_j = \{ \xi \in \mathbb{R}^n : \xi/|\xi| \in \Delta_j \}$ must be flat, otherwise estimates like \cite{LM83} for sharp projection could fail. These two conditions suggest that each $\Delta_j$ is a “$\delta$-regular polyhedron” in $\mathbb{S}^{n-1}$, for which we give the precise definition below.

Definition 1.3. Let $\Delta$ be a subset of $\mathbb{S}^{n-1}$ and $0 < \delta \ll 1$. We say that $\Delta$ is a $\delta$-regular polyhedron of $\mathbb{S}^{n-1}$ if $\Delta$ is surrounded by great $(n-2)$-spheres of $\mathbb{S}^{n-1}$ and there exists $\omega_\Delta \in \mathbb{S}^{n-1}$ such that $\mathbb{S}^{n-1} \setminus B_{c\delta}(\omega_\Delta) \subset \Delta \subset \mathbb{S}^{n-1} \setminus B_{c\delta}(\omega_\Delta)$, where $c \ll C$ are two absolute constants. We call the portion of the great $(n-2)$-sphere that form the boundary of $\Delta$ the face of $\Delta$. Throughout the paper, we assume the polyhedron $\Delta$ has $O(1)$ many faces.

We are interested in the set of disjoint $\delta$-regular polyhedrons. Let us discuss some examples here. The collection $\{ \Delta_j \}$ given by \cite{LM83} is a set of disjoint $N^{-1}$-regular polyhedrons in $\mathbb{S}^1$. In higher dimensions, we can also easily choose a triangulation
of the sphere: $S^{n-1} = \bigcup_j \Delta_j$ such that each $\Delta_j$ is a $\delta$-regular polyhedron ($(n-1)$-simplex) of $S^{n-1}$.

However, it turns out that we still need another condition for the collection $\{\Delta_j\}_j$. In fact, how the normal directions of the faces of each $\Delta_j$ are distributed is critical. Recall that each face of $\Delta_j$ is a portion of a great $(n-2)$-sphere, and the normal direction of the face is the normal direction of the corresponding great $(n-2)$-sphere in $S^{n-1}$. Denote by $\mathcal{N}$ the collection of all the unit normal directions of all the faces of all the $\Delta_j$. One can think of $\mathcal{N}$ as a subset of $S^{n-1}$.

**Definition 1.4.** We call a collection of $\delta$-regular polyhedrons $\{\Delta_j\}$ “one-dimensional”, if $\mathcal{N}$ is contained in $O(1)$ many great circles in $S^{n-1}$.

For instance, the $\{\Delta_j\}$ given in (1.3) is one-dimensional. Another example is the “pyramid” example. Consider the set $[-1, 1]^{n-1} \times \{-1\}$ in $\mathbb{R}^n$. We partition it into the sets of form $D_\delta = \prod_{j=1}^{n-1} \left[ \frac{b_j - 1}{2}, \frac{b_j + 1}{2} \right] \times \{-1\}$, where $b_j = (b_1, \ldots, b_{n-1})$ and $-N \leq b_j \leq N - 1$ are integers. For each $\vec{b} = (b_1, \cdots, b_{n-1})$, we define the small pyramid $\sigma_\delta$ which consists of the points $\xi$ such that the ray $0\xi$ emanating from the origin intersects $D_\delta$. More precisely,

$$\sigma_\delta := \{ \xi \in \mathbb{R}^n : \frac{\xi_n}{|\xi|} < 0, \frac{1}{|\xi_n|} \xi \in D_\delta \}.$$  

We set $\Delta_\delta = S^{n-1} \cap \sigma_\delta$. Then, $\{\Delta_\delta\}_\delta$ is one dimensional.

We can state our result for the sharp operator.

**Theorem 1.5.** Suppose $\{\Delta_j\}_j$ is a set of disjoint $\delta$-regular polyhedrons and is one-dimensional. Define $S_j f = (1_{\Delta_j} (\cdot / |\cdot|) \hat{f} (\cdot))^{\vee}$ which is the Fourier restriction of $f$ to $\{ \xi \in \mathbb{R}^n : \frac{\xi}{|\xi|} \in \Delta_j \}$. Then for any $\varepsilon > 0$ we have

$$\left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|_4 \lesssim \delta^{-\varepsilon} \| f \|_4. \quad (1.5)$$

The one-dimensional assumption on $\{\Delta_j\}_j$ is somewhat necessary. In fact, we will show in the Appendix that without the “one-dimensional” condition, then (1.5) can fail.

**Remark 1.6.** One possible application of Theorem 1.1 as well as Theorem 1.5 is the study of the local smoothing conjecture. Actually, the reverse square function estimate for cone plus the Nikodym maximal estimate for cone together with Theorem 1.1 would imply the local smoothing conjecture for cone. See for instance [TV00].

**Remark 1.7.** After this project was finished, the authors became aware that Francesco Di Plinio and Ioannis Parissis have studied the same problem as in Theorem 1.1. Their result (DPP21) Theorem J will imply our inequality (1.4) (with the same constant $(\log \delta^{-1})^{1/2}$) by a standard interpolation argument (see for example in [Dem10] Lemma 3.1 or [Kat99] Proposition 2.1). Their method is based on the time-frequency analysis, while our method is based on the high-low method developed recently (see [GWZ20], [GSW19], [GMW20]). We also remark that when $n = 2$, both the smooth operator version and the sharp operator version were studied by Accomazzo, Di Plinio, Hagelstein, Parissis and Roncal [ADPH+20].
1.2. Second result. Let us talk about the second result. We consider the cone
\( \Gamma = \{ \xi \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 = \xi_3^2, 1/2 \leq |\xi_3| \leq 1 \} \), and let \( N_{\delta}(\Gamma) \) denote its \( \delta \)-neighborhood. There is a canonical covering of \( N_{\delta}(\Gamma) \) using finitely overlapping planks \( \tau \) of dimensions \( \sim \delta \times \delta^{1/2} \times \delta \). Denote this collection by \( \mathcal{T} = \{ \tau \} \). For each \( \tau \in \mathcal{T} \), choose a smooth bump function \( 1^\ast_{\tau} \) adapted to \( \tau \) so that \( \text{supp}1^\ast_{\tau} \subset 2\tau \) and \( 1^\ast_{\tau} = 1 \) on \( \tau \). Define \( f_\tau := (1^\ast_{\tau} \hat{f})^\vee \) as usual. Guth, Wang and Zhang [GWZ20] proved the following sharp \( L^4 \) reverse square function estimate.

**Theorem 1.8** (Reverse square function estimate, [GWZ20]). Assuming \( \text{supp} \hat{f} \subset N_{\delta}(\Gamma) \), then for any \( \epsilon > 0 \) we have

\[
\|f\|_{L^4(\mathbb{R}^3)} \leq C_\delta \delta^{-\epsilon} \left\| \left( \sum_\tau |f_\tau|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)}.
\]

Or equivalently,

\[
\| \sum_\tau f_\tau \|_{L^4(\mathbb{R}^3)} \leq C_\delta \delta^{-\epsilon} \left\| \left( \sum_\tau |f_\tau|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)}.
\]

In this paper, we prove the sharp \( L^8 \) square function estimate for the cone.

**Theorem 1.9.** Assuming \( \text{supp} \hat{f} \subset N_{\delta}(\Gamma) \), then for any \( \epsilon > 0 \) we have

\[
\left\| \left( \sum_\tau |f_\tau|^2 \right)^{1/2} \right\|_{L^8(\mathbb{R}^3)} \leq C_\delta \delta^{-\epsilon} \|f\|_{L^8(\mathbb{R}^3)}.
\]

**Remark 1.10.** The endpoint \( p = 8 \) is sharp in the sense that if we consider the estimate

\[
\left\| \left( \sum_\tau |f_\tau|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)} \leq C(p, \delta) \|f\|_{L^p(\mathbb{R}^3)}
\]

for \( p > 8 \), then the best constant \( C(p, \delta) \) should be a positive power of \( \delta^{-1} \). We will give a sharp example in the Appendix.

If we remove the condition \( \text{supp} \hat{f} \subset N_{\delta}(\Gamma) \), then the best we can hope is an \( L^4 \)-estimate. This result was implicitly proved by Mockenhaupt, Seeger and Sogge [MSS92]. Actually, by a duality argument, one can reduce the \( L^4 \) square function estimate to a maximal Nikodym estimate which is Lemma 1.4 in [MSS92].

From Theorem 1.1 in this paper, we easily see (1.8) holds for \( p = 4 \). Moreover, if we use trilinear restriction estimate for the cone, then we can prove (1.8) for \( p = 6 \). In order to prove for \( p = 8 \), we need to do more work.

By Littlewood-Paley theory, we can prove a global version of Theorem 1.9. Let us consider \( \mathbb{B}^2 := \mathbb{S}^2 \cap N_{\delta}(\Gamma) \) which is a band of width \( \delta \) in \( \mathbb{S}^2 \). Let \( \{ \Delta_j \} \) be a set of disjoint \( \delta \times \delta^{1/2} \times \delta \)-rectangles that are contained in \( \mathbb{B}^2 \). Also, we choose \( \{ 1^\ast_{\Delta_j} \} \) to be the corresponding smooth bump functions. We see that \( 1^\ast_{\Delta_j}(\xi/|\xi|) \) is a smooth cutoff function in the region \( \{ \xi \in \mathbb{R}^3 : \xi/|\xi| \in \Delta_j \} \). Define \( T_j f := (1^\ast_{\Delta_j}(\xi/|\xi|) \hat{f}(\xi))^\vee \). Our global version is the following.

**Theorem 1.11.** Assuming \( \hat{f} \subset \{ \xi \in \mathbb{R}^3 : \xi/|\xi| \in \mathbb{B}^2 \} \), then for any \( \epsilon > 0 \) we have

\[
\left\| \left( \sum_j |T_j f|^2 \right)^{1/2} \right\|_{L^8(\mathbb{R}^3)} \leq C_\delta \delta^{-\epsilon} \|f\|_{L^8(\mathbb{R}^3)}.
\]
In the end of this section, let us talk about the structure of the paper. In Section 2, we do a global-to-local reduction in the frequency space to reduce the problem to the case that \( \text{supp} \hat{f} \subset \{ |\xi| \sim 1 \} \). In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.3. In Section 5, we prove Theorem 1.9. In the Appendix, we discuss some examples.

Acknowledgement. The authors would like to thank Francesco Di Plinio and Ioannis Parissis for some useful discussions, and for bringing their papers to our attention.

2. THE GLOBAL-TO-LOCAL REDUCTION

The main goal of this section is to reduce Theorem 1.1 and Theorem 1.3 to a local version. That is to say, we only need to prove the case when \( \text{supp} \hat{f} \subset \{ |\xi| \sim 1 \} \).

We may assume \( \Delta_j \) are within the conical region \( \omega \in \mathbb{S}^{n-1} : \omega \cdot e_n \geq 1 \) and \( \{ \Delta_j \} \) are \( 100C \delta \)-separated. Let \( P_k \) be the Littlewood-Paley operator for the dyadic annulus, so that for any function \( f \), \( \hat{P}_k f \) is supported in the annulus \( \{ |\xi| \sim 2^k \} \) and \( f = \sum_{k \in \mathbb{Z}} P_k f \). More precisely, we choose a function \( \rho(r) \) supported in \( [1/4, 4] \) such that \( \sum_{k \in \mathbb{Z}} \rho(r/2^k) = 1 \), and set \( P_k f := (\rho(|\xi|/2^k) \hat{f}(\xi))^\vee \). Choose \( m = 10^n \) which is a large number. For each integer \( 0 \leq i \leq m-1 \), define \( K_i := m \mathbb{Z} + i \), so that there is a partition \( \mathbb{Z} = \bigcup_{0 \leq i \leq m-1} K_i \). We have

\[
\begin{align*}
\| \left( \sum_j \left| T_j f \right|^2 \right)^{1/2} \|_4^4 &= \left\| \left( \sum_j \sum_{k \in \mathbb{Z}} T_j P_k f \right)^{1/2} \right\|_4^4 \\
&\lesssim \sum_{0 \leq i \leq m-1} \left\| \left( \sum_j \sum_{k \in K_i} T_j P_k f \right)^{1/2} \right\|_4^4,
\end{align*}
\]

For simplicity, we just write \( K_i \) as \( K \). The only property of \( K \) we will use is that for any two different numbers \( k_1, k_2 \in K \), we have \( |k_1 - k_2| \geq 10^n \). The goal of Theorem 1.1 is to prove

\[
\left\| \left( \sum_j \left| \sum_{k \in K} T_j P_k f \right|^2 \right)^{1/2} \right\|_4^4 \lesssim (\log \delta^{-1})^2 \| f \|_4^4.
\]

For convenience, let us denote

\[
I = \left\| \left( \sum_j \left| \sum_{k \in K} T_j P_k f \right|^2 \right)^{1/2} \right\|_4^4, \quad II = \left\| \left( \sum_j \sum_{k \in K} T_j P_k f \right)^{1/2} \right\|_4^4.
\]

We begin with the first estimate.

Lemma 2.1. Let \( I \) and \( II \) be as above. We have

\[ I \leq 1000 II. \]

Proof. After expanding the \( L^4 \)-norm, we get

\[
\begin{align*}
I &= \sum_{j_1 \neq j_2} \sum_{k_1, k_2, k_3, k_4 \in K} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f) (T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) \\
&+ \sum_{j_1 = j_2} \sum_{k_1, k_2, k_3, k_4 \in K} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f) (T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f).
\end{align*}
\]
Denote $\vec{k} = (k_1, k_2, k_3, k_4)$. In the following discussion, we would like to find a partition of the set $\mathcal{K}^4$ and hence a partition of the summation $\sum_{\vec{k} \in \mathcal{K}^4}$. We will discuss the two terms (2.5), (2.6) separately.

Case 1: $j_1 \neq j_2$.

When $j_1 \neq j_2$, for the quadruples $\vec{k} = (k_1, k_2, k_3, k_4) \in \mathcal{K}^4$, we consider the following subsets of $\mathcal{K}^4$:

1. $A = \{k_1 = k_3\}$.
2. $B = \{k_2 = k_4\}$.
3. $C = \{k_1 = k_3, k_2 = k_4\}$.
4. $D = \{k_1 \neq k_3, k_2 \neq k_4\}$.

Then we see that $1_{\mathcal{K}^4} = 1_A + 1_B + 1_D - 1_C$.

For $\vec{k} \in D$, we have by Plancherel that

$$\int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f)$$

$$= \int T_{j_1} P_{k_1} f \ast T_{j_2} P_{k_2} f \ast T_{j_1} P_{k_3} f \ast T_{j_2} P_{k_4} f.$$

We claim this integral is 0. First, by definition we have

$$\text{supp} T_{j} P_{k} f \subset \{\xi : \xi/|\xi| \in \Delta_j, |\xi| \sim 2^k\}.$$  

Define

$$\tau_{k,j} := \{\xi : \xi/|\xi| \in \Delta_j, |\xi| \sim 2^k\},$$

then we see that $\tau_{k,j}$ is morally a tube of length $2^k$ and radius $\delta 2^k$, pointing to the direction $c_{\Delta_j} \in S^{n-1}$ ($c_{\Delta_j}$ is the center of $\Delta_j$). We note that the support of the integrand of (2.8) is contained in $(\tau_{j_1, k_1} + \tau_{j_2, k_2}) \cap (\tau_{j_1, k_3} + \tau_{j_2, k_4})$. To show (2.8) $= 0$, it suffices to show $(\tau_{j_1, k_1} + \tau_{j_2, k_2}) \cap (\tau_{j_1, k_3} + \tau_{j_2, k_4}) = \emptyset$, or equivalently

$$(\tau_{j_1, k_1} - \tau_{j_1, k_3}) \cap (\tau_{j_2, k_4} - \tau_{j_2, k_2}) = \emptyset.$$ 

Since $\vec{k} \in D$, we have $k_1 \neq k_3$, so $|k_1 - k_3| \geq 10^n \gg 1$. We may assume $k_1 > k_3$. By an easy geometry, we see that $\tau_{j_1, k_1} - \tau_{j_1, k_3} \subset (1 + 1/10)\tau_{j_1, k_1}$. Here $(1 + 1/10)\tau_{j_1, k_1}$ is the dilation of $\tau_{j_1, k_1}$ with respect to its center (regarding $\tau_{j_1, k_1}$ as a tube). Similarly, we have $\tau_{j_2, k_2} - \tau_{j_2, k_4} \subset (1 + 1/10)\tau_{j_2, \max(k_2,k_4)}$. Note that $(1 + 1/10)\tau_{j_1, k_1}$ is contained in the conical region $\{\xi : \xi/|\xi| \in N_{10C\delta} \Delta_{j_1}\}$, while $(1 + 1/10)\tau_{j_2, \max(k_2,k_4)}$ is contained in the conical region $\{\xi : \xi/|\xi| \in N_{10C\delta} \Delta_{j_2}\}$. Since $j_1 \neq j_2$, we have $\text{dist}(\Delta_{j_1}, \Delta_{j_2}) \geq 100C\delta$, so we prove the disjointness.

As for those quadruples $\vec{k}$ in $A$ or $B$, we have

$$\sum_{j_1 \neq j_2} \left( \sum_{k_1 \in A} + \sum_{k_1 \in B} \right) \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f)$$

$$\leq \int \left( \sum_j \sum_{k \in K} |T_j P_k f|^2 \right) \cdot \left( \sum_j \sum_{k \in K} |T_j P_k f|^2 \right),$$

which, by using Hölder’s inequality, is bounded by

$$[(1/100)I + 100II]/2.$$
Note that for those quadruples \( \vec{k} \in C \),

\[
\sum_{j_1 \neq j_2, k \in C} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) \leq II.
\]

Thus we obtain

(2.14) \( (2.5) \leq (1/100)I + 100II \).

Case 2: \( j_1 = j_2 \).

When \( j_1 = j_2 \), we similarly consider the subsets of \( \mathcal{K}^4 \):

1. \( E_1 = \{k_1 = k_2\}, E_2 = \{k_1 = k_3\}, E_3 = \{k_1 = k_4\}, E_4 = \{k_2 = k_3\}, E_5 = \{k_2 = k_4\}, E_6 = \{k_3 = k_4\} \).
2. \( F'_1 = \{k_1 = k_2, k_3 = k_4\}, F'_2 = \{k_1 = k_3, k_2 = k_4\}, F'_3 = \{k_1 = k_4, k_2 = k_3\} \).
3. \( F_1 = \{k_1 = k_2 = k_3\}, F_2 = \{k_1 = k_2 = k_4\}, F_3 = \{k_1 = k_3 = k_4\}, F_4 = \{k_2 = k_3 = k_4\} \).
4. \( G = \{k_1 = k_2 = k_3 = k_4\} \).
5. \( H = \{k_1, k_2, k_3, k_4\} \) are all different.

Then by inclusion-exclusion principle, one can express \( \mathcal{K}^4 \) as a linear combination of the aforementioned collection of quadruples:

(2.15) \( 1_{\mathcal{K}^4} = \sum_{i=1}^{6} 1_{E_i} - \sum_{i=1}^{3} 1_{F'_i} - 2 \sum_{i=1}^{4} 1_{F_i} + 6 \cdot 1_{G} + 1_{H} \).

Note that when \( j_1 = j_2 \) and \( \vec{k} \in H \), (2.6) is zero.

As for those quadruples in \( E_i \), similar to (2.11) we get

(2.16) \( \sum_{j_1=j_2} \sum_{k \in E_i} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) \leq \left( \sum_{j} \sum_{k \in \mathcal{K}} |T_j P_k f|^2 \right) \left( \sum_{j} \sum_{\mathcal{K}} |T_j P_k f|^2 \right)^2 \leq (1/100)I + 100II) \).

For those \( \vec{k} \in F'_i \) or \( F'_i \), we have:

(2.17) \( \sum_{j_1=j_2} \sum_{k \in F'_i} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) = \int \sum_{j} \sum_{\mathcal{K}} |T_j P_k f|^2 \leq II \).

For those \( \vec{k} \in F_i \), we have:

(2.18) \( \sum_{j_1=j_2} \sum_{k \in F_i} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) = \int \sum_{j} \sum_{\mathcal{K}} (T_j P_k f)^2 \leq II \).

For those quadruples in \( F_i \), for example in \( F_1 \), we get

(2.19) \( \sum_{j_1=j_2} \sum_{k \in F_1} \int (T_{j_1} P_{k_1} f \cdot T_{j_2} P_{k_2} f)(T_{j_1} P_{k_3} f \cdot T_{j_2} P_{k_4} f) = \int \sum_{k} T_j P_k f^2 \cdot T_j P_k f \cdot \sum_{k} T_j P_k f, \)
whose absolute value, by Cauchy-Schwartz inequality, is bounded above by

\[
\sum_j \left( \sum_{k_1} |T_j P_{k_1} f|^4 \right)^{1/2} \left( \sum_{k_4} |T_j P_{k_4} f|^2 \right)^{1/2}
\]

\[
\leq \int \sum_j \sum_{k_1} |T_j P_{k_1} f|^4 + \int \sum_j \sum_{k_1} |T_j P_{k_1} f|^2 \left| \sum_{k_4} T_j P_{k_4} f \right|^2
\]

\[
\leq II + [(1/100)I + 100II]/2.
\]

Finally, for those quadruples in $G$, we can easily get

\[
\sum_j \sum_{\vec{k} \in G} \int (T_j P_{k_1} f \cdot T_j P_{k_2} f)(T_j P_{k_3} f \cdot T_j P_{k_4} f)
\]

\[
= \int \sum_j \sum_k |T_j P_k f|^4 \leq II.
\]

Putting all estimates above together, we reach

\[
\leq 3[(1/100)I + 100II] + 8II + 4[(1/100)I + 100II] + II
\]

\[
\leq (7/100)I + 800II.
\]

Now plug \n(2.14)\n and \n(2.25)\n\n so that

\[
(2.1) = I \leq (8/100)I + 900II,
\]

which gives \n(2.1) = I \leq 1000II. \]

Hence it remains to show

\[
\sqrt{7I} = \left\| \left( \sum_j \sum_{k \in \mathbb{K}} |T_j P_k f|^2 \right)^{1/2} \right\|_4 \lesssim (\log \delta^{-1})^{1/2} ||f||_4.
\]

The desired estimate \n(2.28)\n can be further reduced to the following local estimate.

\[
\left\| \left( \sum_j |T_j P_k f|^2 \right)^{1/2} \right\|_4 \lesssim (\log \delta^{-1})^{1/4} ||f||_4.
\]

It is a local version of \n(1.4).\n We also remark that in the local version, the constant \n(\log \delta^{-1})^{1/4}\n is better than the global version. To deduce \n(2.28)\n from \n(2.29),\n we need the following lemma. It is from \nGRAY20 Proposition 4.2\n (see also \nJSW08 and \nSee88).

**Lemma 2.2.** Let \n\{m_j(\xi)\}_{j \in J}\n be a set of Fourier multipliers on \n\mathbb{R}^n\n, each of which is compactly supported on \n\{\xi : 1/2 \leq |\xi| \leq 2\}\n, and satisfies

\[
\sup_{j \in J} |\partial^\alpha m_j(\xi)| \leq B \text{ for each } 0 \leq |\alpha| \leq n + 1
\]

for some constant \nB. For \n j \in J\n and \n k \in \mathbb{Z},\n write \nT_{j,k}\n the multiplier operator with multiplier \n\text{m}_j(2^{-k}\xi).\n Fix some \np \in [2, \infty).\n Assume that there exists some constant \nA\n such that

\[
\sup_{k \in \mathbb{Z}} \left\| \left( \sum_{j \in J} |T_{j,k} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq A ||f||_{L^p(\mathbb{R}^n)}
\]
for both \( s = p \) and \( s = 2 \). Then

\[
(2.32) \quad \left\| \sum_{k \in \mathbb{Z}} \sum_{j \in J} |T_{j,k}f|^2 \right\|_{L^p(\mathbb{R}^n)}^{1/2} \lesssim A \left| \log \left( 2 + \frac{B}{A} \right) \right|^{1/p} \|f\|_{L^p(\mathbb{R}^n)}.
\]

Let us discuss how to apply Lemma 2.2. First note that

\[
(2.29), \text{ which is the following result}
\]

\[
(2.32)
\]

\[
(2.33)
\]

If we assume (2.29) is true, by rescaling we have for any \( k \in \mathbb{Z} \)

\[
(2.33) \quad \left\| \left( \sum_{j} |T_{j,k}f|^2 \right)^{1/2} \right\|_4 \lesssim (\log \delta^{-1})^{1/4} \|f\|_4.
\]

We choose \( m_j(\xi) = 1_{\Delta_j}(\xi/|\xi|)\rho(\xi) \) in Lemma 2.2, so we have \( T_{j,k}f = T_{j}P_{k}f \). We also choose \( p = 4 \). We can verify the constant \( B = \delta^{-O(1)} \) and \( A = (\log \delta^{-1})^{1/4} \)
will make the conditions (2.30) and (2.31) in the lemma hold. As a result, from (2.32) we obtain

\[
\left\| \left( \sum_{j} \sum_{k \in \mathcal{X}} |T_{j,k}f|^2 \right)^{1/2} \right\|_4 \lesssim A \left| \log \left( 2 + \frac{B}{A} \right) \right|^{1/4} \|f\|_4 \lesssim (\log \delta^{-1})^{1/2} \|f\|_4.
\]

This gives the estimate (2.28).

The proof of (2.29) is given in the next section.

3. PROOF OF THE LOCAL VERSION

Recall we are given a set of disjoint \( \delta \)-balls \( \{ \Delta_j \} \) in \( S^{n-1} \). Also recall the smooth Fourier restriction operator is defined as

\[
T_j f(x) = (1_{\Delta_j}(\xi/|\xi|)\hat{f}(\xi))^\vee(x),
\]

where \( 1_{\Delta_j} \) is a smooth bump function adapted to \( \Delta_j \).

After the global-to-local reduction in the previous section, (1.4) boils down to 2.29, which is the following result

**Theorem 3.1.** For any function \( f \) with \( \text{supp} \hat{f} \subset \{ \xi : 100 \leq |\xi| \leq 101 \} \), we have

\[
(3.1) \quad \left\| \left( \sum_{j} |T_{j}f|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^n)} \lesssim (\log \delta^{-1})^{1/4} \|f\|_{L^4(\mathbb{R}^n)},
\]

where \( C > 0 \) is some universal constant.

Let us first discuss some geometry. For each \( \Delta_j \), we consider the corresponding tube \( \tau_j \) defined as follows

\[
\tau_j := \{ \xi \in \mathbb{R}^n : 100 \leq |\xi| \leq 101, \xi/|\xi| \in \Delta_j \}.
\]

Since \( \text{supp} \hat{f} \subset \{ \xi : 100 \leq |\xi| \leq 101 \} \), we have

\[
T_j f = \left( 1_{\Delta_j}(\xi/|\xi|)\rho(|\xi|)\hat{f}(\xi) \right) ^\vee,
\]

where \( \rho(r) \) is a smooth function supported in \( r \in [99, 102] \) and \( = 1 \) for \( r \in [100, 101] \).

Now we define

\[
\psi_{\tau_j}(\xi) := 1_{\Delta_j}(\xi/|\xi|)\rho(|\xi|),
\]
overlapping covering of tubes. Denote the set of these tubes by \( \Theta \). By definition we know that the tubes are finitely overlapping. Now, let us forget about \( T_J\) and use the new notation \( f_\tau := (\psi_\tau \hat f)^\vee \) for \( \tau \in T \). (3.1) is equivalent to

\[
(3.2) \quad \left< \left\| \left( \sum_{\tau \in T} |f_\tau|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^n)} \right> \lesssim (\log \delta)^{-1/4} \|f\|_{L^4(\mathbb{R}^n)}
\]

For each \( \tau \in T \), the Fourier transform of \( |f_\tau|^2 \) has support in \( \tau - \tau \subset 5\tau_0 \) (here \( \tau_0 \) is the translation of \( \tau \) to the origin). Hence the Fourier transform of \( \sum_{\tau \in T} |f_\tau|^2 \) is supported in \( \bigcup_{\tau \in T} 5\tau_0 \subset B_{10}(0) \). Next, we will partition the frequency ball \( B_{10}(0) \) into tubes and analyze the contribution of \( \sum_{\tau \in T} |f_\tau|^2 \) on each of the partitions.

For any dyadic number \( s \) with \( \delta \leq s \leq 10 \), consider a partition of the annulus \( A_s := \{ \xi \in \mathbb{R}^n : \frac{\delta}{2} \leq |\xi| \leq s \} \) into tubes of dimensions \( \delta \times \cdots \times \delta \times s \) whose centers pass through the origin. More precisely, we choose a set of maximal \( \delta s^{-1} \)-separated points on \( S^{n-1} \), denoted by \( \{\omega_s\} \). For each \( \omega_s \), define

\[
\theta_s = \{ \xi : \frac{s}{2} \leq |\xi| \leq s, \xi/|\xi| \in S^{n-1} \cap B_{\delta s^{-1}}(\omega_s) \}.
\]

Denote the set of these tubes by \( \Theta_s = \{ \theta_s \} \). One can see that \( \Theta_s \) forms a finitely overlapping covering of \( A_s \). Particularly, when \( s = \delta \), we just define \( \Theta_s \) to consist of a single element \( \theta_\delta = \{ \xi \in \mathbb{R}^n : |\xi| \leq \delta \} \) which is a ball of radius \( \delta \) centered at the origin.

Next, we will use \( \theta_s \in \Theta_s \) to give a partition of \( T \). For each \( \theta_s \in \Theta_s \), define

\[
T_{\theta_s} := \{ \tau \in T : \theta_s \cap 10\tau_0 \neq \emptyset \}.
\]

By some elementary geometries, we can see that \( \{T_{\theta_s}\}_{\theta_s \in \Theta_s} \) form a finitely overlapping cover of \( T \).

For each \( \theta_s \), we define \( \theta_s^* \) to be the dual slab of \( \theta_s \). More precisely, \( \theta_s^* \) is a slab centered at the origin with dimensions \( \delta^{-1} \times \cdots \times \delta^{-1} \times s^{-1} \) and with the normal direction the same as the direction of \( \theta_s \).

Now let us start the proof.

**Proof.** Denote the square function by \( g = \sum_{\tau \in T} |f_\tau|^2 \). If \( \xi \in A_s \), then

\[
|\hat g(\xi)|^2 \lesssim \sum_{\theta_s \in \Theta_s} \left< \sum_{\tau \in T_{\theta_s}} (|f_\tau|^2)^\vee(\xi)|^2 \right>.
\]

For each \( \theta_s \), we choose \( \eta_{\theta_s} \) to be a smooth cutoff function at \( \theta_s \) so that \( \eta_{\theta_s} \gtrsim 1 \) in \( \theta_s \) and \( |\eta_{\theta_s}^\vee(x)| \lesssim \frac{1}{|\theta_s|} |1_{\theta_s}(x)| \). Then, we have

\[
|\hat g(\xi)|^2 \lesssim \sum_{\theta_s \in \Theta_s} |\eta_{\theta_s}(\xi)| \sum_{\tau \in T_{\theta_s}} (|f_\tau|^2)^\vee(\xi)|^2,
\]

for \( \xi \in A_s \).

By Plancherel, we get

\[
\int |y|^2 \lesssim \int \sum_{\delta \leq s \leq 10} \sum_{\theta_s \in \Theta_s} |\eta_{\theta_s}^\vee * \sum_{\tau \in T_{\theta_s}} |f_\tau|^2|^2.
\]
Note that $|\eta_\tau(x)| \lesssim \frac{1}{|\partial_\tau|} 1_{\theta}(x)$. This suggests us to tile $\mathbb{R}^n$ by translations of $\theta^*$. We denote this cover by $\{U : U \parallel \theta^*_s\}$. For $x \in U$, we have

$$\sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2 \lesssim |U|^{-1} \int_{2U} \sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2,$$

where $\eta_U(x) = \max_{y \in x + \theta^*_s} |\eta_\tau(x - y)|$ is a smooth bump function supported in $2U$.

Therefore, one has

$$\int |g|^2 \lesssim \sum_{\delta \leq s \leq 10} \sum_{\theta_s \in \Theta_s} \sum_{U \parallel \theta^*_s} |U|^{-1} (\int_{2U} \sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2)^2.$$

By dyadic pigeonholing, there exists an $s$ such that

$$\int |g|^2 \lesssim \log \delta^{-1} \sum_{\theta_s \in \Theta_s} \sum_{U \parallel \theta^*_s} |U|^{-1} (\int_{2U} \sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2)^2.$$

We remark that this is the only place we lose a logarithmic factor $\log \delta^{-1}$.

Now we carefully analyze the integral $\int_{2U} \sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2$. For simplicity, we just write $\theta$ for $\theta_s$ (recall $\theta$ is a tube of dimensions $\delta \times \cdots \times \delta$). For each $\theta \in \Theta$ of form $\theta = \{\xi : \frac{s}{2} \leq |\xi| \leq s, \xi/|\xi| \in S^{n-1} \cap B_{|\delta|-1}(\omega)\}$, we define another tube

$$\vartheta := \{\xi : 100 \leq |\xi| \leq 101, \xi/|\xi| \in S^{n-1} \cap B_{100|\delta|-1}(\omega)\}.$$

This roughly says the radial projection of $\theta$ on $S^{n-1}$ is contained in that of $\vartheta$. From a simple geometric argument, we see that each $\tau \in T_{\theta}$ satisfies $\tau \subset \vartheta$. Let $\Theta = \{\vartheta\}$ be the collection of these $\vartheta$. For each $\vartheta$, we choose a smooth bump function $\psi_\vartheta$ at $\vartheta$ and then define $f_\vartheta := (\psi_\vartheta \tilde{f})^\vee$.

By local $L^2$-orthogonality (see Lemma 5.3), we have

$$\int_{2U} \sum_{\tau \in T_{\theta_s}} |f_{\tau}|^2 \lesssim \int \chi_U |f_\vartheta|^2,$$

where $\chi_U$ is morally a cutoff function at $U$ and decay rapidly outside $U$. Therefore, by Hölder’s inequality, we have

$$\int |g|^2 \lesssim \log \delta^{-1} \sum_{\vartheta \in \Theta} \sum_{U \parallel \theta^*_s} \int (\chi_U)^2 |f_\vartheta|^4 \lesssim \log \delta^{-1} \sum_{\theta_s \in \Theta_s} \int |f_\vartheta|^4.$$

Here we use

$$\sum_{U \parallel \theta^*_s} (\chi_U)^2 \lesssim 1.$$

It remains to prove

$$\sum_{\vartheta \in \Theta} \int |f_\vartheta|^4 \lesssim \int |f|^4.$$

This is just a result of interpolation between the following two inequalities (or see Lemma 5.4):

$$\sup_{\vartheta \in \Theta} \|f_\vartheta\|_{L^s} \lesssim \|f\|_{L^s}.$$

$$\sum_{\vartheta \in \Theta} \|f_\vartheta\|_{L^2}^2 \lesssim \|f\|_{L^2}^2.$$
4. PROOF OF THE SHARP-CUTOFF VERSION

Let us prove Theorem 1.3. Now each \( \Delta_j \) is \( \delta \)-regular, so by definition there is a \( \delta \)-ball \( B_j \subset S^{n-1} \) such that \( cB_j \subset \Delta_j \subset CB_j \). By the disjointness of \( \{ \Delta_j \} \), we see the \( C\delta \)-balls \( \{ CB_j \} \) are finitely overlapping. Without loss of generality, we may assume they are disjoint. (Actually, we can regroup these balls into \( O(1) \) sets so that the \( C\delta \)-balls in each set are disjoint. Then we prove the estimate for each set and sum them up together.) We can choose smooth bump function \( 1_{B_j} \) adapted to \( CB_j \) so that \( 1_{B_j} = 1 \) on \( \Delta_j \). For any function \( h \), we define \( T_j h := (1_{B_j} \hat{h})^\vee \). By the support condition, we see that \( S_j f = S_j T_j f \).

By duality, there is a \( g \in L^2 \) with \( \|g\|_2 = 1 \) so that

\[
\left( \sum_{j} |S_j f|^2 \right)^{1/2} = \int |S_j f|^2 g = \sum_{j} \int |S_j T_j f|^2 g.
\]

Let us look at the integral \( \int |S_j h|^2 g \), where we will plug in \( h = T_j f \) later. Recall that \( S_j h = 1_{\sigma_j} \hat{h} \), where \( \sigma_j = \{ \xi : \xi/|\xi| \in \Delta_j \} \). We want to express \( 1_{\sigma_j} \) in another form. Note that \( \Delta_j \) is a polyhedron on \( S^{n-1} \), so if we denote by \( N_j = \{ \vec{n}_j \} \) the normals of \( \Delta_j \) pointing outward, then

\[
1_{\sigma_j} = \prod_{\vec{n}_j \in N_j} 1_{\{ \xi \cdot \vec{n}_j < 0 \}}.
\]

By the one-dimensional condition (see Definition 1.4), we see that \( N := \cup_j N_j \) is contained in \( O(1) \) many great circles. Define the following maximal functions

\[
M_{\vec{n}}g(x) := \sup_{t>0} \frac{1}{2t} \int_{x+[-t,t]\vec{n}} |g|,
\]

\[
M_{s,\vec{n}}g(x) := M_2 \left( (M_{\vec{n}}|g|^s)^{1/s} \right), \quad M_s g(x) := \sup_{\vec{n} \in \mathcal{N}} M_{s,\vec{n}} g.
\]

Here \( s > 1 \). We see that \( M_s \) is actually a maximal operator associated to one-dimensional directions.

We may assume \( \mathcal{N} \) is contained in one great circle and by rotation we may assume \( \mathcal{N} \) lie in the \( (\xi_1, \xi_2) \)-plane, i.e., \( \mathcal{N} \subset \{ \xi \in S^{n-1} : \xi_1^2 + \xi_2^2 = 1 \} \). Denote \( x = (x_1, x_2, x') \), where \( x' \in \mathbb{R}^{n-2} \). We write

\[
\int |S_j h|^2 g = \int |S_j h(x_1, x_2, x')|^2 g(x_1, x_2, x') dx_1 dx_2 dx'
\]

\[
= \int |K(x_1, x_2) * h(x_1, x_2, x')|^2 g(x_1, x_2, x') dx_1 dx_2 dx'.
\]

Here \( K(x_1, x_2) = (\prod_{\vec{n}_j \in \mathcal{N}} 1_{\{ \xi \cdot \vec{n}_j < 0 \}})^\vee \) is some kernel depending only on \( x_1, x_2 \).

By iterating the weighted estimate of Córdoba-Fefferman [CF76], we have for each \( s > 1 \) the following estimate:

\[
\int |K(x_1, x_2) * h(x_1, x_2, x')|^2 g(x_1, x_2, x') dx_1 dx_2 dx'
\]

\[
\lesssim \int |h|^2 M_{s,\vec{n}_j_1} \circ \cdots \circ M_{s,\vec{n}_j_m} |g|.
\]
Here \( N_j = \{ \vec{n}_{j,1}, \ldots, \vec{n}_{j,m_j} \} \). We assume \( m_j \leq m \) which is a bounded number, so the above inequality is bounded by

\[
\int_{\mathbb{R}^2} |h|^2 M_s^{(m)} |g|,
\]

where \( M_s^{(m)} \) is the composition of \( M_s \) by \( m \) times. We obtain

\[
\int_{\mathbb{R}^n} |S_j h|^2 g \lesssim \int_{\mathbb{R}^n} |h|^2 M_s^{(m)} |g|.
\]

Plugging back to (4.11), we have

\[
\left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|^2_4 \lesssim \int_{\mathbb{R}^n} \sum_j |T_j f|^2 M_s^{(m)} |g| \leq \left\| \left( \sum_j |T_j f|^2 \right)^{1/2} \right\|^2_4 \left\| (M_s^{(m)}) |g| \right\|_2^{-1/2}.
\]

Since \( M_s^{(m)} \) is essentially a maximal operator in the plane, we can use the two-dimensional maximal estimate (see for example in \[Kat99\]) and choose \( s \) very close to 1 to get \( \| M_s^{(m)} |g| \|_2 \lesssim \delta^{-\varepsilon} \|g\|_2 \). This gives

\[
\left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|^2_4 \lesssim \delta^{-\varepsilon} \left\| \left( \sum_j |T_j f|^2 \right)^{1/2} \right\|^2_4.
\]

This boils down to the smooth version that we have already proved in Section 3.

5. \( L^4 \) SQUARE FUNCTION ESTIMATE FOR THE CONE

We prove Theorem 1.9 and Theorem 1.11 in this section. Via a global-to-local reduction that is similar to the one in Section 2, we see that Theorem 1.11 is a corollary of Theorem 1.9. Hence, we will focus on the proof of Theorem 1.9. Let us begin with some elementary tools.

5.1. Some elementary estimates. Let \( R \subset \mathbb{R}^3 \) be a rectangle of dimensions \( a_1 \times a_2 \times a_3 \). We will use \( R^* \) to denote the dual rectangle of \( R \), namely \( R^* \) is the rectangle centered at the origin of dimensions \( a_1^{-1} \times a_2^{-1} \times a_3^{-1} \). Also, we make the convention that if \( R \) lies in the physical space \( \mathbb{R}^3_x \) then \( R^* \) lies in the frequency space \( \mathbb{R}^3_\xi \) and vice versa.

Sometimes we will use the notation that a function \( \varphi \) is a smooth bump function adapted to \( R \). What follows is its precise definition.

**Definition 5.1.** Let \( R \subset \mathbb{R}^3_\xi \) be a rectangle of dimensions \( a_1 \times a_2 \times a_3 \) and let \((e_1,e_2,e_3)\) be the corresponding directions. Denote by \( \xi_R \) the center of \( R \) and write \( \xi \) in the coordinate \((e_1,e_2,e_3)\) as \((\xi_1,\xi_2,\xi_3)\). We say \( \varphi \) is “a smooth bump function adapted” to \( R \), if \( \text{supp} \varphi \subset 2R \), \( \varphi = 1 \) on \( R \) and \( \varphi \) satisfies the following derivative estimate

\[
|D_{e_i}^k \varphi(\xi_R + \xi)| \lesssim k \left( 1 + \frac{|\xi|}{a_i} \right)^{-k},
\]

for \( i = 1, 2, 3 \) and any \( k > 0 \).
Following the notation in the definition above, if $\varphi$ is a smooth bump function adapted to $R$, then
\begin{equation}
|\varphi^\vee(x)| \lesssim_k \frac{1}{|R^*|} (1 + a |x_1| + b |x_2| + c |x_3|)^{-k}.
\end{equation}
This roughly says $|\varphi^\vee| \approx \frac{1}{|R^*|} 1_{R^*}$, which suggests us to define the following indicator function with rapidly decaying tail.

**Definition 5.2.** Let $R$ be a rectangle and $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map such that $A([-1,1]^3) = R$. We define
\begin{equation}
\chi_R(x) := (1 + |A^{-1} x|)^{-10^9}.
\end{equation}
We call $\chi_R$ the indicator function of $R$ with rapidly decaying tail.

The definitions above also work in $\mathbb{R}^n$. Now let us state a weighted $L^2$-estimate. For its proof, see [Cor82] or Lemma 2.3 in [GJW21].

**Lemma 5.3.** Let $\{R\}$ be a set of finitely overlapping congruent rectangles in $\mathbb{R}^n_\xi$, and let $\{\varphi_R(\xi)\}_R$ be the smooth bump functions adapted to them. Then
\begin{equation}
\int_{\mathbb{R}^n} \sum_R |(\varphi_R \hat{f})^\vee|^2 g \lesssim \int_{\mathbb{R}^n} |f|^2 \left( \frac{1}{|R^*|} \chi_{R^*} * |g| \right).
\end{equation}

There is another useful lemma.

**Lemma 5.4.** Let $\{R\}$ be a set of finitely overlapping rectangles in $\mathbb{R}^n_\xi$, and let $\{\varphi_R(\xi)\}_R$ be the smooth bump functions adapted to them. Then
\begin{equation}
\int_{\mathbb{R}^n} \sum_R |(\varphi_R \hat{f})^\vee|^p \lesssim \int_{\mathbb{R}^n} |f|^p,
\end{equation}
for $2 \leq p \leq \infty$.

Lemma 5.4 follows from interpolation between $p = 2$ and $p = \infty$. We also have the local version of Lemma 5.4.

**Lemma 5.5.** Let $\{R\}$ be a set of finitely overlapping congruent rectangles in $\mathbb{R}^n_\xi$, and let $\{\varphi_R(\xi)\}_R$ be the smooth bump functions adapted to them. If $U \subset \mathbb{R}^n_x$ is a rectangle whose translation to the origin contains all the dual rectangles $R^*$, then we have the following estimate:
\begin{equation}
\int \chi_U \sum_R |(\varphi_R \hat{f})^\vee|^p \lesssim \int \chi_U |f|^p,
\end{equation}
for $2 \leq p \leq \infty$.

Again, the proof is by interpolation between $p = 2$ and $p = \infty$. The case $p = \infty$ is easy, let us focus on $p = 2$. We may assume $U$ is centered at the origin. Since $R^* \subset U$, we can partition each $R$ into smaller rectangles that are comparable to $U^*$. Denote the set of all the smaller rectangles coming from the partition by $\{\omega\}$. By local orthogonality, we have
\begin{equation}
\int \chi_U \sum_R |(\varphi_R \hat{f})^\vee|^2 \lesssim \int \chi_U \sum_\omega |(\varphi_\omega \hat{f})^\vee|^2,
\end{equation}
where $\varphi_\omega$ are smooth bump functions adapted to $\omega$. Together with Lemma 5.3 and the fact that $\frac{1}{|U|} \chi_U * \chi_U \lesssim \chi_U$, we prove the result.
Remark 5.6. From the point of view of the so-called “locally constant property”, the lemmas above are obvious, but we still state them carefully for rigor.

5.2. The cutoff replacing property. Let $R, R' \subset \mathbb{R}^3_\xi$ be two rectangles and $\varphi_R, \varphi_{R'}$ be smooth bump functions adapted to them. If $\text{supp} \hat{f} \subset R \cap R'$, then $(\varphi_R \hat{f})' = (\varphi_R \hat{f})'$. In this case, we replace the cutoff $\varphi_R$ by $\varphi_{R'}$, so we call it the cutoff replacing property.

When $\text{supp} \hat{f}$ is not contained in $R \cap R'$, we can still have the cutoff replacing property by choosing $\varphi_R, \varphi_{R'}$ carefully. In the following, we discuss the case that we need in the paper. Let $R = [-a, a] \times [-b, b] \times [-c, c], R' = [-a, a] \times [-b, b] \times [-c', c']$ be two rectangles with $a > b > c' > c$ in the frequency space $\mathbb{R}^3_\xi$. Let $f$ be a function in $\mathbb{R}^3_\xi$ so that $\text{supp} \hat{f} \subset \mathbb{R}^3_\xi \setminus \{\xi_1 \leq a, |\xi_2| \leq b, |\xi_3| > c\}$. We see that $\text{supp} \hat{f}$ is not contained in $R \cap R'$, but we can still construct the smooth bump functions to satisfy the property.

Choose $\varphi(\xi_1, \xi_2)$ to be a smooth bump function adapted to $[-a, a] \times [-b, b]$. Choose $\rho(\xi_3)$ to be a smooth bump function adapted to $[-c, c]$, then $\rho(\hat{f} \xi_3)$ is a smooth bump function adapted to $[-c', c']$. If we set $1_R = \varphi(\xi_1, \xi_2) \rho(\xi_3)$ and $1_{R'} = \varphi(\xi_1, \xi_2) \rho(\hat{f} \xi_3)$, we see that $1_R, 1_{R'}$ are adapted to $R, R'$, and $1_R = 1_{R'}$ on $\mathbb{R}^3_\xi \setminus \{\xi_1 \leq a, |\xi_2| \leq b, |\xi_3| > c\}$ which contains $\text{supp} \hat{f}$. So, we have $(1_R \hat{f})' = (1_R \hat{f})'$. Let us discuss how it works in our proof. Let $\tau$ be a $\delta \times \delta^{1/2} \times 1$-plank in $N_\delta(\Gamma)$ and let $\omega$ be a $\gamma^{-1} \delta \times \delta^{1/2} \times 1$-plank which is the $\gamma^{-1}$-dilation of $\tau$ in the shortest direction (here $\gamma < 1$). Our function $f$ satisfies $\text{supp} \hat{f} \subset N_\delta(\Gamma)$, so it also satisfies a similar condition discussed above after rotation. So, we can find $1_R$ adapted to $\tau$ and $1_R^*$ adapted to $\omega$ such that $(1_R^* \hat{f})' = (1_R^* \hat{f})'$.

Remark 5.7. The reason we want to change the cutoff is that we want to apply Lemma 5.3 and Lemma 5.5.

5.3. A general version of square function. We start the proof of Theorem 5.3. Recall that $\Gamma = \{\xi \in \mathbb{R}^3 : \xi^2_1 + \xi^2_2 = \xi^2_3, 1/2 \leq |\xi_3| \leq 1\},$ and $N_\delta(\Gamma)$ is its $\delta$-neighborhood. There is a canonical covering of $N_\delta(\Gamma)$ using finitely overlapping planks $\tau$ of dimensions $\sim \delta \times \delta^{1/2} \times 1$, denoted by $T = \{\tau\}$. For each $\tau \in T$, choose a smooth bump function $1_\tau^*$ adapted to $\tau$. Define $f_\tau := (1_\tau^* \hat{f})'$ as usual.

To prove the theorem, we need to state the estimate in a more technical way. Fix a dyadic parameter $\delta^{1/2} \leq \gamma \leq 1$. For each $\tau \in T$, we partition it into planks of dimensions $\delta \times \delta^{1/2} \times \gamma$ along the longest side of $\tau$ (see Figure 1). Denote the collection of these sub-planks of $\tau$ by $\Theta_\gamma(\tau)$. For each $\theta \in \Theta_\gamma(\tau)$, there is a smooth cutoff function $1_\theta^*$ adapted to $\theta$ and meanwhile we have $1_\gamma^* = \sum_{\theta \in \Theta_\gamma(\tau)} 1_\theta^*$. We define $f_\theta = (1_\theta^* \hat{f})'$, then $f_\tau = \sum_{\theta \in \Theta_\gamma(\tau)} f_\theta$. Set $\Theta_\gamma = \cup_{\tau \in T} \Theta_\gamma(\tau)$ be the set of all these planks.

We will prove the following general version of the square function estimate.

Theorem 5.8. Assuming $\hat{f} \subset N_\delta(\Gamma)$, then for any $\varepsilon > 0$ we have

$$\left(\sum_{\theta \in \Theta_\gamma} |f_\theta|^2\right)^{1/2} \leq C_\varepsilon (\gamma^{\delta^{-1}})^{\varepsilon} \|f\|_{L^s(\mathbb{R}^3)}.$$

Proof. We prove by induction on $\gamma$. The base case is when $\gamma = \delta^{1/2}$. For each $\theta \in \Theta_\delta^{1/2},$ there is a cube $Q_\theta$ of side length $\delta^{1/2}$ such that $\theta \subset Q_\theta$. By the cutoff
replacing property in Section 5.2, we can choose a smooth bump function \(1^*_Q\) adapted to \(Q\) so that \(1^*_Q \hat{f} = 1^*_Q \hat{f}\). Via the Littlewood-Paley theory for congruent cubes (see [RdF85] or [Lac07]), we obtain

\[
\left\| \left( \sum_{\theta \in \Theta_{\gamma}} |f_{\theta}|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}
\]

for any \(1 < p < \infty\), which in particular implies (5.7) when \(\gamma = \delta^{1/2}\).

Assuming (5.7) is proved for \(\gamma \leq \gamma_0/2\), we are going to look at the case \(\gamma = \gamma_0\). We suggest the reader to pretend \(\gamma_0 = 1\) for the first time of reading. For simplicity, we will omit the subscript \(\gamma\) and write \(\Theta(\tau)\) (or \(\Theta_\gamma(\tau)\)) as \(\Theta(\tau)\) (or \(\Theta(\tau)\)). For each \(\tau \in \mathcal{T}\), define the square function

\[
g_{\tau} := \sum_{\theta \in \Theta(\tau)} |f_{\theta}|^2.
\]

Each term in the summation has Fourier transform supported in \(\theta - \theta\) which is roughly the translation of \(\theta\) to the origin. Note that all the \(\theta \in \Theta(\tau)\) are roughly a translation of each other, so there is a plank of dimensions \(\delta \times \delta^{1/2} \times \gamma\) centered at the origin such that \(\theta - \theta\) is contained in this plank for any \(\theta \in \Theta(\tau)\). We denote this plank by \(\theta_{\tau}\). Now we obtain functions \(\{g_{\tau}\}_{\tau \in \mathcal{T}}\) and planks \(\{\theta_{\tau}\}_{\tau \in \mathcal{T}}\) with \(\text{supp} \hat{g}_{\tau} \subset \theta_{\tau}\). We also see that

\[
\sum_{\theta \in \Theta} |f_{\theta}|^2 = \sum_{\tau} g_{\tau}.
\]

Our goal is to estimate \(\int_{\mathbb{R}^3} |\sum_{\tau} g_{\tau}|^4\).

Next, we will do a high-low frequency decomposition for each \(g_{\tau}\). Fix a large enough constant \(K\) which is to be determined later. Note that \(\theta_{\tau}\) is a plank of dimensions \(\delta \times \delta^{1/2} \times \gamma\) centered at the origin. Define another plank which we call the “low plank” as: \(\theta_{\tau,\text{low}} := \theta_{\tau} \cap B_{CK^{-1}}(0)\). Roughly speaking, \(\theta_{\tau,\text{low}}\) is the portion of \(\theta_{\tau}\) that is centered at the origin and has dimensions \(\delta \times \delta^{1/2} \times K^{-1}\). Choose a smooth bump function \(1^*_{\theta_{\tau}}\) adapted to \(\theta_{\tau}\) and a smooth bump function \(1^*_{\theta_{\tau,\text{low}}}\) adapted \(\theta_{\tau,\text{low}}\). Define

\[
\begin{align*}
g_{\tau,\text{low}} & := (1^*_{\theta_{\tau,\text{low}}} \hat{g}_{\tau})^\vee, \\
g_{\tau,\text{high}} & := (1^*_{\theta_{\tau}} - 1^*_{\theta_{\tau,\text{low}}} \hat{g}_{\tau})^\vee.
\end{align*}
\]

Figure 1. Finer partition
Since $\text{supp} \hat{g}_\tau \subset \theta_\tau$, we have
\[
g_{\tau, \text{low}} + g_{\tau, \text{high}} = (1^*_\theta \hat{g}_\tau)^\vee = g_\tau.
\]
By triangle inequality, we have
\[
(5.11) \quad \int_{\mathbb{R}^3} |\sum_\tau g_\tau|^4 \lesssim \int_{\mathbb{R}^3} |\sum_\tau g_{\tau, \text{low}}|^4 + \int_{\mathbb{R}^3} |\sum_\tau g_{\tau, \text{high}}|^4.
\]
We call the two terms on the right hand side above low term and high term. We consider them separately.

*Estimate for the low term:* For each $\theta \in \Theta$, we cover it by finitely overlapping planks of dimensions $\delta \times \delta^{1/2} \times K^{-1} \gamma$, denoted by $\{\theta'\}$. We use “$\theta' < \theta$” to indicate that $\theta'$ comes from the covering of $\theta$. One observation is that: if $\theta \in \Theta(\tau)$ and $\theta' < \theta$, then $\theta'$ is roughly a translation of $\Theta_{\tau, \text{low}}$. For fixed $\theta$ and $\{\theta'\}_{\theta' < \theta}$, we choose smooth functions $\{1^*_{\theta'}\}_{\theta' < \theta}$ so that each $1^*_{\theta'}$ is a smooth bump function adapted to $\theta'$ and
\[
(5.12) \quad 1^*_\theta = \sum_{\theta' < \theta} 1^*_{\theta'}
\]
on $N_\delta(\Gamma)$.

As usual, we define $f_{\theta'} := (1^*_{\theta'} \hat{f})^\vee$. Since $\text{supp} \hat{f} \subset N_\delta(\Gamma)$, we have
\[
(5.13) \quad f_\theta = \sum_{\theta' < \theta} f_{\theta'}.
\]
Let us look at the low term. By definition,
\[
(5.14) \quad \hat{g}_{\tau, \text{low}} = 1^*_\theta_{\tau, \text{low}} \sum_{\theta \in \Theta(\tau)} |\hat{f}_\theta|^2 = 1^*_\theta_{\tau, \text{low}} \sum_{\theta \in \Theta(\tau)} (|\sum_{\theta' < \theta} f_{\theta'}|^2)^\wedge
\]
\[
(5.15) \quad = 1^*_\theta_{\tau, \text{low}} \sum_{\theta \in \Theta(\tau)} \sum_{\theta' < \theta} \hat{f}_{\theta'} \ast \overline{\hat{f}_{\theta'_2}}.
\]
Note that $\text{supp}(1^*_\theta_{\tau, \text{low}} \cdot \hat{f}_{\theta'_1} \ast \overline{\hat{f}_{\theta'_2}})$ is contained in $\theta_\tau \cap (\theta'_1 - \theta'_2)$, so $1^*_\theta_{\tau, \text{low}} \cdot \hat{f}_{\theta'_1} \ast \overline{\hat{f}_{\theta'_2}}$ is nonzero only when $\theta'_1$ and $\theta'_2$ are roughly adjacent. We use $\theta'_2 \sim \theta'_1$ to denote that $\theta'_2$ and $\theta'_1$ are adjacent. As a result, we have
\[
(5.16) \quad \hat{g}_{\tau, \text{low}} = 1^*_\theta_{\tau, \text{low}} \sum_{\theta \in \Theta(\tau)} (\sum_{\theta'_1 < \theta} \sum_{\theta'_2 \sim \theta'_1} f_{\theta'_1} \overline{f_{\theta'_2}})^\wedge.
\]
So, we have
\[
(5.17) \quad g_{\tau, \text{low}} = (1^*_\theta_{\tau, \text{low}})^\vee \ast \sum_{\theta \in \Theta(\tau)} \sum_{\theta'_1 < \theta} \sum_{\theta'_2 \sim \theta'_1} f_{\theta'_1} \overline{f_{\theta'_2}}
\]
\[
(5.18) \quad \lesssim |(1^*_\theta_{\tau, \text{low}})^\vee| \ast \sum_{\theta \in \Theta(\tau)} \sum_{\theta' < \theta} |f_{\theta'}|^2
\]
\[
(5.19) \quad \lesssim \frac{1}{|\theta^*_{\tau, \text{low}}|} \chi_{\theta^*_{\tau, \text{low}}} \ast \sum_{\theta \in \Theta(\tau)} \sum_{\theta' < \theta} |f_{\theta'}|^2.
\]
where the second-last inequality is by the fact that for each $\theta'_1$ there are $O(1)$ many $\theta'_2$ adjacent to $\theta'_1$. 
Since supp $\hat{f}_{\theta'} \subset \theta'$, we have $|f_{\theta'}|^2$ is locally constant on any translation of $\theta'$. Also noting that $\theta^*_{\tau, low}$ and $\theta^*_{\tau, high}$ are roughly the same, we have $|f_{\theta'}|$ is locally constant on any translation of $\theta^*_{\tau, low}$. So, we actually have

$$
\sum_{\theta \in \Theta(\tau)} \sum_{\theta' < \theta} |f_{\theta'}|^2 = \sum_{\theta' \in \Theta_{\tau, K-1}} |f_{\theta'}|^2.
$$

By induction hypothesis, we have the following estimate for the low term.

$$
\int_{\mathbb{R}^3} |\sum_{\tau} g_{\tau, low}|^4 = \int_{\mathbb{R}^3} (\sum_{\theta' \in \Theta_{\tau, K-1}} |f_{\theta'}|^2)^4 \leq \left(C \epsilon (\gamma K^{-1} \delta^{-1})^2 \|f\|_{L^8}\right)^8.
$$

**Estimate for the high term:** We consider the truncated cone

$$
\Gamma_\gamma := \{\xi \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 = \xi_3^2, \gamma / K \leq |\xi_3| \leq \gamma\},
$$

and its $C\gamma^{-1}\delta$-neighborhood $N_{C\gamma^{-1}\delta} \Gamma_\gamma$. For simplicity, we will omit the constant $C$ and just write as $N_{\gamma^{-1}\delta} \Gamma_\gamma$. By definition, the support of $\hat{g}_{\tau, high}$ is contained in $\theta_\tau \setminus \theta_{\tau, low}$ which consists of two planks symmetric with respect to the origin and of dimensions $\delta \times \delta^{1/2} \times \gamma$. We denote them by $\theta^{+}_{\tau, high}$ and $\theta^{-}_{\tau, high}$, where $\theta^{+}_{\tau, high}$ lies in $\{\xi_3 > 0\}$ and $\theta^{-}_{\tau, high}$ lies in $\{\xi_3 < 0\}$. By a simple geometry, we see that supp $\hat{g}_{\tau, high} \subset N_{\gamma^{-1}\delta} \Gamma_\gamma$. We choose a finitely overlapping covering of $N_{\gamma^{-1}\delta} \Gamma_\gamma$ by $\gamma^{-1}\delta \times \delta^{1/2} \times \gamma$-planks $\omega$, denoted by

$$
N_{\gamma^{-1}\delta} \Gamma_\gamma = \bigcup \omega.
$$

For each $\hat{g}_{\tau, high}$, there exists $\omega$ such that $\theta^{+}_{\tau, high} \cup \theta^{-}_{\tau, high} \subset \omega \cup \omega_{re}$ and hence supp $\hat{g}_{\tau, high} \subset \omega \cup \omega_{re}$. Here $\omega_{re} = \{-\xi : \xi \in \omega\}$ denotes the reflection of $\omega$ with respect to the origin. We associate $\tau$ to $\omega$ (if there are multiple choices, we choose one). The reader can also check each supp $\hat{g}_{\tau, high}$ can intersect a bounded number of sets from $\{\omega \cup \omega_{re}\}$. The relation between $\omega$ and $\theta^{+}_{\tau, high}$ is given in Figure 2.
Remark 5.9. It’s not harmful to the proof if we ignore \( \theta^-_{r, \text{high}} \) (the other end of \( \theta_r \setminus \theta_{r, \text{low}} \)) and think of \( \text{supp} \hat{g}_{\theta_{r, \text{high}}} \subset \theta^+_{r, \text{high}} \). It’s convenient to assume all the \( \omega \) lie in the upper half space \( \{ \xi_3 > 0 \} \).

Define \( T(\omega) \) to be the set of \( \tau \)'s that are associated to \( \omega \). For each \( \omega \), we define our function
\[
(5.23) \quad h_\omega := \sum_{\tau \in T(\omega)} g_{\tau, \text{high}}.
\]
We see that \( \text{supp} \hat{h}_\omega \subset \omega \cup \omega_{r_s} \). We have
\[
(5.24) \quad \int_{\mathbb{R}^3} |\sum_{\tau} g_{\tau, \text{high}}|^4 = \int_{\mathbb{R}^3} |\sum_{\omega} h_\omega|^4.
\]
By rescaling of the factor \( \gamma^{-1} \) in all directions, we see that \( N_{\gamma^{-1} \xi} \Gamma_\gamma \) becomes \( N_{\gamma^{-1} \xi} \Gamma_1 \) and \( \omega \) becomes a \( \gamma^{-2} \delta \times \gamma^{-1} \delta^{1/2} \times 1 \)-plank. We want to apply Guth-Wang-Zhang’s reverse square function estimate \( (1.7) \). Before doing so, we give a remark.

Remark 5.10. In the setting of \( (1.7) \), we use the cone \( \Gamma = \{ \xi_1^2 + \xi_2^2 = \xi_3^3, 1/2 \leq |\xi_3| \leq 1 \} \), but we can also use the cone \( \Gamma_1 \) defined in \( (5.22) \) at the cost of a factor \( K^{O(1)} \) in \( (1.7) \). That is
\[
\| \sum_{\tau} f_\tau \|_{L^4(\mathbb{R}^3)} \leq C_{\epsilon'} K^{O(1)} \delta^{-\epsilon'} \left( \sum_{\tau} |f_\tau|^2 \right)^{1/2}. \tag{5.25}
\]
We have from \( (1.7) \) that
\[
(5.25) \quad \int_{\mathbb{R}^3} |\sum_{\tau} g_{\tau, \text{high}}|^4 = \int_{\mathbb{R}^3} |\sum_{\omega} h_\omega|^4 \leq (C_{\epsilon'} K^{O(1)} (\gamma^{-2} \delta)^{-\epsilon'})^4 \int_{\mathbb{R}^3} (\sum_{\omega} |h_\omega|^2)^2.
\]
To estimate \( \int_{\mathbb{R}^3} (\sum_{\omega} |h_\omega|^2)^2 \), we will use a general version of Lemma 1.4 in \( [GWZ20] \). Before stating the lemma, we introduce some notations.

Fix \( R = \gamma \delta^{-1} \). We see that \( \{ \omega \} \) are \( R^{-1} \gamma \times R^{-1/2} \gamma \times \gamma \)-planks that form a finitely overlapping covering of \( N_{R^{-1} \Gamma_\gamma} \). Suppose we are given a set of functions \( \{ h_\omega \} \) with \( \text{supp} \hat{h}_\omega \subset \omega \). For any dyadic \( s \) in the range \( R^{-1/2} \leq s \leq 1 \), let \( \Omega_s = \{ \omega_s \} \) be \( s^2 \gamma \times s \gamma \times \gamma \)-planks that form a finitely overlapping covering of \( N_{s \delta \Gamma_\gamma} \). For any \( \omega_s \), we define \( U_{\omega_s} \) to be the plank centered at the origin of dimensions \( R \gamma^{-1} \times R s \gamma^{-1} \times R s^2 \gamma^{-1} \). Here, the edge of \( U_{\omega_s} \) with length \( R \gamma^{-1} \) (respectively \( R s \gamma^{-1} \), \( R s^2 \gamma^{-1} \)) has the same direction as the edge of \( \omega_s \) with length \( s^2 \gamma \) (respectively \( s \gamma \), \( \gamma \)). The motivation for the definition of \( U_{\omega_s} \) is that \( U_{\omega_s} \) is the smallest plank that contains the dual plank of \( \omega \) for all \( \omega \subset \omega_s \). Later we will do rescaling \( \xi \mapsto \gamma^{-1} \xi \) in the frequency space, so that \( \{ \omega \} \) becomes standard \( R^{-1} \times R^{-1/2} \times 1 \)-planks that cover the \( N_{R^{-1} \Gamma_1} \).

We tile \( \mathbb{R}^3 \) by translated copies of \( U_{\omega_s} \). We write \( U \parallel U_{\omega_s} \) to denote that \( U \) is one of the copies, and define \( S_U h \) by
\[
(5.26) \quad S_U h = \left( \sum_{\omega \subset \omega_s} |h_\omega|^2 \right)^{1/2} |U|.
\]

Remark 5.11. Different from \( [GWZ20] \), we don’t require there is a common function \( h \) for all the \( h_\omega \) so that \( h_\omega \) is the Fourier restriction of \( h \) to \( \omega \). The only condition we need is \( \text{supp} \hat{h}_\omega \subset \omega \). One can compare the notation \( S_U h \) here to a similar one in \( [GWZ20] \).
We state the following lemma which is a general version of Lemma 1.4 in [GWZ20].

**Lemma 5.12.** Let the notation be given above. Then we have

\[
\int_{\mathbb{R}^3} (\sum_{\omega} |h_\omega|^2)^2 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{\omega_1, \omega_2} \sum_{U \parallel U_{\omega_1}} |U|^{-1} \| S_U h \|_{L^2}^4.
\]

To prove Lemma 5.12, we first do the rescaling $\xi \mapsto \gamma^{-1} \xi$ in the frequency space and correspondingly $x \mapsto \gamma x$ in physical space. Then, the proof is identical to the proof of Lemma 1.4 in [GWZ20] which we do not reproduce here.

Let us come back to (5.25). By Lemma 5.12 we have

\[
\int_{\mathbb{R}^3} (\sum_{\omega} |h_\omega|^2)^2 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{\omega_1, \omega_2} \sum_{U \parallel U_{\omega_1}} |U|^{-1} \| S_U h \|_{L^2}^4 = \sum_{R^{-1/2} \leq s \leq 1} \sum_{\omega_1, \omega_2} \sum_{U \parallel U_{\omega_1}} |U|^{-1} (\int_U \sum_{\omega \subset U_{\omega_1}} |h_\omega|^2)^2.
\]

Recalling the definition (5.23) and (5.10), the above formula equals

\[
\sum_{R^{-1/2} \leq s \leq 1} \sum_{\omega_1, \omega_2} |U|^{-1} (\int_U \sum_{\omega \subset U_{\omega_1}} (1_{\tilde{\theta}_\tau} - 1_{\tilde{\theta}_{\tau, low}})^\vee * g_\tau)^2.
\]

Before proceeding further, we give another definition. For an $\omega$, we see that $U_{\tau} \in T(\omega) \tau$ is a union of $\sim \gamma^{-1}$ many $\delta \times \delta^{1/2} \times 1$-planks, so $U_{\tau} \in T(\omega)$ is contained in a $\delta \gamma^{-2} \times \delta^{1/2} \gamma^{-1} \times 1$-plank. By abuse of notation, we also use $T(\omega)$ to denote this plank. We define $f_{T(\omega)}$ to be the smooth Fourier restriction of $f$ to this plank, i.e., $f_{T(\omega)} := (1_{T(\omega)} f)^\vee$, where $1_{T(\omega)}$ is a smooth bump function adapted to $T(\omega)$. The property we will use is: $1_{\tilde{\theta}} = 1_{\tilde{\theta}} 1_{T(\omega)}$ if $\tau \in T(\omega)$ and $\theta \in \Theta(\tau)$. Heuristically, one may think $f_{T(\omega)} = \sum_{\tau \in T(\omega)} f_{\tau}$.

Fix an $U \parallel U_{\omega_1}$. Let us look at the integral $\int_U |\sum_{\tau \in T(\omega)} (1_{\tilde{\theta}_\tau} - 1_{\tilde{\theta}_{\tau, low}})^\vee * g_\tau|^2$ in the above formula (5.30). Recall (5.3) that $g_\tau = \sum_{\theta \in \Theta(\tau)} |f_\theta|^2$. We will show there is a local orthogonality for $\{g_\tau\}_{\tau \in T(\omega)}$ on $U$, that is, we claim the following estimate:

\[
\int_U |\sum_{\tau \in T(\omega)} (1_{\tilde{\theta}_\tau} - 1_{\tilde{\theta}_{\tau, low}})^\vee * g_\tau|^2 \lesssim \int \chi_U |f_{T(\omega)}|^4,
\]

where $\chi_U$ is the indicator function of $U$ with rapidly decaying tail.

Our first step is to rewrite $(1_{\tilde{\theta}_\tau} - 1_{\tilde{\theta}_{\tau, low}})^\vee * g_\tau$. If $\tau \in T(\omega)$, then by definition $\theta_\tau \setminus \theta_{\tau, low} \subset \omega \cup \omega_{\tau}$. If we denote by $\tilde{\omega}$ the translation of $\omega$ to the center, from a simple geometry we have $\theta_\tau \subset 100\tilde{\omega}$. We may omit the constant 100 and write it as $\theta_\tau \subset \tilde{\omega}$. For reader’s convenience, we remark that $\theta_\tau$ is of dimensions $\delta \times \delta^{1/2} \times \gamma$, and $\omega$ is of dimensions $\gamma^{-1} \delta \times \delta^{1/2} \times \gamma (= R^{-1} \gamma \times R^{-1/2} \gamma \times \gamma)$. Like the definition of $\omega_{\tau, low}$, we define $\omega_{\tau, low}$ to be the portion of $\tilde{\omega}$ that centered at the origin of dimensions $R^{-1} \gamma \times R^{-1/2} \gamma \times R^{-1} \gamma$ (actually $\tilde{\omega}$ and $\omega_{\tau, low}$ are comparable since $K$ is a large constant). Next, we will use the cutoff replacing property (see Section 5.2). Noting that $\tilde{\theta}_\tau \subset \theta_\tau$, we can choose two bump functions $1_{\tilde{\omega}}, 1_{\tilde{\omega}_{\tau, low}}$ adapted to $\tilde{\omega}, \omega_{\tau, low}$ respectively, so that $1_{\tilde{\theta}_\tau} = 1_{\tilde{\omega}} 1_{\tilde{\theta}_{\tau, low}} = 1_{\tilde{\omega}} 1_{\omega_{\tau, low}}$ on $\supp g_\tau$. As a result, we
have \((1^*_\theta - 1^*_\theta, \omega)\) \* \(g_T = (1^*_\theta - 1^*_\omega, \omega)\) \* \(g_T\). In other words, one can rewrite the convolution as
\[
(1^*_\theta, \omega) \* \omega = \varphi^\flat \* \omega,
\]
for some \(\varphi\) adapted to \(\tilde{\omega}\), by taking advantage that \(\text{supp} \tilde{\omega} \subset \theta_T\). Thus,
\[
|\varphi^\flat \* \omega| \lesssim \frac{1}{|\omega^*|} \chi_{\omega^*} \* g_T.
\]
where \(\chi_{\omega^*}\) is the indicator function of \(\omega^*\) with rapidly decaying tail.

Let us prove (5.31). By applying Lemma 5.3, we denote this translated copy by \(\tilde{\omega}\), so each \(\varphi\) is contained in a translated copy of \(\omega\) such that
\[
\int (\tilde{\varphi} \* \omega) \cdot (1^*_\omega - 1^*_\theta, \omega)\) \* \(g_T\) \lesssim \int \chi_{\omega^*} \sum_{\tau \in \Omega(\omega)} \sum_{\theta \in \Theta(\tau)} |f_\theta|^2 \lesssim \int \chi_{\omega^*} \sum_{\tau \in \Omega(\omega)} \sum_{\theta \in \Theta(\tau)} |f_\theta|^2.
\]
In the last inequality, we used \(\frac{1}{|\omega^*|} \chi_{\omega^*} \* 1_U \lesssim \chi_U\) since \(\omega^*\) is contained in the translation of \(U\) to the origin.

Next, we will apply Lemma 5.3. Note that \(\theta_T\) is contained in a translated copy of \(\omega\), so each \(\theta\) is contained in a translated copy of \(\omega\). To indicate its relationship with \(\theta\), we denote this translated copy by \(\omega_\theta\) so that \(\theta \subset \omega_\theta\). One easily sees that \(\{\omega_\theta\}_\theta\) are finitely overlapping. Also, we can find a smooth bump function \(1^*_\omega\) adapted to \(\omega_\theta\) such that \(1^*_\omega = 1^*_\omega\) on \(\text{supp} \tilde{f}\), by taking the advantage that \(\text{supp} \tilde{f} \subset N_\theta(\Gamma)\) and the cutoff replacing property. By duality, we choose a function \(g\) with \(\|g\|_2 = 1\), such that
\[
\left( \int \chi_U \left( \sum_{\tau \in \Omega(\omega)} \sum_{\theta \in \Theta(\tau)} |f_\theta|^2 \right)^{1/2} \right)^2 = \int g^{1/2} \sum_{\tau \in \Omega(\omega)} \sum_{\theta \in \Theta(\tau)} |f_\theta|^2.
\]
Note that \(f_\theta = (1^*_\theta, \tilde{f})^\flat = (1^*_\theta, \tilde{f}_T(\omega))^\flat = (1^*_\theta, \tilde{f}_T(\omega))^\flat = (1^*_\theta, \tilde{f}_T(\omega))^\flat\). Since \(\{\omega_\theta\}\) is a set of congruent rectangles, by applying Lemma 5.3 to the function \(f_T(\omega)\), (5.31) is bounded by
\[
\int \frac{1}{|\omega^*|} \chi_{\omega^*} \* (g^{1/2} \chi^{1/2}_U) \cdot |f_T(\omega)| \lesssim \int \frac{1}{|\omega^*|} \chi_{\omega^*} \* (g^{1/2} \chi^{1/2}_U) \cdot |f_T(\omega)| \cdot \chi^{1/2}_U \left( \int \chi_U |f_T(\omega)| \right)^{1/2}.
\]
To finish the estimate, we just note that
\[
\int \left( \frac{1}{|\omega^*|} \chi_{\omega^*} \* (g^{1/2} \chi^{1/2}_U) \right)^2 \lesssim \int |g|^{2} \chi_U \cdot \left( \frac{1}{|\omega^*|} \chi_{\omega^*} \* \chi^{1/2}_U \right) \lesssim \int |g|^2 = 1.
\]
Here in the last inequality, we use the fact that \(\chi^{1/2}_U \sim \frac{1}{|\omega^*|} \chi_{\omega^*} \* \chi^{1/2}_U\), since \(\omega^*\) is contained in a translation of \(U\). We finish the proof of (5.31).
Plugging back to (5.25) and by Lemma 5.12, we have
\[
\int_{\mathbb{R}^3} \left| \sum_{\tau} g_{\tau, \text{high}} \right|^4 \leq (C_c K^{O(1)} (\gamma^{-2}\delta)^{-\varepsilon'})^4 \sum_{R^{-1/2} \lesssim s \leq 1} \sum_{\omega_s \in \Omega_s} \sum_{U \| U_\omega} |U|^{-1} \left( \int \chi_U \sum_{\omega \in U_\omega} |f_{T(\omega)}|^4 \right)^2.
\]
By Lemma 5.5, we have
\[
\int \chi_U \sum_{\omega \in U_\omega} |f_{T(\omega)}|^4 \lesssim \int \chi_U |f_{T(\omega)}|^4.
\]
Here \(f_{T(\omega)}\) has the similar definition as \(f_{T(\omega)}\) does. So, we have
\[
\int_{\mathbb{R}^3} \left| \sum_{\tau} g_{\tau, \text{high}} \right|^4 \leq (C_c K^{O(1)} (\gamma^{-2}\delta)^{-\varepsilon'})^4 \sum_{R^{-1/2} \lesssim s \leq 1} \sum_{\omega_s \in \Omega_s} \sum_{U \| U_\omega} |U|^{-1} \left( \int \chi_U |f_{\omega_s}|^4 \right)^2
\]
\[
\lesssim (C_c K^{O(1)} (\gamma^{-2}\delta)^{-\varepsilon'})^4 \sum_{R^{-1/2} \lesssim s \leq 1} \sum_{\omega_s \in \Omega_s} \int \chi_U |f_{\omega_s}|^8
\]
\[
\sim (C_c K^{O(1)} (\gamma^{-2}\delta)^{-\varepsilon'})^4 \sum_{R^{-1/2} \lesssim s \leq 1} \sum_{\omega_s \in \Omega_s} \int |f_{\omega_s}|^8
\]
\[
\lesssim (C_c K^{O(1)} (\gamma^{-2}\delta)^{-\varepsilon'})^4 \log \delta^{-1} \int_{\mathbb{R}^3} |f|^8,
\]
where the last inequality is by Lemma 5.12.
Combining the estimate for the low term (5.21), we just need to show
\[
C_c (\gamma K^{-1} \delta^{-1}) \varepsilon + C_c K^{O(1)} (\gamma^{-2} \delta^{-1}) \varepsilon' \log \delta^{-1} \leq \frac{1}{C} C_c (\gamma \delta^{-1}) \varepsilon
\]
in order to close the induction.
By choosing \(K\) large enough and \(\varepsilon' \ll \varepsilon\) (for example \(\varepsilon' = \varepsilon^2\)), we close the induction. \(\square\)

**Appendix A. Examples**

In the appendix, we give some examples. Before doing that, we discuss a property of wave packet which will be used to construct examples. Our argument here is heuristic, but is not hard to be made rigorous.

Let \(R\) be a rectangle in the frequency space \(\mathbb{R}^n_\omega\). After rotation, we may write it as \(R = c_R + \prod_{i=1}^n [-a_i, a_i]\), where \(c_R\) is the center of \(R\). Correspondingly, its dual rectangle is given by \(R^* = \prod_{i=1}^n [-1/a_i, 1/a_i] \subset \mathbb{R}^n_\omega\). One heuristic we will use in the rest of Appendix is:
\[
(\frac{1}{|R|})^1_{R^*}(x) \approx e^{2\pi i x \cdot c_R} 1_{R^*}(x).
\]
By adding a phase to \(1_R\), we also have
\[
(e^{-2\pi i \xi \cdot \frac{1}{|R|}} 1_R(\xi))^1_{R^*}(x) \approx e^{2\pi i (x - x_0) \cdot c_R} x_{0 + R^*}(x).
\]
In other words, there is a function whose support lie in \(R\) and whose inverse Fourier transform, after taking absolute value, is roughly the indicator function of a translation of \(R^*\).

**Remark A.1.** \(e^{2\pi i (x - x_0) \cdot c_R} x_{0 + R^*}(x)\) is referred to as a wave packet at \(x_0 + R^*\).
Let us discuss a trick called wave packet dilation. Given \( \theta' = c + \prod_{i=1}^{n}[a_i, a_i] \), we specify a direction, for example, \( e_n \) (equivalently, the \( \xi_n \)-direction), and then let \( \theta = c + \prod_{i=1}^{n-1}[a_i, a_i] \times [0, a_n] \) be the upper half of \( \theta' \). We see that the dual \( \theta^* \) is the 2-dilation of \( \theta'' \) along \( e_n \). Write \( \theta'' = \text{Dil}_2 e_n \theta^* \). When the direction of the dilation is clear (usually the direction is along the longest side or the second longest side), we just denote it by \( \theta'' = \text{Dil}_2 \theta^* \). When we look for the examples of the map \( T : f \mapsto (\hat{f}_\theta)^\vee \), the auxiliary rectangle \( \theta' \) will be very helpful. If we choose the test function \( f = \frac{1}{|\xi|}(e^{-2\pi i x \cdot \xi} 1_{\theta'} )^\vee \), then

\[
|f| \approx 1_T, \quad |Tf| \approx 1_{\text{Dil}_2 T},
\]

where \( T = x_0 + \theta'' \) is a translation of \( \theta'' \). It means that after the action of \( T \) on the single wave packet \( f \), the resulting new wave packet \( Tf \) is two times longer.

By using this idea, we can quickly give the sharp example for Bochner-Riesz conjecture. First, let us recall the Bochner-Riesz conjecture. Let \( N_{R^{-1}}(S^{n-1}) \) be the \( \delta \)-neighborhood of the unit sphere in \( \mathbb{R}^n \). Let \( 1_{N_{R^{-1}}(\partial A)} \) be a smooth bump function adapted to \( N_{R^{-1}}(S^{n-1}) \). Define the operator \( Sf := (1_{N_{R^{-1}}(\partial A)} f)^\vee \). We are interested in the following estimate

\[
(1.3) \quad \|Sf\|_{L^p(\mathbb{R}^n)} \lesssim R^{C_{n,p}} \|f\|_{L^p(\mathbb{R}^n)}.
\]

The conjecture is: \( (1.3) \) holds for \( p > \frac{2n}{n-1} \) and \( C_{n,p} > \frac{n-1}{p} \). We construct an example for \( (1.3) \). First, we write \( 1_{N_{R^{-1}}(\partial A)} = \sum_\theta 1_{\theta} \). Where \( \{\theta\} \) is a set of \( R^{-1/2} \times \cdots \times R^{-1/2} \times \delta \)-slabs that cover \( N_{R^{-1}}(S^{n-1}) \). For each \( \theta \), we define \( \theta' \) to be a \( R^{-1/2} \times \cdots \times R^{-1/2} \times 100^{-1} \)-slab that contains \( \theta \). Actually, \( \theta' \) is the 100-dilation of \( \theta \) along the normal direction of \( \theta \). Heuristically, we may assume \( \{\theta'\} \) are disjoint. For each \( \theta' \), we choose a function \( f_\theta \) such that \( \text{supp} f_\theta \subseteq \theta' \) and \( |f_\theta| \approx 1_{T_{\theta'}} \), \( |(1_{\theta'} f_\theta)^\vee| \approx 1_{\text{Dil}_2 T_{\theta'}} \), where \( T_{\theta'} \) is a \( R^{1/2} \times \cdots \times R^{1/2} \times 100^{-1} \)-tube dual to \( \theta' \). Now we just choose these tubes so that \( \{T_{\theta'}\} \) are disjoint and \( \{\text{Dil}_2 T_{\theta'}\} \) intersect at the origin.

We choose our example \( f = \sum_\theta a_\theta f_\theta \) where \( a_\theta \in \mathbb{C} \) are to be determined. Since \( \{T_{\theta'}\} \) are disjoint, we have \( \|f\|_p \approx \| \sum_\theta 1_{T_{\theta'}} \|_p \sim R^\frac{n-1}{p} \). Since \( \{\theta'\} \) are disjoint, we have \( Sf = \sum_\theta (1_{\theta'} f_\theta)^\vee = \sum_\theta (a_\theta 1_{\theta'} f_\theta)^\vee \). We can make it have a constructive interference at the unit ball \( B_1(0) \) by properly choosing \( a_\theta \). As a result, \( \|Sf\|_p \gtrsim \# \theta \sim R^{\frac{n-1}{2}} \). Plugging into \( (1.3) \) verifies \( C_{n,p} \geq \frac{n-1}{2} - \frac{n}{p} \).

A.1. We show that if we remove the “one-dimensional” condition in Theorem 1.3 then \( (1.3) \) is no longer true. See in Figure 3 where we plot our \( \{\Delta_i\} \) (blue \( \delta \)-cube) on the sphere \( S^{n-1} \) whose normal directions will be specified latter. There are also some red slabs of dimensions \( \delta \times \cdots \times \delta \times \delta^2 \), each of which is attached to only one \( \Delta_j \), i.e., one half of the red slab lies in \( \Delta_j \) and another half is outside \( \Delta_j \). We denote the slab attached to \( \Delta_j \) by \( \bar{R_j} \), and the set of them by \( \{R_j\} \). We consider the corresponding regions in \( \mathbb{R}^n \). Define

\[
\widetilde{\Delta}_j = \{\xi \in \mathbb{R}^n : |\xi|/|\xi| \in \Delta_j, 1 - \delta \leq |\xi| \leq 1\},
\]

\[
\bar{R_j} = \{\xi \in \mathbb{R}^n : |\xi|/|\xi| \in R_j, 1 - \delta \leq |\xi| \leq 1\}.
\]

We see that \( \{\widetilde{\Delta}_j\} \) is a set of \( \delta \)-cubes in \( \mathbb{R}^n \), and \( \{\bar{R_j}\} \) is a set of \( \delta \times \cdots \times \delta \times \delta^2 \)-slabs. Denote the normal direction of \( \bar{R_j} \) by \( \vec{n}_j \). We may choose \( \{\Delta_j\} \) and \( \{R_j\} \) carefully so that \( \{\vec{n}_j\} \) form a \( \delta \)-dense subset of \( S^{n-1} \).
Now, for each $\Delta_j$, we choose a function $f_j$ so that $\hat{f}_j \subset \tilde{R}_j$ and
\[
|f_j| \approx 1_{T_j}, \quad |S_j f_j| \approx |(1_{\tilde{\Delta}_j} \hat{f}_j)^\vee| \approx 1_{Dil_2 T_j},
\]
where the dilation $Dil_2 T_j$ is along $\vec{n}_j$, and $T_j$ is a $\delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2}$-tube dual to $\tilde{R}_j$. We choose the tubes $\{T_j\}$ so that they are disjoint and their dilations $\{Dil_{100} T_j\}$ intersect the origin. See in Figure 4 where the blue tubes are the $T_j$’s and they intersect $B_{\delta^{-1}}$ at the origin.

We choose the test function $f = \sum_j f_j$ and plug into
\[
\| (\sum_j |S_j f_j|^2)^{1/2} \|_p \lesssim \| f \|_p.
\]

Since $\{T_j\}$ are disjoint, we have $\|f\|_p \approx \| \sum_j 1_{T_j} \|_p \sim \delta^{-\frac{2n}{p'}}$. Since $\{\tilde{\Delta}_j\}$ are disjoint, we have $|S_j f| = |S_j f_j| \approx 1_{Dil_2 T_j}$. We have
\[
\| (\sum_j |S_j f_j|^2)^{1/2} \|_p \gtrsim \| (\sum_j 1_{T_j})^{1/2} \|_{L^p(B_{\delta^{-1}})} \gtrsim \#(\Delta_j)^{1/2} \delta^{-\frac{2}{p'}} \sim \delta^{-\frac{n+1}{2} - \frac{n}{p'}}.
\]

We see the constant $C$ in (1.4) should be greater than $\delta^{-\frac{n+1}{2} + \frac{n}{p'}}$ for which the threshold is $p = \frac{2n}{n+1}$.

A.2. Finally, we give the sharp example for Theorem 1.9. For a $\delta \times \delta^{1/2} \times 1$-slab $\tau$ contained in $N_\delta(\Gamma)$, we use $c(\tau)$, $n(\tau)$ and $t(\tau)$ to denote the light direction, normal direction and tangent direction of $\tau$. More precisely, if $\xi \in \tau \cap \Gamma \cap \{\xi_3 = 1\}$, then we roughly have $c(\tau) = (\xi_1, \xi_2, 1)$, $n(\tau) = (\xi_1, \xi_2, -1)$ and $t(\tau) = (-\xi_2,\xi_1,0)$. In the condition of the Theorem 1.9, we assumed $\tilde{f} \subset N_\delta(\Gamma)$, so we cannot dilate the wave packet in the $n(\tau)$-direction, which is the longest direction of $\tau^*$. However,
the “wave packet dilation” trick still works because we can dilate in the second longest direction \( t(\tau) \).

Let \( \{\tau\} \) be \( \delta \times \delta^{1/2} \times \delta \)-slabs contained in \( N_\delta(\Gamma) \) such that \( \{100\tau\} \) are disjoint. As what we did in the previous example, we can choose \( f = \sum_\tau f_{100\tau} \), such that
\[
\hat{f}_{100\tau} \subset 100\tau \quad \text{and} \quad |f_{100\tau}| \approx \frac{1}{P\tau},
\]
where the dilation \( \text{Dil}_{100}P_\tau \) is along \( t(\tau) \), and \( P_\tau \) is a \( 1 \times \delta^{1/2} \times \delta \)-plank dual to \( \tau \). Now, we carefully choose \( \{P_\tau\} \) so that \( \{P_\tau\} \) are disjoint but \( \{\text{Dil}_{100}P_\tau\} \) intersect the unit ball at the origin. See in Figure 5. We arrange \( \{P_\tau\} \) into the \( \delta^{-1/2} \)-neighborhood of the hyperboloid \( \{x \in \mathbb{R}^3: x_1^2 + x_2^2 - x_3^2 = \delta^{-1}\} \). Each \( P_\tau \) intersect \( \{x_3 = 0\} \) at a \( 1 \times \delta^{1/2} \)-rectangle lying in \( \{(x_1, x_2): \delta^{-1/2} \leq \sqrt{x_1^2 + x_2^2} \leq 2\delta^{-1/2}\} \) and pointing to the origin.

The total measure of all the planks is \( \delta^{-2} \), so \( \|f\|_p \approx \delta^{-\frac{2}{p}} \). On the other hand,
\[
\|(\sum_\tau |\hat{f}_{100\tau}^\ast|^2)^{1/2}\|_p \gtrsim \|(\sum_\tau 1_{\text{Dil}_{100}P_\tau})^{1/2}\|_{L^p(B_1(0))} \sim \delta^{-1/4}.
\]
Plugging into
\[
\|(\sum_\tau |\hat{f}_{100\tau}^\ast|^2)^{1/2}\|_p \leq C\|f\|_p,
\]
we get \( C \geq \delta^{-\frac{1}{4} + \frac{2}{p}} \), yielding that \( p = 8 \) is the critical exponent.
Figure 5. Planks arranged along a hyperboloid

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