Motion of dark solitons in a non-uniform flow of Bose-Einstein condensate

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We study motion of dark solitons in a non-uniform one-dimensional flow of Bose-Einstein condensate. Our approach is based on Hamiltonian mechanics applied to the particle-like behavior of dark solitons in a slightly non-uniform and slowly changing surrounding. In one-dimensional geometry, the condensate’s wave function undergoes the jump-like behavior across the soliton and this leads to generation of the counterflow in the background condensate. For correct description of soliton’s dynamics, the contributions of this counterflow to the momentum and energy of the soliton are taken into account. The resulting Hamilton equations are reduced to the Newton-like equation for the soliton’s path and this Newton equation is solved in several typical situations. The analytical results are confirmed by numerical calculations.

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The idea that localized solitary waves behave in external fields like point particles of the classical mechanics is very old and it was studied and used in a number of articles. However, its application to dynamics of dark solitons has some peculiarities which are not still fully understood. For example, from naive point of view a dark soliton in a cloud of Bose-Einstein condensate (BEC) is just a dip in the density distribution and, consequently, its motion must result in the motion of the center of mass in the opposite direction. In reality both experiments and numerical solutions of the Gross-Pitaevskii equation demonstrate different laws of motion of the dark soliton and the center of mass of the cloud of a quasi-one-dimensional condensate confined in a harmonic trap. This means that the motion of dark solitons is accompanied by some counterflow in the BEC cloud which contributes to the location of its center of mass. Correspondingly, the contributions of the counterflow to the momentum and the energy of dark solitons must be taken into account. Previously, this idea was applied to description of dark solitons motion in a quiescent condensate’s cloud. In this paper, we extend it to the general case of slightly non-uniform and slowly changing background distributions. The resulting Hamilton equations for the soliton’s motion are reduced to the Newton-like equation and this equation is solved for several typical situations of the background evolutions, such as a rarefaction wave, a self-similar expansion of the BEC cloud, a hydraulic flow of BEC past a wide obstacle. Solutions of the Newton equation agree very well with numerical solutions of the full Gross-Pitaevskii equation.

1. INTRODUCTION

The name of solitons was coined by Zabusky and Kruskal due to their particle-like behavior: two solitons interact elastically and pass through one another without losing their identity. This concept of solitons as particle-like nonlinear wave excitations was confirmed and amplified by the properties of solitons propagation in a non-uniform external field (see, e.g. [23]). In particular, bright soliton clouds of attractive BEC oscillate in the external trap potential \( U = m\omega_0^2 x^2 / 2 \) (\( m \) is an atomic mass) with the trap frequency \( \omega_0 \). However, behavior of dark solitons in repulsive BEC has some peculiarities. For example, in case of BEC confined in a quasi-one-dimensional (1D) harmonic potential trap, they propagate along a stationary Thomas-Fermi profile of the density and oscillate with the frequency \( \omega_0/\sqrt{2} \), although the center of mass of the condensate still oscillates with the trap frequency \( \omega_0 \) (see [6]). This means that the counterflow of the condensate should be taken into account. Such a counterflow is caused by the condensate’s phase jump across a dark soliton [18] and it leads to the difference between the canonical momentum \( p \) of the soliton quasi-particle and its ‘naive’ mechanical momentum [19]. As a result, the canonical momentum of a dark soliton, its energy \( \varepsilon \) and its velocity \( V \) are related by the classical formula

\[
V = \frac{\partial \varepsilon}{\partial p} \tag{1}
\]

of Hamiltonian mechanics, and this explicitly demonstrates that a dark soliton behaves like an effective particle (quasi-particle). In particular, conservation of the energy \( \varepsilon \) in a stationary field leads immediately [11] to the frequency \( \omega_0/\sqrt{2} \) of dark solitons oscillations in a harmonic potential. Similar approach yields easily the theory of evolution of ring solitons in cylindrically symmetrical trap [19], and this theory agrees very well with numerical simulations.

The aim of this paper is to extend the above approach to situations with a non-uniform and stationary or time-dependent flow of BEC. The Hamilton equations are reduced to the Newton-like equation and in this way we reproduce the results obtained in Ref [6] by the perturbation method as well as the results derived in Ref [13] from the soliton limit of the Whitham modulation equations. In a stationary case, the Hamiltonian approach provides the non-trivial energy conservation law.

We illustrate our approach by solutions of the dark soliton motion equation for such typical situations as its motion along a rarefaction wave [16] along a self-similar expanding cloud of BEC [14,15], along a stationary 1D flow of BEC past an obstacle...
in the subcritical regime. Solutions of the Newton equation agree very well with numerical solutions of the Gross-Pitaevskii (GP) equation.

II. HAMILTONIAN APPROACH TO DARK SOLITONS MOTION

One-dimensional dynamics of BEC is described with high accuracy by the GP equation which can be written in the non-dimensional form as
\[
\imath \psi_t + \frac{1}{2} \psi_{xx} - |\psi|^2 \psi = U(x) \psi,
\]
where \( U(x) \) is the external potential. Transition from the BEC wave function \( \psi \) to the more convenient hydrodynamic variables, namely, the condensate’s density \( \rho \) and its flow velocity \( u \), is performed by means of the substitution
\[
\psi(x,t) = \sqrt{\rho(x,t)} \exp \left(i \int_0^x u(x', t) \, dx'\right),
\]
so that the GP equation is cast to the system
\[
\rho_t + (\rho u)_x = 0, \quad u_t + uu_x + \rho_x = -U_x. \tag{4}
\]

We assume that the external potential \( U(x) \) forms a non-uniform distribution of the background density which changes at the characteristic distances of the order of magnitude much greater than the healing length of BEC. Since the typical soliton’s width has the order of magnitude of the healing length, in derivation of the soliton solution with account of the flow velocity we can neglect the potential term \( U(x) \) in the above equations. It is well known (see, e.g. [33]) that the traveling wave solution of Eqs. (4) without the potential term can be written in the form \( (\rho = \rho(\xi), \ u = u(\xi), \ \xi = x - Vt) \)
\[
\rho = \rho_1 + (\rho_2 - \rho_1) \frac{\sin^2 \left( \sqrt{\rho_3 - \rho_1} (x - Vt) \right)}{\sqrt{\rho_3 - \rho_1}}, \quad u = V \pm \frac{\sqrt{\rho_1 \rho_2 \rho_3}}{\rho}, \quad \rho_1 \leq \rho_2 \leq \rho_3,
\]
where
\[
m = \frac{\rho_2 - \rho_1}{\rho_3 - \rho_1}
\]
and \( \rho(\xi) \) oscillates in the interval
\[
\rho_1 \leq \rho \leq \rho_2. \tag{7}
\]
Two signs in the formula for \( u(\xi) \) correspond to two directions of the velocity of the background flow along which the wave propagates. We have the same density profiles for both directions of the flow.

We are interested in the soliton solution, so we turn to the soliton limit \( \rho_3 \to \rho_2 \) \((m \to 1)\) and obtain
\[
\rho = \rho_2 - \frac{\rho_2 - \rho_1}{\cosh^2 \left( \sqrt{\rho_2 - \rho_1} (x - Vt) \right)}, \quad u = V \pm \frac{\rho_2 \sqrt{\rho_1}}{\rho}. \tag{8}
\]

Far enough from the soliton’s center the distributions tend to their asymptotic background values \( \rho \to \rho_0, \ u \to u_0 \), that is \( \rho_1 = (V - u_0)^2 \), so we get the soliton solution in terms of the physical variables
\[
\rho = \rho_0 - \frac{\rho_0 - (V - u_0)^2}{\cosh^2 \left( \sqrt{\rho_0 - (V - u_0)^2} (x - Vt) \right)},
\]
\[
u = V - \frac{\rho_0(V - u_0)}{\rho_0 - (V - u_0)^2}.
\]

Consequently, the absolute value of the soliton’s speed relative to the BEC flow is smaller than the sound velocity \( c_0 = \sqrt{\rho_0} \) in a uniform condensate with the density \( \rho_0 \):
\[
|u_0 - c_0| \leq V \leq u_0 + c_0, \tag{10}
\]
where \( V \) is defined in the ‘laboratory’ reference system. As is clear from Eq. (9), the amplitude of the dark soliton is related with its velocity by the formula
\[
a = \rho_0 - (V - u_0)^2. \tag{11}
\]

We consider a dark soliton as a quasi-particle [32], that is a localized excitation propagating through the moving condensate. To describe dynamics of such a soliton, we need to find expressions for its energy and canonical momentum. As is known [2], the excitation of a dark soliton in BEC is accompanied by generation of the counterflow in the background condensate: this counterflow compensates the jump of the phase in BEC \( \Delta \phi = \int u dx \) corresponding to the solution (5). Consequently, the canonical momentum of a dark soliton must include the term resulting from the phase jump contribution. In case of the background at rest the expression for the canonical momentum reads
\[
p = -2(V \sqrt{\rho_0 - V^2} + 2 \rho_0 \text{arccos} \frac{V}{\sqrt{\rho_0}}. \tag{12}
\]

If the background moves with velocity \( u_0 \), then we have to subtract \( u_0 \) from the soliton’s velocity \( V \) to obtain the expression
\[
p = -2(V - u_0) \sqrt{\rho_0 - (V - u_0)^2} + 2 \rho_0 \text{arccos} \frac{V - u_0}{\sqrt{\rho_0}} \tag{13}
\]
for the canonical momentum of dark solitons in the moving background. Then from Eq. (1) we have
\[
\varepsilon = \int V dp = \int \frac{dp}{dV} dV
\]
and substitution of (13) followed by simple integration yields
\[
\varepsilon = \frac{4}{3} [\rho_0 - (V - u_0)^2]^{3/2} + uu_0 p = \varepsilon^{(0)} + uu_0 p, \tag{14}
\]
where \( \varepsilon^{(0)}(V) = \frac{4}{3} (\rho_0 - V^2)^{3/2} \) is the well-known expression for the dark soliton’s energy in the quiescent BEC. Formula
corresponds to the Galilean transformation of the quasi-particle’s energy in agreement with the Landau approach to the superfluidity theory. The last term in Eq. (14) can also be considered as the Doppler shift of the frequency due to motion of the medium.

So far we have considered dark solitons moving along a uniform condensate with the density \( \rho_0 \) and the flow velocity \( u_0 \). Now we assume that the distributions of the density \( \rho = \rho(x,t) \) and the flow velocity \( u = u(x,t) \) are slow functions of the space coordinate \( x \) and time \( t \), that is they change little along the distance of the order of magnitude of the soliton’s width. Then we can introduce the coordinate \( x = x(t) \) of the soliton and in the main approximation \( \rho(x(t),t), u(x(t),t) \) represent the density and the flow velocity of the background distributions at the soliton’s location. In this approximation, if we make the replacements

\[
\rho_0 \rightarrow \rho(x,t), \quad u_0 \rightarrow u(x,t), \quad V \rightarrow \dot{x}(t) = \frac{dx}{dt}
\]

in Eqs. (13) and (14), then we arrive at the expressions for the canonical momentum

\[
p = -2(\dot{x} - u)\sqrt{\rho - (\dot{x} - u)^2} + 2\rho \arccos \frac{\dot{x} - u}{\sqrt{\rho}}
\]

and the energy

\[
\varepsilon = \frac{4}{3} \left[ \rho - (\dot{x} - u)^2 \right]^{3/2} - 2u(\dot{x} - u)\sqrt{\rho - (\dot{x} - u)^2} + 2u\rho \arccos \frac{\dot{x} - u}{\sqrt{\rho}}
\]

(15)

(16)

of the particle-like dark soliton located at the moment of time \( t \) at the point \( x \), where \( \rho = \rho(x,t), u = u(x,t) \). Hamilton equations for the dark soliton motion have the standard form

\[
\frac{dx}{dt} = \left( \frac{\partial \varepsilon}{\partial p} \right)_x, \quad \frac{dp}{dt} = -\left( \frac{\partial \varepsilon}{\partial x} \right)_p,
\]

(17)

where it is implied that the velocity \( \dot{x} \) is excluded from the energy (16) with the use of Eq. (15).

Since it is impossible to express the energy (16) as a function of \( p \) and \( x \) in an explicit form, it is convenient to transform the Hamilton equations (17) to the Newton-like equation for the soliton’s path \( x(t) \). To this end, we differentiate (15) with respect to time \( t \) and find the expression for the left-hand side of the second equation (17),

\[
\frac{dp}{dt} = -4(\dot{x} - u_\varepsilon \dot{x} - u_\varepsilon)\sqrt{\rho - (\dot{x} - u)^2} + 2\rho_\varepsilon \arccos \frac{\dot{x} - u}{\sqrt{\rho}}.
\]

(18)

In Hamiltonian mechanics the velocity \( \dot{x} = V \) is considered as a function \( V = V(p,x) \) of the momentum \( p \) and coordinate \( x \) defined implicitly by Eq. (15). Therefore the derivative of the right-hand side of the second equation (17) can be written as

\[
\left( \frac{\partial \varepsilon}{\partial x} \right)_p = \left. \frac{\partial \varepsilon^{(0)}}{\partial \rho} \right|_{V,u} \rho_\varepsilon + \left. \frac{\partial \varepsilon^{(0)}}{\partial u} \right|_{V,\rho} u_\varepsilon

+ \left. \frac{\partial \varepsilon^{(0)}}{\partial V} \right|_{V,\rho} \frac{\partial V}{\partial x} + u_\varepsilon p,
\]

\[
\left( \frac{\partial \varepsilon}{\partial p} \right)_x = \left. \frac{\partial \varepsilon^{(0)}}{\partial \rho} \right|_{V,u} \rho_\varepsilon + \left. \frac{\partial \varepsilon^{(0)}}{\partial u} \right|_{V,\rho} u_\varepsilon + \left. \frac{\partial \varepsilon^{(0)}}{\partial V} \right|_{V,\rho} \frac{\partial V}{\partial x} + u_\varepsilon p,
\]

(19)

where \( \varepsilon^{(0)} = \frac{4}{3} \rho - (V - u)^2 \). The derivatives of \( \varepsilon^{(0)} \) with respect to \( \rho, u, V \) are calculated without any difficulty and the derivative \( \frac{\partial V}{\partial x} \) is to be calculated by differentiation of Eq. (15) with \( \dot{x} = V(p,x) \) under the condition that \( p = \text{const} \). After simple calculation we obtain

\[
\left( \frac{\partial V}{\partial x} \right)_p = u_\varepsilon + \frac{\rho_\varepsilon}{2\sqrt{\rho - (V - u)^2}} \arccos \frac{V - u}{\sqrt{\rho}}.
\]

(20)

Substitution of all derivatives into Eq. (19) followed by substitution of Eqs. (18) and (20) into the second equation (17) yields the equation

\[
2\ddot{x} = \rho_\varepsilon + (\rho u)_x u_\varepsilon + 2u_t

+ \frac{\rho_\varepsilon + (\rho u)_x}{\sqrt{\rho - (V - u)^2}} \arccos \frac{V - u}{\sqrt{\rho}}.
\]

(21)

This equation describes the dynamics of a soliton with account the source (pumping) and absorption of the condensate. In our case we assume that a dark soliton propagates along a smooth background whose evolution obeys the dispersionless limit of Eqs. (4),

\[
\rho_\varepsilon + (\rho u)_x = 0, \quad u_t + uu_\varepsilon + \rho_\varepsilon = -U_\varepsilon,
\]

(22)

we can cast Eq. (21) to the form

\[
2\ddot{x} = -2U_\varepsilon - \rho_\varepsilon + (\dot{x} - u)u_\varepsilon
\]

(23)

or

\[
2\ddot{x} = -U_\varepsilon + u_t + u_\varepsilon \dot{x} = -U_\varepsilon + \dot{u}.
\]

(24)

Equation (24) was derived in framework of the perturbation theory in Ref. 18.

Let us list several important particular cases. (i) If there is no external potential and a dark soliton propagates along a large scale wave obeying the Euler equations (22) with \( U = 0 \), then Eq. (24) gives at once the integral of motion

\[
2\ddot{x} = -U_\varepsilon + u_t \dot{x}
\]

(25)

which constant value is determined by the initial velocity \( V_0 \) and the value of the flow velocity \( u(x_0) \) at the initial point \( x_0 \) of the soliton’s path. This integral follows also from the soliton limit of the Whitham equations for the GP equation periodic solutions with slowly changing parameters (see Ref. 15).

(ii) If the flow is stationary, that is the distributions \( \rho = \rho(x), u = u(x) \) do not depend on time \( t \), then the motion equation

\[
2\ddot{x} = -U_\varepsilon + u_\varepsilon \dot{x}
\]

(26)

has the integral of energy \( \varepsilon(\dot{x},x) = \text{const} \), see Eq. (16), as it follows immediately from the Hamilton equations (17).

(iii) If there is no flow \( (u = 0) \) and a dark soliton propagates along the stationary Thomas-Fermi distribution \( p(x) + U(x) = \text{const} \), then its motion obeys the Newton equation

\[
2\ddot{x} = -U_\varepsilon.
\]

(27)
In case of condensate confined in a harmonic trap \( U(x) = \omega_0^2 x^2/2 \) such a soliton oscillates with the frequency \( \omega_0/\sqrt{2} \) as was predicted in Ref.\(^6\) and confirmed in the experiment.\(^{19,20}\)

Now we turn to consideration of several typical examples a dark soliton propagation along a non-uniform background.

### III. MOTION OF DARK SOLITONS ALONG A RAREFACTION WAVE

One of the simplest non-uniform time-dependent solutions of equations \((22)\) for the background flow with \( U = 0 \) is a rarefaction wave. Although the motion of dark solitons along such a wave has already been considered in Ref.\(^{13}\) by the Whitham method, we shall consider it briefly for completeness by our method with some additions and simplifications.

First, we introduce the Riemann invariants
\[
\gamma_+ = \frac{u}{2} + \sqrt{\rho}, \quad \gamma_- = \frac{u}{2} - \sqrt{\rho},
\]
and transform Eqs. \((22)\) with \( U = 0 \) to the diagonal Riemann form
\[
\frac{\partial \gamma_-}{\partial t} + v_- \frac{\partial \gamma_-}{\partial x} = 0, \quad \frac{\partial \gamma_+}{\partial t} + v_+ \frac{\partial \gamma_+}{\partial x} = 0,
\]
where
\[
v_- = \frac{1}{2} (3\rho_+ + \rho_-), \quad v_+ = \frac{1}{2} (\rho_+ + 3\rho_-).
\]

The physical variables are related with the Riemann invariants by the formulas
\[
u = \gamma_+ + \gamma_-, \quad \rho = \frac{1}{4} (\gamma_+ - \gamma_-)^2.
\]

A rarefaction wave is a particular self-similar solution of Eqs. \((29)\) with one of the Riemann invariants constant. To be definite, we assume that \( \gamma_- \) is constant and the self-similar rarefaction wave evolves from the initial discontinuity, located at \( x = 0 \), of the Riemann invariant \( \gamma_+ = u/2 + \sqrt{\rho} \). Let the density equal to \( \rho_- \) and the flow velocity equal to zero, \( u_- = 0 \) on the left side of the initial discontinuity. We denote the density on the right side of the discontinuity as \( \rho_+ \), \( \rho_+ > \rho_- \), and then from the condition that \( \gamma_- \) is constant across the discontinuity, \( \gamma_+ = u/2 - \sqrt{\rho} = -\sqrt{\rho_-} \), we find the flow velocity on the right from the discontinuity is equal to
\[
u_+ = 2(\sqrt{\rho_+} - \sqrt{\rho_-}).
\]

The first equation \((29)\) is already satisfied and the self-similar solution \( \gamma_+ = \gamma_+(x/t) \) of the second equation gives \( v_+ = (3\rho_+ + \rho_-)/2 = x/t \), or, with account of \( \gamma_- = -\sqrt{\rho_-} \), we get \( v_+ = u + \sqrt{\rho_-} \). As a result we obtain the distribution of the flow velocity in the rarefaction wave
\[
u(x,t) = \begin{cases} 0, & x < c_- t, \\ \frac{2}{3} (\sqrt{\rho_+} - c_-), & c_- t < x < (3c_+ - 2c_-) t, \\ \frac{2}{3} (\sqrt{\rho_+} - c_-), & x > (3c_+ - 2c_-) t, \end{cases}
\]
where \( c_\pm = \sqrt{\rho_\pm} \).

Let at the initial moment of time \( t = 0 \) a dark soliton be located on the right from the discontinuity at the point \( x_+ > 0 \), let it move with the velocity \( V_+ \), so that its amplitude equals to (see Eq. \((9)\))
\[
a_+ = \rho_+ - (V_+ - u_+)^2.
\]

Since the right edge of the rarefaction wave propagates with velocity \( 3c_+ - 2c_- \), it catches up the soliton at the moment
\[
t_+ = \frac{x_+}{3c_+ - 2c_- - V_+}
\]
provided the denominator of this formula is positive what will be assumed in what follows.

For \( t > t_+ \) the soliton moves along the rarefaction wave and we can use the conservation law \((25)\), which in our case becomes the linear differential equation
\[
\frac{dx}{dt} - \frac{x}{3t} = V_- - c_+ + \frac{2}{3} c_-.
\]

It can be readily solved with the initial condition \( x(t_-) = x_+ + V_+ t_- \), so we get the formula for the soliton’s path,
\[
x(t) = (V_+ - x_+ / 2t_-) t + \frac{3x_+}{2} \left( \frac{t}{t_-} \right)^{1/3}, \quad t > t_+.
\]

The soliton reaches the left edge of the rarefaction wave at the moment \( t_- \) defined by the condition \( x(t_-) = c_- t_- \) which gives
\[
t_- = \frac{1}{\sqrt{2x_+ / c_+ - V_+}} \left( \frac{x_+}{c_+ - V_+} \right)^{1/3},
\]

Consequently, the soliton passes through the rarefaction wave if only \( c_+ > V_+ \) and in this case Eq. \((25)\) gives at once that for \( t > t_- \) the soliton’s velocity becomes equal to \( V_- = c_- - (c_+ - V_+) \). Its amplitude \( a_- = \rho_- - V_-^2 \) can be transformed with the help of Eq. \((34)\) to
\[
a_- = a_+ - 2(c_+ - c_-) \left( c_- \pm \sqrt{c_+ - a_+} \right),
\]
where the choice of the sign is determined by the sign of the relative velocity \( V_- - u_+ \).

If \( c_+ = V_+ \), then the soliton’s path is given by
\[
x(t) = c_- t + \frac{3x_+}{2} \left( \frac{t}{t_-} \right)^{1/3}, \quad t > t_- = \frac{x_+}{2(c_+ - c_-)},
\]
that is it propagates ahead of the rear (left) edge of the rarefaction wave.

If \( c_+ < V_+ \), the soliton also remains forever inside the rarefaction wave, but this time its velocity tends according to Eq. \((36)\) to the constant value
\[
V_\infty = c_- + \frac{3}{2} (V_+ - c_+),
\]
which is greater than the velocity \( c_- \) of the left edge of the rarefaction wave. Along the limiting trajectory \( x = V_\infty t \) the background flow has the values \( u_\infty = V_+ - c_- \), \( \rho_\infty = [c_- + (V_+ - c_+)^2] / 2 \). Hence, the relative soliton’s velocity \( V_\infty - u_\infty \) is equal to the local sound velocity \( c_\infty = \sqrt{\rho_\infty} \), that is the soliton’s amplitude tends to zero in the limit \( t \to \infty \).
IV. DARK SOLITON’S MOTION ALONG EXPANDING CONDENSATE

Now we assume that the initial background state is formed by a harmonic potential \( U(x) = \omega_0^2 x^2 / 2 \) (in non-dimensional variables \( m = 1 \)) and the distribution of density is given by the Thomas-Fermi formula \( \rho(x,0) = (2\mu - \omega_0^2 x^2) / \sqrt{2\mu} \), where \( \mu \) is the chemical potential of BEC, and the initial flow velocity is equal to zero everywhere. At the initial moment of time \( t = 0 \) the trap is switched off and the condensate starts its expansion. The dispersionless equations (22) can be solved exactly in this case, the solution was found in Ref.\[13\] for the focusing NLS equation and it was adapted to the present repulsive BEC situation in Ref.\[15\]. It is given by the formulas

\[
\begin{align*}
\rho(x,t) &= \frac{1}{f(t)} \left( \mu - \frac{x^2}{\phi_0 f^2(t)} \right), \quad u(x,t) = xg(t) \\
g(t) &= f'(t) = \frac{2}{\phi_0} \sqrt{\frac{f(t) - 1}{f^2(t)}}, \quad \phi_0 = \sqrt{\frac{2}{\omega_0}},
\end{align*}
\]

(42)

where the function \( f(t) \) satisfies the differential equation

\[
\frac{df}{dt} = \frac{2}{\phi_0} \sqrt{\frac{f - 1}{f}}, \quad f(0) = 1
\]

(43)

and it can be defined in implicit form by the equation

\[
\begin{align*}
t(f) &= \frac{\phi_0}{2} \left[ \sqrt{f(f-1)} + \ln(\sqrt{f-1} + \sqrt{f}) \right].
\end{align*}
\]

(44)

Let the dark soliton be formed at the initial moment of time \( t = 0 \) at the point \( x = 0 \) with the velocity \( V_0 > 0 \). Then its path can be found by solving the equation (25) which in our case reads

\[
\frac{dx}{dt} - \frac{1}{2} g(t)x = V_0,
\]

(45)

with the initial condition \( x(0) = 0 \). This linear differential equation can be easily solved and with the use of the expressions for \( g(t) \) and \( df/dt \) the solution can be cast to the form

\[
x(f) = V_0 \phi_0 \sqrt{f(f-1)}.
\]

This formula together with Eq. (44) define the soliton’s path \( x = x(t) \) in parametric form. The soliton’s velocity is to be found from Eq. (45).

\[
\dot{x}(f) = V_0 \left( 2 - \frac{1}{f} \right).
\]

(47)

Typical condensate profiles at different moments of time \( t \) are shown in Fig. 1 for \( \mu = 1, \omega_0 = 0.1 \) and \( V_0 = 0.5 \). Plots of \( x(f) \) and \( V = \dot{x}(f) \) are shown in Fig. 2. In the limit \( t \to \infty \), when \( f \to \infty \), the velocity tends to its limiting value \( \dot{x}(\infty) = 2V_0 \) equal to the value of the flow velocity \( u = x/t = 2V_0 \) along the limiting path \( x(t) = 2V_0 t \). Consequently, at asymptotically large times the soliton becomes black and it is transferred by the flow.

V. DARK SOLITON’S MOTION ALONG A HYDRAULIC FLOW OF BEC THROUGH A PENETRABLE BARRIER

Now we shall consider a stationary flow of BEC past a wide barrier whose action is described by the potential \( U(x) \). The large scale dependence of the density \( \rho \) and the flow velocity \( u \) is determined by the stationary version of Eqs. (22),

\[
(\rho u)_x = 0, \quad uu_x + \rho_x + U = 0,
\]

(48)

which are readily integrated to give

\[
\rho u = \rho_0 u_0, \quad \frac{1}{2} u^2 + \rho + U = \frac{1}{2} u_0^2 + \rho_0
\]

(49)

where \( \rho_0 \) and \( u_0 \) are the density and the flow velocity of the condensate at \( x \to \pm \infty \). We imply here that the potential is
localized around the point \( x = 0 \) and its distance of action \( \sigma \) is much greater than the healing length which is equal to unity in our dimensionless notation. Elimination of \( \rho = \rho_0 u_0 / u \) yields the equation

\[
\frac{1}{2} \left[ u_0^2 - u^2(x) \right] + \rho_0 \left( 1 - \frac{u_0}{u(x)} \right) = U(x), \tag{50}
\]

which defines in implicit form the function \( u = u(x) \) for the space dependence of the background flow velocity. When it is found, the distribution of the background density is given by the formula

\[
\rho(x) = \frac{\rho_0 u_0}{u(x)}. \tag{51}
\]

Equation \( \text{(50)} \) is cubic and its solution should be chosen in such a way that it satisfies the boundary condition \( u(x) \to u_0 \) at both infinities \( x \to \pm \infty \). As was noticed in Refs. \[16,17\] this imposes a very important condition on possible values of \( u_0 \). Indeed, at the point where the potential \( U(x) \) reaches its maximal value \( U_m = \max \{ U(x) : -\infty < x < \infty \} \), the function in the left-hand side of Eq. \( \text{(50)} \) has the maximum equal to \( \frac{1}{2} u_0^2 - \frac{3}{2} (\rho_0 u_0)^{2/3} + \rho_0 \) at \( u(x) = u_m = (u_0 \rho_0)^{1/3} \). Consequently, the solution \( u(x) \) for all values of \( x, -\infty < x < \infty \), exists if only

\[
U_m \leq \frac{1}{2} u_0^2 - \frac{3}{2} (\rho_0 u_0)^{2/3} + \rho_0. \tag{52}
\]

This inequality becomes equality for \( u_0 \) equal to two roots \( u_\pm \) of the equation

\[
\frac{1}{2} u_0^2 - \frac{3}{2} (\rho_0 u_0)^{2/3} + \rho_0 = U_m. \tag{53}
\]

Then it is easy to find that the smooth solution \( u = u(x) \) exists for all \( x \) in the subcritical \( u_0 < u_- \) and supercritical \( u_0 > u_+ \) regimes of the flow past an obstacle described by the potential \( U(x) \). We assume that \( u_0 > 0 \), so the root \( u_- \) should be put equal to zero for \( U_m \geq \rho_0 \). If \( U_m \ll \rho_0 \), then \( u_0 \) must be close to the asymptotic value of the sound velocity \( \sqrt{\rho_0} \), so one can easily find that up to the second order in the small parameter \( (U_m / \rho_0)^{1/2} \) we have

\[
u_\pm \approx \sqrt{\rho_0} \left( 1 \pm \frac{3 U_m}{2 \rho_0} + \frac{U_m}{12 \rho_0} \right). \tag{54}
\]

In the transcritical regime \( u_- < u_0 < u_+ \), the condition \( \text{(52)} \) does not hold. In this case, the flow past a wide barrier leads to generation of dispersive shock waves in this finite interval of the flow velocities \[16,17\]. We are only interested in the situations when dispersive shock waves are not formed and the soliton passes through a stationary fairly smooth profile of the condensate.

FIG. 2. (a) Dark soliton’s path \( x(t) \) during its motion in the expanding condensate. (b) Velocity of the dark soliton \( V = \dot{x}(t) \) as a function of time. The red curves show the results of numerical calculations and the dash-dotted blue curves show the analytical solution. Dotted lines correspond to \( V = V_0 \) and dashed lines correspond to \( V = 2V_0 \). Here \( \mu = 1, \omega_0 = 0.1 \) and \( V_0 = 0.5 \).

FIG. 3. Soliton’s paths for fixed values of the parameters \( \rho_0, u_0 \) of the flow at infinity \( x \to \pm \infty \), the fixed potential barrier shape and different values of the soliton’s initial velocity \( V_0 \). At some critical value of the velocity \( V_0 \) the paths passing through the barrier change to the paths reflected from it. The plot corresponds to the values \( \rho_0 = 1.0, u_0 = 0.1, U_m = 0.4, \) and \( \sigma = 300 \) (see Eq. \( \text{(55)} \)).
We fix the parameters $\rho_0, u_0$ of the flow at infinity $x \to \pm \infty$ and calculate solitons paths for different values of the soliton’s initial velocity $V_0$ in the range from 0.5 to 0.8. At some critical value of the soliton’s velocity $V_0 \approx 0.61$ the paths switch from those passing through the barrier to the paths reflected from it. At the critical value $a_0 = a_{cr}$, which separates these two regimes, the trajectory reaches the point $x = 0$ of the potential maximum with zero velocity, that is $\dot{x}(x = 0) = 0$ at the point with $u(0) = u_m$, $\rho(0) = \rho_m$. Substitution of these values of into Eq. (16) gives the soliton’s critical energy

$$\varepsilon_{cr} = \frac{2}{3} \sqrt{\rho_m^3} \left( \frac{2\rho_m^3 + \rho_0^3 \rho_m^3}{\rho_m^3 - \rho_0^3 u_0^2} \right)$$

$$+ 2u_0 \rho_0 \left( \frac{\pi}{2} + \arcsin \frac{\rho_0 u_0}{\rho_m^{3/2}} \right).$$

Equating this soliton’s energy at the turning point to its initial energy, we arrive at the equation

$$\frac{4}{3} a_{cr}^{3/2} - 2u_0 \sqrt{a_{cr} (\rho_0 - a_{cr})} + 2u_0 \rho_0 \arcsin \sqrt{\frac{a_{cr}}{\rho_0}} = \varepsilon_{cr}. \tag{57}$$

If we define the variables

$$A = \frac{a_{cr}}{\rho_0}, \quad M = \frac{u_0}{\sqrt{\rho_0}}, \tag{58}$$

then we can write this equation in the form

$$M(A) = \frac{4A^{3/2} - 3\varepsilon_{cr}/\rho_0^{3/2}}{6 \left( \sqrt{A (1 - A)} - \arcsin \sqrt{A} \right)}. \tag{59}$$

It defines in implicit form the dependence of the non-dimensional critical amplitude on the Mach number of the flow far from the obstacle. In particular, for the values $\rho_0 = 1.0, u_0 = 0.1, U_m = 0.4$, we obtain $\rho_{min} = 0.591$ and Eq. (59)
yields $A_e = 0.738$ ($V_0 = 0.612$) in reasonable agreement with the value obtained in numerical solutions of the GP equation.

If we turn to the trajectories with reflection of solitons from the obstacle, then we see that the initial and final amplitudes of such a soliton are different due to change of sign of the velocity:

$$a_+ = \rho_0 - (V_+ - u_0)^2, \quad a_- = \rho_0 - (V_- - u_0)^2,$$

where both velocities are defined far enough from the obstacle. Let the soliton start its downstream motion at $x = -\infty$ with the velocity $V_+ > 0$, so its initial energy is given by the expression

$$\varepsilon = \frac{4}{3} a_+^{3/2} - 2u_0 \sqrt{a_+(\rho_0 - a_+)} + 2u_0 \rho_0 \arcsin \sqrt{\frac{a_+}{\rho_0}},$$

and it must be equal to its energy after reflection when it moves upstream with the velocity $V_- < 0$ so that its energy is given by

$$\varepsilon = \frac{4}{3} a_-^{3/2} + 2u_0 \sqrt{a_- (\rho_0 - a_-)} + 2u_0 \rho_0 \left( \pi - \arcsin \sqrt{\frac{a_-}{\rho_0}} \right).$$

Equating these two expressions and introducing again the variables $A_+ = a_+ / \rho_0$, $A_- = a_- / \rho_0$, $M = u_0 / \sqrt{\rho_0}$, we get the equation

$$\frac{2}{3} A_+^{3/2} + M \sqrt{A_-(1 - A_+)} + M \left( \pi - \arcsin \sqrt{A_-} \right) = \frac{2}{3} A_-^{3/2} - M \sqrt{A_+(1 - A_-)} + M \arcsin \sqrt{A_+},$$

which determines the final amplitude of the soliton in terms of its initial amplitude and the Mach number of the flow.

We compared our analytical findings with numerical solutions of the GP equation. An example of soliton’s propagation...
for a subcritical ($u_0 < u_-$) background flow velocity is shown in Fig. 4. The soliton moves downstream the condensate flow. Fig. 5 shows the soliton’s trajectory calculated according to Eq. (26) and by means of numerical solution of the GP equation (2). As one can see, the approximate analytical theory (dash-dotted blue line) agrees perfectly well with the exact numerical solution (red line). Similar agreement between two approaches was found also for supercritical flow of the background BEC and for the upstream initial motion of solitons.

Another case, when the soliton is reflected from the barrier, is illustrated in Fig. 6. The solution of Eq. (26) agrees perfectly well with the exact numerical solution of the GP equation, as is shown in Fig. 7. One can clearly see that the soliton’s amplitude changes after reflection. Fig. 8 shows the dependence of the amplitude on $t$. As one can see, the numerical amplitude of the soliton after the collision coincides with the analytical one obtained from Eq. (61).

VI. CONCLUSION

The motion of dark solitons is an interesting problem of nonlinear physics because such a motion cannot be completely separated from the motion of the background medium. Most spectacularly this phenomenon is demonstrated by the difference in the motion of the center of mass of BEC confined in a trap and the motion of the soliton itself described by its mean coordinate. Consequently, the soliton’s motion is accompanied by some counterflow in the background condensate which leads to a proper redistribution of the density. In 1D geometry such a counterflow is necessary for topological reasons: the flow velocity of the condensate is a gradient of the phase of the condensate’s wave function, so formation of a dark soliton leads to the jump of phase across the soliton and, to keep the wave function single-valued, this jump must be compensated by the flow outside the soliton.2 We show that the contribution of the counterflow into the momentum and the energy of a dark soliton is crucially important for correct description of its motion as a localized quasi-particle by the methods of Hamiltonian mechanics. In this way we have reproduced the results obtained earlier by the methods of perturbation theory3 and in framework of the Whitham theory4 (in the last case only for situations without external potential). The advantage of the Hamiltonian approach is that it provides immediately the quite non-trivial energy conservation law for the soliton’s motion in stationary background flows and it would be difficult to find this law by other methods. We have illustrated our approach by several examples and confirmed the analytical results by their comparison with numerical solutions of the GP equation.

We believe that the suggested here Hamiltonian approach can be applied to other interesting situations, such as, for example, interaction of solitons with non-convex flows or “magnetic” solitons in BEC, when the main contribution into the soliton’s canonical momentum is made by the counterflow.

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1N. J. Zabusky, M. D. Kruskal, Phys. Rev. Lett. 15, 240 (1965).
2V. I. Karpman, E. M. Maslov, Sov. Phys. JETP, 46, 281 (1977).
3V. I. Karpman, E. M. Maslov, Sov. Phys. JETP, 48, 252 (1978).
4D. J. Kaup and A. C. Newell, Proc. Roy. Soc. London, A 361, 413 (1978).
5A. M. Kosevich, Physica D 41, 253 (1990).
6Th. Busch, J. R. Anglin, Phys. Rev. Lett. 84, 2298-2301 (2000).
7S. I. Shevchenko, Sov. J. Low Temp. Phys. 14, 553 (1988).
8L. P. Pitaevskii, S. Stringari, Bose-Einstein Condensation, (Clarendon, Oxford, 2003).
9Yu. S. Kivshar, X. Yang, Phys. Rev. E 49, 1657 (1994).
10Yu. S. Kivshar, W. Krölikowski, Optics Commun., 114, 353-362 (1995).
11V. V. Konotop, L. P. Pitaevskii, Phys. Rev. Lett. 93, 240403 (2004).
12A. M. Kamchatnov, S. V. Korneev, Phys. Lett. A, 374, 4625 (2010).
13P. Sprenger, M. A. Hoefer, G. A. El, Phys. Rev. E 97, 032218 (2018).
14V. I. Talanov, JETP Lett., 2, 138 (1965).
15V. A. Brazhnyi, A. M. Kamchatnov, V. V. Konotop, Phys. Rev. A 68, 035603 (2003).
16V. Hakim, Phys. Rev. E 55, 2835 (1997).
17A. M. Leszczyszyn, G. A. El, Yu. G. Gladush, A. M. Kamchatnov, Phys. Rev. A 79, 063607 (2009).
18A. M. Kamchatnov, Nonlinear Periodic Waves and Their Modulations—An Introductory Course, (World Scientific, Singapore, 2000).
19Ch. Becker, S. Stellner, P. Soltan-Panahi, S. Dörscher, M. Baurett, E.-M. Richter, J. Kronjäger, K. Bongs, K. Sengstock, Nature Phys., 4, 496-501 (2008).
20A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, P. G. Kevrekidis, Phys. Rev. Lett., 101, 130401 (2008).
21K. van der Sande, G. A. El, M. A. Hoefer, J. Fluid Mech., 928, A21 (2021).
22C. Qu, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 116, 160402 (2016).
23L. P. Pitaevskii, Phys.-Uspekhi, 59, 1028 (2016).