Some Formulas for Invariant Phases of Unitary Matrices by Jarlskog

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Abstract

We describe calculations of Jarlskog’s determinant in the case of \( n = 3, 4 \) in detail. Next, we investigate some formulas for invariant phases of unitary matrices and derive some explicit relations of them.

1 Introduction

CP violation is expected in the standard model of particle physics with three or more families [1], [2]. Therefore it is an important problem that what is the measure of CP violation with such families which is invariant under the action of phase factors.

To construct invariants for matrix action, the determinant is a useful tool. In the previous paper [3], C. Jarlskog succeeded to define invariants of CP violation by using the determinant for commutator of the quark mass matrices. For the case of 3 families, it is relatively easy to calculate it. Moreover, in that case, her determinant is proportional to an invariant phase of unitary matrices. Then she discussed invariant quantities for 4 families by using projection operators and the trace of some matrices, but she did not deal with her determinant itself for \( n = 4 \) [4]. Therefore the problem is still remained. An approach to this problem is to use a parametrization for

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unitary matrices. See [5], [6], [7], [8] and their references within. Geometric constructions of them are also studied, see, for example, [9], [10], [11].

In this paper, we show the explicit calculation of Jarlskog’s determinant in the case of 4 families and we find that it is hard to use the determinant for investigations of CP violation in the case of \( n = 4 \). Next, we study Jarlskog’s invariant phases of unitary matrices instead of the determinant. we give some useful formulas for them and derive the detailed dependency of them which was simply described in the previous paper [6].

2 Jarlskog’s Determinant

Let \( H, H' \) be \( n \times n \)-matrices (“mass matrices”), their eigenvalues (“masses of quarks”) \( a_i, b_i \) (all multiplicities are 1), and their diagonalizations \( H = UDU^\dagger, H' = U'D'U'^\dagger \) (where \( U, U' \) are unitary matrices). Then, we define Jarlskog’s determinant \( \det[H, H'] \) as follows [3]; we put a “quark mixing matrix” \( V = U^\dagger U' \), then we have

\[
\det[H, H'] = \det(HH' - H'H) = \det(UDU^\dagger U'D'U'^\dagger UDU^\dagger) = \det(DU^\dagger U'D'U'^\dagger U - U^\dagger U'D'U'^\dagger UD) = \det(DV D^\dagger V^\dagger - V D^\dagger V^\dagger D). \tag{2.1}
\]

We denote \( V_{ij} \) as the components of the unitary matrix \( V \), then by a straightforward calculation, we have

\[
(DV D^\dagger V^\dagger - V D^\dagger V^\dagger D)_{ij} = (a_i - a_j) \sum_k b_k V_{ik} V_{jk}.
\]

(I) First we consider her determinant (2.1) in the case of \( n = 3 \). This is a result of Jarlskog [3].

Proposition 1 (Jarlskog).

\[
\det[H, H'] = 2i \ T \ B \ \text{Im} \ (V_{11} V_{22} \bar{V}_{12} \bar{V}_{21}) \tag{2.2}
\]

where

\[
T = \prod_{(i,j)=(1,2),(2,3),(3,1)} (a_i - a_j), \quad B = \prod_{(i,j)=(1,2),(2,3),(3,1)} (b_i - b_j).
\]
Proof

First, we have
\[
\det(DV D' V^\dagger - V D' V^\dagger D) = 2 \sum_{i,j=(1,2),(2,3),(3,1)} b_k V_{ik} \tilde{V}_{jk} - \sum_{k=1}^{3} (a_i - a_j) \sum_{k=1}^{3} b_k V_{jk} \tilde{V}_{ik}.
\]

Therefore, we obtain
\[
\det(DV D' V^\dagger - V D' V^\dagger D) / T = 2i \text{ Im} \prod_{(i,j)=(1,2),(2,3),(3,1)} \sum_{k=1}^{3} b_k (V_{ik} \tilde{V}_{jk} - V_{jk} \tilde{V}_{ik})
\]
\[
= 2i \text{ Im} \prod_{(i,j)=(1,2),(2,3),(3,1)} \left( \sum_{k=1}^{2} b_k V_{ik} \tilde{V}_{jk} + b_3 V_{i3} \tilde{V}_{j3} \right)
\]
\[
= 2i \text{ Im} \prod_{(i,j)=(1,2),(2,3),(3,1)} \left( \sum_{k=1}^{2} b_k V_{ik} \tilde{V}_{jk} + b_3 (\delta_{ij} - V_{i1} \tilde{V}_{j1} - V_{i2} \tilde{V}_{j3}) \right)
\]
\[
= 2i \text{ Im} \prod_{(i,j)=(1,2),(2,3),(3,1)} \left( \sum_{k=1}^{2} (b_k - b_3) V_{ik} \tilde{V}_{jk} \right).
\]
\[
= 2i \text{ Im} \left\{ \sum_{k_1=1}^{2} (b_{k_1} - b_3) V_{1k_1} \tilde{V}_{2k_1} \right\} \left\{ \sum_{k_2=1}^{2} (b_{k_2} - b_3) V_{2k_2} \tilde{V}_{3k_2} \right\} \left\{ \sum_{k_3=1}^{2} (b_{k_3} - b_3) V_{3k_3} \tilde{V}_{1k_3} \right\}
\]
\[
= 2i \text{ Im} \sum_{k_1, k_2, k_3=1}^{2} (b_{k_1} - b_3) (b_{k_2} - b_3) (b_{k_3} - b_3) V_{1k_1} \tilde{V}_{2k_1} V_{2k_2} \tilde{V}_{3k_2} V_{3k_3} \tilde{V}_{1k_3}
\]
\[
= 2i \text{ Im} \sum_{k_1, k_2=1}^{2} (b_{k_1} - b_3) (b_{k_2} - b_3) \left( b_{k_3} - b_3 \right)
\times \left\{ V_{1k_1} \tilde{V}_{2k_1} V_{2k_2} \tilde{V}_{1k_2} \left( \delta_{k_2 k_3} - \tilde{V}_{1k_2} x_{1k_3} - \tilde{V}_{2k_2} x_{2k_3} \right) \right\}
\]
\[
= 2i \text{ Im} \sum_{k_1, k_2=1}^{2} (b_{k_1} - b_3) (b_{k_2} - b_3)^2 V_{1k_1} \tilde{V}_{2k_1} V_{2k_2} \tilde{V}_{1k_2} V_{2k_2} \tilde{V}_{1k_2}
\]
\[
- 2i \text{ Im} \sum_{k_1, k_2, k_3=1}^{2} (b_{k_1} - b_3) (b_{k_2} - b_3) (b_{k_3} - b_3)
\times \left\{ V_{1k_1} \tilde{V}_{2k_1} V_{2k_2} \tilde{V}_{1k_2} x_{1k_3} \right\}^2 + V_{1k_1} \tilde{V}_{2k_1} x_{2k_3} \tilde{V}_{1k_3} \left\{ V_{2k_2} \right\}^2 \right\}.
\]
We calculate all sums explicitly and we remark that $V_{11} \bar{V}_{21} V_{22} V_{12} + V_{12} \bar{V}_{22} V_{21} V_{11}$ is real and its imaginary part vanishes, then we have

$$2i \text{ Im} \left\{ (b_1 - b_3) (b_2 - b_3)^2 V_{11} \bar{V}_{21} V_{22} V_{12} + (b_2 - b_3) (b_1 - b_3)^2 V_{12} \bar{V}_{22} V_{21} V_{11} \right\}$$

$$= 2i \text{ Im} \left\{ (b_1 - b_3) (b_2 - b_3)^2 V_{11} \bar{V}_{21} V_{22} V_{12} - (b_2 - b_3) (b_1 - b_3)^2 V_{11} \bar{V}_{21} V_{22} V_{12} \right\}$$

$$= 2i \text{ Im} \left\{ (b_1 - b_3) (b_2 - b_3)^2 - (b_2 - b_3) (b_1 - b_3)^2 \right\} V_{11} \bar{V}_{21} V_{22} V_{12}$$

Therefore we conclude the proof.

(II) Next, we consider (2.1) in the case of $n = 4$. As mentioned above, Jarlskog did not deal with her determinant (2.1) itself for $n = 4$. Therefore we calculate it directly.

Here we put

$$u_{ij} := (D V D' \bar{V})_{ij} = (a_i - a_j) \sum_k b_k \bar{V}_{ik} \bar{V}_{jk}, \quad \text{where } u_{ji} = -\bar{u}_{ij}.$$  

Then we have

$$\text{det} (D V D' \bar{V} - V D' \bar{V}) = \begin{vmatrix}
0 & u_{12} & u_{13} & u_{14} \\
-\bar{u}_{12} & 0 & u_{23} & u_{24} \\
-\bar{u}_{13} & -\bar{u}_{23} & 0 & u_{34} \\
-\bar{u}_{14} & -\bar{u}_{24} & -\bar{u}_{34} & 0
\end{vmatrix}$$

$$= u_{12} u_{34} \bar{u}_{12} \bar{u}_{34} + u_{13} u_{24} \bar{u}_{13} \bar{u}_{24} + u_{14} u_{23} \bar{u}_{14} \bar{u}_{23}$$

$$- \left\{ u_{12} u_{24} \bar{u}_{13} \bar{u}_{34} + u_{13} u_{34} \bar{u}_{12} \bar{u}_{24} + u_{14} u_{23} \bar{u}_{13} \bar{u}_{24} + u_{13} u_{24} \bar{u}_{14} \bar{u}_{23} \right\}$$

$$+ u_{14} u_{12} \bar{u}_{23} \bar{u}_{34} + u_{12} u_{23} u_{34} \bar{u}_{14}.$$
Here we put

\[
\begin{align*}
&= \left\{ \left( \prod_{(i,j)=(1,2),(3,4),(2,1),(4,3)} + \prod_{(i,j)=(1,3),(2,4),(3,1),(4,2)} + \prod_{(i,j)=(1,4),(2,3),(4,1),(3,2)} \right) \\
&\quad - \left( \prod_{(i,j)=(1,2),(2,4),(3,1),(4,3)} + \prod_{(i,j)=(1,3),(3,4),(2,1),(4,2)} + \prod_{(i,j)=(1,4),(2,3),(3,1),(4,2)} + \prod_{(i,j)=(1,3),(2,4),(4,1),(3,2)} \right) \right\} (a_i - a_j) \sum_{k=1}^{4} b_k V_{ik} \tilde{V}_{jk} \\
&= \Re \left\{ \left( \prod_{(i,j)=(1,2),(3,4),(2,1),(4,3)} + \prod_{(i,j)=(1,3),(2,4),(3,1),(4,2)} + \prod_{(i,j)=(1,4),(2,3),(4,1),(3,2)} \right) \\
&\quad - 2 \left( \prod_{(i,j)=(1,2),(2,4),(3,1),(4,3)} + \prod_{(i,j)=(1,3),(2,4),(4,1),(3,2)} + \prod_{(i,j)=(1,2),(2,3),(3,4),(4,1)} \right) \right\} (a_i - a_j) \sum_{k=1}^{4} b_k V_{ik} \tilde{V}_{jk}
\right. \\
&= (a_i - a_j) \sum_{k=1}^{4} b_k V_{ik} \tilde{V}_{jk}.
\end{align*}
\]

Here we put

\[
T_{(ij)(kl)} := (a_i - a_j)^2(a_k - a_l)^2, \quad T_{(ijkl)} := (a_i - a_j)(a_j - a_k)(a_k - a_l)(a_l - a_i)
\]

\[
B_{(ij)(kl)} := (b_i - b_j)^2(b_k - b_l)^2, \quad B_{(ijkl)} := (b_i - b_j)(b_j - b_k)(b_k - b_l)(b_l - b_i)
\]

\[
b_{k4} := b_k - b_4
\]

and we remark

\[
V_{i4} \tilde{V}_{j4} = \delta_{ij} - \sum_{k=1}^{3} V_{ik} \tilde{V}_{jk} = - \sum_{k=1}^{3} V_{ik} \tilde{V}_{jk} \quad (i \neq j),
\]
Here we introduce some notations:

\[ [\alpha; jk] := V_{\alpha j} V_{j k} \bar{V}_{\alpha k} \bar{V}_{\beta j}, \quad (a b c) = (a b c; k_1 k_2 k_3) := V_{\alpha k_1} \bar{V}_{b k_1} V_{c k_2} \bar{V}_{c k_3} \bar{V}_{\alpha k_3}. \]

Moreover by using a relation

\[ V_{j k} \bar{V}_{k i} = \delta_{k_i k_i} - V_{j k_1} \bar{V}_{k_1} - V_{j k_2} \bar{V}_{k_2} - V_{j k_3} \bar{V}_{k_3}, \]
then, for example, the coefficient of \( T_{(12)(34)} \) is

\[
\sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} V_{1k_1} \bar{V}_{2k_1} V_{3k_2} \bar{V}_{4k_2} V_{2k_3} \bar{V}_{1k_3} V_{4k_4} \bar{V}_{3k_4}
\]

\[
= \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_3] [34; k_2 k_4]
\]

\[
= \sum_{k_1, k_2, k_3=1}^3 b_{k_1} b_{k_2}^2 b_{k_3} [12; k_1 k_3] |V_{3k_2}|^2 - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_3] [31; k_2 k_4]
\]

\[
- \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_3] [32; k_2 k_4] - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_3] [33; k_2 k_4]
\]

\[
= \sum_{k_1, k_2, k_3=1}^3 b_{k_1} b_{k_2} b_{k_3}^2 [12; k_1 k_2] |V_{3k_3}|^2 - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_2] [13; k_3 k_4]
\]

\[
- \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_2] [23; k_3 k_4] - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_3] |V_{3k_2}|^2 |V_{3k_4}|^2
\]

Furthermore, the coefficient of \( -2T_{(1243)} \) is

\[
\sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} V_{1k_1} \bar{V}_{2k_1} V_{2k_2} \bar{V}_{4k_2} V_{3k_3} \bar{V}_{4k_4} \bar{V}_{3k_4}
\]

\[
= \sum_{k_1, k_2, k_3=1}^3 b_{k_1} b_{k_2}^2 b_{k_3} (123; k_1 k_2 k_3) - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_2] [31; k_3 k_4]
\]

\[
- \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} ((123; k_1 k_2 k_3) |V_{2k_2}|^2 + (123; k_1 k_2 k_3) |V_{3k_4}|^2)
\]

\[
= \sum_{k_1, k_2, k_3=1}^3 b_{k_1} b_{k_2} b_{k_3}^2 (312) - \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_2] [13; k_3 k_4]
\]

\[
- \sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} (123) (|V_{2k_2}|^2 + |V_{3k_4}|^2)
\]

Then, we can sum up the coefficient of \( -\sum_{k_1, \ldots, k_4=1}^3 b_{k_1} \cdots b_{k_4} [12; k_1 k_2] [13; k_3 k_4] \);

\[
T_{(12)(34)} + T_{(13)(24)} - 2T_{(1243)} = T_{(14)(23)}
\]
similary, the coefficient of $- \sum_{k_1, \ldots, k_4 = 1}^3 b_{k_14} \cdots b_{k_4} [12; k_1k_2] [23; k_3k_4]$;

$$T_{(12)(34)} + T_{(14)(23)} - 2T_{(1234)} = T_{(13)(24)},$$

and the coefficient of $- \sum_{k_1, \ldots, k_4 = 1}^3 b_{k_14} \cdots b_{k_4} [13; k_1k_2] [23; k_3k_4]$,

$$T_{(13)(24)} + T_{(14)(23)} - 2T_{(1324)} = T_{(12)(34)}$$

Therefore we obtain the following theorem:
Theorem 2.
\[
\det(DV D' V^\dagger - V D' V^\dagger D) = \text{Re} \left[ T_{(12)(34)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 \ [12; k_1 k_2] |V_{3k_3}|^2 \right.
\]
\[
+ T_{(13)(24)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 \ [13; k_1 k_2] |V_{2k_3}|^2 \right.
\]
\[
+ T_{(14)(23)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 \ [23; k_1 k_2] |V_{1k_3}|^2 \right.
\]
\[
- 2 \left( T_{(1243)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 (312) + T_{(1324)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 (132) + T_{(1234)} \sum_{k_1, k_2, k_3 = 1} b_{k_1} b_{k_2} b_{k_3}^2 (123) \right)
\]
\[
- \left\{ T_{(12)(34)} \sum_{k_1, k_2, k_3, k_4 = 1} b_{k_1} \cdots b_{k_4} \left( [13; k_1 k_2] [23; k_3 k_4] + [12; k_1 k_2] |V_{3k_3}|^2 |V_{3k_4}|^2 \right) \right. \right. \nonumber
\]
\[
+ T_{(13)(24)} \sum_{k_1, \cdots, k_4 = 1} b_{k_1} \cdots b_{k_4} \left( [12; k_1 k_2] [23; k_3 k_4] + [13; k_1 k_2] |V_{2k_3}|^2 |V_{2k_4}|^2 \right) \right. \nonumber
\]
\[
+ T_{(14)(23)} \sum_{k_1, \cdots, k_4 = 1} b_{k_1} \cdots b_{k_4} \left( [12; k_1 k_2] [13; k_3 k_4] + [23; k_1 k_2] |V_{1k_3}|^2 |V_{1k_4}|^2 \right) \right. \nonumber
\]
\[
- 2 \left( T_{(1243)} \sum_{k_1, \cdots, k_4 = 1} b_{k_1} \cdots b_{k_4} (312) (|V_{2k_3}|^2 + |V_{3k_4}|^2) \right. \right. \nonumber
\]
\[
+ T_{(1324)} \sum_{k_1, \cdots, k_4 = 1} b_{k_1} \cdots b_{k_4} (132) (|V_{1k_3}|^2 + |V_{2k_4}|^2) \right. \right. \nonumber
\]
\[
+ T_{(1234)} \sum_{k_1, \cdots, k_4 = 1} b_{k_1} \cdots b_{k_4} (123) (|V_{1k_3}|^2 + |V_{3k_4}|^2) \right) \right]\]

Remark We have a relation
\[
T_{(12)(34)} + T_{(13)(24)} + T_{(14)(23)} - 2T_{(1243)} - 2T_{(1324)} - 2T_{(1234)} = 0.
\]

However, we cannot sum up this determinant to more compact form any more. Therefore we conclude that, in case of \( n = 4 \), it is hard to use Jarlskog’s determinant for investigations of CP violation.
3 Invariant Phases of Unitary Matrices

To study CP violation, we need quantities which are invariant under the action

$$V \rightarrow \text{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n}) V \text{diag}(e^{i\theta'_1}, \cdots, e^{i\theta'_n}).$$

One of them is Jarlskog’s determinant \( \det[H, H'] \) and in the case of \( n = 3 \) it has a simple form

$$\det[H, H'] = 2i \, TB \, \text{Im} \, (V_{11}V_{22}V_{12}V_{21}).$$

However, as we showed in the previous section, the determinant is much complicated and it is hard to use it in the case of \( n \geq 4 \). Therefore, according to [3], we introduce invariant phases of unitary matrices.

**Definition 3.**

\[(\alpha\beta; jk) := \text{Im} \, (V_{\alpha j}V_{\beta k}V_{\alpha k}V_{\beta j}),\]

\[< \alpha\beta; jk > := \text{Re} \, (V_{\alpha j}V_{\beta k}V_{\alpha k}V_{\beta j}).\]

First we have

**Lemma 4.**

\[(\alpha\beta; kj) = -(\alpha\beta; jk), \quad (\beta\alpha; jk) = -(\alpha\beta; jk) \quad \text{antisymmetric w.r.t. } \alpha \text{ and } \beta, \ j \text{ and } k\]

\[< \alpha\beta; kj > = < \alpha\beta; jk >, \quad < \beta\alpha; jk > = < \alpha\beta; jk > \quad \text{symmetric w.r.t. } \alpha \text{ and } \beta, \ j \text{ and } k.\]

**Proof:** The proof is easy.

In case of \( n = 3 \), we have already showed \( \det[H, H'] = 2i \, TB \, (12; 12) \).

To investigate relations of \( (\alpha\beta; jk)s \) or \( < \alpha\beta; jk >s \), the following proposition is fundamental.

**Proposition 5.** (I) (Unitary relations of imaginary part)

\[\sum_{\alpha=1}^{n} (\alpha\beta; jk) = 0, \quad \sum_{\beta=1}^{n} (\alpha\beta; jk) = 0 \quad \text{(row unitary relations)},\]

\[\sum_{j=1}^{n} (\alpha\beta; jk) = 0, \quad \sum_{k=1}^{n} (\alpha\beta; jk) = 0 \quad \text{(column unitary relations)}.\]

(II) (Unitary relations of real part)

\[\sum_{\alpha=1}^{n} < \alpha\beta; jk > = \delta_{jk}|V_{\beta j}|^2, \quad \sum_{\beta=1}^{n} < \alpha\beta; jk > = \delta_{jk}|V_{\alpha j}|^2 \quad \text{(row unitary relations)},\]
\[
\sum_{j=1}^{n} < \alpha \beta; jk > = \delta_{\alpha \beta} |V_{ak}|^2, \quad \sum_{k=1}^{n} < \alpha \beta; jk > = \delta_{\alpha \beta} |V_{aj}|^2 \quad \text{(column unitary relations)}.
\]

Proof: We only prove \( \sum_{\alpha=1}^{n} (\alpha \beta; jk) = 0 \). Noting that unitary relations \( \sum_{\alpha=1}^{n} V_{\alpha j} \tilde{V}_{ak} = \delta_{jk} \), we have

\[
\sum_{\alpha=1}^{n} (\alpha \beta; jk) = \text{Im} \sum_{\alpha=1}^{n} (V_{\alpha j} \tilde{V}_{ak} \tilde{V}_{\beta j}) = \text{Im} \left( \sum_{\alpha=1}^{n} (V_{\alpha j} \tilde{V}_{ak}) \tilde{V}_{\beta j} \right)
\]

\[
= \text{Im} (\delta_{jk} \tilde{V}_{\beta j}) = 0.
\]

We can prove other relations in a similar way. Therefore we conclude the proof.

By using the proposition \( \text{in case of } n = 3 \), if we remark relations \((\alpha \beta; jj) = 0\), then we have

\[
(12; 13) = -(12; 11) - (12; 12) = -(12; 12), \quad (12; 23) = -(12; 21) = (12; 12), \\
(13; 12) = -(12; 12), \quad (13; 13) = -(13; 12) = (12; 12), \\
(13; 23) = -(13; 21) = (12; 21) = -(12; 12), \\
(23; 12) = -(21; 12) = (12; 12), \quad (23; 13) = -(23; 12) = (21; 12) = -(12; 12), \\
(23; 23) = -(21; 23) = (21; 21) = (12; 12).
\]

Therefore we have one independent invariant phase \((12; 12)\).

Next, in case of \( n = 4 \), because of \((\alpha, \beta), (j, k) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\), we have \( 6 \times 6 = 36 \) possibilities. In view of the theory for CP violation, we would like to find only three invariant phases in the case of \( n = 4 \). To show this, according to \( \text{[6]} \), we put

\[
R_{\alpha j} := < \alpha, \alpha + 1; j, j + 1 > \quad (\alpha, j = 1, 2, 3),
\]

\[
J_{\alpha j} := (\alpha, \alpha + 1; j, j + 1) \quad (\alpha, j = 1, 2, 3).
\]

Then we have the following proposition.
Proposition 6. We put
\[ J = [J_{\alpha j}] \{ \alpha = 1, 2, 3 \}, \quad A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \]
then
\[ 36 (\alpha \beta; jk) \text{ are expressed by} \]
\( J, JA, AJ, AJA. \)

More explicitly,
\[ [(\alpha \beta; jk)] \{ (\alpha j) = (12), (23), (34) \} = \begin{bmatrix} (12; 24) & (12; 14) & (12; 13) \\ (23; 24) & (23; 14) & (23; 13) \\ (34; 24) & (34; 14) & (34; 13) \end{bmatrix} = JA, \]
\[ [(\alpha \beta; jk)] \{ (\alpha j) = (24), (14), (13) \} = AJ, \quad [(\alpha \beta; jk)] \{ (\alpha j) = (24), (14), (13) \} = AJA. \]

Proof: By using the proposition 5 we have
\[ 0 = (12; 21) + (12; 23) + (12; 24) = -J_{11} + J_{12} + (12; 24) \]
\[ (12; 24) = J_{11} - J_{12}. \] (3.1)
\[ 0 = (12; 31) + (12; 32) + (12; 34) = -(12; 13) - J_{12} + J_{13} \]
\[ (12; 13) = -J_{12} + J_{13}. \] (3.2)

By using (3.1),
\[ 0 = (12; 41) + (12; 42) + (12; 43) = -(12; 14) - (12; 24) - J_{13} \]
\[ = -(12; 14) - J_{11} + J_{12} - J_{13} \]
\[ (12; 14) = -J_{11} + J_{12} - J_{13}. \] (3.3)

In a similar way, we find that
\[ 36 (\alpha \beta; jk) \text{ are expressed by the linear combination of} \]
\[ 9 J_{\alpha j} \text{ s. Therefore we conclude the proof.} \]

By this proposition 6, we have only to show
\[ 9 J_{\alpha j} \text{ as combinations of three of them, say} \]
\( J_{11}, J_{22}, J_{33}, \text{ we need more nonlinear relations between} \]
\[ \text{these invariant phases. The following proposition gives the relations of them.} \]
Proposition 7.  (I)
\[ < \alpha \beta ; jk > (\alpha \beta ; jl) + < \alpha \beta ; kl > (\alpha \beta ; jk) = < \alpha \beta ; kk > (\alpha \beta ; jk) \] (3.4)
\[ < \alpha \beta ; jk > (\beta \gamma ; jk) + < \beta \gamma ; jk > (\alpha \beta ; jk) = < \beta \beta ; jk > (\alpha \gamma ; jk) \] (3.5)

(II)
\[ < \alpha \beta ; jk > < \alpha \beta ; lm > - < \alpha \beta ; jm > < \alpha \beta ; kl > = (\alpha \beta ; jk)(\alpha \beta ; kl) \] (3.6)
\[ < \alpha \beta ; jk > < \gamma \delta ; jk > - < \alpha \delta ; jk > < \beta \gamma ; jk > = (\alpha \gamma ; jk)(\beta \delta ; jk) \] (3.7)

Proof: We can prove them by a straightforward calculation.

For example, we put \((\alpha \beta) = (12), j = 1, k = 2, l = 3\) in (3.4), then
\[
< 12; 12 > (12; 23) + < 12; 23 > (12; 12) = < 12; 22 > (12; 13)
\]
\[
R_{11}J_{12} + R_{12}J_{11} = |V_{12}V_{22}|^2(-J_{12} + J_{13})
\]
\[
-(|V_{12}V_{22}|^2 + R_{11})J_{12} + |V_{12}V_{22}|^2J_{13} = R_{12}J_{11}
\]

Thus, we obtain the relations in (6) as follows;
\[
\begin{bmatrix}
-(|V_{12}V_{22}|^2 + R_{11}) & |V_{12}V_{22}|^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(|V_{21}V_{22}|^2 + R_{11}) & 0 & 0 & 0 \\
|V_{52}V_{33}|^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & |V_{23}V_{33}|^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -R_{22}
\end{bmatrix}
\begin{bmatrix}
J_{12} \\
J_{13} \\
J_{21} \\
J_{31} \\
J_{32} \\
J_{33}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
R_{12}J_{11} \\
R_{23}J_{33} \\
R_{21}J_{11} \\
R_{32}J_{33} \\
(|V_{32}V_{33}|^2 + R_{32})J_{22} \\
(|V_{23}V_{33}|^2 + R_{23})J_{22}
\end{bmatrix}
\]

Since the rank of the coefficient matrix are generally six, we showed that 36 \((\alpha \beta; jk)s\) are expressed by \(J_{11}, J_{22}\) and \(J_{33}\).

4 Discussion

In this paper, we showed the explicit calculation of Jarlskog’s determinant in the case of 4 families and we realized that it was hard to use the determinant for investigations of CP violation in the case of \(n \geq 4\). Next, we studied Jarlskog’s invariant phases of unitary matrices instead of the determinant. Then we gave some useful formulas for them and derived the detailed dependency of them. Mathematically, a generalization of proposition (6) is an interesting problem. It is a future task.
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