Closed–open morphisms on periodic Floer homology

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Abstract

In this note, we investigate homomorphisms from the periodic Floer homology (PFH) to the quantitative Heegaard Floer homology. We call the homomorphisms closed–open morphisms. Under certain assumptions on the Lagrangian link, we first follow R. Lipshitz’s idea to give a cylindrical formulation of the quantitative Heegaard Floer homology. Then we construct the closed–open morphisms from the PFH to the quantitative Heegaard Floer homology. Moreover, we show that the morphisms are non–vanishing. As an application, we deduce a relation between the PFH–spectral invariants and the link spectral invariants.

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1 Introduction and Main results

Given a symplectic surface $(\Sigma, \omega)$ and a symplectic morphism $\varphi \in \text{Symp}(\Sigma, \omega)$, M. Hutchings introduces an invariant $\widetilde{PFH}(\Sigma, \varphi, \gamma_0)$ for the triple $(\Sigma, \omega, \varphi)$ called (twisted) periodic Floer homology $[13, 15]$ (abbreviated as PFH). Here $\gamma_0$ is a reference 1–cycle. PFH is a sister version of a more well–known invariant called embedded contact homology (abbreviated as ECH).

Recently, D. Cristofaro-Gardiner, V. Humilière and S. Seyfaddini extract a sequence numerical invariants from PFH, called PFH–spectral invariant. The PFH–spectral invariants satisfy a kind of “Weyl law” called Calabi property. It is an analogy of the “volume property” for ECH $[4]$. The Calabi property was first verified for a special class of Hamiltonian functions on the two–sphere $[10]$. The general case is proved by O. Edtmair and Hutchings $[12]$, also by D. Cristofaro-Gardiner, R. Prasad and B. Zhang independently $[7]$.

Let $\Lambda = \bigsqcup_{i=1}^{d} \Lambda_i$ be a link consisting of $d$ pairwise disjoint circles on $(\Sigma, \omega)$. Under certain monotonicity assumptions, D. Cristofaro-Gardiner, V. Humilière, C. Mak, S. Seyfaddini and I. Smith define another invariant called quantitative Heegaard Floer homology $[8]$, denoted by $HF(Sym^d \varphi(\Lambda), Sym^d \Lambda)$. It is a variant of the usual Heegaard Floer homology. For a suitable choice of symplectic form, $Sym^d \varphi(\Lambda)$ and $Sym^d \Lambda$ are Lagrangian submanifolds in $Sym^d \Sigma$. Roughly speaking, the quantitative Heegaard Floer homology is the Lagrangian Floer homology of $(Sym^d \varphi(\Lambda), Sym^d \Lambda)$. Similar as the PFH case, they extract a numerical invariant from the quantitative Heegaard Floer
homology called link spectral invariant. The link spectral invariants satisfy the parallel properties of the PFH–spectral invariants, especially the Calabi property.

It is interesting to study the relation of these two homological theories, especially the relation between the PFH–spectral invariants and link spectral invariants. To this end, our goal is to establish a homomorphism from the PFH to the quantitative Heegaard Floer homology which is an analogy of the usual closed–open morphism from symplectic Floer homology to Lagrangian Floer homology [1]. We still call these homomorphisms closed–open morphisms.

The closed–open morphism on PFH in fact is not a new story. A variant of such morphisms have been constructed in a slight different setting by V. Colin, P. Ghiggini, and K. Honda when they prove the equivalence between ECH and Heegaard Floer homology [11]. The ground of the results here has been set up in their works.

In this note, we obtain some partial results on constructing the closed–open morphisms. Let Σ be a closed surface with genus \( g \) and \( \omega \) a volume form of volume 1. Let \( \Lambda = \bigsqcup_{i=1}^{d} \Lambda_i \) be a \( d \)–disjoint union of simple closed curves in \( \Sigma \). We called \( \Lambda \) a link on \( \Sigma \). Throughout this note, we fix a constant \( \eta \geq 0 \) and assume that the link \( \Lambda \) satisfies the following properties:

A.1 \( \Lambda \) consists of \( k > 1 \) disjoint contractile components \( \bigsqcup_{i=1}^{k} \Lambda_i \) and \( g \) meridians, one for each genus. In particular, \( d = k + g \).

A.2 Let \( \Sigma - \Lambda = \bigsqcup_{i=1}^{k+1} \hat{B}_i \), where \( \hat{B}_i \) is a disk for \( 1 \leq i \leq k \) and \( \hat{B}_{k+1} \) is a planar domain with \( 2g + k \) boundary components. Let \( B_i \) denote the closure of \( \hat{B}_i \). For \( 1 \leq i \leq k \), each circle \( \Lambda_i \) is the boundary of \( B_i \).

A.3 \( \hat{B}_i \cap \hat{B}_j = \emptyset \) for \( i \neq j \).

A.4 For \( 0 \leq i < j \leq k \), we have \( \int_{B_i} \omega = \int_{B_j} \omega = \lambda \). Also, \( \lambda = 2\eta(2g + k - 1) + \int_{B_{k+1}} \omega \).

We the link satisfying the above conditions \( \eta \)-admissible. A typical picture of a link satisfying the above conditions is shown in Figure 2. Due to technical reasons, our assumptions here are much stronger than [5].

The quantitative Heegaard Floer homology is defined by counting holomorphic disks in a symplectic manifold of dimension \( 2d \) while the PFH is defined by counting holomorphic curves in a four–dimensional symplectic manifold. It is hard to establish the closed–open morphisms directly when \( d > 1 \). Therefore, the first task of this note is to follow R. Lipshitz’s idea [24] to define a homology \( HF_\ast(\Sigma, \varphi_H, \Lambda, \mathbf{x}) \) whose differential is defined by counting “open holomorphic curves” in a four–manifold. Moreover, we show that this homology is isomorphic to the quantitative Heegaard Floer homology. The precise statement is given as follows:
Theorem 1. Fix a base point \( x \in \text{Sym}^d \Lambda \). Given a non–degenerate Hamiltonian symplectic morphism \( \varphi_H \), we construct a homology \( HF_*(\Sigma, \varphi_H, \Lambda, x) \). Moreover, there is an isomorphism

\[
\Phi_H : HF_*(\Sigma, \varphi_H, \Lambda, x) \to HF_*(\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x)
\]

preserving the action filtration, where the latter is the quantitative Heegaard Floer homology defined in [8]. In particular, we have \( HF_*(\Sigma, \varphi_H, \Lambda, x) \cong H^*(T^d, F[T^{-1}, T]) \).

Then we construct the closed–open morphisms from PFH to the homology \( HF(\Sigma, \varphi_H, \Lambda, x) \).

Theorem 2. Fix a reference 1–cycle \( \gamma_0 \) with degree \( d \). Let \( Z_{\text{ref}} \in H_2(W, \gamma_0, x_H) \) be a reference relative homology class. Let \( (W, \Omega_H, L_{\Lambda_H}) \) be the closed-open symplectic cobordism defined in Section 5.1. Then for a generic admissible almost complex structure \( J \in J(W, \Omega_H) \), there exists a homomorphism

\[
(\text{CO}_{Z_{\text{ref}}} (W, \Omega_H, L_{\Lambda_H})_J)_* : \widetilde{PFH}(\Sigma, \varphi_H, \gamma_0)_J \to HF(\Sigma, \varphi_H, \Lambda, x)_J
\]

satisfying the following properties:

- (Partial invariance) Suppose that \( \varphi_H, \varphi_G \) satisfy the following conditions: (see Definition 2.1)

  - \( \spadesuit.1 \) Each periodic orbit of \( \varphi_H \) with degree less than or equal \( d \) is either \( d \)–negative elliptic or hyperbolic.
  - \( \spadesuit.2 \) Each periodic orbit of \( \varphi_G \) with degree less than or equal \( d \) is either \( d \)–positive elliptic or hyperbolic.

Let \( (X, \Omega_X) \) be the symplectic cobordism defined in [5.33]. Fix reference relative homology classes \( Z_{\text{ref}} \in H_2(X, \gamma_1, \gamma_0) \), \( Z_0 \in H_2(W, \gamma_0, x_G) \) and \( Z_1 \in H_2(W, \gamma_1, x_H) \) satisfying \( -(\pi_{\text{ref}}) Z_1 \# Z_{\text{ref}} = Z_0 \), where \( \pi_{\text{ref}} \in H_2(M, x_H, x_G) \) is the relative homology class defining \( I^H_{G, J_H} \). Then for generic admissible almost complex structures \( J_H \in J(W, \Omega_H) \) and \( J_G \in J(W, \Omega_G) \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\left(\text{PFH}_Z^{\text{sym}}(X, \omega_X)\right)_J & \cong & \left(\text{PFH}_Z^{\text{sym}}(X, \omega_X)\right)_J \\
\downarrow & & \downarrow \\
\left(\text{PFH}_Z^{\text{sym}}(X, \omega_X)\right)_J & \cong & \left(\text{PFH}_Z^{\text{sym}}(X, \omega_X)\right)_J
\end{array}
\]

Here \( \text{PFH}_Z^{\text{sym}}(X, \omega_X) \) is the cobordism map induced by \( (X, \Omega_X) \) and \( I^H_{G, J_H} \) is the continuous morphism on \( HF \) defined in Section 3.3.
• (Non–vanishing) If \( \varphi_H \) satisfies the condition \( \clubsuit.1 \), then there are classes \( \sigma_H \in PFH_*(\Sigma, \varphi_H, \gamma_0) \) and \( e \in HF_*(\Sigma, \Lambda) \) such that

\[
(\text{CO}_{\text{ref}}(W, \Omega_H, L_{\Lambda H}))_*(\sigma_H) = (j_H^\Lambda)^{-1}(e),
\]

where \( j_H^\Lambda \) is the canonical isomorphism \( \clubsuit.22 \) and \( e \neq 0 \in HF_*(\Sigma, \Lambda) \) is the unit of HF (see Definition \( \clubsuit.7 \)). In particular, the closed–open morphism is non–vanishing.

The class \( \sigma_H \) plays the role of unit in PFH; see Remark \( \clubsuit.1 \). A direct application of the closed–open morphisms is that we can deduce a relation on the PFH–spectral invariants and link spectral invariants. Let \( c_{\Lambda, \eta}^{\text{link}} \) and \( c_{\Lambda, \eta}^{\text{pfh}} \) denote the link spectral invariant and PFH spectral invariant.

**Corollary 1.1.** Suppose that the link \( \Lambda \) is 0-admissible. Fix a base point \( x \in \text{Sym}^d\Lambda \). Let \( \gamma_H^\Lambda := \Psi_H(S^1 \times \{x\}) \), where \( \Psi_H \) is the trivialization \( \clubsuit.2 \). Let \( \sigma_H^\gamma \in PFH_*(\Sigma, \varphi_H, \gamma_H^\Lambda) \) be the class defined \( \clubsuit.49 \). Then we have

\[
c_{\Lambda, \eta=0}^{\text{link}}(H, j_H^\Lambda \circ \Phi_H((j_H^\Lambda)^{-1}(e)) \leq c_{\sigma_H^\gamma}^{\text{pfh}}(H, \gamma_H^\Lambda) + \int_0^1 H_t(x)dt,
\]

where \( j_H^\Lambda \) is the canonical isomorphism \( \clubsuit.11 \) and \( H_t(x) \) is short for \( \sum_{i=1}^d H_t(x_i) \).

We expect that the class \( j_H^\Lambda \circ \Phi_H((j_H^\Lambda)^{-1}(e)) \) is unit of \( HF_*(\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x) \). But we cannot confirm this point in this paper because we don’t know whether the isomorphism in Theorem \( \clubsuit \) preserves the unit.

**Remark 1.1.** To keep the link spectral invariants with the same form of the PFH–spectral invariants, the definition here differs with the one in \([8]\) by a factor \( d \). More precisely, we have \( c_{\Lambda, \eta}^{\text{link}}(H, a) = dc_{\Lambda, \eta}(H, a) \), where \( c_{\Lambda, \eta}(H, a) \) is the link spectral invariant defined in \([8]\).

**Remark 1.2.** The HF–spectral invariant \( c_{\Lambda}^{\text{link}}(H, j_H^\Lambda \circ \Phi_H((j_H^\Lambda)^{-1}(e)) \) and the right hand side of \( (1.1) \) in fact are independent of the choice of \( x \). Also, to define the right hand side of \( (1.1) \), we can take \( x \in \text{Sym}^d\Sigma \) and not necessarily in \( \text{Sym}^d\Lambda \). For the details, please see the discussion in Section \( \clubsuit \).

**Remark 1.3.** Analogy to Theorem \( \clubsuit \), it is conjectured that the periodic Floer homology is isomorphic to the symplectic Floer homology of \( (\text{Sym}^d\Sigma, \text{Sym}^d \varphi) \) via the tautological correspondence. Provided that the conjecture is true, the closed–open morphisms defined here should be regarded as an alternative formulation of the usual closed–open morphisms defined in \([1]\). Also, Corollary 1.1 corresponds to Corollary 7.3 in \([8]\).

The following two remarks concerns the assumptions \( \clubsuit.1 \) and \( \clubsuit.2 \).
Remark 1.4. Due to certain technical issues, the cobordism maps on PFH cannot be defined by holomorphic curves so far. The current definition of the PFH cobordism maps relies heavily on the Seiberg–Witten theory [22] and Lee–Taubes’s isomorphism “SWF=PFH” [27]. However, in the proof of Theorem 2, we indeed need the holomorphic curve definition of the cobordism maps. The assumptions ♠.1 and ♠.2 are used to ensure that the cobordism maps on PFH can be defined in terms of holomorphic curves (see Theorem 2 of [9]). We expect that these two assumptions can be removed.

Remark 1.5. Fix a Hamiltonian function $H$, a metric $g_Y$ on $S^1 \times \Sigma$, and positive numbers $\delta > 0, d$. By Proposition 3.7 in [9], we can find a Hamiltonian function $H'$ satisfying the following properties:

1. The symplectomorphism $\varphi_H'$ satisfies the condition ♠.1).
2. $|H - H'| \leq \delta$ and $|dH - dH'|_{g_Y} \leq \delta$.

The condition ♠.1 can be replaced by ♠.2 depending on what we need in the context.

Coefficient Throughout this note, we use $F = \mathbb{Z}/2\mathbb{Z}$ coefficient.

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2 Preliminarily

2.1 Twisted periodic Floer homology

In this section, we review those aspects of periodic Floer homology that we need. For more details, please refer to [15, 16, 27].

Given $\varphi \in \text{Symp}(\Sigma, \omega)$, we define the mapping torus by

$$Y_{\varphi} := [0, 1] \times \Sigma / \sim, \ (1, x) \sim (0, \varphi(x)),$$

Obviously, $\pi : Y_{\varphi} \to S^1$ is a surface bundle over the circle. Let $\xi := \ker \pi_*$ denote the vertical bundle. The volume form $\omega$ and the vector field $\partial_t$ descend to a closed two form $\omega_{\varphi}$ and a vector field $R$ on $Y_{\varphi}$ respectively.

Given a Hamiltonian function $H : S^1 \times \Sigma \to \mathbb{R}$, we define the Hamiltonian vector field $X_H$ by $\omega(X_H, \cdot) = d\Sigma H$. Let $\varphi_H^t$ be the flow generated by $X_H$; it is called the Hamiltonian flow. Set $\varphi_H := \varphi_H^1$. A symplectomorphism $\varphi$ is called
Hamiltonian if \( \varphi = \varphi_H \) for some \( H \). Suppose that \( \varphi = \varphi_H \). Then we have the following global trivialization

\[
\Psi_H : S^1_t \times \Sigma \to Y_{\varphi H} \\
(t, x) \mapsto (t, (\varphi_H^t)^{-1}(x)).
\]

Moreover, \( \Psi_H^*(\omega_{\varphi H}) = \omega + dH \wedge dt \) and \( (\Psi_H^*)_t^{-1}(R) = \partial_t + X_H \). For our purpose, we assume that \( \varphi \) is a Hamiltonian symplectomorphism generated by some \( H \); unless otherwise stated.

A periodic orbit is a smooth map \( \gamma : \mathbb{R}/q\mathbb{Z} \to Y_\varphi \) satisfying the ODE \( \partial_t \gamma = R \circ \gamma \) for some \( q > 0 \). The number \( q \) is called period or degree of the periodic orbit. It is easy to show that \( q \) is the intersection number \( [\gamma] \cdot [\Sigma] \).

We say that a periodic orbit \( \gamma \) is non-degenerate if 1 is not an eigenvalue of the linearized return map. It is called elliptic if the eigenvalues of the linearized return map are on the unit circle, positive hyperbolic if the eigenvalues of are real positive numbers, and negative hyperbolic if the eigenvalues are real negative numbers. The symplectomorphism \( \varphi \) is called \( d \)-non-degenerate if if every closed orbit with degree at most \( d \) is non-degenerate. We assume that the symplectomorphism is \( d \)-non-degenerate throughout.

An orbit set \( \alpha = \{ (\alpha_i, m_i) \} \) is a finite collection of periodic orbits, where \( \alpha_i \) are distinct, non-degenerate, irreducible embedded periodic orbits and \( m_i \) are positive integers. An orbit set \( \alpha \) is called a PFH generator if \( m_i = 1 \) whenever \( \alpha_i \) is a hyperbolic orbit. Sometime we write an orbit set using multiplicative notation \( \alpha = \Pi_i \alpha_i \); we allow \( \alpha_i = \alpha_j \) for \( i \neq j \).

The terminology in the assumptions \( \clubsuit.1 \) \( \clubsuit.2 \) are explained in the following definition.

Definition 2.1. (see \cite{17} Definition 4.1) Fix \( d > 0 \). Let \( \gamma \) be an embedded elliptic orbit with degree \( q \leq d \).

- \( \gamma \) is called \( d \)-positive elliptic if the rotation number \( \theta \) is in \((0, \frac{q}{d}) \mod 1\).
- \( \gamma \) is called \( d \)-negative elliptic if the rotation number \( \theta \) is in \((-\frac{q}{d}, 0) \mod 1\).

Holomorphic currents Let \( (X, \Omega_X) = (\mathbb{R}_s \times Y_\varphi, \omega_\varphi + ds \wedge dt) \) be the symplectization of \( Y_\varphi \). An almost complex structure is called admissible if \( J(\partial_s) = R \), \( J \) preserving \( \xi \) and \( J|_\xi \) is compatible with \( \omega|_\xi \). The set of admissible almost complex structures is denoted by \( \mathcal{J}(Y_\varphi, \omega_\varphi) \).

A holomorphic current \( C = \sum_a d_a C_a \) from \( \alpha \) to \( \beta \) is a finite formal sum of holomorphic curves such that \( C \) is asymptotic to \( \alpha \) as \( s \to \infty \) and asymptotic to \( \beta \) as \( s \to -\infty \).
as \( s \to -\infty \) in current sense, where \( C_a \) are distinct, irreducible, somewhere injective \( J \)-holomorphic curves with finite energy \( \int_{C_a} \omega_J < \infty \) and \( d_a \) are positive integers.

To distinguish with the HF–curves defined later, we call a holomorphic curve in \((X, \Omega_X)\) a PFH–curve.

**The ECH index** Given orbit sets \( \alpha = \{(\alpha_i, m_i)\} \) and \( \beta = \{(\beta_j, n_j)\} \) on \( Y_\varphi \), define \( H_2(Y_\varphi, \alpha, \beta) \) to be the set of 2–chains in \( Y_\varphi \) such that \( \partial Z = \sum_i m_i \alpha_i - \sum_j n_j \beta_j \), modulo the boundary of 3–chains. Note that \( H_2(Y_\varphi, \alpha, \beta) \) is an affine space over \( H_2(Y_\varphi, Z) \cong \mathbb{Z}[\alpha] \oplus (H_1(S^1, \mathbb{Z}) \otimes H_1(\Sigma, \mathbb{Z})) \). An element in \( H_2(Y_\varphi, \alpha, \beta) \) is called a relative homology class.

Given \( Z \in H_2(Y_\varphi, \alpha, \beta) \) and trivializations \( \tau \) of \( \xi|_{\alpha} \) and \( \xi|_{\beta} \), the ECH index is defined by

\[
I(\alpha, \beta, Z) = c_\tau(\xi|Z) + Q_\tau(Z) + \sum_i \sum_{p=1}^{m_i} CZ_\tau(\alpha_i^p) - \sum_j \sum_{q=1}^{n_j} CZ_\tau(\beta_j^q),
\]

where \( c_\tau(Z) \) and \( Q_\tau(Z) \) are respectively the relative Chern number and the relative self–intersection number (see [14, 16]), and \( CZ_\tau \) is the Conley–Zehnder index. The ECH index also can be defined for punctured holomorphic curves in a general symplectic cobordism [14].

**Fredholm index** Let \( u : C \to X \) be a \( J \)-holomorphic curve from \( \alpha = \{(\alpha_i, m_i)\} \) to \( \beta = \{(\beta_j, n_j)\} \). For each \( i \), let \( k_i \) denote the number of ends of \( u \) at \( \alpha_i \), and let \( \{p_{a_i}\}_{a=1}^{k_i} \) denote their multiplicities. Likewise, for each \( j \), let \( l_j \) denote the number of ends of \( u \) at \( \beta_j \), and let \( \{q_{j_b}\}_{b=1}^{l_j} \) denote their multiplicities. Then the Fredholm index of \( u \) is defined by

\[
\text{ind}_u = -\chi(C) + 2c_\tau(u^*\xi) + \sum_{a=1}^{k_i} \sum_{p=1}^{m_i} CZ_\tau(\alpha_i^{p_a}) - \sum_{b=1}^{l_j} \sum_{q=1}^{n_j} CZ_\tau(\beta_j^{q_b}).
\]

**\( J_0 \) index** Besides the ECH index, Hutchings also introduces another topological index \( J_0 \) which measures the Euler characteristic of the holomorphic curves [14]. Given \( Z \in H_2(Y_\varphi, \alpha, \beta) \), the \( J_0 \) index is defined by

\[
J_0(\alpha, \beta, Z) = -c_\tau(\xi|Z) + Q_\tau(Z) + \sum_i \sum_{p=1}^{m_i} CZ_\tau(\alpha_i^p) - \sum_j \sum_{q=1}^{n_j} CZ_\tau(\beta_j^q).
\]

**Partition conditions** Given a simple periodic orbit \( \gamma \) and a holomorphic curve \( C \), suppose that \( C \) has positive ends at covers of \( \gamma \) whose total covering multiplicity is \( m \). The multiplicities of these covers form a partition of the integer \( m \), denoted by \( p_+(\gamma^m, C) \). Similarly, we define \( p_-(\gamma^m, C) \) for negative ends.
Definition 2.2. Let $u : C \to X$ be a $J$-holomorphic curve from $\alpha$ to $\beta$ without $\mathbb{R}$-invariant cylinders. We say that $u$ satisfies the ECH partition conditions if $p_+(\gamma^m, C) = p_+(m, \gamma)$ and $p_-(\gamma^m, C) = p_-(m, \gamma)$, where $p_+(m, \gamma)$ and $p_-(m, \gamma)$ are defined in [13, 16].

Definition 2.3. A connector $u : C \to X$ is a union of branched covers of trivial cylinders. A connector is called trivial if it is a union of unbranched covers of trivial cylinders; otherwise, it is called non–trivial.

The precise definitions of $p_+(m, \gamma)$ and $p_-(m, \gamma)$ are not required in this note. We only need following facts about the connectors and ECH partition conditions.

F.1 Let $\gamma$ be an elliptic orbit. For any partition $p(m, \gamma)$, there are no nontrivial $ind = 0$ connectors from $p_+(m, \gamma)$ to $p(m, \gamma)$. Likewise, there are no nontrivial $ind = 0$ connectors from $p(m, \gamma)$ to $p_-(m, \gamma)$.

F.2 Suppose that $u$ is a simple holomorphic curve. Then $u$ satisfies the ECH partition conditions if and only if $I(u) = ind(u)$. For more details, see [13, 14]. This property also holds for the PFH–HF curves defined later; see Theorem 5.6.9 of [11].

F.3 The connector has non–negative Fredholm index.

Definition of periodic Floer homology Now let us return to the definition of PFH. Fix a reference 1–cycle $\gamma_0$ transversed to $\xi$. We require that $d(\gamma_0) > g(\Sigma)$, where $d(\gamma_0) := [\gamma_0] \cdot [\Sigma]$ is the degree of $\gamma_0$.

The PFH chain complex $\widetilde{PFH}(\Sigma, \varphi, \gamma_0)$ is a free module generated by $(\alpha, [Z])$, where $\alpha$ is a PFH generator with degree $d = d(\gamma_0)$, and $[Z]$ is an equivalent class in $H_2(Y_{\varphi}, \alpha, \gamma_0)/\ker \omega_{\varphi}$. Note that $H_2(Y_{\varphi}, \alpha, \gamma_0)/\ker \omega_{\varphi}$ is an affine space over $\mathbb{Z}[\Sigma]$.

Fix a generic $J \in \mathcal{J}(Y_{\varphi}, \omega_{\varphi})$ on $X$. Let $\mathcal{M}^J_1(\alpha, \beta, Z)$ denote the moduli space of holomorphic currents from $\alpha$ to $\beta$ with ECH index $I = i$ and relative homology class $Z$. The PFH differential is defined by

$$\partial_J(\alpha, [Z]) = \sum_{(\beta, [Z'])} \left( \# \mathcal{M}^J_1(\alpha, \beta, Z - Z')/\mathbb{R} \right) (\beta, [Z'])$$

Note that the right hand side of the above equation is a finite sum because $[\gamma_0]$ is negative monotone with respective to $[\omega_{\varphi}]$ in the sense of Definition 1.1 of [27]. The obstruction gluing argument in [19, 20] show that $\partial_J^2 = 0$. The homology of $(\widetilde{PFH}(\Sigma, \varphi, \gamma_0), \partial_J)$ is called twisted periodic Floer homology, denoted by $\widetilde{PFH}(\Sigma, \varphi, \gamma_0)_J$.

By Lee and Taubes’s isomorphism [27], we know that $PFH(\Sigma, \varphi, \gamma_0)_J$ is independent of the choices of $J$ and the Hamiltonian isotropy of $\varphi$. 

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Let $\gamma_1$ be another choice of reference 1-cycle with $d(\gamma_1) = d(\gamma_0)$. Fix a relative homology class $Z_0 \in H_2(Y, \gamma_0, \gamma_1)$. Then $Z_0$ induces an isomorphism

$$\Psi_{Z_0} : \widetilde{PFH}_*(\Sigma, \varphi, \gamma_0) \to \widetilde{PFH}_*(\Sigma, \varphi, \gamma_1) \quad (2.6)$$

by sending $(\alpha, [Z])$ to $(\alpha, [Z + Z_0])$.

**Cobordism maps** Let $(X, \Omega_X)$ be a complete symplectic manifold such that

$$(X-K, \Omega_X) \cong (\{R_0, \infty\} \times Y_{\varphi_+}, \omega_{\varphi_+} + ds \wedge dt) \cup ((-\infty, -R_0] \times Y_{\varphi_-}, \omega_{\varphi_-} + ds \wedge dt), \quad (2.7)$$

where $K \subset X$ is a compact subset. Fix a relative reference homology class $Z_{ref} \in H_2(X, \gamma_0, \gamma_1)$. Then $(X, \Omega_X, Z_{ref})$ induces a homomorphism

$$PFH_{Z_{ref}}^w(X, \Omega_X) \to \widetilde{PFH}(\Sigma_+, \varphi_+, \gamma_0) \to \widetilde{PFH}(\Sigma_-, \varphi_-, \gamma_1).$$

Such a homomorphism is called a **cobordism map**. The superscript “$sw$” indicates that the cobordism map is defined by the Seiberg–Witten equations via the isomorphism “$SWF=PFH$” in [27]. Even these maps are defined by Seiberg–Witten equations, they ensure that the existence of holomorphic curves in $X$ when the cobordism maps are non–trivial. These crucial properties are called **holomorphic axioms**. We put the precise statements in the appendix (see Theorem A.1).

In this note, we only use a special case that $\pi_X : X \to \mathbb{R} \times S^1$ is a surface bundle over the cylinder (see (5.33)). An $\Omega_X$–compatible almost complex structure $J_X$ is called **admissible** if it satisfies

1. $J_X$ agrees with some admissible almost complex structures $J_\pm \in \mathcal{J}(Y_{\varphi_\pm}, \omega_{\varphi_\pm})$ over the cylindrical ends;

2. (Compatible with the fibration) $J_X$ preserves $TX^{vert}$ and $TX^{hor}$, where $TX^{vert} = \ker(\pi_X)_*$ and $TX^{hor}$ is the $\Omega_X$–orthogonal complement.

Suppose that $\varphi_{H_+}$ satisfies $\blacklozenge.1$ and $\varphi_{H_-}$ satisfies $\blacklozenge.2$. Fix a generic admissible almost complex structure $J_X$. According to Theorem 2 in [9], we can define the cobordism map

$$PFH_{Z_{ref}}^w(X, \Omega_X)_{J_X} : \widetilde{PFH}(\Sigma, \varphi_{H_+}, \gamma_0) \to \widetilde{PFH}(\Sigma, \varphi_{H_-}, \gamma_1).$$

alternatively by counting embedded $J_X$–holomorphic curves with $I = 0$. We remove the superscript “$sw$” to indicate that the map is defined by counting holomorphic curves. Moreover, the holomorphic curve definition coincides with the Seiberg–Witten definition (see Theorem 3 of [9]).
PFH–spectral invariants  The PFH complex admits a natural action functional. It induces a filtration on the complex and gives us the PFH–spectral invariants \[5, 12\]. Imitating the action functional in \[8\](2.14), here we define the action functional on the PFH complex perturbed by the \(J_0\) index.

Let \(H\) be a Hamiltonian function such that \(\varphi = \varphi_H\) is non–degenerate. Let \(\eta \geq 0\) be the fixed constant. Given a generator \((\alpha, [Z])\), its action is defined by

\[
\mathcal{A}_H^{\eta}(\alpha, [Z]) = \int_Z \omega_{\varphi_H} + \eta J_0(Z).
\]

When \(\eta = 0\), we simply write \(\mathcal{A}_H = \mathcal{A}_H^{\eta=0}\). Define a complex \(\tilde{PF}C_L(\Sigma, \varphi_H, \gamma_0)\) generated by the generators \((\alpha, [Z])\) with \(\mathcal{A}_H^{\eta} < L\). To see this is a subcomplex, we need the following lemma.

**Lemma 2.4.** Let \(J \in \mathcal{J}(Y_\varphi, \omega_\varphi)\) be an admissible almost complex structure in the symplectization of \(\mathbb{R} \times Y_\varphi\). Let \(C \in M^J(\alpha_+, \alpha_-)\) be a holomorphic current in \(\mathbb{R} \times Y_\varphi\) without closed component. Then \(J_0(C) \geq 0\).

**Proof.** Let \(C\) be a simple \(J\)-holomorphic curve. By Corollary 6.11 of \[14\], we have

\[
J_0(C) \geq 2(g(C) - 1 + \delta(C)) + |\alpha_+| + |\alpha_-|,
\]

where \(|\alpha_\pm|\) are the quantities in Definition 6.4 of \[14\]. By definition, \(|\alpha_\pm| \geq 1\) if \(\alpha_\pm\) is nonempty. Assume that \(C\) is not closed. Thanks to the fibration structure, \(C\) has at least one positive end and one negative end. Thus, \(|\alpha_\pm| \geq 1\). In particular, \(J_0(C) \geq 0\).

If \(C = \mathbb{R} \times \gamma\) is a trivial cylinder, then \(J_0(mC) = 0\) for any \(m \geq 1\) by definition.

Write \(C = C_0 \cup C_1\), where \(C_0\) consists of trivial cylinders and \(C_1\) does not contain trivial cylinder component. Now \(J\) is \(\mathbb{R}\) invariant. We can assume the all the components in \(C_1\) are distinct by the translation trick in \[13\]. By Proposition 6.14 of \[14\], we have \(J_0(C) \geq 0\). \(\square\)

**Remark 2.1.** In contrast, there may be holomorphic plane in \((\mathbb{R} \times Y, d(e^s\lambda))\) with \(J_0 = -1\) in contact cases. Therefore, perturbing the contact action by \(J_0\) index will not result in a new filtration on ECH in general.

By index reason (see \[13\]), the holomorphic current contributed to \(\partial\) cannot involve closed components. **Lemma 2.4** implies that the differential \(\partial\) decreases \(J_0\). We also know that \(\partial\) decreases \(\mathcal{A}_H\). Hence, \(\partial\) decreases \(\mathcal{A}_H^{\eta}\). The homology of \(\tilde{PF}C^L(\Sigma, \varphi_H, \gamma_0)\) is well defined, denoted by \(\tilde{PFH}^L(\Sigma, \varphi_H, \gamma_0)\). Obviously, the natural inclusion induces a homomorphism

\[
i_L : \tilde{PFH}^*_L(\Sigma, \varphi_H, \gamma_0) \to \tilde{PFH}^*_L(\Sigma, \varphi_H, \gamma_0)\cdot J.
\]
Definition 2.5. Take $\eta = 0$. Fix a class $\sigma \in \widehat{PFH}(\Sigma, \varphi_H, \gamma_0)_J$. The $PFH$ spectral invariant is defined by

$$c_{\sigma}^{pfh}(H, \gamma_0) := \inf\{L \in \mathbb{R} | \sigma \text{ belongs to the image of } i_L \}.$$ 

In the case that $\varphi_H$ is degenerate, we find a sequence $\{H_n\}_{n=1}^{\infty}$ such that $H_n$ converges in $C^\infty$ to $H$ and each $\varphi_{H_n}$ is non-degenerate. Then we define

$$c_{\sigma_n}^{pfh}(H, \gamma_0) := \lim_{n \to \infty} c_{\sigma_n}^{pfh}(H_n, \gamma_0),$$

where $\sigma_n \in \widehat{PFH}^*(\Sigma, \varphi_{H_n}, \gamma_0)$ is the class corresponding to $\sigma$. The limit is well defined by Corollary 4.4 of [12].

Let $\gamma_1$ be another choice of reference cycle with $d(\gamma_1) = d(\gamma_0)$. Fix a relative homology $Z_0 \in H_2(X, \gamma_1, \gamma_2)$. Then $Z_0$ induces an isomorphism $\Psi_{Z_0}$ on the $PFH$ groups. Then the $PFH$ spectral invariants satisfy the following relation: (see Proposition 4.2 of [12])

$$c_{\Psi_{Z_0}(\sigma)}^{pfh}(H, \gamma_1) = c_{\sigma}^{pfh}(H, \gamma_0) + \int_{Z_0} \omega_{\varphi_H}. \quad (2.8)$$

Remark 2.2. When $\eta > 0$, we could use $\mathbb{A}_H^\eta$ to define $PFH$ spectral invariants $c_{\sigma,\eta}^{pfh}(H, \gamma_0, J)$ as in Definition 2.5. Also, Theorem 2 could be used to deduce a relation between $c_{\Lambda,\eta}^{link}$ and $c_{\sigma,\eta}^{pfh}(H, \gamma_0, J)$ as in Corollary 1.1 under the assumption $\spadesuit$. However, the conclusion in Lemma 2.4 may not true in the cobordism case. As a result, the methods in [5] cannot show that $c_{\sigma,\eta}^{pfh}(H, \gamma_0, J)$ satisfying the Hofer continuity. We even don’t know whether $c_{\sigma,\eta}^{pfh}(H, \gamma_0, J)$ is independent of the almost complex structure $J$. Hence, $c_{\sigma,\eta}^{pfh}(H, \gamma_0, J)$ is not very useful at least at this moment.

2.2 Quantitative Heegaard Floer homology

Let $M = Sym^d \Sigma$ be the $d$-symmetric product of $\Sigma$. The product symplectic form $\omega^\times d$ on $(\Sigma)^\times d$ descends to a singular Kähler current $\omega_M$ on $M$. $\omega_M$ is smooth away from the diagonal $\Delta$. Fix a small neighbourhood $V$ of $\Delta$. According Perutz’s result (Section 7 of [28]), we obtain a smooth Kähler form $\omega_V$ on $M$ such that

- $[\omega_V] = [\omega_M] \in H^2(M; \mathbb{R})$;
- $\omega_V = \omega_M$ on $M - V$.

Then $Sym^d \varphi_H(\Lambda)$ and $Sym^d \Lambda$ are $\omega_V$–Lagrangian submanifolds. In fact, they are monotone Lagrangian submanifolds in the sense of Lemma 4.19 of [8]. The Hamiltonian symplectic morphism $\varphi_H$ is called non-degenerate (with respect to $\Lambda$) if
$\text{Sym}^d\varphi_H(\Lambda)$ intersects $\text{Sym}^d\Lambda$ transversely. We assume that $\varphi_H$ is non-degenerate; unless otherwise stated.

Fix a base point $x \in \text{Sym}^d\Lambda$. Define a reference chord
\[ x_H(t) := (\text{Sym}^d\varphi_H) \circ (\text{Sym}^d\varphi_H^{-1})(x). \]

For any $y \in \text{Sym}^d\varphi_H(\Lambda) \cap \text{Sym}^d\Lambda$, a capping is a smooth map $\hat{y} : [0, 1]_s \times [0, 1]_t \to M$ such that $\hat{y}(1, t) = x_H(t)$, $\hat{y}(0, t) = y$, and $\hat{y}(s, i) \in \text{Sym}^d\varphi_H^{-1}(\text{Sym}^d\Lambda)$, $i \in \{0, 1\}$. Two capping $\hat{y}_0$ and $\hat{y}_1$ are equivalent if $y_0 = y_1$ and $\int_{y_0} \omega_\Delta + \eta \Delta \cdot \hat{y}_0 = \int_{y_1} \omega_\Delta + \eta \Delta \cdot \hat{y}_1$. The equivalent class of the capping is denoted by $\mathcal{S}$.

**Definition 2.6** (Definition 13.1 of [24]). A path of almost complex structures $\{J_t\}_{t \in [0, 1]}$ on $\text{Sym}^d\Sigma$ is called a quasi–nearly–symmetric almost complex structure if it satisfies

1. $J_t$ is $\omega_M$–tame over $\text{Sym}^d\Sigma - V$.

2. For each $t$, there exists a complex structure $j_t$ on $\Sigma$ such that $J_t = \text{Sym}^d(j_t)$ over $V$.

Let $CF(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x)$ be a free module generated by $(y, \mathcal{S})$. Fix a generic quasi–nearly–symmetric almost complex structure $\{J_t\}_{t \in [0, 1]}$. Let $\mathcal{M}_1(J_t)(y_+, y_-)$ be the moduli space of holomorphic curves $u : \mathbb{R} \times [0, 1] \to M$ satisfying
\[
\begin{aligned}
\partial_s u + J_t(u)\partial_t u &= 0 \\
u(s, 0) &\in \text{Sym}^d\varphi_H(\Lambda), \ u(s, 1) \in \text{Sym}^d\Lambda \\
\lim_{s \to \pm \infty} u(s, t) &= y_\pm \\
\text{ind} u &= 1.
\end{aligned}
\]

The differential is defined by
\[
m_1(y_+, \mathcal{S}_+) = \sum_{y_-} \sum_{u \in \mathcal{M}_1(J_t)(y_+, y_-)/\mathbb{R}} \varepsilon(u)(y_-, [u]\#\mathcal{S}_+).
\]

The quantitative Heegaard Floer homology $HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x)$ is the homology of $(CF(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x), m_1)$. By Lemma 6.6 in [5], the homology is well defined.

The quantitative Heegaard Floer homology is independent of the choices of Hamiltonian function and almost complex structure. Moreover, Lemma 6.10 in [5] shows that
\[
HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x) \cong H^*(\mathbb{T}^d, \mathbb{F}[T^{-1}, T]). \tag{2.9}
\]

For different choices of base points $x, x'$, we can construct an isomorphism $\Psi_{x, x'} : HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x) \to HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x')$. 

Moreover, this isomorphism preserves the action filtration. The construction of \( \Psi_{S_{x,x'}} \) is the same as \([3.21]\), so we do not repeat it here. It is easy to verify that \( \Psi_{S_{x,x'}} \) satisfies the following diagram:

\[
\begin{array}{ccc}
HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x) & \overset{\Psi_{S_{x,x'}}}{\longrightarrow} & HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x') \\
\downarrow \gamma^H & & \downarrow \gamma^H \\
HF_*(\text{Sym}^d\varphi_G(\Lambda), \text{Sym}^d\Lambda, x) & \overset{\Psi_{S_{x,x'}}}{\longrightarrow} & HF_*(\text{Sym}^d\varphi_G(\Lambda), \text{Sym}^d\Lambda, x').
\end{array}
\] (2.10)

Here \( \gamma^H \) is the continuous morphism on the Lagrangian Floer homology. Therefore, we have an abstract Floer group \( HF(\text{Sym}^d\Lambda) \) and a canonical isomorphism

\[
\jmath^H_f : HF_*(\text{Sym}^d\varphi_H(\Lambda), \text{Sym}^d\Lambda, x) \rightarrow HF(\text{Sym}^d\Lambda). \] (2.11)

**Unit** There is a unique class \( 1_{hf} \in HF(\text{Sym}^d\Lambda) \) called the unit. It is the unit with respective to the quantum product. The unit is defined in the following way.

Let \((D_0, \omega_{D_0}, j_0)\) be a disk with one boundary puncture, where \((\omega_{D_0}, j_0)\) is a Kähler structure. We assume that \((D_0, \omega_{D_0}, j_0)\) is a disk with a negative strip-like end, i.e., there exists a neighbourhood \( U \) of the puncture of such that

\[
(U, \omega_{D_0}, j_0) \cong (]-\infty, 0[ \times [0, 1], ds \wedge dt, j),
\] (2.12)

where \( j \) is the standard complex structure that maps \( \partial_s \) to \( \partial_t \).

Let \( \mathcal{E} = D_0 \times \mathbb{M} \). Let \( \Omega_{\mathcal{E}} \) be a symplectic form such that \( \Omega_{\mathcal{E}} = \omega_V + ds \wedge dt \) over the strip-like end. Let \( \mathcal{L} \subset \partial D_0 \times \mathbb{M} \) be a Lagrangian submanifold such that

\[
\mathcal{L}|_{s \leq -R} = \mathbb{R}_{s \leq -R} \times \{0\} \times \text{Sym}^d \varphi_H(\Lambda) \cup \{1\} \times \text{Sym}^d \Lambda).
\]

Similar as defining the PFH cobordism maps, we need to fix a reference homology class \( \mathcal{S}_{ref} \in H_2(\mathcal{E}, \emptyset, \mathbb{L}_H) \). Let us clarify the choice of \( \mathcal{S}_{ref} \). Let \( D_0 \) be punctured disk with a positive strip-like end. Let \( \overline{\mathcal{E}} = \overline{D}_0 \times \mathbb{M} \) and \( \overline{\mathcal{L}} = \partial \overline{D}_0 \times \text{Sym}^d \Lambda \). Let \( (\mathbb{R}_{s_0} \times [0, 1] \times \mathbb{M}, L_0) \) be a Lagrangian cobordism from \((\text{Sym}^d \Lambda, \Lambda)\) to \((\Lambda, \Lambda)\) given by Lemma 3.11. Then we define the composition of this three pairs

\[
(\mathcal{E}_R, \mathbb{L}_R) := (\mathcal{E}, \mathbb{L})|_{s \geq -R \cup s = -R \sim s_0 = R(\mathbb{R}_{s_0} \times [0, 1] \times \mathbb{M}, L_0)|_{s \leq R \cup s_0 = -R \sim s_0 = R(\overline{\mathcal{E}}, \overline{\mathcal{L}})|_{s \leq R}.
\]

Pick a section \( S \) represented \( \mathcal{S}_x \in H_2(\mathbb{M}, \mathbb{L}_H, x) \), where \( \mathcal{S}_x \) is the relative homology class defined in Lemma 3.11. The composition \( \mathcal{S}_{ref} \# \mathcal{S}_x \# |\overline{D}_0 \times \{x\}| \) gives a class in \( H_2(\mathbb{E}_R, \mathbb{L}_R, \mathbb{Z}) \). Note that \( H_2(\mathbb{E}_R, \mathbb{L}_R, \mathbb{Z}) \cong H_2(D \times \mathbb{M}, \partial D \times \text{Sym}^d \Lambda, \mathbb{Z}) \), where \( D \) is a closed disk. Under this identification, we choose \( \mathcal{S}_{ref} \) such that

\[
\mathcal{S}_{ref} \# \mathcal{S}_x \# |\overline{D}_0 \times \{x\}| = D \times \{x\} \in H_2(D \times \mathbb{M}, \partial D \times \text{Sym}^d \Lambda, \mathbb{Z}).
\]
The triple \((E, \Omega_E, L)\) induces a cobordism map
\[
HF_{S_{ref}}(E, \Omega_E, L) : \mathbb{F}[T^{-1}, T] \to HF_* (\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x) \tag{2.13}
\]
The cobordism map \(HF_{S_{ref}}(E, \Omega_E, L)\) is induced by a homomorphism
\[
CF_{S_{ref}}(E, \Omega_E, L)(1) := \sum_{(y, S)} \# M^d_J(\emptyset, y, S_{ref} # S)(y, S)
\]
in chain level, where \(M^d_J(\emptyset, y, S_{ref} # S)\) is the moduli space of \(\text{ind} = 0\) holomorphic sections \(s : D_0 \to E\) such that \(s|_{\partial D_0} \subset L\) and \([s] = S_{ref} # S\). Also, the negative end of \(s\) is asymptotic to \(y \in \text{Sym}^d \varphi_H(\Lambda) \cap \text{Sym}^d \Lambda\). The cobordism maps \(HF_{S_{ref}}(E, \Omega_E, L)\) only depends on \((\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda)\).

**Definition 2.7.** For any \(H\), define \(I^*_H := HF_{S_{ref}}(E, \Omega_E, L)(1) \in HF_* (\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x)\). The unit \(1_{hf} \in HF(\text{Sym}^d \Lambda)\) is defined by \(1_{hf} := j^*_{H}(I^*_H)\).

The above definition is well defined because of the functorial properties of Lagrangian Floer homology and the diagram \((2.10)\). From \((2.9)\), we know that the quantitative Heegaard Floer homology is non–vanishing, so is the unit \(1_{hf}\).

In Section 6.3, we will describe the unit when \(H\) is a small Morse function.

**Action functional and HF–spectral invariants** Given a generator \((y, S)\), its action is defined by
\[
\mathcal{A}^n_H(y, S)_x = -\int_S \omega_V + \int_0^1 \text{Sym}^d H_t(x) dt - \eta \Delta \cdot S, \tag{2.14}
\]
where \(\Delta \cdot S\) is the algebraic intersection number with the diagonal. We omit the subscript “\(x\)” when the context is clear.

Let \(CF^L(\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x)\) be the complex generated by the capping with \(\mathcal{A}^n_H < L\). As in the PFH case, the action filtration induces a homomorphism
\[
i_L : HF^L(\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x) \to HF(\text{Sym}^d \varphi_H(\Lambda), \text{Sym}^d \Lambda, x)
\]
on the chain complex (see Lemma 6.12 of [3]). Fix a class \(a \neq 0 \in HF(\text{Sym}^d \Lambda)\). The **link spectral invariant** is defined by
\[
c^\text{link}_{\Lambda,H}(H, a) := \inf\{L \in \mathbb{R}|(j^*_H)^{-1}(a) \text{ belongs to the image of } i_L\}.
\]
Note the link spectral invariant is independent of the base point \(x\).
3 A Heegaard Floer type homology

In this section, we construct a Heegaard Floer type homology by following Lipshitz’s approach [24]. The main difference is that the disk bubbles could appear, but we can deal with them easily as the monotone Lagrangian Floer homology.

**Definition 3.1.** Fix a non-degenerate Hamiltonian symplectomorphism \( \varphi_H \). A **Reeb chord** of \( \varphi_H \) is a union of paths

\[
y = [0,1] \times \{y_1, \ldots, y_d\} \subset [0,1] \times \Sigma
\]

so that \( y_i \in \varphi_H(\Lambda_i) \cap \Lambda_{\sigma(i)} \), where \( \sigma \) is a permutation of \( \{1, \ldots, d\} \).

Note that there is a 1–1 correspondence between the Reeb chords and the intersection points \( \text{Sym}^d \varphi_H(\Lambda) \cap \text{Sym}^d \Lambda \).

Consider the symplectic manifold \( M = \mathbb{R} \times [0,1] \times \Sigma \) with the product symplectic form \( \Omega = \omega + ds \wedge dt \). Let \( L_0 = \mathbb{R} \times \{0\} \times \varphi_H(\Lambda) \) and \( L_1 = \mathbb{R} \times \{1\} \times \Lambda \). Note that they are Lagrangian submanifolds of \( (M, \Omega) \).

**HF–curves** Now we define the “open” holomorphic curves in \( M \). Given two Reeb chords \( y_\pm \), define \( Z_{y_+, y_-} := L_0 \cup L_1 \cup (\{\infty\} \times y_+) \cup (\{-\infty\} \times y_-) \).

**Definition 3.2.** Let \( (F, j) \) be a Riemann surface (possibly disconnected) with boundary marked points \( \{p_1^\pm, \ldots, p_d^\pm\}_{i=1}^d \). Let \( \hat{F} = F - \{p_1^\pm, \ldots, p_d^\pm\} \). A **d–multisection** is a smooth map \( u : (\hat{F}, \partial \hat{F}) \to (M, Z_{y_+, y_-}) \) such that

1. \( u(\partial \hat{F}) \subset L_0, L_1 \). For each \( 1 \leq i \leq d \), \( u^{-1}(L_0^i) \), \( u^{-1}(L_1^i) \) consists of exactly one component of \( \partial \hat{F} \), where \( L_0^i = \mathbb{R} \times \{0\} \times \varphi_H(\Lambda_i) \) and \( L_1^i = \mathbb{R} \times \{1\} \times \Lambda_i \).

2. \( u \) is asymptotic to \( y_\pm \) as \( s \to \pm \infty \).

3. \( \int_{\hat{F}} u^* \omega < \infty \).

Let \( H_2(M, y_+, y_-) \) be the set of continuous maps \( u : (F, \partial F) \to (M, Z_{y_+, y_-}) \) satisfying the conditions 1, 2) mod out the relation \( \sim \). Here \( u_1 \sim u_2 \) if and only if the compactifications \( \bar{u}_1 \) and \( \bar{u}_2 \) are equivalent in \( H_2(M, Z_{y_+, y_-}; \mathbb{Z}) \). An element in \( H_2(M, y_+, y_-) \) is called a **relative homology class**.

Fix a base point \( z_j \) in each connected component of \( \Sigma - \Lambda - \varphi_H(\Lambda) \). For each \( A \in H_2(M, y_+, y_-) \), we have a well–defined intersection number

\[
n_j(A) := \#(A \cap (\mathbb{R} \times [0,1] \times z_j)).
\]

(3.15)

A relative homology class \( A \) is called **positive** if \( n_j(A) \geq 0 \) for each \( j \). Note that if \( A \) admits a holomorphic representative, then \( A \) is positive.
Almost complex structure  An $\Omega$–compatible almost complex structure $J$ on $M$ is called **admissible** if it satisfies the following properties:

1. $J$ is $\mathbb{R}_s$–invariant.
2. $J(\partial_s) = \partial_t$.
3. $J$ preserves the vertical bundle of $M$.

The space of admissible almost complex structures is denoted by $\mathcal{J}_M$.

A $J$–holomorphic $d$–multisection $u : (F, \partial F, j) \to (M, Z_{y_+, y_-})$ is called a $(J$–holomorphic) **HF–curve**.

Fredholm index  To define the Fredholm index of a HF–curve $u$, we first need to fix a trivialization of $u^*\Sigma$ as follows. Fix a non–singular vector $v$ on $\Lambda$. Then $(\varphi_*(v), j\varphi_*(v))$ and $(v, j(v))$ give trivializations on $T\Sigma|_{L_0}$ and $T\Sigma|_{L_1}$ respectively. We extend the trivialization arbitrarily along $y_\pm$. Such a trivialization is denoted by $\tau$.

Define a real line bundle $\mathcal{L}$ over $\partial F$ as follows. We set $\mathcal{L}|_{\partial F} = u^*T\varphi(\Lambda), u^*T\Lambda$. Extend $\mathcal{L}$ along $\partial F - \partial F$ by rotating in counterclockwise direction from $u^*T\varphi(\Lambda)$ and $u^*T\Lambda$ by the minimum account possible. Then $(u^*\Sigma, \mathcal{L})$ forms a bundle pair over $\partial F$. With respect to the trivialization $\tau$, we have well-defined

1. Maslov index: $\mu_\tau(u) = \mu(u^*\Sigma, \mathcal{L}, \tau)$;
2. relative Chern number: $c_1(u^*\Sigma, \tau)$.

Note that $2c_1(u^*\Sigma, \tau) + \mu_\tau(u)$ is independent of $\tau$.

The **Fredholm index** of a HF–curve is defined by

$$\text{ind}_u := -\chi(F) + d + 2c_1(u^*\Sigma, \tau) + \mu_\tau(u).$$

ECH index  The concept of ECH index also can be generalized to the current setting. Given $A \in H_2(M, y_+, y_-)$, an oriented immersed surface $C \subset M$ is a $\tau$–representative of $A$ if

1. $C$ intersects the fibers positively along $\partial C$;
2. $\pi_{[0, 1]\times \Sigma}|_C$ is an embedding near infinity;
3. $C$ satisfies the $\tau$–trivial conditions in the sense of Definition 4.5.2 in [11].

Given a class $A \in H_2(M, y_+, y_-)$ with a $\tau$–representative $C$, we define the relative **self–intersection number** as follows. Let $\nu_C$ be the normal bundle of $C$. Let $\psi \in$
\( \Gamma(\nu_C) \) be a generic section such that \( \psi|_{\partial C} = J \tau \), where \( J \) is an admissible almost complex structure. Let \( C' \) be a push off of \( C \) along the direction \( \psi \). Then we define

\[ Q_\tau(A) := \#(C \cap C'). \]

The **ECH index** (see Definition 4.5.11 in [11]) is defined by

\[ I(A) := c_1(T\Sigma|_A, \tau) + Q_\tau(A) + \mu_\tau(A). \]

The Fredholm index and ECH index satisfy the following relation.

**Theorem 3.3** (see Theorem 4.5.13 of [11]). Fix an admissible almost complex structure \( J \) (not necessarily generic). Then for a \( J \)-holomorphic HF–curve \( u \), we have

\[ I(u) = \text{ind}u + 2\delta(u), \]

where \( \delta(u) \) is a sign count of the double points and singularities of \( u \). If \( J \) is generic, then \( I(u) \geq 0 \); \( I(u) = 0 \) if and only if \( u \) is a trivial strip.

### 3.1 Moduli space of HF–curves

Fix a relative homology class \( A \in H_2(M, y_+, y_-) \). Let \( \mathcal{M}^J(y_+, y_-, A) \) denote the moduli space of HF–curves with relative homology class \( A \).

**Bubbles analysis**  First of all, we study the possible bubbles in the compactification of the moduli space of HF–curves and their contribution to the ECH index.

Note that \( H_2(M, y_+, y_-) \) is an affine space over \( H_2([0, 1] \times \Sigma, \{0\} \times \varphi_H(\Lambda) \cup \{1\} \times \Lambda; \mathbb{Z}) \). By the Mayer–Vietoris sequence, we know that the latter is spanned by classes \( [\Sigma] \), \( \{0\} \times \varphi_H(B_i) \}_{i=1}^k \) and \( \{1\} \times B_i \}_{i=1}^k \). To simplify the notation, we write \( \{0\} \times \varphi_H(B_i) \), \( \{0\} \times \varphi_H(B_i^c) \), \( \{1\} \times B_i \) and \( \{1\} \times B_i^c \) as \( [\varphi_H(B_i)], [\varphi_H(B_i^c)], [B_i] \) and \( [B_i^c] \) respectively. Here \( B_i^c \) denote the complement of \( B_i \) in \( \Sigma \). Therefore, for \( A, A' \in H_2(M, y_+, y_-) \), we have

\[ A' - A = m[\Sigma] + \sum_{i=1}^k c_i[B_i] + c'_i[\varphi_H(B_i)]. \]

Since \( \sum_{i=1}^{k+1} [B_i] = \sum_{i=1}^{k+1} [\varphi_H(B_i)] = [\Sigma] \), we can also write

\[ A' - A = \sum_{i} c_i[B_i] + c'_i[\varphi_H(B_i)]. \]
Lemma 3.4. Let $A, A' \in H_2(M, y_+, y_-)$ be positive homology classes such that $A' - A = m[\Sigma] + \sum_{i=1}^{k} c_i[B_i] + \epsilon[B_i] + d_i[B_i] + d_i' [\varphi_H(B_i)]$ and $c_i, c_i', d_i, d_i', m \geq 0$. Then we have
\[
I(A') = I(A) + \sum_{i} (2c_i + 2c_i') + 2k(d_i + d_i') + 2m(k + 1). \tag{3.16}
\]

Proof. Let $u$ be a $\tau$–trivial representative of $A$. Suppose that $d_i = d_i' = m = 0$ first. As $u$ is $\tau$–trivial, we can assume that the ends of $u$ are just trivial strips. Let $E_i = [-1, \infty) \times [0, 1] \times y_i$ be a positive end of $u$ and $y_i \in \varphi_H(A_i)$ for $1 \leq i \leq k$.

We perform a modification on $E_i$ as follows. Let $I = [0, 1]$. Define a path $p_0 : I_s \to I_s \times \{0\} \times \varphi_H(B_i)$ by $p_0(s) = (s, 0, r = 1, \theta_0 + 2\pi f(s))$, where $(r, \theta)$ is the polar coordinates of $\varphi_H(B_i)$, $f(s)$ is a non–decreasing function such that $f = 0$ near $s = 0$ and $f = 1$ near $s = 1$, and $(1, \theta_0)$ is the polar coordinates of $y_i$. Note that $p_0$ wraps the boundary $\partial \varphi_H(B_i)$ counterclockwise one time. Define $p_1 : I_s \to I_s \times \{1\} \times \varphi_H(B_i)$ by $p_1(s) = (s, 1, 1, \theta_0)$. Let $p_{-}(t) = (0, t, 1, \theta_0)$ and $p_{+}(t) = (1, t, 1, \theta_0)$. Then we have an embedded map $v : I_s \times I_t \to I_s \times I_t \times \varphi_H(B_i)$ such that
\[
v(s, t) = (s, t, v_B(s, t));
v(s, 0) = p_0, \ v(s, 1) = p_1;
v(0, t) = p_{-}, \ v(1, t) = p_{+}.
\]

Such a map $v$ exists because the topology of $I_s \times I_t \times \varphi_H(B_i)$ is trivial. The loop $p_0 + p_+ - p_- - p_+$ bounds a disk.

We replace a segment $[0, 1] \times [0, 1] \times y_i$ of the end $E_i$ by $v$, the result is called $u'$. Note that $[u'] = [u] + [\varphi_H(B_i)]$. Let $\psi \in \Gamma(u^{*}T\Sigma)$ be a generic section such that $\psi = \tau$ along $u(\partial \hat{F})$, where $\tau$ is induced by a non–vanishing vector filed on $\varphi_H(A_i)$. We use the notation $\tau$ to denote the trivialization of $u^{*}T\Sigma$ induced by the same vector filed on $\varphi_H(A_i)$. Also, we choose $\tau$ such that it agrees with the one on $u^{*}T\Sigma$ outside the segment $v$. Then the the real line bundle $L$ still is a constant along $u'(\partial \hat{F})$ (with respect to $\tau$). Therefore, we have
\[
\mu_{\tau}(u') = \mu_{\tau}(u).
\]

Take a generic section $\psi' \in \Gamma(u^{*}T\Sigma)$ such that $\psi' = \psi$ outside the region $v$ and $\psi'|_{\partial v} = \tau|_{\partial v}$. Note that $\psi'^{-1}(0) = \psi^{-1}(0)$ outside the segment $v$, but $\psi'$ has extra zeros inside $v$. The zeros $\#(\psi'^{-1}(0)|_v) = \text{wind}(\psi'|_{\partial v})$, where $\text{wind}(\psi'|_{\partial v})$ is the winding number of $\psi'|_{\partial v}$ in $\Sigma \times \varphi_H(B_i)$. We have $\#(\psi'^{-1}(0)) = 1$ because $p_0$ wraps $\varphi_H(B_i)$ one time. As a result, we have
\[
c_1(u^{*}T\Sigma, \tau) = c_1(u^{*}T\Sigma, \tau) + 1.
\]
Note that after the modification, the ends of \( u' \) still intersect the fibers positively. Therefore, we can take \( u'' \tau \Sigma \) to be the normal bundle. As before, we have \( c_1(N_{u'}, J\tau) = c_1(N_u, J\tau) + 1 \). Since the other positive ends of \( u' \) are \([-1, \infty) \times [0, 1] \times \{ y_j \} \), where \( y_j \in \varphi_H(\Lambda_j) \) (\( j \neq i \)), and the image of \( v \) is contained in \( I_s \times I_t \times \varphi_H(B_i) \), the modification does not introduce any new double points. Hence, \( \delta(u') = \delta(u) \). By Lemma 4.5.8 of [11], we have

\[
Q_\tau(u') = Q_\tau(u) + 1.
\]

We perform this operation \( c'_i \) times for \( \varphi_H(B_i) \). For each \( B_i \), \( 1 \leq i \leq k \), we perform a similar operation. But in this case, we need to take \( p_0 \) to be a constant path and take \( p_1 \) to be a path wraps \( \{ 1 \} \times \partial B_i \) one time. Then we get

\[
I(A) = I(A') + \sum_i (2c_i + 2c'_i).
\]

If \( m \neq 0 \), we first modify \( u \) to a \( u' \) represented the class \( A + \sum_i c_i[B_i] + c'_i[\varphi_H(B_i)] \). Let \( u'' \) be a union of \( u' \) with \( m \) copies of \( \Sigma \). By definition of the ECH index, we have

\[
I(u'') = I(u') + I(m\Sigma) + 2m\#(u' \cap \Sigma) = I(u') + 2m(d - g + 1) = I(u') + 2m(k + 1).
\]

Since \( I(u + [\Sigma]) = I(u + [B_i] + [B_i]) = I(u + [B_i]) + 2 \) and \( I(u + [\Sigma]) = I(u) + 2(k + 1) \), we have \( I(u + [B_i]) = I(u) + 2k \). By the same argument, we know that the class \( [\varphi_H(B_i)] \) contribute \( 2k \) to the ECH index.

Lemma 3.5. Fix an admissible almost complex structure \( J \in J_M \). Let \( u : F \to M \) be an irreducible \( J \)-holomorphic curve with Lagrangian boundary conditions (if \( \partial F \neq \emptyset \)). If \( u(F) \) is compact, then \( u \) is contained in a fiber. If \( \partial F = \emptyset \), then it is a branched covering of \( \Sigma \). If \( \partial F \neq \emptyset \) and \( u(\partial F) \subset \{ s \} \times \{ 1 \} \subset \Lambda_i \) or \( \{ s \} \times \{ 0 \} \subset \varphi_H(\Lambda_i) \) for some \( s \in \mathbb{R} \), then we have the following possibilities:

1. \( u \) is a branched covering of \( \{ s \} \times \{ 1 \} \times B_i \) or \( \{ s \} \times \{ 0 \} \times \varphi_H(B_i) \) for \( 1 \leq i \leq k \);

2. \( u \) is a branched covering of \( \{ s \} \times \{ 1 \} \times B_i^c \) or \( \{ s \} \times \{ 0 \} \times \varphi_H(B_i^c) \) for \( 1 \leq i \leq k \), where \( B_i^c = \Sigma - B_i \);

3. \( u \) is a branched covering of \( \{ s \} \times \{ 0 \} \times \Sigma - \varphi_H(\Lambda_i) \) or \( \{ s \} \times \{ 1 \} \times \Sigma - \Lambda_i \), \( k + 1 \leq i \leq d \). In these two cases, the homology class of \( u \) is \( m[\Sigma] \).

In either cases, the homology class \( [u] \) is a linear combination of \( \cup_i^k[B_i], \cup_i^k[\varphi_H(B_i)] \) and \( [\Sigma] \).

Proof. Note that \( \pi \circ u : F \to \mathbb{R} \times [0, 1] \) is holomorphic because \( \pi \) is complex linear. The image \( \text{Image}(\pi \circ u) \) is a compact subset by the assumption. Then the open mapping theorem implies that \( \pi \circ u \) is a constant, i.e., \( u \) lies inside a fiber. It is easy to get the rest of conclusions.
Transversality and compactness

**Lemma 3.6.** There is a Baire subset $\mathcal{J}_{M}^{reg} \subset \mathcal{J}_M$ such that for $J \in \mathcal{J}_{M}^{reg}$, any HF–curve $u \in \mathcal{M}^J(y_+, y_-, A)$ without closed irreducible components is Fredholm regular.

**Proof.** Just note that the HF–curves cannot be multiply–covered because of the asymptotic behaviours. Then the proof is standard. For the details, please refer to Section 3 of [24] and Lemma 4.7.2 of [11].

**Lemma 3.7.** Suppose that $I(A) = 1$. Then for a generic $J \in \mathcal{J}_{M}^{reg}$, the moduli space $\mathcal{M}^J(y_+, y_-, A)$ is compact. In particular, $\mathcal{M}^J(y_+, y_-, A)/\mathbb{R}$ is a finite set of points.

**Proof.** Let $\{u_n : F_n \to M\}$ be a sequence of HF–curves in $\mathcal{M}^J(y_+, y_-, A)$. Note that $u_n$ cannot contain closed irreducible components. Otherwise, Lemma 3.4 implies that $I(u_n) \geq 2(k + 1)$. By the relative adjunction formula (see Lemma 4.5.9 of [11]), the genus of $F_n$ have a uniform upper bound. We may assume that the topology type of $\{F_n\}$ is fixed. Then we apply the Gromov compactness in [2].

The curves $u_n$ can be broken in the following two ways: 1) Bubbles comes from pinching an arc or an interior simple curve in $F_n$; 2) breaking along the necks. Since $\cup_i \Lambda_i$ are pairwise disjoint, if an irreducible component $v$ of the bubbles comes from pinching an arc $a$, then the ends points of $a$ must lie inside the same component of $\Lambda_i$. Therefore, $\partial v \subset \{s\} \times \{0\} \times \varphi_H(\Lambda_i)$ or $\{s\} \times \{1\} \times \Lambda_i$.

The limit of $u_n$ can be written as $u_\infty = \{u^1, \cdots u^N\}$. Each level $u^i$ is a union of HF–curves and the bubbles described above. By Lemma 3.5 and the above discussion, we know that the homology classes of the bubbles are combination of $\cup_i [B_i], \cup_{i=1}^k [\varphi_H(B_i)], \cup_i [B_i], \cup_{i=1}^k [\varphi_H(B_i)]$ and $[\Sigma]$. Let $u^i_0$ be the component of $u^i$ by removing the bubbles. Then

$$[u^i] = [u^i_0] + \sum_j c_{ij}[B_i] + c'_{ij}[\varphi_H(B_i)] + d_{ij}[B_i^+] + d'_{ij}[\varphi_H(B_i^+)] + m_i[\Sigma]$$

for some $c_{ij}, c'_{ij}, d_{ij}, d'_{ij}, m_i$. Moreover, $c_{ij}, c'_{ij}, d_{ij}, d'_{ij}, m_i \geq 0$ because of the energy reason. By Lemma 3.4, we have

$$1 = I(A) = I(u_\infty) = I(u^1) \cdots + I(u^N) = \sum_i \left(I(u^i_0) + \sum_j 2c_{ij} + 2c'_{ij} + 2d_{ij}k + 2d'_{ij}k + 2m_i(k + 1)\right).$$

(3.17)

Hence, $c_{ij} = c'_{ij} = d_{ij} = d'_{ij} = m_i = 0$, i.e., the bubbles cannot exist. Moreover, the index reason implies that $u_\infty = u^1$ cannot be broken. By Theorem 3.3, $\text{ind} u^1 = I(u^1) = 1$ and $u^1$ is embedded. By Lemma 3.6, $\mathcal{M}^J(y_+, y_-, A)/\mathbb{R}$ is a compact zero–dimensional manifold. 

\[\square\]
Lemma 3.8. Suppose that $I(A) = 2$. Then for a generic $J$, $\partial \mathcal{M}^J(y_+,y_-,A)$ consists of the following two types of broken holomorphic curves:

- $u = (u_1,u_2)$, where $u_1 \in \mathcal{M}^J(y_+,y_0,A_1)$ and $u_1 \in \mathcal{M}^J(y_0,y_-,A_2)$. Here $A_1$ and $A_2$ satisfy $A = A_1 \# A_2$ and $I(A_1) = I(A_2) = 1$.

- $u = u_1 \cup u_2$, where $u_1$ is a union of trivial holomorphic strips and $u_2$ is a simple index 1 holomorphic disk whose image is $\{s\} \times \{1\} \times B_i$ or $\{s\} \times \{0\} \times \varphi_H(B_i)$ for some $s$ and $1 \leq i \leq k$. Also, $u_1$ intersects $u_2$ at a point. In particular, $y_+ = y_-$ in this case.

Proof. Note that Equation (3.17) still holds. As $I(u_i^0) \geq 0$, $m_i = d_{ij} = d'_{ij} = 0$ (recall that we assume $k > 1$). If there exists some $u_i^0$ with $I = 1$, then $c_{ij} = c'_{ij} = 0$ as well. By the index reason, there is another level with $I = 1$ and all the other levels have $I = 0$. Then we get the first conclusion.

If $I(u_i^0) = 0$ for all $i$, then $u_0$ is just a union of trivial strips. By Equation (3.17), $c_{ij} = 1$ or $c'_{ij} = 1$ for some $(i,j)$ and all the other $c_{ij}, c'_{ij}$ are zero. Also, $d_{ij} = d'_{ij} = m_i = 0$. Then $u$ can be written as a union of trivial strips $u_1$ and a holomorphic curve $u_2$. By Lemma 3.5, $u_2$ is contained in some fiber and its image is $\{s\} \times \{1\} \times B_i$ or $\{s\} \times \{0\} \times \varphi_H(B_i)$ for some $1 \leq i \leq k$. Since $c_{ij} = 1$ or $c'_{ij} = 1$, $u_2$ cannot be branched covered. Hence, $u_1$ intersects $u_2$ at a point.

\[ \square \]

$J_0$-index for HF curves In the current situation, we imitate Hutchings to define the $J_0$ index. For more insights about the $J_0$ in open setting, please refer to [22]. Let $A \in H_2(M,y_+,y_-)$. The $J_0$ index is defined by

\[ J_0(A) := -c_1(T\Sigma|_A,\tau) + Q_\tau(A). \]

Lemma 3.9. The $J_0$ index satisfies the following properties:

1. Let $u : F \to M$ be an irreducible HF-curve, then

\[ J_0(u) = -\chi(F) + d + 2\delta(u). \]

2. If the class $A$ supports a HF-curve, then $J_0(A) \geq 0$.

3. Let $A,A' \in H_2(M,y_+,y_-)$. Suppose that $A' - A = m[\Sigma] + \sum_{i=1}^k c_i[B_i] + c'_{i}[\varphi_H(B_i)]$. Then

\[ J_0(A') = J_0(A) + 2m(d + g - 1). \]
Proof. The item (1) follows directly from the relative adjunction formula (Lemma 4.5.9) in [11].

If \( A \) supports an irreducible HF-curve \( u \), then \( J_0(A) \geq 0 \) by item (1). In general, if \( u \) consists of several components, without loss of generality, \( u = u_0 \cup u_1 \) has two irreducible components. By definition, we have

\[
J_0(u) = J_0(u_0) + J_0(u_1) + 2 \#(u_0 \cap u_1).
\]

By intersection positivity of holomorphic curves, \( \#(u_0 \cap u_1) \geq 0 \). Therefore, the three terms in right hand side of the above equation are nonnegative.

Item (3) follows directly from the computation in Lemma 3.4.

The following lemma is an analogy of Lemma 4.19 in [8]. It illustrates that the admissible link is monotone.

Lemma 3.10. Let \( A, A' \in H_2(M, y_+, y_-) \). Then we have

\[
\left( \int_A \omega - \int_{A'} \omega \right) + \eta (J_0(A) - J_0(A')) = \frac{\lambda}{2} (I(A) - I(A')).
\]

Proof. Suppose that \( A' - A = m[\Sigma] + \sum_{i=1}^k c_i[B_i] + c'_i[\varphi_H(B_i)] \). Then

\[
\int_A \omega - \int_{A'} \omega = \sum_{i=1}^k (c_i + c'_i) \lambda + m.
\]

Combining the above equation with Lemmas 3.4, 3.9 and the assumption A.4 then we get the result.

3.2 A Heegaard Floer type homology

With the preparation in the previous section, now we define the cylindrical formulation of the quantitative Heegaard Floer homology.

Fix a base point \( x = (x_1, \cdots, x_d) \), where \( x_i \in \Lambda_i \). As before, we fix a reference chord \( x_H(t) := \varphi_H \circ (\varphi_H^t)^{-1}(x) \) from \( \varphi_H(\Lambda) \) to \( \Lambda \). Let \( y \) be a Reeb chord. Replacing the Reeb chord \( y_+ \) in Definition 3.2 by \( x_H \), define the space equivalent classes of \( d \)-multisections from \( x_H \) to \( y \), denoted by \( H_2(M, x_H, y) \). A capping of \( y \) is a relative homology class \( A \in H_2(M, x_H, y) \). Two capping \( A_1, A_2 \) are equivalent if and only if \( \int_{A_1} \omega + \eta J_0(A_1) = \int_{A_2} \omega + \eta J_0(A_2) \). Let \([A]\) denote the equivalent class.

The chain complex is a free module generated by the Reeb chords and capping, i.e.,

\[
CF_*(\Sigma, \varphi_H, \Lambda, x) = \oplus F(y, [A]).
\]
By the monotone assumption (A.4), all the classes $[B_i]$ and $[\varphi_H(B_i)]$ are equivalent, written as $B$. It induces a $\mathbb{Z}$–action on $\text{CF}_*(\Sigma, \varphi_H, \Lambda, x)$ by $(y, [A]) \mapsto (y, [A\#B])$.

Fix a generic $J \in \mathcal{J}^\text{reg}_M$. We define the differential by

$$d_J(y_+, [A_+]) = \sum_{A \in H_2(M, y_+, y_-, I(A), \mathbb{R})} \#M^{J_1}(B_i) - \#M^{J_0}(\varphi_H(B_i)) (y, [A\#B]),$$

By Lemma 3.7 the differential is well defined. Also, (3.18) implies that the right hand side is a finite sum.

According to Lemma 3.8 and the gluing argument in [24], we have

$$d^2_J(y, [A]) = \sum_{i=1}^k (\#M^{J_1}(B_i) - \#M^{J_0}(\varphi_H(B_i))) (y, [A\#B]) = 0,$$

where $M^{J_1}(B_i)$ and $M^{J_0}(\varphi_H(B_i))$ are moduli spaces of holomorphic disks in $\Sigma$ with homology classes $[B_i]$ and $[\varphi_H(B_i)]$ respectively. Obviously, the Hamiltonian symplectic morphism $\varphi_H^{-1}$ gives a diffeomorphism from $M^{J_0}(\varphi_H(B_i))$ to $M^{J_1}(B_i)$. In sum, the homology of $(\text{CF}_*(\Sigma, \varphi_H, \Lambda, x), d_J)$ is well defined, denoted by $HF_*(\Sigma, \varphi_H, \Lambda, x)_J$.

### 3.3 Invariance

The purpose of this section is to show that the homology $HF_*(\Sigma, \varphi_H, \Lambda, x)_J$ is independent of the choices of several datum.

**Lemma 3.11.** [Lemma 6.1.1 of [10]] Given Hamiltonian functions $H_-$ and $H_+$, there exists a triple $(\Omega, L_0, L_1)$ on $M$ such that

1. $\Omega$ is a symplectic form such that $\Omega = \omega + ds \wedge dt$ when $|s| \geq R_0$.
2. $L_1 = \mathbb{R} \times \{1\} \times \Lambda$, $L_0 \subset \mathbb{R} \times \{0\} \times \Sigma$ and
   $$L_{0 \mid s \geq R_0} = \mathbb{R}_{s \geq R_0} \times \{0\} \times \varphi_H^+(\Lambda),$$
   $$L_{0 \mid s \leq -R_0} = \mathbb{R}_{s \leq -R_0} \times \{0\} \times \varphi_H^-(\Lambda),$$
   where $R_0 > 0$ is a positive constant.
3. $L_0$ and $L_1$ are disjoint union of Lagrangian submanifolds with respective to $\Omega$.

**Proof.** There are several ways to construct the triple $(\Omega, L_0, L_1)$. Here we use a construction that is different with [10]. The advantage of this construction is that we can describe the reference relative homology easily.

Let $\chi(s) : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing cut-off function such that

$$\chi(s) = \begin{cases} 0 & \text{if } s \leq -R_0 \\ 1 & \text{if } s \geq R_0. \end{cases}$$

(3.19)
Let $H^s := \chi(s)H_+ + (1 - \chi(s))H_-$. Define a diffeomorphism

$$F : \mathbb{R} \times [0, 1] \times \Sigma \to \mathbb{R} \times [0, 1] \times \Sigma$$

$$(s, t, x) \mapsto (s, t, \varphi_{H^s} \circ (\varphi_{H^s}^{-1}(x))).$$

Let

$$L := F(\mathbb{R} \times \{0, 1\} \times L)$$

$$\omega_E := (F^{-1})^*(\omega + d(H^s_t dt))$$

and

$$\Omega_E = \omega + ds \wedge dt.$$
converges to a broken curve $u_\infty = (u_1, u_2)$. One level is a HF–curve with $I = 1$ and another level is a curve in the Lagrangian cobordism with $I = 0$. The gluing argument (see Appendix in [21]) shows that $CF_{A_{ref}}(\Omega, L_0, L_1)_J$ is a chain map. Therefore, it induces a homomorphism

$$HF_{A_{ref}}(\Omega, L_0, L_1)_J : HF_*(\Sigma, \varphi_{H_+}, \Lambda, x)_J \to HF_*(\Sigma, \varphi_{H_-}, \Lambda, x)_J.$$ 

Moreover, the usual homotopy argument shows that $HF_{A_{ref}}(\Omega, L_0, L_1)_J$ is unchanged if we deform $(L_0, L_1, \Omega, J)$ over a compact subset of $M$. To simplify the notation, we denote $HF_{A_{ref}}(\Omega, L_0, L_1)_J$ by $I_{H_-, J_-}$.

Let $(M_{s_+}, \Omega, L_0, L_1)$ and $(M_{s_-}, \Omega', L'_0, L'_1)$ be Lagrangian cobordisms from $(\varphi_{H_+}(\Lambda), \Lambda)$ to $(\varphi_{H_0}(\Lambda), \Lambda)$ and from $(\varphi_{H_0}(\Lambda), \Lambda)$ to $(\varphi_{H_-}(\Lambda), \Lambda)$ respectively, where $s_{\pm}$ denote the $\mathbb{R}$–coordinates. Define a family of cobordisms by

$$(M, \Omega_R, L^0_R, L^1_R) := (M_{s_+}, \Omega, L_0, L_1)|_{s_+ \geq -R} \cup_{s_+ = -R} \cdots \cup_{s_- = -R} (M_{s_-}, \Omega', L'_0, L'_1)|_{s_- \leq R}$$

Note that $(L^0_R, L^1_R)$ are $\Omega_R$–Lagrangian submanifolds. By the usual neck stretching and homotopy arguments, we have

$$I_{H_+, J_+} = HF_{A_{ref}}(\Omega_R, L^0_R, L^1_R)_s = I_{H_0, J_0} \circ I_{H_+, J_+}.$$ (3.20)

Take $(H_+, J_+)$ to $(H_-, J_-)$. Then the above equation implies that $I_{H_+, J_+}$ is an isomorphism.

**Different choice of base points** Let $x' = (x'_1, \cdots, x'_d)$ be another base point, where $x'_i \in \Lambda_i$. As in the PFH case, any relative homology class $A \in H_2(M, x'_H, x_H)$ induces an isomorphism between the the homologies defined by the different base points.

We take a special relative homology class $A_{x, x'}$ as follows: Take a path $\eta : \sqcup_{i}[0, 1], \Lambda \to \Lambda$ such that $\eta(0) = x$ and $\eta(1) = x'$. Define $u(s, t) := (s, t, \varphi_H \circ (\varphi_H')^{-1}(\eta(s)))$ and $A_{x, x'} = [u] \in H_2(M, x'_H, x_H)$. Then $u$ induces an isomorphism

$$\Psi_{A_{x, x'}} : HF_*(\Sigma, \varphi_H, \Lambda, x)_J \to HF_*(\Sigma, \varphi_H, \Lambda, x')_J$$ (3.21)

by sending $(y, [A])$ to $(y, [A\#u])$.

The map $I_{H_+, J_+}$ defined in the Proposition 3.12 is called a **continuous morphism**. By the relation (3.20) and (3.21), we define $HF(\Sigma, \Lambda)$ to be the direct limit with respect to the continuous morphisms and $\Psi_{A_{x, x'}}$. Therefore, we have a canonical isomorphism

$$j^x_H : HF(\Sigma, \varphi_H, \Lambda, x) \to HF(\Sigma, \Lambda).$$ (3.22)
**Action functional** We define the action functional on the chain complex by
\[ A_H(y, [A]) = - \int A \omega + \int_0^1 H_t(x) dt - \eta J_0(A). \]
Later, we will see that the perturbation term \( \eta J_0 \) is corresponding to the \( \Delta \cdot \) in (2.14). By Lemma 3.9, it is easy to check that the differential decreases the action functional. Therefore, we get a filtration on the chain complex and can define the spectral invariant as before. Let \( c_{\Lambda, \eta} : C^\infty([0,1] \times \Sigma) \times HF(\Sigma, \Lambda) \to \mathbb{R} \) denote the spectral invariant defined by \( HF(\Sigma, \Lambda) \) and \( A_H^0 \).

Recall the class \( A_{x,x'} \in H_2(M, x_H, x_H) \). in (3.21). The advantage of our choice of the class \( A_{x,x'} \) is that \( \Psi_{A_{x,x'}} \) preserves the action functional. To see this, note that the energy of \( u \) is
\[ \int_{\cup_i [0,1]_{a_i} \times [0,1]} u^* \omega = \int_{\cup_i [0,1]_{a_i} \times [0,1]} \omega(X_H \circ \eta(s), \partial_s \eta) ds \wedge dt = \int_{\cup_i [0,1]_{a_i} \times [0,1]} d_{\Sigma} H_t(\partial_s \eta) ds \wedge dt = \int_0^1 H_t(x') dt - \int_0^1 H_t(x) dt, \]
where \( H_t(x) \) is short for \( \sum_{i=1}^d H_t(x_i) \) and similar for \( H_t(x') \). By definition, it is easy to show that \( J_0(A_{x,x'}) = 0 \). Therefore, we get
\[ A_H^0(y, [A]) = A_H^0(y, [A \# u]). \]
Because the isomorphism in Theorem 1 preserves the action filtration, we have
\[ c_{\Lambda, \eta}(H, a) = c_{j^{ink}}(H, j_H^x \circ \Phi_H((j_H^x)^{-1}(a))) \] (3.24)

### 4 Equivalence of two formulations

In this section, we prove the promised isomorphism in Theorem 1. The main idea is the same as [24] what Lipshitz done for the usual Heegaard Floer homology. One may copy the argument in [24] to prove Theorem 1. Here we provide an alternative way to compare the ECH index of HF curves and Fredholm index of holomorphic sections in \( \mathbb{R} \times [0,1] \times Sym^d \Sigma. \)

#### 4.1 Index comparison

First of all, we need to construct a nice representative of \( A \in H_2(M, y_+, y_-) \) which is like a holomorphic curve. Lipshitz needs such a representative in his argument as well. We apply his arguments in Lemma 4.2’ and Lemma 4.9’ of [25] to our setting directly. Then we have the following lemma.
Lemma 4.1. Fix a positive relative homology class \( A \in H_2(M, y_+, y_-) \). Then we have a representative \( u : F \to M \) of \( A + [\Sigma] \) with the following properties:

1. \( u \) is holomorphic in a neighborhood of \( \partial \dot{F} \);
2. \( \pi \circ u : F \to \mathbb{R} \times [0,1] \) is a branched covering;
3. \( u \) is holomorphic near the branch points of \( \pi \circ u \);
4. \( u \) is embedded except for finitely many double points.

Proof. Let \( A \in H_2(M, y_+, y_-) \) be a positive relative homology class. It associates a domain \( \sum_j n_j(A)D_j \), where \( n_j(A) \geq 0 \) is the intersection number in [3.15] and \( \{D_j\} \) is the complement of \( \Lambda \cup \varphi_H(\Lambda) \).

The construction consists of the following steps: First, we follow Lemma 4.2’ of [25] to glue \( \{D_j\} \) together according to the coefficients \( \{n_j(A)\} \). This gives us a surface \( F \) with coners and a map \( u_\Sigma : F \to \Sigma \). Secondly, we modify \( u_\Sigma : F \to \Sigma \) such that

1. \( F \) has coners \( \{p_i^\pm\}_{i=1}^d \) and \( u_\Sigma(p_i^\pm) = y_i^\pm \).
2. The coners are acute.
3. Each component of \( \partial F \) is mapped to some \( \varphi_H(\Lambda_i) \) or \( \Lambda_i \) so that each \( \varphi_H(\Lambda_i) \), \( \Lambda_i, 1 \leq i \leq d \), is used exactly once.

All the modifications in this step are local. Therefore, the argument in Lemma 4.2’ of [25] can be applied to our case directly. The surface \( F \) maybe disconnected. We can glue it with \( \Sigma \) so that the surface \( F \) is connected (see the proof of Lemma 4.9’ in [25]). Moreover, the new map \( u_\Sigma : F \to \Sigma \) still satisfies the previous properties. Finally, we use the argument in Lemma 4.9’ of [25] to construct a branched covering \( u_\Sigma : F \to \mathbb{R} \times [0,1] \).

Define \( u := u_\Sigma \times u_\Sigma : F \to M \). We perturb \( u \) such that it is embedded except for finitely many double points. Then \( u : F \to M \) satisfies all the requirements. \( \square \)

Let \( u \) be the \( d \)-multisection provided by Lemma 4.1. Note that \( u \) intersects each fiber \((s, t) \times \Sigma \) with \( d \) points (counting multiplies) because \( \pi \circ u \) is a branched covering, where \( \pi : M \to \mathbb{R} \times [0,1] \) is the projection. These \( d \) intersection points give us an element \( S(s, t) \in Sym^d \Sigma \). Then we define a section \( s_u \) on \( \mathbb{R} \times [0,1] \times Sym^d \Sigma \) by sending \((s, t)\) to \( S(s, t) \). The section \( s_u \) is called the **tautological correspondence** of \( u \). In terms of \( u \), note that the intersections of \( s_u \) with the diagonal arise in the following two ways:

1. Branch points of \( \pi \circ u \).
2. Double points of $u$.

The main goal of this section is to prove the following proposition.

**Proposition 4.2.** Let $u$ and $s_u$ be the above. Then we have

$$I(u) = \text{inds}_u \text{ and } J_0(u) = \Delta \cdot s_u.$$ 

Define a real subbundle $\mathcal{L}_d$ of $s^* u T \Sigma d^d \Sigma$ by

$$\mathcal{L}_d |\partial u = s_u^* \left( T(\mathbb{R} \times \{0\} \times \text{Sym}^d \varphi_H(\Lambda)) \cap TM \right) \cup \left( s_u^* T(\mathbb{R} \times \{1\} \times \text{Sym}^d \varphi_H(\Lambda)) \cap TM \right).$$

We extend $\mathcal{L}_d$ along $\{\infty\} \times y_+$ and $\{-\infty\} \times y_-$ by the minimal rotation as before. Therefore, the real subbundle $\mathcal{L}_d$ is $\text{Sym}^d \mathcal{L}$. Fix a trivialization $\tau$ of $u^*T\Sigma$ as before. Let $x = [x_1, \ldots , x_d] \in \text{Sym}^d \varphi_H(\Lambda)$ or $\text{Sym}^d \varphi_H(\Lambda)$. Since $\{\Lambda_i\}_{i=1}^d$ are pairwise disjoint, we can identify $T_x \text{Sym}^d \Sigma$ with $T x_1 \Sigma \times \cdots \times T x_d \Sigma$. Therefore, $\tau$ induces a trivialization $\tau_d$ of $T \Sigma$ along $\text{Sym} d^d \Sigma$ and $\text{Sym}^d \varphi_H(\Lambda)$ or $\text{Sym}^d \varphi_H(\Lambda)$. Since the Reeb chords $\{y = \cup_i [0,1] \times y_i | y_i \in \varphi_H(\Lambda_i) \cap \Lambda_{s(i)}\}$ are pairwise disjoint, $\tau$ induces trivializations of $\text{Sym} d^d \Sigma |_{\{\infty\} \times y_0}$ (still dented by $\tau_d$) similarly. The following lemma can be obtained directly from the definition.

**Lemma 4.3.** We have $\mu(u^* T\Sigma, \mathcal{L}, \tau) = \mu(s^*_u T \Sigma d^d \Sigma, \mathcal{L}_d, \tau_d)$.

The next step is to compare the relative Chern number of $u$ and $s_u$. We take a generic smooth section of $\psi \in \Gamma(u^* T\Sigma)$ such that $\psi = \tau$ along $\partial u$. We choose $\psi$ such it satisfies the following condition: Let $q$ be a branch point $\pi \circ u : F \rightarrow \mathbb{R} \times [0,1]$. We identify a neighbourhood of $u(q)$ with $D_w \times D_z$, where $z$ is the holomorphic coordinate of the fiber and $w$ is the coordinate pull back from the base. Similarly, if $q$ is a double point, then we can take the coordinates $(w, z)$ around $u(q)$ as the above. We take $\psi = \partial_z$ in terms of these coordinates. In particular, $\psi$ has no zeros near the branch points and the double points.

Away from the diagonal, the section $\psi$ induces a section $\psi_d$ over $(\Lambda^{\max} \text{Sym} d^d \Sigma)^{\otimes 2}|_{s_u}$ in the following way:

$$\psi_d([x_1, \ldots , x_d]) = (\psi(x_1) \wedge \cdots \psi(x_d)) \otimes (\psi(x_1) \wedge \cdots \psi(x_d)).$$

Note that the right hand side of the above formula is independent of the order of $(x_1, \ldots , x_d)$.

Now we extend $\psi_d$ over the whole $s_u$ by the following way: Let $q$ be a branch point of $\pi \circ u : F \rightarrow \mathbb{R} \times [0,1]$. Assume that the degree of $q$ is $d$ firstly. Reintroduce the coordinates $D_w \times D_z$ in the above. Then $Im u \cap (\pi^{-1}(w)) = [z_1, \ldots , z_d] \in \text{Sym}^d C_z$ near $u(q)$. When $w \neq 0$, then $\{z_1 \ldots z_d\}$ are distinct points in $C_z$. We abuse $z_i$ to denote
We extend the function $\Delta$ the diagonal. Then $|\partial z|_{z_i} = \partial z_i$. Let
\[
\sigma_1(z_1 \ldots z_d) = \sum_i z_i, \ldots, \sigma_d(z_1 \ldots z_d) = \Pi_i z_i
\]
be the elementary symmetric functions. These functions are coordinates of $Sym^d \mathbb{C}$. We have
\[
\psi|_{z_i} = \partial z|_{z_i} = \partial z_i = \sum_j \frac{\partial \sigma_j}{\partial z_i} \partial \sigma_j.
\]
It is well known that the Jacobian of the elementary symmetric functions is $\Delta = \Pi_{i<j}(z_i - z_j)$. Therefore, we have
\[
\psi_d|_{z_i \ldots z_d} = (\partial z|_{z_1} \wedge \cdots \wedge \partial z|_{z_d})^\otimes 2 = (\partial z_1 \wedge \cdots \wedge \partial z_d)^\otimes 2 = \Delta^2(\partial z_1 \wedge \cdots \wedge \partial z_d)^\otimes 2.
\]
(4.25)

We extend the function $\Delta^2$ generically over the whole $D_w \times Sym^d \mathbb{D}_z$, still denoted by $\Delta^2$. We set $\psi_d$ over $D_w \times Sym^d \mathbb{D}_z$ to be the most right hand side of (4.25).

For the general case, the argument is the same. Let $q_1, \ldots, q_N$ be branch points of $\pi \circ u : F \rightarrow \mathbb{R} \times [0,1]$ such that $\pi \circ u(q_k) = p$. Let $d_k$ denote the degree of $q_k$. Then $d = \sum_{k=1}^N d_k$. Let $(w, z_k)$ be the local coordinates around $u(q_k)$ as before. A neighbourhood of $[\pi^{-1}(p) \cap \text{Im} u] \subset \mathbb{R} \times [0,1] \times Sym^d \Sigma$ can be identified with $D_w \times Sym^{d_1} \mathbb{D}_z_1 \times \cdots Sym^{d_N} \mathbb{D}_z_N$. By the discussion above, the vector field $\partial_{z_k}$ gives rise to vectors $(\sum_r \frac{\partial \sigma_k}{\partial z_i} \partial \sigma_k^r j_{1=1}^N$ over $\{d \leq |w| \leq 1\} \times Sym^{d_k} \mathbb{D}_z_k$. Then
\[
\psi_d = (\Pi_k \Delta_k^2)\Pi_k(\partial_{\sigma_k^1} \cdots \partial_{\sigma_k^d})^\otimes 2
\]
over $\{d \leq |w| \leq 1\} \times Sym^{d_1} \mathbb{D}_z_1 \times \cdots Sym^{d_N} \mathbb{D}_z_N$. We extend $\Delta_k$ generically and set $\psi_d$ to be $(\Pi_k \Delta_k^2)\Pi_k(\partial_{\sigma_k^1} \cdots \partial_{\sigma_k^d})^\otimes 2$ over the whole $D_w \times Sym^{d_1} \mathbb{D}_z_1 \times \cdots Sym^{d_N} \mathbb{D}_z_N$. If $u(q)$ is a double point, then we can extend $\psi_d$ over a neighbourhood of $s_u(\pi(u(q)))$ in the above way.

**Lemma 4.4.** The section $\psi_d$ is non–vanishing over the ends of $s_u$. Moreover,
\[
2c_1(s_u^* T Sym^d \Sigma, \tau_d) = \# \psi_d^{-1}(0) = 2c_1(u^* T \Sigma, \tau) + b + 2\delta(u),
\]
where $b$ is the sum over all the branch points of the order of multiplicity minus one and $\delta(u)$ is the sign count of the double points.

**Proof.** The contribution of $\# \psi_d^{-1}(0)$ comes from two parts $\psi_{d,1}^{-1}(0)$ and $\psi_{d,2}^{-1}(0)$, where $\psi_{d,1}$ is the restriction of $\psi_d$ away from diagonal and $\psi_{d,2}$ is the restriction of $\psi_d$ near the diagonal.

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Firstly, consider the case that away from diagonal. By definition, for any \( p \in \mathbb{R} \times [0,1] \) such that \( \psi_{1,d}(s_u(p)) = 0 \), then we have \( \#\psi_{1,d}(0)|_p = 2 \sum_i \#\psi_i^{-1}(0)|_{q_i} \), where \( \pi^{-1}(p) \cap Imu = \{q_1 \ldots q_d\} \). Therefore, we obtain \( \#\psi_{1,d}^{-1}(0) = 2\#\psi^{-1}(0) \) away from the double points and branch points. Since \( \psi \) has no zeros near the branch points and double points, we have

\[
\#\psi_{1,d}^{-1}(0) = 2\#\psi^{-1}(0) = 2c_1(u^*T\Sigma, \tau).
\]

Let \( q \) be a branch point of \( \pi \circ u : F \to \mathbb{R} \times [0,1] \). To simplify the notation, we assume that the degree of \( q \) is \( d \). Reintroduce the coordinates \( D_w \times D_z \) around \( u(q) \). Under coordinates \( D_w \times D_z \) around \( u(q) \), we write \( u \) as \((x^d, f(x))\), where \( x \) is the holomorphic coordinate on \( F \). Since \( u \) is holomorphic near the branch points, we know that \( f \) is a holomorphic function. \( u \) is embedded near \( u(q) \) implies that \( f(x) = c_1 x + O(x^2) \) and \( c_1 \neq 0 \). The contribution of \( \psi_{2,d}^{-1}(0) \) near \( u(q) \) is the same as the winding number of \( \Delta^2|_{|w| = \epsilon} \). Let \( x_1, \ldots x_d \) be the distinct branches of \( w^\frac{1}{2} \). Then we have

\[
\Delta^2|_{s_u(w)} = c_0 w^{d-1} + o(|w|^{d-1})
\]

with \( c_0 \neq 0 \). Hence, \( \text{wind}\Delta^2|_{|w| = \epsilon} = d - 1 \). For the general case, the argument is similar.

If \( u(q) \) is a double point, then locally \( Imu \) is a graph of two functions \( f, g \). Then \( \Delta^2 = (f(w) - g(w))^2 \) under the local coordinates. We may assume \( f(w) = aw \) and \( g(w) = bw \) if the double points is positive. Otherwise, we assume that \( f(w) = aw \) and \( g(w) = bw \). Hence, each positive (negative) double point contribute \( 2(-2) \) to the \( \#\psi_{2,d}^{-1}(0) \).

In sum, \( \#\psi_{2,d}^{-1}(0) = b + 2\delta(u) \) and this finishes the proof of the lemma.

Proof of Proposition 4.2. By the Riemann–Hirwitz formula (see Corollary 3.2 of [24]), we have \( e(F) = de(D) - b \), where \( e(F), e(D) \) are the Euler measure. We know that \( e(F) = \chi(F) - \frac{d}{2} \) and \( e(D) = \chi(D) - \frac{1}{2} = \frac{1}{2} \). By Lemmas 4.3 4.4 we have

\[
\text{inds}_u = 2c_1(s^*_u T\text{Sym}^d \Sigma, \tau_d) + \mu(s^*_u T\text{Sym}^d \Sigma, L_d, \tau_d)
\
= b + 2c_1(u^* T\Sigma, \tau) + \mu_c(u) + 2\delta(u)
\
= -\chi(F) + d + 2c_1(u^* T\Sigma, \tau) + \mu_c(u) + 2\delta(u)
\
= \text{ind}_u + 2\delta(u) = I(u).
\]

By the argument in Lemma 4.9’ of [25], we can arrange that all the branch points of \( u \) are order 2. Then we have \( \#(\Delta \cap s_u) = b + 2\delta(u) \). On the other hand, \( \chi(F) = d - b \) by the Riemann-Hirwitz formula. Therefore,

\[
\Delta \cdot s_u = -\chi(F) + d + 2\delta(u) = J_0(u).
\]

\( \square \)
4.2 Tautological correspondence

In this subsection, we use the tautological correspondence to prove Theorem 1.

**Lemma 4.5.** The tautological correspondence induces a \( \mathbb{Z} \)-module isomorphism

\[
\Psi_* : H_2(M, y_+, y_-) \to H_2(Sym^d \Sigma, y_+, y_-).
\]

**Proof.** Fix a positive relative homology class \( A \in H_2(M, y_+, y_-) \). Let \( u \) be a representative provided by Lemma 4.1. Then the tautological correspondence gives a class \( \Psi_*[A + \Sigma] = [s_u] \in H_2(Sym^d \Sigma, y_+, y_-) \).

Recall that \( H_2(M, y_+, y_-) \) is an affine space over \( \mathbb{Z} < [\varphi_H(B_i)], [B_i] \rangle_{i=1}^{k+1} \). On the other hand, \( H_2(Sym^d \Sigma, y_+, y_-) \) is an affine space over \( H_2(Sym^d \Sigma, Sym^d \varphi_H(\Lambda)) \) and \( H_2(Sym^d \Sigma, Sym^d \Lambda) \). We extend \( \Psi_* \) linearly by

\[
\Psi_*(A + [\Sigma] + \sum_i c_i[B_i] + \sum_i c_i[\varphi_H(B_i)]) = \Psi_*(A + [\Sigma]) + \sum_i c_i \Psi_*(B_i) + \sum_i c_i \Psi_*([\varphi_H(B_i)]).
\]

The definition of \( \Psi_*([B_i]) \) and \( \Psi_*([\varphi_H(B_i)]) \) are given in the next paragraph.

For each \( B_i \), we define the class \( \Psi_*([B_i]) \in H_2(Sym^d \Sigma, Sym^d \Lambda) \) as follows: For \( 1 \leq i \leq k \), define a surface \( F = \bigcup_{j=1}^{d} D_j \). Let \( u^i_{\sigma} : F \to \Sigma \) be a map such that \( u^i_{\sigma}|_{D_j} \) is a constant at \( \Lambda_j \) when \( j \neq i \) and \( u^i_{\sigma}|_{D_i} \) is a biholomorphism to \( B_i \). Take \( u^i_{\Sigma} : F \to \Sigma \) to be the \( d \)-fold trivial covering. In the case that \( i = k + 1 \), let \( F = B_{k+1} \cup \bigcup_{i=k+1}^{d} D_j \). Take \( u^k+1_{\Sigma} : F \to \Sigma \) such that \( u^k+1_{\Sigma}|_{B_{k+1}} \) is a biholomorphism and \( u^k+1_{\Sigma}|_{D_j} \) is a constant at \( \Lambda_j \). Take \( u^k+1_{\Sigma} : F \to \Sigma \) to be a \( d \)-fold branched covering such that \( u^k+1_{\Sigma}|_{B_{k+1}} \) is a \( k \)-fold branched covering and \( u^k+1_{\Sigma}|_{\bigcup_{j=k+1}^{d} D_j} \) is the trivial covering. For each \( u^i = u^i_{\Sigma} \times u^i_{\Sigma} \), we obtain a map \( s_u^i : D \to (Sym^d \Sigma, Sym^d \Lambda) \) via the tautological correspondence. Define \( \Psi_*([B_i]) \) to be the homology class of \( s_u^i \). The construction is similar for \( [\varphi_H(B_i)] \).

By Lemma 4.10 in [8], the above construction give a 1–1 correspondence

\[
\mathbb{Z} < [\varphi_H(B_i)] >_{i=1}^{k+1} \cong H_2(Sym^d \Sigma, Sym^d \varphi_H(\Lambda)) \quad \text{and} \quad \mathbb{Z} < [B_i] >_{i=1}^{k+1} \cong H_2(Sym^d \Sigma, Sym^d \Lambda).
\]

Therefore, \( \Psi_* \) is an isomorphism.

\[\Box\]

**Lemma 4.6.** Let \( A \in H_2(M, y_+, y_-) \) be a positive relative homology class. Then

\[
I(A) = ind \Psi_*(A), J_0(A) = \Psi_*(A) \cdot \Delta \quad \text{and} \quad \int_A \omega = \int_{\Psi_*(A)} \omega_N.
\]

**Proof.** By Lemma 4.15 in [8], \( \mu([B_i]) = 2 \). Therefore, we obtain

\[
ind \Psi_*(A + \Sigma) = ind \Psi_*(A) + \sum_i \mu([B_i]) = ind \Psi_*(A) + 2(k + 1).
\]
Combine the above equation with Proposition 4.2 and Lemma 3.4 then we get the first identity.

By Proposition 4.2 it suffices to check that \( J_0(A') - J_0(A) = \Delta \cdot (\Psi_*(A') - \Psi_*(A)) \).

By Lemma 4.16 in [8], \( \Delta \cdot \Psi_*([B_i]) = 0 \) for \( 1 \leq i \leq k \) and \( \Delta \cdot \Psi_*([B_{k+1}]) = 2(2g + k - 1) = 2(d + g - 1) \). Combine these facts with item (3) in Lemma 3.9 then we obtain the result.

To prove the second statement, we first need to know how to recover \( u \) from \( s_u \). Write \( s_u = (s, t, u') \). Lift \( u' \) to \( \tilde{u}' : \tilde{F} \to (\Sigma)^x \) satisfying the diagram (4.26), where \( \tilde{F} = \{(x, y) \in \mathbb{D} \times (\Sigma)^x : u'(x) = p(y)\} \). Let \( S_{d-1} \) be the permutation group fixing the first factor of \((\Sigma)^x \). Note that \( S_{d-1} \) also acts on \( \tilde{F} \). Let \( \pi_1 \) be the projection of \((\Sigma)^x \) to its first factor. Then \( \pi_\Sigma \circ u = \pi_1 \circ u'/S_{d-1} \).

By the diagram (4.26) and the fact that \( \tilde{u}\mathbb{D} \) is a degree \( d! \)-covering, we have

\[
\int_{\tilde{F}} \tilde{u}^* \omega^x = \int_{\tilde{F}} \tilde{u}^* p^* \omega_M = \int_{\tilde{F}} \tilde{u}^* u^* \omega_M = d! \int_{\mathbb{D}} u^* \omega_M = d! \int_{\mathbb{D}} s_u^* \omega_M.
\]

On the other hand, by the fact that \( \tilde{F} \to F = \tilde{F}/S_{d-1} \) is a \((d - 1)!\)-covering, we have

\[
(d - 1)! \int_F u^* \omega = \int_{\tilde{F}} \tilde{u}^* \pi_1^* \omega = \frac{1}{d} \int_{\tilde{F}} \tilde{u}^* \omega^x.
\]

The above two equations imply that \( \int_F u^* \omega = \int_{\mathbb{D}} s_u^* \omega_M \).

By the definition of \( \Psi_*([B_i]) \) in Lemma 4.5 and the similar argument, we have \( \int_{B_i} \omega = \int_{\Psi_*([B_i])} \omega_M \). As the energy is additivity, we get the second statement of the lemma.

\[\square\]

Proof of Theorem 1. By the isomorphism (3.21), it suffices to prove the result for a special choice of \( x \). Take the base point \( x \) to be \( y_0 = (y_1 \cdots, y_d) \in \varphi_H(A) \cap A \).

First of all, let us identify the moduli space of HF-curves and the moduli space of holomorphic sections in \( \mathbb{R} \times [0, 1] \times Sym^d \Sigma \).

Since \( J \in J_M \) is \( \mathbb{R} \)-invariant, it is determined by a path of complex structures \( \{ j_t \}_{t \in [0, 1]} \) on \( \Sigma \). Therefore, each \( J \in J_M \) gives raise to a path of quasi–nearly–symmetric almost complex structure \( j_t := Sym^d j_t \). We extend it to be an almost complex structure \( j \) on \( \mathbb{R} \times [0, 1] \times Sym^d \Sigma \) by setting \( j(\partial_s) = \partial_t \). If \( u \) is a \( J \)-holomorphic HF–curve, then \( s_u \) is a \( j \)-holomorphic section.

Let \( J \) be a generic almost complex structure such that the curves in \( M^j(y_+^*, y_-^*, A) \) and \( M^j(y_+^*, y_-^*, \Psi_*(A)) \) are Fredholm regular. Fix a positive class \( A \in H_2(M, y_+, y_-) \) with \( I(A) = 1 \). The tautological correspondence gives a map

\[ \Psi : M^j(y_+^*, y_-^*, A) \to M^j(y_+^*, y_-^*, \Psi_*(A)). \]

Obviously, \( \Psi \) is injective. By Lemma 4.6 the curves in \( M^j(y_+^*, y_-^*, \Psi_*(A)) \) have Fredholm index 1.
Conversely, given a holomorphic section \( s = (s, t, u') \in \mathcal{M}^d(y_+, y_-, \Psi_*(A)) \) with \( \text{inds} = 1 \), by the same argument in Proposition 13.2 of [24], we lift \( u' \) to a map \( \tilde{u}' : \tilde{F} \to (\Sigma)^{\times d} \) satisfying the following diagram:

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\tilde{u}'} & (\Sigma)^{\times d} \\
\downarrow \tilde{u}_D & & \downarrow p \\
\mathbb{D} & \xrightarrow{u'} & Sym^{d\Sigma}
\end{array}
\] (4.26)

Here \( \tilde{u}_D : \tilde{F} \to \mathbb{D} \) is a covering map with degree \( d! \). Moreover, the maps in the diagram are holomorphic. Recall the \( S_{d-1} \)-action in Lemma 4.6. Then \( \tilde{u}' \) is \( S_{d-1} \)-equivariant.

The map \( \pi_1 \circ \tilde{u}' \) descends to a map \( u_\Sigma : F \to \Sigma \), where \( F := \tilde{F}/S_{d-1} \). Let \( u_D := \tilde{u}_D/S_{d-1} : F \to \mathbb{D} \). Then we define the inverse \( u = \Psi^{-1}(s) \) by \( u := u_D \times u_\Sigma : F \to M \).

As in [24], we have \( \Psi \circ \Psi^{-1} = id \). Again by Lemma 4.6, we have \( 1 = \text{inds} = I(u) \). Hence, the map \( \Psi \) is 1–1 onto.

We extend the isomorphism in Lemma 4.5 to

\[ \Psi_* : H_2(M, y, y_{0H}) \to H_2(Sym^d\Sigma, y, y_{0H}). \]

Let \( S_0 \in H_2(Sym^d\Sigma, y, y_{0H}) \) be a class represented by a capping \( s : \mathbb{R}_s \times [0, 1], \to (Sym^d\Sigma, Sym^d\varphi_H(\Lambda), Sym^d\Lambda) \). Because the diagonal \( \Delta \) is codimension 2, we assume that \( s \) intersects \( \Delta \) finite many points. Also, assume that \( s \) is holomorphic near the intersection points. Using the argument in the above paragraph, we can construct a surface \( u = \Psi^{-1}(s) : F \to M \). Then \( u \) represents a relative homology class \( A_0 \in H_2(M, y, y_{0H}) \). Then define \( \Psi_*(A \# A_0) = \Psi_*(A) \# S_0 \), where \( A \in H_2(M, y_+, y_-) \).

Define \( \Phi_H : CF_*(\varphi_H, \Lambda, y_0) \to CF(Sym^d\varphi_H(\Lambda), Sym^d\Lambda, y_0) \) by sending \( (y, [A]) \) to \( (y, [\Psi_*(A)]) \). By the above discussion, we know that \( \Phi_H \) is a chain map and it induces an isomorphism in homology level.

To see that the isomorphism preserves the action filtration, by Lemma 4.6, it suffices to show that \( \int_{A_0} \omega = \int_{S_0} \omega_V \) and \( J_0(A_0) = \Delta \cdot S_0 \). Fix another class \( S'_0 \in H_2(Sym^d\Sigma, y_{0H}, y) \). Let \( A'_0 \in H_2(M, y, y_{0H}) \) be the corresponding class via the tautological correspondence. Let \( S' \) and \( S \) be sections represented \( S'_0 \) and \( S_0 \). Also, assume that they intersect the diagonal transversally. Let \( C' \) and \( C \) be their tautological correspondence. By Lemma 4.6, we have \( \Delta \cdot S + \Delta \cdot S' = J_0(C) + J_0(C') \). Since \( S, S' \) are disjoin from the diagonal along \( y_{0H} \), there are no contributions to \( \Delta \cdot \) and \( J_0 \) from the pieces near \( y_{0H} \). Hence, we must have \( \Delta \cdot S = J_0(C) \) and \( \Delta \cdot S' = J_0(C') \). By the same argument in Lemma 4.6, we also have \( \int_{A_0} \omega = \int_{S_0} \omega_V \).

\( \square \)
5 Closed–open morphisms

In this section, we construct the closed–open morphisms and study their properties.

The construction is very similar to the construction of the PFH cobordism maps. It is defined by counting \( I = 0 \) holomorphic curves in a “closed–open” symplectic manifold \( W \). As pointed out by Hutchings (Section 5.5 of [16]), the main difficulty of defining the PFH cobordism maps is that the appearance of holomorphic curves with negative ECH index. These curves will violate the compactness and transversality of the moduli space. However, in the “closed–open” setting, the HF–ends ensure that the holomorphic curves cannot be multiply–covered. A key observation is that if a holomorphic curve in \( W \) with at least one end, then it has at least one positive end and one negative end. This is due of the fibration structure of \( W \). By the ECH index inequality, the “closed–open” curves have non–negative ECH index. Moreover, as in Lemma 3.4, the bubbles contribute at least 2 to the ECH index; we can rule out them by the index reason. Therefore, we can define closed–open morphisms by using the classical techniques.

5.1 Closed–open curves and their indexes

Closed–open cobordism First of all, we introduce the closed–open symplectic manifold. Define a surface \( B \subset \mathbb{R} \times (\mathbb{R} / (2\mathbb{Z})) \) by \( B := \mathbb{R}_s \times (\mathbb{R}_t / (2\mathbb{Z})) - B^c \), where \( B^c = \mathbb{R}_s \times (-\infty, -2) \times [1, 2] \) with the corners rounded. The picture of \( B \) is shown in Figure 1.

![Figure 1:](image)

Define the mapping torus \( Y_{\varphi_H} = [0, 2] \times \Sigma / (0, \varphi_H(x)) \sim (2, x) \). Then \( \pi : \mathbb{R}_s \times Y_{\varphi_H} \to \mathbb{R} \times (\mathbb{R} / (2\mathbb{Z})) \) is a surface bundle over the cylinder. Define \( W_H := \pi^{-1}(B) \). Obviously,
\( \pi_W : W_H \to B \) is a surface bundle over the surface \( B \). The symplectic form \( \Omega_H \) on \( W_H \) is defined to be the restriction of \( \omega_{\varphi_H} + ds \wedge dt \).

We place a copy of \( \Lambda \) on the fiber \( \pi_B^{-1}(-3, 1) \) and take its parallel transport along \( \partial B \) using the symplectic connection. The parallel transport sweeps out a Lagrangian submanifold \( L_{\Lambda_H} \) of \( (W_H, \Omega_H) \). Note that \( L_{\Lambda_H} \) consists of \( d \) disjoint connected components. Moreover, we have

\[
L_{\Lambda_H}|_{s \leq -3} = \mathbb{R} \times \{0\} \times \varphi_H(\Lambda)
\]
\[
L_{\Lambda_H}|_{s \leq -3} = \mathbb{R} \times \{1\} \times \Lambda.
\]

The triple \( (W_H, \Omega_H, L_{\Lambda_H}) \) is called a closed–open symplectic cobordism.

**Remark 5.1.** By using the trivialization \( \{2, 3\} \), we can identify \( W_H \) as a bundle over \( \pi^{-1}(B) \subset \mathbb{R} \times S^1 \times \Sigma \), where \( \pi : \mathbb{R} \times S^1 \times \Sigma \to \mathbb{R} \times S^1 \) is the projection. Therefore, for any two Hamiltonian functions \( H, G \), we have a diffeomorphism from \( W_H \) to \( W_G \) preserving the fibration structure. When the context is clear, we suppress the subscript “\( H \)” from the notation.

**Remark 5.2.** We can define a slight different trivialization of \( W_H \); the advantage of the following trivialization is that we can describe the Lagrangian submanifold \( L_{\Lambda_H} \) easily. We shift the coordinate by \(-\frac{1}{2} \) and define

\[
\mathbb{R} \times Y_{\varphi_H} := \mathbb{R} \times \left[ -\frac{1}{2}, \frac{3}{2} \right] \times \Sigma / (s, -\frac{1}{2}, \varphi_H(x)) \sim (s, \frac{3}{2}, x).
\]

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a non–decreasing cutoff function such that \( \chi = 1 \) when \( t \geq 1 \) and \( \chi = 0 \) when \( t \leq 0 \). Define a flow \( \varphi_t := \varphi_H^{(t)} \) and extend it to be \( \text{id} \) for \( t \leq 0 \) and be \( \varphi_H \) for \( t \geq 1 \). Define a trivialization by

\[
\Psi'_H : \mathbb{R} \times \left[ -\frac{1}{2}, \frac{3}{2} \right] / (\frac{1}{2}, \frac{3}{2}) \times \Sigma \to \mathbb{R} \times Y_{\varphi_H}
\]
\[
(s, t, x) \mapsto (s, t, \varphi_H \circ \varphi_t^{-1}(x)).
\]

The restriction of \( \Psi'_H \) to \( B \times \Sigma \) gives a trivialization of \( W_H \). Moreover, we have \( (\Psi'_H)^* \omega_{\varphi_H} = \omega + d(\dot{\varphi}_H) \wedge dt \) and \( \Psi'_H(\partial B \times \Lambda) = L_{\Lambda_H} \).

**PFH–HF curves**

**Definition 5.1.** Fix a Reeb chord \( y \) and an orbit set \( \gamma \) with degree \( d \). Let \((F, j)\) be a Riemann surface (possibly disconnected) with punctures. A \( d \)-multisection is a smooth map \( u : (F, \partial F) \to W \) such that

1. \( u(\partial \hat{F}) \subset L_{\Lambda_H} \). Let \( \{L^i_{\Lambda_H}\}_{i=1}^d \) be the connected components of \( L_{\Lambda_H} \). For each \( 1 \leq i \leq d \), \( u^{-1}(L^i_{\Lambda_H}) \) consists of exactly one component of \( \partial \hat{F} \).
2. $u$ is asymptotic to $y$ as $s \to -\infty$.

3. $u$ is asymptotic to $\gamma$ as $s \to +\infty$.

4. $\int_F u^* \omega < \infty$.

Fix an orbit set $\gamma$ and a Reeb chord $y$. Let $Z_{\gamma,y} := L_{\Lambda_H} \cup \{\infty\} \times \gamma \cup \{-\infty\} \times y \subset W$.

We denote $H_2(W,\gamma,y)$ the equivalence classes of continuous maps $u : (F,\partial F) \to (W, Z_{\gamma,y})$ satisfying 1), 2), 3) in the above definition. Two maps are equivalent if their difference is trivial in $H_2(W, Z_{\gamma,y}; \mathbb{Z})$. Similar to the previous case, $H_2(W,\gamma,y)$ is an affine space over $H_2(W, L_{\Lambda_H}; \mathbb{Z})$ and the difference of any two classes can be written as

$$Z' - Z = \sum_{i=1}^k c_i B_i + m[\Sigma] + [S],$$

where $B_i$ is the class represented by a symplectic parallel translation of $\{-3\} \times \{1\} \times B_i$, $i = 1, \cdots, k$, and $[S]$ is the $H_1(S^1, \mathbb{Z}) \otimes H_1(\Sigma, \mathbb{Z})$-component of $H_1(Y_{\varphi_H}, \mathbb{Z})$.

**Fredholm index**

Similar as before, we fix a non-singular vector field on $\Lambda$ and it induces a trivialization $\tau$ of $T\Sigma$ along $\Lambda$. We extend $\tau$ by parallel transport along $L_{\Lambda_H}$ and arbitrarily along $y$ and $\gamma$. Define a subbundle $L$ of $u^* T\Sigma_{|\partial F}$ as follows. Set $L = u^* (TL_{\Lambda_H} \cap T\Sigma)$ along $\partial \hat{F}$. We extend $L$ over $\partial F - \partial \hat{F}$ by rotating in clockwise direction from $T\Lambda$ to $T\varphi_H(\Lambda)$ by minimum amount possible. Let $\mu_{\tau}(u)$ denote the Maslov index of the bundle pair $(u^* T\Sigma_{|\partial F}, L)$.

The Fredholm index is defined by

$$\text{ind} u := -\chi(\hat{F}) + 2c_1(u^* T\Sigma, \tau) + \mu_{\tau}^{\text{ind}}(\gamma) + \mu_{\tau}(u),$$

where $\mu_{\tau}^{\text{ind}}(\gamma)$ is the combination of the Conley–Zehnder index defined in (2.4).

**ECH index**

Fix $Z \in H_2(W,\gamma,y)$. Let $u : F \to W$ be a $\tau$–trivial representative in the sense of Definition 5.6.1 of [11]. We can define the relative self–intersection $Q_{\tau}(Z)$ as before. Then the ECH index (Definition 5.6.6 [11]) is

$$I(Z) := c_1(TW|_{Z}, \tau) + Q_{\tau}(Z) + \mu_{\tau}^{\text{ech}}(\gamma) + \mu_{\tau}(Z),$$

where $\mu_{\tau}^{\text{ech}}(\gamma)$ is the combination of the Conley–Zehnder index defined in (2.3).

According to Theorem 5.6.9 of [11], the ECH inequality still holds for the PFH-HF curves, i.e., we have

$$I(u) \geq \text{ind} u + 2\delta(u).$$

Moreover, $I(u) = \text{ind} u$ if and only if the positive ends of $u$ satisfy the ECH partition conditions. By the above inequality, we have $I(u) \geq 0$ for a generic $J$. 

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Again, the concept of $J_0$ index can be generalized to the closed-open setting. Given $Z \in H_2(W, \gamma, y)$, the $J_0$ index is

$$J_0(Z) := -c_1(TW|Z, \tau) + Q_\tau(Z) + \mu_{\tau}^{J_0}(\gamma),$$

(5.29)

where $\mu_{\tau}^{J_0}(\gamma) = \sum_i \sum_{p=1}^{m_i-1} \mu_\tau(\gamma_i^p)$.

**Lemma 5.2.** Let $u$ be an irreducible PFH-HF curve. Then

$$J_0(u) \geq 2(g(F) - 1 + \delta(u)) + \#\partial F + |\gamma|.$$  

In particular, $J_0(u) \geq 0$. Moreover, $J_0(u) \geq 0$ still holds if $u$ has several irreducible components.

**Proof.** We first assume that $u$ is irreducible. By the Relative adjunction formula (Lemma 5.6.3 of [11]), we have

$$J_0(u) = 2g(F) - 2 + \#\partial F + \#\Gamma_+ - w_\tau(u) + \mu_{\tau}^{J_0}(\gamma) + 2\delta(u),$$

where $\Gamma_+$ is the positive punctures of $u$. By Equation (6.2) in [14], we have $\#\Gamma_+ - w_\tau(u) + \mu_{\tau}^{J_0}(\gamma) \geq |\gamma|$.

If $u$ has multiple components, without loss of generality, $u = u_0 \cup u_1$, then by the same argument in [14], we have

$$J_0(u) \geq J_0(u_0) + J_0(u_1) + 2\#(u_0 \cap u_1) + E,$$

where $E \geq 0$ is the quantities defined in [14]. Hence, the inequality still holds and we have $J_0(u) \geq 0$.

**5.2 Moduli space of PFH-HF curves**

Let $J_{tame}(W, \Omega_H)$ be the set of $\Omega_H$-tame almost complex structures $J$ such that $J$ agrees with the admissible almost complex structures on the ends, and the projection $\pi_W$ is $(J,j_B)$ complex linear. Let $M^J(\gamma, y, Z)$ be the moduli space of $J$-holomorphic PFH-HF curves.

Apart from the ECH inequality, the ECH index and Fredholm index also satisfy the following relation. It is an analogy of Proposition 1.6 (c) in [13].

**Lemma 5.3.** Let $u \in M^J(\gamma, y, Z)$ be a PFH-HF curve. Assume that $\gamma = \{(\gamma_i, m_i)\}$ is a PFH generator. Then

$$I(u) = \text{ind}_u \mod 2.$$
Proof. By the relative adjunction formula (see Lemma 5.6.3 of [11]), we have

\[ I(u) - \text{ind}u = 2\delta(u) - w^+_\tau(u) + \mu^{\text{ech}}_\tau(\gamma) - \mu^{\text{ind}}_\tau(\gamma), \]

where \( w^+_\tau(u) \) is the writhe number defined in Section 3.1 of [13].

We may assume that the positive ends of \( u \) are \( \tau \)-trivial in the sense of Definition 2.3 of [13]. Otherwise, we modify \( u \) such that the positive ends are \( \tau \)-trivial. Note that such a modification doesn’t change the relative homology class.

Suppose that \( u \) has positive ends at a simple periodic orbit \( \gamma_i \) with total multiplicity \( m_i \). Because the positive ends of \( u \) are \( \tau \)-trivial, \( u \) has \( m_i \) ends at \( \gamma_i \) and \( w^+_\tau(u) = 0 \).

If \( \gamma_i \) is hyperbolic, then \( \mu^{\text{ech}}_\tau(\gamma_i) = \mu^{\text{ind}}_\tau(\gamma_i) \) since \( m_i = 1 \). If \( \gamma_i \) is elliptic, then \( \mu^{\text{ech}}_\tau(\gamma_i) - \mu^{\text{ind}}_\tau(\gamma_i) = \sum_{q=1}^{m_i} (CZ_\tau(\gamma_i^q) - CZ_\tau(\gamma_i)) = \text{even} \).

\[ \text{(Lemma 5.4)} \]

Assume that \( I(Z) = 0 \) and \( J \in \mathcal{J}_{\text{tame}}(W, \Omega_H) \) is a generic almost complex structure. Then the moduli space \( \mathcal{M}^J(\gamma, y, Z) \) is a compact zero–dimensional manifold.

In other words, \( \mathcal{M}^J(\gamma, y, Z) \) is a set of finite points.

Proof. The SFT–compactness in closed–open cobordisms can be found in Section 6.1 and Section 7.3 in [11]. Let \( u = \{u_{N_+}, \ldots u_0, \ldots u_{N_-}\} \) be a broken holomorphic curve from the limit of curves in \( \mathcal{M}^J(\gamma, y, Z) \), where \( u_i \) are holomorphic curves in \( \mathbb{R} \times Y_{\varphi_H} \) for \( i > 0 \), \( u_0 \) is a curve in \( W \) and \( u_j \) are curves in \( M \) for \( j < 0 \). By the ECH inequality and the fact \( \text{F.2} \), the curves in \( \mathcal{M}^J(\gamma, y, Z) \) satisfy the ECH partition conditions. Therefore, the positive ends of \( u_{N_+} \) also satisfy the ECH partition conditions.

Similar as in Lemma 3.5 by the fact that \( \pi_W : W \to B \) is complex linear and the open mapping theorem, the bubbles lie inside the fiber of \( W \). The bubbles arise in the following two ways:

1. Since \( \{L^i_{\Lambda_H}\}_{i=1}^d \) are pairwise disjoint, if an irreducible component \( v \) of the bubbles comes from pinching an arc \( a \), then the end points of \( a \) must lie inside the same component of \( L_{\Lambda_H} \). Then \( v \) is a branched covering of \( \mathcal{P}_{\tau}^\tau(B_i) \) for some \( \tau \) and \( i \), where \( \mathcal{P}_{\partial B}|_{\tau \in \mathbb{R}} \) is the parallel transport of the symplectic connection and \( B_i \) is one of the following surfaces:

   \begin{enumerate}
   \item[(a)] \( B_i = \{-3\} \times \{1\} \times B_i \) or \( \{-3\} \times \{1\} \times B_i^c \) for some \( 1 \leq i \leq k \).
   \item[(b)] \( B_i = \{-3\} \times \{1\} \times (\Sigma - \Lambda_i) \) for some \( k + 1 \leq i \leq d \). In this case, the homology class of \( B_i \) is \( [\Sigma] \).
   \end{enumerate}

2. If \( v \) comes from pinching an interior simple closed curve, then the homology class of \( v \) is \( m[\Sigma] \) for some \( m \geq 0 \).
In sum, we can write \([u_0] = [u_{00}] + m_0[\Sigma] + \sum_j c_{0j}[B_j] + d_{0j}[B_j^c]\), where \([u_{00}]\) is a PFH–HF curve without closed irreducible components and the rest are homology classes of the bubbles. Observe that each irreducible component of \([u_{00}]\) has at least one positive end and at least one negative. Thus \([u_{00}]\) cannot be multiply-covered and \(I([u_{00}]) \geq 0\). Also, \(m_0, c_{0j}, d_{0j} \geq 0\). The same argument in Lemma 3.4 can be used to show that

\[
I([u_0]) = I([u_{00}]) + 2m_0(k + 1) + \sum_j 2c_{0j} + \sum_j 2kd_{0j}.
\]

The holomorphic curves in \(\mathbb{R} \times Y_{\varphi_H}\) always have non-negative ECH index, thus

\[
0 = I(Z) = I(u) = I(u_{N_+}) \cdots I(u_0) + \cdots I(u_{N_-}) \geq \sum_{i \leq 0} \left( I(u_{0j}) + \sum_j 2c_{ij} + 2c'_{ij} + 2kd_{ij} + 2kd'_{ij} \right) + \sum_i 2m_i(k + 1).
\]

We have \(c_{ij} = d_{ij} = c'_{ij} = d'_{ij} = m_i = 0\); in other words, no fiber bubbles appear. The inequality (5.31) also implies that \(I(u_i) = 0\) for each \(i\). The Fredholm indices of \(u_i\) are adding to zero. By F.3, we have \(ind u_i = 0\) for each \(i\). For \(i < 0\), \(u_i\) are trivial strips which are ruled out in the holomorphic buildings. For \(i > 0\), \(u_i\) are connectors with zero Fredholm index. By the fact F.1 and the ECH partition conditions, they must be trivial. Thus these curves also are ruled out. In sum, \(\mathcal{M}^J(\gamma, y, Z)\) is compact.

\[
\text{Fix a reference class } Z_{\text{ref}} \in H_2(W, \gamma_0, x_H). \text{ We define a homomorphism }
\]

\[
\text{CO}_{Z_{\text{ref}}}(W, \Omega_H, L_{\Lambda_H}, J) : \widehat{PF}(\Sigma, \varphi_H, \gamma_0) \to CF(\Sigma, \varphi_H, \Lambda, x)
\]

by

\[
\text{CO}_{Z_{\text{ref}}}(W, \Omega_H, L_{\Lambda_H}, J)(\gamma, [Z]) = \sum_{(y, [A]), I(Z) = 0} \#\mathcal{M}^J(\gamma, y, Z)(y, [A]),
\]

where \(A\) is determined by the relation \(Z \# Z_{\text{ref}} \# A = Z\). For a generic \(J \in J_{\text{tame}}(W, \Omega_H)\), Lemma 5.4 implies that the above definition make sense.

**Lemma 5.5.** \(\text{CO}_{Z_{\text{ref}}}(W, \Omega_H, L_{\Lambda_H}, J)\) is a chain map.

**Proof.** Consider the moduli space \(\mathcal{M}^J(\gamma, y, Z)\) with \(I(Z) = 1\), where \(\gamma\) is a PFH generator. By the ECH index inequality and Lemma 5.3, the curves in \(\mathcal{M}^J(\gamma, y, Z)\) have Fredholm index \(\text{ind} = 1\). The fact F.2 implies that the curves satisfy the ECH partition conditions. Let \(u_\infty\) be a broken holomorphic curve from the limit of curves in \(\mathcal{M}^J(\gamma, y, Z)\). The inequality (5.31) still holds. Thus the bubbles can be ruled out.

Suppose that \(u\) is broken. The same argument in Lemma 5.4 shows that it either consists of
• A PFH–curve with $I = \text{ind} = 1$ in the top level;

• A PFH–HF curve in $W$ with $I = \text{ind} = 0$;

• The $\text{ind} = 0$ connectors in the middle level;

• There are no negative level;

or

• There is no positive level;

• A PFH–HF curve in $W$ with $I = \text{ind} = 0$;

• A HF–curve with $I = 1$ in the negative level.

The conclusion follows from Hutchings–Taubes’s obstruction gluing analysis [19, 20].

5.3 Homotopy invariance

Even we only define the closed–open morphism for a tuple $(W_H, \Omega_H, L_{\Lambda_H}, J)$, the construction still holds for a slight general situation. If we vary the tuple $(\Omega_H, L_{\Lambda_H}, J)$ over a compact subset of $W_H$ such that the almost complex structure is still generic, then the closed–open morphism is still well defined by counting $I = 0$ PFH–HF curves as before. The purpose of this subsection is to show that this variation doesn’t change the closed–open morphism in homology level.

Let $\{ (\Omega_{\tau}, L_{\tau}, J_{\tau}) \}_{\tau \in [0,1]}$ be a family of triples on $W$ such that

1. $(\Omega_{\tau}, L_{\tau}, J_{\tau})_{|s| \geq 10} = (\Omega_H, L_{\Lambda_H}, J)$.

2. For each $\tau$, $L_{\tau}$ is a $\Omega_{\tau}$–Lagrangian submanifold over $\pi^{-1}(\partial B)$ and $L_{\tau} \cap (s, t) \times \Sigma$ is a $d$ disjoint union of circles. The link $L_{\tau} \cap (s, t) \times \Sigma$ is Hamiltonian isotropic to $\Lambda$.

3. For each $\tau$, $J_{\tau}$ is an $\Omega_{\tau}$–tame almost complex structure such that the projection $\pi_W : W \rightarrow B$ is complex linear with respect to $(J_{\tau}, j_B)$.

By the previous argument, if $J_{\tau}$ is generic, then we can define a homomorphism $(CO_{Z,ref}(W, \Omega_{\tau}, L_{\tau}))_{\tau}$ by counting PFH–HF curves with $I = 0$.

Consider the moduli space $\mathcal{M}^{\{J_{\tau}\}}(\gamma, Y, Z) := \{ (u, \tau) | u \in \mathcal{M}^{J_{\tau}}(\gamma, Y, Z) \}$. Due to the appearance of HF–ends, the curves in $\mathcal{M}^{\{J_{\tau}\}}(\gamma, Y, Z)$ cannot be multiply covered. Therefore, if the path $\{ J_{\tau} \}_{\tau \in [0,1]}$ is generic, then $\mathcal{M}^{\{J_{\tau}\}}(\gamma, Y, Z)$ is a smooth manifold of expected dimension.
Before we go on, note that it is a little misleading to denote the set of relative homology classes by \( H_2(W, \gamma, \mathbf{y}) \) because its definition also depend on the Lagrangian boundary. Now the Lagrangians \( L_\tau \) changes as \( \tau \) varies, the group \( H_2(W, \gamma, \mathbf{y}) \) should change as well. But the family \( \{(W, L_\tau)\}_{\tau \in [0,1]} \) also induces an isomorphism between the groups \( H_2(W, \gamma, \mathbf{y}) \) defined by different \( \tau \). Thus, here we use \( \tau \)-independent notation to denote \( H_2(W, \gamma, \mathbf{y}) \) and the relative homology class.

**Lemma 5.6.** Suppose that \( \{J_\tau\}_{\tau \in [0,1]} \) is a generic family of almost complex structures. Let \( \{(u_n, \tau_n)\}_{n=1}^\infty \) be a sequence of PFH–HF curves in \( \mathcal{M}^{\{J_\tau\}}(\gamma, \mathbf{y}, \mathcal{Z}) \). The sequence converges to a broken holomorphic curve \( u_\infty \) in the sense of SFT. Then

1. If \( I(\mathcal{Z}) = -1 \), then \( u_\infty \) is unbroken.

2. Assume that \( \gamma \) is a PFH generator. If \( I(\mathcal{Z}) = 0 \) and \( \tau \to \tau_* \), then

   (a) If \( J_{\tau_*} \) is generic, then \( u_\infty \) is unbroken, i.e., \( u_\infty \in \mathcal{M}^{J_{\tau_*}}(\gamma, \mathbf{y}, \mathcal{Z}) \).

   (b) If \( J_{\tau_*} \) is not generic, then we have the following two possibilities:

- \( u_\infty = \{u^+, v_1, \cdots v_k, u^-\} \), where \( u^+ \) is a \( J \)-holomorphic curve in \( \mathbb{R} \times Y_{\varphi_H} \) with \( I(u^+) = 1 \), \( v_i \) are \( \text{ind} = 0 \) connectors and \( u^- \) is a \( J_{\tau_*} \)-holomorphic PFH–HF curve in \( W \) with \( I = -1 \).

- \( u_\infty = \{u^+, u^-\} \), where \( u^+ \) is a \( J_{\tau_*} \)-holomorphic PFH–HF curve in \( W \) with \( I = -1 \) and \( u^- \) is a \( J \)-holomorphic HF–curve with \( I = 1 \).

**Proof.** Let \( u \in \mathcal{M}^{\{J_\tau\}}(\gamma, \mathbf{y}, \mathcal{Z}) \) be a PFH–HF curve with \( I(u) = -1, 0 \). Note that \( u \) doesn’t contain closed components; otherwise, the ECH index is at least \( 2k - 1 \geq 1 \). Since the path \( \{J_\tau\}_{\tau \in [0,1]} \) is generic and \( u \) is not multiply–covered, we have \( \text{ind} u \geq -1 \). The ECH inequality (it still holds even \( J_\tau \) is not generic) implies that

\[
-1 \leq \text{ind} u + 2\delta(u) \leq I(\mathcal{Z}) = -1 \text{ or } 0.
\]

If \( I(\mathcal{Z}) = -1 \), then we have \( \text{ind} u = I(u) = -1 \). If \( I(\mathcal{Z}) = 0 \) and \( \gamma \) is a PFH generator, then \( \text{ind} u = 0 \). Here \( \text{ind} u \neq -1 \) since \( \text{ind} u = I(u) \mod 2 \) by Lemma 5.3. In any case, the PFH ends of \( u \) satisfy the ECH partition conditions.

Write \( u_\infty = \{v_k, \cdots v_1, u_0, v_{-1}, \cdots v_{-m}\} \), where \( v_i \) are curves in \( \mathbb{R} \times Y_{\varphi_H} \), and \( u_0 \) is a \( J_{\tau_*} \)-holomorphic curve in \( W \) with Lagrangian boundary conditions \( L_{\tau_*} \), and \( v_{-j} \) are HF–curves in \( M \) with Lagrangian boundary conditions. Moreover, the total ECH index of \( u_\infty \) is \( I(\mathcal{Z}) \).

Write \( u_0 = u_0^0 \cup u_0^1 \) and \( v_i = v_i^0 \cup v_i^1 \), where \( u_0^0 \) is a PFH–HF curve, \( v_i^0 \) is a PFH–curve or a HF–curve, and \( u_0^1, v_i^1 \) consist of the bubbles in fibers. Since the path \( \{J_\tau\}_{\tau \in [0,1]} \) is generic, we have \( \text{ind} u_0^0 \geq -1 \). Therefore, \( I(u_0^0) \geq -1 \). By (5.31) and Lemma 3.4 each
bubble contributes at least 2 to the ECH index. Therefore, when $I(Z) = 0$ or $-1$, the bubbles can be ruled out.

Suppose that $I(Z) = -1$. Since the other levels have non-negative ECH index, we must have $I(u_0) = -1$ and $I(v_i) = 0$. Hence, $v_i$ are connectors for $i \geq 1$ and $v_j$ are trivial strips for $j \leq -1$. Since the Fredholm index is additivity, we have $indu_0 = -1$ and $indv_j = 0$. The connectors can be ruled out by the ECH partition conditions and the fact $\text{F.1}$.

Consider the case that $I(Z) = 0$. The index reason implies that either $u_\infty$ is unbroken or there is a positive or negative level with $I = 1$, $I(u_0) = -1$ and all other levels have $I = 0$. Suppose that $I(v_k) = 1$ for some $k > 0$. Then $indv_k = 1$ and all the other positive levels are connectors with $ind = 0$. The ECH partition conditions and the fact $\text{F.1}$ imply that $v_j$ are trivial connectors for $j > k$, which are ruled out. The negative levels are trivial strips, which are ruled out as well. In the case that $I(v_k) = 1$ for some $k < 0$, then $k = -1$ and the positive levels must be trivial connectors by the ECH partition conditions and the fact $\text{F.1}$. 

**Lemma 5.7.** Let $\{(\Omega_\tau, L_\tau, J_\tau)\}_{\tau \in [0, 1]}$ be a family of triples on $W$ such that $(\Omega_\tau, L_\tau, J_\tau)|_{|s| \geq 10} = (\Omega_H, L_{\Lambda_H}, J)$. Suppose that the family of almost complex structures $\{J_\tau\}_{\tau \in [0, 1]}$ is generic. Then there is a chain homotopy

$$K : \widetilde{PFC}(\Sigma, \varphi_H, \gamma_0) \to CF(\Sigma, \varphi_H, \Lambda, x)$$

such that

$$CO_{Z\text{ref}}(W, \Omega_1, L_1)_{\gamma_1} - CO_{Z\text{ref}}(W, \Omega_0, L_0)_{\gamma_0} = K \circ \partial_J + d_J \circ K.$$ 

In particular, we have $(CO_{Z\text{ref}}(W, \Omega_1, L_1)_{\gamma_1})_* = (CO_{Z\text{ref}}(W, \Omega_0, L_0)_{\gamma_0})_*$. 

**Proof.** Define the homomorphism $K$ by

$$K(\gamma, Z) = \sum_{y, I(Z) = -1} #M^{\{J_\tau\}}(\gamma, y, Z)(y, [A]),$$

where $A$ is determined by the relation $Z \# Z_{\text{ref}} \# A = Z$. By Lemma 5.6, the homomorphism $K$ is well defined.

By Lemma 5.6 and Hutchings–Taubes’s obstruction gluing argument [19, 20], then we get the result. Note that the ECH partition conditions ensure that the gluing coefficient is 1.

Here we explain a little more about the gluing argument. Hutchings and Taubes’s original obstruction gluing argument can be modified to glue a PFH–HF curve and a PFH curve with connectors in between, see Section 6.5 of [11]. Let us outline the key steps for gluing a pair $(u_+, u_-)$, where $u_+$ is an embedded PFH–curve with $I = \text{ind} = 1$ and $u_-$ is a PFH–HF curve with $I = \text{ind} = 0$. Then we will explain the changes in our situation. The gluing argument includes the following three steps:
1. Let $M$ be the moduli space of $ind = 0$ branched covering of the trivial cylinders whose ends are determined by the ECH partition conditions. Given $u_0 \in M$, construct a preglued curve $u^* : F^* \to W$. Let $\psi_{\pm}$ be sections of normal bundles of $u_{\pm}$ and $\psi_0$ be a complex function over $u_0$. Using these sections, we can deform $u^*$ to a new curve $e_{u^*}(\beta_+ \psi_+, \beta_0 \psi_0, \beta_- \psi_-)$, where $\beta_{\pm}, \beta_0$ are suitable cut--off functions.

2. The deformed curve $e_{u^*}(\beta_+ \psi_+, \beta_0 \psi_0, \beta_- \psi_-)$ is $J$–holomorphic if and only if

$$\beta_+ \Theta_+(\psi_+, \psi_0) + \beta_0 \Theta_0(\psi_+, \psi_0, \psi_-) + \beta_- \Theta_-(\psi_-, \psi_0) = 0. \quad (5.32)$$

Hence, it suffices to solves $\Theta_{\pm} = 0$ and $\Theta_0 = 0$. Fix a sufficiently small $\psi_0$. Since $u_{\pm}$ are Fredholm regular, we can find solutions $\psi_{\pm}$ such that $\Theta_{\pm} = 0$, where $\psi_{\pm}$ depends on $\psi_0$. Then we can reduce solving $\Theta_0 = 0$ to finding zeros of a section $s$ of the obstruction bundle $O \to M$.

3. Finally, Hutchings and Taubes show that $s^{-1}(0)$ only depends on the asymptotic expansion of the ends of $u_0$. Moreover, they give a combinatorial formula for $\#s^{-1}(0)$.

In our situation, we need to glue a $J_{r_0}$–holomorphic PFH–HF curve with $I = ind = -1$ and a $I = ind = 1$ PFH curves with connectors in between. The difference between the case here and the above is that $(J_{\tau}, L_{\tau})$ changes as $\tau$ vary. This difference only influences the second steps as the other steps take over the ends of holomorphic curves where $(J_{\tau}, L_{\tau})$ is fixed. As before, the deformed curve is $J_{r_0+\tau}$–holomorphic if it satisfies the Equation (5,32). Over the region of $|s| \leq 10$, $\Theta_-$ should be replaced by $\Theta_-(\psi_-, \tau) = D\bar{\partial}_{J_{r_0}, u_-} \psi_- + \tau \dot{J}_{r_0}(u_-) \circ d u \circ j + F_-(\psi_-, \tau)$, where $F_-(\psi_-, \tau)$ is type 2–quadratic (see Definition 5.1 of [20]) with respect to $\psi_-$ and $\tau$. Over the region $|s| > 10$, $\Theta_-$ is the same as before. As our $\{J_{\tau}\}_{\tau \in [0,1]}$ is generic, the operator $D\bar{\partial}_{J_{r_0}, u_-} \psi_- + \tau \dot{J}_{r_0}(u_-) \circ d u \circ j$ is a bijection for suitable Sobolev spaces. Hence, for sufficiently small $\psi_0$, we can find a unique solution $(\psi_-, \tau)$ depending on $\psi_0$. The rest of argument is the same as before.

5.4 Proof of the partial invariance

To prove the first property of Theorem 2, we want to apply the neck stretching argument. However, currently the PFH cobordism maps only can be defined via Seiberg–Witten equations which is not compatible with our setting here. To ensure that the cobordism maps can be defined by holomorphic curves, we assume that $\varphi_H$ and $\varphi_G$ satisfy $\spadesuit.1$ and $\spadesuit.2$ respectively.
Define a symplectic cobordism by

\[ X = \mathbb{R} \times S^1 \times \Sigma, \quad \omega_X = \omega + dH_{s,t} \wedge dt, \quad \text{and} \quad \Omega_X = \omega_X + ds \wedge dt, \quad (5.33) \]

where \( H_{s,t} = \chi(s)H_t + (1 - \chi(s))G_t \), where \( \chi \) is the cut-off function in (3.3). Then \((X, \Omega_X)\) gives a symplectic cobordism from \((Y_{\varphi_H}, \omega_{\varphi_H})\) to \((Y_{\varphi_G}, \omega_{\varphi_G})\) under the trivialization (2.2).

Let \((W, \Omega_G, L_{\Lambda_G})\) be a closed–open cobordism defined by a Hamiltonian \( G \) as before. Let \((M, \Omega, L_0, L_1)\) be the Lagrangian cobordism from \((\varphi_G(\Lambda), \Lambda)\) to \((\varphi_H(\Lambda), \Lambda)\) provided by Lemma 3.11. For simplicity, assume that the constant \( R_0 = 1 \) in Lemma 3.11 and (5.33). We glue \((X, \Omega_X), (W, \Omega_G, L_{\Lambda_G})\) and \((M, \Omega, L_0, L_1)\) in the following way: Let \( s_+, s_0 \) and \( s_- \) be the \( \mathbb{R} \)-coordinates of \( X, W \) and \( M \) respectively. Define

\[ (W_{r,R}, \Omega_{r,R}) := (M, \Omega)|_{s_- \leq r} \cup_{s_- = r} \sim_{s_0 = -r} (W, \Omega_G)|_{-r \leq s_0 \leq R} \cup_{s_0 = R} \sim_{s_+ = -r} (X, \Omega_X)|_{s_+ \geq -R}. \]

Note that \( \pi : W_{r,R} \to B \) still is a surface bundle over \( B \). We glue the Lagrangian submanifolds together similarly. Define

\[ L_{r,R} := (L_0 \cup L_1)|_{s_- \leq r} \cup_{s_- = r} \sim_{s_0 = -r} L_{\Lambda_G}|_{-r \leq s_0}. \]

Then \( L_{r,R} \) is a \( \Omega_{r,R} \)-Lagrangian submanifold and it consists of \( d \) disjoint connected components. Moreover, the Lagrangian \( L_{r,R} \subset \pi^{-1}(\partial B) \) satisfies

- \( L_{r,R}|_{s \leq -2r-1 \times \{0\}} = \mathbb{R}_{s \leq -2r-1} \times \{0\} \times \varphi_H(\Lambda) \),
- \( L_{r,R}|_{s \leq -2r-1 \times \{1\}} = \mathbb{R}_{s \leq -2r-1} \times \{1\} \times \Lambda \),
- and \( L|_{\{s\} \leq r} = L_{\Lambda_G} \).

Let \((W', \Omega', L') := (M, \Omega, (L_0 \cup L_1))|_{s_- \leq 3} \cup_{s_- = 3} \sim_{s_0 = -3} (W, \Omega_G, L_{\Lambda_G})|_{-3 \leq s_0} \). Set

\[ (W_R, \Omega_R, L_R) := (W_{r,R}, \Omega_{r,R}, L_{r,R})|_{r = -3}. \]

Let \( s \) denote the \( \mathbb{R} \) coordinate on \( W_{r,R} \) such that \( s = s_0 \) over \( W_G \). Take a generic path of almost complex structures \( \{J_R\}_{R \geq 1} \) on \( W_R \) such that

1. \( J_1 = J_R|_{R=1} \) is generic;
2. \( J_R = J_H \in \mathcal{J}(Y_{\varphi_H}, \omega_{\varphi_H}) \) when \( s \geq 2R + 10 \) and \( s \leq -10 \).
3. \( J_R = J_G \in \mathcal{J}(Y_{\varphi_G}, \omega_{\varphi_G}) \) when \( 10 \leq s \leq 2R - 10 \).
4. For each \( R \), \( J_R \) is an \( \Omega_R \)-tame almost complex structure such that the projection \( \pi_R : W_R \to B \) is complex linear with respect to \( (J_R, j_B) \).
5. As \( R \to \infty \), \( J_R|_{W'} \) converges in \( C^\infty_{\operatorname{loc}} \) to a generic \( J' \) and \( J_R|_X \) converges in \( C^\infty_{\operatorname{loc}} \) to a generic admissible almost complex structure \( J_X \).

We define the moduli space \( \mathcal{M}^{(J_R)}(\gamma, y, Z) := \{(u, R)|u \in \mathcal{M}^{J_R}(\gamma, y, Z)\} \).

**Lemma 5.8.** Assume that \( I(Z) = 0 \). Let \( \gamma \) be a PFH generator. Let \( \{(u_n, R_n)\}_{n=1}^\infty \) be a sequence of curves in \( \mathcal{M}^{(J_R)}(\gamma, y, Z) \). The sequence converges to a broken holomorphic curve \( u_\infty \) in the sense of SFT. Then

- If \( R_n \to \infty \), then \( u_\infty = \{u^+, v_1, \ldots, v_k, u^-\} \), where \( u^+ \) is an embedded \( J_X^- \) holomorphic curve in \( X \) with \( I = 0 \), \( v_i \) are \( \text{ind} = 0 \) connectors and \( u^- \) is an embedded \( J' \)-holomorphic PFH–HF curve in \( W' \) with \( I = 0 \).

- If \( R_n \to R_\ast \) and \( J_R \) is not a generic almost complex structure, then we have the following to possibilities:

  1. \( u_\infty = \{u^+, v_1, \ldots, v_k, u^-\} \), where \( u^+ \) is a \( J_H^- \) holomorphic curve in \( \mathbb{R} \times Y_{\varphi_H} \) with \( I(u^+) = 1 \), \( v_i \) are connectors and \( u^- \) is a \( J_{R_\ast} \)-holomorphic PFH–HF curve in \( W_{R_\ast} \) with \( I = -1 \).

  2. \( u_\infty = \{u^+, u^-\} \), where \( u^+ \) is a \( J_{R_\ast}^- \)-holomorphic PFH–HF curve in \( W_{R_\ast} \) with \( I = -1 \) and \( u^- \) is a \( J_H^- \)-holomorphic HF–curve with \( I = 1 \).

**Proof.** Consider the case that \( R_n \to \infty \). Then

\[
u_\infty = \{u^{N^+}, \ldots, u^1, u^+, v_1, \ldots, v_k, u^-, v_{-1}, \ldots, v_{-m}\}
\]

is a broken holomorphic curve with total zero ECH index, where \( u^i \) are \( J_H^- \)-holomorphic curves in \( \mathbb{R} \times Y_{\varphi_H} \), \( u^+ \) is a \( J_X^- \)-holomorphic curve in \( X \), \( v_i \) are \( J_G^- \)-holomorphic curves in \( \mathbb{R} \times Y_{\varphi_G} \), \( u^- \) is a \( J' \)-holomorphic curve in \( W' \), and \( v_{-j} \) are \( J_H^- \)-holomorphic curves in \( M \) with Lagrangian boundary conditions \( \mathbb{R} \times \{0\} \times \varphi_H(\Lambda) \cup \mathbb{R} \times \{1\} \times \Lambda \).

As we choose the almost complex structures so that the projections are complex linear, the same argument in Lemma 3.5 shows that the bubbles lie in the fibers. Hence, the inequality [5.31] still holds. In particular, \( u^- \) and \( v_{-j} \) have non-negative ECH index. By the assumptions on \( \varphi_H, \varphi_G \) and Corollary 7.4 in [9], we have \( I(u^+) \geq 0 \). Hence, each level of \( u_\infty \) has zero ECH index and no bubbles appear. The total Fredholm index is zero as well. Lemma 7.7 in [9] also implies that each level of \( u_\infty \) has zero Fredholm index.

The zero ECH index implies that \( v_{-j} \) are trivial strips and \( u^i, v_j \) are connectors. The ECH partition conditions and [F.1] can be used to show that \( u^i \) are trivial connectors. In particular, \( u^i \) and \( v_{-j} \) are ruled out.

For the case \( R_n \to R_\ast \), the proof is the same as Lemma 5.6. \( \square \)
By Lemma 5.8 and Hutchings–Taubes’s obstruction gluing argument [19, 20], we have
\[
\mathcal{CO}_{Z_1}(W_1, \Omega_1, L)_{J_1} = \mathcal{CO}_{Z_0}(W', \Omega_W', L')_{J_r} \circ \text{PF} \mathcal{C}Z_0(X, \Omega_X)_{J_X} + d_{J_H} \circ K + K \circ \partial_{J_H}.
\]
The morphism \( K : \mathcal{PFC}(\Sigma, \varphi_H, \gamma_0)_{J_H} \to \mathcal{CF}(\Sigma, \varphi_H, \Lambda, x)_{J_H} \) is defined by counting the moduli space \( \mathcal{M}^{(J_H)}(\gamma, y, Z) \) with \( I(Z) = -1 \). The cobordism map \( \text{PF} \mathcal{C}Z_0(X, \Omega_X)_{J_X} \) is defined by counting the \( I = 0 \) PFH-curves in \((X, \Omega_X)\). In particular, we have
\[
(\mathcal{CO}_{Z_1}(W_1, \Omega_1, L))_{J_1} = (\mathcal{CO}_{Z_0}(W', \Omega_W', L)_{J_r}) \circ \text{PF} \mathcal{C}Z_0(X, \Omega_X)_{J_X}.
\]
Recall that \((\Omega_1, L, J_1)_{|s| \geq 20} = (\Omega_H, L_{\Lambda_H}, J_H)\). Under the trivialization in Remark 5.2, it is easy to find a homotopy \( \{(\Omega_r, L_r, J_r)\}_{r \in [0, 1]} \) such that \((\Omega_r, L_r, J_r)\) \(\mid_{r=0} = (\Omega_H, L_{\Lambda_H}, J_H)\) and \((\Omega_r, L_r, J_r)\) \(\mid_{r=1} = (\Omega_1, L, J_1)\). By Lemma 5.7, we have
\[
(\mathcal{CO}_{Z_1}(W_H, \Omega_H, L_{\Lambda_H})_{J_H}) = (\mathcal{CO}_{Z_0}(W', \Omega_W', L)_{J_r}) \circ \text{PF} \mathcal{C}Z_0(X, \Omega_X)_{J_X}.
\]
According to Theorem 3 in [11], we can replace \( \text{PF} \mathcal{C}Z_0(X, \Omega_X)_{J_X} \) by \( \text{PF} \mathcal{C}^wZ_0(X, \Omega_X) \).

**Lemma 5.9.** We have \((\mathcal{CO}_{Z_0}(W', \Omega_W', L)_{J_r}) = I_{H, J_H} \circ (\mathcal{CO}_{Z_0}(W_G, \Omega_G, L_{\Lambda_G})_{J_G})\) for \( J_r \) such that \( I = 0 \).

**Proof.** The proof also use the neck stretching argument, but we stretch the end of the other side. Let
\[
(W_r, \Omega_r, L_r) := (M, \Omega, (L_0 \cup L_1))_{|s_1 \leq r + 3 \cup s_2 \geq r + 3 \sim s_0 = -r - 3} (W, \Omega_G, L_{\Lambda_G})_{-r - 3 \leq s_0}.
\]
Let \( \{J_r\}_{r \geq 0} \) be a generic family of almost complex structures such that
1. \( J_r = J' \) on \( W_G \) converges to \( J_G \) and \( J_r \) on \( M \) converges to \( J'' \).
2. \( J_r = J_G \) over the region \(-2r + 10 \leq s_0 \leq -10\).
3. \( J_r \) is \( \Omega_r \)-tame. Also, the projection \( \pi_r : W_r \to B \) is complex linear with respect to \( (J_r, J_B) \).

Let \( \{u_n\}_{n=1}^\infty \) be a sequence of \( J_{r_n} \)-holomorphic PFH–HF curves in \( W_{r_n} \). Using the same argument in Lemma 5.8, then we get the following results:
- If \( r_n \to \infty \), then \( u_\infty = \{u^+, u^\} \), where \( u^+ \) is an embedded \( J_G \)-holomorphic PFH–HF curve in \( W_G \) with \( I = 0 \) and \( u^- \) is an embedded \( J'' \)-holomorphic curve in \( M \) with \( I = 0 \).
- If \( r_n \to r_* \) and \( J_{r_*} \) is not a generic almost complex structure, then we have the following two possibilities:
1. $u_\infty = \{u^+, v_1, \cdots, v_k, u^-, \}$, where $u^+$ is a $J_G$–holomorphic curve in $\mathbb{R} \times Y_{\varphi_G}$ with $I(u^+) = 1$, $v_i$ are connectors and $u^-$ is a $J_{r_*}$–holomorphic PFH–HF curve in $W_{r_*}$ with $I = -1$.

2. $u_\infty = \{u^+, u^-\}$, where $u^+$ is a $J_{r^*}$–holomorphic PFH–HF curve in $W_{r^*}$ with $I = -1$ and $u^-$ is a $J_H$–holomorphic HF–curve with $I = 1$.

By the gluing argument, we finish the proof of the statement.

Combine Lemma 5.9 and (5.34); then we finish the proof of the partial invariance in Theorem 2.

6 Computations

In this section, we first need to fix a Morse function as follows. Let $f_{\Lambda_i} : \Lambda_i \to \mathbb{R}$ be a perfect Morse function with a minimum $y^-_i$ and a maximum $y^+_i$. We extend $\cup_i f_{\Lambda_i}$ to be a Morse function $f : \Sigma \to \mathbb{R}$ such that

1. $(f, g_\Sigma)$ satisfies the Morse–Smale condition, where $g_\Sigma$ is a fixed metric on $\Sigma$.

2. $f = f_{\Lambda_i} - \frac{1}{2} y^2$ in a neighbourhood of $\Lambda_i$, where $y$ is the coordinate of the normal direction.

3. $\{y^+_i\}_{i=1}^d$ are the only maximum points of $f$. Also, $f(y^+_i) = 0$ for any $1 \leq i \leq d$.

The picture of $f$ is shown in Figure 2. Define a function $H_\epsilon := \epsilon f$, where $0 < \epsilon < 1$ is a small constant.

We prove the non–vanishing results in this section. To this end, the idea is to compute the closed–open morphism for a special Hamiltonian $G$ and then using the partial invariance in Theorem 2 to deduce the non–vanishing for other $H$.

The Morse function $H_\epsilon$ is a nice candidate for computation. In the next subsection, we show that the Reeb chords and periodic orbits of $\varphi_{H_\epsilon}$ correspond to the critical points of $f_\Lambda$ and $f$ respectively. Also, the ECH index and the energy of holomorphic curves in $(W, \Omega_{H_\epsilon}, L_{\Lambda_{H_\epsilon}})$ are computable. Combining the energy and index reasons, we could compute $CO_{Zrcf}(W, \Omega_{H_\epsilon}, L_{\Lambda_{H_\epsilon}})\ast$. However, to apply the partial invariance in Theorem 2 we require that $H_\epsilon$ satisfies assumption $\clubsuit.1$ or $\spadesuit.2$ while $H_\epsilon$ does not. Therefore, we need to modify $H_\epsilon$ to $H'_\epsilon$ satisfying $\spadesuit.2$. After this modification, the periodic orbits of $\varphi_{H'_\epsilon}$ is no longer corresponding to the critical points any more. We need to do a bit more works. For example, it is not clear which PFH generators represented the unit of PFH. Thus we need to compute the PFH cobordism maps.
6.1 Reeb chords and period orbits of $\varphi_{H_\epsilon}$

**Lemma 6.1.** If $0 < \epsilon \ll 1$ is sufficiently small, then $y = [0, 1] \times \{ y_1^\epsilon, \ldots, y_d^\epsilon \}$ are the only Reeb chords of $\varphi_{H_\epsilon}$, where $\epsilon \in \{+, -\}$.

**Proof.** Note that $\text{dist}(\varphi_{H_\epsilon}(x), x) \leq c_0|\nabla f|$. We may assume that $\varphi_{H_\epsilon}(\Lambda_i)$ lies inside a small neighbourhood of $\Lambda_i$, $1 \leq i \leq d$. Let $(x, y)$ be the coordinates near $\Lambda_i$, where $x$ is the coordinate of $\Lambda_i$ and $y$ is the coordinate of the normal direction. We may assume that the symplectic form $\omega = dx \wedge dy$ under these coordinates. It is easy to check that the Hamiltonian vector field is $X_{H_\epsilon} = -\epsilon y \partial_x - \epsilon f'_{\Lambda_i}(x) \partial_y$. Therefore, the Hamiltonian flow is

$$
\begin{align*}
\dot{x} &= -\epsilon y \\
\dot{y} &= -\epsilon f'_{\Lambda_i}(x).
\end{align*}
$$

(6.35)

Take a neighbourhood $\mathcal{U}_{2\delta_0} = \mathcal{U}_{2\delta_0}^0 \cup \mathcal{U}_{2\delta_0}^1 \subset \Lambda_i$ of $f_{\Lambda_i}$’s critical points such that

- $f'_{\Lambda_i}(x) \geq \delta_0$ or $f'_{\Lambda_i}(x) \leq -\delta_0$ away from a smaller neighbourhood $\mathcal{U}_{\delta_0} \subset \mathcal{U}_{2\delta_0}$.

- $f_{\Lambda_i}|_{\mathcal{U}_{2\delta_0}^0} = m_p + \frac{1}{2}x^2$ and $f_{\Lambda_i}|_{\mathcal{U}_{2\delta_0}^1} = -\frac{1}{2}x^2$.

Suppose that for any $\epsilon_n \to 0$, we can find a non–constant solution $(x_n, y_n)$ to (6.35) satisfying $(x_n(0), 0), (x_n(1), 0) \in \Lambda_i$. Take $n \to \infty$; $(x_n, y_n)$ converges in $C^\infty$ to a constant $(x_0, 0)$. If $x_0$ is outside $\mathcal{U}_{\delta_0}$, then we have $f'_{\Lambda_i}(x_0) \geq \frac{\delta_0}{2}$ or $f'_{\Lambda_i}(x_0) \leq -\frac{\delta_0}{2}$ for $n$ large enough. Equations (6.35) implies that $y_n(1) \leq -\frac{\delta_0\epsilon_n}{2}$ or $y_n(1) \geq \frac{\delta_0\epsilon_n}{2}$. This contradicts with $y_n(1) = 0$.

If $x_0$ lies in the $\delta_0$–neighbourhood of the maximum, we may assume that $(x_n, y_n)$ lies inside $\mathcal{U}_{2\delta_0}^1 \times (-2\delta_0, 2\delta_0)$ where $f_{\Lambda_i}(x) = -\frac{1}{2}x^2$. Then it is easy to check that

$$
\begin{align*}
x_n(1) &= \cosh(\epsilon_n)x_n(0) - \sinh(\epsilon_n)y_n(0) \\
y_n(1) &= -\sinh(\epsilon_n)x_n(0) + \cosh(\epsilon_n)y_n(0).
\end{align*}
$$

Again, $(0, 0)$ is the only solution satisfying $y_n(0) = y_n(1) = 0$.

\[\square\]

**First modification on $H_\epsilon$:** The purpose of this modification is to ensure that there exist periodic orbits with large period near the local minimum of $H_\epsilon$. Even these orbits do not contribute to the chain complex $\widehat{PFC}(\Sigma, \varphi_{H_\epsilon}, \gamma_0)$, they are useful when we compute the cobordism maps and closed–open morphisms.
Let \((x, y)\) be the local coordinates near a local minimum \(p\) of \(f\) such that \(\omega = dx \wedge dy\). Under these coordinates, we choose \(f\) such that \(f = m_p + x^2 + y^2\). Let \((r, \theta)\) be the polar coordinates. Let \(\mathcal{U}_p^\delta = \{(r, \theta)| r < \delta\}\) denote the \(\delta\)–neighbourhood of \(p\). Define \(\mathcal{U}^\delta = \bigcup_p \mathcal{U}_p^\delta\), where \(p\) run over all the local minimums of \(f\). Fix positive constants \(\delta, \delta_0 > 0\). Take a function \(\varepsilon(r)\) such that

- \(\varepsilon(r) = \epsilon\) on the regions \(\mathcal{U}^{\delta+\delta_0}\) and \(\Sigma - \mathcal{U}^{\delta+2\delta_0}\). Here \(\epsilon > 0\) is a small constant. Also, \(\varepsilon(r) \geq \epsilon\) on the whole \(\Sigma\).

- Over the region \(\{\delta + \delta_0 \leq r \leq \delta + 2\delta_0\}\), \(\varepsilon\) is a function that only depends on \(r\). The maximum of \(\varepsilon(r)\) is denoted by \(\epsilon_0\). We choose \(\varepsilon(r)\) such that \(0 < \frac{\epsilon_0 - \epsilon}{\delta_0} \ll 1\) is small enough.

Replace \(H_\varepsilon\) by \(H_\varepsilon := \varepsilon(r)f\). Then we have

\[
(\varepsilon(r)f(r))' = \varepsilon'(r)(m_p + r^2) + 2\varepsilon(r)r \geq 2\epsilon(\delta + \delta_0) - c_0\frac{\epsilon_0 - \epsilon}{\delta_0} > 0.
\]

Therefore, the replacement doesn’t introduce any new critical point and \(H_\varepsilon\) is still a Morse function satisfying the Morse–Smale condition. The Hamiltonian vector field of \(H_\varepsilon\) over \(\{r \leq \delta + 2\delta_0\}\) is

\[
X_{H_\varepsilon} = -\frac{1}{r} H_\varepsilon'(r)\partial_\theta.
\]

For \(r_0 \in (\delta + \delta_0, \delta + 2\delta_0)\) such that \(\frac{1}{r_0} H_\varepsilon'(r_0) = \frac{p}{q} \neq 0\) is a rational number \((p, q\) are relatively prime), then \(H_\varepsilon\) has a family of periodic orbits of the form

\[
\gamma_{r_0, \theta_0}(\tau) = (\tau, r_0, \theta_0 - \frac{p}{q}\tau),
\]

where \(\tau \in \mathbb{R}/(q\mathbb{Z})\). Since \(\varepsilon(r)\) and \(\varepsilon'(r)\) are very small, the period \(q > d\).

Suppose that \(\epsilon \leq \tilde{\epsilon}\) is sufficiently small, then the periodic orbits of \(\varphi_{H_\varepsilon}\) with period less than or equal \(d\) are still \(1\)–\(1\) corresponding to the iterations of the constant orbits at critical points of \(H_\varepsilon\). The critical points of \(H_\varepsilon\) are described in Figure 2

Let \(p \in \text{Crit}(H_\varepsilon)\) and \(\gamma_p\) denote the constant simple periodic orbit at \(p\). Note that the degree of \(\gamma_p\) is 1. Let \(\alpha = \gamma_{p_1} \cdots \gamma_{p_d}\) and \(\beta = \gamma_{q_1} \cdots \gamma_{q_d}\) be two orbit sets. Here \(\{p_i\}, \{q_i\}\) could repeat. Let \(\eta: \bigcup_{i=1}^d \mathbb{R}s_i \to \Sigma\) be a path with \(d\) components such that \(\lim_{s_i \to -\infty} \eta(s_i) = p_i\) and \(\lim_{s_i \to -\infty} \eta(s_i) = q_i\). Define a relative homology class

\[
Z_{\alpha, \beta} := [S^1 \times \eta] \in H_2(Y_{\varphi_{H_\varepsilon}}, \alpha, \beta).
\]

Define a chain \(c_\varnothing = \sum (\alpha_I, Z_I) \in \overline{PFC}(\Sigma, \varphi_{H_\varepsilon}, \gamma_0)\). The notations are explained as follows:

**C.1** \(\alpha_I = \gamma_{i_1}^{+} \cdots \gamma_{i_d}^{+}\) and \(I = (i_1, \cdots, i_d)\). The labels \(i_1, \cdots, i_d\) could repeat.
Figure 2: The red points are the index 2 critical points. The green points are saddle points. The blue points are index 0 critical points. The union of the red circles is the link. The function $f$ is a small perturbation of the height function.

**C.2** $Z_I \in H_2(Y_{\varphi_{H_\varepsilon}}, \alpha_I, \gamma_0)$. For any $I, I'$, $Z_I - Z_{I'} = Z_{\alpha_I, \alpha_{I'}}$.

When $I = (1, 2, \cdots d)$, we write $\alpha_I$ as $\alpha_\bigtriangleup = \gamma_{y_1^+} \cdots \gamma_{y_d^+}$.

**Lemma 6.2.** The chain $c_\bigtriangleup = \sum_I (\alpha_I, Z_I)$ is a cycle. Moreover, it represents a non-zero class $\sigma_{H_\varepsilon} \in \tilde{PFH}(\Sigma, \varphi_{H_\varepsilon}, \gamma_0)$.

**Proof.** As remarked in Page 11 of [12], the holomorphic current in $\mathbb{R} \times Y_{\varphi_{H_\varepsilon}}$ contributed to the differential is a union of trivial cylinders and a unique non-trivial holomorphic cylinder. For sufficiently small $\varepsilon \ll 1$, such a non-trivial cylinder corresponds to a Morse flow line of $H_\varepsilon$. Therefore, the computation of the PFH differential is reduced to the computation of the Morse differential. The latter is easy to be figure out for our $H_\varepsilon$. (In Figure 2 for any two maximums adjacent to each other, there is a unique flow line from the maximum to the saddle point in the middle.)

**Remark 6.1.** The class $\sigma_{H_\varepsilon}$ here can be regarded as the “unit” of the periodic Floer homology. Similar to the unit of the quantitative Heegaard Floer homology, the class $\sigma_{H_\varepsilon}$ is equal to $\text{PFH}^{sw}_Z(\Sigma, \varphi_{H_\varepsilon}, \gamma_0)(1)$. Here $X = B_+ \times \Sigma$ is a cobordism from empty set to $Y_{\varphi_{H_\varepsilon}}$; the surface $B_+$ is a punctured sphere with a negative cylindrical end. One could show that $c_\bigtriangleup$ is a cycle by computing the cobordism map $\text{PFH}^{sw}_Z(\Sigma, \varphi_{H_\varepsilon}, \gamma_0)(1)$.

**Second modification on $H$:** We expect that the closed–open morphism maps $c_\bigtriangleup$ to a cycle represented the unit in HF. However, to apply the partial invariance in Theorem 2, we need to perturb $\varphi_{H_\varepsilon}$ so that it satisfies the condition $\bigdiamondsuit.1$ or $\bigdiamondsuit.2$.

According to the Proposition 3.7 of [9], we modify $H_\varepsilon$ to $H'_\varepsilon$ such that $\varphi_{H'_\varepsilon}$ satisfies the condition $\bigdiamondsuit.2$. Such a modification takes over in an arbitrary small neighbourhood
of the periodic orbits with degree less than \(d\) which violates \(\bullet.2\). This modification may create new periodic orbits that violate \(\bullet.2\) in the neighbourhood. But these new orbits have larger period. We repeat the modification in an arbitrary small neighbourhood of the orbits violated \(\bullet.2\) until the new created orbits have period greater than \(d\).

The modifications are local, hence we can arrange that \(X_{H_\varepsilon} = X_{H'_\varepsilon}\) outside \(U^\delta\). In particular, \(\gamma_{\varepsilon,\theta_0}\) in (6.37) are still periodic orbits of \(\varphi_{H'_\varepsilon}\). According to the construction, \(H'_\varepsilon\) satisfies the estimates in Remark 1.5.

### 6.2 Some computations on the PFH cobordism maps

In this section, we find a cycle which represents the “unit” in \(\widetilde{PFH}(\Sigma, \varphi_{H'_\varepsilon}, \gamma_0)\).

Let \((X_+, \Omega_{X_+})\) be the symplectic cobordism from \((Y_{\varphi_{H'_\varepsilon}}, \varphi_{H'_\varepsilon})\) to \((Y_{\varphi_{H_\varepsilon}}, \varphi_{H_\varepsilon})\) defined by (5.33). Keep in mind that \(\Omega_{X_+}\) is \(\mathbb{R}\)-invariant in the region \(\mathbb{R} \times (Y_{\varphi_{H'_\varepsilon}} - S^1 \times U^\delta) \subset X_+\). This region is called a product region. Here we identify \(S^1 \times U^\delta\) as a subset of \(Y_{\varphi_{H_\varepsilon}}\) implicitly by using the trivialization (2.2).

**Lemma 6.3.** Let \(\alpha_+\) be the PFH generator of \((Y_{\varphi_{H'_\varepsilon}}, \varphi_{H'_\varepsilon})\) defined in C.1. Let \(J_+\) be an admissible almost complex structure on \(X_+\) such that it is \(\mathbb{R}\)-invariant in the product region \(\mathbb{R} \times (Y_{\varphi_{H'_\varepsilon}} - S^1 \times U^\delta) \subset X_+\). Let \(Z_{\text{ref}^+} = [\mathbb{R} \times \gamma_0] \in H_2(X_+, \gamma_0, \gamma_0)\) be the reference relative homology class. Then we have

\[
PFC^{\text{ref}^+}_Z(X_+, \Omega_{X_+}, J_+)(\alpha_+, Z_1) = (\alpha_+, Z_1).
\]

**Proof.** Let \(\alpha_- = \gamma_{p_1} \cdots \gamma_{p_d}\) be a PFH generator of \(\widetilde{PFH}(\Sigma, \varphi_{H_\varepsilon}, \gamma_0)\). Let \(\eta \subset \Sigma\) be a union of paths starting from the critical points \(\{p_i\}_{i=1}^d\) and ending at critical points \(\{y_i^+\}_{i=1}^d\). Let \(S_{\eta} = \eta \times S^1\). This gives us a reference relative homology class \(Z_{\alpha_+, \alpha_-} = [S_{\eta}] \in H_2(X_+, \alpha_+, \alpha_-)\). Any other class \(Z \in H_2(X_+, \alpha_+, \alpha_-)\) can be written as \(Z = Z_{\alpha_+, \alpha_-} + M(Z)[\Sigma] + [S_Z]\), where \([S_Z] \in H_1(S^1, \varepsilon) \otimes H_1(\Sigma, Z)\). It is easy to check the ECH index of \(Z \in H_2(X_+, \alpha_+, \alpha_-)\) is

\[
I(\alpha_+, \alpha_-, Z) = 2d - h(\alpha_-) - 2e_+(\alpha_-) + 2M(Z)(k + 1), \tag{6.39}
\]

where \(h(\alpha_-)\) is the number of hyperbolic orbits in \(\alpha_-\) and \(e_+(\alpha_-)\) is the total multiplicities of periodic orbits \(\gamma_{y_i^+}\) in \(\alpha_-\). The \(\omega_{X_+}\)-energy of \(Z\) is

\[
E_{\omega_{X_+}}(Z) = \int_Z \omega_{X_+} = H_\varepsilon(\alpha_+) - H_\varepsilon(\alpha_-) + M(Z).
\]

Suppose that \(< PFC^{\text{ref}^+}_Z(X_+, \Omega_{X_+}, J_+)(\alpha_+, Z_1), (\alpha_-, Z_-) > \neq 0\). Then the holomorphic curve axiom in Theorem \(A.1\) provides us a \(J_+\)-holomorphic current \(C\) with
\( I(\mathcal{C}) = 0 \). Moreover, we have the following estimates:

\[
E_{\omega_{X_+}}(\mathcal{C}) = \int_{\mathcal{C}} \omega + d\Sigma H_s \wedge dt + \partial_s H_s ds \wedge dt \\
\geq \int_{\mathcal{C}} \omega + d\Sigma H_s \wedge dt - d \int_{S^1} \max_{\Sigma} H'_\epsilon - H_\epsilon dt \\
\geq - dc_0 \delta.
\]

The last step follows from the fact that the term \( \int_{\mathcal{C}} \omega + d\Sigma H_s \wedge dt \) is non-negative (see Lemma 3.8 in [5]).

Let \( Z \) be the relative homology class of \( \mathcal{C} \). Since \( M^J(\alpha_1, \alpha_-; Z) \neq \emptyset \), we have \( M(Z) \geq 0 \); otherwise, \( E_{\omega_{X_+}}(Z) \leq -1 + O(\delta) < - dc_0 \delta \) which contradicts (6.40). By (6.39), \( I(Z) = 0 \) implies that \( M(Z) = h(\alpha_-) = 0 \) and \( e_+(\alpha_-) = d \). In other words, \( Z = Z_{\alpha_1, \alpha_{1'}} \) and \( \alpha_- \) only consists of periodic orbits \( \gamma_{y_i^+} \). By the next lemma (Lemma 6.4) and the holomorphic axioms in Theorem A.1, we get the results.

**Lemma 6.4.** Let \( J_+ \) be an admissible almost complex structure on \( X_+ \) such that it is \( \mathbb{R} \)-invariant in the product region \( \mathbb{R} \times (Y - S^1 \times U^d) \subset X_+ \). Let \( \alpha_{1'} = \gamma_{y_i^{1'}} \cdots \gamma_{y_i^{d'}} \). Let \( Z_{\alpha_1, \alpha_{1'}} \in H_2(X_+, \alpha_1, \alpha_{1'}) \) be the relative class defined in Lemma 6.3. Then the moduli space \( M^J(\alpha_1, \alpha_{1'}; Z_{\alpha_1, \alpha_{1'}}) \neq \emptyset \) if and only if \( \alpha'_1 = I' \). Moreover, \( \mathbb{R} \times \alpha_1 \) is the unique element in \( M^J(\alpha_1, \alpha_{1'}; Z_{\alpha_1, \alpha_{1'}}) \).

**Proof.** Let \( \mathcal{C} \in \overline{M^J(\alpha_1, \alpha_{1'}; Z_{\alpha_1, \alpha_{1'}})} \) be a broken holomorphic current. The intersection number \( \#(\mathcal{C} \cap \mathbb{R} \times \gamma_{r_0, \theta_0}) \) is well defined. Moreover, the intersection number only depends on the relative homology class of \( \mathcal{C} \). Therefore, we have

\[
\#(\mathcal{C} \cap \mathbb{R} \times \gamma_{r_0, \theta_0}) = \#(S \cap \mathbb{R} \times \gamma_{r_0, \theta_0}) = 0.
\]

By our choice of \( J_+ \), \( \mathbb{R} \times \gamma_{r_0, \theta_0} \) is a \( J_+ \)-holomorphic curve. According to the intersection positivity of holomorphic curves, \( \mathcal{C} \) doesn’t intersect \( \mathbb{R} \times \gamma_{r_0, \theta_0} \). Since \( \theta_0 \) is arbitrary, \( \mathcal{C} \) must lie inside \( \mathbb{R} \times (Y - (S^1 \times U^{d+\delta_0})) \).

Note that \( E_{\omega_{X_+}}(\mathcal{C}) = E_{\omega_{X_+}}(Z_{\alpha_1, \alpha_{1'}}) = 0 \). Since \( J_+ \) is \( \mathbb{R} \)-invariant over \( \mathbb{R} \times (Y - (S^1 \times U^{d+\delta_0})) \), we can conclude that \( \mathcal{C} = \mathbb{R} \times \alpha_1 \) (see Proposition 9.1 of [13]). In particular, \( \alpha_{1'} = \alpha_1 \).

Let \( (X_-, \omega_{X_-}) \) be the symplectic cobordism from \( (Y_{\varphi_{H_1^+}}; \varphi_{H_1}) \) to \( (Y_{\varphi_{H_1^+}}; \varphi_{H_1}) \) defined as in (5.33). Let \( J_- \) be an admissible almost complex structure such that \( J_- = J_+ \) on the product region \( \mathbb{R} \times (Y - (S^1 \times U^{d+\delta_0})) \subset X_- \). Let \( Z_{ref_+} = [\mathbb{R} \times \gamma_0] \in H_2(X_-, \gamma_0, \gamma_0) \) be the reference relative homology class. By the composition law of the PFH cobordism maps, we have

\[
PFC_{Z_{ref_-}}^{sw}(X_-, \Omega_{X_-}) \circ PFC_{Z_{ref_+}}^{sw}(X_+, \Omega_{X_+} \circ J_+ = Id + K \circ \partial' + \tilde{\partial} \circ K. \quad (6.41)
\]
By Lemma 6.3 and (6.41), we get a cycle

\[ c \circ + K \circ \partial' c \in \widetilde{PFC} (\Sigma, \varphi_{H'_e}, \gamma_0), \]

where \( c \circ = \sum (\alpha_I, Z_I) \) is defined by C.1 C.2. Also, it represents a non–vanishing class of \( PFH (\Sigma, \varphi_{H'_e}, \gamma_0) \) because the cobordism map \( PFH^{sw}_{\text{ref}} (X_-, \Omega_{X_-} J_-) \) is an isomorphism.

**Lemma 6.5.** Write \( K \circ \partial' (\alpha_I, Z_I) = \sum (\beta, Z) \). Then we have

\[ E_{\omega_{\varphi_{H'_e}}} (Z_1 - Z) = H'_{\varphi_{H'_e}} (\alpha, Z_1) - H'_{\varphi_{H'_e}} (\beta, Z) > -dC_0 \delta. \]

**Proof.** If \( < K (\beta, Z), (\beta', Z') > \neq 0 \), then the holomorphic curve axioms (Theorem A.1) imply that there exists a \( J+ \circ_R J_- \)-holomorphic current (possibly broken) \( C' \) in \( X_+ \circ_R X_- \). (The definition of \( X_+ \circ_R X_- \) and \( J+ \circ_R J_- \) can be found in the Appendix.) By the same computation as in (6.40), we have

\[ E_{\omega_{X_+ \circ_R X_-}} (Z - Z') = E_{\omega_{X_+ \circ_R X_-}} (C') \geq -d \max \left| H_\varepsilon - H'_\varphi \right| \geq -dC_0 \delta. \]

Note that \( \omega_{X_+ \circ_R X_-} = \omega_{\varphi_{H'_e}} + d(\alpha \wedge dt) \), where \( \alpha \) is supported on \( |s| \leq R \). By Stokes’s theorem, the above equation is equivalent to

\[ E_{\omega_{\varphi_{H'_e}}} (Z - Z') \geq -dC_0 \delta. \] (6.42)

On the other hand, if \( < \partial' (\alpha_I, Z_I), (\beta, Z) > \neq 0 \), then

\[ E_{\omega_{\varphi_{H'_e}}} (Z_1 - Z) > 0. \] (6.43)

Then the conclusion follows from inequalities (6.43) and (6.42) \( \square \).

### 6.3 Unit of HF

In this subsection, we describe the unit for in \( HF (Sym^d \varphi_{H_e} (\Lambda), Sym^d \Lambda) \).

Recall that \( D_0 \) is a disk with a negative strip-like end (see 2.12). Let \( \mathbb{E} := D_0 \times M \), where \( M = Sym^d \Sigma \). Obviously, \( \pi_M : \mathbb{E} \to D_0 \) is a fiber bundle over \( D_0 \). To begin with, let us construct a symplectic form and a Lagrangian explicitly over \( E \). Let \( K \) be an automatous Hamiltonian function on \( M \). Let \( s \) be the coordinate on the end of \( D_0 \). Let \( \chi : D_0 \to \mathbb{R} \) be a non-increasing cut-off function such that \( \chi = 1 \) when \( s \leq -R_0 \) and \( \chi = 0 \) when \( s \geq -1 \). Define a 2-form

\[ \omega_0 := \omega_V + d(\chi(s)K \wedge dt). \]

Define a diffeomorphism

\[ \Phi : \mathbb{R}_- \times [0, 1] \times M \to \mathbb{R}_- \times [0, 1] \times M \]

\[ (s, t, x) \to (s, t, (\varphi_{K}^{\chi(s)})^{-1}(x)). \]
Because $\Phi = id$ when $s \geq -1$, we can extend it to be $id$ over the rest of $E$. Let $\varphi^t = \varphi^t_K$.

Note that $(\varphi^t)^*K = K$ because $K$ is automatous. By a direct computation, we have

\[
\begin{align*}
\Phi^{-1}_s(\partial_s) &= \partial_s + t\chi(s)X_K \circ \varphi^s\chi(s)t \\
\Phi^{-1}_s(\partial_t) &= \partial_t + \chi(s)X_K \circ \varphi^s\chi(s)t \\
\Phi^{-1}_s(v) &= \varphi^s\chi(s)t(v).
\end{align*}
\]

Combining these ingredients, we get a 2-form

\[
\omega_E := (\Phi^{-1})^*\omega_0 = \omega_V + t\chi(s)ds \wedge dK + \chi Kds \wedge dt.
\]

Note that $\omega_E = \omega_V$ when $s \leq -R_0$. The symplectic form on $E$ is defined by $\Omega_E := \omega_E + \omega_{D_0}$. Also, $L := \Phi(\partial D_0 \times \varphi_K(Sym^d\Lambda))$ is a $\Omega_E$-Lagrangian such that

\[
L|_{s \leq -R_0} = \mathbb{R}|_{s \leq -R_0} \times ((\{0\} \times \text{Sym}^d\Lambda \cup \{1\} \times \varphi_K(Sym^d\Lambda)).
\]

Now we take $K$ to be a Hamiltonian which is compatible with $Sym^dH_\epsilon$, i.e., $K$ satisfies

\[
\begin{align*}
K &= \text{Sym}^dH_\epsilon \text{ outside the diagonal } \Delta, \\
K &= \text{ is a (t-dependent) constant near diagonal.}
\end{align*}
\]

We can choose $K$ to be an automatous Hamiltonian function such that $K \leq 0$ and the only maximum point of $K$ is $\gamma = \{y_1^+,..,y_d^+\}$. Note that $\varphi_K(Sym^d\Lambda) = Sym^d\varphi_{H_\epsilon}(\Lambda)$. The intersection points of $\varphi_E(Sym^d\Lambda) \cap \text{Sym}^d\Lambda$ are still described by Lemma 6.1.

Let $J_E$ denote the set of $\Omega_E$-compatible almost complex structures on $E$ satisfying the following conditions:

1. $d\pi_E \circ J = j_0 \circ d\pi_E$.

2. $J$ preserves the vertical bundle $TM$. Also, $J|_{TM}$ is compatible with $\Omega_E|_{TM}$.

3. Over the end of $E$, $J$ is $\mathbb{R}$-invariant, $J(\partial_s) = \partial_t$.

For each $y = [0,1] \times \{y_1,..,y_d\}$, we construct a relative homology class $S_y$ as follows:

Let $\eta = \bigcup_{i=1}^d \eta_i : \oplus_i [0,1] \to \Lambda$ be a $d$-union of paths in $\Lambda$, where $\eta_i \subset \Lambda_i$ satisfies $\eta_i(0) = y_i$ and $\eta_i(1) = x_i$. Let $u_i(s,t) = (s,t,(\varphi^{t}_{H_\epsilon})^{-1} \circ \varphi_{H_\epsilon}(\eta_i(s)))$. Then $u = \bigcup_{i=1}^d u_i$ is a $d$-multisection and it represents a class $A_y \in H_2(M,\mathbf{x}_{H_\epsilon},y)$. Using the tautological correspondence, $u$ gives arise a relative homology class $S_y \in H_2(M,\mathbf{x}_{H_\epsilon},y)$. Note that $u_i \cap u_j = \emptyset$ for $i \neq j$, the tautological correspondence can be constructed easily. By the same computation in [3.23], we know that $\int_{S_y} \omega = K(x) - K(y) = H_\epsilon(x) - H_\epsilon(y)$.

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Lemma 6.6. Take $\mathbb{K}$ be the above function. Let $y_\mathcal{O} = [0, 1] \times (y_1^+, \ldots, y_d^+)$. Let $J \in J_\mathbb{K}$ be a generic almost complex structure. Let $S_{\text{ref}}$ be the reference class that is represented by $\Phi(D_0 \times \varphi_\mathbb{K}(x))$. Then we have

$$CF_{S_{\text{ref}}}(E, \Omega_E, L)_J(1) = (y_\mathcal{O}, [S_{y_\mathcal{O}}]).$$

In particular, $(y_\mathcal{O}, [S_{y_\mathcal{O}}])$ is a cycle represented the unit.

Proof. Let $\mathcal{M}^d(\emptyset, y, S)$ be the moduli space of holomorphic sections in $E$ with Lagrangian boundary condition $L$. Let $u \in \mathcal{M}^d(\emptyset, y, S)$ be a curve contributed to $CF_{S_{\text{ref}}}(E, \Omega_E, L)_J(1)$. Note that

$$\int_{S_{\text{ref}}} \omega_E = \int_{D_0 \times \varphi_\mathbb{K}(x)} \omega_V + d(\chi(s)K \wedge dt) = -K(x) \text{ and } S_{\text{ref}} \cdot \Delta = 0.$$

Let $S_0 \in H_2(M, x_H, y)$ be the class determined by $S = S_{\text{ref}} \# S_0$. Then

$$\int u^* \omega_E = \int |d^{\text{vert}}u|^2 + \omega_E(d^{\text{hor}} u, J^{\text{hor}} d^{\text{hor}} u) = -\mathfrak{A}_K(y, S_0), \quad u \cdot \Delta = S_{\text{ref}} \cdot \Delta + \Delta \cdot S_0 = \Delta \cdot S_0$$

(6.44)

where $d^{\text{vert}}u \in T^{\text{vert}}E$ and $d^{\text{hor}} u \in T^{\text{hor}}E$ are respectively the vertical and horizontal components of $du$. It is easy to check that $T^{\text{hor}}E = \text{span}\{\partial_s - t\chi X_\mathbb{K}, \partial_t\}$. Therefore, $\omega_E|_{T^{\text{hor}}E} = \chi K \omega_{D_0}$. Hence, $\int u^* \omega_E = \int |d^{\text{vert}}u|^2 + \chi K |d^{\text{hor}} u|^2 \geq 0$. We can choose $J$ such that $J = \text{Sym}^d J_2$ near the diagonal. Therefore, $u \cdot \Delta \geq 0$ by intersection positivity. In particular,

$$\int u^* \omega_E + \eta u \cdot \Delta = -\mathfrak{A}_K^0(y, S_0) \geq 0.$$

(6.45)

Moreover, it is easy to check that

$$0 = \text{ind} u = n(y) + \sum_{i=1}^{k+1} 2c_i$$

$$\mathfrak{A}_K^0(y, [S_0]) = \mathfrak{A}_K(y, [S_0]) - \eta \Delta \cdot S_0$$

$$= H_\varepsilon(y) - \sum_{i=1}^{k} c_i \lambda - c_{k+1} \left( \int_{B_{k+1}} \omega + 2\eta(d + g - 1) \right)$$

$$= H_\varepsilon(y) - \lambda \sum_{i=1}^{k} c_i, \quad \lambda = \sum_{i=1}^{k+1} 2c_i$$

(6.46)

where $n(y)$ is the number of $y_i^-$-components. By (6.44), (6.45) and (6.46), we know that $y = y_\mathcal{O}$, $\int u^* \omega_E = 0$ and $d^{\text{vert}}u = 0$. Therefore, the horizontal section $u = D_0 \times \{y_\mathcal{O}\}$ is the only holomorphic curve contributed to $CF_{S_{\text{ref}}}(E, \Omega_E, L)_J(1)$.
By the construction of the isomorphism $\Phi_{H_e}$ in Theorem 1, the above lemma implies that $\Phi_{H_e}([((y\circ,Ay\circ))]) = [((y\circ,SY\circ))].$ Then $(y\circ,Ay\circ)$ is a cycle and represents a non-zero class $e^*_{H_e} \in HF(\Sigma, \varphi_{H_e}(\Lambda), \Lambda, x).$

**Definition 6.7.** The class $e^*_{H_e}$ induces a class $e \in HF(\Sigma, \Lambda)$ via the canonical isomorphism (3.22). We call $e$ the unit of $HF(\Sigma, \Lambda)$.

### 6.4 Proof of the non–vanishing result

Before we prove the result, first note that we have the following diagram:

$$
\begin{array}{ccc}
\widetilde{PFH}_*(\Sigma, \varphi_{H_e}, \gamma_1)_{J_H} & \xrightarrow{(CO_z_1(W,\Omega_H,L\Lambda_H)_{J_H})^*} & HF_*(\Sigma, \varphi_{H_e}, \Lambda, x)_{J_H} \\
\downarrow \Psi_Z & & \downarrow \Psi_{\Lambda, x, x'} \\
\widetilde{PFH}_*(\Sigma, \varphi_{H_e}, \gamma_0)_{J_H} & \xrightarrow{(CO_z_0(W,\Omega_H,L\Lambda_H)_{J_H})^*} & HF_*(\Sigma, \varphi_{H_e}, \Lambda, x')_{J_H},
\end{array}
$$

where $Z \in H_2(Y_{\varphi_{H_e}}, \gamma_1, \gamma_0)$. The reference relative classes satisfy the relation $Z_0 \# Z = \Lambda_x \# \# \Lambda_1$. Therefore, it suffices to prove the non-vanishing result for a particular choice of base point $x$ and reference 1-cycle $\gamma_0$.

In this subsection, we take $x = y\circ$ and $\gamma_0 = \Psi_{H_e'}(S^1 \times y\circ)$, where $\Psi_{H_e'}$ is the trivialization (2.2).

Reintroduce the closed-open symplectic cobordism $(W, \Omega_{H'}, L_{\Lambda_{H'}})$. Let $J(W, \Omega_H) \subset J_{tame}(W, \Omega_H)$ be the set of almost complex structures which are the restriction of admissible almost complex structures in $J(Y_{\varphi_{H_e}}, \omega_{\varphi_{H_e}})$. Take a $J \in J(W, \Omega_{H'}).$ Let $u_{y_i'} = B \times \{y_i'\}/(0, y_i') \sim (2, y_i')$. Then it is a $J$–holomorphic PFH–HF curve in $\mathcal{M}^J(\gamma_{y_i'}, y_i')$. It is called a horizontal section of $(W, \Omega_{H'}, L_{\Lambda_{H'}}, J)$. Moreover, it is to check that $ind_{\#u_{y_i}} = 0$ and $E_{\omega_{\varphi_{H_e}}}(u_{y_i'}) = 0$ from the definition.

**Lemma 6.8.** Let $u$ be a $J$–holomorphic PFH–HF curve in $(W, \Omega_H, L_{\Lambda_H})$ and $J \in J(W, \Omega_H).$ Then

$$
E_{\omega_{\varphi_{H}}}(u) = \int_F u^* \omega_{\varphi_{H}} \geq 0.
$$

Moreover, when $H = H_{\varepsilon}'$, $E_{\omega_{\varphi_{H_{\varepsilon}}}}(u) = 0$ if and only if $u$ is a union of the horizontal sections.

**Proof.** Note that the almost complex structure $J = \begin{bmatrix} J_{hh} & 0 \\ 0 & J_{vv} \end{bmatrix}$ with respect to the splitting $TW = TW_{hor} \oplus TW_{vert}$, where $TW_{vert} = \ker(\pi_W)_*$ and $TW_{hor}$ is the $\Omega_H$–orthogonal complement. By direct computations, we have

$$
\int_F u^* \omega_{\varphi_{H}} = \int_F |(du)^{vert}|^2 \geq 0.
$$
In the case that \( H = H^\epsilon \) and \( E_{\omega_{H^\epsilon}}(u) = 0 \), we have \( du \in T W^{hor} \). Therefore, the negative ends of \( u \) must lie inside the trivial strips. This implies that \( u = u^{\epsilon}_{y_i} \) provided that \( u \) is irreducible.

**Lemma 6.9.** Let \( Z \in H_2(W_{H^\epsilon}, \alpha, y) \) be a relative homology class. Then the ECH index, \( J_0 \) index and the energy are

\[
I(Z) = n(y) + 2m(k + 1) + 2 \sum_{i=1}^k c_i, \\
J_0(Z) = 2m(d + g - 1), \\
E_{\omega_{H^\epsilon}}(Z) = H^\epsilon_\ast(\alpha I) - H^\epsilon_\ast(y) + m + \sum_{i=1}^k \lambda c_i,
\]

where \( n(y) \) is the number of \( y_i^- \) in \( y \). Moreover, if \( I(Z) = 0 \), then \( E_{\omega_{H^\epsilon}}(Z) + \eta J_0(Z) \leq 0 \) and “=” holds if and only if \( n(y) = 0 \).

**Proof.** The relative homology classes \( Z_{\alpha, \alpha'} \in H_2(Y_{\varphi_{H^\epsilon}}, \alpha, \alpha') \) satisfy \( I(Z_{\alpha, \alpha'}) = J_0(Z_{\alpha, \alpha'}) = E_{\omega_{H^\epsilon}}(Z_{\alpha, \alpha'}) = 0 \). By the additivity of ECH index, \( J_0 \) index and energy, it suffices to prove the statement for \( \alpha = \alpha' \).

Let \( \eta : \mathbb{R} \to \Lambda \) be a path in the hemicycle such that \( \lim_{s \to \infty} \eta(s) = y^+_i \) and \( \lim_{s \to -\infty} \eta(s) = y^-_i \). Then \( u^+_i(s, t) = (s, t, \varphi_{H^\epsilon} \circ (\varphi_{H^\epsilon}^{-1}(\eta(s))) \) represents a class in \( H_2(M, y^+_i, y^-_i) \). By the similar computation in (3.23), the energy of \( u^+_i \) is

\[
E(u_i^+) = \int (u_i^+)^\ast \omega = H^\epsilon_\ast(y^+_i) - H^\epsilon_\ast(y^-_i).
\]

Moreover, it is easy to check that \( \text{ind} u^+_i = 1 \) and \( J_0(u^+_i) = 0 \).

Let \( u^-_i \) be the trivial strip from \( y^+_i \) to itself. Let \( u = \cup_i u'_i \), where \( \epsilon \in \{+, -\} \). Then we have \( I(u) = n(y) \), \( J_0(u) = 0 \) and \( E(u) = H^\epsilon_{\ast}(y^-_i) - H^\epsilon_{\ast}(y) \). Let \( u = \cup_i u_{y_i^+} \) be a union of the horizontal sections from \( \alpha' \) to \( y \). Then \( u \# u \) represents a relative homology class \( Z_0 \in H_2(W_{H^\epsilon}, \alpha', y) \). By the additivity of the ECH index, \( J_0 \) index and energy, we have

\[
I(Z_0) = n(y), J_0(Z_0) = 0, \quad \text{and} \quad E_{\omega_{H^\epsilon}}(Z_0) = H^\epsilon_\ast(y^-_i) - H^\epsilon_\ast(y).
\]

The first statement of the lemma follows from (5.30), the result corresponding to Lemma 3.9 and the fact that any \( Z \in H_2(W_{H^\epsilon}, \alpha', y) \) can be written as \( Z = Z_0 + m[\Sigma] + \sum_{i=1}^k \epsilon_1[B_i] + [S] \).

If \( I(Z) = 0 \), then we have

\[
E_{\omega_{H^\epsilon}}(Z) + \eta J_0(Z) = H^\epsilon_\ast(y^-_i) - H^\epsilon_\ast(y) - \frac{\lambda n(y)}{2}.
\]

If \( n(y) \geq 1 \), then \( E_{\omega_{H^\epsilon}}(Z) + \eta J_0(Z) < 0 \). If \( n(y) = 0 \), then \( y = y^-_i \) and \( E_{\omega_{H^\epsilon}}(Z) + \eta J_0(Z) = 0 \).
Lemma 6.10. $\mathcal{C}O_{z_{ref}}(W, \Omega_{H^*_C}, L_{\Lambda_{H^*_C}})J(K \circ \partial'(\alpha_1, Z_I)) = 0$.

Proof. Suppose that the statement is not true, then we get a chain of holomorphic curves $\mathcal{C} = (C_1, C_2, C_3)$ from $\alpha_1$ to $y$ with total zero ECH index, where $C_1$ is a holomorphic current in $\mathbb{R} \times Y_{\Lambda_{H^*_C}}$ with $I = 1$, $C_2$ is a $J_+ \circ_R J_-$ broken holomorphic current in $X_+ \circ_R X_-$ with $I = -1$ and $C_3$ is a PFH–HF curve with $I = 0$. By Lemmas 6.5 and 6.8 we have

$$E_{\omega_{\varphi_H^*}}(C_1 \# C_2 \# C_3) = E_{\omega_{\varphi_H^*}}(C_1) + E_{\omega_{\varphi_H^*}}(C_2) + E_{\omega_{\varphi_H^*}}(C_3) > -dc_0\delta.$$  

(6.47)

The curve $C_1 \# C_2 \# C_3$ gives us a class $Z \in H_2(W, \alpha_1, y)$.

Reintroduce the periodic orbits $\gamma^i_{r_0, \theta_0}$ (6.37) near the local minimums of $H_z$. Here the superscript “$i$” indicates that the local minimum lies in the domain $B_i$. In particular, $\gamma^i_{r_0, \theta_0}$ lies in $S^1 \times B_i$. Define a curve $v_i := (\mathbb{R} \times \gamma^i_{r_0, \theta_0}) \cap W$. Note that it is $J$–holomorphic and $\partial v_i$ is disjoint from the Lagrangian $L_{\Lambda_{H^*_C}}$. Then for any relative class $Z' \in H_2(W, \alpha_1, y)$, we have a well–defined intersection number

$$n_i(Z') := #((Z' \cap v_i).$$

The relative homology class $Z \in H_2(W, \alpha_1, y)$ in the first paragraph can be written as

$$Z = Z_{\alpha_1, \alpha_\gamma} \# Z_{\text{hor}} \# (\bigcup_i u^i_1) + \sum_{i=1}^k c_i[B_i] + m[\Sigma] + [S],$$

where $Z_{\text{hor}}$ is the class represented by the union of horizontal sections and $u^i_1$ are the curves defined in Lemma 6.9. Let $q_i$ denote the period of $\gamma^i_{r_0, \theta_0}$. Note that the period $q_i$ is determined by the formula in (6.36) and $r_0$, we can choose $\varepsilon$ and $r_0$ such that $q_i = q$ for any $i$. By definition, we have

$$n_i(Z_{\alpha_1, \alpha_\gamma} \# Z_{\text{hor}} \# (\bigcup_i u^i_1)) = 0, \quad n_i([B_j]) = \delta_{ij}q, \quad n_i([S]) = 0 \quad \text{and} \quad n_i([\Sigma]) = q$$

(6.48)

for $1 \leq i, j \leq k + 1$. From (6.48), we know that

$$\#((\mathcal{C} \cap (\bigcup_{i=1}^{k+1} v_i)) = \sum_{i=1}^{k+1} n_i(\mathcal{C}) = \sum_{i=1}^{k} c_iq + (k + 1)mq = -\frac{qn(y)}{2}.$$

The last step is because $I(Z) = 0$. By the intersection positivity of holomorphic curves, we have $n(y) = 0$ and $y = y_{\gamma'}$. Moreover, $C_2$ doesn’t intersect $\mathbb{R} \times \gamma^i_{r_0, \theta_0}$. In particular, $C_2$ lies inside the product region of $X_+ \circ_R X_-$ and $E_{\omega_{\varphi_H^*}}(C_2) \geq 0$. Then the inequality (6.47) becomes

$$E_{\omega_{\varphi_H^*}}(Z) = E_{\omega_{\varphi_H^*}}(C_1) + E_{\omega_{\varphi_H^*}}(C_2) + E_{\omega_{\varphi_H^*}}(C_3) > 0.$$  

By the argument in Lemma 2.4 we have $J_0(C_2) \geq 0$. By Lemma 5.2 we have $J_0(Z) = J_0(\mathcal{C}) \geq 0$. Hence, $E_{\omega_{\varphi_H^*}}(Z) + \eta J_0(Z) > 0$. This contradicts with Lemma 6.9.

\[\square\]
Thus the right hand side of (1.1) is independent of the choice of isomorphism (3.21). Note that HF curve with \( y \) in the symplectic cobordism (5.33). This gives a reference homology class for \( \alpha I \neq \alpha \). Moreover, we have

\[
\text{CO}_{Z_{\text{ref}}}(W, \Omega_{H'_e}, L_{\Lambda_{H'_e}})J((\alpha, Z_I)) = 0
\]

for \( \alpha I \neq \alpha \).

**Proof.** By Lemmas 5.2, 6.8, 6.9 if \( < \text{CO}_{Z_{\text{ref}}}(W, \Omega_{H'_e}, L_{\Lambda_{H'_e}})J((\alpha, [Z_I])), (y, [A]) > \neq 0 \), then \( y \) must be \( y_{\varnothing} \) and any PFH–HF curve \( u \) contributed to the closed–open morphism has \( E_{\omega_{\varnothing} H'_e}(u) = 0 \). According to Lemma 6.8, \( u_{\varnothing} = \bigcup_{i=1}^{d} u_{y_i} \) and it is the unique PFH–HF curve with \( I(u) = E_{\omega_{\varnothing} H'_e}(u) = 0 \). Therefore, we have \( \alpha I = \alpha \) and

\[
\text{CO}_{Z_{\text{ref}}}(W_{H'_e}, L_{\Lambda_{H'_e}}, \Omega_{H'_e})J((\alpha, Z_{\alpha \varnothing}, A_{\varnothing})) = (y_{\varnothing}, A_{\varnothing})
\]

\( \square \)

In sum, we have

\[
\text{CO}_{Z_{\text{ref}}}(W_{H'_e}, L_{\Lambda_{H'_e}}, \Omega_{H'_e})J(\varepsilon_{\varnothing} + K \circ \partial \varepsilon_{\varnothing}) = (y_{\varnothing}, A_{\varnothing})
\]

Since \( \varnothing_{H'_e} = \varnothing_{H_e} \) outside the region \( \Lambda^d \), \( HF(\Sigma, \varnothing_{H'_e}, \Lambda, x) \) is canonical isomorphic to \( HF(\Sigma, \varnothing_{H_e}, \Lambda, x) \). By Lemma 6.6, \( (y_{\varnothing}, A_{\varnothing}) \) represents the unit \( j_{H_e}^{-1}(e) \). Combine this with the partial invariance; we get the non-vanishing result for the closed-open morphisms. Up to now, we finish the proof of Theorem 2.

### 7 Compare two spectral invariants

In this section, we assume that the link \( \Lambda \) is 0-admissible. Fix a base point \( x = (x_1, \ldots, x_d) \). Define a reference 1-cycle \( \gamma \) := \( \Psi_H(S^1 \times x) \). Let \( \mathbb{R} \times S^1 \times x \) be cylinders in the symplectic cobordism (5.33). This gives a reference homology class \( Z_{H, G}^x = [\mathbb{R} \times S^1 \times x] \in H_2(X, \gamma_H, \gamma_G) \). Let \( x' \) be another base point. Let \( \eta \) be \( d \) union of paths starting from \( x \) and ending at \( x' \). Let \( Z_{x', x} = [S^1 \times \eta] \in H_2(X, \gamma_{x'}, \gamma_Y) \).

Let \( \sigma_{\varnothing} \in PFH(\Sigma, \varnothing_{H'}, \gamma'_{Y_{\varnothing}}) \) be the class represented by the chain \( \varepsilon_{\varnothing} + K \circ \partial \varepsilon_{\varnothing} \).

For any \( H \) and \( x \), define a class \( \sigma_{\varnothing, H}^x \in \overline{PFH}(\Sigma, \varnothing_{H}, \gamma_X) \) by

\[
\sigma_{\varnothing, H}^x := \Psi_{Z_{y_{\varnothing}, x} \circ (PFH^{w}_{Z_{y_{\varnothing}, x}}(X, \Omega_X))^{-1}(\sigma_{\varnothing})}
\]

where \( (X, \Omega_X) \) is the cobordism from \( (Y_{\varnothing_{H}}, \omega_{\varnothing_{H}}) \) to \( (Y_{\varnothing_{H'}}, \omega_{\varnothing_{H'}}) \) and \( \Psi_{Z_{y_{\varnothing}, x}} \) is the isomorphism (3.21). Note that \( \sigma_{\varnothing, H} = \Psi_{Z_{x', x} \circ (\sigma_{\varnothing, H})} \). Then (2.8) implies that

\[
\sigma_{\varnothing, H}^{pfh} (H, \gamma_{x'}) + \int_0^1 H_t(x')dt = \sigma_{\varnothing, H}^{pfh} (H, \gamma_{x}) + \int_0^1 H_t(x)dt.
\]

Thus the right hand side of (1.1) is independent of the choice of \( x \).
Proof of Corollary 1.1. By the above discussion, we can set \( x = y \). First of all, suppose that \( H \) satisfies (♠.1). Let \( \mathcal{Z}_{\text{ref}} \in H_2(W, \gamma_H^X, x_H) \) be a relative homology represented by \( \Psi_H' \), where \( \Psi_H' \) is the trivialization in Remark 5.2. Therefore, \( \int_{\mathcal{Z}_{\text{ref}}} \omega_{\varphi_H} = 0 \).

For any \( \delta > 0 \), we find a cycle \( c = \sum (\alpha, [Z]) \) such that it represents \( \sigma_{\varphi_H} \) and \( \mathcal{A}_H(\alpha, [Z]) \leq c_{\sigma_{\varphi_H} J}(H, \gamma_H^X, J) + \delta \). The commutative diagram in Theorem 2 implies that

\[
\text{CO}_{\mathcal{Z}_{\text{ref}}} (W_H, L_{\Lambda_H}, \Omega_H)_*(\sigma_{\gamma_H}^X) = (j_H^X)^{-1}(e),
\]

Therefore, \( \text{CO}_{\mathcal{Z}_{\text{ref}}} (W_H, L_{\Lambda_H}, \Omega_H)_*(\sigma_{\gamma_H}^X) = \sum (y, [A]) \) is a cycle that represents \( (j_H^X)^{-1}(e) \).

By the definition of the link spectral invariant and (3.24), we have

\[
c_{A, y = 0}^{\text{link}} (H, j_H^X \circ \Phi_H((j_H^X)^{-1}(e))) \leq c_{\sigma_{\varphi_H} J}(H, \gamma_H^X, J) + \delta - \mathcal{A}_H(y, A) + \int_0^1 H_t(x) dt.
\]

By the definition of the link spectral invariant and (3.24), we have

\[
c_{A, y = 0}^{\text{link}} (H, j_H^X \circ \Phi_H((j_H^X)^{-1}(e))) \leq c_{\sigma_{\varphi_H} J}(H, \gamma_H^X, J) + \delta - \mathcal{A}_H(y, A) + \int_0^1 H_t(x) dt + \delta.
\]

Then we get the conclusion by taking \( \delta \to 0 \).

For a general Hamiltonian function \( H \) and any fixed \( \delta > 0 \), according to Proposition 3.2 in [9], we can find a Hamiltonian function \( H'_\delta \) satisfying (♠.1) and the estimates in Remark 1.5. Then the conclusion of Corollary 1.1 holds for \( H'_\delta \). Take \( \delta \to 0 \). Since the spectral invariants satisfy the Hofer continuity (see Theorem 3.1 of [6] and Theorem 1.13 of [8]), the conclusion still holds for \( H \).

\[\square\]

A Cobordism maps on PFH

In this appendix, we give the statements about the holomorphic curve axioms of the PFH cobordism maps. They are slight refinements of Theorem 1 in [9]. The statements are parallel to those for ECH (Proposition 6.2 of [18] and [21]). Now we do not assume that \( \varphi \) is Hamiltonian. \( \varphi \in \text{Symp}(\Sigma, \omega) \) can be any symplectic morphism such that the periodic orbits are non–degenerate.

A symplectic cobordism from \((Y_{\varphi_+}, \omega_{\varphi_+})\) to \((Y_{\varphi_-}, \omega_{\varphi_-})\) is a compact symplectic manifold \((X, \Omega_X)\) such that \( \partial X = Y_{\varphi_+} \sqcup (-Y_{\varphi_-}) \) and \( \Omega_X|_{Y_{\varphi_+} \sqcup Y_{\varphi_-}} = \omega_{\varphi_+} \). Let \((X, \Omega_X)\)
be the symplectic completion via adding cylindrical ends (see Section 2.3 of [9]). The completion \((\overline{X}, \Omega_X)\) satisfies (2.7).

Let \((X_+, \Omega_{X_+})\) be a symplectic cobordism from \((Y_{\varphi_+}, \omega_{\varphi_+})\) to \((Y_{\varphi_0}, \omega_{\varphi_0})\) and \((X_-, \Omega_{X_-})\) be another symplectic cobordism from \((Y_{\varphi_0}, \omega_{\varphi_0})\) to \((Y_{\varphi_-}, \omega_{\varphi_-})\). For \(R \geq 0\), we define **stretched composition** by

\[
X_+ \circ_R X_- = X_+ \cup_{Y_{\varphi_0}} \left([-R, R] \times Y_{\varphi_0}\right) \cup X_-.
\]

An \(\Omega_X\)-compatible almost complex structure \(J\) on \(\overline{X}\) is called **cobordism–admissible** if it agrees with some admissible almost complex structure of \(\mathbb{R} \times Y_{\varphi_\pm}\) over the cylindrical ends. The set of cobordism–admissible almost complex structures is denoted by \(\mathcal{J}_{\text{comp}}(X, \Omega_X)\). Given \(J_{\pm} \in \mathcal{J}_{\text{comp}}(X_{\pm}, \Omega_{X_{\pm}})\) such that \(J_+|_{\mathbb{R} \times Y_{\varphi_0}} = J_-|_{\mathbb{R} \times Y_{\varphi_0}}\), then we can glue them together and obtain a cobordism–admissible almost complex structure \(J_+ \circ_R J_-\) on \(\overline{X} \circ_R \overline{X}\). (See Section 2.5 of [9].)

Let \(\phi \in \text{Hom}(\text{PFC}(\Sigma_+, \varphi_+, \gamma_+), \text{PFC}(\Sigma_-, \varphi_-, \gamma_-))\) be a linear map. Following the terminology in [18], we say that \(\phi\) counts **J–holomorphic currents** if \(\phi(\alpha_+, Z_+), (\alpha_-, Z_-) \neq 0\) implies that the moduli space \(\mathcal{M}_J(\alpha_+, \alpha_-, Z_+ \# \mathcal{Z}_{\text{ref}} \# Z_-)\) is non–empty.

Following Hutchings and Taubes' idea for ECH [21], the author defines the PFH cobordism maps via the isomorphism “SWF=PFH” [27] and Seiberg–Witten theory [23]. (See Theorem 1 of [9].) The following theorem covers three situations which do not appear in Theorem 1 of [9], but they can be obtained by the same method. The parallel statement for ECH cobordism maps can be found in Proposition 6.2 of [18].

**Theorem A.1.** Let \((X, \Omega_X)\) be a symplectic cobordism from \((Y_{\varphi_+}, \omega_{\varphi_+})\) to \((Y_{\varphi_-}, \omega_{\varphi_-})\). Assume that \(d(\gamma_\pm) > g(\Sigma_\pm)\). Fix a reference relative homology class \(\mathcal{Z}_{\text{ref}} \in H_2(X, \gamma_+, \gamma_-)\). Then \((X, \Omega_X, Z_{\text{ref}})\) induces a module homomorphism

\[
\text{PFH}^{\text{sw}}_{Z_{\text{ref}}}(X, \Omega_X) : \text{PFH}_*(\Sigma_+, \varphi_+, \gamma_+) \to \text{PFH}_*(\Sigma_-, \varphi_-, \gamma_-)
\]

satisfying the following holomorphic curve axioms:

1. **(Holomorphic curves)** Given a cobordism–admissible almost complex structure \(J \in \mathcal{J}_{\text{comp}}(X, \Omega_X)\) such that \(J_\pm = J|_{\mathbb{R} \times Y_\pm}\) are generic, then there is a chain map

\[
\text{PFC}^{\text{sw}}_{Z_{\text{ref}}}(X, \Omega_X)J : \text{PFC}_*(\Sigma_+, \varphi_+, \gamma_+) \to \text{PFC}_*(\Sigma_-, \varphi_-, \gamma_-)
\]

inducing \(\text{PFH}^{\text{sw}}_{Z_{\text{ref}}}(X, \Omega_X)\) and \(\text{PFC}^{\text{sw}}_{Z_{\text{ref}}}(X, \Omega_X)J\) counts \(J\–holomorphic currents\) with \(I = 0\).

2. **(Composition rule)** Let \((X_+, \Omega_{X_+})\) and \((X_-, \Omega_{X_-})\) be symplectic cobordisms from \((Y_{\varphi_+}, \omega_{\varphi_+})\) to \((Y_{\varphi_0}, \omega_{\varphi_0})\) and from \((Y_{\varphi_0}, \omega_{\varphi_0})\) to \((Y_{\varphi_-}, \omega_{\varphi_-})\) respectively. If \((X, \Omega_X)\)
is the composition of \((X_+, \Omega_{X_+})\) and \((X_-, \Omega_{X_-})\), then there exists a chain homotopy
\[
K : PFC_*(\Sigma_+, \varphi_+, \gamma_+) \to PFC_*(\Sigma_-, \varphi_-, \gamma_-)
\]
such that
\[
PFC_{\text{sw}}^{\text{ref}}(X_-, \Omega_{X_-}) J_+ \circ PFC_{\text{sw}}^{\text{ref}}(X_+, \Omega_{X_+}) J_+ = PFC_{\text{sw}}^{\text{ref}}(X, \Omega_{X}) J + K \circ \partial_+ + K \circ \partial_-
\]
and \(K\) counts \(J_+ \circ_R J_-\)-holomorphic currents with \(I = -1\). Here \(\text{Z}_{\text{ref}} = Z_{\text{ref}} - \#Z_{\text{ref}}\).

3. (Homotopy invariance) Let \(\{(\Omega^t_X, J_t)\}_{t \in [0,1]}\) be a family of symplectic 2–forms and cobordism–admissible almost complex structures such that \(\Omega_X = \omega_{\varphi_{\pm}} + ds \wedge dt\) and \(J = J_{\pm} \in J(\varphi_{\pm}, \omega_{\varphi_{\pm}})\) on the cylindrical ends. Then there exists a chain homotopy
\[
K : PFC_*(\Sigma_+, \varphi_+, \gamma_+) \to PFC_*(\Sigma_-, \varphi_-, \gamma_-)
\]
such that
\[
PFC_{\text{sw}}^{\text{ref}}(X, \Omega^0_X) J_0 = PFC_{\text{sw}}^{\text{ref}}(X, \Omega^1_X) J_1 + K \circ \partial_+ + K \circ \partial_-
\]
and \(K\) counts \(J_t\)-holomorphic currents with \(I = -1\).

4. (Trivial cylinders) Suppose that \(\mathbb{R} \times U\) is a product region contained in \(\overline{X}\), i.e., \(\Omega_X|_{\mathbb{R} \times U} = \omega + ds \wedge dt\). Let \(J\) be a cobordism–admissible almost complex structure on \(\overline{X}\) such that \(J\) is \(\mathbb{R}\)-invariant over \(\mathbb{R} \times U\). Suppose that a union of trivial cylinders is the only element in \(\mathcal{M}_0^J(\alpha_+, \alpha_-)\), then
\[
< PFC_{\text{sw}}^{\text{ref}}(X, \Omega_X) J(\alpha_+, Z_+), (\alpha_-, Z_-) > = 1.
\]

**Proof.** According to Lee–Taubes’s isomorphism [27], we can identify the PFH chain complex with the Seiberg–Witten chain complex (after the \(d-\delta\) flat perturbation). Under this identification, the cobordism map \(PFC_{\text{sw}}^{\text{ref}}(X, \Omega_X) J\) in chain level is defined by counting the solutions to the Seiberg–Witten equations ((5–8) in [9]) perturbed by \(r \Omega_X\). Note that (5–8) in [9] depends on \(J\) as well. In other words, \(PFH_{\text{sw}}^{\text{ref}}(X, \Omega_X) J\) is defined to be the Seiberg–Witten cobordism map.

Suppose that \(PFC_{\text{sw}}^{\text{ref}}(X, \Omega_X) J\) is non–trivial. Then we obtain a solution to Seiberg–Witten equations for any \(r \gg 1\). As \(r \to \infty\), Proposition 5.12 in [9] shows that the solutions converges to a \(J\)-holomorphic current in certain sense. Then we obtain the first statement. For a more general version of Taubes’s “SW to Gr” degeneration, please see [26].

The second and the third statements follow from the composition rule and homotopy invariance of the Seiberg–Witten cobordism maps (see Chapter VII, VIII of [23]). The
chain homotopy $K$ counts the perturbed Seiberg–Witten equations in $(X_+ \circ_R X_-, \Omega_R)$ or $(\overline{X}, \Omega_X)$ accordingly. Therefore, $K \neq 0$ implies that the existence of $r$–perturbed Seiberg–Witten equations in the corresponding setting for any $r \gg 1$. Again, by Proposition 5.12 in [9], we obtain a holomorphic current $C$. Moreover, the ECH index of $C$ follows from Theorem 5.1 of [3].

For the last statement, we need to show that the $I = 0$ moduli space of holomorphic curves is diffeomorphic to the $\ind = 0$ moduli space of solutions to the Seiberg–Witten equations. A special case that $X = \mathbb{R} \times Y$ and $(\Omega_X, J)$ is $\mathbb{R}$–invariant has been proved in [27]. For another special case that the holomorphic current $C$ is embedded, Taubes’s argument can be applied to the cobordism case equally well [9]. The proofs are mirror modifications of the argument in Taubes’ s series papers [29], [30], [31].

Since Taubes’ arguments require many hard analysis on the Seiberg–Witten equations which beyond scope of this paper. We suggest the readers to read [27] and [9] for the relevant settings and more details on Taubes’ s idea. Also see Proposition 6.3 of [21] for the parallel arguments on ECH. Here we just outline the main idea as follow.

a) Let $C = \mathbb{R} \times \alpha$ be the unique holomorphic current with $I = 0$. We follow Section 5a in [29] to build a complex line bundle $E$ by gluing the normal bundle of $C$ with the trivial line bundle away from $C$. The spin–c structure on $X$ is defined by $S_+ = E \oplus EK_X^{-1}$. We can construct an approximation solution $(A^*_r, \psi^*_r)$ (closed to solve (5-8) in [9]) associated to $C$ (see Section 5a of [29]). Away from the trivial cylinders $C$, $(A^*_r, \psi^*_r)$ is just the trivial solution. Let $(\gamma, m)$ be a component of $\alpha$. Near the trivial cylinder $\mathbb{R} \times \gamma$, $(A^*_r, \psi^*_r)$ is determined by a map $\mathbf{v} : \mathbb{R} \times \gamma \to \mathcal{C}_m$, where $\mathcal{C}_m$ is the moduli space of $m$–vortices. The vortex equations can be regarded as the 2–dimensional Seiberg–Witten equations. In the case that $C$ is the trivial cylinders, there is a canonical way to choose the map $\mathbf{v}$. The analysis in [29] can be used to perturb $(A^*_r, \psi^*_r)$ to be a true solution $(A_r, \psi_r)$.

b) The argument in [30] can be used to show that $(A_r, \psi_r)$ is non–degenerate. By Theorem 5.1 of [31], the index of $(A_r, \psi_r)$ is zero.

c) Let $(A'_r, \psi'_r)$ be another solution to the $r\Omega_X$–perturbed Seiberg–Witten equations with $\ind = 0$. By Proposition 5.12 of [9], $(A'_r, \psi'_r)$ converges to the trivial cylinders $C = \mathbb{R} \times \alpha$ as $r \to \infty$ because it is the unique holomorphic current with $I = 0$. For any $\delta > 0$, the convergence implies that $1 - |\psi'_r| < \delta$ away from $C$. Intuitively, this means $(A'_r, \psi'_r)$ is close to the trivial solution away from $C$. The arguments in Section 6 of [31] can be carried over to show that $(A'_r, \psi'_r)$ is gauge equivalent to $(A_r, \psi_r)$.

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