ACCELERATION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS THROUGH PRECISE REMAINDER ASYMPTOTICS

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Abstract. We express the asymptotics of the remainders of the partial sums \( \{ s_n \} \) of the generalized hypergeometric function \( \sum_{q+1} F_q \left( \frac{\alpha_1, \ldots, \alpha_{q+1}}{\beta_1, \ldots, \beta_q} \mid z \right) \) through an inverse power series \( z^n n^\lambda \sum c_k n^k \), where the exponent \( \lambda \) and the asymptotic coefficients \( \{ c_k \} \) may be recursively computed to any desired order from the hypergeometric parameters and argument. From this we derive a new series acceleration technique that can be applied to any such function, even with complex parameters and at the branch point \( z = 1 \). For moderate parameters (up to approximately ten) a C implementation at fixed precision is very effective at computing these functions; for larger parameters an implementation in higher than machine precision would be needed. Even for larger parameters, however, our C implementation is able to correctly determine whether or not it has converged; and when it converges, its estimate of its error is accurate.

1. Introduction

The generalized hypergeometric function \( p F_q \left( \frac{\alpha_1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q} \mid z \right) \) is ubiquitous in applied mathematics; a wide array of special functions are particular cases of this function. Hence the numerical evaluation of this function is an important problem. In many instances, specialized methods for a particular special function are the most computationally efficient, but there are still situations where the generalized hypergeometric function must be evaluated for computationally challenging choices of parameters and argument. In this paper we present a new algorithm that is able to evaluate many of those challenging cases, and we describe its numerical implementation and testing.

1.1. Analytic properties of the generalized hypergeometric function. To understand the difficulties, we briefly recall the definition and analytic properties of the generalized hypergeometric function; for more details see the reference site the Digital Library of Mathematical Functions [1] or its print version [2], the monograph of Slater [3], or the text of Graham et al. [4]. Such functions are characterized by a set of upper and lower parameters and a single argument; in the most general case the argument and any of the parameters may be complex. The function can be defined through a Taylor series about the origin, where that converges:

\[
p F_q \left( \frac{\alpha_1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q} \mid z \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k \beta_1 \cdots \beta_q}{k!} z^k.
\]
Here \((a)_k = a(a + 1) \cdots (a + k - 1)\) is the Pochhammer symbol. The function \(\binom{2}{1}\) is sometimes called simply the hypergeometric function or Gauss’ hypergeometric function; the generalized functions are then all of those with other numbers of parameters. In this paper we will refer to any function given by \((1.1)\) as a generalized hypergeometric function, or sometimes simply a hypergeometric function.

As a function of the complex argument \(z\), the analytic characteristics of \(\binom{p}{q}\) depend on the relative number of upper and lower parameters. When \(p \leq q\), the function is entire and so the power series converges everywhere. When \(p = q + 1\)—the case we will we concerned with in this paper—the series converges for \(z < 1\). On the unit circle the behavior of the series \((1.1)\) with \(p = q + 1\) depends crucially on the real part of the parameter \(\sigma\), defined as:

\[
(1.2) \quad \sigma = \sum_{k=0}^{q+1} \alpha_k - \sum_{k=0}^q \beta_k.
\]

Then the series converges at \(z = 1\) if \(\Re \sigma < 0\), and it converges elsewhere on the unit circle if \(\Re \sigma < 1\).

Outside of the unit circle, \(\binom{q+1}{q}\) may be defined through analytic continuation. This may be effected through the defining differential equation satisfied by the generalized hypergeometric function (see [1]); that equation is singular at \(z = 1\) and the function has a branch point there (logarithmic if \(\sigma \in \mathbb{Z}\); algebraic otherwise). The branch cut is conventionally taken along the positive real axis from \(z = 1\) to infinity.

For the Gaussian hypergeometric function \(\binom{2}{1}\) we can in fact evaluate the function at the branch point in closed form, thanks to Gauss’ formula

\[
(1.3) \quad \binom{2}{1} \left( \frac{\alpha_1, \alpha_2}{\beta_1} \right) = \frac{\Gamma(\beta_1) \Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1) \Gamma(\beta_1 - \alpha_2)}
\]

that is valid whenever \(\Re(\beta_1 - \alpha_1 - \alpha_2) > 0\). This formula will prove very useful later for testing our method.

Finally, though we will not be concerned with the case \(p > q + 1\) in this paper, we do note that in this case the radius of convergence of the series is zero.

With this background, consider now the numerical computation of \(\binom{q+1}{q}\). The advice given in the standard reference Numerical Recipes [5] is to simply use the series \((1.1)\) directly when \(|z| \ll 1\), and to integrate the defining differential equation of the generalized hypergeometric function elsewhere, of course taking care not to cross the branch cut, and avoiding the neighborhood of the branch point. This leaves open, however, the question of how to evaluate the function at or near the branch point, where the differential equation is singular and the series slowly convergent.

1.2. Summary of results. We address that problem in this paper through a novel series acceleration technique. For any \(\binom{q+1}{q}\), we will show that the sequence \(\{s_n\}\) of partial sums:

\[
(1.4) \quad s_n := \sum_{k=0}^{n-1} \frac{(\alpha_1)_k \cdots (\alpha_{q+1})_k z^k}{(\beta_1)_k \cdots (\beta_q)_k k!}
\]

The series converges at \(z = 1\) if \(\Re \sigma < 0\), and it converges elsewhere on the unit circle if \(\Re \sigma < 1\).
satisfies an asymptotic expansion of the form:

\[ s_n \sim s + \mu n^\lambda \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty, \]

for undetermined constants \( s \) and \( \mu \) but for asymptotic coefficients \( c_k \) that can be computed to any desired order recursively from the hypergeometric parameters \( \alpha_1, \ldots, \alpha_{q+1}, \beta_1, \ldots, \beta_q \) and the complex argument \( z \). The exponent \( \lambda \) will be shown to be:

\[ \lambda = \begin{cases} 
\sigma & \text{when } z = 1 \\
\sigma - 1 & \text{otherwise}
\end{cases} \]

with \( \sigma \) defined as in (1.2).

Using the asymptotic expansion (1.5), we may use any two successive computed partial sums \( s_n \) and \( s_{n+1} \) to estimate the undetermined coefficients \( s \) and \( \mu \); the estimate for \( s \) then becomes the accelerated estimate for the sum and hence the function \( {}_{q+1}F_q \left( \alpha_1, \ldots, \alpha_{q+1} \mid \beta_1, \ldots, \beta_q \right) \) itself. We describe in some detail a numerical implementation of this technique, which estimates both truncation and floating point errors to determine either that the algorithm has converged to the user specified tolerance, or that convergence is impossible at standard machine precision. Even in the latter case, however, the algorithm itself is still capable of accelerating convergence; it just must be implemented at higher than machine precision. We present tests of this algorithm to show that it is robustly able to either accelerate convergence (in many cases dramatically) or correctly conclude that a higher working precision is needed. The algorithm is particularly effective at the branch point \( z = 1 \).

There are two key features of this work that deserve highlighting:

1. Our emphasis is on a robust algorithm that can handle complex parameters and argument. We want it to succeed as often as possible—without user intervention to determine convergence—and to reliably indicate failure when it has not converged. Considerable effort has therefore been spent in designing error estimation and stopping criteria, and the tests summarized in section 3.2 are designed to thoroughly probe how well the algorithm meets these criteria.

2. The key novelty of the algorithm is its analytic calculation of the remainder asymptotics. This is possible only because we have an analytic expansion of the term ratio in inverse powers. Thus, this algorithm requires detailed analytic knowledge about the series accelerated. This is a strength of the method in that we might hope (and in fact will see) that specific analytic knowledge about our series allows our method to succeed where other methods fail. But it is also a limitation, since there is no obvious way to generalize the method to series that are only known numerically (certainly a very important class). Nevertheless, even within this limitation we believe that there is more to be explored, as many functions and series beyond the \( {}_{q+1}F_q \) functions we consider here may be amenable to this approach.

1.3. Previous work. The literature on computing the generalized hypergeometric function depends on the restrictions placed on the number and values of the parameters. For real parameters and argument to \( {}_2F_1 \left( \alpha_1, \alpha_2 \mid \beta_1 \right) \), it is often efficient to piece together different approximations based on the values of the parameters
and the argument; this is the approach taken, for example, by the popular GNU
Scientific Library [6]. A more detailed analysis of the algorithms appropriate for
different (real) parameters may be found in the work of Muller [7] for the particular
case of the confluent hypergeometric function \(_1F_1\). Pearson’s master’s thesis [8] con-
siders both the confluent hypergeometric function and \(_2F_1\), and moreover considers
complex parameters and argument. The paper [9] of Forrey describes software that
uses functional transformations and difference equations to evaluate the Gaussian
hypergeometric function for arbitrary real argument, and Becken and Schmelcher
[10] consider analytic continuation formulæ to again customize the computation
based on the range of the argument. Chatterjee and Roy [11] consider a modific-
ation of standard Levin-type methods tailored to the hypergeometric function, and
Gautschi [12] considers evaluation of both Gaussian and confluent hypergeometric
functions for complex arguments, but real parameters, using Gaussian quadrature
to evaluate integral representations of the functions. Weniger looked at using tra-
ditional series acceleration methods but irregular input data in [13] and considered
divergent hypergeometric series at \(z = -1\) using a method tailored to alternating
series in [14]. Finally Kalmykov [15] and Kalmykov et al. [16] consider an expansion
of the Gaussian function near integer values of its parameters.

For complex parameters and argument for the generalized hypergeometric func-
tion \(_{q+1}F_q(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_q | z)\), the literature is much more sparse. Skorokhodov con-
siders analytic continuation via symbolic manipulations for the generalized hyper-
geometric function in the neighborhood of \(z = 1\) in [17, 18]. Ferreira et al. [19]
consider an expansion valid for larger lower parameters of the Gaussian function.
Aside from the recommendation of [5] already given above, Perger et al. [20] im-
plement the defining series (1.1) directly, in higher than machine precision. While
this can protect against some instances of floating point error, it does not speed
the convergence of the series itself. The most sophisticated software package to
handle the generalized hypergeometric function that the present author has en-
countered is Johansson’s \texttt{mpmath} Python module [21]. That package uses a mix
of direct series calculation and analytic continuation, as well as the Shanks’ series
acceleration method near the unit circle but away from \(z = 1\). Near the branch
point Euler-Maclaurin summation is used.

Series acceleration has a long history; we will give a brief review and more point-
ers to the literature in section 2.1. Here we only mention a few recent techniques
that have been specifically applied to hypergeometric functions. Wozny and Nowak
[22] and Wozny [23] consider a new series acceleration technique that they ap-
ply (among other examples) to some instances of the generalized hypergeometric
function. Their approach is based on finding certain difference operators that ap-
proximately annihilate the remainder term of the series. Paszkowski [24] considers
several acceleration algorithms and how they may be modified if an asymptotic
form for the partial sums is known. While it is mentioned that the generalized
hypergeometric functions belong to this class, an explicit expression for the asymp-
totic coefficients is not given, and only a few low order expansions are considered
in examples. Likewise Lewanowicz and Paszkowski [25] consider an acceleration
method based on the asymptotic expansion of the term ratio, as we will, but those
authors do not apply it to the asymptotics of the partial sum, and they are only
able to accelerate certain parameter choices for \(_3F_2\) functions at \(z = \pm 1\). In Sko-
rokhodov [26] and Bogolubsky and Skorokhodov [27] the authors use an asymptotic
expansion of the terms—rather than the term ratios—to calculate an approximant to the truncation error that they evaluate using Hurwitz zeta functions. We shall compare the performance of our method to theirs in section 4.1.

Perhaps closest in spirit to the present work is that of Weniger [28], which is based on finding asymptotic approximations to the remainder terms of a partial series summation through symbolic linear algebra. That method can in some cases yield analytic expressions, as for the Dirichlet series of the Riemann zeta function (section 5 of [28]) or the divergent Euler series for the exponential integral (as in Borghi [29]). However, the method presented in [28] is challenging even for \( _2F_1 \), particularly as compared to the complete asymptotic series that we will give in this paper.

Finally, we mention that Bühring has considered both the behavior of the generalized hypergeometric functions near unity [30], seeking to find the analogue of Gauss’ formula (1.3) for higher order functions; and separately considered the asymptotic behavior of the partial sums of generalized hypergeometric functions at unity [31]. In each case, the expressions derived are nested infinite sums of hypergeometric functions of lower order, so the results do not seem well adapted to numeric computation. More practically useful for us is his work in [32, 33] and the earlier work of Norlund [34], which give expansions valid near the branch point. They can conceivably be leveraged to take an efficient evaluation method at the branch point and evaluate a hypergeometric function near the branch point using expansions valid in a neighborhood of the branch point.

2. Accelerating the convergence of the series

The well-known Euler’s method shows that series acceleration dates to at least the eighteenth century, and both Knopp [35] and Tweddle [36] cite Stirling as the earliest to develop a series acceleration method, but the last several decades have seen the development of a variety of sophisticated techniques. For reviews, see the articles of Brezinski [37], Homeier [38], and Weniger [39], and the monographs of Brezinski [40, 41], Brezinski and Redivo Zaglia [42], Sidi [43], Walz [44] and Wimp [45]. We will give just enough background in the next section to place our new method in context.

2.1. Review of series acceleration methods. The basic idea of any series acceleration technique is to use the expected form of the partial sums \( \{s_n\} \) of a series—which by themselves may be slowly convergent or even divergent—to create a new sequence \( \{s'_n\} \) that converges to the same limit \( s \) (or, in the case of a divergent series, antilimit), but that does so more rapidly, in the sense that:

\[
\lim_{n \to \infty} \frac{s'_n - s}{s_n - s} = 0.
\]

To motivate these transformations, we start from the explicit expression of the sequence \( \{s_n\} \) of partial sums by means of the terms \( \{t_k\} \) of the series:

\[
s_n := \sum_{k=0}^{n-1} t_k.
\]
Now partition the partial sum into its (anti-)limit $s$ and the remainder $\rho_n$:

$$\rho_n := -\sum_{k=n}^{\infty} t_k$$

so that:

$$s_n = s + \rho_n.$$

Various algorithms can then be devised by approximating the remainder as $\rho_n \approx \omega_n \mu_n$, where $\omega_n$ is an explicit remainder estimate, and $\mu_n$ is an $O(1)$ correction factor containing $m$ free parameters. The order of the transformation is $m$.

For instance, if the terms of the series are alternating, then a natural estimate of the remainder would be the first term not included, $t_n$. If we choose $\omega_n = t_n$ and choose:

$$\mu_n = \sum_{k=0}^{m-1} \frac{c_k}{(n + \beta)^k}$$

for some positive $\beta$, and undetermined coefficients $c_k$, then from any sequence of $m + 1$ successive partial sums, we may determine values for the $m$ coefficients $c_k$ and the limit $s$. This choice of remainder estimate $\omega_n$ and correction factor $\mu_n$ gives rise to Smith and Ford’s modification [46] of Levin’s $t$ transformation [47]; other choices for either $\omega_n$ or $\mu_n$ give other sequence transformations; for details see the review articles and monographs mentioned earlier. The actual estimation of $s$ from the $m + 1$ partial sums can be expressed as a ratio of determinants, but is more commonly implemented recursively [38, 39].

Among all of these methods, we will single out one known as the $E$-method [42, 48–50], because it largely subsumes all of the others as special cases: starting from a set of functions $\{g_i(n)\}_{i=1}^{m}$, one requires:

$$s_n = s + \sum_{i=0}^{m} c_i g_i(n).$$

From the $m + 1$ sums $s_n, s_{n+1}, \ldots, s_{n+m}$ we can calculate $s$ and the $c_i$; the value of $s$ is then the accelerated sum determined by the $E$-method for that particular choice of functions $g_i$ and those partial sums. We will compare the performance of this method to that of this paper in section 4.3.

The effectiveness of such algorithms depends on the nature of the series to be summed; for instance, we mentioned above that the $t$ transformation was designed with alternating series in mind. One central characteristic of convergent series that influences the effectiveness of acceleration methods is whether that convergence is linear or logarithmic. The former means that if the limit of the sequence of partial sums is $s$, then

$$\lim_{n \to \infty} \frac{s_{n+1} - s}{s_n - s} = \ell$$

with $0 < |\ell| < 1$. On the other hand, if $\ell = 1$, then the convergence is logarithmic [39].

The distinction is important because while several series acceleration methods can be shown to accelerate the convergence of any linearly convergent series [46], it is known from the work of Delhaye and Germain-Bonne [51] that no method is able to accelerate the convergence of all logarithmically convergent series. Yet at
the branch point of the generalized hypergeometric function (as we will show in the next section) the convergence of the series is logarithmic. Section 14 of [39] contains a summary of methods that may be applied to logarithmically convergent series; while alternating series are often tractable, with a generic choice of complex parameters the terms in the generalized hypergeometric series are complex and exhibit no particular sign pattern (see Sidi [52] for examples of the effect of irregular sign patterns on series acceleration techniques). Moreover, some series acceleration techniques are quite sensitive to the details of the asymptotic form of the remainders [39]; they are able to accelerate convergence when \( \rho_n \sim n^{-k} \) for an integer \( k \), but fail for non-integral but real exponents. For complex exponents—as we will encounter for generalized hypergeometric sums—very little indeed seems to be known.

Thus, while there is a wide array of sophisticated series acceleration methods able to speed the convergence, often substantially, of many series, and even to sum many strongly divergent series, there does not seem to be a method that is broadly applicable to the series expansion of the generalized hypergeometric function, particularly at its branch point. Even very new methods, such as those in [22, 23, 28], typically consider only real hypergeometric parameters.

Instead, we propose a method that can be applied to any generalized hypergeometric of the form \( q+1 \, F_q \). Unlike conventional series acceleration methods, instead of a simple choice for the remainder estimate \( \omega_n \) and a complicated choice for the correction \( \mu_n \) (so that several successive partial sums are needed to calculate each \( s'_n \)) we will use a very precise remainder estimate but just a constant for our correction factor. Thus, we will need only two successive partial sums to compute each estimate of the series limit. We turn now to the derivation of that method.

2.2. Derivation of the partial sum asymptotics. By comparing the general form (2.2) to the expression (1.4) for the partial sums of the generalized hypergeometric and making use of the definition of the Pochhammer symbol, we may write the ratio of two successive terms of the generalized hypergeometric function as:

\[
\frac{t_{k+1}}{t_k} = \frac{(\alpha_1 + k) \cdots (\alpha_q + 1 + k)}{(\beta_1 + k) \cdots (\beta_q + k)(1 + k)} := z \, r(k);
\]

that is, the ratio of two successive terms is a rational function of the index \( k \). This property of generalized hypergeometric functions is very well known; indeed, any function whose term ratio is a rational function of the term index may be expressed in terms of generalized hypergeometric functions [4].

Because the ratio of terms satisfies a first-order recurrence relation, both the remainders \( \{\rho_n\} \) and the partial sums \( \{s_n\} \) satisfy a second-order recurrence; in fact, the two sequences satisfy the same second order recurrence. In equations, we have:

\[
\frac{t_{n+1}}{t_n} = z \, r(n) = \frac{s_{n+2} - s_{n+1}}{s_{n+1} - s_n} = \frac{\rho_{n+2} - \rho_{n+1}}{\rho_{n+1} - \rho_n}
\]

and therefore:

\[
s_{n+2} - (1 + z \, r(n))s_{n+1} + z \, r(n)s_n = 0
\]

\[
\rho_{n+2} - (1 + z \, r(n))\rho_{n+1} + z \, r(n)\rho_n = 0
\]

The key point is that the remainders and partial sums each satisfy a linear, homogeneous, second-order difference equation, and the asymptotic solutions of
such equations are known very precisely. In principle, we could use either of equation (2.10) or (2.11) as our starting point, and typically in series acceleration it would be more natural to focus on the behavior of the remainders. However, it is slightly more convenient to use (2.10) and base our results directly on the asymptotics of the partial sums themselves. That is because there are two linearly independent solutions to our difference equation, and we will find that one of those two solutions is always a constant, and the other approaches zero as $n \to \infty$ when $|z| < 1$, but diverges as $n \to \infty$ when $|z| > 1$. Were we to focus solely on the remainders, all we could say was that when $|z| < 1$ only the decreasing solution can be present, since we know the remainders go to zero in that case. By directing our attention instead to the partial sums, we see that there is always a constant term (the value of the function we are trying to find) as well as a remainder (the second solution) that diverges when we are outside the radius of convergence, and goes to zero inside the radius of convergence. Thus, we show directly that our asymptotic acceleration not only speeds the convergence of the series when it does converge, but also slows the divergence when it does not.

So consider the general problem of the asymptotic solutions (valid for large $n$) of a linear, homogeneous, second-order difference equation that may be written in the form:

\begin{equation}
    w_{n+2} + a(n)w_{n+1} + b(n)w_n = 0
\end{equation}

where the coefficient functions $a(n)$ and $b(n)$ are themselves known, and have asymptotic expansions:

\begin{equation}
    a(n) \sim \sum_{k=0}^{\infty} \frac{a_k}{n^k}, \quad b(n) \sim \sum_{k=0}^{\infty} \frac{b_k}{n^k}, \quad n \to \infty.
\end{equation}

This problem has been considered by several authors, beginning with Birkhoff [53, 54], Adams [55], and Birkhoff and Trjintzinsky [56]. More recently, this body of work has been reviewed and summarized by both Wimp and Zeilberger [57] and Wong and Li [58], who all find Birkoff’s work notable for its complexity. For us, by far the most useful reference will be [58], since the authors carefully analyze the several possible cases that may arise when finding asymptotic solutions to (2.12), and also provide explicit, recursive formulas for arbitrary asymptotic coefficients. That will be crucial. We follow the terminology of [58], but not in general the notation.

The analysis of [58] only considers the case in which $b_0$ of (2.13) is not zero. As we will soon see, that will limit the $_pF_q$ that our acceleration can handle to those for which $p = q + 1$. In a later paper [59] the same authors consider the more general case, and so likewise we will consider the analysis of generalized hypergeometrics where $p \neq q + 1$ in a separate work.

With that restriction on $b_0$, the authors of [58] show that there are two linearly independent asymptotic solutions of (2.12), which fall into one of three cases (depending on the lowest few asymptotic coefficients of $a(n)$ and $b(n)$) as follows:

**The normal case.** When the two roots $\xi_1$ and $\xi_2$ to the characteristic equation

\begin{equation}
    \xi^2 + a_0 \xi + b_0 = 0
\end{equation}
are distinct, then the two linearly independent solutions to (2.12) are each of the form:

\[
\omega_n \sim \xi^n n^\lambda \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty,
\]

for the two values of \( \xi \), and the exponent \( \lambda \) for each solution depends on \( \xi \) through:

\[
\lambda = \frac{a_1 \xi + b_1}{a_0 \xi + 2b_0}.
\]

**The subnormal case.** When the roots of the characteristic equation do coincide, but the double root is not the zero of the auxiliary equation \( a_1 \xi + b_1 = 0 \), then the two solutions are of the form:

\[
\omega_{\pm,n} \sim \xi^n e^{\pm \gamma \sqrt{n}} n^\lambda \sum_{k=0}^{\infty} \frac{c_{\pm,k}}{n^{k/2}}, \quad n \to \infty,
\]

where \( \gamma \) and \( \lambda \) may be explicitly determined, but we will not need them.

**The exceptional case.** When the roots of the characteristic equation coincide and the double root is also the root of the auxiliary equation, then the two linearly independent solutions are again given by equation (2.15), but now the two values of \( \lambda \) are given not by (2.16), but rather as the two roots of the indicial equation:

\[
\lambda(\lambda - 1)\xi^2 + (a_1 \lambda + a_2)\xi + b_2 = 0.
\]

There are some further complications considered in [58] when the two roots of this equation either coincide or differ by a positive integer, but we will not need those subtleties.

In each of these three cases, the leading coefficient \( c_0 \) of the asymptotic expansion may be taken, without loss of generality, to be one, and the higher coefficients are then determined recursively from the \( \{a_n\} \) and \( \{b_n\} \) through formulas that we will quote from [58] later. We will not discuss the derivation of these cases and the corresponding formulas, except to say that superficially the method is much like the series solution of differential equations: one proposes a form of the solution and inserts it into the equation, and this recursively determines all of the higher coefficients. But showing that the resulting formal solutions are indeed asymptotic is far from trivial, and for details the reader is referred to [58] and the rest of the literature cited.

We see immediately that the recursion (2.10) satisfied by the partial sums \( s_n \) is of the form (2.12), provided we take:

\[
a(n) := -\left(1 + z r(n)\right), \quad b(n) := z r(n).
\]

To apply the results of [58] we need an asymptotic expansion for \( r(n) \). But for large \( n \), that is easy; because \( p = q + 1 \) we see from (2.8) that \( r(n) \) is a rational function whose numerator degree equals its denominator degree, and so we divide both numerator and denominator by \( n^{q+1} \) and write:

\[
r(x) := \frac{(1 + a_1 x) \cdots (1 + a_{q+1} x)}{(1 + \beta_1 x) \cdots (1 + \beta_q x)(1 + x)}
\]
where we have defined $x$ as $1/n$. This rational function (we deliberately use the same symbol) has a convergent Taylor series expansion in a neighborhood of zero, and the coefficients of that Taylor series will coincide with the asymptotic coefficients of $r(n)$ as $n \to \infty$. We will need arbitrarily many of these coefficients for our full acceleration method, and efficiently calculating those is not trivial, so we defer it to the next subsection. However, to determine which of the three cases above we fall under, we need only the lowest two, and elementary calculus yields:

\begin{align}
  r_0 &= 1 \\
  r_1 &= \sum_{k=1}^{q+1} \alpha_k - \sum_{k=0}^{q} \beta_k - 1 = \sigma - 1.
\end{align}

Finally, we will also need:

\begin{align}
  a_n &= \begin{cases} 
  -1 - z & \text{if } n = 0 \\
  -z r_n & \text{otherwise}
  \end{cases} \\
  b_n &= z r_n
\end{align}

and the consequent identity $a_n = -b_n$ whenever $n \geq 1$. Note that $b_0 \neq 0$ precisely because we assume $p = q + 1$.

Having considered the generalities, we now turn to precise formulas for the asymptotics of the $\{s_n\}$. As those depend on whether or not we are at the branch point $z = 1$, we take up those two cases in turn. Before beginning that discussion, though, we point out that none of the results of the next two subsections apply when one of the upper or lower parameters is a non-positive integer. That is because in those situations the recursion (2.10) does not really describe the asymptotics of the partial sums as $n \to \infty$; rather, in the first situation the hypergeometric is a terminating polynomial, and in the second, it is undefined. So although our results will not apply, either case is easy to identify and handle without series acceleration techniques.

2.2.1. Partial sum asymptotics away from the branch point. For our recurrence (2.10), we have from (2.21–2.24) that the characteristic equation is:

\begin{equation}
  \xi^2 - (1 + z)\xi + z = 0,
\end{equation}

and the roots of this equation are 1 and $z$. Thus, provided we are not at the branch point of the hypergeometric function, the recurrence equation is in the normal case of the three listed above. We can also find the corresponding exponents from (2.16), and we easily determine that the exponent corresponding to $\xi = 1$ in fact vanishes, whereas the exponent corresponding to the root $\xi = z$ is $r_1 = \sigma - 1$.

To find the asymptotic coefficients for each of these cases, we need the recursion that those coefficients satisfy. That is given in equation (2.3) of [58], and in our notation is:

\begin{equation}
  \sum_{j=0}^{k-1} \left[ \xi^2 2^{k-j} \binom{\lambda - j}{k - j} + \xi \sum_{i=j}^{k} \binom{\lambda - j}{i - j} \alpha_{k-i} + b_{k-j} \right] c_j = 0.
\end{equation}

It is convenient to rewrite this equation so that it explicitly gives $c_k$ in terms of $\{c_0, \ldots, c_{k-1}\}$. Rearranging terms, making use of equations (2.23) and (2.24), and
renaming the index of summation yields

\[ c_k = \frac{1}{k(1-z)} \sum_{j=0}^{k-1} \left[ 2^{k+1-j} \binom{-j}{k+1-j} + \sum_{i=j}^{k+1} \binom{-j}{i-j} a_{k+1-i} + b_{k+1-j} \right] c_j, \]

for the root \( \xi = 1 \) (and therefore \( \lambda = 0 \)), and

\[ c_k = -\frac{1}{k(1-z)} \sum_{j=0}^{k-1} \left[ (2^{k+1-j} - 1)z - 1 \right] \binom{\sigma - 1 - j}{k + 1 - j} \]

\[ -z \sum_{i=j}^{k} \binom{\sigma - 1 - j}{i-j} r_{k+1-i} + r_{k+1-j} \]

for the root \( \xi = z \) (and \( \lambda = \sigma - 1 \)). Each of these equations is valid for all \( k \geq 1 \).

In equation (2.28) we have expressed the recursion in terms of the asymptotic coefficients \( r_n \) of \( r(n) \), rather than the \( a_n \) and \( b_n \), but we have not troubled to do that for the asymptotic coefficients for the \( \xi = 1 \) root. That is because all of the coefficients \( c_k \) in equation (2.27) are zero when \( k \geq 1 \). To see this, first consider the coefficient of \( c_0 \) on the right hand side of (2.27). The binomial coefficients \( \binom{0}{l} \) for any integer \( l \) vanish, except for \( \binom{0}{0} \) which equals one. Thus, since \( j < k+1 \), the first term in the square brackets vanishes because the binomial coefficient does, as do all of the terms in the inner sum over \( i \), except for the \( i = j = 0 \) term. That term survives to give \( a_{k+1} \), but that in turn cancels the term \( b_{k+1} \). Hence, the entire coefficient of \( c_0 \) vanishes.

But by the recursion (2.27), that means that \( c_1 \) vanishes, and then by induction that all of the \( c_k \) vanish when \( k > 0 \), for the \( \xi = 1 \) root of the characteristic equation. Hence, comparing to (2.15), we see that the first asymptotic solution to the difference equation (2.10) is simply a constant. That is not true for the second solution, however, and so since the partial sum \( s_n \) is a linear combination of these two asymptotic solutions, we have shown:

\[ s_n \sim s + \mu z^n n^{\sigma-1} \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty, \]

with \( \sigma \) given by (1.2) and the \( c_k \) determined recursively from \( c_0 = 1 \) by equation (2.28).

This result holds so long as \( z \neq 1 \); in particular, it holds outside the radius of convergence \(|z| = 1\). In that case, however, (2.29) shows that the remainder term diverges as \( n \to \infty \); the series is not convergent there. It clearly converges whenever \(|z| < 1\), and on the circle of convergence the remainder is a decreasing function of \( n \) provided the real part of the exponent of \( n \) is negative; that is, provided \( \Re \sigma < 1 \). So, our asymptotic formula correctly reproduces the analytic properties of \( {}_qF_{q+1} \) whenever \( z \neq 1 \); we next consider \( z = 1 \).

2.2.2. Partial sum asymptotics at the branch point. When \( z = 1 \) we are no longer in the normal case of [58]. To decide between the subnormal and exceptional cases, we must examine the auxiliary equation \( a_1 \xi + b_1 = 0 \). As \( a_1 = -b_1 \) and \( \xi = z = 1 \), our double root is in fact a root of the auxiliary equation, so we are in the exceptional case. The indicial equation when \( \xi = 1 \) is:

\[ \lambda(\lambda - 1) + (a_1 \lambda + a_2) + b_2 = 0. \]
Since $a_2 + b_2 = 0$, we get:

(2.31) \quad \lambda = 0 \quad \text{or} \quad \lambda = r_1 + 1 = \sigma

as our two possible exponents.

Just as for the normal case, we must now examine the recursive equations that determine the asymptotic coefficients $c_n$ in terms of the $\{r_n\}$ for each of these two possible values of $\lambda$. When we do so we again find that only the leading, constant term survives for $\lambda = 0$, while for $\lambda = \sigma$ the entire series is nontrivial. Similar considerations allow us to avoid some special sub-cases of the exceptional case alluded to above. Generically, the solutions to the recursion equation can have terms logarithmic in $n$ when the difference between the two roots of the indicial equation are an integer, but for all of the cases where the hypergeometric series at the branch point converges (that is, where $\Re \sigma < 0$) we can show through inductive arguments similar to that above that the logarithmic term vanishes. Thus, the most general asymptotic solution is of the form:

(2.32) \quad s_n \sim s + \mu n^\sigma \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty,

where again $c_0 = 1$, and the higher coefficients can be shown from equation (7.2) of [58] to satisfy:

(2.33) \quad c_k = \frac{1}{k(\sigma - k)} \sum_{j=0}^{k-1} \left( \binom{\sigma - j}{k + 2 - j} - \sum_{i=j}^{k+1} \binom{\sigma - j}{i - j} r_{k+2-i} + r_{k+2-j} \right) c_j.

As with regular points of the function, we see that the asymptotic expansion at the branch point reproduces the correct analytic behavior of the $_{q+1}F_q$ function. Specifically, the remainder is a decreasing function precisely when $\Re \sigma < 0$, the condition we saw in section 1.1 was needed to ensure that the series converged at $z = 1$. We also see explicitly from (2.32) that the convergence of the series at $z = 1$ is logarithmic, with a complex critical exponent (in general).

2.3. Recursively computing the asymptotic coefficients. The two asymptotic expansions (2.29) and (2.32), together with the respective recursions (2.28) and (2.33), are a complete solution for the asymptotics of the partial sums of the generalized hypergeometric function. To be numerically useful, however, we must be able to calculate arbitrary asymptotic coefficients $r_k$ of the rational term ratio function $r(n)$. It is here that we can make use of the explicit analytic knowledge we have of our series; it comes to us not just as a numerical sequence of terms. As we noted, the $r_k$ are the same as the Taylor coefficients of the expansion around zero of the rational function $r(x)$ of equation (2.20), but that in and of itself is not a practical solution, if we have no better means to calculate the Taylor coefficients than by evaluating high order derivatives at zero.

Fortunately, there is a much more effective procedure. Define the functions $P(x)$ and $Q(x)$ as:

(2.34) \quad P(x) = (1 + \alpha_1 x) \cdots (1 + \alpha_{q+1} x); \quad Q(x) = \frac{1}{(1 + \beta_1 x) \cdots (1 + \beta_q x)(1 + x)}
so that \( r(x) = P(x)Q(x) \). It is obvious that the Taylor coefficients of \( r(x) \) satisfy

\[
(2.35) \quad r_k = \sum_{j=0}^{k} P_j Q_{k-j}
\]

when \( \{P_k\} \) and \( \{Q_k\} \) are the Taylor coefficients of the respective functions.

The coefficients \( P_k \) and \( Q_k \) can be calculated effectively because we have already factored the functions \( P(x) \) and \( Q(x) \). Specifically, rearranging some results of [60] gives:

\[
(2.36) \quad P(x) = \sum_{k=0}^{q+1} e_k(\alpha_1, \ldots, \alpha_{q+1}) x^k
\]

and:

\[
(2.37) \quad Q(x) = \sum_{k=0}^{\infty} (-1)^k h_k(1, \beta_1, \ldots, \beta_q) x^k.
\]

In these equations, \( e_k \) and \( h_k \) are the elementary symmetric polynomials and the complete homogeneous symmetric polynomials, respectively. These may be defined in several ways, but the most computationally useful method defines them recursively from the power sums of the upper and lower parameters (where the lower parameters must be augmented with the implicit parameter one that is part of the definition of the generalized hypergeometric function). The \( k \)-th power sum \( p_k(x_1, \ldots, x_n) \) of any set \( \{x_i\} \) of \( n \) numbers is:

\[
(2.38) \quad p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k.
\]

In terms of the power sums, the elementary symmetric polynomials satisfy the recurrence:

\[
(2.39) \quad ke_k(\alpha_1, \ldots, \alpha_{q+1}) = \sum_{i=1}^{k} (-1)^{k-i} e_{k-i}(\alpha_1, \ldots, \alpha_{q+1}) p_i(\alpha_1, \ldots, \alpha_{q+1})
\]

and the complete homogeneous symmetric polynomials satisfy the recurrence:

\[
(2.40) \quad kh_k(1, \beta_1, \ldots, \beta_q) = \sum_{i=1}^{k} h_{k-i}(1, \beta_1, \ldots, \beta_q) p_i(1, \beta_1, \ldots, \beta_q).
\]

These recurrences do not determine \( e_0 \) or \( h_0 \); both are unity.

It is now straightforward to calculate the Taylor coefficients \( r_k \) for any desired \( k \). We first calculate the power sums \( p_i \) of the upper and augmented lower parameters for \( i \) from zero to \( k \). Then we use the recurrence (2.39) to find the elementary symmetric polynomials of the upper parameters, and the recurrence (2.40) to find the complete homogeneous symmetric polynomials of the augmented lower parameters. Finally, combining equations (2.35–2.37) gives:

\[
(2.41) \quad r_k = \sum_{i=0}^{k} (-1)^{k-i} e_i(\alpha_1, \ldots, \alpha_{q+1}) h_{k-i}(1, \beta_1, \ldots, \beta_q).
\]
Table 1. Stability results for recursive computation of the asymptotic coefficients at the branch point. All errors are geometric means across the sample of the maximum error among the $m$ coefficients.

| $R$ | $m = 30$ | $m = 45$ | $m = 30$ | $m = 45$ |
|-----|----------|----------|----------|----------|
| 1   | $3.9 \times 10^{-11}$ | $5.5 \times 10^{-14}$ | $2.6 \times 10^{-16}$ | $2.6 \times 10^{-16}$ |
| 5   | $4.9 \times 10^{-12}$ | $1.6 \times 10^{-10}$ | $7.5 \times 10^{-15}$ | $7.8 \times 10^{-15}$ |
| 10  | $3.9 \times 10^{-13}$ | $8.7 \times 10^{-12}$ | $2.3 \times 10^{-13}$ | $1.7 \times 10^{-12}$ |
| 50  | $1.9 \times 10^{-14}$ | $4.1 \times 10^{-14}$ | $1.9 \times 10^{-14}$ | $4.1 \times 10^{-14}$ |
| 100 | $1.8 \times 10^{-14}$ | $2.7 \times 10^{-14}$ | $1.8 \times 10^{-14}$ | $2.7 \times 10^{-14}$ |

2.4. Stability of the recursions. The expressions for the asymptotic coefficients (2.28) and (2.33), together with the formulas of the last subsection for calculating the Taylor coefficients, provide a determination of the asymptotic coefficients through several nested recursions. It is therefore important to study the stability of these recursions. Given the nesting of the recursions, an analytic study is daunting, so we investigate the stability numerically. Even though most of our numeric results on the performance of our algorithm are presented in the next section, we digress briefly to study the stability of the coefficient calculation here. That is both because this study was quite different from the analysis of section 3, as it was conducted in much higher than machine precision, and also because we need an assurance of the stability of the asymptotic coefficient calculation before we can examine the overall method.

Accordingly, we used the mpmath [21] package already mentioned in the introduction to implement the calculation of the coefficients in fixed but arbitrary precision. We compared the calculation at a precision of 53 bits (standard double precision, as we will use in the C implementation discussed in the next section) to that calculated with twice the precision, at 106 bits. We expect that the accuracy of the computation will depend on the size of the parameters, the value of $z$, and the number of coefficients calculated, since error presumably accumulates throughout the recursion. To investigate these effects, we studied both $z = 1$, and $z$ chosen uniformly at random in the unit disk. We used four upper and three lower parameters, with real and imaginary parts each chosen uniformly in the range $(-R, R)$, for $R \in \{1, 5, 10, 50, 100\}$. Unlike the actual implementation of the full algorithm itself that we study in the next section, we did not restrict ourselves to only cases where the hypergeometric series itself converges. To examine the effect of the number of coefficients, we considered both $m = 30$ and $m = 45$ coefficients. For each choice of $m$ and $R$, we then chose 1000 sets of seven parameters (and also $z$ if not testing the branch point) at random as described above.

To quantify the results, for each coefficient we found the relative error between the value calculated in 53 bit precision and that in 106 bit precision. We then took the maximum of this relative error out of all $m$ coefficients. To obtain some measure of central tendency among the thousand distinct random samples for a given $m$ and $R$, we took the geometric mean of these thousand worst-case errors, since it is the order of magnitude of the result that is most important. The results of this calculation are shown in the second and third columns of table 1 for the
branch point cases (the unit disk cases are similar and are not shown). We can see already that there is only a weak dependence of these results on \( m \), indicating that nested recursions are in fact quite stable as we consider more and more asymptotic coefficients. Somewhat surprisingly, the average worst-case relative error is seen to be higher when \( R \) is smaller, even though as we will see in the next section it is larger values of \( R \) that make the overall computation of the generalized hypergeometric function more difficult. We might also be surprised that the typical worst-case errors can be as large as \( 10^{-10} \), since we will see results in the next section that indicate we can typically calculate the hypergeometric function itself to higher accuracy in those ranges of \( R \).

This is because the relative errors in the asymptotic coefficients \( c_k \) are not by themselves what is most relevant. More important is the error estimate \( \omega_n \) that we calculate from those coefficients. If our error is largest in the highest coefficients, then that error will be suppressed when we divide by \( n^k \). Thus, in the fourth and fifth columns we show the relative error in \( \omega_{10} \). Note that 10 is a very conservative number of terms to sum; for large parameter values we may typically see many tens or even hundreds of terms that must be summed to calculate the hypergeometric function. Yet we see that we are already very close to just a couple of orders of magnitude above machine precision for almost all values of \( R \), and in particular the relative errors of \( 1 \times 10^{-12} \) and \( 2 \times 10^{-14} \) that we shall use in the next section as relative tolerances to demand of \( \eta+1F_\eta \) are reasonable. More importantly, the rather remarkable precision with which we can calculate \( \omega_n \) is the key reason that the method of this paper is much more stable than the \( E \)-method, as we will see in section 4.3.

3. Implementation and results

We now have all of the pieces in place for an acceleration algorithm. To assemble them, we truncate either equation (2.32) or equation (2.29) for the asymptotics of our partial sums:

\[
s_n = s + \mu \omega_n^{(m)} + O(n^{-m})
\]

where we have defined the truncated remainder estimate:

\[
\omega_n^{(m)} := z^n n^\lambda \sum_{k=0}^{m-1} c_k \frac{1}{n^k}.
\]

Here \( \lambda = \sigma - 1 \) away from the branch point, and \( \sigma \) at \( z = 1 \). Likewise, the asymptotic coefficients \( c_k \) are given by either (2.28) or (2.33), as appropriate. We call \( m \) the order of our method, and we use equation (2.41) to calculate the asymptotic coefficients \( r_k \) regardless of whether or not \( z = 1 \). We will need \( m + 1 \) of the \( r_k \) to calculate \( m \) coefficients \( c_k \); though it might seem from equation (2.33) that we would need \( m + 2 \) of the \( r_k \) when at the branch point, in fact the different appearances of \( r_{m+2} \) cancel.

We emphasize that despite our detailed analytic knowledge of \( \omega_n^{(m)} \), it is not by itself a remainder estimate; we must also know the factor \( \mu \). That cannot be fixed by any of our analytic calculations; it must be estimated directly from the sequence of partial sums. This is a somewhat subtle point, but distinguishes our method from either Euler-Maclaurin summation [61] or methods based on zeta-functions [26, 27, 62]. Those methods each compute an approximation to the remainder
directly, and need the partial sums only to combine with that approximation; our method requires two computed partial sums so that we may fix \( \mu \). It is in this respect more like fixed-order series acceleration methods, such as Shanks’ method, though we continue to refer to \( m \) as the order for our method, as it is \( m \) that determines the rate of acceleration.

From equation (3.1) and any two successive computed partial sums \( s_n \) and \( s_{n+1} \), we may calculate an accelerated sum \( s_n^{(m)} \) that is hopefully a better approximation to \( s \) than either \( s_n \) or \( s_{n+1} \):

\[
 s_n^{(m)} = s_n \omega_n^{(m)} + s_{n+1} \omega_{n+1}^{(m)} - \omega_n \omega_{n+1}^{(m)}.
\]

This equation encapsulates our basic method; it is defined only for \( m > 0 \).

Of course, since our estimates for the remainders are only asymptotic, it may require several computed partial sums before the asymptotic behavior of the remainder is reached; we expect this number to increase with increasing \( m \), and also to depend on the parameters and argument of the hypergeometric. Even once we have reached the asymptotic regime for some particular value of \( m \), we may need to continue calculating estimates for larger \( n \) to ensure that the \( O(z^n n^{\lambda - m}) \) error term is smaller than whatever tolerance is desired; again, we expect that the number of such additional iterations will depend on \( z \) and the hypergeometric parameters. Nevertheless, the basic algorithm contained in equation (3.3) is ripe for an example, so consider the following evaluation, which we can do in closed form thanks to Gauss’ formula (1.3):

\[
 2F1 \left( \begin{array}{c} 1 + 4i, 1.5 + 4.5i \\ 3 + i \end{array} \right) \bigg| 1 = -0.003206491294324765 - 0.006293652031968077i.
\]

We expect this to be challenging for any series summation technique: all of the parameters are complex and we evaluate the series at the branch point; the rate of decay of the remainders is only \( n^{-1/2} \).

As we can see in Figure 1, however, our method is indeed able to accelerate the convergence of this hypergeometric function dramatically. The topmost curve, which shows the relative error \( \delta s_n = |1 - s_n|/s \) with no acceleration, has no correct digits even after summing ten thousand terms, but the acceleration with 30 asymptotic coefficients produces ten digit accuracy with roughly ten terms. This figure also shows clearly how the order \( m \) of the method affects the speed of convergence; the linear slopes of the \( \delta s_n^{(m)} \) on a log-log scale illustrate the power law fall-off predicted by equation (2.32). Finally, we see that the approach to convergence need not be monotonic: notice how \( \delta s_n^{(2)} \) jumps above the unaccelerated series before it becomes asymptotic.

However, even a small change in the parameters can spoil this behavior. As a second example, consider the acceleration of

\[
 2F1 \left( \begin{array}{c} 1 + 20i, 1.5 + 25i \\ 3 + 15i \end{array} \right) \bigg| 1 = (-1.508618716765084 + 2.168373234294654i) \times 10^{-20}.
\]

We plot this in Figure 2 using 45 asymptotic coefficients. We do initially see very rapid decrease in the relative error of the accelerated sum. However, the relative
Figure 1. The relative error $\delta s_n^{(m)}$ for different orders $m$ of acceleration of the hypergeometric function of (3.4).

Figure 2. Relative error in the partial sums and accelerated partial sums of equation (3.5).
error then levels off at about $10^{23}$. This is despite the very small change made to the parameters: we have increased the imaginary parts of each parameter by roughly a single order of magnitude, and we have not changed the real parts at all.

3.1. Numerical considerations. Of course there is really no mystery here: the largest magnitude that the partial sum in the evaluation of (3.5) reaches is roughly $6.5 \times 10^{17}$, or thirty-seven orders of magnitude greater than the true value of the function. Certainly no method implemented in fixed IEEE 754 precision can accomplish such an acceleration.

Obviously we could simply implement the algorithm in a symbolic algebra program that can use arbitrary precision. However, in at least some cases this may not be desirable. For instance, the author’s original interest in this project grew out of a problem in computational quantum gravity that requires the evaluation of thousands of ${}_4F_3(1)$ with complex parameters as part of a larger program. Even where higher precision is available it is useful to have our algorithm correctly determine when that higher precision is really needed. Finally, because of our precise knowledge of the asymptotics of the remainder, we can provide an excellent estimate of the error the algorithm makes when it does converge.

The essential problem is that the accuracy of the algorithm is determined by its truncation error $O(z^n n^{\lambda-m})$ in equation (3.1), but before that truncation error becomes sufficiently small, it may be overwhelmed by the floating point error. Thus, we need to estimate both sources of error as accurately as we can, and compare them to decide whether our accelerated summation has converged to a prescribed accuracy. In the next two sections we discuss each of these errors in turn.

3.1.1. Estimating truncation error. In many numerical problems the best estimate of truncation error may simply be the change between successive estimates, at least when that change begins to exhibit convergence. However, because we know from equation (3.1) precisely how the truncation error behaves asymptotically, we can provide a sharper estimate. Subtracting successive estimates gives:

$$s^{(m)}_{n+1} - s^{(m)}_n \approx A z^n n^{\lambda-m} \left( z \left( 1 + \frac{1}{n} \right)^{\lambda-m} - 1 \right)$$

for some constant $A$. But in fact $A z^n n^{\lambda-m}$ is the leading order of the truncation error, so this suggests that a more accurate estimate of the truncation error would be:

$$\Delta_{tr}s^{(m)}_n = \frac{s^{(m)}_{n+1} - s^{(m)}_n}{\left| z \left( 1 + \frac{1}{n} \right)^{\lambda-m} - 1 \right|}.$$  

In our implementation, however, we use the modified estimate:

$$\Delta_{tr}s^{(m)}_n = \frac{s^{(m)}_{n+1} - s^{(m)}_n}{z \left( 1 + \frac{1}{n} \right)^{-m} - 1}.$$  

Using this estimate greatly reduces the number of false positives (where the algorithm believes it has converged but has not) when inside the unit disk $|z| < 1$ but with $\Re \lambda \gg 0$ (see section 3.2.2 for details). Note that it is only when $|z| < 1$ that the real part of the critical exponent can be positive; at the branch point such series do not converge and the hypergeometric function is undefined there.
Likewise, even though the simpler estimate $\Delta_{tr}s_n^{(m)} = (s_{n+1}^{(m)} - s_n^{(m)})$ has similar rates of convergence or non-convergence to those we will present in section 3.2.2, the estimated error is not nearly as accurate as with the estimate (3.8) above, and so we prefer (3.8) for that reason also.

3.1.2. Estimating floating point error. The estimation of floating point error is more complex. We must be careful, because overestimation of the floating point error can be just as dangerous as underestimation: it will cause us to abandon a calculation that may well have been converging. The very simplest estimates—for instance, comparing $s_n^{(m)}/\mu$ to the floating point precision—are disastrous for all but the smallest ranges of parameters, so we consider a more detailed analysis. Examining equation (3.3), we can identify three primary sources of potentially significant floating point error in the calculation of $s_n^{(m)}$:

1. Subtractive cancellation in either the numerator or denominator of (3.3); in each we subtract two nearby floating point numbers that we expect will only grow closer to each other as $n$ increases.
2. Underflow in the calculation of $\omega_n^{(m)}$, particularly if $\Re\lambda \ll 0$ or $|z| \ll 1$.
3. Accumulated floating point error in $s_n$ and $s_{n+1}$, which are calculated recursively using:

\begin{align*}
s_{n+1} &= s_n + t_n, \quad (3.9) \\
t_{n+1} &= zr(n) t_n. \quad (3.10)
\end{align*}

We will need some notation for our analysis. We denote the computed value of some exact quantity $x$ by $\hat{x}$. The absolute error between $x$ and $\hat{x}$ is $\Delta x$, while the relative error is $\delta x$, so that:

\begin{equation}
\hat{x} = x + \Delta x = x(1 + \delta x).
\end{equation}

Applying this to the formula (3.3) for $s_n^{(m)}$, and calling the numerator of that formula $\mathcal{N}_n$ and its denominator $\mathcal{D}_n$, we get:

\begin{equation}
(\mathcal{D}_n + \Delta \mathcal{D}_n)(s_n^{(m)} + \Delta s_n^{(m)}) = (\mathcal{N}_n + \Delta \mathcal{N}_n)
\end{equation}

implying:

\begin{equation}
\Delta s_n^{(m)} = \frac{1}{\mathcal{D}_n}(\Delta \mathcal{N}_n - s_n^{(m)} \Delta \mathcal{D}_n).
\end{equation}

To derive an estimate from this formula, we must approximate the errors $\Delta \mathcal{N}_n$ and $\Delta \mathcal{D}_n$. We estimate the former by assuming that it is dominated by the error in calculating the partial sums:

\begin{equation}
\Delta \mathcal{N}_n \approx \Delta s_n |\omega_{n+1}| + \Delta s_{n+1} |\omega_n|.
\end{equation}

We estimate $\Delta \mathcal{D}_n$ in terms of the relative error of the remainder estimates, rather than absolute. That is because we wish to vary our estimate for the remainder error based on whether or not the calculation underflowed:

\begin{equation}
\delta \omega_n = \begin{cases} 
\epsilon_p & \text{if } z^n n^\lambda \text{ is a normalized floating point number}, \\
\ |f_{\text{norm}}| z^{-n} n^{-\lambda} | & \text{otherwise}.
\end{cases}
\end{equation}

In this equation, $\epsilon_p$ is the machine precision at which $\omega_n$ is computed, and $f_{\text{norm}}$ is the smallest normalized number in that precision. This subtlety is needed because when $\omega_n$ becomes sufficiently small, it will no longer be represented by a floating
point number with a relative accuracy of $\epsilon_p$, but rather with a relative accuracy that
decreases from $\epsilon_p$ down to zero. Such floating point numbers are called subnormal,
and $f_{\text{norm}}$ is the smallest floating point number that is still normalized (possessing
the full relative precision of $\epsilon_p$).

Of course, these estimates involve many approximations. For example, the relative
error in $\omega_n$ is likely larger than given by (3.15) when it is a normalized number,
as it is itself calculated by a non-trivial chain of floating point computations. But
in those cases it is not a dominant source of error overall, and so (3.15) suffices.
Combining equations (3.13–3.15), approximating unknown exact quantities by their
computed equivalents when needed, and taking absolute values to allow for an upper
bound on the error, we arrive at the final estimate used in our implementation:

\begin{equation}
\Delta f_p s_n^{(m)}(m) = \frac{1}{|\hat{\omega}_{n+1} - \hat{\omega}_n|} \left( |\Delta s_n| + |\delta s_n^{(m)}| \delta \omega_{n+1} \right) \\
+ |\hat{\omega}_n| \left( |\Delta s_{n+1}| + |\delta s_n^{(m)}| \delta \omega_{n+1} \right).
\end{equation}

This equation is not complete until we specify how we estimate $\Delta s_n$. We choose
a simple type of a posteriori estimate; for details, see chapter 3 of Higham’s mono-
graph [63]. The advantage of such estimates over the more straightforward a priori
estimates is that they take into account cancellation that occurs during the com-
putation, and are therefore less prone to overestimate the error (though they can
be more expensive to compute).

Our estimate starts from equation (3.9) and treats $t_n$ as an exact quantity,
accounting only for the accumulation of error in the recursive sum in $s_n$. Then it
is not hard to show that the appropriate a posteriori estimate starts from $\Delta s_0 = 0$
and recursively calculates

\begin{equation}
\Delta s_{n+1} = \Delta s_n + \epsilon_p |s_{n+1}|
\end{equation}

where again $\epsilon_p$ is the machine precision at which the computation is carried out.

Of course, in reality $t_n$ is also corrupted by ever-growing error, since it too
is calculated recursively. It is possible to apply a similar a posteriori analysis and
derive an estimate for $\Delta s_n$ that takes this into account. However when this method
was implemented, it tended to severely overestimate the error, greatly reducing the
convergence rate and increasing the number of false negatives (where the algorithm
does not believe it has converged, even though it has). At the same time it was
more expensive to compute and even though it could terminate some non-converging
cases more rapidly, its overall performance was slower, even for parameter ranges
where a substantial majority of cases did not converge. Hence, for the rest of this
paper we consider only the estimate (3.17) when estimating the error in the partial
sums, and we use that in our overall estimate (3.16) for the floating point error in
our accelerated sum.

3.1.3. Final algorithm. With our estimates for truncation and floating point errors
in hand, we can finally state our complete algorithm, which seeks to approximate
a generalized hypergeometric function to specified accuracy, or determine that this
cannot be done without higher precision. It takes as input not only the hyper-
geometric parameters and argument, but also the order $m$, the desired relative
tolerance $\varepsilon$, and a maximum number of iterations $N$. The errors are estimated as
described in the preceding two subsections. In pseudocode we have:
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Input: $z$, $q$, $\{\alpha_i\}_{i=1}^{q+1}$, $\{\beta_i\}_{i=1}^{q}$, $m$, $N$, and $\varepsilon$.
Calculate: Exponent $\lambda$ and $\{c_k\}_{k=0}^{n-1}$
Initialize: $s_0$, $s_1$, and $s_2$; $\omega_1$ and $\omega_2$; $s^{(m)}_0$; $\Delta s_0$
do: Calculate $s_{n+1}$, $\Delta s_{n+1}$, $r(n)$, and $\omega_{n+1}$
From $s_n$, $s_{n+1}$, $\omega_n$ and $\omega_{n+1}$, calculate $s^{(m)}_{n+1}$
From $s_n$, $s_{n+1}$, $\omega_n$, $\omega_{n+1}$, $\Delta s_n$, and $\Delta s_{n+1}$ calculate $\Delta_{fp}s^{(m)}_n$
From $s^{(m)}_n$ and $s^{(m)}_{n+1}$ calculate $\Delta_{tr}s^{(m)}_n$
$n \rightarrow n + 1$, $s_0 \rightarrow s_n$, $\Delta s_{n+1} \rightarrow \Delta s_n$, and $\omega_{n+1} \rightarrow \omega_n$
while $n < N$ and $\tau \Delta_{fp}s^{(m)}_n < \Delta_{tr}s^{(m)}_n$ and $\Delta_{tr}s^{(m)}_n > \varepsilon |s^{(m)}_n|$ if $\Delta_{tr}s^{(m)}_n \leq \varepsilon |s^{(m)}_n|$ return “Success”, $s^{(m)}_n$, and $\Delta_{tr}s^{(m)}_n / |s^{(m)}_n|$ else return “Maximum iterations reached”

Note in particular that the order of the return statements is such that if on the same iteration we simultaneously reach a tolerance less than the specified tolerance but also the floating point error exceeds the truncation error, we nonetheless deem the acceleration to have converged. This is the less conservative approach, but when these boundary cases are instead treated as failure, we greatly increase the number of false negatives, without avoiding any additional false positives. The parameter $\tau$ is an empirical “fudge factor” to lower the estimate of the floating point error; in our final implementation it was set to 0.1 as this somewhat decreases the false negative rate.

3.2. Testing the method. The algorithm that we have now derived and described in some detail was implemented in C, to test its effectiveness as a general purpose computational strategy for $q+1F_q\left(\begin{array}{c} \alpha_1,\ldots,\alpha_{q+1} \\ \beta_1,\ldots,\beta_q \end{array} \mid z \right)$. Of course we should like to validate it through testing, but we immediately face the problem of what to test it against, since the generalized hypergeometric function can be so challenging to compute in the cases we have in mind.

One choice of course is to test the calculation of $3F_2\left(\begin{array}{c} \alpha_1,\alpha_2 \\ \beta_1,\beta_2 \end{array} \mid z \right)$ at the branch point, since that is simultaneously non-trivial for a series based computation, yet easily benchmarked against Gauss’ formula (1.3). That indeed forms the bulk of our test suite, but we also examined some $3F_2$ and $4F_3$ functions inside the circle of convergence. We compared these against the corresponding calculations from the Python package mpmath [21], which as mentioned in the introduction takes a fairly sophisticated approach to calculating generalized hypergeometric functions. However it was still too slow (at high precision, which we used to ensure accuracy) to test $3F_2$ and $4F_3$ functions at the branch point for a large number of randomly generated cases.

All of our test results below will be presented in terms of a parameter $R$, which sets the scale for choosing the parameters. The precise selection was as follows:

Test cases in the unit disk: We choose a point $z$ with uniform probability inside the disk $|z| \leq 1$. We then choose $q+1$ upper parameters and $q$ lower parameters, with the real and imaginary parts of each chosen uniformly at random between $-R$ and $R$.

Test cases at the branch point: We choose the upper and all but one of the lower parameters with both real and imaginary parts chosen uniformly in $(-R, R)$, but we choose the last lower parameter so that $\Re \sigma < 0$. To
do this, if the real part of the sum of the upper parameters less the sum of the already chosen lower parameters is less than zero, we simply choose the real part of the last lower parameter uniformly between that value and $R$. But if that sum is positive, we choose uniformly between that value and either $R$ or that value plus $0.1R$, whichever is greater. So the last lower value may have a real part greater than $R$.

With that procedure in mind, there are several questions we wish to investigate empirically: What is the optimal choice of order $m$? What is the accuracy of the method, and how often does it terminate correctly? How accurate is the estimation of error? What are optimal choices for the maximum number of allowed terms, $N$? How fast is the method? We present results on all of these questions in the following sections; for most of these we only consider $\text{2}_1F_1(1)$. However when we examine the overall accuracy and termination we also consider cases in the unit disk, as described above.

3.2.1. Convergence as a function of order. We begin with an investigation of the effect of the order on the convergence of the method. We have already seen in a simple example that the order does indeed dramatically affect the rate of convergence, yet at the same time higher order requires more precomputation and slows the execution of the method, so we only wish to invest in this when it is helpful. But if we choose too low an order all of the rest of our tests will be essentially meaningless.

Thus, we considered $\text{q}_1F_q(1)$ for $q = 1$, 2 and 3. We generated $10^4$ random cases for each value of $q$ and for each $R$ in $\{1, 5, 10, 50, 100\}$. We required a relative tolerance of $2 \times 10^{-14}$, and then for each randomly chosen parameter set we called the algorithm with each value of $m$ from 5 to 50. For each parameter set for which at least one of these calls converged, we then observed for which $m$ the fewest number $n$ of partial sums were needed to achieve the specified convergence; we called this the optimal order $m_{\text{opt}}$. We will not present all of the results here, but simply an illustrative example for the $\text{4}_1F_3$ functions; the results for other choices of $q$ are similar.

We can see from figure 3 that there is a dependence on $R$, but note that regardless of $R$ the maximum optimal $m$ is 45, even though we tested up to 50; at these ranges of $R$, at least, there is simply no point in using more than 45 asymptotic coefficients. Of course, this does not prove that using more coefficients will lead to faster execution, since the increased precomputation may offset the need for fewer partial sums. We will examine the speed of the algorithm in section 3.2.5. But we do conclude from this plot and the similar plots for $\text{2}_1F_1$ and $\text{3}_1F_2$ that 45 is a reasonable upper choice for the order; when we wish to compare the effect of order on other tests we will also present results for $m = 30$.

3.2.2. Convergence as a function of the hypergeometric parameters. By far the most important question for our method is whether or not it converges, and whether or not it reliably determines when it has. Our most extensive testing was on this question.

To study it, we again considered $R \in \{1, 5, 10, 50, 100\}$, $m = 30$ and $m = 45$, and a relative tolerance $\varepsilon$ of either $1 \times 10^{-12}$ or $2 \times 10^{-14}$. For each permutation of these parameter choices, we generated $10^5$ random samples and compared the computed with the exact value of $\text{2}_1F_1(1)$. We considered four possible scenarios:
Figure 3. Distribution of optimal order for varying choices of $R$ for the $4F_3(1)$ functions

Convergence: The algorithm claimed that it converged to the specified tolerance, and the true relative error was within a factor of ten of that specified tolerance.

False positive: The algorithm claimed that it had converged, but the true relative error was more than ten times the tolerance.

No Convergence: The algorithm claimed it did not converge, and its true relative error was more than the tolerance.

False negative: The algorithm claimed it did not converge, yet its true relative error was less than the tolerance.

The factor of ten is somewhat arbitrary but merely reflects the uncertainty in our truncation error estimate; in this preliminary investigation it is too stringent to demand that true error be strictly less than than the tolerance, though we will see in section 3.2.3 that this is almost always the case. Finally, in addition to these four scenarios (which are mutually exclusive) we also show the percentage of each sample that reached the maximum number of allowed iterations ($2 \times 10^4$ in this set of tests) without converging. These samples are not really cases that could have converged had they been given more time, but rather cases where the floating point error was underestimated and the algorithm did not terminate early with failure as it should have.

For the case $m = 45$, these results are summarized in tables 2 and 3. We do not present the results for $m = 30$ since they differ very little from these; typically at most a percent. We can see from these tables that the convergence rate depends strongly on $R$, as we would expect from our earlier examples. But the false positive rate is extremely low; and even that overstates the issue: all of
the cases labeled as false positives in fact converged but with a slightly higher ratio between the true and estimated error. Had we chosen our adjustment factor to be 200 instead of 10, the false positive rate would be zero in all cases. Thus, at the branch point we conclude that when the algorithm terminates with success, it is essentially always reliable. The false negative rate is also low, though not nearly so small as the false positive rate. It is also somewhat ambiguous, since it tells us only that when the algorithm terminated with failure, it was in fact really within the prescribed tolerance; it does not identify scenarios where had the calculation continued further, convergence would have been achieved. A wide variation in this rate is therefore possible based on the choice of floating point error estimate; our implementation uses the choice that gave the smallest false negative rate of those estimates we considered. Comparing table 2 to table 3, we can see that requiring higher accuracy does decrease the percentage of convergent cases, with an effect most pronounced for values of $R$ in the middle of the range we considered; at high $R$, the percentage of convergent cases is small enough that most of the effect of the choice of $\varepsilon$ is masked.

We also investigated the accuracy for points chosen in the unit disk. Here for simplicity we considered only $m = 45$ and $\varepsilon = 2 \times 10^{-14}$, but we were able to examine $3 F_2$ and $4 F_3$ functions as well, though for smaller ranges in $R$. These results are shown in table 4. We see that as we might expect the overall convergence rates are higher than for the corresponding value of $R$ at the branch point, and though the data is somewhat limited there does not seem to be a strong dependence on the order $q$ of the hypergeometric; certainly not nearly as strong as the dependence on $R$. One subtlety not shown in this table is that it is no longer true that the (albeit rare) false positives are necessarily benign; roughly half of the false positives for the $R = 100$ cases of $2 F_1$, for instance, had fewer than half of the digits correct; in many cases no digits correct. This always happens when the exponent $\lambda$ has a large positive real part; as already mentioned in section 3.1.1, our choice of estimate for the truncation error reduces the fraction of these false positives. We can now quantify that assertion: had we used the truncation error estimate (3.7) instead

| $R$ | C  | FP  | NC  | FN  | $n_{\text{max}}$ |
|-----|----|-----|-----|-----|-----------------|
| 1   | 100.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 5   | 98.90% | 0.012% | 1.08% | 0.01% | 0.93% |
| 10  | 81.39% | 0.096% | 17.97% | 0.54% | 14.64% |
| 50  | 29.96% | 0.019% | 68.66% | 1.36% | 12.26% |
| 100 | 18.53% | 0.009% | 80.70% | 0.77% | 9.20% |

| $R$ | C  | FP  | NC  | FN  | $n_{\text{max}}$ |
|-----|----|-----|-----|-----|-----------------|
| 1   | 100.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 5   | 93.82% | 0.046% | 5.91% | 0.23% | 5.21% |
| 10  | 78.89% | 0.073% | 20.24% | 0.80% | 13.18% |
| 50  | 36.40% | 0.014% | 62.57% | 1.02% | 11.90% |
| 100 | 22.62% | 0.006% | 76.56% | 0.81% | 9.13% |

The cases labeled as false positives in fact converged but with a slightly higher ratio between the true and estimated error. Had we chosen our adjustment factor to be 200 instead of 10, the false positive rate would be zero in all cases. Thus, at the branch point we conclude that when the algorithm terminates with success, it is essentially always reliable. The false negative rate is also low, though not nearly so small as the false positive rate. It is also somewhat ambiguous, since it tells us only that when the algorithm terminated with failure, it was in fact really within the prescribed tolerance; it does not identify scenarios where had the calculation continued further, convergence would have been achieved. A wide variation in this rate is therefore possible based on the choice of floating point error estimate; our implementation uses the choice that gave the smallest false negative rate of those estimates we considered. Comparing table 2 to table 3, we can see that requiring higher accuracy does decrease the percentage of convergent cases, with an effect most pronounced for values of $R$ in the middle of the range we considered; at high $R$, the percentage of convergent cases is small enough that most of the effect of the choice of $\varepsilon$ is masked.

We also investigated the accuracy for points chosen in the unit disk. Here for simplicity we considered only $m = 45$ and $\varepsilon = 2 \times 10^{-14}$, but we were able to examine $3 F_2$ and $4 F_3$ functions as well, though for smaller ranges in $R$. These results are shown in table 4. We see that as we might expect the overall convergence rates are higher than for the corresponding value of $R$ at the branch point, and though the data is somewhat limited there does not seem to be a strong dependence on the order $q$ of the hypergeometric; certainly not nearly as strong as the dependence on $R$. One subtlety not shown in this table is that it is no longer true that the (albeit rare) false positives are necessarily benign; roughly half of the false positives for the $R = 100$ cases of $2 F_1$, for instance, had fewer than half of the digits correct; in many cases no digits correct. This always happens when the exponent $\lambda$ has a large positive real part; as already mentioned in section 3.1.1, our choice of estimate for the truncation error reduces the fraction of these false positives. We can now quantify that assertion: had we used the truncation error estimate (3.7) instead.
Table 4. Accuracy of $q+1 F_q(z)$ calculations for $|z| < 1$ with $m = 45$ and $\varepsilon = 2 \times 10^{-14}$.

| $R$  | $C$  | $FP$ | $NC$ | $FN$ | $n_{\text{max}}$ |
|------|------|------|------|------|-----------------|
| $2F_1$ | 1    | 99.86% | 0.002% | 0.14% | 0.0% | 0.0% |
|      | 5    | 94.63% | 0.15% | 5.10% | 0.12% | 0.011% |
|      | 10   | 85.98% | 0.13% | 13.70% | 0.19% | 0.030% |
|      | 50   | 49.32% | 0.12% | 50.36% | 0.20% | 0.17% |
|      | 100  | 31.73% | 0.11% | 68.03% | 0.13% | 0.11% |
| $3F_2$ | 1    | 99.76% | 0.03% | 0.21% | 0.0% | 0.0% |
|      | 5    | 92.85% | 0.20% | 6.78% | 0.10% | 0.01% |
|      | 10   | 84.88% | 0.13% | 14.83% | 0.16% | 0.03% |
| $4F_3$ | 1    | 99.74% | 0.0% | 0.25% | 0.010% | 0.0% |
|      | 5    | 91.35% | 0.12% | 8.44% | 0.09% | 0.03% |

of (3.8), then even with an adjustment factor of one thousand instead of ten, our percentage of false positives for the $|z| < 1$, $R = 100$ case of $2F_1$ would be 11.22%, or one hundred times greater. The false positives that still persist even using (3.8) are those where the terms of the series temporarily become much smaller (by many orders of magnitude) before again increasing. This causes the series to appear to be rapidly converging when it is not, and our acceleration method is unable to distinguish this from true convergence. But neither are other methods; even commercial algebraic packages were found to falsely return convergence on these cases, unless specifically instructed to calculate results to very high precision. We know of no reliable way of deciding that this will happen; we must just fortuitously choose to calculate at sufficiently high precision.

In summary, except for those very few cases just mentioned, the method is highly effective at either accelerating the series or determining that a higher precision is needed; the error estimate it returns is almost always accurate to within a factor of ten. If the arguments to the function are themselves much larger than about ten, then it is increasingly unlikely that the method will converge in double precision, but it will correctly identify that failure, and a sufficiently high precision implementation should succeed.

3.2.3. Accuracy of the error estimate. The results of the previous section clearly indicate that our truncation error estimate is quite reliable, but it is useful to consider this in more depth. Thus, from each of the trials that converged or were false positives in the tests of the previous section, we can examine the ratio of the true relative error to that estimated by the algorithm from equation (3.8). As an example of this behavior, we show in figure 4 a relative cumulative frequency plot for each of the five values of $R$ we tested, for the $2F_1 (1)$ function with $m = 30$ and $\varepsilon = 2 \times 10^{-14}$. As this figure shows, the error estimate is excellent and only weakly depends on $R$; in 90% or more of cases the estimated error is less than the true error, and for essentially all cases it is within a factor of two.

3.2.4. Number of partial sums needed. Another important parameter of the method is the maximum number of iterations allowed before the method is deemed to have failed, regardless of the error estimate. Since up to 20% of cases may reach this limit, it is important not to have it unnecessarily high, as otherwise we waste time
Figure 4. The cumulative relative frequency of the ratio of the true error $\epsilon_{\text{true}}$ to the estimated error among those of the randomly chosen calculations that converged, when the order $m$ is 30 and the requested relative tolerance is $2 \times 10^{-14}$.

on an unsuccessful calculation. A sample of such behavior—again as a cumulative relative frequency plot—is shown in figure 5; results for other choices of $m$ or $\epsilon$ are similar. Unlike the error ratio, we see a much stronger dependence on the size of the parameters, as measured by $R$, but we can see that at least for $R$ up to 100 choosing the maximum $N$ to be about one thousand is quite conservative; for smaller parameter ranges this can be reduced even further.

3.2.5. Speed of the algorithm. Finally, apart from the accuracy of the algorithm its most essential characteristic will be its speed. To examine this, we generated one thousand sets of parameters for each of our usual set of five $R$ values, and then averaged the time of each parameter set from one hundred runs on a 1000 MHz AMD Athlon processor under Linux. We report separately the time for all runs versus just those runs that converged, and we set the relative tolerance at $2 \times 10^{-14}$ and the maximum number of allowed iterations at $2 \times 10^3$. For evaluations at the branch point, we obtain the results shown in table 5.

We see from table 5 that as we would expect, increasing the order of the algorithm does increase the execution speed, but more so for the smaller parameter choices, where the algorithm converges quite quickly and there is little to be gained by using more asymptotic coefficients. Within a given order $m$, we also see the time increase with $R$, by roughly a factor of two as we move from $R = 5$ to $R = 100$. The results for cases inside the unit disk are similar, except that the increase in running time as we increase $m$ is more significant; at least a factor of two for all values of $R$, not just small values.
Figure 5. The cumulative relative frequency of the number of terms $n$ that were needed among those of the randomly chosen calculations that converged, when the order $m$ is 30 and the requested relative tolerance is $2 \times 10^{-14}$.

Table 5. Speed of the algorithm at the branch point, when $N = 2 \times 10^3$ and $\varepsilon = 2 \times 10^{-14}$.

| Function $F_k(1)$ | $R$ | $m = 30$ | | $m = 45$ | |
|------------------|----|---------|---|---------|---|
|                  | All Only converged | All Only converged | |
| $2F_1(1)$        | 1  | 0.154 ms 0.0951 ms | 0.307 ms 0.250 ms | |
|                  | 5  | 0.675 ms 0.115 ms | 0.858 ms 0.292 ms | |
|                  | 10 | 0.911 ms 0.145 ms | 1.15 ms 0.339 ms | |
|                  | 50 | 0.982 ms 0.332 ms | 1.25 ms 0.505 ms | |
|                  | 100| 1.11 ms 0.539 ms | 1.38 ms 0.728 ms | |
| $3F_2(1)$        | 1  | 0.119 ms 0.109 ms | 0.338 ms 0.330 ms | |
|                  | 5  | 0.680 ms 0.124 ms | 0.843 ms 0.307 ms | |
|                  | 10 | 0.966 ms 0.157 ms | 1.19 ms 0.323 ms | |
|                  | 50 | 1.15 ms 0.330 ms | 1.39 ms 0.485 ms | |
|                  | 100| 1.21 ms 0.406 ms | 1.46 ms 0.546 ms | |
| $4F_3(1)$        | 1  | 0.118 ms 0.0981 ms | 0.322 ms 0.310 ms | |
|                  | 5  | 0.726 ms 0.118 ms | 0.855 ms 0.258 ms | |
|                  | 10 | 1.05 ms 0.148 ms | 1.29 ms 0.308 ms | |
|                  | 50 | 1.21 ms 0.286 ms | 1.46 ms 0.438 ms | |
|                  | 100| 1.33 ms 0.381 ms | 1.58 ms 0.546 ms | |
4. Comparison with other methods

While the tests of the previous section are a convincing validation of the method of this paper, it is also important to compare the method to others. Here we are somewhat hampered: most of the literature on computing the generalized hypergeometric function gives only a few examples, and not the kind of large, randomly selected test cases we used for testing. Also interest is often focused on obtaining the greatest accuracy with the fewest terms summed, and robustness considerations (such as automatic termination) are rarely mentioned. For these reasons, in this section we use a different implementation of our algorithm, in Python, using mpmath [21] to provide arbitrary precision arithmetic. By using an interpreted language and software implementations of higher than machine precision, we of course pay a large performance penalty (typically a factor of roughly a thousand). However most of the other methods we consider here are also implemented in such systems, so a comparison between the two is still meaningful.

Moreover, our interest here is centered on methods that can handle generic, complex parameters. There can certainly be particular choices of parameters (and argument) for which other methods are more efficient, but when all of the parameters and the argument are permitted to be complex, the number of possible "special cases" grows bewilderingly large. So we focus on two methods that are proposed as general-purpose algorithms for hypergeometric functions, and also on the E-method (mentioned in section 2.1) which bears superficial similarity to the method of this paper.

4.1. Zeta function acceleration. This method [26, 27] is designed to evaluate $q_{+1}F_q(1)$, and as such is perhaps still something of a special-purpose algorithm. But the branch point is the most challenging case, and the authors consider complex parameters as well as several optimizations of their method. That method is based on directly summing the first $N$ terms of the series, and then approximating the remainder in terms of $m$ Hurwitz zeta functions. The method of [27] extends that of [26] by allowing for a complex parameter $\alpha$ that is determined through a symbolic algebra problem requiring the solution of a nonlinear optimization through Gröbner bases. The complexity of this optimization problem grows with $m$, so the authors do not consider values of $m$ as large as those we can easily handle with our method.

In the optimized work [27], a few examples are considered, and here we compare two of them with our method. The timing results of [27] were for an AMD Athlon64 3500+ processor; our processor is comparable if a little faster (it is a 4600+ model).

The first example considered in [27] are $\sum 3F_2(1.6 + 7i, 2.4 - i, \sqrt{2}; 3 + i, \sqrt{6} + i; 1)$. The method of that paper is able to evaluate that function to ten digit accuracy with $N = 35$ in 0.3 seconds; our method required $N = 17$ and 0.37 seconds. But as the required precision is increased, the advantage of our method grows: 15 digit accuracy with the zeta-function method required $N = 100$ and 1.1 seconds, but for our method $N = 25$ and still 0.37 seconds; 35 digit accuracy required $N = 3500$ and 14 seconds for them, but $N = 110$ and still just 0.37 seconds for us.

In fact in every case considered in [27] our method out-performed that method, often substantially; to keep our discussion brief we consider just one more example. The most challenging example considered in that paper was $\sum 4F_3(2.4 + 30i, -0.3 + 0.5i, 2.2 - i, 0.5 + i; 1.8, 1.1 - i, 2 + 17i; 1)$. The authors could achieve 10 digit accuracy with roughly 1000 terms summed, whereas we achieve the same with only
136 terms. To achieve 20 digit accuracy they required approximately 6000 terms, whereas we needed only 290. It is true that we can use a larger value of $m$ than those authors (we used 30 in our tests; they used either 10 or 15) but that is again because it is easy for us to increase $m$, as there is no system of $m$ polynomials to solve in our method.

4.2. Euler-Maclaurin summation. Euler-Maclaurin summation is based on a specific analytic form of the remainder of a series, expressed as an integral and a weighted sum of derivatives of the terms \( \text{with respect to the term index} \) [61]:

\[
\rho_n = \int_{n+1}^{\infty} t(k) \, dk + \frac{1}{2} t(n + 1) - \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} t^{(2j-1)}(n + 1).
\]

Here the $B_{2j}$ are the Bernoulli numbers, and we have assumed that not only do the terms $t_k$ go to zero as $k \to \infty$, but so also do all of the derivatives of the terms with respect to the term index; the method is easily generalized when that does not hold.

It is this need to integrate and differentiate terms with respect to the term index that makes this method challenging. For the Riemann zeta function, the authors of [61] could carry this out analytically, but for generalized hypergeometric functions an analytic solution seems intractable. However a numerical implementation of this method underlies the \texttt{mpmath} [21] calculation of generalized hypergeometric functions near the branch point, so we make our comparison with that implementation.

The first test case considered in [21] is \( _4F_3 \left( \begin{array}{c} 1, 1, 3, 2 \end{array} ; 1, 5, 11 \right) \). For \texttt{mpmath} and 25 digit accuracy, its Euler-Maclaurin based summation method requires 2.4 seconds, while on the same machine our method requires only 0.5 seconds. But if we try to extend the test cases of the previous subsection, then the Euler-Maclaurin based approach is completely incapable of grappling: the first and simplest test case we considered runs for several minutes before returning a failure to converge. It fares even worse in the other test cases.

4.3. \textit{E}-method. Unlike the method of this paper, the previous two methods require the computed partial sum only so they can add their estimates of the remainder to that partial sum; the remainder itself they calculate without direct reference to the sequence of partial sums. Our method requires two successive partial sums, because both $s$ and $\mu$ in equation (3.1) are unknown; we are effectively solving a $2 \times 2$ linear equation. At the other extreme, we could use only our knowledge of the leading behavior of the remainder, and rather than precomputing the asymptotic coefficients $c_k$, we can determine the coefficients $\tilde{c}_k$ in:

\[
s_n = s + \sum_{k=1}^{m} \tilde{c}_k \frac{z^n n^\lambda}{n^{\mu} - 1}.
\]

Here the coefficients $\tilde{c}_k$ are related to our asymptotic coefficients $c_k$ through $\tilde{c}_k = \mu c_k - 1$.

This approach is the \textit{E}-method already mentioned in the introduction, for the particular choice of functions $g_k(n) = z^n / n^{k-\lambda-1}$. That method is described in the monograph [42] of Brezinski and Redivo Zaglia and was independently discovered by Schneider [64], Hävie [50] and Brezinski [48]; a stable numerical implementation is described by Brezinski in [49].
In fact, the E-method is perhaps more properly thought of as a class of methods; most existing methods can be subsumed by specializing to a particular choice of the \( g_k(n) \). Indeed Levin’s original work [47] on nonlinear sequence transformations can be analyzed as a specialization of the E-method to an asymptotic expansion in inverse powers, with differing simple remainder estimates that enable application to a variety of different sequences. The algorithm we analyze now is another such specialization, where the remainder estimate is given by our analytic knowledge of only the leading order of the asymptotic truncation error. Thus the comparisons of this section can also be considered a comparison to a variant of Levin’s methods. Not only is such a specification necessary to have a complete algorithm, but particular specializations will often allow simpler implementations than those described in the references above for the general-purpose E-method. In our present case, the special form of our \( g_k \) makes it simpler to use the recursive scheme in section 7.2 of Weniger’s review article [39], itself based on the work of Fessler et al. in [65].

We compare first the complexity of the two approaches. If we assume that \( m \gg q \), then the complexity of the algorithm of this paper is roughly \( \frac{1}{2}m^3 + 10m^2 + (2m + 7)N \), while for this implementation of the E-method it is \( 6m^2N \). Thus, as \( N \) grows beyond \( m \), the method of this paper has clearly better complexity; even when \( m \approx N \) it is somewhat superior. Note that the E-method cannot have \( N < m \), since we must always consider at minimum \( m + 1 \) partial sums.

The real advantage of our method, however, is in its stability. To illustrate that, we return to the first example we considered, from equation (3.4) and as shown in

**Figure 6.** The relative error \( \delta s_n^{(m)} \) for different orders \( m \) of the E-method applied to (3.4).
ACCELERATION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

figure 1. Figure 6 shows the corresponding plot for the $E$-method using calculations in 80 bit (long double) precision. Comparing to figure 1, we note several differences:

1. The overall error is larger.
2. The floating point error grows much more quickly with the order $m$ of the transformation.
3. Once the minimum error is reached, the error rapidly begins increasing again, so that automatic termination would be much more challenging to implement.

We can understand this overall loss of stability if we consider that the $E$-method is essentially solving the linear system:

$$
\begin{bmatrix}
1 & z^n/n^{-\lambda} & z^n/n^{1-\lambda} & \ldots & z^n/n^{m-1-\lambda} \\
1 & z^{n+1}/(n+1)^{-\lambda} & z^{n+1}/(n+1)^{1-\lambda} & \ldots & z^{n+1}/(n+1)^{m-1-\lambda} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & z^{n+m}/(n+m)^{-\lambda} & z^{n+m}/(n+m)^{1-\lambda} & \ldots & z^{n+m}/(n+m)^{m-1-\lambda}
\end{bmatrix}
\begin{bmatrix}
s \\
\tilde{c}_1 \\
\vdots \\
\tilde{c}_m
\end{bmatrix}
= \begin{bmatrix}
s_n \\
s_{n+1} \\
\vdots \\
s_{n+m}
\end{bmatrix}.
$$

Of course, this system is not explicitly solved at each step, since it is only $s$ that we need, but it is still the stability of the underlying system (4.3) that dictates the stability of the recursive scheme for $s$. We can quantify that through condition numbers. Examining figure 1, we see that we would expect to achieve 10 digit accuracy when $n = 10$ if our order is $m = 30$. Yet the condition number of the matrix of (4.3) for those choices is $1.3 \times 10^{28}$, far too large to allow a solution in long double precision. If we try to avoid this by decreasing $m$, then we must also increase $n$; again from figure 1 we estimate that if $m = 15$ we would need roughly $n = 40$; now the condition number is $3 \times 10^{89}$.

These large condition numbers are not coincidental. We saw that our method begins to lose precision whenever $|s| \ll |\mu|$, since then the unknown that we care about in our linear system becomes much smaller than the other unknown; this must spring from instability in the underlying $(2 \times 2)$ system we are solving. With the $E$-method, the same problem can arise if $|s| \ll \tilde{c}_k$ for any of the $c_k$ that we solve for. But that will happen generically: the coefficients are asymptotic and grow rapidly with $m$; for the example we consider here we have $|c_{30}| = 10^{18}$. Hence the corresponding linear systems must be unstable.

From this perspective, the chief advantage of the method of this paper is that it bypasses such an unstable linear system. Instead, as we saw in section 2.4, we can determine the $\omega_n$ to high accuracy, and that accuracy does not decrease rapidly with $m$, and actually increases with $n$. For this reason, our method is much more stable.

5. Conclusions

Summarizing, we have shown that it is possible to derive the asymptotics of the remainders of the partial sums of the generalized hypergeometric function $_{q+1}F_q$ to any desired order in inverse powers of $n$. We have given explicit formulas for
the remainders in terms of the hypergeometric parameters and argument. This analytic result is the basis for a new series acceleration technique that can dramatically accelerate the convergence of the generalized hypergeometric series, making it feasible to evaluate these for complex arguments, even at the branch point \( z = 1 \).

As implemented in C, the algorithm can be limited by the fixed precision of standard floating point types, but even in this case the precise asymptotic knowledge available enables us to determine correctly when the acceleration has converged. The method seems much more efficient and robust than any others we have found in the literature that are applicable to \( q_{+1}F_{q} \) at generic complex arguments and parameter.

There are still some open issues, which are natural starting points for future research:

- At present, the algorithm is very slow near the branch point, much more so than at the branch point. As shown by Bühring [30–33] and Norlund [34], there is a close connection between the asymptotics of the partial sums at the branch point and the behavior of the function near the branch point. It would be interesting to see if this can be leveraged to evaluate the function near the branch point using the (faster) evaluation at the branch point; of course this is not just a simple series expansion precisely because we are near a singularity.

- As we have noted throughout this paper, our C implementation is limited by fixed floating point precision. Of course it is straightforward to implement the algorithm in any of the free or commercial symbolic computational programs that support arbitrary precision, but it could also be useful to continue development of an arbitrary precision routine in a low-level language, by taking advantage of existing higher precision libraries like MPFR [66] or QD [67]. Moreover, the acceleration presented here is almost certainly not ideal for all inputs; for small \(|z|\), for instance, there is no need to use any acceleration at all. Even when the method of this paper is best, a more automatic implementation should choose the order \( m \) and maximum iterations \( N \) to minimize the computation required to achieve the desired accuracy. All of these considerations together could lead to a reliable and fast library for generalized hypergeometric evaluations.

- We have focused in this paper on the case \( p = q + 1 \) because that restriction was needed to apply the results of [58]. However the general case can be handled by including the further results of those same authors in [59]; work on this extension is already underway.

- We have also only described an algorithm in which we use the precise remainder estimates and a constant correction factor \( \mu \), in contrast to traditional series acceleration techniques that use simple remainder estimates and sophisticated correction factors. But the two choices are not exclusive, and it would be interesting to investigate the performance of a method that combines the remainder estimates of this paper with the correction factors \( \mu_n \) of traditional series acceleration techniques.

- We have limited ourselves to asymptotic expansions in inverse powers of \( n \), because those are the asymptotic functions in which the results of [58] are couched. However, Weniger has found [28, 68] that inverse factorial series can also be very useful—in some cases much more powerful—than
inverse powers, and it is worth investigating if a similar expansion would be effective here. In particular, it is shown in [28, 68] that an expansion in inverse powers can be transformed to an expansion in inverse factorial series, so the question is really how the stability and efficiency of such a scheme compares with the method presented here.

- Finally, it is worth investigating how well the method of this paper performs when applied to other functions for which an analytic asymptotic expansion of the term ratio is available. Though many series of practical interest in the sciences are available only numerically or as expensive computations (e.g., perturbation series), there are still other series of practical interest where we have available the necessary analytic knowledge, and would like to take advantage of that knowledge to efficiently and robustly evaluate the functions.

Acknowledgements

This research was supported in part by the Math/Science research fund of Abilene Christian University, and that support is gratefully acknowledged. I also thank the Max-Planck-Institut für Gravitationsphysik, Hannover, Germany, for their hospitality and support while this paper was written. I also thank Rafał Nowak for pointing out a sign error in an earlier draft of this paper, and the two anonymous referees for their several suggestions, including in particular the suggestion of several references, and other methods with which to compare this work.

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