Non-hyperbolic closed geodesics on positively curved Finsler spheres

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Abstract

In this paper, we prove that for every Finsler $n$-dimensional sphere $(S^n, F)$, $n \geq 3$ with reversibility $\lambda$ and flag curvature $K$ satisfying $(1 + \frac{1}{1+\lambda})^2 < K \leq 1$, there exist at least three distinct closed geodesics and at least two of them are elliptic if the number of prime closed geodesics is finite. When $n \geq 6$, these three distinct closed geodesics are non-hyperbolic.

Key words: Positively curved, closed geodesic, non-hyperbolic, Finsler metric, spheres.

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1 Introduction and main result

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold $(M, F)$, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifold, the inverse curve $c^{-1}$ of a closed geodesic $c$ on a irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. On a reversible Finsler (or Riemannian) manifold, two closed geodesics $c$ and $d$ are called geometrically distinct if $c(S^1) \neq d(S^1)$, i.e., their image sets in $M$ are distinct. We shall omit the word distinct when we talk about more than one prime closed geodesic.

For a closed geodesic $c$ on $n$-dimensional manifold $(M, F)$, denote by $P_c$ the linearized Poincaré map of $c$. Then $P_c \in \text{Sp}(2n - 2)$ is symplectic. For any $M \in \text{Sp}(2k)$, we define the elliptic height $e(M)$ of $M$ to be the total algebraic multiplicity of all eigenvalues of $M$ on the unit circle $U = \{z \in \mathbb{C} | |z| = 1\}$ in the complex plane $\mathbb{C}$. Since $M$ is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$.

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A closed geodesic $c$ is called **elliptic** if $e(P_c) = 2(n - 1)$, i.e., all the eigenvalues of $P_c$ locate on $U$; **hyperbolic** if $e(P_c) = 0$, i.e., all the eigenvalues of $P_c$ locate away from $U$; **non-degenerate** if 1 is not an eigenvalue of $P_c$. A Finsler manifold $(M, F)$ is called **bumpy** if all the closed geodesics on it are non-degenerate.

There is a famous conjecture in Riemannian geometry which claims the existence of infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved for many cases, but not yet for compact rank one symmetric spaces except for $S^2$. The results of Franks [16] in 1992 and Bangert [3] in 1993 imply that this conjecture is true for any Riemannian 2-sphere (cf. [14] and [15]). But once one moves to the Finsler case, the conjecture becomes false. It was quite surprising when Katok [17] in 1973 found some irreversible Finsler metrics on spheres with only finitely many closed geodesics and all closed geodesics are non-degenerate and elliptic (cf. [34]).

Recently, index iteration theory of closed geodesics (cf. [5] and [21]) has been applied to study the closed geodesic problem on Finsler manifolds. For example, Bangert and Long in [4] show that there exist at least two closed geodesics on every $(S^2, F)$. After that, a great number of multiplicity and stability results have appeared (cf. [8]-[11], [12], [22], [23], [28]-[29], [30]-[33] and therein).

In [27], Rademacher has introduced the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold defined by

$$
\lambda = \max \{ F(-X) \mid X \in TM, F(X) = 1 \} \geq 1.
$$

Then Rademacher in [28] has obtained some results about multiplicity and the length of closed geodesics and about their stability properties. For example, let $F$ be a Finsler metric on $S^n$ with reversibility $\lambda$ and flag curvature $K$ satisfying \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\), then there exist at least $n/2 - 1$ closed geodesics with length $< 2n\pi$. If $\frac{9\lambda^2}{4(1+\lambda)^2} < K \leq 1$ and $\lambda < 2$, then there exists a closed geodesic of elliptic-parabolic, i.e., its linearized Poincaré map split into 2-dimensional rotations and a part whose eigenvalues are $\pm 1$. Some similar results in the Riemannian case are obtained in [1] and [2].

Recently, Wang in [30] proved that for every Finsler $n$-dimensional sphere $S^n$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exists one elliptic closed geodesics whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\pi i \mu)$ with an irrational $\mu$. Wang in [33] proved that for every Finsler $n$-dimensional sphere $S^n$ for $n \geq 6$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exists $[\frac{n}{2}] - 2$ closed geodesics possessing irrational mean indices. Furthermore, assume that this metric $F$ is bumpy, in [31], Wang showed that there exist $2[\frac{n+1}{2}]$ closed geodesics on $(S^n, F)$. Also in [31], Wang showed that for every bumpy Finsler metric $F$ on $S^n$ satisfying $\frac{9\lambda^2}{4(1+\lambda)^2} < K \leq 1$, there exist two prime elliptic closed geodesics provided the number of closed geodesics on $(S^n, F)$ is finite.
Very recently, the author in [7] proved that for every Finsler $n$-dimensional sphere $(S^n, F)$ for $n \geq 2$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1 + \lambda}\right)^2 < K \leq 1$, either there exist infinitely many closed geodesics, or there exist at least two elliptic closed geodesics and each linearized Poincaré map has at least one eigenvalue of the form $e^{\sqrt{-1} \theta}$ with $\theta$ being an irrational multiple of $\pi$.

In this paper, we generalize some above results to the following theorem.

**Theorem 1.1.** For every Finsler metric $F$ on the $n$-dimensional sphere $S^n$, $n \geq 3$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1 + \lambda}\right)^2 < K \leq 1$, either there exist infinitely many closed geodesics, or there exist always three prime closed geodesics and at least two of them are elliptic. When $n \geq 6$, these three distinct closed geodesics are non-hyperbolic.

Also note that Wang in [30] obtained the existence of three prime closed geodesics on $(S^3, F)$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1 + \lambda}\right)^2 < K \leq 1$. In Section 3, we will reprove this case of $n = 3$ in a more simple argument by our method. In addition, when $n \geq 10$, Theorem 1.1 is included in Theorem 1.2 of [33].

Our proof of Theorem 1.1 in Section 3 contains mainly three ingredients: the common index jump theorem of [24], Morse theory and some new symmetric information about index jump. In addition, we also follow some ideas from our recent preprints [7] and [11].

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. For an $S^1$-space $X$, we denote by $\overline{X}$ the quotient space $X/S^1$. We define the functions

$$\begin{align*}
[a] &= \max\{k \in \mathbb{Z} \mid k \leq a\}, \quad E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}, \\
\varphi(a) &= E(a) - [a], \quad \{a\} = a - [a].
\end{align*}$$

Equation (1.1)

Especially, $\varphi(a) = 0$ if $a \in \mathbb{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$.

## 2 Morse theory and Morse indices of closed geodesics

### 2.1 Morse theory for closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold $(M, F)$, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 \, dt.$$  \hspace{1cm} (2.1)

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along
any closed geodesic $c$ on $M$, which we denote by $E'(c)$. As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. In the following, we denote by

$$\Lambda^+ = \{ d \in A \mid E(d) \leq \kappa \}, \quad \Lambda^- = \{ d \in A \mid E(d) < \kappa \}, \quad \forall \kappa \geq 0. \quad (2.2)$$

For a closed geodesic $c$ we set $\Lambda(c) = \{ \gamma \in A \mid E(\gamma) < E(c) \}$.

Recall that respectively the mean index $\hat{i}(c)$ and the $S^1$-critical modules of $c^m$ are defined by

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}, \quad \overline{\mathcal{C}}_s(E, c^m) = H_\ast \left( (\Lambda(c^m) \cup S^1 \cdot c^m) / S^1, \Lambda(c^m) / S^1 ; \mathbb{Q} \right). \quad (2.3)$$

We call a closed geodesic satisfying the isolation condition, if the following holds:

**Iso** For all $m \in \mathbb{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of $E$.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

If $c$ has multiplicity $m$, then the subgroup $\mathbb{Z}_m = \{ \frac{n}{m} \mid 0 \leq n < m \}$ of $S^1$ acts on $\overline{\mathcal{C}}_s(E, c)$. As studied in p.59 of [26], for all $m \in \mathbb{N}$, let $H_s(X, A)^{\pm \mathbb{Z}_m} = \{ [\xi] \in H_s(X, A) \mid T_s[\xi] = \pm[\xi] \}$, where $T$ is a generator of the $\mathbb{Z}_m$-action. On $S^1$-critical modules of $c^m$, the following lemma holds:

**Lemma 2.1.** (cf. Satz 6.11 of [26] and [4]) Suppose $c$ is a prime closed geodesic on a Finsler manifold $M$ satisfying (Iso). Then there exist $U_{c^m}$ and $N_{c^m}$, the so-called local negative disk and the local characteristic manifold at $c^m$ respectively, such that $\nu(c^m) = \dim N_{c^m}$ and

$$\overline{\mathcal{C}}_s(E, c^m) = H_\ast \left( (\Lambda(c^m) \cup S^1 \cdot c^m) / S^1, \Lambda(c^m) / S^1 \right)$$

$$= \left( H_{i(c^m)}(U_{c^m} \cup \{ c^m \}, U_{c^m}^\ast) \otimes H_{q-i(c^m)}(N_{c^m} \cup \{ c^m \}, N_{c^m}) \right)^{\pm \mathbb{Z}_m},$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{\mathcal{C}}_s(E, c^m) = \begin{cases} \mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \text{ and } q = i(c^m), \\ 0, & \text{otherwise,} \end{cases}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\overline{\mathcal{C}}_s(E, c^m) = H_{q-i(c^m)}(N_{c^m} \cup \{ c^m \}, N_{c^m})^{\epsilon(c^m)} \mathbb{Z}_m,$$

where $\epsilon(c^m) = (-1)^{i(c^m) - i(c)}$.

Define

$$k_j(c^m) = \dim H_j(N_{c^m} \cup \{ c^m \}, N_{c^m}), \quad k_j^\pm(c^m) = \dim H_j(N_{c^m} \cup \{ c^m \}, N_{c^m})^{\pm \mathbb{Z}_m}. \quad (2.4)$$

Then we have

**Lemma 2.2.** (cf. [26], [23], [30]) Let $c$ be a prime closed geodesic on a Finsler manifold $(M, F)$. Then

(i) For any $m \in \mathbb{N}$, $k_j(c^m) = 0$ for $j \notin [0, \nu(c^m)]$. 


(ii) For any \( m \in \mathbb{N} \), \( k_0(c^m) + k_{\nu(c^m)}(c^m) \leq 1 \) and if \( k_0(c^m) + k_{\nu(c^m)}(c^m) = 1 \) then \( k_j(c^m) = 0 \) for \( j \in (0, \nu(c^m)) \).

(iii) For any \( m \in \mathbb{N} \), there hold \( k_0^{+1}(c^m) = k_0(c^m) \) and \( k_0^{-1}(c^m) = 0 \). In particular, if \( c^m \) is non-degenerate, there hold \( k_0^{+1}(c^m) = k_0(c^m) = 1 \), and \( k_0^{-1}(c^m) = k_j^{+1}(c^m) = 0 \) for all \( j \neq 0 \).

(iv) Suppose for some integer \( m = np \geq 2 \) with \( n \) and \( p \in \mathbb{N} \) the nullities satisfy \( \nu(c^m) = \nu(c^n) \). Then there hold \( k_j(c^m) = k_j(c^n) \) and \( k_j^{+1}(c^m) = k_j^{+1}(c^n) \) for any integer \( j \).

Let \((M, F)\) be a compact simply connected Finsler manifold with finitely many closed geodesics. It is well known that for every prime closed geodesic \( c \) on \((M, F)\), there holds either \( \hat{i}(c) > 0 \) and then \( \hat{i}(c^m) \rightarrow +\infty \) as \( m \rightarrow +\infty \), or \( \hat{i}(c) = 0 \) and then \( \hat{i}(c^m) = 0 \) for all \( m \in \mathbb{N} \). Denote those prime closed geodesics on \((M, F)\) with positive mean indices by \( \{c_j\}_{1 \leq j \leq k} \). H.-B. Rademacher in [25] and [26] established a celebrated mean index identity relating all the \( c_j \)'s with the global homology of \( M \) (cf. Section 7, specially Satz 7.9 of [26]) for compact simply connected Finsler manifolds. Here we give a brief review on this identity.

**Theorem 2.3.** (Satz 7.9 of [26], cf. also [10], [23] and [30]) Assume that there exist finitely many closed geodesic on \((S^n, F)\) and denote prime closed geodesics with positive mean indices by \( \{c_j\}_{1 \leq j \leq k} \) for some \( k \in \mathbb{N} \). Then the following identity holds

\[
\sum_{j=1}^{k} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n, 1) = \begin{cases} \frac{n+1}{2(n-1)}, & n \text{ odd,} \\ -\frac{n}{2(n-1)}, & n \text{ even,} \end{cases} \tag{2.5}
\]

where

\[
\hat{\chi}(c_j) = \frac{1}{n(c_j)} \sum_{0 \leq l \leq 2(n-1)} \chi(c_j^m) = \frac{1}{n(c_j)} \sum_{0 \leq l \leq 2(n-1)} (-1)^{i(c_j^m)+l} k_l^{\epsilon(c_j^m)}(c_j^m) \in \mathbb{Q} \tag{2.6}
\]

and the analytical period \( n(c_j) \) of \( c_j \) is defined by (cf. [23])

\[
n(c_j) = \min \{ l \in \mathbb{N} \mid \nu(c_j^m) = \max_{m \geq 1} \nu(c_j^m), \; i(c_j^m) - i(c_j^{m+l}) \in 2\mathbb{Z}, \; \forall m \in \mathbb{N} \}. \tag{2.7}
\]

Set \( \overline{X} = X^0 S^n = \{ \text{constant point curves in } S^n \} \cong S^n \). Let \((X, Y)\) be a space pair such that the Betti numbers \( b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbb{Q}) \) are finite for all \( i \in \mathbb{Z} \). As usual the Poincaré series of \((X, Y)\) is defined by the formal power series \( P(X, Y) = \sum_{i=0}^{\infty} b_i t^i \). We need the following well known version of results on Betti numbers and the Morse inequality.

**Lemma 2.4.** (cf. Theorem 2.4 and Remark 2.5 of [25] and [13], cf. also Lemma 2.5 of [10]) Let \((S^n, F)\) be a \( n \)-dimensional Finsler sphere.

(i) When \( n \) is odd, the Betti numbers are given by

\[
b_j = \text{rank} H_j(AS^n/S^1, A^0 S^n/S^1; \mathbb{Q}) = \begin{cases} 2, & j \in \mathcal{K} \equiv \{ k(n-1) \mid 2 \leq k \in \mathbb{N} \}, \\ 1, & j \in \{ n-1 + 2k \mid k \in \mathbb{N}_0 \} \setminus \mathcal{K}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}
\]
When \( n \) is even, the Betti numbers are given by

\[
\begin{aligned}
b_j &= \text{rank} H_j(\Lambda S^n/S^1, \Lambda^0 S^n/S^1; \mathbb{Q}) \\
&= \begin{cases} \\
2, & \text{if } j \in \mathcal{K} \equiv \{k(n-1) | 3 \leq k \in 2\mathbb{N} + 1\}, \\
1, & \text{if } j \in \{n - 1 + 2k | k \in \mathbb{N}_0\} \setminus \mathcal{K}, \\
0, & \text{otherwise}.
\end{cases}
\end{aligned}
\]  

(2.9)

**Theorem 2.5.** (cf. Theorem I.4.3 of [6]) Let \((M, F)\) be a Finsler manifold with finitely many closed geodesics, denoted by \(\{c_j\}_{1 \leq j \leq k}\). Set

\[
M_q = \sum_{1 \leq j \leq k, \ m \geq 1} \dim C_q(E, c_j^m), \quad q \in \mathbb{Z}.
\]

Then for every integer \( q \geq 0 \) there holds

\[
M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq b_q - b_{q-1} + \cdots + (-1)^q b_0,
\]

(2.10)

\[
M_q \geq b_q.
\]

(2.11)

## 2.2 Index iteration theory of closed geodesics

In [19] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [20] of 2000. Note that this index iteration formulae works for Morse indices of iterated closed geodesics (cf. [18] and Chap. 12 of [21]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [18], the initial Morse index of a closed geodesic on a Finsler \( S^n \) coincides with the index of a corresponding symplectic path.

As in [20], denote by

\[
N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \text{for } \lambda = \pm 1, \ b \in \mathbb{R},
\]

(2.12)

\[
D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0, \pm 1\},
\]

(2.13)

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi),
\]

(2.14)

\[
N_2(e^{\theta\sqrt{-1}}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and }
\]

\[
B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \text{ and } b_2 \neq b_3.
\]

(2.15)

Here \( N_2(e^{\theta\sqrt{-1}}, B) \) is non-trivial if \((b_2 - b_3) \sin \theta < 0\), and trivial if \((b_2 - b_3) \sin \theta > 0\).
As in [20], the o-sum (direct sum) of any two real matrices is defined by

\[
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}_{2i \times 2i} \circ
\begin{pmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{pmatrix}_{2j \times 2j} =
\begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.
\]

For every \( M \in \text{Sp}(2n) \), the homotopy set \( \Omega(M) \) of \( M \) in \( \text{Sp}(2n) \) is defined by

\[
\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U = \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M), \forall \omega \in \Gamma \},
\]

where \( \sigma(M) \) denotes the spectrum of \( M \), \( \nu_\omega(M) \equiv \text{dim}_C \ker(M - \omega I) \) for \( \omega \in U \). The component \( \Omega^0(M) \) of \( P \) in \( \text{Sp}(2n) \) is defined by the path connected component of \( \Omega(M) \) containing \( M \).

**Theorem 2.6.** (cf. Theorem 7.8 of [19], Theorems 1.2 and 1.3 of [20], cf. also Theorem 1.8.10, Lemma 2.3.5 and Theorem 8.3.1 of [21]) For every \( P \in \text{Sp}(2n-2) \), there exists a continuous path \( f(0) = P \) and

\[
f(1) = N_1(1, 1)^{p_0} \circ I_{2p_0} \circ N_1(1, -1)^{q_0} \circ (-I_{2q_0}) \circ N_1(-1, 1)^{\bar{q}_0} \circ N_2(\epsilon^{-1} \sqrt{-1}, A_1) \circ \cdots \circ N_2(\epsilon^{\alpha_j} \sqrt{-1}, A_j) \circ N_2(\epsilon^{\beta_j} \sqrt{-1}, B_1) \circ \cdots \circ N_2(\epsilon^{\beta_0} \sqrt{-1}, B_0) \circ R(\theta_1) \circ \cdots \circ R(\theta_r) \circ R(\theta_{r+1}) \circ \cdots \circ R(\theta_r) \circ H(2) \circ h,
\]

where \( \frac{\theta_j}{2\pi} \in Q \cap (0, 1) \) for \( 1 \leq j \leq r \) and \( \frac{\theta_j}{2\pi} \notin Q \cap (0, 1) \) for \( r' + 1 \leq j \leq r \); \( N_2(\epsilon^{\alpha_j} \sqrt{-1}, A_j) \)'s are non-trivial and \( N_2(\epsilon^{\beta_j} \sqrt{-1}, B_j) \)'s are trivial, and non-negative integers \( p_-, p_0, p_+, q_-, q_0, q_+, r, r_0, h \) satisfy the equality

\[
p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_0 + 2r_0 + h = n - 1.
\]

Let \( \gamma \in \mathcal{P}_r(2n-2) = \{ \gamma \in C([0, \tau], \text{Sp}(2n-2)) \mid \gamma(0) = I \} \). Denote the basic normal form decomposition of \( P \equiv \gamma(\tau) \) by (2.16). Then we have

\[
i(\gamma^m) = m(i(\gamma) + p_- + p_0 - r) + 2 \sum_{j=1}^r E \left( \frac{m\theta_j}{2\pi} \right) - r
\]

\[
- p_- - p_0 - \frac{1}{2}(1 - 1)^m(q_0 + q_+) + 2 \sum_{j=1}^r \varphi \left( \frac{m\alpha_j}{2\pi} \right) - 2r_0,
\]

\[
\nu(\gamma^m) = \nu(\gamma) + \frac{1}{2}(1 - 1)^m(q_- + 2q_0 + q_+) + 2 \varsigma(m, \gamma(\tau)),
\]

where we denote by

\[
\varsigma(m, \gamma(\tau)) = r - \sum_{j=1}^r \phi \left( \frac{m\theta_j}{2\pi} \right) + r_0 - \sum_{j=1}^{r_0} \phi \left( \frac{m\beta_j}{2\pi} \right).
\]

The following is the common index jump theorem of Long and Zhu in [24].
Theorem 2.7. (cf. Theorems 4.1-4.3 of [24] and [21]) Let $\gamma_k, k = 1, \ldots, q$ be a finite collection of symplectic paths and $M_k = \gamma_k(\tau_k) \in Sp(2n - 2)$. Suppose $\hat{i}(\gamma_k, 1) > 0$, for all $k = 1, \ldots, q$. Then there exist infinitely many $(N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}$ such that

$$
\nu(\gamma_k, 2m_k - 1) = \nu(\gamma_k, 1),
$$

(2.21)

$$
\nu(\gamma_k, 2m_k + 1) = \nu(\gamma_k, 1),
$$

(2.22)

$$
i(\gamma_k, 2m_k - 1) + \nu(\gamma_k, 2m_k - 1) = 2N - \left(\hat{i}(\gamma_k, 1) + 2S^+_{M_k}(1) - \nu(\gamma_k, 1)\right),
$$

(2.23)

$$
i(\gamma_k, 2m_k + 1) = 2N + \hat{i}(\gamma_k, 1),
$$

(2.24)

$$
i(\gamma_k, 2m_k) \geq 2N - \frac{e(M_k)}{2},
$$

(2.25)

$$
i(\gamma_k, 2m_k) + \nu(\gamma_k, 2m_k) \leq 2N + \frac{e(M_k)}{2},
$$

(2.26)

for every $k = 1, \ldots, q$, where $S^+_{M_k}(1)$ is the splitting number of $M_k$.

More precisely, by (4.10) and (4.40) in [24], we have

$$
m_k = \left(\left\lceil \frac{N}{M\hat{i}(\gamma_k, 1)} \right\rceil + \chi_k\right) M, \quad 1 \leq k \leq q,
$$

(2.27)

where $\chi_k = 0$ or $1$ for $1 \leq k \leq q$ and $\frac{m_0}{\chi} \in \mathbb{Z}$ whenever $e^{\sqrt{-1} \theta} \in \sigma(M_k)$ and $\frac{m_0}{\chi} \in \mathbb{Q}$ for some $1 \leq k \leq q$. Furthermore, given $M_0 \in \mathbb{N}$, by the proof of Theorem 4.1 of [24], we may further require $M_0 | N$ (since the closure of the set $\{Nv : N \in \mathbb{N}, M_0 | N\}$ is still a closed additive subgroup of $\mathbb{T}^h$ for some $h \in \mathbb{N}$, where we use notations as (4.21) in [24]. Then we can use the proof of Step 2 in Theorem 4.1 of [24] to get $N$).

3 Proof of Theorem 1.1

Firstly we make the following assumption

(FCG) Suppose that there exist only finitely many closed geodesics $c_k (k = 1, \ldots, q)$ on $(S^n, F)$ with reversibility $\lambda$ and flag curvature $K$ satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$.

If the flag curvature $K$ of $(S^n, F)$ satisfies $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, then every non-constant closed geodesic $c$ must satisfy

$$
i(c) \geq n - 1,
$$

(3.1)

$$\hat{i}(c) > n - 1,
$$

(3.2)

where (3.1) follows from Theorem 3 and Lemma 3 of [27], (3.2) follows from Lemma 2 of [28]. Thus it follows from Theorem 2.2 of [24] (or, Theorem 10.2.3 of [21]) that

$$
i(c^{m+1}) - i(c^m) - \nu(c^m) \geq i(c) - \frac{e(P_c)}{2} \geq 0, \quad \forall m \in \mathbb{N}.
$$

(3.3)

Here the last inequality holds by (3.1) and the fact that $e(P_c) \leq 2(n - 1)$. 

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It follows from (3.2) and Theorem 2.7 that there exist infinitely many \((q+1)\)-tuples \((N, m_1, \cdots, m_q)\in \mathbb{N}^{q+1}\) such that for any \(1 \leq k \leq q\), there holds
\[
i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) = 2N - \left(i(c_k) + 2S_{M_k}^+(1) - \nu(c_k)\right),
\]
\[
i(c_k^{2m_k}) \geq 2N - \frac{e(P_{c_k})}{2},
\]
\[
i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \leq 2N + \frac{e(P_{c_k})}{2},
\]
\[
i(c_k^{2m_k+1}) = 2N + i(c_k).
\]

Note that by List 9.1.12 of [21] and the fact \(\nu(c_k) = p_{k_-} + 2p_{k_0} + p_{k_+}\), we obtain
\[
2S_{M_k}^+(1) - \nu(c_k) = 2(p_{k_-} + p_{k_0}) - (p_{k_-} + 2p_{k_0} + p_{k_+}) = p_{k_-} - p_{k_+}.
\]

So by (3.1)-(3.8) and the fact \(e(P_{c_k}) \leq 2(n - 1)\) it yields
\[
i(c_k^m) + \nu(c_k^m) \leq 2N - i(c_k) - p_{k_-} + p_{k_+}, \quad \forall 1 \leq m < 2m_k,
\]
\[
i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \leq 2N + \frac{e(P_{c_k})}{2} \leq 2N + n - 1,
\]
\[
2N + n - 1 \leq i(c_k^m), \quad \forall m > 2m_k.
\]

In addition, the precise formulae of \(i(c_k^{2m_k})\) and \(i(c_k^{2m_k}) + \nu(c_k^{2m_k})\) can be computed as follows (cf. (3.16) and (3.21) of [7] for the details)
\[
i(c_k^{2m_k}) = 2N - S_{M_k}^+(1) - C(M_k) + 2\Delta_k,
\]
\[
i(c_k^{2m_k}) + \nu(c_k^{2m_k}) = 2N + p_{k_0} + p_{k_+} + q_{k_-} + q_{k_0} + 2r_{k_0}' - 2(r_{k_+} - r_{k_-}') + 2r_k' - r_k + 2\Delta_k, \quad k = 1, \cdots, q,
\]
where \(r_{k_0}', r_{k_+}', r_{k_-}'\) and \(r_{k_0}'\) denote the number of normal forms \(R(\theta), N_2(e^{\alpha\sqrt{-1}\theta}, A)\) and \(N_2(e^{\beta\sqrt{-1}\theta}, B)\) with \(\theta, \alpha, \beta\) being the rational multiple of \(\pi\) in (2.16) of Theorem 2.6 respectively, and
\[
\Delta_k \equiv \sum_{0 < \{m_k \theta/\pi\} < \delta} S_{M_k}^{-}\left(e^{\sqrt{-1}\theta}\right) \leq r_k - r_k' + r_{k_+} - r_{k_-}', \quad C(M_k) \equiv \sum_{\theta \in (0, 2\pi)} S_{M_k}^{-}\left(e^{\sqrt{-1}\theta}\right),
\]
where \(\delta > 0\) is a small enough number (cf. (4.43) of [24]) and the estimate of \(\Delta_k\) follows from the inequality (3.18) of [7].

Under the assumption \((FCG)\), Theorem 1.1 of [7] shows that there exist at least two elliptic closed geodesics \(c_1\) and \(c_2\) on \((S^n, F)\) whose flag curvature satisfies \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\). Next Lemma lists some properties of these two closed geodesics which will be useful in the proof of Theorem 1.1.

**Lemma 3.1.** (cf. Section 3 of [7]) Under the assumption \((FCG)\), there exist at least two elliptic closed geodesics \(c_1\) and \(c_2\) on \((S^n, F), n \geq 2\) whose flag curvature satisfies \(\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1\).
1. Moreover, there exist infinitely many pairs of $(q + 1)$-tuples $(N, m_1, m_2, \ldots, m_q) \in \mathbb{N}^{q+1}$ and $(N', m_1', m_2', \ldots, m_q') \in \mathbb{N}^{q+1}$ such that

\[
i(c_1^{2m_1}) + \nu(c_1^{2m_1}) = 2N + n - 1, \quad \overline{C}_{2N+n-1}(E, c_1^{2m_1}) = Q, \quad (3.15)\]
\[
i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = 2N' + n - 1, \quad \overline{C}_{2N'+n-1}(E, c_2^{2m_2}) = Q, \quad (3.16)\]
\[
p_k = q_{k+} = r_{k-} = r_{k0} - r_{k0}' = h_k = 0, \quad k = 1, 2, \quad (3.17)\]
\[
r_1 - r_1' = \Delta_1 \geq 1, \quad r_2 - r_2' = \Delta_2' \geq 1, \quad (3.18)\]
\[
\Delta_k + \Delta_k' = r_k - r_k', \quad k = 1, 2, \quad (3.19)\]

where we can require $(n-1)|N$ or $(n+1)|N'$ as remarked in Theorem 2.7 and

\[
\Delta_k' = \sum_{0 \leq \{m'_k\theta / \pi\} < \delta} S_{M_k}(e^{\overline{\Gamma}^\theta}), \quad k = 1, 2. \quad (3.20)\]

**Proof.** In fact, all these properties have already been obtained in Section 3 of [7] and here we only list references. More precisely, (3.15) follows from Claim 1 and arguments between (3.25) and (3.26) in [7]. (3.16) follows from Claim 3 and similar arguments between (3.25) and (3.26) as those of $c_1$ in [7]. (3.17) and (3.18) follow from (3.25), Claim 2 and Claim 3 in [7]. (3.19) follows from (3.31) of [7] and (3.17). In a word, the properties of $c_1$ and $c_2$ is symmetric. 

**Lemma 3.2.** Under the assumption (FCG), for these two elliptic closed geodesics $c_1$ and $c_2$ found in Lemma 3.1, the following further properties hold:

\[
i(c_k^m) + \nu(c_k^m) \leq 2N - i(c_k) + p_{k+} \leq 2N - 1, \quad \forall 1 \leq m < 2m_k, k = 1, 2, \quad (3.21)\]
\[
i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \leq 2N + n - 3, \quad (3.22)\]
\[
i(c_1^{2m_1'}) + \nu(c_1^{2m_1'}) \leq 2N' + n - 3, \quad (3.23)\]

where the equalities in (3.22) and (3.23) hold respectively if and only if

\[
p_{k0} + p_{k+} + q_{k-} + q_{k0} + 2r_{k0}' + r_{k}' = n - 2, \quad r_k - r_k' = 1, \quad k = 1, 2. \quad (3.24)\]

**Proof.** Here we only give the proof of $c_2$. And the proof of $c_1$ is the same by using some information of $N'$ and $2m_1'$ instead of those of $N$ and $2m_2$ in the following arguments.

In fact, by (3.18) and (3.19), there holds $\Delta_2 = 0$. Then, together with the fact $r_{2*} = 0$ from (3.17), it follows from (3.13) that

\[
i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = 2N + (p_{20} + p_{2+} + q_{2-} + q_{20} + 2r_{20}' + r_{2}') - (r_2 - r_2'). \quad (3.25)\]

Note that by (2.17) we have

\[
p_{20} + p_{2+} + q_{2-} + q_{20} + 2r_{20}' + r_{2}' + (r_2 - r_2') = p_{20} + p_{2+} + q_{2-} + q_{20} + 2r_{20}' + r_2 \leq n - 1. \quad (3.26)\]
Therefore by (3.18) we get
\[ p_{20} + p_{2+} + q_{2-} + q_{20} + 2r_{20} + r'_{2} \leq n - 1 - (r_2 - r'_2) \leq n - 2, \]  
(3.27)
which, together with (3.25), yields
\[ i(c_2^{2m_2}) + \nu(c_2^{2m_2}) = 2N + (p_{20} + p_{2+} + q_{2-} + q_{20} + 2r'_{20} + r'_2) - (r_2 - r'_2) \leq 2N + n - 3, \]  
(3.28)
where the equality holds if and only if
\[ p_{20} + p_{2+} + q_{2-} + q_{20} + 2r'_{20} + r'_2 = n - 2, \quad r_2 - r'_2 = 1. \]  
(3.29)
So (3.22) and (3.23) follow from (3.28) and (3.29). Then (3.21) follows from (3.1) and (3.27).

This completes the proof of Lemma 3.2.

**Lemma 3.3.** Under the assumption \((FCG)\), for these two elliptic closed geodesics \(c_1\) and \(c_2\) found in Lemma 3.1, there holds
\[ k^{\nu(c_k^{n(c_k)})} \left( c_k^{n(c_k)} \right) = 1, \quad k^{\nu(c_k^{n(c_k)})} \left( c_k^{n(c_k)} \right) = 0, \quad \forall \ 0 \leq j < \nu(c_k^{n(c_k)}), \ k = 1, 2. \]  
(3.30)

**Proof.** We only give the proof for \(c_1\). The proof for \(c_2\) is the same.

Firstly, by (3.15) and Lemma 2.1, we have
\[ 1 = \dim \overline{C}_{2N+n-1}(E, c_1^{2m_1}) = \dim H_{2N+n-1-i(c_1^{2m_1})} \left( N_{c_1^{2m_1}} \cup \{ c_1^{2m_1} \}, N_{c_1^{2m_1}} \right)^{\epsilon(c_1^{2m_1})} \mathbb{Z}_{2m_1} \]
\[ = \dim H_{\nu(c_1^{2m_1})} \left( N_{c_1^{2m_1}} \cup \{ c_1^{2m_1} \}, N_{c_1^{2m_1}} \right)^{\epsilon(c_1^{2m_1})} \mathbb{Z}_{2m_1} \]
\[ = k^{\epsilon(c_1^{2m_1})} \left( c_1^{2m_1} \right), \]  
(3.31)
which implies that \(k^{\epsilon(c_1^{2m_1})} \left( c_1^{2m_1} \right) = 0\), for any \(0 \leq j < \nu(c_1^{2m_1})\) by (ii) of Lemma 2.3. In addition, note that \(n(c_1)|2m_1\) and \(\nu(c_1^{2m_1}) = \nu(c_1^{n(c_1)})\) by (2.7) and (2.27), there holds \(k^{\epsilon(c_1^{2m_1})} \left( c_1^{2m_1} \right) = k^{\epsilon(c_1^{n(c_1)})} \left( c_1^{n(c_1)} \right)\) for any \(0 \leq j \leq \nu(c_1^{2m_1})\) by (iv) of Lemma 2.2. Thus (3.30) holds.

**Proof of Theorem 1.1:**
In order to prove Theorem 1.1, we make the following assumption

\((TCG)\) Suppose that there exist exactly two elliptic closed geodesics \(c_1\) and \(c_2\) on \((S^n, F)\) with reversibility \(\lambda\) and flag curvature \(K\) satisfying \(\left( \frac{\lambda}{\lambda + K} \right)^2 < K \leq 1.\)

**Claim 1:** Under the assumption \((TCG)\), \(M_{2N+n-3}\) only can be contributed by \(c_2^{2m_2}\), i.e.,
\[ M_{2N+n-3} = \sum_{1 \leq j \leq 2, \ m \geq 1} \dim \overline{C}_{2N+n-3}(E, c_j^n) = \dim \overline{C}_{2N+n-3}(E, c_2^{2m_2}), \]  
(3.32)
and

\[ i(c_2^{m_2}) + \nu(c_2^{m_2}) = 2N + n - 3, \quad (3.33) \]
\[ i(c_1^{m_1}) + \nu(c_1^{m_1}) = 2N' + n - 3, \quad (3.34) \]
\[ p_k + p_k' + q_k + q_k' = n - 2, \quad r_k - r_k' = 1, \quad k = 1, 2. \quad (3.35) \]

In fact, note that \( i(c_k^m) + \nu(c_k^m) \leq 2N - 1 \) by (3.21), this implies that all iterates \( c_k^m \) with \( 1 \leq m \leq 2m_k - 1 \), \( k = 1, 2 \) have no contribution to \( M_q \), \( q \geq 2N \) by (i) of Lemma 2.2. On the other hand, by (3.11) we know that all iterates \( c_k^m \) with \( m \geq 2m_k + 1 \), \( k = 1, 2 \) have no contribution to \( M_q \), \( q \leq 2N + n - 2 \) by (i) of Lemma 2.2. By (3.30), the iterate \( c_1^{2m_1} \) only contributes 1 to \( M_{2N+n-1} \) and has no contribution to \( M_q \) with \( q \neq 2N + n - 1 \).

If \( i(c_2^{m_2}) + \nu(c_2^{m_2}) < 2N + n - 3 \) by (3.22), then \( c_2^{m_2} \) has no contradiction to \( M_{2N+n-3} \) by Lemma 2.2. So there holds \( M_{2N+n-3} = 0 \). However, it yields \( b_{2N+n-3} \geq 1 \) by Lemma 2.4. This gives a contradiction 0 = \( M_{2N+n-3} \geq b_{2N+n-3} \geq 1 \) by Theorem 2.5. Thus \( i(c_2^{m_2}) + \nu(c_2^{m_2}) = 2N + n - 3 \) and (3.32) holds.

Similarly, (3.34) can be obtained by using \( N' \) and \( 2m'_1 \) instead of \( N \) and \( 2m_2 \). Then, together with (3.24), this completes the proof of Claim 1.

**Claim 2:** Under the assumption (TCG), there exists at least one closed geodesic, without loss of generality, saying \( c_1 \), satisfying \( p_{1+} = n - 2 \) and \( i(c_1) = n - 1 \). And then there must be \( i(c_2) = n - 1 \) and the analytic period defined by (2.7) satisfies \( n(c_2) \neq 1 \).

In fact, if \( i(c_k) > n - 1 \) or \( p_{k+} < n - 2 \) by (3.1) and (3.27), then by (3.17), (3.9) becomes

\[ i(c_k^m) + \nu(c_k^m) \leq 2N - i(c_k) + p_{k+} \leq 2N - 2, \quad 1 \leq m < 2m_k, \quad k = 1, 2. \quad (3.36) \]

Thus it yields \( M_{2N-1} = 0 \) and \( M_{2N+n-3} = 1 \) by (3.11) and Claim 1.

When \( n = 3 \), we obtain a contradiction 1 = \( M_{2N} \geq b_{2N} = 2 \) by Lemma 2.4 and Theorem 2.5.

When \( n \geq 4 \), we obtain a contradiction 0 = \( M_{2N-1} + M_{2N} \geq b_{2N-1} + b_{2N} \geq 1 \) by Lemma 2.4 and Theorem 2.5.

So, without loss of generality, there holds \( p_{1+} = n - 2 \) and \( i(c_1) = n - 1 \). Then the analytic period \( n(c_1) = 1 \) by (2.7), which, together with Lemma 3.3 and \( i(c_1^m) \geq i(c_1) \geq n - 1 \), yields

\[ \overline{C}_{n-1}(E, c_1^m) = k_{n-1}^{i(c_1^m)}(c_1^m) = 0, \quad m \geq 1, \]

which shows that \( c_1^m \) has no contribution to non-zero \( M_{n-1}(\geq b_{n-1} = 1) \). Thus there must be \( i(c_2) = n - 1 \) and \( n(c_2) > 1 \). In fact, if \( n(c_2) = 1 \) or \( i(c_2) > n - 1 \), then it can be shown that \( c_2^m \) has no contribution to \( M_{n-1} \) by Lemma 3.3 or Lemma 2.2. This contradiction completes this proof.

Next we will carry out the proof according to the value of \( n \) for \( S^m_0 \) in four cases.

**Case 1:** \( n = 3 \).
The existence of at least three closed geodesics in this case has been proved in [30]. Here we give a new and more simple proof.

In this case, there holds and \( b_{2N} = 2 \) by Lemma 2.4. In addition, it follows from (3.11), (3.21) and Claim 1 that among all iterates \( c_k^m, m \geq 1 \) of \( c_k \), \( k = 1, 2 \), only \( c_2^{2m_1} \) contributes 1 to \( M_{2N} \). So, under the assumption (TCG), there holds \( M_{2N} = 1 \), which contradicts to the Morse inequality \( 1 = M_{2N} \geq b_{2N} = 2 \) and completes the proof in the case of \( n = 3 \).

**Case 2:** \( n = 4 \).

According to (3.35) and Claim 2, there exists a continuous path \( f \in \Omega^0(P_{c_1}) \) such that \( f(0) = P_{c_1} \) and \( f(1) = N_1(1, -1)^{g_2} \circ R(\theta_1), \frac{\theta_1}{\pi} \in \mathbb{Q} \). Then by Theorem 2.6 we obtain

\[
i(c_1) = 3, \quad i(c_1^m) = 2m + 2E\left(\frac{m\theta_1}{2\pi}\right) - 1, \quad \nu(c^m) = 2, \quad \forall m \geq 1. \tag{3.37}
\]

So, for the closed geodesic \( c_1 \), by Lemma 3.3 and (3.37) we have

\[
\hat{\chi}(c_1) = \sum_{i=0}^{2} (-1)^{i(c_1)+i} k_i^+(c_1) = -k_0(c_1) + k_1^+(c_1) - k_2^+(c_1) = -1. \tag{3.38}
\]

Note that \( 2 \times 2 \) identity matrix \( I_2 \) and \(-I_2 \) can be viewed as a rotation matrix \( R(\theta) \) with \( \theta = 2\pi \) and \( \theta = \pi \), respectively. Since \( p_2 + p_2 + q_2 + q_2 + 2r_2 + r'_2 + 2 \) by (3.35), we only consider \( p_2 + q_2 + 2r_2 + r'_2 = 2 \) and \( r_2 - r'_2 = 1 \). By Claim 2, \( n(c_2) = 1 \) implies \( p_2 \neq 2 \). Therefore

\[
p_2 + q_2 + 2r_2 + r'_2 = 2, \quad p_2 \neq 2, \tag{3.39}
\]

which yields

\[
4 \geq \nu(c_2^{2m_1}) = \nu(c_2^{n(c_2)}) = p_2 + q_2 + 2r_2 + 2r'_2 \geq 2. \tag{3.40}
\]

If \( \nu(c_2^{n(c_2)}) = 4 \), there must be \( r'_2 = 2 \) by (3.40). In this case, there holds \( i(c_2^m) \in 2\mathbb{N} - 1, \forall m \geq 1 \) by (4.7) and (4.8) of [4] and the symplectic additivity (cf. Theorem 9.1.10 of [21]), and

\[
\nu(c_2^m) \leq 2, \quad \forall 1 \leq m < n(c_2) \text{ since } n(c_2) > 1 \text{ by Claim 2.}
\]

If \( \nu(c_2^{n(c_2)}) = 3 \), there holds either \( r'_2 = 1 \) and \( p_2 = 1 \), or \( r'_2 = 1 \) and \( q_2 = 1 \) by (3.40). In either case, by (4.3), (4.6)-(4.8) of [4] and the symplectic additivity, for any \( 1 \leq m < n(c_2) \), either there holds \( \nu(c_2^m) \leq 1 \), or there holds \( \nu(c_2^m) = 2 \) and \( i(c_2^m) \in 2\mathbb{N} - 1 \) which only happens in the case of \( r'_2 = 1 \) and \( q_2 = 1 \).

If \( \nu(c_2^{n(c_2)}) = 2 \), there must be \( r'_2 = 1 \) or \( q_2 = 2 \) by (3.40). In this case, \( n(c_2) > 1 \) and

\[
\nu(c_2^m) = 0, \quad \forall 1 \leq m < n(c_2).
\]

In summary, only one of the following two cases can happen:

(i) \( \nu(c_2^m) \leq 1 \) for some \( 1 \leq m < n(c_2) \),

(ii) \( \nu(c_2^m) = 2, i(c_2^m) + \nu(c_2^m) \in 2\mathbb{N} - 1 \) for some \( 1 \leq m < n(c_2) \).
Firstly, by Lemma 3.3 and (3.33) we have

$$\chi(c_2^{n(c_2)}) = \sum_{l=0}^{\nu(c_2^{n(c_2)})} (-1)^l c_2^{n(c_2)} l k_l(c_2^{n(c_2)}) (c_2^{n(c_2)}) = -k_l(c_2^{n(c_2)}) (c_2^{n(c_2)}) = -1, \quad (3.41)$$

If (i) happens, then for some $1 \leq m < n(c_2)$ satisfying $\nu(c_2^m) \leq 1$, by (ii) of Lemma 2.2, it yields

$$\chi(c_2^m) = \sum_{l=0}^{\nu(c_2^m)} (-1)^l c_2^{n(c_2)} l k_l(c_2^m) (c_2^m) = -1 l k_l(c_2^m) (c_2^m) \geq -1. \quad (3.42)$$

If (ii) happens, then for some $1 \leq m < n(c_2)$ satisfying $\nu(c_2^m) = 2$, by (ii) of Lemma 2.2, it yields

$$\chi(c_2^m) = \sum_{l=0}^{2} (-1)^l c_2^{n(c_2)} l k_l(c_2^m) (c_2^m) = -k_0(c_2^m) + k_1(c_2^m) - k_2(c_2^m) \geq -1. \quad (3.43)$$

Now by (3.41)-(3.43), we obtain

$$\hat{\chi}(c_2) = \frac{1}{n(c_2)} \sum_{m=1}^{n(c_2)} \chi(c_2^m) \geq -1. \quad (3.44)$$

Note that $\hat{i}(c_k) > 3, k = 1, 2$ by (3.2), so it follows from (3.38) and (3.44) that

$$\frac{\hat{\chi}(c_1)}{\hat{i}(c_1)} = -\frac{1}{\hat{i}(c_1)} > -\frac{1}{3}, \quad \frac{\hat{\chi}(c_2)}{\hat{i}(c_2)} \geq -\frac{1}{\hat{i}(c_2)} > -\frac{1}{3}, \quad (3.45)$$

which, together with Theorem 2.3, yields

$$-\frac{2}{3} = \frac{\hat{\chi}(c_1)}{\hat{i}(c_1)} + \frac{\hat{\chi}(c_2)}{\hat{i}(c_2)} > -\frac{2}{3}. \quad (3.46)$$

This contradiction completes the proof of Theorem 1.1 in the Case of $n = 4$.

**Case 3:** $n = 5$.

In this case, note that $(n - 1)|N$, there holds $b_{2N} = 2$ by Lemma 2.4. In addition, it follows from (3.11), (3.21) and Claim 1 that all iterates $c_k^m, m \geq 1$ of $c_k, k = 1, 2$ have no contribution to $M_{2N}$. So, under the assumption (TCG), there holds $M_{2N} = 0$, which contradicts to the Morse inequality $0 = M_{2N} \geq b_{2N} = 2$ and completes the proof in the case of $n = 5$.

**Case 4:** $n \geq 6$.

In this case, note that $(n - 1)|N$, there holds $b_{2N+n-5} = 1$ by Lemma 2.4. In addition, it follows from (3.11), (3.21) and Claim 1 that all iterates $c_k^m, m \geq 1$ of $c_k, k = 1, 2$ have no contribution to $M_{2N+n-5}$. So, under the assumption (TCG), there holds $M_{2N+n-5} = 0$, which contradicts to the Morse inequality $0 = M_{2N+n-5} \geq b_{2N+n-5} = 1$. Thus there must exist the third closed geodesic $c_3$ such
that the iterates $c_3^m$, $m \geq 1$ have contributed at least 1 to $M_{2N+n-5}$. On the other hand, it follows from (3.1), (3.9) and (3.11) that $M_{2N+n-5}$ can be contributed only by the iterate $c_3^{2m_3}$. So we have

$$\overline{C}_{2N+n-5}(E, c_3^{2m_3}) \neq 0.$$  

(3.47)

If $c_3$ is a hyperbolic closed geodesic which implies that $S^+_{M_3}(1) = C(M_3) = \Delta_3 = 0$, then by (3.13) it yields $i(c_3^{2m_3}) + \nu(c_3^{2m_3}) = 2N$. Then by Lemma 2.2 we obtain $\overline{C}_{2N+n-5}(E, c_3^{2m_3}) = 0$ since $2N + n - 5 \geq 2N + 1$ in this case. This contradiction with (3.47) shows that the closed geodesic $c_3$ must be non-hyperbolic.

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**References**

[1] W. Ballmann, G. Thobergsson and W. Ziller, Closed geodesics on positively curved manifolds. *Ann. of Math.* 116 (1982), 213-247.

[2] W. Ballmann, G. Thobergsson and W. Ziller, Existence of closed geodesics on positively curved manifolds. *J. Diff. Geom.* 18 (1983), 221-252.

[3] V. Bangert, On the existence of closed geodesics on two-spheres. *Inter. J. Math.* 4 (1993), 1-10.

[4] V. Bangert and Y. Long, The existence of two closed geodesics on every Finsler 2-sphere. *Math. Ann.* 346 (2010) 335-366.

[5] R. Bott, On the iteration of closed geodesics and the Sturm intersection theory. *Comm. Pure Appl. Math.* 9 (1956), 171-206.

[6] K. C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems. Birkhäuser. Boston. 1993.

[7] H. Duan, Two elliptic closed geodesics on positively curved Finsler spheres. arXiv:1504.00245v2 [math.DG].

[8] H. Duan and Y. Long, Multiple closed geodesics on bumpy Finsler $n$-spheres. *J. Diff. Equa.* 233 (2007) 221-240.

[9] H. Duan and Y. Long, Multiplicity and stability of closed geodesics on bumpy Finsler 3-spheres. *Cal. Variations and PDEs.* 31 (2008) 483-496.

[10] H. Duan and Y. Long, The index growth and mutiplicity of closed geodesics. *J. of Funct. Anal.* 259 (2010) 1850-1913.
[11] H. Duan and Y. Long, Common index periodicity theorem for symplectic paths and non-hyperbolic closed geodesics on complex projective spaces. Preprint. 2015.

[12] H. Duan, Y. Long and W. Wang, Two closed geodesics on compact simply-connected bumpy Finsler manifolds. Submitted. 2014.

[13] N. Hingston, Equivariant Morse theory and closed geodesics. *J. Differential Geom.* 19 (1984), no. 1, 85-116.

[14] N. Hingston, On the growth of the number of closed geodesics on the two-sphere. *Inter. Math. Research Notices.* 9(1993) 253-262.

[15] N. Hingston, On the length of closed geodesics on a two-sphere. *Proc. Amer. Math. Soc.* 125 (1997) 3099-3106.

[16] J. Franks, Geodesics on $S^2$ and periodic points of annulus diffeomorphisms. *Invent. Math.* 108 (1992), 403-418.

[17] A. B. Katok, Ergodic properties of degenerate integrable Hamiltonian systems. *Izv. Akad. Nauk SSSR.* 37 (1973) (Russian), *Math. USSR-Isv.* 7 (1973), 535-571.

[18] C. Liu, The relation of the Morse index of closed geodesics with the Maslov-type index of symplectic paths. *Acta Math. Sinica.* 21 (2005), 237-248.

[19] Y. Long, Bott formula of the Maslov-type index theory. *Pacific J. Math.* 187 (1999), 113-149.

[20] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Advances in Math.* 154 (2000), 76-131.

[21] Y. Long, Index Theory for Symplectic Paths with Applications. Progress in Math. 207, Birkhäuser. 2002.

[22] Y. Long, Multiplicity and stability of closed geodesics on Finsler 2-spheres. *J. Euro. Math. Soc.* 8 (2006), 341-353.

[23] Y. Long and H. Duan, Multiple closed geodesics on 3-spheres. *Advances in Math.* 221 (2009) 1757-1803.

[24] Y. Long and C. Zhu, Closed charateristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$. *Ann. of Math.* 155 (2002), 317-368.

[25] H.-B. Rademacher, On the average indices of closed geodesics. *J. Diff. Geom.* 29 (1989) 65-83.

[26] H.-B. Rademacher, Morse Theorie und geschlossene Geodatische. *Bonner Math. Schriften* Nr. 229 (1992).
[27] H.-B. Rademacher, A sphere theorem for non-reversible Finsler metric. *Math. Ann.* 328 (2004) 373-387.

[28] H.-B. Rademacher, Existence of closed geodesics on positively curved Finsler manifolds. *Ergod. Th. & Dynam. Sys.* 27 (2007), 957-969.

[29] H.-B. Rademacher, The second closed geodesic on Finsler spheres of dimension $n > 2$. *Trans. Amer. Math. Soc.* 362 (2010), 1413-1421.

[30] W. Wang, Closed geodesics on positively curved Finsler spheres. *Adv. Math.* 218 (2008) 1566-1603.

[31] W. Wang, On a conjecture of Anosov. *Adv. Math.* 230 (2012), 1597-1617.

[32] W. Wang, Closed geodesics on Finsler spheres. *Cal. Variations and PDEs.* 45 (2012) 253-272.

[33] W. Wang, On the average indices of closed geodesics on positively curved Finsler spheres. *Math. Ann.* 355 (2013) 1049-1065.

[34] W. Ziller, Geometry of the Katok examples. *Ergod. Th. & Dynam. Sys.* 3 (1982), 135-157.