Numerical Analysis of the 1-D Parabolic Optimal Transport Problem

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Abstract

Numerical methods for the optimal transport problem is an active area of research. Recent work of Kitagawa and Abedin shows that the solution of a time-dependent equation converges exponentially fast as time goes to infinity to the solution of the optimal transport problem. This suggests a fast numerical algorithm for computing optimal maps; we investigate such an algorithm here in the 1-dimensional case. Specifically, we use a finite difference scheme to solve the time-dependent optimal transport problem and carry out an error analysis of the scheme. A collection of numerical examples is also presented and discussed.

1 Introduction

1.1 The Optimal Transport Problem

The centuries old optimal transport problem asks how to find the cheapest way to transport materials from a given source to a target location [6]. In the 1-dimensional case and for the quadratic cost function, the mathematical formulation of the problem is as follows. Let \([A, B], [C, D] \subset \mathbb{R}\) be bounded intervals, representing, respectively, the source and target domains. Consider two positive functions \(f : [A, B] \to \mathbb{R}\) and \(g : [C, D] \to \mathbb{R}\) satisfying the condition

\[
\int_A^B f(x) \, dx = \int_C^D g(x) \, dx = 1.
\]

Define the class of admissible transport maps

\[
\mathcal{M} = \left\{ S : [A, B] \to [C, D] \text{ satisfying } \int_{S^{-1}(E)} f(x) \, dx = \int_E g(y) \, dy \text{ for any open set } E \subset [C, D] \right\}.
\]
For any \( S \in \mathbb{M} \), define the total cost of \( S \) to be the quantity

\[
C(S) = \int_A^B |x - S(x)|^2 \, dx.
\]

The optimal transport problem is to find a map \( T \in \mathbb{M} \) that minimizes the total cost among all maps \( S \in \mathbb{M} \), i.e.

\[
C(T) = \min_{S \in \mathbb{M}} C(S).
\]

By a celebrated result of Brenier \[4\], under appropriate conditions on \( f \) and \( g \), the optimal map \( T \) exists and is unique. In addition, \( T(x) = u'(x) \) where \( u : [A, B] \to \mathbb{R} \) is a convex function that satisfies the boundary-value problem

\[
\begin{aligned}
&u''(x) = \frac{f(x)}{g(u'(x))}, \\
u'(A) = C, \ u'(B) = D.
\end{aligned}
\]  

(O-T)

Notice that (O-T) implies

\[
\int_A^B f(p) \, dp = \int_A^B g(u'(p))u''(p) \, dp = \int_{u'(A)}^{u'(B)} g(q) \, dq = \int_C^D g(q) \, dq.
\]

If we define the cumulative distribution functions of \( f \) and \( g \), respectively, as

\[
F(x) := \int_A^x f(p) \, dp, \quad G(y) := \int_C^y g(q) \, dq,
\]

we then have the relation

\[
F(x) = G(u'(x)) \quad \text{for all } x \in [A, B].
\]

Since \( g \) is positive on \([C, D]\), we have \( G'(y) = g(y) > 0 \), so \( G \) is strictly increasing, hence invertible. Therefore, the optimal map \( T \) can be expressed in terms of \( F \) and \( G \) as

\[
T(x) = u'(x) = G^{-1}(F(x)) \quad \text{for all } x \in [A, B].
\]  

(1.1.2)

In practice, given \( f \) and \( g \), it is difficult to compute \( F \) and \( G \) analytically. This provides motivation to develop alternate numerical methods of obtaining the optimal map \( T \). Much work has been done on the numerical approximation of optimal maps in low dimensions \[2\ [3\ [7\]. Here we consider an approach based on a time-dependent version of (O-T) studied in \[1\ [5\] and referred to as the parabolic optimal transport problem. In our setting, this problem can be stated as follows: find a time-dependent function \( v(t, x) \) that satisfies

\[
\begin{aligned}
v_t &= \log(v_{xx}) - \log \left( \frac{f(x)}{g(v_x)} \right) \quad \text{in } (0, \infty) \times (A, B), \\
v_x(t, A) = C, \ v_x(t, B) = D \quad \text{for all } t \geq 0, \\
v(0, x) = u_0(x), \\
v(t, \cdot) \text{ strictly convex for all } t \geq 0.
\end{aligned}
\]

(Parabolic O-T)

Here, \( u_0(x) \) is a given convex function on \([A, B]\) that satisfies \( u_0'(A) = C \) and \( u_0'(B) = D \). It is shown in \[5\] that \( \lim_{t \to \infty} v(t, x) = u(x) \) where \( u \) solves (O-T). The more recent work \[1\] shows the convergence is exponentially fast in \( t \).

### 1.2 Discretization of the Problem

The purpose of this paper is to carry out a numerical approximation of (Parabolic O-T) and study a number of examples. To numerically approximate (Parabolic O-T), we choose to use a finite difference scheme. This requires discretizing the interval \([A, B]\) using \( J \in \mathbb{N} \) grid points

\[
x_j = A + \frac{j(B - A)}{J}, \quad j = -1, \ldots, J + 1.
\]
The range of indices is chosen this way in order to provide an extra point outside each end of the interval \([A, B]\).

We will use the notation
\[
\Delta x = \frac{B - A}{J}
\]
to denote the spatial grid resolution and use the short-hand \(\Delta x^2 := (\Delta x)^2\). To discretize the time interval \([0, \infty)\), we let \(\{t_n\}_{n=0}^{\infty}\) be a non-negative sequence of strictly increasing time values with \(t_0 = 0\). Denote the \(n\)-th time step by
\[
\Delta t_n = t_{n+1} - t_n.
\]
We denote by \(G\) the set of all grid points \(\{(t_n, x_j) : n \in \mathbb{N}, j \in \{-1, \ldots, J+1\}\}\).

In order to motivate the finite difference scheme, we recall the following consequences of the Taylor Remainder Theorem:
\[
\begin{align*}
v_x(t_n, x_j) &= \frac{v(t_n, x_{j+1}) - v(t_n, x_{j-1})}{2\Delta x} - \frac{v_{xxx}(t_n, x_j + \psi) + v_{xxx}(t_n, x_j - \psi)}{12}\Delta x^2 \quad \text{for some } \psi \in (0, \Delta x), \\
v_{xx}(t_n, x_j) &= \frac{v(t_n, x_{j+1}) + v(t_n, x_{j-1}) - 2v(t_n, x_j)}{\Delta x^2} - \frac{v_{xxxx}(t_n, x_j + \gamma) + v_{xxxx}(t_n, x_j - \gamma)}{24}\Delta x^2 \quad \text{for some } \gamma \in (0, \Delta x), \\
v_t(t_n, x_j) &= \frac{v(t_{n+1}, x_j) - v(t_n, x_j)}{\Delta t_n} - \frac{v_{tt}(t_n + \kappa, x_j)}{2}\Delta t_n \quad \text{for some } \kappa \in (0, \Delta t_n).
\end{align*}
\]

Our goal is to construct a grid function \(U : G \to \mathbb{R}\) such that \(U(t_n, x_j) \approx v(t_n, x_j)\) for \(j = 0, \ldots, J\) and \(n \in \mathbb{N}\), where \(v\) is the solution of \([(\text{Parabolic O-T})\] with the exact values for these derivatives as given by the Neumann boundary conditions,
\[
\begin{align*}
C &= v_x(t_n, A) = \frac{v(t_n, x_1) - v(t_n, x_{-1})}{2\Delta x} - \frac{v_{xxx}(t_n, A + \psi) + v_{xxx}(t_n, A - \psi)}{12}\Delta x^2 \quad \text{for some } \psi \in (0, \Delta x) \\
D &= v_x(t_n, B) = \frac{v(t_n, x_{J+1}) - v(t_n, x_{-J-1})}{2\Delta x} - \frac{v_{xxx}(t_n, B + \psi) + v_{xxx}(t_n, B - \psi)}{12}\Delta x^2 \quad \text{for some } \psi \in (0, \Delta x).
\end{align*}
\]

Utilizing the definition of the finite difference method presented in [S], we implement the following centered difference approximation of \([(\text{Parabolic O-T})\]
\[
\begin{align*}
\nabla U_{j+1} &= \left(\log \left(\Delta \phi_j^u\right) - \frac{f(x_j)}{\phi_j^u}\right)\Delta t_n + U_j^n, \quad j = 0, \ldots, J, \\
U_{j-1}^0 &= C, \quad \nabla U_0^n = D, \\
U_{J}^n &= U_{J-1}^n - 2C\Delta x, \quad U_{J+1}^n = U_{J-1}^n + 2D\Delta x, \\
U_0^n &= U_0(x_j).
\end{align*}
\]

\[\text{(N-E)}\]
1.3 Structure of the Paper

The remainder of this paper is structured as follows. In Section 2 we show that the error between our numerical approximation and the true solution of \(\text{Parabolic O-T}\) is bounded by quantities depending on previous time-steps. In Section 3 we show how to measure the asymptotic closeness of \(\text{N-E}\) to \(\text{O-T}\). Finally in Section 4 we discuss the code for implementation of the finite difference scheme as well as numerical findings and applications to quantile functions. In the Appendix, the proofs of explicit derivative bounds for the solution of \(\text{Parabolic O-T}\) are given.

2 Error Analysis of Finite Difference Scheme

In order for a numerical approximation to be effective the error at a given time step must be bounded by quantities known from the previous steps. In this section we establish such error bounds for \(\text{N-E}\). Recall \(\Delta^n U\) is a finite approximation of the second derivative and therefore has some error. Although we would expect \(\Delta^n U\) to stay positive because it is approximating a convex function, how to guarantee this is not yet clear. Therefore we must assume the condition of \(\Delta^n U\) staying positive for all \(n\) in the following error analysis.

We first define the infinity norm \(\|\cdot\|_{\infty}\) to be the maximum value of \(\cdot\) at time step \(n\) for all points in \(G\). We will also now define several bounds on the derivatives of \(v\).

**Definition 2.1.** Define the derivative bounds \(K, \Gamma, \Psi, \delta_1, \text{ and } \delta_2\) to satisfy

\[
K \geq |v_{tt}(t,x)|, \\
\Gamma \geq |v_{xxxx}(t_n,x_j)|, \\
\Psi \geq |v_{xxx}(t_n,x_j)|, \\
0 < \delta_1 \leq v_{xx}(t,x) \leq \delta_2.
\]

for all \(t \in [0, \infty)\) and \(x \in [A,B]\).

For a discussion of the value of these error bounds see the appendix. We are now prepared to state our error bound for our numerical scheme.

**Theorem 2.1.** Assuming \(\Delta^n U\) remains positive at every time step \(n\), positive time and spacial grid steps, and that the following is true for \(\Delta x\) and \(\Delta t\):

\[
\Delta x = \min \left\{ \frac{3 \min \delta_1}{2 \Psi}, \sqrt{\frac{6 \delta_1 \Gamma}{1}} \right\}, \\
\frac{\max |g'(y)| \Delta t_n}{2 \min g(y) \Delta x} \leq \frac{\Delta t_n}{\Delta x^2 \min \left\{ \frac{1}{2} \delta_1, \min \{ \Delta^n U \} \right\}} \leq \frac{1}{2}, \\
\text{ and } \Delta x^2 \min \{ \Delta^n U \} \leq \frac{1}{2}.
\]

\(\text{N-E}\) has the following maximum error bound on the interior points at time step \(n\).

\[
||U^n - v^n||_{\infty} \leq \sum_{i=0}^{n-1} \left( \Delta t_i \left( \frac{\Delta t_i}{2} K + \frac{\Delta x^2}{6 \delta_1} \Gamma \right) + \frac{\max |g'|}{\min g} \left( \frac{\Delta x^2}{6 \Psi} \right) \right)
\]

This theorem shows that our finite difference scheme is close to the real solution of \(\text{Parabolic O-T}\) for all \(t_n\). We will spend the rest of Section 2 proving Theorem 2.1. For a more formal discussion of the efficacy of finite difference schemes see [8].

2.1 Calculation of Local Error

In this section we prove the first necessary lemma for proving Theorem 2.1. We start with several definitions.
Definition 2.2. Define the local approximation

\[ V_j^{n+1} := \frac{1}{\Delta t_n} \left( \log (\Delta_j^n v) - \log \left( \frac{f(x_j)}{g(\nabla_j^n v)} \right) \right) \Delta t_n + v_j^n, \quad j = 1, \ldots, J - 1. \]

Definition 2.3. Define the local error \( \tau \) at grid point \((x_j, t_n)\),

\[ \tau_j^n := V_j^n - v_j^n. \]

With these definitions in hand we state the first lemma necessary for proving Theorem 2.1.

Lemma 2.1. Assuming the conditions (2.0.1), the local error \( \tau \) for any point on time step \( n \) has an upper bound

\[ ||\tau^{n+1}||_\infty \leq \Delta t_n \left( \frac{\Delta t_n \max |v_{tL}|}{2} + \frac{\Delta x^2 \max |v_{xxx}|}{6 \min |v_x|} + \frac{\Delta x^2 \max |g'| \max |v_{xxx}|}{6 \min g} \right). \]

Proof. We first define another error term.

Definition 2.4. Define the local discretization error \( \theta \) at grid point \((x_j, t_n)\)

\[ \theta_j^n := \frac{v_j^{n+1} - v_j^n}{\Delta t_n} - \log (\Delta_j^n v) + \log \left( \frac{f(x_j)}{g(\nabla_j^n v)} \right). \tag{2.1.1} \]

This definition provides a useful identity

\[ V_j^{n+1} - v_j^{n+1} = \tau_j^{n+1} = -\Delta t_n \theta_j^n. \tag{2.1.2} \]

Therefore, to calculate the bounds on the local error, \( \tau_j^{n+1} \), we first need to estimate \( \theta_j^n \). Using Parabolic O-T, we can substitute for \( f(x_j) \) and get

\[ \theta_j^n = \frac{v_j^{n+1} - v_j^n}{\Delta t_n} - v_t(t_n, x_j) - (\log (\Delta_j^n v) - \log(v_{xx}(t_n, x_j))) - (\log (g(\nabla_j^n v)) - \log (g(v_x(t_n, x_j)))). \tag{2.1.3} \]

From (1.2.3), we have

\[ \frac{v_j^{n+1} - v_j^n}{\Delta t_n} - v_t(t_n, x_j) = \frac{v_{tt}(t_n + \kappa, x_j)}{2} \Delta t_n. \tag{2.1.4} \]

To simplify \( \log (\Delta_j^n v) - \log(v_{xx}(t_n, x_j)) \) we use the Mean Value Theorem to find a number \( \eta \) between \( v_{xx}(t_n, x_j) \) and \( \Delta_j^n v \) such that

\[ \log (\Delta_j^n v) - \log(v_{xx}(t_n, x_j)) = \frac{1}{\eta} (\Delta_j^n v - v_{xx}(t_n, x_j)). \]

The Taylor approximation (1.2.2) then shows

\[ \log (\Delta_j^n v) = \log(v_{xx}(t_n, x_j)) + \frac{\Delta x^2}{\eta} \left( \frac{v_{xxx}(t_n, x_j + \gamma) + v_{xxx}(t_n, x_j - \gamma)}{24} \right). \tag{2.1.5} \]

The Mean Value Theorem implies there exists a number \( \mu \) between \( g(v_x(t_n, x_j)) \) and \( g(\nabla_j^n v) \), and a number \( \chi \) between \( g'(v_x(t_n, x_j)) \) and \( g'(\nabla_j^n v) \) such that

\[ \log (g(\nabla_j^n v)) - \log (g(v_x(t_n, x_j))) = \frac{1}{\mu} (g(\nabla_j^n v) - g(v_x(t_n, x_j))) \approx \frac{\chi}{\mu} (\nabla_j^n v - v_x(t_n, x_j)). \]

Using the Taylor expansion (1.2.1), we find that

\[ \log (g(\nabla_j^n v)) - \log (g(v_x(t_n, x_j))) = \frac{\chi \Delta x^2}{\mu} \left( \frac{v_{xxx}(t_n, x_j + \psi) + v_{xxx}(t_n, x_j - \psi)}{12} \right). \tag{2.1.6} \]

Substituting (2.1.4), (2.1.5) and (2.1.6) into (2.1.3) gives us
\( \tau_{n+1} = -\Delta t_n \theta^n \)
\[ = \Delta t_n \left( -\frac{\Delta t_n}{2} v_{t(t_n + \kappa, x_j)} + \frac{\Delta x^2}{24 \eta} (v_{xxx}(t_n, x_j + \gamma) + v_{xxx}(t_n, x_j - \gamma)) + \frac{\Delta x^2}{12 \mu} (v_{xxx}(t_n, x_j + \psi) + v_{xxx}(t_n, x_j - \psi)) \right). \] (2.1.7)

Using the derivative bounds in (2.1.7) shows
\[ |\tau_{n+1}| \leq \Delta t_n \left( \frac{K \Delta t_n}{2} + \frac{\Delta x^2 \Gamma}{6 \delta_1} + \frac{\Delta x^2 \Psi}{6 \mu} \right). \] (2.1.8)

If we now choose \( \Delta x \) to satisfy
\[ \Delta x^2 \leq \frac{6 \delta_1}{\Gamma}, \] (2.1.9)
we obtain the inequality
\[ \frac{1}{12} \max_{x,t} |v_{xxx}| \Delta x^2 \leq \frac{1}{2} \min_{x,t} v_{xx}. \]

Then by the Taylor expansion (1.2.2), we obtain
\[ \min_j \{\Delta^n_{\theta} v\} \geq \frac{1}{2} \min_{x,t} v_{xx}, \]

hence
\[ \eta \geq \min_j \{\Delta^n_{\theta} v, v_{xx}(t_n, x_j)\} \geq \frac{\delta_1}{2}. \]

Additionally we know \( \chi \) is bounded from above by \( \max |g'| \) and \( \mu \) is bounded from below by \( \min g \). Substituting the constants \( K, \Gamma, \Psi, \) and \( \delta_1 \) into (2.1.8) and using the triangle inequality with our \( g \) bounds we obtain
\[ |\tau_{n+1}| \leq \Delta t_n \left( \frac{\Delta t_n K}{2} + \frac{\Delta x^2 \Gamma}{6 \delta_1} + \frac{\Delta x^2 \max |g'| \Psi}{6 \min g} \right) \] for all \( j \in \{1, \ldots, J-1\} \). (2.1.10)

under the restriction (2.1.9). Finally we can take the max norm of \( \tau_{n+1} \) over all \( j = 1, \ldots, J-1 \) to obtain
\[ ||\tau^{n+1}||_{\infty} \leq \Delta t_n \left( \frac{\Delta t_n K}{2} + \frac{\Delta x^2 \Gamma}{6 \delta_1} + \frac{\Delta x^2 \max |g'| \Psi}{6 \min g} \right) \] (2.1.11)

\[ \square \]

2.2 Calculation of Total Error

In this section we prove an additional lemma. We must first define another error term.

Definition 2.5. Define the error term \( \varepsilon \)
\[ \varepsilon^n_{j} := U_{j}^{n+1} - V_{j}^{n+1}. \]

Lemma 2.2. Assuming \( \Delta^n_{\theta} U \) stays positive for all \( n \) and the conditions in 2.0.1, the \( \varepsilon \) is bounded by:
\[ ||\varepsilon^{n}||_{\infty} \leq \sum_{i=0}^{n-1} \Delta t_i \left( \frac{\Delta t_i K}{2} + \frac{\Delta x^2 \Gamma}{6 \delta_1} + \frac{\Delta x^2 \max |g'| \Psi}{6 \min g} \right). \]

Proof. Note by definition
\[ \varepsilon^n_{j} = (\log(\Delta^n_{\theta} U) - \log(\Delta^n_{\theta} v) - (\log (g (\nabla^n_{\theta} v)) - \log (g (\nabla^n_{\theta} U)))) \Delta t_n + (U^n_{j} - v^n_{j}). \] (2.2.1)
By the Mean Value Theorem, there is some number $\rho$ between $g(\nabla^n_U)$ and $g'(\nabla^n_U)$, and another number $\omega$ between $g'(\nabla^n U)$ and $g'(\nabla^n v)$ such that

$$\log g(\nabla^n_U) - \log g(\nabla^n v) = \frac{\omega}{2\Delta x \rho} (v^n_{j+1} - U^n_{j+1} + v^n_{j-1} - U^n_{j-1}).$$

(2.2.2)

Similarly, there is some number $\xi$ between $\Delta^n_U$ and $\Delta^n_v$ such that

$$\log(\Delta^n_U) - \log(\Delta^n v) = \frac{1}{\xi}(\Delta^n_U - \Delta^n v).$$

(2.2.3)

Substituting (2.2.3) and (2.2.2) into (2.2.1) shows

$$\varepsilon_{n+1} = \left(\frac{1}{\xi}(\Delta^n_U - \Delta^n v) + \frac{\omega}{2\Delta x \rho} (v^n_{j+1} - U^n_{j+1} + v^n_{j-1} - U^n_{j-1})\right) \Delta t_n + (U^n_j - V^n_j).$$

To continue we must bound $\xi$ from below by known values. Note that the following inequality holds true. If (2.1.9) holds, then $\min_j \{\Delta^n v\}$.

$$\xi \geq \min \left\{ \min_j \{\Delta^n_U\}, \min_j \{\Delta^n v\} \right\}$$

If (2.2.4) holds, then $\min_j \{\Delta^n v\} \geq \frac{1}{2} \delta_1$. Therefore, $\frac{1}{\xi} \leq \frac{1}{\min \left\{ \frac{1}{2} \delta_1, \min_j \{\Delta^n v\} \right\}}$.

Let us define the quantities

$$r := \frac{\Delta t_n}{\Delta x (\frac{1}{2} \delta_1, \min_j \{\Delta^n v\})}, \quad s := \frac{\max[g'|\Delta t_n]}{2 \min g \Delta x}.$$

Notice that we can bound $\rho$ from above by $\max|g'|$ and bound $\omega$ from below by $\min g$. Using the definition of the second order difference operator $\Delta^n_U$ and the triangle inequality, we find that

$$|\varepsilon_{n+1}^n| \leq |r - s||U^n_{j+1} - v^n_{j+1}| + |1 - 2r||U^n_j - v^n_j| + |r + s||U^n_{j-1} - v^n_{j-1}|.$$

From the triangle inequality we also know $|U^n_j - v^n_j| \leq |e_j^n| + |\tau_j^n|$, thus

$$|\varepsilon_{n+1}^n| \leq |r - s|(|e_j^n| + |\tau_j^n|) + |1 - 2r|(\|e_j^n\| + \|\tau_j^n\|) + |r + s|(\|e_j^n\| + \|\tau_j^n\|).$$

Replacing all quantities of $e^n_j$ and $\tau^n_j$ with their maximum norms

$$|\varepsilon_{n+1}^n| \leq (|r - s| + |1 - 2r| + |r + s|)(\|e^n\| + \|\tau^n\|).$$

Now if we choose $\Delta t_n$ and $\Delta x$ such that $s < r \leq \frac{1}{2}$, then

$$\varepsilon_{n+1}^n \leq \|e^n\| + \|\tau^n\|.$$

Iterating the inequality over $n$ shows

$$\|e^n\| \leq \|e^0\| + \|\tau^{n-1}\| + \cdots + \|\tau^0\|$$

for all $n \geq 1$.

Lastly recall that by definition $U^0 - V^0 = 0$ therefore $\|e^0\| = 0$ and we have

$$\|e^n\| \leq \|\tau^{n-1}\| + \cdots + \|\tau^0\|.$$

Using the bound for $r$ from (2.1.11), we finally get

$$\|e^n\| \leq \sum_{i=0}^{n-1} \Delta t_i \left( \frac{\Delta t_i K}{2} + \frac{\Delta x^2 \Gamma}{6 \delta_1} + \frac{\Delta x^2 \max[g'|\Psi]}{6 \min g} \right).$$

(2.4.2)
We now have all the tools at hand to prove Theorem 2.1.

**Proof of Theorem 2.1.** Using the triangle inequality and taking the maximum norm we can say

\[ ||U^n_j - v(t_n, x_j)||_\infty \leq ||r^n_j||_\infty + ||\epsilon^n_j||_\infty \]

By Lemmas 2.1 and 2.2

\[ ||U^n - v^n||_\infty \leq \Delta t_n \left( \frac{\Delta t_n}{2} K + \frac{\Delta x^2}{6\delta_1} \Gamma \right) + \frac{\max |g'|}{\min g} \left( \frac{\Delta x^2}{6} \psi \right) \]  \hspace{1cm} (2.2.5)

2.3 Addressing Boundary Conditions

Due to the Neumann boundary conditions we must analyze error conditions at the boundaries separately. This process is nearly identical as sections 2.1 and 2.2. Additionally calculations on the left and right boundary are almost identical, and we will just focus on the left boundary point.

We begin with our calculation of the local error. Using the same method of section 3.1, the error term from 1.2.3, and 1.2.4 gives us

\[ \tau^a_0 = \Delta t_n \left( -\frac{\Delta t_n}{2} \left( v(t_n + \kappa, 0) \right) + \frac{1}{\lambda} \left( \frac{\Delta x}{3} \left( v_{xxx}(t_n, \psi) \right) + \chi \left( \frac{\Delta x^2}{6} v_{xxx}(t_n, x_j + \gamma) \right) \right) \right), \]

with \( \lambda \) being some point between \( \Delta_0^n v \) and \( v_{xxx}(t_n, 0) \). We assume \( \Delta x \) in the boundary case must also satisfy

\[ \frac{1}{2} v_{xxx} |\Delta x| \leq \frac{1}{2} v_{xx}, \text{ which leads to } \]

\[ \min_j \{ \Delta^n_j v \} \geq \frac{1}{2} \min \{ v_{xx} \}. \]

This implies that \( \lambda < \frac{1}{2} \delta_1 \) by similar methods as used before, and results in a bound on \( \tau^a_0 \) at boundary point \( a \)

\[ |\tau^a_0| \leq \Delta t_n \left( \frac{\Delta t_n}{2} K + \frac{2 \Delta x}{3 \delta_1} \psi + \max |g'| \left( \frac{\Delta x^2}{6} \psi \right) \right). \]

Now calculating our total error at the boundary begins the same way as in section 3.2. If we let \( r \) and \( s \) be the same quantities then, by use of a similar log approximation as well as the triangle inequality

\[ |\varepsilon^a_0| \leq |r + s| U_1^{n-1} - V_1^{n-1}| + |1 - r - s| U_0^{n-1} - V_0^{n-1}|. \]

We can now replace \( |U_0^a - V_0^a| \) terms with \( |\varepsilon^a_0| + |\tau^a_0| \) terms by another triangle inequality

\[ |\varepsilon^a_0| \leq |r + s| (|\varepsilon_1^{n-1}| + |\tau_1^{n-1}|) + |1 - r - s| (|\varepsilon_0^{n-1}| + |\tau_0^{n-1}|). \]

If \( s < r \leq \frac{1}{2} \) as was necessary for (2.2) then it is implied that \( r + s < 1 \). Taking maximums over \( j \) in \( \{ 0, 1 \} \) gives us

\[ |\varepsilon^a_0| \leq \max_j \{ \tau_j^{n-1} \} + ... + \max_j \{ \tau_j^0 \}. \]

This allows us to show a final bound on the boundary conditions,

\[ |U_0^a - v_0^a| \leq \sum_{i=0}^{n-1} \left( \Delta t_i \left( \frac{\Delta t_i}{2} K + \frac{\Delta x}{3 \delta_1} \psi + \max |g'| \left( \frac{\Delta x^2}{6} \psi \right) \right) \right). \]
3 Asymptotic Error Analysis

To ensure that our implementation of the code provides an accurate numerical approximation of the optimal map $T(x) = u'(x)$, where $u$ solves (O-T), we must show that $\max_j |\nabla^n U - u'(x_j)|$ is within a desired tolerance. In this section we show the error between (O-T) and (N-E) is controlled by a quantity that can be calculated at each time step.

Let $\Omega \subset [A,B]$ and let $S : \Omega \to [C,D]$ be an increasing function. Recall the definition of the cumulative distribution functions $F$ and $G$ in (1.1.1). Define the error function of $S$ as

$$E(S,x) := |F(x) - G(S(x))|, \quad x \in \Omega.$$  

Notice that if $T$ is the optimal map between $f$ and $g$, then $E(T,x) = 0$ for all $x \in [A,B]$. Next, we see that for any $x \in \Omega$,

$$E(S,x) = |F(x) - G(S(x))| = |G(T(x)) - G(S(x))| \geq \left( \min_{[C,D]} g \right) |T(x) - S(x)|.$$  

Therefore,

$$\max_{x \in \Omega} |T(x) - S(x)| \leq \frac{\max_{x \in \Omega} E(S,x)}{\min_{[C,D]} g}.$$  

It follows that for any $\sigma > 0$,

$$\max_{x \in \Omega} E(S,x) \leq \sigma \min_{[C,D]} g \quad \Rightarrow \quad \max_{x \in \Omega} |T(x) - S(x)| \leq \sigma. \quad (3.0.1)$$

**Theorem 3.1.** Let $T(x) = u'(x)$ be the optimal map, where $u$ solves (O-T). Assuming $\Delta^n U$ is strictly positive for all $n$. Given a tolerance $\sigma > 0$ the following is true at a large enough time step $n$.

$$\max_{j=0,\ldots,J} E(\nabla^n U, x_j) \leq \sigma \quad \Rightarrow \quad \max_{j=0,\ldots,J} |T(x_j) - \nabla^n U| \leq \frac{\sigma}{\min_{[C,D]} g} \quad (3.0.2)$$

**Proof.** We apply the estimate (3.0.1) to the finite-difference scheme (N-E). Let $\Omega = \{x_0, \ldots, x_J\} \subset [A,B]$ be the set of spatial grid points. For each $n \in \mathbb{N}$, denote the map $S_n : \Omega \to [C,D]$ as

$$S_n(x_j) := \nabla^n U, \quad x_j \in \Omega.$$  

Therefore, we have $S_n(x_0) = C$ and $S_n(x_J) = D$. We check that $S_n$ is an increasing function on $\Omega$, and hence maps into $[C,D]$. This is a condition of the optimal map as implied by (O-T).

Assume $j \in \{1, \ldots, J\}$. Then

$$S_n(x_j) - S_n(x_{j-1}) = \nabla^n U - \nabla^{n-1}_j U$$

$$= \left( \frac{U^n_{j+1} - U^n_{j-1}}{2\Delta x} \right) - \left( \frac{U^n_{j} - U^n_{j-2}}{2\Delta x} \right)$$

$$= \frac{\Delta x}{2} \left[ \left( \frac{U^n_{j+1} - U^n_{j-1}}{\Delta x^2} \right) - \left( \frac{U^n_{j} - U^n_{j-2}}{\Delta x^2} \right) \right]$$

$$= \frac{\Delta x}{2} \left[ \left( \frac{U^n_{j+1} + U^n_{j-1} - 2U^n_{j}}{\Delta x^2} \right) - \left( \frac{U^n_{j} - U^n_{j-2}}{\Delta x^2} \right) \right]$$

$$= \frac{\Delta x}{2} \left[ \left( \frac{U^n_{j+1} + U^n_{j-1} - 2U^n_{j}}{\Delta x^2} \right) + \left( \frac{-U^n_{j-1} + 2U^n_{j} - U^n_{j+1}}{\Delta x^2} \right) \right]$$

$$= \frac{\Delta x}{2} \left[ \left( \frac{U^n_{j+1} + U^n_{j-1} - 2U^n_{j}}{\Delta x^2} \right) + \left( \frac{U^n_{j} + U^n_{j-2} - 2U^n_{j-1}}{\Delta x^2} \right) \right]$$

$$= \frac{\Delta x}{2} \left( \Delta^n U + \Delta^n_{j-1} U \right) \geq 0,$$

as long as $\Delta^n U \geq 0$ for all $j = 0, \ldots, J$ and $n \geq 0$. Notice that if the code does not encounter a domain error at time step $n+1$, then $\Delta^n_{j+1} U > 0$ for all $j$. \(\square\)
The theorem above shows that given a tolerance $\sigma > 0$, if there exists some $N(\sigma) \in \mathbb{N}$ such that
\[
\max_{j=0, \ldots, J} E(\nabla_j^n U, x_j) \leq \sigma \quad \text{for all } n \geq N(\sigma),
\]
it then follows that
\[
\max_{j=0, \ldots, J} |T(x_j) - \nabla_j^n U| \leq \frac{\sigma}{\min_{[C, D]} g} \quad \text{for all } n \geq N(\sigma). \quad (3.0.3)
\]
Since the quantity $\max_j E(\nabla_j^n U, x_j)$ can be computed at each time step $n$, we can run our code to the time step $n = N(\sigma)$ for which $\max_j E(\nabla_j^n U, x_j)$ is less than a specified tolerance. We are not able to guarantee our scheme will always be able to reach such a specified tolerance in a finite number of steps, but if this tolerance is reached, then we can conclude using (3.0.3) that the map $\nabla_j^n U$ is equal to the optimal map on the grid points $x_1, \ldots, x_J$ up to a quantifiable error. It should be noted that in practice it is often simpler to use numerical integration to evaluate $E(\nabla_j^n U, x_j)$. Therefore, (3.0.3) will hold up to the accuracy of the numerical integration method used.

4 Computational Examples and Results

This section is dedicated to describing the code used for implementing (N-E) and certain relevant numerical examples computed using this code. We first note that empirically when $\Delta t$ and $\Delta x$ are chosen to satisfy (2.0.1) then $\Delta_j^n U$ stays bounded above $\frac{1}{2}\delta_1$. Additionally from [5], we know that (Parabolic O-T) converges exponentially to the actual solution of the optimal transport problem. In the following examples this fast convergence can be observed as the results are graphed over time using a uniformly spaced color gradient. Exponential convergence is observed due to the relatively small change in approximation at later time steps.

In testing our code for functionality, we attempted to cover a variety of situations using appropriate choices of $f$ and $g$. Some of the more interesting cases tested have been shown here. For simplicity all cases were run with initial choice $u_0(x) = \frac{1}{2} x^2$. We chose not to graph the function $U_j^n$, as the function that is relevant for the optimal transport theory is $\nabla_j^n U$, which is meant to approximate the function $u'(x)$ for $u$ solving (O-T). We note that the theory for (Parabolic O-T) only guarantees convergence to the solution of (O-T) when $f(x)$, $g(y)$ are continuous and bounded away from zero and infinity on $[A, B]$ and $[C, D]$, respectively. Some of our numerical examples test the limits of the theory by considering cases where $g$ is only piece-wise continuous and also where $f$ gets very close to zero.

Before discussing the examples we will briefly discuss the algorithm. The full implementation in python is available at [https://github.com/manuelarturosantana/ParabolicOptimalTransport](https://github.com/manuelarturosantana/ParabolicOptimalTransport)

4.1 Algorithm

**Result:** Returns the Approximated Solution of OT

Calculate $\Delta x, \Delta t_0, \delta_1, \delta_2, \Psi, K, \Gamma$

current row = initial row based off $v_0$

while $\text{Max Error} > \text{Tolerance}$ do

$\Delta t_n$

current row = calculate next row

Calculate Max Error

end

Algorithm 1: Simple Finite Difference Algorithm

Calculating each row follows the finite difference scheme (N-E). The boundaries and the interior points are calculated separately. Checking the error at every grid point for every time-step is computationally expensive, so a well-spaced subset of $G$ at time step $n$ can be used to measure $E(S_n, x_j)$. Once all grid points in the subset are in tolerance, the time iteration is continued until all grid points at time step $n$ tested are within tolerance.
4.2 ‘Nice’ Functions

In testing our code we tried an initial variety of computationally nice functions for both $f$ and $g$. These functions are bounded well above 0 ($> 0.1$), continuous, and did not change convexity more than twice. Such examples include logarithmic, exponential, linear, quadratic, constant, and concave cosine functions all modified to fit the conditions of Parabolic O-T. This collection was used to determine a baseline functionality of the finite difference scheme N-E. The following is an example of numerical output using two functions from this set:

Example 4.2.1.

$$f(x) = \frac{\log(x + 2)}{3\log(3) + 2} + \frac{2}{3\log(3) + 2}, \quad g(x) = \frac{1}{2}x^2 + \frac{1}{3}, \quad [A, B] = [C, D] = [-1, 1].$$

![Graphs of 4.2.1](image)

Figure 1: Graphs of 4.2.1, 152.9s to Reach Tolerance $\epsilon = 0.01$, $\max_{j=0,...,J} |T(x_j) - \nabla^n_j U| \leq 0.03$

| Tolerance | Iterations | $t_{total}$ | CPU Time (s) |
|-----------|------------|-------------|--------------|
| 0.1       | 814        | 0.0053      | 1.06         |
| 0.01      | 119880     | 0.7815      | 150          |
| 0.001     | 289020     | 1.88        | 360          |
| 0.0001    | 459807     | 2.997       | 573          |

Table 1: The computational time and iterations to reach tolerance. The sum of time steps is also given.

As expected the graph in Figure 1 showed exponential convergence to the optimal map.

4.3 High Frequency Functions

Example 4.3.1.

$$f(x) = \frac{50}{\sin(100) + 200}(\cos(100x) + 2), \quad g(x) = \frac{1}{4}(x + 2)$$

This case has frequent convexity changes of the initial mass function, $f(x)$. Testing this case allows us to know that our code is able to handle more complex smooth cases.
Example 4.3.2.

\[
f(x) = \frac{1}{4}(x + 2), \quad g(x) = \frac{50}{\sin(100) + 200} (\cos(100x) + 2)
\]

This example switches \(f(x)\) and \(g(x)\) in Example 4.3.1. According to the optimal transport theory, the corresponding optimal map will be the inverse of the optimal map from Example 4.3.1. We also expected the runtime to be longer in this case, due to the high oscillation in the term \(g(\nabla^U U)\) from (N-E).

Table 2: Numerics for high frequency function to quadratic.

| Tolerance | Iterations | \(t_{total}\) | CPU Time |
|-----------|------------|---------------|----------|
| 0.1       | 35877      | 0.1006        | 41.31    |
| 0.01      | 354534     | 0.9947        | 414      |
| 0.001     | 692928     | 1.944         | 817      |

Table 3: Numerics for quadratic function to high frequency.

| Tolerance | Iterations | \(t_{total}\) | CPU Time |
|-----------|------------|---------------|----------|
| 0.1       | 83180      | 0.1001        | 596      |
| 0.01      | 786271     | 0.9454        | 5195     |
| 0.001     | 1254499    | 1.509         | 8635     |
As expected, this example required more computational time and iterations to reach tolerance. Furthermore, we observe that the graph of $\nabla^n U$ in (Figure 3(a)) is the inverse of the graph of $\nabla^j U$ in (Figure 2(a)), which is predicted by the optimal transport theory. Due to the large differences in computational time, it would likely be more efficient to let the initial mass distribution function $f$ be the more complicated one, and then computing the inverse of the optimal map between $f$ and $g$ if that is what one needs. However, it is worth keeping in mind the limitations of inverting a grid function, as the inverse is not necessarily defined on a well distributed set of grid points. This is illustrated in the next example.

### 4.4 Quantile Example

Note that (1.1.2) provides a way to calculate the inverse of $G(x)$ if $F(x) = x$; that is, if we let $f \sim \text{Unif}[A,B]$. The numerical scheme (N-E) thus provides a way to compute the quantile function of any probability distribution that is supported on a bounded interval and stays away from zero.

**Example 4.4.1.**

\[
\begin{align*}
  f(x) &= 1, \\
  g(x) &= \frac{e^{\kappa \cos(x)}}{2\pi I_0(\kappa)}
\end{align*}
\]

The function $g$ is an example of a von Mises distribution with $\mu = 0$ and $\kappa = 1$. Recall that $I_0$ is the modified Bessel function of the first kind and of order zero. In this case $f : [0,1] \to \mathbb{R}$ and $g : [-\pi, \pi] \to \mathbb{R}$ with $u_0 = \pi(x^2 - x)$.

![Figure 4: Graphs 4.4.1](image)

We note that this method of computing the quantile function allows to calculate $G^{-1}$ on a uniform grid. Inverting a grid function can result in a non-uniform grid for the numerical inverse, and if $G$ has large derivative, then the grid points of the domain of $G^{-1}$ will be concentrated along the points where the derivative of $G$ is large. The optimal transport method thus provides better resolution in this case, though it is computationally much slower.

### 4.5 Mapping Functions That Are Close To Zero

**Example 4.5.1.**

\[
\begin{align*}
  f(x) &= \frac{9}{20} x + \frac{1}{2}, \\
  g(x) &= \frac{1}{2}
\end{align*}
\]

This is a case where our initial mass distribution, $f(x)$, is quite close to 0. Although the theory implies that any smooth function bounded away from 0 will work for $f$ and $g$, cases such as this prove to be more
computationally difficult when using numerical methods. Even so, when the bounds on the error conditions are implemented correctly the finite difference scheme is able to converge within tolerance to the real solution of (O-T).

Figure 5: Graphs of \( \nabla^2 U \) Over Time

(a) \( \nabla^2 U \) Over Time

(b) \( \Delta^2 U \) Over Time

\[ \epsilon = 0.01, \quad \max_j |T(x_j) - \nabla^2 U| \leq 0.02 \]

| Tolerance | Iterations | \( t_{total} \) | CPU Time (s) |
|-----------|------------|----------------|-------------|
| 0.1       | 263683     | 0.3087         | 280         |
| 0.01      | 922803     | 1.0806         | 975         |
| 0.001     | 1568238    | 1.8364         | 1653        |
| 0.0001    | 2212321    | 2.59074        | 2326        |

Table 4: Numerics for Near Zero Function

Though the numerics show many interesting examples the computational complexity makes this scheme difficult to use for some choices of mass distribution. Consider the more general case of (4.5.1); \( f: [-1, 1] \to \mathbb{R} \) such that \( f(x) = \beta x + \frac{1}{2} \). If \( \beta \) is very close to \( \frac{1}{2} \) then \( \min f \) is very close to zero. It follows from Theorem 2.1 that \( \Delta t \) gets very close to zero. In practice this has meant millions of iterations to reach convergence and a very slow program.

4.6 Mapping Piecewise Functions

In this section we discuss results involving mass distributions that are not guaranteed by (Parabolic O-T) to converge to the solution of (O-T). Yet, experimentally, with our finite difference scheme, we were able to show for some of these examples that the numerical solution can approach a desired tolerance and hence is close to the solution of (O-T). The following example uses a piecewise constant function to show this.

Example 4.6.1.

\[
f(x) = \begin{cases} 
0.3, & \text{if } x \leq -0.5 \\
0.6, & \text{if } -0.5 < x \leq 0 \\
0.2, & \text{if } 0 < x \leq 0.5 \\
0.9, & \text{if } x > 0.5
\end{cases}
\]

\[ g(x) = \frac{1}{2} \]
We were also motivated to test whether a piecewise function would be able to converge within tolerance when mapped to another piecewise function, with discontinuities at different points. Furthermore, we wanted to see the effects of functions that were not piecewise constant. This led us to test the following example:

Example 4.6.2.

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{1}{2} \log(x+2) + \frac{11}{12} - 2 \log(2), & \text{if } x \leq 0 \\
    \frac{1}{4}x^2 + \frac{1}{3}, & \text{if } x > 0
  \end{cases}, \\
  g(x) &= \begin{cases} 
    \frac{3}{10}x + \frac{7}{10}, & \text{if } x \leq -\frac{1}{3} \\
    \frac{1}{2}, & \text{if } -\frac{1}{3} < x \leq \frac{1}{3} \\
    \frac{3}{10}x + \frac{7}{10}, & \text{if } \frac{1}{3} < x
  \end{cases}
\end{align*}
\]

Table 5: Numerics for Piecewise Function

| Tolerance | Iterations | \(t_{total}\) | CPU Time (s) |
|-----------|------------|---------------|--------------|
| 0.1       | 31325      | 0.1088        | 38.47        |
| 0.01      | 327070     | 1.1356        | 400          |
| 0.001     | 672739     | 2.3356        | 821          |

We were also motivated to test whether a piecewise function would be able to converge within tolerance when mapped to another piecewise function, with discontinuities at different points. Furthermore, we wanted to see the effects of functions that were not piecewise constant. This led us to test the following example:

Example 4.6.2.

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{1}{2} \log(x+2) + \frac{11}{12} - 2 \log(2), & \text{if } x \leq 0 \\
    \frac{1}{4}x^2 + \frac{1}{3}, & \text{if } x > 0
  \end{cases}, \\
  g(x) &= \begin{cases} 
    \frac{3}{10}x + \frac{7}{10}, & \text{if } x \leq -\frac{1}{3} \\
    \frac{1}{2}, & \text{if } -\frac{1}{3} < x \leq \frac{1}{3} \\
    \frac{3}{10}x + \frac{7}{10}, & \text{if } \frac{1}{3} < x
  \end{cases}
\end{align*}
\]
Table 6: Numerics for Piecewise Function

| Tolerance | Iterations | $t_{total}$ | Numerical | Analytical |
|-----------|------------|-------------|------------|------------|
| 0.1       | 1          | 7.616e-06   | 0.1283     | 0.0709     |
| 0.01      | 64445      | 0.4909      | 355        | 71.4       |
| 0.001     | 184774     | 1.408       | 1166       | 202        |

From this we were able to see that our scheme seems to also converge within tolerance even when given a piecewise function for both $f$ and $g$. Although only the points of discontinuity in $f$ are seen in $\nabla^n U$, we see all points of discontinuity in both $f$ and $g$ appear in the graph of $\Delta^n U$. This aligns with the expectation that $\Delta^n U$ approximates $u''$. We observed that piecewise functions tend to take more computational time to reach tolerance compared against smooth cases with similar upper and lower bounds on $f$ and $g$. Even so, experimentally we found that closeness to zero had more of an effect on computational time.

Standard numerical integrators can have difficulties integrating discontinuous functions accurately and efficiently. Therefore, in cases involving piecewise functions it may be necessary to alter the error tolerance methods. One solution may be to implement function for the exact integral, calculated analytically if possible. Another would be to implement a specialized numerical integrator capable of handling piecewise functions. Both methods can also significantly improve computational time. See Table 6 for CPU time differences of the standard numerical integrator and exact analytical integrator for 4.6.2.

Conclusion and Outlook

We have shown error bounds on the finite difference scheme for the 1-D parabolic optimal transport problem and provided relevant numerical examples. Our research was only carried out for one spatial dimension. Further research is necessary to devise robust numerical methods for the optimal transport problem in two dimensions and higher. Additional work needs to be done to understand why piecewise functions used in the (Parabolic O-T) are able to converge to (O-T) and proven mathematically.

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A Derivative Estimates

By differentiating \([\text{Parabolic O-T}]\) in \(t\), we find that the function \(w(t, x) := v_t(t, x)\) solves the linearized equation

\[
\begin{cases}
w_t - \left( \frac{1}{v_x} \right) w_{xx} - \left( \frac{g'(v_x)}{g(v_x)} \right) w_x = 0 & \text{in } (0, \infty) \times (A, B), \\w_x(t, A) = 0, \quad w_x(t, B) = 0 & \text{for all } t \geq 0.
\end{cases}
\]

\([L-E]\)

A.1 Bounds on \(v_{xx}\)

Let \(w = v_t\) as above. Then \(w\) satisfies \([L-E]\). Since \(v_{xx} \geq 0\), the parabolic maximum principle and Hopf’s lemma implies

\[
\max_{x \in [A, B], t \geq 0} w(t, x) = \max_{x \in [A, B]} w(0, x), \quad \min_{x \in [A, B], t \geq 0} w(t, x) = \min_{x \in [A, B]} w(0, x).
\]

In terms of \(v_t\), this means

\[
\max_{x \in [A, B], t \geq 0} v_t(t, x) = \max_{x \in [A, B]} v_t(0, x), \quad \min_{x \in [A, B], t \geq 0} v_t(t, x) = \min_{x \in [A, B]} v_t(0, x).
\]

Evaluating \([\text{Parabolic O-T}]\) at \(t = 0\), we get

\[
v_t(0, x) = \log(v_{xx}(0, x)) - \log \left( \frac{f(x)}{g(v_x(0, x))} \right) = \log(u''_0(x)) - \log \left( \frac{f(x)}{g(v_x(0, x))} \right) = \log \left( u''_0(x) g(v_x(0, x)) / f(x) \right).
\]

Consequently,

\[
\min_{x \in [A, B]} u''_0(x) g(v_x(0, x)) / f(x) \leq e^{v_t(0, x)} \leq \max_{x \in [A, B]} u''_0(x) g(v_x(0, x)) / f(x).
\]

In particular,

\[
\frac{\min_{x \in [A, B]} u''_0(x) \min_{y \in [C, D]} g(y)}{\max_{x \in [A, B]} f(x)} \leq \min_{x \in [A, B], t \geq 0} e^{v_t(t, x)} \leq \max_{x \in [A, B], t \geq 0} e^{v_t(t, x)} \leq \frac{\max_{x \in [A, B]} u''_0(x) \max_{y \in [C, D]} g(y)}{\min_{x \in [A, B]} f(x)}.
\]

Since \([\text{Parabolic O-T}]\) implies

\[
v_{xx}(t, x) = \frac{e^{v_t(t, x)} f(x)}{g(v_x)} \quad \text{for all } (t, x) \in (0, \infty) \times (-1, 1),
\]

we conclude that

\[
\min_{x \in [A, B]} u''_0(x) \left( \frac{\min_{x \in [A, B]} f(x)}{\max_{x \in [A, B]} f(x)} \right) \left( \frac{\min_{y \in [C, D]} g(y)}{\max_{y \in [C, D]} g(y)} \right) \leq v_{xx}(t, x) \leq \max_{x \in [A, B]} u''_0(x) \left( \frac{\max_{x \in [A, B]} f(x)}{\min_{x \in [A, B]} f(x)} \right) \left( \frac{\max_{y \in [C, D]} g(y)}{\min_{y \in [C, D]} g(y)} \right).
\]

(A.1.1)
A.2 Bounds on $v_{xxx}$

We let $w := v_t$ and $\phi := v_{xxx}$. Differentiating \textit{(Parabolic O-T)} w.r.t $x$ gives us the relation

$$w_x = \frac{\phi_x}{\phi} - F(x) + G(v_x)\phi, \tag{A.2.1}$$

where $F(x) := \frac{f_t(x)}{g(x)}$ and $G(y) := \frac{g'(y)}{y}$. Since $\phi$ is uniformly bounded, it follows that an estimate for $w_x$ yields an estimate for $\phi_x = v_{xxx}$ under appropriate assumptions on $f$ and $g$.

Recall the linearized equation \textit{[L-E]}

\begin{align*}
L(w) &:= w_t - \phi^{-1}w_{xxx} = G(v_x)w_x \quad \text{in } (0, \infty) \times (A, B), \\
w_x(t, A) &= 0, \quad w_x(t, B) = 0 \quad \text{for all } t \geq 0.
\end{align*}

Differentiating the equation $L(w) = 0$ w.r.t $x$ gives

$$L(w_x) = -\frac{\phi_x w_{xx}}{\phi^2} + G'(v_x)\phi w_x + G(v_x)w_{xx}$$

$$= -w_{xxx}(w_x + F - G(v_x)\phi) + G'(v_x)\phi w_x + G(v_x)w_{xx}$$

$$= \left(2G(v_x) - \frac{(w_x + F)}{\phi}\right)w_{xx} + G'(v_x)\phi w_x.$$

Consider the auxiliary function

$$\eta = \psi_1(w_x) + \psi_2(w),$$

where $\psi_1, \psi_2$ are functions to be determined. We then have

- $\eta_t = \psi'_1(w_x)w_{xt} + \psi'_2(w)w_t$
- $\eta_x = \psi'_1(w_x)w_{xx} + \psi'_2(w)w_x,$
- $\eta_{xx} = \psi''_1(w_x)w_{xx}^2 + \psi'_1(w_x)w_{xxx} + \psi''_1(w)w_x^2 + \psi'_2(w)w_{xx}.$

Consequently,

$$L(\eta) = \eta_t - \phi^{-1}\eta_{xx}$$

$$= \psi'_1(w_x)w_{xt} + \psi'_2(w)w_t - \frac{1}{\phi}(\psi''_1(w_x)w_{xx}^2 + \psi'_1(w_x)w_{xxx} + \psi''_1(w)w_x^2 + \psi'_2(w)w_{xx})$$

$$= \psi'_1(w_x)L(w_x) + \psi'_2(w)L(w) - \frac{1}{\phi}(\psi''_1(w_x)w_{xx}^2 + \psi''_2(w)w_x^2)$$

$$= \psi'_1(w_x)\left(2G(v_x) - \frac{(w_x + F)}{\phi}\right)w_{xx} + (\psi'_1(w_x)G'(v_x)\phi + \psi'_2(w)G(v_x))w_x - \frac{1}{\phi}(\psi''_1(w_x)w_{xx}^2 + \psi''_2(w)w_x^2).$$

Suppose now that $\eta$ attains a maximum value at a point $(t_0, x_0) \in [0, \infty) \times [A, B]$. We assume $\psi_1$ is increasing and satisfies $\lim_{x \to 0} \psi_1(s) = -\infty$, and that $\psi_2$ is bounded on compact sets.

**Case 1:** $t_0 \geq 0$, $x_0 = A$ or $B$.

In this case, since $w$ is uniformly bounded and $w_x(t, A) = w_x(t, B) = 0$, it follows that $\lim_{x \to A+} \eta(t, x) = \lim_{x \to B-} \eta(t, x) = -\infty$.

**Case 2:** $t_0 > 0$, $x_0 \in (A, B)$.

We have $\eta_x(t_0, x_0) = 0$ and $L(\eta) \geq 0$ at $(t_0, x_0)$. This implies

$$\psi'_1(w_{xx})w_{xx} = -\psi'_2(w)w_x \quad \text{at } (t_0, x_0).$$
Substituting this into the equation for $L(\eta)$ yields
\[
0 \leq \left[ \psi_x''(w) \left( \frac{(w_x + F)}{\phi} - G(v_x) \right) + \psi_x'(w_x)G'(v_x) \phi \right] w_x - \frac{1}{\phi} \left( \psi_x''(w) \frac{\psi_x'(w)^2}{\psi_x'(w_x)^2} + \psi_x''(w) \right) w_x^2.
\]
We now choose
\[
\psi_1(s) = \frac{1}{2} \log(s^2), \quad \psi_2(s) = \alpha s, \quad \alpha \text{ constant}.
\]
Then since
\[
\psi_x'(s) = \frac{1}{s}, \quad \psi_x''(s) = -\frac{1}{s^2}, \quad \psi_2'(s) = \alpha, \quad \psi_2''(s) = 0,
\]
we find that
\[
0 \leq \left[ \alpha \left( \frac{(w_x + F)}{\phi} - G(v_x) \right) + \frac{1}{w_x}G'(v_x) \phi \right] w_x - \frac{\alpha^2 w_x^2}{\phi}.
\]
Rearranging terms, we get
\[
0 \leq \alpha(1 + \alpha)w_x^2 + \alpha (F - G(v_x)\phi)w_x + G'(v_x)\phi^2.
\]
Letting $\alpha = -\frac{1}{2}$ yields
\[
w_x^2 + 2(F - G(v_x)\phi)w_x - 4G'(v_x)\phi^2 \leq 0 \quad \text{at} \quad (t_0, x_0).
\]
This implies
\[
(w_x + F - G(v_x)\phi)^2 \leq (F - G(v_x)\phi)^2 + 4|G'(v_x)|\phi^2.
\]
Consequently,
\[
|w_x(t_0, x_0)| \leq |F - G(v_x)\phi| + \sqrt{(F - G(v_x)\phi)^2 + 4|G'(v_x)|\phi^2} \leq C_1(u_0, f, g).
\]
Since $\eta(t, x) \leq \eta(x_0, t_0)$, we have
\[
\log(|w_x(t, x)|) \leq \log(|w_x(t_0, x_0)|) + \frac{1}{2} |w(t, x) - w(t_0, x_0)| \leq \log(C_1(u_0, f, g)) + \max_{x \in [A, B]} |v_t(0, x)|.
\]
Exponentiating this gives
\[
|w_x(t, x)| \leq C_1(u_0, f, g)e^{\max_{x \in [A, B]} |v_t(0, x)|} \quad \text{for all} \quad (x, t) \in [A, B] \times [0, \infty).
\]

**Case 3: $t_0 = 0, x_0 \in (A, B)$**

For any $t \geq 0$ and $x \in [A, B]$
\[
\eta(t, x) \leq \eta(0, x_0) = \psi_1(w_x(0, x_0)) + \psi_2(w(0, x_0))
\]
\[
= \psi_1 \left( \frac{w_0''(x_0)}{w_0'(x_0)} - F(x_0) + G(u_0(x_0))u_0''(x_0) \right) + \psi_2(v_t(0, x_0))
\]
\[
= \log \left( \left| \frac{w_0''(x_0)}{w_0'(x_0)} - F(x_0) + G(u_0(x_0))u_0''(x_0) \right| \right) - \frac{\psi_t(0, x_0)}{2}
\]
\[
\leq \log(C_2(u_0, f, g)) - \frac{\psi_t(0, x_0)}{2}.
\]
Consequently,
\[
|w_x(t, x)| \leq C_2(u_0, f, g)e^{\max_{x \in [A, B]} |v_t(0, x)|} \quad \text{for all} \quad (x, t) \in [A, B] \times [0, \infty).
\]

**A.3 Bounds on $v_{xxxx}$**

Let $z = w_x$. Differentiating the relation \([A.2.1]\) w.r.t $x$ shows that
\[
z_x = w_{xx} = \frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2} - F'(x) + G'(v_x)\phi^2 + G(v_x)\phi_x.
\]
Consequently, an estimate for $z_x$ combined with an estimate for $\phi_x = v_{xxxx}$ implies an estimate for $\phi_{xx} = v_{xxxx}$ under appropriate assumptions on $f$ and $g.$
We first bound $|z_x|$ on the boundary. Define the linear operator

$$\tilde{L} := \partial_t - \phi^{-1}\partial_{xx}^2 - \beta \partial_x$$

where $\beta := G(v_z) - \frac{\phi_x}{\phi}$. Let $\mu := (G(v_z))_x z$. Then $z$ satisfies the initial and boundary value problem

$$\begin{cases}
\tilde{L}(z) = \mu, \\
z(t, A) = z(t, B) = 0 \quad \text{for all } t \geq 0, \\
z(0, \cdot) = z_0.
\end{cases}$$

Consider the barrier function

$$\eta(x) := \gamma(e^{\alpha(x-B)} - 1).$$

where $\alpha, \gamma > 0$ to be determined. Notice that

$$\eta(B) = 0 \quad \text{and} \quad \eta(A) = \gamma(e^{-\alpha(B-A)} - 1) \leq 0.$$

Since

- $\eta_z = \gamma \alpha e^{\alpha(x-B)}$
- $\eta_{xx} = \gamma \alpha^2 e^{\alpha(x-B)}$

we have

$$\tilde{L}(\eta - z) = \tilde{L}(\eta) + \mu = -\phi^{-1} \gamma \alpha^2 e^{\alpha(x-B)} - \beta \gamma \alpha e^{\alpha(x-B)} + \mu = -\gamma \alpha e^{\alpha(x-B)}(\phi^{-1}\alpha + \beta) + \mu.$$

Let $\alpha > 0$ be chosen so that $\phi^{-1}\alpha + \beta \geq 1$. Then

$$-\gamma \alpha e^{\alpha(x-B)}(\phi^{-1}\alpha + \beta) \leq -\gamma \alpha e^{\alpha(A-B)}.$$

We can now choose $\gamma$ so that $\gamma \alpha e^{\alpha(A-B)} \geq \max |\mu|$ to get $\tilde{L}(\eta - z) \leq 0$.

We now show that $\eta \leq z$ on the parabolic boundary. First, we have $\eta(t, B) = 0 = z(t, B)$ and $\eta(t, A) \leq 0 = z(t, A)$ for all $t > 0$. Next, let $\psi(x) = \eta(0, x) - z_0(x)$. Then $\psi(B) = 0$ and $\psi'(x) = \gamma \alpha e^{\alpha(x-B)} - z'_0(x)$. By Taylor’s theorem, for each $x \in [A, B]$ there exists $\xi_x \in (x, B)$ such that

$$\psi(x) = \psi(B) + \psi'(\xi_x)(x - B) = (\gamma \alpha e^{\alpha(\xi_x - B)} - z'_0(\xi_x))(x - B).$$

Now since $e^{\alpha(x-B)} \geq e^{\alpha(A-B)}$ for all $x \in [A, B]$, we have

$$\gamma \alpha e^{\alpha(\xi_x - B)} - z'_0(\xi_x) \geq \gamma \alpha e^{\alpha(A-B)} - \max |z'_0| \geq 0 \quad \text{if } \gamma = \alpha^{-1} e^{\alpha(B-A)} \max |z'_0|.$$

This implies $(\gamma \alpha e^{\alpha(\xi_x - B)} - z'_0(\xi_x))(x - B) \leq 0$ and so $\phi(x) \leq 0$ for all $x \in [A, B]$.

We have thus shown that $\eta \leq z$ on the parabolic boundary and $\tilde{L}(\eta - z) \leq 0$. The parabolic maximum principle thus implies $\eta \leq z$ everywhere. In particular, for any $t \geq 0$, since $x - B \leq 0$, we have

$$\frac{z(t, x) - z(t, B)}{x - B} \leq \frac{\eta(t, x) - \eta(t, B)}{x - B} \rightarrow \gamma \alpha \quad \text{as } x \rightarrow B,$$

giving an upper bound on $z_x(t, B)$. The same argument with $z$ replaced by $-z$ give a lower bound of $-\gamma \alpha$, in particular

$$|z_x(t, B)| \leq \gamma \alpha, \quad \forall t \geq 0.$$

The argument works in a similar fashion for the endpoint $x = A$. 

A.3.1 Boundary Estimate
A.3.2 Interior Estimate

Recall that
\[
L(z) = \left(2G(v_x) - \frac{(z + F)}{\phi} \right) z_x + G'(v_x) \phi z.
\]

Differentiating this equation w.r.t. \(x\) gives
\[
L(z_x) = \left(2G(v_x) - \frac{(z + F)}{\phi} \right) z_{xx} + \left(3G'(v_x) \phi - \frac{(z_x + F')}{\phi} + \frac{(z + F) \phi_x}{\phi^2} \right) z_x + (G'(v_x) \phi)_z z.
\]

Consider the auxiliary function
\[
\eta = \psi_1(z_x) + \psi_2(z),
\]
where \(\psi_1, \psi_2\) are functions to be determined. Then as before, we have
\[
L(\eta) = \psi'_1(z_x)L(z_x) + \psi'_2(z)L(z) - \frac{1}{\phi} \left( \psi''_1(z_x) z_{xx}^2 + \psi''_2(z) z_x^2 \right)
\]
\[= \psi'_1(z_x) \left[ 2G(v_x) - \frac{(z + F)}{\phi} - \frac{\phi_x}{\phi^2} \right] z_{xx} + \left( 3G'(v_x) \phi - \frac{(z_x + F')}{\phi} + \frac{(z + F) \phi_x}{\phi^2} \right) z_x + (G'(v_x) \phi)_z z \]
\[+ \psi'_2(z) \left[ 2G(v_x) - \frac{(z + F)}{\phi} \right] z_x + (G'(v_x) \phi) z - \frac{1}{\phi} \left( \psi''_1(z_x) z_{xx}^2 + \psi''_2(z) z_x^2 \right).
\]

Suppose now that \(\eta\) attains a maximum value at a point \((t_0, x_0) \in (0, \infty) \times (A, B)\). We have \(\eta_0(t_0, x_0) = 0\) and \(L(\eta) \geq 0\) at \((t_0, x_0)\). This implies
\[
\psi'_1(z_x) z_{xx} = -\psi'_2(z) z_x \quad \text{at} \quad (t_0, x_0).
\]

Substituting into the equation for \(L(\eta)\) yields
\[
0 \leq \left( \frac{\psi'_1(z_x) z_{xx}}{\phi} \right) z_x + \psi'_1(z_x) \left[ 3G'(v_x) \phi - \frac{(z_x + F')}{\phi} + \frac{(z + F) \phi_x}{\phi^2} \right] z_x + (G'(v_x) \phi)_z z
\]
\[+ \psi'_2(z) G'(v_x) \phi z - \frac{1}{\phi} \left( \psi''_1(z_x) \psi'_2(z) z_{xx}^2 + \psi''_2(z) z_x^2 \right).
\]

We now choose
\[
\psi_1(s) = \frac{1}{2} \log(s^2), \quad \psi_2(s) = \frac{\alpha s^2}{2}, \quad \alpha \text{ constant.}
\]

Then since
\[
\psi'_1(s) = \frac{1}{s}, \quad \psi''_1(s) = -\frac{1}{s^2}, \quad \psi'_2(s) = \alpha s, \quad \psi''_2(s) = \alpha,
\]
we have
\[
0 \leq \left( \frac{\alpha z \phi_x}{\phi^2} \right) z_x + 3G'(v_x) - \frac{(z_x + F')}{\phi} + \frac{(z + F) \phi_x}{\phi^2} + (G'(v_x) \phi)_x \left( \frac{z}{z_x} \right) + \alpha G'(v_x) \phi z^2 - \frac{\alpha}{\phi} (1 - \alpha z^2) z_x^2.
\]

Therefore,
\[
\alpha (1 - \alpha z^2) z_x^2 + \left( 1 - \frac{\alpha z \phi_x}{\phi} \right) z_x \leq \frac{(z + F) \phi_x}{\phi} - F' + 3 \phi G'(v_x) + \phi (G'(v_x) \phi)_x \left( \frac{z}{z_x} \right) + \alpha G'(v_x) \phi z^2.
\]

Let \(M := \max |v_x|\). Since \(|z| \leq M\), if we choose \(\alpha = \frac{1}{2M^2}\), then \(1 - \alpha z^2 \geq 1 - \alpha M^2 = \frac{1}{4}\). Consequently,
\[
\frac{1}{4M^2} z_x^2 + \left( 1 - \frac{z \phi_x}{2M^2 \phi} \right) z_x \leq \frac{(z + F) \phi_x}{\phi} - F' + 3 \phi G'(v_x) + \phi (G'(v_x) \phi)_x \left( \frac{z}{z_x} \right) + \frac{G'(v_x) \phi z^2}{2M^2}.
\]

Multiplying through by \(4M^2\) and then completing the square gives us
\[
\left( z_x + 1 - \frac{z \phi_x}{\phi} \right)^2 \leq 4M^2 \left[ \frac{(z + F) \phi_x}{\phi} - F' + 3 \phi G'(v_x) + \phi (G'(v_x) \phi)_x \left( \frac{z}{z_x} \right) + \frac{G'(v_x) \phi z^2}{2M^2} \right] + \left( 1 - \frac{z \phi_x}{\phi} \right)^2.
\]

If \(|z_x(t_0, x_0)| \leq 1\), then we have for any \((t, x) \in (0, \infty) \times (A, B)\)
\[
\eta(t, x) = \frac{1}{2} \log(z_x(t, x)^2) + \frac{z(t, x)^2}{4M^2} \leq \frac{1}{2} \log(z_x(t_0, x_0)^2) + \frac{z(t_0, x_0)^2}{4M^2} \leq \frac{1}{4}.
\]
This, in turn, implies $|z_x(t, x)| \leq e^{\frac{1}{2}}$ for all $(t, x) \in (0, \infty) \times (A, B)$. Therefore, we may assume $|z_x(t_0, x_0)| \geq 1$. This implies

$$\left( z_x + 1 - \frac{z\phi_x}{\phi} \right)^2 \leq 4M^2 \left[ \frac{|(z + F)\phi_x|}{\phi} + |F'| + 3\phi|G'(v_x)| + \phi|(G'(v_x)\phi)_x| + \frac{|G'(v_x)|\phi^2}{2} \right] + \left( 1 - \frac{z\phi_x}{\phi} \right)^2.$$ 

We conclude that

$$|z_x| \leq \left| 1 - \frac{z\phi_x}{\phi} \right| + \sqrt{4M^2 \left[ \frac{|(z + F)\phi_x|}{\phi} + |F'| + 3\phi|G'(v_x)| + \phi|(G'(v_x)\phi)_x| + \frac{|G'(v_x)|\phi^2}{2} \right] + \left( 1 - \frac{z\phi_x}{\phi} \right)^2}.$$