A dynamic scheme basing on equation for T-matrix momentum transfer spectral density and integral representation for Jost function is proposed for local Dirac Hamiltonians in arbitrary N-dimension spaces and for Schrödinger one with singular or nonlocal generalized Yukawa-type potentials.

A generalization of the off-shell-Jost function method for that Hamiltonians and universal renormalization procedure of Jost function calculation for singular and nonlocal potentials is proposed.

1 Introduction

It is well known, that determinant
\[ d(W) = \det \left[ G_0(W)G_V^{-1}(W) \right] = \det \left[ I - G_0(W)V \right] = \det \left[ I + G_V(W)V \right]^{-1}, \]
with Green function (resolvent) \( G_V(W) = [W - H_V]^{-1} \), accumulate all observable information about spectra of the stationary Hamiltonian \( H_V = H_0 + V \) in most economical form [1]:
\[ d(W) = \prod_{n=1}^{n_{\text{max}}} \left( 1 - \frac{W_n}{W} \right) \exp \left\{ -\frac{1}{\pi} \int_0^\infty \frac{d\varepsilon}{\varepsilon} \frac{\delta(\varepsilon)}{\varepsilon - W} \right\} \]
which makes it very convenient for solving both direct and inverse scattering problems [2, 3, 4], and for finding a different sums on spectra \( H_V \) [5]. It arise also in one-loop calculations for different quantum effects in external fields \( V(x) \) or in the semiclassical quantization of field theory near nontrivial classical solutions [6, 7, 8]. However, derivation of this determinant usually imply solution of two different eigenvalue problems for finding characteristics of discrete \( W_n \) and continuous \( \delta(\varepsilon) \) spectra separately. That makes their calculation and utilization much more complicated. This circumstance stimulates search of another ways for construction determinant does not requiring any information about \( H_V \) eigenvalues.

On the other hand, in the case of spherically symmetrical Hamiltonian \( V(\vec{x}) = V(r) \), using Green function’s partial expansion onto irreducible representations of rotation group SO(N), for instance, for Dirac operator:
where: \( \vec{x} = r\vec{n} \); \( \vec{y} = y\vec{\omega} \), and \( \Pi_{\kappa \xi}(\vec{n},\vec{\omega}) \) is projector onto subspace with fixed orbital \( l^{(N)}_\xi \) and total \( J_N \) angular momentum (see Appendix); one can formally factorize \( d(W) \) into infinite product:

\[
\mathbf{d}^{(N)}_{Dr}(W\vec{z}(ib)) = \prod_{J_N=\lambda_N}^{\infty} \prod_{\xi=\pm 1} \left[ F_{\kappa \xi}(b) \right]^{\Delta(N,J_N)}
\]

where dimension of the SO(N)-representation for this case is \(^1\):

\[
Tr\{\Pi_{\kappa \xi}\} = \Delta(N,J_N) = 2^{((N-1)/2)} \frac{(J_N + \lambda_N)!}{(J_N - \lambda_N)!(N-2)!};
\]

and the following notations are accepted hereafter:

\[
a_N = \frac{1}{2}(3 - N); \quad \lambda_N = \frac{1}{2} - a_N = \frac{N}{2} - 1; \quad \kappa \equiv \kappa_\xi = \xi \left(J_N + \frac{1}{2}\right);
\]

\[
L_\xi = J_N + \frac{\xi}{2} = l^{(N)}_\xi - a_N; \quad \xi = \pm 1.
\]

with the numbers \( l^{(N)}_\xi = 0, 1, 2, \ldots \), and \( J_N = \lambda_N, \lambda_N + 1, \lambda_N + 2, \ldots \) defining eigenvalues of squared orbital and squared total angular momentum respectively \(^9\) (see Appendix)

\[
\frac{1}{2}(L \cdot L) \Rightarrow l^{(N)}_\xi \left(l^{(N)}_\xi + 2\lambda_N\right); \quad \frac{1}{2}(J \cdot J) \Rightarrow \left(J_N + \frac{1}{2}\right)^2 - \frac{1}{8}(N-1)(N-2).
\]

The partial determinants or Jost functions \( F_{\kappa \xi}(b) \) are defined by the same formulae \(^1\),\(^2\) with partial Green function which matrix elements are \( G_{\kappa \xi}^{ij}(b;r,y) \), and with scattering phase \( \delta_{\kappa \xi}(\varepsilon) \) \(^10\) respectively. In contrast with \( d(W) \) \(^1\), they are well defined for arbitrary local potential which is less singular at \( r = 0 \), than appropriate free Hamiltonian \( H_0 \), and disappear sufficiently fast at \( r \to \infty \) \(^2,3\).

There is still one problem on this way, of finding an integral representation determined a general form of Jost function’s \( l^{(N)}_\xi, J_N \) - dependence. Such representation may be useful both for field theoretical calculations mentioned above and for Regge phenomenology \(^12\). There was an attempt made in \(^11\) for Schrödinger case with \( N=3 \). It led to representation with two variable’s weight function which satisfies to complicate nonlinear integral equation and does not have any known physical meaning \(^2,11\).

A quite different integral representation for Jost function (matrix) was established recently in \(^13,14,15\) for the Dirac operator with \( N=3 \), and for the Schrödinger one in arbitrary N-dimension space and in model with N strongly coupled channels. It play a role, analogous to Froissart-Gribov representation for partial amplitudes, but define the Jost functions in all analytical region over complex variables \( J_S \) and \( b \) in terms of quadratures from half-off-shell T-matrix spectral density over momentum transfer, with energetic variables, analytically continued from the continuum to the bound state region, and provides a group-theoretical interpretation directly for the Jost function. Together with linear Volterra-type integral equation for the spectral density this representation forms a dynamic scheme, from

\(^1\)The minimal gamma-matrix representation is assumed.
which all Jost functions (matrices) are found via solution of one regular problem, which has nothing to do with eigenvalue one.

Present work gives a generalization of this scheme for a wide class of operators, including N-dimension regular Dirac operator, singular or nonlocal Schrodinger operators, and the last with relativistic corrections to potential.

2 Equation for spectral densities

The aim of this section is to derive equations for T-matrix spectral densities over momentum transfer, constituting the foundation for the dynamic scheme in question, and to elucidate their analytical properties.

We define a family of Dirac operators in $\mathbb{R}^N$:

$$H_0 = (\vec{\Gamma} \cdot \vec{P}) + \Gamma_0 m; \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\delta_\mu\nu; \quad \vec{P} = -i\vec{\nabla}_N; \quad (7)$$

$$a) H_V = H_0 + IV; \quad b) H_V = H_0 + \Gamma_0 V; \quad (8)$$

with local Yukawa-type potentials

$$V(r) = \frac{4\pi}{\Omega_N \pi^{N/2}} \int_{\mu_0}^{\infty} d\nu \Sigma(N)(\nu) \left(\frac{r}{2\nu}\right)^a \chi_a(\nu r), \quad (9)$$

or in momentum representation [16] $^2$:

$$<\vec{q}|V|\vec{p}> = \frac{2}{\pi \Omega_N} \int_{\mu_0}^{\infty} d\nu \frac{\Sigma(N)(\nu)}{[\nu^2 + (\vec{q} - \vec{p})^2]} + \text{(subtractions)}. \quad (10)$$

The normalization conditions are:

$$<\vec{q}|\vec{p}> = \delta_N(\vec{q} - \vec{p}); \quad <\vec{q}|\vec{p}> = \exp\left(-\frac{1}{2}i\pi a_N\right)e^{i(\vec{p} \cdot \vec{x})}(2\pi)^{-N/2};$$

and the following notations are used hereafter: $\Omega_N = 2\pi^{N/2}/\Gamma(N/2)$; $N \geq 2$; $\vec{q} = q\vec{r}$; $\vec{p} = p\vec{v}$;

$$\chi_l(\beta r) = (\frac{2}{\pi} \beta r)^{1/2} K_{l+\frac{1}{2}}(\beta r); \quad \chi_0(\beta r) = e^{-\beta r}; \quad (11)$$

where $K_m(z)$ are McDonald function [17]. Choosing for $\Gamma$-matrices the following representation [18]:

$$(\overline{\Gamma})_k = \Gamma_k = \begin{pmatrix} O & \sigma_k \\ \sigma_k^\dagger & O \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}; \quad (12)$$

where matrices $\sigma_k$ for $k, j = 1, 2, ... N$ satisfy to conditions:

$$\sigma_j \sigma_k^\dagger + \sigma_k \sigma_j^\dagger = \sigma_j^\dagger \sigma_k + \sigma_k^\dagger \sigma_j = 2\delta_{jk}, \quad (13)$$

we have a complete set of eigenfunctions for operator (7):

$$H_0(\vec{p}) \ u_\zeta(\vec{p}, [\lambda]) = w_\zeta(\vec{p}) \ u_\zeta(\vec{p}, [\lambda]);$$

$^2$Here subtractions lead to ultralocal terms $\Delta^n \delta_N(\vec{x})$ in (9) corresponding to regularization of the potential in the sense of distributions. For Dirac Hamiltonian such singular potential is unstable with respect to particle creation, and we assume the absence of subtractions for that case.
It is not difficult to see from (10), (20), that it possess spectral representation:

$$ w \equiv \sum_{[\lambda]} u_{\zeta}(\vec{p}, [\lambda]) \otimes u_{\zeta}^*(\vec{p}, [\lambda]) = \varepsilon(p) + \zeta H_0(\vec{p}) $$

$$ (u_{\zeta}^*(\vec{p}, [\mu]) \cdot u_{\zeta}(\vec{p}, [\lambda])) = 2 \varepsilon(p) \delta_{\zeta, \zeta} \delta_{[\mu],[\lambda]}, \quad (14) $$

where following definitions are used:

$$ \varepsilon(p) = \pm \sqrt{p^2 + m^2}; \quad w_{\zeta}(p) = \zeta \varepsilon(p); \quad W_{\zeta}(ib) = \overline{\varepsilon} \sqrt{m^2 - b^2}; \quad \zeta, \overline{\zeta} = \pm 1. \quad (15) $$

Spinors $w_{[\lambda]}(\vec{r})$ with quantum numbers $[\lambda]$ on group SO(N) realize its spinor representation of half dimension than $u_{\zeta}(\vec{p}, [\lambda])$, and satisfy to the following conditions:

$$ (w_{[\mu]}^*(\vec{r}) \cdot w_{[\lambda]}(\vec{r})) = \delta_{[\mu],[\lambda]}; \quad \sum_{[\lambda]} w_{[\lambda]}(\vec{r}) \otimes w_{[\lambda]}^*(\vec{r}) = \mathbf{1} \quad (16) $$

We consider also Schrodinger operators for N=3 with relativistic correction to potential $V(r)$ (here $\sigma_{1,2,3}$ are Pauli matrices):

$$ H_V = \mathbf{P}^2(2m)^{-1} + U(\vec{r}), \quad (17) $$

$$ U(\vec{r}) = V(r) - \frac{1}{2}(2m)^{-1} \left[ (\vec{\sigma} \cdot \vec{P}), \left[ (\vec{\sigma} \cdot \vec{P}), V(r) \right] \right], \quad (18) $$

and with nonlocal interaction:

$$ U(\vec{r}) = V_1(r) + (2m)^{-2}(\mathbf{P}^2 V_2(r) + V_2(r) \mathbf{P}^2). \quad (19) $$

Using definitions (14) and the Lippman-Schwinger (LS) equation

$$ \mathbf{T}(W) = \mathbf{V} + \mathbf{V} \mathbf{G}_0(W) \mathbf{T}(W), \quad (20) $$

with the help of free Green function's decomposition:

$$ \mathbf{G}_0(W_{\overline{\zeta}}; k) = (W_{\overline{\zeta}} + H_0(k)) \left[ (W_{\overline{\zeta}})^2 - k^2 - m^2 - i0 \right]^{-1} = \sum_{\zeta = \pm 1} \sum_{[\lambda]} \frac{u_{\zeta}(k, [\lambda]) \otimes u_{\zeta}^*(k, [\lambda])}{W_{\overline{\zeta}} - \zeta' \varepsilon(k) - i0}, $$

we define for Hamiltonian (7),(8) $T$-operator acting on spinors (16):

$$ w_{[\mu]}^*(\vec{r}) (q, \zeta' | \mathbf{T}(W_{\overline{\zeta}})| \vec{p}, \zeta) w_{[\lambda]}(\vec{r}) = u_{\zeta}^*(q, [\mu]) < q | \mathbf{T}(W_{\overline{\zeta}}) | \vec{p} > u_{\zeta}(\vec{p}, [\lambda]), \quad (21) $$

with symmetry properties:

$$ (q, \zeta' | \mathbf{T}(W_{\overline{\zeta}}) | \vec{p}, \zeta) = ( - q, \zeta' | \mathbf{T}(W_{\overline{\zeta}}) | - \vec{p}, \zeta) = \left( (\vec{p}, \zeta | \mathbf{T}(W_{\overline{\zeta}}) | q, \zeta') \right)^\dagger. \quad (22) $$

It is not difficult to see from (10),(20), that it possess spectral representation:

$$ (q, \zeta' | \mathbf{T}(W_{\overline{\zeta}}(ib)) | \vec{p}, \zeta) = \frac{1}{\pi \Omega_N m} \int_0^\infty \frac{d\nu}{\nu^2 + (q - \vec{p})^2} \cdot \left[ \begin{array}{c} \mathcal{D}(1) \zeta\zeta' (\nu; -ip, b^2, -iq) \overline{\zeta'} + (\vec{\sigma} \cdot \vec{\tau})(\vec{\sigma} \cdot \vec{\nu}) \end{array} \right. $$

$$ \left. \cdot \left( \mathcal{D}(2) \zeta\zeta' (\nu; -ip, b^2, -iq) \overline{\zeta'} + (\nu \cdot \vec{\tau}) \overline{\zeta'} \right) + \text{(subtractions)} \right]. \quad (23) $$

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where for the Born term:

\[ (N) \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \Rightarrow \Sigma^{(N)}(\nu) A^{(2)}(\zeta, q, p) = \Sigma^{(N)}(\nu) \cdot \]

\[ \cdot \left\{ \frac{\zeta}{1} \right\} \left[ (\varepsilon(q) + m/\zeta) (\varepsilon(p) + m/\zeta) \right]^{1/2} = \Sigma^{(N)}(\nu) A^{(2)}(\zeta, q, p) = \]

\[ = \Sigma^{(N)}(\nu) \zeta \left[ \frac{\varepsilon(p) + m/\zeta}{\varepsilon(q) + m/\zeta} \right]^{1/2} q \left\{ \left( \eta^{\zeta}(q) \right)^{-1} \right\} \eta^{\zeta}(p) \]

\[ \equiv \Sigma^{(N)}(\nu) \zeta \left[ \frac{\varepsilon(p) + m/\zeta}{\varepsilon(q) + m/\zeta} \right]^{1/2} q M_{\zeta}^{(2)}(\zeta, q, p); \]

\[ \eta^{\zeta}(p) \equiv (w^{\zeta}(p) - m)/p \equiv p/(w^{\zeta}(p) + m); \eta^{-\zeta}(p) = -(\eta^{\zeta}(p))^{-1}. \] (25)

It is convenient to pass to the quantities, depending from quantum numbers \( \zeta \), \( \zeta \)
only via sheet's indices of the functions \( w^{\zeta}(p) \), \( w^{-\zeta}(q) \) i.e. via branch indices
\( \zeta = \text{sgn}(\text{Re} w(p)) \) of analytic functions \( w(p) = (p^2 + \mu^2)\quad w(q) \). This may
be achieved by putting in accordance with (24) for \( i = 1, 2 \):

\[ (N) \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \Rightarrow \]

\[ = \zeta \left[ \frac{\varepsilon(p) + m/\zeta}{\varepsilon(q) + m/\zeta} \right]^{1/2} q \left\{ \left( \frac{\eta^{\zeta}(q)}{\eta^{\zeta}(p)} \right)^{(N)} \right\} \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \]

\[ = \zeta \left[ \frac{\varepsilon(q) + m/\zeta}{\varepsilon(p) + m/\zeta} \right]^{1/2} p \left\{ \left( \frac{\eta^{-\zeta}(q)}{\eta^{-\zeta}(p)} \right)^{(N)} \right\} \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \] (27)

Then the symmetry (22) takes the form: \( i = 1, 2 \)

\[ (N) \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \Rightarrow \frac{\eta^{\zeta}(q)}{\eta^{\zeta}(p)} \right\} \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \]

Now following to Fubini and Stroffolini [19] and to [13, 14] we calculate discontinuity over momentum transfer \( t = -(\bar{q} - \bar{p})^2 \) from both sides of LS equation (20) with the help of the arbitrary dimension's relation:

\[ \int \frac{d\Omega_N(\vec{u})}{|X - (\vec{p} \cdot \vec{u})||Y - (\vec{q} \cdot \vec{u})|} = \frac{(-1)^{1} 4 \pi \lambda^{1+1}}{2 \Gamma(l + \lambda)} \Xi^{[M]}_l \left( \tilde{\tau} \frac{\partial}{\partial X} + \tilde{\nu} \frac{\partial}{\partial Y} \right) . \]

\[ \int_{Z_{+}(X,Y)}^{\infty} \frac{dZ}{Z(W(X,Y,Z))^{1/2}|Z - (\vec{p} \cdot \vec{v})|} \left( \frac{W(X,Y,Z)}{Z^2 - 1} \right)^{l-\alpha_N} ; \] (28)

\[ W(X,Y,Z) = X^2 + Y^2 + Z^2 - 2XYZ - 1; \]
\[ Z_{+}(X,Y) = XY \pm [(X^2 - 1)(Y^2 - 1)]^{1/2}; \]

for spherical function \( \Xi^{[M]}_l(\vec{u}) \) on group SO(N) [20], and came to the following system of equation for T-matrix spectral density over momentum transfer for operator (8a):

\[ (N) \frac{D}{\zeta} (\mu; -ip, b^2, -iq) \Rightarrow \Sigma^{(N)}(\nu) M_{\zeta}^{(2)}(\zeta, q, p) = \] (29)
\[
\begin{align*}
&= \frac{(N-2)}{2m\pi} \left[ \frac{\Delta(q^2, p^2, -\nu^2)}{\nu^2} \right]^{\alpha} \int_0^{\nu} d\gamma \Sigma(N)(\gamma) \int_0^{\omega_{-}} d\mu \int_{\omega_{-}}^{\omega_{+}} d\gamma (\nu, \gamma; q, p) \, dk^2, \\
\cdot \left[ (\omega^+ - k^2)(k^2 - \omega_{-}) \right]^{-\frac{3}{2} - \alpha} k \sum_{\zeta = \pm 1} g_{\zeta} (-ik; b) \tilde{c}.
\end{align*}
\]

which is independent from subtractions in (23). Here we put \( \mu_0 = 0 \) for simplification of the formulas and the following notations are accepted hereafter:

\[
g^{\zeta'}(-ik; b) = \left( \frac{-1}{(k^2 + b^2)} \right)^{1/2} \left[ 1 + \frac{W^{\zeta'}(ib)}{w^{\zeta'}(k)} \right]^{-1};
\]

\[
X_{\gamma} = \frac{q^2 + k^2 + \gamma^2}{2k}; \quad Y_{\mu} = \frac{p^2 + k^2 + \mu^2}{2p}; \quad Z_{\nu} = Z(qp|\nu) = \frac{q^2 + k^2 + \nu^2}{2qp};
\]

\[
2\nu^2 \omega^+(\nu; \mu, \gamma; q, p) = \nu^2(\nu^2 - \mu^2 - \gamma^2) + q^2(\nu^2 + \mu^2 - \gamma^2) + p^2(\nu^2 - \mu^2 + \gamma^2) \pm \left[ \Delta(\nu^2, \mu^2, \gamma^2) \Delta(q^2, p^2, -\nu^2) \right]^{1/2};
\]

\[
\Delta(a, b, c) = (a + b - c)^2 - 4ab.
\]

For case (8b) we must change \( M^{(2)} \Rightarrow -M^{(2)} \). For the same function \( V(r) \) with different dimensions of \( r^{(N)} \) and \( r^{(D)} \) its solutions are connected by simple Weyl's integral transformation: \( i = 1, 2 \)

\[
\begin{align*}
&= \frac{\Gamma(N/2)}{\Gamma(D/2)^{2} \nu} \left( \frac{d}{dr^2} \right)^n \int_0^{\nu} d\gamma \gamma^{(\mu - \gamma^2)(a_d - a_n + n - 1)} \left( \frac{\nu^2 - \gamma^2}{\nu^2 - a_n} \right) D^{(n)}(\gamma; \cdots), \quad (31)
\end{align*}
\]

where integer number \( n \) is restricted only by convergence condition of this integral: \( n \geq \text{max}[(D - N)/2; 0] \). This transformation is identical with Schrodinger case, and may be checked by the same way [14].

For the Hamiltonian (18) the formulas may be simplified by choosing helicity representation for the spinors \( w(\vec{u}) : (\vec{\sigma} \cdot \vec{u})|w_{\lambda}(\vec{u}) \rangle = 2\lambda|w_{\lambda}(\vec{u}) \rangle \). Then, instead (24), we have:

\[
< \vec{q}, \mu| U| \vec{p}, \lambda > = D^{(1/2)}(\lambda \mu) \left[ 1 - \frac{q^2 + p^2 - 2\mu 2\lambda 2q p}{2(2m)^2} \right] \frac{1}{2m} < \vec{q}| V| \vec{p} >; \quad (32)
\]

where potential \( V(r) \) (10) for \( N=3 \) also is assumed to be regular (without subtractions). Separating in (20),(23) spin-rotation matrix

\[
D^{(1/2)}(\lambda \mu) = \langle \vec{w}_\mu | \vec{v} \rangle |w_{\lambda}(\vec{v}) \rangle,
\]

one can decompose the spectral density matrix onto the sum of two orthogonal projectors with coefficients \( D^{(1)}(\cdot\cdot) [13] \):

\[
D_{\mu \lambda}(\cdot\cdot) = D^{(1)}(\cdot\cdot) + 2\mu 2\lambda D^{(2)}(\cdot\cdot); \quad (33)
\]
Then discontinuity calculation like above give the same system like (29) for N=3 with changed normalization

\begin{align*}
\frac{q}{2m} D^{(N)}_{\zeta, \zeta'} (\nu; -ip, b^2, -iq) \Rightarrow D^{(i)}(\nu; \zeta, \zeta); \quad \frac{q}{2m} M^{(i)}(q, p) \rightarrow A^{(i)}(q, p);
\end{align*}

and with substitutions:

\begin{equation}
\sum_{\zeta'=\pm 1} g^{\nu}(\zeta; k^2, b^2) \rightarrow (k^2 + b^2)^{-1};
\end{equation}

\begin{align*}
A^{(1)}(q, p) &= 1 - \frac{1}{2} (q^2 + p^2)(2m)^{-2}; \quad A^{(2)}(q, p) = qp(2m)^{-2}.
\end{align*}

Spectral density’s equation for Hamiltonian (19) has more simple form [14] with substitution:

\begin{equation}
\Sigma^{(3)}(\nu) \Rightarrow \Sigma_1(\nu) + \Sigma_2(\nu)(q^2 + p^2)(2m)^{-2}
\end{equation}

and in particular case \( V_1 = V_2 \) appears from last system, if one put on them \( D^{(2)} = A^{(2)} = 0 \) eliminating all dependence from spin.

Let now shortly consider analytic properties of the spectral density over energetic variables \( q, p \). It may be shown [21], that due to volterrian property of eq.(29) providing convergence of its iteration serie, the spectral density possess analytic continuation to the domain [14]

\begin{equation}
p = i\varrho, \quad q = iu, \quad k = i\alpha; \quad \varrho > 0; \quad 0 < \nu < u - \varrho;
\end{equation}

\begin{equation}
\left\{ \begin{array}{l}
\Delta(q^2, p^2, -\nu^2) \Delta(u^2, \varrho^2, \nu^2) \frac{1}{2m} \Delta(u^2, \varrho^2, \nu^2) \frac{1}{2} \Delta(u^2, \varrho^2, \nu^2) \\
\omega_+^{(\nu, \mu, \gamma; q, p)} = e^{i\pi} \Lambda_+^{(\nu, \mu, \gamma; u, \varrho)};
\end{array} \right.
\end{equation}

where we put:

\begin{equation}
\Lambda_+^{(\nu, \mu, \gamma; u, \varrho)} = \Lambda_0^{(\nu, \mu, \gamma; u, \varrho)} \pm \frac{1}{2m} \Delta(u^2, \varrho^2, \gamma^2) \Delta(u^2, \varrho^2, \nu^2)^{1/2} ;
\end{equation}

\begin{align*}
2\nu^2 \Lambda_0^{(\nu, \mu, \gamma; u, \varrho)} &= \nu^2(\mu^2 + \gamma^2 - \nu^2) + u^2(\nu^2 + \mu^2 - \gamma^2) + \\
&+ \varrho^2(\nu^2 - \mu^2 + \gamma^2),
\end{align*}

and that continued functions satisfy to system (29) continued to this domain (see bellow).

### 3 Generalizations of the off-shell Jost function method for Dirac operator

Let us now turn to generalizations of the off-shell Jost functions method. Such preliminaries is necessary to establish the relation in question between Jost function and T-matrix momentum transfer spectral density. There are two ways to introduce off-shell Jost functions (OSJF). The first one derive it only for local potential from solution of nonhomogeneous radial Schrodinger (or Dirac) equations i.e. off-shell Jost solution (OSJS). The second one relate OSJF with half-off-shell partial amplitude. Both this ways are equivalent obviously for local nonsingular
potentials, successfully added each other for singular and nonlocal potentials. Although the ideas of this method is not new [22, 23], we outline here its main points in modified form, convenient for our aims to get its generalization on complex value of total angular momentum \( J_N \) and demonstrate its applicability for a wide class of operators.

We begin with the second way [14] introducing off-shell partial amplitudes by expansion of T-matrix (23):

\[
\langle \vec{q}, \vec{\zeta} \mid (N) \mid W^\zeta (ib) \rangle |\vec{p}, \zeta \rangle = -\frac{2(qp)^n}{\pi q} \zeta^{-} \left[ \frac{\varepsilon(q) + m\zeta^{-}}{\varepsilon(p) + m\zeta^{-}} \right]^{1/2} .
\]

\[
\cdot \sum_{J=\lambda_N}^{\infty} \sum_{k=\pm 1} \Pi_{\nu_k}(\vec{r}, \vec{\nu}) \cdot T_{\nu_k}^{\zeta} \zeta(q, p; b^2) \zeta.
\]

For sufficiently large value of \( J_N \) it possess a Froissart-Gribov integral representation:

\[
T_{\nu_k}^{\zeta} \zeta(q, p; b^2) \zeta = \frac{4\pi e^{-i\pi a}}{4m\Omega_N \pi a} \int_{\mu_0}^{\infty} d\nu \left[ Q_{L_\perp}^{(N)} (Z_{\nu}) \sum_{\zeta=\pm 1} T_{\nu_k}^{\zeta} \zeta(q, p; b^2) \zeta \right] + \left[ \Delta(q^2, p^2, -\nu^2) \right]^{-a/2},
\]

(\( Z_{\nu} \) is defined in (30)) which for Born term (see (24)) take place without restriction. The OSJF \( F_{\nu_k}^{\zeta} (q, -ik) \zeta \) is introduced as two variable’s function analytic in the domain

\[
(q, \zeta; k, \zeta': \text{Re} \; q > 0, q \not\in [m, +\infty), \zeta = \pm 1; |Im k| < \mu_0, \pm ik \not\in [m, +\infty), \zeta = \pm 1)
\]

which decompose the partial half-off-shell amplitude

\[
T_{\nu_k}^{\zeta} \zeta(q, k; b^2) \zeta \big|_{b=0+ik} = T_{\nu_k}^{\zeta} \zeta(\pm)(q, k)
\]

according to [22]:

\[
T_{\nu_k}^{\zeta} \zeta(\pm)(q, k) = \left( \frac{k}{q} \right)^{L_\perp} \left[ F_{\nu_k}^{\zeta} (iq, -ik) \zeta - F_{\nu_k}^{\zeta} (-iq, -ik) \zeta \right] \left[ 2iF_{\nu_k}^{\zeta} (\mp ik) \right]^{-1}.
\]

It means, that Jost function is simply related with OSJF:

\[
F_{\nu_k}^{\zeta} (\mp ik, -ik) \zeta = F_{\nu_k}^{\zeta} (\mp ik).
\]

However, inversion of the decomposition (42) now is not as straightforward as for Schrodinger case [14]. One can see, that all mentioned above properties of OSJF hold for the following ansatz:

\[
F_{\nu_k}^{\zeta} (q, -ik) \zeta = Z_{\nu_k}^{\zeta} (q^2, k^2) \zeta = F_{\nu_k}^{\zeta} (\mp ik) \frac{1}{\pi} \int_0^{\infty} ds^2 \left( \frac{s}{k} \right)^{L_\perp} .
\]

\[
\cdot \sum_{\zeta' = \pm 1} g_{\zeta'}^{\zeta'} (-is; q) \zeta' N_{\zeta'}^{\zeta'} (q, -is) \zeta' T_{\nu_k}^{\zeta} \zeta(\pm) (s, k);
\]

(44)
Substituting the partial LS-equation (which is a Fredholm-type equation) disappearing in difference (42) and satisfying to condition:

\[ Z_{\kappa\varepsilon}(-k^2, k^2) \bigg|_{loc} = 1. \]

Due to this condition the ansatz (44) for \( \varrho = \mp ik \) convert in accordance with (43) to general representation for Jost function. The last follows directly from abstract definition (1) with the help of known reasoning [24], using decomposition of partial Green function (3) into Volterrian and separable parts (see Appendix), and the relation for physical solution of radial Dirac equation (see below) which reads:

\( \kappa = \kappa_{\varepsilon} \),

\[ -\frac{V(r)}{\eta^V_{\kappa}(k)} \psi_{\kappa V}^{(\pm)\varepsilon}(k, r) = \frac{1}{\pi} \int_0^\infty ds^2 \sum_{\xi = \pm 1} \phi_{\kappa\varepsilon}^{(\xi)(s, r)} \frac{1}{2w^V_{\kappa } (s)} \eta^V_{\kappa \varepsilon} (s) T^V_{\kappa \varepsilon} (s, k). \]

Substituting the partial LS-equation (which is a Fredholm-type equation)

\[ T^\varphi_{\kappa \varepsilon} (q, k) - F^\varphi_{\kappa \varepsilon} (q, k) = -\frac{1}{\pi} \int_0^\infty ds^2 \sum_{\xi = \pm 1} g^\varphi (-is; \mp ik) T^\varphi_{\kappa \varepsilon} (s, k), \]

to the right hand side of ansatz (44), and using its particular form for \( \varrho = \mp ik \) (clf.(43)) in the first of appearing items, one has for this r.h.s. the expression:

\[ Z_{\kappa \varepsilon}(-k^2, k^2) H^\varphi_{\kappa \varepsilon} (q, k) - F^\varphi_{\kappa \varepsilon} (q, k) \frac{1}{\pi} \int_0^\infty ds^2 \left( \frac{s}{k} \right)^{L_{\xi}} \sum_{\xi = \pm 1} g^\varphi (-is; \mp ik) T^\varphi_{\kappa \varepsilon} (s, k); \]

where the auxiliary kernel is introduced:

\[ H^\varphi_{\kappa \varepsilon} (q, k) = \frac{1}{\pi} \int_0^\infty ds^2 \left( \frac{s}{k} \right)^{L_{\xi}} \sum_{\xi = \pm 1} g^\varphi (-is; q) N^\varphi_{\kappa \varepsilon} (-is) T^\varphi_{\kappa \varepsilon} (s, k). \]

The relation which following [21] from formula (70) (see below) for \( Re \ j > -1 - \alpha N \); \( T(u\varrho|\nu) = (u^2 + \varrho^2 - \nu^2) / 2u\varrho \):

\[ \int_0^\infty \frac{d\alpha}{\Delta(u^2, \alpha^2, \nu^2)^{\alpha/2} (\alpha^2 + k^2)} \left( \frac{u}{\alpha} \right)^j = \int_0^\infty \frac{d s^2}{2\pi k} \left[ \Delta(s^2, k^2, -\nu^2)^{\alpha/2} (s^2 + u^2)^j \right]; \]
and easily verifying formulae

$$
\sum_{\zeta' = \pm 1} \frac{1}{2} \left( 1 + \frac{w^\zeta(k)}{w^\zeta(s)} \right) \left( \frac{w^\zeta(s) \pm m}{W^\zeta(k) \pm m} \right)^{\zeta/(1-\zeta)} = 1;
$$

$$
\sum_{\zeta' = \pm 1} \frac{1}{2} \left( 1 + \frac{w^\zeta(k)}{w^\zeta(s)} \right) \left( \frac{w^{\zeta'}(s) \pm m}{W^{\zeta'}(k) \pm m} \right)^{\zeta'/(1-\zeta)} \left[ \eta^{\zeta'}(s) \right]^{\pm 1} = \left( \frac{k}{s} \right)^{\xi} \left[ \eta^\zeta(k) \right]^{\pm 1};
$$

(52)

allow to rewrite the auxiliary kernel (50) as:

$$
H^\zeta_{\alpha}(g, k) = \int_{\theta + \mu_0}^{\infty} d\alpha \left( \frac{\alpha}{\alpha} \right)^{L_{\alpha}} \sum_{\zeta' = \pm 1} \frac{1}{\eta^{\zeta'}(iu)} K_L(u, g) + \eta^{\zeta'}(iu) K_{L-}(u, g)
$$

(53)

where the new Volterrian kernels are introduced for the case (8a):

$$
K_j(u, g) = \frac{4\pi}{\Omega N^2} \int_{\mu_0}^{u-g} d\nu P_j(T(u\hat{g}\nu)) \frac{\Sigma(N)(\nu)}{\Delta(u^2, \hat{g}^2, \nu^2)^{\alpha/2}};
$$

(54)

$$
K^\zeta_{\alpha, \beta}(u, g) = \frac{iu}{2m} \left[ \frac{1}{\eta^{\zeta'}(iu)} K_L(u, g) + \eta^{\zeta'}(iu) K_{L-}(u, g) \right].
$$

(55)

Here for the case (8b) the second term has opposite sign (cf. remark after (30)) and the branch \( w^\zeta(p) \) takes value at \( p = i\theta + \theta > m > m : w^{\zeta}(i\theta) = i\zeta\sqrt{\theta^2 - m^2}. \) This choice is conventional and does not affect on sum over the sheets \( \zeta = \pm 1, \) for which kinematical cuts \( \pm \theta > m \) disappears. Substitution of the (53) and repeating use of ansatz (44) under the \( \alpha \)-integral, converts the relations (44), (49) to the following Volterra-type equation for OSJF:

$$
F^\zeta_{\alpha, \beta}(g, -ik) - Z^\zeta_{\alpha, \beta}(g^2, k^2 - g^2) = \int_{\theta + \mu_0}^{\infty} d\alpha \left( \frac{\alpha}{\alpha} \right)^{L_{\alpha}} \sum_{\zeta' = \pm 1} \frac{1}{\eta^{\zeta'}(iu)} F^\zeta_{\alpha, \beta}(u, -ik) - Z^\zeta_{\alpha, \beta}(a^2, k^2)^{\zeta} + N^\zeta_{\alpha, \beta}(a, -ik)
$$

(56)

A natural choice of OSJF’s normalization now is given by the relation:

$$
Z^\zeta_{\alpha, \beta}(g^2, k^2)^{\zeta} \bigg|_{\text{loki,nonsin.}} = N^\zeta_{\alpha, \beta}(g, -ik)^{\zeta},
$$

(57)

where function in the right hand side obviously satisfy to all conditions (41),(46) written out for the left one. It transforms the equation (56) for \( b = -ik \) to the following form:

$$
F^\zeta_{\alpha, \beta}(g, b) - N^\zeta_{\alpha, \beta}(g, b)^{\zeta} = \int_{\theta + \mu_0}^{\infty} d\alpha \left( \frac{\alpha}{\alpha} \right)^{L_{\alpha}} \sum_{\zeta' = \pm 1} \frac{1}{\eta^{\zeta'}(iu)} F^\zeta_{\alpha, \beta}(u, b)^{\zeta}. \]

(58)
Here the second line expresses solution of the first line’s equation via Volterrian resolvent kernel $a_{\kappa\xi}^\zeta(u, \varrho; b^2)^\zeta$ which therefor satisfy in its turn to Volterra equations:

$$a_{\kappa\xi}^\zeta(u, \varrho; b^2)^\zeta - K_{\kappa\xi}^\zeta(u, \varrho) = \int_{v+\mu_0}^{u-\mu_0} d\alpha \sum_{\zeta'=\pm 1} g_{\zeta'}(\alpha; b)^\zeta.$$ (59)

Here the second line expresses solution of the first line’s equation via Volterrian resolvent kernel $a_{\kappa\xi}^\zeta(u, \alpha; b^2)^\zeta$, $a_{\kappa\xi}^\zeta(u, \alpha; b^2)^\zeta = K_{\kappa\xi}^\zeta(u, \alpha; b^2)^\zeta$; compatible with the following symmetry properties (cmp.(27)):

$$a_{\kappa\xi}^\zeta(u, \varrho; b^2)^\zeta = \frac{\eta^{\prime}(i\varrho)}{\eta^{\prime}(i\varrho)} a_{\kappa\xi}^{\zeta'}(-\varrho, -u; b^2)^\zeta.$$ (60)

Now in the equation (59) it is possible to make an exact factorization of dependency from $J_\kappa$ which is the main observation for this work. Its form is prompted by the expression for the kernel (55) and reads:

$$a_{\kappa\xi}^\zeta(u, \varrho; b^2)^\zeta = \frac{4\pi}{\Omega_N \pi^{a/2}} \int_{\mu_0}^{u-\varrho} d\nu \left[ P_{\kappa\xi}^\zeta(T(u\varrho|\nu)) \right] \frac{(N)}{\xi} \frac{(1)}{\xi} \frac{(1)}{\xi} (\nu; \varrho, b^2, u)^\zeta +$$

$$+ P_{\kappa\xi}^\zeta(T(u\varrho|\nu)) \frac{(2)}{\xi} \frac{(1)}{\xi} \frac{(2)}{\xi} (\nu; \varrho, b^2, u)^\zeta \left[ \Delta(u^2, \varrho^2, \nu^2) \right]^{-a/2}.$$ (61)

It may be checked by induction with the help of relations for arbitrary integrable function $\mathcal{H}(\alpha)$, for arbitrary complex $l$, and $Re a < 1/2$ [14, 21):

$$\int_{v+\mu}^{u-\gamma} d\alpha \frac{P_{\alpha}^\zeta(X(u\alpha|\gamma))}{\Delta(\alpha^2, \varrho^2, \gamma^2)^{a/2}} = \frac{P_{\alpha}^\zeta(Y(\nu\alpha|\mu))}{\Delta(\alpha^2, \varrho^2, \mu^2)^{a/2}} \mathcal{H}(\alpha) =$$

$$= \int_{\gamma+\mu}^{u-\gamma} d\nu P_{\nu}^\zeta(T(u\nu|\gamma)) \left[ \Delta(u^2, \varrho^2, \nu^2) \right]^{a/2} \frac{\Gamma(\lambda)\sqrt{\pi}}{(\nu^2)^{a/2}} \int_{\Lambda^+}^{\Lambda^-} d\alpha^2 \mathcal{H}(\alpha) \cdot$$

$$\cdot \left( (\Lambda^+ - \alpha^2)^{(a/2)} - (\Lambda^- - \alpha^2)^{(a/2)} \right) \left( (\Lambda^+ - \alpha^2)^{a/2} - (\Lambda^- - \alpha^2)^{a/2} \right) \left( (\Lambda^+ - \alpha^2)^{-a/2} - (\Lambda^- - \alpha^2)^{-a/2} \right);$$ (62)

and its analogy for the products $P_{l+1}^\zeta(X_\nu) \cdot P_{l+1}^\zeta(Y_\mu)$, following by differentiation from multiplication formula and recurrence relations for Legendre functions $P_{l}^\zeta(T)$ [17, 20]. With this relations, substitution of(61) into equations (59) leads to the system for independent from $J_\kappa$ and $L_\zeta$ functions:

$$\frac{(N)}{\xi} \frac{(1)}{\xi} \frac{(2)}{\xi} (\nu; \varrho, b^2, u)^\zeta - \Sigma^{(N)}(\nu) M^{(2)}_{\kappa\xi}(iu, i\varrho) =$$

$$= \frac{(N-2)}{2m\pi} \frac{\Delta(u^2, \varrho^2, \nu^2)}{\mu^2} \int_{\mu_0}^{u-\gamma} d\mu \Sigma^{(N)}(\nu) \int_{\mu_0}^{\gamma-\mu} d\gamma \sum_{\zeta'=\pm 1} \mathcal{H}(\alpha) \cdot$$

$$
\cdot \left[ (\Lambda^+ - \alpha^2)^{(a/2)} - (\Lambda^- - \alpha^2)^{(a/2)} \right]^{a/2} \cdot \frac{\eta^{\prime}(i\varrho)}{\eta^{\prime}(i\varrho)} a_{\kappa\xi}^{\zeta'}(-\varrho, -u; b^2)^\zeta.$$ (63)

$$+ M^{(2)}_{\zeta\zeta}(iu, i\alpha) + M^{(2)}_{\zeta\zeta}(iu, i\alpha) \left( \frac{T_{\nu} Y_\mu - X_{\nu}}{T_{\nu} - 1} \right) \frac{(N)}{\xi} \frac{(1)}{\xi} \frac{(1)}{\xi} (\mu; \varrho, b^2, \alpha)^\zeta$$

$$+ M^{(2)}_{\zeta\zeta}(iu, i\alpha) \left( \frac{T_{\nu} X_{\mu} - Y_{\nu}}{T_{\nu} - 1} \right) \frac{(N)}{\xi} \frac{(1)}{\xi} \frac{(2)}{\xi} (\mu; \varrho, b^2, \alpha)^\zeta \right].$$
Here: \( X_\gamma = X(\omega \| \gamma), Y_\mu = Y(\omega \| \mu) \) are defined analogously to \( T_\nu = T(u \| \nu) \) in (51), and \( \Lambda^+ \) are given in (38). Comparing this system with the one in (29), and keeping in mind formulas of the analytic continuation (37), it is not difficult to see that the systems and theirs solutions are analytic continuations of one another:

when \( p = i \varphi; \; q = i \psi; \; k = i \alpha; \; \omega > 0; \; u - \varphi > \nu > 0; \; Z(qp|\nu) = T(u \varphi|\nu), \) etc., then for \( i = 1, 2 \)

\[
\begin{equation}
D^{(N)} \zeta (\mu; -ip, b^2, -iq)^\gamma = D^{(N)} \zeta (\mu; q, b^2, u)^\gamma. \tag{64}
\end{equation}
\]

Eq. (58),(61) give the following representation for the Jost function:

\[
\begin{equation}
F_{\kappa \xi}^\gamma (b) = F_{\kappa \xi}^\gamma (b, b)^\gamma = 1 + \int_{b + \mu_0}^\infty du \left( \frac{b}{u} \right) L_{\xi} \sum_{\zeta' = \pm 1} g^{\zeta'}(u, b)^\gamma N^\gamma_{\xi}(u, b)^\gamma \Phi_{\kappa \xi}^\gamma(u, b); \tag{65}
\end{equation}
\]

where we put:

\[
\Phi_{\kappa \xi}^\gamma(u, b) = A_{\kappa \xi}^\gamma(u; b^2)^\gamma. \tag{66}
\]

The results obtained here may be confirmed independently via the first of mentioned ways, dealing with off-shell Jost solution (OSJS) which satisfy to radial Dirac equation: (here \( \sigma_{1,2,3} \) are usual Pauli matrices)

\[
\begin{align}
\left( L_{\kappa \xi}^\gamma (b) - V \right) J_{\kappa \xi}^\gamma (b, \rho; r)^\gamma &= \left[ W_{\kappa \xi}^\gamma (ib) - w_{\kappa \xi}^\gamma (i \varphi) \right] X_{\kappa \xi}^\gamma (b, r); \\
\end{align}
\]

\[
\left( L_{\kappa \xi}^\gamma (b) - V \right) = \left( i \sigma_2 \right) \partial_\rho - \sigma_1 \kappa \xi r^{-1} - \sigma_3 \rho W_{\kappa \xi}^\gamma (ib) - V(r), \tag{67}
\]

with potential \( V(r) \) defined in (8a,b), and with boundary condition at \( r \to \infty: \)

\[
J_{\kappa \xi}^\gamma (b, \rho; r)^\gamma \to X_{\kappa \xi}^\gamma (b, r) \to e^{-\varphi r}; \; \left| \frac{1}{i \kappa \xi (i \varphi)} \right| ,
\]

where \( X_{\kappa \xi}^\gamma (b, r) \) is corresponding free solution (see Appendix). One may check that from (67),(59) for OSJS the relations follow:

\[
\begin{align}
J_{\kappa \xi}^\gamma (b, \rho; r)^\gamma - X_{\kappa \xi}^\gamma (b, r) &= \int_{b + \mu_0}^\infty du \sum_{\zeta' = \pm 1} g^{\zeta'}(u, b)^\gamma \\
\end{align}
\]

\[
\begin{align}
\left\{ \begin{array}{l}
K_{\kappa \xi}^\gamma (u, \varphi) J_{\kappa \xi}^\gamma (u, b; r)^\gamma \\
a_{\kappa \xi}^\gamma (u; b^2)^\gamma X_{\kappa \xi}^\gamma (u, r) \\
\end{array} \right\} \\
V(r) J_{\kappa \xi}^\gamma (b, \rho; r)^\gamma = \int_{b + \mu_0}^\infty du \sum_{\zeta' = \pm 1} \frac{1}{2 w_{\kappa \xi}^\gamma (ib)} X_{\kappa \xi}^\gamma (u, r) a_{\kappa \xi}^\gamma (u; b^2)^\gamma. \tag{69}
\end{align}
\]

The Born version of the last relation corresponding to substitutions:

\[
J_{\kappa \xi}^\gamma (b, \rho; r)^\gamma \to X_{\kappa \xi}^\gamma (b, r); \; a_{\kappa \xi}^\gamma (u; b^2)^\gamma \to K_{\kappa \xi}^\gamma (u, \varphi),
\]

follows directly [21] from definitions (54),(55) and the formula:

\[
\int_{\varphi + \nu}^\infty du \frac{P_{\varphi}^u (T(u \varphi|\nu))}{[\Lambda(u^2, \nu^2)]^{\nu/2}} \chi_j(wr) = \frac{1}{\nu} \left( \frac{r}{2 \nu} \right)^u \chi_{-\nu}(\nu r) \chi_j(\varphi r). \tag{70}
\]
In the case (19) we have for kernel $K$ property of radial Green function (see Appendix): ans (8a,b) respectively. It follows directly from definition (1) using easily verified (3) density (76) instead $\Sigma$.

Corresponding Volterrian kernel may be expressed either in the form like (54) with auxiliary kernel (50), (53) do not exist, their difference has a definite value: ($\not{\text{not difficult to see, that although for considering interactions the integrals defining auxiliary kernel (50),(53) do not exist, their difference has a definite value: (l = l_\xi),}}$

$$F_{\kappa_\xi}^\Sigma(g, b; r) = \lim_{r \to 0} \frac{\sqrt{r}}{\Gamma(|\kappa\xi| + \frac{1}{2})} \left( \frac{or}{2} \right)^{|\kappa\xi|} \left\{ \left( \left( S^\Sigma_{\kappa_\xi} (g, b; r) \right)^2 \right) . \right.$$ (71)

We end this section by observation of the simple consequence of CPT-symmetry for the Jost function:

$$a) F_\Sigma^\Sigma(b|g) = F_\Sigma^\Sigma(b|-g); \quad b) F_\Sigma^\Sigma(b|g) = F_\Sigma^\Sigma(b|g); \quad$$ (72)

where $g$ is an interaction constant extracted from potential $V(r)$ for Hamiltonians (8a,b) respectively. It follows directly from definition (1) using easily verified property of radial Green function (see Appendix):

$$G_{-\kappa_\xi}^\Sigma(b; r, y) = -\sigma_1 G_{\kappa_\xi}^\Sigma(b; r, y)\sigma_1.$$

### 4 Nonlocal and singular interactions

There is a close connection between nonlocal and singular interaction. Its become apparent, on the one hand, via construction of the selfadjoint extension for non-selfadjoint singular Hamiltonian with the help of nonlocal interaction [25], on the other hand, via existence of Hamiltonians admitting dual interpretation, as nonlocal from one point of view, and as local but singular from the other one. We start with observation, that for Hamiltonian (18) in spite of its manifest dependence from momentum, pointing its nonlocality, the expression for each partial wave has local form:

$$< JIM|2mU(\xi)|JIM> = U_{\kappa_\xi}(r) = \int_{\mu_0}^\infty d\nu \Sigma_{\kappa_\xi}(\nu) \frac{e^{-\nu r}}{r} - \frac{I^\Sigma_0}{2} \frac{\delta(r)}{2(2m)^{2r^2}}; \quad$$ (73)

with singular behavior at $r \to 0$:

$$U_{\kappa_\xi}(r) = I^\Sigma_0 \frac{(1 + \xi)}{(2m)^{2r^3}} - I^\Sigma_0 \frac{\delta(r)}{2} \frac{1}{r} \left[ I^\Sigma_0 - I^\Sigma_2 \frac{(\kappa\xi + 1/2)}{(2m)^2} \right], \quad$$ (74)

where:

$$I^\Sigma_0 = \int_{\mu_0}^\infty d\nu \Sigma(\nu) \nu^n; \quad$$ (75)

and

$$\Sigma_{\kappa_\xi}(\nu) = \Sigma(\nu) + (2m)^{-2} \left[ \frac{\nu^2}{2} \Sigma(\nu) + (1 + \kappa_\xi) \nu \int_{\mu_0}^\nu d\gamma \Sigma(\gamma) \right]. \quad$$ (76)

Corresponding Volterrian kernel may be expressed either in the form like (54) with density (76) instead $\Sigma^{(3)}(\nu)$, or in the form like (55) as (clf.(35)):

$$K_{l\xi, l}(u, \varrho) = \tilde{A}^{(1)}(iu, i\varrho) \, K_{l\xi, l}(u, \varrho) + \tilde{A}^{(2)}(iu, i\varrho) \, K_{l, -\xi}(u, \varrho). \quad$$ (77)

In the case (19) we have for kernel $K_{l}(u, \varrho)$ the form (54) with density (36). It is not difficult to see, that although for considering interactions the integrals defining auxiliary kernel (50),(53) do not exist, their difference has a definite value: ($l = l_\xi$),

$$H_{lJ}(\varrho, s) - H_{lJ}(\varrho, k) = (k^2 - s^2) \int_{\varrho + \mu_0}^\infty d\alpha \frac{K_{lJ}(\alpha, \varrho)}{(\alpha^2 + s^2)(\alpha^2 + k^2)} \left( \frac{\varrho}{\alpha} \right)^1. \quad$$ (78)
That prompt to turn for this case to ansatz (44) subtracted in the point \( \varrho = \Lambda \), where \( \Lambda \to \infty \). Repeating for it all transformations of the previous section with use of subtracted functions (written for simplicity for case (19)):

\[
\mathcal{H}_l^\Lambda(\varrho, k) = \left( \frac{\Lambda^2 + k^2}{\Lambda^2 - \varrho^2} \right) [\mathcal{H}_l(\varrho, k) - \mathcal{H}_l(\Lambda, k)],
\]

\[
\mathcal{M}_l^\Lambda(\varrho^2, k^2) = \left[ F_l(\Lambda, -ik) - Z_l(\Lambda^2, k^2) + Z_l(-k^2, k^2) \right] (\Lambda^2 - \varrho^2)^{-1} h(\Lambda^2),
\]

\[
\mathcal{F}_l^\Lambda(\varrho, -ik) = \left[ F_l(\varrho, -ik) - Z_l(\varrho^2, k^2) + Z_l(-k^2, k^2) \right] (\Lambda^2 - \varrho^2)^{-1} h(\Lambda^2),
\]

where \( h(\Lambda^2) \) is some appropriate function choosing bellow, we get for limiting value

\[
\mathcal{F}_l(\varrho, -ik) = \lim_{\Lambda \to \infty} \mathcal{F}_l^\Lambda(\varrho, -ik)
\]

Volterra-type equation similar to Schrodinger version [14] of eq.(58):

\[
\mathcal{F}_l(\varrho, -ik) = \lim_{\Lambda \to \infty} \left[ \mathcal{M}_l^\Lambda(\varrho^2, k^2) + \int_{\varrho + \nu_0}^{\infty} du \frac{K_l(u, \varrho)}{(u^2 + k^2)} \left( \varrho^2 - k^2 \right) \mathcal{F}_l(u, -ik) \right].
\]

For any finite \( \Lambda \) its iteration serie convergent under the conditions:

\[
I_0^{S_2} = 0; \quad |I_2^{S_2}| < \infty,
\]

and lead to solution in familiar form:

\[
\mathcal{F}_l(\varrho, -ik) = \lim_{\Lambda \to \infty} \left[ \mathcal{M}_l^\Lambda(\varrho^2, k^2) + \int_{\varrho + \nu_0}^{\infty} du \frac{a_l(u, \varrho; -k^2)}{(u^2 + k^2)} \left( \varrho^2 - k^2 \right) \mathcal{M}_l^\Lambda(u^2, k^2) \right].
\]

Here resolvent \( a_l(u, \varrho; -k^2) \) satisfy to independent from \( \Lambda \) Schrodinger version of equations (59) [14] with corresponding kernel \( K_{l,l}(u, \varrho) \) or \( K_l(u, \varrho) \) (c.f. substitutions (34)). Moreover, it admit also exact factorization of \( l, J \) dependence in terms of \( T \)-matrix momentum transfer spectral density, repeating the form of (61) for the case (18), and for the case (19) repeating a simple form [14]. It is well known [24] that for nonlocal interaction the right hand side of simple normalization condition (46) is replaced to the determinant \( \mathcal{M}_l(k^2) \) which has the same form (1) with Volterrian Green function \( B_{l0}(k^2) \) (c.f. Appendix) instead \( G_{l0}(W) \), and play a role of the measure of nonlocality \(^3\). However such uncertainty of normalization did not affect on scattering phase. If one can choose the function \( h(\Lambda^2) \) so, that the limit (80),(83) exist, then the function \( \mathcal{F}_l(\varrho, -ik) \) is renormalized OSIF for nonlocal "potentials" (18),(19) and the respective limit of \( \mathcal{M}_l^\Lambda(\varrho^2, k^2) \) is renormalized determinant \( \mathcal{M}_l(k^2) \) (c.f.(79)).

For Schrodinger operator \(^4\)

\[
(L_r^l(b) - V(r)) = \partial_r^2 - l(l + 1)r^{-2} - b^2 - V(r)
\]

with singular local potential the definition (1) make no sense, and we left only with possibility to define Jost function as Wronskian [26]:

\[
\mathcal{W}_l(\mp ik) = \left( \mathcal{F}_l(\mp ik, r) \right) \partial_r \varphi_l(k^2, r).
\]

---

\(^3\)So, for Schrodinger Hamiltonian with local nonsingular potential (9) we have instead (57) the natural choice \( Z_l(\varrho^2, k^2) = M_l^\Lambda(\varrho^2, k^2) \equiv 1 \) [14].

\(^4\)Without loss of generality here we can restrict ourself by the case \( N = 3 \) [14].
Here \( f_i(\mp ik, r), \varphi_l(k^2, r), I_l(k^2, r) \) are respectively the Jost, regular, and irregular solutions of Schrodinger equation:

\[
\left( f_i(\pm ik, r) \frac{\partial}{\partial r} f_i(-ik, r) \right) = 2ik, \quad \left( I_l(k^2, r) \frac{\partial}{\partial r} \varphi_l(k^2, r) \right) = 1. \quad (86)
\]

It is not difficult to show [21], that for \( \Re \rho > |\Im k| \) independing from type of potential’s singularity, this definition of Jost function is equivalent to representation:

\[
f_i(\rho) = (\rho^2 + k^2) \int_0^\infty dr \varphi_l(k^2, r) f_i(\rho, r), \quad (87)
\]

(where r.h.s. really don’t depend from \( k^2 \) in this domain), and as a consequence, it is related with the following OSJF:

\[
f_i(\rho) = \lim_{k \to i\rho} T_l(\rho, -ik), \quad (88)
\]

\[
T_l(\rho, -ik) = (\rho^2 + k^2) \int_0^\infty dr \varphi_l(k^2, r) \chi_l(\rho r), \quad (89)
\]

which in its turn is well defined by the natural generalization of Schrodinger version of (71) for singular repulsive potential:

\[
T_l(\rho, -ik) = \lim_{r \to 0} \left[ I_l(k^2, r) \right]^{-1} J_l(\rho, -ik; r). \quad (90)
\]

Therefor from the first line of Schrodinger variant of (68) [14, 15], we obtain for it a homogeneous Volterra equation:

\[
T_l(\rho, b) = \int_{\rho+\mu_0}^\infty du \frac{K_l(u, \rho)}{(u^2 - b^2)} T_l(u, b). \quad (91)
\]

It is clear that nontrivial solution of such equation originated from singular behavior of its kernel \( K_l(u, \rho) \) for \( u \to \infty \). Putting in this limit for resolvent:

\[
a_l(u, \rho; b^2) \Rightarrow A_l(u, b^2) C_l(\rho, b^2); \quad (92)
\]

one can see, that if:

\[
K_l(u, \rho) / A_l(u, b^2) \Rightarrow 0; \quad (93)
\]

then from the second line of the Schrodinger version of resolvent’s equation (59) [14] the equation for \( C_l(\rho, b^2) \) follows which is identical to the (91). Thus we can identify supposing for a moment the continuity on \( u \):

\[
T_l(\rho, b) = C_l(\rho, b^2) = \lim_{u \to \infty} a_l(u, \rho; b^2) / A_l(u, b^2). \quad (94)
\]

The function \( A_l(u, b^2) \) may be found independently from the first line of resolvent equation in the limit \( u \to \infty \), where the kernel (54) \( (N = 3) \) is changed by its asymptotic degenerate form: \( K_l(u, \rho) \Rightarrow U_l(u) R_l(\rho) \equiv K_l^\infty(u, \rho) \). The asymptotical solution reads:

\[
A_l(u, b^2) = U_l(u) \exp[O_l(u, b^2)], \quad O_l(u, b^2) = \int_0^\infty d\alpha \frac{K_l^\infty(\alpha, \alpha)}{(\alpha^2 - b^2)}. \quad (95)
\]
Because the resolvent $a_l(u, \varrho; -k^2)$ is in generally a distribution over $u - \varrho$, we need in corresponding integral form of relation (94). It is easy to verify that such form is nothing but eq.(83), multiplied by $\varrho^{-l}$, with the following choice of regulator function:

$$\mathcal{M}_l^\Lambda(\varrho^2, k^2) \Rightarrow \frac{1}{\Lambda} \exp\left\{ -\frac{1}{\Lambda} \exp \left[ O_l(\varrho, -k^2) \right] \right\}$$

The own limit of regulator function in that case is zero in accordance with homogeneous character of eq.(91). One can notice, that for interaction (18) the convergence conditions (82) lead to nonsingular behavior at the origin. We want to emphasize, that iteration series for spectral density (63), resolvent (59) and OSJS (68) converge under essentially more weak restrictions on the potential, than the serie for OSJF. The corresponding estimations [21] show, that the first are entire functions of coupling constant $g$ (and angular momentum $l$ or $J_N$) for arbitrary potential considering here, but the second has well-known essential singularity at $g = 0$ for singular potential [26], and for regular one it is regular only at $Re J_N \geq \lambda_N$. So, for singular or nonlocal interactions we may considering eq.(83), (88) as definition of OSJF and Jost function respectively. The question now is only to choose the regulator function $\mathcal{M}_l^\Lambda(\varrho^2, k^2)$ so that corresponding limit should exist. For singular repulsive potential the answer is given by the eq.(96). For nonlocal interaction (19) one can make it choice only if the manifest form of resolvent $a_l(u, \varrho; -k^2)$ is known.

5 Conclusions

In this work the generalization of the OSJF method for Dirac Hamiltonian in arbitrary dimension space is given. New integral representations for OSJF, OSJS and Jost determinant via T-matrix momentum transfer spectral density are found, which together with the linear Volterrian integral equation for it constitute a closed dynamic system, allowing to calculate all observable quantities without dealing with eigenvalue problem.

The OSJF for singular and nonlocal potential are constructed through this spectral density with the help of common renormalization procedure, which naturally generalizes such representation for local nonsingular potential.

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6 Appendix

We used the following manifest form of partial Green functions:

$$G_{\kappa \xi 0}^\zeta(-ik; r, y) = \theta(y - r) B_{\kappa \xi 0}^\zeta(-ik; r, y) - \frac{(-1)^L\zeta}{L!} \frac{X_{\kappa \xi}(ik, r) (T) \phi_{\kappa \xi}^0(k, y)}{\eta(k)}$$

$$B_{\kappa \xi 0}^\zeta(\pm ik; r, y) = \frac{1}{2\eta(k)} \left[ X_{\kappa \xi}^\zeta(ik, r) X_{\zeta}^\xi(-ik, y) - [k \rightarrow -k] \right]$$

where $\kappa = \kappa_\xi, L = L_\xi, L - \xi = L_{-\xi}, (2\kappa_\xi + 1) = \xi(2L_\xi + 1)$, and free solutions are:

$$\psi_{j0}(kr) = \left( \frac{\pi kr}{2} \right)^{1/2} J_{J + \frac{1}{2}}(kr)$$
using the relation for the Legendre function of second kind, for the following relations may be checked:

\[
\phi_{\kappa,0}(k,r) = \left[ \xi \psi_{L,0}(kr) \eta^*(k) \psi_{L-\kappa,0}(kr) \right] = \frac{1}{2i} \left[ (-i)^L \chi_{\kappa}^{L+\xi}(-ik,r) - i^L \chi_{\kappa}^{L-\xi}(ik,r) \right];
\]

\[
X_{\kappa}^{\xi}(\varphi, r) = \left[ i\eta^*(i\varphi) \chi_{-\kappa}(i\varphi) \right] = \left[ i\eta^*(i\varphi) \chi_{L}(i\varphi) \right];
\]

Relation (28) may be obtained [21] analogously with it particular case \( l = 0 \), in [14] using the relation [17] for the Legendre function of second kind, for \( l = l - a_N \):

\[
S_{j}^{\alpha}(Z) = \frac{e^{-i\pi a}}{(Z^2 - 1)^{a/2}} \frac{(-1)^j}{2j+1} \int_{-1}^{1} dt \left( \frac{1 - t^2}{Z - t} \right). \quad (100)
\]

The projector onto the states with given \( l^{(N)} \), \( J \) reads:

\[
\Pi_{\kappa}^{(N)}(\varphi, \vec{v}) = \frac{\xi}{\Omega_{N}} \left[ C_{l-1}^{\lambda+1}((\varphi \cdot \vec{v}) (\varphi \cdot \vec{v})^\dagger - C_{l-1}^{\lambda+1}(\varphi \cdot \vec{v}) \right] =
\]

\[
= \frac{\xi}{\Omega_{N}} \left[ (l + \lambda (1 - \xi)) C_{l-1}^{\lambda}((\varphi \cdot \vec{v}) + 2i\lambda \xi (\varphi \cdot \Omega \cdot \vec{v}) C_{l-1}^{\lambda+1}(\varphi \cdot \vec{v}) \right]; \quad (101)
\]

where:

\[
\Omega_{jk} = \frac{1}{2i} \left( \sigma_j \sigma^\dagger_k - \sigma_k \sigma^\dagger_j \right); \quad \tilde{\Omega}_{jk} = \frac{1}{2i} \left( \sigma^\dagger_j \sigma_k - \sigma^\dagger_k \sigma_j \right). \quad (102)
\]

It is convenient for separation of variables to use total antisymmetry of tensor \( E_{jknl} \) and relation with it:

\[
E_{jknl} = \frac{i}{2} \left[ \Gamma_n \{ \Gamma_j, \Sigma_{jk} \} \right];
\]

\[
\{ \Sigma_{jk}, \Sigma_{nl} \} = 2(\delta_{nj}\delta_{kt} - \delta_{nj}\delta_{kn}) - E_{jknl}; \quad (103)
\]

\[
\Sigma_{jk} = \frac{1}{2i} \left[ \Gamma_j, \Gamma_k \right] = \left( \begin{array}{cc} \Omega_{jk} & 0 \\ 0 & \tilde{\Omega}_{jk} \end{array} \right).
\]

Operators of orbital and total angular momentum are defined as:

\[
\mathbf{L}_{jk} = x_j \mathbf{P}_k - x_k \mathbf{P}_j; \quad \mathbf{J}_{jk} = \mathbf{L}_{jk} + \frac{1}{2} \Sigma_{jk}. \quad (104)
\]

Introducing the notations:

\[
n_k = \cos \theta_{k-1} \prod_{j=k}^{N-1} \sin \theta_j; \quad \nabla_N = \vec{\nabla} + \frac{1}{r} \partial_r; \quad (105)
\]

the following relations may be checked:

\[
\frac{1}{2}(\mathbf{L} \cdot \Omega) = -(\vec{\sigma} \cdot \vec{\eta})(\vec{\sigma}^\dagger \cdot \vec{\partial}_r); \quad (\vec{\eta} \cdot \vec{\partial}_r) = 0; \quad (106)
\]

which together with addition theorem and recurrence relations for Gegenbauer polynomials [17]:

\[
\frac{d}{dz} C_{l}^{\lambda}(z) = 2\lambda C_{l-1}^{\lambda+1}(z); \quad zC_{l-1}^{\lambda+1}(z) - C_{l-1}^{\lambda+1}(z) = \frac{\lambda}{2\lambda} \left[ l + \lambda(1 - \xi) \right] C_{l}^{\lambda}(z); \quad \xi = \pm 1; \quad (107)
\]
help to verify the following useful properties:

\[
\int d\Omega_N(\mathbf{\bar{n}}) \, \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\nu}}) \, \Pi_{\tau\sigma}(\mathbf{\bar{n}}, \mathbf{\bar{\nu}}) = \delta_{\xi\xi} \delta_{|\kappa|,|\tau|} \, \Pi_{\kappa\xi}(\mathbf{\bar{\omega}}, \mathbf{\bar{\nu}}); \quad (108)
\]

\[
\frac{4\pi}{(2\pi)^{3\alpha}} \sum_{J_N=\lambda_N}^{\infty} \sum_{\xi=\pm 1} S_{\xi}(Z) \left\{ \begin{array}{c}
\Pi_{\kappa\xi}(\mathbf{\bar{\nu}}, \mathbf{\bar{\nu}}) \\
\Pi_{\kappa\xi}(\mathbf{\bar{\nu}}, \mathbf{\bar{\nu}})
\end{array} \right\} = \left( \mathbf{\bar{\sigma}} \cdot \mathbf{\bar{\tau}} (\mathbf{\bar{\sigma}} \cdot \mathbf{\bar{\nu}}) \right) \left( \frac{1}{Z - (\mathbf{\bar{\tau}} \cdot \mathbf{\bar{\nu}})} \right);
\]

\[
\left[ \frac{N-1}{2} + \frac{1}{2} (\mathbf{L} \cdot \mathbf{\Omega}) \right] \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}}) = -\kappa_{\xi} \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}});
\]

\[
\left[ \frac{N-1}{2} + \frac{1}{2} (\mathbf{L} \cdot \mathbf{\Omega}) \right] (\mathbf{\bar{\sigma}} \cdot \mathbf{\bar{n}})^{\dagger} \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}}) = \kappa_{\xi} (\mathbf{\bar{\sigma}} \cdot \mathbf{\bar{n}})^{\dagger} \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}}); \quad (110)
\]

\[
\frac{1}{2} (\mathbf{L} \cdot \mathbf{L}) \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}}) = \left( \frac{l^N_{\kappa}}{l^N_{\xi}} \right) \left( \frac{l^N_{\xi}}{l^N_{\kappa}} + 2\lambda_N \right) \Pi_{\kappa\xi}(\mathbf{\bar{n}}, \mathbf{\bar{\omega}}); \quad (111)
\]

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