Nonlinear dynamic response and global stability of an air compressor vibration system

Yuanping Li¹,²,³, Xuemin Wang³, Siyu Chen⁴ and Hongbing Lu³

Abstract
This study proposes a strategy for the vibration isolation mounting of an air compressor to attenuate the vibration near the primary resonance region by using a system with dynamic negative stiffness. The vibration system is modelled as a parametric pendulum system. The nonlinear dynamic responses, including the global stability of the air compressor vibration system, are investigated analytically. The efficiency of the proposed vibration isolation strategy is numerically demonstrated over the original device. To analyse the bifurcation of the nonlinear response of the pendulum system, the phase portrait, bifurcation diagram and maximum Lyapunov exponent of the pendulum system are obtained numerically. Furthermore, the Floquet multiplier level is obtained by solving the perturbation equation numerically and can be used to determine the global stability of the air compressor vibration system.

Keywords
Compressor, parametric pendulum system, bifurcation, chaos

Introduction
A compressed air generator, or air compressor, is mounted under the body of a railway vehicle. This component supplies circulated air for passengers and provides power for braking and other operations. Thus, the air compressor is an important subsystem in a high-speed train. However, the air compressor generates uncomfortable high-intensity vibration and noise. In addition, the vibration in the compressed air generator influences the stability of a high-speed train and produces fatigue loading, leading to a reduction in the vehicle’s service life. Thus, the vibrations must be isolated to achieve improved performance.

Figure 1(a) shows a type of compressed air generator that is widely used in high-speed trains. To analyse its dynamic response, the system is illustrated as several elements, including a frame tied with a vehicle, vibration suspension parts (consisting of rubber, a spring rope, and viscoelastic damping materials), and a mass, as shown in Figure 1(b). In general, the excitations of a compressed air generator are categorized into internal excitation and external excitation. Internal excitation is due to the internal movement of the rotor excitation, the airflow fluctuations and the excitation resulting from the running motor. Rough roads and wind loads at high speeds induce external excitation. A comprehensive model should account for both excitations.

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The mathematical model of an air compressor can be simplified as a parametric pendulum system. In this case, a suspension mass is connected to a vertical spring and two horizontal springs, and a pendulum is added as an unbalanced mass to simulate the internal eccentricity of a compressed air generator. The system is generally configured as a geometrical negative stiffness to reduce the dynamic stiffness. A period doubling bifurcation route to chaos occurs in different regions of excitation frequency in this system, and the dynamic response is strongly influenced by the stiffness ratio of the vertical and horizontal springs. When the vertical stiffness is close to the stiffness in the horizontal direction, resonance will occur with the emergence of chaotic motion. Reducing the stiffness in the horizontal direction to increase the stiffness ratio can improve the dynamic response of the vibration system; however, the horizontal spring cannot be removed because it will change the negative stiffness configuration.

Parametric pendulum systems, particularly those that are nonlinear, have been of great interest in the past few decades because of their rich dynamic behaviours. Ansari and Khan modelled a slider-crank mechanism by modelling the engine connection rod as a pendulum; the resulting governing equations had two degrees of freedom. Mouchet et al. studied the dynamical tunnelling between symmetry-related stable modes in a periodically driven pendulum and presented strong evidence that the tunnelling process was governed by nonlinear resonances that manifest within the regular phase-space islands on which the stable modes were localized. Cartmell summarized the mechanics of three different problems in which pendulum motions can occur; the research indicated that pendulum motion can be a principal but specific feature of the dynamic behaviour of three different mechanical systems, and these differences require different control objectives. Warminski and Kecik studied the dynamics of a parametric pendulum system and found that the pendulum can be applied as a dynamic absorber to overcome the chaotic motion near the primary parametric resonance and instability; a magneto rheological damper and nonlinear spring were proposed to improve the dynamics.

Complex pendulum systems have been studied in recent years. Náprstek and Fischer modelled a pendulum vibration damper as a two-degree-of-freedom, strongly nonlinear auto-parametric system. They found that in certain domains of pendulum and excitation parameters, the semi-trivial solution does not exist and various post-critical three-dimensional regimes occur. Horton et al. studied the effects of a small ellipticity in driving, perturbing the classical parametric pendulum. They found that rotation could potentially increase with ellipticity, and the most characteristic feature of the classical bifurcation scenario of a parametrically driven pendulum is the resonance tongue. Han and Cao studied various types of bifurcations and limit cycles of a rotating pendulum under nonlinear perturbation. Zhen et al. calculated the transition curves and periodic solutions of a parametric pendulum system by employing the energy method and proved that nonlinearity does not significantly change the area of the stable regions in the parametric plane but alters the positions of the stable regions considerably. Furthermore, the position of the stable regions is strongly related to the amplitude of the periodic vibrations of the pendulum, particularly when the angular displacement of the pendulum is sufficiently large.

Yuanyuan et al. have investigated the resonance vibration in the double-layer semi-active isolation system of marine auxiliary machinery by using averaging method. Their result indicates that the damping of a revised Bingham magneto-rheological damper and the control force have a significant effect on the vibration transmissibility in the resonance region. The model in Yuanping and Siyu’s paper can effectively simulate the vibration of a compressed air generator, but it is a simplified model. The coupling effect between the springs in the X- and Y-directions is neglected. The X-direction spring is restricted in the horizontal direction, the Y-direction spring is restricted in the vertical direction, and the X- and Y-direction springs have no effect on each other, which is not
completely consistent with the actual situation. Thus, a new model is developed in this paper. In this new model, the primary mass $M$ is also constrained along the horizontal and vertical directions by two massless springs, but the horizontal and vertical springs can have translational movements. A coupling effect exists between the horizontal and vertical springs. The system is allowed to move in a two-dimensional plane, as shown in Figure 2.

The remainder of this paper is organized as follows. The following section introduces the mathematical model derived by the Lagrangian approach. In the subsequent section, the nonlinear characteristics of bifurcation and chaos under lower stiffnesses are calculated by the Runge–Kutta method; approximate solutions are obtained; the stability is analysed based on Floquet theory; and the influence of stiffness is investigated through the root mean square (RMS) and maximum Lyapunov exponent. Finally, suggestions for engineering applications are provided in the final section.

**Mathematical model**

The stiffnesses of vertical and horizontal springs are denoted as $k_x$ and $k_y$, respectively. The initial lengths of the vertical and horizontal springs are $L_1$ and $L_2$, respectively. The unbalanced mass $m$ due to the generator is modelled as a pendulum hanging from the suspension mass $M$. The suspension length of the pendulum is $l_2$, and the angular displacement of the pendulum is $\theta$. The displacements of the suspension mass along the horizontal and vertical directions are denoted as $x$ and $y$, respectively.

The kinetic energy $T_M$ of suspended mass $M$ is

$$T_M = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2)$$

and the potential energy is

$$V_M = M g y + \frac{1}{2} k_y \left( \sqrt{(L_2 + y)^2 + x^2} - L_2 \right)^2 + \frac{1}{2} k_x \left( \sqrt{(L_1 + x)^2 + y^2} - L_1 \right)^2 + \frac{1}{2} k_x \left( \sqrt{(L_1 - x)^2 + y^2} - L_1 \right)^2$$

Similarly, the kinetic energy of suspended mass $m$ is

$$T_m = \frac{1}{2} m \left[ (\dot{x} + l_2 \dot{\theta} \cos \theta)^2 + (\dot{y} + l_2 \dot{\theta} \sin \theta)^2 \right]$$
The potential energy of mass $m$ is

$$V_m = mgy - mgl_2\cos\theta$$  \hspace{1cm} (4)

By combining equations (1) and (3), the total kinetic energy of the system is given by

$$T = T_M + T_m = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\left[(\dot{x} + l_2\dot{\theta}\cos\theta)^2 + (\dot{y} + l_2\dot{\theta}\sin\theta)^2\right]$$  \hspace{1cm} (5)

By combining equations (2) and (4), the total potential energy of the system becomes

$$V = V_M + V_m = (M + m)gy - mgl_2\cos\theta + \frac{1}{2}k_y\left(\sqrt{(L_2 + y)^2 + x^2 - L_2}\right)^2 + \frac{1}{2}k_x\left[\left(\sqrt{(L_1 + x)^2 + y^2 - L_1}\right)^2 + \left(\sqrt{(L_1 - x)^2 + y^2 - L_1}\right)^2\right]$$

By combining equations (2) and (4), the total potential energy of the system becomes

$$V = V_M + V_m = (M + m)gy - mgl_2\cos\theta + \frac{1}{2}k_y\left(\sqrt{(L_2 + y)^2 + x^2 - L_2}\right)^2 + \frac{1}{2}k_x\left[\left(\sqrt{(L_1 + x)^2 + y^2 - L_1}\right)^2 + \left(\sqrt{(L_1 - x)^2 + y^2 - L_1}\right)^2\right]$$  \hspace{1cm} (6)

The governing equations are derived by the Lagrange equation as follows

$$\frac{d}{dt}\left(\frac{\partial(T - V)}{\partial\dot{q}_i}\right) - \frac{\partial(T - V)}{\partial q_i} = Q_i$$  \hspace{1cm} (7)

Then, the governing equations are obtained as follows

$$(M + m)\ddot{x} + ml_2\cos\theta\ddot{\theta} - ml_2\sin\theta\dot{\theta}^2 + (k_x + k_y)x$$

$$-k_xL_1\left(\frac{L_1 + x}{2\sqrt{(L_1 + x)^2 + y^2}} + \frac{L_1 - x}{2\sqrt{(L_1 - x)^2 + y^2}}\right) - k_yL_2\frac{x}{\sqrt{(L_2 + y)^2 + x^2}} = Q_x$$  \hspace{1cm} (8)

$$(M + m)\ddot{y} + ml_2\sin\theta\ddot{\theta} + ml_2\cos\theta\dot{\theta}^2 + (k_y + k_x)y + (M + m)g + k_yL_2$$

$$-k_xL_1\frac{y}{2\sqrt{(L_1 + x)^2 + y^2}} - k_yL_1\frac{y}{2\sqrt{(L_1 - x)^2 + y^2}} - k_yL_2\frac{L_2 + y}{\sqrt{(L_2 + y)^2 + x^2}} = Q_y$$  \hspace{1cm} (9)

$$ml_2^2\ddot{\theta} + ml_2\cos\theta\ddot{x} + ml_2\sin\theta\ddot{y} + mgl_2\sin\theta = Q_\theta$$  \hspace{1cm} (10)

There are two types of excitation for a compressed air generator. The first type of excitation is external excitation induced by road roughness that excites the train bogie. This type of excitation is generally applied to the base. The other type of excitation is internal excitation generated when the compressed air generator is in operation. In this work, a simple external excitation is considered; it is assumed that the external force is applied only on the suspension mass along the vertical direction, namely

$$Q_x = 0, \quad Q_y = A_0\sin\Omega t, \quad Q_\theta = 0$$  \hspace{1cm} (11)

Moreover, the dissipation due to the vertical and horizontal dampers is modelled in the form of viscous damping; therefore

$$(M + m)\ddot{x} + c_x\dot{x} + (k_x + k_y)x + ml_2\cos\theta\ddot{\theta} - ml_2\sin\theta\dot{\theta}^2$$

$$-k_xL_1\left(\frac{L_1 + x}{2\sqrt{(L_1 + x)^2 + y^2}} + \frac{L_1 - x}{2\sqrt{(L_1 - x)^2 + y^2}}\right) - k_yL_2\frac{x}{\sqrt{(L_2 + y)^2 + x^2}} = 0$$  \hspace{1cm} (12)
\[(M + m)\ddot{y} + c_1 \dot{y} + (k_x + k_y)y + ml_2\sin\theta \ddot{\theta} + ml_2\cos\theta \dot{\theta}^2 + (M + m)g\]
\[+k_xL_x - k_yL_y\left(\frac{y}{2\sqrt{(L_1 + x)^2 + y^2}} + \frac{y}{2\sqrt{(L_1 - x)^2 + y^2}}\right)\]
\[-k_yL_y\frac{L_2 + y}{\sqrt{(L_2 + y)^2 + x^2}} = A\sin\Omega t\]

The dimensionless form of equations (12)–(14) is

\[\dot{X} + 2\zeta X + (1 + \kappa)X + m_1\cos\theta \dot{\theta} - m_1\sin\theta \ddot{\theta}^2 - \kappa\lambda_2 \frac{X}{\sqrt{(\lambda_2 + Y)^2 + X^2}} = 0\]
\[-\lambda_1 \left(\frac{\lambda_1 + X}{2\sqrt{(\lambda_1 + X)^2 + Y^2}} + \frac{\lambda_1 - X}{2\sqrt{(\lambda_1 - X)^2 + Y^2}}\right) = 0\]

\[\dot{Y} + 2\zeta Y + (1 + \kappa)Y + m_1\sin\theta \dot{\theta} + m_1\cos\theta \dot{\theta}^2 - \kappa\lambda_2 \frac{Y}{\sqrt{(\lambda_2 + Y)^2 + X^2}} + \kappa\lambda_2 = f_0 + f_1\sin\tau\]

\[\cos\theta \dot{X} + \sin\theta \dot{Y} + \ddot{\theta} + 2\zeta \dot{\theta} - f_0\sin\theta = 0\]

where \(X = x/l_2\), \(Y = y/l_2\), \(\tau = \omega_x t\) and the natural frequency is defined as \(\omega_x = \sqrt{k_x/(M + m)}\), the damping ratios are \(\zeta_x = c_x/(2(M + m)\omega_x, \zeta_y = c_y/(2(M + m)\omega_x\), and \(\zeta_\theta = c_\theta/2ml_2\omega_x\). The stiffness ratios between the x- and y-direction is \(\kappa = k_y/k_x\); the length ratio corresponding to length of pendulum are \(\lambda_1 = L_1/l_2\) and \(\lambda_2 = L_2/l_2\), and the mass ratio is \(m_1 = m/(M + m)\), and \(X\), \(Y\), and \(\theta\) are derivatives with respect to dimensionless time \(\tau\).

Equations (15) to (17) are written in matrix form as

\[M\ddot{q} + C\dot{q} + Kq + NF(q, \dot{q}) = F\]

where

\[q = [X, Y, \theta]^T\]

\[M = \begin{bmatrix}
1 & 0 & m_1\cos\theta \\
0 & 1 & m_1\sin\theta \\
\cos\theta & \sin\theta & 1
\end{bmatrix}\]

\[C = \begin{bmatrix}
2\xi_\theta & 0 & 0 \\
0 & 2\xi_\theta & 0 \\
0 & 0 & 2\xi_\theta
\end{bmatrix}\]

\[K = \begin{bmatrix}
1 + \kappa & 0 & 0 \\
0 & 1 + \kappa & 0 \\
0 & 0 & 0
\end{bmatrix}\]
Nonlinear characteristics

Bifurcation and chaos

In this section, the dynamic responses of the suspension pendulum system are determined. The amplitude and frequency of the external excitation are selected as control parameters. The stiffness ratio $\kappa$ is taken as 1, and the other dimensionless parameters are given as $f_0 = -0.1$, $m_1 = 0.09$, $\zeta = 0.05$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.5$.

In the first case, $f_1 = 0.05$ is considered; the bifurcation diagram and maximum Lyapunov exponent of the pendulum system with respect to excitation frequency $\omega$ are shown in Figures 3 and 4, respectively. These figures indicate that there are large variations in the periodic and chaotic motions in this case. When the excitation frequency is lower than 0.56, the system undergoes periodic motion. As the excitation frequency increases, a periodic double bifurcation phenomenon appears around a frequency of 0.56, and the maximum Lyapunov exponent intersects with zero. The system undergoes period-2 motion when $\omega \in (0.56, 0.66)$, chaotic motion appears for $\omega \in (0.6624, 0.68)$, and the maximum Lyapunov exponent is greater than zero in this region. The phase diagrams of the period-1 motion, period-2 motion, period-4 motion and chaotic motion for the four selected excitation frequencies are shown in Figure 5. In these figures, the cycle shown in red indicates the Poincare points indicative of the period of a motion.

Next, the amplitude of the external excitation is increased to $f_1 = 0.1$; the bifurcation diagram and maximum Lyapunov exponent of the pendulum system with respect to excitation frequency $\omega$ are shown in Figures 6 and 7, respectively. The system still experiences bifurcation and chaotic motion near the primary resonance region with increases in the external excitation frequency. Notably, aside from the chaotic motion, an unbounded motion occurs in the resonance region, as shown in the shaded region in Figure 6. In the unbounded motion region, the assumption of a small-amplitude vibration is not suitable for the pendulum, as shown in Figure 8. The phase portraits for $\omega = 0.65$ and $\omega = 0.85$ are shown in the figure. The pendulum will rotate circularly and deteriorate the stability of the overall system.

Global stability

To provide a comprehensive understanding of the stability of the pendulum vibration system, this section explores the global stability of the system in parameter plane $(\omega, f_1)$ via Floquet theory.

The determinant of equation (20) is

$$dM = \det(M) = 1 - m_1$$

Here, $m_1 \neq 1$; therefore, $dM \neq 0$ will be satisfied. Thus, equation (18) can be transformed into equation (26) by multiplying by $M^{-1}$

$$\ddot{q} + M^{-1}C\dot{q} + M^{-1}Kq + M^{-1}NF(q, \dot{q}) = M^{-1}F$$

$$NF(q, \dot{q}) = \begin{bmatrix}
-m_1\sin \theta \dot{\theta}^2 - \kappa \lambda_2 \frac{X}{\sqrt{(\lambda_2 + Y)^2 + X^2}} - \lambda_1 \left( \frac{\lambda_1 + X}{2\sqrt{(\lambda_1 + X)^2 + Y^2}} + \frac{\lambda_1 - X}{2\sqrt{(\lambda_1 - X)^2 + Y^2}} \right) \\
m_1\cos \theta \dot{\theta}^2 + \kappa \lambda_2 - \kappa \lambda_2 \frac{\dot{\theta}^2 + Y}{\sqrt{(\lambda_2 + Y)^2 + X^2}} - \lambda_1 \left( \frac{Y}{2\sqrt{(\lambda_1 + X)^2 + Y^2}} + \frac{Y}{2\sqrt{(\lambda_1 - X)^2 + Y^2}} \right) \\
-f_0 \sin \theta 
\end{bmatrix}$$

$$F = \begin{bmatrix}
0 \\
f_0 + f_1 \sin \omega t \\
0
\end{bmatrix}$$

$$\det(M) = 1 - m_1$$

$$dM = \det(M) = 1 - m_1$$

$\ddot{q} + M^{-1}C\dot{q} + M^{-1}Kq + M^{-1}NF(q, \dot{q}) = M^{-1}F$
Let $\tilde{q}(t) = [\tilde{X}(t), \tilde{Y}(t), \tilde{\theta}(t)]^T$ represent a known periodic solution of equation (26); then, the perturbation $\Delta q(t) = [\Delta x(t), \Delta y(t), \Delta \theta(t)]^T$ for the periodic solution is given as

$$q(t) = \tilde{q}(t) + \Delta q(t)$$

(27)

Figure 3. Bifurcation map with $\kappa = 1$ and $f_1 = 0.05$: (a) X-direction displacement versus excitation frequency $\omega$; (b) Y-direction displacement versus excitation frequency $\omega$; and (c) pendulum rotation angle $\theta$ versus excitation frequency $\omega$. 

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Substituting equation (27) into equation (26) and by Taylor expansion, the perturbation equation is obtained as

$$Dq + \tilde{M}^{-1} \tilde{C} \Delta q + \tilde{M}^{-1} \tilde{K} \Delta q = 0$$

(28)

Figure 4. Maximum Lyapunov exponent with $\kappa = 1$ and $f_1 = 0.05$.

Figure 5. Phase portrait and Poincare points at different excitation frequencies $\omega$ for $\kappa = 1$ and $f_1 = 0.05$.

Substituting equation (27) into equation (26) and by Taylor expansion, the perturbation equation is obtained as

$$\Delta \dot{q} + \tilde{M}^{-1} \tilde{C} \Delta q + \tilde{M}^{-1} \tilde{K} \Delta q = 0$$

(28)

$$\tilde{M} = \begin{bmatrix} 1 & 0 & m_1 \cos \hat{\theta} \\ 0 & 1 & m_1 \sin \hat{\theta} \\ \cos \hat{\theta} & \sin \hat{\theta} & 1 \end{bmatrix}$$

(29)

$$\tilde{C} = \begin{bmatrix} 2\zeta & 0 & -2m_1 \hat{\theta} \sin \hat{\theta} \\ 0 & 2\zeta & 2m_1 \hat{\theta} \cos \hat{\theta} \\ 0 & 0 & 2\zeta \end{bmatrix}$$

(30)
\[ K = \begin{bmatrix}
1 + \kappa - \kappa \lambda_2 \left( \frac{\ddot{X} + \ddot{Y}}{AA} \right) - \lambda_1 \left( \frac{\ddot{Y}^2}{BB} - \frac{\ddot{Y}^2}{CC} \right), \kappa \lambda_2 - \lambda_1 \left( \frac{\ddot{Y} (\dot{X} + \ddot{X})}{BB} - \frac{\ddot{Y} (\dot{X} - \ddot{X})}{CC} \right), -m_1 \sin \ddot{\theta} - m_1 \cos \ddot{\theta}^2 \\
-\kappa \lambda_2 \left( \frac{\ddot{X} + \ddot{Y}}{AA} \right) + \lambda_1 \left( \frac{\ddot{Y} (\dot{X} + \ddot{X})}{BB} + \frac{\ddot{Y} (\dot{X} - \ddot{X})}{CC} \right), 1 + \kappa - \kappa \lambda_2 \frac{\ddot{X}^2}{AA} - \lambda_1 \left( \frac{(\dot{X} + \ddot{X})^2}{BB} + \frac{(\dot{X} - \ddot{X})^2}{CC} \right), -m_1 \sin \ddot{\theta}^2 + m_1 \cos \ddot{\theta} \\
0, 0, -\sin \ddot{\theta} \dddot{X} + \cos \dddot{\theta} \dddot{Y} - f_0 \cos \ddot{\theta}\
\end{bmatrix}
\] (31)

**Figure 6.** Bifurcation map with \( \kappa = 1 \) and \( f_0 = 0.1 \): (a) X-direction displacement versus excitation frequency \( \omega \); (b) Y-direction displacement versus excitation frequency \( \omega \); (c) pendulum rotation angle \( \theta \) versus excitation frequency \( \omega \).
Assuming that $Q(t)$ is a fundamental solution of equation (28), there is a constant matrix $A$ that satisfies $Q(t + T) = Q(t)A$. Then, $A$ is called a Floquet transition matrix, and its eigenvalues are Floquet multipliers. It is difficult to obtain the Floquet transition matrix in practice. Thus, a numerical method is used to determine the transition matrix and its eigenvalues by simultaneously solving equations (18) and (28).16

In the following, we limit the excitation frequency in the range from 0.2 to 1.1 and the excitation amplitude $f_1$ from 0.01 to 0.18. Then, the parameter plane $(\omega, f_1)$ is divided into 100 × 100 grids; then, the Floquet multiplier is calculated at every point in the plane $(\omega, f_1)$. The Floquet multiplier level of the vibration system is shown in Figure 9, where unstable region is labelled as U, and B is the boundary between the stable and unstable region. When the excitation amplitude $f_1 < 0.04$, the system undergoes stable periodic motion in all considered excitation frequency regions. With increases in the excitation amplitude $f_1$, the instable motion first appears in the primary resonance region, and the instable region becomes larger along the primary resonance bone curve. For example, the instability first appears within the primary resonance region $\omega = 0.8 \sim 0.9$ when $\kappa = 0.5$, as shown in Figure 9(a), and the instable region becomes larger along the primary resonance bone curve with increases in the excitation amplitude $f_1$. However, the instability first appears within $\omega = 0.6 \sim 0.7$ when $\kappa = 1$ and $\kappa = 3$, as shown in Figure 9(b) and (c). The results indicate that the stability of the pendulum system is sensitive to the stiffness ratio; the vibration of the air compressor can be reduced and the stability and reliability of the air compressor can be improved by tuning the suspension stiffness. However, a larger $\kappa$ does not necessarily yield better performance. To obtain stable periodic motion of the pendulum system, a suitable $\kappa$ must be chosen in response to the scope of the external excitation frequency and amplitude. A detailed analysis of the influence of stiffness is provided in the next section.

\[
AA = \sqrt{\left[(\dot{\omega}_2 + \ddot{Y})^2 + \dot{X}^2\right]^3}, \quad BB = 2\sqrt{\left[(\dot{\omega}_1 + \dot{X})^2 + \ddot{Y}^2\right]^3}; \quad \text{and} \quad CC = 2\sqrt{\left[(\dot{\omega}_1 - \dot{X})^2 + \ddot{Y}^2\right]^3}
\]
Influence of stiffness

To further identify the effect of the stiffness ratio $\kappa$ on the vibration characteristics, the stiffness ratio is set as a control parameter to evaluate the dynamic response of the pendulum system. The non-dimensional parameters $\kappa$ are set as $0.5$, $1$, and $3$, as shown in Figure 9.

**Figure 9.** Floquet multiplier level curve in the $(\omega, f_1)$ plane for (a) $\kappa = 0.5$, (b) $\kappa = 1$ and (c) $\kappa = 3$.

**Influence of stiffness**

To further identify the effect of the stiffness ratio $\kappa$ on the vibration characteristics, the stiffness ratio is set as a control parameter to evaluate the dynamic response of the pendulum system. The non-dimensional parameters...
used in the analysis are $f_0 = -0.1$, $f_1 = 0.05$, $\zeta = 0.05$, $m_1 = 0.09$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.5$. The RMS of displacements $X$ and $Y$ and the angular displacement $\theta$ as a function of the excitation frequency $\omega$ for stiffness ratios $\kappa = 0.5, 0.8, 1, 3, 7$ are shown in Figure 10, from which the following observation can be made:

1. When the stiffness ratio is, $k = 0.5$ the maximum values of the dynamic response in the X- and Y-directions appear in the region $(0.5, 1)$, and two clear peaks appear in this region. However, the oscillation of mass $m$ has another trend when the excitation frequency increases, and the rotational solution occurs in the region $(0.8, 0.9)$. This result indicates that there is a high-amplitude vibration in the primary resonance region, but the rotational motion is limited to a small region.

Figure 10. Response diagram for $\kappa = 0.5, 0.8, 1, 3, 7$. (a) RMS in the X-direction versus excitation frequency $\omega$; (b) RMS in the Y-direction versus excitation frequency $\omega$; and (c) RMS of the pendulum rotation $\theta$ versus excitation frequency $\omega$. 
Figure 11. Max Lyapunov exponent in parameter space $(\omega, \kappa) = (0.2, 1.2) \times (0.5, 4)$ with $f_1 = 0.05$.

Figure 12. Contour map of the Max Lyapunov exponent in parameter space $(\omega, \kappa) = (0.2, 1.2) \times (0.5, 4)$. 
2. When the stiffness ratio increases to 0.8 and 1, minor changes occur compared to when $k=0.5$. However, the rotational motion of mass $m$ disappears.
3. As the stiffness ratio $k$ increases further to 3 and 7, the rotational motion appears again, and the region is confined within $(0.6, 0.7)$, but the peak is considerably higher than in the previous cases, it shows that the stiffness ratio has a reasonable range, not the bigger the better.

To provide a comprehensive understanding of the effect of stiffness ratio $k$ on the vibration characteristics, the maximum Lyapunov exponent in parameter space $(\omega, k) = (0.2, 2) \times (0.5, 5)$ under $f_1 = 0.05$ is calculated, and the results are shown in Figures 11 and 12, from which the following observation can be made:

1. For $k \in [0.5, 0.892)$, chaos occurs in the region $\omega \in (0.8, 0.9)$, and the scope of the chaos decreases with the increasing of stiffness ratio $k$;
2. When $k = 0.892$, the maximum Lyapunov exponent equals zero, which is a critical state separating the stable state from the unstable state; for $k \in [0.892, 1)$, chaos is not observed;
3. When $k \geq 1$, chaos always exists, and chaotic interval is restricted to the region of $\omega \in (0.6, 0.7)$.
4. The results indicate that the stiffness ratio $k$ has a significant influence on the dynamic response of the air compressor system. A value of $k$ that is either too small or too high will lead to high-amplitude rotational motion and deteriorates the stability and reliability of the air compressor system. Such behaviour is quite different from the model in Yuanping and Siyu’s paper.1 The difference is likely induced by the coupling effects in the springs along the X- and Y-directions.

Conclusions and remarks
A vibration isolation mounting strategy for an air compressor is proposed to attenuate the vibration near the primary resonance region by using a system with dynamic negative stiffness. The coupling effect of the horizontal and vertical springs of a compressed air generator is included in the model. The phase portrait, Poincare map, bifurcation diagram and Floquet multiplier are obtained numerically to analyse the bifurcation and stability of the nonlinear response in this nonlinear vibration system. The results indicate the following:

1. When the external excitation amplitude is $f_1 = 0.05$, periodic double bifurcation and the chaos phenomenon occur with the change of external excitation frequency $\omega$; when the amplitude of the external excitation is $f_1 = 0.1$, unbounded motion is encountered in the resonance region apart from periodic double bifurcation and chaotic motion, which should be avoided in operation due to the potentially high risks involved.
2. When the amplitude of external excitation is $f_1 < 0.04$, the system undergoes stable periodic motion in all excitation frequency regions considered; as the amplitude of the external excitation increases, the unstable motion first appears in the primary resonance region, and the unstable region becomes larger along the primary resonance bone curve.
3. The stability of the pendulum system is sensitive to the stiffness ratio $k$. Proper parameters can be selected to attenuate the vibration near the primary resonance region to avoid chaotic and instable motion. When the external excitation amplitude $f_1 = 0.05$, the optimal stiffness ratio is $k \in (0.892, 1)$.

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References
1. Yuanping L and Siyu C. Periodic solution and bifurcation of a suspension vibration system by incremental harmonic balance and continuation method. Nonlinear Dyn 2015; 83: 941–950.
2. Leven RW and Koch BP. Chaotic behaviour of a parametrically excited damped pendulum. Phys Lett A 1981; 86: 71–74.
3. Whitaker RJ. Types of two-dimensional pendulums and their uses in education. Sci Educ 2004; 13: 401–415.
4. Bishop SR, Sofroniou A and Shi P. Symmetry-breaking in the response of the parametrically excited pendulum model. Chaos Solitons Fractals 2005; 25: 257–264.
5. Trueba JL, Baltanás JP and Sanjuán MAF. A generalized perturbed pendulum. Chaos Solitons Fractals 2003; 15: 911–924.
6. Ansari KA and Khan NU. Nonlinear vibrations of a slider-crank mechanism. Appl Math Model 1986; 10: 114–118.
7. Mouchet A, Eltschka C and Schlagheck P. Influence of classical resonances on chaotic tunneling. Phys Rev E 2006; 74: 026211.
8. Cartmell MP. On the need for control of nonlinear oscillations in machine systems. Meccanica 2003; 38: 185–212.
9. Warminski J and Kecik K. Autoparametric vibrations of a nonlinear system with a pendulum and magnetorheological damping. In: Warminski J, Lenci S and Cartmell MP (eds) Nonlinear dynamic phenomena in mechanics. Dordrecht: Springer, 2012, 382: 1–61.
10. Warminski J and Kecik K. Instabilities in the main parametric resonance area of a mechanical system with a pendulum. J Sound Vib 2009; 322: 612–628.
11. Náprstek J and Fischer C. Auto-parametric semi-trivial and post-critical response of a spherical pendulum damper. Comput Struct 2009; 87: 1204–1215.
12. Horton B, Sieber J, Thompson JMT, et al. Dynamics of the nearly parametric pendulum. Int J Nonlinear Mech 2011; 46: 436–442.
13. Han N and Cao Q. Global bifurcations of a rotating pendulum with irrational nonlinearity. Commun Nonlinear Sci Numer Simul 2016; 36: 431–445.
14. Zhen B, Xu J and Song Z. Influence of nonlinearity on transition curves in a parametric pendulum system. Commun Nonlinear Sci Numer Simul 2017; 42: 275–284.
15. Yuanyuan F, Yanyan Z and Zhaowang X. Vibration transmission analysis of nonlinear floating raft isolation system with magneto-rheological damper. J Low Freq Noise Vib Active Control 2018; 0(0): 1–11.
16. Qin W, Zhang J and Ren X. Response and bifurcation of rotor with squeeze film damper on elastic support. Chaos Solitons Fractals 2009; 39: 188–195.