Group Theoretical Structure of $N = 1$ and $N = 2$ Two-Form Supergravity

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ABSTRACT

We clarifies the group theoretical structure of $N = 1$ and $N = 2$ two-form supergravity, which is classically equivalent to the Einstein supergravity. $N = 1$ and $N = 2$ two-form supergravity theories can be formulated as gauge theories. By introducing two Grassmann variables $\theta^A$ ($A = 1, 2$), we construct the explicit representations of the generators $Q^i$ of the gauge group, which makes to express any product of the generators as a linear combination of the generators $Q^i Q^j = \sum_k f^{ij}_k Q^k$. By using the expression and the tensor product representation, we explain how to construct finite-dimensional representations of the gauge groups. Based on these representations, we construct the Lagrangeans of $N = 1$ and $N = 2$ two-form supergravity theories.
1 Introduction

The Einstein gravity theory might be an effective theory of a more fundamental theory e.g. superstring theory since the action of the Einstein gravity is not renormalizable. Two-form gravity theory is known to be classically equivalent to the Einstein gravity theory and is obtained from a topological field theory, which is called BF theory [1], by imposing constraint conditions [2]. The BF theory has a large local symmetry called the Kalb-Ramond symmetry [3]. Since the Kalb-Ramond symmetry is very stringy symmetry, the fundamental gravity theory is expected to be a kind of string theory [4]. The extension of the two-form gravity theory to the supergravity theory was considered in Ref.[5]. Furthermore the supergravity theory which has a cosmological term or \( N = 2 \) supersymmetry was proposed in Ref.[6] and the group theoretical structure of these supergravity theory was discussed in Ref.[7]. \( N = 1 \) and \( N = 2 \) two-form supergravity theories can be formulated as gauge theories. In this paper, we call the gauge algebras as \( N = 1 \) and \( N = 2 \) topological superalgebras (TSA), which are the subalgebras of \( N = 1 \) and \( N = 2 \) Neveu-Schwarz algebras whose generators are \((L_0, L_{\pm 1}, G_{\pm \frac{1}{2}})\) and \((L_0, L_{\pm 1}, G_{\pm \frac{1}{2}}, J_0)\), respectively. \( N = 1 \) topological superalgebra is nothing but \( osp(1, 2) \) algebra.

In this paper, by introducing two Grassmann variables \( \theta^A \) (\( A = 1, 2 \)), we construct the explicit representations of the generators \( Q^i \), which makes to express any product of the generators as a linear combination of the generators \( Q^i Q^j = \sum_k f^{ij}_k Q^k \). By using the expression and the direct product representation, we explain how to construct finite-dimensional representations of the gauge groups. The representation theory makes it possible to construct a general action of two-form \( N = 1 \) and \( N = 2 \) supergravity theories, which is expected to give a clue for the non-perturbative analysis of the supergravity. The non-perturbative analysis is also partially given in this paper.

This paper is organized as follows: In section 2, we give the representation theories of \( N = 1 \) and \( N = 2 \) topological superalgebra. By using the representation, we construct the Lagrangeans of \( N = 1 \) and \( N = 2 \) two-form supergravities in section 3. In section 4, we investigate the symmetry of the system and consider the non-perturbative effect. The last section is devoted to summary.
2 The Representations of $N = 1$ and $N = 2$ Topological Superalgebras

By using two Grassmann (anti-commuting) variables $\theta^A (A = 1, 2)$, we define the following generators,

$$G_A = \frac{\partial}{\partial \theta^A}, \quad T^a = \theta^A T^a_B \frac{\partial}{\partial \theta^B}$$
$$I = \theta^A \frac{\partial}{\partial \theta^A}, \quad H^A = \theta^A \theta^B \frac{\partial}{\partial \theta^B}$$

(1)

Here

$$T^a_B = \frac{1}{2} \sigma^a_A$$

(2)

and $\sigma^a$s ($a = 1, 2, 3$) are Pauli matrices. These generators make the following algebra, which we call topological superconformal algebra (TSCA) in this paper:

$$\{G_A, H^B\} = \frac{1}{2} \delta^B_A I - 2 T^a_A T^a_B$$
$$[G_A, T^a] = T^a_B G_B, \quad [T^a, H^A] = T^a_B H^B$$
$$[T^a, T^b] = i \epsilon^{abc} T^c, \quad \{G_A, G_B\} = \{H^A, H^B\} = 0$$

(3)

This algebra contains a closed subalgebra, which is found by defining an operator $\hat{G}_A$:

$$\hat{G}_A \equiv G_A + \alpha \epsilon_{AB} H^B$$

(4)

Here $\alpha$ is a parameter which can be absorbed into the redefinition of the operators but we keep $\alpha$ as a free parameter for later convenience. Then $\hat{G}_A$ and $T^a$ make a closed algebra:

$$\{\hat{G}_A, \hat{G}_B\} = -4 \alpha T^a_{AB} T^a$$
$$[\hat{G}_A, T^a] = T^a_B \hat{G}_B, \quad [T^a, T^b] = i \epsilon^{abc} T^c$$

(5)

Here $T^a_{AB}$ is defined by

$$T^a_{AB} \equiv \epsilon_{BC} T^a_A$$

(6)

and

$$\epsilon^{AB} = -\epsilon^{BA}$$
$$\epsilon_{AB} = -\epsilon_{BA}$$
$$\epsilon^{12} = \epsilon_{21} = 1$$

(7)
The algebra (5) is nothing but osp(1, 2) algebra which is the subalgebra of the Neveu-Schwarz algebra whose generators are $L_0$, $L_{\pm 1}$ and $G_{\pm \frac{1}{2}}$. In this paper, we call this algebra (5) as $N = 1$ topological superalgebra (TSA). As we will see later, $\hat{G}_A$ generates left-handed supersymmetry.

By defining the following operators,

$$J = -\frac{8}{3} \alpha T^a T^a + \epsilon^{AB} \hat{G}_A \hat{G}_B$$
$$G_A^1 = \hat{G}_A$$
$$G_A^2 = \frac{4}{3} i T^a_B (T^a \hat{G}_B + \hat{G}_B T^a) ,$$

we can also construct an algebra, which we call $N = 2$ topological superalgebra $(k,l = 1,2)$

$$\{ G^k_A, \hat{G}^l_B \} = -4 \delta^{kl} \alpha T^a AB T^a + i \epsilon^{kl} \epsilon_{AB} J$$

$$\{ G^k_A, T^a \} = T^a_B G^k_B, \quad [T^a, T^b] = i \epsilon^{abc} T^c$$

$$[J, G^i_A] = i \alpha \epsilon^{ij} G^j_A, \quad [T^a, J] = 0$$

This algebra is the subalgebra of $N = 2$ Neveu-Schwarz algebra whose generators are $L_0$, $L_{\pm 1}$, $G_{\pm \frac{1}{2}}$ and $J_0$.

Since all the operators are explicitly given in terms of $\theta^A$ and $\frac{\partial}{\partial \theta^A}$, we can find that the product of operators is given by a linear combination of the operators:

$$G^k G^l = \delta^{kl} \left\{ -2 \alpha T^a AB T^a - \epsilon_{AB} \left( \frac{3}{2} J + 2 \alpha P \right) \right\}$$

$$+ i \epsilon^{kl} \left( -2 \alpha T^a AB T^a + \frac{1}{2} \epsilon_{AB} J \right)$$

$$T^a T^b = \frac{i}{2} \epsilon^{abc} T^c + \delta^{ab} \left( \frac{1}{4 \alpha} J + \frac{1}{2} P \right)$$

$$J^2 = 3 \alpha J + 2 \alpha^2 P$$

$$G^k_A T^a = T^a G^k_A + T^a_B G^k_B = \frac{1}{2} T^a_A B G^k_B - i \epsilon^{kl} T^a_A B G^l_B$$

$$J G^k_A = G^k_A J + i \alpha \epsilon^{kl} G^l_A = - \frac{1}{2} \alpha G^k_A + \frac{3}{2} i \alpha \epsilon^{kl} G^l_A$$

$$J T^a = T^a J = - \alpha T^a$$

(10)
Here $P$ is a projection operator

$$P^2 = P$$  \hspace{1cm} (11)$$
defined by

$$P \equiv -\frac{1}{2\alpha}\epsilon^{AB}\hat{G}_A\hat{G}_B + 2T^aT^a$$  \hspace{1cm} (12)$$
and $P$ acts as unity on the operators

$$PG^k_A = G^k_A P = G^k_A$$
$$PT^a = T^a P = T^a$$
$$PJ = JP = J$$  \hspace{1cm} (13)$$

Therefore the invariant trace of the product of the operators can be defined by the coefficients of $P$ in Equation (10):

$$\text{tr}G^k_A G^l_A = -2\alpha\delta^{kl}\epsilon_{AB}, \quad \text{tr}T^aT^b = \frac{1}{2}\delta^{ab}, \quad \text{tr}J^2 = 2\alpha^2$$
$$\text{tr}G^k_A T^a = \text{tr}T^a G^k_A = \text{tr}J G^k_A = \text{tr}G^k_A J T^a = \text{tr}T^a J = 0$$  \hspace{1cm} (14)$$

We express the product law (11) of the operators $Q^i$ ($Q^i = T^a$, $G^k_A$, $J$ and $P$) by

$$Q^i Q^j = \sum_k f^{ij}_k Q^k .$$  \hspace{1cm} (15)$$

Especially the expression of the invariant trace (14) is given by

$$\text{tr}Q^i Q^j = g^{ij} \equiv f^{ij}_P$$  \hspace{1cm} (16)$$

The representation of $N = 1$ superalgebra is given by a doublet of the representations $(p, p + \frac{1}{2})$ in $SU(2)$ ($p$ is an integer or half-integer), which is generated by $T^a$ and that of $N = 2$ is given by a quartet $(p, p + \frac{1}{2}, p + \frac{1}{2}, p + 1)$.

$(\frac{1}{2}, 1)$ representation of $N = 1$ superalgebra is given by $(\hat{G}_A, T^a)$ and $(0, \frac{1}{2})$ is given by $(J, G^2_A)$. $(J, G^k_A, T^a)$ makes the $(0, \frac{1}{2}, \frac{1}{2}, 1)$ representation of $N = 2$ superalgebra.

$(1, \frac{3}{2})$ and $(\frac{3}{2}, 2)$ representations of $N = 1$ superalgebra are given by a tensor product, where $\hat{G}_A$ and $T^a$ are replaced by $\hat{G}_A \otimes P + P \otimes \hat{G}_A$ and $T^a \otimes P + P \otimes T^a$. 

5
\( K_{AB} = -T_{AB}^a (J \otimes T^a + T^a \otimes J) - i \frac{1}{2} \epsilon_{kl} G^k_{(A} \otimes G^l_{B)} \)

\( N_{ABC}^2 = T_{AB}^a (G^2_{C} \otimes T^a + T^a \otimes G^2_{C}) \)  

(17)

Here \((AB \cdots X)\) means a symmetrization with respect to the indeces \( AB \cdots X \).

\( K_{AB}, N_{ABC}^1, M_{ABCD} \) makes \( (1, \frac{3}{2}, 2) \) representation of \( N = 2 \) superalgebra. The commutator of \( G^k_A \) with \( (K_{AB}, N_{ABC}^1, M_{ABCD}) \) are given by

\[
\begin{align*}
[G^k_E, M_{ABCD}] &= -\frac{1}{2} \epsilon_{E(A} N_{BCD)}^k \\
\{G^k_D, N^l_{ABC}\} &= -8 \alpha \delta^{kl} M_{ABCD} - i \epsilon^{kl} \epsilon_{D(A} K_{BC)} \\
[G^k_C, K_{AB}] &= -3 i \epsilon^{kl} N_{ABC}^l.
\end{align*}
\]

(19)

The coefficient of \( P \otimes P \) in the product of \( K_{AB}, N_{ABC}^1 \) and \( M_{ABCD} \) gives the invariant trace

\[
\begin{align*}
\text{tr} M_{ABC}^D M_{A'B'C'D'} &= \frac{1}{2} \epsilon_{A(A'} \epsilon_{BB'} \epsilon_{CC'} \epsilon_{DD')} \\
\text{tr} N_{ABC}^k N_{A'B'C'}^l &= \alpha \epsilon_{A(A'} \epsilon_{BB'} \epsilon_{CC')} \\
\text{tr} K_{AB} K_{A'B'} &= \alpha^2 \epsilon_{A(A'} \epsilon_{BB')} 
\end{align*}
\]

(20)

Here \((AB \cdots \hat{F} \cdots Z)\) means the symmetrization with respect to the indeces \( AB \cdots Z \) except \( F \).

\footnote{\( G_A^k \) in Equation (19) is understood to be \( G_A^k \otimes P + P \otimes G_A^k \).}
3 The Lagrangeans of $N = 1$ and $N = 2$ Two-Form Supergravities

In order to construct the Lagrangean of $N = 1$ two-form supergravity theory, we introduce the gauge field $A_\mu$ which is $(\frac{1}{2}, 1)$ representation

$$A_\mu = \psi^A_\mu \hat{G}_A + \omega^a_\mu T^a$$

and define the field strength as follows

$$R_{\mu \nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$$

$$= \{\partial_\mu \psi^A_\nu - \partial_\nu \psi^A_\mu + T^a_B A(\psi^B_\mu \omega^a_\nu - \psi^B_\nu \omega^a_\mu)\} \hat{G}_A$$

$$+ \{\partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + i \epsilon^{abc} \omega^b_\mu \omega^c_\nu + 4 \alpha T^a_{AB} \psi^A_\mu \psi^B_\nu\} T^a$$

The left-handed supersymmetry transformation law of the gauge fields is given by

$$\delta G_\mu = [\epsilon^A \hat{G}_A, \partial_\mu + A_\mu]$$

$$= (-\partial_\mu \epsilon^A + T^a_B \epsilon^B \omega^a_\mu) \hat{G}_A + 4 \alpha T^a_{AB} \psi^A_\mu \psi^B_\nu T^a$$

We also introduce the two-form field $X_{\mu \nu}$ which is $(\frac{1}{2}, 1)$ representation:

$$X_{\mu \nu} = \chi^A_{\mu \nu} \hat{G}_A + \Sigma^a_{\mu \nu} T^a$$

Then the Lagrangean $L_{BF}$ of the so-called BF theory with $N = 1$ local supersymmetry is given by

$$L_{BF} = \epsilon^{\mu \nu \rho \sigma} \left\{ \frac{1}{g} \text{tr} R_{\mu \nu} X_{\rho \sigma} + \Lambda \text{tr} X_{\mu \nu} X_{\rho \sigma} \right\}$$

$$\text{tr} R_{\mu \nu} X_{\rho \sigma} = 2 \alpha \epsilon_{AB} \{\partial_\mu \psi^B_\nu - \partial_\nu \psi^B_\mu + T^a_B A(\psi^B_\mu \omega^a_\nu - \psi^B_\nu \omega^a_\mu)\} \lambda^B_{\rho \sigma}$$

$$+ \frac{1}{2} \{\partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + i \epsilon^{abc} \omega^b_\mu \omega^c_\nu + 4 \alpha T^a_{AB} \psi^A_\mu \psi^B_\nu\} \Sigma^a_{\rho \sigma}$$

$$\text{tr} X_{\mu \nu} X_{\rho \sigma} = 2 \alpha \epsilon_{AB} \chi^A_{\mu \nu} \lambda^B_{\rho \sigma} + \frac{1}{2} \Sigma^a_{\mu \nu} \Sigma^a_{\rho \sigma}$$

Here $g$ is a gauge coupling constant and $\Lambda$ is a cosmological constant. In order to obtain $N = 1$ two-form supergravity theory, we need to introduce the multiplier field $\Phi$ which is $(\frac{3}{2}, 2)$ representation:

$$\Phi = \kappa^{ABC} N_{ABC} + \phi^{ABCD} M_{ABCD}$$
The Lagrangean $\mathcal{L}$ of $N = 1$ two-form supergravity is given by adding the constraint term to the Lagrangean $\mathcal{L}_{BF}$:

\[
\mathcal{L} = \mathcal{L}_{BF} + \varepsilon^{\mu\nu\rho\sigma} \text{tr} \Phi(X_{\mu\nu} \otimes X_{\rho\sigma})
\]

\[
\text{tr} \Phi(X_{\mu\nu} \otimes X_{\rho\sigma}) = -\alpha T_{AB}^a \varepsilon_{CD}^{\alpha\beta} (X_{\mu\nu}^{D} \sum_{\rho\sigma}^a + \sum_{\mu\nu}^a X_{\rho\sigma}^D)
\]

\[
+ T_{AB}^a T_{CD}^b \phi_{ABC} \sum_{\mu\nu}^a \sum_{\rho\sigma}^b
\]

\[
(29)
\]

The Lagrangean of $N = 2$ theory is also given by introducing gauge field which is $(0, \frac{1}{2}, \frac{1}{2}, 1)$ representation

\[
A_\mu = B_\mu J + \psi^k A G^k + \omega^a T^a
\]

(30)

and defining the field strength

\[
R_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]
\]

\[
= \{\partial_\mu B_\nu - \partial_\nu B_\mu - i \varepsilon_{ABC} \varepsilon^{kl} \psi^k_{\mu} A \psi^l_{\nu} \}
\]

\[
+ \{\partial_\mu \psi^k_{\nu} - \partial_\nu \psi^k_{\mu} + T^a_{B} A (\psi^k_{\mu} B \omega^a - \psi^k_{\nu} B \omega^a)
\]

\[
- i \varepsilon^{kl} (B_{\mu} \psi^l_{\nu} - B_{\nu} \psi^l_{\mu}) \}
\]

\[
\Gamma^A
\]

\[
+ \{\partial_\mu \omega^a_{\mu} - \partial_\nu \omega^a_{\nu} + i \varepsilon^{abc} \omega^b_{\mu} \omega^c_{\nu} + 4 \alpha T^a_{AB} \psi^k_{\mu} A \psi^k_{\nu} B \}
\]

(31)

The gauge transformation law of the gauge field has the following form:

\[
\delta A_\mu = [a J + \epsilon^k A G^k + \delta^a T^a, \partial_\mu + A_\mu]
\]

\[
= (-\partial_\mu a - i \varepsilon_{ABC} \epsilon^{kl} \psi^k_{\mu} B) J
\]

\[
+ \{(-\partial_\mu \epsilon^k A + T^a_{B} A (\epsilon^kB \omega^a + i \delta^a \psi^k B) - i \alpha \epsilon^{kl} (a \psi^l A - e^{lA} B) \}
\]

\[
G^k_A
\]

\[
+ (-\partial_\mu \delta^a + i \varepsilon^{abc} \delta^b \omega^c + 4 \alpha T^a_{AB} \epsilon^k A \psi^k B) T^a
\]

(32)

The two-form field in $N = 2$ theory is $(0, \frac{1}{2}, \frac{1}{2}, 1)$ representation

\[
X_{\mu\nu} = \Pi_{\mu\nu} J + \chi^k A G^k + \sum_{\mu\nu} T^a
\]

(33)

Then the Lagrangean of $N = 2$ BF theory is given by

\[
\mathcal{L}_{BF} = \varepsilon^{\mu\rho\sigma} \left\{ \frac{1}{g} \text{tr} R_{\mu\nu} X_{\rho\sigma} + \Lambda \text{tr} X_{\mu\nu} X_{\rho\sigma} \right\}
\]

\[
\text{tr} R_{\mu\nu} X_{\rho\sigma} = 2 \alpha^2 \{\partial_\mu B_\nu - \partial_\nu B_\mu - i \varepsilon_{ABC} \epsilon^{kl} \psi^k_{\mu} A \psi^l_{\nu} B\} \Pi_{\rho\sigma}
\]

(34)
\[
+2\alpha\{\partial_\mu\psi_\nu^k A - \partial_\nu\psi_\mu^k A + T^a_B A(\psi_\mu^k B\omega^a_\nu - \psi_\nu^k B\omega^a_\mu) \\
-\epsilon^{kl}(B_\mu\psi_\nu^l A - B_\nu\psi_\mu^l A)\}\chi^{kB} \\
+\frac{1}{2}\{\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a + i\epsilon^{abc}\omega_\mu^b\omega_\nu^c + 4\alpha T^a_A\psi_\mu^k A\psi_\nu^k B\}\Sigma^a_{\rho\sigma} \tag{35}
\]

\[
\text{tr}X_{\mu\nu}X_{\rho\sigma} = 2\alpha^2\Pi_{\mu\nu}\Pi_{\rho\sigma} + 2\alpha\epsilon_{AB}\chi^{kA}_{\mu\nu}\chi^{kB}_{\rho\sigma} + \frac{1}{2}\Sigma^a_{\mu\nu}\Sigma^a_{\rho\sigma} \tag{36}
\]

The Lagrangean \(\mathcal{L}\) of \(N = 2\) two-form supergravity theory is given by introducing the multiplier field which is \((1, \frac{3}{2}, \frac{3}{2}, 2)\) representation

\[
\Phi = \lambda^{AB}K_{AB} + \kappa^{kABC}N^k_{ABC} + \phi^{ABCD}M_{ABCD} \tag{37}
\]

and adding the term which gives the constraint on the two-form field

\[
\mathcal{L} = \mathcal{L}_{\text{BF}} + \epsilon^{\mu\nu\rho\sigma}\text{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma}) \tag{38}
\]

\[
\text{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma}) = \alpha^2\lambda^{AB}\{-T^a_A(\Pi_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\Pi_{\rho\sigma}) + i\epsilon^{kl}\epsilon_{BCD}\chi^{kD}_{\mu\nu}\chi^{kC}_{\rho\sigma}\} \\
-\alpha T^a_A\epsilon_{BCD}\kappa^{kABC}(\chi^{kD}_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\chi^{kD}_{\rho\sigma}) \\
+T^a_A T^b_B \phi^{ABCD}\Sigma^a_{\mu\nu}\Sigma^b_{\rho\sigma} \tag{39}
\]

4 \hspace{1em} \text{The Symmetry of the Lagrangeans}

We now consider the right-handed supersymmetry. The Lagrangeans of the \(N = 1\) and \(N = 2\) have the following form

\[
\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\left\{\frac{1}{g}\text{tr}R_{\mu\nu}X_{\rho\sigma} + \Lambda\text{tr}X_{\mu\nu}X_{\rho\sigma} + \text{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma})\right\} \tag{40}
\]

On the other hand the Lagrangeans of the corresponding BF theory have the following form

\[
\mathcal{L}_{\text{BF}} = \epsilon^{\mu\nu\rho\sigma}\left\{\frac{1}{g}\text{tr}R_{\mu\nu}X_{\rho\sigma} + \Lambda\text{tr}X_{\mu\nu}X_{\rho\sigma}\right\} \tag{41}
\]

The Lagrangean \((41)\) has the large local symmetry which is called Kalb-Ramond symmetry. The parameter of the transformation \(C_\mu\) is \((\frac{1}{2}, 1)\) representation in \(N = 1\) theory and \((0, \frac{1}{2}, \frac{1}{2}, 1)\) representation in \(N = 2\) theory.
and the transformation law of the Kalb-Ramond symmetry is given by
\begin{align*}
\delta_{KR} A_\mu &= -g \Lambda C_\mu \\
\delta_{KR} X_{\mu\nu} &= \frac{1}{2} (D_\mu C_\nu - D_\nu C_\mu)
\end{align*} (42)

Here the covariant derivative $D_\mu$ is defined by
\begin{equation}
D_\mu \cdot = [\partial_\mu + A_\mu, \cdot] \tag{43}
\end{equation}

Now we consider the Kalb-Ramond like transformation for the Lagrangean (40):
\begin{align*}
\delta_{KR} A_\mu &= -g \Lambda C_\mu - g \Phi \times C_\mu \\
\delta_{KR} X_{\mu\nu} &= \frac{1}{2} (D_\mu C_\nu - D_\nu C_\mu)
\end{align*} (44)

Here the product $R \times S$ of two operators $R = \sum_{ij} r_{ij} Q^i \otimes Q^j$, which is $\left(\frac{3}{2}, 2\right)$ representation in $N = 1$ theory and $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$ representation in $N = 2$ theory, and $S = \sum_i s_i Q^i$, which is $\left(\frac{3}{2}, 1\right)$ representation in $N = 1$ theory and $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$ representation in $N = 2$ theory, is defined by
\begin{equation}
R \times S \equiv \sum_{ijk} s_i r_{jk} g^{ik} G^j \tag{45}
\end{equation}

Here $g^{ik}$ is defined in Equation (16). The product $R \times S$ is $\left(\frac{3}{2}, 2\right)$ representation in $N = 1$ theory and $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$ representation in $N = 2$ theory. Then the change of the Lagrangean (40) is given by
\begin{equation}
\delta_{KR} \mathcal{L} = -\epsilon^{\mu\nu\rho\sigma} \text{tr} D_\mu \Phi C_\nu \Sigma_{\rho\sigma} + \text{total derivative} \tag{46}
\end{equation}

This tells that the Lagrangean (40) is invariant if the parameter $C_\mu$ satisfies the equation
\begin{equation}
0 = \epsilon^{\mu\nu\rho\sigma} B_\nu \otimes \Sigma_{\rho\sigma} \left|\left(\frac{3}{2}, 2\right)\right. \text{ or } \left(1, \frac{3}{2}, \frac{3}{2}, 2\right) \text{ part} \tag{47}
\end{equation}

Equation (47) has non-trivial solutions and the fermionic part of the solution corresponds to right-handed supersymmetry [6]. The commutator of the right-handed supersymmetry transformation and the left-handed one contains the general coordinate transformation.
When \( \alpha \neq 0 \), the parameter \( \alpha \) can be absorbed into the redefinition of the operators or fields as follows:

\[
\begin{align*}
\omega^a_\mu &\to \omega^a_\mu, \quad \psi^A_\mu \to \alpha^{-1} \psi^A_\mu, \quad B_\mu \to \alpha^{-1} B_\mu \\
\Sigma^{a}_{\mu \nu} &\to \Sigma^{a}_{\mu \nu}, \quad \chi^A_{\mu \nu} \to \alpha^{-1} \chi^A_{\mu \nu}, \quad \Pi_{\mu \nu} \to \alpha^{-1} \Pi_{\mu \nu} \\
\phi^{ABC \!D} &\to \alpha^{-1} \phi^{ABC \!D}, \quad \kappa^{ABC} \to \alpha^{-1} \kappa^{ABC}, \quad \lambda^{AB} \to \alpha^{-1} \lambda^{AB}.
\end{align*}
\] (48)

Then \( N = 1 \) Lagrangean has the following form

\[
\begin{align*}
\mathcal{L} &= \epsilon^{\mu \nu \rho \sigma} \frac{1}{g} \left\{ 2 \epsilon_{AB} \{ \partial_\mu \psi^A_\nu - \partial_\nu \psi^A_\mu + T^a_{B \mu} (\psi^B_\mu \omega^a_\nu - \psi^B_\nu \omega^a_\mu) \} \chi^B_{\rho \sigma} \\
&+ \frac{1}{2} \left[ \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + i \epsilon^{abc} \omega^b_\mu \omega^c_\nu + 4 T^a_{AB} \psi^A_\mu \psi^B_\nu \right] \Sigma^a_{\rho \sigma} \right\} \\
&+ \Lambda \left\{ 2 \epsilon_{AB} \chi^A_{\mu \nu} \chi^B_{\rho \sigma} + \frac{1}{2} \Sigma^a_{\mu \nu} \Sigma^a_{\rho \sigma} \right\} \\
&- T^a_{AB} \epsilon^{CD} \kappa^{ABC} \left( \chi^D_{\mu \nu} \Sigma^a_{\rho \sigma} + \Sigma^a_{\mu \nu} \chi^D_{\rho \sigma} \right) \\
&+ \left[ T^a_{AB} T^b_{CD} \phi^{ABC \!D} \Sigma^a_{\mu \nu} \Sigma^b_{\rho \sigma} \right]
\end{align*}
\] (49)

and \( N = 2 \) Lagrangean the following form

\[
\begin{align*}
\mathcal{L} &= \epsilon^{\mu \nu \rho \sigma} \frac{1}{g} \left\{ 2 \{ \partial_\mu B_\nu - \partial_\nu B_\mu - i \epsilon_{AB} \epsilon^{kl} \psi^k_\mu \psi^l_\nu \} \Pi_{\rho \sigma} \\
&+ 2 \{ \partial_\mu \psi^k_\nu - \partial_\nu \psi^k_\mu + T^a_{B \mu} (\psi^k_\mu B^a_\nu - \psi^k_\nu B^a_\mu) \\
&- i \epsilon_{kl} (B^a_\mu \psi^l_\nu - B^l_\mu \psi^l_\nu) \} \chi^{k \!B}_{\rho \sigma} \\
&+ \frac{1}{2} \left[ \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + i \epsilon^{abc} \omega^b_\mu \omega^c_\nu + 4 T^a_{AB} \psi^k_\mu \psi^B_\nu \right] \Sigma^a_{\mu \nu} \right\} \\
&+ \Lambda \left\{ 2 \Pi_{\mu \nu} \Pi_{\rho \sigma} + 2 \epsilon_{AB} \chi^A_{\mu \nu} \chi^{k \!B}_{\rho \sigma} + \frac{1}{2} \Sigma^a_{\mu \nu} \Sigma^a_{\rho \sigma} \right\} \\
&+ \lambda^{AB} \left\{ - T^a_{AB} (\Pi_{\mu \nu} \Sigma^a_{\rho \sigma} + \Sigma^a_{\mu \nu} \Pi_{\rho \sigma}) + i \epsilon_{kl} \epsilon_{AC} \epsilon_{BD} \chi^{k \!C \!D}_{\mu \nu} \chi^{l \!D}_{\rho \sigma} \right\} \\
&- T^a_{AB} \epsilon^{CD} \kappa^{ABC} \left( \chi^{k \!A \!B \!C \!D}_{\mu \nu} \Sigma^a_{\rho \sigma} + \Sigma^a_{\mu \nu} \chi^{k \!A \!B \!C \!D}_{\rho \sigma} \right) \\
&+ \left[ T^a_{AB} T^b_{CD} \phi^{ABC \!D} \Sigma^a_{\mu \nu} \Sigma^b_{\rho \sigma} \right]
\end{align*}
\] (50)

The Lagrangeans (49) and (50) are nothing but the Lagrangeans found in Ref.③. These Lagrangeans are invariant under the following \( U(1) \) “symmetry”

\[
\omega^a_\mu \to \omega^a_\mu, \quad \psi^k_\mu \to \psi^k_\mu, \quad B_\mu \to B_\mu
\]
\[ \Sigma^a_{\mu\nu} \rightarrow e^{2\phi} \Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \rightarrow e^{\phi} \chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \rightarrow e^{\phi} \Pi_{\mu\nu} \]

\[ \phi^{ABCD} \rightarrow e^{-2\phi} \phi^{ABCD}, \quad \kappa^{ABC} \rightarrow e^{-2\phi} \kappa^{ABC}, \quad \lambda^{AB} \rightarrow e^{-2\phi} \lambda^{AB} \]

\[ g \rightarrow e^{2\phi} g, \quad \Lambda \rightarrow e^{-2\phi} \Lambda \quad (51) \]

We can also consider \( \alpha \rightarrow 0 \) theory by redefining the fields as follows:

\[ \omega^a_{\mu} \rightarrow \omega^a_{\mu}, \quad \psi^kA_{\mu} \rightarrow \psi^kA_{\mu}, \quad B_{\mu} \rightarrow B_{\mu} \]

\[ \Sigma^a_{\mu\nu} \rightarrow \Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \rightarrow \alpha^{-1} \chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \rightarrow \alpha^{-2} \Pi_{\mu\nu} \]

\[ \phi^{ABCD} \rightarrow \phi^{ABCD}, \quad \kappa^{ABC} \rightarrow \kappa^{ABC}, \quad \lambda^{AB} \rightarrow \lambda^{AB} \]

\[ g \rightarrow g, \quad \Lambda \rightarrow \begin{cases} \alpha \Lambda & (N = 1) \\ \alpha^2 \Lambda & (N = 2) \end{cases} \quad (52) \]

then \( N = 1 \) Lagrangean is rewritten by:

\[ \mathcal{L} = \epsilon_{\mu
u\rho\sigma} \frac{1}{g} \left\{ 2\epsilon_{AB} \left[ \partial_{\mu} \psi^A_{\nu} - \partial_{\nu} \psi^A_{\mu} + T^a_B \left( \psi^B_{\mu} \omega^a_{\nu} - \psi^B_{\nu} \omega^a_{\mu} \right) \right] \chi^{B}_{\rho\sigma} + \frac{1}{2} \left( \partial_{\mu} \omega^a_{\nu} - \partial_{\nu} \omega^a_{\mu} + i \epsilon^{abc} \omega^b_{\mu} \omega^c_{\nu} \right) \Sigma^a_{\rho\sigma} \right\} + 2\Lambda \epsilon_{AB} \chi^{A}_{\mu\nu} \chi^B_{\rho\sigma} - T^a_B \epsilon^{ABC} \left( \chi_D^{a} \Sigma^a_{\mu\nu} + \Sigma^a_{\mu\nu} \chi_D^{a} \right) + T^a_B T^b_C \phi^{ABC} \Sigma^a_{\mu\nu} \Sigma^b_{\rho\sigma} + \alpha \Lambda \Pi_{\mu\nu} \Pi_{\rho\sigma} + \lambda^{AB} \left\{ - T^a_B \left( \Pi_{\mu\nu} \Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu} \Pi_{\rho\sigma} \right) + i \epsilon^{kA} \epsilon^{BC} \chi^{kA}_{\mu\nu} \chi^B_{\rho\sigma} \right\} + T^b_C \phi^{ABC} \kappa^{a}_{\mu\nu} \Sigma^a_{\rho\sigma} + T^a_B T^b_C \phi^{ABC} \Sigma^a_{\mu\nu} \Sigma^b_{\rho\sigma} \quad (53) \]

The above Lagrangean with \( \Lambda = 0 \) was found in Ref.\[5\]. On the other hand the \( N = 2 \) Lagrangean has the following form:

\[ \mathcal{L} = \epsilon_{\mu
u\rho\sigma} \left[ \frac{1}{g} \left\{ 2\epsilon_{AB} \left[ \partial_{\mu} \psi^A_{\nu} - \partial_{\nu} \psi^A_{\mu} - i \epsilon_{AB} \epsilon^{kl} \psi^k_{\mu} \psi^l_{\nu} \right] \Pi_{\rho\sigma} + 2\left[ \partial_{\mu} \psi^k_{\nu} A - \partial_{\nu} \psi^k_{\mu} A + T^a_B \left( \psi^B_{\mu} \omega^a_{\nu} - \psi^B_{\nu} \omega^a_{\mu} \right) \right] \chi^{kA}_{\rho\sigma} + \frac{1}{2} \left( \partial_{\mu} \omega^a_{\nu} - \partial_{\nu} \omega^a_{\mu} + i \epsilon^{abc} \omega^b_{\mu} \omega^c_{\nu} \right) \Sigma^a_{\rho\sigma} \right\} + 2\alpha \Pi_{\mu\nu} \Pi_{\rho\sigma} + \lambda^{AB} \left\{ - T^a_B \left( \Pi_{\mu\nu} \Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu} \Pi_{\rho\sigma} \right) + i \epsilon^{kA} \epsilon^{BC} \chi^{kA}_{\mu\nu} \chi^B_{\rho\sigma} \right\} + T^b_C \phi^{ABC} \kappa^{a}_{\mu\nu} \Sigma^a_{\rho\sigma} + T^a_B T^b_C \phi^{ABC} \Sigma^a_{\mu\nu} \Sigma^b_{\rho\sigma} \right\} + \alpha \Lambda \Pi_{\mu\nu} \Pi_{\rho\sigma} + \lambda^{AB} \left\{ - T^a_B \left( \Pi_{\mu\nu} \Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu} \Pi_{\rho\sigma} \right) + i \epsilon^{kA} \epsilon^{BC} \chi^{kA}_{\mu\nu} \chi^B_{\rho\sigma} \right\} + T^b_C \phi^{ABC} \kappa^{a}_{\mu\nu} \Sigma^a_{\rho\sigma} + T^a_B T^b_C \phi^{ABC} \Sigma^a_{\mu\nu} \Sigma^b_{\rho\sigma} \quad (54) \]
The Lagrangeans (53) and (54) have two kinds of $U(1)$ “symmetries”, one of which is given by

\begin{align*}
\omega^a \rightarrow \omega^a, \quad \psi^{kA}_\mu \rightarrow \psi^{kA}_\mu, \quad B_\mu \rightarrow B_\mu \\
\Sigma^\mu_\nu \rightarrow e^{\varphi} \Sigma^\mu_\nu, \quad \lambda^{kA}_\mu \rightarrow e^{\varphi} \lambda^{kA}_\mu, \quad \Pi_{\mu\nu} \rightarrow e^{\varphi} \Pi_{\mu\nu} \\
\phi^{ABCD} \rightarrow e^{-2\rho} \phi^{ABCD}, \quad \kappa^{ABC} \rightarrow e^{-2\rho} \kappa^{ABC}, \quad \lambda^{AB} \rightarrow e^{-2\rho} \lambda^{AB} \\
g \rightarrow e^{\varphi} g, \quad \Lambda \rightarrow e^{-2\varphi} \Lambda
\end{align*}

(56)

We call the another $U(1)$ “symmetry” as $U(1)_R$ “symmetry” since the symmetry corresponding to the scale transformation of the Grassmann number $\theta^A$. The $U(1)_R$ “symmetry” is given by

\begin{align*}
\omega^a \rightarrow \omega^a, \quad \psi^{kA}_\mu \rightarrow e^{\rho} \psi^{kA}_\mu, \quad B_\mu \rightarrow e^{2\rho} B_\mu \\
\Sigma^\mu_\nu \rightarrow \Sigma^\mu_\nu, \quad \lambda^{kA}_\mu \rightarrow e^{-\rho} \lambda^{kA}_\mu, \quad \Pi_{\mu\nu} \rightarrow e^{-2\rho} \Pi_{\mu\nu} \\
\phi^{ABCD} \rightarrow \phi^{ABCD}, \quad \kappa^{ABC} \rightarrow e^{\rho} \kappa^{ABC}, \quad \lambda^{AB} \rightarrow e^{2\rho} \lambda^{AB} \\
g \rightarrow g, \quad \Lambda \rightarrow \left\{ \begin{array}{ll}
e^{2\rho} \Lambda & (N = 1) \\
e^{4\rho} \Lambda & (N = 2) \end{array} \right.
\end{align*}

(57)

If we assume the above $U(1)$ symmetries survive in the quantum theory, the form of the effective Lagrangean is restricted. If we started from the theory which does not has a cosmological term ($\Lambda = 0$), the gauge symmetry including the left-handed supersymmetry restricts the form of the terms appearing in the effective Lagrangean as $g \left( \frac{1}{g} R \right)^m X^n$ after integrating the multiplier field $\Phi$ (Here we abbreviated the Lorentz indeces). The $U(1)$ symmetry and Lorentz symmetry give the further restrictions:

\begin{align*}
l - m + n &= 0 \quad (58) \\
m + n &= 2 \quad (59)
\end{align*}

i.e.,

\begin{align*}
l &= 2m - 2 \quad (60)
\end{align*}

It would be natural to assume the theory has the good weak coupling limit ($g \rightarrow 0$), which gives $l \geq 0$. We also assume $m \geq 0$ since $R$ contains the derivative. Then there does not appear the cosmological term even in the quantum theory. The term proportional to $R^m$ appears only perturbatively. Since there does not appear the higher derivative terms perturbatively, the
possible terms are \((l, m, n) = (0, 1, 1), (2, 2, 0)\). Therefore if the term of \((l, m, n) = (2, 2, 0)\) do not appear at the order of \(g^2\), only the term in the original Lagrangean \(i.e.,\) the term of \((l, m, n) = (0, 1, 1)\) can appear. This might tell only that there is no quantum correction and the Einstein theory is the unique infrared theory.

5 Summary

In this paper, we have considered the group theoretical structure of \(N = 1\) and \(N = 2\) two-form supergravity theories based on \(N = 1\) and \(N = 2\) topological superalgebras (TSA), which are the subalgebras of \(N = 1\) and \(N = 2\) Neveu-Schwarz algebras whose generators are \((L_0, L_{\pm 1}, G_{\pm 1}^\pm)\) and \((L_0, L_{\pm 1}, G_{\pm 1/2}^\pm, J_0)\), respectively. By introducing two Grassmann variables \(\theta^A (A = 1, 2)\), we have found the explicit representations of the generators \(Q^i\) and we found that any product of the generators is given by a linear combination of the generators; \(Q^i Q^j = \sum_k f_k^{ij} Q^k\). By using the expression and the direct product representation, it has been explained how to construct finite-dimensional representation of the gauge groups. It is expected that this gives a clue for the non-perturbative analysis of the supergravity.

Acknowledgement

I would like to appreciate A. Sugamoto for the useful discussions.
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