EULER CHARACTERISTIC OF COHERENT SHEAVES ON SIMPLICIAL TORICS VIA THE STANLEY-REISNER RING

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Abstract. We combine work of Cox on the total coordinate ring of a toric variety and results of Eisenbud-Mustata-Stillman and Mustata on cohomology of toric and monomial ideals to obtain a formula for computing $\chi(O_X(D))$ for a Weil divisor $D$ on a complete simplicial toric variety $X_\Sigma$. The main point is to use Alexander duality to pass from the toric irrelevant ideal, which appears in the computation of $\chi(O_X(D))$, to the Stanley-Reisner ideal of $\Sigma$, which is used in defining the Chow ring of $X_\Sigma$.

1. Introduction

For a divisor $D$ on a smooth complete variety $X$, the Hirzebruch-Riemann-Roch theorem describes the Euler characteristic of $O_X(D)$ in terms of intersection theory:

$$\chi(O_X(D)) = \int ch(D) \cdot Td(X).$$

The divisor $D$ corresponds to a class $[D]$ in the Chow ring of $X$, and $ch(D)$ consists of the first $n = \dim(X)$ terms of the formal Taylor expansion of $e^D$. The Todd class of $D$ is defined similarly, but using the Taylor expansion for $D - e^D$.

Let $\Sigma \subseteq \mathbb{R}^n$ be a complete simplicial rational polyhedral fan with $d = |\Sigma(1)|$ rays, $X_\Sigma$ the associated toric variety, and $D \in \text{Cl}(X_\Sigma)$ a Weil divisor on $X_\Sigma$. We combine Alexander duality and the Cox ring with results of Mustata [11] on monomial ideals to obtain a formula for the Euler characteristic of the associated rank one reflexive sheaf $O_{X_\Sigma}(D)$. Put $Z = \{0, 1\}^d$ and $1 = \{1\}^d$. Then for $l \gg 0$,

$$\chi(O_X(D)) = \sum_{m \in Z \setminus 0} (-1)^{|m| - d + n} \dim_k(S/I_\Sigma)^{1-m} \cdot \dim_k S_{l \cdot \phi(m)} + D.$$

Here $I_\Sigma$ denotes the Stanley-Reisner ideal, and $\mathbb{Z}^d \xrightarrow{\phi} \text{Cl}(X_\Sigma)$ is the standard surjection of $\mathbb{Z}^d$ onto the class group. The Cox ring $S$ is a polynomial ring, graded by $\text{Cl}(X_\Sigma)$; on $S/I_\Sigma$ we use the $\mathbb{Z}^n$ grading. We recall the definitions of these objects in §2. Any coherent sheaf on a nondegenerate toric variety corresponds to a finitely generated $\text{Cl}(X_\Sigma)$-graded $S$-module (see [3] for the simplicial case, and [17] for the general case), so such a sheaf has a resolution by rank one reflexive sheaves, and Equation (1) yields a formula for $\chi(F)$ for any coherent sheaf $F$. Bounds on $l$ are determined by Eisenbud-Mustata-Stillman in [6], and are discussed in §2.

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Connections to physics and some history. The methods which are used to prove Equation 1 have applications to computations arising in mathematical physics: in a recent preprint [1], Blumenhagen, Jurke, Rahn and Roschy conjectured an algorithm for computing the cohomology of line bundles on a toric variety. Their motivation was to compute massless modes in Type IIB/F and heterotic compactifications, on a complete intersection in a toric variety. A strong form of the algorithm is established by Maclagan and Smith in Corollary 3.4 of [14]; later proofs appear in Jow [9] and Rahn-Roschy [20]. In all these papers Alexander duality and results of [6] play a key role, as they do in the proof of Equation 1. The original motivation for this work was to find a toric proof for the Hirzebruch-Riemann-Roch theorem.

The first toric interpretation of Hirzebruch-Riemann-Roch is due to Khovanskii [11]. In [12], [13], Pukhlikov-Khovanskii study additive measures on virtual polyhedra, and obtain a Riemann-Roch formula for integrating sums of quasipolynomials on virtual polytopes. Pommersheim [15] and Pommersheim and Thomas [19] obtain results on Todd classes of simplicial torics, and in [2], Brion-Vergne prove an equivariant Riemann-Roch for simplicial torics.

The results of Eisenbud-Mustață-Stillman in [6] show that in the toric setting, \( \chi(O_X(D)) \) may be calculated via certain \( Ext \) modules over the Cox ring of \( X \). On the other hand, evaluating the expression \( \int ch(D) \cdot Td(X) \) involves a computation in the Chow ring of \( X \), and the Cox and Chow rings of a simplicial toric variety are connected by Alexander duality.

The paper is structured as follows: in §2 we recall the results of [6] and the computation of cohomology via the Cox ring. In §3 we introduce the Chow ring, recall that the Stanley-Reisner ideal of \( \Sigma \) is the Alexander dual of the toric irrelevant ideal of \( \Sigma \), and use results of Mustață and Stanley to connect the parts. Equation 1 is proved in §4, and illustrated on the Hirzebruch surface \( \mathcal{H}_2 \).

Toric facts. Let \( N \cong \mathbb{Z}^n \) be a lattice, with dual lattice \( M \), and let \( \Sigma \subseteq N \otimes \mathbb{R} \cong \mathbb{R}^n \) be a complete simplicial rational polyhedral fan (henceforth, simply fan), with \( \Sigma(i) \) denoting the set of \( i \)-dimensional faces of \( \Sigma \), and let \( X_\Sigma \) be the associated toric variety. A Weil divisor on \( X_\Sigma \) is of the form

\[
D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \text{ with } a_\rho \in \mathbb{Z}.
\]

Let \( d = |\Sigma(1)| \). The class group of \( X_\Sigma \) has a presentation

\[
0 \longrightarrow M \xrightarrow{\psi} \mathbb{Z}^d \xrightarrow{\phi} \text{Cl}(X_\Sigma) \longrightarrow 0,
\]

where \( \psi \) is defined by

\[
\chi^m \mapsto \sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle D_\rho, \text{ where } v_\rho \text{ is a minimal lattice generator for } \rho.
\]

In [3], Cox introduced the total coordinate ring (henceforth called the Cox ring) of \( X_\Sigma \). This is a polynomial ring, graded by the class group \( \text{Cl}(X_\Sigma) \).

Definition 1.1. \( S = \mathbb{K}[x_\rho \mid \rho \in \Sigma(1)] = \bigoplus_{\alpha \in \text{Cl}(X_\Sigma)} S_\alpha \).

The utility of this grading is that for \( \alpha \cong D \in \text{Cl}(X_\Sigma) \), \( H^0(O_X(D)) \cong S_\alpha \). For more background on toric varieties, see [4], [5], or [7].
2. Cohomology and the Cox ring

The Cox ring has a distinguished ideal, the toric irrelevant ideal

\[ B(\Sigma) = \langle x^\sigma \mid \sigma \in \Sigma \rangle, \]

where \( x^\sigma = \prod_{\rho \notin \sigma(1)} x_\rho. \)

Note that \( B(\Sigma) \) is generated by monomials corresponding to the complements of the maximal faces of \( \Sigma. \) For an ideal \( I = \langle f_1, \ldots, f_m \rangle \) let

\[ I[l] = \langle f_1^l, \ldots, f_m^l \rangle. \]

In [6], Eisenbud-Mustata-Stillman show that for \( D \in \text{Cl}(X_\Sigma), i \geq 1 \) and \( l \gg 0, \)

\[ H^i(\mathcal{O}_X(D)) \simeq \text{Ext}^{i+1}_S(S/B(\Sigma)[l], S(D))_0, \]

They also obtain a bound for \( l. \) Fix a basis for \( M, \) and let \( A \) be a \( d \times n \) matrix with a row for each ray \( u_\rho \in \Sigma(1), \) written with respect to the fixed basis. Define

\[ a = \max(|\text{entries of } A|) \]

\[ b = \max(|(n-1) \times (n-1) \text{ minors of } A|) \]

\[ c = \min(|\text{nonzero } n \times n \text{ minors of } A|). \]

Corollary 3.3 of [6] shows that if \( D = \sum_\rho a_\rho D_\rho, \) then Equation 2 holds for

\[ l \geq n^2 \max_{\rho \in \Sigma(1)} (|a_\rho|)ab/c \]

For brevity, we use lower case to denote \( \dim_k \) of an object, e.g. \( s_\alpha = \dim_k S_\alpha. \)

**Lemma 2.1.** For \( l \gg 0 \) and \( D \in \text{Cl}(X_\Sigma), \)

\[ \chi(\mathcal{O}_X(D)) = \sum_{i=0}^{n} (-1)^i h^i(D) = s_D - \sum_{i=0}^{n+1} (-1)^i \text{Ext}_S^{i+1}(S/B(\Sigma)[l], S(D))_0. \]

**Proof.** \( \text{Ext}^0_S(S/B(\Sigma)[l], S) = \text{Ext}^1_S(S/B(\Sigma)[l], S) = 0, \) so this follows from [6]. \( \square \)

**Lemma 2.2.** If \( F_* \) is a free resolution for \( S/B(\Sigma)[l], \) then

\[ \sum_{i=0}^{n+1} (-1)^i \text{Ext}_S^{i+1}(S/B(\Sigma)[l], S(D))_0 = \sum_{i=0}^{d} (-1)^i \dim_k F^\vee_i(D)_0 \]

\[ = \sum_{i=0}^{d} (-1)^i \dim_k(F_i)^\vee_D. \]

**Proof.** Take Euler characteristics. \( \square \)

**Lemma 2.3.** If \( F_* \) is a minimal free resolution for \( S/B(\Sigma)[l], \) then

\[ \dim_k(F_i)^\vee_D = \sum_{D' \in \text{Cl}(X_\Sigma)} \text{tor}^S_i(S/B(\Sigma)[l], k)_{D'} \cdot s_{D'+D}. \]

**Proof.** Let \( F_* \) be a minimal free resolution for \( S/B(\Sigma)[l], \) and

\[ r_i(D') = \text{tor}^S_i(S/B(\Sigma)[l], k)_{D'}. \]

Then

\[ F_i = \bigoplus_{D' \in \text{Cl}(X_\Sigma)} S(-D')^{r_i(D')}. \]

Now dualize and take the shift by \( D \) into account. \( \square \)
3. Combinatorial commutative algebra

Taylor resolution. We now observe that the multigraded betti numbers \( r_i(D') \) of \( S/B(\Sigma)^{[l]} \) can be replaced with related numbers which arise from a Taylor resolution for \( S/B(\Sigma) \). The Taylor resolution [23] of a monomial ideal is a variant of the Koszul complex, which takes into account the LCM’s of the monomials involved.

Let \( I = \langle m_1, \ldots, m_k \rangle \) be a monomial ideal, and consider a complete simplex with vertices labelled by the \( m_i \), and each \( n \)-face \( F \) labelled with the LCM of the \( n+1 \) monomials corresponding to vertices of \( F \). Define a chain complex where the differential on an \( n \)-face \( F = [v_{i_0}, \ldots, v_{i_n}] \) is

\[
d(F) = \sum_{j=0}^{n} (-1)^j \frac{m_F^{\nu_j} F \setminus v_{i_j}}{m_F \setminus v_{i_j}},
\]

with \( m_F \) denoting the monomial labelling face \( F \). As shown by Taylor, this complex is actually a resolution (though often nonminimal) of \( I \). When the \( m_i \) are squarefree, the LCM of a subset of \( l \)-th powers is the \( l \)-th power of the LCM of the original monomials, hence the Taylor resolution for \( I^{[l]} \) is given by the \( l \)-th power of the Taylor resolution for \( I \), in the sense that a summand \( S(-\alpha) \) in the free resolution for \( I \) is replaced with \( S(-l \cdot \alpha) \) in the resolution for \( I^{[l]} \).

Thus, the Taylor resolution of \( S/B(\Sigma) \) determines the Taylor resolution of \( S/B(\Sigma)^{[l]} \). The formula in Lemma 2.3 requires a minimal free resolution, which the Taylor resolution is generally not. However, this is no obstacle:

**Lemma 3.1.** If \( F_* \) is a free resolution for \( S/B(\Sigma) \), then

\[
\sum_{i=0}^{n+1} (-1)^i \text{ext}_S^i(S/B(\Sigma)^{[l]}, S(D))_0 = \sum_{i=0}^{d} (-1)^i \sum_{D' \in \text{Cl}(X)} \text{tor}_i^S(S/B(\Sigma), \mathbb{K})_{D' \cdot s_1 \cdot D' + D}.
\]

**Proof.** If \( F_* \) is a minimal resolution of \( S/B(\Sigma)^{[l]} \), then Lemmas [22] and [23] yield

\[
\sum_{i=0}^{n+1} (-1)^i \text{ext}_S^i(S/B(\Sigma)^{[l]}, S(D))_0 = \sum_{i=0}^{d} (-1)^i \sum_{D' \in \text{Cl}(X)} \text{tor}_i^S(S/B(\Sigma)^{[l]}, \mathbb{K})_{D' \cdot s_1 \cdot D' + D}.
\]

Lemma [23] shows that the \( l \)-th power of a Taylor resolution for \( S/B(\Sigma) \) can be used to compute the left-hand side. Furthermore, when \( F_* \) is non-minimal, in the expression

\[
\sum_{i=0}^{d} (-1)^i \text{dim}_S(F_i)^S
\]

the nonminimal summands cancel out, hence we may pass back to the description in terms of Tor, yielding the result. \( \square \)

**Alexander duality and monomial ideals.** Let \( \Delta \) be a simplicial complex on vertex set \( \{1, \ldots, d\} \). Let \( S = \mathbb{Z}[x_1, \ldots, x_d] \) be a polynomial ring, with variables corresponding to the vertices of \( \Delta \).

**Definition 3.2.** The Stanley-Reisner ideal \( I_{\Delta} \subseteq S \) is the ideal generated by all monomials corresponding to nonfaces of \( \Delta \):

\[
I_{\Delta} = \langle x_{i_1} \cdots x_{i_k} | [i_1, \ldots, i_k] \text{ is not a face of } \Delta \rangle.
\]
Figure 1. The fan for $H_2$

The Stanley-Reisner ring is $S/I_{\Delta}$. The intersection of a complete simplicial fan $\Sigma \subseteq \mathbb{R}^n$ with the unit sphere $S^{n-1}$ gives a simplicial complex we denote by $P_\Sigma$; define $I_{\Sigma}$ as the Stanley-Reisner ideal of $P_\Sigma$.

**Definition 3.3.** If $\Delta$ is a simplicial complex on $[d] = \{1, \ldots, d\}$, then the Alexander dual $\Delta^\vee$ is a simplicial complex consisting of the complements of the nonfaces of $\Delta$:

$$\Delta^\vee = \{[d] \setminus \sigma | \sigma \not\in \Delta\}.$$

**Example 3.4.** The Hirzebruch surface $H_2$ corresponds to the fan in Figure 1. Since $[u_2, u_4]$ and $[u_1, u_3]$ are nonfaces of $\Sigma$, and every other nonface such as $[u_1, u_2, u_4]$ contains them, the Stanley-Reisner ideal is

$$I_{\Sigma} = \langle x_1 x_3, x_2 x_4 \rangle.$$

The Alexander dual $\Sigma^\vee$ contains all $\rho \in \Sigma(1)$. Since $\widehat{u_1 u_3} = [u_2, u_4]$ and $\widehat{u_2 u_4} = [u_1, u_3]$, $\Sigma^\vee(2) = \{[u_2, u_4], [u_1, u_3]\}$. So

$$I_{\Sigma^\vee} = \langle x_1 x_2, x_1 x_4, x_2 x_3, x_3 x_4 \rangle.$$

**Lemma 3.5.** The toric irrelevant ideal $B(\Sigma)$ is Alexander dual to the Stanley-Reisner ideal $I_{\Sigma}$.

**Proof.** The Alexander dual $I_{\Sigma^\vee}$ to $I_{\Sigma}$ is obtained by monomializing ([15], Proposition 1.35) a primary decomposition for $I_{\Sigma}$. If $MC(\Sigma)$ denotes the set of minimal cofaces of $\Sigma$, then the primary decomposition of $I_{\Sigma}$ is

$$I_{\Sigma} = \bigcap_{[i_1, \ldots, i_k] \in MC(\Sigma)} \langle x_{i_1}, \ldots, x_{i_k} \rangle.$$

The ideal $I_{\Sigma^\vee}$ is generated by monomials corresponding to minimal cofaces, which are complements to maximal faces, hence $I_{\Sigma^\vee} = B(\Sigma)$. \qed

**Theorem 3.6** (Danilov [5], Jurkiewicz [10]). For a complete simplicial fan $\Sigma$, let $J = \langle div(x^m) \rangle_{m \in M}$. The rational Chow ring $\text{Ch}(X_{\Sigma})$ is the rational Stanley-Reisner ring of $\Sigma$, modulo $J$.

The ideal $J$ is minimally generated by a regular sequence: it is these linear forms which encode the geometry of $\Sigma$. To interpret the Euler characteristic of $\mathcal{O}_X(D)$ in terms of intersection theory, we must change computations involving the toric irrelevant ideal into computations involving the Stanley-Reisner ideal. For a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_d]$ endowed with the fine (also called $\mathbb{Z}^d$) grading.
Lemma 3.7. For a complete fan \( \Sigma \subseteq \mathbb{N} \otimes \mathbb{R} \approx \mathbb{R}^n \) with \( |\Sigma(1)| = d \),

(1) \( \text{Ext}^j(S/I_\Sigma, S) = 0 \) for all \( j \neq d - n \).

(2) In the \( \mathbb{Z}^d \) grading, \( \text{Ext}^{d-n}(S/I_\Sigma, S) \simeq S/I_\Sigma(1) \).

Proof. From Definition 3.2 \( I_\Sigma \) is the Stanley-Reisner ideal of the simplicial sphere \( P_\Sigma \), which is Gorenstein by Corollary II.5.2 of [22]. Since \( \dim P_\Sigma = n - 1 \),

\[ \text{codim}(I_\Sigma) = (d - 1) - (n - 1) = d - n. \]

Everything follows from this, save that \( S/I_\Sigma \) is shifted by \( 1 \). The Gorenstein property means the minimal free resolution of \( S/I_\Sigma \) is of the form

\[ 0 \longrightarrow S(-\alpha) \stackrel{\partial_{d-n}}{\longrightarrow} \bigoplus_{j=1}^{k} S(-\beta_j) \stackrel{\partial_{d-n-1}}{\longrightarrow} \cdots \longrightarrow \bigoplus_{j=1}^{k} S(-\gamma_j) \stackrel{|I_\Sigma|}{\longrightarrow} S \longrightarrow S/I_\Sigma \longrightarrow 0, \]

where \( \partial_{d-n} \) is (up to signs) the transpose of the matrix of minimal generators \( |I_\Sigma| \).

To show that the shift in \( \text{Ext}^{d-n} \) is \( 1 \), we use a result of Hochster. For a complex \( \Delta \) and weight \( \alpha \), let \( \Delta_\alpha = \{ \sigma \in \Delta \mid \sigma \subseteq \alpha \} \). Equating the multidegree \( 1 \) with the full simplex on all vertices of \( \Delta \), Hochster’s formula (5.12 of [15]) yields

\[ \text{Tor}^S_{d-n}(S/I_\Sigma, \mathbb{K})_1 = \tilde{H}^{n-1}(\Sigma|_1, \mathbb{K}). \]

Since \( \Sigma|_1 \simeq P_\Sigma \simeq S^{n-1} \), the result follows. \( \square \)

Example 3.8. The Stanley-Reisner ring for the fan \( \Sigma \) of Example 3.4 has a \( \mathbb{Z}^4 \) graded minimal free resolution

\[ 0 \longrightarrow S(-1, -1, -1, -1) \begin{bmatrix} -x_2x_4 \\ x_1x_3 \end{bmatrix} \oplus \begin{bmatrix} x_1x_3 \\ x_2x_4 \end{bmatrix} \longrightarrow S(0, 0, 0, -1) \longrightarrow S/I_\Sigma. \]

Thus, \( \text{Ext}^2(S/I_\Sigma, S) \simeq S(1, 1, 1, 1)/I_\Sigma \). The simplicial complex \( P_\Sigma \) consists of vertices \( [1], [2], [3], [4] \) and edges \([12], [23], [34], [41]\) and is homotopic to \( S^1 \). Since the multidegrees are all smaller than \( 1 \) in the pointwise order, \( \Sigma|_1 = P_\Sigma \), so

\[ \mathbb{K} = \tilde{H}^1(S^1, \mathbb{K}) = \tilde{H}^1(\Sigma|_1, \mathbb{K}) = \text{Tor}^S_2(S/I_\Sigma, \mathbb{K})_1, \]

showing the shift \( \alpha \) in the last step of the free resolution of \( S/I_\Sigma \) is \( S(-1) \).
We now prove Equation \([1]\) By Equation \([5]\)
\[
\chi(\mathcal{O}_X(D)) = s_D - \sum_{i=0}^{n+1} (-1)^i \text{ext}^i_S(S/B(\Sigma)^{[i]}, S(D))_0.
\]

Let \(\gamma(m) = s_{t \cdot \phi(m) + D}\) and \(E = \sum_{i=0}^{n+1} (-1)^i \text{ext}^i_S(S/B(\Sigma)^{[i]}, S(D))_0\). It suffices to show
\[
E = s_D + \sum_{m \in \mathbb{Z} \setminus 0} (-1)^{|m| - d + n + 1} \dim_K(S/I_{\Sigma})_{1-m} \cdot \gamma(m).
\]

First, observe that
\[
E = \sum_{i=0}^{d} (-1)^i \sum_{D^i \in \text{Cl}(X_D)} \left( \sum_{\phi(m) = D^i} \text{tor}^S_i(S/B(\Sigma), \mathbb{K})_m \right) \cdot \gamma(m).
\]

For \(i \geq 0\),
\[
\text{Tor}^S_i(B(\Sigma), \mathbb{K}) \simeq \text{Tor}^S_{i+1}(S/B(\Sigma), \mathbb{K}),
\]
so using Equation \([7]\) we may rewrite the last line of Equation \([9]\) as
\[
(10) \quad s_D + \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{m \in \mathbb{Z} \setminus 0} \text{ext}^{m-i}_S(S/I_{\Sigma}, S)_{-m} \cdot \gamma(m).
\]

By Lemma \([3.7]\) \(\text{Ext}^{m-i}_S(S/I_{\Sigma}, S)\) is nonzero iff \(|m| - i = d - n\), and
\[
\text{Ext}^{d-n}_S(S/I_{\Sigma}, S) \simeq S/I_{\Sigma}(1).
\]

Since the only nonzero terms in Equation \([10]\) occur for \(i = |m| - d + n\) we rewrite Equation \([10]\) as
\[
(11) \quad s_D + \sum_{m \in \mathbb{Z} \setminus 0} (-1)^{|m| - d + n + 1} \text{ext}^{d-n}_S(S/I_{\Sigma}, S)_{-m} \cdot \gamma(m)
\]

This shows that
\[
E = s_D + \sum_{m \in \mathbb{Z} \setminus 0} (-1)^{|m| - d + n + 1} \dim_K(S/I_{\Sigma})_{1-m} \cdot \gamma(m),
\]
and Equation \([1]\) follows. \(\square\)
Example 4.1. Consider the divisor $D = 3D_3 - 5D_4$ on the Hirzebruch surface $\mathcal{H}_2$ from Figure 1. Since the support function for $D$ is not convex, $D$ is not nef. Thus, computing $\chi(\mathcal{O}_{\mathcal{H}_2}(D))$ involves more than a simple global section computation. A direct calculation with Riemann-Roch for surfaces shows that

$$\chi(\mathcal{O}_{\mathcal{H}_2}(D)) = 4.$$ 

Using the methods of §9.4 of [4], it can be shown that $h^0(D) = 0$, $h^1(D) = 2$, and $h^2(D) = 6$. Now we illustrate how to apply Equation 1. Let

$$\phi = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$ 

so that the Class group is given by

$$\mathbb{Z}^4 \xrightarrow{\phi} \mathbb{Z}^2 \simeq Cl(\mathcal{H}_2) \longrightarrow 0.$$ 

The Eisenbud-Mustaţă-Stillman bound of Equation 1 is $l = 80$, but a careful analysis (see Example 3.6 of [6]) shows that in this case taking $l = 4$ is sufficient. Then for example with $m = (0, 1, 0, 1)$ we have $\phi(m) = (-2, 2)$ so since $D = (3, -5),

$$S_4 \cdot \phi(m) + D = S_{(-5,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(-5, 3)),$$

and the dimension of this space is two. However,

$$(S/I_\Sigma)_{(-0,1,0,1)} = (S/I_\Sigma)_{(1,0,1,0)} = 0,$$

since $x_1x_3 \in I_\Sigma$. A check shows that all terms in the summation vanish, save when

$m \in \{(1, 1, 0, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$

For the first two values, $\phi(m) = (-1, 2)$, and we compute

$$S_4 \cdot \phi(m) + D = S_{(-1,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(-1, 3)),$$

which has dimension twelve. Since $1 - m$ is either $(0, 0, 1, 0)$ or $(1, 0, 0, 0)$, for these two values of $m,$

$$\dim_K(S/I_\Sigma)_{1 - m} = 1$$

Since $|m| - d + n = 1$, these two weights contribute $(-1) \cdot 2 \cdot 12 = -24$ to the Euler characteristic. For the remaining weight $m = (1, 1, 1, 1)$, the Stanley-Reisner ring is one dimensional in degree $1 - m = (0, 0, 0, 0)$, and $\phi(1, 1, 1, 1) = (0, 2)$ and

$$S_4 \cdot \phi(m) + D = S_{(3,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(3, 3)),$$

which has dimension 28. Since $|m| - d + n = 2$ the contribution is positive, thus

$$\chi(\mathcal{O}_{\mathcal{H}_2}(3D_3 - 5D_4)) = -24 + 28 = 4.$$ 

Problem As noted in the introduction, this work began as an attempt to find a toric proof of Hirzebruch-Riemann-Roch using Equation 1 it would be interesting to find such a proof. A proof of Equation 1 also follows from results of Maclagan-Smith [14], I thank Greg Smith for noting this.

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