THE CLIFFORD ALGEBRA OF A FINITE MORPHISM

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Abstract. We develop a general theory of Clifford algebras for finite morphisms of schemes and describe several applications to the theory of Ulrich bundles and connections to period-index problems for curves of genus 1.

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1. Introduction

The goal of this paper is to develop a general theory of Clifford algebras for finite morphisms of schemes, with a view toward the theory of Ulrich bundles and period-index theorems for genus 1 curves.

A construction of Roby [Rob69], defines a Clifford algebra, denoted \( C(f) \), associated to a homogeneous form \( f \) of degree \( d \) in \( n \) variables on a vector space \( V \) (the classical Clifford algebra arising in the case that \( d = 2 \)). The behavior of this algebra is intimately connected with the geometry of the hypersurface \( X \) defined by the equation \( x_0^d - f(x_1, \ldots, x_n) \) in \( \mathbb{P}^n \). The classical results leads one to believe that perhaps the Clifford algebra of the form \( f \) is intrinsic to the variety \( X \).

As we explain here, the Clifford algebra of a form is really a structure associated not to a scheme \( X \), but to a finite morphism \( \phi : X \to Y \), designed to (co)represent a functor on the category of locally free algebras over the base scheme \( S \). Roughly speaking, the Clifford algebra of \( \phi \) is a locally free sheaf \( \mathcal{C} \) of (not necessarily commutative) \( \mathcal{O}_S \)-algebras such that maps from \( \mathcal{C} \) into any locally free \( \mathcal{O}_S \)-algebra \( \mathcal{B} \) are the same as maps from \( \phi_* \mathcal{O}_X \) into \( \mathcal{B}|_Y \). Taking \( \mathcal{B} \) to be a matrix algebra, we see that the representations of such a \( \mathcal{C} \) parameterize sheaves on \( X \) with trivial pushforward to \( Y \), yielding a connection to Ulrich bundles. Making this idea work is somewhat delicate and requires various hypotheses on \( X \) and \( Y \) that we describe in the text.

In the case of the classical Clifford algebra of a form described above, if one takes \( Y \) to be the projective space \( \mathbb{P}^{n-1} \) and the morphism \( \phi : X \to \mathbb{P}^{n-1} \) to be given by dropping the \( x_0 \)-coordinate, then one obtains
a natural identification $C(\phi) \cong C(f)$. Our construction also generalizes other constructions of Clifford algebras, as in [HH07, Kuo11, CK15] which do not come directly from a homogeneous form.

1.1. Structure of paper. In Section 2, we construct the Clifford algebra of a morphism satisfying certain conditions (see Definition 2.3.11, Theorem 2.2.8). In Section 3 we study the representations of the Clifford algebra and their relations to Ulrich bundles in a general context. In particular, we show that a natural quotient of the Clifford algebra (the so-called “reduced Clifford algebra,” of Definition 3.0.14) is Azumaya, and its center is the coordinate ring for the coarse moduli space of its representations of minimal degree (see Theorem 3.0.15), generalizing results of Haile and Kulkarni [Hai84, Kul03]. Sections 2 and 3 work over an arbitrary base scheme $S$. In Section 4 we specialize to the case of a finite morphism from a curve to the projective line, extending results of [Cos11, Kul03] and relating the Clifford algebra and its structure to the period-index problem for genus 1 curves. Finally, in Appendix A, we give more explicit constructions in the case of morphisms of subvarieties of weighted projective varieties, relating our construction to the more classical perspective of Clifford algebras associated to forms.

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2. General definition and existence

2.1. Notation. Fix throughout the section a base scheme $S$. For an $S$-scheme $X$, we will write $\pi_X : X \to S$ for the structure morphism, or simply write $\pi = \pi_X$ if the context is clear. We write $\text{Shv}/S$ for the category of sheaves of sets on $S$. For a sheaf of unital $\mathcal{O}_S$-algebras $\mathcal{A}$, we write $\epsilon_{\mathcal{A}} : \mathcal{O}_S \to \mathcal{A}$ for the $\mathcal{O}_S$-algebra structure map. We assume that all algebras (and sheaves of algebras) are unital and associative. We do, however, allow the possibility of the 0-ring, containing a single element in which the elements 0 and 1 coincide.

For a quasicoherent sheaf $\mathcal{N}$ of $\mathcal{O}_X$-modules, we write $\mathcal{N}^\vee$ to denote the dual sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)$. Let $\text{LF}/\mathcal{O}_S$ denote the category of locally free quasi-coherent sheaves of $\mathcal{O}_S$-modules (with arbitrary, possibly infinite, local rank) and $\text{FLF}/\mathcal{O}_S$ the category of finite locally free sheaves (i.e., those with rank an element of $\Gamma(S, \mathbb{Z})$). Let $\mathcal{Alg}/\mathcal{O}_S$ denote the category of quasi-coherent sheaves of $\mathcal{O}_S$-algebras, and $\mathcal{LFAlg}/\mathcal{O}_S$ denote the full subcategory of those which are locally free as $\mathcal{O}_S$-modules.

2.2. The Clifford functor and Clifford algebra. Associated to a morphism $\phi$ of $S$-schemes, we will define a Clifford functor $\mathcal{CF}_\phi$, and under certain assumptions show that it is representable by an algebra which we refer to as the Clifford algebra of the morphism $\phi$.

Definition 2.2.1. Let $\phi : X \to Y$ be a morphism of $S$-schemes. We define the Clifford functor of $\phi$

$\mathcal{CF}_\phi : \mathcal{LFAlg}/\mathcal{O}_S \to \text{Shv}/S$

by the formula

$\mathcal{CF}_\phi(B) = \text{Hom}_{\mathcal{Alg}/\mathcal{O}_S}(\phi_!, \mathcal{O}_X, \pi_Y^*B)$

This functor is not representable in general, as we will see. We show in Theorem 2.2.8 we show that under certain natural hypotheses we may find a sheaf of $\mathcal{O}_S$-algebras $\mathcal{C}_\phi$ such that $\mathcal{CF}_\phi(B) = \text{Hom}_{\mathcal{Alg}/\mathcal{O}_S}(\mathcal{C}_\phi, B)$, however in general we do not know that the algebra we construct is locally free over $\mathcal{O}_S$. On the other hand, this is clearly true in case $S$ is the spectrum of the field, and we show that, the formation of $\mathcal{C}_\phi$ commutes with pullbacks on $S$, showing that the Clifford algebra exists in certain more general settings as well.

Before we discuss the representability of this functor in more general situations, we consider a few examples. Later in A, we will exhibit explicit presentations for the Clifford algebra in various situations.

Example 2.2.2. Let $k$ be a field, $R$ a commutative $k$-algebra, free and of finite rank over $k$, and $S$ a finitely generated commutative $R$-algebra. Let $\phi : \text{Spec}(S) \to \text{Spec}(R)$ be the structure morphism. Then the Clifford functor of $\phi$ is representable by a $k$-algebra $C$.
Proof. Let \( r_1, \ldots, r_n \) be a basis for \( R \) over \( k \) with structure coefficients \( r_{ij}^p \) defined by \( r_i r_j = \sum_p r_{ij}^p r_p \), and choose a presentation \( S = R[t_1, \ldots, t_m]/(f_1, \ldots, f_t) \). We note that for any associative \( k \)-algebra \( B \), a morphism of \( R \)-algebras \( S \to B \otimes R \) is given by the images of \( t_i \) which are given as expressions of the form \( \beta_i = \sum_{j=1}^n b_{ij} \otimes r_j \), which must satisfy \( f_p(\beta_1, \ldots, \beta_m) = 0 \) for \( p = 1, \ldots, t \). Expanding these expressions and examining each of the coefficients of the \( r_j \) gives a number of identities, in the form of noncommutative polynomials \( P_{p,q} \) in the \( b_{i,j} \) which must be satisfied for this map to be a homomorphism of algebras. The Clifford algebra is then seen to be represented by the free associative algebras in the variables \( x_{i,j} \) modulo the ideal generated by the polynomials \( P_{p,q} \) in the \( x_{i,j} \).

It is good to keep in mind that when the Clifford algebra exists it need not be nonzero, as the following example shows.

Example 2.2.3. Let \( k \) be a commutative ring, \( R = k[x], S = k[x, x^{-1}] \) and \( \phi : \text{Spec}(S) \to \text{Spec}(R) \) the structure map. Then the Clifford algebra of \( \phi \) exists and is the zero algebra.

Proof. Suppose \( B \) is an associative \( k \)-algebra such that we have an \( R \)-algebra map \( S \to B \otimes R \). In particular, we must map \( x^{-1} \) to some element \( \sum_{i=0}^n b_i \otimes x^i \) which must satisfy \( 1 = x \sum b_i \otimes x^i = \sum_{i=0}^n b_i \otimes x^{i+1} \). Comparing coefficients, we find that in the coefficient of \( x^0 \) that we need to have \( 0 = 1 \) in \( B \). Hence any such \( B \) must receive a map from the 0-ring, and must therefore be the 0-ring itself. As all relevant maps are forced to be zero, it must also satisfy the universal property as desired.

Finally, we give an example to illustrate that the Clifford algebra need not exist in general.

Example 2.2.4. Let \( k \) be a field, \( R = k[x], S = k[x, y] \) and \( \phi : \text{Spec}(S) \to \text{Spec}(R) \) the structure map. Then the Clifford algebra of \( \phi \) does not exist. That is, the Clifford functor is not representable.

Proof. Suppose \( C \) is the Clifford algebra of \( \phi \) with associated morphism \( \psi : k[x, y] \to C \otimes_k k[x] \). Such a morphism is defined by where \( y \) is sent, say \( y \mapsto \sum_{i=0}^n c_i \otimes x^i \). We claim that we can find some \( k \)-algebra \( B \), together with a map \( \gamma : k[x, y] \to B \otimes_k k[x] \) such that there is no algebra map \( \rho : C \to B \) such that \( \gamma = (\rho \otimes_k k[x])\psi \). For this, we set \( B = k \) and note that a map \( k[x, y] \to B \otimes_k k[x] = k[x] \) of \( k[x] \)-algebras is given by an arbitrary element of \( k[x] \) as an image of \( y \), since \( k[x, y] \) is a free commutative \( k[x] \)-algebra with one generator, \( y \). Consequently, we can choose \( \gamma(y) = x^{n+1} \). But for any \( \rho : C \to B \), we find that \( (\rho \otimes_k k[x])\psi \) takes \( y \) to a polynomial in \( x \) of degree at most \( n \), by definition of \( \psi \). Consequently \( \gamma \) is not of this form, and \( C \) cannot represent the functor as claimed.

We now describe some useful hypotheses which will help us describe the circumstances under which our Clifford algebras will exist.

Definition 2.2.5. We say that a sheaf of \( \mathcal{O}_Y \)-modules \( \mathcal{N} \) is friendly with respect to \( \pi_Y = \pi \) if it is a locally free coherent sheaf of finite rank, \( \pi_*(\mathcal{N}^\vee) \) is also locally free of finite rank, and if for any pullback square with \( f : S' \to S \) an arbitrary morphism of schemes,

\[
\begin{array}{ccc}
Y' & \xrightarrow{f_Y} & Y \\
\pi' \downarrow & & \pi \downarrow \\
S' & \xrightarrow{f} & S
\end{array}
\]

the base change map gives an isomorphism: \( f^* \pi_*(\mathcal{N}^\vee) \cong (\pi')_*((f')^*(\mathcal{N}^\vee)) \).

We note that by [Har77, Cor 12.9, Thm 12.11], the base change map being an isomorphism holds if the map \( p \mapsto \dim(\mathcal{N}(p)) \) from \( S \) to \( \mathbb{Z} \) is locally constant. For example, this holds trivially if \( S \) is the spectrum of a field. We note also that the property of being friendly is itself preserved by base change with respect to morphisms \( S' \to S \).

Notation 2.2.6 (Property (C)). Let \( \phi : X \to Y \) be a morphism of \( S \)-schemes. We say “\( \phi \) has property (C)” if \( (\phi, \mathcal{O}_X) \) and \( (\phi, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (\phi, \mathcal{O}_X) \) are friendly with respect to \( \pi_Y \).

Remark 2.2.7. In particular, if \( S \) is the spectrum of a field, these conditions will be satisfied when \( \phi \) is a finite flat morphism and \( Y \) is proper over \( S \).
For a morphism $\phi$ with property (C) an algebra $\mathcal{C}_\phi$ is described in Definition 2.3.11, which we will refer to as the Clifford algebra, although in general we are not able to verify our algebra is locally free over $\mathcal{O}_S$.

**Theorem 2.2.8.** If $\phi : X \to Y$ satisfies condition (C) then there is a sheaf of $\mathcal{O}_S$-algebras $\mathcal{C}_\phi$ and an isomorphism of functors from the category $\mathbf{LFAAlg}/\mathcal{O}_S$ to the category of sets.

$$\mathcal{H}om_{\mathbf{LFAAlg}/\mathcal{O}_S}(\mathcal{C}_\phi, -) \cong \mathcal{C}_\phi.$$ 

Further, if $f : S' \to S$ is a morphism of schemes and $\phi' : X' \to Y'$ the fiber product of $\phi$ with $S'$ then $\phi'$ also satisfies condition (C), and we have a canonical identification $f^*\mathcal{C}_\phi \cong \mathcal{C}_{\phi'}$.

The proof of Theorem 2.2.8 is constructive and occupies Section 2.3 below. In particular, for a morphism $\phi$ with property (C) the Clifford algebra $\mathcal{C}_\phi$ is constructed in Definition 2.3.11 below.

**Remark 2.2.9.** In particular, if $\mathcal{C}_\phi$ is itself locally free (for example, in the case $S$ is the spectrum of a field), then it represents the functor $\mathcal{C}_\phi$, and is the unique locally free algebra (up to canonical isomorphism) that does so.

### 2.3. Construction of the Clifford algebra

To define the Clifford algebra, we first describe a construction on quasi-coherent sheaves, having a somewhat analogous property.

**Notation 2.3.1.** Given an $S$-scheme $\pi : Y \to S$, and $\mathcal{N}$ a sheaf of $\mathcal{O}_Y$-modules, let

$$\mathcal{N}_\pi = (\pi_*(\mathcal{N}^\vee))^\vee = \mathcal{H}om_{\mathcal{O}_S}\left(\pi_*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{O}_Y)), \mathcal{O}_S\right).$$

The association $\mathcal{N} \mapsto \mathcal{N}_\pi$ defines a covariant functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_S$-modules. In general, this functor has no good properties (e.g., it rarely sends quasi-coherent sheaves to quasi-coherent sheaves). We will show that, under certain assumptions, this construction has a universal property and behaves well with respect to base change.

**Proposition 2.3.2.** Suppose that $\pi : Y \to S$ and $\mathcal{N}$ is a friendly sheaf of $\mathcal{O}_Y$-modules. Then:

1. There is a canonical natural isomorphism of functors (in $\mathcal{M} \in \mathbf{LF}/\mathcal{O}_S$)

   $$\Xi : \pi_*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, \pi^*\mathcal{M}) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{N}_\pi, \mathcal{M}).$$

   In particular, there is a canonical arrow $\eta : \mathcal{N} \to \pi^*\mathcal{N}_\pi$ such that $\Xi$ sends an arrow $\mathcal{N} \to \mathcal{M}$ to the composition

   $$\mathcal{N} \xrightarrow{\eta} \pi^*\mathcal{N}_\pi \xrightarrow{\pi^*\Xi} \pi^*\mathcal{M}.$$

2. Given a pullback square

   $$\begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ \downarrow{\pi'} & & \downarrow{\pi} \\ S' & \xrightarrow{f} & S \end{array}$$

   we have $f^*(\mathcal{N}_\pi) = (f^*\mathcal{N})^\vee$.

Before proving Proposition 2.3.2 we require a few lemmas.

**Lemma 2.3.3.** Let $X$ be a quasi-compact topological space, $\Lambda$ a cofiltered category, and $\mathcal{F}_\lambda : \Lambda \to \mathbf{Ab}_X$ a cofiltered system of Abelian sheaves on $X$. Set $\mathcal{F} = \lim_{\lambda} \mathcal{F}_\lambda$. Then $\Gamma(\mathcal{F}, X) = \lim_{\lambda} \Gamma(\mathcal{F}_\lambda, X)$.

**Proof.** By definition, we have that $\mathcal{F}$ is the sheafification of the presheaf which associates to each open set $U$, the set $\lim_{\lambda} \mathcal{F}_\lambda$. Let $\mathcal{F}$ be the category whose elements are open covers of $X$, and with morphisms

$$\{U_i \subset X\}_{i \in I} \to \{V_j \subset X\}_{j \in J}$$

given by refinements— that is by maps of sets $\phi : I \to J$ such that we have inclusions $U_i \subset V_{\phi(i)}$. This is a filtered category, via common refinements. By definition of the sheafification, writing $U$ for a cover $\{U_i\}_{i \in I}$, we have:

$$\mathcal{F}(X) = \lim_{U \in \mathcal{U}} \ker \left( \prod_{i \in I} \lim_{\lambda} \mathcal{F}_\lambda(U_i) \to \prod_{i,j \in I^2} \lim_{\lambda} \mathcal{F}_\lambda(U_i \cap U_j) \right).$$

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Since $X$ is quasi-compact, if we set $\mathcal{X}'$ to be the subcategory of $\mathcal{X}$ consisting of finite coverings, we find that $\mathcal{X}'$ is cofinal in $\mathcal{X}$, and so we can take limits over $\mathcal{X}'$ instead of $\mathcal{X}$. In particular, we find that for a cover $U = \{U_i\}$ in $\mathcal{X}'$, products and coproducts (direct sums) coincide over the finite index sets $I$ and $I'$. Therefore, we have:

$$\mathcal{F}(X) = \lim_{U \in \mathcal{X}'} \ker \left( \bigoplus_{i \in I} \mathcal{F}_\lambda(U_i) \to \bigoplus_{i,j \in I'} \mathcal{F}_\lambda(U_i \cap U_j) \right).$$

In particular, since the direct sum is a colimit, it commutes with the colimit taken over $\lambda \in \Lambda$. Since the kernel is a finite limit, it also commutes with the cofiltered colimit in $\lambda$, and finally, the two colimits described by $U$ and $\lambda$ commute. We therefore have

$$\mathcal{F}(X) = \lim_{\lambda \in \Lambda} \lim_{U \in \mathcal{X}'} \ker \left( \bigoplus_{i \in I} \mathcal{F}_\lambda(U_i) \to \bigoplus_{i,j \in I'} \mathcal{F}_\lambda(U_i \cap U_j) \right)$$

$$= \lim_{\lambda \in \Lambda} \lim_{U \in \mathcal{X}'} \ker \left( \prod_{i \in I} \mathcal{F}_\lambda(U_i) \to \prod_{i,j \in I'} \mathcal{F}_\lambda(U_i \cap U_j) \right)$$

$$= \lim_{\lambda \in \Lambda} \lim_{\lambda' \in \Lambda} \ker \left( \prod_{i \in I} \mathcal{F}_{\lambda'}(U_i) \to \prod_{i,j \in I'} \mathcal{F}_{\lambda'}(U_i \cap U_j) \right)$$

where the last equality follows from the fact that $\mathcal{F}_\lambda$ is a sheaf. \[ \square \]

**Lemma 2.3.4.** Let $\pi : Y \to S$ be a quasi-compact morphism, $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, and $\mathcal{G} \in \mathcal{L}/\mathcal{O}_S$. Then the natural morphism of sheaves of $\mathcal{O}_S$-modules:

$$\pi_*(\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{G} \to \pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^* \mathcal{G})$$

is an isomorphism.

**Proof.** Since tensor product and pushforward commute with flat base change, we may work locally on $S$ and assume that

$$\mathcal{G} = \mathcal{O}_S^{\oplus I}$$

for some index set $I$ (the exponent indicating direct sum indexed by the elements of $I$). We have

$$\pi_*(\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{G} = \pi_*(\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_S^{\oplus I} = \pi_*(\mathcal{F}^{\oplus I})$$

On the other hand, we have:

$$\pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^* \mathcal{G}) = \pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^* \mathcal{O}_S^{\oplus I})$$

Since tensor and direct sum commute, we have

$$\pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^* \mathcal{O}_S^{\oplus I}) = \pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_S^{\oplus I}) = \pi_*(\mathcal{F}^{\oplus I})$$

But since we can write the direct sum as a cofiltered colimit of finite direct sums, by Lemma 2.3.3 we can identify $\pi_*(\mathcal{F}^{\oplus I})$ with $\pi_*(\mathcal{F}^{\oplus I})$, completing the proof. \[ \square \]

**Proof of Proposition 2.3.2.** For part 1, using the fact that $\mathcal{N}$ is finite locally free, we have

$$\pi_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, \pi^* \mathcal{M}) \cong \pi_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{N}^\vee \otimes_{\mathcal{O}_Y} \pi^* \mathcal{M})$$

$$\cong \pi_*(\mathcal{N}^\vee \otimes_{\mathcal{O}_Y} \pi^* \mathcal{M})$$

and, since $\mathcal{M}$ is locally free, we have, by Lemma 2.3.4 that

$$\pi_*(\mathcal{N}^\vee \otimes_{\mathcal{O}_Y} \pi^* \mathcal{M}) \cong \pi_*(\mathcal{N}^\vee) \otimes_{\mathcal{O}_S} \mathcal{M}.$$
Finally, since $\pi_*(\mathcal{N}^\vee)$ is finite locally free, we have
\[
\pi_*(\mathcal{N}^\vee) \otimes_{\mathcal{O}_S} \mathcal{M} \cong \mathcal{H} \text{om}_{\mathcal{O}_S}(\mathcal{O}_S, \pi_*(\mathcal{N}^\vee) \otimes_{\mathcal{O}_S} \mathcal{M}) \\
\cong \mathcal{H} \text{om}_{\mathcal{O}_S}(\pi_*(\mathcal{N}^\vee)^\vee, \mathcal{M}) \\
\cong \mathcal{H} \text{om}_{\mathcal{O}_S}(\mathcal{N}_\pi, \mathcal{M}),
\]
as desired.

For part 2, as $\mathcal{N}$ is friendly with respect to $\pi$, we have
\[
f^*\pi_*(\mathcal{N}^\vee) \cong \pi'_*f^*_Y(\mathcal{N}^\vee) \\
\cong \pi'_*f^*_Y\mathcal{H} \text{om}_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{O}_Y)
\]
and since $\mathcal{N}$ is finite locally free, we have
\[
\pi'_*f^*_Y\mathcal{H} \text{om}_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{O}_Y) \cong \pi'_*\mathcal{H} \text{om}_{\mathcal{O}_Y}(f^*_YN, \mathcal{O}_Y) \\
= \pi'_*(f^*_YN)^\vee.
\]
Since $\pi_*(\mathcal{N}^\vee)$ is finite locally free,
\[
f^*(\mathcal{N}_\pi) = f^*(\pi_*(\mathcal{N}^\vee))^\vee = f^*\mathcal{H} \text{om}_{\mathcal{O}_S}(\pi_*(\mathcal{N}^\vee), \mathcal{O}_S) \\
= \mathcal{H} \text{om}_{\mathcal{O}_{S'}}(f^*(\pi_*(\mathcal{N}^\vee)), \mathcal{O}_{S'}) \\
= \mathcal{H} \text{om}_{\mathcal{O}_{S'}}(\pi'_*(f^*_YN)^\vee, \mathcal{O}_{S'}) \\
= \pi'_*((f^*_YN)^\vee)^\vee \\
= (f^*_YN)^\pi^\vee,
\]
as claimed. 

To use this module in the construction of the Clifford algebra, we first introduce a “relative free algebra construction.”

**Definition 2.3.5.** A **unital $\mathcal{O}_S$-module** is a quasi-coherent sheaf of $\mathcal{O}_S$-modules $\mathcal{N}$, together with a $\mathcal{O}_S$-module morphism $\epsilon_\mathcal{N} : \mathcal{O}_S \to \mathcal{N}$, referred to as the **unit morphism**. A morphism of unital $\mathcal{O}_S$-modules is simply an $\mathcal{O}_S$-module morphism which commutes with the unit morphisms. We let $\text{UQC}/\mathcal{O}_S$ denote the category of unital $\mathcal{O}_S$-modules.

**Notation 2.3.6.** For any scheme $Z$, let
\[
\text{Un} : \text{Alg}/\mathcal{O}_Z \to \text{UQC}/\mathcal{O}_Z
\]
denote the canonical forgetful functor that sends an $\mathcal{O}_Z$-algebra $\mathcal{A}$ to the unital module given by the underlying $\mathcal{O}_Z$-module of $\mathcal{A}$ together with the identity element $\mathcal{O}_Z \to \mathcal{A}$.

The functor $\text{Un}$ has a left adjoint. We note that there is a forgetful map from the category of quasi-coherent sheaves of $\mathcal{O}_S$-algebras to the category of unital $\mathcal{O}_S$-modules. The left adjoint to this is constructed as follows:

**Lemma 2.3.7.** Let $S$ be a scheme. There is a “free algebra” functor
\[
\mathcal{N}, \epsilon \mapsto F(\mathcal{N}, \epsilon) : \text{UQC}/\mathcal{O}_S \to \text{Alg}/\mathcal{O}_S
\]
that is left adjoint to $\text{Un}$. Moreover, for a morphism $f : T \to S$, we have
\[
f^*F(\mathcal{N}, \epsilon) = F(f^*\mathcal{N}, f^*\epsilon).
\]
We will usually omit $\epsilon$ from the notation and write $F(\mathcal{N})$.

**Remark 2.3.8.** The counit of the adjunction yields a canonical morphism
\[
\xi : \mathcal{N} \to \text{Un}F(\mathcal{N})
\]
of unital $\mathcal{O}_S$-modules such that a morphism of $\mathcal{O}_S$-algebras $F(\mathcal{N}) \to \mathcal{B}$ is associated to the composition of $\mathcal{O}_S$-module maps $\mathcal{N} \to \text{Un}F(\mathcal{N}) \to \text{Un}\mathcal{B}$. 

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Proof. We construct \( F(\mathcal{N}) \) as the sheafification of a presheaf as follows. For an affine open set \( U = \text{Spec} \ R \subset S \), write \( \mathcal{M}(U) = N \), and \( \iota(U) = i : R \to N \). We consider the algebra \( F(N) \) to be the free associative \( R \)-algebra \( R(\langle N \rangle) \) (the tensor algebra) modulo the ideal \( I \) generated by the expressions of the form \( i(r) - r \) where \( r \in R \), the element \( r \) being viewed on the right as taken from the coefficients of the tensor algebra. It is clear that this presheaf of algebras has the corresponding universal property among presheaves of algebras, and hence by the universal property of sheafification the resulting sheafified algebra has the correct universal property as well.

The assertion concerning the behavior under pullback will follows from the fact that, on the level of affine schemes, this description is preserved by tensor products with respect to a homomorphism of rings \( R \to R' \) and the construction of the tensor algebra commutes with base change to \( R' \). \( \square \)

In classical constructions of the Clifford algebra of a homogeneous form (see, for example A), the Clifford algebra is defined as a free associative algebra, generated by an underlying vector space of the form, modulo a certain ideal. The construction above gives an analog of this free algebra; we will now describe the construction of the corresponding ideal in the relative case.

**Definition 2.3.9.** Let \( \pi : Y \to S \) be a morphism of schemes, \( \mathcal{A} \) a quasi-coherent sheaf of \( \mathcal{O}_Y \)-algebras. An agreeable algebra for \( \mathcal{A} \) is a quasi-coherent sheaf of \( \mathcal{O}_S \)-algebras \( \mathcal{B} \) together with a morphism \( v_{\mathcal{B}} : \mathcal{A} \to \pi^* \mathcal{B} \) of unital \( \mathcal{O}_Y \)-modules in the sense of Definition 2.3.5. A morphism of agreeable algebras is a morphism of sheaves of algebras \( f : \mathcal{B} \to \mathcal{D} \) such that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{v_{\mathcal{B}}} & \pi^* \mathcal{B} \\
\downarrow & & \downarrow f \\
\pi^* \mathcal{D} & \xrightarrow{v_{\mathcal{D}}} & \pi^* \mathcal{D}
\end{array}
\]

We let \textbf{Agree}/\( \mathcal{A} \) denote the category of agreeable algebras for \( \mathcal{A} \). We say that an agreeable map \( f : \mathcal{B} \to \mathcal{D} \) is a compromise for \( (\mathcal{B}, v_{\mathcal{B}}) \) if \( v_{\mathcal{D}} : \mathcal{A} \to \pi^* \mathcal{D} \) is a morphism of \( \mathcal{O}_Y \)-algebras. We write \textbf{Comp}_{\mathcal{A}}(\mathcal{B}, \mathcal{D}) \) for the set of compromises from \( (\mathcal{B}, v_{\mathcal{B}}) \) to \( \mathcal{D} \).

**Lemma 2.3.10.** Let \( \pi : Y \to S \) be a morphism of schemes, \( \mathcal{A} \) a sheaf of \( \mathcal{O}_Y \)-algebras, such that \( \mathcal{A} \) and \( \mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \) are friendly with respect to \( \pi \). Let \( \mathcal{B}, v : \mathcal{A} \to \pi^* \mathcal{B} \) an agreeable algebra for \( \mathcal{A} \). Then there is a sheaf of ideals \( \mathcal{I}(v) \subset \mathcal{B} \) such that

1. the induced map \( \mathcal{A} \to \pi^* (\mathcal{B}/\mathcal{I}(v)) \) is a map of \( \mathcal{O}_Y \)-algebras, that is, it is a compromise for \( (\mathcal{B}, v) \),
2. for any other quasi-coherent sheaf of ideals \( \mathcal{J} \subset \mathcal{B} \) such that the induced map \( \mathcal{A} \to \pi^* (\mathcal{B}/\mathcal{J}) \) is a map of \( \mathcal{O}_Y \)-algebras, \( \mathcal{I}(v) \subset \mathcal{J} \),
3. for any quasi-coherent sheaf of \( \mathcal{O}_S \)-algebras \( \mathcal{D} \), we have a natural bijection

\[
\text{Comp}_{\mathcal{A}}(\mathcal{B}, \mathcal{D}) = \text{Hom}_{\text{Alg}/\mathcal{O}_S}(\mathcal{B}/\mathcal{I}(v), \mathcal{D})
\]

4. for \( f : T \to S \) a morphism of schemes, we have \( f^*(\mathcal{B}/\mathcal{I}(v)) \cong (f^*\mathcal{B})/f^*\mathcal{I}(f^*v) \)

**Proof of Lemma 2.3.10.** Let \( (\mathcal{D}, v_{\mathcal{D}}) \) be an agreeable algebra for \( \mathcal{A} \). Via the multiplication maps on \( \mathcal{D} \) and \( \mathcal{A} \), we may define two maps of \( \mathcal{O}_Y \)-modules:

\[
m_1^\mathcal{D} : \mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \xrightarrow{m_1_{\mathcal{D}}} \mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \xrightarrow{v_{\mathcal{A}}} \pi^* \mathcal{D}
\]

\[
m_2^\mathcal{D} : \mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \xrightarrow{v_{\mathcal{A}}} \pi^* \mathcal{D} \otimes_{\mathcal{O}_Y} \mathcal{A} \xrightarrow{\pi^* m_{\mathcal{D}}} \pi^* \mathcal{D}
\]

and by definition, \( v_{\mathcal{D}} \) is a map of \( \mathcal{O}_Y \)-algebras if \( m_1^\mathcal{D} = m_2^\mathcal{D} \). Consider the difference \( \delta_{\mathcal{D}} = m_1^\mathcal{D} - m_2^\mathcal{D} \in \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A}, \pi^* \mathcal{D}) \). It follows from the construction that this is natural in the sense that if we are given \( f : (\mathcal{D}, v_{\mathcal{D}}) \to (\mathcal{D}', v_{\mathcal{D}'}) \), a morphism of agreeable algebras for \( \mathcal{A} \), then we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{\delta_{\mathcal{D}}} & \pi^* \mathcal{D} \\
\downarrow & & \downarrow \pi^* f \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{\delta_{\mathcal{D}'}} & \pi^* \mathcal{D}'
\end{array}
\]
and we see that a morphism $(\mathcal{D}, v_{\mathcal{D}}) \to (\mathcal{D}', v_{\mathcal{D}'})$ is a compromise (i.e., $v_{\mathcal{D}'}$ is an $\mathcal{O}_Y$-algebra map) if and only if $\delta_{\mathcal{D}'} = 0$.

Since $\mathcal{A}$ and $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A}$ are both friendly, given a morphism as before $f : (\mathcal{D}, v_{\mathcal{D}}) \to (\mathcal{D}', v_{\mathcal{D}'})$ of agreeable algebras for $\mathcal{A}$, the diagram above corresponds to a diagram of $\mathcal{O}_S$-modules:

$$
\begin{array}{ccc}
(\mathcal{A} \otimes \mathcal{A})_\pi & \xrightarrow{(\delta_{\mathcal{D}})_\pi} & \mathcal{D} \\
\downarrow \phi & & \downarrow f \\
(\delta_{\mathcal{D}'})_\pi & \xleftarrow{(\delta_{\mathcal{D}})_\pi} & \mathcal{D}'
\end{array}
$$

and in particular, $v_{\mathcal{D}}$ is an $\mathcal{O}_Y$-algebra map if and only if the map $(\delta_{\mathcal{D}})_\pi$ is the zero map. Define $\mathcal{I}(v_{\mathcal{D}})$ to be the sheaf of ideals generated by the image of $(\delta_{\mathcal{D}})_\pi$. It follows from that construction that for any morphism $f : (\mathcal{D}, v_{\mathcal{D}}) \to (\mathcal{D}', v_{\mathcal{D}'})$ of agreeable algebras, we have $f(\mathcal{I}(v_{\mathcal{D}})) \subset \mathcal{I}(v_{\mathcal{D}'})$. In particular, we find that considering the quotient map $(\mathcal{B}, v) \to (\mathcal{B}/\mathcal{I}(v), \overline{v})$ we obtain Part 3, and consequently this quotient map is a compromise, verifying part 1.

Part 2 follows from the fact that for an ideal sheaf $\mathcal{J} \trianglelefteq \mathcal{B}$, we have seen that the induced map $(\mathcal{B}, v) \to (\mathcal{B}/\mathcal{J}, \overline{v})$ is a compromise if and only if $(\delta_{\mathcal{J}})_\pi$ is zero, which by the commutativity of diagram (1) happens if and only if $\mathcal{I}(v) \subset \mathcal{J}$ as desired.

Finally, part 4 follows from the fact that the formation of the image sheaf theoretic image commutes with base change, as does the operation of an ideal sheaf generated by a subsheaf. □

We are now prepared to give the definition of the Clifford algebra.

**Definition 2.3.11.** Let $\phi : X \to Y$ be a morphism of $S$-schemes satisfying condition (C)(see Notation 2.2.6). By Proposition 2.3.2(1), we have a morphism

$$
\eta : \phi_* \mathcal{O}_X \to \pi^* ( (\phi_* \mathcal{O}_X)_\pi )
$$

Which we can consider as a map of unital $\mathcal{O}_Y$-modules via the structure map $\mathcal{O}_Y \to \phi_* \mathcal{O}_X$. Consider the counit map

$$
\xi : \pi^* ( (\phi_* \mathcal{O}_X)_\pi ) \subset F(\pi^* ( (\phi_* \mathcal{O}_X)_\pi )) = \pi^* F((\phi_* \mathcal{O}_X)_\pi)
$$

of Remark 2.3.8, and let $v_\phi$ be the composition:

$$
\phi_* \mathcal{O}_X \xrightarrow{\eta} \pi^* ( \phi_* \mathcal{O}_X )_\pi \xrightarrow{\xi} \pi^* F((\phi_* \mathcal{O}_X)_\pi)
$$

Letting $\mathcal{I}(v_\phi) \trianglelefteq F((\phi_* \mathcal{O}_X)_\pi)$ be the ideal as defined in Lemma 2.3.10, we then define the Clifford algebra $\mathcal{C}_\phi$ to be the sheaf of $\mathcal{O}_S$-algebras given by

$$
\mathcal{C}_\phi = F((\phi_* \mathcal{O}_X)_\pi) / \mathcal{I}(v_\phi)
$$

We now show that this construction behaves well with respect to pullbacks.

**Lemma 2.3.12.** Let $\phi : X \to Y$ be a morphism of $S$-schemes satisfying condition (C). If $f : T \to S$ is any morphism, and $\phi_T : X_T \to Y_T$ the pullback morphism, then there is a natural isomorphism of sheaves of algebras

$$
f^* \mathcal{C}_\phi \cong \mathcal{C}_{\phi_T}.
$$

**Proof.** By Lemma 2.3.7, we have

$$
f^* F((\phi_* \mathcal{O}_X)_\pi) = F(f^* ((\phi_* \mathcal{O}_X)_\pi)).
$$

Since the hypotheses imply that $(\phi_* \mathcal{O}_X)_\pi$ is finite and locally free, it follows from Proposition 2.3.2(2) that

$$
f^* F((\phi_* \mathcal{O}_X)_\pi) = F((\phi_T)_* \mathcal{O}_{X_T}).
$$

The result now follows from Lemma 2.3.10(4). □

We now prove that the algebra constructed above has the desired properties.
Proof of Theorem 2.2.8. The fact that the hypotheses are preserved by base change follows from the analogous fact for friendly sheaves, and the fact that the construction is preserved by base change follows from Lemma 2.3.12. Without loss of generality, via changing the base, we may reduce to the case $S$ is affine and hence check this by simply comparing global sections on each side. We have

$$\text{Hom}_{\mathcal{O}_S-\text{alg}}(\phi_*\mathcal{O}_X, \pi^*\mathcal{B}) = \{ \rho \in \text{Hom}_{\mathcal{O}_Y}(\phi_*\mathcal{O}_X, \pi^*\mathcal{B}) \mid \rho \text{ an alg. hom.} \}$$

and using Proposition 2.3.2(1), this is identified with

$$\{ \psi \in \text{Hom}_{\mathcal{O}_S}(\phi_*\mathcal{O}_X, \pi^*\mathcal{B}) \mid \phi_*\mathcal{O}_X \xrightarrow{\eta} \pi^*((\phi_*\mathcal{O}_X)_\pi) \xrightarrow{\psi} \pi^*\mathcal{B} \text{ an alg. hom.} \}$$

using Lemma 2.3.7, this is in bijection with

$$\{ \psi \in \text{Hom}_{\mathcal{O}_S-\text{alg}}(F(\phi_*\mathcal{O}_X), \mathcal{B}) \mid \phi_*\mathcal{O}_X \xrightarrow{\psi} F(\pi^*((\phi_*\mathcal{O}_X)_\pi))(\psi) \text{ an alg. hom.} \}$$

which we can identify with $\text{Comp}_{\phi_*\mathcal{O}_X}(F(\pi^*((\phi_*\mathcal{O}_X)_\pi)), \mathcal{B})$. Finally, by Lemma 2.3.10(3), this is in bijection with

$$\text{Hom}_{\mathcal{O}_S-\text{alg}}(F(\pi^*((\phi_*\mathcal{O}_X)_\pi))/\mathcal{I}_{\mathcal{O}_S}, \mathcal{B}) = \text{Hom}_{\mathcal{O}_S-\text{alg}}(\mathcal{C}_\phi, \mathcal{B})$$

as desired. \qed

3. Representations and Ulrich Bundles

Having discussed the Clifford algebra, in this section we describe its representations. Since we will be considering a morphism of $S$-schemes $\phi : X \to Y$ and representations on sheaves of $\mathcal{O}_S$-modules, there are pullback functors on categories of representations induced by base changes $T \to S$. Consequently, the various categories of representations, as the base changes, will fit together into a stack, and this will be the natural way to describe the arithmetic and geometry of these representations.

For considering representations of Clifford algebras, again we fix a base scheme $S$, and it will be useful to restrict to only certain morphisms $\phi : X \to Y$ of $S$-schemes. We strengthen property (C) to also require our base variety to be proper and have connected fibers:

Definition 3.0.1. Let $\phi : X \to Y$ be a morphism of $S$-schemes. We say that $\phi$ has property (C*) if

1. $\pi_Y$ is proper, flat, and of finite presentation;
2. $\phi$ is finite locally free and surjective;
3. $\mathcal{O}_X, \phi_*\mathcal{O}_X$ and $\phi_*\mathcal{O}_X \otimes \mathcal{O}_Y$ are all friendly with respect to $\pi_Y$.

Note that $\mathcal{O}_Y$ being friendly with respect to $\pi_Y$ coincides with the notation that $\pi_Y$ is cohomologically flat in dimension 0. In applications, $S$ will often be the spectrum of a field and $Y$ will be a geometrically integral proper variety over $S$.

Definition 3.0.2. Let $S$ be a scheme, and $\mathcal{A}$ a sheaf of $\mathcal{O}_S$-algebras. We define $\text{Rep}_{\mathcal{A}}$ to be the category whose objects are pairs $(f, W)$ where $W$ is a locally free sheaf of $\mathcal{O}_S$-modules of finite rank, and where $f : \mathcal{A} \to \mathcal{O}_S(W)$ is a homomorphism of $\mathcal{O}_S$-algebras. A morphism $(f, W) \to (g, U)$ in $\text{Rep}_{\mathcal{A}}$ is a morphism of $\mathcal{O}_S$-modules $\lambda : W \to U$ such $\lambda(f(a))(w) = g(a)(\lambda(w))$ for sections $w, a$ of $W$ and $\mathcal{A}$ respectively. We let $\text{Rep}_{\mathcal{A}}^n$ denote the subcategory of pairs $(f, W)$ where $W$ has rank $n$ over $\mathcal{O}_S$.

From these, we obtain stacks $\text{Rep}_{\mathcal{A}}$ (respectively $\text{Rep}_{\mathcal{A}}^n$) defined over $S$ which associates to $U \to S$ the (isomorphisms in the) category $\text{Rep}_{\mathcal{A}_U}$ (respectively $\text{Rep}_{\mathcal{A}_U}^n$). In the case of a morphism $\phi : X \to Y$ of $S$-schemes, we simply write $\text{Rep}_{\phi}$ and $\text{Rep}_{\phi}^n$ to denote $\text{Rep}_{\mathcal{A}_\phi}$ and $\text{Rep}_{\mathcal{A}_\phi}^n$ respectively.

Definition 3.0.3. Let $\phi : X \to Y$ be a morphism of $S$-schemes. We say that a coherent sheaf $V$ of $\mathcal{O}_X$-modules is Ulrich for $\phi$ if there is a fppf covering $S' \to S$ (that is, a surjective flat map, locally of finite presentation) and a section $r \in H^0(S', \mathbb{Z})$ such that

$$\phi_*V_{S'} \cong \pi_{S'} \mathcal{O}_S^{\oplus r}.$$

The Ulrich sheaves (respectively, the Ulrich sheaves with pushforward of rank $m$ for a fixed integer $m$) form a full subcategory of the category of coherent sheaves on $X$, which we denote by $\text{Ulrr}_\phi$ (respectively $\text{Ulrr}_\phi^m$).
Ulrich sheaves form a substack $\mathcal{U}_{lr}$ of the $S$-stack of coherent sheaves on $X$ by setting $\mathcal{U}_{lr}(U) = \mathcal{U}_{lr}(U^c)$ for $U$ an $S$-scheme. (Of course, one should add additional hypotheses, such as flatness over the base, if one hopes to produce an algebraic stack but this is inessential here.)

The following proposition is a generalization of [VdB87, Proposition 1].

**Proposition 3.0.4.** Let $\phi : X \to Y$ be a morphism satisfying condition $(C*)$. Then there is an isomorphism of stacks $\theta : \mathcal{R}_{\phi} \to \mathcal{U}_{lr}$, inducing equivalences of categories $\mathcal{R}_{\phi} \cong \mathcal{U}_{lr}$. In the case that $\phi$ has constant degree, this gives for every positive integer $m$ an equivalence $\mathcal{R}_{\phi}^m \cong \mathcal{U}_{lr}^m$. In particular, every representation of the Clifford algebra has rank a multiple of $d$.

**Lemma 3.0.5.** Suppose that $\pi : Y \to S$ is a morphism of schemes such that the natural map

$$\mathcal{O}_S \to \pi_* \mathcal{O}_Y$$

is an isomorphism. Given an affine morphism $H \to S$, the natural map

$$\text{Hom}_S(S, H) = \text{Hom}_S(Y, H)$$

is bijective.

**Proof.** This is an immediate consequence of [Gro60, Ch. I, Prop. 2.2.4].

**Proof of Proposition 3.0.4.** By the universal property of the Clifford algebra, it is easy to see that $\mathcal{R}_{\phi}$ is equivalent to the stack whose objects over $T$ consist of pairs $(W, \psi)$ where $W$ is a finite locally free sheaf of $\mathcal{O}_T$-modules, and where $\psi : \phi_* \mathcal{O}_X \to \text{End}_{\mathcal{O}_Y}(\pi^*W)$ is a $\phi_* \mathcal{O}_X$-module structure on $\pi^*W$, and where morphisms must preserve the $\pi_* \mathcal{O}_X$-module structure. This in turn, is equivalent to the stack whose objects over $T$ consist of triples $(W, V, f)$, where

- $W$ is a finite locally free sheaf of $\mathcal{O}_T$-modules,
- $V$ is a coherent sheaf of $\mathcal{O}_Y$-modules,
- $f : \phi_* V \to \pi^* W$ is an isomorphism of $\mathcal{O}_Y$-modules, and
- morphisms $(W, V, f) \to (W', V', f')$ are given by maps $\alpha : W \to W'$ and $\beta : V \to V'$ such that the diagram

$$\begin{array}{ccc}
\phi_* V & \xrightarrow{\phi_* \beta} & \phi_* V' \\
\downarrow f & & \downarrow \beta' \\
\pi^* W & \xrightarrow{\pi^* \alpha} & \pi^* W'
\end{array}$$

commutes.

We define $\theta : \mathcal{R}_{\phi} \to \mathcal{U}_{lr}$ by sending $(W, V, f)$ to $V$. Essential surjectivity of $\theta$ follows from the definition of the Ulrich stack.

Now consider $\mathcal{R}_{\phi}$ as a fibered category over $\mathcal{U}_{lr}$. To see that we have an equivalence of stacks, it suffices to show that, for an object $V \in \mathcal{U}_{lr}$, that the fiber category over $V$ is a groupoid such that for every pair of objects $a, b \in \mathcal{R}_{\phi}(V)$, the morphism set $\text{Hom}_{\mathcal{R}_{\phi}(V)}(a, b)$ consists of a single element. To verify this, we choose $a = (W_1, V, \alpha_1)$, $b = (W_2, V, \alpha_2)$. Define $\gamma : \pi^* W_1 \to \pi^* W_2$ as the unique isomorphism of $\mathcal{O}_Y$-modules making the diagram

$$\begin{array}{ccc}
\pi^* W_1 & \xrightarrow{\alpha_1} & \gamma \\
\downarrow \alpha_2 & & \downarrow \gamma \\
\pi^* W_2
\end{array}$$

commute. We claim that the map $\gamma$ comes from a unique isomorphism $\tilde{\gamma} : W_1 \to W_2$. To do this, we consider the sheaf $H = \mathcal{H} \text{Hom}_{\mathcal{O}_S}(W_1, W_2)$. Local freeness of $W_i$ implies that we have $\pi^* H = \mathcal{H} \text{Hom}_{\mathcal{O}_Y}(\pi^* W_1, \pi^* W_2)$. Setting $H \to S$ to be the underlying affine scheme of the vector bundle $H$, we then have $H \times_S Y$ is the underlying affine scheme of the vector bundle $\pi^* H$. We would like to show that a given section $\gamma : Y \to H \times_S Y$ comes via pullback from a unique section $\tilde{\gamma} : S \to H$. But this is exactly Lemma 3.0.5. 

We can similarly define projectively Ulrich bundles and Azumaya representations as follows.

**Definition 3.0.6.** Let \( S \) be a scheme, and \( \mathcal{A} \) a sheaf of \( \mathcal{O}_S \)-algebras. The category of Azumaya representations, denoted \( \text{AzRep}_{\mathcal{A}} \) (respectively, Azumaya representations of degree \( n \), denoted \( \text{AzRep}^n_{\mathcal{A}} \)) is the category whose objects are pairs \((f, B)\) where \( B \) is a sheaf of Azumaya algebras over \( S \) (respectively, Azumaya algebras over \( S \) of degree \( n \)), and where \( f : \mathcal{A} \rightarrow B \) is an isomorphism of \( \mathcal{O}_S \)-algebras. A morphism \((f, B) \rightarrow (g, C)\) will be a morphism of \( \mathcal{O}_S \)-algebras \( B \rightarrow C \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & B \\
\downarrow g & & \downarrow \\
C & \xrightarrow{g} & C
\end{array}
\]

As before, we obtain a stack \( \text{AzRep}^n_{\mathcal{A}} \) (respectively \( \text{AzRep}^n_{\mathcal{A}} \)) defined over \( S \) which associates to \( U \rightarrow S \) the category \( \text{AzRep}_{\mathcal{A}_U} \) (respectively \( \text{AzRep}^n_{\mathcal{A}_U} \)). This stack carries a universal sheaf of Azumaya algebras \( \mathcal{A}_{\mathcal{A}} \) and a universal representation \( \mathcal{A}_{\text{AzRep}^n_{\mathcal{A}}} \rightarrow \mathcal{A}_{\mathcal{A}} \). We note that there is a natural morphism of stacks \( \text{Rep}^n_{\mathcal{A}} \rightarrow \text{AzRep}^n_{\mathcal{A}} \) which gives \( \text{Rep}^n_{\mathcal{A}} \) the structure of a \( G_m \)-gerbe over \( \text{AzRep}^n_{\mathcal{A}} \), whose class over an object \((f, B)\) is precisely the Brauer class of \( B \). In particular, the global class of the gerbe is given by the algebra \( \mathcal{A}_{\mathcal{A}} \).

**Definition 3.0.7.** Let \( \phi : X \rightarrow Y \) be morphisms of \( S \)-schemes. The category of projectively Ulrich bundles \( \text{PrUlr}_\phi \) is the category of sheaves of modules \( V \) on \( X \) such that the projective bundle \( \mathbb{P}(\phi_! V) \rightarrow Y \) is isomorphic over \( Y \) to \( P \times_S Y \rightarrow Y \), where \( P \rightarrow S \) is a Brauer-Severi scheme.

This is to say, we require that \( P \) is isomorphic \( \text{fppf} \)-locally on \( S \) to \( \mathbb{P}^n_S \).

Comparing automorphism groups, for a morphism \( \phi : X \rightarrow Y \) satisfying condition (\( C_\ast \)), we have a diagram of stacks that commutes up to 2-isomorphism

\[
\begin{array}{ccc}
\text{Rep}^{md}_{\phi} & \xrightarrow{\alpha} & \text{AzRep}^{md}_{\phi} \\
\downarrow & & \downarrow \\
\text{Ulir}^{m}_{\phi} & \xrightarrow{\alpha} & \text{Prulir}^{m}_{\phi}
\end{array}
\]

in which the vertical arrows are equivalences.

**Remark 3.0.8.** Note that the horizontal arrows in the above diagram are \( G_m \)-gerbes, so the stacks in each row of the diagram have isomorphic sheafifications.

**Definition 3.0.9.** We say that a representation \((f, W) \in \text{Rep}_{\phi}\) is a specialization if \( f : \mathcal{A}_\phi \rightarrow \text{End}(W) \) is surjective (and similarly for objects in \( \text{AzRep}_{\phi}\)).

Recall that a vector bundle \( V/X \) is called simple if its automorphism sheaf over \( S \) is \( G_m \).

**Definition 3.0.10.** Let \( \text{SplUlr}_{\phi}, \text{PrSplUlr}_{\phi} \) denote the categories of simple Ulrich bundles and projectively bundles respectively. Let \( \text{SplUlr}^m_{\phi}, \text{PrSplUlr}^m_{\phi} \) denote the associated substacks of \( \text{ulir}^m_{\phi}, \text{prulir}^m_{\phi} \).

**Remark 3.0.11.** If a representation \((f, W)\) is a specialization, then it follows that its associated Ulrich bundle is simple (from the fact that any automorphism of a vector space which commutes with every linear transformation must be central and hence scalar multiplication).

**Remark 3.0.12.** Since the objects of \( \text{SplUlr}_{\phi} \) have automorphism group \( G_m \), it follows that we can identify \( \text{PrSplUlr}_{\phi} \) with the sheafification of \( \text{SplUlr}_{\phi} \).

To examine the representations of the Clifford algebra, it is natural to consider imposing the identites of \( d \times d \) matrices. Let us first recall how this is done, following [Row80, Prop 1.3.10]. Consider the polynomial ring \( \mathbb{Z}[\xi] \) in countably many (commutative) indeterminates \( \xi_{i,j} \) and let \( m^{(\ell)} \in M_n(\mathbb{Z}[\xi]) \) be the matrix with entries \( (\xi_{i,j}^{(\ell)}) \). Consider \( \mathbb{Z}[\widetilde{\alpha}] \) the free associative algebra in countably many (noncommutative) indeterminates \( a_k \). Let \( \widetilde{I}_d \) denote the kernel of the map \( \mathbb{Z}[\widetilde{\alpha}] \rightarrow \mathbb{Z}_n[\xi] \) defined by sending \( a_\ell \) to the “generic matrix” \( m^{(\ell)} \).
Definition 3.0.13. For \( p \in \mathcal{I}_d \), and for \( \vec{b} \) a tuple of elements in \( B \), we consider the expression \( p(\vec{b}) \) in \( B \) obtained by specializing \( a_\ell \) to \( b_\ell \). We define \( I_d(B) \), the ideal of identities of \( d \times d \) matrices in \( B \), to be the ideal of \( B \) generated by all elements of the form \( p(\vec{b}) \) for \( p \in \mathcal{I}_d \) and all tuples \( \vec{b} \in B^d \).

Definition 3.0.14. Let \( \phi : X \to Y \) be a morphism of \( S \)-schemes satisfying condition \((C*)\), with \( \phi \) constant rank \( d \). We let the **reduced Clifford algebra**, denoted \( C^\text{red} \), be the quotient of \( C \) by the sheaf of ideals \( I_d(C) \) generated locally by all the identities of \( d \times d \) matrices.

Theorem 3.0.15. Let \( \phi : X \to Y \) be a morphism of \( S \)-schemes satisfying condition \((C*)\), with \( \phi \) of constant degree \( d \). Let \( C = C^\text{red} \), and \( \mathcal{Z} = \mathcal{Z}(C) \) its center. Then

1. there is an equivalence of categories \( \text{Rep}_C \cong \text{Rep}_C^d \) that is functorial with respect to base change on \( S \), inducing an isomorphism of stacks \( \text{Rep}_C^d \cong \text{Rep}_C^d \).
2. \( C \) is Azumaya over \( \mathcal{Z} \) of rank \( d \); and \( \mathcal{Z} = \mathcal{Z}(C) \) its center. Then
3. there is a natural isomorphism \( \text{Spec}(\mathcal{Z}) \cong \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \), and every Azumaya representation of degree \( d \) is a specialization;
4. if \( v \in \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \) then the class of the gerbe \( \text{Rep}_C^d \to \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \) lying over \( v \) is exactly \( C \otimes \mathcal{Z} k(v) \), where \( k(v) \) is the residue field of \( v \).

Proof. Part 1 follow immediately from the fact that any \( d \times d \) identities are automatically in the kernel of these representations.

For part 2 the result is a consequence of Artin’s characterization of Azumaya algebras via identities in [Art69, Theorem 8.3] or [Row80, Theorem 1.8.48(i/v)], since by Proposition 3.0.4, no homomorphic image of \( C \) can lie inside a matrix algebra of degree smaller than \( d \).

For parts 3 and 4, we construct mutually inverse morphisms \( \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \to \text{Spec} \mathcal{Z} \), and show that the universal Azumaya algebras on the left coincides with the Azumaya algebra \( C \) on the right.

Note that as in 3.0.12, since the Ulrich bundles under consideration are line bundles (via the numerology of Proposition 3.0.4), the representations of degree \( d \) have automorphism group \( G_m \).

Let \( \text{Spec} R \) be a \( k \)-algebra, and consider an object of \( \text{Azumaya} \mathcal{Z} \text{Rep}_C^d(R) \) described as a representation \( C \to B \) where \( B \) is a degree \( d \) Azumaya algebra over an \( k \)-algebra \( R \). This gives a homomorphism of commutative \( k \)-algebras \( \mathcal{Z} \to R \) and hence an object of \( \text{Spec} \mathcal{Z} \)(\( R \)). Since by part 2, \( C \) is Azumaya of degree \( d \), it follows that \( C \otimes \mathcal{Z} R \to B \) is a homomorphism of Azumaya algebras over \( R \) of the same rank, and hence must be an isomorphism. Therefore the Azumaya algebra \( C \) on \( \text{Spec} \mathcal{Z} \) pulls back to the canonical Azumaya algebra on \( \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \).

In the other direction, since a homomorphism \( \mathcal{Z} \to R \) yields a representation \( C \to C \otimes \mathcal{Z} R \) which is a rank \( d \) Azumaya algebra over \( R \), we obtain an inverse morphism \( \text{Spec} \mathcal{Z} \to \text{Azumaya} \mathcal{Z} \text{Rep}_C^d \) as desired. \( \square \)

Corollary 3.0.16. Let \( k \) be a field and \( \phi : X \to Y \) a finite faithfully flat degree \( d \) morphism of proper integral \( k \)-schemes of finite type. Then \( C^\text{red} \) is an Azumaya algebra of degree \( d \) over its center. In particular any \( k \)-linear map \( C \to D \) for a central simple \( k \)-algebra \( D \) of degree \( d \) must coincide with a (surjective) specialization of the Azumaya algebra \( C^\text{red} \) with respect to a \( k \) point of \( \text{Spec} \mathcal{Z} \).

Proof. Note that since \( k \) is a field, the condition above ensures that \( \phi \) will satisfy condition \((C*)\). This is then an immediate consequence of Theorem 3.0.15. \( \square \)

4. Clifford algebras for curves

In this section, we specialize to the case that \( S = \text{Spec} k \) is the spectrum of the field, \( X \) is a smooth projective \( k \)-curve \( Y = \mathbb{P}^1 \). In this case, we will find that the Clifford algebra is in some sense not sensitive to the choice of the particular morphism, but only on its degree (see Corollary 4.0.5), and that the period-index obstruction for the curve gives some structural information about the Clifford algebra (see Corollary 4.2.4).

Let us begin with some preliminary concepts and language. Let \( X/k \) be a smooth, projective, geometrically connected curve. We recall that the index of \( X \), denoted \( \text{ind} X \) is the minimal degree of a \( k \)-divisor on \( X \). Let \( \text{Pic} X \) denote the Picard group of \( X \), \( \overline{\text{Pic}}_X \) the Picard stack of line bundles on \( X \), and \( \mathcal{Pic}_X \) its coarse moduli
space. Write $\mathcal{P}ic^n_X$, $\mathcal{P}ic^0_X$ for the components of line bundles of degree $n$. The Jacobian variety $J(X) = \mathcal{P}ic^0_X$ has the structure of an Abelian variety under which the spaces $\mathcal{P}ic^n_X$ are principal homogeneous spaces. The period of $X$, per $X$, is the order of $\mathcal{P}ic^1_X$, considered as a principal homogeneous space over the Jacobian of $X$. The index can be considered as the minimal $n$ such that $\mathcal{P}ic^n_X$ has a rational point.

We recall that we have a natural map from the $k$-rational points on the Picard scheme of $X$ to the Brauer group of $k$, giving us an exact sequence

$$\text{Pic} X \to \mathcal{P}ic_X(k) \to \text{Br}(k) \to \text{Br}(k(X))$$

and identifying the image of the Picard scheme with the relative Brauer group $\text{Br}(k(X)/k)$ defined simply as the kernel of the map above on the right (see, for example [Cia05, Section 3] or [CK, Theorem 2.1]). In [CK, Theorem 3.5] it is shown that this map can be describe as being obtained from specializing a Brauer class $\alpha_X \in \text{Br}(\mathcal{P}ic_X)$, described in [CK, Lemma 3.2, Remark 3.4]. We define the subgroup $\text{Br}_0(k(X)/k) \subset \text{Br}(k(X)/k)$ to be those elements which are images of degree 0 classes, i.e. $k$-points of the Jacobian of $X$. From [CK, Theorem 2.1], we have an isomorphism

$$\frac{\text{Br}(k(X)/k)}{\text{Br}_0(k(X)/k)} \cong \frac{\mathbb{Z}}{(\text{ind } X/\text{per } X)\mathbb{Z}}$$

**Definition 4.0.1.** Suppose that $\alpha \in \text{Br}(k)$ which represents a nontrivial element of the cyclic group $\text{Br}(k(X)/k)/\text{Br}_0(k(X)/k)$. Then, following [O’N02] we call $\alpha$ an (period-index) obstruction class for $X$.

**Remark 4.0.2.** Let $m$ be the period of $X$. If $p \in \mathcal{P}ic^m_X(k)$ and $q \in \mathcal{P}ic^{m'}_X(k)$, then it follows that

$$\alpha_X|_q = \alpha_X|_{rp} + \alpha_X|_{q-rp} = r\alpha_X|_p + \alpha_X|_{q-rp},$$

and so the class of $\alpha_X|_q$ is equal to the class of $\alpha_X|_{rp}$ in $\text{Br}(X/k)/\text{Br}_0(X/k)$, and consequently the image of any point in $\mathcal{P}ic^m_X(k)$ is a period-index obstruction class.

4.0.1. **Relation between the universal Clifford representation space and the universal gerbe.** Note that since the leftmost map $\text{Pic} X \to \mathcal{P}ic_X(k)$ can be identified with the sheafification/coarse moduli map of stacks $\mathcal{P}ic_X \to \mathcal{P}ic_X$ on objects defined over $k$ (a $\mathbb{G}_m$-gerbe, as in Remark 3.0.12), the Brauer class $\alpha_x$ corresponding to a $k$-point $x \in \text{Pic} X(k)$ is split if and only if the $\mathbb{G}_m$-gerbe on $x$ obtained by pullback is split. By a result of Amitsur ([(Am55, Theorem 9.3)]), it follows that the Brauer class corresponding to the gerbe $\mathcal{P}ic_X \to \mathcal{P}ic_X$ and the Brauer class $\alpha_X$ defining the obstruction map generate the same cyclic subgroup in the Brauer group of each component of $\mathcal{P}ic_X$. If $X \to Y$ is a finite morphism of degree $d$, then via the identification of the gerbe $\mathcal{P}ic_X \to \mathcal{P}ic_X$ with the restriction of the gerbe $\mathbb{R}ep^d \to \mathbb{A}_2\mathbb{R}ep^d$ of rank $d$ representations of the Clifford algebra, which are necessarily specializations (see Corollary 3.0.16) and the algebra $\mathcal{C}_\phi^{\text{red}}$, we find that the degree $d$ specializations of the Clifford algebra consists exactly of those central simple algebras of degree $d$ which are obstruction classes for some Ulrich line bundle on $X$ with respect to the $k$-morphism $\phi : X \to Y$.

4.0.2. **Stability and semistability.** For a $X$ a smooth projective curve over a field $k$ and a coherent sheaf $V/X$, we write $\deg V = \deg c_1(V) = \int c_1(V)$ and $\mu V = \deg V/\text{rank } V$. Recall that a coherent sheaf is called **semistable** if for every subsheaf $W \leq V$, we have $\mu W \leq \mu V$ and **stable** if for every proper subsheaf $W < V$, we have $\mu W < \mu V$.

We would like to characterize in a natural way, which coherent sheaves on $X$ will be Ulrich with respect to a finite morphism $\phi : X \to \mathbb{P}^1$. To do this, we have the following fact, closely following [VdB87, Sections 2.1, 2.2] and [Cos11].

**Proposition 4.0.3.** Let $X$ be a smooth projective geometrically connected curve of genus $g$ over a field $k$. If $\phi : X \to \mathbb{P}^1$ is a finite morphism of degree $d$, then a coherent sheaf $V/X$ is Ulrich with respect to $\phi$ if and only if

1. $V$ is a semistable vector bundle on $X$ of slope $\mu V = d + g - 1$,
2. $H^0(X, V(-1)) = 0$. 

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Remark 4.0.4. We note that the condition $H^0(X, V(-1)) = 0$ can be interpreted as saying that the vector bundle $V(-1)$, which has slope $g - 1$, lies in the complement of a “generalized $\Theta$-divisor,” (see for example [Cos11, Kul03]). Recall for example, that in the classical case, the $\Theta$-divisor is the subvariety of $Pic_{X}^{g-1}$ whose $k$-points correspond to classes of effective divisors, and hence those for which $H^0$ is nontrivial.

Proof of Proposition 4.0.3. First, suppose that $V/X$ is Ulrich with respect to $\phi$. Since $X$ and $\mathbb{P}^1$ are regular of dimension 1 and $\phi$ is flat, the sheaf $V$ is locally free if and only if $\phi_*V$ is locally free. (Indeed, the torsion subsheaf would have torsion pushforward.) The fact that $V$ must be semistable is a consequence of [VdB87, Lemma 1]. Now, suppose that $V$ is rank $r$ so that $\phi_*V \cong \mathcal{O}_{\mathbb{P}^1}^r$, we use Hirzebruch-Riemann-Roch to see

$$\chi(V) = h^0V - h^1V = \deg V - r(g - 1)$$

and using the fact that $\chi(V) = \chi(\phi_*V)$, and $h^0\phi_*V = h^0\mathcal{O}_{\mathbb{P}^1}^r = rd$, and $h^1\phi_*V = 0$, we find $\chi(V) = rd$.

Consequently, we have

$$\deg V = rd + r(g - 1) = r(d + g - 1)$$

and so $\mu(V) = d + g - 1$, as claimed. For the other condition, we note that

$$H^0(X, V(-1)) = H^0(\mathbb{P}^1, \phi_*V(-1)) = H^0(\mathbb{P}^1, \mathcal{O}(-1)^r) = 0.$$

For the converse, let us assume that $V/X$ has slope $d + g - 1$ and $H^0(X, V(-1)) = 0$. By the result of Birkhoff-Grothendieck-Hazewinkel-Martin [HM82, Theorem 4.1], we can write

$$\phi_*V \cong \oplus \mathcal{O}(n_i)$$

for some collection of integers $n_i$. We claim that all the $n_i$ are equal to 0, which would imply the result.

The condition that $H^0(X, V(-1)) = 0$ tells us that all the $n_i$ are nonpositive. It therefore follows that we have $h^0(V)$ is precisely the number of indices $i$ such that $n_i$ is equal to 0. It follows from Hirzebruch-Riemann-Roch that

$$\chi(V) = \deg -r(g - 1) = rd + r(g - 1) - r(g - 1) = rd,$$

and therefore $h^0V = rd + h^1V$, which implies that $h^0V \geq rd$. But this implies that at least $rd$ of the integers $n_i$ are nonnegative, and therefore all are 0, as claimed. □

Corollary 4.0.5. Suppose that $\phi, \phi' : X \to \mathbb{P}^1$ are two degree $d$ morphisms of curves. Then there exists a line bundle $\mathcal{N}$ on $X$ such that tensoring by $\mathcal{N}$ gives an equivalence between the Ulrich bundles with respect to $\phi$ and the Ulrich bundles with respect to $\phi'$.

Proof. Let $\mathcal{L}, \mathcal{L}'$ be the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(-1)$ under $\phi$ and $\phi'$ respectively. Since these are both degree $d$ line bundles, we can write $\mathcal{L} \otimes \mathcal{N} = \mathcal{L}'$ for some line bundle $\mathcal{N}$ of degree 0. By Proposition 4.0.3, we then find that a coherent sheaf $V/X$ is Ulrich with respect to $\phi$ if and only if $H^0(X, V \otimes \mathcal{L}'') = 0$. But

$$V \otimes \mathcal{L}'' = V \otimes (\mathcal{L}' \otimes \mathcal{N}''')' = (\mathcal{N} \otimes V) \otimes \mathcal{L}'''
$$

and so $H^0(X, V \otimes \mathcal{L}'') = H^0(X, (\mathcal{N} \otimes V) \otimes \mathcal{L}'')$ and we see that $V/X$ is Ulrich with respect to $\phi$ if and only if $\mathcal{N} \otimes V$ is Ulrich with respect to $\phi'$. □

It follows that the the stack of Ulrich bundles is independent of the specific morphism $\phi$, and only depends on its degree $d$. In particular, by Theorem 3.0.15(4), the center of the reduced Clifford algebra $\mathcal{C}^\text{red}_\phi$ and its Brauer class over its center only depend on $d$ and not on the specific choice of $\phi$.

The relative Brauer map and related period-index obstruction (Definition 4.0.1) have been the subject of a great deal of arithmetic investigations (see, for example [LT58, Sal61, CK, Cla05, Lic68, Roq76]). An interesting aspect of the study of the Clifford algebra is that it gives another concrete interpretation of this morphism, and understanding its specializations can yield nontrivial arithmetic information about a curve.

In this direction we give a result on the specializations of the Clifford algebra of a curve. In [HH07] and [Hai84], it is shown that in certain cases of a Clifford algebra associated to a genus 1 hyperelliptic or plane cubic curve, the Clifford algebra specializes to any division algebra of degree 2 or 3 respectively, which is split by the function field of the genus 1 curve. The following result gives a natural generalization of this result for general curves.

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Proposition 4.0.6. Let $X/k$ be a geometrically integral smooth projective curve of index $d$ over $k$, which admits a degree $d$ morphism $\phi : X \to \mathbb{P}^1$, and suppose $D$ is a division algebra of degree $d$ over its center $k$, such that $D_{k(X)}$ is split. Then there is a specialization $\mathcal{C}_\phi \to D$.

Remark 4.0.7. It follows from 4.0.3 that such a class $[D]$ must arise as the obstruction class for the gerbe $\mathbb{P}ic_X^{-d+g-1} \to \mathbb{P}ic_X^{-d+g-1}$.

Proof of Proposition 4.0.6. Let $\mathcal{G} \to \text{Spec} k$ be a $\mu_d$-gerbe representing the Brauer class of $D$, and write $D = \text{End}(W)$ for a $\mathcal{G}$-twisted sheaf $W$ of rank $d$. Let $\mathcal{P} \to \mathbb{P}^1$ and $\mathcal{X} \to X$ be the pullbacks of $\mathcal{G}$ to $\mathbb{P}^1$ and $X$. Let $\mathcal{L}$ be a $\mathcal{X}$-twisted invertible sheaf, and let $V := \phi_* \mathcal{L}$ be the pushforward $\mathcal{P}$-twisted sheaf.

We claim that there is an integer $n$ and an isomorphism $V(n) \to W_{\mathcal{P}}$. Note that $V$ is naturally a $\phi_* \mathcal{O}_{\mathcal{X}}$-module, giving a map $\phi_* \mathcal{O}_{\mathcal{X}} \to \text{End}(W)$. The claim thus yields a map $\phi_* \mathcal{O}_{\mathcal{X}} \to \text{End}(W_{\mathcal{P}}) = D_{\mathcal{P}}$. Since both sheaves have trivial inertial action, this is the pullback of a unique map $\phi_* \mathcal{O}_{\mathcal{X}} \to D_{\mathbb{P}^1}$, which by Corollary 3.0.16, comes from a specialization of the Clifford algebra $\mathcal{C}_{X/\mathbb{P}^1} \to D$.

It remains to prove the claim. For this, note that $V$ is a $\mathcal{P}$-twisted sheaf of minimal rank (as the index of $\mathcal{P}$ is equal to $d$, which is the rank of $V$). In particular, it follows that $V$ must be stable of some slope $\mu$, since otherwise the Jordan-Hölder filtration will yield a twisted sheaf of smaller rank. Further, it follows by the same reasoning that $V^n$ must be equal to its $\mu$-socle, the sum of its $\mu$-stable subsheaves. Hence, $V$ must be geometrically polystable, which, implies that $W \otimes \bar{k} = \mathcal{L}(m)^d$ for some fixed $m$ and an invertible $\mathcal{P} \otimes \mathcal{I}$-twisted sheaf $\mathcal{L}$ of degree $0$.

On the other hand, $W_{\mathcal{P}}$ is also a locally free $\mathcal{P}$-twisted sheaf of rank $d$, hence also geometrically polystable. It follows that there is an integer $n$ such that $V(n) \otimes \bar{k}$ and $W_{\mathcal{P}} \otimes \bar{k}$ are isomorphic over $\mathcal{P} \otimes \bar{k}$. The space $I := \text{Isom}(V(n), W_{\mathcal{P}})$ is thus a right $\text{Aut}(W)$-torsor which is open in the (positive-dimensional) affine space $\text{Hom}(V(n), W)$. If $k$ is infinite, it follows that $I$ has a rational point (as the rational points are dense in any open subset of an affine space); if $k$ is finite, then $I$ has a rational point because any torsor under a smooth connected $k$-group scheme of finite type is split by Lang’s theorem [Lan56, Theorem 2]. In either case, we see that $V(n)$ and $W_{\mathcal{P}}$ are isomorphic, verifying the claim.

4.1. Genus 1 curves. For the remainder, we will focus on the case where $X$ is a curve of genus 1 over $k$. Let us begin with the following theorem, which illustrated the connection between our Clifford algebras and the arithmetic of genus 1 curves:

Theorem 4.1.1. Suppose that $X/k$ is a genus 1 curve of index $d > 1$. Then

1. $X$ admits a degree $d$ finite morphism $\phi : X \to \mathbb{P}^1$.
2. For such a morphism, the Ulrich locus of $\mathbb{P}ic_X$ lies within the component $\mathbb{P}ic_X^d$.
3. We have an equivalence of stacks $\mathbb{P}ic_X^d \cong \mathbb{P}ic_X^0$.
4. No specialization of the Clifford algebra is a period-index obstruction algebra for $X$ (as in Definition 4.0.1).

Proof. For part 1, we note that for such a curve, if we choose a divisor $D$ of degree $d$, then by Riemann-Roch, $h^0(D) = \deg D \geq 2$, and so we see that we can find a pencil of effective divisors providing a degree $d$ morphism $\phi : X \to \mathbb{P}^1$.

Part 2 follows from Proposition 4.0.3(1).

For part 3 observe that, since $X$ has an effective divisor of degree $d$ defined over $k$, by adding and subtracting that point, we obtain an equivalence of stacks $\mathbb{P}ic_X^d \cong \mathbb{P}ic_X^0$.

Finally, part 4 is now a consequence of Remark 4.0.7.

4.2. Decomposability of the period-index obstruction. We consider now the concept of decomposability of algebras. We recall that a central simple algebra $A$ is decomposable if it can be written as $A \cong B \otimes C$ for two nontrivial algebras $B, C$. Circumstantial evidence would seem to suggest that algebras which have index “maximally different” from the period should be decomposable. Results of this type have been obtained by Suresh ([Sur10, Theorem 2.4], see also [BMT11, Remark 4.5]), over fields such as $\mathbb{Q}_p(t)$. We begin by giving the following weakening of the standard notion of decomposability:

Definition 4.2.1. We say that a central simple algebra $A$ is weakly decomposable if there exist central simple algebras $B, C$ of degree greater than 1, such that $\deg B, \deg C$ divide but are strictly less than $\deg A$ and $A$ is Brauer equivalent to $B \otimes C$. 

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Remark 4.2.2. We note that in the case that $A$ is a weakly decomposable algebra of degree $p^2$, then $A$ is in fact decomposable.

Proposition 4.2.3. Let $X/k$ be a smooth genus $1$ curve. Then every class in the relative Brauer group $\text{Br}(k(X)/k)$ can be written as $[B \otimes D]$ where $\text{ind } B, \text{ind } D \mid \text{per } X$. In particular, if $\text{ind } X \neq \text{per } X$, and $A$ is a division algebra of index $d$ whose class is in $\text{Br}(k(X)/k)$, then $A$ is weakly decomposable.

Proof. If $A$ is a central simple algebra of degree $d$ which is not division then the result is immediate. Therefore we may assume that $A$ is a central division algebra of degree $d$, and let $D$ be any obstruction class for $X$. By Proposition 4.0.6, $A$ is a specialization of the Clifford algebra, and hence by Theorem 4.1.1 its class must lie in $\text{Br}_0(X/k)$ (see the discussion at the beginning of Section 4), and therefore $A \otimes D^{\text{op}}$ must be a period-index obstruction class. By [O’N02, Proposition 2.3], we find that $\text{ind } D, \text{ind } A \otimes D^{\text{op}} \mid \text{per } X < d$. Setting $B$ to be a division algebra Brauer equivalent to $A \otimes D^{\text{op}}$, we find $A \sim D \otimes B$, showing that $A$ is weakly decomposable as desired.

Using this, we find that the Clifford algebra is also weakly decomposable in this situation.

Corollary 4.2.4. Let $X/k$ be a smooth genus $1$ curve, and let $\phi : X \to \mathbb{P}^1$ be a morphism of degree $\text{ind } X$. Let $C$ be the specialization of the reduced Clifford algebra $\mathcal{C}^\text{red}_\phi$ to the generic point of $\text{Pic}^\text{ind}_X$. If $\text{per } X \neq \text{ind } X$ then $C$ is weakly decomposable.

Proof. Let $L = k(\text{Pic}^\text{ind}_X)$ be the function field of $\text{Pic}^\text{ind}_X$, and let $\eta \in \text{Pic}^\text{ind}_X(L)$ be the generic point. Regarding $C$ as the specialization of $\mathcal{C}^\text{red}_\phi$ to $\eta \in \text{Pic}^\text{ind}_X(L) = \text{Pic}^\text{ind}_X(L)$, it follows from Section 4.0.1 that $C \in \text{Br}(X_L/L)$, and the result now follows from Proposition 4.2.3. □

Appendix A. Explicit constructions

In this appendix, we relate our Clifford algebra functor (and therefore our constructed Clifford algebras) to the more classical constructions in the literature and their natural generalizations. With this in mind, we present Clifford algebras in a number of generalizations of previously seen contexts, each time showing how previous constructions fit within this description. It turns out that all the existing descriptions of Clifford algebras can be all seen as particular examples of the Clifford algebra associated to projection of a hypersurfaces in certain weighted projective spaces. We finish by giving an explicit version of the general existence proof for such Clifford algebras, giving an explicit presentation for such Clifford algebras.

To set notational conventions, we will assume that all rings and algebras are associative and unital and their homomorphism are unital. For a ring $R$, we let $R\langle x_1, ..., x_n \rangle$ denote the free associative algebra over $R$ generated by the $x_i$.

A.1. The Clifford algebra of a homogeneous polynomial. Recall, if $f$ is a degree $d$ homogeneous polynomial in the variables $x_1, ..., x_n$, following [Rob69], we define the Clifford algebra of $f$, denoted $C(f)$ by

$$C(f) = k\langle a_1, ..., a_n \rangle / I$$

where $I$ is the ideal generated by the coefficients of the variables $x_i$ in the expression

$$(a_1 x_1 + ... + a_n x_n)^d - f(x_1, ..., x_n) \in k\langle a_1, ..., a_n \rangle [x_1, ..., x_n].$$

Proposition A.1.1. Suppose $f$ is a degree $d$ homogeneous polynomial in the variables $x_1, ..., x_n$. Let $X$ be the hypersurface defined by the equation $x_0^d - f(x_1, ..., x_n)$, and let $\phi : X \to \mathbb{P}^{n-1}$ be the degree $d$ morphism given by dropping the $x_0$-coordinate. Then $\mathcal{C}_\phi$ is represented by the algebra $C(f)$.

Proof. This follows from Theorem A.4.1 □

A.2. Weighted Clifford algebras of homogeneous polynomials. This construction is a generalization of the hyperelliptic Clifford algebras introduced by Haile and Han in [HH07].

For positive integers $m, d$, let $f$ is a degree $md$ homogeneous polynomial in the variables $x_1, ..., x_n$. We define the Clifford algebra of $f$, weighted by $m$ denoted $C_m(f)$ by

$$C_m(f) = k\langle a_1, ..., a_n \rangle / I$$
This follows from Theorem A.4.1.

**Proof.** This follows from Theorem A.4.1.

A.3. Non-diagonal Clifford algebras of homogeneous polynomials. This version of a Clifford algebra construction is due to Pappacenca [Pap00]. Particularly interesting case are the non-diagonal Clifford algebras of a binary cubic form, studied by Kuo in [Kuo11], and in somewhat more generality by Chapman and Kuo in [CK15].

For a positive integer $d$, suppose that we are given $f_1, f_2, \ldots, f_d \in k[x_1, x_n]$ where $f_i$ is homogeneous of degree $i$. We define $C(f_1, \ldots, f_n)$ to be the associative $k$-algebra given by

$$C(f_1, \ldots, f_n) = k\langle a_1, \ldots, a_n \rangle / I$$

where $I$ is the ideal generated by the coefficients of the variables $x_i$ in the expression

$$(a_1x_1 + \cdots + a_n x_n)^d - (a_1x_1 + \cdots + a_n x_n)^{d-1} f_1(x_1, \ldots, x_n)$$

$$(a_1x_1 + \cdots + a_n x_n)^{d-2} f_2(x_1, \ldots, x_n) - \cdots - f_d(x_1, \ldots, x_n) \in k\langle a_1, \ldots, a_n \rangle [x_1, \ldots, x_n].$$

**Proposition A.3.1.** Suppose we are given polynomials $f_1, \ldots, f_n$ in the variables $x_1, \ldots, x_n$, where $f_i$ is homogeneous of degree $i$. Let $X$ be the hypersurface in $\mathbb{P}^n$ defined by the equation

$$x_0^d = x_0^{d-1} f_1 + x_0^{d-2} f_2 + \cdots + f_d,$$

and let $\phi : X \to \mathbb{P}^{n-1}$ be the degree $d$ morphism given by dropping the $x_0$-coordinate. Then $\mathcal{C}(\phi)$ is represented by the algebra $C_m(f).$

**Proof.** This follows from Theorem A.4.1.

A.4. Weighted non-diagonal Clifford algebras of homogeneous polynomials. This construction is a common generalization of the previous constructions. Suppose that we are given $f_m, f_{2m}, \ldots, f_{dm} \in k[x_1, x_n]$ where $f_i$ is homogeneous of degree $i$. We define $C(f_m, \ldots, f_dm)$ to be the associative $k$-algebra given by

$$C(f_m, \ldots, f_dm) = k\langle a_j \rangle_{|j|=m} / I$$

where $I$ is the ideal generated by the coefficients of the variables $x_i$ in the expression

$$\left( \sum_{|j|=m} a_j x^j \right)^d = \sum_{\ell=1}^m \left( \sum_{|j|=m} a_j x^j \right)^{d-\ell} f_{\ell m}(x_1, \ldots, x_n) \in k\langle a_j \rangle_{|j|=m} [x_1, \ldots, x_n].$$

**Theorem A.4.1.** Suppose that we are given $f_m, f_{2m}, \ldots, f_{dm} \in k[x_1, x_n]$ where $f_i$ is homogeneous of degree $i$. Let $X$ be the hypersurface in the weighted projective space $\mathbb{P} = \mathbb{P}_{m,1,\ldots,1}$ defined by the degree $md$ homogeneous equation

$$x_0^d = x_0^{d-1} f_m + x_0^{d-2} f_{2m} + \cdots + f_{dm},$$

and let $\phi : X \to \mathbb{P}^{n-1}$ be the degree $d$ morphism given by dropping the $x_0$-coordinate. Then $\mathcal{C}(\phi)$ is represented by the algebra $C(f_m, \ldots, f_dm).$
Proof. To begin, let us examine the morphism \( \phi \) in local coordinates. Let \( U_i \cong \mathbb{A}^{n-1} \) be the affine open set of \( \mathbb{P}^{n-1} \) defined by the nonvanishing of the coordinate \( x_i \), so that \( U_i = \text{Spec}(R_i) \) where \( R_i = k[x_i/x_i, \ldots, x_n/x_i] \subset k(\mathbb{P}^{n-1}) \). Similarly, let \( V_i \subset \mathbb{P} \) be defined by the nonvanishing of the \( x_i \) coordinate on \( \mathbb{P} \). If we write \( V_i = \text{Spec}(S_i) \) then we have

\[
S_0 = k[x^j/x_0][j = m]
\]

where \( x^j = x_1^j \cdots x_n^j \) is a monomial of degree \( |J| = m \). For \( i \neq 0 \), we have

\[
S_i = k[x_0/x_i^m, x_1/x_i, \ldots, x_n/x_i]
\]

which we note is just a polynomial ring in one variable, represented by \( x_0/x_i^m \) over the ring \( R_i \). If we let \( X_i = X \cap V_i \), then we see that (via homogenization) \( X_0 \) is cut out by the equation

\[
1 = f_m/x_0 + f_{2m}/x_0^2 + \cdots + f_{dm}/x_0^d.
\]

We claim that in fact \( X_0 \subset \cup_{i=1}^n X_i \), or in other words, \( X \) is contained in the union of the open sets \( V_1, \ldots, V_n \). To see this, suppose that \( p \in X_0(L) \) is a point for some field extension \( L/F \). It follows that for some \( J \) with \( |J| = m \), we have \( x^j/x_0(p) \neq 0 \), so since otherwise we would have \( f_m/x_0^d(p) = 0 \) for each \( \ell \) contradicting the equation above. Now, we can choose \( i \) with \( j_i \neq 0 \) — i.e., so that \( x_i \) appears with a nonzero multiplicity in the monomial \( x^j \). We claim that \( p \in X_i \). But this follows by construction: \( p \) does not lie in the zero set of the homogeneous polynomial \( x_i \), the ideal of which on the affine set \( V_i \) contains the term \( x_i^0 \).

For \( i \neq 0 \), in the affine set \( V_i = \text{Spec}(S_i) = \text{Spec}(k[x_0/x_i^m, x_1/x_i, \ldots, x_n/x_i]) \), the closed subscheme \( X_i \) is cut out by the equation

\[
(x_0/x_i^m)^d = (x_0/x_i^m)^{d-1}f_m(x_1/x_i, \ldots, x_n/x_i) + \cdots + f_{dm}(x_1/x_i, \ldots, x_n/x_i).
\]

Let \( A = k(a_1, \ldots, a_n)/I \), where the ideal \( I \) is as described above. We will show that \( A \) represents the functor \( \mathcal{C}(\phi) \). For the first direction, note that we have a homomorphisms of sheaves of \( \mathcal{O}_{\mathbb{P}^{n-1}} \)-algebras \( \phi_* \mathcal{O}_X \to A \otimes_k \mathcal{O}_{\mathbb{P}^{n-1}} \) given on the open set \( U_i \) by the inclusion

\[
S_i \to A \otimes_k R_i
\]

defined by sending \( x_0/x_i^m \) to \( \sum_{|J| = m} a_J \otimes x^j \). It follows from multiplying the defining equation (2) by \( x_i^{-md} \) that this defines a homomorphism of \( R_i \)-algebras.

Conversely, suppose that \( B \) is any \( k \)-algebra, and that we have a homomorphism of sheaves of \( \mathcal{O}_{\mathbb{P}^{n-1}} \)-algebras \( \phi_* \mathcal{O}_X \to B \otimes_k \mathcal{O}_{\mathbb{P}^{n-1}} \). Over the open set \( U_i \) write \( \beta_i \) for the image of \( x_0/x_i^m \in S_i \) in \( B \otimes_k R_i \), we see that since \( b_i x_i^m/x_j^m = \beta_j \in B \otimes R_j \), it follows that \( \beta_i \), considered as a polynomial in \( x_1/x_i, \ldots, x_n/x_i \) can have degree no larger than \( m \). In particular, we may write

\[
b_i x_i^m = \sum_{|J| = m} \beta_{J,i} x^j
\]

and using the identity \( b_i x_i^m = b_j x_j^m \), it follows that the elements \( \beta_{J,i} = \beta_j \in B \) do not in fact depend on \( i \). But now, \( a_J \to \beta_J \) defines a homomorphism \( A \to B \) such that \( S_i \to B \otimes R_i \) factors as

\[
S_i \to A \otimes R_i \to B \otimes R_i
\]
as desired. \( \square \)

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