OPTIMAL REGULARITY AND EXPONENTIAL STABILITY FOR THE BLACKSTOCK–CRIGHTON EQUATION IN $L_p$-SPACES WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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Abstract. The Blackstock–Crighton equation models nonlinear acoustic wave propagation in thermo-viscous fluids. In the present work we investigate the associated inhomogeneous Dirichlet and Neumann boundary value problems in a bounded domain and prove long-time well-posedness and exponential stability for sufficiently small data. The solution depends analytically on the data. In the Dirichlet case, the solution decays to zero and the same holds for Neumann conditions if the data have zero mean.

We choose an optimal $L_p$-setting, where the regularity of the initial and boundary data are necessary and sufficient for existence, uniqueness and regularity of the solution. The linearized model with homogeneous boundary conditions is represented as an abstract evolution equation for which we show maximal $L_p$-regularity. In order to eliminate inhomogeneous boundary conditions, we establish a general higher regularity result for the heat equation. We conclude that the linearized model induces a topological linear isomorphism and then solve the nonlinear problem by means of the implicit function theorem.

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1. Introduction

An acoustic wave propagates through a medium as a local pressure change. Nonlinear effects typically occur in case of acoustic waves of high amplitude which are used for several medical and industrial purposes such as lithotripsy, thermotherapy, ultrasound cleaning or welding and sonochemistry. Research on mathematical aspects of nonlinear acoustic wave propagation is therefore not only interesting from a mathematicians point of view. In fact, in case of medical applications, enhancement of the mathematical understanding of the underlying models should lead to a considerable reduction of complication risks.

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The present work aims to provide a mathematical analysis of the Blackstock–Crighton–Kuznetsov equation
\begin{equation}
(\alpha\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left( \frac{1}{c^2} \frac{B}{2A}(u_t)^2 + |\nabla u|^2 \right)_t
\end{equation}
and the Blackstock–Crighton–Westervelt equation
\begin{equation}
(\alpha\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left( \frac{1}{c^2} (1 + \frac{B}{2A}) (u_t)^2 \right)_t
\end{equation}
for the acoustic velocity potential \( u \), where \( c \) is the speed of sound, \( b \) is the diffusivity of sound and \( a \) is the heat conductivity of the fluid. Note that \( a = \nu \Pr \), where \( \nu \) is the is kinematic viscosity and \( \Pr \) denotes the Prandtl number. Alternatively, (1.1) and (1.2) can be expressed in terms of the acoustic pressure via the pressure density relation \( \rho u_t = p \), where \( \rho \) denotes the mass density. The quantity \( B/A \) is known as the parameter of nonlinearity and is proportional to the ratio of the coefficients of the quadratic and linear terms in the Taylor series expansion of the variations of the pressure in a medium in terms of variations of the density. Note that (1.2) is obtained from (1.1) by neglecting local nonlinear effects in the sense that the expression \( c^2|\nabla u|^2 - (u_t)^2 \) is sufficiently small. For a detailed introduction to the theory and applications of nonlinear acoustics we refer to [HB98].

Equations (1.1) and (1.2) result from two evolution equations of fourth order governing finite-amplitude sound in thermoviscous relaxing fluids, namely
\begin{equation}
-c^2\alpha\Delta^2 u + (a + b) \Delta u_{tt} + \left( c^2\Delta u - u_{ttt} \right) = \left( |\nabla u|^2_t + \frac{B}{2A} u_t \Delta u_t \right)_t,
\end{equation}
\begin{equation}
(\alpha\Delta - \partial_t)(u_{tt} - c^2\Delta u) = \left( |\nabla u|^2_t + \frac{B}{2A} u_t \Delta u_t \right)_t,
\end{equation}
which have been derived by Blackstock [Bla63] from the basic equations describing the general motion of thermally relaxing, viscous fluids (continuity equation, momentum equation, entropy equation and an arbitrary equation of state) and also appear as equations (11) and (13) in Crighton’s work [Cri79] on nonlinear acoustic models. We replace \( \Delta u \) in the last term of (1.3) and (1.4) by \( \frac{1}{c^2} u_{tt} \), which can be justified by the main part of the differential operator corresponding to the wave equation \( u_{tt} - c^2\Delta u = 0 \). Moreover, in (1.4) we consider potential diffusivity as in (1.3). Therewith, we arrive at equation (1.1) for which in [Bru15] the name Blackstock–Crighton–Kuznetsov equation has been introduced. For a more rigorous derivation of (1.1) we refer to Section 2 in [Bru15].

The classical models (1.1) and (1.2) are enhanced models in nonlinear acoustics, the Kuznetsov
\begin{equation}
\frac{1}{2}(u_{tt} - b\Delta u_t - c^2\Delta u) = \left( \frac{1}{c^2} \frac{B}{2A}(u_t)^2 + |\nabla u|^2 \right)_t
\end{equation}
and the Westervelt equation
\begin{equation}
\frac{1}{2}(u_{tt} - b\Delta u_t - c^2\Delta u) = \left( \frac{1}{c^2} (1 + \frac{B}{2A}) (u_t)^2 \right)_t,
\end{equation}
are classical, well-accepted and widely used models governing sound propagation in fluids. As (1.1) and (1.2), they are derived from the basic equations in fluid mechanics. The Kuznetsov equation is the more general one of these classical models, in particular the Westervelt equation is obtained from the Kuznetsov equation by neglecting local nonlinear effects. Moreover, for a small ratio of \( \nu \) and \( \Pr \), that is, for small heat conductivity, (1.5) and (1.6) can be regarded as simplifications of (1.1) and (1.2), respectively.

The classical models (1.5) and (1.6) have recently been extensively investigated. In particular, results on well-posedness for the Kuznetsov and the Westervelt equation with homogeneous Dirichlet [KL10] and inhomogeneous Dirichlet [KLV11],[KL12] and Neumann [KL11] boundary conditions have recently been shown in an \( L_2(\Omega) \)-setting on spatial domains \( \Omega \subset \mathbb{R}^n \).
of dimension $n \in \{1, 2, 3\}$. Moreover, there are results on optimal regularity and long-time behavior of solutions for the Westervelt equation with homogeneous Dirichlet boundary conditions in $L_p(\Omega)$-spaces where the spatial domain $\Omega \subset \mathbb{R}^n$ is of arbitrary dimension.

On the contrary, mathematical research on higher order partial differential equations arising in nonlinear acoustics is still in an early stage. Well-posedness and exponential decay results for the homogeneous Dirichlet boundary value problems associated with (1.1) and (1.2) in an $L_2(\Omega)$-setting where $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$, have been shown in [Bru15] and [BK14], respectively. In the present work we consider (1.1) and (1.2) with inhomogeneous Dirichlet and Neumann boundary conditions in $L_p(\Omega)$-spaces where the spatial domain $\Omega$ is of dimension $n \in \mathbb{N}$. We show global well-posedness and long-time behavior of solutions in an optimal functional analytic setting in the sense that the regularity of the solution is necessary and sufficient for the regularity of the initial and boundary data. While in [Bru15] and [BK14] the results were proved by means of appropriate energy estimates and the Banach fixed-point theorem, the techniques used in the present paper are based on maximal $L_p$-regularity for parabolic problems and the implicit function theorem in Banach spaces.

We suppose that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain, i.e., an open, connected and bounded subset of the $n$-dimensional Euclidean space, with smooth boundary $\Gamma$. Let $J = (0, T)$ for some finite $T > 0$ or $J = \mathbb{R}_+ = (0, \infty)$. We consider the inhomogeneous Dirichlet boundary value problem

$$\begin{align*}
(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 u) &= (k(u_t)^2 + s|\nabla u|^2)_{tt} \quad \text{in } J \times \Omega, \\
(u, \Delta u) &= (g, h) \quad \text{on } J \times \Gamma, \\
(u, u_t, u_{tt}) &= (u_0, u_1, u_2) \quad \text{on } \{t = 0\} \times \Omega,
\end{align*}$$

(1.7)

and the inhomogeneous Neumann boundary value problem

$$\begin{align*}
(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 u) &= (k(u_t)^2 + s|\nabla u|^2)_{tt} \quad \text{in } J \times \Omega, \\
(\partial_{\nu} u, \partial_{\nu} \Delta u) &= (g, h) \quad \text{on } J \times \Gamma, \\
(u, u_t, u_{tt}) &= (u_0, u_1, u_2) \quad \text{on } \{t = 0\} \times \Omega,
\end{align*}$$

(1.8)

where $u_0, u_1, u_2 : \Omega \to \mathbb{R}$ and $g, h : J \times \Gamma \to \mathbb{R}$ are given, $u : J \times \Omega \to \mathbb{R}$ is the unknown, $u(t, x)$, and $a, b, c$ and $k$ are positive constants. Moreover, $\partial_{\nu} u = \nu \cdot \nabla u|_{\Gamma}$ where $\nu$ is the outer normal unit vector denotes the normal derivative of $u$. The parameter $s \in [0, 1)$ allows us to switch between (1.1) and (1.2).

We point out that the present work extends the results from [Bru15] in several ways. First, while in [Bru15] the Blackstock–Crighton equation was considered with homogeneous Dirichlet boundary conditions, we also allow for inhomogeneous Dirichlet as well as Neumann boundary conditions. We are able to remove the restriction $n \in \{1, 2, 3\}$ on the dimension of the spatial domain $\Omega$. Instead of $L_2(\Omega)$, we consider (1.1) and (1.2) in $L_p(\Omega)$ where $p \in (1, \infty)$ in case of the linearized equation and $p > \max\{n/4 + 1/2, n/3\}$ in case of the nonlinear equations (1.7) and (1.8). In particular, we require $p \in (5/4, \infty)$ in case $n = 3$ and then $p = 2$ is admissible. Moreover, most notably, our conditions on the regularity of the data $(g, h, u_0, u_1, u_2)$ are necessary and sufficient for the existence of a unique solution of the Blackstock–Crighton equation (within a certain regularity class/a certain subspace of $L_p(J \times \Omega)$).

Our strategy for solving (1.7) and (1.8) is to prove that their linearizations induce isomorphisms between suitable Banach spaces and to apply the implicit function theorem. In some
sense these linearizations can be considered as a composition of a heat problem and another linearized problem for the Westervelt equation. While the linearized Westervelt equation can be handled similar as in [MW11] [MW13], the heat equation has to be solved with higher regularity conditions.

The paper is organized as follows. The purpose of Section 2 is to recall several facts we need on our way to global well-posedness and exponential stability of (1.7) and (1.8). In particular, we mention all function spaces we use, provide facts about the homogeneous Dirichlet and Neumann Laplace operator and list some important embeddings and traces. We also give a short review of the concept of maximal $L^p$-regularity for parabolic problems. Furthermore, we recall respectively prove optimal regularity results for the heat equation and the linearized Westervelt equation with inhomogeneous Dirichlet and Neumann boundary conditions.

Section 3 is devoted to the inhomogeneous Dirichlet boundary value problem (1.7). First of all we consider the corresponding linear problem and represent it as an abstract parabolic evolution equation for which we show maximal $L^p$-regularity. This gives us optimal regularity for the homogeneous linear version of (1.7). Based on the optimal regularity results for the heat and the linearized Westervelt equation from Section 2 we prove optimal regularity for the linear inhomogeneous Dirichlet boundary value problem in Proposition 3.5. The main result in this section is Theorem 3.6 which states global well-posedness of (1.7) and immediately implies exponential stability (Theorem 3.7).

In Section 4 we treat the inhomogeneous Neumann boundary value problem (1.8). Here, we proceed analogously to Section 3. Theorem 4.8 provides local well-posedness for (1.8). In the Neumann case, global well-posedness is shown for data having zero mean (Theorem 4.9). Moreover, Theorem 4.11 states long-time behavior of solutions.

In Appendix A we collect several facts about the homogeneous Neumann Laplacian. We outline how one finds a realization of the Laplacian with homogeneous Neumann boundary conditions such that its spectrum is contained in the positive half-line.

In Appendix B we first study the temporal trace operator acting on a class of anisotropic Sobolev spaces. We present its mapping properties, provide a right-inverse and thus obtain its precise range space. Moreover, we construct functions with prescribed higher-order initial data. Second, we prove some so-called mixed derivative embeddings which are often used for checking the continuity of differential operators acting on anisotropic spaces.

In Appendix C we prove some higher regularity results for the heat equation with inhomogeneous Dirichlet or Neumann boundary conditions and inhomogeneous initial conditions in a far more general framework than needed in the main text. In particular, we state explicitly all necessary compatibility conditions between initial and boundary data and show how they are used to contruct a solution with high regularity.

2. Preliminaries

The purpose of this section is to introduce the notation we are going to use throughout the paper and to recall several important facts and results we need to prove global well-posedness and long-time behavior of solutions for (1.7) and (1.8). As already mentioned in Section 1 we always assume that the spatial domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is bounded and has smooth boundary $\Gamma = \partial \Omega$. We write $J$ for a time interval and consider either $J = (0, T)$ for some finite time horizon $T > 0$ or $J = \mathbb{R}_+ = (0, \infty)$.

2.1. Function spaces, operators, embeddings and traces. The space $BUC^k(\Omega)$ contains all $k$-times Fréchet differentiable functions $\Omega \to \mathbb{R}$, whose derivatives up to order $k$ are
bounded and uniformly continuous. For $p \in (1, \infty)$, let $L_p(\Omega)$ denote the space of (equivalence classes of) Lebesgue measurable $p$-integrable functions $\Omega \to \mathbb{R}$. We write $W^m_p(\Omega)$ for the Sobolev–Slobodeckij space and $H^m_p(\Omega)$ for the Bessel potential space of order $m \in [0, \infty)$, where we have $W^m_p(\Omega) = H^m_p(\Omega)$ if $m \in \mathbb{N}_0$. Moreover, $W^m_p(\Omega; X)$ and $H^m_p(\Omega; X)$ denote the vector valued versions and $0W^1_p(J; X)$ denotes the space of all functions $u \in W^1_p(J; X)$ with $u(0) = 0$. For $p \in [1, \infty)$, $q \in [1, \infty]$, $s \in \mathbb{R}_+$, the Besov space $B^s_{p,q}(\Omega)$ is defined as $(L_p(\Omega), W^m_p(\Omega))_{s/m,q}$ where $m = [s]$ and $(\cdot, \cdot)_{s/m,q}$ indicates real interpolation. It holds that $B^s_{p,p}(\Omega) = W^s_p(\Omega)$ if $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $B^s_{p,q}(\Omega) = W^s_p(\Omega)$ if $p = q = 2$. Moreover,

\[(2.1) \quad B^s_{p,q}(\Omega) = (W^k_p(\Omega), W^m_p(\Omega))_{\Theta, q}, \quad \text{where} \; 0 \leq k < s \leq m \text{ and } s = (1 - \Theta)k + \Theta m.\]

We always write $X \hookrightarrow Y$ if the Banach space $X$ is continuously embedded into the Banach space $Y$. Moreover, let $L(X, Y)$ be the space of all bounded linear operators between $X$ and $Y$. A linear operator $A: X \to Y$ is called an isomorphism if it is bounded and bijective. Then the closed graph theorem implies that $A^{-1}: Y \to X$ is also bounded and therefore $A: X \to Y$ is a homeomorphism. Now, let $X$ and $\mathcal{X}$ be Banach spaces such that $X \hookrightarrow L_{1,loc}(J; X)$ where $L_{1,loc}(J; X)$ is the space of locally integrable functions $J \to X$. For any $\omega \in \mathbb{R}$ we define the exponentially weighted space

\[e^{\omega X} = \{u \in L_{1,loc}(J; X); e^{-\omega t}u \in \mathcal{X}\},\]

equipped with the norm $\|u\|_{e^{\omega X}} = \|e^{-\omega t}u\|_{\mathcal{X}}$ where $e^{-\omega t}u$ denotes the mapping $[t] \mapsto e^{-\omega t}u(t)$.

Let $-\Delta_D: D(\Delta_D) \to L_p(\Omega)$, $u \mapsto -\Delta u$ denote the negative Dirichlet Laplacian with domain $D(\Delta_D) = \{u \in W^2_p(\Omega): u = 0 \text{ on } \Gamma\}$. We recall that the spectrum $\sigma(-\Delta_D)$ is a discrete subset of $(0, \infty)$ consisting only of eigenvalues $\lambda_n^2 = \lambda_n(-\Delta_D)$, $n \in \mathbb{N}_0$ with finite multiplicity. We write $\lambda_n^2 > 0$ for the smallest eigenvalue of $-\Delta_D$. Moreover, the negative Neumann Laplacian $-\Delta_N: D(\Delta_N) \to L_p(\Omega)$, $u \mapsto -\Delta u$ with domain $D(\Delta_N) = \{u \in W^2_p(\Omega): \partial_N u = 0 \text{ on } \Gamma\}$ has a discrete spectrum $\sigma(-\Delta_N) \subset (0, \infty)$ which contains only eigenvalues $\lambda_n^N = \lambda_n(-\Delta_N)$, $n \in \mathbb{N}_0$, of finite multiplicity. Here, $\lambda_0^N = 0$ is an isolated point of $\sigma(-\Delta_N)$ which can be removed when introducing the space $L_{p,0}(\Omega) = \{u \in L_p(\Omega): \int_\Omega u \, dx = 0\}$ and considering $-\Delta_{N,0}: D(\Delta_{N,0}) \to L_{p,0}(\Omega)$, $u \mapsto -\Delta u$ with $D(\Delta_{N,0}) \supseteq D(\Delta_N) \cap L_{p,0}(\Omega)$. We then have $\sigma(-\Delta_{N,0}) \subset (0, \infty)$ where $\lambda_1^N = \lambda_1(-\Delta_N) > 0$ is the smallest eigenvalue of $-\Delta_{N,0}$. For details we refer to Appendix A.

We shall use the embeddings $W^1_p(J) \hookrightarrow BUC(J)$ and $W^s_p(\Omega) \hookrightarrow W^t_p(\Omega)$ for $s \geq t$. We always write $\gamma_D = |\Gamma$ and $\gamma_N = \partial_N \cdot |r = \nu \cdot (\nabla \cdot)\Gamma$ for the Dirichlet and the Neumann trace, respectively. Moreover, $\gamma_l = |l = t(0)$ denotes the temporal trace. Let $B \in \{D, N\}$, $j_D = 0$, $j_N = 1$. For $p \in (1, \infty)$, $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ the spatial trace

\begin{align*}
   u \mapsto \gamma_B u: W^{k+l}_p(J; L_p(\Omega)) \cap W^k_p(J; W^{2l}_p(\Omega)) \\
   \quad \to W^{k+l-j_B/2-1/2p}_p(J; L_p(\Gamma)) \cap W^k_p(J; W^{2l-2j_B-1/p}(\Gamma))
\end{align*}

is bounded, see Appendix B. Furthermore, the trace

\[(2.3) \quad u \mapsto \gamma_B u: W^s_p(\Omega) \to B^{s-j_B-1/p}_p(\Gamma)\]

is bounded for every $s \in (j_B + 1/p, \infty)$, cf. [Tri83] Theorem 3.3.3. The temporal trace

\[(2.4) \quad \gamma_l : u \mapsto u|_{l=0} : W^s_p(J; W^s_p(\Omega)) \cap L_p(J; W^{s+2\alpha}_p(\Omega)) \to B^{s+\alpha-2/p}_p(\Omega)\]

is bounded for $\alpha \in (1/p, 1]$, $s \in [0, \infty)$ and $s + 2\alpha \notin \mathbb{N}$ for $\alpha < 1$. The same holds when the domain $\Omega$ is replaced by its boundary $\Gamma$. Finally, we mention that for $p \in (1, \infty)$, $l, s \in \mathbb{N}_0$,
\[2.3.1.\] Dirichlet boundary conditions.

Given \( \gamma, \sigma \in \mathbb{N}, \theta \in (0, 1) \) we have the mixed derivative embedding

\[
\begin{align*}
W^{\gamma+}(J; W^\sigma_p(\Omega)) \cap W^1(J; W^\sigma_p(\Omega)) &\hookrightarrow W^{\gamma+\theta}(J; W^{\sigma+\theta}(\Omega)),
\end{align*}
\]

where again \( \Omega \) can be replaced by \( \Gamma \). For more general embeddings of this form we refer to Appendix B.

2.2. Maximal \( L_p \)-regularity. Let \( J = (0, T) \) or \( J = \mathbb{R}^+ = (0, \infty) \) and assume \( p \in (1, \infty) \).

We say that a closed linear operator \( A : \mathcal{D}(A) \to X \) with dense domain \( \mathcal{D}(A) \) in a Banach space \( X \) admits maximal \( L_p \)-regularity on \( J \) if for each \( F \in L_p(J; X) \) the abstract Cauchy problem

\[
(2.6) \quad v_t(t) + A v(t) = F(t), \quad t \in J, \quad v(0) = v_0,
\]

admits a unique solution \( u \in E(J) = W^1_p(J; X) \cap L_p(J; \mathcal{D}(A)) \) for \( v_0 = 0 \).

Furthermore, the abstract inhomogeneous Cauchy problem \( (2.6) \) is said to admit maximal \( L_p \)-regularity, if

\[
(2.7) \quad (\partial_t + A, \gamma_t) : E(J) \to L_p(J; X) \times \text{tr} E(J), \quad v \mapsto (F, v_0)
\]

is a homeomorphism. Then its inverse is the solution map

\[
(2.8) \quad (\partial_t + A, \gamma_t)^{-1} : L_p(J; X) \times \text{tr} E(J) \to E(J), \quad (F, v_0) \mapsto v.
\]

If \( A : \mathcal{D}(A) \to X \) has maximal \( L_p \)-regularity on \( J \), then the abstract Cauchy problem \( (2.6) \) has maximal \( L_p \)-regularity on \( J \), cf. Section III.1.5 in [Ama95]. The following result is very useful and will be used several times throughout this paper.

**Lemma 2.1** (cf. [Ama95 Proposition III.1.5.3]). Let \( \alpha \in \mathbb{R} \). Suppose that

\[
(\partial_t + A, \gamma_t) : E(J) \to L_p(J; X) \times \text{tr} E(J)
\]

is a homeomorphism. Then

\[
(\partial_t + A, \gamma_t) : e^{\alpha t} E(J) \to e^{\alpha t} L_p(J; X) \times \text{tr} E(J)
\]

is a homeomorphism.

2.3. Optimal regularity results. In order to prove our results on optimal regularity for the linearized versions of (1.7) and (1.8), we need optimal regularity results for the heat equation and the linearized Westervelt equation. We always let \( a, b, c \in (0, \infty) \).

2.3.1. Dirichlet boundary conditions. Recall that \( \lambda_0^D > 0 \) always denotes the smallest eigenvalue of the negative Dirichlet Laplacian in \( L_p(\Omega) \).

**Lemma 2.2** ([LPS06 Proposition 8]). Let \( p \in (1, \infty) \) and \( \omega \in (0, a\lambda_0^D) \). Then the initial boundary value problem for the heat equation

\[
(2.9) \quad \begin{cases}
    u_t - a \Delta u = f, & \text{in } \mathbb{R}^+ \times \Omega, \\
    u = g & \text{on } \mathbb{R}^+ \times \Gamma, \\
    u = u_0 & \text{on } \{t = 0\} \times \Omega,
\end{cases}
\]

has a unique solution

\[
u \in e^{-\omega} H_u, \quad H_u = W^1_p(\mathbb{R}^+; L_p(\Omega)) \cap L_p(\mathbb{R}^+; W^2_p(\Omega)), \]

if and only if the given data \( f, g \) and \( u_0 \) satisfy the regularity conditions

\[ \begin{align*}
(i) \quad f &\in e^{-\omega} L_p(\mathbb{R}^+ \times \Omega), \\
(ii) \quad u_0 &\in W^{2-2/p}_p(\Omega),
\end{align*} \]
(iii) \( g \in e^{-\omega}H_\Gamma, \quad H_\Gamma = W^{1-1/2p}([-1,1]; L^p(\Gamma)) \cap L^p([-1,1]; W^{-1/p}((\Gamma)), \)

(iv) \( u_0|_\Gamma = g|_{t=0} \) in the sense of traces.

**Lemma 2.3.** (MW13, Lemma 5). Suppose \( p \in (1, \infty), \quad p \neq 3/2, \quad n \in \mathbb{N} \) and define \( \omega_0 = \min\{bL_0^2/2, \epsilon^2/b\} \). Then for every \( \omega \in (0, \omega_0) \) there exists a unique solution

\[
\begin{align*}
\int_\Omega u(t,x) & = \int_\Omega u_0(x) \\
\int_\Gamma u(t,x) & = \int_\Gamma g(x) \\
\int_\Omega \Delta u(t,x) & = \int_\Omega f(t,x)
\end{align*}
\]

of the linear initial boundary value problem

\[
\begin{cases}
\partial_t u(t,x) - \Delta u(t,x) = f(t,x) & \text{in } \Omega_t, \\
\partial_t u(t,x) = g(t,x) & \text{on } \Gamma_t,
\end{cases}
\]

\[ u(0,x) = u_0(x) \quad \text{in } \Omega, \]

if and only if the data satisfy the following conditions:

(i) \( f \in e^{-\omega}L^p([-1,1] \times \Omega), \)

(ii) \( u_0 \in L^p([-1,1] \times \Omega), \)

(iii) \( g \in e^{-\omega}L^p([-1,1] \times \Gamma), \)

(iv) \( g|_{t=0} = u_0|_{t=0} \) and if \( p > 3/2 \) also \( g|_{t=0} = u_0|_{t=0} \) in the sense of traces.

2.3.2. Neumann boundary conditions. Now we prove optimal regularity results for the heat equation and the linearized Westervelt equation with Neumann boundary conditions. Recall that \( \lambda_1^N > 0 \) denotes the smallest eigenvalue of the negative homogeneous Neumann Laplacian in \( L^p(\Omega) \). Let \( \bar{u} = |\Omega|^{-1} \int_\Omega u \, dx \) denote the mean of a function \( u : \Omega \to \mathbb{R} \) and let \( \bar{g} = |\Gamma|^{-1} \int_\Gamma g \, d\Gamma \) for \( g : \Gamma \to \mathbb{R} \).

**Lemma 2.4.** Let \( p \in (1, \infty) \setminus \{3\} \) and \( \omega \in [0, a\lambda^N_1) \). Then the inhomogeneous Neumann boundary value problem for the heat equation

\[
\begin{cases}
\partial_t u(t,x) - \Delta u(t,x) = f(t,x) & \text{in } \Omega_t, \\
\partial_n u(t,x) = g(t,x) & \text{on } \Gamma_t,
\end{cases}
\]

 admits a unique solution of the form \( u(t,x) = \bar{u}(t) + v(t,x) \) with

\[
v \in e^{-\omega}H_\nu, \quad H_\nu = W^{1/2}([-1,1] \times \Omega) \cap L^2([-1,1] \times \Omega) \cap L^p([-1,1] \times \Omega), \quad w \in e^{-\omega}L^p([-1,1] \times \Omega),
\]

if and only if the data satisfy the following conditions:

(i) \( f \in e^{-\omega}L^p([-1,1] \times \Omega), \)

(ii) \( u_0 \in W^{1/2}([-1,1] \times \Omega), \)

(iii) \( g \in e^{-\omega}H_\nu, \quad H_\nu = W^{1/2}([-1,1] \times \Omega) \cap L^2([-1,1] \times \Omega) \cap L^p([-1,1] \times \Omega), \)

(iv) \( g|_{t=0} = \partial_n u_0|_{t=0} \) in the sense of traces if \( p > 3/2 \).

If in addition \( f(t,\cdot) \), \( u_0 \), \( g(t,\cdot) \) have mean value zero over \( \Omega \) resp. \( \Gamma \) for all \( t \), then \( w = 0 \).

**Proof.** We first let \( \omega = 0 \). By [DHP03, Theorem 8.2], Lemma A.5 and [Dor93, Theorem 2.4], the Neumann Laplacian in \( L^p(\Omega) \) with domain \( D(\Delta_{N,0}) = D(\Delta \cap L^p(\Omega) \) has maximal regularity on \( \Omega \). We therefore obtain a unique solution \( u_3 \in H^1(\Omega) \) for the problem

\[
\begin{cases}
\partial_t u_3 - \Delta u_3 = f_3 & \text{in } \Omega_t, \\
\partial_n u_3 = 0 & \text{on } \Gamma_t,
\end{cases}
\]

for every given \( f_3 \in L^p(\Omega) \). Furthermore, problem (2.11) admits at most one solution. Indeed, let us construct it as \( u = u_1 + u_2 + u_3 \) where we first solve

\[
\begin{cases}
\partial_t u_1 + \mu u_1 - \Delta u_1 = 0 & \text{in } \Omega_t, \\
\partial_n u_1 = g & \text{on } \Gamma_t,
\end{cases}
\]

\[ u_1(0) = u_0 \text{ in } \Omega, \]
for some sufficiently large $\mu > 0$ with [DHP07, Theorem 2.1]. Next, we let $u_2$ solve the ordinary differential equation
\[
\partial_t u_2(t) = \tilde{f}(t) + \mu \bar{u}_1(t), \quad u_2(0) = 0,
\]
Finally, with $f_3 = f - \tilde{f} + \mu(u_1 - \bar{u}_1)$, we obtain $u_3$ as above. It is easy to check that $v = u_1 + u_3$ and $w = u_2$ satisfy the assertion. The case $\omega > 0$ can be reduced to the previous one by multiplying the functions $u$, $f$, $g$ with $t \mapsto e^{\omega t}$ and using that the spectrum of $-a\Delta_{N,0} + \omega$ is contained in $(0, \infty)$.

**Lemma 2.5.** Let $p \in (1, \infty) \setminus \{3\}$ and $\omega \in (0, b\lambda_1^N)$. Then the inhomogeneous Neumann boundary value problem

\[
\begin{cases}
  u_{tt} - b\Delta u_t = f & \text{in } \mathbb{R}_+ \times \Omega, \\
  \partial_\nu u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\
  (u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega,
\end{cases}
\]

admits a unique solution of the form $u(t, x) = v(t, x) + w(t)$ with
\[
v \in e^{-\omega W_{u,0}}, \quad W_{u,0} = W^2_p(\mathbb{R}_+; L_p(\Omega)) \cap W^1_p(\mathbb{R}_+; W_p^2(\Omega) \cap L_p(0, \Omega)),
\]
\[
w \in L_p, \text{loc}(0, \infty), \quad w_{tt} \in e^{-\omega L_p(\mathbb{R}_+)},
\]
if and only if the data satisfy the following conditions:

1. $f \in e^{-\omega L_p(\mathbb{R}_+ \times \Omega)}$,
2. $u_0 \in W^2_p(\Omega)$, $u_1 \in W^{2-2/p}(\Omega)$,
3. $g \in e^{-\omega W_{u,\nu}}$, $W_{u,\nu} = W^{3/2-1/2p}_p(\mathbb{R}_+; L_p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{1-1/p}_p(\Gamma))$,
4. $g|_{t=0} = \partial_\nu u_0|\Gamma$ and if $p > 3$ also $g_t|_{t=0} = \partial_\nu u_1|\Gamma$ in the sense of traces,
5. $\int_0^\infty f(t) \, dt = b\Delta u_0 - u_1$.

Moreover, $u$ satisfies $u_{tt}(t) = \tilde{f}(t) + b|\Gamma||\Omega|^{-1}\tilde{g}_t(t)$ with $w(0) = \bar{u}_0$ and $w_t(0) = \bar{u}_1$.

**Proof.** We start by proving sufficiency. First, note that $\partial_t^2 \varphi = f$ on $\mathbb{R}_+ \times \Omega$, $\partial_\nu \varphi = g_t$ on $\mathbb{R}_+ \times \Gamma$, $\varphi(0) = u_1$ in $\Omega$, admits a unique solution of the form $\varphi(t, x) = \varphi_1(t, x) + \varphi_2(t)$ such that $\varphi_1 \in e^{-\omega H_{u,0}}$ and $\partial_\nu \varphi_2 \in e^{-\omega L_p(\mathbb{R}_+)}$. In particular, since $\varphi_1$ has zero mean over $\Omega$, we have $\varphi_2(0) = \varphi(0) = \bar{u}_1$ and $\varphi_1(0) = u_1 - \bar{u}_1$. For $x \in \Omega$ and $t \in \mathbb{R}_+$ we define $u(t, x) = v(t, x) + w(t)$, where
\[
v(t, x) = -\int_0^\infty \varphi_1(s, x) \, ds \quad \text{and} \quad w(t) = -\int_t^\infty \varphi_2(s) \, ds.
\]
Clearly $u_t = \varphi$, hence $u_{tt} - b\Delta u_t = f$ in $\Omega$. Integrating the latter with respect to space, multiplying with $|\Omega|^{-1}$ and using the identity $\int_\Omega \Delta u \, dx = \int_\Gamma \partial_\nu u \, dS$, we deduce that $w$ solves the ordinary differential equation
\[
w_{tt}(t) = \tilde{f}(t) + b|\Gamma||\Omega|^{-1}\tilde{g}_t(t), \quad w(0) = \bar{u}_0, \quad w_t(0) = \bar{u}_1.
\]
This implies $v_{tt} - b\Delta v_t = f - \tilde{f} - b|\Gamma||\Omega|^{-1}\tilde{g}_t$ in $\Omega$. In what follows, we abbreviate $v(t) = v(t, \cdot)$, $\varphi_1(t) = \varphi_1(t, \cdot)$ etc. and we let $\chi_{\mathbb{R}_-}(t) = 1$ for $t < 0$ and $\chi_{\mathbb{R}_-}(t) = 0$ for $t > 0$. Using $\omega > 0$, $v_t = \varphi_1$ and the identity
\[
e^{\omega t} v(t) = -\int_t^\infty e^{\omega(t-s)} e^{\omega s} \varphi_1(s) \, ds = -(e^{\omega \cdot \chi_{\mathbb{R}_-}} * (e^{\omega \cdot \varphi_1}))(t)
\]
together with Young’s inequality implies \( v \in e^{-\omega}W_{u,0} \). Moreover, we have
\[
\partial_v v(t) = \partial_v u(t) = - \int_t^\infty \partial_v \varphi(s) \, ds = - \int_t^\infty g(s) \, ds = g(t),
\]
\[
v_t = \varphi_1(t) = - \int_t^\infty \partial_s \varphi_1(s) \, ds = - \int_t^\infty (f(s) + b \Delta \varphi_1(s) - \partial_s \varphi_2(s)) \, ds
\]
\[
= - \int_t^\infty f(s) \, ds + b \Delta v(t) - \varphi_2(t)
\]
and \( v_t(0) = \varphi_2(0) = u_1 - \bar{u}_1 \). Altogether, \( b \Delta v(0) = v_t(0) + \bar{u}_1 + \int_0^\infty f(s) \, ds = b \Delta (u_0 - \bar{u}_0) \) and \( \partial_v v(0) = g(0) = \partial_v (u_0 - \bar{u}_0) \) which implies that \( v(0) = u_0 - \bar{u}_0 \) in \( \Omega \).

To verify necessity of (i)–(v), we assume that \( u(t, x) = v(t, x) + w(t) \) with \( v \in e^{-\omega}W_{u,0} \) and \( w_{tt} \in e^{-\omega}L^p(\mathbb{R}^n) \) is a solution of (2.12). We have \( e^{\omega t} f = e^{\omega t} v_{tt} + e^{\omega t} w_{tt} + b \Delta (e^{\omega t} v_t) \). Since
\[
e^{\omega t} v_t = (e^{\omega t} v_t) - \omega e^{\omega t} v_t \in L^p(\mathbb{R}^n; W^2_p(\partial \Omega) \cap L^p(\Omega)) \]
and \( e^{\omega t} v_{tt} = (e^{\omega t} v_{tt}) - 2 \omega e^{\omega t} v_t - \omega^2 e^{\omega t} v \in L^p(\mathbb{R}^n; L^p(0,\Omega)) \) we conclude \( e^{\omega t} f \in L^p(\mathbb{R}^n \times \Omega) \) and (i) is verified. Concerning (ii) note that exponential weights do not affect the initial regularity. Due to \( W^1_p(\mathbb{R}^n) \hookrightarrow BUC(\mathbb{R}^n) \) we infer \( v_{t=0} \in W^2_p(\partial \Omega) \cap L^p(0,\Omega) \), hence \( u_{t=0} = v_{t=0} + w_{t=0} \in W^2_p(\Omega) \). Furthermore, we have \( e^{\omega t} v_t \in W^1_p(\mathbb{R}^n; L^p(0,\Omega)) \cap L^p(\mathbb{R}^n; W^2_p(\Omega) \cap L^p(0,\Omega)) \). Applying the temporal trace (2.4) with \( \alpha = 1 \) and \( s = 0 \) implies \( v_{t=0} \in B^{2-2/p}_p(\Omega) = W^{2-2/p}_p(\Omega) \), hence \( u_{t=0} = v_{t=0} + w_{t=0} \in W^{2-2/p}_p(\Omega) \). In order to check (iii), we apply the spatial trace (2.2) with \( k = l = 1 \) to \( e^{\omega t} v \in W_{u,0} \) which gives us \( e^{\omega t} \partial_v u(t) \in \mathcal{L} \) as claimed. Next, note that (2.3) applied to \( u_0 \in W^2_p(\Omega) \) implies \( \partial_v u_0 \in B^{1-1/p}_p(\Omega) = W^{1-1/p}_p(\Omega) \). Moreover, we have \( g \in W^p(\mathbb{R}^n; W^{1-1/p}_p(\Gamma)) \hookrightarrow BUC(\mathbb{R}^n, W^{1-1/p}_p(\Gamma)) \), hence \( \partial_v u_0 | \Gamma = g_{t=0} \in W^{1-1/p}_p(\Gamma) \). From \( u_1 \in W^{2-2/p}_p(\Omega) \) we obtain \( \partial_v u_1 \in B^{1-3/p}_p(\Gamma) = W^{1-3/p}_p(\Gamma) \) for \( p > 3 \) and, from \( g_t \in \mathcal{H}_v \), using (2.4) with \( \alpha = 1/2 - 1/2p \) and \( s = 0 \), we get \( g_{t=0} \in B^{1-3/p}_p(\Gamma) = W^{1-3/p}_p(\Gamma) \) if \( p > 3 \). Altogether, \( \partial_v u_0 | \Gamma = g_{t=0} \in W^{1-3/p}_p(\Gamma) \) for \( p > 3 \). For (v), note that integrating \( u_{tt}(t) - b \Delta u_t(t) = f(t) \) with respect to time yields \( u_t(t) - b \Delta u(t) = - \int_t^\infty f(s) \, ds \), hence in particular \( u_t(0) - b \Delta u(0) = - \int_0^\infty f(s) \, ds \). Therewith the proof of necessity is complete.

Finally, we show that (2.12) has at most one solution. To this end, suppose we have given two solutions of (2.12). Their difference \( \hat{u} \) solves
\[
\hat{u}_{tt} - b \Delta \hat{u}_t = 0 \quad \text{in } \mathbb{R}^n \times \Omega, \quad \partial_\nu \hat{u} = 0 \quad \text{on } \mathbb{R}^n \times \Gamma, \quad \hat{u}(0) = \hat{u}_t(0) = 0 \quad \text{in } \Omega.
\]
Furthermore, \( \hat{u} = \hat{u}_t \) solves the heat problem
\[
\hat{u}_t - b \Delta \hat{u}_t = 0 \quad \text{in } \mathbb{R}^n \times \Omega, \quad \partial_\nu \hat{u} = 0 \quad \text{on } \mathbb{R}^n \times \Gamma, \quad \hat{u} = 0 \quad \text{in } \Omega,
\]
which implies \( \hat{u}_t = 0 \). Hence \( \hat{u} \) is constant which together with \( \hat{u}(0) = 0 \) implies \( \hat{u} = 0 \).

\textbf{Lemma 2.6.} Let \( p \in (1, \infty) \setminus \{3\} \) and \( \omega_0 = \min \{b \lambda^N_3/2, c^2/b \} \). Then for every \( \omega \in (0, \omega_0) \) the initial boundary value problem
\[
\begin{cases}
\begin{aligned}
 u_{tt} - b \Delta u_t - c^2 \Delta u &= f, & \text{in } \mathbb{R}^n \times \Omega, \\
 \partial_\nu u &= 0, & \text{on } \mathbb{R}^n \times \Gamma, \\
 (u, u_t) &= (u_0, u_1) & \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\end{cases}
\]
has a unique solution
\[
u \in e^{-\omega}W_{u,0}, \quad W_{u,0} = W^2_p(\mathbb{R}^n; L^p(\Omega)) \cap W^1_p(\mathbb{R}^n; W^2_p(\Omega) \cap L^p(0,\Omega))
\]
if and only if

(i) \( f \in e^{-\omega L_p(\mathbb{R}_+; L_{p,0}(\Omega))} \),

(ii) \( u_0 \in W^2_p(\Omega) \cap L_{p,0}(\Omega) \) and \( u_1 \in W^{2-2/p}_p(\Omega) \cap L_{p,0}(\Omega) \) such that \( \partial_\nu u_0|_\Gamma = 0 \) and if \( p > 3 \) additionally \( \partial_\nu u_1|_\Gamma = 0 \).

Proof. In [MW11] the result was established with Dirichlet instead of Neumann boundary conditions. Here, we just point out the main steps of the proof. We represent \( 2.13 \) as

\[
\begin{pmatrix}
  u_t \\
  u_{tt}
\end{pmatrix} + \begin{pmatrix}
  0 & -I \\
  -c^2\Delta_{N,0} & -b\Delta_{N,0}
\end{pmatrix} \begin{pmatrix}
  u \\
  u_t
\end{pmatrix} = \begin{pmatrix}
  0 \\
  f
\end{pmatrix}, \quad \begin{pmatrix}
  u(0) \\
  u_t(0)
\end{pmatrix} = \begin{pmatrix}
  u_0 \\
  u_1
\end{pmatrix}.
\]

Let us consider the space \( \tilde{X} = D(\Delta_{N,0}) \times L_{p,0}(\Omega) \) and the operator \( \tilde{A}: D(\tilde{A}) \to \tilde{X} \) given by

\[
\tilde{A} = \begin{pmatrix}
  0 & -I \\
  -c^2\Delta_{N,0} & -b\Delta_{N,0}
\end{pmatrix}, \quad D(\tilde{A}) = D(\Delta_{N,0}) \times D(\Delta_{N,0}).
\]

First we show that there is some \( \nu > 0 \) such that \( \nu + \tilde{A} \) admits maximal \( L_p \)-regularity on \( \mathbb{R}_+ \) by a perturbation argument. Choosing a decomposition of \( \tilde{A} = \tilde{A}_1 + \tilde{A}_2 \) with

\[
\tilde{A}_1 = \begin{pmatrix}
  \alpha I & -I \\
  0 & -b\Delta_{N,0}
\end{pmatrix} \quad \text{and} \quad \tilde{A}_2 = \begin{pmatrix}
  -\alpha I & 0 \\
  -c^2\Delta_{N,0} & 0
\end{pmatrix}
\]

for some \( \alpha > 0 \), it turns out that operator \( \tilde{A}_1: D(\tilde{A}) \to \tilde{X} \) admits maximal regularity on \( \mathbb{R}_+ \) due to Lemma 2.4 and the fact that the bounded operator \( (\partial_t + \alpha): 0W^1_p(\mathbb{R}_+; D(\Delta_{N,0})) \to L_p(\mathbb{R}_+; D(\Delta_{N,0})) \) is invertible. Moreover, since \( \tilde{A}_2: \tilde{X} \to \tilde{X} \) is bounded, on the strength of Proposition 4.3 and Theorem 4.4 in [DHP03] there exists some \( \nu > 0 \) such that \( \nu + \tilde{A}_1 + \tilde{A}_2 \) admits maximal regularity on \( \mathbb{R}_+ \).

Next we claim that the spectral bound \( s(-\tilde{A}) = \sup\{\Re(\lambda): \lambda \in \sigma(-\tilde{A})\} \) of \( -\tilde{A} \) is given by \( s(-\tilde{A}) = -\omega_0 \). This follows analogously to [MW11] Lemma 2.4 if one replaces the Dirichlet eigenvalues by the ones of \( -\Delta_{N,0} \).

Since \( s(-\tilde{A}) = -\omega_0 < 0 \), the spectral bound of \( -\tilde{A} + \omega \) equals \( \omega - \omega_0 \) which is strictly negative as long as \( \omega \in [0, \omega_0) \). Hence, for each \( \omega \in [0, \omega_0) \) the operator \( -\tilde{A} + \omega \) has maximal \( L_p \)-regularity on \( \mathbb{R}_+ \) by [Dor93] Theorem 2.4, that is

\[
(\partial_t + (-\tilde{A} + \omega, \gamma_t): W^1_p(\mathbb{R}_+; \tilde{X}) \cap L_p(\mathbb{R}_+; D(\tilde{A})) \to L_p(\mathbb{R}_+; \tilde{X}) \times (\tilde{X}, D(\tilde{A}))_{1-1/p,p}
\]

is an isomorphism. Employing Lemma 2.4 we conclude that for every \( \omega \in [0, \omega_0) \) the operator

\[
(\partial_t + (\tilde{A}, \gamma_t): e^{-\omega}(W^1_p(\mathbb{R}_+; \tilde{X}) \cap L_p(\mathbb{R}_+; D(\tilde{A}))) \to e^{-\omega}L_p(\mathbb{R}_+; \tilde{X}) \times (\tilde{X}, D(\tilde{A}))_{1-1/p,p}
\]

is an isomorphism.

It is easy to check that \( (u, u_t) \in e^{-\omega}(W^1_p(\mathbb{R}_+; \tilde{X}) \cap L_p(\mathbb{R}_+; D(\tilde{A}))) \) implies \( u \in e^{-\omega}W_u,0 \).

Finally we arrive at our optimal regularity result for the linearized Westervelt equation with inhomogeneous Neumann boundary conditions.
Lemma 2.7. Let \( p \in (1, \infty) \setminus \{3\} \) and set \( \omega_0 = \min\{b\lambda_1^N/2, c^2/b\} \). Then for every \( \omega \in (0, \omega_0) \) the linear initial boundary value problem

\[
\begin{aligned}
&u_{tt} - b\Delta u_t - c^2\Delta u = f, & \text{in } \mathbb{R}_+ \times \Omega, \\
&\partial_\nu u = g, & \text{on } \mathbb{R}_+ \times \Gamma, \\
&(u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\]  

(2.15)

admits a unique solution of the form \( u(t, x) = v(t, x) + w(t) \), where \( v \in e^{-\omega}W_{u,0} \), \( W_{u,0} = W^2_p(\mathbb{R}_+; L^p_0(\Omega)) \cap W^1_p(\mathbb{R}_+; W^2_p(\Omega) \cap L^p_0(\Omega)) \), \( w_{tt} \in e^{-\omega}L^p_p(\mathbb{R}_+) \) if and only if the data satisfy the following conditions:

(i) \( f \in e^{-\omega}L^p_p(\mathbb{R}_+ \times \Omega) \),

(ii) \( u_0 \in W^2_p(\Omega), \ u_1 \in W^{2-2/p}_p(\Omega) \),

(iii) \( g \in e^{-\omega}W^p_p, \ W^p_p = W^{3/2-1/2p}_p(\mathbb{R}_+; L^p_p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{1-1/p}_p(\Gamma)) \),

(iv) \( g|_{t=0} = \partial_\nu u_0 |_{\Gamma} \) and if \( p > 3 \) additionally \( g_t|_{t=0} = \partial_\nu u_1 |_{\Gamma} \) in the sense of traces.

Proof. From Lemma 2.6 we obtain uniqueness. In order to show necessity of (i)–(iv) one proceeds as in the proof of Lemma 2.5. It therefore remains to show sufficiency. Let \( \delta > \omega \). From Lemma 2.5 we obtain that

\[
\begin{aligned}
&\varphi_{tt} - b\Delta \varphi_t = f - f_\delta \quad \text{in } \mathbb{R}_+ \times \Omega, \\
&\partial_\nu \varphi = g, \quad \text{on } \mathbb{R}_+ \times \Gamma, \\
&(\varphi, \varphi_t) = (u_0, u_1) \quad \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\]

where \( f_\delta = \delta e^{-\delta t}(\int_0^\infty f(s) \, ds + u_1 - b\Delta u_0) \), admits a unique solution \( \varphi(x, t) = \varphi_v(x, t) + \varphi_w(t) \) such that \( \varphi_v \in e^{-\omega}W_{u,0} \) and \( \partial^2_\nu \varphi_w \in e^{-\omega}L^p_p(\mathbb{R}_+) \). Next, Lemma 2.6 implies that

\[
\begin{aligned}
&\theta_{tt} - b\theta_{vt} - c^2\Delta \theta_v = f_\delta - \bar{f}_\delta + c^2\Delta \varphi_v - c^2\Delta \varphi_v \quad \text{in } \mathbb{R}_+ \times \Omega, \\
&\partial_\nu \theta_v = 0, \quad \text{on } \mathbb{R}_+ \times \Gamma, \\
&(\theta_v, \theta_{vt}) = (0, 0) \quad \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\]

has a unique solution \( \theta \in e^{-\omega}W_{u,0} \). Furthermore, we define \( \theta_w \) as the solution of the ordinary differential equation

\[
\theta_{w,tt} = c^2\Delta \varphi_v(t) + \bar{f}_\delta(t), \quad \theta_w(0) = 0, \quad \theta_{w,t}(0) = 0.
\]

Then \( v = \varphi_v + \theta_v \) and \( w = \varphi_w + \theta_w \) satisfy the assertion and we are done.

Remark 2.8. If we consider (2.15) on a finite time interval \( J = (0, T) \) instead of \( \mathbb{R}_+ \), we may set \( \omega = 0 \) and obtain a unique solution \( u \in W^2_p(J; L^p_0(\Omega)) \cap W^1_p(J; W^2_p(\Omega)) \) if and only if conditions (i)–(iv) (with \( \omega = 0 \)) hold.

2.4. Analysis in Banach spaces. For later use in the proof of global well-posedness of (1.7) and (1.8) we will now recall the concept of analytic mappings in Banach spaces and the analytic version of the implicit function theorem. The remainder of this section is collected from Section 15.1 in [Dei85].

Let \( X \) and \( Y \) be Banach spaces over the same field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \) and let \( U \subset X \) be open. Then \( F: U \to Y \) is called analytic at \( x_0 \in U \) if there is some \( r > 0 \) and continuous symmetric \( k \)-linear operators \( F_k: X^k = X \times \cdots \times X \to Y \) for \( k \geq 1 \) such that

\[
\sum_{k=1}^\infty \|F_k\| \|h\|^k < \infty \quad \text{and} \quad F(x_0 + h) = F(x_0) + \sum_{k=1}^\infty F_k(h^k).
\]
for \( h \in X, \|h\| < r \). Here, \( \|F_k\| = \sup\{\|F_k(h^k)\| : \|h\| \leq 1\} \). We then necessarily have \( F_k = \frac{1}{h} F(h^k)(x_0) \). The map \( F: U \to Y \) is called analytic if \( F \) is analytic at every \( x_0 \in U \). In particular, every bounded linear map \( F: X \to Y \) is analytic.

**Theorem 2.9** (Implicit Function Theorem, cf. [Dei85, 15.1]). Let \( X, Y \) and \( Z \) be Banach spaces over the same field \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \). Assume \( U \subset X \) and \( V \subset Y \) are neighborhoods of \( x_0 \in X \) and \( y_0 \in Y \), respectively. Furthermore, suppose

(i) \( F: U \times V \to Z, (x, y) \mapsto F(x,y) \) is continuous,

(ii) the Fréchet derivative \( F_x: U \times V \to L(X,Z) \) of \( F \) with respect to \( x \) is continuous,

(iii) \( F(x_0,y_0) = 0 \) and \( F_x(x_0,y_0): X \to Z \) is an isomorphism.

Then there exist balls \( B_p(x_0) \subset U \) and \( B_p(y_0) \subset V \) and a unique map \( \varphi: B_p(y_0) \to B_p(x_0) \) such that \( \varphi(y_0) = x_0 \) and \( F(\varphi(y),y) = 0 \) for all \( y \in B_p(y_0) \). The map \( \varphi \) is continuous. If furthermore, \( F \) is analytic, then \( \varphi \) is analytic in some neighborhood of \( y_0 \), in particular on some (possibly smaller) ball \( B_{\kappa}(y_0) \subset B_p(y_0) \).

### 3. The Dirichlet boundary value problem

In this section we prove global well-posedness and exponential stability for (1.7). First of all, we consider the linearized equation for the inhomogeneous Dirichlet boundary value problem and represent it as an abstract evolution equation. We show that this abstract equation admits maximal \( L_p \)-regularity and derive an optimal regularity result for the linearized equation associated with (1.7). Then we use the implicit function theorem to construct a solution of the nonlinear problem (1.7). Exponential decay of this solution is an immediate consequence.

#### 3.1. Maximal \( L_p \)-regularity for the linearized equation

Suppose \( J = (0, T) \) or \( J = \mathbb{R}_+ \). For \( f \in L_p(J \times \Omega) \) we consider the boundary value problem

\[
\begin{aligned}
 \left\{
 (a\Delta - \partial_t)(u_t) - b\Delta u_t - c^2\Delta u &= f & \text{in } J \times \Omega, \\
 (u, \Delta u) &= (g, h) & \text{on } J \times \Gamma, \\
 (u, u_t, u_{tt}) &= (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\]

where \( u_0, u_1, u_2: \Omega \to \mathbb{R} \) and \( g, h: J \times \Gamma \to \mathbb{R} \) are the given initial and boundary data, respectively. In order to address the problem of maximal \( L_p \)-regularity for the linearized equation, we represent (3.1) with \( g = h = 0 \) as an abstract Cauchy problem

\[
\begin{pmatrix}
 \partial_t + \begin{pmatrix}
 0 & -I & 0 \\
 -c^2\Delta_D & -b\Delta_D & -I \\
 0 & 0 & -a_D\Delta
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
 u \\
 u_t \\
 u_{tt} - b\Delta_D u_t - c^2\Delta u_D
\end{pmatrix} = \begin{pmatrix}
 0 \\
 0 \\
 -f
\end{pmatrix}.
\]

This motivates us to consider the Banach space

\[
X^D = \mathcal{D}((\Delta_D)^2) \times \mathcal{D}((\Delta_D)^2) \times L_p(\Omega)
\]

and the densely defined linear operator \( A^D: \mathcal{D}(A^D) \to X^D \) given by

\[
A^D = \begin{pmatrix}
 0 & -I & 0 \\
 -c^2\Delta_D & -b\Delta_D & -I \\
 0 & 0 & -a\Delta_D
\end{pmatrix}, \quad \mathcal{D}(A^D) = \mathcal{D}((\Delta_D)^2) \times \mathcal{D}((\Delta_D)^2) \times \mathcal{D}(\Delta_D).
\]

Therewith, we may write (3.1) as an abstract evolution equation

\[
\partial_t v^D + Av^D = F, \quad v^D(0) = v_0^D
\]
if we define

\[
(3.4) \quad v^D = \begin{pmatrix} u \\ u_t \\ u_{tt} - b\Delta_D u_t - c^2 \Delta_D u \end{pmatrix}, \quad v_0^D = \begin{pmatrix} u_0 \\ u_1 \\ u_2 - b\Delta_D u_1 - c^2 \Delta_D u_0 \end{pmatrix}, \quad F \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}.
\]

First of all we will treat the issue of maximal $L_p$-regularity of $A^D : D(A^D) \to X^D$ on $\mathbb{R}_+$. 

**Proposition 3.1.** Let $p \in (1, \infty)$. There is a constant $\mu > 0$ such that $\mu + A^D$ has maximal $L_p$-regularity on $\mathbb{R}_+$.

**Proof.** Let $\alpha > 0$. We decompose $A^D$, $A^D = A_1^D + A_2^D$, where

\[
A_1^D = \begin{pmatrix} \alpha I & -I & 0 \\ 0 & -b\Delta_D & -I \\ 0 & 0 & -a\Delta_D \end{pmatrix} \quad \text{and} \quad A_2^D = \begin{pmatrix} -\alpha I & 0 & 0 \\ -c^2 \Delta_D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

First we show that $A_1^D : D(A^D) \to X^D$ has maximal $L_p$-regularity. To this end, we consider the Cauchy problem $v_t + A_1^D v = F$, $v_0 = 0$ and show that for each $F \in L_p(\mathbb{R}_+; X^D)$ there exists a unique solution $v \in W^1_p(\mathbb{R}_+; X^D) \cap L_p(\mathbb{R}_+; D(A^D))$. With $v = (v_1, v_2, v_3)^T$ and $F = (f_1, f_2, f_3)^T$, we explicitly have

\[
\begin{align*}
\partial_t v_1 + \alpha v_1 - v_2 &= f_1, & v_1(0) &= 0, \\
\partial_t v_2 - b\Delta_D v_2 - v_3 &= f_2, & v_2(0) &= 0, \\
\partial_t v_3 - a\Delta_D v_3 &= f_3, & v_3(0) &= 0.
\end{align*}
\]

Since we know from Lemma 222 that the homogeneous heat equation admits maximal $L_p$-regularity, we obtain that for all $f_3 \in L_p(\mathbb{R}_+ \times \Omega)$ there exists a unique solution

\[v_3 \in W^1_p(\mathbb{R}_+; L_p(\Omega)) \cap L_p(\mathbb{R}_+; D(\Delta_D)).\]

Moreover, as $f_2 + v_3 \in L_p(\mathbb{R}_+; D(\Delta_D))$, Lemma C.5 implies that there is a unique solution

\[v_2 \in W^1_p(\mathbb{R}_+; D(\Delta_D)) \cap L_p(\mathbb{R}_+; D((\Delta_D)^2)).\]

Now, note that for $\alpha > 0$ the operator $(\partial_t + \alpha) : W^1_p(\mathbb{R}_+; D((\Delta_D)^2)) \to L_p(\mathbb{R}_+; D((\Delta_D)^2))$ is invertible. Since $f_1 + v_2 \in L_p(\mathbb{R}_+; D((\Delta_D)^2))$ we obtain a unique solution

\[v_1(t) = \int_0^t e^{-\alpha(t-s)}(f_1(s) + v_2(s)) \, ds,
\]

which satisfies $v_1 \in W^1_p(\mathbb{R}_+; D((\Delta_D)^2))$. Altogether, we conclude that $A_1 : D(A^D) \to X^D$ admits maximal $L_p$-regularity.

Moreover, by the fact that $A_2^D : X^D \to X^D$ is a bounded linear operator, Proposition 4.3 and Theorem 4.4 in [DHP03] imply that there exists some $\mu > 0$ such that $\mu + A_1^D + A_2^D$ has maximal regularity which concludes the proof. \qed

In order to show maximal regularity for the operator $A^D : D(A^D) \to X^D$, we need the following result on its spectrum.

**Lemma 3.2** (cf. [BK14, Lemma 3.10]). The spectral bound $s(-A^D) = \sup \{ \text{Re}(\lambda) : \lambda \in \sigma(-A^D) \}$ of $-A^D$ is given by $s(-A^D) = -\omega_0^D$, where $\omega_0^D = \min \{ a\lambda_0^D, b\lambda_0^D/2, c^2/b \}$. In particular, if $\text{Re}(\lambda) < \omega_0^D$, then $\lambda \in \rho(A^D)$. 

Theorem 3.3. Let $p \in (1, \infty)$ and $\omega \in [0, \omega_0)$ where $\omega_0^D = \min\{a\lambda_p^D, b\lambda_0^D/2, c^2/b\}$. Then $A^D : \mathcal{D}(A^D) \to X^D$ has maximal $L_p$-regularity on $\mathbb{R}^+$ in the sense that

\[
(\partial_t + A^D, \gamma_t) : e^{-\omega}(W^1_p(\mathbb{R}^+; X^D) \cap L_p(\mathbb{R}^+; \mathcal{D}(A^D))) \to e^{-\omega}L_p(\mathbb{R}^+; X^D) \times (X^D, \mathcal{D}(A^D))_{1-1/p, p}
\]
is an isomorphism.

Proof. We follow the proof of Theorem 2.5 in [MW11]. From Proposition 3.1 we know $\mu + A^D$ admits maximal regularity on $\mathbb{R}^+$ for some $\mu > 0$. Multiplying $v_t^D + A^D v^D = F$ by $e^{-\mu t}$ shows that $A^D$ has maximal $L_p$-regularity on bounded intervals $J = (0, T)$. Lemma 3.2 tells us that spectral bound $s(-A^D) = -\omega_0$ of $-A^D$ is strictly negative. Hence $s(-A^D + \omega) = \omega - \omega_0^D < 0$ as long as $\omega \in [0, \omega_0^D)$. From [Dor93 Theorem 2.4] we deduce that $A^D - \omega$ admits maximal $L_p$-regularity on $\mathbb{R}^+$ for every $\omega \in (0, \omega_0^D)$, that is,

\[
(\partial_t + A^D - \omega, \gamma_t) : W^1_p(\mathbb{R}^+; X^D) \cap L_p(\mathbb{R}^+; \mathcal{D}(A^D)) \to L_p(\mathbb{R}^+; X^D) \times (X^D, \mathcal{D}(A^D))_{1-1/p, p}
\]
is an isomorphism. Now Lemma 2.1 implies the result. \hfill \Box

Corollary 3.4. Let $p \in (1, \infty) \setminus \{3/2\}$ and define $\omega_0^D = \min\{a\lambda_p^D, b\lambda_0^D/2, c^2/b\}$. Then for every $\omega \in (0, \omega_0)$ the linear Dirichlet boundary value problem

\[
\begin{cases}
(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = f & \text{in } \mathbb{R}^+ \times \Omega, \\
(u, \Delta u) = (0, 0) & \text{on } \mathbb{R}^+ \times \Gamma, \\
(u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega,
\end{cases}
\tag{3.5}
\]

admits maximal $L_p$-regularity in the sense that there exists a unique solution

\[
u \in e^{-\omega}E_u, \quad E_u = W^3_p(\mathbb{R}^+; L_p(\Omega)) \cap W^1_p(\mathbb{R}^+; W^4_p(\Omega)),
\]
if and only if

(i) $f \in e^{-\omega}L_p(\mathbb{R}^+ \times \Omega)$,

(ii) $u_0 \in W^4_p(\Omega), u_1 \in W^{4-2/p}_p(\Omega), u_2 \in W^{2-2/p}_p(\Omega)$ with $u_0|\Gamma = \Delta u_0|\Gamma = u_1|\Gamma = 0$ and, if $p > 3/2$, also $\Delta u_1|\Gamma = u_2|\Gamma = 0$ in the sense of traces.

Proof. Based on the choices of $X^D$ and $\mathcal{D}(A^D)$ in (3.2) and (3.3), it is straightforward to check that the condition $v^D \in e^{-\omega}(W^1_p(\mathbb{R}^+; X^D) \cap L_p(\mathbb{R}^+; \mathcal{D}(A^D)))$ where $v^D$ is given by (3.4) implies $u \in e^{-\omega}(E_u \cap W^2_p(\mathbb{R}^+; W^4_p(\Omega)))$. Since the mixed derivative embedding gives us $E_u \subseteq W^2_p(\mathbb{R}^+; W^4_p(\Omega))$, we arrive at $u \in e^{-\omega}E_u$. Next, we determine $(X, \mathcal{D}(A))_{1-1/p, p}$. It is trivial that $(\mathcal{D}((\Delta_D)^2), \mathcal{D}((\Delta_D)^2))_{1-1/p, p} = \mathcal{D}((\Delta_D)^2)$, i. e. we have $u_0 \in W^4_p(\Omega)$ with $u_0|\Gamma = \Delta u_0|\Gamma = 0$. Moreover, since for $p \in (1, \infty)$ we have $2/p \in \mathbb{R} \setminus \mathbb{N}$ unless $p = 2$, (2.1) gives us

\[
(W^2_p(\Omega), W^4_p(\Omega))_{1-1/p, p} = B^{4-2/p}_p(\Omega) = W^{4-2/p}_p(\Omega),
\]

\[
(L_p(\Omega), W^2_p(\Omega))_{1-1/p, p} = B^{2-2/p}_p(\Omega) = W^{2-2/p}_p(\Omega).
\]

Moreover, interpolation with boundary conditions as in [Ama09 Section 4.9] yields $u_1|\Gamma = \Delta u_1|\Gamma = u_2 - b\Delta_D u_1 - c^2\Delta_D u_0|\Gamma = 0$. Hence, we have $u_0 \in W^4_p(\Omega), u_1 \in W^{4-2/p}_p(\Omega)$ and $u_2 - c^2\Delta_D u_0 - b\Delta_D u_1 \in W^{2-2/p}_p(\Omega)$ which is equivalent to $u_0 \in W^4_p(\Omega), u_1 \in W^{4-2/p}_p(\Omega)$ and $u_2 \in W^{2-2/p}_p(\Omega)$. The result now follows from Theorem 3.3. \hfill \Box
We now arrive at the final result for this section and prove optimal regularity for the linear initial boundary value problem (3.1) in the sense that the regularity of the data \( f, g, h, u_0, u_1 \) and \( u_2 \) are necessary and sufficient for the existence of a unique solution \( u \in e^{-\omega E_u} \).

**Proposition 3.5.** Let \( p \in (1, \infty) \setminus \{3/2\} \) and define \( \omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D / 2, e^2 / b\} \). Then for every \( \omega \in (0, \omega_0) \) the linear initial boundary value problem

\[
\begin{cases}
(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 \Delta u) = f & \text{in } \mathbb{R}_+ \times \Omega, \\
(u, \Delta u) = (g, h) & \text{on } \mathbb{R}_+ \times \Gamma, \\
(u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{ t = 0 \} \times \Omega,
\end{cases}
\]

has a unique solution

\[
u \in e^{-\omega E_u}, \quad E_u = W^3_p(\mathbb{R}_+; L^p(\Omega)) \cap W^1_p(\mathbb{R}_+; W^2_p(\Omega)),
\]

if and only if the data satisfy the conditions

(i) \( f \in e^{-\omega L_p}(\mathbb{R}_+ \times \Omega), \)

(ii) \( u_0 \in W^4_p(\Omega), \) \( u_1 \in W^{4-2/p}_p(\Omega), \) \( u_2 \in W^{2-2/p}_p(\Omega), \)

(iii) \( g \in e^{-\omega F_{g,\Gamma}}, \) \( F_{g,\Gamma} = W^{3-1/2p}_p(\mathbb{R}_+; L^p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{4-1/p}_p(\Gamma)), \)

\[
h \in e^{-\omega F_{h,\Gamma}}, \quad F_{h,\Gamma} = W^{2-1/2p}_p(\mathbb{R}_+; L^p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{2-1/p}_p(\Gamma)),
\]

(iv) \( u_0|_ \Gamma = g|_{t=0}, u_1|_ \Gamma = g_1|_{t=0}, \Delta u_0|_ \Gamma = h_1|_{t=0} \) and, if \( p > 3/2, \) also \( \Delta u_1|_ \Gamma = h_t|_{t=0} \) and \( u_2|_ \Gamma = g_\Gamma|_{t=0} \) hold in the sense of traces.

Moreover, the solution fulfills the estimate

\[
\|u\|_{e^{-\omega E_u}} \leq \|f\|_{e^{-\omega L_p}} + \|g\|_{e^{-\omega F_{g,\Gamma}}} + \|h\|_{e^{-\omega F_{h,\Gamma}}} + \|u_0\|_{W^4_p} + \|u_1\|_{W^{4-2/p}_p} + \|u_2\|_{W^{2-2/p}_p}.
\]

**Proof.** First we show necessity of (i)–(iv) for the existence of a unique solution \( u \in e^{-\omega E_u} \) of (3.6). In the proof of Corollary 3.4 we already mentioned that \( E_u \hookrightarrow W^2_p(\mathbb{R}_+ \times \Omega). \)

Since \( (e^{\omega t}u)_t = \omega e^{\omega t}u + e^{\omega t}u_t, (e^{\omega t}u)_{tt} = \omega^2 e^{\omega t}u + 2\omega e^{\omega t}u_t + e^{\omega t}u_{tt} \) and \( (e^{\omega t}u)_{ttt} = \omega^3 e^{\omega t}u + 3\omega^2 e^{\omega t}u_t + 3\omega e^{\omega t}u_{tt} + e^{\omega t}u_{ttt}, \) implies that \( u \in e^{-\omega E_u} \) implies \( e^{\omega t}u \in L^p(\mathbb{R}_+; W^4_p(\Omega)) \cap W^1_p(\mathbb{R}_+; W^2_p(\Omega)) \), \( e^{\omega t}u_{tt} \in L^p(\mathbb{R}_+; W^1_p(\Omega)) \cap W^1_p(\mathbb{R}_+; L^p(\Omega)) \) and \( e^{\omega t}u_{ttt} \in L^p(\mathbb{R}_+ \times \Omega) \), hence \( e^{\omega t}f = e^{\omega t}(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 \Delta u) \in L^p(\mathbb{R}_+ \times \Omega) \).

Next, we show (ii). The embedding \( W^1_p(\Omega) \hookrightarrow BUC(J) \) implies \( u_0 \in W^4_p(\Omega) \) whereas the temporal trace (2.4) with \( \alpha = 1, s = 2 \) and \( \alpha = 1, s = 0 \) gives us the desired regularities of \( u_1 \) and \( u_2 \), respectively.

For \( u \in e^{-\omega E_u} \) the spatial trace (2.2) with \( k = 1 \) and \( l = 2 \) implies \( u|_ \Gamma = g \in e^{-\omega F_{g,\Gamma}} \) and for \( \Delta u \in e^{-\omega W^1_u} \) the choice \( k = l = 1 \) gives us \( \Delta u|_ \Gamma = h \in e^{-\omega F_{h,\Gamma}} \). This shows (iii).

Using \( W^1_p(\Omega) \hookrightarrow BUC(J), \) the spatial trace (2.3), the temporal trace (2.4) and the mixed derivative embedding (2.5) one shows (iv). We have

\[
\begin{align*}
u_0|_ \Gamma &= g|_{t=0} \in W^{4-1/p}_p(\Gamma), \quad &\Delta u_0|_ \Gamma &= h|_{t=0} \in W^{2-1/p}_p(\Gamma), \\
u_1|_ \Gamma &= g_1|_{t=0} \in B^{4-3/p}_{p,p}(\Gamma), \quad &\Delta u_1|_ \Gamma &= h_t|_{t=0} \in B^{2-3/p}_{p,p}(\Gamma) \text{ if } p > 3/2,
\end{align*}
\]

It remains to show that conditions (i)–(iv) imply the existence of a unique solution \( u \in e^{-\omega E_u} \) of (3.6). Since we are dealing with a linear partial differential equation with constant coefficients, we may interchange the order of differentiation on the left-hand side and consider
the subproblems
\begin{align}
\begin{cases}
  w_{tt} - b\Delta w_t - c^2 \Delta w = f & \text{in } \mathbb{R}_+ \times \Omega, \\
  \quad w = ah - g_t & \text{on } \mathbb{R}_+ \times \Gamma, \\
  \quad (w, \dot{w}) = (a\Delta u_0 - u_1, a\Delta u_1 - u_2) & \text{on } \{t = 0\} \times \Omega,
\end{cases}
\end{align}
(3.7)

and
\begin{align}
\begin{cases}
  a\Delta u - u_t = w & \text{in } \mathbb{R}_+ \times \Omega, \\
  \quad u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\
  \quad u = u_0 & \text{on } \{t = 0\} \times \Omega.
\end{cases}
\end{align}
(3.8)

From condition (ii) we obtain \(a\Delta u_0 - u_1 \in W^2_p(\Omega)\) and \(a\Delta u_1 - u_2 \in W^{2-2/p}_p(\Omega)\). Furthermore, (iii) implies \(ah - g_t \in e^{-\omega_0 t} \mathbb{H}_1\). On the strength of Lemma 2.3 we obtain that (3.7) admits a unique solution \(w \in e^{-\omega_0 t} \mathbb{W}_u\). Now we use Corollary C.4 with \(l = 1\) and \(k = 2\) to solve (3.8) and obtain that (3.8) has a solution \(u \in e^{-\omega_0 t} \mathbb{E}_u\). This concludes the proof of sufficiency. Uniqueness follows from Corollary 3.4.

3.2. Global well-posedness and exponential stability. Based on Proposition 3.3 we now show that there exists a unique global solution of the nonlinear initial boundary value problem (1.7) which depends continuously (in fact, even analytically) on the (sufficiently small) initial and boundary data. Moreover, we prove that the equilibrium \(u = 0\) is exponentially stable.

**Theorem 3.6** (Global well-posedness - the Dirichlet case). Let \(p > \max\{n/4 + 1/2, n/3\}\), \(p \neq 3/2\) and define \(\omega_0^p = \min\{a\lambda_D^0, b\lambda_D^0 / 2, c^2 / b\}\). Suppose

\begin{align}
\begin{cases}
  u_0 \in W^1_p(\Omega), & u_1 \in W^{4-2/p}_p(\Omega), & u_2 \in W^{2-2/p}_p(\Omega), \\
  g \in e^{-\omega_0 t} \mathbb{F}_g, & \mathbb{F}_g, \mathbb{F}_g, \mathbb{F}_g = W^{3-1/2p}_p(\mathbb{R}_+; L_p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{4-1/p}_p(\Gamma)), \\
  h \in e^{-\omega_0 t} \mathbb{F}_h, & \mathbb{F}_h, \mathbb{F}_h, \mathbb{F}_h = W^{2-1/2p}_p(\mathbb{R}_+; L_p(\Gamma)) \cap W^1_p(\mathbb{R}_+; W^{2-1/p}_p(\Gamma))
\end{cases}
\end{align}
(3.9)

with \(u_0|_\Gamma = g|_{t=0}, u_1|_\Gamma = g|_{t=0}, \Delta u_0|_\Gamma = h|_{t=0}\) and, if \(p > 3/2\), also \(\Delta u_1|_\Gamma = h|_{t=0}\). Then for every \(\omega \in (0, \omega_0)\) there exists some \(\rho > 0\) such that if

\[ \|g\|_{e^{-\omega t} \mathbb{F}_g, \mathbb{F}_g, \mathbb{F}_g} + \|h\|_{e^{-\omega t} \mathbb{F}_h, \mathbb{F}_h, \mathbb{F}_h} + \|u_0\|_{W^1_p} + \|u_1\|_{W^{4-2/p}_p} + \|u_2\|_{W^{2-2/p}_p} < \rho, \]

the nonlinear initial boundary value problem (1.7) admits a unique solution

\[ u \in e^{-\omega t} \mathbb{E}_u, \quad \mathbb{E}_u = W^3_p(\mathbb{R}_+; L_p(\Omega)) \cap W^1_p(\mathbb{R}_+; W^4_p(\Omega)) \]

which depends analytically on the data (3.9) with respect to the corresponding topologies. Moreover, conditions (3.9) are necessary for the regularity of the solution given in (3.10).

**Proof.** Employing the results on the linearized problem (3.1) from Section 3.1 we will now construct a solution of the nonlinear initial boundary value problem (1.7) which we linearize at \(u = 0\). Hence, the solution will be of the form \(u = u_* + u_\bullet\), where \(u_*\) solves the linearized problem (3.1) for the data \((f = 0, g, h, u_0, u_1, u_2)\) and \(u_\bullet\) satisfies homogeneous boundary and initial conditions. We will find the (small) deviation \(u_\bullet\) from \(u_*\) by application of the implicit function theorem to the map

\[ G : e^{-\omega t} \mathbb{E}_{u,h} \times e^{-\omega t} \mathbb{E}_u \rightarrow e^{-\omega t} L_p(\mathbb{R}_+ \times \Omega), \]

\[ (u_\bullet, u_*) \mapsto D(\partial_t, \Delta) u_* - (k((u_* + u_\bullet)_t)^2 - s |\nabla (u_* + u_\bullet)|^2)_{tt}. \]
where the differential expression $D(\partial_t, \Delta)$ is given by $D(\partial_t, \Delta) = (a \Delta - \partial_t)(\partial_t^2 - b \Delta \partial_t - c^2 \Delta)$ and $E_{u,h} = \{ u \in E_u : u(0) = u_h(0) = 0, u|_\Gamma = \Delta u|_\Gamma = 0 \}$. Explicitly, we have

$$D(\partial_t, \Delta)u_\ast = -u_{\ast,ttt} + (a + b)\Delta u_{\ast,t} + c^2 \Delta u_{\ast,t} - ab \Delta^2 u_{\ast,t} - ac^2 \Delta^2 u_{\ast}.$$  

**Step 1:** The implicit function theorem applies. First of all, we will now verify the assumptions of the implicit function theorem (Theorem 2.4.1).

**Step 1(a):** $G$ is analytic. The mixed derivative embedding (B.1a) implies that the linear maps

$$u_\ast \mapsto -u_{\ast,ttt} : \quad e^{-\omega}E_{u,h} \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega),$$

$$u_\ast \mapsto \Delta u_{\ast,t} : \quad e^{-\omega}E_{u,h} \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega),$$

$$u_\ast \mapsto \Delta u_{\ast,t} : \quad e^{-\omega}E_{u,h} \to e^{-\omega}(L_p(\mathbb{R}_+; W^2_p(\Omega)) \cap W^1_p(\mathbb{R}_+; L_p(\Omega))),$$

$$u_\ast \mapsto \Delta^2 u_{\ast,t} : \quad e^{-\omega}E_{u,h} \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega),$$

$$u_\ast \mapsto \Delta^2 u_{\ast} : \quad e^{-\omega}E_{u,h} \to e^{-\omega}W^1_p(\mathbb{R}_+; L_p(\Omega)),$$

are bounded and therefore analytic. Hence $u_\ast \mapsto D(\partial_t, \Delta)u_\ast : e^{-\omega}E_{u,h} \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$ is analytic. Next, note that for $p > 1/2 + n/4$ the embedding

$$E_u \hookrightarrow W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \cap W_p^2(\mathbb{R}_+ \times \Omega)$$

$$\hookrightarrow H_p^{1/2 + p + \varepsilon}(\mathbb{R}_+; H_p^{4 - 2/p - 2\varepsilon}(\Omega)) \hookrightarrow BUC(\mathbb{R}_+; BUC(\Omega))$$

holds. In particular, it holds if $\varepsilon > 0$ is sufficiently small and $4 - 2/p - 2\varepsilon - n/p > 0$. Such an $\varepsilon > 0$ exists if $p > (n + 2)/4$. Moreover, on the strength of the mixed derivative embedding theorem and the Sobolev embedding theorem we conclude similar as in the proof of Lemma 6 in [MW13] that

$$E_u \hookrightarrow W_p^2(\mathbb{R}_+; W_p^2(\Omega)) \cap W_p^3(\mathbb{R}_+; L_p(\Omega))$$

$$\hookrightarrow H_{p}^{1/2 + \Theta - \varepsilon/2}(\mathbb{R}_+; H_{p}^{2 + 2\Theta + \varepsilon}(\Omega))$$

$$\quad \text{for } \Theta \in [0, 1] \text{ and } \varepsilon > 0,  $$

$$\hookrightarrow W_p^{2 + \Theta - \varepsilon}(\mathbb{R}_+; W_p^{2 - 2\Theta}(\Omega))$$

$$\quad \text{for } \varepsilon > 0,  $$

$$\hookrightarrow W_p^{2 + \Theta - \varepsilon}(\mathbb{R}_+; W_p^{2 - 2\Theta}(\Omega))$$

$$\quad \text{for } \Theta \geq 1/p + \varepsilon,  $$

$$\hookrightarrow W_p^{2 + \Theta - \varepsilon}(\mathbb{R}_+; L_p(\Omega))$$

$$\quad \text{for } \Theta \leq 1 - n/4p,  $$

provided $\varepsilon > 0$ is sufficiently small and $p > 1/2 + n/4$. Furthermore, we observe that $e^{-2\omega}L_p(\mathbb{R}_+ \times \Omega) \hookrightarrow e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$ since $e^{\omega t} \leq e^{2\omega t}$ for $\omega \geq 0$. Prepared like that, we estimate

$$||f_t g_t||_{L_p} \leq ||f_t||_{L_{2p}} ||g_t||_{L_{2p}} \lesssim ||f||_{E_u} ||g||_{E_u},$$

$$||f_t g_t||_{L_p} \leq ||f_{tt}||_{L_p} ||g_t||_{L_{\infty}} + ||f_t||_{L_{\infty}} ||g_{tt}||_{L_p} \lesssim ||f||_{E_u} ||g||_{E_u},$$

$$||f_t g_t u_t||_{L_p} \leq ||f_{ttt}||_{L_p} ||g_t||_{L_{\infty}} + 2 ||f_t||_{L_{\infty}} ||g_{ttt}||_{L_{2p}} + ||f_t||_{L_{\infty}} ||g_{ttt}||_{L_p} \lesssim ||f||_{E_u} ||g||_{E_u},$$

and conclude that $(f, g) \mapsto f_t g_t : E_u \times E_u \to W_p^2(\mathbb{R}_+; L_p(\Omega))$ is bilinear and bounded, thus analytic. Setting $w = u_\ast + u_\ast$ in

$$e^{2\omega t}((w_t)^2)_{tt} = \frac{1}{2}((e^{\omega t} w_t^2)^2)_{tt} - 3\omega((e^{\omega t} w_t^2)^2)_{tt} + 6\omega^2 (e^{\omega t} w_t^2)^2$$

and choosing $f = e^{\omega t} u_\ast$ and $g = e^{\omega t} u_\ast$ proves that

$$(u_\ast, u_\ast) \mapsto (((u_\ast + u_\ast)_{tt})^2_{tt} : e^{-\omega}E_{u,h} \times e^{-\omega}E_u \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$$
is analytic. It remains to show analyticity of the map
\[(u_*, u_+) \mapsto (|\nabla (u_+ + u_*)|^2)_{tt}: e^{-\omega}E_{u,h} \times e^{-\omega}E_u \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega).\]

Note that we have the embeddings
\[
E_u \hookrightarrow W^1_p(\mathbb{R}_+; W^2_p(\Omega)) \hookrightarrow BUC(\mathbb{R}_+; BUC^1(\Omega)), \quad p > n/3
\]
\[
E_u \hookrightarrow W^1_p(\mathbb{R}_+; W^2_p(\Omega)) \cap W^2_p(\mathbb{R}_+ \times \Omega) \hookrightarrow H^{n/2 + \varepsilon}(\mathbb{R}_+; H^{n/2 - \varepsilon}_p(\Omega))
\]
\[
\hookrightarrow W^1_p(\mathbb{R}_+; H^{n/2 - \varepsilon}_p(\Omega)) \hookrightarrow W^1_p(\mathbb{R}_+ \times \Omega), \quad p > n/6 + 1/3
\]

Therewith, we obtain the estimates
\[
\|\nabla f \cdot \nabla g\|_{L^p} \leq \|\nabla f\|_{L^2_p}\|g\|_{L^2_p} \lesssim \|f\|_{E_u}\|g\|_{E_u},
\]
\[
\|(\nabla f \cdot \nabla g)\|_{L^p} \leq \|(\nabla f)\|_{L^2_p}\|\nabla g\|_{L^2_p} + \|\nabla f\|_{L^2_p}\|(\nabla g)\|_{L^2_p} \lesssim \|f\|_{E_u}\|g\|_{E_u},
\]
\[
\|(\nabla f \cdot \nabla g)_{tt}\|_{L^p} \lesssim \|\nabla f\|_{L^\infty}\|\nabla g\|_{L^2_p} + 2\|(\nabla f)\|_{L^2_p}\|(\nabla g)\|_{L^2_p} + \|\nabla f\|_{L^\infty}\|(\nabla g)\|_{L^2_p} \lesssim \|f\|_{E_u}\|g\|_{E_u},
\]
and conclude that \((f, g) \mapsto \nabla f \cdot \nabla g: E_u \times E_u \to W^2_p(\mathbb{R}_+; L^p(\Omega))\) is bilinear and bounded, thus analytic. Moreover, we have
\[
e^{2\omega t}(\nabla w^2)_{tt} = ((e^{\omega t} \nabla w)^2)_{tt} - 4\omega ((e^{\omega t} \nabla w)^2)_t + 2\omega^2 (e^{\omega t} \nabla w)^2).
\]

By setting \(w = u_+ + u_+\), \(f = e^{\omega t} u_+\) and \(g = e^{\omega t} u_+\) we are done. Altogether, we have that \(G: e^{-\omega}E_{u,h} \to e^{-\omega}E_{u,h} \to e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)\) is analytic.

**Step 1(b):** \(D_{u_*} G(0,0): e^{-\omega}L_p(\mathbb{R}_+ \times \Omega) \to e^{-\omega}E_{u,h}\) is an isomorphism. The Fréchet derivative of \(G\) with respect to \(u_*\) at \((0, 0)\) is given by
\[
D_{u_*} G(0,0)|\overline{\tau} = (a\Delta - \partial_t)(\overline{\tau}_{tt} - c^2\Delta \overline{\tau} - b\Delta \overline{\tau}_t).
\]

The map \(D_{u_*} G(0,0): e^{-\omega}L_p(\mathbb{R}_+ \times \Omega) \to e^{-\omega}E_{u,h}\) is an isomorphism since, according to Corollary \[\ref{cor:iso} \] for every \(\overline{f} \in e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)\) the equation \((a\Delta - \partial_t)(\overline{\tau}_{tt} - c^2\Delta \overline{\tau} - b\Delta \overline{\tau}_t) = \overline{f}\) admits a unique solution \(\overline{\tau} \in e^{-\omega}E_{u,h}\).

**Step 2: Construction of the solution.** On the strength of the Implicit Function Theorem there exists a ball \(B_p(0) \subset e^{-\omega}E_u\) with sufficiently small radius \(\rho > 0\) and an analytic map \(\varphi: B_p(0) \subset e^{-\omega}E_u \to e^{-\omega}E_{u,h}, u_* \mapsto u_* = \varphi(u_*)\) satisfying \(\varphi(0) = 0\) and \(G(\varphi(u_*), u_*) = 0\) for all \(u_* \in B_p(0)\). Hence, whenever \(u_*\) satisfies the boundary conditions \(u_*|_{\Gamma} = g, \Delta u_*|_{\Gamma} = h\) and initial conditions \(u_*|_{t=0} = u_0, u_*|_{t=0} = u_1, u_*|_{t=0} = u_2\) which is the case if we define \(u_* \in e^{-\omega}E_u\) to be the unique solution of \(\ref{system}\) with \((f = 0, u_0, u_1, u_2, g, h)\), then \(u_* + u_+ = \varphi(u_*) + u_*\) solves \(\ref{system}\).

**Step 3: Dependence of the solution on the data.** It remains to show that the solution \(u \in e^{-\omega}E_u\) depends analytically on \((g, h, u_0, u_1, u_2)\). To this end, we define the spaces
\[
\mathcal{D} := e^{-\omega}E_g \times e^{-\omega}E_h \times W^4_p(\Omega) \times W^{4/2}_p(\Omega) \times W^{2-2/p}_p(\Omega),
\]
\[
\mathcal{D} := \{(g, h, u_0, u_1, u_2) \in \mathcal{D}: u_0|_{\Gamma} = g|_{t=0}, u_1|_{\Gamma} = g_t|_{t=0}, u_2|_{\Gamma} = g_{tt}|_{t=0} \text{ if } p > 3/2, \Delta u_0|_{\Gamma} = h|_{t=0}, \Delta u_1|_{\Gamma} = h_t|_{t=0} \text{ if } p > 3/2\}.
\]

From Proposition \[\ref{prop:dependence}\] with \(f = 0\) we obtain that \(u_*\) depends linearly and continuously and thus analytically on \((g, h, u_0, u_1, u_2) \in \mathcal{D}\). Moreover, \(u_* \mapsto u_* = \varphi(u_*)\) is analytic on \(B_p(0)\) and therefore \(u_* \in e^{-\omega}E_{u,h}\) depends analytically on the data \((g, h, u_0, u_1, u_2) \in \mathcal{D}\). Altogether, \(u = u_* + u_+\) enjoys the same property which concludes the proof. \qed
An immediate consequence of Theorem 3.6 is that the global solution \( u \in e^{-\omega t}E_u \) of (1.7) decays to zero at an exponential rate.

**Theorem 3.7** (Exponential stability - the Dirichlet case). Under the same assumptions as in Theorem 3.6 the solution \( u \) decays exponentially fast to zero as \( t \to \infty \), in the sense that

\[
\|u(t)\|_{W_p^4} + \|u_t(t)\|_{W_p^{4-2/p}} + \|u_{tt}(t)\|_{W_p^{2-2/p}} \leq C e^{-\omega t}, \quad t \geq 0,
\]

for some \( C \geq 0 \) depending on the boundary and initial data \( g, h, u_0, u_1 \) and \( u_2 \).

**Proof.** We have \( u \in e^{-\omega W_p^4(\mathbb{R}^+; W_p^4(\Omega))} \to e^{-\omega} BUC(\mathbb{R}^+; W_p^4(\Omega)) \), hence

\[
u(t)\|_{W_p^4} \leq C_1 e^{-\omega t} \text{ with } C_1 = \|e^{-\omega}u\|_{BUC(\mathbb{R}^+; W_p^4)}.
\]

Furthermore, \( \nabla_j^2 u_t \in H(\mathbb{R}^+) \to BUC(\mathbb{R}^+; W_p^{2-2/p}(\Omega)) \) for \( j \in \{0, 1, 2\} \). Therefore, we obtain

\[
u_t(t)\|_{W_p^{4-2/p}} \leq C_2 e^{-\omega t} \text{ with } C_2 = \|e^{-\omega}u_t\|_{BUC(\mathbb{R}^+; W_p^{4-2/p})},
\]

Finally, from \( u_{tt} \in H(\mathbb{R}^+) \) we deduce that

\[
u_{tt}(t)\|_{W_p^{2-2/p}} \leq C_3 e^{-\omega t} \text{ with } C_3 = \|e^{-\omega}u_{tt}\|_{BUC(\mathbb{R}^+; W_p^{2-2/p})},
\]

and the claim follows. \( \Box \)

### 4. The Neumann boundary value problem

In this section we treat the inhomogeneous Neumann boundary value problem (1.8). We proceed analogously to the Dirichlet case, that is, we first consider the linearized equation and then construct a solution of the nonlinear problem (1.8) by means of the implicit function theorem.

Note that, in the Dirichlet case, the fact that the operator \(-A^D: D(A^D) \to X^D\) defined by (3.3) has a strictly negative spectral bound (Lemma 3.2) was crucial in order to show that the linearized equation (3.1) admits maximal regularity on \( \mathbb{R}^+ \), see the proof of Theorem 3.3. In the Neumann case, due to the zero eigenvalue of \(-\Delta_N: D(\Delta_N) \to L_p(\Omega)\) with \( D(\Delta_N) = \{u \in W_p^2(\Omega): \partial_\nu u = 0 \text{ on } \Gamma\} \), we cannot expect to obtain maximal regularity on \( \mathbb{R}^+ \).

For this reason we consider \(-\Delta_{N,0}: D(\Delta_{N,0}) \to L_p(0,\Omega)\), where \( D(\Delta_{N,0}) = D(\Delta_N) \cap L_p(0,\Omega) \).

The spectrum of \(-\Delta_{N,0}\) is contained in \((0, \infty)\), therefore we can prove maximal regularity of the homogeneous linear Neumann boundary problem on \( \mathbb{R}^+ \) analogously to the Dirichlet case. However, if we restrict ourselves to finite time intervals \( J = (0, T) \), then we do not necessarily need to use the realization \(-\Delta_{N,0}\) in \( L_p,0(\Omega) \). In case of finite time intervals we use \(-\Delta_N\). As a consequence, we will prove global well-posedness of (1.8) only if the data \( u_0, u_1, u_2 \) and \( g, h \) have zero mean whereas local well-posedness holds also for data with non-zero mean.

#### 4.1. Maximal \( L_p \)-regularity for the linearized equation

As in Section 3.1 let \( J = (0, T) \) or \( J = \mathbb{R}^+ \) and assume \( p \in (1, \infty) \). Here, for \( f \in L_p(J \times \Omega) \) we consider

\[
\begin{aligned}
(\alpha \Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 \Delta u) &= f & \text{in } J \times \Omega, \\
(\partial_t u, \partial_t \Delta u) &= (g, h) & \text{on } J \times \Gamma, \\
(u, u_t, u_{tt}) &= (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega,
\end{aligned}
\]

(4.1)
where \( u_0, u_1, u_2 : \Omega \to \mathbb{R} \) and \( g, h : J \times \Gamma \to \mathbb{R} \) are the given initial and boundary data, respectively. Analogously to the Dirichlet case we first represent (4.1) with \( g = h = 0 \) as an abstract evolution equation of the form
\[
\partial_t v^N + A^N v^N = F, \quad v^N(0) = v_0^N
\]
by setting
\[
v^N = \begin{pmatrix} u \\ u_t \\ u_{tt} - c^2 \Delta_N u - b \Delta_N u_t \end{pmatrix}, \quad v_0^N = \begin{pmatrix} u_0 \\ u_1 \\ u_2 - c^2 \Delta_N u_0 - b \Delta_N u_1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix},
\]
introducing the Banach space
\[
X^N = \mathcal{D}((\Delta_N)^2) \times \mathcal{D}(\Delta_N) \times L^p(\Omega)
\]
and defining the coefficient operator \( A^N : \mathcal{D}(A^N) \to X^N \) via
\[
A^N = \begin{pmatrix} 0 & -I & 0 \\ -c^2 \Delta_N & -b \Delta_N & -I \\ 0 & 0 & -a \Delta_N \end{pmatrix}, \quad \mathcal{D}(A^N) = \mathcal{D}((\Delta_N)^2) \times \mathcal{D}((\Delta_N)^2) \times \mathcal{D}(\Delta_N).
\]

On one hand, in the following we will show maximal regularity of \( A^N \) on finite time intervals. On the other hand, as already pointed out, we are going to use the realization \( -\Delta_N \alpha \) of the homogeneous Neumann Laplacian. For this reason we introduce the Banach space
\[
X^{N,0} = \mathcal{D}((\Delta_N,0)^2) \times \mathcal{D}(\Delta_N,0) \times L^p_{\text{loc}}(\Omega),
\]
and the densely defined operator \( A^{N,0} : \mathcal{D}(A^{N,0}) \to X^{N,0} \), where \( \Delta_N \) has to be replaced by \( \Delta_{N,0} \) in (4.3).

**Proposition 4.1.** Let \( p \in (1, \infty) \). There exists some \( \nu > 0 \) such that the operators \( \nu + A^N \) and \( \nu + A^{N,0} \) admit maximal regularity on \( \mathbb{R}_+ \).

**Proof.** The result can be proved similarly to Proposition 3.1. For some \( \alpha > 0 \) consider
\[
A_1^N = \begin{pmatrix} \alpha I & -I & 0 \\ 0 & \alpha I - b \Delta_N & -I \\ 0 & 0 & \alpha I - a \Delta_N \end{pmatrix}, \quad \text{and} \quad A_2^N = \begin{pmatrix} -\alpha I & 0 & 0 \\ -c^2 \Delta_N & -\alpha I & 0 \\ 0 & 0 & -\alpha I \end{pmatrix},
\]
Clearly, the operator \( A_2^N : X^N \to X^N \) is bounded. Moreover, \( A_1^N : \mathcal{D}(A_1^N) \to X^N \) has maximal \( L^p \)-regularity on \( \mathbb{R}_+ \) which is seen as in the proof of Proposition 3.1 by considering \( v_1 + A_1^N v = F, \quad v_0 = 0 \) for \( v = (v_1, v_2, v_3)^\top \) and \( F = (f_1, f_2, f_3)^\top \). Explicitly, we have
\[
\partial_t v_1 + \alpha v_1 - v_2 = f_1, \quad v_1(0) = 0,
\]
\[
\partial_t v_2 + \alpha v_2 - b \Delta_N v_2 - v_3 = f_2, \quad v_2(0) = 0,
\]
\[
\partial_t v_3 + \alpha v_3 - a \Delta_N v_3 = f_3, \quad v_3(0) = 0.
\]
Let \( F \in L^p(\mathbb{R}_+; X^N) \). Now one solves stepwise the equations above, starting with the last one, to get a unique solution \( v_3 \in W^1_p(\mathbb{R}_+; L^p(\Omega)) \cap L^p(\mathbb{R}_+; \mathcal{D}(\Delta_N)) \). Then \( f_2 + v_3 \in L^p(\mathbb{R}_+; \mathcal{D}(\Delta_N)) \). Here, we need to employ Lemma 3.1 in order to obtain a unique solution \( v_2 \in W^1_p(\mathbb{R}_+; \mathcal{D}(\Delta_N)) \cap L^p(\mathbb{R}_+; \mathcal{D}((\Delta_N)^2)) \). As in the Dirichlet case, the first equation gives us a unique solution \( v_1 \in W^1_p(\mathbb{R}_+; \mathcal{D}(\Delta_N)^2) \). Altogether, since the condition \( F \in L^p(\mathbb{R}_+; X^N) \) implies existence of a unique solution \( v \in W^1_p(\mathbb{R}_+; X^N) \cap L^p(\mathbb{R}_+; \mathcal{D}(A^N)) \) we conclude that \( A_1^N : \mathcal{D}(A^N) \to X^N \) admits maximal \( L^p \)-regularity on \( \mathbb{R}_+ \). Finally, as in the
proof of Proposition 3.1, a perturbation argument implies that there exists some $\nu > 0$ such that $\nu + A_1^N + A_2^N = \nu + A^N$ has the property of maximal $L_p$-regularity on $\mathbb{R}_+$.

Maximal $L_p$-regularity of $\nu + A^{N,0}$ follows analogously by considering the operators $A_1^{N,0} = A_1^N$ and $A_2^{N,0} = A_2^N$, which are equal to $A_1^N$ and $A_2^N$ upon replacement of $\Delta_N$ by $\Delta_{N,0}$ and proceeding as above. 

Since $\nu + A^N$ has maximal $L_p$-regularity on $\mathbb{R}_+$, multiplication of (4.2) with $e^{-\nu t}$ shows that $A^N$ has maximal $L_p$-regularity on bounded intervals $J = (0, T)$.

**Theorem 4.2.** Suppose $J = (0, T)$ is a finite time interval and let $p \in (1, \infty)$. Then $A^N : D(A^N) \to X^N$ has maximal $L_p$-regularity on $J$ and therefore

$$
(\partial_t + A^N, \gamma_t) : W^1_p(J; X^N) \cap L_p(J; D(A^N)) \to L_p(J; X^N) \times (X^N, D(A^N))_{1-1/p,p}
$$

is an isomorphism.

Next, observe that $A^{N,0}$ has a strictly negative spectral bound. This can be shown likewise to Lemma 3.2 since the spectrum of the negative Neumann Laplacian in $L_p(\Omega)$ is contained in $(0, \infty)$ and consists only of eigenvalues of finite multiplicity.

**Lemma 4.3.** The spectral bound of $-A^{N,0}$ is given by $s(-A^{N,0}) = -\omega_0^N$, where $\omega_0^N = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$. Here, $\lambda_1^N$ denotes the smallest non-zero eigenvalue of $-\Delta_{N,0}$. In particular, if $\text{Re}(\lambda) < \omega_0^N$, then $\lambda \in \rho(A^{N,0})$.

By means of Lemma 4.3 one shows that $A^{N,0}$ has maximal $L_p$-regularity on $\mathbb{R}_+$. For details we refer to the proof of Theorem 3.3.

**Theorem 4.4.** Let $p \in (1, \infty)$ and $\omega \in [0, \omega_0^N)$. Then $A^{N,0} : D(A^{N,0}) \to X^{N,0}$ has maximal $L_p$-regularity on $\mathbb{R}_+$ and therefore

$$
\begin{aligned}
(\partial_t + A^{N,0}, \gamma_t) : & e^{-\omega t}(W^1_p(J; X^{N,0}) \cap L_p(J; D(A^{N,0}))) \\
& \to e^{-\omega t}L_p(J; X^{N,0}) \times (X^{N,0}, D(A^{N,0}))_{1-1/p,p}
\end{aligned}
$$

is an isomorphism.

Theorems 4.2 and 4.4 immediately yield optimal regularity for (1.1) with homogeneous boundary conditions, i.e. $g = h = 0$.

**Corollary 4.5.** Let $p \in (1, \infty) \setminus \{3\}$ and consider the homogeneous Neumann boundary value problem

$$
\begin{cases}
(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = f & \text{in } J \times \Omega, \\
(\partial_\nu u, \partial_\nu \Delta u) = (0, 0) & \text{on } J \times \Gamma, \\
(u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega.
\end{cases}
$$

(i) If $J = (0, T)$ is finite, then (4.4) admits optimal regularity in the sense that there exists a unique solution

$$
u \in E_u(J), \quad E_u(J) = W^3_p(J; L_p(\Omega)) \cap W^4_p(J; W^4_p(\Omega))
$$

if and only if $f \in L_p(J \times \Omega)$, $u_0 \in W^4_p(\Omega)$, $u_1 \in W^{4-2/p}_p(\Omega)$, $u_2 \in W^{2-2/p}_p(\Omega)$ and the initial and boundary data are compatible, that is, we have $\partial_\nu u_0|_\Gamma = \partial_\nu u_1|_\Gamma = 0$ and, if $p > 3$, also $\partial_\nu \Delta u_1|_\Gamma = \partial_\nu u_2|_\Gamma = 0$ in the sense of traces.
Proof. Assertion (i) follows immediately from Theorem 4.2. Analogously to the proof of Corollary 3.3 one verifies that \( u_N \in W^2_p(\mathbb{R}_+ \times \Omega) \cap W^1_p(\mathbb{R}_+ ; W^4_p(\Omega) \cap L_{p,0}(\Omega)) \)

if and only if \( f \in e^{-\omega}L_p(\mathbb{R}_+ ; L_{p,0}(\Omega)) \), \( u_0 \in W^4_p(\Omega) \cap L_{p,0}(\Omega) \), \( u_1 \in W^{4-2/p}_p(\Omega) \cap L_{p,0}(\Omega) \), \( u_2 \in W^{2-2/p}_p(\Omega) \cap L_{p,0}(\Omega) \) and the initial and boundary data are compatible. \( \square \)

Finally we arrive at our global optimal regularity result for (4.1). As in the Dirichlet case, sufficiency is shown by a combination of an optimal regularity result for the linearized Westervelt equation and a higher regularity result for the heat equation.

**Proposition 4.6.** Let \( p \in (1, \infty) \setminus \{3\} \) and define \( \omega_0 = \min\{a\lambda^N, b\lambda^N/2, c^2/b\} \). Then for every \( \omega \in (0, \omega_0) \) the linear initial boundary value problem

\[
\begin{cases}
  (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = f & \text{in } \mathbb{R}_+ \times \Omega, \\
  (\partial_t u, \partial_t^2 u) = (g, h) & \text{on } \mathbb{R}_+ \times \Gamma, \\
  (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega.
\end{cases}
\]

has a unique solution of the form \( u(t, x) = v(t, x) + w(t) \), where

\[
v \in e^{-\omega}E_{a,0}, \quad E_{a,0} = W^3_p(\mathbb{R}_+ ; L_{p,0}(\Omega)) \cap W^1_p(\mathbb{R}_+ ; W^4_p(\Omega) \cap L_{p,0}(\Omega)), \quad \partial_t^3 w \in e^{-\omega}L_p(\mathbb{R}_+)
\]

if and only if the data satisfy the conditions

(i) \( f \in e^{-\omega}L_p(\mathbb{R}_+ \times \Omega) \),
(ii) \( u_0 \in W^4_p(\Omega) \cap L_{p,0}(\Omega) \), \( u_1 \in W^{4-2/p}_p(\Omega) \cap L_{p,0}(\Omega) \), \( u_2 \in W^{2-2/p}_p(\Omega) \cap L_{p,0}(\Omega) \),
(iii) \( g \in e^{-\omega}F_{g,\nu} \), \( F_{g,\nu} = W^{5/2-1/2p}_p(\mathbb{R}_+ ; L_p(\Gamma)) \cap W^1_p(\mathbb{R}_+ ; W^{3-1/p}_p(\Gamma)) \),
(\( h \in e^{-\omega}F_{h,\nu} \), \( F_{h,\nu} = W^{3/2-1/2p}_p(\mathbb{R}_+ ; L_p(\Gamma)) \cap W^1_p(\mathbb{R}_+ ; W^{1-1/p}_p(\Gamma)) \)),
(iv) \( \partial_\nu u_0|_\Gamma = g|_{t=0} \), \( \partial_\nu \Delta u_0|_\Gamma = h|_{t=0} \), \( \partial_\nu u_1|_\Gamma = g_t|_{t=0} \) and, if \( p > 3 \), also \( \partial_\nu \Delta u_1|_\Gamma = h_t|_{t=0} \), \( \partial_\nu u_2|_\Gamma = g_{tt}|_{t=0} \) in the sense of traces.

Moreover, the solution fulfills the estimate

\[
\|u\|_{e^{-\omega}E_u} \lesssim \|f\|_{e^{-\omega}L_p} + \|g\|_{e^{-\omega}F_{g,\nu}} + \|h\|_{e^{-\omega}F_{h,\nu}} + \|u_0\|_{W^4_p} + \|u_1\|_{W^{4-2/p}_p} + \|u_2\|_{W^{2-2/p}_p}.
\]

**Proof.** It is not surprising that the proof of necessity can be done similarly to Proposition 3.5. Assume that \( u(t, x) = v(t, x) + w(t, x) \) is a solution of (4.5) with \( v \in e^{-\omega}E_u \) and \( \partial_t^3 w \in e^{-\omega}L_p(\mathbb{R}_+) \). Since, apart from having zero mean, \( v \) has the same regularity as \( u \) in Proposition 3.5, we can conclude that \( e^{\omega t}f = -e^{\omega t}(v + w)_{ttt} - ab\Delta^2(e^{\omega t}v_t) - ac^2\Delta(e^{\omega t}v) + (a + b)\Delta(e^{\omega t}vt_t) + c^2\Delta^2(e^{\omega t}v_t) \in L_p(\mathbb{R}_+ \times \Omega) \) and (i) is readily checked. Moreover, \( w, v_t \) and \( v_{tt} \) are just time-dependent and thus constant at \( t = 0 \), hence the regularity of the initial values (ii) can be shown as in the Dirichlet case. Moreover, the regularity of the boundary data (iii) is obtained from the spatial trace (2.2) with the same choices of \( k \) and \( l \) as in the proof of
Proposition 4.3 and setting \( j_B = 1 \). Concerning (iv) it is straightforward to show

\[
\begin{align*}
\partial_t u_0 |_{\Gamma} &= g |_{t=0} \text{ in } W^{-3-1/p}(\Gamma), \\
\partial_t \Delta u_0 |_{\Gamma} &= h |_{t=0} \text{ in } W^{-1-1/p}(\Gamma), \\
\partial_t u_1 |_{\Gamma} &= g |_{t=0} \text{ in } W^{-3-3/p}(\Gamma), \\
\partial_t \Delta u_1 |_{\Gamma} &= h |_{t=0} \text{ in } W^{-1-3/p}(\Gamma) \text{ if } p > 3, \\
\partial_t u_2 |_{\Gamma} &= g |_{t=0} \text{ in } W^{-1-3/p}(\Gamma) \text{ if } p > 3.
\end{align*}
\]

Hence the compatibility condition (C.7) becomes

\[
(4.9)
\]

In order to apply Corollary C.4, we first note that the right-hand side \( a \partial_t \varphi - \varphi \) is satisfied. Therefore Corollary C.4 yields a unique solution

Next, we show that conditions (i)–(iv) are sufficient for the existence of a unique solution \( u(t, x) = v(t, x) + w(t) \) of (4.5) such that \( v \in e^{-\omega}E_u \) and \( w_{tt} \in e^{-\omega}L_p(R_+) \). As in the Dirichlet case, we interchange the order of differentiation on the left-hand side and consider the subproblems

\[
(4.6)
\]

From condition (ii) we obtain \( a \Delta u_0 - u_1 \in W^2_p(\Omega) \) and \( a \Delta u_1 - u_2 \in W^{2-2/p}(\Omega) \). Furthermore, (iii) implies \( ah - g_t \in e^{-\omega}W_p \). On the strength of Lemma 2.7 we obtain that (4.9) admits a unique solution of the form \( \varphi(t, x) = \varphi_1(t, x) + \varphi_2(t) \) with \( \varphi_1 \in e^{-\omega}W_{p,0} \) and \( \partial_t^2 \varphi_2 \in e^{-\omega}L_p(R_+) \). We now make the ansatz \( u(t, x) = v(t, x) + w(t) \) such that \( v(\cdot, t) = 0 \). Applying \( |\Omega|^{-1} \int_{\Omega} a \Delta u - u_t = \varphi \) we deduce that \( w \) solves the ordinary differential equation \( w_t = -\partial_t^2 \varphi + a |\Omega|^{-1} \tilde{g} \) with \( w(0) = \tilde{u}_0 \). Hence \( \partial_t^3 w \in e^{-\omega}L_p(R_+) \). Moreover, \( v \) is a solution of

\[
(4.8)
\]

In order to apply Corollary C.4, we first note that the right-hand side \( \varphi_1 + a |\Omega|^{-1} \tilde{g} \) belongs to \( e^{-\omega}W_u \) since \( \tilde{g} \) only depends on time and belongs to \( e^{-\omega}W^2_p(R_+) \). The rescaled function \( v_a(t, x) = av(t/a, x) \) should solve the system

\[
(4.9)
\]

Hence the compatibility condition (C.7) becomes

\[
- \int_{\Omega} (\varphi_1 + a |\Omega|^{-1} \tilde{g}) \, dx + \int_{\Gamma} ag \, dS = 0
\]

and is clearly satisfied. Therefore Corollary C.4 yields a unique solution \( v_a \in e^{-\omega}E_u,0 \) of problem (4.9) and thus \( u = v + w \) solves problem (4.5).

Finally, uniqueness follows by considering two solutions of (4.5), the difference \( \varphi \) of which solves (4.4) with \( f = 0 \) and \( u_0 = u_1 = u_2 = 0 \), hence \( \varphi = 0 \) and the proof is complete. \( \square \)

Proposition 4.7. Let \( p \in (1, \infty) \setminus \{3\} \) and let \( J = (0, T) \) be a finite interval. Then the linear initial boundary value problem (4.1) has a unique solution

\[
u \in W^{3}_{p}(J; L_{p}(\Omega)) \cap W^{1}_{p}(J; W^{3}_{p}(\Omega))
\]

if and only if conditions (i)–(iv) from Proposition 4.6 hold with \( R_+ \) replaced by \( J \).
Proof. Uniqueness and necessity are shown likewise to Proposition 4.6. For the proof of sufficiency one considers (4.16) and (4.17) on (0, T) instead of $\mathbb{R}_+$. Remark 2.8 and Corollary 3.4 then imply existence of a unique solution $u \in W^3_p(J; L^p_0(\Omega)) \cap W^1_p(J; W^4_p(\Omega))$. □

4.2. Local well-posedness, global well-posedness and exponential stability. We now arrive at our well-posedness results for the Neumann problem (1.8). If we allow the (sufficiently small) initial and boundary data to have non-zero mean, we are only able to prove well-posedness of (1.8) on finite time intervals $J = (0, T)$. On the other hand, if the (sufficiently small) data have zero mean, that is, $\bar{u}_0 = \bar{u}_1 = \bar{u}_2 = 0$ and $\bar{g} = \bar{h} = 0$, then we obtain a globally well-posed solution which decays exponentially fast to zero.

Theorem 4.8 (Local well-posedness). Let $J = (0, T)$ for some $T < \infty$ and $p > \max\{n/4 + 1/2, n/3\}$, $p \neq 3$. Suppose

$$
\begin{align*}
&u_0 \in W^4_p(\Omega), \quad u_1 \in W^{4-2/p}_p(\Omega), \quad u_2 \in W^{2-2/p}_p(\Omega), \\
&g \in e^{-\omega F_{g,\nu}(J)} F_{g,\nu}(J) = W^{5/2-1/2p}_p(J, L^p(\Gamma)) \cap W^1_p(J, W^{3-1/p}_p(\Gamma)), \\
&h \in e^{-\omega F_{h,\nu}(J)} F_{h,\nu}(J) = W^{3/2-1/2p}_p(J, L^p(\Gamma)) \cap W^1_p(J, W^{1-1/p}_p(\Gamma)),
\end{align*}
$$

such that $\partial_\nu u_0|_\Gamma = g|_{t=0}$, $\partial_\nu \Delta u_0|_\Gamma = h|_{t=0}$, $\partial_\nu u_1|_\Gamma = g|_{t=0}$ and, if $p > 3$, also $\partial_\nu \Delta u_1|_\Gamma = h|_{t=0}$, $\partial_\nu u_2|_\Gamma = g|_{t=0}$ in the sense of traces.

There exists some $\rho > 0$ such that if

$$
\|g\|_{F_{g,\nu}} + \|h\|_{F_{h,\nu}} + \|u_0\|_{W^4_p} + \|u_1\|_{W^{4-2/p}_p} + \|u_2\|_{W^{2-2/p}_p} < \rho,
$$

the nonlinear Neumann boundary value problem (1.8) admits a unique solution

$$
\begin{align*}
u \in \mathbb{E}_u(J) = W^3_p(J; L^p_0(\Omega)) \cap W^1_p(J; W^4_p(\Omega))
\end{align*}
$$

which depends analytically (in particular continuously) on the data (4.10) with respect to the corresponding topologies. Moreover, conditions (4.10) are necessary for the regularity of the solution given in (4.11).

Theorem 4.9 (Global well-posedness - the Neumann case). Let $p > \max\{n/4 + 1/2, n/3\}$, $p \neq 3$ and define $\omega_0 = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$. Suppose

$$
\begin{align*}
&u_0 \in W^4_p(\Omega) \cap L^p_{p,0}(\Omega), \quad u_1 \in W^{4-2/p}_p(\Omega) \cap L^p_{p,0}(\Omega), \quad u_2 \in W^{2-2/p}_p(\Omega) \cap L^p_{p,0}(\Omega), \\
&g \in e^{-\omega F_{g,\nu,0}} F_{g,\nu,0} = W^{5/2-1/2p}_p(\mathbb{R}_+, L^p_{p,0}(\Gamma)) \cap W^1_p(\mathbb{R}_+, W^{3-1/p}_p(\Gamma) \cap L^p_{p,0}(\Gamma)), \\
&h \in e^{-\omega F_{h,\nu,0}} F_{h,\nu,0} = W^{3/2-1/2p}_p(\mathbb{R}_+, L^p_{p,0}(\Gamma)) \cap W^1_p(\mathbb{R}_+, W^{1-1/p}_p(\Gamma) \cap L^p_{p,0}(\Gamma)),
\end{align*}
$$

such that $\partial_\nu u_0|_\Gamma = g|_{t=0}$, $\partial_\nu \Delta u_0|_\Gamma = h|_{t=0}$, $\partial_\nu u_1|_\Gamma = g|_{t=0}$ and, if $p > 3$, also $\partial_\nu \Delta u_1|_\Gamma = h|_{t=0}$, $\partial_\nu u_2|_\Gamma = g|_{t=0}$ in the sense of traces.

Then for every $\omega \in (0, \omega_0)$ there exists some $\rho > 0$ such that if

$$
\|g\|_{e^{-\omega F_{g,\nu,0}}} + \|h\|_{e^{-\omega F_{h,\nu,0}}} + \|u_0\|_{W^4_p} + \|u_1\|_{W^{4-2/p}_p} + \|u_2\|_{W^{2-2/p}_p} < \rho,
$$

then the nonlinear Neumann boundary value problem (1.8) admits a unique solution

$$
\begin{align*}
u \in e^{-\omega \mathbb{E}_u,0} \quad \mathbb{E}_u,0 = W^3_p(\mathbb{R}_+, L^p_{p,0}(\Omega)) \cap W^1_p(\mathbb{R}_+, W^4_p(\Omega) \cap L^p_{p,0}(\Omega)),
\end{align*}
$$

which depends analytically on the data (4.12) with respect to the corresponding topologies. Moreover, conditions (4.12) are necessary for the regularity of the solution given in (4.13).
Remark 4.10. In case of Theorem 4.8 we define $u_*$ to be the solution according to Proposition 4.7 which satisfies (4.1) for the data $(f = 0, g, h, u_0, u_1, u_2)$ and suppose $u_*$ satisfies homogeneous boundary and initial conditions. The solution is of the form $u = u_0 + u_*$ and $u_*$ is found by the implicit function theorem. The claim then follows likewise to the proof of Theorem 4.6.

However, if we want to prove global well-posedness, we need to use Proposition 4.6 for the linearized equation, where for given data $(f = 0, g, h, u_0, u_1, u_2)$ according to (ii)–(iv) the solution is of the form $u(t, x) = v(t, x) + w(t)$, where $v \in e^{-\omega t}E_\omega, 0$ has zero mean and $w$ is only time-dependent. If $w \neq 0$, the term $((u_\sigma)^2)_{t t}$ in the nonlinear right-hand side of (1.8) causes problems. Recall (3.12) and note that due to Proposition 4.6 we in fact have $u_* = v_* + w_*$ with $v_* \in e^{-\omega t}E_\omega, 0$ and $\partial_\nu w_* \in e^{-\omega t}L_p(R_+)$. Then $w_*$, $\partial w_*$ and $\partial_\nu^2 w_*$ are in general not contained in $e^{-\omega t}L_p(R_+)$ and thus (3.13) fails. However, if we assume that the data $(g, h, u_0, u_1, u_2)$ have zero mean then $w_* = 0$ and, since $u_* = v_*$ in this case, Theorem 4.9 follows analogously to the result on global well-posedness for the Dirichlet boundary value problem (Theorem 3.6).

Finally, provided the data have zero mean, we obtain the following result on exponential stability for the Neumann problem (1.8).

Theorem 4.11 (Exponential stability - the Neumann case). Under the same assumptions as in Theorem 4.9 the solution $u$ decays exponentially fast to zero as $t \to \infty$ in the sense that
\[
\|u(t)\|_{W_\sigma^p} + \|u(t)\|_{W_\sigma^{4-2/p}} + \|u_{t t}(t)\|_{W_\sigma^{2-2/p}} \leq Ce^{-\omega t}, \quad t \geq 0,
\]
for some $C \geq 0$ depending on the boundary and initial data $g, h, u_0, u_1$ and $u_2$.

Proof. Note that we have $u \in e^{-\omega t}E_\omega, 0 \hookrightarrow e^{-\omega t}E_\omega$, therefore the result follows likewise to Theorem 3.7. \qed

Appendix A. The Neumann Laplace operator

Let $p \in (1, \infty)$ and assume, as always, that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \partial \Omega$. The homogeneous Neumann-Laplacian is given by

$$-\Delta_N: D(\Delta_N) \to L_p(\Omega),$$

where $D(\Delta_N) = \{u \in W_\sigma^2(\Omega) : \partial_\nu u = 0 \text{ on } \Gamma\}$. It is well-known that $-\Delta_N$ has compact resolvent and that its spectrum $\sigma(-\Delta_N)$ is a discrete subset of $(0, \infty)$ consisting only of eigenvalues $(\lambda_1^N)_n \geq 0$ with finite multiplicity. In particular, $0 = \lambda_1^N \in \sigma(-\Delta_N)$ is an isolated eigenvalue of $-\Delta_N$. We seek for a realization of the Laplace operator with homogeneous Neumann boundary conditions such that the spectrum is contained in $[\lambda_1^N, \infty)$ where $\lambda_1^N > 0$ is the smallest non-zero eigenvalue of $-\Delta_N$.

In order to remove the zero eigenvalue we will use several results from Appendix A in [Lin95]. In what follows, $A: D(A) \subset X \to X$ denotes a linear closed linear operator whose domain $D(A)$ is dense in the real or complex Banach space $X \neq \{0\}$. We say that a subset $\sigma_1 \subset \sigma(A)$ is a spectral set if both, $\sigma_1$ and $\sigma(A) \setminus \sigma_1$ are closed in $\mathbb{C}$. Let $\sigma_1$ be a bounded spectral set and let $\sigma_2 = \sigma(A) \setminus \sigma_1$. Since $\text{dist}(\sigma_1, \sigma_2) > 0$, there exists a bounded open set $O$ such that $\sigma_1 \subset O$ and $\overline{O} \cap \sigma_2 = \emptyset$. We may assume that the boundary $\gamma$ of $O$ consists of a finite number of rectifiable closed Jordan curves, oriented counterclockwise and define a
linear bounded operator $P$ by
\begin{equation*}
P = \frac{1}{2\pi i} \int_{\gamma} R(\xi, A) \, d\xi.
\end{equation*}
The following result shows how to find a realization $A_2$ of $A$ such that $\sigma(A_2) = \sigma(A) \setminus \sigma_1$.

**Proposition A.1** ([Lun95 Proposition A.1.2]). Let $\sigma_1$ be a bounded spectral set. Then the operator $P$ is a projection and $P(X)$ is contained in $\mathcal{D}(A^n)$ for every $n \in \mathbb{N}$. Moreover, if we set $X_1 = P(X)$, $X_2 = (I - P)(X)$ and define the operators
\begin{align*}
A_1 & : X_1 \to X_1, u \mapsto Au \\
A_2 & : \mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2 \to X_2, u \mapsto Au
\end{align*}
then
\begin{equation*}
\sigma(A_1) = \sigma_1 \quad \text{and} \quad \sigma(A_2) = \sigma_2.
\end{equation*}

The crucial point is thus to determine the space $X_2$. In case $\sigma_1 = \{\lambda_0\}$ where $\lambda_0$ is an isolated point of $\sigma(A)$ and a pole of $R(\cdot, A)$ the following result helps to determine the spaces $X_1$ and $X_2$.

**Proposition A.2** ([Lun95 Proposition A.2.2 and Corollary A.2.4]). If $\lambda_0$ is an isolated point of $\sigma(A)$ and a pole of $R(\cdot, A)$, then the following are equivalent:
\begin{enumerate}[(i)]
\item $X_1 = N(\lambda_0 I - A)$
\item $X_2 = R(\lambda_0 I - A)$
\item $\lambda_0$ is a simple pole of $\lambda \mapsto R(\lambda, A)$
\item $R(\lambda_0 I - A)$ is closed and $X = N(\lambda_0 I - A) \oplus R(\lambda_0 - A)$
\item $N(\lambda_0 I - A) = N(\lambda_0 I - A)^2$
\end{enumerate}

We now apply the foregoing results to the strong Neumann-Laplacian $-\Delta_N$ and set $\lambda_0 = \lambda_1^N = 0$, hence $\sigma_1 = \{0\}$ and $\sigma_2 = \sigma(-\Delta_N) \setminus \{0\}$. Then we clearly have $\sigma_2 \subset [\lambda_1^N, \infty)$, where $\lambda_1^N$ is the smallest non-zero eigenvalue of $-\Delta_N$.

**Lemma A.3.** The spectrum $\sigma(-\Delta_N)$ of $-\Delta_N$ consists only of poles of $\lambda \mapsto R(\lambda, -\Delta_N)$.

**Proof.** Since $-\Delta_N$ is closed, densely defined and has compact resolvent, the result is an immediate consequence of Corollary IV.1.19 in [EN00]. \qed

We introduce the space $\mathbb{K}_u = \{u \in L_p(\Omega): u \text{ is constant}\}$ and start with the following observation.

**Lemma A.4.** We have $N(\Delta_N) = \mathbb{K}_u$.

**Proof.** Let $u \in N(\Delta_N)$, i.e. $u \in W_p^2(\Omega)$, $\partial_\nu u = 0$ on $\Gamma$ and $-\Delta u = 0$. For sufficiently large $\mu > 0$ the map $\mu - \Delta_N : \mathcal{D}(\Delta_N^{j+1}) \to \mathcal{D}(\Delta_N^j)$ is an isomorphism for every $j \in \mathbb{N}_0$. We write $0 = \Delta_N u = \mu u - (\mu - \Delta_N)u$ and obtain $u = (\mu - \Delta_N)^{-1}\mu u$. Hence, if $u \in \mathcal{D}(\Delta_N)$ then we have $u \in \mathcal{D}(\Delta_N)$ and altogether conclude $u \in \mathcal{D}(\Delta_N)$. The Sobolev embedding $W_p^k(\Omega) \hookrightarrow W_2^2(\Omega)$ holds for sufficiently large $k$. Therefore we in fact have $u \in W_2^2(\Omega)$ and calculate
\begin{equation*}
0 = -\int_{\Omega} \Delta u \overline{u} \, dx = \int_{\Omega} \nabla u \cdot \nabla \overline{u} \, dx - \int_{\Gamma} \partial_\nu u \overline{u} \, dS = \|\nabla u\|_{L_2}^2,
\end{equation*}

hence $u \in \mathbb{K}_u$. Conversely, every function $u \in \mathbb{K}_u$ trivially satisfies $-\Delta u = 0$. \qed

**Lemma A.5.** We have $N(\Delta_N) = N(\Delta_N^2)$.
Proof. Let \( u \in N((\Delta_N)^2) \), i.e. \( u \in \mathcal{D}((\Delta_N)^2) \) and \((-\Delta)^2 u = 0\). Then
\[
0 = \int_{\Omega} (-\Delta)^2 u \, dx = -\int_{\Omega} \nabla(u) \cdot \nabla u \, dx + \int_{\Gamma} \partial_{\nu}(\Delta u) \, dS
\]
\[
= \int_{\Omega} \Delta u \Delta u \, dx = \|\Delta u\|^{2}_{L^2},
\]
hence \(-\Delta u = 0\) and thus \( u \in N(\Delta_N) \). Conversely, let \( u \in N(\Delta_N) \). Then \((-\Delta)^2 u = 0\) and \(-\Delta u \in \mathcal{D}(\Delta_N)\).

Before we proceed, let us recall the space \( L_{p,0}(\Omega) = \{ u \in L_{p}(\Omega): \int_{\Omega} u = 0 \} \).

Lemma A.6. We have \( L_{p}(\Omega) = K_{u} \oplus L_{p,0}(\Omega) \) as a topological direct sum.

Proof. We consider the map \( P: L_{p}(\Omega) \to L_{p,0}(\Omega) \), \( u \mapsto u - \langle u \rangle_{\Omega} \) where \( \langle u \rangle_{\Omega} = \|\Omega\|^{-1} \int_{\Omega} u \, dx \).

It is straightforward to verify that for \( u \in L_{p}(\Omega) \) we indeed have \( \int_{\Omega} u - \langle u \rangle_{\Omega} \, dx = 0 \) and hence \( Pu \in L_{p,0}(\Omega) \). Furthermore, \( P^{2}u = Pu - \langle Pu \rangle_{\Omega} = Pu \) implies that \( P \) is a projection. Finally \( N(P) = K_{u} \) since \( u - \langle u \rangle_{\Omega} = 0 \) if and only if \( u \in L_{p}(\Omega) \) is constant.

Therewith, by means of Proposition A.2 have determined the space \( X_{2} = L_{p,0}(\Omega) \) which gives us the following positive realization of the Neumann Laplacian.

Theorem A.7. The spectrum of the closed and densely defined operator
\[
-\Delta_{N,0}: \mathcal{D}(\Delta_{N,0}) \to L_{p,0}(\Omega),
\]
\[
u \mapsto -\Delta u,
\]
with \( \mathcal{D}(\Delta_{N,0}) = \mathcal{D}(\Delta_{N}) \cap L_{p,0}(\Omega) \) is a discrete subset of \( [\lambda_{1}^{N}, \infty) \), where \( \lambda_{1}^{N} > 0 \) is the smallest non-zero eigenvalue of \(-\Delta_{N}\). Moreover, \( \sigma(-\Delta_{N,0}) \) consists only of eigenvalues with finite algebraic multiplicity.

Appendix B. Traces and mixed derivatives

In this section we consider the temporal trace operator in some anisotropic fractional Sobolev spaces. Furthermore, we present some mixed derivative embeddings for such spaces which are needed for proving suitable mapping properties of differential operators.

A bounded linear operator \( r: X \to Y \) between Banach spaces \( X \) and \( Y \) is called a retraction, if there is a bounded linear map \( r^{c}: Y \to X \) such that \( rr^{c} = I_{Y} \). Thus \( r \) is surjective and \( r^{c} \) is a bounded right-inverse for \( r \). The map \( r^{c} \) is called a co-retraction for \( r \).

The following trace theorem can be derived from [DB84 Lemma 11], [Lun95 Section 2.2.1], [Ama95 Proposition III.4.10.3].

Theorem B.1. Let \( A \) be the generator of a bounded analytic semigroup \( (e^{-tA})_{t \geq 0} \) in a Banach space \( X \) such that \( A: \mathcal{D}(A) \to X \) has a bounded inverse, let \( p \in (1, \infty) \) and let \( \mathcal{D}_{A}(\alpha, p) := (X, \mathcal{D}(A))_{\alpha, p} \) for \( \alpha \in (0, 1) \) and \( \mathcal{D}_{A}(1, p) := \mathcal{D}(A) \).

Then, for every \( \alpha \in (1/p, 1] \), the trace operator
\[
\gamma_{t} = |_{t=0}: W_{p}^{\alpha}(\mathbb{R}_{+}; X) \cap L_{p}(\mathbb{R}_{+}; \mathcal{D}_{A}(\alpha, p)) \to \mathcal{D}_{A}(\alpha - 1/p, p)
\]
is a retraction, the operator
\[
R_{A}: u_{0} \mapsto (t \mapsto e^{-tA}u_{0}) \,, \quad \mathcal{D}_{A}(\alpha - 1/p, p) \to W_{p}^{\alpha}(\mathbb{R}_{+}; X) \cap L_{p}(\mathbb{R}_{+}; \mathcal{D}_{A}(\alpha, p))
\]
is a co-retraction for \( \gamma_{t} \) and the following embedding is continuous.
\[
W_{p}^{\alpha}(\mathbb{R}_{+}; X) \cap L_{p}(\mathbb{R}_{+}; \mathcal{D}_{A}(\alpha, p)) \hookrightarrow BUC(\mathbb{R}_{+}; \mathcal{D}_{A}(\alpha - 1/p, p)).
\]
This theorem can be applied to the spaces $W_p^\gamma(\mathbb{R}_+; W_p^s(\Omega; E)) \cap L_p(\mathbb{R}_+; W_p^{s+2\alpha}(\Omega; E))$ for $\alpha \in (1/p, 1]$, $s \in [0, \infty)$, provided that in the case $\alpha < 1$ the number $s + 2\alpha$ is not an integer and provided that $s + 2\alpha \leq 2k$. Here and in the following, we assume that $E$ is a Banach space of class $\mathcal{H}_T$ and has property (α), where we refer to [KW04] and [KS12] for the definitions of such spaces and additional information. For instance, any Hilbert space is of class $\mathcal{H}_T$ with property (α) and these properties are inherited to closed subspaces and isomorphic spaces. Moreover, the space $L_q(\Omega, \mathcal{A}, \mu; E)$ on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ with $q \in (1, \infty)$ is of class $\mathcal{H}_T$ and has property (α). For $s \in (0, \infty)$ and $p, q \in (1, \infty)$, the Sobolev-Slobodeckii spaces $W^s_p(\mathbb{R}_n; E)$, the Bessel potential spaces $H^s_p(\mathbb{R}_n; E)$ and the Besov spaces $B^s_{p,q}(\mathbb{R}_n; E)$ and are also of class $\mathcal{H}_T$ with property (α).

Let us indicate how Theorem [B.1] can be applied. Let first $\mathcal{E}_\Omega$ be an extension operator from $\Omega$ to $\mathbb{R}_n$ which acts as a bounded linear operator $W^t_p(\Omega; E) \to W^t_p(\mathbb{R}_n; E)$ for all $t \in [0, 2k]$. Such extension operators are defined in [AF03] for $t \in \mathbb{N}_0$ and their boundedness for $t \notin \mathbb{N}_0$ follows from real interpolation. Then it remains to study $\mathcal{E}_\Omega u$ in $\mathbb{R}_+ \times \mathbb{R}_n$. In this situation the operator $A = 1 - \Delta$ in $X = W^s_p(\mathbb{R}_n; E)$ with domain $W^{s+2}(\mathbb{R}_n; E)$ has the required properties. Indeed, [DHP03], Section 5 covers the case $s = 0$ and an abstract result of Dore [Dor99] covers the case $s \in (0, 2) \setminus \{1\}$. The remaining cases follow by means of isomorphic mappings, interpolation and taking fractional powers. Hence $\mathcal{D}_A(\alpha, p) = W^{s+2}(\mathbb{R}_n; E)$ and $\mathcal{D}_A(\alpha - 1/p, p) = B^{s+2\alpha-2/p}(\mathbb{R}_n; E)$. Then the temporal trace operator can be rewritten as $\gamma_t: u \mapsto \left(\mathcal{E}_\Omega u\right)_{|t=0}$ and acts as a bounded linear operator

$$W^\alpha_p(\mathbb{R}_+; W^s_p(\Omega; E)) \cap L_p(\mathbb{R}_+; W^{s+2\alpha}_p(\Omega; E)) \to B^{s+2\alpha-2/p}_p(\mathbb{R}_n; E).$$

For the boundary spaces $W^\alpha_p(\mathbb{R}_+; W^s_p(\Gamma; E)) \cap L_p(\mathbb{R}_+; W^{s+2\alpha}_p(\Gamma; E))$ we use a common retraction $r: W^t_p(\mathbb{R}_n; E)^N \to W^t_p(\Gamma; E)$ for all $t \in [0, 2k]$ with some $N \in \mathbb{N}$. A co-retraction for $r$ can be constructed by means of a partition of unity for $\Gamma$ and local parametrizations of $\Gamma$ over subsets of $\mathbb{R}_n$ as in the proof of Lemma [C.6]. Then the temporal trace operator can be rewritten as $\gamma_t: u \mapsto r((r^c u)_{|t=0})$ and maps

$$W^\alpha_p(\mathbb{R}_+; W^s_p(\Gamma; E)) \cap L_p(\mathbb{R}_+; W^{s+2\alpha}_p(\Gamma; E)) \to B^{s+\alpha-2/p}_p(\Gamma; E).$$

In order to construct functions with prescribed initial values, we consider an operator $A: \mathcal{D}(A) \subset X \to X$ as in Theorem [B.1] and define the spaces

$$\mathcal{D}_A(k + \alpha, p) := A^{-k}\mathcal{D}_A(\alpha, p) = (\mathcal{D}(A^k), \mathcal{D}(A^{k+1}))_{\alpha,p} \quad \text{for} \quad k \in \mathbb{N}_0, \alpha \in [0, 1], p \in (1, \infty).$$

Then Theorem [B.1] and the identity $\partial_t e^{-uA} = -A e^{-uA} = e^{-uA}A$ yield the following result.

**Corollary B.2.** Let $k \in \mathbb{N}_0$, $\alpha \in (1/p, 1]$ and $p \in (1, \infty)$. Then the operator

$$R_{A}: u \mapsto (t \mapsto e^{-tA}u), \quad \mathcal{D}_A(k + \alpha - 1/p, p) \to W^k_{p+\alpha}(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}_A(k + \alpha, p))$$

is a bounded right-inverse for $\gamma_t$.

We next deal with higher order initial conditions.

**Lemma B.3.** Let $\gamma_t^j := (\partial_t^j \gamma_t)_{|t=0}$ and let $l \in \mathbb{N}_0$, $m \in \mathbb{N}$ with $m \geq l + 1$. Then the operator

$$(\gamma_t^0, \gamma_t^1, \ldots, \gamma_t^l): W^m_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^m)) \to \prod_{j=0}^l \mathcal{D}_A(m - j - 1/p, p)$$

is a retraction.
Proof. For \( j \in \{0, 1, \ldots, l\} \) and \( x \in X \) we define
\[
(S_j^A x)(t, \cdot) := \sum_{i=0}^{l} c_{ij} e^{-t(1+i)A} A^{-j} x \quad \text{for } t \geq 0,
\]
where, for each \( j \), the \( l+1 \) numbers \( c_{ij} \) (\( i \in \{0, 1, \ldots, l\} \)) solve the linear system
\[
\sum_{i=0}^{l} c_{ij} (-1+i)^m = \delta_{mj} \quad \text{for } m \in \{0, 1, \ldots, l\}.
\]
By using Vandermonde's matrix
\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1+k \\
\vdots & \vdots & & \vdots \\
1^k & 2^k & \cdots & (1+k)^k
\end{pmatrix}
\]
the numbers \( c_{ij} \) are given by \( (c_{0j}, \ldots, c_{lj})^T := V^{-1} e_j^T \). Hence
\[
(\partial_t^m S_j^A x)(0) = \sum_{i=0}^{l} c_{ij} (-1+i)^m x = \delta_{mj} x \quad \text{for } m \in \{0, 1, \ldots, l\}.
\]
From Corollary B.2 we infer that \( S_j^A \) acts as a bounded linear operator
\[
S_j^A : D_A(m-j-1/p, p) \to W_p^m(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^m)) \quad \text{for } m \in \mathbb{N}, m \geq j+1.
\]
Therefore the desired co-retraction is given by
\[
S^A(x_0, x_1, \ldots, x_l) := \sum_{j=0}^{l} S_j^A x_j.
\]

Theorem B.4 (Mixed derivative embeddings). Let \( n \in \mathbb{N}, p \in (1, \infty), t, s \in [0, \infty), \tau, \sigma \in (0, \infty), \theta \in (0, 1), E \in \mathcal{HT}, J = \mathbb{R} \) or \( J = (0, T) \) for \( T \in (0, \infty) \) and let \( \Omega \) be the whole space \( \mathbb{R}^n \) or a bounded domain with smooth boundary or a compact smooth hypersurface of \( \mathbb{R}^n \).

Then the following embeddings are continuous.

(B.1a) \( H^{t+\tau}(J; H^s_p(\Omega; E)) \cap H^t_p(J; H^{s+\sigma}_p(\Omega; E)) \hookrightarrow H^{t+\theta \tau}(J; H^{s+(1-\theta)\sigma}_p(\Omega; E)), \)

(B.1b) \( B^{t+\tau}_p(J; H^s_p(\Omega; E)) \cap H^t_p(J; B^{s+\sigma}_p(\Omega; E)) \hookrightarrow B^{t+\theta \tau}_p(J; H^{s+(1-\theta)\sigma}_p(\Omega; E)), \)

(B.1c) \( B^{t+\tau}_p(J; H^s_p(\Omega; E)) \cap H^t_p(J; B^{s+\sigma}_p(\Omega; E)) \hookrightarrow B^{t+\theta \tau}_p(J; B^{s+(1-\theta)\sigma}_p(\Omega; E)). \)

Proof. We adapt the proof of [MS12, Proposition 3.2]. It is sufficient to consider the case \( J \times \Omega = \mathbb{R} \times \mathbb{R}^n \) since the other spaces are retracts of corresponding spaces over \( \mathbb{R} \times \mathbb{R}^n \).

In the ground space \( X := H^t_p(H^s_p) := H^t_p(\mathbb{R}; H^s_p(\mathbb{R}^n; E)) \) we consider the operators
\[
A = (1 - \partial_t^2)^{t/2}, \quad \mathcal{D}(A) = H^{t+\tau}_p(\mathbb{R}; H^s_p(\mathbb{R}^n; E)),
\]
\[
B = (1 - \Delta_n)^{s/2}, \quad \mathcal{D}(B) = H^t_p(\mathbb{R}; H^{s+\sigma}_p(\mathbb{R}^n; E)).
\]
Here \( \Delta_n \) denotes the Laplacian in \( \mathbb{R}^n \) with respect to the spatial variable. It follows from [DHP03, Theorem 5.5] (see also [MS12, Lemma 3.1]) that the operator
\[
J := (1 - \Delta_n)^{r/2}: H^{r+\rho}_p(\mathbb{R}^m; F) \to H^{r}_p(F; F)
\]
is invertible and has a bounded \( \mathcal{H}^\infty \) functional calculus and thus bounded imaginary powers in \( H^r_p(\mathbb{R}^m; F) \) for all \( m \in \{1, n\} \), \( r, \rho \in [0, \infty) \) and all Banach spaces \( F \) of class \( \mathcal{HT} \).

The latter property implies that its fractional powers \( J_{\theta}^\rho (\theta \in (0, 1)) \) have the domains \( \mathcal{D}(J_{\theta}^\rho) = [H^r_p(\mathbb{R}^m; F), H^{r+\rho}_p(\mathbb{R}^m; F)]_{\theta} = H^{r+\theta \rho}_p(\mathbb{R}^m; F) \) by [DHP03, Theorem 2.5] and complex interpolation.
By choosing \((m, r, \rho, F) = (1, t, \tau, H^\mu_p(\mathbb{R}^n; E))\) or \((m, r, \rho, F) = (n, s, \sigma, H^\nu_{1/2}p(\mathbb{R}; E))\) and employing Fubini's theorem, we see that \(A: D(A) \to X\) and \(B: D(B) \to X\) are invertible and have bounded \(H^\infty\) functional calculi. Sobolevskii’s mixed derivative theorem [Sol75, Theorem 6] implies that \(D(A) \cap D(B) \hookrightarrow D(A^\theta B^{1-\theta}) \cap D(B^{1-\theta}A^\theta)\) which proves (B.1a).

For proving (B.1b) we apply (B.1a) and real interpolation to the space

\[
H^{t+\tau \pm \epsilon \tau}(H^s_p) \cap H^{t}(H^s+\sigma \pm \epsilon \sigma)_p
\]

for sufficiently small \(\epsilon > 0\). For \(\tau_\pm := (1 \pm \epsilon)\tau\) and \(\sigma_\pm := (1 \pm \epsilon)\sigma\) and \(\theta_\pm \in (0, 1)\), we obtain

\[
H^{t+\tau \pm \epsilon \tau}(H^s_p) \cap H^{t}(H^s+\sigma \pm \epsilon \sigma)_p = H^{t+\tau \pm \sigma}(H^s_p) \cap H^{t}(H^s+\sigma \pm \epsilon \sigma)_p \hookrightarrow H^{t+\theta \pm \epsilon \tau}(H^s_p)(1-\theta)\sigma_\pm).
\]

We choose \(\theta_\pm\) such that \((1-\theta_\pm)\sigma_\pm = (1-\theta)\sigma\), that is, \(\theta_\pm := (\theta \pm \epsilon)/(1 \pm \epsilon)\). Then \(\theta_\pm \epsilon_\pm = \theta \epsilon \pm \epsilon \tau\) and it remains to apply the real interpolation functor \((\cdot, \cdot)_{1/2,p}\) to

\[
Z_\pm := H^{t+\tau \pm \epsilon \tau}(H^s_p) \cap H^{t}(H^s+\sigma \pm \epsilon \sigma)_p \hookrightarrow H^{t+\theta \pm \epsilon \epsilon \tau}(H^s_p)((1-\theta)\sigma).
\]

Indeed, interpolation of the right-hand side yields

\[
(Z_-, Z_+)(1/2,p) \hookrightarrow \left( H^{t+\theta \pm \epsilon \tau}(H^s_p)((1-\theta)\sigma), H^{t+\theta \pm \epsilon \epsilon \tau}(H^s_p)((1-\theta)\sigma) \right)_{1/2,p} = B^{t+\theta \epsilon \tau}(H^s_p)((1-\theta)\sigma).
\]

For an interpolation of the left-hand side we write \(Z_+ = D(L)\) and \(Z_- = D(L(1-\epsilon)/(1+\epsilon))\) where the operator \(L = (1-\partial^2_\xi)(1-\epsilon)/(1+\epsilon)\) is considered in the ground space \(Z_0 := H^0_p(H^s_p)\). Then the reiteration theorem ([Lun95, Remark 1.2.16]) yields

\[
(Z_-, Z_+)(1/2,p) = (D(L(1-\epsilon)/(1+\epsilon)), D(L))_{1/2,p}
\]

\[
= D_L (1/2 + (1-\epsilon)/(2 + 2\epsilon), p) = B^{t+\tau}(H^s_p) \cap H^0_{1/2,p}(B^{s+\sigma})_{1/2,p}.
\]

Hence (B.1b) is proved. The proof of (B.1a) is similar and therefore omitted. \(\square\)

**APPENDIX C. HIGHER REGULARITY FOR THE HEAT EQUATION**

We study the regularity of solutions of the heat problem

\[
\begin{aligned}
& (\partial_t + \mu_B - \Delta)u = f & \quad & \text{in } J \times \Omega, \\
& \gamma_B u = g & \quad & \text{on } J \times \Gamma, \\
& u|_{t=0} = u_0 & \quad & \text{in } \Omega.
\end{aligned}
\]

Here \(J\) is a bounded interval \((0, T)\) or the half line \((0, \infty)\) and \(\Omega\) is a bounded domain in \(\mathbb{R}^n, n \in \mathbb{N}\), with smooth boundary \(\Gamma\). For \(B \in \{D, N\}\), let \(\mu_B\) be a real number and let

\[
\gamma_D := |\Gamma|, \quad \gamma_N := (\partial_\nu \cdot \nu)|\Gamma\nu \cdot (\nabla \cdot \nu)|\Gamma|, \quad \gamma_l := (\partial_t (\gamma_l))|_{t=0}, \quad \gamma_l := \gamma_l^0
\]

denote the Dirichlet, the Neumann, and the temporal trace operators, respectively. Again we let \(\lambda_D > 0\) denote the smallest eigenvalue of \(-\Delta_D\) and \(\lambda_N > 0\) denote the smallest non-zero eigenvalue of \(-\Delta_N\). We will prove the following regularity result.

**Theorem C.1.** Let \(B \in \{D, N\}, j_D = 0, j_N = 1, \mu_D \in (-\lambda_D^0, \infty), \mu_N \in (0, \infty), l \in \mathbb{N}_0, k \in \mathbb{N}\) and \(p \in (1, \infty)\) such that \(j_D/2 + 3/2p \neq 1\). Then problem (C.1) has a unique solution

\[
u \in \mathbb{E}^l_k := W^{l+k}_p(J; L_p(\Omega)) \cap W^l_p(J; W^{2k}_p(\Omega)),
\]

\[
\begin{aligned}
& (\partial_t + \mu_B - \Delta)u = f & \quad & \text{in } J \times \Omega, \\
& \gamma_B u = g & \quad & \text{on } J \times \Gamma, \\
& u|_{t=0} = u_0 & \quad & \text{in } \Omega.
\end{aligned}
\]
if and only if the data \((f,g,u_0)\) satisfy the regularity conditions

\[
\begin{align*}
(C.3a) & \quad f \in E^{l,k-1} = W_p^{l+k-1}(J; L_p(\Omega)) \cap W_p^l(J; W_p^{2k-2}(\Omega)), \\
(C.3b) & \quad g \in \gamma_B E^{l,k} = W_p^{l+k-j/2-1/2p}(J; L_p(\Gamma)) \cap W_p^l(J; W_p^{2k-j/2-1/p}(\Gamma)), \\
(C.3c) & \quad u_0 \in \gamma E^{l,k} = \begin{cases} W_p^{2k}(\Omega) & \text{if } l \geq 1, \\
W_p^{2k-2/p}(\Omega) & \text{if } l = 0,
\end{cases}
\end{align*}
\]

and the compatibility conditions

\[
\begin{align*}
(C.4a) & \quad u_j := \gamma_t^{j-1} f + (\Delta - \mu_B) u_{j-1} \in \begin{cases} W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1,l-1], \\
W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l,l+k-1],
\end{cases} \\
(C.4b) & \quad \gamma_t^j g = \gamma_B u_j \quad \text{for } j \in \mathbb{N}_0, \ j \leq l + k - j_B/2 - 3/2p.
\end{align*}
\]

Remark C.2.  
(i) The space \(E^{0,1} \) \((l = 0, k = 1)\) is the standard parabolic solution space.
(ii) If \(l = 0\) and \(J \times \Omega\) is the half space \(\mathbb{R}_+ \times \mathbb{R}^n\) or the wedge \(\mathbb{R}_+ \times \mathbb{R}^n_+\), then \(E^{l,k}\) is the anisotropic space \(H^{2k/p}_\nu(J \times \Omega)\) with weight \(\nu = (2,1,\ldots,1)\) in the sense of [Ama09]. This fact will be used in the construction of functions with prescribed boundary values.
(iii) We exclude the case \(j_B/2 + 3/2p = 1\) in order to avoid the more complicated trace spaces \(\gamma_D W_\nu^{2/3}(\Omega)\) and \(\gamma_N W_\nu^{4/3}(\Omega)\).
(iv) The additional regularity conditions \((C.4a)\) follow from the non-triangular structure of the space \(E^{l,k}\) in the case \(l \geq 1\) and are derived in Subsection C.1. For \(j \geq l + 1\), the formula \((C.4a)\) does not contain additional regularity conditions and should be merely understood as the definition of the functions \(u_j\), which appear in \((C.4b)\).
(v) Every solution satisfies the higher order boundary conditions

\[
(C.5) \quad \gamma_B \Delta^{j+1} u = (\partial_t + \mu_B) g_j - \gamma_B \Delta^j f =: g_{j+1} \quad \text{for } j \in \mathbb{N}_0 \cap [0,k-2], \quad \text{with } g_0 := g.
\]

With the temporal trace theorem and \(u_i = \gamma_i^l u\) we obtain

\[
(C.6) \quad \gamma_B \Delta^j u_i = \gamma_i^l g_j \quad \text{for } j \in \mathbb{N}_0 \cap [0,k-1], \ i \in \mathbb{N}_0 \cap [0,l + k - j - j_B/2 - 3/2p].
\]

These equations are no additional regularity or compatibility conditions but follow from \((C.4), (C.5)\), by induction over \(j \in \mathbb{N}_0\). Indeed, suppose that \(\gamma_B \Delta^j u_i = \gamma_i^j g_j\) for all \(i\) and some \(j\). Then \((C.5)\), the induction hypothesis, and \((C.4)\) yield

\[
\begin{align*}
\gamma_i^j g_{j+1} &= \gamma_i^j ((\partial_t + \mu_B) g_j - \gamma_B \Delta^j f) = \gamma_i^{j+1} g_j + \mu_B \gamma_i^j g_j - \gamma_B \gamma_i^j \Delta^j f \\
&= \gamma_B \Delta^j u_{i+1} + \mu_B \gamma_B \Delta^j u_i - \gamma_B \gamma_i^j \Delta^j f \\
&= \gamma_B \Delta^j (\gamma_i^j f + (\Delta - \mu_B) u_i) + \mu_B \gamma_B \Delta^j u_i - \gamma_B \gamma_i^j \Delta^j f = \gamma_B \Delta^{j+1} u_i.
\end{align*}
\]

(vi) In the case \(l = 1, k = 2, B = D\), the compatibility conditions read as

\[
\begin{align*}
u_1 & := f \big|_{t=0} + (\Delta - \mu_D) u_0 \in W_p^{2k}(\Omega), \\
u_2 & := \partial_t f \big|_{t=0} + (\Delta - \mu_D) u_1 \in W_p^{2k-2/p}(\Omega), \\
g \big|_{t=0} & = u_0 |_{\Gamma}, \\
\partial_t g \big|_{t=0} & = f \big|_{t=0,\Gamma} + ((\Delta - \mu_D) u_0) |_{\Gamma}, \\
\partial_t^2 g \big|_{t=0} & = \partial_t f \big|_{t=0,\Gamma} + ((\Delta - \mu_D) u_1) |_{\Gamma} \quad \text{if } p > 3/2.
\end{align*}
\]
(vii) The corresponding result for $B = N$ has the compatibility conditions

\[ u_1 := f|_{t=0} + (\Delta - \mu_N)u_0 \in W^{2k}_p(\Omega), \]
\[ u_2 := \partial_t f|_{t=0} + (\Delta - \mu_N)u_1 \in W^{2k-2/p}_p(\Omega), \]
\[ g|_{t=0} = \partial_{\nu} u_0|_{\Gamma}, \]
\[ \partial_t g|_{t=0} = \partial_{\nu} f|_{t=0}|_{\Gamma} + \partial_{\nu}((\Delta - \mu_N)u_0)|_{\Gamma}, \]
\[ \partial_{t}^{2} g|_{t=0} = \partial_{\nu} \partial_{t} f|_{t=0}|_{\Gamma} + \partial_{\nu}((\Delta - \mu_N)u_1)|_{\Gamma} \quad \text{if } p > 3. \]

Aiming at stability for Neumann boundary conditions, we will also prove the following result, where we consider the subspace $L_{p,0}(\Omega) := \{ f \in L_{p}(\Omega) : \int_{\Omega} f(x) dx = 0 \}$.

**Corollary C.3.** Let $B = N, \mu_N \in (-\lambda_1^N, \infty), l \in \mathbb{N}_0, k \in \mathbb{N}, p \in (1, \infty), p \neq 3$. Then problem (C.1) has a unique solution

\[ u \in E_{0}^{l,k} := E^{l,k} \cap L_{p}(J; L_{p,0}(\Omega)), \]

if and only if the data $(f, g, u_0)$ satisfy the regularity conditions (C.3) and the compatibility conditions (C.4) and

\[ \int_{\Omega} u_0(x) dx = 0, \quad \int_{\Omega} f(t, x) dx + \int_{\Gamma} g(t, x) dS(x) = 0 \text{ for } t \in J. \]

Next, we study the original heat problem

\[ (\partial_t - \Delta) u = f \quad \text{in } \mathbb{R}_{+} \times \Omega, \]
\[ \gamma_B u = g \quad \text{on } \mathbb{R}_{+} \times \Gamma, \]
\[ u|_{t=0} = u_0 \quad \text{in } \Omega. \]

**Corollary C.4.** (i) Let $B \in \{D, N\}, \mu_D \in (-\lambda_0^D, \infty), \mu_N \in (0, \infty), l \in \mathbb{N}_0, k \in \mathbb{N}, p \in (1, \infty)$ such that $j_B/2 + 3/2p \neq 1$. Then problem (C.8) has a unique solution $u \in \mathbb{E}_{B}^{l,k}$ if and only if the data $(f, g, u_0)$ satisfy the regularity conditions

\[ (f, g, u_0) \in \mathbb{E}_{B}^{l,k-1} \times \mathbb{E}_{B}^{l,k} \times \left\{ \begin{array}{ll}
W_p^{2k}(\Omega) & \text{if } l \geq 1, \\
W_p^{2k-2/p}(\Omega) & \text{if } l = 0,
\end{array} \right. \]

and the compatibility conditions

\[ u_j := \gamma_{t}^{j-1} f + \Delta u_{j-1} \in \left\{ \begin{array}{ll}
W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1, l - 1], \\
W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l, l+k-1],
\end{array} \right. \]
\[ \gamma_{t}^{j} g = \gamma_B u_j \quad \text{for } j \in \mathbb{N} \cap [0, l + k - j_B/2 - 3/2p]. \]

(ii) Let $B = N, \mu_N \in (-\lambda_1^N, \infty).$ Then problem (C.8) has a unique solution $u \in \mathbb{E}_{N}^{l,k}$ if and only if the data $(f, g, u_0)$ satisfy the regularity conditions (C.9) and the compatibility conditions (C.10), (C.7).

**Proof.** In problem (C.1) we multiply $f, g$ with $e^{\mu_B t}$, so that

\[ e^{\mu_B t} f = e^{\mu_B t}(\partial_t + \mu_B - \Delta) u = (\partial_t - \Delta)e^{\mu_B t} u, \quad e^{\mu_B t} g = \gamma_B e^{\mu_B t} u. \]

This shows that $e^{\mu_B t} u$ solves (C.8) for $(e^{\mu_B t} f, e^{\mu_B t} g, u_0)$ if and only if $u$ solves the shifted problem (C.1) for $(f, g, u_0)$. Hence Theorem C.1 and Corollary C.3 yield the assertions. □
C.1. Compatibility conditions. By means of the results from Section \[13\] it is not difficult to verify that the regularity conditions (C.3) are indeed necessary for \( u \in \mathbb{E}^{l,k} \). Let us now derive the remaining compatibility conditions.

First, any function \( u \in \mathbb{E}^{l,k} \) satisfies the initial regularity conditions
\[
\gamma_l^j u \in \begin{cases} 
W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N}_0 \cap [0, l - 1], \\
W_p^{2l+2k-2j-2/p}(\Omega) & \text{for } j \in \mathbb{N}_0 \cap [l, l + k - 1]. 
\end{cases}
\]

If \( u \) solves (C.1) with data \((f, g, u_0)\), then an application of \( \partial_t^{j-1} \) to the heat equation yields
\[
\partial_t^{j-1} u = \partial_t^{j-1} f + (\Delta - \mu_B)\partial_t^{j-1} u.
\]

In particular, the initial values \( u_j := \partial_t^j u|_{t=0} \) are given in terms of \( f \) and \( u_0 \) by
\[
u_j = \sum_{i=0}^{j-1} (\Delta - \mu_B)\gamma_l^{j-i} f + (\Delta - \mu_B)\gamma_l^{j-1} u_0 \quad \text{for } j \in \mathbb{N} \cap [1, l + k - 1].
\]

Then (C.11) implies that the data \((f, u_0)\) and \( u_j \) must satisfy
\[
u_j = \gamma_l^{j-1} f + (\Delta - \mu_B)u_{j-1} \in \begin{cases} 
W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1, l - 1], \\
W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l, l + k - 1]. 
\end{cases}
\]

For \( j \in [1, l] \) this is indeed an additional condition, since \( f \) merely satisfies
\[
\gamma_l^{j-1} f \in \begin{cases} 
W_p^{2k-2}(\Omega) & \text{for } j \in \mathbb{N} \cap [0, l], \\
W_p^{2l+2k-2j-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l + 1, l + k - 1]. 
\end{cases}
\]

The conditions for \( l + 1 \leq j \leq l + k - 1 \) then follow from the regularity of \( f, u_0, \ldots, u_l \) and could therefore be omitted in (C.4a), but we keep them there as a definition of \( u_{l+1}, \ldots, u_{l+k-1} \). Indeed, these functions still admit traces on \( \Gamma \). By differentiating the boundary condition \( \gamma_B u = g \) with respect to time, we obtain
\[
\gamma_l^j g = \gamma_B u_j \in \begin{cases} 
W_p^{2k-j-3/lB-1/p}(\Gamma) & \text{for } j \in \mathbb{N} \cap [0, l - 1], \\
W_p^{2(l+k-j)-j-3/lB-3/p}(\Gamma) & \text{for } j \in \mathbb{N} \cap [l, l + k - j - 2 - 3/2p]. 
\end{cases}
\]

This shows that (C.4b) is a necessary condition.

We conclude that the necessity part of Theorem C.1 is true, that is, if problem (C.1) has a solution \( u \in \mathbb{E}^{l,k} \) with data \((f, g, u_0)\), then (C.3) and (C.4) are satisfied. We next prepare the proof of the existence part.

C.2. Interior regularity and initial conditions. From [DHP03] Theorem 8.2] we deduce that for \( B \in \{D, N\} \) and \( E \in \mathcal{H} \) there exists \( \mu_B \geq 0 \) such that the realization
\[
\mu_B - \Delta_B \text{ with domain } \mathcal{D}(\Delta_B) = \{ u \in W_p^2(\Omega; E) : \gamma_B u = 0 \} \text{ in } L_p(\Omega; E),
\]
has maximal regularity of type \( L_p(\mathbb{R}^+; L_p(\Omega; E)) \). Thus the operator
\[
\mu_B + \partial_t - \Delta_B : _0W_p^1(\mathbb{R}^+; L_p(\Omega; E)) \cap L_p(\mathbb{R}^+; \mathcal{D}(\Delta_B)) \to L_p(\mathbb{R}^+; L_p(\Omega; E))
\]
is invertible for \( B \in \{D, N\} \). Here and in the following, the space \(_0W_p^s(\mathbb{R}^+; E)\) is the closure of \( C_c^\infty(\mathbb{R}^+; E) \) in \( W_p^s(\mathbb{R}^+; E) \). For \( k < s - 1/p < k + 1 \) with \( k \in \mathbb{N}_0 \), they consist precisely of those functions with vanishing initial traces \( \partial_t^j u(0) = 0 \) for \( 0 \leq j \leq k \), see [Ama09] Theorem 4.7.1. By using [Dor93] Theorem 2.4] and a spectral theoretic argument as in [MW11] we may even allow for \( \mu_D \in (-\lambda_0^D, \infty) \), \( \mu_N \in (0, \infty) \).
In order to obtain higher regularity results we consider the spaces
\[ X_B^k := (\mu_B - \Delta_B)^{-k} L_p(\Omega; E), \quad \|u\|_{X_B^k} := \|(\mu_B - \Delta_B)^k u\|_{L_p(\Omega; E)} \quad \text{for } k \in \mathbb{N}_0. \]

These spaces can be easily characterized by
\[ X_B^k = \{ u \in W_p^{2k}(\Omega; E) : \gamma_B \Delta^j u = 0 \text{ for } 0 \leq j \leq k - 1 \}. \]

By commuting the operator \( \mu_B + \partial_t - \Delta_B \) with \((\mu_B - \Delta_B)^k\) it follows that \( \mu_B - \Delta_B \) has maximal regularity of type \( L_p(\mathbb{R}_+; X_B^k) \) for every \( B \in \{D, N\}, k \in \mathbb{N}_0 \), that is,
\[ \mu_B + \partial_t - \Delta : 0W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \to L_p(\mathbb{R}_+; X_B^k) \]
is a topological linear isomorphism. Moreover, the map
\[ \epsilon + \partial_t : 0W_p^{l+1}(\mathbb{R}_+; E) \to 0W_p^l(\mathbb{R}_+; E) \]
is a topological linear isomorphism for every \( \epsilon > 0 \) and every \( l \in \mathbb{N}_0 \), see e.g. [MS12]. Hence, by commuting \( \mu_B + \partial_t - \Delta_B \) with \( \epsilon + \partial_t \), we obtain the following result.

**Lemma C.5.** Let \( \mu_D \in (-\lambda_0^D, \infty), \mu_N \in (0, \infty), l \in \mathbb{N}_0, k \in \mathbb{N}_0, B \in \{D, N\} \). Then the map
\[ \mu_B + \partial_t - \Delta_B : 0W_p^{l+1}(\mathbb{R}_+; X_B^k) \cap 0W_p^l(\mathbb{R}_+; X_B^{k+1}) \to 0W_p^l(\mathbb{R}_+; X_B^k) \]
is a topological linear isomorphism.

We next comment on function spaces for the initial data. From [Ama09, Section 4.9] we derive the characterization
\[ (X_B^k, X_B^{k+1})_{1-1/p,p} = \{ u \in W_p^{2k+2-2/p}(\Omega; E) : \gamma_B \Delta^j u = 0 \text{ for } 0 \leq j \leq k - j_B/2 - 3/2p \}. \]

Then the temporal trace operator
\[ \gamma_t : W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \to (X_B^k, X_B^{k+1})_{1-1/p,p} \]
is a bounded and surjective and therefore
\[ (\mu_B + \partial_t - \Delta_B, \gamma_t) : W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \to L_p(\mathbb{R}_+; X_B^k) \times (X_B^k, X_B^{k+1})_{1-1/p,p} \]
is also a topological linear isomorphism for \( B \in \{D, N\}, k \in \mathbb{N}_0 \).

**C.3. Boundary conditions.** We will use the following result for constructing a function with prescribed boundary conditions [C55].

**Lemma C.6.** Let \( l \in \mathbb{N}_0, k \in \mathbb{N}, p \in (1, \infty), \) let \( \gamma^j : (\partial_B^l \cdot ) |_\Gamma \) in the sense of traces and let
\[ 0E^{l,m/2}_p(\mathbb{R}_+ \times \Gamma) : = 0W_p^{l+m/2-1/2p}(\mathbb{R}_+; L_p(\Gamma; E)) \cap 0W_p^l(\mathbb{R}_+; W_p^m(\Gamma; E)) \quad \text{for } m \in \mathbb{N}. \]

Then \( \gamma^j : 0E^{l,k}(\mathbb{R}_+ \times \Omega) \to 0G^{l,k-j/2}(\mathbb{R}_+ \times \Gamma) \) is a retraction and the operator
\[ B_{l,k} : = (\gamma^0_D, \gamma^1_D, \gamma^0_N, \gamma^1_N, \ldots, \gamma^0_j, \gamma^1_j) : 0E^{l,k}(\mathbb{R}_+ \times \Omega) \to \prod_{j=0}^{2k-1} 0G^{l,k-j/2}(\mathbb{R}_+ \times \mathbb{R}^n) \]
is a retraction.

**Proof.** In the case \( \Omega = \mathbb{R}^n_+ \), \( l = 0 \) we infer from [Ama09, Theorem 4.11.6] that
\[ \overline{B}_{0,k} := \{ (-1)^j \gamma^j_D, 0G^{l,k}(\mathbb{R}_+ \times \mathbb{R}^n) \to \prod_{j=0}^{2k-1} 0G^{l,k-j/2}(\mathbb{R}_+ \times \mathbb{R}^{n-1}) \]
is a retraction. Let \( \overline{B}_{0,k} \) denote a co-retraction for \( \overline{B}_{0,k} \).
In the case \( \Omega = \mathbb{R}^n_+ \), \( l \in \mathbb{N}_0 \) we use the fact that \( (\epsilon + \partial_t)^j : \partial W^{s+j}_p(\mathbb{R}_+; F) \to \partial W^s_p(\mathbb{R}_+; F) \) is invertible for every \( \epsilon > 0 \), \( j \in \mathbb{N} \), \( s \in [0, \infty) \) and every Banach space \( F \) of class \( \mathcal{H}\mathcal{T} \). Therefore a co-retraction is given by \( B^c_{i,k} = (\epsilon + \partial_t)^{-1} \Theta_{0,k}(\epsilon + \partial_t)^i \).

For bounded smooth domains we define such operators by a localization technique. It is well-known (see e. g. \[GT01\] Section 14.6., \[PS13\]) that the tubular neighborhood map

\[
X \colon (x,t) \mapsto x + t v_T(x), \quad \Gamma \times (-R,R) \to B_R(\Gamma) := \{ x \in \mathbb{R}^n : \text{dist}(x,\Gamma) < R \}
\]

is a homeomorphism for some \( R > 0 \). Let \( \{ U_j : j \in I \} \) be a finite open covering of \( \Gamma \) in \( \mathbb{R}^n \) and let \( \{ \varphi_j : j \in I \} \subset C^\infty_c(\Gamma) \) be a partition of unity subordinate to \( \{ U_j \cap \Gamma : j \in I \} \). Then there exists \( r \in (0,R) \) such that \( B_r(\Gamma) \) is covered by \( \{ U_j : j \in I \} \). For given \( \chi \in C^\infty((-r,r)) \) with \( 0 \leq \chi \leq 1 \) and \( \chi(t) = 1 \) for \( |t| \leq r/2 \) we extend \( \varphi_j \) to \( \mathbb{R}^n \) by means of \( \varphi_j(X(x,t)) := \varphi_j(x) \chi(t) \) for \( (x,t) \in \Gamma \times (-r,r) \) so that \( \text{supp} \varphi_j \subset U_j \) and \( \partial_t \varphi_j = 0 \) near \( \Gamma \) for all \( m \geq 1 \).

In addition, let \( U_j = B_r(x^{(j)}) \) with \( x^{(j)} \in \Gamma \) for some \( r \in (0,R) \) and choose rigid transformations \( \Xi_j : x \mapsto x^{(j)} + Q_j x \) with \( Q_j \) orthogonal such that \( Q_j(-e_n) = \nu_T(x^{(j)}) \). There exist \( \omega_j \in C^\infty_c(\mathbb{R}^{n-1}) \) with \( \omega_j(0) = |\nabla \omega_j(0)| = 0 \) such that for \( \theta_j(x',x_n) := (x',x_n + \omega_j(x')) \) we have \( U_j \cap \Omega = U_j \cap \Xi_{j}(\theta_j(\mathbb{R}^n_+)) \) and thus \( U_j \cap \Gamma = U_j \cap \Xi_j(\theta_j(\Gamma_0)) \) with \( \Gamma_0 := \mathbb{R}^{n-1} \times \{0\} \). Let us construct smooth diffeomorphisms \( \Theta_j \) of \( \mathbb{R}^n \) such that \( U_j \cap \Omega = U_j \cap \Theta_j(\mathbb{R}^n_+) \) and \( U_j \cap \Gamma = U_j \cap \Theta_j(\mathbb{R}^{n-1} \times \{0\}) \). Given \( r \in (0, R/2) \), \( \psi \in C^\infty_c(B_{2r}(0)) \) with \( \psi = 1 \) on \( B_r(0) \), let

\[
\Theta_j(x) = \begin{cases} 
\psi(x) \left[ \Xi_j(\theta_j(x',0)) - x_n \nu_T(\Xi_j(\theta_j(x',0))) \right] + (1 - \psi(x)) \Xi_j(x) & \text{for } |x| \leq 2r, \\
\Xi_j(x) & \text{for } |x| \geq 2r.
\end{cases}
\]

If \( r \in (0, R/2) \) is sufficiently small, then \( \Theta_j \) is a diffeomorphism since \( \partial_n \Theta_j(x) \to Q_j \) as \( r \to 0 \), uniformly on \( \mathbb{R}^n \). Moreover, \( \Theta_j \) has the asserted properties and satisfies \( -\partial_n \Theta_j(x',0) = \nu_T(\Theta_j(x',0)) \) and \( \partial_t \Theta_j(x',0) = 0 \) for all \( m \geq 2 \) and \( x' \in B_r(0) \).

Choose smooth cut-off functions \( \psi_j \in C^\infty_c(\Theta_j^{-1}(U_j)) \) with \( \psi_j = 1 \) on \( \Theta_j^{-1}(\text{supp} \varphi_j) \) and define the multiplication operator \( M_j : u \mapsto \psi_j u \). With the pull-back \( \Theta_j^* : u \mapsto u \circ \Theta_j \) and the push-forward \( \Theta_j* : u \mapsto u \circ \Theta_j^{-1} \) we define a co-retraction for \( B_{i,k} \) by

\[
B^c_{i,k}g := \sum_{j \in I} \Theta_j^* M_j B^c_{i,k} \Theta_j^*(\varphi_j g) \quad \text{for } g \in \prod_{j=0}^{2k-1} \partial^l \mathbb{G}^{l,k-j/2}(\mathbb{R}_+ \times \Gamma).
\]

By means of the chain rule, Hölder’s inequality and the mixed derivative embeddings, it can be shown that the linear operators \( g \mapsto \varphi_j g, \Theta_j^*, M_j \) and \( \Theta_j* \) act continuously in the relevant spaces and the properties of \( \Theta_j \) and \( \varphi_j \) with respect to the normal direction imply that indeed \( B_{i,k}B^c_{i,k}g = g \). This concludes the proof of Lemma \[C.6\].

C.4. Proof of Theorem \[C.1\] We have already discussed the necessity of the regularity conditions and the compatibility conditions on \( (f,g,u_0) \). It remains to prove the uniqueness and existence of a solution \( u \in \mathbb{E}^{l,k} \) for given data \( (f,g,u_0) \) subject to these conditions.

In order to prove uniqueness, it suffices to consider the most general case \( l = 0, k = 1 \), where \( \mathbb{E}^{l,k} = \mathbb{W}^1_p(J; L_p(\Omega)) \cap L_p(J; W^{2}_p(\Omega)) \) and \( (f,g,u_0) = 0 \). If further \( \mu_0 \) is sufficiently large, then the general result of \[DHP03\] implies that \( \mu_0 - \Delta \) has maximal regularity of type \( L_p(\mathbb{R}_+; L_p(\Omega)) \) and this yields \( u = 0 \) in case \( \mu_B \geq \mu_0 \).

Next, we employ spectral theory to cover the case \( \mu_B \in (-\lambda^B_0, \infty) \), where \( \lambda^B_0 = \lambda_0(-\Delta_B) \geq 0 \) denotes the smallest eigenvalue of \( -\Delta_B \). It is well known that, since \( D(\Delta_B) \) is compactly embedded into \( L_p(\Omega) \), the spectrum of \( \Delta_B \) is discrete and consists only of eigenvalues with
finite multiplicity. As for Lemma $[\Lambda.4]$ we see that its eigenfunctions belong to
\[\mathcal{D}(\Delta_B^m) = \{ u \in W^{2m}_p(\Omega) : \gamma_B(\Delta^j u) |_{\Gamma} = 0 \text{ on } \Gamma \text{ for } j \leq m-1 \}\]
for every $m \in \mathbb{N}$ and hence belong to $W^2_p(\Omega)$. An integration by parts implies that the spectrum of $\Delta_B$ is contained in $(-\infty, -\lambda_B^0]$. A result of Dore $[\text{Dor03}]$ Theorem 2.4) implies that $\mu_B - \Delta_B$ has maximal regularity of type $L_p(\mathbb{R}^+; L_p(\Omega))$ for each $\mu_B \in (-\lambda_B^0, \infty)$ and this ensures uniqueness.

Existence. We construct a solution $u = u^1 + u^2 + u^3 \in \mathbb{E}^{l,k}$ such that
\[
(\partial_t + \mu - \Delta) u^1 =: f^1, \quad \gamma_l^i u^1 = u_i,
\]
\[
(\partial_t + \mu - \Delta) u^2 =: f^2, \quad \gamma_B \Delta^j u^2 = g_j - \gamma_B \Delta^j u^1, \quad \gamma_l^i u^2 = 0,
\]
\[
(\partial_t + \mu - \Delta) u^3 = f - f^2 - f^1, \quad \gamma_B \Delta^j u^3 = 0, \quad \gamma_l^i u^3 = 0,
\]
for all $i, j$ with $0 \leq i \leq l + k - 1$ and $0 \leq j \leq k - 1$. This means that we first construct $u^1 \in \mathbb{E}^{l,k}$ with prescribed initial data $u_i$. Then we construct $u^2$ with prescribed boundary data $g_j - \gamma_B \Delta^j u^1$ and we finally we construct $u^3$ with prescribed interior data $f - f^2 - f^1$.

Here the functions $u_i$ and $g_j$ are defined according to (C.4a) and (C.5) by
\[
(C.12) \quad u_i := \gamma_l^i f + (\Delta - \mu_B) u_{i-1} \quad \text{for } i \in \mathbb{N} \cap [1, l + k - 1],
\]
\[
(C.13) \quad g_j := -\gamma_B \Delta^j f + (\partial_t + \mu_B) g_{j-1} \quad \text{for } j \in \mathbb{N} \cap [1, k - 1], \quad g_0 := g.
\]

Construction of $u^1$. Let $r_\Omega^t$ be a common co-retraction for the restriction $r_\Omega : W^{1}_{p}(\mathbb{R}^n) \to W^{l}_{p}(\Omega)$ for all $t \in [0, 2k]$ (cf. $[\text{AF03}]$ Theorem 5.22)). With the co-retraction $S^A$ for the operator $(\gamma_l^0, \ldots, \gamma_l^{l+k-1})$ from Lemma $[\text{B.3}]$ we define
\[
u^1 := r_\Omega S^l(r_\Omega^0 u_0, r_\Omega^1 u_1, \ldots, r_\Omega^k u_{k-1}, 0, \ldots, 0)
\]
\[
+ r_\Omega S^l - \Delta(0, \ldots, 0, r_\Omega^0 u_1, \ldots, r_\Omega^k u_{k-1}).
\]
Here we consider the identity operator $I : \mathcal{D}(I) \to X$ with $X = \mathcal{D}(I) = W^{2k}_{p}(\mathbb{R}^n)$ so that the first summand of $u^1$ belongs to $(t \mapsto e^{-t})BUC^\infty(\mathbb{R}^+; W^{2k}_{p}(\Omega)) \hookrightarrow \mathbb{E}^{l,k}$. In the second summand, we consider the operator $I - \Delta : \mathcal{D}(\Delta) \to X$ in $X = L_p(\mathbb{R}^n)$ with domain $\mathcal{D}(\Delta) = W^{2}_{p}(\mathbb{R}^n)$ so that $r_\Omega^t u \in \mathcal{D}(\Delta(k-1/p, p))$ and thus $S^l - \Delta(0, \ldots, 0, r_\Omega^0 u_1, \ldots, r_\Omega^k u_{k-1}) \in W^{l+k}_{p}(\mathbb{R}^+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}^+; W^{2l+k}_{p}(\mathbb{R}^n))$. Therefore $u^1$ belongs to $\mathbb{E}^{l,k}$, satisfies $\gamma_l^i u^1 = u_i$ and depends continuously on the data $(f, u_0)$ with respect to the norms induced by the regularity conditions on $(f, u_0)$ and the compatibility condition (C.4a).

Construction of $u^2$. From (C.6) it follows that $\gamma_l^i g_j - \gamma_B \Delta^j u_i = 0$ for all $i, j \in \mathbb{N}_0$ with $j \leq k - 1, i \leq l + k - j$. Thus $g_j - \gamma_B \Delta^j u^1$ belongs to $\gamma_B^0 \mathbb{E}^{l+k-j}$. Near $\Gamma$ we can split the Laplacian into $\Delta = \Delta_\Gamma + H_\Gamma \partial_\nu + \Delta_\nu^2$ where the Laplace-Beltrami operator $\Delta_\Gamma = \text{div}_\Gamma \nabla_\Gamma$ and some $H_\Gamma \in \mathcal{C}^\infty(\Gamma)$. The operator $\Delta_\Gamma$ commutes with $\partial_\nu$, since it only depends on tangential derivatives. Therefore the normal traces $h_j := \langle \partial_\nu^j u^2 \rangle |_{\Gamma}$ ($j \in \{0, \ldots, 2k - 1\}$) of the desired solution $u^2 \in \mathbb{E}^{l,k}$ are uniquely determined by requiring that $h_{2j+j_b+1} = 0$ and
\[
\gamma_B(\Delta_\Gamma + H_\Gamma \partial_\nu + \Delta_\nu^2) u^2 = g_j - \gamma_B \Delta^j u^1 \quad \text{for } 0 \leq j \leq k - 1.
\]
With the co-retraction $B_{l,k}$ from Lemma $[\text{C.6}]$ we define $u^2 := B_{l,k}(h_0, \ldots, h_{2k-1})$ which depends linearly and continuously on $(f, g, u_0)$ with respect to the norms induced by (C.3) and (C.4).

Construction of $u^3$. Finally, we shall construct a function
\[
u^3 \in W^{1+k}_{p}(\mathbb{R}^+; L_p(\Omega)) \cap_0 W^{l}_{p}(\mathbb{R}^+; \mathcal{D}(\Delta_B^l)),
\]
which solves the equation \((\partial_t + \mu_B - \Delta)u^3 = f^3 := f - f^1 - f^2\), where \(f^m := (\partial_t + \mu_B - \Delta)u^m\) for \(m \in \{1, 2\}\). The compatibility conditions yield \(\gamma_B \Delta^j f = 0\) for \(0 \leq j \leq k - 2\) and thus
\[
f^3 \in 0 W^{l+k-1}_p(J; L_p(\Omega)) \cap 0 W^l_p(J; D(\Delta^{k-1}_B)).
\]
Hence \(u^3 = (\partial_t + \mu_B - \Delta)^{-1} f^3\) is well-defined by Lemma C.5 and the map \(f^3 \mapsto u^3\) is continuous. The proof of Theorem C.4 is complete.

Proof of Corollary C.8. With the same arguments as above and Theorem A.7 we see that \(\mu_N - \Delta_N\) has maximal regularity of type \(L_p(\mathbb{R}_+; L_{p,0}(\Omega))\). Hence problem (C.1) has at most one solution within the space \(E^{l,k}_0\). Analogously as for Lemma C.5 we conclude that
\[
\partial_t + \mu_N - \Delta_N: 0 W^{l+k}_p(\mathbb{R}_+; L_{p,0}(\Omega)) \cap 0 W^l_p(\mathbb{R}_+; D(\Delta^k_N)) \rightarrow 0 W^{l+k-1}_p(\mathbb{R}_+; L_{p,0}(\Omega)) \cap 0 W^l_p(\mathbb{R}_+; D(\Delta^{k-1}_N))
\]
is a topological linear isomorphism. For the proof of existence, we modify the above construction of the solution \(u = u^1 + u^2 + u^3\). For \(i \in \{1, 2\}\) we may replace \(u^i\) by \(u^i - \bar{u}^i \in E^{l,k}_0\), since \(\partial_t \bar{u}^i = 0\) and \(\bar{u}^i(0) = \bar{u} = 0\). Then we obtain \(\bar{f}^3(t) = \bar{f}(t) + |\Gamma| |\Omega|^{-1} \bar{g}(t) = 0\). Hence \(u^3 := (\partial_t + \mu_N - \Delta)^{-1} f^3\) is well-defined in \(E^{l,k}_0\). □

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