Some Processes Associated with Fractional Bessel Processes

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Abstract

Let \( B = \{ (B_1^i, \ldots, B_d^i), t \geq 0 \} \) be a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \) and let \( R_t = \sqrt{(B_1^1)^2 + \cdots + (B_d^d)^2} \) be the fractional Bessel process. Itô’s formula for the fractional Brownian motion leads to the equation

\[
R_t = \sum_{i=1}^d \int_0^t B_i^i \frac{dB_i^i}{R_s} + H(d - 1) \int_0^t \frac{s^{2H-1}}{R_s} ds
\]

In the Brownian motion case (\( H = 1/2 \)), \( X_t = \sum_{i=1}^d \int_0^t B_i^i dB_i^i \) is a Brownian motion. In this paper it is shown that \( X_t \) is not a fractional Brownian motion if \( H \neq 1/2 \). We will study some other properties of this stochastic process as well.

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1 Introduction

Let \( B = \{(B^1_t, \ldots, B^d_t), t \geq 0\} \) be a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). That is, the components of \( B \) are independent one-dimensional fractional Brownian motions with Hurst parameter \( H \in (0, 1) \).

Denote the fractional Bessel process by \( R_t = \sqrt{(B^1_t)^2 + \cdots + (B^d_t)^2} \). In the standard Brownian motion case there is an extensive literature on this process, see for example [7]. It is natural and interesting to study this process for any other parameter \( H \). If \( d \geq 2 \) and \( 1/2 < H < 1 \), using the Itô’s formula for the fractional Brownian motion we obtain

\[
R_t = \sum_{i=1}^{d} \int_{0}^{t} \frac{B^i_s}{R_s} dB^i_s + H(d - 1) \int_{0}^{t} \frac{s^{2H-1}}{R_s} ds \tag{1}
\]

and for \( d = 1 \) we have

\[
|B_t| = \int_{0}^{t} \text{sign}(B_t) dB_t + H \int_{0}^{t} \delta_0(B_s)s^{2H-1} ds, \tag{2}
\]

where \( \delta_0 \) is the Dirac delta function, and the stochastic integrals are interpreted in the divergence form. Equation (1) have been proved in [1] in the case \( H > 1/2 \), and for Equation (2) we refer to [1], [4], [5] and [6].

In the classical Brownian motion case it is well-known from the Lévy’s characterization theorem that the first term in the decomposition (1)

\[
X_t = \begin{cases} 
\sum_{i=1}^{d} \int_{0}^{t} \frac{B^i_s}{R_s} dB^i_s & \text{when } d \geq 2 \\
\int_{0}^{t} \text{sign}(B_t) dB_t & \text{when } d = 1
\end{cases}
\]

is a classical Brownian motion. It is then natural and interesting to ask whether for any other \( H \), the process \( X = \{X_t, t \geq 0\} \) is a fractional Brownian motion or not. The difficulty is that there is no characterization as convenient as Lévy’s one for general fractional Brownian motion (\( H \neq 1/2 \)). It is then difficult to show whether a stochastic process is a fractional Brownian motion or not. In this paper, we shall prove that if \( H \neq 1/2 \), then \( \{X_t, t \geq 0\} \) is NOT a fractional Brownian motion. Our approach to show this fact is based on the Wiener chaos expansion (see for example [3] and [5]).
It seems to be the natural method to be used here since there is no other powerful tool available.

Although \( \{X_t, t \geq 0\} \) is not a fractional Brownian motion, it enjoys some properties that the fractional Brownian motion has, such as self-similarity and long range dependence \((H > 2/3)\). We will study these and some other properties of the process \(X\).

Section 2 will recall some preliminary results. Section 3 will study the case \(d = 1\), namely, the process \(\int_0^t \text{sign}(B_t)dB_t\) and Section 4 is devoted to the study of general dimension, i.e., the process \(\sum_{i=1}^d \int_0^t \frac{B_i}{\sigma_i} dB_i\).

2 Preliminaries

Let \( B = \{B_t, t \geq 0\} \) be a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). That is, \( B \) is a zero mean Gaussian process with the covariance function

\[
R_H(t, s) = E(B_tB_s) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t-s|^{2H} \right).
\]

We denote by \( K_H(t, s) \) the square integrable kernel such that

\[
R_H(t, s) = \int_0^{t \wedge s} K_H(t, u)K_H(s, u)du.
\]

Fix a time interval \([0, T]\), and let \( \mathcal{H} \) be Hilbert space defined as the closure of the set of step functions on \([0, T]\) with respect to the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]

The mapping \( 1_{[0,t]} \rightarrow B_t \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space \( H_1(B) \) associated with \( B \). We will denote this isometry by \( \varphi \rightarrow B(\varphi) \).

The operator defined by

\[
(K_H^*1_{[0,t]})(s) = K_H(t, s)1_{[0,t]}(s).
\]

can be extended to a linear isometry between \( \mathcal{H} \) and \( L^2(0, T) \). This operator can be expressed in terms of fractional operators. More precisely, if \( H > \frac{1}{2} \) we have

\[
(K_H^*\varphi)(s) = c_H \Gamma \left( H - \frac{1}{2} \right) s^{rac{1}{2} - H} \left( I_{T -}^{H - \frac{1}{2}} u^{H - \frac{1}{2}} \varphi(u) \right)(s)
\]
and if $H < \frac{1}{2}$

$$(K_H^* \varphi)(s) = c_H \Gamma(H + \frac{1}{2}) s^{\frac{1}{2} - H} (D_{T-}^H u^{H - \frac{1}{2}} \varphi(u))(s),$$

where $c_H = \sqrt{\frac{2H}{(1-2H)(1-2H)H + 1/2}}$, and for any $\alpha > 0$ we denote by $I_{T-}^\alpha$ (resp. $D_{T-}^\alpha$) the fractional integral (resp. derivative) operator given by

$$I_{T-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) \, ds$$

(resp. $D_{T-}^\alpha f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} \, ds \right)$).

We denote by $D$ and $\delta$ the derivative and divergence operators that can be defined in the framework of the Malliavin calculus with respect to the process $B$. Let $\mathbb{D}^{k,p}$, $p > 1$, $k \in \mathbb{R}$, be the corresponding Sobolev spaces. We recall that the divergence operator $\delta$ is defined by means of the duality relationship

$$\mathbb{E}(F \delta(u)) = \mathbb{E} \langle DF, u \rangle_{\mathcal{H}},$$

where $u$ is a random variable in $L^2(\Omega; \mathcal{H})$. We say that $u$ belongs to the domain of the divergence, denoted by Dom $\delta$, if there is a square integrable random variable $\delta(u)$ such that (3) holds for any $F \in \mathbb{D}^{1,2}$.

The domain of the divergence operator is sometimes too small. For instance, in [1] it is proved that the process $u = B$ belongs to $L^2(\Omega; \mathcal{H})$ if and only if $H > \frac{1}{4}$. On the other hand, in [2] it is proved that for all $t \geq 0$, the process $\text{sign}(B_t)$ belongs to the domain of the divergence when $H > \frac{1}{3}$.

Following the approach of [1] it is possible to extend the domain of the divergence operator to processes whose trajectories are not necessarily in the space $\mathcal{H}$. Set $\mathcal{H}_2 = (K_H^*)^{-1} (K_H^{*,a})^{-1} (L^2(0,T))$, where $K_H^{*,a}$ denotes the adjoint of the operator $K_H^*$. Denote by $\mathcal{S}_\mathcal{H}$ the space of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)),$$

where $n \geq 1$, $f \in C_6^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}_2$. 

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Definition 1 Let \( u = \{u_t, t \in [0, T]\} \) be a measurable process such that
\[
\mathbb{E} \left( \int_0^T u_t^2 dt \right) < \infty.
\]
We say that \( u \in \text{Dom}^*_T \delta \) (extended domain of the divergence in \( [0, T] \)) if there exists a random variable \( \delta(u) \in L^2(\Omega) \) such that for all \( F \in \mathcal{S}_H \) we have
\[
\int_0^T \mathbb{E}(u_t K_H^{*, u} K_H^* D_t F) dt = \mathbb{E}(\delta(u) F).
\]

In [1] it is proved that for any \( H \in (0, 1) \), the process \( \text{sign}(B_t) \) belongs to the extended domain of the divergence in any time interval \( [0, T] \) and the following version of Tanaka’s formula holds
\[
|B_t| = \int_0^t \text{sign}(B_s) dB_s + H \int_0^t \delta_0(B_s) s^{2H-1} ds. \tag{5}
\]
(see also [5], [6] for this and a more general formula). In this formula \( L_t^a = H \int_0^t \delta_0(B_s) s^{2H-1} ds \) is the the density of the occupation measure
\[
\Gamma \mapsto 2H \int_0^t 1_\Gamma(B_s) s^{2H-1} ds.
\]

3 The process \( \int_0^t \text{sign}(B_t) dB_t \)

Define the process \( X = \{X_t, t \geq 0\} \), by
\[
X_t = \int_0^t \text{sign}(B_s) dB_s.
\]

In the case of the classical Brownian motion \((H = \frac{1}{2})\), the process \( X \) turns out to be a Brownian motion. We will show first that for any \( H \in (0, 1) \), \( X \) is a \( H \)-self-similar process, that is, for all \( a > 0 \) the processes \( \{X_{at}, t \geq 0\} \) and \( \{a^{H}X_t, t \geq 0\} \) have the same law.

Proposition 2 The process \( X = \{X_t, t \geq 0\} \) is \( H \)-self-similar.
Proof. Using the self-similarity property of the fractional Brownian motion and Tanaka’s formula \((5)\) yields that for any \(a > 0\)

\[
X_{at} = |B_{at}| - H \int_0^{at} \delta_0(B_s)s^{2H-1}ds
\]

\[
= |B_{at}| - H \int_0^{t} \delta_0(B_{au})(au)^{2H-1}adu
\]

\[
\overset{d}{=} a^H |B_t| - a^{2H} H \int_0^{t} \delta_0(a^H B_u)u^{2H-1}du
\]

\[
= a^H X_t,
\]

where the symbol \(\overset{d}{=}\) means that the distributions of both processes are the same. This completes the proof. ■

Then, it is natural to conjecture that for any \(H\), the process \(X_t\) is a fractional Brownian motion of Hurst parameter \(H\). We will see that this is no longer true if \(H \neq \frac{1}{2}\), although the process \(X_t\) shares some of the properties of the fractional Brownian motion.

Let us first find the Wiener chaos expansion of the process \(\text{sign}(B_t)\). We will denote by \(I_n\) the multiple Wiener integral with respect to the process \(B_t\).

Lemma 3 Let \(0 < H < 1\). We have the following chaos expansion for \(\text{sign}(B_t)\):

\[
\text{sign}(B_t) = \sum_{k=0}^{\infty} b_{2k+1} I_{2k+1}(1),
\]

where

\[
b_{2k+1} = \frac{2(-1)^k}{(2k+1)\sqrt{2\pi t}(2k+1)H}.
\]

Proof. Denote by \(p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/\varepsilon}, x \in \mathbb{R}, \varepsilon > 0\), the heat kernel. The function

\[
f_\varepsilon(x) = 2 \int_{-\infty}^{x} p_\varepsilon(y)dy - 1
\]

converges to \(\text{sign}(x)\) as \(\varepsilon\) tends to zero. Hence, \(f_\varepsilon(B_t)\) converges to \(\text{sign}(B_t)\) in \(L^2(\Omega)\) as \(\varepsilon\) tends to zero. The application of Stroock’s formula yields

\[
f_\varepsilon(B_t) = \sum_{n=0}^{\infty} a_n^\varepsilon(t) \int_{0<s_1<\cdots<s_n<t} dB_{s_1} \cdots dB_{s_n},
\]

(7)
where
\[
 a_n^\varepsilon(t) = \mathbb{E} [D^n(f_\varepsilon(B_t))] = 2\mathbb{E} [p_\varepsilon^{(n-1)}(B_t)] \\
= 2(-1)^{n-1} \frac{\partial^{n-1}}{\partial y^{n-1}} \mathbb{E} [p_\varepsilon(B_t-y)] |_{y=0} \\
= 2(-1)^{n-1} p_\varepsilon^{(n-1)}(0).
\]

Taking the limit of (7) in $L^2(\Omega)$ as $\varepsilon$ tends to zero we obtain
\[
\text{sign}(B_t) = \sum_{n=0}^{\infty} a_n(t) \int_{0< s_1< \cdots< s_n< t} dB_{s_1} \cdots dB_{s_n},
\]
where $a_n(t) = \lim_{\varepsilon \downarrow 0} a_n^\varepsilon(t) = 2(-1)^{n-1} p_\varepsilon^{(n-1)}(0)$. As a consequence, $a_n(t) = 0$ if $n$ is even and
\[
a_n(t) = \frac{2(-1)^k(2k)!}{\sqrt{2\pi} t^{nH} k! 2^k}
\]
if $n = 2k + 1$. \[\blacksquare\]

Using Stirling’s formula we obtain
\[
\mathbb{E} [I_{2k+1}(b_{2k+1})]^2 = \frac{4(2k+1)! t^{2H+1}}{(2k+1)2\pi t^{2H+1} (k! 2^k)^2} \\
= \frac{4(2k)!}{(2k+1)2\pi (k! 2^k)^2} \\
\lesssim C k^{-3/2},
\]
and we have proved the following proposition.

**Proposition 4** For any $0 < H < 1$, the random variable $\text{sign}(B_t)$ belongs to the Sobolev space $\mathbb{D}^{\alpha,2}$ for any $\alpha < \frac{1}{2}$.

Now it is easy to obtain the chaos expansion of $\int_0^t \text{sign}(B_s) dB_s$.

**Proposition 5** For any $0 < H < 1$,
\[
\int_0^t \text{sign}(B_s) dB_s = \sum_{k=1}^{\infty} c_k I_{2k}(h_{2k}),
\]
where
\[ c_k = \frac{(-1)^{k-1}}{\sqrt{2\pi(2k-1)(k-1)!}2^{k-2}} \]
and
\[ h_{2k}(s_1, \ldots, s_{2k}) = (s_1 \lor s_2 \lor \cdots \lor s_{2k})^{-(2k-1)H} \].

A consequence of this proposition is the following

**Proposition 6** For any \( 0 < H < 1 \) and \( t > 0 \), the random variable \( \int_0^t \text{sign}(B_s)dB_s \) belongs to the Sobolev space \( \mathbb{D}^{\alpha,2} \) for any \( \alpha < 1/2 \).

**Proof.** It is easy to check that there is a constant \( C > 0 \) such that
\[
\mathbb{E} [I_{2k}(h_{2k})]^2 \leq C \frac{(2k)!}{(2k-1)^2 [(k-1)!]^2} 2^{2k}.
\]
Therefore
\[
\mathbb{E} [c_k I_{2k}(h_{2k})]^2 \leq C \frac{(2k)!}{(k!2k)^2} \leq Ck^{-3/2}.
\]
This proves the proposition. 

The next proposition states that \( \int_0^t \text{sign}(B_t)dB_t \) is not a fractional Brownian motion.

**Proposition 7** The process \( X = \{X_t, t \geq 0\} \) is not a fractional Brownian motion.

**Proof.** Suppose that \( X \) is a fractional Brownian motion. Then it is a fractional Brownian motion with Hurst parameter \( H \) since it is self-similar with parameter \( H \). Then, the process
\[ Y_t = \int_0^t \eta_H(t,r) dX_r \]
must be a standard Brownian motion with respect to the filtration generated by \( X \), where
\[ \eta_H(t,r) = (K_H^*)^{-1} (1_{[0,t]})(r). \]
We claim that

\[ Y_t = Z_t, \quad (8) \]

where

\[ Z_t = \int_0^t \eta_H(t, r) \text{sign}(B_r) dB_r. \quad (9) \]

In fact, set \( t^n_k = \frac{ tk}{n}, \ k = 0, \ldots, n \), and consider the approximations

\[ Y^n_t = \sum_{k=1}^n \eta_H(t, t^n_{k-1}) \left( X^n_{t^n_k} - X^n_{t^n_{k-1}} \right). \]

We know that \( Y^n_t \) converges in \( L^2(\Omega) \) to \( Y_t \), because the functions

\[ \sum_{k=1}^n \eta_H(t, t^n_{k-1}) 1_{[t^n_{k-1}, t^n_k)}(r) \]

converge to \( \eta_H(t, r) 1_{[0,t)}(r) \) in the norm of the Hilbert space \( \mathcal{H} \). On the other hand, by Definition 1, for any smooth and cylindrical random variable \( F \in \mathcal{S}_\mathcal{H} \) we have

\[
\mathbb{E}(F Y^n_t) = \mathbb{E} \left( \left< D_r F, \sum_{k=1}^n \eta_H(t, t^n_{k-1}) 1_{[t^n_{k-1}, t^n_k)}(r) \text{sign}(B_r) \right>_{\mathcal{H}} \right) \\
= \mathbb{E} \left( \left< \Gamma_{H,T}^*, \Gamma_{H,T} D_r F, \sum_{k=1}^n \eta_H(t, t^n_{k-1}) 1_{[t^n_{k-1}, t^n_k)}(r) \text{sign}(B_r) \right>_{L^2(0,T)} \right).
\]

As before this converges to

\[
\mathbb{E} \left( \left< K^*, \eta_H(t, r) 1_{[0,t)}(r) \text{sign}(B_r) \right>_{L^2(0,T)} \right) \\
= \mathbb{E}(F Z_t),
\]

as \( n \) tends to infinity. So (8) holds.

We can write, using Lemma 3

\[ Z_t = \sum_{k=0}^{\infty} b_k \int_0^t \eta_H(t, r) r^{-(2k+1)} I_{2k+1}(1_{[0,r]}(dB_r), \quad \text{(10)} \]

where

\[ b_k = \frac{(-1)^k}{\sqrt{2\pi}(2k+1)k!2^{k-1}}. \]
So,
\[ Z_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}(f_{2k+2}), \]
where
\[
\begin{align*}
  f_{2k+2}(t, s_1, \ldots, s_{2k+2}) &= \text{symm} \left( \eta_H(t, s_{2k+2}) I_{2k+2}^{-(2k+1)H} 1_{[0,2k+2]}(s_1) \cdots 1_{[0,2k+2]}(s_{2k+1}) \right) \\
  &= \frac{1}{2k+1} \eta_H(t, s_1 \vee \cdots \vee s_{2k+2}) (s_1 \vee \cdots \vee s_{2k+2})^{-(2k+1)H},
\end{align*}
\]
and \( I_{2k+2,t}(f) \) denotes \( I_{2k+2}^{(2k+2)}(f) \). We can transform these multiple stochastic integrals into integrals with respect to a standard Brownian motion, using the operator \( K_H^* \). In this way we obtain
\[
I_{2k+2,t}(f_{2k+2}) = I_{2k+2,t}^{W} (K_H^{*\otimes(2k+2)} f_{2k+2}),
\]
and the process
\[
Z_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}^{W} (K_H^{*\otimes(2k+2)} f_{2k+2})
\]
is a Brownian motion with respect to the filtration generated by \( W \). Hence, every component of the chaos expansion is a martingale with respect to the filtration generated by \( W \). In particular, this implies that the coefficient of the second chaos \( K_H^{*\otimes2} f_2(t, s_1, s_2) \) must not depend on \( t \).

For \( H > \frac{1}{2} \) we have
\[
K_H^{*\otimes2} f_2(t, s_1, s_2) = d_H^2 (s_1 s_2)^{\frac{1}{2} - H} \times \left[ I_{t-}^{(H-\frac{1}{2})\otimes2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2)
\]
where \( d_H = c_H \Gamma(H - \frac{1}{2}) \). We have used the fact that
\[
\left( I_{t-}^{\alpha} f \right) 1_{[0,t]} = I_{t-}^{\alpha} \left( f 1_{[0,t]} \right).
\]
Then
\[
\begin{align*}
  &\left[ I_{t-}^{(H-\frac{1}{2})\otimes2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2) \\
  &= \frac{1}{\Gamma(H - \frac{1}{2})^2} \int_{s_2}^{t} \int_{s_1}^{t} \left( (u_1 - s_1) (u_2 - s_2) \right)^{-\frac{1}{2}} \\
  &\quad \times \eta_H(t, u_1 - s_1) \vee (u_2 - s_2) du_1 du_2.
\end{align*}
\]
Taking $t = \max(s_1, s_2)$, we would have $K^{\otimes 2}_H f_2(t, s_1, s_2) = 0$, because

$$\eta_H(t, r) \leq C t^{H - \frac{1}{2}} r^{H - \frac{1}{2}} (t - r)^{\frac{1}{2} - H}. $$

Hence, $\eta_H(t, u_1 \vee u_2) = 0$, which leads to a contradiction.

Suppose now that $H < \frac{1}{2}$. In this case we have

$$K^{\otimes 2}_H f_2(t, s_1, s_2) = e_H^2 (s_1 s_2)^{\frac{1}{2} - H}
\times \left[ D_L \left( \left( \frac{1}{2} - H \right)^\otimes (u_1 u_2) \right) \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2),$$

where $e_H = c_H \Gamma(H + \frac{1}{2})$, using again that $(D^\alpha f) 1_{[0,t]} = D^\alpha_T f 1_{[0,t]}$. Notice that

$$\eta_H(t, r) = \frac{1}{e_H \Gamma(\frac{1}{2} - H)} r^{\frac{1}{2} - H} \int_r^t (y - r)^{-\frac{1}{2} - H} y^{H - \frac{1}{2}} dy.$$

As a consequence, $\eta_H(t, r)$ behaves as $C r^{\frac{1}{2} - H} (t - r)^{\frac{1}{2} - H}$. We have

$$\left[ D_L \left( \left( \frac{1}{2} - H \right)^\otimes (u_1 u_2) \right) \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2)
= \frac{1}{\Gamma(H + \frac{1}{2})^2} D^\frac{1}{2} \left( \frac{(s_1 s_2)^{\frac{1}{2} - H} \eta_H(t, s_1 \vee s_2)}{(t - s_1)^{\frac{1}{2} - H} (t - s_2)^{\frac{1}{2} - H}}
+ \left( \frac{1}{2} - H \right) \int_{s_1}^t \frac{(s_1 s_2)^{\frac{1}{2} - H} \eta_H(t, s_1 \vee s_2) - (y s_2)^{\frac{1}{2} - H} \eta_H(t, y \vee s_2)}{(y - s_1)^{\frac{1}{2} - H} (t - s_2)^{\frac{1}{2} - H}} dy
+ \left( \frac{1}{2} - H \right) \int_{s_2}^t \frac{(s_1 s_2)^{\frac{1}{2} - H} \eta_H(t, s_1 \vee s_2) - (y s_1)^{\frac{1}{2} - H} \eta_H(t, y \vee s_1)}{(y - s_2)^{\frac{1}{2} - H} (t - s_1)^{\frac{1}{2} - H}} dy
+ \left( \frac{1}{2} - H \right) \int_{s_1}^t \int_{s_2}^t \frac{(y - s_1)^{\frac{1}{2} - H} (z - s_2)^{\frac{1}{2} - H} (s_1 s_2)^{\frac{1}{2}} \eta_H(t, s_1 \vee s_2) - (y s_2)^{\frac{1}{2} - H} \eta_H(t, y \vee s_2)}{(y - s_2)^{\frac{1}{2} - H} (t - s_1)^{\frac{1}{2} - H}} dy dz
- (y s_2)^{\frac{1}{2} - H} \eta_H(t, y \vee s_2) - (z s_1)^{\frac{1}{2} - H} \eta_H(t, z \vee s_1) + (y z)^{\frac{1}{2} - H} \eta_H(t, y \vee z) \right] dy dz. $$

Taking again $t = \max(s_1, s_2)$, we would have $K^{\otimes 2}_H f_2(t, s_1, s_2) = 0$ which leads to a contradiction. 

Consider the covariance between two increments of the process $X$:

$$r(n) := \mathbb{E} \left[ (X_{a+1} - X_a) (X_{n+1} - X_n) \right],$$

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where $0 < a \leq n$. We say that $X$ is long-range dependent if for any $a > 0$,
\[
\sum_{n \geq a} |r(n)| = \infty.
\]

The next proposition studies the long-range dependence properties of the process $X$. We see that this property differs from that of fractional Brownian motion.

**Proposition 8** Let $X_t = \int_0^t \text{sign}(B_s) dB_s$. If $H \geq 2/3$, then $X_t$ is long-range dependent and if $1/2 < H < 2/3$, then $X_t$ is not long-range dependent.

**Proof.** From Lemma, we can deduce the Wiener chaos expansion of the random variable $X_t$. In fact, we have
\[
X_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}(h_{2k+2}),
\]
where $b_k$ is defined in (10) and
\[
h_{2k+2}(s_1, \ldots, s_{2k+2}) = (s_1 \lor \cdots \lor s_{2k+2})^{-(2k+1)H}.
\]

Let us compute the covariance of $X_s$ and $X_t - X_r$, where $0 < r < t$. From the Itô isometry of multiple stochastic integrals it follows that
\[
\mathbb{E} \left[ X_s (X_t - X_r) \right] = \sum_{k=0}^{\infty} b_k^2 \mathbb{E} \left[ I_{2k+2,s}(h_{2k+2}) \left( I_{2k+2,t}(h_{2k+2}) - I_{2k+2,r}(h_{2k+2}) \right) \right].
\]
We have
\[
\mathbb{E} \left[ X_s (X_t - X_r) \right] = \sum_{k=0}^{\infty} b_k^2 (2k+2)! \left\langle h_{2k+2} \mathbf{1}_{[0,s]}, h_{2k+2} \left( \mathbf{1}_{[0,t]} - \mathbf{1}_{[0,r]} \right) \right\rangle_{H^{2k+2}}
\geq 2b_0^2 \left\langle h_0 \mathbf{1}_{[0,s]}, h_2 \left( \mathbf{1}_{[0,t]} - \mathbf{1}_{[0,r]} \right) \right\rangle_{H^2}
\geq \frac{b_0^2}{2} \int_r^t \int_0^s \int_0^s \int_0^s \int_0^s \int_0^s \phi(s_1, t_1) \phi(s_2, t_2) ds_1 ds_2 dt_1 dt_2,
\]
where $\phi(s, t) = H(2H - 1)|t - s|^{2H-2}$. 

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Thus let $s = 1$, $r = n$, and $t = n + 1$ and we have

$$r(n) := \mathbb{E}[(X_{a+1} - X_a)(X_{n+1} - X_n)]$$

$$\geq C \int_{n+1}^{n+1} \int_{n}^{n+1} \int_{a}^{a+1} s_2^{-H} t_2^{-H} (t_1 - s_1)^{2H-2} (t_2 - s_2)^{2H-2} ds_1 ds_2 dt_1 dt_2$$

$$\geq C \int_{n}^{n+1} \int_{n+1}^{n+1} \int_{a}^{a+1} s_2^{-H} t_2^{-H} (t_1 - a)^{2H-2} (t_2 - a - 1)^{2H-2} ds_1 ds_2 dt_1 dt_2$$

$$\approx C n^{3H-3},$$

as $n$ tends to infinity. Thus if $H \geq 2/3$, $\sum_{n \geq a} r(n) = \infty$.

If $H < 2/3$, then we use another approach. Set, as before

$$r(n) = \mathbb{E} \left[ \left( \int_{a}^{a+1} \text{sign} B_t dB_t \right) \left( \int_{n}^{n+1} \text{sign} B_t dB_t \right) \right].$$

Using the formula for the expectation of the product of two divergence integrals we obtain

$$r(n) = \alpha_H \int_{a}^{a+1} \int_{n}^{n+1} \mathbb{E} (\text{sign} B_s \text{sign} B_t) |s - t|^{2H-2} ds dt$$

$$+ 4 \alpha_H^2 \int_{a}^{a+1} \int_{n}^{n+1} \int_{0}^{s} \int_{0}^{t} \mathbb{E} (\delta_0(B_s) \delta_0(B_t))$$

$$\times |s - \sigma|^{2H-2} |\theta - t|^{2H-2} d\sigma d\theta ds dt,$$

where $\alpha_H = H(2H - 1)$. This formula can be proved by approximating the function sign$(x)$ by smooth functions and then taking the limit in $L^2(\Omega)$. We have

$$\int_{0}^{t} |s - \sigma|^{2H-2} d\sigma = \frac{1}{2H - 1} (s^{2H-1} + |s - t|^{2H-1} \text{sign}(t - s)).$$

Hence,

$$r(n) = \alpha_H \int_{a}^{a+1} \int_{n}^{n+1} \mathbb{E} (\text{sign} B_s \text{sign} B_t) |s - t|^{2H-2} ds dt$$

$$+ 4 \alpha_H^2 \int_{a}^{a+1} \int_{n}^{n+1} \mathbb{E} (\delta_0(B_s) \delta_0(B_t))$$

$$\times (s^{2H-1} + (t - s)^{2H-1}) (t^{2H-1} - (t - s)^{2H-1}) ds dt$$

$$= a_n + b_n.$$
For the second term we have

\[ b_n = \frac{4H^2}{2\pi} \int_a^{a+1} \int_n^{n+1} \frac{(s^{2H-1} + (t-s)^{2H-1})(t^{2H-1} - (t-s)^{2H-1})}{[(st)^{2H} - \frac{1}{4} (t^{2H} + s^{2H} - |t-s|^{2H})^2]^{1/2}} dsdt \]

Therefore,

\[ b_n \leq \frac{4H^2(2H-1)}{2\pi} \left[ \frac{(a+1)^{2H-1} + (n+1)^{2H-1}}{(an)^{2H} - \frac{1}{4} ((n+1)^{2H} + (a+1)^{2H} - |n-a|^{2H})^2} \right]^{1/2} \leq Cn^{3H-3}. \]

To estimate \( a_n \), we have from (4)

\[ \mathbb{E} [\text{sign}(B_u)\text{sign}(B_v)] = \sum_{k=0}^{\infty} \frac{4(2k)!}{(2k+1)^2 \pi (uv)^{2k+1}} (u^{2H} + v^{2H} - |u-v|^{2H})^{2k+1} \leq C \int_a^{a+1} \int_n^{n+1} |u-v|^{2H-2} \frac{u^{2H} + v^{2H} - |u-v|^{2H}}{(uv)^H} dvdu \leq C_1 \frac{u^{2H} + v^{2H} - |u-v|^{2H}}{(uv)^H}. \]

Therefore

\[ a_n \leq C_2 \int_a^{a+1} \int_n^{n+1} |u-v|^{2H-2} u^{2H} + v^{2H} - |u-v|^{2H} dvdu \leq C_3 n^{3H-3}. \]

As a consequence, if \( H < 2/3 \), then \( \sum_{n \geq 1} r(n) < \infty. \)

4 General Dimension

In this section we consider a \( d \)-dimensional fractional Brownian motion \( B = \{ (B^1_t, \ldots, B^d_t), t \geq 0 \} \), with Hurst parameter \( H > \frac{1}{2} \). Let \( R_t = |B_t| \) be the fractional Bessel process associated to the \( d \)-dimensional fBm \( B \).

Suppose first that \( H > \frac{1}{2} \). Fix a time interval \([0, T]\), and define the derivative and divergence operators, \( D^{(i)} \) and \( \delta^{(i)} \), with respect to each component.
$B^{(i)}$, as in Section 2. We assume that the Sobolev spaces $\mathbb{D}^{1,p}_i$ include functionals of all the components of $B$ and not only of component $i$. For each $p > 1$, let $\mathbb{L}^{1,p}_{H,i}$ be the set of processes $u \in \mathbb{D}^{1,p}_i(\mathcal{H})$ such that

$$E\left[\|u\|_{L^{1/H}([0,T])}^p\right] + E\left[\|\mathcal{D}^{(i)}u\|_{L^{1/H}([0,T]^2)}^p\right] < \infty.$$ 

It has been proved in [1] that $\left\{\frac{R^i_s}{R_s}, s \in [0,T]\right\}$ belongs to the space $\mathbb{L}^{1,1/H}_{H,i}$ for each $i = 1, \ldots, d$ and

$$R_t = \sum_{i=1}^{d} \int_0^t \frac{B^i_s}{R_s} dB^i_s + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds. \quad (11)$$

In the case $H < \frac{1}{2}$ the following result holds.

**Proposition 9** If $H < \frac{1}{2}$, the process $\frac{B^i_s}{R_s}$ belongs to the extended domain of the divergence operator $\text{Dom}^*_i \delta^i$ on any time interval $[0,t]$, and (11) holds.

**Proof.** For any test random variable $F \in \mathcal{S}_H$ we have

$$\int_0^t E\left(\frac{B^i_s}{R_s} K^*_H K^{*}_H D^{(i)} F\right) ds = \lim_{\varepsilon \to 0} \int_0^t E\left(h'_{\varepsilon}(R^2_s) B^i_s K^*_H K^{*}_H D^{(i)} F\right) ds$$

where, for any $\varepsilon > 0$,

$$h_{\varepsilon}(x) = \begin{cases} \frac{3}{8} \varepsilon \sqrt{x} + \frac{3}{4\sqrt{x}} & \text{if } x < \varepsilon \\ \frac{1}{8\varepsilon} & \text{if } x \geq \varepsilon \end{cases}.$$ 

We have $h_{\varepsilon}(x) \in C^2(\mathbb{R})$ and $\lim_{\varepsilon \to 0} h_{\varepsilon}(x) = \sqrt{x}$ for all $x \geq 0$. By Itô’s formula for the fractional Brownian motion in the case $H < \frac{1}{2}$, we have

$$h_{\varepsilon}(R^2_t) - h_{\varepsilon}(0) = \sum_{i=1}^{d} \int_0^t h'_{\varepsilon}(R^2_s) B^i_s dB^i_s + J_{\varepsilon},$$
where
\[
J_\varepsilon = H(d-1) \int_0^t 1\{R_s^2 \geq \varepsilon\} \frac{s^{2H-1}}{R_s} ds \\
+ H \int_0^t 1\{R_s^2 \leq \varepsilon\} \frac{1}{2\sqrt{\varepsilon}} \left[ 3d - (d+2) \frac{R_s^2}{\varepsilon} \right] s^{2H-1} ds.
\]

The process
\[
X_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s} B_s^i ds
\]
(12)
is $H$-self-similar, Hölder continuous of order $\alpha < H$, and it has the same $1/H$-variation as the fractional Brownian motion. Nevertheless, as we will show in the next proposition it is not a fractional Brownian motion with Hurst parameter $H$.

For $h \in \mathcal{H}^\otimes n$, we denote
\[
I_{j_1,\ldots,j_n}(h) = \int_{0< s_1,\ldots,s_n < t} h(s_1,\ldots,s_n) dB_{s_1}^{j_1} \cdots dB_{s_n}^{j_n},
\]
(13)
First we find the chaos expansion of $\sum_{i=1}^d \int_0^t f_i(B_s) dB_s^i$, where $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ are smooth functions with polynomial growth.

**Proposition 10** The following chaos expansion holds for $Z_t = \sum_{i=1}^d \int_0^t f_i(B_s) dB_s^i$
\[
Z_t = \sum_{i=1}^d \sum_{n=1}^\infty \sum_{1 \leq j_1,\ldots,j_n \leq d} I_{j_1,\ldots,j_n,i} \left( g_{j_1,\ldots,j_n}^i(s_1,\ldots,s_{n+1}) \right),
\]
where
\[
g_{j_1,\ldots,j_n}^i(s_1,\ldots,s_{n+1}) = \frac{(-1)^n (s_1 \vee \cdots \vee s_{n+1})^{-nH}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left[ \frac{\partial^n}{\partial y_{j_1} \cdots \partial y_{j_n}} e^{-|y|^2/2} \right] f_i(y(s_1 \vee \cdots \vee s_{n+1})^H) dy.
\]

**Proof.** Using Stroock’s formula yields for each $i = 1,\ldots,d$
\[
f_i(B_s) = \sum_{n=0}^\infty \sum_{1 \leq j_1,\ldots,j_n \leq d} \frac{1}{n!} I_{j_1,\ldots,j_n} \left( f_{j_1,\ldots,j_n}^i(s)^{1_\otimes n}[0,s] \right),
\]
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where
\[
f_{j_1,\ldots,j_n}(s) = \mathbb{E}(D_{j_1} \cdots D_{j_n}(f_i(B_s)))
\]
\[
= \mathbb{E}\left(\frac{\partial^n f_i}{\partial z_{j_1} \cdots \partial z_{j_n}}(B_s)\right)
\]
\[
= \frac{1}{(2\pi s^{2H})^{d/2}} \int_{\mathbb{R}^d} \frac{\partial^n f_i}{\partial z_{j_1} \cdots \partial z_{j_n}}(z)e^{-\frac{|z|^2}{2s^{2H}}} \, dz
\]
\[
= \frac{s^{-nH}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial^n f_i}{\partial y_{j_1} \cdots \partial y_{j_n}}(ys^H)e^{-\frac{|y|^2}{2s^H}} \, dy
\]
\[
= \frac{(-1)^n s^{-nH}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f_i(ys^H)\frac{\partial^n}{\partial y_{j_1} \cdots \partial y_{j_n}}e^{-\frac{|y|^2}{2s^H}} \, dy.
\]

Finally, the result follows from
\[
Z_t = \sum_{i=1}^d \int_0^t f_i(B_s)dB_s^i
\]
\[
= \sum_{i=1}^d \sum_{n=0}^{\infty} \sum_{1 \leq j_1,\ldots,j_n \leq d} I_{j_1,\ldots,j_n,d} \left(\text{symm}\left( f_{j_1,\ldots,j_n}(s) \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]}(s) \right) \right)
\]
\[
= \sum_{i=1}^d \sum_{n=0}^{\infty} \sum_{1 \leq j_1,\ldots,j_n \leq d} I_{j_1,\ldots,j_n,d} \left( f_{j_1,\ldots,j_n}(s_1 \vee \cdots \vee s_{n+1}) \prod_{i=1}^{n+1} \mathbb{1}_{[0,t]}(s_i) \right),
\]
which completes the proof of the proposition. \(\blacksquare\)

Now let \(f_i(x) = \frac{x_i}{\sqrt{x_1^2 + \cdots + x_d^2}}\). Then it is easy to check \(f_i(tx) = f_i(x)\) for all \(t > 0\). Hence, for such \(f_i\), we have
\[
g_{j_1,\ldots,j_n}(s_1, \ldots, s_{n+1}) = b_{j_1,\ldots,j_n}(s_1 \vee \cdots \vee s_{n+1})^{-nH},
\]
where
\[
b_{i,j_1,\ldots,j_n} = \frac{(-1)^n}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial^n}{\partial y_{j_1} \cdots \partial y_{j_n}}e^{-\frac{|y|^2}{2s^H}} f_i(y) \, dy.
\]

Then the chaos expansion of \(f_i(B_t)\) is given by
\[
f_i(B_t) = \sum_{n=0}^{\infty} \sum_{1 \leq j_1,\ldots,j_n \leq d} b_{i,j_1,\ldots,j_n} t^{-nH} \int_{\{0<s_1<\cdots<s_{n}<t\}} dB_{s_1}^{j_1} \cdots dB_{s_n}^{j_n},
\]

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and the chaos expansion of the divergence of this process is

\[
\int_0^t f_i(B_s)dB^i_s = \sum_{n=1}^{\infty} \sum_{1 \leq j_1, \ldots, j_n \leq d} b_{i,j_1,\ldots,j_n} \times \int_{\{0<s_1,\ldots,s_{n+1}<t\}} (s_1 \lor \cdots \lor s_{n+1})^{-nH} dB^i_{s_1} \cdots dB^i_{s_n} dB^i_{s_{n+1}}. 
\]

Using these results we can prove the following proposition.

**Proposition 11** The process \( Y = \{Y_t, t \geq 0\} \) defined in (12) is not a fractional Brownian motion.

**Proof.** If \( Y_t = \sum_{i=1}^{d} \int_0^t f_i(B_s)dB^i_s \) is a fractional Brownian motion, then as in previous section one can show that

\[
Z_t = \sum_{i=1}^{d} \int_0^t \eta_H(t,s)f_i(B_s)dB^i_s
\]

is a classical Brownian motion. But

\[
Z_t = \sum_{i=1}^{d} \sum_{n=1}^{\infty} \sum_{1 \leq j_1, \ldots, j_n \leq d} b_{i,j_1,\ldots,j_n} \int_{\{0<s_1,\ldots,s_{n+1}<t\}} h(t,s_1,\ldots,s_{n+1}) dB^i_{s_1} \cdots dB^i_{s_n} dB^i_{s_{n+1}},
\]

where

\[
h(t,s_1,\ldots,s_{n+1}) = \eta(t,s_1 \lor s_2 \lor \cdots \lor s_{n+1})(s_1 \lor s_2 \lor \cdots \lor s_{n+1})^{-nH}.
\]

In a similar way to one dimensional case, one can show that \( \{Z_t, t \geq 0\} \) is not a martingale. 

**Proposition 12** Let \( Y = \{Y_t, t \geq 0\} \) be the process defined in (12). If \( H \geq 2/3 \), then \( Y_t \) is long range dependent and if \( 1/2 < H < 2/3 \), then \( Y_t \) is not long range dependent.

**Proof.** Set

\[
\rho_n = \mathbb{E} \left[ \left( \sum_{i=1}^{d} \int_0^1 \frac{B^i_s}{|B_s|} dB^i_s \right) \left( \sum_{i=1}^{d} \int_n^{n+1} \frac{B^i_s}{|B_s|} dB^i_s \right) \right].
\]
By the formula for the expectation of the product of two divergence integrals we can write

\[
\rho_n = \sum_{i,j=1}^{d} \alpha_{H} \int_{0}^{1} \int_{n}^{n+1} \mathbb{E} \left( \frac{B_{s} B_{t}}{|B_{s}| |B_{t}|} \right) |t - s|^{2H-2} ds dt \\
+ \sum_{i,j=1}^{d} \alpha_{H}^{2} \int_{0}^{1} \int_{n}^{n+1} \int_{0}^{t} \int_{0}^{s} \mathbb{E} \left( D_{\theta} \left( \frac{B_{s}^{i}}{|B_{s}|} \right) D_{\sigma} \left( \frac{B_{t}^{j}}{|B_{t}|} \right) \right) \\
\times |\theta - t|^{2H-2} |\sigma - s|^{2H-2} d\theta d\sigma ds dt : = \rho_{1}^{n} + \rho_{2}^{n}.
\]

In order to estimate the term \( \rho_{1}^{n} \) we make use of the orthogonal decomposition

\[ B_{t} = \frac{R(t, s)}{s^{2H}} \beta_{s,t} Y, \]

where

\[ \beta_{s,t}^{2} = \frac{(st)^{2H} - R(t, s)^{2}}{s^{2H}}, \]

and \( Y \) is a \( d \)-dimensional standard normal random variable independent of \( B_{s} \). Set

\[ \lambda_{st} = \frac{R(t, s)}{\beta_{s,t}s^{2H}} = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right) s^{H} \left[ (st)^{2H} - \frac{1}{4} (t^{2H} + s^{2H} - |t - s|^{2H})^{2} \right]^{1/2} \]

As \( t \) tends to infinity and \( s \) belongs to \((0,1)\), the term \( \lambda_{st} \) behaves as \( Hs^{-2H} t^{H-1} \). Hence, by Lemma 13

\[
\mathbb{E} \left( \frac{\langle B_{s}, B_{t} \rangle}{|B_{s}| |B_{t}|} \right) = \mathbb{E} \left( \frac{\langle B_{s}, \lambda_{st} B_{s} + Y \rangle}{|B_{s}| |\lambda_{st} B_{s} + Y|} \right) \\
= \mathbb{E} \left( \frac{\langle B_{1}, s^{H} \lambda_{st} B_{1} + Y \rangle}{|B_{1}| s^{H} \lambda_{st} B_{1} + Y|} \right) \\
= s^{H} \lambda_{st} \mathbb{E} \left( \frac{|B_{1}|^{2} |Y|^{2} - \langle B_{1}, Y \rangle^{2}}{|B_{1}| |Y|^{3}} \right) + o(t^{H-1}).
\]

Hence,

\[
\mathbb{E} \left( \frac{\langle B_{s}, B_{t} \rangle}{|B_{s}| |B_{t}|} \right) \approx HC s^{-H} t^{H-1},
\]

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where
\[ C = \mathbb{E}\left( \frac{|B_1|^2 |Y|^2 - \langle B_1, Y \rangle^2}{|B_1| |Y|^3} \right) > 0. \]

This implies that the term \( \rho_1 \) behaves as \( n^{3H-3} \).

For the term \( \rho_2 \) we have
\[
\rho_2 = H^2 \sum_{i,j=1}^d \int_0^1 \int_n^{n+1} \mathbb{E}\left( \frac{\delta_{ij} - B_i^i B_s^i}{|B_s|} \right) \left( \frac{\delta_{ij} - B_t^i B_t^i}{|B_t|} \right) \times (s^{2H-1} + (t - s)^{2H-1}) (t^{2H-1} - (t - s)^{2H-1}) dsdt
\]
\[
= H^2 \int_0^1 \int_n^{n+1} \mathbb{E}\left( \frac{d}{|B_s|} - \frac{|B_t|^2}{|B_s| |B_t|^3} - \frac{|B_s|^2}{|B_s|^3 |B_t|} + \frac{\langle B_s, B_t \rangle^2}{|B_s|^3 |B_t|^3} \right) \times (s^{2H-1} + (t - s)^{2H-1}) (t^{2H-1} - (t - s)^{2H-1}) dsdt
\]
\[
= H^2 \int_0^1 \int_n^{n+1} \mathbb{E}\left( \frac{d - 2}{|B_s| |B_t|} + \frac{\langle B_s, B_t \rangle^2}{|B_s|^3 |B_t|^3} \right) \times (s^{2H-1} + (t - s)^{2H-1}) (t^{2H-1} - (t - s)^{2H-1}) dsdt.
\]

The term
\[
\mathbb{E}\left( \frac{1}{|B_s| |B_t|} \left( d - 2 + \frac{\langle B_s, B_t \rangle^2}{|B_s|^2 |B_t|^2} \right) \right)
\]
behaves as \( Kt^{-H} \) as \( t \) tends to infinity, where
\[
K = \mathbb{E}\left( \frac{1}{|B_1||Y|} \left( d - 2 + \frac{\langle B_1, Y \rangle^2}{|B_1|^2 |Y|^2} \right) \right) > 0.
\]

Hence, the term \( \rho_2 \) behaves also as \( n^{3H-3} \). This completes the proof taking into account that the constants \( C \) and \( K \) are positive. ■

**Lemma 13** Let \( X \) and \( Y \) be independent \( d \)-dimensional standard normal random variables. Then as \( \varepsilon \) tends to zero we have
\[
\mathbb{E}\left( \frac{\langle X, \varepsilon X + Y \rangle}{|X| |\varepsilon X + Y|} \right) = \varepsilon \mathbb{E}\left( \frac{|X|^2 |Y|^2 - \langle X, Y \rangle^2}{|X| |Y|^3} \right) + o(\varepsilon).
\]
Proof. We have
\[
\mathbb{E} \left( \frac{\langle X, \varepsilon X + Y \rangle}{|X||\varepsilon X + Y|} \right) = \mathbb{E} \left( \frac{\langle X, \varepsilon X + Y \rangle - \langle X, Y \rangle}{|X||\varepsilon X + Y| - |X||Y|} \right) \\
= \mathbb{E} \left( \frac{\langle X, \varepsilon X + Y \rangle |Y| - \langle X, Y \rangle |\varepsilon X + Y|}{|X||\varepsilon X + Y| |Y|} \right) \\
= \mathbb{E} \left( \frac{\varepsilon |X|^2 |Y| - \varepsilon \langle X, Y \rangle^2 / |Y| + o(\varepsilon)}{|X||\varepsilon X + Y| |Y|} \right),
\]
and that yields the desired estimation. ■

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