Quasi-shuffle products revisited

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Abstract

Quasi-shuffle products, introduced by the first author, have been useful in studying multiple zeta values and some of their analogues and generalizations. The second author, together with Kajikawa, Ohno, and Okuda, significantly extended the definition of quasi-shuffle algebras so it could be applied to multiple $q$-zeta values. This article extends some of the algebraic machinery of the first author’s original paper to the more general definition, and demonstrates how various algebraic formulas in the quasi-shuffle algebra can be obtained in a transparent way. Some applications to multiple zeta values, interpolated multiple zeta values, multiple $q$-zeta values, and multiple polylogarithms are given.

1 Introduction

This article revisits the construction of quasi-shuffle products in [7]. In [12], the construction of [7] was put in a more general setting that had two chief
advantages: (i) it simultaneously applied to multiple zeta and multiple zeta- 
star values and their extensions; and (ii) it could be applied to the $q$-series 
version of multiple zeta values studied in \cite{14}. Here we show that some of 
the algebraic machinery developed in \cite{17}, particularly the coalgebra struc-
ture and linear functions induced by formal power series (not considered in 
\cite{18}), can be carried over to the more general setting and used to make 
the calculations in the quasi-shuffle algebra, including many of \cite{18}, more 
transparent. We also describe some applications of quasi-shuffle algebras not 
considered in \cite{18}, including applications to the interpolated multiple zeta 
values introduced in \cite{16}.

The original quasi-shuffle product was inspired by the multiplication of 
multiple zeta values, i.e.,

$$\sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}},$$

(1)

with $i_1 > 1$ to insure convergence. One can associate to the series (1) the 
monomial $z_{i_1} \cdots z_{i_k}$ in the noncommuting variables $z_1, z_2, \ldots$; then we write 
the value (1) as $\zeta(z_{i_1} \cdots z_{i_k})$. For any monomials $w = z_i w'$ and $v = z_j v'$, 
define the product $w \ast v$ recursively by

$$w \ast v = z_i (w' \ast v) + z_j (w \ast v') + z_{i+j} (w' \ast v').$$

(2)

Then $\zeta(w) \zeta(v) = \zeta(w \ast v)$, where we think of $\zeta$ as a linear function on 
monomials. As we shall see in the next section, the recursive rule (2) is a 
quasi-shuffle product on monomials in the $z_i$ derived from the product $\diamond$ on 
the vector space of $z_i$’s given by $z_i \diamond z_j = z_{i+j}$.

In \cite{14} the multiple $q$-zeta values were defined as

$$\sum_{n_1 > \ldots > n_k \geq 1} \frac{q^{(i_1-1)n_1} \cdots q^{(i_k-1)n_k}}{[n_1]_q^{i_1} \cdots [n_k]_q^{i_k}},$$

(3)

where $[n]_q = 1 + q + \ldots + q^{n-1} = (1 - q^n)/(1 - q)$. If we denote (3) by 
$\zeta_q(z_{i_1} \cdots z_{i_k})$, then to have $\zeta_q(w) \zeta_q(v) = \zeta_q(w \ast v)$ the recursion (2) must 
be significantly modified: in place of $z_i \diamond z_j = z_{i+j}$ we must have

$$z_i \diamond z_j = z_{i+j} + (1 - q)z_{i+j-1}.$$

This means that to have a theory of quasi-shuffle algebras that applies to 
multiple $q$-zeta values, two restrictions in the original construction of [7]
must be removed: that the product \( a \diamond b \) of two letters be a letter, and that the operation \( \diamond \) preserve a grading. This was done in [12]. The same paper also addressed the relation between multiple zeta values (1) and the multiple zeta-star values

\[
\zeta^\ast(z_1 \cdots z_k) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}. \tag{4}
\]

This relation can be expressed in terms of a linear isomorphism (here denoted \( \Sigma \)) from the vector space of monomials in the \( z_i \)'s to itself. The function \( \Sigma \) acts on monomials as, e.g.,

\[
\Sigma(z_i z_j z_k) = z_i z_j z_k + (z_i \diamond z_j) z_k + z_i (z_j \diamond z_k) + z_i \diamond z_j \diamond z_k
\]

and then \( \zeta^*(w) = \zeta(\Sigma(w)) \). If we define analogously multiple \( q \)-zeta-star values \( \zeta_q^*(w) \), then \( \zeta_q^*(w) = \zeta_q(\Sigma(w)) \).

Important properties of \( \Sigma \) were established in [12], though some of the inductive proofs are tedious. Here we make use of two aspects of the theory developed in [7] not used in [12]. First, for any formal power series

\[
f = c_1 t + c_2 t^2 + \cdots,
\]

it is possible to define a linear function (but not necessarily an algebra homomorphism) \( \Psi_f \) from (the vector space underlying) the quasi-shuffle algebra to itself. This process respects composition (i.e., \( \Psi_{fg} = \Psi_f \Psi_g \)), and many important isomorphisms can be represented this way, e.g., \( \Sigma = \Psi_{\frac{t}{1-t}} \). Second, the quasi-shuffle algebra together with the “deconcatenation” coproduct is a Hopf algebra: in fact, it turns out that its antipode is closely related to \( \Sigma \).

This paper is organized as follows. In §2 we define the quasi-shuffle products \( \ast \) and \( \star \) on the vector space \( k\langle A \rangle \), where \( A \) is a countable set of letters, and \( kA \) is equipped with a commutative product \( \diamond \). Then in §3 we explain how to obtain linear isomorphisms from \( k\langle A \rangle \) to itself using formal power series: as noted above, this gives a useful representation of \( \Sigma \). In §4 we describe three Hopf algebras: the ordinary Hopf algebras \( (k\langle A \rangle, \ast, \Delta) \) and \( (k\langle A \rangle, \star, \Delta) \), and the infinitesimal Hopf algebra \( (k\langle A \rangle, \diamond, \tilde{\Delta}) \), where \( \Delta \) is deconcatenation, \( \tilde{\Delta}(w) = \Delta(w) - w \otimes 1 - 1 \otimes w \), and \( \diamond \) is an extension of the original operation on \( A \) to a (noncommutative) product on \( k\langle A \rangle \). Each of these Hopf algebras is associated with a representation of \( \Sigma \) via the antipode. In §5 we apply the machinery of the preceding two sections to obtain many of
the algebraic formulas of [12] (and generalizations thereof) in a transparent way. Finally, in §6 we illustrate some of these algebraic formulas (particularly Corollary 5.1 below in each case) for five different homomorphic images of quasi-shuffle algebras.

2 The quasi-shuffle products

We start with a field $k$ containing $\mathbb{Q}$, and a countable set $A$ of “letters”. We let $kA$ be the vector space with $A$ as basis, and suppose there is an associative and commutative product $\cdot$ on $kA$.

Now let $k\langle A \rangle$ be the noncommutative polynomial algebra over $A$. So $k\langle A \rangle$ is the vector space over $k$ generated by “words” (monomials) $a_1 a_2 \cdots a_n$, with $a_i \in A$: a word $w = a_1 \cdots a_n$ has length $\ell(w) = n$. (We think of 1 as the empty word, and set $\ell(1) = 0$.) Following [12], we define two $k$-bilinear products $\ast$ and $\star$ on $k\langle A \rangle$ by making 1 $\in k\langle A \rangle$ the identity element for each product, and requiring that $\ast$ and $\star$ satisfy the relations

\begin{align*}
aw \ast bv &= a(w \ast bv) + b(aw \ast v) + (a \diamond b)(w \ast v) \quad (5) \\
aw \star bv &= a(w \star bv) + b(aw \star v) - (a \diamond b)(w \ast v) \quad (6)
\end{align*}

for all $a, b \in A$ and all monomials $w, v$ in $k\langle A \rangle$. As in [7] we have the following result.

**Theorem 2.1.** If equipped with either the product $\ast$ or the product $\star$, the vector space $k\langle A \rangle$ becomes a commutative algebra.

**Proof.** We prove the result for $\ast$, as the proof for $\star$ is almost identical. It suffices to show that $\ast$ is commutative and associative. For commutativity, it is enough to show that $u_1 \ast u_2 = u_2 \ast u_1$ for words $u_1, u_2$: we proceed by induction on $\ell(u_1) + \ell(u_2)$. This is trivial if either $u_1$ or $u_2$ is empty, so write $u_1 = aw$ and $u_2 = bv$ for $a, b \in A$ and words $w, v$. Then by Eq. (5),

\[ u_1 \ast u_2 - u_2 \ast u_1 = (a \diamond b)(w \ast v) - (b \diamond a)(v \ast w), \]

and the right-hand side is zero by the induction hypothesis and the commutativity of $\diamond$.

Similarly, to prove associativity it is enough to show that $u_1 \ast (u_2 \ast u_3) = (u_1 \ast u_2) \ast u_3$ for words $u_1, u_2, u_3$, and this can be done by induction on $\ell(u_1) + \ell(u_2) + \ell(u_3)$. The required identity is trivial if any of $u_1, u_2, u_3$ is 1,
so we can write \( u_1 = aw, u_2 = bv, \) and \( u_3 = cy \) for \( a, b, c \in A \) and words \( w, v, y \). Then

\[
\begin{align*}
 u_1 \ast (u_2 \ast u_3) - (u_1 \ast u_2) \ast u_3 &= a(w \ast b(v \ast cy) + b(aw \ast (v \ast cy))) + (a \circ b)(w \ast (v \ast cy)) \\
&+ a(w \ast c(bv \ast y)) + c(aw \ast (bv \ast y)) + (a \circ c)(w \ast (bv \ast y)) \\
&+ a((w \ast bv) \ast cy) - c(a(w \ast bv) \ast y) - (a \circ c)((w \ast bv) \ast y) \\
&- b((aw \ast v) \ast cy) - c(b(aw \ast v) \ast y) - (b \circ c)((aw \ast v) \ast y) \\
&- (a \circ b)((w \ast bv) \ast cy) - c((a \circ b)(w \ast bv) \ast y) - ((a \circ b) \circ c)((w \ast bv) \ast y) = \\
a(w \ast (bv \ast cy)) + c(aw \ast (bv \ast y)) - a((w \ast bv) \ast cy) - c((aw \ast bv) \ast y) = 0,
\end{align*}
\]

by the induction hypothesis and the associativity of \( \circ \). \( \square \)

If the product \( \circ \) is identically zero, then \( \ast \) and \( \ast \) coincide with the usual shuffle product \( \shuffle \) on \( k\langle A \rangle \). We call both \( \ast \) and \( \ast \) quasi-shuffle products.

We note that \( \circ \) can be extended to a product of on all of \( k\langle A \rangle \) by defining \( 1 \circ w = w \circ 1 = w \) for all words \( w \), and \( w \circ v = w'(a \circ b)v' \) for nonempty words \( w = w'a \) and \( v = bv' \) (where \( a, b \) are letters). Then \( (k\langle A \rangle, \circ) \) is a noncommutative algebra that contains the commutative subalgebra \( k1 + kA \).

## 3 Linear maps induced by power series

In this section we show how any formal power series \( f \in tk[[t]] \) induces a linear function \( \Psi_f \) from \( k\langle A \rangle \) to itself, and also how various isomorphisms among the algebras \( (k\langle A \rangle, \shuffle) \), \( (k\langle A \rangle, \ast) \) and \( (k\langle A \rangle, \ast) \) can be recognized as being of the form \( \Psi_f \). This has been treated in an operadic context by Yamamoto [17].

Let \( a_1, a_2, \ldots, a_n \in A \). If \( w = a_1a_2 \cdots a_n \), and \( I = (i_1, \ldots, i_m) \) is a composition of \( n \) (i.e., a sequence of positive integers whose sum is \( n \)), define (as in [7])

\[
I[w] = (a_1 \circ \cdots \circ a_{i_1})(a_{i_1+1} \circ \cdots \circ a_{i_1+i_2}) \cdots (a_{i_1+\cdots+i_{m-1}+1} \circ \cdots \circ a_n). \tag{7}
\]

We call \( n = |I| \) the weight of the composition of \( I \), and \( m = \ell(I) \) its length. Note that the parentheses in Eq. (7) are not really necessary: the right-hand side is simultaneously an \( m \)-fold product in the concatenation algebra \( k\langle A \rangle \) and a product of length

\[
1 + (i_1 - 1) + (i_2 - 1) + \cdots + (i_m - 1) = n + 1 - m
\]
in the algebra $(k\langle A\rangle, \diamond)$. If we define

$$I\langle w \rangle = a_1 \cdots a_{i_1} \diamond a_{i_1+1} \cdots a_{i_1+i_2} \diamond \cdots \diamond a_{i_1+\cdots+i_{m-1}+1} \cdots a_n$$

so that, e.g.,

$$(2, 1, 2)[a_1 a_2 a_3 a_4 a_5] = a_1 \diamond a_2 a_3 a_4 \diamond a_5 = (1, 3, 1)\langle a_1 a_2 a_3 a_4 a_5 \rangle,$$

then $I[w] = I^*[w]$ defines an involution $*$ on compositions such that $|I^*| = |I|$ and $\ell(I^*) = |I| + 1 - \ell(I)$.

Now we consider formal power series

$$f = c_1 t + c_2 t^2 + c_3 t^3 + \cdots \in tk[[t]]. \quad (8)$$

Any two “functions” $f, g \in tk[[t]]$, say $f = \sum_{i \geq 1} c_i t^i$ and $g = \sum_{i \geq 1} d_i t^i$, have a “functional composition”

$$f \circ g = \sum_{i \geq 1} c_i g^i = c_1 (d_1 t + d_2 t^2 + \cdots) + c_2 (d_1 t + d_2 t^2 + \cdots)^2 + \cdots$$

$$= c_1 d_1 t + (c_1 d_2 + c_2 d_1^2) t^2 + \cdots \in tk[[t]].$$

Writing $[t^i]f$ for the coefficient of $t^i$ in $f \in k[[t]]$, it is evident that

$$[t^k]f \circ g = \sum_{j=1}^{k} [t^j]f[t^k]g^j. \quad (9)$$

Clearly $f = t$ is the identity for functional composition; and if $P \subset tk[[t]]$ is the set of power series invertible under functional composition, it is not hard to see that $f \in P$ if and only if $[t]f \neq 0$.

For $f$ given by Eq. (8), we define the $k$-linear map $\Psi_f : k\langle A\rangle \rightarrow k\langle A\rangle$ by

$$\Psi_f(1) = 1$$

and

$$\Psi_f(w) = \sum_{I=(i_1,\ldots,i_m) \in C(\ell(w))} c_{i_1} \cdots c_{i_m} I[w], \quad (10)$$

for nonempty words $w$, where $C(n)$ is the set of compositions of $n$. The following result generalizes Lemma 2.4 of [7].

**Theorem 3.1.** For $f, g \in tk[[t]]$, $\Psi_f \Psi_g = \Psi_{f \circ g}$. 

6
\textit{Proof.} Since

\[ \Psi_g(w) = \sum_{I=(i_1, \ldots, i_m) \in \mathcal{C}(\ell(w))} [t^{i_1}]g \cdots [t^{i_m}]gI[w] \]

we have

\[ \Psi_f \Psi_g(w) = \sum_{m=1}^{\ell(w)} \sum_{J=(j_1, \ldots, j_l) \in \mathcal{C}(m)} \sum_{I=(i_1, \ldots, i_m) \in \mathcal{C}(\ell(w))} [t^{j_1}]f \cdots [t^{j_l}]f[t^{i_1}]g \cdots [t^{i_m}]gJ[I[w]]. \]

On the other hand,

\[ \Psi_f \circ g(w) = \sum_{K=(k_1, \ldots, k_l) \in \mathcal{C}(\ell(w))} [t^{k_1}]f \circ g \cdots [t^{k_l}]f \circ gK[w], \]

so we need to show that, for all compositions \( K = (k_1, \ldots, k_l) \in \mathcal{C}(n) \),

\[ [t^{k_1}]f \circ g \cdots [t^{k_l}]f \circ g = \sum_{m=1}^{n} \sum_{J=(j_1, \ldots, j_l) \in \mathcal{C}(m)} \sum_{I=(i_1, \ldots, i_m) \in \mathcal{C}(n)} \sum_{JI=K} [t^{j_1}]f \cdots [t^{j_l}]f[t^{i_1}]g \cdots [t^{i_m}]g \quad (11) \]

where

\[ JI = (i_1 + \cdots + i_{j_1}, i_{j_1}+1 + \cdots + i_{j_1+j_2}, \ldots, i_{j_1+\cdots+j_{l-1}+1} + \cdots + i_m) \]

is the obvious “composition” of the compositions \( I = (i_1, \ldots, i_m) \) and \( J = (j_1, \ldots, j_l) \), with \( J \in \mathcal{C}(m) \). Now the right-hand side of Eq. (11) can be rewritten

\[
\begin{align*}
&\sum_{J=(j_1, \ldots, j_l)} \sum_{I=(i_1, \ldots, i_j) \in \mathcal{C}(n)} \prod_{s=1}^{l} [t^{j_s}]f[t^{i_{j_1}+\cdots+j_{s-1}+1}]g \cdots [t^{i_{j_1}+\cdots+j_s}]g \\
&= \sum_{J=(j_1, \ldots, j_l)} \prod_{s=1}^{l} [t^{j_s}]f[t^{k_s}]g^{j_s},
\end{align*}
\]

from which Eq. (11) follows by use of (9). \( \square \)
3.1 The isomorphisms $T$ and $\Sigma$

From the preceding result, $\Psi_f$ is an isomorphism when $f \in P$. We now consider some particular examples. First, it is immediate from Eq. (10) that $\Psi_t$ is the identity homomorphism of $k\langle A \rangle$. Now, following [12], consider

$$T = \Psi_{-t} \quad \text{and} \quad \Sigma = \Psi_{\frac{t}{1+t}}.$$  

(The function we call $\Sigma$ is written $S$ in [12], but as in [7] we wish to reserve $S$ for a Hopf algebra antipode.) For words $w$ of $k\langle A \rangle$, $T(w) = (-1)^{\ell(w)} w$ and

$$\Sigma(w) = \sum_{I \in C(\ell(w))} I[w].$$

Evidently $T$ is an involution, and $\Sigma^{-1} = \Psi_{\frac{t}{1+t}}$ is given by

$$\Sigma^{-1}(w) = \sum_{I \in C(\ell(w))} (-1)^{\ell(w)-\ell(I)} I[w].$$

We note that, for letters $a$ and words $w \neq 1$,

$$T(aw) = -aT(w) \quad (12)$$

$$\Sigma(aw) = a\Sigma(w) + a \circ \Sigma(w) \quad (13)$$

$$\Sigma^{-1}(aw) = a\Sigma^{-1}(w) - a \circ \Sigma^{-1}(w) \quad (14)$$

and (as in [12]) the property (13) can be used to define $\Sigma$. The functions $\Sigma$ and $T$ are not inverses, but we have the following result.

**Corollary 3.1.** The functions $\Sigma$ and $T$ satisfy $T \Sigma T = \Sigma^{-1}$, and (if the product $\circ$ is nonzero) generate the infinite dihedral group.

**Proof.** From Theorem 3.1 we have $\Sigma^n = \Psi_{\frac{n}{1+n}}$, so all powers of $\Sigma$ are distinct (unless $\circ = 0$, in which case $\Sigma = \text{id}$). We have also $T \Sigma T = \Psi_{\frac{t}{1+t}} = \Sigma^{-1}$. \[ \square \]

It follows immediately that $T \Sigma$ and $\Sigma T$ are involutions (cf. [12] Prop. 2]). For future reference we note that the equation $\Sigma^p = \Psi_{\frac{t}{1+pt}}$ defines $\Sigma^p$ for any $p \in k$: from Theorem 3.1 we have $\Sigma^p \Sigma^q = \Sigma^{p+q}$, and $\Sigma^p$ is the $p$th iterate of $\Sigma$ when $p$ is an integer.

From [7] we have the (inverse) functions $\exp = \Psi_{e^{-t}}$ and $\log = \Psi_{\log(1+t)}$. As shown in [7] Theorem 2.5, $\exp$ is an algebra isomorphism from $(k\langle A \rangle, \sqcup)$ to $(k\langle A \rangle, \ast)$. The functions $\exp$ and $\log$ are related to $\Sigma$ and $T$ as follows.
Corollary 3.2. $\Sigma = \exp T \log T$.

Proof. This is immediate from Theorem 3.1 since $\exp T = \Psi_{e^{-t-1}}$, $\log T = \Psi_{\log(1-t)}$, and $\log(1-t)$ composed with $e^{-t} - 1$ gives
\[
\frac{1}{1-t} - 1 = \frac{1-(1-t)}{1-t} = \frac{t}{1-t}.
\]

We now turn to the algebraic properties of $T$ and $\Sigma$.

Proposition 3.1. $T : (k\langle A \rangle, \ast) \to (k\langle A \rangle, \ast)$ and $T : (k\langle A \rangle, \ast) \to (k\langle A \rangle, \ast)$ are homomorphisms.

Proof. We prove the first statement; the second then follows because $T$ is an involution. We shall show that $T(u_1 \ast u_2) = T(u_1) \ast T(u_2)$ for any words $u_1, u_2$ by induction on $\ell(u_1) + \ell(u_2)$. The result is immediate if $u_1$ or $u_2$ is 1, so write $u_1 = aw$ and $u_2 = bv$ for letters $a, b$ and words $w, v$. Then
\[
T(u_1 \ast u_2) = T(a(w \ast bv) + b(aw \ast v) + (a \diamond b)(w \ast v))
\]
\[
= -a(T(w) \ast T(bv)) - b(T(aw) \ast T(v)) - (a \diamond b)(T(w) \ast T(v))
\]
\[
= a(T(w) \ast bT(v)) + b(aT(w) \ast T(v)) - (a \diamond b)(T(w) \ast T(v))
\]
\[
= aT(w) \ast bT(v) = T(u_1) \ast T(u_2),
\]
where we have used the induction hypothesis and Eq. (12).

The following result was proved as Theorem 1 in [12] in a much less direct way.

Corollary 3.3. The linear isomorphism $\Sigma : (k\langle A \rangle, \ast) \to (k\langle A \rangle, \ast)$ is an algebra isomorphism.

Proof. This follows from Corollary 3.2 since $\Sigma$ is the composition
\[
(k\langle A \rangle, \ast) \xrightarrow{T} (k\langle A \rangle, \ast) \xrightarrow{\log} (k\langle A \rangle, \sqcup) \xrightarrow{T} (k\langle A \rangle, \sqcup) \xrightarrow{\exp} (k\langle A \rangle, \ast)
\]
of homomorphisms (that $T$ is an endomorphism of $(k\langle A \rangle, \sqcup)$ follows by taking $\diamond$ to be the zero product in Proposition 3.1).
In fact, the following is a commutative diagram of algebra isomorphisms:

\[
\begin{array}{ccc}
(k\langle A\rangle, \ast) & \xrightarrow{\exp} & (k\langle A\rangle, \ast) \\
(k\langle A\rangle, \sqcup) & \xrightarrow{\Sigma} & (k\langle A\rangle, \ast) \\
& \xleftarrow{T \log T} & \end{array}
\]

\[\text{(15)}\]

**Corollary 3.4.** The involutions \(\Sigma T : (k\langle A\rangle, \ast) \to (k\langle A\rangle, \ast)\) and \(T \Sigma : (k\langle A\rangle, \ast) \to (k\langle A\rangle, \ast)\) are algebra automorphisms.

**Proof.** Immediate from Proposition 3.1 and Corollary 3.3 \(\square\)

### 3.2 A one-parameter family of automorphisms

Let \(p \neq 0\) be an element of \(k\), and set

\[H_p = \exp \Psi_{pt} \log\]

Evidently \(H_1 = \text{id}\) and \(H_p H_q = H_{pq}\), so this is a one-parameter family of isomorphisms of the vector space \(k\langle A\rangle\). We can write \(H_p = \Psi_{(1+t)^p-1}\), where \((1+t)^p - 1\) is the power series

\[
\sum_{n \geq 1} \left(\frac{p}{n}\right) t^n = pt + \frac{p(p-1)}{2!} t^2 + \frac{p(p-1)(p-2)}{3!} t^3 + \ldots .
\]

From Corollary 3.2 \(H_{-1} = \Sigma T\), so

\[H_{-1}(w \ast v) = \Sigma T(w \ast v) = \Sigma(T(w) \ast T(v)) = H_{-1}(w) \ast H_{-1}(v)\]

for any words \(w, v\). In fact, this property holds for all \(p\).

**Theorem 3.2.** For all \(p \neq 0\) and words \(w, v\), \(H_p(w \ast v) = H_p(w) \ast H_p(v)\).

**Proof.** Since

\[\Psi_{pt}(w) = p^\ell(w) w\]

for all words \(w\), it follows that

\[\Psi_{pt}(w \sqcup v) = \Psi_{pt}(w) \sqcup \Psi_{pt}(v)\]
for all words $w, v$. Hence, since $\log(w * v) = \log w \sqcup \log v$,

$$H_p(w * v) = \exp(\Psi_{pt}(\log w \sqcup \log v)) = \exp(\Psi_{pt}(\log w) \sqcup \Psi_{pt}(\log v)) = H_p(w) * H_p(v).$$

Thus, $H_p$ is an automorphism of the algebra $(k\langle A \rangle, \ast)$. Note also that $T H_p T = \Psi_{-(1-t)p}$ is an automorphism of the algebra $(k\langle A \rangle, \ast)$.

### 4 Hopf algebra structures

As in [7] we define a coproduct $\Delta$ on $k\langle A \rangle$ by

$$\Delta(w) = \sum_{uv = w} u \otimes v,$$

for words $w$, where the sum is over all pairs $(u, v)$ of words with $uv = w$ including $(1, w)$ and $(w, 1)$, and a counit $\epsilon : k\langle A \rangle \to k$ by $\epsilon(1) = 1$ and $\epsilon(w) = 0$ for $\ell(w) > 0$. It will also be convenient to define the reduced coproduct $\tilde{\Delta}$ by $\tilde{\Delta}(1) = 0$ and $\tilde{\Delta}(w) = \Delta(w) - w \otimes 1 - 1 \otimes w$ for nonempty words $w$.

The coproduct can be used to define a convolution product on the set $\text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$ of $k$-linear maps from $k\langle A \rangle$ to itself, which we denote by $\circ$: for $L_1, L_2 \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$ and words $w$ of $k\langle A \rangle$,

$$L_1 \circ L_2(w) = \sum_{uv = w} L_1(u)L_2(v).$$

(The reader is warned that this is not the usual convolution for either of the Hopf algebras $(k\langle A \rangle, \ast, \Delta)$ or $(k\langle A \rangle, \ast, \Delta)$ defined below.) The convolution product $\circ$ has unit element $\eta_e$, where $\eta : k \to k\langle A \rangle$ is the unit map (i.e., it sends $1 \in k$ to $1 \in k\langle A \rangle$). It is easy to show that any $L \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$ with $L(1) = 1$ has a convolutional inverse, which we denote by $L^{\circ(-1)}$.

We call $C \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$ a contraction if $C(1) = 0$ and $C(w)$ is primitive (i.e., $\tilde{\Delta}C(w) = 0$) for all words $w$, and $E \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$ an expansion if $E(1) = 1$ and $E$ is a coalgebra map. If $C$ is a contraction and $E$ is an expansion, we say $(E, C)$ is an inverse pair if

$$E = (\eta e - C)^{\circ(-1)} = \eta e + C + C \circ C + C \circ C \circ C + \cdots$$

(16)
or equivalently
\[ C = \eta \varepsilon - E \otimes (-1) \]  
(17)

**Proposition 4.1.** Suppose \( C \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle) \) is a contraction and \( E \) is given by Eq. (16). Then \((E, C)\) is an inverse pair. Conversely, if \( E \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle) \) is an expansion and \( C \) is given by Eq. (17), then \((E, C)\) is an inverse pair.

**Proof.** Suppose first that \( C \) is a contraction. Evidently \( E(1) = 1 \) from Eq. (16), so it suffices to show \( E \) a coalgebra map. Now Eq. (16) implies
\[ E(w) = \sum_{u_1 \cdots u_n = w} C(u_1) \cdots C(u_n) \]
for words \( w \neq 1 \), where the sum is over all decompositions \( w = u_1 \cdots u_n \) into subwords \( u_i \neq 1 \). Hence
\[ \Delta E(w) = E(w) \otimes 1 + 1 \otimes E(w) + \sum_{u_1 \cdots u_n = w, n \geq 2} \sum_{i=1}^{n-1} C(u_1) \cdots C(u_i) \otimes C(u_{i+1}) \cdots C(u_n), \]
which can be seen to agree with \((E \otimes E)\Delta(w)\).

Now suppose \( E \) is an expansion. Eq. (17) implies \( C(1) = 0 \), so it suffices to show \( C(w) \) primitive for words \( w \). We proceed by induction on \( \ell(w) \). Suppose \( C \) primitive on all words of length \( < n \), and let \( \ell(w) = n \). Then Eq. (17) implies
\[ C(w) = E(w) - \sum_{uv = w, u \neq 1} C(u)E(v), \]
and by the induction hypothesis it follows that \( \Delta C(w) \) can be written
\[ \Delta E(w) - \sum_{uv = w, u \neq 1} (C(u) \otimes 1)\Delta E(v) - \sum_{uv = w, u \neq 1} 1 \otimes C(u)E(v) = C(w) \otimes 1 + \sum_{uv = w, u \neq v} E(u) \otimes E(v) - \sum_{uv_1 v_2 = w, v_2 \neq 1} C(u)E(v_1) \otimes E(v_2) + 1 \otimes C(w). \]
Then
\[ \hat{\Delta} C(w) = \sum_{uv = w, u \neq 1} \left[ E(u) - \sum_{u_1 u_2 = u} C(u_1)E(u_2) \right] \otimes E(v), \]
and the quantity in brackets is zero by Eq. (17). \( \square \)
Now let \( f = c_1t + c_2t^2 + \cdots \in tk[[t]] \), and let \( \Psi_f \) be the corresponding linear map of \( k\langle A \rangle \) as defined in §3. Define the linear map \( C_f : k\langle A \rangle \rightarrow kA \) by \( C_f(1) = 0 \) and \( C_f(a_1a_2 \cdots a_n) = c_na_1 \odot a_2 \odot \cdots \odot a_n \) for \( a_1, a_2, \ldots, a_n \in A \). Then we have the following result.

**Theorem 4.1.** For any \( f \in tk[[t]] \), \((\Psi_f, C_f)\) is an inverse pair.

**Proof.** It is evident from definitions that \( \Psi_f(1) = 1 \) and

\[
\Psi_f(w) = \sum_{k=1}^{n} C_f(a_1 \cdots a_k)\Psi_f(a_{k+1} \cdots a_n).
\]

for \( w = a_1 \cdots a_n \) with \( a_i \in A \) and \( n \geq 1 \). Stated in terms of the convolution product, this is

\[
\Psi_f = C_f \odot \Psi_f + \eta \epsilon,
\]

from which Eq. (16) (with \( E = \Psi_f, C = C_f \)) follows. Since evidently \( C_f \) is a contraction, the result follows. \( \square \)

For a word \( w \) of \( k\langle A \rangle \), say \( w = a_1 \cdots a_n \) with the \( a_i \in A \), the “reverse” of \( w \) is \( R(w) = a_na_{n-1} \cdots a_1 \). If we set \( R(1) = 1 \), then \( R \) extends to a linear map from \( k\langle A \rangle \) to itself, which is evidently an involution. While \( R \) is not a coalgebra map for \( \Delta \) (despite the incorrect statement in §8), the following is true.

**Proposition 4.2.** \( R \) is an automorphism of both \((k\langle A \rangle, \ast)\) and \((k\langle A \rangle, \star)\).

**Proof.** In view of the next result and the commutative diagram (15), it suffices to prove that \( R \) is an automorphism of \((k\langle A \rangle, \sqcup)\). But this is evident from the well-known description of the shuffle product as

\[
a_1a_2 \cdots a_k \sqcup a_{k+1}a_{k+2} \cdots a_{k+l} = \sum_{\sigma \in S_{k,l}} a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(k+l)}
\]

where \( a_1, \ldots, a_{k+l} \) are letters and \( S_{k,l} \) is the subgroup of the symmetric group consisting of all permutations \( \sigma \) with \( \sigma(1) < \sigma(2) < \cdots < \sigma(k) \) and \( \sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l) \). \( \square \)

**Proposition 4.3.** \( R \) commutes with \( \Psi_f \) for all \( f \in tk[[t]] \).

**Proof.** For a composition \( I = (i_1, \ldots, i_l) \), let \( \bar{I} = (i_l, \ldots, i_1) \). Then evidently \( R(I[w]) = \bar{I}[R(w)] \). Since \( I \mapsto \bar{I} \) is an involution of \( C(n) \), the conclusion then follows from the definition (10) of \( \Psi_f \). \( \square \)
**Theorem 4.2.** \((k\langle A\rangle, *, \Delta)\) and \((k\langle A\rangle, *, \Delta)\) are Hopf algebras, with respective antipodes \(S_* = \Sigma T R\) and \(S_* = T \Sigma R\).

**Proof.** The inductive argument in [7, Theorem 3.1] that \(\Delta\) is a homomorphism for \(*\) works equally well for \(*\), so \((k\langle A\rangle, *, \Delta)\) and \((k\langle A\rangle, *, \Delta)\) are bialgebras. Although these bialgebras are not necessarily graded, they are filtered by word length: \(k\langle A\rangle^n\) is the subspace generated by words of length at most \(n\). Since \(k\langle A\rangle^0 = k1\), these bialgebras are filtered connected, and thus automatically Hopf algebras (see, e.g., [15]). In fact, the proof of the explicit formula for \(S_*\) in [7, Theorem 3.2] (by induction on word length) carries over to this setting, giving

\[
S_*(w) = (-1)^n \sum_{I \in C(n)} I[a_n a_{n-1} \cdots a_1]
\]

for a word \(w = a_1 a_2 \cdots a_n\) in \(k\langle A\rangle\), i.e., \(S_*(w) = \Sigma T R(w)\). The antipode \(S_*\) of the Hopf algebra \((k\langle A\rangle, *, \Delta)\) is uniquely determined by the conditions

\[
S_*(1) = 1 \quad \text{and} \quad \sum_{uv = w} S_*(u) * v = 0 \quad \text{for words} \ w \neq 1. \quad (18)
\]

Now \(S_*\) satisfies

\[
\sum_{uv = w} S_*(Tu) * Tv = 0
\]

for \(w \neq 1\); apply \(T\) both sides to get

\[
\sum_{uv = w} TS_*T(u) * v = 0,
\]

from which we see that \(S_* = TS_*T\) satisfies (18). Since \(T\) commutes with \(R\), this means that \(S_* = T \Sigma R\). \(\square\)

Since \(S_*\) and \(S_*\) are antipodes of commutative Hopf algebras, they are involutions and algebra automorphisms of \((k\langle A\rangle, *)\) and \((k\langle A\rangle, *)\) respectively. Since \(R\) commutes with \(\Sigma\) and \(T\), this gives another proof of Corollary 3.4. Note also that \(S_*S_* = \Sigma^2\) and \(S_*S_* = \Sigma^{-2}\).

For any \(f \in k[[t]]\), \(\Psi_f\) is a coalgebra map by Theorem 4.1. In particular, the maps \(H_p\) of the last section are automorphisms of the Hopf algebra \((k\langle A\rangle, *, \Delta)\), and [15] is a commutative diagram of Hopf algebra isomorphisms.

Recall that \((k\langle A\rangle, \circ)\) is a noncommutative algebra. We will now show that \((k\langle A\rangle, \circ, \Delta)\) is an infinitesimal Hopf algebra (see [11] for definitions).
Theorem 4.3. \((k\langle A \rangle, \circ, \tilde{\Delta})\) is an infinitesimal Hopf algebra, with antipode \(S_\circ = -\Sigma^{-1}\).

Proof. First we show that \((k\langle A \rangle, \circ, \tilde{\Delta})\) is an infinitesimal bialgebra, i.e., that
\[
\tilde{\Delta}(w \circ v) = \sum_v (w \circ v_{(1)}) \otimes v_{(2)} + \sum_w w_{(1)} \otimes (w_{(2)} \circ v),
\]
for words \(w, v\), where
\[
\tilde{\Delta}(w) = \sum_w w_{(1)} \otimes w_{(2)} \quad \text{and} \quad \tilde{\Delta}(v) = \sum_v v_{(1)} \otimes v_{(2)}.
\]

Eq. (19) is immediate if \(w\) or \(v\) is 1, so we can assume both are nonempty. Write \(w = a_1 \cdots a_n\) and \(v = b_1 \cdots b_m\), where the \(a_i\) and \(b_i\) are letters. If \(n = 1\) we have \(\tilde{\Delta}(w) = 0\), so Eq. (19) becomes
\[
\tilde{\Delta}(a_1 \circ v) = \sum_v (a_1 \circ v_{(1)}) \otimes v_{(2)},
\]
which is evidently true. The case \(m = 1\) is similar, so we can assume that \(n, m \geq 2\). Then
\[
\tilde{\Delta}(w \circ v) = \sum_{j=1}^{n-1} a_1 \cdot \cdots \cdot a_j \otimes a_{j+1} \cdots \cdot a_{n-1} a_n \circ b_1 b_2 \cdots b_m + \sum_{i=1}^{m-1} a_1 \cdots a_{n-1} a_n \circ b_1 b_2 \cdots b_i \otimes b_{i+1} \cdots b_m,
\]
and the right-hand side evidently agrees with that of Eq. (19).

To show \((k\langle A \rangle, \circ, \tilde{\Delta})\) an infinitesimal Hopf algebra we now need to show that it has an antipode, i.e., a function \(S_\circ \in \text{Hom}_k(k\langle A \rangle, k\langle A \rangle)\) with
\[
\sum_w S_\circ(w_{(1)}) \circ w_{(2)} + S_\circ(w) + w = 0 = \sum_w w_{(1)} \circ S_\circ(w_{(2)}) + w + S_\circ(w)
\]
for any \(w \in k\langle A \rangle\), where \(\tilde{\Delta}(w) = \sum_w w_{(1)} \otimes w_{(2)}\). This follows from [1, Prop. 4.5], but we shall prove that \(S_\circ = -\Sigma^{-1}\) by showing that \(-\Sigma^{-1}\) satisfies the defining property. We prove that the equation
\[
\sum_w \Sigma^{-1}(w_{(1)}) \circ w_{(2)} = -\Sigma^{-1}(w) + w
\]
\(15\)
holds for all words $w$ by induction on the length of $w$. Evidently Eq. (21) is true if $\ell(w) \leq 1$. Now suppose Eq. (21) holds for $w \neq 1$: we prove it for $aw$, $a \in A$. Since $\Delta(aw) = (a \otimes 1)\Delta(w) + a \otimes w$, we must show that

$$
\sum_w \Sigma^{-1}(aw(1)) \diamond w(2) + \Sigma^{-1}(a) \diamond w = -\Sigma^{-1}(aw) + aw
$$

Using Eq. (14), this is

$$
a \sum_w \Sigma^{-1}(w(1)) \diamond w(2) - a \diamond \sum_w \Sigma^{-1}(w(1)) \diamond w(2) + a \diamond w
$$

$$
= -a\Sigma^{-1}(w) + a \diamond \Sigma^{-1}(w) + aw.
$$

The conclusion then follows by use of the induction hypothesis (21). The proof that

$$
\sum_w w(1) \diamond \Sigma^{-1}(w(2)) = w - \Sigma^{-1}(w)
$$

is similar, except that in place of Eq. (14) one needs the identity

$$
\Sigma^{-1}(wa) = \Sigma^{-1}(w)a - \Sigma^{-1}(w) \diamond a
$$

for words $w$ and letters $a$.

The algebra $(k\langle A \rangle, \diamond)$ has the canonical derivation $D = \diamond\Delta$, i.e. $D(w) = 0$ for words $w$ with $\ell(w) \leq 1$ and

$$
D(a_1a_2 \cdots a_n) = \sum_{i=1}^{n-1} a_1 \cdots a_i \diamond a_{i+1} \cdots a_n
$$

for letters $a_1, \ldots, a_n$, $n \geq 2$. We note that $D^n(w) = 0$ whenever $n \geq \ell(w)$, so that

$$
e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} = \text{id} + D + \frac{D^2}{2!} + \cdots
$$

makes sense as an element of $\text{Hom}_k(k\langle A \rangle, k\langle A \rangle)$, and similarly for $e^{-D}$. By [11] Prop. 4.5], $\Sigma^{-1} = -S_\Sigma = e^{-D}$. In fact, this can be sharpened as follows.

**Corollary 4.1.** For any $r \in k$, $\Sigma^r = e^{rD}$. 

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Proof. By definition
\[
\Sigma^r(w) = \Psi_{1,\eta^r}(w) = \sum_{|I| = \ell(w)} r^{|I| - \ell(I)} I[w] \tag{22}
\]
for any word \(w\) of \(k\langle A\rangle\). On the other hand, by [1, Prop. 4.4]
\[
\frac{D^k}{k!} = \phi^{(k)} \Delta^{(k)},
\]
where \(\phi^{(k)} : k\langle A\rangle^{\otimes (k+1)} \to k\langle A\rangle\) and \(\Delta^{(k)} : k\langle A\rangle \to k\langle A\rangle^{\otimes (k+1)}\) are respectively the iterated \(\otimes\)-product and coproduct maps. Now for a word \(w\) of \(k\langle A\rangle\),
\[
\phi^{(k)} \Delta^{(k)}(w) = \sum_{\ell(I) = k+1, |I| = \ell(w)} I\langle w \rangle = \sum_{\ell(I) = k+1, |I| = \ell(w)} I^*[w]
\]
(where we recall the definition of \(I\langle w \rangle\) and \(I^*\) from §3), and so
\[
e^{rD}(w) = \sum_{k \geq 0} r^k \phi^{(k)} \Delta^{(k)}(w) = \sum_{k \geq 0} r^k \sum_{\ell(I) = k+1, |I| = \ell(w)} I^*[w] = \sum_{|I| = \ell(w)} r^{|I|} I^*[w] = \sum_{|I| = \ell(w)} r^{|I| - \ell(I)} I[w],
\]
which agrees with the right-hand side of Eq. (22). \qed

**Corollary 4.2.** For any \(r \in k\), \(\Sigma^r\) is an automorphism of \((k\langle A\rangle, \phi)\).

*Proof.* The exponential of a derivation is an automorphism [14, sect. I.2], so this follows from the preceding result. \qed

## 5 Exponentials and logarithms

Let \(\lambda\) be a formal parameter, \(\bullet\) any of the symbols \(*\), \(*\), \(\sqcup\), or \(\otimes\), and let \(f \in tk[[t]]\) be given by Eq. (8). Set
\[
f_{\bullet}(\lambda w) = \sum_{i \geq 1} \lambda^i c_i w^\bullet i \in k\langle A\rangle[[\lambda]],
\]
for \(w \in k\langle A\rangle\), and use this to define a map \(f_{\bullet}\) from \(\lambda k\langle A\rangle[[\lambda]]\) to itself. We write \(\exp_{\bullet}(u)\) for \(1 + f_{\bullet}(u)\), where \(f = e^t - 1\); and \(\log_{\bullet}(1 + u)\) for \(f_{\bullet}(u)\), where \(f = \log(1 + t)\). Then for any \(w \in k\langle A\rangle\),
\[
\log_{\bullet}(\exp_{\bullet}(\lambda w)) = \lambda w \quad \text{and} \quad \exp_{\bullet}(\log_{\bullet}(1 + \lambda w)) = 1 + \lambda w;
\]
and for \(w, v \in k\langle A \rangle\) for \(\bullet = \ast\) or \(\bullet = \circ\), and \(w, v \in kA\) for \(\bullet = \diamond\),
\[
\exp_\bullet(\lambda(w + v)) = \exp_\bullet(\lambda w) \bullet \exp_\bullet(\lambda v).
\]  
(23)

We extend the automorphisms \(\Phi_f\) of \(k\langle A \rangle\) to \(kA[[\lambda]]\) by setting \(\Phi_f(\lambda) = \lambda\).

The following result generalizes Lemma 3 of [12].

**Theorem 5.1.** For any \(f = c_1t + c_2t^2 + \cdots \in tk[[t]]\) and \(z \in kA[[\lambda]]\),
\[
\Phi_f\left(\frac{1}{1 - \lambda z}\right) = \frac{1}{1 - f_\circ(\lambda z)}.
\]

**Proof.** In fact, we shall show that
\[
E\left(\frac{1}{1 - \lambda z}\right) = \frac{1}{1 - C(\lambda z + \lambda^2 z^2 + \cdots)}
\]  
(24)

for any inverse pair \((E, C)\): the conclusion then follows by Theorem 4.1, noting that \(f_\circ(\lambda z) = C_f(\lambda z + \lambda^2 z^2 + \cdots)\). We can write the left-hand side of Eq. (24) as
\[
(\eta C + C \odot C + \cdots)(1 + \lambda z + \lambda^2 z^2 + \cdots) = 1 + \sum_{n \geq 1} \sum_{k \leq n} C \odot k(\lambda^n z^n),
\]
which we will denote by \(\square\). Evidently each term except 1 in \(\square\) has an initial factor of form \(C(\lambda^k z^k)\), so
\[
\square - 1 = C(\lambda z)\square + C(\lambda^2 z^2)\square + \cdots = C(\lambda z + \lambda^2 z^2 + \cdots)\square,
\]
and Eq. (24) follows. \(\square\)

Since \(\exp : (k\langle A \rangle, \cup) \to (k\langle A \rangle, \ast)\) is an algebra isomorphism, we have \(\exp f_\cup = f_\ast \exp\) for any \(f \in tk[[t]]\). For such \(f\) we also have \(\Sigma f_\ast = f_\ast \Sigma\) and \(T f_\ast = f_\ast T\). In particular, for \(z \in kA[[\lambda]]\), \(\Sigma f_\ast(\lambda z) = f_\ast(\lambda z)\) and \(T f_\ast(\lambda z) = f_\ast(-\lambda z)\). For \(z \in kA[[\lambda]]\) we also have
\[
\exp_\ast(\lambda z) = \exp(\exp_\cup(\lambda z)) = \exp\left(\frac{1}{1 - \lambda z}\right),
\]  
(25)

where we have used the identity
\[
\exp_\cup(\lambda z) = 1 + \lambda z + \lambda^2 z^2 + \lambda^3 z^3 + \cdots = \frac{1}{1 - \lambda z},
\]
which in turn follows from \(z^m = n!z^n\) for \(z \in kA[[\lambda]]\). We can now give a quick proof of the following result (cf. [13 Prop. 4] and [12 Prop. 3]).
Corollary 5.1. For $z \in kA[[\lambda]]$,

$$
\exp_*(\log_0(1 + \lambda z)) = \frac{1}{1 - \lambda z} \quad \text{and} \quad \exp_*(- \log_0(1 + \lambda z)) = \frac{1}{1 + \lambda z}.
$$

Proof. In view of Eq. (25), the first identity is equivalent to

$$
\frac{1}{1 - \log_0(1 + \lambda z)} = \log \left( \frac{1}{1 - \lambda z} \right),
$$

which is just Theorem 5.1 applied to the formal power series $f = \log(1 + t)$. To get the second identity, apply $T$ to both sides of the first.

Remark. By applying Theorem 5.1 to the formal power series $e^t - 1$, we get

$$
\exp \left( \frac{1}{1 - \lambda z} \right) = \frac{1}{1 - (\exp_0(\lambda z) - 1)},
$$

or $\exp_*(\lambda z) = (2 - \exp_0(\lambda z))^{-1}$, as in [13, Prop. 4].

Here are some further corollaries of Theorem 5.1.

Corollary 5.2. For $p \in k$ and $z \in kA[[\lambda]]$,

$$
H_p \left( \frac{1}{1 - \lambda z} \right) * p.
$$

Proof. Using Theorem 3.2 and Corollary 5.1, the left-hand side of the identity can be written

$$
H_p(\exp_*(\log_0(1 + \lambda z))) = \exp_*(H_p(\log_0(1 + \lambda z))).
$$

Now $\log_0(1 + \lambda z) \in kA[[\lambda]]$, so the latter quantity is

$$
\exp_*(p \log_0(1 + \lambda z)) = (\exp_*(\log_0(1 + \lambda z)))^p = \left( \frac{1}{1 - \lambda z} \right)^p.
$$

Corollary 5.3. For any $p \in k$, $f = c_1t + c_2t^2 + \cdots \in tk[[t]]$ and $z \in kA[[\lambda]]$,

$$
\Psi_g \left( \frac{1}{1 + \lambda z} \right) * \left( \Psi_f \left( \frac{1}{1 - \lambda z} \right) \right)^p = 1,
$$

where $(1 + g(t))(1 + f(-t))^p = 1$, i.e., $g = ((1 + t)^{-p} - 1) \circ f \circ (-t)$. 

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Proof. By Theorem 3.1 \( \Psi_g = H_p \Psi_f T \), so the conclusion can be written as
\( \xi = H \Psi_f T \).

Now Theorem 5.1 says that \( \xi = \frac{1}{1 - \lambda u} \) for
\[
\begin{align*}
u & = c_1 z + \lambda c_2 z^2 + \lambda^2 c_3 z^3 + \cdots \in kA[[\lambda]],\end{align*}
\]
so we can apply Corollary 5.2 to obtain the conclusion.

Remark. In particular, taking \( p = 1 \) and \( f = \frac{t}{1 - st} \) in the preceding result gives
\[
\Sigma^s \left( \frac{1}{1 - \lambda z} \right) * \Sigma^{1 - s} \left( \frac{1}{1 + \lambda z} \right) = 1 \tag{27}
\]
for any \( s \in k \), generalizing Corollary 1 of [12].

**Corollary 5.4.** For \( y, z \in kA[[\lambda]] \),
\[
\frac{1}{1 - \lambda y} * \frac{1}{1 - \lambda z} = \frac{1}{1 - \lambda y - \lambda z - \lambda^2 y \diamond z}
\]
and
\[
\frac{1}{1 + \lambda y} * \frac{1}{1 + \lambda z} = \frac{1}{(1 + \lambda y) \diamond (1 + \lambda z)}.
\]

*Proof.* Using Corollary 5.1 the left-hand side of the first identity is
\[
\exp_* (\log_\circ (1 + \lambda y)) * \exp_* (\log_\circ (1 + \lambda z)) = \exp_* (\log_\circ (1 + \lambda y + \log_\circ (1 + \lambda z))
\]
\[
= \exp_* (\log_\circ ((1 + \lambda y) \diamond (1 + \lambda z))) = \exp_* (\log_\circ (1 + \lambda y + \lambda z + \lambda^2 y \diamond z))
\]
\[
= \frac{1}{1 - \lambda y - \lambda z - \lambda^2 y \diamond z},
\]
so the identity follows. To get the second identity, apply \( T \) to both sides of the first. \( \square \)

**Corollary 5.5.** For any \( r \in k \) and \( z \in kA[[\lambda]] \),
\[
\Sigma^r \left( \frac{1}{1 - \lambda z} \right) * \frac{1}{1 + r \lambda z} = \frac{1}{1 - (1 - r) \lambda z}.
\]
Proof. By Theorem 5.1
\[ \sum^r \left( \frac{1}{1 - \lambda z} \right) = \frac{1}{1 - f_\circ(\lambda z)} \]
for \( f = \frac{t}{1 - rt} = t + rt^2 + r^2t^3 + \ldots \). Thus
\[ \sum^r \left( \frac{1}{1 - \lambda z} \right) \cdot \frac{1}{1 + r\lambda z} = \frac{1}{1 - f_\circ(\lambda z) + r\lambda z + \lambda rz \circ f_\circ(\lambda z)} \]
by Corollary 5.4. But evidently \( \lambda rz \circ f_\circ(\lambda z) = f_\circ(\lambda z) - \lambda z \), so the denominator on the right-hand side is \( 1 + r\lambda z - \lambda z \) and the conclusion follows. \( \square \)

We also have the following result, which is proved in [12, Prop. 4] by another method.

**Theorem 5.2.** For \( a, b \in A \),
\[ \Sigma \left( \frac{1}{1 - \lambda ab} \right) = \frac{1}{1 - \lambda ab} \cdot \Sigma \left( \frac{1}{1 - \lambda a \circ b} \right). \]

Proof. Using Eq. (27) with \( s = 1 \), the conclusion can be written as
\[ \Sigma \left( \frac{1}{1 - \lambda ab} \right) \cdot \frac{1}{1 + \lambda a \circ b} = \frac{1}{1 - \lambda ab}. \]
Now use Corollary 3.2 and apply log to both sides to make this
\[ T \log \left( \frac{1}{1 - \lambda ab} \right) \sqcup \log \left( \frac{1}{1 + \lambda a \circ b} \right) = \log \left( \frac{1}{1 - \lambda ab} \right). \quad (28) \]

Now
\[ \log \left( \frac{1}{1 - \lambda ab} \right) = 1 + \sum_{i \geq 1} \lambda^i \sum_{\substack{I = (I_1, \ldots, I_n) \\ |I| = 2i}} (-1)^n \frac{I[(ab)^i]}, \]
and applying \( T \) simply eliminates the signs. Further,
\[ \log \left( \frac{1}{1 + \lambda a \circ b} \right) = 1 + \sum_{i \geq 1} \lambda^i \sum_{\substack{J = (J_1, \ldots, J_k) \\ |J| = i}} (-1)^k \frac{J[(a \circ b)^i]}, \]
so to prove (28) and hence the conclusion it suffices to show

\[ \sum_{i=0}^{m} \left( \sum_{I=(i_{1}, \ldots, i_{n})} \frac{1}{i_{1} \cdots i_{n}} I[(ab)^{i}] \right) = \sum_{I=(i_{1}, \ldots, i_{n})} \frac{(-1)^{k}}{i_{1} \cdots i_{n}} J[(a \circ b)^{m-i}] \]

\[ = \sum_{I=(i_{1}, \ldots, i_{n})} \frac{(-1)^{n}}{i_{1} \cdots i_{n}} I[(ab)^{m}]. \quad (29) \]

To prove the latter equation, we consider an arbitrary term of the form

\[ (i_{1}, i_{2}, \ldots, i_{n})[(ab)^{m}], \quad i_{1} + \cdots + i_{n} = 2m, \quad (30) \]

and note that every even \( i_{h} = 2j \) produces a factor \((a \circ b)^{o_{j}}\). Write \((i_{1}, \ldots, i_{n})\) as \((t_{1}^{p_{1}}, \ldots, t_{s}^{p_{s}})\), where the exponents mean repetition, and let \((t_{u_{1}}, \ldots, t_{u_{f}}) = (2j_{u_{1}}, \ldots, 2j_{u_{f}})\) be the subsequence of even \( t_{i}'s \). Then (30) appears on the right-hand side of Eq. (29) with coefficient

\[ \sum_{0 \leq q_{u_{h}} \leq p_{u_{h}}} \frac{(-1)^{p_{1}^{u} + \cdots + p_{s}^{u}}}{t_{1}^{p_{1}} \cdots t_{s}^{p_{s}}} J^{u_{1} \cdots u_{f}}(p_{u_{1}}^{q_{1}}) \cdots (p_{u_{f}}^{q_{f}}), \]

and on the left-hand side of Eq. (29) with coefficient

\[ \sum_{0 \leq q_{u_{h}} \leq p_{u_{h}}} \frac{(-1)^{q_{1}^{u} + \cdots + q_{f}^{u}}}{t_{1}^{q_{1}} \cdots t_{s}^{q_{s}}} (p_{u_{1}}^{q_{1}}) \cdots (p_{u_{f}}^{q_{f}}), \]

where

\[ p_{i}^{'} = \begin{cases} p_{i}, & \text{if } t_{i} \text{ is odd;} \\ p_{i} - q_{i}, & \text{if } t_{i} \text{ is even.} \end{cases} \]

Since \( j_{u_{h}}^{q_{u_{h}}} = 2^{-q_{u_{h}}} t_{u_{h}}^{q_{u_{h}}} \) for \( h = 1, 2, \ldots, f \), we can write the latter coefficient as

\[ \frac{1}{t_{1}^{p_{1}} \cdots t_{s}^{p_{s}}} \sum_{0 \leq q_{u_{h}} \leq p_{u_{h}}} (-2)^{q_{1}^{u} + \cdots + q_{f}^{u}} (p_{u_{1}}^{q_{1}}) \cdots (p_{u_{f}}^{q_{f}}), \]

which by the binomial theorem agrees with (31). \[ \square \]
6 Applications

To demonstrate the scope of applications of quasi-shuffle products, in this section we will outline five types of objects that are homomorphic images of quasi-shuffle algebras: multiple zeta values, multiple t-values, (finite) multiple harmonic sums, multiple q-zeta values, and values of multiple polylogarithms at roots of unity. In each case we show how Corollary 5.1 can be applied. In several cases we also apply Theorem 5.2.

6.1 Multiple zeta values

Consider the case $A = \{z_1, z_2, \ldots \}$ and $z_i \diamond z_j = z_{i+j}$ (see [7, Ex. 1]). Let $\tilde{\mathfrak{H}}^1 = k\langle A \rangle$, and let $\tilde{\mathfrak{H}}^0 \subset \tilde{\mathfrak{H}}^1$ be the subspace generated by monomials that don’t start in $z_1$. Then there is a homomorphism $\zeta : (\tilde{\mathfrak{H}}^0, *) \to \mathbb{R}$ given as in the introduction:

$$\zeta(z_{i_1}z_{i_2}\cdots z_{i_k}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}}.$$

(The restriction that $i_1 \neq 1$ is necessary for convergence of the series.) Also, if we define $\zeta^*(z_{i_1}z_{i_2}\cdots z_{i_k})$ by Eq. (4), then $\zeta^* : (\tilde{\mathfrak{H}}^0, *) \to \mathbb{R}$ is a homomorphism.

From Corollary 5.1 we have

$$\exp^*(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i z_k^{i^2}) = \sum_{n=0}^{\infty} z_k^n \lambda^n$$

and applying $\zeta$ to both sides gives

$$\exp\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(i)\right) = \sum_{n=0}^{\infty} \zeta(z_k^n)\lambda^n.$$

That is, the MZV $\zeta(k, \ldots, k)$ (with $n$ repetitions of $k \geq 2$) is the coefficient of $\lambda^n$ in

$$Z_k(\lambda) = \exp\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(i)\right).$$

(33)

This is a well-known result: it goes back at least to [2] (see Eq. (11)). To obtain the counterpart for zeta-star values, replace $\lambda$ with $-\lambda$ in the second
part of Corollary \ref{5.1} and set $z = z_k$ to get
\[
\exp, \left( \sum_{i \geq 1} \frac{\lambda^i z_k^i}{i} \right) = \sum_{n=0}^{\infty} z_k^n \lambda^n. \tag{34}
\]
Now apply $\zeta^*$ to both sides:
\[
\exp \left( \sum_{i=1}^{\infty} \frac{\lambda^i \zeta(ik)}{i} \right) = \sum_{n=0}^{\infty} \zeta^*(z_k^n) \lambda^n.
\]
Henceforth any string enclosed by $\{ \}$ is understood to be repeated $n$ times, so the preceding equation implies that $\zeta(\{k\}_n)$ is the coefficient of $\lambda^n$ in
\[
\exp \left( \sum_{i=1}^{\infty} \frac{\lambda^i \zeta(ik)}{i} \right) = \frac{1}{Z_k(-\lambda)},
\]
where $Z_k$ is given by Eq. \eqref{33}. In particular, from \cite[Cor. 2.3]{6} (and also \cite[Eq. (36)]{2}) we have
\[
\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n+1)!}, \quad \text{hence} \quad Z_2(\lambda) = \frac{\sinh(\pi \sqrt{\lambda})}{\pi \sqrt{\lambda}}, \tag{35}
\]
and thus
\[
\sum_{n=0}^{\infty} \zeta^*(\{2\}_n) \lambda^n = \frac{\pi \sqrt{\lambda}}{\sin(\pi \sqrt{\lambda})}. \tag{36}
\]
It follows that
\[
\zeta^*(\{2\}_n) = \frac{(-1)^{n-1} 2(2^{n-1} - 1) B_{2n} \pi^{2n}}{(2n)!};
\]
\cite[p. 203]{12}. Similarly, we have from \cite[Eq. (37)]{2} that
\[
\zeta(\{4\}_n) = \frac{2^{n+1} \pi^{4n}}{(4n+2)!}, \quad \text{hence} \quad Z_4(\lambda) = \frac{\cosh(\sqrt{2\pi \sqrt{\lambda}}) - \cos(\sqrt{2\pi \sqrt{\lambda}})}{2\pi^2 \sqrt{\lambda}},
\]
and thus
\[
\sum_{n=0}^{\infty} \zeta^*(\{4\}_n) \lambda^n = \frac{1}{Z_4(-\lambda)} = \frac{2\pi^2 i \sqrt{\lambda}}{\cosh(\sqrt{2\pi e^{i\frac{\pi}{4}} \sqrt{\lambda}}) - \cos(\sqrt{2\pi e^{i\frac{\pi}{4}} \sqrt{\lambda}})}
\]
\[
= \frac{\pi^2 \sqrt{\lambda}}{\sinh(\pi \sqrt{\lambda})} = 1 + \frac{\pi^4 \lambda}{90} + \frac{13\pi^8 \lambda^2}{113400} + \frac{4009\pi^{12} \lambda^3}{3405402000} + \cdots .
\]
Applying $\zeta$ to Theorem 5.2 (with $a = z_i$, $b = z_j$) gives

$$\sum_{n=0}^{\infty} \zeta^*(\{i, j\}_n) \lambda^n = \sum_{p=0}^{\infty} \zeta(\{i, j\}_p) \lambda^p \sum_{q=0}^{\infty} \zeta^*(\{i + j\}_q) \lambda^q$$

(37)

for any positive integers $i, j$ with $i \geq 2$. In particular, taking $i = 2$ and $j = 1$ gives

$$\sum_{n=0}^{\infty} \zeta^*(\{2, 1\}_n) \lambda^n = \sum_{p=0}^{\infty} \zeta(\{2, 1\}_p) \lambda^p \sum_{q=0}^{\infty} \zeta^*(\{3\}_q) \lambda^q = \frac{Z_3(\lambda)}{Z_3(-\lambda)}$$

(38)

since $\zeta(\{2, 1\}_p) = \zeta(\{3\}_p)$ by duality of multiple zeta values. Now

$$\frac{Z_3(\lambda)}{Z_3(-\lambda)} = \exp\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(3i)\right) \exp\left(\sum_{i \geq 1} \frac{\lambda^i \zeta(3i)}{i}\right)$$

$$= \exp\left(\sum_{i \geq 1 \text{ odd}} \frac{2 \lambda^i \zeta(3i)}{i}\right) = \prod_{i \geq 1 \text{ odd}} \sum_{j=0}^{\infty} \frac{2^j \lambda^j \zeta(3i)^j}{i^j j!},$$

so it follows from Eq. (38) that

$$\zeta^*(\{2, 1\}_n) = \sum_{i_1+3i_3+5i_5+\ldots=n} \frac{2^{i_1+i_3+i_5+\ldots} \zeta(3)^{i_1} \zeta(9)^{i_3} \zeta(15)^{i_5} \ldots}{1^{i_1} 3^{i_3} 5^{i_5} i_5! \ldots}.$$  

(39)

Similarly, using the Zagier-Broadhurst identity [3, Theorem 1]

$$\zeta(\{3, 1\}_n) = \frac{2\pi^{4n}}{(4n + 2)!},$$

we obtain from Eq. (37) with $i = 3, j = 1$,

$$\sum_{n=0}^{\infty} \zeta^*(\{3, 1\}_n) \lambda^n = \sum_{p=0}^{\infty} \zeta(\{3, 1\}_p) \lambda^p \sum_{q=0}^{\infty} \zeta^*(\{4\}_q) \lambda^q = \frac{Z_4(\lambda)}{Z_4(-\lambda)}$$

$$= \frac{\cosh(\pi \sqrt[4]{\lambda}) - \cos(\pi \sqrt[4]{\lambda})}{\sinh(\pi \sqrt[4]{\lambda}) \sin(\pi \sqrt[4]{\lambda})} = 1 + \frac{\pi^4 \lambda}{72} + \frac{53\pi^8 \lambda^2}{362880} + \frac{15107\pi^{12} \lambda^3}{10059033600} + \ldots$$
S. Yamamoto [16] defines interpolated multiple zeta values \( \zeta^r(i_1, \ldots, i_k) \) as, in effect, \( \zeta(\Sigma^r(z_{i_1} \cdots z_{i_k})) \). Note that \( \zeta^0 = \zeta \) and \( \zeta^1 = \zeta^* \). By applying \( \zeta \) to both sides of Corollary 5.5 with \( z = z_k, \ k \geq 2 \), we obtain

\[
\sum_{n=0}^{\infty} \zeta^r(\{k\}_n) \lambda^n = \frac{Z_k((1 - r)\lambda)}{Z_k(-r\lambda)}. \tag{40}
\]

In particular, taking \( r = \frac{1}{2} \) in Eq. (40) gives

\[
\sum_{n=0}^{\infty} \zeta^{\frac{1}{2}}(\{k\}_n) \lambda^n = \frac{Z_k(\frac{\lambda}{2})}{Z_k(-\frac{\lambda}{2})} = \exp \left( \sum_{i \geq 1 \text{ odd}} \frac{\lambda^i \zeta(ik)}{i 2^{i-1}} \right) = \prod_{i \geq 1 \text{ odd}} \sum_{j=0}^{\infty} \frac{\lambda^i \zeta(ik)^j}{j! 2^{j(0-1)j!}}
\]

from which follows

\[
\zeta^{\frac{1}{2}}(\{k\}_n) = \sum_{i_1 + 3i_3 + \cdots = n} \frac{2^{i_1 + i_3 + \cdots} \zeta(k)^{i_1} \zeta(3k)^{i_3} \zeta(5k)^{i_5} \cdots}{i_1!^i_1 i_3!^i_3 i_5!^i_5 \cdots}. \tag{41}
\]

Comparing the case \( k = 3 \) of Eq. (41) with Eq. (39) above gives

\[
\zeta^{\frac{1}{2}}(\{3\}_n) = \frac{1}{2^n} \zeta^*(\{2,1\}_n),
\]

an instance of the two-one formula discussed in [16] and recently proved in [20].

The sum theorem for multiple zeta values [5] states that \( \zeta(S(k, l)) = \zeta(k) \) for \( 1 \leq l \leq k - 1 \), where \( S(k, l) \) is the sum of all monomials in \( \delta^k \) of degree \( k \) and length \( l \). Now it is a straightforward combinatorial exercise to show that \( DS(k, l) = (k - l)S(k, l - 1) \), where \( D \) is the canonical derivation of \( \delta^k \).

Then using Corollary 4.1

\[
\zeta^r(S(k, l)) = \zeta(\Sigma^r S(k, l)) = \zeta \left( \sum_{n=0}^{l-1} \frac{r^n D^n}{n!} S(k, l) \right) = \zeta \left( \sum_{n=0}^{l-1} \frac{r^n (k - l - 1 + n)}{n} S(k, l - n) \right)
\]

\[
= \sum_{n=0}^{l-1} \frac{r^n (k - l - 1 + n)}{n} \zeta(k),
\]

giving a new proof of [16] Theorem 1.1.

In [9] it is proved that if

\[
e(2n, k) = \sum_{(i_1, \ldots, i_k) \in C(n)} z_{2i_1} \cdots z_{2i_k} \tag{42}
\]

26
where the sum is over all compositions \((i_1, \ldots, i_k)\) of \(n\) having \(k\) parts, then the generating functions

\[
F(t, s) = 1 + \sum_{n \geq k \geq 1} \zeta(e(2n, k)) t^n s^k, \quad F^*(t, s) = 1 + \sum_{n \geq k \geq 1} \zeta^*(e(2n, k)) t^n s^k
\]

have the closed forms

\[
F(t, s) = \frac{\sin(\pi \sqrt{(1 - s)t})}{\sqrt{1 - s \sin(\pi \sqrt{t})}}, \quad F^*(t, s) = \frac{\sqrt{1 + s \sin(\pi \sqrt{t})}}{\sin(\pi \sqrt{(1 + s)t})}.
\]

If we define

\[
F(t, s; r) = 1 + \sum_{n \geq k \geq 1} \zeta^r(e(2n, k)) t^n s^k,
\]

then this result can be generalized to

\[
F(t, s; r) = \frac{\sin(\pi \sqrt{(1 - s + rs)t}) \sqrt{1 + rs}}{\sin(\pi \sqrt{(1 + rs)t}) \sqrt{1 - s + rs}} = \frac{F(t, (1 - r)s)}{F(t, -rs)}.
\]  \(\text{(43)}\)

To prove Eq. \(\text{(43)}\), we first note that \((k\langle A \rangle, *)\) is isomorphic to the algebra \(\text{QSym}\) of quasi-symmetric functions. Further, \(e(2n, k)\) is the image under the degree-doubling map \(\mathcal{D} : \text{QSym} \rightarrow \text{QSym}\) that sends \(z_{i_1} \cdots z_{i_k}\) to \(z_{2i_1} \cdots z_{2i_k}\) of the symmetric function

\[
N_{n, k} = \sum_{\text{partitions } \lambda \text{ of } n \text{ with } k \text{ parts}} m_{\lambda},
\]

and, as shown in [9],

\[
1 + \sum_{n \geq k \geq 1} N_{n, k} t^n s^k = E((s - 1)t) H(t),
\]

where \(E(t), H(t)\) are respectively the generating functions of the elementary and complete symmetric functions. From Corollary 5.5 we have, for any rational \(p\),

\[
\Sigma^p \left( \frac{1}{1 - t z_1} \right) = \frac{1}{1 - (1 - p) t z_1} * \left( \frac{1}{1 + p t z_1} \right)^{-*},
\]

which translated into the language of symmetric functions is

\[
\Sigma^p E(t) = E((1 - p)t) E(-pt)^{-1} = E((1 - p)t) H(pt).
\]
Then
\[ \Sigma^r \left( 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k \right) = \Sigma^r E((s - 1)t) H(t) = \Sigma^{r+1/2} E(st) = \]
\[ \Sigma^{r+1/2} E(st) = E((s - rs - 1)t) H((1 + rs)t). \]

From Eqs. (35) and (36) above we have
\[ \zeta \mathcal{D} E(t) = \sinh(\pi \sqrt{t}) \quad \text{and} \quad \zeta \mathcal{D} H(t) = \frac{\pi \sqrt{t}}{\sin(\pi \sqrt{t})} \]
and so, since \( \mathcal{D} \) commutes with \( \Sigma^r \),
\[ F(t, s; r) = \zeta \Sigma^r \mathcal{D} \left( 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k \right) = \]
\[ \zeta(\mathcal{D}(E((s - rs - 1)t) H((1 + rs)t))) = \frac{\sinh(\pi \sqrt{(s - rs - 1)t}) \sqrt{(1 + rs)t}}{\sqrt{(s - rs - 1)t} \sin(\pi \sqrt{(1 + rs)t})}, \]
from which Eq. (43) follows.

**6.2 Multiple \( t \)-values**

As in [10], let \( t(i_1, \ldots , i_k) \) be the sum of those terms in the series (11) for \( \zeta(i_1, \ldots , i_k) \) with odd denominators. Then one has a homomorphism \( t : (\mathcal{H}^0, *) \to \mathbb{R} \) defined by \( t(z_{i_1} \cdots z_{i_k}) = t(i_1, \ldots , i_k) \) for \( i_1 > 1 \). We can also define multiple \( t \)-star values by
\[ t^*(i_1, \ldots , i_k) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1, \text{ } n_i \text{ odd}} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}} \]
for \( i_1 > 1 \), so that there is a homomorphism \( t^* : (\mathcal{H}^0, *) \to \mathbb{R} \) given by \( t^*(z_{i_1} \cdots z_{i_k}) = t^*(i_1, \ldots , i_k) \). Applying \( t \) to both sides of Eq. (32) gives
\[ \exp \left( \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i t(ik) \right) = \sum_{n=0}^{\infty} t(z_k^n) \lambda^n. \quad (44) \]

Now
\[ t(ik) = \sum_{n \geq 1 \text{ odd}} \frac{1}{n^{ik}} = \left( 1 - \frac{1}{2^{ik}} \right) \zeta(ik) \]
so that the left-hand side of Eq. (44) is
\[
\exp \left( \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left( 1 - \frac{1}{2^i} \right) \lambda^i \zeta(ik) \right) = \\
\exp \left( \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(ik) \right) \exp \left( - \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left( \frac{\lambda}{2^i} \right)^i \zeta(ik) \right)
\]
and thus
\[
t(\{k\}_n) = \text{coefficient of } \lambda^n \text{ in } \frac{Z_k(\lambda)}{Z_k(\lambda/2^k)},
\]
where \(Z(\lambda)\) is given by Eq. (33); cf. [10, Theorem 8]. Similarly
\[
t^\star(\{k\}_n) = \text{coefficient of } \lambda^n \text{ in } \frac{Z_k(-\lambda/2^k)}{Z_k(-\lambda)}.
\]
In particular, from the preceding subsection we have
\[
\sum_{n=0}^{\infty} t(\{2\}_n) \lambda^n = \cosh \left( \frac{\pi}{2} \sqrt{\lambda} \right), \quad \sum_{n=0}^{\infty} t^\star(\{2\}_n) \lambda^n = \sec \left( \frac{\pi}{2} \sqrt{\lambda} \right), \quad (45)
\]
and
\[
\sum_{n=0}^{\infty} t(\{4\}_n) \lambda^n = \frac{1}{2} \left[ \cosh \left( \frac{\pi \sqrt{\lambda}}{\sqrt{2}} \right) + \cos \left( \frac{\pi \sqrt{\lambda}}{\sqrt{2}} \right) \right],
\]
\[
\sum_{n=0}^{\infty} t^\star(\{4\}_n) \lambda^n = \sec \left( \frac{\pi \sqrt{\lambda}}{2} \right) \text{sech} \left( \frac{\pi \sqrt{\lambda}}{2} \right).
\]
As with multiple zeta values, we can define interpolated multiple \(t\)-values
\[
t^r(i_1, \ldots, i_k) = t(\Sigma^r(z_{i_1} \cdots z_{i_k})).
\]
Reasoning as before, we obtain from Corollary 5.5 an analogue of Eq. (40):
\[
\sum_{n=0}^{\infty} t^r(\{k\}_n) \lambda^n = \frac{Z_k((1-r)\lambda)Z_k(-r\lambda/2^k)}{Z_k((1-r)\lambda/2^k)Z_k(-\lambda)}.
\]
In the case \(r = \frac{1}{2}\) this gives a formula like Eq. (41):
\[
t^{\frac{1}{2}}(\{k\}_n) = \sum_{i_1 + 3i_3 + \cdots = n} \frac{2^{i_1+i_3+i_5-\cdots} t(k)^{i_1} t(3k)^{i_3} t(5k)^{i_5} \cdots}{i_1! i_3! i_5! \cdots}.
\]
Also, we can generalize the result of [19] that
\[
1 + \sum_{n \geq k \geq 1} t(e(2n, k))\tau^n \sigma^k = \frac{\cos\left(\frac{\pi}{2} \sqrt{(1 - \sigma)\tau}\right)}{\cos\left(\frac{\pi}{2} \sqrt{\tau}\right)}
\]
(where \(e(2n, k)\) is given by Eq. (42) above) to
\[
1 + \sum_{n \geq k \geq 1} t'(e(2n, k))\tau^n \sigma^k = \frac{\cos\left(\frac{\pi}{2} \sqrt{(1 - \sigma + r\sigma)\tau}\right)}{\cos\left(\frac{\pi}{2} \sqrt{(1 + r\sigma)\tau}\right)}
\]
using the reasoning of the last subsection and Eqs. (45).

6.3 Multiple harmonic sums

If one defines, for fixed \(n\), the finite sums
\[
A_{(k_1, \ldots, k_l)}(n) = \sum_{n \geq m_1 > m_2 > \ldots > m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}}
\]
and
\[
S_{(k_1, \ldots, k_l)}(n) = \sum_{n \geq m_1 \geq m_2 \geq \ldots \geq m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},
\]
then there are homomorphisms \(\zeta_{\leq n} : (\mathcal{H}^1, *) \to \mathbb{R}\) and \(\zeta^*_\leq n : (\mathcal{H}^1, *) \to \mathbb{R}\) given by
\[
\zeta_{\leq n}(z_{k_1} \cdots z_{k_l}) = A_{(k_1, \ldots, k_l)}(n)
\]
and
\[
\zeta^*_{\leq n}(z_{k_1} \cdots z_{k_l}) = S_{(k_1, \ldots, k_l)}(n).
\]

Applying these homomorphisms to the Eqs. (32) and (34) above, we obtain
\[
\zeta_{\leq n}(\{k\}_r) = \text{coefficient of } \lambda^r \text{ in } \exp\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i A_{ik}(n)\right).
\]
and
\[
\zeta^*_{\leq n}(\{k\}_r) = \text{coefficient of } \lambda^r \text{ in } \exp\left(\sum_{i \geq 1} \frac{\lambda^i A_{ik}(n)}{i}\right). \tag{46}
\]
Note that $k$ can be 1 in these formulas since the sums involved are finite. In particular, it is well-known that

$$\zeta_{\leq n}(\{1\}_r) = \frac{1}{n!} \left[\begin{array}{c} n+1 \\ r+1 \end{array} \right],$$

where $\left[\begin{array}{c} n \\ k \end{array} \right]$ is the number of permutations of $\{1, 2, \ldots, n\}$ with $k$ disjoint cycles (unsigned Stirling number of the first kind). Eq. (16) can be compared to the explicit formula given by [8, Eq. (21)].

### 6.4 Multiple $q$-zeta values

As in the preceding examples let $A = \{z_1, z_2, \ldots\}$, but now define the product $\diamond$ by

$$z_i \diamond z_j = z_{i+j} + (1 - q)z_{i+j-1}. \quad (47)$$

Here we take as our ground field $k = \mathbb{Q}(1 - q)$. Let $\mathcal{H}_q^1 = k \langle A \rangle$, $\mathcal{H}_q^0$ the subspace generated by words not starting with $z_1$. Then we have homomorphisms $\zeta_q : (\mathcal{H}_q^0, \ast) \to \mathbb{Q}[[q]]$ and $\zeta_q^\ast : (\mathcal{H}_q^0, \ast) \to \mathbb{Q}[[q]]$ given by

$$\zeta_q(z_{k_1} z_{k_2} \cdots z_{k_l}) = \sum_{m_1 > m_2 > \cdots > m_l \geq 1} \frac{q^{m_1(k_1-1)+m_2(k_2-1)+\cdots+m_l(k_l-1)}}{[m_1]^{k_1} [m_2]^{k_2} \cdots [m_l]^{k_l}},$$

and

$$\zeta_q^\ast(z_{k_1} z_{k_2} \cdots z_{k_l}) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_l \geq 1} \frac{q^{m_1(k_1-1)+m_2(k_2-1)+\cdots+m_l(k_l-1)}}{[m_1]^{k_1} [m_2]^{k_2} \cdots [m_l]^{k_l}},$$

where $[m] = (1 - q^m)/(1 - q)$.

Formulas like those obtained in the last three examples are complicated by presence of the extra term in Eq. (47). Iteration of (47) gives

$$z_k^\diamond i = \sum_{j=0}^{i-1} \binom{i-1}{j} (1 - q)^j z_{ik-j}.$$

Then, as in [12, Ex. 4], we can apply $\zeta_q$ and $\zeta_q^\ast$ to the equations

$$\exp_* \left( \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i z_k^\diamond i \right) = \sum_{i=0}^\infty z_k^i \lambda^i \quad \text{and} \quad \exp_* \left( \sum_{i \geq 1} \frac{\lambda^i}{i} z_k^\diamond i \right) = \sum_{i=0}^\infty z_k^i \lambda^i.$$
to get, for $k \geq 2$,
\[
\zeta_q(\{k\}_r) = \text{coefficient of } \lambda^r \text{ in } \exp \left[ \sum_{i \geq 1} \frac{(-1)^{i-1} \lambda^i}{i} \left( \sum_{j=0}^{i-1} \binom{i-1}{j} (1-q)^j \zeta_q(ik - j) \right) \right]
\]
and
\[
\zeta_q^* (\{k\}_r) = \text{coefficient of } \lambda^r \text{ in } \exp \left[ \sum_{i \geq 1} \frac{\lambda^i}{i} \left( \sum_{j=0}^{i-1} \binom{i-1}{j} (1-q)^j \zeta_q(ik - j) \right) \right].
\]

### 6.5 Multiple polylogarithms at roots of unity

Fix $r \geq 2$, and let $\omega = e^{\frac{2\pi i}{r}}$. Then for an integer composition $I = (i_1, \ldots, i_k)$, the values of the multiple polylogarithm $L_I$ at $r$th roots of unity are given by

\[
L_I(\omega^{j_1}, \ldots, \omega^{j_k}) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{\omega^{n_1 j_1} \cdots \omega^{n_k j_k}}{n_1^{i_1} \cdots n_k^{i_k}},
\]
and the series converges provided $\omega^{j_1 i_1} \neq 1$. We can define the multiple "star-polylogarithms" by

\[
L_I^*(\omega^{j_1}, \ldots, \omega^{j_k}) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{\omega^{n_1 j_1} \cdots \omega^{n_k j_k}}{n_1^{i_1} \cdots n_k^{i_k}}.
\]

Here we let $A = \{z_{i,j} : i \geq 1, 0 \leq j \leq r-1\}$ and $z_{i,j} \diamond z_{p,q} = z_{i+p,j+q}$, where the second subscript is understood mod $r$. The algebra $(k\langle A \rangle, \star)$ is called the Euler algebra in [7] (see Ex. 2). Let $E_r = k\langle A \rangle$, $E_r^0$ the subalgebra of $k\langle A \rangle$ generated by words not starting in $z_{1,0}$. Then there are homomorphisms $Z : (E_r^0, \star) \to \mathbb{C}$ and $Z^* : (E_r^0, \star) \to \mathbb{C}$ given by

\[
Z(z_{i_1,j_1} \cdots z_{i_k,j_k}) = L^*_{(i_1, \ldots, i_k)}(\omega^{j_1}, \ldots, \omega^{j_k}),
\]

\[
Z^*(z_{i_1,j_1} \cdots z_{i_k,j_k}) = L^*_{(i_1, \ldots, i_k)}(\omega^{j_1}, \ldots, \omega^{j_k}).
\]

From Corollary 5.1 we have

\[
\exp \left( \sum_{i \geq 1} \frac{(-1)^i \lambda^i}{i} z_{s,t}^i \right) = \sum_{i=0}^{\infty} z_{s,t}^i \lambda^i.
\]
Now \( z_{s,t}^{oi} = z_{si,ti} \), and if \( t \) is relatively prime to \( r \) the preceding equation is

\[
\exp_* \left( \sum_{j=0}^{r-1} \sum_{i \geq 1 \mod r, ti \equiv j} \frac{(-1)^{i-1} \lambda^i}{i} z_{is,j} \right) = \sum_{i=0}^{\infty} z_{s,t}^i \lambda^i,
\]

with fewer terms on the left-hand side if \( t \) has factors in common with \( r \).

Applying \( Z \) to both sides, we have

\[
\text{Li}_{k} (s, \ldots, s) (\omega^t, \ldots, \omega^t) = \text{coefficient of } \lambda^k \text{ in } \exp \left( \sum_{j=0}^{r-1} \sum_{i \geq 1 \mod r, ti \equiv j} \frac{(-1)^{i-1} \lambda^i}{i} \text{Li}_{is}(\omega^j) \right). \tag{48}
\]

The counterpart for multiple star-polylogarithms is

\[
\text{Li}_{k}^* (s, \ldots, s) (\omega^t, \ldots, \omega^t) = \text{coefficient of } \lambda^k \text{ in } \exp \left( \sum_{j=0}^{r-1} \sum_{i \geq 1 \mod r, ti \equiv j} \frac{\lambda^i}{i} \text{Li}_{is}(\omega^j) \right). \tag{49}
\]

The simplest case is \( r = 2 \). Here \( \omega = -1 \) and images under \( Z \) and \( Z^* \) are the alternating or “colored” MZVs (which lie in \( \mathbb{R} \)). In this case we can streamline the notation and write, e.g., \( \zeta(2, 1) \) instead of \( \text{Li}_{(2,1)}(-1, 1) \). Note that \( \zeta(1) = -\log 2 \) and \( \zeta(s) = (2^{1-s} - 1)\zeta(s) \) for \( s \geq 2 \). For \( r = 2, t = 1 \) and \( s \geq 2 \), Eq. (48) above simplifies to

\[
\sum_{k=0}^{\infty} \zeta((s)_{k}) \lambda^k = \exp \left( - \sum_{i \text{ even}} \frac{\lambda^i}{i} \zeta(is) + \sum_{i \text{ odd}} \frac{\lambda^i}{i} \zeta(is) \right)
\]

\[
= \exp \left( - \sum_{i=1}^{\infty} \frac{\lambda^i}{i} \zeta(is) + 2 \sum_{i \text{ odd}} \left( \frac{\lambda^i}{2^s} \right)^i \frac{\zeta(is)}{i} \right)
\]

\[
= Z_{s}(-\lambda) \exp \left( \sum_{i \text{ odd}} \left( \frac{\lambda^i}{2^s} \right)^i \frac{\zeta(is)}{i} \right)^2,
\]
where \( Z_s(\lambda) \) is defined by Eq. (33) above. Now
\[
\sum_{i \text{ odd}} \left( \frac{\lambda}{2^s} \right)^i \frac{\zeta(is)}{i} = \sum_{i=1}^{\infty} \left( \frac{\lambda}{2^s} \right)^i (-1)^{i-1} \frac{\zeta(is)}{i} + \sum_{i=1}^{\infty} \left( \frac{\lambda}{2^s} \right)^{2i} \frac{\zeta(2is)}{2i}
\]
so that
\[
\exp \left( \sum_{i \text{ odd}} \left( \frac{\lambda}{2^s} \right)^i \frac{\zeta(is)}{i} \right) = \frac{Z_s \left( \frac{\lambda}{2^s} \right)}{\sqrt{Z_{2s} \left( -\frac{\lambda^2}{4s} \right)}}
\]
and thus
\[
\zeta(\{s\}_k) = \text{coefficient of } \lambda^k \text{ in } \frac{Z_s(-\lambda)Z_s \left( \frac{\lambda}{2^s} \right)^2}{Z_{2s} \left( -\frac{\lambda^2}{4s} \right)};
\]
cf. [2, Eq. (12)]. In particular
\[
\sum_{k=0}^{\infty} \zeta(\{2\}_k) \lambda^k = \frac{Z_2(-\lambda)Z_2 \left( \frac{\lambda}{2^s} \right)^2}{Z_4 \left( -\frac{\lambda^2}{16} \right)} = \frac{2 \cos \left( \frac{\pi}{2} \sqrt{\lambda} \right) \sinh \left( \frac{\pi}{2} \sqrt{\lambda} \right)}{\pi \sqrt{\lambda}}.
\]
Similarly, for \( r = 2, t = 1 \) and \( s \geq 2 \), Eq. (49) above gives
\[
\sum_{k=0}^{\infty} \zeta^{\ast}(\{\bar{s}\}_k) \lambda^k = \exp \left( \sum_{i \text{ even}} \frac{\lambda^i}{i} \zeta(i) + \sum_{i \text{ odd}} \frac{\lambda^i}{i} \zeta(i) \right) - \sum_{i=1}^{\infty} \frac{(-\lambda)^i}{i} \zeta(i) + 2 \sum_{i \text{ odd}} \left( \frac{\lambda}{2^s} \right)^i \frac{\zeta(is)}{i}
\]
which for \( s = 2 \) is
\[
\sum_{k=0}^{\infty} \zeta^{\ast}(\{2\}_k) \lambda^k = \frac{Z_2 \left( \frac{\lambda}{2} \right)^2}{Z_2(\lambda)Z_4 \left( -\frac{\lambda^2}{16} \right)} = \frac{\pi \sqrt{\lambda}}{2 \cosh \left( \frac{\pi}{2} \sqrt{\lambda} \right) \sin \left( \frac{\pi}{2} \sqrt{\lambda} \right)}.
\]
In the case \( s = 1 \) we have
\[
\sum_{k=0}^{\infty} \zeta(\{1\}_k) \lambda^k = \exp \left( -\sum_{i \text{ even}} \frac{\lambda^i}{i} \zeta(i) + \sum_{i \text{ odd}} \frac{\lambda^i}{i} \zeta(i) \right) - \sum_{i=2}^{\infty} \frac{\lambda^i}{i} \zeta(i) - \lambda \log 2 + 2 \sum_{i \geq 3 \text{ odd}} \left( \frac{\lambda}{2} \right)^i \frac{\zeta(i)}{i}
\]
\[
\exp \left( -\sum_{i=2}^{\infty} \frac{\lambda^i}{i} \zeta(i) - \lambda \log 2 + 2 \sum_{i \geq 3 \text{ odd}} \left( \frac{\lambda}{2} \right)^i \frac{\zeta(i)}{i} \right) = \frac{Z_1(-\lambda)Z_1 \left( \frac{\lambda}{2} \right)^2}{Z_2 \left( -\frac{\lambda^2}{4} \right)}.
\]
with $Z_1(\lambda)$ interpreted as
\[
\exp \left( \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(i) \right) = \frac{1}{e^{\gamma \lambda} \Gamma(1 + \lambda)},
\]
where $\gamma$ is Euler’s constant and $\Gamma$ is the gamma function. Using the duplication and reflection formulas for the gamma function, Eq. (50) gives
\[
\sum_{k=0}^{\infty} \zeta(\{1\}_k) \lambda^k = \frac{\sqrt{\pi}}{\Gamma \left( \frac{1 - \lambda}{2} \right) \Gamma \left( \frac{1 + \lambda}{2} \right)},
\]
cf. [2, Eq. (13)]. Similarly
\[
\sum_{k=0}^{\infty} \zeta^*(\{1\}_k) \lambda^k = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1 + \lambda}{2} \right) \Gamma \left( \frac{1 - \lambda}{2} \right).
\]
The remarkable identity
\[
\zeta(\{2,1\}_n) = \frac{1}{8^n} \zeta(\{3\}_n), \tag{51}
\]
was conjectured in [2] but only proved by J. Zhao [18] 13 years later. As in Subsection 6.1 above, we can apply Theorem 5.2 (this time with $a = z_{2,1}$ and $b = z_{1,0}$) to get
\[
\sum_{n=0}^{\infty} \zeta^*(\{2,1\}_n) \lambda^n = \sum_{p=0}^{\infty} \zeta(\{2,1\}_p) \lambda^p \sum_{q=0}^{\infty} \zeta^*(\{3\}_q) \lambda^q,
\]
which by Eq. (51) and the results above is
\[
\sum_{n=0}^{\infty} \zeta^*(\{2,1\}_n) \lambda^n = \frac{Z_3 \left( \frac{\lambda}{8} \right)^3}{Z_3(\lambda) Z_6 \left( -\frac{\lambda^2}{64} \right)} = \exp \left( \sum_{i \geq 1 \text{ odd}} \frac{\zeta(3i) \lambda^i}{i 8^i} (3 - 8^i) \right) \exp \left( \sum_{i \geq 2 \text{ even}} \frac{\zeta(3i) \lambda^i}{i 8^i} (8^i - 1) \right).
\]
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