RADEMACHER–CARLITZ POLYNOMIALS

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Abstract. We introduce and study the Rademacher–Carlitz polynomial

\[ R(u, v, s, t, a, b) := \sum_{k=\lceil s \rceil}^{b-1} u^{\lfloor ka+t \rfloor} v^k \]

where \( a, b \in \mathbb{Z}_{>0}, s, t \in \mathbb{R} \), and \( u \) and \( v \) are variables. These polynomials generalize and unify various Dedekind-like sums and polynomials; most naturally, one may view \( R(u, v, s, t, a, b) \) as a polynomial analogue (in the sense of Carlitz) of the Dedekind–Rademacher sum

\[ r_t(a, b) := \sum_{k=0}^{b-1} \left( \frac{ka+t}{b} \right) \left( \frac{k}{b} \right), \]

which appears in various number-theoretic, combinatorial, geometric, and computational contexts. Our results come in three flavors: we prove a reciprocity theorem for Rademacher–Carlitz polynomials, we show how they are the only nontrivial ingredients of integer-point transforms

\[ \sigma(x, y) := \sum_{(j, k) \in P \cap \mathbb{Z}^2} x^j y^k \]

of any rational polyhedron \( P \), and we derive a novel reciprocity theorem for Dedekind–Rademacher sums, which follows naturally from our setup.

1. Introduction

While studying the transformation properties of \( \eta(z) := e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz}) \) under \( \text{SL}_2(\mathbb{Z}) \), Richard Dedekind, in the 1880’s [10], naturally arrived at what we today call the Dedekind sum

\[ s(a, b) := \sum_{k=0}^{b-1} \left( \frac{ka+t}{b} \right) \left( \frac{k}{b} \right), \]

where \( a \) and \( b \) are positive integers and

\[ ([x]) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases} \]

The Dedekind sum and its generalizations have since intrigued mathematicians from various areas such as analytic (see, e.g., [1, 3]) and algebraic number theory (see, e.g., [9, 17, 22]), topology (see, e.g., [13, 15]), algebraic (see, e.g., [7, 12, 19]) and combinatorial geometry (see, e.g., [6, 16]), and algorithmic complexity (see, e.g., [14]).

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Almost a century after the appearance of Dedekind sums, Leonard Carlitz introduced a polynomial analogue, the \textit{Dedekind–Carlitz polynomial}

\[
c(u, v, a, b) := \sum_{k=1}^{b-1} u\left\lfloor \frac{ka}{b} \right\rfloor v^{k-1}.
\]

Here \(u\) and \(v\) are indeterminates and \(a\) and \(b\) are positive integers. Undoubtedly the most important basic property for any Dedekind-like sum is \textit{reciprocity}. For the Dedekind–Carlitz polynomials, it says that if \(a\) and \(b\) are relatively prime then \[8\]

\[
(v - 1) c(u, v, a, b) + (u - 1) c(v, u, b, a) = u^{a-1} v^{b-1} - 1.
\]

Carlitz’s reciprocity theorem generalizes that of Dedekind \[10\], which states that for relatively prime positive integers \(a\) and \(b\),

\[
s(a, b) + s(b, a) = -\frac{1}{4} + \frac{1}{12} \left( \frac{a}{b} \right)^2 + \frac{1}{ab} + \frac{b}{a}.
\]

Dedekind reciprocity follows from (1) by applying the operators \(u \partial u\) twice and \(v \partial v\) once to Carlitz’s reciprocity identity.

Dedekind sums have many generalizations. One of the earliest will play a central role in this paper: for \(a, b \in \mathbb{Z}_{>0}\), and \(t \in \mathbb{R}\), we define the \textit{Dedekind–Rademacher sum} \[20\]

\[
r_t(a, b) := \sum_{k=0}^{b-1} \left( \left( \frac{k}{b} \right) \right) \left( \left( \frac{k}{b} + t \right) \right).
\]

Our goal is to introduce and study an analogue of this sum in the world of polynomials: for \(a, b \in \mathbb{Z}_{>0}\), \(s, t \in \mathbb{R}\), and variables \(u\) and \(v\), we define the \textit{Rademacher–Carlitz polynomial}

\[
R(u, v, s, t, a, b) := \sum_{k=\lceil s \rceil}^{[s]+b-1} u\left\lfloor \frac{ka+t}{b} \right\rfloor v^{k}.\]

Naturally, Dedekind–Carlitz polynomials are special cases of Rademacher–Carlitz polynomials, in the sense that \(v c(u, v, a, b) = R(u, v, 0, 0, a, b) - 1\). It will be handy to abbreviate the linear function \(f(x) := \frac{ax+t}{b}\) which appears in the exponent of \(u\), and so we will typically use the notation

\[
R(u, v, s, f) := \sum_{k=\lceil s \rceil}^{[s]+b-1} u\left\lfloor f(k) \right\rfloor v^{k}.
\]

with the understanding that \(b\) equals the denominator in the linear function \(f\).

Our motivation to study Rademacher–Carlitz polynomials is twofold: first, they seem natural generalizations of Dedekind–Carlitz polynomials and, as we will see below, they give rise not only to new reciprocity theorems but also new results on old constructs, such as Dedekind–Rademacher sums. Our second motivation stems from the fact that Rademacher–Carlitz polynomials appear naturally—as we will also show below—in the \textit{integer-point transforms}

\[
\sigma P(x, y) := \sum_{(m, n) \in P \cap \mathbb{Z}^2} x^m y^n
\]

of 2-dimensional rational polyhedra \(P\), in particular, 2-dimensional cones/polygons with rational vertices. In fact, our paper extends some of the methods introduced in \[4\], which showed that Dedekind–Carlitz polynomials are natural ingredients for 2-dimensional \textit{lattice} polyhedra, i.e., those with integral vertices. Carlitz’s reciprocity theorem \[11\] was a natural by-product of the geometric
approach of [4], and our first result, which mirrors the geometric setup of [4], is a reciprocity theorem for Rademacher–Carlitz polynomials.

**Theorem 1.** Let \( f(x) := \frac{ax + b}{b} \) be a linear function with relatively prime \( a, b \in \mathbb{Z}_{>0} \), \( t \in \mathbb{R} \), and let \((p, q) \in \mathbb{R}^2 \) be a point on the graph of \( f \). Then

\[
v(1-u) R(v, u, p, f) + u(1-v) R(u, v, q, f^{-1}) = u[p]v[q] \left( 1 - u^b v^a \right) - u^c v^d (1-u)(1-v),
\]

where \((c, d)\) is the unique lattice point on the half-open line segment \([(p, q), (p+b, q+a))\); if there are no integer points on the graph of \( f \) (and so \((c, d)\) does not exist), the last term on the right-hand side needs to be omitted.

We give a proof in Section 2, where we will also show how (1) follows as a corollary. One can phrase the conditions in Theorem 1 in purely number-theoretic terms as follows.

**Corollary 2.** Let \( a, b \in \mathbb{Z}_{>0} \) be relatively prime and \( p, q \in \mathbb{R} \). Then

\[
v(1-u) R(v, u, p, bq - ap, a, b) + u(1-v) R(u, v, q, ap - bq, b, a) = u[p]v[q] \left( 1 - u^b v^a \right) - u^c v^d (1-u)(1-v),
\]

where \( c \in \mathbb{Z} \) is (uniquely) determined by the conditions

\[
ac \equiv ap - bq \pmod{b} \quad \text{and} \quad p \leq c < p + b,
\]

and \( d := \frac{ac + bq - ap}{b} \). If \( ap - bq \notin \mathbb{Z} \) then the last term on the right-hand side needs to be omitted.

Returning to our second motivation, we remark that the evaluation \( \sigma_\mathcal{P}(1,1) \) of an integer-point transform yields the number of integer lattice points in \( \mathcal{P} \). Ehrhart [11] famously proved in the 1960s that the counting function

\[
ehr_\mathcal{P}(t) := \# (t\mathcal{P} \cap \mathbb{Z}^d)
\]

is a polynomial in the positive integer variable \( t \) when \( \mathcal{P} \) is a lattice polytope, and a quasipolynomial when \( \mathcal{P} \) is a rational polytope (see, e.g., [6] for more on Ehrhart quasipolynomials). It is a natural question how to compute Ehrhart (quasi-)polynomials and integer-point transforms, both in a computational complexity sense and in terms of ingredients for possible formulas. We will only briefly touch on the computational aspect, which is governed by Barvinok’s theorem [2]. The ingredients of degree-2 Ehrhart polynomials are easy; they essentially follow from Pick’s theorem [18] (of which Ehrhart’s theorem can be viewed as a far-reaching generalization). The classification question for degree-2 Ehrhart quasipolynomials, i.e., stemming from rational polygons was answered much more recently [3]; here Dedekind–Rademacher sums play a crucial role as the only nontrivial ingredients. The analogous classification question for integer-point transforms of lattice polygons was answered in [3], and Dedekind–Carlitz polynomials played the role here of the nontrivial ingredients. Our next result provides formulas for the integer-point transforms of rational polygons; it can be viewed as a common generalization (and combination) of the classification results in [4] and [5], and indeed, from this point of view, it should come as no surprise that Rademacher–Carlitz polynomials make an appearance.

**Theorem 3.** Let \( a, b, c, d, e, f, g, h \in \mathbb{Z}_{>0} \), and let \( \Delta \) denote the triangle with vertices \((\frac{f}{f}, \frac{g}{g}), (\frac{a}{a}, \frac{b}{b})\) and \((\frac{e}{e}, \frac{h}{h})\). Moreover, we define \( \alpha := dh(b - af) \), \( \beta := bf(ch - dg) \), and \( l(x) := \frac{2}{a}x + \frac{2}{e} - \frac{2}{f}\beta \). Then the integer-point transform of \( \Delta \) equals

\[
\sigma_\Delta(x, y) = \frac{x \left[ \frac{f}{f}y \right]}{(1-x)(1-y)} + \frac{R(x, y, \frac{g}{g}, l^{-1})}{(1-x^{-1})(1-x^{a}y^{b})} + \frac{R(y, x, \frac{e}{e}, l)}{(1-y^{-1})(1-x^{-a}y^{-\beta})}.
\]
We give a proof in Section 3. Theorem 3 suffices to provide formulas for the integer-point transform of any rational polygon: we can triangulate a given rational polygon, hence we only have to treat the case of rational triangles and rational line segments, whose integer-point transforms are relatively straightforward to compute. Using a simple geometric argument (which we will see in Section 3), we can reduce the case of rational triangles to rational right triangles with edges parallel to \( x \)- and \( y \)-axis, which are the contents of Theorem 3.

Our final result is a pleasant by-product of the geometric treatment of Dedekind-like sums; it turns out that we obtain the following reciprocity theorem for Dedekind–Rademacher sums, which seems to be new.

**Theorem 4.** Let \( a \) and \( b \) be relatively prime positive integers with \( a < b \), and let \( t \in \mathbb{R} \) with \( 0 \leq t < b \). Then

\[
    r_{-t}(a,b) + r_t(b,a) = 12 \left( \frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right) - \frac{1}{4} + \frac{1}{2ab} |t| (|t| + 1) - \frac{1}{2} \left\lfloor \frac{t}{a} \right\rfloor - \frac{\chi}{2} \left( \left( \frac{a^{-1}t}{b} \right) + \left( \frac{b^{-1}t}{a} \right) \right),
\]

where \( \chi \) equals 1 or 0 depending on whether or not \( t \in \mathbb{Z} \), \( a a^{-1} \equiv 1 \mod b \), and \( b b^{-1} \equiv 1 \mod a \).

Note that the conditions on \( a \), \( b \), and \( t \) do not constitute a restriction for practical purposes, as

\[
    r_t(a,b) = r_{t \mod b}(a \mod b, b).
\]

At any rate, our proof of Theorem 4 which we give in Section 4 contains reformulations without the conditions \( a < b \) and \( 0 \leq t < b \).

Dedekind’s reciprocity theorem (2) follows naturally from Theorem 4 by setting \( t = 0 \). However, the more interesting comparison is with Rademacher’s reciprocity theorem, which he stated as follows [20]: For \( a, b \in \mathbb{Z} \) and \( x, y \in \mathbb{R} \), let

\[
    s(a,b; x, y) := \sum_{k=0}^{b-1} \left( \left( \frac{(k+y)a}{b} + x \right) \left( \frac{k+y}{b} \right) \right).
\]

Then, if \( a \) and \( b \) are relatively prime and \( x \) and \( y \) are not both integers,

\[
    s(a,b; x, y) + s(b,a; y, x) = (x)((y)) + \frac{1}{2} \left( \frac{a}{b} B_2(y) + \frac{1}{ab} B_2(ay + bx) + \frac{b}{a} B_2(x) \right),
\]

where \( B_2(x) := \{x\}^2 - \{x\} + \frac{1}{6} \) is the periodized second Bernoulli polynomial. A moment’s thought reveals that any sum of the form (3) can be expressed in the form (4) and vice versa. Indeed, setting \( y = 0 \) and \( x = \frac{t}{b} \) gives

\[
    s \left( a, b; \frac{t}{b}, 0 \right) = \sum_{k=0}^{b-1} \left( \left( \frac{ka + t}{b} \right) \left( \frac{k}{b} \right) \right) \quad \text{and} \quad s \left( b, a; 0, \frac{t}{b} \right) = \sum_{k=0}^{a-1} \left( \left( \frac{kb + t}{a} \right) \left( \frac{k}{b} \right) \right).
\]

The latter sum equals \( \sum_{k=0}^{a-1} \left( \left( \frac{kb+t}{a} \right) \left( \frac{k}{b} \right) \right) \) plus some trivial terms. So Rademacher’s reciprocity theorem expressed in terms of \( r_t(a,b) \) says that

\[
    r_t(a,b) + r_t(b,a)
\]

equals a simple expression in terms of \( a \), \( b \), and \( t \). Theorem 4, on the other hand, says that

\[
    r_t(a,b) + r_{-t}(b,a)
\]

equals a simple expression, and so it gives a statement complementary to Rademacher reciprocity. As far as we can tell, the only overlap of the two reciprocity theorems is the case \( t = 0 \), i.e., Dedekind’s reciprocity theorem.
2. The reciprocity theorem for Rademacher–Carlitz polynomials

Proof of Theorem 1. As mentioned in the introduction, we follow the ideas of [4] which gave a novel geometric proof of (1). Let

\[ f(x) := ax + bt \]

with \(a, b \in \mathbb{Z}_{>0}\), where \(\gcd(a, b) = 1\), and \(t \in \mathbb{R}\), and let \((p, q) \in \mathbb{R}^2\) be a point on the graph of \(f\). Consider the half-open cones

\[
\mathcal{K}_1 := \{(p, q) + \lambda_1(1, 0) + \lambda_2(b, a) : \lambda_1 > 0, \lambda_2 \geq 0\}
\]

\[
\mathcal{K}_2 := \{(p, q) + \lambda_1(0, 1) + \lambda_2(b, a) : \lambda_1 > 0, \lambda_2 \geq 0\}
\]

and the ray

\[
\mathcal{K}_3 := \{(p, q) + \lambda(b, a) : \lambda \geq 0\}.
\]

These three objects give a disjoint conic decomposition of the shifted first quadrant, shown in Figure 1:

\[
\{(p, q) + \lambda_1(1, 0) + \lambda_2(0, 1) : \lambda_1, \lambda_2 \geq 0\} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3,
\]

and our goal is to compute the integer-point transforms on both sides. For the shifted first quadrant, this integer-point transform is

\[
\sigma_{\mathcal{K}_1}(u, v) = \frac{u|v|_1[q]}{(1 - u)(1 - v)}.
\]

By a simple tiling argument (see, for example, [6, Chapter 3]), the integer-point transform \(\sigma_{\mathcal{K}_1}(u, v)\) of the half-open cone \(\mathcal{K}_1\) is

\[
\sigma_{\mathcal{K}_1}(u, v) = \frac{\sigma_{\Pi_1}(u, v)}{(1 - u)(1 - u^b v^a)}
\]

where

\[
\Pi_1 := \{(p, q) + \lambda_1(1, 0) + \lambda_2(b, a) : 0 < \lambda_1 \leq 1, 0 \leq \lambda_2 < 1\},
\]

the fundamental parallelogram of \(\mathcal{K}_1\). Since it has width 1, there is exactly one integer point in \(\Pi_1\) for each \(y\) running from \(\lceil q \rceil\) to \(\lceil q \rceil + a - 1\). The \(x\)-coordinate of this integer point is \(\lceil f^{-1}(y) \rceil + 1\). Thus

\[
\sigma_{\Pi_1}(u, v) = \sum_{k = \lceil q \rceil}^{\lceil q \rceil + a - 1} u^{\lceil f^{-1}(k) \rceil + 1} v^k = u R(u, v, q, f^{-1}).
\]

With a similar argument, changing the roles of the axes, we obtain our second integer-point transform:

\[
\sigma_{\mathcal{K}_2}(u, v) = \frac{\sigma_{\Pi_2}(u, v)}{(1 - v)(1 - u^b v^a)}
\]
where

\[ \sigma_{\Pi_2}(u, v) = \sum_{k=\lceil p \rceil}^{\lceil p \rceil + b - 1} u^k v^{\lfloor f(k) \rfloor + 1} = v^R(v, u, p, f). \]

It remains to compute the integer-point transform of the ray \( K_3 \). It is clear that any two lattice points on \( K_3 \) differ by a multiple of \((b, a)\) and so

\[ \sigma_{K_3}(u, v) = u^{c} v^{d} \]

where \((c, d)\) is the lattice point on \( K_3 \) with the smallest coordinates, if there is a lattice point on \( K_3 \) at all—otherwise \( \sigma_{K_3}(u, v) \) will simply not appear in our formulas.

Thus (5) translates into the identity of rational generating functions

\[ \frac{u^p v^q}{(1-u)(1-v)} = \frac{u^R(u, v, q, f^{-1})}{(1-u)(1-u^b v^a)} + \frac{v^R(v, u, p, f)}{(1-v)(1-u^b v^a)} + \frac{u^c v^d}{1-u^b v^a}, \]

where the last term only appears if \( K_3 \) contains lattice points. Clearing denominators gives Theorem 1.

Carlitz’s reciprocity theorem (1) follows as an immediate corollary by choosing \( t = p = q = 0 \): note that then \( c = d = 0 \), and so Theorem 1 gives in this special case

\[ v(1-u) R(v, u, 0, f) + u(1-v) R(u, v, 0, f^{-1}) = 1 - u^b v^a - (1-u)(1-v). \]

We rewrite the expression on the left to see Dedekind–Carlitz polynomials appear:

\[ v(1-u) R(v, u, 0, f) - 1 + u(1-v) R(u, v, 0, f^{-1}) - 1 = -u^b v^a + uv. \]

Dividing by \(-uv\) gives (1).

We finish this section with a remark about computational complexity. In the introduction we hinted at Barvinok’s theorem [2], which says that in fixed dimensions, the integer-point transform \( \sigma_{\mathcal{P}}(x_1, \ldots, x_d) \) of a rational polyhedron \( \mathcal{P} \) can be computed as a sum of short rational functions in \( x_1, x_2, \ldots, x_d \) in time polynomial in the input size of \( \mathcal{P} \). Thus (say) \( \sigma_{\Pi_2}(u, v) \) can be computed efficiently, which means we can compute Rademacher–Carlitz sums efficiently. (This is a nontrivial statement, since Rademacher–Carlitz sums have exponentially many terms when measured in the input size of its parameters.)

### 3. Integer-point transforms of rational polygons

In this section, we give the details behind our claim that Theorem 3 suffices to characterize the integer-point transform of any rational polygon, and we will prove Theorem 3.

As mentioned in the introduction, any rational polygon can be triangulated, and so we can compute its integer-point transform in an inclusion-exclusion fashion from integer-point transforms of rational line segments and rational triangles. Furthermore, we can embed an arbitrary triangle in a rectangle in such a way that we can express the triangle as a set union/subtraction of rectangles and right triangles with edges parallel to \( x- \) and \( y- \)axis, as suggested by Figure 3 if the triangle was rational to begin with, so will be the rectangles and right triangles. The integer-point transforms of rectangles are easy, and thus it remains to compute integer-point transforms of right triangles with edges parallel to \( x- \) and \( y- \)axis, which (by a harmless lattice transformation) we may assume to be in the first quadrant with its right angle in the southwestern vertex. That is, it remains to prove Theorem 3.
Proof of Theorem 3. As stated in the conditions, we assume that \( \Delta \) looks like in Figure 3. To compute the integer-point transform of \( \Delta \), we use Brion’s theorem [7], which says that \( \sigma_{\Delta}(x,y) \) equals the sum of the integer-point transforms of the three vertex cones of \( \Delta \). (The vertex cone of a polytope \( \mathcal{P} \) at a vertex \( v \) is the smallest cone with apex \( v \) that contains \( \mathcal{P} \).) Thus we need to compute the integer-point transforms of the vertex cones

\[
\begin{align*}
V_1 &:= \left\{ \left( \frac{e}{f}, \frac{g}{h} \right) + \lambda_1 (1,0) + \lambda_2 (0,1) : \lambda_1, \lambda_2 \geq 0 \right\} \\
V_2 &:= \left\{ \left( \frac{a}{b}, \frac{g}{h} \right) + \lambda_1 (-1,0) + \lambda_2 \left( -dh(be - af), bf(ch - dg) \right) : \lambda_1, \lambda_2 \geq 0 \right\} \\
V_3 &:= \left\{ \left( \frac{e}{f}, \frac{c}{d} \right) + \lambda_1 (0,-1) + \lambda_2 \left( -dh(be - af), -bf(ch - dg) \right) : \lambda_1, \lambda_2 \geq 0 \right\}.
\end{align*}
\]

To shorten notation, we define, as in the statement of Theorem 3, \( \alpha := dh(be - af) \) and \( \beta := bf(ch - dg) \). The integer-point transform of \( V_1 \) is straightforward:

\[
\sigma_{V_1}(x,y) = \sum_{k \geq \left\lceil \frac{e}{f} \right\rceil, j \geq \left\lceil \frac{g}{h} \right\rceil} x^k y^j = \frac{x^{\left\lceil \frac{e}{f} \right\rceil} y^{\left\lceil \frac{g}{h} \right\rceil}}{(1-x)(1-y)}.
\]

For the other two vertex cones, we use a tiling argument similar to the one in the proof of Theorem 1. This gives,

\[
\begin{align*}
\sigma_{V_2}(x,y) &= \frac{\sigma_{\Pi_2}(x,y)}{(1-x^{-1})(1-x^{-\alpha}y^{-\beta})} \\
\sigma_{V_3}(x,y) &= \frac{\sigma_{\Pi_3}(x,y)}{(1-y^{-1})(1-x^{-\alpha}y^{-\beta})}
\end{align*}
\]
where

$$\Pi_2 := \left\{(a, \frac{g}{h}) + \lambda_1 (-1, 0) + \lambda_2 (\alpha, \beta) : 0 \leq \lambda_1, \lambda_2 < 1 \right\}$$

$$\Pi_3 := \left\{(e, \frac{f}{g}) + \lambda_1 (0, -1) + \lambda_2 (-\alpha, -\beta) : 0 \leq \lambda_1, \lambda_2 < 1 \right\}$$

are the fundamental parallelograms of $V_2$ and $V_3$, respectively. To compute the integer-point transform of $\Pi_2$, we note that the linear function $l(x) := \frac{\beta}{\alpha} x + \frac{c}{d} - \frac{e \alpha}{f \beta}$ given in the statement of Theorem 1 describes the line that contains the hypotenuse of $\Delta$. Since $\Pi_2$ has height 1 and is half open, for every integral $y$-coordinate between $\lceil \frac{g}{h} \rceil$ and $\lceil \frac{g}{h} \rceil + \beta - 1$ there is exactly one point with integral $x$-coordinate, namely $\lfloor l^{-1}(y) \rfloor$, and so

$$\sigma_{\Pi_2}(x, y) = \sum_{k=\lceil \frac{g}{h} \rceil}^{\lceil \frac{g}{h} \rceil + \beta - 1} x^{l^{-1}(k)} y^k = R \left( x, y, \frac{g}{h}, l^{-1} \right).$$

A parallel argumentation yields

$$\sigma_{\Pi_3}(x, y) = \sum_{k=\lfloor \frac{e}{f} \rfloor}^{\lfloor \frac{e}{f} \rfloor + \alpha - 1} x^{l^{-1}(k)} y^{l(k)} = R \left( y, x, \frac{e}{f}, l \right).$$

Brion’s theorem says

$$\sigma_{\Delta}(x, y) = \sigma_{V_1}(x, y) + \sigma_{V_2}(x, y) + \sigma_{V_3}(x, y),$$

which, using (6)–(10), yields Theorem 1. □

4. A novel reciprocity theorem for Dedekind–Rademacher sums

Our goal in this section is to prove Theorem 4. We will need a few identities that are slightly technical but straightforward. For $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{>0}$, we denote by $[x]_m$ the smallest nonnegative real number congruent to $x$ mod $m$.

**Lemma 5.** Let $a$ and $b$ be positive relatively prime integers, and let $t \in \mathbb{R}$.

(a) $\sum_{k=0}^{b-1} \left\{ \frac{ak + t}{b} \right\} = \frac{b - 1}{2} + \{t\}.$

(b) $\sum_{k=0}^{b-1} k \left\{ \frac{ak + t}{b} \right\} = b \tau_t(a, b) + \frac{1}{2} b (b - 1) + \frac{1}{2} b \{t\} - \frac{1}{2} [t]_b + \frac{1}{2} b \left( \left( \frac{ta^{-1}}{b} \right) \right)$

where $\chi$ equals 1 or 0 depending on whether or not $t$ is an integer.

**Proof.** (a) is essentially Raabe’s formula (see, e.g., [21 Lemma 1]).

(b) We compute

$$\frac{1}{b} \sum_{k=0}^{b-1} k \left\{ \frac{ak + t}{b} \right\} = \frac{b-1}{2} \sum_{k=1}^{b-1} \left\{ \frac{k}{b} \right\} \left\{ \frac{ak + t}{b} \right\}$$

$$= \sum_{k=1}^{b-1} \left( \left( \frac{k}{b} \right) \left( \left( \frac{ak + t}{b} \right) \right) \right) + \frac{1}{2} \sum_{k=1}^{b-1} \left\{ \frac{ak + t}{b} \right\} + \frac{1}{2} \sum_{k=1}^{b-1} \left\{ \frac{k}{b} \right\} - \frac{b - 1}{4} + \chi \left( \left( \frac{ta^{-1}}{b} \right) \right).$$
Lemma 5. For all $x$, we start by applying the operators $u \partial u$ twice and $v \partial v$ once to the identity in Theorem 1, which yields

\[
\sum_{k=0}^{[p]+b-1} k \left( \frac{ak + t}{b} \right) = \sum_{k=0}^{[p]+b-1} k \left( \frac{ak + t}{b} \right) - \sum_{k=0}^{[p]+b-1} k \left( \frac{ak + t}{b} \right)
\]

\[
= \frac{1}{3} ab^2 + ab[p] + a[p]^2 - \frac{1}{2} ab - a[p] + \frac{1}{2} bt + [p]t + \frac{1}{6} a - \frac{1}{2} t - \sum_{k=0}^{[p]+b-1} (k + [p]) \left( \frac{a(k + [p]) + t}{b} \right)
\]

\[
= \frac{1}{3} ab^2 + ab[p] + a[p]^2 - \frac{1}{2} ab - a[p] + \frac{1}{2} bt + [p]t + \frac{1}{6} a - \frac{1}{2} t + \frac{1}{2} [a[p] + t] b
\]

\[
- \frac{1}{4} a b^2 - \frac{1}{2} b[p] + \frac{1}{4} b + \frac{1}{2} p - b \chi_{a[p] + t}(a, b) - \frac{1}{2} b \left( \frac{[p] + ta^{-1}}{b} \right),
\]

where again $a a^{-1} \equiv 1 \text{ mod } b$. (Note that $a[p] + t \in \mathbb{Z}$ if and only if $t \in \mathbb{Z}$.)
Finally, we have

\[
\sum_{k=\lfloor p \rfloor}^{\lfloor p \rfloor + b-1} \left\lfloor \frac{ak + t}{b} \right\rfloor = \sum_{k=\lfloor p \rfloor}^{\lfloor p \rfloor + b-1} \frac{ak + t}{b} - \sum_{k=\lfloor p \rfloor}^{\lfloor p \rfloor + b-1} \left\{ \frac{ak + t}{b} \right\} = \frac{1}{2} a(b - 1) + a\lfloor p \rfloor + t - \sum_{k=0}^{b-1} \left\{ \frac{k + t}{b} \right\} = \frac{1}{2} (a - 1)(b - 1) + a\lfloor p \rfloor + \lfloor t \rfloor.
\]

Analogously,

\[
\sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left\lfloor \frac{bk - t}{a} \right\rfloor = \frac{1}{2} (a - 1)(b - 1) + b\lfloor q \rfloor + \lfloor -t \rfloor.
\]

Finally,

\[
\sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left\lfloor \frac{bk - t}{a} \right\rfloor^2 = \sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left( \frac{bk - t}{a} \right)^2 - 2 \sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \frac{bk - t}{a} \left\{ \frac{bk - t}{a} \right\} + \sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left\{ \frac{bk - t}{a} \right\}^2 = \frac{1}{3} ab^2 + b^2 \lfloor q \rfloor - \frac{1}{2} b^2 - bt + \frac{1}{a} \left( b^2 \lfloor q \rfloor^2 - b^2 \lfloor q \rfloor - 2b\lfloor q \rfloor t + \frac{1}{6} b^2 + bt + t^2 \right) - \frac{2b}{a} \sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left\{ \frac{bk - t}{a} \right\} + \frac{2t}{a} \sum_{k=\lfloor q \rfloor}^{\lfloor q \rfloor + a-1} \left\{ \frac{k - t}{a} \right\} + \sum_{k=0}^{a-1} \left( \frac{k + \lfloor -t \rfloor}{a} \right)^2 = \frac{1}{3} ab^2 - \frac{1}{2} ab + \frac{1}{3} a + b^2 \lfloor q \rfloor - bt - \frac{1}{2} b^2 + \frac{1}{2} b - \frac{1}{2} \lfloor -t \rfloor - b\lfloor q \rfloor + \frac{1}{a} \left( b^2 \lfloor q \rfloor^2 - b^2 \lfloor q \rfloor + 2b\lfloor q \rfloor \lfloor -t \rfloor + \frac{1}{6} b^2 + bt + \frac{1}{6} \lfloor -t \rfloor^2 + \lfloor -t \rfloor + b\lfloor q \rfloor \right) - 2b \frac{r_{[q]} - t}{[a]} - \frac{b}{a} \lfloor -at \rfloor + b \frac{b\lfloor q \rfloor - t}{a} - \chi \left( \left( \frac{q - tb^{-1}}{a} \right) \right),
\]

where \( bb^{-1} \equiv 1 \mod a \). (Note that \( b\lfloor q \rfloor - t \in \mathbb{Z} \) if and only if \( t \in \mathbb{Z} \).)

We are all set to substitute the expressions we found back into (111). Simplifying terms such as \( \{t\} + \{-t\} \) (which equals 1 if \( t \not\in \mathbb{Z} \) and 0 if \( t \in \mathbb{Z} \)) and \( \frac{r_{[p]} + t}{a} = \left\{ \frac{a}{b} \right\} \) gives

\[
\frac{a\lfloor p \rfloor^2}{2b} - \frac{a\lfloor p \rfloor}{2b} + b\lfloor q \rfloor^2 + b\lfloor q \rfloor + \frac{b}{12a} + \frac{a}{12a + \frac{1}{a} + [\lfloor q \rfloor - t]} + \frac{[\lfloor q \rfloor + t]}{b} + \frac{\lfloor p \rfloor}{2b} + \frac{a}{2a} - \frac{t}{2b} - \frac{\lfloor p \rfloor \lfloor q \rfloor}{2} + \frac{3}{4} + \frac{\lfloor -t \rfloor^2}{2ab} + \frac{\lfloor -t \rfloor}{2ab} + \frac{1}{2} \left\{ \frac{a\lfloor p \rfloor + t}{b} \right\} + \frac{1}{2} \left\{ \frac{b\lfloor q \rfloor - t}{a} \right\} + \chi \left( \frac{1}{2} \left( \left( \frac{a^{-1}(a\lfloor p \rfloor + t)}{b} \right) \right) - \frac{1}{2} \left( \left( \frac{b^{-1}(b\lfloor q \rfloor - t)}{a} \right) \right) + \frac{1}{2} - \frac{c}{b} \right).
\]
Now we use the relation $bq = ap + t$, which simplifies the left-hand side to

$$r_{a(p) - b(q)}(a, b) + r_{b(q) - a(p)}(b, a).$$

But this means we might as well choose $p$ and $q$ in some interval of length 1; it is easiest to assume $-1 < p, q \leq 0$, since this will simplify the right-hand side most easily:

$$r_{bq - ap}(a, b) + r_{ap - bq}(b, a) =$$

$$\frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{|ap - bq|^2}{2ab} + \frac{|ap - bq|}{2ab} - \frac{1}{2} \left\lfloor \frac{ap - bq}{a} \right\rfloor - \frac{1}{2} \frac{bq - ap}{b}$$

$$+ \chi \left( \frac{1}{2} - c \right) - \frac{1}{2} \left\lfloor \frac{a - 1}{b} \right\rfloor - \frac{1}{2} \left\lfloor \frac{b - 1}{a} \right\rfloor.$$

Recall that $c$ is the unique integer satisfying

$$c \equiv a^{-1}(ap - bq) \pmod{b} \quad \text{and} \quad p \leq c < p + b,$$

Since $-1 < p \leq 0$, this condition simply says that $c$ is the smallest nonnegative integer congruent to $a^{-1}(ap - bq)$ modulo $b$, that is,

$$c = b \left\lfloor \frac{a^{-1}(ap - bq)}{b} \right\rfloor = -b \left( \left\lfloor \frac{a^{-1}(ap - bq)}{b} \right\rfloor \right) + (1 - \mu) \frac{b}{2},$$

where $\mu = 1$ if $b|ap - bq$ and $\mu = 0$ otherwise. This yields

$$r_{bq - ap}(a, b) + r_{ap - bq}(b, a) =$$

$$\frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{|ap - bq|^2}{2ab} + \frac{|ap - bq|}{2ab} - \frac{1}{2} \left\lfloor \frac{ap - bq}{a} \right\rfloor - \frac{1}{2} \frac{bq - ap}{b}$$

$$+ \chi \left( \frac{\mu}{2} + \frac{1}{2} \left\lfloor \frac{a^{-1}(ap - bq)}{b} \right\rfloor \right) - \frac{1}{2} \left\lfloor \frac{b^{-1}(ap - bq)}{a} \right\rfloor.$$

Now we set $q = 0$ and assume that $a < b$, for which the above identity simplifies to

$$r_{bq}(a, b) + r_{-bq}(b, a) =$$

$$\frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{|-bq|^2}{2ab} + \frac{|-bq|}{2ab} - \frac{1}{2} \left\lfloor \frac{-bq}{a} \right\rfloor - \frac{1}{2} \left\lfloor q \right\rfloor$$

$$+ \chi \left( \frac{\mu}{2} - \frac{1}{2} \left\lfloor \frac{a^{-1}(-bq)}{b} \right\rfloor \right) - \frac{1}{2} \left\lfloor \frac{b^{-1}(-bq)}{a} \right\rfloor,$$

where $\chi$ equals 1 or 0 depending on whether or not $bq$ is an integer, $\mu$ equals 1 or 0 depending whether or not $q = 0$, $a a^{-1} \equiv 1 \pmod{b}$, and $b b^{-1} \equiv 1 \pmod{a}$. Noticing that $\left\lfloor q \right\rfloor = -1$ unless $q = 0$, and setting $t = -bq$ (which is a real number in the interval $[0, b]$) yields Theorem 3.

We strongly suspect that there exists a more direct proof of Theorem 3. We leave it as a challenge to the reader to find one.

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