Carleson embeddings with loss for Bergman–Orlicz spaces of the unit ball

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ABSTRACT
We prove Carleson embeddings for Bergman–Orlicz spaces of the unit ball that extend the lower triangle estimates for the usual Bergman spaces.

1. Introduction
Our setting is the unit ball $B^n$ of $\mathbb{C}^n$. We denote by $H(B^n)$, the space of all holomorphic functions on $B^n$. Let us denote by $d\nu$ the Lebesgue measure on $B^n$.

A surjective function $\Phi : [0, \infty) \to [0, \infty)$ is a growth function, if it is continuous and non-decreasing. We note that this implies that $\Phi(0) = 0$.

The growth function $\Phi$ satisfies the $\Delta_2$-condition if there exists a constant $K > 1$ such that, for any $t \geq 0$,

$$\Phi(2t) \leq K\Phi(t).$$

For $\alpha > -1$, we denote by $\nu_\alpha$ the normalized Lebesgue measure on $B^n$ defined by $d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z)$, where $c_\alpha$ is the normalizing constant. Given a growth function $\Phi$ satisfying the $\Delta_2$-condition, we denote by $L^{\Phi}_\alpha(B^n)$ the space of all functions $f$ such that

$$||f||_{\Phi, \alpha} = ||f||_{L^{\Phi}_\alpha} := \int_{B^n} \Phi(|f(z)|) d\nu_\alpha(z) < \infty.$$

The weighted Bergman–Orlicz space $A^{\Phi}_\alpha(B^n)$ is the subspace of $L^{\Phi}_\alpha(B^n)$ consisting of all holomorphic functions. The Luxembourg gauge on $L^{\Phi}_\alpha(B^n)$ is defined by

$$||f||_{\Phi, \alpha}^{\text{lux}} = ||f||_{L^{\Phi}_\alpha}^{\text{lux}} := \inf\left\{ \lambda > 0 : \int_{B^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_\alpha(z) \leq 1 \right\}.$$
We observe that for \( \Phi \) a convex growth function, \( \| \cdot \|_{\Phi, \alpha}^{\text{lux}} \) is a norm while for \( \Phi \) a concave growth function, it is a quasi-norm. In both case, if moreover \( \Phi \) is \( C^1 \), we have that \( \| f \|_{\Phi, \alpha}^{\text{lux}} \) is finite if \( f \in A_{\alpha}^{\Phi} (\mathbb{B}^n) \) (see [1, Remark 1.4]). Moreover, \( \| f \|_{\Phi, \alpha}^{\text{lux}} = 0 \) implies that \( f = 0 \) (see [1, p.569]).

The usual weighted Bergman space \( A_{\alpha}^p (\mathbb{B}^n) \) corresponds to \( \Phi(t) = t^p \) and is defined as the set of all \( f \in H(\mathbb{B}^n) \) such that

\[
\| f \|_{p, \alpha}^p := \int_{\mathbb{B}^n} |f(z)|^p \, d\nu_{\alpha}(z) < \infty.
\]

Given \( 0 < p, q < \infty \), we consider the question of the characterization of the positive measures \( \mu \) on \( \mathbb{B}^n \) such that the embedding \( I_{\mu} : A_{\alpha}^p (\mathbb{B}^n) \to L^q (\mathbb{B}^n, d\mu) \) is continuous. That is, there exists a constant \( C > 0 \) such that the following inequality

\[
\int_{\mathbb{B}^n} |f(z)|^q \, d\mu(z) \leq C \| f \|_{p, \alpha}^q
\]

holds for any \( f \in A_{\alpha}^p (\mathbb{B}^n), \alpha > -1 \). In the setting of Bergman spaces of the unit disk, for \( 0 < q < p < \infty \), this question was answered by Hastings [2] and the answer to the case \( 0 < q < p < \infty \) (estimation with loss) was obtained by Luecking [3]. The extensions of these results to the unit ball are due to Cima and Wogen [4] and Luecking [5,6]. When this holds, we speak of Carleson embedding (or estimate) for \( A_{\alpha}^p (\Omega) \); we also say that \( \mu \) is a \( q \)-Carleson measure for \( A_{\alpha}^p (\Omega) \).

Let \( \Phi \) be a growth function. For a function \( f \in H(\mathbb{B}^n) \), we define

\[
A_{\Phi, \mu}(f) := \inf \left\{ \lambda > 0 : \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) \, d\mu(z) \leq 1 \right\}.
\]

In the setting of Bergman–Orlicz spaces, the definition of Carleson measures is as follows.

**Definition 1.1:** Let \( \Phi_1 \) and \( \Phi_2 \) be two growth functions, and let \( \alpha > -1 \). Let \( \mu \) be a positive measure on \( \mathbb{B}^n \). We say \( \mu \) is a \( \Phi_2 \)-Carleson measure for \( A_{\alpha}^{\Phi_1} (\mathbb{B}^n) \), if there exists a constant \( C > 0 \) such that for any \( f \in A_{\alpha}^{\Phi_1} (\mathbb{B}^n) \),

\[
A_{\Phi_2, \mu}(f) \leq C \| f \|_{\Phi_1, \alpha}^{\text{lux}}.
\]

**Remark 1.1:** We observe that (3) is equivalent to saying that there is a constant \( C > 0 \) such that for any \( f \in A_{\alpha}^{\Phi_1} (\mathbb{B}^n), f \neq 0 \),

\[
\int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{\| f \|_{\Phi_1, \alpha}^{\text{lux}}} \right) \, d\mu(z) \leq C.
\]

Our concern in this paper is the characterization of the positive measures \( \mu \) such that (4) holds, when \( \Phi_1 \) and \( \Phi_2 \) are in some appropriate sub-classes of growth functions and such that \( \Phi_2 \circ \Phi_1^{-1} \) is non-increasing. This corresponds exactly to the case \( 0 < q < p < \infty \) for the usual Carleson measures described above. The case where \( \Phi_2 \circ \Phi_1^{-1} \) is non-decreasing has been answered in [7–9].

Carleson measures are an important tool usually used in several problems in mathematical analysis and its applications. Among these problems, one has the questions of the boundedness of Composition operators and Toeplitz operators to name a few.
2. Settings and presentation of the main result

We recall that the growth function $\Phi$ is of upper type $q$ if we can find $q > 0$ and $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$\Phi(st) \leq Ct^q \Phi(s).$$  \hspace{1cm} (5)

We denote by $\mathcal{U}^q$ the set of growth functions $\Phi$ of upper type $q$, (with $q \geq 1$), such that the function $t \mapsto (\Phi(t)/t)$ is non-decreasing. We write

$$\mathcal{U} = \bigcup_{q \geq 1} \mathcal{U}^q.$$  

We also recall that $\Phi$ is of lower type $p$ if we can find $p > 0$ and $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq Ctp^{1/p} \Phi(s).$$  \hspace{1cm} (6)

We denote by $\mathcal{L}_p$ the set of growth functions $\Phi$ of lower type $p$, (with $p \leq 1$), such that the function $t \mapsto (\Phi(t)/t)$ is non-increasing. We write

$$\mathcal{L} = \bigcup_{0 < p \leq 1} \mathcal{L}_p.$$  

Remark that we may always suppose that any $\Phi \in \mathcal{L}$ (resp. $\mathcal{U}$), is concave (resp. convex) and that $\Phi$ is a $C^1$ function with derivative $\Phi'(t) \approx (\Phi(t)/t)$.

The complementary function $\Psi$ of the growth function $\Phi$, is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t \in \mathbb{R}_+} \{ts - \Phi(t)\}. \hspace{1cm} (7)

A growth function $\Phi$ is said to satisfy the $\nabla_2$-condition whenever both $\Phi$ and its complementary function satisfy the $\Delta_2$-conditon.

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we let

$$\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

which gives $|z|^2 = \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2$.

For $z \in \mathbb{B}^n$ and $\delta > 0$ define the average function

$$\hat{\mu}_\delta(z) := \frac{\mu(D(z, \delta))}{\nu_\alpha(D(z, \delta))}$$

where $D(z, \delta)$ is the Bergman metric ball centered at $z$ with radius $\delta$.

The Berezin transform $\tilde{\mu}$ of the measure $\mu$ is the function defined for any $w \in \mathbb{B}^n$ by

$$\tilde{\mu}(w) := \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^{n+1+\alpha}}{[1 - \langle z, w \rangle]^{2(n+1+\alpha)}} d\mu(z).$$

Our main result can be stated as follows.

**Theorem 2.1:** Let $\Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U}$, $\alpha > -1$. Assume that
(i) \( \Phi_1 \circ \Phi_2^{-1} \) satisfies the \( \nabla_2 \)-condition;
(ii) \( ((\Phi_1 \circ \Phi_2^{-1}(t))/t) \) is non-decreasing.

Let \( \mu \) be a positive measure on \( \mathbb{B}^n \), and let \( \Phi_3 \) be the complementary function of \( \Phi_1 \circ \Phi_2^{-1} \).
Then the following assertions are equivalent.

(a) There is a constant \( C > 0 \) such that for any \( f \in A_\alpha^{\Phi_1}(\mathbb{B}^n) \), \( f \neq 0 \), the inequality (4) holds.
(b) For any \( 0 < \delta < 1 \), the average function \( \hat{\mu}_\delta \) belongs to \( L^{\Phi_2}(\mathbb{B}^n, d\nu_\alpha) \).
(c) The Berezin transform \( \tilde{\mu} \) of the measure \( \mu \) belongs to \( L^{\Phi_3}(\mathbb{B}^n, d\nu_\alpha) \).

We observe that the condition (ii) insures that the growth function \( \Phi_1 \circ \Phi_2^{-1} \) belongs to \( \mathcal{U} \). If \( \Phi_1(t) = t^p \) and \( \Phi_2(t) = t^q \), then \( ((\Phi_1 \circ \Phi_2^{-1}(t))/t) = t^{(p/q) - 1} \) and for this to be non-decreasing, one should have \( q \leq p \). Hence the above result is an extension of the embedding \( I_\mu : A_\alpha^{\Phi_2}(\mathbb{B}^n) \to L^q(\mathbb{B}^n, d\mu) \) when \( 0 < q < p < \infty \).

As mentioned previously, the power function version of the above result is due to Luecking [10]. In his proof, Luecking used a method that appeals to the atomic decomposition of Bergman spaces and Khintchine’s inequalities. We will use the same approach here. We will prove and use a generalization of Khintchine’s inequalities to growth functions, and take advantage of the recent characterization of the atomic decomposition of Bergman–Orlicz spaces obtained in [11]. There are some other technicalities that require a good understanding of the properties of growth functions.

For \( \phi : \mathbb{B}^n \to \mathbb{B}^n \) holomorphic, the composition operator \( C_\phi \) is the operator defined for any \( f \in H(\mathbb{B}^n) \) by
\[
C_\phi(f)(z) = (f \circ \phi)(z), \quad z \in \mathbb{B}^n.
\]
We say the operator \( C_\phi : A_\alpha^{\Phi_1}(\mathbb{B}^n) \to L^{\Phi_2}(\mathbb{B}^n, d\mu) \) is bounded, if there is a constant \( K > 0 \) such that for any \( f \in A_\alpha^{\Phi_1}(\mathbb{B}^n) \), \( f \neq 0 \),
\[
\int_{\mathbb{B}^n} \Phi_2 \left( \frac{|C_\phi(f)(z)|}{\|f\|_{l_\infty^{\Phi_1, \alpha}}} \right) \, d\mu(z) \leq K.
\] (8)
The characterization of the symbols \( \phi \) such that (8) holds, relies essentially on the characterization of Carleson embeddings for Bergman–Orlicz spaces (see [7,8]).

Let \( \phi \) be as above, and let \( \beta > -1 \). Define the measure \( \mu_{\phi, \beta} \) by
\[
\mu_{\phi, \beta}(E) = \nu_\beta(\phi^{-1}(E))
\]
for any Borel set \( E \subseteq \mathbb{B}^n \).

Then it follows from classical arguments (see, e.g. [8]) and Theorem 4.2 that the following holds.

**Corollary 2.1:** Let \( \Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U} \), \( \alpha, \beta > -1 \). Assume that

(i) \( \Phi_1 \circ \Phi_2^{-1} \) satisfies the \( \nabla_2 \)-condition;
(ii) \( ((\Phi_1 \circ \Phi_2^{-1}(t))/t) \) is non-decreasing.
Let $\Phi_3$ be the complementary function of $\Phi_1 \circ \Phi_2^{-1}$. Then for any holomorphic self mapping $\phi$ of $\mathbb{B}^n$, the following conditions are equivalent.

(a) $C_\phi$ is bounded from $A^\Phi_1(\mathbb{B}^n)$ to $A^\Phi_2(\mathbb{B}^n)$.

(b) The measure $\mu_{\phi, \beta}$ is a $\Phi_2$-Carleson measure for $A^\Phi_1(\mathbb{B}^n)$.

(c) For any $0 < \delta < 1$, the function $z \to ((\mu_{\phi, \beta}(D(z, \delta)))/(\nu(\delta,D(z, \delta))))$ belongs to $L^\Phi_3(\mathbb{B}^n, d\nu)$.

The paper is organized as follows. In the next section, we present some useful tools needed to prove our result. Section 3 is devoted to the proof of our result.

As usual, given two positive quantities $A$ and $B$, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there is an absolute positive constant $C$ such that $A \leq CB$ (resp. $A \geq CB$). When $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say $A$ and $B$ are equivalent. Finally, all over the text, $C$ or $K$ will denote a positive constants not necessarily the same at distinct occurrences.

3. Preliminary results

In this section, we give some fundamental facts about growth functions, Bergman metric and Bergman–Orlicz spaces. These results are needed in the proof of our result.

3.1. Growth functions and Hölder-type inequality

Let $\Phi$ be a $C^1$ growth function. Then the lower and the upper indices of $\Phi$ are, respectively, defined by

$$a_\Phi := \inf_{t > 0} \frac{t\Phi(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t > 0} \frac{t\Phi(t)}{\Phi(t)}.$$

It is well known that if $\Phi$ is convex, then $1 \leq a_\Phi \leq b_\Phi < \infty$ and, if $\Phi$ is concave, then $0 < a_\Phi \leq b_\Phi \leq 1$. We observe that a convex growth function satisfies the $\nabla_2$-condition if and only if $1 < a_\Phi \leq b_\Phi < \infty$ (see [12, Lemma 2.6]).

We also observe that if $\Phi$ is a $C^1$ growth function, then the functions $((\Phi^{-1}(t))/((t^{1/b_\Phi})))$ are increasing. We then deduce the following.

Lemma 3.1: Let $\Phi \in \mathcal{L}_p$. Then the growth function $\Phi_p$, defined by $\Phi_p(t) = \Phi(t^{1/p})$ is in $\mathcal{U}_q$ for some $q \geq 1$.

When writing $\Phi \in \mathcal{U}_q$ (resp. $\Phi \in \mathcal{L}_p$), we will always assume that $q$ (resp. $p$) is the smallest (resp. biggest) number $q_1$ (resp. $p_1$) such that $\Phi$ is of upper type $q_1$ (resp. lower type $p_1$). We note that $a_\Phi$ (resp. $b_\Phi$) coincides with the biggest (resp. smallest) number $p$ such that $\Phi$ is of lower (resp. upper) type $p$.

We also make the following observation (see [13, Proposition 2.1]).

Proposition 3.1: The following assertion holds:

$$\Phi \in \mathcal{L} \quad \text{if and only if} \quad \Phi^{-1} \in \mathcal{U}.$$ 

We recall the following Hölder-type inequality (see [14, p.58]).
Lemma 3.2: Let $\Phi \in U$, $\alpha > -1$. Denote by $\Psi$ the complementary function of $\Phi$. Then
\[
\int_{\mathbb{B}^n} |f(z)g(z)| \, d\nu_\alpha(z) \leq 2 \left( \int_{\mathbb{B}^n} \Phi(|f(z)|) \, d\nu_\alpha(z) \right) \left( \int_{\mathbb{B}^n} \Psi(|g(z)|) \, d\nu_\alpha(z) \right).
\]

3.2. The Bergman metric and atomic decomposition

For $a \in \mathbb{B}^n$, $a \neq 0$, let $\varphi_a$ denote the automorphism of $\mathbb{B}^n$ taking 0 to $a$ defined by
\[
\varphi_a(z) = \frac{a - P_a(z) - (1 - |z|^2)^{1/2}Q_a(z)}{1 - \langle z, a \rangle}
\]
where $P_a$ is the projection of $\mathbb{C}^n$ onto the one-dimensional subspace span of $a$ and $Q_a = I - P_a$ where $I$ is the identity. It is easy to see that
\[
\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \varphi_a \circ \varphi_a(z) = z.
\]

The Bergman metric $d$ on the unit ball $\mathbb{B}^n$ is defined by
\[
d(z, w) = \frac{1}{2} \log \left( \frac{1 + \varphi_z(w)}{1 - \varphi_z(w)} \right), \quad z, w \in \mathbb{B}^n.
\]

For $\delta > 0$, we denote by
\[
D(z, \delta) = \{ w \in \mathbb{B}^n : d(z, w) < \delta \},
\]
the Bergman ball centered at $z$ with radius $\delta$. It is well known that for any $\alpha > -1$, and for $w \in D(z, \delta)$,
\[
\nu_\alpha(D(z, \delta)) \approx |1 - \langle z, w \rangle|^{n+1+\alpha} \approx (1 - |w|^2)^{n+1+\alpha}.
\]

We recall that a sequence $a = \{a_k\}_{k \in \mathbb{N}}$ in $\mathbb{B}^n$ is said to be separated in the Bergman metric $d$, if there is a positive constant $r > 0$ such that
\[
d(a_k, a_j) > r \quad \text{for} \ k \neq j.
\]

We refer to [15, Theorem 2.23] for the following result.

Theorem 3.1: Given $\delta \in (0, 1)$, there exists a sequence $\{a_k\}$ of points of $\mathbb{B}^n$ called $\delta$-lattice such that

(i) the balls $D(a_k, \delta/4)$ are pairwise disjoint;
(ii) $\mathbb{B}^n = \bigcup_k D(a_k, \delta)$;
(iii) there is an integer $N$ (depending only on $\mathbb{B}^n$) such that each point of $\mathbb{B}^n$ belongs to at most $N$ of the balls $D(a_k, 4\delta)$.

Let $\alpha > -1$ and $\Phi$ a growth function satisfying the $\Delta_2$-condition. For a fixed $\delta$-lattice $a = \{a_k\}_{k \in \mathbb{N}}$ in $\mathbb{B}^n$, we define by $l_{a, \alpha}^\Phi$, the space of all complex sequences $c = \{c_k\}_{k \in \mathbb{N}}$ such that
\[
\sum_k (1 - |a_k|^2)^{n+1+\alpha} \Phi(|c_k|) < \infty.
\]

The following was observed in [16, Proposition 2.8].
Lemma 3.3: Let $\Phi \in \mathcal{U}$, and $\alpha > -1$. Assume that $\Phi$ satisfies the $\nabla_2$-condition and denote by $\Psi$ its complementary function. Then, the dual space $(\ell^\Phi_{a,\alpha})^*$ of the space $\ell^\Phi_{a,\alpha}$ identifies with $\ell^\Psi_{a,\alpha}$ under the sum pairing

$$\langle c, d \rangle_{a,\alpha} = \sum_k (1 - |a_k|^2)^{n+1+\alpha} c_k d_k,$$

where $c = \{c_k\}_{k \in \mathbb{N}} \in \ell^\Phi_{a,\alpha}$, $d = \{d_k\}_{k \in \mathbb{N}} \in \ell^\Psi_{a,\alpha}$ for $a = \{a_k\}_{k \in \mathbb{N}}$ a fixed $\delta$-lattice in $\mathbb{B}$.

For $\Phi$ a growth function, define

$$p_{\Phi} = \begin{cases} 1 & \text{if } \Phi \in \mathcal{U} \\ p & \text{if } \Phi \in \mathcal{L}_p. \end{cases} \quad (9)$$

We refer to [11, Proposition 3.1 and Proposition 3.2] for the following result.

Proposition 3.2: Let $\Phi \in \mathcal{L} \cup \mathcal{U}$, $\alpha > -1$ and $b > ((n + 1 + \alpha)/p_{\Phi})$. Let $a = \{a_k\}_{k \in \mathbb{N}}$ be a $\delta$-separated sequence in $\mathbb{B}$, then for any sequence $c = \{c_k\}_{k \in \mathbb{N}}$ of complex numbers that satisfy the condition

$$\sum_k (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right) < \infty,$$

the series $\sum_{k=1}^{\infty} (c_k/((1 - \langle z, a_k \rangle)^b))$ converges in $A^\Phi_{a}(\mathbb{B})$ to a function $f$ and

$$\int_{\mathbb{B}} \Phi \left( \sum_{k=1}^{\infty} \frac{c_k}{(1 - \langle z, a_k \rangle)^b} \right) \, dv_\alpha(z) \lesssim \sum_k (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right). \quad (10)$$

Let us close this subsection with the following estimate.

Lemma 3.4: Let $r > 0$, $\Phi \in \mathcal{L} \cup \mathcal{U}$ and $\alpha > -1$. Then there exists a constant $K > 0$ such that for any $f \in H(\mathbb{B})$,

$$\Phi(|f(z)|) \leq K \int_{D(z,r)} \Phi(|f(\zeta)|) \frac{dv(w)}{(1 - |w|^2)^{n+1}}. \quad (11)$$

Proof: For $\Phi$ a growth function, define $p_{\Phi}$ as in (9). Then by [15, Lemma 2.24], there exists a constant $C > 0$ such that

$$|f(z)|^{p_{\Phi}} \leq \frac{C}{(1 - |z|)^{n+1+\alpha}} \int_{D(z,r)} |f(\zeta)|^{p_{\Phi}} \, dv_\alpha(w) \lesssim \int_{D(z,r)} |f(\zeta)|^{p_{\Phi}} \frac{dv_\alpha(w)}{v_\alpha(D(z,r))}.$$ 

Observing that $\Phi_p(t) = \Phi(t^{1/p_{\Phi}})$ is in $\mathcal{U}$, we obtain applying Jensen’s inequality to the above estimate that

$$\Phi(|f(z)|) \leq C \int_{D(z,r)} \Phi(|f(\zeta)|) \frac{dv_\alpha(w)}{v_\alpha(D(z,r))} \leq K \int_{D(z,r)} \Phi(|f(\zeta)|) \frac{dv(w)}{(1 - |w|^2)^{n+1}}.$$ 

\[\blacksquare\]
3.3. Averaging functions and Berezin transform

Let \( \mu \) be a positive measure on \( \mathbb{B}^n \) and \( \alpha > -1 \). For \( w \in \mathbb{B}^n \), the normalized reproducing kernel at \( w \) is given by

\[
k_{\alpha,w}(z) = \frac{(1 - |w|^2)^{(n+1+\alpha)/2}}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.
\]

The Berezin transform \( \tilde{\mu} \) of the measure \( \mu \) is the function defined for any \( w \in \mathbb{B}^n \) by

\[
\tilde{\mu}(w) := \int_{\mathbb{B}^n} |k_{\alpha,w}(z, w)|^2 d\mu(z).
\]

When \( d\mu(z) = f(z) d\nu_\alpha(z) \), we write \( \tilde{\mu} = \tilde{f} \) and speak of the Berezin transform of the function \( f \).

It is well known that the Berezin transform \( (f \mapsto \tilde{f}) \) is bounded on \( L^p_\alpha(\mathbb{B}^n) \) if and only if \( p > 1 \) (see [17]). It follows from this and the interpolation result [12, Theorem 4.3] that the following holds.

**Lemma 3.5:** Let \( \alpha > -1 \). Then the Berezin transform is bounded on \( L^\Phi_\alpha(\mathbb{B}^n) \) for any \( \Phi \in \mathcal{U} \) that satisfies the \( \nabla_2 \)-condition.

For \( z \in \mathbb{B}^n \) and \( \delta \in (0, 1) \), we define the average of the positive measure \( \mu \) at \( z \) by

\[
\hat{\mu}_\delta(z) = \frac{\mu(D(z, \delta))}{\nu_\alpha(D(z, \delta))}.
\]

The function \( \hat{\mu} \) is very useful in the characterization of Carleson embeddings with loss and some other operators (see [3, 18, 19] and the references therein).

The following lemma follows as in the power functions case [20, Proposition 3.6] (see also [21, Lemma 2.9]).

**Lemma 3.6:** Let \( \Phi \in \mathcal{L} \cup \mathcal{U}, \alpha > -1, \) and \( r, s \in (0, 1) \). Assume that \( \mu \) is a positive Borel measure on \( \mathbb{B}^n \). Then the following assertions are equivalent.

(i) The function \( \mathbb{B}^n \ni z \mapsto (\mu(D(z, r))/\nu_\alpha(D(z, r))) \) belongs to \( L^\Phi_\alpha(\mathbb{B}^n) \).

(ii) The function \( \mathbb{B}^n \ni z \mapsto (\mu(D(z, s))/\nu_\alpha(D(z, s))) \) belongs to \( L^\Phi_\alpha(\mathbb{B}^n) \).

The following is an elementary exercise (see [19, p.16]).

**Lemma 3.7:** Let \( \delta \in (0, 1) \). Then there exists a constant \( C_\delta > 0 \) such that

\[
\hat{\mu}_\delta(z) \leq C_\delta \tilde{\mu}(z), \quad \text{for any } z \in \mathbb{B}^n.
\]

We also observe the following.

**Lemma 3.8:** Given \( \delta \in (0, 1) \), there exists \( C_\delta > 0 \) such that

\[
\tilde{\mu}(z) \leq C_\delta \hat{\mu}_\delta(z), \quad \text{for any } z \in \mathbb{B}^n.
\]
Proof: Using Lemma 3.4 and Fubini’s lemma, we obtain
\[
\hat{\mu}(z) = \int_{B^n} |k_{\alpha,z}(\xi)|^2 \, d\mu(\xi)
\]
\[
\lesssim \int_{B^n} \frac{1}{\nu_\alpha(D(\xi, \delta))} \int_{D(\xi, \delta)} |k_{\alpha,z}(w)|^2 \, d\nu_\alpha(w) \, d\mu(\xi)
\]
\[
= \int_{B^n} \frac{1}{\nu_\alpha(D(\xi, \delta))} \int_{B^n} |k_{\alpha,z}(w)|^2 \chi_{D(\xi, \delta)}(w) \, d\nu_\alpha(w) \, d\mu(\xi)
\]
\[
\approx \int_{B^n} \frac{1}{\nu_\alpha(D(\xi, \delta))} \int_{B^n} |k_{\alpha,z}(w)|^2 \chi_{D(\xi, \delta)}(w) \, d\nu_\alpha(w) \, d\mu(\xi)
\]
\[
= \hat{\mu}_\delta(z).
\]
We have used that \(\chi_{D(\xi, \delta)}(w) = \chi_{D(\xi, \delta)}(w)\) and that as \(w \in D(\xi, \delta)\), \(\nu_\alpha(D(\xi, \delta)) \approx \nu_\alpha(D(w, \delta))\).

We then have the following result.

Lemma 3.9: Let \(\Phi \in \mathcal{L} \cup \mathcal{U}, \alpha > -1\), and \(r, s \in (0, 1)\). Let \(a = \{a_j\}_{j \in \mathbb{N}}\) be a \(r\)-lattice in \(B^n\). Then the following assertions are equivalent.

(i) \(\hat{\mu}_s \in L^\Phi_a(B^n)\).

(ii) \(\{\hat{\mu}_r(z_j)\}_{j \in \mathbb{N}} \in L^\Phi_{a, \alpha}\). If moreover, \(\Phi \in \mathcal{U}\) and satisfies the \(\nabla_2\)-condition, then the above assertions are both equivalent to the following,

(iii) The Berezin transform \(\hat{\mu}\) of the measure \(\mu\) belongs to \(L^\Phi(B^n, d\nu_\alpha)\).

Proof: The equivalence (i) \(\iff\) (ii) follows as in the power functions case (see [20, Theorem 3.9] or [21, Lemma 2.12]). That (i) \(\implies\) (ii) is Lemma 3.8 together with Lemma 3.5. That (iii) \(\implies\) (i) follows from Lemma 3.7.

4. Carleson measures for weighted Bergman–Orlicz spaces

In this section, we present the proof of our characterization of Carleson measures for weighted Bergman–Orlicz spaces. We start by recalling the classical Khinchine’s inequality. We recall that the Rademacher functions \(r_n\) in \((0, 1]\) are defined as follows
\[
r_n(t) = \text{sgn}(\sin 2^n \pi t), \quad n = 0, 1, 2 \ldots
\]

Lemma 4.1 (Khinchine’s inequality): For \(0 < p < \infty\) there exist constants \(0 < A_p \leq B_p < \infty\) such that for any sequence of complex numbers \(x = \{x_k\} \in \ell^2\),
\[
A_p \left( \sum_k |x_k|^2 \right)^{1/2} \leq \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^p \, dt \right)^{1/p} \leq B_p \left( \sum_k |x_k|^2 \right)^{1/2}.
\]
We next show how to extend this result to the case where the power function \( t^p \) is replaced by any growth function in the classes considered in this note.

**Theorem 4.1 (Extended Khinchine's inequality):** For \( \Phi \in \mathcal{L} \cup \mathcal{U} \), there exist constants \( 0 < A_\Phi \leq B_\Phi < \infty \) such that, for any sequence \( x = \{x_k\} \in \ell^2 \),

\[
A_\Phi \left( \sum_k |x_k|^2 \right)^{1/2} \leq \Phi^{-1} \left( \int_0^1 \Phi \left( \left| \sum_k x_k r_k(t) \right|^p \right) \, dt \right) \leq B_\Phi \left( \sum_k |x_k|^2 \right)^{1/2}.
\]

**Proof:** Let us first prove the left inequality in (13). If \( \Phi \in \mathcal{L} \cup \mathcal{U} \), then if \( p_\Phi \) is defined as in (9), we recall with Lemma 3.1 that the growth function \( \Phi_{p_\Phi}(t) = \Phi(t^{1/p_\Phi}) \) belongs to \( \mathcal{U} \). Then applying \( \Phi \) to the left inequality in (12) with \( p = p_\Phi \), and using Jensen's inequality, we obtain

\[
\Phi \left( A_p \left( \sum_k |x_k|^2 \right)^{1/2} \right) \leq \Phi \left( \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^p \, dt \right)^{1/p} \right) = \Phi_{p_\Phi} \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^p \, dt \right) \leq \int_0^1 \Phi_{p_\Phi} \left( \left| \sum_k x_k r_k(t) \right|^p \right) \, dt = \int_0^1 \Phi \left( \left| \sum_k x_k r_k(t) \right|^p \right) \, dt.
\]

Let us now prove the right inequality. If \( \Phi \in \mathcal{L} \), then by Proposition 3.1, \( \Phi^{-1} \in \mathcal{U} \). It follows using the Jensen's inequality and the right hand side of (12) with \( p = 1 \) that

\[
\Phi^{-1} \left( \int_0^1 \Phi \left( \left| \sum_k x_k r_k(t) \right| \right) \, dt \right) \leq \int_0^1 \left| \sum_k x_k r_k(t) \right| \, dt \leq B_1 \left( \sum_k |x_k|^2 \right)^{1/2}.
\]

If \( \Phi \in \mathcal{U} \), and if \( q \) is its upper indice, then the growth function \( \Phi_q(t) = \Phi(t^{1/q}) \) belongs to \( \mathcal{L} \). Hence using the right hand side of (12) with \( p = q \), we first obtain

\[
\Phi_q \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^q \, dt \right) = \Phi \left( \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^q \, dt \right)^{1/q} \right) \leq \Phi \left( B_q \left( \sum_k |x_k|^2 \right)^{1/2} \right).
\]
As $\Phi_q$ is concave, it follows that

$$\int_0^1 \Phi_q \left( \left| \sum_k x_k r_k(t) \right|^q \right) \, dt \leq \Phi_q \left( \int_0^1 \left| \sum_k x_k r_k(t) \right|^q \, dt \right).$$

Hence

$$\int_0^1 \Phi \left( \left| \sum_k x_k r_k(t) \right| \right) \, dt \leq \Phi \left( B_q \left( \sum_k |x_k|^2 \right)^{1/2} \right).$$

The proof is complete. ■

Proof of Theorem 2.1: We observe that the equivalence $(b) \iff (c)$ is given in Lemma 3.9. It is then enough to prove that $(a) \iff (b)$. This is given in the result below. ■

We have the following embedding with loss.

Theorem 4.2: Let $\Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U}$, $\alpha > -1$. Assume that

(i) $\Phi_1 \circ \Phi_2^{-1}$ satisfies the $\nabla_2$-condition;
(ii) $((\Phi_1 \circ \Phi_2^{-1}(t))/t)$ is non-decreasing.

Let $\mu$ is a positive measure on $\mathbb{B}^n$, and let $\Phi_3$ be complementary function of $\Phi_1 \circ \Phi_2^{-1}$. Then the following assertions are satisfied.

(a) If the function

$$\mathbb{B}^n \ni z \mapsto \frac{\mu(D(z, \delta))}{\nu_\alpha(D(z, \delta))}$$

belongs to $L^{\Phi_3}_\alpha(\mathbb{B}^n)$, for some $\delta \in (0, 1)$, then there exists a constant $C > 0$ such that for any $f \in A^{\Phi_1}_\alpha(\mathbb{B}^n)$, with $f \neq 0$,

$$\int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{\|f\|_{\Phi_1, \alpha}^{lux}} \right) \, d\mu(z) \leq C. \quad (14)$$

(b) If (14) holds, then for any $0 < \delta < 1$, the average function $\hat{\mu}_\delta$ belongs to $L^{\Phi_3}_\alpha(\mathbb{B}^n, d\nu_\alpha)$.

Proof: Let us start with the proof of assertion (a). Let $K$ be the constant in (11). Then for $1 > r > \delta > 0$ fixed, we obtain using (11) that

$$M := \int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{\|f\|_{\Phi_1, \alpha}^{lux}} \right) \, d\mu(z)$$

\begin{align*}
&\leq K \int_{\mathbb{B}^n} \left( \int_{D(z, \delta)} \Phi_2 \left( \frac{|f(z)|}{\|f\|_{\Phi_1, \alpha}^{lux}} \right) \, d\nu_\alpha(w) \left(1 - |w|^2\right)^{n+1} \right) \, d\mu(z) \\
&\leq K \int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{\|f\|_{\Phi_1, \alpha}^{lux}} \right) \, d\mu(z).
\end{align*}
\[= K \int_{\mathbb{B}^n} \left( \int_{\mathbb{B}^n} \chi_D(z, \delta) \, d\mu(z) \right) \Phi_2 \left( \frac{|f(z)|}{\|f\|_{\phi_1, \alpha}} \right) \frac{dv(w)}{(1 - |w|^2)^{n+1}} \]
\[\leq K \int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(w)|}{\|f\|_{\phi_1, \alpha}} \right) \mu(D(w, r)) \frac{d\nu_\alpha(w)}{(1 - |w|^2)^{n+1 + \alpha}} \]

It follows from Lemma 3.2 and the hypothesis that

\[M \leq 2K \left( \int_{\mathbb{B}^n} \Phi_1 \left( \frac{|f(w)|}{\|f\|_{\phi_1, \alpha}} \right) d\nu_\alpha(w) \right) \left( \int_{\mathbb{B}^n} \Phi_3 \left( \frac{\mu(D(w, r))}{(1 - |w|^2)^{n+1 + \alpha}} \right) d\nu_\alpha(w) \right) \]
\[\leq 2K \left( \int_{\mathbb{B}^n} \Phi_3 \left( \frac{\mu(D(w, r))}{(1 - |w|^2)^{n+1 + \alpha}} \right) d\nu_\alpha(w) \right) .\]

Hence (14) holds with constant

\[C = 2K \left( \int_{\mathbb{B}^n} \Phi_3 \left( \frac{\mu(D(w, r))}{(1 - |w|^2)^{n+1 + \alpha}} \right) d\nu_\alpha(w) \right) .\]

Proof of (b): Assume that (14) holds for any \( 0 \neq f \in A^\phi_1(\mathbb{B}^n) \). We recall with Proposition 3.2 that for any sequence \( c = \{c_k\}_{k \in \mathbb{N}} \) of complex numbers that satisfies the condition

\[\sum_k (1 - |a_k|^2)^{n+1 + \alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^{b}} \right) < \infty, \quad (15)\]

where \( a = \{a_k\} \) is some \( \delta \)-lattice in \( \mathbb{B}^n \), and \( b > (n+1+\alpha)/\phi_1 \), the series

\[\sum_{k=1}^\infty \frac{c_k}{(1 - \langle z, a_k \rangle)^b}\]

converges in \( A^\phi_\alpha(\mathbb{B}^n) \) to a function \( f \).

For simplicity, we may assume that \( f \) is such that \( \|f\|_{\phi_1, \alpha} = 1 \). Thus

\[\int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_{k=1}^\infty \frac{c_k}{(1 - \langle z, a_k \rangle)^b} \right) \right) \, d\mu(z) \leq C.\]

Replacing \( c_k \) by \( c_k r_k(t) \), this gives us

\[\int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_{k=1}^\infty \frac{c_k r_k(t)}{(1 - \langle z, a_k \rangle)^b} \right) \right) \, d\mu(z) \leq C. \quad (16)\]

By the extended Kinchines inequalities, we have

\[\Phi_2 \left( A_{\Phi_2} \left( \sum_{k=1}^\infty \frac{|c_k|^2}{(1 - \langle z, a_k \rangle)^{2b}} \right)^{1/2} \right) \leq \int_0^1 \Phi_2 \left( \left( \sum_{k=1}^\infty \frac{c_k r_k(t)}{(1 - \langle z, a_k \rangle)^b} \right) \right) \, dt.\]
We can also assume that $A_{\Phi_2} = 1$. From the last inequality and (16), we obtain
\[
\int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_{k=1}^{\infty} \frac{|c_k|^2}{|1 - (z, a_k)|^{2b}} \right)^{1/2} \right) d\mu(z) \leq C. \tag{17}
\]

Put $d_k = (c_k/((1 - (z, a_k))^b))$ and observe that the sequence $\{\Phi_2(|d_k|)\}$ belongs to $\ell_{a,\alpha}^{\Phi_1 \circ \Phi_2^{-1}}$ whenever $c = \{c_k\}$ satisfies (15). Define $D_k = D(a_k, \delta)$. Then
\[
\langle \{\Phi_2(|d_k|)\}, \left\{ \frac{\mu(D_k)}{(1 - |a_k|^2)^{n+1+\alpha}} \right\} \rangle_{\alpha} = \sum_k \Phi_2(|d_k|) \mu(D_k) \leq \int_{\mathbb{B}^n} \sum_k \Phi_2(|d_k|) \chi_{D_k}(z) d\mu(z). \tag{18}
\]

We observe that if $\tilde{\Phi}_2(t) = \Phi_2(t^{1/2})$ is in $\mathcal{U}$, then
\[
\sum_k \Phi_2(|d_k|) \chi_{D_k} \leq \tilde{\Phi}_2 \left( \sum_k |d_k|^2 \chi_{D_k} \right) \leq \Phi_2 \left( \left( \sum_k |d_k|^2 \chi_{D_k} \right)^{1/2} \right).
\]

If $\Phi_2 \in \mathcal{L}_s$, then as $\tilde{\Phi}_2(t^{1/s})$ is in $\mathcal{U}$, we obtain
\[
\sum_k \Phi_2(|d_k|) \chi_{D_k} \leq \tilde{\Phi}_2 \left( \left( \sum_k |d_k|^2 \chi_{D_k} \right)^{1/s} \right) \leq \tilde{\Phi}_2 \left( \left( \sum_k |d_k|^2 \chi_{D_k} \right) \left( \sum_k \chi_{D_k} \right)^{(1-s)/s} \right) \leq \Phi_2 \left( \left( \sum_k |d_k|^2 \chi_{D_k} \right)^{1/2} \right).
\]

Taking these observations in (18) and using (17), we obtain
\[
L := \left\{ \langle \Phi_2(|d_k|), \left\{ \frac{\mu(D_k)}{(1 - |a_k|^2)^{n+1+\alpha}} \right\} \rangle_{\alpha} \right\} \leq K \int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_k |d_k|^2 \chi_{D_k}(z) \right)^{1/2} \right) d\mu(z)
\]
\[
= K \int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_k \frac{|c_k|^2}{(1-|a_k|^2)^{2b}} \chi_{D_k}(z) \right)^{1/2} \right) d\mu(z)
\]
\[
\leq K \int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_k \frac{|c_k|^2}{(1-|a_k|^2)^{2b}} \left( 1 - \frac{|a_k|^2}{2b} \right) \right)^{1/2} \right) d\mu(z)
\]
\[
= K \int_{\mathbb{B}^n} \Phi_2 \left( \left( \sum_k \frac{|c_k|^2}{1-\langle z, a_k \rangle^2} \right)^{1/2} \right) d\mu(z)
\]
\[
\leq KC.
\]
As this holds for any the sequence \( \{\Phi_2(|d_k|)\} \) belonging to \( \ell^{\Phi_1 \circ \Phi_2^{-1}}_{a,\alpha} \), we deduce that the sequence \( \{(\mu(D_k))/(1-|a_k|^2)^{n+1+a})\} \) belongs to \( \ell^{\Phi_{a,\alpha}} \). By Lemma 3.9, this is equivalent to saying that the average function \( \hat{\mu}_\delta \) belongs to \( L^{\Phi_2}(\mathbb{B}^n, d\nu_\alpha) \). The proof is complete. 

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