Non-linear phonon Peltier effect in dissipative quantum dot systems

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Solid state thermoelectric cooling is based on the electronic Peltier effect, which cools via an electronic heat current in the absence of an applied temperature gradient. In this work, we demonstrate that equivalently, a phonon Peltier effect may arise in the non-linear thermoelectric transport regime of a dissipative quantum dot thermoelectric setup described via Anderson-Holstein model. This effect leads to an electron induced phonon heat current in the absence of a thermal gradient. Utilizing the modification of quasi-equilibrium phonon distribution via charge induced phonon accumulation, we show that in a special case the polarity of the phonon heat current can be reversed so that setup can dump heat into the hotter reservoirs. In further exploring possibilities that can arise from this effect, we propose a novel charge-induced phonon switching mechanism that may be incited via electrostatic gating.

The theoretical formulation of this effect is based on the Anderson-Holstein model, which describes the interplay between electronic and phononic degrees of freedom in a quantum dot. The model Hamiltonian for this setup is given by

\[ H = H_{\text{dot}} + H_{\text{leads}} + H_{\text{res}} + H_{\text{interaction}} \]

where

- \( H_{\text{dot}} \) is the Hamiltonian of the quantum dot
- \( H_{\text{leads}} \) is the Hamiltonian of the metallic leads
- \( H_{\text{res}} \) is the Hamiltonian of the phonon reservoirs
- \( H_{\text{interaction}} \) is the electron-phonon interaction term

The dynamics of such a system can be explored using quantum transport theory, which allows for the calculation of electron and phonon currents and temperatures. In the present paper, we address possible areas of practical interest that can arise out of this effect to propose a charge induced phonon switch.

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On the other hand, the reservoir Hamiltonian \( \hat{H}_R \), corresponding to the \( \epsilon_{\alpha}d_{\alpha}^\dagger d_{\alpha} \) of energies \( \epsilon_{\alpha} \), drives the phonons out of equilibrium. The non-equilibrium phonon temperature is estimated using a thermometer bath weakly coupled to the dot. The black dotted arrows represent direct electron tunneling processes, which do not change the dot phonon number. The red bold and dashed arrows represent phonon assisted tunneling processes which change the dot phonon count. The black dotted arrows represent direct electron tunneling processes, which do not change the dot phonon numbers of the state. The modification of phonon coupling energy \( \gamma_{\alpha} \) and the mode independent coupling energy between the dot and the reservoir phonons is \( \lambda_{\alpha} \). The renormalization of electron coupling energies \( \tau_0 \) and \( \tau_{n,q} \) is neglected considering the weak coupling between the dot and the phonon reservoirs. The rate of electron tunneling \( \dot{\gamma}_{\alpha} \) and phonon relaxation \( \beta_{\alpha} \) are evaluated using the Fermi’s golden rule: 

\[
\gamma_{\alpha} = \frac{2\pi}{\hbar} \sum_{k'q} |\epsilon_{k'q}|^2 \rho_{\alpha,k'q}^\dagger \rho_{\alpha,k'q} \quad \text{and} \quad \beta_{\alpha} = \frac{2\pi}{\hbar} |\epsilon_{k'q}|^2 D_{\alpha}.
\]

The modification of phonon coupling energy \( \tau_{n,q} \) is neglected considering the weak coupling between the dot and the reservoir phonons. Using these, the rate of electron tunneling \( \gamma_{\alpha} \) and phonon relaxation \( \beta_{\alpha} \) are evaluated using the Fermi’s golden rule: 

\[
\gamma_{\alpha} = \frac{2\pi}{\hbar} \sum_{k'q} |\epsilon_{k'q}|^2 \rho_{\alpha,k'q}^\dagger \rho_{\alpha,k'q} \quad \text{and} \quad \beta_{\alpha} = \frac{2\pi}{\hbar} |\epsilon_{k'q}|^2 D_{\alpha}.
\]

The black bold arrows represent reservoir assisted phonon transitions.
Transport equations. We first describe physical implications imposed by our setup which leads to the current formulation of the transport equations. First, we use the approximation $\hbar \gamma_\alpha \gg \hbar \lambda_\alpha$, so that we can ignore the system damping and assume that each reservoir generates phonon currents independent of other reservoirs. Second, we also set $\hbar \omega_\nu \gg \hbar \gamma_\alpha$, so that any overlap between two adjacent phonon sidebands is excluded and subsequently, the electron tunneling events are completely uncorrelated. This assumption gives us enough scope to ignore phonon reservoir memory. Lastly, the sequential tunneling limit is assumed such that $k_B T \gg \hbar \gamma_\alpha$, where, charge and heat transport are described via the master equation framework by evaluating the steady state probability $P_{n,q}$ of an electron-phonon many-body state $|n, q\rangle$. The electronic tunneling between the dot and the electrodes ($\alpha_1 \in L, R$) causes transitions between two electron-phonon states $|n, q\rangle$ and $|n \pm 1, q\rangle$, with the following rates:

$$R_{n,q} \rightarrow (n+1,q) = \sum_{\alpha_1 \in L,R} \gamma_{\alpha_1} |\langle n, q | d_{\alpha_1}^\dagger | n+1, q \rangle|^2 \times f_{\alpha_1} \left( \frac{E_{n+1,q} - E_{n,q} - \mu_1}{k_B T_{\alpha_1}} \right),$$

(5)

$$R_{n,q} \rightarrow (n-1,q) = \sum_{\alpha_1 \in L,R} \gamma_{\alpha_1} |\langle n, q | d_{\alpha_1}^\dagger | n-1, q \rangle|^2 \times \left( 1 - f_{\alpha_1} \left( \frac{E_{n+1,q} - E_{n,q} - \mu_1}{k_B T_{\alpha_1}} \right) \right)$$

(6)

where $d_{\alpha_1}^\dagger (d_{\alpha_1})$ is the renormalized creation (annihilation) operator for dot electrons following the polaron transformation. Similarly, the rate of phonon transition between the dot and reservoirs ($\alpha_2 \in H, C$) causes a transition between the states $|n, q\rangle$ and $|n, q \pm 1\rangle$, with the following rates that abide by the Boltzman ratio:

$$R_{n,q} \rightarrow (n,q+1) = \sum_{\alpha_2 \in H,C} \beta_{\alpha_2} (q + 1) \exp \left( -\frac{\hbar \omega_q}{k_B T_{\alpha_2}} \right)$$

(7)

$$R_{n,q} \rightarrow (n,q-1) = \sum_{\alpha_2 \in H,C} \beta_{\alpha_2} (q + 1)$$

(8)

Then the master equation for the probabilities $P_{n,q}$ of an electron-phonon state $|n, q\rangle$ can be written as follows:

$$\frac{dP_{n,q}}{dt} = \sum_{q} \left[ R_{n,q}^{d\downarrow} \rightarrow (n,q+1) P_{n,q+1} - R_{n,q}^{d\downarrow} \rightarrow (n,q) P_{n,q} - R_{n,q}^{ph} \rightarrow (n,q+1) P_{n,q+1} \right]$$

(9)

In the steady state $\frac{dP_{n,q}}{dt}$ becomes 0 and we can evaluate the steady state probabilities $P_{n,q}$ using which we can calculate the charge and heat currents associated with the electrodes $\alpha_1 \in L, R$ and the reservoirs $\alpha_2 \in H, C$ as

$$I_{n,q} = \sum_{n,q,q'} q \left[ R_{n,q}^{d\downarrow} \rightarrow (n,q') P_{n,q'} - R_{n,q'}^{d\uparrow} \rightarrow (n,q) P_{n,q} \right] \delta(n+1, q'),$$

(10)

$$I_{n,q}^{ph} = \sum_{n,q,q'} \hbar \omega_q \left[ R_{n,q}^{ph} \rightarrow (n,q') P_{n,q'} - R_{n,q'}^{ph} \rightarrow (n,q) P_{n,q} \right] \delta(q+1, q'),$$

(11)

Since the electron tunneling rate $\gamma_{n,q}$ is greater than the phonon relaxation rate $\beta_{n,q}$, the steady state probabilities $P_{n,q}$ and hence the phonon heat current $I_{n,q}^{ph}$ are primarily determined by the charge transport. However the electron-phonon interaction parameter $\lambda_q$ plays an important role to modify the electronic tunneling rate $\gamma_{n,q}$ to be discussed in detail in the following section.

Inter-coupled electron and phonon transport. We note that in the presence of electron-phonon interaction, the effective electron tunneling rate $\gamma_{n,q}^{eff}$ between the two states $(n, q)$ and $(n \pm 1, q)$ is modified by the $\nu$ dependent Frank-Condon $FC_{q,q'}$ overlapping factor between them, and it is denoted as

$$\gamma_{n,q}^{eff} = \gamma_{n,q} |\langle n, q | d_{\alpha_1}^\dagger | n' q' \rangle|^2 \delta(n', n-1) + |\langle n', q \delta_{\alpha_1}^- | n+1, q' \rangle|^2 \delta(n', n+1)$$

(12)

$$\gamma_{n,q}^{eff} = \gamma_{n,q} |FC_{q,q'}|^2$$

(13)

(14)

$$\gamma_{n,q}^{eff} = \gamma_{n,q} |FC_{q,q'}|^2$$

(15)

$$\gamma_{n,q}^{eff} = \gamma_{n,q} |FC_{q,q'}|^2$$

(16)

where $L_k^{K-k}$ is the associated Laguerre polynomial with $k = \min(q, q')$ and $K = \max(q, q')$. When $q = q'$ the charge transport leads to phonon generation (or absorption) in the dot with a rate $G_{q,q'}^{ph} = \sum_{n,q} \sum_{n \pm 1} (q' - q) P_{n,q} R_{n,q}^{ph} \rightarrow (n \pm 1, q')$. One should note from (12), that when $\lambda_q = 0$, $\gamma_{n,q}^{eff}$ becomes zero.
unless \( q = q' \). It implies that there is no charge assisted phonon generation in the absence of electron-phonon interaction. The generated phonons are further removed by the reservoirs with a rate of \( RE_{ph}^{\alpha} \) given by:\(^{45}\)

\[
RE_{ph}^{\alpha} = \beta_{\alpha} \frac{\langle N_{ph}^{\alpha} \rangle - \langle N_{eq}^{\alpha} \rangle}{1 + \langle N_{eq}^{\alpha} \rangle}.
\]

(13)

In the above equation, \( \langle N_{ph}^{\alpha} \rangle = [\exp(h\omega_n/k_BT_{eq}) - 1]^{-1} \) is the average equilibrium phonon number of the reservoirs. The average dot phonon occupation is defined as \( \langle N_{ph} \rangle = \sum_n q P_{n,ph} \). When the system is in equilibrium (\( \lambda = 0 \)), \( \langle N_{ph} \rangle \) equals to \( \langle N_{eq} \rangle \) due to no charge assisted phonon generation and \( RE_{ph}^{\alpha} \) nullifies and we get zero \( I_{Q,ph}^{I} \). In this case, the phonon distributions obey the following Boltzmann’s ratio:\(^{45}\)

\[
P_{n,ph} = \exp\left[-\frac{h\omega_n}{k_B T_{eq}}\right]
\]

(14)

On the other way, in the non-equilibrium condition (\( \lambda = 0 \)), \( GE_{ph}^{\alpha} \) becomes finite. If \( GE_{ph}^{\alpha} \) exceeds \( RE_{ph}^{\alpha} \), phonons accumulate in the dot and the dot temperature deviates from the reservoir temperature. The expression for the dot temperature \( T_{M} \), can be approximated from the Boltzmann’s ratio with a quasi-equilibrium assumption as:\(^{46}\)

\[
T_{M} = \frac{h\omega_{eq}}{k_B} \ln \left[ \frac{P_{n,ph}}{P_{n,ph+1}} \right]^{-1}
\]

(15)

The charge transport assisted phonon transport gives rise to some interesting outcomes which we are going to elaborate now. From now on, we will denote \( \lambda, I_{ph}, I_{Q,ph}^{I}, GE_{ph}^{\alpha}, RE_{ph}^{\alpha} \) as \( \lambda, I, I_{ph}, G_{ph}, R_{ph} \), respectively, to maintain notational simplicity. Throughout the work, we will maintain the fact that no thermal gradient is applied across the electrodes (i.e \( T_{L} = T_{R} = T \)) as well as across the reservoirs (i.e \( T_{M} = T_{L} = T_{R} \)). We also assume a symmetric electronic coupling between the dot and the electrodes (\( \gamma_{L} = \gamma_{R} = \gamma \)) and a symmetric phonon coupling between the dot and reservoirs (\( \beta_{\alpha} = \beta_{C} = \beta \)).

Results

Voltage stimulated phonon heat current. We start by investigating the voltage response of \( I_{Q,ph}^{I} \) considering that the electrodes and reservoirs are at same temperature (\( T_{L} = T_{R} = T \)) as shown in Fig. 2(a). Figure 2(b), notes that in equilibrium condition (\( \lambda = 0 \)), the phonon heat current \( I_{Q,ph}^{I} \) vanishes for all voltages. It is consistent with the fact that in the absence of electron-phonon interaction, charge assisted phonon generation freezes and hence, the dot phonons remain in equilibrium with the reservoir phonons. This leads to the nullification of \( R_{ph}^{eq} \) as well as \( I_{ph}^{I} \). The scenario changes when we turn on, a finite electron-phonon interaction (\( \lambda \neq 0 \)), where a non-trivial voltage dependence of \( I_{Q,ph}^{I} \) results, as seen in Fig. 2(b). In this case, \( I_{Q,ph}^{I} \) first shoots up and then levels off as voltage increases. When voltage becomes large, higher order phonon assisted tunneling events are suppressed due to limited Frank-condon overlap. This leads to the saturation of both \( I \) and \( G_{ph} \). Consequently, \( I_{Q,ph}^{I} \) also levels off at large voltage.

As explained earlier, a finite \( \lambda \) facilitates phonon generation and the average dot phonon distribution \( \langle N_{ph} \rangle \) deviates from \( N_{eq} \) as shown in Fig. 2(d). Hence, as dictated by (13), the phonon removal rate changes with voltage ultimately leading to the voltage dependence of \( I_{ph}^{I} \). Throughout this work we maintain \( \gamma_{L(R)} > \beta_{C} \), resulting in \( G_{ph} \) exceeding \( R_{ph} \) as noticed in Fig. 2(e) and (f). It leads to a phonon accumulation inside the dot with \( \langle N_{ph} \rangle \) always exceeding \( \langle N_{eq} \rangle \). As a result, \( I_{ph}^{I} \) remains negative, i.e., \( I_{ph}^{I} \) flows away from the dot toward the reservoirs. From this discussion it can be concluded that the voltage dependence of \( I_{Q,ph}^{I} \) primarily depends on \( \langle N_{eq} \rangle \), which is a function of \( \lambda \) and voltage.

An important point to be noted is that the variation of \( I_{Q,ph}^{I} \) takes place only in the non-linear voltage range, i.e., when \( q V_{app} \gg k_{B}T \). In the linear response regime, phonon current abides with the Onsager’s reciprocity. The voltage dependence of \( I_{Q,ph}^{I} \) immediately hints at a physically new effect which can be thought of as the phonon Peltier effect rather than the conventional electronic Peltier effect.

Phonon Peltier effect. The phonon Peltier coefficient is calculated as \( \tau_{ph} = I_{Q,ph}^{I}/I \), provided there is no temperature gradient (i.e., \( T_{M} = T_{C} \)). Here we maintain the analogy with electronic Peltier coefficient \( \tau_{el} = I_{Q,el}^{I}/I \). The variation of \( \tau_{ph} \) is plotted as a function of voltage in Fig. 3(a). In our simulations, we plot \( \tau_{ph} = \frac{dI_{Q,ph}^{I}}{dI} \) to avoid the discontinuity at the short-circuit (\( V_{C} = 0 \)) point and \( \tau_{ph} \) is extracted from the gradient of the \( I_{Q,ph}^{I} \) plot shown in the inset of Fig. 3(a). We immediately note that \( \tau_{ph} \) vanishes when there is no \( \lambda \). When \( \lambda \) becomes finite, \( \tau_{ph} \) increases almost linearly within a small voltage range and levels off at large voltages along with the charge current. Typically, the Peltier effect causes a temperature gradient when subject to an electrical excitation. In this case, we note a temperature difference between the dot and the reservoir phonons, which we estimate via the dot temperature \( T_{M} \) using a thermometer probe. Figure 3(b) shows a clear deviation of \( T_{M} \) from the reservoir temperature \( T_{R} \). Since \( \gamma_{L} > \beta_{C} \), phonons accumulate in the dot and hence \( T_{M} \) is always greater than \( T_{R} \). Consequently, the phonon current will flow away from the dot as shown in the inset of Fig. 3(b) and signifies Peltier effect.
Charge controlled phonon distribution. So far we noticed that charge transport manipulates the non-equilibrium ($\lambda \neq 0$) phonon distribution. However, it is evident from (14), that the phonon distribution finds a strong dependence on the temperature of reservoirs and hence we test the phonon distribution in three cases: (i) Case H: where, the temperature of the reservoirs is greater than the temperature of electrodes ($T_{TTB} > T$), (ii) Case E: where, the temperature of the reservoirs is equal to the temperature of electrodes ($T_{TTB} = T$) and (iii) Case C: where, the temperature of the reservoirs is less than the temperature of electrodes ($T_{TTB} < T$). The inset of Fig. 4(b) shows that in Case H, $\langle N_{ph} \rangle$ is greater than $\langle N_{\text{ph}}^0 \rangle$ at short-circuit point ($V = 0$). When voltage increases, $\langle N_{\text{ph}} \rangle$ shoots up and overcomes $\langle N_{\text{ph}}^0 \rangle$ at $|V|$. Figure 4(b) notes that at $V = |V|$, the dot phonon distribution equals the equilibrium phonon distribution. In the Case E,
Figure 4. Manipulation of phonon equilibrium distribution by electron flow. (a) We consider three cases: (i) Case H: Reservoir temperature $T_{R} = T_{H}$, greater than the electrode temperature $T$, (ii) Case E: Reservoir temperature $T_{R} = T_{E}$, equal to the electrode temperature $T$ and (iii) Case C: Reservoir temperature $T_{R} = T_{C}$, less than the electrode temperature $T$. (b,c) and (d) show the voltage dependence of the phonon probability distribution under non-equilibrium ($\lambda \neq 0$) and equilibrium ($\lambda = 0$) conditions for Cases H, E, and C respectively. Insets present the respective deviations of $(N_{ph})$ from $(N_{ph}^{eq})$ and depict the voltage dependence of $I_{ph}^{h}$ under non-equilibrium ($\lambda \neq 0$) and equilibrium ($\lambda = 0$) conditions for Cases H, E, C respectively. Each profile of $I_{ph}^{h}$ is consistent with the temperature gradient between the dot and reservoir as shown in the insets of (e,f) and (g). The polarity reversal of $I_{ph}^{h}$ is noticed at $\pm V_{0}$ and it is schematically presented in (h). For $V > |V_{0}|$, the dot dumps heat into the hotter reservoirs.

Charge induced phonon switching. We note that if one wishes to switch phonon currents, voltage is not a suitable control parameter because $I_{ph}^{h}$ cannot even turn on when $\lambda = 0$. Even in the finite $\lambda$ case, $I_{ph}^{h}$ can be turned on but not turned off. However, switching of phonon currents can be achieved by varying $\lambda$ at a constant voltage. In Fig. 5(a), we investigate the variation of $I_{ph}^{h}$ as a function of $\lambda$ at fixed voltage. The phonon current first shoots up, levels off and finally falls as $\lambda$ is ramped up. Therefore, $I_{ph}^{h}$ can be switched on and off via the modulation of $\lambda$. According to (12), the phonon generation rate associated with the charge transfer between two phonon states with phonon number $q_{1}$ and $q_{2}$ is proportional to $(q_{2} - q_{1}) |FC_{q_{1},q_{2}}|^{2}$. In Fig. 5(b), we plot the variation of $FC_{q_{1},q_{2}}$ as a function of $\lambda$ for different values of $q_{1}$, with $q_{2} = 0$. It is observed that for the small values of $\lambda$, $FC_{q_{1},q_{2}}$ shoots up and the switch is turned ON as we increase $\lambda$. As $\lambda$ is increased further, $FC_{q_{1},q_{2}}$ starts to decay. The decay starts at earlier $\lambda$ for smaller values of $q_{2}$ and shifts to larger values of $\lambda$ with increasing $q_{2}$. At intermediate $\lambda$, $I_{ph}^{h}$ levels off and the switch remains ON. At large $\lambda$ all the processes fall sharply, and the switch is turned OFF. Recently, a pioneering demonstrated that $\lambda$ can be turned ON and OFF in a small bandgap suspended carbon nanotube quantum dot by controlling the location of quantum dot along the tube by applying voltage through a
series of gates. In addition to that, specific electronic and phonon modes can be coupled in real space. Theoretically this kind of selective coupling leads to the tuning of \( \lambda \). The electron-induced thermal switch that we have proposed can thus be accomplished using such kinds of experimental set ups.

Phonon computation is an emerging area that aims to convert waste heat (phonon currents in our case) to electrical signals which can be re-implemented in logic design. A major challenge in that course is to modulate the phonon current by electrical means. Our proposal for phonon switching can accomplish such a goal. This switching can be used to design thermal pulse generators and thermal multi-vibrators. For efficient switching, the ‘ON’ region should be more broadened, and that can be achieved by increasing the number of phonon sidebands in the voltage window as shown in the inset of Fig. 5(b). The incrementation of sidebands is feasible by reducing the frequency of phonon mode.

**Discussion**

The non-linear transport regime in a quantum dot heat engine described by Anderson-Holstein model was investigated in detail. It was shown that a finite electron-phonon interaction leads to a charge induced phonon generation that stimulates a phonon current even in the absence of a thermal gradient. This gave rise to the non-linear phonon Peltier effect which increases with the electron-phonon interaction. Utilizing the manipulation of phonon distribution via charge induced phonon accumulation, we demonstrated that the polarity of the phonon heat current can be reversed so that setup can dump heat into hotter reservoirs. In further exploring possibilities that can arise from this effect, we proposed a charge-induced phonon switching mechanism as a building block for phonon computation.

**Methods**

In this work, the evaluation of the probability \( P_{n,q} \) of an electron-phonon many-body state \( |n, q\rangle \), the charge current \( I_1 \) and the phonon heat current \( I_{Q2} \) is carried out using matrix computation. We enlist below the crucial steps that are necessary to be elucidated regarding the simulation method.

1. The parameters used are as follows. The on-site energy is set to \( \varepsilon = 1 \text{ meV} \), so that \( \varepsilon < 2k_B T (T = 10 \text{ K}) \), which is a relevant for thermolectric operations\(^{35}\). The Coulomb interaction \( U \) is set to 0.1 eV. We chose large \( U (U \gg k_B T) \) to avoid double occupancy in the dot within the allowed voltage range. The rate of electron tunneling (\( \gamma = 10^{-3}/\text{h} \text{S}^{-1} \)) is set higher than the phonon relaxation rate (\( \beta = 10^{-6}/\text{h} \text{S}^{-1} \)) to ensure that the phonon reservoirs can change the dot phonon number by \( \pm 1 \) when a single electron tunnels into or out of the dot. The phonon frequency is set higher than \( \gamma (\hbar \omega > \hbar \gamma, \omega = 1.5 \times 10^{11} \text{ sec}^{-1}) \), to exclude the overlap between the neighboring phonon sidebands.

2. The steady state probability \( P_{n,q} \) of \( |n, q\rangle \) is evaluated from (9), using the condition \( \sum_{n,q} P_{n,q} = 1 \). In this context, simultaneous algebraic equations are solved by matrix inversion method. Using \( P_{n,q} \), we can derive the charge current \( I_1 \) and the phonon current \( I_{Q2} \) from (10) and (11) respectively.

3. In Fig. 2(a), \( I_{ph}^{Fr} \) is calculated considering all the reservoirs and electrodes at the same temperature \( (T_{hi} = T_c = T_L = T_R = 10 \text{ K}) \). The same operating condition is maintained in Fig. 3(a), where the voltage dependence of \( \pi_{ph} \) is plotted.

4. The dot temperature \( T_D \) in Fig. 3(d) is estimated from a phonon thermometer bath. The estimation is based on the principle that the voltage dependent phonon current between the dot and thermometer vanishes at
the temperature $T_p$. While estimating the temperature, the phonon thermometer is weakly coupled to the dot as compared to the other reservoirs ($\hbar\beta_p = 10^{-15}$ eV and $\hbar\beta_{HC} = 10^{-6}$ eV) so that it does not interrupt the other transport processes.

5. The role of the temperature of reservoirs in controlling $I_{ph}^0$ is examined in three cases. In Case H, the reservoirs are hotter than the electrodes ($T_{HC} = 30\,\text{K}$ and $T = 10\,\text{K}$). In Case E, the reservoirs and electrodes are at the same temperature ($T_{HC} = T = 10\,\text{K}$). In Case C, the reservoirs are colder than the electrodes ($T_{BC} = 5\,\text{K}$ and $T = 10\,\text{K}$). In all the cases, $I_{ph}^0$ is derived from (11).

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Additional Information
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