Reduction of Event Structures under History Preserving Bisimulation

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Abstract. Event structures represent concurrent processes in terms of events and dependencies between events modelling behavioural relations like causality and conflict. Since the introduction of prime event structures, many variants of event structures have been proposed based on different behavioural relations and thus providing a different expressive power. In particular, a single event in a more expressive structure can correspond to several event occurrence in a prime event structure. This leads to the problem of finding a minimal representation of a process using some brand of event structure. The paper faces this problem focusing on generalisations of prime event structures featuring non-symmetric conflict, like asymmetric event structures and disjunctive causality, like flow event structures. We provide techniques for identifying sets of events which can be seen as occurrences of the same action, and thus can be folded together by keeping the behaviour unchanged with respect to history-preserving bisimilarity. By iterating the folding, any event structure can be reduced to a an minimal form (not unique in general), behaviourally equivalent to the original one.

1 Introduction

The concept of concurrent process is pervasive in computer science, with applications in a multitude of distinct fields, and a wide range of formalisms and techniques have been developed for the modelling and analysis of processes. For instance, in the field of business process modelling, it is well known how to translate any process in BPMN notation into a Petri net \cite{1}. Then the techniques developed for the analysis of Petri nets becomes available for the analysis of the modelled business processes. For instance, it is possible to find deadlocks and livelocks, or to study equivalences between processes. Different representations of a process allows for structural or behavioural analyses at different levels of abstraction. When one is interested in dependencies between activities, like causal dependencies, choices, possibility of parallel execution, a well established model, more abstract than Petri nets, is given by event structures \cite{2}.

Event structures represent processes by means of events and behavioural relations. Events can be seen as (occurrences of) atomic actions. Behavioural
relations, which differ in the various event structure models, explains how events
depend on each other. E.g., the execution of an event could require that some
events have already been executed (causality) and it could be prevented by the
execution of other events (conflict or choice). The seminal work [3,2] introduces
prime event structures (PESs), where dependencies between events are reduced
to causality and conflict. Since then many different types of event structures have
been proposed, based on more expressive relations, like flow event structures
(FESs) [4] and bundle event structures (BESs) [5], featuring a disjunctive form
of causality (the causes of an event can be chosen from conflictual events) and
asymmetric event structures (AESs) [6], where conflict is allowed to be non-
symmetric.

Every type of event structure uses different behavioural relations, providing
a different expressive power. For instance, every event in a PES has a uniquely
determined set of causes, whilst this is not the case for AESs, BESs or FESs. A
single event in a more expressive event structure could need to be represented
by different event instances in a PES in order to obtain the same behaviour
(under a suitable behavioural equivalence). This naturally leads to the problem
of finding the smallest representation for a process using some brand of event
structure. The paper faces this problem focusing on event structures based on
binary behavioural relations, and specifically on PESs, AESs and FESs.

We provide a technique for identifying sets of events which can be seen as
occurrences of the same activity, with the property that they can be replaced by
a single event, an operation referred to as folding. The folding operation does not
alter the behaviour of an event structure in the sense that the original and folded
event structure are behaviourally equivalent. The notion of equivalence adopted
is history preserving bisimilarity [7,8,9], one of the classical behavioural equiva-
lences in the true concurrent spectrum. The iteration of the folding operation on
a given finite model eventually produces a minimal event structure, equivalent
with the original one. Unfortunately the minimal model is not unique, in general
and thus it does not provide a canonical representation of the behaviour: the
same process can have non-isomorphic (and irreducible) foldings using either
AESs or FESs.

Interestingly, the studied event structures can be transformed back to Petri
nets. For instance, flow event structures correspond to an specific set of nets, re-
ferred to as flow nets [4]. In the context of business process models, the possible
minimisation of the behaviour would allow the reduction (and possible elimina-
tion) of duplicate activities present in the original model and the computation
of a compact representation of a process useful for a behavioural comparison,
among others.

In order to give a more precise idea of the kind of structures the paper deals
with and of the results we aim at, consider the event structures depicted in Fig.1.
In all cases, the labelled nodes represent events and they are interconnected with
a set of relations.

In Fig.1a a PES is represented. The black straight arrows represent causality
relation, which say that some events are required by others to be executed. For
instance, given that the events $a, b$ are in causality (flow) relation, we have that “event $a$ must be executed before $b$”. Causality is transitive, although, graphically only direct relations are represented. The annotated dotted lines represent the conflict relation. For instance, $d$ and $b$ are in conflict and thus “either $d$ or $b$ occurs in a computation but not both”.

For AESs (Figure 1b), symmetric conflict is replaced by an asymmetric relation called asymmetric conflict. It is represented with a dashed arrow. The fact that, $b$ is in asymmetric conflict with $c$, means that “the occurrence of event $c$ avoids the occurrence of event $b$”. The same can be expressed by saying that “whenever $b$ and $c$ occur in a computation, then the execution of event $b$ will precede the execution of $c$”, hence also asymmetric conflict establishes an order of execution. If two events are in asymmetric conflict in both directions, then they cannot appear in the same computation (as each should precede the other). Hence symmetric conflict is represented by cycles of asymmetric conflict.

Finally, in FESs (Figure 1c) causality is replaced by the flow relation, still represented by straight arrows. The flow relation is not transitive. Intuitively, events in the flow relation with another represent the set of its potential direct causes and, in order to be executed, an event needs the prior execution of a maximal set of its potential causes, free of conflicts. For instance, in the example, the leftmost event with label $c$ must be preceded either by $\{e, d\}$ or $\{b\}$.

It is possible to see that the three event structures depicted in Figures 1(a)-(c) represent the same set of computations, but with a different number of events. In fact, AES and FES can take advantage from their greater expressiveness in order to avoid some duplication of event $c$. Clearly, PESs can be seen as special AESs (where asymmetric conflict is symmetric) or as special FESs (where the flow relation is transitive and potential causes do not contain conflicts). The folding technique in the paper allows to obtain the AESs and FESs in Fig. 1b and 1c starting from the PES in Fig. 1a.

The way in which AESs and FESs increase the expressive power of PES is somehow orthogonal (asymmetric conflict and disjunctive causality). As a consequence they allow for conceptually different reductions, sets of combinable events in each of the structures are different. We will discuss a translation from AESs to FESs which is possible, at the price of inserting additional ($\tau$) silent events. With such transformation, it is possible to show that any combinable
set of events in an AES is also a combinable set in the corresponding FES. This allows one to take advantage from the combined expressiveness of AESs and FESs. The transformation to FES and folding from the AES in Figure 1 is depicted in Figure 2.

The organisation of the paper is as follows. Section 2 introduces basic concepts about the notation, the event structures used and the adopted equivalence. The folding technique over AES is presented in Section 3. The folding technique defined for FES is presented in Section 4. Section 5 contains the proposed transformation from AES to FES. Finally, Section 6 draws some conclusions and proposes possible avenues for future work.

2 Prime event structures and history preserving bisimilarity

This section recalls the basics of prime event structures and introduces the notion of history-preserving bisimilarity, that will provide a foundation to discussion in the following sections. The basics of asymmetric and flow event structures will be provided in the section discussing their reduction.

We shall first recall some basic notation on sets and relations. Let \( R \subseteq X \times X \) be a binary relation and let \( Y \subseteq X \), then \( R|_Y \) denotes the restriction of \( R \) to \( Y \), i.e., \( R|_Y = R \cap (Y \times Y) \). We say that \( R \) is well-founded if it has no infinite descending chain, i.e., \( \langle e_i \rangle_{i \in \mathbb{N}} \) such that \( e_{i+1} R e_i \), \( e_i \neq e_{i+1} \), for all \( i \in \mathbb{N} \). The relation \( R \) is acyclic if it has no “cycles” \( e_0 R e_1 R \ldots R e_n R e_0 \), with \( e_i \in X \). In particular, if \( R \) is well-founded, then it has no (non-trivial) cycles. Relation \( R \) is a preorder, if it is reflexive and transitive; it is a partial order if it is also antisymmetric.

2.1 Prime Event structures

We recalling the formal definition of prime event structures [2] which complements the informal description provided in the introduction. Hereafter \( \Lambda \) denotes a fixed set of labels.

**Definition 1 (prime event structure).** A (labelled) prime event structure (PES) is a tuple \( \mathcal{P} = (E, \preceq, \#; \lambda) \), where \( E \) is a set of events, \( \preceq \) and \( \# \) are binary relations on \( E \) called causality and conflict, and \( \lambda : E \to \Lambda \) is a labelling function, such that
– ≤ is a partial order and \( \{ e' \in E \mid e' \leq e \} \) is finite for all \( e \in E \);
– # is irreflexive, symmetric and hereditary with respect to causality, i.e., for all \( e, e', e'' \in E \), if \( e \# e' \leq e'' \) then \( e \# e'' \)

An event \( e \in E \) labelled with \( a \) represents the occurrence of an action \( a \) in a run of the system, \( e < e' \) means that \( e \) is a prerequisite for the occurrence of \( e' \) and \( e \# e' \) means that \( e \) and \( e' \) cannot both happen in the same run. In order to lighten the notation, whenever this does not cause confusion, we will often confuse events and event labels.

The computations in an event structure are usually described in terms of configurations, i.e., sets of events which are closed with respect to causality and conflict free. Formally, a configuration of a PES \( P = (E, \leq, \#, \lambda) \) is a finite set of events \( C \subseteq E \) such that

– for all \( e \in C \), \( \{ e \} \subseteq C \) and
– for all \( e, e' \in C \), \( - (e \# e') \).

Configurations come equipped with an extension order, \( C_1 \subseteq C_2 \) meaning that a configuration \( C_1 \) can evolve into \( C_2 \). For PESs, the extension order is simply subset inclusion.

### 2.2 History preserving bisimilarity

Behavioural equivalences for event structures can be defined on the transition system where states are configurations and transitions are given by the extension relation. In this paper we focus on history preserving bisimilarity [789], a classical equivalence in the true-concurrent spectrum. As for interleaving bisimilarity, an event of an event structure must be simulated by an event of the other, with the same label, and vice-versa, but additionally, the two events are required to have the same “causal history”.

**Definition 2 (history preserving bisimilarity).** Let \( E_1, E_2 \) be two PESs. A history preserving (hp-)bisimulation is a set \( R \) of triples \((C_1, f, C_2)\), where \( C_1 \) and \( C_2 \) are configurations of \( E_1 \) and \( E_2 \), respectively, and \( f \) is an isomorphism, such that \((\emptyset, \emptyset, \emptyset) \in R \) and \( \forall (C_1, f, C_2) \in R \)

\[
\begin{align*}
& a) \text{ if } C_1 \cup \{ e_1 \} \in \text{Conf}(E_1), \text{ for an event } e_1 \in E_1, \text{ there exists } e_2 \in E_2 \text{ such that } \\
& \lambda_1(e_1) = \lambda_2(e_2) \text{ and } (C_1 \cup \{ e_1 \}, f', C_2 \cup \{ e_2 \}) \in R; \\
& b) \text{ if } C_2 \cup \{ e_2 \} \in \text{Conf}(E_2), \text{ for an event } e_2 \in E_2, \text{ there exists } e_1 \in E_1 \text{ such that } \\
& \lambda_1(e_1) = \lambda_2(e_2) \text{ and } (C_1 \cup \{ e_1 \}, f', C_2 \cup \{ e_2 \}) \in R.
\end{align*}
\]

The definition of hp-bisimilarity, although given for PESs is completely generic, and it will immediately adapt to the variants of event structures used in the paper.

### 3 Behaviour-Preserving Reduction of AESs

In this section we describe a technique for reducing the size of asymmetric event structures, in a way that preserves their behaviour.
3.1 Basics of asymmetric event structures

We briefly review the basics of asymmetric event structures.

**Definition 3 (asymmetric event structure).** A (labelled) asymmetric event structure \((AES)\) is a tuple \(A = \langle E, \leq, \triangleright, \lambda \rangle\), where \(E\) is a set of events, \(\leq\) and \(\triangleright\) are binary relations on \(E\) called causality and asymmetric conflict, and \(\lambda: E \to \Lambda\) is a labelling function, such that

1. \(\leq\) is a partial order and \([e] = \{e' \in E \mid e' \leq e\}\) is finite for all \(e \in E\);  
2. \(\triangleright\) satisfies, for all \(e, e', e'' \in E\)
   - \(e < e' \Rightarrow e \triangleright e'\),
   - if \(e \triangleright e'\) and \(e' < e''\) then \(e \triangleright e''\);
3. \(\|\{e\}\) is acyclic;
4. if \(\|\{e\}\) is cyclic then \(e \triangleright e'\).

AESs generalise PESs by allowing a conflict relation which is no longer symmetric. As hinted at in the introduction, the asymmetric conflict relation has a double interpretation, that is \(a \triangleright b\) can be understood as (i) the occurrence of \(a\) is prevented by \(b\), or (ii) \(a\) precedes \(b\) in all computations where both appear. Condition 1 of Definition 3 arises from the fact that, according to the interpretation \(\|\{e\}\) of the asymmetry conflict relation, \(\triangleright\) can be seen as a weak form of causality, hence it is natural to ask that it is included in \(<\). In the graphical representation of an AES, \(\leq\) takes precedence over \(\triangleright\) and, therefore, when both holds a solid edge is used. Condition 2 is a form of hereditarity of asymmetric conflict along causality: if \(e \triangleright e'\) and \(e' < e''\) then \(e\) is necessarily executed before \(e''\) when both appear in the same computation, hence \(e \triangleright e''\) (see Fig. 3(a)).

Concerning conditions 3 and 4, observe that events forming a cycle of asymmetric conflict cannot appear in the same run, since each event in the cycle should occur before itself in the run. This leads to a notion of conflict over sets of events \#X, defined by the following rules

\[
\frac{e_0 \triangleright e_1 \triangleright \ldots \triangleright e_n \triangleright e_0}{\#{\{e_0, \ldots, e_n\}}} \quad \frac{\#{\{X \cup \{e\}\}} \leq e'}{\#{\{X \cup \{e'\}\}}}
\]

In this view, condition 3 corresponds to irreflexiveness of conflict in PES, while condition 4 requires that binary symmetric conflict are represented by asymmetric conflict in both directions.

We recall that every PES can be seen as a special AES by replacing every binary conflict relation with asymmetric conflict relations in both directions.

In the following, direct relations, namely immediate causality and conflicts that are not inherited, will play a special role.

**Definition 4 (direct relations).** Let \(A\) be an AES and let \(e, e' \in E\). We say that \(e'\) is an immediate cause of \(e\), denoted \(e' <_\mu e\), when \(e' < e\) and there is no \(e''\) such that \(e' < e'' < e\). An asymmetric conflict \(e \triangleright e''\) is called direct, written \(e \triangleright \mu e''\) when there is no \(e'\) such that \(e \triangleright e' < e''\). A binary conflict \(e \# e'\) is called direct, written \(e \# \mu e'\), when \(e \triangleright \mu e'\) and \(e' \triangleright \mu e\).
For instance, in Fig. 3(a) \( e \nsucc_{\mu} e' \) and \( e \nsucc e'' \) while it is not the case that \( e \nsucc_{\mu} e'' \). In Fig. 3(b) we have that \( e'' \nsucc_{\mu} e' \) and \( e \nsucc_{\mu} e' \), hence \( e \#_{\mu} e'' \).

Fig. 3: Hereditarity of \( \nsucc \)

Configurations in AESs are defined, as in PESs, as causally closed and conflict free set of events. More precisely a configuration of \( \mathcal{A} = \langle E, \leq, \nsucc, \lambda \rangle \) is a set of events \( C \subseteq E \) such that 1) for any \( e \in C \), \( [e] \subseteq C \) (causal closedness) 2) \( \nsucc | C \) is acyclic (or equivalently, \( \neg(e \#_{\mu} e') \) for all \( e, e' \in C \)). The set of all configurations of \( \mathcal{A} \) is denoted by \( \text{Conf}(\mathcal{A}) \).

Differently from what happens for PESs, the extension order on configurations is not simply set-inclusion, since a configuration \( C \) cannot be extended with an event which is prevented by some of the events already present in \( C \). More formally, if \( C_1, C_2 \in \text{Conf}(\mathcal{A}) \) are configurations, we say that \( C_2 \) extends \( C_1 \), written \( C_1 \sqsubseteq C_2 \), if \( C_1 \subseteq C_2 \) and for all \( e \in C_1 \), \( e' \in C_2 \setminus C_1 \), \( \neg(e' \nsucc e) \).

A fundamental notion is that of history of an event in a configuration.

**Definition 5 (history and possible histories).** Let \( \mathcal{A} \) be an AES and let \( e \in E \) be an event in \( \mathcal{A} \). Given a configuration \( C \in \text{Conf}(\mathcal{A}) \) such that \( e \in C \), the history of \( e \) in \( C \) is defined as \( C[[e]] = \{ e' \in C \mid e' \nsucc (\nsucc | C)^* e \} \). The set of possible histories of \( e \), denoted by \( \text{hist}(e) \), is then defined as

\[
\text{hist}(e) = \{ C[[e]] \mid C \in \text{Conf}(\mathcal{A}) \land e \in C \}
\]

We will write \( \tilde{H}(e) = \bigcup \text{hist}(e) \) to represent the the set of all events possibly occurring in a history of event \( e \). Moreover, given a history \( h \in \text{hist}(e) \), we define \( h^- = h \setminus \{ e \} \).

Roughly speaking, \( C[[e]] \) consists of the events which necessarily must occur before \( e \) in the configuration \( C \). While in the case of PESs, each event \( e \) has a unique history, i.e., the set \( [e] \), in the case of AESs, an event \( e \) may have several histories. For example, the event \( c_{0.2} \) in the AES \( \mathcal{A}_2 \) (Figure 4(c)) has four different histories, \( \text{hist}(c_{0.2}) = \{ \{c_{0.2}\}, \{d, c_{0.2}\}, \{e, c_{0.2}\}, \{d, e, c_{0.2}\} \} \).

The notation above will be often used on sets. E.g., we write \( \text{hist}(X) = \bigcup_{e \in X} \text{hist}(e) \) to denote the set of events in the history of a set of events \( X \).

### 3.2 Reduction of AESs

The technique for behaviour preserving reduction of AESs consists in iteratively identifying a set of events carrying the same label, i.e., intuitively referring to
the same action, and replacing all the events in the set with a single event. Such
a substitution is called a folding. However, the configurations of the AES should
remain “essentially” unchanged after the folding or, more precisely, the original
and the folded AES should be hp-bisimilar.

In order to understand the intuition behind folding, consider the sample AESs
in Figure 4 where events are named using their label, possibly with subscripts
(e.g., \(c_0\) is an event labelled by \(c\)). The AES \(A_1\) can be thought of as a reduction
of \(A_0\) obtained by folding two \(c\)-labelled events \(c_0\) and \(c_1\), the first in conflict with
\(d\) and the second caused by \(d\), into a single event \(c_{0,1}\), in asymmetric conflict
with \(d\). The dependencies \(d \# c_0\) and \(d < c_1\) in \(A_0\) give rise to an asymmetric
conflict, i.e., \(d \triangleright c_{0,1}\) in \(A_1\), as a side effect of the substitution.

![Fig. 4: AESs such that \(A_0 \equiv_{hp} A_1\) but \(A_0 \not\equiv_{hp} A_2\).](attachment:image.png)

The configurations of \(A_0\) and \(A_1\), are \(\text{Conf}(A_0) = \{(c_0), \{d, c_1\}, \{d, e, c_2\}\}\)
and \(\text{Conf}(A_1) = \{(c_{0,1}), \{d, c_{0,1}\}, \{d, e, c_2\}\}\), and it is not difficult to see that the two
AESs are hp-bisimilar.

Also \(A_2\) could look as a reduced version of \(A_0\) where \(c_0\) and \(c_2\) are folded into
\(c_{0,2}\). However, this folding would not preserve the behaviour. In fact, \(\text{Conf}(A_2) = \{(c_{0,2}), \{d, c_1\}, \{e, c_{0,2}\}, \{d, c_1\}, \{d, e, c_{0,2}\}\}\)
contains an additional configuration not in \(\text{Conf}(A_0)\). This immediately implies that \(A_0\) is not hp-bisimilar to \(A_2\).

We next identify sets of events that can be safely folded. For this we need
some further notation. Given a set \(X\) of events, whenever it can be folded, the
resulting merged event will have as causes the common causes of all the events
in \(X\), while events which are causes or weak causes of only some of the events
in \(X\) will become weak causes of the merged event. More precisely, given a set
of events \(X\), we define its strict causes \(S(X) = \bigcap \text{hist}(X) \setminus X\) and the weak causes
as \(W(X) = \{e'' \mid \exists e \in X. e'' \triangleright e \land \lnot(e \triangleright e'')\} \setminus (S(X) \cup X)\).

As anticipated, the set of weak causes of \(X\) consists of all direct \(\triangleright\)-predecessors
of any event \(e \in X\) that are not strong causes (and not in conflict with \(e\), whence
the last requirement). For instance, in Fig. 4 we have that \(S(\{c_0, c_1\}) = \emptyset\) and
\(W(\{c_0, c_1\}) = \{d\}\). Instead, \(S(\{c_1, c_2\}) = \{d\}\) and \(W(\{c_1, c_2\}) = \{e\}\).

A first notion is that of compatible events.

**Definition 6 (compatible events).** Let \(\mathcal{A} = \langle E, \leq, \triangleright, \lambda \rangle\) be an AES. A set of
events \(X \subseteq E\) is called compatible if for all \(e, e' \in X\):
1. \( \lambda(e) = \lambda(e') \) and \( e \neq e' \)
2. for all \( e'' \notin \mathcal{H}(e') \), if \( e \mathrel{\triangleright} e'' \) then \( e' \mathrel{\triangleright} e'' \);
3. for all \( e'' \in \mathcal{H}(e') \), if \( e \mathrel{\triangleright} e'' \) then \( e' \mathrel{\triangleright} e \) (i.e., \( e \neq e'' \));
4. for all \( e'' \in W(\{e, e'\}) \), if \( e'' \mathrel{\triangleright}_{\mu} e \) then \( e'' \mathrel{\triangleright}_{\mu} e' \).

![Fig. 5: Three AES such that \( \mathcal{A}_0 \equiv_{hp} \mathcal{A}_1 \) and \( \mathcal{A}_0 \not\equiv_{hp} \mathcal{A}_2 \).](image)

Intuitively, events to be folded should represent different occurrences of the same activity, hence the first condition is that they need to have the same label and be in conflict. Conditions 2, 3 and 4 roughly asks that all events in \( X \) have, essentially, the same asymmetric conflicts (with the exception of those involving events in the histories). More precisely, given two events \( e, e' \in X \), if for an event \( e'' \) we have \( e \mathrel{\triangleright} e'' \) conditions 2 requires that also \( e' \mathrel{\triangleright} e'' \) unless \( e'' \) is not part of some history of \( e' \). In this latter case, by condition 3, \( e \) is required to be in conflict with \( e'' \). This can be understood as follows: we would like to see \( e \) and \( e' \) as occurrences of the same activity with different histories, hence \( e'' \) plays the role of a weak cause, inserted in the history of \( e' \) and incompatible with the history of \( e \). Finally, condition 4 requires that \( e \) and \( e' \) have the same sets of \( \mathrel{\triangleright}_{\mu} \)-predecessors (weak causes).

Examples motivating conditions 2, 3 and 4 are in Figs. 5a, 5b, 5c, which present situations where the merging of \( a, a' \) does not preserve the behaviour and hence should not be allowed.

An event resulting as the merging of a set of compatible events \( X \) will have \( W(X) \) as weak causes. As a consequence, all consistent subsets of \( W(X) \) will give rise to different possible histories, which, order not to modify the overall behaviour, should be almost present in the original PES. may not be depicted in the original process. This is formalised by the definition of the combinable set of events.

**Definition 7 (combinable set of events).** Let \( \mathcal{A} = (E, \leq, \triangleright, \lambda) \) be an AES. A set of events \( X \subseteq E \) of equivalent events is combinable if \( \forall Y \subseteq W(X) \) consistent, \( \exists e \in X : \exists h_e \in \text{hist}(e), h_e \subseteq S(X) \cup [Y] \).

Armed with the above definitions, we can now formally introduce the folding of an AES.

**Definition 8 (folding of an AES).** Let \( \mathcal{A} \) be an AES, \( X \) be a set of combinable events. The folding of \( \mathcal{A} \) on \( X \) is the AES \( \mathcal{A}/_X = (E/_X, </_X, \triangleright/_X, \lambda/_X) \) where
\[ E_{/X} = E \setminus X \cup \{ e_X \}, \]
\[ \leq_{X} = \leq_{(E \setminus X) \cup \{ (e_X, e) \ | \ \forall e \in S(X) \} \cup \{ (e, e) \ | \ \exists e' \in X \land e' < e \}}, \]
\[ \rhd_{X} = \rhd_{(E \setminus X) \cup \{ (e', e_X) \ | \ \forall e \in X : e' \rhd e \} \cup \{ (e_X, e') \ | \ \forall e \in X : e \rhd e' \}}, \]
\[ \lambda_{/X} = \lambda_{(E \setminus X) \cup \{ (e_X, e_X) \} \cup \{ (e, e) \ | \ \forall e \in X \},} \]
\[ \lambda_{/X}(e_X) = \lambda(e) \text{ for an event } e \in X. \]

In words, the folding of \( \mathcal{A} \) is obtained by replacing the set \( X \) of events with a single event \( e_X \), with the same label as those in \( X \). The causes of \( e_X \) are the common causes \( S(X) \) of the events in \( X \), and \( e_X \) is a cause for all events caused by at least one event in \( X \). The asymmetric conflicts for \( e_X \) are exactly those of the events in \( X \).

It can be shown that \( \mathcal{A}_{/X} \) is indeed a properly defined AES (the proof can be found in the Appendix).

In order to show that the folding operation preserves the behaviour, i.e., that the original and folded AESs are hp-bisimilar, we rely on the notion of AES-morphism [6]. Intuitively, an AES-morphism is a mapping between AESs which shows how the target AES can simulate the source AES.

**Definition 9 (folding morphisms).** Let \( \mathcal{A} = (E, \leq, \rhd, \lambda) \) be an AES and let \( X \subseteq E \) be combinable. The folding function \( f : E \to E_{/X} \) is defined as follows:

\[
f(e) = \begin{cases} 
eq_{/X} & \text{if } e \in X \\
eq & \text{otherwise} \end{cases}
\]

It can be shown that the folding morphism is indeed an AES-morphism, and as such it maps configurations of \( \mathcal{A} \) into configurations of \( \mathcal{A}_{/X} \).

Actually, according to the next lemma, the folding morphism have very special properties in a way that it can be seen as a hp-bisimilarity between \( \mathcal{A} \) and \( \mathcal{A}_{/X} \). Proofs can be found in the Appendix.

**Lemma 1.** Let \( \mathcal{A} = (E, \leq, \rhd, \lambda) \) be an AES, and let \( \mathcal{A}_{/X} = (E_{/X}, \leq_{/X}, \rhd_{/X}, \lambda_{/X}) \) be the folding of \( \mathcal{A} \) on the set of events \( X \). Let \( f : \mathcal{A} \to \mathcal{A}_{/X} \) be the folding morphism. Then

\[
R = \{(C_1, f(C_1)) \mid C_1 \in \text{Conf}(\mathcal{A})\}
\]

is a hp-bisimulation.

**Corollary 1 (folding does not change the behavior).** The folding operation of AESs preserves hp-bisimilarity.
By iteratively applying folding to a given finite AES we can thus obtain a minimal AES hp-bisimilar to the given one. Unfortunately, this does not provide a canonical minimal representative of the behaviour as there can be non-isomorphism minimal hp-bisimilar AESs. For instance, consider the AES in Figure 4(a). There exist two possible folded AESs, presented side-by-side in Fig. 6, which are minimal in the sense that they cannot be further folded.

4 Behaviour preserving reduction of FESs

In this section we develop a behaviour preserving reduction technique for flow event structures. As for AESs, the basic idea is that of folding events representing different instances of the same activity, although technically there are relevant differences.

4.1 Basics of flow event structures

We start by recalling the formal definition of (labelled) flow event structures [4].

Definition 10 (flow event structure). A (labelled) flow event structure (FES) is a tuple \( \mathcal{F} = (E, #, \prec, \lambda) \) where

1. \( E \) is a set of events,
2. \( \lambda : E \to \Lambda \) is a labelling function, and
3. \( \prec \subseteq E \times E \), the flow relation, is irreflexive.
4. \( # \subseteq E \times E \), the conflict relation, is a symmetric relation.

Note that the flow relation is not required to be transitive. The \( \prec \)-predecessors of an event \( e \in E \), are defined as \( \bullet e = \{ e' \mid e' \prec e \} \). Similarly, for a set of events \( X \) we write \( \bullet X = \bigcup \{ \bullet e \mid e \in X \} \). The \( \prec \)-successors of \( e \) and \( X \) are defined as \( e\bullet = \{ e' \mid e \prec e' \} \) and \( X\bullet = \bigcup \{ e\bullet \mid e \in X \} \), accordingly.

The flow predecessors of an event \( e \), i.e., \( \bullet e \), can be seen as a set of possible immediate causes for \( e \). Conflicts can exist in \( \bullet e \), and, in order to be executed, \( e \) needs to be preceded by a maximal and conflict free subset of \( \bullet e \).

Formally, a configuration of a FES \( \mathcal{F} = (E, #, \prec, \lambda) \) is a finite subset \( C \subseteq E \) such that

1. \( C \) is conflict free,
2. \( C \) has no flow cycles, i.e. \( \prec_C \) is a partial order,
3. for all \( e \in C \) and \( e' \notin C \) s.t. \( e' \prec e \), there exists an \( e'' \in C \) such that \( e' \# e'' \prec e \).

We denote by \( Conf(\mathcal{F}) \) the set of configurations of \( \mathcal{F} \). The extension order, as for PESs, is simply subset inclusion.

Since in FESs the flow relation is not transitive and the conflict relation does not adhere to the principle of heredity: even though two events are not in conflict they might not appear together in any configuration, and an event could be not executable at all. More precisely, define the semantic conflict \( \#_s \) as \( e\#_s e' \) when for any configuration \( C \in Conf(\mathcal{F}) \), it does not hold that \( \{ e, e' \} \subseteq C \). Then clearly \( \# \subseteq \#_s \) and in general the inclusion is strict.

In line with the authors of [4], hereafter we restrict to the subclass of FES, where for which:
1. semantics conflict $\#$ coincides with conflict $\#$ (faithfulness),
2. conflict is irreflexive (fullness), hence all events are executable.

Observe that FESs generalise AESs in that, clearly, every PES can be seen as a special FES where the flow relation is transitive and the $\prec$-predecessors of any event are conflict free.

4.2 Reduction of FESs

As in the case of AESs, we identify sets of events which can be seen as instances of the same activity and which can be merged into a single event. As mentioned before, the way in which FES generalise PES is somehow orthogonal to that of AESs. As a consequence, at a technical level the conditions which define combinable events are quite different.

Consider, for instance, the example in Fig. 7a. First, if we take events $c_0$ and $c_1$ and try to merge them into a single event $c_{0,1}$, there would be no way of updating the dependency relations while keeping the behaviour unchanged (the resulting dependency between $b$ and the merged event $c_{0,1}$ would be an asymmetric conflict which is not representable in FESs). Instead, we can merge events $c_1$ and $c_2$ in $F_0$ into a single event $c_{1,2}$, thus obtaining the FES in Fig. 7b. In this case, the folding is possible because the original events $c_1$ and $c_2$ are enabled by $\{b\}$ and $\{d,e\}$, respectively, and since $b \not\# d$, $b \not\# e$, after the merge the same situation is properly represented as an disjunctive causality.

![Diagram of FESs](image)

In order to define combinable events we need some further notation. Given a set of events $Z$, we denote by $\mathbb{C}(Z)$ the set of maximal and consistent (i.e., conflict free) subsets of $Z$. Given an event $e \in E$, we write $\#(e)$ for the set of events in conflict with $e$, i.e., $\#(e) = \{e' \mid e' \in E \land e' \# e\}$. Additionally, as for the case of AESs we need to single out conflicts which are direct.

**Definition 11 (direct conflict).** Let $F$ be a FES. The events $e, e' \in E$ are in direct conflict, denoted as $e \# d e'$, if $e \# e'$ and $\exists Y \in \mathbb{C}(\bullet e)$ s.t. $Y \cap \#(e') = \emptyset$.

Intuitively, a conflict $e \# e'$ is direct when there is a way of reaching a configuration $e$ is enabled, without disabling $e'$.

We can now define the notion of combinable set of events for FESs.
Definition 12 (combinable set of events). Let $\mathcal{F}$ be a FES. A set of events $X \subseteq E$ is combinable if for all $x, x' \in X$ and $e, e' \in E \setminus X$ the following holds

1. $\lambda(x) = \lambda(x')$ and $x \# x'$,
2. $x \# e \Rightarrow x' \# e$,
3. $x \prec e \Rightarrow x' \prec e \lor x' \# e$,
4. $e \prec x \Rightarrow \bullet x' \neq \emptyset \land (x \prec x' \lor e \# x')$,
5. $e \in (x \bullet \cap x' \bullet) \land e' \prec e \land (e' \# x') \land (e' \# x')
   \Rightarrow \forall Y' \in C(\bullet e) \cdot x' \in Y' : (Y' \setminus X) \cap \#(e') \neq \emptyset$

Roughly speaking, condition 1 requires that the events in $X$ are occurrences of the same activity (they have the same label and they are in conflict). Condition 2 requires that events in $X$ have essentially the same conflicts. Conditions 3 and 4 state that predecessors and successors are preserved among events in $X$ or they can be turned into conflicts. Finally, condition 5 takes into account the situation in which two (or more) events $x, x' \in X$ are potential causes of an event $e$, but they have different conflicts with another potential cause $e'$ of the same event: $x \# e'$, while $\neg(e' \# e')$. This is problematic, since after the merging the conflict in disagreement will be lost. The condition says that folding can be still possible if the conflict $x' \# e'$ is not essential when forming the maximal consistent sets of $\prec$-predecessors for $e$. For example, the FES depicted in Fig. 8a illustrates a situation in which condition 5 fails. Please note that in Fig. 8a events corresponding to those in condition 5 have a subscript aimed at facilitating the analysis. Merging $a_x$ and $a_{x'}$ would lead to the FES in Figure 8b, which is not behaviourally equivalent to $\mathcal{F}_2$. In particular, observe that $c$ is no longer executable since $b \# e$, but it is not the case that $b \# a$ (hence such FES is not faithful). The conflict $b \# a$ could not be imposed in the folded FES $\mathcal{F}_3$ otherwise a configuration corresponding to $\{d, a_x, b_{x'}\}$ in $\mathcal{F}_2$ would be lost. A legal folding is shown in Figures 8c-8d in $\mathcal{F}_5$, after the execution of $e$ or $d$, it is possible to have a maximal and consistent set of $\prec$-predecessors for the event $c$, i.e., $\{a_{0,1}, f\}$ or $\{a_{0,1}, b\}$.

We can now formally define the folding of FESs as follows.

Definition 13 (folding of FES). Let $\mathcal{F}$ be a FES, $X$ be a set of combinable events. The folding of $\mathcal{F}$ on $X$ is the FES $\mathcal{F}_{/X} = \langle E_{/X}, \#_{/X}, \prec_{/X}, \lambda_{/X} \rangle$ where

\[
\begin{align*}
E_{/X} &= E \setminus X \cup \{e_X\}, \\
\#_{/X} &= \#(E_{/X}) \cup \{(e, e_X) \mid \forall e' \in X : e \# e'\}, \\
\prec_{/X} &= \prec(E_{/X}) \cup \{(e, e_X) \mid \exists e' \in X : e \prec e'\} \cup \{(e, e_X) \mid \exists e' \in X : e < e'\}, \\
\lambda_{/X} &= \lambda, \lambda_{/X}(e_X) = \lambda(e) \text{ for an event } e \in X.
\end{align*}
\]

The rest of the section is dedicated to show that the folding operation on FESs preserves $hp$-bisimilarity. As a first step we note a immediate consequence of Definition 12 as captured in the following corollary.

Corollary 2. Let $\mathcal{F}$ be a FES and let $X \subseteq E$ be a combinable set of events. Then the following holds:

\[\forall Y \in C(\bullet X) \Leftrightarrow \exists x \in X : Y \subseteq \bullet x.\]
Proof. Follows directly from condition 4 in Definition 12.

The idea, which underlies also AES folding, is that events which are merged are occurrences of the same activity with different histories. They can be merged if the histories are compatible and after merging the possible histories remain the same. Since a FES an event can occur after a maximal and consistent set of \( \preceq \)-predecessors (i.e., once all the conflict among its predecessors has been resolved). By Corollary 2 above, after merging a set of combinable events this maximal subsets of consistent events remains unchanged. This will be at the basis of the proof that the merging does not alter the behaviour.

As in the case of AESs, we rely on the folding morphism.

Definition 14 (folding morphism). Let \( \mathcal{F} \) be a FES and let \( X \subseteq E \) be combinable. The folding function \( f : E \to \mathcal{F}_{/X} \) is defined as follows:

\[
f(e) = \begin{cases} \bar{e} & \text{if } e \in X \\ e & \text{otherwise} \end{cases}
\]

We can prove that the folding morphism is an instance of a more general notion of FES-morphism and, as such, it maps configurations into configurations. The folding morphism has special properties, so that, besides reflecting conflicts it also satisfies a form of conflict preservation, as expressed below.

Corollary 3. Let \( \mathcal{F} \) be a FES, \( X \subseteq E \) be a combinable set of events and \( \mathcal{F}_{/X} = \langle \mathcal{E}_{/X}, \#_{/X}, \prec_{/X}, \lambda_{/X} \rangle \) be the folded FES. For any pair of events \( x, x' \in \mathcal{E}_{/X} \) the following holds.

Fig. 8: Example FESs to illustrate Condition 5 in Definition 12
\[\neg(x \# x') \Rightarrow \exists e, e' \in E : f(e) = x \land f(e') = x' \land \neg(e \# e').\]

The above can be used to show that the FES resulting from a folding is faithful and full (see Appendix for the detailed proof).

**Lemma 2.** Let \( F \) be a FES, \( F_X = (E_X, \#, \prec_X, \lambda_X) \) be the folded FES of \( F \) and \( f : F \to F_X \) be the folding morphism. The FES \( F_X \) is 1) faithful, and 2) full.

Building on the previous technical results we can finally prove that the folding morphism \( f \) can be seen as a hp-bisimulation.

**Lemma 3.** Let \( F = (E, \#, \prec, \lambda) \) be a FES and \( F_X = (E_X, \#, \prec_X, \lambda_X) \) be a folded FES for a set of combinable events \( X \subseteq E \). Let \( f : F \to F_X \) be the folding morphism. Then

\[
R = \{(C_1, f(C_1), f(C_1)) \mid C_1 \in \text{Conf}(F)\}
\]

is a hp-bisimulation.

**Corollary 4 (folding does not change the behavior).** The folding operation of FESs preserves hp-bisimilarity.

As for AESs the iterative application of folding to a given finite FES allows one to minimise the given FES while preserving the behaviour. Also in this case, there is no canonical representative, i.e., there can be several minimal non-isomorphic FESs.

## 5 Transformation from AESs to FESs

In the previous sections we discussed suitable techniques for reducing the number of events in AESs and FESs, without changing the behaviour (up to hp-bisimilarity). We already noted that AESs and FESs can be seen as orthogonal extensions of PESs, and thus the kinds offoldings they allow are conceptually different. Here we show that a translation from AESs to FESs is possible by allowing the presence of silent, unobservable events which can be used to encode asymmetric conflicts in FESs. This translation can be used to take advantage of the combined expressiveness of AESs and FESs when looking for a minimised representation of the behaviour.

First, notice that the causality and binary symmetric conflict (cycles of asymmetric conflict) in AESs can be represented in primitively represented in FESs. The idea for modelling asymmetric conflicts in FESs is the following. Given an event \( e \), if \( e' \not\prec e \) then \( e' \) acts as a weak cause of \( e \), namely it can either be included or not in the history of \( e \). The same effect can be obtained in a FES by letting \( e' \prec e \) and inserting a new silent event \( \tau \), with \( \tau \prec e \) and \( \tau \# e' \). In this way, \( e' \) can be excluded from an history of \( e \) by the presence of \( \tau \), which is invisible. Consider for instance the AES \( A_4 \) in Fig. 9c. Its encoding as a FES is reported
as $\mathcal{F}_9$ in Fig. 9d. The corresponding configurations are in Figs. 9e and 9f, and observe that they have a close correspondence once silent events are removed.

Since asymmetric conflicts can emerge at each folding of an AES. Therefore, it is necessary not just to define the $\tau$-events that would be used to simulate the asymmetric conflict relation; but, to characterise the cases when additional $\tau$-events are required for allowing further folding.

Now, consider the PESs $\mathcal{P}_1$ and $\mathcal{P}_2$ in Figs. 9a and 9b. We have that $\mathcal{P}_1$ can be folded as an AES, thus obtaining $\mathcal{A}_4$ in Figure 9c, but not as a FES. Instead, $\mathcal{P}_2$ where can be folded into the FES $\mathcal{F}_9$ of Fig. 9d. This suggests that, exploiting the encoding of asymmetric conflicts into FESs, we can use the insertion of $\tau$-events into FESs simulate the asymmetric conflict and thus enable further folding of a given FES.

We next formalise these ideas. We consider FESs with silent events, where the labelling function is $\lambda: E \rightarrow \Lambda \cup \{\tau\}$. An event $e \in E$ is called silent when $\lambda(e) = \tau$ and visible otherwise. The insertion of $\tau$’s for the encoding of asymmetric conflicts does not preserve bisimilarity, as the presence of $\tau$ determines the possibility of a silent choice. For instance, considering $\mathcal{A}_4$ in Fig. 9c and $\mathcal{F}_9$ in Fig. 9d one can see that configuration $\{\tau\}$ $\text{Conf}(\mathcal{A}_4)$ has no “counterpart” in $\text{Conf}(\mathcal{F}_9)$, and as a consequence $\mathcal{A}_4$ and $\mathcal{F}_9$ are not bisimilar (hence not hp-bisimilar).

In this context, we thus take a reference behavioural equivalence weaker than hp-bisimilarity, namely visible-pomset equivalence [8]. Given a set of events $X \subseteq E$, we denote by $X^L = \{e \in X \mid \lambda(e) \neq \tau\}$. The same notation is used when $X$ is a pomset, i.e., it carries also a order $\leq_X$. In this case $X^A$ carries also the restriction of the order on $X$ and it is called the visible pomset underlying $X$. Given an event structure $\mathcal{E}$ we denote by $\text{Conf}(\mathcal{E})^A$ the set of visible pomsets underlying its configurations, i.e., $\text{Conf}(\mathcal{E})^A = \{C^A : C \in \text{Conf}(\mathcal{E})\}$, with the
extension order inherited from $\text{Conf}(E)$. The definition below is given for a
generic brand of event structure (as it only depends on the set of configurations).

**Definition 15 (pomset equivalence [8]).** The event structures $E_1$ and $E_2$
are called visible-pomset equivalent, written $E_1 \equiv_{pt} E_2$, when $\text{Conf}(E_1)^A \sim \text{Conf}(E_2)^A$.

### 5.1 Translation of the asymmetric conflict

The asymmetric conflict relations represent weak causes of an event, which
can be either inserted or not in the history of the event. Hence, if an event
has several weak causes, any causally closed and conflict free) combination of
them can occur in some history. For instance, consider the AES $\mathcal{A}_5$ (Fig. 10a).
The event $e$ can be preceded by any subset of its weak causes $\{e_1, e_2\}$, i.e.,
$\text{hist}(e) = \{\{e_1, e_2, e\}, \{e_1, e\}, \{e_2, e\}, \{e\}\}$. The same visible behaviour can be
represented by the FES in Fig. 10b where three $\tau$-events allow to selectively
inhibit any possible subset of weak causes. More precisely $\tau_3$ inhibits the occurrence of $\{e_1, e_2\}$
leading to configuration $\{e\}$. Similarly, $\tau_1$ inhibits event $\{e_2\}$
and $\tau_2$ inhibits $\{e_1\}$, and thus lead to the configurations $\{e_1, e\}$ and $\{e_2, e\}$,
respectively.

More generally, when translating an AES into a FES, asymmetric conflicts
can be turned into $\prec$-dependencies. Then, intuitively, for any possible history
$h$ of an event $e$, we insert a $\tau$-event in conflict with those weak causes of $e$
not included in $h$. The inclusion of such event in the history of $e$ will shape
appropriately the history, ruling out the undesired weak causes. The following
definition provides the basic concepts used later on. Hereafter, we use $\prec$ to
denote the strict asymmetric conflict, i.e., we write $e \prec e'$ as a shortcut for
$e \not\sim e'$ and not $e \leq e'$.

**Definition 16 (sources and targets of asymmetric conflict).** Let $\mathcal{A}$ be an
AES. We denote by $\zeta(\mathcal{A}) = \{e' \in E \mid \exists e'' \in E. e'' \not\sim e' \land (e'' \not\prec e')\}$, and by
$\omega(e) = \{e' \in E \mid e' \not\sim e \land (\lnot(e' \not\prec e')\}$. 
In words, \( \varsigma(A) \) is the set of events which have weak causes, while \( \omega(e) \) is the set of weak causes of event \( e \).

The next definition makes the above consideration on \( \tau \)-events, precise.

**Definition 17 (sets of events to prevent).** Let \( A \) be an AES, \( e \in E \) be an event in \( A \). A possible set of weak causes for \( e \) is a set \( Z \subseteq \omega(e) \) which is \( \leq \)-closed and conflict free. In this situation we denote by \( \bar{Z} \) the corresponding excluded events, i.e.,

\[
\bar{Z} = \{ e' \mid e' \in \omega(e) \land \neg \exists e'' \in Z. e' \neq e'' \}
\]

We denote by \( \theta(e) \) the collection of possible sets of weak causes for \( e \) and by \( \nu(e) = \{ \bar{Z} \mid Z \in \theta(e) \land \bar{Z} \neq \emptyset \} \).

For any event \( e \) each conflict-free and causally closed subset \( Z \) of weak causes of \( e \) determines a possible different history for \( e \). Correspondingly, a \( \tau \)-event must be introduced in conflict with the other weak causes of \( e \) not in \( Z \), which could potentially be inserted in the history as they are not in conflict with any event in \( Z \). This is the set which above, is denoted by \( \bar{Z} \).

Consider the AES in Figure 10a where \( \varsigma(A) = \{ e \} \) and \( \omega(e) = \{ e_1, e_2 \} \), then \( \theta(e) = \{ \{ e_1, e_2 \}, \{ e_1 \}, \{ e_2 \}, \emptyset \} \), and \( \nu(e) = \{ \{ e_2 \}, \{ e_1 \}, \{ e_1, e_2 \} \} \). Indeed, the translation in Figure 10b introduces a \( \tau \)-event for each of the elements of in \( \nu(Z) \).

### 5.2 Required \( \tau \)-events

As already mentioned, given an AES, \( \tau \)-events can be introduced to represent existing asymmetric conflicts. However, further \( \tau \)-events can be introduced which allows the representation in the FES of an asymmetric conflict which would arise as an effect of a folding in the AES.

As an example, consider the AES \( A' \) (Fig. 11a) where the pair of events with label \( a \) are combinable. If we just view \( A' \) as a FES the events with label \( a \) are no longer combinable. The reason is that the folding of this two events in \( A' \) leads to a new asymmetric conflict (as shown in Fig. 11c), where the event \( b \) would be a weak cause of the event \( a_{0,1} \).

If in the FES representation we insert a \( \tau \)-event preceding \( a_1 \) as shown in \( F_9' \) (Fig. 11b), then the events labelled by \( a \) are combinable. The FES resulting from the folding of \( F_9' \) (see Fig. 11d) is exactly the encoding of the AES resulting as the folding of \( A' \). In this specific example, the insertion of the \( \tau \)-event was done with respect to event \( a_1 \).

More generally, we identify a sets of **events to complete**, i.e., events for which an additional \( \tau \)-event predecessor is inserted.
Definition 18 (events to complete in AES). Let \( \mathcal{A} \) be an AES. An event \( e \in E \) needs to be completed if there is an event \( e' \in E \) such that \( \lambda(e) = \lambda(e') \), \( e \neq e' \) and there exists an event \( e'' \in E \) such that

\[-e \#_\mu e'' \land \neg(e' \#_\mu e'') \], and
\[-e'' \succ_\mu e' \lor e'' <_\mu e'\]

The set of events to complete in \( \mathcal{A} \) is referred to as \( \mathcal{C}(\mathcal{A}) \).

5.3 Transformation from AES to FES

The rest of this section describes the transformation of an AES into a FES. As already mentioned, causality and (strict) asymmetric conflicts become \( \prec \)-relations, additional \( \tau \)-events are inserted for any set \( Z \in \rho(e) \), for some event \( e \), and for any event in \( \mathcal{C}(\mathcal{A}) \). We still need to describe in detail the newly inserted \( \tau \)-events relates to the rest of the events. For easing the transformation, any event \( e_1 \) in the FES is represented as a triple \( e_1 = (e, \rho, l) \), where \( e \) represents an event in the original AES, \( \rho \) is the set of events in conflict with \( e_1 \), and \( l \) is the label for the event \( e_1 \) (\( \tau \) if it is a silent event). By definition, consider any event as self-concurrent, and thus the following transformation computes the relations between pairs of distinct events.

Definition 19 (Transformation from AES to FES). Let \( \mathcal{A} \) be an AES. Let \( \#(e) \) denote the set of events in conflict relation with event \( e \), i.e. \( \#(e) = \{ e' \in E \mid e' \# e \} \). The FES encoding of \( \mathcal{A} \) is denoted \( \delta(\mathcal{A}) \) and defined by

\[ E_F = \{ (e, \#(e), \lambda(e)) \mid e \in E \} \cup \{ (e, \#(e), \tau) \mid e \in \mathcal{C}(\mathcal{A}) \} \cup \{ (e, \#(e) \cup \rho, \tau) \mid e \in \mathcal{C}(\mathcal{A}) \land \rho \in \rho(e) \}, \]
\[ \#_F = \{ (((d, \rho, l), (e, \rho', l')) \mid e \in \rho \lor (d = e \land \rho \neq \#(e) \land \rho' \neq \#(e)) \lor (\exists d' \in \rho \land (d <_\mu e)) \}, \]
\[ \prec_F = \{ (((d, \rho, l), (e, \rho', l')) \mid (d <_\mu e \lor d \succ_\mu e) \lor (l = \tau \land d = e \land \rho' = \#(e)) \}, \]
\[ \lambda_F((d, \rho, l)) = l \]

An event \( e \) in an AES can lead to the introduction of a number of \( \tau \)-events in the corresponding FES. We next define a function \( \delta \) that associates any event in AES to the corresponding non-silent event in the FES encoding.

Definition 20 (AES2FES function). Let \( \mathcal{A} \) be an AES and \( \delta(\mathcal{A}) = \langle E_F, \#_F, \prec_F, \lambda_F \rangle \) be its FES encoding. The function \( \delta : E_F \to E \) is defined as \( \delta(e, \rho, l) = e \) when \( l \neq \tau \) and \( \delta(e, \rho, l) = \perp \), otherwise.

The transformation preserves the immediate causality and conflict relation between any pair of events in AESs. Moreover it can be shown that the set of configurations are preserved. Using these facts one can show that the source AES and its encoding as a FES are pomset-equivalent.
**Corollary 5.** Let \( \mathcal{A} \) be an AES. Then \( \mathcal{A} \) and its FES encoding \( \delta(\mathcal{A}) \) are pomset-trace equivalent, i.e., \( \mathcal{A} \equiv_{pt} \delta(\mathcal{A}) \).

**Proof.** See the Appendix. \( \square \)

Finally, the following lemma shows that as a result of the transformation from AES to FES, any set of events combinable in AES is also combinable in FES.

**Lemma 4.** Let \( \mathcal{A} \) be an AES, \( \delta(\mathcal{A}) = \mathcal{F} = (E_F, \#_F, \prec_F, \lambda_F) \) be its corresponding FES, and \( \delta : E_F \rightarrow E \) be the mapping function. For any set of combinable events \( X \subseteq E \) there is a combinable set of events \( Y \subseteq E_F \), such that \( \delta(Y) = X \).

The overall transformation from AES to FES is graphically represented in Figure 12. The functors taken from existing literature are depicted in gray, i.e., from PES to AES [6] and from PES to FES [4]. The functor \( f \) represents the folding of AES and FES respectively; whereas \( \delta \) is the functor from AES to FES.

![Fig. 12: Overall transformation process from AES to FES.](image)

The example in Figure 13 depicts the foldings and different transformations of the PES \( P_0 \). The edges between event structures use the notation presented in Figure 12. The AESs \( \mathcal{A}_5 \) and \( \mathcal{A}_6 \) are two possible foldings of \( P_0 \), note that both event structures cannot be longer folded, uncovering the non-canonical representation of the behaviour in AESs. On the other hand, \( F_{11} \) and \( F_{12} \) are the FESs resulting from the translation of \( \mathcal{A}_5 \) and \( \mathcal{A}_6 \), correspondingly. The unobservable events inserted to simulate the asymmetric conflict are labelled as \( \tau' \), whereas the events labelled as \( \tau \) were introduced to *complete the event c*. Note the that the non-canonical representation is, as expected, present in the transformation to FES also. Finally, both FESs (\( F_{11} \) and \( F_{12} \)) can be further folded to \( F_{13} \) and \( F_{14} \). This example shows that the non-canonicity of the behaviour can be carried all the way, using either AES or FES, through the different set of transformations and foldings.
Fig. 13: Transformations from PES to FES, through the folding of AESs
6 Conclusion and future work

This paper presents reduction techniques, referred to as folding, for AESs and FESs which allow one to reduce the number of events in an event structure without changing the behaviour. The folding operation merge sets of events that are intended to represent instances of the same activity. The equivalence notion adopted is history preserving bisimulation, a standard equivalence in the true concurrent spectrum. Due to the different expressive power of AESs and FESs, tailored folding techniques have been proposed for the two brands of event structures.

It turned out that neither AESs nor FESs offer a canonical representation of the behaviour of a process. More specifically, the same process can have non-isomorphic (and not further reducible) foldings both in terms of AESs and FESs. Additionally, since AESs and FESS extends PESs in orthogonal ways, and the sets of combinable events in each of the structures are generally different. A transformation from AESs to FESs is proposed, which at the price of inserting unobservable events, allows one to take advantage of the combined expressiveness of AESs and FESs ensuring that any folding in an AES is a possible folding in its corresponding FES.

The interest on finding a smaller representation of a process is manifold. For instance, in the field of business process modelling, it is well known how to translate any process in BPMN notation into a Petri net [1]. In this case, the computation of a prime event structure from the Petri net representation of a process is straightforward by [2]. Therefore, a possible minimisation of the behaviour would allow the reduction (and possible elimination) of duplicate activities present in the original model, finding a canonical representation of a process for behavioural comparison, among others.

Future work includes the assessment of performance (accuracy, efficiency) of the presented technique for process model differencing in real world process model collections. Naturally, it is planned to extend this work to cover cases with cycles. Finally, a promising avenue is the use of folding of FESs for approaching problems like process mining and elimination of duplicates in process models. An additional advantage of FESs is that they can easily transformed into a certain type of Petri nets, flow nets.

The minimisation of the behaviour of a process can be translated into some kind of minimisation problem for automata or labelled transition system. Most available techniques focus on interleaving behavioural equivalences (like language or trace equivalence or various forms of bisimilarity). We are not aware of approaches for the minimisation of event structures or partially ordered models of computation. In some cases, given a Petri net or an event structure a special transition system can be extracted, on which minimisation is performed. For instance in [10] the authors propose an encoding of safe Petri nets into a causal automata, in a way which preserves hp-bisimilarity. The causal automata can be transformed into a standard labelled transition system (LTS). In this way, the LTS representation can be used to check the equivalence between a pair of processes or to find a minimal representation of the behaviour. However, once a
Petri net has been transformed into a causal automaton, then it is not possible to obtain the Petri net representation back, which can be of interest in some specific applications. In [11], the author uses a state transition diagram referred to as process graph, for the representation of the behaviour of a Petri net. Again, the transition diagram could be minimised with some technique for LTSs with structured states, but not direct approach is proposed.

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