Method of functional integration in the problem of line width of parametric X-ray relativistic electron radiation in a Crystall

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Abstract

The coherent and non-coherent scattering effects on "backward" parametric X-ray radiation by relativistic electrons in a crystal on the basis of the method of functional integration is investigated. A comparison of contributions of these effects to parametric X-ray radiation line width has been considered. It is shown that in a number of cases the major contribution to the line width of parametric X-ray radiation is made by non-coherent multiple scattering.

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1. The different coherent and interference radiation effects are possible when a relativistic electron moves at a small angle with respect to one of the crystallographic axes in a crystal [12]. Due to these effects the spectral-angular radiation density has sharp maxima with high radiation intensity. One of these effects is Parametric X-ray Radiation (PXR) [3,4,5]. This radiation is caused by particle field scattering on nonuniformities of lattice

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permitivity and mainly concentrated in direction close to the Bragg direction of particle field reflection from crystalline planes of atoms. Of special interest is "backward" PXR when an electron falls at a small incidence angle to one of the crystallographic axes (z-axis), because in this case the contribution of other kinds of radiation such as coherent and channeling radiation are considerably suppressed. The coherent and channeling radiation are largely concentrated in the line of particle motion. Narrow lines then appear in the spectral-angular radiation density as a result of the interference of irradiated waves by electron at interaction with crystalline planes of atoms oriented normally to z-axis. Experimental studies of PXR line widths have shown that the line widths were, however, much larger than the natural PXR line width. Performed analysis of experimental data in [5] is shown that the multiple scattering must make considerable influence on line width of PXR.

In [6] a theory of the line width of "backward" PXR based on the method of functional integration was suggested. On the basis of this theory in [6] the simplest case of the multiple scattering effect on the PXR spectral-angular characteristics was considered in which multiple scattering was studied only in continuous field of the crystal atomic strings.

This paper is devoted to investigation of non-coherent multiple scattering effect of particles on the line width in a crystal. Offered averaging method in [6] over PXR spectral-angular density is generalized to the case, when coherent and non-coherent effects in particle multiple scattering on atomic strings are took into account. It is shown that this problem in many respects is similar to the problem of the Landau-Pomeranchuk-Migdal effect of multiple scattering on coherent bremsstrahlung of high energy electrons in a crystal and in an amorphous medium [7,8,9,10]. On the basis of obtained formulas a comparison of contributions of coherent and non-coherent multiple scattering of particles in a crystal to PXR line width is verified. It is considered the case for which the non-coherent multiple scattering makes the major contribution to the PXR line width.

2. We shall investigate "backward" PXR when relativistic electrons fall at a small incidence angle $\psi$ with respect to one of the crystallographic axes (z-axis). Narrow lines then appear in the spectral-angular radiation density caused by particle field reflection from the crystallographic planes of atoms
oriented transversely to z-axis on equal distance \( a \) from each other. Spectral-angular radiation density corresponding to the line width frequency \( \omega_n \) is given by the equation [6]

\[
\frac{dE}{d\omega d\omega} = \frac{e^2 \omega_n^2}{4\pi^2 a^2} \left| \epsilon_{\omega_n, g} \right|^2 \left[ \int_0^L dt e^{2i(\omega - \omega_n)} \frac{\gamma^2}{\gamma^2 + \left( \theta - \psi \right)^2 - \frac{2i}{\omega_n} \frac{\partial}{\partial \theta} \Phi(\mathbf{v}_\perp(t))} \right]_0^{\mu \rightarrow 0}^2
\]  

(1)

where \( L \) is target thickness, \( \psi = (\psi_x, 0) \) is two-dimensional vector in the \((x, y)\) plane normally to z-axis defining angle \( \psi = |\psi| \) between falling particle velocity vector and z-axis (it is proposed that beam falls to the crystal in the \((x, y)\) plane) and \( \theta = (\theta_x, 0) \) is the angle at which radiation occurs. (we are interested in radiation in the \((x, y)\) plane in the region of angles \( \theta \) close to the direction of Bragg reflection of waves; \( \theta \) counts off from perpendicular to the crystal surface on which beam falls).

The value \( \epsilon_{\omega_n, g} \) in (1) is the Fourier component of lattice permittivity. In our case the permittivity nonuniformity along \( x \)- and \( y \)-axes is inessential to radiation. \( \epsilon_{\omega_n, g} \) is, therefore, formed only by permittivity non-uniformity along the z-axis

\[
\epsilon_{\omega_n, g} = \int_0^a dz \epsilon_n \exp(-igz) ,
\]  

(2)

where \( g = \frac{2\pi n}{a} \) and \( n \) are integers.

The functional \( \Phi(\mathbf{v}_\perp(t)) \) in (1) determines influence of multiple scattering on PXR spectral-angular density

\[
\Phi(\mathbf{v}_\perp(t)) = \exp \left\{ \mu \mathbf{v}_\perp(t) + \frac{i\omega_n}{2} \int_0^t dt' \mathbf{v}_\perp^2(t') - i\omega_n(\theta - \psi) \int_0^t dt' \mathbf{v}_\perp(t') \right\}
\]  

(3)

where \( \mathbf{v}_\perp(t) \) is the transverse component of electron velocity vector at time \( t \). The value \( \omega_n \) in our geometry is determined by relation

\[
\omega_n = vg \cos \psi (1 + v \cos(\theta + \psi))^{-1}.
\]  

(4)

Eq. (1) was derived taking into account both target thickness and small deviations of the particle trajectory in crystal from linear one. Obtaining (1) it was supposed that the medium permittivity is close to unity and also photon absorption was neglected. The finite size of crystal leads us, as is well [5, 6], to the natural line width of PXR. Small deviations of electron
trajectory in crystal from linear one is caused by both particle plane channeling effect and multiple scattering on non-uniformities of lattice potential. The plane channeling effect is absent when electrons fall at a small incidence angle with respect to one of the crystallographic axes (see $\Pi$ for example). In this case multiple scattering on atomic strings of crystal parallel to the $z$-axis leads us to the deviation of particle trajectory from linear one.

Under conditions $|\theta - \psi| \gg |v_\perp(t)|$ the dependence of preexponential factor on random value $v_\perp(t)$ can be ignored. Such dependence in (1) appears as a result of action of differential operators on the functional $\Phi(v_\perp(t))$.

In this simplest situation formula (1) takes the next form

$$\frac{dE}{d\omega} = \frac{e^2 \omega_n^2 |\epsilon_{\omega_n,g}|^2}{4\pi^2 a^2} \frac{(\theta - \psi)^2}{[\gamma^2 + (\theta - \psi)^2]^2} \left| \int_0^L dt e^{2i(\omega_\perp(t))(t-t')} \Phi(v_\perp(t)) \right|^2$$

Eq. (5) should be averaged over random angle value of particle scattering in crystal. Separating out in (5) function subject to averaging we obtain the following formula for average value of PXR spectral-angular radiation distribution

$$\langle \frac{dE}{d\omega} \rangle = \frac{e^2 \omega_n^2 |\epsilon_{\omega_n,g}|^2}{4\pi^2 a^2} \frac{(\theta - \psi)^2}{[\gamma^2 + (\theta - \psi)^2]^2} L^2 F(L, \omega - \omega_n)$$

where

$$F(L, \omega - \omega_n) = \frac{2}{L^2} \mathcal{R} \int_0^L dt \int_0^t dt' e^{2i(\omega_\perp(t))(t-t')} \langle \Phi(t, t') \rangle$$

and

$$\Phi(t, t') = \exp \left\{ \frac{i\omega_n}{2} \int_{t'}^t d\tau \dot{v}_\perp(\tau) - i\omega_n(\theta - \psi) \int_{t'}^t d\tau \nabla_\perp(\tau) \right\}$$

Relativistic electron scattering in a crystal when an electron falls at a small incidence angle $\psi$ to crystallographic axis is caused by both coherent and non-coherent particles scattering on lattice atoms. Coherent scattering is mainly along azimuthal angle in the $(x,y)$ plane, orthogonal to the $z$-axis $\Pi$. A redistribution of particles over this angle occurs as a result of multiple scattering by different atomic strings. If $\psi \gg \psi_c$, the multiple scattering is a Gaussian process with the mean square of multiple scattering angle $\theta^2 = q_cL$ (here $\psi_c = \sqrt{\frac{4Ze^2}{cd}}$ is the critical angle of axial channeling, $d$
is the interatomic distance along $z$-axis, $\epsilon$ is the particle energy) \cite{1}. Non-coherent scattering at $\psi \gg \psi_c$ mainly occurs as a result of particle scattering angle by thermal displacement of atom positions in lattice. In this case the mean square of multiple scattering angle is close to the value of this one in an amorphous medium $\theta^2_a = q_a L$. Thus, particle trajectory deviation along the $y$-axis, orthogonal to the $(z, v)$ plane, is caused by both coherent and non-coherent particle scattering in crystal with mean square of multiple scattering angle $\theta^2_y = q_y L$ where $q_y = q_c + q_a/2$. Particle deviation then along $x$-axis is caused by non-coherent scattering with mean square of multiple scattering angle $\theta^2_x = q_x L$, where $q_x = q_a$.

Random values $v_{\perp x}(\tau)$ and $v_{\perp y}(\tau)$ in (8) can be factorized. The average value of function $\Phi(t, t')$ then can be written as the product of average magnitudes

$$
\langle \Phi \rangle = \langle \Phi_x \rangle \cdot \langle \Phi_y \rangle.
$$

Since in our case scattering process is a Gaussian, we can write $\langle \Phi_x \rangle$ in terms of functional integration

$$
\langle \Phi_x \rangle = \lim_{N \to \infty} \int \cdots \int \frac{dv_1 \cdots dv_N}{(2\pi q_x \Delta)^{N/2}} \exp \left\{ -\frac{v_1^2}{2q_x \Delta} - \cdots - \frac{(v_N - v_{N-1}^2)}{2q_x \Delta} - i\omega_n \theta_r \sum_{n=k}^N v_n + \frac{i\omega_n}{2} \Delta \sum_{n=k}^N v_n^2 \right\} (9)
$$

where $t = N \Delta, t' = k \Delta, v_n$ is scattering angle at time $t_n = n \Delta$ and $\theta_r = \theta - \psi$. (Here we use $v \approx 1$.)

In our geometry $\theta$ and $\psi$ are situated in the $(x, y)$ plane. In this case the value $\langle \Phi_y \rangle$ can be derived from $\langle \Phi_x \rangle$ if we replace $q_x$ in $\langle \Phi_x \rangle$ by $q_y$ and use $\theta_r = 0$.

The functional integral (9) has the same structure as the corresponding integral in the theory of Landau-Pomeranchuk-Migdal multiple scattering effect on bremsstrahlung of high energy electrons in an amorphous medium. Integral (9) can be therefore calculated by the method developed in \cite{9,10}.

We then obtain function $F(L, \omega - \omega_n)$ in the next form

$$
F(L, \omega - \omega_n) = 2Re \int_0^1 dx \int_0^x du \, e^{2i(\omega - \omega_n)Lu} \times \\
\times \frac{1}{\sqrt{B_x B_y}} \exp \left\{ -\alpha^2 \frac{u^2}{B_x} \right\} (10)
$$

where $\alpha^2 = \frac{\sigma_x^2}{q_x L^2}, \sigma_x = \omega_n q_x L^2, \sigma_y = \omega_n q_y L^2, B_y = 1 - i\sigma_y u(x - u) + ...$
\[ + \frac{\sigma_y^2}{3}u^3(x - u) \] and \[ B_x = 1 - i\sigma_x u(x - u) + \frac{\sigma_y^2}{3}u^3(x - u). \]

Here we put in dimensionless quantities \( u = (t - t')/L \) and \( x = t/L \).

3. Eq. (10) shows that the influence of multiple scattering on "backward" PXR is determined by the parameters \( \sigma_x, \sigma_y \) and \( \alpha \). We shall consider some extreme cases.

If \( \sigma_x \to 0 \) and \( \sigma_y \to 0 \) the multiple scattering influence on PXR can be ignored. The function \( F = F_0 \) then determines the natural line width of PXR

\[ F_0 = \frac{\sin^2(\omega - \omega_n)L}{(\omega - \omega_n)^2} \] (11)

The line width in this case is given by \( \Delta \omega \sim 1/L \) to within the order of magnitude.

At \( \sigma_x \to 0 \) Eq. (10) gets over in corresponding results [6]. (we note that \( q_y \) is related in the Eq. (13) of the work [6] by means of ratio \( q_y = q/2 \). The value \( \overline{\theta^2} \) then in the [6] must be determined by the ratio \( \overline{\theta^2} = qL/2 = q_yL \)

In this case if \( \sigma_y \gg 1 \) the line width to within the order of magnitude is given by

\[ \Delta \omega \sim \frac{1}{L} \left( \frac{\sigma_y^2}{3} \right)^{1/3} \] (12)

If \( \sigma_y \sim \sigma_x \gg 1 \) the major contribution to integral over \( u \) in (10) at \( \alpha \gg 1 \) can be made by \( u_{\text{eff}} \sim 1/\alpha \sigma_x \ll 1 \). (We use here that the inequality \( \alpha \gg 1 \) is the validity condition of Eq. (5).) Neglecting the values of order \( 1/\alpha \sigma_x \) we can rewrite (10) in the next form

\[ F(L, \omega - \omega_n) = 2\text{Re} \int_0^1 dx \int_0^x du e^{2i(\omega - \omega_n)Lu} \exp \left( -\alpha^2 \sigma_x^2 u^2 x \right). \] (13)

According to (13) the line width to within the order of magnitude is given by

\[ \Delta \omega \sim \frac{1}{L} \alpha \sigma_x. \] (14)

Thus, at \( \sigma_x \sim \sigma_y \gg 1 \) and \( \alpha \gg 1 \) the non-coherent multiple scattering causes substantial broadening of PXR lines compared with natural line width \( \Delta \omega \sim 1/L \). Under the conditions of experiment [5] in which electrons with the energy \( \epsilon = 855 \) MeV moved in a silicon crystal at a small angle to the \( \langle 111 \rangle \) axis, the value \( \alpha \sigma_x \) at \( \theta_r \sim 5 \cdot 10^{-3} \) rad is \( \alpha \sigma_x \sim 40n \).
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