Gear Method for Solving Differential Equations of Gear Systems

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Abstract. It is very difficult to obtain perfect analytical solutions of differential equations of gear system dynamics. The dynamic model which describes the torsional vibration behaviors of gear system has been introduced accurately in this paper. The differential equation of gear system nonlinear dynamics exhibiting combined nonlinearity influence such as time-varying stiffness, tooth backlash and dynamic transmission error (DTE) has been proposed by using a polynomial of degree 7 to fit the nonlinear backlash function and using time-varying stiffness Fourier transform to obtain its harmonic forms with 5 orders for the first time. The theory of GEAR method has been presented. Contrasting with other numerical methods, GEAR method has higher precision and calculation efficiency, especially in solving stiff differential equations. Based on GEAR method, the numerical calculation method for solving differential equations of gear system dynamics has been presented in this paper. Numerical calculation results proved that the numerical solutions by using Gear method is validated by comparison with experimental measurements and can be used to solve all kinds of differential equations, especially for large differential equations.

1. Introduction
The theory of linear oscillation systems has been improved greatly, but for nonlinear oscillation systems, especially for strongly nonlinear systems, it's difficult to obtain perfect analytical solutions. For weakly nonlinear oscillation systems, some asymptotic methods have been developed such as Lindstedt-Poincare method, KBM method, multiple scales method and harmonic balance method. For strongly nonlinear oscillation systems, some improvement methods have been developed by many scholars. S. E. Jones studied free oscillation of Duffing equations by using parameters changing method. M. Benton studied the dynamics stability of single degree of freedom (SOF) gear systems by using phase plane method and numerical simulation. Kahraman and Singh proposed a SDOF oscillation equation with invariant stiffness, backlash and transmission error parameter excitation and obtained jump up and down frequencies, sub-harmonic resonances and chaos phenomena using harmonic balance method in 1991. However, all these asymptotic methods are adapted to their own oscillation systems for both weakly and strongly nonlinear systems, namely they are not validated in general. No method can be used for solving all kinds of differential equations. Only for a few simple nonlinear oscillation equations, accurate analytical solutions can be obtained successfully [1].

It is too difficult to obtain perfect analytical solutions of the nonlinear dynamics differential equations of gear systems. So only some numerical solutions have been obtained for complicated and high order nonlinear differential equations. The general numerical methods are Euler method, fourth order Runge-Kutta method, ADAMS method, state-space method, and so on. But Euler method has so
large errors that it is only used in theoretical analysis. The error of fourth order Runge-Kutta
method is $O(h^4)$. In order to obtain more precise solution, more complicated iterative formula must be
developed. ADAMS method uses Lagrange interpolate polynomials to substitute integrate functions
and integrates the polynomials to obtain differentiate formula. State-space method is based on physical
space and proposes state space model to calculate the implicit analytical solution numerically. All
these methods are not validated for stiff differential equations. Gear method is presented for solving
nonlinear dynamics differential equations in this paper. Compared with other numerical methods, Gear
method can not only obtain higher calculation precision and efficiency, but also can change step size
automatically. GEAR method is validated especially for stiff differential equations and is a general
method of numerical calculation [2,3].

2. Nonlinear differential equation of gear system dynamics

The dynamic model of a gear pair with backlash supported by rigid mounts as shown in figure 1.

Given viscous damping coefficient $c_e$, driving gear base radius $R_p$, driven gear base radius $R_g$, the
equation of the torsional model of the gear pair can be given by

$$
I_p \frac{d^2 \theta_p}{dt^2} + R_p c_e (R_p \frac{d\theta_p}{dt} - R_g \frac{d\theta_g}{dt} - \frac{de}{dt}) + R_g k_e (\hat{\theta}_p R_p - \hat{\theta}_g R_g - \bar{\theta}) = T_p (\hat{\theta})
$$

(1)

$$
I_g \frac{d^2 \theta_g}{dt^2} - R_g c_e (R_p \frac{d\theta_p}{dt} - R_g \frac{d\theta_g}{dt} - \frac{de}{dt}) - R_g k_e (\hat{\theta}_p R_p - \hat{\theta}_g R_g - \bar{\theta}) = -T_g (\hat{\theta})
$$

(2)

where $I_p, I_g$ are the polar mass moment of inertia of driving and driven gears respectively (kg·m²), $\theta_p, \theta_g$ are
torsional displacement of driving and driven gears, $T_p (\hat{\theta})$, $T_g (\hat{\theta})$ are the torque of driving and
-driven gears (N·m), $\bar{\theta}$ is static transmission error of the gear pair (m), $k_e (\hat{\theta})$ is time variant meshing
stiffness of the gear pair (N/m) which can be given by harmonic form

$$
k_e (\hat{\theta}) = k_m + \sum_{j=1}^{5} A_j \cos(j \omega \hat{\theta} + \phi_j)
$$

Figure 1. Dynamic model of a gear pair.

Given the differentiation $\bar{\tau} (\hat{\theta})$ between dynamic and static transmission errors of gear pair, then

$$
\bar{\tau} (\hat{\theta}) = \bar{\tau}_d (\hat{\theta}) - \bar{\tau} (\hat{\theta}) = R_p \hat{\theta}_p (\hat{\theta}) - R_g \hat{\theta}_g (\hat{\theta}) - \bar{\theta} (\hat{\theta})
$$

(3)
According to formula (3), the equations (1) and (2) can be reduced to a single DOF model, namely the differential equation can be given by

\[ m_c \frac{d^2 \bar{\bar{\alpha}}}{dt^2} + c_c \frac{d\bar{\alpha}}{dt} + k_c (\bar{\bar{\alpha}}) \delta(\bar{\bar{\alpha}}) = F_m + \bar{F}_{af} \cos(\bar{\omega}_{af} \bar{\tau} + \phi_f) + m_c \bar{\omega}^2 \bar{\omega}_{ch} \cos(\bar{\omega}_{ch} \bar{\tau} + \phi_h) \]  

(4)

where \( m_c \) is the system equivalent mass, namely \( m_c = \frac{1}{J_e} \left( \frac{I_e R_x^2 + I_e R_y^2}{T_e} \right) \), \( F_m \) is the mean load of the gear pair, namely \( F_m = \frac{T_{pm}}{R_p} = \frac{T_{gm}}{R_g} \), \( \bar{F}_{af}(\bar{\tau}) \) is the external excitation load determined by the variant torque component \( \bar{F}_{af}(\bar{\tau}) \), namely \( \bar{F}_{af}(\bar{\tau}) = \frac{m_c R_p \bar{F}_{af}(\bar{\tau})}{I_p} \), \( \bar{F}_{ab}(\bar{\tau}) \) is internal excitation load

\[ \bar{F}_{ab}(\bar{\tau}) = -m_e \frac{d^2 \bar{\alpha}}{dt^2} \]

The external excitation can be expressed in terms of harmonic function by

\[ \bar{F}_{af}(\bar{\tau}) = \bar{F}_{af} \cos(\bar{\omega}_{af} \bar{\tau} + \phi_f) \]  

(5)

where \( \bar{F}_{af} \), \( \bar{\omega}_{af} \) are amplitude and base frequency of external excitation.

The internal excitation can be expressed in terms of harmonic function by

\[ \bar{F}_{ab}(\bar{\tau}) = m_e \bar{\omega}^2 \bar{\omega}_{ch} \cos(\omega_{ch} \bar{\tau} + \phi_h) \]  

(6)

\[ \bar{F}_{ab}(\bar{\tau}) = m_e \bar{\omega}^2 \bar{\omega}_{ch} \cos(\omega_{ch} \bar{\tau} + \phi_h) \]

(7)

where \( \bar{\omega} \), \( \bar{\omega}_{ch} \) are amplitude and fundamental frequency of internal excitation.

Given \( x = \bar{x} \) \((b \) is the tooth face width), \( \omega_n = \frac{k_m}{m_e} \), \( \zeta = \frac{c_c}{2 m_e} \), \( t = \omega_{ch} \bar{\tau} \), \( F_m = \frac{F_m}{b k_m} \),

\( F_{af} = \bar{F}_{af} b k_m \), \( F_{ab} = \bar{F}_{ab} b \), \( B_j = A_j / m_e \omega_n^2 \), \( \omega_c = \bar{\omega} \), \( \omega_T = \frac{\bar{\omega}_{af}}{\omega_n} \), \( \omega_{ch} = \frac{\bar{\omega}_{ch}}{\omega_n} \), then equation (4) can be expressed by a dimensionless equation

\[ \ddot{x}(t) + 2 \zeta \dot{x}(t) + [1 + \sum_{j=1}^{5} B_j \cos(j \omega_{ch} t + \phi_j)] f(x) = F_m + F_{af} \cos(\omega_{ch} t + \phi_f) + F_{ab} \omega_{ch}^2 \cos(\omega_{ch} t + \phi_h) \]  

(8)

where \( f(x) \) —— dimensionless backlash function \( f(x) = \frac{\delta(\bar{\bar{\alpha}})}{b} = \begin{cases} x(t) - 1 & x(t) > 1 \\ 0 & -1 < x(t) < 1 \\ x(t) + 1 & x(t) < -1 \end{cases} \)

be fitted to a polynomial of degree 7, namely \( f(x) = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 \).

Then, according to equation (8), the equation of motion of a gear system can be given by

\[ \ddot{x}(t) + 2 \zeta \dot{x}(t) + [1 + \sum_{j=1}^{5} B_j \cos(j \omega_{ch} t + \phi_j)] (a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7) = F_m + F_{af} \cos(\omega_{ch} t + \phi_f) + F_{ab} \omega_{ch}^2 \cos(\omega_{ch} t + \phi_h) \]  

(9)

3. Gear method for solving differential equations

For an ordinary differential equation with initial values

\[ \begin{align*}
\dot{x}(t) &= f(t, x) \\
x(t_0) &= x_0
\end{align*} \]

(10)

Euler method can be obtained by differentiating equation (10), namely substituting the forward differentiation form \( x(t+h) \approx x(t) \) into \( \dot{x}(t) \), then

\[ x(t + h) = x(t) + hf(t, x(t)) \]  

(11)
Gear method can improve the precision formula (11) by selecting more accurate numerical integration algorithm. For equation (10), given \( t = t_0 + ih, \ x_i = x(t_i), \ f_i = f(i, x(t_i)) \), then the three data series are expressed by

\[
\begin{align*}
T_m, T_{m-1}, T_{m-2}, \ldots, T_{m-k} \\
x_m, x_{m-1}, x_{m-2}, \ldots, x_{m-k} \\
f_m, f_{m-1}, f_{m-2}, \ldots, f_{m-k}
\end{align*}
\]  

(12)  

(13)  

(14)

Gear method using equation (12) and (13) to form a Lagrange interpolation polynomials \( q_m(t) \) in terms of \( k \) power of \( x(t) \), if \( x(t) \) can be differentiate \( k+1 \) times in success at interval \([t_0,T]\) and the rest term is \( s_{m,k}(t) \), then

\[
x(t) = q_{m,k}(t) + s_{m,k}(t)
\]

\[
= \sum_{i=0}^{k} \left( \prod_{j=0}^{k} \frac{t - t_{m-j}}{t_{m-i} - t_{m-j}} \right) x_{m-i} + \frac{x^{(k+1)}(\xi)}{(k+1)!} \prod_{j=0}^{k} (t - t_{m-j})
\]

(15)

Substituting equation (15) into (10), then timing \( h \) in both sides, \( t = t_m \), then

\[
h \dot{q}_{m,k}(t_m) + h \dot{s}_{m,k}(t_m) = h f[T_m, x(t_m)]
\]

(16)

Neglecting the rest term and substituting \( x_i \) into \( x(t_i) \) \( (i = m-k, m-k+1, \ldots, m, \ldots, m) \), then

\[
\sum_{i=0}^{k} c_{k,i} x_{m-i} = h f_m
\]

(17)

where

\[
\tilde{c}_{k,i} = h \prod_{j=0}^{k} \frac{t - t_{m-j}}{t_{m-i} - t_{m-j}} \delta_{i} = \begin{cases} 
1, & i = 0, \\
\left(-1\right)^{i+1} \frac{\binom{k}{i}}{\binom{k}{i}}, & i > 0.
\end{cases}
\]

For calculation, equation (17) can be rewritten as

\[
x_m + \sum_{i=1}^{k} c_{k,i} x_{m-i} = h g_k f_m
\]

(18)

where

\[
c_{k,i} = \frac{\tilde{c}_{k,i}}{\tilde{c}_{k,0}}, \quad g_k = \frac{1}{\tilde{c}_{k,0}}.
\]

Equation (18) is known as \( k \) step algorithm of GEAR method.

According to formula (15), the truncating errors of GEAR method can be given by

\[
\tilde{R}_{m,k} = h \dot{s}_{m,k}(t_m) = h \dot{s}^{(k+1)}(\xi) \prod_{j=0}^{k} (t - t_{m-j}) \delta_{i} = \frac{x^{(k+1)}(\eta)}{k+1} h^{k+1}, \quad t_{m-k} \leq t \leq t_m
\]

(19)

According to formula (19), the truncating errors of \( k \) step algorithm of GEAR method is \( O(h^{k+1}) \).

The flow chart of GEAR method for program can be obtained in reference [5].

4. Calculation examples

4.1. Nonlinear pendulum

In figure 2, mass \( m \) is attached to a lever with \( l \) length. The corresponding equation of motion of the mass is given by

\[
ml \ddot{\theta} + mg \sin \theta = 0
\]

(20)

namely

\[
\ddot{\theta} + \omega_0^2 \sin \theta = 0
\]

(21)

where

\[
\omega_0^2 = g / l.
\]
Given $l=1\text{m}$, $g=9.8\text{m/s}^2$, $\theta(0)=0$ rad, $\dot{\theta}(0) = 0.3 \text{rad/s}$, the pendulum equation can not simplified as a linear differential equation by $\sin \theta \approx \theta$ for the large motion. But according to the conservation theory of mechanical energy $(1/2)m\dot{\theta}^2 = mgll(1 - \cos \theta_{\text{max}})$, the analytical maximum angle $\theta_{\text{max}} = 0.0958681928\text{rad}$.

![Figure 2. Nonlinear pendulum.](image)

Table 1 shows the results of GEAR method and other numerical calculation methods. It proves that GEAR method can obtain higher efficiency and calculation precision for solving nonlinear differential equations.

| methods                     | step(s) | time(s) | Maximum angle | Error(%)   |
|-----------------------------|---------|---------|----------------|------------|
| Instantaneous linear algorithm | 0.001   | 171.8   | 0.0958681920   | 0.00000095 |
| Taylor method               | 0.001   | 85.5    | 0.0958681920   | 0.00000094 |
| State space method          | 0.001   | 21.4    | 0.0958681922   | 0.00000073 |
| GEAR method                 | 0.001   | 6.2     | 0.0958681928   | 0.00000001 |

4.2. Numerical solution of differential equations of gear system dynamics

Equation (9) is a second order nonlinear differential equation with variant coefficients. The calculation results of equation (9) has been presented in this paper by using GEAR method. The teeth number of gears is 50, module is 3mm, tooth face with is 20mm, the speed of gear is 3000rpm, the mesh period of gear system (the time of one tooth meshing cycle) is 0.0004s.

The calculation results of steady state response of gear system during four meshing periods can be given by figure 3. These results are in accordance with experimental measurements obtained from reference [5].

![Figure 3. The calculation results of differential equations of gear dynamics.](image)
5. Conclusion

The numerical solutions of dynamic differential equation of gear system have been obtained by using GEAR method. Compared with other numerical methods, GEAR method has higher calculation efficiency and precision.

GEAR method is an auto-adaptive algorithm which can select step size automatically and change orders. GEAR method especially can be used in solving high order differential equations and can use highly steady iteration formula. GEAR method takes fewer calculation times in each step for solving implicit equations than most other numerical calculation methods. GEAR method can not only solve ordinary differential equations with initial values, but also can solve stiff differential equations successfully. GEAR numerical method is validated in general in solving all kinds of complicated nonlinear differential equations.

References

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