The Heavy Dirac Monopole

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We present a model for the Dirac magnetic monopole, suitable for the strong coupling regime. The magnetic monopole is static, has charge $g$ and mass $M$, occupying a volume of radius $R \approx O(g^2/M)$. It is shown that inside each $n$-monopole there exist infinite multipoles. It is given an analytical proof of the existence of monopole-antimonopole bound state. Theses bound-states might give additional strong light to light scattering in the $p\bar{p}$ process and in $e^+e^- \rightarrow Z \rightarrow 3\gamma$ process.

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In the present letter we extend the Dirac theory of pointlike monopole, to the domain of high energy physics. We have used the same global Wu-Yang approach, adapted to the strong limit coupling, . The parametric equation of the vector potential $A = (A_1, A_2, A_3)$, shows a singular self-energy with toric configurations inside. The corresponding potential 1-form gives the magnetic field 2-form as an infinite sum of Legendre Polynomials. The dependence of magnetic field $B$, on the additional dimension $\tau$, gives the quantized magnetic flux over a $\pi/2$-cycle. In the low energy limit, the theory simplifies to the Wu-Yang monopole theory.

The existence of the monopole is one of the open questions in particle physics. If monopoles exist, then the elementary magnetic and electric charge ($g, e$) are linked by the relation $ge = 2\pi n$ where $n$ is an integer. If free quarks exist the magnetic charge is increased by a factor three.

A pair of high energy real photons may be produced in a virtual monopole loop emerging from the proton-antiproton collisions. The contribution of pointlike monopole to such diphoton production was calculated in .

Fermilab researchers at the D0 detector, have looked for signs of heavy pointlike monopole among the same data set used to discover the top quark . No evidence for the monopole was found but lower limits on the mass of the monopole were established: 610 GeV, 870 GeV, or 1580 GeV, if the pointlike Dirac monopole has spin 0, 1/2 or 1.

A similar monopole loop production occurs also in the process $e^+e^- \rightarrow Z \rightarrow 3\gamma$ and was tested at the CERN $e^+e^-$ collider LEP .

Fermilab’s paper it was stressed that further theoretical work is desirable, to upgrade the theory of pointlike monopoles. One of the reasons is that the non-observation of a new extra dimension in QED implies that the monopole mass should exceed 100 TeV. Existing or planned particle accelerators will not have enough energy to produce such monopole.

Recently new features on the observable effects of the virtual monopole loop were reported in , considering only pointlike monopole. In that paper there is a call for a new theory of heavy monopole having the standard $SU(2) \times U(1)$ theory as the lower energy limit. This challenging statement is one of the main motivations for our work.

We begin by recalling the fact that the strong coupling regime is incompatible with pointlike structure of the particles, so the monopole must have a definite volume of radius $R \approx O(g^2/M)$ to accommodate the self energy of the strong regime coupling. The size of the monopole being large, compared with the quantum length scale, permits a classical description.

First of all we have to answer the following question: If the monopole occupies a volume of radius $R \approx O(g^2/M)$, what class of compact 3-D space $\Omega$, is capable to enclose the monopole such that $\oint_{\partial \Omega} B = 4\pi g$?

The following statements were keys to discover the appropriate $\Omega$:

1. The magnetic charge density cannot be a smooth distribution. If it were, then the Maxwell equations would be substituted by the Yang-Mills-Higgs equations, to allow finite-energy action monopole solutions.

2. The quantum consistency for QED or $SU(2) \times U(1)$, requires that the magnetic charge cannot be spread out over a length scale larger than the minimum for which the standard model is accurate .

3. The magnetic charges must be compacted to produce the “cones over cones” stratification of the configuration space of the Yang-Mills-Higgs $SU(2) \times U(1)$ theory on $\mathbb{R}^3 \times S^1$ space .

We consider $\Omega$ a regular spatial domain bounded by a surface $\partial \Omega$. By “regular” we mean that $\Omega$ may have an infinite number of isolated singularities.

The exterior space is the complement in the Euclidean space $\mathcal{E} = \mathbb{R}^3 - \Omega$. The spaces $\mathcal{E}$ and $\Omega$ have a common boundary, homeomorphic to a Riemannian sphere $S^2$. The region $\Omega$, in contrast to $\mathcal{E}$, must be as small as possible to describe the strong coupling regime. We have found that in the strong coupling regime, the geometry of $\Omega$ must be sub-Riemannian to satisfy the above requirements. Euclidean geometry holds locally in $\mathcal{E}$, where the
monopole can be considered pointlike, in the low energy limit.

We have chosen the geometry of the compact 3-D Heisenberg group \( \mathcal{H}(3) \) as the geometry of \( \Omega \). We have done so for several reasons. The non-compact 3-D Heisenberg group \( H(3) \), (also known as Weyl group \( [11] \)) is the only connected nilpotent non-Abelian Lie group, homeomorphic (as manifold) to \( \mathbb{R}^3 \), \( [2] \). It is represented by upper triangular matrices \( a = (x_1, x_2, x_3) \) with one in each diagonal entry. The group product is defined as \( a \cdot a' = (x_1 + x_1', x_1 + x_2', x_1 + x_3' + x_2 x_1' - x_1 x_2') \).

The discrete subgroup \( D \), is generated by the matrices \( D_n = (1, 0, 0), (0, 1, 0), (0, 0, 1/m) \), where \( m = 1, 2, \ldots \). The compact group is the coset \( \mathcal{H}(3) = H(3)/D \). As a compact Lie group, \( \mathcal{H}(3) \), has an invariant Haar measure which provides a small finite volume. \( [3] \). Its Lie algebra satisfies the relations

\[
[X_1, X_2] = X_3; \quad [X_3, X_1] = 0; \quad [X_3, X_2] = 0 \quad (1)
\]

Therefore due to the nonintegrability, there exist a vector field in this Lie algebra with undefined line integral. \( \oint A \cdot dx \neq 0 \), around an unshrinkable loop \( [14] \).

The 3-D Heisenberg ball has a complicated boundary \( [15,16] \), \( \partial \Omega = \partial \Omega_0 \cup \partial \Omega_0^+ \cup \partial \Omega_0^- \). The boundary extends to the interior with an infinite number of smooth small cones with two covers, \( \partial \Omega_0^\pm, m = 1, 2, \ldots \). These cones are the charges of the \( SU(2) \) theory. In our model, each small cone may receive a magnetic charge \( q = \pm 2\pi n/e \) along its surface. The great exterior boundary \( \partial \Omega_0 \), expands to the interior and terminates at two conic singularities (the north and south pole). This boundary is topologically equivalent to a double covering of a Riemannian sphere \( S^2 \).

In the Heisenberg ball the distance scales as \( d_H(S(X_1), S(X_2)) = \lambda d_3(x_1, x_2) \), the volume as \( \lambda^4 \), \( [17] \). The relation between the Heisenberg distance and the Euclidean distance is \( d_3 = \sqrt{d_H^2} \). Therefore the maximal radius of the monopole is \( \varepsilon = \sqrt{q}/M^{1/2} \).

The shape of the boundary \( \partial \Omega \), is determined by the set of end points of the geodesics of \( \Omega \). The geodesics of the Heisenberg ball are given by the minimizers of the Heisenberg distance in the interval \( [0, \tau] \):

\[
d_{\mathcal{H}(3)} = \int_0^\tau \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} \, dt \quad (2a)
\]

\[
\dot{x}_3 + x_1 \dot{x}_2 - x_2 \dot{x}_1 = 0 \quad (2b)
\]

The set of end points of the geodesics of the system \( [2] \), is given by the parametric equations, \( [18] \):

\[
x_1 = \varepsilon \left[ \frac{\sin(\theta + \phi) - \sin \phi}{\theta} \right] \quad (3a)
\]

\[
x_2 = \varepsilon \left[ \frac{- \cos \phi + \cos(\theta + \phi)}{\theta} \right] \quad (3b)
\]

\[
x_3 = \varepsilon^2 \left[ \frac{\theta - \sin \theta}{\theta^2} \right] \quad (3c)
\]

\(-2(k + 1)\pi \leq \theta \leq 2(k + 1)\pi; \quad k = 0, 1, 2, \ldots \quad 0 \leq \phi \leq 2\pi\).

In fig.1 is shown parametric plot of the north-hemisphere \( \partial \Omega^+ \), with three magnetic charges \( 0 \leq \theta \leq 8\pi \). In fig.2 is shown in detail one magnetic charge, \( 2\pi \leq \theta \leq 4\pi \).

Let us now consider the map \( A : \partial \Omega^+ \to \partial S \subset \mathcal{M} \), where \( \mathcal{M} \) is the space of all potential vectors modulo gauge transformation. In order to work with the three components of \( A = (A_1, A_2, A_3) \) we impose the gauge constraint \( \nabla \cdot A = 0 \), and \( V = 0 \) for the scalar potential.

The energetic configuration space \( S_3 \), is compact and has also a complicated boundary \( \partial S = \partial S_0 \cup T_m \). The boundary \( \partial S_0 \) has a pear-like shape without north-pole, with a singularity at the south-pole. This configuration is the self-energy of the monopole. The other configurations \( T_m \), inside \( \partial S_0 \), are “tori inside tori” in harmonic correspondence with the “cones over cones” structures inside \( \partial \Omega^+_m \).

As in the Hopf map between Riemannian spheres, we identify the great circles in \( \partial \Omega^+_m \), with points in \( \partial S_0 \), \( [19] \). In the present case we identify the equator of \( \partial \Omega^+_m \) (\( \theta = 0 \)) with the polar circle of \( \partial S_0 \). The singularity at the north of \( \partial \Omega^+_m \) (\( \theta = 2\pi \)), with the singularity at the south of \( \partial S_0 \). In order to obtain such correspondence the components of the potential vector \( A \) must satisfy the relation

\[
A_1^2 + A_2^2 + A_3^2 = M(x_1^2(2\theta) + x_2^2(2\theta)) \quad (4)
\]

where we have taken into account that one turn in the meridional plane of \( \partial \Omega^+_m \), corresponds to two turns in the meridional plane of \( \partial S_0 \). Using equations \( (2a) \) and \( (2b) \) in \( (4) \), we obtain the parametric equations of the potentials for \( \rho_0 < r \leq \varepsilon \):

\[
A_1 = \frac{g}{\sqrt{2}} \sqrt{1 - \cos 2\theta} \sin \theta \cos \phi \quad (5a)
\]

\[
A_2 = \frac{g}{\sqrt{2}} \sqrt{1 - \cos 2\theta} \sin \theta \sin \phi \quad (5b)
\]

\[
A_3 = \frac{g}{\sqrt{2}} \sqrt{1 - \cos 2\theta} \cos \theta \quad (5c)
\]

\( -2(k + 1)\pi \leq \theta \leq (k + 1)\pi; \quad k = 0, 1, 2, \ldots \).

Equations \( (3) \) are the parametric equations of the energetic configuration corresponding to the charge distribution on the Heisenberg ball \( \Omega \). The positive branch of these solutions is associated with the monopole, while the negative branch is associated with the anti-monopole, like
in the electron-positron theory \[20,21\]. In fig. 3 is shown the parametric plot of the self-energy, \(0 \leq \theta \leq \pi\). In fig.4 is show the configuration corresponding to three magnetic charges, \(0 \leq \theta \leq 4\pi\). The two branched solutions are connected through the branch cut in the complex plane beginning at the threshold for pair productions. Bellow and near this point, there exist bound states of the scattering matrix connecting \(-(1 + k)\pi\) solutions to \((1 + k)\pi\) solutions. In fig.5 it is shown the parametric plot of a bound-pair between \(-\pi \leq \theta \leq 2\pi\).

To obtain the magnetic field of such configurations we must write the potentials 1-forms, for \(r_0 < r < \varepsilon\), where \(r_0\) is the length beyond of which we must use the Bogomolny equations of the SU(2) × U(1) Yang-Mills-Higgs theory.

The Jacobian of (3) gives the Haar measure of \(S\): \(\omega(\theta) = 2\pi r^2 (\sin^4 \theta/\theta^4) d\theta \equiv 2\pi r^2 f(\theta) d\theta\). Let us take \(A_r = A_\theta = 0\), and

\[
A_\phi = \frac{g}{\sqrt{2}} \frac{1}{r f(\theta)} \sqrt{1 - \cos 2\theta}
\]

(6)

It is easy to verify that \(\lim_{\theta \to 0} A_\phi = g/r\) and \(\lim_{\theta \to \pi} A_\phi = \infty\). With the above choice, the phase factor of the gauge theory becomes well defined around the singularity. Indeed, the integral of the vector potential around the loop defined by the line element \(d\mathbf{x} = 2\pi r f(\theta) d\theta d\phi\), gives

\[
\oint \mathbf{A} \cdot d\mathbf{x} = 2\pi \frac{g}{\sqrt{2}} \int_0^\pi \sqrt{1 - \cos 2\theta} d\theta = 4\pi g
\]

(7)

Thus the potential 1-form must be

\[
\mathbf{A}^+ = \frac{2g}{\sqrt{2r}} (\sqrt{1 - \cos 2\theta}) d\phi; \quad 0 \leq \theta < (1 + k)\pi
\]

(8)

corresponding to monopole solutions.

The magnetic field of the monopole is given by

\[
\mathbf{B}^+ = d\mathbf{A}^+ = \frac{4g}{\sqrt{2}} \left( \frac{\sin \theta \cos \theta}{\sqrt{1 - \cos 2\theta}} \right) d\theta \wedge d\phi
\]

(9)

or

\[
\mathbf{B}^+ = 4g (\sin \theta \cos \theta) \left( \sum_{k=0}^{\infty} P_k(\cos 2\theta) \right) d\theta \wedge d\phi
\]

(10)

where \(P_k(\cos 2\theta)\) is the Legendre polynomial. In fig.6 is shown the parametric plot of the magnetic field in the "instant" \(\tau = 1\), considering the first two terms of the expansion \(k = 0, k = 1\).

We remark that the modulator factor is the field of the Wu-Yang monopole, the field of the self-energy.

The integration of (10), over a \(\pi/2\)-cycle, gives the flux associated to the magnetic field. It is easy to verify, that only the self-energy is responsible to the flux. Using the orthogonality properties of the Legendre Polynomials, we obtain

\[
\int_0^{\pi/2} (\sin \theta \cos \theta) \left( \sum_{k=0}^{\infty} P_k(\cos 2\theta(\tau)) \right) d\theta = \frac{1}{2}
\]

(11)

So the flux of the \(n\)-monopole is:

\[
\Phi^+ = \oint_{\partial \Omega} \mathbf{B}^+ = 4\pi g = \frac{8\pi^2 n}{e}; \quad n = 1, 2, \ldots
\]

(12)

In conclusion, we have shown that the extension of the pointlike monopole to strong coupling regime is the \(\mathcal{H}(3) \times SU(2) \times U(1)\) gauge theory, where \(\mathcal{H}(3)\) is the compact 3-D Heisenberg group. One of the features of our theory, not present in the pointlike theory, is the bound state solutions with toroidal distribution of energy around a very thin tube of quantized flux, as shown in fig.5. The bound pair is confined. Indeed, due to the flux tube linking the poles, any tentative to separate them leads to the energy rise, because by construction this configuration is the minimum energy. Thus if the virtual monopole loop exists, then a Bohm-Aharonov effect might change the momentum of the protons (antiprotons) in the transverse direction by just a quantum of magnetic flux, like in the Bohm-Aharonov experiment for electrons [22].

Finally we remark that the Haar measure of the \(\mathcal{H}(3)\) group reveals, structures inside structures, measurable depending on the energetic scale used to measure, [23].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{\(\partial \Omega^+\) with three charges. \(-0 \leq \theta \leq 8\pi; \quad 0 \leq \phi \leq 1.3\pi\)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The magnetic charge. \(2\pi \leq \theta \leq 4\pi; \quad 0 \leq \phi \leq 2\pi\)}
\end{figure}
This paper is dedicated to the memory of Professor Guido Beck and Professor Carlos Marcio do Amaral.