Electron dynamics in the vicinity of Lower Hybrid antennas

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Abstract. Heating and current drive in tokamaks with Lower Hybrid (LH) waves appears to generate hot spots magnetically connected to LH antennas. Correlated to these spurious patterns of heat deposition, fast electron populations are also reported experimentally. A Vlasov–Poisson (1D, 1V) model is used to address this problem. External drive by waves with appropriate spectrum to describe the physics induced by LH antennas is considered. Fast electron tails, symmetric in velocity, are shown to build-up. In the range considered in this work, the energy content of these tails is found to be proportional to the energy of the excitation LH waves.

1. Introduction
In tokamaks using Lower Hybrid (LH) waves for current drive, hot spots can be observed on components magnetically connected to the LH antennas [1, 2]. These hot spots are believed to be caused by fast electrons accelerated in front of the LH antennas. Indeed, fast electrons have been detected in the Scrape-off-layer (SOL) on magnetic field lines connected to LH antennas by Langmuir probes and soft X-ray measurements on TdeV [3], and by RFA measurements on Tore-Supra [4] and EAST [5]. Modeling indicates however that ions could be accelerated in electric fields that result from charge separation during the electron acceleration process [6], yet without experimental evidence to support this result.

This work presents the first steps towards modeling the interaction of electrons with the $k$-spectrum of the electric field of an LH antenna. The ultimate goal is to predict the heat flux generated by this process and to find mechanisms for mitigating the effect.

LH antennas are composed of toroidal rows of several tens of wave-guides, stacked in the poloidal direction. If one assumes that the electron density in front of the antenna is uniform in the poloidal direction, each row can be considered identical. The hot spots appear to be produced radially within typically one centimeter [4]. Experimental results are consistent with ballistic transport of accelerated electrons along magnetic field lines until they impinge onto plasma facing components (PFCs) [2, 7, 8]. Therefore, we consider here only the dynamics of electrons along a magnetic field line connected to the LH antenna.

This region of interest can be split into three parts: (i) the front of LH antennas where the edge plasma absorbs a small part – between 1% and 10% [3] – of the power, (ii) magnetic field lines where heat flux propagates from LH antennas to PFCs and (iii) strike points where magnetic fields lines intersect PFCs and heat flux is deposited. In this first step, we address the excitation part, which allows one to consider periodic boundary conditions. These are easier to handle.
numerically. Furthermore, we reduce the simulation domain to two wave guides with opposite phase and given frequency $\omega_{\text{ext}}$. The important part of the LH-spectrum in wave vector, figure 1, is then governed by the sharp transition from one wave guide to another, leading to a broadband spectrum at high refraction index (up to typically 100), figure 1. Conversely, the useful part of the spectrum is governed by the synergy between the series of wave guides which yields a refraction index of order 2. While the former part of the spectrum is taken into account by the model, the latter is not. This is consistent with the fact that, in the SOL, particle resonance with the LH waves will only occur in the high refraction index region, figure 1. With the given choice of LH set-up, the simulation domain is rather homogeneous and strong charge separation governed by the expansion of the fast electron cloud will not occur. In agreement with results from the linearized system, it is therefore sufficient to consider frozen ions uniformly distributed at this stage.

The paper is organized as follows, the model and simulation tool are introduced in Section 2, benchmark with respect to the standard Landau damping in Section 3. The excitation by a standing wave is addressed in Section 4 and results for a broad band, wave-guide shaped electric field are presented in Section 5. Discussion and conclusion close the paper, Section 6.

2. Modelling LH wave coupling to SOL plasma

Given the operation range for LH antennas in the SOL plasma, the ratio of antenna size $L$ to collisional mean free path in the parallel direction, $\nu^{*}_{\text{LH}}$, ranges from $10^{-1}$ to $10^{-3}$ so that a kinetic model is required to describe the plasma. In a first step, collisional damping of kinetic filamentation is not taken into account, $\nu^{*}_{\text{LH}} \to 0$. One also expects that the electric field varies on scales smaller or comparable to the Debye length so that the Poisson equation is required to determine the electrostatic potential generated by the plasma. Finally, on short time scales, consistent with the high frequency of LH waves, we assume the asymptotic limit $\sqrt{m_e/m_i} \to 0$ where $m_e$ and $m_i$ are the electron and ion mass respectively. For this frozen ion background we assume a uniform ion density, which is used to normalize electron and ion densities. Let $z$ be the parallel position normalized by the Debye length $\lambda_D$, $t$ time normalized by $1/\omega_p$, where $\omega_p$ is the plasma frequency, and $v$ the electron velocity normalized by the electron thermal velocity $v_{\text{th}} = \sqrt{T_0/m_e}$, with $T_0$ the initial electron thermal energy. The Vlasov–Poisson system for the normalized electric potential, normalized by $T_0/e$, $e$ being the elementary charge, and normalized electron distribution function $f$ is then:

$$\partial_t f + v \partial_z f + (\partial_z \phi_p + \partial_z \phi_{\text{ext}}) \partial_v f = 0,$$

(1a)

$$\partial^2_z \phi_p = \int_{-\infty}^{+\infty} f \, dv - 1,$$

(1b)
where $\phi_{\text{ext}}$ is the electric potential generated by the LH antenna. Note that the density only appears via the definition of the normalizing scale $\lambda_D$ and consequently normalizing frequency $\omega_p$. We consider in this paper a periodic system in $z$ of length $L$ and global quasi-neutrality, namely that the electron density averaged in $z$ is equal to 1. To analyze the effect of LH wave coupling to the SOL electron population, it is interesting to determine the evolution of the electron current $j = -n u$, where $n$ and $u$ are respectively the density and mean electron velocity, and total energy $\mathcal{E} = n (T + u^2) / 2$, where $T$ is the electron thermal energy normalized by $T_0$:

$$\partial_t j - 2\partial_z \mathcal{E} = n E, \quad \partial_t \mathcal{E} + \partial_z Q = jE. \quad (2)$$

In these balance equations, $E$ is the total electric field, sum of the self-consistent Poisson $E_p$ and external $E_{\text{ext}}$ contributions, and $Q = \int dv (v^3/2)f$ stands for the energy flux. It readily appears that the spatial averages of both $j$ and $\mathcal{E}$, $\langle j \rangle_L$ and $\langle \mathcal{E} \rangle_L$, evolve due to the coupling of the electric field with density and current, respectively:

$$\partial_t \langle j \rangle_L = \langle nE \rangle_L, \quad \partial_t \langle \mathcal{E} \rangle_L = \langle jE \rangle_L. \quad (3)$$

The balance equation for $\langle n \rangle_L$ is trivially satisfied.

The non-linear dynamics of the driven Vlasov–Poisson system eq. (1) is addressed numerically with the CALVA code (Code for Acceleration of eLectrons in Vicinity of Antennas), which originates from the code developed in [9]. The code adopts the Particle-In-Cell (PIC) method to evolve the distribution function while a second order finite difference scheme is used to solve the Poisson equation. The distribution function $f$ is described by a set of $M$ macro-particles [10] that are coupled via the Poisson equation:

$$f(z,v,t) = \sum_{m=0}^{M-1} M_m S_z(z - z_m(t)) S_v(v - v_m(t)), \quad (4)$$

$M_m$ is the assigned (constant) weight of the $m$-th macro-electron, located in phase space by $S_z$ the shape function in position and $S_v$ the shape function in velocity. The Vlasov equation (1a) can then be replaced by the motion equations of the macro-electrons:

$$\text{d}_t z_m = v_m(t), \quad \text{d}_t v_m = -E_m(t), \quad (5)$$

$E_m(t)$ is the Lagrangian total electric field, i.e. the electric field weighted by the finite-size macro-electron:

$$E_m(t) = \int_0^L (-\partial_z \phi(z,t)) S_z(z - z_m(t)) \, dz, \quad \phi = \phi_p + \phi_{\text{ext}}. \quad (6)$$

3. **Simulation of a damped Langmuir wave**

In the absence of any excitation source, i.e. $\phi_{\text{ext}} = 0$, eqs. (1a,1b) reduce to the well-known 2D Vlasov-Poisson system for Langmuir waves. These waves are known to exhibit collisionless Landau damping [11, 12]. Let us linearize the system eqs. (1a,1b) at vanishing exciting field ($\phi_{\text{ext}} = 0$) and close to a Maxwellian equilibrium. The equilibrium electric field is null. When expressed in Fourier space ($f, \phi = \sum_k \int d\omega \exp[i(kz - \omega t)]$), linear solutions are such that angular frequency $\omega$ and wave vector $k$ satisfy the following dispersion relation:

$$D(k,\omega) = k^2 + 1 + \zeta Z(\zeta) = 0 \quad (7)$$
where $\zeta = \omega/(|k| \sqrt{2})$ and $Z(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\theta^2) \, d\theta$ is the plasma dispersion function [13]. In the case of a periodic system, $k$ is real. Conversely, $\omega = \omega_\perp + i \gamma$ has a non-vanishing imaginary part $\gamma$, the Landau damping rate, which accounts for the resonant interaction between waves and particles. It can be computed numerically. This mechanism is suspected to play a dominant role in electron heating by LH waves. In addition, recovering the analytical values of $\gamma$ constitutes a stringent test for numerical codes. As detailed below, this test is used to benchmark our PIC code.

At $t = 0$, the distribution function is prepared so that the density field is perturbed by a single mode $k$, $n(z,t = 0) = 1 + \varepsilon \cos(kz)$, leading to the following initial electric potential: $\phi_p(z,t = 0) = -\varepsilon/k^2 \cos(kz)$. The small magnitude of the perturbation $\varepsilon \ll 1$ ensures that the system remains in the linear regime. The initial perturbation evolves towards the following damped stationary wave:

$$\phi_p(z,t) = -\frac{\varepsilon}{k^2} \exp\left(\frac{\gamma t}{k^2}\right) \cos(kz) \cos(\omega_t t).$$

Here, $(k, \pm \omega_\perp + i \gamma)$ is the solution of the dispersion relation eq. (7) (both signs of $\omega_\perp$ are solutions, as a result of the symmetry property of the dispersion function, $Z(\zeta^*) = Z^*(-\zeta)$) with the smallest imaginary part, other solutions with larger values of $|\gamma|$ being damped away on very short time scales.

In figure 2, results of a simulation performed with the CALVA code are compared with theoretical predictions. About $M \approx 26 \times 10^6$ macro-electrons were used, with $k = 0.4$ (the domain is $L = 2\pi/k \approx 15.7$ long) and $\varepsilon = 0.005$. $k = 0.4$ corresponds to the high-$k$ / low phase velocity component of LH antennas. Both $\omega_\perp$ and $\gamma$ are successfully recovered with relative errors of $0.08\%$ and $0.3\%$, respectively. The exponential decay does not reach the floating number precision $\approx 10^{-16}$ in the simulation run. Indeed, it is lower bounded by the intrinsic statistical noise of the PIC method, which decreases as $M^{-1/2}$. In the present case, the exponential decay evolves into a stochastic behavior for $t \gtrsim 50$.

![Figure 2](image)

**Figure 2.** Blue lines are from a CALVA simulation. (a) Fourier transform of the electrical potential $\phi_p(z = L/2, t)$ (time window: $0 \leq t \leq 60$). Red line: theoretical prediction for $\omega_\perp$. (b) Time evolution of $\phi_p(z = L/2, t)$. Dashed red: theoretical exponential decay.

### 4. Excitation by a monochromatic plane wave

When forced by an external electric potential $\phi_{\text{ext}}$, the system eqs. (1a,1b) exhibits a more complex behaviour: Landau damping now competes with the excitation source. Before addressing the case of a broad excitation spectrum, reminiscent of that of the LH antenna, let us consider the simpler case of an excitation by a monochromatic plane wave $\phi_{\text{ext}}(z,t) = \phi_{\text{ext},k\omega} \cos(\omega_{\text{ext}} t - k_{\text{ext}} z)$.

Provided the source amplitude remains small enough $\phi_{\text{ext},k\omega} \ll 1$, the plasma response can be studied within the linear framework. In this case, one can compute the linear gain $G(k, \omega)$,
defined as the ratio between the amplitude of the response $\phi_{p,k\omega}$ and the amplitude of the excitation $\phi_{ext,k\omega}$. If the initial distribution function is Maxwellian, it reads as follows:

$$G(k, \omega) = \frac{\phi_{p,k\omega}}{\phi_{ext,k\omega}} = \frac{1 + \zeta Z(\zeta)}{k^2 + 1 + \zeta Z(\zeta)}$$

(9)

Here, $k = k_{ext}$ and $\omega = \omega_{ext}$ (we recall that $\zeta = \omega/(|k|\sqrt{2})$ are the wave vector and angular frequency of the excitation monochromatic wave. The gain $G$, which is a complex function, is displayed in fig. 3. Both 2D $(k, \omega)$ contours and 1D curves are plotted for its magnitude and argument. Green lines correspond to Langmuir wave solutions of eq. (7), namely $(k, \omega_r(k))$. The dashed black line corresponds to the maximum of $|G|$ with respect to $\omega$.

Complexity of the gain function $G(k, \omega)$ stems directly from the balance between real and imaginary parts of the plasma dispersion function $Z(\zeta)$. One can however draw a few properties. In the fluid limit $\omega/k \gg v_{th}$ (with $v_{th} = 1$ the normalized electron thermal velocity), the imaginary part of $Z(\zeta)$ tends to be small, so that the denominator of $G(k, \omega)$ will be small for $\omega$ solution of the Landau dispersion relation. For small $k$ this yields the Bohm-Gross relation, $\omega_{BG}^2 = 1 + 3k^2$. This property locates the maximum amplitude of $G(k, \omega)$. In this limit, one can also readily show that the plasma response, i.e. the electrostatic potential $\phi_p$, is in phase with the external potential $\phi_{ext}$, while $|G(k, \omega)|$ is proportional to $1/\omega^2$. Conversely, in the adiabatic electron limit $\omega/k \ll v_{th}$, one finds that $\phi_p$ is opposite to $\phi_{ext}$ and that $|G(k, \omega)| \approx 1/(1 + k^2)$, of order unity for $k \to 0$. The electric field driven by the electrons then cancels out the external electric field.

For a multi-species plasma, the expression of the gain in the linear limit is comparable $G(k, \omega) = -\sum_a g_a(\zeta_a)/(k^2 + \sum_a g_a(\zeta_a))$, with $g_a(\zeta_a) = c_aZ_a^2(1 + \zeta_aZ(\zeta_a))$. Here, the following definitions have been introduced: $c_a = n_a/n_0$, $Z_a = q_a/e$ and $\zeta_a = \sqrt{m_a/m_e}\ zeta$, with $n_a$, $q_a$ and $m_a$ the density, charge and mass of particles of species $a$ respectively. In this expression the thermal energies are assumed the same, waving-off this assumption is straightforward. For $\zeta \approx 1$, when electrons are resonant, one thus obtains $\zeta_a \gg 1$. The ions are therefore in the fluid limit with negligible response. This linear analysis result justifies considering the frozen ion limit to address the plasma response to LH drive on short time scales. It also indicates the range of wave vectors required to obtain a large ion response. Indeed with LH, as discussed in the following, $\omega$ is fixed while the $k$-spectrum is governed by the wave-guide shaping. Hence if for the main $k_{LH}$ of the LH launcher one has $\omega/k_{LH}$ ranging between 1 and 10, then ion resonance effects will be driven by wave vectors $k$ of order $k_{LH}\sqrt{m_i/m_e}$, typically 60 times larger, and consequently with a smaller amplitude, at least by a factor $\sqrt{m_e/m_i}$ for a square shaped, hence “perfect”, wave guide. Regarding direct acceleration of particles by LH near electric field, it seems therefore reasonable to only consider the electrons.

The spectrum of LH antennas is quasi-monochromatic in frequency and broad in wave-vector. It would therefore make sense to perform a scan in $k$ at fixed excitation frequency $\omega_{ext}$. However, changing the $k$ value while keeping constant the number of grid points per period would require to change the domain size $L = 2\pi/k$ of the simulation. At constant accuracy, the number of macro-electrons should then be modified accordingly. Indeed, in PIC simulations, statistical errors scale like the inverse square-root of the number of macro-electrons $M_{cell}$ per phase–space cell $\Delta v \Delta z$. Given $\Delta v$ and $\Delta z$, the number of macro-electrons $M$ should then scale like $M \propto L$. Scanning the excitation pulsation $\omega_{ext}$ instead of the wave vector constitutes an easier (especially less costly in terms of numerical resources) way to benchmark CALVA against linear theory. The scan presented below is performed at $k_{ext} = 0.4$. At this wave vector, the linear gain $|G|$ is maximum at angular frequency $\omega_G(k_{ext}) \approx 1.285$. The plasma response $\phi_p$ exhibits a non-monochromatic spectrum, as exemplified in Fig. 4 for
\begin{align*}
\omega_{\text{ext}} & \approx 0.964, \text{ which corresponds to } \omega_{\text{ext}} / \omega (k_{\text{ext}} = 0.75). \text{ The Fourier spectrum peak is close to the excitation frequency, but other frequencies are also present. Also, the phase shift (between } \phi_p \text{ and } \phi_{\text{ext}} \text{) is } \approx -0.63\pi \text{ while } \text{Arg}[G(k_{\text{ext}}, \omega_{\text{ext}})] \approx -0.86\pi \text{ is predicted by linear theory. The relative error reaches about } 25\%. \text{ The discrepancy is of the same order of magnitude for the amplitude of the gain (} \sim 1.2 \text{ in the simulation, as compared to } \sim 1.6 \text{ from theory).}
\end{align*}

\begin{align*}
\text{Figure 3.} & \text{ Modulus and argument of the linear gain, eq. (9). } \omega \text{ and } k \text{ are the pulsation and wave vector of the excitation field } \phi_{\text{ext}}.
\end{align*}

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\end{align*}

\begin{align*}
\text{Figure 4.} & \text{ (a) Time evolution at } z = L/2 \text{ of excitation } \phi_{\text{ext}} (\phi_{\text{ext}}(L/2,t) = 0.005) \text{ and response } \phi_p, \text{ and of the magnitude of the Fourier component of } \phi_p \text{ at } (k_{\text{ext}} = 0.4, \omega_{\text{ext}} = 0.964). \text{ (b) Fourier spectrum of } \phi_p \text{ at } z = L/2. \text{ Red line: } \omega_{\text{ext}}. \text{ Dashed black: angular frequency at the maximum of } |G|.
\end{align*}

However, it is worth mentioning that the matching between simulations and linear theory predictions is improved when the excitation frequency \( \omega_{\text{ext}} \) departs from the resonant frequency, \textit{i.e.} from the neighborhood of the maximum of \( |G| \). This characteristic feature is exemplified in Fig. 5, where the gain is scanned with respect to \( \omega_{\text{ext}} \), and compared to the linear prediction, eq. (9). Although the magnitude and phase of the gain depart from the linear analytical theory,
the resonant frequency matches the analytical one.

![Figure 5](image_url)

**Figure 5.** Open circles: gain computed from CALVA simulations for various excitation frequencies \( \omega_{\text{ext}} \) (fixed wave vector \( k_{\text{ext}} = 0.4 \)). Plain curve: analytical linear prediction, eq. (9).

5. **Excitation by a shaped electric field**

The electric field in front of LH antennas is expected to exhibit a broad Fourier spectrum, as exemplified in Fig. 1. In the present work, we consider a simplified expression of the electric field. It captures those features which are essential to explain the acceleration of electrons up to suprathermal velocities.

In this framework, one considers an LH antenna made of 2 wave-guides only, of width \( d = 50 \) (in Debye units), separated by a zero-width septum and launching the same and uniform power. The width of the periodic simulation domain is thus \( L = 2d \). For the sake of simplicity, the phase shift of the electric field from one wave-guide to the other is set at \( \pi \). This allows one considering a smaller antenna, made of two wave-guides, hence requiring less numerical resources. In this case, the electric field has the following expression:

\[
E_{\text{ext}}(z,t) = E_0 \cos (\omega_{\text{ext}} t - \varphi(z)), \quad \varphi(z) = \text{int} \left( \frac{2z}{L} \right) \pi, \tag{10}
\]

where \( \text{int}(x) \) stands for the integer part of \( x \). \( E_{\text{ext}} \) can also be expressed as superimposed monochromatic waves \( E_{\text{ext}}(z,t) = E_0 \sum_{j \in \mathbb{Z}} \frac{4}{k_jL} \cos (\omega_{\text{ext}} t - k_j z + \pi/2) \), with \( k_j = (1 + 2j)2\pi/L \) the wave vector of wave \( j \). One then finds a series of resonances at resonant velocities \( \omega_{\text{ext}}/k_j = v_j \) and resonant width \( \Delta v_j = 2\omega_{b,j}/k_j \), where \( \omega_{b,j}^2 = 4E_j/L \) and \( E_j = E_0 |1 + G(\omega_{\text{ext}}, k_j)| \) takes into account the plasma response via the gain function for each wave-vector. At each resonance a plateau is expected to develop on the width \( \Delta v_j \) with characteristic time \( 1/\omega_{b,j} \), the trapping time in the island.

With time, electrons can experience large velocity excursions from one resonance location to another, leading to the formation of plateaus of the distribution function in the velocity space. This overlapping effect can lead to merging of the plateaus. As exemplified in Fig. 6(a), the initial Maxwellian then develops heavy tails. There, the plateau extends up to 3 times the thermal speed at \( t \approx 200 \). This generic mechanism is suspected to be responsible for the acceleration of electrons up to several times the thermal velocity in front of LH antennas. The power transferred from the wave to the electrons, and given by the product of the electron current and the electric field \( \langle jE \rangle_L \) eq. (3), scales like the energy injected by the exciting field. This proportionality is evident in Fig. 6(b), where each symbol corresponds to a different simulation at a prescribed magnitude of \( E_{\text{ext}} \).
Figure 6. (a) Spatial average of the distribution function at $t = 0$ (dashed blue) and $t \approx 200$ (bold red) in the case $E_0 = 0.3$. Black horizontal lines give the location and width of resonances (see text). (b) Exchanged power between waves and electrons as a function of the injected energy.

6. Discussion and Conclusion

The present analysis of the electron response to the near electric field of Lower Hybrid launchers in the SOL plasma indicates that tails of accelerated electrons can be formed, ranging from one to three times the thermal velocity. In agreement with work published on this issue, these plateaus in velocity space are observed to be driven by an overlap effect of different resonances. The wave vectors and electric field amplitude associated to the latter are governed by the square-shape of the electric field pattern, hence, as expected, by the geometry of the LH launcher. We show here that the kinetic energy of the accelerated electron population is proportional to the LH power. On the short times of the present simulations, the frozen ion assumption is justified. The fast electron population then tends to be trapped by the ions. Acceleration remains possible given the periodic conditions. When addressing the physics that could allow these fast electrons to escape from the launcher vicinity and create hot spots, more demanding self consistent electron-ion simulations are required, including the near antenna region along the field lines. Work along this line is in progress.

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