Laplace Transformations of Submanifolds

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Dedicated to

Pi-mei and Godelieve,

with our sincere thanks for their devotion to our families, their permanent support to our work and their hospitality to our visiting colleagues and their families.
PREFACE

After corresponding for some time on various problems on the differential geometry of submanifolds, the authors started their joint research on submanifold theory at the Michigan State University (MSU) at East Lansing during the academic year 1975-1976, when the second author, as a Fulbright-Hays Grantees, Nato Research Fellow and Researcher of the Belgian National Science Foundation, did post-doctoral work there, under the guidance of the first author, who then was a young Professor at MSU.

We are happy to say that, ever since, we stayed in close scientific contact, and that we continuously had opportunities over the past 20 years to effectively do joint research at East Lansing and at Leuven-Brussel, thanks to the support of mainly MSU, the Katholieke Universiteiten of Leuven and Brussel (KUL and KUB) - most in particular through their Research Councils -, the NSF of Belgium and the Fulbright Programm of the CEE between the USA and Belgium. Moreover, we are very glad that over the years our scientific contacts naturally extended to many of our coworkers, of whom, in particular, we would like to mention here the first author’s colleagues at MSU and both our friends D.E. Blair and G. Ludden, and the second author’s colleagues and former students at KUL and KUB, of whom we specially would like to mention at this stage both our friends P. Verheyen, F. Dillen and L. Vrancken. Through the frequent visits of the first author to Leuven-Brussel, the second author would like to emphasize that most of his ± 20 doctoral students till now, from Belgium and several other countries, certainly were largely influenced by the highly innovative work of the first named author, and that also many of them enjoyed the great hospitality of the MSU-geometers and their families at East Lansing.

One more side effect of our cooperation, we would like to mention here is the following. Around New Year 1986, J.M. Morvan and the authors discussed at MSU, the organisation of international meetings on differential geometry. This later developed into what lately became the yearly congresses in “Pure and Applied Differential Geometry”, the last four being held at Leuven-Brussel. We are very grateful for all the work done by many to make these events successful, scientifically and socially, and in particular would like to mention here J.M. Morvan, A. West and S. Carter, U. Simon and M. Magid, and from the Center for Pure and Applied Differential Geometry at KUL-KUB (PADGE) : F. Dillen and I. Van de Woestijne.

In the spring term of 1991, at MSU, the authors continued their joint work on the theory of submanifolds of finite type, which was created some decade before by the first named author, and also worked on biharmonic submanifolds. In the course of this work, we got engaged in the study of the Laplace transformations
of submanifolds and some related topics. A first version of this work, just stating part of our results on this at that time, was written down in a paper for the book “Differential Geometry in honor of Radu Rosca”, published by the Department of Mathematics of the KUL, dedicated to both our friend, and first teacher in research of the second author, at the occasion of his 80th birthday. The present monograph is an extension of this work, including also the proofs of the results. We are convinced that many more results could be obtained on the topics initiated in this booklet and we hope that its reading would inspire some other mathematicians to continue research on them.

The KUB indeed is, at this moment, but a small Flemish University at Brussels, which employs only a small number of mathematicians and physicists, and one chemist in its Group of Exact Sciences, mainly for teaching in their Faculty of Economical and Applied Economical Sciences. The second author is as such teaching there, together with I. Van de Woestijne, the first year’s course on mathematics to the economical engineers. The mathematicians doing research at the KUB on submanifold theory at the moment are I. Van de Woestijne, J. Walrave, J. Vestruelaen and the second author. As member of the group of Exact Sciences at the KUB, the second author is very happy to state here that, despite the rather limited possibilities in general of this university, the working atmosphere there is extremely positive, the contacts with the colleagues and students are truly enjoyable and all Deans of this Faculty so far as well as the former and present Rector F. Van Hemelrijk and F. Gotzen, have always fully supported the research activities of our Group. In particular, the last three meetings so far of the above mentioned series of congresses in differential geometry, would not have been possible without their support.

The second author would like to make a general comment concerning the research activities of PADGE, which basically is an international group of pure and applied mathematicians, centered essentially at the Department of Mathematics of the KUL (in its section with the same name) and at the Group of Exact Sciences of the KUB: namely, we see our Center as sort of a prototype of effective cooperation on research and teaching between the KUL and KUB. Besides our indebtedness to the authorities of the KUB mentioned before, we would also like to thank very much A. Warrinier, the Chairman of the Department of Mathematics of the KUL, and A. Oosterlinck, the Director of the Group of Exact Sciences of the KUL, for their support to PADGE since its creation.

The Group of Exact Sciences of the KUB intends to publish a series of monographs on the work in which they are involved, and of which this is the first volume. We hope, also in this way, to express our deep gratitude towards the authorities of the KUB for their generous support of our work.
Chapter I: INTRODUCTION

Let \( x : M^n \rightarrow \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional \( (n > 0) \) connected Riemannian manifold \( M^n \) into a Euclidean \( m \)-space. Denote by \( \Delta \) the Laplace operator of \( M^n \) with respect to the Riemannian structure. Then \( \Delta \) gives rise to a differentiable map \( L : M^n \rightarrow \mathbb{E}^m \), called the Laplace map, defined by \( L(p) = (\Delta x)(p) \), for any point \( p \in M^n \), where \( x \) denotes the immersion of \( M^n \) into \( \mathbb{E}^m \) as well as the position vector field of the submanifold \( M^n \) in \( \mathbb{E}^m \). We call the image \( L(M^n) \) of this map the Laplace image, and we call the transformation \( L : M^n \rightarrow L(M^n) \) from \( M^n \) onto its Laplace image \( L(M^n) \) via \( \Delta \) the Laplace transformation of the immersion \( x : M^n \rightarrow \mathbb{E}^m \) or of the submanifold \( M^n \) in \( \mathbb{E}^m \). Similar definitions of course can be given for submanifolds in pseudo-Euclidean spaces exactly in the same way. Many of the results mentioned in this article can be easily extended to submanifolds in pseudo-Euclidean spaces.

Let \( H \) denote the mean curvature vector field of the immersion \( x : M^n \rightarrow \mathbb{E}^m \) or the submanifold \( M^n \). As is well-known, \( H \) is a natural and canonically determined normal vector field on \( M^n \) in \( \mathbb{E}^m \). Its length \( \alpha = ||H|| \) is called the mean curvature function of \( x \) or of \( M^n \), or simply, their mean curvature. Much of the geometry of the submanifold \( M^n \) is determined by the properties of \( H \) and \( \alpha \). One has the following fundamental formula of Beltrami:

\[
\Delta x = -nH;
\]

(for pseudo-Riemannian submanifolds in pseudo-Euclidean ambient spaces, see [C5, C7]). From the Beltrami formula, it follows that the Laplace image \( L(M^n) \) of a submanifold \( M^n \) in \( \mathbb{E}^m \) is obtained by first parallel translating the mean curvature vector field \( H \) of \( M^n \) to a vector field with the origin 0 as its initial point, and by then performing a homothety in \( \mathbb{E}^m \) with center 0 and factor \( -n \). Of course, the geometries of the Laplace image \( L(M^n) \) and of the submanifold in \( \mathbb{E}^m \) generated by the mean curvature vector field \( H \) of \( M^n \) when centered at the origin 0, which we could call the \( H \)-image of \( M^n \) in \( \mathbb{E}^m \), are basically the same.

From the Beltrami formula, one knows, for instance, that the immersion \( x \) is minimal, or that \( M^n \) is a minimal submanifold in \( \mathbb{E}^m \), if and only if their Laplace image \( L(M^n) \) is the point 0 (namely the origin of the ambient space \( \mathbb{E}^m \)). Also, it is clear that \( x \) or \( M^n \) have non-vanishing constant mean curvature function \( \alpha \) if and only if their Laplace image \( L(M^n) \) is contained in a hypersphere of \( \mathbb{E}^m \) centered at 0. In the same way, for Riemannian or pseudo-Riemannian submanifolds in, for instance, a Minkowski space-time, it is clear that they are pseudo-minimal in the sense of R. Rosca, \( i.e. \), \( H \neq 0 \) but \( ||H|| = 0 \), if and only if their Laplace image lies on the light-cone minus the origin.
Our main purpose of this article is to initiate the study of the following geometric problem:

“To what extent do the properties of the Laplace transformation and/or the Laplace image of the immersion \(x : M^n \rightarrow E^m\) determine the immersion?”

and we will report here on the results of our joint work on this problem: Announcements of some of these results were made in the papers [CV1,CV2].

Here, we would like to mention that the formula of Beltrami plays a rather important role in the theory of submanifolds of finite type, which was originated by the first author (see, for instance, [C3-C7]). And the main topic of our present study actually came up when studying questions on finite type submanifolds and biharmonic submanifolds; the latter one is in fact one of the off-springs of the theory of finite type.

Following up on some studies initiated in this work, further research has been done in the mean time by various people. For instance, following up on our study of submanifolds with harmonic mean curvature function in Chapter VII, further research has been done by e.g. also O. Garay, M. Barros and G. Zafindratafa, and also, this theory now fits in as a special case in the variational theory of \(k\)-minimal submanifolds, as studied first by the authors together with F. Dillen and L. Vrancken, and later also by D. E. Blair, M. Petrovic and J. Vrancken.

The contents of this monograph is as follows. In Chapter II, we recall some basic facts and formulas on submanifolds in general, as far as useful in the work later on. In particular, this chapter offers an introduction to the theory of submanifolds of finite type and several related notions. The main purpose of Chapter III is to study submanifolds whose Laplace maps have small rank, or more precisely, have a constant rank smaller than their dimension. In particular, we consider submanifolds whose Laplace maps have constant rank 1. Also, the Laplace maps of ruled surfaces is studied. In Chapter IV, we study submanifolds for which the Laplace transformations is homothetic. Among others, we do so in relation with the theory of finite type. Also, classification results are obtained for surfaces and hypersurfaces with homothetic Laplace transformations. Chapter V deals with submanifolds with conformal Laplace transformations. Characterizations are given for such submanifolds to be minimal in hyperspheres and for their Laplace images to be minimal in hyperspheres centered at the origin. Also, in this context, conformally flat hypersurfaces are studied. Then, in particular, surfaces with conformal Laplace transformations are studied, revealing among others relations with the property to be biharmonic. In Chapter VI, we study the geometry of Laplace images of submanifolds. As such, we study, e.g. surfaces whose Laplace images lie in a plane, a sphere, a cylinder or a cone, and also curves whose Laplace images lie in a line or a circle. Finally, for totally real surfaces, we study when their Laplace images are also totally real.
In Chapter VII, we point out the relation between submanifolds having harmonic Laplace maps and biharmonic submanifolds. In particular, an old result of the first author, according to which every biharmonic surface in $\mathbb{E}^3$ is minimal, is also included here. Then we study curves and submanifolds with parallel mean curvature vector field having harmonic Laplace transformations. Finally, we initiate the study of the (no-compact) submanifolds with harmonic mean curvature functions, and, in particular, classify the flat surfaces in $\mathbb{E}^3$ with harmonic mean curvature functions. In Chapter VIII, we study curves, surfaces and hypersurfaces whose Laplace-Gauss-transformations, for short $LG$-transformations, are conformal or, in particular, homothetic. Here, by the $LG$-transformation of a submanifold $M^n$ in $\mathbb{E}^m$, we mean the natural map $LG : L(M^n) \to G(M^n)$ from the Laplace image $L(M^n)$ of $M^n$ to the Gauss image $G(M^n)$, i.e. the image of the Gauss map of $M^n$. Lastly, in Chapter IX, we study the following, first, we study the problem when the spherical Laplace map of a submanifold $M^n$ in $\mathbb{E}^m$ with nonzero constant mean curvature $\alpha$ is harmonic; here, the spherical Laplace map $LS$ is the natural function $LS : M^n \to S^{m-1}(n\alpha) \subset \mathbb{E}^m$, observing that for such submanifolds the Laplace image $L(M^n)$ is always contained in a hypersphere of $\mathbb{E}^m$ with radius $n\alpha$ and centered at the origin. Then we study mostly 2-type submanifolds in relation with their Laplace map, in the course of which we introduce the notions of conjugate and dual 2-type submanifolds. In this work we follow the notations of [C3,C4] closely.

This work was started when the second author was visiting the Michigan State University at East Lansing in 1991. We had further discussions on it when the first author was visiting the Katholieke Universiteit of Leuven and of Brussel in 1991, 1993 and 1994. Also the second author worked on it when visiting our colleagues P. Embrechts and K. Voss at the ETH Zürich in 1994. We would like to thanks the ETH Zürich and our own universities for their support making this work possible. Moreover, we would like to thank the Research Council of the Katholieke Universiteit Brussel for its support to the mathematicians of its Group of Exact Sciences in general, and, in particular, for publishing this monograph. (March 4, 1995)
Chapter II: SUBMANIFOLDS OF FINITE TYPE

The main purpose of this chapter is to review some results on submanifolds of finite type for later use. For the general references, see for instances, [C5,C6,C18].

§1. Basic formulas.

Throughout this monograph, submanifolds are assumed to be connected, smooth, and of positive dimension unless mentioned otherwise.

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional Riemannian manifold \( M \) into \( \mathbb{E}^m \). Let \( \nabla \) and \( \tilde{\nabla} \) denote the Levi-Civita connections of \( M \) and \( \mathbb{E}^m \), respectively. For any vector fields \( X, Y \) tangent to \( M \), the Gauss formula is given by

\[
(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),
\]

where \( h \) denotes the second fundamental form of \( M \) in \( \mathbb{E}^m \). The mean curvature vector \( H \) is given by

\[
H = \frac{1}{n} \text{trace} h, \quad n = \dim M.
\]

The length of the mean curvature vector is called the mean curvature.

Let \( A \) denote the Weingarten map. Then \( h \) and \( A \) are related by

\[
(2.2) \quad <h(X,Y), \xi> = <A_\xi X, Y>.
\]

for \( X, Y \) tangent to \( M \) and \( \xi \) normal to \( M \), where \( <, > \) is the inner product in \( \mathbb{E}^m \). Let \( D \) denote the normal connection of \( M \) in \( \mathbb{E}^m \). Then the Weingarten formula is given by

\[
(2.3) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi.
\]

Let \( R \) denote the Riemannian curvature tensor of \( M \). The equation of Gauss is then given by

\[
(2.4) \quad <R(X,Y)Z,W> = <h(X,W), h(Y,Z)> - <h(X,Z), h(Y,W)>.
\]

For the second fundamental form \( h \), the covariant derivative of \( h \) is defined by

\[
(2.5) \quad (\tilde{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]

The equation of Codazzi is given by

\[
(2.6) \quad (\tilde{\nabla}_X h)(Y,Z) = (\tilde{\nabla}_Y h)(X,Z).
\]

If we denote by \( R^D \) the curvature tensor associated with the normal connection \( D \) of \( M \) in \( \mathbb{E}^m \), then the equation of Ricci is given by

\[
(2.7) \quad <R^D(X,Y)\xi, \eta> = <[A_\xi, A_\eta]X, Y>.
\]

for \( X, Y \) tangent to \( M \) and \( \xi, \eta \) normal to \( M \). Since \( M \) is Riemannian, one may choose orthonormal local frame fields \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\} \) on \( M \) such that
$e_1, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, \ldots, e_m$ are normal to $M$. Let $\omega^1, \ldots, \omega^n$ be the fields of dual 1-forms of $e_1, \ldots, e_n$. The connection form $(\omega^B_A)$ is then given by

$$\tilde{\nabla}_e A = \sum_{B=1}^{m} \omega^B_A e_B, \quad \omega^B_A = -\omega^A_B, \quad A, B, C = 1, \ldots, m.$$  

From (2.8) we find

$$d\omega^B_A = -\sum_{C=1}^{m} \omega^C_A \wedge \omega^B_C.$$  

Put

$$h = \sum_{i,j,r} h^r_{ij} \omega^i \omega^j e_r, \quad i, j = 1, \ldots, n, \quad r = n + 1, \ldots, s.$$  

Then we have

$$\omega^r_i = \sum_j h^r_{ij} \omega^j, \quad <A_r e_i, e_j> = \epsilon_r h^r_{ij},$$

where $A_r = A_{e_r}$.

An $n$–dimensional submanifold $M$ of a Riemannian manifold $\tilde{M}$ is called a totally umbilical submanifold if the Weingarten map $A_\xi$ is proportional to the identity map for any normal vector $\xi$. It is known that totally umbilical submanifolds of a Euclidean $E^m$ are open portions of $n$–dimensional affine subspaces and open portions of hyperspheres of $(n + 1)$–dimensional affine subspaces of $E^m$. An $n$–dimensional submanifold $M$ of a Riemannian manifold $\tilde{M}$ is called a pseudo–umbilical submanifold if mean curvature vector $H$ is nowhere zero and the Weingarten map $A_H$ at $H$ is proportional to the identity map.

Here, we mention two results concerning pseudo–umbilical submanifolds for later use.

**Theorem 1.1.** [CY] Let $x : M \to E^m$ be an isometric immersion of an $n$–dimensional Riemannian manifold $M$ into $E^m$. Then $M$ is a pseudo–umbilical submanifold with parallel mean curvature vector if and only if $M$ is immersed as a minimal submanifold in a hypersphere of $E^m$. □

**Theorem 1.2.** [C1] Let $x : M \to E^{n+2}$ be an isometric immersion of an $n$–dimensional Riemannian manifold $M$ into $E^{n+2}$. Then $M$ is a pseudo–umbilical submanifold with constant mean curvature if and only if $M$ is immersed as a minimal submanifold in a hypersphere of $E^m$. □

We need the following formula for later use, too.

**Theorem 1.3.** [C5,C6] Let $M$ be an $n$–dimensional submanifold of $E^m$. Then we have

$$\Delta H = \Delta^D H + ||A_{n+1}||^2 H + a(H) + \text{trace}(\nabla A_H),$$
where
\[ \nabla A_H = \nabla A_H + A_{DH}, \]
\[ \text{trace}(\nabla A_H) = \sum_{i=1}^{n} \{ A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i \}. \]
\[ a(H) = \sum_{r=n+2}^{m} \text{trace}(A_H A_r) e_r. \]

(1.24) \[ \Delta H = \Delta^D H + \sum_{i=1}^{n} h(A_H e_i, e_i) + \frac{n}{2} \text{grad} < H, H > + 2 \text{trace} A_{DH}, \]

where \( \Delta^D \) is the Laplacian operator associated with the normal connection \( D \).

If \( M \) is a spherical submanifold, we have the following \([C5, C6, C8]\).

**Proposition 1.4.** Let \( M \) be an \( n \)-dimensional submanifold of a hypersphere \( S^{m-1}(r) \) of radius \( r \) and centered at the origin of \( \mathbb{E}^m \). Then we have

\[ \Delta H = \Delta^D H + \sum_{i=1}^{n} h(A_H e_i, e_i) + \frac{n}{2} \text{grad} < H, H > + 2 \text{trace} A_{DH}, \]

where \( \bar{H} \) is the mean curvature vector of \( M \) in \( S^{m-1}(r) \), \( \xi \) the unit vector in the direction of \( \bar{H} \), and \( \bar{D}, A(\bar{H}) \) the normal connection and the allied mean curvature vector of \( M \) in \( S^{m-1}(r) \), respectively.

\[ \Delta H = \Delta^D H + n + ||A(\bar{H})||^2 \bar{H} + \bar{A}(\bar{H}) + \text{trace}(\nabla A_H) - \frac{na^2}{r^2} x, \]

where \( \bar{H} \) is the mean curvature vector of \( M \) in \( S^{m-1}(r) \), \( \xi \) the unit vector in the direction of \( \bar{H} \), and \( \bar{D}, A(\bar{H}) \) the normal connection and the allied mean curvature vector of \( M \) in \( S^{m-1}(r) \), respectively.

\[ \Delta H = \Delta^D H + n + \sum_{i=1}^{n} h(A_H e_i, e_i) + \frac{n}{2} \text{grad} < H, H > + 2 \text{trace} A_{DH}, \]

\[ a(H) = \sum_{r=n+2}^{m} \text{trace}(A_H A_r) e_r. \]

§2. Order of submanifolds.

An algebraic manifold or an algebraic variety is defined by algebraic equations. Thus, one may define the notion of degree of an algebraic manifold by its algebraic structure (which can also be defined by using homology). The concept of degree is both important and fundamental in algebraic geometry. On the other hand, one cannot talk about the degree of an arbitrary submanifold in a Euclidean m–space \( \mathbb{E}^m \). In this section, we will use the induced Riemannian structure on a submanifold \( M \) of \( \mathbb{E}^m \) to introduce two well–defined numbers \( p, q \) associated with the submanifold \( M \); more precisely, \( p \) is a positive integer and \( q \) is either \( +\infty \) or an integer \( \geq p \). We call the pair \( [p, q] \) the order of the submanifold \( M \); moreover, \( p \) is called the lower order and \( q \) the upper order of the submanifold \( M \). The submanifold \( M \) is said to be of finite type if the upper order \( q \) is finite. And \( M \) is of infinite type if the upper order \( q \) is \( \infty \).

Although it remains possible to define the notion of submanifolds of finite type and the related notions of order,... for those submanifolds, for simplicity, we define them here only for compact submanifolds.
Let \((M, g)\) be a compact Riemannian manifold, denote the Levi–Civita connection of \((M, g)\) by \(\nabla\) and let \(\Delta = -\text{trace} \nabla^2\) be the Laplacian of \((M, g)\) acting as a differential operator on \(C^\infty(M) \subset L^2(M, \mu_g)\). It is well–known that the eigenvalues of \(\Delta\) form a discrete infinite sequence:

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \rightarrow \infty.
\]

Let \(V_k = \{ f \in C^\infty(M) : \Delta f = \lambda_k f \}\) be the eigenspace of \(\Delta\) with eigenvalue \(\lambda_k\). Then \(V_k\) is finite–dimensional. We define an inner product \(( , )\) on \(C^\infty(M)\) by

\[
(f, h) = \int_M fh dV
\]

where \(dV\) is the volume element of \((M, g)\).

Then \(\sum_{k=0}^\infty V_k\) is dense in \(C^\infty(M)\) (in \(L^2\)–sense). Denoting by \(\hat{\oplus}V_k\) the completion of \(\sum V_k\), we have

\[
C^\infty(M) = \hat{\oplus}k V_k.
\]

For each function \(f \in C^\infty(M)\) let \(f_t\) be the projection of \(f\) onto the subspace \(V_t\) \((t = 0, 1, 2, \ldots)\). Then we have the following spectral decomposition:

\[
f = \sum_{t=0}^\infty f_j \quad (\text{in } L^2\text{-sense}).
\]

Because \(V_0\) is 1–dimensional, for any non–constant function \(f \in C^\infty(M)\) there is a positive integer \(p \geq 1\) such that \(f_p \neq 0\) and

\[
f - f_0 = \sum_{t \geq p} f_t,
\]

where \(f_0 \in V_0\) is a constant. If there are infinite \(f_t\)'s which are nonzero, we put \(q = \infty\), Otherwise, there is an integer \(q, q \geq p\), such that \(f_q \neq 0\) and

\[
f - f_0 = \sum_{t=p}^q f_t. \tag{2.1}
\]

If we allow \(q\) to be \(\infty\), we have the decomposition (2.1) in general.

The set

\[
T(f) = \{ t \in N_0 : f_t \neq 0 \}
\]

is called the type of \(f\). A function \(f\) in \(C^\infty(M)\) is said to be of finite type if \(T(f)\) is a finite set, i.e., if its spectral decomposition contains only finitely many non–zero terms. Otherwise \(f\) is said to be of infinite type. The smallest element in \(T(f)\) is called the lower order of \(f\) (notation: \(\text{l.o.}(f)\)) and the supremum of \(T(f)\) is called the upper order of \(f\) (notation: \(\text{u.o.}(f)\)). \(f\) is said to be of \(k\)-type if \(T(f)\) contains exactly \(k\) elements.
For an isometric immersion \( x : M \to \mathbb{E}^m \) of a compact Riemannian manifold \( M \) into a Euclidean \( m \)-space, we put
\[
x = (x_1, \ldots, x_m)
\]
where \( x_A \) is the \( A \)-th Euclidean coordinate function of \( M \) in \( \mathbb{E}^m \). For each \( x_A \) we have
\[
x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t, \quad A = 1, \ldots, m.
\]
For the isometric immersion \( x : M \to \mathbb{E}^m \), we put
\[
p = \inf_A \{p_A\}, \quad q = \sup_A \{q_A\}
\]
where \( A \) ranges among all \( A \) such that \( x_A - (x_A)_0 \neq 0 \). It is easy to see that \( p \) and \( q \) are well-defined geometric invariants such that \( p \) is a positive integer and \( q \) is either \( \infty \) or an integer \( \geq p \). By using the above notation, we have the following spectral decomposition of \( x \) in vector form:
\[
(2.3) \quad x = x_0 + \sum_{t=p}^q x_t.
\]
We define \( T(x) \) by
\[
T(x) = \{t \in \mathbb{N}_0 : x_t \neq 0\}.
\]
The immersion \( x \) or the submanifold \( M \) is said to be of \( k \)-type if \( T(x) \) contains exactly \( k \) elements. Similarly we can define the lower order of \( x \) and the upper order of \( x \). The immersion \( x \) is said to be of finite type if \( q \), the upper order of \( x \), is finite; and \( x \) is of infinite type if \( q \) is \( \infty \).

**Remark 2.1.** It is easy to see that \( T(x) = T(y) \) if \( x \) and \( y \) are congruent isometric immersions of \( (M, g) \) in \( \mathbb{E}^m \), so that all these notions do in fact have geometric significance. \( \square \)

**Remark 2.2** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion and \( \bar{x} : M \to \mathbb{E}^m \subset \mathbb{E}^\bar{m} \). Then we have \( T(x) = T(\bar{x}) \). \( \square \)

We give the following lemmas for later use.

**Lemma 2.1.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold \( M \) into \( \mathbb{E}^m \). Then \( x_0 \) is the center of mass of \( M \) in \( \mathbb{E}^m \).

Consider the spectral decomposition:
\[
x = x_0 + \sum_{t=p}^q x_t, \quad \Delta x_t = \lambda_t x_t.
\]
By Hopf’ lemma, we have
\[
\int_M x_t \, dV = \frac{1}{\lambda_t} \int_M \Delta x_t \, dV = 0, \quad t \neq 0.
\]
Since \( x_0 \) is a constant vector in \( E^m \), we obtain that
\[
x_0 = \frac{1}{\text{vol}(M)} \int_M x \, dV.
\]
This shows that \( x_0 \) is the center of mass of \( M \) in \( \mathbb{E}^m \). □

For two \( \mathbb{E}^m \)-valued functions \( v, w \) on \( M \), we define the inner product \( v, w \) by
\[
(v, w) = \int_M \langle v, w \rangle \, dV
\]
where \( \langle v, w \rangle \) denotes the Euclidean inner product of \( v, w \).

We then have the following.

**Lemma 2.2.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold \( M \) into \( \mathbb{E}^m \). Then we have
\[
(x_t, x_s) = 0, \quad t \neq s,
\]
where \( x_t \) is the \( t \)-th component of \( x \) with respect to the spectral decomposition of \( x \).

**Proof.** Since \( \Delta \) is self–adjoint, we have
\[
\lambda_t (x_t, x_s) = (\Delta x_t, x_s) = (x_t, \Delta x_s) = \lambda_s (x_t, x_s).
\]
Because \( \lambda_t \neq \lambda_s \), we obtain the lemma. □

For general Riemannian manifolds, in general one cannot make a spectral decomposition of a function. However, it remains possible to define the notion of a function of finite type and the related notions of order, etc. for those functions. A function \( f \) in general is said to be of finite type if it is a finite sum of eigenfunctions of the Laplacian. More precisely, a function of finite type can be written as
\[
f = f_0 + \sum_{i=1}^k f_i
\]
where \( \Delta f_0 = 0 \), \( f_i \) is a nonzero eigenfunction of \( \Delta \) with nonzero eigenvalue \( \lambda_{t_i} \) and all \( \lambda_{t_i} \) are mutually distinct. We define \( T(f) \) to be the set \( \{ \lambda_{t_1}, \ldots, \lambda_{t_k} \} \), u.o.(\( f \)) = sup \( T(f) \), l.o.(\( f \)) = inf \( T(f) \). Similar notions can be defined for an isometric immersion from a general Riemannian manifold into a Euclidean space.

Here we mention the following two well-known results.

**Proposition 2.3.** (E. Beltrami) Let \( x : M \to \mathbb{E}^m \) be an isometric immersion. Then we have
\[
(2.6) \quad \Delta x = -nH,
\]
where \( H \) denotes the mean curvature vector of \( x \).

**Proof.** Let \( h, \nabla, \tilde{\nabla} \) be the second fundamental form of \( x \), the Levi–Civita connection of \( M \) and the Levi–Civita connection of \( \mathbb{E}^m \), respectively. Denote by
$e_1, \ldots, e_n$ an orthonormal local frame fields of $M$. Then we have
\[
\Delta x = -\sum_{i=1}^{n} (e_i e_i x - (\nabla e_i, e_i) x) = -\sum_{i=1}^{n} h(e_i, e_i) = -nH. \quad \square
\]

**Theorem 2.4.** [Ta1] Let $x : M \to \mathbb{E}^m$ be an isometric immersion. If $\Delta x = \lambda x$, $\lambda \neq 0$, then

(i) $\lambda > 0$;

(ii) $x(M) \subset S_0^{m-1}(r)$, where $S_0^{m-1}(r)$ is a hypersphere of $\mathbb{E}^m$ centered at the origin $0$ and with radius $r = \sqrt{\frac{n}{\lambda}}$;

(iii) $x(M) \subset S_0^{m-1}(r)$ is a minimal immersion; moreover, if $x(M) \subset S_0^{m-1}(r)$ is a minimal immersion, then $\Delta x = (\frac{\lambda}{r^2}) x$.

**Proof.** If $\Delta x = \lambda x$, $\lambda \neq 0$, then we have $H = -\lambda x$. Let $X$ be a vector field tangent to $M$. We have $\langle X, x \rangle = 0$. Thus, $X \langle x, x \rangle = 2 \langle X, x \rangle = 0$. Therefore, $\langle X, x \rangle$ is constant on $M$. This proves that $M$ is immersed into a hypersphere $S^{m-1}(r)$ of $\mathbb{E}^m$ centered at the origin with radius $r$. Let $h$, $h'$ and $\tilde{h}$ be the second fundamental forms of $M$ in $\mathbb{E}^m$, $M$ in $S^{m-1}(r)$, and $S^{m-1}(r)$ in $\mathbb{E}^m$, respectively. Then we have
\[
h(X, Y) = h'(X, Y) + \tilde{h}(X, Y)
\]
for $X, Y$ tangent to $M$. Thus, the mean curvature vector $H = H' = -\frac{1}{r} x$. Since $x$ is perpendicular to $S^{m-1}(r)$, and $H$ is parallel to $x$, we have $H' = 0$. Thus $M$ is minimal in $S^{m-1}(r)$. Furthermore, we have
\[
\Delta x = -nH = \frac{n}{r^2} x.
\]
Therefore, $\lambda = \frac{\lambda}{r^2}$. This proves statements (i)–(iii). The remaining part is easy to verify. \quad \square

**Remark 2.3.** The pseudo-Riemannian version of Theorem 2.4 was obtained in [C9]. \quad \square

In terms of finite type submanifolds, Proposition 2.3 and Theorem 2.4 give the following result.

**Proposition 2.5** Let $x : M \to \mathbb{E}^m$ be an isometric immersion. Then $x$ is of 1–type if and only if either $M$ is a minimal submanifold of $\mathbb{E}^m$ or $M$ is a minimal submanifold of a hypersphere of $\mathbb{E}^m$. \quad \square

In particular, Proposition 2.5 shows that a linear subspace and a hypersphere of a Euclidean space are of 1–type.

We give some more examples of finite type submanifolds.

**Example 2.1.** (Product submanifolds) If $M$ is a finite type submanifold of $\mathbb{E}^m$ and $N$ a finite type submanifold of $\mathbb{E}^n$, then the product submanifold $M \times N$ in $\mathbb{E}^{m+n}$ is of finite type.
In particular, the product of two plane circles $S^1(a) \times S^1(b)$ is of finite type in $\mathbb{E}^4$. In fact, it is of 1–type if $a = b$ and is of 2–type if $a \neq b$. Also a circular cylinder $R \times S^1$ is a finite type surface in $\mathbb{E}^3$, in fact, a 2–type surface in $\mathbb{E}^3$. □

**Example 2.2.** (Diagonal immersions) Let $x_i : M \rightarrow \mathbb{E}^{n_i}$, $i = 1, \ldots, k$ be $k$ isometric immersions from a Riemannian manifold $M$ into $\mathbb{E}^{n_i}$ respectively. For any $k$ real numbers $c_1, \ldots, c_k$ with $c_1^2 + \ldots + c_k^2 = 1$, the immersion:

$$\tilde{x} = (c_1 x_1, \ldots, c_k x_k) : M \rightarrow \mathbb{E}^{n_1 + \cdots + n_k}$$

is an isometric immersion from $M$ into $\mathbb{E}^{n_1 + \cdots + n_k}$. Such an immersion is called a diagonal immersion. If $x_i$ are all of finite type, then also $\tilde{x}$ is of finite type. □

There are ample examples of finite type submanifolds which are not of the above kinds. Here we give some such examples.

**Example 2.3.** (2–type curves in $\mathbb{E}^3$) For each positive number $\epsilon$ we put

$$\gamma_\epsilon(s) = \frac{1}{\epsilon^2 + 36} (\epsilon \sin s, -\epsilon^2 \cos s + \cos 3s, -\frac{\epsilon^2}{12} \sin s + \sin 3s).$$

Then $\gamma_\epsilon$ is of 2–type. According to a result of [CDV], up to homothetic transformations of $\mathbb{E}^3$, a 2–type curve in $\mathbb{E}^3$ is either a right circular helix or it is congruent to the curve $\gamma_\epsilon$ for some positive number $\epsilon$. □

**Example 2.4.** (A flat torus in $\mathbb{E}^6$) Consider the flat torus

$$T_{ab} = S^1(a) \times S^1(b), \quad a^2 + b^2 = 1.$$ 

Let $x : T_{ab} \rightarrow \mathbb{E}^6$ be defined by

$$x = x(s, t) = (a \sin s, b \sin s \sin \frac{t}{b}, b \sin s \cos \frac{t}{b}, a \cos s, b \cos s \sin \frac{t}{b}, b \cos s \cos \frac{t}{b}).$$

By a direct computation, we can see that $T_{ab}$ is of 2–type in $\mathbb{E}^6$. □

This example was first given by N. Ejiri [Ej1] in answering an open question of Weiner.

**Example 2.5.** (Spheres in $\mathbb{E}^m$) Let $S^n$ be the unit hypersphere of $\mathbb{E}^{n+1}$ defined by $y_1^2 + y_2^2 + \ldots + y_{n+1}^2 = 1$, where $(y_1, \ldots, y_{n+1})$ is a Euclidean coordinate system of $\mathbb{E}^{n+1}$. Let

$$x = (x_1, \ldots, x_m) : S^n \rightarrow \mathbb{E}^m$$

be an isometric immersion of $S^n$ into $\mathbb{E}^m$. Then the coordinate functions

$$x_A = x_A(y_1, \ldots, y_{n+1})$$

are functions of $y_1, \ldots, y_{n+1}$. Since the eigenspace of $\Delta$ of $S^n$ associated with $\lambda_k$ is spanned by harmonic homogeneous polynomials of degree $k$ on $\mathbb{E}^{n+1}$, restricted to $M$, the isometric immersion $x : M \rightarrow \mathbb{E}^m$ is thus of finite type if and only if each of $x_A(y_1, \ldots, y_{n+1})$ is a polynomial in $y_1, \ldots, y_{n+1}$. Moreover, when $x : M \rightarrow \mathbb{E}^m$ is of finite type, then the lower order $p$ of the immersion $x$ is nothing but the lowest
(nonzero) degree of \( \{ x_A(y_1, \ldots, y_{n+1}) : A = 1, \ldots, m \} \) and the upper order \( q \) of the immersion \( x \) is the highest degree of \( \{ x_A(y_1, \ldots, y_{n+1}) : A = 1, \ldots, m \} \).

Therefore, in this case, the order of the immersion is nothing but the degree. □

**Example 2.6.** (Symmetric Spaces in \( \mathbb{E}^m \))

For a compact rank one symmetric space, we have the following two theorems of [CDV1].

**Theorem 2.6.** If \( x : M \to \mathbb{E}^m \) is an isometric immersion of a compact rank one symmetric space \( M \) into a Euclidean space, then \( x \) is of finite type if and only if all geodesics of \( M \) are curves of finite type in \( \mathbb{E}^m \) via \( x \). □

**Theorem 2.7.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of finite type of a compact rank one symmetric space \( M \) with upper order \( q \). Then

(i) every geodesic is mapped to a curve of finite type of upper order at most \( q \);

and

(ii) through each point of \( M \), there is a geodesic of upper order \( q \). □

Theorem 2.7 was generalized by J. Deprez in [De1] to the following.

**Theorem 2.8.** Let \( M \) be the Riemannian product of a symmetric space of compact type and a number of circles. Then an isometric immersion \( x : M \to \mathbb{E}^m \) is of finite type if and only if \( x : M \to \mathbb{E}^m \) maps all geodesics to curves of finite type.

§3. Minimal polynomial and examples.

The following criterion of [C5,C6] for finite type submanifolds is quite useful.

**Theorem 3.1.** (Minimal Polynomial Criterion) Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold \( M \) into \( \mathbb{E}^m \) and \( H \) the mean curvature vector of \( M \) in \( \mathbb{E}^m \). Then

(i) \( M \) is of finite type if and only if there is a nontrivial polynomial \( Q(t) \) such that \( Q(\Delta)H = 0 \);

(ii) if \( M \) is of finite type, there is a unique monic polynomial \( P(t) \) of least degree with \( P(\Delta)H = 0 \);

(iii) if \( M \) is of finite type, then \( M \) is of \( k \)-type if and only if \( \deg(P) = k \).

The same results holds if \( H \) is replaced by \( x - x_0 \).
Proof. Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold \( M \) into \( E^m \). Consider the spectral decomposition:

\[
x = x_0 + \sum_{t=p}^{q} x_t, \quad \Delta x_t = \lambda_t x_t.
\]

If \( M \) is of finite type, then \( q < \infty \). Because \( \Delta x = -nH \), we have

\[
-n\Delta^i H = \sum_{t=p}^{q} \lambda_t^{i+1} x_t, \quad i = 0, 1, \ldots
\]

Let

\[
c_1 = -\sum_{t=p}^{q} \lambda_t, \quad c_2 = \sum_{t<s} \lambda_t \lambda_s, \ldots, c_{q-p+1} = (-1)^{q-p+1} \lambda_p \cdots \lambda_q.
\]

Then by direct computation we find

\[
\Delta^k H + c_1 \Delta^{k-1} H + \ldots + c_k H = 0,
\]

where \( k = q-p+1 \). Conversely, if \( H \) satisfies \( Q(\Delta)H = 0 \) for some monic polynomial

\[
Q(t) = t^k + c_1 t^{k-1} + \ldots + c_{k-1} t + c_k,
\]

then from (2.2) we have

\[
\sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \ldots + c_k) x_t = 0.
\]

For each positive integer \( s \), (3.4) yields

\[
\int_M \langle x_s, x_t \rangle dV = \sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \ldots + c_k) x_t = 0.
\]

Therefore, by applying Lemma 1.2, we obtain

\[
(\lambda_s^k + c_1 \lambda_s^{k-1} + \ldots + c_k) ||x_s||^2 = 0,
\]

where \( ||x_s||^2 = \langle x_s, x_s \rangle \). If \( x_s \neq 0 \), then (3.5) implies

\[
\lambda_s^k + c_1 \lambda_s^{k-1} + \ldots + c_k = 0.
\]

Since this equation has at most \( k \) real solutions and equation (3.5) holds for any positive integer \( s \), at most \( k \) of the \( x_t \)'s are nonzero. Thus the decomposition (3.1) is in fact a finite decomposition. Consequently, the immersion \( x \) is of finite type. This proves (i).

Suppose that \( M \) is of finite type. Let \( P = P(t) \) be a monic polynomial of least degree with \( P(\Delta)H = 0 \). If \( K \) is another such polynomial, then \( \deg(P) = \deg(K) \). Since \( R = P - K \) is a polynomial of smaller degree satisfying \( R(\Delta)H = 0 \), we have \( R = 0 \). This implies that \( P = K \). So (ii) is proved.

(iii) follows from the proof of (i).

The same proof applies if we replace \( H \) by \( x - x_0 \). \( \square \)
**Definition 3.1.** The unique monic polynomial $P$ given in (ii) of Theorem 2.1 is called the **minimal polynomial** of the finite type submanifold $M$. □

If the submanifold $M$ is not compact, then the existence of a nontrivial polynomial $Q$ such that $Q(\Delta)H = 0$ does not imply that $M$ is of finite type in general. However, we have the following three results of [CP].

**Theorem 3.2.** Let $\gamma$ be a curve in $E^m$. If there is a nontrivial polynomial $Q$ of one variable such that $Q(\Delta)H = 0$, then $\gamma$ is a curve of finite type. □

**Theorem 3.3.** Let $x: M \rightarrow E^m$ be an isometric immersion. If there exists a polynomial $Q$ such that $Q(\Delta)H = 0$, then either $M$ is of infinitely type or is of $k$–type with $k \leq \deg P + 1$. □

**Theorem 3.4.** Let $x: M \rightarrow E^m$ be an isometric immersion. If there exist a vector $c \in E^m$ and a polynomial $P(t) = \prod_{i=1}^{k}(t - \ell_i)$ with mutually distinct $\ell_1, \ldots, \ell_k$ such that $P(\Delta)(x - c) = 0$, then $M$ is of finite type. □

**Remark 3.1.** It is clear that congruent submanifolds of finite type have the same minimal polynomial. Furthermore, if two finite type submanifolds $M$ and $N$ in $E^m$ have the same minimal polynomial, then (1) they are of the same type, (2) the Laplacians of $M$ and $N$ have at least $k$ common eigenvalues given by the roots of the minimal polynomial and (3) the order of $M$ and $N$ are the same whenever $M$ and $N$ are isospectral. As a consequence, the minimal polynomial provides us some important information on the spectrum of a finite type submanifold. □

**Definition 3.2.** Let $M$ be a manifold and $G$ a closed subgroup of the group $I(M)$ of the isometries acting transitively on $M$ and $\tilde{M}$ a Riemannian manifold with group $I(\tilde{M})$ as the group of isometries. An immersion $f: M \rightarrow \tilde{M}$ of $M$ into $\tilde{M}$ is called $G$–equivariant if there is a homomorphism $\zeta: G \rightarrow I(\tilde{M})$ such that

$$f(a(p)) = \zeta(a)f(p)$$

for $a \in G$ and $p \in M$. □

**Theorem 3.5.** [C5] Let $M$ be a compact homogeneous Riemannian manifold. If $M$ is equivariantly isometrically immersed in $E^m$, then $M$ is of finite type. Moreover, $M$ is of $k$–type with $k \leq m$.

**Proof.** Let $p$ be an arbitrary point of $M$. Then the $m + 1$ vectors $H, \Delta H, \ldots, \Delta^m H$ at $p$ are linearly independent. Thus, there is a polynomial $Q(t)$ of degree $\leq m$ such that $Q(\Delta)H = 0$ at $p$. Because $M$ is equivariantly isometrically immersed in $E^m$, $Q(\Delta)H = 0$ at every point of $M$. Thus, by Theorem 2.1, $M$ is of finite type. Moreover, because the minimal polynomial of $M$ is of degree $\leq m$, $M$ is of $k$–type with $k \leq m$. □
In [De1, Ta2], J. Deprez and T. Takahashi studied equivariant isometric immersions of compact irreducible homogeneous Riemannian manifolds and they proved the following.

**Theorem 3.6.** $M$ be a compact homogeneous Riemannian manifold with irreducible isotropy action. Then an equivariant isometric immersion $x : M \to \mathbb{E}^m$ of $M$ into $\mathbb{E}^m$ is the diagonal immersion of some $1$–type isometric immersions of $M$. □

A submanifold $M$ of a hypersphere $S^{m-1}$ of $\mathbb{E}^m$ is said to be mass-symmetric if the center of gravity of $M$ coincides with the center of the hypersphere $S^{m-1}$ in $\mathbb{E}^m$.

For mass–symmetric 2–type immersions of a topological sphere we have the following result of M. Kotani.

**Theorem 3.7.** [Ko] Any mass–symmetric 2–type immersion of a topological 2–sphere into a hypersphere of $\mathbb{E}^m$ is the diagonal immersion of two 1–type immersions. □

Following from this theorem of Kotani, we have the following two consequences.

**Corollary 3.8.** [Ko] If the immersion in Theorem 3.7 is full, then $m$ is odd and greater than 5. □

**Corollary 3.9.** [Ko] If a 2–sphere admits a mass–symmetric 2–type immersion into $S^9$, then the 2–sphere is of constant curvature. □

Although there do exist mass–symmetric 2–type surfaces in $S^3$ and in $S^5$, it is interesting to point out that for surfaces in $S^4$, we have the following non–existence theorem.

**Theorem 3.10.** [BC1] There exists no compact mass–symmetric 2–type surfaces which lie fully in $S^4 \subset \mathbb{E}^5$. □

In view of Theorem 3.7 of Kotani, it is interesting to point out that there exist 2–type isometric immersions of an ordinary 2–sphere into $\mathbb{E}^{10}$ with order $[1,3]$ that are not diagonal immersions. (cf. [CDV3]).

We mention the following result of [C11,C12] for later use.

**Theorem 3.11** Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an $n$–dimensional Riemannian manifold into $\mathbb{E}^m$. Then the mean curvature vector field of $x$ is an eigenvector of the Laplace operator, i.e., $\Delta H = \lambda H$ for some constant $\lambda$, if and only if $M$ is one of the following submanifolds:

1. a 1-type submanifold of $\mathbb{E}^m$;
2. a biharmonic submanifold; or
(3) a null 2-type submanifold.

In particular, a surface $M$ in $E^3$ satisfies the condition $\Delta H = \lambda H$ for some constant $\lambda$ if and only if $M$ is either a minimal surface or an open portion of circular cylinder.

□

**Remark 3.2.** In fact, the exact proof given in [C12] works for pseudo-Riemannian submanifolds in pseudo-Euclidean space as well. Hence, the pseudo-Riemannian version of Theorem 3.11 also holds (cf. [C12,C23]). □

**Remark 3.3.** The classifications of submanifolds in hyperbolic spaces and in de Sitter space-times which satisfy condition $\Delta H = \lambda H$ are obtained in [C21] and in [C23], respectively.

**Remark 3.4.** For the classification of spherical 2-type surfaces with constant sectional curvature, see [Mi]; and for the classification of flat 2-type spherical Chen surfaces, see [G1].

§4. Order, mean Curvature, and isoparametric hypersurfaces.

In this section, we will give some relations between order, mean curvature and type for spherical submanifolds. For simplicity, we will do this here only for spherical hypersurfaces.

In the following, we denote by $S^{n+1}$ the unit hypersphere of $E^{n+2}$ centered at the origin.

**Theorem 4.1.** Let $M$ be a compact hypersurface of $S^{n+1}$. Then $M$ has nonzero constant mean curvature in $S^{n+1}$ and constant scalar curvature if and only if $M$ is mass–symmetric and of 2–type, unless $M$ is a small hypersphere of $S^{n+1}$.

**Proof.** First we give the following [C8].

**Lemma 4.2.** Let $M$ be an $n$–dimensional hypersurface of $S^{n+1}$. Then we have

\begin{equation}
\text{trace}(\nabla A_H) = \frac{n}{2} \text{grad} \alpha^2 + 2 \text{trace} A_{DH}.
\end{equation}

**Proof of Lemma 4.2.** Let $\xi$ be a unit normal vector field of $M$ in $S^{n+1}$ and $e_1, \ldots, e_n$ orthonormal eigenvectors of $A_\xi$ with eigenvalues $\rho_1, \ldots, \rho_n$, respectively. Denote by $H'$ the mean curvature vector field of $M$ in $S^{n+1}$ and $H' = \alpha' \xi$. Then we have

\begin{equation}
A_{H'} e_i = \alpha' \rho_i e_i.
\end{equation}

We put $\nabla e_j = \sum \omega^k_j e_k$. Then by the Codazzi equation we may obtain

\begin{equation}
e_j \rho_i = (\rho_i - \rho_j) \omega^i_j (e_i), \quad j \neq i.
\end{equation}
On the other hand, by applying
\[ H = H' - x, \quad A_x = -I, \quad D_x = 0, \]
and
\[ (\nabla_{e_i} A_{H'}) e_j = (e_i \alpha') \rho_j e_j + \alpha'(e_i \rho_j) e_j + \sum \alpha'(\rho_j - \rho_k) \omega^k_j (e_i) e_k, \]
we may obtain
\[ \text{trace}(\nabla A_{H'}) = \sum_i \{ 2(e_i \alpha') \rho_i + \alpha'(e_i \rho_i) + \sum_k \alpha'(\rho_k - \rho_i) \omega^k_i (e_k) \} e_i. \]
Substituting (4.3) into (4.4) and making a direct computation, we may obtain (4.1).
\[ \square \]
Now, we really start the proof of Theorem 4.1.

Combining Lemma 3.4 with Lemma 4.2 we find
\[ \Delta H = \Delta^D H + a(H) + ||A_{n+1}||^2 H + \frac{n}{2} \text{grad} \alpha^2 + 2 \text{trace} A_{DH}. \]
Since \( H = H' - x = \alpha' \xi - x \), we may also obtain from (4.5) the following.
\[ \Delta H = (\Delta \alpha') \xi + (||A_\xi||^2 + n)H' - n|H|^2 x + \frac{n}{2} \text{grad} \alpha^2 + 2 \text{trace} A_{DH}. \]
In particular, if \( M \) is mass–symmetric and of 2–type, then we have the following spectral decomposition:
\[ x = x_p + x_q, \quad \Delta x_p = \lambda_p x_p, \quad \Delta x_q = \lambda_q x_q. \]
This implies
\[ n\Delta H = n(\lambda_p + \lambda_q) H + \lambda_p \lambda_q x. \]
Because \( H = H' - x \) and since \( \xi, x \) are orthonormal, (4.6) and (4.7) imply
\[ |H|^2 = 1 + (\alpha')^2 = \frac{\lambda_p + \lambda_q}{n} - \frac{\lambda_p \lambda_q}{n^2}, \]
which implies that \( M \) has constant mean curvature. And hence (4.6) and (4.7) also yield
\[ ||h||^2 = ||A_\xi||^2 + n = \lambda_p + \lambda_q. \]
Consequently, by the Gauss equation, we conclude that the scalar curvature of \( M \) is given by
\[ \tau = \frac{1}{n}(\lambda_p + \lambda_q) - \frac{1}{n(n - 1)} \lambda_p \lambda_q. \]
Conversely, if \( M \) has constant mean curvature and constant scalar curvature, then by (4.6) we have
\[ \Delta^2 x = -n\Delta H = -n||h||^2 H' + n^2 |H|^2 x = ||\sigma||^2 \Delta x + (n^2 |H|^2 - n ||\sigma||^2) x. \]
Since \( ||h|| \) and \( |H| \) are constant, this implies that \( M \) has a minimal polynomial of degree \( \leq 2 \). Hence, by Theorem 2.1, \( M \) is either of 1–type or of 2–type.
If $M$ is of 1–type and $M$ is not minimal in $S^{n+1}$, then $M$ is contained in the intersection of two hyperspheres of $E^{n+2}$. Thus, $M$ is a small hypersphere of $S^{n+1}$.

If $M$ is of 2–type, then (4.11) and Hopf’s lemma imply that $M$ is mass-symmetric in $S^{n+1}$. □

**Remark 4.1.** Theorem 4.1 was first proved in [C5,C8] (see also [C14]). Recently, it was proved in [HV2] that Theorem 4.1 is fact of local natural, so that Theorem 4.1 still holds if the compactness and the mass-symmetric condition in Theorem 4.1 were omitted.

For spherical 2–type hypersurfaces, we have the following [C5,C8].

**Theorem 4.3.** Let $M$ be a compact 2–type hypersurface of $S^{n+1}$. Then the geometric invariants: the mean curvature, the scalar curvature and the length of second fundamental form of $M$ in $E^{n+2}$ are completely determined by the order of $M$; more precisely, we have

(i) the mean curvature $\alpha$ of $M$ in $E^{n+2}$ is constant and given by

$$\alpha^2 = \frac{1}{n}(\lambda_p + \lambda_q) - \frac{1}{n^2}\lambda_p \lambda_q;$$

(ii) the scalar curvature $\tau$ of $M$ is constant and given by

$$\tau = \frac{1}{n}(\lambda_p + \lambda_q) - \frac{1}{n(n-1)}\lambda_p \lambda_q;$$

(iii) the length of the second fundamental form $\sigma$ of $M$ in $E^{n+2}$ is constant and given by

$$||h||^2 = \lambda_p + \lambda_q.$$

Theorem 4.3 shows that the notion of order is fundamental for submanifolds in Euclidean spaces.

A hypersurface $M$ in $S^{n+1}$ is called an isoparametric hypersurface if $M$ has constant principal curvatures. H. F. Münzner [Mi] proved that the number $g$ of distinct principal curvatures of an isoparametric hypersurface is 1, 2, 3, 4 or 6. É. Cartan has shown that an isoparametric hypersurface with at most 3 distinct principal curvature is homogeneous. R. Takagi and T. Takahashi [TT] classified all homogeneous isoparametric hypersurfaces. For $g = 4$ or 6, there exist non–homogeneous examples. In fact, H. Ozeki and M. Takeuchi [OT] constructed two infinite series of non–homogeneous isoparametric hypersurfaces. D. Ferus, H.Karcher and H. F. Münzner [Mi] found, for $g = 4$, a new type of examples constructed from representations of a Clifford algebra. J. Dorfmeister and E. Neher [DN] gave another algebraic approach to isoparametric hypersurfaces in spheres.

Since isoparametric hypersurfaces have constant mean curvature and constant scalar curvature, Theorem 4.1 implies immediately the following [C5,C8].
Theorem 4.4. Every isoparametric hypersurface of \( S^{n+1} \) is either of 1–type or of 2–type. \( \square \)

Remark 4.2. Theorem 4.2 shows that there exist abundant examples of mass–symmetric 2–type hypersurfaces in \( S^{n+1} \). \( \square \)

Although H. B. Lawson proved that there exists many examples of minimal surfaces in \( S^3 \) (i.e., 1–type surfaces in \( S^3 \)). The following result of [C5,BG,HV1] shows that standard tori are the only 2–type surfaces in \( S^3 \).

Theorem 4.5. The only 2–type surfaces in \( S^3 \) are open pieces of the product of circles: \( S^1(a) \times S^1(b) \), \( a \neq b \). \( \square \)

The following complete classification of compact 2-type hypersurfaces in the unit 4-sphere is obtained in [C20] (see, also [C22]).

Theorem 4.6. A compact hypersurface in \( S^4(1) \subset E^5 \) is of 2-type if and only if it is one of the following hypersurfaces:

1. \( S^1(a) \times S^2(b) \subset S^4(1) \subset E^5 \) with \( a^2 + b^2 = 1 \) and \( (a,b) \neq (\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}) \) imbedded in the standard way.
2. A tubular hypersurface with constant radius \( r \neq \frac{\pi}{2} \) about the Veronese minimal imbedding of the real projective plane \( \mathbb{RP}^2(\frac{1}{3}) \) of constant curvature \( \frac{1}{3} \) in \( S^4(1) \). \( \square \)

For 3-type hypersurfaces we have the following results.

Theorem 4.7. [C13,CL] Every 3-type hypersurface in a hypersphere \( S^{n+1} \subset E^{n+2} \) have non-constant mean curvature. \( \square \)

Theorem 4.8. [HV4] A 3-type surface in the Euclidean 3-space \( E^3 \) has non-constant mean curvature. \( \square \)

For finite type surfaces in \( S^3 \) we have the following [CD1].

Theorem 4.9. Let \( M(c) \) be a compact surface with constant curvature \( c \) in \( S^3 \). Then \( M(c) \) is of finite type if and only if either (i) \( c \geq 1 \) and \( M(c) \) is totally umbilical in \( S^3 \) or (ii) \( c = 0 \) and \( M = S^1(a) \times S^1(b) \) with \( a^2 + b^2 = 1 \). \( \square \)

Note that the surfaces in (i) are of 1–type and the surfaces in (ii) are of 2–type unless \( a = b \), in which case they are of 1–type.

§5. Classification of submanifolds of finite type.

Let \( \gamma : S^1 \rightarrow E^m \) be an isometric immersion from a circle with radius 1 into a Euclidean \( m \)-plane \( E^m \). Denote by \( s \) the arclength of \( S^1 \).
For a periodic continuous function \( f = f(s) \) with period \( 2\pi \), \( f(s) \) has a Fourier series expansion given by
\[
f(s) = \frac{a_0}{2} + a_1 \cos s + b_1 \sin s + a_2 \cos 2s + b_2 \sin 2s + \cdots,
\]
where \( a_k, b_k \) are the Fourier coefficients given by
\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \cos ks \, ds, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \sin ks \, ds.
\]

In terms of Fourier series expansion, we have the following

**Proposition 5.1.** Let \( \gamma : S^1 \to \mathbb{R}^m \) be a closed smooth curve in \( \mathbb{R}^m \). Then \( \gamma \) is of finite type if and only if the Fourier series expansion of each coordinate function \( x_A \) of \( \gamma \) has only finite nonzero terms.

**Proof.** Because \( \Delta = -\frac{d^2}{ds^2} \) in this case, we have
\[
\Delta^j H = (-1)^j a^{(2j+2)}, \quad j = 0, 1, 2, \ldots,
\]
where \( x^{(j)} = \frac{d^j x}{ds^j} \). If \( \gamma \) is of finite type in \( \mathbb{R}^m \), then each Euclidean coordinate function \( x_A \) of \( \gamma \) in \( \mathbb{R}^m \) satisfies the following homogeneous ordinary differential equation with constant coefficients:
\[
x_A^{(2k+2)}+c_1 x_A^{(2k)}+\cdots+c_k x_A^{(2)}=0,
\]
for some integer \( k \geq 1 \) and some constants \( c_1, \ldots, c_k \). Because the solutions of (5.1) are periodic with period \( 2\pi \), each solution \( x_A \) is a finite linear combination of the following particular solutions:
\[
1, \cos n_i s, \sin m_i s
\]
where \( n_i, m_i \) are positive integers. Therefore, each \( x_A \) is of the following forms:
\[
x_A = c_A + \sum_{p}^{q} \{ a_A(t) \cos ts + b_A(t) \sin ts \}
\]
for some suitable constants \( a_A(t), b_A(t), c_A \) and some positive integers \( p_A, q_A; A = 1, \ldots, m \). Therefore, each coordinate function \( x_A \) has a Fourier series expansion which has only finite nonzero terms.

Conversely, if each \( x_A \) has a Fourier series expansion which has only finite nonzero terms, then the position vector \( x \) of \( \gamma \) in \( \mathbb{R}^m \) takes the following form:
\[
x = c + \sum_{t=p}^{q} \{ a_t \cos ts + b_t \sin ts \}
\]
for some constant vectors \( a_t, b_t, c \) in \( \mathbb{R}^m \). Let \( x_t = a_t \cos ts + b_t \sin ts \). Since \( \Delta = -\frac{d^2}{ds^2} \), we have \( \Delta x_t = t^2 x_t \). This shows that \( \gamma \) is of finite type. \( \square \)

The basic results concerning finite type curves are given in the following.
Theorem 5.2. Let $\gamma$ be a plane curve of finite type. Then $\gamma$ is a part of a circle or a part of a straight line. □

In views of Proposition 5.1, Theorem 5.2 can be stated as follows: the Fourier series expansion of all unit speed curves in $\mathbb{E}^2$, except the circles and lines, are always infinite. For space curves, the situation is completely different. In fact, for any $k \in \{2, 3, 4, \cdots \}$, infinitely many distinct $k$–type curves are constructed in [CDDVV].

Theorem 5.3. For every $k \in \{2, 3, 4, \cdots \}$ there exist infinitely many non–equivalent curves of $k$–type in $\mathbb{E}^3$. □

Theorem 5.4. Every closed space curve of finite type which is contained in a 2–dimensional sphere is of 1–type, and hence a circle. □

Theorem 5.5. Every closed space curve of finite type which is contained in a 3–dimensional sphere is of 1–type, and hence a circle, or a 2–type $W$–curve. □

For higher dimensional spheres, a similar result is no longer true. One can have finite type curves in 4–dimensional spheres that are not $W$–curves.

Theorem 5.6. Every closed curve of finite type in $\mathbb{E}^3$ having constant curvature is of 1–type, and hence a circle. □

Remark 5.1. Theorem 5.2 was first proved in [C5] for closed plane curves. In [C12], it was pointed out that a finite type non–closed curve in a plane is a part of a straight line without giving detailed proof. The detailed proof was then given in [CDDVV]. Theorems 5.3–5.6 are also given in [CDDVV]. □

2–type curves were classified in [CDV] and [DPVV]. It is known that every 2–type curve in $\mathbb{E}^3$ lies on a hyperboloid of revolution of one sheet or on a cone of revolution [DDV]. Curves of finite type in $\mathbb{E}^3$ whose image is contained in a quadratic surface, other than spheres, were treated in [DDV]. In particular, we have the following.

Theorem 5.7. [DDV] Let $\gamma$ be a closed curve of finite type in $\mathbb{E}^3$ whose image is contained in a quadratic surface $Q$. Then

(1) $\gamma$ is a circle, or
(2) $Q$ is one of the following surfaces:
   (2.a) an ellipsoid of revolution which is not a sphere;
   (2.b) a (one– or two–sheeted) hyperboloid of revolution, or
   (2.c) a circular cone. □

Examples of curves of finite type lying on certain ellipsoids of revolution, on circular cones or on one–sheeted hyperboloids of revolution are given in [DDV]. It is not known whether there are curves of finite type on two–sheeted hyperboloids of revolution.
Curves of finite type in the context of affine differential geometry were studied by J. Copaert and L. Verstraelen (cf. [V1]). Quite remarkable is that all affine curve of finite type are affinely equivalent to skew lines. In Euclidean spaces, a similar statement is false.

Now, we present some classification theorems for surfaces of finite type in Euclidean spaces.

**Theorem 5.8.** The only tubes of finite type in $\mathbb{E}^3$ are circular cylinders. □

**Theorem 5.9.** A ruled surface $M$ in $\mathbb{E}^m$ is of finite type if and only if $M$ is a cylinder on a curve of finite type or $M$ is a part of a helicoid in an affine subspace $\mathbb{E}^3$. □

**Theorem 5.10.** A ruled surface $M$ in $\mathbb{E}^3$ is of finite type if and only if $M$ is a part of a plane, a circular cylinder or a helicoid. □

**Corollary 5.11.** A flat surface in $\mathbb{E}^3$ is of finite type if and only if it is a part of plane or a circular cylinder. □

**Remark 5.2.** Theorem 5.8 was proved in [C10] and Theorem 5.9, Theorem 5.10 and Corollary 5.11 are due to Chen–Dillen–Verstraelen–Vrancken [CDVV1]. □

For algebraic surfaces of degree 2 we have the following result of [CD2].

**Theorem 5.12.** The only quadrics in $\mathbb{E}^3$ which is of finite type are the spheres and the circular cylinders. □

The complete classification of finite type algebraic hypersurfaces of degree 2 is obtained in [CDS].

In 1822, C. Dupin defined a cyclide to be a surface $M$ in $\mathbb{E}^3$ which is the envelope of the family of spheres tangent to three fixed spheres. This was shown to be equivalent to requiring that both sheets of the focal set degenerate into curves. The cyclides are equivalently characterized by requiring that the lines of curvatures in both families be arcs of circles or straight lines. Thus, one can obtain three obvious examples: a torus of revolution, a circular cylinder and a circular cone. It turns out that all cyclides can be obtained from these three by inversions in a sphere in $E^3$.

Recently, F. Defever, R. Deszcz and L. Verstraelen [DDV1,DDV2] proved the following.

**Theorem 5.13.** Cyclides of Dupin are of infinite type. □

All of the above results support the following conjecture.

**Conjecture 5.1.** The only compact surfaces of finite type in $\mathbb{E}^3$ are spheres. □
From Theorem 5.2 we know that, in one dimension case, this is true: namely, the circles are the only closed planar curves of finite type.

**Theorem 5.2'.** If $\gamma : S^1 \rightarrow \mathbb{E}^2$ is an isometric immersion from $S^1$ into $\mathbb{E}^2$. Then $\gamma$ is of finite type if and only if $\gamma$ is a circle. $\square$

§6. Linearly independent and orthogonal immersions.

Let $x : M \rightarrow \mathbb{E}^m$ be an immersion of $k$-type. Suppose

\begin{equation}
(6.1) \quad x = c + x_1 + \ldots + x_k, \quad \Delta x_i = \lambda_i x_i, \quad \lambda_1 < \ldots < \lambda_k,
\end{equation}

is the spectral decomposition of the immersion $x$, where $c$ is a constant vector and $x_1, \ldots, x_k$ are non-constant maps. For each $i \in \{1, \ldots, k\}$ we put

\[ E_i = \text{Span}\{x_i(p) : p \in M\}. \]

Then each $E_i$ is a linear subspace of $\mathbb{E}^m$.

Here, we recall the notions of linearly independent and orthogonal immersions first introduced in [C15].

**Definition 6.1.** Let $x : M \rightarrow \mathbb{E}^m$ be an immersion of $k$-type whose spectral decomposition is given by (6.1). Then the immersion $x$ is said to be **linearly independent** if the subspaces $E_1, \ldots, E_k$ are linearly independent, that is, the dimension of the subspace spanned by all vectors in $E_1 \cup \ldots \cup E_k$ is equal to $\dim E_1 + \ldots + \dim E_k$. And the immersion $x$ is said to be **orthogonal** if the subspaces $E_1, \ldots, E_k$ are mutually orthogonal in $\mathbb{E}^m$.

Obviously, every 1-type immersion is an orthogonal immersion and hence a linearly independent immersion. There exist ample examples of orthogonal immersions and ample examples of linearly independent immersions which are not orthogonal. For instances, every $k$-type curve lying fully in $\mathbb{E}^{2k}$ is an example of linearly independent immersion and the immersion of the curve is orthogonal if and only if it is a $W$-curve. Moreover, the product of any two linearly independent immersions is again a linearly independent immersion.

Let $x : M \rightarrow \mathbb{E}^m$ be an immersion of $k$-type whose spectral decomposition is given by (6.1). We choose a Euclidean coordinate system $(u_1, \ldots, u_m)$ on $\mathbb{E}^m$ with $c$ as its origin. Then we have the following spectral decomposition of $x$:

\begin{equation}
(6.2) \quad x = x_1 + \ldots + x_k, \quad \Delta x_i = \lambda_i x_i, \quad \lambda_1 < \ldots < \lambda_k.
\end{equation}

For each $i \in \{1, \ldots, k\}$ we choose a basis $\{c_{ij} : j = 1, \ldots, m_i\}$ of $E_i$, where $m_i$ is the dimension of $E_i$. Put $\ell = m_1 + \ldots + m_k$ and let $E^\ell$ denote the subspace of $\mathbb{E}^m$ spanned by $E_1, \ldots, E_k$. If the immersion $x$ is linearly independent, then the vectors $\{c_{ij} : i = 1, \ldots, k; j = 1, \ldots, m_i\}$ are $\ell$ linearly independent vectors in $\mathbb{E}^\ell$. 
Furthermore, we choose the Euclidean coordinate system \((u_1, \ldots, u_m)\) on \(\mathbb{E}^m\) such that \(\mathbb{E}^\ell\) is defined by \(u_{\ell+1} = \ldots = u_m = 0\). Regard each \(c_{ij}\) as a column \(\ell\)-vector. We put

\[
S = (c_{11}, \ldots, c_{1m_1}, \ldots, c_{k1}, \ldots, c_{km_k}).
\]

Then the matrix \(S\) is a nonsingular \(\ell \times \ell\) matrix. Let \(D\) denote the diagonal \(\ell \times \ell\) matrix given by

\[
D = \text{Diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_k),
\]

where \(\lambda_i\) repeats \(m_i\)-times. If we put \(A = SDS^{-1}\), then \(Ac_{ij} = \lambda_i c_{ij}\) for any \(i \in \{1, \ldots, k\}\) and \(j \in \{1, \ldots, m_i\}\). Therefore, we have

\[
(6.5)' \quad \Delta x = Ax
\]

for the immersion \(x : M \to \mathbb{E}^\ell\) induced from the original immersion \(x : M \to \mathbb{E}^m\). By regarding the \(\ell \times \ell\) matrix \(A\) as an \(m \times m\) matrix in a natural way (with zeros for each of the additional entries), we obtain

\[
(6.5) \quad \Delta x = Ax, \quad A = (a_{ij})
\]

for the immersion \(x : M \to \mathbb{E}^m\).

If \(x : M \to \mathbb{E}^m\) is a minimal immersion, then we have \(A = 0\). If \(x : M \to \mathbb{E}^\ell\) \((\subset \mathbb{E}^m)\) is a non-minimal full immersion, then the Euclidean coordinate functions \(u_1, \ldots, u_\ell\) of \(\mathbb{E}^\ell\), restricted to \(M\), do not satisfy any linear equation. Thus, the coordinate functions \(u_1|_M, \ldots, u_\ell|_M\) of \(M\) in \(\mathbb{E}^\ell\) are linearly independent functions. Therefore, if \(B\) is any \(\ell \times \ell\) matrix such that \(\Delta x = Bx\), then \(A = B\). Hence, the \(\ell \times \ell\) matrix \(A\) in \((6.5)'\) defined above is unique. Consequently, if the (original) immersion \(x : M \to \mathbb{E}^m\) is a non-minimal, linearly independent immersion, then the \(m \times m\) matrix \(A\) given in \((6.5)\) is also uniquely defined (with respect to the Euclidean coordinate system so chosen).

By using this \(m \times m\) matrix \(A\) in \((6.5)\) defined above, the first author introduced in [C15] the notion of adjoint hyperquadric as follows.

**Definition 6.2** Let \(x : M \to \mathbb{E}^m\) be a non-minimal, linearly independent immersion whose spectral decomposition is given by \((6.1)\). Let \(u = (u_1, \ldots, u_m)\) be a Euclidean coordinate system on \(\mathbb{E}^m\) with \(c\) as its origin and let \(A\) be the \(m \times m\) matrix in \((6.5)\) associated with the immersion \(x\) defined above. Then, for any point \(p \in M\), the equation

\[
(6.6) \quad \langle Au, u \rangle := \sum_{i,j} a_{ij} u_i u_j = c_p, \quad (c_p = \langle Ax, x \rangle (p))
\]

defines a hyperquadric \(Q_p\) in \(\mathbb{E}^m\). We call the hyperquadric \(Q_p\) the adjoint hyperquadric of the immersion \(x\) at \(p\). In particular, if \(x(M)\) is contained in an adjoint hyperquadric \(Q_p\) of \(x\) for some point \(p \in M\), then all of the adjoint hyperquadrics
\{Q_p : p \in M\} give a common adjoint hyperquadric, denoted by Q. We call the
hyperquadric Q the adjoint hyperquadric of the linearly independent immersion x.
△

The following result from [C15] provides us a necessary and sufficient condition
for a compact, linearly independent submanifold to lie in its adjoint hyperquadric.

**Theorem 6.1.** Let $x : M \to \mathbb{E}^m$ be a linearly independent immersion from a
compact manifold M into $\mathbb{E}^m$ whose spectral decomposition is given by (6.2). Then
M is immersed into the adjoint hyperquadric of x if and only if $x(M)$ is contained
in a hypersphere of $\mathbb{E}^m$ centered at the origin.

**Proof.** Assume that M is immersed into a hypersphere $S^{m-1}(r)$ of $\mathbb{E}^m$ with
radius r centered at the origin. Denote by $H$ and $\bar{H}$ the mean curvature vectors of
M in $\mathbb{E}^m$ and of M in $S^{m-1}(r)$, respectively. Then we have

$$H = \bar{H} - \frac{1}{r} x. \tag{6.7}$$

This implies $\langle H, x \rangle = -r$. Since $\Delta x = -nH, n = \dim M$, we obtain $\langle \Delta x, x \rangle = nr$.
Therefore, by (6.5), we conclude that M is immersed into the adjoint hyperquadric
defined by $\langle Au, u \rangle = nr$, where $nr$ is a constant.

Conversely, suppose that $x : M \to \mathbb{E}^m$ is a linearly independent immersion of a
compact manifold M such that $x(M)$ is contained in an adjoint hyperquadric $Q_p$
for some point p. Then we have $\langle Ax, x \rangle = c_p$ where $c_p$ is the constant given by
$c_p = \langle Ax, x \rangle (p)$. Since $Ax = \Delta x = -nH$, we have

$$\langle nH, x \rangle = -c_p. \tag{6.8}$$

Because M is compact, we also have

$$\int_M \{1 + \langle H, x \rangle\} * 1 = 0. \tag{6.9}$$

Formulas (6.8) and (6.9) imply $c_p = -n$. Therefore,

$$\Delta \langle x, x \rangle = 2 \langle \Delta x, x \rangle - 2n = -2n(\langle H, x \rangle + 1) = 0.$$

Thus, by Hopf’s lemma, $\langle x, x \rangle$ is a constant and M is immersed into a hypersphere
of $\mathbb{E}^m$ centered at the origin. □

**Remark 6.1.** Although the implication ($\Leftarrow$) in Theorem 6.1 holds in general
without the assumption of compactness, the implication ($\Rightarrow$) does not hold in
general if M is not compact. For example, let M be the product surface of a unit
plane circle and a line. Then the inclusion map $x$ of M in $E^3$ defined by

$$x(u, v) = (\cos u, \sin u, v)$$

is a non-spherical, linearly independent immersion whose adjoint hyperquadric is
given by $u_1^2 + u_2^2 = 1$. It is clear that M coincides with the adjoint hyperquadric
Q of M in $\mathbb{E}^3$. □
The following result from [C15] provides us a necessary and sufficient condition for a linearly independent immersion to be an orthogonal immersion in terms of the adjoint hyperquadric.

**Theorem 6.2.** Let \( x : M \to \mathbb{E}^m \) be a non-minimal, linearly independent immersion. Then \( M \) is immersed by \( x \) as a minimal submanifold of the adjoint hyperquadric if and only if the immersion \( x \) is an orthogonal immersion. \( \square \)

In general a submanifold obtained from an equivariant immersion of a compact homogeneous space is not a minimal submanifold of any hypersphere. However, the following result from [C15] shows that such a submanifold is always a minimal submanifold in its adjoint hyperquadric.

**Theorem 6.3.** Let \( x : M \to \mathbb{E}^m \) be an equivariant isometric immersion of a compact \( n \)-dimensional Riemannian homogeneous space \( M \) into \( \mathbb{E}^m \). Then \( M \) is immersed as a minimal submanifold of the adjoint hyperquadric. \( \square \)

The class of 1-type immersions has been classified by T. Takahashi. In fact, as stated before, he showed that the submanifolds of \( \mathbb{E}^m \) for which
\[
\Delta x = \lambda x
\]
are precisely either the minimal submanifolds of \( \mathbb{E}^m (\lambda = 0) \) or the minimal submanifolds of the hyperspheres \( S^{m-1} \) in \( \mathbb{E}^m \) (the case when \( \lambda \neq 0 \), actually \( \lambda > 0 \)).

As a generalization of Takahashi’s result, Garay studied the hypersurfaces \( M^n \) in \( \mathbb{E}^{n+1} \) for which
\[
\Delta x = Ax,
\]
where \( A \) is a diagonal matrix.

Dillen, Pas and Verstraelen observed that Garay’s condition is not coordinate-invariant and they considered the submanifolds in \( \mathbb{E}^m \) for which
\[
\Delta x = Ax + b
\]
where \( A \in \mathbb{E}^{m \times m} \) and \( b \in \mathbb{E}^m \). This setting generalizes Takahashi’s condition in a way which is independent of the choice of coordinates. In 1988 Garay proved that if a hypersurface \( M \) in \( \mathbb{E}^{n+1} \) satisfies his condition, it is either minimal in \( \mathbb{E}^{n+1} \), or it is a hypersphere, or it is a spherical hypercylinder. In 1990 Dillen, Pas and Verstraelen proved that a surface in \( \mathbb{E}^3 \) satisfies their condition if and only if it is an open part of a minimal surface, a sphere, or a circular cylinder.

In [CP], Chen and Petrovic proved the following.

**Theorem 6.4.** (Characterization) Let \( x : M \to \mathbb{E}^m \) be an immersion of finite type. Then the immersion \( x \) is linearly independent if and only if \( x \) satisfies \( \Delta x = Ax + b \) for some \( A \in \mathbb{E}^{m \times m} \) and \( b \in \mathbb{E}^m \). \( \square \)
Theorem 6.5. (Characterization) Let \( x : M \to \mathbb{E}^m \) be an immersion of finite type. Then the immersion \( x \) is orthogonal if and only if \( x \) satisfies \( \Delta x = Ax + b \) for some symmetric matrix \( A \in \mathbb{E}^{m \times m} \) and \( b \in \mathbb{E}^m \). □

Theorem 6.6. (Classification) Let \( x : M \to \mathbb{E}^{n+1} \) be an isometric immersion from an \( n \)-dimensional manifold \( M \) in \( \mathbb{E}^{n+1} \). Then the immersion \( x \) is linearly independent if and only if \( M \) is an open portion of minimal hypersurface, a hypersphere \( S^n \), or a spherical hypercylinder \( S^\ell \times \mathbb{E}^{n-\ell}, \ell \in \{1, 2, \ldots, n-1\} \). □

Theorem 6.6 was obtained in [CP], also independently in [HV3].

The class of linearly independent immersions is contained in a much larger class of immersions, namely the class of immersions of restricted type. A submanifold of a Euclidean (or pseudo-Euclidean) space is said to be of restricted type if its shape operator with respect to the mean curvature vector is the restriction of a fixed linear transformation of the ambient space to the tangent space of the submanifold at every point of the submanifold. The notion of immersions of restricted type is first introduced in [CDVV3]. Further results can be found, e.g. in [DVVW] and [BBCD].

§7. Variational minimal principle.

In this section we mention the fundamental relationship between the theory of finite type and calculus of variations obtained in [CDVV2,CDVV4].

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold \( M \) in a Euclidean space \( \mathbb{E}^m \). Associated to each \( \mathbb{E}^m \)-valued vector field \( \xi \) defined on \( M \), there is a deformation \( \phi_t \), defined by

\[
\phi_t(p) := x(p) + t\xi(p), \quad p \in M, \quad t \in (-\epsilon, \epsilon),
\]

where \( \epsilon \) is a sufficiently small positive number. For each \( t \), \( \phi_t \) gives rise to a submanifold \( M_t = \phi_t(M) \). Let \( A(t) \) denote the area of \( M_t \).

Let \( \mathcal{D} \) denote the class of all deformations acting on the submanifold \( M \) and let \( \mathcal{EE} \) denote a nonempty subclass of \( \mathcal{D} \). A compact submanifold \( M \) in \( \mathbb{E}^m \) is said to satisfy the variational minimal principle in the class \( \mathcal{EE} \) if \( M \) is a critical point of the volume functional for all deformations in \( \mathcal{EE} \), i.e., for each deformation in \( \mathcal{EE} \), one has \( A'(0) = 0 \). A compact submanifold \( M \) in \( \mathbb{E}^m \) is called stable in the class \( \mathcal{EE} \) if it satisfies the variational minimal principle in the class \( \mathcal{EE} \) and if \( A''(0) \geq 0 \) for each deformation in \( \mathcal{EE} \).

We consider one type of deformations which occurs frequently, namely the directional deformations. Such deformations occur when an object moves in a fixed direction.
Directional deformations are defined as follows: let $c$ be a fixed vector in $\mathbb{E}^m$ and let $f$ be a smooth function defined on the submanifold $M$. Then we have a deformation given by

$$
\phi^f_t(p) := x(p) + tf(p)c, \quad p \in M, \quad t \in (-\epsilon, \epsilon).
$$

Such a deformation is called a directional deformation in the direction $c$.

For each $q \in \mathbb{N}$, we define $C_q$ to be the class of all directional deformations given by smooth functions $f \in \sum_{i \geq q} V_i$. It is clear that $C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_k \supset \cdots$.

Therefore, if a compact submanifold $M$ of $\mathbb{E}^m$ satisfies the variational minimal principle for one class $C_k$, then it automatically satisfies the variational minimal principle in $C_\ell$ for $\ell \geq k$.

Theorem 7.1. [CDVV2,CDVV4] We have:

(a) There are no compact submanifolds in $\mathbb{E}^m$ which satisfy the variational minimal principle in the classes $C_0$ and $C_1$.

(b) A compact submanifold $M$ of $\mathbb{E}^m$ is of finite type if and only if it satisfies the variational minimal principle in the class $C_q$ for some $q \geq 2$.

Proof. Let $c \in \mathbb{E}^m$ and $f \in C^\infty(M)$. Consider the directional deformation $\phi^f_t$ defined by (7.1). Let $e_1, \ldots, e_n$ be an orthonormal local frame field on $M$. Extended $e_1, \ldots, e_n$ by $(\phi_t)_* e_1, \ldots, (\phi_t)_* e_n$. Put

$$
g_{ij}(t) = ((\phi_t)_* e_i, (\phi_t)_* e_j).
$$

Then

$$
A(t) = \int_M \sqrt{\det(g_{ij}(t))} dA.
$$

On the other hand, from (7.1), we have

$$
(\phi_t)_* e_i = e_i + t(f_i)_c, \quad \frac{\partial}{\partial t} = fc,
$$

where $f_i = e_i f$. From (7.2) and (7.3) we get

$$
g_{ij}(t) = \delta_{ij} + t(f_i)_c(e_j) + f_j \langle c, e_i \rangle + t^2 f_i f_j \langle c, c \rangle.
$$

By using (7.3) and (7.5), we may obtain

$$
A(t) = \int_M \{1 + 2t \langle c, \nabla f \rangle + t^2 (\langle c^+, c^+ \rangle |\nabla f|^2 + \langle c, \nabla f \rangle^2)\}^\frac{1}{2} dA,
$$

where $\langle \ , \ \rangle$ is the inner product of $\mathbb{E}^m$, $c^+$ the normal component of $c$, $\nabla f$ the gradient of $f$ and $dA$ the volume element of $M$.

From (7.6) we obtain

$$
A'(0) = \int_M \langle c, \nabla f \rangle dA.
$$
Let $c^T$ denote the tangential component of $c$ and $(c^T)^\sharp$ the one-form on $M$ dual to $c^T$. Then we have
\begin{equation}
\tag{7.8}
(c^T)^\sharp = dh_c,
\end{equation}
where $h_c$ denotes the height function of $M$ in $\mathbb{E}^m$ with respect to $c$.

Denote by $d$ and $\delta$ the differential and the co-differential operators of $M$. By using (7.8) we find
\begin{equation}
\tag{7.9}
\delta(c^T)^\sharp = \Delta h_c = \langle \Delta x, c \rangle,
\end{equation}
where $x$ is the position vector field of $M$ in $\mathbb{E}^m$. On the other hand, the well-known Beltrami formula yields
\begin{equation}
\tag{7.10}
\Delta x = -nH, \quad n = \dim M,
\end{equation}
where $H$ is the mean curvature vector of $M$ in $\mathbb{E}^m$. Hence, by using (7.7), (7.9) and (7.10), we have
\begin{equation}
\tag{7.11}
A'(0) = \int_M H_c f dA,
\end{equation}
where $H_c = \langle H, c \rangle$.

If $M$ is a compact submanifold in $\mathbb{E}^m$ which satisfies the variational minimal principle in the class $C_1$, then (7.11) yields
\begin{equation}
\tag{7.12}
\langle H_c, f \rangle = \int_M H_c f dA = 0.
\end{equation}
Since (7.12) holds for any $f \in \sum_{t=1}^{\infty} V_t$ and any $c \in \mathbb{E}^m$, the mean curvature vector field $H$ is a constant vector. Therefore, can conclude that $H = 0$, which is a compact. the Weingarten operator $A_H$ with respect to $H$ vanishes. Taking the trace of $A_H = 0$, we obtain $H = 0$, which means that the submanifold would be minimal. This is impossible since $M$ is compact. Consequently, $M$ cannot satisfy the variational minimal principle in the class $C_1$. Hence, it does not satisfy the variational minimal principle in the class $C_0$. This proves (a).

Now we prove (b). First, assume that $M$ is a compact submanifold of finite type. Then the position vector field of $M$ in $\mathbb{E}^m$ has a finite spectral decomposition:
\begin{equation}
\tag{7.13}
x = x_0 + x_{i_1} + \cdots + x_{i_k},
\end{equation}
where $x_0$ is a constant vector and $\Delta x_{i_j} = \lambda_{i_j} x_{i_j}$ for $j = 1, \ldots, k$. Assume $\lambda_{i_1} < \cdots < \lambda_{i_k}$. From (7.10) and (7.13) we obtain
\begin{equation}
\tag{7.14}
-nH = \lambda_{i_1} x_{i_1} + \cdots + \lambda_{i_k} x_{i_k}.
\end{equation}
Equation (7.14) implies that $H_c \in \sum_{j\leq \ell} V_j$, where $\ell$ is the upper order of $M$. Put $q = \ell + 1$. Then (7.11) implies that $A'(0) = 0$ for any deformation in $C_q$. Hence $M$ satisfies the variational minimal principle in $C_q$. Obviously $q \geq 2$. 
Conversely, if $M$ satisfies the variational minimal principal in $C_q$, for some $q \geq 2$, then (7.12) implies that $H_c \in \sum_{i<q} V_i$. Since this holds for any $c$, $H$ has finite spectral decomposition $H = H_0 + H_1 + \cdots + H_{q-1}$, $\Delta H_j = \lambda_j H_j$. We put $P(u) = u \prod_{j=1}^{q-1} (u - \lambda_j)$. Then $P(\Delta)H = 0$. Hence, by applying the minimal polynomial criterion (Theorem 3.1), $M$ is of finite type.

**Theorem 7.2.** Every compact submanifold $M$ of finite type in a Euclidean space is stable in the class $C_q$ for any $q \geq \text{u.o.}(M) + 1$.

**Proof.** Let $M$ be a compact submanifold of finite type in $\mathbb{E}^m$, then, from the proof of Theorem 7.1, we see that $M$ satisfies the variational minimal principal in the class $C_q$ for any $q \geq \text{u.o.}(M) + 1$. On the other hand, from (7.6), we have

$$A''(0) = \int_M \langle c^\perp, c^\perp \rangle |\nabla f|^2 dA \geq 0.$$ 

Therefore, $M$ is stable in the class $C_q$ for any $q \geq \text{u.o.}(M) + 1$. □

Now, we give some consequences of Theorem 7.1.

**Corollary 7.1.** A compact hypersurface in a Euclidean space is a hypersphere if and only if it satisfies the variational minimal principle in the class $C_2$. □

**Corollary 7.2.** Circles in $\mathbb{E}^2$ are the only closed planar curves which satisfy the variational minimal principle in the class $C_q$ for some $q \geq 2$. □

**Corollary 7.3.** Every $k$-th standard immersion of a compact homogeneous Riemannian manifold in $\mathbb{E}^m$ satisfies the variational minimal principle in the class $C_{k+1}$. □

For hypersurfaces in $\mathbb{E}^{n+1}$ we may prove the following.

**Theorem 7.3.** [CDVV4] A compact embedded hypersurface $M$ in $\mathbb{E}^{n+1}$ has finite type Gauss map if and only if the volume enclosed by $M$ is invariant under the directional deformations in a class $C_q$, for some $q \geq 2$.

**Proof.** If $M$ is an embedded hypersurface in $\mathbb{E}^{n+1}$, $M$ can be regarded as the boundary $\partial \Omega$ of a domain $\Omega \subset \mathbb{E}^{n+1}$. For any fixed unit vector $c \in \mathbb{E}^{n+1}$, we choose a Euclidean coordinate system $(u_1, u_2, \ldots, u_{n+1})$ such that $c = (1, 0, \ldots, 0)$. Let $X = u_1 c$. Then $X$ is a vector field defined on $\mathbb{E}^{n+1}$. According to the divergence theorem, we know the volume $V$ enclosed by $M$ is given by

$$V = \int_\Omega (\text{div} X) dV = \int_{M=\partial \Omega} \langle \xi, X \rangle dA,$$

where $\xi$ is the unit outward normal vector field of $M$ in $\mathbb{E}^{n+1}$ and $dA$ denotes the area element of $M$. Let $e_1, \ldots, e_n$ be an positive-oriented orthonormal local frame field of the tangent bundle of $M$. Then, by (7.16), we have

$$V = \int_M u_1 \langle e_1 \times \cdots \times e_n, c \rangle dA,$$
where \( e_1 \times \cdots \times e_n \) denotes the vector product of \( e_1, \ldots, e_n \) in \( \mathbb{E}^{n+1} \).

For a given function \( f \) on \( M \), consider the directional deformation given by
\[
\phi_t(p) = x(p) + tf(p)c, \quad t \in (-\epsilon, -\epsilon).
\]
Let \( (u_1)_t \) denote the \( u_1 \)-component of \( M_t = \phi_t(M) \). Then
\[
(u_1)_t = u_1 + tf, \quad (\phi_t)_* e_i = e_i + t(e_i f)c.
\]
Let \( V(t) \) denote the volume enclosed by \( M_t \). Then, from (7.17) and (7.18), we find
\[
(7.18) \quad V(t) = V(0) + \int_M \langle \xi, c \rangle fdA.
\]
Formula (7.19) implies
\[
(7.20) \quad V'(t) = \int_M \langle \xi, c \rangle fdA.
\]
We recall that the Gauss map \( \nu \) of \( M \) in \( \mathbb{E}^{n+1} \) is given by \( \nu(p) = \xi(p) \) and the Gauss map \( \nu \) is mass-symmetric in the unit hypersphere of \( \mathbb{E}^{n+1} \) centered at the origin.

If the Gauss map \( \nu \) of \( M \) is of finite type, then \( \nu \) has a finite spectral decomposition:
\[
(7.21) \quad \nu = \nu_1 + \cdots + \nu_k
\]
where \( \nu_1, \ldots, \nu_k \) are \( \mathbb{E}^{n+1} \)-valued eigenfunctions of \( \Delta \). Let \( \ell \) is the upper order of the Gauss map \( \mu \). Put \( q = \ell + 1 \). Then (7.20) implies that \( V'(0) = 0 \) for any deformation in the class \( C_q \). Thus, according to (7.19), \( V(t) = V(0) \) for any \( t \in (-\epsilon, -\epsilon) \). This means that the volume enclosed by \( M \) is invariant under directional deformations in the class \( C_q \).

Conversely, if the volume enclosed by \( M \) is invariant under directional deformations in the class \( C_q \), for some \( q \geq 2 \), then \( V(t) = V(0) \) for \( t \in (-\epsilon, -\epsilon) \) and \( f \in \sum_{i \geq 0} V_i \). Thus, by applying (7.19), we get \( \langle \xi, c \rangle \in \sum_{i < q} V_i \). Since this is true for any \( c \in \mathbb{E}n+1 \), we get \( \nu = \nu_1 + \cdots + \nu_{q-1}, \Delta \nu_i = \lambda_1 \nu_i \). We put \( P(t) = \prod_{j=1}^{q-1} (t - \lambda_j) \). Then \( P(\Delta)\nu = 0 \). Therefore, by Theorem 2.2 of [CPi], the Gauss map of \( M \) is of finite type. 

Theorem 7.4. [CDVV4] A compact embedded hypersurface \( M \) in a Euclidean space \( \mathbb{E}^{n+1} \) is a hypersphere if and only if the volume enclosed by \( M \) is invariant under directional deformations in \( C_2 \).

Proof. If \( M \) is a hypersphere of \( \mathbb{E}^{n+1} \), then the Gauss map of \( M \) satisfies \( \Delta \nu = \lambda_1 \nu \), where \( \lambda_1 \) is the first nonzero eigenvalue of \( \Delta \). Thus, from the proof of Theorem 7.3, we see that the volume enclosed by \( M \) is invariant under directional deformations in \( C_2 \).

Conversely, if the volume enclosed by \( M \) is invariant under directional deformations in \( C_2 \), then \( \nu = \nu_1, \Delta \nu_1 = \lambda_1 \nu_1 \), since \( \nu \) is mass-symmetric. In particular,
this implies that the Gauss map is of 1-type. Therefore, by applying Theorem 4.2 of [CP1], $M$ is a hypersphere in $\mathbb{E}^{n+1}$. □

For maps of finite type we have the following results [CDV4].

**Theorem 7.5.** A smooth map $\phi : M \to \mathbb{E}^m$ of a compact Riemannian manifold $M$ in $\mathbb{E}^m$ is of finite type if and only if it satisfies the variational minimal principle in the class $\mathcal{C}_q$ for some $q \in \mathbb{N}$. □

**Theorem 7.6.** A smooth map $\phi : M \to \mathbb{E}^m$ of a compact Riemannian manifold $M$ in $\mathbb{E}^m$ is a constant map if and only if it satisfies the variational minimal principle in the class $\mathcal{C}_1$. □
Chapter III: LAPLACE MAPS OF SMALL RANK

§1. Curves in $\mathbb{E}^m$.

In this section we discuss the Laplace map of a regular curve in a Euclidean $m$–space.

Consider a regular curve $\beta : I \rightarrow \mathbb{E}^m$ from an open interval $I$ into $\mathbb{E}^m$ parametrized by the arclength $s$. Then the Laplacian operator $\Delta$ of $\beta$ is given by $\Delta = -\frac{d^2}{ds^2}$. Therefore, the Laplace map of $\beta$ is given by $L(s) = \Delta \beta = -\beta''(s)$. Let $t = \beta' = \frac{d\beta}{ds}$ be the unit tangent vector of $\beta$. The differential of the Laplace map $L$ is given by $L^*(t) = -\beta'''(s)$. Recall that the curve $\beta$ is said to be regular if $\beta'(s)$ is nowhere zero on $I$. In this monograph, by a 3–regular curve we mean a regular curve $\beta : I \rightarrow \mathbb{E}^m$ such that $\beta'''(s)$ is nowhere zero.

If $\beta(s)$ is a regular plane curve, then there is a unique unit normal vector field $n(s)$ such that $\{t(s), n(s)\}$ gives a right handed orthonormal basis of $\mathbb{E}^2$ for each $s$. The plane curvature $\kappa(s)$ of $\beta$ is defined by $\kappa(s) = \langle \beta''(s), n(s) \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{E}^2$. It is easy to see that a regular plane curve is 3–regular if the plane curvature and its derivative do not vanish simultaneously. Clearly, there exist regular 3-regular plane curves whose derivative of its planar curvature vanishes at a point; e.g. the plane polynomial spirals (cf. [D1]) with curvature function $\kappa(s) = as^2 + bs + c$ such that the discriminant $b^2 - 4ac \neq 0$.

For curves in higher dimensional Euclidean spaces, we explain the Frenet curvatures and Frenet vectors of a regular curve $\beta : I \rightarrow \mathbb{E}^m$ as follows.

Let $\beta_1 = \beta'$ be the unit tangent vector and put $\kappa_1 = \|\nabla_{\beta'} \beta_1\|$. If $\kappa_1$ vanishes on $I$, then the curve $\beta$ is said to be of rank one. If $\beta_1$ is not identically zero, then we define $\beta_2$ by

$$\nabla_{\beta'} \beta_1 = \kappa_1 \beta_2, \quad \text{on} \quad I_1 = \{s \in I : \kappa_1(s) \neq 0\},$$

$\beta_2$ is called the principal normal vector of $\beta$.

Put

$$\kappa_2 = \|\nabla_{\beta'} \beta_2 + \kappa_1 \beta_1\|.$$  

If $\kappa_2 \equiv 0$ on $I_1$, then $\beta$ is said to be of rank 2. If $\kappa_2$ is not identically zero on $I_1$, then we define $\beta_3$ by

$$\nabla_{\beta'} \beta_2 = -\kappa_1 \beta_1 + \kappa_2 \beta_3,$$

on $I_2 = \{s \in I : \kappa_2(s) \neq 0\}$. For $m = 3$, $\beta_3$ is called the binormal vector of $\beta$. Inductively, we put

$$\kappa_i = \|\nabla_{\beta'} \beta_i + \kappa_{i-1} \beta_{i-1}\|.$$
and if \( \kappa_i \equiv 0 \) on \( I_{i-1} \), then \( \beta \) is said to be of rank \( i \). \( \kappa_i \) is then called the \( i \)-th Frenet curvature of \( \beta \) and \( \beta_i \) the \( i \)-th Frenet vector.

A curve in \( \mathbb{E}^m \) is called a \( W \)-curve if its Frenet curvatures are constant.

**Lemma 1.1.** If \( \beta \) is a regular curve in \( \mathbb{E}^m \) whose first Frenet curvature is nowhere zero, then the Laplace map of \( \beta \) is regular.

**Proof.** Let \( s \) denote an arclength parameter of the curve \( \beta \). Then \( t = \frac{d\beta}{ds} \) is a unit tangent vector field of \( \beta \). Denote by \( \kappa_i \) and \( \beta_i \) the \( i \)-th Frenet curvature and \( i \)-th Frenet normal vector of \( \beta \), respectively. Then we have

\[
L(s) = -\beta'' = -\kappa_1 \beta_2.
\]

Therefore,

\[
(1.5) \quad L \ast (t) = -\beta'' = \kappa_1(s)^2 \beta_1 - (\kappa_1(s))' \beta_2 - \kappa_1(s)\kappa_2(s)\beta_3
\]

where the last term occurs only when \( m \geq 3 \). From (1.5), we obtain the Lemma. \( \square \)

**Proposition 1.2.** Let \( \beta(s) \) be a curve parametrized by arclength \( s \) in \( \mathbb{E}^m \). Then the Laplace transformation of \( \beta \) is homothetic if and only if (1) \( \beta \) has nonzero first Frenet curvature function \( \kappa_1(s) \) and (2) the first and the second Frenet curvature functions satisfy the relation: \( \kappa_1^4 + (\kappa_1')^2 + \kappa_2^2 = c \) for some positive constant \( c \).

**Proof.** From (1.5) we have

\[
(1.6) \quad \langle dL(t), dL(t) \rangle = \kappa_1^4 + (\kappa_1')^2 + \kappa_2^2.
\]

From (1.6) we obtain the Proposition. \( \square \)

In particular, if \( \beta \) is a planar curve, this Proposition yields the following.

**Corollary 1.3.** A planar curve \( \beta(s) \) has homothetic Laplace transformation if and only if its curvature function \( \kappa \) satisfies \( \kappa^4 + (\kappa')^2 = c \) for some positive constant. \( \square \)

**Remark 1.1.** We will come back to this situation in Chapter IV, section 3. \( \square \)

From the Proposition 1.2 we also have the following

**Corollary 1.4** Every \( W \)-curve in \( \mathbb{E}^m \) has homothetic Laplace transformation.

§2. Laplace maps of small rank.

Let \( x : M^n \to \mathbb{E}^m \) be an isometric immersion. Denote by \( dL \) the differential of the Laplace map \( L : M^n \to \mathbb{E}^m \) associated with the isometric immersion \( x \).
The main purpose of this section is to study isometric immersions whose Laplace map has rank \( < n = \dim M \).

First we give the following result which classifies submanifolds whose Laplace map is of rank 0.

**Proposition 2.1.** Let \( x : M^n \to \mathbb{E}^m \) be an isometric immersion. Then the Laplace image \( L(M^n) \) of the immersion \( x \) is a point if and only if \( x \) is a minimal immersion.

**Proof.** If the Laplace image is a point, then the mean curvature vector field \( H \) is a constant vector, say \( c \). Thus, by the Weingarten formula, we have

\[
\nabla_X H = -A_H X + D_X H = 0
\]

for any vector \( X \) tangent to \( M^n \). Thus \( A_H = 0 \) which implies that \( H = c = 0 \). The converse is trivial. \( \square \)

For hypersurfaces with singular Laplace map we have the following.

**Proposition 2.2.** Let \( x : M^n \to \mathbb{E}^{n+1} \) be an isometric immersion. Then the Laplace map of \( x \) satisfies \( \text{rank}(dL) \equiv k \), for some \( k \) with \( 0 < k < n \), if and only if \( M^n \) is foliated by \( (n-k) \)-dimensional submanifolds such that

(a) the second fundamental form \( h \) of \( M^n \) in \( \mathbb{E}^{n+1} \), restricted to each leaf \( N^{n-k} \), vanishes identically and

(b) the mean curvature vector \( H \) of \( x \) is constant along each leaf \( N^{n-k} \).

**Proof.** Assume that the Laplace map of \( x \) satisfies \( \text{rank}(dL) \equiv k \), \( 0 < k < n \). Then, for any \( k+1 \) vectors \( X_1, \ldots, X_{k+1} \) tangent to \( M^n \), we have

\[
\nabla_{X_1} H \wedge \ldots \wedge \nabla_{X_{k+1}} H = 0.
\]

Let \( e_1, \ldots, e_n \) be an orthonormal frame such that \( e_i, i = 1, \ldots, n \), are principal vectors of \( M^n \) in \( \mathbb{E}^{n+1} \) with principal curvatures \( \kappa_i, i = 1, \ldots, n \), respectively. Then we have

\[
\nabla_{e_i} H = -\alpha \kappa_i e_i + (e_i \alpha) e_{n+1},
\]

where \( H = \alpha e_{n+1} \). Thus from (2.1) and the fact that \( \alpha \neq 0 \) by the previous Proposition, we may assume that the orthonormal frame \( e_1, \ldots, e_n \) satisfies

\[
\kappa_1 \ldots \kappa_k \kappa_r = 0, \quad \kappa_1 \ldots \kappa_k \kappa_r \alpha = 0, \quad r = k+1, \ldots, n.
\]

Consequently, also taking into account that \( \text{rank}(dL) \equiv k \), without loss of generality, we may assume that one of the following two cases occurs:

(2.2) \( \kappa_1, \ldots, \kappa_k \neq 0, \quad \kappa_{k+1} = \ldots = \kappa_n = 0, \quad e_{k+1} \alpha = \ldots = e_n \alpha = 0, \)

or

(2.3) \( \kappa_1, \ldots, \kappa_{k-1} \neq 0, \quad \kappa_k = \ldots = \kappa_n = 0, \quad e_k \alpha \neq 0, \quad e_{k+1} \alpha = \ldots = e_n \alpha = 0. \)
From (2.2) and (2.3) we have

\[
\tilde{\nabla}_{e_{k+1}} H = \ldots = \tilde{\nabla}_{e_n} H = 0.
\]

Let \( \mathcal{D} = \{ Z \in TM^n : \tilde{\nabla}_Z H = 0 \} \) and let \( Z, W \) be any two vector fields in \( \mathcal{D} \). Then from the Codazzi equation, we have

\[
B([Z, W]) = B(\nabla_Z W) - B(\nabla_W Z) = (\nabla_W B)Z - (\nabla_Z B)W = 0,
\]

where \( B = A_{e_{n+1}} \). From the definition of \( \mathcal{D} \) we also have

\[
[Z, W] \alpha = ZW \alpha - WZ \alpha = 0.
\]

From (2.5) and (2.6) we conclude that \( \mathcal{D} \) is completely integrable. So \( M^n \) is foliated by \((n - k)\)-dimensional submanifolds. Furthermore, from the definition of \( \mathcal{D} \), we see that each leaf of \( \mathcal{D} \) satisfies conditions (a) and (b). The converse is clear. \( \square \)

In particular, if \( \text{rank}(dL) \equiv 1 \), we have the following.

**Proposition 2.3.** Let \( x : M^n \to \mathbb{E}^{n+1} \) be an isometric immersion. Then the Laplace map of \( x \) satisfies \( \text{rank}(dL) \equiv 1 \) if and only if \( M^n \) is locally a hypercylinder on a \( 3 \)-regular curve.

**Proof.** Assume that \( \text{rank}(L_*) \equiv 1 \). Then, from the proof of Proposition 2.2, we see that there exists an orthonormal frame \( e_1, \ldots, e_n \) of principal vectors such that their corresponding principal curvatures satisfy the following conditions:

\[
\kappa_1 = n \alpha, \quad \kappa_2 = \ldots = \kappa_n = 0, \quad e_2 \alpha = \ldots = e_n \alpha = 0.
\]

Let \( \mathcal{D} = \{ Z \in TM : \tilde{\nabla}_Z H = 0 \} \). Then \( \mathcal{D} \) is integrable from the proof of Proposition 3.2. Let \( X \) be any vector field in \( D^\perp = \text{Span}\{e_1\} \) and \( Z \in \mathcal{D} \). Then we have

\[
\nabla_Z (BX) = (\nabla_Z B)X + B(\nabla_Z X) = (\nabla_X A)Z + B(\nabla_Z X) = B(\nabla_Z X) - B(\nabla_X Z) = B([Z, X]).
\]

This implies that \( \nabla_Z e_1 \in \mathcal{D} \). Consequently, each leaf of \( \mathcal{D} \) is totally geodesic in \( \mathbb{E}^{n+1} \); and hence it is an open portion of a linear subspace of \( \mathbb{E}^{n+1} \). Since \( D^\perp \) is of rank one, \( \mathcal{D}^\perp \) is trivially integrable. Because the second fundamental form of \( M^n \) in \( \mathbb{E}^{n+1} \) satisfies \( h(D, D^\perp) = \{0\} \), a lemma of Moore implies that \( M^n \) is locally a hypercylinder on a planar curve. Moreover, since \( \text{rank}(dL) \equiv 1 \), this curve is \( 3 \)-regular. The converse is easy to verify. \( \square \)

For surfaces in a Euclidean space of higher codimension, we have the following.

**Proposition 2.4.** Let \( x : M^2 \to \mathbb{E}^m \) be an isometric immersion. If the Laplace map of \( x \) satisfies \( \text{rank}(dL) \equiv 1 \), then \( M^2 \) is non-positively curved, i.e., the Gauss curvature \( K \) of \( M^2 \) is \( \leq 0 \).

In particular, if \( K = 0 \), then \( M^2 \) is a cylinder on a \( 3 \)-regular curve.
Proof. Let \( x : M^2 \to \mathbb{E}^m \) be an isometric immersion whose Laplace map satisfies \( \text{rank} \,(dL) \equiv 1 \). Then we have

\[
(2.8) \quad \tilde{\nabla}_{X_1} H \wedge \tilde{\nabla}_{X_2} H = 0,
\]
for any vectors \( X_1, X_2 \) tangent to \( M^2 \).

Let \( U = \{ p \in M^2 : H(p) \neq 0 \} \). Then \( U \) is an open subset of \( M^2 \). It is easy to see from the Gauss equation that the Gauss curvature \( K \leq 0 \) on \( M - U \).

On \( U \), we choose an orthonormal frame \( e_1, \ldots, e_m \), such that, restricted to \( U \), \( e_1 \) and \( e_2 \) are tangent to \( M^2 \) and \( H = \alpha e_3 \). Furthermore, we may also assume that 
\[
A_H e_1 = \kappa_1 e_1 \quad \text{and} \quad A_H e_2 = \kappa_2 e_2.
\]
From (2.8) we have
\[
(-\alpha \kappa_1 e_1 + (e_1 \alpha) e_3 + \alpha D_{e_1} e_3) \wedge (-\alpha \kappa_2 e_2 + (e_2 \alpha) e_3 + \alpha D_{e_2} e_3) = 0.
\]
From this we may assume without loss of generality the following:

\[
(2.9) \quad \kappa_1 \neq 0, \quad \kappa_2 = 0, \quad e_2 \alpha = D_{e_2} e_3 = 0. \]

Since \( e_3 \) is parallel to the mean curvature vector \( H \), we have trace \((A_r) = 0\) for \( r = 4, \ldots, m \). Thus, by (2.9), the Gauss curvature \( K \) of \( M^2 \) is \( \leq 0 \).

In particular, if \( K \equiv 0 \), then we have

\[
(2.10) \quad A_3 = \begin{pmatrix} 2\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \ldots = A_m = 0.
\]

From (2.10) have

\[
(2.11) \quad h(e_1, e_1) = 2\alpha e_3 \quad h(e_1, e_2) = h(e_2, e_2) = 0.
\]

From (2.9), (2.11) we may obtain

\[
(2.12) \quad (\tilde{\nabla}_{e_1} h)(e_1, e_2) = -\omega^2_1 (e_1) h(e_1, e_1), \quad (\tilde{\nabla}_{e_2} h)(e_1, e_1) = 0.
\]

By (3.10) and the equation of codazzi we obtain

\[
(2.13) \quad \omega_1^2 = f \omega^2,
\]
for some function \( f \) on \( M^2 \). By taking the exterior derivative of \( \omega_3^2 = 0 \) and by applying (2.8) and (2.13), we may get \( f = 0 \). This implies \( \nabla e_1 = \nabla e_2 = 0 \). Hence, by combining this with (2.11), we see that each integral curve of \( e_2 \) is an open portion of a straight line in \( \mathbb{E}^m \) and so \( M^2 \) is a cylinder on a 3–regular curve. □

Remark. There do exist negatively-curved surfaces in \( \mathbb{E}^m \) whose Laplace maps \( L \) satisfy \( \text{rank}(dL) \equiv 1 \). □

From Proposition 2.2 and Proposition 2.4, we obtain the following

Corollary 2.5. Let \( x : M^n \to \mathbb{E}^m \) be an isometric immersion of a compact Riemannian manifold. Then
(1) if \( m = n + 1 \), the Laplace map is regular (i.e., it is of maximal rank) on a non-empty open submanifold of \( M^n \) and

(2) if \( n = 2 \) and \( M^2 \) is diffeomorphic to a sphere, a real projective plane, a Klein bottle or a torus, then the Laplace map is regular on a non-empty open subset. □

§3. Ruled surfaces in \( \mathbb{E}^m \).

In this section we study the Laplace map of ruled surfaces in \( \mathbb{E}^m \).

**Theorem 3.1.** Let \( M^2 \) be a ruled surface in a Euclidean \( m \)-space \( \mathbb{E}^m \). Then the restriction of \( dL \) in the direction of the rulings vanishes if and only if either \( M^2 \) is an open portion of a helicoid or it is an open portion of a cylinder over a curve.

**Proof.** We consider the two cases separately.

**Case 1.** \( M \) is a cylinder.

Suppose that the surface \( M \) is a cylinder over a curve \( \gamma \) in an affine hyperplane \( \mathbb{E}^{n-1} \), which we can choose to have the equation \( x_m = 0 \). Assume that \( \gamma \) is parametrized by its arc length \( s \). Then a parametrization of \( M \) is given by

\[
(3.1) \quad x(s, t) = \gamma(s) + te_m.
\]

The Laplacian \( \Delta \) of \( M \) is given in terms of \( s \) and \( t \) by

\[
(3.2) \quad \Delta = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}.
\]

Thus the Laplace map \( L \) of the cylinder is given by \( L(s, t) = -\gamma''(s) \); and hence \( dL(\frac{\partial}{\partial s}) = 0 \) i.e., the restriction of \( dL \) in the direction of the rulings vanishes identically.

**Case 2:** \( M \) is not cylindrical.

If the ruled surface \( M \) is not cylindrical, we can decompose \( M \) into open pieces such that on each piece we can find a parametrization \( x \) of the form:

\[
(3.3) \quad x(s, t) = \alpha(s) + t\beta(s)
\]

where \( \alpha \) and \( \beta \) are curves in \( \mathbb{E}^m \) such that

\[
\langle \alpha', \beta \rangle = 0, \quad \langle \beta, \beta \rangle = 1, \quad \langle \beta', \beta' \rangle = 1.
\]

We have \( x_s = \alpha' + t\beta' \) and \( x_t = \beta \). We define functions \( q, u \) and \( v \) by

\[
(3.4) \quad q = \|x_s\|^2 = t^2 + 2ut + v, \quad u = \langle \alpha', \beta' \rangle, \quad v = \langle \alpha', \alpha' \rangle.
\]
The Laplacian $\Delta$ of $M$ can be expressed as follows:

\[ \Delta = \frac{\partial^2}{\partial t^2} - \frac{1}{q} \frac{\partial^2}{\partial s^2} + \frac{1}{2} \frac{\partial q}{\partial s} \frac{\partial^2}{\partial s^2} - \frac{1}{2} \frac{\partial q}{\partial t} \frac{\partial}{\partial t}. \tag{3.5} \]

From (3.3) and (3.5) we see that the Laplace map is given by

\[ L(s,t) = \frac{1}{2q^2} \{ -2qq''(s) + q_s \alpha'(s) - 2tq\beta''(s) + q_s \beta'(s) - qtq\beta(s) \} \]

where $q_s = \frac{\partial q}{\partial s}$, $q_t = \frac{\partial q}{\partial t}$. This implies that $dL(\frac{\partial q}{\partial t}) = 0$ if and only if

\[ 2qq'' + (qq_{st} - 2q_s q_t)\alpha' + 2q(tq_t - q)\beta'' + (qq_{st} - 2q_s q_t)\beta' + q(q_t^2 - qq_{tt})\beta = 0, \tag{3.6} \]

where $q_s = \frac{\partial q}{\partial s}$, $q_{st} = \frac{\partial^2 q}{\partial s \partial t}$, and $q_{tt} = \frac{\partial^2 q}{\partial t^2}$.

From (3.4) and (3.6) we obtain

\[ (\beta'' + \beta')t^4 + 2(2u\beta + u\beta'' - \alpha'')t^3 - 3(2u\alpha'' - 2u^2 \beta + u' \alpha' + u' \beta')t^2 - \{2(v'' + u')\alpha' + 2uv\beta'' + 2(v' + uu')\beta'\}t - \{2uv'' - u' v)\alpha' + v^2 \beta'' + 2uv\alpha'' + (2uv' - u' v)\beta' + (v^2 - 2u^2 v)\beta \} = 0. \tag{3.7} \]

From (3.7) we obtain

\[ \beta'' + \beta = 0, \tag{3.8} \]

\[ \alpha'' = u\beta, \tag{3.9} \]

\[ u'(\alpha' + \beta') = (uu' + v')(\alpha' + \beta') = 0, \tag{3.10} \]

\[ (2uv' - u' v)(\alpha' + \beta') = 0. \tag{3.11} \]

If $\alpha' + \beta' = 0$, then $\alpha(s) = -\beta(s) + c$ for some constant vector $c \in \mathbb{E}^m$. On the other hand, (3.8) implies that $\beta$ is a unit speed curve of 1--type. Hence, $\beta$ is a plane circle. Consequently, by using $\alpha(s) = -\beta(s) + c$, we may conclude that $x(s,t) = \alpha(s) + t\beta(s)$ is an open portion of a plane which is a special case of cylinder.

If $\alpha' + \beta' \neq 0$, then (3.10) yields

\[ v' = -uu', \quad u'v = -2uv. \]

Thus $(2u^2 + v)u' = 0$. Since $v = \langle \alpha', \alpha' \rangle > 0$, $u$ is a constant. Hence

\[ \alpha'' = -\lambda \beta. \tag{3.12} \]
for some nonzero constant $\lambda$. From (3.8), (3.12) and $\langle \beta, \beta \rangle = \langle \beta', \beta \rangle = 1$, we may choose a Euclidean coordinate system on $E^m$ such that $\alpha$ and $\beta$ are given respectively by

\[(2.13) \quad \beta(s) = (\cos s, \sin s, 0, \ldots, 0).\]

\[(3.14) \quad \alpha(s) = (\lambda \cos s + c_1 s, \lambda \sin s + c_2 s, c_3 s, 0, \ldots, 0)\]

for some constants $c_1, c_2, c_3$. From (3.13) and (3.14) we see that the ruled surface $x(s, t) = \alpha(s) + t\beta(s)$ is an open portion of a helicoid. Conversely, since a helicoid is a minimal surface, its Laplace map $L$ is a constant map. Thus $dL = 0$ identically. □

In the remaining part of this section we investigate the Laplace map of flat ruled surfaces in $E^m$. It is well known that flat ruled surfaces in $E^m$ are “in general” cylinders, cones or tangential developables of curves. As we already have seen, the differential $dL$ of the Laplace map of a cylinder in $E^m$ has rank $\leq 1$.

Now, we study the Laplace maps of cones and tangential developables.

**Proposition 3.2.** The Laplace map of a cone in $E^m (m \geq 3)$ is a cone.

**Proof.** Let $x : M \rightarrow E^m$ be a cone in $E^m$. Without loss of generality, we may assume the cone has its vertex at the origin and it is parametrized by

\[(3.15) \quad x(t, s) = t\beta(s), \quad |\beta| = |\beta'| = 1.\]

Put

\[(3.16) \quad e_1 = \frac{1}{t} \frac{\partial}{\partial s} \quad e_2 = \frac{\partial}{\partial t}.\]

Then, by direct computation, the second fundamental form $h$ of $M$ in $E^m$ satisfies

\[(3.17) \quad h(e_1, e_1) = \frac{1}{t} \beta'' - \frac{1}{t} \langle \beta'', \beta \rangle \beta, \quad h(e_2, e_2) = 0.\]

From (3.17) we see that the Laplace map of $x$ is given by

\[L(t, s) = \frac{1}{t} \langle (\beta''(s), \beta(s)) \beta(s) - \beta''(s) \rangle.\]

On the other hand, (3.15) yields $\langle \beta'', \beta \rangle = -1$. Thus

\[(3.18) \quad L(t, s) = -\frac{1}{t} (\beta(s) + \beta''(s)),\]

which implies that the Laplace map $L$ is also a cone with vertex at the origin. □

**Proposition 3.3.** The Laplace map of a tangential developable surface in $E^m (m \geq 3)$ is a cone.

**Proof.** Assume $M$ is a tangential developable surface in $E^m$ given by the tangent lines of a unit speed curve $\beta(s)$ in $E^m$. Then $M$ is parametrized by

\[(3.19) \quad x(s, t) = \beta(s) + t\beta'(s), \quad |\beta'(s)| = 1.\]
Let $\kappa_1$ denote the first Frenet curvature of $\beta$ in $\mathbb{E}^m$.

Put

$$e_1 = \frac{1}{t\kappa_1} \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right), \quad e_2 = \frac{\partial}{\partial t}. \tag{3.20}$$

Then, by a direct computation, the second fundamental form $h$ of $M$ in $\mathbb{E}^m$ satisfies

$$h(e_1, e_1) = \frac{1}{t\kappa_1} \beta_3, \quad h(e_2, e_2) = 0, \tag{3.21}$$

where $\beta_3$ is the third Frenet vector. (3.21) implies that the Laplace map of $M$ is given by

$$L(s, t) = -\frac{1}{t\kappa_1} \beta_2(s). \tag{3.22}$$

Therefore, the Laplace map is a cone with vertex at the origin. □
Chapter IV: HOMOTHETIC LAPLACE TRANSFORMATIONS.

§1. Some general results.

Let \( x : M \rightarrow \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional connected Riemannian manifold \( M \) into a Euclidean \( m \)-space. Denote by \( L : M^n \rightarrow \mathbb{E}^m \) the Laplace map and by \( L(M^n) \) the Laplace image of the immersion \( x \). Recall that the transformation \( L : M \rightarrow L(M) \) from \( M^n \) onto its Laplace image \( L(M) \) is called the Laplace transformation of the immersion \( x \).

The main purpose of this section is to give some general properties concerning isometric immersions \( x : M^n \rightarrow \mathbb{E}^m \) with homothetic Laplace transformation.

First we give the following general results.

**Lemma 1.1.** Let \( x : M \rightarrow \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional Riemannian manifold into \( \mathbb{E}^m \). Then the Laplace transformation \( L : M \rightarrow L(M) \) is homothetic if and only if

\[
\langle A_H X, A_H Y \rangle + \langle D_X H, D_X H \rangle = c^2 \langle X, Y \rangle
\]

holds for all vectors \( X, Y \) tangent to \( M \), where \( c \) is a positive constant.

**Proof.** Because the Laplace map is given by \( L(p) = (\Delta x)(p) = -nH(p) \), the differential of the Laplace map satisfies

\[
dL(X) = nA_H X - nD_X H.
\]

From (1.2) we obtain

\[
\langle dL(X), dL(Y) \rangle = \langle A_H X, A_H Y \rangle + \langle D_X H, D_H Y \rangle
\]

which implies the Lemma. \( \Box \)

In the following, by \( S^{m-1}(r) \) we mean the hypersphere of \( \mathbb{E}^m \) centered at the origin and with radius \( r \).

**Lemma 1.2.** If

\[
x : M \rightarrow S^{m-1}(r) \subset \mathbb{E}^m
\]

is an isometric immersion which immerses \( M \) into the hypersphere \( S^{m-1}(r) \) as a minimal submanifold, then \( x \) has homothetic Laplace transformation.

**Proof.** If

\[
x : M \rightarrow S^{m-1}(r) \subset \mathbb{E}^m
\]

is an isometric immersion which immerses \( M \) into the hypersphere \( S^{m-1}(r) \) as a minimal submanifold, then the Laplace map \( L \) of \( x \) is given by \( L = \frac{\Delta}{\Delta} x \). Thus, \( dL(X) = \frac{\Delta}{\Delta} X \) for any vector \( X \) tangent to \( M \). Hence, \( x \) has homothetic Laplace transformation. \( \Box \)
Lemma 1.3. Let \( x : M \to \mathbb{E}^m \) and \( y : N \to \mathbb{E}^r \) be two isometric immersions whose Laplace maps are given by \( L_M : M \to \mathbb{E}^m \) and \( L_N : N \to \mathbb{E}^r \), respectively. Then

1. the Laplace map of the product immersion \((x, y) : M \times N \to \mathbb{E}^{m+r}\) is given by \( (L_M, L_N) : M \times N \to \mathbb{E}^{m+r} \);
2. the Laplace image of the product immersion \((x, y) : M \times N \to \mathbb{E}^{m+r}\) is given by \( L_M (M) \times L_N (N) \); and
3. the Laplace transformation \( L_{M \times N} : M \times N \to L_{M \times N} (M \times N) \) of the product immersion is homothetic if and only if the Laplace transformations of \( x \) and \( y \) are homothetic transformations with the same constant homothetic factor.

Proof. This Lemma follows easily from the fact that the Laplace operator of the Riemannian product \( M \times N \) of two Riemannian manifolds \( M \) and \( N \) is given by

\[
\Delta_{M \times N} = \Delta_M \times \Delta_N. \quad \square
\]

Lemma 1.2 and Lemma 1.3 show that there exist ample examples of submanifolds with homothetic Laplace transformation.

If \( x : M \to \mathbb{E}^m \) is an isometric immersion of a Riemannian manifold \((M, g)\) with Riemannian metric \( g \) and \( c \) a positive number, then the immersion \( x^c \) defined by \( (x^c)(p) = cx(p) \), for \( p \in M \), is an isometric immersion from the Riemannian manifold \((M, c^2 g)\) into \( \mathbb{E}^m \).

Lemma 1.4. Let \( x : M \to \mathbb{E}^m \) be an isometric immersion. Then the Laplace map \( L^c : M \to \mathbb{E}^m \) of the isometric immersion \( x^c \) is given by \( L^c (p) = c^{-2} L(p) \) for any point \( p \in M \).

Proof. Follows from the fact that the Laplace operator \( \Delta^c \) of \((M, c^2 g)\) and the Laplace operator \( \Delta \) of \((M, g)\) are related by

\[
\Delta^c = c^{-2} \Delta. \quad \square
\]

Let \( y_i : M \to \mathbb{E}^{m_i} \), \( i = 1, \ldots, k \), be \( k \) maps of \( M \) into \( \mathbb{E}^{m_i} \), respectively, and let \( a_1, \ldots, a_k \) be \( k \) positive numbers. Then

\[
y(p) = (a_1 y_1 (p), \ldots, a_k y_k (p)), \quad p \in M,
\]

is a map of \( M \) into \( \mathbb{E}^{m_1 + \ldots + m_k} \), which is called a diagonal map of \( y_1, \ldots, y_k \). In particular, if \( y_1, \ldots, y_k \) are isometric immersions and \( a_1, \ldots, a_k \) satisfy \( a_1^2 + \ldots + a_k^2 = 1 \), then the diagonal map \( y \) of \( y_1, \ldots, y_k \) is an isometric immersion, called a diagonal immersion of \( y_1, \ldots, y_k \).
Lemma 1.5. Let \( x : M \to \mathbb{E}^m \) be a diagonal immersion of \( k \) isometric immersions. Then the Laplace map \( L : M \to \mathbb{E}^m \) of \( x \) is a diagonal map.

Proof. From the definition of diagonal immersion, we have
\[
(\Delta x)(p) = (a_1 (\Delta x_1)(p), \ldots, a_k (\Delta x_k)(p))
\]
which implies that the Laplace map \( L \) of the diagonal immersion \( x \) is related to the Laplace map \( L_i \) of \( x_i \) by
\[
L(p) = (a_1 L_1(p), \ldots, a_k L_k(p)), \quad p \in M.
\]
Thus, the Laplace map of a diagonal immersion is also a diagonal map.  

If \( M \) is a compact (connected) Riemannian homogeneous manifold, let \( G = \text{Iso}(M) \) be the identity component of the group of all isometries of \( M \). \( G \) is a compact Lie group acting on \( M \) transitively. A map \( x : M \to E^m \) from \( M \) into \( E^m \) is said to be equivariant if there exists a Lie homomorphism \( \varphi \) of \( G \) into the isometry group \( I(E^m) \) of \( E^m \) such that
\[
x(\sigma(p)) = \varphi(\sigma)(x(p))
\]
for any \( \sigma \in G \) and \( p \in M \).

For the Laplace map of an equivariant isometric immersion of a compact homogeneous space, we have the following general result.

Lemma 1.6. Let \( x : M \to \mathbb{E}^m \) be an equivariant isometric immersion of a compact Riemannian homogeneous manifold \( M \) into \( \mathbb{E}^m \). Then the Laplace map \( L : M \to \mathbb{E}^m \) of \( x \) is also equivariant.

Proof. This lemma follows from the fact that the Laplacian operator \( \Delta \) is a Riemannian invariant.  

Lemma 1.6 implies the following.

Lemma 1.7. Let \( x : M \to \mathbb{E}^m \) be an equivariant isometric immersion of a compact Riemannian homogeneous manifold \( M \) into \( \mathbb{E}^m \). Then the immersion \( x \) has constant mean curvature function and the Laplace map \( L : M \to \mathbb{E}^m \) of \( x \) is spherical.  

For equivariant immersions we also have the following

Theorem 1.8. Let \( x : M^n \to \mathbb{E}^m \) be an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold. Then the associated Laplace transformation \( \mathcal{L} : M^n \to L(M^n) \) is a homothetic transformation.

Proof. Let \( x : M \to \mathbb{E}^m \) be an equivariant isometric immersion of a compact connected Riemannian homogeneous space \( M \) into \( \mathbb{E}^m \). Without loss of generality we may assume the immersion is full. From Lemma 1.7, the immersion \( x \) is
spherical. Thus \( x(M) \) is contained in a hypersphere \( S^{m-1} \) of \( \mathbb{E}^m \). Without loss of generality, we may also assume that \( S^{m-1} \) is centered at the origin of \( \mathbb{E}^m \). Then there is a Lie homomorphism \( \phi : G \to SO(\mathbb{E}^m) \) such that \( x(g(p)) = \phi(g)(x(p)) \) for every \( g \in G \) and \( p \in M \). Because \( (\phi, \mathbb{E}^m) \) is a representation of the compact Lie group \( G, (\phi, \mathbb{E}^m) \) is the direct sum of some irreducible subrepresentations \( (\phi_1, E_1), \ldots, (\phi_k, E_k) \) such that \( \mathbb{E}^m \) is the Euclidean direct sum \( E_1 \oplus \ldots \oplus E_k \) of \( E_1, \ldots, E_k \). Let \( x_i \) denote the \( E_i \)-component of \( x \). Then we have

\[
(1.7) \quad x_i(g(p)) = \phi_i(g)(x_i(p)), \quad g \in G, \; p \in M, \; i = 1, \ldots, k,
\]

where \( E_1, \ldots, E_k \) are mutually orthogonal in \( \mathbb{E}^m \).

Now we claim that each \( x_i \) is a 1-type map, that is, \( \Delta x_i = \lambda_i x_i, \; i \in \{1, \ldots, k\} \), for some real numbers \( \lambda_i \). In order to do so, we choose a fixed point \( o \in M \). Denote by \( K \) the isotropy subgroup of \( G \) at \( o \). Then \( M \) can be identified with \( G/K \) in a natural way. Consider a biinvariant Riemannian metric on the compact Lie group \( G \) such that the projection \( \pi : G \to M = G/K \) is a Riemannian submersion. Let \( e_1, \ldots, e_N \) be any orthonormal basis of the Lie algebra \( \mathfrak{g} = T_e G \) of \( G \), where \( e \) is the identity element of \( G \).

For each \( i \in \{1, \ldots, k\} \), denote also by \( \phi_i \) the homomorphism \( \mathfrak{g} \to \mathfrak{so}(E_i) \) induced from \( \phi_i : G \to E_i \), where \( \mathfrak{so}(E_i) \) is the Lie algebra of \( SO(E_i) \). Then each \( \phi_i(e_a) \) is a skew-symmetric linear transformation of \( E_i \) and \( \sum_{a=1}^N \phi_i(e_a)^2 \) is a symmetric linear transformation. Let \( Ad : G \to GL(\mathfrak{g}) \) be the adjoint representation of \( G \). Since \( ad(g)(h) = ghg^{-1} \) and \( \phi_i \) is a Lie homomorphism, we have

\[
\phi_i(g)\phi_i(X)\phi_i(g^{-1}) = \phi_i(Ad(g)X)
\]

for any \( X \in \mathfrak{g}, g \in G \) and \( i \in \{1, \ldots, k\} \). Therefore we find

\[
\phi_i(g)\sum_{a=1}^N \phi_i(e_a)^2\phi_i(g^{-1}) = \sum_{a=1}^N \phi_i(Ad(g)e_a)^2 = \sum_{a=1}^N \phi_i(e_a)^2
\]

for any \( g \in G \). This shows that \( \sum_{a=1}^N \phi_i(e_a)^2 \) lies in the centralizer of \( \phi_i(G) \). Since the representation \( (\phi_i, E_i) \) is irreducible, Schur’s lemma in representation theory implies that

\[
(1.8) \quad \sum_{a=1}^N \phi_i(e_a)^2 = -\lambda_i I_i,
\]

for some constants \( \lambda_i \), where \( I_i \) is the identity transformation on \( E_i \). On the other hand, it is known that the Laplacian of \( x_i \) is given by

\[
(1.9) \quad \Delta x_i(p) = -\sum_{a=1}^N \frac{d^2}{dt^2} x_i(exp te_a)|_{t=0} = -\sum_{a=1}^N \phi_i(e_a)^2 (x_i(p)).
\]

Therefore, by (1.8) and (1.9), we have

\[
(1.10) \quad \Delta x_i = \lambda_i I_i, \quad i = 1, \ldots, k,
\]
for some constants $\lambda_1, \ldots, \lambda_k$. In particular, this shows that $x$ is a diagonal immersion of 1–type maps $x_i : M \to E_i$, $i = 1, \ldots, k$. Therefore, the Laplace map of the equivariant immersion $x$ is given by

\[(1.11) \quad L(p) = \lambda_1 x_1(p) + \cdots + \lambda_k x_k(p).\]

Since $E_1, \ldots, E_k$ are mutually orthogonal in $\mathbb{E}^m$, (1.11) implies

\[(1.12) \quad \langle dL, dL \rangle = \lambda_1^2 \langle dx_1, dx_1 \rangle + \cdots + \lambda_k^2 \langle dx_k, dx_k \rangle.\]

If $M$ is an irreducible homogenous Riemannian manifold, the linear isotropy representation is irreducible. So, in this case, each $\langle dx_i, dx_i \rangle$ is a constant multiple of the original metric on $M$. Thus, by (1.12), the Laplace transformation of the equivariant immersion $x$ is a homothetic transformation. \(\square\)

For compact submanifolds with homothetic Laplace transformation, we have the following

**Proposition 1.9.** Let $x : M^n \to \mathbb{E}^m$ be an isometric immersion of a compact Riemannian manifold $M^n$ into $\mathbb{E}^m$. Then we have:

1. the Laplace map $L : M^n \to \mathbb{E}^m$ has center of gravity at the origin 0 of $\mathbb{E}^m$ and
2. if the Laplace transformation $L : M^n \to L(M^n)$ of $x$ is homothetic, or more generally volume-element preserving, then the center of gravity of the Laplace image $L(M^n)$ (with respect to the induced metric) in $\mathbb{E}^m$ is the origin 0 of $\mathbb{E}^m$.

**Proof.** Since $M$ is compact and $L = \Delta x$, Hopf’s lemma implies that the center of gravity of the Laplace map is the origin. This proves statement (1). Statement (2) follows from statement (1) easily. \(\square\)

**Remark 1.1.** Proposition 1.9 shows that not every submanifold in $\mathbb{E}^m$ can be realized as the Laplace image of some submanifolds in $\mathbb{E}^m$. In particular, every compact submanifold in $\mathbb{E}^m$ cannot be realized as the Laplace image of any submanifold if its center of gravity differs from the origin. \(\square\)

The following results establish some relations between Laplace transformations and the notion of submanifolds of finite type.

**Proposition 1.10.** Let $x : M^n \to \mathbb{E}^m$ be an isometric immersion. If $x$ has homothetic Laplace transformation $L : M^n \to L(M^n)$, then

1. if the immersion $x$ is of finite type, then the Laplace map $L : M^n \to \mathbb{E}^m$ is of finite type;
2. if $M$ is compact, then the Laplace map $L$ is of $k$–type if and only if the immersion $x$ is of $k$–type;
if the immersion $x$ is of finite type, then the Laplace map is of non-null finite type; in particular, if $x$ is of non-null $k$–type, then $L$ is of non-null $k$–type; and if $x$ is of null $k$–type, then $L$ is of non-null $(k-1)$–type.

Proof. If the Laplace transformation is homothetic, then the Laplacian operator $\Delta_L$ of the Laplace image is related with the Laplacian operator of $M$ by $\Delta_L = c\Delta$ for some positive constant $c$.

(1) If the immersion $x$ is of finite type, then we have
\begin{equation}
(1.13) \quad x = x_1 + \cdots + x_k, \quad \Delta x_i = \lambda_i x_i, \quad i = 1, \ldots, k
\end{equation}
for some natural number $k$ and real numbers $\lambda_1, \ldots, \lambda_k$ and non-constant maps $x_1, \ldots, x_k$. From (1.10) we get
\begin{equation}
(1.14) \quad L = L_1 + \cdots + L_k,
\end{equation}
where $L_i = \lambda_i x_i$. Because $\Delta_L L_i = c\Delta L_i = c\lambda_i L_i$, (1.14) implies that the Laplace map $L$ of $x$ is also of finite type.

(2) Suppose $M$ is compact and the Laplace map $L$ of $x$ is of $k$–type. Let $Q(t)$ be the minimal polynomial of $L$. Then $\deg Q = k$ and $Q(\Delta_L)L = 0$. Because $L = \Delta x$ and $\Delta_L = c\Delta$ for some positive constant $c$, there exists a polynomial $\bar{Q}(t)$ such that $\deg \bar{Q} = \deg Q$ and $P(\Delta)x = 0$ where $P(t) = t\bar{Q}$. Therefore, by Theorem 3.1 of Chapter 1, the immersion $x$ is of $\ell$–type with $\ell \leq k+1$.

If $x$ is of $(k+1)$–type, then
\begin{equation}
(1.15) \quad x = x_1 + \cdots + x_{k+1}, \quad \Delta x_i = \lambda_i x_i, \quad i = 1, \ldots, k+1
\end{equation}
for non-constant maps $x_1, \ldots, x_{k+1}$. Because, $M$ is compact, all of $\lambda_1, \ldots, \lambda_{k+1}$ are non-zero. This implies that $L$ is of $(k+1)$–type which is a contradiction. Similarly, if $x$ is of $\ell$–type with $\ell < k$, then $L$ is of $\ell$–type with $\ell < k$. The remaining part of statement (2) follows from the proof of statement (1).

(3) If $x$ is of non-null $k$–type whose spectral decomposition is given by (1.13), then (1.14) is the spectral decomposition of the Laplace map $L$. Since $\Delta_L L_i = c\lambda_i L_i$ and $c, \lambda_i$ are nonzero constants, (1.14) shows that $L$ is of non-null $k$–type. If $x$ is of null $k$–type, then one of $\lambda_1, \ldots, \lambda_k$ is zero. Without loss of generality, we may assume $\lambda_1 = 0$. Then the spectral decomposition of the Laplace map is given by
\begin{equation*}
L = L_2 + \cdots + L_k, \quad \Delta_L L_i = c\lambda_i L_i, \quad i = 2, \ldots, k
\end{equation*}
which implies that $L$ is non-null $(k-1)$–type. This proves statement (3). □

Proposition 1.11. Let $x : M \to \mathbb{E}^n$ be an isometric immersion with homothetic Laplace transformation $L : M \to L(M)$. Then

(1) if $x$ is linearly independent, then the Laplace map is also linearly independent; and
(2) If $x$ is orthogonal, then the Laplace map is also orthogonal.

**Proof.** Let $x : M \to \mathbb{E}^m$ be an immersion of $k$-type and let

$$\begin{align*}
  x &= c + x_1 + \ldots + x_k, \\
  \Delta x_i &= \lambda_i x_i, \quad \lambda_1 < \ldots < \lambda_k
\end{align*}$$

be the spectral decomposition of the immersion $x$, where $c$ is a constant vector and $x_1, \ldots, x_k$ are non-constant maps. For each $i \in \{1, \ldots, k\}$ we put $E_i = \text{Span}\{x_i(p) : p \in M\}$. Then each $E_i$ is a linear subspace of $\mathbb{E}^m$.

(1) If the immersion $x$ is linearly independent, then the subspaces $E_1, \ldots, E_k$ are linearly independent, that is, the dimension of subspace spanned by all vectors in $E_1 \cup \ldots \cup E_k$ is equal to $\dim E_1 + \ldots + \dim E_k$. Since the Laplace transformation is homothetic, (1.16) yields

$$L = L_1 + \ldots + L_k, \quad \Delta_L L_i = c\lambda_i L_i, \quad i = 1, \ldots, k,$$

where $L_i = 0$ when $\lambda_i = 0$. Because $\text{Span}\{L_i(p) : p \in M\} = \text{Span}\{x_i(p) : p \in M\} = E_i$ for $L_i \neq 0$, (1.17) implies that the Laplace map is also linearly independent.

(2) If the immersion $x$ is orthogonal, then the subspaces $E_1, \ldots, E_k$ are mutually orthogonal in $\mathbb{E}^m$. Because $L$ has the spectral decomposition given by (1.17), statement (2) can be proved in a way similar to statement (1).

**Theorem 1.12.** Let $x : M \to \mathbb{E}^m$ be an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold. Then, with respect to the induced metric, the Laplace map $L : M \to \mathbb{E}^m$ is a homothetic immersion of finite type.

**Proof.** If $x : M \to \mathbb{E}^m$ be an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold, then $x$ is of finite type [C5, page 258]. On the other hand, by Theorem 1.8, we know that the Laplace transformation of $x$ is homothetic. Thus, by applying Proposition 1.10, the Laplace map $L$ is a homothetic immersion of finite type.

§2. Some classification theorems.

In this section we assume $x : M^n \to \mathbb{E}^m$ is an isometric immersion whose Laplace transformation is given by $L : M^n \to L(M^n)$. If $L$ is homothetic, then the Laplacian operator $\Delta_L$ of the Laplace image is related with the Laplace operator of $M$ by $\Delta_L = c\Delta$ for some positive number $c$.

First we give the following.
**Theorem 2.1.** Let \( x : M^n \rightarrow \mathbb{E}^m \) be an isometric immersion with homothetic Laplace transformation \( L : M^n \rightarrow L(M^n) \). Then the Laplace image lies in a hypersphere of \( \mathbb{E}^m \) as a minimal submanifold if and only if either

1. \( M^n \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) or
2. \( x \) is of null 2-type.

**Proof.** Let \( x : M^n \rightarrow \mathbb{E}^m \) be an isometric immersion with homothetic Laplace transformation. If the Laplace image lies in a hypersphere \( S^{m-1} \) of \( \mathbb{E}^m \) centered at the origin as a minimal submanifold, then the Laplace map \( L \) of \( x \) is of non-null 1–type by Theorem 2.4 of Chapter II. Thus

\[
\Delta^2 x = \Delta L = \frac{1}{c} \Delta_L L = \frac{\lambda}{c} L = \frac{\lambda}{c} \Delta x,
\]

for some positive number \( \lambda \). Put

\[ x_1 = \frac{c}{\lambda} x, \quad x_0 = x - x_1. \]

Then, by (2.1), we have

\[
\Delta x_1 = \frac{c}{\lambda} \Delta^2 x = \Delta x = \frac{\lambda}{c} x_1, \quad \Delta x_0 = \Delta x - \Delta x_1 = 0.
\]

Thus, \( x \) is either of 1–type (when \( x_0 = 0 \)) or of null 2–type (when \( x_0 \neq 0 \)). If the first case occurs, \( M^n \) is immersed by \( x \) as a minimal submanifold of a hypersphere of \( \mathbb{E}^m \). \( \square \)

**Proposition 2.2.** Let \( x : M^n \rightarrow \mathbb{E}^m \) be an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold. Then

1. the immersion \( x \) is of 1-type if and only if the Laplace image \( L(M^n) \) is a minimal submanifold of a hypersphere; and
2. if the immersion \( x \) is not of 1-type, then the Laplace image is a minimal submanifold in some hyperquadric of \( \mathbb{E}^m \).

**Proof.** If \( x : M^n \rightarrow \mathbb{E}^m \) is an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold, then \( x \) has homothetic Laplace transformation according to Theorem 1.8. Thus, by applying Theorem 2.1 we obtain statement (1), since \( M^n \) is compact.

In general, Lemma 1.4 implies that the Laplace map \( L \) of \( x \) is an equivariant homothetic immersion. By multiplying the metric on \( M \) by a suitable constant, \( L \) becomes an equivariant isometric immersion. Therefore, by applying Theorem 6.3 of Chapter II, we see that the Laplace image of \( M \) is a minimal submanifold of some hyperquadric of \( \mathbb{E}^m \). \( \square \)

The following result classifies all submanifolds with homothetic Laplace transformation and with parallel mean curvature vector.
Theorem 2.3. Let \( x : M \to \mathbb{E}^m \) be an isometric immersion with parallel mean curvature vector. Then the Laplace transformation \( \mathcal{L} : M \to L(M) \) of \( x \) is homothetic if and only if \( M \) is immersed as a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) via \( x \).

Proof. Assume the immersion \( x : M \to \mathbb{E}^m \) has parallel mean curvature and homothetic Laplace transformation. Then by Lemma 1.1 we have

\[
\langle A_H X, A_H Y \rangle = c^2 \langle X, Y \rangle,
\]
for vectors \( X, Y \) tangent to \( M \), where \( c \) is a positive number. Put

\[
U = \{ p \in M : H(p) \neq 0 \}.
\]

Then \( U \) is a dense open subset of \( M \), since \( \mathcal{L} \) is homothetic. On \( U \) we choose a local orthonormal frame field \( e_1, \ldots, e_n \) of the tangent bundle and a local orthonormal frame field \( e_{n+1}, \ldots, e_m \) of the normal bundle such that \( e_1, \ldots, e_n \) are eigenvectors of \( A_H \) and \( e_{n+1} \) is in the direction of the mean curvature vector \( H \) on \( U \). Hence we have

\[
A_{n+1} e_i = \kappa_i e_i, \quad i = 1, \ldots, n,
\]
where \( A_r = A_{e_r} \) is the Weingarten map of \( M \) and \( \kappa_1, \ldots, \kappa_n \) are eigenvalues of \( A_{n+1} \). From (2.2) and (2.3) we may assume

\[
\kappa_1 = \ldots = \kappa_k = \lambda, \quad \kappa_{k+1} = \ldots = \kappa_n = -\lambda,
\]
where \( \lambda = n\alpha/(2k - n) \) and \( \alpha^2 = \langle H, H \rangle \) are constant.

Assume \( 0 < k < n \). We put

\[
D_1 = \{ X \in TM : A_{n+1} X = \lambda X \},
\]

(2.5) \[
D_2 = \{ X \in TM : A_{n+1} X = -\lambda X \}.
\]

Because \( DH = 0 \), the equation of Codazzi implies that \( D_1, D_2 \) are integrable distributions on \( U \). Furthermore, by applying Codazzi's equation, it is easy to show that leaves of \( D_1 \) and \( D_2 \) are totally geodesic submanifolds of \( U \). Therefore, locally \( U \) is the Riemannian product \( M_1 \times M_2 \) of some integrable submanifolds of \( D_1 \) and \( D_2 \).

From (2.3) and (2.4) we know that \( A_{n+1} \) takes the following form:

\[
(2.6) \quad A_{n+1} = \begin{pmatrix} \lambda I_k & 0 \\ 0 & -\lambda I_{n-k} \end{pmatrix}.
\]

On the other hand, since the mean curvature vector \( H \) is parallel, the equation of Ricci yields \( [A_{n+1}, A_r] = 0 \). Therefore, by using (2.6), the second fundamental form \( h \) of \( U \) in \( \mathbb{E}^m \) satisfies \( h(D_1, D_2) = \{0\} \). Consequently, by applying a lemma of Moore, we know that \( M \) is locally a product submanifolds, say

\[
(2.7) \quad x = (y, z) : M_1 \times M_2 \to \mathbb{E}^{m_1} \times \mathbb{E}^{m-m_1} = \mathbb{E}^m.
\]
Let $\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}, \ldots, \bar{e}_m$ be a local orthonormal frame field such that $\bar{e}_{n+1}$ is in the direction of the mean curvature vector $H_1$ of $M_1$ in $\mathbb{E}^{m_1}$, $\bar{e}_{n+2}$ is in the direction of the mean curvature vector $H_2$ of $M_2$ in $\mathbb{E}^{m_2-m_1}$, $\bar{e}_1, \ldots, \bar{e}_k$ are tangent to $M_1$ and $\bar{e}_{k+1}, \ldots, \bar{e}_n$ are tangent to $M_2$, respectively. Then we have

\begin{equation}
H = \frac{1}{n} \{(\text{trace} A \bar{e}_{n+1}) \bar{e}_{n+1} + (\text{trace} A \bar{e}_{n+2}) \bar{e}_{n+2}\}. \tag{2.8}
\end{equation}

From (2.8) we get

\begin{equation}
A_H = \frac{1}{n} \{(\text{trace} A \bar{e}_{n+1}) A \bar{e}_{n+1} + (\text{trace} A \bar{e}_{n+2}) A \bar{e}_{n+2} \}. \tag{2.9}
\end{equation}

In particular, if $\bar{e}_1, \ldots, \bar{e}_k$ are tangent to $M_1$ and eigenvectors of $A \bar{e}_{n+1}$ and $\bar{e}_{k+1}, \ldots, \bar{e}_n$ are tangent to $M_2$ and are eigenvectors of $A \bar{e}_{n+2}$, then we have

\begin{equation}
A_H = \frac{1}{n} \begin{pmatrix}
(\text{trace} A \bar{e}_{n+1}) B & 0 \\
0 & (\text{trace} A \bar{e}_{n+2}) C
\end{pmatrix}, \tag{2.10}
\end{equation}

where $B$ and $C$ are diagonal matrices given by

\begin{equation}
B = \text{Diag}(\delta_1, \ldots, \delta_k), \quad C = \text{Diag}(\beta_{k+1}, \ldots, \beta_n). \tag{2.11}
\end{equation}

Since $H = \alpha \bar{e}_{n+1}$, (2.6), (2.10) and (2.11) imply

\begin{align*}
\delta_1 = \ldots = \delta_k = \delta, & \quad \beta_{k+1} = \ldots = \beta_n = \beta.
\end{align*}

Therefore, we find $n \alpha \lambda = k \delta^2$ and $-n \alpha \lambda = (n - k) \beta^2$ which is impossible, since $\alpha$ and $\lambda$ are non–zero on $U$. Consequently, either $k = 0$ or $k = n$. Thus, $U$ is a pseudo–umbilical submanifolds with parallel mean curvature vector. Hence, $U = M$. Therefore, by applying a result of [CY] (i.e. Theorem 1.1 of Chapter II), we conclude that $M$ is immersed by $x$ as a minimal submanifold of a hypersphere of $\mathbb{E}^m$.

The converse of this is given by Lemma 1.2. □

In the following we study hypersurfaces with homothetic Laplace transformation. For simplicity, here we classify hypersurfaces of dimension 2 and 3 only.

**Theorem 2.4.** Let $x : M \to \mathbb{E}^3$ be a surface in $\mathbb{E}^3$. Then the Laplace transformation $L : M \to L(M)$ of the surface is homothetic if and only if $M$ is an open portion of an ordinary sphere in $\mathbb{E}^3$.

**Proof.** If $M$ is an open portion of an ordinary sphere in $\mathbb{E}^3$, it is easy to see that it has homothetic Laplace transformation.

Conversely, suppose $M$ is a surface of $\mathbb{E}^3$ with homothetic Laplace transformation. It suffices to show that $M$ has constant mean curvature according to Theorem 2.3.

Let $U = \{p \in M : d\alpha^2 \neq 0 \text{ at } p\}$. Then $U$ is an open subset of $M$. In order to prove that $U$ is an empty set, we choose a local orthonormal frame field
$e_1, e_2, e_3$ such that $e_1, e_2$ are tangent vector fields of $U$ which are eigenvectors of the Weingarten map $A_3 = A e_3$. So, we have

$$A_3 e_i = \kappa_i e_i, \quad i = 1, 2,$$

where $\kappa_1, \kappa_2$ are principal curvatures.

From Lemma 1.1 and (2.12) we get

$$\begin{align*}
(e_1 \alpha)(e_2 \alpha) &= 0.
\end{align*}$$

Therefore, without loss of generality, we may choose $e_1$ such that $e_1$ is in the direction of the gradient of $\alpha$, $\nabla \alpha$. So we have

$$e_2 \alpha = 0.$$

From Lemma 1.1 and (2.12) we also have

$$\begin{align*}
\alpha^2 \kappa_1^2 + (e_i \alpha)^2 &= c^2,
\end{align*}$$

where $c$ is a positive number. Thus (2.14) yields

$$\begin{align*}
\alpha^2 \kappa_2^2 &= c^2.
\end{align*}$$

So, by choosing suitable $e_3$, we have

$$\kappa_2 = \frac{c}{\alpha} > 0.$$

From (2.15) and (2.17), we have

$$\begin{align*}
\omega_1^3 &= (2\alpha - \frac{c}{\alpha})\omega^1, \quad \omega_2^3 = \frac{c}{\alpha}\omega^2,
\end{align*}$$

$$\begin{align*}
(e_1 \alpha)^2 &= 4\alpha^2 \alpha^2 (c^2 - \alpha^2).
\end{align*}$$

Put

$$\begin{align*}
\omega_1^2 &= f_1 \omega^1 + f_2 \omega^2.
\end{align*}$$

Then we have

$$\begin{align*}
d\omega^1 &= f_1 \omega^1 \land \omega^2, \quad d\omega^2 = f_2 \omega^1 \land \omega^2.
\end{align*}$$

By taking exterior derivative of (2.18) and using (2.21) and the structure equations, we obtain

$$\begin{align*}
\alpha^2 &= cf_1, \quad e_1 \alpha = 2(\alpha - \frac{\alpha^3}{e})f_2.
\end{align*}$$

Combining (2.19) and (2.22) we find

$$\begin{align*}
(c - \alpha^2)f_2^2 &= c^2.
\end{align*}$$

(2.22) and (2.23) yield

$$\begin{align*}
e_2 f_1 = e_2 f_2 = 0.
\end{align*}$$
Taking the exterior derivative of (2.20), we find

\begin{equation}
(2.25) \quad f_1^2 + f_2^2 = \frac{c(c - 2\alpha^2)}{\alpha^2} - e_1 f_2.
\end{equation}

On the other hand, (2.22) and (2.23) imply

\begin{equation}
(2.26) \quad e_1 f_2 = \frac{e^5 - 4c^4 \alpha^2 + 2c^3 \alpha^4 - c\alpha^8 + \alpha^8}{\alpha^2 (c - \alpha^2)}.
\end{equation}

From (2.23) we get

\begin{equation}
(2.27) \quad f_2(e_1 f_2) = \frac{c^2 \alpha (e_1 \alpha)}{(c - \alpha^2)^2}.
\end{equation}

Using (2.19), (2.23) and (2.27) we find

\begin{equation}
(2.28) \quad (e_1 f_2)^2 = \frac{4c^2 \alpha^4}{(c - \alpha^2)^2}.
\end{equation}

Combining (2.26) and (2.28), we conclude that \( U \) is an empty set. Thus, \( M \) has constant mean curvature and it is an open portion of an ordinary sphere in \( \mathbb{E}^3 \).

**Theorem 2.5.** Let \( x : M \to \mathbb{E}^4 \) be a hypersurface of \( \mathbb{E}^4 \). Then the Laplace transformation \( \mathcal{L} : M \to L(M) \) of the hypersurface is homothetic if and only if \( M \) is an open portion of a hypersphere of \( \mathbb{E}^4 \).

**Proof.** Suppose \( M \) is a hypersurface of \( \mathbb{E}^4 \) with homothetic Laplace transformation. It suffices to show that \( M \) has constant mean curvature according to Theorem 2.3.

Let \( U = \{ p \in M : \partial \alpha^2 \neq 0 \text{ at } p \} \). Then \( U \) is an open subset of \( M \). In order to prove that \( U \) is an empty set, we choose a local orthonormal frame field \( e_1, e_2, e_3, e_4 \) such that \( e_1, e_2, e_3 \) are tangent vector fields of \( U \) which are eigenvectors of the Weingarten map \( A_4 = A_{e_4} \). So, we have

\begin{equation}
(2.29) \quad A_{n+1} e_i = \kappa_i e_i, \quad i = 1, 2, 3,
\end{equation}

where \( \kappa_1, \kappa_2, \kappa_3 \) are principal curvatures.

From Lemma 1.1 and (2.29) we get

\begin{equation}
(2.30) \quad (e_i \alpha)(e_j \alpha) = 0, \quad i \neq j.
\end{equation}

Therefore, without loss of generality, we may choose \( e_1 \) such that \( e_1 \) is in the direction of the gradient of \( \alpha \), \( \nabla \alpha \). So we have

\begin{equation}
(2.31) \quad e_2 \alpha = e_3 \alpha = 0.
\end{equation}

From Lemma 1.1 and (2.29) we have

\begin{equation}
(2.32) \quad \alpha^2 \kappa_1^2 + (e_1 \alpha)^2 = c^2,
\end{equation}

where \( c \) is a positive number. Thus (2.31) yields

\begin{equation}
(2.33) \quad \alpha^2 \kappa_2^2 + \alpha^2 \kappa_3^2 = c^2.
\end{equation}
So, by choosing a suitable $e_4$, we have either
\[ (2.34) \quad \kappa_1 = 3\alpha, \quad \kappa_2 = \frac{c}{\alpha}, \quad \kappa_3 = -\frac{c}{\alpha}, \]
or
\[ (2.35) \quad \kappa_1 = 3\alpha - \frac{2c}{\alpha}, \quad \kappa_2 = \kappa_3 = \frac{c}{\alpha}, \]

We treat these two cases separately.

**Case 1.** $(2.34)$ holds. In this case, we put $\mu = \frac{c}{\alpha}$. From $(2.32)$ and $(2.34)$ we have
\[ (2.36) \quad (e_1 \alpha)^2 = c^2 - 9\alpha^4. \]
\[ (2.37) \quad \omega_1^4 = 3\alpha \omega^1, \quad \omega_2^4 = \mu \omega^2, \quad \omega_3^4 = -\mu \omega^3. \]

By taking the exterior derivative of $(2.37)$ and applying $(2.37)$ and the structure equations, we obtain
\[ (2.38) \quad \omega_1^2(e_1) = \omega_3^2(e_1) = 0. \]
on $U$. Hence, integral curves of $e_1$ in $U$ are geodesics of $M$.

Since $\alpha \mu = c$, $(2.31)$ yields
\[ (2.39) \quad e_2 \mu = e_\mu = 0. \]

By taking exterior derivatives of the last two equations of $(2.37)$ and using $(2.37)$, $(2.39)$ and the structure equations, we obtain
\[ (2.40) \quad e_1 \mu = (3\alpha - \mu) \omega_2^2(e_2) = -(3\alpha + \mu) \omega_3^3(e_3), \]
\[ (2.41) \quad \omega_1^2(e_1) = \omega_3^2(e_1) = \omega_2^3(e_2) = \omega_2^3(e_3) = 0, \]
\[ (2.42) \quad (3\alpha - \mu) \omega_1^2(e_3) = 2\mu \omega_2^3(e_1), \quad (3\alpha + \mu) \omega_3^3(e_2) = 2\mu \omega_2^3(e_2). \]

From $(2.41)$ we have
\[ (2.43) \quad \omega_1^2 = f_2 \omega^2 + f_3 \omega^3, \]
\[ (2.44) \quad \omega_3^2 = g_2 \omega^2 + g_3 \omega^3, \]
\[ (2.45) \quad \omega_2^3 = h \omega^3, \]
for some functions $f_2, f_3, g_2, g_3, h$. Moreover, by $(2.40)$, and $(2.42)$, we have
\[ (2.46) \quad e_1 \alpha = \frac{\alpha}{c}(c - 3\alpha^2)f_2 = \frac{\alpha}{c}(c + 3\alpha^2)g_3, \]
\[ (2.47) \quad (3\alpha^2 + c)g_2 = 2ch, \quad (3\alpha^2 - c)f_3 = 2ch. \]
Also, from (2.46) and (2.47), we have

\[(2.48)\quad g_3 = \frac{c - 3\alpha^2}{c + 3\alpha^2}f_2, \quad g_2 = -\frac{c - 3\alpha^2}{c + 3\alpha^2}f_3.\]

By taking exterior derivative of (2.34) we obtain

\[(2.49)\quad e_2 h = e_3 h = 0, \quad dh = (e_1 h)\omega^1,\]

\[(2.50)\quad \alpha^2 h(g_2 - f_3) = c^2 + \alpha^2 (g_2 f_3 - g_3 f_2).\]

Similarly, by taking exterior derivative of (2.43) and (2.44), we find

\[(2.51)\quad e_1 f_2 = hg_2 - 3c - f_2^2 + f_3(h - g_2),\]

\[(2.52)\quad e_1 f_3 = hg_3 - f_3 g_3 - (f_3 + h) f_2,\]

\[(2.53)\quad e_3 f_2 = e_2 f_3,\]

\[(2.54)\quad e_1 g_2 = -hf_2 - f_2 g_2 + (h - g_2) g_3,\]

\[(2.55)\quad e_1 g_3 = -hf_3 + 3c - (f_3 + h) g_2 - g_3^2,\]

\[(2.56)\quad e_3 g_2 = e_2 g_3.\]

From (2.31) and (2.46) we have

\[e_2 e_1 \alpha = \frac{\alpha}{c}(c + 3\alpha^2)(e_2 g_3), \quad e_1 e_2 \alpha = 0.\]

Therefore, using (2.41), we get

\[\frac{\alpha}{c}(c + 3\alpha^2)(e_2 g_3) = [e_2, e_1] \alpha = -(\nabla_{e_1} e_2) \alpha = 0.\]

Hence, by (2.56), we get

\[(2.57)\quad e_2 g_3 = e_3 g_2 = 0.\]

Similarly by using

\[e_3 e_1 \alpha = \frac{\alpha}{c}(c + 3\alpha^2)(e_3 g_3), \quad e_1 e_3 \alpha = 0,\]

we obtain

\[(2.58)\quad e_3 g_3 = 0.\]

Similarly using \[e_1 \alpha = \frac{\alpha}{c}(c - 3\alpha^2)f_2,\] we also obtain

\[(2.59)\quad e_2 f_2 = e_3 f_2 = 0, \quad e_2 f_3 = 0.\]

From (2.47) and (2.57), we also find

\[(2.60)\quad e_2 h = e_3 h = 0.\]

Consequently, from (2.31), (2.47) and (2.60), we obtain

\[(2.61)\quad e_i g_j = e_i f_j = e_i h = 0, \quad i, j = 2, 3.\]
By using (2.37) and a long computation we derive

\[(2.62)\]
\[e_1 g_2 = \frac{f_2 f_3 (c - 3\alpha^2)}{c + 3\alpha^2}.\]

On the other hand, substituting (2.47) and (2.48) into (2.54) yields

\[(2.63)\]
\[e_1 g_2 = \frac{f_2 f_3 (c - 3\alpha^2)}{c(c + 3\alpha^2)^2}(2c^2 + 3c\alpha^2 + 9\alpha^4).\]

Combining (2.51) and (2.52), we obtain \(f_2 f_3 = 0\).

If \(f_2 = 0\), then \(g_3 = 0\) according to (2.48). Thus, (2.46) and \(e_1 \alpha = 0\) imply \(M\) has constant mean curvature.

If \(f_3 = 0\), (2.47) and (2.48) yield \(h = g_2 = 0\). Thus, (2.51) implies

\[(2.64)\]
\[e_1 f_2 = -3c - f_2^2.\]

On the other hand, by taking derivative of \(\omega^2_1 = f_2 \omega^2\) and using (2.32), (2.43), (2.45) together with \(h = f_3 = 0\), we have

\[(2.65)\]
\[e_1 f_2 = \frac{c^2}{\alpha^2} - f_2^2.\]

Combining (2.64) and (2.65) we see that \(U\) is an empty set which implies that \(M\) has constant mean curvature \(M\) is an open portion of a hypersphere.

**Case 2.** (2.35) holds. In this case, we have

\[(2.66)\]
\[\omega^4_1 = (3\alpha - \frac{2c}{\alpha})\omega^1, \quad \omega^4_2 = \frac{c}{\alpha} \omega^2, \quad \omega^4_3 = \frac{c}{\alpha} \omega^3,\]

\[(2.67)\]
\[(e_1 \alpha)^2 = 3(c - \alpha^2)(c - 3\alpha^2).\]

By taking exterior derivatives of the three equations in (2.66), we may prove after a long computation that

\[(2.68)\]
\[\omega^2 = \omega^3 = 0.\]

Thus, \(d\omega^1 = 0\). Hence, if we put \(D_1 = \text{Span}\{e_1\}\) and \(D_2 = \text{Span}\{e_2, e_3\}\) on \(U\), then \(D_1, D_2\) are totally geodesic integrable distributions on \(U\). Hence, \(U\) is locally the Riemannian product of a curve and a surface \(N\). Moreover, since the second fundamental form \(h\) satisfies \(h(D_1, D_2) = \{0\}\), a lemma of Moore implies that \(U\) is locally a product hypersurface of a line and a surface in a affine 3–subspace \(E^3\) of \(E^4\). This is a contradiction since such a hypersurface does not have a homothetic Laplace transformation. Consequently, \(U\) is an empty set. Thus, \(M\) is an open portion of a hypersphere of \(E^4\).

The converse of this is trivial. □
Remark 2.1. Similar but much more complicated arguments can be used to classify hypersurfaces of a Euclidean space of any dimension with homothetic Laplace transformation. □

§3. Surfaces with homothetic Laplace transformation.

The purpose of this section is to study surfaces with homothetic Laplace transformation.

In the following, a submanifold $M$ of a Riemannian manifold is said to have parallel normalized mean curvature vector if the mean curvature vector $H$ is nonzero and the unit vector field in the direction of $H$ is parallel in the normal bundle. It is clear that a non-minimal hypersurface has parallel normalized mean curvature vector. For submanifolds with parallel normalized mean curvature vector, we have the following.

**Theorem 3.1.** Let $x : M \to \mathbb{E}^m$ be an isometric immersion with parallel normalized mean curvature vector of a surface $M$. Then the Laplace transformation $L : M \to L(M)$ of $x$ is homothetic if and only if $M$ is immersed as a minimal surface in a hypersphere of $\mathbb{E}^m$ via $x$.

**Proof.** Assume $M$ is a surface in $\mathbb{E}^m$ with parallel normalized mean curvature vector and homothetic Laplace transformation. Denote by $\alpha$ the mean curvature of $M$ in $\mathbb{E}^m$. We choose an orthonormal local frame field $e_1, e_2, e_3, \ldots, e_m$ such that $e_1, e_2$ are eigenvectors of $A_H$ and $H = \alpha e_3$. We put $A_3 e_i = \kappa_i e_i, i = 1, 2$. Then we have $(e_1 \alpha)(e_2 \alpha) = 0$. So, we may choose $e_1, e_2$ such that $e_1$ is in the direction of the gradient of $\alpha^2$. Let $U$ be the open subset of $M$ on which the gradient of $\alpha^2$ is non-zero. Thus, we have

$$e_2 \alpha = 0.$$  

Hence, from Lemma 1.1 and (3.1), we have

$$\kappa_1 = 2\alpha - \frac{c}{\alpha}, \quad \kappa_2 = \frac{c}{\alpha}, \quad \omega^3_1 = \kappa_1 \omega^1, \quad \omega^3_2 = \kappa_2 \omega^2,$$

$$\omega^3_1 = \cdots = \omega^3_m = 0.$$  

where $c$ is a positive constant. (3.3) implies $c > \alpha^2$ on $U$. Because $M$ has parallel normalized mean curvature vector, we have $De_3 = 0$; and hence

$$\omega^3_1 = \cdots = \omega^3_m = 0.$$  

From the equation of Ricci and (3.4), we get

$$[A_3, A_r] = 0, \quad r = 4, \ldots, m.$$  

Since the gradient of $\alpha^2$ is nowhere zero on $U$, $c \neq \alpha^2$. Therefore, (3.5) implies that each $A_r$ is diagonalized with respect to $e_1, e_2$. Because $e_3$ is parallel to $H$,
this implies that one may choose $e_4, \ldots, e_m$ in such a way that $A_4, \ldots, A_m$ take the following forms:

\[(3.6)\quad A_4 = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \quad A_5 = \cdots = A_m = 0.\]

By taking exterior derivatives of $\omega_1^3$ and $\omega_2^3$ and using the structure equations and (3.2), we find

\[(3.7)\quad (2\alpha - \frac{2c}{\alpha})f_1 = 0,\]
\[(3.8)\quad e_1 \alpha = \frac{2\alpha}{c}(c - \alpha^2)f_2,\]

where $\omega_2^2 = f_1 \omega^1 + f_2 \omega^2$. Since $\alpha$ is nowhere constant on $U$, (3.7) yields $f_1 = 0$. Put $f = f_2$. Then $\omega_2^2 = f \omega^2$.

Taking exterior derivative of $\omega_4^1 = \beta \omega^1$ and $\omega_2^4 = -\beta \omega^2$ we find

\[(3.9)\quad e_1 \beta = -2\beta f, \quad e_2 \beta = 0.\]

Combining (3.3) and (3.8), we get

\[(3.10)\quad \omega_1^2 = f \omega^2 = \pm \frac{c}{\sqrt{c - \alpha^2}} \omega^2.\]

By taking exterior derivative of (3.10) and applying (3.8) and (3.10), we obtain

\[(3.11)\quad \beta^2 = 2c - \frac{c^2}{\alpha^2} + \frac{2c\alpha^2 + c^2}{c - \alpha^2}.\]

Taking exterior derivative of (3.11) and using (3.8) we get

\[(3.12)\quad \beta(e_1 \beta) = \pm 2c \alpha \sqrt{c - \alpha^2} \left\{ \frac{2c}{\alpha^3} + \frac{6c\alpha}{(c - \alpha^2)^2} \right\}.\]

From (3.9), (3.11) and (3.12), we may conclude that $U$ is an empty set. Thus, $M$ has constant mean curvature. Because $M$ has parallel normalized mean curvature vector, $M$ has parallel mean curvature vector. Therefore, by Theorem 2.3, $M$ is immersed as a minimal surface in a hypersphere of $E^m$. The converse follows from Lemma 1.2. □

**Theorem 3.2.** Let $x : M \to E^4$ be a surface in $E^4$ with constant mean curvature. Then $M$ has homothetic Laplace transformation if and only if $M$ is a minimal surface in a hypersphere of $E^4$.

**Proof.** Assume $M$ has constant mean curvature and homothetic Laplace transformation. Then the mean curvature is nonzero. If $M$ has parallel mean curvature vector, then $M$ is a minimal surface in a hypersphere of $E^4$ by Theorem 3.1. So, we assume that $U = \{ p \in M : DH \neq 0 \text{ at } p \}$ is non-empty. It is clear that $U$ is an open subset of $M$. 

On $U$, we choose a local orthonormal frame field $e_1, e_2, e_3, e_4$ such that $e_3$ is in the direction of $H$ and $e_1, e_2$ are eigenvectors of $A_3$ with eigenvalues $\kappa_1, \kappa_2$, respectively. Since $DH = \alpha D e_3 = \alpha \omega_3^3$, Lemma 1.1 implies $\omega_3^4(e_1) \omega_4^3(e_2) = 0$. Without loss of generality, we may assume $\omega_3^4(e_1) = 0$. Then we have

$$D_{e_1}H = 0, \quad \omega_3^4 = \mu \omega^2,$$

for some local function $\mu$. So, by Lemma 1.1, we have

$$\alpha^2 \kappa_1^2 = c^2, \quad \alpha^2 \kappa_2^2 + \alpha^2 \mu^2 = c^2,$$

where $c$ is a positive constant. Because $M$ has constant mean curvature, (3.15) implies $\kappa_1, \kappa_2$ and $\mu$ are constant. Put

$$A_4 = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}.$$ 

Taking exterior derivative of $\omega_3^2 = \kappa_1 \omega^1$ and $\omega_2^2 = \kappa_2 \omega^2$, respectively, we obtain

$$e_1 \delta + 2 \gamma f_1 = \kappa_1 \mu + e_2 \gamma - 2 f_2 \delta,$$

$$e_2 \delta + 2 f_2 \gamma = -e_1 \gamma + 2 f_1 \delta,$$

where $\omega_1^2 = f_1 \omega^1 + f_2 \omega^2$. From (3.17) and (3.19) we find

$$f_1 \delta + \gamma f_2 = 0.$$

Taking exterior derivative of $\omega_3^2 = \mu \omega^2$, we find

$$\mu f_2 = \delta (\kappa_2 - \kappa_1).$$

From (3.19) and (3.22) we get

$$\{ \mu^2 + (\kappa_1 - \kappa_2)^2 \} f_2 = 0.$$

If $\kappa_1 \equiv \kappa_2$ on $U$, then $U$ is a pseudo-umbilical surface with non-zero constant mean curvature in $E^4$. Thus, by applying a result of [C1] (cf. Theorem 1.2 of Chapter II), we know that $U$ has parallel mean curvature vector. This is a contradiction. Therefore, $V = \{ p \in U : \kappa_1 \neq \kappa_2 \text{ at } p \}$ is a non-empty open subset of $U$. On $V$, (3.19) and (3.23) yield $f_2 = 0$ and $\delta \mu = 0$. Since $DH \neq 0$ on $U$, we obtain $\delta = 0$.

Since $f_2 = 0$, we have $\omega_1^2 = f_1 \omega^1$. So, by taking exterior derivative of $\omega_1^2$ and applying $\delta = 0$, we find

$$e_2 f_1 = f_1^2 - \gamma^2 + \kappa_1 \kappa_2.$$
On the other hand, (3.17) implies
\[
(3.25) \quad e_2 f_1 = \left( \frac{\mu}{\kappa_1 - \kappa_2} \right) (2\gamma f_1 - \kappa_1 \mu).
\]
Combining (3.17), (3.24) and (3.25) we conclude that \( \gamma \) is constant. Thus, by (3.17) and (3.22), we conclude that \( f_1 \) is also a constant and
\[
(3.26) \quad f_1^2 = \kappa_1 \kappa_2 - \gamma^2.
\]
Moreover, using (3.17) and (3.18), we find
\[
(3.27) \quad f_1^2 = \frac{\kappa_1 \mu^2}{2(\kappa_1 - \kappa_2)}
\]
Because \( \kappa_1 \neq 0 \), (3.26) and (3.27) give
\[
(3.28) \quad \mu^2 = (\kappa_1 - \kappa_2)(3\kappa_2 - \kappa_1).
\]
From (3.15) and (3.28) we obtain \( \kappa_1 = \kappa_2 \) on \( V \) which is a contradiction. Consequently, \( M \) has parallel mean curvature vector. Hence, \( M \) is immersed as a minimal surface of a hypersphere of \( \mathbb{E}^4 \).

The converse of this is clear. \( \square \)

It is interesting to point out that Theorem 3.1 and Theorem 3.2 are best possible. In fact we have the following results.

**Proposition 3.3.** Let \( C_1 \) and \( C_2 \) be two planar curves parametrized by arclength. Then the product surface \( M = C_1 \times C_2 \) in \( \mathbb{E}^4 \) has homothetic Laplace transformation if and only if the curvature functions of \( C_1, C_2 \) are of the following form:
\[
(3.29) \quad \kappa(s) = c(1 + c^4 a e^{-8c^2 s})^{-\frac{1}{4}}
\]
for some constants \( a \) and \( c > 0 \).

**Proof.** Assume \( C_1 \times C_2 \) is given by
\[
(3.30) \quad X(u, v) = (x(u), y(u), z(v), w(v)),
\]
where \( u, v \) are arclength parametrizations of \( C_1 \) and \( C_2 \), respectively. Then the mean curvature vector \( H \) of \( M \) in \( \mathbb{E}^4 \) is given by
\[
(3.31) \quad H = \frac{1}{2}(x''(u), y''(u), z''(v), w''(v)).
\]
Denote \( e_1 = \frac{\partial}{\partial u}, e_2 = \frac{\partial}{\partial v} \). Then we have
\[
(3.32) \quad L_*(e_1) = (-x''(u), -y''(u), 0, 0),
\]
\[
L_*(e_2) = (0, 0, -z''(v), -w''(v)).
\]
Therefore, by Lemma 1.1, the plane curvatures \( \kappa \) of \( C_1, C_2 \) satisfy the following differential equation:
\[
(3.33) \quad (\kappa'(s))^2 + \kappa(s)^4 = c^2,
\]
for some positive constant $c$, where $s$ is the arc-length parametrization of the curve. (3.32) implies $c \geq \kappa^2$. Solving equation (3.33), we obtain

$$
(3.34) \quad \kappa(s)^4 = \frac{c^2}{1 + c^2 ae^{-8c^2 s}},
$$

where $a$ is a constant.

The converse of this is easy to verify. □

**Remark 3.1.** Proposition 3.3 still holds if we replace $C_1 \times C_2$ by the product of $k$ plane curves in $\mathbb{E}^{2k}$, for any $k = 1, 3, 4, \ldots$, (see also Corollary 1.3 of Chapter III). □

**Remark 3.2.** The plane curve whose curvature function satisfies (3.29) is an open portion of a circle if and only if $a = 0$. When $a \neq 0$, then a plane curve whose curvature function satisfies condition (3.29) is congruent to a curve given by

$$
(3.35) \quad \left( \int \cos \theta(s) ds, \int \sin \theta(s) ds \right),
$$

where

$$
(3.36) \quad \theta(s) = c_1 - \frac{1}{8c^2} \{ \ln(\ell(s) - 1) - \ln(\ell(s) + 1) + 2 \tan^{-1} \ell(s) \}, \quad c_1 \in \mathbb{R},
$$

$$
(3.37) \quad \ell(s) = \left( 1 + c^2 ae^{-8c^2 s} \right)^{\frac{1}{4}}.
$$

**Remark 3.3.** From Proposition 3.3 and Remark 3.2 it follows that there exist ample examples of surfaces which do not lie in any hypersphere of $\mathbb{E}^4$ but with homothetic Laplace transformation. □

The following two Propositions also show that Theorems 3.1 and 3.3 are best possible.

**Proposition 3.4.** Let $M^2$ be a surface in $\mathbb{E}^5$ defined by

$$
(3.38) \quad x(u, v) = \left( au, b^2 \cos u, b^2 \sin u, \frac{1}{b}(a^2 + b^4)^{3/4} \cos v, \frac{1}{b}(a^2 + b^4)^{3/4} \sin v \right)
$$

for some nonzero constants $a$ and $b$. Then the surface $M^2$ in $\mathbb{E}^5$ satisfies the following properties:

1. the Laplace transformation is homothetic;
2. the mean curvature function is a nonzero constant;
3. $M^2$ has non-parallel normalized mean curvature vector field; and
4. $M^2$ is not contained in any hypersphere of $\mathbb{E}^5$.

**Proposition 3.5.** Let $M^2$ be the product of two circular helices in $\mathbb{E}^6$ defined by

$$
(3.39) \quad x(u, v) = (au, av, c \cos u, c \sin u, c \cos v, c \sin v)
$$
for some nonzero constants $a$ and $c$. Then the surface satisfies the following properties:

(1) the Laplace transformation is homothetic;
(2) the mean curvature function is a nonzero constant;
(3) the surface is pseudo-umbilical;
(4) $M^2$ has non-parallel normalized mean curvature vector field;
(5) $M^2$ is not contained in any hypersphere of $\mathbb{E}^6$; and
(6) the immersion $x$ is of null 2-type.

Propositions 3.4 and 3.5 can be proved by direct computation.
§1. Some general results.

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional connected Riemannian manifold \( M \) into a Euclidean \( m \)-space. Denote by \( L : M^n \to E^m \) the Laplace map and by \( L(M^n) \) the Laplace image of the immersion \( x \). As before, we denote the Laplace transformation by \( L : M \to L(M) \). The Laplace transformation \( L \) is said to be conformal (respectively, weakly conformal) if there exists a function \( \rho > 0 \) (respectively, \( \rho \geq 0 \)) such that
\[
\langle dL(X), dL(Y) \rangle = \rho^2 \langle X, Y \rangle
\]
for all vectors \( X, Y \) tangent to \( M \).

The main purpose of this section to give some general results concerning submanifolds with conformal Laplace transformation.

First we give the following general lemma.

**Lemma 1.1.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional Riemannian manifold into \( \mathbb{E}^m \). Then the Laplace transformation \( L : M \to L(M) \) is conformal (respectively, weakly conformal) if and only if
\[
\langle A_H X, A_H Y \rangle + \langle D_X H, D_Y H \rangle = \rho^2 \langle X, Y \rangle,
\]
holds for vectors \( X, Y \) tangent to \( M \), where \( \rho \) is a strictly positive function (respectively, positive function) on \( M \).

**Proof.** Because the Laplace map is given by \( L(p) = (\Delta x)(p) = -nH(p) \), the differential of the Laplace map satisfies
\[
dL(X) = nA_H X - nD_X H.
\]
From (1.2) we obtain
\[
\langle dL(X), dL(Y) \rangle = \langle A_H X, A_H Y \rangle + \langle D_X H, D_Y H \rangle
\]
which implies the Lemma. \( \square \)

**Theorem 1.2.** Let \( M \) be a submanifold in \( \mathbb{E}^m \). Then \( M \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) if and only if \( M \) has conformal Laplace transformation and the mean curvature vector field of \( M \) in \( \mathbb{E}^m \) is parallel in the normal bundle.

**Proof.** If \( M \) has parallel mean curvature vector and conformal Laplace transformation, then Lemma 1.1 yields
\[
\langle A_H X, A_H Y \rangle = \rho^2 \langle X, Y \rangle,
\]
for some strictly positive function \( \rho \). It is clear that \( H \) is nowhere zero.

Let \( e_1, \ldots, e_n \) be a local orthonormal frame field given by eigenvectors of \( A_H \) satisfying \( A_H e_i = \mu_i e_i, i = 1, \ldots, n \). Then (1.4) implies \( \mu_i^2 = \rho^2 \). Therefore, \( A_H \)
has at most two distinct eigenvalues given by \( \rho, -\rho \) with multiplicities, say \( k, n-k \), respectively.

If \( k = 0 \) or \( k = n \), then \( M \) is a pseudo–umbilical submanifold with parallel mean curvature. Thus, by applying Theorem 1.1 of Chapter II, \( M \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \).

If \( 0 < k < n \), then, without loss of generality, we may put

\[
\mu_1 = \cdots = \mu_k = \rho, \quad \mu_{k+1} = \cdots = \mu_n = -\rho. \tag{1.5}
\]

Because \( M \) has parallel mean curvature vector, the mean curvature of \( M \) is constant. Thus, (1.5) implies that \( \rho \) is a positive constant. Therefore, \( M \) has homothetic Laplace transformation. Hence, by Theorem 2.3 of Chapter IV, \( M \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \).

The converse of this follows immediately from Theorem 2.3 of Chapter IV. □

Theorem 1.2 implies the following

**Corollary 1.3.** Let \( M \) be a hypersurface of \( \mathbb{E}^{n+1} \). Then \( M \) is an open part of a hypersphere in \( \mathbb{E}^{n+1} \) if and only if \( M \) has constant mean curvature function and conformal Laplace transformation. □

It is easy to see that the Laplace image of a minimal submanifold \( M \) of a hypersphere of \( \mathbb{E}^m \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) centered at the origin. So it is natural to ask the following problem.

**Problem 1.1.** When is the Laplace image (with the induced metric) of an isometric immersion \( x : M \to \mathbb{E}^m \) a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) centered at the origin? □

The following result gives a solution to Problem 1.1.

**Theorem 1.4.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional Riemannian manifold into \( \mathbb{E}^m \) with conformal Laplace transformation. Then the Laplace image \( L(M) \) (with respect to its induced metric) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) centered at the origin if and only if \( \Delta H = fH \) for some nonzero function \( f \) on \( M \).

**Proof.** Assume the Laplace transformation \( \mathcal{L} : M \to L(M) \) is conformal. Then the Laplace operator \( \tilde{\Delta} \) of \( L(M) \) (with respect to its induced metric) is given by \( \tilde{\Delta} = \rho^{-2} \Delta \) for some positive function \( \rho \). Hence the mean curvature vector \( \tilde{H} \) of \( L(M) \) in \( \mathbb{E}^m \) is given by \( \tilde{H} = \rho^{-2} \Delta H \).

If \( L(M) \) is immersed as a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) centered at the origin, then the mean curvature vector \( \tilde{H} \) of \( L(M) \) in \( \mathbb{E}^m \) satisfies \( \tilde{H} = c\bar{x} \),
where \( \bar{x} \) denotes the position vector field of \( L(M) \) in \( \mathbb{E}^m \) and \( c \) is a nonzero constant. Since \( \bar{x} = Lx = -nH \), we have \( \Delta H = \rho^2 \bar{H} = cp^2 \bar{x} = fH \), where \( f = -ncp^2 \neq 0 \).

Conversely, if \( \Delta H = fH \) for some non-zero function \( f \), then we get \( \bar{H} = \rho^{-2} \Delta H = fH = -\frac{c}{\rho^2} \bar{x} \). This shows that the position vector field of \( L(M) \) in \( E^m \) is normal to \( L(M) \). Hence, \( \langle \bar{x}, \bar{x} \rangle \) is a constant function on \( L(M) \). Thus, \( L(M) \) is contained in a hypersphere \( S^{m-1}(r) \) of \( E^m \) with radius \( r \) and centered at the origin. Let \( \bar{H} \) denote the mean curvature vector of \( L(M) \) in \( S^{m-1}(r) \). Then we have \( \bar{H} = \bar{H} - \frac{1}{\rho^2} \bar{x} \). Because \( \bar{H} \) is parallel to \( \bar{x} \), we get \( \bar{H} = 0 \), i.e., \( L(M) \) is a minimal submanifold of \( S^{m-1}(r) \). \( \square \)

Combining Lemma 1.3 of Chapter II and Theorem 1.4, we obtain the following.

**Corollary 1.5.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion with conformal Laplace transformation. Then the Laplace image \( L(M) \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) centered at the origin if and only if

1. \( \text{trace}( \bar{\nabla}A_H ) = 0 \) and
2. \( \Delta^D H + a(H) \) is parallel to \( H \), where \( a(H) \) denotes the allied mean curvature vector field of \( x \), \( D \) denotes the normal connection and \( \Delta^D \) the Laplace operator of the normal bundle. \( \square \)

From Theorem 1.4 we also have the following.

**Corollary 1.6.** If \( x : M \to \mathbb{E}^m \) is a null 2-type isometric immersion with conformal Laplace transformation, then the Laplace image lies in a hypersphere of \( \mathbb{E}^m \) as a minimal surface.

**Proof.** This follows from the fact that every null 2-type isometric immersion satisfies \( \Delta H = \lambda H \) for some nonzero constant. \( \square \)

Now, we give the following

**Lemma 1.7.** Let \( M \) be a hypersurface of \( \mathbb{E}^{n+1} \). If \( M \) has conformal Laplace transformation, then

1. the gradient of the square of mean curvature function, \( \text{grad} \alpha^2 \), is a principal direction on the open subset \( U = \{ p \in M : \nabla \alpha^2 \neq 0 \text{ at } p \} \) and
2. \( M \) has at most 3 distinct principal curvatures of the following forms:

\[
\nu, \quad \sqrt{\nu^2 + |\text{grad} (\ln \alpha)|^2}, \quad -\sqrt{\nu^2 + |\text{grad} (\ln \alpha)|^2},
\]

where the multiplicity of \( \nu \) is one on the open subset \( U \).

**Proof.** If \( M \) is a hypersurface of \( \mathbb{E}^{n+1} \) with conformal Laplace transformation, then Lemma 1.1 implies

\[
(1.6) \quad \langle A_H X, A_H Y \rangle + (X\alpha)(Y\alpha) = \rho^2 \langle X, Y \rangle,
\]
for $X,Y$ tangent to $M$, where $\rho$ is a positive function. Let $e_1, \ldots, e_n$ be a local orthonormal frame field given by eigenvectors of $A_H$ with eigenvalues $\mu_1, \ldots, \mu_n$, respectively. Then, from (1.6), we have $(e_i \alpha)(e_j \alpha) = 0$, $i \neq j$. Without loss of generality, we may assume
\begin{equation}
(1.7) \quad e_2 \alpha = \cdots = e_n \alpha = 0.
\end{equation}
Then $e_1$ is parallel to the gradient of $\alpha^2$, which proves statement (1).

(2) If $M$ has constant mean curvature, then Corollary 1.3 implies that $M$ has exactly one principal curvature. In this case statement (2) holds. So, we assume that the mean curvature is not constant, i.e., $U \neq \emptyset$. From (1.6) we have
\begin{equation}
(1.8) \quad \mu_i^2 + (e_i \alpha)^2 = \mu_j^2 + (e_j \alpha)^2, \quad i \neq j.
\end{equation}
Combining (1.7) and (1.8), we obtain
\begin{equation}
1.9 \quad \mu_j^2 = \mu^2 + (e_1 \alpha)^2, \quad \mu = \mu_1, \quad j = 2, \ldots, n.
\end{equation}
Let $\kappa_1, \ldots, \kappa_n$ be the principal curvatures of $M$ in $E^{n+1}$. Then $\mu_i = \alpha \kappa_i, i = 1, \ldots, n$. Thus, from (1.9), we have statement (2). □

If $M$ is a conformally flat hypersurface of dimension $n > 3$ in $E^{n+1}$, then, according to a well-known result of É. Cartan, the shape operator of $M$ in $E^{n+1}$ has an eigenvalue $\beta$ of multiplicity $\geq n - 1$. Let $\gamma$ denote the other eigenvalue of the shape operator at the points where the multiplicity of $\beta$ is $n - 1$ and let $\gamma = \beta$ when the multiplicity is $n$. Then the mean curvature function of $M$ is given by $n \alpha = (n - 1) \beta + \gamma$.

The following result characterizes the class of conformally flat hypersurfaces with conformal Laplace transformation.

**Theorem 1.8.** Let $M$ be an $n$-dimensional ($n > 3$) conformally flat hypersurface of $E^{n+1}$. Then $M$ has conformal Laplace transformation if and only if

- the two eigenvalues of the shape operator satisfy the following relation:
\begin{equation}
(1) \quad \alpha^2 \beta^2 = \alpha^2 \gamma^2 + |\nabla \alpha|^2
\end{equation}
on $U = \{p \in M : \nabla \alpha \neq 0 \text{ at } p\}$, and
- the gradients of the eigenvalues $\beta$ and $\gamma$ are parallel.

**Proof.** (1) follows easily from statement (2) of Lemma 1.6.

(2) Let $e_1, \ldots, e_n$ be a local orthonormal frame field of $M$ given by principal directions with
\begin{equation}
1.10 \quad A_{n+1} e_1 = \gamma e_1, \quad A_{n+1} e_i = \beta e_i, \quad i = 2, \ldots, n.
\end{equation}
Denote by $S$ and $\tau$ the Ricci tensor and the scalar curvature of $M$, respectively. Then we have

\begin{equation}
S(X,Y) = \sum_{i=1}^{n} <R(e_i, X)Y, e_i>,
\end{equation}

\begin{equation}
\tau = \sum_{i=1}^{n} S(e_i, e_i),
\end{equation}

where $R$ is the Riemann curvature tensor of $M$.

We put

\begin{equation}
L(X,Y) = -\frac{1}{n-2} S(X,Y) + \frac{\tau}{2(n-1)(n-2)} g(X,Y).
\end{equation}

Let $\hat{L}$ denote the $(1,1)$–tensor associated with $L$. It is well–known that if $M$ is a conformally flat manifold of dimension $n > 3$, then we have (cf. for instance, [C2])

\begin{equation}
(\nabla_X \hat{L})Y = (\nabla_Y \hat{L})X,
\end{equation}

for $X, Y$ tangent to $M$.

From (1.10), it follows that the Weingarten map of $M$ in $\mathbb{E}^{n+1}$ is given by

\begin{equation}
A_{n+1} = \beta I + (\gamma - \beta) \omega^1 \otimes e_1,
\end{equation}

where $I$ is the identity map.

Moreover, (1.10) also implies

\begin{equation}
\hat{L} = -\frac{\beta^2}{2} I + \beta(\beta - \gamma) \omega^1 \otimes e_1.
\end{equation}

From the Codazzi equation, we have

\begin{equation}
(\nabla_X A_{n+1})Y = (\nabla_Y A_{n+1})X.
\end{equation}

(1.15) and (1.17) yields

\begin{equation}
\hat{L} + \beta A_{n+1} = \frac{\beta^2}{2} I.
\end{equation}

From (1.14), (1.15), (1.17) and (1.18), we obtain

\begin{equation}
(X\beta)A_{n+1} Y - \beta(X\beta) Y = (Y\beta)A_{n+1} X - \beta(Y\beta) X.
\end{equation}

Setting $X = e_1, Y = e_i$ with $i = 2, \ldots, n$ in (1.19), we find $e_2\beta = \cdots = e_n\beta = 0$. Therefore, the gradient of $\beta$ is parallel to $e_1$. Consequently, by statement (1), we conclude that the gradients of the eigenvalues $\beta$ and $\gamma$ are parallel. □

Remark 1.1. Hypersurfaces of revolution $M$ in $\mathbb{E}^{n+1}$ automatically satisfy condition (2) of Theorem 1.8, whereas condition (1) amounts to an ordinary differential equation of order 3 to be satisfied by the planar profile curve of $M$; hence, many hypersurfaces of revolution do have conformal Laplace transformations. On the other hand, for instance, the hypercatenoids $M$ in $\mathbb{E}^{n+1}$, i.e. the hypersurfaces
of revolution whose profile curve is a catenary curve, do not satisfy condition (1); hence, the hypercatenoids have non-conformal Laplace transformations. □

By using Lemma 1.7 we obtain the following

**Theorem 1.9.** Let $M$ be a hypersurface of $\mathbb{E}^{n+1}$ $(n > 3)$. If $M$ has conformal Laplace transformation and positive semi–definite Ricci tensor, then $M$ is conformally flat.

**Proof.** If $M$ has conformal Laplace transformation, Lemma 1.4 implies that $M$ has at most 3 distinct principal curvatures of the following forms:

$$\nu, \delta, -\delta,$$

with multiplicities $1, k$ and $n-k-1$, respectively, where

$$\delta = \sqrt{\nu^2 + |\text{grad} \ln \alpha|^2}.$$

Without loss of generality, we may assume $\nu > 0$.

If $k = 0$ or $k = n - 1$, $M$ is a quasi–umbilical hypersurface. In this case, $M$ is conformally flat.

If $0 < k < n - 1$, then (1.20) implies that the Ricci tensor $S$ of $M$ satisfies

$$S(e_1,e_1) = (2k + 1 - n)\nu\beta, \quad S(e_n,e_n) = -\nu\beta + (n - 2k - 2)\beta^2.$$

Since $M$ is assumed to have positive semi–definite Ricci tensor, (1.21) gives $2k + 2 < n \leq 2k + 1$ which is impossible. □

§2. Surfaces in $\mathbb{E}^m$ with conformal Laplace transformation.

In this section we investigate surfaces of general codimension with conformal Laplace transformation.

First we give the following result.

**Proposition 2.1.** Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian surface into $\mathbb{E}^m$. If the Laplace transformation $\mathcal{L} : M \to L(M)$ is conformal, then the Laplace image $L(M)$ (with respect to its induced metric) is a minimal surface of $\mathbb{E}^m$ via $L$ if and only if the immersion $x$ is biharmonic.

**Proof.** Assume the Laplace transformation $\mathcal{L} : M \to L(M)$ is conformal. Let $g$ and $\bar{g}$ denote the Riemannian metrics of $M$ and $L(M)$, respectively. Then we have $\mathcal{L}^* \bar{g} = \rho^2 g$ for some positive function $\rho$ on $M$. Denote the Laplace operator of $(M,g)$ and $(M,\bar{g})$, respectively, by $\Delta$ and $\bar{\Delta}$. Then we have $\bar{\Delta} = \rho^{-2} \Delta$. With respect to $\bar{g}$, the Laplace map $L : (M,\bar{g}) \to L(M) \subset \mathbb{E}^m$ is isometric. Hence, we have

$$-2\bar{H} = \rho^{-2} \Delta H,$$
where $H$ and $\bar{H}$ denote the mean curvature vector of $x$ and of $L$, respectively. From (2.1) we conclude that $L(M)$ is a minimal surface in $E^m$ if and only if the original immersion $x$ is biharmonic. □

For flat ruled surfaces in $E^m$ we have the following.

**Proposition 2.2.** If $M$ is a flat ruled surface in $E^m$, then its Laplace transformation is not conformal.

**Proof.** As we mentioned in Chapter III, flat ruled surfaces in $E^m$ are “in general” cylinders, cones or tangential developables of curves. If $M$ is a cylinder in $E^m$, the differential $dL$ (or $L_*$) of its Laplace map has rank $\leq 1$. So, its Laplace transformation is not conformal.

Let $M$ be a cone in $E^m$. Without loss of generality, we may assume the vertex of the cone is at the origin and the cone is parametrized by

\begin{equation}
(2.2) \quad x(t,s) = t\beta(s), \quad |\beta| = |\beta'| = 1.
\end{equation}

From the proof of Proposition 3.2 of Chapter III, we know that the Laplace map of $M$ is also a cone given by

\begin{equation}
(2.3) \quad L(t,s) = \frac{1}{t}((\beta''(s),\beta(s))\beta(s) - \beta''(s)),
\end{equation}

which implies

\begin{equation}
(2.4) \quad dL\left(\frac{\partial}{\partial t}\right) = (\beta''(s),\beta(s))\beta(s) - \beta''(s),
\end{equation}

\begin{equation}
(2.5) \quad dL\left(\frac{\partial}{\partial s}\right) = \frac{1}{t}((\beta''(s),\beta(s))\beta(s) + (\beta'',\beta)\beta' - \beta'''(s)).
\end{equation}

(2.4) and (2.5) show that the metric tensor of $L(M)$ is given by

\begin{equation}
\bar{g}_{11} = (\beta'',\beta)(1 - 2(\beta'',\beta)) + |\beta''|^2,
\end{equation}

\begin{equation}
\bar{g}_{12} = \frac{1}{t}((\beta'',\beta') - (\beta'',\beta)(\beta'',\beta)),
\end{equation}

\begin{equation}
\bar{g}_{22} = \frac{1}{t^2}((|\beta''|^2 - (\beta'',\beta)^2 + |\beta'''|^2 - (\beta'',\beta)(\beta',\beta'''))).
\end{equation}

On the other hand, (2.2) implies that the metric tensor of $M$ is given by

\begin{equation}
(2.7) \quad g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = t^2.
\end{equation}

Comparing (2.6) and (2.7) we conclude that the Laplace transformation of $M$ is not conformal.

If $M$ is a tangential developable surface in $E^m$ given by the tangent lines of a unit speed curve $\beta(s)$ in $E^m$, then $M$ can be parametrized by

\begin{equation}
(2.8) \quad x(s,t) = \beta(s) + t\beta'(s), \quad |\beta'(s)| = 1.
\end{equation}
Let \( \kappa_1 \) denote the first Frenet curvature and \( \beta_2 \) the second Frenet vector of \( \beta \) in \( \mathbb{E}^m \). From the proof of Proposition 3.3 of Chapter III we know that the Laplace map of \( M \) is given by

\[
L(s, t) = -\frac{1}{t\kappa_1} \beta_2(s).
\]

By computing the metric tensors of \( M \) and \( L(M) \) and using (2.8) and (2.9), we conclude that the Laplace transformation of a tangential developable surface is also not conformal. \( \square \)

The following result shows that if a submanifold \( M \) of \( \mathbb{E}^m \) has a conformal Laplace transformation and if its Laplace image \( L(M) \) is a minimal surface of a hypersphere of \( \mathbb{E}^m \), then \( M \) is not necessary a minimal submanifold of a hypersphere of \( \mathbb{E}^m \).

**Proposition 2.3.** If \( M \) is a surface in \( \mathbb{E}^6 \) defined by

\[
x(u, v) = (au, av, b\cos u, b\sin u, b\cos v, b\sin v)
\]

for some nonzero constants \( a \) and \( b \), then

1. \( M \) has homothetic Laplace transformation,
2. the Laplace image \( L(M) \) is a minimal submanifold of a hypersphere of \( \mathbb{E}^6 \) centered at the origin and,
3. the surface \( M \) is not spherical.

**Proof.** This follows from straight-forward computations. \( \square \)

As an application of Corollary 1.5 we give the following

**Theorem 2.3.** Let \( x : M \to \mathbb{E}^4 \) be an isometric immersion with conformal Laplace transformation. Then the Laplace image \( L(M) \) of \( x \) is a minimal surface of a hypersphere of \( \mathbb{E}^4 \) centered at the origin if and only if \( M \) is a minimal surface of a hypersphere.

**Proof.** Assume \( M \) is a minimal surface of a hypersphere \( S^3 (r) \) of \( \mathbb{E}^4 \). Then without loss of generality we may assume \( S^3 (r) \) is centered at the origin. In this case, we have \( L = \frac{2}{r} x \). This implies that the Laplace image \( L(M) \) lies in the hypersphere \( S^3 (\frac{2}{r}) \) as a minimal surface.

Conversely, assume that the Laplace transformation is conformal and the Laplace image \( L(M) \) lies in a hypersphere centered at the origin as a minimal surface. Then \( M \) has constant mean curvature in \( \mathbb{E}^4 \), since \( L = -2H \).

Let \( e_1, e_2, e_3, e_4 \) be a local orthonormal frame field defined on \( M \) such that \( H = \alpha e_3 \) and \( e_1, e_2 \) are eigenvectors of \( A_3 \) with \( A_3 e_1 = \kappa_1 e_1, A_3 e_2 = \kappa_2 e_2 \). If the Laplace transformation is conformal, then Lemma 1.1 implies \( (D_{e_1} e_3)(D_{e_2} e_3) = 0 \). Since \( e_3 \) is perpendicular to both \( D_{e_1} e_3, D_{e_2} e_3 \), at least one of \( D_{e_1} e_3, D_{e_2} e_3 \)
vanishes. Without loss of generality, we may assume $D_e, e_3 = 0$. From Lemma 1.1 we find
\begin{equation}
\alpha^2 \kappa_1^2 = \alpha^2 \kappa_2^2 + \phi^2, \quad \phi = \omega_3 (e_2).
\end{equation}

On the other hand, since $M$ has constant mean curvature, Corollary 1.5 implies
\[ 0 = \text{trace}(\nabla A_H) = \text{trace}(A_{DH}) = \phi A_4 e_2. \]
Therefore, either $\phi = 0$ or $A_4 e_2 = 0$.

If $\phi = 0$, then $DH = 0$ and (2.10) implies $\kappa_1 = \pm \kappa_2$. Since the Laplace transformation is conformal, $\alpha$ is nonzero. Thus, $\kappa_1 = \kappa_2$. Hence, $M$ is a pseudo–umbilical surface with parallel mean curvature vector. By applying Theorem 1.1 of Chapter II, $M$ is contained in a hypersphere of $E^4$ as a minimal surface.

If $\phi \neq 0$, then $A_4 e_2 = 0$. Hence, by trace $A_4 = 0$, we get $A_4 = 0$. So, by applying Codazzi’s equation, we obtain $\kappa_1 = 0$. Combining this with (2.10) we find $\phi = 0$, which is a contradiction. □

§3. Surfaces in $E^3$ with conformal Laplace transformation.

In this section we investigate surfaces in $E^3$ with conformal Laplace transformation. Since such surfaces are abundant, the complete classification of such surfaces is formidable.

Proposition 3.1. Let $M$ be a surface in $E^3$ with nonzero mean curvature function. Then $M$ has conformal Laplace transformation if and only if the following three conditions hold:

1. the gradient $\nabla \alpha^2$ of the mean curvature function is a principal direction on the open subset $U$ of $M$ where $\nabla \alpha \neq 0$;
2. the Gauss curvature $K$ of $M$ is given by $K = \alpha^2 - \frac{1}{16\alpha^6} |\nabla \alpha|^4$; and
3. the mean curvature function satisfies the following equation:
\[ \Delta \alpha^2 = 4\alpha^4 - 5|\nabla \alpha|^2. \]

Proof. If $M$ is a surface in $E^3$ with conformal Laplace transformation, then Lemma 1.7 implies that the gradient $\nabla \alpha^2$ is a principal direction on $U = \{ p \in M : \nabla \alpha^2 \neq 0 \text{ at } p \}$. Moreover, from Lemma 1.7, $A_H$ also satisfies
\begin{equation}
A_H e_1 = \mu e_1, \quad A_H e_2 = \pm \sqrt{\mu^2 + |\nabla \alpha|^2} e_2,
\end{equation}
where $e_1$ is choosen in the direction of $\nabla \alpha^2$. (3.1) yields
\[ 2\alpha^2 = \mu \pm \sqrt{\mu^2 + |\nabla \alpha|^2}. \]
Thus we obtain

\begin{equation}
\mu = \alpha^2 - \frac{|\nabla \alpha|^2}{4\alpha^2},
\end{equation}

\begin{equation}
\mu^2 + |\nabla \alpha|^2 = (\alpha^2 + \frac{|\nabla \alpha|^2}{4\alpha^2})^2.
\end{equation}

From (3.2) and (3.3) we get

\[
A_3 e_1 = \left( \alpha - \frac{|\nabla \alpha|^2}{4\alpha^3} \right) e_1, \quad A_3 e_2 = \pm \left( \alpha + \frac{|\nabla \alpha|^2}{4\alpha^3} \right) e_2.
\]

Because \( \alpha \neq 0 \) and the two eigenvalues of \( A_3 \) satisfy \( 2\alpha = \kappa_1 + \kappa_2 \), we must have

\begin{equation}
\kappa_1 = \alpha - \frac{|\nabla \alpha|^2}{4\alpha^3}, \quad \kappa_2 = \alpha + \frac{|\nabla \alpha|^2}{4\alpha^3}.
\end{equation}

Therefore, the Gauss curvature \( K \) of \( M \) is given by

\begin{equation}
K = \alpha^2 - \frac{1}{16\alpha^6} |\nabla \alpha|^4.
\end{equation}

Put \( \omega_1 = f_1 \omega^1 + f_2 \omega^2 \). Then, by Codazzi’s equation, (3.5), \( e_2 \alpha = 0 \), and a direct computation, we find

\begin{equation}
e_2 e_1 \alpha = f_1 e_1 \alpha,
\end{equation}

\begin{equation}
e_1 e_1 \alpha = \frac{3}{2\alpha} (e_1 \alpha)^2 - f_2 e_1 \alpha - 2\alpha^3.
\end{equation}

From (3.6), (3.7) and \( e_2 \alpha = 0 \), we get

\begin{equation}
\Delta \alpha = -e_1 e_1 \alpha - f_2 e_1 \alpha = 2\alpha^3 - \frac{3}{2\alpha} (e_1 \alpha)^2.
\end{equation}

Therefore

\[
\Delta \alpha^2 = 2\alpha \Delta \alpha - 2(e_1 \alpha)^2 = 4\alpha^4 - 5|\nabla \alpha|^2.
\]

Conversely, assume \( M \) is a surface in \( \mathbb{E}^3 \) satisfying conditions (1) and (2) of the Proposition. Then we have

\[
e_2 \alpha = 0
\]
on \( U \) and

\begin{equation}
\kappa_1 \kappa_2 = \alpha^2 - \frac{(e_1 \alpha)^4}{16\alpha^6}.
\end{equation}

Because \( \kappa_1 = 2\alpha - \kappa_1 \), (3.9) yields

\begin{equation}
\kappa_1 = \alpha \pm \frac{(e_1 \alpha)^2}{4\alpha^3},
\end{equation}

\begin{equation}
\kappa_2 = \alpha \mp \frac{(e_1 \alpha)^2}{4\alpha^3}.
\end{equation}

If \( \kappa_1 = \alpha - \frac{(e_1 \alpha)^2}{4\alpha^3} \) \( \kappa_2 = \alpha + \frac{(e_1 \alpha)^2}{4\alpha^3} \), then it is easy to see that \( \alpha^2 \kappa_1^2 + (e_1 \alpha)^2 = \alpha^2 \kappa_2^2 \).

Thus, by using \( e_2 \alpha = 0 \), we conclude that \( M \) has conformal Laplace transformation by Lemma 1.1. If \( \kappa_1 = \alpha + \frac{(e_1 \alpha)^2}{4\alpha^3} \) \( \kappa_2 = \alpha - \frac{(e_1 \alpha)^2}{4\alpha^3} \), then, by using Codazzi’s equation
and \(e_2 \alpha = 0\), we conclude that the mean curvature does not satisfy condition (3) of the Proposition. Consequently, we conclude that if conditions (1), (2) and (3) of the Proposition hold, then \(M\) has conformal Laplace transformation. \(\square\)

If \(M\) is a compact surface, we have the following characterization theorem.

**Theorem 3.2.** Let \(M\) be a compact surface in \(\mathbb{E}^3\) whose mean curvature function is nonzero. Then \(M\) has conformal Laplace transformation if and only if we have:

1. \(\nabla \alpha^2\) is a principal direction on the open subset \(U\) of \(M\) where \(\nabla \alpha^2 \neq 0\), and
2. the Gauss curvature \(K\) of \(M\) is given by \(K = \alpha^2 - \frac{1}{16\alpha^2} |\nabla \alpha|^4\).

**Proof.** Let \(M\) be a compact surface in \(\mathbb{E}^3\) whose mean curvature function is nonzero. If \(M\) has conformal Laplace transformation, Proposition 3.1 implies (1) and (2) hold.

Conversely, if \(M\) is compact and (1) and (2) hold, then from the proof of Proposition 3.1 we know that either \(M\) has conformal Laplace transformation or

\[
\begin{align*}
\kappa_1 &= \alpha + (e_1 \alpha)^2, \\
\kappa_2 &= \alpha - (e_1 \alpha)^2, \\
de_2 \alpha &= 0.
\end{align*}
\]

Put \(\omega_1 = f_1 \omega^1 + f_2 \omega^2\) as before. Then, by (3.12), Codazzi’s equation and direct computation, we obtain

\[
e_2 e_1 \alpha = f_1 e_1 \alpha, \quad e_1 e_1 \alpha = 2\alpha^3 - f_2 e_1 \alpha - \frac{3(e_1 \alpha)^2}{2\alpha}.
\]

Thus, we get

\[
\Delta \alpha = 2\alpha^3 + \frac{3(e_1 \alpha)^2}{2\alpha}.
\]

Consequently, we find

\[
\Delta \alpha^2 = 2\alpha \Delta \alpha - 2(e_1 \alpha)^2 = 4\alpha^4 + |\nabla \alpha|^2
\]

Because \(M\) is assumed to be compact, this implies \(\alpha \equiv 0\) which is a contradiction. \(\square\)

The following result shows that examples of surfaces in \(\mathbb{E}^3\) with conformal Laplace transformation are abundant.

**Proposition 3.3.** There exist infinitely many surfaces of revolution in \(\mathbb{E}^3\) whose Laplace transformations are conformal, but not homothetic.

**Proof.** Let \(M\) be a surface of revolution parametrized by

\[
x(t, \theta) = (t, f(t) \cos \theta, f(t) \sin \theta).
\]
Then

\[(3.16) \quad \frac{\partial}{\partial t} = (1, f'(t) \cos \theta, f'(t) \sin \theta), \quad \frac{\partial}{\partial \theta} = (0, -f(t) \sin \theta, f(t) \cos \theta).\]

From (3.15) and (3.16) we obtain

\[(3.17) \quad L(t, \theta) = \frac{1 + f'^2 - ff''}{f(1 + f'^2)^2}(-f', \cos \theta, \sin \theta).\]

By using (3.17) we can prove that the Laplace transformation is conformal if and only if \(f = f(t)\) satisfies the following third order ordinary differential equation:

\[(3.18) \quad Af''' + Bf'' + C = 0,\]

where

\[
A = f^4(1 + f'^2)^3f''', \quad 
B = 2f^2(1 + f'^2)((1 + f'^2)(2ff'' + f'(1 + f'^2)) - 4f^2 f' f''
+ f'^2(1 + f'^2)^2 + 2f f''(f'^4 + f f'' - 1 - 3f f'^2 f''))
\]

\[(3.19) \quad C = \{(1 + f'^2)(2ff'' + f'(1 + f'^2)) - 4f^2 f'^2 f''\}^2
+ \{f'^2(1 + f'^2)^2 + 2ff''(f'^4 + f f'' - 1 - 3f f'^2 f'')\}^2
- (1 + f'^2)^3(1 + f'^2 - f f''^2).\]

It is easy to verify that, for infinitely many values of \(a = f(0), b = f'(0), c = f''(0)\), the value of the discriminant \(D = D(f, f', f'') = B^2 - 4AC\) at \(t = 0\) is positive.

Consider the third order ordinary differential equation:

\[(3.20) \quad f''' = F(f, f', f''), \quad f = \frac{1}{2A}(-B + \sqrt{B^2 - 4AC}).\]

For \(a = f(0), b = f'(0), c = f''(0)\) with \(a \neq 0\) and \(D(a, b, c) > 0\), the initial value problem:

\[f''' = F(f, f', f''), \quad f(0) = a, \quad f'(0) = b, \quad f''(0) = c\]

has a unique solution \(f = f(t)\), in an open neighborhood of 0. Such a solution \(f = f(t)\) of the initial value problem gives rise to a surface of revolution in \(E^3\) with conformal Laplace transformation. Generically, such surfaces of revolution do not have homothetic Laplace transformation. \(\Box\)

Finally, we give the following.

**Proposition 3.4.** The Laplace image of a surface \(M\) in \(E^3\) with conformal Laplace transformation is not a minimal surface in \(E^3\) (with respect to its induced metric).

**Proof.** Assume \(M\) is a surface in \(E^3\) with conformal Laplace transformation. Then the Laplace operator of the Laplace image \(L(M)\) satisfies \(\Delta = \rho^{-2} \Delta\) for some positive function \(\rho\) and the mean curvature vector \(\vec{H}\) of \(L(M)\) in \(E^3\) is given by \(\vec{H} = \rho^{-2} \Delta H\). Thus, the Laplace image \(L(M)\) is a minimal surface of \(E^3\) if and only
if $M$ is a biharmonic surface. Since the only biharmonic surfaces in $\mathbb{E}^3$ are minimal surfaces and minimal surfaces in $\mathbb{E}^3$ do not have conformal Laplace transformation, the Laplace image $L(M)$ cannot be a minimal surface in $\mathbb{E}^3$. □
Chapter VI: GEOMETRY OF LAPLACE IMAGES

§1. Immersions whose Laplace images lie in a cone.

From the proof of Propositions 3.2 and 3.3 of Chapter III, we know if \( M \) is a cone in \( \mathbb{E}^m \) with vertex at the origin or a tangential developable surface, then the Laplace image of \( M \) lies in a cone in \( \mathbb{E}^m \) with vertex at the origin. So, it is natural to ask the following problem:

**Problem 1.1.** “When the Laplace image of a surface lies in a cone?”

The purpose of this section is to investigate this Problem.

First, we prove the following

**Theorem 1.1.** Let \( M \) be a surface in \( \mathbb{E}^3 \) with regular Laplace map. Then the Laplace image of \( M \) lies in a cone with vertex at the origin if and only if \( M \) is locally a cone with vertex at the origin or a tangential developable surface.

**Proof.** Let \( M \) be a surface in \( \mathbb{E}^3 \) with regular Laplace map. If the Laplace image of \( M \) lies in a cone with vertex at the origin. Then the Laplace image can be locally reparametrized by

\[
L(s, t) = t\beta(s).
\]

Since \( L \) is regular, \( s, t \) can be considered as local coordinates of \( M \), too. With respect to \((s, t)\) we have

\[
dL \left( \frac{\partial}{\partial s} \right) = 2A_H \left( \frac{\partial}{\partial s} \right) - 2 \frac{\partial}{\partial s} e_3 = t\beta'(s),
\]

\[
dL \left( \frac{\partial}{\partial t} \right) = 2A_H \left( \frac{\partial}{\partial t} \right) - 2 \frac{\partial}{\partial t} e_3 = \beta(s),
\]

where \( H = \alpha e_3 \) and \( e_3 \) is a unit normal vector of \( M \) in \( \mathbb{E}^3 \). Because \( L = -2H \), (1.1) implies that \( \beta \) is normal to the surface \( M \) in \( \mathbb{E}^3 \). Thus, (1.2) yields

\[
A_H \left( \frac{\partial}{\partial t} \right) = 0.
\]

Since \( L \) is assumed to be regular, \( H \neq 0 \). Thus, (1.3) implies \( M \) is a flat surface in \( \mathbb{E}^3 \). Hence, \( M \) is locally a cylinder, a cone or a tangential developable surface. Because \( L \) is regular, \( M \) cannot be a cylinder. Therefore, \( M \) is locally a tangential developable surface or a cone. If it is a cone with vertex at a point \( p \), then from the proof of Proposition 3.2 of Chapter III we see that the Laplace image lies in a cone with vertex also at \( p \). So, in this case, \( M \) is a cone with vertex at the origin.

The converse follows from Propositions 3.2 and 3.3 of Chapter III. □
Remark 1.1. In view of Theorem 1.1, it is interesting to point out that there exist non-flat surfaces in $E^3$ whose Laplace images are contained in a cone with vertex at a point other than the origin. In fact, by applying the fundamental theorem of ordinary differential equations, we can prove that there exist surfaces of revolution which are non-flat and whose Laplace images are cones with vertices at points other than the origin of $E^3$. □

The following theorem yields a general result for surfaces in $E^m$ whose Laplace images lie in cone with vertex at the origin.

**Theorem 1.2.** Let $x : M \to E^m$ be a surface in $E^m$ with regular Laplace map. If the Laplace image of $M$ lies in a cone with vertex at the origin, then $M$ is non-positively curved, i.e., the Gauss curvature $K$ of $M$ is $\leq 0$. Furthermore, if $K = 0$, then locally $M$ is either a cone with vertex at the origin or a tangential developable surface.

**Proof.** Let $M$ be a surface in $E^m$ with regular Laplace map. If the Laplace image of $M$ lies in a cone with vertex at the origin, then the Laplace image can be locally parametrized by

$$L(s, t) = t\beta(s), \quad |\beta| = |\beta'| = 1.$$ 

From (1.4) we may choose $e_3 = \beta$ with $H = \alpha e_3$ and $t = -2\alpha$.

Since $L$ is regular, $s, t$ can be considered as local coordinates of $M$, too. With respect to $(s, t)$ we have

$$dL\left(\frac{\partial}{\partial s}\right) = 2A_H\left(\frac{\partial}{\partial s}\right) + tD_\alpha e_3 = t\beta'(s),$$

$$dL\left(\frac{\partial}{\partial t}\right) = 2A_H\left(\frac{\partial}{\partial t}\right) + e_3 + tD_\alpha e_3 = \beta(s).$$

(1.5) yields

$$A_H\left(\frac{\partial}{\partial t}\right) = 0, \quad D_\alpha e_3 = 0,$$

(1.6)

$$\beta' = -A_3\left(\frac{\partial}{\partial s}\right) + D_\alpha e_3.$$ 

By choosing $e_2$ in the direction of $\frac{\partial}{\partial \tau}$, and $e_1$ a unit tangent vector of $M$ perpendicular to $e_2$, we have

$$A_3 e_1 = -t e_1, \quad A_3 e_2 = 0.$$ 

(1.8)

Since $e_3$ is parallel to the mean curvature vector, we have

$$\text{trace } A_4 = \cdots = \text{trace } A_m = 0,$$

(1.9)

where $A_r = A_r e_r$ and $e_3, \ldots, e_m$ is a local orthonormal frame field of the normal bundle of $M$ in $E^m$. Thus, $\det(A_r) \leq 0, r = 4, \ldots, m$. Hence, by (1.8) and (1.9), we conclude that the Gauss curvature $K$ of $M$ satisfies $K \leq 0$. 


If $K = 0$, then (1.8) and (1.9) yield

\[ A_4 = \cdots = A_m = 0. \]

From (1.6), (1.8) and (1.10) we have

\[ \omega_3^1 = -t\omega^1, \quad \omega_3^2 = 0, \quad \omega_r^i = 0, \quad i = 1, 2, \quad r = 4, \ldots, m. \]

Taking exterior derivative of $\omega_3^2 = 0$ and applying (1.11) and the structure equations, we find $\omega_3^2(e_2) = 0$. Thus, $t$-curves on $M$ are geodesics. Hence, by using (1.8) and (1.10), we conclude that $t$-curves on $M$ are straight lines in $E^m$. Hence, in this case, $M$ is a flat ruled surface. Therefore, locally, $M$ is a cylinder, a cone, or a tangential developable surface. Because the Laplace map is assumed to be regular, $M$ cannot be a cylinder. It $M$ is a cone, then its vertex must be at the origin, since the Laplace image lies in a cone with vertex at the origin. \( \square \)

In view of Theorem 1.2, it is interesting to give the following.

**Proposition 1.3.** There exist infinitely many negatively curved surfaces in $E^4$ whose Laplace images lie in a cone with vertex at the origin.

**Proof.** Let $b$ and $c$ be real numbers such that $b > 0$ and $c > 4b^2$. Let $D$ be the unit disk of $E^2$ centered at $(0, 1)$ with Riemannian metric given by

\[ g = \frac{1}{4v^2} du^2 + \frac{1}{\mu^2 v^2} dv^2, \quad \mu^2 = v^2(c - b^2 v^2). \]

Put

\[ e_1 = -2v \frac{\partial}{\partial u}, \quad e_2 = \mu v \frac{\partial}{\partial v}. \]

Denote by $\omega^1, \omega^2$ the dual frame of $e_1, e_2$. Then we have

\[ \omega^1 = -\frac{1}{2v} du, \quad \omega^2 = \frac{1}{\mu v} dv, \quad \omega_i^1 = \mu \omega^1. \]

Let $F = D \times E^2$ be the rank 2 trivial bundle over $D$. With the usual Euclidean metric on fibres, $F$ is a Riemannian vector bundle with usual connection $D$. Let $e_3, e_4$ be the canonical orthonormal frame of the fibre $E^2$.

We define a bilinear map $h$ by

\[ h(e_1, e_1) = 2ve_3 + bv^2 e_4, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = -bv^2 e_4. \]

Then, by direct computation, we can prove that $(D, g)$ together with $D, h$ satisfy the equations of Gauss, Codazzi and Ricci. Hence, by the fundamental theorem of submanifolds, we conclude that there exists an isometric immersion of $(D, g)$ into $E^4$ with $F$ as its normal bundle and $h$ as its second fundamental form.

Condition (1.15) says that the Gauss curvature of $D$ is given by $K = -b^2 v^2 < 0$.

Now, we claim that the Laplace image of such a surface lies in a cone with vertex at the origin. This can be seen as follows.
From (1.15) we know that the Laplace map of such a surface is given by

\[(1.16) \quad L(u, v) = -2ve_3.\]

From (1.15) and \(De_3 = 0\) we get \(\frac{\partial e_3}{\partial u} = 0\), which shows that \(e_3\) is a function of \(u\). Therefore, (1.16) implies that the Laplace image is contained in a cone with vertex at the origin. □

**Proposition 1.4.** Every surface constructed in Proposition 1.3 is the locus of a planar curve \(\gamma(s)\) moving along a space curve; moreover the planar curves are congruent to curves of the following form:

\[(1.17) \quad \left(\frac{b}{c} \sinh^{-1}\left(\frac{cs}{b}\right), -\frac{1}{c} \sqrt{b^2 + c^2 s}\right).\]

where \(b\) and \(c\) are nonzero constants.

**Proof.** First we observe from (1.13) and \(\omega_1^2 = \mu \omega^1\) in (1.14) that \(v\)-curves of the surface are geodesics. Let \(\gamma(v)\) be a \(v\)-curve with \(s\) as its arclength parameter.

Denote by \(\nabla\) the Euclidean connection of \(\mathbb{E}^4\). From (1.14) and (1.15) we get

\[(1.17) \quad \nabla_{e_2} e_2 = -bv^2 e_4, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = bv^2 e_2.\]

From (1.17) it follows that the \(v\)-curve \(\gamma\) is a plane curve whose curvature function is given by

\[(1.18) \quad \kappa = -bv^2.\]

From (1.12) and (1.13) we see that \(s\) and \(v\) are related by

\[(1.19) \quad s = \int \frac{dv}{v^2 \sqrt{c - b^2 v}} = -\frac{\sqrt{c - b^2 v}}{cv} + a.\]

where \(a\) is an integration constant. Without loss of generality, we may assume \(a = 0\). From (1.18) and (1.19) we find

\[(1.20) \quad \kappa = -\frac{bc}{b^2 + c^2 s^2}.\]

Put

\[(1.21) \quad \gamma'(s) = (\cos \theta(s), \sin \theta(s)).\]

Then

\[(1.22) \quad \frac{d\theta}{ds} = -\frac{bc}{b^2 + c^2 s^2}.\]

So we obtain

\[(1.23) \quad \theta(s) = -\tan^{-1}\left(\frac{cs}{b}\right) + c_2,\]

where \(c_2\) is a constant. Without loss of generality, we may assume \(c_2 = 0\). From (1.23) we have

\[(1.24) \quad \gamma'(s) = \left(\frac{b}{\sqrt{b^2 + c^2 s^2}}, -\frac{cs}{\sqrt{b^2 + c^2 s^2}}\right).\]
By taking integration of $\gamma'(s)$ with respect to $s$, we obtain the Proposition. □ The final result of this section is the following theorem.

**Theorem 1.5.** Let $M$ be a non-flat surface in $\mathbb{E}^m$ with regular Laplace map and parallel normalized mean curvature vector. If the Laplace image of $M$ lies in a cone with vertex at the origin, then $M$ is a locus of planar curves moving along a space curve, moreover, the planar curves are congruent to curves of the following form:

$$\left( \frac{b}{c} \sinh^{-1} \left( \frac{cs}{b} \right), -\frac{1}{c} \sqrt{b^2 + c^2 s} \right),$$

where $b$ and $c$ are real numbers with $b > 0$.

**Proof.** Let $M$ be a surface in $\mathbb{E}^m$ with regular Laplace map and parallel normalized mean curvature vector. If the Laplace image of $M$ lies in a cone with vertex at the origin, then the Laplace image can be locally parametrized by

$$(1.25) \quad L(s, t) = t\beta(s), \quad |\beta| = |\beta'| = 1.$$  

From (1.25), we may choose $e_3 = \beta$ with $H = \alpha e_3$ and $t = -2\alpha$. Since $M$ has parallel normalized mean curvature vector, $De_3 = 0$.

Since $L$ is regular, $s, t$ can be considered as local coordinates of $M$, too. With respect to $(s, t)$ we have

$$(1.26) \quad dL \left( \frac{\partial}{\partial s} \right) = 2AH \left( \frac{\partial}{\partial s} \right) = t\beta'(s),$$

$$dL \left( \frac{\partial}{\partial t} \right) = 2AH \left( \frac{\partial}{\partial t} \right) + e_3 = \beta(s),$$

The first equation in (1.26) shows that $\beta'$ is tangent to $M$ in $\mathbb{E}^m$. The second equation in (1.26) yields

$$(1.27) \quad AH \left( \frac{\partial}{\partial t} \right) = 0.$$  

By choosing $e_2$ in the direction of $\frac{\partial}{\partial t}$, and $e_1$ a unit tangent vector of $M$ perpendicular to $e_2$, we have

$$(1.28) \quad A_3 e_1 = -te_1, \quad A_3 e_2 = 0.$$  

Since $\beta'(s)$ is tangent to $M$ and $\partial^2 \beta'/\partial t^2 = 0$, the equation of Gauss yields

$$(1.29) \quad \nabla_{\frac{\partial}{\partial s}} \beta' = 0, \quad h(\beta', e_2) = 0.$$  

Put $\frac{\partial}{\partial s} = a_1 e_1 + a_2 e_2$. Then

$$t\beta' = 2AH \left( \frac{\partial}{\partial s} \right) = 2a_1 A_H e_1 + 2a_2 A_H e_2 = 4a_1 \alpha^2 e_1.$$  

Hence

$$(1.30) \quad \frac{\partial}{\partial s} = \frac{1}{t} e_1 + a_2 e_2, \quad e_1 = \beta'.$
(1.28) and $De_3 = 0$ imply that $M$ has flat normal connection, i.e., $R^D = 0$. Hence, we may choose $e_3, \ldots, e_m$ such that

\[(1.31) \quad \omega_1^4 = \eta \omega^1, \quad \omega_2^4 = -\eta \omega^2, \quad A_5 = \cdots = A_m = 0.\]

From (1.28), (1.29) and (1.31) we have

\[(1.32) \quad h(e_1, e_1) = 2 \alpha e_3 + \eta e_4, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = -\eta e_4.\]

Also, from (1.28) and (1.30), we obtain $\omega_2^2(e_2) = 0$, which implies $t$–curves of $M$ are geodesics. We put

\[(1.33) \quad \omega_1^2 = f \omega^1.\]

From (1.32), (1.33), $De_3 = 0$ and the equation of Codazzi, we obtain

\[(1.34) \quad e_1 \eta = 0, \quad e_2 \eta = 2f \eta, \quad e_2 \alpha = f \alpha,\]

\[(1.35) \quad \eta De_2 e_4 = \eta De_1 e_4 = 0.\]

Because $M$ is assumed to be non–flat, $\eta \neq 0$. So, (1.35) implies

\[(1.36) \quad De_4 = 0.\]

By using (1.32), (1.33) and (1.36) we obtain

\[(1.37) \quad \tilde{\nabla} e_2 e_1 = 0, \quad \tilde{\nabla} e_2 e_2 = -\eta e_4, \quad \tilde{\nabla} e_2 e_3 = 0, \quad \tilde{\nabla} e_2 e_4 = \eta e_2.\]

Because $d\omega^2 = -\omega_1^2 \wedge \omega^1 = 0$ by (1.33), $\omega^2 = d\sigma$ for some function $\sigma$. We put

\[(1.38) \quad d\sigma = \sigma_s ds + \sigma_t dt.\]

Since $\omega^2(e_1) = 0$, (1.30) yields $a_2 = \sigma_s$. Therefore, (1.30) gives

\[(1.39) \quad \frac{\partial}{\partial s} = \frac{1}{t} e_1 + \sigma_s e_2.\]

Similarly, we may also obtain

\[(1.40) \quad \frac{\partial}{\partial t} = \sigma_t e_2.\]

Because $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$, (1.39) and (1.40) imply $\sigma_t = \frac{1}{tf}$. Consequently, $e_2 = tf \frac{\partial}{\partial t}$. Therefore, the second equation in (1.34) implies $\eta = bt^2$ for some nonzero constant $b$ along a $t$–curve in $M$. Combining this with (1.37) we may conclude as in the proof of Proposition 1.4 that $t$–curves are plane curves which are congruent to the curve mention in the Proposition. □

§2. Laplace images of surfaces of revolution.

In this section we study the Laplace images of surfaces of revolution.

First we make the following observation.
Proposition 2.1. Let $M$ be a surface of revolution in $\mathbb{E}^3$ about an axis $L$. If $M$ is non-minimal, then the Laplace map of $M$ is also a surface of revolution about the same axis.

Proof. Let $M$ be a surface of revolution in $\mathbb{E}^3$. Without loss of generality, we may assume the axis of $M$ is the $x$-axis. So, $M$ can be parametrized by

$$x(t, \theta) = (t, f(t) \cos \theta, f(t) \sin \theta).$$

By a direct computation, we see that the Laplace operator is given by

$$\Delta = -\frac{1}{1 + f'^2} \frac{\partial^2}{\partial t^2} - \frac{f' + f'^3 - f f''}{f(1 + f'^2)^2} \frac{\partial}{\partial t} - \frac{1}{f'^2} \frac{\partial^2}{\partial \theta^2}.$$

By (2.2) we may conclude that the Laplace map of the surface in $\mathbb{E}^3$ is given by

$$L(t, \theta) = \left(\frac{1 + f'^2 - f f''}{f(1 + f'^2)^2}\right)(-f', \cos \theta, \sin \theta).$$

From (2.3) it follows that the surface of revolution $M^2$ in $\mathbb{E}^3$ is minimal if and only if $f$ satisfies the differential equation $1 + f'^2 - f f'' = 0$. If $M^2$ is non-minimal, then, by applying (2.3), we see that the Laplace map of $M$ is also a surface of revolution with the same axis. □

The following result classifies surfaces of revolution whose Laplace image lies in a cylinder.

Proposition 2.2. Let $M$ be a surface of revolution in $\mathbb{E}^3$ with the $x$-axis as its axis. Then the Laplace image of $M$ in $\mathbb{E}^3$ lies in a cylinder if and only if, after multiplying the Euclidean coordinates of $\mathbb{E}^3$ by a suitable constant, $M$ is of the following form:

$$x(t, \theta) = \left(-\frac{t}{2} \sqrt{a^2 t^2 - 1} + \frac{1}{2a} \ln(at + \sqrt{a^2 t^2 - 1}) - b, t \cos \theta, t \sin \theta\right)$$

for some suitable constant $a, b$.

Proof. Assume $M$ is a surface of revolution parametrized by

$$x(u, \theta) = (u, f(u) \cos \theta, f(u) \sin \theta).$$

Then the Laplace map of $M$ is given by

$$L(u, \theta) = \left(\frac{1 + f'^2 - f f''}{f(1 + f'^2)^2}\right)(-f', \cos \theta, \sin \theta).$$

Formula (2.6) implies that the Laplace image of $M$ lies in a cylinder if and only if

$$1 + f'^2 - f f'' = cf(1 + f'^2)^2$$

where $c$ is a nonzero constant. By applying a suitable constant to each coordinate, we may obtain $c = 1$ in the following. So, we may assume $c = 1$. Thus, (2.7) yields

$$f f'' + f'^2(1 + f'^2) = 0.$$
By solving this second order ordinary differential equation, we obtain the solution of (2.8) which is given by

\[
(2.9) \quad u = -\frac{f}{2} \sqrt{a^2 f^2 - 1} + \frac{1}{2a} \ln(a + \sqrt{a^2 f^2 - 1}) - b,
\]

where \(a, b\) are integration constants. (2.4) now follows from (2.5) and (2.9) by letting \(t = f(u)\).

The converse can be verified by direct computation. □

From the definition of Laplace map, it is easy to see that the Laplace image of a surface in \(\mathbb{E}^3\) with non-zero constant mean curvature lies in a sphere. So it is natural to ask the following

**Problem 2.1.** “Beside surfaces of constant mean curvature, do there exist surfaces in \(\mathbb{E}^3\) whose Laplace image lies in some sphere?”

Concerning this problem we have the following solution.

**Proposition 2.3.** There exist infinitely many surfaces of revolution in \(\mathbb{E}^3\) with non-constant mean curvature whose Laplace images are contained in a sphere which passes through the origin of \(\mathbb{E}^3\).

**Proof.** Let \(M\) be a surface of revolution parametrized by (2.5). Then the Laplace map of \(M\) is given by (2.6) which implies that if the Laplace image of \(M\) lies in a sphere, then the center of the sphere \(S^2\) must be a point on the \(x\)-axis. Assume \(S^2\) is centered at \((r, 0, 0)\) and with radius \(R\). Then we have

\[
\left( \frac{f'(1 + f'^2 - f f'')}{f(1 + f'^2)^2} - r \right)^2 + \left( \frac{1 + f'^2 - f f''}{f(1 + f'^2)^2} \right)^2 = R^2,
\]

which is equivalent to

\[
(1 + f'^2 - f f'')^2 - 4rf f'(1 + f'^2)(1 + f'^2 - f f'') + 4(r^2 - R^2)f^2(1 + f'^2)^3 = 0.
\]

In particular, if the sphere passes through the origin, then we have

\[
(1 + f'^2 - f f'')(1 + f'^2 - f f'') = 0.
\]

Since \(M\) is minimal if and only if \(1 - f'^2 - f f'' = 0\), the hypothesis yields

\[
(2.11) \quad 1 + f'^2 - f f'' = 0.
\]

Put

\[
F(u, f, v) = \frac{1}{f}(1 + v^2 - 4rvf(1 + v^2)).
\]

It is easy to see that \(F, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial f}, \frac{\partial F}{\partial v}\) are continuous functions on any open set of the \((u, f, v)\)-space on which \(f \neq 0\).

By applying the existence theorem of second order ordinary differential equations, we know that for any given point \(x_0\) with \(f(x_0) \neq 0\), differential equation
(2.11) together with initial conditions: \( f(x_0) = f_0, f'(x_0) = f'_0 \) has a unique local solution about \( x_0 \). Generically, for such a solution \( f = f(u) \), the corresponding surface of revolution defined by (2.5) does not have constant mean curvature. Moreover, the Laplace image of such a surface lies in the sphere centered at \((r,0,0)\) with radius \( r \). □

Concerning totally geodesic Laplace images, we have the following

**Theorem 2.4.** Let \( M \) be a surface of revolution in \( \mathbb{E}^3 \). Then the Laplace image of \( M \) lies in a plane if and only if, up to rigid motions of \( \mathbb{E}^3 \), the surface is either a circular cylinder or a surface of revolution given by
\[
x(t, \theta) = \left( \frac{1}{a} \ln |at + \sqrt{a^2 t^2 - 1}|, t \cos \theta, t \sin \theta \right),
\]
where \( a \) is a positive constant.

**Proof.** Assume \( M \) is a surface of revolution parametrized by (2.5). Then the Laplace map of \( M \) is given by
\[
(2.12) \quad L(u, \theta) = \left( \frac{1 + f'^2 - ff''}{f(1 + f'^2)^2} \right) (-f', \cos \theta, \sin \theta).
\]
From (2.12) we see that the Laplace image of \( M \) lie in a plane \( P \) if and only if
\[
(2.13) \quad f'(1 + f'^2 - ff'') = cf(1 + f'^2)^2
\]
for some constant \( c \).

Without loss of generality, we may assume that the plane \( P \) is the \( yz \)–plane and hence \( c = 0 \). Then, from (2.13), we have
\[
(2.14) \quad f'(ff'' - 1 - f'^2) = 0.
\]
If \( f' = 0 \), then \( M \) is a circular cylinder. In this case, the Laplace image is a circle in the plane \( P \).

If \( f' \neq 0 \), then (2.14) yields
\[
(2.15) \quad ff'' = 1 + f'^2.
\]
From (2.15) we get
\[
(2.16) \quad f'^2 = a^2 f^2 - 1
\]
for some constant \( a > 0 \). Solving (2.16) we get
\[
(2.17) \quad u = \pm \frac{1}{a} \ln |af + \sqrt{a^2 f^2 - 1}| + b
\]
where \( b \) is a constant. Without loss of generality, we may assume
\[
(2.18) \quad u = \frac{1}{a} \ln |af + \sqrt{a^2 f^2 - 1}|.
\]
By putting \( f(u) = t \), (2.5) and (2.18) show that the surface of revolution is given by

\[
x(t, \theta) = \left( \frac{1}{a} \ln |at + \sqrt{a^2 t^2 - 1}|, t \cos \theta, t \sin \theta \right).
\]

The converse follows from direct computations. □

§3. Laplace images of curves.

In this section we study Laplace image of curves.

It is clear that the Laplace image of a circle is a circle and the Laplace image of a line is a point. So, it is natural to ask the following.

Problem 3.1. “Besides circles and lines in \( \mathbb{E}^2 \), do there exist other planar curves whose Laplace images lie in a circle?”

The following result gives an affirmative answer to this Problem.

Proposition 3.1. The Laplace image of a unit speed planar curve \( \gamma(s) \) lies in a circle if and only if locally it is either a line, a circle, or, up to similarity transformations of \( \mathbb{E}^2 \), it is a curve given by

\[
\gamma(s) = \left( s - \frac{1}{2} \ln(1 + c^2 e^{4s}), \tan^{-1}(ce^{2s}) \right),
\]

where \( c \) is a positive constant.

Proof. Let \( \gamma(s) = (x(s), y(s)) \) be a unit speed curve in the \( xy \)-plane and \( \theta(s) \) be the angle between \( \gamma'(s) \) and the positive direction of the \( x \)-axis. Then we have

\[
\gamma'(s) = (\cos \theta(s), \sin \theta(s)).
\]

Because the Laplace operator of \( \gamma \) is given by \( \Delta = -\frac{d^2}{ds^2} \), the Laplace map of \( \gamma \) is given by

\[
L(s) = \frac{d\theta}{ds}(\sin \theta(s), -\cos \theta(s)).
\]

Assume the Laplace image lies in a circle. By applying a suitable similarity transformation on the plane, the center of the circle can be chosen to be \((1, 0)\) and its radius to be one. Therefore, by using (3.3) we obtain

\[
(\theta'(s))^2 - 2(\sin \theta)\theta'(s) = 0.
\]

If \( \gamma \) is not contained in a line or in a circle, then \( \theta'(s) \neq 0 \). Thus (3.4) yields \( \theta'(s) = 2 \sin \theta \). By solving this differential equation, we obtain

\[
\theta(s) = 2 \tan^{-1}(ce^{2s})
\]
for some positive constant \(c\). Combining (3.2) and (3.5), we may obtain

\[
\gamma'(s) = \left(\frac{1 - c^2 e^{4s}}{1 + c^2 e^{4s}}, \frac{2ce^{2s}}{1 + c^2 e^{4s}}\right),
\]

from which we obtain (3.1).

The converse is easy to verify. □

Similar to Problem 3.1, we would like to know whether there exist curves whose Laplace images lie in a line.

**Proposition 3.2.** Let \(\beta(s)\) be a curve in \(E^m\) parametrized by arclength. Then the Laplace image of \(\beta\) lies in a line if and only if either

**(1)** \(\beta\) is a planar curve whose curvature function \(\kappa(s)\) satisfies the differential equation:

\[
\kappa\kappa'' - \kappa^4 = 3(\kappa')^2
\]

or

**(2)** \(\beta\) is a helix in an affine 3-space whose first and second Frenet curvature functions \(\kappa_1\) and \(\kappa_2\) satisfy the equations:

\[
\kappa_2 = c\kappa_1, \quad \kappa_1\kappa_1'' - (1 + c^2)\kappa_1^4 = 3(\kappa_1')^2,
\]

where \(c\) is a nonzero constant.

**Proof.** The Laplace map of the unit speed curve \(\beta\) is given by \(L(s) = -\beta''(s)\).

(1) If \(\beta\) is a planar curve, then

\[
L'(s) = \kappa^2\beta_1 - \kappa'\beta_2, \quad L''(s) = 3\kappa\kappa'\beta_1 + (\kappa^3 - \kappa'')\beta_2,
\]

where \(\kappa\) is the plane curvature and \(\beta_1 = \beta'\) and \(\beta_2\) is a unit normal vector field. Because the Laplace image of \(\beta\) is contained in a line if and only if \(L'(s)\) and \(L''(s)\) are linearly dependent, (3.8) implies statement (1).

(2) Let \(\beta\) be a unit speed space curve in \(E^m\) with \(m \geq 3\). If \(m = 3\), we may consider \(E^3\) as a linear subspace of \(E^4\). Thus, without loss of generality, we may assume \(m \geq 4\). Denote the \(i\)-the Frenet curvature by \(\kappa_i\) and the \(i\)-the Frenet vector by \(\beta_i\). Then we have

\[
L'(s) = \kappa_1^2\beta_1 - \kappa_1'\beta_2 - \kappa_1\kappa_2\beta_3,
\]

\[
L''(s) = 3\kappa_1\kappa_1'\beta_1 - (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)\beta_2
\]

\[
-2(\kappa_1'\kappa_2 - \kappa_1\kappa_2')\beta_3 - \kappa_1\kappa_2\kappa_3\beta_4.
\]

From (3.9) and (3.10) we conclude that the Laplace image of the space curve \(\beta\) lies in a line if and only if the following four conditions hold:

\[
\kappa_1\kappa_1'' - \kappa_1^4 - \kappa_1^2\kappa_2^2 = 3(\kappa_1')^2,
\]
\[ \kappa'_1 \kappa_2 = \kappa_1 \kappa'_2, \]

(3.12)

\[ 2 \kappa_1^2 \kappa_2 + \kappa_1 \kappa'_1 \kappa'_2 + \kappa_1^4 \kappa_2 + \kappa_1^2 \kappa_2^3 = \kappa''_1 \kappa_1 \kappa_2, \]

(3.13)

and

\[ \kappa_2^2 \kappa_2 \kappa_3 = \kappa_1 \kappa'_1 \kappa_2 \kappa_3 = \kappa'_1 \kappa_2^2 \kappa_3 = 0. \]

(3.14)

(3.12) implies \( \kappa_2 = c \kappa_1 \) for some constant \( c \). Thus, from (3.11) as well as from (3.13), we obtain (3.7). From (3.14), we obtain \( \kappa_3 = 0 \). Thus \( \beta \) is a curve in an affine 3-space with \( \kappa_2 = c \kappa_1 \). Hence, \( \beta \) is a helix.

Conversely, if \( \beta \) is a unit speed helix in \( \mathbb{E}^3 \) whose Frenet curvatures satisfy (3.7), then conditions (3.11) and (3.12) hold. Condition (3.14) holds automatically since \( \beta \) is a curve in \( \mathbb{E}^3 \). Furthermore, because \( \kappa_2 = c \kappa_1 \), equation (3.13) is nothing but the second equation in (3.7). Thus, the Laplace image of \( \beta \) lies in a line. \( \Box \)

§4. Laplace images of totally real submanifolds.

Let \( x : M \to \mathbb{C}^n \) be a map from an \( n \)-dimensional Riemannian manifold \( M \) into the complex number \( m \)-space \( \mathbb{C}^m \) equipped with the standard flat Kaehler metric. Denote by \( J \) the almost complex structure of \( \mathbb{C}^m \). The map \( x \) is said to be a complex map if each tangent space of \( M \) is invariant under the action of \( J \), i.e.,

\[ J(dx(T_p M)) \subset dx(T_p M), \quad p \in M. \]

The map \( x \) is said to be a totally real map if

\[ J(dx(T_p M)) \subset (dx(T_p M))^\perp, \quad p \in M, \]

where \((dx(T_p M))^\perp\) is the orthogonal complementary subspace of \( dx(T_p M) \) in \( \mathbb{C}^m \).

For a totally real isometric immersion from \( M \) into \( \mathbb{C}^m \), it is natural to ask the following question:

**Problem 4.1.** “When is the Laplace image of a totally real submanifold totally real?”

Concerning this question we have the following results.

**Lemma 4.1.** Let \( x : M \to \mathbb{C}^n \) be a totally real isometric immersion of an \( n \)-dimensional Riemannian manifold \( M \) into \( \mathbb{C}^n \). Then the Laplace map \( L : M \to \mathbb{C}^n \) of the immersion \( x \) is totally real if and only if the shape operator \( A_H \) in the direction of \( H \) and the normal connection \( D \) of \( x \) satisfy the condition:

\[ \langle A_H X, JD_Y H \rangle = \langle A_H Y, JD_X H \rangle, \]

(4.1)

for any \( X,Y \) tangent to \( M^n \).
Proof. The Laplace map of the isometric totally real immersion $x$ is given by $L = -nH$. For any vector $X$ tangent to $M$, we have

$$dL(X) = nA_H X - nD_X H.$$ 

Thus the Laplace map $L$ is totally real if and only if

$$\langle A_H X - D_X H, JA_H Y - JD_Y H \rangle = 0$$

for all $X, Y$ tangent to $M$. Since the immersion $x$ is totally real and $m = n$, (4.2) is equivalent to condition (4.1). □

Recall that a submanifold $M$ is said to have parallel normalized mean curvature vector if $M$ has nonzero mean curvature and the unit normal vector field in the direction of the mean curvature vector field is a parallel normal vector field.

**Theorem 4.2.** Let $x : M \to \mathbb{C}^m$ be a totally real isometric immersion of an $n$–dimensional Riemannian manifold into $\mathbb{C}^m$. Then

1. if the mean curvature vector field $H$ of $x$ is parallel in the normal bundle, then the Laplace map is totally real; and
2. if $M$ has parallel normalized mean curvature vector field and $m = n$, then the Laplace map is totally real if and only if $A_H JH$ is parallel to the gradient of the mean curvature function, $\nabla \alpha$.

**Proof.** Let $x : M \to \mathbb{C}^m$ be a totally real isometric immersion of an $n$–dimensional Riemannian manifold into $\mathbb{C}^m$ for which $H$ is parallel in the normal bundle. Then the Laplace map satisfies $dL(X) = nA_H X$. Thus, for any $X$ tangent to $M$, $dL(X)$ is also a tangent vector. Since $x$ is a totally real isometric immersion, this implies $\langle JdL(X), dL(Y) \rangle = 0$ for any $X, Y$ tangent to $M$. Hence, by definition, the Laplace map is totally real.

(2) Assume $x$ has parallel normalized mean curvature vector. Then the mean curvature function is nonzero, i.e., $\alpha \neq 0$, and $D\xi = 0$, where $\xi$ is the unit normal vector field in the direction of $H$. Thus, for any vector $X$ tangent to $M$, we have

$$dL(X) = nA_H X - n(X\alpha)\xi.$$ 

Hence, the Laplace map is a totally real map if and only if

$$\langle Y\alpha \rangle \langle A_H X, J\xi \rangle = \langle X\alpha \rangle \langle A_H Y, J\xi \rangle,$$

for any $X, Y$ tangent to $M$.

If we choose a local orthonormal frame field $e_1, \ldots, e_n$ of $M$ such that $e_1$ is in the direction of $\nabla \alpha$, then $e_2\alpha = \cdots = e_n\alpha = 0$. Let $U$ be the open subset of $M$ on which $\nabla \alpha \neq 0$. Then (4.4) implies

$$\langle A_H e_i, J\xi \rangle = 0, i = 2, \ldots, n,$$
which is equivalent to
\[ \langle e_2, A_H (J\xi) \rangle = \cdots = \langle e_n, A_H (J\xi) \rangle = 0. \]
Therefore, \( A_H (JH) \) is parallel to \( \nabla \alpha \).

Conversely, if \( A_H (JH) \) is parallel to \( \nabla \alpha \), then
\[ (4.5) \quad \langle A_H e_i, J\xi \rangle = 0, \quad i = 2, \ldots, n, \]
where \( e_1 \) is in the direction of \( \nabla \alpha \). Because \( e_2 \alpha = \cdots = e_n \alpha = 0 \), (4.5) implies (4.4) for any \( X, Y \) tangent to \( M \). Since \( M \) is assumed to have parallel normalized mean curvature vector, from (4.4) we see that condition (4.1) holds, too. Therefore, by applying Lemma 4.1, we may conclude that the Laplace map of \( x \) is totally real. □

Remark 4.1. Statement (1) of Theorem 4.2 implies that there exist infinitely many totally real submanifolds in \( \mathbb{C}^n \) whose Laplace map are also totally real. □
Chapter VII: SUBMANIFOLDS WITH HARMONIC LAPLACE MAPS AND TRANSFORMATIONS

§1. Submanifolds with harmonic Laplace map.

Let \( \phi : M \to N \) be a differentiable map between Riemannian manifolds \( M \) and \( N \). Denote by \( \nabla \) and \( \tilde{\nabla} \) the Levi–Civita connections of \( M \) and \( N \), respectively. Then the second fundamental form \( h_\phi \) and the energy density \( e(\phi) \) of the map \( \phi \) are given respectively by

\[
(1.1) \quad h_\phi(X,Y) = \tilde{\nabla}_X(\phi^*Y) - \phi^*(\nabla_X Y),
\]

\[
(1.2) \quad e(\phi) = \frac{1}{2} ||d\phi||^2 = \frac{1}{2} \text{trace}(\phi^* g'),
\]

where \( X,Y \) are tangent vectors of \( M \), \( g' \) is the Riemannian metric on \( N \), \( \phi^* = d\phi \), and \( \phi^* \) is the induced map of \( \phi \).

If \( M \) is compact, the energy \( E(\phi) \) of \( \phi \) is defined by

\[
E(\phi) = \int_M e(\phi) * 1.
\]

The Euler-Lagrange operator associated with \( E \) will be written \( \tau(\phi) = \text{div}(d\phi) \) and called the tension field of \( \phi \). A map is said to be harmonic if its tension field vanishes identically. For a differentiable map \( \psi : M^n \to \mathbb{E}^m \), one has (cf. [EL1,2])

\[
(1.3) \quad \Delta \psi = -\tau(\psi).
\]

Furthermore, let \( i : N \to P \) be an isometric immersion and \( \Phi : M \to P \) be the composition of \( \phi \) and \( i \). Then the tension field \( \tau(\phi) \) is the orthogonal projection of \( \tau(\Phi) \) onto the tangent bundle \( T(N) \). More precisely, we have (cf. [EL1,EL2])

\[
(1.4) \quad \tau(\Phi) = \tau(\phi) + \text{trace} h(d\phi, d\phi),
\]

where \( h \) is the second fundamental form of \( N \) in \( P \). In particular, \( \phi : M \to N \) is harmonic if and only if \( \tau(\Phi) \) is perpendicular to \( N \).

Concerning harmonic maps, it is natural to ask the following problem:

**Problem 1.1.** “When is the Laplace map \( L : M^n \to \mathbb{E}^m \) of an isometric immersion \( x : M^n \to \mathbb{E}^m \) harmonic?” □

From Beltrami’s formula and formula (1.3), it follows that the Laplace map \( L : M^n \to \mathbb{E}^m \) of the immersion \( x \) is a harmonic map if and only if the immersion \( x \) is biharmonic. Biharmonic submanifolds were first studied by the first author as one of the off–springs of the theory of finite type. In fact, the first author obtained in 1985 (unpublished then, see [Di1] for details) that if \( x : M^2 \to \mathbb{E}^3 \) is an isometric immersion, then the mean curvature vector field \( H \) is harmonic, i.e., \( \Delta H = 0 \), (or equivalently, \( x \) is biharmonic: \( \Delta^2 x = 0 \)) if and only if the immersion is minimal. In
his 1989 doctoral thesis at Michigan State University, I. Dimitric generalized this result to some more general classes of submanifolds (cf. [Di1]). Furthermore, it is shown in [CI1] that space–like biharmonic surfaces in the Minkowski space–time \( \mathbb{E}^3_1 \) are minimal surfaces, too.

In terms of the Laplace map, this provides us the following first solution to Problem 1.1.

**Theorem 1.1.** Let \( M \) be a surface in \( \mathbb{E}^3 \). Then the Laplace map \( L : \mathbb{M}^2 \to \mathbb{E}^3 \) is a harmonic map if and only if \( \mathbb{M}^2 \) is a minimal surface.

**Proof.** Let \( x : M \to \mathbb{E}^3 \) be a surface whose Laplace map is a harmonic map, then the isometric immersion \( x \) is biharmonic. We put \( H = \alpha e_3 \). Then, from Theorem 1.3 of Chapter II, we know that

\[
\Delta H = \Delta^D H + ||h||^2 H + \nabla \alpha^2 + 2A_3(\nabla \alpha),
\]

where \( ||h|| \) is the length of the second fundamental form \( h \). Since \( M \) is biharmonic, we have

\[
\Delta \alpha + ||h||^2 = 0,
\]

\[
A_3(\text{grad} \alpha) = -\alpha(\text{grad} \alpha).
\]

We choose \( e_1, e_2 \) which diagonalize \( A_3 \). Thus we have

\[
h(e_1, e_1) = \beta e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \gamma e_3
\]

for some functions \( \beta, \gamma \). From the equation of Codazzi and (1.8) we have

\[
e_2 \beta = (\beta - \gamma) \omega_1^2(e_1), \quad e_1 \gamma = (\beta - \gamma) \omega_1^2(e_2).
\]

Let \( U = \{ p \in M \mid \nabla \alpha^2 \neq 0 \text{ at } p \} \). Then \( U \) is an open subset of \( M \). Assume \( U \neq \emptyset \). Then, by (1.9), \( \nabla \alpha \) is an eigenvector of \( A_3 \) with eigenvalue \( -\alpha \) on \( U \). We choose \( e_1 \) in the direction of \( \nabla \alpha \) on \( U \). Then we have \( e_2 \alpha = 0 \) on \( U \). Moreover, the Weingarten map \( A_3 \) satisfies

\[
\beta = -\alpha, \quad \gamma = 3\alpha, \quad ||h||^2 = 10\alpha^2.
\]

Since \( e_2 \alpha = 0 \), (1.9) and (1.10) imply

\[
\omega_1^2(e_1) = 0, \quad d\omega_1 = 0.
\]

Thus, locally, \( \omega_1 = du \) for some function \( u \). Since \( d\alpha \wedge \omega_1 = d\alpha \wedge du = 0 \), \( \alpha \) is a function of \( u \). We denote by \( \alpha' \) and \( \alpha'' \) the first two derivatives of \( \alpha \) with respect to \( u \). From (1.9), (1.10), and (1.11), we have

\[
4\alpha\omega_1^2 = -3\alpha' \omega^2.
\]

Also from \( e_2 \alpha = 0 \) and (1.12), we have

\[
4\alpha \Delta \alpha = 3(\alpha')^2 - 4\alpha \alpha''.
\]
By using (1.6), (1.10), and (1.13), we get
\begin{equation}
4a\alpha'' - 3(\alpha')^2 - 40\alpha^4 = 0. \tag{1.14}
\end{equation}
From (1.14) we may obtain
\begin{equation}
(\alpha')^2 = 8\alpha^4 + C\alpha^{3/2} \tag{1.15}
\end{equation}
for some constant $C$. On the other hand, the equation of Gauss, (1.8) and (1.12) imply
\begin{equation}
a\alpha'' - \frac{7}{4}(\alpha')^2 + 4\alpha^4 = 0. \tag{1.16}
\end{equation}
From (1.14) and (1.16) we get
\begin{equation}
(\alpha')^2 = 16\alpha^4. \tag{1.17}
\end{equation}
Combining (1.15) and (1.17) we conclude that $\alpha$ is constant on $U$ which is a contradiction. Therefore, $U$ is empty. So, $M$ has constant mean curvature. Thus, by applying (1.6), we obtain $\alpha = 0$. Hence the immersion $x$ is a minimal immersion.
\[\square\]

Next, we consider the following problem.

**Problem 1.2.** “When is the Laplace transformation $L : M \to L(M)$ of an isometric immersion $x : M \to \mathbb{R}^m$ harmonic?”

Since a (weakly) conformal map between two Riemannian surfaces is a harmonic map, Proposition 3.3 of Chapter V shows that there exist many surfaces of revolution which have harmonic Laplace transformations and which have non-constant mean curvature functions. Therefore, this seems to be a difficult problem in general. However, for curves in $\mathbb{R}^m$, this Problem is much easier to solve.

First, it is easy to see that lines and circles in a plane have harmonic Laplace transformations. If the curve is neither a line or a circle, we have the following results.

**Proposition 1.2.** Let $\beta(s)$ be a curve parametrized by arclength $s$. If $\beta$ is neither a line or a circle, then the Laplace transformation of $\beta$ is a harmonic map if and only if $\beta$ is a curve whose first and second Frenet curvature functions satisfy the equation:
\begin{equation}
k_1'(2k_1^3 + k_1'') + k_1k_2(k_1'k_2 + k_1k_2') = 0. \tag{1.18}
\end{equation}

In particular, if $\beta$ is a planar curve, then the Laplace transformation of $\beta$ is a harmonic map if and only if the curvature function $\kappa$ of $\beta$ satisfies the differential equation: $\kappa'' = -2\kappa^3$. 


Proof. Let $\beta(s)$ be a unit speed curve. Then the Laplace map $L$ is given by $L(s) = -\beta''(s)$. Thus

$$\Delta L = -3\kappa_1 \kappa_1' \beta_1 - (\kappa^3 - \kappa_1'' + \kappa_1 \kappa_2^2) \beta_2$$

$$+ (2\kappa_1' \kappa_2 + \kappa_1 \kappa_1') \beta_3 + \kappa_1 \kappa_2 \kappa_3 \beta_4,$$

where $\beta_i$ is the $i$-th Frenet vector of $\beta$. On the other hand, we have

$$dL \left( \frac{d}{ds} \right) = \kappa_2' \beta_1 - \kappa_1' \beta_2 - \kappa_1 \kappa_2 \beta_3.$$  

It is known that the Laplace transformation $L$ of $\beta$ is a harmonic map if and only if the component of $\Delta L$ in the direction of $dL \left( \frac{d}{ds} \right)$ vanishes. Therefore, by using (1.19) and (1.20), we may conclude that the Laplace transformation $L$ is a harmonic map if and only if

$$-3\kappa_1^3 \kappa_1' + \kappa_1' (\kappa_3^2 - \kappa_1'' + \kappa_1 \kappa_2^2) + \kappa_1 \kappa_2 (-2\kappa_1' \kappa_2 - \kappa_1 \kappa_1') = 0.$$  

It is easy to verify that (1.21) is equivalent to (1.18).

If $\beta$ is a planar curve, then (1.18) is equivalent to $\kappa' (2\kappa^3 + \kappa'') = 0$ where $\kappa$ is the plane curvature of $\beta$. Because $\beta$ is neither a line or a circle, this yields $\kappa'' = -2\kappa^3$.

The converse is easy to verify. □

From this Proposition we know that the class of planar curves whose Laplace transformations are harmonic maps depends on two parameters. Moreover, by combining Proposition 3.3 and Remark 3.1 of Chapter IV and the last Proposition, we have the following

**Corollary 1.3.** The Laplace transformation of a planar curve $\beta$ is homothetic if and only if the curve $\beta$ has nonzero curvature function and it has harmonic Laplace transformation.

Proof. From Proposition 3.3 and Remark 3.1 of Chapter IV we know that a unit speed planar curve $\beta(s)$ has homothetic Laplace transformation if and only if the plane curvature of $\beta$ satisfies

$$\left( \kappa'(s) \right)^2 + \kappa(s)^4 = c^2,$$

for some positive constant $c$. By taking the derivative of (1.22) with respect to $s$, we obtain

$$\kappa' (\kappa'' + 2\kappa^3) = 0.$$  

If $\kappa' = 0$, then (1.22) implies that $\beta$ is a circle which has harmonic Laplace transformation. If $\beta$ is not a circle, then (1.23) yields $\kappa'' = -2\kappa^3$. Thus, by applying Proposition 1.2, we conclude that the Laplace transformation of $\beta$ is a harmonic map.

Conversely, if $\beta$ is a planar curve which has nonzero curvature function and harmonic Laplace transformation, then either $\beta$ is a circle or the plane curvature of
β satisfies $\kappa'' = -2\kappa^3$. By solving this differential equation, we may obtain (1.22), which implies β has homothetic Laplace transformation. □

For submanifolds with parallel mean curvature vector field, we have the following result.

**Theorem 1.4.** Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an $n$-dimensional Riemannian manifold into $\mathbb{E}^m$ with parallel mean curvature vector field. Then the Laplace transformation of $x$ is a harmonic map.

**Proof.** From Theorem 1.3 of Chapter II we have the following fundamental formula:

$$\Delta H = \Delta^D H + \sum_{i=1}^n h(A_H e_i, e_i) + \frac{n}{2} \text{grad} <H, H> + 2 \text{trace} A_D H,$$

where $\Delta^D$ is the Laplacian associated with the normal connection $D$. If $x$ has parallel mean curvature vector field, then (1.24) yields

$$\Delta L = -n \sum_{i=1}^n h(A_H e_i, e_i).$$

On the other hand, from $D H = 0$, we have

$$L_\ast (X) = n A_H (X),$$

for any vector $X$ tangent to $M$. Comparing (1.25) and (1.26) we conclude that the Laplace transformation of $x$ is a harmonic map. □

§2. Submanifolds with harmonic mean curvature function.

In this section, we would like to consider submanifolds whose mean curvature functions are harmonic. From Hopf’s lemma we know that if a compact submanifold has harmonic mean curvature function, then the mean curvature function is constant. So in this section we are only interested in non-compact submanifolds.

First we give the following

**Lemma 2.1.** The only planar curves with harmonic curvature function are open parts of a line, a circle, or a Cornu spiral. (By a Cornu spiral curve, we mean a planar curve whose curvature function in terms of an arc–length parameter $s$ is given by $\kappa(s) = as + b$ for some real numbers, $a \neq 0$ and $b$.)

**Proof.** The Laplace operator of a unit speed curve $\gamma(s)$ is given by $\Delta = -\frac{d^2}{ds^2}$. Thus, $\gamma$ has harmonic curvature function if and only if $\kappa''(s) = 0$, or equivalently, $\kappa(s) = as + b$ for some constants $a, b$. If $a = b = 0$, the curve is an open part of a line. If $a = 0$ and $b \neq 0$, $\gamma$ is an open part of a circle. If $a \neq 0$, $\gamma$ is an open part of a Cornu spiral curve. □
Let $M$ be a flat surface in $\mathbb{E}^3$. Then, locally, $M$ is a cylinder, a cone, or a tangential developable surface. We shall consider these three cases separately.

**Theorem 2.2.** The only hypercylinders in $\mathbb{E}^{n+1}$ with harmonic mean curvature function are open parts of the following hypersurfaces:

1. hyperplanes;
2. circular hypercylinders; and
3. hypercylinders $C \times \mathbb{E}^{n-1}$, where $C$ is a Cornu spiral.

**Proof.** Let $M$ be a hypercylinder in $\mathbb{E}^{n+1}$ parametrized by

$$x(s,t_2,\ldots,t_n) = (u(s),v(s),t_2,\ldots,t_n),$$

where $\gamma(s) = (u(s),v(s))$ is a planar curve with $s$ as its arclength parameter. Then the Laplace operator of $M$ is given by

$$\Delta = -\frac{\partial^2}{\partial s^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial t_i^2}.$$

It is easy to see that the mean curvature function $\alpha$ of $M$ in $\mathbb{E}^{n+1}$ is related with the plane curvature function $\kappa$ of $\gamma$ by $\kappa(s) = n\alpha$. Thus, by the expression of the Laplace operators of $M$ and $\gamma$, we know that the hypercylinder $M$ has harmonic mean curvature function if and only if $\gamma$ has harmonic mean curvature function. Thus, by applying Lemma 2.1, we obtain the Theorem. □

**Lemma 2.3.** Open parts of a plane are the only tangential developable surfaces in $\mathbb{E}^3$ with harmonic mean curvature function.

**Proof.** Let $M$ be a tangential developable surface parametrized by

$$x(s,t) = \beta(s) + t\beta'(s),$$

where $\beta(s)$ is a unit speed curve in $\mathbb{E}^3$. Then, by a direct computation, we have

$$\Delta = -\frac{1}{t\kappa_1} \left\{ \frac{\partial}{\partial s} \left( \frac{1}{t\kappa_1} \frac{\partial}{\partial s} \right) - \frac{\partial}{\partial s} \left( \frac{1}{t\kappa_1} \frac{\partial}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{1}{t\kappa_1} \frac{\partial}{\partial t} \right) \right\} + \frac{\partial}{\partial t} \left( \frac{1}{t\kappa_1} \frac{\partial}{\partial t} \right).$$

From (2.3) we obtain

$$\frac{\partial x}{\partial s} = \beta_1 + t\kappa_1 \beta_2, \quad \frac{\partial x}{\partial t} = \beta_1.$$ 

Thus, $\beta_1, \beta_2$ form an orthonormal frame field on $M$. Therefore $\xi = \beta_3$ is a unit normal vector field of the surface. By direct computation, we have

$$A_\xi \beta_1 = 0, \quad A_\xi \beta_2 = \frac{\kappa_2}{t\kappa_1} \beta_2.$$
This implies that the mean curvature function of $M$ in $\mathbb{E}^3$ is given by
\begin{equation}
\alpha = \frac{\kappa_2}{2t\kappa_1}.
\end{equation}
By using (2.4) and (2.7) we conclude that $\Delta \alpha = 0$ if and only if the Frenet curvatures of $\beta$ satisfy
\begin{equation}
(\kappa_1' \kappa_2'' - \kappa_1 \kappa_2'' - 3\kappa_1' \kappa_2' + 3\kappa_1^2 \kappa_2'' + \kappa_1^4 \kappa_2')t^2
+ (2\kappa_1^2 \kappa_2' - 4\kappa_1' \kappa_2 + \kappa_1^2 \kappa_2)t + 3\kappa_1^2 \kappa_2 = 0.
\end{equation}
Since (2.8) holds for any $t$, we obtain $\kappa_2 = 0$. Therefore, $\beta$ is a planar curve; and hence the tangential developable surface is an open part of a plane.

**Lemma 2.4.** Besides open parts of planes, the only other cones in $\mathbb{E}^3$ with harmonic mean curvature function are those whose rays cut the unit sphere $S^2_p(1)$ centered at $p$ in a curve $\beta(s)$ of which the curvature in $S^2_p(1)$ is given by $c_1 \cos s + c_2 \sin s$, where $s$ is an arclength parameter of $\beta$ and $c_1$ and $c_2$ are two nonzero constants.

**Proof.** It is clear that open parts of planes have harmonic mean curvature functions and circular cones do not have harmonic mean curvature functions.

Let $M$ be a cone parametrized by
\begin{equation}
x(s, t) = t\beta(s), \quad <\beta, \beta> = <\beta', \beta'>' = 1.
\end{equation}
We assume $M$ is neither an open part of a plane or of a circular cone. We put $\rho = \kappa_1^{-1}$ and $\sigma = \kappa_2^{-1}$. It is well-known from classical differential geometry that the unit speed spherical curve $\beta$ satisfies
\begin{equation}
\beta = -\rho \beta_2 - (\sigma \beta') \beta_3.
\end{equation}
From (2.9) and a direct computation, we have
\begin{equation}
\frac{\partial x}{\partial s} = t\beta_1, \quad \frac{\partial x}{\partial t} = \beta.
\end{equation}
Put
\begin{equation}
\xi = (\sigma \beta') \beta_2 - \rho \beta_3.
\end{equation}
Then $\xi$ is a normal vector field of $M$ in $\mathbb{E}^3$. Moreover, by direct computation, we obtain
\begin{equation}
\nabla_{\beta_2} \xi = -\kappa_1 \sigma \rho' \beta_1, \quad \nabla_{\beta_1} \xi = 0.
\end{equation}
Therefore, the mean curvature function of $M$ is given by
\begin{equation}
\alpha = -\frac{\kappa_1'}{2t\kappa_1 \kappa_2}.
\end{equation}
By direct computation, we also have
\begin{equation}
\Delta = -\frac{1}{t} \left\{ \frac{\partial}{\partial s} \left( \frac{1}{t} \frac{\partial}{\partial s} \right) + \frac{\partial}{\partial t} \left( t \frac{\partial}{\partial t} \right) \right\}.
\end{equation}
Therefore, by using (2.14) and (2.15), we conclude that $M$ has harmonic mean curvature function if and only if the Frenet curvatures of $\beta$ satisfy

\[(2.16) \quad \left( \frac{\kappa'_1}{\kappa_1 \kappa_2} \right)'' + \frac{\kappa'_1}{\kappa_1 \kappa_2} = 0.\]

From (2.9) and (2.10) we obtain

\[(2.17) \quad \kappa^2_1 \kappa^2_2 (\kappa^2_1 - 1) = \kappa'_1^2.\]

Since the curvature function $\bar{\kappa}$ of $\beta(s)$ in $S^2(1)$ is given by $\bar{\kappa} = \sqrt{\kappa^2_1 - 1}$, (2.17) yields

\[(2.18) \quad \bar{\kappa} = \pm \frac{\kappa'_1}{\kappa_1 \kappa_2}.\]

After solving the second order differential equation (2.16), we find from (2.18) that $\bar{\kappa}(s) = c_1 \cos{s} + c_2 \sin{s}$ for some constants $c_1, c_2$. □

**Theorem 2.5.** Let $M$ be a flat surface in $E^3$. Then $M$ has harmonic mean curvature function if and only if $M$ is given by open parts of the following surfaces:

1. the planes;
2. the circular cylinders;
3. the Cornu cylinders $C \times \mathbb{R}$, i.e., the right cylinders on the Cornu spiral curves;
4. the cones with vertex at a point $p$ whose rays cut the unit sphere $S^2_p(1)$ centered at $p$ in a curve $\beta(s)$ of which the curvature in $S^2_p(1)$ is given by $c_1 \cos{s} + c_2 \sin{s}$, where $s$ is an arclength parameter of $\beta$ and $c_1$ and $c_2$ are two nonzero constants.

**Proof.** This Theorem follows easily from Theorem 2.2, Lemma 2.3 and Lemma 2.4. □

**Remarks 2.1.**

(i) Surfaces given in (1) and (2) of Theorem 2.5 are of course surfaces with constant mean curvature; surfaces in (3) and (4) have non-constant harmonic mean curvature. □

(ii) The Cornu spirals are planar curves with harmonic curvature; the curves $\beta(s)$ mentioned in (4) are spherical curves with harmonic curvature. □

(iii) In view of Theorem 2.5, it is interesting to point out that the only complete flat surfaces in $E^3$ with harmonic mean curvature functions are planes, circular cylinders and Cornu cylinders.

We recall that Delaunay’s surfaces in $E^3$ are surfaces of revolution with constant mean curvature function. For surfaces of revolution we have the following result.

**Theorem 2.6.** We have the following:
(1) If $M$ is a surface of revolution in $\mathbb{E}^3$ defined by
\begin{equation}
(2.19) \quad x(t, \theta) = (t, f(t) \cos \theta, f(t) \sin \theta),
\end{equation}
then $M^2$ has harmonic mean curvature function if and only if the generating function $f$ satisfies the following ordinary differential equation of third order:
\begin{equation}
(2.20) \quad (1 + f'^2) \left\{ f'^3 + f' + ff'' \right\} - 3f'^2 f''^2 + cf(1 + f'^2)^3 = 0,
\end{equation}
where $c$ is an arbitrary constant; and

(2) besides the surfaces of Delaunay, there exist infinitely many surfaces of revolution in $\mathbb{E}^3$ with harmonic mean curvature function.

**Proof.** Let $M$ be a surface of revolution in $\mathbb{E}^3$ defined by (2.19). Then the Laplace operator and the Laplace map of $M$ are respectively given by
\begin{equation}
(2.21) \quad \Delta = -\frac{1}{1 + f'^2} \frac{\partial^2}{\partial t^2} - \frac{f' + f'^3 - ff'f''}{f(1 + f'^2)^2} \frac{\partial}{\partial t} - \frac{1}{f'^2} \frac{\partial^2}{\partial \theta^2},
\end{equation}
\begin{equation}
(2.22) \quad L(t, \theta) = \left( \frac{1 + f'^2 - ff''}{f(1 + f'^2)^2} \right) (-f', \cos \theta, \sin \theta).
\end{equation}
Thus the mean curvature function of $M$ is given by
\begin{equation}
(2.23) \quad \alpha = \frac{1 + f'^2 - ff''}{2f(1 + f'^2)^{3/2}}.
\end{equation}
(2.21) and (2.23) imply that the mean curvature function is harmonic if and only if
\begin{equation}
(2.24) \quad \frac{f}{\sqrt{1 + f'^2}} \frac{d}{dt} \left( \frac{1 + f'^2 - ff''}{f(1 + f'^2)^{3/2}} \right) = c,
\end{equation}
where $c$ is a constant.

It is easy to verify that (2.24) is equivalent to (2.20). This proves (1).

(2) From the existence and uniqueness theorem of ordinary differential equations, we know that for a given initial point $t_0$, there exists a unique solution $f = \phi(t)$ defined on some interval about $t_0$ of the differential equation defined by (2.20) that satisfies the prescribed initial conditions:
\begin{equation}
(2.25) \quad f(t_0) = f_0, \quad f'(t_0) = f_0', \quad f''(t_0) = f_0''.
\end{equation}
Furthermore, by using (2.23), we see that the class of surfaces of revolution with constant mean curvature depends on three parameters: $t_0, f(t_0), f'(t_0)$; and the class of surfaces of revolutions with harmonic mean curvature function depends on four parameters $t_0, f(t_0), f'(t_0), f''(t_0)$. From these we conclude that, besides the surfaces of Delaunay, there exist infinitely many surfaces of revolution in $\mathbb{E}^3$ with harmonic mean curvature function. $\Box$
Chapter VIII: LAPLACE AND GAUSS IMAGES

§1. Laplace and Gauss images of submanifolds in $E^m$.

Let $x : M^n \rightarrow E^m$ be an isometric immersion of an $n$-dimensional, oriented Riemannian manifold $M^n$ into $E^m$ and $e_{n+1}, \ldots, e_m$ a local oriented orthonormal frame of the normal bundle $T^\perp(M)$. Then the Gauss map $G : M^n \rightarrow G(n, m - n)$, defined by

$$G(p) = (e_{n+1} \wedge \ldots \wedge e_m)(p), \quad p \in M,$$

is a differentiable map from $M$ into the real Grassmannian of oriented $(m - n)$-planes in $E^m$. We call the image $G(M^n)$ of the Gauss map $G$ the Gauss image of the immersion $x$. Whenever the Laplace map of $x$ is an immersion, then, locally, we have a map, denoted by

$$LG : L(M^n) \rightarrow G(M^n),$$

from the Laplace image into the Gauss image defined by $LG(L(p)) = G(p)$ for $p \in M$. We call this map the Laplace-Gauss transformation of $x$, or for short, LG-transformation.

For simplicity, we shall assume throughout this section that the Laplace map of $x$ is an immersion. The purpose of this section is to study the geometry of LG-transformations of such immersions.

First we consider curves in Euclidean spaces.

Proposition 1.1. Let $\beta$ be a curve in $E^m$ parametrized by arclength. Then

(1) if $\beta$ is a planar curve, then the LG-transformation of $\beta$ is homothetic if and only if its curvature function $\kappa$ is of the following form:

$$\kappa(s) = \frac{1}{2} \left( \sqrt{a^2 e^{-s^2} + 4(c + c^2)} - ae^{-s} \right),$$

for some positive constants $a, c$;

(2) the LG-transformation of $\beta$ is homothetic if and only if the first and the second Frenet curvatures of $\beta$ satisfy the differential equation

$$\kappa_2^2 = c\{\kappa_1^2 + (\ln \kappa_1)'^2\},$$

where $c$ is a positive number.

Proof. (1) If $\beta(s)$ is a planar curve parametrized by arclength, then the Laplace and Gauss maps are given respectively by

$$L(s) = -\kappa(s)\beta_2, G(s) = \beta_2(s).$$
Since the Laplace map is assumed to be regular, \( \kappa(s) \neq 0 \). Moreover, we have from (1.4) that the induced metrics of the Laplace and Gauss images of \( \beta \) are given respectively by

\[
(1.5) \quad g_L = \kappa^4 + \kappa'^2, \quad g_G = \kappa^2.
\]

Therefore, the \( LG \)-transformation of the plane curve is homothetic if and only if \( \kappa'^2 = c^2 \kappa^2 - \kappa^4 \) for some positive number \( c \). By solving this differential equation, we obtain (1.3).

(2) If \( \beta \) is a curve in \( \mathbb{E}^m \) with \( m \geq 3 \), then the induced metrics of the Laplace and Gauss images of \( \beta \) are given respectively by

\[
(1.6) \quad g_L = \kappa_1^4 + \kappa_1'^2 + \kappa_2^2 \kappa_2'^2, \quad g_G = \kappa_1^2.
\]

This implies that the \( LG \)-transformation is homothetic if and only if the first and second Frenet curvatures of \( \beta \) satisfy \( \kappa_2^2 = c(\kappa_1^2 + (\ln \kappa_1)'^2) \).

**Remark 1.1.** Proposition 1.1 implies that there exist infinitely many curves whose \( LG \)-transformation are homothetic. In particular, statement (1) of Proposition 1.1 says that, up to Euclidean motions, the class of planar curves with homothetic \( LG \)-transformation depends on two parameters \( a, c > 0 \).

For hypersurfaces of dimension \( > 1 \) we have the following result.

**Theorem 1.2.** Let \( x : M \to \mathbb{E}^{n+1} \) be an isometric immersion. Then the \( LG \)-transformation \( LG : L(M^n) \to G(M^n) \) of \( x \) is weakly conformal if and only if, locally, either \( M \) has constant mean curvature function in \( \mathbb{E}^{n+1} \) or \( M \) is the product of a planar curve and an affine \( (n-1) \)-space \( \mathbb{E}^{n-1} \).

**Proof.** Let \( M \) be a hypersurface of \( \mathbb{E}^{n+1} \) with \( n > 1 \). We choose a local orthonormal frame field \( e_1, \ldots, e_n, e_{n+1} \) such that \( e_1, \ldots, e_n \) are tangent vectors given by eigenvectors of \( A_{n+1} \) with \( A_{n+1} e_i = \kappa_i e_i, i = 1, \ldots, n \). Then the Laplace and the Gauss maps of \( M \) satisfy

\[
(1.7) \quad L_* (e_i) = n \alpha \kappa_i e_i - (e_i \alpha) e_{n+1}, \quad G_* (e_i) = -\kappa_i e_i, \quad i = 1, \ldots, n.
\]

Therefore, the induced metrics on the Laplace and the Gauss images of \( M \) are given by

\[
(1.8) \quad g_L (e_i, e_j) = n^2 \alpha^2 \kappa_i \kappa_j \delta_{ij} + (e_i \alpha) (e_j \alpha), \quad g_G (e_i, e_j) = \kappa_i \kappa_j \delta_{ij}.
\]

If the \( LG \)-transformation is weakly conformal, then (1.8) implies

\[
(1.9) \quad n^2 \alpha^2 \kappa_i \kappa_j \delta_{ij} + (e_i \alpha) (e_j \alpha) = f \alpha \kappa_i \kappa_j \delta_{ij}, \quad i, j = 1, \ldots, n,
\]

for some function \( f \). In particular, we have \( (e_i \alpha) (e_j \alpha) = 0 \) when \( i \neq j \). Therefore, the gradient of \( \alpha, \nabla \alpha \) is parallel to an eigenvector of \( A_{n+1} \). Without loss of generality, we may assume \( \nabla \alpha \) is parallel to \( e_1 \). Then, by (1.9), we have

\[
(1.10) \quad n^2 \alpha^2 \kappa_1^2 + |\nabla \alpha|^2 = f \alpha \kappa_1^2, \quad (n^2 \alpha^2 - f^2) \kappa_i^2 = 0, \quad i = 2, \ldots, n.
\]
Case 1. If at least one of $\kappa_2, \ldots, \kappa_n$ is nonzero, then the second equation of (1.10) yields $f^2 = n^2\alpha^2$ and hence, by the first equation of (1.10), $\alpha$ is constant.

Case 2. If $\kappa_2 = \cdots = \kappa_n = 0$, then
\begin{equation}
A_{n+1} e_1 = n\alpha e_1, \quad A_{n+1} e_2 = \cdots = A_{n+1} e_n = 0.
\end{equation}
Put $U = \{p \in M : H(p) \neq 0\}$. If $U$ is an empty set, then $M$ is an open part of a hyperplane. So, we assume $U \neq \emptyset$ and let $W$ be a connected component of $U$.

We claim that $W$ is an open part of the product of a planar curve and an affine $(n-1)$–space $E^{n-1}$. This can be seen as follows.

Let $\mathcal{D}_1 = \text{Span}\{e_1\}$ and $\mathcal{D}_2 = \text{Span}\{e_2, \ldots, e_n\}$ on $W$. If $X, Y$ are vector fields in $\mathcal{D}_2$, then we have $A_{n+1} X = A_{n+1} Y = 0$. Thus, by the equation of Codazzi, we have
\begin{equation}
A_{n+1}([X,Y]) = (\nabla_X A_{n+1})(Y) - (\nabla_Y A_{n+1})(X) = 0.
\end{equation}
Hence, $\mathcal{D}_2$ is integrable. Moreover, since $\mathcal{D}_1$ is 1–dimensional, $\mathcal{D}_1$ is trivially integrable.

Furthermore, since $\nabla \alpha$ is parallel to $e_1$, (1.12) and the equation of Codazzi imply that
\begin{equation}
0 = (\nabla_{e_1} A_{n+1})(e_1) - (\nabla_{e_i} A_{n+1})(e_1) = n\alpha \omega^1_1 (e_1) e_1 - n\alpha \nabla_{e_i} e_1,
\end{equation}
for $i = 2, \ldots, n$. Therefore
\begin{equation}
\nabla_{e_1} e_1 = 0, \quad \nabla_{e_i} e_1 = 0.
\end{equation}
(1.14) imply that integral submanifolds of $\mathcal{D}_1, \mathcal{D}_2$ are totally geodesic submanifolds of $W$. Hence, by the de Rham decomposition theorem, $W$ is locally the Riemannian product of a geodesic of $M$ and an $(n-1)$–dimensional totally geodesic submanifold of $M$. Moreover, by (1.11) and a lemma of Moore, we may conclude that $W$ is the product submanifold of a planar curve and an affine $(n-1)$–space $E^{n-1}$.

The converse is easy to verify by using (1.8). □

Theorem 1.2 implies immediately the following.

Corollary 1.3. Surfaces of Delaunay (i.e., the surfaces of revolution in $\mathbb{E}^3$ with constant mean curvature functions) have homothetic LG-transformations. □

For hypersurfaces with homothetic LG–transformation, we have the following.

Theorem 1.4. Let $x : M \to \mathbb{E}^{n+1}$ be an isometric immersion. Then the LG-transformation $LG : L(M^n) \to G(M^n)$ of $x$ is homothetic if and only if, locally, either $M$ has constant mean curvature function in $\mathbb{E}^{n+1}$ or $M$ is the product of an affine $(n-1)$–space $E^{n-1}$ with a planar curve whose curvature function is given by
\begin{equation}
\kappa(s) = \frac{n}{2}(\sqrt{a^2e^{-s^2} + 4(c + c^2)} - ae^{-s}),
\end{equation}
for $a, c > 0$. □
for some positive constants $a, c$.

**Proof.** If $M$ is a hypersurface of $\mathbb{E}^{n+1}$ with homothetic $LG$-transformation, then Theorem 1.2 implies locally either $M$ has constant mean curvature or $M$ is the product of an affine $(n-1)$-space $\mathbb{E}^{n-1}$ and a planar curve $\gamma(s)$. Moreover, from the proof of Theorem 1.2, we know that if the second case occurs, the curvature function $\kappa(s)$ of the planar curve satisfies the differential equation:

\begin{equation}
\kappa^4 + n^2 \kappa'^2 = n^2 c^2 \kappa^2,
\end{equation}

for some positive constant $c$. By solving differential equation (1.16), we obtain (1.15).

The converse can be easily verified. □

§2. **$LG$-transformation of spherical submanifolds.**

In this section we study the $LG$-transformation of spherical hypersurfaces.

**Theorem 2.1.** Let $x : M \to S^{n+1} \subset E^{n+2}$ be a hypersurface of a hypersphere $S^{n+1}$ of $E^{n+2}$. Then $M$ has constant mean curvature function and homothetic $LG$-transformation if and only if either

1. $M$ is an open part of the product of two spheres: $M^k(a) \times M^{n-k}(b)$ with some suitable radii $a$ and $b$; or
2. $M^n$ is an open part of a hypersphere of $S^{n+1}$.

**Proof.** Assume $M$ is a hypersurface of $S^{n+1}$ of $E^{n+2}$ with constant mean curvature and homothetic $LG$-transformation. Without loss of generality, we may assume the hypersphere is centered at the origin and with radius one. Denote by $\bar{\alpha}$ the mean curvature of $M$ in $S^{n+1}$ and by $\xi$ a unit normal vector of $M$ in $S^{n+1}$. Then we have

\begin{equation}
H = \bar{\alpha} \xi - x.
\end{equation}

Let $e_1, \ldots, e_n$ be a local orthonormal frame field of $M$ such that

\begin{equation}
A_\xi e_i = \mu_i e_i, \quad i = 1, \ldots, n.
\end{equation}

Then from (2.1) and (2.2) we have

\begin{equation}
dL(e_i) = n(\bar{\alpha} \mu_i + 1)e_i, \quad i = 1, \ldots, n.
\end{equation}

From the definition of the Gauss map, we have $G(p) = \xi \wedge x$. Thus, we have

\begin{equation}
dG(e_i) = -\mu_i e_i \wedge x + \xi \wedge e_i, \quad i = 1, \ldots, n.
\end{equation}
From (2.3) and (2.4) we see that the $LG$-transformation of $M$ is homothetic if and only if
\[(2.5) \quad (\bar{\alpha} \mu_i + 1)^2 = c^2 (\mu^2_i + 1), \quad i = 1, \ldots, n,\]
for some positive constant $c$. (2.5) implies that $M$ has at most two constant principal curvatures in $S^{n+1}$. If $M$ has exactly one constant principal curvature, $M$ is an open portion of a hypersphere of $S^{n+1}$. If $M$ has exactly two constant principal curvatures, then $M$ is an open portion of the product of two spheres with some suitable radii which are completely determined by the two roots of the quadratic equation given by (2.5).

The converse is easy to verify. □

If $M$ has non–constant mean curvature function in $S^{n+1}$, then we have the following result.

**Theorem 2.2.** Let $M$ be a hypersurface of a hypersphere $S^{n+1}$ of $\mathbb{E}^{n+2}$. If $M$ has non-constant mean curvature function and the $LG$-transformation $LG : L(M) \to G(M^n)$ is conformal, then

1. the gradient of the mean curvature function is an eigenvector of the shape operator $A_\xi$ of $M^n$ in $S^{n+1}$;
2. the shape operator $A_\xi$ has at most three distinct eigenvalues; and
3. the eigenvalue corresponding to the eigenvector given by the gradient of the mean curvature function is of multiplicity one.

**Proof.** Assume $M$ is a hypersurface of $S^{n+1}$ of $\mathbb{E}^{n+2}$ with non–constant mean curvature and conformal $LG$–transformation. Without loss of generality, we may assume the hypersphere is centered at the origin and with radius one. Denote by $\bar{\alpha}$ the mean curvature of $M$ in $S^{n+1}$ and by $\xi$ the a unit normal vector of $M$ in $S^{n+1}$ as before. And let $e_1, \ldots, e_n$ be a local orthonormal frame field of $M$ such that
\[(2.6) \quad A_\xi e_i = \mu_i e_i, \quad i = 1, \ldots, n.\]

Then we have
\[(2.7) \quad dL(e_i) = n(\bar{\alpha} \mu_i + 1)e_i - n(e_i \bar{\alpha}) \xi, \quad i = 1, \ldots, n.\]

Also for the Gauss map, we have as before the following
\[(2.8) \quad dG(e_i) = -\mu_i e_i \wedge x + \xi \wedge e_i, \quad i = 1, \ldots, n.\]

From (2.7) and (2.8) we see that the induced metrics of the Laplace and Gauss images satisfy
\[(2.9) \quad g_\Sigma (e_i, e_j) = n^2 (\bar{\alpha} \mu_i + 1)(\bar{\alpha} \mu_j + 1) \delta_{ij} + n^2 (e_i \bar{\alpha})(e_j \bar{\alpha}),\]
\[(2.10) \quad g_\sigma (e_i, e_j) = (\mu_i \mu_j + 1) \delta_{ij}.\]
From (2.9) and (2.10), we see that $\nabla \bar{\alpha}$ is an eigenvector of $A_\xi$. Without loss of generality, we may assume $\nabla \bar{\alpha}$ is parallel to $e_1$. Thus, from (2.9) and (2.10), we may get

\begin{align}
(\bar{\alpha} \mu_1 + 1)^2 + |\nabla \bar{\alpha}|^2 &= f^2 (\mu_1^2 + 1), \\
(\bar{\alpha} \mu_i + 1)^2 &= f^2 (\mu_i^2 + 1), \quad i = 2, \ldots, n,
\end{align}

for some positive function $f$. From (2.11) and (2.12) we conclude that the shape operator $A_\xi$ has at most three distinct eigenvalues and the eigenvalue $\mu_1$ is of multiplicity one. $\square$

For surfaces, we prove the following

**Theorem 2.3.** Let $M$ be a surface of a hypersphere $S^3$ of $\mathbb{E}^4$. Then $M$ has constant mean curvature and conformal LG-transformation if and only if $M$ is either a totally umbilical surface or a minimal surface in $S^3$;

**Proof.** Without loss of generality, we may assume the radius of $S^3$ is one.

If $M$ is a totally umbilical surface in $S^3$, then $\mu_1 = \mu_2 = \bar{\alpha}$ which is a constant. Thus, $M$ has constant mean curvature; moreover, (2.9) and (2.10) imply that the $LG$–transformation is homothetic.

If $M$ is a minimal surface in $S^3$, then $\mu_1 = -\mu_2$. Thus, (2.9) implies $M$ has conformal $LG$–transformation.

Conversely, if $M$ has constant mean curvature and conformal $LG$–transformation, then (2.9) and (2.10) imply

\begin{align}
(\bar{\alpha} \mu_1 + 1)^2 &= f^2 (\mu_1^2 + 1), \\
(\bar{\alpha} \mu_2 + 1)^2 &= f^2 (\mu_2^2 + 1),
\end{align}

for some positive function $f$. Combining (2.13) and (2.14) we get

\begin{align}
(\mu_1^2 + 1)(\bar{\alpha} \mu_2 + 1)^2 &= (\mu_2^2 + 1)(\bar{\alpha} \mu_1 + 1)^2.
\end{align}

Simplifying (2.15) we find

\begin{align}
(\mu_1 - \mu_2)\bar{\alpha} = 0
\end{align}

which implies $M$ is neither a totally umbilical or a minimal surface in $S^3$. $\square$

Recall that a Clifford torus is the product of two plane circles with the same radius.

**Theorem 2.4.** Let $M$ be a surface of a hypersphere $S^3$ of $\mathbb{E}^4$. Then $M$ has constant Gauss curvature and homothetic $LG$–transformation if and only if $M$ is either a totally umbilical surface or a Clifford torus in $S^3$. 

Proof. Assume $M$ is a surface in a unit hypersphere $S^3$ with constant Gauss curvature and homothetic $LG$-transformation. From Theorem 2.2 we may choose $e_1, e_2$ to be eigenvectors of $A_\xi$ such that $\nabla \bar{\alpha}$ is parallel to $e_1$. From the proof of Theorem 2.2, we obtain

\begin{align}
(\bar{\alpha} \mu_1 + 1)^2 + |\nabla \bar{\alpha}|^2 &= c^2 (\mu_1^2 + 1), \tag{2.17} \\
(\bar{\alpha} \mu_2 + 1)^2 &= c^2 (\mu_2^2 + 1), \tag{2.18}
\end{align}

for some positive constant $c$. Since the Gauss curvature $G$ of $M$ is constant, we get

\begin{equation}
\mu_1 \mu_2 + 1 = a, \tag{2.19}
\end{equation}

where $G = a$ is constant.

Case 1. If $\mu_2 = 0$, then (2.18) yields $c = 1$. Hence, by (2.17) and $\mu_1 = \bar{\alpha}$, we obtain $\mu_1 = 0$. Thus, in this case, $M$ is a totally geodesic surface in $S^3$; hence $M$ is an open part of a great sphere of $S^3$.

Case 2. If $\mu_2 \neq 0$, then $\mu_1 = \mu_2^2 (a - 1)$. Hence, by (2.18), we get $(\mu_2^2 + a + 1)^2 = 4c^2 (\mu_2^2 + 1)$, from which we conclude that $\mu_2$ is constant. Hence $\mu_1$ and $\bar{\alpha}$ are also constant. It is known that the only surfaces in $S^3$ with constant mean curvature and constant Gauss curvature are open parts of the product of two circles or open parts of a totally umbilical surface of $S^3$. If $M$ is an open part of the product of two circles, then, according to Theorem 2.3, $M$ is a minimal surface of $S^3$. Since the only minimal surfaces of $S^3$ with constant Gauss curvature are either open portions of a great sphere or open portions of a Clifford torus, we conclude that $M$ is an open part of a Clifford torus.

The converse is easy to verify. □
Chapter IX: LAPLACE MAP AND 2-TYPE IMMERSIONS

§1. Spherical Laplace map.

If an isometric immersion \( x : M \rightarrow \mathbb{E}^m \) has nonzero constant mean curvature function \( \alpha \), then the Laplace image \( L(M) \) is contained in a hypersphere \( S^{m-1}(n\alpha) \) centered at the origin and with radius \( n\alpha \). In this case we have an associated spherical map

\[
L_S : M \rightarrow S^{m-1}(n\alpha).
\]

We call this spherical map the spherical Laplace map of the immersion.

In this section we study the following problem.

**Problem 1.1** “When is the spherical Laplace map \( L_S : M^n \rightarrow S^{m-1} \) of an isometric immersion harmonic?”

First we give the following result.

**Theorem 1.1.** Let \( x : M \rightarrow \mathbb{E}^m \) be an isometric spherical immersion from an \( n \)-dimensional Riemannian manifold \( M \) into \( \mathbb{E}^m \). If \( x \) has constant mean curvature function \( \alpha \), then

1. the associated spherical Laplace map \( L_S : M \rightarrow S^{m-1}(n\alpha) \) of \( x \) has positive constant energy density;
2. the spherical Laplace map \( L_S \) of \( x \) is a harmonic map if and only if \( M \) is immersed as a minimal submanifold of a hypersphere of \( \mathbb{E}^m \) via \( x \).

**Proof.** Let \( x : M \rightarrow \mathbb{E}^m \) be an isometric spherical immersion with nonzero constant mean curvature function \( \alpha \). Without loss of generality, we may assume \( M \) is immersed in the hypersphere \( S^{m-1}(r) \) centered at the origin and with radius \( r \). Since a spherical submanifold in \( \mathbb{E}^m \) has nonzero mean curvature function, by the constancy of the mean curvature, we may assume \( n\alpha = 1 \) for simplicity. In this case, the Laplace map is the composition \( j \circ L_S \) of the spherical Laplace map \( L_S \) followed by the inclusion \( j \) of the unit hypersphere \( S^{m-1}(1) \) in \( \mathbb{E}^m \). The second fundamental forms \( h_L, h_{LS} \) and \( h_j \) of the maps \( L, L_S \) and \( j \), respectively, satisfy

\[
h_L(X,Y) = j_* h_{LS}(X,Y) + h_j(dL_S(X), dL_S(Y))
\]

for \( X,Y \) tangent to \( M \). Thus we have

\[
\Delta L = -\tau(L) = -j_* \tau(L_S) - \sum_{i=1}^n h_j(dL_S(e_i), dL_S(e_i)),
\]

where \( e_1, \ldots, e_n \) is an orthonormal local frame field of \( M \), and \( \tau(L), \tau(L_S) \) are the tension fields of the Laplace map and the spherical Laplace map, respectively. Since
$j$ is a totally umbilical isometric immersion, (1.3) yields
\[ \Delta L = -j^* \tau(L_S) + 2e(L_S)L, \]
where $e(L_S)$ is the energy density of the spherical Laplace map.

On the other hand, since $M$ is isometrically immersed into the hypersphere $S^{m-1}(r)$ and the immersion $x$ has constant mean curvature, the mean curvature of $x$ is a positive constant. Moreover, from Theorem 1.3 of Chapter II, we have
\[ \Delta H = \Delta \bar{D} \bar{H} + (n + ||A\xi||^2)\bar{H} + A(\bar{H}) + 2\text{trace} A_{DH} - \frac{n\alpha^2}{r^2}x, \]
where $\bar{H}$ is the mean curvature vector of $M$ in $S^{m-1}(r)$, $\xi$ the unit vector in the direction of $\bar{H}$, and $D$, $A(\bar{H})$ the normal connection and the allied mean curvature vector of $M$ in $S^{m-1}(r)$, respectively.

Because $L = -nH = -n\bar{H} + \frac{2}{r}x$, by comparing the $x$-components of (1.4) and (1.5), we obtain $e(L_S) = \frac{2\alpha^2}{r^2}$ which is a nonzero constant. This proves (1).

(2) If the spherical Laplace map $L_S$ is harmonic, then (1.4) implies $\Delta H = cH$, where $c = 2e(L_S)$ is a constant by statement (1). Thus, by Theorem 3.11 of Chapter II, $M$ is either immersed as a minimal submanifold of a hyperphere of $E^m$ via $x$, or immersed as a biharmonic submanifold of $E^m$, or immersed as a null 2-type submanifold. Because $x$ is spherical, the second and the third cases cannot occur (cf. [C16]). □

As an immediate consequence of Theorem 1.1, we have the following

**Corollary 1.2.** Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion with nonzero constant mean curvature. Then the spherical Laplace map $L_S$ is a harmonic map if and only if $M^n$ is an open part of a hypersphere of $\mathbb{E}^{n+1}$. □

For curves we have the following.

**Proposition 1.3.** Let $\beta(s)$ be a curve in $\mathbb{E}^m$ whose first Frenet curvature function is a nonzero constant. Then the spherical Laplace map of $\beta$ is a harmonic map if and only either $\beta$ is an open part of a circle in a plane or $\beta$ is an open part of a circular helix lying in an affine 3-space of $\mathbb{E}^m$.

**Proof.** Without loss of generality, we may assume $\beta$ is a unit speed curve in $\mathbb{E}^m$ with $m \geq 4$. Denote by $\kappa_i$ and $\beta_i$ the $i$–the Frenet curvature and the $i$–the Frenet vector of $\beta$. By the hypothesis and direct computation, we have
\[ \tau(L_S) = \kappa_1 (\kappa_2 \beta_3 + \kappa_2 \kappa_3 \beta_4). \]
From this we see that the second Frenet curvature $\kappa_2$ is a constant. If $\kappa_2 = 0$, $\beta$ is an open part of a circle in a plane. And if $\kappa_2 \neq 0$, then $\kappa_3 = 0$. In this case, $\beta$ is an open part of a circular helix. □
§2. 2–type immersions.

The main purpose of this section is to give some special properties of 2–type immersions.

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of \( k \)–type with spectral decomposition given by

\[
x = c + x_1 + \cdots + x_k, \quad \Delta x_1 = \lambda_1 x_1, \ldots, \Delta x_k = \lambda_k x_k, \quad \lambda_1 < \cdots < \lambda_k,
\]
where \( c \) is a constant map and \( x_1, \ldots, x_k \) are non–constant maps. Put

\[
E_i = \text{Span}\{x_i(p) : p \in M\}.
\]

Recall that the immersion \( x \) is said to be \emph{orthogonal} if the subspaces

\[
E_1, \ldots, E_k
\]

are mutually orthogonal in \( \mathbb{E}^m \).

For orthogonal 2–type immersions, we have the following result.

\textbf{Proposition 2.1.} Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a compact manifold \( M \) into \( \mathbb{E}^m \). Assume \( x \) is an orthogonal 2–type immersion with spectral decomposition given by

\begin{equation}
(2.1) \quad x = x_1 + x_2, \quad \Delta x_1 = \lambda_1 x_1, \quad \Delta x_2 = \lambda_2 x_2.
\end{equation}

Then

\begin{enumerate}
\item \( M \) lies in a hypersphere \( S^{m-1}(r) \) centered at the origin with radius, say \( r \), as a mass–symmetric submanifold;
\item \( M \) has constant mean curvature \( \alpha \) in \( \mathbb{E}^m \) given by
\[
\alpha^2 = \frac{1}{n}(\lambda_1 + \lambda_2) - \left(\frac{r}{n}\right)^2 \lambda_1 \lambda_2;
\]
\item both \( x_1, x_2 \) are mass–symmetric spherical maps;
\item \( x_1, x_2 \) and the Laplace map of \( x \) satisfy
\begin{equation}
(2.2) \quad x_1 = \frac{\lambda_2 x - L}{\lambda_2 - \lambda_1}, \quad x_2 = \frac{\lambda_1 x - L}{\lambda_1 - \lambda_2};
\end{equation}
and
\item the spherical maps \( \bar{x}_1, \bar{x}_2 \), induced from \( x_1, x_2 \), are harmonic maps which have constant energy density.
\end{enumerate}

\textbf{Proof.} Let \( x : M \to \mathbb{E}^m \) be an orthogonal, 2–type, isometric immersion of a compact manifold \( M \) into \( \mathbb{E}^m \) whose spectral decomposition is given by (2.1). For any vector \( X \) tangent to \( M \), we put

\begin{equation}
(2.3) \quad X = X_1 + X_2, \quad X_1 \in E_1, \quad X_2 \in E_2.
\end{equation}
Then, by the orthogonality of $E_1, E_2$, we have

(2.4) \[ \nabla_X x_i = X_i, \; i = 1, 2. \]

We put

(2.5) \[ f = \langle \Delta x, x \rangle. \]

Then, from (2.5), Beltrami’s formula and the hypothesis, we have

\[
Xf = X(\lambda_1 \langle x_1, x_1 \rangle + \lambda_2 \langle x_2, x_2 \rangle) \\
= 2(\lambda_1 \langle x_1, X_1 \rangle + \lambda_2 \langle x_2, X_2 \rangle) \\
= 2 \langle \Delta x, X \rangle = 0.
\]

Therefore, $f$ is a constant. Hence,

(2.6) \[ \Delta \langle x, x \rangle = 2 \langle \Delta x, x \rangle - 2n = 2f - 2n \]

is a constant. Because $M$ is compact, this implies that $\langle x, x \rangle$ is a constant. Thus, $M$ is contained in a hypersphere $S^{m-1}(r)$ of $\mathbb{E}^m$ centered at the origin and with radius, say $r$. This shows that $M$ is a mass–symmetric, 2–type spherical submanifold which proves (1).

(2) Follows from (1) and a Theorem 4.1 of [C5, page 274].

(3) From (1) we have

(2.7) \[ \langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle = r^2. \]

On the other hand, by using Beltrami’s formula, we also have

(2.8) \[ \lambda_1 \langle x_1, x_1 \rangle + \lambda_2 \langle x_2, x_2 \rangle = \langle -nH, x \rangle = n. \]

Combining (2.7) and (2.8) we obtain

(2.9) \[ \langle x_1, x_1 \rangle = \frac{\lambda_2 r^2 - n}{\lambda_2 - \lambda_1}, \quad \langle x_2, x_2 \rangle = \frac{\lambda_1 r^2 - n}{\lambda_1 - \lambda_2}, \]

which implies (3).

(4) Follows from (2.1) and the fact: $L = \lambda_1 x_1 + \lambda_2 x_2$.

(5) From (3) we have

(2.10) \[ x_1 : M \to S^{m-1}(r_1) \subset \mathbb{E}^m, \]

where

\[ r_1 = \sqrt{(\lambda_2 r^2 - n)/(\lambda_2 - \lambda_1)}. \]

Denote by $h_j$ the second fundamental form of $S^{m-1}(r_1)$ in $\mathbb{E}^m$. Then (2.10) implies that the tension fields of $x_1$ is given by

(2.11) \[ \tau(x_1) = j_\ast \tau(\bar{x}_1) - 2e(\bar{x}_1) x_1, \]

where $\bar{x}_1$ is the map $M \to S^{m-1}(r_1)$ induced from $x_1 : M \to \mathbb{E}^m$, $j$ the inclusion of $S^{m-1}(r_1)$ into $\mathbb{E}^m$, and $e(\bar{x}_1)$ the energy density of $\bar{x}_1$. 
On the other hand, we have
\[ \Delta x_1 = -\tau(x_1) = \lambda_1 x_1. \] Comparing (2.11) and (2.12), we conclude that \( \bar{x}_1 \) is a harmonic map and it has constant energy density. The same argument can be applied to \( x_2 \). □

**Remark 2.1.** The condition of compactness in Proposition 2.1 is essential. For instance, the circular cylinder defined by
\[ x(t, s) = (t, \cos s, \sin s) \]
is an orthogonal 2–type surface in \( \mathbb{E}^3 \) which is not spherical. □

If \( x : M \to \mathbb{E}^m \) is a map of 2-type, we define the notion of *conjugation* of \( x \) as follows.

**Definition 2.1.** Let \( x \) be a map of 2-type whose spectral decomposition takes the following form:
\[ x = c + x_1 + x_2, \quad \Delta x_1 = \lambda_1 x_1, \quad \Delta x_2 = \lambda_2 x_2, \quad \lambda_1 < \lambda_2, \]
where \( c \) is a constant map and \( x_1 \) and \( x_2 \) are non–constant maps. Then the map \( \bar{x} : M^n \to \mathbb{E}^m \) given by
\[ \bar{x} = c + x_1 - x_2 \]
is called the conjugate of the 2–type map \( x \). □

The conjugate of a 2–type isometric immersion is not isometric, in general. For example, for the conjugate of a 2–type curve in \( \mathbb{E}^m \), we have the following result.

**Proposition 2.2.** The conjugate \( \bar{\beta} \) of a 2-type unit speed curve \( \beta(s) \) in \( \mathbb{E}^m \) is a unit speed curve if and only if \( \beta \) is either an open part of a circular helix lying in an affine 3–space or an open part of the diagonal immersion of two circles.

**Proof.** First we recall that if \( \beta(s) \) is curve of finite type, then \( \beta \) can be written as (cf. [CDVV1])
\[ \beta(s) = a_0 + b_0 s + \sum_{t=1}^{k} (a_t \cos(p_t s) + b_t \sin(p_t s)). \]
We may assume \( a_0 = 0 \) by choosing \( a_0 \) as the origin of \( \mathbb{E}^m \). Thus, if \( \beta \) is of 2–type, then \( \beta \) can be expressed as one of the following forms:
\[ \beta(s) = b_0 s + (a_1 \cos(ps) + b_1 \sin(ps)), \]
or
\[ \beta(s) = \sum_{t=1}^{2} (a_t \cos(p_t s) + b_t \sin(p_t s)), \]
where \( b_0, a_1, b_2, a_2, b_1, b_2 \) are constant vectors in \( \mathbb{E}^m \) and \( p, p_1, p_2 \) are nonzero constants.
If $\beta(s)$ is a unit speed curve whose conjugate is also a unit speed curve, then we have
\begin{equation}
\langle \beta'(s), \beta'(s) \rangle = \langle \bar{\beta}'(s), \bar{\beta}'(s) \rangle = 1.
\end{equation}

**Case 1.** If (2.15) holds, then, by (2.17), we may prove that $b_0, a_1, b_1$ are mutually orthogonal and $|a_1| = |b_1|$. Hence, in this case, $\beta$ is an open part of a circular helix.

**Case 2.** If (2.16) holds, then (2.17) implies that $a_1, a_2, b_1, b_2$ are mutually orthogonal and, moreover, $|a_1| = |b_1|, |a_2| = |b_2|$. Hence, $\beta$ is the diagonal immersion of two circles.

The converse is easy to verify. □

Similar to the notion of orthogonal immersions, we give the following

**Definition 2.2.** Let $x : M \rightarrow \mathbb{E}^m$ be an immersion of $k$–type whose spectral decomposition is given by
\begin{equation}
x = c + x_1 + \cdots + x_k, \quad \Delta x_1 = \lambda_1 x_1, \ldots, \Delta x_k = \lambda_k x_k, \quad \lambda_1 < \cdots < \lambda_k,
\end{equation}
where $c$ is a constant map and $x_1, \ldots, x_k$ are non–constant maps. Then the immersion $x$ is said to be pointwise orthogonal (respectively, strongly pointwise orthogonal) if for each $p \in M, x_1(p), \ldots, x_k(p)$ are mutually orthogonal (respectively, the image subspaces $x_{1*}(T_p M), \ldots, x_{k*}(T_p M)$ are mutually orthogonal) in $\mathbb{E}^m$. □

For 2-type isometric immersions, we have the following result.

**Theorem 2.3.** Let $x : M \rightarrow \mathbb{E}^m$ be a 2-type isometric immersion, then

1. $x$ is strongly pointwise orthogonal if and only if the conjugate $\bar{x}$ of $x$ is an isometric immersion;
2. $x$ is pointwise orthogonal if the isometric immersion $x$ is a mass-symmetric spherical immersion; and
3. $x$ is orthogonal if and only if the conjugate $\bar{x}$ of $x$ is congruent to immersion $x$ up to an orthogonal transformation of $\mathbb{E}^m$.

**Proof.** (1) Let $x$ be a 2–type isometric immersion whose conjugate $\bar{x}$ is also an isometric immersion. Assume the spectral decomposition of $x$ is given by (2.13). Then we have
\begin{equation}
\langle dx, dx \rangle = \langle dx_1, dx_1 \rangle + \langle dx_2, dx_2 \rangle + 2 \langle dx_1, dx_2 \rangle,
\end{equation}
\begin{equation}
\langle d\bar{x}, d\bar{x} \rangle = \langle dx_1, dx_1 \rangle + \langle dx_2, dx_2 \rangle - 2 \langle dx_1, dx_2 \rangle.
\end{equation}
Since both $x$ and $\bar{x}$ are isometric immersions, (2.18) and (2.19) yield
\[\langle dx_1, dx_2 \rangle = 0\]
identically. This implies that the immersion $x$ is strongly pointwise orthogonal.

Conversely, if $x$ is strongly pointwise orthogonal, then (2.18) and (2.19) imply $\langle dx, dx \rangle = \langle d\bar{x}, d\bar{x} \rangle$. Therefore, the conjugate of $x$ is isometric.

(2) Let $x: M \to S^{m-1}(r) \subset \mathbb{R}^m$ be a spherical, mass–symmetric, 2–type isometric immersion. Assume the spectral decomposition of $x$ is given by

$$x = x_1 + x_2, \quad \Delta x_1 = \lambda_1 x_1, \quad \Delta x_2 = \lambda_2 x_2, \quad \lambda_1 < \lambda_2,$$

Then we have

$$\langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle + 2 \langle x_1, x_2 \rangle = r^2.$$  

By Beltrami’s formula, we have

$$\lambda_1 x_1 + \lambda_2 x_2, x = \langle \Delta x, x \rangle = \langle -nH, x \rangle = n.$$  

Furthermore, by Lemma 4.2 of [C5, p.273], we have

$$\langle \lambda_1^2 x_1 + \lambda_2^2 x_2, x \rangle = -n \langle \Delta H, x \rangle = \left( \frac{na}{r} \right)^2.$$  

From (2.20), (2.21) and (2.22) we obtain $\langle x_1, x_2 \rangle = 0$. Thus, $x$ is pointwise orthogonal.

(3) Let $x$ be a 2–type immersion. Without loss of generality, we may assume the spectral decomposition of $x$ is given by (2.1). Assume, up to orthogonal transformations of $E^m$, the conjugate $\bar{x}$ of $x$ is congruent to the immersion $x$. Then there is $A \in O(m)$ such that $Ax = \bar{x}$. Thus, we have

$$\langle x_1 - x_1, Ax_2 + x_2 \rangle = 0.$$  

Because $\Delta(Ax_i) = A(\Delta x_i) = \lambda_i Ax_i$, (2.23) yields

$$\lambda_1 (Ax_1 - x_1) + \lambda_2 (Ax_2 + x_2) = 0.$$  

From (2.23) and (2.24) we get

$$Ax_1 = x_1, \quad Ax_2 = -x_2.$$  

Let $E_i = \text{Span}\{x_i(p) : p \in M\}, i = 1, 2,$ and let $V_1, V_2$ be the eigenspaces of $A$ with eigenvalues $1, -1$, respectively. Then (2.25) implies

$$E_1 \subset V_1, \quad E_2 \subset V_2.$$  

Because $A$ is an orthogonal transformation, (2.26) implies that, for any $u \in V_1, v \in V_2$, we have $\langle u, v \rangle = \langle Au, Av \rangle = -\langle u, v \rangle$. Therefore, $V_1$ and $V_2$ are orthogonal in $E^m$. Consequently, by (2.26), we conclude that $x$ is an orthogonal immersion.

The converse of this is clear. □
§3. Laplace map of 2–type immersions.

The purpose of this section is to study relations between the Laplace maps of a
2–type immersion and its conjugate.

**Lemma 3.1.** Let $x : M \to \mathbb{E}^m$ be a 2-type isometric immersion. Then, up to
±1, the Laplace map of $x$ and the Laplace map of the conjugate $\bar{x}$ of $x$ in $\mathbb{E}^m$ are
equal if and only if the immersion $x$ is of null 2-type.

**Proof.** Let $x$ be a 2–type immersion with spectral decomposition given by

\[
x = c + x_1 + x_2, \quad \Delta x_1 = \lambda_1 x_1, \quad \Delta x_2 = \lambda_2 x_2.
\]

Then the Laplace map of $x$ and its conjugate $\bar{x}$ of $x$ are given by

\[
L = \lambda_1 x_1 + \lambda_2 x_2, \quad \bar{L} = \lambda_1 x_1 - \lambda_2 x_2.
\]

Therefore, $L = \bar{L}$ (respectively, $L = -\bar{L}$) if and only if $\lambda_2 = 0$ (respectively,
$\lambda_1 = 0$). Hence, $L = \bar{L}$ if and only if immersion $x$ is of null 2–type. □

**Lemma 3.2.** If $x : M \to \mathbb{E}^m$ is a 2-type isometric immersion whose conjugate $\bar{x}$
is an isometric immersion, then the Laplace map of the conjugate $\bar{x}$ is the conjugate
of the Laplace map.

**Proof.** If $x : M \to \mathbb{E}^m$ is a 2-type isometric immersion whose conjugate $\bar{x}$
is an isometric immersion, then the Laplace operator $\Delta$ on $M$ induced from the
conjugate $\bar{x}$ equals the Laplace operator $\Delta$ of $M$. Hence, the Laplace map of $\bar{x}$ is
given by $\lambda_1 x_1 - \lambda_2 x_2$ which is nothing but the conjugate of the Laplace map of $x$.
□

We now introduce the following

**Definition 3.1.** A 2–type immersion (respectively, a 2–type map) $x : M \to \mathbb{E}^m$ is called a dual 2-type immersion (respectively, a dual 2–type map) if the
spectral decomposition of immersion $x$ takes the form: $x = c + x_1 + x_2$ such that
$\Delta x_1 = -\lambda x_1$ and $\Delta x_2 = \lambda x_2$, where $c$ is a constant map and $x_1$ and $x_2$ are two
nonconstant maps and $\lambda$ is a positive number. □

Geometrically, up to a constant, a dual 2-type immersion is an immersion whose
the Laplace transformation is involutive.

**Lemma 3.3.** Let $x : M \to \mathbb{E}^m$ be a 2-type isometric immersion. Then, up
to nonzero constants, the Laplace map of $x$ is the conjugate $\bar{x}$ of $x$ if and only if
immersion $x$ is of dual 2-type.

**Proof.** Let $x$ be a 2–type immersion with spectral decomposition given by (3.1).
If there is a constant $\mu \neq 0$ such that the Laplace map $L$ of $x$ and the conjugate $\bar{x}$
of \( x \) satisfy \( L = \mu \bar{x} \), then we have

\[(3.3) \quad (\lambda_1 - \mu)x_1 + (\lambda_2 + \mu)x_2 = \mu c.\]

From (3.3), we get

\[(3.4) \quad \lambda_1 (\lambda_1 - \mu)x_1 + \lambda_2 (\lambda_2 + \mu)x_2 = 0,\]

\[(3.5) \quad \lambda_1^2 (\lambda_1 - \mu)x_1 + \lambda_2^2 (\lambda_2 + \mu)x_2 = 0.\]

Since \( \lambda_1 \neq \lambda_2 \), (3.3), (3.4) and (3.5) imply

\[(3.6) \quad c = 0, \quad \lambda_1 = -\lambda_2.\]

Thus, the immersion \( x \) is of dual 2–type.

Conversely, if \( x \) is of dual 2–type, then

\[(3.7) \quad x = c + x_1 + x_2, \quad \Delta x_1 = -\lambda x_1, \quad \Delta x_2 = \lambda x_2.\]

From (3.7), we get \( L = -\lambda(x_1 - x_2) = \lambda \bar{x}. \quad \square \)

**Remark 3.1.** For an isometric immersion \( x : M \to \mathbb{E}^m \) of a Riemannian manifold \((M^n, g)\) into \( \mathbb{E}^m \) and for a positive real number \( c \), we have the immersion \( x^c \) defined by \( x^c = cx \). The induced metric \( g^c \) on \( M \) with respect to \( x^c \) is given by \( g^c = c^2 g \) and the Laplace operator \( \Delta^c \) of \((M, g^c)\) is given by \( \Delta^c = c^{-2} \Delta \). It is easy to see that the Laplace image of \( cx \) is given by \( c L(M) \). Therefore, when the Laplace transformation \( L : M^n \to L(M) \) is homothetic, one may multiply a suitable constant to the immersion \( x \) to obtain an isometric immersion whose Laplace transformation is also an isometry. \( \square \)

In terms of conjugation, we also have the following result.

**Lemma 3.4.** Let \( x : M \to \mathbb{E}^m \) be a dual 2-type isometric immersion. If the Laplace transformation \( \mathcal{L} \) of \( x \) is isometric, then

1. the immersion \( x \) is constructed from eigenspaces of \( \Delta \) belonging to eigenvalues 1 and \(-1\);
2. the immersion \( x \) is strongly pointwise orthogonal;
3. up to sign, the Laplace map \( L : M \to \mathbb{E}^m \) is an immersion which is given by the conjugation of the immersion \( x : M \to \mathbb{E}^m \);
4. the immersion \( x : M \to \mathbb{E}^m \) is the conjugate of the Laplace immersion \( L : M \to \mathbb{E}^m \) of \( x \); and
5. the Laplace transformation \( \mathcal{L} \) is idempotent.

**Proof.** Let \( x = c + x_1 + x_2 \) be a dual 2-type isometric immersion. Then \( \Delta x_1 = \lambda x_1, \Delta x_2 = -\lambda x_2 \), for some number \( \lambda \neq 0 \). Thus, the Laplace map is given by

\[(3.8) \quad L = \lambda(x_1 - x_2).\]
Assume the Laplace transformation $\mathcal{L} : M \to L(M)$ is isometric. Then we have
\[
\lambda^2 (\langle dx_1, dx_1 \rangle - 2 \langle dx_1, dx_2 \rangle + \langle dx_2, dx_2 \rangle ) \\
= \langle dx_1, dx_1 \rangle + 2 \langle dx_1, dx_2 \rangle + \langle dx_2, dx_2 \rangle .
\]
Let $X$ be a tangent vector of $M$ such that $(x_1)_* X \neq 0$ and $(x_2)_* X = 0$. Then from (3.9), we find $\lambda^2 = 1$. This proves (1).

The remaining parts of this Lemma follow easily from (1). □
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