Simplicial fibrations

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Abstract
We undertake a systematic study of fibrations in the setting of abstract simplicial complexes, where the concept of “homotopy” has been replaced by that of “contiguity”. Then, a fibration will be a simplicial map satisfying the “contiguity lifting property”. This definition turns out to be equivalent to that introduced by Minian, established in terms of a cylinder construction $K \times I_m$. This allows us to prove several properties of simplicial fibrations which are analogous to the classical ones in the topological setting, for instance: all the fibers of a fibration with connected base have the same strong homotopy type and any fibration with a strongly collapsible base is fibrewise trivial. We also introduce the concept of “simplicial finite-fibration”, that is, a simplicial map which has the contiguity lifting property only for finite complexes. Then, we prove that the path fibration $PK \to K \times K$ is a finite-fibration, where $PK$ is the simplicial complex of Moore paths introduced by Grandis. This result allows us to prove that any simplicial map factors through a finite-fibration, up to a P-homotopy equivalence. Moreover, we prove a simplicial version of a Varadarajan result for fibrations, relating the LS-category of the total space, the base and the generic fiber. Finally, we introduce a definition of “Švarc genus” of a simplicial map and we are able to compare the Švarc genus of path fibrations with the notions of simplicial LS-category and simplicial topological complexity introduced by the authors in several previous papers.

Keywords Simplicial complexes · Contiguous simplicial maps · Fibrations · LS-category · Topological complexity

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1 Introduction

In recent years there has been a renovated interest in abstract simplicial complexes, as a setting which is well suited for discretizing topological invariants and for designing computer algorithms. Under this point of view and boosted by the increasing computer capacities, sev-
eral classical theories have been developed, thus providing new powerful tools like persistent homology [8] or discrete Morse theory [14], which are being applied in robotics, neural networks or big data mining.

However, there is a lack of development of other ideas in this new field of “applied algebraic topology”, like Lusternik-Schnirelmann category or topological complexity, which classically needed the use of notions such as homotopy, fibrations or cofibrations.

In the framework of abstract simplicial complexes, the classical notion of “contiguity” between simplicial maps [21] plays the role that “homotopy” plays in the context of topological spaces. This notion has received new attention in the last years after the work of Barmak and Minian [5]. They showed that the equivalence under contiguity classes is the same as the equivalence by “strong collapses”, a highly interesting idea which is related on one hand with the classical Whitehead collapses, and on the other hand with the theory of posets and finite topological spaces [4].

Using the ideas above, several of the authors have recently introduced a notion of LS-category in the simplicial setting, which generalizes the well known notion of “arboricity” in graph theory [11,13]. Moreover, we also introduced a notion of topological complexity, defined in purely combinatorial terms [12]. Both invariants have similar properties to the classical ones and also new results arise.

As a collateral result, cofibrations were studied in [13], but a systematic study of the notion of simplicial fibration was lacking. This study is the aim of the present paper.

The contents are as follows:

In Sect. 2 we recall the basic notions of simplicial complexes and contiguity classes.

In Sect. 3 we introduce the notion of simplicial fibration in terms of a contiguity lifting property and we show that in fact it is equivalent to the notion introduced by Minian in [19]: a simplicial map \( p: E \to B \) is a simplicial fibration if for any simplicial map \( H: K \times I_m \to B \) and any simplicial map \( f: K \times \{0\} \to E \), there is a simplicial map \( \tilde{H}: K \times I_m \to E \) such that \( p \circ \tilde{H} = H \) and \( \tilde{H} \circ i_0^m = f \) (see Definition 7 and Proposition 3). There is a less restrictive notion of simplicial finite-fibration if we limit the lifting property to finite complexes \( K \) in the definition above. The finiteness condition will be necessary in order to obtain several key results that involve the path fibration studied in Sect. 5, as Theorems 1 and 4.

In Sect. 4 we give several basic examples and constructions, including products and pullbacks of simplicial fibrations. Then, we introduce (Sect. 5) the notion of Moore path and the space \( P_K \) of Moore paths on a simplicial complex \( K \). This notion has been developed in [15] by Grandis. The main result of this section is that the path map \( P_K \to K \times K \) is a simplicial finite-fibration (Theorem 1).

In Sect. 6 we prove the important result that all the fibers of a simplicial fibration have the same strong homotopy type. In the same line, we adapt another classical result by showing (Sect. 7) that a simplicial fibration with a collapsible base is trivial. Here, “collapsible” means that there is a finite sequence of strong collapses and expansions transforming the base onto a point.

In the next section we introduce one new idea based on a notion of “P-homotopy” modelled on Moore paths (see Definition 13), which allows us to prove the P-equivalence of the complexes \( K \) and \( P_K \) and to give a general result about the factorization of any map into a P-equivalence and a finite-fibration (Sect. 8).

Our next result (Sect. 9) is a simplicial version of Varadarajan’s theorem (see Theorem 5) relating the LS-category of the total space, the base and the generic fiber of a fibration.

On the other hand, the P-homotopy equals the usual contiguity property for finite complexes. This allows us to define in Sect. 10 a general notion of Švarc genus for simplicial maps.
and to discuss its relationship with the simplicial LS-category and the discrete topological complexity introduced in our previous papers [11–13].

2 Simplicial complexes and contiguity

We start by briefly recalling the notions of simplicial complex and contiguity. We are assuming that the reader is familiarized with these notions as well as others that will be appearing throughout the paper (see, for instance, [16,21] for more details on this topic).

Definition 1 An (abstract) simplicial complex is a set $V$ together with a collection $K$ of finite subsets of $V$ such that if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.

Notice that $K$ is not necessarily finite in the above definition. As usual, $K$ will denote the simplicial complex and $V(K)$ the corresponding vertex set.

Definition 2 Given two simplicial complexes $K$ and $L$, a simplicial map from $K$ to $L$ is a set map $\varphi : V(K) \to V(L)$ such that if $\sigma \in K$ then $\varphi(\sigma) \in L$.

Definition 3 Let $K$, $L$ be two simplicial complexes. Two simplicial maps $\varphi, \psi : K \to L$ are contiguos [21, p. 130] if, for any simplex $\sigma \in K$, the set $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of $L$; that is, if $v_0, \ldots, v_k$ are the vertices of $\sigma$, then the vertices $f(v_0), \ldots, f(v_k), g(v_0), \ldots, g(v_k)$ span a simplex of $L$.

This relation, denoted by $\varphi \sim_c \psi$, is reflexive and symmetric, but in general it is not transitive. In order to overcome this fact we use the notion of contiguity class.

Definition 4 Two simplicial maps $\varphi, \psi : K \to L$ are in the same contiguity class with $m$ steps, denoted by $\varphi \sim \psi$, if there is a finite sequence

$$\varphi = \varphi_0 \sim_c \cdots \sim_c \varphi_m = \psi$$

of contiguous simplicial maps $\varphi_i : K \to L$, $0 \leq i \leq m$.

Now we recall a formal notion of combinatorial homotopy introduced by Minian in [19]. First of all we need a triangulated version of the real interval $[0, n]$.

Definition 5 For $n \geq 1$, let $I_n$ be the one-dimensional simplicial complex whose vertices are the integers $\{0, \ldots, n\}$ and the edges are the pairs $\{j, j+1\}$, for $0 \leq j < n$.

Definition 6 [16, Definition 4.25] Let $K$ and $L$ be two simplicial complexes. The categorical product $K \times L$ is the simplicial complex whose set of vertices is $V(K \times L) = V(K) \times V(L)$, and whose simplices are given by the rule: $\sigma \in K \times L$ if and only if $pr_1(\sigma) \in K$ and $pr_2(\sigma) \in L$, where $pr_1$, $pr_2$ are the canonical projections.

Given two simplicial maps, Minian proved that belonging to the same contiguity class is equivalent to the existence of a simplicial homotopy between them, modelled by a simplicial cylinder.

Proposition 1 [19, Prop. 2.16] Two simplicial maps $f, g : K \to L$ are in the same contiguity class, with $m \geq 1$ steps, $f \sim g$, if and only if there exists a simplicial map $H : K \times I_m \to L$ such that $H(v, 0) = f(v)$ and $H(v, m) = g(v)$, for all vertices $v \in K$. 

\[ \text{Springer} \]
Remark 1 The preceding proposition holds when we consider $K \times I_m$ to be the categorical product, but notice that the proof does not work for the more usual notion of simplicial product, namely the so-called simplicial cartesian product (see [16]).

Barmak and Minian introduced [4,5] the so-called strong homotopy type for simplicial complexes. Two simplicial complexes have the same strong homotopy type, denoted by $K \sim L$, if they are related by a finite sequence of two kind of simplicial moves, namely, strong collapses and expansions. An elementary strong collapse consists of removing the open star around a dominated vertex, where a vertex $v$ is dominated by another vertex $v'$ if every maximal simplex that contains $v$ also contains $v'$. A complex is called strongly collapsible if it has the same strong homotopy type that a point.

Strong homotopy type is deeply related to the notion of contiguity between simplicial maps. More precisely, the following result holds:

Proposition 2 [5, Cor. 2.12] Two simplicial complexes $K$ and $L$ have the same strong homotopy type if and only if there are simplicial maps $\varphi: K \to L$ and $\psi: L \to K$ such that $\psi \circ \varphi \sim 1_K$ and $\varphi \circ \psi \sim 1_L$.

3 Definition of simplicial fibration

The goal of this section is to establish a notion of fibration in the simplicial context. As we shall see, there are several options of doing this, depending on the particular kind of lifting property we deal with.

The definition of fibration we give below corresponds to simplicial maps with the contiguity lifting property with respect to any simplicial complex.

Definition 7 A simplicial map $p: E \to B$ is a simplicial fibration if for any simplicial complex $K$, for any two simplicial maps $f, g: K \to B$ in the same contiguity class, $f \sim g$, with $m$ steps, and for any map $\tilde{f}: K \to E$ such that $p \circ \tilde{f} = f$, there exists a simplicial map $\tilde{g}: K \to E$ such that $\tilde{f}$ and $\tilde{g}$ are in the same contiguity class with $m$ steps, $\tilde{f} \sim \tilde{g}$, and for the sequences of contiguous simplicial maps

$$f = \varphi_0 \sim_c \cdots \sim_c \varphi_m = g$$

it holds that $p \circ \tilde{\varphi}_i = \varphi_i$, $0 \leq i \leq m$.

We can obtain an equivalent notion of simplicial fibration if we consider the notion of homotopy for simplicial complexes introduced by Minian (see Prop. 1).

Proposition 3 A simplicial map $p: E \to B$ is a simplicial fibration if and only if, given simplicial maps $H: K \times I_m \to B$ and $\varphi: K \times \{0\} \to E$ as in the following commutative diagram:

\[
\begin{array}{ccc}
K \times I_m & \xrightarrow{H} & B \\
\downarrow{i_0} & & \downarrow{p} \\
K \times \{0\} & \xrightarrow{\varphi} & E
\end{array}
\]

(1)

\[\text{ Springer}\]
there exists a simplicial map \( \tilde{H} : K \times I_m \to E \) such that \( \tilde{H} \circ i_0^m = \varphi \) and \( p \circ \tilde{H} = H \).

**Proof** First, let us assume that the simplicial map \( p : E \to B \) is a simplicial fibration. Consider a simplicial complex \( K \) and simplicial maps \( \varphi : K \times \{0\} \to E \) and \( H : K \times I_m \to B \) such that Diagram (1) is commutative.

Now, we define \( \tilde{\varphi}_i : K \to B \) by \( \tilde{\varphi}_i(v) = H(v, i) \), where \( v \in V(K) \) and \( i \in V(I_m) \). These maps are contiguous because given \( \tau \in K \) the following fact holds true:

\[
\varphi_i(\tau) \cup \varphi_{i+1}(\tau) = H(\tau \times \{i, i+1\}) \in B
\]

since \( H \) is a simplicial map. Then, by hypothesis, there exists a finite chain of simplicial maps \( \tilde{\varphi}_i : K \to E \) and direct contiguities, with \( m \) steps, \( \tilde{\varphi}_0 \sim_c \tilde{\varphi}_1 \sim_c \cdots \sim_c \tilde{\varphi}_m \), such that \( p \circ \tilde{\varphi}_i = \varphi_i \), for all \( i = 0, \ldots, m \).

Hence, the map \( \tilde{H} : K \times I_m \to E \) given by \( \tilde{H}(v, i) = \tilde{\varphi}_i(v) \) is simplicial, by an argument analogous to (2), and satisfies \( \tilde{H} \circ i_0^m = \varphi \) and \( p \circ \tilde{H} = H \).

Reciprocally, let us assume that the simplicial map \( p : E \to B \) satisfies the condition indicated above and let us prove that \( p \) is a simplicial fibration. Consider a simplicial complex \( K \) and two maps \( f, g : K \to B \) in the same contiguity class with \( m \) steps.

If \( m = 1 \), that is, \( f \) and \( g \) are contiguous, by Proposition 1, there exists a homotopy \( H : K \times I_1 \to B \) such that \( H(v, 0) = f(v) \) and \( H(v, 1) = g(v) \) for all \( v \in V(K) \).

Consider a simplicial map \( \tilde{f} : K \to E \) such that \( p \circ \tilde{f} = f \). By hypothesis, there is a simplicial map \( \tilde{H} : K \times I_1 \to E \) such that \( \tilde{H} \circ i_0^1 = \varphi \), where \( p \circ \varphi(v, 0) = \tilde{f}(v) \) for all \( v \in V(K) \), and \( p \circ \tilde{H} = H \). Let \( \tilde{g} : K \to E \) given by \( \tilde{g}(v) = \tilde{H}(v, 1) \), where \( v \in V(K) \). By Proposition 1, we conclude that \( \tilde{f} \sim_c \tilde{g} \).

If \( m > 1 \), that is, there exists a finite sequence of direct contiguities

\[
\tilde{f} = \varphi_0 \sim_c \varphi_1 \sim_c \cdots \sim_c \varphi_m = g,
\]

when iterating the same argument for every \( \tilde{\varphi}_i \) with \( i = 1, \ldots, m \), we obtain a finite sequence of direct contiguities, \( \tilde{f} = \tilde{\varphi}_0 \sim_c \tilde{\varphi}_1 \sim_c \cdots \sim_c \tilde{\varphi}_m = \tilde{g} \), where \( p \circ \tilde{\varphi}_i = \varphi_i, i = 1, \ldots, m \), that conclude the proof. \( \square \)

**Remark 2** Observe that, for a simplicial map to be a simplicial fibration, it is enough that the map satisfies the lifting property for contiguous maps.

**Remark 3** Notice that the complex \( K \) that we considered in Definition 7 and Proposition 3 may not be finite.

It is possible to restrict the notion of simplicial fibration and the proposition above to the cases where \( K \) is finite. This allows us to introduce the corresponding notion of *simplicial finite-fibration* and the corresponding equivalent notions given by the finite versions of Proposition 3 and Remark 2.

### 4 Examples and properties

In this section we will introduce some important examples of simplicial fibrations. All the proofs are standard in algebraic homotopy theory [3] and therefore they are omitted and left to the reader. The following proposition will give us the first basic ones. Notice that we will use the characterization of simplicial fibration given by Proposition 3.

**Proposition 4** (i) Any simplicial isomorphism is a simplicial fibration.
(ii) If $\ast$ denotes the one-vertex simplicial complex, then the constant simplicial map $E \to \ast$ is a simplicial fibration, for any simplicial complex $E$.

(iii) The composition of simplicial fibrations is a simplicial fibration.

For the next result we need to recall the pullback construction for simplicial complexes. Given any pair of simplicial maps $f : K \to M$ and $g : L \to M$ their pullback is given as the following diagram:

\[
\begin{array}{ccc}
K \times_M L & \xrightarrow{f'} & L \\
\downarrow{g'} & & \downarrow{g} \\
K & \xrightarrow{f} & M
\end{array}
\]

where $K \times_M L$ is the full simplicial subcomplex of $K \times L$ whose underlying vertex set is given by those pairs of vertices $(v, w) \in K \times L$ satisfying $f(v) = g(w)$. The induced simplicial maps $f'$ and $g'$ are given by $f'(v, w) = w$ and $g'(v, w) = v$. It is plain to check that this construction is the pullback of $f$ and $g$ in the category of simplicial complexes.

**Proposition 5** Let $p : E \to B$ be a simplicial fibration and $f : K \to B$ any simplicial map. Then the simplicial map $p' : K \times_B E \to K$ induced by $p$ in the pullback

\[
\begin{array}{ccc}
K \times_B E & \xrightarrow{f'} & E \\
\downarrow{p'} & & \downarrow{p} \\
K & \xrightarrow{f} & B
\end{array}
\]

is also a simplicial fibration.

As a consequence, taking into account Proposition 4 (ii), we obtain the next result.

**Corollary 1** Let $K$ and $L$ be simplicial complexes and $K \times L$ be their categorical product. Then the canonical projections $\text{pr}_1 : K \times L \to K$ and $\text{pr}_2 : K \times L \to L$ are simplicial fibrations.

Another interesting example of simplicial fibration is given by the product of simplicial fibrations. Recall that, if $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ are simplicial maps, then one can construct their product simplicial map:

\[f_1 \times f_2 : K_1 \times K_2 \to L_1 \times L_2\]

defined as $(f_1 \times f_2)(v_1, v_2) := (f_1(v_1), f_2(v_2))$, for any vertex $(v_1, v_2) \in K_1 \times K_2$.

**Proposition 6** Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be simplicial fibrations. Then their product $p_1 \times p_2$ is also a simplicial fibration.

5 The path complex $PK$

Now we study the complex $PK$ of Moore paths. The concept of Moore path and most of the content of subsections 5.1 and 5.2 are originally due to Grandis [15, Sections 2.2 and 2.5].
5.1 Moore paths

Consider the one-dimensional simplicial complex $\mathbf{Z}$, whose vertices are all the integers $i \in \mathbb{Z}$ and whose 1-simplices are all the consecutive pairs $\{i, i+1\}$, that is, $\mathbf{Z}$ is a triangulation of the real line.

**Definition 8** ([15]) Let $K$ be a simplicial complex. A Moore path in $K$ is a simplicial map $\gamma : \mathbf{Z} \rightarrow K$ which is eventually constant on the left and eventually constant on the right, i.e., there exist integers $i^-, i^+ \in \mathbb{Z}$ satisfying the two following conditions:

(i) $\gamma(i) = \gamma(i^-)$, for all $i \leq i^-$,

(ii) $\gamma(i) = \gamma(i^+)$, for all $i \geq i^+$.

Obviously, if $i^- = i^+$ we have the constant map. For a non constant Moore path $\gamma : \mathbf{Z} \rightarrow K$ we can consider the integers

$$\gamma^- := \max\{i^- : \gamma(i) = \gamma(i^-), \text{ for all } i \leq i^-,\}$$

$$\gamma^+ := \min\{i^+ : \gamma(i) = \gamma(i^+), \text{ for all } i \geq i^+\}.$$

Observe that $\gamma^- < \gamma^+$.

**Definition 9** The images $\alpha(\gamma) := \gamma(\gamma^-)$ and $\omega(\gamma) := \gamma(\gamma^+)$ are called the initial vertex and final vertex of $\gamma$, respectively. When $\gamma$ is constant we set $\gamma^- = 0 = \gamma^+$.

If $a, b \in \mathbb{Z}$ with $a \leq b$, $[a, b]$ will denote the full subcomplex of $\mathbf{Z}$ generated by all vertices $i$ with $a \leq i \leq b$. Considering this notation, any Moore path $\gamma$ in $K$ may be identified with the restricted simplicial map $\gamma : [\gamma^-, \gamma^+] \rightarrow K$. The interval $[\gamma^-, \gamma^+]$ will be called the support of $\gamma$.

If $\gamma$ is a Moore path in $K$ with support $[\gamma^-, \gamma^+]$, then one can take the reverse Moore path $\overline{\gamma}$ as

$$\overline{\gamma}(i) = \gamma(-i),$$

whose support is $[-\gamma^+, -\gamma^-]$. Notice that this reparametrization describes $\gamma$ in the opposite direction.

If $\gamma$ is a Moore path in $K$ with support $[\gamma^-, \gamma^+]$ such that $\gamma^+ - \gamma^- = m$, then we define one normalized Moore path $|\gamma| : I_m \rightarrow K$ as

$$|\gamma|(i) = \gamma(i + \gamma^-).$$

The advantage of this reparametrization is that the support of $|\gamma|$ is $[0, m]$, and therefore it will be more manageable when dealing with simplicial fibrations.

**Definition 10** Given $\gamma, \delta$ Moore paths in $K$ such that $\omega(\gamma) = \alpha(\delta)$, the product path $\gamma \ast \delta$ is defined as

$$(\gamma \ast \delta)(i) = \begin{cases} 
\gamma(i - \delta^-), & i \leq \gamma^+ + \delta^- \\
\delta(i - \gamma^+), & i \geq \gamma^+ + \delta^-. 
\end{cases}$$

It is not difficult to see that the support of $\gamma \ast \delta$ is $[\gamma^- + \delta^-, \gamma^+ + \delta^+]$. The product of Moore paths is strictly associative, that is, given Moore paths $\gamma, \delta, \varepsilon$ such that $\omega(\gamma) = \alpha(\delta)$ and $\omega(\delta) = \alpha(\varepsilon)$, then $\gamma \ast (\delta \ast \varepsilon) = (\gamma \ast \delta) \ast \varepsilon$.

Moreover, if $c_v$ denotes the constant path in a vertex $v \in K$, then it is immediate to check that $\gamma \ast c_v = \gamma = c_v \ast \gamma$ where $v = \alpha(\gamma)$ and $w = \omega(\gamma)$.
5.2 The path complex

Next we will consider a suitable notion of Moore path complex associated to the simplicial complex $K$. In order to do so we need to recall some categorical properties in the category $SC$ of simplicial complexes and simplicial maps.

Indeed, if $K$ and $L$ are simplicial complexes, we define the simplicial complex $L^K$, whose vertices are all simplicial maps $f: K \to L$ and where we consider as simplices the finite sets $\{f_0, \ldots, f_p\}$ of simplicial maps $K \to L$ such that $\bigcup_{i=0}^{p} f_i(\sigma) \in L$, for any simplex $\sigma \in K$.

It is not difficult to check that this definition induces a structure of simplicial complex in $L^K$.

Moreover, denoting by $\times$ the categorical product in $SC$, we have that the evaluation map $ev: L^K \times K \to L$, $(f, v) \mapsto f(v)$,

is simplicial. This fact allows us to establish a natural bijection $\text{SC} (M \times K, L) \equiv \text{SC} (M, L^K).$ (3)

Observe that for a simplicial map $h: L \to L'$, there is a well defined map $h^K: L^K \to (L')^K$, which preserves the identities and the compositions, that is, we have a functor $(-)^K: SC \to SC$. More is true, the functor $(-) \times K: SC \to SC$ is left adjoint to the functor $( -)^K: SC \to SC$.

**Definition 11** Let $K$ be a simplicial complex. We define the Moore path complex of $K$, denoted by $P^K$, as the full subcomplex of $K^Z$ generated by all Moore paths $\gamma: Z \to K$.

Then, $\{\gamma_0, \ldots, \gamma_p\} \subset P^K$ defines a simplex in $P^K$ if and only if $\{\gamma_0(i), \ldots, \gamma_p(i), \gamma_0(i+1), \ldots, \gamma_p(i+1)\}$ is a simplex in $K$, for any integer $i \in Z$.

An interesting property of $P^K$ is that, for any bounded interval $[a, b] \subset Z$, the complex $K^{[a,b]}$ is, in fact, a full subcomplex of $P^K$. Moreover, given a simplicial map $f: K \to L$, since the composite $f \circ \gamma$ is a Moore path in $L$ for any Moore path $\gamma$ in $K$, we obtain the Moore path complex functor $P: SC \to SC$. One can check that this functor preserves binary products and equalizers. Therefore $P$ preserves finite limits and, in particular, pullbacks. In general $P$ does not preserve limits; for instance, $P$ does not preserve infinite products [15]. Hence, $P$ has not a left adjoint functor.

**Definition 12** The initial and final vertices of any given Moore path $\gamma$ define simplicial maps $\alpha: PK \to K$ and $\omega: PK \to K$.

**Remark 4** Taking into account Proposition 5 and the above comments, it is immediately seen that the category of simplicial complexes satisfies axioms P1, P2 and P5 of a $P$-category (or category with a natural path object) in the sense of Baues [3]. Besides, by Proposition 4, axiom P3 is partially satisfied. However, as the path functor $P$ has not a left adjoint we cannot pass from certain diagrams to others in which some kind of cylinder could be involved in order to use our notion of simplicial fibration. Moreover, the simplicial complex $PK$ is always infinite, even in the case that $K$ is finite. These two facts prevent us to prove that any simplicial fibration satisfies the homotopy lifting property with respect to $P$ (in the sense of Baues). For identical reasons we can not prove P4 axiom (relative path axiom).

We also observe that, with our notion of simplicial fibration, if a hypothetical $P$-category structure held on the category of simplicial complexes, then this would not give rise to another fibration category structure agreeing with the former on finite simplicial complexes, as the functor $P: SC \to SC$ does not restrict to finite complexes.
5.3 The path fibration

The aim of this subsection is to establish the following important example of simplicial finite-fibration (see Remark 3).

**Theorem 1** If \( K \) is any simplicial complex, then the following simplicial map

\[
\pi = (\alpha, \omega) : PK \to K \times K
\]

is a simplicial finite-fibration where \( \alpha \) and \( \omega \) are the maps given in Definition 12.

**Proof** Let \( L \) be a finite simplicial complex, \( m \geq 1 \), and a commutative diagram of simplicial maps

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi} & PK \\
\downarrow{i_0^m} & & \downarrow{\pi=(\alpha,\omega)} \\
L \times I_m & \xrightarrow{(G,H)} & K \times K
\end{array}
\]

We recall that, as \( L \) is finite, there exists a factorization of \( \varphi \) of the form

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi} & PK \\
\downarrow{i_0^m} & & \downarrow{\pi=(\alpha,\omega)} \\
K[a,b] & \xrightarrow{\psi} & PK
\end{array}
\]

Indeed, if \([a(v), b(v)]\) denotes the support of \( \varphi(v) \) for any vertex \( v \in L \), then we may take \( a = \min\{a(v) : v \in L\} \) and \( b = \max\{b(v) : v \in L\} \) due to the fact that \( L \) is finite. We therefore obtain a commutative diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi} & K[a,b] \\
\downarrow{i_0^m} & & \downarrow{(\psi)} \\
L \times I_m & \xrightarrow{(G,H)} & K \times K \\
& \xrightarrow{(ev_a,ev_b)} & \downarrow{\pi=(\alpha,\omega)} \\
& & K \times K
\end{array}
\]

where \( ev_a \) and \( ev_b : K[a,b] \to K \) denote the evaluation simplicial maps at \( a \) and \( b \), respectively. We define the simplicial map

\[
\Psi : L \times I_m \to K[a-m, b+m] \subset PK
\]

whose expression is

\[
\Psi(v, i)(j) = \begin{cases} 
G(v, i), & a - m \leq j \leq a - i, \\
G(v, a - j), & a - i \leq j \leq a, \\
\varphi(v)(j), & a \leq j \leq b, \\
H(v, j - b), & b \leq j \leq b + i, \\
H(v, i), & b + i \leq j \leq b + m.
\end{cases}
\]

Taking into account the exponential law in the simplicial setting and the fact that \( \varphi, G \) and \( H \) are simplicial maps, it is not difficult to check that \( \Psi \) is indeed a simplicial map.
Moreover, $\Psi$ fits in the following commutative diagram

$$
\begin{array}{ccccccccc}
L & \xrightarrow{\psi} & K_{[a,b]} & \xrightarrow{\pi} & PK \\
\downarrow{\imath_0^\phi} & & \downarrow{(ev_{a-m}, ev_{b+m})} & & \downarrow{\pi} \\
L \times I_m & \xrightarrow{(G,H)} & K \times K & & K \times K
\end{array}
$$

The result follows.

5.4 P-homotopy

The maps $\alpha$ and $\omega$ of Definition 12 allow us to introduce the following notion of homotopy, originally due to Grandis [15, p. 122]:

**Definition 13** Given $f, g : K \to L$ simplicial maps, we will say that $f$ is $P$-**homotopic** to $g$, denoted by $f \simeq g$, when there exists a simplicial map $H : K \to PL$ such that $\alpha \circ H = f$ and $\omega \circ H = g$.

This relationship is certainly reflexive and symmetric. More is true, it is compatible with left and right compositions. However we do not know whether it is transitive. In [15, p. 123] it is said to be presumably non transitive.

**Definition 14** Let $f : K \to L$ be a simplicial map. Then $f$ is said to be a $P$-**homotopy equivalence** if there exists a simplicial map $g : L \to K$ such that $g \circ f \simeq 1_K$ and $f \circ g \simeq 1_L$.

Taking into account the links between strong homotopy type and contiguity classes established by Barmak and Minian in [4] and [5], notice that if P-homotopies are substituted with finite sequences of contiguous maps in the above definition, then we obtain the notion of strong equivalence.

An alternative equivalent form for the notion of P-homotopy is given in the following result, which is implicit in [15, Section 3.1]:

**Proposition 7** Let $f, g : K \to L$ be simplicial maps. Then $f \simeq g$ if and only if there exists a sequence of simplicial maps $\{f_i : K \to L\}_{i \in \mathbb{Z}}$ indexed by the integer numbers, such that:

(i) $f_i \sim_c f_{i+1}$ are contiguous maps;
(ii) For every vertex $v \in K$ there exist integers $n_v$ and $m_v$ such that $f_i(v) = f(v)$, for all $i \leq n_v$ and $f_i(v) = g(v)$ for all $i \geq m_v$.

**Proof** If $H : K \to PL$ is a homotopy between $f$ and $g$ then we define the simplicial map $f_i := H(-)(i) : K \to L$. Obviously, as $H$ is simplicial and $[i, i+1]$ is a simplex in $\mathbb{Z}$, we have that $f_i$ and $f_{i+1}$ are contiguous, so condition (i) holds. Moreover, since for any $v \in K$ we have that $H(v) \in PL$ is a Moore path with support $[n_v, m_v]$, condition (ii) is also fulfilled as $\alpha \circ H = f$ and $\omega \circ H = g$.

Conversely, given a sequence of simplicial maps $\{f_i : K \to L\}_{i \in \mathbb{Z}}$ satisfying (i) and (ii) we define $H : K \to PL$ as $H(v)(i) := f_i(v)$. It is straightforward to check from (i) that $H$ is simplicial and from (ii) that $\alpha \circ H = f$ and $\omega \circ H = g$. \hfill $\square$
Now we will see the relationship between the homotopy $\simeq$ and the class of contiguity $\sim$. It is clear that Proposition 1 can be rewritten as follows: $f$ and $g$ are in the same class of contiguity if and only if there exist integers $a \leq b$ and a simplicial map $H : K \times [a, b] \to L$ such that $H(v, a) = f(v)$ and $H(v, b) = g(v)$, for every vertex $v \in K$.

**Proposition 8** Let $f, g : K \to L$ be simplicial maps. Then

(i) If $f \sim g$, then $f \simeq g$.
(ii) If $K$ is finite and $f \simeq g$, then $f \sim g$.

In particular, if $K$ is finite, then $f$ and $g$ are in the same contiguity class, $f \sim g$, if and only if they are $P$-homotopic, $f \simeq g$.

**Proof** First consider $H : K \times [a, b] \to L$ such that $H(v, a) = f(v)$ and $H(v, b) = g(v)$. Taking the composition of the adjoint simplicial map $K \to L^{[a, b]}$, given by the bijection (3), with the inclusion $L^{[a, b]} \subset PL$, we obtain a simplicial map $H' : K \to PL$ satisfying $\alpha \circ H' = f$ and $\omega \circ H' = g$.

Conversely, suppose that $K$ is finite and consider a simplicial map $G : K \to PL$ satisfying $\alpha \circ G = f$ and $\omega \circ G = g$. For any $v \in K$ we have that $G(v)$ is a Moore path in $L$ with support $[a_v, b_v]$. Taking $a = \min\{a_v : v \in K\}$ and $b = \max\{b_v : v \in K\}$ we obtain a factorization:

![Diagram](image)

Considering the adjoint of $K \to L^{[a, b]}$ we obtain a simplicial map $G' : K \times [a, b] \to L$ such that $G'(v, a) = f(v)$ and $G'(v, b) = g(v)$, for all $v \in K$. Hence, this means that $f \sim g$. $\square$

The last paragraph of Proposition 8 is explicitly stated (with a different notation and without proof) in [15, Section 3.2]. We have the following immediate consequence:

**Corollary 2** Let $f : K \to L$ be a simplicial map between finite simplicial complexes. Then $f$ is a $P$-homotopy equivalence if and only if $f$ is a strong equivalence.

We now include an example, taken from [15, p. 123], showing that for an infinite complex the two notions are different.

**Example 1** Let $K = \mathbb{Z}$. By using Proposition 7, the identity $\text{id}_\mathbb{Z}$ is $P$-homotopic to the constant zero map $0_\mathbb{Z}$ by means of the telescopic homotopy, which is given by the sequence of contiguous maps $f_i, i \in \mathbb{Z}$, where $f_i = \text{id}_\mathbb{Z}$ if $i \leq 0$, and $f_i = f \circ \cdots \circ f$, if $i \geq 1$, where $f : \mathbb{Z} \to \mathbb{Z}$ is defined as

$$f(n) = \begin{cases} n - 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ n + 1 & \text{if } n \leq -1. \end{cases}$$

However it is clear that $\text{id}_\mathbb{Z}$ and $0_\mathbb{Z}$ are not in the same contiguity class. Otherwise, since two contiguous simplicial maps $f, g$ must verify $|f(n) - g(n)| \leq 1$, if there is a sequence of $m$ contiguous maps between $\text{id}_\mathbb{Z}$ and $0_\mathbb{Z}$ it should be $|n| \leq m$ for all $n \in \mathbb{Z}$. 

$\text{Springer}$
6 Homotopy fiber theorem

The notion of subdivision map between finite intervals is crucial for the proofs of the main results in the following sections. The main references are Minian’s papers [18,19]. These techniques allow one to adapt to the simplicial setting the classical results on fibrations of topological spaces.

Recall that for any natural number \( n \), \( I_n \) is the full subcomplex of \( \mathbb{Z} \) generated by all vertices \( 0 \leq i \leq n \). If \( n' \geq n \), then a subdivision map \( t: I_{n'} \rightarrow I_n \) is any simplicial map satisfying \( t(0) = 0 \) and \( t(n') = n \). The proof of the following Lemma can be found in [19, Lemma 3.6].

**Lemma 1** Given natural numbers \( n, m \), there exist \( n', m' \) with \( n' \geq n \) and \( m' \geq m \) and a simplicial map \( \phi: I_{n'} \times I_{m'} \rightarrow I_n \times I_m \) with the following sketches on the boundaries:

\[
\begin{array}{ccc}
\text{a'} & \cdots & \text{e'} \\
\text{d'} & \cdots & \text{f'} \\
\end{array}
\]

where \( a' \rightarrow a, b' \rightarrow b \), etc. are subdivision maps. Moreover, there exist \( n'' \geq n', m'' \geq m' \) and a simplicial map \( \phi': I_{n''} \times I_{m''} \rightarrow I_{n'} \times I_{m'} \) with the opposite sketches on the boundaries such that the composition

\[
\phi \circ \phi': I_{n''} \times I_{m''} \rightarrow I_n \times I_m
\]

has the form \( \phi \circ \phi' = t_1 \times t_2 \), where \( t_1: I_{n''} \rightarrow I_n \) and \( t_2: I_{m''} \rightarrow I_m \) are subdivision maps.

The next straightforward result is called Simplicial Pasting Lemma.

**Lemma 2** Let \( U, V \) be subcomplexes of a simplicial complex \( K \). Let \( U \cup V \) denote the subcomplex of \( K \) consisting of the simplices in \( U \) or in \( V \) and let \( f: U \rightarrow L, g: V \rightarrow L \) be simplicial maps such that \( f(v) = g(v) \), for every vertex \( v \in U \cap V \). Then, the vertex function \( h: U \cup V \rightarrow L \), defined as

\[
h(v) = \begin{cases} 
  f(v), & \text{if } v \in U, \\
  g(v), & \text{if } v \in V,
\end{cases}
\]

is a simplicial map.

Given \( v, v' \) vertices in a simplicial complex \( K \) and a Moore path \( \gamma \) joining \( v \) and \( v' \), that is, \( \alpha(\gamma) = v \) and \( \omega(\gamma) = v' \), let us consider the Moore path \( \overline{\gamma} := [\overline{\gamma}] \), that is, the normalized of the reverse Moore path of \( \gamma \). It is plain to check that the simplicial map \( H: I_{2n} \times I_n \rightarrow B \) defined as

\[
H(i, j) := \begin{cases} 
  \gamma(\max\{n-i, j\}), & 0 \leq i \leq n, \\
  \gamma(\max\{i-n, j\}), & n \leq i \leq 2n,
\end{cases}
\]

gives a homotopy \( H \) between \( \overline{\gamma} \ast \gamma \) and the constant Moore path \( c_b \) relative to \( \{0, n\} \). Similarly, one can also check that \( \gamma \ast \overline{\gamma} \sim c_b \) rel. \( \{0, n\} \).
Using this language, a simplicial complex $B$ is connected if and only for any pair of vertices $b, b' \in B$ there exists a normalized Moore path $\gamma: [0, n] \to B$ such that $\gamma(0) = b$ and $\gamma(n) = b'$.

Given a simplicial fibration $f: E \to B$ and a vertex $b_0 \in B$, the simplicial fiber of $p$ over $b_0$, denoted by $F_{b_0}$, is the full subcomplex of $E$ generated by all the vertices $e \in E$ such that $p(e) = b_0$. In other words, $F_{b_0} = p^{-1}(b_0)$.

The following theorem is one of the main results of this paper, since it shows that our notion of simplicial fibration has nice properties such as the homotopy invariance of the fiber.

**Theorem 2** Let $p: E \to B$ be a simplicial fibration where $B$ is a connected simplicial complex. Then any two simplicial fibers of $p$ have the same strong homotopy type.

**Proof** Let $\gamma: I_n \to B$ be a Moore path such that $\gamma(0) = b$ and $\gamma(n) = b'$. Let us first check that there is a simplicial map $F_b \to F_{b'}$. Indeed, if $i_b$ denotes the inclusion $F_b \subset E$ and $\text{pr}_2: F_b \times I_n \to I_n$ the natural projection, then take a lift in the diagram:

\[
\begin{array}{ccc}
F_b \times \{0\} & \xrightarrow{i_b} & E \\
\downarrow & \searrow \gamma & \downarrow p \\
F_b \times I_n & \xrightarrow{\gamma \circ \text{pr}_2} & B \\
\end{array}
\]

Clearly, we have that $\widetilde{\gamma}(v, n) \in F_{b'}$, for all $v \in F_b$. Therefore there is an induced simplicial map $\gamma^\sharp: F_b \to F_{b'}$ given by $\gamma^\sharp(v) := \widetilde{\gamma}(v, n)$.

We will prove that $\gamma^\sharp$ is independent, up to contiguity class, of the chosen lift $\widetilde{\gamma}$.

Suppose $\delta: I_n \to B$ is another Moore path satisfying that $\delta(0) = b$, $\delta(n) = b'$ and there is $H: I_n \times I_m \to B$ such that $H: \gamma \sim \delta$ rel. $\{0, n\}$. First, let us consider the commutative diagram:

\[
\begin{array}{ccc}
F_b \times J_{nm} & \xrightarrow{f} & E \\
\downarrow & \searrow H \circ \text{pr} & \downarrow p \\
F_b \times I_n \times I_m & \xrightarrow{g} & B \\
\end{array}
\]

where:

- $J_{nm} = (I_n \times \{0\}) \cup ([0] \times I_m) \cup (I_n \times \{m\})$,
- $\text{pr}: F_b \times I_n \times I_m \to F_b \times I_n$ is the projection on the two first complexes,
- $f$ is the simplicial map (see Lemma 2 above) defined as
  
  $f(v, i, 0) = \widetilde{\gamma}(v, i)$,
  
  $f(v, 0, j) = v$,
  
  $f(v, i, m) = \delta(v, i)$.

Now, take the simplicial maps $\phi: I_n' \times I_{m'} \to I_n \times I_m$ and $\phi': I_n' \times I_{m'} \to I_n' \times I_{m'}$ given by Lemma 1. As $\phi(I_n' \times \{0\}) \subset J_{nm}$, we have a lift $\tilde{H}: F_b \times I_{n'} \times I_{m'} \to E$, represented by the dotted arrow in the composite diagram:
Taking into account that \( \phi'(J_{n''m''}) \subset I_{n'} \times \{0\} \) and that \( \phi \circ \phi' = t_1 \times t_2 \), we have that the simplicial map
\[
H' := \widetilde{H} \circ (1_{F_b} \times \phi') : F_b \times I_{n''} \times I_{m''} \to E
\]
satisfies that \( H'(v, n'', j) \in F_b \), for all \( v \in F_b \) and \( j \in I_{m''} \). Therefore, there is an induced simplicial map
\[
\overline{H} := H'(-, n'', -) : F_b \times I_{m''} \to F_{b'}.
\]
It is not difficult to see that \( \overline{H}(v, 0) = \gamma^\natural(v) \) and \( \overline{H}(v, n'') = \delta^\natural(v) \); that is, \( \gamma^\natural \sim \delta^\natural \) are in the same contiguity class. It is immediate to check that, if \( c_b : I_n \to B \) denotes the constant Moore path in \( b \in B \), then \( c_b^\natural \) is (up to contiguity class) the identity \( 1_{F_b} : F_b \to F_b \). Moreover, if \( \gamma' : I_n \to B \) and \( \delta' : I_m \to B \) are Moore paths such that \( \gamma(0) = b, \gamma(n) = b' = \delta(0) \) and \( \delta(m) = b'' \), then we can prove that \( (\gamma * \delta)^\natural : F_b \to F_{b'} \) belongs to the same contiguity class of the composition \( \delta^\natural \circ \gamma^\natural \). Indeed, the simplicial map \( \overline{\gamma * \delta} : F_b \times I_{n+m} \to E \) defined as
\[
(\overline{\gamma * \delta})(v, i) = \begin{cases} 
\widetilde{\gamma}(v, i), & 0 \leq i \leq n, \\
\widetilde{\delta}(\overline{\gamma}(v, n), i - n), & n \leq i \leq n + m,
\end{cases}
\]
gives a lift:
\[
F_b \times \{0\} \xrightarrow{i_b} E \xrightarrow{p} B
\]
\[
F_b \times I_{n+m} \xrightarrow{\overline{\gamma \ast \delta} \circ \text{pr}_2} B
\]
This proves that \( (\gamma \ast \delta)^\natural \) and \( \delta^\natural \circ \gamma^\natural \) are in the same contiguity class.

Using the above reasonings and the fact that \( \overline{\gamma * \gamma} \sim c_b \) rel. \( \{0, n\} \) and \( \gamma * \overline{\gamma} \sim c_b \) rel. \( \{0, n\} \), we conclude the proof of the result. \( \square \)

7 Collapsible base

Minian’s Lemma (Lemma 1) and the Simplicial Pasting Lemma (Lemma 2) will be crucial for the next results. These results should be compared with those included in [21]. We start with this fairly general proposition.

Proposition 9 Let \( p : E \to B \) be a simplicial fibration and let \( F_0, F_1 : K \times I_n \to E \) be simplicial maps such that \( p \circ F_0 \) and \( p \circ F_1 \) are in the same contiguity class with \( m \) steps. Let \( H : p \circ F_0 \sim p \circ F_1 \) be the map given by Proposition 1 and analogously, let \( G : F_0|_{K \times \{0\}} \sim F_1|_{K \times \{0\}} \) be the map given by Proposition 1. Consider the following commutative diagram:
Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

**Proof** Consider the commutative diagram

Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

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Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

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Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

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Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

**Proof** Consider the commutative diagram

Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.

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**Proof** Consider the commutative diagram

Then, for suitable $q \geq n$ and $p \geq m$, there exist subdivision maps $t_1 : I_q \rightarrow I_n$ and $t_2 : I_p \rightarrow I_m$ and a simplicial map $H' : K \times I_q \times I_p \rightarrow E$ such that $H'$ defines a homotopy $F_0 \circ (1_K \times t_1) \sim F_1 \circ (1_K \times t_1)$, which is an extension of $G \circ (1_K \times \{0\} \times t_2)$.
Corollary 3 Let $p: E \to B$ be a simplicial fibration and let $F_0, F_1: K \times I_n \to E$ be two lifts of the same map

\[ \begin{array}{c c c}
K \times \{0\} & \xrightarrow{f} & E \\
\downarrow{F_0} & & \downarrow{p} \\
K \times I_n & \xrightarrow{F_1} & B
\end{array} \]

Then, for a suitable $q \geq n$, there exists a subdivision map $t: I_q \to I_n$ such that $F_0 \circ (1_K \times t) \sim_p F_1 \circ (1_K \times t)$.

Proof. Just apply Proposition 9 to the simplicial maps $G, H$ defined as $G(v, 0, j) = f(v, 0)$ and $H(v, i, j) = (p \circ f)(v, 0)$, for all $v \in K$ and $j \in I_n$. \(\square\)

Observe that the previous corollary also holds true when we consider $K \times \{n\}$ in the diagram instead of $K \times \{0\}$.

For our next result we will consider the notion of having the same type of fibrewise contiguity.

Definition 16 We say that two simplicial fibrations $p_1: E_1 \to B$, $p_2: E_2 \to B$ have the same type of fibrewise contiguity when there exist simplicial maps $f: E_1 \to E_2$ and $g: E_2 \to E_1$ such that $p_2 \circ f = g$, $p_1 \circ g = f$ and $g \circ f \sim_p \id_{E_1}$ and $f \circ g \sim_p \id_{E_2}$.

The following important result asserts that simplicial maps in the same class of contiguity induce simplicial fibrations having the same type of fibrewise contiguity.

Theorem 3 Let $p: E \to B$ be a simplicial fibration and let $f_0, f_1: K \to B$ simplicial maps. Consider, for each $i = 1, 2$, the pullback of $f_i$ along $p$:

\[ \begin{array}{c c c}
E_i & \xrightarrow{f'_i} & E \\
p_i \downarrow & & \downarrow p \\
K & \xrightarrow{f_i} & B
\end{array} \]

If $f_0$ and $f_1$ are in the same class of contiguity, then the simplicial fibrations $p_0$ and $p_1$ have the same type of fibrewise contiguity.

Proof. Consider a simplicial map $F: K \times I_n \to B$ such that $F(v, 0) = f_0(v)$ and $F(v, n) = f_1(v)$, for all $v \in K$. Take $F'_0, F'_1$ lifts of the following diagrams (for the second diagram see Remark 5):

\[ \begin{array}{c c c}
E_0 \times \{0\} & \xrightarrow{f'_0} & E \\
\downarrow{F'_0} & & \downarrow{p} \\
E_0 \times I_n & \xrightarrow{F_{\circ(1)}} & B \\
\downarrow{F_{\circ(1)}} & & \downarrow{F_{\circ(1)}} \\
E_1 \times \{n\} & \xrightarrow{f'_1} & E \\
\downarrow{F'_1} & & \downarrow{p} \\
E_1 \times I_n & \xrightarrow{F_{\circ(1)}} & B
\end{array} \]

By the pullback property there are well defined simplicial maps $g_0: E_0 \to E_1$ and $g_1: E_1 \to E_0$ satisfying the commutativities given in the following diagrams:
Lemma 3. Indeed, remember that, in the statement of such theorem, we have pullbacks

\[ E_0 \xrightarrow{g_0} E_1 \xrightarrow{f_1} E \]

We finish this section with an interesting property whose proof relies on the proof of

Corollary 3 assures the existence of a suitable subdivision map \( t : I_{n'} \to I_n \) and a simplicial

map \( G : E_1 \times I_{n'} \times I_m \to E \) satisfying that

\[ G : F_0' \circ (g_1 \times 1) \circ (1 \times t) \sim_p F_1' \circ (1 \times t) \]

Again, using the pullback property, there is an induced simplicial map \( \tilde{G} : E_1 \times I_m \to E_1 \)
satisfying \( f_1' \circ \tilde{G} = G(-, n', -) \) and \( p_1 \circ \tilde{G} = p_1 \circ \text{pr} \). A simple inspection proves that

\[ \tilde{G} : g_0 \circ g_1 \sim_{p_1} 1_{E_1} \]

Analogously, by applying Corollary 3 to the diagram \( p \circ f_1' \circ g_0 = F \circ (p_0 \times 1)|_{E_0 \times [n]} \) for the common lifts \( F_1' \circ (g_0 \times 1) \) and \( F_0' \), and taking into account Remark 5, one can find a simplicial map \( H \) satisfying \( H : F_1' \circ (g_0 \times 1) \circ (1 \times s) \sim_p F_0' \circ (1 \times s) \) for a suitable subdivision map \( s : I_{n''} \to I_n \). Taking \( H \) the simplicial map characterized by the equalities \( f_0' \circ \tilde{H} = G(-, 0, -) \) and \( p_0 \circ \tilde{H} = p_0 \circ \text{pr} \), we obtain \( \tilde{H} : g_1 \circ g_0 \sim_{p_0} 1_{E_0} \). □

Corollary 4. Let \( p : E \to B \) be a simplicial fibration where \( B \) is strongly collapsible. Then \( p \)
has the same class of fibrewise contiguity of the trivial fibration \( B \times p^{-1}(b_0) \to B \), for any

vertex \( b_0 \in B \).

Proof. Just take into account that \( 1_B : B \to B \) and the constant path \( c_{b_0} : B \to B \) are in the
same class of contiguity and use Theorem 3. □

We finish this section with an interesting property whose proof relies on the proof of

Theorem 3. Indeed, remember that, in the statement of such theorem, we have pullbacks

\[ p \circ f_i^t = f_i \circ p_i \quad (i = 0, 1) \]

where \( f_0 \sim f_1 \) (i.e., \( f_0 \) and \( f_1 \) are in the same class of contiguity). From the proof one obtains simplicial maps \( g_0 : E_0 \to E_1 \) and \( g_1 : E_1 \to E_0 \) (with

\[ p_1 \circ g_0 = p_0 \] and \( p_0 \circ g_1 = p_1 \)) and Minian simplicial homotopies \( F_0' : E_0 \times I_n \to E \) and

\( F_1' : E_1 \times I_n \to E \) satisfying \( F_0' \circ f_0' \sim f_1' \circ g_0 \) and \( F_1' : f_0' \circ g_1 \sim f_1' \). Moreover, \( g_0 \) and \( g_1 \)
satisfy \( g_1 \circ g_0 \sim_{p_0} 1_{E_0} \) and \( g_0 \circ g_1 \sim_{p_1} 1_{E_1} \). In particular \( g_1 \circ g_0 \sim 1_{E_0} \) and \( g_0 \circ g_1 \sim 1_{E_1} \).

We easily have the following lemma:

Lemma 3. Consider the pullback of a simplicial fibration \( p : E \to B \) along a simplicial map

\( f : B \to B \) such that \( f \sim 1_B \). If \( f' : E' \to E \) denotes the base change of \( f \), then \( f' \circ g_0 \)
and \( f' \circ g_1 \sim 1_E \). In particular, \( f' \) is a strong equivalence with \( g_1 : E \to E' \) as a homotopy
inverse.
**Proof** Just use the argument above to the pullback of \( p \) along the identity \( 1_B : B \to B \). \( \square \)

And finally our result. Observe that such result completes Proposition 5.

**Proposition 10** Consider the pullback of a simplicial fibration \( p : E \to B \) along a simplicial map \( f : K \to B \):

\[
\begin{array}{ccc}
E' & \xrightarrow{f'} & E \\
p' & & p \\
\downarrow & & \downarrow \\
K & \xrightarrow{f} & B
\end{array}
\]

If \( f \) is a strong equivalence, then \( f' \) is also a strong equivalence.

**Proof** Suppose \( g : B \to K \) is a simplicial map such that \( g \circ f \sim 1_K \) and \( f \circ g \sim 1_B \). Consider first the following diagram, where \( E'' \) is the pullback of \( p' \) along \( g \):

\[
\begin{array}{ccc}
E'' & \xrightarrow{g'} & E' \\
p'' & & p' \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & K \\
\end{array}
\]

As \( f \circ g \sim 1_B \), by the lemma above (note that the composite of pullbacks is a pullback) one can find simplicial maps \( g_0 : E'' \to E \) and \( g_1 : E \to E'' \) such that \( g_1 \circ g_0 \sim 1_{E''} \), \( g_0 \circ g_1 \sim 1_E \), \( f' \circ g' \sim g_0 \) and \( f' \circ g' \circ g_1 \sim 1_E \). Therefore, \( g' \) has a left homotopy inverse and \( f' \) has a right homotopy inverse: \( (g_1 \circ f') \circ g' \sim 1_{E''} \) and \( f' \circ (g' \circ g_1) \sim 1_E \).

Similarly, from the diagram of pullbacks

\[
\begin{array}{ccc}
E''' & \xrightarrow{f''} & E'' \\
p''' & & p'' \\
\downarrow & & \downarrow \\
K & \xrightarrow{f} & B \\
\end{array}
\]

we have that \( g' \) has a right homotopy inverse. It straightforwardly follows that \( g' \) is a strong equivalence. Since \( f' \circ g' \sim g_0 \) we conclude that \( f' \) is a strong equivalence. \( \square \)

### 8 Factorization

In this section we will see that any simplicial map may be considered, in a homotopical sense, as a simplicial finite-fibration.

For our main result in this section we will use the fact that \((\alpha, \omega) : PK \to K \times K\) is a simplicial finite-fibration, for any simplicial complex \( K \) (see Theorem 1).

**Theorem 4** Let \( f : K \to L \) be a simplicial map. Then there is a factorization \( f = p \circ j \), where \( j \) is a \( P \)-homotopy equivalence and \( p \) is a simplicial finite-fibration.
Proof We consider $P_f = K \times L$ the pullback of $f : K \to L$ along $\alpha : PL \to L$, where $\alpha$ is the initial vertex map. In this way we have a commutative square:

$$
\begin{array}{ccc}
P_f & \xrightarrow{f'} & PL \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
K & \xrightarrow{f} & L
\end{array}
$$

As we have previously observed, the simplicial complex $P_f$ is defined as the full subcomplex of $K \times PL$ whose set of vertices is given as

$$
P_f = \{(v, \gamma) \in K \times PL : f(v) = \alpha(\gamma)\}
$$

and $\alpha', f'$ are the restrictions of the obvious projections. We define the simplicial map $j : K \to P_f$ as $j(v) := (v, c_f(v))$, where $c_f(v) : \mathbb{Z} \to L$ denotes the constant Moore path at $f(v)$. On the other hand, the simplicial map $p : P_f \to L$ is defined as $p(v, \gamma) := \omega(\gamma)$, where $\omega$ is the final vertex map. Obviously, we have that $p \circ j = f$.

Let us first check that $p$ is a simplicial finite-fibration. Indeed, consider the following commutative diagram:

$$
\begin{array}{ccc}
P_f & \xrightarrow{f'} & PL \\
\downarrow{(\alpha', p)} & & \downarrow{(\alpha, \omega)} \\
K \times L & \xrightarrow{f \times 1_L} & L \times L \\
\downarrow{\text{pr}_1} & & \downarrow{\text{pr}_1} \\
K & \xrightarrow{f} & L
\end{array}
$$

It is not difficult to check that the bottom diagram is a pullback. As, by construction, the composition diagram is also a pullback, we obtain that the top diagram must be a pullback. But then, being $(\alpha, \omega)$ a simplicial finite-fibration, the simplicial map $(\alpha', p) : P_f \to K \times L$ must be also a simplicial finite-fibration. Therefore, the composition of the finite-fibrations $p = \text{pr}_2 \circ (\alpha', p) : P_f \to L$ is a finite-fibration.

Next we check that $j : K \to P_f$ is a P-homotopy equivalence. As $\alpha' \circ j = 1_K$ it only remains to see that $j \circ \alpha' \simeq 1_{P_f}$. We directly define the homotopy $H : P_f \to PP_f$ as $H(v, \gamma)(i) := (v, \gamma_i)$ where $\gamma_i \in PL$ is defined as

$$
\gamma_i(j) = \begin{cases} 
\gamma(j), & j \leq i, \\
\gamma(i), & j \geq i,
\end{cases}
$$

for any $\gamma \in PL$ and $i \in \mathbb{Z}$. Using the exponential law for simplicial complexes one can see that $H$ is indeed a well-defined simplicial map defining a homotopy between $j \circ \alpha'$ and $1_{P_f}$. We leave the details to the reader. \hfill \Box

It could be interesting to compare the above theorem and its proof with Theorem 29 of [19].

As a consequence of this theorem new finite-fibrations appear.

Example 2 For instance, for any simplicial complex $K$ and a base vertex $v_0 \in K$ one can consider the full subcomplex of $PK$, $P_0K = \{\gamma \in PK : \alpha(\gamma) = v_0\}$. Observe that this
simplicial complex is nothing else than the construction $P_f$ given in the theorem above, where $f : \{v_0\} \hookrightarrow K$ is the inclusion map. Then, the simplicial finite-fibration associated to $f$ is precisely the map $P_0 K \rightarrow K$, where $\gamma \mapsto \omega(\gamma)$.

**Example 3** There is also a factorization that we are specially interested in. Although it does not come from the general construction of the above theorem, one can check that the following diagram gives a factorization of the diagonal map $\Delta : K \rightarrow K \times K$ through a P-homotopy equivalence followed by a simplicial finite-fibration

$$
\begin{array}{ccc}
K & \xrightarrow{\Delta} & K \times K \\
& \downarrow c & \\
PK & \xrightarrow{(\alpha, \omega)} & PK
\end{array}
$$

We already know that $(\alpha, \omega)$ is a simplicial finite-fibration. On the other hand, $c : K \rightarrow PK$ is defined as the simplicial map sending each $v \in K$ to the constant Moore path at $v$, $c(v) := c_v$. This map is, indeed, a P-homotopy equivalence: the simplicial map $\alpha : PK \rightarrow K$ satisfies $\alpha \circ c = 1_K$. Moreover, the homotopy $H : PK \rightarrow P^2 K$ defined as $H(\gamma)(i) = \gamma_i$, where $\gamma_i$ is the $i$-truncated Moore path of $\gamma$, satisfies $\alpha \circ H = c \circ \alpha$ and $\omega \circ H = 1_{PK}$; that is, $H : c \circ \alpha \simeq 1_{PK}$.

### 9 Varadarajan’s theorem

As an application of Theorem 2 about the strong homotopy type of the fibers of a fibration, we prove in this section a simplicial version of a well-known result from Varadarajan for topological fibrations [22], establishing a relationship between the LS-categories of the total space, the base and the homotopic fiber. A more general result for smooth foliations was proved by Colman and Macías-Virgós in [6].

**Definition 17** The simplicial LS-category $\text{scat}(K)$ of the simplicial complex $K$ is the least integer $n \geq 0$ such that $K$ can be covered by $n + 1$ subcomplexes $K_j$ such that each inclusion $\iota_j : K_j \subset K$ belongs to the contiguity class of some constant map $c_vj : K_j \rightarrow K$.

This notion is the simplicial version of the well known homotopic invariant $\text{cat}(X)$, the so-called Lusternik-Schnirelmann category of the topological space $X$ [7]. It has been introduced by the authors in [11,13], its most important property being the invariance by strong homotopy equivalences.

Accordingly to Theorem 2, all the fibers $p^{-1}(v), v \in B$, of a fibration with connected base $p : E \rightarrow B$ have the same strong homotopy type, so we call generic fiber $F$ of the fibration any simplicial complex into that equivalence class, and its simplicial category $\text{scat}(F)$ is well defined.

**Theorem 5** Let $p : E \rightarrow B$ be a simplicial fibration with connected base $B$ and generic fiber $F$. Then $\text{scat}(E) + 1 \leq (\text{scat}(B) + 1)(\text{scat}(F) + 1)$.

**Proof** Let $\text{scat} B = m$, and take a categorical covering $U_0, \ldots, U_m$ of $B$. From Theorem 2 we know that all the fibers have the same strong homotopy type. We identify $F$ to the fiber $p^{-1}(v)$ over some base point $v \in B$. Let $\text{scat} F = n$, with $V_0, \ldots, V_n$ a categorical covering of $F$.
For each $i \in \{0, \ldots, m\}$ let $i: U_i \subset B$ be the inclusion. By definition of simplicial category, the map $i$ is in the contiguity class of a constant map, say $c_i: U_i \to B$. Since $B$ is path connected, we can assume that $c_i$ is the constant map corresponding to the base point $v$. Consider the map $p_i = i \circ p: p^{-1}(U_i) \to B$. If $\epsilon_i: p^{-1}(U_i) \subset E$ is the inclusion, then $p \circ \epsilon_i = p_i$. Now, from $i \sim c_v$ it follows that $p_i \sim c_v$, the latter being the constant map with domain $p^{-1}(U_i)$. By the contiguity lifting property, there exists a simplicial map $G_i: p^{-1}(U_i) \to E$ such that $\epsilon_i \sim G_i$ and $p \circ G_i = c_v$. The latter means that the map $G_i$ takes its values in $F$. We denote $g_i: p^{-1}(U_i) \to F$ the map given by $g_i(v) = G_i(v)$. In this way $G_i = \tau_F \circ g_i$, where $\tau_F: F \subset E$ is the inclusion. For each $i \in \{0, \ldots, m\}$, $j \in \{0, \ldots, n\}$, we take the subcomplex $W_{ij} = g_i^{-1}(V_j) \subset p^{-1}(U_i) \subset E$. Since $B = U_0 \cup \cdots \cup U_m$ implies $E = p^{-1}(U_0) \cup \cdots \cup p^{-1}(U_m)$, and since $F = V_0 \cup \cdots \cup V_n$ implies $p^{-1}(U_i) = W_{i0} \cup \cdots \cup W_{in}$, it follows that $\{W_{ij}\}$ is a covering of $E$.

It only remains to prove that each $W_{ij}$ is categorical in $E$. Let $g_{ij}: W_{ij} \to V_j$ be the restriction of $g_i$ to $W_{ij} \subset p^{-1}(U_i)$. We know that each $V_j \subset F$ is categorical, so the inclusion $J_j: V_j \subset F$ is in the contiguity class of some constant map $c_j: V_j \to F$, whose image is some vertex $f_j$ of $F$. Then the composition

$$W_{ij} \xrightarrow{g_{ij}} V_j \subset F$$

belongs to the contiguity class of the constant map $f_j: W_{ij} \to F$ since we have that $J_j \circ g_{ij} \sim c_j \circ g_{ij} = f_j$. Let $\epsilon_{ij}: W_{ij} \subset E$ be the inclusion, that is, the restriction of $\epsilon_i: p^{-1}(V_j) \subset E$ to $W_{ij}$. Since $\epsilon_i \sim G_i$ it follows that $\epsilon_{ij} \sim G_{ij}$, the latter being the restriction of $G_i$ to $W_{ij}$. Finally we have $\epsilon_{ij} \sim G_{ij} = \tau_F \circ J_j \circ g_{ij} = \tau_F \circ f_j$, so the inclusion $W_{ij} \subset E$ is in the contiguity class of a constant map. \hfill $\square$

10 Švarc genus

In the classical topological setting, the Lusternik-Schnirelmann category can be seen as a particular case of the so-called “Švarc genus” or sectional category of a continuous map. In this section we recall the simplicial version of this invariant introduced by the authors in [12] and establish some key links with simplicial fibrations.

**Definition 18** The simplicial Švarc genus of a simplicial map $\varphi: K \to L$ is the minimum integer $n \geq 0$ such that $L$ is the union $L_0 \cup \cdots \cup L_n$ of $n + 1$ subcomplexes, and for each $j$ there exists a section $\sigma_j$ of $\varphi$, that is, a simplicial map $\sigma_j: L_j \to K$ such that $\varphi \circ \sigma_j$ is the inclusion $\iota_j : L_j \subset L$.

We denote this genus by $Sg(\varphi)$. A slight modification in the above definition is to change the equality by “being in the same contiguity class”.

**Definition 19** The homotopy simplicial Švarc genus of a simplicial map $\varphi: K \to L$, denoted by $hSg(\varphi)$, is the minimum $n \geq 0$ such that $L = L_0 \cup \cdots \cup L_n$, with $L_j \subset L$ subcomplex, and for each $j \in \{0, \ldots, n\}$ there exists an “up to contiguity class” simplicial section $\sigma_j$ of $\varphi$, that is, a simplicial map $\sigma_j: L_j \to K$ such that $\varphi \circ \sigma_j \sim \iota_j$.

**Remark 6** Note that when the complex $L$ is finite, the condition $\varphi \circ \sigma_j \sim \iota_j$ (same contiguity class) can be changed to $\varphi \circ \sigma_j \simeq \iota_j$ (P-homotopy), by Proposition 8.

Obviously, $hSg(\varphi) \leq Sg(\varphi)$. The equality holds for some particular classes of maps.
Theorem 6 Let $p: E \to B$ be a simplicial fibration. Then $hSg(p) = Sg(p)$.

Proof We only have to prove that $Sg(\varphi) \leq hSg(\varphi)$, so let $hSg(\varphi) = n$ and $L_0, \ldots, L_n$ be a covering of $B$ by subcomplexes such that there exist simplicial maps $\sigma_j: L_j \to E$ with $p \circ \sigma_j \sim \iota_j$, as in Definition 19. By Proposition 3, for each $j$ we have simplicial maps $H_j: L_j \times [0, m_j] \to B$ with $H_j(\cdot, 0) = p \circ \sigma_j$ and $H_j(\cdot, m_j) = \iota_j$ the inclusion $L_j \subset B$. Take a lift $\tilde{H}_j: L_j \times [0, m_j] \to E$ in the following diagram,

$\begin{array}{c}
L_j \times \{0\} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
L_j \times [0, m_j] \\
\tilde{H}_j \\
p \\
E \\
\sigma_j \\
\end{array}$

in such a way that $p \circ \tilde{H} = H$ and $\tilde{H}(v, 0) = \sigma_j(v)$. Then the map $\xi_j: L_j \to E$ given by $\xi_j(v) = \tilde{H}(v, m_j)$ is simplicial and verifies

$p \circ \xi_j(v) = p \circ \tilde{H}(v, m_j) = H(v, m_j) = \iota_j(v) = v,$

so it is a true section of $p$. Then $Sg(\varphi) \leq n$. \qed

Remark 7 Note that if $p: E \to B$ is a simplicial finite-fibration, then Theorem 6 above is still true when the base $B$ is finite.

10.1 Simplicial L-S category

Taking into account the notion of simplicial LS-category (Definition 17), what we want to see is that it equals (as in the classical case) the Švarc genus of a certain fibration. Assume that the complex $K$ is connected. Then every two constant maps $c_{v_i}, c_{v_j}$, belong to the same contiguity class, as can be easily seen by considering a path $\gamma: [0, m] \to K$ connecting $v_i$ and $v_j$ and the sequence of contiguous maps given by the truncated paths $\gamma_k: [0, k] \to K$. So we can choose a base point $v_0$ and assume that all the constant maps in Definition 17 equal $c_{v_0}$.

Consider $P_0K$ the full subcomplex of the path complex $PK$ consisting on all Moore paths $\gamma \in PK$ whose initial point is the base vertex $v_0$ (see Example 2). The map $\omega: P_0K \to K$, sending each path to its final point, is then the pullback of the simplicial finite-fibration $(\alpha, \omega): PK \to K \times K$ by the map $f$, where $f(v) = (v_0, v)$:

$\begin{array}{c}
P_0K \\
\downarrow \quad \downarrow \\
K \\
\omega \\
PK \\
(\alpha, \omega) \\
K \times K \\
f \\
\end{array}$

Theorem 7 Let $K$ be a connected finite complex. Then the simplicial LS-category $scat(K)$ equals the Švarc genus of the simplicial finite-fibration $\omega: P_0K \to K$.

Proof We have to prove that $scat(K) = Sg(\omega)$, but we know that the latter equals $hSg(\omega)$. So, first assume that $scat(K) = n$ and let $K = K_0 \cup \cdots \cup K_n$ such that each inclusion $\iota_j: K_j \subset K$ belongs to the contiguity class of the constant map $v_0$, that is, there is a Minian.
simplicial homotopy $H_j : K_j \times [0, m_j] \to K$ with $H_j(v, 0) = v_0$ and $H_j(v, m_j) = v$. Take a lift $\tilde{H}_j$ of $H_j$ in the following diagram:

$$
\begin{array}{c}
K_j \times [0, m_j] \xrightarrow{c_j} P_0K \\
\downarrow \quad \downarrow \quad \downarrow \\
K_j \times [0, m_j] \xrightarrow{H_j} K
\end{array}
$$

where $c_j(v)$ is the constant Moore path at $v_0$, for all $v$. Define $\sigma_j : K_j \to P_0K$ as $\sigma_j(v) = \tilde{H}_j(v, m_j)$. Then $\omega \circ \sigma_j(v) = H_j(v, m_j) = v$, so $\sigma_j$ is a section of the fibration. The map $\sigma_j$ is simplicial since

$$
v \in K_j \mapsto (v, m_j) \in K_j \times [0, m_j]
$$

is a simplicial map. We have then proved that $\text{Sg}(\omega) \leq n$.

Conversely, if $s : L \to P_0K$ is a section of $\omega$, each path $s(v)$ has initial vertex $v_0$ and final vertex $v$. Denote by $[v^-, v^+]$ the support of $s(v)$. Then, since $K$ is finite, there is an interval $[m, n]$ containing all the supports, so we can define $H : L \times [m, n] \to K$ as $H(v, i) = s(v)(i)$. Then $H(v, m) = s(v)(m) = s(v)(v^+) = \alpha(s(v)) = v_0$, while $H(v, m) = s(v)(n) = s(v)(v^-) = \omega(s(v)) = v$, showing that the inclusion $L \subseteq K$ belongs to the contiguity class of the constant map $v_0$.

So each covering of $K$ by subcomplexes $L$ verifying the Definition 18 gives the same covering verifying Definition 17. Then $\text{scat}(K) \leq \text{Sg}(\omega)$.

\[\square\]

### 10.2 Discrete topological complexity

In [12], a subset of the authors introduced a notion of *discrete topological complexity* which is a version of Farber’s topological complexity [10], adapted to the simplicial setting. By using well known equivalences in the topological setting, the simplicial definition avoids the use of any path complex.

**Definition 20** [12] The *discrete topological complexity* $\text{TC}(K)$ of the simplicial complex $K$ is the least integer $n \geq 0$ such that $K \times K$ can be covered by $n + 1$ “Farber subcomplexes” $\Omega_j$, where the latter means that there exist simplicial maps $\sigma_j : \Omega_j \to K$ such that $\Delta \circ \sigma_j$ is in the contiguity class of the inclusion $\iota_j : \Omega_j \subseteq K \times K$. Here $\Delta : K \to K \times K$ denotes the diagonal map $v \mapsto (v, v)$.

In other words:

**Proposition 11** The discrete topological complexity of the abstract simplicial complex $K$ is the homotopic Švarc genus of the diagonal map $\Delta : K \to K \times K$, i.e., $\text{TC}(K) = h\text{Sg}(\Delta)$.

Our main result in this section is the following one.

**Theorem 8** Let $K$ be a finite complex. The discrete topological complexity of $K$ equals the Švarc genus of the finite-fibration $(\alpha, \omega) : PK \to K \times K$.

**Proof** As it was stated in Proposition 7, contiguity and $P$-homotopy are equivalent notions for finite complexes. Therefore, $\iota_i : K_i \subseteq K$ is categorical if and only if there exists $H_i : K_i \to PK$ such that $\alpha \circ H_i = c_{v_0}$ and $\omega \circ H_i = \iota_i$, where $v_0$ is the base point indicated at the beginning of subsection 10.1. In other words, $(\alpha, \omega) \circ H_i = f \circ \iota$ where $f : K \to K \times K$
is given by $f(v) = (v_0, v)$. Then, by the universal property of pullbacks, there exists a unique $\tilde{H}_i: K_i \to P_0 K$, which is, by commutativity of the involved diagram, a section of $\omega: P_0 K \to K$. Hence $\text{scat}(K) \geq Sg(w)$.

Conversely, if $\tilde{H}_i: K_i \to P_0 K$ is a section of $\omega: P_0 K \to K$, then

$$H_i: K_i \to P_0 K \hookrightarrow P K$$

makes $(\alpha, \omega) \circ H_i = f \circ \iota$, what shows that $\iota_i$ and $c_{v_0}$ are $P$-homotopic. Hence, $\text{scat}(K) \leq Sg(w)$. $\square$

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