REGULARIZATION OF $\ell_1$ MINIMIZATION FOR DEALING WITH OUTLIERS AND NOISE IN STATISTICS AND SIGNAL RECOVERY

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Abstract. We study the robustness properties of $\ell_1$ norm minimization for the classical linear regression problem with a given design matrix and contamination restricted to the dependent variable. We perform a fine error analysis of the $\ell_1$ estimator for measurements errors consisting of outliers coupled with noise. We introduce a new estimation technique resulting from a regularization of $\ell_1$ minimization by inf-convolution with the $\ell_2$ norm. Concerning robustness to large outliers, the proposed estimator keeps the breakdown point of the $\ell_1$ estimator, and reduces to least squares when there are not outliers. We present a globally convergent forward-backward algorithm for computing our estimator and some numerical experiments confirming its theoretical properties.

Key words. $\ell_1$ norm minimization, robust regression, sparse reconstruction, breakdown point, inf-convolution, forward-backward algorithm.

AMS subject classifications. 90C31, 62F35, 65K05, 94B35

1. Introduction. In this paper we adress the problem of recovering a vector $f \in \mathbb{R}^p$ from a set of $n$ measurements ($p < n$),

$$y = Xf + \delta,$$

(1.1)

where $y \in \mathbb{R}^n$ is the vector of measurements or observations, $X$ is an $n \times p$ matrix of full rank, whose rows are realizations of the explicative variables, and $\delta \in \mathbb{R}^n$ is an error term.

In classical linear regression, a vector of observations or dependent variables $y \in \mathbb{R}^n$ is given along with the same number of explicative variables $x_1, \ldots, x_n \in \mathbb{R}^p$. We assume that the random variables $x_1, \ldots, x_n$ and $y$ are related through a linear model, which implies the existence of a vector $f \in \mathbb{R}^p$ such that

$$(\forall i \in \{1, \ldots, n\}) \quad y_i = x_i^\top f + \delta_i,$$

(1.2)

where $(\delta_i)_{1 \leq i \leq n}$ are i.i.d. random variables independent of the $x_i$s with zero mean and finite variance. The objective in linear regression is to estimate $f$. The Least Squares Estimator (LSE) of $f$ is defined as the solution to

$$\min_{g \in \mathbb{R}^p, r \in \mathbb{R}^n} \sum_{i=1}^n r_i^2 \
s.t. \quad r = y - Xg,$$

(1.3)

where $r$ denotes the vector of residuals. Under the common assumption that the errors $\delta_i$ are gaussian, the LSE is the best linear unbiased estimator of $f$ [22]. However, it is very sensitive to deviations from normality, even moderated ones. As the hypothesis of normality is often violated in practice, there is a great interest in developing statistical procedures that are robust face to different error distributions.

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In robust regression, model (1.2) is enlarged by considering that errors come from 
contaminated distributions
\[ \delta \sim (1 - \varepsilon)F_I + \varepsilon F_{II}, \]
where \( F_I \) is a light-tail distribution, usually normal, and \( F_{II} \) is an arbitrary distribution, supposed to model outliers. The quantity \( 0 < \varepsilon < 1 \) represents the fraction of contamination. The ability of an estimation method to give reasonable results under model (1.4) can be measured by the Regression Breakdown Point (RBP), defined as the maximum fraction of the components of \( \delta \) that can diverge while keeping the estimator bounded. The LSE has an asymptotic RBP of 0\%, since a single divergent observation can completely mislead the fit, independently of the sample size. There exists many robust estimators with the highest possible RBP (see [21, 18] for a comprehensive treatment of the subject), but all of them involve solving hard global and/or combinatorial optimization problems. The M-estimator [14, 15] is the first attempt to obtain robust and efficiently computable estimators. They are a generalization of (1.3), defined as a solution to
\[ \min_{g \in \mathbb{R}^p, r \in \mathbb{R}^n} \sum_{i=1}^{n} \rho(r_i) \]  
\[ \text{s.t } r = y - Xg, \]
for some differentiable pair function \( \rho : \mathbb{R} \to \mathbb{R}_+ \) which is non-decreasing in \( \mathbb{R}_+ \). The first order optimality conditions of problem (1.5) yields
\[ \sum_{i=1}^{n} w_i x_i = 0, \]
where \( w_i := \rho'(r_i) \) acts as a weight of the influence of each observation on the fit. Hence, if the function \( \rho \) is additionally convex the observations with large residuals have a higher weight. This implies that the M-estimator is sensitive to outliers in this case. In the opposite case, if the function \( \rho \) has non-increasing derivative, we face a nonconvex optimization problem, which are beyond the capabilities of the state-of-the-art of optimization methods, even for problems of modest size.

The border case is the \( \ell_1 \) estimator, also called Least Absolute Deviations, which is defined as a solution to
\[ \min_{g \in \mathbb{R}^p, r \in \mathbb{R}^n} \sum_{i=1}^{n} |r_i| \]  
\[ \text{s.t } r = y - Xg. \]
It does not fit in the framework of (1.5) since the function to minimize is not differentiable. Nonetheless, it satisfies equation (1.6) for \( w_i \) equal to one if \( r_i > 0 \), equal to minus one if \( r_i < 0 \), and between \( -1 \) and \( 1 \) for null residuals. Therefore, the \( \ell_1 \) estimator gives a bounded weight to each observation while keeping the estimation problem convex.

Despite of the remarkable properties of the \( \ell_1 \) estimator, it has been difficult to find its place in robust regression. In fact, most of the literature on the subject adopts the notion of breakdown point of Donoho-Huber [8], that considers the effect of replacing a subset of pairs \((x_i, y_i)\) of observations by arbitrary ones. In [17] the Donoho-Huber breakdown point of the \( \ell_1 \) estimator was shown to be 0, just as for
the LSE. This result leaves the impression that the \( \ell_1 \)-estimator is not robust at all, at least for random carriers.

The quantitative study of the robustness properties of the \( \ell_1 \)-estimator for non-random carriers (also called fixed design), i.e., for a deterministic \( X \), start with [12]. In this work, the authors introduce a finite-sample measure of performance for regression estimators based on tail behaviour. For the \( \ell_1 \)-estimator as well as for a class of M-estimators, their tail performance measure turns out to be equal to the RBP and they give a simple characterization of it in terms of the design configuration. In particular, they show that the RBP of the \( \ell_1 \) estimator can be positive for non-random carriers. The same expression for the RBP is obtained by Ellis and Morgenthaler [10], who also study its role as a leverage measure. Interestingly, these characterizations have been recently rediscovered in the context of the theory of compressed sensing, as we shall see in Section 2. From an optimization point of view, the same problem is studied by Giloni and Padberg [11], who provide a characterization of the RBP by using the concepts of \( q \)-strength and \( s \)-stability of a matrix, introduced by themselves. Additionally, they discuss uniqueness issues and their implications for the RBP. These results reopen the discussion on the robustness of the \( \ell_1 \) estimator.

Problem (1.1) is reconsidered in [3] by signal processing specialists. Their work lies in the fixed design framework and they suppose, as in [12, 10, 11], that contamination is restricted to the dependent variable \( y \). Moreover, they assume that the vector \( \delta \) in (1.1) is sparse, i.e., only a small fraction of the observations is contaminated and the rest is completely free of errors. This hypothesis, that would horrify any statistician, permits to solve this problem via the successful theory of sparse solutions to linear systems. This theory provides sufficient conditions for exact recovery of a signal from corrupted measurements. The sufficient condition is known as the restricted isometry property and it is verified with high probability for random normal matrices \( X \) when \( n \) and \( p \) go to infinity in a proper ratio.

Later, in [1], a modification of \( \ell_1 \) minimization for linear regression is put forward in order to deal with outliers and noise. The sufficient conditions for the noiseless case are adapted to this more realistic context. However, their conditions are only sufficient and in the particular instance when \( X \) is normal random and has orthonormal columns. A thorough study of \( \ell_1 \) minimization for struggling against noise coupled with outliers in linear regression is missing.

We perform a detailed error analysis of the \( \ell_1 \) estimator when the errors in (1.1) take the form \( \delta = z + e \), where \( z \) is a noise term and \( e \) is a sparse vector. As a consequence, we show that the RBP of the \( \ell_1 \) estimator characterizes the critical sparsity level of \( e \) in order to exactly recovering \( f \) in (1.1) by solving (1.7) when \( z = 0 \). The general conclusion of this analysis is that \( \ell_1 \) minimization manages remarkably well the presence of sparse outliers, but has a poor response to noise.

We introduce a new robust estimator that inherits the good properties of \( \ell_1 \) estimation and LSE for dealing simultaneously with outliers and noise, for a general matrix \( X \). Our estimator is defined by a minimization problem involving the inf-convolution of the \( \ell_1 \) and \( \ell_2 \) norms of the residuals. A globally convergent algorithm for computing our estimator is proposed. A fine error analysis and numerical experiments corroborates that our estimator actually have a better behavior than LSE and \( \ell_1 \) estimator in face to noise and outliers. Moreover, in the absence of outliers or noise, our estimator reduces to LSE or \( \ell_1 \) estimator, respectively.

This paper is organized as follows. In Section 2 we recall the contributions from the theory of sparse recovery to robust linear regression. In Section 3 we expound a
of a vector, often called the \( \ell_\infty \parallel \cdot \parallel \) norm or “cardinality. For a vector \( x \in \mathbb{R}^n \), we denote by \( \text{supp}(x) \) its support, i.e., the index set of nonzero components, \( \text{supp}(x) = \{ i \in N \mid x_i \neq 0 \} \). The cardinality of the support of a vector, often called the “\( \ell_0 \)-norm” or “cardinality norm”, is denoted as \( \|x\|_0 \); thus
\[
\|x\|_0 = |\{ i \in N \mid x_i \neq 0 \}|.
\]
For a subset \( M \) of \( N \) and \( p \in [1, +\infty[ \), we define \( \| \cdot \|_{p,M} : x \mapsto (\sum_{i \in M} |x_i|^p)^{1/p} \) and \( \| \cdot \|_{\infty,M} : x \mapsto \max_{i \in M} |x_i| \). Moreover, for every \( x \in \mathbb{R}^n \) and \( p \in [1, +\infty[ \), we denote
\[
\|x\|_p = \|x\|_{p,N} \text{ and } \|x\|_\infty = \|x\|_{\infty,N}.
\]
Let \( \phi : \mathbb{R}^n \to ]-\infty, +\infty[ \) be a lower semicontinuous convex function which is proper in the sense that \( \text{dom} \phi = \{ x \in \mathbb{R}^n \mid \phi(x) < +\infty \} \neq \emptyset \). The subdifferential operator of \( \phi \) is
\[
\partial \phi : \mathbb{R}^n \to 2^{\mathbb{R}^n} : x \mapsto \{ u \in \mathbb{R}^n \mid (\forall y \in \mathbb{R}^n) u^\top (y - x) + \phi(x) \leq \phi(y) \}
\]
and we have \[\text{[13, Theorem 2.2.1]}\]
\[
x \in \text{Argmin}_{y \in \mathbb{R}^n} \phi(x) \iff 0 \in \partial \phi(x). \tag{1.8}
\]
The proximal mapping associated with \( \phi \) is defined by
\[
\text{prox}_\phi : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto \text{argmin}_{u \in \mathbb{R}^n} \left( \phi(u) + \frac{1}{2} \| u - x \|^2 \right), \tag{1.9}
\]
From \[\text{[L8]}\] we obtain
\[
p = \text{prox}_\phi x \iff x - p \in \partial \phi(p),
\]
and, since \( \phi + \| \cdot \| \cdot -x \|^2/2 \) is strongly convex, \( \text{prox}_\phi(x) \) exists and is unique for all \( x \in \mathbb{R}^n \).

The following lemma will be useful throughout the paper.

**Lemma 1.1.** Let \( \gamma \in ]0, +\infty[ \) and let \( \phi : \mathbb{R}^n \to \mathbb{R} : x \mapsto \phi(x) = \gamma \|x\|_1 = \gamma \cdot \sum_{i=1}^n |x_i| \). Then the following hold.

(i) For every \( x \in \mathbb{R}^n \),
\[
\partial \phi(x) = \bigcap_{i=1}^n \partial \gamma \mid \cdot \mid(x_i),
\]
where
\[
(\forall \xi \in \mathbb{R}) \ \partial \gamma \mid \cdot \mid(\xi) = \begin{cases} 
\gamma, & \text{if } \xi > 0; \\
[-\gamma, \gamma], & \text{if } \xi = 0; \\
-\gamma, & \text{if } \xi < 0.
\end{cases}
\]
For every $x \in \mathbb{R}^n$,
\[
\text{prox}_{\gamma \phi} x = (\text{prox}_{\gamma |.|}(x_i))_{1 \leq i \leq n},
\]
where
\[
(\forall \xi \in \mathbb{R}) \quad \text{prox}_{\gamma |.|}(\xi) = \begin{cases} 
\xi - \gamma & \text{if } \xi > \gamma; \\
0 & \text{if } \xi \in [-\gamma, \gamma]; \\
\xi + \gamma & \text{if } \xi < -\gamma.
\end{cases}
\]

**Proof.** The results follow from [6, Lemma 2.1, Lemma 2.9, and Example 2.16].

Recall the unique orthogonal decomposition of $z \in \mathbb{R}^n$ as
\[ z = X\hat{g} + \hat{b}, \quad \hat{b} \in \ker(X^\top). \]

When $X$ has full rank the hat matrix
\[ H = X(X^\top X)^{-1}X^\top \]
is well defined and it holds that
\[ \hat{b} = (I - H)z, \quad \hat{g} = (X^\top X)^{-1}X^\top z, \quad \text{and } X\hat{g} = H z. \]

2. Connections with sparse reconstruction and compressed sensing. In [3], the problem of recovering an input $f$ from corrupted measurements
\[
y = Xf + e, \tag{2.1}
\]
when the error term $e$ is sparse, is considered. The goal was to solve this problem by exploiting recent advances in the study of Sparse Reconstruction Problems (SRP), which consist in finding the sparsest solution to underdetermined linear systems.

If we consider a matrix $F$ such that $\ker(F) = \text{ran}(X)$, then from (2.1) we obtain $Fy = Fe$. Let us denote $\hat{y} = Fy$, and consider the following SRP
\[
\min_{s \in \mathbb{R}^n} \|s\|_0 \quad \text{s.t.} \quad Fs = \hat{y}. \tag{2.2}
\]

Clearly, $e$ is a feasible point for Problem (2.2). If it was the unique solution then it would be possible to recover the signal $f$ from $e$ by solving the system
\[ Xf = y - e. \]

This is indeed the case, under very mild assumptions on the sparsity of $e$, as the following Lemma shows.

**Lemma 2.1.** Let $\kappa(X) = \max\{|M| : \exists \theta \in \mathbb{R}^p, \ X\theta \neq 0, \ s.t \ x_i^\top \theta = 0 \ \forall i \in M \subset \{1, \ldots, n\}\}$, let $F$ be such that $\ker(F) = \text{ran}(X)$, and set $\hat{y} = Fy$. If
\[ \|e\|_0 \leq (n - \kappa(X) - 1)/2, \]
then (2.2) has unique solution $e$. 

Proof. Suppose that there is a vector \( d \neq e \) with less than \((n - \kappa(X) - 1)/2\) nonzero components such that \( Fd = \hat{y} = Fe \). This implies that \( F(d - e) = 0 \), i.e., \( d - e \in \ker(F) = \text{ran}(X) \). Hence, there exists \( \theta \in \mathbb{R}^p \) such that \( d - e = X\theta \neq 0 \). Since \( \|d\|_0 \leq (n - \kappa(X) - 1)/2 \) and \( \|e\|_0 \leq (n - \kappa(X) - 1)/2 \), \( \|d - e\|_0 \leq n - \kappa(X) - 1 \). Therefore, \( d - e = X\theta \) has at least \( \kappa(X) + 1 \) null components, which is in contradiction with the maximality of \( \kappa(X) \).

Under the conditions provided in Lemma 2.1, the problem of recovering a signal from very incomplete information can be solved via an SRP. Unfortunately, the problem of finding the sparsest solution to linear systems is NP-hard (see, e.g., [3]). Therefore, a common approach consists in replacing the \( \ell_0 \)-norm by the \( \ell_1 \)-norm, which results in a convex (linear) optimization problem that can be efficiently solved. The problem of determining if this relaxation gives the sparsest solutions have been studied in [2, 3, 4, 9] with positive results. In these works, the authors provide sufficient conditions in order to obtain the sparsest solution via \( \ell_1 \) minimization. In [3], Candes and Tao prove that actually is the unique solution to the convex problem

\[
\min_{s \in \mathbb{R}^n} \|s\|_1 \quad \text{subject to} \quad F(s - y) = 0, \tag{2.3}
\]

provided that \( F \) satisfies the following restricted isometry property (RIP)

\[
(\exists q \in \{1, \ldots, n\}) \quad \delta_q + \theta_q, q + \theta_{q, 2q} \leq 1,
\]

where

\[
\delta_q = \max_{|J| \leq q, c \in \mathbb{R}^{|J|}} \left| \frac{\|F_{Jc}\|^2}{\|c\|^2} - 1 \right| \quad \text{and} \quad \theta_{q, q'} = \max_{|J| \leq q, c \in \mathbb{R}^{|J|}} \left| \frac{\langle F_{Jc}, F_{J'c'} \rangle}{\|c\| \|c'\|} \right|
\]

are the restricted isometry constants of \( F \) and \( F_J \) denotes the sub matrix of the columns of \( F \) indexed by \( J \). Let us further define

\[
m_R(F) = \max\{q \in N \mid \delta_q + \theta_q, q + \theta_{q, 2q} \leq 1\}.
\]

The following result gives a relation between the solution to (2.3) under condition RIP and the reconstruction of \( f \).

**Theorem 2.2** (Candes and Tao [3], Theorem 1.4). Let \( y = Xf + e \) where \( f \in \mathbb{R}^p \) and \( e \in \mathbb{R}^n \), and let \( F \) be a matrix such that \( FX = 0 \). If \( \|e\|_0 \leq m_R(F) \), then \( f \) is the unique solution to the problem

\[
\min_{g \in \mathbb{R}^p} \|y - Xg\|_1. \tag{2.4}
\]

Since then, Theorem 2.2 has been the object of several improvements. In [16, Theorem 1] it is shown that \( e \) is the unique solution to (2.3) for any \( \|e\|_0 \leq k \) if and only if \( \hat{\gamma}_k(F) < 1/2 \), where \( \hat{\gamma}_k(F) \) is defined as

\[
\hat{\gamma}_k(F) = \max_{s \in \mathbb{R}^n} \max_{|M| \leq k} \left\{ \sum_{i \in M} |s_i| : \|s\|_1 \leq 1, Fs = 0 \right\}. \tag{2.5}
\]

This result extends Theorem 2.2 by giving necessary and sufficient conditions for any given, deterministic matrix.
However, model (2.1) is too simple. In practice one expects that all observations carry some noise. A more realistic model is
\[ y = Xf + z + e, \]
where \( z \) is a dense, presumably small, vector of noise and \( e \) is an arbitrary sparse vector. Under this model, exact recovery is no longer possible. The goodness of an estimator is measured by its distance to some reference point, which can be the true parameter \( f \), or some estimator of it. If there is a bound on that distance which is finite for any \( e \) such that \( \|e\|_0 \leq k \), then the RBP of the estimator is at least \( k \).

In [1], the estimation problem is studied for the error model (2.6), in the particular case of a matrix \( X \) with orthonormal columns. They prove that the vector \( f \) can be estimated from noisy measurements up to an additive factor by solving the convex problems (for \( r = 2 \) or \( r = \infty \))
\[
\begin{align*}
\text{minimize}_{(g,b) \in \mathbb{R}^r \times \mathbb{R}^n} & \quad \|y - Xg - b\|_1 \\
\text{s.t} & \quad \|b\|_r \leq \sigma \\
& \quad X^\top b = 0.
\end{align*}
\]
provided that \( \|\bar{b}\|_r \leq \sigma \) and that additional conditions on the restricted isometry constants of \( \sqrt{n/p} X^\top \) hold.

Nevertheless, a RIP-based analysis of this problem results unsatisfactory. It provides only sufficient conditions, which are very conservative in practice. Moreover, the only known matrices with a high value of \( m_R \) are random matrices from normal or Bernoulli distributions. Also, it is not stable under linear transformations. For any given matrix \( X \) one can find an invertible matrix \( G \) in such a way that the RIP constants of \( GX \) are arbitrarily bad, independently of those of \( X \) [24]. This point is particularly serious since a closer look at [1] shows that if the matrix \( X \) does not have orthonormal columns, as is the case in statistical applications, the analysis would rely on the RIP constants of the matrix \( GX^\top \), for \( G = (X^\top X)^{-1} \).

In Section 3 we obtain sharp bounds on the estimation error of the \( \ell_1 \) estimator when the errors follow model (2.6) for a general matrix \( X \). Our treatment is simple, transparent, and covers the cases with and without noise in a unified way. It serves as the basis for the improvement of the \( \ell_1 \) estimator presented in Section 4.

### 3. Characterization of the behavior of the \( \ell_1 \)-estimator faced to sparse outliers and noise.

In this section we aim to study the problem of estimating, by \( \ell_1 \) minimization, the vector \( f \) from observations of the form (2.6). In our case, the matrix \( X \) is only assumed to be of full rank and we provide deterministic and non-asymptotic error bounds for the estimator of \( f \). In order to achieve these goals let us introduce some definitions and useful properties.

For a \( n \times p \) matrix \( X \), define for every \( k \in \{1, \ldots, n\} \) the leverage constants \( c_k \) of \( X \) as
\[
c_k(X) = \min_{M \subseteq N, \ |M| = k} \min_{g \in \mathbb{R}^p, \ g \neq 0} \frac{\sum_{i \in M} |x_i^\top g|}{\sum_{i \in N \setminus M} |x_i^\top g|} = \min_{M \subseteq N} \min_{g \in \mathbb{R}^p, \ |g|_2 = 1} \frac{\sum_{i \in M} |x_i^\top g|}{\sum_{i \in N} |x_i^\top g|} \quad (3.1)
\]
and
\[
m(X) = \max \left\{ k \in N \mid c_k(X) > 1/2 \right\}. \quad (3.2)
\]
Note that the two minima in (3.1) are achieved since the feasible set in both cases is compact and the objective function is continuous. When there is not place for confusion, we shall omit the dependency of the constants \(c_k\) on \(X\).

**Lemma 3.1.** We have \(c_0 = 1, c_n = 0\) and, for every \(k \in \{1, \ldots, n\}\), \(c_k \leq c_{k-1}\).

**Proof.** It is clear that \(c_0 = 1\) and that \(c_n = 0\). Let \(k \in \{1, \ldots, n\}\), let \(g \in \mathbb{R}^p \setminus \{0\}\), and let \(M\) with \(|M| = k - 1\) such that

\[
c_{k-1} = \frac{\sum_{i \in N \setminus M} |x_i^\top g|}{\sum_{i \in N} |x_i^\top g|}.
\]

Now let \(i_0 \in N \setminus M\) and let \(\hat{M} = M \cup \{i_0\}\). We have \(|\hat{M}| = k\) and, from (3.1) we obtain

\[
c_{k-1} = \frac{\sum_{i \in N \setminus M} |x_i^\top g| + |x_{i_0} g|}{\sum_{i \in N} |x_i^\top g|} \geq \frac{\sum_{i \in N \setminus M} |x_i^\top g|}{\sum_{i \in N} |x_i^\top g|} \geq c_k,
\]

which yields the result. \(\square\)

**Remark 3.2.** Let \(F\) be such that \(\ker(F) = \text{ran}(X)\) as in Section 2. Let \(\hat{\gamma}_k(F)\) be defined in (2.5) and \(s_+(F) = \max \{k \in N \mid \hat{\gamma}_k(F) < \frac{1}{2}\}\). These constants are related to \(c_k(X)\) and \(m(X)\) via \(c_k(X) = 1 - \hat{\gamma}_k(F)\) and \(m(X) = s_+(F)\).

Many of the results in this article rely on the following fundamental \(\ell_1\) error estimate, which is an extension and refinement of [12, Lemma 5.2].

**Lemma 3.3.** Let \(X\) be a \(n \times p\) real matrix, let \((c_k)_{1 \leq k \leq n}\) and \(m(X)\) be defined as in (3.1) and (3.2), respectively. In addition, let \(M \subseteq N\), and let \(y, b^* \in \mathbb{R}^n\) and \(g^*, g \in \mathbb{R}^p\) be arbitrary. Then the following hold.

(i) Suppose that \(|M| = k < m(X)\). Then,

\[
\|y - X g - b^*\|_1 - \|y - X g^* - b^*\|_1 \geq (2c_{k-1})\|X(g - g^*)\|_1 - 2 \sum_{i \in N \setminus M} |y_i - x_i^\top g^* - b_i^*|.
\]

(ii) Suppose that \(|M| = 0\). Then we have, for every \(b \in \mathbb{R}^n\),

\[
\|y - X g - b\|_1 - \|y - X g^* - b^*\|_1 \geq \|X(g - g^*) + b - b^*\|_1 - 2 \sum_{i \in N} |y_i - b_i^* - x_i^\top g^*|.
\]

**Proof.** (i) Let \(y, b^* \in \mathbb{R}^n\) and \(g^*, g \in \mathbb{R}^p\). We have,

\[
\|y - X g - b^*\|_1 = \sum_{i \in N} |y_i - x_i^\top g - b_i^*|
\]

\[
= \sum_{i \in N} |(y_i - x_i^\top g^* - b_i^*) - (x_i^\top g - x_i^\top g^*)|
\]

\[
= \sum_{i \in N} |(x_i^\top g - x_i^\top g^*) - (y_i - x_i^\top g^* - b_i^*)|
\]

\[
+ \sum_{i \in M} |(y_i - x_i^\top g^* - b_i^*) - (x_i^\top g - x_i^\top g^*)|
\]

\[
\geq (2c_{k-1})\|X(g - g^*)\|_1 - 2 \sum_{i \in N \setminus M} |y_i - x_i^\top g^* - b_i^*|.
\]
and using the reverse triangle inequality \(|u - v| \geq ||u| - |v|| \geq |u| - |v|\) we obtain
\[
\|y - Xg - b^*\|_1 \geq 2 \sum_{i \in N \setminus M} |x_i^T (g - g^*)| - \sum_{i \in N} |x_i^T (g - g^*)| + \sum_{i \in M} |y_i - x_i^T g^* - b_i| - 2 \sum_{i \in N \setminus M} |y_i - x_i^T g^* - b_i^*|.
\] (3.3)

It follows from (3.2) and (3.1) that \(c_k > 1/2\) and there exists \(g_k \neq g^*\) such that
\[
(\forall g, g^* \in \mathbb{R}^p) \text{ s.t. } g \neq g^* \quad \frac{\sum_{i \in N \setminus M} |x_i^T (g - g^*)|}{\sum_{i \in N} |x_i^T (g - g^*)|} \geq \frac{\sum_{i \in N \setminus M} |x_i^T (g_k - g^*)|}{\sum_{i \in N} |x_i^T (g_k - g^*)|} = c_k.
\]

Thus,
\[
\sum_{i \in N \setminus M} |x_i^T (g - g^*)| \geq c_k \sum_{i \in N} |x_i^T (g - g^*)|.
\]

By replacing in (3.3) we obtain:
\[
\|y - Xg - b^*\|_1 - \|y - Xg^* - b^*\|_1 \geq (2c_k - 1)\|X(g - g^*)\|_1 - 2 \sum_{i \in N \setminus M} |y_i - x_i^T g^* - b_i^*|
\]
and the result holds.

\[\blacksquare\]

The result is a direct consequence of the triangle inequality of the L1 norm.

Next, we provide an estimate for the reconstruction error of a solution \(f_1\) to the \(\ell_1\) minimization problem (2.3) depending on the level of contamination, including both noise and outliers. Since the least squares estimator is optimal in the absence of outliers, we measure the reconstruction error by comparing \(f_1\) with \(f_n\), which is the least squares estimator in this case. More precisely, if \(y_n := y - e = Xf + z\) is the noisy part of the data, without outliers, and \(z = X\overline{g} + \overline{b}\), with \(\overline{g} \in KerX^\top\) is the orthogonal decomposition of the noise, the LSE on the data \(y_n\) is \(f_n = (X^\top X)^{-1}X^\top y_n = f + \overline{g}\).

**Theorem 3.4.** Let \(y = Xf + z + e\) and \(M = \text{supp}(e)\). Suppose that \(|M| = k \leq m(X)\). Consider the unique decomposition of \(z\) as \(z = X\overline{g} + \overline{b}\), where \(\overline{g} \in \mathbb{R}^p\) and \(\overline{b} \in KerX^\top\), and let \(f_n = f + \overline{g}\) as discussed above. Then the following hold for \(f_1\).

(i) If \(\|\overline{b}\|_{\infty, N \setminus M} = 0\), then \(f_1 = f_n\).

(ii) If \(\|\overline{b}\|_{\infty, N \setminus M} > 0\), then
\[
\|X(f_1 - f_n)\|_1 \leq \frac{1}{2c_k - 1} \left(\|\overline{b}\|_{1, N \setminus M} + \frac{\|\overline{b}\|_{2, N \setminus M}^2}{\|\overline{b}\|_{\infty, N \setminus M}}\right).
\] (3.4)

**Proof.** Using Lemma [X.1] with \(b^* = 0, g = f_1, g^* = f_n\) we obtain
\[
\|y - Xf_1\|_1 - \|y - Xf_n\|_1 \geq (2c_k - 1)\|X(f_1 - f_n)\|_1 - 2 \sum_{i \in N \setminus M} |y_i - x_i^T f_n|.
\]

Since, by hypothesis, \(y_i = x_i^T (f + \overline{g}) + \overline{b}_i = x_i^T f_n + \overline{b}_i\) for \(i \in N \setminus M\) we have
\[
(2c_k - 1)\|X(f_1 - f_n)\|_1 \leq 2\|\overline{b}\|_{1, N \setminus M} + \|y - Xf_1\|_1 - \|y - Xf_n\|_1.
\] (3.5)
First note that, since \( f_1 \) is a minimizer, \( \| y - X f_1 \|_1 - \| y - X f_n \|_1 \leq 0 \), thus if \( \| \bar{b} \|_{\infty, N \setminus M} = 0 \) it follows from (3.5) and the hypothesis of full rank on \( X \) that \( f_1 = f_n \).

Now suppose that \( \| \bar{b} \|_{\infty, N \setminus M} > 0 \). By LP duality [11, p. 1031-1032],

\[
\| y - X f_1 \|_1 = \min_{y \in \mathbb{R}^p} \| y - X g \|_1 = \max_{d \in P^*} d^T y,
\]

where \( P^* = \{ d \in \text{ker} X^T \mid \| d \|_{\infty} \leq 1 \} \). Thus,

\[
\| y - X f_1 \|_1 - \| y - X f_n \|_1 = \max_{d \in P^*} d^T (e + \bar{b}) - \| e + \bar{b} \|_1.
\]

Hence, by using Lemma 7.1 we obtain

\[
\| y - X f_1 \|_1 - \| y - X f_n \|_1 \leq \| e + \bar{b} \|_{1, M} + \frac{\| \bar{b} \|_{2, N \setminus M}^2}{\| \bar{b} \|_{\infty, N \setminus M}} - \| e + \bar{b} \|_1
\]

\[= -\| \bar{b} \|_{1, N \setminus M} + \frac{\| \bar{b} \|_{2, N \setminus M}^2}{\| \bar{b} \|_{\infty, N \setminus M}}\]

which altogether with (3.5) yields (3.2).

**Remark 3.5.** Note that, by Hölder inequality, \( \| \bar{b} \|_{2, N \setminus M} \leq \| \bar{b} \|_{1, N \setminus M} \| \bar{b} \|_{\infty, N \setminus M} \) then

\[
\| \bar{b} \|_{1, N \setminus M} + \frac{\| \bar{b} \|_{2, N \setminus M}^2}{\| \bar{b} \|_{\infty, N \setminus M}} \leq 2\| \bar{b} \|_{1, N \setminus M} \leq 2\| \bar{b} \|_1.
\]

In the particular case when only sparse errors are present \((z = 0)\), the following result is a characterization of the exact recovery property, which improves Theorem 2.2 (see also [10, Theorem 1] and [24, Proposition 2.3] for related results).

**Theorem 3.6.** Let \( f \in \mathbb{R}^p, e \in \mathbb{R}^n \), and set \( y = X f + e \). Then \( f \) is the unique solution of the problem

\[
\min_{g \in \mathbb{R}^p} \| y - X g \|_1.
\]

for any \( \| e \|_0 \leq k \) if and only if \( k \leq m(X) \).

**Proof.** First note that, in this case, \( f_n = f \). If \( \| e \|_0 \leq m(X) \), by using Theorem 3.4 with \( z = 0 \), we obtain that \( X(f_1 - f_n) = X(f_1 - f) = 0 \) and, since \( X \) has full rank, we conclude that \( f_1 = f \). Now let us show that for \( k = \| e \|_0 > m(X) \) we can find an instance of the problem for which \( f \), whether is not a solution, or it is not the unique solution. Let \( f \in \mathbb{R}^p \) be arbitrary. From the definition of \( c_k \), there exists \( g_k \in \mathbb{R}^p \) such that \( \| g_k \|_2 = 1 \) and \( \| M \| = k \) such that

\[
\sum_{i \in N \setminus M} |x_i^T g_k| \leq \sum_{i \in M} |x_i^T g_k|.
\]

Now define, for \( \alpha > 0 \),

\[
\pi_i = \alpha \cdot \begin{cases} x_i^T g_k, & \text{if } i \in M; \\ 0, & \text{otherwise} \end{cases}
\]
and \( \mathbf{y} = Xf + \mathbf{e} \). Then,

\[
\|\mathbf{y} - Xf\|_1 = \alpha \sum_{i \in M} |x_i^\top g_k|
\]

\[
\|\mathbf{y} - X(f + \alpha g_k)\|_1 = \alpha \sum_{i \in N \setminus M} |x_i^\top g_k|.
\]

Hence, it follows from (3.6) that \( \|\mathbf{y} - X(f + \alpha g_k)\|_1 \leq \|\mathbf{y} - Xf\|_1 \), then \( f + \alpha g_k \) is a minimizer.

The proof of Theorem 3.6 shows that if \( k > m(X) \), for any \( \alpha > 0 \) we can find a vector \( \mathbf{e} \) such that \( \|\mathbf{e}\|_0 = k \) and the \( \ell_1 \) estimator \( f_1 \) on the data \( y = Xf + \alpha \mathbf{e} \) satisfies \( \|f_1 - f\|_2 = \alpha \). Combined with Theorem 3.4, this shows that the RBP of the \( \ell_1 \) estimator equals \( m(X) \), recovering results from [11, 19]. At the same time, it shows the close relation between the concepts of regression breakdown point and exact recovery of sparse signals. The most important consequence of this relation is the approximation of the RBP of a given matrix using SemiDefinite Programming (SDP). Indeed, in [7, 16] we can find SDP bounds on \( \hat{\gamma}_k(F) \) that, in view of Remark 3.2, give lower bounds on the quantities \( c_k(X) \), and thus on \( m(X) \), which characterizes the RBP of a given matrix.

In the next section we introduce a new estimator for the model including sparse errors and noise. We also verify that this new estimator has a better performance compared to the \( \ell_1 \) estimator when dense noise and sparse errors are present.

4. A robust estimator against sparse outliers and noise. In this section, we derive a new technique for estimating \( f \) from

\[
y = Xf + z + e,
\]

where \( z \) is a noise and \( e \) is an arbitrary sparse vector. Our estimator keeps the robustness of the \( \ell_1 \) estimator while improving its response to noisy observations. In particular, in the absence of outliers, it reduces to the LSE.

Theorem 3.6 proves the efficacy of the \( \ell_1 \) estimator when dealing with sparse errors. In contrast, Theorem 3.4 highlights the drawbacks of this estimator when facing noisy observations. Since the LSE is the optimal choice when facing gaussian noise, it is natural to aim at combining their main strengths. The previous discussion motivates the following definition.

**Definition 4.1.** Let \( \sigma > 0 \), let \( y \in \mathbb{R}^n \), and let \( X \) be a \( n \times p \) real matrix with full rank. The \( \ell_1 \Box \ell_2 \) estimator is defined as the first component \( \hat{g} \) of a solution to

\[
\min_{(g,b,s) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n} \sigma \|s\|_1 + \frac{1}{2} \|b\|_2^2
\]

\[
st. \quad y = Xg + b + s,
\]

where \( b \) and \( s \) are optimization variables estimating \( z \) and \( e \), respectively, and \( \sigma \) is an estimate of the magnitude of the noise.

**Remark 4.2.** Note that (4.1) can be set in the form of (3.1) with

\[
\rho: r \mapsto \rho(r) = \inf_{b \in \mathbb{R}^n} \sigma \|r - b\|_1 + \frac{1}{2} \|b\|_2^2 = \frac{1}{2} \| \cdot \|_2 \Box \| \cdot \|_1(r),
\]

where \( h_1 \Box h_2 = \inf_u \{h_1(\cdot - u) + h_2(u)\} \) denotes the inf-convolution of \( h_1 \) and \( h_2 \) [20, 13]. In other words, Definition 4.1 amounts to defining the \( \ell_1 \Box \ell_2 \) estimator as
a minimizer of the inf-convolution of the $\ell_1$ norm and the squared $\ell_2$ norm of the residuals. That is the reason for using the notation $\ell_1 \square \ell_2$ for our estimator.

Problem (4.1) can be reduced by isolating $b$ or $s$ from the linear constraint. This brings up the following two equivalent problems:

$$
\begin{align*}
\text{minimize} & \quad (g, b) \in \mathbb{R}^p \times \mathbb{R}^n \\
\psi(g, b) := & \sigma \|y - Xg - b\|_1 + \frac{1}{2} \|b\|_2^2.
\end{align*}
\tag{4.2}
$$

and

$$
\begin{align*}
\text{minimize} & \quad (g, s) \in \mathbb{R}^p \times \mathbb{R}^n \\
\phi(g, s) := & \sigma \|s\|_1 + \frac{1}{2} \|y - Xg - s\|_2^2.
\end{align*}
\tag{4.3}
$$

Problems (4.2) and (4.3) are equivalent to Problem (4.1). The existence of solutions is ensured by the full rank condition on $X$, and the coercivity and continuity of the objective functions.

Problem (4.2) is more advantageous for analysis of theoretical properties of solutions, while Problem (4.3) is better adapted to be numerically solved. For these reasons we state and proof here the optimality conditions of Problem (4.2) and postpone the analysis of Problem (4.3) for Section 5.

**Lemma 4.3.** The following hold.

(i) $(\hat{g}, \hat{b})$ is a solution to (4.2) if and only if

$$
(\forall i \in \{1, \ldots, n\}) \quad \hat{b}_i = \begin{cases} 
\sigma, & \text{if } y_i - x_i^\top \hat{g} > \sigma; \\
y_i - x_i^\top \hat{g}, & \text{if } y_i - x_i^\top \hat{g} \in [-\sigma, \sigma]; \\
-\sigma, & \text{if } y_i - x_i^\top \hat{g} < -\sigma.
\end{cases}
\tag{4.4}
$$

In particular $\|\hat{b}\|_\infty \leq \sigma$.

(ii) The dual of (4.2) is

$$
\gamma := \max_{u \in \sigma \mathcal{P}_*} u^\top y - \frac{1}{2} \|u\|_2^2,
\tag{4.5}
$$

where $\mathcal{P}_* = \{u \in \ker X^\top \mid \|u\|_\infty \leq 1\}$ and $\min_{(g, b) \in \mathbb{R}^p \times \mathbb{R}^n} \psi(g, b) = \gamma$.

**Proof.** Note that $\psi(g, b)$ can be equivalently written as

$$
\psi(g, b) = \sigma \|y - [X \ I_n] \left( \begin{array}{c} g \\ b \end{array} \right)\|_1 + \frac{1}{2} \|[0_p \ I_n] \left( \begin{array}{c} g \\ b \end{array} \right)\|_2^2.
\tag{4.6}
$$

where $I_n$ denotes the identity matrix of size $n \times n$ and $0_p$ the zero matrix of size $p \times p$.

Since (4.2) is convex, a necessary and sufficient conditions for a solution $(\hat{g}, \hat{b})$ to Problem (4.2) is

$$
0 \in \partial \psi(\hat{g}, \hat{b}).
\tag{4.7}
$$

Hence, by using [13, Theorem 4.2.1] in (4.6) (qualification conditions are trivially satisfied), (4.7) is equivalent to

$$
\left( \begin{array}{c} 0 \\ 0 \end{array} \right) \in - \left[ \begin{array}{c} X^\top \\ I_n \end{array} \right] \partial \sigma \| \cdot \|_1 (y - X\hat{g} - \hat{b}) + \left( \begin{array}{c} 0 \\ 0 \end{array} \right).
$$

Therefore, there exists $u \in \partial \sigma \| \cdot \|_1 (y - X\hat{g} - \hat{b})$ such that

$$
\begin{align*}
X^\top u &= 0, \\
\hat{b} &= u.
\end{align*}
$$
or, equivalently,
\[
\begin{aligned}
  \hat{b} &\in \partial \sigma \cdot \|1\|_1(y - X\hat{g} - \hat{b}), \\
  X^T \hat{b} &= 0.
\end{aligned}
\]

Hence
\[
y - X\hat{g} - \hat{b} = \text{prox}_{\sigma\|\cdot\|_1}(y - X\hat{g}),
\]
and the result follows from Lemma 1.1(ii). (ii): Problem (4.2) is equivalent to (4.1) and, applying Lagrangian duality, the dual is
\[
\max_{u \in \mathbb{R}^p} \min_{(g, b, s) \in \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n} \sigma\|s\|_1 + \frac{1}{2}\|b\|_2^2 + u^T (y - Xg - b - s)
\]
or, equivalently,
\[
\max_{u \in \mathbb{R}^p} \left( u^T y + \left( \min_{b \in \mathbb{R}^n} \frac{1}{2}\|b\|_2^2 - u^T b \right) + \left( \min_{s \in \mathbb{R}^n} \sigma\|s\|_1 - u^T s \right) - \max_{g \in \mathbb{R}^p} g^T (X^T u) \right). \tag{4.8}
\]

The optimality conditions associated to the convex optimization problem
\[
\min_{b \in \mathbb{R}^n} \frac{1}{2}\|b\|_2^2 - u^T b
\]
yields \(b = u\), hence \(\min_{b \in \mathbb{R}^n} \frac{1}{2}\|b\|_2^2 - u^T b = -\frac{1}{2}\|u\|_2^2\). The second minimization problem can be written as
\[
\min_{s \in \mathbb{R}^n} \sigma\|s\|_1 - u^T s = \sum_{i=1}^n \min_{s_i \in \mathbb{R}} \sigma|s_i| - u_i s_i = \begin{cases} -\infty, & \text{if } \|u\|_\infty > \sigma; \\ 0, & \text{if } \|u\|_\infty \leq \sigma. \end{cases}
\]

Finally, we have
\[
\max_{g \in \mathbb{R}^p} g^T (X^T u) = \begin{cases} +\infty, & \text{if } u \notin \ker X^\top; \\ 0, & \text{if } u \in \ker X^\top. \end{cases}
\]

Altogether, it follows from (4.8) that the dual to (4.2) is given by (4.5) and the absence of duality gap follows from the Slater qualification condition and the existence of multipliers [13, section 4]. □

Let us show that, in the absence of sparse errors (\(e = 0\)), the solution to Problem 4.1 actually coincides with the LSE.

**Proposition 4.4.** Let \(y = Xf + z\) and consider the unique decomposition of \(z\) as \(z = X\bar{\sigma} + \bar{b}\), where \(\bar{\sigma} \in \mathbb{R}^p\) and \(\bar{b} \in \text{Ker} X^\top\). If \(\|\bar{b}\|_\infty < \sigma\), then \((f_n, \bar{b})\) is the unique solution to (4.2).

**Proof.** Let us first prove that \((f_n, \bar{b})\) is a solution. By definition \(X^\top \bar{b} = 0\), therefore it is enough to prove that \(\bar{b} \in \partial \sigma \cdot \|1\|_1(y - Xf_n - \bar{b})\).

Since \(y = X(f + \bar{g}) + \bar{b} = Xf_n + \bar{b}\), then \(y - Xf_n - \bar{b} = 0\) and \(\sigma \partial \cdot \|1\|_1(y - Xf_n - \bar{b}) = [-\sigma, \sigma]^n\). By the hypothesis on \(\bar{b}\) we conclude that \(\bar{b} \in \partial \sigma \cdot \|1\|_1(y - Xf_n - \bar{b})\). Now let us prove that \((f_n, \bar{b})\) is the unique solution. Let \(\varphi : \mathbb{R}^p \to \mathbb{R}\) be the continuous function defined by \(\varphi(g) = \|y - X\hat{g}\|_\infty\). By hypothesis, \(\varphi(f_n) = \|\bar{b}\|_\infty < \sigma\) and the
continuity of \( \varphi \) yields the existence of a neighbourhood \( V \) of \( f_n \) such that \( \varphi(g) < \sigma \) for every \( g \in V \). Now let \((g, b)\) be a pair in \( V \times \mathbb{R}^n \) satisfying (4.4). Then, since (4.4) yields \( y = Xg + b = Xf_n + \hat{b} \), it follows from (4.2) that

\[
\psi(g, b) - \psi(f_n, \hat{b}) = \sigma(\|y - Xg - b\|_1 - \|y - Xf_n - \hat{b}\|_1) + \frac{1}{2}\|\hat{b}\|_2^2 - \frac{1}{2}\|\hat{b}\|_2^2
\]

\[
= \frac{1}{2}\|y - Xg\|_2^2 - \frac{1}{2}\|\hat{b}\|_2^2
\]

\[
= \frac{1}{2}\|X(f_n - g)\|_2^2 + \hat{b}^T X(f_n - g)
\]

\[
= \frac{1}{2}\|X(f_n - g)\|_2^2 \geq 0.
\]

Since this inequality is valid for any candidate to solution close enough to \((f_n, \hat{b})\), the uniqueness follows from the convexity of \( \psi \) and the full rank of \( X \). \( \square \)

Proposition 4.4 provides an interpretation of \( \sigma \) as a threshold of the significance of outliers. Indeed, the part of the residuals that is below \( \sigma \) is considered as noise, and the rest as outlier. If most of the outliers are comparable to \( \sigma \) in magnitude, they can be perceived as noise, and the \( \ell_1 \)-\ell_2 estimator is close to the LSE. Moreover, as \( \sigma \) goes to 0, our estimator tends to \( f_1 \).

We pursue the study of our estimator by showing that the additional term \( b \), which makes the difference between our estimator and the \( \ell_1 \) estimator, improves its error bounds. The numerical simulations performed in Section 6 confirm that the additional term actually plays an important role reducing the bias induced by noise.

**Theorem 4.5.** Let \( y = Xf + z + e \), let \( M = \text{supp}(e) \), and suppose that \(|M| = k \leq m(X)\). Consider the unique decomposition of \( z \) as \( z = X\bar{\varphi} + \bar{b} \), where \( \bar{\varphi} \in \mathbb{R}^p \) and \( \bar{b} \in \text{Ker}X^\top \). Then any solution \((\hat{g}, \hat{b})\) to (4.2) satisfies

\[
\|X(\hat{g} - f_n)\|_1 \leq \frac{1}{2c_k - 1}\left(\|\bar{b} - \hat{b}\|_{1,N \setminus M} + \frac{\|\bar{b} - \hat{b}\|_2}{\|\hat{b}\|_{\infty,N \setminus M}}\right),
\]

where \( f_n = f + \bar{f} \) is the LSE on \( y_n = Xf + z \).

**Proof.** From Lemma 3.8[1] and (4.2) we deduce

\[
\psi(\hat{g}, \hat{b}) - \psi(f_n, \hat{b}) \geq \sigma(2c_k - 1)\|X(\hat{g} - f_n)\|_1 - 2\sigma\|y - Xf_n - \hat{b}\|_{1,N \setminus M}.
\]

Hence, it follows from \( f_n = f + \bar{g} \) that, for every \( i \in \{1, \ldots, n\} \), \( y_i - x_i^\top f_n = e_i + \bar{b}_i \) and, thus, \( \psi(f_n, \hat{b}) = \sigma(e_i + \bar{b}_i) + \|\bar{b}_i\|_2^2 / 2 \). Therefore, since \( e_i = 0 \) for any \( i \in N \setminus M \),

\[
\sigma(2c_k - 1)\|X(\hat{g} - f_n)\|_1 \leq 2\sigma\|\bar{b} - \hat{b}\|_{1,N \setminus M} - \sigma\|e + \bar{b} - \hat{b}\|_1 + \psi(\hat{g}, \hat{b}) - \frac{1}{2}\|\hat{b}\|_2^2. \quad (4.9)
\]

From Lemma 4.3[1] the dual problem to (4.2) is

\[
\max_{u \in \sigma F^*} u^\top (e + \hat{b}) - \frac{1}{2}\|u\|_2^2
\]

and \( \psi(\hat{g}, \hat{b}) = \max_{u \in \sigma F^*} u^\top (e + \hat{b}) - \frac{1}{2}\|u\|_2^2 \). Therefore

\[
\psi(\hat{g}, \hat{b}) - \frac{1}{2}\|\hat{b}\|_2^2 = \max_{u \in \sigma F^*} u^\top (e + \hat{b}) - \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|\hat{b}\|_2^2
\]

\[
= \max_{u \in \sigma F^*} u^\top (e + \hat{b} - \hat{b}) - \frac{1}{2}\|u - \hat{b}\|_2^2
\]

\[
\leq \max_{u \in \sigma F^*} u^\top (e + \hat{b} - \hat{b}).
\]
Hence, it follows from Lemma 7.1 that
\[ \psi(\hat{g}, \hat{b}) - \frac{1}{2}\|\hat{b}\|_2^2 \leq \sigma\|e + \bar{b} - \hat{b}\|_{1,M} + \frac{\sigma}{\|\bar{b} - \hat{b}\|_{\infty,N\setminus M}}\|\bar{b} - \hat{b}\|_{2,N\setminus M}^2, \]
which, combined with (4.9), yields
\[ (2c_k - 1)\|X(\hat{g} - f_n)\|_1 \leq 2\|\bar{b} - \hat{b}\|_{1,N\setminus M} - \|e + \bar{b} - \hat{b}\|_1 + \|e + \bar{b} - \hat{b}\|_{1,M} \]
\[ + \frac{1}{\|\bar{b} - \hat{b}\|_{\infty,N\setminus M}}\|\bar{b} - \hat{b}\|_{2,N\setminus M}^2 = \|\bar{b} - \hat{b}\|_{1,N\setminus M} + \frac{1}{\|\bar{b} - \hat{b}\|_{\infty,N\setminus M}}\|\bar{b} - \hat{b}\|_{2,N\setminus M}^2.\]
as claimed.

Note that the bound in Theorem 4.5 depends only on the data of the problem and, in particular, it does not depend explicitly on \( \sigma \). The only dependency is through \( \hat{b} \), which is bounded by \( \sigma \).

The following result gives a connection between the RBP of the \( \ell_1 \square \ell_2 \) estimator and that of the \( \ell_1 \) estimator. We recall that the regression breakdown point of an estimator is the maximum number of components of the data \( y \) that may diverge while keeping the estimator bounded.

**Corollary 4.6.** The RBP of the \( \ell_1 \square \ell_2 \) estimator is at least \( m(X) \).

**Proof.** Let us prove that any minimizer \( \hat{g} \) of the function \( \psi \) defined in (4.2) is bounded, no matter how large the sparse term \( e \) is. From Theorem 4.5 and Lemma 4.3(i) if \( k = \|e\|_0 \leq m(X) \) we obtain the estimate
\[ \|X(\hat{g} - f_n)\|_1 \leq \frac{n\sigma}{c_k - 1/2} + \|\bar{b}\|_1 < +\infty, \]
and the result follows.

### 5. Algorithm

In this section we propose and study an algorithm for computing the estimator introduced in the previous section, which is an application of the forward-backward splitting method [5, 6]. Note that problem (4.3) can be written equivalently as
\[ \min_{s \in \mathbb{R}^n} \left( \sigma\|s\|_1 + \min_{g \in \mathbb{R}^p} \frac{1}{2}\|y - Xg - s\|_2^2 \right). \]  

(5.1)
The first-order optimality condition of the inner problem in \( g \) yields \( X^\top(Xg + s - y) = 0 \) or equivalently
\[ g = (X^\top X)^{-1}X^\top(y - s). \]

(5.2)
Hence, from (5.1) we obtain
\[ \min_{s \in \mathbb{R}^n} \sigma\|s\|_1 + \frac{1}{2}\|(I - H)(y - s)\|_2^2, \]
where \( H = X(X^\top X)^{-1}X^\top \) is the hat matrix. Moreover, since the objective function in (5.3) is the sum of a general convex function and a differentiable convex function...
with Lipschitz gradient, the solutions of (5.3) are characterized [6, Proposition 3.1] by

\[(\forall \gamma > 0) \quad s = P_{\gamma\sigma}(s - \gamma(I - H)(y - s)), \quad (5.4)\]

where \(P_{\gamma\sigma} := \text{prox}_{\gamma\sigma\|\cdot\|_1}\) is the proximal operator associated to the function \(\gamma\sigma\|\cdot\|_1\), defined in (1.9). By using Lemma 1.1 we obtain that for every \(\gamma > 0\),

\[P_{\gamma} : \mathbb{R}^n \to \mathbb{R}^n\]

\[(\xi_1, \ldots, \xi_n) \mapsto (\text{sign}(\xi_1) \max\{\gamma - |\xi_1|, 0\}, \ldots, \text{sign}(\xi_n) \max\{\gamma - |\xi_n|, 0\}). \quad (5.5)\]

Combining the fixed-point characterization (5.4) with the expression for the proximal mapping (5.5) and adding relaxation steps \((\lambda_k)_{k \in \mathbb{N}}\), we obtain Algorithm 1.

**Algorithm 1** The forward-backward algorithm for solving (5.3).

Choose \(s_0 \in \mathbb{R}^n\) and set \(k := 0\). Iterate:

1. Choose \(0 < \lambda_k \leq 1\) and \(0 < \gamma_k < 2/\|I - H\|\).
2. Let

\[s_{k+1} = s_k + \lambda_k (P_{\sigma\gamma_k}(s_k - \gamma_k(I - H)(s_k - y)) - s_k).\]

3. If a stopping criterion is satisfied, stop. Otherwise set \(k := k + 1\) and go to step 1.

The convergence properties of Algorithm 1 are stated in the following Theorem.

**Theorem 5.1.** Let \((\gamma_k)_{k \in \mathbb{N}}\) be a sequence in \([0, +\infty[\) such that \(0 < \inf_{k \in \mathbb{N}} \gamma_k \leq \sup_{k \in \mathbb{N}} \gamma_k < 2/\|I - H\|\) and let \((\lambda_k)_{k \in \mathbb{N}}\) be a sequence in \([0, 1]\) such that \(\inf_{k \in \mathbb{N}} \lambda_k > 0\). Let \(s_0 \in \mathbb{R}^n\) be arbitrary, and let \((s_k)_{k \in \mathbb{N}}\) be the sequence of iterates obtained from Algorithm 1. Then the following hold.

(i) \((s_k)_{k \in \mathbb{N}}\) converges to a solution \(\hat{s}\) to (5.3).

(ii) \(\sum_{k \in \mathbb{N}} \|I - H\|(s_k - \hat{s})\|^2_2 < +\infty\).

Proof. Note that \(\|\cdot\|_1\) is a convex continuous function and \(\frac{1}{2}\|I - H\|(y - \cdot)\|_2^2\) is a convex differentiable function with gradient

\[\nabla \frac{1}{2}\|I - H\|(y - \cdot)\|_2^2 = (I - H)(\cdot - y), \quad (5.6)\]

where the last equality follows from the projector property of \(H\), \((I - H)^\top(I - H) = (I - H)^2 = (I - H)\). We deduce from (5.6) that \(\nabla \frac{1}{2}\|I - H\|(y - \cdot)\|_2^2 = (I - H)(\cdot - y)\) is \(L\)-Lipschitz continuous with \(L = \|I - H\|\). Moreover it follows from (5.5) that, for every \(\gamma \in [0, +\infty[\), \(\text{prox}_{\gamma\|\cdot\|_1} = P_{\gamma}\). Altogether the results follow from [6, Theorem 3.4].

**Theorem 5.1** asserts that Algorithm 1 approximates a solution \(\hat{s}\) to (5.3). Since (5.3) is equivalent to (4.1), it follows from (5.2) that the \(\ell_1\ell_2\) estimator can be computed as

\[\hat{g} = (X^\top X)^{-1}X^\top(y - \hat{s}).\]
6. Numerical Experiments. As announced in Section 4, numerical experiments confirm that the new estimator have lower bias when compared to \( \ell_1 \) or LSE estimation. In this section we describe the experimental setup and present numerical results.

The matrix \( X \) is generated randomly with independent entries drawn from a standard normal distribution. Its size is \( n \times p = 512 \times 128 \). The vector of data is generated according to

\[
y = Xf + z + e,
\]

with \( f = 0 \) and \( z \) standard normal, for different types and levels of contamination.

We estimate \( f \) by three different methods: LSE, \( \ell_1 \), and \( \ell_1 \sqcup \ell_2 \), with \( \sigma = \sqrt{\chi^2_{1}(0.95)} \). The size of the support of \( e \) ranges from 1 to \( (n - p - 1)/2 \), which means that the maximum fraction of contamination is close to 40%. We consider three types of sparse contamination. In the first and second types, each non-zero component of \( e \) is drawn i.i.d. from a Normal (light-tailed) and Laplace (heavy-tailed) distribution with mean 0 and standard deviation 5, respectively. The last type of sparse error is considered to be very large and adversarial. For generating the adversarial contamination we first create the vector \( \tilde{e} = X\mathbb{1}_p \), where \( \mathbb{1}_p \) is the vector of ones of size \( p \times 1 \). Then the sparse errors are obtained by selecting some components of \( \tilde{e} \) randomly and by multiplying them by 50.

For each type of contamination, for every \( k \in \{1, \ldots, (n - p - 1)/2\} \), we repeat 1000 times the following:

1) Choose randomly a subset \( M \) of \( N \) of size \( k \).
2) Construct the sparse vector \( e \) by filling the entries indexed by \( M \) with the corresponding type of large errors.
3) Generate \( z \) with independent \( N(0, 1) \) entries.
4) Set \( y = z + e \) and estimate \( f = 0 \) by LSE, \( \ell_1 \), and \( \ell_1 \sqcup \ell_2 \) methods.

For computing the \( \ell_1 \sqcup \ell_2 \) estimator the algorithm described in Section 5 is used. The code is available at \texttt{http://www.dim.uchile.cl/~sflores}. The \( \ell_1 \) estimator is computed by solving an equivalent linear program using the GNU solver \texttt{glpk}.

In Figure 6.1 the bias for data with gaussian noise and sparse contamination is plotted. For each percentage of outliers the bias is quantified by the mean of the quotients \( \|\hat{f} - f_n\|_2/\|f_n\|_2 \), where \( \hat{f} \) is the estimation of \( f \) obtained by each of the three methods and \( f_n = (X^T X)^{-1}X^T z \). In the figure on the left the bias is plotted for different levels of contamination with light-tailed outliers. We perceive that LSE outperforms \( \ell_1 \sqcup \ell_2 \) estimator when the vector of outliers is very sparse (less than 5% of contamination) and, hence, the gaussian noise predominates. However, the \( \ell_1 \sqcup \ell_2 \) estimator has a lower bias in general. Notice that the difference of the bias between LSE and \( \ell_1 \) estimator decreases as the percentage of contamination raises. In the figure on the right the bias is plotted for different levels of contamination with heavy-tailed outliers. In this case we observe the much better performance of the \( \ell_1 \sqcup \ell_2 \) estimator with respect to LSE even for very low levels of sparse contamination. Notice that, in this case, the \( \ell_1 \) estimator outperforms LSE for almost any percentage of contamination. In Figure 6.2 we plot the bias under gaussian noise and very large adversarial sparse errors. On the left, we observe that the \( \ell_1 \sqcup \ell_2 \) estimator outperforms dramatically LSE for any level of contamination and, on the right, we focus on the low contamination zone for perceiving the better performance of the \( \ell_1 \sqcup \ell_2 \) estimator with respect to the \( \ell_1 \) estimator. In addition, we appreciate a clear breakdown phenomenon when the level of contamination exceeds the 30% approximately.
Fig. 6.1. Relative error $\|\hat{f} - f_n\|/\|f_n\|$ for different percentage of outliers with gaussian noise. On the left, the contamination is drawn from a $N(0,5)$ distribution and on the right from a Laplace $(0,5)$ distribution.

Fig. 6.2. Plot of relative error $\|\hat{f} - f_n\|/\|f_n\|$, with $z$ standard gaussian, for different fractions of gross errors; at left, with adversarial contamination in the order of 50; at right a closeup comparing $\ell_1 \square \ell_2$ and $\ell_1$ on the zone of low contamination.

In summary, we perceive the high sensitivity of LSE with respect to the percentage of outliers and, in special, with respect to heavy-tailed and adversarial ones. In every examined case we confirm the better performance of the $\ell_1 \square \ell_2$ estimator with respect to the $\ell_1$ estimator, as expected in view of Theorem 4.5.
7. Conclusions. We have studied in deep the connections between robust regression and sparse reconstruction. This link between apparently unrelated areas is of great interest for specialists as it permits to feed from each other of results, techniques as well as of new problems and questions. The results presented in this article are quantitative, in contrast with the qualitative (bounded/unbounded) character of the results prevailing in robust statistics. We provide necessary and sufficient conditions, deterministic and for general data, in contrast with previous works based on restricted isometries. Our approach is simple and transparent, but powerful enough to treat the noisy case without modifications.

We have introduced a new estimator that combines the robustness of the ℓ₁ estimator with the nice properties of the LSE. Numerical experiments show that the ℓ₁□ℓ₂ estimator behaves like the ℓ₁ estimator concerning robustness to large outliers, but is less influenced by noise.

Appendix. Lemma 7.1. Let \( b \in \mathbb{R}^n \), \( e \in \mathbb{R}^n \) and let \( M = \text{supp}(e) \). Suppose that \( |M| \leq m(X) \) and \( \max_{i \in N \setminus M} |b_i| > 0 \). Let us define

\[
P^* = \{ d \in \ker X^\top \mid \|d\|_\infty \leq 1 \}.
\]

Then, for every \( \sigma > 0 \),

\[
\max_{d \in \sigma P^*} d^\top (e + b) \leq \sigma \|e + b\|_{1,M} + \frac{\sigma}{\|b\|_\infty, N \setminus M} \|b\|^2_{2,N \setminus M}.
\]

Proof. Let

\[
\tilde{b}_i = \begin{cases} 
0, & \text{if } i \in M; \\
\tilde{b}_i, & \text{otherwise},
\end{cases}
\quad
\tilde{e}_i = \begin{cases} 
\tilde{b}_i + e_i, & \text{if } i \in M; \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( \text{supp}(\tilde{e}) = M \), \( b + e = \tilde{b} + \tilde{e} \), \( \|b + e\|_1 = \|\tilde{b}\|_1 + \|\tilde{e}\|_1 \), and

\[
\max_{d \in \sigma P^*} d^\top (e + b) = \max_{d \in \sigma P^*} d^\top (\tilde{e} + \tilde{b}) \leq \max_{d \in \sigma P^*} d^\top \tilde{e} + \max_{d \in \sigma P^*} d^\top \tilde{b}.
\]

On one hand, it follows from Lemma [3.3.1] with \( y = \tilde{e}, g^* = 0 \), and \( b^* = 0 \) that, for every \( g \in \mathbb{R}^p \), \( \|\tilde{e}\|_1 \leq \|\tilde{e} - Xg\|_1 \), hence \( 0 \in \arg\min_{g \in \mathbb{R}^p} \|\tilde{e} - Xg\|_1 \) and from the first order optimality condition \( 0 \in X^\top \partial \|_1(\tilde{e}) \) or, equivalently,

\[
(\exists u \in P^*) \quad u^\top \tilde{e} = \|\tilde{e}\|_1.
\]

Since, for every \( u \in P^* \), \( u^\top e \leq \|e\|_1 \) we hence deduce that \( \max_{u \in P^*} u^\top \tilde{e} = \|\tilde{e}\|_1 \).

Therefore, by considering the change of variables \( u = d/\sigma \), we obtain

\[
\max_{d \in \sigma P^*} d^\top \tilde{e} = \sigma \cdot \max_{u \in P^*} u^\top \tilde{e} = \sigma \|\tilde{e}\|_1.
\]

On the other hand,

\[
\max_{d \in \sigma P^*} d^\top \tilde{b} \leq \max_{\|d\|_1 \leq \sigma} \|d\|_\infty \tilde{b} = \frac{\sigma}{\|b\|_\infty} \|\tilde{b}\|^2_2.
\]

Therefore, by replacing \( (7.4) \) and \( (7.5) \) in \( (7.3) \), the result follows from \( (7.2) \). □
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