Superfluid density of the ultra-cold Fermi gas in optical lattices

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Abstract
In this paper, we study the superfluid density of the two-component Fermi gas in optical lattices with population imbalance. Three different types of phases, the BCS state (Bardeen, Cooper and Schrieffer), the FFLO state (Fulde, Ferrel, Larkin and Ovchinnikov) and the Sarma state, are considered. We show that the FFLO superfluid density differs from the BCS/Sarma superfluid density in an important way. Although there are dynamical instabilities in the FFLO phase, when the interaction is strong or densities are high, the FFLO phase is found to be stable on the weak coupling limit.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The field of ultra-cold Fermi gases offers a great tool to study many different problems of correlated quantum systems. For example, in recent experiments [1–6] polarized Fermi gases were considered. These systems make it possible to study physics in the presence of mismatched Fermi surfaces, and non-BCS (Bardeen, Cooper and Schrieffer) type pairing such as that appearing in FFLO (Fulde, Ferrel, Larkin and Ovchinnikov) states [7, 8] or Sarma states [9, 10]. These possibilities have been considered extensively in condensed-matter, nuclear and high-energy physics [11].

It is also possible to study many different physical problems with close analogues in the field of solid state physics using optical lattices. However, unlike solid state systems, ultra-cold gases in optical lattices provide a very clean environment. These systems have very few imperfections and if one is interested in imperfections, they can be easily imposed on the system. Optical lattices are made with lasers; thus, the lattice geometry is easy to modify [12–15] by changing the properties of the intersecting laser beams. For these reasons, one can study various quantum many-body problems, such as Mott insulators, phase coherence and superfluidity, in optical lattices. The possibility of a superfluid alkali atom Fermi gas in an optical lattice has been recently studied both theoretically [16–22] and experimentally [23].

The pairing predicted by the BCS theory does not always mean that the gas is a superfluid [24]. When a linear phase is imposed on the order parameter, the phase gradient corresponds to superfluid velocity, and a part of the gas, which has superfluid velocity, is called superfluid and the coefficient of the inertia of this moving part is called superfluid density [25–27]. However, if the gas is normal the order parameter vanishes and thus the phase shift does not affect the normal gas. If the superfluid density is positive in all directions, the gas is a superfluid. Negative superfluid density implies dynamical instability of the gas [28, 29].

There are a fair number of studies on the superfluid density of Fermi gas in the free space as well as in a trap [30–33] and few studies on the superfluid density in optical lattices [29, 34]. However, those papers have focused on population-balanced cases, i.e. cases where the densities of the components are the same. In this paper, we study the superfluid density of an ultra-cold two-component Fermi gas in optical lattices at finite temperatures and with finite polarization. We are motivated by the fact that by calculating the superfluid density, we can draw more conclusions on whether the gas is actually superfluid or not. Furthermore, we investigate if the superfluid density can be used as an indicator of different superfluid phases. The phases we consider are the BCS phase, i.e. the densities of the component are equal; the one-mode FFLO phase, i.e. the densities are not equal and the phase of the pairing gap modulates as a function of the position; and the
Sarma phase, i.e. the densities are not equal and the phase of the pairing gap is constant. We show that there are qualitative differences between the BCS/Sarma superfluid density and the one-mode FFLO superfluid density. We also study the stability of different phases. We also demonstrate that there can be dynamical instabilities in the one-mode FFLO phase.

This paper is organized as follows. In section 2, we discuss the physical system and present the Hamiltonian of the system. In subsection 2.1, the mean-field approximation and the ans{"a}tze we consider are presented. In section 3, we determine the superfluid density tensor. In section 4, we present the numerical results and end with some concluding remarks in section 5.

2. Lowest band fermions in an optical lattice

We consider a two-component Fermi gas, whose components are two different hyperfine states of the same isotope, and we call them $\uparrow$ state and $\downarrow$ state. The Hamiltonian of the system is given by

$$\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \int \mathrm{d} \mathbf{r} \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\sigma}(\mathbf{r}) - \mu_{\sigma} \right) \tilde{\Psi}_{\sigma}(\mathbf{r})$$

$$+ g \sum_{\sigma=\uparrow,\downarrow} \int \mathrm{d} \mathbf{r} \tilde{\tilde{\Psi}}_{\sigma}(\mathbf{r}) \tilde{\tilde{\Psi}}_{\sigma}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \tilde{\tilde{\Psi}}_{\sigma}(\mathbf{r}') \tilde{\tilde{\Psi}}_{\sigma}(\mathbf{r}) ,$$

where $\hbar = h/2\pi$, $h$ is Planck’s constant, $\Psi_{\sigma}(\mathbf{r})$ and $\tilde{\Psi}_{\sigma}(\mathbf{r})$ are the fermionic creation and annihilation field operators of the component $\sigma$ respectively, $\mu_{\sigma}$ is the chemical potential of the component $\sigma$ and $\delta(\mathbf{r} - \mathbf{r}')$ is Dirac’s $\delta$-function. The interaction strength is related to the $s$-wave scattering length $a_s$ through

$$g = \frac{4\pi \hbar^2 a_s}{m} .$$

The lattice potential has a cubic structure and is given by

$$V_{\sigma}(\mathbf{r}) = E_r \sum_{\alpha} s_{\alpha,\sigma} \sin^2(k x_{\alpha}) ,$$

where $E_r = \hbar^2 k^2 / 2m$ is the recoil energy ($k = (\pi / d)$ where $d$ is a lattice constant) and $s_{\alpha,\sigma}$ is the lattice depth in the $\alpha$-direction for component $\sigma$.

When we assume that only the lowest band is occupied, the field operators can be expanded by using the localized Wannier functions in the following way:

$$\tilde{\tilde{\Psi}}_{\sigma}(\mathbf{r}) = \sum_{i} w_{\sigma,i}(\mathbf{r}) \tilde{c}_{\sigma,i} ,$$

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where $w_{\sigma,i}(\mathbf{r})$ is a Wannier function, which is localized at a lattice point $i$, and $\tilde{c}_{\sigma,i}$ is an annihilation operator.

The lowest band Hubbard model is valid, when the lattice is deep enough. In other words, it is valid when the Wannier functions decay within a single lattice constant and when the effective interaction between atoms is much smaller than the bandgap between bands. These two conditions imply that one has to take into account only the onsite interactions. In this case overlap integrals of the kinetic energy operator between the next nearest neighbours are small compared to overlap integrals of the kinetic energy operator between the nearest neighbours [35], and consequently one needs to take into account only hopping between the nearest neighbours. Then the lowest band Hubbard Hamiltonian is given by

$$\hat{H} = - \sum_{\sigma=\uparrow,\downarrow} \left( J_{\sigma,x} \sum_{\langle i,j \rangle} + J_{\sigma,y} \sum_{\langle i,j \rangle} + J_{\sigma,z} \sum_{\langle i,j \rangle} \right) \tilde{c}_{\sigma,i} \tilde{c}_{\sigma,j}$$

$$- \sum_{i} \left( \mu_{\uparrow} \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} + \mu_{\downarrow} \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i} \right)$$

$$+ U \sum_{i} \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i} ,$$

where $(i, j)_{\alpha}$ means the sum over the nearest neighbours in the $\alpha$-direction, and the hopping strength is defined by

$$J_{\sigma,\alpha} = - \int \mathrm{d} \mathbf{r} w_{\sigma,i}(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\sigma}(\mathbf{r}) \right) w_{\sigma,i+\delta\alpha}(\mathbf{r}) .$$

The coupling strength of the above Hubbard model is given by

$$U = g \int \mathrm{d} \mathbf{r} \left| w_{\uparrow,i}(\mathbf{r}) \right|^2 \left| w_{\downarrow,i}(\mathbf{r}) \right|^2 .$$

2.1. Mean-field approximation

Because the interaction term $U \sum_{i} \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i}$ is hard to handle, we can approximate it by using mean-field approximation. Under this approximation, the interaction part becomes

$$\sum_{i} \Delta(i) \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} + \Delta^*(i) \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i} = \frac{|\Delta(i)|^2}{U} ,$$

where the pairing gap $\Delta(i) = U(\tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i})$. With respect to position dependence, we only consider the case in which only the phase of the gap $\Delta(i) = |\Delta| e^{i q \cdot \mathbf{R}}$ (where $\mathbf{R}$ is a lattice vector) can depend on the lattice site $i$. This ansatz is called one-mode FFLO [7, 8]. This ansatz includes also the BCS ansatz and the Sarma-state ansatz as special cases. In these phases, the momentum $\mathbf{q}$ is simply zero. Under this mean-field approximation, the Hamiltonian is given by

$$\hat{H}_0 = - \sum_{\sigma=\uparrow,\downarrow} \left( J_{\sigma,x} \sum_{\langle i,j \rangle} + J_{\sigma,y} \sum_{\langle i,j \rangle} + J_{\sigma,z} \sum_{\langle i,j \rangle} \right) \tilde{c}_{\sigma,i} \tilde{c}_{\sigma,j}$$

$$- \sum_{i} \left( \mu_{\uparrow} \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} + \mu_{\downarrow} \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i} \right)$$

$$+ \sum_{i} \frac{|\Delta|}{U} e^{i \mathbf{q} \cdot \mathbf{R}} \tilde{c}_{\uparrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\downarrow,i} \tilde{c}_{\uparrow,i} .$$

By minimizing the free energy $F_0 = \Omega_0 + \mu_{\uparrow} N_{\uparrow} + \mu_{\downarrow} N_{\downarrow}$ ($\Omega_0$ is the grand canonical potential of the mean-field Hamiltonian) with respect to $|\Delta|$ and $\mathbf{q}$, and solving the number equations $\partial F_0 / \partial \mu_{\sigma} = 0$ simultaneously, one finds the pairing gap, the chemical potentials and the momentum $\mathbf{q}$ as functions of the temperature and the particle numbers $N_{\sigma}$.
3. Superfluid density

Landau’s two-component model for a superfluid gas states that the superfluid gas consists of two components: the normal component and the superfluid component. In the free space, the superfluid density is the density of the superfluid component. The case is a little bit different in lattices; thus, the energy difference between the twisted system and the system without the twisting phase can depend on the direction of the phase shift. This is related to the fact that the effective masses can depend on the direction (the effective masses are related to the hopping strengths). Thus in the lattice, superfluid density is more like a tensor than a scalar.

In the free space, when the superfluid gas flows, the kinetic energy of the gas is given by

\[ E_k = \frac{1}{2} \int \text{d}r \bar{\rho}_s(r)v_s(r)^2, \]

where \( \bar{\rho}_s(r) \) is the superfluid density and \( v_s(r) \) is the superfluid velocity. The superfluid velocity is defined by

\[ v_s(r) = \frac{\hbar}{2m} \nabla \phi(r). \]

where \( \phi(r) \) is the phase of the order parameter.

To impose the linear phase variation on the order parameter, one can impose a linear phase variation \( \Theta \cdot \mathbf{R}_\alpha = (\Theta_x/(M, d), \Theta_y/(M, d), \Theta_z/(M, d)) \cdot \mathbf{R} \) [26, 27, 36] on the Hamiltonian. Here \( M_\alpha \) indicates the number of lattice sites in the \( \alpha \) direction, i.e. the total volume of the lattice \( V = M_xM_yM_zd^3 \). This variation corresponds to small \( (\Theta_\alpha \cdot d) \) (small) superfluid velocity, which is given by

\[ v_s = \frac{\hbar}{2m}(\Theta_x/(M, d), \Theta_y/(M, d), \Theta_z/(M, d)). \]

Thus this imposed phase gradient gives the system a kinetic energy, which corresponds to the free energy difference \( F_\Theta - F_0 \), where \( F_\Theta \) is the free energy within the phase variation and \( F_0 \) is the free energy without the phase variation. The superfluid fraction in this case can be determined as [26, 27]

\[ \rho_{\text{eff}} = \lim_{\Theta \to 0} \frac{1}{N} \int_{\Theta \cdot \mathbf{R} \cdot \mathbf{R}'} \frac{\partial^2 F_{\Theta_\alpha \cdot \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha'}}{\partial \Theta_{\alpha'} \partial \Theta_{\alpha'}} \bigg|_{\Theta = 0}, \]

where \( N \) is the total number of particles and \( J_k = (J_{k,x} + J_{k,y})/2 \). This \( J_k \) corresponds to the effective mass

\[ m_{\text{eff}} = \frac{\hbar^2}{2J_kd^2}. \]

When one takes the limit \( \Theta \to 0 \), the temperature and the number of particles (and of course the interaction strength) should be kept as constants.

Of course, one could define the superfluid fraction as

\[ \rho_{\text{eff}} = \lim_{\Theta \to 0} \frac{1}{N} \int \frac{F_\Theta - F_0}{\Theta_{\alpha} \cdot \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha'}, \]

where \( J_{\alpha' \cdot \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha'} = (J_{k,x} + J_{k,y})/2 \), but in this case, how to define the off-diagonal elements is a bit unclear. This definition is not directly connected to the energy difference between the twisted system and the untwisted system.

By imposing the phase variation on the order parameter, one finds

\[ \hat{H}_\Theta = \sum \left( J_{\sigma,x} \sum_{(i,j)} + J_{\sigma,y} \sum_{(i,j)} + J_{\sigma,z} \sum_{(i,j)} \right) \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,j} - \sum \left( \mu_1 \chi_{\sigma,i} \hat{\chi}_{\sigma,i} + \mu_2 \chi_{\sigma,i}^2 \right) + \sum_i |\Delta| e^{i(qx+\Theta_1x) R} \langle \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,i} | + |\Delta| e^{-i(qx+\Theta_1x) R} \hat{c}_{\sigma,i} \hat{c}_{\sigma,i}. \] (8)

One can make the unitary transformation [26, 27, 36], which corresponds to the local transformation \( \hat{c}_{\sigma,i} \to \hat{c}_{\sigma,i} e^{i\Theta_\alpha \cdot \mathbf{R}_\alpha} \), and the twisted Hamiltonian becomes

\[ \hat{H}_\Theta = \sum \left( \mu_1 \chi_{\sigma,i} \hat{\chi}_{\sigma,i} + \mu_2 \chi_{\sigma,i}^2 \right) + \sum_i |\Delta| e^{i\Theta_1 \cdot \mathbf{R}_\alpha} \langle \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,i} | + |\Delta| e^{-i\Theta_1 \cdot \mathbf{R}_\alpha} \hat{c}_{\sigma,i} \hat{c}_{\sigma,i} \right). \] (9)

The twist angles \( \Theta_\alpha \cdot \mathbf{R}_\alpha \) have to be sufficiently small to avoid effects other than the collective flow of the superfluid component. Since \( \Theta_\alpha \cdot d \) is small, we can expand up to second order

\[ e^{\pm i\Theta_\alpha \cdot \mathbf{R}_\alpha} \approx 1 \pm \frac{i}{M_{\alpha}} \Theta_\alpha \cdot \mathbf{R}_\alpha, \]

In this way, we can write the twisted Hamiltonian as

\[ \hat{H}_\Theta = \hat{H}_0 + \Delta H \approx \hat{H}_0 \]

\[ + \sum \left( \frac{\Theta_\alpha^2}{2M_{\alpha}} \sum_{\sigma} J_{\sigma,x} \left( \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,n+d\hat{x}} + \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{x}} \right) \right), \]

\[ + \sum \left( \frac{\Theta_\alpha^2}{2M_{\alpha}} \sum_{\sigma} J_{\sigma,y} \left( \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,n+d\hat{y}} + \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{y}} \right) \right), \]

\[ + \sum \left( \frac{\Theta_\alpha^2}{2M_{\alpha}} \sum_{\sigma} J_{\sigma,z} \left( \hat{c}_{\sigma,i}^\dagger \hat{c}_{\sigma,n+d\hat{z}} + \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{z}} \right) \right), \]

\[ - \frac{1}{M_{\alpha}} \sum_{\sigma} \sum_{\sigma} J_{\sigma,x} \left( \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{x}} \right), \]

\[ - \frac{1}{M_{\alpha}} \sum_{\sigma} \sum_{\sigma} J_{\sigma,y} \left( \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{y}} \right), \]

\[ - \frac{1}{M_{\alpha}} \sum_{\sigma} \sum_{\sigma} J_{\sigma,z} \left( \hat{c}_{\sigma,n}^\dagger \hat{c}_{\sigma,n-d\hat{z}} \right). \] (10)

In the momentum space, this formula becomes

\[ \hat{H}_\Theta \approx \hat{H}_0 + \sum_{\sigma} \left( \frac{\Theta_\alpha^2}{2M_{\alpha}} \sum_{\mathbf{k}} 2J_{\sigma,\mathbf{k}} \cos(k_d) \hat{c}_{\sigma,k}^\dagger \hat{c}_{\sigma,k} \right), \]

\[ + \frac{\Theta_\alpha}{M_{\alpha}} \sum_{\mathbf{k}} 2J_{\sigma,\mathbf{k}} \sin(k_d) \hat{c}_{\sigma,k}^\dagger \hat{c}_{\sigma,k} \]

\[ = \hat{H}_0 + \hat{T} + \hat{J}, \] (11)
where

\[
\hat{T} = \sum_{\sigma,\alpha} \left[ \frac{\Theta_{\alpha}}{2M_{\alpha}} \sum_{k} 2J_{\sigma,\alpha}(k_d) \hat{c}_{\sigma,k}^\dagger \hat{c}_{\sigma,k} \right] \tag{12}
\]

and

\[
\hat{J} = \sum_{\sigma,\alpha} \left[ \frac{\Theta_{\alpha}}{M_{\alpha}} \sum_{k} 2J_{\sigma,\alpha}(k_d) \hat{c}_{\sigma,k}^\dagger \hat{c}_{\sigma,k} \right]. \tag{13}
\]

These two terms are proportional to the current and number operator of the system, respectively. These two terms commute with the number operators; thus when one imposes the perturbation in the system, the number of particles is conserved. By using the same method as that in [30], it can be shown that in the mean-field level

\[
\lim_{[\Theta_1]=0} \frac{\partial^2 F(\Theta)}{\partial \Theta_\alpha \partial \Theta_{\alpha'}} = \lim_{[\Theta_1]=0} \left( \frac{\partial^2 \Omega(\Theta)}{\partial \Theta_\alpha \partial \Theta_{\alpha'}} \right)^\Delta,_{\beta},_{\beta'},_{\beta_1}. \tag{14}
\]

Therefore, we can use the grand canonical potential.

The grand potential of the system can be written with a perturbing phase gradient as a series [37]:

\[
\Omega_0 = \Omega_0 - \sum_{n=1}^{\infty} \frac{(-1)^n}{\beta n h^2} \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \int_0^{\beta h} d\tau'' \tau J_\tau (\tau) \tag{15}
\]

where \( \beta = 1/k_B T \), \( k_B \) is Boltzmann’s constant and \( \Omega_0 \) is the grand canonical potential in the absence of the perturbation. The symbol \( T_\tau \) orders the operators in such a way that \( \tau \) decreases from left to right. The brackets \( \{ \cdots \}_0 \) = \( \text{Tr} \{ \exp(-\beta(\hat{H}_0 - \mu_j \hat{N}_j - \mu_i \hat{N}_i)) \cdots \} \) mean the thermodynamic average evaluated in the equilibrium state of the unperturbed system at temperature \( T \). Because all the twisted angles \( \Theta_0 \) are small, we can safely ignore terms of order higher than \( \Theta_0^2 \). With this approximation, the grand potential becomes

\[
\Omega_\alpha \approx \Omega_0 - \frac{1}{\beta h} \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \tau J_\tau (\tau) J(\tau'). \tag{16}
\]

The first perturbation term is easy to calculate and it is given by

\[
\frac{1}{\beta h} \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \tau J_\tau (\tau) \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \tau J_\tau (\tau) J(\tau'). \tag{17}
\]

where \( N_{\sigma,k} \) is the number of particles of the \( \sigma \)-component in the momentum state \( k \). Using Wick’s theorem, the second perturbation term is given by

\[
-\frac{1}{2\beta h^2} \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \tau J_\tau (\tau) J(\tau'). \tag{18}
\]

These two terms are proportional to the current and number operator of the system, respectively. These two terms commute with the number operators; thus when one imposes the perturbation in the system, the number of particles is conserved. By using the same method as that in [30], it can be shown that in the mean-field level

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\]
\begin{align}
\begin{split}
+2J_{\sigma,A}J_{\sigma,A'} \sin((k_a - q_a/2)d) \sin((k_a - q_a/2)d) \\
\times \left[ 2|v_{k,q}|^4|v_{k,q}|^2 \frac{f(E_{+k,q}) + f(E_{-k,q}) - 1}{E_{+k,q} + E_{-k,q}} \\
- \beta |v_{k,q}|^4 f(E_{+k,q}) (1 - f(E_{+k,q})) \right. \\
- \beta |v_{k,q}|^4 f(E_{-k,q}) (1 - f(E_{-k,q})) \\
+ 4J_{\sigma,A}J_{\sigma,A'} \sin((k_a + q_a/2)d) \sin((k_a - q_a/2)d) \\
\times |v_{k,q}|^2|v_{k,q}|^2 \left( \frac{2(f(E_{+k,q}) - f(E_{-k,q}))}{E_{+k,q} - E_{-k,q}} - \frac{2(f(E_{+k,q}) - 1)}{2E_{+k,q}} \right. \\
\left. \left. - \frac{2(f(E_{-k,q}) - 1)}{2E_{-k,q}} \right) \\
\rho_{\alpha\alpha'} = 0,
\end{split}
\end{align}

where \( f(E) \) is the Fermi–Dirac distribution. In order to handle the limit \( \lim_{q \to k} \), we have implicitly assumed in \( (26) \) that the lattice is large, i.e. \( M_a \gg 1 \).

Now the twisted grand canonical potential can be written as

\begin{equation}
\Omega_\alpha \approx \Omega_0 + \sum_{\alpha' \alpha} \delta \Omega_{\alpha \alpha'} \frac{\Theta_{\alpha} \Theta_{\alpha'}}{M_\alpha M_{\alpha'}}.
\end{equation}

In all the cases we consider \( \delta \Omega_{\alpha \alpha'} = 0 \) when \( \alpha \neq \alpha' \). The off-diagonal terms can be non-zero only when the single particle dispersion curves the different directions. In our case where the directions are independent, the \( k \) sums in different directions are independent, and because \( \sin \) is an antisymmetric function these sums vanish. Therefore, the grand potential becomes

\begin{equation}
\Omega_\alpha \approx \Omega_0 + \sum_\alpha \delta \Omega_{\alpha \alpha} \frac{\Theta^2_{\alpha}}{M_\alpha}.
\end{equation}

We can now determine the components of the dimensionless superfluid fraction tensor as

\begin{equation}
\rho_{\alpha\alpha'} = \frac{\delta \Omega_{\alpha \alpha'}}{J_s (N_1 + N_1')}.
\end{equation}

As a formula, these components are given by

\begin{equation}
\rho_{\alpha\alpha'} = \frac{1}{N} \sum_{k,\sigma} J_{\sigma,A} \cos(k_a d) N_{\alpha,k} \\
+ \frac{1}{N} \sum_k \left[ J_{\sigma,A}^2 \sin^2((k_a + q_a/2)d) \\
\times \left( 2|v_{k,q}|^4|v_{k,q}|^2 \frac{f(E_{+k,q}) + f(E_{-k,q}) - 1}{E_{+k,q} + E_{-k,q}} \\
- \beta |v_{k,q}|^4 f(E_{+k,q}) (1 - f(E_{+k,q})) \right. \\
- \beta |v_{k,q}|^4 f(E_{-k,q}) (1 - f(E_{-k,q})) \\
+ 4J_{\sigma,A} J_{\sigma,A'} \sin((k_a + q_a/2)d) \sin((k_a - q_a/2)d) \\
\times |v_{k,q}|^2|v_{k,q}|^2 \left( \frac{2(f(E_{+k,q}) - f(E_{-k,q}))}{E_{+k,q} - E_{-k,q}} - \frac{2(f(E_{+k,q}) - 1)}{2E_{+k,q}} \right. \\
\left. \left. - \frac{2(f(E_{-k,q}) - 1)}{2E_{-k,q}} \right) \right].
\end{equation}

The off-diagonal part of the gap equation; in \( (39) \) the uniform case \( (\text{the external potential is zero}) \), without the lattice. In figure 1(b), we show the BCS superfluid fraction divided by \( |\Delta|^2 \) as a function of the temperature. From figure 1(a) one can see that the BCS superfluid fraction is 1 at zero temperature, and from figure 1(b) it can be seen that the superfluid fraction is almost proportional to \( |\Delta|^2 \), but the temperature dependence of the superfluid fraction differs somewhat from the temperature dependence of \( |\Delta|^2 \). The standard BCS result is that the superfluid fraction is proportional to \( |\Delta|^2 \) in the limit \( T \to T_c \). [39] When we calculated the gap, we used the renormalization, i.e. we have cancelled out the divergent part of the gap equation; in [39] the cut-off energy is used. This is the reason for a small difference between the results.

4. Results

4.1. BCS/Sarma-phase results

In the continuum, the formula of the BCS superfluid fraction is the well-known Landau’s formula. Figure 1(a) shows the BCS superfluid fraction as a function of the temperature, in the uniform case (the external potential is zero), without the lattice. In figure 1(b), we show the BCS superfluid fraction divided by \( |\Delta|^2 \) as a function of the temperature. From figure 1(a) one can see that the BCS superfluid fraction is 1 at zero temperature, and from figure 1(b) it can be seen that the superfluid fraction is almost proportional to \( |\Delta|^2 \), but the temperature dependence of the superfluid fraction differs somewhat from the temperature dependence of \( |\Delta|^2 \). The standard BCS result is that the superfluid fraction is proportional to \( |\Delta|^2 \) in the limit \( T \to T_c \). [39] When we calculated the gap, we used the renormalization, i.e. we have cancelled out the divergent part of the gap equation; in [39] the cut-off energy is used. This is the reason for a small difference between the results.
Figure 1. (a) The BCS superfluid fraction as a function of the temperature in a free space (absence of the lattice). (b) The BCS superfluid fraction divided by $|\Delta|^2$ as a function of the temperature. The interaction strength $k_F a_s = -0.30$, where $k_F$ is the Fermi wave vector.

Figure 2. (a) The BCS superfluid fraction $\rho = \rho_{xx} = \rho_{yy} = \rho_{zz}$ as a function of the total filling fraction $n_\uparrow + n_\downarrow = 2n_\uparrow = 2n_\downarrow$ at zero temperature. (b) $\rho(n_\uparrow + n_\downarrow)$ as a function of the total filling fraction. (c) and (d) show $\Delta/J$ and $|\Delta|^2/J^2$, respectively as functions of the total filling fraction. All the hopping strengths are the same, i.e. $J = J_{1,\alpha} = J_{1,\alpha'}$, and $-U/J = 0.6$.

When all the hopping strengths are the same, i.e. $J = J_{1,\alpha} = J_{1,\alpha'}$, filling fractions $n_\sigma = N_\sigma/M_xM_yM_z$ are the same and the average filling fraction $n_\sigma = (n_\uparrow + n_\downarrow)/2$ goes to zero, then the superfluid fraction should be 1 at zero temperature. For low filling fractions the atoms occupy only the lowest momentum states and the single particle dispersions $\epsilon_{\sigma,k} = \sum_\alpha 2J_{\sigma,\alpha}(1 - \cos(k_\alpha d)) - \mu_\sigma$ approaches to form $J(kd)^2 - \mu_\sigma$, which is the free space dispersion. Therefore, the results on the limit of low filling fractions become the free space result and we know from Landau’s formula that the BCS superfluid fraction in the free space is 1 at zero temperature (see figure 1(a)).

In figure 2(a), we show the BCS superfluid fraction as a function of the total filling fraction in the lattice. As we can see, the BCS superfluid fraction approaches 1 when the filling fractions become small. In the free space the superfluid fraction is 1 at zero temperature, but in the lattice this no longer holds. In figure 2(b) we show $\rho(n_\uparrow + n_\downarrow)$ as a function of the total filling fraction and as one can see from it, the BCS superfluid density, which is not divided by the total filling
fraction, is symmetric with respect to half-filling. The cause for this is the particle-hole symmetry of the lattice model. The particle-hole symmetry of the lattice model can be seen also from figures 2(c) and (d). In figures 2(c) and (d) we show $\Delta/J$ as a function of the temperature, with three different average filling fractions. The cause is symmetric with respect to half-filling. The cause is symmetric with respect to half-filling. The cause is symmetric with respect to half-filling.

Figure 3(a) shows the BCS superfluid fraction as a function of the temperature, with three different average filling fractions. When one compares this figure to figure 1(a), one can note that $\rho(n_{\uparrow} + n_{\downarrow})$ is not directly proportional to $\Delta$ or $\Delta^2$.

Figure 3(a) shows the BCS superfluid fraction as a function of the temperature, with three different average filling fractions. When one compares this figure to figure 1(a), one can note that $\rho(n_{\uparrow} + n_{\downarrow})$ is not directly proportional to $\Delta$ or $\Delta^2$.

Since it is experimentally easy to study anisotropic lattices and use them to explore dimensional crossovers, let us explore superfluid fractions in anisotropic lattices. In figure 4(a), we show the BCS superfluid fraction as a function of the temperature in the case where the hopping strengths are different in different directions. As one can see from the figure, the components of the superfluid fraction tensor are different in different directions. Furthermore, we see that the superfluid fraction is larger in the direction of large hopping strength. This is easy to understand, and is due to the fact that the free energy difference in the $\alpha$ direction is proportional to $J_{\alpha}$ as we see from equations (28) and (29). If the superfluid fraction was defined as in equation (7), then the components of the...
superfluid fraction would be equal in every direction. In other words, the effective masses are different in different directions. Figure 4(c) shows that in the case where the hopping strengths are different in different directions, $\rho_{\alpha\alpha}/|\Delta|^2$ is almost a constant as a function of the temperature.

Figure 4(b) shows the BCS superfluid fraction as a function of the temperature in the case where the hopping strengths are different for the components. This can be seen from figure 4(d), where the pairing gaps are shown as a function of the temperature in the cases where the hopping strengths are different for the components. One can also note that when $J_7/J_t \rightarrow \infty$, then $\rho \rightarrow 0$ (in this case, the critical temperature also goes to zero). When $J_t/J_7$ increases but $2U/(J_t + J_7)$ remains constant, the lattice becomes deeper and deeper for the $\downarrow$ component. Thus, the atoms of component $\downarrow$ are more localized and the atoms do not move easily.

Figure 5(a) shows the superfluid fraction as a function of polarization $P = (n_{\uparrow} - n_{\downarrow})/(n_{\uparrow} + n_{\downarrow})$, at a constant temperature $k_B T/J = 0.75$. When the polarization is larger than zero, the state is called the Sarma state. One sees that when the polarization increases, the superfluid fraction decreases. Figure 5(b) shows the superfluid fraction divided by $|\Delta|^2$ as a function of polarization at a constant temperature. When $P$ is about 0.35, the superfluid fraction divided by $|\Delta|^2$ drops to zero suddenly. This happens because the pairing gap vanishes at this polarization. From figure 5(b), it is seen that the superfluid fraction at a constant temperature is almost proportional to $|\Delta|^2$. Figure 5(c) shows the Sarma state superfluid fraction as a function of the temperature, at a constant polarization ($P = 0.10$). Figure 5(d) shows the Sarma state superfluid fraction divided by $|\Delta|^2 (\rho/|\Delta|^2)$. We note from figures 5(b) and (d) that $\rho/|\Delta|^2$ is almost a constant as a function of the temperature and polarization, i.e. $\rho = c(T, P) |\Delta|^2$, where $c(T, P)$ depends weakly on the temperature and the polarization.

4.2. Results for the FFLO phase

When the lattice is cubic and all the hopping strengths are equal, the one-mode FFLO superfluid fraction is symmetric like the BCS/Sarma superfluid fraction. This is due to the following fact: when one varies $q$, it turns out that the free energy is minimized and when $q$ lies alongside the axis, i.e. $q = q_0 \hat{a}_0$, the system does not favour any of these axes.

The BCS and the Sarma state superfluid fractions are almost directly proportional to $|\Delta|^2$, i.e. $\rho \sim |\Delta|^2$. This
Figure 5. The BP/Sarma phase superfluid fraction. (a) The superfluid fraction as a function of the polarization $P = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$. (b) The superfluid fraction divided by $|\Delta|^2$. (c) The superfluid fraction as a function of the temperature. (d) The superfluid fraction divided by $|\Delta|^2$. (a) and (b) $k_B T/J = 0.75$; (c) and (d) $P = 0.10$. In all the figures, all the hopping strengths are the same, i.e. $J_{\sigma,\alpha} = J_{\sigma',\alpha'} = J$, the average filling fraction $n_{av} = 0.50$, $U/J = -6.0$, and $q = 0$.

Figure 6. The FFLO superfluid fraction at zero temperature. (a) The FFLO superfluid fraction as a function of the total filling fraction, with three different interactions at zero temperature. (b) $\rho_{xx}(n_\uparrow + n_\downarrow)$ as a function of the total filling fraction at zero temperature, when $U/J = -6.0$. In both the figures, all the hopping strengths are the same, i.e. $J_{\sigma,\alpha} = J_{\sigma',\alpha'} = J$, $q = q_x \hat{x}$ and $P \approx 0.30$. The horizontal lines in the figures show value 0.

changes for the one-mode FFLO superfluid fraction as we will now demonstrate. In figure 6(a), we show the $\rho_{xx}$ component of the FFLO superfluid fraction as a function of the total filling fraction, with three different interactions $U/J = -4.0, -5.0, -6.0$, at zero temperature. Figure 6(b) shows $\rho_{xx}(n_\uparrow + n_\downarrow)$ as a function of the total filling fraction, when $U/J = -6.0$, at zero temperature. The data which are used in figures 6(a) and (b) have been calculated just above the Clogston limit [40]. The Clogston limit is the limit for the chemical potential difference below which one can find only the BCS-type solution when one minimizes the grand potential. The numerical value of this limit is roughly $\delta \mu \approx \sqrt{2} \Delta_0$, where $\Delta_0$ is the pairing gap at zero temperature when $\delta \mu = 0$. As one can see from figure 6(a) when $|U/J|$ increases the superfluid fraction decreases and eventually becomes negative. One can also see that as the total
Figure 7. (a) The Sarma and FFLO phase superfluid fractions as a function of the polarization at constant temperature. (b) The Sarma and FFLO phase superfluid fractions divided by $|\Delta|^2$. The parameters, which were used, are $k_B T/J \approx 0.23$, $U/J \approx -5.1$, and the average filling fraction is $n_{av} = (n_\uparrow + n_\downarrow)/2 = 0.2$. All the hopping strengths are equal. Below $P \approx 0.15$, $q = 0$ and the phase is the Sarma phase. Above $P \approx 0.15$, $q \neq 0$ and the phase is the one-mode FFLO phase. The sudden drop in (b) at $P \approx 0.5$ indicates that the system becomes normal for higher polarizations.

Figure 8. Phase diagrams with four different interaction strengths, when the average filling fraction $n_{av} = 0.2$. The interaction strengths are from top left to bottom right: $U/J \approx -3.7$, $U/J \approx -4.4$, $U/J \approx -5.1$ and $U/J \approx -6.3$. All the hopping strengths are equal. The colours denote the following: BCS/Sarma = blue (black), stable FFLO = yellow (light grey), unstable FFLO = red (dark grey), normal gas = white.

The superfluid density, i.e. $\rho_{xx}(n_\uparrow + n_\downarrow)$, is symmetric as a function of the total filling fraction, with the value $n_\uparrow + n_\downarrow = 1.0$; this is shown in figure 6(b). This is again due to the particle-hole symmetry of the lattice model.

Figure 7(a) shows the superfluid fraction as a function of polarization at a constant temperature ($k_B T/J \approx 0.23$), and figure 7(b) shows the superfluid fraction divided by $|\Delta|^2$ as a function of polarization. There is a second-order phase transition between the Sarma/BCS phase and the FFLO phase, when polarization $P \approx 0.15$; this can be clearly seen from the bend in the superfluid fraction. We also see from figure 7(b) that the FFLO superfluid fraction is not proportional to $|\Delta|^2$. When one writes $\rho = c(T, P)|\Delta|^2$, the polarization dependence of $c(T, P)$ is very different in the FFLO phase compared to that in the Sarma phase.

In figures 8 and 9 we present phase diagrams with four different interaction strengths, when the average filling fractions are $n_{av} = 0.2$ and $n_{av} = 0.5$, respectively. The figures show that there can be unstable regions in the FFLO
Figure 9. Phase diagrams with four different interaction strengths, when the average filling fraction $n_{av} = 0.5$. The interaction strengths are from top left to bottom right: $U/J \approx -3.7$, $U/J \approx -4.4$, $U/J \approx -5.1$ and $U/J \approx -6.3$. All the hopping strengths are equal. The colours denote the following: BCS/Sarma = blue(black), stable FFLO = yellow (light grey), unstable FFLO = red (dark grey), normal gas = white.

5. Conclusions

In this paper we have presented, at the mean-field level, the superfluid density of the two-component Fermi gas in an optical lattice. We have shown that the BCS superfluid density in optical lattices differs from the free space results. We have also shown that the one-mode FFLO superfluid density differs crucially from the BCS/Sarma superfluid density. In the BCS/Sarma phase the gas is always a stable superfluid, but in the FFLO phase dynamical instabilities can appear. However, the FFLO phase is stable, when $|U/J| < 4.0$, i.e. on the BCS limit. Even in the BCS phase, the superfluid density can be different in different directions depending on the lattice structure.

Although the one-mode FFLO phase is stable when $|U/J| < 4.0$, it is unlikely that one-mode FFLO will be the free energy minimum of the system; calculations made by using the Bogoliubov–de Gennes equations show that the spatially modulating gap might be more favourable [41–43]. Such states are also, quite likely, more stable than the one-mode FFLO phase.

The methods that we have used in this paper to calculate the superfluid density work, in principle, for many different systems. For example, one could use the methods used here to calculate the superfluid density in the system where another component of the two-component Fermi gas occupies the first excited band [46] or in the case where an optical lattice is in a trap. These methods can also be used to calculate the superfluid density of modulating gap states.

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