Global higher integrability for a doubly nonlinear parabolic system

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Abstract. In this paper we establish a higher integrability result up to the boundary of weak solutions to doubly nonlinear parabolic systems. We show that the spatial gradient of a weak solution with vanishing lateral boundary values is integrable to a larger power than the natural power $p$, where the statement holds for parameters in the subquadratic case $\max\{\frac{2n}{n+2}, 1\} < p \leq 2$.

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1. Introduction

This paper is concerned with the global higher integrability of weak solutions to the following Cauchy-Dirichlet problem with vanishing lateral boundary values:

$$\begin{cases} 
\partial_t(|u|^{p-2}u) - \text{div} A(x, t, u, Du) = \text{div}(|F|^{p-2}F) & \text{in } \Omega_T, \\
u = 0 & \text{on } \partial\Omega \times (0, T), \\
u = u_0 & \text{on } \Omega \times (0, T). 
\end{cases}$$

The prototype is the homogeneous doubly nonlinear equation

$$\partial_t(|u|^{p-2}u) - \text{div}(|Du|^{p-2}Du) = \text{div}(|F|^{p-2}F) \quad \text{in } \Omega_T,$$

which is a special case of the fully doubly nonlinear equation

$$\partial_t(|u|^{m-1}u) - \text{div}(|Du|^{p-2}Du) = \text{div}(|F|^{p-2}F) \quad \text{in } \Omega_T. \quad (1.2)$$

$\Omega_T := \Omega \times (0, T)$ is a space-time cylinder consisting of an open, bounded set $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ and $T > 0$. $\partial_t$ denotes the derivative with respect to time, while $D$ and div denote the spatial gradient and the spatial divergence, respectively.
The vector field \( A : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn} \) is a Carathéodory function, meaning it is measurable in \( \Omega_T \) for all \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}\) and continuous in \( \mathbb{R}^N \times \mathbb{R}^{Nn} \) for a.e. \((x, t) \in \Omega_T\). \( A \) is required to satisfy the growth and coercivity conditions

\[
A(x, t, u, \xi) \cdot \xi \geq L_1 |\xi|^p,
\]

\[
|A(x, t, u, \xi)| \leq L_2 |\xi|^{p-1}
\]

in \( \Omega_T \) for \( 0 < L_1 \leq L_2 \). The parabolic boundary of \( \Omega_T \) is denoted by \( \partial_{\text{par}} \Omega_T := \Omega \times \{0\} \cup \partial \Omega \times (0, T) \). In general, \( p \) can be a value in \((1, \infty)\). For the right hand side \( F \), one would naturally require that \( F \in L^p(\Omega_T) \), while for the initial boundary datum the condition \( u_0 \in W^{1,p}_0(\Omega) \) is appropriate. To obtain our higher integrability result, these conditions must be slightly stronger.

The fully doubly nonlinear equation (1.2) shows different behaviour depending on whether \( m < p - 1 \), called the slow diffusion case, or \( m \geq p - 1 \), which is termed the fast diffusion case. The case \( m = p - 1 \) thus represents the threshold between these cases.

The spatial gradient of a weak solution to the system (1.1) naturally admits the integrability condition \( Du \in L^p(\Omega_T) \). The aim of this paper is to show that there exists a constant \( \varepsilon > 0 \) such that \( Du \in L^p(1+\varepsilon)(\Omega_T) \). In particular, we want to show that the self-improving property of integrability holds up to the boundary. We cover the range of exponents given by \( \max\{\frac{2n}{n+2}, 1\} < p \leq 2 \). The lower bound is analogous to the higher integrability result for parabolic \( p \)-Laplace systems [22].

We continue with a historical overview. In [26], Elcrat & Meyers started by observing this self-improving property in the setting of the elliptic \( p \)-Laplace systems, based on Gehring [13]. We also refer to [16, Chapter 11, Theorem 1.2] and [20, Sect. 6.5]. The higher integrability has consequently been used to derive further regularity results, see for example [17,18]. Kilpeläinen and Koskela were able to show in [21] that for equations of \( p \)-Laplace type this self-improving property holds up to the boundary.

Giaquinta and Struwe were able to expand onto parabolic systems in [19]. In [22], Kinnunen and Lewis treated more general parabolic systems fulfilling a \( p \)-growth condition. They continued to successfully consider very weak solution in [23]. The used approach utilises intrinsic cylinders as introduced by DiBenedetto [9–11] to compensate for inhomogeneous behaviour of nonlinear parabolic equations. The higher integrability for the parabolic \( p \)-Laplacian was expanded onto the boundary by Parviainen in [28,29]. Higher orders are covered in [2,8] by Bögelein and Parviainen. To obtain the result up to the boundary, it has been shown already in [21] that the natural regularity condition for the domain is the uniform \( p \)-thickness of the complement of the domain. This is being reaffirmed in [2,8]. Adimurthi and Byun proved global higher integrability even for very weak solution of parabolic \( p \)-Laplace equations in [1].

For the porous medium equation, obtained by setting \( p = 1 \) in (1.2), Gianazza and Schwarzacher [14] used the technique of expansion of positivity to obtain a higher integrability result in the interior of the domain. To treat
systems, and thus also signed solutions, another approach was used in [5]. Higher integrability for porous medium type systems up to the boundary is shown in [27]. For the singular case indicated by the condition $m < 1$, we refer to [7,15]. The interior case for the higher integrability for doubly nonlinear systems with $p = m + 1$ in (1.2) is covered in [4].

The approach of using the intrinsic geometry introduced by DiBenedetto [9–11] has been used with various deviations for example in the previously mentioned articles related to higher integrability [5,14,22,27] and also in [30]. We continue in similar fashion in the present article.

The structure of this paper is as follows: In Sect. 2, we present the main theorem and a collection of auxiliary material. In Sect. 3, we prove energy estimates in both the lateral and the initial case, in addition to a gluing Lemma. The intrinsic geometry will be introduced in Sect. 4 and used to prove parabolic Sobolev-Poincaré type inequalities. In the following Sect. 5, this will be extended to reverse Hölder inequalities. Finally, in Sect. 6 we construct cylinders on which the previous work will be applied, culminating in the proof of the gradient estimate.

2. Preliminaries

2.1. Setting and main result

Throughout the paper, we will use the notation $dz = dx dt$. We start by defining the boundary term

$$b[u, v] := \frac{1}{p}|v|^p - \frac{1}{p}|u|^p - |u|^{p-2}u \cdot (v - u),$$

(2.1)

for $u, v \in \mathbb{R}^N$. For our main theorem we consider weak solutions, which are defined as follows:

**Definition 2.1.** Let $A$ fulfil the growth and coercivity conditions (1.3). A measurable function $u : \Omega_T \rightarrow \mathbb{R}^N$ with

$$u \in C^0([0,T]; L^p(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}_0(\Omega, \mathbb{R}^N))$$

is called a weak solution to the Cauchy-Dirichlet problem of the doubly nonlinear parabolic system (1.1) if and only if

$$\iint_{\Omega_T} [p^-2 u \cdot \partial_t \varphi - A(x,t,u,Du) \cdot D\varphi] \, dz = \iint_{\Omega_T} |F|^{p-2} F \cdot D\varphi \, dz$$

(2.2)

holds for all $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$ and

$$\frac{1}{h} \int_0^h \int_{\Omega} b[u, u_0] \, dx dt \rightarrow 0 \text{ as } h \downarrow 0.$$  (2.3)

The condition $u \in L^p(0,T; W^{1,p}_0(\Omega, \mathbb{R}^N))$ also contains the information $u \in W^{1,p}_0(\Omega, \mathbb{R}^n)$ for a.e. $t \in (0,T)$, treating the lateral boundary condition, while (2.3) is for the initial boundary. For a center point $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$,
a radius \( \varrho > 0 \) and a scaling parameter \( \mu > 0 \), we define the respective space-time cylinder by

\[
Q^{(\mu)}_\varrho(z_0) := B_\varrho(x_0) \times \Lambda^{(\mu)}_\varrho(t_0),
\]

where

\[
\Lambda^{(\mu)}_\varrho(t_0) := (t_0 - \mu^{p-2}\varrho^p, t_0 + \mu^{p-2}\varrho^p), \quad \Lambda^{(1)}_\varrho(t_0) := \Lambda^{(1)}_\varrho(t_0).
\]

Also, a cylinder which contains only parts with positive time is labelled as

\[
Q^{(\mu)}_{\varrho,+}(z_0) := B_\varrho(x_0) \times [\Lambda^{(\mu)}_\varrho(t_0) \cap (0, T)].
\]

Further, define \( Q_{r,s}(z_0) := B_r(x_0) \times (t_0 - s, t_0 + s) \). This notation will only be used in Sect. 3. For such cylinders and a function \( v \in L^1(Q^{(\mu)}_\varrho, \mathbb{R}^N) \), define the mean value of \( v \) on \( Q^{(\mu)}_\varrho \) as

\[
\langle v \rangle^{(\mu)}_\varrho := \frac{\\int^{t_0}_{t_0} v \, dz}{Q^{(\mu)}_\varrho} = \frac{1}{|Q^{(\mu)}_\varrho|} \int_{Q^{(\mu)}_\varrho} v \, dz
\]

and the respective mean value of \( v \) on a time slice \( t \in \mathbb{R} \) as

\[
\langle v \rangle_\varrho(t) := \frac{\int_{B_\varrho} v(\cdot, t) \, dx}{|B_\varrho|} = \frac{1}{|B_\varrho|} \int_{B_\varrho} v(\cdot, t) \, dx.
\]

The latter means exist for all times \( t \) if the function \( v \) is continuous with respect to time. Furthermore, we sometimes write \( u(t) := u(\cdot, t) \). The notation \( v^\alpha \) for the power of a vector with \( \alpha > 0 \) is defined by \( v^\alpha := |v|^{\alpha-1}v \) for \( v \in \mathbb{R}^n \setminus \{0\} \) and \( v^0 = 0 \) for \( v = 0 \).

We will require some form of boundary regularity, more specifically the following property concerning the variational \( p \)-capacity \( \text{cap}_p \), which will be inspected later on, see Sect. 2.3.

**Definition 2.2.** Let \( 1 < p < \infty \). A set \( E \subseteq \mathbb{R}^n \) is called uniformly \( p \)-thick, if there exist constants \( \nu, \varrho_0 > 0 \) such that

\[
\text{cap}_p \left( E \cap \overline{B}_\varrho(x_0), B_{2\varrho}(x_0) \right) \geq \nu \text{cap}_p \left( \overline{B}_\varrho(x_0), B_{2\varrho}(x_0) \right)
\]

for all \( x_0 \in E \) and for all \( 0 < \varrho < \varrho_0 \).

The main objective is to prove the following theorem:

**Theorem 2.3.** Let \( \max \{ \frac{2n}{n+2}, 1 \} < p \leq 2 \) and \( \Omega \subseteq \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus \Omega \) is uniformly \( p \)-thick with corresponding constants \( \nu, \varrho_0 \). Let \( A \) be a Carathéodory function fulfilling the growth and coercivity conditions (1.3). Assume that for an \( \varepsilon_2 > 0 \), \( F \in L^{p(1+\varepsilon_2)}(\Omega_T) \) and \( u_0 \in W^{1,p(1+\varepsilon_2)}(\Omega) \).

Then there exists \( \varepsilon_1 = \varepsilon_1(n, p, N, \nu, \varrho_0) > 0 \), such that for any weak solution \( u \) to the Cauchy-Dirichlet problem (1.1) there holds

\[
Du \in L^{p(1+\varepsilon_2)}(\Omega_T, \mathbb{R}^{Nn})
\]

for \( \varepsilon_0 = \min\{ \varepsilon_1, \varepsilon_2 \} \). Moreover, for any parabolic cylinder \( Q_{2R}(z_0) \subseteq \mathbb{R}^n \times (-T, T) \) with \( z_0 \in \Omega_T \cup \partial_{\text{par}} \Omega_T \) and any \( \varepsilon \in (0, \varepsilon_0] \) we have
Global higher integrability for a doubly nonlinear PDE

Lemma 2.6.

For any \( p \geq 1 \), there holds:

1. If \( v \in L^p(\Omega_T, \mathbb{R}^N) \), then \( \|v\|_{L^p(\Omega_T, \mathbb{R}^N)} \to 0 \) as \( h \to 0 \)

2. If \( v \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N)) \), then \( \|v\|_{L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))} \to 0 \) as \( h \to 0 \)

3. If \( v \in L^p(0, T; W^{1,p}_0(\Omega, \mathbb{R}^N)) \), then \( \|v\|_{L^p(0, T; W^{1,p}_0(\Omega, \mathbb{R}^N))} \to 0 \) as \( h \to 0 \)

4. If \( v \in L^p(0, T; L^p(\Omega, \mathbb{R}^N)) \), then \( \|v\|_{C([0, T]; L^p(\Omega, \mathbb{R}^N))} \to 0 \) as \( h \to 0 \)

5. The weak time derivative \( \partial_t[v]_h \) exists in \( \Omega_T \) and can be computed by

\[
\partial_t[v]_h = \frac{1}{h} (v - [v]_h).
\]

What follows now is a collection of useful vector inequalities, taken from [4, p.6]
Lemma 2.7. Let $\alpha > 0$ and $N \in \mathbb{N}$. There exists $c = c(\alpha) > 0$ such that for any $a, b \in \mathbb{R}^N$ such that

$$\frac{1}{c} |b^\alpha - a^\alpha| \leq (|a| + |b|)^{\alpha - 1}|b - a| \leq c|b^\alpha - a^\alpha|.$$  

Further, if $\alpha \geq 1$, there exists $c = c(\alpha) > 0$ such that for any $a, b \in \mathbb{R}^N$ such that

$$|b - a|^\alpha \leq c|b^\alpha - a^\alpha|.$$  

For the boundary term defined in (2.1), we require the following estimate [4, Lemma 3.4]:

Lemma 2.8. Let $p \geq 1$ and $N \in \mathbb{N}$. There exists $c = c(p) > 0$ such that for any $u, v \in \mathbb{R}^N$, there holds

$$\frac{1}{c} b[u, v] \leq |u|^p - |v|^p \leq c b[u, v].$$  

An additional fact we use is the quasi-minimality of the mean-value integral. For the proof we refer to [4, Lemma 3.5].

Lemma 2.9. Let $p \geq 1, \alpha \geq \frac{1}{p}$ and $k \in \mathbb{N}$. There exists $c = c(p) > 0$ such that for any $A \subseteq B \subseteq \mathbb{R}^k$ with $|A|, |B| < \infty$, any $u \in L^p(B, \mathbb{R}^N)$ and any $a \in \mathbb{R}^N$, there holds

$$\frac{1}{c} \int_B |u^\alpha - (u)_A^\alpha|^p \, dx \leq c \frac{|B|}{|A|} \int_B |u^\alpha - a^\alpha|^p \, dx.$$  

Moreover we take the following well known Iteration Lemma from [20, Lemma 6.1]. It will be essential in absorbing certain quantities.

Lemma 2.10. Let $0 < \vartheta < 1, A, C \geq 0$ and $\alpha > 0$. There exists $c = c(\alpha, \vartheta) > 0$ such that for any bounded non-negative function $\phi : [r, \varrho] \to [0, \infty)$ with $0 < r < \varrho$ that satisfies

$$\phi(t) \leq \vartheta \phi(s) + \frac{A}{(s-t)^\alpha} + C \quad \text{for all } r \leq t < s \leq \varrho,$$

there holds

$$\phi(r) \leq c \left[ \frac{A}{(\varrho - r)^\alpha} + C \right].$$  

2.3. $p$-capacity

Now, we present a selection of properties of the variational $p$-capacity. It plays a role in the boundary condition that is the uniform $p$-thickness in Definition 2.2. We state the definition of the capacity from [27]:

For $1 < p < \infty$ and a compact set $D \subset \mathbb{R}^n$, the $p$-capacity of $C \subset \mathbb{R}^n$ is defined by

$$\text{cap}_p(C, D) := \inf_f \int_D |Df|^p \, dx,$$

where the infimum is taken over all functions $f \in C_0^\infty(D)$ such that $f \geq 1$ in $C$. The $p$-capacity of an open set $D \subset E$ is defined as the supremum of
capacities of compact sets $C \subset D$. At last, the variational $p$-capacity of a set $D \subset \mathbb{R}^n$ is defined as the infimum of capacities of open sets $O \supset D$.

As seen for example in [12, Theorem 4.15(iv)], the capacity of a ball can be computed as

$$\text{cap}_p (B_2(x_0), B_{2\varepsilon}(x_0)) = c\varepsilon^{n-p},$$

with $c(n,p) > 0$. We continue by observing properties of the uniform $p$-thickness.

**Lemma 2.11.** Let $1 < p < \infty$ and $E \subseteq \mathbb{R}^n$ compact and uniformly $p$-thick. Then $E$ is also uniformly $\tilde{p}$-thick for all $\tilde{p} \geq p$.

The uniform thickness admits a self-improving property which was proven in [25] and is as follows:

**Theorem 2.12.** Let $1 < p \leq n$ and $E \subseteq \mathbb{R}^n$ uniformly $p$-thick. There exists $q = q(n,p,\nu) \in (1,p)$ such that $E$ is uniformly $q$-thick.

In the case $p > n$, every non-empty set is uniformly $p$-thick, see [28, p.340]. We conclude this section with the following Lemma, taken from [28, Lemma 3.8].

**Lemma 2.13.** Let $\Omega \subset \mathbb{R}^n$ be bounded and open such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick. For a point $y \in \Omega$ with $B_{4\varepsilon/3}(y) \setminus \Omega \neq \emptyset$, there exists $\nu = \nu(n,\nu,\varrho,0,p) > 0$ such that

$$\text{cap}_p (B_{2\varepsilon}(y) \setminus \Omega, B_{4\varepsilon}(y)) \geq \bar{\nu} \text{cap}_p (B_{2\varepsilon}(y), B_{4\varepsilon}(y)).$$

### 3. Energy estimate

Recall that in this section, the notation $Q_{r,s}(z_0) := B_r(x_0) \times (t_0 - s, t_0 + s)$ is used. We start with an energy estimate which is valid even outside of the spatial region $\Omega$. It thus takes care of the lateral case, but also of cylinders that intersect both the lateral and the initial boundary.

#### 3.1. Lateral boundary

**Lemma 3.1.** Let $u$ be a weak solution to the Cauchy-Dirichlet problem (1.1), let $0 < R < 1$ as well as $S > 0$ and let $A$ satisfy (1.3). There exists $c = c(n,p,L_1,L_2) > 0$, such that for any $z_0 = (x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$ and $Q_{r,s}(z_0) \subset Q_{R,S}(z_0) \subset \mathbb{R}^n \times \mathbb{R}$ with $r \in [R/2,R)$, $s \in [S/2,S)$, there holds

$$\sup_{t \in A_s(t_0) \cap (0,T)} \frac{1}{|B_r|} \int_{B_r(x_0) \cap \Omega} \frac{|u^\#(t) - u_0^\#|^2}{s} \, dx + \frac{1}{|Q_{r,s}|} \int_{Q_{r,s}(z_0) \cap \Omega_T} |Du|^p \, dz \leq \frac{c}{|Q_{R,S}|} \int_{Q_{R,S}(z_0) \cap \Omega_T} \left[ \frac{|u^\# - u_0^\#|^2}{s - s} + \frac{|u - u_0|^p}{(R-r)^p} + |F|^p \right] \, dz$$

$$+ \frac{c}{|B_R|} \int_{B_R \cap \Omega} |Du_0|^p \, dz.$$

**Proof.** We may assume, $Q_{r,s}(z_0) \cap \Omega_T \neq \emptyset$, since otherwise the estimate is obvious. The first step is to define suitable cutoff functions:
Let $\eta \in C^1_0(B_R(x_0), [0, 1])$ a cutoff function with $\eta = 1$ on $B_r(x_0)$ and $|D\eta| \leq \frac{2}{r^2}$.

Let $t_1 \in \Lambda_S(t_0) \cap (0, T)$ and $\varepsilon > 0$ so small such that $t_1 + \varepsilon < \min\{T, t_0 + S\}$. Define $\psi_\varepsilon \in W^{1,\infty}((0, t_0 + S), [0, 1])$ with

$$\psi_\varepsilon(t) := \begin{cases} 
0 & t \in (0, \varepsilon], \\
\frac{t-\varepsilon}{\varepsilon} & t \in (\varepsilon, 2\varepsilon], \\
1 & t \in (2\varepsilon, t_1], \\
1 - \frac{1}{\varepsilon}(t - t_1) & t \in (t_1, t_1 + \varepsilon), \\
0 & t \in [t_1 + \varepsilon, t_0 + S).
\end{cases}$$

Let $\zeta \in W^{1,\infty}(\Lambda_S(t_0), [0, 1])$ with

$$\zeta(t) := \begin{cases} 
t - t_0 + S & t \in (t_0 - S, t_0 - s), \\
S - s & t \in [t_0 - s, t_0 + s), \\
1 & t \in [t_0 + s, t_0 + S).
\end{cases}$$

The method consists of inserting $\varphi(x, t) := \eta^p(x)\zeta(t)\psi_\varepsilon(t)(u(x, t) - u_0(x))$ into the mollified system (2.4). Due to the boundary conditions of $u$, $\varphi$ does indeed vanish when approaching the boundary of the space-time cylinder $\Omega_T$. Thus $\varphi$ is an admissible test function.

**Estimating the parabolic part**

With $w^{p-1} := \|w^{p-1}\|_h$, the parabolic term in (2.4) reads

$$\iint_{\Omega_T} \partial_t [w^{p-1}]_h \cdot \varphi \, dz = \iint_{Q_{R,S} \cap \Omega_T} \eta^p \psi_\varepsilon \partial_t w^{p-1} \cdot [w - u_0 + (u - w)] \, dz.$$  

Lemma 2.6 implies $\partial_t w^{p-1} = \frac{1}{h}(w^{p-1} - w)$ and thus the second term can be estimated from below by zero. The remaining term can be rewritten due to $(\partial_t w^{p-1})w = \frac{p-1}{p}\partial_t |w|^p$, $\partial_t u_0 = 0$, $\partial_t |u_0|^p = 0$ and $\psi_\varepsilon(0) = 0 = \psi_\varepsilon(T)$, yielding the following estimate:

$$\iint_{\Omega_T} \partial_t [w^{p-1}]_h \cdot \varphi \, dz \geq \iint_{Q_{R,S} \cap \Omega_T} \eta^p \psi_\varepsilon \partial_t w^{p-1} \cdot (w - u_0) \, dz = \iint_{Q_{R,S} \cap \Omega_T} \eta^p \psi_\varepsilon \partial_t [\frac{p-1}{p} |w|^p + \frac{1}{p} |u_0|^p - w^{p-1} \cdot u_0] \, dz = \iint_{Q_{R,S} \cap \Omega_T} \eta^p \psi_\varepsilon \partial_t \mathbf{b}[w, u_0] \, dz = -\iint_{Q_{R,S} \cap \Omega_T} \eta^p (\zeta_\varepsilon' + \zeta_\varepsilon'' \psi_\varepsilon) \mathbf{b}[w, u_0] \, dz.$$  

Now we are able to let $h \downarrow 0$ by using the convergence properties of the mollification in Lemma 2.6. That way, we obtain
\[
\liminf_{h \downarrow 0} \iint_{\Omega_T} \partial_t [u^{p-1}]_h \cdot \varphi \, dz \geq \iint_{Q_{R,S} \cap \Omega_T} -\eta^p (\zeta \psi'_\varepsilon + \zeta' \psi_\varepsilon) b[u, u_0] \, dz =: I_\varepsilon + II_\varepsilon.
\]

For the first term, there holds
\[
I_\varepsilon = -\iint_{Q_{R,S} \cap \Omega_T} \eta^p \zeta' \psi_\varepsilon b[u, u_0] \, dz
\]
\[
= -\frac{1}{\varepsilon} \int_\varepsilon^{t_1+\varepsilon} \iint_{B_R \cap \Omega} \eta^p \zeta b[u, u_0] \, dz + \frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} \iint_{B_R \cap \Omega} \eta^p \zeta \xi b[u, u_0] \, dz
\]
\[
\overset{\varepsilon \downarrow 0}{\longrightarrow} \int_{(B_R \cap \Omega) \times \{t_1\}} \eta^p b[u, u_0] \, dx,
\]
as \(\zeta = 1\) for \(t_1 \in \Lambda_s(t_0) \cap (0, T)\). The initial condition (2.3) took care of the first term when \(\varepsilon \downarrow 0\). The term \(II_\varepsilon\) on the other hand can be estimated by
\[
II_\varepsilon = -\iint_{\Omega_T \cap Q_{R,S}} \eta^p \zeta' \psi_\varepsilon b[u, u_0] \, dz
\]
\[
\geq -\iint_{\Omega_T \cap Q_{R,S}} |\zeta'| |b[u, u_0]| \, dz
\]
\[
\geq -\iint_{\Omega_T \cap Q_{R,S}} \frac{1}{S-s} b[u, u_0] \, dz,
\]
which is independent of \(\varepsilon\). We combine the previous estimates and obtain
\[
\liminf_{\varepsilon \downarrow 0} \liminf_{h \downarrow 0} \iint_{\Omega_T} \partial_t [u^{p-1}]_h \cdot \varphi \, dz
\]
\[
\geq \int_{(B_R \cap \Omega) \times \{t_1\}} \eta^p b[u, u_0] \, dx - \iint_{\Omega_T \cap Q_{R,S}} \frac{b[u, u_0]}{S-s} \, dz.
\]

**Estimating the elliptic part and right hand side**

The second term of the left hand side in the mollified equation (2.4) can be computed as
\[
\liminf_{h \downarrow 0} \int_{\Omega_T} \|A(x, t, u, Du)\|_h \cdot D\varphi \, dz
\]
\[
= \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_\varepsilon A(x, t, u, Du) \cdot [\eta^p (Du - Du_0) + p\eta^{p-1}(u - u_0) \otimes D\eta] \, dz
\]
\[
\geq L_1 \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_\varepsilon \eta^p |Du|^p \, dz
\]
\[
- \int_{\Omega_T \cap Q_{R,S}} pL_2 \zeta \psi_\varepsilon \eta^{p-1} |Du||u - u_0| |Du|^{p-1} + L_2 \zeta \psi_\varepsilon \eta^p |Du|^{p-1} |Du_0| \, dz
\]
\[
\geq \frac{L_1}{2} \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_\varepsilon \eta^p |Du|^p \, dz - c(L_1, L_2, p) \int_{\Omega_T \cap Q_{R,S}} \frac{|u - u_0|^p}{(R - r)^p} + |Du_0|^p \, dz,
\]
with the usage of the growth and coercivity conditions (1.3), Young’s inequality and $|D\eta| \leq \frac{1}{R-r}$. The right hand side in (2.4) can be estimated by
\[
\lim_{h \to 0} \left| \int_{\Omega_T} \left[ |F|^{p-2} F \right]_h \cdot D\varphi \, dz \right|
\leq \int_{\Omega_T \cap \Omega_{R,S}} \zeta \psi_\varepsilon \left| F \right|^{p-1} \left| \eta^p |Du - Du_0| + p\eta^{p-1}|u - u_0||D\eta| \right| \, dz
\leq \frac{L_1}{4} \int_{\Omega_T \cap \Omega_{R,S}} \zeta \psi_\varepsilon \eta^p |Du|^p \, dz
\]
\[
+ c(L_1, p) \int_{\Omega_T \cap \Omega_{R,S}} \frac{|u - u_0|^p}{(R-r)^p} + |F|^p + |Du_0|^p \, dz,
\]
with the same arguments as above. The final term on the right hand side of the mollified system (2.4) vanishes when $h \to 0$ due to $\varphi(\cdot, 0) = 0$. The convergence properties of Lemma 2.6 are also applied.

Combination of the previous estimates
We combine the previous estimates, letting $\varepsilon \to 0$ and obtain
\[
\int_{(B_r \cap \Omega) \times \{t_1\}} b[u, u_0] \, dx + \int_{t_0-s}^{t_1} \int_{B_r \cap \Omega} |Du|^p \, dz
\leq c \int_{Q_{R,S} \cap \Omega_T} \frac{b[u, u_0]}{S - s} + \frac{|u - u_0|^p}{(R-r)^p} + |F|^p + |Du_0|^p \, dz
\]
with $c = c(n, p, L_1, L_2) > 0$. We let $t_1 \to t_0 + s$ for the second term, while taking the supremum over $t_1 \in \Lambda_s(t_0)$ for the first term and dividing by $|Q_{R,S}|$. To get the measure of the smaller cylinder on the left hand side, use $r \in [R/2, R)$, $s \in [S/2, S)$, also inducing an $n$-dependency of the constant $c$. At last, Lemma 2.8 replaces the boundary term $b[u, u_0]$. □

3.2. Initial boundary
It will be necessary to extend $u$ beyond the initial time $t = 0$. In [27], this has been done by reflecting the boundary values. Here, we define
\[
\hat{u}(x, t) := \begin{cases} 
  u(x, t) & t > 0, \\
  u_0(x) & t \leq 0.
\end{cases} \tag{3.1}
\]
Due to the vanishing lateral boundary values, we extend $\hat{u}$ outside of $\Omega$ by zero, for any time $t$. We do the same for $u_0 \in W^{1,p}_0(\Omega)$. Also, we recall the notation $Q_{R,S,+} := B_R \cap (\Lambda_S \cap (0, T))$ for the part of the cylinder with positive time.

We follow up with an energy estimate valid close to the initial boundary, so restricted inside of the spatial domain $\Omega$. In this case we have $Q_{R,S,+} = Q_{R,S} \cap \Omega_T$.

Lemma 3.2. Let $u$ be a weak solution to the Cauchy-Dirichlet problem (1.1) and let $A$ satisfy (1.3). There exists $c = c(n, p, L_1, L_2) > 0$, such that for any
For the second term \( II \), with the same cutoff functions as in the previous Lemma, we have

\[
\sup_{t \in \Delta_s(t_0)} \int_{B_r(z_0)} \frac{|\hat{u}^2(t) - a\hat{u}^2_s|^2}{S} \, dx + \frac{1}{|Q_{r,s}|} \int_{Q_{r,s}+}(z_0) |Du|^p \, dz \\
\leq c \frac{1}{|Q_{r,s}|} \int_{Q_{r,s}+}(z_0) \left[ \frac{|u - a|^p}{(R-r)^p} + |F|^p \right] \, dz \\
+ c \int_{Q_{r,s}(z_0)} \frac{|\hat{u}^2(t) - a\hat{u}^2_s|^2}{S - s} \, dz.
\]

**Proof.** With the same cutoff functions as in the previous Lemma, \( \varphi = \zeta \psi \eta^p (u - a) \) is admissible as a testing function in the mollified equation (2.4), since the ball \( B_R \) lies within \( \Omega \) and \( t_0 > 0 \).

**Estimating the parabolic part**

With \( w^{p-1} := [w^{p-1}]_h \), the parabolic term reads

\[
\int_{\Omega_T} \partial_t [w^{p-1}]_h \cdot \varphi \, dz = \int_{Q_{r,s}(z_0)} \eta^p \zeta \psi \eta^p \partial_t w^{p-1} \cdot [w - a + (u - w)] \, dz.
\]

The properties of the mollification, contained in Lemma 2.6, will be used repeatedly. Once again the second term can be estimated from below by zero, while the first term can be computed as follows:

\[
\int_{\Omega_T} \partial_t [w^{p-1}]_h \cdot \varphi \, dz \geq \int_{Q_{r,s}(z_0)} \eta^p \zeta \psi \eta^p \partial_t w^{p-1}(w - a) \, dz \\
= \int_{Q_{r,s}(z_0)} \eta^p \zeta \psi \eta^p \partial_t \left[ \frac{p-1}{p} |w|^p - w^{p-1} a \right] \, dz \\
= \int_{Q_{r,s}(z_0)} \eta^p \zeta \psi \eta^p \partial_t b[w, a] \, dz \\
= - \int_{Q_{r,s}(z_0)} \eta^p (\zeta \psi' + \psi \zeta') b[w, a] \, dz \\
\longrightarrow - \int_{Q_{r,s}(z_0)} \eta^p (\zeta \psi' + \psi \zeta') b[u, a] \, dz,
\]

as \( h \downarrow 0 \). These terms will be referred to as \( I_\varepsilon, II_\varepsilon \). The first term \( I_\varepsilon \) converges as follows when \( \varepsilon \downarrow 0 \):

\[
I_\varepsilon = - \frac{1}{\varepsilon} \int_{B_R} \eta^p \zeta b[u, a] \, dz + \frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} \eta^p \zeta \int_{B_R} b[u, a] \, dz \\
\longrightarrow - \zeta(0) \int_{B_R} \eta^p b[u_0, a] \, dx + \int_{B_R \times \{t_1\}} \eta^p \zeta b[u, a] \, dx.
\]

For the second term \( II_\varepsilon \) one can use that the boundary term is non-negative to obtain
\[ |\Pi_\varepsilon| \leq \int_{\Omega_T \cap Q_{R,S}} |\zeta'| b[u, a] \, dz \leq \int_{\Omega_T \cap Q_{R,S}} \frac{1}{S-s} b[u, a] \, dz. \]

Together, this yields the estimate
\[
\liminf_{\varepsilon \to 0} \liminf_{h \to 0} \int_{\Omega_T} \partial_t [u^{p-1}]_h \cdot \varphi \, dz 
\geq -\zeta(0) \int_{B_R} \eta^p b[u_0, a] \, dx + \int_{B_R \times \{t_1\}} \eta^p \zeta b[u, a] \, dx 
- \int_{\Omega_T \cap Q_{R,S}} \frac{b[u, a]}{S-s} \, dz.
\]

**Estimating the elliptic part and the right hand side**

Using \( Du = D(u-a) \), the second term in the mollified equation (2.4) can be computed similarly as in the previous Lemma:
\[
\lim_{h \to 0} \int_{\Omega_T} [A(x, t, u, Du)]_h \cdot D\varphi \, dz 
= \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_x A(x, t, u, Du) \cdot [\eta^p D(u-a) + p\eta^{p-1}(u-a) \otimes D\eta] \, dz 
\geq \frac{L_1}{2} \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_x \eta^p |Du|^p \, dz - c \int_{\Omega_T \cap Q_{R,S}} \frac{|u-a|^p}{(R-r)^p} \, dz.
\]

Likewise, for the right hand side in (2.4) there holds the estimate
\[
\lim_{h \to 0} \left| \int_{\Omega_T} \|F|^{p-2} F\|_h \cdot D\varphi \, dz \right| 
\leq \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_x |F|^{p-1} [\eta^p |D(u-a)| + p\eta^{p-1}|u-a||D\eta|] \, dz 
\leq \frac{L_1}{4} \int_{\Omega_T \cap Q_{R,S}} \zeta \psi_x \eta^p |Du|^p \, dz + c \int_{\Omega_T \cap Q_{R,S}} \frac{|u-a|^p}{(R-r)^p} + |F|^p \, dz.
\]

**Combination of the previous estimates**

As in to the previous Lemma and by using the estimates for the boundary term in Lemma 2.8, this yields the estimate
\[
\int_{B_r}^t \int_{B_r} |Du|^p \, dz + \int_{B_r \times \{t_1\}} |u^T - a^T|^2 \, dx 
\leq c \int_{\Omega_T \cap Q_{R,S}} \frac{|u^T - a^T|^2}{S-s} + \frac{|u-a|^p}{(R-r)^p} + |F|^p \, dz + c\zeta(0) \int_{B_R} \eta^p b[u_0, a] \, dx.
\]

The last term can be treated in the following way.

If \( t_0 - S \geq 0 \), then \( \zeta(0) = 0 \) and the last term vanishes. Otherwise, \( t_0 - S < 0 \) and since \( \bar{u}(x, t) = u_0(x) \) for \( t < 0 \) there holds
Global higher integrability for a doubly nonlinear Cauchy-Dirichlet problem (Gluing Lemma)

Lemma 3.4. Let $p > 1$ and $u$ be a (local) weak solution, so fulfilling merely equation (2.2). Then, on any cylinder $Q_{R,S}(z_0) \subseteq \Omega_T$ with $R,S > 0$ there exists $\hat{r} \in [\frac{R}{2}, R)$ such that for all $t_1,t_2 \in \Lambda_S(t_0)$, we have

$$\langle u^{p-1} \rangle_{x_0, \hat{r}(t_2)} - \langle u^{p-1} \rangle_{x_0, \hat{r}(t_1)} \leq c \frac{S}{R} \iint_{Q_{R,S}(z_0)} [|Du|^{p-1} + |F|^{p-1}] \, dz$$

where $c = c(L_2)$.

This can be extended for times $t < 0$ as follows:

Lemma 3.3. Gluing Lemma

We quote the Gluing Lemma in the local case, taken from [4, Lemma 4.2]. This result holds true also in our context if both occurring times are positive.
\[ |\langle \hat{u}^{p-1}_{x_0} f(t_2) - \langle \hat{u}^{p-1}_{x_0} f(t_1) \rangle | \leq c \frac{S}{R} \frac{1}{|Q_{R,S}|} \int_{Q_{R,S,+}(z_0)} [D\hat{u}^{p-1} + |F|^{p-1}] \, dz \]

with \( c = c(n, L_2) > 0 \).

**Proof.** If \( t_1, t_2 \leq 0 \), the left hand side vanishes since \( \hat{u}(x, t_1) = \hat{u}(x, t_2) = u_0(x) \). If \( t_1, t_2 > 0 \), the respective result from Lemma 3.3 yields the claim. So let \( t_1 \leq 0 \) and \( t_2 > 0 \). Then \( \langle \hat{u}^{p-1} \rangle (t_1) = \langle u_0 \rangle^{p-1} \). Define as testing function \( \varphi(x, t) := \xi(t) \eta(x) e_i \), where \( e_i \) is the \( i \)-th canonical basis vector in \( \mathbb{R}^N \), \( \eta(x) = \eta_0(x) := \zeta_\delta(|x - x_0|) \) and

\[
\xi(t) = \xi_\varepsilon(t) := \begin{cases} 
0 & t \in (0, \varepsilon), \\
\frac{t - \varepsilon}{\varepsilon} & t \in (\varepsilon, 2\varepsilon), \\
1 & t \in (2\varepsilon, t_2), \\
1 - \frac{t}{\varepsilon}(t - t_2) & t \in (t_2, t_2 + \varepsilon), \\
0 & t \in (t_2 + \varepsilon, t_0 + \varepsilon).
\end{cases}
\]

and

\[
\zeta_\delta(s) := \begin{cases} 
1 & s \in (0, r), \\
1 - \frac{1}{\delta}(s - r) & s \in (r, r + \delta), \\
0 & s \in (r + \delta, R)
\end{cases}
\]

for some \( r \in \left[ \frac{R}{2}, R \right) \). The first term obtained by inserting the test function \( \varphi \) into the weak formulation (2.2) is given by

\[
\int_{\Omega_T} u^{p-1} \cdot \partial_t \varphi \, dz = \int_{\Omega_T} \eta \partial_t \xi u^{p-1} \cdot e_i \, dz
\]

by the initial boundary condition (2.3). The remaining terms of (2.2) are given by

\[
\int_{\Omega_T} \left[ A(x, t, u, Du) + |F|^{p-2} F \right] \cdot D\varphi \, dz
\]

\[
\int_{\partial B_r(x_0)} \left[ A(x, t, u, Du) + |F|^{p-2} F \right] \cdot e_i \otimes \frac{x - x_0}{|x - x_0|} \, d\mathcal{H}^{n-1} \, dt
\]

\[
\int_{\partial B_r(x_0)} \left[ A(x, t, u, Du) + |F|^{p-2} F \right] \cdot e_i \otimes \frac{x - x_0}{|x - x_0|} \, d\mathcal{H}^{n-1} \, dt
\]

\[
\int_{\partial B_r(x_0)} \left[ A(x, t, u, Du) + |F|^{p-2} F \right] \cdot e_i \otimes \frac{x - x_0}{|x - x_0|} \, d\mathcal{H}^{n-1} \, dt.
\]
We multiply the weak formulation by $e_i$, sum over $i = 1, \ldots, N$ while also replacing $u_0(\cdot)$ by $\hat{u}(\cdot, t_1)$. This way, we obtain

$$\left| \int_{B_r(x_0)} \hat{u}^{p-1}(t_2) - \hat{u}^{p-1}(t_1) \, dx \right|$$

$$= \left| \int_0^{t_2} \int_{\partial B_r(x_0)} [A(x, t, u, Du) + |F|^{p-2}F] \frac{x - x_0}{|x - x_0|} \, d\mathcal{H}^{n-1} \, dt \right|$$

$$\leq \int_0^{t_0+S} \int_{\partial B_r(x_0)} [L_2|Du|^{p-1} + |F|^{p-1}] \, d\mathcal{H}^{n-1} \, dt$$

In the last step, (1.3) was used. We abbreviate $I := \int_{(0,t_0+S)} [L_2|Du|^{p-1} + |F|^{p-1}] \, dt$. Due to

$$\int_{B_R(x_0)} I \, dx = \int_0^R \int_{\partial B_r(x_0)} I \, d\mathcal{H}^{n-1} \, dr \geq \int_{R/2}^R \int_{\partial B_r(x_0)} I \, d\mathcal{H}^{n-1} \, dr,$$

there must exist a $\hat{r} \in [R/2, R)$ such that

$$\int_{\partial B_{\hat{r}}(x_0)} I \, d\mathcal{H}^{n-1} \leq \frac{2}{R} \int_{B_R(x_0)} I \, dx.$$

By choosing $\hat{r}$ in (3.2) and taking mean values on both sides yields

$$|\langle \hat{u}^{p-1}_{x_0;\hat{r}}(t_2) - \langle \hat{u}^{p-1}_{x_0;\hat{r}}(t_1) \rangle| \leq c \frac{2}{R} \frac{SR^n}{\hat{r}^n} \frac{1}{|Q_{R,S}|} \int_{Q_{R,S,+}(z_0)} |Du|^{p-1} + |F|^{p-1} \, dz$$

$$\leq c \frac{S}{R} \frac{1}{|Q_{R,S}|} \int_{Q_{R,S,+}(z_0)} |Du|^{p-1} + |F|^{p-1} \, dz,$$

since $\hat{r} \geq R/2$ and where $c = c(n, L_2) > 0$. \hfill \Box

4. Parabolic Sobolev-Poincaré type inequalities

For convenience, we write $X_{\Omega_T}$ for the characteristic function of the set $\Omega_T$, so $X_{\Omega_T}(x, t) = 1$ if $(x, t) \in \Omega_T$ and $X_{\Omega_T}(x, t) = 0$ otherwise. We consider scaled cylinders $Q_\varrho^{(\mu)}(z_0) \subseteq \mathbb{R}^{n+1}$ and when $B_\varrho(x_0) \subseteq \Omega$, we may write $Q_\varrho^{(\mu)}(z_0)$ instead of $Q_\varrho^{(\mu)}(z_0) \cap \Omega_T$.

4.1. Lateral boundary

If for $\varrho, \mu > 0$ and some $K \geq 1$ there holds

$$\frac{\iint_{Q_\varrho^{(\mu)}(z_0)} |\hat{u} - u_0|^p + |u_0|^p \varrho^p \, dz}{\iint_{Q_\varrho^{(\mu)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^pX_{\Omega_T} \, dz} \leq K \mu^p,$$  \hspace{1cm} (4.1)
then we call such a cylinder $\mu$-sub-intrinsic. If on the other hand

$$\mu^p \leq K \frac{\iint_{Q^{(\mu)}_{\rho}(z_0)} |\dot{u} - u_0|^p + |u_0|^p \, dz}{\iint_{Q^{(\mu)}_{\rho}(z_0)} |D\dot{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz} \quad \text{or} \quad \mu^p \leq K \quad (4.2)$$

holds, then the cylinder is called $\mu$-super-intrinsic. The first and second inequality from (4.2) will be individually referenced as (4.2)$_1$ and (4.2)$_2$, respectively. A cylinder fulfilling both super- and sub-intrinsic properties is called $\mu$-intrinsic. First, we quote [27, Lemma 4.1], which in turn is an adaptation of [8, Lemma 4.2].

**Lemma 4.1.** Let $v \in W^{1,p}_0(\Omega)$ for a.e. $t \in (0,T)$ and $\mathbb{R}^n \setminus \Omega$ be uniformly $p$-thick with corresponding constants $\nu$ and $\varrho_0$. Consider a cylinder $Q^{(\mu)}_{\varrho}(z_0) \subset \mathbb{R}^{n+1}$ with $B_{\varrho/3}(x_0) \setminus \Omega \neq \emptyset$. Then there exists a constant $\gamma = \gamma(n,\nu) \in (1,p)$ such that for any $\gamma \leq \varrho \leq p$ there holds

$$\int_{B_{\varrho} \cap \Omega} |v(\cdot,t)|^\varrho \, dx \leq c \varrho \int_{B_{\varrho} \cap \Omega} |Dv(\cdot,t)|^\varrho \, dx$$

for a.e. $t \in (0,T)$ and also

$$\iint_{Q^{(\mu)}_{\varrho} \cap \Omega_T} |v|^\varrho \, dz \leq c \varrho \iint_{Q^{(\mu)}_{\varrho} \cap \Omega_T} |Dv|^\varrho \, dz$$

with $c = c(n,N,\nu, \varrho_0, \varrho) > 0$.

Combining this inequality with a slice-wise application of Sobolev’s inequality, we are able to derive the following parabolic Sobolev-Poincaré type inequality in the lateral case. We delay the application of the intrinsic property of the cylinder until Lemma 5.1 to obtain a better bound for the scaling parameter $\mu$.

**Lemma 4.2.** Let $1 < p \leq 2$. Let $u$ be a weak solution to the Cauchy-Dirichlet problem (1.1) and let $\mathbb{R}^n \setminus \Omega$ be $p$-thick with corresponding constants $\nu$ and $\varrho_0$. Let $q = \max\{2n/(n+2), \gamma, 1\} \in (1,p)$ with $\gamma = \gamma(n,\nu)$ from Lemma 4.1. On a cylinder $Q^{(\mu)}_{\varrho}(z_0) \subset \mathbb{R}^{n+1}$ with $\text{dist}(B_{\varrho}(x_0), \partial \Omega) = 0$ and for any $\varepsilon \in (0,1]$, there holds

$$\iint_{Q^{(\mu)}_{\varrho}(z_0)} \frac{|\dot{u}^\varrho - u_0^\varrho|^2}{\mu^{p-2} \varrho^p} \, dz \leq \varepsilon c \sup_{t \in \Lambda^{(\mu)}_{\varepsilon,\varrho}} \iint_{B_{\varrho}(x_0)} \frac{|\dot{u}^\varrho - u_0^\varrho|^2}{\mu^{p-2} \varrho^p} \, dx$$

$$+ \frac{c \mu^{2-p}}{\varepsilon^{2-q} \varrho^p} \left[ \iint_{Q^{(\mu)}_{\varrho}} |D\dot{u} - Du_0|^q \, dz \right]^{\frac{p}{q}}$$

with $c = c(p,n,N,\nu, \varrho_0) > 0$. 
Proof. We omit the center point \( z_0 \) and start by calculating

\[
\int_{Q_{\hat{e}}^{(\mu)}} \frac{|\hat{u}^{\frac{\varrho}{p}} - u_0^{\frac{p}{2}}|^2}{\mu^{p-2} q^p} \, dz = \int_{Q_{\hat{e}}^{(\mu)}} \left( \int_{B_\varrho} \frac{|\hat{u}^{\frac{\varrho}{p}} - u_0^{\frac{p}{2}}|^2}{\mu^{p-2} q^p} \, dx \right)^{\frac{2-p}{2}} \mu^{(2-p)\frac{q}{2}} \left( \int_{B_\varrho} |\hat{u} - u_0|^p q^p \, dx \right)^{\frac{q}{2}} \, dt \\
\leq c \int_{Q_{\hat{e}}^{(\mu)}} \left( \sup_{t \in Q_{\hat{e}}^{(\mu)}} \int_{B_\varrho} \frac{|\hat{u}^{\frac{\varrho}{p}} - u_0^{\frac{p}{2}}|^2}{\mu^{p-2} q^p} \, dx \right)^{\frac{2-q}{2}} \mu^{(2-p)\frac{q}{2}} \left( \int_{B_\varrho} |\hat{u} - u_0|^p q^p \, dx \right)^{\frac{q}{2}} \, dt,
\]

where we use Lemma 2.7 in the second last line. The condition dist \((B_\varrho(x_0), \partial \Omega) = 0\) implies that \( B_{4\varrho/3} \backslash \Omega \neq \emptyset \). Thus Lemma 4.1 can be applied for the ball \( B_{4\varrho} \). For the last term we can thus use Sobolev’s inequality (note that \( \frac{p}{q} \geq 1 \)), Lemma 4.1 with \( \varrho = q \) for the ball \( B_{4\varrho} \) as well as Hölder’s inequality with \( \frac{q}{p} \geq 1 \). It follows that

\[
\int_{Q_{\hat{e}}^{(\mu)}} \left( \int_{B_\varrho} \frac{|\hat{u} - u_0|^p}{q^p} \, dx \right)^{\frac{q}{2}} \, dt \leq c \int_{Q_{\hat{e}}^{(\mu)}} \left( \int_{B_\varrho} \frac{|\hat{u} - u_0|^q}{q^q} + |D\hat{u} - Du_0|^q \, dx \right)^{\frac{q}{2}} \, dt \\
\leq c \int_{Q_{\hat{e}}^{(\mu)}} \left( \int_{B_{4\varrho}} |D\hat{u} - Du_0|^q \, dx \right)^{\frac{q}{2}} \, dt \\
\leq c \left( \int_{Q_{\hat{e}}^{(\mu)}} |D\hat{u} - Du_0|^q \, dz \right)^{\frac{q}{2}}
\]

with \( c = c(p, n, N, \nu, q_0) > 0 \). Inserting this estimate into (4.3) and by applying Young’s inequality with \( \frac{2}{2-q} \) and \( \frac{2}{q} \) we obtain

\[
\int_{Q_{\hat{e}}^{(\mu)}} \frac{|u^{\frac{\varrho}{p}} - u_0^{\frac{p}{2}}|^2}{\mu^{p-2} q^p} \, dz \leq \varepsilon c \sup_{t \in Q_{\hat{e}}^{(\mu)}} \int_{B_\varrho} \frac{|\hat{u}^{\frac{\varrho}{p}} - u_0^{\frac{p}{2}}|^2}{\mu^{p-2} q^p} \, dx \\
+ \frac{c\mu^{2-p}}{\varepsilon^{\frac{2-q}{2}}} \left[ \int_{Q_{\hat{e}}^{(\mu)}} |D\hat{u} - Du_0|^q \, dz \right]^{\frac{q}{q}}
\]

with \( c = c(p, n, N, \nu, q_0) > 0 \). \( \square \)

**Lemma 4.3.** Let \( 1 < p \leq 2 \). Let \( u \) be a weak solution to the Cauchy-Dirichlet problem (1.1) and let \( \mathbb{R}^n \backslash \Omega \) be \( p \)-thick with corresponding constants \( \nu \) and \( q_0 \). On a cylinder \( Q_{\hat{e}}^{(\mu)} \subseteq \mathbb{R}^{n+1} \) satisfying the sub-intrinsic coupling (4.1), there
exists \( c = c(K, p, n) > 0 \) such that for any \( \delta \in (0, 1] \), there holds
\[
\iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u} - u_0|^p}{\varrho^p} \, dz \leq c\delta \iint_{Q_{\nu}^{(\mu)}} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
+ \frac{c}{\delta^{2-p}} \iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u}_0^\nu - u_0^\nu|^2}{\mu^{p-2} \varrho^p} \, dz.
\]

Proof. We start by using Lemma 2.7 with \( b = \hat{u}^\nu, a = u_0^\nu, \alpha = \frac{2}{p} \) and Hölder’s inequality. This way, it follows that
\[
\iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u} - u_0|^p}{\varrho^p} \, dz \leq c \iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u}_0^\nu - u_0^\nu|^p}{\varrho^p} \left[ |\hat{u}| + |u_0| \right]^\frac{p(2-p)}{2} \, dz
\]
\[
\leq c \left[ \iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u}_0^\nu - u_0^\nu|^2}{\varrho^p} \, dz \right]^\frac{2}{2-p} \left[ \iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u}| + |u_0|^p}{\varrho^p} \, dz \right]^\frac{2-p}{2}.
\]
The latter integral can be estimated via the sub-intrinsic property (4.1). Hence,
\[
\iint_{Q_{\nu}^{(\mu)}} \frac{|\hat{u}| + |u_0|^p}{\varrho^p} \, dz \leq c\mu \iint_{Q_{\nu}^{(\mu)}} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
with \( c = c(K, p, n) > 0 \). After shifting the appearing power of \( \mu \) to the first integral the proof can be finished by applying Young’s inequality. \( \square \)

4.2. Initial boundary

Similar as in the lateral case we call a cylinder \( \mu \)-sub-intrinsic, if for \( \varrho, \mu > 0 \) and some \( K \geq 1 \) there holds
\[
\iint_{Q_{\nu}^{(\mu)}(z_0)} \frac{|\hat{u}|^p}{\varrho^p} \, dz \leq K\mu^p, \quad (4.4)
\]
and
\[
\iint_{Q_{\nu}^{(\mu)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
If on the other hand
\[
\mu^p \leq K \iint_{Q_{\nu}^{(\mu)}(z_0)} \frac{|\hat{u}|^p}{\varrho^p} \, dz \quad \text{or} \quad \mu^p \leq K \quad (4.5)
\]
holds, then the cylinder is called \( \mu \)-super-intrinsic. As before, a cylinder fulfilling both super- and sub-intrinsic properties is called \( \mu \)-intrinsic. As we consider \( B_\varrho(x_0) \subseteq \Omega \) in the current case, we may write \( Q_{\varrho, +}^{(\mu)}(z_0) \) instead of \( Q_{\nu}^{(\mu)}(z_0) \cap \Omega_T \). In this subsection we will follow the strategy as in the local setting [4, Chapter 5].

Lemma 4.4. Let \( 1 < p \leq 2 \). Assume that \( u \) is a weak solution to the Cauchy-Dirichlet problem (1.1). For \( z_0 \in \Omega_T \) and \( \varrho, \mu > 0 \), consider a cylinder \( Q_{\varrho}^{(\mu)}(z_0) \)
\[ \subseteq \Omega \times (-T, T) \] that fulfils the sub-intrinsic property (4.4). Then there exists \( c = c(p, K) > 0 \) such that
\[
\iint_{Q^\rho_0} \frac{|\hat{u} - (\hat{u})_{\rho_0}^{(\mu)}|^p}{\rho^p} \, dz \leq c \left[ \iint_{Q^\rho_0} \frac{|\hat{u}^\frac{2}{p} - (\hat{u})_{\rho_0}^{(\mu)}|^2}{\mu^{p-2} \rho^p} \, dz \right]^{\frac{2}{p}} \]
\[
\iint_{Q^\rho_0} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{2-p}{p}}.
\]

**Proof.** Write \( Q^\rho_0 = Q^\rho_0(z_0) \). Applying Lemma 2.7 with \( \alpha = \frac{2}{p} \) and Hölder’s inequality leads to the estimate
\[
\iint_{Q^\rho_0} \frac{|\hat{u} - (\hat{u})_{\rho_0}^{(\mu)}|^p}{\rho^p} \, dz \leq c \iint_{Q^\rho_0} \frac{|\hat{u}^\frac{2}{p} - (\hat{u})_{\rho_0}^{(\mu)}|^2}{\rho^p} \, dz \leq c \left[ \iint_{Q^\rho_0} |\hat{u}| + |(\hat{u})_{\rho_0}^{(\mu)}|^p \, dz \right]^{\frac{2}{p}} \left[ \iint_{Q^\rho_0} |\hat{u}| + |(\hat{u})_{\rho_0}^{(\mu)}|^p \, dz \right]^{\frac{2-p}{p}},
\]
where \( c = c(p) > 0 \). Due to Jensen’s inequality and the sub-intrinsic scaling (4.4) it follows that
\[
\iint_{Q^\rho_0} \left| |\hat{u}| + |(\hat{u})_{\rho_0}^{(\mu)}| \right|^p \frac{\rho^p}{\rho^p} \, dz \leq 2^p \iint_{Q^\rho_0} \frac{|\hat{u}|^p}{\rho^p} \, dz \leq 2^p K \mu^p \iint_{Q^\rho_0} \left[ |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \right] \, dz.
\]
Together, this yields the desired estimate with \( c = c(p, K) > 0 \). \[\square\]

**Lemma 4.5.** Let \( 1 < p \leq 2 \). Assume that \( u \) is a weak solution to the Cauchy-Dirichlet problem (1.1). For \( z_0 \in \Omega_T \) and \( \rho, \mu > 0 \), consider a cylinder \( Q^\rho_0(z_0) \subseteq \Omega \times (-T, T) \) that fulfils the sub-intrinsic property (4.4). For any \( q \in [1, p] \), there exist \( c = c(n, p, L_2, K) > 0 \) such that
\[
\iint_{Q^\rho_0} \frac{|\hat{u} - (\hat{u})_{\rho_0}^{(\mu)}|^q}{\rho^q} \, dz \leq c \left[ \iint_{Q^\rho_0} |D\hat{u}|^q + |Du_0|^q + |F|^q \chi_{\Omega_T} \, dz \right]^{p-1} \left[ \iint_{Q^\rho_0} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{q(p-1)}{p}}.
\]

**Proof.** Let \( \hat{\rho} \) be the radius \( \hat{r} \) from the Gluing Lemma 3.4. By using the quasi-minimality of the mean value for \( \hat{u} \) from Lemma 2.9, it follows that
where Poincaré’s inequality this implies that Lemma 2.9 with the value \( \alpha = \frac{1}{p-1} \geq \frac{1}{q} \) on every time slice. The quotient of measures \( |B_{\hat{\varrho}}|/|B_{\varrho}| \) is compensated by the fact that \( \hat{\varrho} \in [\varrho, \varrho] \). Together with Poincaré’s inequality this implies that

\[
I \leq c \iint Q_{\varrho}^{(\mu)} \left| \frac{\hat{u} - \langle \hat{u} \rangle_{\varrho}(t)}{\varrho} \right|^q \, dz \leq c \iint Q_{\varrho}^{(\mu)} \left| D\hat{u} \right|^q \, dz.
\]

The constant in Poincaré’s inequality depends continuously on \( q \). Since \( q \in [1, p] \), we can thus write the constant as \( c = c(n, p) > 0 \). Lemma 2.7 with \( \alpha = \frac{1}{p-1} \geq 1 \) and Hölder’s inequality imply that

\[
\begin{align*}
II & \leq \frac{c}{\varrho^q} \int_{\Lambda_{\varrho}^{(\mu)}} \left[ \left| \langle \hat{u}^{p-1} \rangle_{\varrho}(t) \rangle + \left| \langle \hat{u}^{p-1} \rangle_{\varrho}^{(\mu)} \right| \right] \frac{q(2-p)}{p-1} \left| \langle \hat{u}^{p-1} \rangle_{\varrho}^{(\mu)} \right| |t - \langle \hat{u}^{p-1} \rangle_{\varrho}^{(\mu)}| \, dt \\
& \leq \frac{c}{\varrho^q} \sup_{t, \tau \in \Lambda_{\varrho}^{(\mu)}} \left| \langle \hat{u}^{p-1} \rangle_{\varrho}(t) - \langle \hat{u}^{p-1} \rangle_{\varrho}(\tau) \right|^q \\
& \quad \int_{\Lambda_{\varrho}^{(\mu)}} \left[ \left| \langle \hat{u}^{p-1} \rangle_{\varrho}(t) \rangle + \left| \langle \hat{u}^{p-1} \rangle_{\varrho}^{(\mu)} \right| \right] \frac{q(2-p)}{p-1} \, dt \\
& \leq \frac{c}{\varrho^{q(p-1)}} \sup_{t, \tau \in \Lambda_{\varrho}^{(\mu)}} \left| \langle \hat{u}^{p-1} \rangle_{\varrho}(t) - \langle \hat{u}^{p-1} \rangle_{\varrho}(\tau) \right|^q \left[ \iint Q_{\varrho}^{(\mu)} \left| \hat{u} \right|^p \frac{1}{\varrho^p} \, dz \right] \frac{q(2-p)}{p},
\end{align*}
\]

where \( c = c(p) \). The Gluing Lemma 3.4 takes care of the supremum term, while the sub-intrinsic scaling (4.4) allows the estimation of the final term. This way, the appearing powers of \( \mu \) cancel each other out. Together with an application of Jensen’s inequality, noting that \( q/(p-1) > 1 \), it follows that

\[
\begin{align*}
II & \leq c \left[ \frac{1}{Q_{\varrho}^{(\mu)}} \int_{Q_{\varrho}^{(\mu)}} \left[ |D\hat{u}|^{p-1} + |F|^{p-1} \right] \, dz \right]^q \left[ \iint Q_{\varrho}^{(\mu)} \left| D\hat{u} \right|^p + |F|^p \chi_{\Omega_T} \, dz \right] \frac{q(2-p)}{p} \\
& \leq c \left[ \iint Q_{\varrho}^{(\mu)} \left| D\hat{u} \right|^{p-1} + |D\hat{u}|^{p-1} + |F|^{p-1} \, dz \right]^q \left[ \iint Q_{\varrho}^{(\mu)} \left| D\hat{u} \right|^p + |D\hat{u}| + |F|^p \chi_{\Omega_T} \, dz \right] \frac{q(2-p)}{p}.
\end{align*}
\]
\[
\leq c \left[ \iint_{Q_{q}^{(\mu)}(\lambda)} |D \hat{u}|^q + |Du_0|^q + |F|^q \chi_{\Omega_T} \, dz \right]^{p-1} \\
\left[ \iint_{Q_{q}^{(\mu)}(\lambda)} |D \hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{q(2-p)}{p}}
\]

with \( c = c(p, L_2, K) \). Inserting these inequalities into the initial equation at the start of the proof yields the desired result. \( \square \)

At one point in the final chapter we will work with a cylinder fulfilling an adapted version of the sub-intrinsic property (4.4), namely

\[
\iint_{Q_{q}^{(\mu)}} |\hat{u}|^p \frac{d\rho}{\rho^p} \, dz \leq K \mu^p \lambda^p
\]

for some \( \lambda > 0 \). In that instance we will further use the exponent \( q = p \). Note that the sub-intrinsic property is applied merely once in the proof of the previous Lemma. Inserting the adapted version instead results in the following inequality:

\[
\iint_{Q_{q}^{(\mu)}} |\hat{u} - (\hat{u})^{(\mu)}_{z_0, \varrho}|^p \frac{d\rho}{\rho^p} \, dz \leq c \iint_{Q_{q}^{(\mu)}} |D \hat{u}|^p \, dz \\
+ c \left[ \iint_{Q_{q}^{(\mu)}} |D \hat{u}|^p + |Du_0|^p + |F|^p \, dz \right]^{p-1} \lambda^{p(2-p)}.
\]

(4.6)

**Lemma 4.6.** Let \( \max\{\frac{2n}{n+2}, 1\} < p \leq 2 \). Assume that \( u \) is a weak solution to the Cauchy-Dirichlet problem (1.1). For \( z_0 \in \Omega_T \) and \( \varrho, \mu > 0 \), consider a cylinder \( Q_{q}^{(\mu)}(z_0) \subseteq \Omega \times (-T, T) \) that fulfils both the sub-intrinsic property (4.4) and the super-intrinsic property (4.5). For any \( \varepsilon \in (0, 1] \) and for \( q = \max\{\frac{2n}{n+2}, 1\} \) there exists \( c = c(n, p, L_2, K) > 0 \) such that

\[
\iint_{Q_{q}^{(\mu)}} \left[ \frac{\hat{u}^\frac{q}{p} - [(\hat{u})^{(\mu)}_{z_0, \varrho}]^\frac{q}{p}}{\mu^{p-2} \rho^p} \right]^2 \, dz \\
\leq \varepsilon \left[ \sup_{t \in \Lambda_{q}^{(\mu)}(t_0)} \int_{B_{q}(x_0)} \left[ \frac{\hat{u}^\frac{q}{p} - (\hat{u})^{(\mu)}_{z_0, \varrho}]^\frac{q}{p}}{\mu^{p-2} \rho^p} \right]^2 \, dz + \iint_{Q_{q}^{(\mu)}} |D \hat{u}|^p + |Du_0|^p \, dz \right] \\
+ \frac{c}{\varepsilon^\frac{q(p-1)}{p-2}} \left[ \iint_{Q_{q}^{(\mu)}} |D \hat{u}|^q + |Du_0|^q \, dz \right]^\frac{q}{q} + \frac{1}{|Q_{q}^{(\mu)}|} \iint_{Q_{q}^{(\mu)}} |F|^p \, dz \right].
\]

**Proof.** Write \( a := (\hat{u})^{(\mu)}_{\varrho} \). Expanding the left hand side with the powers \(-q/2 + q/2\) and applying Hölder’s inequality for the latter part, while taking the supremum over the time slices in the first part yields
\[
\iint_{Q_{\nu}^{(\epsilon)}} \left| \frac{\hat{u}^\frac{p}{2} - a^\frac{2}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dz 
\leq \left[ \sup_{t \in \Lambda_{\nu}^{(\epsilon)}} \int_{B_\epsilon} \left| \frac{\hat{u}^p(t) - a^\frac{p}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dx \right]^{\frac{2}{2-p}} \left[ \int_{\Lambda_{\nu}^{(\epsilon)}} \left[ \int_{B_\epsilon} \left| \frac{\hat{u}^p - a^\frac{p}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dx \right]^{\frac{2}{p}} \, dt \right]^{\frac{p}{2}}
=:\Gamma^{\frac{2}{2-p}} II.
\]

Estimating II can be achieved by inspecting \(\mu^{(2-p)q/2} \) with the help of the super-intrinsic scaling \((4.5)\).

This condition consists of two cases.

**Case** \((4.5)_1\): First note that by using Lemma 4.5 with \(q = p\) one has

\[
\left[ \iint_{Q_{\nu}^{(\epsilon)}} |\hat{u}|^p \, \varrho^p \, dz \right]^{\frac{1}{p}} \leq \left[ \iint_{Q_{\nu}^{(\epsilon)}} \left| \hat{u} - a^\frac{p}{2} \right|^p \, \varrho^p \, dz \right]^{\frac{1}{p}} + |a| \varrho
\]

\[
\leq c \left[ \iint_{Q_{\nu}^{(\epsilon)}} |\hat{D}u|^p + |D u_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{1}{p}} + |a| \varrho.
\]

The super-intrinsic scaling \((4.5)_1\) then yields

\[
\mu \leq c + \frac{c|a|}{\varrho \left[ \iint_{Q_{\nu}^{(\epsilon)}} |\hat{D}u|^p + |D u_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{1}{p}}},
\]

**Case** \((4.5)_2\): Here, the condition \(\mu^p \leq K\) immediately implies the previous inequality with \(c = K^{\frac{1}{p}}\).

In turn, the following conclusion holds in both cases:

\[
\II \leq c \left[ \int_{\Lambda_{\nu}^{(\epsilon)}} \left[ \int_{B_\epsilon} \left| \frac{\hat{u}^\frac{p}{2} - a^\frac{2}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dx \right]^{\frac{2}{2-p}} \, dt \right]
+ c \left[ \int_{\Lambda_{\nu}^{(\epsilon)}} \left[ \int_{B_\epsilon} \left| \frac{|a|^{2-p} |\hat{u}^\frac{p}{2} - a^\frac{2}{2}|^2}{\varrho^2} \right| \, dx \right]^{\frac{2}{2}} \, dt \right]
\]

\[
\left[ \iint_{Q_{\nu}^{(\epsilon)}} |\hat{D}u|^p + |D u_0|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{\frac{(2-p)q}{2p}}
=:\II_1 + \II_2.
\]

For the first of these terms, use Lemma 2.8 with the exponent \(\alpha = \frac{2}{p} \geq 1\), then Sobolev’s and Hölder’s inequality to conclude that

\[
\II_1 \leq c \int_{\Lambda_{\nu}^{(\epsilon)}} \left[ \int_{B_\epsilon} \left| \frac{\hat{u} - a}{\varrho \varrho^p} \right|^p \, dx \right]^{\frac{2}{p}} \, dt
\leq c \int_{\Lambda_{\nu}^{(\epsilon)}} \left[ \int_{B_\epsilon} |\hat{D}u|^q + \left| \frac{\hat{u} - a}{\varrho^q} \right|^q \, dx \right]^{\frac{2}{q}} \, dt
\leq c \left[ \iint_{Q_{\nu}^{(\epsilon)}} |\hat{D}u|^q + \left| \frac{\hat{u} - a}{\varrho^q} \right|^q \, dz \right]^{\frac{2}{q}}.
\]

(4.7)
By expanding the first of these mean value integrals with the exponents $p - 1$ and $1 - p$, we obtain

$$\iint_{Q_0} |D\hat{u}|^q \, dz \leq \left[ \iint_{Q_0} |D\hat{u}|^q + |Du_0|^q + |F|^q X_{\Omega_T} \, dz \right]^{p-1} \left[ \iint_{Q_0} |D\hat{u}|^q + |Du_0|^q + |F|^q X_{\Omega_T} \, dz \right]^{2-p}.$$ 

For the latter integral, use Hölder’s inequality with $\frac{p}{q} > 1$ to obtain a right hand side that equals, up to a constant, the right hand side of Lemma 4.5. By inserting this and estimating the second part of the right hand side of (4.7) with Lemma 4.5, it follows that

$$\Pi_1 \leq c \left[ \iint_{Q_0} |D\hat{u}|^q + |Du_0|^q + |F|^q X_{\Omega_T} \, dz \right]^{\frac{(p-1)p}{2}} \left[ \iint_{Q_0} |D\hat{u}|^p + |Du_0|^p + |F|^p X_{\Omega_T} \, dz \right]^{\frac{(2-p)q}{2p}}.$$ 

For $\Pi_2$, use Lemma 2.7 with $\alpha = \frac{2}{p} \geq 1$ and once again Sobolev’s inequality. This way, we obtain

$$\Pi_2 \leq c \frac{\int_{\Lambda_{q_0}} \left[ \int_{B_{q_0}} \frac{|\hat{u} - a|^2}{\theta^2} \, dx \right]^{\frac{q}{2}} \, dt}{\left[ \iint_{Q_0} |D\hat{u}|^p + |Du_0|^p + |F|^p X_{\Omega_T} \, dz \right]^{\frac{(2-p)q}{2p}}}.$$ 

With Lemma 4.5 it follows that

$$\Pi_2 \leq c \left[ \iint_{Q_0} |D\hat{u}|^q + |Du_0|^q + |F|^q X_{\Omega_T} \, dz \right]^{p-1} \left[ \iint_{Q_0} |D\hat{u}|^p + |Du_0|^p + |F|^p X_{\Omega_T} \, dz \right]^{\frac{(2-p)q}{2p}}.$$ 

For $\tilde{q} \in \{p, q\}$, we abbreviate

$$\mathcal{F}(\tilde{q}) := \iint_{Q_0} |D\hat{u}|^{\tilde{q}} + |Du_0|^{\tilde{q}} + |F|^{\tilde{q}} X_{\Omega_T} \, dz.$$
Together with Young’s inequality with the exponents $\frac{2}{2-q}$ and $\frac{2}{q}$, the previous estimates imply that

$$\iint_{Q_\rho(\mu)} \left| \frac{\hat{u}^\frac{\mu}{2} - a^\frac{\mu}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dz \leq \varepsilon \sup_{t \in \Lambda_{\rho}} \int_{B_\rho} \left| \frac{\hat{u}^\frac{\mu}{2}(t) - a^\frac{\mu}{2}}{\mu^{p-2} \varrho^p} \right|^2 \, dx$$

$$+ c\varepsilon^{-\frac{2-q}{q}} \left[ F(q)^{(p-1)\frac{\mu}{2}F(p)(2-p)^\frac{\mu}{2} + F(q)^{p-1} F(p)(2-p)^\frac{\mu}{2} \right]^\frac{1}{q}$$

Now one can apply Young’s inequality twice on the products on the right hand side to gain control over the terms of the form $F(p)$. Finally, use Hölder’s inequality for the integral with $|F|^q$ to obtain the power $|F|^p$. □

5. Reverse Hölder inequalities

The goal of this section is to obtain reverse Hölder type inequalities. Similarly to [27], we must distinguish whether the cylinder is close to the initial or the lateral boundary. Yet, in contrast to the same reference, an intrinsic coupling will be available in any case for our setting.

5.1. Lateral boundary

In the lateral case, we consider cylinders $Q_\rho^{(\mu)}(z_0) \subset Q_{\varrho p}^{(\nu)}(z_0) \subseteq \mathbb{R}^n \times (-T,T)$ for some $\varrho, \mu > 0$ with $\text{dist}(B_\varrho(x_0), \partial \Omega) = 0$. For some $K \geq 1$, we impose the sub-intrinsic condition

$$\iint_{Q_\rho^{(\mu)}(z_0)} \frac{|\hat{u} - u_0|^p + |u_0|^p}{(4\varrho)^p} \, dz \leq K \mu^p \quad (5.1)$$

and the super-intrinsic condition

$$\mu^p \leq K \frac{\iint_{Q_\rho^{(\mu)}(z_0)} |\hat{u} - u_0|^p + |u_0|^p}{\varrho^p} \, dz \quad \text{or} \quad \mu^p \leq K. \quad (5.2)$$

These conditions imply the sub- and super-intrinsic conditions (4.1) and (4.2) for every $s \in [\varrho, 2\varrho]$, respectively.

Lemma 5.1. Let $\max\{1, \frac{2n}{n+2}\} < p \leq 2$ and $u$ be a weak solution to the Cauchy-Dirichlet Problem (1.1). Let $q = \max\{2n/(n+2), \gamma, 1\} \in (1, p)$ with $\gamma = \gamma(n, \nu)$ from Lemma 4.1. Then, on a cylinder $Q_{\varrho p}^{(\nu)}(z_0) \subseteq \mathbb{R}^n \times (-T,T)$ with $\text{dist}(B_\varrho(x_0), \partial \Omega) = 0$ satisfying the sub-intrinsic coupling (5.1) and the super-intrinsic coupling (5.2), there exists a constant $c = c(K, p, n, N, L_1, L_2, \nu, \varrho_0) > 0$ such that
\[
\frac{1}{|Q^e_{\delta}|} \int_{Q^e_{\delta} \cap \Omega} |Du|^p \, dz \leq c \left( \int_{Q^e_{\delta}} |\hat{u}|^q + |u_0|^q \, dz \right)^{\frac{p}{q}} + c \int_{Q^e_{\delta}} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz.
\]

**Proof.** Assume \( \rho \leq r < s \leq 2\rho \). The center point \( z_0 \) will be omitted throughout this proof. First note that the energy estimate from Lemma 3.1 thus yields

\[
\sup_{t \in K^{(\mu)} \cap (0, T)} \int_{B_r} \frac{|\hat{u}^s(t) - u_0|^2}{\mu^{p-2} r^p} \, dx + \frac{1}{|Q^e_{\delta}|} \int_{Q^e_{\delta} \cap \Omega_T} |Du|^p \, dz 
\leq c \frac{1}{|Q^s_{\delta}|} \int_{Q^s_{\delta} \cap \Omega_T} \frac{|\hat{u}^s - u_0|^2}{\mu^{p-2} (s-r)^p} \, dz + c \int_{Q^s_{\delta}} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz =: I + II + III.
\]

To estimate the first and second term we define \( \mathcal{R}_{r,s} := s/(s-r) \). By using \( (s-r)^p \leq s^p - r^p \), this leads to

\[
I \leq \frac{c \mathcal{R}_{r,s}^p}{|Q^s_{\delta}|} \int_{Q^s_{\delta} \cap \Omega_T} \frac{|\hat{u}^s - u_0|^2}{\mu^{p-2} s^p} \, dz
\]

and by Lemma 4.3 to

\[
II \leq c \mathcal{R}_{r,s}^p \int_{Q^s_{\delta} \cap \Omega_T} \frac{|\hat{u} - u_0|^p}{s^p} \, dz
\]

\[
\leq c \mathcal{R}_{r,s}^p \left[ \delta \int_{Q^s_{\delta}} |\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz + \frac{c}{\delta^{2\frac{p-2}{p}}} \int_{Q^s_{\delta}} \frac{|\hat{u}^s - u_0|^2}{\mu^{p-2} s^p} \, dz \right]
\]

for any \( \delta \in (0, 1] \). Moreover, we use Lemma 4.1 for \( \hat{u} - u_0 \) and \( u_0 \), respectively, with \( \vartheta = p \)

\[
\int_{Q^s_{\delta}} |\hat{u} - u_0|^p + |u_0|^p \, dz \leq c \int_{Q^s_{\delta}} |\hat{u} - u_0|^p + |u_0|^p \, dz
\]

\[
\leq c \int_{Q^s_{\delta}} |\hat{u} - Du_0|^p + |Du_0|^p \, dz
\]

\[
\leq c \int_{Q^s_{\delta}} |\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]

and the super-intrinsic coupling (5.2) to show \( \mu^p \leq c(K, n, p) \). Altogether by applying Lemma 4.2 this leads to

\[
I + II \leq \delta c \mathcal{R}_{r,s}^p \int_{Q^s_{\delta}} |\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz + \left( 1 + \delta^{\frac{p-2}{p}} \right) c \mathcal{R}_{r,s}^p \int_{Q^s_{\delta}} \frac{|\hat{u}^s - u_0|^2}{\mu^{p-2} s^p} \, dz
\]

\[
\leq \delta c \mathcal{R}_{r,s}^p \int_{Q^s_{\delta}} |\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
Lemma 2.10 to get
\[ \|u^\pm(t) - u_0^\pm\|_{L^{p-2}_s} \leq c \varepsilon \sup_{s \in \Lambda_s^{(N)}} \int_{B_s} \frac{|\hat{\nabla}^\pm(t) - \hat{\nabla}^\pm|^2}{\mu^{p-2}s^p} dz \]

for any $\delta, \varepsilon \in (0, 1]$. Now we choose $\delta$ and $\varepsilon$ small enough, i.e.
\[ \delta = \frac{1}{2} c R_{r,s}, \quad \varepsilon = \frac{1}{2} \left( 1 + \delta^{\frac{p-2}{q}} c R_{r,s} \right)^{-1} \]
and end up with
\[
\sup_{t \in \Lambda^{(N)}(0,T)} \frac{1}{|B_r|} \int_{B_r \cap \Omega} \frac{|\hat{\nabla}^\pm(t) - u_0^\pm|^2}{\mu^{p-2}r^{p-2}} dx + \frac{1}{|Q_s^{(N)}| \cap \Omega_T} \int_{Q_s^{(N)} \cap \Omega_T} |Du|^p dz \leq \frac{1}{2} \sup_{t \in \Lambda^{(N)}(0,T)} \frac{1}{|B_r|} \int_{B_r \cap \Omega} \frac{|\hat{\nabla}^\pm(t) - u_0^\pm|^2}{\mu^{p-2}r^{p-2}} dx + \frac{1}{|Q_s^{(N)}| \cap \Omega_T} \int_{Q_s^{(N)} \cap \Omega_T} |Du|^p dz \]
\[
c \left( \frac{1}{(s-r)^\alpha} \left( \frac{|\hat{\nabla}^\pm|}{\mu^{p-2}} + |Du| |Du_0|^{q} \right) \right)^{\frac{p}{q}} + c \sup_{Q_{s}^{(N)} \cap \Omega_T} |Du_0|^p + |F|^p \chi_{\Omega_T} d z \]
with $\alpha = \left( p + \frac{(2-p)p}{2} \right) \left( 1 + \frac{2-q}{q} \right)$. This allows to apply the iteration lemma Lemma 2.10 to get
\[
\sup_{t \in \Lambda^{(N)}(0,T)} \frac{1}{|B_r|} \int_{B_r \cap \Omega} \frac{|\hat{\nabla}^\pm(t) - u_0^\pm|^2}{\mu^{p-2}r^{p-2}} dx + \frac{1}{|Q_s^{(N)}| \cap \Omega_T} \int_{Q_s^{(N)} \cap \Omega_T} |Du|^p dz \leq c \left( \sup_{Q_{s}^{(N)} \cap \Omega_T} |Du|^p + |Du_0|^{q} + |F|^p \chi_{\Omega_T} d z \right) \]
and finish the proof. \[\square\]

### 5.2. Initial boundary

In this case, we consider a pair of cylinders $Q_{\rho}^{(N)}(z_0) \subset Q_{\varrho}^{(N)}(z_0) \subseteq \mathbb{R}^{n+1}$ with $\varrho, \mu > 0$. For some $K \geq 1$, we impose the sub-intrinsic condition
\[
\sup_{Q_{\varrho}^{(N)}(z_0)} \left( \frac{1}{(2\varrho)^p} \right) d z \leq K \mu^p \quad (5.3)
\]
and the super-intrinsic condition
\[
\mu^p \leq K \inf_{Q_{\varrho}^{(N)}(z_0)} \left( \frac{1}{(2\varrho)^p} \right) d z \quad \text{or} \quad \mu^p \leq K. \quad (5.4)
\]

As we are inspecting the initial boundary case, we replace $Q_{\varrho}^{(N)}(z_0) \cap \Omega_T$ with $Q_{\varrho,\mu}(z_0)$ once again.
Lemma 5.2. Let $\max\{1, \frac{2n}{n+2}\} < p \leq 2$, $q := \max\{\frac{2n}{n+2}, 1\} \in (1, p)$ and $u$ be a weak solution to the Cauchy-Dirichlet Problem (1.1). Then, on a cylinder $Q_{2\varrho}(z_0) \subseteq \Omega \times (-T, T)$ with $z_0 \in \Omega_T$ satisfying the intrinsic coupling (5.3) and (5.4), there exists a constant $c = c(n, p, L_1, L_2, K) > 0$ such that

$$
\frac{1}{|Q_{\varrho}^{(\mu)}(z_0)|} \iint_{Q_{\varrho}^{(\mu)}(z_0)} |Du|^p \, dz \leq c \left[ \iint_{Q_{2\varrho}^{(\mu)}(z_0)} |D\hat{u}|^q + |Du_0|^q \, dz \right]^{\frac{p}{q}} + c \iint_{Q_{2\varrho}^{(\mu)}(z_0)} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz.
$$

Proof. Assume $\varrho \leq r < s \leq 2\varrho$. The center point $z_0$ will be omitted throughout this proof. By inserting mean values as the vector $a$ in the energy estimate from Lemma 3.2, we get

$$
sup_{t \in \Lambda^{(\mu)}_r} \int_{B_r} \frac{|\hat{u}^{\frac{p}{s}}(t) - [(\hat{u})^{(s)}(\mu)]^{\frac{p}{s}}|^2}{\mu^{p-2} s^p} \, dx + \frac{1}{|Q^{(\mu)}_r|} \iint_{Q_r^{(\mu)}} |Du|^p \, dz \leq c \int_{Q^{(\mu)}_{s+r}} \left[ \frac{|u - (\hat{u})^{(s)}(\mu)|}{s - r} \right] \, dz \leq c \iint_{Q^{(\mu)}_r} \left[ \frac{|\hat{u}^{\frac{p}{s}}(t) - [(\hat{u})^{(s)}(\mu)]^{\frac{p}{s}}|^2}{\mu^{p-2}(s^p - r^p)} \right] \, dz + c \frac{1}{|Q^{(\mu)}_s|} \iint_{Q^{(\mu)}_{s+r}} |F|^p \, dz
$$

As in the lateral case, set $R_{r,s} := s/(s - r)$. For the first term, we start by applying the quasi-minimality of the mean value integral as in Lemma 2.9 with $\alpha = 1$. Note that we can control the quotient of the size of the occuring cylinders by a power of 2, since $|Q_r^{(\mu)}| \geq \frac{1}{2} |Q_r^{(\mu)}|$. Further, since the modified scaling conditions (5.3) and (5.4) imply the sub- and super-intrinsic properties (4.4) and (4.5) for every $s \in [\varrho, 2\varrho]$ we can use Lemma 4.4. Lastly, we apply Young’s inequality. This way, we obtain

$$
I = R_{r,s}^p \frac{1}{|Q^{(\mu)}_s|} \iint_{Q^{(\mu)}_{s+r}} \frac{|u - (\hat{u})^{(s)}(\mu)|}{s^p} \, dz \leq c R_{r,s}^p \iint_{Q^{(\mu)}_s} \frac{|\hat{u} - (\hat{u})^{(s)}(\mu)|}{s^p} \, dz \leq c R_{r,s}^p \iint_{Q^{(\mu)}_s} \left[ \frac{|\hat{u}^{\frac{p}{s}} - [(\hat{u})^{(s)}(\mu)]^{\frac{p}{s}}|^2}{\mu^{p-2}s^p} \right] \, dz \leq c R_{r,s}^p \delta \iint_{Q^{(\mu)}_s} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
$$

$$
+ \frac{1}{\delta^{\frac{2-p}{p}}} R_{r,s}^p \iint_{Q^{(\mu)}_s} \left| \frac{|\hat{u}^{\frac{p}{s}} - [(\hat{u})^{(s)}(\mu)]^{\frac{p}{s}}|^2}{\mu^{p-2}s^p} \right| \, dz.
$$
for every $\delta \in (0, 1]$. For the second term, Lemma 2.9 and $(s - r)^p \leq s^p - r^p$ implies

$$\II \leq c\mathcal{R}^p_{r,s} \ii_{Q^{(s)}_{\delta}} \frac{\hat{u}^{\frac{p}{2}} - [(\hat{u})^{(s)}]^{\frac{p}{2}}}{\mu^{p-2}s^{p}} \, dz.$$ 

Now, by adding these last two inequalities and applying the parabolic Sobolev-Poincaré type inequality from Lemma 4.6 with $\varepsilon = \delta^{\frac{1}{2}}$, it follows that

$$\I + \II \leq c\mathcal{R}^p_{r,s} \delta \ii_{Q^{(s)}_{\delta}} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz$$

$$+ c\mathcal{R}^p_{r,s} \frac{1}{\delta^{2-p}} \ii_{Q^{(s)}_{\delta}} \frac{\hat{u}^{\frac{p}{2}} - [(\hat{u})^{(s)}]^{\frac{p}{2}}}{\mu^{p-2}s^{p}} \, dz$$

$$\leq \mathcal{R}^p_{r,s} \delta \ii_{Q^{(s)}_{\delta}} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz$$

$$+ c\delta \mathcal{R}^p_{r,s} \sup_{t \in \Lambda_{r,s}(t_0)} \ii_{B_s(x_0)} \frac{\hat{u}^{\frac{p}{2}}(t) - [(\hat{u})^{(s)}]^{\frac{p}{2}}}{\mu^{p-2}s^{p}} \, dz$$

$$+ \mathcal{R}^p_{r,s} \delta^{1 - \frac{2}{p+1}} \left[ \ii_{Q^{(s)}_{\delta}} |D\hat{u}|^q + |Du_0|^q \, dz \right]^{\frac{1}{q}} + \frac{1}{|Q^{(s)}_{\delta}|} \ii_{Q^{(s)}_{\delta}} |F|^p \, dz$$

for the exponent $q = \max\{\frac{2p}{p+2}, 1\}$. Note that for $t \leq 0$ by definition $\hat{u} = u_0$ and thus the integral over the part of the cylinder with $t \leq 0$ can be shifted from the $|D\hat{u}|^p$-term, creating an integral with $|Du_0|^p$. Putting these estimates together and choosing sufficiently small $\delta > 0$, i.e. $\delta = \frac{1}{4\varepsilon} \mathcal{R}^{-p}_{r,s}$ we end up with

$$\sup_{t \in \Lambda_{r,s}} \ii_{B_s} \frac{\hat{u}^{\frac{p}{2}}(t) - [(\hat{u})^{(s)}]^{\frac{p}{2}}}{\mu^{p-2}s^{p}} \, dx + \frac{1}{|Q^{(s)}_{\delta}|} \ii_{Q^{(s)}_{\delta}} |Du|^p \, dz$$

$$\leq \frac{1}{2} \left( \sup_{t \in \Lambda_{r,s}} \ii_{B_s} \frac{\hat{u}^{\frac{p}{2}}(t) - [(\hat{u})^{(s)}]^{\frac{p}{2}}}{\mu^{p-2}s^{p}} \, dx + \frac{1}{|Q^{(s)}_{\delta}|} \ii_{Q^{(s)}_{\delta}} |Du|^p \, dz \right)$$

$$+ c\mathcal{R}^p_{r,s} \sup_{t \in \Lambda_{r,s}} \ii_{Q^{(s)}_{\delta}} |Du_0|^p \, dz + \frac{1}{(s - r)^{\frac{4}{q(p-1)}}} \left( |Q^{(s)}_{\delta}| \ii_{Q^{(s)}_{\delta}} |F|^p \, dz \right).$$

Therefore we are able to apply the Iteration Lemma 2.10 to absorb the supremum term as well as the term involving $|Du|^p$ into the left hand side. As a result, we obtain
respectively, and hence we either have
\(\lambda \rightarrow \infty\) since the integrals tend to zero while the right hand side tends to infinity as
\(Q\).

6. Proof of higher integrability

We consider \(Q_{SR} := Q_{SR,R}^+(\tilde{z}_0) \subset \mathbb{R}^n \times (-T, T)\) with \(R \in (0, 1]\) and \(\tilde{z}_0 \in \Omega_T \cup \partial \Omega_T\), where we omit the center, since it will be fixed during this chapter. Let
\[
\lambda \geq \lambda_0 \geq 1 + \left[ \iint_{Q_{SR}} 2^p |\hat{u} - u_0|^p + |u_0|^p \left( \frac{R}{8R} \right)^p \, dz \right]^{1/p} \tag{6.1}
\]
and for \(z_0 = (x_0, t_0) \in Q_{2R}\) we define \(d_0 := \frac{1}{2} \text{dist}(x_0, \partial \Omega)\). Moreover we recall the definition of \(Q_{\rho}^{(\mu)}(z_0)\)
\[
Q_{\rho}^{(\mu)}(z_0) := B_{\rho}(x_0) \times (t_0 - \mu^{-2} \rho, t_0 + \mu^{-2} \rho)
\]
and remark that \(Q_{\rho}^{(\mu)}(z_0) \subset Q_{\rho}^{(\kappa)}(z_0)\) whenever \(\kappa \leq \mu\).

6.1. Construction of a non-uniform system of cylinder

Following the approach in [4] let \(z_0 \in Q_{2R} \cap \Omega_T\) and define for \(\rho \in (0, R]\)
\[
\tilde{\mu}_{z_0; \rho}^{(\lambda)} := \begin{cases}
\inf \left\{ \mu \in [1, \infty): \frac{1}{|Q_{\rho}|} \iint_{Q_{\rho}^{(\mu)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz \leq \mu^{2p-2} \lambda^p \right\}, & \text{if } \rho < d_0 \\
\inf \left\{ \mu \in [1, \infty): \frac{1}{|Q_{\rho}|} \iint_{Q_{\rho}^{(\mu)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{\rho^p} \, dz \leq \mu^{2p-2} \lambda^p \right\}, & \text{if } \rho \geq d_0
\end{cases}
\]
where the set for which the condition of the infimum is satisfied is not empty, since the integrals tend to zero while the right hand side tends to infinity as \(\mu \rightarrow \infty\). For better readability we will write \(\tilde{\mu}_\rho\) instead of \(\tilde{\mu}_z^{(\lambda)}\) for fixed \(z_0\) and \(\lambda\). Moreover, by the definition of \(Q_{\rho}^{(\mu)}(z_0)\) the estimate is equivalent to
\[
\iint_{Q_{\rho}^{(\mu)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz \leq \mu^p \lambda^p \quad \text{and} \quad \iint_{Q_{\rho}^{(\mu)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{\rho^p} \, dz \leq \mu^p \lambda^p
\]
respectively, and hence we either have
\[
\tilde{\mu}_\rho = 1 \quad \text{and} \quad \begin{cases}
\iint_{Q_{\rho}^{(\tilde{\mu}_\rho)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz \leq \tilde{\mu}_\rho^p \lambda^p = \lambda^p, & \text{if } \rho < d_0, \\
\iint_{Q_{\rho}^{(\tilde{\mu}_\rho)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{\rho^p} \, dz \leq \tilde{\mu}_\rho^p \lambda^p = \lambda^p, & \text{if } \rho \geq d_0.
\end{cases}
\]
or

\[\tilde{\mu}_p > 1 \quad \text{and} \quad \left\{ \begin{array}{ll}
\iint_{Q_\varepsilon^{(\bar{\mu}_\varepsilon)^-}(z_0)} |\hat{u}|^p \frac{1}{\rho^p} \, dz = \bar{\mu}_\varepsilon^p \lambda^p, & \text{if } \rho < d_0, \\
2^p \iint_{Q_\varepsilon^{(\bar{\mu}_\varepsilon)^+}(z_0)} |\hat{u} - u_0|^p + |u_0|^p \, dz = \bar{\mu}_\varepsilon^p \lambda^p, & \text{if } \rho \geq d_0.
\end{array} \right. \] (6.2)

For \( \rho = R \) this means either \( \tilde{\mu}_R = 1 \) or with (6.1)

\[\frac{-2p^2}{\mu_R^{2p-2}} = 2^p \frac{1}{\lambda^p |Q_R|} \iint_{Q_R^{(\bar{\mu}_R)^+}(z_0)} \frac{|\hat{u} - u_0|^p + |u_0|^p}{R^p} \, dz \leq 2^p \frac{1}{\lambda^p} \iint_{Q_R(z_0)} \frac{|\hat{u} - u_0|^p + |u_0|^p}{R^p} \, dz \leq \frac{8^{n+2p} \lambda_0^p}{\lambda^p} \leq 8^{n+2p}, \]

such that in any case

\[\tilde{\mu}_R \leq 8^{\frac{n+2p}{R-\rho}}. \] (6.3)

Our first aim is to show that the mapping \((0, R] \ni \rho \mapsto \tilde{\mu}_\rho \) is continuous.

**Lemma 6.1.** The mapping \( \rho \mapsto \tilde{\mu}_\rho \) is continuous on \((0, d_0)\) as well as \([d_0, R]\) and moreover

\[\lim_{\rho \searrow d_0} \tilde{\mu}_\rho \leq \lim_{\rho \searrow d_0} \tilde{\mu}_\rho.\]

**Proof.** Consider \( \varepsilon \in (0, R] \) and define \( \mu_\pm := \tilde{\mu}_\rho \pm \varepsilon. \) Note first that \( Q_\varepsilon^{(\mu_+)}(z_0) \) \( \subset Q_\varepsilon^{(\bar{\mu}_\varepsilon)^+}(z_0) \) \( \subset Q_\varepsilon^{(\mu_-)}(z_0) \) since \( \mu_- < \tilde{\mu}_\rho < \mu_+ \). Hence there exists \( \delta = \delta(\varepsilon, \rho) > 0 \) such that

\[\mu_- \leq \tilde{\mu}_r \leq \mu_+ \]

for every \( r \) in the same subinterval as \( \rho \) with \( |r - \rho| \leq \delta. \) This can be shown in the following way.

If \( \tilde{\mu}_\rho = 1 \), i.e. \( \mu_- \leq 1 \), then the left hand side of the former inequality holds true trivially. Otherwise, by the definition of \( \tilde{\mu}_\rho \) we have for \( r = \rho \)

\[\mu_-^{2p-2} \lambda^p < \tilde{\mu}_\rho^{2p-2} \lambda^p = \frac{1}{|Q_\varepsilon|} \iint_{Q_\varepsilon^{(\bar{\mu}_\rho)^-}(z_0)} |\hat{u}|^p \frac{1}{\rho^p} \, dz \leq \frac{1}{|Q_\varepsilon|} \iint_{Q_\varepsilon^{(\mu_-)}(z_0)} |\hat{u}|^p \frac{1}{\rho^p} \, dz, \quad \text{if } \rho < d_0 \]

and

\[\mu_-^{2p-2} \lambda^p < \tilde{\mu}_\rho^{2p-2} \lambda^p \leq \frac{1}{|Q_\varepsilon|} \iint_{Q_\varepsilon^{(\mu_-)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{\rho^p} \, dz \leq \frac{1}{|Q_\varepsilon|} \iint_{Q_\varepsilon^{(\mu_-)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{\rho^p} \, dz, \quad \text{if } \rho \geq d_0. \]
Note that both sides are continuous with respect to the radius and hence this implies
\[
\mu_{p-2}^2 \lambda^p < \begin{cases} 
\frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz, \\
\text{for every } r \in (0, d_0) \text{ with } |r - \rho| < \delta \text{ if } \rho < d_0, \\
\frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz, \\
\text{for every } r \in (d_0, R) \text{ with } |r - \rho| < \delta \text{ if } \rho \geq d_0.
\end{cases}
\]

In the same manner as above we have
\[
\mu_{p-2}^2 \lambda^p \gg \tilde{\mu}_{\rho}^2 \gg \frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz, \\
\text{if } \rho < d_0,
\]
\[
\frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz, \\
\text{if } \rho \geq d_0,
\]
which by the continuity of both sides implies
\[
\mu_{p+}^2 \lambda^p \gg \begin{cases} 
\frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} \frac{|\hat{u}|^p}{\rho^p} \, dz, \\
\text{for every } r \in (0, d_0) \text{ with } |r - \rho| < \delta \text{ if } \rho < d_0, \\
\frac{1}{|Q|} \iint_{Q^{(\mu_\rho)}(z_0)} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz, \\
\text{for every } r \in (d_0, R) \text{ with } |r - \rho| < \delta \text{ if } \rho \geq d_0.
\end{cases}
\]

Altogether this shows the above claim, since otherwise we have a contradiction to the definition of \( \tilde{\mu}_\rho \). Finally the limit value observation is a direct consequence of the definition of \( \tilde{\mu}_\rho \) and \( |\hat{u}|^p \leq 2^p (|\hat{u} - u_0|^p + |u_0|^p) \). \( \square \)

Unfortunately the mapping \( \rho \mapsto \tilde{\mu}_\rho \) might not be monotone or continuous at \( \rho = d_0 \). Therefore we define
\[
\mu_\rho = \mu_{\rho, R}^{(\lambda)} := \max_{r \in [\rho, R]} \tilde{\mu}_{\rho_0, R}^{(\lambda)},
\]
where we again omit \( z_0 \) and \( \lambda \) if they are fixed. By construction the mapping \( (0, R] \ni \rho \mapsto \mu_\rho \) is continuous and monotonically decreasing. Moreover cylinders with scaling parameter \( \mu_\rho \) are sub-intrinsic, which is the topic of the next Lemma.

**Lemma 6.2.** Cylinders \( Q_s^{(\mu_\rho)}(z_0) \) are \( \mu \)-subintrinsic with constant \( K = 1 \) in the sense that for every \( 0 < \rho \leq s \leq R \)
\[
\iint_{Q_s^{(\mu_\rho)}(z_0)} \frac{|\hat{u}|^p}{s^p} \, dz \leq \mu_\rho^p \lambda^p, \quad \text{if } \rho < d_0 \quad \text{and}
\]
\[
\iint_{Q_s^{(\mu_\rho)}(z_0)} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{s^p} \, dz \leq \mu_\rho^p \lambda^p, \quad \text{if } \rho \geq d_0.
\]

**Proof.** By the definition of \( \mu_\rho \) we have \( \tilde{\mu}_s \leq \mu_s \leq \mu_\rho \), hence \( Q_s^{(\mu_\rho)}(z_0) \subset Q_s^{(\tilde{\mu}_s)}(z_0) \) and therefore
\[ \iiint_{Q_s^{(\varrho)}} |\hat{u}|^p \, dz \leq \left( \frac{\mu_\varrho}{\mu_s} \right)^{2-p} \iiint_{Q_s^{(\tilde{\varrho})}} |\hat{u}|^p \, dz \]
\[ \leq \left( \frac{\mu_\varrho}{\mu_s} \right)^{2-p} \bar{\mu}_s \lambda^p = \mu_\varrho^{2-p} \bar{\mu}_s^{p-2} \lambda^p \]
\[ \leq \mu_\varrho^p \lambda^p \]

in the case \( \varrho < d_0 \) as well as
\[ \iiint_{Q_s^{(\varrho)}} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz \leq \left( \frac{\mu_\varrho}{\mu_s} \right)^{2-p} \iiint_{Q_s^{(\tilde{\varrho})}} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz \]
\[ \leq \left( \frac{\mu_\varrho}{\mu_s} \right)^{2-p} \bar{\mu}_s \lambda^p = \mu_\varrho^{2-p} \bar{\mu}_s^{p-2} \lambda^p \leq \mu_\varrho^p \lambda^p \]

for \( \varrho \geq d_0 \).

Next we define
\[ \tilde{\varrho} := \begin{cases} R, & \text{if } \mu_\varrho = 1 \\ \inf \{ s \in [\varrho, R] : \mu_s = \bar{\mu}_s \}, & \text{if } \mu_\varrho > 1 . \end{cases} \quad (6.4) \]

It is easy to see that \( \mu_s = \bar{\mu}_\varrho \) for every \( s \in [\varrho, \tilde{\varrho}] \), especially \( \mu_\varrho = \tilde{\mu}_\varrho \).

Our next aim is to show that for every \( s \in (\varrho, R) \) we can estimate
\[ \mu_\varrho \leq \left( \frac{s}{\varrho} \right)^{n+2p} \frac{\mu_s}{s^{p-2}}. \quad (6.5) \]

Therefore we first consider the cases \( \mu_\varrho = 1 \) as well as \( \mu_\varrho > 1 \), \( s \in (\varrho, \tilde{\varrho}) \). Then \( \mu_s = 1 = \mu_\varrho \) or \( \mu_s = \bar{\mu}_\varrho = \mu_\varrho \) respectively and the claim follows directly. The remaining case \( (\mu_\varrho > 1, s \in (\varrho, R]) \) can be handled by the definition of \( \tilde{\mu}_\varrho \), the monotonicity of \( s \mapsto \mu_s \) and Lemma 6.2

\[ \mu_\varrho^{2p-2} = \bar{\mu}_\varrho^{2p-2} = \frac{1}{\lambda^p |Q_{\varrho}|} \iiint_{Q_{\varrho}^{(\varrho)}} |\hat{u}|^p \, dz = \frac{1}{\lambda^p |Q_{\varrho}|} \iiint_{Q_{\varrho}^{(\tilde{\varrho})}} |\hat{u}|^p \, dz \]
\[ = \left( \frac{s}{\varrho} \right)^{n+2p} \frac{1}{\lambda^p |Q_{\tilde{\varrho}}|} \iiint_{Q_{\varrho}^{(\tilde{\varrho})}} \frac{|\hat{u}|^p}{s^p} \, dz \]
\[ \leq \left( \frac{s}{\varrho} \right)^{n+2p} \frac{1}{\lambda^p |Q_{\tilde{\varrho}}|} \iiint_{Q_{\varrho}^{(\tilde{\varrho})}} \frac{|\hat{u}|^p}{s^p} \, dz \leq \left( \frac{s}{\varrho} \right)^{n+2p} \mu_s^{2p-2}, \quad \text{if } \varrho < d_0, \]

and

\[ \mu_\varrho^{2p-2} = \bar{\mu}_\varrho^{2p-2} = \frac{1}{\lambda^p |Q_{\varrho}|} \iiint_{Q_{\varrho}^{(\varrho)}} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz \]
\[ = \frac{1}{\lambda^p |Q_{\varrho}|} \iiint_{Q_{\varrho}^{(\tilde{\varrho})}} 2^p |\hat{u} - u_0|^p + |u_0|^p \, dz \]
\[ = \left( \frac{s}{\varrho} \right)^{n+2p} \frac{1}{\lambda^p |Q_{\tilde{\varrho}}|} \iiint_{Q_{\varrho}^{(\tilde{\varrho})}} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{s^p} \, dz \]
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Global higher integrability for a doubly nonlinear

\[ \left( \frac{s}{\varrho} \right)^{n+2p} \frac{1}{\lambda^p |Q_s|} \int_{Q_s^{(\mu_s)}} 2^p |\dddot{u} - u_0|^p + |u_0|^p \, dz \]

\[ \leq \left( \frac{s}{\varrho} \right)^{n+2p} \mu_s^{2p-2}, \quad \text{if } \varrho \geq d_0, \]

where we use \( s \geq \tilde{\varrho} \) and \( \mu_s \leq \mu_{\tilde{\varrho}} \) to enlarge the area of integration in the penultimate estimate. Therefore (6.5) holds true in any mentioned case. For \( s = R \) we can use the estimate from (6.3) as well as \( \mu_R = \tilde{\mu}_R \) to get an upper bound for \( \mu_\varrho \) by

\[ \mu_\varrho \leq \left( \frac{R}{\varrho} \right)^{n+2p} \mu_R \leq \left( \frac{8R}{\varrho} \right) \left( \frac{n+2p}{2p-2} \right). \]  

(6.6)

In what follows we consider a family of cylinders \( Q_\varrho^{(\mu_{\tilde{\varrho}})}(z_0) \) with \( \varrho \in (0, R] \), \( z_0 \in Q_{2R} \), which are nested in the sense that

\[ Q_r^{(\mu_{\tilde{\varrho}})}(z_0) \subset Q_s^{(\mu_{\tilde{\varrho}})}(z_0) \quad \text{for every } 0 < r < s \leq R. \]  

(6.7)

6.2. Covering property

We want to show in this subsection that the cylinders constructed above fulfill a Vitali covering property. More precisely there exists a constant \( \hat{c} = \hat{c}(n, p) \geq 80 \) such that for any \( \lambda \geq \lambda_0 \) and every collection \( \mathcal{F} \) of cylinders \( Q_{8r_1}^{(\mu_{\tilde{\varrho}})}(z) \) with \( r \in (0, \frac{R}{\hat{c}}] \) there exists a countable subcollection \( \mathcal{G} \subset \mathcal{F} \) of disjoint cylinders with

\[ \bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{\hat{Q} \in \mathcal{G}} \hat{Q}, \]  

(6.8)

where \( \hat{Q} := Q_{\frac{8r_1}{\hat{c}}}^{(\mu_{\tilde{\varrho}})}(z) \) denotes the \( \frac{1}{\hat{c}} \)-times enlarged cylinder \( Q_{8r_1}^{(\mu_{\tilde{\varrho}})}(z) \).

Following again the proof of [4, Lemma 7.1] we define for \( \hat{c} \) to be specified later on

\[ \mathcal{F}_j := \left\{ Q_{8r_1}^{(\mu_{\tilde{\varrho}})} \in \mathcal{F} : \frac{R}{2^j \hat{c}} < r \leq \frac{2R}{2^j \hat{c}} \right\}. \]

We now select \( \mathcal{G}_j \subset \mathcal{F}_j \) as follows:

- Let \( \mathcal{G}_1 \) be any maximal disjoint collection of cylinders in \( \mathcal{F}_1 \).
- When \( \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k-1} \) have been selected for some \( k \in \mathbb{N}_{\geq 2} \) then let \( \mathcal{G}_k \) be a maximal disjoint collection in \( \{ Q \in \mathcal{F}_k : Q \cap \hat{Q} = \emptyset \} \) for any \( \hat{Q} \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \).

Note that by the definition of \( \mathcal{F}_1 \) as well as the definition of the cylinders the Lebesgue measure of \( Q \in \mathcal{G}_1 \) is bounded from below and hence \( \mathcal{G}_1 \) contains finitely many cylinders. Therefore by setting \( \mathcal{G} := \bigcup_{j=1}^{\infty} \mathcal{G}_j \) we constructed a countable sub-collection of disjoint cylinders in \( \mathcal{F} \).

It remains to show that for every \( Q = Q_{8r_1}^{(\mu_{\tilde{\varrho}})}(z) \in \mathcal{F} \) there exists \( Q^* = Q_{8r_1}^{(\mu_{\tilde{\varrho}})}(z_*) \in \mathcal{G} \) such that \( Q \cap Q^* \neq \emptyset \) and \( Q \subset \mathcal{Q}^* \). For the first statement consider an arbitrary \( Q \in \mathcal{F} \), hence \( Q \in \mathcal{F}_j \) for some \( j \in \mathbb{N} \). If in addition
Q ∈ G one can choose Q* = Q. Otherwise Q ̸∈ G and the maximality of G ensures the existence of Q* ∈ ∪j=1 Gj with Q ∩ Q* ̸= ∅. Furthermore we have r ≤ \frac{2R}{2j} ≤ 2r*, in both cases.

Next we will present an estimate for μ\(z, r^\ast\) in terms of μ\(z, r\) which is the main difficulty in showing Q ⊂ \(\hat{Q}\).

**Lemma 6.3.** Let Q ∈ F and Q* ∈ G as above. Then μ\(z, r^\ast\) is bounded from above in terms of μ\(z, r\) by μ\(z, r^\ast\) ≤ (8η)\(\frac{n+2p}{2p-2}\)μ\(z, r\) with η = 25.

**Proof.** To prove the claim, we have to distinguish different cases. Consider first μ\(z, r^\ast\) = 1, then by the definition of μ and \(\tilde{\mu}\) it is obvious that μ\(z, r^\ast\) = 1 ≤ μ\(z, r\). For the other case, i.e. μ\(z, r^\ast\) > 1, let \(\tilde{r}^\ast\) be as in (6.4). We know by (6.2) that

\[
(\mu\(z, r^\ast\))^2 - (\mu\(z, r^\ast\))^2 = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{|\hat{u}|^p}{r^\ast} d\mathbf{z} = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{|\hat{u}|^p}{r^\ast} d\mathbf{z}, \quad \text{if } \varrho < d_0,
\]

and

\[
(\mu\(z, r^\ast\))^2 - (\mu\(z, r^\ast\))^2 = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{r^\ast} d\mathbf{z} = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{r^\ast} d\mathbf{z}, \quad \text{if } \varrho \geq d_0.
\]

For \(\tilde{r}^\ast > \frac{R}{\eta}\) we can use this inequality to estimate

\[
(\mu\(z, r^\ast\))^2 = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{|\hat{u}|^p}{r^\ast} d\mathbf{z} = \left(\frac{8R}{\tilde{r}^\ast}\right)^p \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{|\hat{u}|^p}{r^\ast} d\mathbf{z} \leq \left(\frac{8R}{\tilde{r}^\ast}\right)^p \frac{|Q_{8R}|}{|Q_{r^\ast}|} \left(\frac{8R}{\tilde{r}^\ast}\right) \frac{|Q_{8R}|}{|Q_{r^\ast}|} \left(8\eta\right)^{n+2p},
\]

if \(\varrho < d_0\) as well as for \(\varrho \geq d_0\)

\[
(\mu\(z, r^\ast\))^2 = \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{r^\ast} d\mathbf{z} = \left(\frac{8R}{\tilde{r}^\ast}\right)^p \frac{1}{\lambda^p |Q_{r^\ast}|} \iiint_{Q_{r^\ast}} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{r^\ast} d\mathbf{z}.
\]
\[
\left( \frac{8R}{\tilde{r}_*} \right)^p \frac{|Q_{8R}|}{|Q_{8R}|} \lambda^p \int_{Q_{8R}} 2^p \frac{|\hat{u} - u_0|^p + |u_0|^p}{(8R)^p} \, dz \\
\leq \left( \frac{8R}{\tilde{r}_*} \right)^p \frac{|Q_{8R}|}{|Q_{8R}|} = \left( \frac{8R}{\tilde{r}_*} \right)^{n+2p} \leq (8\eta)^{n+2p},
\]
which yields the claim. If otherwise \( \tilde{r}_* \leq \frac{R}{\eta} \) we assume without loss of generality \( \mu_{z,\eta;r}^{(\lambda)} \leq \mu_{z,\eta;r_*}^{(\lambda)} \), since otherwise the claim follows immediately. Then the monotonicity of the mapping \( \eta \mapsto \mu_{z,\eta;r}^{(\lambda)} \) as well as \( r \leq 2r_* \leq 2\tilde{r}_* \leq \eta\tilde{r}_* \) imply
\[
\mu_{z,\eta;r}^{(\lambda)} \leq \mu_{z,\eta;r_*}^{(\lambda)} \leq \mu_{z,\eta;r_*}^{(\lambda)}.
\]
Since \( \tilde{r}_* \geq r_* \) and \( |x_* - x| \leq 8r + 8r_* \leq 24r_* \), we know that \( B_{\tilde{r}_*}(x_* \in B_{\eta \tilde{r}_*}(x) \). Moreover as \( p < 2 \)
\[
(\mu_{z,\eta;r_*}^{(\lambda)})^{p-2} \tilde{r}_*^p + |t_* - t| \leq (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2} \tilde{r}_*^p + (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p + (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p \leq (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(\eta \tilde{r}_*^p)
\]
holds true, therefore \( \Lambda_{\frac{(\mu_{z,\eta;r_*}^{(\lambda)})}{\eta^p}} \subset \Lambda_{\eta \tilde{r}_*}^{(\mu_{z,\eta;r_*}^{(\lambda)})} \) and altogether \( Q_{\frac{(\mu_{z,\eta;r_*}^{(\lambda)})}{\eta^p}}(z_*) \subset Q_{\eta \tilde{r}_*}^{(\mu_{z,\eta;r_*}^{(\lambda)})} (z) \). We end up with
\[
(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2} = \frac{1}{\lambda^p |Q_{\tilde{r}_*}|} \int_{Q_{\frac{(\mu_{z,\eta;r_*}^{(\lambda)})}{\eta^p}}(z_*)} \frac{|\hat{u}|^p}{\tilde{r}_*^p} \, dz \leq \frac{\eta^p}{\lambda^p |Q_{\tilde{r}_*}|} \int_{Q_{\eta \tilde{r}_*}^{(\mu_{z,\eta;r_*}^{(\lambda)})}} \frac{|\hat{u}|^p}{(\eta \tilde{r}_*^p)} \, dz \leq \eta^{n+2p}(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2} \leq \eta^{n+2p}(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2},
\]
or rather
\[
(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2} = \frac{1}{\lambda^p |Q_{\tilde{r}_*}|} \int_{Q_{\frac{(\mu_{z,\eta;r_*}^{(\lambda)})}{\eta^p}}(z_*)} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{\tilde{r}_*^p} \, dz \leq \frac{\eta^p}{\lambda^p |Q_{\tilde{r}_*}|} \int_{Q_{\eta \tilde{r}_*}^{(\mu_{z,\eta;r_*}^{(\lambda)})}} \frac{2^p |\hat{u} - u_0|^p + |u_0|^p}{(\eta \tilde{r}_*^p)} \, dz \leq \eta^{n+2p}(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2} \leq \eta^{n+2p}(\mu_{z,\eta;r_*}^{(\lambda)})^{2p-2},
\]
for the different cases of \( \eta \), where we used Lemma 6.2 in the penultimate line.

The previous result \( \mu_{z,\eta;r_*}^{(\lambda)} \leq (8\eta)^{\frac{n+2p}{p-2}}(\mu_{z,\eta;r_*}^{(\lambda)}) \) with \( \eta = 25 \) implies \( \Lambda_{\frac{(\mu_{z,\eta;r_*}^{(\lambda)})}{\eta^p}}(t) \subset \Lambda_{\tilde{c}}^{(\mu_{z,\eta;r_*}^{(\lambda)})}(t) \) for suitable large \( \tilde{c} = \tilde{c}(n, p) \) by
\[
(\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p + |t - t_*| \leq (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p + (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p + (\mu_{z,\eta;r_*}^{(\lambda)})^{p-2}(8r_*)^p
\]
\[ \leq \left[ 2^{p+1} (64\eta)^{2p(2-p)} \right] \left( \mu_\alpha^{\lambda}(\mathcal{R}) \right)^{p-2} (8r_*)^p \]

Moreover, if we choose \( \hat{c} \geq 40 \), we also have \( B_{8r}(x) \subset B_{16r+8r_*}(x_*) \subset B_{40r_*}(x_*) \subset B_{\hat{c}r_*}(x_*). \) This shows that \( Q = Q_{r_*}(\mu_\alpha^{\lambda})(z) \subset Q_{r_*}(\mu_\alpha^{\lambda})(z_*) = Q_* \) and finishes the proof of the Vitali type covering property of the constructed cylinders.

### 6.3. Stopping time argument

First of all we define \( \lambda_0 \) by

\[ \lambda_0 := 1 + \left[ \iint_{Q_{8R}} 2^p \left| \hat{u} - u_0 \right|^p + \left| u_0 \right|^p + \left| \nabla \hat{u} \right|^p + \left| Du_0 \right|^p + \left| F \right|^p \chi_{\Omega_T} \, dz \right]^{\frac{1}{p}}, \]

(6.9)

such that (6.1) is fulfilled. Furthermore we identify with \( E(r, \lambda) \) for \( \lambda \geq \lambda_0 \) and \( r \in (0, 2R] \) the superlevel set for \( Du \), i.e.

\[ E(r, \lambda) := \{ z \in Q_r \cap \Omega_T : z \text{ is Lebesgue point of } |Du| \text{ and } |Du|(z) > \lambda \}, \]

where the Lebesgue point has to be understood with respect to the cylinders constructed in Sect. 6.1. For radii \( R \leq R_1 < R_2 \leq 2R \) we consider concentric cylinders \( Q_R \subset Q_{R_1} \subset Q_{R_2} \subset Q_{2R} \) and fix \( z_0 \in E(R_1, \lambda) \). For \( s \in (0, R] \) we shorten the notation by \( \mu_s = \mu_s^{\lambda_0}; s \). Lebesgue’s differentiation theorem shows

\[ \liminf_{s \to 0} \iint_{Q_{s}^{\mu_s}(z_0)} \left| \nabla \hat{u} \right|^p + \left| Du_0 \right|^p + \left| F \right|^p \chi_{\Omega_T} \, dz \geq \left| Du \right|^p(z_0) > \lambda^p. \]

(6.10)

By \( \hat{c} = \hat{c}(n, p) \) we denote the constant from Sect. 6.2 and consider \( \lambda \) satisfying \( \lambda \geq B \lambda_0 \) with \( B := \left( \frac{8\hat{c}R}{R_2 - R_1} \right)^{2p-2} > 1 \). By (6.6) and the definition of \( \lambda_0 \) we have for every \( \frac{R_2 - R_1}{\hat{c}} \leq s \leq R \) and \( 1 \leq \alpha \leq 8 \)

\[ \iint_{Q_{s}^{\mu_s}(z_0)} \left| \nabla \hat{u} \right|^p + \left| Du_0 \right|^p + \left| F \right|^p \chi_{\Omega_T} \, dz \]

\[ \leq \frac{\left| Q_{sR} \right|}{\left| Q_{s}^{\mu_s} \right|} \iint_{Q_{sR}} \left| \nabla \hat{u} \right|^p + \left| Du_0 \right|^p + \left| F \right|^p \chi_{\Omega_T} \, dz \]

\[ \leq \frac{\left| Q_{sR} \right|}{\left| Q_{s}^{\mu_s} \right|} \lambda_0^p = \left( \frac{8\hat{c}R}{R_2 - R_1} \right)^{2p-2} \lambda_0^p \leq \left( \frac{8R}{s} \right)^{p(n+2)} \lambda_0^p \]

(6.11)

By (6.10) it is possible to find small \( 0 < s < \frac{R_2 - R_1}{\hat{c}} \) such that

\[ \iint_{Q_{s}^{\mu_s}(z_0)} \left| \nabla \hat{u} \right|^p + \left| Du_0 \right|^p + \left| F \right|^p \chi_{\Omega_T} \, dz > \lambda^p. \]

The continuity of the mapping \( \rho \mapsto \mu_s \), (6.11) with \( \alpha = 1 \) and the absolute continuity of the integral ensure the existence of a maximal radius \( 0 < \rho_{z_0} < \)
\[
\frac{R_2 - R_1}{c} \quad \text{with}
\]
\[
\iint_{Q_{\varrho_0}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz = \lambda^p, \quad (6.12)
\]
whereby the maximality of \( \varrho_0 \) guarantees that
\[
\iint_{Q_{\varrho_0}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz < \lambda^p, \quad \text{for every } \varrho_0 < s \leq R.
\]
(6.13)
The monotonicity of \( \varrho \mapsto \mu_\varrho \) together with (6.5) leads to
\[
\mu_s \leq \mu_{\varrho_0} \leq \left( \frac{s}{\varrho_0} \right)^{\frac{n+2p}{2p-2}} \mu_s
\]
for every \( \varrho_0 < s \leq R \) and hence
\[
\iint_{Q_{s}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
\leq \frac{|Q_{\varrho_0}^{(\mu_0 \varrho_0)}|}{|Q_{s}^{(\mu_0 \varrho_0)}|} \iint_{Q_{s}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
= \left( \frac{\mu_{\varrho_0}}{\mu_s} \right)^{2-p} \iint_{Q_{s}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
< \left( \frac{s}{\varrho_0} \right)^{\frac{n+2p}{2p-2}(2-p)} \lambda^p
\]
(6.14)
for every \( \varrho_0 < s \leq R \).

Finally, since \( R_1^p + (R_2 - R_1)^p \leq R_2^p \), we have \( Q_{\tilde{\varrho}_0}^{(\mu_{\tilde{\varrho}_0})}(z_0) \subset Q_{\varrho_0}^{(\mu_{\varrho_0})}(z_0) \subset Q_{R_2} \).

### 6.4. Reverse Hölder inequalities

Consider \( z_0 \in E(R_1, \lambda) \) with \( R_1 \) and \( \lambda \) as in Sect. 6.3. By the former construction we have \( 0 < \varrho_0 < \frac{R_2 - R_1}{c} \) and abbreviate as before \( \mu_{\varrho_0} := \mu_{\varrho_0}^{(\lambda)} \). Moreover, according to (6.4) we define \( \tilde{\varrho}_0 \in [\varrho_0, R] \) such that \( \mu_s = \mu_{\varrho_0} = \mu_{\tilde{\varrho}_0} \) for every \( s \in [\varrho_0, \tilde{\varrho}_0] \).

**Lateral case \( \varrho_0 \geq d_0 \):** In this case our aim is to apply Lemma 5.1 on the cylinder \( Q_{2\varrho_0}^{(\mu_0 \varrho_0)}(z_0) \). Therefore we use the (6.14) with \( s = 8\varrho_0 \leq \frac{R_2 - R_1}{5} < R \) and (6.12) to get
\[
\iint_{Q_{8\varrho_0}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \leq c(n, p) \lambda^p
\]
\[
= c(n, p) \iint_{Q_{2\varrho_0}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
\leq c(n, p) 2^{n+p} \iint_{Q_{2\varrho_0}^{(\mu_0 \varrho_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
(6.15)
for a constant $c(n,p) > 1$. The second inequality in Lemma 6.2 with $s = 8\varrho_{z_0}$ together with the second part of the former inequality shows

$$\frac{\iint_{Q_{2\varrho_{z_0}}}(\varrho_{z_0})}{} \left(\frac{|\hat{\mu} - u_0|^p + |u_0|^p}{(8\varrho_{z_0})^p} \right) \leq c(n,p)2^{n+p}\mu_{2\varrho_{z_0}}$$

and hence the hypothesis (5.1) is fulfilled for the cylinder $Q_{2\varrho_{z_0}}(z_0)$ with $K = c(n,p)2^{n+p}$.

Next, we assume $\varrho_{z_0} < R$ since otherwise $\mu_{4\varrho_{z_0}} = 1$ holds true, satisfying (5.2). We can use (6.2) and apply Lemma 4.1 with $\vartheta = p$ to the cylinder $Q_{4\varrho_{z_0}}(z_0)$. Afterwards we can either use the maximality of $\varrho_{z_0}$ in (6.13) with $s = 4\varrho_{z_0}$ if $4\varrho_{z_0} < R$, or (6.11) with $s = \varrho_{z_0} > 4\varrho_{z_0}$ if $\varrho_{z_0} < R$, and $c = 4$ to get

$$\mu_{\varrho_{z_0}} = \mu_{\varrho_{z_0}} \lambda = \left(\iint_{Q_{\varrho_{z_0}}(\varrho_{z_0})} \frac{|\hat{\mu} - u_0|^p + |u_0|^p}{\varrho_{z_0}^p} \right)^{\frac{1}{p}}$$

$$= 2 \left(\iint_{Q_{\varrho_{z_0}}(\varrho_{z_0})} \frac{|\hat{\mu} - u_0|^p + |u_0|^p}{\varrho_{z_0}^p} \right)^{\frac{1}{p}}$$

$$\leq c \left(\iint_{Q_{\varrho_{z_0}}(\varrho_{z_0})} \frac{|\hat{\mu} - u_0|^p + |u_0|^p}{(4\varrho_{z_0})^p} \right)^{\frac{1}{p}}$$

$$\leq c(n,p) \left(\iint_{Q_{4\varrho_{z_0}}(\varrho_{z_0})} |\hat{\mu} - u_0|^p + |u_0|^p \right)^{\frac{1}{p}}$$

$$\leq c\lambda$$

with constant $c = c(n,p,N,\nu,\varrho_{z_0})$. Note that for the application of (6.13) we have to change the cylinder to $Q_{4\varrho_{z_0}}(\varrho_{z_0})$ which is possible by (6.5). This verifies (5.2) with the cylinder $Q_{2\varrho_{z_0}}(\varrho_{z_0})$ with $K = c$ in the lateral case. Since moreover $2\varrho_{z_0} > 2d_0$, hence dist$(B_{2\varrho}(x_0), \partial \Omega) = 0$, we can apply Lemma 5.1 on $Q_{2\varrho_{z_0}}(\varrho_{z_0})$ and end up with

$$\frac{1}{|Q_{2\varrho_{z_0}}(\varrho_{z_0})|} \iint_{Q_{2\varrho_{z_0}}(\varrho_{z_0}) \cap \Omega_T} |D\hat{\mu}|^p \leq c \left(\iint_{Q_{\varrho_{z_0}}(\varrho_{z_0})} |\hat{\mu}|^q + |D\hat{u}|^q \right)^{\frac{p}{q}}$$

$$+ c \iint_{Q_{4\varrho_{z_0}}(\varrho_{z_0})} |D\hat{\mu}|^p + |F|^p \chi_{\Omega_T} \leq d_0.$$  

**Initial case** $\varrho_{z_0} < d_0$: Here, we first want to consider the case where also $\varrho_{z_0} < d_0$. Moreover we distinguish between the non-degenerate case $\varrho_{z_0} \leq 2\varrho_{z_0}$ and the degenerate case $2\varrho_{z_0} < \varrho_{z_0}$. In the former case our aim is to apply
Lemma 5.2 to the cylinder $Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)$. Since $2\tilde{\varrho}_{z_0} \leq 4\varrho_{z_0} \leq \frac{R_2 - R_1}{5} < R$ we can apply the first inequality in Lemma 6.2 with $s = 2\tilde{\varrho}_{z_0}$ together with (6.12) to get
\[
\int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} \frac{|\hat{u}|^p}{(2\tilde{\varrho}_{z_0})^p} \, dz \leq \mu_{\tilde{\varrho}_{z_0}}^p \lambda^p
\]
\[
= \mu_{\tilde{\varrho}_{z_0}}^p \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
\leq \mu_{\varrho_{z_0}}^p |Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)| \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
\leq \mu_{\varrho_{z_0}}^p c(n,p) \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz.
\]
This shows that (5.3) is fulfilled. Next, (6.14) with $2\tilde{\varrho}_{z_0} \leq R$ and $\tilde{\varrho}_{z_0} \leq 2\varrho_{z_0}$ leads to
\[
\int\int_{Q_{2\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz < c(n,p)\lambda^p.
\]
Combining this with (6.2) we achieve
\[
\mu_{\varrho_{z_0}}^p = \tilde{\mu}_{\varrho_{z_0}}^p = \frac{1}{\lambda^p} \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |\hat{u}|^p \, dz = \frac{1}{\lambda^p} \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |\hat{u}|^p \, dz
\]
\[
\leq \frac{c(n,p)}{\int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz}
\]
and therefore (5.4) is satisfied with $K = c(n,p) > 1$. Since in addition $B_{2\tilde{\varrho}_{z_0}} \subset \Omega$ we can apply Lemma 5.2 to the cylinder $Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)$ and end up with
\[
\frac{1}{|Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)|} \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |Du|^p \, dz \leq c(n,p) \frac{1}{|Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)|} \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |Du|^p \, dz
\]
\[
\leq c \left[ \int\int_{Q_{2\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^q + |Du_0|^q \, dz \right]^\frac{p}{q} + c \int\int_{Q_{2\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz
\]
\[
\leq c \left[ \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |D\hat{u}|^q + |Du_0|^q \, dz \right]^\frac{p}{q} + c \int\int_{Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz.
\]
For the second case, i.e. the degenerate one, we want to apply Lemma 5.2 to the cylinder $Q_{\tilde{E}_{x_0}}^{(\mu_{\varepsilon_0})}(z_0)$. By the first inequality in Lemma 6.2 with $s = \varrho_{z_0}$ together with (6.12) we have
\[
\iint_{Q^{(\mu z_0)}(z_0)} \frac{|\hat{u}|^p}{(2z_0)^p} \, dz \leq \mu_{\tilde{z}_0}^p \lambda^p
\]

\[
= \mu_{\tilde{z}_0}^p \iint_{Q^{(\mu z_0)}(z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz,
\]

showing that (5.3) is satisfied. To verify the super-intrinsic property in this case, note first that the cylinder \(Q^{(\mu z_0)}(z_0)\) is sub-intrinsic in the sense of Lemma 6.2. Hence we can adapt the proof of Lemma 4.5 with \(q = p\) to these cylinder by replacing the original sub-intrinsic condition (4.4), resulting in an inequality as seen in (4.6). Additionally, we use \(\mu_{\tilde{z}_0} = \mu_{\tilde{z}_0}\) and apply (6.13) with \(s = \tilde{z}_0\) to estimate

\[
\iint_{Q^{(\mu z_0)}(z_0)} \frac{|\hat{u} - (\hat{u})_{Q^{(\mu z_0)}(z_0)}|}{\tilde{z}_0^p} \, dz 
\leq c \iint_{Q^{(\mu z_0)}(z_0)} |D\hat{u}|^p \, dz + c \left[ \iint_{Q^{(\mu z_0)}(z_0)} |D\hat{u}|^p + |F|^p \chi_{\Omega_T} \, dz \right]^{p-1} \lambda^{p(2-p)}
\leq c(n, p, L_2) \lambda^p.
\]

Combining Lemma 2.9, Jensen’s inequality, Lemma 6.2 with \((\varrho, s) = (\varrho_{z_0}, \frac{1}{2} \varrho_{z_0})\) as well as the former inequality leads to

\[
\mu_{\varrho_{z_0}} \lambda = \tilde{\mu}_{\tilde{z}_0} \lambda
\leq \left( \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u}|^p}{\varrho_{z_0}^p} \, dz \right)^{\frac{1}{p}} = \left( \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u}|^p}{\varrho_{z_0}^p} \, dz \right)^{\frac{1}{p}}
\leq \left( \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u} - (\hat{u})_{Q^{(\mu \varrho_{z_0})}(z_0)}|^p}{\varrho_{z_0}^p} \, dz \right)^{\frac{1}{p}} + \left( \frac{|(\hat{u})_{Q^{(\mu \varrho_{z_0})}(z_0)}|}{\varrho_{z_0}} \right)^{\frac{1}{p}}
\leq \left[ c(p) \frac{Q^{(\mu \varrho_{z_0})}(z_0)}{Q^{(\mu \varrho_{z_0})}(z_0)} \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u} - (\hat{u})_{Q^{(\mu \varrho_{z_0})}(z_0)}|^p}{\varrho_{z_0}^p} \, dz \right]^{\frac{1}{p}}
\leq \left[ \frac{1}{2} \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u}|}{\varrho_{z_0}} \, dz \right]^{\frac{1}{p}} + \left[ c(n, p) \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u} - (\hat{u})_{Q^{(\mu \varrho_{z_0})}(z_0)}|^p}{\varrho_{z_0}^p} \, dz \right]^{\frac{1}{p}}
\leq c(n, p) \left( \iint_{Q^{(\mu \varrho_{z_0})}(z_0)} \frac{|\hat{u} - (\hat{u})_{Q^{(\mu \varrho_{z_0})}(z_0)}|^p}{\varrho_{z_0}^p} \, dz \right)^{\frac{1}{p}}.
\]
\[ + \frac{1}{2} \left( \frac{\sqrt[p]{\int_{Q_{\bar{z}_{z_0}}(z_0)} |\hat{u}(z_0)|^p \, dz}}{\frac{1}{2} \bar{\varrho}_{z_0}} \right) \leq c(n, p) \left( \frac{\sqrt[p]{\int_{Q_{\bar{z}_{z_0}}(z_0)} \left| \hat{u} - (\hat{u})_{Q_{\bar{z}_{z_0}}(z_0)} \right|^p \, dz}}{\frac{1}{2} \bar{\varrho}_{z_0}} \right) + \frac{1}{2} \mu_{z_0} \lambda \]

Absorbing the second term on the right hand side we end up with

\[ \frac{1}{2} \mu_{z_0} \lambda \leq c(n, p, L_2) \lambda, \]

verifying the super-intrinsic property (5.4) in this case. Since in addition

\[ B_2 \varrho_{z_0} \subset \Omega \]

can be applied to the cylinder \( Q_{\bar{z}_{z_0}}(z_0) \) and end up with

\[ \frac{1}{|Q_{\bar{z}_{z_0}}(z_0)|} \int_{Q_{\bar{z}_{z_0}}(z_0)} |D\hat{u}|^p \, dz \]

Finally we have to consider the remaining case, i.e. \( \varrho_{z_0} < d_0 \leq \bar{\varrho}_{z_0} \). As in the
degenerate case we want to apply Lemma 5.2 to the cylinder \( Q_{\bar{z}_{z_0}}(z_0) \) and by the same argument as above one can see that (5.3) is satisfied in this case. The super-intrinsic property (5.4) can be achieved in the same way as in the lateral case, since \( \varrho \geq d_0 \). Altogether we end up with (6.16).

**Conclusion**: Altogether we achieve the estimate

\[ \int_{Q_{\varrho_{z_0}}(z_0)} |D\hat{u}|^p \, dz \leq c \left[ \int_{Q_{\bar{z}_{z_0}}(z_0)} |\hat{u}|^q + |D\hat{u}_0|^q \, dz \right] \frac{p}{q} + c \int_{Q_{\bar{z}_{z_0}}(z_0)} |D\hat{u}_0|^p + |F|^p \chi_{\Omega_T} \, dz \]

in any case for the exponent \( q = \max\{2n/(n + 2), \gamma, 1\} \) with \( \gamma = \gamma(n, \nu) \in (1, p) \) from Lemma 4.1. The constant \( c \) depends on \( n, p, N, L_1, L_2, \nu, \varrho_0 \). Note that we integrate over the whole cylinder on the left hand side, in contrast to the former statements resulting from the Lemmata 5.1 and 5.2. But the remaining piece of the cylinder can be added on both sides of the inequality, as \( |D\hat{u}|^p \) is either zero (outside of \( \Omega \)) or equal to \( |D\hat{u}_0|^p \) (for \( t \leq 0 \)).
On the right hand side of (6.17), we aim for an integration only on the interior of $\Omega_T$. We start by seeing that due to $u \in W_0^{1,p}(\Omega)$ for a.e. $t \in (0,T)$, $\hat{u} = u_0$ for $t \leq 0$ and $z_0 \in \Omega_T$ it follows that

$$
\iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \setminus \Omega_T} |D\hat{u}|^q \, dz = \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \cap \{t \leq 0\}} |Du_0|^q \, dz 
\leq c \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \cap \{t \geq 0\}} |Du_0|^q \, dz 
= c \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)} |Du_0|^q \chi_{\Omega_T} \, dz
$$

Thus we can estimate the right hand side of (6.17) further, also using Jensen’s inequality to obtain

$$
\iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)} |D\hat{u}|^p \, dz 
\leq c \left[ \frac{1}{|Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)|} \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \cap \Omega_T} |Du|^q + |Du_0|^q \, dz \right]^{\frac{p}{q}} 
+ c \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)} |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz 
\leq c \left[ \frac{1}{|Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)|} \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \cap \Omega_T} |Du|^q \, dz \right]^{\frac{p}{q}} 
+ c \iint_{Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)} (|Du_0|^p + |F|^p) \chi_{\Omega_T} \, dz.
$$

(6.18)

6.5. Estimate on super-level set

The aim of this subsection is to show a reverse Hölder inequality on super-level sets. Therefore we define

$$
F(r, \lambda) := \{ \, z \in Q_r \cap \Omega_T : \text{z is Lebesgue point of } |Du_0|^p + |F|^p \text{ and } (|Du_0|^p + |F|^p)(z) > \lambda^p \, \}.
$$

Moreover we have shown so far, that if $\lambda \geq B\lambda_0$ with $B = \left( \frac{8\|R\|}{R_2 - R_1} \right)^{\frac{n+2}{mp}} > 1$ and $\hat{c}(n,p) \geq 80$ then for every $z_0 \in E(R_1, \lambda)$ there exists a cylinder $Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0)$ such that $Q_{\varrho z_0}^{(\mu \varrho z_0)}(z_0) \subset Q_{R_2}$ and (6.12), (6.14) and (6.18) hold true, where we abbreviate $\mu_{\varrho z_0} = \mu_{\varrho z_0}^{(\lambda)}$, as before. We introduce the new parameter $\eta \in (0,1]$, which we specify later. We will apply (6.12), (6.18), Hölder’s inequality and (6.14) with $s = 8\varrho_{2z_0}$, $\alpha = 1$. Note that both $E(R_2, \eta\lambda), F(R_2, \eta\lambda) \subset \Omega_T$ by definition. This way, it follows that
\[ \lambda^p = \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \]

\[ \leq c \left[ \frac{1}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \right] \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |Du|^q \, dz \] 

\[ + c \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \chi_{\Omega_T} \, dz \]

\[ \leq cr^p \lambda^p + c \left[ \frac{1}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \right] \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |Du|^q \, dz \]

\[ + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \, dz \]

\[ \leq cr^p \lambda^p + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \left( \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |Du|^q \, dz \right) \]

\[ \left[ \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |D\hat{u}|^q \, dz \right]^{\frac{p}{q}-1} \]

\[ + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \, dz \]

\[ \leq cr^p \lambda^p + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \left( \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |Du|^q \, dz \right) \]

\[ \left[ \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |D\hat{u}|^p \, dz \right]^{1-\frac{q}{p}} \]

\[ + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \, dz \]

\[ \leq cr^p \lambda^p + \frac{c \lambda^{p-q}}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} |Du|^q \, dz \]

\[ + \frac{c}{|Q_{\delta x_0}^{(n \varepsilon x_0)}} \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \, dz \]

with \( c = c(n, p, N, \nu, \varrho_0) \). We now choose \( \eta := \left( \frac{1}{2c} \right)^p \) to absorb the first term on the right hand side. Therefore we have

\[ \lambda^p |Q_{\delta x_0}^{(n \varepsilon x_0)}} \leq c \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} \lambda^{p-q} |Du|^q \, dz \]

\[ + c \iint_{Q_{\delta x_0}^{(n \varepsilon x_0)}} (|Du_0|^p + |F|^p) \, dz. \]
Applying (6.14) with \( s = \hat{c}g_{z_0} \leq R_2 \leq R \) and \( \alpha = 1 \) we can estimate the left hand side from below by

\[
\lambda^p |Q_{g_{z_0}}^{(\mu_{z_0})}| \geq \left( \frac{g_{z_0}}{\hat{c}g_{z_0}} \right)^{\frac{n+2p}{2p-n}} \|Q_{g_{z_0}}^{(\mu_{z_0})}\| \int \int_{Q_{c\hat{c}g_{z_0}}^{(\mu_{z_0})} (z_0)} |D\hat{u}|^p + |Du_0|^p + |F|^p \chi_T \, dz \\
\geq c(n, p) \int \int_{Q_{c\hat{c}g_{z_0}}^{(\mu_{z_0})} (z_0)} |D\hat{u}|^p \, dz
\]

and hence we have

\[
\int \int_{Q_{c\hat{c}g_{z_0}}^{(\mu_{z_0})} (z_0)} |D\hat{u}|^p \, dz \leq c \int \int_{Q_{c\hat{c}g_{z_0}}^{(\mu_{z_0})} (z_0) \cap E(2, \eta)} \lambda^{p-q} |Du|^q \, dz \\
+ c \int \int_{Q_{c\hat{c}g_{z_0}}^{(\mu_{z_0})} (z_0) \cap F(2, \eta)} |Du_0|^p + |F|^p \, dz.
\]

(6.19)

Up to now \( z_0 \in E(R_1, \lambda) \) is arbitrary and hence for \( \lambda > B\lambda_0 \) we have a family of cylinders \( \mathcal{F} = \left\{ Q_{g_{z_0}}^{(\mu_{z_0})} (z_0) \right\} \) covering \( E(R_1, \lambda) \) such that each cylinder is contained in \( Q_{R_2} \) and fulfills (6.19). By the Vitali covering from Sect. 6.2 there exists a countable subfamily \( \left\{ Q_{g_{z_i}}^{(\mu_{z_i})} (z_i) \right\}_{i \in \mathbb{N}} \subset \mathcal{F} \) of pairwise disjoint cylinders such that

\[
E(R_1, \lambda) \subset \bigcup_{i \in \mathbb{N}} Q_{g_{z_i}}^{(\mu_{z_i})} (z_i) 
\subset Q_{R_2}.
\]

Together with (6.19) we get

\[
\int \int_{E(R_1, \lambda)} |Du|^p \, dz \leq \sum_{i=1}^{\infty} \int \int_{Q_{g_{z_i}}^{(\mu_{z_i})} (z_i)} |D\hat{u}|^p \, dz \\
\leq c \sum_{i=1}^{\infty} \int \int_{Q_{g_{z_i}}^{(\mu_{z_i})} (z_i) \cap E(R_2, \eta)} \lambda^{p-q} |Du|^q \, dz \\
+ c \sum_{i=1}^{\infty} \int \int_{Q_{g_{z_i}}^{(\mu_{z_i})} (z_i) \cap F(R_2, \eta)} |Du_0|^p + |F|^p \, dz \\
\leq c \int \int_{E(R_2, \eta)} \lambda^{p-q} |Du|^q \, dz + c \int \int_{F(R_2, \eta)} |Du_0|^p + |F|^p \, dz.
\]

Moreover on \( E(R_1, \eta) \setminus E(R_1, \lambda) \) we have \( \eta \lambda < |Du| \leq \lambda \) and hence

\[
\int \int_{E(R_1, \eta) \setminus E(R_1, \lambda)} |Du|^p \, dz \leq \int \int_{E(R_1, \eta) \setminus E(R_1, \lambda)} \lambda^{p-q} |Du|^q \, dz \\
\leq \int \int_{E(R_2, \eta) \setminus E(R_1, \lambda)} \lambda^{p-q} |Du|^q \, dz.
\]
Combining the last two inequalities shows
\[ \iint_{E(R_1, \eta \lambda)} |Du|^p \, dz \leq c \iint_{E(R_2, \eta \lambda)} \lambda^{p-q} |Du|^q \, dz + c \iint_{F(R_2, \eta \lambda)} |Du_0|^p + |F|^p \, dz \]
and replacing \(\eta \lambda\) by \(\lambda\) with \(\eta \leq 1\) leads to
\[ \iint_{E(R_1, \lambda)} |Du|^p \, dz \leq c \iint_{E(R_2, \lambda)} \left( \frac{\lambda}{\eta} \right)^{p-q} |Du|^q \, dz \\
+ c \iint_{F(R_2, \lambda)} |Du_0|^p + |F|^p \, dz \tag{6.20} \]
for any \(\lambda > \eta B \lambda_0 =: \lambda_1\) with constant \(c = c(n, p, N, L_1, L_2, \nu, \varrho_0)\), which is the required reverse Hölder inequality on super-level sets.

### 6.6. Proof of the gradient estimate

To finish the gradient estimate of the main result we first consider some \(k > \lambda_1\) und define the truncation of \(|Du|\) by \(|Du|_k := \min\{|Du|, k\}\) and for \(r \in (0, 2R]\) the corresponding super-level set by \(E_k(r, \lambda) := \{z \in Q_r \cap \Omega_T : |Du|_k > \lambda\}\). Note that \(|Du|_k \leq |Du|\) almost everywhere as well as \(E_k(r, \lambda) = \emptyset\) for \(k \leq \lambda\) and \(E_k(r, \lambda) = E(r, \lambda)\) for \(k > \lambda\). Therefore (6.20) shows
\[ \iint_{E_k(R_1, \lambda)} |Du|^p \, dz \leq \iint_{E_k(R_1, \lambda)} |Du|^p \, dz \\
\leq c \iint_{E(R_2, \lambda)} \lambda^{p-q} |Du|^q \, dz \\
+ c \iint_{F(R_2, \lambda)} |Du_0| + |F|^p \, dz. \]

For some \(\varepsilon \in (0, 1]\) to be specified later on we multiply the previous inequality with \(\lambda^{\varepsilon p-1}\) and integrate the result with respect to \(\lambda\) over \((\lambda_1, \infty)\)
\[ \int_{\lambda_1}^{\infty} \lambda^{\varepsilon p-1} \iint_{E_k(R_1, \lambda)} |Du|^p \, dz \, d\lambda \]
\[ \leq c \int_{\lambda_1}^{\infty} \lambda^{p-q+\varepsilon p-1} \iint_{E(R_2, \lambda)} |Du|^q \, dz \, d\lambda \tag{6.21} \]
\[ + c \int_{\lambda_1}^{\infty} \lambda^{\varepsilon p-1} \iint_{F(R_2, \lambda)} |Du_0|^p + |F|^p \, dz \, d\lambda. \]

Next we apply Fubini’s theorem to exchange the order of integration in all terms. We start with the left hand side, for which we have
\[
\int_{\lambda_1}^{\infty} \lambda^{p-q-1} \iint_{E_k(R_1, \lambda)} |Du|^{p-q} |Du|^q \, dz \, d\lambda \\
= \iint_{E_k(R_1, \lambda_1)} |Du|^{p-q} |Du|^q \lambda^{p-1} \, d\lambda \, dz \\
= \iint_{E_k(R_1, \lambda_1)} |Du|^{p-q} |Du|^q \left[ \frac{1}{\varepsilon p} |Du|^{\varepsilon p} - \frac{1}{\varepsilon p} \lambda^{\varepsilon p} \right] \, dz.
\]

Going to the right hand side of (6.21), the first term reads

\[
\int_{\lambda_1}^{\infty} \lambda^{p-q+\varepsilon p-1} \iint_{E(R_2, \lambda)} |Du|^q \, dz \, d\lambda \\
= \iint_{E(R_2, \lambda)} |Du|^q \lambda^{p-q+\varepsilon p-1} \, d\lambda \, dz \\
= \iint_{E(R_2, \lambda_1)} |Du|^q \left[ \frac{1}{p-q+\varepsilon p} |Du|^{p-q+\varepsilon p} - \frac{1}{p-q+\varepsilon p} \lambda_1^{p-q+\varepsilon p} \right] \, dz \\
\leq \frac{1}{p-q} \iint_{E(R_2, \lambda_1)} |Du|^q |Du|^{p-q+\varepsilon p} \, dz.
\]

Continuing on (6.21), the \(|F|^p\)-term on the right hand side transforms as follows:

\[
\int_{\lambda_1}^{\infty} \lambda^{\varepsilon p-1} \iint_{F(R_2, \lambda)} |F|^p \, d\lambda \, dz = \iint_{F(R_2, \lambda)} |F|^p \lambda^{\varepsilon p-1} \, d\lambda \, dz \\
= \iint_{F(R_2, \lambda_1)} |F|^p \left[ \frac{1}{\varepsilon p} |F|^{\varepsilon p} - \frac{1}{\varepsilon p} \lambda_1^{\varepsilon p} \right] \, dz \\
\leq \frac{1}{\varepsilon p} \iint_{Q_{2R} \cap \Omega_T} |F|^{(1+\varepsilon)p} \, dz.
\]

Lastly, for the \(|Du_0|^p\)-term in (6.21) there holds

\[
\int_{\lambda_1}^{\infty} \lambda^{\varepsilon p-1} \iint_{F(R_2, \lambda)} |Du_0|^p \, d\lambda \, dz = \iint_{F(R_2, \lambda)} |Du_0|^p \lambda^{\varepsilon p-1} \, d\lambda \, dz \\
= \iint_{F(R_2, \lambda_1)} |Du_0|^p \left[ \frac{1}{\varepsilon p} |Du_0|^{\varepsilon p} - \frac{1}{\varepsilon p} \lambda_1^{\varepsilon p} \right] \, dz \\
\leq \frac{1}{\varepsilon p} \iint_{Q_{2R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} \, dz.
\]

Altogether this yields
\[
\iint_{E_k(R_1, \lambda_1)} |Du|^{p-q+\varepsilon p}|Du|^q \, dz \leq \lambda_1^{\varepsilon p} \iint_{E_k(R_1, \lambda_1)} |Du|^{p-q}|Du|^q \, dz \\
+ c^{\frac{\varepsilon p}{p-q}} \iint_{E(R_2, \lambda_1)} |Du|^q |Du|^{p-q+\varepsilon p} \, dz \\
+ c \iint_{Q_{2R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz.
\]

Moreover, on \((Q_{R_1} \cap \Omega_T) \setminus E_k(R_1, \lambda_1)\) we have \(|Du|_k \leq \lambda_1\) and hence
\[
\iint_{(Q_{R_1} \cap \Omega_T) \setminus E_k(R_1, \lambda_1)} |Du_k|^{p-q+\varepsilon p} |Du|^q \, dz \\
\leq \lambda_1^{\varepsilon p} \iint_{(Q_{R_1} \cap \Omega_T) \setminus E_k(R_1, \lambda_1)} |Du_k|^{p-q} |Du|^q \, dz
\]
and therefore
\[
\iint_{Q_{R_1} \cap \Omega_T} |Du_k|^{p-q+\varepsilon p} |Du|^q \, dz \leq c \lambda_1^{\frac{\varepsilon p}{p-q}} \iint_{Q_{R_2} \cap \Omega_T} |Du|^q |Du_k|^{p-q+\varepsilon p} \, dz \\
+ \lambda_1^{\varepsilon p} \iint_{Q_{2R} \cap \Omega_T} |Du|^p \, dz \\
+ c \iint_{Q_{2R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz.
\]

Applying the iteration result from Lemma 2.10 shows
\[
\iint_{Q_R \cap \Omega_T} |Du_k|^{p-q+\varepsilon p} |Du|^q \, dz \leq c \lambda_0^{\varepsilon p} \iint_{Q_{2R} \cap \Omega_T} |Du|^p \, dz \\
+ c \iint_{Q_{2R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz
\]
with \(c = c(n, p, N, \nu, \varrho_0)\). Finally we apply Fatou’s lemma to the left hand side and the definition of \(\lambda_0\) to end up with
\[
\iint_{Q_R \cap \Omega_T} |Du|^{(1+\varepsilon)p} \, dz \\
\leq c \lambda_0^{\varepsilon p} \iint_{Q_{2R} \cap \Omega_T} |Du|^p \, dz \\
+ c \iint_{Q_{2R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz \\
\leq c \left[ 1 + \iint_{Q_{8R}} 2^p |\hat{u} - u_0|^p + |u_0|^p \right] \iint_{Q_{8R}} |D\hat{u}|^p 
\]
\[ + |Du_0|^p + |F|^p \chi_{\Omega_T} \, dz \varepsilon \int_{Q_2R \cap \Omega_T} |Du|^p \, dz \]
\[ + c \int_{Q_2R \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz. \]

As the center point \( \tilde{z}_0 \) of \( Q_{8R} \) fulfils \( \tilde{z}_0 \in \Omega_T \cup \partial_{\text{par}} \Omega_T \) and \( u_0 \in W^{1,p}_0(\Omega) \) does not depend on time, it follows that
\[ \int_{Q_{8R}} |Du_0|^p \, dz \leq c \int_{Q_{8R}} |Du_0|^p \chi_{\Omega_T} \, dz. \]

Recalling that \( \hat{u}(\cdot, t) = 0 \) outside of \( \Omega \) for \( t \in (-T, T) \), while \( \hat{u}(x, t) = u_0(x) \) for \( t \leq 0 \), we further estimate
\[ \int_{Q_R \cap \Omega_T} |Du|^{(1+\varepsilon)p} \, dz \]
\[ \leq c \left[ 1 + \frac{1}{|Q_{8R}|} \int_{Q_{8R} \cap \Omega_T} 2^p \frac{|u - u_0|^p + |u_0|^p}{(8R)^p} \right] \]
\[ + |Du|^p + |Du_0|^p + |F|^p \, dz \varepsilon \int_{Q_2R \cap \Omega_T} |Du|^p \, dz \]
\[ + c \int_{Q_2R \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz \]
\[ \leq c \left[ 1 + \frac{1}{|Q_{8R}|} \int_{Q_{8R} \cap \Omega_T} \frac{|u - u_0|^p + |u_0|^p}{(8R)^p} + |Du|^p \, dz \varepsilon \int_{Q_2R \cap \Omega_T} |Du|^p \, dz \right] \]
\[ + c \int_{Q_{8R} \cap \Omega_T} |Du_0|^{(1+\varepsilon)p} + |F|^{(1+\varepsilon)p} \, dz \]

At last, we repeat a covering argument as in [27]: We cover the cylinder \( Q_R \) with finitely many cylinders \( Q_{R/8}(z_i) \) and use the previous estimate on these smaller cylinders. The sum of these inequalities finishes the proof of Theorem 2.3.

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