Synchronization of Networked Jahn-Teller Systems in Circuit QED

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We consider the nonlinear effects in Jahn-Teller system of two coupled resonators interacting simultaneously with flux qubit using Circuit QED. Two frequency description of Jahn Teller system that inherits the networked structure of both nonlinear Josephson Junctions and harmonic oscillators is employed to describe the synchronous structures in multifrequency scheme. Emergence of dominating mode is investigated to analyze frequency locking by eigenvalue spectrum. Rabi Supersplitting and asymmetry of side peaks in power spectrum is tuned for coupled and uncoupled synchronous configurations in terms of frequency entrainment switched by coupling strength between resonators. Second order coherence functions are employed to investigate self-sustained oscillations in resonator mode and qubit dephasing. Synchronous structure between correlations of privileged mode and qubit is obtained in localization-delocalization and photon blockade regime controlled by the population imbalance.

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I. INTRODUCTION

Following the program of simulating a physical system by means of another, advances in control and flexibility of quantum mechanical systems leads to the the era of quantum simulators ultracold atoms[1, 2], ion traps [3, 4] and cavity qed arrays[5, 6]. Exploring fundamental quantum mechanics in lattice arrays of Circuit QED [7, 8] and embedding the artificial atoms into open transmission line resonators [9, 10] bring another suitability criteria in manipulation of strong and ultrastrong regimes for quantum information processing [11, 12]. Emergence of cooperativity and synchronization in collective behavior of many body coupled systems trigger quantum classical behaviors of particles trapped in local minima of double well potentials warped by the nonlinearities governing both number and phase dynamics [29]. Circuit QED architectures inherits the nonlinear characteristics of josephson junctions leading to the emergence of spatial and temporal transitions in coupled resonator schemes. Two regimes of coherent oscillations and self trapping by controlled nonlinearities is analyzed in hybrid system [30] composed of bosonic Josephson Junction [31, 32] and localization delocalization transitions is shown in photon Josephson Junction in Circuit QED setup [33] as an unit of Jaynes-Cummins (JC) lattice array [34, 35].

In Josephson Junction Arrays (JJA), due to the collective behavior of cooper pairs , synchronization desynchronization transitions comes out in phase coherence patterns [36, 37]. In spatially extended systems, cluster of oscillator networks have the ability of tuning transitions locally in multifrequency schemes having distributions of coupling strengths relaxing towards the localized dominating node[39]. Optomechanical systems, as lumped model of two coupled harmonic oscillator via light, appear as test bed for reduced form of the effective Kuramoto model in dissipative environment and reconfigurable synchronous oscillator networks [40, 41] in the absence of quantum fluctuations.

Our aim is to employ cavity and Circuit QED realization of JT models [12, 33] in describing the effect of nonlinearities in multifrequency coupled resonator schemes. Coupled modes of resonators over which the JT coupling distribution can be tuned so that networked two frequency description become feasible to manipulate synchronization of qubit dephasing and population imbalance in terms of normal modes conveying JJ nonlinearities [43, 45]. Quadratic interactions, responsible for warping in JT systems networked to the outer circuitry, appear as the nonlinear Josephson inductance coupling between the flux qubit and the plasma mode [46, 48] describing the effect of thermal fluctuation on qubit dephas-
This paper is organized as follows. In Sec.II we introduce the coupled model with quadratic interactions and use effective single mode transformation. The results and discussions are presented in Sec.III. Finally, we give conclusions in Sec. VI.

II. MODEL

Circuit QED simulations of JT-models requires both multi-frequency description of vibrational interactions and going beyond the Rotating Wave Approximation (RWA) due to the ultrastrong coupling regime. Our model hamiltonian is \( \hbar = 1 \)

\[
H = \frac{\omega}{2} \sigma_z + \sum_{i=1,2} \omega_i a_i^\dagger a_i + (\lambda_i(a_i + a_i^\dagger)) + g_i (a_i + a_i^\dagger)^2 \sigma_x \tag{1}
\]

\( \omega \) and \( \omega_{1,2} \) are the qubit and resonator frequencies and \( a_{1,2}(a_{1,2}^\dagger) \) are the annihilation and creation operators, \( \sigma_x \) is the Pauli operator. This hamiltonian shows the coupling between flux qubit and two plasma mode in both linear and nonlinear interaction strengths \( \lambda_{1,2} \) and \( g_{1,2} \) respectively. To go beyond RWA, Circuit QED realization of our system is mapped to two frequency JT model and described as

\[
H = H_q + H_r + H_{JT} + H_{NL} \tag{2}
\]

where

\[
H_q = \frac{\omega}{2} \sigma_z \tag{3}
\]

\[
H_r = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 \tag{4}
\]

are the qubit and resonator hamiltonians with frequencies. The Jahn-Teller interaction is given by

\[
H_{JT} = k_1 \omega_1 (a_1 + a_1^\dagger) \sigma_x + k_2 \omega_2 (a_2 + a_2^\dagger) \sigma_x \tag{5}
\]

where \( k_{1,2} \) are the dimensionless JT scaling factors. In the absence of Nonlinear term, our system behaves as an effective single mode model where the qubit coupled to resonators asymmetrically due to the concentration of JT interaction in privileged mode.

Quadratic interactions terms appears due to the nonlinear Josephson inductance in SQUID phase leading to the occurrence of second order terms corresponding to the fluctuations of dynamical variables controlled by the external parameters. In Circuit network analysis of SQUID, each dissipative element of JJ modelled as bath of harmonic oscillators sitting in its own node networked to the circuit of lumped superconducting system. On the other hand, in JT systems, quadratic interactions is determined empirically warping the APES and depends on the symmetry lowering configurations of crystal geometries. Then, quadratic interactions is written as

\[
H_{NL} = [\omega_1 (a_1 + a_1^\dagger)^2] + [\omega_2 (a_2 + a_2^\dagger)^2] \sigma_x \tag{6}
\]

which makes the system networked to outer crystal structure surrounding the host lattice containing impurity and described as bath of harmonic oscillators with natural frequencies \( \omega_{1,2} = g_{1,2} \) in spin-boson treatment. Using both linear and nonlinear coupling, our system in two frequency effective JT model becomes

\[
H = H_{JT} + H_{NL} \tag{7}
\]

where

\[
H_{JT} = \frac{\omega}{2} \sigma_z + \omega' \alpha_2^\dagger \alpha_2 + J(\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2) + \omega_{eff} [\alpha_1^\dagger a_1 + k_{eff} (\alpha_2 + \alpha_2^\dagger) \sigma_x] + c_2 (\alpha_1^\dagger \alpha_2 + k_{eff} (\alpha_2^\dagger + \alpha_2) \sigma_z) \tag{8}
\]

and

\[
H_{NL} = [\omega_{eff} (\alpha_1 + \alpha_1^\dagger)^2] + [\omega^2 (\alpha_2 + \alpha_2^\dagger)^2] + J(\alpha_1 + \alpha_1^\dagger)(\alpha_2 + \alpha_2^\dagger) \sigma_x \tag{9}
\]

with the frequency of effective mode

\[
\omega_{eff} = \frac{\omega_1 k_1^2 + \omega_2 k_2^2}{k_{eff}} \tag{10}
\]

and qubit-resonator coupling strength

\[
k_{eff}^2 = k_1^2 + k_2^2. \tag{11}
\]

The frequency of disadvataged mode is given by

\[
\omega' = \frac{\omega_1 k_2^2 + \omega_2 k_1^2}{k_{eff}} \tag{12}
\]

and coupled to the priviledged mode with

\[
c_2 = \frac{\Delta k_1 k_2}{k_{eff}^2}. \tag{13}
\]

Rayleigh’s theorem makes it possible to bring JT systems and networks of harmonic oscillator on the same footing at the aspect of stability and damping effects in terms of normal modes. In the former case, the constraint for multi frequency JT system is

\[
\sum_i \frac{k_i^2}{\omega_i - \omega} = 0 \tag{14}
\]

for which \( i = 1,2 \) corresponds to eqn.(12) and represent the disappearance of interacting terms in \( n-1 \) frequencies of \( \omega \) except with the priviledged mode. In two frequency JT system, tunability of phonon-phonon coupling distribution is described by \( c_2^2 = \sigma^2 - \sigma^2' \) and transformed in terms of frequency difference and scaling factors as
eqn.(13). Following the same reasoning, in the latter case, conditions for synchronous scheme is obtained by controllability of coupling strength of frozen mode corresponding to privileged mode in JT systems.

$E \otimes (b_1 + b_2)$ JT systems, in it’s splitted structure $E \otimes b_1$ and $E \otimes b_2$ corresponding to tetrahedral distortions in definite directions, makes it possible the anisotropic complexes composed networks of oscillating normal modes in which effective mode frequencies appears weighted by the scale factors $k_i^2/\sum_i k_i^2$ for which $i = 1, 2$ results in eqn.(10). Tetrahedral networks allows switchable configurations of JT centers due to the distortions driven by the corner sharing spins playing a central role in cooperativity of elongated or compressed lattice structure [23, 25].

For simulation purposes, two parameters $(k, \Delta)$, JT scaling factors describing coupling regime to go beyond RWA and frequency difference of the resonators representing the asymmetry of linear and quadratic interactions are used for Circuit QED Hamiltonian written as

$$H = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \sigma_z + \frac{\Delta}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

$$+ \sqrt{2} k_1 (\hat{a}_1^\dagger + \hat{a}_1) + (\hat{a}_1^\dagger + \hat{a}_1)^2$$

$$+ \frac{\Delta}{2} ((\hat{a}_2^\dagger + \hat{a}_2) + (\hat{a}_2^\dagger + \hat{a}_2)^2)) \sigma_x. \tag{15}$$

where $k_1 = k_2 = k$, $\lambda_1 = (\omega_1 + \omega_2)k/\sqrt{2}$, $\lambda_2 = \Delta k/\sqrt{2}$, and $c_2 = \Delta/2$. We present the coupling of two resonator with the hopping parameter $J = c_2$.

In this manner, we consider our coupled system as coupling of privileged mode interacting simultaneously with qubit and disadvantaged mode. Correlations of privileged mode, dephasing qubit and imbalance between subpopulations give rise to cooperative and synchronous JT systems in Circuit QED.

### III. RESULTS

Wiring up the linear JT models with nonlinearities controlled externally, makes the coupled systems plausible for emergence of cooperativity and synchronization in singled out mode lying out on a 2D manifold of potential tuning the coupling strengths asymmetric in both strong and ultrastrong regime.

In Josephson Junction arrays, distribution of frequencies is modulated so as to fraction of nonlinear oscillators become frequency locked in the weak coupling regime corresponds to the Kuramoto model of mean field theories [37]. In two frequency JT model, as an minimal coupled model of nonlinear oscillator populations, appearance of singled out effective mode dominating the ones representing perturbative effect on the system still makes the coupling $c_i$ distribution controllable with nonlinear interaction terms. Competition between dynamical coupling strength and intrinsic disorder in Kuramoto model is simulated in Circuit QED setup in terms of scaling factors $k$ dominating priviledged mode and frequency difference $\Delta$ corresponding to coupling strength of perturbations.

![Figure 1](image1.png)

**FIG. 1:** (Color online) Emergence of frequency locking for two-mode JT system shown in spectrum of the lowest five eigenvalues depending on the frequency difference $\Delta$. (a) At $\Delta = 0$ Rabi splitting of first energy levels occurs for $k = 0.1/\sqrt{2}$ and avoiding crossing is seen at $\Delta = 1.9$ due to nonlinearities. Interaction between privileged and disadvantaged mode can be tuned up to $\Delta = 0.1$ in single effective mode. (b) Range of single mode regime extends up to $\Delta = 0.5$ in ultrastrong regime $k = 1.0/\sqrt{2}$

We examined the eigenenergies of 5 lowest lying in spectrum of our system where each resonator is described with Fock space dimension 2 plotted in Fig.2. We present the tendency of frequency locked structure of our system in both strong and ultrastrong regime. In Fig.2(a) our system is in single privileged mode only for frequency difference $|\Delta| < 0.1$ and $k = 0.1/\sqrt{2}$. Effect of perturbative coupling leads to pure Rabi splitting of first excited level for $|\Delta| = 0$. Nonlinearities is seen in repelling the level of Rabi splitting and avoiding crossing between ground and first excited states at $|\Delta| = 1.9$. In Fig.2(b), when we are in ultrastrong regime for $k = 1.0/\sqrt{2}$ the range of single mode structure extends up to $|\Delta| < 0.5$ and avoiding crossing is replaced with a level repelling. The pattern of eigenvalue spectrum mixed by $|\Delta|$ and smoothed by $k$ is used as
an essence of competition between linear and nonlinear terms.

Circuit QED realizations of vacuum Rabi splitting is detected by the transmitted amplitude of field quadratures corresponding to voltage and current in an array of transmon qubits coupled with a common resonator in which supersplitting is described in a reduced two-level model [13]. Linear JT model of two mode coupled systems shows frequency conversion modulated by nonlinear susceptibility [14]. Circuit QED setup is chosen so as to make it appropriate to detectable transmission measurement and macroscopic quantum coherence resulting in localization in quantum classical transition [51, 54].

Open system dynamics is governed by

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}\rho,$$

(17)

where the Liouvillian superoperator $\mathcal{L}$ is given by

$$\mathcal{L}\rho = \sum_{j=1,2} (1 + n_{th})\kappa D[\hat{\alpha}_j]\rho + n_{th}\kappa D[\hat{\alpha}_j^\dagger]\rho$$

$$+ \gamma D[\sigma]\rho + \gamma_\phi D[\sigma_\phi]\rho,$$

(18)

with $n_{th}$ is average thermal photon number taken as $n_{th} = 0.15$ corresponding to 100 mK [43, 54]. $D$ denotes the Lindblad type damping superoperators. $\kappa$ is the cavity photon loss rate. Qubit relaxation and dephasing rates are, respectively, $\gamma$ and $\gamma_\phi$. We use balanced dissipation where resonator decay parameters $\kappa_1 = \kappa_2 = 0.001$ and qubit relaxation and dephasing parameters $\gamma = 0.001, \gamma_\phi = 0.01$ with the thermal occupation number $n_{th} = 0.15$.

Fig. 2 shows nonlinear vacuum response of the cavity field for hopping parameter $J = 0$ and $J = 0.5$ values corresponding to single mode JC (blue) and two mode JC (red) regimes. Fig. 2(a) presents asymmetric Rabi peaks in Lorentzian line shape when the system is in single mode JC regime. Increasing the coupling strength reveals the emergence of supersplitting of each vacuum Rabi peak into a doublet with a higher amplitude. Fig. 2(b) shows the Rabi supersplitting for $k = 0.05$. Going beyond the single mode JC regime increase the central dip in each peaks. Effect of hopping parameters is seen in multi-photon transitions.

Characteristic changes occur in power spectrum when the system is in strong coupling regime since the quadrature operator appears as the quasi-particle corresponding to dark polariton [13] and displacement operator [50] in Raman spectrum of Cu$^{2+}$:CaO compounds. Based on the realization of dark state by stimulated Raman adiabatic passage, optical selection rules appear in artificial atoms [52] allowing frequency conversion in which symmetric modes plays the role of privileged mode coupled nonlinearly with asymmetric mode [44, 45].

From the point of synchronization, when the number of subpopulations in variants of Kuramoto model is equal to the degrees of freedom of the system under considerations, one can obtain reconfigurations of coupled and uncoupled schemes by tuning frequency mismatch with $(k, \Delta)$ parameters. In coupled resonator scheme, $\Delta \neq 0$, hopping terms appears as the inter-cavity control parameter resulting in cooperative and synchronous structure. Field quadratures are used in describing measure of quantum analogue of frequency entrainment and locking in optomechanical systems and harmonically driven Van der Pol oscillator [55, 56]. Self-sustained oscillation is shown in amplitude locking with quadratic coupling leading to multipeak field spectrum [57].

Fig. 3 presents how the coupling regime dominates the inherited features of nonlinear JJ and networked structure depending on the configurations. Fig. 3(a) shows
the shift of splitted peaks from each other and raise of extra peak around \( \omega = 0 \) for \( k = 0.1 \) where the system in two mode JC regime indicates of synchronization entrainment although their amplitude is still different. In Fig.(3b), at intermediate coupling regime \( k = 0.5 \), spectrum evolve into a triplet where the asymmetry of peaks are due to cavity coupling strengths and are easy to control in terms of relative coupling between privileged and disadvantaged mode. Emergence of stokes and anti-stokes peaks are due to field quadrature operator behaves as qubit-polariton operator revealing Raman process. Coherent evolution is modulated by the multilevel structure of atomic states carrying the nonlinearities of JJs intrinsically. In view of oscillator networks, the frequency amplitude of coupled and uncoupled scheme gets closer and is coincident in both central and side peaks in definite frequencies. Networked structure of two resonators shows switchable synchronous configurations as an indication of synchronization entrainment between the pure two mode JT and the effective privileged mode model.

Two frequency realization of Circuit QED architectures appears as a platform to simulate the self-sustained oscillator behavior of tetrahedral networks distorted by corner sharing spin where nonlinearities is induced by the lattice restoring energy. Coupled resonator schemes is seen as qubit sharing oscillators residing in it’s own JT centers. Switching symmetric and asymmetric mode configurations leads to the transverse and longitudinal prolongation of host lattice arrangement mimicking the rhythmic behavior of diamond shaped crystal geometries. Accumulate and fire oscillators description gives way to slow growth of correlations and its resetting similar to the van der Pol relaxation oscillator \[55, 56\]. In our model, self-sustained oscillation of each normal mode is described by correlations revealing delocalization and trapping regimes of coupled system.

In order to see the correlations of distortions, we use the second order coherence functions of field and atomic states

\[
g_i^{(2)} = \frac{\rho_i^+(t)\rho_i^+(t+\tau)\rho_i(t)\rho_i(t+\tau)}{\rho_i^+(t)\rho_i(t)} \tag{19}
\]

where \( i = r, q \) is used in place of resonator and qubit.

The condition \( g_{r,q}^{(2)} \ll 1 \) corresponds to antibunching, and used for the indication of photon blockade and energetic localization of qubit. Another central quantity of couple cavity system is the photon population imbalance \( z(t) = n_1 - n_2 \) where \( n_i = \text{Tr} \alpha_i^\dagger \alpha_i \rho \) for \( i = 1, 2 \) corresponds to the two cavities described by privileged and disadvantaged mode containing single and zero photons respectively. Initially qubit is taken as excited state.

In Fig.4 we show the correlations functions of resonator and qubit in weak, strong and ultrastrong regimes. In the first two top panel, we present the second order coherence functions of privileged mode (blue) and the qubit (red) and in the third panel shows population imbalance (green). Fig.4.a shows correlations and poulation imbalance in weak coupling regime, \( k = 0.01/\sqrt{2} \). Population imbalance is in oscillating regime and synchronous with the photon correlation. Self-sustained oscillation is seen via decreasing of population imbalance while resettling of antibunching of privileged mode at two different time scales corresponding to accumulate and fire oscillator in the sense of van der Pol relaxation. As \( \tau \) increases population imbalance reaches zero and qubit correlation with beats in anharmonic time intervals becomes stable. Starting with \( g_i^{(2)}(0) = 0 \) corresponding to the photon blockade mode, photon correlations reach stable point with decreasing peaks while onsite repulsion increasing. In strong regime \( k = 0.1/\sqrt{2} \), fig.4.b shows the delocalization-localization transitions in population imbalance. Contrary to the weak coupling regime, qubit
and photon correlation becomes synchronous representing simultaneous firing and damping of correlations as $\tau$ increases. Although accumulation gets diminished in blockade regime, there is still firing of qubit and privileged mode correlation due to the multilevel transitions by the inherited nonlinearities of JJs. Fig.4.c presents the fully localized regime for population imbalance and all the quantities reach a stable point in ultra-strong regime, $k = 1.0/\sqrt{2}$. Effect of qubit dephasing is shown by quenching thermal fluctuations by taking $\gamma_\phi = 0.1$ and leaving the other parameters the same.

These results suggest that two frequency description JT system can be used to investigate cooperative and synchronous behaviors of circuit QED schemes by modulating the qubit anharmonicities due to JJs with networked nonlinearities.

IV. CONCLUSION

In conclusion, we have shown that nonlinearities plays a central role in describing the cooperativity and synchronization in Circuit QED architectures by the inherited nonlinearities of JJs. In our model, flux qubit simultaneously coupled to two resonator with both linear and quadratic interaction terms. We performed the eigenenergy and power spectrum calculations frequency locking and synchronization entrainment regimes respectively. Nonlinearities gives way to fundamental exploring of quantum mechanics such as Rabi Super-splitting. Tedrahedral structures of JT systems opens the way of constructing networked oscillators which can be translated to coupled resonator schemes of circuit QED. Correlation functions of normal modes and qubit indicates cooperative and synchronous structures in localization delocalization and photon blockade regimes as powerfull tools for qubit readout schemes.

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