COLORING DISTANCE GRAPHS ON THE INTEGERS

GLENN G. CHAPPELL

Department of Mathematics, Southeast Missouri State University

Abstract. Given a set \( D \) of positive integers, the associated distance graph on the integers is the graph with the integers as vertices and an edge between distinct vertices if their difference lies in \( D \). We investigate the chromatic numbers of distance graphs. We show that, if \( D = \{d_1, d_2, d_3, \ldots\} \), with \( d_n \mid d_{n+1} \) for all \( n \), then the distance graph has a proper 4-coloring. We further find the exact chromatic numbers of all such distance graphs. Next, we characterize those distance graphs that have periodic proper colorings and show a relationship between the chromatic number and the existence of periodic proper colorings.

1. Introduction

What is the least number of classes into which the integers can be partitioned, so that no two members of the same class differ by a square? What if “square” is replaced by “factorial”? Questions like these can be formulated as graph coloring problems. Given a set \( D \) of positive integers, the distance graph \( Z(D) \) is the graph with the integers as vertices and an edge between distinct vertices if their difference lies in \( D \); we call \( D \) the distance set of this graph. A proper coloring of a graph is an assignment of colors to the vertices so that no two vertices joined by an edge receive the same color. The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the least number of colors in a proper coloring. We abbreviate \( \chi(Z(D)) \) by \( \chi(D) \). We refer to [1, 11] for graph-theoretic terminology not defined here.

When \( D \) is the set of all positive squares, we call \( Z(D) \) the square distance graph. When \( D \) is the set of all factorials, we obtain the factorial distance graph. The questions at the beginning of this section ask for the chromatic numbers of these two graphs. We will study the chromatic numbers of these and other distance graphs on the integers.

Distance graphs on the integers were introduced by Eggleton, Erdős, and Skilton in [6]. In [6, 7], the problem was posed of characterizing...
those distance sets $D$, containing only primes, such that $\chi(D) = 4$. This problem was studied in [4, 9, 12, 13]; see also [8]. More recently, [2, 3] have discussed the chromatic numbers of more general distance graphs with distance sets having 3 or 4 elements.

In this paper, we are primarily interested in distance graphs for which the distance set is infinite, although our results apply to finite distance sets as well. We begin in Section 2 with some easy lemmas on connectedness and bounds on the chromatic number. In Section 3, we consider distance graphs for which the distance set is totally ordered by the divisibility relation. We determine the chromatic numbers of all such graphs; in particular, we prove that they are all 4-colorable. In Section 4, we study periodic proper colorings of distance graphs and their relationship to the chromatic number.

Throughout this paper we will use standard notation for intervals to denote sets of consecutive integers. For example, $[2, 6]$ denotes the set $\{2, 3, 4, 5, 6\}$.

2. Basic Results

In this section, we establish some basic facts about the connectedness and chromatic number of distance graphs. The results of this section have all been at least partially stated in earlier works.

Our first result characterizes those distance sets for which the distance graph is connected. This result has been partially stated or implicitly assumed in a number of earlier works; see [6, p. 95]. For $D$ a set of positive integers, we note that $\gcd(D)$ is well defined when $D$ is infinite. Given a real number $k$ and a set $D$, we denote by $k \cdot D$ the set $\{kd : d \in D\}$.

Lemma 2.1. Let $D$ be a nonempty set of positive integers. The graph $\mathbb{Z}(D)$ is connected if and only if $\gcd(D) = 1$. Further, each component of $\mathbb{Z}(D)$ is isomorphic to $\mathbb{Z}\left(\frac{1}{\gcd(D)} \cdot D\right)$.

Proof. There is a path between vertices $k$ and $k + 1$ if and only if there exist $d_1, \ldots, d_a, e_1, \ldots, e_b \in D$ such that $d_1 + \cdots + d_a - e_1 - \cdots - e_b = 1$. This happens precisely when $\gcd(D) = 1$, and so the first statement of the lemma is true.

For the second statement, one isomorphism is the function $\varphi: \gcd(D) \cdot \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(k) = \frac{k}{\gcd(D)}$. \qed

When we determine the chromatic numbers of distance graphs, Lemma 2.1 will often allow us to assume that the GCD of the distance set is 1.
Next, we prove a useful upper bound on the chromatic number. This result is a slight generalization of a result of Chen, Chang, and Huang [2, Lemma 2].

**Lemma 2.2.** Let $D$ be a nonempty set of positive integers, and let $k$ be a positive integer. If $\frac{1}{\gcd(D)} \cdot D$ contains no multiple of $k$, then $\chi(D) \leq k$.

**Proof.** Let $D$ and $k$ be as stated. By Lemma 2.1 we may assume that $\gcd(D) = 1$. Thus, we assume that $D$ contains no multiple of $k$. We color the integers with colors $[0, k - 1]$, assigning to each integer $i$ the color corresponding to the residue class of $i$ modulo $k$. Two integers will be assigned the same color precisely when they differ by a multiple of $k$. Since no multiple of $k$ occurs in $D$, this is a proper $k$-coloring of $\mathbb{Z}(D)$. □

The converse of Lemma 2.2 holds when $k = 2$. This gives us a characterization of bipartite distance graphs: $\mathbb{Z}(D)$ is bipartite precisely when $\frac{1}{\gcd(D)} \cdot D$ contains no multiple of 2, that is, when all elements of $D$ have the same power of 2 in their prime factorizations. This result has been partially stated in earlier works; see [6, Thms. 8 & 10] and [2, Thms. 3 & 4].

**Proposition 2.3.** Let $D$ be a set of positive integers. The graph $\mathbb{Z}(D)$ is bipartite if and only if there exists a non-negative integer $k$ so that $\frac{1}{2} \cdot D$ contains only odd integers.

**Proof.** We may assume $D \neq \emptyset$. Since a graph is bipartite if and only if each component is bipartite, we may also assume, by Lemma 2.1, that $\gcd(D) = 1$. For such $D$ we show that $\mathbb{Z}(D)$ is bipartite if and only if each element of $D$ is odd.

$(\implies)$ Since $\gcd(D) = 1$, $D$ must have an odd element $d$. Suppose that $D$ has an even element $e$. If we begin at 0, take $e$ steps in the positive direction, each of length $d$, ending at $de$, and then take $d$ steps in the negative direction, each of length $e$, ending at 0, then we have followed a closed walk of odd length. Formally, the set

$$\{0, d, d \cdot 2, \ldots, d(e - 1), de, (d - 1)e, (d - 2)e, \ldots, 2e, e\}$$

is the vertex set of an odd circuit, and so $\mathbb{Z}(D)$ is not bipartite.

$(\impliedby)$ If every element of $D$ is odd, then $\chi(D) \leq 2$, by Lemma 2.2. □

The converse of Lemma 2.2 does not hold when $k > 2$. For example, let $k > 2$, and let $D = \{1, k\}$. Then $\frac{1}{\gcd(D)} \cdot D$ contains a multiple of $k$, and yet $\chi(D) \leq 3 \leq k$ (this is not hard to show; it will also follow from Lemma 3.3). As with general graphs, it appears to be quite difficult to
determine when a distance graph has a proper $k$-coloring, for $k \geq 3$. However, when $D$ is finite, there does exist an algorithm to determine $\chi(D)$. This was proven for $D$ a finite set of primes by Eggleton, Erdős, and Skilton [9, Corollary to Thm. 2]; essentially the same proof works for more general sets.

**Theorem 2.4.** There exists an algorithm to determine $\chi(D)$ for $D$ a finite set of positive integers.

*Proof.* (Outline—see [9, Thm. 2]) Let $q = \max(D)$. Then $\chi(D) \leq q+1$, by Lemma 2.2. We consider the colorings of the subgraph of $\mathbb{Z}(D)$ induced by $S = [1, q^a + q]$. We show that, for $k \leq q$, if $S$ has a proper $k$-coloring, then $\chi(D) \leq k$; thus, $\chi(D)$ can be determined by a bounded search.

Let $k \leq q$, and suppose that $S$ has a proper $k$-coloring. The number of $k$-colorings of a block of $q$ consecutive integers is at most $q^a$. Since $S$ contains $q^a + 1$ such blocks, two such blocks contained in $S$ (say $[a, a + q - 1]$ and $[b, b + q - 1]$, with $a < b$) receive the same pattern of colors. We extend the coloring of $[a, b + q - 1]$ to a coloring $f$ of $\mathbb{Z}$ using the rule $f(i + a - b) = f(i)$, for all $i$. We can show that this is a proper coloring if $\mathbb{Z}(D)$, and so $\chi(D) \leq k$. □

While an algorithm exists to determine $\chi(D)$ for finite $D$, we do not know whether there is an efficient algorithm. For finite graphs, determining whether the chromatic number is at most $k$ is NP-complete [10]. We conjecture that this is also true for distance graphs with finite distance sets.

**Conjecture 2.5.** Let $k \geq 3$. Determining whether $\chi(D) \leq k$ for finite sets $D$ is NP-complete. □

3. Divisibility Chains

We now focus on a particular class of distance graphs: those in which the distance set is totally ordered by divisibility. We show that all such graphs are 4-colorable, and we determine their chromatic numbers.

A *divisibility chain* is a set of positive integers that is totally ordered by the divisibility relation. When $D$ is a (finite or infinite) divisibility chain we denote the elements of $D$ by $d_1, d_2, \ldots$, where $d_1 | d_2 | \cdots$. The *ratios* of $D$ are the numbers $r_i = \frac{d_{i+1}}{d_i}$, for each $i$. When determining $\chi(D)$, we may, by Lemma 2.1, assume that $\gcd(D) = d_1 = 1$. Thus, $\chi(D)$ depends only on the ratios. We may also assume that all the $d_i$'s are distinct, that is, that none of the ratios is equal to 1.
A string over \( \{1, 2\} \) is a finite sequence of 1’s and 2’s, written without spaces or separators. For example, \( \alpha = 1211 \) is a string of length 4 with \( \alpha_1 = 1, \alpha_2 = 2 \), etc.

For \( k \) a positive integer, a string \( \alpha \) is \( k \)-compatible with a distance set \( D \) if there is a proper \( k \)-coloring of \( \mathbb{Z}(D) \) with colors \([0, k-1]\) such that the differences, modulo \( k \), between colors of consecutive vertices form repeated copies of \( \alpha \). Below is part of such a coloring with \( k = 4 \) and \( \alpha = 1211 \).

| vertex | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|--------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| color  | 0 | 1 | 3 | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 1  | 2  | 3  |
| difference | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 1  | 1  | 1  |

We see that 1211 is not 4-compatible with \{3\}, since, for example, 1 and 4 receive the same color; this is because the sum of three consecutive entries of the repeated copies of \( \alpha \) is divisible by 4 (i.e., \( 2 + 1 + 1 = 4 \)).

Generally, a string \( \alpha \) is \( k \)-compatible with \( \{d\} \) if the concatenation of repeated copies of \( \alpha \) contains no \( d \) consecutive entries whose sum is a multiple of \( k \).

**Theorem 3.1.** If \( D \) is a divisibility chain, then \( \chi(D) \leq 4 \).

**Proof.** We may assume that the \( d_i \)'s are all distinct, and that \( d_1 = 1 \). We use notation such as \( \alpha^n \) to denote a string; the superscript does not denote exponentiation or concatenation.

**Claim.** For \( n = 1, 2, 3, \ldots \), there exist strings \( \alpha^n, \beta^n \) of length \( d_n \) over \( \{1, 2\} \) such that

1. \( \alpha^n, \beta^n \) differ only in the first entry, with \( \alpha^n_1 = 1 \), and \( \beta^n_1 = 2 \),
   and
2. if \( \gamma \) is a string resulting from the concatenation of any number of copies of \( \alpha^n \) and/or \( \beta^n \), in any order, then \( \gamma \) is 4-compatible with \( \{d_1, d_2, d_3, \ldots, d_n\} \).

Before we prove the claim, we show that the theorem follows from it. If the claim holds, then, for each \( n \), \( \alpha^n \) is 4-compatible with \( \{d_1, d_2, \ldots, d_n\} \), and so \( \mathbb{Z}(\{d_1, d_2, \ldots, d_n\}) \) has a proper 4-coloring. Since every finite subgraph of \( \mathbb{Z}(D) \) is isomorphic to a finite subgraph of \( \mathbb{Z}(\{d_1, \ldots, d_n\}) \) for some \( n \), every finite subgraph of \( \mathbb{Z}(D) \) is 4-colorable, and we may conclude that \( \chi(D) \leq 4 \), by a compactness argument. Hence, it suffices to prove the claim.

**Proof of Claim.** We proceed by induction on \( n \). For \( n = 1 \), we assumed that \( d_n = 1 \). Let \( \alpha^n = 1 \), and let \( \beta^n = 2 \); these satisfy the claim for \( n = 1 \).
Now suppose that \( n \geq 1 \), and that the claim holds for \( n \). Define \( s \) and \( t \) as follows.

\[
s := \sum_{i=1}^{d_n} \alpha_i^{n+1}, \quad t := \sum_{i=1}^{d_n} \beta_i^{n+1} = s + 1.
\]

We show first that \([r_n \cdot s, r_n \cdot t]\) contains integers \( w, w + 1 \), neither a multiple of 4. If \( r_n > 2 \), then this is true since there are at least 4 consecutive integers in \([r_n \cdot s, r_n \cdot t]\). On the other hand, if \( r_n = 2 \), then \( r_n \cdot s \) and \( r_n \cdot t \) are both even. Exactly one of the two is divisible by four. If \( 4 \mid (r_n \cdot s) \), then let \( w = r_n \cdot s \); otherwise, let \( w = r_n \cdot s \).

Now we choose \( a \geq 1, b \geq 0 \) so that \( a + b = r_n \) and \( as + bt = w \); let \( b = w - r_n \cdot s \), and let \( a = r_n - b \). We define \( \alpha^{n+1} \) to be the concatenation of \( a \) copies of \( \alpha^n \) followed by \( b \) copies of \( \beta^n \). We let \( \beta^{n+1} \) be the concatenation of \( \beta^n \) followed by \( a - 1 \) copies of \( \alpha^n \) followed by \( b \) copies of \( \beta^n \); equivalently, \( \beta^{n+1} \) is \( \alpha^{n+1} \) with its first entry replaced by 2.

Now, \( \alpha^{n+1} \) and \( \beta^{n+1} \) both have length \( d_{n+1} \), since \( a + b = r_n \), and \( \alpha^{n+1} \) and \( \beta^{n+1} \) differ only in the first entry. Let \( \gamma \) be a concatenation of copies of \( \alpha^{n+1}, \beta^{n+1} \). Then \( \gamma \) is a concatenation of copies of \( \alpha^n \) and \( \beta^n \), and so, by the induction hypothesis, \( \gamma \) is 4-compatible with \( \{d_1, d_2, \ldots, d_n\} \).

In order to prove that \( \alpha^{n+1}, \beta^{n+1} \) satisfy the claim, it remains only to show that \( \gamma \) is 4-compatible with \( \{d_{n+1}\} \). This is true if the concatenation of repeated copies of \( \gamma \) has no \( d_{n+1} \) consecutive entries whose sum is a multiple of 4. Since \( \alpha^{n+1} \) and \( \beta^{n+1} \) differ in only one entry, the sum of \( d_{n+1} \) consecutive entries of repeated copies of \( \gamma \) is equal either to the sum of the entries of \( \alpha^{n+1} \) or to the sum of the entries of \( \beta^{n+1} \); that is, it is equal

\[
either to \sum_{i=1}^{d_{n+1}} \alpha_i^{n+1} = as + bt = w, \\
or to \sum_{i=1}^{d_{n+1}} \beta_i^{n+1} = t + (a - 1)s + bt = w + 1.
\]

Neither of these is a multiple of 4.

Thus, the claim is proven. \( \square \)

The bound in Theorem 3.1 is sharp: if \( D = \{1, 2, 6\} \), then the subgraph of \( Z(D) \) induced by \([1, 7]\) has no proper 3-coloring. On the other hand, graphs satisfying the hypotheses of the theorem need not have chromatic number 4, even if \( D \) is infinite. For example, if \( D \) is the set
of all powers of 3, then every element of \( D \) is odd, and so \( \chi(D) = 2 \), by Proposition 2.3.

**Example 3.2.** Let \( D = \{d_1, d_2, d_3, \ldots \} \), where \( d_i = i! \) for each \( i \). We use the technique of the above proof to produce part of a proper 4-coloring of \( \mathbb{Z}(D) \), the factorial distance graph.

Let \( \alpha^1 = 1 \) and \( \beta^1 = 2 \). We find consecutive nonmultiples of 4 in \([2 \cdot 1, 2 \cdot 2] = \{2, 3, 4\} \): let \( w = 2 \), so that \( w + 1 = 3 \). So, \( a = 2 \), and \( b = 0 \). The string \( \alpha^2 \) is 2 copies of \( \alpha^1 \) followed by 0 copies of \( \beta^1 \). That is, \( \alpha^2 = 11 \), and so \( \beta^2 = 21 \).

Continuing, we find consecutive nonmultiples of 4 in \([3 \cdot 2, 3 \cdot 3] = \{6, 7, 8, 9\} \): let \( w = 6 \), so that \( w + 1 = 7 \). So, \( a = 3 \), and \( b = 0 \). The string \( \alpha^3 \) is 3 copies of \( \alpha^2 \) followed by 0 copies of \( \beta^2 \). That is, \( \alpha^3 = 111111 \), and so \( \beta^3 = 211111 \).

Once again, we find consecutive nonmultiples of 4 in \([4 \cdot 6, 4 \cdot 7] = [24, 28]\): let \( w = 25 \), so that \( w + 1 = 26 \). So, \( a = 3 \), and \( b = 1 \). The string \( \alpha^4 \) is 3 copies of \( \alpha^3 \) followed by 1 copy of \( \beta^3 \). That is, \( \alpha^4 = 111111111111111112111111 \), and so \( \beta^4 = 211111111111111121111111 \).

The coloring of \([1, 24]\) obtained from \( \alpha^4 \) is the following.

\[
0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0.
\]

□

In *almost* the entire proof of Theorem 3.1, “4” can be replaced by “3”; that is, we use 3-compatibility instead of 4-compatibility, we find a 3-coloring instead of a 4-coloring, and we find consecutive nonmultiples of 3 instead of 4. The one place where 4 is required is the argument in the proof showing the existence of two consecutive nonmultiples when \( r_n = 2 \). Thus, if we require that \( r_n \neq 2 \) for each \( n \), then we can replace 4 by 3 in the proof, and we have the following result.

**Lemma 3.3.** Let \( D \) be a divisibility chain, with ratios \( r_1, r_2, \ldots \). If \( r_i \neq 2 \) for all \( i \), then \( \chi(D) \leq 3 \). □

Again, the bound in this result is sharp: if \( D = \{1, 4\} \), then the subgraph of \( \mathbb{Z}(D) \) induced by \([1, 5]\) has no proper 2-coloring.

We now find \( \chi(D) \) for every divisibility chain \( D \).

**Theorem 3.4.** Let \( D \) be a divisibility chain, with ratios \( r_1, r_2, \ldots \). All of the following hold.

1. \( \chi(D) \leq 4 \).
2. \( \chi(D) \leq 3 \) if and only if there do not exist \( i, j \) with \( i < j \), \( r_i = 2 \), and \( 3 \mid r_j \).
3. \( \chi(D) \leq 2 \) if and only if \( r_i \) is odd, for each \( i \).
4. \( \chi(D) = 1 \) if and only if \( D = \emptyset \).
Proof. Statement (1) follows from Theorem 3.1, statement (3) follows from Proposition 2.3, and statement (4) holds because a graph is 1-colorable precisely when it has no edges. It remains to prove statement (2). We may assume that \( d_1 = 1 \).

\[ \implies \text{ Suppose that there exist } i \text{ and } j \text{ with } i < j, r_i = 2, \text{ and } 3 \mid r_j. \]

Then \( d_{i+1} = 2d_i \), and \( d_{j+1} \) is divisible by \( 3d_i \). Suppose that \( \mathbb{Z}(D) \) has a proper 3-coloring. Consider the colors assigned to the multiples of \( d_i \).

Since \( d_i, 2d_i \in D \), vertices \( 0, d_i, \text{ and } 2d_i \) induce a complete subgraph and so must be assigned 3 different colors. Similarly, \( d_i, 2d_i, \text{ and } 3d_i \) must receive 3 different colors, and so 0 and \( 3d_i \) have the same color. Continuing this argument, all multiples of \( 3d_i \) must receive the same color, including 0 and \( d_{j+1}, \) which is impossible.

\[ \iff \text{ Suppose there do not exist } i \text{ and } j \text{ with the properties specified in statement (2); that is, every ratio divisible by 3 precedes every ratio equal to 2 in the list } \{ r_1, r_2, \ldots \}. \]

If there exist infinitely many ratios that are divisible by 3, then, by our assumption, there exists no ratio equal to 2, and so \( \chi(D) \leq 3 \), by Lemma 3.3. Thus, we may assume that there are only finitely many ratios that are divisible by 3.

Let \( c \) be the least positive integer such that \( 3 \nmid r_i \), for all \( i \geq c \). Then none of \( r_1, r_2, \ldots, r_{c-1} \) is equal to 2. Thus, by Lemma 3.3, the graph \( \mathbb{Z}(\{d_1, d_2, \ldots, d_c\}) \) has a proper 3-coloring. By the proof of Lemma 3.3—that is, the proof of Theorem 3.1, as modified to prove Lemma 3.3—there is a string \( \alpha^c \) of length \( d_c \) over \( \{1, 2\} \) such that \( \alpha^c \) is 3-compatible with \( \{d_1, d_2, \ldots, d_c\} \).

We claim that \( \alpha^c \) is 3-compatible with \( D \). To see this, first note that

\[ \sum_{i=1}^{d_c} \alpha^c_i \]

is not a multiple of 3, since \( \alpha^c \) is 3-compatible with \( \{d_c\} \). Thus, if integers \( x \) and \( y \) differ by a multiple of \( d_c \), then, in a 3-coloring whose differences, modulo 3, form repeated copies of \( \alpha^c \), \( x \) and \( y \) receive the same color precisely when their difference is a multiple of \( 3d_c \). Now, no \( r_i \) with \( i \geq c \) is divisible by 3; thus, no \( d_i \) with \( i \geq c \) is divisible by \( 3d_c \). We conclude that, for each \( i \geq c \), no two integers with difference \( d_i \) receive the same color, and so \( \alpha^c \) is 3-compatible with \( \{d_c, d_{c+1}, d_{c+2}, \ldots\} \).

Thus, \( \alpha^c \) is 3-compatible with \( D \), and we have \( \chi(D) \leq 3. \)

By Theorem 3.4, the chromatic number of the factorial distance graph is 4. We will have more to say about this graph in the next section.
4. Periodic Colorings

In this section, we consider periodic proper colorings of distance graphs. We characterize those distance graphs that have no periodic proper coloring, and we find a relationship between the chromatic number and the nonexistence of periodic proper colorings. Periodic colorings have been previously studied in [9].

**Lemma 4.1.** Let $D$ be a set of positive integers, and let $k$ be a positive integer. If $D$ contains no multiple of $k$, then $\mathbb{Z}(D)$ has a periodic proper $k$-coloring.

**Proof.** We may assume $D \neq \emptyset$. The proof of Lemma 2.2 gives a periodic proper $k$-coloring of each component of $\mathbb{Z}(D)$; this results in a periodic proper $k$-coloring of the graph. □

We can use Lemma 4.1 to characterize those distance graphs that have no periodic proper coloring. The following result generalizes an observation of Eggleton [5] that the square distance graph has no periodic proper coloring.

**Proposition 4.2.** Let $D$ be a set of positive integers. The graph $\mathbb{Z}(D)$ has no periodic proper coloring if and only if $D$ contains a multiple of every positive integer.

**Proof.** ($\Rightarrow$) If there is some positive integer $k$ such that $D$ contains no multiple of $k$, then, by Lemma 4.1, $\mathbb{Z}(D)$ has a periodic proper coloring.

($\Leftarrow$) Let $D$ contain a multiple of every positive integer. Let $\mathbb{Z}(D)$ be colored in a periodic manner; say this coloring has period $k$. Every pair of vertices whose difference is a multiple of $k$ will have the same color. Since $D$ contains some multiple of $k$, this cannot be a proper coloring. □

**Remark 4.3.** It follows from Theorem 3.4 that the chromatic number of the factorial distance graph is 4. However, by Proposition 4.2, the factorial distance graph has no periodic proper coloring. □

Now we examine the effect of the existence of uniquely colorable subgraphs on proper colorings of distance graphs. We prove a useful lower bound on the chromatic number based on uniquely colorable subgraphs and periodic colorings.

**Proposition 4.4.** Let $D$ be a set of positive integers, and let $k$ be a positive integer. If $\mathbb{Z}(D)$ has a finite, uniquely $k$-colorable subgraph, then every proper $k$-coloring of $\mathbb{Z}(D)$ is periodic.
Proof. Suppose that \( H \) is a uniquely \( k \)-colorable subgraph of \( \mathbb{Z}(D) \). We may assume that the least integer that is a vertex of \( H \) is 1. Let \( n \) be the greatest-numbered vertex of \( H \). Since \( H \) is uniquely \( k \)-colorable, every \( k \)-coloring of \([1, n-1]\) that can be extended to a proper \( k \)-coloring of \( \mathbb{Z}(D) \) has a unique extension to a proper \( k \)-coloring of \([1, n]\).

In short, once we have \( k \)-colored \([1, n-1]\), the color of vertex \( n \) is forced. But, \([2, n+1]\) also contains a copy of \( H \), and so once we have colored \([2, n]\), the color of vertex \( n+1 \) is forced. By an inductive argument, we can see that \( k \)-coloring \([1, n-1]\) completely determines the coloring of \([1, \infty)\).

Essentially the same argument works in the opposite direction: \( k \)-coloring \([2, n]\) forces a certain color to occur at vertex 1. Hence, \( k \)-coloring any set of \( n-1 \) consecutive vertices determines the coloring of all of \( \mathbb{Z} \).

Now, there are only a finite number of \( k \)-colorings of \( n-1 \) consecutive integers. Since the colorings of blocks of \( n-1 \) consecutive integers must eventually repeat, every proper \( k \)-coloring of the distance graph is periodic. \( \square \)

Proposition 4.2 and Proposition 4.4 have nearly opposite conclusions; the former concludes that the graph has no periodic proper coloring, while the latter concludes that every proper \( k \)-coloring of the distance graph is periodic. Suppose that a distance graph satisfies the hypothesis of both propositions, that is, the distance set contains a multiple of every positive integer, and the graph has a finite, uniquely \( k \)-colorable subgraph. Then the conclusions of both propositions must be true: there is no periodic proper coloring, and yet every proper \( k \)-coloring is periodic. We can only conclude that the distance graph must have no proper \( k \)-coloring at all, and so we have the following result.

**Theorem 4.5.** Let \( D \) be a set of positive integers, and let \( k \) be a positive integer. If \( D \) contains a multiple of every positive integer, and \( \mathbb{Z}(D) \) has a finite, uniquely \( k \)-colorable subgraph, then \( \chi(D) \geq k+1 \). \( \square \)

We can use Theorem 4.5 to place a lower bound on the chromatic number of the square distance graph. Let \( D \) be the set of all positive squares. Any Pythagorean triple gives a \( K_3 \) in the square distance graph. For example, the vertices 0, \( 3^2 \), \( 5^2 \) induce a \( K_3 \), since \( 3, 4, 5 \) is a Pythagorean triple. Since \( \mathbb{Z}(D) \) has a \( K_3 \) subgraph, \( \chi(D) \geq 3 \). Furthermore, \( K_3 \) is uniquely 3-colorable, and \( D \) contains a multiple of every positive integer. Thus, \( \chi(D) \geq 4 \), by Theorem 4.5. Eggleton [5] has found a \( K_4 \) in the square distance graph: the vertices are 0, 672, 680, and 697. We have \( 680^2 - 672^2 = 104^2 \), \( 697^2 - 680^2 = 153^2 \), and
697^2 - 672^2 = 185^2. Since the square distance graph has a uniquely 4-colorable subgraph, we have the following result.

**Corollary 4.6.** The chromatic number of the square distance graph is at least 5. □

We do not know whether the square distance graph contains a $K_5$ or whether its chromatic number is greater than 5.

**Problem 4.7.** What is the chromatic number of the square distance graph? Equivalently, what is the least number of classes into which the integers can be partitioned, so that no two members of the same class differ by a square? □

It seems likely that no finite number of colors suffices.

We can ask similar questions about the distance graph resulting when $D$ is the set of all positive $n$th powers, for $n \geq 3$. We know that these graphs contain no $K_3$ (this is equivalent to “Fermat’s Last Theorem”, proven by Wiles [14]), that they do not have periodic proper colorings, by Proposition 4.2, and that their chromatic numbers are all at least 3, by Theorem 4.5 (or Proposition 2.3). It seems likely that these graphs have infinite chromatic number as well.

As noted in Section 1, determining which distance graphs have chromatic number at most $k$, for a given $k \geq 3$, appears to be difficult. A similar problem, whose difficulty we cannot estimate at this time, is the following.

**Problem 4.8.** Characterize those sets $D$ such that $\chi(D)$ is infinite. □

No coloring requiring an infinite number of colors is periodic. Thus, by Proposition 4.2, a necessary condition for such sets $D$ is that they contain a multiple of every integer. However, this condition is not sufficient, by Remark 4.3.

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