Cut ideals of $K_4$-minor free graphs
are generated by quadrics.

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Abstract

Cut ideals are used in algebraic statistics to study statistical models defined by graphs. Intuitively, topological restrictions on the graphs should imply structural statements about the corresponding cut ideals. Several theorems and many computer calculations support that.

Sturmfels and Sullivant conjectured that the cut ideal is generated by quadrics if and only if the graph is free of $K_4$-minors. Parts of the conjecture has been resolved by Brennan and Chen, and later by Nagel and Petrović.

We prove the full conjecture by introducing a new type of toric fiber product theorem.

1 Introduction

The theory of cut ideals was initiated by Sturmfels and Sullivant [7]. To every graph $G$ they associated a toric ideal $I_G$ called the cut ideal of $G$. From theorems about similar constructions, and computer calculations, it is reasonable to believe that topological properties of $G$ should be reflected in algebraic properties of $I_G$.

Theorem (Conjectured by Sturmfels and Sullivant [7]) The cut ideal is generated by quadrics if and only if $G$ is free of $K_4$ minors.
Partial results on the conjecture was proved by Brennan and Chen [1], and then by Nagel and Petrović [6].

The conjecture follows as a corollary of Theorem 3.6, which is a fiber product type theorem. In the same way as the fiber product theorems in [2] and [7] could be generalized in [8], we will present a more general form of Theorem 3.6 in [4]. Methods from this paper were used on ideals of graph homomorphisms in Engström and Norén’s paper [5].

2 Cut ideals

A cut of a graph $G$ is a partition of its vertex set into two sets. An edge is in the cut if its vertices belongs to different parts. As defined in [7]:

$$\phi_G : \mathbb{K}[q] \to \mathbb{K}[s,t], \quad q_{A|B} \mapsto \prod_{ij \in A|B} s_{ij} \cdot \prod_{ij \notin A|B} t_{ij},$$

where $A \subseteq V(G)$, and $A \mid B$ and $B \mid A$ is the same cut. The cut ideal $I_G$ is the kernel of $\phi_G$. The largest degree of a minimal generator of $I_G$ is $\mu(G)$. By Corollary 3.3 of [7] the contraction of an edge or deletion of a vertex cannot increase $\mu$. In Theorem 2.1 of [7] it is proved that if $G$ is glued together from two graphs $G_1$ and $G_2$ over a complete graph with zero, one, or two vertices, then the cut ideal $I_G$ is generated by lifts of generators of $I_{G_1}$ and $I_{G_2}$; and quadratic binomials for sorting cuts. The main theorem of this note is a variation on Theorem 2.1 of [7] when gluing over an edge.

3 Decompositions of graphs and ideals

The induced subgraph of $G$ on $S$ is denoted $G[S]$.

**Definition 3.1** Let $u, v$ be two vertices of $G$ and $A_1 \mid B_1, A_2 \mid B_2, \ldots, A_n \mid B_n$ a list of cuts. The height, $h_{u,v}(q)$, of

$$q = q_{A_1|B_1} q_{A_2|B_2} \cdots q_{A_n|B_n}$$

with respect to $u$ and $v$ is the number of cuts in the list with $u$ and $v$ in different parts.

If there is an edge between $u$ and $v$ in $G$ then $h_{u,v}(q)$ is the degree of $s_{uv}$ in $\phi_G(q)$. Another way to define the height of $q$ with respect to $u$ and $v$ is as the degree of $s_{uv}$ in $\phi_{G+uv}(q)$, and that is a good way to think of it.

**Definition 3.2** A set of generators

$$q_{A_1|B_1} q_{A_2|B_2} \cdots q_{A_n|B_n}$$

of $I_G$ is slow-varying with respect to the vertices $u$ and $v$ of $G$ if

$$\left| h_{u,v}(q_{A_1|B_1} \cdots q_{A_n|B_n}) - h_{u,v}(q'_{A_1|B_1} \cdots q'_{A_n|B_n}) \right| \leq 2$$

for all $i$. 

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Lemma 3.3 If $w_1 - w_2 - \cdots - w_k$ is a path in $G$ then

$$h_{w_1, w_k}(q_{A|B}) \equiv \sum_{i=1}^{k-1} (s_{w_i, w_{i+1}} - \text{degree of } \phi_G(q_{A|B}))$$

modulo 2.

PROOF: A walk on the path from $w_1$ to $w_k$ crosses the cut an odd number of times if and only if $w_1$ and $w_k$ are in different parts. \qed

Lemma 3.4 If there is a path in $G$ from $u$ to $v$ and $\phi_G(q) = \phi_G(q')$ then $h_{u,v}(q) \equiv h_{u,v}(q')$ modulo 2.

PROOF: Use Lemma 3.3. \qed

Proposition 3.5 Any set of generators of $I_G$ validating $\mu(G) \leq 2$, is slow-varying with respect to any vertex pair.

PROOF: Clear. \qed

Theorem 3.6 Let $G$ be a graph with two special non-adjacent vertices $u$ and $v$. Assume that $G$ almost can be decomposed into a left and right part: There are $L, R \subseteq V(G)$ such that $L \cup R = V(G)$, $L \cap R = \{u, v\}$, and $E(G) = E(G[L]) \cup E(G[R])$.

If there is a path from $u$ to $v$ both in $G[L]$ and in $G[R]$, and there are slow-varying generators of both $I_{G[L]}$ and $I_{G[R]}$ with respect to $u$ and $v$, then

$$\mu(G) = \max\{\mu(G[L]), \mu(G[R]), \mu(G[L] + uv), \mu(G[R] + uv)\}.$$ 

The cut ideal of $G$ is generated by a union of

(i) lifts of generators of $I_{G[L]+uv}$,

(ii) lifts of generators of $I_{G[R]+uv}$,

(iii) joins of generators $q_1 - q_2$ of $I_{G[L]}$ and $q_3 - q_4$ of $I_{G[R]}$ such that $|h_{u,v}(q_1) - h_{u,v}(q_2)| = |h_{u,v}(q_3) - h_{u,v}(q_4)| = 2$,

(iv) quadratic binomials to reorder with.

PROOF: The reader who is not familiar with the proof of Theorem 2.1 in \cite{7} should study that first, since this proof builds on a modification of its basic structure.

First we prove the $\leq$ case.

We will prove the theorem by an explicit construction of generators for $I_G$. Let

$$q = \prod_{i=1}^{n} q_{A_i|B_i} \quad \text{and} \quad q' = \prod_{i=1}^{n} q_{A_i'|B_i'}$$
be two elements of $\mathbb{K}[A|B \mid A \sqcup B = V(G)]$ with $\phi_G(q) = \phi_G(q')$. If we for any such $q$ and $q'$ can construct a sequence of moves from $q$ to $q'$, then we can generate $I_G$. A move from $q_1$ to $q_2$ is a composition of a $q_3$ with a binomial generator $q_4 - q_5$ such that

$$q_1 - q_2 = q_3(q_4 - q_5).$$

We can assume that $h_{u,v}(q) \geq h_{u,v}(q')$

**Main idea:** To construct the sequence from $q$ to $q'$ we use sequences from $q_L$ to $q'_L$ and from $q_R$ to $q'_R$. ($q_L$ is $q$ induced on $L$ and similar for $q_R$.) If we simply took a sequence from $q_L$ to $q'_L$ given by $I_{G[L]}$ and a corresponding one on $R$ and tried to glue them together it would sometimes not work on the vertex pair $u$ and $v$. The thing that goes wrong is that the number of cuts with $u$ and $v$ in different parts does not need to be the same. That is, the height $h_{u,v}$ could be different on the left and the right side. But we know that the height is the same for $q_L$ and $q_R$ in the begining of the sequence, and for $q'_L$ and $q'_R$ in the end of the sequence.

In the sequence $q_L,\ldots,q'_L$ the number of cuts with $u$ and $v$ in different parts can look like the fat gray line in Figure 1. If it changes, it changes by an even number by Lemma 3.3. It never changes by more than 2 since $I_{G[L]}$ is slow-varying. Since the height of the sequence $q_R,\ldots,q'_R$ does not have to have the same shape as the grey line, we need to normalize the sequences.

**How to normalize the sequence $q_L,\ldots,q'_L$:** We do this as described in Figure 1. Let $q''_{L,h}$ be the last element in the sequence with height $h$ for $h = h_{u,v}(q_L), h_{u,v}(q_L) - 2,\ldots,h_{u,v}(q'_L) + 2, h_{u,v}(q'_L)$. Let $q_{L,h}$ be the element after $q''_{L,h+2}$ in the sequence for $h = h_{u,v}(q_L) - 2,\ldots,h_{u,v}(q'_L) + 2, h_{u,v}(q'_L)$. And let $q_{L,h}(q_L) = q_L$. In our normalized sequence we still go from $q''_{L,h}$ to $q_{L,h-2}$ by using a generator of $I_{G[L]}$. But from $q_{L,h}$ to $q''_{L,h}$ we build up the sequence by using generators of $I_{G[L]+uv}$, this is possible since the heights of $q_{L,h}$ and $q''_{L,h}$ are the same. For our normalized sequence the height is never increasing.
Let $F$ be assumed to have an order such that the extension is needed to allow binomial generators of different degree from $F$ to $F'$.

Thus we need four kinds of moves:

- **(F₁)** all from $I_{G[L]+uv}$,
- **(F₂)** all from $I_{G[R]+uv}$,
- **(F₃)** those from $I_{G[L]}$ and $I_{G[R]}$ that change height by 2,
- **(F₄)** reorderings to match cuts.

Let $F_L$, $F_{L+uv}$, $F_R$, and $F_{R+uv}$ be the binomial generating sets of $I_{G[L]}$, $I_{G[L]+uv}$, $I_{G[R]}$, and $I_{G[R]+uv}$. If the maximal degree of a binomial in $F_L$ or $F_R$ is $M$ then extend $F_L$ to

$$
\tilde{F}_L = \{ q_1(q_2 - q_3) \mid \text{degree of } q_1q_2 \leq M \text{ and } q_2 - q_3 \in F_L \}
$$

and $F_R$ to

$$
\tilde{F}_R = \{ q_1(q_2 - q_3) \mid \text{degree of } q_1q_2 \leq M \text{ and } q_2 - q_3 \in F_R \}.
$$

The extension is needed to allow binomial generators of different degree from the left and right side to be joined when the height decreases by two. In the definitions of $F_1, F_2,$ and $F_3$, any product of the type

$$
\prod_{i=1}^{m} q_{C_i | D_i}
$$

is assumed to have an order such that

$$
h_{u,v}(q_{C_1 | D_1}) \geq \cdots \geq h_{u,v}(q_{C_m | D_m}).
$$

Let

- **(F₁)**

\[
F_1 = \left\{ \prod_{i=1}^{m} q_{C_i | D_i} - \prod_{i=1}^{m} q_{C'_i | D'_i} \in \mathbb{K}[q] \mid \prod_{i=1}^{m} q_{C_i \cap L | D_i \cap L} - \prod_{i=1}^{m} q_{C_i \cap L | D'_i \cap L} \in F_{L+uv} \right. \\
\left. C_i \cap R = C'_i \cap R \text{ for } i = 1, \ldots, m \right\}
\]

- **(F₂)**

\[
F_2 = \left\{ \prod_{i=1}^{m} q_{C_i | D_i} - \prod_{i=1}^{m} q_{C'_i | D'_i} \in \mathbb{K}[q] \mid \prod_{i=1}^{m} q_{C_i \cap R | D_i \cap R} - \prod_{i=1}^{m} q_{C_i \cap R | D'_i \cap R} \in F_{R+uv} \right. \\
\left. C_i \cap L = C'_i \cap L \text{ for } i = 1, \ldots, m \right\}
\]

- **(F₃)**

\[
F_3 = \left\{ \prod_{i=1}^{m} q_{C_i | D_i} - \prod_{i=1}^{m} q_{C'_i | D'_i} \in \mathbb{K}[q] \mid \prod_{i=1}^{m} q_{C_i \cap L | D_i \cap L} - \prod_{i=1}^{m} q_{C_i \cap L | D'_i \cap L} \in \tilde{F}_L \\
\prod_{i=1}^{m} q_{C_i \cap R | D_i \cap R} - \prod_{i=1}^{m} q_{C_i \cap R | D'_i \cap R} \in \tilde{F}_R \\
h_{u,v}(\prod_{i=1}^{m} q_{C_i | D_i}) \neq h_{u,v}(\prod_{i=1}^{m} q_{C_i | D_i}) \right. \\
\left. C_1 \cap L = C'_1 \cap L, \quad C_2 \cap L = C'_2 \cap L \\
C_1 \cap R = C'_1 \cap R, \quad C_2 \cap R = C'_2 \cap R \right\}
\]

- **(F₄)**

\[
F_4 = \left\{ \prod_{i=1}^{2} q_{C_i | D_i} - \prod_{i=1}^{2} q_{C'_i | D'_i} \in \mathbb{K}[q] \mid C_1 \cap L = C'_1 \cap L, \quad C_2 \cap L = C'_2 \cap L \\
C_1 \cap R = C'_1 \cap R, \quad C_2 \cap R = C'_2 \cap R \right\}.
\]
We have that $F = F_1 \cup F_2 \cup F_3 \cup F_4$ is a generating set of $I_G$. From that we get:

$$
\mu(G) \leq \max\{2, \mu(G[L]), \mu(G[R]), \mu(G[L] + uv), \mu(G[R] + uv)\}
$$

In $G[L]$ there is an induced path from $u$ to $v$ with more than one edge. For the path with two edges we have $\mu = 2$ and thus by contraction $\mu \geq 2$ for any path, which shows that $\mu(G[L]) \geq 2$. The 2 can be removed to get:

$$
\mu(G) \leq \max\{\mu(G[L]), \mu(G[R]), \mu(G[L] + uv), \mu(G[R] + uv)\}
$$

We are left with proving the $\geq$ inequality. Removing vertices do not increase $\mu$, so $\mu(G) \geq \mu(G[L]), \mu(G[R])$. Contracting edges also do not increase $\mu$. Start with $G$ and repeatedly contract any edge $uv$ if $w \notin L$. Contractions will be possible until a graph $G[L \cup R'] + uv$ is reached where $R'$ are the vertices of $G[R]$ not in the same component as $u$ and $v$. We get the edge $uv$ from the path between $u$ and $v$ in $G[R]$. Removing $R'$ we get $\mu(G) \geq \mu(G[L \cup R'] + uv) \geq \mu(G[L] + uv)$. In the same way we get that $\mu(G) \geq \mu(G[R] + uv)$ and can conclude that

$$
\mu(G) = \max\{\mu(G[L]), \mu(G[R]), \mu(G[L] + uv), \mu(G[R] + uv)\}
$$

since $\mu(G) \geq \mu(G[L]), \mu(G[R]), \mu(G[L] + uv), \mu(G[R] + uv)$.

\[\square\]

**Corollary 3.7** Let $H_1$ and $H_2$ be two graphs on different vertex sets satisfying:

- $u_1, v_1$ are two distinct non-adjacent vertices of $H_1$,
- $u_2, v_2$ are two distinct non-adjacent vertices of $H_2$,
- $H_1$ and $H_2$ are connected,
- $\mu(H_1), \mu(H_2), \mu(H_1 + u_1v_1), \mu(H_2 + u_2v_2) \leq 2$.

Then $\mu \leq 2$ for the graph we get by gluing $u_1 = u_2$ and $v_1 = v_2$ in $H_1 \cup H_2$.

**PROOF:** Insert Proposition 3.5 into Theorem 3.6 \[\square\]

For a definition and basic material on series-parallel graphs, in particular on the gluing constructions, we refer to [3].

**Corollary 3.8 (Conjecture 3.5 of [7])** The cut ideal is generated by quadrics if and only if $G$ is free of $K_4$ minors.

**PROOF:** We prove that if $G$ is series-parallel then $\mu(G) \leq 2$. The other direction was proved in [7]. We only need to prove it for connected series-parallel graphs.

The proof is by induction on the number of vertices of $G$. If there are less than four vertices then $\mu(G) \leq 2$ by explicit calculations in [7].

Now assume that $G$ has at least four vertices. If $G$ is constructed by two graphs $H_1$ and $H_2$ put in series and glued at one point, then $\mu(G) = \max\{\mu(H_1), \mu(H_2)\} \leq 2$ by the fiber construction in [7].
If $G$ is constructed by two graphs $H_1$ and $H_2$ glued parallel together in two points we have two cases.

The first case: However subgraphs $H_1$ and $H_2$ are chosen to be glued together in parallel to create $G$, one of them will only be an edge.

Assume that $H_2$ is only the edge $uv$, and that $uv$ is not in $H_1$. If $H_1$ came from a parallel gluing of $H'_1$ and $H''_1$ at $u$ and $v$, then $G$ could be parallel constructed from $H'_1$ and $H''_1 + uv$ and none of them is only an edge, which is a contradiction. So $H_1$ is from a series gluing at some vertex $w \notin \{u, v\}$. Both graphs glued together to get $H_1$ cannot be only edges, since then $G$ is a triangle, and we assumed $G$ to have more than 3 vertices. Thus we can assume that the part of $H_1$ between $v$ and $w$ have more than two vertices. But then $G$ can be formed as a parallel construction glued at $v$ and $w$ where none of the parts is only an edge, and that situation is the second case.

The second case: The graph $G$ can be created by a parallel construction at $u, v$ of two graphs $H_1$ and $H_2$ and both of them have more than two vertices. If $uv$ is an edge of $G$ then $\mu(G) = \max\{\mu(H_1 + uv), \mu(H_2 + uv)\} \leq 2$ since $H_1$ and $H_2$ are series-parallel. If there is no edge between $u$ and $v$ in $G$ we use that $H_1, H_2, H_1 + uv, H_2 + uv$ are series-parallel and Corollary 3.7 to get that $\mu(G) \leq 2$. □

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