HYPERCYCLIC TOEPLITZ OPERATORS

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Abstract. We study hypercyclicity of the Toeplitz operators in the Hardy space $H^2(D)$ with symbols of the form $p(z) + \varphi(z)$, where $p$ is a polynomial and $\varphi \in H^\infty(D)$. We find both necessary and sufficient conditions for hypercyclicity which almost coincide in the case when $\deg p = 1$.

1. Introduction and Main Results

Toeplitz operators with antianalytic symbols are among the basic examples of hypercyclic operators. In 1968, S. Rolewicz showed that the operator $T_{\alpha z}$ (a multiple of the backward shift) is hypercyclic on the Hardy space $H^2$ whenever $|\alpha| > 1$. Later, G. Godefroy and J. Shapiro [6] showed that for a function $\varphi \in H^\infty$ the antianalytic Toeplitz operator $T_{\varphi}$ is hypercyclic if and only if $\varphi(D) \cap T \neq \emptyset$. Here, as usual, $D$ and $T$ denote the unit disc and the unit circle, respectively. On the other hand, it is obvious that there are no hypercyclic Toeplitz operators with analytic symbols (i.e., among multiplication operators).

However, it seems that hypercyclicity phenomena for general Toeplitz operators are much less studied, and the hypercyclicity criteria are not known. This problem was explicitly stated by Shkarin [10] who described hypercyclic Toeplitz operators with symbols of the form $\Phi(z) = az + b + cz$ (i.e., with tridiagonal matrix).

The aim of this note is to give new examples of hypercyclic Toeplitz operators. We give necessary or sufficient conditions for hypercyclicity of $T_{\Phi}$ in the case when

$$\Phi(z) = p\left(\frac{1}{z}\right) + \varphi(z),$$

where $p$ is a polynomial and $\varphi$ is in $H^\infty$ (sometimes we will assume that $\varphi$ belongs to the disc-algebra $A(D)$). In the case when $p(z) = \gamma z$ (i.e., $\Phi \in \bar{z}H^\infty$) the gap between the necessary and sufficient conditions becomes especially small.

A novel feature of these conditions is the role of univalence or $N$-valence (where $N$ is the degree of $p$) of the symbol. It seems that such conditions did not appear in the linear dynamics before, with one notable exception: in [3] Bourdon and Shapiro studied Bergman space Toeplitz operators with antianalytic symbols and in some of their results the univalence of the symbol plays a role.

Let us state the main results of the paper. In what follows we denote by $\overline{D}$ the closed unit disc and put $\overline{D} = \mathbb{C} \setminus \mathbb{T}$.

Our first result applies to the case when the antianalytic part has degree 1.

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Theorem 1.1. Let $\gamma \in \mathbb{C}$, let $\varphi \in H^\infty$ and let $\Phi(z) = \frac{z}{\gamma} + \varphi(z)$.

1. If $T_\Phi$ is hypercyclic, then
   (a) the function $\Phi$ is univalent in $D \setminus \{0\}$;
   (b) $\overline{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$ and $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$.

2. Assume that $\varphi \in A(D)$ and that
   (a′) the function $\Phi$ is univalent in $\overline{D} \setminus \{0\}$;
   (b′) $D \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$ and $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$.
   Then $T_\Phi$ is hypercyclic.

The gap between the necessary and sufficient conditions is related only to the boundary behaviour of $\Phi$. While it is necessary that $\Phi$ is univalent in $D$, we ask for univalence up to the boundary in the sufficient condition. Also, while the necessary condition requires the spectrum $\sigma(T_\Phi) = \mathbb{C} \setminus \Phi(D)$ to intersect the unit circle, in the sufficient condition we need a stronger assumption that the set $\mathbb{C} \setminus \Phi(D)$ (which is, essentially, the point spectrum of $T_\Phi$) intersects the open disc $D$.

In our second result $p$ is a polynomial of degree $N$. Recall that an analytic function $h$ in the domain $D$ is said to be $N$-valent in $D$ if the equation $h(z) = w$ has at most $N$ solutions in $D$ counting multiplicities. Note that $\Phi(z) \sim c_N z^{-N}$, $z \to 0$, and so $\Phi(z) = w$ has exactly $N$ solutions when $|w|$ is sufficiently large. Put

$$\Phi(D, N) = \{ w \in \mathbb{C} : \text{equation } \Phi(z) = w \text{ has exactly } N \text{ solutions in } D \},$$

where the solutions are counted according to their multiplicities.

Theorem 1.2. Let $p$ be a polynomial of degree $N \geq 1$, let $\varphi \in H^\infty$ and let $\Phi$ be given by (1).

1. If $T_\Phi$ is hypercyclic, then
   (a) the function $\Phi$ is $N$-valent in $D \setminus \{0\}$;
   (b) $\overline{D} \cap (\mathbb{C} \setminus \Phi(D, N)) \neq \emptyset$ and $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(D, N)) \neq \emptyset$.

2. Assume that $\varphi \in A(D)$ and that
   (a′) for any $w \in \Phi(D) \setminus \{0\}$ the equation $\Phi(z) = w$ has exactly $N$ solutions in $\mathbb{D} \setminus \{0\}$;
   (b′) $D \cap (\mathbb{C} \setminus \Phi(D))$ and $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$.
   Then $T_\Phi$ is hypercyclic.

Note that condition (a′) implies, in particular, that $\Phi(D) = \Phi(D, N)$.

The proofs of Theorems 1.1 and 1.2 are essentially elementary (modulo some basic results about polynomial approximation, like Mergelyan’s theorem).

2. Preliminaries

Recall that a continuous linear operator $T$ in a separable Banach (or Fréchet) space $X$ is said to be hypercyclic if there exists $x \in X$ such that the set $\{T^n x, n \in \mathbb{N}_0\}$ is dense in $X$ (here $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$).

One of the most basic sufficient conditions of hypercyclicity is the so-called Godefroy–Shapiro criterion (see [6] or [1, 7]). Suppose that, for a continuous linear operator $T$,
the subspaces
\[ X_0 = \text{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{C}, |\lambda| < 1\}, \]
\[ Y_0 = \text{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{C}, |\lambda| > 1\}, \]
are dense in \( X \). Then \( T \) is hypercyclic.

Let \( H^2 \) denote the standard Hardy space in \( \mathbb{D} \). Recall that for a function \( \psi \in L^\infty(\mathbb{T}) \) the Toeplitz operator \( T_\psi \) with the symbol \( \psi \) is defined as \( T_\psi f = P_+(\psi f) \), where \( P_+ \) stands for the orthogonal projection from \( L^2(\mathbb{T}) \) onto \( H^2 \).

In this section we always assume that \( T_\Phi \) is the Toeplitz operator with the symbol \( \Phi \), where \( \Phi \) is a polynomial of degree \( N \geq 1 \) and \( \varphi \in H^\infty \). Without loss of generality we assume that \( p(0) = 0 \).

First we show that \( D \)-valence of \( \Phi \) in \( D \) is necessary for hypercyclicity.

**Proposition 2.1.** Assume that for some \( \mu \in \mathbb{C} \), the equation \( \Phi(z) = \mu \) has at least \( N + 1 \) solutions counting multiplicities. Then \( (T_\Phi)^* = \overline{T_\Phi} \) has an eigenvector and, in particular, \( T_\Phi \) is not hypercyclic.

**Proof.** Denote by \( k_\lambda \) the Cauchy kernel (reproducing kernel of \( H^2 \)): \( k_\lambda(z) = \frac{1}{1-\bar{z}\lambda} \). It is well known that for any antianalytic Toeplitz operator we have \( T_\Phi k_\lambda = \varphi(\lambda) k_\lambda \).

Assume, for simplicity, that the equation \( \Phi(z) = \mu \) has \( N + 1 \) distinct solutions \( z_1, z_2, \ldots, z_{N+1} \) in \( \mathbb{D} \). We will construct the eigenvector of \( (T_\Phi)^* \) in the form \( f = \sum_{j=1}^{N+1} \alpha_j k_{z_j} \), where \( \alpha_j \) are some complex coefficients. If \( p(z) = \sum_{k=1}^{N} c_k z^k \), then \( T_\Phi f = \overline{p(z) + \varphi(z)} \), where \( \overline{p(z)} = \sum_{k=1}^{N} \overline{c_k} \bar{z}^k \). Hence, using the fact that \( p(1/z_j) + \varphi(z_j) = \mu \), we get

\[
(2) \quad T_\Phi f(z) = \sum_{j=1}^{N+1} \alpha_j \left( \frac{p(z)}{1 - \bar{z}_j z} + \frac{\varphi(z_j)}{1 - \bar{z}_j z} \right) = \tilde{p} f(z) + \sum_{j=1}^{N+1} \alpha_j \frac{\overline{p(z)} - \overline{p(1/z_j)} \overline{1 - \bar{z}_j z}}{1 - \bar{z}_j z}.
\]

The functions
\[
\frac{\overline{p(z)} - \overline{p(1/z_j)} \overline{1 - \bar{z}_j z}}{1 - \bar{z}_j z} = \frac{\overline{\tilde{p}(z)} - \overline{\tilde{p}(1/z_j)} \overline{1 - \bar{z}_j z}}{1 - \bar{z}_j z}, \quad j = 1, \ldots, N + 1,
\]
are polynomials of degree \( N - 1 \). Hence, there exist nontrivial coefficients \( \alpha_j \) such that the last sum in (2) is identically zero, and so \( T_\Phi f = \tilde{\mu}_f \).

In the case of a zero \( z_j \) of multiplicity \( m_j \), consider the linear combination of the functions \( (1 - \bar{z}_j z)^{-l}, 1 \leq l \leq m_j \). We omit the straightforward computations. \( \square \)

Next we study the spectrum \( \sigma(T_\Phi) \), the point spectrum \( \sigma_p(T_\Phi) \) and its eigenvectors. Note that \( T_{\bar{z}} = T_{1/\bar{z}} \) is the backward shift operator \( S^* \) on \( H^2 \), that is,
\[
T_{\bar{z}} f(z) = \frac{f(z) - f(0)}{z} \quad \text{and} \quad T_{\bar{z}^k} f(z) = \frac{1}{z^k} \left( f(z) - \sum_{j=0}^{k-1} \frac{f(j)(0)}{j!} z^j \right).
\]

In the proof of the next proposition we will need the basic results on inner-outer (Nevanlinna) factorization of the functions in the Hardy spaces (see, e.g., \[5\] Chapter 2 or \[3\] Chapter IV]).

**Proposition 2.2.** Assume that \( \Phi \) is \( D \)-valent in \( \mathbb{D} \). Then
\[
\sigma(T_\Phi) = \mathbb{C} \setminus \Phi(\mathbb{D}, N), \quad \sigma_p(T_\Phi) = \mathbb{C} \setminus \overline{\Phi(\mathbb{D})}.
\]
If \( \lambda \in \mathbb{C} \setminus \Phi(\mathbb{D}) \), then the corresponding eigenspace has dimension \( N \) and the eigenvectors are given by
\[
f_\lambda(z) = \frac{q(z)}{z^N \Phi(z) - \lambda z^N},
\]
where \( q \) is an arbitrary polynomial of degree at most \( N - 1 \).

**Proof.** First we prove the inclusion \( \sigma(T_\Phi) \subset \mathbb{C} \setminus \Phi(\mathbb{D}, N) \). Namely, we show that any \( \lambda \in \Phi(\mathbb{D}, N) \) is a regular point for \( T_\Phi \), i.e., the equation \( T_\Phi f - \lambda f = g \) has the unique solution \( f \in H^2 \) for any \( g \in H^2 \).

Let \( p(z) = \sum_{k=1}^{N} c_k z^k \). Then the equation \( T_\Phi f - \lambda f = g \) may be rewritten as
\[
\sum_{k=1}^{N} \frac{c_k}{z^k} \left( f(z) - \sum_{j=0}^{k-1} \frac{f^{(j)}(z)}{j!} z^j \right) + \varphi(z) f(z) - \lambda f(z) = g(z),
\]
or, equivalently,
\[
f(z) \left( \sum_{k=1}^{N} c_k z^{-k} + z^N \varphi(z) - \lambda z^N \right) = z^N g(z) + \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} \frac{f^{(j)}(z)}{j!} z^{-k+j}.
\]

If \( \lambda \in \Phi(\mathbb{D}, N) \), then the expression in brackets (which equals \( z^N \Phi(z) - \lambda z^N \)) has exactly \( N \) zeros in \( \mathbb{D} \) counting multiplicities, say, \( z_1, z_2, \ldots, z_N \). Moreover, it is clear that \( |\Phi(z) - \lambda| \geq \delta > 0 \) for some \( \delta > 0 \) and almost every \( z \in \mathbb{T} \). Consider the (unique) polynomial \( q \) of degree \( N - 1 \) such that \( z^N g(z_j) + q(z_j) = 0, \ j = 1, \ldots, N \) (with obvious modification for multiple zeros). Then, for this choice of \( q \), the function
\[
f(z) = \frac{z^N g(z) + q(z)}{z^N \Phi(z) - \lambda z^N}
\]
belongs to \( H^2 \). Note that we necessarily have
\[
q(z) = \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} \frac{f^{(j)}(z)}{j!} z^{-k+j}
\]
(just compare the Taylor coefficients), and so \( f \) is indeed the unique solution of the equation \( T_\Phi f - \lambda f = g \). Thus, we have shown that \( \sigma(T_\Phi) \subset \mathbb{C} \setminus \Phi(\mathbb{D}, N) \).

For the proof of the converse inclusion \( \mathbb{C} \setminus \Phi(\mathbb{D}, N) \subset \sigma(T_\Phi) \) we will need the following observation:

**Claim.** If \( \lambda \) is a regular point for \( T_\Phi \), then \( (\Phi - \lambda)^{-1} \in L^\infty(\mathbb{T}) \) and the Nevanlinna factorization of the function \( \Psi(z) = z^N \Phi(z) - \lambda z^N \in H^\infty \) contains no nontrivial singular inner factor.

**Proof of the Claim.** Assume that the equation \( T_\Phi f - \lambda f = g \) has the unique solution for any \( g \in H^2 \). Then \( f \) is of the form (3) where the polynomial \( q \) is given by (4). Note that for a function \( \gamma \) on \( \mathbb{T} \) we have the inclusion \( \gamma H^2 = \{ \gamma h : h \in H^2 \} \subset L^2(\mathbb{T}) \) if and only if \( \gamma \in L^\infty(\mathbb{T}) \). Since the function \( f \) in (3) is in \( H^2 \) for any \( g \in H^2 \), while \( |q|_\infty \leq C \|f\|_2 \leq C_1 \|g\|_2 \) for some constants \( C, C_1 \) independent from \( g \), we conclude that \( (z^N \Phi(z) - \lambda z^N)^{-1} \in L^\infty(\mathbb{T}) \).

If \( \Psi \) has a nontrivial singular inner factor, then, taking, \( g \equiv 1 \), we obtain a function \( f \) of the form \( f = \frac{u_1 B_1}{u_2 B_2} \), where \( u_1, u_2 \) are outer functions, \( B_1, B_2 \) Blaschke products.
and $I$ a nontrivial singular inner function. Hence, $f \notin H^2$, a contradiction. The Claim is proved.

Now we return to the proof of the inclusion $\mathbb{C} \setminus \Phi(\mathbb{D}, N) \subset \sigma(T_\Phi)$. Let $\lambda \notin \Phi(\mathbb{D}, N)$. We will show that $\lambda \in \sigma(T_\Phi)$. From now on we assume that $(\Phi - \lambda)^{-1} \in L^\infty(\mathbb{T})$ and the Nevanlinna factorization of $\Psi$ contains no nontrivial singular inner factor (otherwise, we already know from the Claim that $\lambda \in \sigma(T_\Phi)$).

Assume first that $\lambda \notin \Phi(\mathbb{D})$. Then $\Psi \neq 0$ in $\mathbb{D}$, $\Psi$ has no singular inner factor and so $\Psi$ is an outer $H^\infty$ function. Since $\Psi^{-1} \in L^\infty(\mathbb{T})$, we conclude that $\Psi^{-1} \in H^\infty(\mathbb{D})$. Hence, the function

$$f_\lambda(z) = \frac{Q(z)}{\Psi(z)} = \frac{Q(z)}{z^N \Phi(z) - \lambda z^N}$$

is in $H^2$ and is an eigenvector of $T_\Phi$ for any choice of the polynomial $Q$ of degree at most $N - 1$.

Finally, if $\lambda \in \Phi(\mathbb{D}) \setminus \Phi(\mathbb{D}, N)$, then $\Psi$ has $m$ zeros $z_1, z_2, \ldots, z_m$ in $\mathbb{D}$ counting multiplicities, where $m < N$ (recall that $\Phi$ is $N$-valent in $\mathbb{D}$). Therefore, for any polynomial $Q$ which vanishes at $z_j$, the function (5) will be an eigenfunction of $T_\Phi$. Thus, $\mathbb{C} \setminus \Phi(\mathbb{D}, N) \subset \sigma(T_\Phi)$.

The inclusion $\mathbb{C} \setminus \Phi(\mathbb{D}) \subset \sigma_p(T_\Phi)$ is easy. If $\lambda \in \mathbb{C} \setminus \Phi(\mathbb{D})$, then, for some $\delta > 0$, we have $|\Psi(z)| \geq \delta$, $z \in \mathbb{D}$, whence $\Psi^{-1} \in H^\infty(\mathbb{D})$, and so any function $f$ of the form (5) is an eigenvector.

**Remark 2.3.** Note that we have shown in the proof of Proposition 2.2 that $\sigma_p(T_\Phi)$ contains all points $\lambda \in \Phi(\mathbb{D}) \setminus \Phi(\mathbb{D}, N)$ such that $(\Phi - \lambda)^{-1} \in L^\infty(\mathbb{T})$ and $\Psi$ has no singular inner factor (this is the case, e.g., if there exist $r \in (0, 1)$, $\delta > 0$ such that $|\Psi(z)| \geq \delta$, $r < \mid z \mid < 1$).

### 3. Proofs of main results

We start with the proof of necessity parts of Theorems 1.1 and 1.2.

**Proof of Statement 1 of Theorems 1.1 and 1.2.** By Proposition 2.1, if $T_\Phi$ is hypercyclic, then $\Phi$ is $N$-valent in $\mathbb{D}$. In particular, $\Phi$ is univalent in $\mathbb{D}$ when $N = 1$. Property (a) is proved.

Clearly, if $\hat{\mathbb{D}} \subset \Phi(\mathbb{D}, N)$, then for any $\zeta \in \mathbb{T}$ for which the nontangential boundary value $\Phi(\zeta)$ exists, we have $|\Phi(\zeta)| \leq 1$. Indeed, otherwise there exist $z_1, \ldots, z_N \in \mathbb{D}$ such that $\Phi(z_j) = \Phi(\zeta)$ and the equation $\Phi(z) = w$ will have at least $N + 1$ solutions for some $w$ sufficiently close to $\Phi(\zeta)$. Hence, $|\Phi| \leq 1$ a.e. on $\mathbb{T}$ and so $\|T_\Phi\| \leq 1$, a contradiction to hypercyclicity.

Finally, if $T_\Phi$ is hypercyclic, then $\sigma(T_\Phi) \cap \mathbb{T} \neq \emptyset$. By Proposition 2.2, $\sigma(T_\Phi) = \mathbb{C} \setminus \Phi(\mathbb{D}, N)$ and, in particular, $\sigma(T_\Phi) = \mathbb{C} \setminus \Phi(\mathbb{D})$ when $N = 1$. This completes the proof of (b).

The following proposition plays a key role in the proof of sufficient conditions in Theorems 1.1 and 1.2.
Proposition 3.1. 1. Let \( h \in A(\mathbb{D}) \) be injective in \( \overline{\mathbb{D}} \) (i.e. univalent up to the boundary). Then the system \( \{h^k\}_{k \geq 0} \) is complete in \( H^2 \).

2. Let \( h \in A(\mathbb{D}) \) be \( N \)-valent in \( \overline{\mathbb{D}} \) and, moreover, assume that for any \( w \in h(\overline{\mathbb{D}}) \) the equation \( h(z) = w \) has exactly \( N \) solutions in \( \overline{\mathbb{D}} \). Then the system of functions \( \{z^j h^k : k \geq 0, j = 0, 1, \ldots, N - 1\} \) is complete in \( H^2 \).

Proof. 1. Let \( \Omega = h(\mathbb{D}) \), \( \Gamma = \partial \Omega \), \( g = h^{-1} : \Omega \to \mathbb{D} \). Clearly, \( g \) admits a continuation to a continuous function on \( \overline{\Omega} = \Omega \cup \Gamma \). Since \( \Gamma \) is a closed Jordan curve (without intersections), the complement \( \mathbb{C} \setminus \overline{\Omega} \) is connected and, by Mergelyan’s theorem, any function \( f \) in \( H^\infty(\Omega) \cap C(\overline{\Omega}) \) may be uniformly approximated by analytic polynomials, \( p_n(u) \to f(u) \) uniformly in \( u \in \overline{\Omega} \). Hence, \( p_n(h(z)) \to f(h(z)) \) uniformly in \( z \in \mathbb{D} \), whence any function from \( H^\infty \cap C(\mathbb{D}) \) may be approximated by polynomials in \( h \).

2. It is not difficult to show that the hypothesis implies that for any function \( f \) which is sufficiently smooth up to the boundary (say, \( f \in C^N(\mathbb{D}) \)) there exist functions \( f_j \in H^\infty(\Omega) \cap C(\overline{\Omega}) \), \( j = 0, 1, \ldots, N - 1 \), such that

\[
\begin{align*}
    f(z) &= f_0(h(z)) + z f_1(h(z)) + \cdots + z^{N-1} f_{N-1}(h(z)).
\end{align*}
\]

Here, as above, \( \Omega = h(\mathbb{D}) \). Indeed, for a point \( w \) with \( N \) distinct preimages \( z_1, \ldots, z_N \) consider the system of linear equations \( f(z_l) = \sum_{j=0}^{N-1} z_l^j f_j(w), l = 1, 2, \ldots, N \), with the unknown \( f_j(w) \). Since \( z_l \) are locally analytic functions of \( w \), we conclude that \( f_j \) are locally analytic at such points \( w \); it is easy to show that the functions \( f_j \) have removable singularities at \( w \) in the case of multiple zeros, and so are analytic in the whole \( \Omega \) and continuous up to the boundary.

Now it remains to note that the “exact \( N \)-valence up to the boundary” condition implies that \( \Omega \) is a Jordan domain, \( \mathbb{C} \setminus \overline{\Omega} \) is connected and, by Mergelyan’s theorem, each function \( f_j \) is a uniform limit of polynomials \( p_{j,m}, m \to \infty \), in \( \mathbb{D} \). Hence, the sum \( \sum_{j=0}^{N-1} z^j p_{j,m}(h(z)) \) converges to \( f \) uniformly in \( \mathbb{D} \). Thus, any sufficiently smooth \( f \) belongs to the uniform closure in \( \overline{\mathbb{D}} \) of the linear span of \( \{z^j h^k : k \geq 0, j = 0, 1, \ldots, N - 1\} \). Hence, this system is complete also in \( H^2 \). \( \square \)

Remark 3.2. The problem of completeness of systems \( \{h^k\}_{k \geq 0} \) in \( H^2(\mathbb{D}) \) or (essentially) equivalent problem of density of polynomials in the Hardy space \( H^2(\Omega), \Omega = h(\mathbb{D}) \), is in general, a deep problem for which no explicit answer exists (see [9, 4, 2]). Clearly, univalence of \( h \) in \( \mathbb{D} \) is necessary. On the other hand, Caughran [4] showed that if the polynomials are dense in \( H^2(\Omega) \) and \( h \in A(\mathbb{D}) \), then \( \Omega \) is a Jordan domain, and so \( h \) is univalent in \( \mathbb{D} \) up to the boundary. In the general case it is a result by Bourdon [2] that the density of polynomials in \( H^2(\Omega) \) implies that \( h \) is univalent almost everywhere on \( \mathbb{T} \).

Proof of Statement 2 of Theorems 1.1 and 1.2. We first consider the case \( N = 1 \), \( p(z) = \gamma z \). Since, by Proposition 2.2, \( \sigma_p(T_\Phi) \supset \mathbb{C} \setminus \Phi(\mathbb{D}) \), condition \((b')\) implies that we have open sets \( U_1 \subset \mathbb{D} \) and \( U_2 \subset \mathbb{D} \) of eigenvalues. By the Godefroy–Shapiro criterion, it remains to show that the corresponding eigenvectors are complete in \( H^2 \). Fix some \( \lambda_0 \in U_1 \) and let

\[
h(z) = \frac{z}{\gamma - \lambda_0 z + z \varphi(z)} = \frac{1}{\Phi(z) - \lambda_0}.
\]
By the conditions on $\Phi$ we have that $h \in A(\mathbb{D})$ and $h$ is injective in $\mathbb{D}$. Now note that for $\lambda$ in a small neighborhood $\{|\lambda - \lambda_0| < \delta\}$ of $\lambda_0$

$$f_\lambda(z) = \frac{1}{\gamma - \lambda z + z \varphi(z)} = \sum_{k=0}^{\infty} \frac{(\lambda - \lambda_0)^k z^k}{(\gamma - \lambda_0 z + z \varphi(z))^{k+1}}.$$  

Thus, if $f \perp f_\lambda$, $|\lambda - \lambda_0| < \delta$, then

$$f \perp (\gamma - \lambda_0 z + z \varphi(z))^{-1} h^k, \quad k \geq 0.$$  

By Statement 1 of Proposition 3.1 the system $\{h^k\}_{k \geq 0}$ is complete in $H^2$. The additional factor $1/(\gamma - \lambda_0 z + z \varphi(z))$ is an invertible element of $H^\infty$ and so the system

$$\{(\gamma - \lambda_0 z + z \varphi(z))^{-1} h^k\}_{k \geq 0}$$  

is also complete. We conclude that the eigenvectors corresponding to $\lambda \in U_1$ are complete, the proof for $\lambda \in U_2$ is the same.

Now let $N > 1$. As above, $(b')$ guarantees that we have open sets $U_1 \subset \mathbb{D}$ and $U_2 \subset \mathbb{D}$ such that $U_1, U_2 \subset \mathbb{C} \setminus \Phi(\mathbb{D}) = \mathbb{C} \setminus \Phi(\mathbb{D}, N)$ and, thus, consist of eigenvalues. Fix $\lambda_0 \in U_1$. In this case we have, by Proposition 2.2, $N$ eigenvectors corresponding to $\lambda_0$:

$$f_{\lambda_0, j}(z) = \frac{z^j}{z^N \Phi(z) - \lambda_0 z^N}, \quad j = 0, 1, \ldots, N - 1.$$  

Using, as above, the Taylor expansion for $\lambda$ close to $\lambda_0$, we conclude that if $f$ is orthogonal to the eigenvectors corresponding to $\lambda$ in a small neighborhood of $\lambda_0$, then

$$f \perp \frac{z^j h^k(z)}{z^N \Phi(z) - \lambda_0 z^N}, \quad k \geq 0, 0 \leq j \leq N - 1,$$  

where

$$h(z) = \frac{z^N}{z^N \Phi(z) - \lambda_0 z^N} = \frac{1}{\Phi(z) - \lambda_0}.$$  

By the assumptions on $\Phi$, $h$ is $N$-valent and for any $w \in h(\mathbb{D})$ the equation $h(z) = w$ has exactly $N$ solutions in $\mathbb{D}$ counting multiplicities. Hence, by Statement 2 of Proposition 3.1 the system $\{z^j h^k : k \geq 0, j = 0, 1, \ldots, N - 1\}$ is complete in $H^2$. We conclude that any function $f$ satisfying (7) is zero. 

4. Shkarin’s characterization of tridiagonal Toeplitz operators

In [10] Shkarin characterized hypercyclic Toeplitz operators with symbols of the form $\Phi(z) = \frac{a}{z} + b + cz$:

**Proposition 4.1.** [10, Proposition 5.10] The Toeplitz operator $T_\Phi$ with $\Phi(z) = \frac{a}{z} + b + cz$ is hypercyclic if and only if

(a) $|a| > |c|$;  
(b) $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(\mathbb{D})) \neq \emptyset$ and $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(\mathbb{D})) \neq \emptyset$.

In fact, in [10] condition (b) is replaced by $\min_{z \in T} |\Phi(z)| < 1 < \max_{z \in T} |\Phi(z)|$, but this condition is obviously incorrect. If we take $a = 2$, $b = c = 0$, then $T_\Phi = 2S^*$ is hypercyclic, but the estimate $\min_{z \in T} |\Phi(z)| < 1$ does not hold. It is however clear from the proof that the author means the correct condition (b).
Let us show how to deduce this result from our Theorem \ref{main1}. It is clear that $\Phi$ is univalent in $D$ if and only if $|a| \geq |c|$ and $\Phi$ is univalent in $D'$ if and only if $|a| > |c|$. Hence, sufficiency of $(a)$ and $(b)$ follows immediately from Statement 2 of Theorem \ref{main1}.

Let us show the necessity of $(a)$ and $(b)$. Univalence of $\Phi$ implies that $|a| \geq |c|$. To show the strict inequality we need to apply the argument from \cite{grivaux}, if $|a| = |c|$, then $T_\Phi$ is a normal operator, and hence is not hypercyclic. In general this argument is not applicable. On the other hand, in \cite{grivaux} the case $|a| < |c|$ is excluded by appealing to the theory of hyponormal operators. It seems that this kind of argument can not be used for more general operators of the form $T_{\gamma z + \phi(z)}$.

The property $\hat{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$ is obvious. To show that $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset$ one has again to use an ad hoc argument from \cite{grivaux} which uses the very special form of the symbol. Assume, on the contrary, that $\sigma(T_\Phi) = \mathbb{C} \setminus \Phi(D) \subset \{z : |z| \geq 1\}$. Note that in our case $\sigma(T_\Phi)$ is a convex set (some ellipse) and so it can be separated from the unit disc. Thus, there exists $\theta \in \mathbb{R}$ such that $\text{Re}(e^{i\theta}\Phi(z)) \geq 1, z \in \mathbb{T}$. Hence,

$$\text{Re}(e^{i\theta}T_\Phi f, f) = \int_{\mathbb{T}} \text{Re}(e^{i\theta}\Phi)|f|^2 dm \geq \|f\|_2^2,$$

and so $T_\Phi$ is an expansion. However, in general, $\mathbb{C} \setminus \Phi(D)$ need not be convex.

5. Some open questions

We conclude this note with several open questions.

**Question 1.** Let $\Phi = \frac{a}{z} + \varphi(z)$ and assume that $T_\Phi$ is hypercyclic. Does it follow that

$$\mathbb{D} \cap \sigma(T_\Phi) = \mathbb{D} \cap (\mathbb{C} \setminus \Phi(D)) \neq \emptyset?$$

It is true in the case of Toeplitz operators $T_{\psi}$ with antianalytic symbols since if $\sigma(T_{\psi}) \cap \mathbb{D} = \emptyset$, then $|\psi| > 1$ in $\mathbb{D}$ and so its inverse $T_{\psi^{-1}}$ is a contraction, a contradiction. In the case $\Phi(z) = \frac{a}{z} + b + cz$ another argument was suggested by Shkarin (see Section 4). However, these methods do not seem to apply in general.

Let us mention on the other hand that there are no general obstacles for a hypercyclic operator $T$ to satisfy $\sigma(T) \cap \mathbb{D} = \emptyset$ and the intersection $\sigma(T) \cap \mathbb{T}$ may be a one-point set. Answering a question of the first author, Sophie Grivaux constructed an example of a hypercyclic operator $T$ such that $\sigma(T) = \overline{B(2, 1)}$ and $\sigma_p(T) = B(2, 1)$ (by $B(z_0, r)$ we denote the disc of radius $r$ centered at $z_0$).

**Question 2.** Is the univalence of $\Phi$ up to the boundary necessary in the Statement 2 of Theorem \ref{main1}? Apparently, it is necessary for the completeness of the functions of the form $\{h^k\}_{k \geq 0}$ for any individual function $h = \frac{1}{\Phi - \lambda_0}$. However, it seems that it is not necessary for completeness of eigenvectors with small or large eigenvalues. Namely, the following is true: assume that $\Phi$ is univalent in $D$ and $\mathbb{C} \setminus \Phi(D)$ consists of finite number of connected components $U_j, 1 \leq j \leq m$. If any component $U_j$ intersects $\mathbb{D}$ and $\hat{D}$, then $T_\Phi$ is hypercyclic. To what extent are these conditions necessary?

**Question 3.** What are sufficient conditions of hypercyclicity in the case when the valence of $\Phi$ changes inside $D$? One can show that the representation \cite{godefroy} need not be true anymore. Therefore, it is not clear, which approximation problem corresponds to an application of the Godefroy–Shapiro criterion in this case.
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