On Some Physical Aspects of Planck-Scale Relativity: A Simplified Approach

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Abstract

The kinematics of the two-scale relativity theory (new relativity) is revisited using a simplified approach. This approach allows us not only to derive the dispersion equation introduced earlier by Kowalski-Glikman, but to find an additional dispersion relation, and, even more important, to provide a physical basis for such relations. They are explained by the fact that in the observer invariant two-scale relativity (new relativity) the Planck constant does not remain constant anymore, but depends on the universal length scale. This leads to the correct relation between energy and frequency at any scale.

1 Introduction

The pioneering papers By G.Amelino-Camelia [1],[2] have explicitly and unambiguously introduced what can be called a "two-scale special relativity" (or a new relativity). The crucial feature of this new relativity is the introduction of an observer-independent absolute scale (possibly the Planck scale $\Lambda \sim 1.6 \times 10^{-35} m$) in addition to the existing in special relativity observer-independent universal velocity scale, the speed of light. The investigation of a structure of the emerging space-time at the Planck scale (different from Minkowski space-time) is based upon $\kappa$-Poincare algebra ([3] and references therein). It should be mentioned that consequences of the finite universal length scale on the resulting mechanics have also been studied by L.Nottale, C.Castro, and others researchers (e.g., [4], [5], and references therein), albeit
from a different point of view.

J.Kowalski-Glikman [6] investigated the space-time transformations and found that the metric structure of the respective space-time events has the Minkowski distance as an invariant. He also proposed the dispersion relation which removes a seemingly paradoxical discrepancy between the infinite group velocity of a wave and the finite velocity of a massless particle. However the author emphasized that this relation is required a justification "by more solid arguments".

We have to emphasize that here we are not going to dwell on the algebra (κ-Poincare) underlying the respective space, but rather try via a simplify approach to revisit some physical points of the problem.

In view of this, we provide a simplified derivation of the dispersion relation and show that there is an additional dispersion relation, consistent with the finite velocity of a massless particle. We also provide an explanation of the physical reason behind these relations.

The first of these relations corresponds to the boost transformation of position and time carried out in the space of the boost parameter ξ, independent of the universal invariant length scale λ. In this case the transformation does not coincide with the special relativistic transformation, and neither \( p_0 \) (energy) is proportional to the frequency nor the momentum \( p_\alpha, \alpha = 1, 2, 3 \) is proportional to the wave number.

On the other hand, the second transformation carried out in the space of the boost parameter \( z \) which is a function of the length-scale-independent boost parameter ξ and the invariant length scale itself. In this space the transformation coincides with the special-relativistic boost, the momentum is proportional to the wave number, but energy is not proportional to the frequency, requiring a certain modification of the relation between them.

The explanation of this situation (and equally of the situation with the previous case) can be as follows. Since in quantum mechanics energy is given by \( \hbar \times \textit{frequency} \), and now frequency is not proportional to energy, this means that the introduction of the additional invariant parameter (invariant length scale \( \lambda \)) into a description of mechanics results in other parameters (including \( \hbar \)) becoming functions of \( \lambda \). This leads to the emergence of the so-called effective Planck constant (e.g. ref[3]) and the accompanying restoration of the proportionality between energy and frequency.
In addition, by examining the metric structure of the space-time with the help of simple arguments we explicitly find the boundaries of time-like/space-like intervals. It has turned out that these boundaries indicate two mutually exclusive possibilities:

1) **physical**. Massive particles cannot reach the speed of light, in a world with two independent universal observer-independent scales,

2) **hypothetical**. Massive particles can move with velocities reaching the speed of light, but two universal scales of the theory are not independent.

Finally, we investigate the uncertainty relation connected with the 2-scale relativity and compare it with the respective relation in the string theory.

### 2 Dispersion Relation in Planck-Scale Relativity

We begin our analysis by considering the Minkowski distance (written in the differential form), which we assume to be invariant as in special relativity:

\[
\frac{dx_0^2}{ds^2} - \frac{dx_\alpha^2}{ds^2} = \frac{dx'_0^2}{ds^2} - \frac{dx'_\alpha^2}{ds^2} \tag{1}
\]

where \(x_0, x_\alpha, \alpha = 1, 2, 3\) are the time and space components of the four-vector \(x_i (i = 0, 1, 2, 3)\), the primes denote a transformed (under the boost) system, and \(ds\) is the appropriate space-time interval.

In the system at rest \(dx_\alpha^2/ds^2 = p_\alpha = 0\) and \(\text{[6]}\)

\[
\left(\frac{dx_0}{ds}\right)^2 = \left[\frac{1}{\lambda} \sinh(p_0\lambda)\right]^2 \tag{2}
\]

Here \(p_0\) and \(p_\alpha\) are the energy and momentum components of the momentum four-vector \(p_0, p_\alpha\). This means that in the rest frame the interval \(ds\) has the following dependence on \(dx_0\):

\[
ds = dx_0 \frac{\lambda}{\sinh(m\lambda)}
\]

where \(m\) is the mass (rest energy). From the relation between \(p_0, p,\) and \(m\) found in \(\text{[7]}\) on the basis of the group-theoretical arguments

\[
\left[\frac{2}{\lambda} \sinh\left(\frac{m\lambda}{2}\right)\right]^2 = \left[\frac{2}{\lambda} \sinh\left(\frac{p_0\lambda}{2}\right)\right]^2 - p_\alpha^2 e^{p_0\lambda} \tag{3}
\]
we find that in the rest frame
\[
\left[ \frac{1}{\lambda} \sinh \left( \frac{p_0 \lambda}{2} \right) \right]^2 = \left[ \frac{2}{\lambda} \sinh \left( \frac{m \lambda}{2} \right) \right]^2
\] (4)

Using (4) in (1) we obtain the following expression for the time-space interval \( ds \).
\[
(ds)^2 = \left[ \frac{\lambda}{\sinh (m \lambda)} \right]^2 (dx_0^2 - dx_0^2)
\] (5)

For the following we drop the prime denoting the transformed system and introduce action \( S \) for a free particle similar to the one in special relativity:
\[
S = -a \int ds = -a \frac{\lambda}{\sinh (m \lambda)} \int dx_0 \sqrt{1 - V_\alpha^2}
\] (6)

where \( a \) is some constant to be determined later. The Lagrange function for the particle is then
\[
\mathcal{L} = -a \frac{\lambda}{\sinh (m \lambda)} \sqrt{1 - V_\alpha^2}
\] (7)

Here \( V_\alpha \) is a component of the three-velocity of the particle (in the particular frame).

Using (7) we find the momentum \( p_\alpha \) of the particle
\[
p_\alpha = \frac{\partial \mathcal{L}}{\partial V_\alpha} = a \frac{\lambda}{\sinh (m \lambda)} \frac{V_\alpha}{\sqrt{1 - V_\alpha^2}}
\] (8)

If we introduce a reduced mass
\[
\bar{m} = \frac{\sinh (m \lambda)}{\lambda}
\]
then Eq.(8) looks exactly as the respective relation in the conventional special relativity:
\[
p_\alpha = \frac{\bar{m} V_\alpha}{\sqrt{1 - V_\alpha^2}}
\] (9)

I. First, we consider all the relations expressed in terms of the length-scale independent boost parameter \( \xi \). With this in mind, let us boost a massive particle (of mass \( m \)), initially at rest. Its momentum \( p_\alpha \) is then
\[
p_\alpha = \bar{m} \sinh \xi \frac{\hat{p}_\alpha}{\lambda \bar{m} \cosh \xi + \sqrt{1 + \bar{m}^2 \lambda^2}}
\] (10)
where $\hat{p}_\alpha$ is the unit vector of $p_\alpha$.

Comparing (8) and (10) we obtain

$$a \frac{V_\alpha}{\sqrt{1 - V_\alpha^2}} = \hat{p}_\alpha \bar{m}^2 \frac{\sinh \xi}{\lambda \bar{m} \cosh \xi + \sqrt{1 + (\bar{m} \lambda)^2}}$$

(11)

From the dimensional considerations we set the value of $a$

$$a = \bar{m}^2.$$  

Solving (11) with respect to $V_\alpha$ we get the following two cases depending on the sign of $\lambda$

a) $\lambda > 0$

$$|V_\alpha| = \frac{\sinh \xi}{cosh \xi \sqrt{1 + \lambda^2 \bar{m}^2 + \sqrt{1 + \bar{m}^2 \lambda^2}}}$$

(12)

This is exactly the expression obtained in [6] on the basis of different considerations. For the future analysis we denote the upper bound on the velocity $|V_\alpha|$, reached in this case at $\xi \to \infty$, as the speed of "light" $c_\lambda$

$$c_\lambda \equiv 1/cosh(\lambda \bar{m}) \equiv 1/\sqrt{1 + \bar{m}^2 \lambda^2}$$

From (12) we can easily see that time-like (and space-like) regions of the respective Minkowski space for a massive particle are not fixed anymore, but depend on its mass:

$$x_0 = x \sqrt{1 + (\bar{m} \lambda)^2}.$$  

Amusingly enough, with the increase of mass $m$ the time-like region decreases, and in the hypothetical case of $m \lambda \to \infty$ shrinks to zero!

b) Another case corresponds to the values $\lambda \leq 0$. In this case the limiting value of $|V_\alpha|$ can be equal to 1:

$$|V_\alpha| = \frac{\sinh \xi}{cosh \xi \sqrt{1 + (\bar{m} \lambda)^2 - \bar{m} |\lambda|}} = 1$$

From this expression follows that the limiting value of $\xi$ is

$$cosh(\xi) = coth(m|\lambda|) \equiv \frac{\sqrt{1 + \bar{m}^2 \lambda^2}}{\bar{m} |\lambda|} \equiv \frac{1}{\sqrt{1 - c_\lambda^2}}.$$
The respective value of the momentum $p \to \infty$ as in the case of the conventional special relativity. What is most interesting about this case that now two scales, speed of light and $\lambda$, are not independent, hinting at the existence of only one observer-invariant scale, length scale $\lambda$.

Since expression (11) for $p_\alpha$ as a function of $V_\alpha$ formally looks exactly like its special relativistic counterpart, this gives us a clue for undertaking the task of casting the appropriate equations into the form found in special relativity. We begin with the relation between energy $p_0$ 

$$p_0 = \frac{1}{\lambda} \log[\bar{m}\lambda \cosh \xi + \sqrt{1 + (\bar{m}\lambda)^2}]$$

and the momentum $p_\alpha$ given by Eq.(10).

Since the identity $\cosh^2 \xi - \sinh^2 \xi \equiv 1$ (yielding in particular the relation $E^2 = p^2 + m^2$) allows to reach our goal, we have to express $\cosh$ and $\sinh$ in terms of $p_0$ and $p_\alpha$ using (10) and (13). For convenience sake, in the following I will drop the subscript at $p_\alpha$.

Comparing (10) and (13), we find that

$$\sinh \xi = \frac{pe^{p_0\lambda}}{\bar{m}}$$

On the other hand, from (13) we obtain

$$\cosh \xi = \frac{1}{\bar{m}} \frac{e^{p_0\lambda} - \sqrt{1 + \bar{m}^2\lambda^2}}{\lambda}$$

Now it is easy to identify what we call the effective frequency $\bar{\omega}$ and the effective wave number $\bar{k}$:

$$\bar{k} = pe^{p_0\lambda}$$

$$\bar{\omega} = \frac{e^{p_0\lambda} - \sqrt{1 + \bar{m}^2\lambda^2}}{\lambda}$$
According to (14) and (15), the relation between them is the exact copy of the respective relation in special relativity:

\[ \bar{\omega}^2 = \bar{k}^2 + \bar{m}^2 \]  

(18)

Thus we have arrived at the dispersion relation by identifying the frequency not with the energy part \( p_0 \) of the four-momentum but with a certain function (to be found) of energy and momentum, which we denote by \( \bar{\omega} \), and using the same procedure for the momentum part \( p \) (by identifying the wave number with a certain function \( k(\bar{p}, \lambda) \)).

The physical justification of this is as follows. On the scales much exceeding \( \lambda \) the Planck constant is not affected by the existence of the universal length scale. However, at the scales comparable with \( \lambda \) this is not true anymore, and as a result the Planck “constant” becomes the effective Planck function \( h_{\text{eff}} \) which varies with \( \lambda \), momentum \( p \), and energy \( p_0 \) (e.g., reference [3]). In the limit of \( \lambda \to 0 \) \( h_{\text{eff}} \to \hbar \). Thus despite the fact that now frequency is a function of \( p \), \( p_0 \), and \( \lambda \) the overall linear relation between energy \( p_0 \) and frequency is preserved. This means that since \( \omega = p_0 \) (in the units where the Planck constant \( \hbar = 1 \)):

\[ \text{energy} = h_{\text{eff}}^0 p_0 = \bar{\omega}; \quad \text{momentum} = h_{\text{eff}}^1 p = \bar{k} \]  

(19)

What seems very interesting is that the emerging effective Planck numbers (or more correctly functions) in such 2-scale relativity have to transform differently for the spatial and temporal parts of the four-momentum to be consistent with the fact that \( V_{m=0} = 1 \). It looks as if the Planck function also becomes sort of 4-vector. We investigate the relations (19) in more detail in the next section.

Now the group velocity \( v_g \)

\[ v_g = \frac{d\bar{\omega}}{dk} = \frac{\bar{k}}{\sqrt{\bar{m}^2 + \bar{k}^2}} \]

is always less, and the phase velocity \( v_p \)

\[ v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{\sqrt{\bar{m}^2 + \bar{k}^2}}{\bar{k}} \]
is always greater than the maximum speed \( c_\lambda \) attainable by a massive particle

\[
c_\lambda \equiv \frac{1}{\cosh(m\lambda)} = \frac{1}{\sqrt{1 + (\bar{m}\lambda)^2}}.
\]

For a massless particle both the group velocity and phase velocity reach the speed of light, in full agreement with the expression for \( V_\alpha \) (12). The well-known theorem of the conventional relativity stating that

\[
v_{\text{group}}^2 v_{\text{phase}}^2 = 1
\]

holds true.

The effective frequency \( \bar{\omega} \) is given in terms of \( p_0 \) and \( \lambda m \), Eq.(17). Using the basic relation (3) between \( m, p, \) and \( p_0 \) we derive the relation between \( \bar{\omega} \), Eq.(17) and \( p_0 \) by inserting the value of \( \cosh(m\lambda) \equiv \sqrt{1 + (\bar{m}\lambda)^2} \) from (3) into (17). This would immediately give us the following

\[
\bar{\omega} = \frac{\sinh(\lambda p_0)}{\lambda} + \frac{\lambda p^2 e^{\lambda p_0}}{2} \tag{20}
\]

The derived relations (16) and (20) are the relations introduced in [8] without a proof. Moreover, we have provided a physical justification for these relations.

Now we will recast the expressions for the spatial part of 4-velocity (12) into the form similar to special relativity. Recalling that in special relativity \( \sinh \xi \) corresponds to the spatial part \( u \) of the four-velocity, equation (14) can be written in terms of the reduced four-velocity \( \bar{u} \):

\[
\frac{\bar{p}}{\bar{m}} = \sinh \xi = \bar{u} \tag{21}
\]

We express \( \bar{u} \) in terms of \( V \) with the help of (12), use the definition of the ”speed of light” \( c_\lambda \), and denote \( pe^{\lambda p_0} = \bar{p} \). As a result, we get

\[
\bar{u} = \pm \frac{V c_\lambda}{\sqrt{1 - V^2} \mp \sqrt{1 - c_\lambda^2}} \tag{22}
\]
In the limit of \( c_\lambda \to 1 \) (that is in the limit yielding the conventional special relativity) we get the familiar expression for the spatial part of the four velocity in special relativity:

\[
\bar{u} = u = \pm \frac{V}{\sqrt{1 - V^2}}.
\]

Note the symmetric character of the "forward" (+) and "backward" (−) velocity in this case. This feature is lost in the general expression \( (22) \).

In another limit of \( V \to c_\lambda \) we get

\[
\bar{u}^+ \to \infty, \\
\bar{u}^- \to -\frac{c_\lambda^2}{2\sqrt{1 - c_\lambda^2}}
\]

indicating that the region of "backward" velocities is bounded from below for all the values of \( c_\lambda \neq 1 \).

Using the definition of the reduced velocity according to

\[
\bar{V} = \tanh \xi = \frac{\bar{u}}{\sqrt{1 + \bar{u}^2}}
\]

we easily obtain expression for \( \bar{V} \) as a function of \( V \):

\[
\bar{V}^+ = V \frac{c_\lambda}{1 - \sqrt{(1 - V^2)(1 - c_\lambda^2)}} \\
\bar{V}^- = -V \frac{c_\lambda}{1 + \sqrt{(1 - V^2)(1 - c_\lambda^2)}}
\]

In the limit of \( c_\lambda \to 1 \) we obtain the usual results of the special relativity with a complete symmetry between "backward" and "forward" velocities. In general, however these two values are not symmetrical. For example, in the limit \( V \to c_\lambda \) the respective values of \( \bar{V}^+ \) and \( \bar{V}^- \) are

\[
\bar{V}^+ \to 1, \\
\bar{V}^- \to -\frac{c_\lambda^2}{2 - c_\lambda^2}
\]
To find the law of velocity composition we notice that for $\bar{V}$ it looks exactly like the one found in special relativity. This allows us to write the respective law for $V$ for two particles having the same mass. After performing some algebra, we obtain:

$$V_{1,2} = \frac{V_1 + V_2 - \sqrt{1 - c_\lambda^2(w_1 + w_2)}}{1 + V_1V_2 + \sqrt{1 - c_\lambda^2[(w_1 + w_2 - w_1w_2)(1 + \sqrt{1 - c_\lambda^2}) - (1 - c_\lambda^2)]}}$$  \hspace{1cm} (25)$$

where

$$w_k = 1 - \sqrt{1 - V_k^2},$$

From (25) follows that if either $V_1$ or $V_2$ (or both) go to $c_\lambda$ the combined velocity also tends to $c_\lambda$ as could be expected.

II. Here we investigate another approach to the problem of the dispersion equation, based on working directly in the space of the boost parameter $z(\xi, \lambda)$ which depends on both the "bare" boost parameter $\xi$ and the universal length scale $\lambda$. We return to the Minkowski distance and write down the respective spatio-temporal transformation from one inertial system $K$ to a system $K'$ moving with respect to $K$ with a constant velocity $V$.

We require (in full agreement with the invariance of Minkowski distance) this transformation to be exactly similar to the special relativistic relations (cf.[6]):

$$x = x'\cosh z(\xi, \lambda) + x'_0\sinh z(\xi, \lambda), \quad x_0 = x'\sinh z(\xi, \lambda) + x'_0\cosh z(\xi, \lambda)$$  \hspace{1cm} (26)$$

where the boost parameter $z(\xi, \lambda)$ has to be determined.

To find parameter $z(\xi, \lambda)$ we use the fact that the velocity of $K'$ with respect to $K$ is

$$V = \tanh z,$$

as follows from (26). Introducing this expression for $V$ into (26) we obtain

$$\tanh z(\xi) = \frac{\sinh \xi}{\cosh \xi \sqrt{1 + \lambda^2 \bar{m}^2} + \bar{m} \lambda} \equiv \frac{2e^{-\lambda\tanh(\frac{\xi}{2})}}{1 + e^{-2\lambda\tanh^2(\frac{\xi}{2})}} \equiv \tanh(2y)$$  \hspace{1cm} (27)$$

where we denote $y \equiv e^{-\lambda\tanh(\frac{\xi}{2})}$. As a result, the explicit form of the function $z(\xi)$ is

$$z = 2y \equiv 2\tanh^{-1}[e^{-\lambda\tanh(\frac{\xi}{2})}]$$  \hspace{1cm} (28)$$
This function was obtained in [4] on the basis of group-theoretical arguments, with the only difference that in [4] was a typographical error showing an additional factor $e^{m\lambda}$ at $\tanh^{-1}$.

Here we would like to briefly comment on the following. It has turned out that starting with the assumption of the formal coincidence of the spatio-temporal transformation from one inertial system to another with special-relativistic relations, and using the kinematical relation (for $V$) obtained with the help of the momentum sector of the phase space one can arrive at the basic relation of the algebra (a member of a class of algebras in ref. [3]) studied in [4]. In fact by using Eq. (13), differentiating Eq. (28) and using the result in transformations (26) we arrive at the basic relation of the above algebra (in two dimensions) [4]:

$$\delta x_0 = e^{-p_0 \lambda} x; \quad \delta x = e^{-p_0 \lambda} x_0.$$ 

The momentum $p$ is given by (9), which in terms of the boost parameter is

$$p = \vec{m} \frac{V}{\sqrt{1 - V^2}} = \vec{m} \cosh z.$$ 

In view of this it is clear that defining energy as $p_0$ [Eq. (13)] is not going to give us the relation between $p$ and $p_0$ consistent with the $V(m = 0) = 1$. The analogous case was considered above, but there we dealt with the transformation space defined directly by the parameter $\xi$ and not yielding the transformation in the form (26). The problem there has been solved by the introduction of both effective wave number and effective frequency.

Here however we look for another transformation which will keep the wave number as $p$ (in the appropriate units), but transform $p_0$ in such a way as to produce the required effective frequency. To this end we require such an effective frequency $\hat{\omega}$ to behave like the temporal component (vs. spatial component $p \leftrightarrow \vec{k}$) of the 4-vector of coordinates defined by (28)

$$\hat{\omega}^2 = \vec{m}^2 + \vec{k}^2 \tag{29}$$

Note that in this case (as we have already indicated), as in the previous case the effective mass $\vec{m}_e$ is the same. On the other hand, the effective wave number ($\vec{k} \leftrightarrow p$) and the effective frequency are different as compared to case 1.
Since $\bar{k} = p$ we have to find the explicit expression for the frequency $\bar{\omega}$. Using the expression for $p$, Eq.(10) we get
\[
\bar{\omega} = \frac{cosh(m\lambda)cosh\xi + sinh(m\lambda)}{sinh(m\lambda)cosh\xi + cosh(m\lambda)}
\]
(30)

By combining Eq.(30) with the formula for $p_0$, Eq.(13), we obtain
\[
\bar{\omega} = \frac{cosh(m\lambda) - e^{-p_0\lambda}}{\lambda}
\]
(31)

Upon substitution of $cosh(m\lambda)$ from Eq.(3) we get the dependence of $\omega$ on $p_0$:
\[
\bar{\omega} = \frac{sinh(p_0\lambda)}{\lambda} - \frac{\lambda}{2} p^2 e^{p_0\lambda}
\]
(32)

In contradistinction to case I, the effective frequency $\bar{\omega}$ is finite when the momentum $p \leftrightarrow \bar{k} \to 1/\lambda$. This is especially well seen if we rewrite the dispersion relation Eq.(32) in the equivalent form:
\[
\bar{\omega} = \sqrt{cosh^2(m\lambda) - (1 - \lambda^2\bar{k}^2)}
\]
(33)

Let us indicate that Eq.(20) of the case I has the same form.

Note that if we use the Planck number, common for $k$ and $\omega$, we would not be able to reconcile the dispersion relations with a massless case where
\[
\bar{\omega} = \bar{k},
\]

implying (according to $\hbar\omega = p_0$ and $\hbar k = p$) that $p_0 = p$ which is patently not the case. This justifies the introduction of two Planck numbers $\hbar_{eff}^0$ and $\hbar_{eff}^1$ which was done earlier, Eq.(19).

We still have to choose the dispersion equation out of (20) and (33) on the basis of their physical meaning. Let us consider a massless case, $m = 0$. Using Eqs.(33) and (19) we obtain that in this case energy is
\[
\text{energy} = p \Leftrightarrow \bar{k}
\]

and since $0 \leq p \leq 1/\lambda$, the energy does not tend to $\infty$ with $p \to 1/\lambda$, as predicted by relation (3).

Therefore the case II is not physical in this sense, which leaves us with case I and the respective dispersion equation (20).
3 Uncertainty Relation

We use this equation to explicitly obtain the value of the effective Planck number \( \hbar_{\text{eff}} \) needed for the comparison of the uncertainty relations here and in string theory. Inserting the effective Planck "number" (or rather vector), defined according to Eq. (19), into (33) we get

\[
\hbar_{\text{eff}}^0 = \frac{1}{p_0 \lambda} \sqrt{\cosh^2(m \lambda) + (1 - \lambda^2 \bar{k}^2)}
\]

\[
\hbar_{\text{eff}}^1 = e^{p_0 \lambda}; \quad \bar{k} = p e^{p_0 \lambda}
\] (34)

Using this expression for \( \hbar_{\text{eff}}^1 \), we can compare the uncertainty relation

\[
\delta x \delta p \geq \hbar_{\text{eff}}^1
\] (35)

which follow from this value and the uncertainty relation according to the string theory. We restrict our attention to a massless case \( m = 0 \).

We find that now (3) yields

\[
p_0 = -\frac{1}{\lambda} \log(1 - \lambda \delta p)
\] (36)

Inserting (34) and (36) into (35) we arrive at the following expression

\[
\delta x \geq -\frac{1}{\delta p (1 - \lambda \delta p)}
\] (37)

where we use \( p \geq \delta p \).

For small values of \( \lambda \) we expand (37) in Taylor series and retain the terms up to the second power of \( \lambda \). This yields the following relation:

\[
\delta x \geq \left[ \lambda + \frac{1}{\delta p} + \lambda^2 \delta p \right]
\] (38)

The analogous relations have been obtained previously [3], [8], [9], [10], although the full uncertainty relation for a massless particle have not been investigated from a position of the dispersion relation dictating the introduction of the effective Planck number (or a vector) varying with both the
Figure 1: Uncertainty relations $\delta x \text{ vs. } \lambda \delta p$ for a massless particle in string theory (lower curve) and in new relativity (upper curve); note the unbounded growth of $\delta x$ at the finite value of $\delta p = 1/\lambda$. 

universal length scale and the momentum uncertainty.

Comparing (38) with the string uncertainty relation (see, for example [12])

$$\delta x \geq \frac{1}{\delta p} + \lambda^2 \delta p,$$

we notice that in the (truncated) uncertainty relation (38) there is an extra term, linear in $\lambda$, whereas in string theory this term is absent. Moreover, since we consider the truncated form of (37), it does not give us the unbounded increase of $\delta x$ when $\delta p \to \lambda$. Nonetheless, the general character of the dependence (37) is in agreement with the string relation, with the minimum $\delta x_{min} = 2\lambda$, reached at $\delta p = .5/\lambda$. The graph with both the string uncertainty relation and the string uncertainty relation (37) is given in Fig.1.

4 Conclusion

We have undertaken a task of deriving and to physically justify by simple means the consistent (with the velocity of a massless particle) two-scale relativity (new relativity [1]) dispersion relations. It has turned out that in
addition to the one relation introduced in [6] there exists another relation which is compatible with the velocity of a massless particle to be \( V(m = 0) = 1 \). However, it has turned out that this relation is inconsistent with the the infinite energy corresponding to the maximum value of the momentum \( p = 1/\lambda \).

The physical (and even non-physical one) dispersion relation resolves an apparent paradox arising from the fact that if one would take the frequency to be equal to energy and wave number to be equal to the momentum (in the appropriate units), then the basic relation between energy and momentum cannot be translated into a consistent dispersion relation. The seemingly puzzling behavior of frequency-energy and momentum-wave number has a rather straightforward explanation.

At length scales comparable to the universal length scale \( \lambda \) the Planck constant stops to be constant and becomes a function of both the wave number and the universal length scale. Moreover, this function has two components (in 2 dimensions), one corresponding to the wave number \( \hbar_{\text{eff}}^1 \), and another one \( \hbar_{\text{eff}}^0 \). Since we assume that the 2-scale relativity is based upon the two universal invariants, thus indicating that at the scales comparable to \( \lambda \) the quantities considered constant at larger scales, become dependent on the two invariants and dynamic variables. This restores the linear character of frequency-energy and momentum-wave number relations. Moreover, such an explanation is in full agreement with the string theory, where the uncertainty relation clearly indicates a variable character of the Planck number at scales comparable to \( \lambda \).

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