MUMFORD DIVISORS

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Abstract. We define the notion of Mumford divisors, argue that they are the natural divisors to study on reduced but non-normal varieties and prove a structure theorem for the Mumford class group.

In higher dimensional algebraic geometry it is necessary to work with pairs $(X, \Delta)$ where $X$ is not normal. This happens especially frequently in moduli problems and in proofs that use induction on the dimension. While the main interest is in semi-log-canonical pairs, for a general theory the right framework seems to be demi-normal and, more generally, seminormal varieties; see [Kol13, Sec.5.1]. The aim of this note is to discuss the basics of the divisor theory on such non-normal schemes. The main result, Theorem 21, shows that the divisor class group of seminormal varieties is much larger than for normal varieties.

For singular curves a theory of generalized Jacobians was worked out by Severi [Sev47] and Rosenlicht [Ros54]; an exposition is given by Serre [Ser59]. This theory starts with a smooth, projective curve $C$ and a finite subscheme $P \subset C$. The basic objects are divisors $B$ that are contained in the open curve $C \setminus P$, and two divisors $B', B''$ are called linearly equivalent modulo $P$ if there is a rational function $\phi$ on $C$ such that $(\phi) = B' - B''$ and $1 - \phi$ vanishes on $P$. The linear equivalence classes of degree 0 divisors form a semi-abelian variety.

This approach does not generalize well to higher dimensions, since the condition that $\phi|_P$ be constant is too stringent if $\dim P > 0$.

A more direct precursor of our definition is Mumford’s observation that in order to get a good theory of pointed curves $(C, P)$, the curve should be nodal and the marked points $P = \{p_1, \ldots, p_n\}$ should be smooth points of $C$. Two such point sets $P', P''$ are then considered linearly equivalent iff the corresponding sheaves $\mathcal{O}_C(P')$ and $\mathcal{O}_C(P'')$ are isomorphic. Note that these sheaves are locally free since the marked points are smooth.

For moduli purposes, the best higher dimensional generalization of pointed curves is the class of semi-log-canonical pairs $(X, \Delta)$. Here $X$ is allowed to have normal crossing singularities in codimension 1, but none of the irreducible components of $\text{Supp} \Delta$ is contained in $\text{Sing} X$. Equivalently, $X$ is smooth at the generic points of $\text{Supp} \Delta$. The latter is a key feature of such pairs and leads to our concept of Mumford divisors in Definition 1.

Another important property of semi-log-canonical pairs is that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. However $\Delta$ need not be $\mathbb{Q}$-Cartier and even if both $K_X + \Delta$ and $\Delta$ are $\mathbb{Q}$-Cartier, the log-pluricanonical sheaves $\omega_X^{[m]}(|m\Delta|)$ are frequently not. For me the immediate reason for studying Mumford divisors was to understand the deformation invariance of log-plurigenera of semi-log-canonical pairs [Kol18a, Kol18b].

If $X$ is normal then there are ways to reduce the general setting to the $\mathbb{Q}$-Cartier case by a small modification [Kol18a, Cor.21], but in the non-normal case this is not
possible. Thus it seems necessary to study the divisor theory of semi-log-canonical varieties without any \(\mathbb{Q}\)-Cartier conditions. This has also long been the viewpoint advocated by Shokurov; see \cite{Sho92, Amb03, Fuj17}.

1. Mumford divisors

Recall that a \textit{Weil divisor} on a scheme \(X\) is a formal linear combination \(B = \sum m_i B_i\) of codimension 1, irreducible, reduced subschemes \(B_i \subset X\). The set of all Weil divisors with coefficients in a ring \(R\), also called \textit{Weil \(R\)-divisors}, forms an \(R\)-module, denoted by \(\text{WDiv}_R(X)\).

We also write \(\text{WDiv}(X)\) to denote \(\text{WDiv}_{\mathbb{Z}}(X)\).

Combining the notion of Weil divisors with Mumford’s observation gives the following definition in higher dimensions.

**Definition 1.** Let \(X\) be a scheme. A \textit{Mumford divisor} on \(X\) is a Weil divisor \(B\) such that \(X\) is regular at all generic points of \(\text{Supp} B\). The set of all Mumford divisors with coefficients in a ring \(R\), also called \textit{Mumford \(R\)-divisors}, forms an \(R\)-module, denoted by \(\text{MDiv}_R(X)\). We also write \(\text{MDiv}(X)\) to denote \(\text{MDiv}_{\mathbb{Z}}(X)\). It is a subgroup of \(\text{WDiv}(X)\).

On a normal scheme every Weil divisor is a Mumford divisor, thus the latter notion is of interest mainly on non-normal schemes.

More generally, let \(X\) be a scheme and \(D = \{D_1, \ldots, \} \) a set of codimension 1 irreducible, reduced subschemes \(D_j \subset X\) such that \(X\) is regular at all codimension 1 points of \(X \setminus D\). A \textit{Mumford divisor} on \((X,D)\) is a Weil divisor \(B = \sum m_i B_i\) on \(X\) such that none of the \(B_i\) is among the \(D_j\). As before, the set of all Mumford divisors with coefficients in a ring \(R\) forms an \(R\)-module, denoted by \(\text{MDiv}_R(X,D)\). It can be viewed as a subgroup of \(\text{WDiv}_R(X \setminus D)\), though this is not very useful.

Thus a Mumford divisor on \(X\) is the same as a Mumford divisor on the pair \((X,\text{Sing} X)\). (This is somewhat sloppy notation. We should consider the pair \((X, D(\text{Sing} X))\) where \(D(\text{Sing} X)\) is the set of closures of all codimension 1 singular points of \(X\).) Thus \(\text{MDiv}(X) = \text{MDiv}(X,\text{Sing} X)\).

While the definition makes sense if \(X\) is arbitrary and even if \(D\) is infinite, in order to avoid various pathologies we only consider schemes \(X\) that satisfy the following.

**Assumption 1.1.** \(X\) is noetherian, reduced, separated, pure dimensional and \(\text{Sing} X\) is a nowhere dense, closed subscheme. (Note that if \(X\) is also excellent then it satisfies the latter assumption on \(\text{Sing} X\).)

Two Mumford divisors \(B', B''\) on \((X,D)\) are called \textit{linearly equivalent modulo \(D\)} if there is a rational function \(\phi\) such that \((\phi) = B' - B''\) and \(\phi\) is regular (and nonzero) at the generic points of \(\eta_j \in D\). The linear equivalence classes of Mumford divisors form a group, denoted by \(\text{MCl}(X,D)\). It is called the \textit{Mumford class group of \((X,D)\)}.

The Mumford class group of \((X,\text{Sing} X)\) is called the \textit{Mumford class group of \(X\)} and denoted by \(\text{MCl}(X)\).

**Warning 1.2.** In the usual definition of rational equivalence of divisors and \(\text{Chow}_{n-1}(X)\), one allows functions that are not regular at the generic points of \(\text{Sing} X\), cf. \cite[Sec.1.2]{Ful84}. This implies that rational equivalence is preserved by push-forward \cite[Sec.1.4]{Ful84}. Therefore, if \(X\) is a proper variety of dimension \(n\) with normalization \(\pi: \tilde{X} \to X\), then \(\pi_*: \text{Chow}_{n-1}(\tilde{X}) \to \text{Chow}_{n-1}(X)\) has finitely generated kernel and finite cokernel.
By contrast, linear equivalence modulo $D$ is \textit{not} preserved by push-forward and we will see that the difference between $\text{MCl}(X)$ and $\text{MCl}(\bar{X}) \cong \text{Cl}(\bar{X})$ is huge.

Thus the condition that $\phi$ be regular at the generic points of $\eta_j \in D$ turns out to have a decisive influence on the structure of $\text{MCl}(X)$.

Our main interest is to study $\text{MCl}(X)$ for demi-normal schemes. The main result, Theorem \ref{main-theorem}, shows that if $\dim X \geq 2$ and $X$ is not normal, then $\text{MCl}(X)$ is much larger than class groups of normal varieties. In particular, if we work over $\mathbb{C}$, then the discrete part of $\text{MCl}(X)$ always has uncountably infinite rank. First we note that on a normal variety, we do not get anything new.

\textbf{Lemma 2.} Let $X$ be a noetherian, normal scheme and $D$ a finite set of divisors. Then $\text{MCl}(X, D) \cong \text{Cl}(X)$.

Proof. By Lemma \ref{linear-equivalence} every Weil divisor $B$ is linearly equivalent to Weil divisor $B'$ whose support does not contain any of the $D_j$. Thus the natural map $\text{MCl}(X, D) \to \text{Cl}(X)$ is surjective and it is clearly an injection. \hfill $\square$

Extending the correspondence between Cartier divisors and line bundles to Mumford divisors leads to divisorial sheaves.

\textbf{Definition 3} (Divisorial sheaves). A coherent sheaf $F$ on a scheme $X$ is called a \textit{divisorial sheaf} if there is a closed subset $Z \subset X$ (depending on $F$) such that

1. $\text{depth}_ZF \geq 2$,
2. $F|_{X \setminus Z}$ is locally free of rank 1 and
3. $X \setminus Z$ is $S_2$.

Let $j : U \hookrightarrow X$ be a dense, open, $S_2$ subscheme and $L_U$ an invertible sheaf on $U$. If $j_* (L_U)$ is coherent then it is a divisorial sheaf.

If $X$ is pure dimensional and excellent, then $j_* (L_U)$ is coherent iff $\text{codim}_X(X \setminus U) \geq 2$. This is the only case that we use; see \cite{Kol17} for the general setting.

The divisorial sheaves form a group, where the group operation is the $S_2$-hull of the tensor product. This is obvious from the second definition in the pure dimensional, excellent case. In general it follows from \cite{Kol17}. This group is denoted by $\text{DSh}(X)$.

\textbf{Proposition 4.} Let $X$ be a noetherian, reduced, separated, pure dimensional, excellent scheme. Then $B \mapsto \mathcal{O}_X(B)$ gives an isomorphism

$$\text{MCl}(X) \cong \text{DSh}(X).$$

Proof. Let $B$ be a Mumford divisor on $X$. Then $Z := \text{Supp} B \cap \text{Sing} X$ is a closed subset of codimension $\geq 2$ and $B|_{X \setminus Z}$ is Cartier. Thus

$$\mathcal{O}_X(B) := j_* (\mathcal{O}_{X \setminus Z}(B|_{X \setminus Z}))$$

is a divisorial sheaf, where $j : X \setminus Z \hookrightarrow X$ is the natural open embedding. This defines $\text{MCl}(X) \to \text{DSh}(X)$ and it is clearly an injection.

Let $L$ be a divisorial sheaf and $x_1, \ldots, x_r$ the generic points of $\text{Sing} X$ that have codimension 1 in $X$. By Lemma \ref{linear-equivalence} $L$ has a rational section $s$ that is regular and nonzero at the points $x_1, \ldots, x_r$. Then $(s)$ is a Mumford divisor and $\mathcal{O}_X((s)) \cong L$. Thus the map is also surjective. \hfill $\square$

\textbf{Lemma 5.} Let $X$ be a reduced, separated, noetherian scheme, $z_1, \ldots, z_r$ codimension 1 points of $X$, $x_1, \ldots, x_s$ codimension 1 regular points of $X$ and $m_1, \ldots, m_s \in \mathbb{Z}$ such that $\sum 2m_i \geq 2$. Then

$$\text{MCl}(X) \cong \text{DSh}(X).$$
Let $L$ be a line bundle on $X$. Then $L$ has a rational section $\phi$ that is regular at the points $z_1, \ldots, z_r$ and vanishes to order $m_1, \ldots, m_s$ at the points $x_1, \ldots, x_s$.

Proof. By [Sta13, Tag 09NN] there is an open affine subscheme $U \subset X$ that contains the points $z_1, \ldots, z_r$ and $x_1, \ldots, x_s$. We can thus assume that $X$ is affine. We can then choose a global section $s \in H^0(X, L)$ that does not vanish at any of the points $z_1, \ldots, z_r$ and $x_1, \ldots, x_s$ and regular functions $\phi_1, \ldots, \phi_s$ such that $\phi_i$ vanishes at $x_i$ to order 1 and does not vanish at any of the other points. Then $\phi := s \cdot \prod_i \phi_i^{m_i}$ works. \qed

6 (S2-hulls). Let $X$ be a reduced, noetherian, affine scheme such that $\text{Sing } X$ is closed. Let $g = 0$ be an equation of $\text{Sing } X$ that is nonzero at all generic points. Let $p_1 \in X$ be the embedded primes of $\mathcal{O}_X/(g)$ and $Z \subset X$ the union of their closure. Then $\text{codim}_X Z \geq 2$ and $X \setminus Z = S_2$. We call $Z$ the non-$S_2$-locus of $X$ and $X \setminus Z$ the $S_2$-locus of $X$.

Let $j : X \setminus Z \to X$ be the natural embedding. If $j_* \mathcal{O}_{X \setminus Z}$ is coherent (for example, $X$ is excellent) then set $X^* := \text{Spec } j_* \mathcal{O}_{X \setminus Z}$. Thus $X^*$ is $S_2$, the projection $\pi : X^* \to X$ is finite and an isomorphism outside subsets of codimension $\geq 2$. This $X^*$ is unique and called the $S_2$-hull of $X$.

For a reduced, noetherian, excellent scheme $X$ we can glue the local $S_2$-hulls together to get the $S_2$-hull $\pi : X^* \to X$. If $X$ is seminormal then so is $X^*$.

Since the notion of Mumford divisors is not sensitive to closed subsets of codimension $\geq 2$ we see that $\pi_*$ induces isomorphisms

\[
\text{MDiv}(X^*) \cong \text{MDiv}(X) \quad \text{and} \quad \text{MCl}(X^*) \cong \text{MCl}(X).
\]

7 (Pulling back divisors). Let $g : Y \to X$ be a morphism and $B$ a Weil divisor on $X$ that is Cartier on an open set $U \subset X$. Then $g^*(B|_U)$ is a Cartier divisor on $g^{-1}(U)$. If $Y \setminus g^{-1}(U)$ has codimension $\geq 2$ then the closure of $g^*(B|_U)$ is a well defined Weil divisor on $Y$, denoted by $g^*(B)$. (Probably $g^{(i)}(B)$ would be a better notation.)

Thus we need to understand morphisms $g : Y \to X$ for which $\text{codim}_X Z \geq 2 \Rightarrow \text{codim}_Y g^{-1}(Z) \geq 2$. To guarantee this, we consider the following setting.

Assumption 7.1. $X$ is pure dimensional, the morphism $g$ is finite and dominant on irreducible components. That is, every irreducible component of $Y$ dominates some irreducible component of $X$.

If, in addition, $X$ and $Y$ are normal then the pull-back is defined on all Weil divisors and we get $g^* : \text{WDiv}(X) \to \text{WDiv}(Y)$ and $g^* : \text{Cl}(X) \to \text{Cl}(Y)$.

In particular, if $\pi : \tilde{X} \to X$ is the normalization of $X$ then we get

\[
\pi^* : \text{MDiv}(X) \to \text{WDiv}(\tilde{X}) \quad \text{and} \quad \pi^* : \text{MCl}(X) \to \text{Cl}(\tilde{X}),
\]

where the surjectivity follows using Lemma 6.

8 (Restriction of Mumford divisors). Let $S \subset X$ be a closed subscheme. $B$ is called a Mumford divisor along $S$ if

1. $\text{Supp } B$ does not contain any irreducible component of $S$,
2. $B$ is Cartier at all generic points of $S \cap \text{Supp } B$ and
3. $S$ is regular at all generic points of $S \cap \text{Supp } B$. 


These imply that $B|_S$ is a well-defined Mumford divisor on $S$. (It seems that (1–3) are all necessary for this.) Furthermore, there is a subset $Z \subset S$ of codimension $\geq 2$ such that $B$ is Cartier at all points of $S \setminus Z$. The restriction sequence

$$0 \to \mathcal{O}_X(B)(-S) \to \mathcal{O}_X(B) \to \mathcal{O}_S(B|_S) \to 0,$$

is left exact everywhere and right exact on $X \setminus Z$.

Assume next that $S \subset X$ is a Cartier divisor. Then (8.4) gives a natural injection

$$r : \mathcal{O}_X(B)|_S \to \mathcal{O}_S(B|_S),$$

which is an isomorphism on $S \setminus Z$. Since $\mathcal{O}_S(B|_S)$ is $S_2$ by definition, $r$ is an isomorphism everywhere if $\mathcal{O}_X(B)|_S$ is $S_2$. This is equivalent to $\mathcal{O}_X(B)$ being $S_3$ along $S$. (Recall that a sheaf $F$ is $S_3$ along a closed subscheme $Z$ if $\text{depth}_x F \geq \min\{3, \dim_x F\}$ holds for every $x \in Z$.) We thus obtain the following observation.

**Claim 8.6.** If $S \subset X$ is a Cartier divisor then the sequence (8.4) is exact iff $\mathcal{O}_X(B)$ is $S_3$ along $S$. \qed

**Definition 9** (Principal divisors). Let $V$ be a normal scheme. The group of *principal divisors* on $V$ is denoted by $\text{PDiv}(V)$. It is a subgroup of $\text{WDiv}(V)$. The latter group is free by definition, hence so is $\text{PDiv}(V)$; cf. [Lan02, p.880]. If $V$ is a $k$-scheme of finite type and $\dim V \geq 1$ then the rank of $\text{PDiv}(V)$ is $|k|$ if $k$ is infinite and countable infinite if $k$ is finite. We can write this as $|k|$.

Let $V$ be a reduced $k$-scheme. The algebraic closure of $k$ in $k(X)$ is the algebra of *locally constant functions* on $V$. We denote it by $k^{lc}(V)$. Thus $k^{lc}(V) \subset k(V)$ is a finite dimensional, reduced $k$-algebra. Note that both $k(V)$ and $k^{lc}(V)$ are birational invariants. If $V$ is proper and normal, then $k^{lc}(V) = k[V] = H^0(V, \mathcal{O}_V)$.

If $V$ is reduced and proper with normalization $V^n \to V$ and $\eta_V \in V$ is the generic point then we have identifications

$$\text{PDiv}(V^n) = k(V)^*/k^{lc}(V)^* = k(\eta_V)^*/k^{lc}(\eta_V)^*. \hspace{1cm} (\text{9}1)$$

Next let $g : W \to V$ be a finite morphism of normal and proper $k$-schemes that is dominant on irreducible components. Let $\eta_W \in W$ and $\eta_V \in V$ denote the generic points. Then

$$\frac{\text{PDiv}(W^n)}{g^* \text{PDiv}(V^n)} = \frac{k(W)^*}{\langle g^* k^{lc}(V)^*, k^{lc}(W)^* \rangle} = \frac{k(\eta_W)^*}{\langle g^* k(\eta_V)^*, k^{lc}(\eta_V)^* \rangle}. \hspace{1cm} (\text{9}2)$$

We study the structure of these quotient groups in Paragraph 91.

**Remark 93.** It does not seem possible to define a sensible analog of $k^{lc}(V)$ for non-reduced schemes $V$. The problem is that if $\phi$ is a nilpotent global section of $\mathcal{O}_V$ then $1 + \phi$ satisfies the equation $(x - 1)^n = 0$ for some $n > 0$. Thus there can be too many global sections of $\mathcal{O}_V$ that satisfy a polynomial in $k[x]$ with leading coefficient $= 1$.

**10** (Algebraic equivalence of Mumford divisors). Let $D \subset X \times C$ be an algebraic family of divisors over a connected curve $C$ such that every fiber $D_c \subset X_c$ is a Mumford divisor. By normalization of $X$ we get $D^n \subset \tilde{X} \times C$, which is again an algebraic family of divisors. Thus 2 Mumford divisors $D_1, D_2$ on $X$ are algebraically equivalent iff their pull-backs $\pi^* D_1, \pi^* D_2$ are algebraically equivalent on the normalization $\pi : \tilde{X} \to X$. Thus

$$\text{MCl}(X)/\text{MCl}^{\text{alg}}(X) \cong \text{Cl}(\tilde{X})/\text{Cl}^{\text{alg}}(\tilde{X}) \cong \sum_i \text{Cl}^{\text{ns}}(\tilde{X}).$$
2. SEMINORMAL AND DEMI-NORMAL SCHEMES

11 (Seminormal schemes). A morphism of schemes \( p : Y \to X \) is an isomorphism on points if \( p^*_x : k(x) \to k(\text{red } p^{-1}(x)) \) is an isomorphism for every \( x \in X \). If \( X, Y \) are \( \mathbb{C} \)-schemes then this holds iff \( p \) is a bijection on \( \mathbb{C} \)-points.

A morphism of schemes \( p : X' \to X \) is a partial seminormalization if \( X' \) is reduced, \( p \) is finite and an isomorphism on points. A scheme \( X \) is called seminormal if every partial seminormalization \( p : X' \to X \) is an isomorphism. If the normalization \( X^n \to X \) is finite (for example, if \( X \) is excellent) then there is a unique factorization \( X^n \to X^{sn} \to X \) where \( X^{sn} \) is seminormal and \( X^{sn} \to X \) is a partial seminormalization. Then \( X^n \to X \) is called the seminormalization of \( X \).

See [AN67, AB69, Tra70] for the origin of this notion and [Kol13, Sec.10.2] for more recent details.

12 (Conductor). Let \( \pi : X' \to X \) be a finite, birational morphism of reduced schemes. The conductor ideal sheaf is \( I_{X'/X} := \text{Hom}_X(\pi_*\mathcal{O}_{X'}, \mathcal{O}_X) \subset \mathcal{O}_X \). It is the largest ideal sheaf on \( X \) that is also an ideal sheaf on \( X' \). Somewhat sloppily, I refer to either of the subschemes \( V(I_{X'/X}) \subset X \) and \( V(I_{X'/X}) \subset X' \) as the conductor of \( \pi \).

The most important special case is when \( \pi : \bar{X} \to X \) is the normalization. The conductor ideal \( I_X := \text{Hom}_X(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \subset \mathcal{O}_X \) of \( \pi \) is then called the conductor ideal of \( X \). We write \( D := V(I_X) \subset X \) and \( \bar{D} := V(I_{\bar{X}}) \subset \bar{X} \) and call both of them the conductor of \( \pi \). Note that \( \pi_D : D \to D \) is a finite, surjective morphism and \( X \) is uniquely determined by \( \bar{X} \) and the morphism \( \pi_D : \bar{D} \to D \). Indeed, let \( \pi_{\bar{X}/\bar{D}} : \mathcal{O}_{\bar{X}} \to \mathcal{O}_{\bar{D}} \) be the restriction map. By push-forward we get \( \pi_X / \bar{D} : \pi_*\mathcal{O}_{\bar{X}} \to \pi_*\mathcal{O}_{\bar{D}} \).

There is a natural injection \( j : \mathcal{O}_D \hookrightarrow \pi_*\mathcal{O}_{\bar{D}} \) and
\[
\mathcal{O}_X = R_{\bar{X}/\bar{D}}(j(\mathcal{O}_{\bar{D}})),
\] (12.1)
see [Kol96, Sec.7.2] for details.

It is easy to see that if \( X \) is seminormal then \( \bar{D} \) is reduced (cf. [Kol96, I.7.2.5]) and if \( X \) is \( S_2 \) then \( \bar{D} \subset X \) is pure of codimension 1 (cf. [Kol13, 10.14]). Thus if \( X \) is seminormal and \( S_2 \) then \( \pi_D : D \to D \) is a finite, surjective morphism of reduced schemes that is dominant on irreducible components. Thus the induced map on the function algebras
\[
\pi_D^* : k(D) \to k(\bar{D}),
\]
is a finite extension of direct sums of fields. In this case \( X \) is uniquely determined by \( \bar{X} \) and by \( \pi_D^* : k(D) \to k(\bar{D}) \), cf. [Kol13, 10.14].

**Notation 13.** Let \( X \) be a seminormal, \( S_2 \) scheme with normalization \( \pi : \bar{X} \to X \). We write \( D \subset X \) and \( \bar{D} \subset \bar{X} \) for its conductors. We also call \( D \) the singular divisor of \( X \). The normalization, and its restriction to \( \bar{D} \), are usually written as
\[
\pi : (\bar{X}, \bar{D}) \to (X, D) \quad \text{and} \quad \pi_D := \pi|_D : D \to \bar{D}.
\] (13.1)
We write
\[
X = \bigcup_{i \in I} X_i \quad \text{and} \quad D = \bigcup_{j \in J} D_j
\] (13.2)
as the union of their irreducible components. Thus \( \bar{X} = \bigcup_{i \in I} \bar{X}_i \) and \( \pi_i : \bar{X}_i \to X_i \) are the normalizations.

Similarly we have a decomposition \( \bar{D} = \bigcup_{j \in J} \bar{D}_j \) and finite maps \( \pi_j : \bar{D}_j \to D_j \).

Note that the \( \bar{D}_j \) may be reducible.
In general, neither the $D_j$ nor the $\bar{D}_j$ are normal, their normalizations are denoted by $D^n_j$ and $\bar{D}^n_j$. Thus we get finite morphisms $\pi^n_j : \bar{D}^n_j \to D^n_j$, that are dominant on irreducible components. Set

$$Z := \text{Sing } D \cup \text{(non-$S_2$-locus of } X) \cup \pi(\text{Sing } \bar{X} \cup \text{Sing } \bar{D})$$

and define the semi-regular locus of $(X, D)$ as

$$(X^{sr}, D^{sr}) := (X \setminus Z, D \setminus Z).$$

Note that $X^{sr}$ is $S_2$, the schemes $\bar{X}^{sr}, D^{sr}, \bar{D}^{sr}$ are all regular and $\pi^n_j : \bar{D}^{sr} \to D^{sr}$ is flat. If $\bar{X}^{sr}, D^{sr}, \bar{D}^{sr}$ are all smooth, we also call this the semi-smooth locus.

**Definition 14** (Demi-normal schemes). A scheme $X$ is called demi-normal if it is seminormal, $S_2$ and $\pi_D$ has degree 2 over every irreducible component of $D$. Equivalently, $X$ is $S_2$ and its codimension 1 points are either regular or nodal (see [Kol13, 1.31] for details).

Using Notation 13 we write $X = \bigcup_{i \in I} X_i$ and $D = \bigcup_{j \in J} D_j$ as the union of their irreducible components and $\bar{D} = \bigcup_{j \in J} \bar{D}_j$. The induced maps $\pi_j : \bar{D}_j \to D_j$ have degree 2. As before, the $\bar{D}_j$ may be reducible.

Let $\eta_j \in D_j$ and $\bar{\eta}_j \in \bar{D}_j$ denote the generic points. Thus $k(\bar{\eta}_j)$ is a reduced $k(\eta_j)$-algebra of dimension 2. There are 3 possibilities.

1. (Reducible) $k(\bar{\eta}_j) \cong k(\eta_j) + k(\eta_j)$. Let $\tau_j$ denote the involution that interchanges the summands.

2. (Separable) $k(\bar{\eta}_j)/k(\eta_j)$ is a degree 2 separable field extension with Galois involution $\tau_j$.

3. (Inseparable) $k(\bar{\eta}_j)/k(\eta_j)$ is a degree 2 inseparable field extension. This can happen only if $\text{char } k(\eta_j) = 2$.

In cases (1–2) $\tau_j$ induces an involution on $\bar{D}^n_j$, which we also denote by $\tau_j$. All the $\tau_j$ together determine an involution $\tau : D^n \to D^n$ called the gluing involution.

Thus $D^n = D^n/\langle \tau \rangle$ and $X$ is uniquely determined by the data $(\bar{X}, \bar{D}, \tau)$; see [Kol13, 5.3] for details.

3. The structure of the class group

Let us recall the known general results about the class group of normal schemes and of nodal curves.

**Definition 15** (Class group of a normal variety). Let $X$ be a normal and proper $k$-variety. Let $\text{Cl}^\circ(X) \subset \text{Cl}(X)$ denote the group of divisors that are algebraically equivalent to 0. I call the quotient $\text{Cl}^{\text{ns}}(X) := \text{Cl}(X)/\text{Cl}^\circ(X)$ the Néron-Severi class group of $X$. Note that there are natural inclusions

$$\text{Pic}^\circ(X) \subset \text{Cl}^\circ(X) \quad \text{and} \quad \text{NS}(X) \subset \text{Cl}^{\text{ns}}(X).$$

Both of these are equalities iff every Weil divisor is Cartier, for example if $X$ is regular.

Note that if $X$ is a proper $k$-scheme with Picard scheme $\text{Pic}^\circ(X)$ then there is a natural injection $\text{Pic}^\circ(X) \to \text{Pic}^\circ(X)(k)$ which is an isomorphism if $X(k) \neq \emptyset$ and $H^0(X, \mathcal{O}_X) \cong k$, cf. [Gro62, V.2.1]. Furthermore, $\text{NS}(X)$ is a finitely generated abelian group. In the complex case these go back to Picard [Pic95] and Severi [Sev1906]; see Mumford’s Appendix V in [Zar71] for a discussion of the (quite convoluted) history of the general case. The most complete references may be the
hard-to-find papers of Matsusaka [Mat52] and of Néron [Né52]. The latter also proves the analogous results for the class groups.

More recent results on various aspects of the class group of singular varieties are discussed in [BVS93, BYRS09, RS09]. It is quickest to construct the scheme $\text{Cl}^p(X)$ using the Albanese variety.

**Definition 16** (Albanese variety). Let $X$ be a geometrically normal and proper $k$-variety. If $X(k) \neq \emptyset$ then, following the classical definition, the *Albanese map* is the universal rational map $\text{alb}_X : X \dasharrow \text{Alb}(X)$ to an Abelian variety. This definition differs from the one in [Gro62, VI.3.3], which requires $\text{alb}_X$ to be a morphism but assumes universality only for morphisms $X \to (\text{Abelian variety})$. The 2 versions are the same if $X$ has rational singularities.

In general one should define the *Albanese map* as the universal rational map $\text{alb}_X : X \dasharrow \text{Alb}(X)$ to an Abelian torsor. Here an *Abelian torsor* is a principal homogeneous space under an Abelian $k$-variety. (Equivalently, a $k$-scheme $A_k$ such that $A_k^0$ is isomorphic to an Abelian variety for some separable extension $K/k$. In this case $A_k^0$ is a principal homogeneous space under $\text{Aut}^0(A_k)$.)

Over $C$, their construction of the Albanese variety is usually attributed to [Alb32, Alb34] (see [Alb96] for a more accessible source), but in retrospect much of it is already in [Sev1913]. Over any field the existence of the Albanese variety is proved by Matsusaka [Mat52].

We can now state the structure theory of the class group in the following form.

**Theorem 17** (Class group of a normal variety). Let $X$ be a geometrically normal and proper $k$-variety. Then

1. there is a natural injection $\text{Cl}^p(X) \hookrightarrow \text{Pic}^p(\text{Alb}(X))(k)$ which is an isomorphism if $X(k) \neq \emptyset$, and
2. $\text{Cl}^p(X)$ is a finitely generated abelian group. \qed

If $X$ has a resolution of singularities $Y \to X$ then the following can be used to compute $\text{Cl}(X)$ in terms of the better known $\text{Pic}(Y)$; cf. [BN12].

**Lemma 18.** Let $g : Y \to X$ be a proper, birational morphism between normal varieties and $\{E_i : i \in I\}$ the exceptional divisors. Then push-forward by $g$ gives isomorphisms

$$\text{Cl}^p(Y) \cong \text{Cl}^p(X) \quad \text{and} \quad \text{Cl}^p(Y)/\oplus_i \mathbb{Z}[E_i] \cong \text{Cl}^p(X). \quad (18.1)$$

Proof. The push-forward is clearly surjective, thus we only need to prove the following.

**Claim 18.2**. If $D$ is a divisor on $Y$ and $g_* (D)$ is algebraically equivalent to 0 then $D$ is algebraically equivalent to a sum of exceptional divisors.

If $g_* (D)$ is Cartier then $g^* (g_* (D)) - D$ is exceptional and algebraically equivalent to $D$. Thus we claim that even if $g_* (D)$ is not Cartier, we can still define $g^* (g_* (D))$.

If $g_* (D)$ is linearly equivalent to 0, then $g_* (D) = (\phi)$ for some rational function $\phi$ on $X$. Thus $D - (\phi \circ g)$ is $g$-exceptional. Consider next a family of algebraically equivalent divisors $D^X \subset X \times C$ and its birational transform $D^Y \subset Y \times C$ for some irreducible, smooth, projective curve $C$. Let $0 \in C$ be a point and $D_0$ a divisor on $Y$ such that $g_*(D_0) = D^X_0$ and $D_0$ is algebraically equivalent to a sum of exceptional divisors. Note that $g_*(D^Y_0) = g_*(D_0)$, thus $D^Y_0 - D_0$ is $g$-exceptional,
hence $D_0^\gamma \overset{\text{alg}}{\sim} \sum a_i E_i$ for some $a_i \in \mathbb{Z}$. Therefore, for any other points $c \in C$ we have

$$D_0^\gamma - \sum a_i E_i \overset{\text{alg}}{\sim} D_0^\gamma - \sum a_i E_i \overset{\text{alg}}{\sim} 0.$$ 

Such families generate algebraic equivalence, thus (18) holds. Finally note that the classes of exceptional divisors are linearly independent by [KM98, 3.39].

Combining Lemma 18 and (17) gives the following useful observation.

**Corollary 19.** Let $X$ be a normal and proper $k$-variety and $k$ perfect. Then

1. $\text{Cl}^p(X)$ is a birational invariant of $X$ and
2. there is a normal and proper $k$-variety $X'$ that is birational to $X$ such that $\text{Cl}^p(X') = \text{Pic}^C(X')$.

20 (Picard group of nodal curves). Let $C$ be a proper, connected nodal curve over $\mathbb{C}$ and $\pi : \bar{C} \to C$ its normalization. Then $\text{MCl}(C) \cong \text{Pic}(C)$, $\text{Cl}(\bar{C}) = \text{Pic}(\bar{C})$ and there is an exact sequence

$$0 \to (\mathbb{C}^*)^m \to \text{MCl}(C) \overset{\pi^*}{\longrightarrow} \text{Cl}(\bar{C}) \to 0,$$

where $m = h^1(C, \mathcal{O}_C) - h^1(\bar{C}, \mathcal{O}_{\bar{C}})$; see [ACG11, Sec.X.2] for details. More invariantly, there is a topological $1$-complex $D(C)$ whose points are the irreducible components of $\bar{C}$ and for each node $p \in C$ we add an edge that connects the 2 irreducible components passing through $p$. (We add a loop if they are the same.) Then the improved exact sequence is

$$0 \to H^1(D(C), \mathbb{C}^*) \to \text{MCl}(C) \overset{\pi^*}{\longrightarrow} \text{Cl}(\bar{C}) \to 0.$$ (20.1)

Our aim is to generalize this sequence to higher dimensions. Thus let $X$ be a proper, seminormal variety over $\mathbb{C}$ with normalization $\pi : \bar{X} \to X$. Then $\text{Pic}(\bar{X})$ should be replaced by $\text{Cl}(\bar{X})$ and again there is a natural topological $1$-complex $D_1(X)$, giving a subgroup $H^1(D_1(X), \mathbb{C}^*) \hookrightarrow \text{MCl}(X)$. However, the sequence

$$H^1(D_1(X), \mathbb{C}^*) \to \text{MCl}(X) \overset{\pi^*}{\longrightarrow} \text{Cl}(\bar{X})$$ (20.3)

is no longer exact. In (20) we define subgroups of the Mumford class group

$$\text{MCl}^{\text{et}}(X) := \ker[\pi^* : \text{MCl}(X) \to \text{Cl}(\bar{X})]$$ and

$$\text{MCl}^{\text{pt}}(X) := \text{im}[H^1(D_1(X), \mathbb{C}^*) \to \text{MCl}(X)],$$

but the key new problem is to understand the quotient

$$\frac{\text{MCl}^{\text{et}}(X)}{\text{MCl}^{\text{pt}}(X)} = \frac{\ker[\pi^* : \text{MCl}(X) \to \text{Cl}(\bar{X})]}{H^1(D_1(X), \mathbb{C}^*)},$$

which turns out to be essentially a free abelian group of uncountable rank.

4. The Mumford class group

The following result describes the Mumford class group of seminormal varieties over an arbitrary base field. As we noted in Paragraph 5, it is enough to consider $k$-schemes that are seminormal and $S_2$.

**Theorem 21.** Let $X$ be a proper, geometrically semi-normal, $S_2$, pure dimensional $k$-scheme with normalization $\pi : \bar{X} \to X$. Following Notation 18 let $\{X_i : i \in I\}$ be the irreducible components of $X$, $\{D_j : j \in J\}$ the irreducible components of the
conductor $D \subset X$ and $\bar{D} = \cup_{j \in J} \bar{D}_j$. Then the Mumford class group $\text{MCl}(X)$ has a natural filtration by subgroups

$$0 \subset \text{MCIP}^t(X) \subset \text{MCIP}^c(X) \subset \text{MCIP}(X),$$

and the successive quotients have the following descriptions.

(2) $\text{MCIP}^t(X) \cong \text{coker} [k^{lc}(\bar{X})^* \times k^{lc}(D)^* \to k^{lc}(\bar{D})^*]$ as in (21.2–3), where $k^{lc}$ denotes locally constant functions as in Definition 2.

(3) $\text{MCIP}^c(X) = \text{MCIP}^t(X)$ iff either $X$ is normal or $\dim X = 1$.

(4) If $\dim X \geq 2$ and $X$ is not normal then $\text{MCIP}^c(X)/\text{MCIP}^t(X)$ has the following properties.

(a) There is a natural isomorphism

$$\text{MCIP}^c(X)/\text{MCIP}^t(X) \cong \bigoplus_j k(\bar{D}_j)^*/(k(D_j)^*, k^{lc}(\bar{D}_j)^*).$$

(b) If $\text{char } k \neq 2$ then there is a non-canonical isomorphism

$$\text{MCIP}^c(X)/\text{MCIP}^t(X) \cong \mathbb{Z}[\bar{k}] + \text{(finite group)}.$$

(c) If $\text{char } k = 2$ then there is a non-canonical isomorphism

$$\text{MCIP}^c(X)/\text{MCIP}^t(X) \cong \mathbb{Z}[\bar{k}] + T_2 + \text{(finite group)},$$

where $T_2$ is a $2^n$-torsion group of cardinality $|\bar{k}|$ for some $m$ and either one (but not both) of the first 2 summands may be missing.

Comments 22. A useful feature of these formulas is that each quotient can be computed using only small parts of the description of $X$ as the push-out of the maps $D \to X$ and $\bar{D} \to D$.

(1) $\text{MCl}(X)/\text{MCIP}^t(X)$ is computable from the normalization $\bar{X}$,

(2) $\text{MCIP}^c(X)/\text{MCIP}^t(X)$ from the generic points of $\bar{D} \to D$ and

(3) $\text{MCIP}^t(X)$ needs only the generic points of $\bar{X}$ and of the conductors.

Furthermore, we see during the proof that in (21.5.b–c) the order of every element of the torsion summands divides $\text{lcm}(\deg(\bar{D}_j/D_j) : j \in J)$.

Warning 22.4. There does not seem to be any sensible way to identify the quotient $\text{MCIP}^c(X)/\text{MCIP}^t(X)$ with the $k$-points of an algebraic group.

Also, as far as I can tell, $\text{MCl}(X)$ does not contain any algebraic subgroup that maps onto $\bigoplus_i \text{Cl}^0(\bar{X}_i)$.

Example 23. Set $S := (xy = 0) \subset \mathbb{P}^3$ with double curve $D = (x = y = 0) \cong \mathbb{P}^1$. Then $\text{MCIP}^t(S) = 0$ and $\text{MCl}(S)$ sits in an exact sequence

$$0 \to k(\mathbb{P}^1)^*/k^* \to \text{MCl}(S) \to \text{Cl}(\mathbb{P}^2)^2 \cong \mathbb{Z}^2 \to 0.$$

A free generating set of $k(\mathbb{P}^1)^*/k^*$ is given by the $\text{Gal}(\bar{k}/k)$-orbits on $\mathbb{P}^1(\bar{k})$ or as the set of $\text{Gal}(\bar{k}/k)$-orbits on $\bar{k}$ plus the point at infinity. Thus

$$\text{MCl}(S) \cong \mathbb{Z}[\bar{k}]/\text{Gal}(\bar{k}/k) + \mathbb{Z}^3$$

is a free abelian group of rank $|\bar{k}|$.

Example 24. Set $S := (xyz = 0) \subset \mathbb{P}^2_{xyz} \times \mathbb{P}^1_{uv}$ with double curves $D_1, D_2, D_3$ isomorphic to $\mathbb{P}^1_{uv}$. Then

(1) $\text{MCl}(S)/\text{MCIP}^c(S) \cong \text{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)^3 \cong \mathbb{Z}^6$.

(2) $\text{MCIP}^t(S) \cong k^*$.

(3) There is an isomorphism $\text{MCIP}^t(S)/\text{MCIP}^t(S) \cong (k(\mathbb{P}^1)^*/k^*)^3$. 
Taken together we get non-canonical isomorphisms
\[ \text{MCl}(S) \cong k^* + (k(\mathbb{P}^1)^*/k^*)^3 + \mathbb{Z}^6 \cong k^* + \mathbb{Z}^{[k]} \].

**Example 25** (Higher pinch point). Set \( S := (x^m y = z^m) \subset \mathbb{P}^3 \) and assume that \( \text{char } k \not| m \). It is singular along the line \( D = (x = y = 0) \) and has 2 higher pinch points on it at the coordinate vertices. Its normalization is the ruled surface \( \bar{S}_{m-1} \); the line \( (z = w = 0) \) gives the negative section. We obtain that
\[ \text{MCl}(S) \cong \mathbb{Z}/m\mathbb{Z} + \mathbb{Z}^{[k]} \].

The \( m \)-torsion is given by the difference of lines \( \bar{B} \) so \( mB \), so it does not give a linear equivalence for \( B \). Moreover, \( \bar{\tau} = (\bar{\tau})^m \), but \( \bar{\tau} \) is not regular along the singular line \( D \), so it does not give a linear equivalence for \( B \).

Next we define the two subgroups that appear in \( (211) \).

**26** (Component-wise trivial Mumford divisors). Let \( X \) be a reduced scheme with normalization \( \pi : \tilde{X} \to X \) and conductor \( D \subset \tilde{X} \). As in \( (7.3) \) we have a natural pull-back map on divisors
\[ \pi^*: \text{MDiv}(X) \to \text{MDiv}(\tilde{X}, \tilde{D}), \tag{26.1} \]
which descends to a pull-back map on divisor classes
\[ \pi^*: \text{MCl}(X) \to \text{MCl}(\tilde{X}, \tilde{D}) \cong \text{Cl}(\tilde{X}), \tag{26.2} \]
where the last isomorphism is by Lemma \( 2 \). The kernel of this map is called the group of *component-wise trivial* Mumford divisors. We denote it by \( \text{MCl}^t(X) \). Thus there is an exact sequence
\[ 0 \to \text{MCl}^t(X) \to \text{MCl}(X) \to \bigoplus_i \text{Cl}(\tilde{X}_i) \to 0 \tag{26.3} \]
and \( \text{MCl}(X)/\text{MCl}^t(X) \) is the quotient of \( \text{MCl}(X) \) that is easiest to study.

**27** (Piecewise trivial divisorial sheaves, geometric case). The geometrically simplest case is when \( \tilde{X} \) and \( \tilde{D} \) are both smooth and each \( D_j \subset D \) is contained in 2 irreducible components \( X_{i(j,1)} \) and \( X_{i(j,2)} \). Correspondingly we get that \( D_j = D_j^\prime \cup D_j^\prime\prime \) has 2 irreducible components for every \( j \), where \( D_j^\prime \) lies on \( \tilde{X}_{i(j,1)} \) and \( D_j^\prime\prime \) lies on \( \tilde{X}_{i(j,2)} \). It is convenient to fix an ordering of the \( X_i \) and always choosing \( i(j, 1) < i(j, 2) \).) Thus the gluing involution is given by isomorphisms \( \tau_j : D_j^\prime \cong D_j^\prime\prime \).

A *piecewise trivial line bundle* on \( X \) is given by the trivial line bundle on \( \tilde{X} \) with gluing data
\[ \mu_j : O_{D_j^\prime} \cong \tau_j^* O_{D_j^\prime\prime}, \tag{27.1} \]
where \( \mu_j \) is multiplication by a constant \( \mu_j \in k^* \). We can change the trivialization of \( O_{\tilde{X}} \) by different multiplicative constants \( \nu_i \) on the irreducible components \( \tilde{X}_i \). This changes \( \mu_j \) by the quotient \( \nu_{i(j,1)}/\nu_{i(j,2)} \).

The end result is best expressed in terms of the *dual complex* of \( X \). This is a topological 1-complex \( D_1(X) \) whose points are the irreducible components \( X_i \subset X \) and for each irreducible component \( D_j \subset D \) we add an edge that connects \( X_{i(j,1)} \) and \( X_{i(j,2)} \). (In general we add a loop if an irreducible component of \( X \) has self-intersection along \( D_j \).) Then the above considerations say that
\[ \text{MCl}^t(X) \cong H^1(D_1(X), k^*). \tag{27.2} \]
If $X$ is a simple normal crossing scheme or, more generally, a semi-dlt pair, then $D_1(X)$ is the 1-skeleton of the true dual complex of $X$, cf. [dFKX12].

If $(X, D)$ is an arbitrary seminormal pair over $\mathbb{C}$ then in general $\bar{X}$ and $\bar{D}$ are not smooth. However, as we noted in Notation [13] there is a closed subset $Z \subset X$ of codimension $\geq 2$ such that the smoothness assumptions are satisfies by $(X \setminus Z, D \setminus Z)$. We can then use the above construction to get $\text{MCl}^{\text{pt}}(X \setminus Z)$ and then pushing forward from $X \setminus Z$ to $X$ to obtain

$$\text{MCl}^{\text{pt}}(X) := \text{MCl}^{\text{pt}}(X \setminus Z).$$

(27.3)

Note that while the divisors in $\text{MCl}^{\text{pt}}(X)$ are Cartier on $X \setminus Z$, they need not be Cartier on $X$.

Example (27.3). Let $S := (x_1x_2x_3 = 0) \subset \mathbb{P}^3$ and consider the divisor $B := L_1 + L_2 + L_3 = (x_0 = 0)$ where each $L_i$ is a line through the origin contained in the plane $(x_i = 0)$ but different from the coordinate axes. Then $B$ is in $\text{MCl}^{\text{pt}}(S)$ but it is Cartier only if the 3 lines $L_1, L_2, L_3$ are coplanar.

Here we can take $Z = (x_1 = x_2 = x_3 = 0)$ and $\mathcal{D}(S \setminus Z)$ is a triangle. Thus $\text{MCl}^{\text{pt}}(S) \cong k^*$. The map from the above divisors to $k^*$ can be given as follows. Write $L_i = (x_i = a_ix_{i+1} + b_ix_{i+2} = 0)$ using indices modulo 3. Then

$$L_1 + L_2 + L_3 - (x_0 = 0) \mapsto \frac{a_1b_2b_3}{b_1b_2b_3} \in k^*$$

gives the isomorphism $\text{MCl}^{\text{pt}}(S) \cong k^*$.

28 (Trivializations). Let $X$ be a scheme. A trivialization of a line bundle $L$ on $X$ is an isomorphism $\sigma : \mathcal{O}_X \cong L$.

Let $X$ be a $k$-scheme. Two trivializations $\sigma_1 : \mathcal{O}_X \cong L$ are called projectively equivalent if $\sigma_2^{-1} \sigma_1 : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is multiplication by a locally constant function. A projective trivialization of a line bundle $L$ is an equivalence class of projectively equivalent trivializations.

Note that if $X$ is proper and normal, more generally, if $X$ is the complement of codimension $\geq 2$ subset in a proper and normal $k$-scheme, then any two trivializations are projectively equivalent.

In our applications this will hold for $X$ but not necessarily for its conductor $D$. The codimension 2 subset $Z \subset X$ that we remove in (27.3) usually has codimension 1 in $D$.

29 (Piecewise trivial divisorial sheaves, general case). Let $(X, D)$ be a seminormal $k$-scheme with normalization $\pi : (\bar{X}, \bar{D}) \rightarrow (X, D)$. Assume first that $D, \bar{D}$ are normal and $\pi_D : \bar{D} \rightarrow D$ is flat.

A piecewise trivial line bundle on $X$ is a line bundle $L$ on $X$ endowed with projective trivializations of $L|_D$ and of $\pi^* L$ that are compatible over $\bar{D}$.

If we fix the trivializations of $\pi^* L$ and of $L|_D$, then we are left with locally constant isomorphisms

$$\text{Isom}^{\text{lc}}(\pi^* \mathcal{O}_D, \mathcal{O}_X|_D) \cong k^{\text{lc}}(\bar{D})^*.$$  \hfill (29.1)

Changing the trivializations of $\pi^* L$ and of $L|_D$ gives that the group $\text{Pic}^{\text{pt}}(X)$ of piecewise trivial line bundles on $X$ is isomorphic to

$$\text{coker}\left[ (j_D^* \times \pi_D^* ) : k^{\text{lc}}(\bar{X})^* \times k^{\text{lc}}(\bar{D})^* \rightarrow k^{\text{lc}}(\bar{D})^* \right].$$  \hfill (29.2)
where \(j_D : \bar{D} \hookrightarrow \bar{X}\) is the natural injection. Observe that both \(T_1 := k^c(\bar{X})^* \times k^c(D)^*\) and \(T_2 := k^c(\bar{D})^*\) are \(k\)-tori, and then

\[
\text{Pic}^{pl}(X) \cong \ker \left[ T_1(k) \to T_2(k) \right].
\]  

(Note that although \(\ker [T_1 \to T_2] \) is a \(k\)-torus, usually we have a strict containment \(\ker [T_1 \to T_2] \subset \ker [T_1 \to T_2](k)\), so \(\text{Pic}^{pl}(X)\) is not the group of \(k\)-points of a \(k\)-torus.)

Let now \(X\) be an arbitrary proper, seminormal \(k\)-scheme. A piecewise trivial divisorial sheaf on \(X\) is a divisorial sheaf on \(X\) obtained as the push-forward of a piecewise trivial line bundle on its semi-regular locus \(X^\text{sr}\). They form a group, for which we use either the divisor or the sheaf notation

\[
\text{MCl}^{pl}(X) := \text{DSH}^{pl}(X) := \text{Pic}^{pl}(X^\text{sr}).
\]

\textbf{Note.} Another variant of this definition would allow line bundles \(L\) on \(X^\text{sr}\) that are assumed to be trivial only on \(X^\text{sr}\). In this case \(L|_{D^\text{sr}}\) need not be trivial, but it is a torsion element of \(\text{Pic}(D^\text{sr})\). This is similar to the usual minor difference between \(\text{Pic}^0(X)\) and \(\text{Pic}^\ast(X)\).

\textbf{30 (Proof of Theorem 21).} The surjective homomorphism

\[
\pi^* : \text{MCl}(X) \to \text{MCl}(\bar{X}, \bar{D}) \cong \oplus_i \text{Cl}(\bar{X}_i),
\]

was described in Paragraph 20 proving (212).

If \(X\) is normal then \(\text{MCl}(X) \cong \text{Cl}(X)\) by Lemma 7, and the curve case is described in Paragraph 27. Thus assume from now on that \(\dim X \geq 2\) and \(X\) is not normal.

Let \(B = \{B_i : i \in I\}\) be a Mumford divisor such that \([B] \in \ker \pi^*\). Then on each \(\bar{X}_i\) we have a rational function \(\tilde{\phi}^B\) such that \((\tilde{\phi}^B)^* = \pi_i^* \cdot B_i\). Together they give a rational function \(\tilde{\phi}^B\) on \(\bar{X}\) that is regular and nonzero at all generic points of \(\bar{D}\). Thus \(\tilde{\phi}^B\) pulls back to a rational function \(\tilde{\phi}_D^B\) on \(\bar{D}\). If \(\tilde{\phi}_D^B\) is the pull-back of a rational function \(\phi_D^B\) on \(D^n\) then \(\phi_D^B\) descends to a rational function \(\phi^B\) on \(X\) by (121) and \((\tilde{\phi}^B)^* = B\).

Note that \(\tilde{\phi}^B\) and \(\tilde{\phi}_D^B\) are not uniquely determined by \(B\), we can multiply them by constants. Thus the well-defined object is the divisor \((\tilde{\phi}_D^B)^*\) of \(\tilde{\phi}_D^B\) on \(\bar{D}\), and

\[
(\tilde{\phi}_D^B)^* \in k(D^n)^*/k^c(D^n)^* \cong \oplus_j k(D^n)^*/k^c(D^n)^*.
\]

Taking a further quotient by \(\pi_D\) on \(\bar{D}\) we get classes

\[
[\tilde{\phi}_D^B]^*_{\{j\}} \in k(\bar{D}^n)^*/(k(\bar{D}^n)^*, k^c(\bar{D}^n)^*)
\]
as invariants of \([B] \in \ker \pi^*\). Since the restriction map \(k(\bar{X}) \to k(\bar{D})\) is surjective, these give a surjective map

\[
\rho_D : \text{MCl}^d(X) \to \oplus_j k(\bar{D}^n)^*/(k(D^n)^*, k^c(D^n)^*).
\]

Once we prove that its kernel is \(\text{MCl}^{pl}(X)\), we get the isomorphism in (215.a). (The definitions of \(k(D)\) and of \(k^c(D)\) are set up to be birational invariants of \(D\).)

If \(\tilde{\phi}_D^B\) is in the kernel of \(\rho_D\) then \(\tilde{\phi}_D^B|_{D^n} = \psi_j \cdot \bar{c}_j\) where \(\psi_j \in k(D^n)^*\) and \(\bar{c}_j \in k^c(\bar{D}^n)^*\). Moreover, the product decomposition is unique up to a factor \(c_j \in k^c(\bar{D}^n)^*\). Thus \(\psi_j\) determines a projective trivialization \(28\) of a line bundle \(L_j\) on \(D^n\) whose pull-back to \(\bar{D}^n\) is isomorphic to \(\mathcal{O}_{\bar{X}}(B)|_{\bar{D}^n}\). Thus the kernel of \(\rho_D\) consists of piecewise trivial divisorial sheaves on \(X\), as described in (209). This shows (215.a) which in turn implies (215.b–c) by the computations of Paragraph 31.
Finally \((\text{21})5.b-c\) imply \((\text{21})4\). \(\square\)

**31** (The multiplicative group of a function field). Let \(V\) be a normal, irreducible, proper \(k\)-variety of dimension \(\geq 1\) with field of constants \(k^{\text{lc}}(V)\). We saw in Definition \(9\) that

\[
k(V)^*/k^{\text{lc}}(V)^* \cong \text{PDiv}(V) \cong \mathbb{Z}^{[k]}.
\]

Let next \(W \rightarrow V\) be an irreducible, separable cover of proper, normal varieties. For any prime divisor \(P \subset V\) let \(P_1, \ldots, P_{r(P)}\) be the prime divisors on \(W\) lying over \(P\). Write \(\pi^*P = \sum_i e_i P_i\) and set \(e_P := \gcd(e_1, \ldots, e_{r(P)})\). Taking the divisor of a function gives

\[
q_{W/V} : k(W)^*/(k(V)^*, k^{\text{lc}}(W)^*) \rightarrow \sum_P \frac{\sum [P_i]}{\sum (e_i/e_P) P_i}.
\]

The kernel of \(q_{W/V}\) consists of \(\phi \in k(W)\) whose divisor \((\phi)\) is the pull-back of a \(\mathbb{Q}\)-divisor on \(V\). If \(\deg(W/V) = m\) and \(\phi \in \ker q\) then

\[
(\phi^m) = \pi^*\pi_*(\phi) = \pi^*(\text{norm}_{W/V}(\phi)).
\]

In particular, the kernel of \(q_{W/V}\) is \(m\)-torsion. Moreover, whether or not a \(\mathbb{Q}\)-divisor on \(V\) is a principal \(\mathbb{Z}\)-divisor is invariant under separable field extensions. We can thus bound the torsion after base change to the separable closure \(K \supset k\). In this case we have a factorization \(W_K \rightarrow W^{\text{ab}}_K \rightarrow V_K\) where \(W^{\text{ab}}_K \rightarrow V_K\) corresponds to the maximal abelian subfield of \(K(W) \supset K(W^{\text{ab}}) \supset K(V)\). Then Kummer theory tells us that the torsion in \(K(W^*)/K(V)^*\) is isomorphic to \(\text{Gal}(K(W^{\text{ab}})/K(V))\); cf. [Ser79 Sec.X.3].

We can be especially explicit in the degree 2 case if \(\text{char } k \neq 2\). Then \(k(W)/k(V)\) is a Galois extension and there is a unique \(\psi \in k(W)^*/k(V)^*\) such that \(\psi^2 \in k(V)^*\). Thus, in this case

\[
k(W)^*/(k(V)^*, k^{\text{lc}}(W)^*) \cong \begin{cases} \mathbb{Z}^{[k]} & \text{if } k^{\text{lc}}(W) \neq k^{\text{lc}}(V) \\
\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}^{[k]} & \text{if } k^{\text{lc}}(W) = k^{\text{lc}}(V).
\end{cases}
\]

If \(W \rightarrow V\) is purely inseparable of degree \(q\) then \(k(W)^q \subset k(V)\), hence in this case

\[
k(W)^*/(k(V)^*, k[W]^*) \cong (q\text{-torsion group of cardinality } [k]).
\]

Finally an arbitrary \(W \rightarrow V\) can be written as the composite of a purely inseparable map (say of degree \(q\)) and of a separable map (say of degree \(m\)) and then combining \((\text{31}4–5)\) and elementary arguments as in [Lan02 p.880] show that

\[
k(W)^*/(k(V)^*, k[W]^*) \cong (\text{finite } m\text{-torsion}) + (q\text{-torsion}) + \mathbb{Z}^{[k]}.
\]

Riemann-Roch tells us that the Euler characteristic of a line bundle \(L\) on a proper variety \(X\) can be computed from it class \([L] \in \text{NS}(X)\). The following example shows that for a Mumford divisor \(B\), its pull-back \(\tilde{B}\) does not carry enough information to compute \(\chi(X, \mathcal{O}_X(B))\).

**Example 32.** Let \((X, D)\) be semi-smooth and \(B\) a Mumford divisor on \(X\). Etale locally we can write \(X\) as

\[
X = (x_1 x_{12} = 0) \subset \mathbb{A}^{n+1}(x_{11}, x_{12}, x_2, \ldots, x_n).
\]

Its irreducible components are \(X_1 := (x_{12} = 0)\) and \(X_2 := (x_{11} = 0)\). The divisors \(B_i\) on \(X_i\) are given by local equations \(B_i = (F_i(x_{11}, x_2, \ldots, x_n) = 0)\) where the \(F_i\)
are rational functions. The sections of $\mathcal{O}_X(B_i)$ are locally of the form $G_i/F_i$ where $G_i$ is regular. Set
\[ f_i := F_i(0, x_2, \ldots, x_n) \quad \text{and} \quad g_i := G_i(0, x_2, \ldots, x_n). \tag{32.2} \]

Two sections $G_i/F_i$ patch to a section of $\mathcal{O}_X(B)$ iff
\[ g_1(x_2, \ldots, x_n)/f_1(x_2, \ldots, x_n) = g_2(x_2, \ldots, x_n)/f_2(x_2, \ldots, x_n). \tag{32.3} \]

It is easier to express this in terms of divisors. Write $B_i|_D = \sum_j m_{ij}A_j$ where the $A_j$ are prime divisors on $D$. Define
\[ \min\{B|_D\} := \sum_j \min\{m_{1j}, m_{2j}\}A_j. \tag{32.4} \]

Thus we get an exact sequence
\[ 0 \to \pi_*\mathcal{O}_X(\bar{B} - \bar{D}) \to \mathcal{O}_X(B) \to \mathcal{O}_D(\min\{B|_D\}) \to 0. \tag{32.5} \]

**Claim 32.6.** Fix $X$ and $[\bar{B}] \in \text{Cl}(\bar{X})$. Then $\min\{B|_D\}$ can be arbitrary.

**Proof.** Pick any divisor $A$ on $D$. Choose $H$ sufficiently ample on $X$ such that $H|_{X - B_i}$ and $H|_D - A$ are very ample. Choose $H_i \sim H|_X$ such that $H_i|_D = A + R_i$ and the $R_i$ are general. Then choose general $G_i \sim H|_{X - B_i}$. Then $B'_i := H_i - G_i \sim B_i$ and
\[ \min\{B'|_D\} = \min\{(H_1 - G_1)|_D, (H_2 - G_2)|_D\} = \min\{A + R_1 - G_1|_D, A + R_2 - G_2|_D\} = A. \quad \Box \]

Loosely speaking, this says that the group $\text{MCl}^\text{ct}(X)/\text{MCP}^\text{ct}(X)$ is really important in cohomological questions.

**Informal corollary 32.7.** Let $X$ be a proper, demi-normal, non-normal variety and $B$ a Mumford divisor on $X$. Knowing $X$ and $[\bar{B}] \in \text{Cl}(\bar{X})$ tells essentially nothing about the Euler characteristic or the cohomology groups of $\mathcal{O}_X(B)$.

**Informal corollary 32.8.** Let $X$ be a proper, demi-normal, non-normal variety of dimension $n$, $B$ a Mumford divisor on $X$ and $H$ an ample Cartier divisor. Then $X$ and $[\bar{B}] \in \text{Cl}(\bar{X})$ determine the $2$ highest coefficients of the Hilbert polynomial
\[ \chi(X, \mathcal{O}_X(B + mH)), \]

but give no information about the others.

\[ \Box \]

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