Higher-degree Smoothness of Perturbations I

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1 Introduction

In [14], we developed the global perturbation method for constructing virtual moduli cycles for general symplectic manifolds. It was used to extend the definitions of Gromov-Witten invariants and Floer homology from the case of semi-positive symplectic manifolds to general case. This method is in the setting of infinite dimensions and is relatively simple (comparing with the other infinite dimensional methods in [4]).

However, there was an analytic difficulty in this method. Recall that the global perturbations are constructed as follows. First the local perturbations on local uniformizers/slices $S_f$ were constructed and by multiplying smooth cut-off functions on the local slices, these local perturbations are globalized. Here $S_f$ is the local slice of the local chart $\tilde{W}(f)$ under the (local) group action of $G$, where $G$ is the group of the reparametrizations of the domains, and $\tilde{W}(f)$ consists of stable $L^p_k$-maps near the smooth center $f$. Since the action of $G$ on $\tilde{W}(f)$ is only continuous not even of class $C^1$, the transition functions between different local slices are only of class $C^0$. Consequently, the smooth local perturbations and cut-off functions on one slice may not be smooth anymore viewed in the other slices (see a discussion for this in [15]).

To overcome this difficulty, for the genus zero case with fixed domain $S^2$ and $G \cong PSL(2, \mathbb{C})$, several methods were developed [1, 9, 10]. All these methods start with a smooth section $\xi = \xi_{S_f}$ with certain special properties on a local slice $S_f$, end with proving that its $G$-equivariant extension $\xi_{O_{S_f}}$ on the $G$-orbit $O_{S_f}$ is smooth as well, utilizing the special properties on the initial section $\xi$.

In this paper, we approach the problem of the lack of smoothness above with rather different perspectives.
(i) We give necessary and sufficient conditions on when the $G$-equivariant extension is smooth, instead of only focusing on proving the smoothness of the $G$-equivariant extension for the special sections used in Gromov-Witten and Floer theory. (ii) We reproduce the simple conceptual proof of the smoothness of the cut-off function and its $G$-equivariant extension in [10] (compare the proofs in [1, 2, 3, 9]), and give a new proof of the $C^{m_0}$-smoothness of the total evaluation map as a corollary of Theorem 1.1 (compare the proof in [8]).

(iii) Above all we find a new point view on lack of smoothness of $G$-action that makes this negative aspect into a positive and crucial analytic input stated in Theorem 1.1, from which all analytic results used in GW/Floer theory related to the lack of smoothness, such as the results in (i) and (ii) above can be treated in a uniform and simple manner.

(iv) In such a treatment, the proofs should be given in such a way that they can be easily generalized.

It is our hope that our treatment here paves the way to generalize part of the work of global analysis in [16] for mapping spaces by allowing deformations of domains with various topological types as well as actions of the groups of reparametrizations.

We now explain the new point view that this work is based upon.

The prototype of the usual lack of smoothness is the statement that the translation action map $\Psi = \Psi_0 : G \times L^p_k \rightarrow L^p_k$ with $G = \mathbb{R}^1$ and $L^p_k = L^p_k(\mathbb{R}^1)$ is only continuous but not uniformly continuous and certainly not $C^1$. We refer to this statement as $S_0$. Statement $S_0$, a simple analytic fact has a nature generalization proved in [11] that $\Psi_m : G \times L^p_k \rightarrow L^p_{k-m}$ is of class $C^m$ if $m \leq k$. We refer to this generalization as statement $S_m$.

Statement $S_m$, the more complete and precise version of the smoothness/lack of smoothness of the action maps $\Psi_m$, in more general context, can be viewed in two opposite ways.

In the negative direction, the usual version of the lack of smoothness stated in $S_0$ leads to the well-known analytic difficulties in the infinite dimensional method in the foundation of GW theory.

On the other hand, statement $S_m$ above is the prototype of the key analytic input in this paper on the $C^m$-smooth of the action maps $\Phi_m$ stated below, which naturally leads to the above mentioned criterion on the existence of smooth equivariant extensions (i). Combing this with the simple proof of the smoothness of cut-off functions, the above analytic difficulties can be resolved in a simple and direct manner.
Theorem 1.1 Consider the action map $\Phi : G_e \times L_{k}^p \to L_{k}^p$ induced by the map $\phi$ below, defined by $\Phi(\gamma, \xi) = \xi \circ \gamma$, where $L_{k}^p := L_{k}^p(\Sigma, R^N)$. Its restriction $\Phi_m : G_e \times L_{k+m}^p \to L_{k}^p$ is of class $C^m$. Here we only assume that the "reparametrization" map $\phi : G_e \times \Sigma \to \Sigma$ is a smooth family of diffeomorphisms of class $C^\infty$ parametrized by $G_e \simeq B^{\epsilon}_e$ of an $\epsilon$-ball in $R^n$, not necessarily a (local) group action.

We now explain how above theorem natural leads the necessary and sufficient conditions for the smoothness of $G_e$-equivariant extensions in the particular case that the ambient space is $L_{k}^p$.

To this end, let $\tilde{W} = \tilde{W}_{k,p}$ be a small neighborhood of a smooth $f \in L_{k}^p$ above and $S_f = (S_f)_{k,p}$ be a local slice of the $G_e$-action. We may assume that $\tilde{W}$ is the same as the $G_e$-orbit $O_{S_f}$. Then for any $k \in \tilde{W}$, there exists a $T^{-1}(k) \in G_e$ such that $k \circ T^{-1}(k) \in S_f$. It was proved in sec. 3 of this paper and [9] that $T : \tilde{W} \to G_e$ is of class $C^{m_0}$ with $m_0 = [k - 2/p]$. Let $\eta : S_f \to L_{k}^p$ be a smooth map. Clearly the $G_e$-equivariant extension $\eta_{O_{S_f}}(k) = \eta_{S_f}(k \circ T^{-1}(k)) \circ T(k)$ for $k \in \tilde{W}(f, H)$. It is decomposed as four maps

$$
\begin{align*}
  k &\to (T^{-1}(k), k) \\
      &\to (T(k), k \circ T^{-1}(k)) \\
      &\to (T(k), \eta(k \circ T^{-1}(k))) \\
      &\to (\eta(k \circ T^{-1}(k)) \circ T(k)).
\end{align*}
$$

Hence $\eta_{O_{S_f}} = \Phi_m^L \circ (Id_{G_e} \times \eta_{-m,m}) \circ (IN_{G_e} \times \Phi_m^W) \circ (T^{-1}, Id_{W}).$ The first and the second maps ($T^{-1}, Id_{W}) : \tilde{W}_{k,p} \to G_e \times \tilde{W}_{k,p}(f)$ and $IN_{G_e} \times \Phi_m^W : G_e \times \tilde{W}_{k,p}(f) \to G_e \times (S_f)_{k-m,p}$ are of class $C^m$ by the Theorem 1.1, where $IN_{G_e} : G_e \to G_e$ is the inverse map. Note that the map $\Phi_m^W : G_e \times \tilde{W}_{k,p}(f) \to \tilde{W}_{k-m,p}(f)$ should have the image $\tilde{W}_{k,p}(f)$ in general, but for the case here for the elements $(T^{-1}(k), k)$, the corresponding image is inside the submanifold $(S_f)_{k-m,p}$. The last map $\Phi_m^L : G_e \times L_{k+m}^p \to L_{k}^p$ is of class $m$ by the Theorem 1.1 again. The third map $Id_{G_e} \times \eta_{-m,m} : G_e \times (S_f)_{k-m,p}(f) \to G_e \times L_{k+m}^p$ is of class $C^m$ only when we require that the map $\eta : (S_f)_{k,p} \to L_{k}^p$ can be extended into a $C^m$-map $\eta_{-m,m} : (S_f)_{k-m,p}(f) \to L_{k+m}^p$ with image lying in $L_{k+m}^p \subset L_{k}^p$.

This gives the necessary and sufficient condition for the $C^m$-smoothness of $\eta_{O_{S_f}}$ in the above particular case. The proof in next section is to deduce
the general case to the special case here.

To state the necessary and sufficient condition for the case used in Gromov-Witten theory, we recall some notations in [14] first. Let \((M, \omega, J)\) be a compact smooth symplectic manifold with a \(\omega\)-compatible almost complex structure \(J\). It gives rise a Riemannian metric \(g_J(-,-) = \omega(-, J-)\) on \(M\). Denote the space of \(L^p_k\)-maps \(f : \Sigma = S^2 \to M\) by \(B = : \tilde{B}_{k,p}\). The group \(G = \text{PSL}(2, \mathbb{C})\) acts on \(\tilde{B}\) as the group of reparametrizations. Let \(\tilde{B}^s\) be the collection of stable \(L^p_k\)-maps in the sense of [12]. Note that the action of \(G\) on \(\tilde{B}^s\) is locally free in \(G\). Since for a fix element in \(G\), its action is a smooth automorphism of \(\tilde{B}^s\) and related structures, for the purpose here, it is sufficient to only consider local action where \(G\) is replaced by a small neighborhood \(G_e\) of the identity (the notation used in the above theorem).

For \(f \in \tilde{B}^s\), let \(W(f, H)\) (defined in Sec. 3) be a local uniformizer inside a local chart \(\tilde{W}(f)\) of \(\tilde{B}^s\). Then \(W(f, H)\) is a (local) slice of the action \(G_e\) on \(\tilde{W}(f)\), denoted also by \(S_f\). We may assume that its \(G_e\)-orbit \(\mathcal{O}_{S_f} = \tilde{W}(f)\).

Throughout this paper we assume that (1) \(m_0 = [k - 2/p] > 0\) where \(k - 2/p\) is the Sobolev smoothness of \(L^p_k\)-maps. Note that under this assumption, the space of \(L^p_k\)-functions on \(\Sigma\) is a Banach algebra; (2) the center \(f\) of each local chart is of class \(C^\infty\) or sufficiently smooth.

Let \(\mathcal{L} = : \mathcal{L}_{k,p} \to \tilde{B}\) be the Banach bundle with the fiber \(\mathcal{L}_h = L^p_k(\Sigma, \wedge^1_\Sigma(h^*TM))\) for \(h \in \tilde{B}\).

For any \(h \in \tilde{W}(f)\) and \(\xi \in \mathcal{L}_f\), the local trivialization \(\Pi_f : \tilde{W}(f) \times \mathcal{L}_f \to \mathcal{L}|_{\tilde{W}(f)}\) is given by \(\Pi_f(h, \xi)(x) = P_{h(x)f(x)} \xi(x)\) for \(x \in S^2\) where \(P_{h(x)f(x)}\) is the map induced by the \(J\)-invariant parallel transport from \(f(x)\) to \(h(x)\) along the shortest geodesic. The restriction of this local trivialization to \(W(f, H)\) gives a \(C^\infty\)-smooth local trivialization of the bundle \(\mathcal{L} \to W(f, H)\).

For a section \(\eta : S_f \to \mathcal{L}_{k,p}|_{S_f}\) of class \(C^{m_0}\), with respect to the local trivialization \(\mathcal{L}|_{S_f} \simeq S_f \times L^p_k(\Sigma, \wedge^1_\Sigma(f^*TM)), \eta(h) = (h, [\eta](h))\). Here \([\eta] : S_f \to \mathcal{L}_f = L^p_k(\Sigma, \wedge^1_\Sigma(f^*TM))\) is the corresponding \(C^{m_0}\)-function.

Now we impose the conditions \(C_1\) and \(C_2\) on \([\eta]/\eta\) defined as follows.

\(C_1 (= C_1(m))\) : The section \(\eta : S_f = (S_f)_0 \to \mathcal{L}|_{S_f} = (\mathcal{L}|_{S_f})_0\) can be extended into a \(C^m\)-smooth section \(\eta_m : (S_f)_m \to (\mathcal{L}|_{S_f})_0\) for some non-negative integer \(m \leq m_0\). Here \((S_f)_m = : W_{k-m,p}(f, H)\) consists of the \(L^p_{k-m}\)-maps.

\(C_2 (= C_2(m))\) : For \(h \in (S_f)_m\), the image of \(\eta_m(h)\) is lying in the fiber of \(L^p_{k+m}\)-sections in the sense that \([\eta_m](h) \in L^p_{k+m}(\Sigma, \wedge^1_\Sigma(f^*(TM)))\) for some \(m \leq m_0\).
Note that here we have used the fact that the bundle \( f^*(TM) \to \Sigma \) is of class \( C^\infty \) or sufficiently smooth so that \( L^p_{k,m}(\Sigma, \wedge^1_{\Sigma}(f^*(TM))) \) is well-defined.

Hence the map \([\eta_{-m,m}] : (S_f)_{-m} \to L^p_{k+m}(\Sigma, \wedge^1_{\Sigma}(f^*(TM)))\) and the corresponding section \( \eta_{-m,m} \) are of class \( C^m \).

**Theorem 1.2** Under the conditions \( C_1 \) and \( C_2 \) on the smooth section \( \eta =: \eta_{S_f} \) defined on the local slice \( S_f \), the \( G \)-equivariant extension \( \eta_{O_{S_f}} \) is at least of class \( C^m \) for any \( m \leq m_0 \). Moreover, these conditions are necessary.

**Remark 1.1** Let \( \eta_{-m,n} : (S_f)_{k-m,p} \to \mathcal{L}_{k+n,p} \) be a smooth section of class \( C^d \). It gives rise a section \( \eta = \eta_{0,0} : (S_f)_{k,p} \to \mathcal{L}_{k,p} \) by first restricting \( \eta_{-m,n} \) to \( (S_f)_{k,p} \) then composing with the obvious inclusion map. Then \( \eta \) satisfies the condition \( C_1(m) \) and \( C_2(n) \). In the case that \( \eta_{-m,n} \) is a bounded linear operator between the corresponding Sobolev spaces, this is exactly the so called smoothing operator of degree \( m + n \). As for our case, since \( \eta_{-m,n} \) may not be linear, the smoothness of \( \eta_{-m,n} \) is imposed as an independent condition.

Thus the sections satisfying the conditions \( C_1(m) + C_2(n) \) can be identified exactly with sufficiently smooth (possibly non-linear) smoothing operators of degree \( m + n \). There are sufficient such sections in order to achieve transversality. In fact for the applications in Gromov-Witten and Floer theories, it is sufficient only considering the constant sections in [14].

Now recall the definition of ”constant” sections \( \xi_{S_f} \) in [9].

Given a \( C^\infty \) element \( \xi_0 \in L^p_k(\Sigma, \wedge^1_{\Sigma}(f^*TM)) \) of the central fiber, let \([\xi_{S_f}] : S_f \to L^p_k(\Sigma, \wedge^1_{\Sigma}(f^*TM))\) be the constant function defined by \([\xi_{S_f}](h) = \xi_0\).

Then the ”constant” section \( \xi_{S_f} \) is defined to be the corresponding section under the local trivialization \( \mathcal{L}|_{S_f} \cong S_f \times L^p_k(\Sigma, \wedge^1_{\Sigma}(f^*TM)) \). In [14, 11], \( \xi_0 \) is obtained from the elements in the cokernel \( K_f \) of the linearization at \( f \) of the \( \overline{\partial}_f \)-operator in GW/Floer theory. These particular constant sections are important for GW/Floer theory as they are used to achieve the transversality for the perturbed \( \overline{\partial}_f \)-operator.

It is easy to see that the conditions \( C_1 \) and \( C_2 \) are satisfied for \( \xi_{S_f} \). Indeed, the extended map \([([\xi_{S_f}]_{-m,m}] : (S_f)_{-m} \to L^p_{k+m}(\Sigma, \wedge^1_{\Sigma}(f^*(TM)))\) corresponding to the section \( \xi_{S_f} \) is still the constant map, \([([\xi_{S_f}]_{-m,m}](h) = \xi_0 \in C^\infty(\Sigma, \wedge^1_{\Sigma}(f^*(TM)))\) for any \( h \in (S_f)_{-m} \).

We refer the readers to [10] for the definition of the geometric sections \( \eta_{S_f} \). These sections are obtained by pulling-back a ”constant”-section of \( TM \to M \).
over a small ball $D(x_0) \subset M$ centered at $x_0$. Hence $\eta_{S_f}$ has similar properties as $\xi_{S_f}$. In particular, one can easily show that the conditions $C_1$ and $C_2$ are satisfied.

The following corollary recovers the results in [1] and [10].

**Corollary 1.1** Both the constant sections $\xi_{S_f}$ and the geometric sections $\eta_{S_f}$ are of class $C^{m_0}$ viewed in any local slices.

Now assume that $p$ is an even integer. It was proved in last section and [1, 9, 10] that both the $p$-th power $N_{S_f} : S_f \to \mathbb{R}$ of the $L^p_k$-norm and its $G$-equivariant extension are of class $C^{m_0}$. Using this a $C^{m_0}$-smooth cut-off function supported on $S_f$ can be constructed such that it is still $C^{m_0}$-smooth viewed in any other local slices. Hence it gives rise a ”globally” defined $C^{m_0}$-smooth cut-off function. By multiplying such a cut-off function $\beta_{S_f}$ with $\xi_{S_f}$, we get the desired $C^{m_0}$-smooth perturbations for the case of the fixed domain $S^2$.

**Corollary 1.2** For any section $\eta_{S_f}$ satisfying $C_1+C_2$, the perturbation $\beta_{S_f} \eta_{S_f}$ is of class $C^{m_0}$ viewed in any local slices. Hence in this sense it is a $C^{m_0}$-smooth global section.

Applying this to Gromov/Witten and Floer theories, the analytic difficulty in our global perturbation method [14] is resolved for the case of the fixed domain $S^2$.

The general genus zero case is treated in [6, 7]. However, the results in this paper alone already essentially contain the ”hard” analytic facts needed for the global perturbation method in [14]. To a large extend, the proofs in the sequel of this paper [6, 7] are the reductions from the general cases to the case treated in this paper.

This paper is organized as follows. Section 2 gives the proof for Theorem 1.2. Section 3 proves the Theorem 1.1. Section 4 gives a new proof that the evaluation map $E : \Sigma \times \widetilde{W}(f) \to M$ define by $E(x, g) = g(x)$ is of class $C^{m_0}$ as a corollary of Theorem 1.1. This result was proved in [8] by a different method. Section 5 reproduces the simple proofs in [10] that both the $p$-th power $N_{S_f}$ of the $L^p_k$-norm and its $G$-equivariant extension are of class $C^{m_0}$.

In order to concentrate on the main issues related to lack of smoothness of $G$-action, in this paper the basis facts in the ’usual’ analytic foundation of Gromov-Witten/Floer theories are assumed. Other than that, this paper is written only using the basic facts on Sobolev spaces and standard calculus on Banach spaces. They can be found in [16, 5].
2 The Proof of the Theorem 1.2

We make a reduction first. Fix an embedding $i : M \to \mathbb{R}^N$. Then we have the splitting of $E = M \times \mathbb{R}^N = TM \oplus \nu_M$ of the trivial bundle $E$ into sub bundles, where $\nu_M$ is the normal bundle of $M$ of the embedding. For a $C^\infty$-map $f : \Sigma \to M$, we get the corresponding splitting of $L_k^p$-bundles: $f^*E \otimes \bigwedge^1_\Sigma = (f^*TM \otimes \bigwedge^1_\Sigma) \oplus (f^*\nu_M \otimes \bigwedge^1_\Sigma)$. By taking $L_k^p$-sections, we get the induced splitting of the Banach bundles $\mathcal{E}|_{\tilde{W}_{k,p}(f)} \simeq \tilde{\mathcal{L}}_{\tilde{W}_{k,p}(f)} \oplus \mathcal{V}_{\tilde{W}_{k,p}(f)}$. Here the fibers at $f$ are the spaces of $L_k^p$-sections of the above corresponding finite dimensional $C^\infty$-bundles. There are two parallel transports defined on the trivial bundle $E$: the trivial ‘flat’ one and the one induced from the parallel transports on $TM$ and $\nu_M$. By the usual process in GW-theory, using these two parallel transports, we get the corresponding local trivializations of $\mathcal{E}|_{\tilde{W}_{k,p}(f)}$: the ‘trivial’ one $\mathcal{E}|_{\tilde{W}_{k,p}(f)} \simeq \tilde{W}_{k,p}(f) \times L_k^p(\Sigma, f^*E \otimes \bigwedge^1_\Sigma) = \tilde{W}_{k,p}(f) \times (\Omega^1_\Sigma \otimes_{C^\infty} L_k^p(\Sigma, \mathbb{R}^N))$ and the one with respect to the splitting $\mathcal{E}|_{\tilde{W}_{k,p}(f)} \simeq \mathcal{L}_{\tilde{W}_{k,p}(f)} \oplus \mathcal{V}_{\tilde{W}_{k,p}(f)}$.

**Proposition 2.1** The two trivializations on $\mathcal{E}|_{\tilde{W}_{k,p}(f)}$ are $C^\infty$-equivalent. Moreover, their restrictions to $S_f$ are $C^\infty$-equivalent trivializations.

The proof of this proposition is similar to the proof of Proposition 2.2 [10] that identifies the two local trivializations for $\mathcal{TB}_{k,p}$ of the tangent bundle of $\mathcal{B}_{k,p}$ there. The details will be given in [13].

Now a section $\xi : \tilde{\mathcal{B}}_{k,p} \to \mathcal{L}_{k,p}$ becomes a section $\xi^E : \tilde{\mathcal{B}}_{k,p} \to \mathcal{E}_{k,p}$.

The following corollary is an immediate consequence of the proposition above.

**Corollary 2.1** The inclusion map $\mathcal{L}_{k,p} \to \mathcal{E}_{k,p}$ between Banach bundles is of class $C^\infty$. Consequently, the section $\xi$ is of class $C^m$ if and only if $\xi^E$ is.

**Lemma 2.1** Given a section $\gamma : S_f \to \mathcal{L}|_{S_f}$, let $\gamma_{\mathcal{O}_{S_f}} : \mathcal{O}_{S_f} = \tilde{W}|_{k,p}(f) \to \mathcal{L}|_{\mathcal{O}_{S_f}}$ be its $G_e$ equivariant extension. Then $(\gamma^E)_{\mathcal{O}_{S_f}} = (\gamma_{\mathcal{O}_{S_f}})^E$.

**Proof:**

For $h \in \tilde{W}(f)$, $(\gamma_{\mathcal{O}_{S_f}})^E(h) = i(\gamma_{\mathcal{O}_{S_f}}(h)) = i(\gamma(h \circ T^{-1}(h)) \circ T(h)) = i(\gamma(h \circ T^{-1}(h))) \circ T(h) = \gamma^E(h \circ T^{-1}(h)) \circ T(h) = (\gamma^E)_{\mathcal{O}_{S_f}}(h)$.

Here $i : h^*(TM) \otimes \bigwedge^1_\Sigma \to h^*(E) \otimes \bigwedge^1_\Sigma$ is the inclusion map.
Note that \( i(s) \circ \phi = i \circ s \circ \phi = i(s \circ \phi) \) for any \( \phi \in G_e \) and any section \( s : \Sigma \rightarrow h^*(TM) \), which was used above.

□

Applying this to our case, we have that the \( G_e \)-equivariant extension \( (\eta^E_{S_f})_{\circ S_f} = (\eta_{S_f})_{E} \).

Thus \( \eta_{S_f} \) is of class \( C^m \) if and only if \( (\eta^E_{S_f})_{\circ S_f} \) is.

Hence we only need to prove Theorem 1.2 for a section \( \eta = \eta_{S_f} : S_f \rightarrow E \).

Now the embedding \( i : M \rightarrow \mathbb{R}^N \) induces the corresponding \( C^\infty \)-embedding \( i_k : \tilde{B}_{k,p} \rightarrow L^p_{k} =: L^p_{k}(\Sigma, \mathbb{R}^N) \) as submanifold. It was proved in Proposition 2.3 of [10] that the embedding \( i_k \) is splitting over an small tubular neighborhood of \( \tilde{B}_{k,p} \). Consequently any map \( \Psi : \mathcal{C} \rightarrow \tilde{B}_{k,p} \) is smooth if and only if \( i_k \circ \Psi \) is.

Now the maps \( \Phi \) and \( i \) commute each other: \( i_{k-m,p} \circ \Phi^k_m = \Phi^m \circ i_{k,p} \).

Let \( \tilde{V}(f) \) be the small neighborhood of \( f \) in \( L^p_{k} \) such that \( \tilde{W}(f) = \tilde{V}(f) \cap \tilde{B}_{k,p} \). Then Theorem 1.1 implies the following

**Corollary 2.2** The corresponding action map \( \Phi^W_m : G_e \times \tilde{W}_{k,p}(f) \rightarrow \tilde{W}_{k-m,p}(f) \) is of class \( C^m \).

**Theorem 2.1** Under the conditions \( C_1 \) and \( C_2 \) for \( \eta : S_f \rightarrow E \), the section \( \eta_{S_f} \) is of class \( C^m \).

**Proof:**

Under the ‘canonical’ trivialization, the section \( \eta : S_f \rightarrow E|_{S_f} \) becomes \( \eta(h) = (h, [\eta](h)) \). Here \([\eta] : S_f \rightarrow (\Omega^1_{\Sigma}) \otimes_{C^\infty} L^p_{k}(\Sigma, \mathbb{R}^N) \) is the corresponding map.

Then \( \eta_{S_f}(k) = (\eta(k \circ T^{-1}(k))) \circ T(k) = (k, ([\eta](k \circ T^{-1}(k))) \circ T(k)) \). Thus we only need to show that \( k \rightarrow ([\eta](k \circ T^{-1}(k))) \circ T(k) \) is of class \( C^m \).

Now the map \( k \rightarrow ([\eta](k \circ T^{-1}(k))) \circ T(k) \) is decomposed as four maps

\[
\begin{align*}
k \ &\rightarrow \ (T^{-1}(k), k) \\
&\rightarrow \ (T(k), k \circ T^{-1}(k)) \\
&\rightarrow \ (T(k), [\eta](k \circ T^{-1}(k))) \\
&\rightarrow \ ([\eta](k \circ T^{-1}(k))) \circ T(k).
\end{align*}
\]

The first map \( k \rightarrow (T^{-1}(k), k) \) given by \( (T^{-1}, Id^W) : \tilde{W}_{k,p}(f) = \mathcal{O}_{S_f} \rightarrow G_e \times \tilde{W}_{k,p}(f) \) is of class \( C^{m_0} \); the second \( (T^{-1}(k), k) \rightarrow (T(k), k \circ T^{-1}(k)) \) given
by $\text{IN}_{G_e} \times \Phi_m^W : G_e \times \hat{W}_{k,p}(f) \to G_e \times (S_f)_{k-m,p} \subset G_e \times \hat{W}_{k-m,p}(f)$ is of class $C^m$ by the corollary above, where $\text{IN}_{G_e} : G_e \to G_e$ is the inverse map; the third $(T(k), k \circ T^{-1}(k)) \to (T(k), [\eta](k \circ T^{-1}(k)))$ given by $\text{Id}_{G_e} \times [\eta]_{m,m} : G_e \times (S_f)_{k-m,p}(f) \to G_e \times (\Omega^1_{\Sigma} \otimes C^\infty L^p_{k+m}(\Sigma, \mathbb{R}^N))$ is of class $C^m$ by the condition $C_1 + C_2$ on $\eta$ and the last map $(T(k), [\eta](k \circ T^{-1}(k))) \to ([\eta](k \circ T^{-1}(k)))$ given by $\Phi_m^E : G_e \times (\Omega^1_{\Sigma} \otimes C^\infty L^p_{k+m}(\Sigma, \mathbb{R}^N)) \to \Omega^1_{\Sigma} \otimes C^\infty L^p_k(\Sigma, \mathbb{R}^N)$ is of class $m$ by the Theorem 1.1.

Now in the decomposition of $[\eta_{O_{S_f}}]$ above, the stated degree of smoothness for each map is already maximal.

In order to get the required smoothness for $\eta_{O_{S_f}}$, the conditions $C_1$ and $C_2$ on $\eta$ have to be satisfied. In this sense, these conditions are necessary.

\[ \square \]

3 Proof of the Theorem 1.1

First recall the definition of natural coordinate chart $\hat{W}(f)$. For simplicity, assume that $f$ is of class $C^\infty$. It is defined to be $\text{Exp}_f : \hat{W} \to \hat{W}(f) =: \hat{W}(f)$, where $\hat{W}$ consists of all $L^p_k$-sections $\xi$ of the bundle $f^*(TM) \to \Sigma$ with $\|\xi\|_{k,p} < \epsilon$, and $(\text{Exp}_f \xi)(x) = \exp_{f(x)} \xi(x)$.

Here the exponential map is taken with respect to an $f$-dependent metric defined as follows. Let $x = \{x_1, x_2, x_3\} = \{0, 1, \infty\}$ be the three standard marked points of $S^2$ and $H = \{H_1, H_2, H_3\}$ be the three local hypersurfaces of codimension 2 of $M$ that are transversal to $f$ at the three points $f(x_1), f(x_2)$ and $f(x_3)$. Then the $f$-dependent metric on $M$ is the one such that each $H_i, i = 1, 2, 3$ is totally geodesic. This can be done by assuming that the metric near $f(x_i)$ is flat.

Let $h_i \subset T_{f(x_i)}M, i = 1, 2, 3$ be the tangent spaces of $H_i$ at $f(x_i)$. Since each $H_i$ is totally geodesic at $f(x_i)$, $\exp_{f(x_i)}(h_i) = H_i$ locally. Define $\hat{W}(f, h)$ to be the subspace of $\hat{W}(f)$ consisting of the elements $\eta$ such that $\eta(x_i) \in h_i, i = 1, 2, 3$.

Then $W(f, H) =: \text{Exp}_{f(x)} \hat{W}(f, h)$ is the collection of the maps $g$ in $\hat{W}(f)$ such that $f(x) \in H$. Since the evaluation map $E_x : \hat{W}(f) \to M^3$ for fixed $x$ is smooth and obvious transversal to any pint of $M^3, W(f, H) = E_x^{-1}(H)$ is a $C^\infty$ submanifold of $\hat{W}(f)$ with finite co-dimension. It is a local uniformizer of $\mathcal{B}^s$, where $\mathcal{B}^s$ is the space of unparametrized stable $L^p_k$-maps. For simplicity, we assume that the elements in $\mathcal{B}^s$ are somewhere injective so that the action
of the reparametrization group on the space $\tilde{B}^s$ of the parametrized stable $L_k^p$-maps has local slices given by $W(f, H)$(see [12] for a general discussion on stability).

The following lemma gives the definition of $T^{-1} : \tilde{W}(f) \to G_e$ used in Sec. 1. It is a consequence of the implicit function theorem applying to the evaluation map $E : \Sigma^3 \times \tilde{W}(f) \to M^3$ define by $E(x, g) = (g(x_1), g(x_2), g(x_3))$, assuming the $C^{m_0}$-smoothness of the evaluation map (proved in [8] and next section).

**Lemma 3.1** When $\epsilon$ is small enough, there is a $C^{m_0}$-smooth function $T^{-1} : \tilde{W}(f) = \tilde{W}_e(f) \to G = \text{PSL}(2, \mathbb{C})$ such that for any $k \in \tilde{W}(f)$, $k \circ T^{-1}(k) \in W(f, H)$.

**Proof:**

Indeed since the evaluation map $E : \Sigma^3 \times \{f\} \to M^3$ is transversal to the local hypersurfaces $H = \{H_1, H_2, H_3\}$ at $(f(x_1), f(x_2), f(x_3)) \in M^3$. By implicit function theorem, the equation $E(y, k) \in H$ has an unique solution $y = y(k)$ for $k \in \tilde{W}_e(f)$ such that $y = y(k)$ is of class $C^{m_0}$ and $\epsilon$-close to $x$ when $\epsilon$ is sufficiently small. Then there is an unique element $\gamma(y) \in G_e$ smooth in $y$ given by $x \to y$ so that $(k \circ \gamma(y))(x) \in H$. Define $T^{-1}(k) = \gamma(y(k))$. \hfill \Box

As above, we also denote the local uniformizer $W(f, H)$ by $S_f$.

We restate Theorem 1.1 in the following form.

**Theorem 3.1** The action map, denoted by $\Phi_m : G_e \times L_k^p(\Sigma, \mathbb{R}^N) \to L_k^{p-m}(\Sigma, \mathbb{R}^N)$ given by $\Phi_m(\phi, h) = h \circ \phi$ is of class $C^m$ assuming that $k - m - 2/p > 0$.

The proof of this theorem will occupy the rest of this section.

Clearly we can replace $\mathbb{R}^N$ by $\mathbb{R}^1$.

We need make another reduction. Let $V = \cup_{i \in I} V_i$ and $U = \cup_{i \in I} U_i$ be two finite open coverings of $\Sigma$ with each $V_i \subset \subset U_i, i \in I$ being open disks and $\alpha_i$ and $\beta_i$ being two partitions of unit subordinate to $V$ and $U$ respectively.

Assume that $|I| = l$. Then the maps $I_{\alpha} : L_k^p(\Sigma, \mathbb{R}^1) \to \oplus_{i \in I} L_k^p(U_i, \mathbb{R}^1) \subset \oplus_{i \in I} L_k^p(T^2, \mathbb{R}^1)$ given by $I_{\alpha}(\xi) = (\alpha_1 \cdot \xi, \cdots, \alpha_l \cdot \xi)$ and $J_{\beta} : \oplus_{i \in I} L_k^p(T^2, \mathbb{R}^1) \to L_k^p(\Sigma, \mathbb{R}^1)$ given by $J_{\beta}(\eta_1, \cdots, \eta_l) = \Sigma_l \beta_i \eta_i$ are bi-linear ( with respect to $(\xi, \alpha)/ (\eta, \beta)$) and continuous, and hence are smooth. Here each $U_i$ is embedded into $T^2$ and $\beta_i$ is also considered as a function on $T^2$. Similarly
by considering each $\alpha_i$ is as a function on $T^2$, $I_\alpha$ extends into a function $I_\alpha : L^p_k(T^2, \mathbb{R}^1) \to \bigoplus_{i \in I} L^p_k(U_i, \mathbb{R}^1) \subset \bigoplus_{i \in I} L^p_k(T^2, \mathbb{R}^1)$. We may assume that $\beta_i = 1$ on $V_i'$ for some fixed $V_i \subset V_i' \subset U_i$. Then $J_\beta \circ I_\alpha \xi = J_\beta((\alpha_1 \cdot \xi, \ldots, \alpha_l \cdot \xi)) = \Sigma_i^l \beta_i(\alpha_i \cdot \xi) = \Sigma_i^l \alpha_i \cdot \xi = \xi$ is the identity map of $L^p_k(\Sigma, \mathbb{R}^1)$ as a subspace of $L^p_k(T^2, \mathbb{R}^1)$. Here we have used the fact that $\beta_i = 1$ on $V_i'$.

When $|G_e|$ is small enough, we may assume that the action $G_e \times V_i \subset V_i'$, $i \in I$.

Then for $\xi \in L^p_k(\Sigma, \mathbb{R}^1)$, $\Phi(\phi, \xi) = \xi \circ \phi = J_\beta \circ I_\alpha(\xi \circ \phi) = J_\beta(\alpha_1 \cdot (\xi \circ \phi), \ldots, \alpha_l \cdot (\xi \circ \phi)) = J_\beta(((\alpha_1 \circ \phi^{-1}) \cdot \xi) \circ \phi, \ldots, ((\alpha_l \circ \phi^{-1}) \cdot \xi) \circ \phi) = J_\beta\{J_{\alpha \circ \phi^{-1}}(\xi) \circ \phi\} = J_\beta \Phi(\phi, J_{\alpha \circ \phi^{-1}}(\xi))$. Here $\Phi(\phi, J_{\alpha \circ \phi^{-1}}(\xi))$ stands for $l$-tuple of the actions. Since $\alpha$ is of class $C^\infty$, $\alpha = (\alpha_1, \ldots, \alpha_l) \to \alpha \circ \phi^{-1}$ is smooth in $(\alpha, \phi)$ as a map in the relevant spaces. Hence the map $(\alpha, \phi, \xi) \to J_{\alpha \circ \phi^{-1}}(\xi)$ is smooth as well. Now $J_{\alpha \circ \phi^{-1}}(\xi)$ is a $l$-tuple, each is supported in $U_i$ and hence becomes an element in $L^p_k(T^2, \mathbb{R}^1)$). The action of $\phi$ in the expression $\Phi(\phi, J_{\alpha \circ \phi^{-1}}(\xi))$ then is ‘supported’ inside $U_i$ accordingly for each $i = 1, \ldots, l$ so that it can be extended into an action on $T^2$, denoted by $\hat{\phi}_i$ by requiring that $\hat{\phi}_i = id$ outside $V_i'$. Let $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_l)$. It is easy to see that the extension $\hat{\phi}$ can be defined in such a way that the map $\phi \to \hat{\phi}$ is of class $C^\infty$ in the corresponding spaces. Thus $\Phi(\phi, J_{\alpha \circ \phi^{-1}}(\xi)) = \Phi^{T^2}(\hat{\phi}, J_{\alpha \circ \phi^{-1}}(\xi))$, and we only need to prove the theorem for $\Phi^{T^2}(\hat{\phi}, \eta)$.

Thus we can indeed replace the domain $\Sigma$ by $T^2$ (or $\mathbb{R}^2$) in the proof of the Theorem 1.1. The purpose of this reduction is to have global ”coordinate chart” of the domain.

In the following proof, we will still use $\Sigma$ to denote $\mathbb{R}^2$ or $T^2$.

In the following the smooth (action) map is denoted by $T : G_e \times \Sigma \to \Sigma$ and $T_a = T_{\{a\} \times \Sigma} : \Sigma \to \Sigma$; and the gradient $\nabla \eta = (\partial x_1 \eta, \partial x_2 \eta)$ of $\eta$ is given using the global coordinate $x = (x_1, x_2)$ of $\Sigma$.

Recall that we define $||\xi||_{k,p} = (\Sigma_{i=0}^k \int_{\Sigma} |\nabla^i \xi|^p d\mu) \frac{1}{p}$ rather than $\Sigma_{i=0}^k (\int_{\Sigma} |\nabla^i \xi|^p d\mu) \frac{1}{p}$.

**Lemma 3.2** For $p > 1$, $\left(\int_0^1 |f(x, t)| dt\right)^p \leq \int_0^1 |f(x, t)|^p dt$. Consequently $\left\| \int_0^1 f(x, t) dt \right\|_{k,p}^{p} \leq \int_0^1 \left\| f(x, t) \right\|_{\Sigma \times \{t\}}^{p} dt$. In particular, $\left\| \int_0^1 \xi \circ T_t dt \right\|_{k,p} \leq C_2(||T_t||_{C^k}) \cdot ||\xi||_{k,p}$.

**Proof:**
The inequality \( \left( \int_0^1 |f(x,t)| dt \right)^p \leq \int_0^1 |f(x,t)|^p dt \) follows from the convexity of the function \( g(x) = x^p \) with \( x > 0 \) and \( p > 1 \).

This implies that
\[
\int_0^1 f(x,t) dt \leq \int_0^1 |f(x,t)| dt.
\]

Similarly for a \( L^p \)-function on \( \Sigma \) and a smooth family of diffeomorphism \( T_t : \Sigma \to \Sigma \),
\[
\int_0^1 \xi \circ T_t dt \leq C \left( \| T_t \|_{C^k} \right) \cdot \| \xi \|_{k,p},
\]

for some constant \( C \) depending on \( \| T_t \|_{C^k} \). Here \( F(\text{Jac}_{T_t}) \) is a polynomial in \( \text{Jac}_{T_t} \), \( \text{Jac}_{T_t}^{-1} \) and their \( j \)-th derivatives with \( j \leq i \), and similarly for \( F_1(T_t) \).

Then \( \int_0^1 \xi \circ T_t dt \leq C \left( \| T_t \|_{C^k} \right) \cdot \| \xi \|_{k,p} \), hence
\[
\int_0^1 \xi \circ T_t dt \leq C \left( \| T_t \|_{C^k} \right) \cdot \| \xi \|_{k,p} = C \| \xi \|_{L^0}.
\]

Similar proofs to the last inequality in the lemma above are used throughout the rest of this section.
In the rest of this section, if there is no confusion, we denote \( \Phi_m \) by \( \Phi \), \( L^p_k(\Sigma, \mathbb{R}) \) by \( L_0 \) and \( L^p_{k-m}(\Sigma, \mathbb{R}) \) by \( L_{-m} \) for short; the norms of the functions \( F(Jac_{T_1}, Jac_{T_1}^{-1}) \) in the proofs below will be denoted by \( \| Jac_{T_1} \| \) or \( \| Jac_{T_1}^{-1} \| \).

Let \( D_2 \Phi \) be the partial derivative of \( \Phi \) along \( L_0 \)-direction. Since \( \Phi(\phi, \xi) = \xi \circ \phi \) is linear in \( \xi \), we have \( (D_2 \Phi)(\phi, \xi)(\gamma) = \gamma \circ \phi \). Hence \( D_2 : G_e \times L_0 \to L(L_0, L_{-m}) \) is given by \( (\phi, \xi) \to D_2 \Phi(\phi, \xi) = \{ \Phi_\phi = : \Phi(\phi, -) : L_0 \to L_{-m} \} \in L(L_0, L_{-m}) \) with \( \Phi_\phi(\eta) = \Phi(\phi, \eta) = \eta \circ \phi \). Note that \( (D_2 \Phi)(\phi, \xi) \) is independent of \( \xi \) so that it is factorized as \( D_2 \Phi = \Psi \circ \pi_1 : G_e \times L_0 \to G_e \to L(L_0, L_{-m}) \) where \( \pi_1 : G_e \times L_0 \to G_e \) is the natural projection map and \( \Psi : G_e \to L(L_0, L_{-m}) \) is defined by \( \Psi(\phi) = \Phi_\phi \in L(L_0, L_{-m}) \). Clearly \( D_2 \Phi \) and \( \Psi \) has the same degree of the smoothness.

Then we have to show the following proposition.

**Proposition 3.1** \( \Psi = \Psi_{m-1} : G_e \to L(L_0, L_{-m}) \) defined by \( \Psi(a) = \eta \circ T_a \) for \( \eta \in L_0 \) is of class \( C^{m-1} \). Consequently \( D_2 \Phi \) \((=D_2 \Phi_m)\) is of class \( C^{m-1} \) as well.

**Proof:**

Let \( g \) be the 'Lie algebra' of \( G \). We identify \( G_e \) with an open set in \( g \). Let \( a = (a^1, \cdots, a^l) \) be the coordinate of \( g \) with respect to a fixed basis \( a_j, j = 1, \cdots, l \). Then the \( n \)th derivative \( D^n \Psi : G_e \to L^n(\mathbf{g}_1 \times \cdots \times \mathbf{g}_n, L(L_0, L_{-m})) \) can be identified with the collection of all \( n \)-th partial derivatives at \( a \in G_e \), \( \partial^\alpha \Psi(a) \) as elements in \( L(L_0, L_{-m}) \). Here \( \mathbf{g}_i \) is the \( i \)-th copy of \( g \), \( L^n(\mathbf{g}_1 \times \cdots \times \mathbf{g}_n, L(L_0, L_{-m})) \) is the set of \( n \) multi-linear maps from \( \mathbf{g}_1 \times \cdots \times \mathbf{g}_n \) to \( L(L_0, L_{-m}) \), \( \alpha = (\alpha_1, \cdots, \alpha_l) \) with \( |\alpha| = |\alpha_1| + \cdots + |\alpha_l| = n \) and \( \partial^\alpha \Psi = \partial_{a_1}^{\alpha_1} \cdots \partial_{a_l}^{\alpha_l} \Psi \).

The proposition is proved in the following steps.

(I) \( \Psi : G_e \to L(L_0, L_{-1}) \) is continuous.

**Proof:**

Indeed

\[
\| \Psi(a_2) - \Psi(a_1) \|_{L(L_0, L_{-1})} = \max_{\| \eta \|_{L_0} \leq 1} \| \Psi(a_2)(\eta) - \Psi(a_1)(\eta) \|_{L_{-1}}
\]

\[
= \max_{\| \eta \|_{L_0} \leq 1} \| \eta \circ T_{a_2} - \eta \circ T_{a_1} \|_{L_{-1}} = \max_{\| \eta \|_{L_0} \leq 1} \left\| \int_0^1 \frac{d}{dt}(\eta \circ T_{a_1 + t(a_2 - a_1)}) dt \right\|_{L_{-1}}
\]
Proof:

Note that continuous.

$$\nabla$$

above.

$$\partial$$

by shifting the index by one and replacing

$$a$$

by a harmless factor, $$\partial_j$$ exists and

$$\eta$$

(II) The map $$\Phi^D : G_e \to L(L_0, L_{-1}) \oplus L(L_0, L_{-1})$$ given by $$\Phi^D(a)(\eta) = \nabla \eta \circ (T_a)$$ exists for $$\eta \in L_0$$. Moreover, $$\Phi^D : G_e \to L(L_0, L_{-2}) \oplus L(L_0, L_{-2})$$ is continuous.

Note that $$\eta \to \nabla \eta = (\partial_{x_1} \eta, \partial_{x_2} \eta)$$ is considered as a map: $$L_0 \to L_{-1} \oplus L_{-1}$$.

Proof:

Existence part follows from the definition. Indeed denote $$L(L_0, L_{-1}) \oplus L(L_0, L_{-1})$$ by $$\oplus^2 L(L_0, L_{-1})$$ for short.

Then for any $$a \in G_e$$,

$$\|\Phi^D(a)\|_{\oplus^2 L(L_0, L_{-1})} = \max_{\|\eta\|_{L_0} \leq 1} \|\Phi^D(a)(\eta)\|_{L_{-1} \oplus L_{-1}}$$

$$= \max_{\|\eta\|_{L_0} \leq 1} \|\nabla \eta \circ (T_a)\|_{L_{-1} \oplus L_{-1}} \leq \|\text{Jac}^{-1}_{T_a}\|_{C^k} \cdot \max_{\|\eta\|_{L_0} \leq 1} \|\nabla \eta\|_{L_{-1} \oplus L_{-1}}$$

for some constant only depending on $$T$$.

In fact we have proved that $$\Psi_1$$ above is Lipschitz, though we do not need this.

(II) The partial derivatives $$\partial_j \Psi =: \partial_{a_j} \Psi : G_e \to L(L_0, L_{-2})$$ exists and $$\partial_j \Psi(a)(\eta) = \{\nabla \eta \circ (T_a)\} \cdot (\partial_j T)_a = \{\Phi^D(a)(\eta)\} \cdot (\partial_j T)_a$$. In other words, upto a harmless factor, $$\partial_j \Psi$$ is essentially just $$\Phi^D$$ so that it is continuous by (II) above.
Proof:
Indeed
\[ \| \Psi(a + ta_j) - \Psi(a) - \partial_j \Psi(a)(ta_j) \|_{L(L_0, L_{-2})} \]
\[ = \max_{\| \eta \|_{L_0} \leq 1} \| \eta \circ T_{a + ta_j} - \eta \circ T_{a} - t \cdot \{ \nabla \eta \circ (T_{a}) \} \cdot (\partial_j T_{a}) \|_{L_{-2}} \]
\[ = \max_{\| \eta \|_{L_0} \leq 1} \| \int_0^1 \left\{ \frac{d}{ds} (\eta \circ T_{a + sta_j}) - t \cdot \{ \nabla \eta \circ (T_{a}) \} \cdot (\partial_j T_{a}) \right\} ds \|_{L_{-2}} \]
\[ \leq |t| \max_{\| \eta \|_{L_0} \leq 1} \int_0^1 \| \{ \nabla \eta \circ T_{a + sta_j} \} \cdot (\partial_j T)_{a + sta_j} - \{ \nabla \eta \circ (T_{a}) \} \cdot (\partial_j T_{a}) \|_{L_{-2}} ds \]
\[ = |t| \max_{\| \eta \|_{L_0} \leq 1} \int_0^1 \| \int_0^1 \frac{d}{dv} \{ \nabla \eta \circ T_{a + vsta_j} \cdot (\partial_j T)_{a + vsta_j} \} dv \|_{L_{-2}} ds \]
\[ \leq |t| \max_{\| \eta \|_{L_0} \leq 1} \int_0^1 \| \int_0^1 \frac{d}{dv} \{ \nabla \eta \circ T_{a + vsta_j} \cdot (\partial_j T)_{a + vsta_j} \} \|_{L_{-2}} dv ds \]
\[ \leq |t|^2 \max_{\| \eta \|_{L_0} \leq 1} \int_0^1 \int_0^1 \| \nabla^2 \eta \circ T_{a + vsta_j} \|_{L_{-2}} \cdot \| (\partial_j T)_{a + vsta_j} \|_{C^{k-2}}^2 \]
\[ + \| \nabla \eta \circ T_{a + vsta_j} \|_{L_{-2}} \cdot \| (\partial_j T)_{a + vsta_j} \|_{C^{k-1}} dv ds \]
\[ \leq |t|^2 \max_{\| \eta \|_{L_0} \leq 1} \int_0^1 \int_0^1 \| \eta \|_{L_0} \cdot \| \text{Jac}_{a + vsta_j}^{-1} \|_{C^{k-2}} \cdot (1 + \| T_{a + vsta_j} \|_{C^k})^2 dv ds \]
\[ = |t|^2 \cdot \int_0^1 \int_0^1 \| \text{Jac}_{a + vsta_j}^{-1} \|_{C^{k-2}} \cdot (1 + \| T_{a + vsta_j} \|_{C^k})^2 dv ds. \]
(IV) Inductively, we assume that the \((m - 1)\)-th partial derivatives with \(|\alpha| = m - 1\) and \(\alpha = (\alpha_1, \cdots, \alpha_l)\) is a map \(\partial_\alpha^{|\alpha|}\Psi : \partial_{j_1}^{\alpha_1} \cdots \partial_{j_l}^{\alpha_l}\Psi : G_e \to L(L_0, L_{-m})\) and that for \(a \in G_e\) and \(\eta \in L_0\), \(\partial_\alpha^{|\alpha|}\Psi(a)(\eta)\) is a summation of the terms \(\partial^\gamma \eta \circ T_a =: \partial_{x_1}^\gamma \cdots \partial_{x_l}^\gamma \eta\) multiplying with \((\partial^\beta T)_a = (\partial_{j_1}^{\beta_1} \cdots \partial_{j_l}^{\beta_l})T_a\) with \(\partial^\beta T\) along \(G_e\)-directions. Here \(|\gamma|\) and \(|\beta|\) is less than or equal to \(m - 1\). We have to show that

(A) \(\partial_\alpha^{|\alpha|}\Psi : G_e \to L(L_0, L_{-m})\) with \(|\alpha| = m - 1\) is continuous.

(B) \(\partial_j \partial_\alpha^{|\alpha|}\Psi : G_e \to L(L_0, L_{-m-1})\) with \(|\alpha| = m - 1\) is a summation of the terms \(\partial^\gamma \eta \circ T_a\) multiplying with \((\partial^\beta T)_a\) with \(\partial^\beta T\) along \(G_e\)-directions. Here \(|\gamma|\) and \(|\beta|\) is less than or equal to \(m\). The proof of A and B will complete the induction.

To prove A and B, note that the factor \((\partial^\beta T)_a\) do not affect anything here. Denote the term \(\partial^\gamma \eta \circ T_a\) by \(\Phi^{D,\gamma}(a)(\eta)\). Then \(\Phi^{D,\gamma} : G_e \to L(L_0, L_{-|\gamma|})\). Note that \(\Phi^{D,\gamma}\) has essentially the same type of form as \(\Phi^D\) has in (II) above. By (II) with an obvious degree shifting, (i) the image of \(\Phi^{D,\gamma}\) is indeed in \(L(L_0, L_{-|\gamma|})\) and (ii) \(\Phi^{D,\gamma} : G_e \to L(L_0, L_{-|\gamma|-1})\) is continuous. This proves A by letting \(|\gamma| = m - 1\).

Similarly, \(\Phi^{D,\gamma}\) and \(\Psi\) are also the ”same type”. Hence if we shift the degree of the target above by one further and consider \(\partial_j \Phi^{D,\gamma} : G_e \to L(L_0, L_{-m-1})\). The essentially the same argument of III shows that B is true for \(\partial_j \Phi^{D,\gamma}\) for \(|\gamma| = m - 1\). This implies that B is true for \(\partial_j \partial_\alpha^{|\alpha|}\Psi\) by the definition of \(\Phi^{D,\gamma}\) above.

\[\square\]

Next consider the partial derivative \(D_1 \Phi = D_G \Phi\) along \(G_e\)-direction for \(\Phi_n : G_e \times L_0 \to L_{-n}\). Since \(G_e\) is finite dimensional so that \(L(g, L_{-n})\) is a finite sum of \(L_{-n}\), by the discussion before, the partial derivative \(D_1 \Phi_n : G_e \times L_0 \to L(g, L_{-n})\) can be identified with the collection of \(\partial_{a_j} \Phi_n =: \partial_j \Phi_n : G_e \times L_0 \to L_{-n}, j = 1, \cdots, l\). Denote \(\partial_{a_j} \Phi_k : G_e \times L_0 \to L_{-k}, j = 1, \cdots, l\), by \((\partial_j \Phi)_k\).

We have to show the following lemma.

**Lemma 3.3** For each \(j\), \((\partial_j \Phi)_n : G_e \times L_0 \to L_{-n}\) is of class \(C^{n-1}\).

**Proof:**

We claim that as a map \((\partial_j \Phi) =: (\partial_j \Phi)_0 : G_e \times L_0 \to L_{-1}\), \(\partial_j \Phi(a, \eta) = (\nabla \eta) \circ (T_a) \cdot \partial_{a_j}(T_a)\).
Indeed for \((a, \eta) \in G_e \times L_0, \)

\[
\|\Phi(a + ta, \eta) - \Phi(a, \eta) - t \cdot (\nabla \eta) \circ (T_a) \cdot \partial_{a_j}(T)_a\|_{k-1,p}
\]

\[
= |t| \cdot \| \int_0^1 \{(\nabla \eta) \circ (T_{a+\nu t a_j})\} \cdot \partial_{a_j}(T)_{a+\nu t a_j} - \{(\nabla \eta) \circ (T_a) \cdot \partial_{a_j}(T)_a\} d\nu \|_{k-1,p}
\]

\[
\leq |t| \cdot \| \int_0^1 \{(\nabla \eta) \circ (T_{a+\nu t a_j})\} \cdot \partial_{a_j}(T)_{a+\nu t a_j} - \{(\nabla \eta) \circ (T_a) \cdot \partial_{a_j}(T)_a\} d\nu \|_{k-1,p}.
\]

Now for fixed \(\eta\) with \((\nabla \eta) \in L_{-1} = L_{k-1}^p\), the continuity of the map \(\Phi: G_e \times L_{-1} \to L_{-1}\) given \((\xi, a) \to \xi \circ T_a\) implies that the integrand above goes to zero as \(t \to 0\) uniformly for \(\nu\) since \(|\nu \cdot t| \leq |t|\). This implies that the integral above goes to zero as \(t \to 0\), and hence the claim is proved.

Thus up a harmless factor \(\partial_{a_j}(T)_a, \partial_{a_j} \Phi\) is essentially given by \(\Phi_j: G_e \times L_0 \to L_{-1}\) defined by \(\Phi_j(a, \eta) = (\nabla \eta) \circ (T_a)\). Now \(\Phi_j\) is of the same type as the original \(\Phi: G_e \times L_0 \to L_0\) with a degree shifting by 1 and replacement of \(\eta\) by \(\nabla \eta\). Hence we are in the position to apply induction to conclude that \(\Phi_j: G_e \times L_0 \to L_{-n}\) is of class \(C^{n-1}\) so that \((\partial_j \Phi)_n: G_e \times L_0 \to L_{-n}\) is of class \(C^{n-1}\).

This finishes the proof of the main theorem of this section under the assumption of the continuity of \((\Phi)_0\).

\[\square\]

**Proposition 3.2** \((\Phi)_0: G_e \times L_0 \to L_0\) is continuous.

**Proof:**

\[
\|\Phi(a + t, \eta + \gamma) - \Phi(a, \eta)\|_{L_0}
\]

\[
\leq \|\Phi(a + t, \eta + \gamma) - \Phi(a + t, \eta)\|_{L_0} + \|\Phi(a + t, \eta) - \Phi(a, \eta)\|_{L_0} = A + B.
\]

Now

\[
A = \|(\eta + \gamma) \circ T_{a+t} - \eta \circ T_{a+t}\|_{L_0}
\]

\[
= \|\gamma \circ T_{a+t}\|_{k,p} \leq C(J(T), \|T\|_{C^k}) \cdot \|\gamma\|_{k,p}
\]

for a constant \(C(J(T), \|T\|_{C^k})\) depending only on \(J(T)\) and \(\|T\|_{C^k}\). Here and below \(J(T)\) is the Jacobian of \(T\). Hence \(A < \epsilon/8\) if we choose \(\|\gamma\|_{L_0} < \delta_1 = \epsilon/(8 \cdot C(J(T))\|T\|_{C^k})\).

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To estimate $B$, choose a smooth $\xi$ such that $\|\xi - \eta\|_{k,p} < \delta_2$. Then

$$B \leq \|\Phi(a + t, \xi) - \Phi(a, \xi)\|_{L_0} + \|\Phi(a + t, \eta) - \Phi(a + t, \xi)\|_{L_0} + \|\Phi(a, \eta) - \Phi(a, \xi)\|_{L_0}$$

$$= B_1 + B_2 + B_3.$$ 

Clear the estimate for $A$ above is applicable to both $B_2$ and $B_3$ so that for $i = 2, 3$, $B_i \leq C(J(T), ||T||_{C^k}) \cdot \|\xi - \eta\|_{k,p} \leq C(J(T), ||T||_{C^k}) \cdot \delta_2 \leq \epsilon/8$ if we choose $\delta_2 = \epsilon/(8 \cdot C(J(T)||T||_{C^k})).$

Now $B_1 = \|\xi \circ T_{a+t} - \xi \circ T_a\|_{k,p} = \|\int_0^1 \frac{d}{dt} \{\xi \circ T_{a+vt}\} d\nu\|_{k,p}$. Applying the estimate in $I$ or $III$ above (or its argument) to the case here, we get $B_1 \leq C(J(T), ||T||_{C^{k+1}}) \cdot ||t\| \cdot ||\xi||_{k+1,p} < \epsilon/8$ if we choose $|t| < \delta_3 = \epsilon/(8 \cdot C(J(T), ||T||_{C^{k+1}}) \cdot ||\xi||_{k+1,p})$.

$\Box$

4 Smoothness of the evaluation map

The following theorem was proved in [8] by a different method. Here we reprove it as a corollary of the above Theorem 1.1.

**Theorem 4.1** Assume that $p > 2$. The $m$-fold total evaluation map: $E_m : \mathcal{B} \times \Sigma^m \to M^m$ is of class $C^{m_0}$, where $\mathcal{B} = \mathcal{B}_{k,p}$ is the collection of $L^p_k$ maps from $\Sigma \simeq S^2$ to $M$.

**Proof:**

We make some reductions. (i) It is sufficient to prove the theorem for $E_1$ and $\widetilde{W}(f)$; (ii) It is reduced further to show that for any $x_0 \in \Sigma$, the map $E_{D(x_0)} = E_1|_{\tilde{w}(f) \times D(x_0)} : \tilde{W}(f) \times D(x_0) \to M$ is of class $C^{m_0}$, where $D(x_0)$ is the small disc on $\Sigma$ centered at $x_0$.

Let $\phi =: \phi_{D(x_0)} : D(x_0) \times \Sigma \to \Sigma$ be the smooth ”action” map such that for any $x \in D(x_0)$, the restriction $\phi_x : \{x\} \times \Sigma \to \Sigma$ is a diffeomorphism with the property that (a) $\phi_x : D(x_0) \to D(x)$ by a ”translation”, in particular $\phi_x(x_0) = x$ and (b) $\phi_x$ is the identity map outside a larger disc centered at $x_0$.

Denote $D(x_0)$ by $G_e$. Consider the corresponding action map induced by $\phi$ above, $\Phi_{k-1} : G_e \times \tilde{W}(f) = D(x_0) \times \tilde{W}(f) \to (\tilde{W}(f))_{-(k-1)}$. Then by Theorem 1.1 and Corollary 2.1 $\Phi_{k-1}$ is of class $C^{m_0}$ since $m_0 = [k - 2/p] = k - 1$. 

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Now \((\tilde{W}(f))_{(k-1)}\) consists of \(L^p_k\)-maps that are continuous by the assumption \(p > 2\), so that the evaluation map at a fixed point \(x_0\), \(E_{x_0} : (\tilde{W}(f))_{(k-1)} \to M\) is well-defined. Clearly \(E_{x_0}\) is smooth of class \(C^\infty\): in the local charts \(\text{Exp}_f : \tilde{W}(f) \to \tilde{W}(f)\) and \(\exp_{f(x_0)} : B(f(x_0)) \to M\), the map \(E_{x_0}\) is corresponding to \(\hat{E}_{x_0}\) given by \(\hat{E}_{x_0}(\xi) = \xi(x_0)\), which is linear. Here \(B(f(x_0)) \subset T_{f(x_0)}M\) is the corresponding ball.

Recall that \(\Phi_{k-1}(x, g) = g \circ \phi_x\). Now \(E_{D(x_0)} = E_{x_0} \circ \Phi_{k-1} : G_e \times \tilde{W}(f) \to M\), hence is of class \(C^{m_0}\). Indeed for any \((x, g) \in D(x_0) \times \tilde{W}(f), E_{x_0} \circ \Phi_{k-1}(x, g) = E_{x_0}(g \circ \phi_x) = g(\phi_x(x_0)) = g(x) = E_1(g, x) = E_{D(x_0)}(g, x)\).

\[\square\]

5 Smoothness of cut-off functions

We first prove the smoothness of the \(p\)-th power \(N_k\) of \(L^p_k\)-norm on \(\tilde{B} = \tilde{B}_{k,p}\). It is sufficient to consider the case \(\tilde{B} = L^p_k(\Sigma, \mathbb{R}^M)\).

**Theorem 5.1** Assume that \(p\) is an positive even integer. Then \(N_k : L^p_k(\Sigma, \mathbb{R}^M) \to \mathbb{R}\) defined by \(N_k(f) = \|f\|_{k,p}^p = \Sigma_{i=0}^k \|\nabla^i f\|_p^p\) is smooth.

**Proof:**

Clearly it is sufficient to prove the case that \(k = 0\).

Then \(N_0(f) = \int_\Sigma <f, f>_m dvol_\Sigma\), where \(m = p/2\). Since \(I : L^1(\Sigma, \mathbb{R}) \to \mathbb{R}^1\) given by \(I(h) = \int_\Sigma h dvol_\Sigma\) is linear and hence smooth, we only need to show that the map \(\tilde{P} : L^p(\Sigma, \mathbb{R}^M) \to L^1(\Sigma, \mathbb{R})\) given by \(P(f) = <f, f>_m\) is well-defined and smooth. Indeed, \(P(f) = M \circ \Delta(f)\) where \(\Delta = \Delta_p : L^p(\Sigma, \mathbb{R}^M) \to (L^p(\Sigma, \mathbb{R}^M))^p\) is the \(p\)-fold diagonal map given by \(\Delta(f) = (f_1, \cdots, f_p)\) with \(f_1 = \cdots = f_p = f\), and \(M : (L^p(\Sigma, \mathbb{R}^M))^p \to L^1(\Sigma, \mathbb{R})\) given by \(M(f_1, \cdots, f_p) = \langle f_1, f_2 > \cdots < f_{p-1}, f_p >\). Clearly \(\Delta\) is smooth. We have to show that \(M\) is well-defined and continuous. Indeed, by Holder inequality,

\[
\|M(f_1, \cdots, f_p)\|_{L^1} = \|<f_1, f_2 > \cdots < f_{p-1}, f_p >\|_{L^1} \leq \|f_1\|_{L^p} \cdots \|f_p\|_{L^p}.
\]

Hence \(M\) is well-defined, continuous and multi-linear. This implies that \(M\) is smooth.

\[\square\]
Note: The theorem above was proved in [1, 2, 9] by a different method. This result was mentioned without proof in various articles or books. It was used in Sec. 4 of [14]. Until recently the author found that its first formal proof was given in [3] as a consequence of the general results there. Comparing with all those proofs, the proof above is much simpler.

Now consider the action map \( \phi : G_e \times \Sigma \to \Sigma \subset \mathbb{R}^S \). Here we have fixed an embedding of \( \Sigma \) into \( \mathbb{R}^S \). We require that \( \phi \) is of class \( C^\infty \) as a map into \( \mathbb{R}^S \).

**Lemma 5.1** For any \( k \), the following maps are of class \( C^\infty \): \( Jac : G_e \to C^k(\Sigma, \mathbb{R}) \) defined by \( Jac(g) = Jac(\phi_g) \), \( \phi_G : G_e \to C^k(\Sigma, \mathbb{R}^S) \) defined by \( \phi_G(g) = \phi_g \) and \( D^l\phi_G : G_e \to C^k(\Sigma, \mathbb{R}^S) \) defined by \( (D^l\phi_G)(g) = D^l\phi_g \). Here \( D^l\phi_g \) is the partial derivatives of \( \phi_g \) along \( \Sigma \).

**Proof:**

We leave the direct proof to the readers. It is straightforward but somewhat tedious.

For an indirect proof, note that \( \phi_G(g) = \phi_g = \Phi(id\Sigma, g) \). Since \( id\Sigma : \Sigma \to \mathbb{R}^S \) is of class \( C^\infty \), the conclusion follows from the discussion before on the action map \( \Phi \).

Denote the induced action of \( \phi \) by \( \Phi : G_e \times L^p_k(\Sigma, \mathbb{R}^M) \to L^p_k(\Sigma, \mathbb{R}^M) \) given by \( \Phi(g, f) = f \circ \phi_g \). Then the function \( F_k : G_e \times L^p_k(\Sigma, \mathbb{R}^M) \to \mathbb{R} \) is defined by \( F_k = N_k \circ \Phi \).

**Theorem 5.2** Assume that \( p \) is an positive even integer, and that all \( \phi_g \) are orientation-preserving diffeomorphisms. Then the function \( F_k \) is smooth.

**Proof:**

Since \( F_k(g, f) = \Sigma_{i=0}^k \int_{\Sigma} |D^i(f \circ \phi_g)|^p dvol_\Sigma \), it is a summation of the terms in the form

\[
\int_{\Sigma} |(D^j f) \circ \phi_g|^p \cdot |D^l\phi_g|^p dvol_\Sigma
\]

\[
= \int_{\Sigma} (|(D^j f)| \cdot |D^l\phi_G(g^{-1})|)^p \cdot (Jac^{-1}(g)) dvol_\Sigma
\]

\[
= I((Jac^{-1}(g)) \cdot P((D^j f) \cdot D^l\phi_G(g^{-1}))
\]

with \( j, l \leq k \).
Thus each such term is decomposed as the following smooth maps: (i) First map is \((f, g) \to |(D^j f)|^2 \cdot |D^l \phi_G(g^{-1})|^2\) from \(L^p_k \times G_e \to L^p\). (ii) The next is \(|(D^j f)| \cdot |D^l \phi_G(g)| \to P((D^j f) \cdot D^l \psi_G(g))\) given by the map \(P : L^p_k \to L^1\). (iii) The third map \(P((D^j f) \cdot D^l \psi_G(g)) \to (Jac^{-1}(g)) \cdot P((D^j f) \cdot D^l \psi_G(g))\) is from \(C^k \times L^1 \to L^1\) given by \((h, f) \to h \cdot f\). (iv) The last map is the integration \(I : L^1 \to \mathbb{R}\).

The conclusion follows.

An immediate corollary is the following.

**Corollary 5.1** Let \((N_k)_s\) be the restrictions of \(N_k\) to the \(C^{m_0}\) local slices \(S_f\), then its \(G_e\)-equivariant extension is of class \(C^{m_0}\).

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