Minimal surfaces with limit ends in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

For any $m \geq 1$, we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and $m$ limit ends. We also provide examples with an infinite countable number of limit ends. All these examples are vertical bi-graphs.

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1 Introduction

The theory of properly embedded minimal surfaces with genus zero (i.e. those which are topologically a punctured sphere) in Euclidean space $\mathbb{R}^3$ has been largely studied (see [1, 2, 8, 9] and the references therein). The final classification of such surfaces was given by Bill Meeks, Joaquín Pérez and Antonio Ros in [10]. The only examples with infinite topology are Riemann minimal surfaces. They form a 1-parameter family whose natural limits are the catenoid and the helicoid. When the planar ends are horizontally placed, each Riemann minimal example is invariant by a non-horizontal translation, its intersection with any horizontal plane is either a circle or a straight line, it has infinitely many annular ends asymptotic to horizontal planes, and has exactly two limit ends\(^1\): one top and one bottom limit end.

Laurent Hauswirth [5] constructed Riemann-type minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, which are properly embedded and have genus zero, infinitely many ends asymptotic to horizontal slices and two limit ends: one top and one bottom limit end. It is natural to ask if there are examples of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely

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\(^{1}\)A limit end $e$ of a non-compact surface $M$ is an accumulation point of the set $\mathcal{E}(M)$ of ends of $M$. See [9].
many ends and $m_0$ limit ends, with $m_0 \neq 2$. In this paper we construct examples for any $m_0 \geq 1$, and we also construct examples with an infinite countable number of limit ends.

In a joint work with Filippo Morabito [11], we have recently constructed a $(2k - 3)$-parameter family $F_k$ of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with total (intrinsic) curvature $4\pi(1 - k)$, genus zero and $k$ vertical planar ends (i.e. annular ends asymptotic to vertical geodesic planes), for any $k \geq 2$. We call them \textit{minimal k-noids}. Each surface in this family is invariant by reflection symmetry about the horizontal slice $\mathbb{H}^2 \times \{0\}$. Pyo [13] has constructed independently a 1-parameter family of surfaces with the same properties. The examples given by Pyo, which are included in $F_k$, are also invariant by reflection symmetry about $k$ vertical geodesic planes forming an angle $\pi/k$. The examples with genus zero and infinitely many ends we construct in the present paper are obtained by taking limits when $k \to +\infty$ of certain surfaces $M_k \in F_k$.

The simple ends (i.e. non-limit ends) of the surfaces we construct are vertical planar ends. Fixed an orientation of $\mathbb{H}^2$, we can order these simple ends cyclically. If a limit end can be obtained as accumulation of simple ends ordered following the negative orientation (resp. the positive orientation) but it cannot be obtained as accumulation of simple ends ordered following the positive orientation (resp. negative orientation), we will say that it is a \textit{left} (resp. \textit{right}) limit end. In other case, we will say that it is a \textit{2-sided} limit end.

Now we state the main results of this paper.

\textbf{Theorem 1.1.} For any $m_0 \geq 1$, there exists a properly embedded minimal surface $\Sigma$ in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and $m_0$ limit ends, which is symmetric with respect to a horizontal slice (in fact, it is a vertical bi-graph). Moreover, if we denote by $E^{1}_\infty, \ldots, E^{m_0}_\infty$ the limit ends of $\Sigma$, we can prescribe each $E^{m}_\infty$ to be left, right or 2-sided.

If we take limits of appropriately chosen minimal surfaces in $F_k$, we can also obtain properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends.

\textbf{Theorem 1.2.} There exists a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends $\{E^{m}_\infty\}_{m \in \mathbb{N}}$, which is symmetric with respect to a horizontal slice (in fact, it is a vertical bi-graph). Moreover, we can prescribe each $E^{m}_\infty$ to be left, right or 2-sided.

All the examples constructed in Theorems 1.1 and 1.2 are obtained by reflection symmetry about the horizontal slice $\mathbb{H}^2 \times \{0\}$ from a vertical graph contained in $\mathbb{H}^2 \times [0, +\infty)$ whose boundary lies on $\mathbb{H}^2 \times \{0\}$.

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2 Preliminaries

We consider the half-plane model of $\mathbb{H}^2$, $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, with the hyperbolic metric $g_{-1} = \frac{1}{y^2}(dx^2 + dy^2)$. We denote by $t$ the coordinate in $\mathbb{R}$ and consider in $\mathbb{H}^2 \times \mathbb{R}$ the usual product metric,

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + dt^2.$$

Given an open domain $\Omega \subset \mathbb{H}^2$ and a smooth function $u : \Omega \to \mathbb{R}$, the graph of $u$ is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ when

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0,$$

where all terms are calculated with respect to the metric of $\mathbb{H}^2$.

Finally, we denote by $\partial_\infty \mathbb{H}^2$ the infinite boundary of $\mathbb{H}^2$, i.e.

$$\partial_\infty \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

2.1 Flux of a minimal graph along a curve

Let $u$ be a minimal graph defined on a domain $\Omega \subset \mathbb{H}^2$. Assume $\partial \Omega$ is piecewise smooth and $u$ extends continuously to $\overline{\Omega}$ (possibly with infinite values). We define the flux of $u$ along a curve $\Gamma \subset \partial \Omega$ as

$$F_u(\Gamma) = \int_{\Gamma} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \eta \right) ds,$$

where $\eta$ is the outer normal to $\partial \Omega$ in $\mathbb{H}^2$ and $ds$ is the arc-length of $\partial \Omega$.

In the case $\Gamma \subset \Omega$, we can see $\Gamma$ in the boundary of different subdomains of $\Omega$, with two possible induced orientations. The flux $F_u(\Gamma)$ of $u$ along $\Gamma$ is then well-defined up to sign, and $|F_u(\Gamma)|$ is well-defined.

Given an arc $C \subset \mathbb{H}^2$, we will denote by $|C|$ the length of $C$ in $\mathbb{H}^2$. The proof of the following result can be found in [12], Lemmata 1 and 2.

Lemma 2.1 ([12]). Let $u$ be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$. 

(i) For every subdomain \( \Omega' \subset \Omega \) such that \( \overline{\Omega'} \) is compact, we have \( F_u(\partial \Omega') = 0 \).

(ii) Let \( C \) be a piecewise smooth curve contained in the interior of \( \Omega \), or a convex curve in \( \partial \Omega \) where \( u \) extends continuously and takes finite values. If \( C \) has finite length, then \( |F_u(C)| < |C| \).

(iii) Let \( T \subset \partial \Omega \) be a geodesic arc of finite length such that \( u \) diverges to \(+\infty\) (resp. \(-\infty\)) as one approaches \( T \) within \( \Omega \). Then \( F_u(T) = |T| \) (resp. \( F_u(T) = -|T| \)).

The last statement in Lemma 2.1 admits the following generalization.

**Lemma 2.2 ([12])**. For each \( n \in \mathbb{N} \), let \( u_n \) be a minimal graph on a fixed domain \( \Omega \subset \mathbb{H}^2 \) which extends continuously to \( \overline{\Omega} \), and let \( T \) be a geodesic arc of finite length in \( \partial \Omega \).

(i) If \( \{u_n\}_n \) diverges uniformly to \(+\infty\) on compact subsets of \( T \) while remaining uniformly bounded in compact subsets of \( \Omega \), then \( F_{u_n}(T) \to |T| \).

(ii) If \( \{u_n\}_n \) diverges uniformly to \(+\infty\) in compact subsets of \( \Omega \) while remaining uniformly bounded on compact subsets of \( T \), then \( F_{u_n}(T) \to -|T| \).

### 2.2 Divergence lines

Let \( \Omega \subset \mathbb{H}^2 \) be a polygonal domain (i.e. a domain whose edges are geodesic arcs of \( \mathbb{H}^2 \)) with vertices in \( \mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2 \), possibly infinitely many. Given a sequence \( \{u_k\}_k \) of minimal graphs defined on \( \Omega \), we define its **convergence domain** as

\[
\mathcal{B} = \{ p \in \Omega \mid \{ |\nabla u_k(p)| \}_k \text{ is bounded} \},
\]

and the **divergence set** of \( \{u_k\}_k \) as

\[
\mathcal{D} = \Omega - \mathcal{B}.
\]

From Lemma 4.3 in [7], we know that the divergence set \( \mathcal{D} \) is composed of geodesic arcs contained in \( \Omega \), called **divergence lines**, each one joining two points of \( \partial \Omega \) (including the vertices of \( \Omega \)). The following proposition describes the convergence domain and the divergence set of a sequence of minimal graphs. Its proof can be found in [7], Lemmata 4.2 and 4.3 and Proposition 4.4.

**Proposition 2.3 ([7])**. Let \( \Omega \subset \mathbb{H}^2 \) be a polygonal domain, and \( \{u_k\}_k \) a sequence of minimal graphs on \( \Omega \). Suppose that \( \mathcal{D} \) is a countable set of divergence lines. Then passing to a subsequence we have:
1. $\mathcal{D}$ is composed of pairwise disjoint geodesic arcs contained in $\Omega$ (called divergence lines), each one joining two points in $\partial \Omega$ (including the vertices of $\Omega$).

2. $\{|F_{u_k}(T)|\}_k$ converges to $|T|$ as $k \to +\infty$, for any geodesic arc $T$ with finite length contained in a divergence line $L \subset \mathcal{D}$.

3. $\mathcal{B}$ is an open set. Moreover, for any component $\Omega'$ of $\mathcal{B}$ and any $p \in \Omega'$, $\{u_k - u_k(p)\}_k$ converges uniformly on compact subsets of $\Omega'$ to a minimal graph $u_\infty$.

### 2.3 Jenkins-Serrin graphs on semi-ideal polygonal domains

Let $\Omega$ be a polygonal domain. We say that $\Omega$ is semi-ideal when no two consecutive vertices of $\Omega$ are either in $\mathbb{H}^2$ or at $\partial_\infty \mathbb{H}^2$ (see Figure 1). We call interior vertices of $\Omega$ to those which are contained in $\mathbb{H}^2$; ideal vertices of $\Omega$ to those lying on $\partial_\infty \mathbb{H}^2$; and limit ideal vertices of $\Omega$ to the limit points of ideal vertices of $\Omega$.

Fix a semi-ideal polygonal domain $\Omega$ with a finite number of vertices $p_1, \ldots, p_{2k}$ (cyclically ordered). We can assume the odd vertices $p_{2i-1}$ are ideal, and then the even vertices $p_{2i}$ are interior.
For each $i = 1, \ldots, k$, we call $A_i$ (resp. $B_i$) the geodesic arc joining $p_{2i-1}, p_{2i}$ (resp. $p_{2i}, p_{2i+1}$). We consider a horocycle $H_{2i-1}$ at $p_{2i-1}$. Assume $H_{2i-1} \cap H_{2j-1} = \emptyset$ for any $i \neq j$. Given a polygonal domain $\mathcal{P}$ inscribed in $\Omega$ (i.e. a polygonal domain $\mathcal{P} \subset \Omega$ whose vertices are vertices of $\Omega$, possibly at $\partial_{\infty}\mathbb{H}^2$), we denote by $\Gamma(\mathcal{P})$ the part of $\partial \mathcal{P}$ outside the horocycles (observe that $\Gamma(\mathcal{P}) = \partial \mathcal{P}$ in the case all the vertices of $\mathcal{P}$ are interior). Also let us call

$$\alpha(\mathcal{P}) = \sum_{i=1}^{k} |A_i \cap \Gamma(\mathcal{P})|$$

and

$$\beta(\mathcal{P}) = \sum_{i=1}^{k} |B_i \cap \Gamma(\mathcal{P})|,$$

where we recall that $|\bullet| = \text{length}_{\mathbb{H}^2}(\bullet)$. See Figure 1.

**Definition 2.4.** Let $\Omega$ be a semi-ideal polygonal domain with a finite number of vertices $p_1, \ldots, p_{2k}$, where $p_{2i-1} \in \partial_{\infty}\mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$. We say that $\Omega$ is Jenkins-Serrin if for some choice of horocycles $H_{2i-1}$ as above it holds:

(i) $\alpha(\Omega) = \beta(\Omega)$.

(ii) $2\alpha(\mathcal{P}) < |\Gamma(\mathcal{P})|$ and $2\beta(\mathcal{P}) < |\Gamma(\mathcal{P})|$, for every polygonal domain $\mathcal{P}$ inscribed in $\Omega$, $\mathcal{P} \neq \Omega$.

We remark that condition (i) in the above definition does not depend on the choice of horocycles; and if the inequalities of condition (ii) are satisfied for some choice of horocycles, then they continue to hold for “smaller” horocycles (see the argument given by Pascal Collin and Harold Rosenberg in [3], pages 1884 and 1885). The following result is a particular case of Theorem 4.12 in [7].

**Theorem 2.5 ([7]).** Let $\Omega$ be a semi-ideal polygonal domain with edges $A_1, B_1, \ldots, A_k, B_k$ (cyclically ordered). There exists a solution $u$ for the minimal graph equation (1) in $\Omega$ with boundary values

$$u|_{A_i} = +\infty \quad \text{and} \quad u|_{B_i} = -\infty, \quad \text{for any} \quad i = 1, \ldots, k,$$

if, and only if, $\Omega$ is a Jenkins-Serrin domain. Moreover, such a solution is unique up to an additive constant, when it exists.

We will work with convex semi-ideal polygonal domains $\Omega$ satisfying the following additional condition (see Figures 1 and 2):

(*) There exists a choice of pairwise disjoint horocycles $H_{2i-1}$ at the ideal vertices $p_{2i-1} \in \partial_{\infty}\mathbb{H}^2$ such that

$$\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$$

for any $i = 1, \ldots, k$, using the cyclical notation $p_0 = p_{2k}$.
Figure 2: Example of a convex semi-ideal polygonal domain satisfying condition (⋆) which is not Jenkins-Serrin, since $2\alpha(\mathcal{P}) > |\Gamma(\mathcal{P})|$ because $|A_2 \cap \Gamma(\mathcal{P})| > |L \cap \Gamma(\mathcal{P})|$.

We remark that condition (⋆) does not depend on the choice of horocycles $H_{2i-1}$, and it is equivalent to the existence of a horocycle $C_{2i-1}$ at $p_{2i-1}$ passing through $p_{2i-2}, p_{2i}$, for any $i = 1, \ldots, k$. We call $D_{2i-1}$ the component of $\mathbb{H}^2 - C_{2i-1}$ whose only point of $\partial_{\infty}\mathbb{H}^2$ at its infinite boundary is $p_{2i-1}$ (i.e. $D_{2i-1}$ is the horodisk at $p_{2i-1}$ bounded by $C_{2i-1}$), and $\overline{D_{2i-1}} = D_{2i-1} \cup C_{2i-1}$.

Before finishing this subsection, we describe geometrically when a semi-ideal polygonal domain with a finite number of vertices and satisfying condition (⋆) is a Jenkins-Serrin domain. See Figure 2.

**Lemma 2.6.** Let $\Omega$ be a semi-ideal polygonal domain with vertices $p_1, \ldots, p_k$ cyclically ordered so that $p_{2i-1} \in \partial_{\infty}\mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$, for any $i = 1, \ldots, k$. Suppose $\Omega$ satisfies condition (⋆) above. Then the following assertions are equivalent:

1. $\Omega$ is a Jenkins-Serrin domain.

2. $p_{2j} \in \mathbb{H}^2 - \overline{D_{2i-1}}$, for any $j$ and any $i \notin \{j, j + 1\}$.

**Proof.** Before proving the lemma, let us fix some notation. For any $i = 1, \ldots, k$, consider the nested sequence of horocycles $\{H_{2i-1}(n)\}_n$ at $p_{2i-1}$ contained in $D_{2i-1}$ and converging
to \( p_{2i-1} \) as \( n \to +\infty \), such that \( \text{dist}_{\mathbb{H}^2}(H_{2i-1}(n), C_{2i-1}) = n \), for any \( n \). Then
\[
\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}(n)) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i}(n)) = n.
\]

Let us now prove Lemma 2.6. First suppose \( \Omega \) is Jenkins-Serrin and there exists some \( p_{2j} \in \overline{D}_{2i-1}, \) with \( i \not\in \{j, j+1\} \). We then have
\[
\text{dist}_{\mathbb{H}^2}(p_{2j}, H_{2i-1}(n)) \leq n
\]
for \( n \) large. Let \( L \) be the geodesic arc from \( p_{2j} \) to \( p_{2i-1} \), and \( P \) be the component of \( \Omega- \) containing \( A_i \) on its boundary (see Figure 2). Clearly, \( P \) is a polygonal domain inscribed in \( \Omega \). And, for this choice of horocycles \( H_{2i-1}(n) \), it holds \( |A_\ell \cap \Gamma(P)| = n \) (resp. \( |B_\ell \cap \Gamma(P)| = n \)) for any \( \ell \) such that \( A_\ell \subset \partial P \) (resp. \( B_\ell \subset \partial P \)). Thus \( \beta(P) = \alpha(P) - n \), and then
\[
|\Gamma(P)| = \text{dist}_{\mathbb{H}^2}(p_{2j}, H_{2i-1}(n)) + \alpha(P) + \beta(P) \leq 2\alpha(P).
\]
This holds for every \( n \) large, which contradicts that \( \Omega \) is a Jenkins-Serrin domain. This proves \((1) \Rightarrow (2)\).

Now assume \( p_{2j} \in \mathbb{H}^2-\overline{D}_{2i-1}, \) for any \( j \) and any \( i \not\in \{j, j+1\} \), and let us prove that \( \Omega \) is a Jenkins-Serrin domain. As we have remarked above, we have \( \alpha(\Omega) = \beta(\Omega) \). Suppose there exists an inscribed polygonal domain \( P \) in \( \Omega \), \( P \neq \Omega \), such that
\[
|\Gamma(P)| \leq 2\alpha(P)
\]
(the case \( |\Gamma(P)| \leq 2\beta(P) \) follows similarly). Since \( P \neq \Omega \), there is at least an interior geodesic \( \gamma_1 \) in \( \partial P \) (i.e. \( \gamma_1 \subset \partial P \cap \Omega \)). We can assume there are no two consecutive interior geodesics \( \gamma_1, \gamma_2 \) in \( \partial P \): We would replace \( P \) by another inscribed polygonal domain satisfying the same properties as \( P \) by replacing \( \gamma_1 \cup \gamma_2 \) by the geodesic \( \gamma_3 \) such that \( \gamma_1 \cup \gamma_2 \cup \gamma_3 \) bounds a geodesic triangle contained in \( \Omega \). In a similar way, we can assume that
\[
\partial P = A_{i_1} \cup \gamma_1 \cup \ldots \cup A_{i_j} \cup \gamma_j \cup A_{i_{j+1}} \cup \ldots \cup A_{i_s} \cup \gamma_s,
\]
where each \( \gamma_j \) is either an interior geodesic or a \( B_i \) edge, and at least \( \gamma_1 \subset \Omega \). In particular, each \( \gamma_j \) joins an even vertex \( p_{2i_j} \) to an odd vertex \( p_{2i_{j+1}-1} \). (Observe that, when \( \gamma_j \) is a \( B_i \) edge, then \( \gamma_j = B_{i_j} \) and \( i_{j+1} = i_j + 1 \).)

Hence \( \sum_{j=1}^s |\gamma_j \cap \Gamma(P)| = |\Gamma(P)| - \alpha(P) \leq \alpha(P) = sn \) from where we deduce there must be an interior geodesic \( \gamma_j \subset \partial P \) whose length is smaller than or equal to \( n \). But this implies the vertex \( p_{2i_j} \) lies on \( \overline{D}_{2i_{j+1}-1} \) and \( i_j \not\in \{i_{j+1} - 1, i_{j+1}\} \), a contradiction. \( \square \)
2.4 Conjugate surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this subsection we briefly recall how to obtain minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by conjugation from other known minimal examples. For more details see Daniel [4, Section 4] and Hauswirth, Sa Earp and Toubiana [6].

Let $\Sigma$ be a 2-sided minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. We call height function of $\Sigma$ to the horizontal projection $h : \Sigma \to \mathbb{R}$, which is known to be a real harmonic map. And we denote by $F : \Sigma \to \mathbb{H}^2$ the vertical projection, which is a harmonic map, and by

$$Q = \langle F_z \bar{F}_z \rangle (dz)^2$$

the Hopf differential associated to $F$, where $z$ is a local conformal coordinate on $\Sigma$. Finally, we denote by $N$ a globally defined unit normal vector field on $\Sigma$ and by $\nu = \langle N, \partial_t \rangle$ the angle function of $\Sigma$.

**Theorem 2.7** ([4, 6]). Let $\Sigma$ be a simply-connected minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. There exists a minimal surface $\Sigma^* \subset \mathbb{H}^2 \times \mathbb{R}$, called conjugate surface of $\Sigma$, such that:

1. $\Sigma$ and $\Sigma^*$ are isometric. (If we identify points in $\Sigma$ and $\Sigma^*$ via an isometry, we can assume that the angle function $\nu^*$, the height function $h^*$, the vertical projection $F^*$ of $\Sigma^*$, and the Hopf differential $Q^*$ associated to $F^*$, are all defined on $\Sigma$.)

2. The angle functions $\nu, \nu^*$ coincide.

3. The height functions $h, h^*$ are real harmonic conjugate.

4. $Q^* = -Q$.

The conjugate surface $\Sigma^*$ is well-defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. Finally, the conjugation exchanges the following Schwarz reflections:

- The symmetry with respect to a vertical geodesic plane of $\mathbb{H}^2 \times \mathbb{R}$ containing a geodesic curvature line of $\Sigma$ becomes the rotation of $\mathbb{H}^2 \times \mathbb{R}$ by angle $\pi$ with respect to a horizontal geodesic contained in $\Sigma^*$, and viceversa.

- The symmetry with respect to a horizontal slice containing a geodesic curvature line of $\Sigma$ becomes the rotation by angle $\pi$ with respect to a vertical straight line contained in $\Sigma^*$, and viceversa.

We will use the above correspondence to study the conjugate surface of a minimal graph defined on a convex semi-ideal polygonal domain of $\mathbb{H}^2$. The surface constructed in this way is a minimal graph (and consequently embedded), as ensured by the following Krust-type theorem.
Figure 3: Vertical projection of the conjugate surface $\Sigma^*$ in a symmetric case.

**Theorem 2.8 ([6]).** If $\Sigma$ is a minimal graph over a convex domain $\Omega$ of $\mathbb{H}^2$, then $\Sigma^*$ is also a minimal graph over a (non-necessarily convex) domain $\Omega^* \subset \mathbb{H}^2$.

### 2.5 Minimal $k$-noids of $\mathbb{H}^2 \times \mathbb{R}$

In this subsection we briefly explain the construction of the properly embedded minimal surfaces of $\mathbb{H}^2 \times \mathbb{R}$ given in [11, 13], which have genus zero, $k \geq 2$ vertical planar ends and finite total (intrinsic) curvature $4\pi(1 - k)$. We call them minimal $k$-noids of $\mathbb{H}^2 \times \mathbb{R}$.

Let $\Omega$ be a convex Jenkins-Serrin semi-ideal polygonal domain with $2k$ vertices $p_1, \ldots, p_{2k}$, cyclically ordered, so that the even vertices $p_{2i}$ are located in the interior of $\mathbb{H}^2$, and the odd vertices $p_{2i-1}$ are at $\partial_\infty \mathbb{H}^2$, for $i = 1, \ldots, k$. We call $A_i$ the edge of $\Omega$ whose endpoints are $p_{2i-1}, p_{2i}$, and $B_i$ the edge of $\Omega$ whose endpoints are $p_{2i}, p_{2i+1}$. We also require that $\Omega$ satisfies the condition (⋆) defined in Subsection 2.3.

By Theorem 2.5, there exists a unique solution $u$ to the minimal graph equation (1) defined over $\Omega$ with boundary values $+\infty$ on $A_i$ and $-\infty$ on $B_i$ such that $u(p_0) = 0$, for some fixed point $p_0 \in \Omega$. Denote by $\Sigma$ the graph surface of $u$; $\Sigma$ is bounded by the $k$ vertical straight lines $\Gamma_i = \{p_{2i}\} \times \mathbb{R}$, $i = 1, \ldots, k$.

The conjugate surface $\Sigma^*$ of $\Sigma$ is a minimal graph over a (non-necessarily convex) domain $\Omega^* \subset \mathbb{H}^2$, by Theorem 2.8 (see Figure 3). And $\partial \Sigma^*$ consists of $k$ horizontal geodesic curvature lines $\Gamma_i^*$. In [11] it is proved that $\Gamma_i^* \subset \mathbb{H}^2 \times \{0\}$ for any $i$ and that $\Sigma^*$ is contained in one of the half-spaces determined by $\mathbb{H}^2 \times \{0\}$. By reflecting $\Sigma^*$ with respect to $\mathbb{H}^2 \times \{0\}$, we get a properly embedded minimal surface $M$ with genus zero and
$k$ vertical planar ends, which has total (intrinsic) curvature $4\pi (1-k)$. The ends of $M$ are asymptotic to the vertical geodesic planes $\eta_i^* \times \mathbb{R}$, where the $\eta_i^*$ are the complete geodesics such that $\partial \Omega^* = \Gamma_1^* \cup \eta_1^* \cup \ldots \cup \Gamma_k^* \cup \eta_k^*$ (cyclically ordered).

3 Proof of Theorem 1.1: examples with $m_0$ limit ends

Firstly, let us recall some definitions. A limit end $e$ of a non-compact surface $M$ is an accumulation point of the set $\mathcal{E}(M)$ of ends of $M$. This makes sense since $\mathcal{E}(M)$ can be endowed with a natural topology for which it is a compact, totally disconnected subspace of the real interval $[0,1]$. See [9] for more details. We call simple ends of $M$ to its non-limit ends.

Assume the simple ends of $M$ are asymptotic to vertical geodesic planes (called vertical planar ends) which can be ordered cyclically\footnote{The surface $M$ we want to construct will be obtained as a limit of minimal $k$-noids, and it is got by reflection symmetry from a minimal graph with boundary values $0, +\infty$, alternately. The vertical projection of $M$ will be bounded by strictly concave curves $\Gamma_i^*$ and geodesic curves $\eta_i^*$, disposed alternately and asymptotic at $\partial_{\infty}\mathbb{H}^2$. The ends of $M$ will be asymptotic to the vertical geodesic planes $\eta_i^* \times \mathbb{R}$, which are cyclically ordered.}, fixed an orientation of $\partial_{\infty}\mathbb{H}^2$. If the limit end can be obtained as accumulation of simple ends ordered following the negative orientation (resp. the positive orientation) but it cannot be obtained as accumulation of simple ends ordered following the positive orientation (resp. negative orientation), we will say that it is a left (resp. right) limit end. In other case, we will say that it is a 2-sided limit end.

In this section we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and $m_0$ limit ends, for any $m_0 \geq 1$. Furthermore, we can prescribe the behavior of the limit ends; more precisely, if we denote by $E_{m_0}^1, \ldots, E_{m_0}^{m_0}$ (cyclically ordered) the limit ends of such a surface, we can prescribe if each $E_{m_0}^m$ is either a left, a right or a 2-sided limit end.

Now let us explain our construction. In a first step we will construct, by taking limits of convex Jenkins-Serrin semi-ideal polygonal domains $\Omega_k$ with finitely many vertices, a convex semi-ideal polygonal domain $\Omega_{\infty}$ with an infinite countable set $S$ of ideal vertices and $m_0$ limit points $p_{\infty}^1, \ldots, p_{\infty}^{m_0}$ (cyclically ordered) such that, if $E_{m_0}^m$ is prescribed to be a left (resp. a right or a 2-sided) limit end, then $p_{\infty}^m$ is a left (resp. a right or a 2-sided) limit point, see Definition 3.1 below. We will say that such a limit point $p_{\infty}^m$ is a left (resp. a right or a 2-sided) limit ideal vertex of $\Omega_{\infty}$.

Definition 3.1. Let $S$ be a set of points in $\partial_{\infty}\mathbb{H}^2 = \{y = 0\}$. We will say that $p_{\infty} \in S$ is a limit point if every neighborhood of $p_{\infty}$ in $\partial_{\infty}\mathbb{H}^2$ contains a point of $S$ other than $p_{\infty}$.
itself; i.e. if \( p_\infty = (x_\infty, 0) \), then \( S \cap \{ y = 0, \ 0 < |x - x_\infty| < \varepsilon \} \neq \emptyset \) for every \( \varepsilon > 0 \); or \( p_\infty = \infty \) and \( S \cap \{ y = 0, \ |x| > M \} \neq \emptyset \) for every \( M > 0 \).

A limit point \( p_\infty = (x_\infty, 0) \) of \( S \) is said to be a left (resp. a right) limit point if there exists some \( \varepsilon > 0 \) such that \( S \cap \{ -\varepsilon < x - x_\infty < 0 \} = \emptyset \) (resp. \( S \cap \{ 0 < x - x_\infty < \varepsilon \} = \emptyset \)); and it is said to be a 2-sided limit point in other case.

If \( p_\infty = \infty \) is a limit point of \( S \), we say that it is a left (resp. a right) limit point if there exists some \( M > 0 \) such that \( S \cap \{ x > M \} = \emptyset \) (resp. \( S \cap \{ x < -M \} = \emptyset \)); and it is a 2-sided limit point in other case.

Next we will get a Jenkins-Serrin minimal graph \( \Sigma \) over \( \Omega_\infty \) as a limit of Jenkins-Serrin minimal graphs over the \( \Omega_k \) domains. Finally, we will prove that the conjugate surface of \( \Sigma \) is a minimal graph over the \( \Omega_\infty \) in the desired conditions. We call \( \Omega_\infty \) as a 2-sided limit point in other case.

Finally, we will prove that the conjugate surface of \( \Sigma \) is a minimal graph \( \Sigma^* \subset \mathbb{H}^2 \times [0, +\infty) \) whose boundary, which consists of horizontal geodesic curvature lines, is contained in the horizontal slice \( \mathbb{H}^2 \times \{ 0 \} \). The desired surface is obtained from \( \Sigma^* \) by reflection symmetry about \( \mathbb{H}^2 \times \{ 0 \} \).

### 3.1 Construction of the domains

This subsection deals with the construction of the convex semi-ideal polygonal domain \( \Omega_\infty \) in the argument explained above. We will construct a sequence of convex Jenkins-Serrin semi-ideal polygonal domains \( \Omega_k \) satisfying condition (\( \ast \)) defined in Subsection 2.3, each \( \Omega_k \) with a finite number of vertices, with \( \Omega_k \subset \Omega_{k+1} \) for any \( k \), and such that they converge to a domain \( \Omega_\infty \) in the desired conditions.

Consider \( m_0 \) different ideal points in \( \partial_\infty \mathbb{H}^2 \):

\[
p_1^\infty = \infty, \ p_2^\infty = (x_2^\infty, 0), \ldots, \ p_m^\infty = (x_m^\infty, 0),
\]

with \( -\infty < x_2^\infty < \ldots < x_m^\infty < +\infty \), when \( m_0 \geq 2 \); in the case \( m_0 = 1 \), we only have \( p_1^\infty = \infty \). These points will be the limit ideal vertices of \( \Omega_\infty \).

We call \( \mathcal{M} = \{ m \in \mathbb{N} \mid 1 \leq m \leq m_0 \} \). For any \( m \in \mathcal{M} \), choose two ideal points \( p_m^{-1} = (x_m^{-1}, 0), \ p_m^1 = (x_m^1, 0) \) with \( x_m^1 < x_m^{-1} < x_m^1 < x_m^{m+1} \), where \( x_1^- = -\infty \) and \( x_m^{m+1} = +\infty \).

For any \( m \in \mathcal{M} \) and any \( j \in \{-1, 1, \infty\} \), we call \( C_j^m \) the horocycle at \( p_j^m \) passing through \( P_0 = (0, 1) \). Denote by \( p_0^m \) (resp. \( q_2^m, q_2^m \)) the point in \( C_{-1}^m \cap C_1^m \) (resp. \( C_\infty^m \cap C_{-1}^m, C_1^m \cap C_{m+1}^m \)) which is different from \( P_0 \), see Figure 4. We define \( \Omega_1 \) as the semi-ideal polygonal domain with set of vertices

\[
\{ p_m^m, q_{-2}^m, p_{-1}^m, p_0^m, p_1^m, q_2^m \mid m \in \mathcal{M} \}.
\]

By definition of \( q_2^m, p_0^m, q_2^m \), it is clear that \( \Omega_1 \) satisfies condition (\( \ast \)). Now let us see we can choose the ideal vertices \( p_{-1}^m, p_1^m \) to assure \( \Omega_1 \) is convex. It suffices to choose appropriately \( x_m^{-1}, x_m^1 \) such that \( q_{-2}^m, p_0^m, q_2^m \subset \{ 0 < y < 1 \} \), except for \( q_2^m, q_2^m \in \{ y = 1 \} \).
Figure 4: The shadowed region is an example of $\Omega_1$ for three limit ideal vertices, where $p_1^\infty = \infty$ and $p_i^m = (x_i^m, 0)$ for the remaining values of $i, m$. The interior vertices in white correspond, from the left to the right, to $q_{-2}, P_0, q_2, P_0, q_2, q_{-2}, P_0, q_2$.

- If $x_\infty^m \geq 0$ or $x_{\infty}^{m+1} \leq 0$, then $x_{-1}^m, x_1^m$ can be chosen arbitrarily.
- In the case $x_\infty^m < 0 < x_{\infty}^{m+1}$, we take $\max\{x_\infty^m, -1\} < x_{-1}^m < 0 < x_1^m < \min\{x_\infty^{m+1}, 1\}$.

With the choice above, the domain $\Omega_1$ is convex. Finally, let us check that $\Omega_1$ is a Jenkins-Serrin domain. Using Lemma 2.6 it suffices to get that $p_0^m$ (resp. $q_{-2}^m, q_2^m$) lies outside $C_j^{m'}$, for any $j \in \{-1, 1, \infty\}$ and any $m' \in \mathcal{M}$ such that $C_j^{m'}$ is different from $C_{-1}^{m}, C_1^{m}$ (resp. $C_{\infty}^{m}, C_{\infty}^{m'}, C_1^{m'}, C_1^{m+1}$). By the choice above, this is the case when $C_j^{m'} = C_\infty^{m}$. Let us assume $C_j^{m'} \neq C_\infty^{m}$. We prove it for $p_0^m$ (for $q_{-2}^m, q_2^m$ it can be obtained similarly): If we denote $p_0^m = (x_0^m, y_0^m)$, we have $x_{-1}^m < x_0^m < x_1^m$. If $x_j^{m'} > x_1^m$ (resp. $x_j^{m'} < x_{-1}^m$), then $C_j^{m'}$ divides $C_1^{m}$ (resp. $C_{-1}^{m}$) in two components, one of them containing both $p_0^m$ and $p_1^m$ (resp. $p_{-1}^m$). That says that $p_0^m$ is outside $C_j^{m'}$.

Now we consider the subsets of $\mathcal{M}$ given by

$\mathcal{M}^+ = \{m \in \mathcal{M} \mid E_{\infty}^{m+1} \text{ is prescribed to be either a right or a 2-sided limit end}\}$,

$\mathcal{M}^- = \{m \in \mathcal{M} \mid E_{\infty}^{m} \text{ is prescribed to be either a left or a 2-sided limit end}\}$.

For any $k \geq 2$, we define $\Omega_k$ as the semi-ideal polygonal domain with set of vertices $V_k^- \cup V_0^0 \cup V_k^+$, where

$V_k^- = \{q_{-2k}^m, p_{1-2k}^m, p_{2-2k}^m, \ldots, p_{-3}^m, p_{-2}^m \mid m \in \mathcal{M}^-\} \cup \{q_{-2}^m \mid m \in \mathcal{M} - \mathcal{M}^-\}$,
\[ V^0_k = \{ p^m_\infty, p^m_{-1}, p^m_0, p^m_1 | m \in \mathcal{M} \}, \]
\[ V^+_k = \{ p^m_2, p^m_3, \ldots, p^m_{2k-2}, p^m_{2k-1}, q^m_{2k} | m \in \mathcal{M}^+ \} \cup \{ q^m_{2} | m \in \mathcal{M} - \mathcal{M}^+ \}. \]

and the vertices \( p^m_{\pm i}; q^m_{\pm 2k} \) are defined by induction as follows:

1. Suppose that \( m \in \mathcal{M}^+ \) and that we have defined the ideal vertices
\[ p^m_1 = (x^m_1, 0), \ldots, p^m_{2k-1} = (x^m_{2k-1}, 0), \]
with \( k \geq 1 \) and \( x^m_1 < \ldots < x^m_{2k-1} < x^m_\infty \). These ideal vertices determine the following data: For \( 1 \leq i \leq k \),
   - let \( C^m_{2i-1} \) be the horocycle at \( p^m_{2i-1} \) passing through \( P_0 \);
   - \( p^m_{2i-2} \) is defined as the intersection point in \( C^m_{2i-3} \cap C^m_{2i-1} \) different from \( P_0 \);
   - \( q^m_{2i} = \gamma^m_{2i} \) is the intersection point in \( C^m_{2i-1} \cap C^m_{\infty+1} \) different from \( P_0 \).

This choice of \( p^m_{2i-2}, q^m_{2i} \) will assure that \( \Omega_k \) is a convex Jenkins-Serrin semi-ideal polygonal domain which satisfies condition (\( \ast \)).

Let us now define \( p^m_{2k+1} \). We call \( \Gamma^m_{2k} \) (resp. \( \gamma^m_{2k} \)) the complete geodesic curve with endpoint \( p^m_{2k-1} \) (resp. \( p^m_{\infty} \)) passing through \( q^m_{2k} \). Let \( (a^m_{2k}, 0) \) (resp. \( (b^m_{2k}, 0) \)) be the endpoint of \( \Gamma^m_{2k} \) (resp. \( \gamma^m_{2k} \)) different from \( p^m_{2k-1} \) (resp. \( p^m_{\infty} \)). We take \( p^m_{2k+1} = (x^m_{2k+1}, 0) \) satisfying \( b^m_{2k} \leq x^m_{2k+1} \leq a^m_{2k} \). We consider that property for \( p^m_{2k+1} \) in order to get \( \Omega_k \subset \Omega_{k+1} \).

We remark that both \( p^m_{2k+1} \) and \( p^m_{2k} \) converge to \( p^m_{\infty} \) as \( k \to +\infty \).

2. The corresponding definition for \( m \in \mathcal{M}^- \) follows analogously: Suppose \( m \in \mathcal{M}^- \) and that, for \( k \geq 1 \), we have defined the ideal vertices
\[ p^m_{1-2k} = (x^m_{1-2k}, 0), \ldots, p^m_{-1} = (x^m_{-1}, 0), \]
with \( x^m_{\infty} < x^m_{1-2k} < \ldots < x^m_{-1} \). These ideal vertices determine the following data: For \( 1 \leq i \leq k \),
   - let \( C^m_{-1-2i} \) be the horocycle at \( p^m_{-1-2i} \) passing through \( P_0 \);
   - \( p^m_{2-2i} \) is defined as the intersection point in \( C^m_{1-2i} \cap C^m_{3-2i} \) different from \( P_0 \);
   - \( q^m_{-2i} = \gamma^m_{-2i} \) is the intersection point in \( C^m_{\infty} \cap C^m_{1-2i} \) different from \( P_0 \).
Figure 5: The shadowed region is a piece of $\Omega_2$, with $1 \in M^- - M^+$ and $2 \in M^-$.

Let us now define $p_{m-2}^m$. We call $\Gamma_{m-2}^m$ (resp. $\gamma_{m-2}^m$) the complete geodesic curve with endpoint $q_{m-2}^m$ (resp. $p_{m}^m$) passing through $q_{m-2}^m$. Let $(a_{m-2}^m, 0)$ (resp. $(b_{m-2}^m, 0)$) be the endpoint of $\Gamma_{m-2}^m$ (resp. $\gamma_{m-2}^m$) different from $p_{m-2}^m$ (resp. $p_{m}^m$). We choose $p_{m-2}^m = (x_{m-2}^m, 0)$, with $b_{m-2}^m \leq x_{m-2}^m \leq a_{m-2}^m$. With this choice of ideal vertices, we get $\Omega_k \subset \Omega_{k+1}$ and that both $p_{m-2}^m$ and $p_{m}^m$ converge to $p_{\infty}^m$ as $k \to +\infty$.

By definition of the interior vertices $p_{2i-2}^m, q_{2i}^m$, the semi-ideal polygonal domain $\Omega_k$ satisfies condition (⋆). As $x_{2k-1}^m$ has the same sign as $x_{2k-3}^m$ and $x_{\infty}^m$, then $p_{2i-2}^m, q_{2i}^m \subset \{0 < y < 1\}$. That fact assures that $\Omega_k$ is convex. Moreover, as the horocycles $C_{m'}^j$ can be ordered from the left to the right and they all pass through $P_0$, we can deduce (as in the case of $\Omega_1$) that the interior vertices $p_{2i}^m, q_{2i-2}^m, q_{2i}^m$ are outside the horocycles $C_{m'}^j$, except for those used for defining them (i.e. their consecutive ones). Then $\Omega_k$ is a Jenkins-Serrin domain, by Lemma 2.6.

Finally, we have defined the ideal vertices $p_{2i-2}^m$ to get $\Omega_k \subset \Omega_{k+1}$; for instance, when $m \in M^+$ and $k$ is positive, the geodesics $A_{k}^m, B_{k}^m, \tilde{A}_{k+1}^m$ do not intersect $\tilde{A}_{k}^m$, where $A_{k}^m$ (resp. $B_{k}^m, \tilde{A}_{k+1}^m, \tilde{A}_{k}^m$) is defined as the geodesic arc joining $p_{2k-1}^m, p_{2k}^m$ (resp. $p_{2k}^m, p_{2k+1}^m; p_{2k+1}^m, q_{2k+2}^m; p_{2k-1}^m, q_{2k}^m$).

Let $\Omega_{\infty}$ be the semi-ideal polygonal domain with set of vertices $V_{\infty}^- \cup V^0 \cup V_{\infty}^+$, where

$V_{\infty}^- = \{p_{2k}^m, p_{1-2k}^m \mid m \in M^-, \ k \in \mathbb{N}\} \cup \{q_{2}^m \mid m \in \mathcal{M} - M^-\}$,

$V_{\infty}^+ = \{p_{2k-1}^m, p_{2k}^m \mid m \in M^+, \ k \in \mathbb{N}\} \cup \{q_{2}^m \mid m \in \mathcal{M} - M^+\}$. 

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It is clear that $\Omega_k \to \Omega_\infty$ as $k \to +\infty$. We can deduce, arguing as above for $\Omega_k$, that $\Omega_\infty$ is a convex semi-ideal polygonal domain verifying condition $(\ast)$ (the same condition can be defined for the case of infinitely many vertices).

Since $p_{2k-1} \to p_\infty^m$ as $k \to +\infty$ when $m \in \mathcal{M}^+$, and $p_{2k-1} \to p_\infty^{m-1}$ as $k \to -\infty$ when $m \in \mathcal{M}^-$, then each $p_\infty^m$ is a limit ideal vertex of $\Omega_\infty$ and it is:

- left when $m \in \mathcal{M}^-$ and $m - 1 \notin \mathcal{M}^+$;
- right when $m \notin \mathcal{M}^-$ and $m - 1 \in \mathcal{M}^+$;
- or 2-sided when $m \in \mathcal{M}^-$ and $m - 1 \in \mathcal{M}^+$.

### 3.2 Construction of the Jenkins-Serrin minimal graphs

Let $\Omega_k, \Omega_\infty$ be the domains constructed above. We call $A_i^m$ (resp. $B_i^m$) the geodesic arc joining $p_{2i-1}^m, p_{2i}^m$ (resp. $p_{2i+1}^m, p_{2i}^m$), when they are defined; and $\tilde{A}_i^m$ (resp. $\tilde{B}_i^m$) the geodesic arc joining $p_{2k-1}^m, q_{2k}^m$ (resp. $q_{2k}^m, p_{2k}^m, p_{2k}^m, q_{2k}^m, q_{2k}^m, p_{1-2k}^m$).

By Theorem 2.5, there exists a solution $u_k$ (unique up to an additive constant) for the minimal graph equation (1) in $\Omega_k$ with boundary values $+\infty$ (resp. $-\infty$) on edges $A_i^m, \tilde{A}_i^m, \tilde{A}_{i-1}^m$ (resp. $B_i^m, \tilde{B}_i^m, \tilde{B}_{i-1}^m$) which lie on $\partial \Omega_k$.

Fix a point $P \in \Omega_1$. We translate vertically the Jenkins-Serrin graphs so that $u_k(P) = 0$, for any $k$.

**Lemma 3.2.** The sequence $\{u_k\}_k$ has no divergence lines.

**Proof.** Firstly, let us introduce some notation: For $\mu = 2i - 1$ or $\mu = \infty$, consider a sequence of nested horocycles $H_\mu^m(n)$ at $p_\mu^m$ (in the case $p_\mu^m$ is defined) contained in $C_\mu$ such that $\text{dist}_{H_\mu^m}(H_\mu^m(n), C_\mu^m) = n$ for any $n$. In particular, the horocycles $H_\mu^m(n)$ are pairwise disjoint for $n$ large. Given a polygonal domain $\mathcal{P}_k$ inscribed in $\Omega_k$, denote by $\mathcal{P}_k(n)$ the polygonal domain bounded by the part of $\partial \mathcal{P}_k$ outside the horocycles $H_\mu^m(n)$ together with geodesic arcs joining points in $\partial \mathcal{P}_k \cap ((\cup_{m,i} H_{2i-1}^m(n)) \cup (\cup_{m} H_{\infty}^m(n)))$. Also denote

\[
\alpha_k(n) = \sum_{m=1}^{m_0} \left( \sum_{i=1-k}^{k-1} |A_i^m \cap \partial \mathcal{P}_k(n)| + |\tilde{A}_{i-1}^m \cap \partial \mathcal{P}_k(n)| + |\tilde{A}_i^m \cap \partial \mathcal{P}_k(n)| \right),
\]

\[
\beta_k(n) = \sum_{m=1}^{m_0} \left( \sum_{i=1-k}^{k-1} |B_i^m \cap \partial \mathcal{P}_k(n)| + |\tilde{B}_{i-1}^m \cap \partial \mathcal{P}_k(n)| + |\tilde{B}_i^m \cap \partial \mathcal{P}_k(n)| \right),
\]

\[
\varepsilon_k(n) = |\partial \mathcal{P}_k(n) - \partial \mathcal{P}_k|.
\]
We observe that, for any fixed \( k, \varepsilon_k(n) \to 0 \) as \( n \to +\infty \).

Now, let us prove Lemma 3.2. Suppose there exists a divergence line \( L \) of \( \{u_k\}_k \). As \( \{\Omega_k\}_k \) is a monotone increasing sequence of domains converging to \( \Omega_\infty \), then we can suppose \( k \) is large enough so that \( L \subset \Omega_k \). We denote by \( L(n) \) the geodesic arc in \( L \) outside the horocycles \( H^m_{2i-1}(n), H^m_{\infty}(n) \). By Proposition 2.3, \( |F_{u_k}(L(n))| \to |L(n)| \) as \( k \to +\infty \).

We fix a component \( \mathcal{P}_k \) of \( \Omega_k - L \). By Lemma 2.1,

\[
|F_{u_k}(L(n)) + \alpha_k(n) - \beta_k(n)| \leq \varepsilon_k(n),
\]

where \( \alpha_k(n), \beta_k(n), \varepsilon_k(n) \) are defined as above for this choice of \( \mathcal{P}_k \).

- In the case \( L \) has finite length, we have \( L(n) = L \) for \( n \) large enough. And \( \alpha_k(n) - \beta_k(n) = c \) is constant. Taking limits when \( n \) goes to \( +\infty \), we get \( F_{u_k}(L) = -c \). This contradicts the fact that \( |F_{u_k}(L)| < |L| \) but \( |F_{u_k}(L)| \to |L| \) as \( k \to +\infty \). Then \( L \) must have infinite length.

- If \( L \) joins either two ideal vertices \( p^m_{2i-1}, p^m_{2j-1} \), two limit ideal vertices \( p^\infty_m, p^\infty_m \) or an ideal vertex \( p^\infty_{2i-1} \) to a limit ideal vertex \( p^\infty_m \), then we have \( \alpha_k(n) = \beta_k(n) \) because of the choice of horocycles above. For any compact geodesic arc \( T \subset L(n) \) and \( k \) large, we have \( |F_{u_k}(T)| \leq |F_{u_k}(L(n))| \leq \varepsilon_k(n) \). Taking \( n \to +\infty \), we get \( F_{u_k}(T) = 0 \). But this contradicts \( |F_{u_k}(T)| \to |T| \) as \( k \to +\infty \).

- Then \( L \) must join a vertex \( p^m_\mu \), with \( \mu = 2i - 1 \) or \( \mu = \infty \), to a point \( q \in \partial \Omega_k \cap \mathbb{H}^2 \). Either \( q \) is an interior vertex, and we denote it by \( \tilde{q} \), or it lies on an edge of \( \Omega_k \) and we call \( \tilde{q} \) the interior endpoint of such an edge. We can choose \( \mathcal{P}_k \) to have \( \beta_k(n) \geq \alpha_k(n) \). Hence for \( n \) large enough we have that \( \beta_k(n) - \alpha_k(n) = n - c \), where \( c = \text{dist}_{\mathbb{H}^2}(\tilde{q}, \tilde{q}) \). Then

\[
F_{u_k}(L(n)) = n - c - F_{u_k}(\partial \mathcal{P}_k(n) - \partial \mathcal{P}_k).
\]

(Observable that, in the case \( L \) finishes at \( p^m_{\infty} \), all the vertices \( q^m_{2i} \) are contained in the same horocese \( C^\infty_{\mu} \).) Since \( |L(n)| - n = d \) is constant for \( n \) large, we get \( |L(n)| - |F_{u_k}(L(n))| \to d + c \) as \( n \to +\infty \). On the other hand, \( |F_{u_k}(L(n))| \to |L(n)| \) as \( k \to +\infty \). So it must hold \( c + d = 0 \). Therefore,

\[
\text{dist}_{\mathbb{H}^2}(\tilde{q}, \partial H^\mu_\mu(n)) \leq \text{dist}_{\mathbb{H}^2}(\tilde{q}, q) \leq \text{dist}_{\mathbb{H}^2}(q, \partial H^\mu_\mu(n)) = c + |L(n)| = n,
\]

which implies that \( \tilde{q} \) is contained in the horodisk bounded by \( C^\mu_\mu \), in contradiction with the fact that \( \Omega_k \) is a Jenkins-Serrin domain (see Lemma 2.6).
Proposition 3.3. Passing to a subsequence, \( \{u_k\}_k \) converges uniformly on compact subsets of \( \Omega_\infty \) to a minimal graph \( u_\infty \) such that it goes to \( +\infty \) (resp. \( -\infty \)) as we approach within \( \Omega_\infty \) to each \( A_i^m \) and each \( \widetilde{A}_{-1}^m \) (resp. each \( B_i^m \) and each \( \widetilde{B}_{1}^m \)) in the boundary of \( \Omega_\infty \).

Proof. Since we have translated vertically the graphs \( u_k \) so that \( u_k(P) = 0 \) for any \( k \), we get from Lemma 3.2 and Proposition 2.3 that, after passing to a subsequence, \( \{u_k\}_k \) converges to a minimal graph \( u_\infty \), and the convergence is uniform on compact subsets of \( \Omega_\infty \). It is clear that \( u_\infty(P) = 0 \).

For any bounded geodesic arc \( T \subset A_i^m \) we have \( F_{u_k}(T) = |T| \) by Proposition 2.3; and then \( F_{u_\infty}(T) = |T| \). Hence \( u_\infty \) goes to \( +\infty \) as we approach \( T \) within \( \Omega_\infty \). This proves \( u_\infty|_{A_i^m} = +\infty \). Similarly we get \( u_\infty|_{\widetilde{A}_{-1}^m} = +\infty \), \( u_\infty|_{B_i^m} = -\infty \) and \( u_\infty|_{\widetilde{B}_{1}^m} = -\infty \), which finishes Proposition 3.3. \( \square \)

3.3 Passing to the conjugate surface

Denote by \( \Sigma_\infty \) (resp. \( \Sigma_k \)) the graph surface of \( u_\infty \) (resp. \( u_k \)). Observe that, if \( m \in \mathcal{M}^+ \) (resp. \( m \in \mathcal{M}^- \)) and \( i \geq 1 \) (resp. \( i \leq -1 \)), then the vertical straight line \( \Gamma_i^m = \{p_{2i}^m\} \times \mathbb{R} \) is contained in the boundary of \( \Sigma_\infty \) and of \( \Sigma_k \), for any \( k \) large; and \( \Gamma_i^m = \{p_0^m\} \times \mathbb{R} \subset \partial \Sigma_\infty \cap \partial \Sigma_k \), for any \( m \) and any \( k \). We also denote \( \tilde{\Gamma}_i^m = \{q_{2i}^m\} \times \mathbb{R} \). Then \( \tilde{\Gamma}_i^m \subset \partial \Sigma_k \) and \( \tilde{\Gamma}_{-1}^m \subset \partial \Sigma_\infty \) when \( m \in \mathcal{M}^+ \); and \( \tilde{\Gamma}_{-k}^m \subset \partial \Sigma_k \) and \( \tilde{\Gamma}_1^m \subset \partial \Sigma_\infty \), when \( m \in \mathcal{M}^- \).

We call \( \Sigma_k^* \) the conjugate surface of \( \Sigma_k \). If \( \Gamma \) is a curve in \( \partial \Sigma_k \), then we denote by \( \Gamma(k)^* \) the corresponding curve in \( \Sigma_k^* \). We know (see Subsection 2.5; or [11], section 4) that \( \Sigma_k^* \) is a minimal graph bounded by horizontal geodesic curvature lines contained in the same horizontal slice,

\[
\partial \Sigma_k^* = \bigcup_{m \in \mathcal{M}} \left( \Upsilon_k^{m-} \cup \Upsilon_0^m(k)^* \cup \Upsilon_k^{m+} \right),
\]

where

\[
\Upsilon_k^{m-} = \begin{cases} \tilde{\Gamma}_{k}^m(k)^* \cup \Gamma_{k-1}^m(k)^* \cup \Gamma_{k-2}^m(k)^* \cup \ldots \cup \Gamma_1^m(k)^* & \text{if } m \in \mathcal{M}^- \\ \Gamma_{-k}^m(k)^* & \text{if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}
\]

\[
\Upsilon_k^{m+} = \begin{cases} \Gamma_{k}^m(k)^* \cup \ldots \cup \Gamma_{k-2}^m(k)^* \cup \Gamma_{k-1}^m(k)^* \cup \Gamma_{k}^m(k)^* & \text{if } m \in \mathcal{M}^+ \\ \tilde{\Gamma}_{1}^m(k)^* & \text{if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}
\]
Up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$, we can assume that the horizontal geodesic curvature lines $\Gamma_i^m(k)^*$ are contained in the horizontal slice $\mathbb{H}^2 \times \{0\}$, and $\Sigma_k^* \subset \{ t \geq 0 \}$. Each $\Gamma_i^m(k)^*, \bar{\Gamma}_i^m(k)^* \subset \partial \Sigma_k^*$ corresponds by conjugation, respectively, to $\Gamma_i^m, \bar{\Gamma}_i^m \subset \partial \Sigma_k$.

If we denote by $\Omega_k^*$ the vertical projection of $\Sigma_k^*$ over $\mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$, then

$$\partial \Omega_k^* = \bigcup_{m=1}^{m_0} \left( \Lambda_k^{m_0} \cup \Gamma_0^m(k)^* \cup \eta_0^m(k)^* \cup \Lambda_k^{m_0+} \right),$$
cyclically ordered, where

$$\Lambda_k^{m_0} = \begin{cases} \bar{\Gamma}_k^{-1}(k)^* \cup \eta_k^{-1}(k)^* \cup (\bigcup_{i=1}^{m_0} (\Gamma_i^m(k)^* \cup \eta_i^m(k)^*)) & , \text{ if } m \in \mathcal{M}^- \\ \bar{\Gamma}_k^{-1}(k)^* \cup \eta_1^m(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}$$

$$\Lambda_k^{m_0+} = \begin{cases} (\bigcup_{i=1}^{m_0} (\Gamma_i^m(k)^* \cup \eta_i^m(k)^*)) \cup \bar{\Gamma}_k^{-1}(k)^* \cup \eta_k^{-1}(k)^* & , \text{ if } m \in \mathcal{M}^+ \\ \bar{\Gamma}_k^{-1}(k)^* \cup \eta_k^{-1}(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}$$

and each $\eta_i^m(k)^*$ denotes a complete geodesic curve joining at $\partial_{\infty} \mathbb{H}^2$ the corresponding curves in $\partial \Sigma_k^*$. Furthermore, the curves $\Gamma_i^m(k)^*, \bar{\Gamma}_i^m(k)^*$ are strictly concave with respect to $\Omega_k^*$ (by the maximum principle).

Similarly, we denote by $\Sigma_{\infty}^*$ the conjugate surface of $\Sigma_{\infty}$.

**Proposition 3.4.** $\Sigma_{\infty}^* \subset \{ t \geq 0 \}$ is a minimal graph over a domain $\Omega_{\infty}^* \subset \mathbb{H}^2$. Moreover, $\partial \Sigma_{\infty}^* \subset \mathbb{H}^2 \times \{0\}$ consists of a collection of geodesic curvature lines,

$$\partial \Sigma_{\infty}^* = \bigcup_{m=1}^{m_0} \left( \Upsilon_\infty^{m_0} \cup \Gamma_0^m(k)^* \cup \Upsilon_\infty^{m_0+} \right),$$
cyclically ordered, where

$$\Upsilon_\infty^m = \begin{cases} \bigcup_{i=-\infty}^{m_0} \Gamma_i^m \cup \eta_i^m(k)^* & , \text{ if } m \in \mathcal{M}^- \\ \bar{\Gamma}_1^m & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}$$

$$\Upsilon_\infty^m = \begin{cases} \bigcup_{i=1}^{m_0} \Gamma_i^m \cup \eta_i^m(k)^* & , \text{ if } m \in \mathcal{M}^+ \\ \bar{\Gamma}_1^m & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}$$

Each $\Gamma_i^m(k)^*$ is strictly concave with respect to $\Omega_{\infty}^*$. Moreover,

$$\partial \Omega_{\infty}^* = \bigcup_{m=1}^{m_0} \left( \Lambda_{\infty}^{m_0} \cup \Gamma_0^m(k)^* \cup \eta_0^m(k)^* \cup \Lambda_{\infty}^{m_0+} \right).$$
(cyclically ordered), with

$$
\Lambda_m^-= \begin{cases} 
\bigcup_{i=-\infty}^{-1}(\Gamma_i^m \cup \eta_i^m) 
& \text{if } m \in \mathcal{M}^- \\
\overline{\Gamma}_m^* \cup \eta_{-1}^m 
& \text{if } m \in \mathcal{M} - \mathcal{M}^- 
\end{cases}
$$

$$
\Lambda_m^+= \begin{cases} 
\bigcup_{i=1}^{+\infty}(\Gamma_i^m \cup \eta_i^m) 
& \text{if } m \in \mathcal{M}^+ \\
\overline{\Gamma}_m^* \cup \eta_1^m 
& \text{if } m \in \mathcal{M} - \mathcal{M}^+ 
\end{cases}
$$

where $\eta_i^m$ denotes a complete geodesic curve asymptotic to its consecutive curves of $\partial \Omega_\infty$, at $\partial_\infty \mathbb{H}^2$.

**Proof.** Theorem 2.8 says $\Sigma_\infty^*$ is a minimal graph over certain domain $\Omega_\infty^* \subset \mathbb{H}^2$, because $\Sigma_\infty$ is a minimal graph over a convex domain. Moreover, since $\partial \Sigma_\infty = \bigcup_{i,m} \Gamma_i^m$ and each $\Gamma_i^m$ is a vertical geodesic curve, we get by Theorem 2.7 that the boundary of $\Sigma_\infty^*$ is composed of horizontal geodesic curvature lines $\Gamma_i^m^*$. But we do not know a priori if they are all contained in the same horizontal slice.

Let us prove that $\Sigma_\infty^*$ can be obtained as a limit of a subsequence of the conjugate graphs $\Sigma_k^*$, when $k$ goes to $+\infty$ (in which case the curves $\Gamma_i^m(k)^* \subset \partial \Sigma_\infty^*$ converge to $\Gamma_i^m^*$). This holds by [11, Proposition 2.10], but we give the idea of the proof: If we prove that, after passing to a subsequence, the graphs $\Sigma_k^*$ converge to a surface $S$, then up to isometries of $\mathbb{H}^2 \times \mathbb{R}$ we get $S = \Sigma_\infty^*$ by Theorem 6 in [6] (both $\Sigma_\infty^*, S$ are isometric to $\Sigma_\infty$; and the Hopf differentials associated to their vertical projection coincide with $-Q_\infty$, where $Q_\infty$ is the Hopf differential associated to the vertical projection of $\Sigma_\infty$). So we only have to obtain that the sequence $\{\Sigma_k^*\}$ converges. We know that the convergence domain associated to $\{u_k\}_k$ coincides with $\Omega_\infty$. Then, if we denote by $\nu_k$ the angle function of $\Sigma_k$, then $\{\nu_k\}_k$ is uniformly bounded away from zero on compact subsets. Since the angle function $\nu_k^*$ of $\Sigma_k^*$ coincides with the one of $\Sigma_k$, then the same happens for $\{\nu_k^*\}_k$. We deduce from here that there are no divergence lines for $\{u_k^*\}_k$, and we get the convergence of the graphs $\Sigma_k^*$, passing to a subsequence.

Since the graphs $\Sigma_k^*$ converge to $\Sigma_\infty^*$, then $\Sigma_\infty^* \subset \{t \geq 0\}$ and $\partial \Sigma_\infty^* \subset \{t = 0\}$. We also deduce that the curves $\Gamma_i^m^*$ are cyclically ordered as follows: $\Gamma_i^m^* \leq \Gamma_j^m^*$ if, and only if, $m < m'$ or $m = m'$ and $i \leq j$. By the maximum principle (using vertical geodesic planes), each $\Gamma_i^m^* \subset \partial \Omega_\infty^*$ is strictly concave with respect to $\Omega_\infty^*$.

Let us now prove that $\Gamma_i^m^*, \Gamma_{i+1}^m^*$ cannot finish at the same point $Q$ of $\partial_\infty \mathbb{H}^2$. Suppose this is the case. Since $\Gamma_i^m^*, \Gamma_{i+1}^m^*$ are strictly concave with respect to $\Omega_\infty^*$, we get $\text{dist}_{\mathbb{H}^2}(\Gamma_i^m^*, \Gamma_{i+1}^m^*) = 0$. Consider a triangle $T \subset \Omega_\infty^*$ bounded by subarcs of $\Gamma_i^m^*, \Gamma_{i+1}^m^*$ and a geodesic arc $c'$ joining points in $\Gamma_i^m^*, \Gamma_{i+1}^m^*$. Let $u^*_T : T \rightarrow \mathbb{R}$ define the graph.
Consider $\Sigma^*_\infty$ over $T$. Then $u^*_\infty$ has boundary values 0 on $\Gamma^{m,*}_{i}, \Gamma^{m,*}_{i+1}$ and a bounded continuous function over $c'$. We call $c$ the complete geodesic of $\mathbb{H}^2$ containing $c'$ and we consider the minimal graph $w^+$ (resp. $w^-$) over the component $\Delta$ of $\mathbb{H}^2 - c$ which contains $T$, which has boundary values $+\infty$ (resp. $-\infty$) over $c$ and 0 over $\partial \Delta \cap \partial_{\infty} \mathbb{H}^2$. By the maximum principle, $w^-|_T \leq u^*_\infty \leq w^+|_T$. Hence we deduce that $u^*_\infty$ converges to 0 as we approach $Q$ in any direction, and then $\text{dist}_{\Sigma^*_\infty}(\Gamma^{m,*}_{i}, \Gamma^{m,*}_{i+1}) = 0$. But $\Sigma^*_\infty, \Sigma^*_{\infty}$ are isometric and $\text{dist}_{\Sigma^*_\infty}(\Gamma^{m}_{i}, \Gamma^{m}_{i+1}) \geq \text{dist}_{\mathbb{H}^2}(p^m_{2i}, p^m_{2i+2}) > 0$, a contradiction.

Therefore, the geodesics $\eta^m_i(k)$ in the boundary of $\Omega^*_k$ converge to a geodesic $\eta^m_{\infty} \subset \partial \Omega^*_\infty$ over which $\Sigma^*_\infty$ goes to $+\infty$. Thus $\partial \Omega^*_\infty = \bigcup_{m=1}^{\partial \Omega^*_k} \left( \bigcup_{i=-\infty}^{+\infty} (\Gamma^{m,*}_{i} \cup \eta^m_{i}) \right)$, cyclically ordered. This finishes the proof of Proposition 3.4. 

If we reflect $\Sigma^*_\infty$ with respect to $\mathbb{H}^2 \times \{0\}$, we get a properly embedded minimal surface $M$ of genus zero and infinitely many planar ends in $\mathbb{H}^2 \times \mathbb{R}$. The non-limit ends of $M$ are asymptotic to the vertical geodesic planes $\eta^m_{\infty} \times \mathbb{R}$. We can deduce that there is exactly one limit end from $\eta^m_{0} \times \mathbb{R}$ to $\eta^m_{+\infty} \times \mathbb{R}$, that we call $E^m_{\infty}$, and $E^m_{\infty}$ is a left (resp. right, 2-sided) limit end when $p^m_{\infty}$ is a left (resp. right, 2-sided) limit ideal vertex.

4 Proof of Theorem 1.1: infinite countable case

In this section we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends $\{E^m_{\infty} | m \in \mathbb{N}\}$. Furthermore, as in the finite case, we can prescribe if each limit end $E^k_{\infty}$ is left, right or 2-sided.

We follow the same sketch as in section 3. We firstly construct, by taking limits of a monotone increasing sequence of convex Jenkins-Serrin semi-ideal polygonal domains $\Omega_k$ with finitely many vertices and satisfying condition ($\ast$), a convex semi-ideal polygonal domain $\Omega^*_\infty$ with an infinite countable number of limit ideal vertices $\{p^m_{\infty} | m \in \mathbb{N}\}$ such that, if $E^m_{\infty}$ is prescribed to be a left (resp. a right or a 2-sided) limit end, then $p^m_{\infty}$ is a left (resp. a right or a 2-sided) limit ideal vertex. The remaining part of the construction follows exactly as in Section 3, replacing $m_0$ by $+\infty$ and $\mathcal{M}$ by $\mathbb{N}$.

4.1 Construction of the domains

Consider $p^1_{\infty} = \infty$ and two ideal points $p^2_{\infty} = (x^2_{\infty}, 0)$ and $p^3_{\infty} = (x^3_{\infty}, 0)$, with $-1 < x^2_{\infty} < x^3_{\infty} \leq 1$. These points will be limit ideal vertices of $\Omega^*_\infty$. We call $C^1_{\infty} = \{y = 1\}$ and $C^2_{\infty}$ (resp. $C^3_{\infty}$) the horocycle at $p^2_{\infty}$ (resp. $p^3_{\infty}$) passing through $P_0 = (0,1)$. 

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Let \( q^3 \) be the point in \( C_{3}^\infty \cap C_{1}^\infty \) different from \( P_0 \), \( \alpha^3 \) be the complete geodesic curve with endpoint \( p^3_\infty \) passing through \( q^3 \) and \( \beta^3 = \{ x = b^3 \} \), where \( b^3 \) is the constant for which \( \beta^3 \) passes through \( q^3 \). We take any point \( p^4_\infty = (x^4_\infty, 0) \) with \( b^3 \leq x^4_\infty \leq a^3 \), where \((a^3, 0)\) is the endpoint of \( \alpha^3 \) different from \( p^3_\infty \), see Figure 6.

Let us now define by induction the remaining limit ideal vertices \( p^k_\infty \), \( k \in \mathbb{N} \), of \( \Omega^\infty \). Assume we have define \( p^4_\infty = (x^4_\infty, 0), \ldots, p^{k}_\infty = (x^{k}_\infty, 0) \), with \( x^{3}_\infty < x^4_\infty < \ldots < x^{k}_\infty < +\infty \). For any \( 4 \leq i \leq k \), let \( C^{i}_\infty \) be the horocycle at \( p^i_\infty \) passing through \( P_0 = (0, 1) \), and \( q^i \) be the point in \( C^{i}_\infty \cap C_1^\infty \) different from \( P_0 \). We also consider the complete geodesic curve \( \alpha^i \) with endpoint \( p^i_\infty \) passing through \( q^i \) and \( \beta^i = \{ x = b^i \} \) the geodesic which contains \( q^i \). Denote by \((a^i, 0)\) the endpoint of \( \alpha^i \) different from \( p^i_\infty \). We assume that \( b^{i-1} \leq x^i_\infty \leq a^{i-1} \).

We now define \( p^{k+1}_\infty \) with the same property: We take any point \( p^{k+1}_\infty = (x^{k+1}_\infty, 0) \) such that \( b^k \leq x^{k+1}_\infty \leq a^k \).

We now want to define the non-limit ideal vertices of \( \Omega^\infty \). We consider:

\[ \mathcal{M}^+ = \{ m \in \mathbb{N} \mid E_{m+1}^{\infty} \text{ is prescribed to be either a right or a 2-sided limit end} \}, \]

\[ \mathcal{M}^- = \{ m \in \mathbb{N} \mid E_m^{\infty} \text{ is prescribed to be either a left or a 2-sided limit end} \}. \]

For any \( m \in \mathbb{N} \), we define exactly as in Subsection 3.1 (using this new definition of the sets \( \mathcal{M}^+, \mathcal{M}^- \)) the ideal vertices \( p^{m}_{2i-1} \) of \( \Omega^\infty \) placed from \( p^{m}_\infty \) to \( p^{m+1}_\infty \), the horocycles \( C^{m}_{2i-1} \), the interior vertices \( p^{m}_{2i} \) and the interior points \( q^{m}_{2i} \).

We can now define the monotone sequence of semi-ideal Jenkins-Serrin polygonal domains \( \Omega_k \): We call \( \Omega_1 \) the semi-ideal polygonal domain with vertices

\[ \{ p^{m}_\infty, q^{m}_{-2}, p^{m}_{-1}, p^{m}_{0}, p^{m}_{1}, q^{m}_{2} \mid 1 \leq m \leq 2 \} \cup \{ p^3_\infty, q^3 \}. \]
Figure 7: Sketch of the vertices of $\Omega_2$ in the case $p_1^1$ and $p_3^1$ are right limit ideal vertices, $p_2^2$ is a 2-sided limit ideal vertex and $p_4^4$ is a left limit ideal vertex.

For $k \geq 2$, let $\Omega_k$ be defined as the semi-ideal polygonal domain with vertices

$$\{p_\infty^m, p_{-1}^m, p_0^m, p_1^m \mid 1 \leq m \leq k + 1\} \cup V^-_k \cup V^+_k \cup \{p_{k+2}^k, q_{k+2}^k\},$$

where

$$V^-_k = \{q_{-2k}^m, p_{-2k}^m, p_{-2k}^m, \ldots, p_{-3}^m, p_{-2}^m \mid 1 \leq m \leq k + 1, m \in M^-\} \cup \{q_{-2}^m \mid 1 \leq m \leq k + 1, m \in M - M^-\},$$

$$V^+_k = \{p_2^m, p_3^m, \ldots, p_{2k-2}^m, p_{2k-1}^m, q_{2k}^m \mid 1 \leq m \leq k + 1, m \in M^+\} \cup \{q_{2}^m \mid 1 \leq m \leq k + 1, m \in M - M^+\}.$$  

The domain $\Omega_k$ has $4(k + 1) + 2 + N$ vertices, where $2(k + 1) \leq N \leq 2(k + 1)(2k - 1)$ depends on the number of left, right or 2-sided limit ideal ends in $\{E_1^1, \ldots, E^{k+1}_\infty\}$. As in Subsection 3.1, $\Omega_k$ is a convex Jenkins-Serrin semi-ideal polygonal domain satisfying condition $(*),$ and $\Omega_k \subset \Omega_{k+1}$. When $k$ goes to $+\infty$, $\Omega_k$ converges to the convex semi-ideal polygonal domain $\Omega_{\infty}$ with set of vertices $V^-_{\infty} \cup V^0_{\infty} \cup V^+_{\infty}$, where

$$V^-_{\infty} = \{p_{-2k}^m, p_{-2k}^m \mid k \in \mathbb{N}, m \in M^-\} \cup \{q_{-2}^m \mid m \in M - M^-\},$$

$$V^0_{\infty} = \{p_\infty^m, p_{-1}^m, p_0^m, p_1^m \mid m \in \mathbb{N}\}$$

$$V^+_{\infty} = \{p_{2k-1}^m, p_{2k}^m \mid k \in \mathbb{N}, m \in M^+\} \cup \{q_{2}^m \mid m \in M - M^+\}.$$  

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