Dilatonic Parallelizable NS-NS Backgrounds

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We complete the classification of parallelizable NS-NS backgrounds in type II supergravity by adding the dilatonic case to the result of Figueroa-O’Farrill on the non-dilatonic case. We also study the supersymmetry of these parallelizable backgrounds. It is shown that all the dilatonic parallelizable backgrounds have sixteen supersymmetries.
1. Introduction

The parallelizable pp-wave backgrounds have been recently discussed in [1]. They are a class of the non-dilatonic NS-NS backgrounds in type II supergravity. Immediately, Figueroa-O’Farrill has classified all the parallelizable NS-NS backgrounds with the dilaton field turned off [2]. He also classified all the ten-dimensional parallelizable NS-NS field configurations which do not necessarily satisfy the equations of motion of type II supergravity but with a closed torsion in the same paper [2]. This is the classification of the ten-dimensional parallelized Lorentzian Lie groups with a bi-invariant metric.

In this short note, we will turn on the dilaton field to classify all the dilatonic parallelizable NS-NS backgrounds in type II supergravity. Our strategy is simple: we will find the solution of the dilaton field to the equations of motion which is facilitated by the parallelizability of the geometry. It turns out that all the manifolds in the classification of [2] except for $AdS_3 \times \mathbb{R}^7$ and $CW_4(\lambda) \times S^3 \times S^3$ can be dilatonic parallelizable NS-NS backgrounds and are invariant under half of the supersymmetries. The result in this note together with the classification in [2] are summarized in Table 1. To our knowledge, among the dilatonic backgrounds in Table 1, it seems that the case $\mathbb{R}^{1,1} \times SU(3)$ haven’t appeared in the literature.

The parallelizable NS-NS backgrounds are a simple class of curved spacetimes [3,4,5]. The propagation of strings on them can be described by the conformal field theory, in particular by the Wess-Zumino-Novikov-Witten models. Some backgrounds are interpreted as the near horizon geometry of the (intersecting) NS5-branes and/or NS1-branes. However, not all the backgrounds have been given any interpretation in terms of the brane configuration. Thus, the above $\mathbb{R}^{1,1} \times SU(3)$ might deserve further study. Since the NS-NS part of the supergravity action in this note is the same as the bosonic action of heterotic supergravity with no Yang-Mills background fields, one may be able to generalize the results of this note as well as [2] to the heterotic parallelizable backgrounds.

This note is organized as follows: in section 2, imposing the parallelizability on the metric and the NS three-form in the equations of motion of type II supergravity, we find a system of equations which the dilaton field has to satisfy. In section 3, the number of the supersymmetries is examined for the dilatonic solutions. The main results of this note are summarized in section 4.
2. Dilatonic Parallelizable NS-NS Backgrounds

The NS-NS part of the ten-dimensional type II supergravity action is given by

$$S_{\text{NSNS}} = \frac{1}{l_s^8} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[ R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right],$$

yielding the equations of motion

$$\nabla_\rho (e^{-2\phi} H^{\rho\mu\nu}) = 0,$$

$$\nabla_\mu \nabla_\nu e^{-2\phi} = e^{-2\phi} \left( R_{\mu\nu} + 4\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma} \right),$$

$$\nabla_\rho \nabla^\rho e^{-2\phi} = \frac{1}{6} e^{-2\phi} H_{\mu\nu\rho} H^{\mu\nu\rho},$$

where $\nabla_\mu$ is the covariant derivative with respect to the Levi-Civita connection $\Gamma^\nu_{\mu\rho}$. The field strength $H_{\mu\nu\rho}$ of the NS-NS two-form potential $B_{\mu\nu}$ is given as usual by $H = dB$. Henceforth, we assume that $H_{\mu\nu\rho}$ is closed, i.e., $dH = 0$.

Imposing on the metric $g_{\mu\nu}$ and the field strength $H_{\mu\nu\rho}$ the Ricci-parallelizability

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma} = 0,$$

$$\nabla_\rho H^{\rho\mu\nu} = 0,$$

one find that the equations of motion (2.2) are reduced to

$$H^{\mu\nu\rho} \nabla_\rho e^{-2\phi} = 0,$$

$$\nabla_\mu \nabla_\nu \phi = 0,$$

$$\nabla_\rho \phi \nabla^\rho \phi = \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}.$$

It follows from the above equations (2.4) that non-dilatonic background solutions are viable, if $H_{\mu\nu\rho} H^{\mu\nu\rho}$ is vanishing, or equivalently if the scalar curvature $R$ is zero. This is the case for the parallelizable pp-waves [1,2]. Instead, if $H_{\mu\nu\rho} H^{\mu\nu\rho}$ is a non-zero constant, one can easily solve the equations (2.4) with the linear dilaton field $\phi$ in the direction transverse to $H_{\mu\nu\rho}$. The purpose of this note is to consider the latter case in detail.

In [2], Figueroa-O’Farrill have given the classification of not only the non-dilatonic parallelizable solutions of the supergravity, but also ten-dimensional parallelizable space-times, as in Table 1. Since the scalar curvature of the latter case is non-zero, with the help of the above linear dilaton field, all the members of the latter case, except for the two $AdS_3 \times \mathbb{R}^7$ and $CW_4(\lambda) \times S^3 \times S^3$ become classical solutions of type II supergravity.
Let us look at the dilaton gradient more closely. According to [2], parallelizable spacetimes are composed of some of the Minkowski space, a flat space, $CW_{2n}(\lambda)$ ($n = 2, \cdots, 5$), $AdS_3$, $S^3$, and SU(3) (See table 1). These components are classified into the following three classes:

1. simple groups: $AdS_3$, $S^3$, and SU(3)
2. pp-waves: $CW_{2n}(\lambda)$.
3. flat spacetimes: $\mathbb{R}^{1,m}$, $\mathbb{R}^m$

First, as for the simple group manifolds, since the torsion of the parallelizing connection is proportional to the structure constants of the groups, one cannot introduce the dilaton gradient tangent to the above group manifolds, because of the first equation of (2.4).

Secondly, let us turn to the pp-waves $CW_{2n}(\lambda)$. The metric of this case is written [1,2] as

$$ds^2 = -2dudv - \sum_{i,j=1}^{2n-2} \mu_{ij} x^i x^j du^2 + \sum_{i=1}^{2n-2} dx^i dx^i, \quad \mu_{ij} = \frac{1}{4} \sum_{k=1}^{2n-2} h_{ik} h_{jk},$$

$$H = \frac{1}{2} \sum_{i,j=1}^{2n-2} h_{ij} dx^i \wedge dx^j \wedge du. \quad (2.5)$$

The matrix $h_{ij}$ is assumed to be invertible. Then, the first equation of (2.4) implies that the dilaton gradient can be introduced in the $u$ direction. Moreover, the dilaton must be linear in $u$, because of the second equation of (2.4). Note that this linear dilaton in the $u$ direction doesn’t contribute to the third equation of (2.4), since the $u$ direction is light-like.

Finally, we consider the dilaton field in the flat spacetime. The second equation of (2.4) requires that the dilaton field be linear in the coordinates. If the dilaton gradient is time-like, the left-hand side of the third equation of (2.4) is negative. $H_{\mu\nu\rho} H^{\mu\nu\rho}$ however never becomes negative in this case since the time-like component of $H_{\mu\nu\rho}$ is zero in the flat spacetime. As a result, the third equation of (2.4) cannot be satisfied with the time-like dilaton gradient. On the other hand, the null dilaton gradient doesn’t contribute to $\nabla_\rho \phi \nabla^\rho \phi$ in the third equation of (2.4) and can be turned on. The most interesting case is the space-like linear dilaton. In this case, the dilaton field can balance the contribution from $H_{\mu\nu\rho} H^{\mu\nu\rho}$ in the third equation of (2.4) to give the solutions.

In summary, the dilaton can be introduced only linearly in the $u$ direction of $CW_{2n}(\lambda)$, in the light-like direction of $\mathbb{R}^{1,m}$, and in the space-like of $\mathbb{R}^n$ as well as $\mathbb{R}^{1,n}$. 
| spacetime                        | non-dilatonic SUSYs | dilatonic SUSYs |
|---------------------------------|---------------------|-----------------|
| $AdS_3 \times S^3 \times S^3 \times \mathbb{R}$ | 16                  | 16              |
| $AdS_3 \times S^3 \times \mathbb{R}^4$ | 16                  | 16              |
| $AdS_3 \times \mathbb{R}^7$        | $\times$            |                 |
| $CW_{10}(\lambda)$               | 16, 18(A), 20, 22(A), 24(B), 28(B) | 16              |
| $CW_8(\lambda) \times \mathbb{R}^2$ | 16, 20              | 16              |
| $CW_6(\lambda) \times S^3 \times \mathbb{R}$ | $\times$            |                 |
| $CW_6(\lambda) \times \mathbb{R}^4$ | 16, 24              | 16              |
| $CW_4(\lambda) \times S^3 \times S^3$ | $\times$            | $\times$        |
| $CW_4(\lambda) \times S^3 \times \mathbb{R}^3$ | $\times$            | 16              |
| $CW_4(\lambda) \times \mathbb{R}^6$ | 16                  | 16              |
| $\mathbb{R}^{1,1} \times SU(3)$    | $\times$            | 16              |
| $\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$ | $\times$            | $\times$        |
| $\mathbb{R}^{1,3} \times S^3 \times S^3$ | $\times$            | 16              |
| $\mathbb{R}^{1,6} \times S^3$      | $\times$            | 16              |
| $\mathbb{R}^{1,9}$                | 32                  | 16              |

*Table 1: the classification of ten-dimensional Lorentzian parallelizable spaces in [3]. (A) and (B) indicate the cases occurring only in type IIA and IIB supergravity, respectively. The cross $\times$ indicates that the case in question is not the supergravity background.

3. Supersymmetry of the Dilatonic Parallelizable Backgrounds

The supersymmetry transformation of type II supergravity with no R-R background fields is given by

$$ \delta \psi_\mu = \left[ \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} + \frac{1}{8} H_{\mu ab} \Gamma^{11} \Gamma^{ab} \right] \epsilon, $$

$$ \delta \lambda = \left[ \Gamma^\mu \nabla_\mu \phi + \frac{1}{12} \Gamma^{abc} H_{abc} \Gamma^{11} \right] \epsilon, $$

(3.1)
for type IIA supergravity, and

$$\delta \psi_\mu = \left[ \partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \Gamma_{ab} + \frac{1}{8} H_{\mu ab} \Gamma^{ab} \sigma_3 \right] \epsilon, \quad (3.2)$$

$$\delta \lambda = \left[ \Gamma^\mu \nabla_\mu \phi + \frac{1}{12} \Gamma^{abc} H_{abc} \sigma_3 \right] \epsilon,$$

for type IIB supergravity, where $\sigma_3$ is the generator of the $SL(2, \mathbb{R})$ symmetry of type IIB supergravity.

The supersymmetry transformation for the gravitinos $\psi$ imposes on the Killing spinors either of the conditions

$$\left[ \partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \Gamma_{ab} \pm \frac{1}{8} H_{\mu ab} \Gamma^{ab} \right] \epsilon = 0. \quad (3.3)$$

Since the parallelizability implies that the curvature tensor including the torsion vanishes, it means that the holonomy is locally trivial \[1,2\]. In this note, since our attention will be restricted to simply connected spaces, the locally trivial holonomy guarantees the existence of the solutions to (3.3). Thus, one concludes that the condition (3.3) impose no conditions on the Killing spinors, as in the non-dilatonic case \[1,2\].

We proceed to the supersymmetry transformation for the dilatinos $\lambda$, which imposes on the Killing spinors either of the conditions

$$\left( \Gamma^\mu \nabla_\mu \phi \pm \frac{1}{12} \Gamma^{abc} H_{abc} \right) \epsilon = 0. \quad (3.4)$$

Since the parallelizability means that all the members listed in Table 1 are group manifolds, the field strength $H_{\mu \nu \rho}$ of the parallelizing connections is proportional to the structure constants of the corresponding groups. Therefore, by making use of the Jacobi identity of the structure constants, one find that

$$\left( \Gamma^{abc} H_{abc} \right)^2 = -6 H_{\mu \nu \rho} H^{\mu \nu \rho}. \quad (3.5)$$

First, we consider the case with non-zero $\nabla_\mu \phi \nabla^\mu \phi$. Since $\nabla_\mu \phi$ is transverse to $H_{\mu \nu \rho}$, by multiplying $\Gamma^\mu \nabla_\mu \phi$ with both sides of (3.4), one obtain

$$0 = \left( \nabla_\rho \phi \nabla^\rho \phi \pm \frac{1}{12} \Gamma^{\sigma \mu \nu \lambda} \nabla_\sigma \phi \cdot H_{\mu \nu \lambda} \right) \epsilon \equiv 2 (\nabla_\rho \phi \nabla^\rho \phi) \mathcal{P}_\pm \epsilon, \quad (3.6)$$

1 More careful examination is needed to find the solutions to (3.3) for the discrete quotients of the spaces under consideration. A simple example is provided by the $S^1$ compactified parallelizable pp-waves \[1\], where the condition (3.3) actually reduces the number of the Killing spinors.
where the operators $P_\pm$ are the projectors satisfying that $(P_\pm)^2 = P_\pm$ and that $P_+ + P_- = 1$. These projectors also satisfy $\text{Tr} \left[ \frac{1}{2} (1 \pm \Gamma^{11}) P_\pm \right] = 8$, where the first signs $\pm$ are not related to the second ones. The projectors $P_\pm$ guarantee that half of the supersymmetries survive the condition (3.6).

Next, we turn to the case of the null linear dilatonic background, i.e., $\nabla_\mu \phi \nabla^\mu \phi = 0$. This type of the backgrounds is $CW_{2n}(\lambda) \times \mathbb{R}^{10-2n}$ or $\mathbb{R}^{1,9}$, where only the non-zero component $\nabla_u \phi$ is constant. The dilatino condition (3.4) reads

$$\left( \nabla_u \phi \pm \frac{1}{4} \Gamma^{ij} h_{ij} \right) \Gamma^+ \epsilon = 0. \quad (3.7)$$

Here, the matrix $\Gamma^{ij} h_{ij}$ is an anti-hermitian matrix and its eigenvalues are pure imaginary or zero. This means that the matrix $\left( \nabla_u \phi \pm (1/4) \Gamma^{ij} h_{ij} \right)$ is invertible. It follows from this fact that the ones satisfying that $\Gamma^+ \epsilon = 0$ are the solutions to (3.7): the backgrounds in this case preserve sixteen supersymmetries.

Thus, all the dilatonic solutions are invariant under sixteen supersymmetries.

4. Summary

The main result of this paper is summarized in Table 1. In this section, we will try to elaborate a little more on it.

4.1. $AdS_3 \times M_7$

The anti-de Sitter space $AdS_3$ can be parallelizable. Since the scalar curvature of the anti-de Sitter space is negative, the case $M_7 = \mathbb{R}^7$ cannot be the background of the supergravity. In this case, $H_{\mu \nu \rho} H^{\mu \nu \rho}$ is also negative. As is seen from the last equation of (2.4), the dilaton field $\phi$ would be imaginary. In the two cases of $M_7 = S^3 \times S^3 \times \mathbb{R}$ and $M_7 = S^3 \times \mathbb{R}^4$ [8,9], since $S^3 \simeq SU(2)$ is parallelizable and the curvature of $S^3$ is positive, one can adjust the radii of the three-spheres to set the total scalar curvature to zero. Then, they become the non-dilatonic backgrounds of the supergravity. Even if the contribution of the three-spheres to the total scalar curvature overcomes the one of the $AdS_3$ space, the dilatonic solutions are viable with the space-like linear dilaton field $\phi$. Actually, in the case of $M_7 = S^3 \times S^3 \times \mathbb{R}$, the relation of the radii and the dilaton becomes

$$\nabla_\mu \phi \nabla^\mu \phi + \frac{1}{L_0^2} - \frac{1}{L_1^2} - \frac{1}{L_2^2} = 0, \quad (4.1)$$
where $L_0$ is the radius of $AdS_3$, and $L_1, L_2$ are the radii of two $S^3$. The relation for the case with $M_7 = S^3 \times \mathbb{R}^4$ is obtained by the limit $L_2 \to \infty$ in (4.1). The parallelizing connections of $AdS_3 \simeq SL(2, \mathbb{R})$ and $S^3 \simeq SU(2)$ include the torsion proportional to the structure constant of the corresponding groups. Therefore, as is seen from the equations in (2.4), the dilaton field $\phi$ can linearly depend only on the coordinate in the direction normal to $AdS_3$ and $S^3$. Whether they are dilatonic or not, they are invariant under half of the supersymmetries, as explained above.

4.2. $CW_{2n}(\lambda) \times M_{10−2n}$

The field strength $H_{\mu\nu\rho}$ of the parallelizable pp-waves $CW_{2n}(\lambda)$ doesn’t contribute to $H_{\mu\nu\rho}H^{\mu\nu\rho}$. Indeed, the parallelizable pp-wave backgrounds $CW_{2n}(\lambda)$ are given [1,2] by (2.3). In the case of $CW_4(\lambda) \times S^3 \times \mathbb{R}^3$ and $CW_6(\lambda) \times S^3 \times \mathbb{R}$, thus, the only contribution to $H_{\mu\nu\rho}H^{\mu\nu\rho}$ comes from the three-sphere $S^3$. The first equation of (2.4) allows the dilaton gradient $\nabla_\mu \phi$ to have a non-zero $u$-component and non-zero components in the spatial direction normal to $CW_{2n}(\lambda)$ and $S^3$. However, $\nabla_u \phi$ doesn’t contribute to $\nabla_\rho \phi \nabla^\rho \phi$ while $\nabla_v \phi$ is kept zero. This is the reason why $CW_4(\lambda) \times S^3 \times S^3$ isn’t the NS-NS background of the supergravity. In the case in question, there is the factor $\mathbb{R}$, in the direction of which the dilaton field can be turned on to yield the solution to (2.4).

For the case $M_{10−2n} = \mathbb{R}^{10−2n}$, one can find the non-dilatonic backgrounds [1,2], where the diverse numbers of supersymmetries are available. Furthermore, by turning on the null linear dilaton field, one obtain the dilatonic solutions, which are invariant under half of the supersymmetries.

4.3. $\mathbb{R}^{1,n} \times M_{9−n}$

As is seen from Table 1, $M_{9−n}$ can be $SU(3)$, $S^3 \times S^3$ [10], and $S^3$ [11,12] to yield the supergravity backgrounds. Obviously, the ten-dimensional Minkowski spacetime $\mathbb{R}^{1,9}$ is also the solution. All in the former case have the positive scalar curvature. The field strength $H_{\mu\nu\rho}$ given by the parallelizing connection is proportional to the structure constant of the respective groups. Therefore, the space-like linear dilaton is needed to be turned on in the direction tangent to $\mathbb{R}^n$. Although the ordinary Minkowski spacetime $\mathbb{R}^{1,9}$ is the supergravity solution with all the thirty-two supersymmetries, one can see that the Minkowski spacetime $\mathbb{R}^{1,9}$ with the null linear dilaton is also the solution with half of the supersymmetries.
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Note added: after publishing this paper in Phys. Lett. B568 (2003) 78-82, we were informed from José Figueroa-O’Farrill that the parallelizable manifold $\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$ should be included in the original Table 1.

On this manifold, it follows from the first equation in (2.4) that only the time-like dilaton gradient can be turned on, which can only give the negative contribution to the left-hand side of the third equation in (2.4). Therefore, since the right-hand side of the same equation is positive, the manifold is not a solution to the equations of motion. Thus, our result on the classification of the dilatonic parallelizable backgrounds remains intact.
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