DIFFERENCE OF COMPOSITION-DIFFERENTIATION OPERATORS
FROM HARDY SPACES TO WEIGHTED BERGMAN SPACES VIA
HARMONIC ANALYSIS

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ABSTRACT. In this paper, the boundedness and compactness of the difference of
composition-differentiation operators $D_ϕ − D_ψ$ acting from Hardy spaces $H^p$ to
weighted Bergman spaces $A^q_α$ are completely characterize for all $0 < p, q < ∞$.

Keywords: Hardy space; weighted Bergman space; composition operator; difference; norm.

1. INTRODUCTION

Let $D$ and $\partial D$ denote the open unit disk and the unit circle of the complex plane
$\mathbb{C}$, respectively. In the sequel, $dm = \frac{dm}{2\pi}$ will be the normalized Lebesgue measure
on $\partial D$. The Lebesgue space $L^p(\partial D, m)$ will also be denoted by $L^p(\partial D)$. We denote
by $H(D)$ the class of all functions analytic in $D$.

For $0 < p < ∞$, let $H^p$ be the classical Hardy space of all $f \in H(D)$ such that
$$
\|f\|_p^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{iθ})|^p r dθ < ∞.
$$

Given $α > −1$ and $0 < p < ∞$, a function $f \in H(D)$ belongs to the weighted
Bergman spaces $A^p_α$, if
$$
\|f\|_A^p = \int_D |f(z)|^p dA_α(z) < ∞.
$$
Here $dA$ is the Lebesgue measure on $D$, normalized so that $A(D) = 1$. The measure
$dA_α$ is given by $dA_α = (α + 1)(1 − |z|^2)^α dA(z)$. We refer the reader to [4–6, 27, 28]
for more information of Hardy spaces and weighted Bergman spaces.

Let $ϕ$ be an analytic self-map of $D$. The composition operator $C_ϕ$ on $H(D)$ is
defined by
$$
C_ϕ f = f \circ ϕ, \ f \in H(D).
$$
Let $D_ϕ$ denote the operator given by
$$
D_ϕ f = f' \circ ϕ, \ f \in H(D).
$$
It has been of growing interest to study the difference of composition operators for
the last three decades. The question of when the difference of two composition
operators is compact on Hardy spaces $H^p$ was posed by Shapiro and Sundberg
[23] in 1990. The difference of composition operators has been studied by many
authors on several function spaces. See [7,9,10,12,24–26] for example. In 2004, Nieminen and Saksman [16] showed that the compactness of $C_\varphi - C_\psi$ on $H^p$, with $1 \leq p < \infty$, is independent of $p$. Very recently, in [2], Choe et al. completely characterized compact operators $C_\varphi - C_\psi$ on $H^p$ by using Carleson measures of Bergman spaces. Moorhouse [15] characterized the compactness of $C_\varphi - C_\psi$ on the standard weighted Bergman space $A^2_\alpha$. Saukko [21,22] generalized Moorhouse’s results by characterizing the boundedness and compactness from $A^p_\alpha$ to $A^q_\beta$. But Saukko’s proof contains some serious problems. Very recently, in [11], Lindström, Saukko and the first author of this paper have corrected Saukko’s proof.

In the context of analytic functions on $\mathbb{D}$, it also seems reasonable to consider the difference of composition-differentiation operators. The aim of this paper is to characterize the boundedness and compactness of the difference of composition-differentiation operators $D_\varphi - D_\psi : H^p \to A^q_\alpha$. The Littlewood-Paley formula of standard Bergman spaces

$$\|f\|_{A^p_\alpha} \approx \sum_{j=0}^{n-1} |f^{(j)}(0)| + \|f^{(n)}\|_{A^{p\alpha}_{p\alpha}}$$

implies that the analogous problem for $D_\varphi - D_\psi : A^p_\alpha \to A^q_\beta$ is equivalent to characterize the boundedness and compactness of the difference of composition operators $C_\varphi - C_\psi : A^p_{\alpha+p} \to A^q_\beta$, which have been solved in [11,15,21,22]. However, this method does not work for our setting, because such a Littlewood-Paley formula does not exist for Hardy spaces when $p \neq 2$.

Let us now recall some definitions that will enable us to state the solution to the primary question of this paper. For $a \in \mathbb{D}$, let $\sigma_a(z) = \frac{a-z}{1-\overline{z}a}$ be the disc automorphism that exchanges 0 for $a$. For two points $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance is given by $\rho(z, w) = |\sigma_w(z)| = |\frac{w-z}{1-\overline{z}w}|$. We denote $\Delta(a, r) = \{z \in \mathbb{D} : |\sigma_a(z)| < r\}$ by the pseudo-hyperbolic disk centered at $a$ with radius $r$. Given two holomorphic self-maps $\varphi, \psi$ of $\mathbb{D}$, we put $\rho(z) = \rho(\varphi(z), \psi(z))$ for short. For $0 < p, q < \infty$ and $-1 < \alpha < \infty$, the joint pull-back measure $\mu$ is defined by

$$\mu(E) := \mu_{q,\alpha,\varphi,\psi}(E) = \int_{\varphi^{-1}(E)} \rho(z)^q dA_\alpha(z) + \int_{\psi^{-1}(E)} \rho(z)^q dA_\alpha(z)$$

(1)

for any Borel set $E \subset \mathbb{D}$. Here and henceforth $\mu$ denotes the joint pull-back measure defined as above. $\mu$ is actually the sum of two pull-back measures $\rho^q dA_\alpha \circ \varphi^{-1}$ and $\rho^q dA_\alpha \circ \psi^{-1}$. By a standard argument one can verify that

$$\int_{\mathbb{D}} gd\mu = \int_{\mathbb{D}} (g \circ \varphi + g \circ \psi) \rho^q dA_\alpha$$

(2)

for any positive Borel function $g$ on $\mathbb{D}$.

We state our main result in this paper as follows.
Theorem 1. Suppose $\varphi$ and $\psi$ are analytic self-maps of $\mathbb{D}$. Let $0 < r < 1$, $0 < p, q < \infty$, and let $\mu$ denote the joint pull-back measure induced by $\varphi$ and $\psi$ defined as (1).

(i) If either $0 < p < q < \infty$ or $2 \leq p = q$, then

$$(ia) D_\varphi - D_\psi : H^p \to A^q_d$$

is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{2}{p} + q}} < \infty.$$ 

$(ib) D_\varphi - D_\psi : H^p \to A^q_d$ is compact if and only if

$$\lim_{n \to 1} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{2}{p} + q}} = 0.$$ 

(ii) If $q < \min\{2, p\}$, then the following statements are equivalent:

$(iia) D_\varphi - D_\psi : H^p \to A^q_d$ is bounded;

$(iib) D_\varphi - D_\psi : H^p \to A^q_d$ is compact;

$(iic) \zeta \mapsto \left(\int_{\Gamma(\zeta)} \left(\frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+p}}\right)^{\frac{2}{p}} dA(z) \frac{(1 - |z|^2)^{1+p}}{1 - |z|}\right)^{\frac{2}{2p}}$

belongs to $L^p(\partial \mathbb{D}, m)$.

(iii) If $0 < q = p < 2$, then

$(iii) D_\varphi - D_\psi : H^p \to A^q_d$ is bounded if and only if

$$\left(\frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+p}}\right)^{\frac{2}{p}} dA(z) \frac{(1 - |z|^2)^{1+p}}{1 - |z|}$$

is a Carleson measure.

$(iii) D_\varphi - D_\psi : H^p \to A^q_d$ is compact if and only if

$$\left(\frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+p}}\right)^{\frac{2}{p}} dA(z) \frac{(1 - |z|^2)^{1+p}}{1 - |z|}$$

is a vanishing Carleson measure.

(iv) If $2 \leq q < p < \infty$, then

$(iv) D_\varphi - D_\psi : H^p \to A^q_d$ is bounded if and only if the function

$$\zeta \mapsto \sup_{z \in \Gamma(\zeta)} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}}$$

belongs to $L^{\frac{p}{p-q}}(\partial \mathbb{D}, m)$.

$(iv) D_\varphi - D_\psi : H^p \to A^q_d$ is compact if and only if the function

$$\zeta \mapsto \sup_{z \in \Gamma(\zeta) \cap \{z \geq r\}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}}$$

converges to zero in $L^{\frac{p}{p-q}}(\partial \mathbb{D}, m)$ as $r \to 1^-$. 
The present paper is organized as follows. In Section 2, we collect some notations and standard facts that will be used in Section 3. In Section 4, we give the proof of Theorem 1. For two quantities $A$ and $B$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $c$ (independent of the associated variables) such that $A \leq cB$. We write $A \preccurlyeq B$, if $A \lesssim B \lesssim A$. Given $p \in [1, \infty]$, we will denote by $p' = \frac{p}{p-1}$ its Hölder conjugate. In this paper we agree that $1' = \infty$ and $\infty' = 1$.

2. PREREQUISITES

In this section, we collect some well-known results that will be used throughout the paper.

2.1. Carleson measures. We recall that a positive Borel measure $\mu$ on $\mathbb{D}$ is a Carleson measure if there is a positive constant $C > 0$ such that
\[
\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C\|f\|_{H^p}^p
\]
for any $f \in H^p$. It is well known that, $\mu$ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $C > 0$ such that
\[
\mu(Q_r(\zeta)) \leq Cr
\]
for all $\zeta \in \partial \mathbb{D}$ and $r > 0$, where $Q_r(\zeta) = \{z \in \mathbb{D} : |1 - \overline{\zeta}z| < r\}$ is called a Carleson region at $\zeta$. Obviously every Carleson measure is finite. We say that a positive finite Borel measure $\mu$ is a vanishing Carleson measure if for any sequence $\{f_k\}$ in $H^p$ with $\|f\|_{H^p} \leq 1$ and $f_k \to 0$ uniformly on any compact subset of $\mathbb{D}$,
\[
\lim_{k \to \infty} \int_{\mathbb{D}} |f_k(z)|^p d\mu(z) = 0.
\]
It is well known that, a positive finite Borel measure $\mu$ is a vanishing Carleson measure if and only if
\[
\lim_{r \to 0} \frac{\mu(Q_r(\zeta))}{r} = 0
\]
uniformly for $\zeta \in \mathbb{D}$. We refer the reader to [4, 6, 27, 28] for more information of Carleson measure and vanishing Carleson measure.

2.2. Separated sequences and lattices. Recall that a sequence $\{z_k\} \subset \mathbb{D}$ is called a $\delta$-separated sequence if there exists $\delta > 0$ such that $\rho(z_i, z_j) \geq \delta$ for all $i$ and $j$ with $i \neq j$. A sequence $\{z_k\} \subset \mathbb{D}$ is called a $\delta$-lattice in the pseudo-hyperbolic metric if it is $\delta$-separated and $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \delta)$. The following result is similar with [14, Lemma 4].

Lemma 2. Assume $0 < r < 1$. Then there exists a $\delta$-lattice $\{z_k\}$ such that $\inf_k |z_k| \geq r$ for some $0 < \delta < 1$. 

Proof. Denote $\delta = \frac{2r}{r_1}$. Let $z_1$ be any point with $\rho(z_1, 0) = r$, and let $z_2$ be any point with $\rho(z_1, z_2) = \delta$. Given $z_1, z_2, \ldots, z_{k-1}$, select $z_k \notin \bigcup_{j=1}^{k-1} \Delta(z_j, \delta)$ such that $\rho(z_1, z_k)$ is minimized. This defines the sequence $\{z_k\}$ inductively. Clearly $\rho(z_k, z_n) \geq \delta$ for any $k \neq n$ and $\inf_k |z_k| \geq r$. In fact,

$$|z_k| = \rho(0, z_k) \geq \frac{\rho(z_1, z_k) - \rho(z_1, 0)}{1 - \rho(z_1, z_k)\rho(0, z_k)} \geq \frac{\delta - r}{1 - \delta r} = r,$$

for any $k \neq 1$ and $|z_1| = r$.

We now show that $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \delta)$. We will do this by using a proof by contradiction. Assume that $z \notin \bigcup_{k=1}^{\infty} \Delta(z_k, \delta), z \in \mathbb{D}$. Denote $s = \rho(z_1, z)$ and $\lambda = \frac{2r}{s}$. By the choice of $z_k$ for all $k$ we have $\rho(z_1, z_k) \leq s < \lambda$. On the other hand, by [5, Chapter 2, Lemma 15] or [14, Lemma 3], there exists a constant $K$ such that $z_1$ belongs to at most $K$ discs $\Delta(z_k, \lambda)$, a contradiction. Therefore, $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \delta)$. The proof is complete. \hfill \Box

Remark. By the proof of the above lemma, we known that for any $\delta \in (0, 1)$, there exists a $\delta$-lattice $\{z_k\}$ such that $\inf_k |z_k| \geq r$, where $r = \frac{1-N}{\delta}$. The following lemma formulates a stability property of a separated sequence under a small perturbation.

Lemma 3 ([11, Lemma 2.3]). Let $N > 1$. Suppose that $\{a_k\}$ is a $r_0$-separated sequence of $\mathbb{D}$ with $|a_k| \geq 1 - \frac{1}{2N}$ and $b_k = (1 - N(1 - |a_k|))a_k$. Then $\{b_k\}$ is a $\frac{\delta \delta_0}{N}$-separated sequence of $\mathbb{D}$, where $\delta$ is a constant independent of $r_0$ and $N$.

2.3. Area methods and equivalent norms. For $\gamma > 1$ and $\zeta \in \partial \mathbb{D}$, define the Koranyi(admissible, non-tangential) approach region $\Gamma_{\gamma}(\zeta)$ by

$$\Gamma_{\gamma}(\zeta) = \{z \in \mathbb{D} : |1 - \overline{z}\zeta| < \frac{\gamma}{2}(1 - |z|^2)\}.$$

In this paper we agree that $\Gamma(\zeta) = \Gamma_2(\zeta)$. It is known that for every $0 < r < 1$ and $\gamma > 1$, there exists $\gamma' > 1$ so that

$$\bigcup_{z \in \Gamma_{\gamma'}(\zeta)} \Delta(z, r) \subseteq \Gamma_{\gamma'}(\zeta).$$

Given $z \in \mathbb{D}$, define

$$I(z) = \{\zeta \in \partial \mathbb{D} : z \in \Gamma(\zeta)\} \subset \partial \mathbb{D}.$$

Since $|I(z)| \approx 1 - |z|$, an application of Fubini’s theorem yields the following important formula:

$$\int_{\mathbb{D}} \varphi(z) dv(z) = \int_{\partial \mathbb{D}} \left( \int_{\Gamma_{\gamma}(\zeta)} \varphi(z) \frac{dv(z)}{1 - |z|} \right) dm(\zeta),$$

where $\varphi$ is any non-negative measurable function and $v$ is a finite positive measure.

The following result is the famous Calderón theorem and can be found in [18, Theorem 1.3] or [17, Theorem 5.3].
Lemma 4. Suppose \(0 < p, q < \infty, f \in H^p, f(0) = 0\). Then
\[
\left(\int_{\Delta} \left( \int_{\Gamma(z)} \left| f^n(z) dA(z) \right|^q \right)^{\frac{p}{q}} dm(\zeta) \right)^{\frac{1}{p}} \leq \| f \|_{H^p}.
\]

2.4. Tent spaces. Tent spaces were introduced by Coifman, Meyer and Stein [3] in order to study several problems in harmonic analysis. Luecking [13] used these tent spaces to study embedding theorems for Hardy spaces on \(\mathbb{R}^n\), results that have been obtained in the unit ball \(\mathbb{B}_n\) by Arsenovic and Jevtic [1, 8]. Recently, Peláez and Rättyä [20] showed that tent spaces have natural analogues for Bergman spaces, and they may play a role in the theory of small weighted Bergman spaces similar to that of the original tent spaces in the Hardy space case.

For \(0 < p, q < \infty\) and a positive Borel measure \(\nu\) on \(\mathbb{D}\), finite on compact sets, the tent spaces \(T^p_q(\nu)\) consists of the \(\nu\)-equivalence classes of \(\nu\)-measurable functions \(f\) such that
\[
\| f \|^p_{T^p_q(\nu)} = \int_{\Delta} \left( \int_{\Gamma(z)} \left| f(z) \right|^q \nu(z) \right)^{\frac{p}{q}} dm(\zeta) < \infty.
\]
For \(0 < p < \infty\), define
\[
\| f \|^p_{T^p_q(\nu)} = \int_{\Delta} \left( \nu - ess \sup_{z \in \Gamma(\zeta)} \left| f(z) \right| \right)^p dm(\zeta) < \infty.
\]
For \(0 \neq a \in \mathbb{D}\), we define \(\zeta_a = \frac{a}{|a|}\) and set
\[
Q(a) = \{ z \in \mathbb{D} : |1 - \overline{a}z| < 1 - |a|^2 \}.
\]
We also set \(Q(0) = \mathbb{D}\). The space \(T^\infty_q(\nu)\) consists of \(\nu\)-measureable functions \(f\) on \(\mathbb{D}\) with
\[
\| f \|^p_{T^\infty_q(\nu)} = \sup_{\zeta \in \partial \mathbb{D}} \left( \sup_{a \in Q(\zeta)} \frac{1}{|a|} \int_{Q(a)} |f(z)|^q (1 - |z|) d\nu(z) \right)^{\frac{1}{q}} < \infty,
\]
where \(I_a = \{ \zeta \in \partial \mathbb{D} : |\arg \zeta - \arg a| \leq \frac{1 - |a|}{2} \}\). By Section 5.2 of [27], we notice that \(f \in T^\infty_q(\nu)\) if and only if \(\| f(z)|^q (1 - |z|) d\nu(z)\) is a Carleson measure on \(\mathbb{D}\). The aperture \(\gamma > 0\) of Koranyi region is suppressed from the above notation. Moreover, it is well known that any two apertures generate the same function space with equivalent quasinorms.

The following result gives a description of the dual of \(T^p_q(\nu)\). See [3, 13, 19].

Lemma 5. Let \(1 \leq p, q < \infty\) with \(p + q \neq 2\), and let \(\nu\) be a positive Borel measure on \(\mathbb{D}\), finite on compact sets of \(\mathbb{D}\). Then the dual of \(T^p_q(\nu)\) is isomorphic to \(T^p_q(\nu)\) under the pairing:
\[
\langle f, g \rangle_{T^p_q(\nu)} = \int_{\mathbb{D}} f(z) \overline{g(z)} (1 - |z|) d\nu(z),
\]
and there exists a positive constant \(C\) such that
\[
\| \langle f, g \rangle_{T^p_q(\nu)} \| \leq C \| f \|_{T^p_q(\nu)} \| g \|_{T^p_q(\nu)}
\]
for any \(f \in T^p_q(\nu)\) and \(g \in T^p_q(\nu)\).
If \( \nu = \sum_k \delta_{z_k}(z) \) (where \( \delta_{z_k} \) denotes a unit mass at \( z_k \)), where \( \{z_k\} \) is a separated sequence, then we write \( T_q^p(\nu) = T_q^p([z_k]) \).

The following result gives a description of the dual of \( T_q^p([z_k]) \). See [1, 8, 13, 20].

**Lemma 6.** Let \( 0 < q \leq 1 < p < \infty \) and \( \{z_k\} \) be a separated sequence on \( \mathbb{D} \). Then the dual of \( T_q^p([z_k]) \) is isomorphic to \( T_{q'}^p([z_k]) \) under the pairing:

\[
\langle f, g \rangle_{T_q^p(z_k)} = \sum_k f(z_k) g(z_k)(1 - |z_k|),
\]

and there exists a positive constant \( C \) such that

\[
|\langle f, g \rangle_{T_q^p(z_k)}| \leq C \|f\|_{T_q^p(z_k)} \|g\|_{T_{q'}^p(z_k)}
\]

for any \( f \in T_q^p([z_k]) \) and \( g \in T_{q'}^p([z_k]) \).

We need the following important result, which plays in part a similar role on \( H^p \) as the atomic decomposition on standard Bergman spaces \( A_p^\alpha \).

**Lemma 7** ([19, Lemma D]). Let \( 0 < p < \infty \) and \( \{z_k\} \) be a separated sequence. Define

\[
S_\lambda(f)(z) = \sum_k f(z_k) \left(1 - \frac{|z_k|}{1 - \overline{z_k} z}\right)^\lambda, \quad z \in \mathbb{D}.
\]

Then there exists \( \lambda_0 = \lambda_0(p) \geq 1 \) such that \( S_\lambda : T_q^p([z_k]) \to H^p \) is bounded for all \( \lambda > \lambda_0 \).

### 2.5. Local estimates

We shall also use the following inequality. Here and henceforth \( \Delta \) denotes the Laplacian operator.

**Lemma 8** ([19, Lemma 3]). If \( 2 \leq q < \infty \) and \( 0 < r < 1 \), there is a constant \( C(q, r) \) such that

\[
|f'(z)|^q (1 - |z|^2)^q \leq C(q, r) \int_{\Delta(z, r)} \Delta |f|^q(\zeta) dA(\zeta), \quad z \in \mathbb{D}.
\]

The following lemma plays an essential role in the proof of Theorem 1.

**Lemma 9.** Let \( 0 < p < \infty \), and \( 0 < s < r < 1 \). Then there exists a constant \( C = C(p, s, r) \) such that

\[
|f'(z) - f'(w)|^p \leq C_p(z, w)^p \int_{\Delta(z, r)} |f(\zeta)|^p dA(\zeta) \left(1 - |z|^2\right)^{p+2}
\]

for all \( z \in \mathbb{D} \), \( w \in \Delta(z, s) \) and \( f \in H(\mathbb{D}) \).
**Proof.** Let $0 < s < r_1 < r$, $r_2 = r - r_1$. By [21, Lemma 3.1] or [9] Lemma 2.2, we have

\[
|f'(z) - f'(w)|^p \leq C_p(z, w)\frac{\int_{\Delta(z, r_1)} |f'(\eta)|^p dA(\eta)}{(1 - |z|^2)^2} \tag{4}
\]

for any $f \in H(\mathbb{D})$.

On the other hand, by the mean value property and the fact that $\Delta(\eta, r_2) \subset \Delta(z, r)$ and $1 - |z|^2 \geq 1 - |\eta|^2$ for any $\eta \in \Delta(z, r_1)$, we get

\[
|f'(\eta)|^p \leq \frac{1}{(1 - |\eta|^2)^{p+2}} \int_{\Delta(\eta, r_2)} |f(\zeta)|^p dA(\zeta)
\]

\[
\leq \frac{1}{(1 - |z|^2)^{p+2}} \int_{\Delta(z, r)} |f(\zeta)|^p dA(\zeta).
\]

Then the above estimate together with $A(\Delta(z, r_1)) \approx (1 - |z|^2)^2$ yield the desired result. \qed

**Lemma 10.** Let $0 < p < s \leq q < \infty$ and $0 < r < 1$. Then

\[
\int_{\mathbb{D}} |f'(z)|^q d\mu(z) \leq \left( \int_{\mathbb{D}} |f(\zeta)|^s \frac{(\mu(\Delta(\zeta, r)))^{q - s}}{(1 - |\zeta|^2)^{q - s} + 2} dA(\zeta) \right)^{\frac{q}{s}}
\]

for all $f \in H(\mathbb{D})$.

**Proof.** Recall the known estimate

\[
|f'(z)|^s \leq \frac{1}{(1 - |z|^2)^{s+2}} \int_{\Delta(z, r)} |f(\zeta)|^dA(\zeta), \quad z \in \mathbb{D}, \quad s > 0.
\]

Then, the above inequality and Minkowski’s inequality give

\[
\int_{\mathbb{D}} |f'(z)|^q d\mu(z) \leq \int_{\mathbb{D}} \left( \frac{1}{(1 - |z|^2)^{s+2}} \int_{\Delta(z, r)} |f(\zeta)|^dA(\zeta) \right)^{\frac{q}{s}} d\mu(z)
\]

\[
\leq \left( \int_{\mathbb{D}} |f(\zeta)|^{s} \frac{(\mu(\Delta(\zeta, r)))^{q - s}}{(1 - |\zeta|^2)^{q - s} + 2} dA(\zeta) \right)^{\frac{q}{s}}.
\]

\qed

**Lemma 11.** Let $2 \leq p < \infty$ and $0 < r < 1$. Then

\[
\int_{\mathbb{D}} |f'(z)|^p d\mu(z) \leq \int_{\mathbb{D}} \frac{\Delta |f|^p(\zeta)}{(1 - |\zeta|^2)^p} \mu(\Delta(\zeta, r)) dA(\zeta).
\]

**Proof.** By Lemma [8] Fubini’s theorem and the fact that $1 - |z| \approx 1 - |\zeta|$ for any $\zeta \in \Delta(z, r)$, we get the desired result. \qed

**Lemma 12.** Suppose $0 < q < \infty$ and $0 < r < 1$. Then

\[
\int_{\mathbb{D}} |f'(z)|^q d\mu(z) \leq \int_{\mathbb{D}} |f'(\zeta)|^q \mu(\Delta(\zeta, r)) dh(\zeta),
\]

where $dh(z) = \frac{dA(\zeta)}{(1 - |\zeta|^2)^7}$ denote the hyperbolic measure.
Proof. By the mean value property and Fubini’s theorem and the fact that \(1 - |z|^2 \geq 1 - |\xi|^2\) for any \(z \in \Delta(\zeta, r)\), we have
\[
\int_D |f'(z)|^q d\mu(z) \leq \int_D \frac{1}{(1 - |z|^2)^2} \int_{\Delta(z,s)} |f'(\zeta)|^q dA(\zeta) d\mu(z)
= \int_D |f'(\zeta)|^q \int_{\Delta(z,s)} \frac{1}{(1 - |z|^2)^2} d\mu(z) dA(\zeta)
\leq \int_D |f'(\zeta)|^q d\mu(\Delta(z, r)) dh(\zeta).
\]
\[\Box\]

By [27] Lemma 4.30, for all \(a, z, w \in \mathbb{D}\) with \(\rho(z, w) < r\) and any real \(s\), we have
\[
\left| 1 - \left( \frac{1 - \overline{a}z}{1 - \overline{aw}} \right)^s \right| \leq C(s, r) \rho(z, w),
\]
and therefore, for all \(w, z, a \in \mathbb{D}\) with \(z \in \Delta(a, r)\) and any \(s > 0\),
\[
\left| \frac{1}{(1 - \overline{az})^s} - \frac{1}{(1 - \overline{aw})^s} \right| \leq C(s, r) \rho(z, w) \left| \frac{1}{(1 - \overline{az})^s} \right|.
\]

Although the converse inequality does not hold we have the following result, which can be found in [9] Theorem 2.8 or [26, Lemma D]. Since Koo and Wang’s result is stated for the unit ball setting (see [9]), for the convenience of readers, and in order to offer no doubt of the validity of the result, we give a proof here.

Lemma 13 ([9] Theorem 2.8]). Suppose \(s > 1\) and \(0 < r_0 < 1\). Then there are \(N = N(r_0) > 1\) and \(C = C(s, r_0)\) such that
\[
\left| \frac{1}{(1 - \overline{az})^s} - \frac{1}{(1 - \overline{aw})^s} \right| + \left| \frac{1}{(1 - t_N \overline{az})^s} - \frac{1}{(1 - t_N \overline{aw})^s} \right| \geq C \rho(z, w) \left| \frac{1}{(1 - \overline{az})^s} \right|
\]
for all \(z \in \Delta(a, r_0)\) with \(1 - |a| < \frac{1}{2N}\), \(t_N = 1 - N(1 - |a|)\) and \(w \in \mathbb{D}\).

Proof. For \(a \in \mathbb{D}\backslash\{0\}\) and \(0 < r < 1\), denote \(S(a, r) = \{z \in \mathbb{D} : |1 - \overline{a}| \leq |z| < r\}\). Let \(z \in \Delta(a, r_0)\). Since
\[
|1 - \overline{a}z| = |(1 - |a|^2) + \overline{a}(a - z)| < 2(1 - |a|) + r_0 |1 - \overline{a}z|,
\]
we have
\[
|1 - \overline{a}z| < \frac{2}{1 - r_0}(1 - |a|).
\] (5)

Take \(N = \left(\frac{10}{1 - r_0}\right)^2\). We shall split the proof into two cases.

Case I. \(w \notin S\left(\frac{a}{|a|}, \sqrt{N}(1 - |a|)\right)\). In this case, we have
\[
|1 - \overline{aw}| \geq |a| \left| 1 - \frac{\overline{a}}{|a|}w \right| - (1 - |a|) \geq \left( \frac{\sqrt{N}}{2} - 1 \right)(1 - |a|) \geq \frac{4}{1 - r_0}(1 - |a|).
\]
Combine this with (5), we have
\[
\left| \frac{1}{(1 - \overline{a}z)^s} - \frac{1}{(1 - \overline{a}w)^s} \right| \geq \frac{1}{|1 - \overline{a}|^s} - \frac{1}{|1 - \overline{a}w|^s} \geq \frac{1}{2}
\]

Case II. \( w \in S(\frac{1}{|a|}, \sqrt{N}(1 - |a|)) \). Set \( u(a) = \frac{1 - \rho(z, w)}{1 - \rho(z, w)} \). If \( |u(a)| \leq \frac{1}{2} \) or \( |u(a)| \geq 2 \), by using elementary estimates we easily get that
\[
|1 - u(a)^4| \geq |1 - u(a)|.
\]
Now, assume that \( \frac{1}{2} \leq |u(a)| \leq 2 \). Denote \( u(a) = re^{i\theta} \), where \( \theta = \arg(u(a)) \).

\[
|1 - \overline{aw}| = |a| \left| 1 - \overline{a}w \right| + (1 - |a|) < 2 \sqrt{N}(1 - |a|),
\]
we have
\[
|(1 - t_{N\overline{a}w}) - N(1 - |a|)| \leq |1 - \overline{aw}| \leq 2 \sqrt{N}(1 - |a|).
\]
By (5), we have
\[
|(1 - t_{N\overline{a}z}) - N(1 - |a|)| \leq |1 - \overline{az}| \leq \sqrt{N}(1 - |a|).
\]
Thus, \( 1 - t_{N\overline{a}z} \) and \( 1 - t_{N\overline{a}w} \) are points inside the disc centered at \( N(1 - |a|) \) with radius \( 2 \sqrt{N}(1 - |a|) \). Fix \( 0 < \epsilon < 1 \) small enough and choose \( N(r_0) \) sufficiently large such that
\[
|\arg(1 - t_{N\overline{a}z})| < \frac{\epsilon}{2}, \quad |\arg(1 - t_{N\overline{a}w})| < \frac{\epsilon}{2}
\]
and
\[
|\arg(1 - t_{N\overline{a}z})^s| < \frac{\epsilon}{2}, \quad |\arg(1 - t_{N\overline{a}w})^s| < \frac{\epsilon}{2}.
\]
Then \( \theta \leq \epsilon \) and \( s\theta \leq \epsilon \). Since \( \epsilon \) is small enough, we have
\[
|1 - u(a)^4| \approx |1 - r^s \cos(s\theta) + r^s| \sin(s\theta)|
\]
\[
= |1 - r^s| + r^s(| \sin(s\theta)| + (1 - \cos(s\theta))
\]
\[
\approx |1 - r^s| + sr^s|\theta|
\]
\[
= |1 - r^s| + r(|\theta| - (1 - \cos(\theta))
\]
\[
= |1 - r^s| + r(|\sin(\theta)|)
\]
\[
= |1 - u(a)|.
\]
Therefore
\[
\frac{1}{|1 - t_{N\overline{a}z}|^s} - \frac{1}{(1 - t_{N\overline{a}w})^s} = \frac{1}{|1 - t_{N\overline{a}z}|^s} |1 - u(a)| \geq \frac{1}{|1 - \overline{a}|^s} |1 - u(a)|
\]
\[
\geq \frac{1}{|1 - \overline{a}|^s} |1 - t_{N\overline{a}z}|^s \geq \frac{1}{|1 - \overline{a}|^s} |z - w|^s \geq \frac{1}{|1 - \overline{a}|^s} |1 - \overline{a}w|^s.
\]
where we used the fact that
\[ |1 - t_N \bar{a}z| \leq |1 - \bar{a}z|, |1 - t_N \bar{a}w| \leq |1 - \bar{a}w| \text{ and } |1 - \bar{a}w| \leq |1 - \bar{z}w|. \]
The proof is complete. \( \square \)

3. \( D_\phi - D_\psi \) FROM \( H^p \) TO \( A^q_\alpha \)

We will split Theorem 1 into two theorems, i.e., Theorem 14 and Theorem 15.

**Theorem 14.** Suppose \( \varphi \) and \( \psi \) are analytic self-maps of \( \mathbb{D} \). Let either \( 0 < p < q < \infty \) or \( 2 \leq p = q \), \( 0 < r < 1 \) and \( \mu \) denote the joint pull-back measure induced by \( \varphi \) and \( \psi \) defined as \( \mathcal{M} \).

(a) \( D_\phi - D_\psi : H^p \to A^q_\alpha \) is bounded if and only if
\[
\sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{1}{p} + q}} < \infty.
\]
Moreover,
\[
\|D_\phi - D_\psi\|_{H^p \to A^q_\alpha} = \sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{1}{p} + q}}.
\]

(b) \( D_\phi - D_\psi : H^p \to A^q_\alpha \) is compact if and only if
\[
\lim_{|a| \to 1} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{1}{p} + q}} = 0.
\]

**Proof.** (a) First, consider the lower bound. Suppose that \( D_\phi - D_\psi : H^p \to A^q_\alpha \) is bounded. Let
\[
k_a(z) = \left( \frac{1 - |a|^2}{1 - \bar{a}z^2} \right)^{\frac{1}{2}}, \quad k_{a_N}(z) = \left( \frac{1 - |a|^2}{1 - t_N \bar{a}z^2} \right)^{\frac{1}{2}},
\]
where \( |a| > r_1 = 1 - \frac{1}{2N} \), \( t_N \) and \( N \) are defined as Lemma 13. By Lemma 13 we get
\[
\|D_\phi - D_\psi\|_{A^q_\alpha} \leq \sup_{a \in \mathbb{D}} \left( \|D_\phi - D_\psi\|_{A^q_\alpha} + \|D_\phi - D_\psi\|_{A^q_\alpha} \right)
\]
\[
\geq \int_{\mathbb{D}} \frac{\rho(\xi)^q}{(1 - |\xi|^2)^{\frac{1}{p} + q}} dA_\alpha(\xi) \quad (7)
\]
and
\[
\|D_\phi - D_\psi\|_{A^q_\alpha} \leq \sup_{a \in \mathbb{D}} \left( \|D_\phi - D_\psi\|_{A^q_\alpha} + \|D_\phi - D_\psi\|_{A^q_\alpha} \right)
\]
\[
\geq \int_{\mathbb{D}} \frac{\rho(\xi)^q}{(1 - |\xi|^2)^{\frac{1}{p} + q}} dA_\alpha(\xi) \quad (8)
\]
Therefore,
\[
\|D_\phi - D_\psi\|^q \geq \sup_{a \in \mathbb{D}} \left( \|D_\phi - D_\psi\|_{A^q_\alpha} + \|D_\phi - D_\psi\|_{A^q_\alpha} \right)
\]
\[
\geq \sup_{|a| > r_1} \frac{\mu(\Delta(a, r))}{(1 - |a|^2)^{\frac{1}{p} + q}}.
\]
For $|a| \leq r_1$, take $r_2 = \frac{r_1 + r}{1 - r_1}$, then $\Delta(a, r) \subset \Delta(0, r_2)$. Therefore

\[
\frac{\mu(\Delta(a, r))}{(1 - |a|^2)^{p+q}} \leq \frac{1}{(1 - r_1^2)^{p+q}} \left( \int_{\varphi^{-1}(\Delta(a, r))} \rho(z)^q dA_\alpha(z) + \int_{\psi^{-1}(\Delta(a, r))} |\varphi(z) - \psi(z)|^q dA_\alpha(z) \right) 
\leq \frac{1}{(1 - r_1^2)^{p+q}} \left( \int_{\varphi^{-1}(\Delta(a, r))} \rho(z)^q dA_\alpha(z) + \int_{\psi^{-1}(\Delta(a, r))} |\varphi(z) - \psi(z)|^q dA_\alpha(z) \right) 
\leq \|D_\varphi - D_\psi\|_q [z_{A_q}^q] 
\leq \|D_\varphi - D_\psi\|_q. 
\]

(9)

Next we will consider the upper bound. For $f \in H^p$, consider

\[
\|D_\varphi - D_\psi\|_q [z_{A_q}^q] = \left( \int_{\varphi(z)} + \int_{\psi(z)} \right) |f' \circ \varphi(z) - f' \circ \psi(z)|^q dA_\alpha(z) 
\]

:= $I_1 + I_2$. 

(10)

The first term is uniformly bounded.

\[
I_1 \leq 2^{2q} \int_{\Delta} |f'(z)|^q d\mu(z). 
\]

(11)

We shall split the proof into two cases.

**Case 0 < p < q < \infty.** Let $0 < r < 1$ be fixed. Choose $s \in (p, q]$. By Lemma 9, we get

\[
I_2 \leq \int_{\Delta} \rho(z)^q \left( \int_{\varphi(z, r)} \frac{|f(w)|^q}{(1 - |w|^2)^{s+2}} dA_\alpha(w) \right) dA_\alpha(z). 
\]

So, by Minkowski’s inequality we obtain

\[
I_2 \leq \int_{\Delta} \rho(z)^q \left( \int_{\varphi(z, r)} \frac{|f(w)|^q}{(1 - |w|^2)^{s+2}} dA_\alpha(w) \right) dA_\alpha(z) 
\leq \left( \int_{\Delta} |f(w)|^q \left( \int_{\varphi^{-1}(\Delta(w, r))} \rho(z)^q dA_\alpha(z) \right)^{\frac{q}{s}} dA_\alpha(w) \right) \frac{q}{s} 
\leq \left( \int_{\Delta} |f(w)|^q \frac{(\mu(\Delta(w, r)))^{\frac{q}{s}}}{(1 - |w|^2)^{s+2}} dA_\alpha(w) \right) \frac{q}{s}. 
\]

(12)

Therefore, by (10), (11), (12) and Lemma 10, we get

\[
\|D_\varphi - D_\psi\|_q \leq \left( \int_{\Delta} |f(w)|^q \frac{(\mu(\Delta(w, r)))^{\frac{q}{s}}}{(1 - |w|^2)^{s+2}} dA_\alpha(w) \right) \frac{q}{s}. 
\]

(13)
Since $s > p$, by Duren’s theorem we have $\|f\|_{A^s_p} \leq \|f\|_{H^p}$. Therefore

$$\|D_\varphi - D_\psi\|^q \leq \sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)\frac{2}{p} + q}.$$ 

This completes the proof of the case $0 < p < q < \infty$.

**Case $q = p \geq 2$.** By (4), Lemma 8 and Fubini’s theorem, we get

$$I_2 = \int_{\rho(z) < \frac{1}{|z|^2}} |f' \circ \varphi(z) - f' \circ \psi(z)|^q dA_a(z)$$

$$\leq \int_{\rho(z) < \frac{1}{|z|^2}} \rho(z)^q \left( \int_{\Delta(\varphi(z), s_1)} \frac{|f'(w)|^q}{(1 - |w|^2)^2} dA(w) \right) dA_a(z)$$

$$\leq \int_{\mathbb{D}} \frac{\rho(z)^q}{(1 - |\varphi(z)|^2)^{2+q}} \left( \int_{\Delta(\varphi(z), s_2)} \left( \int_{\Delta(w, s_2)} \Delta|f|^q dA(u) \right) dA(w) \right) dA_a(z)$$

$$\leq \int_{\mathbb{D}} \frac{\rho(z)^q}{(1 - |\varphi(z)|^2)^{2+q}} \left( \int_{\Delta(\varphi(z), r)} \Delta|f|^q dA(u) \right) dA_a(z)$$

where $s_1, s_2 \in (0, 1)$ are chosen sufficiently small depending only on $r$, for example, we can take $s_1 = s_2 = \frac{1 - \sqrt{r}}{r}$. From this with (10), (11) and Lemma 11, we have

$$\|D_\varphi - D_\psi\|_{A^p_a} \leq \int_{\mathbb{D}} \frac{\Delta|f|^q(u)}{(1 - |u|^2)^{\frac{2}{p} - q}} \mu(\Delta(u, r)) dA(u).$$

Using the Hardy-Stein-Spencer identity

$$\|f\|^p_{H^p} = |f(0)|^p + \frac{1}{2} \int_{\mathbb{D}} \Delta|f(z)|^p \log \frac{1}{|z|} dA(z),$$

and the fact that $p = q$, the proof can be finished as in the previous case.

(b) We now turn to the proof of the compactness.

If $D_\varphi - D_\psi : H^p \to A^p_a$ is compact, then one may deduce from (7), (8) that

$$\lim_{|a| \to 1} \frac{\mu(\Delta(a, r))}{(1 - |a|^2)^\frac{2}{p} + q} = 0.$$ 

Now, suppose

$$\lim_{|a| \to 1} \frac{\mu(\Delta(a, r))}{(1 - |a|^2)^\frac{2}{p} + q} = 0.$$ 

To prove the compactness of $D_\varphi - D_\psi : H^p \to A^p_a$, we consider an arbitrary sequence $\{f_n\}$ in $H^p$ such that $\|f_n\|_{H^p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. It is enough to show that

$$\lim_{n \to \infty} \|(D_\varphi - D_\psi)f_n\|_{A^p_a} = 0.$$ (16)
We first consider the case $0 < p < q < \infty$. Let $t \in (0, 1)$. By (13) we have
\[
\|(D_\varphi - D_\psi)f_n\|_{A_p^q}^q \leq \left( \int_{\mathbb{D}} |f_n(w)|^q \frac{(\mu(\Lambda(w, r)))^{\frac{q}{p}}}{(1 - |w|^2)^{s+2}} dA(w) \right)^{\frac{q}{q}}.
\]
(17)

Note that $\mu$ is a finite measure, thus
\[
\sup_{a \in \mathbb{D}} \frac{\mu(\Delta(a, r))}{(1 - |q+r|)^{q+p}} < \infty.
\]

Now, since $f_n \to 0$ uniformly on $\mathbb{D}$, we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{D}} |f_n(w)|^q \frac{(\mu(\Lambda(w, r)))^{\frac{q}{p}}}{(1 - |w|^2)^{s+2}} dA(w) = 0.
\]

On the other hand, we have
\[
\left( \int_{\mathbb{D}\setminus\mathbb{D}} |f_n(w)|^q \frac{(\mu(\Lambda(w, r)))^{\frac{q}{p}}}{(1 - |w|^2)^{s+2}} dA(w) \right)^{\frac{q}{q}} \leq \sup_{z \in \mathbb{D}\setminus\mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{q+p}} \left( \int_{\mathbb{D}} |f_n(w)|^q (1 - |w|^2)^{\frac{q}{p} - 2} dA(w) \right)^{\frac{q}{q}}
\]
\[
\leq \sup_{z \in \mathbb{D}\setminus\mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{\frac{q}{p} + q}}.
\]

Letting $t \to 1$, we get (16), as desired.

The case $2 < p = q < \infty$. By an similar argument, using (15) and the Hardy-Stein-Spencer identity, we deduce (16), as desired. □

**Theorem 15.** Suppose $\varphi$ and $\psi$ are analytic self-maps of $\mathbb{D}$. Let $0 < r < 1$, $0 < q \leq p < \infty$, and let $\mu$ denote the joint pull-back measure induced by $\varphi$ and $\psi$ defined as (1).

(i) If $q < \min\{2, p\}$, then the following conditions are equivalent:

(a) $D_\varphi - D_\psi : H^p \to A^q_p$ is bounded;

(b) $D_\varphi - D_\psi : H^p \to A^q_p$ is compact;

(c) The function
\[
\zeta \mapsto \left( \int_{\Gamma(\zeta)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right)^{\frac{q}{p}} \frac{dA(z)}{(1 - |z|^2)} \right)^{\frac{2-q}{2}}
\]
belongs to $L^{\frac{q}{q-2}}(\partial \mathbb{D}, m)$.

(ii) If $0 < q = p < 2$, then

(a) $D_\varphi - D_\psi : H^p \to A^q_p$ is bounded if and only if
\[
\left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+p}} \right)^{\frac{q}{p}} \frac{dA(z)}{1 - |z|}
\]
is a Carleson measure.
(ii) $D_\varphi - D_\psi : H^p \rightarrow A^q_\alpha$ is compact if and only if

$$\int_D \left( \frac{\mu(\Delta(z, r))}{1 - |z|^2} \right)^{\frac{p}{p-q}} dA(z)$$

is a vanishing Carleson measure.

(iii) If $2 \leq q < p < \infty$, then

(iiiia) $D_\varphi - D_\psi : H^p \rightarrow A^q_\alpha$ is bounded if and only if the function

$$\zeta \mapsto \sup_{z \in \Gamma(z)} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}}$$

belongs to $L^{\frac{p}{p-q}}(\partial \mathbb{D}, m)$.

(iiiib) $D_\varphi - D_\psi : H^p \rightarrow A^q_\alpha$ is compact if and only if the function

$$\zeta \mapsto \sup_{z \in \Gamma(z) \cap \{|z| \geq r\}} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}}$$

converges to zero in $L^{\frac{p}{p-q}}(\partial \mathbb{D}, m)$ as $r \rightarrow 1^-$.

**Proof.** By Lemma[2] there exists a $\delta$-lattice $\{z_k\}$ such that $\inf_k |z_k| > 1 - \frac{1}{2N}$, where $N$ is defined as Lemma[3]. Let

$$CT^p_2((z_k)) = \{ f \in T^p_2((z_k)) : \|f\|_{T^p_2((z_k))} = 1 \}.$$  

For each $\lambda > \lambda_0$ ($\lambda_0$ is that of Lemma[7]), set

$$S_\lambda(f)(z) = \sum_{k=1}^{\infty} f(z_k) \left( \frac{1 - |z_k|}{1 - z_k \bar{z}} \right)^{\lambda}, \quad f \in CT^p_2((z_k)), \ z \in \mathbb{D},$$

and

$$S_{\lambda,N}(f)(z) = \sum_{k=1}^{\infty} f(z_k) \left( \frac{1 - |z_k|}{1 - t_k,N \bar{z}_k} \right)^{\lambda}, \quad f \in CT^p_2((z_k)), \ z \in \mathbb{D},$$

where $t_{k,N} = 1 - N(1 - |z_k|)$. By Lemmas[3] and[7] we have

$$\|S_\lambda(f)\|_{H^p} \leq \|f\|_{T^p_2((z_k))} \quad \text{and} \quad \|S_{\lambda,N}(f)\|_{H^p} \leq \|f\|_{T^p_2((z_k))}.$$  

Therefore,

$$\int_D \left|(D_\varphi - D_\psi)S_\lambda(f)(z)\right|^q dA_\alpha(z) \leq \|D_\varphi - D_\psi\|^q \|f\|^q_{T^p_2((z_k))},$$

Let $g_t \in T^p_2((z_k))$ such that $g_t(z_k) = f(z_k)r_k(t)$, where $r_k$ denotes the $k$th Rademacher function. Replace $f(z_k)$ by $g_t(z_k)$ in the above inequality, and integrate with respect to $t$ we obtain

$$\int_0^1 \int_D \left|(D_\varphi - D_\psi)S_\lambda(g_t)(z)\right|^q dA_\alpha(z) dt \leq \|D_\varphi - D_\psi\|^q \|f\|^q_{T^p_2((z_k))}.$$
Using Fubini’s theorem and Khinchine’s inequality we get

\[ I = \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} |f(z_k)|^2 \left| \frac{(1 - |z_k|)^4}{(1 - z_k \varphi(z))^{4+1}} - \frac{(1 - |z_k|)^4}{(1 - z_k \psi(z))^{4+1}} \right|^{\frac{1}{2}} \right) dA_\sigma(z) \]

\[ \leq \|D_\varphi - D_\psi\|^q \|f\|^q_{T^2(\mathbb{T})}. \]  

(18)

Replace now \( S_\lambda(f) \) by \( S_{\lambda,N}(f) \), we have

\[ II = \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} |f(z_k)|^2 \left| \frac{(1 - |z_k|)^4}{(1 - z_k \varphi(z))^{4+1}} - \frac{(1 - |z_k|)^4}{(1 - z_k \psi(z))^{4+1}} \right|^{\frac{1}{2}} \right) dA_\sigma(z) \]

\[ \leq \|D_\varphi - D_\psi\|^q \|f\|^q_{T^2(\mathbb{T})}. \]  

(19)

Now, for any fixed \( r_0 \in (0, 1) \), by Lemma \[13\]

\[ I + II \]

\[ \leq \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} |f(z_k)|^2 \left| \frac{(1 - |z_k|)^4}{(1 - z_k \varphi(z))^{4+1}} - \frac{(1 - |z_k|)^4}{(1 - z_k \psi(z))^{4+1}} \right|^{\frac{1}{2}} \right) dA_\sigma(z) \]

\[ + \left| \frac{(1 - |z_k|)^4}{(1 - t_k \varphi(z))^{4+1}} - \frac{(1 - |z_k|)^4}{(1 - t_k \psi(z))^{4+1}} \right|^{\frac{1}{2}} dA_\sigma(z) \]

\[ \leq \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} |f(z_k)|^2 \rho(z)\left( \chi_\varphi^{-1}(\Delta(z_k, r_0))(z) + \chi_\psi^{-1}(\Delta(z_k, r_0))(z) \right) \frac{z}{(1 - |z_k|)^2} \right) dA_\sigma(z). \]

If \( q \geq 2 \), using the inequality \( \sum_j a_j^x \leq \left( \sum_j a_j \right)^x \), valid for all \( a_j \geq 0 \) and \( x \geq 1 \), and if \( 0 < q < 2 \), by Hölder’s inequality and the fact that there exists a constant \( K \) such that every \( z \in \mathbb{D} \) belongs to at most \( K \) discs \( \Delta(z_k, r_0) \), we get

\[ I + II \]

\[ \leq \int_{\mathbb{D}} \sum_{k=1}^{\infty} |f(z_k)|^2 \rho(z)\left( \chi_\varphi^{-1}(\Delta(z_k, r_0))(z) + \chi_\psi^{-1}(\Delta(z_k, r_0))(z) \right) \frac{z}{(1 - |z_k|)^q} dA_\sigma(z) \]

\[ \leq \sum_{k=1}^{\infty} |f(z_k)|^2 \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|)^q}. \]

Therefore

\[ \sum_{k=1}^{\infty} |f(z_k)|^q \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|)^q} \leq \|D_\varphi - D_\psi\|^q \|f\|^q_{T^2(\mathbb{T})} \]

\[ = \|D_\varphi - D_\psi\|^q \left( \int_{\partial \mathbb{D}} \left( \sum_{z_k \in \partial \mathbb{G}} \left( |f(z_k)|^q \right)^{\frac{q}{2}} \right)^{\frac{2}{q}} dm(\zeta) \right)^{\frac{2}{q}}. \]

(20)

On the other hand, by \[10\], \[11\], \[14\] and Lemma \[12\] we have
\[
\| (D_\psi - D_\phi)f \|_{A^q_p}^q \\
\leq \int_D |f'(w)|^q \mu(\Delta(w, r))dh(w) \\
= \int_D \left( |f'(w)(1 - |w|^2)|^q \right) \frac{\mu(\Delta(w, r))}{(1 - |w|^2)^{1+q}} (1 - |w|)dh(w),
\]
for any \(0 < r < 1\).

We now treat different cases separately.

(i) It is trivial that \((ib) \implies (ia)\). Now, we prove that \((ia) \implies (ic)\). Since \(0 < q < \min(2, p), s = \frac{q}{t} > 1\) and \(t = \frac{2}{q} > 1\), Lemma \(5\) yields \(T_\gamma^s(\{|z_k\}|^s) \approx T_\gamma^s(\{|z_k\}|)\). Therefore, by \((20)\) we get

\[
\left\{ \int_{\partial\Omega} \left( \sum_{z_k \in \Gamma(\zeta)} \left( \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|^2)^{1+q}} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} \right) dm(\zeta) \right\} \leq \| D_\psi - D_\phi \|^q.
\]

Using \((5)\) and the fact that \(\Delta(z, r) \subset \Delta(z_k, r_0)\) for all \(z \in \Delta(z_k, \delta)\) and all \(k\), where \(\frac{r + \delta}{1 + r_0} \leq r_0 < 1\), we get

\[
\int_{\Gamma(\zeta)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} \frac{dA(z)}{(1 - |z|^2)} \\
\leq \sum_{k: \Delta(z_k, \delta) \cap \Gamma(\zeta) \neq \emptyset} \int_{\Delta(z_k, \delta)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} \frac{dA(z)}{(1 - |z|^2)} \\
\leq \sum_{z_k \in \Gamma(\zeta)} \left( \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|^2)^{1+q}} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} \frac{dA(z)}{(1 - |z|^2)}
\]

and hence

\[
\left( \int_{\partial\Omega} \left( \int_{\Gamma(\zeta)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} \frac{dA(z)}{(1 - |z|^2)} \right) \frac{1}{r_0^{\frac{q}{p} - \frac{q}{t}}} dm(\zeta) \right)^{\frac{p}{q}} \\
\leq \| D_\psi - D_\phi \|^q.
\]

Then the assertion follows.

Finally, let us prove that \((ic) \implies (ib)\). Let \(f_n \in H^p\) such that \(\| f_n \|_{H^p} \leq 1\) and \(f_n \to 0\) uniformly on compact subsets of \(D\). It is enough to show that

\[
\lim_{n \to \infty} \| (D_\psi - D_\phi)f_n \|_{A^q_p} = 0.
\]

Denote

\[
F_n(w) = |f_n'(w)|^q (1 - |w|^2)^q, dh_R = dh_\chi(r < |z| < 1), 0 \leq R < 1
\]
and \( \Phi_\mu(w) = \frac{\mu(\Delta(w, r))}{(1-|w|)^{1+q}}. \) Fix \( \epsilon > 0. \) Since \( \Phi_\mu \in T_{\frac{p}{1+q}}^\mu(h), \) by the dominated convergence theorem there exists \( R_0 \) such that
\[
\sup_{R \geq R_0} \| \Phi_\mu \|_{T_{\frac{p}{1+q}}^\mu(h)} < \epsilon^q.
\]
Next, choose \( k_0 \) such that
\[
\sup_{n \geq k_0} \| f'_n(z) \| \leq \epsilon.
\]
Then, bearing in mind (21), using Lemma 5 and the inequality
\[
\| F_n \|_{T_{\frac{p}{1+q}}^\mu(h)} \leq \| f_n \|_{H^p},
\]
we get
\[
\| (D_\varphi - D_\psi) f_n \|_{A^q_\alpha} \leq \epsilon^q.
\]
Letting \( \epsilon \to 0, \) we get the desired result.

(ii) (iia) Suppose that \( D_\varphi - D_\psi : H^p \to A^q_\alpha \) is bounded. Since \( 0 < q = p < 2, \) then \( s = \frac{q}{p} = 1 \) and \( t = \frac{2+q}{p} > 1. \) So by Lemma 5 (\( T^r_\varphi((z_k)) = T^{\alpha_\varphi}(\{z_k\}) \)) with the equivalence norms. Therefore (20) yields
\[
\sup_{\xi \in D} \sup_{\omega \in \Gamma(\xi)} \left( \frac{1}{|I_a|} \sum_{z \in Q(a)} \frac{\mu(\Delta(z_k, r_0))}{(1-|z_k|)^{1+p}} \right)^{\frac{2}{2+p}} \| f \|_{L^r_{t, \mu}(h)} \leq \| D_\varphi - D_\psi \|_{A^q_\alpha},
\]
that is
\[
\sup_{\omega \in \Gamma(\xi)} \left( \frac{1}{|I_a|} \sum_{z \in Q(a)} \frac{\mu(\Delta(z_k, r_0))}{(1-|z_k|)^{1+p}} |I_{z_k}| \right)^{\frac{2}{2+p}} \| f \|_{L^r_{t, \mu}(h)} \leq \| D_\varphi - D_\psi \|_{A^q_\alpha}.
\]
The point $z \in \mathbb{D}$ for which $Q(a) \cap \Delta(z, r) \neq \emptyset$ is contained in some $Q(a')$, where $\arg a = \arg a'$ and $1 - |a'| \approx 1 - |a|$, for all $|a| > R$, where $R = R(r) \in (0, 1)$. Then

$$\int_{Q(a)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|} \leq \sum_{k : |z_k| \neq 0} \int_{\Delta(z_k, \delta)} \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} dA(z) \frac{1}{1 - |z|}$$

$$\leq \sum_{k \in Q(a')} \left( \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|} \leq \sum_{k \in Q(a')} \left( \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|^2)^1 + p} \right)^{\frac{1}{2p}} (1 - |z_k|),$$

where $r_0 \in (0, 1)$ sufficiently large such that $\Delta(z, r) \subset \Delta(z_k, r_0)$ for all $z \in \Delta(z_k, \delta)$ and all $k$, and hence

$$\left( \frac{1}{|I_a|} \int_{Q(a)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|} \right)^{\frac{1}{2p}} \leq \|D_\varphi - D_\psi\|^p$$

for any $|a| > R$. For $|a| \leq R$, by (23)

$$\frac{1}{|I_a|} \int_{Q(a)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|} \leq \int_{\mathbb{D}} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|} \leq \sum_{k = 1}^{\infty} \int_{\Delta(z_k, \delta)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|}$$

$$\leq \sum_{k = 1}^{\infty} \left( \frac{\mu(\Delta(z_k, r_0))}{(1 - |z_k|^2)^1 + p} \right)^{\frac{1}{2p}} (1 - |z_k|) \leq \|D_\varphi - D_\psi\|^{\frac{2p}{2p}},$$

and then we get that $\left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^1 + p} \right)^{\frac{1}{2p}} dA(z) \frac{1}{1 - |z|}$ is a Carleson measure.

On the other hand, denote

$$F(w) = |f'(w)|^q (1 - |w|^2)^q, \quad \Phi_\mu(w) = \frac{\mu(\Delta(w, r))}{(1 - |w|^2)^1 + q},$$
Then $F \in T^1_{\frac{1}{p}}(h)$ and $\|F\|_{T^1_{\frac{1}{p}}(h)} \leq \|f\|_{H^p}^p$. Therefore, by (21) and Lemma 5 we have

$$\|(D_\varphi - D_\psi)f\|_{A^q_\alpha}^q \leq \int_D \left( \int_D (1 - |z|^2)^q \frac{\varphi(z) \Delta(w,r)}{(1 - |w|^2)^{1+q}}(1 - |w|)dh(w) \right) \mu(\Delta(w,r)) \left( \frac{\mu(\Delta(z,r))}{(1 - |z|^2)^{1+p}} \right) \frac{dA(z)}{1 - |z|} \cdot \|f\|_{H^p}^p.$$ 

Then the assertion follows.

(iib) Let us first consider the sufficiency. Let $\{f_n\}$ be in $H^p$ such that $\|f_n\|_{H^p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $D$ when $n \to \infty$. A standard argument shows that

$$\lim_{|a| \to 1^-} \frac{1}{|I(a)|} \int_{Q(a)} \Phi_\mu(z) \frac{dA(z)}{1 - |z|^2} = 0$$

if and only if

$$\lim_{R \to 1^-} \sup_{a \in D} \frac{1}{|I(a)|} \int_{Q(a)} \Phi_\mu(z) \frac{dA(z)}{1 - |z|^2} = 0.$$ 

(24)

So for a fixed $\epsilon > 0$, there exists $R_0$ such that

$$\sup_{R \geq R_0} \|\Phi_\mu\|_{T^\infty_{\frac{1}{q}}(h_R)} < \epsilon^q.$$ 

Let $k_0$ be such that $\sup_{n \geq k_0} \sup_{|z| \leq R_0} |f_n'(z)| \leq \epsilon$. Then, bearing in mind (21) and the inequality $\|F_n\|_{T^1_{\frac{1}{q}}(h)} \leq \|f_n\|_{H^p}^q$, we get

$$\|(D_\varphi - D_\psi)f_n\|_{A^q_\alpha}^q \leq \epsilon^q \int_{R_0D} (1 - |w|)^q \Phi_\mu(w)(1 - |w|)dh(w)$$

$$+ \int_D F_n(w) \Phi_\mu(w)(1 - |w|)dh_{R_0}(w)$$

$$\leq \epsilon^q \left( \int_{|z| \leq R_0} (1 - |w|)^q \Phi_\mu F_ndh_{R_0} + \int_{|z| \leq R_0} F_n \Phi_\mu dh_{R_0} \right)$$

$$\leq \epsilon^q \left( \|D_\varphi - D_\psi\|_{A^q_\alpha} + \|F_n\|_{T^1_{\frac{1}{q}}(h)} \Phi_\mu \|T^\infty_{\frac{1}{q}}(h_R) \right)$$

$$\leq \epsilon^q.$$ 

Therefore, $D_\varphi - D_\psi : H^p \to A^q_\alpha$ is compact.

Next, we prove the necessity. Since $D_\varphi - D_\psi : H^p \to A^q_\alpha$ is compact, the set $(D_\varphi - D_\psi) \circ S_{\lambda}(CT_{\frac{1}{q}}^p(z_k))$ is relatively compact in $A^q_\alpha$. Consider $f \in T^1_{\frac{1}{q}}(z_k)$ with
\[ \|f\|_{T^p_0(z)} = 1. \] We can choose \( 0 < R_0 < 1 \) such that

\[ \int_{D \setminus R_0 D} \left| (D_\varphi - D_\psi) S_{A}(f)(z) \right|^q \, dA_\alpha(z) \lesssim e^q. \]

Let \( g_r \in T^q_\infty(z_k) \), such that \( g_r(z_k) = f(z_k)r_k(t) \), where \( r_k \) denotes the \( k \)th Rademacher function, replace now \( f(z_k) \) by \( g_r(z_k) \) in the above inequality, and integrate with respect to \( t \) we obtain

\[ \int_0^1 \int_{D \setminus R_0 D} \left| (D_\varphi - D_\psi) S_{A}(g_r)(z) \right|^q \, dA_\alpha(z) \, dt \lesssim e^q. \]

Using Fubini's theorem and Khinchine's inequality we get

\[ I = \int_{D \setminus R_0 D} \left( \sum_{k=1}^\infty |f(z_k)|^2 \left| \frac{(1 - |z_k|)^d}{(1 - t_k z_k \varphi(z))^{d+1}} - \frac{(1 - |z_k|)^d}{(1 - t_k z_k \psi(z))^{d+1}} \right|^2 \right)^{\frac{q}{2}} \, dA_\alpha(z) \lesssim e^q. \]

Replace now \( S_{A}(f) \) by \( S_{A,N}(f) \), we have

\[ II = \int_{D \setminus R_0 D} \left( \sum_{k=1}^\infty |f(z_k)|^2 \left| \frac{(1 - |z_k|)^d}{(1 - t_k z_k \varphi(z))^{d+1}} - \frac{(1 - |z_k|)^d}{(1 - t_k z_k \psi(z))^{d+1}} \right|^2 \right)^{\frac{q}{2}} \, dA_\alpha(z) \lesssim e^q. \]

Now, for any fixed \( r_0 \in (0, 1) \), by Lemma 13

\[ I + II \]

\[ \lesssim \int_{D \setminus R_0 D} \left( \sum_{k=1}^\infty |f(z_k)|^2 \left( \left| \frac{(1 - |z_k|)^d}{(1 - t_k z_k \varphi(z))^{d+1}} - \frac{(1 - |z_k|)^d}{(1 - t_k z_k \psi(z))^{d+1}} \right|^2 + \left| \frac{(1 - |z_k|)^d}{(1 - t_k z_k \varphi(z))^{d+1}} - \frac{(1 - |z_k|)^d}{(1 - t_k z_k \psi(z))^{d+1}} \right|^2 \right)^{\frac{q}{2}} \, dA_\alpha(z) \]

\[ \lesssim \int_{D \setminus R_0 D} \left( \sum_{k=1}^\infty |f(z_k)|^2 \left( \frac{\rho(z)^2 \left( \chi_{\varphi^{-1}(\Delta(z_k, r_0))}(z) + \chi_{\psi^{-1}(\Delta(z_k, r_0))}(z) \right)}{(1 - |z_k|)^2} \right)^q \, dA_\alpha(z) \]

\[ \lesssim \int_{D \setminus R_0 D} \sum_{k=1}^\infty |f(z_k)|^q \left( \frac{\rho(z)^q \left( \chi_{\varphi^{-1}(\Delta(z_k, r_0))}(z) + \chi_{\psi^{-1}(\Delta(z_k, r_0))}(z) \right)}{(1 - |z_k|)^q} \right) \, dA_\alpha(z) \]

\[ = \sum_{k=1}^\infty |f(z_k)|^q \frac{\mu(\Delta(z_k, r_0) \cap (D \setminus R_0 D))}{(1 - |z_k|)^q}. \]

Therefore

\[ \sum_{k=1}^\infty |f(z_k)|^q \frac{\mu(\Delta(z_k, r_0) \cap (D \setminus R_0 D))}{(1 - |z_k|)^q} \lesssim e^q. \] (25)

Since \( 0 < q = p < 2 \), then \( s = \frac{p}{q} = 1 \) and \( t = \frac{2}{p} > 1 \), by Lemma 5, \( T^p_\infty(z_k) = T^{p_0}_T(z_k) \) with equivalence of norms. Hence

\[ \sup_{\zeta \in \partial D} \sup_{a \in \Gamma(z)} \left( \frac{1}{|I(a)|} \sum_{z \in Q(a)} \frac{\mu(\Delta(z_k, r_0) \cap (D \setminus R_0 D))}{(1 - |z_k|)^{1+p}} \right)^{\frac{q}{2}} (1 - |z_k|)^{\frac{q}{2}} \lesssim e^q. \]
Using the fact that the point \( z \in \mathbb{D} \) for which \( Q(a) \cap \Delta(z, r) \neq \emptyset \) is contained in some \( Q(a') \), where \( \arg a = \arg a' \) and \( 1 - |a'| > 1 - |a| \), for all \( |a| > R \), where \( R = R(r) \in (0, 1) \), yields

\[
\int_{Q(a)} \left( \frac{\mu(\Delta(z, r) \cap (\mathbb{D} \setminus R_0 \mathbb{D}))}{(1 - |z|^2)^{1+q}} \right)^{\frac{1}{pq}} dA(z) \leq \frac{1}{I(a)} \int_{Q(a)} \left( \frac{\mu(\Delta(z, r) \cap (\mathbb{D} \setminus R_0 \mathbb{D}))}{(1 - |z|^2)^{1+q}} \right)^{\frac{1}{pq}} dA(z) \leq \varepsilon^q.
\]

It is easy to see that \( \Delta(z, r) \cap \mathbb{D} \setminus R_0 \mathbb{D} = \Delta(z, r) \), for any \( z \in Q(a) \), when \( a \) close enough to 1. Therefore,

\[
\lim_{|a| \to 1^-} \frac{1}{I(a)} \int_{Q(a)} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right)^{\frac{1}{pq}} dA(z) = 0,
\]

which completes the proof of the necessity.

(iii). We only give a detail proof for (iiib) since the proof of (iiia) can be proved by standard modifications of the proof of (iiib) and hence we omitted it.

(iiib) Suppose that \( D_{\varphi} - D_{\psi} : H^p \to A_\varphi^p \) is compact. Since \( 2 \leq q < p \), we have \( s = \frac{p}{q} > 1 \) and \( t = \frac{2}{p} < 1 \). So by Lemma 6 \( (T^s_T((z_k)))^* \approx T^s_{\partial}(\{|z_k|\}) \) with equivalence of norms. Therefore, (25) yields

\[
\int_{\partial \mathbb{D}} \left( \sup_{\zeta \in \Gamma_c} \left( \frac{\mu(\Delta(z, r) \cap (\mathbb{D} \setminus R_0 \mathbb{D}))}{(1 - |z|^2)^{1+q}} \right)^{\frac{1}{pq}} \right) dm(\zeta) \leq \varepsilon^q.
\]

There is a \( 0 < R < 1 \) such that \( \Delta(z, r) \cap \mathbb{D} \setminus R_0 \mathbb{D} = \Delta(z, r) \), for any \( |z| \geq R \). Thus

\[
\lim_{R \to 1} \int_{\partial \mathbb{D}} \left( \sup_{\zeta \in \Gamma_c, |z| \geq R} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right) \right) dm(\zeta) = 0.
\]

Now, we prove the necessity. Let \( \{f_n\} \) be in \( H^p \) such that \( \|f_n\|_{H^p} \leq 1 \) and \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \). We can assume that \( f_n(0) = 0 \). It is enough to show that

\[
\lim_{n \to \infty} \|(D_{\varphi} - D_{\psi}) f_n\|_{A_{\varphi}^p} = 0. \tag{26}
\]
Fix $\epsilon > 0$. There exists $R_0$ such that

\[
\left( \int_{\partial D} \left( \sup_{z \in \Gamma(\zeta) \cap |z| \leq R_0} \left( \frac{\mu(\Delta(z, r))}{(1 - |z|^2)^{1+q}} \right)^p \right)^{\frac{p-q}{p}} d\zeta \right)^{\frac{1}{p}} < \epsilon^q. \tag{27}
\]

Next, choose $k_0$ such that

\[
\sup_{n \geq 0} \sup_{|z| \leq R_0} |f_n''(z)| \leq \epsilon. \tag{28}
\]

Denote $F_n(w) = |f_n''(w)|^q(1 - |w|^2)^q$ and $\Phi_\mu(w) = \frac{\mu(\Delta(w, r))}{(1 - |w|^2)^{1+q}}$. Then, bearing in mind (21), we get

\[
\| (D_\varphi - D_\psi) f_n \|_{A_q^\mu}^q \lesssim \int_{\partial D} F_n(w) \Phi_\mu(w)(1 - |w|) d\zeta(w)
\]
\[
= \int_{\partial D} \left( \int_{\Gamma(\zeta)} F_n(w) \Phi_\mu(w) d\zeta(w) \right) d\zeta
\]
\[
= \int_{\partial D} \left( \int_{\Gamma(\zeta) \cap |w| \leq R_0} F_n(w) \Phi_\mu(w) d\zeta(w) \right) d\zeta
\]
\[
+ \int_{\partial D} \left( \int_{\Gamma(\zeta) \cap |w| > R_0} F_n(w) \Phi_\mu(w) d\zeta(w) \right) d\zeta
\]
\[
:= J_1 + J_2.
\]

Since $\mu$ is a finite measure, by (28),

\[
J_1 \lesssim \epsilon^q \int_{\partial D} \left( \int_{\Gamma(\zeta) \cap |w| \leq R_0} \mu(\Delta(w, r)) \frac{dA(w)}{(1 - |w|^2)^{1+q}} \right) d\zeta \lesssim \epsilon^q.
\]

Let $\delta_1 = \frac{2\delta}{1 + 2\delta}$. Using the fact that $\Delta(z_k, \delta) \subset \Delta(w, \delta_1)$ for any $w \in \Delta(z_k, \delta)$ and (3), we get

\[
J_2 \lesssim \int_{\partial D} \sum_{k \in \Delta(z_k, \delta) \cap \Gamma(\zeta) \neq \emptyset} \int_{\Delta(z_k, \delta) \cap |w| > R_0} F_n(w) \Phi_\mu(w) d\zeta(w) d\zeta
\]
\[
\lesssim \int_{\partial D} \left( \sup_{w \in \Gamma(\zeta) \cap |w| > R_0} \Phi_\mu(w) \right) \cdot \left( \sum_{k \in \Delta(z_k, \delta) \cap \Gamma(\zeta) \neq \emptyset} \sup_{w \in \Delta(z_k, \delta)} F_n(w) \right) d\zeta.
\]

By Lemma, we get

\[
\sup_{w \in \Delta(z_k, \delta)} F_n(w) \lesssim \int_{\Delta(z_k, \delta_1)} \Delta f_n''(z) dA(z).
\]
Therefore, using Hölder’s inequality and (27), we have

\[ J_2 \leq \left( \int_{\partial \Omega} \sup_{w \in \Gamma(z) \cap \{ |w| > R_0 \}} \Phi_{\rho}(w) \, dm(\zeta) \right)^{\frac{1}{p'}} \left( \int_{\partial \Omega} \left( \sum_{k: \Delta(z, \delta) \cap \Gamma(z) \neq \emptyset} \int_{\Delta(z, \delta)} |\Delta[f_n]\|^q(z) \, dA(z) \right)^{\frac{s}{q}} \, dm(\zeta) \right)^{\frac{1}{s}} \]

\[ \leq \epsilon^q \left( \int_{\partial \Omega} \left( \sum_{k: \Delta(z, \delta) \cap \Gamma(z) \neq \emptyset} \int_{\Delta(z, \delta)} |\Delta[f_n]\|^q(z) \, dA(z) \right)^{\frac{s}{q}} \, dm(\zeta) \right)^{\frac{1}{s}}. \]

Let \( z \in \Delta(z_k, \delta) \cap \Gamma(z) \). Then there exists \( 0 < r_1 < 1 \) depending only on \( \delta \) such that \( \Delta(z_k, \delta_1) \subset \Delta(z, r_1) \subset \Gamma_\gamma(\zeta) \).

Meanwhile, there exists a positive integer \( M \), such that for each \( z \in \Gamma_\gamma(\zeta) \) belongs to at most \( M \) pseudo-hyperbolic disks \( \Delta(z_k, \delta) \). Therefore, by Lemma 4

\[ J_2 \leq \epsilon^q \left( \int_{\Delta(z, \delta)} |\Delta[f_n]\|^q(z) \, dA(z) \right)^{\frac{s}{q}} \]

\[ \leq \epsilon^q ||f_n||^q_{H^p} \leq \epsilon^q. \]

Letting \( \epsilon \to 0 \), we get (26). The proof is complete. \( \square \)

By combining Theorems 14 and 15 we give a complete proof for Theorem 1, our main result in this paper.

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