The Wideband Slope of Interference Channels: 
The Large Bandwidth Case

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Abstract

It is well known that minimum received energy per bit $E_{b/N_0} \big|_{\min}$ in the interference channel is \(-1.59dB\) as if there were no interference. Thus, the best way to mitigate interference is to operate the interference channel in the low-SNR regime. However, when the SNR is small but non-zero, $E_{b/N_0} \big|_{\min}$ alone does not characterize performance. Verdu introduced the wideband slope $S_0$ to characterize the performance in this regime. We show that a wideband slope of $S_0 \leq \frac{1}{2}$ is achievable. This result is similar to recent results on degrees of freedom in the high SNR regime, and we use a type of interference alignment using delays to obtain the result. We also show that in many cases the wideband slope is upper bounded by $S_0 \leq \frac{1}{2}$ for large number of users $K$.

Index Terms

Interference channels, wideband slope, interference alignment.

I. INTRODUCTION

Recently there has been much interest in interference channels [1], [2], [3]. In [4] it was shown that in the high-SNR regime, it is possible to achieve $K/2$ degrees of freedom in a $K$-user interference channel (half of the $K$ degrees of freedom if there were no interference). The basic idea is to align interference from all $K-1$ undesired users in half the signal space, and then receive the desired signal in the other half space without interference, an idea pioneered by [5]. The paper [4] has inspired a large body of research on interference alignment in the high-SNR regime, for example [6], [7], [8], [9], [10], [11].

In this paper we consider the interference channel in the low-SNR regime, where explicitly

$$\text{SNR} \triangleq \frac{P}{B\sigma_0^2}, \quad (1)$$

$P$ is the input power, and $B$ is the system bandwidth. While the work in [4] and follow-up work shows impressively that much can be done to mitigate the effect of interference in the high-SNR regime, one could argue that the best way to mitigate the effect of interference is to avoid the high-SNR regime and instead operate in the low-SNR

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regime, when possible. It is well-known (e.g., [12]) that in a point-to-point channel the received minimum energy per bit $E_b N_0 \bigg|_{\min} = -1.59dB$ is achieved as the spectral efficiency (bits/s/Hz) $R \to 0$. It is also known from [12] that this energy is unchanged in the presence of interference. Thus, in this limit the effect of interference is completely eliminated. However, as Verdu pointed out in [13], in practical systems the spectral efficiency must be non-zero, though it might still be small. One way to characterize the effect of this is through the \textit{wideband slope}. The wideband slope is defined by

$$S_0 \triangleq \lim_{E_b N_0 \to E_b N_0 \bigg|_{\min}} \frac{R \left( \frac{E_b}{N_0} \right)}{10 \log_{10} \left( \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0} \bigg|_{\min} \right) 10 \log_{10} 2},$$

(2)

where $R \left( \frac{E_b}{N_0} \right)$ is the spectral efficiency as a function of $\frac{E_b}{N_0}$. The wideband slope essentially represents a second order approximation in the low power regime of the spectral efficiency as a function of SNR, or first order approximation of the spectral efficiency as a function of $\frac{E_b}{N_0}$. For example, we can write

$$R \approx \frac{S_0}{10 \log_{10} 2} \left( 10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0} \bigg|_{\min} \right) + \frac{R}{S_0} \cdot 10 \log_{10} 2.$$

Examples in [13] show that this is a good approximation for many channels up to fairly high spectral efficiencies, e.g., 1 bit/s/Hz. Further, [13] shows that $\frac{E_b}{N_0} \bigg|_{\min}$ and $S_0$ can be determined by the first and second order Taylor expansion coefficients of $R(\text{SNR})$ at SNR = 0, namely

$$\frac{E_b}{N_0} \bigg|_{\min} = \left. \frac{\log 2}{R(0)} \right|,$$

(3)

$$S_0 = \frac{-2 \left( \hat{R}(0) \right)^2}{\tilde{R}(0)}.$$

(4)

where $\hat{R}(0) = \frac{\partial R(\text{SNR})}{\partial \text{SNR}} \bigg|_{\text{SNR}=0}$ and $\tilde{R}(0) = \frac{\partial^2 R(\text{SNR})}{\partial \text{SNR}^2} \bigg|_{\text{SNR}=0}$.

The reference point for wideband slope is the point-to-point AWGN (additive white Gaussian noise) channel, which has a wideband slope of 2. The wideband slope also characterizes the bandwidth required to transmit at a given rate (in the low-SNR regime). For example, if the wideband slope is decreased from 2 to 1, twice the bandwidth is required for transmitting at a given rate.

The wideband slope for interference channels was considered for the 2-user channel in [14] (a generalization to QPSK can be found in [15]). They showed that TDMA (time-division multiple access) is not efficient in the low-SNR regime. In Section III we will extend the results of [14]. However, the main focus of the paper is the $K$-user channel, and in particular how interference alignment as in [4] can be used in the low-SNR regime.

Traditional interference alignment as in [4] does not work in the low-SNR regime. The results in [4] depend on time or frequency selectivity of the channel. However, to achieve the minimum energy per bit in a non-flat channel, all data needs to be transmitted on the strongest channel only – which means that the wideband slope is poor (e.g., $\frac{2}{K}$ for a $K$-user interference channel if only the strongest user transmits). On the other hand, delay differences between different paths can be effectively used. Delay differences for interference alignment was also considered in...
However, delay is a more natural fit for the low-SNR regime. Namely, as the bandwidth $B \to \infty$ even the smallest delay will eventually be magnified to the point of being much larger than the symbol duration. Therefore, delays can be efficiently manipulated and used for high bandwidth.

In this paper we will prove that interference alignment using delays can be used to achieve half the wideband slope of an interference-free channel, similar to losing half the degrees of freedom in the high-SNR regime. We will also show that generally it is difficult to obtain a larger wideband slope. The fact that wideband slope is reduced by only half means that near single-user performance can be obtained in the low-power regime. For example, if it is desired to transmit at $R = 0.5$ spectral efficiency, in the interference-free channel this requires 0.6dB extra energy over the minimum energy per bit for $R = 0$. With interference, 1.2dB, e.g., 0.6dB extra energy is needed to overcome interference, independently of the number of users.

II. SYSTEM MODEL

We consider a scalar complex $K$-user interference channel with Gaussian noise with line-of-sight (LOS) propagation. There are $K$ transmitters, numbered 1 to $K$, and $K$ receivers, also numbered 1 to $K$. Transmitter $j$ needs to transmit a message to receiver $j$, and receiver $j$ has no need for messages from transmitter $i$, $i \neq j$. All transmitters and receivers have one antenna. As the specifics of the wireless model affect the results, we will discuss in more details the physical modeling of the system. The transmitters and receivers are placed in a two or three dimensional space, where the distance from transmitter $i$ to receiver $j$ is denoted $d_{ji}$. Consistent with the LOS model, we assume the wireless signal propagates directly from transmitter $j$ to receiver $i$, and the delay in signal arrival is therefore determined by $d_{ji}$.

While the LOS model is particular, it does apply directly to some real systems, for example fractionated spacecraft [19]. An extension of results to multipath may be possible, but far from straightforward. Therefore, to obtain a concise mathematical theory we restrict attention to the LOS model.

Consider at first a single transmitter-receiver pair, $i$ and $j$. Let the complex discrete-time transmitted signal of transmitter $i$ be $x_i[n]$ and the corresponding baseband (continuous-time) signal be $x_i(t)$ with (two-sided) bandwidth $B$. Specifically, to satisfy a strict band limit we must have

$$x_i(t) = \sum_n x_i[n] \text{sinc}(Bt - n).$$

This is modulated with the carrier signal $c(t) = \exp(i(\omega_0(t - \varsigma_i)))$, where $\omega_0$ is the carrier frequency and $\varsigma_i$ is the delay (phase offset) in the oscillator at transmitter $i$ (and $i = \sqrt{-1}$). The real part is transmitted,

$$s_i(t) = \Re\{\exp i(\omega_0(t - \varsigma_i))x_i(t)\}$$

$$= \cos(\omega_0(t - \varsigma_i))\Re\{x_i(t)\} - \sin(\omega_0(t - \varsigma_i))\Im\{x_i(t)\}.$$

The received signal at receiver $j$ is

$$r_j(t) = A_{ji}\Re\{\exp i(\omega_0(t - \varsigma_i - \tau_{ji}))x_i(t - \tau_{ji})\} + \tilde{z}_j(t)$$

$$\tau_{ji} = \frac{d_{ji}}{c}$$
where $A_{ji}$ is an attenuation factor, $c$ is the speed of light, and $\tilde{z}_j(t)$ is white Gaussian noise with power spectral density $N_0$. This is modulated to baseband by multiplying with $\exp(-j\omega_0(t - \tau_{ji}))$, where $\tau_{ji}$ is the delay in the oscillator at receiver $j$, and using a lowpass filter, resulting in the baseband signal

$$y_j(t) = A_{ji} \exp(j\omega_0(\varsigma_i + \tau_{ji} - v_j))x_i(t - \tau_{ji}) + z_j(t).$$

This expression is valid on the assumption that $\omega_0 > B$. Here $z_j(t)$ is white Gaussian noise filtered to a bandwidth $B$.

Return now to the interference channel. When all users transmit, the received signal at receiver $j$ is

$$y_j(t) = A_{jj} \exp(j\omega_0(\varsigma_j + \tau_{jj} - v_j))x_j(t - \tau_{jj}) + \sum_{i \neq j} A_{ji} \exp(j\omega_0(\varsigma_i + \tau_{ji} - v_j))x_i(t - \tau_{ji}) + z_j(t).$$

This is sampled at the Nyquist frequency $f_s = B$ (as $B$ is the two-sided bandwidth). Let

$$n_{ji} = \lfloor \tau_{ji}B + \frac{1}{2} \rfloor$$

$$\delta_{ji} = \tau_{ji}B - \lfloor \tau_{ji}B + \frac{1}{2} \rfloor$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to $x$. Without loss of generality we can assume that the received signal at receiver $j$ is sampled symbol-synchronous with the desired signal. Then the discrete-time model is

$$y_j[n] = A_{jj} \exp(j\omega_0(\varsigma_j + \tau_{jj} - v_j))x_j[n - n_{jj}] + \sum_{i \neq j} A_{ji} \exp(j\omega_0(\varsigma_i + \tau_{ji} - v_j))\tilde{x}_i[n - n_{ji}] + z_j[n]$$

where $z_j[n]$ is as sequence of i.i.d circularly symmetric random variables, $z_j[n] \sim \mathcal{N}(0, B_{N_0})$, and

$$\tilde{x}_i[n] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m + \delta_{ji}).$$

We will also occasionally make the dependency on the fractional delay explicit as follows

$$\tilde{x}_i[n, \delta_{ji}] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m + \delta_{ji}).$$

By the Shannon sampling theorem, this discrete-time model is equivalent with the original continuous-time model. Results do not change if we normalize the time at each receiver so that $n_{jj} = 0$. And as the carrier frequency is large, the phases $\exp(j\omega_0(\varsigma_i + \tau_{ji} - v_j))$ can be reasonably modeled as independent uniform random variables $\theta_{ji}$ over the unit circle. We therefore arrive at the following expression for the received signal

$$y_j[n] = A_{jj} \exp(j\theta_{jj})x_j[n] + \sum_{i \neq j} A_{ji} \exp(j\theta_{ji})\tilde{x}_i[n - n_{ji}] + z_j[n]$$

$$= C_{jj}x_j[n] + \sum_{i \neq j} C_{ji}\tilde{x}_i[n - n_{ji}] + z_j[n]$$

where $C_{jj} = A_{jj} \exp(j\theta_{jj})$.

Notice that this model makes no assumptions on or approximations of modulation, e.g., it does not assume rectangular waveforms. Transmission in our model is strictly bandlimited to a bandwidth $B$, as opposed to [16].
A. Approaching the Low-SNR Regime: Large B Case and Small B Case

What is interesting is that there are two distinct ways to approach the low-SNR regime, which have very different impacts on the performance of the interference channel defined by (11). Although approaching the low-SNR regime by letting $B \to \infty$ is emphasized in previous papers, it is not the only way. As can be noted from the definition of SNR, SNR approaches zero if either $B \to \infty$ or $P \to 0$. Consider a point-to-point AWGN channel with spectral efficiency

$$R = \log \left( 1 + \frac{P}{BN_0} \right).$$

The low-SNR results are based on a Taylor series of $\log(1 + x)$, as also seen by (3-4); therefore as long as $\text{SNR} = \frac{P}{NB_0} \to 0$ low-power results such as minimum energy per bit and wideband slope are valid. The key is that the spectral efficiency $R \to 0$, not that $B \to \infty$. For the interference channel (11), on the other hand, different results are obtained depending on how the low-SNR regime is approached.

In the first approach, let $B \to \infty$ while $P$ is fixed and finite. We call this approach the large bandwidth case. In this case, the propagation delay is large compared with the symbol duration, i.e., as $B \to \infty$, $n_{ji}$ can become arbitrarily large even for very small $\tau_{ji}$.

In the second approach, let $P \to 0$ while $B$ is fixed and finite. We further assume that the propagation delay is much smaller than the symbol duration, i.e., $\tau_{ij}B \ll 1$. Under this assumption, $n_{ji} = 0$ and $\delta_{ji} \approx 0$. This approach is called the small bandwidth case.

The large bandwidth case is the topic of this paper; the small bandwidth case will be considered in a later paper.

B. Performance criteria

In [14] the whole slope region of the interference region in the 2-user case was analyzed. However, for more than two users it is complicated to compare complete slope regions, and we are therefore looking at a single quantity to characterize performance. We denote the power constraint for each user $P_i$ and the spectral efficiency $R_i$; we further set $\text{SNR}_i = \frac{P_i}{BN_0}$. We consider two different constraints

- **The equal power constraint.** In this case we maximize the sum spectral efficiency $R_s = R_1 + R_2 + \cdots + R_K$ under the constraint $P_1 = P_2 = \cdots = P_K$, i.e., $\text{SNR}_1 = \text{SNR}_2 = \cdots = \text{SNR}_K$. We want to characterize the wideband slope of the sum spectral efficiency $R_s$.

- **The equal rate constraint.** In this case we minimize the total power per Hz $\text{SNR} = \text{SNR}_1 + \text{SNR}_2 + \cdots + \text{SNR}_K$ under the constraint $R = R_1 = R_2 = \cdots = R_K$. We want to characterize the wideband slope of the sum spectral efficiency $R_s = K \cdot R$.

The equal power constraint could correspond to a scenario where each user needs to consume energy at the same rate, e.g., so that batteries last the same for all users. The equal rate constraint could correspond to a scenario where we want to minimize total system energy consumption. Each constraint can be easily generalized to unbalanced cases, e.g. $\mu_1 \text{SNR}_1 = \mu_2 \text{SNR}_2 = \cdots = \mu_K \text{SNR}_K$, but we only consider the balanced case here to keep results concise.
As performance measure we use

$$\Delta S_0 = \frac{S_0}{S_{0,\text{no interference}}}.$$ 

The quantity $S_{0,\text{no interference}}$ is the wideband slope of a $K$-user interference channel where all interference links are nulled, $|C_{ij}| = 0$, $i \neq j$, but the direct links $C_{ii}$ are unchanged. We can interpret $\Delta S_0$ as the loss in wideband slope due to interference, or equivalently $(\Delta S_0)^{-1}$ as (approximately) the additional bandwidth required to overcome interference. Alternatively, if we define

$$\Delta E_b = 10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_{0\min}}$$

as the extra energy required to operate at a spectral efficiency $R > 0$, we have

$$\Delta E_b \approx (\Delta S_0)^{-1} \Delta E_{b,\text{no interference}}$$

for small increases in spectral efficiency. Thus, $(\Delta S_0)^{-1}$ also measures the amount of energy needed to overcome interference.

C. $\frac{E_b}{N_0}_{\text{min}}$ of the Interference Channel

The papers [13] and [20] show that the minimum energy per bit of the interference channel is equal to that of an interference-free channel, achievable by Treating Interference as Noise (TIN) and TDMA. The following theorem gives the transmitted $\frac{E_b}{N_0}_{\text{min}}$ under the two different constraints.

**Theorem 1.** The minimum energy per bit of the interference channel defined by (5) is

$$\frac{E_b}{N_0}_{\text{min}} = \frac{\sum (|C_{jj}|^{-2})}{K} \log_e 2$$

(12)

under the equal rate constraint; and

$$\frac{E_b}{N_0}_{\text{min}} = \frac{K \log_e 2}{\sum_{j=1}^{K} |C_{jj}|^2}$$

(13)

under the equal power constraint.

The best known achievable rate for the interference channel is the Han-Kobayashi region [21]. For the Gaussian interference channel, in particular the idea of transmitting common messages has been shown to be powerful [22]. However, the common message does have a higher $\frac{E_b}{N_0}$ in the low power limit than the minimum, and therefore does not improve the wideband slope. To make fair comparison any bound imposed on the wideband slope must have the correct $\frac{E_b}{N_0}_{\text{min}}$. We emphasize this requirement in the following remark.

**Remark 2.** For a bound on rate to be useful as a bound on wideband slope, it needs to have the correct $\frac{E_b}{N_0}_{\text{min}}$ given in Theorem [1]
III. THE 2-USER CASE

We will start by analyzing the 2-user case as this is instrumental for the K-user case. As we have discussed in Section II-A, the essential difference between large bandwidth case and small bandwidth case is that they impact the behavior of propagation delays differently. It turns out that all results in the 2-user case are independent of delay, thus independent of how the low-SNR regime is approached. This indicates that the capacity region in the 2-user case could be independent of delay in general, but we have not been able to prove so.

A. Achievable Schemes

First we will outline the strategies that can be used for the achievable rate. In order to use these to inner bound the the sum slope, as mentioned in Remark 2, they must have the correct $\frac{E_b}{N_0} \Big|_{\text{min}}$, and that only leaves three strategies

1) **Interference decoding.**

If $|C_{ji}| > |C_{ii}|$, user $j$ can decode the message from user $i$, and the capacity region of the interference channel is equivalent to the capacity region of the multi-access channel formed by transmitter $i$, transmitter $j$, and receiver $j$, which is

\[
R_j \leq \log \left( 1 + |C_{jj}|^2 \text{SNR}_j \right)
\]

\[
R_i \leq \log \left( 1 + |C_{ii}|^2 \text{SNR}_i \right)
\]

\[
R_i + R_j \leq \log \left( 1 + |C_{ji}|^2 \text{SNR}_i + |C_{jj}|^2 \text{SNR}_j \right). \tag{16}
\]

In the low-SNR regime, as $\text{SNR} \to 0$, there always exists some real number $\epsilon > 0$ such that if $\text{SNR}_j, \text{SNR}_i < \epsilon$ the sum slope outer bound given by the summation of (14) and (15) is less than the sum slope outer bound given by (16) because $|C_{ji}| > |C_{ii}|$. Therefore, (16) can be discarded and the multi-access bound is equivalent to the rectangular capacity region of a channel with no interference. Thus, interference does not affect wideband slope in this case.

2) **Treating interference as noise (TIN).**

The transmitters use i.i.d Gaussian code books, and the receivers treat the interference as part of the background noise. Notice that delay does not affect the distribution of interference as $\bar{x}_i[n - n_{ji}]$ has same distribution as $x_i[n]$. The achievable $(R_1, R_2)$ is

\[
R_1 \leq \log \left( 1 + \frac{|C_{11}|^2 \text{SNR}_1}{1 + |C_{12}|^2 \text{SNR}_2} \right) \tag{17}
\]

\[
R_2 \leq \log \left( 1 + \frac{|C_{22}|^2 \text{SNR}_2}{1 + |C_{21}|^2 \text{SNR}_1} \right). \tag{18}
\]

3) **TDMA.**

In time-division multiple access the transmitters use orthogonal time slots. Because of the delay differences, users have to insert buffers with no transmission around each TDMA frame, so that they are orthogonal at both users. However, the length of these buffers is finite, so as the code length converges towards infinity (as
required by capacity analysis), the effect of these buffers on spectral efficiency will converge towards zero.

TDMA therefore achieves the following spectral efficiency also in the case of delays,

\[
R_1 \leq \frac{1}{2} \log \left(1 + 2 |C_{11}|^2 \text{SNR}_1 \right) \quad (19)
\]

\[
R_2 \leq \frac{1}{2} \log \left(1 + 2 |C_{22}|^2 \text{SNR}_2 \right).
\]

The achievable sum slope can easily be straightforwardly calculated from these equations using (4). The expressions are too complex to give much insight, so we will only state them for later reference for a canonical 2-user channel with symmetric interference link gains.

**Theorem 3.** Consider a 2-user interference channel where \(|C_{jj}|^2 = 1\) and \(|C_{ji}|^2 = a, i \neq j\). The sum slope is inner bounded by

\[
S_0 \geq \begin{cases} 
4 & a > 1 \\
2 & \frac{1}{2} < a < 1 \\
\frac{1}{1+2a} & a \leq \frac{1}{2}
\end{cases} \quad (21)
\]

under both the equal power constraint and the equal rate constraint.

**B. Outer Bounds**

In this section, we will state some sum slope outer bounds, and discuss the so-called noisy interference channel, where the exact sum slope is known.

The following theorem generalizes Theorem 2 from [23] to channels with delay.

**Theorem 4** (Kramer’s bound). Suppose that \(|C_{21}| < |C_{11}|\). Then

\[
R_1 \leq \log \left(1 + |C_{11}|^2 \text{SNR}_1 \right) \quad (22)
\]

\[
R_2 \leq \log \left( 1 + \frac{|C_{22}|^2 \text{SNR}_2 + |C_{21}|^2 \text{SNR}_1}{|C_{11}|^2 2R_1 + 1 - \frac{|C_{21}|^2}{|C_{11}|^2}} \right) \quad (23)
\]

\[
R_1 + R_2 \leq \log \left(1 + |C_{11}|^2 \text{SNR}_1 \right) + \log \left(1 + \frac{|C_{22}|^2 \text{SNR}_2}{1 + |C_{21}|^2 \text{SNR}_1} \right) \quad (24)
\]

independent of delay.

**Proof:** Put \(C_{12} = 0\) to enlarge the capacity region. Now assume that, different from the system model (11), receiver 2 also samples the received signal synchronously with the transmitted signal of user 1. A Z-channel with

\footnote{According to the sampling theorem, this does not change the capacity region.}
delay is formed:

\[ y_1'[n] = C_{11}x_1[n] + z_1[n] \]
\[ y_2[n] = C_{22}\tilde{x}_2[n - n_{22}] + C_{21}x_1[n] + z_2[n] \]

(26)

where \( \tilde{x}_2[n] \) is defined by (9).

Next, we show that the capacity region of (26) is independent of delay. The channel (a) and (b) illustrated in Figure 1 have identical capacity regions because \( p(\tilde{y}_2|x_1, \tilde{x}_2) \) and \( p(y_2|x_1, \tilde{x}_2) \) have the same distribution. \( \tilde{z}_2[n] \) is i.i.d Gaussian noise independent of \( z_1[n] \) and the input signals, with power \( \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) N_0B \). Because \( \frac{|C_{21}|^2}{|C_{11}|^2} < 1 \), such \( \tilde{z}_2[n] \) is guaranteed to exist. The argument is identical to (a)–(c) of Figure 6 in [24]. Details are skipped here.

The channel (b) has the form

\[ y_1'[n] = C_{11}x_1[n] + z_1[n] \]
\[ \tilde{y}_2[n] = C_{22}\tilde{x}_2[n - n_{22}] + C_{21}y_1'[n] + \tilde{z}_2[n]. \]

(27)

(28)

Using Fano’s inequality as usual, we can now bound the capacity of this channel by

\[ nR_2 - n\epsilon_n \leq h(\tilde{y}_2^n) - h(\tilde{y}_2^n|\tilde{x}_2) \]

(29)

\[ \leq h(\tilde{y}_2^n) - h(\tilde{y}_2^n|w_2) \]

(30)

\[ = h(\tilde{y}_2^n) - h \left( \frac{C_{21}}{C_{11}}y_1^n + \tilde{z}_2^n | w_2 \right), \]

(31)
where $w_2$ is the message sent by transmitter 2. The step (29) to (28) is from the data processing inequality, as $\tilde{x}_2$ is a function of the transmitted codeword $x_2$, which is a function of $w_2$. The second term in (31) is independent of delay, and can be lower bounded by the entropy power inequality \[25\]. The first term can be upper bounded by the delay-free case. Therefore, the capacity region of (26) is identical to that of the channel without delay. The papers \[24\] and \[23\] show that the capacity region of delay-free channel can be derived from an equivalent degraded broadcast channel. Given Theorem 1 in \[26\] its rate region has upper bound (22). Finally, it is easy to see that the capacity region of (11) is contained within that of the Z-channel. The equation (25) is a restatement of (47) in \[23\].

We use Theorem 4 to obtain a sum slope outer bound under the equal power constraint as follows,

**Corollary 5.** Suppose that $|C_{21}| < |C_{11}|$. Under the equal power constraint, the wideband slope for the sum rate has outer bound

$$S_0 \leq 2 \left( \frac{|C_{11}|^2 + |C_{22}|^2}{2|C_{21}|^2 |C_{22}|^2 + |C_{11}|^4 + |C_{22}|^4} \right)$$

$$\Delta S_0 \leq 2 \frac{|C_{21}|^2 |C_{22}|^2}{|C_{11}|^4 + |C_{22}|^4} + 1$$

independent of delay.

**Proof:** This result can be easily shown combining (25) and the formulas (3) and (4). □

Results similar to Theorem 4 and Corollary 5 can be obtained for $|C_{12}| < |C_{22}|$ case by interchanging the indices '1' and '2'.

For the equal-rate constraint if only one interference link is weak, bound (22) does not have the correct $\frac{E_b}{N_0} \min$ and therefore cannot be used for bounding the wideband slope by Remark 2. If both interference links are weak, we have following corollary.

**Corollary 6.** Suppose that $|C_{21}| < |C_{11}|$ and $|C_{12}| < |C_{22}|$. Under the equal rate constraint, the wideband slope for the sum rate is upper bounded by

$$S_0 \leq 4 \cdot (|C_{11}|^2 + |C_{22}|^2) \left( 1 - \frac{|C_{12}|^2 |C_{21}|^2}{|C_{22}|^2 |C_{11}|^2} \right) \cdot (|C_{11}|^2 + |C_{22}|^2 +$$

$$|C_{21}|^2 \left( 2 - 3 \frac{|C_{12}|^2}{|C_{22}|^2} \right) + |C_{12}|^2 \left( 2 - 3 \frac{|C_{21}|^2}{|C_{11}|^2} \right)^{-1}$$

$$\Delta S_0 \leq (|C_{11}|^2 + |C_{22}|^2) \left( 1 - \frac{|C_{12}|^2 |C_{21}|^2}{|C_{22}|^2 |C_{11}|^2} \right) \cdot (|C_{11}|^2 + |C_{22}|^2 +$$

$$|C_{21}|^2 \left( 2 - 3 \frac{|C_{12}|^2}{|C_{22}|^2} \right) + |C_{12}|^2 \left( 2 - 3 \frac{|C_{21}|^2}{|C_{11}|^2} \right)^{-1}$$

independent of delay.
Proof: \cite{22} gives
\[
|C_{21}|^2 \text{SNR}_1 + |C_{22}|^2 \text{SNR}_2 \geq 2^{R_2} \left( \frac{|C_{21}|^2}{|C_{11}|^2} - \frac{|C_{21}|^2}{|C_{11}|^2} + 1 \right) - 1
\]  
(34)

\[
|C_{11}|^2 \text{SNR}_1 + |C_{12}|^2 \text{SNR}_2 \geq 2^{R_1} \left( \frac{|C_{12}|^2}{|C_{22}|^2} - \frac{|C_{12}|^2}{|C_{22}|^2} + 1 \right) - 1
\]  
(35)

Under the equal rate constraint, \( R_1 = R_2 = \frac{B}{2} \), and our objective is to minimize \( \text{SNR}_1 + \text{SNR}_2 \). We construct the following optimization problem

\[
\begin{align*}
\min & \quad \text{SNR}_1 + \text{SNR}_2 \\
\text{s.t.} & \quad A \begin{pmatrix} \text{SNR}_1 \\ \text{SNR}_2 \end{pmatrix} \geq b \\
& \quad P_j \geq 0
\end{align*}
\]

where \( A = \begin{pmatrix} |C_{21}|^2 & |C_{22}|^2 \\ |C_{11}|^2 & |C_{12}|^2 \end{pmatrix} \) and \( b = \begin{pmatrix} 2^{R_1} \cdot \left( \frac{|C_{21}|^2}{|C_{11}|^2} - \frac{|C_{21}|^2}{|C_{11}|^2} + 1 \right) - 1 \\ 2^{R_2} \cdot \left( \frac{|C_{12}|^2}{|C_{22}|^2} - \frac{|C_{12}|^2}{|C_{22}|^2} + 1 \right) - 1 \end{pmatrix} \). Using simple linear programming principles, one optimal solution can be found at the vertex of the feasible region. That is, \( \text{SNR}_1 + \text{SNR}_2|_{\text{min}} = \text{SNR}_{1o} + \text{SNR}_{2o} \) where \( \begin{pmatrix} \text{SNR}_{1o} \\ \text{SNR}_{2o} \end{pmatrix} = A^{-1}b > 0 \). We solve this simple linear system and get

\[
\text{SNR}_{1o} = |C_{11}|^{-2} \cdot \frac{2^{R_1} \left( \frac{|C_{22}|^2}{|C_{12}|^2} \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) + \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) \left( 2^{R_2} - 1 \right) \right)}{1 - \frac{|C_{21}|^2}{|C_{22}|^2} \frac{|C_{21}|^2}{|C_{11}|^2}}
\]  
(36)

\[
\text{SNR}_{2o} = |C_{22}|^{-2} \cdot \frac{2^{R_2} \left( \frac{|C_{21}|^2}{|C_{11}|^2} \left( 1 - \frac{|C_{22}|^2}{|C_{12}|^2} \right) + \left( 1 - \frac{|C_{22}|^2}{|C_{12}|^2} \right) \left( 2^{R_1} - 1 \right) \right)}{1 - \frac{|C_{22}|^2}{|C_{21}|^2} \frac{|C_{22}|^2}{|C_{11}|^2}}
\]  
(37)

Now we have the expression of sum power as a function of sum rate. The following formulas are equivalent to (3) and (4).

\[
\frac{E_b}{N_0} = \left. \frac{d\text{SNR} (R)}{dR} \right|_{R=0}
\]  
(38)

\[
S = 2 \left. \frac{d^2\text{SNR}(R)}{dR^2} \right|_{R=0} \log 2.
\]  
(39)

They can be proved using a technique similar to (140)-(144) in \cite{13}. Details are skipped here. Combining (36), (37) and (39), we have

\[
\frac{E_b}{N_0} = \left( \frac{|C_{11}|^{-2} + |C_{22}|^{-2}}{2} \right) \log_e 2
\]

\[
S_0 = 4 \cdot (|C_{11}|^2 + |C_{22}|^2) \left( 1 - \frac{|C_{12}|^2}{|C_{22}|^2} \frac{|C_{21}|^2}{|C_{11}|^2} \right) \cdot (|C_{11}|^2 + |C_{22}|^2)
\]

\[
+ |C_{21}|^2 \left( 2 - 3 \frac{|C_{12}|^2}{|C_{22}|^2} \right) + |C_{12}|^2 \left( 2 - 3 \frac{|C_{21}|^2}{|C_{11}|^2} \right)^{-1}
\]

\[\blacksquare\]
For the 2-user interference channel without delay, \[1\], \[3\] and \[2\] show that there exists a class of channels whose optimal sum spectral efficiency can be achieved by i.i.d. Gaussian inputs and treating interference as noise. This class of channel is one of the few where the exact capacity is known, and consequently also the exact sum slope. We here extend these results to channels with delay.

**Theorem 7.** For a 2-user interference channel defined by (11), if there exist complex numbers \(\rho_1, \rho_2\) and positive real numbers \(\sigma_1^2, \sigma_2^2\) such that,

\[
|\rho_1|^2 \leq \frac{\sigma_1^2}{1 - |\rho_2|^2} \leq 1 \quad (40)
\]

\[
|\rho_2|^2 \leq \frac{\sigma_2^2}{1 - |\rho_1|^2} \leq 1 \quad (41)
\]

\[
C_{21} = \frac{\rho_1 C_{11}}{|C_{12}|^2 SNR_2 + 1} \quad (42)
\]

\[
C_{12} = \frac{\rho_2 C_{11}}{|C_{12}|^2 SNR_2 + 1} \quad (43)
\]

then the optimal sum capacity

\[
R_1 + R_2 \leq \log \left( 1 + \frac{|C_{11}|^2 SNR_1^2}{1 + |C_{12}|^2 SNR_2} \right) + \log \left( 1 + \frac{|C_{22}|^2 SNR_2^2}{1 + |C_{21}|^2 SNR_1} \right) \quad (44)
\]

is achievable by i.i.d. Gaussian input and treating interference as noise at the receivers. Further, (40) ~ (43) are satisfied as long as

\[
\sqrt{\frac{|C_{12}|^2}{|C_{22}|^2} (1 + |C_{21}|^2 SNR_1)} + \sqrt{\frac{|C_{21}|^2}{|C_{11}|^2} (1 + |C_{12}|^2 SNR_2)} \leq 1. \quad (45)
\]

**Proof:** Please see Appendix A.

Theorem 7 is identical to the case where there is no delay, which is discussed in \[27\], Theorem 6. We now use Theorem 7 to derive the exact sum slope.

**Corollary 8.** Consider the 2-user interference channel defined by (11). Under the equal power constraint, if the channel coefficients satisfy

\[
\sqrt{\frac{|C_{12}|^2}{|C_{22}|^2}} + \sqrt{\frac{|C_{21}|^2}{|C_{11}|^2}} < 1, \quad (46)
\]

then i.i.d. Gaussian inputs and treating interference as noise achieve the optimal sum slope \(S_0\), which is

\[
S_0 = \frac{2 \left( |C_{11}|^2 + |C_{22}|^2 \right)^2}{|C_{11}|^4 + |C_{22}|^4} \quad (47)
\]

\[
\Delta S_0 = 1 + 2 \frac{\left( |C_{11}|^2 |C_{12}|^2 + |C_{21}|^2 |C_{22}|^2 \right)}{|C_{11}|^4 + |C_{22}|^4}. \quad (48)
\]

**Proof:** Under the equal power constraint where \(SNR_i = \frac{SNR_i}{2}\) there must exist some \(\epsilon > 0\), such that if \(SNR < \epsilon\) then (45) can be satisfied. Because the low-SNR regime is approached as \(SNR \rightarrow 0\), this gives (46). Given (44), under the equal power constraint the sum rate achieved by treating interference as noise is
\[ R_s \leq \log \left( 1 + \frac{|C_{11}|^2 \text{SNR}}{2} \right) \\
+ \log \left( 1 + \frac{|C_{12}|^2 \text{SNR}}{2} \right) + \log \left( 1 + \frac{|C_{22}|^2 \text{SNR}}{2} \right), \] 

(49)

Combining (49) with (3) and (4) we have (47).

Figure 2 illustrates the sum slope region of a 2-user interference channel with unit direct link gain, and symmetric cross link gain, that is, \(|C_{11}|^2 = |C_{22}|^2 = 1\), and \(|C_{12}|^2 = |C_{21}|^2 = a\). In this figure, the inner bound is given by Theorem 3, the inner bounds labeled “Strong Int.”, “Achievable, TDMA”, and “Achievable TIN” are represented by the first, the second and the last line in (21), respectively. The outer bound is given by (32). Given Corollary 8 if \(a \leq \frac{1}{2}\), treating interference as noise achieves optimal the sum slope, i.e., the inner bound is tight, which is also indicated on the figure.

The focal point here is the point \(a = 1\). Just above that, the effect of interference is completely eliminated. Just below that, interference is at its worst. One could wonder if, for the \(K\)-user case, the former fact could be used effectively. It turns out that is not the case. In Section IV-B we will show that in a \(K\)-user interference channel when \(K\) is large, with high probability each user will form an 2-user weak interference pair, where \(a\) is just below 1, with some other user.

Figure 2: Sum slope versus \(\frac{|C_{21}|}{|C_{11}|}\). In the legend, TIN stands for treating interference as noise.

IV. THE \(K\)-USER CASE

A. Achievable Scheme and Inner Bound

For the 2-user case, the achievable rates are unaffected by delay, as seen in Section III-A. However, the \(K\)-user case is very different from the 2-user case, just as for the high SNR case considered in [4]. Similar to the high
SNR case, we can obtain a significant increase in rate by using a variation of interference alignment. The type of interference alignment used in [4] based on time or frequency selectivity does not work in the low-SNR regime; however, propagation delays can be used. Specifically, we show that for any set of delays $\tau_{ji}$, $i, j = 1, \cdots, K$ that are linearly independent over the rational numbers, there exist arbitrarily large $B$ that can make the direct propagation delays $\tau_{ii}$ arbitrarily close to an even integer while the cross-delays $\tau_{ji}$, $j \neq i$ are close to some odd integer. As a result, if we let each user use even time slots in the discrete time baseband channel model, then at the receiver, the desired message and the interference signal are almost orthogonal in the time domain. Therefore, the interference channel can achieve $\Delta S_0 = \frac{1}{2}$.

The idea of interference alignment over time domain is also used in [16], [17], and [18]. However, delay is much more efficient when we let $B \to \infty$, as $n_{ji} = \lfloor \tau_{ji}B \rfloor$ can become arbitrarily large. We therefore do not need to use the approximation $\delta_{ji} \approx 0$ as in [16] or large $K$ as in [17]. The paper [18] mainly discusses how to design an algorithm to place $K$ users in an $N$ dimensional space, $N > 3$, so that interference alignment can be realized. In our work, $N \geq 2$ and user locations are given. Our method for interference alignment works for any user location (with probability 1 for a continuous distribution of user locations).

In order to state the results, we need to refine the definition of wideband slope (2) as

$$
\frac{E_b}{N_0 \min} = \liminf_{B \to +\infty} \frac{P}{R(N_0B)}
$$

(50)

$$
S_0 \triangleq \limsup_{B \to +\infty} \frac{R\left(\frac{E_b}{N_0}\right)}{10\log_{10} \frac{E_b}{N_0} - 10\log_{10} \frac{E_b}{N_0 \min}} 10\log_{10} 2.
$$

(51)

Notice that if the limit exists, (51) is identical to (2), so this is not a new definition, but a widening of the applicability of the wideband slope. We will see through an example in Section IV-A2 that this generalized definition has an operational meaning, as does the wideband slope in [13].

For comparison purposes, we will list the results for the interference-free case, i.e., $C_{ji} = 0$ for $i \neq j$ as follows (directly obtained from [13, Theorem 9]):

$$
S_{0, \text{no interference}} = \frac{2\left(\sum_j |C_{jj}|^2\right)^2}{\sum_j |C_{jj}|^4} \quad \text{equal power constraint};
$$

$$
S_{0, \text{no interference}} = 2K \quad \text{equal rate constraint}.
$$

1) The Achievable Sum Slope: In the following, we will precisely specify the interference alignment scheme we use to obtain the achievable sum slope.

**Definition 9 (Delay-based interference alignment).** Fix the transmission bandwidth $B_0 \leq B$.

1) At transmitter $j$

   - Use a codebook generated from independent Gaussians according to

   $$
   x_j[n] \sim \begin{cases} 
   \mathcal{N}\left(0, \frac{2P}{B_0}\right) & n = 2k \\
   0 & n = 2k + 1 
   \end{cases} \quad k \in \mathbb{Z}.
   $$

   (52)
Generate the baseband transmitted signal according to

\[ x_j(t) = \sum_{n=-\infty}^{+\infty} x_j[n] \text{sinc} (B_0 (t - nT)). \]

Notice that this signal has bandwidth \( B_0 \leq B \).

2) At receiver \( j \):

- Sample the received continuous time baseband signal with rate \( B_0 \) and symbol synchronize with \( x_j[n] \);
- Discard \( y_j[2m+1], m \in \mathbb{Z} \);
- Decode the desired message from \( y_j[2m], m \in \mathbb{Z} \), with typical decoding while treating any remaining interference as noise.

Since we only use every other time slot for transmission in (52), we can at most achieve a wideband slope \( \Delta S_0 = \frac{1}{2} \). In the following we will show that it is possible to choose the transmission bandwidth \( B_0 \) so that most of the interference lines up in the discarded time slots \( y_j[2m+1], m \in \mathbb{Z} \), which in turn means that we can actually achieve \( \Delta S_0 = \frac{1}{2} \).

The concept of linearly independent over rational number will be introduced first.

**Definition 10** ([28]). A set of real numbers \( \theta = \{\theta_1, \theta_2, \ldots, \theta_n\} \) are linearly independent over rational number if \( \sum a_i \theta_i = 0 \) only if \( a_i = 0 \) for all \( a_i \in \mathbb{Z} \).

**Lemma 11.** If \( \tau_{ji}, i, j \in \{1, \ldots, K\}, i \neq j \) are linearly independent over the rational numbers, then for any \( \delta > 0 \), there exist an arbitrarily large real number \( B \), such that

\[ |\tau_{ji}B - 2k_{ji} - 1| \leq \delta \]  

for some integers \( k_{ji}, j, i \in \{1, \ldots, K\} \).

The proof of Lemma 11 is based on the following fundamental approximation results in number theory.

**Theorem 12** ([28, Theorem 7.9, First Form of Kronecker’s Theorem]). If \( \alpha_1, \ldots, \alpha_n \) are arbitrary real numbers, if \( \theta_1, \ldots, \theta_n \) are linearly independent real numbers over the rational numbers, and if \( \epsilon > 0 \) is arbitrary, then there exists an real number \( t \) and integers \( h_1, \ldots, h_n \) such that

\[ |t\theta_i - h_i - \alpha_i| < \epsilon \]  

\( \forall i \in \{1, 2, \ldots, n\} \)

We also have

**Lemma 13.** ([28, Exercise 7.7, page 160]) Under the hypotheses of Theorem 12 if \( T > 0 \) is given, there exists a real number \( t > T \) satisfying the \( n \) inequalities (54).

Now let us prove Lemma 11.
Proof of Lemma 11: Let \( \alpha_1, \ldots, \alpha_n = 0.5, \epsilon = \frac{\delta}{2} \). According to Theorem 12, there exist arbitrarily large real number \( \hat{B} \) and some integers \( n_{ji}, i, j \in \{1, \ldots, K\} \) such that

\[
|\tau_{ji} \hat{B} - k_{ji} - 0.5| \leq \frac{\delta}{2}.
\]

Let \( B = 2 \hat{B} \), we have

\[
|\tau_{ji} B - 2k_{ji} - 1| \leq \delta.
\]

Combining the inequality above with Lemma 13, Lemma 11 is proved.

Lemma 11 shows that using this transmission scheme, the desired signal is almost orthogonal with the interference signal in time domain. However, there is always some interference leaking into the signal time slots. We need to show that as the fractional delay \( \delta_{ji} \to 0 \), the power of this interference become negligible. For this we need the following lemma,

**Lemma 14.** Under the assumptions \( x_j[2m] \) are i.i.d. Gaussian random variable with distribution \( \mathcal{N}(0, 2P_j) \) and \( x_j[2m+1] = 0 \) for all \( j \) and \( m \), \( \mathbb{E}[\tilde{x}_i[n_1, \delta_{ji}]\tilde{x}_i[n_2, \delta_{ji}]] \) is a continuous function of \( \delta_{ji} \) which satisfies

\[
\lim_{\delta_{ji} \to 0} \mathbb{E}[\tilde{x}_i[n_1, \delta_{ji}]\tilde{x}_i[n_2, \delta_{ji}]] = \begin{cases} 
2P_i & \text{if } n_1 = n_2 = 2k, \\
\text{for some integer } k \\
0 & \text{o.w.}
\end{cases}
\]

**Proof:** Please see Appendix B.

Equipped with the interference alignment scheme in definition 9, Lemma 11 and Lemma 14 we proceed to show the main results on the achievable sum slope of the \( K \)-user interference channel.

**Theorem 15.** Suppose that the set of delays \( \tau_{ji}, i, j \in \{1, \ldots, K\}, i \neq j \) are linearly independent over the rational numbers. Then the following wideband slope is achievable

\[
S_0 = \frac{\left(\sum_j |C_{jj}|^2\right)^2}{\sum_j |C_{jj}|^4} \quad \text{equal power constraint};
\]

\[
S_0 = K \quad \text{equal rate constraint}.
\]

Under both constraints,

\[
\Delta S_0 = \frac{1}{2}
\]

is achievable.

**Proof:** Assume that the system uses the transmission scheme proposed in Definition 9. Let

\[
\epsilon_j(B) = \sum_{i \neq j} K \mathbb{E}\left[\left(\tilde{x}_i[2n]\right)^2\right]
\]

denote the power of the leaked interference.
The best rate with this scheme is clearly achieved if the leaked interference power is zero; in that case the channel is an interference-free channel where half the symbols are not used. We can therefore conclude

$$\Delta S_0 \leq \frac{1}{2}$$ (55)

On the other hand, taking into account the leaked interference, the achievable rate at receiver \(j\) is

$$R_j = \frac{1}{2} \log \left( 1 + \frac{|C_{jj}|^2 2P_j}{BN_0} \right)$$

(56)

$$= |C_{jj}|^2 \frac{P_j}{BN_0} - \left( \epsilon_j(B)|C_{jj}|^2 P_j + |C_{jj}|^4 P_j^2 \right) \left( \frac{1}{BN_0} \right)^2 + o \left( \left( \frac{1}{BN_0} \right)^2 \right).$$

(57)

The wideband slope is a continuous function of the coefficients in the first two terms in the Taylor series of \(R_j\) in \(B_j\). According to Lemma 11 for any \(\delta > 0\) there exists some \(B_\delta\) and a set of integers \(k_{ji}\) such that \(n_{ji} = 2k_{ji} + 1\), i.e., the integer part of the delay is an odd number, and the fractional part of the delay satisfies \(|\delta_{ji}| \leq \delta\).

From Lemma 11 and Lemma 14 we can then conclude that there exists a sequence of real numbers \(\{B_{o1}, B_{o2}, \ldots\}\), \(B_{o(k+1)} > B_{ok}\), so that \(k \to \infty\) and \(\epsilon_j(B_{ok}) \to 0\) for all \(j = 1, \ldots, K\). This means that \(\Delta S_0 = \frac{1}{2}\) is a limit point, and together with (55) this shows that \(\Delta S_0 = \frac{1}{2}\) is the limit superior.

The Theorem has the following corollary.

Corollary 16. Suppose that all transmitters and receivers have independent positions and each node position has a continuous distribution. Then the propagation delays \(\tau_{ji}, i, j \in \{1, \ldots, K\}\), are linearly independent over the rational numbers with probability one, and

$$\Delta S_0 = \frac{1}{2}$$

is achievable.

So in practice \(\Delta S_0 = \frac{1}{2}\) is achievable, since transmitters and receivers can never be positioned accurately in a grid; there is always some nano-scale inaccuracy (dither) in positions, at the fundamental level due to quantum mechanics!

2) Practical Implementation and Simulation Results: In this section we will show that the interference alignment ideas of the previous section can be used in a practical system, and show some simulation results. This will also make it clear why the modified definition (51) is needed.

We can see that one key question concerning the transmission scheme defined by Definition 9 is: how to find \(B_o\)? Here we propose an algorithm, stated in the following proposition.

Proposition 17. Assume that both the transmitters and the receivers have perfect channel knowledge.

Initialize \(B\) to be any positive integer. Proceed with the following while loop:

While \(\exists i \neq j : |\tau_{ji}B - 2k_{ji} - 1| > \delta\) {

Increase \(B\) by 1, i.e., \(B = B + 1\).

}
If \( \tau_{ji} \) are linearly independent over the rational numbers the algorithm terminates after a finite number of iterations. The output \( B \) of the algorithm satisfies

\[
\forall i \neq j : |\tau_{ji}B - 2k_{ji} - 1| \leq \delta,
\]

which can therefore be chosen as \( B_o \).

Lemma 18 guarantees that the searching algorithm defined in the proposition above terminates. The proof is almost identical to the proof of Lemma 11. However, the essential difference is that while Lemma 11 only shows the existence of \( B \) satisfying (53) over the set of positive real numbers \( \mathbb{R}^+ \), the results in this section ensure that such \( B \) can be found even if we restrict \( B \) to be integer.

**Lemma 18.** If \( \tau_{ji} \) are linearly independent over the rational numbers, then for any \( \delta > 0 \), there exist an integer \( B \), such that

\[
|\tau_{ji}B - 2k_{ji} - 1| \leq \delta
\]

for some integers \( k_{ji} \). Further, \( B \) can be made arbitrarily large.

The proof of Lemma 18 is based on the second form of Kronecker’s theorem \[28, \text{Theorem 7.10, Second Form of Kronecker’s Theorem}\], which shows that Theorem 12 still holds even if we require \( t \) to be an integer. Details are skipped here.

We can see that the brute force algorithm of searching through all integer \( B \) is guaranteed to find good operating bandwidths. Fig. 3b shows the performance of the proposed achievable scheme when the system operating at a sequence of \( B_\delta, \delta = 0.2 \). However, designing more efficient \( B_o \)-searching algorithm could be a subject of further research.

In the simulation, we consider a 3-user channel with symmetric channel gain: \( |C_{jj}|^2 = 1, |C_{ji}|^2 = 0.8 \). Notice that for channels with symmetric link gains, equal power and equal rate constraints are equivalent. The delays \( \tau_{ji} \) are chosen such that they are linearly independent over the rational numbers.

Fig. 3a also shows why we need the modified definition (51). In this case the limit (2) does not exist; one definition of a limit of a function is that for any sequence \( x_n \rightarrow x \), \( f(x_n) \rightarrow f(x) \). In Fig. 3a the points along the upper envelope and the points along the low envelope, for example, give different slope. However, the \( \lim \sup \) always exists. One sequence that achieves the \( \lim \sup \) is shown in Fig. 3b. What is important to notice that the new definition of the wideband slope is still *operational* as in [13]. That is, it is possible to choose some finite
bandwidth where the performance is close to the wideband approximation. But different from [13] is not enough to use a bandwidth that is sufficiently large. It also has to be chosen very carefully.

![Figure 3: Achievable spectral efficiency versus $\frac{E_b}{N_0}$. The straight line shows the performance approximated to first order by the wideband slope.](image)

B. Outer bounds

In Section IV-A we have seen that $\Delta S_0 = \frac{1}{2}$ can be achieved. Is this the best possible? Clearly no. In the 2-user channel the interference alignment scheme proposed in Definition 9 reduces to TDMA. As we have seen in Section III interference decoding and treating interference as noise can be better than TDMA. For $K > 2$ case, it is also not difficult to construct examples where $\Delta S_0 > \frac{1}{2}$ is achievable. However, in this section we will show that for large $K$ this happens rarely.

Let us first define two concepts: $(1 - \epsilon)$-interference pair and weak $(1 - \epsilon)$-interference pair.

**Definition 19.** We say that users $i$ and $j$ form an $(1 - \epsilon)$-interference pair if

$$1 - \epsilon \leq \frac{|C_{ji}|^2}{|C_{ii}|^2} \cdot \frac{|C_{jj}|^2}{|C_{jj}|^2} < 1,$$

and form a weak $(1 - \epsilon)$-interference pair if

$$1 - \epsilon \leq \frac{|C_{ji}|^2}{|C_{ii}|^2} < 1 \text{ or } 1 - \epsilon \leq \frac{|C_{ij}|^2}{|C_{jj}|^2} < 1.$$

In section IV-B1 we will show that as the number of users $K \rightarrow \infty$, the event

$$\{\text{user } j, \forall j \in \{1, \ldots, K\}, \text{ forms a } (1 - \epsilon)\text{-interference pair with at least one other user}\}$$
happens with high probability. Consider Fig. 2 when two users form an \((1 - \epsilon)\)-interference pair, they operate in the point just below 1 in the figure, where Kramer’s bound bounds each user’s wideband slope by 1. This results in \( \Delta S_0 \leq \frac{1}{2} + \delta, \forall \delta > 0 \) under the equal rate constraint.

Similarly, in section IV-B2 we will show that as \( K \to \infty \), the event
\[
\{ \text{K users form } K \text{ disjoint weak } (1 - \epsilon) \text{-interference pairs} \}
\]
happens with high probability, which gives \( \Delta S_0 \leq \frac{1}{2} + \delta, \forall \delta > 0 \) under the equal power constraint if the distribution of \( C_{ji} \) satisfies some additional conditions.

The outer bounds in this section are proven under the assumption that the channel coefficients \( C_{ji} \) for all \( i, j \in \{1, \cdots, K\} \) are i.i.d. random variables. However, this is not a necessary condition, only a convenient condition to simplify proofs; later in the section we will comment more on this.

1) The Equal Rate Constraint: First consider the equal rate constraint. We assume that the channel coefficients \( C_{ji} \) are i.i.d. and \( E \left[ |C_{ii}|^2 \right] < \infty \); if the latter assumption were not satisfied, \( \lim_{K \to \infty} \frac{1}{K} \sum_{i=1}^{K} P_i = \infty \) even in the interference-free case (see (12)), so the energy per bit and wideband slope would not be well-defined for large \( K \) (see also the comment at the top of page 1325 in [13] about Rayleigh fading).

Let \( F_{|C_{ji}|^2} \) be the CDF of \( |C_{ij}|^2 \); this defines a probability measure on the real numbers through \( \mu_F((a, b]) = F_{|C_{ii}|^2}(b) - F_{|C_{ii}|^2}(a) \) (this is true for any random variable) For \( \forall \epsilon, \hat{\epsilon} > 0 \), define two sets
\[
R_{\hat{\epsilon}} = \{ x \in \mathbb{R} : F_{|C_{ii}|^2}(x) - F_{|C_{ii}|^2}((1 - \epsilon)x) < \hat{\epsilon} \};
\]
\[
D_{\hat{\epsilon}} = \{ \text{all } i \in \{1, \cdots K\} : |C_{ii}|^2 \in R_{\hat{\epsilon}} \}.
\]

The following lemma shows that as the number of users \( K \to \infty \) a user in \( D_{\hat{\epsilon}} \), with high probability forms a \((1 - \epsilon)\)-interference pair with at least one other user.

**Lemma 20.** Given \( \forall \epsilon, \hat{\epsilon} > 0 \), denote
\[
B_{\epsilon, \hat{\epsilon}} = \{ i \in D_{\hat{\epsilon}} : \text{user } i \text{ does not form an } (1 - \epsilon) \text{-interference pair with any other user} \}
\]

Then
\[
\lim_{K \to \infty} \Pr(B_{\epsilon, \hat{\epsilon}} = \emptyset) = 1.
\]

**Proof:** Please see Appendix C

On the other hand users in \( D_{\hat{\epsilon}} \) do not necessarily form \((1 - \epsilon)\)-pairs. The following lemmas are used to show that the probability of the set \( D_{\hat{\epsilon}} \) is small,

**Lemma 21.** Given any infinite sequence \( \hat{\epsilon}_n > 0 \) satisfying \( \hat{\epsilon}_n > \hat{\epsilon}_{n+1} \) and \( \hat{\epsilon}_n \to 0 \), the corresponding sequence of \( R_{\hat{\epsilon}_n} \) satisfies
1) \( R_{\hat{\epsilon}_{n+1}} \subseteq R_{\hat{\epsilon}_n} \);
2) $\mu_F(R_{\epsilon,n}) \to 0$.

**Proof:** Please see Appendix D.

Lemma 22. Let $X$ be a positive random variable with $E[X] < \infty$, and let $\mu_X$ be the measure induced by the CDF of $X$. Let $G_i \subset \mathbb{R}$ be a sequence of measurable sets with $G_{i+1} \subseteq G_i$ and $\lim_{i \to \infty} \mu_X(G_i) = 0$. Define

$$X_i = \begin{cases} X & X \in G_i \\ 0 & X \notin G_i \end{cases}.$$

Then

$$\lim_{i \to \infty} E[X_i] = 0.$$

**Proof:** Please see Appendix E.

Our main result is stated in the following theorem.

Theorem 23. Suppose that the channel coefficient $C_{ij}$ are i.i.d.. Under the equal rate constraint

$$\forall \delta > 0 : \lim_{K \to \infty} \Pr\left(\Delta S_0 \leq \frac{1}{2} + \delta\right) = 1.$$  \hfill (59)

**Proof:** We discuss users in the set $D_\epsilon$ and those in the set $D_\epsilon^c$ separately.

First, let us look at user $j, i \in D_\epsilon$. We assume that each user $j \in D_\epsilon^c$ forms a $(1-\epsilon)$-interference pair with some user $i(j)$. Given Lemma 22, this happens with high probability. Consider a single $(1-\epsilon)$-interference pair $(j, i(j))$. We can get an upper bound on the spectral efficiency, by eliminating all interference links except the links between users $j$ and $i(j)$, so that the received signal is

$$y_j = C_{jj}x_j + C_{ji(j)}x_{i(j)} + z_j$$

$$y_{i(j)} = C_{i(j)j}x_j + C_{i(j)i(j)}x_{i(j)} + z_{i(j)}.$$

Let $\frac{\left|C_{ji(j)}\right|^2}{\left|C_{i(j)i(j)}\right|^2} = 1 - \epsilon_{i(j)}, \frac{\left|C_{jj}\right|^2}{\left|C_{i(j)i(j)}\right|^2} = 1 - \epsilon_{i(j),j}$. Since $\{y_j, y_{i(j)}\}$ is a $(1-\epsilon)$-interference pair, we have

$$0 \leq \epsilon_{ji(j)}, \epsilon_{i(j),j} < \epsilon.$$  \hfill (60)

Applying (36) and (37) to $\{y_j, y_{i(j)}\}$, we have the optimum solution

$$\text{SNR}_{i(j)} = \left|C_{ij(i(j))}\right|^2 \cdot \frac{\left(\frac{\epsilon}{2} - 1\right) \left(1 - \epsilon_{i(j)j}\right)}{1 - \left(1 - \epsilon_{i(j)}\right) \left(1 - \epsilon_{i(j),j}\right)}$$

$$\text{SNR}_j = \left|C_{jj}\right|^2 \cdot \frac{\left(\frac{\epsilon}{2} - 1\right) \left(1 - \epsilon_{ji(j)}\right) \epsilon_{i(j)j} + \epsilon_{j(i(j))} \left(\frac{\epsilon}{2} - 1\right)}{1 - \left(1 - \epsilon_{ji(j)}\right) \left(1 - \epsilon_{i(j)}\right)}.$$  \hfill (61)

And $\text{SNR}_{i(j)} + \text{SNR}_j \geq \text{SNR}_{i(j)} + \text{SNR}_j$. Notice that the RHS of (63) and (64) are monotonically decreasing function of either $\epsilon_{ji(j)}$ or $\epsilon_{i(j),j}$. Thus, given the condition (60), we can relax (63) and (64) by substituting $\epsilon_{ji(j)}$, respectively.
and \( \epsilon_{i(j)} \) by \( \epsilon \),

\[
\text{SNR}_{i(j)} \geq |C_{i(j)i(j)}|^{-2} \cdot \frac{2^{\frac{R_s}{N_0}} \left( 1 - \epsilon \right) 2^{\frac{R_s}{N_0}} + \epsilon}{2 - \epsilon} - 1
\]

(63)

\[
\text{SNR}_{j} \geq |C_{jj}|^{-2} \cdot \frac{2^{\frac{R_s}{N_0}} \left( 1 - \epsilon \right) 2^{\frac{R_s}{N_0}} + \epsilon}{2 - \epsilon} - 1.
\]

(64)

Thus

\[
\text{SNR}_j = \frac{2^{\frac{R_s}{N_0}} \left( 1 - \epsilon \right) 2^{\frac{R_s}{N_0}} + \epsilon}{2 - \epsilon} |C_{jj}|^{-2}, \text{if } j \in D^c_{\hat{\epsilon}}.
\]

(65)

Second, for user \( k, k \in D_{\hat{\epsilon}} \), we treat them as being interference-free. In this case, we have

\[
\text{SNR}_k \geq \left( 2^{\frac{R_s}{N_0}} - 1 \right) |C_{kk}|^{-2}, \text{if } k \in D_{\hat{\epsilon}}.
\]

(66)

Combining (64) and (66), the minimum sum power required for an equal rate system with sum spectral efficiency \( R_s \) is lower bounded by

\[
\text{SNR}_s \geq \frac{2^{\frac{R_s}{N_0}} \left( 1 - \epsilon \right) 2^{\frac{R_s}{N_0}} + \epsilon}{2 - \epsilon} \sum_{j \in D^c_{\hat{\epsilon}}} |C_{jj}|^{-2}
\]

\[
+ \left( 2^{\frac{R_s}{N_0}} - 1 \right) \sum_{k \in D_{\hat{\epsilon}}} |C_{kk}|^{-2}.
\]

(67)

Using (39) on (67) we get

\[
\frac{E_b}{N_0 \min} = \frac{\sum (|C_{jj}|^{-2})}{K} \log 2
\]

(68)

\[
\Delta S_0 = \frac{(2 - \epsilon)}{(4 - 3\epsilon)(1 - \theta) + (2 - \epsilon)\theta}
\]

(69)

where

\[
\theta \triangleq \frac{\sum_{k \in D_{\hat{\epsilon}}} |C_{kk}|^{-2}}{\sum_{j=1}^{K} |C_{jj}|^{-2}} = \frac{1}{K} \sum_{k \in D_{\hat{\epsilon}}} |C_{kk}|^{-2}.
\]

Notice that the outer bound converges to the correct \( \frac{E_b}{N_0 \min} \), and (69) can therefore be used as an outer bound on the slope.

Now, we want to show that \( \forall \epsilon > 0, \theta \) can be made arbitrarily small. Define random variable \( H_{j,\hat{\epsilon}} \) as

\[
H_{j,\hat{\epsilon}} = \begin{cases} |C_{jj}|^{-2} & j \in D_{\hat{\epsilon}} \\ 0 & j \notin D_{\hat{\epsilon}} \end{cases}
\]

Given the fact that \( H_{j,\hat{\epsilon}} \) and \( H_{i,\hat{\epsilon}}, i \neq j \) are independent, and \( \sum_{k \in D_{\hat{\epsilon}}} |C_{kk}|^{-2} = \sum_{j=1}^{K} H_{j,\hat{\epsilon}} \), we can apply the law of large number to \( \theta \), which gives

\[
P \left( \lim_{K \to \infty} \theta = \frac{E(H_{j,\hat{\epsilon}})}{E(|C_{jj}|^{-2})} \right) = 1.
\]

(70)
Combining Lemma 21 and Lemma 22, we have

$$\lim_{\hat{\epsilon} \downarrow 0} E(H_{j, \hat{\epsilon}}) = 0.$$  \hfill (71)

This proves (59) explicitly as follows. For any $\delta > 0$ we can choose $\epsilon, \theta > 0$ sufficiently small to make (69) less than $\frac{1}{2} + \delta$. We can choose $\hat{\epsilon} > 0$ sufficiently small to make $E(H_{j, \hat{\epsilon}}) < \theta$ with high probability and $Pr(B_{j, \hat{\epsilon}} = \emptyset)$ close to 1.

2) The Equal Power Constraint: We now consider the equal power constraint. Assume that the number of users $K$ is an even integer, $K = 2M$. For any $\epsilon > 0$, we define the event

$$A_{\epsilon} \equiv \{K \text{ users can form } M \text{ disjoint weak } (1 - \epsilon) - \text{pairs}\},$$

and denote the indices of users belonging to the same weak $(1 - \epsilon)$-pairs as $\{m_1, m_2\}$.

Let the channel coefficients $C_{ij}, i, j = 1, \ldots, K$ be random variables with a distribution that could depend on $K$. We consider the following property of this sequence of distributions

Property 1. $\forall \epsilon : Pr(A_{\epsilon}) \to 1$ as $K \to \infty$.

Proposition 24. If the channel gains $C_{ij}$ are i.i.d (independent of $K$) with continuous distribution, Property 1 is satisfied.

Proof: Please see Appendix F.

Theorem 25. If property 1 is satisfied and the direct channel gains $C_{jj}$ are i.i.d with finite 4th order moments, then under the equal power constraint

$$\forall \delta > 0 : \lim_{K \to \infty} Pr\left(\Delta S_0 \leq \frac{1}{E[|C_{jj}|^2]} + \delta \right) = 1.$$  \hfill (72)

Proof: For the equal power constraint where $\text{SNR}_j = \frac{\text{SNR}}{K}$, if property 1 is satisfied, then for $K = 2M$ users, $M$ disjoint weak $(1 - \epsilon)$-pairs $\{m_1, m_2\}, m = 1, \cdots, M$ can be formed with high probability, and we will assume this is the case. Applying Kramer’s bound Theorem 4 on each pair, we have

$$R_{m_1} + R_{m_2} \leq \min \left( \log \left( 1 + \frac{|C_{m_1m_1}|^2 \text{SNR}_s}{K} \right) + \log \left( 1 + \frac{|C_{m_2m_2}|^2 \text{SNR}_s}{K} \right) \right),$$

$$\log \left( 1 + \frac{|C_{m_1m_2}|^2 \text{SNR}_s}{K} \right) + \log \left( 1 + \frac{|C_{m_2m_1}|^2 \text{SNR}_s}{K} \right)$$  \hfill (73)
in nats/s. For each weak \((1 - \epsilon)\)-pair, (73) gives
\[
\frac{d (R_{m_1} + R_{m_2})}{d P_s} \bigg|_{P_s=0} = \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2}{K} - \frac{d^2 (R_{m_1} + R_{m_2})}{d {\text{SNR}}_s^2} \bigg|_{P_s=0} \geq \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2 + 2 \min \left\{ |C_{m_1m_2}|, |C_{m_1m_2}|, |C_{m_2m_1}|, |C_{m_2m_1}| \right\}}{K} \geq \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2 + 2 \min \left\{ (1 - \epsilon_1), (1 - \epsilon_2) \right\}}{K^2} \geq \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2}{K^2}
\]
since the \(M\) pairs are disjoint and using the linearity of derivatives, we have
\[
\frac{d R_s}{d {\text{SNR}}_s} \bigg|_{P_s=0} = \sum_{m=1}^{M} \frac{d (R_{m_1} + R_{m_2})}{d {\text{SNR}}_s} \bigg|_{P_s=0} = \frac{\sum_{j=1}^{K} |C_{jj}|^2}{K} - \frac{d^2 R_s}{d {\text{SNR}}_s^2} \bigg|_{P_s=0} \geq \sum_{m=1}^{M} \left( |C_{m_1m_1}|^4 + |C_{m_2m_2}|^4 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \right) \geq \frac{2K \left( \frac{\sum_{j=1}^{K} |C_{jj}|^2}{K} \right)^2}{\sum_{m=1}^{M} \left( |C_{m_1m_1}|^4 + |C_{m_2m_2}|^4 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \right)}.
\]
Therefore
\[
\begin{align*}
\frac{E_0}{N_0} & = \frac{K \log_e 2}{\sum_{j=1}^{K} |C_{jj}|^2} \\
S_0 & \leq \frac{2 \left( \frac{\sum_{j=1}^{K} |C_{jj}|^2}{K} \right)^2}{\sum_{m=1}^{M} \left( |C_{m_1m_1}|^4 + |C_{m_2m_2}|^4 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \right)} \\
& = 2K \frac{\left( \frac{\sum_{j=1}^{K} |C_{jj}|^2}{K} \right)^2}{\frac{\sum_{j=1}^{K} |C_{jj}|^2}{K} \sum_{m=1}^{M} |C_{m_1m_1}|^2 |C_{m_2m_2}|^2}.
\end{align*}
\]
Now
\[
\frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^2 \xrightarrow{P} E \left[ |C_{jj}|^2 \right] \quad \frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^4 \xrightarrow{P} E \left[ |C_{jj}|^4 \right] \quad \frac{1}{M} \sum_{m=1}^{M} |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \xrightarrow{P} E \left[ |C_{jj}|^2 \right]^2
\]
as \(K \to \infty\), where \(\xrightarrow{P}\) stands for convergence in probability since all random variables are positive and the moments.
are assumed to exist. Using standard rules for convergence of transformation, we then obtain

\[ \forall \varepsilon > 0 : \lim_{K \to \infty} \Pr \left( \Delta S_0 \leq \frac{1}{1 - \varepsilon} \frac{(E|C_{jj}|^2)^2}{E|C_{jj}|^4} + 1 \right) = 1 \]

from (74). Equivalently,

\[ \forall \varepsilon > 0 : \lim_{K \to \infty} \Pr \left( \Delta S_0 \leq \frac{1}{1 - \varepsilon} \frac{(E|C_{jj}|^2)^2}{E|C_{jj}|^4} + \delta \varepsilon \right) = 1 \]

where

\[ \delta \varepsilon = \frac{\varepsilon (E|C_{jj}|^2)^2}{(1 - \varepsilon) \frac{(E|C_{jj}|^2)^2}{E|C_{jj}|^4} + (2 - \varepsilon) \frac{(E|C_{jj}|^2)^2}{E|C_{jj}|^4} + 1} \]

Notice that \( \forall \delta, \exists \varepsilon > 0 \) such that \( \delta \varepsilon < \delta \). Therefore, we obtain (72).

It is perhaps illustrative to write (72) in terms of central moments,

\[ \frac{1}{(E|C_{jj}|^2)^2} + 1 = \frac{1}{\mu^4 - 2\sigma^2\mu^2 + \sigma^4} + 1 \]

\[ \mu = E[X] \]

\[ \sigma^2 = \text{var}[X] \]

\[ \gamma_1 = \frac{E[(X - \mu)^3]}{\sigma^3} \]

\[ \gamma_2 = \frac{E[(X - \mu)^4]}{\sigma^4} \]

It can be seen that for \( \sigma \ll \mu \), i.e., a nearly constant distribution, (75) is close to \( \frac{1}{2} \).

We will discuss some implications of these theorems. For the equal rate constraint, (59) essentially states that the wideband slope is bounded by \( \frac{1}{2} \) of that of no interference for large \( K \). Since this is also achievable by Theorem 15 this is indeed the wideband slope, and delay-based interference alignment is optimum. The bound for the equal power constraint is slightly weaker, but is still close to \( \frac{1}{2} \) for some distributions.

Theorem 23 and 25 have been proven under an i.i.d. assumption on all channel coefficients. This can seem restrictive and not that realistic in a line of sight model. However, the i.i.d. assumption is not essential. In Theorem 23 it is used to prove that every user has at least one other user with which it forms an \((1 - \varepsilon)\)-pair with high probability. This might be true under many other model assumptions. It is also used to invoke the law of large numbers, which has a wide range of generalizations. In Theorem 25 the i.i.d. assumption is used to prove that users form disjoint weak \((1 - \varepsilon)\)-pairs, and again for invoking the law of large numbers.

What can be concluded is that for small special examples it is possible to find a better wideband slope by optimizing a combination of interference alignment, interference decoding, and treating interference as noise. However, it probably does not pay off to try to find a general algorithm for optimizing wideband slope: comparing the achievable sum slope given by Theorem 15 and the upper bounds provided by Theorem 23 and 25 we can
see that as the number of users $K$ grows large, the gap between the upper bounds and the inner bounds achieved by the interference alignment scheme defined by Definition 9 becomes arbitrarily small. Furthermore, finding such schemes are hard based on our experimentation.

Another interesting observation is that the outer bounds do not depend on delay, only on the channel gains. Thus, the outer bound depends on the macroscopic location of transmitters and receivers (e.g., if gain is proportional to $d_{ji}^\alpha$ for some $\alpha > 0$), while the inner bounds depend on the microscopic location (i.e., fractional delay differences). This also means that the outer bounds apply to general scalar interference channels, not only LOS channels. However, for non-LOS channels better outer bounds can be proven, which is the subject of a later paper (for initial results, see [29]).

V. CONCLUSIONS

In this paper we have shown that by using interference alignment with delay differences, a wideband slope of half of the interference-free case is achievable. We have also shown that, mostly, it is the best achievable. What it means is that near single-user performance can be achieved in the interference channel in the low-SNR regime. One surprising conclusion is that orthogonalizing interference is (near) optimum in the low-power regime. It is not too surprising that this is optimum in the high-SNR regime [4], since that regime is interference limited. But since the low-power regime is also noise-limited, one could have expected that orthogonalizing interference is sub-optimum. That is indeed the case for a 2-user channel. But for a $K$-user channel, orthogonalizing is near optimum, as shown by Theorem 25.

A number of questions remain open. What if the bandwidth remains fixed, but the transmission rate approaches zero (e.g., in a sensor network)? This case is more complicated, and will be covered in a later paper. How can the delay based interference alignment be implemented in practical systems? As we have seen in section IV-A2 the achievable spectral efficiency is very dependent on choosing the right symbol rate, so this touches on issues of channel knowledge and estimation, and how to optimize symbol rate in a given spectral efficiency region, as well as up to what spectral efficiencies the wideband slope provides a good approximation.

REFERENCES

[1] V. Sreekanth Annapureddy and V. Veeravalli, “Sum capacity of the gaussian interference channel in the low interference regime,” in Information Theory and Applications Workshop, 2008, 27 2008-Feb. 1 2008, pp. 422–427.
[2] A. Motahari and A. Khandani, “Capacity bounds for the gaussian interference channel,” Information Theory, IEEE Transactions on, vol. 55, no. 2, pp. 620–643, Feb. 2009.
[3] X. Shang, G. Kramer, and B. Chen, “A New Outer Bound and the Noisy-Interference Sum-Rate Capacity for Gaussian Interference Channels,” ArXiv e-prints, vol. 712, dec 2007.
[4] V. Cadambe and S. Jafar, “Interference alignment and degrees of freedom of the $k$-user interference channel,” Information Theory, IEEE Transactions on, vol. 54, no. 8, pp. 3425 –3441, aug. 2008.
[5] M. Maddah-Ali, A. Motahari, and A. Khandani, “Signaling over MIMO multiple-base systems: combination of multiple-access and broadcast schemes,” in Information Theory, 2006. ISIT 2006. Proceedings. International Symposium on, 2006.
[6] K. S. Gomadam, V. R. Cadambe, and S. A. Jafar, “Approaching the capacity of wireless networks through distributed interference alignment,” CoRR, vol. abs/0803.3816, 2008.
[7] V. R. Cadambe, S. A. Jafar, and S. Shamai, “Interference alignment on the deterministic channel and application to fully connected gaussian interference networks,” *IEEE Trans. Inf. Theor.*, vol. 55, no. 1, pp. 269–274, 2009.

[8] V. Cadambe, S. A. Jafar, and C. Wang, “Interference alignment with asymmetric complex signaling - settling the host-madsen-nosratinia conjecture,” *IEEE Transactions on Information Theory*, submitted.

[9] A. S. Motahari, S. O. Gharan, and A. K. Khandani, “Real interference alignment with real numbers,” *CoRR*, vol. abs/0908.1208, 2009.

[10] R. H. Etkin and E. Ordentlich, “The degrees-of-freedom of the k-user gaussian interference channel is discontinuous at rational channel coefficients,” *IEEE Trans. Inf. Theor.*, vol. 55, no. 11, pp. 4932–4946, 2009.

[11] A. Ghasemi, A. S. Motahari, and A. K. Khandani, “Interference alignment for the k user mimo interference channel,” *CoRR*, vol. abs/0909.4604, 2009.

[12] S. Verdu, “On channel capacity per unit cost,” *IEEE Transactions on Information Theory*, vol. 36, no. 5, pp. 1019–1030, Sep. 1990.

[13] S. Verdú, “Spectral efficiency in the wideband regime,” *IEEE Transactions on Information Theory*, vol. 48, no. 6, pp. 1319–1343, 2002.

[14] G. Caire, D. Tuninetti, and S. Verdù, “Suboptimality of TDMA in the low-power regime,” *IEEE Transactions on Information Theory*, vol. 50, no. 4, pp. 608–620, April 2004.

[15] M. Wiese, F. Knabe, J. G. Klotz, and A. Sezgin, “The performance of qpsk in low-snr interference channels,” in 2010 *International Symposium on Information Theory and its Applications (ISITA), Taichung, Taiwan, October 2010*.

[16] V. Cadambe and S. Jafar, “Degrees of freedom of wireless networks - what a difference delay makes,” nov. 2007, pp. 133–137.

[17] L. Grokop, D. N. C. Tse, and R. D. Yates, “Interference alignment for line-of-sight channels,” *CoRR*, vol. abs/0809.3035, 2008.

[18] R. Mathar and G. Böcherer, “On spatial patterns of transmitter-receiver pairs that allow for interference alignment by delay,” in *3rd International Conference on Signal Processing and Communication Systems (ICSPCS 2009)*, Omaha, USA, Sep. 2009.

[19] “Fractionated spacecraft,” [http://en.wikipedia.org/wiki/Fractionated_spacecraft](http://en.wikipedia.org/wiki/Fractionated_spacecraft).

[20] G. Caire, D. Tuninetti, and S. Verdú, “Suboptimality of tdma in the low-power regime,” *IEEE Transactions on Information Theory*, vol. 50, no. 4, pp. 608–620, 2004.

[21] T. Iian and K. Kobayashi, “A new achievable rate region for the interference channel,” *IEEE Transactions on Information Theory*, vol. IT-27, no. 1, pp. 49–60, January 1981.

[22] R. H. Etkin, D. N. C. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” *IEEE Transactions on Information Theory*, submitted.

[23] G. Kramer, “Outer bounds on the capacity of gaussian interference channels,” *Information Theory, IEEE Transactions on*, vol. 50, no. 3, pp. 581–586, 2004. [Online]. Available: [http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1273673](http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1273673)

[24] M. Costa, “On the gaussian interference channel,” *Information Theory, IEEE Transactions on*, vol. 31, no. 5, pp. 607–615, 1985. [Online]. Available: [http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1057085](http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1057085)

[25] P. Bergmans, “A simple converse for broadcast channels with additive white Gaussian noise,” *IEEE Transactions on Information Theory*, vol. IT-20, no. 2, pp. 279–280, March 1974.

[26] A. Host-Madsen and A. Nosratinia, “The multiplexing gain of wireless networks,” in *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on*, sept. 2005, pp. 2065 –2069.

[27] X. Shang, B. Chen, G. Kramer, and H. V. Poor, “Capacity regions and sum-rate capacities of vector gaussian interference channels,” 2009. [Online]. Available: [http://www.citebase.org/abstract?id=oai:arXiv.org:0907.0472](http://www.citebase.org/abstract?id=oai:arXiv.org:0907.0472)

[28] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed. New York: Springer-Verlag Inc., 1990, graduate Texts in Mathematics;41.

[29] M. Shen and A. Hø andst Madsen, “Wideband slope of interference channel: Finite bandwidth case,” in *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*, 2011, allerton 2011.

[30] X. Shang, B. Chen, G. Kramer, and H. V. Poor, “On the Capacity of MIMO Interference Channels”, *ArXiv e-prints*, jul 2008.

[31] W. Rudin, *Principles of mathematical analysis*, 3rd ed. New York: McGraw-Hill Book Co., 1976, international Series in Pure and Applied Mathematics.

[32] B. Bollobas, *Random Graphs*, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, and B. Totaro, Eds. Cambridge University Press, 2001.

[33] S. Janson, “Random graphs.” Wiley, 2000.
APPENDIX A

Proof of Theorem[7]

The following lemmas will be used in this proof.

Lemma 26. Let \( \mathbf{X}_n = \{X_1, X_2, \ldots, X_n\} \) be a sequence of random variables satisfying power constraint \( \frac{1}{n} \sum_{i=1}^{n} \text{cov} (X_i) \leq \text{SNR} \). Let \( \mathbf{X}_G = \{X_{1G}, X_{2G}, \ldots, X_{nG}\} \) be a sequence of i.i.d. Gaussian random variable, \( X_G \sim \mathcal{N}(0, \text{SNR}) \). Let \( \mathbf{Z}_1^n \) and \( \mathbf{Z}_2^n \) be two sequence of i.i.d. random variables with distributions \( Z_1 \sim \mathcal{N}(0, \sigma_1^2) \) and \( Z_2 \sim \mathcal{N}(0, \sigma_2^2) \).

Then we have the following inequality
\[
\begin{align*}
    h(X_n + Z_1^n) - h(X_n + Z_1^n + Z_2^n) \leq h(X_G + Z_1) - h(X_G + Z_1 + Z_2).
\end{align*}
\]

Lemma 27. Let \( \mathbf{X}_n = \{X_1, X_2, \ldots, X_n\} \) and \( \mathbf{Y}_n = \{Y_1, Y_2, \ldots, Y_n\} \) be two sequence of random variables. Let \( \hat{X}_G, \hat{Y}_G \) be random variables satisfying
\[
\text{cov} \left( \hat{X}_G, \hat{Y}_G \right) \leq \frac{1}{n} \sum_{i=1}^{n} \text{cov} \left( X_i, Y_i \right) \leq \text{cov} \left( \hat{X}_G, \hat{Y}_G \right).
\]

Then
\[
\begin{align*}
    h(X^n) &\leq nh(\hat{X}_G) \leq nh(\hat{X}_G) \\
    h(Y^n|X_n) &\leq nh(\hat{Y}_G|\hat{X}_G) \leq nh(\hat{Y}_G|\hat{X}_G).
\end{align*}
\]

Proof: This is a special case of [27] Lemma 2.

For the delay-free case where \( \hat{X}_i[n-n_{ji}] = X_i[n] \), this theorem is identical to the previous results in [30], [1], [2], and a later work [27]. Here we use similar technique as the proof of Theorem 6 in [27] to show that this results still hold for channel with non-zero delay.

Assume that the channel coefficients and input power constraints satisfy [45]. Provide side information \( S_1^n \) and \( S_2^n \) to receiver 1 and 2 respectively
\[
\begin{align*}
    S_1^n &= C_{21}X_1^n + W_1^n \\
    S_2^n &= C_{12}X_2^n + W_2^n
\end{align*}
\]
where \( W_j \) are zero mean i.i.d Gaussian noise. And the joint distribution of \( W_j \) and \( Z_j \) is
\[
\begin{align*}
\begin{pmatrix}
    Z_j \\
    W_j
\end{pmatrix}
&\sim \mathcal{N} \left( 0, \begin{pmatrix} 1 & \rho_j \\ \rho_j^* & \sigma_j^2 \end{pmatrix} \right),
\end{align*}
\]
\( \rho_j \) and \( \sigma_j^2 \) satisfy (40) to (43). From Fano’s inequality, we have

\[
\begin{align*}
  n (R_1 + R_2) \\
  \leq & \ I (X_1^n; Y_1^n) + I (X_2^n; Y_2^n) + o(n) \\
  \leq & \ I (X_1^n; Y_1^n, S_1^n) + I (X_2^n; Y_2^n, S_2^n) + o(n) \\
  \overset{(a)}{=} & \ h (S_1^n) - h (S_1^n | X_1^n) + h (Y_1^n | S_1^n) - h (Y_1^n | S_1^n, X_1^n) \\
  & \ + h (S_2^n) - h (S_2^n | X_2^n) + h (Y_2^n | S_2^n) - h (Y_2^n | S_2^n, X_2^n) \\
  \overset{(b)}{=} & \ h (C_{21}X_1^n + W_1) - h (W_1^n) - h \left( C_{12}\hat{X}_2^n + Z_1^n \bigg| W_1^n \right) \\
  & \ + h \left( C_{11}X_1^n + C_{12}\hat{X}_2^n + Z_1^n \bigg| C_{21}X_1^n + W_1^n \right) \\
  & \ + h (C_{12}X_2^n + W_2^n) - h (W_2^n) - h \left( C_{21}\hat{X}_1^n + Z_2^n \bigg| W_2^n \right) \\
  & \ + h \left( C_{21}\hat{X}_1^n + C_{22}X_2^n + Z_2^n \bigg| C_{12}X_2^n + W_2^n \right) + o(n) \\
  \overset{(c)}{\leq} & \ -nh (W_1) + h (C_{12}X_2^n + W_2^n) - h (C_{12}X_2^n, Z_1^n | W_1^n) \\
  & \ + h \left( C_{11}X_1^n + C_{12}\hat{X}_2^n + Z_1^n \bigg| C_{21}X_1^n + W_1^n \right) \\
  & \ -nh (W_2) + h (C_{21}X_1^n + W_1) - h (C_{21}X_1^n, Z_2^n | W_2^n) \\
  & \ + h \left( C_{21}\hat{X}_1^n + C_{22}X_2^n + Z_2^n \bigg| C_{12}X_2^n + W_2^n \right) + o(n) \\
  \overset{(d)}{\leq} & \ -nh (W_1) + nh (C_{12}X_{2G} + W_2) - nh (C_{12}X_{2G} + Z_1 | W_1) \\
  & \ + h \left( C_{11}X_1^n + C_{12}\hat{X}_{2G} + Z_1 \bigg| C_{21}X_{1G} + W_1 \right) \\
  & \ -nh (W_2) + nh (C_{21}X_{1G} + W_1) - nh (C_{21}X_{1G} + Z_2 | W_2) \\
  & \ + h \left( C_{21}\hat{X}_{1G} + C_{22}X_{2G} + Z_2 \bigg| C_{12}X_{2G} + W_2 \right) + o(n) \\
  \overset{(e)}{\leq} & \ -nh (W_1) + nh (C_{12}X_{2G} + W_2) - nh (C_{12}X_{2G} + Z_1 | W_1) \\
  & \ + nh \left( C_{11}X_{1G} + C_{12}\hat{X}_{2G} + Z_1 \bigg| C_{21}X_{1G} + W_1 \right) \\
  & \ -nh (W_2) + nh (C_{21}X_{1G} + W_1) - nh (C_{21}X_{1G} + Z_2 | W_2) \\
  & \ + nh \left( C_{21}\hat{X}_{1G} + C_{22}X_{2G} + Z_2 \bigg| C_{12}X_{2G} + W_2 \right) + o(n) \\
  \overset{(f)}{\leq} & \ -nh (W_1) + nh (C_{12}X_{2G} + W_2) - nh (C_{12}X_{2G} + Z_1 | W_1) \\
  & \ + nh \left( C_{11}X_{1G} + C_{12}\hat{X}_{2G} + Z_1 \bigg| C_{21}X_{1G} + W_1 \right) \\
  & \ -nh (W_2) + nh (C_{21}X_{1G} + W_1) - nh (C_{21}X_{1G} + Z_2 | W_2) \\
  & \ + nh \left( C_{21}\hat{X}_{1G} + C_{22}X_{2G} + Z_2 \bigg| C_{12}X_{2G} + W_2 \right) + o(n) \\
  \overset{(g)}{\leq} & \ -nh (W_1) + nh (C_{12}X_{2G} + W_2) - nh (C_{12}X_{2G} + Z_1 | W_1) \\
  & \ + nh \left( C_{11}X_{1G} + C_{12}\hat{X}_{2G} + Z_1 \bigg| C_{21}X_{1G} + W_1 \right) \\
  & \ -nh (W_2) + nh (C_{21}X_{1G} + W_1) - nh (C_{21}X_{1G} + Z_2 | W_2) \\
  & \ + nh \left( C_{21}\hat{X}_{1G} + C_{22}X_{2G} + Z_2 \bigg| C_{12}X_{2G} + W_2 \right) + o(n)
\end{align*}
\]

where \( \lim_{n \to \infty} o(n)/n = 0 \), \( X_{jG} \) with 'G' subscription means that input at transmitter \( j \) is i.i.d. Gaussian, with distribution \( X_{jG} \sim \mathcal{N}(0, \text{SNR}_j) \). (a) is from chain rule. (c) holds because both \( X_j \) and \( \hat{X}_j \) can be obtained from
sampling the same continuous-time baseband signal $X_j(t)$ at the Nyquist rate, while $Z_j$ and $W_j$ are sampled from white Gaussian noise, so that

$$h\left(C_{21}\tilde{X}_1^n + Z^n_1 \bigg| W^n_1\right) = h\left(C_{21}X^n_1 + Z^n_1 | W^n_1\right) + o(n)$$

$$h\left(C_{12}\tilde{X}_2^n + Z^n_2 \bigg| W^n_2\right) = h\left(C_{12}X^n_2 + Z^n_2 | W^n_2\right) + o(n)$$

because Given (40) and (41), $\text{cov}(W_1^n) \leq \text{cov}(Z^n_1 | W^n_2)$ and $\text{cov}(W_2^n) \leq \text{cov}(Z^n_1 | W^n_2)$. Combining Lemma 26 and [3, Lemma 3], we have

$$h\left(C_{12}X^n_2 + W^n_2\right) - h\left(C_{12}X^n_2 + Z^n_1 | W^n_2\right) \leq nh\left(C_{12}X_{2G} + W_2\right) - nh\left(C_{12}X_{2G} + Z_1 | W_1\right)$$

(83)

and

$$h\left(C_{21}X^n_1 + W^n_1\right) - h\left(C_{21}X^n_1 + Z^n_2 | W^n_2\right) \leq nh\left(C_{21}X_{1G} + W_1\right) - nh\left(C_{21}X_{1G} + Z_2 | W_2\right).$$

(84)

Therefore (d) is true.

(f) is from Lemma [27] where $\tilde{X}_jG$ are i.i.d. Gaussian random variable satisfying $\text{cov}\left(\tilde{X}_jG\right) = \frac{1}{n} \text{Tr} \left(\tilde{X}_j^n \left(\tilde{X}_j^n\right)^\dagger\right)$. Denote $\text{SNR}_j' = \frac{1}{n} \text{Tr} \left(\tilde{X}_j^n \left(\tilde{X}_j^n\right)^\dagger\right)$ as the power of $\tilde{X}_j^n$. We could see that $\text{SNR}_j' \leq \text{SNR}_j$, because time-shifting of a signal sampled at the Nyquist rate does not change signal power. Therefore we have (g). This shows that the sum capacity of a channel with delay is outer bounded by that of a channel without delay.

We could see that the inequality (g) is independent of the propagation delay. It is identical to the first inequality in [27] (89)]. Therefore, from this point on, the proof is the same as in the delay-free case.

**APPENDIX B**
Proof: We have
\[
E [\hat{x}_i^* [n_1, \delta_{ji}] \hat{x}_i [n_2, \delta_{ji}]] = E \left[ \left( \sum_{m=-\infty}^{\infty} x_i [2m] \text{sinc}(n_1 - 2m + \delta_{ji}) \right)^* \left( \sum_{m=-\infty}^{\infty} x_i [2m] \text{sinc}(n_2 - 2m + \delta_{ji}) \right) \right]
\]
\[
= \sum_{m=-\infty}^{\infty} E \left[ |x_i [2m]|^2 \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right]
\]
\[
= \sum_{m=0}^{\infty} E \left[ |x_i [2m]|^2 \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right]
+ \sum_{m=1}^{\infty} E \left[ |x_i [-2m]|^2 \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji}) \right]
= 2P_i \left( \sum_{m=0}^{\infty} \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right)
+ \sum_{m=1}^{\infty} \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji}) \right) .
\]

Define \( f_m (\delta_{ji}) \) and \( g_m (\delta_{ji}) \) as
\[
f_m (\delta_{ji}) \triangleq \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji})
\]
\[
g_m (\delta_{ji}) \triangleq \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji})
\]
and their partial sums \( s_{f,M} (\delta_{ji}) = \sum_{m=0}^{M} f_m (\delta_{ji}) \), \( s_f (\delta_{ji}) = \lim_{M \to \infty} s_{f,M} (\delta_{ji}) \); \( s_{g,M} (\delta_{ji}) = \sum_{m=0}^{M} g_m (\delta_{ji}) \), \( s_g (\delta_{ji}) = \lim_{M \to \infty} s_{g,M} (\delta_{ji}) \). Here
\[
|f_m (\delta_{ji})| = \left| \frac{\sin (\pi(n_1 - 2m + \delta_{ji})) \sin (\pi(n_2 - 2m + \delta_{ji}))}{(n_1 - 2m + \delta_{ji})(n_2 - 2m + \delta_{ji})} \right|
\leq \frac{1}{(n_1 - 2m + \delta_{ji})(n_2 - 2m + \delta_{ji})}
\]
and
\[
|g_m (\delta_{ji})| \leq \frac{1}{(n_1 + 2m + \delta_{ji})(n_2 + 2m + \delta_{ji})} .
\]

Let \( M_{f,k} \triangleq \frac{1}{(n_1 - 2m + \delta_{ji})(n_2 - 2m + \delta_{ji})} \), \( M_{g,k} \triangleq \frac{1}{(n_1 + 2m + \delta_{ji})(n_2 + 2m + \delta_{ji})} \). Because \( \sum_{m=1}^{\infty} M_{f,k} \) and \( \sum_{k=1}^{\infty} M_{g,k} \) converge too. Due to Weierstrass’s test for uniform convergence, \( s_{f,M} (\delta_{ji}) \) and \( s_{g,M} (\delta_{ji}) \) converge uniformly. And using Theorem 7.11 in [31], we have
\[
\lim_{\delta_{ji}\downarrow 0,M \to \infty} s_{f,M} (\delta_{ji}) = \lim_{M \to \infty} \lim_{\delta_{ji}\downarrow 0} s_{f,M} (\delta_{ji})
\]
\[
\lim_{\delta_{ji}\downarrow 0,M \to \infty} s_{g,M} (\delta_{ji}) = \lim_{M \to \infty} \lim_{\delta_{ji}\downarrow 0} s_{g,M} (\delta_{ji}) .
\]
Thus, (85) becomes
\[
\lim_{\delta_{ji} \downarrow 0} E \left[ \tilde{x}^*_{\delta_{ji} \downarrow 0} \tilde{x}_{\delta_{ji} \downarrow 0} \right] = 2P_i \left( \lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{f,M} (\delta_{ji}) + \lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{g,M} (\delta_{ji}) \right)
\]
\[
= \begin{cases} 
2P_i & \text{if } n_1 = n_2 = 2k, \\
0 & \text{o.w.} 
\end{cases}
\]

And given Theorem 7.12 in [31] and the continuity of sinc function, we can conclude that \( E \left[ \tilde{x}^*_{\delta_{ji} \downarrow 0} \tilde{x}_{\delta_{ji} \downarrow 0} \right] \) is a continuous function of \( \delta_{ji} \).

**APPENDIX C**

**PROOF OF LEMMA 20**

Let \( C_{i,\epsilon} \) be the event that user \( i \) does not form an \((1 - \epsilon)\)-pair with any other user. Then
\[
\Pr(B_{\epsilon,\epsilon} \neq \emptyset) = \Pr \left( \bigcup_{i=1}^{K} C_{i,\epsilon} \right)
\]
\[
\leq \sum_{i=1}^{K} \Pr(C_{i,\epsilon})
\]
\[
= K \Pr(C_{1,\epsilon})
\]

and
\[
\Pr(C_{1,\epsilon}) = \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon} \right)
\]
\[
= \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon} \right) + \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \right)
\]
\[
- \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon} \right)
\]
\[
= (1 - p_j^{K-1}) \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon} \right) + p_j^{K-1}
\]

where \( p_j = \Pr \left( \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \right) \in (0, 1) \). Notice that the events \( \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \) are independent for different \( j \), and \( p_j^{K-1} \to 0 \). Thus, \( \Pr(C_{1,\epsilon}) \to \Pr \left( \forall j > 1 : \frac{|C_{jj}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon} \right) \). As the \( |C_{jj}|^2 \)
are independent,
\[
\Pr \left( \forall j > 1 : \frac{|C_{1j}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_\epsilon \right)
\]
\[
= \int_{\mathbb{R}} \prod_{j=2}^{K} \left( 1 - \int_{(1-\epsilon)x}^x dF_{|C_{1j}|^2}(u) \right) dF_{|C_{11}|^2}(x)
\]
\[
= \int_{\mathbb{R}} \left( 1 - F_{|C_{12}|^2} (x) + F_{|C_{11}|^2} ((1 - \epsilon) x) \right)^{K-1} dF_{|C_{11}|^2}(x)
\]
\[
\leq (1 - \hat{\epsilon})^{K-1} \int_{\mathbb{R}} dF_{|C_{11}|^2}(x)
\]
\[
= (1 - \mu_F(R_\epsilon)) (1 - \hat{\epsilon})^{K-1}.
\]
Thus, \( P_\sigma \leq K (1 - \mu_F(R_\epsilon)) (1 - \hat{\epsilon})^{K-1} \), and \( \lim_{K \to \infty} \Pr(B_{\epsilon, \hat{\epsilon}} \neq \emptyset) = 0 \).

**APPENDIX D**

**PROOF OF LEMMA 21**

Given the definition \( \forall x \in R_\epsilon : F_{|C_{11}|^2} (x) - F_{|C_{11}|^2} ((1 - \epsilon) x) \leq \hat{\epsilon} \), and given the fact \( F_{|C_{11}|^2} (x) - F_{|C_{11}|^2} ((1 - \epsilon) x) < \hat{\epsilon}_{n+1} \leq \hat{\epsilon}_n \), it clearly follows that \( R_{\hat{\epsilon}_{n+1}} \subseteq R_{\hat{\epsilon}_n} \). Let \( I_{\hat{\epsilon}_n} (x) \) be the indicator function of \( R_{\hat{\epsilon}_n} \). Using Lebesgue dominated convergence we have
\[
\lim_{n \to \infty} \int_0^\infty I_{\hat{\epsilon}_n} (x) dF_{|C_{11}|^2} (x)
\]
\[
= \int_0^\infty \lim_{n \to \infty} I_{\hat{\epsilon}_n} (x) dF_{|C_{11}|^2} (x)
\]
\[
= \int_0^\infty I_0 (x) dF_{|C_{11}|^2} (x)
\]
where \( I_0 (x) \) is the indicator function of the set \( R_0 = \{ x \in \mathbb{R} : F_{|C_{11}|^2} (x) - F_{|C_{11}|^2} ((1 - \epsilon) x) = 0 \} \). Since we have assumed that \( E \left[ |C_{11}|^{-2} \right] < \infty \), also \( \Pr(|C_{11}|^2 = 0) = 0 \) and clearly \( \mu_F(R_0) = 0 \), and \( \mu(R_{\hat{\epsilon}_n}) \to \mu(R_0) \).

**APPENDIX E**

**PROOF OF LEMMA 22**

To be explicit, let \( X \) be a random variable on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) [2]. Given the definition of \( X_i \), we can conclude that
\[
\lim_{i \to \infty} X_i = 0 \quad \text{w.p. 1}.
\]
Namely, if there is a set \( B \in \mathcal{F} \) with \( \mathbb{P}(B) > 0 \) where \( \lim_{i \to \infty} X_i \neq 0 \) then \( \mathbb{P}(B) \leq \mu_X (\bigcap_{i=1}^{\infty} G_i) \) which contradicts \( \lim_{i \to \infty} \mu_X (G_i) = 0 \).

Now \( X_i \leq X \), and therefore by Lebesgue dominated convergence
\[
\lim_{i \to \infty} E[X_i] = E[\lim_{i \to \infty} X_i] = 0.
\]
Appendix F

Proof of Property 1

Model the interference channel as a graph $G_K$, with $K = 2M$ vertices $u_1, u_2, \ldots, u_{2n}$. Vertices $u_i$ and $u_j$ are connected by edge $E_{ij}$ if they form a weak $(1-\epsilon)$-pair, i.e., $(1-\epsilon) \leq \frac{|C_i|}{|C_j|} \leq 1$ or $(1-\epsilon) \leq \frac{|C_j|}{|C_i|} \leq 1$. Divide vertices into two disjoint classes $V_1 = \{u_1, u_2, \ldots, u_M\}$ and $V_2 = \{u_{M+1}, u_{M+2}, \ldots, u_{2M}\}$. Now define event $\hat{A}_e =$ \{there exists a perfect matching in the bipartite graph $G_{M,M}$\}.

As $\hat{A}_e \subseteq A_e$, $P(A_e) \geq P(\hat{A}_e)$. Thus, if we can show that $P(\hat{A}_e) = 1 - o(1)$ as $K \rightarrow \infty$ then Property 1 holds.

For any bipartite graph, a perfect matching exists if Hall’s condition is satisfied.

Theorem 28 (Hall 1935). Given a bipartite graph $G_{M,M}$ with disjoint vertices classes $V_1$ and $V_2$, $V_1 \bigcup V_2 = V$, $|V_i| = M$, whose set of edges is $E(G_{M,M})$, a perfect matching exists if and only if for every $S \subseteq V_i$, $i = 1 or 2$, $|N(S)| \geq |S|$, where $N(S) = \{y : xy \in E(G_{M,M}) \text{ for some } x \in S\}$.

Any bipartite graph that does not have a perfect matching has following properties

Lemma 29. Suppose $G_{M,M}$ has no isolated vertices and it does not have a perfect matching. Then Hall’s condition must be violated by some set $A \subseteq V_i$, $i = 1 or 2$. And such set with minimal cardinality satisfies the following necessary conditions

(i) $|N(A)| = |A| - 1$;

(ii) $2 \leq |A| \leq \left\lceil \frac{M}{2} \right\rceil$

(iii) the subgraph of $G$ spanned by $A \bigcup N(A)$ is connected, and it has at least $2a - 2$ edges;

(iv) every vertex in $N(A)$ is adjacent to at least two vertices in $A$;

(v) any subsets in $N(A)$ can find a perfect matching in $|A|$;

Proof: (i), (ii), (iii), and (iv) are proved by Lemma 7.12 in [32], and p.82 of [33]. And (iv) is true because if there exists a subset $B$ of $N(A)$ that can not find a perfect match, we could just let $B$ be $\hat{A}$, and its neighbors in $A$ be $N(\hat{A})$. Then $\hat{A}$ violates Hall’s condition, while $|\hat{A}| < a$. This contradicts the assumption that $A$ is the minimal set violating Hall’s condition.

Define the event $F_a$: there is a set $A \subseteq V_i$, $i = 1 or 2$, $|A| = a$, satisfying (i), (ii) and (iii) in Lemma 29 [32] shows that for a graph with no isolated vertex, $P(A_e) = 1 - o(1)$ is equivalent to $P\left(\bigcup_{a=2}^{n\frac{M}{2}} F_a\right) = o(1)$. Define $F_1$ as the event that there exists at least one isolated vertex in $G_{M,M}$. In our case, we want to show that

$$P \left(\bigcup_{a=2}^{n\frac{M}{2}} F_a\right) + P(F_1) = o(1).$$
Using the union bound, we have

\[ P(F_1) \leq \sum_{i=1}^{2M} P(a_i \text{ isolated}) \leq 2M \cdot P(a_1 \text{ isolated}) \leq 2M \cdot (1 - p_{1j})^M \]

\[ \leq o(1) \]

where \( p_{1j} \triangleq P\left(1 - \epsilon \leq \frac{|C_i|^2}{|E_j|^2} \leq 1 \right), j = M+1, \cdots, 2M \). We also define \( p_0 \triangleq P\left(1 - \epsilon \leq \frac{|C_i|^2}{|E_j|^2} \leq 1, \text{ or } (1 - \epsilon) \leq \frac{|C_i|^2}{|E_j|^2} \leq 1 \right) \) for later use. (a) holds because the event \( \frac{|C_i|^2}{|E_j|^2} \notin (1 - \epsilon, 1) \) is independent of \( j \). And it is a necessary condition for \( V_1 \) to be isolated.

Now, let us look into \( F_a \) for \( 2 \leq a \leq \left\lceil \frac{M}{2} \right\rceil \). Let \( A_1 \subset V_1, A_2 \subset V_2 \), and \( |A_1| = |A_2| + 1 = a \). Denote \( P(A_a) \) as the probability that the subgraph of \( G_{M,M} \) spanned by \( A_1 \cup A_2 \) satisfies (i), (ii), and (iii) in Lemma 29. We have

\[
P\left( \bigcup_{a=2}^{\left\lceil \frac{M}{2} \right\rceil} F_a \right) \leq \sum_{a=2}^{\left\lceil \frac{M}{2} \right\rceil} P(F_a) \leq 2 \sum_{a=2}^{\left\lceil \frac{M}{2} \right\rceil} \binom{M}{a} \binom{M}{a-1} P(A_a). \tag{86} \]

(d) is from the union bound; (e) is from the union bound, and from that fact that there are \( \binom{M}{a} \binom{M}{a-1} \) choices for \( A \) with \( |A| = a \), and \( \binom{M}{a-1} \) more choices for \( N(A) \). In [32], [33], the case where edge probabilities are i.i.d, whose value is \( p \), is considered. In [32], \( P(A_a) \) is bounded using condition (i), (ii) and (iii), which gives \( P(A_a) \leq \binom{a}{a-1} \frac{(a-1)(a-2)}{2} p^{2a-2} p^{a(n-a+1)}. \) The term \( p^{a(n-a+1)} \) is the probability that the vertices in \( A_1 \) do not connect to vertices in \( V_2 - A_2 \). And in [33], condition (iv) instead of (iii) are used, which gives \( P(A_a) \leq \binom{a}{a-1} p^{2a-2} p^{a(n-a+1)}. \) However, in our case, any two edges having adjacent vertices are dependent. So we use condition (v). Since for \( N(A) \), a perfect match exists, then the subgraph spanned by \( A \cup N(A) \) has \( a - 1 \) edges that are not adjacent with each other. Thus, \( P(A_a) \) can be bounded by

\[
P(A_a) \leq Pr \text{ (condition (i), (ii) and (iv) are satisfied, vertices in } A_1 \text{ do not connect to vertices in } V_2 - A_2) \quad \text{ (87)}
\]

\[
\binom{a-1}{k} \prod_{k=1}^{a-1} \left( \frac{k}{1} \right) p_A 1_A \bar{A}_2 \quad \text{ (89)}
\]
where

\[ P_{A_1 \bar{A}_2} = P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \notin [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \text{ and } u_j \in (V_2 - A_2) \right) \]

\[ \leq \prod_{u_j \in (V_2 - A_2)} P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \notin [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \right) \]

\[ = \left( P \left( \frac{|C_{1j}|^2}{|C_{jj}|^2} \notin [(1 - \epsilon), 1], i = 1, \ldots, a j = M + a \right) \right)^{M-a+1} \tag{90} \]

notice that the event \( \frac{|C_{ij}|^2}{|C_{jj}|^2} \notin [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \text{ and } u_j \in (V_2 - A_2) \) is an necessary Substitute \( |C_{1j}|^2 \) by \( x_j \), and \( |C_{11}|^2 \) by \( x_1 \), denote their CDF by \( F_x(x) \), and their joint CDF \( F_{X}(x) \). Notice that \( |C_{ij}|^2 \) are i.i.d. distributed. Then

\[ P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \notin [(1 - \epsilon), 1], i = 1, \ldots, a j = M + a \right) \]

\[ = \int_{A_X} dF_{X}(x) \]

\[ = \int_0^\infty f_{x_{M+a}}(x_{M+a}) \prod_{i=1}^a \left( \int_{x_{M+a}} f_{x_i}(x_i) dx_i \right) dx_{M+a} \]

\[ = \int_0^\infty f_{x_1}(x_1) \left( \int_{x_{M+a}} f_{x_{M+1}}(x_{M+1}) dx_{M+1} \right)^a dx_{M+a} \]

\[ \leq (f) \left( \int_0^\infty g^2_{x_1}(x_1) dx_1 \right)^{1/2} \left( \int_0^\infty g^2_{x_1}(x_1) dx_1 \right)^{a/2} \tag{91} \]

where \( g(x_1) \triangleq \int_{x_{M+a}} f_{x_{M+1}}(x_{M+1}) dx_{M+1} \). (f) is from Cauchy-Schwartz inequality. Denote

\[ q_1 \triangleq \left( \int_0^\infty f^2_{x_1}(x_1) dx_1 \right)^{\frac{1}{2\alpha M + a + 1}} \left( \int_0^\infty g^2_{x_1}(x_1) dx_1 \right)^{1/2} \]

\[ q_1 < 1, \text{ and } \lim_{M \to \infty} q_1 = \left( \int_0^\infty g^2_{x_1}(x_1) dx_1 \right)^{1/2}. \]

Notice that this limit value do not depend on the value of \( M \). Now combining (90) and (91), we have

\[ P (A_a) \leq \left( p_0^{\alpha-1} \prod_{k=2}^a \left( \frac{k}{1} \right) \right) q_1^{a(M-a+1)}. \tag{92} \]
Given (86) and (92),

\[
P\left(\bigcup_{a=2}^{\lceil \frac{M}{2} \rceil} F_a\right) \leq 2 \sum_{a=2}^{\lceil \frac{M}{2} \rceil} \binom{M}{a} \binom{M}{a-1} \left(p_0^{a-1} \prod_{k=2}^{a} \binom{k}{1}\right) q_1^a (M-a+1)
\]

\[
\leq 2 \sum_{a=2}^{\lceil \frac{M}{2} \rceil} \left(eM \right)^a \left(\frac{eM}{a-1}\right)^{a-1} a^{a-1} p_0^{a-1} q_1^a (M-a+1)
\]

\[
\leq 2q_1 \sum_{a=2}^{\lceil \frac{M}{2} \rceil} \left(e^2 M^2 \right)^a \left(\frac{a}{a-1}\right)^{a-1} p_0^{a-1} q_1^a (M-a+1)
\]

\[
\leq 2q_1 \sum_{a=2}^{\lceil \frac{M}{2} \rceil} \left(2e^2 p_0 M^2 q_1^a \frac{M}{a}\right)^{a-1}
\]

\[
\leq 2q_1 \frac{M}{2} \left(2e^2 p_0 M^2 q_1^a \frac{M}{a}\right)
\]

\[
= o(1).
\]

This means we can find a perfect matching with high probability, i.e., \(1 - o(1)\).