The Classical and Quantum Analysis of a Charged Particle on the Spacetime Produced by a Global Monopole.

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Abstract

We study the classical and quantum motion of a relativistic charged particle on the spacetime produced by a global monopole. The self-potential, which is present in this spacetime, is considered as an external electrostatic potential. We obtain classical orbits and quantum states for a spin-1/2 and spin-0 particles.

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1. Introduction

Global monopoles are heavy objects probably formed in the early Universe by the phase transition that occur in a system composed of a self-coupling scalar field triplet \( \phi^a \) whose its original global symmetry \( O(3) \) is spontaneously broken to \( U(1) \).

The simplest model that gives rise to a global monopole is described by the Lagrangian density below

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi^a)(\partial^\mu \Phi^a) - \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2
\]

Coupling this matter field with the Einstein equation, Barriola and Vilenkin [1] have shown that the effect produced by this object in the geometry can be approximately represented by a solid angle deficit in \( (3+1) \)-dimensional spacetime, whose the line element of this manifold is given by

\[
ds^2 = -dt^2 + \frac{dr^2}{\alpha^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where the parameter \( \alpha^2 = 1 - 8\pi G \eta^2 \) is smaller than one and depends on the energy scale where the global symmetry is spontaneously broken. The area of a sphere of unit radius in this manifold is \( 4\pi \alpha^2 \).

Recently has been found that a charged particle placed in the spacetime produced by a global monopole becomes subjected to a repulsive electrostatic self-potential [2] given by

\[
U = \frac{K}{r}
\]

where \( r \) is the radial distance from the particle to the monopole and

\[
K = \frac{q^2 S(\alpha)}{2} > 0
\]

being \( S(\alpha) \) a numerical factor which is positive for \( \alpha < 1 \) and negative for \( \alpha > 1 \). This self-potential is consequence of the topology of this spacetime and also its non-vanishing curvature.

Here in this paper we decided to analyse the classical and quantum motion of a charged particle in the spacetime produced by a global monopole, taking
into account the self-potential as an external electrostatic potential in the equations of motion. So, with the objective to develop this analysis in a more general point of view we present this paper as follows: In section 2, we study the classical orbits by using the Hamilton-Jacobi formalism. We analyse the possibility of the test particle become bound to the monopole considering the unphysical case where $\alpha > 1$. In section 3, we study the quantum motion considering the test particle as a bosonic and fermionic one, by using, respectively, the Klein-Gordon and Dirac equations. Finally in section 4 we summarize our main results and make some remarks.

2. Classical Motion

In this section we study the relativistic classical motion of a particle with mass $M$ and electric charge $q$ subjected to the electrostatic self-potential (3) and (4) in the manifold described by in (2). In order to do that, we shall use the Hamilton-Jacobi formalism. The first set of equations to be used is given by

$$M \frac{dx^\mu}{d\lambda} + q A^\mu = g^{\mu\nu} \frac{\partial S}{\partial x^\nu},$$

(5)

where

$$A^\mu = \frac{K}{qr} \delta^\mu_0,$$

(6)

being $\lambda$ a parameter along the trajectory of the particle.

The second set is given by

$$g^{\mu\nu} \left( \frac{\partial S}{\partial x^\nu} - q A_\nu \right) \left( \frac{\partial S}{\partial x^\mu} - q A_\mu \right) = -M^2.$$

(7)

For this system we shall employ the spherically symmetric ansatz for the functional $S$.

$$S = -Et + R(r) + \Theta(\theta) + L_Z \phi,$$

(8)

where $R(r)$ and $\Theta(\theta)$ are unknown functions and $E$ and $L_Z$ constants of motion.
Substituting (8) into (7) we get the following set of differential equations:

$$\left(\frac{d\Theta}{d\theta}\right)^2 = L^2 - \frac{L_z^2}{\sin^2\theta},$$  \hspace{2cm} (9)

and

$$\alpha^2 \left(\frac{dR}{dr}\right)^2 = \left(-E + \frac{K}{r}\right)^2 - M^2 - \frac{L_z^2}{r^2},$$  \hspace{2cm} (10)

where $L^2$ is a positive constant.

We also can write

$$M \frac{dr}{d\lambda} = \alpha^2 \left(\frac{dR}{dr}\right),$$  \hspace{2cm} (11)

which, combining with (10) results in

$$\frac{M^2}{\alpha^2} \left(\frac{dr}{d\lambda}\right)^2 = \left(E - \frac{K}{r}\right)^2 - M^2 - \frac{L_z^2}{r^2}. $$  \hspace{2cm} (12)

Also, from (5) we get

$$M \left(\frac{d\theta}{d\lambda}\right) = \frac{1}{r^2} \left(\frac{d\Theta}{d\theta}\right),$ $

\text{and}$

$$M \left(\frac{d\varphi}{d\lambda}\right) = \frac{1}{r^2 \sin^2\theta} L_z.$$  \hspace{2cm} (14)

Combining (13) and (14) we obtain a differential equation relating the two angular variables, which by integration yields

$$\cot^2\theta = (\lambda^2 - 1) \sin^2\varphi,$$  \hspace{2cm} (15)
with \( \chi = \frac{L}{L_Z} \). We can see from the equation above that the relation between both angular variables does not depend on the parameter \( \alpha \).

In order to analyze the dependence of the radial coordinate on the angular ones, what we call by equation of the classical orbits, we shall derive the function \( r(\varphi) \). In order to simplify our calculation only, let us consider the surface \( \theta = \pi/2 \), and defining \( u = 1/r \) we get after some steps:

\[
\frac{1}{\alpha^2 \chi^2} \left( \frac{du}{d\varphi} \right)^2 + u^2 = \frac{1}{L^2} \left[ (E - Ku)^2 - M^2 \right],
\]

whose solutions are

\[
u(\varphi) = \frac{E K}{L^2 - K^2} \pm \sqrt{(E^2 - M^2) L^2 + M^2 K^2} \cos \left[ \frac{\sqrt{L^2 - K^2}}{L_Z} \alpha (\varphi - \varphi_0) \right],
\]

for \( L^2 > K^2 \),

\[
u(\varphi) = \frac{E K}{K^2 - L^2} \pm \sqrt{(E^2 - M^2) L^2 + M^2 K^2} \times \cosh \left[ \frac{\sqrt{K^2 - L^2}}{L_Z} \alpha (\varphi - \varphi_0) \right],
\]

for \( L^2 < K^2 \), and

\[
u(\varphi) = \frac{M^2}{2EK} - \frac{E \alpha^2 \chi^2 (\varphi - \varphi_0)^2}{2K}
\]

for \( L^2 = K^2 \).

From the equations above we can see that the trajectories described by (17), for the attractive case, \( K < 0 \) (which happens for the unphysical situation when \( \alpha > 1 \)), will be bounded from below by some positive value if \( E < M \), so \( r \) will be bounded from above. If \( E > M \), \( r \) can be infinity. The trajectories given by (18) and (19) are spiral ones, and \( r \) goes to zero when \( \varphi \) goes to infinity. In relativistic classical mechanics finite trajectories in general are not closed but rather rosetteshaped [4]. The former case presents period given by

\[
T = \frac{2\pi L_Z}{\alpha \sqrt{L^2 - K^2}}.
\]
which clearly on the parameter $\alpha$.

There is also a constant solution obtained by setting $du/d\phi = 0$ in (16). This circular motion occurs in relativistic theory just as in classical one.

The relation between $u = 1/r$ and the polar angle $\theta$ can also be obtained. After some minor development from the previous equations we get for $L^2 > K^2$,

$$u(\psi) = -\frac{EK}{L^2 - K^2} + \frac{\sqrt{(E^2 - M^2)L^2 + M^2K^2}}{L^2 - K^2} \times \cos \left[ \frac{\sqrt{L^2 - K^2}}{Lz} \alpha(\psi - \psi_0) \right],$$

where the new angular variable $\psi$ is expressed in terms of $\theta$ by

$$\psi = -\frac{1}{\chi} \arcsin \left( \frac{\chi \cos \theta}{\sqrt{\chi^2 - 1}} \right).$$

We also should note that the equations above depend on the parameter $\alpha$, however these two last relations differ from the previous ones involving the azimuthal angle, where it appears as a factor multiplying $\phi$.

Combining equation (11) with the $\mu = 0$ component of (5) we can obtain an expression for the radial velocity for the test particle.

$$\frac{dr}{dt} = \frac{\alpha^2}{E - K/r} \frac{dR}{dr}.$$

The turning points of the trajectory are given by setting $dr/dt = 0$. As consequence we get the potential curves given by

$$\frac{E}{M} = \left( 1 + \frac{L^2}{M^2r^2} \right)^{1/2} + \frac{K}{Mr}.$$  

For $K > 0$, which means the physical situation with $\alpha < 1$ the right hand side of the above expression presents no extremals, so the particle cannot be bound by the monopole; however for $K < 0$, which is associated with the unphysical situation with $\alpha > 1$, it is possible to have $E/M < 1$ for a finite radius in (24). (In this case the potential curves above coincides with
the circular trajectory mentioned previously). The situation where the self-potential is absent, \( K = 0 \), has been analysed in \([4]\). There it was found no possibility of the test particle be trapped by the monopole.

After this analysis about the classical motion of the test particle in this manifold, where exact solutions were obtained, we shall leave for the next section the relativistic quantum motion study for this system.

### 3. Quantum Motion

This section is devoted to the relativistic quantum analysis of scattering and bound states of a massive charged test particle in the spacetime metric given by \( (2) \), considering the self-potential as an external electrostatic potential. In this sense this section is a natural extension of the previous one, taking into account the quantum effects. In order to make this analysis complete, we shall consider the cases where the test particle is a bosonic and fermionic particle. We start first this analysis considering the former case.

#### A) Bosonic Case

The Klein-Gordon equation written in a covariant form in the presence of an external four-vector potential \( A^\mu \) reads \([3]\)

\[
\Box - \frac{iqA^\mu}{\sqrt{-g}}(\partial_\mu \sqrt{-g}) - iq(\partial_\mu A^\mu) - 2iqA^\mu \partial_\mu - q^2 A^\mu A_\mu - M^2 \right] \varphi(x) = 0, \quad (25)
\]

with

\[
\Box \varphi(x) = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi(x)], \quad (26)
\]

being \( g = det(g_{\mu\nu}) \).

We can also consider in our analysis the so called non-minimal coupling between the scalar field \( \varphi(x) \) with the geometry of the manifold itself. This term, which is invariant, is expressed by \( \Upsilon R \varphi \), where the coefficient \( \Upsilon \) is
an arbitrary coupling constant and R the scalar curvature. (For the massless case the conformal coupling constant is $\Upsilon = 1/6$.)

The inclusion of the non-minimal coupling in our formalism, does not make our analysis more complicated. The reason for this fact is that the scalar curvature associated with the space-time produced by a global monopole is $R = 2(1 - \alpha^2)/r^2$, which presents the similar behaviour of the centrifugal potential energy.

Now, writing down the Klein-Gordon equation in the manifold described by (2), including the self-potential for a charged particle given by (3) and also the non-minimal coupling, we get from (25):

$$[-\partial_t^2 + \frac{\alpha^2}{r^2} \partial_r (r^2 \partial_r) - \frac{L^2}{r^2} - 2i K \partial_t + \frac{K^2}{r^2} - M^2 - \Upsilon \eta \frac{1}{r^2}]\varphi(x) = 0,$$

where

$$\eta = 2(1 - \alpha^2)$$

and $L$ the usual orbital angular momentum operator.

Because our metric tensor is a static one and the self-potential is time independent, we shall adopt for the wave function the form below

$$\varphi(x) = e^{-iEt}R(r)Y^m_l(\theta, \varphi),$$

where $E$ is the energy of the particle. Substituting (29) into (27) we get the radial Klein-Gordon equation.

$$\frac{\alpha^2}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - [l(l+1) + \Upsilon \eta - K^2] \frac{R}{r^2} - \frac{2EK}{r} R + (E^2 - M^2) R = 0,$$

where we can identify the equation above with the Schrödinger one, associating,

$$L^2 \rightarrow \frac{L^2 + \Upsilon \eta - K^2}{\alpha^2}.$$
\[ V(r) \to \frac{EK}{M\alpha^2 r}, \]

and

\[ E \to \frac{E^2 - M^2}{2M\alpha^2}. \]

We shall analyse in this section scattering states mainly; however if we admit the possibility of the parameter \( \alpha^2 \) in (2) be bigger than one, the self-potential will be an attractive one, so bound states can be present in this case. Analysing the behaviour of the wave functions near the origin, and assuming \( E^2 > M^2 \), well defined scattering states can be obtained expressed in terms of confluent hypergeometric functions as:

\[ R(r) = C r^{\lambda_l} e^{i\kappa r} F_1(\lambda_l + 1 + i\beta, 2(\lambda_l + 1); -2i\kappa r), \quad (31) \]

where \( \lambda_l = -\frac{1}{2} + \frac{\sqrt{\alpha^2+4[(l+1)+\Upsilon_\eta-K^2]}}{2\alpha} \), \( \kappa = \frac{\sqrt{E^2-M^2}}{\alpha} \) and \( \beta = \frac{EK\alpha^2}{\alpha^2\kappa} \).

Following the standard procedure [6] we can take the asymptotic form of the confluent hypergeometric function obtain the long distance behaviour for the radial function.

\[ R(r) \sim C \frac{\Gamma(2\lambda_l + 2)e^{\beta\pi/2}}{\Gamma(\lambda_l + 1 + i\beta)} | \frac{\cos(\kappa r - (\pi/2)(\lambda_l + 1) - \beta \ln(2\kappa r) + \gamma_l)}{\kappa r^\gamma_l} |, \quad (32) \]

where \( \gamma_l = \arg \Gamma(\lambda_l + 1 + i\beta) \).

From the above equation it is possible to obtain the phase shift \( \delta_l \), which is the most relevant parameter in the calculation of the scattering amplitude [6].

\[ \delta_l = \frac{\pi}{2} (l - \lambda_l) + \gamma_l. \quad (33) \]

As we can see from the result above, the phase shift presents two contributions: (i) From the modification of the effective angular quantum number
due to the geometry of the manifold itself and (ii) from the presence of the repulsive self-interaction term as another indirect consequence of this non-trivial topology. Taking $K = \eta = 0$ in (33) we reobtain the result found in [7].

Now, if we are inclined to consider the possibility of the test particle to be bound by the global monopole, we have to assume that the parameter $\alpha^2$ in (2) is bigger than one. In this case the induced self-interaction is attractive and bound states can be obtained by taking in (30), $E^2 < M^2$ and $K = -|K|$. Again imposing appropriate boundary condition on the solutions, they can be expressed as follows:

$$R(r) = C\lambda\eta e^{-\kappa r} _1F_1(\lambda_l + 1 - \xi, 2(\lambda_l + 1); 2\kappa r)$$  \hspace{1cm} (34)

where $\kappa = \sqrt{M^2 - E^2}$ and $\xi = \frac{E|K|}{\alpha^2 \kappa}$. In order to obtain bound states we have to choose appropriately the parameters to terminate the series in (34). Admitting a polynomial of degree $n$ for the hypergeometric function, we must impose the condition

$$\frac{E|K|}{\alpha^2 \kappa} - \lambda - 1 = n.$$  \hspace{1cm} (35)

With this condition we get discret values for the self-energy given by:

$$E^{(\alpha)}_{n,l} = M \left[ \frac{\alpha^2(n + \lambda_l + 1)^2}{K^2 + \alpha^2(n + \lambda_l + 1)^2} \right]^{1/2},$$  \hspace{1cm} (36)

with $n = 0, 1, 2...$

Observing the expression above we call attention for two interesting results: (i) The first one refers to the degeneracy problem. Because $\alpha \neq 1$, the self-energy depends on $n + \lambda_l$ which is not in general an integer number, this reduce the degree of the degeneracy of our solutions. (ii) The second one refers with self-energy itself. As we can $E^{(\alpha)}_{n,l}$ depends on the parameter $K^2$ which in general can be a very small quantity, so expanding (33) in powers...
of $K^2$ we get the non-relativistic limit of the energy \[ E_{n,l} \simeq M \left[ 1 - \frac{K^2}{2\alpha^2(n + \lambda_l + 1)^2} \right]. \] (37)

B) Fermionic Case

The Dirac equation written in a covariant form and in presence of an external four-vector potential reads
\[ [i\gamma^\mu(x)(\partial_\mu + iqA_\mu - \Gamma_\mu(x)) - M]\Psi(x) = 0, \] (38)
where $\Gamma_\mu(x)$ is the spinorial connection coefficient given in terms of the tetrads and Christoffel symbols
\[ \Gamma_\mu = -\frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^{\nu}_{(a)}(\partial_\mu e_{(b)}^{\nu} - \Gamma_\lambda^{\nu\mu\nu}e_{(b)\lambda}) \] (39)

The generalized Dirac matrices $\gamma^\mu(x)$ are given in terms of the standard flat spacetime gammas $\gamma^{(a)}$ by the relation
\[ \gamma^\mu(x) = e^\mu_{(a)} \gamma^{(a)} \] (40)

For the metric corresponding to the global monopole we shall use the following tetrads :
\[ e^\mu_{(a)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha \sin \theta \cos \phi & r^{-1} \cos \theta \cos \phi & -(r \sin \theta)^{-1} \sin \phi \\
0 & \alpha \sin \theta \sin \phi & r^{-1} \cos \theta \sin \phi & (r \sin \theta)^{-1} \cos \phi \\
0 & \alpha \cos \theta & -r^{-1} \sin \theta & 0
\end{pmatrix} \] (41)

\[1\]This result would be obtained if we have used the Schrodinger equation defined in the geometry defined in (3), in the presence of an external electrostatic potential (3). (See Ref. [2])
which obey the relation

\[ e^{\mu}_{(a)} e^{\nu}_{(b)} \eta^{(a)(b)} = g^{\mu\nu}, \quad (42) \]

being \( g^{\mu\nu}(x) \) given by (2). (Although there is a simpler basis tetrad to describe the metric tensor associated with a global monopole, the above choice is convenient, as we shall see, in the sense that taking \( \alpha = 1 \) we reobtain the standard expression for the Dirac equation in a flat spacetime.) Now writing the Dirac equation in this spacetime and in presence of the self-potential we get:

\[
\left[ i \gamma^{(0)} \frac{\partial}{\partial t} + i \alpha \gamma^{(r)} \frac{\partial}{\partial r} + \frac{i}{r \sin \theta} \gamma^{(\theta)} \frac{\partial}{\partial \theta} + \frac{i}{r} \gamma^{(\varphi)} \frac{\partial}{\partial \varphi} + i \frac{(\alpha - 1)}{r} \gamma^{(r)} - q \gamma^{(0)} A_0 - M \right] \Psi(x) = 0, \quad (43)
\]

where \( \gamma^{(r)} = \vec{\gamma}. \hat{r}, \gamma^{(\theta)} = \vec{\gamma}. \hat{\theta} \) and \( \gamma^{(\varphi)} = \vec{\gamma}. \hat{\varphi} \), being \( \hat{r}, \hat{\theta} \) and \( \hat{\varphi} \) th standard unit vector along the three spatial directions in the spherical coordinates. We shall use the following representation of the flat space \( \gamma \)-matrices.

\[
\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\gamma^{(i)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}
\]

and for the complete set solutions to the Dirac equation

\[
\Psi(r, t) = \frac{1}{r} \begin{pmatrix} i F_{jm}(r) \Phi_{jm}(\theta, \varphi) \\ G_{jm}(r) (\vec{\sigma}. \vec{r}) \Phi_{jm}(\theta, \varphi) \end{pmatrix} e^{-iEt} \quad (44)
\]

which presents a well defined parity under the transformation \( \vec{r} \rightarrow \vec{r}' = -\vec{r} \). The \( \Phi_{jm}(\theta, \varphi) \) are the spinor spherical harmonics.

Substituting (44) into (43) and after some minor steps we find the set of radial differential equations,

\[
(E - M - K/r) F_{jm} = -\alpha \frac{dG_{jm}}{dr} + \eta \frac{G_{jm}}{r}, \quad (45)
\]
\[(E + M - K/r)G_{jm} = \alpha \frac{dF_{jm}}{dr} + \eta \frac{F_{jm}}{r}\]  

(46)

where \(\eta = \mp (j + 1/2)^2\). Let us first consider the possibility of this system, charged particle and global monopole, presents bound states. This fact, as we have mentioned previously, is only possible when \(K < 0\), which implies \(\alpha > 1\). (Although this situation is really an unphysical one, we decided to consider it below in order to make our analysis a complete one). We shall try solutions of the equations (45) and (46) in the form

\[F_{jm}(r) = C(1 + E/M)^{1/2}e^{-\kappa r}(\kappa r)^{\lambda_j}(F_1(r) + F_2(r)),\]  

(47)

and

\[G_{jm}(r) = C(1 - E/M)^{1/2}e^{-\kappa r}(\kappa r)^{\lambda_j}(F_1(r) - F_2(r)),\]  

(48)

where \(\kappa = \sqrt{\frac{M^2 - E^2}{\alpha}}\) and \(\lambda_j = \sqrt{\left(\frac{j+1/2}{\alpha}\right)^2 - K^2}\).

Using in (45) and (46) \(K = -|K|\) and developing intermediate calculations, we found that the solutions for the unknown functions \(F_1(r)\) and \(F_2(r)\) are expressed in terms of confluent hypergeometric functions as

\[F_2(r) = C_1 F_1(\lambda_j - \xi, 2\lambda_j + 1; 2\kappa r)\]  

(49)

and

\[F_1(r) = C_2 \frac{\alpha^2 \kappa \lambda_j - E|K|}{M|K| - \eta \alpha \kappa} F_1(\lambda_j - \xi + 1, 2\lambda_j + 1; 2\kappa r)\]  

(50)

where \(\xi = \frac{EK}{\alpha^2 \kappa}\). The expression for \(F_1(r)\) has been found using the general relations between the confluent hypergeometric functions [3]

\[(z \frac{d}{dz} + a) F_1(a, b; z) = a F_1(a + 1, b; z).\]

So the complete expressions to the functions \(F\) and \(G\) are given by:

\[F_{jm}(r) = C(1 + E/M)^{1/2}e^{-\kappa r}(\kappa r)^{\lambda_j} \left[ F_1(\lambda_j - \xi, 2\lambda_j + 1; 2\kappa r) + \frac{\alpha^2 \kappa \lambda_j - E|K|}{M|K| - \eta \alpha \kappa} F_1(\lambda_j - \xi + 1, 2\lambda_j + 1; 2\kappa r) \right]\]  

(51)
and

\[ G_{jm}(r) = -C(1 - E/M)^{1/2}e^{-\kappa r}(kr)^{\lambda_j} \left\{ 1_F(\lambda_j - \xi, 2\lambda_j + 1; 2\kappa r) - \frac{\alpha^2\kappa\lambda_j - E|K|}{M|K| - \eta\alpha\kappa} 1_F(\lambda_j - \xi + 1, 2\lambda_j + 1; 2\kappa r) \right\}. \quad (52) \]

From the expressions above we can see the dependence of the radial functions on the parameter \( \alpha \) which define this geometry and also on the external electrostatic potential through the constant \( K \).

In order to obtain discrete values for the self-energies it is necessary to impose the vanishment condition on the wave function when \( \kappa r \to \infty \). From the asymptotic behaviour the confluent hypergeometric function \[9] we get.

\[ \frac{E|K|}{\kappa\alpha^2} - \lambda_j = n, \quad n = 0, 1, 2... \quad (53) \]

Using the values for \( \kappa \) and \( \lambda_j \) given previously we obtain the explicit expression for the self-energy.

\[ E_{n,j} = M \left[ 1 + \frac{K^2}{(n\alpha + \sqrt{(j + 1/2)^2 - K^2})^2} \right]^{-1/2}. \quad (54) \]

Also from the expression above we see the energy depend on the parameter \( \alpha \) in two different ways, through this parameter itself and the constant \( K \) indirectly. (Considering \( \alpha = 1 \) and keeping the constant \( K \neq 0 \), the self-energy above reproduces the values of the self-energies of an electron in the hydrogen atom.) The explicit normalized eigenfunctions \[13\] can also be obtained.

Let us now study scattering states. These states are obtained considering \( E^2 > M^2 \) in (45) and (46). This is the real case because \( K > 0 \).

Again we shall try solutions in the form:

\[ F_{j,m}(r) = C(1 + E/M)^{1/2}e^{ikr}(kr)^{\lambda_j}(F_1(r) + F_2(r)), \quad (55) \]
\[ G_{j,m}(r) = -C(1 - E/M)^{1/2} e^{i\kappa r}(kr)^{\lambda_j}(F_1(r) - F_2(r)), \quad (56) \]

where \( \kappa = \sqrt{E^2 - M^2}/\alpha. \)

Substituting the expressions above in (45) and (46) and developing some intermediate calculations we find that \( F_1(r) \) and \( F_2(r) \) can also be expressed in terms of complex confluent hypergeometric functions as:

\[ F_2(r) = C_1 F_1(\lambda_j + i\beta, 2\lambda_j + 1; -2i\kappa r) \quad (57) \]

and

\[ F_1(r) = \frac{\alpha^2(i\beta + \lambda_j)}{i(\frac{-M\kappa}{\kappa} + i\eta\alpha)} F_1(\lambda_j + 1 + i\beta, 2\lambda_j + 1; -2i\kappa r), \quad (58) \]

where \( \beta = EK/\alpha^2\kappa. \)

The complete expressions for the scattering wave function can be given by

\[ F_{j,m} = C(1 + E/M)^{1/2} e^{i\kappa r} (kr)^{\lambda_j} \left\{ F_1(\lambda_j + i\beta, 2\lambda_j + 1; -2i\kappa r) + \frac{\alpha^2(i\beta + \lambda_j)}{i(\frac{-M\kappa}{\kappa} + i\eta\alpha)} F_1(\lambda_j + 1 + i\beta, 2\lambda_j + 1; -2i\kappa r) \right\}, \quad (59) \]

and

\[ G_{j,m} = C(1 - E/M)^{1/2} e^{i\kappa r} (kr)^{\lambda_j} \left\{ F_1(\lambda_j + i\beta, 2\lambda_j + 1; -2i\kappa r) - \frac{\alpha^2(i\beta + \lambda_j)}{i(\frac{-M\kappa}{\kappa} + i\eta\alpha)} F_1(\lambda_j + 1 + i\beta, 2\lambda_j + 1; -2i\kappa r) \right\}, \quad (60) \]

As in the bosonic case we can also obtain the phase shift by the asymptotic form of the wave function (44), which by its turn depend on the above expressions. So, let us write down the long distance behavior for \( F \) and \( G. \)

\[ F_{jm} \approx C \left(1 + E/M\right)^{1/2} \frac{\Gamma(2\lambda_j + 1)}{|\Gamma(\lambda_j + i\beta)|} \frac{2}{\left| \frac{n}{\alpha} - i \frac{M\kappa}{\kappa} \right|} e^{\beta \pi/2} e^{\frac{1}{2}(\sigma_1 - \sigma_2)} \times \left\{ \cos \left[ \kappa r - \frac{\lambda_j \pi}{2} - \beta \ln(2\kappa r) + \gamma_j + \frac{\sigma_1 - \sigma_2}{2} \right] + \left| \frac{\lambda_j + i\beta}{2\kappa r} \right| \sin \left[ \kappa r - \frac{\lambda_j \pi}{2} - \beta \ln(2\kappa r) + \gamma_j + \frac{\sigma_1 + \sigma_2}{2} \right] + ... \right\} \quad (61) \]
and

\[ G_{jm} \approx -C(1 - E/M)^{\frac{\lambda_j}{2}} \Gamma(2\lambda_j + 1) \left| \frac{2}{\Gamma(\lambda_j + i\beta)} \left( -\frac{\eta}{\alpha} - i\frac{MK}{\kappa\alpha^2} \right) \right| e^{\beta\pi/2} e^{\frac{1}{2} \left( \sigma_1 - \sigma_2 \right)} \times \]

\[ \left\{ \sin \left[ \kappa r - \frac{\lambda_j}{2} - \beta \ln(2kr) + \gamma_j + \frac{\sigma_1 - \sigma_2}{2} \right] - \frac{1}{2\kappa r} \cos \left[ \kappa r - \frac{\lambda_j}{2} - \beta \ln(2kr) + \gamma_j + \frac{\sigma_1 + \sigma_2}{2} \right] + ... \right\} \] (62)

where

\[ \sigma_1 = \arg \left( \frac{1}{\lambda_j - i\beta} \right), \] (63)

\[ \sigma_2 = \arg \left( \frac{1}{-\eta/\alpha - iMK/\kappa\alpha^2} \right) \] (64)

and

\[ \gamma_j = \arg \Gamma(\lambda_j + i\beta). \] (65)

From the above expressions we can obtain the phase shift, which is given by

\[ \delta_j = \frac{\pi}{2} [(j + 1/2) - \lambda_j] + \gamma_j + \frac{\sigma_1 - \sigma_2}{2}. \] (66)

As in bosonic case we can see that the phase shift presents contributions coming from: 

(i) The modification in the effective total angular quantum numbers, \( \lambda_j \), due to the geometry of the manifold itself and (ii) the presence of the induced self-interaction. There is also an additional contribution when we compare (63) with (33), the phase shift for the bosonic scattering states, this difference is due to specific behaviour of the spinor field in presence of an external potential.
4. Concluding Remarks

In this paper we have analysed the relativistic motion at classical and quantum point of view, of a charged particle in the neighbourhood of a global monopole, considering the induced electrostatic self-interaction as the zeroth component of an external four vector potential $A_\mu$. Although the magnitude of the self-interaction can be a small quantity for a typical model of global monopole system coming from a grand unification theory \footnote{For a typical grand unification theory the parameter $\eta$, associated with the scale where the global symmetry is spontaneously broken, is of the order $10^{16}$ Gev and $\Delta = 1 - \alpha^2 = 8\pi G\eta^2 \simeq 10^{-5}$. In Ref.\cite{2} it is estimated the magnitude of the induced self-interaction for small $\Delta$, it is approximately $K \approx \frac{q^2 \pi \Delta}{32}$}, which seems to be the more probable, its effects can be measured by the phase shift for scattering elementary particle by a global monopole.

In order to make our analysis as more complete as possible, we decided to investigate the behaviour of the classical and quantum motion of the charged particle, considering the possibility to have an attractive electrostatic self-interaction. For this case we could observe that it is possible to construct bound states for the system composed by the particle and the global monopole.

The main results of this paper were the obtainment of exact equation for classical orbits, self-energies, phase shifts and complete eigen-function at quantum level, for the movement of a charged particle in the neighbourhood of a global monopole.

Finally we would like to make a comment about the procedure adoptel by us in this paper. Although we have used the induced electrostatic self-interaction given in Eq.\cite{3} to study the relativistic motion a charged particle, this result was obtained in Ref.\cite{2} assuming that the particle was at a fixed position outside the global monopole. The complete procedure to determine the self-interaction due to a moving particle in the presence of a global monopole, should take into account the previous knowledge of the particle’s trajectory. Because this procedure makes a concret analysis about this system almost impossible we decided to adopt the quasistatic approach presented here, i.e, we neglected the corrections on the self-interaction coming from this movement \footnote{In Ref.\cite{10} is presented the induced electromagnetic self-interaction on a charged particle moving along an arbitrary trajectory on a conical space-time}.
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