Measurable operators and the asymptotics of heat kernels and zeta functions

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Abstract

In this note we answer some questions inspired by the introduction in [6, 7], by Alain Connes, of the notion of measurable operators using Dixmier traces. These questions concern the relationship of measurability to the asymptotics of $\zeta$-functions and heat kernels. The answers have remained elusive for some 15 years.\[1\]

Keywords: Dixmier traces, heat kernels, measurable elements, generalized limits, Cesàro operator.

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1. Introduction and Preliminaries

In [7] (see also [6]) Alain Connes described in part the relationship between Dixmier traces, heat kernel asymptotics and the behaviour of $\zeta$–functions at their leading singularity. In that discussion he introduced the notion of a measurable operator. Subsequently these notions have arisen in other contexts and interest has been generated in obtaining a comprehensive picture of how they are related. The present authors were forced to confront these ideas in their attempts to develop tools for semifinite noncommutative geometry in [3], [5]. Similar issues arise also in [2]. In addition, after discussions with many colleagues, it became clear to us that, for applications, extensions of [7, 3] were needed. There has been considerable progress in the last few years in [2, 4, 13, 16, 17, 22, 23, 24, 26, 27]. In this note we provide the final answer to two of the outstanding questions.

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1 We dedicate this paper to the memory of Nigel Kalton.
As we have done previously in [3], [5], we will work in the generality of semifinite von Neumann algebras although even for the more standard case of the bounded operators on Hilbert space the results of this paper are new. Let \( \mathcal{M} \) be a von Neumann algebra equipped with a faithful normal semifinite trace \( \tau \). For every operator \( A \in \mathcal{M} \), let \( E_A(s, \infty) \) denote the spectral measure of \( |A| \), then its distribution function \( d_A \) and rearrangement \( \mu(A) \) are defined by the following formulas:

\[
d_A(s) = \tau(E_A(s, \infty)), \quad s > 0
\]

\[
\mu(t, A) = \inf\{s : d_A(s) \leq t\}, \quad t > 0.
\]

The following sets of operators from \( \mathcal{M} \) are widely used in noncommutative geometry (see [7, 2, 3, 4, 5, 16, 17, 27]). The reader should be aware of the fact that the notation we are using is not that of [7].

\[
\mathcal{M}_{1,\infty} = \{ A \in \mathcal{M} : \sup_{t \in (0, \infty)} \frac{1}{\log(1 + t)} \int_0^t \mu(s, A) ds < \infty \}
\]

and

\[
\mathcal{L}_{1,\infty} = \{ A \in \mathcal{M} : \sup_{t > 0} t \mu(t, A) < \infty \}.
\]

Equipped with the norm

\[
\|A\|_{\mathcal{M}_{1,\infty}} := \sup_{t \in (0, \infty)} \frac{1}{\log(1 + t)} \int_0^t \mu(s, A) ds
\]

the first set is an example of a Marcinkiewicz operator space. The second set is the so-called weak \( \mathcal{L}_1 \) space, which is a linear (non-closed) subspace in \( (\mathcal{M}_{1,\infty}, \| \cdot \|_{\mathcal{M}_{1,\infty}}) \). Recall also that \( \mathcal{L}_{1,\infty} \) is not dense in \( \mathcal{M}_{1,\infty} \) with respect to the norm \( \| \cdot \|_{\mathcal{M}_{1,\infty}} \) (see e.g. [13, Lemma 5.5 in Ch.II.7]).

We need the (multiplicative) Cesaro operator acting on the space \( L_\infty(0, \infty) \) of all essentially bounded Lebesgue measurable functions given by the formula

\[
(Mx)(\nu) = \frac{1}{\log(\nu)} \int_1^\nu x(s) \frac{ds}{s}, \quad (1)
\]

If \( A \in \mathcal{L}_{1,\infty} \), then it follows from [4, Lemma 5.1] that

\[
\sup_{\lambda} \int \frac{1}{\lambda} \tau(e^{-\lambda A} - 1) < \infty.
\]

The inequality (2) does not necessarily hold for \( A \in \mathcal{M}_{1,\infty} \) (see [4, Example on p. 274]). It is implicitly proved in [3] (see also [27, Theorem 40 and Corollary 41] where a much stronger result is established) that

\[
\sup_{\lambda} M(\lambda \rightarrow \int \frac{1}{\lambda} \tau(e^{-(\lambda A)} - 1)) < \infty
\]

for every \( A \in \mathcal{M}_{1,\infty} \) where the notation is a shorthand for taking the supremum of the function obtained from applying \( M \) to \( \lambda \rightarrow \int \frac{1}{\lambda} \tau(e^{-(\lambda A)} - 1) \).

This note is motivated by the following two questions (that we will completely answer here).
Question 1. Suppose that $A \in L^{+}_{1,\infty}$ is such that the limit
\[ \lim_{\lambda \to \infty} \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1}) \]
exists. What information is then available on the distribution function of the operator $A$?

Question 2. Suppose that $A \in M^{+}_{1,\infty}$ is such that the following limit
\[ \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1})) \]
exists. What information is then available on the distribution function of the operator $A$?

Note we are again using an obvious shorthand notation in Question 2. These questions are in fact related to somewhat similar matters studied in [4] (there they are called questions A and B) for the $\zeta$-function and the connection between them can be established via the theory of Dixmier traces (see e.g. [2, 6, 3, 2, 17, 16, 14, 24, 26]). Very briefly, we now recall some basic definitions from that theory.

First, a positive normalised functional on a unital von Neumann algebra is called a state and any state on the algebra $L_{\infty}(0, \infty)$ is called a generalised limit if it vanishes on every function with compact support.

Second, given $s > 0$, a dilation operator $\sigma_s : L_{\infty}(0, \infty) \to L_{\infty}(0, \infty)$ is defined by setting $(\sigma_s x)(t) = x(t/s)$. A generalised limit $\omega$ is said to be dilation invariant if $\omega \circ \sigma_s = \omega$ for every $s > 0$.

Third, if $\omega$ is an arbitrary dilation invariant generalised limit then a Dixmier trace $\tau_\omega$ on $M_{1,\infty}$ is defined (see [14, Definition 9]) by the formula
\[ \tau_\omega(A) := \omega(t \to \frac{1}{\log(1 + t)} \int_0^t \mu(s, A) ds), \quad A \in M_{1,\infty}^+. \]

Recall that a positive linear functional $\varphi$ on $M_{1,\infty}$ is called fully symmetric if for all $0 \leq A, B \in M_{1,\infty}$ such that
\[ \int_0^t \mu(s, B) ds \leq \int_0^t \mu(s, A) ds, \quad \text{for all } t > 0, \]
we have $\varphi(B) \leq \varphi(A)$. In this note the largest possible class of Dixmier traces, namely the class
\[ D := \{ \tau_\omega : \omega \text{ is a dilation invariant generalized limit} \} \]
of all Dixmier traces is needed. (See further possibilities in [2, 17, 5, 16, 24, 26]). That this class is natural is confirmed by the fact that $D$ coincides with the class of all fully symmetric singular functionals on $M_{1,\infty}$.

More precisely, the following assertion follows from [14, Theorem 11] if we set the function denoted by $\psi$ in that theorem to be $\psi(t) = \log(1 + t)$.
Theorem 1. For every fully symmetric functional \( \varphi \) on \( \mathcal{M}_{1,\infty} \), there exists a
dilation invariant generalised limit \( \omega \) such that the Dixmier trace \( \tau_\omega = \varphi \).

Now we establish the notation for, and background to, our main theorem.
Let \( \omega \) be an arbitrary dilation invariant generalised limit. A heat kernel
functional \( \xi_\omega \) is defined (see [27, Sections 1 and 5]) by the formula
\[
\xi_\omega(A) := (\omega \circ M)(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1})), \quad A \in \mathcal{M}_{1,\infty}^+.
\]
See [7, 3, 4, 27] for the reasons for this particular form of the definition of the
heat kernel functional.
Let \( \gamma \) be an arbitrary generalised limit. The \( \zeta \)-function residue (associated
with \( \gamma \)) is defined (see [27, Section 1] and also [7, 3]) by the formula
\[
\zeta_\gamma(A) := \gamma(r \to \frac{1}{r} \tau(A^{1+1/r})).
\]
Evidence that the zeta and heat kernel functionals are closely related comes
from the following theorems, proved in [27].

Theorem 2. [27, Theorem 8]. For every generalised limit \( \gamma \), the \( \zeta \)-function
residue \( \zeta_\gamma \) is a fully symmetric functional on \( \mathcal{M}_{1,\infty} \).

Theorem 3. [27, Theorem 22]. For every dilation invariant generalised limit
\( \omega \), the heat kernel functional \( \xi_\omega \) is a fully symmetric functional on \( \mathcal{M}_{1,\infty} \).

Theorem 4. [27, Theorem 31]. For every fully symmetric functional \( \varphi \) on
\( \mathcal{M}_{1,\infty} \), there exists a dilation invariant generalised limit \( \omega \) such that the heat
kernel functional \( \xi_\omega = \varphi \).

Remark 5. In fact, it is proved in [27, Theorem 31] and [27, Lemma 20] that
for every fully symmetric functional \( \varphi \) on \( \mathcal{M}_{1,\infty} \), there exists a dilation invariant
generalised limit \( \omega \) such that for every \( q > 0 \),
\[
(\omega \circ M)(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-q})) = \Gamma(1 + 1/q) \varphi.
\]
In view of Theorems 1 to 4 it is natural to ask whether the equality \( \tau_\omega = \xi_\omega \)
holds for an arbitrary dilation invariant generalised limit \( \omega \). This is however
not the case, see [27, Theorem 37] where examples of \( \omega \)'s are given for which we
have \( \tau_\omega \neq \xi_\omega \).

Finally, we come to one of the major new notions introduced in this context
in [7] (see also [6]) and generalised in [17, 5, 16, 24, 26].

Definition 6. The operator \( A \in \mathcal{M}_{1,\infty} \) is said to be measurable if and only if
the set \( \{ \tau_\omega(A) : \tau_\omega \in \mathcal{D} \} \) consists of a single point.
In view of the previously cited results and counter-examples our main result, which we now state, is not entirely expected. It answers Question 2 and complements and extends earlier results in [7, 3, 4]. It also provides a very short new proof of the main result from [17] (see the proof of the implication (i) \(\implies\) (ii) below).

**Theorem 7.** Let \(A \in \mathcal{M}_{1,\infty}\) be a positive operator. The following conditions are equivalent.

(i) The operator \(A\) is measurable.

(ii) The limit \(\lim_{t \to \infty} \frac{1}{\log(1+t)} \int_0^t \mu(s, A)ds\) exists.

(iii) The limit \(\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda A)^{-1}}))\) exists.

(iv) The limit \(\lim_{s \to 0} s \tau(A^{1+s})\) exists.

Furthermore, if any of the conditions (i)-(iv) above holds, then we have the coincidence of the three limits

\[
\lim_{t \to \infty} \frac{1}{\log(1+t)} \int_0^t \mu(s, A)ds = \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda A)^{-1}})) = \lim_{s \to 0} s \tau(A^{1+s})
\]

with the value given by \(\{\tau_\omega(A) : \tau_\omega \in \mathcal{D}\}\).

**Remark 8.** Let \(a_1 = \tau_\omega(A)\) for every \(\tau_\omega \in \mathcal{D}\) in (i) in Theorem 7. Let \(a_2, a_3\) and \(a_4\) be the limits in (ii), (iii) and (iv) (respectively) in Theorem 7. For every \(1 \leq i, j \leq 4\), we show in the proof of Theorem 7 that (i) \(\implies\) (j) and \(a_i = a_j\).

The results should be seen in the general context of the continuing study of the notion of measurable operators introduced in [6, 7] and further elaborated in [17, 3, 16]. The main interest remains in the following areas: (i) comparing various modifications of this notion with respect to various subsets of Dixmier traces (as a rule with additional properties of invariance), (ii) finding convenient descriptions of the set of self-adjoint measurable operators, and (iii) determining when a given self-adjoint measurable operator is Tauberian. We remark that this current note is related to progress on these directions which will appear in [24, 26], where it is shown that not every self-adjoint measurable operator is necessarily Tauberian (which is in stark contrast with the case of positive operators). It will also be shown in [26] that the notion of measurability as originally introduced by Connes in [6] and its version considered in [17] actually coincide.

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2. Proof of the main result

The following lemma is well-known. The proof can be found in e.g. [12, Section 6.8].

Lemma 9. Let \( z \in L_\infty(0, \infty) \) be a positive differentiable function. If \( tz'(t) \geq \text{const} \) for every \( t > 0 \), then the following implication holds

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t z(s) ds = C \implies \lim_{t \to \infty} z(t) = C.
\]

The following lemma is also well-known. Due to the lack of a suitable reference we provide a short proof for convenience of the reader.

Lemma 10. Let \( x \in L_\infty(0, \infty) \) and let \( a \in \mathbb{R} \). The following conditions are equivalent.

1. We have the bounds

\[
\liminf_{t \to \infty} x(t) \leq a \leq \limsup_{t \to \infty} x(t).
\]

2. There exists a generalised limit \( \gamma \) such that \( \gamma(x) = a \).

Proof. The implication (2) \( \implies \) (1) follows immediately from the definition of the generalised limit.

In order to prove the implication (1) \( \implies \) (2), define a functional \( \gamma \) on \( \mathbb{R} + x\mathbb{R} \) by setting \( \gamma(\alpha + \beta x) = \alpha + \beta a \). Clearly,

\[
\gamma(z) \leq \limsup_{t \to \infty} z(t), \quad z \in \mathbb{R} + x\mathbb{R}.
\]

The assertion follows now from the Hahn-Banach theorem. \( \square \)

Our next lemma plays an important role in the proof of our main result.

Lemma 11. Let \( z \) be a positive locally integrable function on \((0, \infty)\). If \( Mz \in L_\infty(0, \infty) \), then we have

\[
\lim_{t \to \infty} (M^2 z)(t) = C \implies \lim_{t \to \infty} (M z)(t) = C.
\]

Proof. Set \( x = (Mz) \circ \exp \). We have

\[
(M^2 z)(t) = \frac{1}{\log(t)} \int_1^t (Mz)(u) \frac{du}{u} = \frac{1}{\log(t)} \int_0^{\log(t)} x(s) ds,
\]

where we used the substitution \( u = e^s \) in the second equality. By the assumption, we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds = C.
\]
Let us now verify that the function $t \rightarrow tx'(t)$ satisfies the assumption of Lemma 9. We have

$$tx'(t) = t\left(\frac{1}{t} \int_{0}^{e^{t}} z(s) \frac{ds}{s}\right)' = -\frac{1}{t} \int_{0}^{e^{t}} z(s) \frac{ds}{s} + z(e^{t}).$$

Since $z$ is positive, we have $tx'(t) \geq -(Mz)(e^{t})$ and since $Mz \in L_{\infty}(0, \infty)$, we conclude $tx'(t) \geq \text{const}$. By Lemma 9 we have $\lim_{t \to \infty} x(t) = C$ and hence $\lim_{t \to \infty} (Mz)(t) = C$.

The following remark is well known and can be found in e.g. [7].

**Remark 12.** For every generalised limit $\gamma$, the state $\gamma \circ M$ is a dilation invariant generalised limit.

With these preliminary results in hand we come to the proof of our main result.

**Proof. (Of Theorem 7.)** First, the implication $(ii) \implies (i)$ follows from the definition of $\tau_{\omega}$. Next, the implication $(i) \implies (ii)$ was first proved in [17, Theorem 6.6] (see also [3]). We provide here a new (very short and straightforward) proof.

Let

$$C := \tau_{\omega}(A), \text{ for all } \tau_{\omega} \in \mathcal{D}.$$ 

In particular, by Remark 12 we have $\tau_{\gamma \circ M}(A) = C$ for every generalised limit $\gamma$. That is, we have the equality

$$(\gamma \circ M)(t) \to \frac{1}{\log(1 + t)} \int_{0}^{t} \mu(s, A) ds = C,$$

which, due to Lemma 10 guarantees

$$\lim_{t \to \infty} M(t) \to \frac{1}{\log(1 + t)} \int_{0}^{t} \mu(s, A) ds = C. \quad (4)$$

Set $z(t) := t\mu(t, A)$. Observe that $z$ is a positive measurable, but not necessarily bounded function. However, since $A \in M_{1, \infty}$, the function

$$t \to (Mz)(t) = \frac{1}{\log(t)} \int_{1}^{t} \mu(s, A) ds$$

is bounded. Thus, $Mz \in L_{\infty}(0, \infty)$ and obviously

$$\lim_{t \to \infty} (Mz)(t) - \frac{1}{\log(1 + t)} \int_{0}^{t} \mu(s, A) ds = 0. \quad (5)$$

Combining (4) and (5), and using the (obvious) fact that $\lim_{t \to \infty} (Mz)(t) = 0$ whenever $y \in L_{\infty}(0, \infty)$ satisfies $\lim_{t \to \infty} y(t) = 0$, we infer that

$$\lim_{t \to \infty} (Mz)(t) = C.$$
By Lemma 11, we obtain from the preceding equality

\[ \lim_{t \to \infty} (Mz)(t) = C \]

and the proof of the implication is completed by referring to (3).

(iii) \(\implies\) (i). Let \(C\) be the limit in (iii). By definition of \(\xi_\omega\), we have \(\xi_\omega(A) = C\) for every dilation invariant generalised limit \(\omega\). By Theorems 1 and 4, the class \(D\) coincides with the class of all heat kernel functionals and so, we also have \(\tau_\omega(A) = C\) for every \(\tau_\omega \in D\) and the proof of the implication is completed.

(i) \(\implies\) (iii). Suppose that \(\tau_\omega(A) = C\) for every \(\tau_\omega \in D\). Then the same argument as above shows that \(\xi_\omega(A) = C\) for every dilation invariant generalised limit \(\omega\). In particular, due to Remark 12, we have \(\xi_{\gamma \circ M}(A) = C\) for every generalised limit \(\gamma\). That is,

\[ (\gamma \circ M^2)(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1})) = C. \]

It follows from Lemma 10 that

\[ \lim_{\lambda \to \infty} M^2(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1})) = C. \]

Due to (3), we know that the mapping \(\lambda \to M(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-1}))\) is bounded and therefore the proof of the implication is completed by invoking Lemma 11.

(i) \(\implies\) (iv). Suppose that \(\tau_\omega(A) = C\) for every \(\tau_\omega \in D\). It follows from Theorems 2 and 1 that the class of all \(\zeta\)-function residues is a subclass of \(D\). Hence, for every generalised limit \(\gamma\), we have

\[ \gamma(t \to \frac{1}{t} \tau(A^{1+1/t})) = C. \]

An appeal to Lemma 11 completes the proof of the implication.

Finally, the implication (iv) \(\implies\) (i) is established in [3, Theorem 3.1].

Our methods have a further interesting consequence. Repeating the argument (i) \(\implies\) (iii) in Theorem 7 verbatim (and using Remark 5 instead of Theorem 4), we obtain the following result.

**Proposition 13.** For every measurable positive operator \(A \in \mathcal{M}_{1,\infty}\) and every \(q > 0\), we have

\[ \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-\lambda A}^{-q})) = \Gamma(1 + \frac{1}{q}) \lim_{s \to 0} s \tau(A^{1+s}). \]

3. **Answering Question 1**

The following corollary answers Question 1. Its proof immediately follows from the implication (iii) \(\implies\) (ii) established in Theorem 7.
Corollary 14. Let $A \in \mathcal{L}_{1,\infty}$ be a positive operator. If the limit
\[ \lim_{\lambda \to \infty} \frac{1}{\lambda} \tau(e^{-\lambda A})^{-1} \]
exists then so does the limit
\[ \lim_{t \to \infty} \frac{1}{\log(1 + t)} \int_0^t \mu(s, A) ds. \]

The following example shows that the converse to Corollary 14 does not hold.

Example 15. There exists a positive operator $A \in \mathcal{L}_{1,\infty}$ such that
\[ \lim_{t \to \infty} \frac{1}{\log(1 + t)} \int_0^t \mu(s, A) ds = 0 \] (6)
and
\[ \limsup_{t \to \infty} \frac{1}{\lambda} \tau(e^{-\lambda A})^{-1} > 0. \] (7)

Proof. Define a positive operator $A$ by setting
\[ \mu(s, A) = \begin{cases} s^{-1}, & s \in (e^n, ne^n), \ n \geq 1 \\ e^{-e^{n+1}}, & s \in (ne^n, e^{n+1}), \ n \geq 1 \\ e^{-e}, & s \in (0, e^n). \end{cases} \]

For every $n \geq 1$, we have
\[ \int_0^{e^{n+1}} \mu(s, A) ds = 1 + \sum_{k=1}^{n} \left( \int_{e^k}^{e^{k+1}} \mu(s, A) ds + \int_{e^k}^{e^{k+1}} \mu(s, A) ds \right) = \]
\[ = 1 + \sum_{k=1}^{n} \left( \log(k) + 1 - (k + 1)e^{-(e-1)e^k} \right) = n + \log(n!) + O(1) = O(n \log(n)). \]

Here, the last equality follows from Stirling’s formula
\[ n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}, \ 0 < \theta < 1. \]

For every $t > e$, let $\nu = \nu(t) = [\log(\log(t))]$. It follows that
\[ \int_0^t \mu(s, A) ds \leq \int_0^{e^{\nu+1}} \mu(s, A) ds = O(\nu \log(\nu)) = o(\log(t)), \]
which yields (5).

On the other hand, we have
\[ \frac{1}{\lambda} \tau(e^{-\lambda A})^{-1} \geq \frac{1}{\lambda} \sum_{n=1}^{\infty} \int_{e^n}^{ne^n} e^{-\lambda s} ds = \sum_{n=1}^{\infty} e^{-\lambda e^n} - e^{-n\lambda^{-1}e^n}. \]
For a given \( n \in \mathbb{N} \), set \( \lambda = e^{e^n} \). It follows that
\[
\frac{1}{\lambda} \tau(e^{-(\lambda A)^{-1}}) \geq e^{-\lambda^{-1}e^{e^n}} - e^{-n\lambda^{-1}e^{e^n}} = e^{-1} - e^{-n}.
\]
Therefore,
\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \tau(e^{-(\lambda A)^{-1}}) \geq e^{-1},
\]
yielding (7).

This example has a further interesting consequence.

**Corollary 16.** The limit
\[
\lim_{t \to \infty} \frac{1}{\lambda} \tau(e^{-(\lambda A)^{-1}})
\]
does not exist and hence we cannot omit \( M \) in Theorem [7].

**Proof.** Suppose that the limit in (8) exists and is equal to \( c \). Then, obviously
\[
\lim_{\lambda \to \infty} M(\lambda \to 1) \tau(e^{-(\lambda A)^{-1}}) = c
\]
and by Theorem [7] we obtain that
\[
\lim_{t \to \infty} \frac{1}{\log(1+t)} \int_0^t \mu(s, A) ds = c.
\]
It follows from (6) that \( c = 0 \). Thus, we should then have that the limit in (8) is 0. However, the latter contradicts (7).

Finally we see that this example demonstrates that we are not able to claim any meromorphic continuation property for the zeta function on the basis of our results to this point.

**Lemma 17.** For the operator \( A \) constructed in Example [15], the \( \zeta \)-function
\[
s \to \tau(A^{1+s})
\]
does not have a pole or a removable singularity at 0.

**Proof.** Assume the contrary, that is, the \( \zeta \)-function admits an analytic continuation into the punctured neighborhood of 0 and has an \( n \)-th order pole there. By Theorem [7] and [10], we have \( \lim_{s \to 0} s \tau(A^{1+s}) = 0 \). Therefore, we have \( \lim_{s \to 0} s^n \tau(A^{1+s}) = 0 \), which contradicts the assumption. The \( \zeta \)-function \( s \to \tau(A^{1+s}) \) does not have a removable singularity at 0 because \( A \notin \mathcal{L}_1 \) (that is the limit \( \lim_{s \to 0} \tau(A^{1+s}) \) does not exist).

It is important to observe that we are also in a position to answer analogues of Questions [1] and [2] in the case of arbitrary operators from \( \mathcal{M}_{1,\infty} \) (not necessarily positive). For brevity, we state and prove such analogues for self-adjoint operators.
Theorem 18. Suppose that a self-adjoint operator $A \in \mathcal{M}_{1,\infty}$ is such that the following limit
\[
\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda}(\tau(e^{-\lambda A_+})^{-1} - \tau(e^{-\lambda A_+})^{-1}))
\]
exists. Then the operator $A$ is measurable.

Proof. The proof is a verbatim repetition of the arguments used in the proof of the implication (iii) $\implies$ (i) in Theorem 7. We omit further details. 

It is worth remarking that we cannot ascertain whether, under the assumptions in Theorem 18, the limit
\[
\lim_{t \to \infty} \frac{1}{\log(1 + t)} \int_0^t (\mu(s, A_+) - \mu(s, A_-))ds
\]
exists. However, this can be done, if $A$ belongs to the weak $\mathcal{L}_1$ space.

Theorem 19. Suppose that a self-adjoint operator $A \in \mathcal{L}_{1,\infty}$ is such that the following limit
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda}(\tau(e^{-\lambda A_+})^{-1} - \tau(e^{-\lambda A_+})^{-1})
\]
exists. Then the operator $A$ is measurable, and, in addition, the limit
\[
\lim_{t \to \infty} \frac{1}{\log(1 + t)} \int_0^t (\mu(s, A_+) - \mu(s, A_-))ds
\]
exists.

Proof. The first assertion follows immediately from Theorem 18. The second assertion is provided by [24, Corollary 19].

Remark 20. The assumption $A \in \mathcal{L}_{1,\infty}$ in Theorem 19 above can be further weakened by requesting $\mu(t; A) = o(\frac{\log(1 + t)}{t})$ for sufficiently large $t > 0$. In a sense the latter is the best possible, in particular, the assertion of Theorem 19 fails if the latter condition does not hold. For details, we refer the reader to [24].

4. The case $p > 1$ and examples

4.1. Notations

We firstly say a few words concerning the notations.

In the paper [6] (where the applications of Dixmier traces to noncommutative geometry were first presented) Alain Connes considered the ideal $\mathcal{L}^{1+}$ of all compact operators $T$ on an infinite-dimensional Hilbert space $H$ whose singular values $\{\mu(j, T)\}_{j \in \mathbb{N}}$ satisfy
\[
\sup_{N > 1} \frac{1}{\log N} \sum_{j=1}^N \mu(j, T) < \infty.
\]
This ideal later, in [7, p.303] was denoted by $L^{(1,\infty)}$. Further, in [6, p.677], the ideal $L^{n+}$, whose $n^{th}$ root lies in $L^{1+}$ in $B(H)$, was introduced. It is noted in [7] that the ideals $L^{(p,\infty)}$ correspond to the notion of weak $L^p$-spaces in classical analysis. An alternative notation $L^{p+}$ is also mentioned.

It is now important to realize that there is a small notational discrepancy here, and addressing this discrepancy, we have used another notation for the space $L^{1+}$ in [6] and $L^{(1,\infty)}$ in [7]. Namely, we used the symbol $M_{1,\infty}$. We now explain a little bit more about our choice.

As noted in [7], the Banach space $(M_{1,\infty}, \| \cdot \|_{M_{1,\infty}})$ was probably first considered by Macaev [19] (with yet another notation, which we do not use here at all in order not to confuse the reader) as the dual space to the ideal which is customarily called, a Macaev ideal. For a complete exposition of the theory of these spaces and detailed references, we refer the reader to the books [10, 11]. The reason we used this notation is due to the fact that the space $(M_{1,\infty}, \| \cdot \|_{M_{1,\infty}})$ may be viewed as a noncommutative analogue of a Sargent (sequence) space, see [21]. This fact is explained in the article [20] by A.Pietsch, for which we refer the reader for a fuller treatment of the history of the space $M_{1,\infty}$ and additional references. We follow this notation also because it allows us to reserve $L^{1,\infty}$ for the well-established notion of quasi-normed weak $L^1$-space (which we identify here with a non-closed subspace in $(M_{1,\infty}, \| \cdot \|_{M_{1,\infty}})$).

The classical $p$-convexification procedure for an arbitrary Banach lattice $X$ is described in [18, Section 1.d] and is sometimes termed power norm transformation. It is simply a direct generalization of the procedure of defining $L^p$-spaces from an $L^1$-space. Applying the analogous operation to the ideal $M_{1,\infty}$, we obtain the space $Z_p$, firstly introduced and (alternatively) described in [4]. It is unfortunate that, due to other notations used in [4], the space $Z_p$ was identified there with the notation $L^{p,\infty}$. One of the reasons, we have switched to the notations $M_{p,\infty}$ and $L_{1,\infty}$ is that the notation $L^{p,\infty}$ is then properly associated with the $p$-convexification of the weak $L^1$ space $L_{1,\infty}$. In this way, our usage of the symbol $L^{p,\infty}$ is perfectly compatible with the usage of the same symbol in [4] for all $p > 1$ (excepting $p = 1$ for which we use $M_{1,\infty}$). It is now natural to denote the space $Z_p$ by the symbol $M_{p,\infty}$. Thus, the space $M_{p,\infty}$ is exactly obtained by asking for $p$-th roots in $M_{1,\infty}$ and coincides with the space $L^{p,\infty}$ from [4], whereas the $p$-convexification of its subspace $L_{1,\infty}$ equipped with the weak quasi-norm yields the Banach space $L^{p,\infty}$ and this is exactly the same space from [7] which we cited above, at the beginning of this subsection.

4.2. Results

The following assertion is a consequence of Theorem 7.

**Corollary 21.** Let $A \in Z_p = M_{p,\infty}$ be a positive operator. The following conditions are equivalent.

(i) The operator $A^p$ is measurable.

(ii) The limit $\lim_{{t \to \infty}} \frac{1}{\log(1+t)} \int_0^t \mu(s,A^p)ds$ exists.

(iii) The limit $\lim_{{\lambda \to \infty}} M(\lambda \to \frac{1}{\lambda^p} \tau(e^{-(\lambda A)^{-p}}))$ exists.
(iv) The limit $\lim_{s \to 0} s\tau(A^p + s)$ exists.

Furthermore, if any of the conditions (i)-(iv) above holds, then we have the coincidence of the three limits

$$\lim_{t \to \infty} \frac{1}{\log(1 + t)} \int_0^t \mu(s, A^p) ds = \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda^p} \tau(e^{-(\lambda A)^{-p}})) = \frac{1}{p} \lim_{s \to 0} s\tau(A^p + s)$$

with the value given by $\{\tau_\omega(A^p) : \tau_\omega \in \mathcal{D}\}$.

Proof. Set $B = A^p$. Clearly, $B \in M_{1,\infty}$ and

$$\lim_{s \to 0} s\tau(A^p + s) = p \lim_{s \to 0} s\tau(B^{1+s}).$$

Let $P : L_\infty(0, \infty) \to L_\infty(0, \infty)$ be the operator defined by setting $(Px)(t) = x(t^p)$, $x \in L_\infty(0, \infty)$, $t > 0$. We have $PM = MP$ (see [3, Proposition 1.3(4)]). Hence,

$$\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda^p} \tau(e^{-(\lambda A)^{-p}})) = \lim_{\lambda \to \infty} MP(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda B)^{-1}})) = \lim_{\lambda \to \infty} PM(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda B)^{-1}})) = \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda B)^{-1}})).$$

From this last equality it is clear that the result follows immediately from Theorem 7. □

Remark 22. The implication (iii) $\to$ (i) of Theorem 21 significantly strengthens Proposition 5.3 of [4]. This is because we require here only the existence of the limit in (iii) and not an asymptotic expansion for the heat kernel as is assumed in [4] and furthermore we do not require $\omega$ to be $M-$invariant as is needed in [4].

The next corollary prepares the way for a discussion of heat kernel bounds. It follows from Proposition 13 and parallels [3, Proposition 4.2]. The latter proposition looks similar to the one below, however its proof is totally different.

Corollary 23. Let $A \in \mathbb{Z}_p = \mathcal{M}_{p,\infty}$ be a positive operator such that $A^p$ is measurable. We have

$$\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda^p} \tau(e^{-(\lambda A)^{-p}})) = \frac{1}{2} \Gamma\left(\frac{p}{2}\right) \lim_{s \to 0} s\tau(A^p + s).$$

Proof. Set $B = A^p$. Clearly, $B \in M_{1,\infty}$. Using the same argument as in the proof of Corollary 21 we obtain

$$\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda^p} \tau(e^{-(\lambda A)^{-p}})) = \lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda^{-2/p} A^{-2})})).$$

Now, by Proposition 13 we have

$$\lim_{\lambda \to \infty} M(\lambda \to \frac{1}{\lambda} \tau(e^{-(\lambda B)^{-2/p}})) = \Gamma(1 + \frac{p}{2}) \lim_{s \to 0} s\tau(B^{1+s}).$$

Finally, we write

$$\lim_{s \to 0} s\tau(B^{1+s}) = \frac{1}{p} \lim_{s \to 0} s\tau(A^p + s).$$

□
4.3. Discussion

It would of course be interesting to find examples that flesh out Corollary 21. Recent work on heat kernels on metric spaces (such as fractals) is promising. These examples illustrate that there is a hierarchy of conditions on the asymptotics of zeta functions and heat kernels. In the study of diffusion processes on fractals [15] one is given the generator of the heat semigroup $\Delta$ as a positive self adjoint densely defined operator. Then we assume that $
abla = \tau(\Delta^{-s/2}) < \infty$ for all $s > p$ where $p$ is called the spectral dimension. (For simplicity we are going to assume that $\Delta$ has bounded inverse if not there is a simple remedy [3].)

The weakest condition we can impose is that there are constants $C_0, C_0'$ with:

$$C_0' \leq (s - p)\zeta_\Delta(s) \leq C_0$$

for all $s > p$. It follows from Theorem 4.5 of [4] that the operator $\Delta^{-p/2} \in \mathcal{M}_{1,\infty}$ however, it also follows by an example in [4] (which does not come from any concrete diffusion process but is an artificial counterexample) that this bound is insufficient to obtain heat kernel bounds and that the best we can do is the bound [3] where we need to insert the Cesaro mean.

On the other hand a heat kernel bound of the form

$$C^{-1}t^{-p/2} \leq \tau(e^{-t\Delta}) \leq Ct^{-p/2}, \quad 0 < t < 1$$

(which is known to hold for some diffusion processes on metric spaces and in particular for certain fractals, see for example [15]) is stronger than the zeta function bound (9) as can be seen by the following elementary argument.

Recall that if $B \in \mathcal{M}$ is a positive operator then it follows from the spectral theorem that

$$B^{s/2} = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1}e^{-tB^{-1}}dt.$$  \hspace{1cm} (10)

Next, suppose that (10) holds. Setting $B = \Delta^{-1}$, it follows from (10) that for all $s > p$ we have

$$\int_0^\infty t^{s/2-1}\tau(e^{-t\Delta})dt \leq C \int_0^1 t^{(s-p)/2-1}dt + \int_1^\infty t^{s/2-1}\tau(e^{-t\Delta})e^{-(t-1)\|\Delta^{-1}\|^{-1}dt \leq \frac{2C}{s-p} + e\|\Delta^{-1}\|^{-1}\tau(e^{-\Delta}) \frac{\Gamma(s/2)}{\|\Delta^{-1}\|^{s/2}}. \hspace{1cm} (11)$$

It follows from Fatou lemma that $B^{s/2}$ is trace class for all $s > p$ and so

$$(s - p)\zeta_\Delta(s) \leq 2C + o(1), \quad s \downarrow p.$$  

Similarly, we have

$$\int_0^\infty t^{s/2-1}\tau(e^{-t\Delta})dt \geq C^{-1} \int_0^1 t^{(s-p)/2-1}dt = \frac{2C^{-1}}{s-p}$$

and therefore

$$2C^{-1} \leq (s - p)\zeta_\Delta(s) \quad s \downarrow p.$$
We have assumed that the function \( s \to \tau(\Delta^{-s/2}) \) is analytic in \( s \) for \( \Re(s) > p \) and that it may have a singularity at \( s = p \). However, we saw in Lemma 17 that the nature of this singularity is not obvious in general.

It is well known (and, in the context of the questions discussed here, explained in [4]) how an asymptotic expansion for small \( t \) of the form \( \tau(e^{-t\Delta}) \sim Ct^{-p/2} + O(t^{-\alpha/2}) \), where \( \alpha < p \), implies that the \( \zeta \)-function has a meromorphic continuation to a half plane \( \Re(s) > p - \epsilon \), for some \( \epsilon > 0 \) with the only singularity in this half plane being a simple pole at \( s = p \). However such an assumption is not in line with what has been found for certain fractals.

There is a discussion of the pole structure of the zeta function for certain fractal diffusion processes in [8, 25], and literature cited therein. There we find fractals where \( \xi \) is meromorphic with simple poles on the line \( \{ p + iv | v \in \mathbb{R} \} \). To discuss this situation we can employ here a well known argument similar to that of [4], in particular the ideas introduced in the proof of Theorem 5.2 in that paper (where we used the notation \( T = \Delta^{-1} \)).

We have \( \xi(s) = \int_0^\infty t^{s/2-1} \tau(e^{-t\Delta})dt \) and may split this integral into two parts as in (11). Then only \( \int_0^1 t^{s/2-1} \tau(e^{-t\Delta})dt \) contributes to the singularity at \( s = p \) (as we exploited in [4]). Now suppose that we have a simple asymptotic expansion of the form \( \tau(e^{-t\Delta}) \sim Ct^{-p/2} + O(t^{-\alpha/2}) \) with \( \alpha < p \) for \( 0 < t < 1 \). Then
\[
\int_0^1 t^{s/2-1} \tau(e^{-t\Delta})dt = \frac{2C}{s-p} + G(s)
\]
where \( G(s) = \int_0^1 (t^{s/2-1} \tau(e^{-t\Delta}) - Ct^{(s-p)/2})dt \). Here the integrand is by assumption continuous and \( O(t^{(s-\alpha)/2}) \) and hence \( G(s) \) is analytic for \( \Re(s) > \alpha \) and in particular on the line \( \{ p + iv | v \in \mathbb{R} \} \) which is inconsistent with the assumption of there being poles on this line. Thus the asymptotic behaviour of the trace of the heat kernel must be more complicated for such fractals.

There is a positive result that we obtain from Corollary 23. Setting \( t = \lambda^{-2} \) and \( A = \Delta^{-1/2} \) in the formula in this Corollary gives
\[
\lim_{t \to 0} M(t \to \tau(t^{p/2}e^{-t\Delta})) = \frac{1}{2} \Gamma\left(\frac{p}{2}\right) \lim_{z \to 0} z \tau(\Delta^{-(p+z)/2}).
\]
Connecting with our previous notation we set \( s = p + z \) and see that the presence of a simple pole at \( s = p \) for the \( \zeta \)-function means that \( \lim_{t \to 0} M(t \to \tau(t^{p/2}e^{-t\Delta})) \) exists. This simple pole behavior at \( s = p \) is conjectured in [25] to be a generic feature of a certain class of fractals. Our Corollary 23 suggests that to infer from this, information about the trace of the heat kernel for small \( t \), it is more promising to investigate the asymptotic behavior of \( M(t \to t^{p/2} \tau(e^{-t\Delta})) \).

To illustrate this we use [1]. There it is shown that for the Sierpinski Gasket one has for small \( t \),
\[
\tau(e^{-t\Delta}) = t^{-\beta} \gamma(t) + o(t^{-\beta})
\]
(13) where \( \beta = \log 3/\log 5 \) and
\[ \gamma(t) = \sum_{-\infty}^{\infty} c_n \Gamma(1 + \beta + \frac{2\pi in}{\log 5}) e^{-2\pi in \log t / \log 5}. \] (14)

Numerical evidence supports the conjecture that \( \gamma(t) = a + b \sin \frac{2\pi v}{\log 5} (\log t - c) \) for some real \( a, b, c \). If this conjecture is true we can insert equation (13) into equation (12). Then we make the change of variable \( t = \lambda^{-2} \) and consider the resulting Cesaro mean (1) as a function of the asymptotic variable \( \nu \). We see that it is bounded by \( a + C \log \nu \) for some constant \( C \) as \( \nu \to \infty \) so that the \( t \) independent constant \( a \) in \( \gamma(t) \) gives the required zeta function residue.

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