Entropy generation in radiative magneto-hydrodynamic mixed convective flow of viscoelastic hybrid nanofluid over a spinning disk

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ABSTRACT

The heat and mass relocation properties of magneto-hydrodynamic mixed convective viscoelastic hybrid nanofluid flow induced by an extending spinning disk under the effect of entropy generation, thermal radiation, convective condition, velocity, and concentration slips has been investigated in this study. For hybrid nanofluid, the amalgamation of aluminum nitride and alumina nanoparticles embedded in carboxymethyl cellulose with a volume mass concentration of 0.0%-0.4% is considered for the study. The acquired system of partial differential equations from the intended problem is translated into ordinary differential equations employing resemblance conversion and solved by the Galerkin finite element approach. The main effects of the governing constraints on the velocity field, temperature dispersion, concentration, Bejan number, entropy production, skin friction, Nusselt number, and Sherwood number were detailed and depicted in graphs and tables. The results show that snowballing in viscoelastic constraint and volume fraction of solid non-sized particles of $\text{Al}_2\text{O}_3$ and $\text{AlN}$ can be used to control fluid flow speed. Also, it is observed from the result that an increment in the magnetic field parameter causes a decline in the velocity field. However, the increase in magnetic field constraint and volume fraction of solid non-sized particles of $\text{Al}_2\text{O}_3$ and $\text{AlN}$ causes an upsurge in temperature distribution. Entropy generation in the system can also be regulated by using a higher volume fraction of aluminum nitride and alumina nanoparticles. This numerical and theoretical investigation is more useful in bio-viscoelastic fluids, advanced technology, and industry.

1. Introduction

The fundamental issue that arises in the heat transfer process in many advanced industries and technologies is low thermal conductivity. As a result, researchers are looking for a groundbreaking solution to improve the thermal competence of liquids. As solutions, they have presented a variety of mechanisms. Accordingly, Choi and Eastman [1] have proposed the use of nano-sized ($\leq 100$ nm) particles in conventional fluids and coined the term “nanofluid” to describe them. Nanofluids are widely used in modern industries and technologies such as nanotechnology, biotechnology, electromechanical systems, modern drug delivery systems, paint and nuclear schemes, electronics device cooling, and lubrication, among others. Later, researchers such as [2, 3, 4] have studied the problems associated with nanofluids theoretically and numerically through various aspects of flow geometries, concluding that heat transfer performance improves as the volume fraction of nanoparticles increases. Recently, the attention of investigators is increasing from day to day towards investigating a new class of nanofluid, which is a so-called hybrid nano-liquid, due to its ability to offset any shortcomings of mono nano-fluid by mixing at least two different kinds of nanoparticles into the ordinary fluid to improve the heat relocation. For instance, Hamid et al. [5] investigated the heat relocation properties of TiO$_2$–SiO$_2$–water hybrid nanoparticles and discovered that the maximum rate of heat transfer augmentation is 22.1% at a volumetric fraction of nanoparticles of 3.0%. In addition, Ashorynejad [6] investigated the thermal proficiency improvement by hybrid nanofluid in an open hollow, indicating that heat transfer performance improves as the volumetric fraction of nanoparticles increases. Furthermore, a summary of the heat relocation assets of mono and hybrid nanofluids was presented by Bumataria et al. [7]. Likewise, Sadaf and Abdelsalam [8] theoretically described the heat relocation assets of Ag–$\text{Al}_2\text{O}_3$/blood hybrid nanofluid in an area bounded by two concentric circles. They also discovered that brick, platelet, and cylinder-shaped nanoparticles have features similar to the
trapped bolus. Moreover, Tili et al. [9] numerically investigated the heat conveyance competency of CuO–MgO/methanol hybrid nanofluid across an uneven extending sheet and found that the CuO–MgO/methanol combination is an effective insulator. Tassadig et al. [10] investigated the heat and mass transmission of a CNT–Fe$_3$O$_4$–water hybrid nanofluid formed by a spinning disk, concluding that the heat and mass transfer rate is improved by the combination of CNTs and Fe$_3$O$_4$/water nanoparticles in water. Lund et al. [11] studied the MHD 3D flow of Al$_2$O$_3$–Cu hybrid nanoparticles over a nonlinear shrinking surface and found that the heat transfer rate for the hybrid nanofluid was higher than for the viscous fluid and simple nanofluid. Abdel-Nour et al. [12] also looked at the effects of entropy formation on an Al$_2$O$_3$–Cu/H$_2$O hybrid nanoliquid in an absorbent media. ElZahar et al. [13] investigated Al$_2$O$_3$–Cu/water hybrid-nanoparticles over a horizontal circular cylinder, concluding that Cu-water has a higher heat transfer rate than hybrid nanofluid and Al$_2$O$_3$–H$_2$O. In a permeable canal, Eid and Nafe [14] investigated magnetohydrodynamic (MHD) hybrid nanofluid flow and found that the increase of Fe$_3$O$_4$/EG nanoparticle concentration enhances the heat transfer rate of the hybrid nanofluid in a shrinkable case. Zainal et al. [15] also investigated the flow of Al$_2$O$_3$–Cu/H$_2$O hybrid nanofluid across expanding and shrinking surfaces, and indicated that an upsurge attribution in the heat transfer rate is observed resulting from the increment of Biot number.

Flow across a spinning disk has a significant application in mechanical, architectural, civil, and marine engineering, computer storage devices, geothermal industry, and aeronautical science [16, 17] and such alike flows were early studied by Karman [18]. In addition, [19, 20, 21, 22, 23, 24] have identified more research on fluid flow across a rotating disk. Yin et al. [25] investigated the heat displacement of Cu, Al$_2$O$_3$, and CuO/H$_2$O nanofluid generated by a spinning disk. Micropolar nanofluid flow induced by a spinning disk was examined by Zemed and Ibrahim [26].

Heat relocation in non-Newtonian (Visco-elastic) fluids is particularly relevant because of its practical applications in food processing, oil recovery, and flow through filtering media. As a result, [27, 28, 29] investigated the flow of viscoelastic fluid with various flow geometries, and summarized how much the flow of visco-elastic fluid influenced heat and mass relocation. Mahat et al. [30] examined the flow of visco-elastic nanofluid above a horizontal circular cylinder as well. Khan et al. [31] investigated the swirling flow of visco-elastic nanofluid created by a spinning disk.

Heat transmission effects with entropy creation have been extensively studied since Bejan’s seminal work [32], in which he proposed a method to optimize system degradation and heat transfer effects with entropy formation. The influence of entropy production and an induced magnetic field on the flow of pseudoplastic nanofluid across an elastic surface was examined by Hou et al. [33]. They concluded that increasing the Brinkmann number increases the rate of entropy generation. In addition, Vaidya et al. [34] investigated the influence of homogeneous and heterogeneous responses on the peristaltic flow of MHD Jeffrey fluid with entropy generation, finding that the temperature variance constraint increases the fluid’s entropy generation. The impact of heat and mass distribution on entropy generation due to the MHD flow of nanofluid across a rotating frame was investigated by Mahood et al. [35]. They discovered that the rate of entropy generation and Bejhan number depend upon the magnetic field and the Eckert number. For further information on the linked idea, readers can refer to Ref [36, 37, 38, 39, 40].

As can be seen from all of the above-mentioned works, the problem of entropy generation in the radiative magneto-hydrodynamic mixed convective flow of hybrid nanofluid caused by a spinning disk has not been investigated so far. This investigation aims to examine the entropy generation in the radiative magneto-hydrodynamic mixed convective flow of viscoelastic AlN–Al$_2$O$_3$ hybrid nanofluid across a spinning disk with the convective condition, velocity, and concentration slips. The non-Newtonian (Visco-elastic) fluid model and the Hamilton Crosser type hybrid nanofluid model consisting of AlN–Al$_2$O$_3$ (CMC-water) were also considered in the study. The effects of the most important governing constraints on centrifugal and tangential motion, temperature dispersion, concentration dispersion, Bejhan number, entropy production, skin friction, Nusselt number, and Sherwood number were detailed and represented in graphs and tables. Also, the solution to the designed problem in the study was obtained by the Galerkin finite element approach [41].

2. Physical interpretation and modeling

2.1. Physical interpretation

Undertake the steady laminar flow of an electrically conducting and an incompressible hybrid nanofluid induced by a spinning disk with an angular velocity $\Omega$ around the z-axis as displayed in Fig. 1. To inspect heat relocation attributes, the visco-elastic fluid model, mixed convection flow, entropy generation, thermal radiation, convective condition, velocity, and concentration slips are considered for the study. The amalgamation of AlN and Al$_2$O$_3$ is chosen as the hybrid nanofluid and CMC-water is preferred as the base fluid. The cylindrical coordinate system $(r, \omega, z)$ is chosen with the corresponding velocity $(u, v, w)$. Assume the disk is situated at $z = 0$ and the flow of hybrid nanofluid is confined in the semi-space $z > 0$ and the magnetic field strength $B_0$ is applied perpendicular to the fluid flow. The induced magnetic field and electric field effects are not incorporated under low magnetic Reynolds number assumptions. For the study, the convective heating mode is considered, and for this reason, the temperature and concentration at the surface, respectively, is assumed as $T_s$ and $C_s$, whereas $T_\infty$ and $C_\infty$ respectively, are considered free stream temperature and concentration.

2.2. Mathematical modeling

The major governing equations under the aforesaid supposition [16, 31, 42] are given in (1)–(6).

\[
\frac{u}{r} + \frac{\partial w}{\partial z} = 0
\]  
(1)

\[
\frac{\partial u}{\partial r} + u \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} - \frac{\nu}{\rho_{\text{nf}}} \left( \frac{\partial^2 u}{\partial z^2} \right) = - \sigma_{\text{nf}} B_0^2 \frac{u}{\rho_{\text{nf}}} - g \left( \bar{\beta}_t (T - T_\infty) + \bar{\beta}_c (C - C_\infty) \right)
\]  
(2)

\[
\frac{\partial v}{\partial r} + u \frac{\partial v}{\partial r} + \frac{w}{r} = \frac{\nu}{\rho_{\text{nf}}} \left( \frac{\partial^2 v}{\partial z^2} \right) = \frac{u}{\rho_{\text{nf}}} - g \left( \bar{\beta}_t (T - T_\infty) + \bar{\beta}_c (C - C_\infty) \right)
\]  
(3)

\[
\frac{\partial w}{\partial r} + u \frac{\partial w}{\partial r} + \frac{\partial w}{\partial z} = \frac{\nu}{\rho_{\text{nf}}} \left( \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right)
\]  
(4)
| Physical properties | $\rho$ (kg m$^{-3}$) | $c_p$ (kg/m$^3$K) | $k$ (W/mK) |
|---------------------|----------------------|-------------------|------------|
| CMC-water (0.0%-0.4%) | 997.1                | 4179              | 0.613      |
| Alumina (Al$_2$O$_3$) | 3.970                | 765.0             | 40.000     |
| AlN | 3260                | 735.0             | 180.00     |

\[
\frac{\partial T}{\partial r} + \frac{u}{r} \frac{\partial T}{\partial z} = \frac{k_{\text{hf}}}{(\rho c_p)_{\text{hf}}} \left( \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) + \frac{16\sigma r T_A^4}{k_1(\rho c_p)_{\text{hf}}} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right)
\]

\[
u \frac{\partial C}{\partial r} + \frac{u}{r} \frac{\partial C}{\partial z} = \gamma_{\text{hf}} \left( \frac{\partial^2 C}{\partial z^2} \right)
\]

Where $\gamma_{\text{hf}}$ corresponds to (density, dynamic viscosity, electrical conductivity, thermal conductivity, heat capacity, and mass diffusivity) of the AlN–Al$_2$O$_3$ system, respectively, stands for (acceleration due to gravity, thermal expansion and concentration expansion coefficient).

### 2.3. Boundary stipulations

Boundary conditions [26, 43] are given in (7):

\[
\begin{align*}
\frac{du}{dz} &= \frac{\partial T}{\partial z} = 0, \\
\frac{dv}{dz} &= \frac{\partial u}{\partial z} = 0, \\
\frac{dw}{dz} &= \frac{\partial T}{\partial z} = h_T (T_s - T), \\
C &= C_0 + v \frac{\partial C}{\partial z}
\end{align*}
\]

\[
\begin{align*}
u &= r \Omega + \tilde{R} \frac{du}{dz}, \\
v &= r \Omega + \tilde{R} \frac{\partial u}{\partial z}, \\
w &= 0, \\
\frac{\partial T}{\partial z} &= h_T (T_s - T), \\
C &= C_0 + v \frac{\partial C}{\partial z}
\end{align*}
\]

\[
\begin{align*}
\frac{du}{dz} &= \frac{\partial T}{\partial z} = h_T (T_s - T), \\
C &= C_0 + v \frac{\partial C}{\partial z}
\end{align*}
\]

$\tilde{R}$ is constant and $h_T$ refers the heat transfer coefficient.

### 2.4. Thermo-physical attributes

For the considered hybrid nanofluid, the theoretical models and thermo-physical attributes are as in Table 1 and Table 2, where the sets of $(k_{\alpha}, k_{\beta}, k_{\gamma}), (\phi_2, \phi_3, \phi_4), (\rho_{\alpha}, \rho_{\beta}, \rho_{\gamma}), (c_{\alpha}, c_{\beta}, c_{\gamma})$, correspondingly, refer the thermal conductivity of (Al$_2$O$_3$, AlN and CMC-water), nanoparticle volume fraction of (Al$_2$O$_3$, AlN), density of (Al$_2$O$_3$, CMC-water, and AlN), electrical conductivity of (Al$_2$O$_3$, CMC-water and AlN).

### 2.5. Similarity conversions

The resemblance conversions involved in the study [26, 50] are given in (8):

\[
\begin{align*}
\eta &= \frac{z}{X}, \\
u &= r \Omega f(q), \\
v &= r \Omega f(q), \\
w &= -(2\Omega X)^{\frac{1}{2}} f(q), \\
T &= (T_s - T_{\infty}) \Theta(q) + T_{\infty}, \\
C &= (C_s - C_{\infty}) \Phi(q) + C_{\infty}
\end{align*}
\]
Where, \( \eta \) (dimensionless similarity variable), \( f \) (dimensionless stream velocity), \( g \) (tangential velocity) and \( f' \) (radial velocity), \( \Theta \) (non-dimension temperature), and \( \Phi \) (dimensionless concentration), \( \chi \) (kinematic viscosity)

### 2.6. Non-dimensional form of governing equations

Using (8), the resulting non-dimensional ordinary differential equations are stated in (9)–(13)

\[
2f'''' - \left[ (1 + \phi_A) \right]^{2,5} \left[ (1 - \phi_B) \right]^{2,5} \left[ f'''' - \left[ \left( 1 - \phi_B \right) \left( 1 - \phi_A \right) + \phi_A \left( 1 - \phi_B \right) \right] \frac{f'''}{f'} + \frac{f''}{f'} \right] \left[ \left( f'''' - \frac{f''}{f'} \right) - \frac{GrRe_2}{\sigma_f} \left( \Theta - \Phi \right) - \frac{\eta}{\sigma_f} M f' \right] = 0
\]

\[
2g'''' - \left[ (1 + \phi_A) \right]^{2,5} \left[ (1 - \phi_B) \right]^{2,5} \left[ g'''' - \left[ \left( 1 - \phi_B \right) \left( 1 - \phi_A \right) + \phi_A \left( 1 - \phi_B \right) \right] \frac{g'''}{g'} + \frac{g''}{g'} \right] \left[ \left( g'''' - \frac{g''}{g'} \right) - \frac{GrRe_2}{\sigma_f} \left( \Theta - \Phi \right) - \frac{\eta}{\sigma_f} M g' \right] = 0
\]

\[
\left( \frac{k_{hbf}}{k_f} + \frac{4}{3} Rd \right) \Theta'' + Pr \left[ \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) + \phi_A \frac{(\rho c_p)_{\alpha}}{(\rho c_p)_{\alpha} + \phi_B (\rho c_p)_{\alpha}} \right] \left\{ f \Theta' \right\} + 0 = 0
\]

\[
\Phi'' + Pr \left( \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) \right) L e f \Phi' = 0
\]

with

\[
f (\eta = 0) = 0, \quad f' (\eta = 0) = 0, \quad g (0) = 1 + S g' (\eta = 0), \quad \Theta' (\eta = 0) = \gamma (\Theta (\eta = 0) - 1), \quad \Phi (\eta = 0) = 1 + \gamma \Phi' (\eta = 0)
\]

\[
f' \rightarrow 0, \quad f'' \rightarrow 0, \quad g \rightarrow 0, \quad g' \rightarrow 0, \quad \Theta \rightarrow 0, \quad \Phi \rightarrow 0 \text{ as } \eta \rightarrow \infty
\]

where,

\[
\begin{align*}
Gr \text{ (Grashof number)} &= \frac{\hat{\rho}_h (T_e - T_m)}{\chi^2} \quad \text{Re} \text{ (Reynolds number)} = \frac{\Omega U_w}{\chi} \quad \text{Le} \text{ (Lewis number)} = \frac{a_f}{D_B} \\
N_i \text{ (concentration buoyancy parameter)} &= \frac{\tau_i (C_i - C_m)}{\hat{\rho}_h (T_e - T_m)} \quad \text{Pr} \text{ (Prandtl number)} = \frac{\rho c_p}{\hat{\rho} c_p \chi} \\
M \text{ (Hartman parameter)} &= \frac{\sigma F_{\alpha i}^2}{\Omega \chi} \quad S \text{ (slip parameter)} = \frac{\hat{R} \sqrt{\frac{3}{6} \chi}}{\gamma} \quad \gamma \text{ (Biot number)} = h_i \sqrt{\frac{3}{6} \chi} \\
A \text{ (viscoelastic parameter)} &= \tau_i \frac{\Omega}{\chi}
\end{align*}
\]

### 2.7. Physical quantities

The physical quantities are \( C_f \) (skin friction), \( N_u \) (heat transportation rate), and \( Sh \) (mass transportation rate) and are defined as in (14).

\[
C_f = \frac{\tau_{uv}}{\frac{1}{2} \hat{\rho}_h (ar)^2}, \quad N_u = \frac{q_u}{k_{hbf} (T_e - T_m)} \quad Sh = \frac{q_u}{D_B (C_i - C_m)}
\]

Where, \( \tau_{uv} \) (wall shear stress) = \( \hat{\rho}_h \frac{\partial u}{\partial z} \bigg|_{z=0} \), \( q_u \) (wall heat flux) = \( -k_{hbf} \frac{\partial T}{\partial z} \bigg|_{z=0} \), \( \phi_A \) (mass flux) = \( -D_B \frac{\partial A}{\partial z} \bigg|_{z=0} \).

The dimensionless form of (14) is given in (15).

\[
\frac{1}{2} \left( C_f \left( Re_e \right) \right)^{3/2} = \frac{1}{\left( 1 - \phi_B \right)^{2,5} \left( 1 - \phi_A \right)^{2,5} f''''(0)}, \quad (Re_e)^{-3/2} N_u = -\left( \frac{k_{hbf}}{k_f} + \frac{4}{3} Rd \right) \Theta' (0), \quad (Re_e)^{-6/5} Sh = -\Phi' (0)
\]

### 2.8. Entropy generation

The entropy generation expression for an incompressible viscoelastic AlN–Al2O3 hybrid nanofluids [35, 51] is as given in (16):

\[
S_G = \frac{1}{T_2^2} \left[ k_{hbf} + \frac{2 \sigma T_2^3}{3 k_i} \left( \frac{\partial u}{\partial z} \right)^2 + \frac{\hat{\rho}_h}{T_2} \left\{ \frac{2}{\partial r} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{2}{\partial r} \left( \frac{\partial u}{\partial z} \right)^2 + \frac{2}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 + \frac{2}{\partial z} \left( \frac{\partial u}{\partial r} \right)^2 \right\} \right]
\]

\[
S_G = \text{owing to heat relocation and thermal radiation}
\]

\[
N_G = \frac{1}{T_2} \left( k_{hbf} + \frac{2 \sigma T_2^3}{3 k_i} \right) \left( \frac{\partial u}{\partial z} \right)^2 + \frac{\hat{\rho}_h}{T_2} \left\{ 2 \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right\}
\]

\[
N_G = \frac{2}{k_f (T_e - T_m)} \left( \frac{\hat{\rho}_h (ar)^2}{\chi^2} \right)
\]

The dimensionless form of (17) is (18)
\[ N_G = \delta \left( \frac{k_{hi}}{k_f} + \frac{4Rd}{3} \right) \left( \Theta'^2 \right) + \frac{Br}{\left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5}} \left[ 6f'f'' + r_1^2 \left( f'^2 + g'^2 \right) \right] \] (18)

2.9. Bejan number

Mathematically obtained as in (19)

\[ Be = \frac{\frac{k_{hi}}{k_f} + \frac{2\sigma f^3}{3h_n} \left( \frac{\partial \sigma}{\partial \Theta} \right)^2}{\left( \frac{\Theta}{\Theta_0} \right)^2} \] (19)

The dimensionless form of (19) is stated in (20)

\[ Be = \delta \left( \frac{k_{hi}}{k_f} + \frac{2\sigma f^3}{3h_n} \right) \left( \Theta'^2 \right) + \frac{Br}{\left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5}} \left[ 6f'f'' + r_1^2 \left( f'^2 + g'^2 \right) \right] \] (20)

Where, \( Br \) (Brinkmann number) = \( \frac{k_{hi}H^3}{k_f(1 - \phi_A)} \), \( \delta \) (temperature ratio parameter) = \( \frac{T_r - T_\infty}{T_r} \).

3. Numerical solution

Equations (9)–(12) are naturally nonlinear, and solving such a like system of equations analytically is difficult. As a result, the Galerkin finite element method is used to solve (9)–(12). This approach is so widely used in modern engineering analysis that it is frequently used to solve integral equations, including dynamic computational fluids, and is a highly successful method for tackling a variety of nonlinear problems [41, 52]. To use the Galerkin finite element method, first, make the following assumptions as in (21).

\[ f' = H, \quad g' = P \] (21)

Putting (21) into (9)–(12) and the resulting equation are shown in (22)–(25):

\[ 2H'' - \left( \left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5} \right) \left[ A \left( 7H H'' - 4H'''' \right) - \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) + \phi_A \left( 1 - \phi_B \right) \frac{\rho_A}{\rho_f} + \phi_B \frac{\rho_B}{\rho_f} \right] = 0 \] (22)

\[ 2P' - \left( \left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5} \right) \left[ - \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) + \phi_A \left( 1 - \phi_B \right) \frac{\rho_A}{\rho_f} + \phi_B \frac{\rho_B}{\rho_f} \right] = 0 \] (23)

\[ \left( \frac{k_{hi}}{k_f} + \frac{4}{3}Rd \right) \Theta'' + Pr \left[ \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) + \left( 1 - \phi_B \right) \phi_A \left( \frac{\rho_A}{\rho_f} \right)_A + \phi_B \left( \frac{\rho_B}{\rho_f} \right)_B \right] \left( f\Theta' \right)' = 0 \] (24)

\[ \Theta'' + Pr \left( \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) \right) Lef \Phi' = 0 \] (25)

with boundary conditions are given by (26)

\[ f(0) = 0, \quad f'(0) = Sf''(0), \quad g(0) = +Sg'(0), \quad \Theta'(0) = \gamma (\Theta(0) - 1), \quad \Phi(0) = 1 + \gamma \Theta'(0), \]

\[ f' \rightarrow 0, \quad f'' \rightarrow 0, \quad g \rightarrow 0, \quad g' \rightarrow 0, \quad \Theta \rightarrow 0, \quad \Phi \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \] (26)

3.1. Variational formulation

Based on (21)–(25), the variation form of a particular linear element, \( J_n = (\eta_n, \eta_{n+1}) \) are shown in (27)–(32).

\[ \int_{\eta_n}^{\eta_{n+1}} \{ f' - H \} d\eta = 0 \] (27)

\[ \int_{\eta_n}^{\eta_{n+1}} \left[ 2H'' - \left( \left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5} \right) \left( A \left( 7H H'' - 4H'''' \right) - \left( 1 - \phi_A \right)^{2.5} \left( 1 - \phi_B \right)^{2.5} \right) \right] d\eta = 0 \] (28)

\[ \int_{\eta_n}^{\eta_{n+1}} \left( \left( 1 - \phi_A \right) \left( 1 - \phi_B \right) + \phi_A \left( 1 - \phi_B \right) \frac{\rho_A}{\rho_f} + \phi_B \frac{\rho_B}{\rho_f} \right) \left( H^2 - fH' + g'' \right) - \frac{Gr}{Re^2} \left( \Theta - N_1 \Phi \right) - \frac{\sigma_{hi}}{\sigma_f} M H \right] d\eta = 0 \] (29)

\[ \int_{\eta_n}^{\eta_{n+1}} \{ g' - P \} d\eta = 0 \] (30)
\[
\int_{\alpha_{i+1}}^{\alpha_i} \frac{2P' - \left(1 - \phi_A\right)^2 \left(1 - \phi_B\right)^2}{\eta} \left( \left(1 - \phi_A\right) \left(1 - \phi_B\right) + \frac{\left(\rho_{\psi p} \rho_{\psi p} + \phi_B}{\rho_f} \right) \left(2gH - 2fP' - \frac{Gr}{Re} \left(\Theta - N_i \Phi\right) - \frac{\eta}{\eta_f} M_R \right) \right) \left( \eta \right) = 0 \quad (30)
\]

\[
\int_{\alpha_{i+1}}^{\alpha_i} \left\{ \frac{k_{inf}}{k_f} + \frac{4}{3} R \frac{d}{d\eta} \right\} \Theta' + Pr \left[ \left(1 - \phi_A\right) \left(1 - \phi_B\right) + \left(1 - \phi_B\right) \frac{\left(\rho_{\psi p} \rho_{\psi p} + \phi_B}{\rho_f} \right) + \left(\rho_{\psi p} \rho_{\psi p} \right) f \Phi' \right] \left( \eta \right) = 0 \quad (31)
\]

\[
\int_{\alpha_{i+1}}^{\alpha_i} \Phi'' + \left(1 - \phi_A\right) \left(1 - \phi_B\right) \eta \left( \eta \right) = 0 \quad (32)
\]

Where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \) and \( \alpha_6 \) are weight functions, correspondingly, measured the variation in \( H, f, P, g, \Theta, \) and \( \Phi. \)

### 3.2. Finite element formulation

By applying finite element approximation on (27)–(32), the resulting finite element models are (33):

\[
f = \sum_{j=1}^{3} f_{ij}, \quad H = \sum_{j=1}^{3} H_{ij}, \quad g = \sum_{j=1}^{3} g_{ij}, \quad P = \sum_{j=1}^{3} P_{ij}, \quad \Theta = \sum_{j=1}^{3} \Theta_{ij}, \quad \Phi = \sum_{j=1}^{3} \Phi_{ij} \quad (33)
\]

with \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha = \gamma_i \) (i = 1, 2, 3) and \( Y_1 \) is defined for quadratic element as in (34):

\[
Y_1 = \frac{\left(\eta_{k+1} - \eta\right) \left(\eta_{k+1} + \eta - 2\eta_k\right)}{\left(\eta_{k+1} + \eta_k\right)^2}, \quad Y_2 = \frac{4 \left(\eta - \eta_k\right) \left(\eta_{k+1} + \eta - \eta_k\right)}{\left(\eta_{k+1} + \eta_k\right)^2}, \quad Y_3 = \frac{\left(\eta_{k+1} - \eta\right) \left(\eta_{k+1} + \eta - \eta_k\right)}{\left(\eta_{k+1} + \eta_k\right)^2} \quad (34)
\]

and \( \eta_k \leq \eta \leq \eta_{k+1}. \)

\[
\begin{bmatrix}
(Q_{11}^{(1)}) \\
(Q_{22}^{(1)}) \\
(Q_{33}^{(1)}) \\
(Q_{12}^{(1)}) \\
(Q_{13}^{(1)}) \\
(Q_{23}^{(1)}) \\
(Q_{14}^{(1)}) \\
(Q_{24}^{(1)}) \\
(Q_{34}^{(1)}) \\
(Q_{44}^{(1)}) \\
(Q_{15}^{(1)}) \\
(Q_{25}^{(1)}) \\
(Q_{35}^{(1)}) \\
(Q_{45}^{(1)}) \\
(Q_{55}^{(1)}) \\
(Q_{66}^{(1)})
\end{bmatrix} = 
\begin{bmatrix}
\{f\} \\
\{g\} \\
\{\Theta\} \\
\{\Phi\}
\end{bmatrix} \quad (35)
\]

Where \( [Q_{mn}] \) and \( [f_m], (m, n = 1, 2, 3, 4, 5, 6) \) are matrices and defined as:

\[
Q_{11} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_1}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{22} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_2}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{33} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_3}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{12} = Q_{21} = 0, \quad Q_{13} = Q_{31} = 0, \quad Q_{23} = Q_{32} = 0, \quad Q_{44} = 0, \quad Q_{55} = 0, \quad Q_{66} = 0
\]

\[
Q_{14} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_1}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{24} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_2}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{34} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_3}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{15} = Q_{25} = Q_{35} = Q_{45} = Q_{55} = Q_{65} = 0, \quad Q_{16} = Q_{26} = Q_{36} = Q_{46} = Q_{56} = Q_{66} = 0
\]

\[
Q_{23} = \left(1 - \phi_A\right) \left(1 - \phi_B\right) \int_{\alpha_{i+1}}^{\alpha_i} Y_1 Y_2 d\eta, \quad Q_{24} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{Gr}{Re^2} \left(1 - \phi_A\right) \left(1 - \phi_B\right) \int_{\alpha_{i+1}}^{\alpha_i} Y_1 Y_2 d\eta, \quad Q_{26} = \frac{Gr}{Re^2} \int_{\alpha_{i+1}}^{\alpha_i} \left(1 - \phi_A\right) \left(1 - \phi_B\right) Y_2 d\eta, \quad Q_{25} = \int_{\alpha_{i+1}}^{\alpha_i} \frac{\partial Y_1}{\partial \eta} \left( \frac{\partial \eta_{i,j}}{\partial \eta} \right) d\eta, \quad Q_{23} = Q_{32} = Q_{42} = Q_{52} = Q_{62} = 0
\]
The velocity along the radial and tangential is expressed by the following equations:

\[
Q_{ij}^{13} = -\frac{\sigma_{ij}}{\sigma_f} M \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta,
\]

\[
Q_{ij}^{14} = -\frac{2}{\sigma_f} \int_{\eta_j}^{\eta_{j+1}} \frac{\partial Y_j}{\partial \eta} d\eta - \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

By using the Galerkin finite element method, the velocity along the radial and tangential is expressed by the following equations:

\[
Q_{ij}^{15} = \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \frac{\mathcal{G}_r}{\mathcal{R}_c^2} \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta,
\]

\[
Q_{ij}^{16} = 0, \quad Q_{ij}^{17} = \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \frac{6 \mathcal{E}_c}{\mathcal{R}_c^2} \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

\[
Q_{ij}^{18} = 0, \quad Q_{ij}^{19} = \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \frac{2 \mathcal{E}_c}{\mathcal{R}_c^2} \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

\[
Q_{ij}^{20} = -\frac{k_{hf}}{k_f} + \sqrt{\frac{a}{2 \mathcal{R}_d}} \int_{\eta_j}^{\eta_{j+1}} \frac{\partial Y_j}{\partial \eta} d\eta \left[ (1 - \phi_A) (1 - \phi_B) + (1 - \phi_B) \phi_A \frac{(\rho_p)_{\phi_A}}{(\rho_f)_{\phi_A}} + (\rho_p)_{\phi_B} \right] \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

\[
Q_{ij}^{21} = 0, \quad Q_{ij}^{22} = Q_{ij}^{23} = Q_{ij}^{24} = Q_{ij}^{25} = Q_{ij}^{26} = 0.
\]

\[
Q_{ij}^{27} = \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \left( \frac{6 \mathcal{E}_c}{\mathcal{R}_c^2} \right) \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

\[
Q_{ij}^{28} = -\frac{k_{hf}}{k_f} + \sqrt{\frac{a}{2 \mathcal{R}_d}} \int_{\eta_j}^{\eta_{j+1}} \frac{\partial Y_j}{\partial \eta} d\eta \left[ (1 - \phi_A) (1 - \phi_B) + (1 - \phi_B) \phi_A \frac{(\rho_p)_{\phi_A}}{(\rho_f)_{\phi_A}} + (\rho_p)_{\phi_B} \right] \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

\[
Q_{ij}^{29} = 0, \quad Q_{ij}^{30} = \left( (1 - \phi_A)^2 (1 - \phi_B)^2 \right) \left( \frac{6 \mathcal{E}_c}{\mathcal{R}_c^2} \right) \int_{\eta_j}^{\eta_{j+1}} Y_i Y_j d\eta.
\]

Where,

\[
\mathcal{H} = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}, \quad \mathcal{H}' = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}, \quad \mathcal{P} = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}, \quad \mathcal{P}' = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}, \quad \mathcal{\Theta} = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}, \quad \mathcal{\Phi} = \sum_{j=1}^{3} \frac{\partial Y_j}{\partial \eta}.
\]

4. Outcomes and discussion

In the current investigation, the non-dimensional equations labeled in (9)–(12) with matching boundary conditions in (13) are numerically solved through the Galerkin finite element method. The upshot offered in the current study is acquired with the default values of $N_1 = 0.2, \gamma = 1, \mathcal{P} = 6.2, \mathcal{L} = 3, \mathcal{G}_r = 10, S = \phi_A = \phi_B = 0.01, \mathcal{E}_c = 0.3, \mathcal{R}_d = \Delta = \mathcal{R}_c = \mathcal{M} = 0.1, \gamma_1 = 0.6, \mathcal{R}_s = 5$. Also, for the cases of nanofluid (AIN+ viscoelastic fluid), viscoelastic hybrid nanofluid (AIN+Al$_2$O$_3$+ viscoelastic fluid), in that order, we assume $(\phi_B \neq 0, \phi_A = 0)$ and $(\phi_B = \phi_A \neq 0)$, whereas for only viscoelastic fluid, we presume $(\phi_A = \phi_B = 0)$.

4.1. Effect of Grashof number

The role of the Grashof number on the radial and tangential velocity, temperature, concentration, entropy generation, and Bejan number is confirmed in Fig. 2–Fig. 7. As witnessed in Fig. 2 and Fig. 3, the velocity along the radial and tangential gets bigger for higher values of the Grashof number $(0.1 \leq \mathcal{G}_r \leq 16)$. Physically, improvement in $\mathcal{G}_r (\mathcal{G}_r > 0)$ tends to increase the force of buoyancy, which prompts the movement of both the viscoelastic AIN–Al$_2$O$_3$ and the viscoelastic AIN nanofluid at the side of the rotating surface. The temperature and concentration distributions of the viscoelastic hybrid nanofluid (AIN+Al$_2$O$_3$+ viscoelastic fluid) and the viscoelastic nanofluid (AIN+ viscoelastic fluid) are reduced as $\mathcal{G}_r$ increases. Physically, the buoyancy force fabricated in the flow region owing to a high value of $\mathcal{G}_r$ dominates the inertia force that upsurges the heat relocation rate. It means that when the Grashof number increases, the molecular motion of the fluid elements reduces, resulting in a decrease in temperature dissemination throughout the flow field. Due to this fact, the finier in $\mathcal{G}_r$ causes a major diminution in temperature distribution, as observed in Fig. 4. In the same way, an increase in the values of the Grashof number has the tendency to increase the mass buoyancy effect. This gives rise to an increase in the induced flow and thereby decreases concentration, as seen in Fig. 5. It is observable from Fig. 6 that amplification in $\mathcal{G}_r$ causes augmentation in $N_0$. Intensification in $\mathcal{G}_r$ produces high viscous dissipation because of the increase in the change of velocity that is inclined to boost the entropy generation $(N_0)$. Consequently, an increase in the value of $\mathcal{G}_r$ generates the rate of entropy generation $(N_0)$ at a high level close to the wall of the rotating disk. One can perceive from Fig. 7 that the Bejan number shrinks in $\mathcal{G}_r$. Physically, intensification in the $\mathcal{G}_r$ tends to a decrease in Bejan number inside the region of flow for the reason that a reduction in irreversibility has happened because of fluid friction.
Fig. 2. Effect of Gr on radial velocity profile $f'(\eta)$.

Fig. 3. Impact of Gr on tangential velocity profile $g(\eta)$.

Fig. 4. Impact of Gr on temperature profile $\Theta(\eta)$.

Fig. 5. Effects of Gr on concentration profile $\Phi(\eta)$.

Fig. 6. Effects of Gr on $N_G$.

Fig. 7. Effects of Gr on Be.
4.2. Effect of viscoelastic parameter

Fig. 8 demonstrates that intensification in the viscoelastic parameter (0.1 ≤ A ≤ 0.4) causes a diminution in the radial velocity. Physically, the tensile stress created by viscoelasticity resists fluid motion, which is why the velocity of both the (AlN+Al2O3+ viscoelastic fluid) viscoelastic hybrid nanofluid and the (AlN+ viscoelastic fluid) viscoelastic nanofluid is reduced. Therefore, the flow of fluid assets in the viscoelastic fluid can be constrained by persuading the change in the viscoelastic parameter.

4.3. Effect of volumetric fraction of nanoparticles

Fig. 9–Fig. 14 exemplify the disparity of φ_A (1% to 4% vol. fraction of alumina) and φ_B (1% to 4% vol. fraction of aluminum nitride) on velocities along centrifugal and tangential, temperature dissemination, dissemination of concentration, entropy generation, and Bejan number, correspondingly. Noticed that f’(η) and g(η) are dwindling against the mounting value of φ_A and φ_B. Physically, the finer in φ_A causes the shrinking comportment of the momentum front stratum. Furthermore, the viscosity is enhanced in the flow of (AlN–Al2O3+ viscoelastic fluid) by intensifying from 1% to 4% vol. fraction of particles stretch advanced resistance to the fluid motion. Evidently, such a situation happened due to the incidences of Al2O3 and AlN solid nanoparticles in CMC-water as is seen in Fig. 9 and Fig. 10. However, from Fig. 11 it is illustrated that Θ(η) is improved by escalating in φ_A and φ_B. Besides, as seen in Fig. 12 that escalation in φ_A and φ_B tends to reduce the concentration distribution of (AlN–Al2O3+ viscoelastic fluid) and CMC-water based AlN nanofluid. This phenomenon emerged because a massive volume fraction of Al2O3 (φ_A) and AlN (φ_B) is embedded into the viscoelastic fluid which augments the random motion of nanoparticles. The collision rate of the nanoparticles increases throughout this random motion, lowering the nanoparticle concentration. Also, it is very appealing to perceive that the entropy production of the system is lesser in favor of a larger volume fraction (0.01 ≤ φ_A = φ_B ≤ 0.04) for Al2O3 and AlN nanoparticles as seen in Fig. 13. Furthermore, using the finer of φ_A and φ_B it is possible to control the entropy production in the system. However, from Fig. 14, the opposite tendency is observed in the Bejan number with the varying values of φ_A and φ_B. The same to the preceding clarification in favor of the Bejan number (Be), the graph of
the entropy production close to the wall as large as a consequence of the inertial effects impose to an augment in the fluid friction effect and thus, Bejan number (Be) becomes higher for both (AlN–Al₂O₃ viscoelastic fluid) viscoelastic hybrid nanofluid and (AlN+ viscoelastic fluid) viscoelastic nanofluid.

4.4. Effects of Brinkmann number

The disparity of Br against NG and Be is displayed in Fig. 15 and Fig. 16. A magnification in Brinkmann number (0.3 ≤ Br ≤ 3.4) improves the NG owing to viscous heating of the liquid while it shrinks the Be because heat irreversibility from viscous dissipation dominates over heat irreversibility.

4.5. Effects of radiation parameter

Fig. 17 and Fig. 18 display the impact of Rd on NG and Be. It is observed from the figures that both NG and Bejan number are amplified with intensification in Rd in the range of (0.1 ≤ Rd ≤ 1.8). Physically, the excessive temperature produced in the flow region due to high Rd causes high disorderedness in the system. Accordingly, an increment in the entropy production rate and Bejan number is observed.

4.6. Effects of magnetic field parameter

Fig. 19–Fig. 23 show the effect of the magnetic field parameter on the radial velocity, tangential velocity, temperature, entropy generation rate, and Bejan number. As the magnetic field parameter M is increased, the radial and tangential velocities decrease significantly, as shown in Fig. 19 and Fig. 20. In reality, the Lorentz force is significantly correlated with the magnetic field, with bigger magnetic fields having a stronger Lorentz force and smaller magnetic fields having a weaker Lorentz force. The Lorentz force swells the frictional force, which acts as a retarding force, and as a result, a reduction in the radial and tangential velocities is observed. However, an increase in the magnetic field parameter tends to increase
the temperature distribution, as seen in Fig. 21. Increasing the magnetic field parameter increases the Lorentz force, which results in Joule heating (or Lorentz heating) in the energy equation, which can be used as an extra heat source for the flow system. Manifestly, as seen in Fig. 22, the strengthening in M imposes a rise in the NG. It is rather well-matched with physical laws because the enrichment of Lorentz force due to augment in M results in more friction, which imposes an increase in entropy fabrication rate. However, as shown in Fig. 23, an increase in M generally results in a decrease in the Bejan number. Physically, the Bejan number is a non-dimensional number indicating the fraction of entropy production owed to heat transport to the whole entropy production. Classically, Bejan number presumes values flanked by 0 to 1. For Bejan number is very near to 1, the entropy production owing to heat relocation dominates and thus a diminution in (Be) is observed.

4.7. Confirmation

To validate our proposed numerical procedure, we perform a $-\Theta'(0)$ comparison with the works of [53, 54, 55], as shown in Table 3, and excellent agreement between current and previous result is observed under limiting conditions.

4.8. Skin friction coefficient, Nusselt and Sherwood numbers

Table 4 shows the numerical variation of $Gr, Re, M, A$, and $\phi_B$ on $f''(0)$. One can observe from Table 4 that strengthening in $f''(0)$ is observed in all the cases of viscoelastic $AlN$–$Al_2O_3$ hybrid nanofluid, viscoelastic $AlN$-nanofluid, and viscoelastic fluid for higher in $Gr$, $M$, and $\phi_B$. Also, this table demonstrates that skin friction generated due to the viscoelastic $AlN$–$Al_2O_3$ hybrid nanofluid is much stronger than that for regular viscoelastic $AlN$ - nanofluid and viscoelastic fluid. This is caused by the inclusion of both $AlN$ and $Al_2O_3$ nanoparticles in viscoelastic fluid. However, the reduction in $f''(0)$ is perceived with amplification in Re and $A$. Thus, skin friction can be controlled by increasing Re and $A$. From Table 5, it is found that for all the cases of viscoelastic $AlN$–$Al_2O_3$ hybrid nanofluid and viscoelastic $AlN$-nanofluid, the rate of heat relocation amplifies with finer values of Grashof number, and as a consequence, the temperature transportation declines, but, as predicted, only the viscoelastic
fluid remains unaffected by the disparity of Gr, Re, M, and Ec. Also, from this table, it is seen that the rate of heat relocation $-\Theta'(0)$ declines with an improvement in Re, Ec, and $\phi_B$. Furthermore, the heat transportation rate of AlN–Al$_2$O$_3$ hybrid nanofluid is made stronger than viscoelastic AlN-nanofluid and viscoelastic fluid. This outcome demonstrates that the heat transportation rate is finer in the hybrid nanofluid when compared to the regular nanofluid and fluid. That’s why utilizing hybrid nanofluid as a replacement for the nanofluid is more desirable in thermal administration systems. As explained in Table 6, intensification values of Gr, Le, and $\phi_B$ augment the mass transportation rate whereas, reduction in mass transportation rate is observed for the case of intensification in Re and M.
Table 4. Variation of \( \frac{1}{1-\varphi_A} + \frac{1}{1-\varphi_B} f''(0) \) against \( \text{Re}, \ M, \ Gr, \ Rd, \ A, \ \varphi_B \).

| Gr | Re | M | A | \( \varphi_B \) | Hybrid nanofluid \( \varphi_A = 0.01 \) | Nanofluid \( \varphi_A = 0 \) | Viscous fluid \( \varphi_A = \varphi_B = 0 \) |
|----|----|---|---|---------|----------------|----------------|----------------|
| 10 | 5  | 0.1| 0.1| 0.01    | 1.306755       | 1.268152       | 1.238550       |
| 12 | 1  | 3.57710 | 1.320041 | 1.289107 |
| 14 | 1  | 4.11648 | 1.371734 | 1.340337 |
| 16 | 1  | 4.61005 | 1.422524 | 1.391835 |
| 10 | 6  | 2.26856 | 1.187999 | 1.157516 |
| 7  | 1  | 1.66227 | 1.128302 | 1.097938 |
| 8  | 1  | 1.14286 | 1.106262 | 1.073610 |
| 5  | 0.2| 1.31042 | 1.272339 | 1.241307 |
| 0.3| 1  | 1.314956| 1.276070 | 1.248823 |
| 0.4| 1  | 1.320424| 1.282303 | 1.256006 |
| 0.1| 0.2| 0.527498| 0.509686 | 0.495577 |
| 0.3| 1  | 0.293639| 0.283690 | 0.275136 |
| 0.4| 1  | 0.189760| 0.182974 | 0.177485 |
| 0.1| 0.02| 1.338046| 1.300586 |         |
| 0.03| 1.364778| 1.332640 | -        |
| 0.04| 1.402471| 1.364689 | -        |

Table 5. Variation of \(- \left( \frac{1}{1-\varphi_A} + \frac{1}{1-\varphi_B} \right) \theta''(0) \) via \( \text{Gr}, \ \text{Re}, \ M, \ Ec \) and \( \varphi_B \).

| Gr | Re | M | Ec | \( \varphi_B \) | Hybrid nanofluid \( \varphi_A = 0.01 \) | Nanofluid \( \varphi_A = 0 \) | Viscous fluid \( \varphi_A = \varphi_B = 0 \) |
|----|----|---|----|---------|----------------|----------------|----------------|
| 10 | 5  | 0.1| 0.2| 0.01    | 0.256799       | 0.260389       | 0.263135       |
| 12 | 1  | 0.275789| 0.260785 | 0.263135 |
| 14 | 1  | 0.258458| 0.261166 | 0.263135 |
| 16 | 1  | 0.259262| 0.261529 | 0.263135 |
| 10 | 6  | 0.255295| 0.259755 | 0.263135 |
| 7  | 1  | 0.208536| 0.209583 | 0.210353 |
| 8  | 1  | 0.208340| 0.209498 | 0.210353 |
| 5  | 0.2| 0.257824| 0.260905 | 0.263135 |
| 0.3| 1  | 0.258994| 0.261450 | 0.263135 |
| 0.4| 1  | 0.260250| 0.262042 | 0.263135 |
| 0.1| 0.4| 0.254242| 0.259324 | 0.263135 |
| 0.6| 1  | 0.251787| 0.258261 | 0.263135 |
| 0.2| 0.02| 0.253427| 0.257360 |         |
| 0.03| 0.249983| 0.254049 | -        |
| 0.04| 0.246063| 0.250457 | -        |

Table 6. Variation of \(-\Phi'(0)\) via \( \text{Gr}, \ \text{Re}, \ M, \ Le \) and \( \varphi_B \).

| Gr | Re | M | Le | \( \varphi_B \) | Hybrid nanofluid \( \varphi_A = 0.01 \) | Nanofluid \( \varphi_A = 0 \) | Viscous fluid \( \varphi_A = \varphi_B = 0 \) |
|----|----|---|----|---------|----------------|----------------|----------------|
| 10 | 5  | 0.1| 3  | 0.01    | 0.632497       | 0.629424       | 0.620586       |
| 12 | 1  | 0.633666| 0.630639 | 0.627785 |
| 14 | 1  | 0.634911| 0.631844 | 0.629000 |
| 16 | 1  | 0.636032| 0.630262 | 0.630219 |
| 10 | 6  | 0.636052| 0.627536 | 0.624646 |
| 7  | 1  | 0.569683| 0.567016 | 0.564456 |
| 8  | 1  | 0.569385| 0.566732 | 0.564169 |
| 5  | 0.2| 0.632459| 0.629388 | 0.626546 |
| 0.3| 0.632421| 0.629312 | 0.626507 |         |
| 0.4| 0.632377| 0.629268 | 0.626465 |         |
| 0.1| 0.706728| 0.703403 | 0.700307 |
| 5  | 1  | 0.766803| 0.763435 | 0.760274 |
| 6  | 1  | 0.815980| 0.812674 | 0.809549 |
| 3  | 0.02| 0.635361| 0.632337 |         |
| 0.03| 0.638211| 0.635240 | -        |
| 0.04| 0.641139| 0.638150 | -        |
5. Conclusions

Entropy generation in the radiative magneto-hydrodynamic mixed convective flow of viscoelastic hybrid nanofluid across a spinning disk under the effect of thermal radiation, convective conditions, velocity, and concentration slips has been examined. The study considers a mixture of aluminum nitride and alumina nanoparticles embedded in carboxymethyl cellulose with a volume mass concentration of 0.0% to 0.4% for hybrid nanofluid. The obtained system of partial differential equations from the conservation of momentum, energy, and nanoparticle concentration were transfigured into ordinary differential equations and then solved by the Galerkin finite element approach in Matlab. The effects of various parameters on velocity, temperature, concentration, entropy production, and the Bejan number have been investigated. The following are the key findings:

1. The flow becomes more vigorous when the Grashof number is raised.
2. Centrifugal and tangential velocities can be well-ordered by snowballing in a viscoelastic constraint.
3. A greater Grashof number can control the temperature of the fluid.
4. The temperature transport of hybrid nanofluid is higher than that of mono nanofluid and base fluid.
5. A reduction in concentration dissemination is witnessed for value-added $\phi_A$ and $\phi_B$.
6. With strengthening in Br, Rd, Gr, and M, the entropy production rate is improved.
7. The entropy production rate in the system can be controlled by adding $\phi_A$ and $\phi_B$.
8. Bejan number (Be) can be improved via higher alumina and AlN volume fractions.

Declarations

Author contribution statement

Wubshet Ibrahim, Dachasa Gamachu: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

Additional information

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