Computing Class Groups of Function Fields Using Stark Units

Ming-Deh Huang and Anand Kumar Narayanan

Abstract. Let $k$ be a fixed finite geometric extension of the rational function field $\mathbb{F}_q(t)$. Let $F/k$ be a finite abelian extension such that there is an $\mathbb{F}_q$-rational place $\infty$ in $k$ which splits in $F/k$ and let $O_F$ denote the integral closure in $F$ of the ring of functions in $k$ that are regular outside $\infty$. We describe algorithms for computing the divisor class number and in certain cases for computing the structure of the divisor class group and discrete logarithms between Galois conjugate divisors in the divisor class group of $F$. The algorithms are efficient when $F$ is a narrow ray class field or a small index subextension of a narrow ray class field.

We prove that for all prime $\ell$ not dividing $q(q-1)[F:k]$, the structure of the $\ell$-part of the ideal class group $\text{Pic}(O_F)$ of $O_F$ is determined by Kolyvagin derivative classes that are constructed out of Euler systems associated with Stark units. This leads to an algorithm to compute the structure of the $\ell$-primary part of the divisor class group of a narrow ray class field for all primes $\ell$ not dividing $q(q-1)[F:k]$.

1. Introduction

Fix $k/\mathbb{F}_q(t)$, a finite geometric extension of the rational function field $\mathbb{F}_q(t)$. Let $F/k$ be a finite abelian extension of conductor $m$ such that $F$ has an unramified $\mathbb{F}_q$ rational place $\mathfrak{B}$. Let $\infty$ be the place in $k$ lying below $\mathfrak{B}$. Since $\infty$ splits completely in $F/k$, we call $F$ totally real with respect to $\infty$. Denote by $O_k$ the ring of functions in $k$ regular outside $\infty$. Let $G := \text{Gal}(F/k)$ denote the Galois group of the extension, $\text{deg}(f)$ the degree of an ideal $f \subset O_k$ as the degree of the divisor $f$ and $O_F$ the integral closure of $O_k$ in $F$. Let $H_m$ denote the narrow ray class field of modulus $m \subset O_K$.

Let $D_F$ denote the group of divisors of $F$, which is the free abelian group on the places of $F$. Denote by $D_F^0$, the subgroup of $D_F$ of degree zero divisors and by $P_F$ the subgroup of principal divisors which consists of divisors of functions in $F$. The quotient $\mathcal{C}l_F^0 = D_F^0/P_F$ is the (degree zero) divisor class group of $F$.

The divisor class group $\mathcal{C}l_F^0$ is a finite abelian group and fits in the following exact
All stated algorithmic results that take $F$ as an input assume that an efficient representation of the finite field $F$ is given. An efficient representation is one that allows field addition and multiplication in time polylogarithmic in the field size (see $[16]$ for a formal definition).

The algorithm in theorem 1.1 is efficient when $F$ is of small index in $H_m$. For instance, when $k = F_q(t)$ and $F$ is $H_m$, the running time is polylogarithmic in $h(H_m)$. This is because the genus $g(H_m)$ of $H_m$ grows roughly as $|\mathcal{O}_k/m|^{\times} \log(|\mathcal{O}_k/m|^{\times})$ (see $[9]$ and $[25]$ for an exact expression), $|\mathcal{O}_k/m|^{\times}$ is about $q^{\deg m}$ and the divisor class number $h(H_m)$ is approximately $q^{O(h_m)}$.

Due to Lauder and Wan $[15]$ [Theorem 37], there is an algorithm to compute the divisor class number of an arbitrary finite extension of $F_q(t)$ in time polylogarithmic in the divisor class number if the characteristic of $k$ is fixed. It would be interesting to compare the performance the algorithm of Lauder and Wan when restricted to the family of narrow ray class fields with the algorithm in theorem 1.1.

In $[29]$, Yin defined an ideal $I_F$ in $\mathbb{Z}[G]$ that annihilates $\mathcal{O}_F^0$. The ideal is comprised of Stickelberger elements that arise in the proofs by Deligne $[27]$ and Hayes $[7]$ of the function field analogue of the Brumer-Stark conjecture, and are intimately related to Stark units. When $F$ is either $K_m$, the cyclotomic extension of conductor $m$ or $H_m \subset K_m$, the narrow ray class field of modulus $m$, Yin $[29]$ derived an index
theorem demonstrating that $[\mathbb{Z}[G] : I_F]$ is up to a power of $q - 1$, the degree zero divisor class number of $F$. Ahn, Bae and Jung [2] extended the index theorem to all sub extensions of $K_m$. Since a totally real extension of conductor $m$ is contained in $H_m$, the index theorem applies to $F$ that we consider.

The ideal $I_F$ and the corresponding index theorem are analogues of the Stickelberger ideal in cyclotomic extensions of $\mathbb{Q}$ and the Iwasawa-Sinnott [26] index formula. It is remarkable that the index of $I_F$ relates to the divisor class number in its entirety. In contrast, in cyclotomic extensions over $\mathbb{Q}$, the index of the Stickelberger ideal relates only to the relative part of the class number. As Yin [29] § 1] suggests, it is perhaps appropriate to regard $[\mathbb{Z}[G] : I_F]$ as being composed of both the relative part which is analogous to the Iwasawa-Sinnott index and the real part which corresponds to the Kummer-Sinnott [26] unit index formula. The construction of a large ideal such as $I_F$ that annihilates the divisor class group is possible in part due to the partial zeta functions over function fields being $\mathbb{Q}$ valued when evaluated at 0. In contrast, in cyclotomic extensions over $\mathbb{Q}$, the evaluation of partial zeta functions of the real part of cyclotomic extensions at 0 could be irrational and the Stickelberger ideal corresponding to the real part is the zero ideal.

Based on the construction in [29, 2], the following theorem is proven in § 3.4.

**Theorem 1.2.** There is a deterministic algorithm that given the rational function field $F_q(t)$, a totally real finite abelian extension $F/F_q(t)$ presented as an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$ and a generator for the conductor $m$ of $F$, computes a generating set of $I_F$ in time polynomial in $q^{\deg(m)}$ and the size of $X_F$.

Let $r_F$ be the largest factor of $h(F)$ that is relatively prime to $h(k)|F : k|$. If the $r_F$-torsion $\text{Cl}_F^0[r_F]$ of $\text{Cl}_F^0$ is $\mathbb{Z}[G]$ cyclic, then the structure of $\text{Cl}_F^0[r_F]$ is determined by the Stickelberger ideal $I_F$. This leads to an algorithm to compute the structure of $\text{Cl}_F^0$ resulting in the following theorem proven in § 3.5.

**Theorem 1.3.** There is a deterministic algorithm that given the rational function field $F_q(t)$, a totally real finite abelian extension $F/F_q(t)$ presented as an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$ and a generator for the conductor $m$ of $F$, if $\text{Cl}_F^0[r_F]$ is $\mathbb{Z}[G]$ cyclic, computes the structure of $\text{Cl}_F^0$ in time polynomial in $q^{\deg(m)}$ and the size of $X_F$. If in addition a $\mathbb{Z}[G]$ generator of $\text{Cl}_F^0[r_F]$ is given, the invariant factor decomposition of $\text{Cl}_F^0$ can be computed in time polynomial in $q^{\deg(m)}$ and the size of $X_F$.

Further, given a $\mathbb{Z}[G]$ generator $\gamma$ of $\text{Cl}_F^0[r_F]$, we can project an element in $\mathbb{Z}[G] \gamma$ into the invariant decomposition of $\text{Cl}_F^0$ and hence efficiently compute discrete logarithms between $\gamma$ and its Galois conjugates. See § 3.5 for details.

In section 6.2, we describe an algorithm to compute the structure of the $\ell$ primary part of the regulator $R_F$ for a prime $\ell$ not dividing $q(q - 1)|F : k|$ and prove the below theorem and the corollary that follows.

**Theorem 1.4.** There is a deterministic algorithm that given the rational function field $F_q(t)$, a totally real finite abelian extension $F/F_q(t)$ presented as an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$ and a prime
irreducible $Z$-function field. In a recent work \[18\] furthered the theory of Euler systems and using it proved the main conjecture of [6].

$\ell \nmid q(q - 1)[F : k]$, computes the structure of the $\ell$-primary part of the regulator $R_F$ in time polynomial in $\log(q)$ and the size of $X_F$.

**Corollary 1.5.** There is a deterministic algorithm that given the rational function field $\mathbb{F}_q(t)$, a totally real finite abelian extension $F/\mathbb{F}_q(t)$ presented as an irreducible polynomial $X_F(t) \in \mathbb{F}_q(t)[y]$ such that $F = \mathbb{F}_q(t)[y]/(X_F(t,y))$ and a prime $\ell \nmid q(q - 1)[F : k]$ computes the cardinality of the $\ell$-primary part of the ideal class group $\text{Pic}(\mathcal{O}_F)$ in time polynomial in $\log(q)$ and the size of $X_F$.

By factoring $h(F)$, we can obtain a list of primes that divide all primes dividing $h(\mathcal{O}_F)$. If $(h(F), q[F : k]) = 1$, by determining the cardinality of the $\ell$-primary part of $\text{Pic}(\mathcal{O}_F)$ for every prime $\ell$ in the list, we can determine $h(\mathcal{O}_F)$. Since $h(F) = \mathcal{O}(q^g)$, where $g$ is the genus of $F$, factoring $h(F)$ using the Number Field Sieve takes $\mathcal{O}(q^{g^{1/3}})$ time under heuristics [17].

In particular if $\text{Pic}(\mathcal{O}_F)$ is trivial then $\mathcal{O}_F^0$ is isomorphic to $R_F$, which is $\mathbb{Z}[G]$-cyclic with any prime over $\infty$ as a generator and we can apply Theorem [1,3].

If $\text{Pic}(\mathcal{O}_F)$ is non-trivial, then we deploy full machinery of Euler systems of Stark units to determine the $\ell$-part of $\text{Pic}(\mathcal{O}_F)$ for $\ell$ not dividing $q[F : k]$. We briefly describe this approach below.

The intersection of the multiplicative group $S_F$ generated by the Stark units with the unit group $\mathcal{O}_F^*$ is of finite index in $\mathcal{O}_F^\times$ and this index equals $|\text{Pic}(\mathcal{O}_F)|$ by the Kummer-Sinnott unit index formula. Gras conjecture [6] over function fields is a refinement of the Kummer-Sinnott index formula and it relates the cardinality of the $\ell$-primary part of $\text{Pic}(\mathcal{O}_F)$ to the (finite) index $[\mathcal{O}_F^\times : \mathcal{O}_F^\times \cap S_F]$, where $S_F$ is the group of Stark units. If $\ell$ is a prime not dividing $q[F : k]$, $\chi$ a non-trivial irreducible $\mathbb{Z}_\ell$ character of $G$ and $e(\chi) \in \mathbb{Z}[G]$ the corresponding idempotent, then Gras conjecture claims

$$|e(\chi)(\mathbb{Z}_\ell \otimes \mathbb{Z} \mathcal{O}_F^\times \cap S_F))| = |e(\chi)(\mathbb{Z}_\ell \otimes \mathbb{Z} \text{Pic}(\mathcal{O}_F))|.$$  

In its originally formulated context of cyclotomic extensions of $\mathbb{Q}$, Gras conjecture is known to be true as a consequence of the proof of the main conjecture of Iwasawa theory by Mazur and Wiles [18]. Kolyvagin [14] gave a more elementary proof as an application of the method of Euler systems that he had just developed. Rubin furthered the theory of Euler systems and using it proved the main conjecture of Iwasawa theory over imaginary quadratic extensions [23,24].

Feng and Xu [4] introduced the method of Euler systems to the function field setting and proved the Gras conjecture when $F = H_m$ and $k$ is the rational function field. In a recent work [20, Thm 1.1], Oukhaba and Viguie extended the proof to all totally real abelian extensions $F$ except when $\ell \mid [H_m : F]$ and $F$ contains the $\ell$th roots of unity.

Given $N$ that is a power of a prime $\ell$ not dividing $q[F : k]$, we consider Euler systems of modulus $N$ that starts with an element of $S_F$. From an Euler system $\Psi$, we have a derived system, called a Kolyvagin system, consisting of of functions $\kappa(a) \in F^\times$ indexed by $a \in B_N$, where $B_N$ is the set of all finite free product of a certain infinite set of primes in $\mathcal{O}_k$. The places that appear in the divisor $[\kappa(a)]$ admit
Gras’ conjecture analog of the Cohen-Lenstra heuristics to be large and the ideal class group is expected to be small. The function field function fields with probability of an abelian $\ell$-group $\kappa$ from $\mathbb{Z}$ over the rational function field $k$ of an irreducible $\ell$-th power but not an $\ell^{i+1}$-th power in $F^\times / F^\times N(\chi)$, we define $\ell^t$ to be the $\chi$-index of $\alpha$. We consider a Kolyvagin system of modulus $N$ divisible by $\ell t^2$ starting from a unit $\kappa(t) \in E$ of $\chi$-index $t$, where $t^{\dim(\chi)}$ is the cardinality of the $\chi$-component of $\mathcal{O}_F^\times / (\mathcal{O}_F^\times \cap S_F)$. Write $\kappa(a) \xrightarrow{\chi} a(p)$, if $uT\mathfrak{B} = [e(\chi)|\kappa(ap)|]_{\varphi}$ mod $N$ where $T$ is the $\chi$-index of $\kappa(a)$, $\mathfrak{B}$ is a prime over $p$ in $F$ and $u \in ((\mathbb{Z}/N\mathbb{Z}[G])/(\chi))^\times$. A $\chi$-path starting from $\kappa(t)$:

$$ \kappa(t) \xrightarrow{\chi} \kappa(p_1) \xrightarrow{\chi} \kappa(p_1p_2) \xrightarrow{\chi} \ldots \xrightarrow{\chi} \kappa(p_1p_2\ldots p_n) $$

is complete if the $\chi$-index of the last node $\kappa(p_1p_2\ldots p_n)$ is 1. The following theorem (proven in §4.3) says that the $\chi$-component of $\text{Pic}(\mathcal{O}_F)$ is completely determined by a complete $\chi$-path.

**Theorem 1.6.** Let $\kappa(t) \xrightarrow{\chi} \kappa(p_1) \xrightarrow{\chi} \kappa(p_1p_2) \xrightarrow{\chi} \ldots \xrightarrow{\chi} \kappa(p_1p_2\ldots p_n)$ be a $\chi$-path from $\kappa(t)$. Let $\mathcal{C}_i$ be the subgroup of $\text{Pic}(\mathcal{O}_F)$ generated by all primes of $F$ above $p_1$, $p_2$, ..., $p_i$. Let $t_i$ be the $\chi$-index of $\kappa(p_1p_2\ldots p_i)$. Then $[\mathcal{C}_i(\chi) : \mathcal{C}_{i-1}(\chi)] = (t_{i-1}/t_i)^d$ for $i = 1, ..., n$, and $t_n = 1$ if and only if $\mathcal{C}_n(\chi) = \text{Pic}_\chi(\mathcal{O}_F)(\chi)$. Moreover any $\chi$-path from $\kappa(t)$ can be extended to a complete $\chi$-path.

The characterization of Theorem 1.6 leads to an algorithm that determines the structure of $\text{Pic}_\chi(\mathcal{O}_F)(\chi)$ as a $\mathbb{Z}_\ell[G]$-module, and the following theorem is proven in §4.3.

**Theorem 1.7.** Let $H_m$ denote the narrow ray class field $H_m$ of conductor $m$ over the rational function field $k = \mathbb{F}_q(t)$ and for a prime $\ell$ and a $\mathbb{Z}_\ell$ representation $\chi$, let $t(\chi)$ denote the exponent of the $\chi$ component of $\mathcal{O}_{H_m}^\times / (\mathcal{O}_{H_m}^\times \cap S_{H_m})$. There is a deterministic algorithm, that given a generator for an ideal $m$ in $\mathbb{F}_q[t]$ and a prime $\ell$ not dividing $q(q-1)|H_m : k|$, finds the structure of $\text{Pic}_\chi(\mathcal{O}_{H_m})(\chi)$ as a $\mathbb{Z}_\ell[\text{Gal}(H_m/k)]$ module in time polynomial in $q^{t(\chi)}$ and $[H_m : k]$ for every non trivial irreducible $\mathbb{Z}_\ell$ representation $\chi$ of $\text{Gal}(H_m/k)$.

Since $t(\chi)^{\dim(\chi)}$ is the cardinality of the $\chi$ component of $\mathcal{O}_{H_m}^\times / (\mathcal{O}_{H_m}^\times \cap S_{H_m})$, by Gras’ conjecture $t(\chi)^{\dim(\chi)} = [\text{Pic}_\chi(\mathcal{O}_{H_m})(\chi)]$. The exponential dependence on the size of $\text{Pic}_\chi(\mathcal{O}_{H_m})(\chi)$ would be less of a concern if the regulator is expected to be large and the ideal class group is expected to be small. The function field analog of the Cohen-Lenstra heuristics conjecture that an isomorphism class of an abelian $\ell$-group $H$ occurs as the $\ell$ primary part of the divisor class group of function fields with probability

$$ \frac{1}{\text{Aut}(H)} \prod_{i=1}^{\infty} (1 - \ell^{-i}). $$

In particular, it predicts that the $\ell$-primary part of the divisor class group is more likely to be cyclic than otherwise. If the point $\infty$ is chosen at random from the rational places that split completely in $F$, then it is likely that aforementioned
cyclic subgroup is contained in the subgroup generated by the places in \( F \) above \( \infty \). Thus the regulator is expected to be large and the ideal class group is expected to be small.

2. Stickelberger Elements and Stark Units

2.1. Cyclotomic Extensions. In this subsection, we build notation and recount properties of cyclotomic function fields over global function fields based on the theory of sign-normalized Drinfeld modules developed by Hayes. Refer to [7] for a detailed description and proofs of claims made here.

Let \( k_\infty \) be a completion of \( k \) at \( \infty \). Let \( \mathbb{F}(\infty) \) be the constant field of \( k_\infty \) and \( \Omega \) the completion of an algebraic closure of \( k_\infty \). Let \( \mathcal{V}_\infty \) be the extension of the normalized valuation of \( k_\infty \) at \( \infty \) to \( \Omega \). Fix a sign function \( sgn : k_\infty^\times \to \mathbb{F}(\infty)^\times \), a co-section of the inclusion morphism \( \mathbb{F}(\infty)^\times \hookrightarrow k_\infty^\times \) such that \( sgn(\zeta) = 1 \) for every \( \zeta \) in the group of 1 units \( U^\times_{k_\infty} \). Let \( \rho : O_k \to \Omega(\tau_q) \) be a sign-normalized rank 1 Drinfeld-module. Here, \( \Omega \) is the left twisted polynomial ring with \( \tau_q \) satisfying the relation \( \tau_q x = x^q \tau_q, \forall x \in \Omega \). The image of \( a \in O_k \) under \( \rho \) is denoted by \( \rho_a \).

Let \( H_\epsilon \) denote the maximal real unramified abelian extension of \( k \) and \( K_\epsilon \), the normalizing field with respect to \( sgn \) obtained by adjoining to \( k \) the coefficients of \( \rho_a \) for every \( a \in O_k \). The extension \( H_\epsilon/k \) is contained in \( K_\epsilon/k \) with \( [K_\epsilon : H_\epsilon] = q - 1 \) and the primes in \( H_\epsilon \) above \( \infty \) are totally ramified in \( K_\epsilon/H_\epsilon \).

For an integral ideal \( m \subset O_k \), define the \( m \)-torsion points \( \Lambda_m = \{ w \in \Omega | \rho_a(w) = 0, \forall a \in m \} \). As \( O_k \)-modules, \( \Lambda_m \) is cyclic and isomorphic to \( O_k/mO_k \). We fix a generator \( \lambda_m \in \Lambda_m \) of \( \Lambda_m \) as an \( O_k \)-module as described below.

Associated with every rank 1 \( O_k \)-submodule of \( \Omega(\tau_q) \) is a rank 1 Drinfeld-module \( \xi : \mathbb{F}(\infty)^\times \to O_k^\times \). Let \( \xi(m) \) denote the invariant determined up to an \( \mathbb{F}_{q^\infty}^\times \) multiple by the property that the \( O_k \)-submodule \( \xi(m) m \) corresponds to a sign-normalized Drinfeld-module. Define \( c_m(z) := z \prod_{0 \neq \gamma \in m} \frac{1}{1 - \gamma} \) to be the exponential function associated with the \( O_k \)-submodule \( m \). Then \( \xi(m)c_m(1) \) is determined by \( m \) and \( sgn \) up to an \( \mathbb{F}_{q^\infty}^\times \) multiple and generates \( \Lambda_m \) as an \( O_k \)-module. Set \( \lambda_m := \xi(m)c_m(1) \).

The cyclotomic function field \( K_m \) is obtained by adjoining \( \Lambda_m \) to \( K_\epsilon \). Since \( \lambda_m \) generates \( \Lambda_m \) as an \( O_k \)-module, \( K_m = K_\epsilon(\lambda_m) \). The extension \( K_m/k \) is abelian. For the class of \( a \in O_k \) in \( (O_k/m)^\times \), there is a unique \( \sigma_a \in Gal(K_m/K_\epsilon) \) such that \( \sigma_a(\lambda_m) = \rho_a(\lambda_m) \) and thus \( Gal(K_m/K_\epsilon) \cong (O_k/m)^\times \). The maximal real subfield of \( K_m \) denoted by \( H_m \) is the ray class field modulo \( m \) and is independent of \( sgn \). Further, \( H_m = H_\epsilon(\lambda_m^{-1}) \) and \( [K_m : H_m] = q - 1 \).

From now on, let \( F/k \) be a finite abelian extension of conductor \( m \) and Galois group \( G := Gal(F/k) \). Let \( \epsilon = O_k \) denote the unit ideal. For a non-zero ideal \( \mathfrak{f} \subseteq O_k \), define \( F_{\mathfrak{f}} := K_{\mathfrak{f}} \cap F, F_{\mathfrak{f}}^+ := H_{\mathfrak{f}} \cap F \) and for \( \mathfrak{f} \neq \epsilon, \lambda_{\mathfrak{f}, F} := N_{K_{\mathfrak{f}}/F}(\lambda_\mathfrak{f}) \).

For a finite Galois extension \( L/k \), let \( O_L \) denote the ring of integers of \( L \). Let \( \mathcal{O}_L^0 \) and \( \text{Pic}(O_L) \) refer to the degree zero divisor class group of \( L \) and the ideal class group of \( O_L \) respectively. For \( f \in L \), let \( [f]_L \) denote its divisor. If the field in
question is clear from the context, we will drop the subscript $L$. For a sub extension $\bar{L}/k \subseteq L/k$ and an integral ideal $a \subseteq \mathcal{O}_{\bar{L}}$, let $(a, L/\bar{L}) \in Gal(L/\bar{L})$ denote the Artin symbol and $N_{L/\bar{L}}$ the norm map. The cardinality of the residue class ring $\mathcal{O}_{\bar{L}}/a$ is denoted by $N(a)$. For an integer $b$, let $\mu_b$ indicate the group of $b^{th}$-roots of unity.

2.2. Elliptic and Stark Units. Let $B_F$ be the $\mathbb{Z}[Gal(F_r/k)]$-submodule of $F^\times_r$ generated by $F^\times_q$ and $\{\lambda_{F}\}_s$, where $f$ ranges over non zero integral ideals of $\mathcal{O}_k$. The group of elliptic units $E_F$, which is of finite index in $\mathcal{O}_{F_r}^\times$, is $E_F := B_F \cap \mathcal{O}_{F_r}^\times$.

We next define the group of Stark units, whose intersection with $\mathcal{O}_k^\times$ be the Artin $L$-function attached to $\vartheta$. The product is over all places in $k$ excluding $P_F$, the set of places that ramify in $F/k$.

Extend $\vartheta$ linearly to $\mathbb{C}[G]$. The Stickelberger element $\Theta_F$ is the unique element in $\mathbb{C}[G]$ such that for all non trivial irreducible complex character $\vartheta$ of $G$,

$$\vartheta(\Theta_F) = (q - 1)L_{F/k}(0, \tilde{\vartheta}),$$

where $\tilde{\vartheta}$ is the complex conjugate of $\vartheta$. From the proof of the Brumer-Stark conjecture over function fields [27] Chapter V[7], $\Theta_F \in \mathbb{Z}[G]$ and $\Theta_F$ annihilates $\mathcal{O}_F^0$. Thus, for a divisor $D$ of degree 0 in $F$, there exists $\alpha_D \in F$ uniquely determined up to a root of unity such that $\Theta_F(D) = [\alpha_D]_F$. The following stronger claim is proven in [27][7][Theorem 1.1].

Let $\mathfrak{R}$ be a prime divisor in $F$. There exists $\alpha_{\mathfrak{R}} \in F$ unique up to a root of unity such that

1. $\Theta_F \in \mathbb{Z}[G]$
2. If $P_F$ is of cardinality greater than 1, then $\Theta_F(\mathfrak{R}) = [\alpha_{\mathfrak{R}}]_F$.
   If $P_F = \{b\}$ for a place $b$ in $k$, then $\Theta_F(\mathfrak{R}) + \mathcal{B} = [\alpha_{\mathfrak{R}}]_F$ where $\mathcal{B}$ is the sum of places in $F$ above $b$.
3. $F(\alpha_{2\mathfrak{R}}^{1/(q-1)})/k$ is abelian.

Let $J_F \subset \mathbb{Z}[G]$ denote the annihilator of the roots of unity in $F$. Since $F(\alpha_{2\mathfrak{R}}^{1/(q-1)})/k$ is abelian, from the characterization of $J_F$ in [7][Lemma 2.5], it follows that for an $\eta \in J_F$, there exists $\lambda(\mathfrak{R}, \eta) \in F$ unique up to a root of unity such that $\lambda(\mathfrak{R}, \eta)^{q-1} = \alpha_{\mathfrak{R}}^{q-1}$.

The subgroup $S_F$ of $F^\times$ generated by $F^\times_q$ and $\lambda(\mathfrak{R}, \eta)$ as $\mathfrak{R}$ ranges over prime divisors in $F$ dividing $\infty$ and $\eta$ ranges over $J_F$ is defined as the group of Stark units. The Stark units are thus supported on places above $\infty$ and $\mathfrak{m}$, and are thus $P_F$-units.

Hayes [7] gave an explicit description of the $\alpha_{\mathfrak{R}}$ in terms of the $\mathfrak{m}$-torsion points $\Lambda_{\varphi}(\mathfrak{m})$. In particular, $\lambda_{\mathfrak{m}}^{q-1}$ is a Stark unit in $H_{\mathfrak{m}}$ [7][4.6] and $S_F$ is generated by $\mu(F)$ and $N_{H_{\mathfrak{f}}/F_{\varphi}}(\lambda_{\mathfrak{f}}^{N_{\varphi}(\mathfrak{f} \cap \mathfrak{m}, k)})$ [20][3] where $\mathfrak{f} \subset \mathcal{O}_k$ ranges over non zero ideals and $\mathfrak{g} \subset \mathcal{O}_k$ ranges over non trivial ideals coprime to $\mathfrak{f}$.  


3. Algorithms for Computing the Divisor Class Groups

In this section, we develop the algorithms for the proofs of Theorem 1.1 and Theorem 1.2. In this section, we assume that the point at infinity \( \infty \) in \( k \) chosen is of degree 1. The index theorems concerning the Stickelberger ideal are known to hold only under this assumption. The rest of the algorithms, in particular those for determining the \( \chi \) part of the ideal class number and ideal class group do not require this assumption.

Further, for the following subsection wherein the computation of \( \chi \) is addressed, we restrict ourselves to the case where \( k \) is \( \mathbb{F}_q(t) \).

3.1. Computation of the narrow ray class field and Stark units. For this subsection, we set \( k \) to be the rational function field \( \mathbb{F}_q(t) \). Since \( \infty \) is of degree 1, without loss of generality we may assume that \( \mathcal{O}_k = \mathbb{F}_q[t] \). For otherwise, we can perform an appropriate change of variable from \( t \) to \( s \) to ensure \( k = \mathbb{F}_q(s) \) and \( \mathcal{O}_k = \mathbb{F}_q[s] \).

**Lemma 3.1.** There is a deterministic algorithm that given \( \mathbb{F}_q(t) \) and a generator \( f(t) \in \mathbb{F}_q[t] \) of an ideal \( f \subset \mathbb{F}_q[t] \), computes the minimal polynomial of \( \lambda_f \) over \( \mathbb{F}_q(t) \) in time polynomial in \( q^{\deg(f)} \).

**Proof.** Let \( f = q_1^{c_1}q_2^{c_2}\ldots q_n^{c_n} \) be the factorization into powers of prime ideals. Using Berlekamp’s algorithm [12], we can obtain such a factorization deterministically in time polynomial in \( \deg(f) \) and \( q \). From the factorization, obtain for each \( i \in \{1, 2, \ldots, n\} \), a monic irreducible \( q_i(t) \in \mathbb{F}_q[t] \) such that \( q_i^{c_i} = (q_i^{c_i}(t)) \).

We first describe the computation of \( \lambda_{q_i^{c_i}} \) based on standard results found in [9][22][Chap 12]. For \( k = \mathbb{F}_q(t) \) and \( \mathcal{O}_k = \mathbb{F}_q[t] \), \( \rho \) is the ring homomorphism that maps \( t \) to \( \rho_t = t + \tau_q \) and thus \( \rho_t(y) = t + y^q \). Further, the minimal polynomial of \( \lambda_{q_i^{c_i}} \) over \( \mathbb{F}_q(t) \) is

\[
\rho_{q_i^{c_i}}(t)(y)/\rho_{q_i^{c_i-1}}(t)(y).
\]

Since \( \rho_{q_i^{c_i}}(t) \) and \( \rho_{q_i^{c_i-1}}(t) \) have degrees \( c_i \deg(q_i) \) and \( (c_i - 1) \deg(q) \) in \( \tau_q \) respectively, \( \rho_{q_i^{c_i}}(t)(y) \) is of degree \( q^{c_i \deg(q_i)} \) and \( \rho_{q_i^{c_i-1}}(t)(y) \) is of degree \( q^{(c_i - 1) \deg(q_i)} \). Thus in time polynomial in \( q^{c_i \deg(q_i)} \) we can compute the minimal polynomial of \( \lambda_{q_i^{c_i}} \) over \( \mathbb{F}_q(t) \).

As an induction hypothesis, assume that \( \lambda_a \) and \( \lambda_b \) have been computed for all non trivial and relatively prime ideals \( a \) and \( b \) such that \( \mathfrak{f} = ab \).

Let \( a(t) \) and \( b(t) \) respectively generate \( a \) and \( b \). Using the extended Euclidean algorithm over \( \mathbb{F}_q[t] \), compute \( c(t), d(t) \in \mathbb{F}_q[t] \) such that \( 1 = c(t)a(t) + d(t)b(t) \). Since \( \rho \) is a ring homomorphism, \( \rho_1 = \rho_c(t)\rho_a(t) + \rho_d(t)\rho_b(t) \) and its application on an \( \mathbb{F}_q[t] \) module generator \( \lambda \) of \( \Lambda_f \) yields

\[
\lambda = \rho_c(t)(\rho_a(t)(\lambda)) + \rho_d(t)(\rho_b(t)(\lambda)).
\]

Since the \( \mathbb{F}_q[t] \) submodules of \( \Lambda_f \) generated by \( \rho_a(t)(\lambda) \) and \( \rho_b(t)(\lambda) \) are respectively \( \Lambda_b \) and \( \Lambda_a \), it follows that there exist \( \mathbb{F}_q[t] \) module generators \( \hat{\lambda}_b \) and \( \hat{\lambda}_a \) of \( \Lambda_b \) and \( \Lambda_a \) respectively such that \( \rho_c(t)(\hat{\lambda}_b) + \rho_d(t)(\hat{\lambda}_a) \) generates \( \Lambda_f \) as an \( \mathbb{F}_q[t] \) module.
Given an element in $\Lambda_f$, since we know the factorization of $f$, we can efficiently test if it generates $\Lambda_f$ by testing if a proper factor of $f$ annihilates it. Since there are $q^{\deg f}$ choices for $(\lambda_a, \lambda_b)$ we may try them all to find a $F_q[t]$ module generator $\lambda_f$ of $\Lambda_f$.

As a corollary, given a generator of an ideal $f \in \mathcal{O}_k$, we can construct $K_f$ as $F_q(t)(\lambda_f)$ and $H_f$ as $F_q(t)(-\lambda_f^{-1})$ in time polynomial in $q^{\deg f}$.

For a totally real finite abelian extension $F$ presented as an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$ along with a generator for the conductor $m$ of $F$, we can explicitly compute the inclusion $F \hookrightarrow H_m$ as follows. Factor $X_F(y)$ in $H_m[y]$ and from the resulting splitting express a root of $X_F(y)$ as a polynomial in $-\lambda_m^{-1}$ with $F_q(t)$ coefficients. The factorization takes time polynomial in $q$ and $[H_m : k]$ and the size of $X_F$ [21]. Thus the total running time for computing the inclusion is bounded by a polynomial in $q^{\deg(m)}$. As a consequence for an ideal $f$ dividing $m$, by considering $H_f$ and $F$ as subfields of $H_m$, we can construct $F_f = H_f \cap F$ in time polynomial in $q^{\deg(m)}$ and the size of $X_F$.

**Lemma 3.2.** There is a deterministic algorithm that given $F_q(t)$, a totally real finite abelian extension $F/F_q(t)$ presented as an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$ and a generator of the conductor $m$ of $F$ finds $\lambda_{f,F}$ for all $f$ dividing $m$ in time polynomial in $q^{\deg(m)}$ and the size of $X_F$.

**Proof.** We first obtain the factorization of $m$ into prime power ideals in $F_q[t]$ using Berlekamp’s deterministic polynomial factorization algorithm over finite fields [3].

For each $f$ dividing $m$, let $X_{f,F}(y) \in F_f[y]$ denote the minimal polynomial of $-\lambda_f^{-1}$ over $F_f$ and let $X_f(y) \in k[y]$ denote the minimal polynomial of $-\lambda_f^{-1}$ over $k$.

For each $f$ dividing $m$, factor $X_f(y)$ over $F_f$ thereby obtaining the factorization

$$X_f(y) = \prod_{\theta \in \text{Gal}(F_f/F_q(t))} (X_{f,F}(y))^\theta$$

where $(X_{f,F}(y))^\theta$ denotes $X_{f,F}$ with its coefficients acted on by $\theta$. We can read off $\lambda_{f,F} = \text{N}_{H_f/F_f}(-\lambda_f^{-1})$ as $X_{f,F}(0)$ up to a $\text{Gal}(F_f/F_q(t))$ conjugate.

**3.2. Computation of the Conductor.** In this subsection, we sketch how to compute the conductor of $F$ given an irreducible polynomial $X_F(y) \in F_q(t)[y]$ such that $F = F_q(t)[y]/(X_F(y))$.

First we compute the discriminant $\mathfrak{d}(X_F) \subseteq F_q[t]$ of the minimal polynomial $X_F$ and then find the set of all prime ideals in $F_q[t]$ dividing $\mathfrak{d}(X_F)$. The running time is polynomial in $q$ and the size of $X_F$. Since $m$ divides the discriminant of $F$ which in turn divides $\mathfrak{d}(X_F)$, the set of prime ideals dividing $\mathfrak{d}(X_F)$ contains the set of prime ideals dividing $m$. To compute $m$, for each prime $q$ dividing $\mathfrak{d}(X_F)$, we have to determine the highest non negative integer $e$ such that $q^e$ divides $m$.

Fix a prime ideal $q$ dividing $\mathfrak{d}(X_F)$ and let $e$ denote the highest non negative integer such that $q^e$ divides $m$. 
Since \( q^e \) and \( m/q^e \) are relatively prime, \( H_m \) is the composite \( H_{m/q^e} H_q^e \).

Let \( F_q \) denote the localization of \( F \) at a prime above \( q \) and let \( k_q \) denote the localization of \( k \) at \( q \). For a positive integer \( i \), let \( H_{q,i} \) denote the localization of \( H_q^e \) at a prime above \( q \) and let \( k_q^i \) denote the unique unramified extension of \( k_q \) of degree \( i \).

Since \( F \subseteq H_m = H_{m/q^e} H_q^e \) and the localization of \( H_{m/q^e} \) at a prime above \( q \) is unramified and of degree at most \( [H_{m/q^e} : k] \), there exists a positive integer \( i \) such that \( F_q \subseteq k_q^i H_{q,i} \). Denote by \( n \) the smallest positive integer such that \( F_q \subseteq k_q^i H_{q,i} \). Further, there exists a positive integer \( i \) such that \( F_q \subseteq k_q^i H_{q,j} \) if and only if \( j \geq e \).

This leads to the following algorithm to determine \( e \). Start with \( i = 0 \). If \( F_q \subseteq k_q^i H_{q,i} \) then output \( i \) and terminate. Otherwise increment \( i \).

The algorithm terminates with the output \( i_{out} \) that is at least \( e \) and at most \( \max(n, e) \). Given \( i_{out} \), we can determine \( e \) as the smallest non negative integer \( j \) such that \( F_q \subseteq k_q^{i_{out}} H_{q,j} \). Since \( \max(n, e) \) is bounded by \( q^{\deg(m)} \), we can determine \( e \) with the number of trials bounded by a polynomial in \( q^{\deg(m)} \).

3.3. Computation of the Divisor Class Number: Proof of Theorem 1.1

For an irreducible complex character \( \vartheta \) of \( G \) and \( s \in \mathbb{C} \) with \( \Re(s) > 1 \), let

\[
L_k(s, \vartheta) := \prod_{b \in \mathcal{P}_F(\vartheta)} (1 - \vartheta((F/k, b))(N(b))^{-s})^{-1}
\]

be the Artin \( L \)-function attached to \( \vartheta \). The product is over \( \mathcal{P}_F(\vartheta) \), the set of all places in \( k \) not dividing the conductor of the character \( \vartheta \). The analytic class number formula states that

\[
h(F) = h(k) \prod_{1 \neq \vartheta \in \hat{G}} L_k(0, \vartheta)
\]

where the product is over the group of irreducible complex characters excluding the principal character.

For \( F \) of conductor \( m \), we next describe how the \( L \)-function evaluations \( L_k(0, \vartheta) \) can be derived from the Stark unit \( \lambda_{m,F} \).

Associated with a Galois extension \( L/k \) is the logarithm map

\[
\log_L : L \to \mathbb{Q}[\text{Gal}(L/k)]
\]

\[
h \mapsto \sum_{\sigma \in \text{Gal}(L/k)} V_\infty(L)(h^\sigma)\sigma^{-1}.
\]

Here \( V_\infty(L) \) denotes the valuation at a place \( \infty(L) \) in \( L \) above \( \infty \). By [7, § 6], the Stickelberger element \( \Theta_{H_{l_1}} \) can be computed from the image of \( -\lambda_1 q^{-1} \) under the
polynomial in $q$ logarithm map as

$$\Theta_{H_1} = \frac{1}{q-1} \log_{H_1}(-\lambda_1^{q-1})$$

For finitely abelian extension $L/F$, by [2, Lem 2.3]

$$\log_F(N_{L/F}(z)) = \text{Res}_{L/F}(\log_L(z)), \forall z \in L.$$ 

Since the places that ramify in $H_m/k$ and the places that ramify in $F/k$ have the same support, $\Theta_{F} = \text{Res}_{H_m/F}(\Theta_{H_m})$ and

$$\Theta_{F} = \frac{1}{q-1} \log_{F}(\lambda_{m,F}).$$

For a non principal $\vartheta \in \hat{G}$ linearly extended to $\mathbb{Z}[G]$,

$$L_{F/k}(0, \vartheta) = \frac{1}{q-1} \vartheta(\Theta_{F}) = \vartheta(\log_{F}(\lambda_{m,F}))$$

$$\Rightarrow L_{F/k}(0, \vartheta) = \vartheta(\log_{F}(\lambda_{m,F})).$$

To compute the divisor class number $h(F)$ using the analytic class number formula, we require $L_k(0, \vartheta)$. But the $\vartheta$ component of the Stark unit $\lambda_{m,F}$ under the logarithm map only yields $L_{F/k}(0, \vartheta)$. However, $L_k(0, \vartheta)$ and $L_{F/k}(0, \vartheta)$ are off only by finitely many Euler factors.

In particular, the set of places that divide the conductor of $\vartheta$ is the set of places that are ramified in the extension $F/F_{\vartheta}$ where $F_{\vartheta}$ is the fixed field of $\vartheta$ in $F$. Thus $P_F \setminus P_F(\vartheta)$ consists of the places that ramify in $F/k$ but not in $F/F_{\vartheta}$. Hence for all non principal $\vartheta$,

$$L_k(0, \vartheta) = L_{F/k}(0, \vartheta) \prod_{b \in P_F \setminus P_F(\vartheta)} (1 - \vartheta(F/F_{\vartheta}, b))^{-1}$$

Hence the divisor class number can be derived from the Stark unit $\lambda_{m,F}$ as

$$h(F) = h(k) \prod_{1 \neq \vartheta \in \hat{G}} \left( \vartheta(\log_{F}(\lambda_{m,F})) \prod_{b \in P_F \setminus P_F(\vartheta)} (1 - \vartheta(F/F_{\vartheta}, b))^{-1} \right)$$

Given $\lambda_{m,F}$, the running time of the algorithm to compute $h(F)$ is bounded by a polynomial in $q$ and the size of $X_F$. From lemma [3.2] $\lambda_{m,F}$ can be computed in time polynomial in $q^{\log(m)}$ and theorem [1.1] follows.

**3.4. The Stickelberger Ideal: Proof of Theorem [1.2]** In [29], Yin defined an ideal $I_{F}$ in $\mathbb{Z}[G]$ that annihilates $C^0_{F}$. In addition, an index formula was derived that shows that $[\mathbb{Z}[G] : I_{K_m}]$ is up to a computable power of $(q - 1)$ the divisor class number $h(K_m)$. In [24], the index theorem was extended to hold for all abelian extensions $F/k$ that we consider.

Following [29], we define $Q_{F}$, a $G$ submodule of $\mathbb{Q}[G]$ such that $I_{F} = Q_{F} \cap \mathbb{Z}[G]$. The module $Q_{F}$ is comprised of a ramified part $Q_{F}^{r}$ and an unramified part $Q_{F}^{u}$. For an abelian extension $L/k$ and a subextension $\bar{L}/k$, let $\text{Res}_{L/\bar{L}}$ be the linear extension to $\mathbb{Q}[\text{Gal}(L/k)]$-modules of the restriction map from $\text{Gal}(L/k)$ to $\text{Gal}(\bar{L}/k)$. Likewise, let $\text{Cor}_{L/\bar{L}}$ be linear extension to $\mathbb{Q}[\text{Gal}(L/k)]$-modules of the corestriction map from $\text{Gal}(\bar{L}/k)$ to $\text{Gal}(L/k)$. 
For \( S \subseteq P_F \), let \( F_S \) be the maximal subextension of \( F \) where the primes outside \( S \) are unramified. The ramified part \( Q_F^a \) is defined to be the \( \mathbb{Z}[G] \)-module generated by \( \{ \text{Cor}_{F/F_S}(\Theta_{F_S}) \}_{S \subseteq P_F} \), the Stickelberger elements of the maximal subextensions \( F_S \) under the conorm map [29] \( \S \) \( 2 \). For \( F \subseteq H_m \), \( Q_F^a \) is generated by \( \{ \text{Cor}_{F/F^+_i \cap H_i}(\Theta_{H_i}) \}_{i | m} \), where \( f \) ranges over all non trivial prime ideals dividing \( m \) [2, \( \S \) \( 4 \)].

The unramified part \( Q_F^u \) is the \( \mathbb{Z}[G] \)-module generated by \( \{ \text{Cor}_{F/F_i}(\text{Res}_{H_i/F_i}(\Theta_{H_i})) \}_{i \in R} \) and \( \frac{1}{q-1} \sum_{\sigma \in G} a \) where \( a \) ranges over all prime ideals of \( \mathcal{O}_k \).

Let \( R \) be a finite set of ideals of \( \mathcal{O}_k \) such that \( R \) is mapped surjectively onto \( \text{Gal}(K_c/k) \) under the Artin map which takes \( a \in R \) to \( (a, K_c/k) \). We further require that for every \( a \in R \), \( (a, K_c/k) \) is not the identity. Let \( M = \{ f \in k^\times : \text{sgn}(f) = 1 \} \) be the multiplicative group of positive functions in \( k \). Let \( \mathcal{P} \) be the subgroup of \( K_c^\times \) generated by \( \{(\mathcal{O}_k)/\xi(\mathcal{g})\}_{b \in R} \) where \( \mathcal{g} \) ranges over all ideals of \( \mathcal{O}_k \). Let \( \mathcal{P} \) be the subgroup of \( \mathcal{P} \) generated by \( \{(\mathcal{O}_k)/\xi(h)\}_{b \in R} \). Hayes [10] Equation 1.9 proved that \( P \) is the direct product of \( \mathcal{P} \) and \( M \). Further, \( P \cap k^\times = M \) [10] Cor 2.5. Hence for every \( \eta \in P \), there exists integers \( b_a \) and a \( \gamma \in M \subset k^\times \) such that

\[
\eta = \gamma \prod_{a \in R} \left( \frac{\xi(\mathcal{O}_k)}{\xi(a)} \right)^{b_a}
\]

Further, \( \lambda_{\mathcal{g}, F} = N_{K_c/F_c}(\xi(\mathcal{O}_k)/\xi(\mathcal{g})) \) for all ideals \( \mathcal{g} \) [10] \( \S \) \( 3 \). For every ideal \( \mathcal{a} \) such that \( (\mathcal{a}, K_c/k) \in \text{Gal}(K_c/F_c) \), \( N_{K_c/F_c}(\xi(\mathcal{O}_k)/\xi(\mathcal{g})) \in M \subset k^\times \) [2, Lem 3.4].

For \( \gamma \in M \subset k^\times \), \( \log_{F_F}(\gamma) = c \sum_{\sigma \in G} \sigma \) for some integer \( c \). Hence for every ideal \( \mathcal{g} \), there exists integers \( c_a \) and \( c_0 \) such that

\[
\log_{F_F}(\lambda_{\mathcal{g}, F}) = c_0 + \sum_{a \in R} c_a \log_{F_F}(\lambda_{a, F})
\]

Hence \( Q_F \) is generated as a \( \mathbb{Z}[G] \)-module by \( \{ \text{Cor}_{F/F_i}(\text{Res}_{F/F_i}(\log_{F_F}(\lambda_{F_i, F}))) \}_{\mathcal{g} \in R} \), \( \frac{1}{q-1} \sum_{\sigma \in G} a \) and \( \{ \text{Cor}_{F/F^+_i \cap H_i}(\log_{F^+_i}(\lambda_{F^+_i, F})) \}_{i | m} \).

Let \( C := \{ \mathcal{g} \subset \mathcal{O}_k : \text{deg}(\mathcal{g}) \leq 2 \log_2((K_c : k)) \} \) be a set of prime ideals. From Chebotarev’s density theorem [19], \( C \) is mapped surjectively onto \( \text{Gal}(K_c/k) \) under the Artin map. Compute \( \lambda_{F, F} \) for every \( f \) that either divides \( m \) or is in \( C \) and write down a generating set \( M_F \) for \( Q_F \) as a \( \mathbb{Z}[G] \)-module. Let \( \Theta = \frac{1}{q-1} \sum_{\sigma \in G} \sigma \) and \( Q_F = I_F + \mathbb{Z}[G] \). Further, \( Q_F = I_F + \mathbb{Z}[G] \). For every \( s \in M_F \), define \( i_s := s - z_s \Theta \) where \( z_s \) is the unique integer such that \( 0 \leq z_s < q - 1 \) and \( i_s \in I_F \). Clearly \( Z_F := \{ i_s \}_{s \in M_F} \cup \{(q - 1) \Theta \} \) generates \( I_F \) as a \( \mathbb{Z}[G] \)-module.

For \( k = \mathbb{F}_q(t) \), given \( \lambda_{F, F} \) for \( f \) dividing \( m \), the running time of the algorithm to compute \( M_F \) is bounded by a polynomial in \( q \) and the size of \( X_F \). From lemma 3.2, \( \lambda_{F, F} \) for \( f \) dividing \( m \) can be computed in time polynomial in \( q^{\text{deg}(m)} \) and theorem 1.13 follows.

### 3.5. Stickelberger Ideal and Structure of the Divisor Class Group:

**Proof of Theorem 1.13** In this section, we show that if the divisor class group \( \mathcal{C}_F^0 \) is a cyclic \( \mathbb{Z}[G] \) submodule, then the structure of \( \mathcal{C}_F^0 \) is determined by the
Stickelberger ideal \( I_F \) up to. This leads to an algorithm to compute the structure of \( \text{Cl}_F^0 \) as an abelian group. Further, given a \( \mathbb{Z}[G] \) generator, we can compute the invariant decomposition of \( \text{Cl}_F^0 \).

The index \( [\mathbb{Z}[G] : I_F] \) of the Stickelberger ideal was computed in [2] as \( h(F)/h(k) \) up to a factor that is supported over primes dividing \( [F : k] \). A precise statement of their index theorem follows.

Let \( T_{q,F} \leq G \) be the inertia group of \( q \) and \( \sigma_{q,F} := (q, F/k)^{-1} \left( \sum_{\tau \in T_{q,F}} \tau \right)/(|I_{q,F}|) \).

For an ideal \( f \), define

\[
\alpha_{f,F} := \left( \sum_{\sigma \in \text{Gal}(F/F_f)} \sigma \right) \prod_{q \mid f} (1 - \sigma_{q,F}).
\]

Let \( e_F^+ = \sum_{\tau \in G} \tau \). Let \( V_F \) be the \( \mathbb{Q}[G] \) module generated by \( \{\alpha_{f,F}\}_{\ell|\mathfrak{m}} \) and let \( U_F = V_F + \left( \sum_{\tau \in \text{Gal}(F/F_f)} \tau \right) \mathbb{Z}[G] \). Ahn, Bae, Jung proved that [2] Thm 4.11

\[
[Z[G] : I_F] = \frac{h(F)[F : k][Z[G] : U_F]}{h(k)}
\]

Further, the set of prime divisors of \( (Z[G] : U_F) \) is contained in the set of prime divisors of \( [F : F_f] \). Hence

\[
[Z[G] : I_F] = \frac{h(F)B}{h(k)}
\]

for some \( B \) whose prime divisors are contained in the primes dividing \( [F : k] \).

Let \( r \) be the largest factor of \( [Z[G] : I_F] \) that is relatively prime to \( h(k)[F : k] \).

From the index theorem, it follows that the largest factor of \( h(F) \) that is relatively prime to \( h(K)[F : k] \) is \( r \). Let \( s_1 = [Z[G] : I_F]/r \) and \( s_2 = h(F)/r \). For an abelian group \( K \), let \( K[u] \) denote its \( n \)-torsion. Then,

\[
\text{Cl}_F^0 = \text{Cl}_F^0[r] \oplus \text{Cl}_F^0[s_2].
\]

Assume that \( \text{Cl}_F^0[r] \) is a cyclic \( Z[G] \) module. If \( \gamma \) generates \( \text{Cl}_F^0[r] \) as a \( Z[G] \) module and \( J \subseteq Z[G] \) denotes the annihilator of \( \gamma \), then

\[
Z[G]/J \cong \text{Cl}_F^0[r] = Z[G]\gamma.
\]

Since \( I_F \) annihilates \( \text{Cl}_F^0 \), \( I_F \subseteq J \) and there is a natural surjection

\[
Z[G]/I_F \twoheadrightarrow Z[G]/J
\]

which implies that there is a surjection

\[
\phi : (Z[G]/I_F)[r] \oplus (Z[G]/I_F)[s_1] \twoheadrightarrow \text{Cl}_F^0[r].
\]

Since \( r \) and \( s_1 \) are coprime, \((Z[G]/I_F)[s_1] \) is in the kernel of \( \phi \) and the restriction of \( \phi \) to \((Z[G]/I_F)[r] \) is surjective.

Since \(|(Z[G]/I_F)[r]| = |\text{Cl}_F^0[r]| = r \), the restriction of \( \phi \) to \((Z[G]/I_F)[r] \) is a \( \mathbb{Z}[G] \) module isomorphism.

Since \((Z[G]/I_F)[r] \) and \( \text{Cl}_F^0[r] \) are isomorphic as \( \mathbb{Z}[G] \) modules, they are isomorphic as groups.
Recall that $Z_F$ is a finite set that generates $I_F$ as a $\mathbb{Z}[G]$ module. Thus $Z_F := \{\sigma(z) | \sigma \in G, z \in Z_F\}$ generates $I_F$ as a $\mathbb{Z}$-module and we can determine the structure of $(\mathbb{Z}[G]/I_F)[r]$. Further, given $\gamma$, we can compute the invariant factor decomposition of $\mathcal{C}r_F[r]$ by a Smith normal form computation. Further, the Smith normal form computation yields a unimodular projection matrix that allows us to efficiently project a divisor class written as $\sum_{\sigma \in G} a_{\sigma} \gamma^\sigma$ for $a_{\sigma} \in \mathbb{Z}$ into the the invariant factor decomposition of $\mathcal{C}r_F[r]$.

We next turn our attention to $\mathcal{C}r^0_F[s_2]$.

Let $\ell$ be a prime dividing $s_2$ and let $(\mathcal{C}r^0_F)_\ell$ be the $\ell$-primary component of $\mathcal{C}r^0_F$. Let $\ell^{b_i}$ be the exponent of $(\mathcal{C}r^0_F)_\ell$.

If $A$ is a cyclic subgroup of $(\mathcal{C}r^0_F)_\ell$ of order $\ell^{b_i}$, then $(\mathcal{C}r^0_F)_\ell$ is the direct sum of $A$ and its complement. Thus, if we can find a cyclic subgroup $A$ of $(\mathcal{C}r^0_F)_\ell$ of order $\ell^{b_i}$, then the problem of computing the structure of $(\mathcal{C}r^0_F)_\ell$ is reduced to computing the structure of the complement of $A$ and we can proceed inductively.

There is a set $A_F$ of degree zero divisors of size polynomial in $[F : k]$ and $\log q$ whose divisor classes generate $\mathcal{C}r^0_F$. [Theorem 34]. Further, $A_F$ consists of divisors of pole degree bounded by $O(\log([F : k])$.

Thus,

$$(\mathcal{C}r^0_F)_\ell = \left\{ \frac{h(F)}{\langle (\mathcal{C}r^0_F)_\ell \rangle} H, H \in A \right\}$$

and at least one of the elements in the above generating set of $(\mathcal{C}r^0_F)_\ell$ has order $\ell^{b_i}$. Further, given a degree zero divisor with pole degree $\delta$ and an integer $a$, we can test if it is principal using a Riemann-Roch computation in time polynomial in $[F : k]$, $\log q$, $\delta$ and $\log(\ell^a)$ [12]. Hence we can find the element of maximal order in the generating set and the subgroup generated by it would be the $A$ that we seek.

By computing the structure of $(\mathcal{C}r^0_F)_\ell$ for every $\ell$ dividing $s_2$, we determine $(\mathcal{C}r^0_F)[s_2]$.

Thus we can obtain the invariant decomposition $\mathcal{C}r^0_F$ of the form

$$\mathcal{C}r^0_F = \langle e(1)\gamma \rangle \oplus \langle e(2)\gamma \rangle \oplus \ldots \oplus \langle e([F : k])\gamma \rangle$$

where for $1 \leq i \leq [F : k]$, $e(i) \in \mathbb{Z}[G]$ and $d_i$ the order of $e(i)\gamma$ in $\mathcal{C}r^0_F$ and for $1 \leq i < [F : k]$, $d_i \mid d_{i+1}$. Thus theorem [13] follows.

Given two degree zero divisors $D_1, D_2$, the discrete logarithm problem in $\mathcal{C}r^0_F$ is to compute an integer $x$ such that $D_1 \sim xD_2$ if it exists. The discrete logarithm problem over $\mathcal{C}r^0_F$ is believed to be hard. There are several cryptosystems whose security is reliant on the hardness of solving the discrete logarithm problem, in particular when $F$ is the function field of an elliptic curve.

Assume that $D$ is a degree zero divisor that generates $\mathcal{C}r^0_F$ and that $D$ has $[F : k]$ distinct conjugates. The above decomposition allows us to project an degree
zero divisor in \( \mathbb{Z}[G]D \) in to the invariant decomposition of \( \mathcal{O}_F^0 \). This reduces the discrete logarithm problem between two divisors in \( \mathbb{Z}[G]D \) to inversion in \( \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_r\mathbb{Z} \) which can be solved efficiently using the extended Euclidean algorithm.

4. Euler Systems from Stark Units

Let \( \ell \) be a prime number not dividing \( q(q^d-1)[F:k] \) and \( N \) a power of \( \ell \). Let \( \ell^a \) be the cardinality of \( \text{Pic}_\ell(\mathcal{O}_F) \). Fix a finite set \( \{\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_s\} \) of ideals of \( \mathcal{O}_k \) such that \( \text{Pic}_\ell(\mathcal{O}_k) \), the \( \ell \)-primary part of \( \text{Pic}(\mathcal{O}_k) \) decomposes as

\[
\text{Pic}_\ell(\mathcal{O}_k) = \langle \mathfrak{h}_1 \rangle \times \langle \mathfrak{h}_2 \rangle \times \ldots \times \langle \mathfrak{h}_s \rangle
\]

where for \( 1 \leq i \leq s \), \( \mathfrak{h}_i \) is the class of \( \mathfrak{h}_i \) in \( \text{Pic}(\mathcal{O}_k) \). For \( 1 \leq i \leq s \), let \( n_i \) be the order of \( \langle \mathfrak{h}_i \rangle \). Fix a \( \mathfrak{h}_i \in \mathcal{O}_k \) such that \( \langle \mathfrak{h}_i \rangle^{n_i} = \mathfrak{h}_i \mathcal{O}_k \). Let \( R_N \) be the set of prime ideals of \( \mathcal{O}_k \) that split completely in the extension \( F' := F(\mu_N, h_1^{N_1}, h_2^{N_2}, \ldots, h_s^{N_s})/k \).

Let \( B_N \) be the set of square free products of ideals in \( R_N \). For an \( \mathfrak{a} = \mathfrak{p}_1 \mathfrak{p}_2 \ldots \mathfrak{p}_t \in B_N \) with \( \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t \in R_N \), let \( F(\mathfrak{a}) \) denote the compositum \( F(\mathfrak{p}_1)F(\mathfrak{p}_2) \ldots F(\mathfrak{p}_t) \).

For the unit ideal \( \mathfrak{e} = \mathcal{O}_k \), let \( F(\mathfrak{e}) := F \).

An Euler system of modulus \( N \) is a function \( \Psi : B_N \rightarrow k_\infty^\times \) such that \( \forall \mathfrak{a} \in B_N \) and \( \forall \mathfrak{p} \in R_N \),

1. \( \Psi(\mathfrak{a}) \in F(\mathfrak{a})^\times \)
2. If \( \mathfrak{a} \neq \mathfrak{e} \), then \( \Psi(\mathfrak{a}) \in \mathcal{O}_F^0(\mathfrak{a}) \)
3. \( \mathcal{N}_{F(\mathfrak{ap})/F(\mathfrak{a})}(\Psi(\mathfrak{ap})) = \Psi(\mathfrak{a})^{1-(p,F(\mathfrak{a})/k)^{-1}} \)
4. \( \Psi(\mathfrak{ap}) = \Psi(\mathfrak{a})^{(p,F(\mathfrak{a})/k)^{-1}(N(\mathfrak{p})-1)/N} \) modulo every prime in \( F(\mathfrak{ap}) \) above \( \mathfrak{p} \).

Oukhaba and Vigue [20, § 3] proved that for every non zero coprime ideals \( \mathfrak{f}, \mathfrak{g} \subset \mathcal{O}_k \),

\[
\Psi_{\mathfrak{f},\mathfrak{g}}(\mathfrak{a}) := \mathcal{N}_{H(\mathfrak{f})/F(\mathfrak{a})}(\lambda_{\mathfrak{f},\mathfrak{a}}^{N(\mathfrak{g}) - (\mathfrak{g},K_m/k)})
\]

is an Euler system such that \( \Psi_{\mathfrak{f},\mathfrak{g}}(\mathfrak{e}) = \mathcal{N}_{H(\mathfrak{f})/H(\mathfrak{g})}(\lambda_{\mathfrak{f},\mathfrak{g}}^{N(\mathfrak{g}) - (\mathfrak{g},K_m/k)}) \).

Since \( \{\mathcal{N}_{H(\mathfrak{f})/H(\mathfrak{g})}(\lambda_{\mathfrak{f},\mathfrak{g}}^{N(\mathfrak{g}) - (\mathfrak{g},K_m/k)})\}_{\mathfrak{f},\mathfrak{g}} \) generates \( S_F \) up to roots of unity and the product of two Euler systems is an Euler system, for every \( \alpha \in S_F \), there exists an Euler system \( \Psi \) such that \( \Psi(\mathfrak{e}) = \alpha \).[20 Cor 3.16]. If \( \Psi(\mathfrak{e}) = \alpha \), we call \( \Psi \) an Euler system starting from \( \alpha \).

4.1. Kolyvagin Systems of Derivative Classes. From an Euler system \( \Psi \), a collection of functions \( \kappa(\mathfrak{a}) \in F^\times \) indexed by \( \mathfrak{a} \in B_N \) is derived. The places that appear in the divisor \( [\kappa(\mathfrak{a})] \) admit a precise characterization up to an \( N^{th} \) multiple due to the properties of Euler systems.

For \( \mathfrak{p} \in R_N \), let \( D_\mathfrak{p} := \sum_{i=0}^{N-1} i\sigma_\mathfrak{p} \). For an \( \mathfrak{a} \in B_N \), let \( D_\mathfrak{a} := \prod_{\mathfrak{p} \mid \mathfrak{a}} D_\mathfrak{p} \) where the product is over prime ideals \( \mathfrak{p} \) dividing \( \mathfrak{a} \).
For every $\sigma \in Gal(F(a)/F)$ and every prime $p$ dividing $a$, the class of $\Psi(p)^{\sigma-1}D_a$ in $F(a)^\times/(F(a)^\times)^N$ is fixed by $Gal(F(a)/F)$ [20, Lem 4.1].

The $N^{th}$ roots of unity are trivial in $F(a)$ and the 1-cocycle $C_a : Gal(F(a)/F) \rightarrow F(a)^\times$ that takes $\sigma$ to $\Psi(a)^{\sigma-1}D_a$ is well defined. Hilbert’s theorem 90 implies that there exists a $\beta \in F(a)^\times$ such that $C_a(\sigma) = \beta^{\sigma-1}$ for all $\sigma \in Gal(F(a)/F)$. Set $\kappa(a) := \frac{\Psi(a)^{\sigma}}{\beta^{\sigma}}$. In particular, set

$$\frac{1}{\beta} := \sum_{\sigma \in Gal(F(a)/F)} C_a(\sigma)(e) = \sum_{\sigma \in Gal(F(a)/F)} \left(\Psi(a)^{(\sigma-1)D_a}\right)^\frac{1}{\beta} \sigma(e)$$

Here $e \in F(a)^\times$ is picked such that the term on the right does not vanish. Independence of characters implies the existence of such an $e$. For instance, $e = \lambda_{a,F(a)}$ assures that the term does not vanish. Set $e = \lambda_{a,F(a)}$ and $\kappa(a) := \frac{\Psi(a)^{\sigma}}{\beta^{\sigma}}$.

For every $\sigma \in Gal(F(a)/F)$,

$$\left(\beta^{(\sigma-1)}\right)^N = \Psi(a)^{(\sigma-1)D_a} \Rightarrow \left(\frac{\Psi(a)^{D_a}}{\beta^N}\right)^{(\sigma-1)} = 1 \Rightarrow \kappa(a) \in F^\times$$

Further, $\kappa(a) = \Psi(a)^{D_a}$ modulo $(F^\times)^N$.

Let $I$ be the group of fractional ideals of $O_F$ written additively as a subgroup of the group of divisors of $F$. For a prime ideal $p$ of $O_F$, let $I_p$ be the subgroup of $I$ supported at places in $F$ above $p$. For $f \in F^\times$, let $[f]_p \in I_p$ be the projection of $fO_F$ in $I_p$.

From [20 Prop4.3], for every $p \in R_N$, there exists a $G$-equivariant map

$$\varphi_p : (O_F/p)^\times/((O_F/p)^\times)^N \rightarrow I_p/NI_p$$

unique up to a multiple of $(\mathbb{Z}/N\mathbb{Z})^\times$, that makes the following diagram commute.

$$\begin{array}{ccc}
(O_F/p)^\times & \xrightarrow{\varphi_p} & I_p/NI_p \\
(x \rightarrow \{z^{1-\sigma_p}\}) \downarrow & & \downarrow (x \rightarrow [\kappa_{F(a)/F}(x)]_p) \\
F(p)^\times & \xrightarrow{\psi} & (O_F/p)^\times/((O_F/p)^\times)^N
\end{array}$$

where $d = \frac{N(p)-1}{N}$. Let $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ where $\mathfrak{p}$ is a prime ideal in $F(p)$ above $p$. Let $\mathfrak{B} := \mathfrak{p} \cap O_F$.

Let $Gal(F(p)/F) = \langle \sigma_p \rangle$. Then the image of $\pi^{1-\sigma_p}$ is of order $N$ in $(O_{F(p)}/\mathfrak{B})^\times \cong (O_F/\mathfrak{B})^\times$, and it is independent of the choice of $\pi$. We will denote this image by $\overline{\pi}_\mathfrak{B}$.

The unique $G$-equivariant map $\varphi_p$ that makes the above diagram commute takes

$$\bigoplus_{\mathfrak{B} | p} \overline{\pi}_\mathfrak{B} \rightarrow \sum_{\mathfrak{B} | p} b_{\mathfrak{B}} \mathfrak{B}.$$

The following lemma relates Kolyvagin derivative classes through the $\varphi_p$ map.
Lemma 4.1.  ([20] Lem 4.4]) For \( p \in B_N \), if \( p \nmid a \) then \( [\kappa(a)]_p = 0 \) mod \( N \) and if \( p \mid a \) then \( [\kappa(a)]_p = \varphi_p(\kappa(a/p)) \) mod \( N \).

5. Characterizing Ideal Class Group Using Kolyvagin Systems

Let \( \chi \) be a non trivial irreducible \( \mathbb{Z}_l \)-representation of \( G \) of dimension \( \dim(\chi) \) and \( e(\chi) = \frac{1}{|F:k|} \sum_{\sigma \in G} \text{Tr}(\chi(\sigma))^{e^{-1}} \) the corresponding idempotent in \( Z_l[G] \). For a \( Z_l[G] \) module \( B \), define \( B(\chi) := e(\chi)B \). Let \( U = \mathcal{O}_F^\times \) and \( E = S_F \cap \mathcal{O}_F^\times \). Gras conjecture, proven true by Oukhaba and Vigué in this context, relates the cardinalities of \( U/E \) and \( \text{Pic}(\mathcal{O}_F) \).

Theorem 5.1. (Gras Conjecture [20] Thm 1.1) For every prime \( \ell \) not dividing \( q(q^{d_{\infty}} - 1)|F:k| \), \( |\text{Pic}_l(\mathcal{O}_F)(\chi)| = |(U/E)(\chi)| \) for every non trivial irreducible \( \mathbb{Z}_l \) representation \( \chi \) of \( G \).

Theorem 1.1 in [20] is stronger than what is stated here. It allows \( \ell \) to divide \( q^{d_{\infty}} - 1 \) as long as \( \ell \) does not divide \( [H_m : k] \) and \( \chi \) is of a certain form.

More is known regarding the structure of \( (U/E)(\chi) \). Since \( e(\chi)Z_l[G] \) is isomorphic to the ring of integers of the unramified abelian extension of \( \mathbb{Q}_l \) of degree \( \dim(\chi) \), \( e(\chi)Z_l[G] \) is a discrete valuation ring and every simple torsion \( e(\chi)Z_l[G] \)-module is isomorphic to \( \mathbb{Z}/\ell^t\mathbb{Z}[G]e(\chi) \) for some \( c \). It is proven in [20] Thm 4.8 that \( (U/E)(\chi) \) is \( G \)-isomorphic to \( e(\chi)\mathbb{Z}/t\mathbb{Z}[G] \) for some \( t \) (which is power of \( \ell \)) such that \( t(U/E)(\chi) = 0 \).

5.1. Structure of the \( \ell \)-part of the Class Group: Proof of Theorem 1.6

We present a characterization of the structure of \( \text{Pic}_l(\mathcal{O}_F)(\chi) \) in terms of Kolyvagin’s derivative classes.

Let \( N \) be a power of \( \ell \). For an \( \alpha \in F^\times/F^\times N \), if \( e(\chi)\alpha \) is an \( \ell^t \)-th power but not an \( \ell^{t+1} \)-th power in \( F^\times/F^\times N(\chi) \), we define \( \ell^t \) to be the \( \chi \)-index of \( \alpha \).

Since \( U(\chi) \) modulo the roots of unity is a free rank one \( e(\chi)Z_l[G] \)-module [20 § 4], there exists a \( \lambda \in U \) whose projection in \( e(\chi)(U/U^N) \) has order \( N \). Thus \( \lambda^t \in e(\chi)(E/U^N) \) and \( \lambda^t \) has \( \chi \)-index \( t \). Hence there exists elements in \( E \) of \( \chi \)-index \( t \). Consider a Kolyvagin system of modulus \( N \) starting from a unit \( \kappa(e) \in E \) of \( \chi \) index \( t \).

We introduce a concept that will be useful for extending the reasoning in the proof of Gras Conjecture in [4] [20] to obtain further results.

Write \( \kappa(a) \xrightarrow{\lambda} \kappa(ap) \), if there is a prime \( \mathfrak{p}|p \) in \( F \) and a \( u \in ((\mathbb{Z}/N\mathbb{Z}[G])(\chi))^\times \) such that

\[ uT\mathfrak{p} = [e(\chi)\kappa(ap)]_p \mod N \]

where \( T \) is the \( \chi \)-index of \( \kappa(a) \). If more specific, we write \( \kappa(a) \xrightarrow{\lambda} \kappa(ap) \) through \( \mathfrak{p} \).

Let \( ord_N(q) \) be the order of \( q \) in \( (\mathbb{Z}/N\mathbb{Z})^\times \). The following lemma without the requirement that \( \mathfrak{p} \) is of degree at most \( \max\{ord_N(q), 2 \log_q(\ell^{4n+2}|F:k|)\} \) is proven in [20] Thm 4.7. However, our computation needs the effective version stated below with the degree of \( \mathfrak{p} \) bounded.
Lemma 5.2. Let $A$ be a $\mathbb{Z}[G]$-quotient of $\text{Pic}(\mathcal{O}_F)_\ell(\chi)$. Let $H$ be the abelian extension of $F$ corresponding to $A$. Let $F_N = F(\mu_N)$, $F' = F(\mu_N, h_1^{1/N}, h_2^{1/N}, \ldots, h_s^{1/N})$ and $L = F'(W^{1/N}) \cap H$. Let $\beta \in (F^\times/F^\times N)(\chi)$ and $b$ be the order of $\beta$ in $F^\times/F^\times N$. Let $W$ be a finite cyclic $G$-submodule of $F^\times/F^\times N$ generated by $\beta$. Let $s$ be the number of factors in the primary decomposition of $A$. Then there exists a $\mathbb{Z}[G]$-generator $c'$ of $\text{Gal}(L/F)$ such that for every $c$ in $A$ whose restriction to $L$ is $c'$, there exists a prime $\mathfrak{P}$ of $F$ of degree at most $\max\{\text{ord}_N(q), 2\log_q([F : k]), 2(s+3)\log_q(N)\}$ such that

1. The projection of $\mathfrak{P}$ in $A$ is in $c$.
2. $p \in R_N$, where $p = \mathfrak{P} \cap k$.
3. $[\beta]_p = 0$ and there exists $u \in ((\mathbb{Z}/N\mathbb{Z}[G])(\chi))^\times$ such that $\varphi_p(\beta) = u(N/b)\mathfrak{P}$.

Proof. Everything in the lemma is proven in [20, Thm 4.7] except the degree bound on $\mathfrak{P}$. As argued in the proof in [20, Thm 4.7], there exists $\tau \in \text{Gal}(F'(W^{1/N}/F'))$ that generates $\text{Gal}(F'(W^{1/N}/F')$ over $\mathbb{Z}[\text{Gal}(F_N/k)]$. The restriction $c'$ of $\tau$ to $L$ is a $\mathbb{Z}[G]$-generator of $\text{Gal}(L/F) \cong \text{Gal}(L/F')$. Let $\theta \in \text{Gal}(H/F) = A$ correspond to $c$ where $c$ is an extension of $c'$ to $H$. Choose $\rho \in \text{Gal}(HF'(W^{1/N}/F))$ such that

$$\rho|_H = \theta$$
$$\rho|_{F'(W^{1/N})} = \tau$$

The field of constants of $HF'(W^{1/N})$ is $\mathbb{F}_{q^{\text{ord}_N(q)}}$. Let $m$ be a multiple of $\text{ord}_N(q)$. To ensure $F$ contains places of degree $m$, further assume that $m > \log_q([F : k])$. Let $E$ be the conjugacy class of $\rho$ in $\text{Gal}(HF'(W^{1/N})/F)$. Let $N_m(E)$ denote the cardinality of $S_m(E) := \{s| \deg(s) = m, (HF'(W^{1/N})/F, s) \in E\}$, where $s$ denotes a place in $F$ that is unramified in $HF'(W^{1/N})/F$. By the Chebotarev density theorem [13],

$$\left|N_m(E) - \frac{|E|\text{ord}_N(q)q^m}{m|HF'(W^{1/N}) : F|}\right| \leq 6.5 D \frac{|HF'(W^{1/N}) : F|}{q^{m/2}}$$

where $D$ is the degree of the different of the extension $HF'(W^{1/N})/F$.

The different $D$ is bounded by the genus of $HF'(W^{1/N})$ and $[HF'(W^{1/N}) : F] \leq N^{s+3}$, where $s$ is the number of factors in the primary decomposition of $A$. Pick $m$ to be the smallest multiple of $\text{ord}_N(q)$ such that $m > (s+3)\log_q(N)$ and $m > \log_q([F : k])$, then $N_m(E)$ is non zero. Further, $N_m(E)$ is at least $\frac{1}{N^{s+3}}$ fraction of the number of places in $F$ of degree $m$.

Pick a place $\mathfrak{P}$ in $S_m(E)$. The rest of the proof is exactly the same as [20, Thm 4.7].

For $a \in B_N$, let $C_a$ denote the subgroup of $\text{Pic}(\mathcal{O}_F)_\ell$ generated by primes dividing $a$ in $F$.

Lemma 5.3. Let $\chi$ be an irreducible $\mathbb{Z}_\ell$ representation of $G$. Let $a \in B_N$ and suppose $C_a(\chi)$ is a proper subgroup of $\text{Pic}_\ell(\mathcal{O}_F)(\chi)$. Let $A$ be the $\mathbb{Z}_\ell[G]$-quotient $\text{Pic}_\ell(\mathcal{O}_F)(\chi)/C_a(\chi)$. Then there is a prime $\mathfrak{P}$ of $F$ that projects to a nontrivial
Consider a \( \mathfrak{P} \) such that \( \mathfrak{P} \) is over a prime \( p \in B_N \) with \( p \nmid a \) and \( \kappa(a) \xrightarrow{\psi} \kappa(ap) \) through \( \mathfrak{P} \).

**Proof.** The lemma follows by applying Lemma 5.2 to the finite \( G \)-submodule \( W \) of \( F^\times / F^\times N \) generated by \( \beta = e(\chi)\kappa(a) \).

**Lemma 5.4.** Suppose \( \kappa(a) \xrightarrow{\psi} \kappa(ap) \) through \( \mathfrak{P} \). Let \( T \) be the \( \chi \)-index of \( \kappa(a) \) and \( \mathcal{B} \) the \( \chi \)-index of \( \kappa(ap) \). Then \( B/T \). If \( (N/T)\text{Pic}_c(O_F)(\chi) = 0 \), then the class of \( \mathfrak{P} \) in \( \text{Pic}_c(O_F)(\chi)/C_\alpha(\chi) \) has order dividing \( T/B \).

**Proof.** We have \( e(\chi)\kappa(ap) = (e(\chi)\alpha)^p \) in \( e(\chi)(F^\times / F^\times N) \) for some \( \alpha \in F^\times \). On the other hand, since \( \kappa(a) \xrightarrow{\psi} \kappa(ap) \) through \( \mathfrak{P} \), \( uTe(\chi)\mathfrak{P} = [e(\chi)\kappa(ap)]_p \). So \( Be(\chi)\alpha \equiv uTe(\chi)\mathfrak{P} \mod N \). Therefore \( B \mid T \). Since \( [Be(\chi)\alpha] \equiv [e(\chi)\kappa(ap)] \mod N \) and \( (N/B)\text{Pic}_c(O_F)(\chi) = 0 \), \( (T/B)e(\chi)\mathfrak{P} \equiv 0 \mod C_\alpha(\chi) \) and the lemma follows.

Consider a \( \chi \)-path starting from \( \kappa(e) \):

\[
\kappa(e) \xrightarrow{\psi} \kappa(p_1) \xrightarrow{\psi} \kappa(p_1p_2) \xrightarrow{\psi} \ldots \xrightarrow{\psi} \kappa(p_1p_2\ldots p_n)
\]

We say that the \( \chi \)-path is complete if the \( \chi \)-index of the last node \( \kappa(p_1p_2\ldots p_n) \) is 1.

From now on we assume that \( N = \ell t^2 \).

Suppose in the \( \chi \)-path above, \( \kappa(p_1p_2\ldots p_{i-1}) \xrightarrow{\psi} \kappa(p_1p_2\ldots p_i) \) through \( \mathfrak{P}_i \mid p_i \). Note that for all primes \( \mathfrak{P} \) in \( F \), \( e(\chi)\mathfrak{P}^\sigma = e(\chi)\mathfrak{P} \). Hence for all \( 1 \leq i \leq n 
\]

\[
e(\chi)\mathfrak{P}_i(G) \mid \mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_i) = C_{p_1p_2\ldots p_i}(\chi)
\]

Let \( C_1(\chi) = C_{p_1p_2\ldots p_i}(\chi) \) and let \( d = \dim(\chi) \). Let \( t_i \) be the \( \chi \)-index of \( \kappa(p_1p_2\ldots p_i) \). From Lemma 5.3, we see that \( [C_1(\chi) : 1] \) divides \( (t/t_i)^d \) and for \( i > 1 \), \( [C_i(\chi) : C_{i-1}(\chi)] \) divides \( (t_{i-1}/t_i)^d \). It follows that \( [C_n(\chi) : 1] \) divides \( (t/t_n)^d \). Suppose \( C_n(\chi) = \text{Pic}_c(O_F)(\chi) \), then \( [C_n(\chi) : 1] = t^d \) by Gras Conjecture, hence we must have \( t_n = 1 \).

Conversely, suppose \( t_n = 1 \). Suppose for a contradiction that \( C_n(\chi) \neq \text{Pic}_c(O_F)(\chi) \). By Lemma 5.3, there exists a prime \( \mathfrak{P} \) that projects to a non-trivial class \( c \in \text{Pic}_c(O_F)(\chi)/C_n(\chi) \), such that \( \mathfrak{P} \) is over a prime \( p \in B_N \) not dividing \( p_1p_2\ldots p_n \) and \( \kappa(p_1p_2\ldots p_n) \xrightarrow{\psi} \kappa(p_1p_2\ldots p_n p) \) through \( \mathfrak{P} \). Since \( (N/t_n)\text{Pic}_c(O_F)(\chi) = 0 \), it follows from Lemma 5.3 that the class of \( \mathfrak{P} \) in \( \text{Pic}_c(O_F)(\chi) \) (which is \( c \)), has order modulo \( C_n(\chi) \) dividing \( t_n = 1 \), hence is 1. We have a contradiction. Hence \( C_n(\chi) = \text{Pic}_c(O_F)(\chi) \) and for all \( 1 \leq i \leq n, [C_i(\chi) : C_{i-1}(\chi)] = (t_{i-1}/t_i)^d \).

Suppose \( t_n > 1 \). Then \( C_n(\chi) \neq \text{Pic}_c(O_F)(\chi) \). By Lemma 5.3, there exists prime \( \mathfrak{P} \) that projects to a non-trivial class \( C \in \text{Pic}_c(O_F)(\chi)/C_n(\chi) \) such that \( \mathfrak{P} \) is over a prime \( p \in B_N \) not dividing \( p_1p_2\ldots p_n \) and \( \kappa(p_1p_2\ldots p_n) \xrightarrow{\psi} \kappa(p_1p_2\ldots p_n p) \) through \( \mathfrak{P} \). In this fashion, we may extend the \( \chi \)-path until the \( \chi \)-index of the last element is one or equivalently we have the entire \( \text{Pic}_c(O_F)(\chi) \) constructed. We have thus proven Theorem 1.6.

Since \( C_i(\chi)/C_{i-1}(\chi) \) is \( G \)-cyclic of exponent \( t_{i-1}/t_i \), it follows that \( C_n(\chi) \) is of
exponent dividing $\prod_{i=1}^{n} t_{i-1}/t_i = t_0$, which is the exponent of $(U/E)(\chi)$. Therefore we have the following

**Theorem 5.5.** The exponent of $\text{Pic}_\ell(\mathcal{O}_F)(\chi)$ divides the exponent of $(U/E)(\chi)$.

Theorem 1.4 leads to an iterative procedure to compute $\text{Pic}_\ell(\mathcal{O}_F)(\chi)$, which will be discussed in §6.3

### 6. Computation of the Ideal Class group

We explore the algorithmic implications of Gras conjecture and Theorem 1.6.

#### 6.1. Constructing $\ell$-adic Representations

We begin by constructing the $\mathbb{Z}_\ell$ character $\chi$ of $G$. It is sufficient for our algorithm to compute the associated idempotent $e(\chi)$. Consider the primary decomposition $G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \oplus \ldots \oplus \langle \sigma_s \rangle$ where for $1 \leq i \leq s$, $\sigma_i$ has order $q_i^{b_i}$ in $G$, where $q_i$ is a prime. For each $1 \leq i \leq s$, we describe all irreducible $\mathbb{Z}_\ell$ representations $\chi_i$ of the group $\langle \sigma_i \rangle$. For every choice of irreducible representations $\{ \chi_i \}_i$, the Kronecker product $\bigotimes_i \chi_i$ defines an irreducible $\mathbb{Z}_\ell$ representation on $G$ and every irreducible $\mathbb{Z}_\ell$ representation of $G$ can be obtained in this manner.

Consider the factorization $x^{q_i^{b_i}} - 1 = \prod_j g_j(x)$, where $g_j(x) \in \mathbb{Q}_\ell[x]$ are monic irreducible polynomials. For each such factor $g_j$, we construct an irreducible $\mathbb{Z}_\ell$ representation $\chi_i$ of $\langle \sigma_i \rangle$ as follows. Let $\zeta$ be a root of $g_j$ in an algebraic closure of $\mathbb{Q}_\ell$. Let $\mathcal{O}(\mathbb{Q}_\ell(\zeta))$ denote the ring of integers of $\mathbb{Q}_\ell(\zeta)$, viewed as a $\mathbb{Z}_\ell$ module. Define $\chi_i(\sigma_i)$ to be the $\mathbb{Z}_\ell$ linear automorphism of $\mathcal{O}(\mathbb{Q}_\ell(\zeta))$ that acts on $\mathcal{O}(\mathbb{Q}_\ell(\zeta))$ as left multiplication by $\zeta$. The dimension of the representation equals the degree of the polynomial $g_j$. The fixed space of $\chi_i(\sigma_i)$ is trivial and thus $\chi_i$ is irreducible. Since the factorization $\prod_j g_j(x)$ is square free and since distinct $g_j$ correspond to distinct representations, counting dimensions reveals that we have constructed all the irreducible representations of $\langle \sigma_i \rangle$.

Factor $x^{q_i^{b_i}} - 1$ over $\mathbb{F}_\ell$ and lift the factorization to $\mathbb{Q}_\ell[x]$ by Hensel Lifting. For each factor $g_j \in \mathbb{Q}_\ell[x]$, if $\zeta$ is a root of $g_j$ then $\{1, \zeta, \zeta^2, \ldots, \zeta^{\deg(g_j)-1}\}$ forms a $\mathbb{Z}_\ell$ basis for $\mathcal{O}(\mathbb{Q}_\ell(\zeta))$. Write down $\chi_i(\sigma_i)$ as the $\deg(g_j)$ dimensional square matrix over $\mathbb{Z}_\ell$ that takes the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{\deg(g_j)-1}\}$ to $\{\zeta, \zeta^2, \ldots, \zeta^{\deg(g_j)} = g_j(\zeta) - \zeta^{\deg(g_j)}\}$.

Thus the irreducible $\mathbb{Z}_\ell$ representation of $G$ and the corresponding idempotents can be determined from the primary decomposition of $G$ as described above and the computation takes time polynomial in $[F : k]$.

#### 6.2. Computation of the $\ell$-part of the Regulator: Proof of Theorem 1.4

For every non principal irreducible $\mathbb{Z}_\ell$ representation $\chi$ of $\text{Gal}(F/k)$, the regulator part $e(\chi)R_F$ can be computed as follows. Since $e(\chi)\mathbb{Z}_\ell[G]$ is isomorphic to the ring of integers of the unramified abelian extension of $\mathbb{Q}_\ell$ of degree $\text{dim}(\chi)$, every simple torsion $e(\chi)\mathbb{Z}_\ell[G]$-module is isomorphic to $\mathbb{Z}/\ell^c\mathbb{Z}[G]e(\chi)$ for some $c$. Since $\chi \neq 1$, the idempotent $e(\chi)$ is orthogonal to the principal idempotent $e(\chi)\infty_F$ has degree 0, where $\infty_F$ is a place in $F$ above $\infty$.

Since $e(\chi)R_F$ is the cyclic $\mathbb{Z}_\ell[G]$ module generated by $e(\chi)\infty_F$, the order of $e(\chi)R_F$ is $\ell^b$ if and only if $b$ is the smallest positive integer for which $\ell^be(\chi)\infty_F$ is principal. Using an algorithm of Hess [11], we can test if $\ell^be(\chi)(\infty_F)$ is principal in time
polynomial in $[F : k]$, $\log q$ and $\log(e^b)$. By finding the smallest $b$ for which it is principal, determine the structure of $e(\chi) R_F$ and theorem 1.4 follows.

Let $b_\chi$ be the smallest positive integer for which $\ell^{b_\chi} e(\chi) \infty_F$ is principal. The cardinality of $e(\chi) R_F$ is $\ell^{b_\chi \dim(\chi)}$ where $\dim(\chi)$ is the dimension of the character $\chi$. From the exact sequence

$$0 \longrightarrow R_F \longrightarrow \mathbb{C}_F^0 \longrightarrow \text{Pic}(\mathcal{O}_F) \longrightarrow 0$$

it follows that

$$|\text{Pic}(\mathcal{O}_F)| = \frac{h(F)_\ell}{\prod_\chi \ell^{b_\chi}}$$

where $h(F)_\ell$ is the cardinality of the $\ell$ primary part of $\mathbb{C}_F^0$ and the product is over all non principal irreducible $\mathbb{Z}_\ell$ representations $\chi$ of $\text{Gal}(F/k)$.

The divisor class number $h(F)$ can be computed in $\tilde{O}(p^{12} d^{13} [F : k]^{30})$ time by [15] [Theorem 37] and theorem 1.5 follows.

### 6.3. Computation of the $\ell$-part of the Class Group: Proof of Theorem 1.7

For this subsection we assume that the field $k$ is the rational function field $\mathbb{F}_q(t)$ and $F = H_m$ and present the algorithmic details of the iterative procedure outlined in theorem 1.6. At the end of the section we briefly discuss the algorithmic issues involved removing the assumption that $k = \mathbb{F}_q(t)$ and $F = H_m$.

As in the previous section, we fix a non principal irreducible $\mathbb{Z}_\ell$ character $\chi$ of $G$ and set $N = \ell t^2$ where $t$ is the exponent of $(U/E)(\chi)$.

To begin the iterative procedure, we need to construct an element in $E$ of $\chi$-index $t$ and an Euler system starting from it.

When $k = \mathbb{F}_q(t)$ and $F = H_m$, $S_F/\mu_F$ is generated by $\{\lambda_m^\sigma | \sigma \in G\}$ and the expression for the Euler system $\Psi_{f,g}$ in §4 greatly simplifies. The function $\xi : B_N \longrightarrow k_\infty$ that maps

$$a \longmapsto N_{K_{x=1}/H_a} \left( \lambda_m - \sum_{p|a} \lambda_p \right) \in H_a^\times$$

is an Euler system that starts from $\xi(e) = -\lambda_m^{t-1}$ [4][§2]. The summation in the above expression is over prime $p$ dividing $a$.

Since we have a finite generating set for $S_F/\mu_F$ and can test for identity in $(E/U^N)(\chi)$, we can compute a basis for $(E/U^N)(\chi)$ and one of the basis elements has to have $\chi$-index $t$. Express this basis element as a product of the form

$$\prod_{\sigma \in G_1} \lambda_m^\sigma(e(\chi)) \prod_{\tau \in G_2} \lambda_m^\tau(e(\chi))$$

where $G_1$ and $G_2$ are subsets of $G$ of the same cardinality.

Then the function $\Phi$ that maps $a \in B_N$ to

$$\frac{\prod_{\sigma \in G_1} \xi(a)^\sigma}{\prod_{\tau \in G_2} \xi(a)^\tau}$$

(6.1)
is an Euler system algorithm constructs an element of \( \chi \)-index \( t \).

The iterative algorithm constructs a \( \chi \)-path starting from \( \kappa(\epsilon) \),

\[
\kappa(\epsilon) \xrightarrow{\Phi} \kappa(p_1) \xrightarrow{\Phi} \kappa(p_1p_2) \xrightarrow{\Phi} \ldots \xrightarrow{\Phi} \kappa(p_1p_2\ldots p_n)
\]
such that \( \chi \)-index of \( \kappa(p_1p_2\ldots p_n) \) is 1.

The critical computation at each iteration is to find a \( p_i \) such that

\[
\kappa(p_1p_2\ldots p_{i-1}) \xrightarrow{\Phi} \kappa(p_1p_2\ldots p_i).
\]

The existence of such \( p_i \) of degree bounded by a polynomial in \( N \) and \( \log_q([F : k]) \) is guaranteed by lemma 5.2. Let \( m \) be the multiple of \( \text{ord}_N(q) \) chosen in Lemma 5.2. At each step, we randomly generate a prime \( p_i \) of degree \( m \) and check if \( \kappa(p_1p_2\ldots p_n) \xrightarrow{\Phi} \kappa(p_1p_2\ldots p_i) \). From proof of lemma 5.2, at each step we succeed in finding a \( p_i \) satisfying \( \kappa(p_1p_2\ldots p_n) \xrightarrow{\Phi} \kappa(p_1p_2\ldots p_{i-1}) \) in expected number of trials bounded by a polynomial in \( N \).

Given a choice of \( p_i \), we test if

\[
\kappa(p_1p_2\ldots p_{i-1}) \xrightarrow{\Phi} \kappa(p_1p_2\ldots p_i)
\]
by first computing \( \kappa(p_1p_2\ldots p_i) \) and then computing its \( \chi \)-index as described below.

We assume that we have constructed \( \lambda_1, \lambda_2, \ldots, \lambda_{i-1} \) in the previous iteration.

Compute \( \lambda_{p_i} \) using lemma 5.1 and then compute \( \Phi(p_1p_2\ldots p_i) \) using equation 6.1.

The cyclic extension \( F(p_1p_2\ldots p_i)/F \) can be constructed as the compositum

\[
F(p_1p_2\ldots p_i) = F.H(p_1p_2\ldots p_i)
\]
where \( H(p_1p_2\ldots p_i) \) is the fixed field of \( \text{Gal}(H_{p_1p_2\ldots p_i}/k)^N \).

Once \( F(p_1p_2\ldots p_i) \) and \( \Phi(p_1p_2\ldots p_i) \) are constructed, we can compute \( \kappa(p_1p_2\ldots p_i) \) using equation 4.1. The running time for computing \( \kappa(p_1p_2\ldots p_i) \) is dominated by the construction of the extension \( F(p_1p_2\ldots p_i) \) and the evaluation of equation 4.1 which take time polynomial in \( q^{\deg(p_i)} \) and \([F : k]\).

All that remains is to compute the \( \chi \)-index of \( \kappa(p_1p_2\ldots p_i) \).

To compute the \( \chi \)-index of an \( \alpha \in F^x/F^{xN} \), it suffices to be able to decide if \( e(\chi)\alpha \in Z_t \otimes \mathbb{Z} (F^x/F^{xN}) \) is an \( \ell^\text{th} \) power. Since \( N \) is a power of \( \ell \), \( Z_t \otimes \mathbb{Z} (F^x/F^{xN}) \cong Z/ZN \otimes \mathbb{Z} (F^x/F^{xN}) \) and \( e(\chi)\alpha \) can be expressed in the form \( 1 \otimes \mathbb{Z} f \) and viewed as the function \( f \) in \( F^x/F^{xN} \). Further, \( e(\chi)\alpha \) being an \( \ell^\text{th} \) power in \( Z/ZN \otimes \mathbb{Z} (F^x/F^{xN}) \) is equivalent to \( f \) being an \( \ell^\text{th} \) power in \( F^x/F^{xN} \). Since \( |\text{Pic}_F(\mathcal{O}_F)(\chi)| \) divides \( N \), \( f \) being an \( \ell^\text{th} \) power in \( F^x/F^{xN} \) is equivalent to its lift \( \tilde{f} \) being an \( \ell^\text{th} \) power in \( F^x \). The element \( \tilde{f} \) is an \( \ell^\text{th} \) power in \( F^x \) if and only if the Riemann-Roch space \( L([\tilde{f}]f) \) is nonempty. We can decide if \( L([\tilde{f}]f) \) is empty in time polynomial in \([F : k]\) and polylogarithmic in the pole degree of the divisor of \([\tilde{f}]f \). [12].
Thus the running time at each iteration of the algorithm is bounded by a polynomial in $q^{\text{ord}_N(q)}$ and $[F:k]$.

By computing $\text{Pic}_\ell(O_F)(\chi)$ for every $\chi$, we can determine $\text{Pic}_\ell(O_F)$ in time polynomial in $q^{\text{ord}_N(q)}$ and $[F:k]$. Since $\text{ord}_N(q)$ is at most $N-1$, Theorem 1.7 follows.

We briefly discuss an issue that arise while attempting to turn Theorem 1.6 into an effective algorithm that works not just for $k = \mathbb{F}_q(t)$ and $F = H_m$ but for every $k, H$ that Theorem 1.6 applies to. The generating set $\{N_{H_f/H}F(\lambda_m^{N(q)}-\{g, K_m/k\})\}_{f,g}$ for the Stark units is not finite since $f, g$ range over coprime non zero ideals in $O_F$. Hence, it not obvious as to how to find an element in $E$ of $\chi$ index $t$ and construct an Euler system starting from it. In the generating set, the choice of $f$ can be narrowed to a finite set. It is sufficient to consider $f$ either dividing $m$ to account for the ramified part of $F/k$ and $f$ of bounded degree to account for the unramified part. Since $S_F$ is finitely generated, it should be sufficient to consider $g$ of bounded degree, but this degree bound needs further investigation.

If $k$ is an arbitrary finite geometric extension of $\mathbb{F}_q(t)$ and $F = H_m$, then an element in $E$ of $\chi$ index $t$ and an Euler system starting from it can be efficiently found using [28][Theorem 2.3].

7. Acknowledgements

We would like to thank the two anonymous reviewers for their valuable suggestions.

References

[1] J. D. Achter, The distribution of class groups of function fields, J. Pure and Appl. Algebra 204 (2006), no. 2, 31633.
[2] J. Ahn, S Bae, H Jung, Cyclotomic units and Stickelberger ideals of global function fields, Trans. Amer. Math. Soc. 355 (2003), 1803-1818.
[3] E. R. Berlekamp, Factoring polynomials over large finite fields, Math. Comp. 24 (1970), 713-735.
[4] K. Feng, F. Xu Kolyvagin’s “Euler Systems” in Cyclotomic Function Fields, Journal of Number Theory 57, 114-121.
[5] E. Friedman, L. C. Washington, On the distribution of divisor class groups of curves over a finite eld, Theorie des nombres (Quebec, PQ, 1987), de Gruyter, Berlin,1989, pp. 227-239.
[6] G. Gras, Classes d’ideaux des corps abeliens et nombres de Bernoulli generalises, (French) Ann. Inst. Fourier (Grenoble) 27 (1977), no. 1, ix, 166.
[7] D. Hayes, Stickelberger elements in function fields, Compositio Math. 55, 209-235 (1985)
[8] D. Hayes, Explicit class field theory in global function fields, Studies in algebra and number theory, Vol. 6 (1979), pp. 173-217.
[9] D. Hayes, Explicit class field theory for rational function fields, Trans. Amer. Math. Soc. 189 (1974), 77-91
[10] D. Hayes, Elliptic units in function fields, Progress in Mathematics 26, Birkhauser, (1982)321-340.
[11] F. Hess, Computing relations in divisor class groups of algebraic curves over finite fields, Preprint.
[12] F. Hess, Computing Riemann-Roch Spaces in Algebraic Function Fields and Related Topics, J. Symb. Comput. 33(4): 425-445 (2002)
[13] M. Ishibashi, Effective version of the Tschebotareff density theorem in function fields over finite fields, Bull. London Math. Soc. 24 (1992), 52-56.
[14] V. Kolyvagin, Euler systems, The Grothendieck Festschrift, Vol. II, Progr. Math., 87, Boston, MA: Birkhuser Boston, pp. 435-483 (1990).
[15] A. Lauder, D. Wan, Counting rational points on varieties over finite fields of small characteristic. MSRI Computational Number Theory Proceedings, (2008).
[16] H.W. Lenstra, Finding isomorphism between finite fields, Math. Comp., 56 (1991), pp. 329-347.
[17] A.K. Lenstra, H.W. Lenstra, Jr., M. S. Manasse, J. M. Pollard, The Number Field Sieve, STOC 90, Pages 564-572.
[18] B. Mazur, A. Wiles, Class fields of abelian extensions of Q, Inventiones Mathematicae 76 (2): 179-330.
[19] V. K. Murthy, J. Scherk, Effective versions of the Chebotarev density theorem for function fields, C. R. Acad. Sci. (Paris), 319 (1994), 523-528.
[20] H. Oukhaba, S. Viguie, The Gras conjecture in function fields by Euler systems, Bull. London Math. Soc. (2011) 43 (3): 523-535.
[21] M. Pohst, Factoring polynomials over global fields I, Journal of Symbolic Computation, 39 (2005), p 1325-1339.
[22] M. Rosen, Number theory in function fields. Graduate Texts in Mathematics 210.
[23] K. Rubin, The main conjectures of Iwasawa theory for imaginary quadratic fields, Inventiones Mathematicae 103 (1): 2568(1990).
[24] K. Rubin, Euler Systems. Princeton, NJ: Princeton University Press, 2000.
[25] G.D.V. Salvador, Topics in the Theory of Algebraic Function Fields (Mathematics: Theory and Applications).
[26] W. Sinnott, On the Stickelberger Ideal and the Circular Units of a Cyclotomic Field, The Annals of Mathematics, Vol. 108, No. 1 (Jul., 1978), pp. 107-134.
[27] J. Tate, Le Conjectures de Stark sur les Fonctions L d’Artin en s=0”, Progress in Mathematics (Birkhauser) 47, 1984.
[28] F. Xu, J. Zhao, Euler systems in global function fields, Israel Journal of Mathematics 124(2001), 367-379.
[29] L. Yin, Stickelberger ideals and divisor class numbers. Math. Z. 239 (2002), no. 3, 425-440.