Modified box dimension of trees and hierarchical scale-free graphs

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Abstract. In this paper, we introduce a new concept: the modified box dimension of graph sequences motivated by the notion of fractality of complex networks proposed by Song et al. We show that the definition of fractality cannot be applied to networks with ‘tree-like’ structure and exponential growth rate of neighborhoods. However, we show that the definition of fractal dimension could be modified in a way that takes into account the exponential growth, and with the modified definition, the fractal dimension becomes a proper parameter. We find that this parameter is related to the growth rate of trees. We also generalize the concept of box dimension further and introduce the modified Cesaro box dimension. Using rigorous proofs we determine the optimal box covering and modified box dimension of various models: the hierarchical graph sequence model introduced by Komjáthy and Simon, Song-Havlin-Makse model, spherically symmetric trees, and supercritical Galton-Watson trees.

1 Introduction

The study of complex networks has received immense attention recently, mainly because networks are used in several disciplines of science, such as in Information Technology (World Wide Web, Internet), Sociology (social relations), Biology (cellular networks) etc. Understanding the structure of such networks has become essential since the structure affects their performance, for example the topology of social networks influences the spread of information and disease. In most cases real networks are too large to describe them explicitly. Hence, models must be considered. A network model can be static, i.e., it models a snapshot of the network, such as [6], [12], [21] or dynamic, i.e., the model mimics the evolution of the network on the long term [3].

Most of the classical random graph models (e.g. the Chung-Lu model [6], the configuration model [5] or the preferential attachment model [3]) do not take hierarchical or self-similar structure into account. Barabási, Ravasz and Vicsek introduced deterministic hierarchical scale-free graphs constructed by a method which is common in generating fractals [4]. Ravasz and Barabási improved the original model to obtain similar clustering behavior to many real-world networks [27]. Heydenreich, Hulshof and Jorritsma showed the existence of a hierarchical structure supercritical scale-free percolation model [16]. A similar fractal based approach was introduced by Andrade et al. [1], generalized and intensively investigated by Zhang et al. [35], [37]; since this model is related to Apollonian circle packing it was called Apollonian network. For further fractal related network models see e.g. [10], [17], [36]. The first and last authors of the present paper generalized the models of [4] and [27] by introducing a general hierarchical graph sequence derived from a graph directed self-similar fractal [19].

A different approach to handle the fractality and self-similarity of complex networks was initiated by Song, Havlin and Makse [32]. They proposed that the procedure of identifying the existence of fractality in a complex network must be similar to that of regular fractal objects:
using the box-covering method. Fractality stands for a polynomial relation between the number of boxes needed to cover the network and the size of the box, defining a finite box dimension, that was verified in many real-world networks. The fractality and self-similarity of complex networks was investigated in several further articles [14], [18], [29], [30], [31], [33] and we give a short review of the topic in Section 2. Here we note that while this article is dealing with the box dimension, a number of other notions of fractal dimension have been investigated throughout the years, for a recent survey we refer to [28].

As the main point of this article, we find that for network models with non-polynomial local growth rate a new definition of box dimension is needed. After a short review of the topic of network fractality by Song et al. [32], in Section 2 below we introduce the definition of box dimension for graph sequences, the modified box dimension and a generalized version, the modified Cesaro box dimension. Following the preliminary and shorter version of this article [22], the modified box dimension of models of weighted hierarchical graphs and weighted fractal networks have been already investigated by Dai et al. [7], [8]. In Section 3 we determine the optimal number of boxes needed to cover the hierarchical graph sequence model [19]. In the literature, box covering is determined mostly by approximation algorithms [9], [31], while our method is rigorous. We find that the hierarchical graph sequence model [19] does not have a finite box dimension (based on the usual definition assuming polynomial growth) but the modified box dimension exists (based on our new definition assuming exponential growth). In Section 2 we investigate the optimal boxing and modified box dimension of a fractal network model introduced by Song, Havlin and Makse [33]. In Section 4 we determine the optimal boxing of some deterministic and random trees, in particular, spherically symmetric trees and Galton-Watson branching processes, and show that the modified box dimension of trees is related to the growth rate of trees introduced by Lyons and Peres [20]. Section 5 concludes the work.

2 Fractal scaling in complex networks and concepts of box dimension

In this section we review the concept of fractality proposed by Song et al. in [32] and introduce the new concept of modified box dimension. The technique they proposed for identifying the presence of fractality in complex networks is analogous to that of regular fractals. For ‘conventional’ fractal objects in the Euclidean space (e.g. the attractors of iterated function systems), a basic tool is the box covering method [13], that can be applied to networks. Since the Euclidean metric is not relevant for graphs, it is reasonable to use a natural metric, namely the shortest path length between two vertices. In the case of unweighted graphs this metric is called the graph distance metric.

The method works as follows [31]: For a given network $G$ with $N$ vertices, we partition the vertices into subgraphs (boxes) with diameter at most $\ell - 1$ (it is illustrated in Figure 1). The minimum number of boxes needed to cover the entire network $G$ is denoted by $N_B(\ell)$. Determining $N_B(\ell)$ for any given $\ell \geq 2$ belongs to a family of NP-hard problems but in practice various algorithms are adopted to obtain an approximate solution [31]. In accordance with regular fractals, Song et al. proposed to define the fractal dimension or box dimension $d_B$ of the graph by the approximate relationship:

$$N_B(\ell)/N \approx \ell^{-d_B},$$  

i.e., the required number of boxes scales as a power of the box size, and the dimension is the absolute value of the exponent.

According to this method, the power form of (1) (with a finite $d_B$) can be verified by plotting and fitting in a number of real-world networks such as WWW, actor collaboration network and protein interaction networks [33]. A large class of complex networks (called non-fractal networks) is characterised by a sharp decay of $N_B$ with increasing $\ell$, i.e., has infinite fractal dimension, for example, the Internet at router level or most of the social networks [14], [24] falls into this category. To distinguish these cases, they introduced the concept of fractality as follows [32].
The fractality of a network (also called fractal scaling or topological fractality) means that there exists a power relation between the minimum number of boxes needed to cover the entire network and the size of the boxes.

In other words, equation 1 holds for a wide range of $\ell$. Although it is possible to ascertain the fractal dimension with this description and 1 using approximation methods, here we give a more rigorous mathematical definition shortly below. To motivate our choice of definition, when considering regular fractal objects (that is, sets embedded in $\mathbb{R}^d$ for some integer $d$) the box dimension, or Minkowski-dimension, is defined as the limit of the opposite of the ratio of the logarithm of the number of boxes and the logarithm of the box size, as the box size tends to 0. This definition would make no sense with respect to networks, since the graph distance can not be less than 1. On the other hand, tending to infinity with the box sizes might be a solution if the network itself grows, or is infinite to start with. For this reason, we should consider graph sequences. Several real-world networks (collaboration networks, WWW) grow in size as time proceeds, therefore it is reasonable to consider graphs of growing size, denoted by $\{G_n\}_{n \in \mathbb{N}}$ (where $\mathbb{N}$ stands for the set of natural numbers). Now, we can define the box dimension of a graph sequence rigorously. We need some basic definitions first.

**Definition 1 ($\ell$-box).** Consider two vertices $u, v$ in a graph $G$. Let $\Gamma(u, v)$ denote the set of all paths connecting $u, v$ within $G$. The length of a path $\pi$ is defined as the number of edges on $\pi$ and is denoted by $|\pi|$. The graph distance between two vertices $u, v$ in a graph $G$ is defined as $d_G(u, v) = \min\{|\pi| : \pi \in \Gamma(u, v)\}$. We say that a subgraph $H$ of a graph $G$ is an $\ell$-box if $d_H(u, v) \leq \ell - 1$ holds for all $u, v \in H$.

**Definition 2.** The box dimension $d_B$ of a graph sequence $\{G_n\}_{n \in \mathbb{N}}$ is defined by

$$d_B(\{G_n\}_{n \in \mathbb{N}}) := \lim_{\ell \to \infty} \lim_{n \to \infty} \frac{\log (N^n_B(\ell)/|G_n|)}{-\log \ell}$$

if the limit exists; where $N^n_B(\ell)$ denotes the minimum number of $\ell$-boxes needed to cover $G_n$, and $|G_n|$ denotes the number of vertices in $G_n$.

Although the order of limits is quite natural in the previous definition, the question arises whether the limiting operations can be interchanged. Considering the fact that the number of boxes needed to cover $G_n$ is clearly $N^n_B(\ell) = 1$ if $\ell > \text{diam}(G_n)$, it is meaningless to change the order of the limits.

![Fig. 1](image-url)
It is not hard to see that this definition of fractality cannot be applied to networks with locally 'tree-like' structure and exponential growth rate of neighborhoods. Indeed, in this case the optimal number of boxes does not scale as a power of the box size. On the other hand, the box-covering method gives rise to another interesting parameter if we modify the required functional relationship between the minimal number of boxes and the box size (similarly to the box-covering method gives rise to another interesting parameter if we modify the required functional relationship between the minimal number of boxes and the box size). Namely, we might consider the optimal number of boxes does not scale as a power of the box size. On the other hand, locally 'tree-like' structure and exponential growth rate of neighborhoods. Indeed, in this case

\[
\tau (\{G_n\}_{n \in \mathbb{N}}) := \lim_{\ell \to \infty} \lim_{n \to \infty} \frac{\log (N_B^\ell (\ell) / |G_n|)}{-\ell},
\]

if the limit exists; where \(N_B^\ell (\ell)\) denotes the minimum number of \(\ell\)-boxes needed to cover \(G_n\), and \(|G_n|\) denotes the number of vertices in \(G_n\).

**Remark 1.** We call \(\tau\) modified box dimension or 'growth-constant' since it captures how spread-out neighborhoods of vertices are, on an exponential scale.

We shall see in Section 5.1 that for some models with exponentially growing neighborhood sizes the limit in (1) does not exist but the limit of the Cesaro means give rise to an interesting constant, the modified Cesaro box dimension. We modify Definition 3 by considering the Cesaro-sum instead of the pure limit in \(n\).

**Definition 4.** The modified Cesaro box dimension \(\tau^*\) of a graph sequence \(\{G_n\}_{n \in \mathbb{N}}\) is defined by

\[
\tau^* (\{G_n\}_{n \in \mathbb{N}}) := \lim_{\ell \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\log (N_B^{i+\ell}(\ell) / |G_{n+i}|)}{-\ell},
\]

if the limit exists; where \(N_B^{i+\ell}(\ell)\) denotes the minimum number of \(\ell\)-boxes needed to cover \(G_n\), and \(|G_n|\) denotes the number of vertices in \(G_n\).

In what follows we investigate graph sequences with exponentially growing neighborhood sizes, and determine their modified box as well as modified Cesaro box dimension. These examples shall demonstrate that our definition is a natural one.

### 3 Optimal boxing of a hierarchical scale-free network model based on fractals

#### 3.1 Description of the model

This model was introduced by the first and last author of this article. In this Section we follow the notation of [19]. We start with an arbitrary initial bipartite graph \(G\), the base graph, on \(N\) vertices and we define a hierarchical sequence of deterministic graphs \(\{HM_n\}_{n \in \mathbb{N}}\) in a recursive manner. Let \(V(HM_n)\), the set of vertices of \(HM_n\) be \(\{0, 1, \ldots, N - 1\}^n\). The construction of \(HM_n\) from \(HM_{n-1}\) is achieved by taking \(N\) identical copies of \(HM_{n-1}\), corresponding to the \(N\) vertices of the base graph \(G\). Next, we construct the edges between the copies in a somewhat complicated way described in Definition 5 below. Along these lines, \(HM_n\) contains \(N^{n-1}\) copies of \(HM_1\), connected in a hierarchical way.

Let \(G\), our base graph, be any labeled bipartite graph on the vertex set \(\Sigma = \Sigma_1 = \{0, 1, \ldots, N - 1\}\) with bipartition \(\Sigma = V_1 \cup V_2\), such that one of the end points of any edge in \(G\) is in \(V_1\), while the other one is in \(V_2\). We write \(n_i := |V_i|, i = 1, 2\) and \(E(G)\) for the edge set of \(G\). We denote edges as \(\binom{\Sigma}{2}\), that shall be convenient later. The vertex set of \(HM_n\) is then given by \(\Sigma_n = \{x_1x_2\ldots x_n : x_i \in \Sigma\}\), all words of length \(n\) above the alphabet \(\Sigma\). In order to define the edge set of \(HM_n\), we need to introduce some further definitions [19].
Definition 5.

1. We assign a type to each element of $\Sigma$. Namely, $\text{typ}(x) := \begin{cases} 1, & \text{if } x \in V_1; \\ 2, & \text{if } x \in V_2. \end{cases}$

2. For $i = 1, 2$, we say that the type of a word $z = (z_1z_2\ldots z_n) \in \Sigma^n$ equals $i$ and write $\text{typ}(z) = i$, if $\text{typ}(z_j) = i$, for all $j = 1, \ldots, n$. Otherwise $\text{typ}(z) := 0$.

3. For $x = (x_1 \ldots x_n), y = (y_1 \ldots y_n) \in \Sigma^n$ we denote the common prefix by $x \land y := (z_1 \ldots z_k)$ s.t. $x_i = y_i = z_i, \forall i = 1, \ldots, k$ and $x_{k+1} \neq y_{k+1}$.

4. and the postfixes $\tilde{x}, \tilde{y} \in \Sigma^{n-|x \land y|}$ are determined by 

$$x := (x \land y)\tilde{x}, \quad y := (x \land y)\tilde{y},$$

where the concatenation of the words $a, b$ is denoted by $ab$.

Next we define the edge set $E(HM_n)$. Two vertices $x$ and $y$ in $HM_n$ are connected by an edge if and only if the following criteria hold:

(a) One of the postfixes $\tilde{x}, \tilde{y}$ is of type 1, the other is of type 2,

(b) for each $i > |x \land y|$, the coordinate pair $(i, x_i)$ forms an edge in $G$.

Remark 2 (Hierarchical structure of $HM_n$). For every initial digit $x \in \{0, 1, \ldots, N-1\}$, consider the set $W_x$ of vertices $(x_1 \ldots x_n)$ of $HM_n$ with $x_1 = x$. Then the induced subgraph on $W_x$ is identical to $HM_{n-1}$.

The following two examples satisfy the requirements of our general model.

Example 1 (Cherry). The “cherry” model was introduced in [4], and is presented in Figure 2. Let $V_1 = \{1\}$ and $V_2 = \{0, 2\}$. $E(G) = \{(1, 0), (1, 2)\}$.

Example 2 (Fan). Our second example is called “fan”, and is defined in Figure 3. Note that here $|V_1| > 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{The first three elements of the “cherry” model: $HM_1$, $HM_2$ and $HM_3$. Note that we added all loops to $HM_n$. In particular $HM_1$ differs from $G$ only in the loops. The figure was adapted by the authors from [19].}
\end{figure}
3.2 The optimal box-covering

In this section we determine the optimal box-covering of the hierarchical graph sequence model introduced before. We find that the optimal number of boxes does not scale as a power of the box size, meaning that this graph sequence has no finite box dimension, on the other hand, the modified box dimension is still a well-defined meaningful parameter.

**Theorem 1.** The fractal dimension of the hierarchical graph sequence \( \{HM_n\}_{n \in \mathbb{N}} \) is not well-defined in terms of Definition 2. The modified box dimension \( \tau \) defined in Definition 3 exists and equals

\[
\tau\left(\{HM_n\}_{n\in\mathbb{N}}\right) = \frac{\log N}{2},
\]

where \( N \) denotes the number of vertices in the base graph \( G \) of \( \{HM_n\}_{n\in\mathbb{N}} \).

In the rest of this section we investigate the optimal boxing of the model for certain box sizes, namely those that can be expressed as \( \text{diam}(HM_k) + 1 \). We thus define

\[
\ell_k := \text{diam}(HM_k) + 1, \quad B_n^k := N_n^B(\ell_k),
\]

that is, the minimal number of \( \ell_k \)-boxes that we need to cover \( HM_n \). Using this notation, we prove Theorem 1. The analysis of the box-covering consists of two main parts: giving upper and lower bound on \( B_n^k \).

**Upper bound on the optimal number of boxes.** The following lemma is a useful tool to examine the box dimension of the graph sequence. Here we use the notation of Section 3.

**Lemma 1.** The diameter of the hierarchical graph sequence model \( HM_n \) defined in Section 3.1 is \( \text{diam}(HM_n) = 2(n - 1) + \text{diam}(G) \).

**Proof.** The proof is a rewrite of [19] that we include for completeness. For two arbitrary vertices \( x, y \in \Sigma_n \) we denote the length of their common prefix by \( k = k(x, y) := |x \wedge y| \). Furthermore, let us decompose the postfixes \( \tilde{x}, \tilde{y} \) into longest possible blocks of digits of the same type:

\[
\tilde{x} =: b_1b_2\cdots b_r, \quad \tilde{y} =: c_1c_2\cdots c_q,
\]

with \( \{1, 2\} \ni \text{typ}(b_i) \neq \text{typ}(b_{i+1}) \in \{1, 2\} \) and \( \{1, 2\} \ni \text{typ}(c_j) \neq \text{typ}(c_{j+1}) \in \{1, 2\} \).

We denote the number of blocks in \( \tilde{x}, \tilde{y} \) by \( r \) and \( q \), respectively. From Definition 5 it follows that for any path \( P(x, y) = (x = q^0, \ldots, q^t = y) \), the consecutive vertices on the path only
differ in their postfixes, and these have different types. That is, each consecutive pair of vertices can be written in the form
\[ \forall i, q^i = u^i z^i, \quad q^{i+1} = u^{i+1} z^{i+1}, \] with \( \text{typ}(z^i) \neq \text{typ}(\hat{z}^i) \in \{1, 2\} \).

Now we fix an arbitrary self-map \( p \) of \( \Sigma \) such that
\[ (x, p(x)) \in E(G) \forall x \in G. \]

Most commonly, \( p(p(x)) \neq x \). Note that \( x \) and \( p(x) \) have different types since \( G \) is bipartite. For a word \( z = (z_1 \ldots z_m) \) with \( \text{typ}(z) \in \{1, 2\} \) we define \( p(z) := (p(z_1) \ldots p(z_m)) \). Then, Definition 5 implies that
\[ (t, p(t)) \text{ is an edge in } G_{t+m}, \forall t = (t_1 \ldots t_t). \tag{9} \]

Using (9), we construct a path \( P(x, y) \) between two arbitrary vertices \( x \) and \( y \) that has length at most \( p + q + \text{diam}(G) - 2 \). Starting from \( x \) the first half of the path \( P(x, y) \) is as follows:
\[
\hat{x}^0 = x = (x \wedge y) b_1 \cdots b_r y,
\hat{x}^1 = (x \wedge y) b_1 \cdots b_{r-1} p(b_r),
\ldots
\hat{x}^{r-1} = (x \wedge y) b_1 p(b_2) \cdots p(b_{r-1} p(b_r)).
\]

Starting from \( y \) the first half of the path \( P(x, y) \) is as follows:
\[
\hat{y}^0 = y = (x \wedge y) c_1 c_2 \cdots c_r,
\hat{y}^1 = (x \wedge y) c_1 \cdots c_{r-1} p(c_r),
\ldots
\hat{y}^{q-1} = (x \wedge y) c_1 p(c_2) \cdots p(c_{r-1} p(c_q)).
\]

It follows from (9) that \( P_x := (\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{r-1}) \) and \( P_y := (\hat{y}^{q-1}, \cdots, \hat{y}^1, \hat{y}^0) \) are two paths in \( \text{HM}_n \). To construct \( P(x, y) \), it remains to connect \( \hat{x}^{r-1} \) and \( \hat{y}^{q-1} \) using (9). Using (9) it is easy to see that this can be done with a path \( P_{\Delta} \) of length at most \( \text{diam}(G) \). Indeed, since the postfixes \( c_1 p(c_2) \cdots p(c_{r-1} p(c_q)) \) both have a type, one can connect them in at most as many edges as the diameter of the base graph. Clearly,

\[ \text{Length}(P(x, y)) \leq r + q + \text{diam}(G) - 2 \leq 2(n-1) + \text{diam}(G). \]

For the lower bound on the diameter of \( \text{HM}_n \), we show that we can find two vertices in \( \text{HM}_n \) of distance \( 2(n-1) + \text{diam}(G) \). Pick two vertices with \( |x \wedge y| = 0 \), so \( x_1 \neq y_1 \) so that the distance between \( x_1 \) and \( y_1 \) in \( G \) is exactly \( \text{diam}(G) \), and set each blocks \( b_i \) and \( c_i \) of length 1. Note that in each step on any path between two vertices, the number of blocks in \( \Sigma \) changes by at most one. Further, since \( x_1 \neq y_1 \) to connect \( x \) to \( y \), we have to reach two vertices that have a type. Starting from \( x \) to reach the first vertex \( \hat{a} = (x_1 \ldots) \) of this property, we need at least \( n-1 \) steps on any path \( P \). Similarly, starting from \( y \), we need at least \( n-1 \) steps to reach the first vertex \( \hat{b} = (y_1 \ldots) \) where all the digits are of the same type. Since the distance of \( x_1 \) and \( y_1 \) in \( G \) is \( \text{diam}(G) \), and we can change the first digit of a vertex on a path only to a neighbor digit in \( G \) in one step on any path, we need at least \( \text{diam}(G) \) edges to connect \( \hat{a} \) to \( \hat{b} \). This completes the proof.

One can do this coordinate-wise by using the edge-connection rule described in (b) above: Suppose \( z = z_1 z_2 \ldots z_k \) and \( y = v_1 v_2 \ldots v_k \) are two vertices that both have a type. Then for each coordinate pair \( z_i, v_i \) we choose the shortest path on the base graph \( G \) that connects them, that we denote by \( P_i \), with length \( m_i < \text{diam}(G) \). Then \( \text{dist}(z, y) = \max_i m_i \), and the path can be realized so that each coordinate follows the path \( P_i \) independently. The shorter paths simply stay put at their final vertex \( z_i \) once they are finished.
Recall $\ell_k, B^n_k$ from \cite{7}. The following two lemmas give an upper bound on $B^n_k$, the number of boxes needed to cover $HM_n$ with boxes of diameter at most $\ell_k$. The first lemma reduces the problem of covering with $\ell_k$-boxes to covering with $\ell_1$-boxes:

**Lemma 2 (Upper bound on $B^n_k$).** For all $k \geq n$, $B^n_k = 1$, while for all $n > k$,

$$B^n_k \leq B^1_{n-k+1}. \quad (10)$$

**Proof.** When $k \geq n$, $B^n_k = 1$ follows immediately from the fact that $\ell_k = \text{diam}(HM_k) + 1 \geq \text{diam}(HM_n) + 1$, i.e., all of the vertices of $HM_n$ can be covered by one box. When $k < n$, inequality \cite{10} means that if we can cover $HM_{n-k+1}$ with some number of size $\ell_1$-boxes, then the same number of size $\ell_k$-boxes is sufficient to cover $HM_n$. The proof works with an “anti-projection” procedure, that comes from the hierarchical construction of $HM_n$: namely, we can ‘blow up’ all the vertices of $HM_{n-k+1}$ by a copy a $HM_{k-1}$ in order to create $HM_n$. We show that the same $\ell_1$-boxing that we used in $HM_{n-k+1}$ is an appropriate $\ell_k$-boxing for $HM_n$ (in the sense that $HM_{k-1}$ is substituted for each vertex) since the substitution increases the maximum distance within the box by $2(k-1)$. In other words, let $\mathcal{G} = (x_1, \ldots, x_{n-k+1})$ and $\mathcal{Y} = (y_1, \ldots, y_{n-k+1})$ two arbitrary vertices in $HM_{n-k+1}$, contained in the same $\ell_1$-box, i.e., the distance between $\mathcal{G}$ and $\mathcal{Y}$ is at most $\text{diam}(G)$. During the anti-projection we create two copies of $HM_{k-1}$, with sets of vertices: $\mathcal{X} := \{(\tilde{x}_1, \ldots, \tilde{x}_n) | (\tilde{x}_1, \ldots, \tilde{x}_{n-k+1}) = \mathcal{G}\}$ and $\mathcal{Y} := \{(\tilde{y}_1, \ldots, \tilde{y}_n) | (\tilde{y}_1, \ldots, \tilde{y}_{n-k+1}) = \mathcal{Y}\}$. We show that all vertices within $\mathcal{X}$ and $\mathcal{Y}$ belong to the same $\ell_k$-box. If two vertices are either both in $\mathcal{X}$ or both in $\mathcal{Y}$ then the distance between them is at most $\text{diam}(HM_{k-1})$, since $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic to $HM_{k-1}$. Let $\tilde{\mathcal{X}} \in \mathcal{X}$ and $\tilde{\mathcal{Y}} \in \mathcal{Y}$. Starting from $\tilde{\mathcal{X}}$ it takes at most $k-1$ steps to reach the first vertex $\tilde{\mathcal{a}}$ where all the digits after the $n-k$th position are of the same type. Similarly, starting from $\tilde{\mathcal{Y}}$ we need at most $k-1$ steps to reach the first vertex $\tilde{\mathcal{b}}$ of the same property (the worst case scenario is when $\tilde{\mathcal{X}} = \{(\tilde{x}_1, \ldots, \tilde{x}_n) | (\tilde{x}_1, \ldots, \tilde{x}_{n-k+1}) = \mathcal{G}\}$ and $\text{typ}(\tilde{\mathcal{X}}) = \text{typ}(\tilde{\mathcal{Y}})$). To connect $\tilde{\mathcal{a}}$ and $\tilde{\mathcal{b}}$ we need at most $\text{diam}(G)$ edges, since their “primitives” $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ are in the same $\ell_1$-box in $HM_{n-k+1}$. Thus the distance between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ is not greater than $2(k-1) + \text{diam}(G) = \text{diam}(HM_k)$. This completes the proof.

**Lemma 3 (Upper bound on $B^n_i$).** For every $i \geq 1$ we have:

$$B^i_i \leq N^{i-1}. \quad (11)$$

**Proof.** It is easy to see that we need one $\ell_1$-box to cover $HM_1$. It follows from the hierarchical structure of $HM_i$ (Remark 3) that $HM_i$ contains $N^{i-1}$ copies of $HM_1$, since for every prefix $\mathcal{Z} = (z_1 \ldots z_i) \in \Sigma_i$, consider the set $W_\mathcal{Z}$ of vertices $(x_1 \ldots x_i)$ of $HM_i$ with $(x_1 \ldots x_i) = \mathcal{Z}$. Then the induced subgraph on $W_\mathcal{Z}$ is identical to $HM_i$ and $|\Sigma_i| = N^{i-1}$. This implies that we can cover $HM_i$ with $N^{i-1}$ $\ell_1$-boxes.

Combining Lemma 2 and Lemma 3 we get the following corollary.

**Corollary 1.** For all $n > k$ the following inequality holds:

$$B^n_k \leq B^1_{n-k+1} \leq N^{n-k}. \quad (12)$$

We continue giving lower bounds. First we investigate the case $k = 1$, i.e., $\ell = \ell_1 = \text{diam}(G)$.

**Lemma 4 (Lower bound on $B^n_i$).** For all $i \geq n_i + 1$,

$$B^i_i \geq N^{i-n_i}, \quad (13)$$

where $n_i := |V_i|$, $i = 1, 2$ and we assume that $n_1 \leq n_2$ without loss of generality.

**Proof.** Note that $\text{diam}(G) \leq 2n_1$ since we assumed that $G$ is bipartite and connected. It is enough to show that we can find $N^{i-n_i}$ witness vertices in $HM_i$ ($\forall i \geq n_i + 1$) such that the pairwise distances between these witnesses are greater than $2n_1$ (hence greater than $\text{diam}(G)$).
so they all must be in distinct $\ell_1$-boxes. First we investigate the case when $i = n_1 + 1$. In this case we need $N^{i-n_1} = N$ witnesses. We say that a vertex $\hat{z}_x = \hat{z}_y$ is a witness for $x \in \{0, \ldots, N - 1\}$ if $\hat{z}_x = z_1 \ldots z_i = x$ and $\text{typ}(z_i) \neq \text{typ}(z_j + 1)$ for all $j \leq i$. Let us pick an arbitrary $\hat{z}_x$ witness for every $x \in \{0, 1, \ldots, N - 1\}$. We claim that this collection of vertices is a good witness set, i.e., the distance between any two of them is at least $\ell_1 + 1$. To see this, consider $\hat{z}_x$ and $\hat{z}_y$ for $x \neq y$ and note that $|\hat{z}_x \wedge \hat{z}_y| = 0$ and $q = r = i = n_1 + 1$ in the notation of the proof of Lemma 4 thus we need at least $r - 1 + q - 1 + 1 = 2n_1 + 1$ steps on any path between $\hat{z}_x$ and $\hat{z}_y$. Using $\text{diam}(G) \leq 2N_1$ these witnesses must be in distinct $\ell_1$-boxes, so we need at least $N \ell_1$-boxes to cover $\text{HM}_{n_1+1}$. This proves the lemma for $i = n_1 + 1$.

Note that we can modify this procedure for arbitrary $i \geq n_1 + 1$. Let $i = n_1 + j + 1$ for a $j > 0$. The extensibility of the procedure follows from the hierarchical structure, i.e., $\text{HM}_{n_1+1+j}$ consists of $N^j$ copies of $\text{HM}_{n_1+1}$. Hence, in $\text{HM}_{n_1+1+j}$ the corresponding witnesses can be chosen as follows: For every $y \in \Sigma_j$ consider the witnesses $\hat{y} \hat{z}_x \hat{y} \in \Sigma_{1+j}$ for all $x \in \Sigma$, where $\hat{y} \hat{z}_x$ means the concatenation of $\hat{y}$ and $\hat{z}_x$. It is easy to see that the pairwise distance between any witnesses $\hat{y} \hat{z}_x$ and $\hat{y} \hat{z}_y$ is at least $\text{diam}(G)$. This way we created $N \cdot N^j = N^{i-n_1}$ witnesses. Thus we need at least $N^{i-n_1}$ boxes to cover $\text{HM}_i$: $B_i^y \geq N^{i-n_1}$.

**Lemma 5 (Lower bound on $B_i^y$).** Using the notation of the previous lemma, the following inequality holds if $n - k \geq n_1$:

$$B_i^y \geq N^{n-k-n_1+1} = N^{n-k} \cdot C,$$

where $C = N^{1-n_1}$ is a fix constant determined by the base graph $G$.

**Proof.** We have seen in Lemma 4 that $B_i^y \geq N^{i-n_1}$ for all $i \geq n_1 + 1$. We created $N^{i-n_1}$ vertices in $\text{HM}_i$ with pairwise distance greater than $\text{diam}(G)$. It is enough to show that we can find the same number of witnesses (i.e., $N^{i-n_1}$ many) in $\text{HM}_{k+i-1}$ for all $k \geq 1$, such that the pairwise distances between them are greater than $\text{diam}(\text{HM}_k)$, this implies $B_{k+i-1}^y \geq N^{i-n_1}$. For this, we ‘lift up’ the witnesses for the $\ell_1$-boxes from $\text{HM}_i$ to $\text{HM}_{k+i-1}$: we continue the code of a witness $\hat{z}_x$ in a way that the type is changed at every character (otherwise arbitrarily). More precisely:

$$\hat{z}_x \in \Sigma_1 \mapsto \hat{z}_x \in \Sigma_{i+k-1}; \hat{z}_x = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_i),$$

where $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_i) := \hat{z}_x$ and $\text{typ}(\hat{z}_j) \neq \text{typ}(\hat{z}_{j+1})$, $j \in \{1, 2, \ldots, i, k-1\}$. Due to the choice of $\hat{z}_x$ we also have $\text{typ}(\hat{z}_j) \neq \text{typ}(\hat{z}_{j+1})$, $j \in \{1, 2, \ldots, i, k-1\}$.

A lower bound on the shortest path between $\hat{z}_x$ and $\hat{z}_y$, where $x \neq y \in \Sigma$ can be given using a similar method than that in Lemma 4 and in Lemma 5. Since $|\hat{z}_x \wedge \hat{z}_y| = 0$ and $q = r = i+k-1$, we need at least

$$r - 1 + q - 1 + 1 = 2i + 2k - 3 \geq 2n_1 + 2(k-1) + 1 \geq \text{diam}(G) + 2k - 1 \geq \ell_k + 1$$

steps on any path between $\hat{z}_x$ and $\hat{z}_y$. Hence, these witnesses must be in distinct $\ell_k$-boxes and we need at least $N^{i-n_1}$ $\ell_k$-boxes to cover $\text{HM}_{k+i-1}$, i.e., substituting $n = k + i - 1$ yields that $B_{k+i-1}^y = B_{k+i-1}^{k+n-k} \geq N^{n-k+1-n_1} = N^{n-k} \cdot C$.

Now we can prove the existence of the modified box dimension of the hierarchical graph sequence model and determine the value of $\tau$.

**Proof (Proof of Theorem 4).** The first statement follows since $N_B^\ell / N$ in this case is not polynomial but exponential in $\ell$ by Lemma 5. For the modified box dimension, we note that it is enough to determine a subsequential limit in $\ell$ along the sequence $\ell_k = \text{diam} (\text{HM}_k) + 1$ since $N_B^\ell (n)$ is monotone decreasing in $\ell$. Hence we have

$$\tau ((\text{HM}_n)_{n \in \mathbb{N}}) = \lim_{\ell \to \infty} \lim_{n \to \infty} \frac{\log (N_B^\ell (n) / |\text{HM}_n|)}{-\ell} = \lim_{k \to \infty} \lim_{n \to \infty} \frac{\log (B_k^\ell / |\text{HM}_n|)}{-\ell_k}.$$
\[
\tau (\{HM_n\}_{n \in \mathbb{N}}) \leq \lim_{k \to \infty} \lim_{n \to \infty} \frac{\log (N^n) - \log (N^{n-k+1-n_1})}{\text{diam}(G) - 1 + 2k} = \lim_{k \to \infty} \frac{(k + n_1) \log N}{\text{diam}(G) - 1 + 2k} = \frac{\log N}{2}.
\]  
(14)

Similarly, using Corollary 1 yields a lower bound on the value of \( \tau \):

\[
\tau (\{HM_n\}_{n \in \mathbb{N}}) \geq \frac{\log N}{2}.
\]  
(15)

Combining (14) and (15) we can conclude that

\[
\tau (\{HM_n\}_{n \in \mathbb{N}}) = \frac{\log N}{2}.
\]

4 Song-Havlin-Makse model

In this section we analyze the model proposed by Song, Havlin and Makse in [33] to generate graphs with and without fractal scaling of Eq. (1). The motivation of the model is that the main feature that seems to distinguish the fractal networks is an effective “repulsion” (dissortativity) between nodes with high degree (hubs), this idea was first suggested by Yook et al. based on empirical evidence [34] and developed by Song et al. with analytical and modeling confirmations [33]. To put in other words, the most connected vertices tend to not be directly linked with each other but they prefer to link with less-connected nodes. In contrast, in case of non-fractal networks, hubs are primarily connected to hubs. The model \( \{SHM_p^n\}_{n \in \mathbb{N}} \) defined below can capture the main features (e.g. scale-free [23]) of real-world networks and the presence of fractal scaling is governed by a parameter of the model.

- **Initial condition:** We start at \( n = 0 \) with an arbitrary connected simple graph of a few vertices (e.g. a star shape of five nodes as in Figure 4).
- **Growth:** At each time step \( n + 1 \) we link \( m \cdot \text{deg}_n(v) \) new vertices to every \( v \) vertex that is already present in the network, where \( m \) is an input parameter and \( \text{deg}_n(v) \) is the degree of vertex \( v \) at time \( n \).
- **Rewiring edges:** At each time step \( n+1 \) we rewire the already existing edges as a stochastic combination of Mode I (with probability \( p \)) and Mode II (with probability \( 1-p \))
  - **Mode I:** we keep the old edge generated before time \( n+1 \).
  - **Mode II:** we substitute the edge \((u, v)\) generated in one of the previous time steps by a link between newly added nodes, i.e., by an edge \((u', v')\), where \( u' \) and \( v' \) are newly added neighbors of \( u \) and \( v \) respectively, as shown in Figure 4.

The model evolves by linking new nodes to already existing ones as follows: those nodes that appeared in the earlier stages form the hubs in the network. Consequently, Mode I leaves the direct edges between the hubs leading to hub-hub attraction, i.e. there are edges between vertices with high degrees. On the contrary Mode II leads to hub-hub repulsion or anticorrelation. It is interesting to investigate how the connection mode affects the fractality of the model, what happens if only one of the modes is used (\( p = 0 \) or \( p = 1 \)) or the combination of the two modes (i.e. \( p \in (0, 1) \)).

Here we note that using only Mode I (\( p = 1 \)) results in a tree (assuming that the initial graph contained no cycles) but this tree is not a rooted locally finite tree in contrast with the trees considered in Section 5.

The model with parameter \( p = 0 \) (i.e. using only Mode II) has a finite box dimension with \( d_B = \log(2m+1)/\log 3 \) [33]. On the other hand, with parameter \( p = 1 \) the box dimension is not finite but the modified box dimension is still a valid parameter.

**Theorem 2.** The diameter of the network generation model with parameter \( p = 1 \) is

\[
\text{diam}(SHM_p^n) = \Theta(n).
\]  
(16)

The model with \( p = 1 \) has no finite box dimension \( d_B \) but has finite modified box dimension

\[
\tau \left( \{SHM_p^n\}_{n \in \mathbb{N}} \right) = \log(2m+1).
\]  
(17)
Fig. 4. Different modes of growth with $m = 2$. In (a) the initial stage is illustrated, (b), (c) and (d) demonstrates Mode I, Mode II and the combination of the two modes respectively. The figure was adapted by the authors from [33].

The first half of the assertion above was claimed in [33] with heuristic explanation, here we give a more analytical argument and prove the existence of the modified box dimension.

Remark 3. In words, the above network generation model \{SHM^p_n\}_{n \in \mathbb{N}} with parameter $p = 1$ (i.e. using only Mode I) leads to a small-world network, i.e., the diameter of the graph grows proportionally to the logarithm of the number of vertices, and non-fractal topology, i.e., no finite box dimension $d_B$, but with finite modified box dimension.

Proof (Proof of Theorem 2). First, we establish some basic facts about the graph sequence model \{SHM^p_n\}_{n \in \mathbb{N}} in general. It is easy to see that the number of vertices in the network at time $n$ is deterministic, namely: $V(n) = V(n-1) + 2mE(n-1)$, where $V(n)$ and $E(n)$ denotes the number of nodes and edges in the network at time $n$ respectively. Let us suppose that w.l.o.g. $E(0) = V(0) = c$ for a fix constant $c$ then $V(n) = (2m + 1)^n \cdot c$.

If we use only Mode I ($p = 1$), the diameter increases by 2 in every step, thus $\text{diam}(\text{SHM}^1_{n+1}) = \text{diam}(\text{SHM}^1_n) + 2$, i.e. $\text{diam}(\text{SHM}^1_n) = \text{diam}(\text{SHM}^1_0) + 2k$. It implies that the diameter grows proportionally to the logarithm of the number of nodes in the network, i.e. $\text{diam}(\text{SHM}^1_n) = \Theta(\log N(t))$ leading to a small-world network. In order to handle fractality we should examine the boxing of the graph sequence. It is easy to recognize that we can cover $\text{SHM}^1_n$ with $V(n-1)$ $\ell_0$-boxes, with $\ell_0 = \text{diam}(\text{SHM}^1_0) + 1$, as it is also suggested by Figure 4. Another quite apparent observation (following from the hierarchical structure of the model) is that $\text{SHM}^1_n$ can be covered with $N(n-k-1)$ $\ell_k$-boxes, with $\ell_k := \text{diam}(\text{SHM}^1_k) + 1$ if $n-k-1 \geq 0$ and clearly only one box is enough to cover $\text{SHM}^1_n$ if $n < k + 1$. An appropriate boxing of $\text{SHM}^1_n$ with


\( \ell_k \)-boxes if we choose the centers of the boxes as the vertices generated \((k + 1)\) step ago. Let \( N_B^p(\ell_k) \) denote the minimum number of \( \ell_k \)-boxes needed to cover \( SHM_1^n \), so we have

\[
N_B^p(\ell_k) \leq \begin{cases} 
V(n - k - 1) & \text{if } n \geq k + 1 \\
1 & \text{if } n < k + 1.
\end{cases}
\]

(18)

It is less straightforward to verify that this is the optimal boxing (if diam(\(SHM_0^{1}\)) \leq 2) or at least asymptotically optimal (if diam(\(SHM_0^{1}\)) > 2). We can observe that in case of covering \(SHM_1^n\) with \( \ell_k \)-boxes, with \( \ell_k = \text{diam}(\text{SHM}_1^n) + 1 \), we can find \( V(n - k - [\text{diam}(\text{SHM}_0^{1})/2]) \) witness vertices such that the pairwise distances between the vertices are greater than diam(\(SHM_1^n\)). Here we do not give the exact construction but for the boxing of the hierarchical graph sequence model \(19\) a complete rigorous proof can be found in Section \(3.2\) and the optimal boxing of this model can be proved analogously. This implies that

\[
N_B^p(\ell_k) \geq \begin{cases} 
V(n - k - [\text{diam}(\text{SHM}_0^{1})/2]) & \text{if } n \geq k + [\text{diam}(\text{SHM}_0^{1})/2] \\
1 & \text{if } n < k + [\text{diam}(\text{SHM}_0^{1})/2].
\end{cases}
\]

(19)

Therefore, combining (18) and (19) we can conclude that \( N_B^p(\ell_k) = \Theta(V(n - k)) \), this together with the exponential growth of \( V(n) \) and the linear grow of diam(\(SHM_1^n\)) = diam(\(SHM_0^{1}\)) + 2n = \( \Theta(n) \) yields that no finite \( d_B \) exists in the sense of Definition \(2\) thus Mode I leads to a small-world non-fractal topology.

On the other hand, we can consider the modified box dimension \( \tau \) along the subsequence of box sizes \( \ell_k = \text{diam}(\text{SHM}_1^n) + 1 = \text{diam}(\text{SHM}_0^{1}) + 2k + 1 \) due to the monotonicity of \( N_B^p(\ell) \) in \( \ell \). It is clear from (18) and from (19) combined with the fact that \( V(n) = \Theta((2m + 1)^n) \) that

\[
\tau\left(\{\text{SHM}_1^n\}_{n \in \mathbb{N}}\right) = \lim_{k \to \infty} \lim_{n \to \infty} \log \left(\frac{(2m + 1)^n - k - 1}{2m + 1} / (2m + 1)^n \right) - \ell_k = \log(2m + 1).
\]

Thus we can conclude that the modified box dimension of the graph sequence \( \{\text{SHM}_1^n\} \) is \( \log(2m + 1) \).

Mode 1 produces exponentially growing neighborhoods (with a proper exponential growth rate \( \tau \)), while Mode 2 destroys this exponential growth so much that the growth rate becomes polynomial. It would be interesting to study rigorously the interpolation between these two very different growth rates by studying where the phase transition takes place between the two regimes. Is there such a \( p_c \in [0, 1] \) such that \( \tau\left(\{\text{SHM}_p^n\}_{n \in \mathbb{N}}\right) \) exists and nonzero for all \( p \geq p_c \) while the usual box dimension exists for all \( p < p_c \)? It was claimed in \(33\) with heuristic explanation that \( p_c = 1 \).

5 Boxing of trees and connection to the growth rate

In this section we calculate the modified box dimension \( \tau \) for some rooted evolving trees and compare it to the value of growth rate. We find that the modified box dimension of infinite trees is strongly related to the growth rate. While growth rate is defined only on trees, our concept of modified box dimension is defined on any graph sequences. The growth rate is defined for rooted infinite locally finite trees, i.e., when the degree of each vertex is finite \(20\).

First some notation. Let us denote an infinite tree by \( T_\infty \), and its root by \( \varrho \). Let \( L_n \) denote the set of vertices at distance \( n \) from the root, and its size by \( L_n \). We obtain a sequence \( \{T_n\}_{n \in \mathbb{N}} \) from \( T_\infty \) such that \( T_n = \bigcup_{i \leq n} L_i \) denotes \( T_\infty \) truncated at height \( n \). The generation of a vertex is its graph distance from the root, thus \( L_n \) forms generation \( n \). The subtree of a vertex \( v \), denoted by \( T^\nu(v) \), is defined as the vertices \( w \) that have the property that the shortest path to the root passes through \( v \). We call these the descendants of \( v \). We write \( T^\nu_k = \{w \in T^\nu(v) : d(v, w) \leq k\} \). Observe that the diameter of \( T^\nu_k \) is at most \( 2k \), hence, \( T^\nu_k \) is a \( 2k + 1 \)-box. The vertices in \( T^\nu_1 \setminus \{v\} \) are called the children of \( v \).
Definition 6 (Growth rate of trees, [20]). The growth rate of a tree sequence \( \{T_n\}_{n \in \mathbb{N}} \) obtained from an infinite tree \( T_\infty \) is defined as

\[
\text{gr}(\{T_n\}_{n \in \mathbb{N}}) = \lim_{n \to \infty} L_n^{1/n},
\]

whenever this limit exists. We will shortly write \( \text{gr}(T_\infty) \) for this quantity.

We shall use the notation introduced in the definition above throughout the paper. In the rest of this section we investigate the optimal boxing of various trees for certain box sizes – along the subsequence \( \ell_k := 2k + 1 \) and we write \( B^n_k \) for the minimal number of \( \ell_k \)-boxes that we need to cover \( T_n \). We compare the modified box dimension and the growth rate of some trees. Let us start with some examples; later we will generalise them to spherically symmetric trees.

Example 3 (Complete \( d \)-ary tree). A complete \( d \)-ary tree \( T^d_\infty \) is a rooted tree where each vertex has exactly \( d \) (\( d \geq 2 \)) children.

Example 4 (“2-3”-tree). A “2-3”-tree \( T^{2,3}_\infty \) is a rooted tree such that vertices at even distances from the root have 2 children while all other vertices have 3 children [20].

An important tool for the boxing of trees is the greedy boxing method.

Definition 7 (Greedy boxing starting from the leaves). Let \( n + 1 := a(k + 1) + b \) for some \( a, b \in \mathbb{N}, b < k + 1 \). We define the greedy boxing of a rooted tree \( T_n \) with \( 2k + 1 \)-boxes as follows:

\[
\text{Greedy}(n, k) := \left( \bigcup_{i=1}^{a} \bigcup_{v \in \mathcal{L}_{n+1-i(k+1)}} T_k^{(v)} \right) \cup T_{b-1}^{(\rho)}.
\]

That is, every vertex \( v \) at generation \( n + 1 - i(k+1) \), for \( i = 1, \ldots, a \) and its subtree \( T_k^{(v)} \) forms one box, and the box of the root might be somewhat smaller. When \( b = 0 \) the box of \( \rho \) is included in the first union hence the last box \( T_{b-1}^{(\rho)} \) is not there in the expression.

Claim 3 Consider a rooted tree \( T_n \) where each vertex not in generation \( n \) has at least one child. The minimal number of \( 2k + 1 \)-boxes, \( B^n_k \), satisfies

\[
L_{n-k} \leq B^n_k \leq G(n, k) \leq \sum_{i=0}^{n-k} L_i,
\]

where \( G(n, k) \) is the number of boxes in the greedy boxing, and \( L_i \) is the size of generation \( i \).

Proof. The fact that \( B^n_k \leq G(n, k) \) follows from the optimality of \( B^n_k \). The last inequality is established by observing that \( G(n, k) \) uses vertices in \( \mathcal{L}_{n-k} \) to cover generations \( n - k, \ldots, n \). The sum of earlier generations is a somewhat crude upper bound, since not all vertices in every generation before \( n - k \) are put in a separate box. For the lower bound, we find \( L_{n-k} \) witness vertices, that is vertices with pairwise distance at least \( 2k + 1 \) away. We choose a witness vertex from each box \( T_k^{(v)}, v \in \mathcal{L}_{n-k} \); let \( w(v) \) be any of the vertices in \( T_k^{(v)} \) that is in generation \( n \). We show that when \( u \neq v \), then \( d(w(u), w(v)) \geq 2k + 1 \). Indeed, since \( w(u), w(v) \) are in different subtrees in generation \( n - k \), the shortest path between \( w(u), w(v) \) travels through both \( u \) and \( v \) and so it contains at least \( d(w(u), u) + d(w(v), v) + 2 = 2k + 2 \) edges. This finishes the proof.

Theorem 4. The complete \( d \)-ary tree and “2-3”-tree have finite modified box dimension, namely \( \tau(T^d_\infty) = (\log d)/2 \) and \( \tau(T^{2,3}_\infty) = (\log \sqrt{6})/2 \). The growth rate of these trees are \( d \) and \( \sqrt{6} \) respectively.
Proof. A direct application of Definition \ref{def:box_dimension} yields that \( \text{gr}(T^n_\infty) = d \). For the modified box dimension, by monotonicity it is enough to consider the subsequence \( 2k + 1 \) for \( \ell \). Applying Claim \ref{claim:2} for this case yields that
\[
d^{n-k} \leq B^k_n \leq \sum_{i=0}^{n-k} d^i = \frac{d^{n-k+1} - 1}{d - 1}.
\]

Because \( |V(T^n_\infty)| = (d^{n+1} - 1)/(d - 1) \), using the bounds from Claim \ref{claim:2} we can calculate the limit
\[
\tau(S^n_\infty) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{\log(B^k_n / |V(T^n_\infty)|)}{(2k + 1)} = (\log d)/2.
\]

The proof works similarly for the “2-3”-tree. It is shown in \cite{20} that \( \text{gr}(T^{2,3}_\infty) = \sqrt{6} \). Elementary calculation shows that
\[
|V(T^{2,3}_n)| = \begin{cases} 
3 \cdot (6^{(n+1)/2} - 1)/5 & \text{if } n \text{ is odd} \\
3 \cdot (6^{n/2} - 1)/5 + 6^{n/2} & \text{if } n \text{ is even}
\end{cases}
\]
(24)

A simple calculation yields that
\[
L_n = \begin{cases}
2 \cdot 6^{(n-1)/2} & \text{if } n \text{ is odd} \\
6^{n/2} & \text{if } n \text{ is even}
\end{cases}
\]
(25)

Combining the results above with Claim \ref{claim:2} yields that
\[
\tau(S^{2,3}_n) = (\log \sqrt{6})/2
\]

### 5.1 Spherically symmetric trees

Let \( T^n_\infty \) be a spherically symmetric tree, that is an infinite rooted tree such that for each \( h \), every vertex at distance \( h \) from the root has the same number of children, namely, \( f(h) \) many, \( f(h) \in \mathbb{N}^+ = \{1, 2, \ldots\} \). Examples \ref{example:3} and \ref{example:4} are also spherically symmetric trees. For spherically symmetric trees, \( L_n = \prod_{i=0}^{n-1} f(h) \) and \( |T_n^n| = \sum_{i=0}^{n} L_i \).

In what follows we investigate the assumptions needed on the sequence \( f = (f(h))_{h \in \mathbb{N}} \) so that the modified box dimension of a spherically symmetric tree exists. First, we need a definition.

**Definition 8 (Length of the maximum contiguous subsequence of the same element).** Let \( s = (s_i)_{i=0}^\infty \) be an arbitrary sequence with codomain \( S \) and \( a \in S \). The length of the maximum contiguous subsequence of sequence \( s \) with respect to element \( a \) (i.e., the length of the longest block of consecutive ’a’ s):
\[
\text{lmcs}(s, a) := \max \{n \in \mathbb{N} \mid \exists i \in \mathbb{N} : a = s_i = s_{i+1} = s_{i+2} = \cdots = s_{i+n} \} + 1
\]

**Condition 1 (Regularity assumption)** Let us assume that \( f = (f(h))_{h \geq 0} \) in a spherically symmetric tree \( T^n_\infty \) satisfies for some \( K \in \mathbb{N} \) that
\[
\text{lmcs}(f, 1) = K < \infty.
\]
(26)

**Lemma 6.** Condition \ref{condition:1} implies that for all \( n \geq 1 \) it holds that
\[
L_n \leq |T^n_n| = \sum_{i=0}^{n} L_i \leq 2(\text{lmcs}(f, 1) + 1) \cdot L_n,
\]
(27)

that is, the total size of the tree is the same order as the size of the last generation.

---

\(^5\) The codomain of a sequence is the set into which all of the elements of the sequence is constrained to fall.
Proof (Proof of Lemma 6). The first inequality of the lemma is trivial. Now we express $L_i$ in terms of $L_n$:

$$L_i = L_n \prod_{h=1}^{n-1} f(h)^{-1} \leq L_n \prod_{h=1}^{n-1} f^*(h)^{-1},$$

(28)

where we define $f^* := f^*(f)$ as

$$f^*(h) := \begin{cases} 1, & \text{if } f(h) = 1 \\ 2, & \text{if } f(h) \geq 2. \end{cases}$$

By denoting $\lfloor x \rfloor := \sup\{ y \in \mathbb{Z} : y \leq x \}$ the lower integer part of a real number $x$, and $K := \text{lmcs}(f, 1)$,

$$\prod_{h=i}^{n-1} f^*(h)^{-1} \leq \left( \frac{1}{2} \right)^{\lfloor (n-i)/(K+1) \rfloor}$$

(29)

since the product is minimised when $f^*$ is such that $K$ 1’s are followed by a single 2 periodically. Combining (28) and (29) yields that

$$\sum_{i=0}^{n} L_i \leq L_n \left( 1 + \sum_{i=0}^{n-1} \left( \frac{1}{2} \right)^{\lfloor (n-i)/(K+1) \rfloor} \right) \leq L_n (K+1) \sum_{i=0}^{\infty} 2^{-i},$$

(30)

since $\lfloor (n-i)/(K+1) \rfloor$ takes on each integer $\leq |n/(K+1)|$ at most $K+1$ times. The sum of the geometric series is at most 2 and thus (27) is established.

Remark 4. Condition 1 is not only sufficient but necessary for Lemma 6. If $\text{lmcs}(f, 1) > M$, $\forall M \in \mathbb{N}$, then for some $n$ where $f(n-M) = \cdots = f(n) = 1$; and then $\sum_{i=n-M}^{n} L_i = (M+1)L_n$.

Corollary 2. Under Condition 2 the modified box dimension and growth rate of a spherically symmetric tree $T_{\infty}^f$ can be expressed as follows, if the limits exist:

$$\tau(T_{\infty}^f) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{\sum_{h=n-k}^{n-1} \log(f(h))}{(2k+1)},$$

(31)

$$\log(gr(T_{\infty}^f)) = \lim_{n \to \infty} \frac{1}{\sum_{h=0}^{n-1}} \log(f(h)).$$

(32)

Proof (Proof of Corollary 3). Let us write $c_+ := 2(\text{lmcs}(f, 1) + 1)$. Combining Claim 3 with Lemma 6 yields

$$L_{n-k} \leq B^n_k \leq \sum_{i=0}^{n-k} L_i \leq c_+ L_{n-k}.$$

(33)

Using Lemma 6 once more yields that $L_n \leq |T^f_n| \leq c_+ L_n$ and combining this with (33) we can observe

$$\frac{B^n_k}{|T^f_k|} \in [1/c_+, c_+] L_{n-k}/L_n = [1/c_+, c_+] \prod_{h=n-k}^{n-1} f(h)^{-1}. $$

(34)

Using these bounds in Definition 4 we see that the constant prefactor will vanish when taking logarithm, yielding (31). (32) is a direct consequence of Definition 6.

A natural question is to determine the conditions on $f$ that guarantee that $\tau$ agrees with $\log(gr)/2$. It is clear from (31) and (32) that if the sequence $f(h)$ converges then this is the case. However, a convergent integer sequence is equal to a constant for large enough $h$. On the other hand, we can easily construct an example (see Example 5) when the limit on the rhs of (32) exists but the limit on the rhs of (31) does not, here we provide an unbounded and a bounded counterexample.
Exchanging sums yields that the rhs inside the limits equals the inner limit (using (34)).

**Proof.** By Definition 4 we obtain for any spherically symmetric tree satisfying Condition 1, Let Example 6. Let

\[ \log(f(h)) := \begin{cases} a & \text{if } m(m+1) < h \leq (m+1)^2 \text{ for some } m \in \mathbb{N} \\ b \neq a & \text{otherwise} \end{cases} \]

This means that we have blocks of \(a\)'s and blocks of \(b\)'s with linearly increasing lengths. It is not hard to see that the limit in (32) exists, since it equals

\[ \log \text{gr}(T^f_\infty) = \lim_{m \to \infty} \frac{1}{m^2} \sum_{h=1}^{m} ((h+1)a + hb) = (a+b)/2 \]

On the other hand, the inner limit (as \(n \to \infty\)) on the rhs of (31) does not exist, since we see oscillations of \(a\)'s and \(b\)'s when \(k < m\).

These examples led us to a natural generalization of the modified box dimension in such a way that it agrees the half of the logarithm of the growth rate for spherically symmetric trees under a mild condition on the growth of the degree sequence. This is the modified Cesaro box dimension in Definition 3.

**Theorem 5.** Let \(T^f_\infty\) be a spherically symmetric tree with degrees \(f = (f(h))_{h \geq 1}\) that satisfies Condition 1 and that

\[ \lim_{h \to \infty} \frac{\log f(h)}{h} = 0. \quad (35) \]

Then, \(\tau^*(T^f_\infty) = \log(\text{gr}(T^f_\infty))/2\).

**Remark 5.** The growth condition (35) means that the degrees grow sub-exponentially. For instance, \(f(h) = [\exp(h^\gamma)]\) for any \(\gamma < 1\) satisfies this criterion.

**Proof.** By Definition 3 we obtain for any spherically symmetric tree satisfying Condition 1 using (34)

\[ \tau^*(T^f_\infty) = \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \frac{\log(B_{i+k}^k / |T^f_{i+k}|)}{-(2k+1)} = \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \frac{\log(L_i / L_{i+k})}{-(2k+1)} \quad (36) \]

Observing that \(L_n = \prod_{h=1}^{n-1} f(h)\) and \(\lim_{k \to \infty} (2k)/(2k+1) = 1\) we obtain that

\[ \tau^*(T^f_\infty) = \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{2mk} \sum_{i=1}^{m} \sum_{j=1}^{i+k-1} \log f(j) \quad (37) \]

Exchanging sums yields that the rhs inside the limits equals

\[ \frac{1}{2m} \sum_{j=1}^{m-1} j \log(f(j)) + \frac{1}{2m} \sum_{j=k}^{m-1} \log(f(j)) + \frac{1}{2m} \sum_{j=k}^{m+k-1} \left(1 - \frac{1}{j-m}\right) \log(f(j)), \quad (38) \]

where we define \(1/(j-m) := 0\) for \(j = m\). The first term in (38) tends to zero as evaluating the inner limit \((m \to \infty)\) in (36) for any fixed \(k\). For the second term in (37) we have:

\[ \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{2m} \sum_{j=k}^{m-1} \log(f(j)) = \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{2m} \sum_{j=1}^{m-1} \log(f(j)) - \lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{2m} \sum_{j=1}^{k-1} \log(f(j)) \]

\[ = \lim_{m \to \infty} \frac{1}{2m} \sum_{j=1}^{m-1} \log(f(j)) \quad (39) \]
since the second term tends on the rhs to zero for each fixed $k$, and the first term does not depend on $k$. Regarding the third term in (37), assumption (35) implies that for any fixed $k$,

$$
\lim_{m \to \infty} \frac{1}{2m} \sum_{j=m}^{m+k-1} \left(1 - \frac{1}{j-m}\right) \log(f(j)) = 0.
$$

(40)

Combining (38), (39), (40) with Corollary 2 we obtain that

$$
\tau^* \left( T^*_\infty \right) = \lim_{m \to \infty} \frac{1}{2m} \sum_{j=1}^{m-1} \log(f(j)) = \frac{\log \text{gr}(T_\infty)}{2},
$$

and the proof is finished.

5.2 Supercritical Galton-Watson trees

In this section we will consider the supercritical Galton-Watson trees.

**Definition 9 (Galton-Watson branching process).** Let $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ be an infinite vector of nonnegative real numbers with $\sum_{i=0}^{\infty} q_i = 1$ and let $\mathbb{P}_\mathbf{q}$ be the probability measure on rooted trees such that the number of offspring (children) of each vertex is i.i.d. and the probability that a given vertex has $i$ children is $q_i$.

Let $Z_n$ denote the number of vertices at distance $n$ from the root. We write $\mu$ for the mean of the offspring distribution $\mu = E(Z_1) = \sum_{i=0}^{\infty} iq_i$.

A Galton-Watson branching process (BP) with mean offspring $\mu$ is said to be supercritical if $\mu > 1$, critical if $\mu = 1$ and subcritical if $\mu < 1$.

It is well-known that in the subcritical case ($\mu < 1$) and in the critical case ($\mu = 1$), the BP dies out eventually (i.e. $\exists \ n : Z_n = 0$) with probability 1, while in the supercritical case ($\mu > 1$), the BP survives (i.e., $\forall \ n : Z_n \geq 1$) with positive probability $\mathbb{P}_\mathbf{q}$. When $q_0 = 0$, the BP survives with probability 1, since each vertex has at least one child. For further discussion and characterization of Galton-Watson BPs we refer the reader to [22] and [17]. A supercritical Galton-Watson BP behaves similarly to a deterministic regular tree of the “same growth” that suggests that the modified box dimension should be $\log \mu$ where $\mu$ is the mean of the offspring distribution. When $\mu$ is an integer, this deterministic tree is just the $\mu$-ary tree, but when $\mu$ is nonintegral, it is a virtual tree, in the sense of Pemantle and Peres [25], [26].

**Theorem 6.** Let $GW_\infty$ be a supercritical Galton-Watson tree with the following assumptions $q_0 = 0$, $q_1 \neq 1$ and

$$
E(Z_1 | \log(Z_1))^+ < \infty,
$$

(41)

where $|x|^+ := \max\{x, 0\}$ denotes the positive part of the expression. Then, $\tau(GW_\infty) \to (\log(\mu))/2$ almost surely.

The following elementary fact will be needed to prove Theorem 6.

**Lemma 7.** Let $\xi_k$ be an arbitrary convergent sequence of real numbers, that is, $\lim_{k \to \infty} \xi_k = A$ for some $A \in \mathbb{R}$. For $\beta \in (0, 1)$ let $S_n := \sum_{k=0}^{\infty} \beta^{n-k} \xi_k$. Then

$$
\lim_{n \to \infty} S_n = \sum_{k=0}^{\infty} \beta^k \cdot A = \frac{A}{1-\beta} \in \mathbb{R}.
$$

Proof. Using the convergence of $\xi_k$, by definition $\forall \epsilon > 0 : \exists N^* \ s.t. \ if \ i > N^* \ then \ |\xi_i - A| < \epsilon$. Now we consider the following sum

$$
\sum_{i=N^*}^{n} \xi_i \beta^{n-i} \in (A-\epsilon, A+\epsilon) \cdot \sum_{k=0}^{n-N^*} \beta^k,
$$

and the proof is finished.
Hence

$$
\lim_{n \to \infty} \sum_{i=0}^{n} \xi_i \beta^{n-i} \in (A - \epsilon, A + \epsilon) \cdot \sum_{k=0}^{\infty} \beta^k = ((A - \epsilon)/(1 - \beta), (A + \epsilon)/(1 - \beta)).
$$

(42)

Now let us consider the remaining part of the sum, namely \(\sum_{i=0}^{N^n} \xi_i \beta^{n-i}\). Due to the fact that \(\xi_i\) is convergent thus for some \(K: \xi_i < K, \forall i\) and \(\lim_{n \to \infty} \sum_{k=n-N^n}^{\infty} \beta^k = 0\) we have

$$
\sum_{i=0}^{N^n} \xi_i \beta^{n-i} < K \cdot \sum_{k=n-N^n}^{\infty} \beta^k \xrightarrow{n \to \infty} 0.
$$

(43)

Using (42) and (43) we can conclude that \(\lim_{n \to \infty} S_n = A/(1 - \beta)\).

**Proof (Proof of Theorem 6).** By writing \(W_i := \xi_i \mu^{-i}\), [2] Chapter I, Part C, Theorem 1, the limit \(W = \lim_{i \to \infty} W_i\) exists and is in \((0, \infty)\) almost surely under the \(x \log x\) assumption in [41] and that \(q_0 = 0\). With this notation, we bound the ratio \(B^n_k / |V(GW_n)|\) using Claim 3 (note that \(Z_i\) has the notation \(L_i\) there):

$$
\frac{W_{n-k} \mu^{n-k}}{\sum_{i=0}^{n-k} W_i \mu^i} = \frac{Z_{n-k}}{\sum_{j=0}^{n-k} Z_j} \leq \frac{B^n_k}{|V(GW_n)|} \leq \frac{\sum_{i=0}^{n-k} \xi_{n-i}}{\sum_{j=0}^{n-k} Z_j} \leq \frac{\sum_{i=0}^{n-k} W_i \mu^i}{\sum_{j=0}^{n-k} W_j \mu^j}.
$$

(44)

Dividing both the numerators and denominators with \(\mu^{-n}\), the bound turns into

$$
(1/\mu)^k \sum_{i=0}^{n-k} W_i (1/\mu)^{n-i} \leq \frac{B^n_k}{|V(GW_n)|} \leq (1/\mu)^k \sum_{i=0}^{n-k} W_i (1/\mu)^{n-k-i}.
$$

(45)

Applying now Lemma 7 with \(\xi_k := W_k, \beta := 1/\mu\), we see that the almost sure limits exists for each fixed \(k\)

$$
\lim_{n \to \infty} \sum_{i=0}^{n-k} W_i (1/\mu)^{n-i} = W/(1 - 1/\mu) = 1,
$$

$$
\lim_{n \to \infty} \sum_{i=0}^{n-k} W_i (1/\mu)^{n-k-i} = W/(1 - (1/\mu)) = 1 - (1/\mu).
$$

(46)

Therefore, for all sufficiently large \(n\),

$$
(1/\mu)^k (1 - (1/\mu))/2 \leq \frac{B^n_k}{|V(GW_n)|} \leq 2 (1/\mu)^k.
$$

(47)

Substituting these bounds into Definition 3 of modified box dimension, with \(\ell = 2k + 1\), yields that

$$
\tau \left(\{GW_n\}_{n \in \mathbb{N}}\right) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{\log(B^n_k/|V(GW_n)|)}{-(2k + 1)} = \frac{\log(\mu)}{2}.
$$

(48)

**Remark 6.** It is clear that \(\text{gr} (GW_\infty) \xrightarrow{a.s} \mu\) which implies that the almost sure limit of the modified box dimension and the logarithm of the growth rate of Galton-Watson trees differ only in a factor of 2 under the assumptions of Theorem 6.

**Remark 7.** Under some regularity assumptions the modified box dimension of Galton-Watson trees is well-defined in contrast with spherically symmetric trees where we had to introduce the concept of modified Cesaro box dimension. This phenomena arises from the fact that the random growth of the Galton-Watson branching process is smoother than the deterministic growth of spherically symmetric trees.
6 Conclusion and discussion

In this paper we have investigated the heuristic statement that networks with hierarchical structure are fractal (i.e., self-similar) objects. In particular, we considered a graph sequence with strict hierarchical structure, and investigated its fractal properties. Doing so we showed that the definition of fractality cannot be applied to networks with locally ‘tree-like’ structure and exponential growth rate of neighborhoods. However, the box-covering method gives a parameter that is related to the growth rate of trees. We also introduced a more general concept, the modified Cesaro box dimension. We investigated various models: the hierarchical graph sequence model introduced by Komjáthy and Simon, Song-Havlin-Makse model, spherically symmetric trees, and supercritical Galton-Watson trees. We determined bounds on the optimal box covering and calculated the modified box dimension of the aforementioned models using rigorous techniques. It would be also interesting to apply our method to other locally tree like graphs such as Erdős-Rényi graph, preferential attachment graph or configuration model.

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