Global Well-posedness of Classical Solutions to the Compressible Navier-Stokes-Poisson Equations with Slip Boundary Conditions in 3D Bounded Domains

Yazhou CHEN, Bin HUANG, Xiaoding SHI
College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, P. R. China

Abstract
We consider the initial-boundary-value problem of the isentropic compressible Navier-Stokes-Poisson equations subject to large and non-flat doping profile in 3D bounded domain with slip boundary condition and vacuum. The global well-posedness of classical solution is established with small initial energy but possibly large oscillations and vacuum. The steady state (except velocity) and the doping profile are allowed to be of large variation.

Keywords: Navier-Stokes-Poisson equations; global classical solutions; slip boundary condition; large oscillations; non-flat doping profile.

AMS Subject Classifications: 35Q35, 35B40, 76N10

1 Introduction
In this paper, we consider the compressible Navier-Stokes-Poisson (NSP) equations for the dynamics of charged particles of electrons (see [25]) in a domain \( \Omega \subset \mathbb{R}^3 \), which can be written as

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \rho \nabla \Phi, \\
\Delta \Phi = \rho - \tilde{\rho},
\end{cases}
\]

where \((x, t) \in \Omega \times (0, T]\), \( t \geq 0 \) is time, \( x \in \mathbb{R}^3 \) is the spatial coordinate. The unknown functions \( \rho, u = (u^1, u^2, u^3), P = P(\rho) \) and \( \Phi \) denote the electron density, the particle velocity, pressure and the electrostatic potential, respectively. Here we consider the isentropic flows with \( \gamma \)-law pressure \( P(\rho) = a \rho^\gamma \), where \( a > 0 \) and \( \gamma > 1 \) are some physical parameters. The given function \( \tilde{\rho} = \tilde{\rho}(x) > 0 \) is the doping profile, which describes the density of fixed, positively charged background ions. The constants \( \mu \) and \( \lambda \) are the shear viscosity and bulk coefficients respectively satisfying \( \mu > 0, 2\mu + 3\lambda \geq 0 \). In addition, the system is solved subject to the given initial data

\[\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = \rho_0 u_0(x), \quad x \in \Omega,\]

E-mail addresses: chenyz@mail.buct.edu.cn (Y. Chen), abinhuang@gmail.com (B. Huang), shixd@mail.buct.edu.cn (X. Shi)
and boundary condition
\[ u \cdot n = 0, \quad \text{curl} \times n = 0, \quad \nabla \Phi \cdot n = 0, \]
on \partial \Omega, \quad \text{on} \ \partial \Omega, \quad (1.3) \]
where \( n \) is the unit outer normal to \( \partial \Omega \). The boundary condition (1.3) on the velocity is a special Navier-type slip boundary condition (see [7,37]), in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress. This type of boundary condition was first introduced by Navier in [26], which was followed by great many applications, numerical studies and analysis for various fluid mechanical problems, see, for instance [6,8,14,37,38] and the references therein. For the electrostatic potential, we consider the Neumann boundary condition (1.4), which describes that the boundary of the domain is insulated (see [11]). It should be pointed out that
\[ \int (\rho - \tilde{\rho}) dx = 0. \quad (1.5) \]
Furthermore, we observed that the solvability of the system (1.1)-(1.4) implies that the initial density \( \rho_0 \) and the doping profile \( \tilde{\rho} \) must satisfy
\[ \int (\rho_0 - \tilde{\rho}) dx = 0. \quad (1.6) \]
Moreover, when we consider the Poisson equation with the Neumann boundary condition in a bounded domain, we need the following uniqueness condition on \( \Phi \),
\[ \int \Phi dx = 0. \quad (1.7) \]
Next, we consider the steady state of the system (1.1) with the non-flat doping profile \( \tilde{\rho}(x) \). Assume that \( (\rho_s, u_s, \Phi_s) \) with \( u_s \equiv 0 \) is the stationary solution of (1.1). Then it follows that
\[ \begin{aligned}
\nabla P(\rho_s) &= \rho_s \nabla \Phi_s \quad \text{in} \ \Omega, \\
\Delta \Phi_s &= \rho_s - \tilde{\rho} \quad \text{in} \ \Omega, \\
\nabla \Phi_s \cdot n &= 0 \quad \text{on} \ \partial \Omega.
\end{aligned} \quad (1.8) \]
Let \( Q(\rho) \) be the enthalpy function defined by
\[ Q'(\rho) = P'(\rho)/\rho, \quad (1.9) \]
which can change (1.8) to a quasilinear elliptic PDE and the classical fixed-point theorem or a variational method can be used to prove the existence of stationary solution, without the smallness assumption on the oscillation of \( \tilde{\rho}(x) \) (see [11,15]). We will record the existence and uniqueness of the solution to (1.8) in Lemma 2.1.

The compressible Navier-Stokes-Poisson system (1.1) has been attracted a lot of attention and significant progress has been made in the analysis of the well-posedness and dynamic behavior to the solutions of the system. We briefly review some results related to the global existence of strong (classical) solutions to Cauchy problem of the multi-dimensional compressible Navier-Stokes-Poisson system (1.1)-(1.2). When the doping profile is flat, i.e., \( \tilde{\rho}(x) \equiv \tilde{\rho} > 0 \), the steady state of (1.1) is the trivial constant one \((\tilde{\rho},0,0)\). Global existence and the decay rates of the classical solution to its steady state were studied in [4,12,16,19,20,33,35,11] in the Sobolev spaces.
or Besov spaces framework. This is extended to non-isentropic Navier-Stokes-Poisson systems, see [30,31,36,39,40]. While the doping profile is non-flat, global well-posedness of classical solutions and stability of the steady state were obtained in [10,29], under analogous smallness conditions as the ones in [19,34]. All of the above results need the smallness assumption on the oscillations between the initial data and the steady state. In particular, the initial density is near the non-vacuum steady density, which indicates that the density is uniformly away from the vacuum. Recently, for Cauchy problem of the compressible Navier-Stokes-Poisson system (1.1)-(1.2) subject to large and non-flat doping profile, global existence and uniqueness of strong solutions with large oscillations and vacuum was established in [24], provided the initial data are of small energy and the steady state is strictly away from vacuum.

For the initial-boundary-value problem of the compressible Navier-Stokes-Poisson system (1.1)-(1.2) with non-slip boundary condition for the velocity field, in the case that the doping profile is flat, the local and global existence of weak solution were obtained in [9,18] and the local existence of unique strong solutions with vacuum was proved in [32] with the initial data \( \rho_0 \) and \( u_0 \) satisfy a nature compatibility condition. Recently, for the case that the doping profile is non-flat, global existence of smooth solutions near the steady state for compressible Navier-Stokes-Poisson equations was established with the exponential stability in [23], with the smallness assumption on the oscillations between the initial data and the steady state but larger doping profile. We also mentioned that the global existence of solutions to compressible Navier-Stokes-Poisson equations with the large initial data on a domain exterior to a ball was proved in [22] with the radial symmetry assumption.

However, there are no works about the global existence of the strong (classical) solution to the initial-boundary-value problem (1.1)-(1.4) for general bounded smooth domains \( \Omega \subset \mathbb{R}^3 \) with initial density containing vacuum, at least to the best of our knowledge. Recently, for the barotropic compressible Navier-Stokes equations in \( \Omega \) with slip boundary condition, the global classical solutions with large oscillations and vacuum to the initial-boundary-value problem was established in [7], with some new estimates on boundary integrals related to the slip boundary condition.

Based on the above research works, we study the global existence of the classical solutions with large oscillations and vacuum to the initial-boundary value problem (1.1)-(1.4) subject to large and non-flat doping profile in a bounded domain \( \Omega \subset \mathbb{R}^3 \). Motivated by the works in [7,24], we would like to obtain the time-independent upper bound of the density and the time-dependent higher-norm estimates of \( (\rho, u, \Phi) \), and extend the classical solution globally in time. In our system, we need to conquer the difficulties arising from the coupling of the density \( \rho \) with the electric field \( \nabla \Phi \) and the slip boundary conditions (1.3). Firstly, compared with the previous results (see [17,24]) where they treated the Cauchy problem, it can not get the \( L^p \)-norm \( (2 \leq p \leq 6) \) of \( \nabla u \) by the standard elliptic estimate, due to the bounded domain with the slip boundary condition. To deal with this difficulty, we utilize \( L^p \)-theory for the div-curl system to control \( \nabla u \) by means of div\( u \) and curl\( u \) (Lemma A.3). For the electric field \( \nabla \Phi \), with the help of the classical regularity theory for the Neumann problem of elliptic equation, we obtain the estimates (2.19) and (2.23), which can be used to deal with the coupling between the density and the electric field. Secondly, owing to the coupling term \( (\rho - \rho_s)\nabla \Phi_s \) and \( \rho \nabla (\Phi - \Phi_s) \) in (2.8), we can not obtain the time-independent estimates of \( \|\rho - \rho_s\|_{L^2(0,T;L^2)} \) or \( \|P - P_s\|_{L^2(0,T;L^2)} \). It is also difficult to get time-independent estimates of \( \|\nabla (\Phi - \Phi_s)\|_{L^2(0,T;L^2)} \) due to the ellipticity of the Poisson equation (2.21). As a result, we can not close the time-independent estimates A1(T)
(see (3.1)) in a similar manner as that in [7,17]. In order to overcome this difficult point, we divide the time-independent estimates $A_1(T)$ into two time intervals $(0, \sigma(T))$ and $(\sigma(T), T)$. On the one hand, we can derive the short-time estimates (3.70) with the help of time-weighted estimates (see Lemma 3.5). On the other hand, we obtain the long-time estimates (3.79) by the time-piecewise iterative argument, which was used in [24]. Furthermore, we also obtain the time-dependent estimate $B[t_1, t_2]$, which can be bounded by the initial energy and the factor $t_2 - t_1$ for any $1 \leq t_1 \leq t_2 \leq T$ (see (3.72)).

These estimates help us to derive the uniform (in time) upper bound for the density. Additionally, we also need to pay more attention to control the boundary integrals during we derive the time-independent estimates of $A_1(T)$. It should be pointed out that the boundary condition $u \cdot n = 0$ yields

$$u \cdot \nabla u \cdot n = -u \cdot n \cdot u,$$

which is the key to estimate the integrals on the boundary $\partial \Omega$ and we obtain the estimate of $\dot{u}$ and $\nabla \dot{u}$ (see Lemma 3.3). In order to estimate the high order derivatives of the solutions, we recall the similar Beale-Kato-Majda-type inequality (see Lemma A.5) with the respect to the slip boundary condition to prove the important estimates on the gradients of the density and velocity.

Before formulating our main result, we first explain the notation and conventions used throughout the paper. For integer $k \geq 1$ and $1 \leq q < +\infty$, We denote the standard Sobolev space by $W^{k,q}(\Omega)$ and $H^k(\Omega) \equiv W^{k,2}(\Omega)$. For some $\beta \in (0,1)$, the fractional Sobolev space $H^\beta(\Omega)$ is defined by

$$H^\beta(\Omega) \equiv \left\{ u \in L^2(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\beta}} dxdy < +\infty \right\},$$

with the norm:

$$\|u\|_{H^\beta(\Omega)} \equiv \|u\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\beta}} dxdy \right)^{1/2}.$$

For simplicity, we denote $L^q(\Omega)$, $W^{k,q}(\Omega)$, $H^k(\Omega)$ and $H^\beta(\Omega)$ by $L^q$, $W^{k,q}$, $H^k$ and $H^\beta$ respectively, and set

$$\int f dx \equiv \int_{\Omega} f dx, \quad \int_0^T \int f dx \equiv \int_0^T \int_{\Omega} f dx.$$

For two $3 \times 3$ matrices $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, the symbol $A : B$ represents the trace of $AB^*$, where $B^*$ is the transpose of $B$, that is,

$$A : B \equiv \text{tr}(AB^*) = \sum_{i,j=1}^3 a_{ij}b_{ij}.$$

Finally, for $v = (v^1, v^2, v^3)$, we denote $\nabla_i v \equiv (\partial_i v^1, \partial_i v^2, \partial_i v^3)$ for $i = 1, 2, 3$, and the material derivative of $v$ by $\dot{v} \equiv v_t + u \cdot \nabla v$.

Assume $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ and its smooth boundary $\partial \Omega$ has a finite number of 2-dimensional connected components. The initial total energy of (1.1) is defined as

$$C_0 \equiv \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) + \frac{1}{2} |\nabla (\Phi_0 - \Phi_s)|^2 \right) dx. \quad (1.11)$$
where \( \Phi_0 = \Phi(x, 0) \) satisfies \( \Delta \Phi_0 = \rho_0 - \tilde{\rho} \) in \( \Omega \) and \( \nabla \Phi_0 \cdot n = 0 \) on \( \partial \Omega \), and \( G(\rho) \) is the potential energy density given by
\[
G(\rho) \triangleq \int_{\rho_s}^{\rho} \int_{\rho_s}^{\xi} \frac{P'(\zeta)}{\zeta} d\zeta d\xi = \rho \int_{\rho_s}^{\rho} \frac{P(\zeta) - P_s}{\xi^2} d\xi.
\] (1.12)
where \( P_s = P(\rho_s) \).

Now we can state our main result, Theorem 1.1 concerning existence of global classical solutions to the problem (1.1)-(1.4).

**Theorem 1.1** Let \((\rho_s, \Phi_s)\) be the stationary solutions of (1.8). Assume that the smooth function \( \tilde{\rho}(x) \) satisfy \( 0 < \rho \leq \tilde{\rho}(x) \leq \bar{\rho} \). For \( q \in (3, 6) \) and some given constants \( M > 0, \beta \in \left( \frac{1}{2}, 1 \right), \) and \( \bar{\rho} \geq \bar{\rho} + 1 \), suppose that the initial data \((\rho_0, u_0)\) satisfy (1.3), (1.6) and
\[
(\rho_0, P(\rho_0)) \in H^2 \cap W^{2,q}, \quad u_0 \in H^2, \quad 0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{H^\beta} \leq M,
\] (1.13)
and the compatibility condition
\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div } u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g,
\] (1.15)
for some \( g \in L^2 \). Then there exists a positive constant \( \varepsilon \) depending only on \( \mu, \lambda, \gamma, a, \bar{\rho}, \tilde{\rho}, \beta, \Omega \) and \( M \) such that for the initial energy \( C_0 \) as in (1.11) if
\[
C_0 \leq \varepsilon,
\] (1.16)
the initial-boundary-value problem (1.1)-(1.4), (1.7) has a unique global classical solution \((\rho, u, \Phi)\) in \( \Omega \times (0, \infty) \) satisfying
\[
0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad (x, t) \in \Omega \times (0, \infty),
\] (1.17)
\[
\begin{align*}
(\rho, P) & \in C([0, \infty); H^2 \cap W^{2,q}), \\
\nabla u & \in C([0, \infty); \dot{H}^1) \cap L^\infty_{\text{loc}}(0, \infty; H^2 \cap W^{2,q}), \\
\nabla u & \in L^\infty_{\text{loc}}(0, H^2) \cap H^1_{\text{loc}}(0, \infty; H^1), \\
\nabla \Phi & \in C([0, \infty); H^2 \cap W^{2,q}) \cap L^\infty(0, \infty; H^3 \cap W^{3,q}), \\
\n\nabla \Phi_t & \in C([0, \infty); L^2) \cap L^\infty(0, \infty; H^2).
\end{align*}
\] (1.18)

**Remark 1.1** Compared with the results about global existence of the classical solutions mentioned above (see, for instance, \[18, 23, 29\]), our conclusion does not need the smallness assumption on the oscillations between the initial data and the steady state. Although it has small energy, its oscillations could be arbitrarily large. In particular, the initial vacuum states are allowed.

**Remark 1.2** From (1.18), Sobolev’s inequality and the embedding
\[
L^2(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^2),
\]
the solution obtained in Theorem 1.1 becomes a classical one away from the initial time.
Remark 1.3 When we consider the following general slip boundary for the velocity field
\[ u \cdot n = 0, \quad \text{curl} u \times n = -Au \quad \text{on} \quad \partial \Omega, \quad (1.19) \]
and assume that the matrix \( A \) is smooth and positive semi-definite, and even if the restriction on \( A \) is relaxed to \( A \in H^3 \) and the negative eigenvalues of \( A \) (if exist) are small enough, Theorem 1.1 will still hold. This can be achieved by a similar way as in [7]. Roughly speaking, we generalize the results of [7] to the compressible Navier-Stokes-Poisson equations.

The rest of the paper is organized as follows. In Section 2, the existence theorem of the steady-state solution and the local strong solution, and some key a priori estimates needed in later analysis are collected. Section 3 is devoted to deriving the necessary a priori estimates on classical solutions which can guarantee the local classical solution to be a global classical one. In Section 4, the proof of Theorem 1.1 will be completed. In Appendix A, we list some elementary inequalities and important lemmas that we use intensively in the paper.

2 Auxiliary lemma

In this section, we recall the steady-state solution and the local strong (classical) solution of the system (1.1)-(1.4). We also derive some key a priori estimates, which will be used frequently later.

First, similar to the proof of [11,15], we have the following existence and uniqueness of the solution to (1.8).

Lemma 2.1 Assume that the smooth function \( \tilde{\rho}(x) \) satisfy \( 0 < \rho \leq \tilde{\rho}(x) \leq \bar{\rho} \). Then the problem (1.8) has a uniqueness classical solution \( (\rho_s, \Phi_s) \). Moreover

\[ \rho \leq \rho_s(x) \leq \bar{\rho}, \quad (2.1) \]
\[ \| \nabla \rho_s \|_{H^3} + \| \nabla \Phi_s \|_{H^4} \leq C, \quad (2.2) \]

where \( C \) depends only on \( a, \gamma, \tilde{\rho}(x) \) and \( \Omega \).

The following local existence theorem of classical solution of (1.1)-(1.4) can be proved in a similar manner as that in [32], base on the standard contraction mapping principle.

Lemma 2.2 Assume that the initial date \( (\rho_0, u_0) \) satisfy the conditions (1.13) and (1.15). Then there exist a positive time \( T_0 > 0 \) and a unique classical solution \( (\rho, u, \Phi) \) of the system (1.1)-(1.4) in \( \mathbb{R}^3 \times (0, T_0] \), satisfying that \( \rho \geq 0 \), and that for \( \tau \in (0, T_0) \),

\[
\begin{align*}
\rho & \in C([0, T_0]; H^2), \\
\nabla u & \in C([0, T_0]; H^1) \cap L^\infty(\tau, T_0; H^2 \cap W^{2, q}), \\
\nabla \Phi & \in C([0, T_0]; H^2 \cap W^{2, q}), \\
\nabla \Phi & \in C([0, T_0]; H^2 \cap W^{2, q}), \\
\n\Phi & \in C([0, T_0]; H^2 \cap W^{2, q}).
\end{align*}
\]
Next, we denote the effective viscous flux $F$ and the vorticity $\omega$ by

$$F \triangleq (\lambda + 2\mu)\text{div} u - (P - P_s), \quad \omega \triangleq \nabla \times u,$$  

which plays an important role in our following analysis, similarly to that for the compressible Navier-Stokes equations (see [13, 17, 21]). For $F$, $\omega$ and $\nabla u$, we give the following conclusion, which is a key to a priori estimates.

**Lemma 2.3** Let $(\rho, u, \Phi)$ be a smooth solution of (1.1)-(1.4). Then for any $p \in [2, 6]$, there exists a positive constant $C$ depending only on $p$, $\mu$, $\lambda$ and $\Omega$ such that

\[
\|\nabla F\|_{L^p} \leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p}),
\]

\[
\|\nabla \omega\|_{L^p} \leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p} + \|\nabla u\|_{L^2}),
\]

\[
\|F\|_{L^p} \leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p} + \|\nabla u\|_{L^2})^{\frac{3p-6}{2p}}(\|\nabla u\|_{L^2} + \|P - P_s\|_{L^2})^{\frac{6-p}{2p}} + C\|\nabla u\|_{L^2} + \|P - P_s\|_{L^p},
\]

Moreover,

\[
\|\nabla u\|_{L^p} \leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p})^{\frac{3p-6}{2p}}(\|\nabla u\|_{L^2} + \|P - P_s\|_{L^2})^{\frac{6-p}{2p}} + C\|\nabla u\|_{L^2} + \|P - P_s\|_{L^p}.
\]

**Proof.** By (1.1), one can find that the viscous flux $F$ satisfies

\[
\begin{aligned}
\frac{\partial F}{\partial t} &= \text{div}(\rho \mu \nabla u - \rho \nabla (\Phi - \Phi_s) - (\rho - \rho_s) \nabla \Phi_s) \quad \text{in } \Omega, \\
\frac{\partial F}{\partial n} &= \rho \mu \nabla \times \mu \quad \text{on } \partial \Omega.
\end{aligned}
\]  

It follows from Lemma 4.27 in [28], for $1 < q < +\infty$, that

\[
\|\nabla F\|_{L^q} \leq C(\|\rho \mu\|_{L^q} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^q} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^q}),
\]

so that (2.5) holds. On the other hand, one can rewrite (1.1) as

\[
\mu \nabla \times \omega = \nabla F - \rho \mu \nabla (\Phi - \Phi_s) + (\rho - \rho_s) \nabla \Phi_s.
\]

Notice that $\omega \times n = 0$ on $\partial \Omega$ and $\text{div} \omega = 0$, by Lemma A.1, we get

\[
\|\nabla \omega\|_{L^q} \leq C(\|\nabla \omega\|_{L^q} + \|\omega\|_{L^q})
\]

\[
\leq C(\|\rho \mu\|_{L^q} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^q} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^q} + \|\omega\|_{L^q}),
\]

where we have taken advantage of (2.11). By Sobolev’s inequality and (2.13), for $p \in [2, 6],

\[
\|\nabla \omega\|_{L^p} \leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p} + \|\nabla \omega\|_{L^2} + \|\omega\|_{L^2})
\]

\[
\leq C(\|\rho \mu\|_{L^p} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^p} + \|\rho - \rho_s\|\nabla \Phi_s\|_{L^p} + \|\nabla u\|_{L^2}),
\]

which implies (2.6).

Furthermore, one can deduce from (A.1) and (2.5) that for $p \in [2, 6],

\[
\|F\|_{L^p} \leq C\|F\|_{L^2}^{\frac{6-p}{2p}} \|\nabla F\|_{L^2}^{\frac{3p-6}{2p}} + C\|F\|_{L^2}
\]
Proof of any integer $k$ holds. Moreover, for any integer $k$ and $q > 1$, there exists a positive constant $C$ depending only on $k$, $q$ and $\Omega$ such that

\[ \|\nabla(\Phi - \Phi_s)\|_{W^{k+1,q}} \leq C\|\rho - \rho_s\|_{W^{k,q}}, \quad (2.19) \]

and

\[ \|\nabla(\Phi - \Phi_s)\|_{W^{k+1,q}} \leq C(||pu||_{L^q} + ||\rho u||_{W^{k,q}}). \quad (2.20) \]

Proof. From (1.1), (1.4) and (1.8), we have

\[
\begin{aligned}
\begin{cases}
\Delta(\Phi - \Phi_s) = \rho - \rho_s & \text{in } \Omega, \\
\frac{\partial(\Phi - \Phi_s)}{\partial n} = 0 & \text{on } \partial\Omega,
\end{cases}
\end{aligned}
\]

which, in view of the classical regularity theory for the Neumann problem of elliptic equation [5], leads to (2.19). From (1.1) and (2.21), we obtain

\[
\begin{aligned}
\begin{cases}
\Delta(\Phi - \Phi_s) = -\text{div}(pu) & \text{in } \Omega, \\
\frac{\partial(\Phi - \Phi_s)}{\partial n} = 0 & \text{on } \partial\Omega.
\end{cases}
\end{aligned}
\]

Using the same method as Lemma 2.3, we obtain, for $1 < q < +\infty$, that

\[ \|\nabla(\Phi - \Phi_s)\|_{L^q} \leq C\|pu\|_{L^q}, \quad (2.23) \]

Moreover, for any integer $k \geq 0$,

\[ \|\nabla(\Phi - \Phi_s)\|_{W^{k+1,q}} \leq C(||pu||_{L^q} + ||\rho u||_{W^{k,q}}). \quad (2.24) \]

The proof of Lemma 2.4 is completed. \qed
Remark 2.2 It follows from (2.23), if \( \rho \leq 2 \hat{\rho} \), that
\[
\| \nabla (\Phi - \Phi_s) \|_{L^2} \leq C \| \mu u \|_{L^2} \leq C \| \nabla u \|_{L^2},
\]
where \( C \) depends on \( \hat{\rho} \) and \( \Omega \).

3 A priori estimates

In this section, we will establish some necessary a priori bounds for smooth solutions to the problem (1.1)-(1.4) to extend the local classical solutions guaranteed by Lemma 2.2.

Let \( T > 0 \) be a fixed time and \( (\rho, u, \Phi) \) be a smooth solution to (1.1)-(1.4) on \( \Omega \times (0, T] \) with smooth initial data \( (\rho_0, u_0) \) satisfying \( u_0 \in H^\beta \) for some \( \beta \in (\frac{1}{2}, 1] \) and \( 0 \leq \rho_0 \leq 2 \hat{\rho} \). Set \( \sigma = \sigma(t) \triangleq \min\{1, t\} \), we define
\[
A_1(T) \triangleq \sup_{0 \leq t \leq T} \left( \sigma \| \nabla u \|_{L^2}^2 + \sigma^3 \| \sqrt{\rho} u \|_{L^2}^2 \right),
\]
\[
A_2(T) \triangleq \sup_{0 \leq t \leq T} \int \rho |u|^3 dx,
\]
\[
B[t_1, t_2] \triangleq \int_{t_1}^{t_2} \left( \sigma \| \sqrt{\rho} u \|_{L^2}^2 + \sigma^3 \| \nabla u \|_{L^2}^2 \right) dt,
\]
where \( 0 \leq t_1 < t_2 \leq T \) and \( \dot{v} = v_t + u \cdot \nabla v \) is the material derivative.

Now we will give the following key a priori estimates in this section, which guarantees the existence of a global classical solution of (1.1)-(1.4).

Proposition 3.1 Under the conditions of Theorem 1.1, for \( \delta_0 \triangleq \frac{2 \beta - 1}{4 \beta} \in (0, \frac{1}{4}] \), there exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \alpha, \gamma, \hat{\rho}, \beta, \Omega \) and \( M \) such that if \( (\rho, u, \Phi) \) is a smooth solution of (1.1)-(1.4) on \( \Omega \times (0, T] \) satisfying
\[
\sup_{\Omega \times [0, T]} \rho \leq 2 \hat{\rho}, \quad A_1(T) \leq 2 C_0^{1/2}, \quad A_2(\sigma(T)) \leq 2 C_0^{\delta_0},
\]
then the following estimates hold
\[
\sup_{\Omega \times [0, T]} \rho \leq 7 \hat{\rho}/4, \quad A_1(T) \leq C_0^{1/2}, \quad A_2(\sigma(T)) \leq C_0^{\delta_0},
\]
provided \( C_0 \leq \varepsilon \).

Proof. Proposition 3.1 is a consequence of the following Lemmas 3.6, 3.8.

In the following, we will use the convention that \( C \) denotes a generic positive constant depending on \( \mu, \lambda, \gamma, \alpha, \rho_s, \hat{\rho}, \beta, \Omega \) and \( M \) and use \( C(\alpha) \) to emphasize that \( C \) depends on \( \alpha \). We begin with the following standard energy estimate for \( (\rho, u, \Phi) \).

Lemma 3.2 Let \( (\rho, u, \Phi) \) be a smooth solution of (1.1)-(1.4) on \( \Omega \times (0, T] \) satisfying \( \rho \leq 2 \hat{\rho} \). Then there is a positive constant \( C \) depending only on \( \mu, \lambda, \alpha, \gamma, \hat{\rho} \) and \( \Omega \) such that
\[
\sup_{0 \leq t \leq T} \left( \frac{1}{2} \| \sqrt{\rho} u \|_{L^2}^2 + \| G(\rho) \|_{L^1} + \frac{1}{2} \| \nabla (\Phi - \Phi_s) \|_{L^2}^2 \right)
\]
\[ + \int_0^T (\lambda + 2\mu)\|\text{div} u\|^2_{L^2} + \mu\|\omega\|^2_{L^2}) dt \leq C_0, \] (3.6)

and

\[ \sup_{0 \leq t \leq T} \|\rho - \rho_s\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{L^2} dt \leq CC_0. \] (3.7)

**Proof.** Note that \(-\Delta u = -\nabla \text{div} u + \nabla \times \omega\) and (1.8), we rewrite (1.1) as

\[ \rho u + \rho u \cdot \nabla u - (\lambda + 2\mu)\text{div} u + \mu \nabla \omega + \nabla (P - P_s) = \rho \nabla (\Phi - \Phi_s) + (\rho - \rho_s) \nabla \Phi. \] (3.8)

Multiplying (3.8) by \(u\) and (1.1) by \(G'(\rho)\) respectively, integrating over \(\Omega\), summing them up, by (1.3)-(1.4) and (2.21), we have

\[ \left( \int (G(\rho) + \frac{1}{2}\rho|u|^2) dx \right) + (\lambda + 2\mu) \int (\text{div} u)^2 dx + \mu \int |\omega|^2 dx \]
\[ = \int \text{div}(\rho u)G'(\rho) dx + \int u \cdot \nabla (P - P_s) dx + \int (\rho - \rho_s) u \cdot \nabla \Phi dx + \int \rho u \cdot \nabla (\Phi - \Phi_s) dx \]
\[ = \int \rho u \cdot \nabla Q(\rho) + \int u \cdot \nabla P dx - \left( \int \frac{1}{2} |\nabla (\Phi - \Phi_s)|^2 dx \right)_t \]
\[ = - \left( \int \frac{1}{2} |\nabla (\Phi - \Phi_s)|^2 dx \right)_t, \] (3.9)

where we have used the fact \(G'(\rho) = Q(\rho) - Q(\rho_s)\) and \(Q'(\rho) = P'(\rho)/\rho\). Then integrating (3.9) over \((0, T)\) yields (3.6). Finally, it is easy to check that there exists a positive constant \(C\), depending only on \(a, \gamma\) and \(\rho_s\), such that

\[ C^{-1} (\rho - \rho_s)^2 \leq G(\rho) \leq C (\rho - \rho_s)^2, \]

which together with (A.6) and (3.6) gives (3.7). The proof of Lemma 3.2 is completed. \(\Box\)

Next we need the estimates on the material derivative of \(u\). Since \(u \cdot n = 0\) on \(\partial \Omega\), it follows that

\[ (\dot{u} + (u \cdot \nabla n) \times u^\perp) \cdot n = 0 \text{ on } \partial \Omega, \] (3.10)

where \(u^\perp \triangleq -u \times n\) on \(\partial \Omega\). In view of this observation, we review the following Poincaré-type inequality of \(\dot{u}\) (see [7], Lemma 3.2).

**Lemma 3.3** If \((\rho, u, \Phi)\) is a smooth solution of (1.1) with slip condition (1.3)-(1.4), then there exists a positive constant \(C\) depending only on \(\Omega\) such that

\[ \|\dot{u}\|_{L^6} \leq C(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|^2_{L^2}), \] (3.11)
\[ \|\nabla \dot{u}\|_{L^2} \leq C(\|\text{div} \dot{u}\|_{L^2} + \|\text{curl} \dot{u}\|_{L^2} + \|\nabla u\|^2_{L^2}). \] (3.12)

Then, we give the estimate of \(A_1(T)\) and \(B[0, T]\).

**Lemma 3.4** Let \((\rho, u, \Phi)\) be a smooth solution of (1.1)-(1.4) satisfying (3.4). Then there is a positive constant \(C\) depending only on \(\mu, \lambda, \gamma, \rho_s, \rho, \dot{\rho}\) and \(\Omega\) such that

\[ A_1(T) + B[0, T] \leq CC_0 + C \int_0^T (\sigma \|\nabla u\|^2_{L^2} + \sigma^2 \|\nabla u\|^4_{L^2}) dt. \] (3.13)
Similarly, we conclude that where in last term on the right-hand side of (3.16) we have used the fact (1.10) on $\partial \Omega$ and we conclude that

$$
\int_{\partial \Omega} \sigma^m (P - P_s) u \cdot \nabla u \cdot nds \leq C \int_{\partial \Omega} \sigma^m |u|^2 ds \leq C \sigma^m ||u||^2_{L^2}.
$$

Hence,

$$
I_1 \leq \left( \int \sigma^m (P - P_s) \text{div} u dx \right)_t + C \sigma^m ||\nabla u||^2_{L^2} + C m \sigma^{m-1} ||\rho - \rho_s||^2_{L^2}.
$$

Similarly,

$$
I_2 \leq (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} (u \cdot \nabla u \cdot n) ds - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\nabla u)^2 dx \right)_t
$$

$$
C \sigma^m ||u||^3_{L^3} + C m \sigma^{m-1} ||\nabla u||^2_{L^2}.
$$

For the first term on the righthand side of (3.17), by (2.1) and Lemma 2.3 we obtain

$$
(\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} (u \cdot \nabla u \cdot n) ds
$$

$$
= - \int_{\partial \Omega} \sigma^m F u \cdot \nabla u \cdot nds - \int_{\partial \Omega} \sigma^m (P - P_s) u \cdot \nabla u \cdot nds
$$

$$
\leq C \sigma^m (||\nabla F||_{L^2} ||u||^2_{L^4} + ||F||_{L^6} ||u||_{L^6} ||\nabla u||_{L^2} + ||F||_{L^2} ||u||^2_{L^4}) + C \sigma^m ||\nabla u||^2_{L^2}.
$$
\[ \leq \frac{1}{2} \sigma^m \| \rho \dot{u} \|_{L^2}^2 + C \sigma^m \| \nabla u \|_{L^2}^2 (\| \nabla u \|_{L^2}^2 + 1), \]  
(3.20)

Therefore,
\[ I_2 \leq -\frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\text{div} u)^2 dx \right) + C \sigma^m \| \nabla u \|_{L^3}^3 + \frac{1}{4} \sigma^m \| \rho \dot{u} \|_{L^2}^2 \]
\[ + C \sigma^m (\| \nabla u \|_{L^2}^2 + 1) \| \nabla u \|_{L^2}^2. \]
(3.21)

Next, by (1.3), a straightforward computation shows that
\[ I_3 = -\frac{\mu}{2} \left( \int \sigma^m |\omega|^2 dx \right) + \frac{\mu m}{2} \sigma^m \| \omega \|_{H^1}^2 - \mu \int \sigma^m \omega \cdot \text{curl}(u \cdot \nabla u) dx \]
\[ \leq -\frac{\mu}{2} \left( \int \sigma^m |\omega|^2 dx \right) + C \sigma^m \| \nabla u \|_{L^2}^2 + C \sigma^m \| \nabla u \|_{L^3}^3. \]
(3.22)

In view of (2.19) and (2.25), it follows that
\[ I_4 = \left( \int \sigma^m \rho u \cdot \nabla (\Phi - \Phi_s) dx \right) - m \sigma^m \| \nabla (\Phi - \Phi_s) dx \]
\[ - \int \sigma^m \rho u \cdot \nabla (\Phi - \Phi_s)_t dx - \int \sigma^m \rho u \cdot \nabla^2 (\Phi - \Phi_s) \cdot u dx \]
\[ \leq \left( \int \sigma^m \rho u \cdot \nabla (\Phi - \Phi_s) dx \right) + C m \sigma^m \| \nabla (\Phi - \Phi_s) \|^2_{L^2} + C \sigma^m \| \nabla u \|_{L^2}^2, \]
(3.23)

and similarly,
\[ I_5 \leq \left( \int \sigma^m (\rho - \rho_s) u \cdot \nabla \Phi_s dx \right) + C m \sigma^m \| \rho - \rho_s \|^2_{L^2} + C \sigma^m \| \nabla u \|_{L^2}^2, \]
(3.24)

Making use of the results (3.18), (3.21), (3.22) and (3.23), it follows from (3.14) that
\[ \left( \lambda + 2\mu \right) \int \sigma^m (\text{div} u)^2 dx + \mu \int \sigma^m |\omega|^2 dx \right) + \int \sigma^m \rho |\dot{u}|^2 dx \]
\[ \leq \left( \int \sigma^m \left( (P - P_s) \text{div} u + (\rho - \rho_s) u \cdot \nabla \Phi_s + \rho u \cdot \nabla (\Phi - \Phi_s) \right) dx \right) + C \sigma^m \| \nabla u \|_{L^3}^3 \]
\[ + C \sigma^m \| \nabla u \|_{L^2}^2 (1 + \| \nabla u \|_{L^2}^2) + C m \sigma^m \| \nabla (\Phi - \Phi_s) \|^2_{L^2} + \| \rho - \rho_s \|^2_{L^2}, \]
(3.25)

integrating over \((0, T]\), choosing \(m = 1\), by (A.6), (3.4), Lemma 3.2 and Young’s inequality, we conclude that
\[ \sup_{0 \leq t < T} \sigma \| \nabla u \|_{L^2}^2 + \int_0^T \sigma \| \sqrt{\rho} \dot{u} \|_{L^2}^2 dt \leq CC_0 + C \int_0^T \sigma \| \nabla u \|_{L^3}^3 dt. \]
(3.26)

Next, operating \(\sigma^m \dot{u} [\partial / \partial t + \text{div}(u \cdot \cdot)] \) to (3.8)\(^j\), summing with respect to \(j\), and integrating over \(\Omega\), together with (1.13.1), we get
\[ \left( \frac{\sigma^m}{2} \int \rho |\dot{u}|^2 dx \right) - \frac{m}{2} \sigma^m \| \nabla u \|_{L^3}^3 \int \rho |\dot{u}|^2 dx \]
\[ = \int \sigma^m (\dot{u} \cdot \nabla F_t + \dot{u}^j \text{div}(u \partial_j F)) dx \]
+ \mu \int \sigma^m (\nabla \times \omega) + \nabla \times \omega^i) dx \\
+ \int \sigma^m (\partial_t \Phi_s (\rho - \rho_s) + \nabla \times \omega^i) \partial_t \Phi_s u) dx \\
+ \int \sigma^m (\partial_t (\rho \Phi_s - \Phi_s) t + \nabla \times \omega^i \partial_t (\rho \Phi_s u)) dx = \sum_{i=1}^4 J_i. \quad (3.27)

Let us estimate $J_1$, $J_2$, $J_3$ and $J_4$. By (1.3) and (3.15), a direct computation yields

$$J_1 = \int_{\partial \Omega} \sigma^m F_t \nabla u \cdot n ds - (\lambda + 2\mu) \int \sigma^m \div(\nabla u) + \lambda + 2\mu \int \sigma^m \div(\nabla u) \nabla u \cdot \nabla u dx$$

$$+ \gamma \int \sigma^m P_{\div(\nabla u)} + \sigma^m \div u \cdot \nabla F dx - \int \sigma^m \div u \cdot \nabla u \cdot \nabla F dx$$

$$- \int \sigma^m \div u \cdot \nabla P_s dx$$

$$\leq \int_{\partial \Omega} \sigma^m F_t \nabla u \cdot n ds - (\lambda + 2\mu) \int \sigma^m \div(\nabla u) + \lambda + 2\mu \int \sigma^m \div(\nabla u) \nabla u \cdot \nabla u dx$$

$$+ \frac{\delta}{12} \sigma^m \div(\nabla u) \nabla u \cdot \nabla F dx + C \sigma^m \|\nabla u\|^4_{L^2} + C \sigma^m (\|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \quad (3.28)$$

where in the first equality we have used

$$F_t = (2\mu + \lambda) \div u - (2\mu + \lambda) \nabla u \cdot u - \nabla F + \gamma P_{\div(\nabla u)} + u \cdot \nabla P_s.$$

For the first term on the right-hand side of (3.28), we have

$$\int_{\partial \Omega} \sigma^m F_t \nabla u \cdot n ds = - \int_{\partial \Omega} \sigma^m F_t (u \cdot \nabla n \cdot u) ds$$

$$= - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right) + m \sigma^{m-1} \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds$$

$$+ \int_{\partial \Omega} \sigma^m (F_t \nabla n \cdot u + F \nabla n \cdot u) ds$$

$$- \int_{\partial \Omega} \sigma^m (F_t \nabla n \cdot u + F \cdot \nabla n \cdot u) ds$$

$$\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right) + C m \sigma^{m-1} \|\nabla u\|^2_{L^2}$$

$$+ C \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla F\|^2_{H^1} + C \sigma^m \|\nabla u\|^4_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} \quad (3.29)$$

Combining these estimates of the boundary terms with (3.28), using (2.5), (3.11) and (3.12) gives

$$J_1 \leq - (\lambda + 2\mu) \int \sigma^m (\div(\nabla u)) dx - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)$$

$$+ \frac{\delta}{4} \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla u\|^4_{L^2} + C \sigma^m \|\sqrt{\rho} \nabla u\|^2_{L^2}$$

$$+ C \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla u\|^4_{L^2}$$

$$+ C m \sigma^{m-1} \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} \quad (3.30)$$

Next, by $\omega_t = \curl u - u \cdot \nabla \omega - \nabla u^i \times \partial_i u$ and a straightforward calculation leads to

$$J_2 = -\mu \int \sigma^m \curl u^i dx + \mu \int \sigma^m (\omega \cdot \nabla u^i) \cdot \nabla u^i dx$$
\[
- \mu \int \sigma^m \text{curl}\hat{u} \cdot (\nabla u^i \times \nabla_i u) dx - \mu \int \sigma^m \text{div}\omega \cdot \text{curl}\hat{u} dx \\
\leq -\mu \int \sigma^m |\text{curl}\hat{u}|^2 dx + \frac{\delta}{4} \sigma^m \|
abla \hat{u}\|_{L^2}^2 + C\sigma^m \|
abla u\|_{L^4}^4. \tag{3.31}
\]

In view of (2.2) and (3.11), we obtain that
\[
J_3 = \int \sigma^m ((\rho - \rho_s)u \cdot \nabla^2 \Phi_s \cdot \hat{u} dx - \int \sigma^m \hat{u} \cdot \nabla \Phi_s \text{div}(\rho u) dx \\
\leq C\sigma^m (\|
abla \hat{u}\|_{L^2} \|
abla \Phi_s\| \rho - \rho_s\|_{L^2} + \|
abla \hat{u}\|_{L^2} \|
abla u\|_{L^2} + \|
abla \hat{u}\|_{L^2} \|
abla u\|_{L^2}) \|
abla \Phi_s\|_{H^2} \\
\leq \frac{\delta}{4} \sigma^m \|
abla \hat{u}\|_{L^2}^2 + C\sigma^m \|
abla u\|_{L^2}^2 + C\sigma^m \|
abla u\|_{L^2}^4, \tag{3.32}
\]

and similarly, by (3.11), (2.19), (2.25) and (2.23), we have
\[
J_4 = \int \sigma^m \rho \hat{u} \cdot \nabla (\Phi - \Phi_s) dx + \int \sigma^m \rho u \cdot \nabla^2 (\Phi - \Phi_s) \cdot \hat{u} dx \\
\leq C\sigma^m (\|
abla \hat{u}\|_{L^2} \|
abla (\Phi - \Phi_s)\| \rho - \rho_s\|_{L^2} + \|
abla \hat{u}\|_{L^2} \|
abla u\|_{L^2}) \|
abla (\Phi - \Phi_s)\|_{H^2} \\
\leq \frac{\delta}{4} \sigma^m \|
abla \hat{u}\|_{L^2}^2 + C\sigma^m (\|
abla \hat{u}\|_{L^2} + \|
abla u\|_{L^2}^4). \tag{3.33}
\]

Combining (3.30), (3.31), (3.32) and (3.33) with (3.27), by (3.12) and choosing \(\delta\) small enough, we have
\[
(\sigma^m \|
abla \hat{u}\|_{L^2}^2)_t + \sigma^m \|
abla \hat{u}\|_{L^2}^2 \\
\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) Fds \right)_t + C\sigma^m \|
abla u\|_{L^4}^4 + C\sigma^m \|
abla \hat{u}\|_{L^2}^2 \\
+ C\sigma^m \|
abla \hat{u}\|_{L^2} (\|
abla u\|_{L^2}^2 + \|
abla u\|_{L^2}^4) + C\sigma^m \|
abla u\|_{L^2}^4 (1 + \|
abla u\|_{L^2}^2 + \|
abla u\|_{L^2}^4) \\
+ C\sigma^{m-1} \sigma^m (\|
abla \hat{u}\|_{L^2}^2 + \|
abla u\|_{L^2}^2) \tag{3.34}
\]

For the boundary term in the right-hand side of (3.34), from Lemma 2.3, we have
\[
\int_{\partial \Omega} (u \cdot n \cdot u) Fds \leq C \|
abla u\|_{L^2}^2 \|F\|_{H^1} \leq \frac{1}{2} \|
abla \hat{u}\|_{L^2}^2 + C(\|
abla u\|_{L^2}^2 + \|
abla u\|_{L^2}^4). \tag{3.35}
\]

Now integrating (3.34) with \(m = 3\) over \((0, T)\), using (3.4), (3.26) and (3.35), yields
\[
\sup_{0 \leq t \leq T} \sigma^3 \|
abla \hat{u}\|_{L^2}^2 + \int_0^T \sigma^3 \|
abla \hat{u}\|_{L^2}^2 dt \leq CC_0 + C \int_0^T (\sigma \|
abla \hat{u}\|_{L^2}^2 + \sigma^3 \|
abla u\|_{L^4}^4) dt. \tag{3.36}
\]

Combining (3.26) with (3.36), we immediately obtain (3.13) and complete the proof of Lemma 3.4. \(\square\)

**Lemma 3.5** Assume that \((\rho, u, \Phi)\) is a smooth solution of (1.1)-(1.4) satisfying (3.1) and the initial data condition (1.14), then there exist positive constants \(C, \varepsilon_1\) depending only on \(\mu, \lambda, \gamma, a, \rho_s, \hat{\rho}, \beta, \Omega\) and \(M\) such that
\[
\sup_{0 \leq t \leq T} t^{1-s} \|
abla u\|_{L^2}^2 + \int_0^\sigma(T) t^{1-s} \int_0^t \rho |\hat{u}|^2 dx dt \leq C(\hat{\rho}, M), \tag{3.37}
\]
\[
\sup_{0 \leq t \leq T} t^{2-s} \int_0^t \rho |\hat{u}|^2 dx + \int_0^{\sigma(T)} t^{2-s} \|
abla \hat{u}\|_{L^2}^2 dx dt \leq C(\hat{\rho}, M), \tag{3.38}
\]
provide that \(C_0 < \varepsilon_1\).
Proof. Suppose \( w_1(x,t) \), \( w_2(x,t) \) and \( w_3(x,t) \) solve problems

\[
\begin{aligned}
\begin{cases}
Lw_1 = 0 & \text{in } \Omega, \\
w_1(x,0) = w_{10}(x) & \text{in } \Omega, \\
w_1 \cdot n = 0 \text{ and } \text{curl} w_1 \times n = 0 & \text{on } \partial \Omega, 
\end{cases}
\end{aligned}
\quad (3.39)
\]

\[
\begin{aligned}
\begin{cases}
Lw_2 = -\nabla (P - P_s) & \text{in } \Omega, \\
w_2(x,0) = 0 & \text{in } \Omega, \\
w_2 \cdot n = 0 \text{ and } \text{curl} w_2 \times n = 0 & \text{on } \partial \Omega, 
\end{cases}
\end{aligned}
\quad (3.40)
\]

and

\[
\begin{aligned}
\begin{cases}
Lw_3 = \rho \nabla (\Phi - \Phi_s) + (\rho - \rho_s)\nabla \Phi_s & \text{in } \Omega, \\
w_3(x,0) = 0 & \text{in } \Omega, \\
w_3 \cdot n = 0 \text{ and } \text{curl} w_3 \times n = 0 & \text{on } \partial \Omega, 
\end{cases}
\end{aligned}
\quad (3.41)
\]

where \( Lf \triangleq \rho \dot{f} - \mu \Delta f - (\lambda + \mu) \nabla \text{div} f \).

A similar way as for the proof of (3.36) shows that

\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_1|^2 dx + \int_0^{\sigma(T)} \int |\nabla w_1|^2 dx dt \leq C \int |w_{10}|^2 dx, \quad (3.42)
\]

\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_2|^2 dx + \int_0^{\sigma(T)} \int |\nabla w_2|^2 dx dt \leq C C_0, \quad (3.43)
\]

and

\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_3|^2 dx + \int_0^{\sigma(T)} \int |\nabla w_3|^2 dx dt \leq C C_0. \quad (3.44)
\]

Multiplying (3.39) by \( w_{1t} \) and integrating over \( \Omega \), by (3.4), Sobolev’s and Young’s inequalities, we obtain

\[
\left( \frac{\lambda + 2\mu}{2} \int (\text{div} w_1)^2 dx + \frac{\mu}{2} \int |\text{curl} w_1|^2 dx \right) + \int \rho |\dot{w}_1|^2 dx
\leq C \| \sqrt{\rho} \dot{w}_1 \|_{L^2} \| \rho^{\frac{1}{2}} w_1 \|_{L^6} \| \nabla w_1 \|_{L^6} \leq C_1 C_0^{\frac{5}{6}} (\| \sqrt{\rho} \dot{w}_1 \|_{L^2}^2 + \| \nabla w_1 \|_{L^2}^2), \quad (3.45)
\]

since it follows from (3.39), with the similar methods to obtain (2.11) as in Lemma (2.3) that

\[
\| \nabla w_1 \|_{L^6} \leq C \| w_1 \|_{W^{2,6}} \leq C (\| \rho \dot{w}_1 \|_{L^2} + \| \nabla w_1 \|_{L^2}). \quad (3.46)
\]

Together (3.45) with (3.42), and by Gronwall’s inequality and Lemma A.3 it yields

\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_1|^2 dx dt \leq C \| \nabla w_{10} \|_{L^2}^2, \quad (3.47)
\]

and

\[
\sup_{0 \leq t \leq \sigma(T)} t \| \nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} t \int \rho |\dot{w}_1|^2 dx dt \leq C \| w_{10} \|_{L^2}^2, \quad (3.48)
\]
provided $C_0 < \hat{c}_1 \triangleq (2C_1)^{-\frac{3}{8}}$. Since the solution operator $w_{10} \mapsto w_1(\cdot,t)$ is linear, by the standard Stein-Weiss interpolation argument [3], one can deduce from (3.47) and (3.48) that for any $\theta \in [0,1]$,

$$
\sup_{0 \leq t \leq \sigma(T)} t^{-\theta} \| \nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |\dot{w}_1|^2 dx dt \leq C \|w_{10}\|_{H^\theta}^2,
$$

(3.49)

with a uniform constant $C$ independent of $\theta$.

Multiplying (3.40) by $w_{2t}$ and integrating over $\Omega$ give that

$$
\left( \frac{\lambda + 2\mu}{2} \int (\text{div} w_2)^2 dx + \frac{\mu}{2} \int |\text{curl} w_2|^2 dx - \int (P - P_s) \text{div} w_2 dx \right)_t + \int \rho |\dot{w}_2|^2 dx
$$

$$
= \int \rho \dot{w}_2 \cdot (u \cdot \nabla w_2) dx - \frac{\lambda + 2\mu}{2} \int (P - P_s)(F_{w_2} \text{div} u + \nabla F_{w_2} \cdot u) dx
$$

$$
- \frac{1}{2} \left( \int (P - P_s)^2 \text{div} u dx + \gamma \int \text{div} \text{div} w_2 dx + \int u \cdot \nabla P_s \text{div} w_2 dx \right)
$$

$$
\leq C(\|\sqrt{\rho} \dot{w}_2\|_{L^2}^2 + \frac{1}{4} \|\sqrt{\rho} \dot{w}_2\|_{L^2}^2 + C(\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|F_{w_2}\|_{L^2} + \|P - P_s\|_{L^6}^2),
$$

(3.50)

where we have utilized (3.41), (3.15) and the following estimates

$$
\|\nabla F_{w_2}\|_{L^2} \leq C(\|\rho \dot{w}_2\|_{L^2} + \|\nabla w_2\|_{L^2}),
$$

(3.52)

$$
\|\nabla w_2\|_{L^6} \leq C(\|\rho \dot{w}_2\|_{L^2} + \|\nabla w_2\|_{L^2} + \|P - P\|_{L^6}),
$$

(3.53)

where $F_{w_2} = (\lambda + 2\mu) \text{div} w_2 - (P - P)$. As a result,

$$
\left( \frac{\lambda + 2\mu}{2} \|\text{div} w_2\|_{L^2}^2 + \mu \|\text{curl} w_2\|_{L^2}^2 - 2 \int (P - P_s) \text{div} w_2 dx \right)_t + \int \rho |\dot{w}_2|^2 dx
$$

$$
\leq C \left( \|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P - P_s\|_{L^2}^2 \right),
$$

(3.54)

provide that $C_0 < \hat{c}_2 \triangleq (4C_2)^{-\frac{3}{8}}$. Integrating (3.54) over $(0,\sigma(T))$, with (3.43) and Lemmas 3.3, 3.2 one has

$$
\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_2\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_2|^2 dx dt \leq C C_0^{1/3}.
$$

(3.55)

Similarly, multiplying (3.41) by $w_{3t}$ and integrating over $\Omega$ give that

$$
\left( \frac{\lambda + 2\mu}{2} \int (\text{div} w_3)^2 dx + \frac{\mu}{2} \int |\text{curl} w_3|^2 dx \right)_t + \int \rho |\dot{w}_3|^2 dx
$$

$$
= \left( \int (\rho - \rho_s) w_3 \cdot \nabla \Phi_s dx \right)_t + \int \text{div}((\rho - \rho_s) u) w_3 \cdot \nabla \Phi_s dx
$$

$$
+ \int \text{div}(\rho_s u) w_3 \cdot \nabla \Phi_s dx + \int \rho_s w_3 \cdot \nabla (\Phi - \Phi_s) dx
$$

$$
+ \int \rho_s u \cdot \nabla w_3 \cdot \nabla (\Phi - \Phi_s) dx + \int \rho u \cdot \nabla w_3 \cdot \dot{w}_3 dx
$$

where
\[ \Delta \triangleq \frac{d}{dt}L_0 + \sum_{i=1}^{5} L_i. \tag{3.56} \]

By (2.2) and (A.1), we have

\[ L_0 \leq C\|\nabla \Phi_s\|_{L^1}\|\rho - \rho_s\|_{L^2}\|w_3\|_{L^6} \leq \frac{\mu}{4}\|\nabla w_3\|_{L^2}^2 + CC_0. \tag{3.57} \]

Using the similar methods to obtain (2.11) as in Lemma (2.3), it follows from (3.41) that

\[ \|\nabla w_3\|_{L^6} \leq C(\|\rho \dot{w}_3\|_{L^2} + \|(\rho - \bar{\rho})\nabla \Phi_s\|_{L^2} + \|\rho \nabla (\Phi - \Phi_s)\|_{L^2} + \|\rho w_3\|_{L^2}) \]

\[ \leq C(\|\rho \dot{w}_3\|_{L^2} + \|\rho w_3\|_{L^2} + C\sigma^{1/2}_0). \tag{3.58} \]

Then, by (2.2), (3.6) and (3.58), a directly computation yields

\[ L_1 = -\int (\rho - \rho_s)(u \cdot \nabla w_3 \cdot \nabla \Phi_s + u \cdot \nabla^2 \Phi_s \cdot w_3) dx \]

\[ \leq C(\|\rho - \rho_s\|_{L^2}\|u\|_{L^6}(\|w_3\|_{L^6} + \|\nabla w_3\|_{L^6})\|\nabla \Phi_s\|_{H^2}) \]

\[ \leq \frac{1}{4}\|\sqrt{\rho \dot{w}_3}\|_{L^2} + C(\|\nabla u\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2 + C_0). \tag{3.59} \]

Similarly, using (2.2), (2.19), (2.23), (3.6) and (3.4), we obtain

\[ \sum_{i=2}^{5} L_i \leq C(\|\nabla u\|_{L^2}\|\nabla \Phi_s\|_{L^3} + \|\nabla \rho_s\|_{L^2}\|\nabla \Phi_s\|_{L^6}\|u\|_{L^6})\|w_3\|_{L^6} \]

\[ + C\|\sqrt{\rho \dot{w}_3}\|_{L^2}\|\nabla (\Phi - \Phi_s)\|_{L^2} \]

\[ + C\|\rho^{1/3}u\|_{L^3}\|\nabla \Phi_s\|_{L^6}(\|\nabla (\Phi - \Phi_s)\|_{L^2} + \|\sqrt{\rho \dot{w}_3}\|_{L^2}) \]

\[ \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2 + C_0) + \left(\frac{1}{4} + CC_0^{1/3}\right)\|\sqrt{\rho \dot{w}_3}\|_{L^2}. \tag{3.60} \]

Putting (3.57), (3.59) and (3.60) into (3.56) yields

\[ \left(\|\nabla w_3\|_{L^2}^2\right) + \int \rho |\dot{w}_3|^2 dx \]

\[ \leq C C_0^{1/3}\|\sqrt{\rho \dot{w}_3}\|_{L^2} + C(\|\nabla w_3\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2 + C_0). \tag{3.61} \]

Thus, integrating it over (0, \sigma(T)], choosing \( C_0 \leq \varepsilon_3 = (2C_3)^{-3/b_0} \), we obtain

\[ \sup_{0 \leq t \leq \sigma(T)} \|\nabla w_3\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_3|^2 dx dt \leq CC_0. \tag{3.62} \]

Now let \( w_1 = u_0 \), so that \( w_1 + w_2 + w_3 = u \), we derive (3.37) from (3.49), (3.55) and (3.62) directly under certain condition \( C_0 < \varepsilon_1 \leq \min\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3\} \).

In order to prove (3.38), taking \( m = 2 - s \) in (3.34), and integrating over (0, \sigma(T)] instead of (0, T], in a similar way as we have gotten (3.36), we obtain

\[ \sup_{0 \leq t \leq \sigma(T)} \sigma^{2-s}\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \int_0^{\sigma(T)} \sigma^{2-s}\|\nabla \dot{u}\|_{L^2}^2 dt \]

\[ \leq C \int_0^{\sigma(T)} \sigma^{2-s}\|\nabla u\|_{L^4}^2 dt + C(\hat{\rho}, M). \tag{3.63} \]
where we have taken advantage of (3.37). By (3.4) and (3.37), we have
\[
\int_0^{\sigma(T)} \sigma^{2-s} \|\nabla u\|_{L^3}^2 dt
\]
\[
\leq C \int_0^{\sigma(T)} t^{2-s} \|\sqrt{\rho} \dot{u}\|_{L^2} \|\nabla u\|_{L^2} dt + C \int_0^{\sigma(T)} t^{2-s} \|\nabla u\|_{L^2}^2 dt + C
\]
\[
\leq C \int_0^{\sigma(T)} \frac{t^{2-s}}{2} \bigg( (t^{1-s} \|\nabla u\|_{L^2}^2)^{\frac{1}{2}} (t^{2-s} \|\sqrt{\rho} \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \bigg) dt + C
\]
\[
\leq C(\hat{\rho}, M) \left( \sup_{0 \leq t \leq \sigma(T)} \frac{t^{2-s}}{2} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^{\frac{1}{2}} + C,
\]
(3.64) which together with (3.63) yields (3.38). The proof of Lemma 3.5 is completed. 

**Lemma 3.6** If \((\rho, u, \Phi)\) is a smooth solution of (1.1) - (1.4) satisfying (3.4) and the initial data condition (1.11), then there exists a positive constant \(\varepsilon_2\) depending only on \(\mu, \lambda, \gamma, a, \hat{\rho}, \hat{\rho}, \beta, \Omega,\) and \(M\) such that
\[
A_3(\sigma(T)) \leq C_0^6 \varepsilon_2, \quad (3.65)
\]
provided \(C_0 < \varepsilon_2\).

**Proof.** Multiplying (1.1) by \(3|u|u\), and integrating the resulting equation over \(\Omega\), lead to
\[
\left( \int \rho|u|^3 dx \right)_t + 3(\lambda + 2\mu) \int \text{div} u \text{div}(|u|u) dx + 3\mu \int \omega \cdot \text{curl}(|u|u) dx
\]
\[
- 3 \int (P - P_s) \text{div}(|u|u) dx - 3 \int (\rho \nabla (\Phi - \Phi_s) + (\rho - \rho_s) \nabla \Phi_s) \cdot |u|u dx = 0. \quad (3.66)
\]
By (2.2), (2.9), (2.19), (3.4) and (3.6), it follows that
\[
\left( \int \rho|u|^3 dx \right)_t
\]
\[
\leq C \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^3} + C \|P - P_s\|_{L^3} \|u\|_{L^6} \|\nabla u\|_{L^2}
\]
\[
+ C \|\rho u\|_{L^3}^2 \|\nabla (\Phi - \Phi_s)\|_{L^3} + \|\rho - \rho_s\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \Phi_s\|_{L^6}
\]
\[
\leq C \|\nabla u\|_{L^2}^2 \|\rho \dot{u}\|_{L^2}^2 + C \|\nabla u\|_{L^2}^3 + CC_0^{\frac{5}{4}} \|\nabla u\|_{L^2}^2
\]
\[
+ CC_0^{\frac{1}{4}} \|\nabla u\|_{L^2}^2 + CC_0^{2\delta_0} + \frac{1}{2}. \quad (3.67)
\]
Hence, integrating (3.67) over \((0, \sigma(T))\) and using (3.4), we get
\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho|u|^3 dx
\]
\[
\leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^\frac{5}{2} \|\rho \dot{u}\|_{L^2}^\frac{1}{2} dt + C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^3 dt
\]
\[
+ CC_0^{\frac{1}{2}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^\frac{5}{2} dt + CC_0^{2\delta_0} + \int \rho_0 |u_0|^3 dx
\]
and Lemma 3.5, one can check that
\[ \epsilon \leq C \left( t^{1-\beta} \| \rho u \|_{L^2}^2 \right) \frac{1}{2} \left( t^{1-\beta} \| \nabla u \|_{L^2}^2 \right) \frac{5-8\beta}{4} \| \nabla u \|_{L^2}^{\frac{3}{2}} \frac{\beta}{2} dt \]
\[ + C \int_0^{\sigma(T)} (t^{1-\beta} \| \rho u \|_{L^2}^2) \frac{1}{2} \| \nabla u \|_{L^2}^2 t \frac{\beta}{2} dt \]
\[ + CC_0^2 \int_0^{\sigma(T)} (t^{1-\beta} \| \nabla u \|_{L^2}^2) \frac{5}{2} t^{\frac{5(\beta-1)}{4}} dt + CC_0^{2\delta_0} + \int \rho_0 |u_0|^3 dx \]
\[ \leq CC_0^{2\delta_0} + \int \rho_0 |u_0|^3 dx \leq C_4 C_0^{2\delta_0}, \] (3.68)
where we have used the fact \( \delta_0 = \frac{2\beta-1}{4\beta} \in (0, \frac{1}{4}) \), \( \beta \in (1/2, 1] \) and
\[ \int \rho_0 |u_0|^3 dx \leq C \rho_0^\frac{1}{2} u_0 L^2 \| u_0 \|^{3(2\beta-1)/2\beta} \| u_0 \|^{3/2\beta} \leq CC_0^{3\delta_0}. \] (3.69)

Finally, set \( \epsilon_2 \triangleq \min \{ \epsilon_1, (C_4)^{-\frac{1}{\gamma_0}} \} \), we get (3.65) and the proof of Lemma 3.6 is completed.

Lemma 3.7 Let \((\rho, u, \Phi)\) be a smooth solution of (1.1)–(1.4) on \( \Omega \times (0, T) \) satisfying (3.4) and the initial data condition \( \| u_0 \|_{H^\beta} \leq M \) in (1.14). Then there exists a positive constant \( C \) and \( \epsilon_3 \) depending only on \( \mu, \lambda, \gamma, a, \beta, \rho, \bar{\rho}, M \) and \( \Omega \) such that
\[ A_1(\sigma(T)) + B[0, \sigma(T)] \leq CC_0^{\frac{3}{\gamma}}, \]
\[ A_1(T) \leq C_0^\frac{3}{\gamma}, \] (3.70)
provided \( C_0 \leq \epsilon_3 \). Furthermore, if \( T > 1 \), then for any \( 1 \leq t_1 < t_2 \leq T \),
\[ B[t_1, t_2] \leq CC_0^{3/4} + CC_0(t_2 - t_1). \] (3.72)

Proof. The proof proceeds in two steps. First, for \( t \in (0, \sigma(T)) \), \( T \geq 1 \), by (2.9), (3.4) and Lemma 3.5, one can check that
\[ \int_0^{\sigma(T)} \sigma \| \nabla u \|_{L^2}^3 dt + \int_0^{\sigma(T)} \sigma^3 \| \nabla u \|_{L^4}^4 dt \]
\[ \leq C \int_0^{\sigma(T)} t \left( \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \| \nabla u \|_{L^2}^\frac{3}{2} + \| \nabla u \|_{L^2}^3 + C_0^\frac{3}{2} \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} + C_0^\frac{3}{2} \| \nabla u \|_{L^2}^\frac{3}{2} + C_0 \right) dt \]
\[ + C \int_0^{\sigma(T)} t^\frac{3}{2} \left( \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \| \nabla u \|_{L^2}^\frac{3}{2} + \| \nabla u \|_{L^2}^3 + C_0^\frac{1}{2} \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} + C_0 \| \nabla u \|_{L^2} + C_0 \right) dt \]
\[ \leq C \int_0^{\sigma(T)} t^\frac{\beta-2}{1-\beta} \left( \| \nabla u \|_{L^2}^2 \right) \frac{1}{2} \left( t^{2-\beta} \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \right) + CC_0^\frac{3}{4} \int_0^{\sigma(T)} t^\frac{\beta-1}{1-\beta} \| \sqrt{\rho} \|_{L^2}^2 dt \]
\[ + C \int_0^{\sigma(T)} t^\frac{\beta}{1-\beta} \left( t^3 \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \right) \frac{1}{2} \left( t^{2-\beta} \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \right) dt + CC_0^\frac{3}{4} \]
\[ + CC_0^\frac{3}{4} \int_0^{\sigma(T)} t^\frac{\beta}{1-\beta} \left( t^3 \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \right) \frac{1}{2} \left( t^{2-\beta} \| \sqrt{\rho} \|_{L^2}^\frac{3}{2} \right) dt + CC_0^\frac{3}{4} \]
\[ \leq CC_0^\frac{3}{4}, \] (3.73)
which, along with (3.13), gives (3.70).
Second, for \( t \in (\sigma(T), T), T \geq 1 \), we shall show that (3.71) holds on each small time-interval. It should be pointed out that (3.70) implies that on \( t = 1 \),
\[
\|\nabla u(1)\|_{L^2}^2 + \|\sqrt{\rho \dot{u}(1)}\|_{L^2}^2 \leq CC_0^{3/4}.
\]
Integrating (3.25) and (3.34) over \([1, 3]\), summing them up, from (3.4), (2.9) and (3.70), we can obtain that there exists a positive \( C \), independent of \( T \), such that
\[
\sup_{1 \leq t \leq 3} (\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho \dot{u}}\|_{L^2}^2) + B[1, 3]
\leq \|\nabla u(1)\|_{L^2}^2 + \|\sqrt{\rho \dot{u}(1)}\|_{L^2}^2 + CC_0 + C \int_1^3 \|\nabla u\|_{L^2}^2 dt + C \int_1^3 \|\nabla u\|_{L^2}^2 dt
\leq CC_0^{3/4} + C \int_1^3 \|\sqrt{\rho \dot{u}}\|_{L^2}^2 \|\nabla u\|_{L^2} dt + C \int_1^3 \|\sqrt{\rho \dot{u}}\|_{L^2}^2 \|\nabla u\|_{L^2} dt
\leq CC_0^{3/4},
\]
(3.75)
For \( T \geq 3 \), Let \([T]\) be the largest integer less or equal to \( T \). For each integer \( k = 2, 3, \ldots, [T] - 1 \), we introduce the function \( \sigma_k(t) = \sigma(t + 1 - k) = \min\{1, t + 1 - k\} \). Then, for \( t \in [k - 1, k + 1] \), by replacing \( \sigma(t) \) with \( \sigma_k(t) \) and repeating the process of Lemma 3.4, we obtain that (3.25) and (3.34) still holds with \( \sigma_k(t) \) instead of \( \sigma(t) \). Therefore, integrating them over \([k - 1, k + 1]\) respectively, summing them up, from (3.4), (2.9) and (3.70), we can obtain that there exists a positive \( C \), independent of \( k \) and \( T \), such that
\[
\sup_{k - 1 \leq t \leq k + 1} (\sigma_k(t)\|\nabla u\|_{L^2}^2 + \sigma_k^3(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2) + \int_{k - 1}^{k + 1} (\sigma_k(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \sigma_k^3(t)\|\nabla u\|_{L^2}^2) dt
\leq CC_0^{3/4} + C \int_{k - 1}^{k + 1} \sigma_k(t)\|\nabla u\|_{L^2}^2 dt + C \int_{k - 1}^{k + 1} \sigma_k^3(t)\|\nabla u\|_{L^2}^2 dt
\leq CC_0^{3/4} + C \int_{k - 1}^{k + 1} \sigma_k(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2 \|\nabla u\|_{L^2} dt + C \int_{k - 1}^{k + 1} \sigma_k^3(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2 \|\nabla u\|_{L^2} dt
\leq CC_0^{3/4} + C \sup_{1 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho \dot{u}}\|_{L^2}^2) \int_0^T \|\nabla u\|_{L^2}^2 dt + \frac{1}{2} \int_{k - 1}^{k + 1} \sigma_k(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2 dt
\leq CC_0^{3/4} + \frac{1}{2} \int_{k - 1}^{k + 1} \sigma_k(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2 dt,
\]
(3.76)
which implies that
\[
\sup_{k - 1 \leq t \leq k + 1} (\sigma_k(t)\|\nabla u\|_{L^2}^2 + \sigma_k^3(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2)
\leq CC_0^{3/4}, \quad \text{for } k = 2, \ldots, [T] - 1.
\]
(3.77)
Similarly, choosing \( \sigma_{[T]}(t) = \sigma(t + 1 - [T]) = \min\{1, t + 1 - [T]\} \), one has
\[
\sup_{[T] - 1 \leq t \leq T} (\sigma_k(t)\|\nabla u\|_{L^2}^2 + \sigma_k^3(t)\|\sqrt{\rho \dot{u}}\|_{L^2}^2)
\leq CC_0^{3/4}.
\]
(3.78)
Since $\sigma_k(t) = 1$ for $t \in [k, k+1]$ and $\sigma_{[T]}(t) = 1$ for $t \in [[T], T]$, (3.77) and (3.78) yields that
\[
\sup_{1 \leq t \leq T} (\|\nabla u\|^2_{L^2} + \|\sqrt{\rho_0}u\|^2_{L^2}) \leq C C_0^{3/4}.
\] (3.79)
Combining (3.70) and (3.72) yields that
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla u\|^2_{L^2} + \sigma^3 \|\sqrt{\rho_0}u\|^2_{L^2}) \leq C C_0^{3/4}.
\] (3.80)
Set $\varepsilon_3 \triangleq \min\{\varepsilon_2, (C_0^{-4})\}$, (3.71) holds when $C_0 < \varepsilon_3$. Noting that $[0, \infty) = \bigcup_{k=1}^{\infty} [k - 1, k + 1]$ and the estimate (3.77) holds for each $k = 2, 3, \ldots$, we thus conclude that (3.71) is valid in the case when $T = \infty$.

Finally, we proceed to prove (3.72). Integrating (3.25) and (3.34) over $(t_1, t_2)$ with $1 \leq t_1 < t_2 \leq T$, summing them up, using (2.9), (3.71) and (3.79), yields
\[
B[t_1, t_2] \leq C C_0^{3/4} + C \int_{t_1}^{t_2} (\|\nabla u\|^3_{L^2} + \|\nabla u\|^4_{L^4}) dt
\leq C C_0^{3/4} + C \int_{t_1}^{t_2} \|\sqrt{\rho_0}u\|^3_{L^2} \|\nabla u\|^3_{L^2} dt + C \int_{t_1}^{t_2} \|\sqrt{\rho_0}u\|^3_{L^2} \|\nabla u\|_{L^2} dt
\leq C C_0^{3/4} + C C_0(t_2 - t_1),
\] (3.81)
and finishes the proof of Lemma 3.7.

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher order estimates and thus to extend the classical solution globally.

**Lemma 3.8** There exists a positive constant $\varepsilon_4$ depending on $\mu$, $\lambda$, $\gamma$, $a$, $\rho$, $\rho_0$, $\beta$, $\Omega$, and $M$ such that, if $(\rho, u, \Phi)$ is a smooth solution of (1.1)-(1.4) on $\Omega \times (0, T)$ satisfying (3.4) and the initial data condition (1.11), then
\[
\sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{7 \hat{\rho}}{4}.
\] (3.82)
provided $C_0 \leq \varepsilon_4$.

**Proof.** First, the equation of mass conservation (1.1)_1 can be equivalently rewritten in the form
\[
D_t \rho = g(\rho) + b'(t),
\] (3.83)
where
\[
D_t \rho \triangleq \rho_t + u \cdot \nabla \rho, \quad g(\rho) \triangleq \frac{\rho(P - P_s)}{2 \mu + \lambda}, \quad b(t) \triangleq \frac{1}{2 \mu + \lambda} \int_0^t \rho F dt.
\] (3.84)
Naturally, we shall prove our conclusion by Lemma A.2. It is sufficient to check that the function $b(t)$ must verify (A.3) with some suitable constants $N_0$, $N_1$.

For $t \in [0, \sigma(T)]$, one deduces from (A.1), (A.2), (2.5), (2.6), (3.11), (3.4) and Lemmas 3.2, 3.5 that for $\delta_0$ as in Proposition 3.1 and for all $0 \leq t_1 \leq t_2 \leq \sigma(T)$,
\[
|b(t_2) - b(t_1)| = \frac{1}{\lambda + 2 \mu} \left| \int_{t_1}^{t_2} \rho F dt \right| \leq C \int_0^{\sigma(T)} \|F\|_{L^\infty} dt
\]
\[ \leq C \int_0^{\sigma(T)} \| F \|_{L^2} \| \nabla F \|_{L^2} \frac{1}{2} dt + C \int_0^{\sigma(T)} \| F \|_{L^2} dt \]

\[ \leq C \int_0^{\sigma(T)} (\| \sqrt{p} \|_{L^2} + C_0^\frac{3}{2}) (\| \sqrt{\rho} \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2}^2 + C \sigma_0^{\frac{3}{2}}) \frac{1}{2} dt + CC_0^\frac{3}{2} \]

\[ \leq C \int_0^{\sigma(T)} (\| \sqrt{p} \|_{L^2} \| \sqrt{\rho} \|_{L^2} + \| \nabla u \|_{L^2} + C \sigma_0^{\frac{3}{2}}) \frac{1}{2} dt + CC_0^{\frac{1}{2}} \]

\[ \leq C \sigma_0^{\frac{3}{2}}. \quad (3.85) \]

Combining (3.85) with (3.83) and choosing \( N_0 = 0, N_0 = C_0 \sigma_0^{\frac{3}{2}}, \bar{\zeta} = \bar{\rho} \) in Lemma [A.2] gives

\[ \sup_{t \in [0, \sigma(T)]} \| \rho \|_{L^\infty} \leq \bar{\rho} + C_1 C_0^{\frac{3}{2}} \leq \frac{3 \bar{\rho}}{2}, \quad (3.86) \]

provided \( C_0 \leq \bar{\epsilon}_4 = \min \{ \bar{\epsilon}_3, \left( \frac{\bar{\rho}}{\sigma_0^{\frac{3}{2}}} \right)^{\frac{1}{2}} \} \).

On the other hand, for any \( 1 \leq t_1 \leq t_2 \leq T \), it follows from (2.5), (3.11), (3.4) and Lemma [3.2] that

\[ |b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} \| F \|_{L^\infty} dt \]

\[ \leq C \int_{t_1}^{t_2} \| F \|_{L^2} \| \nabla F \|_{L^2} \frac{1}{2} dt + C \int_{t_1}^{t_2} \| F \|_{L^2} dt \]

\[ \leq CC_0^{\frac{3}{2}} \int_{t_1}^{t_2} (\| \sqrt{\rho} \|_{L^2} + 1) dt \]

\[ \leq C_7 C_0^{\frac{3}{4}} + C_8 C_0^{\frac{3}{2}} (t_2 - t_1). \quad (3.87) \]

Now we choose \( N_0 = C_7 C_0^{3/4}, N_1 = C_8 C_0^{\frac{3}{2}} \) in (A.3) and set \( \tilde{\zeta} = \frac{3 \bar{\rho}}{2} \) in (A.4). Since for all \( \zeta \geq \tilde{\zeta} = \frac{3 \bar{\rho}}{2} > \rho_s + 1, \)

\[ g(\zeta) \leq -\frac{a}{\lambda + 2\mu} \leq -C_8 C_0^{\frac{3}{2}} = -N_1. \]

Together with (3.83) and (3.87), by Lemma [A.2] we have

\[ \sup_{t \in [\sigma(T), T]} \| \rho \|_{L^\infty} \leq \frac{3 \bar{\rho}}{2} + C_7 C_0^{3/4} \leq \frac{7 \bar{\rho}}{4}, \quad (3.88) \]

provided \( C_0 \leq \bar{\epsilon}_4 = \min \{ \bar{\epsilon}_3, (\frac{\bar{\rho}}{\sigma_0^{\frac{3}{2}}})^{\frac{1}{4}}, (\frac{\bar{\rho}}{2\mu + \lambda \sigma_0})^{12} \} \). The combination of (3.86) with (3.88) completes the proof of Lemma [3.8] \( \square \)
The following Lemmas deal with some necessary higher order estimates, which make sure that one can extend the strong solution globally in time. The proofs are similar to the ones in [7,13,17,21], and are sketched here for completeness. From now on, we always assume that the initial energy $C_0 \leq \varepsilon_0$, and the positive constant $C$ may depend on $T$, $\mu$, $\lambda$, $a$, $\gamma$, $\rho$, $\hat{\rho}$, $\Omega$, $\rho_s, \Phi_s$ and $g$, where $g \in L^2(\Omega)$ is given as in (1.15).

**Lemma 3.9** There exists a positive constant $C$, such that

\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2) dt \leq C, \tag{3.89}
\]

\[
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^6} + \|u\|_{H^2}) + \int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^6}) dt \leq C, \tag{3.90}
\]

\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^T \int |\nabla u_t|^2 dx dt \leq C. \tag{3.91}
\]

**Proof.** First, taking $s = 1$ in (3.37) along with (3.71) and (3.72) gives

\[
\sup_{t \in [0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \rho |\dot{u}|^2 dx dt \leq C. \tag{3.92}
\]

Choosing $m = 0$ in (3.34), integrating it over $(0, T)$, by (2.19), (3.6), (3.92) and the compatibility condition (1.15), we have

\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \leq C + C \int_0^T \|\rho u_t\|_{L^2}^2 dt \leq C + \frac{1}{2} \sup_{0 \leq t \leq T} \|\sqrt{\rho} u\|_{L^2}^2,
\]

which along with (3.92) gives (3.89). Based on the Beale-Kato-Majda type inequality (see Lemma A.5), we can derive (3.90), in arguments similar to [7], and we omit the details. (3.89) and (3.90) directly yields (3.91). This finishes the proof. \[\square\]

**Lemma 3.10** There exists a positive constant $C$ such that

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^2} + \|P\|_{H^2} + \|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho u_t\|_{L^2}^2 + \|P u_t\|_{L^2}^2) dt \leq C, \tag{3.93}
\]

\[
\sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma \|\sqrt{\rho} u_{tt}\|_{L^2}^2 dt \leq C, \tag{3.94}
\]

\[
\sup_{0 \leq t \leq T} (\|\nabla (\Phi - \Phi_s)\|_{H^3} + \|\nabla \Phi_t\|_{H^2} + \|\nabla \Phi_{tt}\|_{L^2}) \leq C. \tag{3.95}
\]

**Proof.** From (2.17), (2.18), (2.19), (3.11) and Lemma 3.9 we have

\[
\|\nabla^2 u\|_{L^2} \leq C \|\nabla \dot{u}\|_{L^2} + C \|\nabla^2 P\|_{L^2} + C,
\]

which can help us to get (3.93) by the same method as that in [7]. So is (3.94) and we omit the details. From (1.1), we have

\[
\Delta \Phi_t = \rho_t, \quad \Delta \Phi_{tt} = - \text{div}(\rho_t u + \rho u_t). \tag{3.97}
\]

With the similar arguments used in Lemma 2.4 combining (3.93) with (2.19) and (3.97) leads to (3.95). The proof is completed. \[\square\]
Lemma 3.11 There exists a positive constant \( C \) so that for any \( q \in (3, 6) \),
\[
\sup_{t \in [0, T]} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \left( \|\nabla u\|_{H^2}^2 + \|\nabla^2 u\|_{W^{1, q}}^2 + \sigma \|\nabla u_t\|_{H^1}^2 \right) \, dt \leq C, \tag{3.98}
\]
\[
\sup_{t \in [0, T]} (\|\rho - \rho_s\|_{W^{2, q}} + \|P - P_s\|_{W^{2, q}} + \|\nabla (\Phi - \Phi_s)\|_{W^{1, q}}) \leq C, \tag{3.99}
\]
where \( p_0 = \frac{9q - 6}{10q - 12} \in (1, \frac{7}{3}) \).

**Proof.** Let’s start with (3.98). By Lemma 3.9, 3.94, 3.95 and Poincaré’s, Sobolev’s inequalities, one can check that
\[
\|\nabla^2 u\|_{H^1} \leq C (\|\rho \dot{u}\|_{H^1} + \|\rho \nabla (\Phi - \Phi_s)\|_{H^1} + \|\rho - \rho_s\|_{H^1} + \|\nabla \Phi_s\|_{H^1} + \|P - P_s\|_{H^2} + \|u\|_{L^2}) \leq C + C \|\nabla u_t\|_{L^2}. \tag{3.100}
\]
It then follows from (3.100), (3.91) and (3.94) that
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \|\nabla u\|_{H^2}^2 \, dt \leq C. \tag{3.101}
\]
Next, we deduce from Lemmas 3.9, 3.10 that
\[
\|\nabla^2 u_t\|_{L^2} \leq C (\|\rho \dot{u}\|_{L^2} + \|\nabla P_t\|_{L^2} + \|\rho \nabla \Phi_t\|_{L^2} + \|u_t\|_{L^2}) \leq C \|\nabla u_t\|_{L^2} + C, \tag{3.102}
\]
where in the first inequality, we have utilized the \( L^p \)-estimate for the following elliptic system
\[
\begin{cases}
\mu \Delta u_t + (\lambda + \mu) \nabla \text{div} u_t = (\rho \dot{u})_t + \nabla P_t + (\rho \nabla \Phi)_t & \text{in } \Omega, \\
u_t \cdot n = 0 \text{ and } \omega_t \cdot n = 0 & \text{on } \partial \Omega.
\end{cases} \tag{3.103}
\]
Together with (3.102) and (3.94) yields
\[
\int_0^T \sigma \|\nabla u_t\|_{H^1}^2 \, dt \leq C. \tag{3.104}
\]
By Sobolev’s inequality, (3.11), (3.90) and (3.94), we get for any \( q \in (3, 6) \),
\[
\|\nabla (\rho \dot{u})\|_{L^q} \leq C \|\nabla \rho\|_{L^q} (\|\nabla \dot{u}\|_{L^q} + \|\nabla \ddot{u}\|_{L^q} + \|\nabla \nabla u\|_{L^2}^2) + C \|\nabla u\|_{L^q} \leq C (\|\nabla u_t\|_{L^2} + 1) + C \|\nabla u_t\|_{L^2}^{6-q} \|\nabla u_t\|_{L^2}^{q-2} \|\nabla \nabla u\|_{L^2} \leq C \sigma^{-\frac{q}{2}} + C \|\nabla u\|_{H^2} + C \sigma^{-\frac{q}{2}} (\sigma \|\nabla u_t\|_{H^1}^2) \frac{3(q-2)}{4q} + C. \tag{3.105}
\]
Integrating this inequality over \([0, T]\), by (3.89) and (3.104), we have
\[
\int_0^T \|\nabla (\rho \dot{u})\|_{L^q}^{p_0} \, dt \leq C. \tag{3.106}
\]
On the other hand, the combination of (3.15) with (2.17), (2.18), (3.89) and (3.94) gives
\[
(\|\nabla^2 P\|_{L^q})_t \leq C \|\nabla u\|_{L^\infty} \|\nabla^2 P\|_{L^q} + C \|\nabla^2 u\|_{W^{1, q}}.
\]
\[ \leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla^2 P\|_{L^q} + C(1 + \|\nabla u_t\|_{L^2}) + C\|\nabla (\rho \dot{u})\|_{L^q}, \quad (3.107) \]

where in the last inequality we have used the following simple fact that
\[ \|\nabla^2 u\|_{W^{1,q}} \leq C(1 + \|\nabla u_t\|_{L^2} + \|\nabla (\rho \dot{u})\|_{L^q} + \|\nabla^2 P\|_{L^q}), \quad (3.108) \]
due to (2.17), (2.18), (3.90) and (3.94). Hence, applying Gronwall’s inequality in (3.107), we deduce from (3.90), (3.91) and (3.106) that
\[ \sup_{t \in [0,T]} \|\nabla^2 P\|_{L^q} \leq C, \quad (3.109) \]
which along with (3.91), (3.94) and (3.108) also gives
\[ \sup_{t \in [0,T]} \|P - P_s\|_{W^{2,q}} + \int_0^T \|\nabla^2 u\|_{W^{1,q}}^2 \, dt \leq C. \quad (3.110) \]
Similarly, one has
\[ \sup_{0 \leq t \leq T} \|\rho - \rho_s\|_{W^{2,q}} \leq C, \quad (3.111) \]
which together with (3.110) and (2.19) gives (3.99). The proof of Lemma 3.11 is finished.

**Lemma 3.12** There exists a positive constant $C$ such that
\[ \sup_{0 \leq t \leq T} \sigma (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{H^1} + \|\nabla u\|_{W^{2,q}}) + \int_0^T \sigma^2 \|\nabla u_t\|_{L^2}^2 \, dt \leq C, \quad (3.112) \]
for any $q \in (3, 6)$.

**Proof.** Differentiating (3.8) with respect to $t$ twice, multiplying it by $u_{tt}$, and integrating over $\Omega$ lead to
\[
\frac{d}{dt} \int \frac{\rho}{2} |u_{tt}|^2 \, dx + (\lambda + 2\mu) \int (\text{div} u_{tt})^2 \, dx + \mu \int |\omega_{tt}|^2 \, dx \\
= -4 \int \rho u_{tt} u \cdot \nabla u_{tt} \, dx - \int (\rho u_t) \cdot [\nabla (u_t \cdot u_{tt}) + \nabla u_t \cdot u_{tt}] \, dx - \int (\rho_{tt} + 2\rho u_t) \cdot \nabla u \cdot u_{tt} \, dx \\
- \int (\rho_{tt} \cdot \nabla u \cdot u_{tt} - P_{tt} \text{div} u_{tt}) \, dx - \int (\rho_t \nabla \Phi + 2\rho_t \nabla \Phi_t + p \nabla \Phi_t) \, u_{tt} \, dx \\
\Delta \sum_{i=1}^5 R_i, \quad (3.113) \]
Let us estimate $R_i$ for $i = 1, \cdots, 5$. Hölder’s inequality, (3.89), (3.90), (3.91), (3.93) and (3.94), give
\[
\sum_{i=1}^4 R_i \leq \delta \|u_{tt}\|_{L^2}^2 + C(\delta) \|\sqrt{\rho} u_{tt}\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^3 + C(\delta) \|\nabla u_t\|_{L^2}^2 \\
+ C(\delta) \|\rho_{tt}\|_{L^2}^2 + C(\delta) \|P_{tt}\|_{L^2}^2, \quad (3.114) \]
By (3.89), (3.91), (3.93) and (3.94), we conclude that
\[ R_5 \leq C \left( \|\rho t\|_{L^4} \|\nabla \Phi\|_{L^3} + \|\rho t\|_{L^3} \|\nabla \Phi_t\|_{L^6} \right) \|u_{tt}\|_{L^6} + \|\nabla \Phi_t\|_{L^2} \|u_{tt}\|_{L^2} \]
\[ \leq \delta \|u_{tt}\|_{L^2}^2 + C(\delta) (\|\rho t\|_{L^2}^2 + \|\nabla \Phi_t\|_{L^2}^2 + \|\nabla \Phi_t\|_{L^2}^2 + \|\Phi_t\|_{L^2}^2) \]
(3.115)
Substituting these estimates into (3.113), utilizing the fact that
\[ \|\nabla u_{tt}\|_{L^2} \leq C (\|\text{div} u_{tt}\|_{L^2} + \|\omega_{tt}\|_{L^2}) \]
(3.116)
due to Lemma A.3 since \( u_{tt} \cdot n = 0 \) on \( \partial \Omega \), and then choosing \( \delta \) small enough, we can get
\[ \frac{d}{dt} \|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 \]
\[ \leq C (\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \|\rho t\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 + C) \]
(3.117)
which together with (3.93), (3.94), and by Gronwall’s inequality yields that
\[ \sup_{0 \leq t \leq T} \sigma^2 \|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \int_0^T \sigma^2 \|\nabla u_{tt}\|_{L^2}^2 dt \leq C. \]
(3.118)
Furthermore, it follows from (2.18), (3.102) and (3.94) that
\[ \sup_{0 \leq t \leq T} \sigma \|\nabla^2 u_t\|_{L^2} \leq C \sigma \left( 1 + \|\sqrt{\rho}u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} \right) \leq C. \]
(3.119)
Finally, we deduce from (3.94), (3.99), (3.98), (3.105), (3.108), (3.118) and (3.119) that
\[ \sigma \|\nabla^2 u\|_{W^{1,q}} \leq C \sigma \left( 1 + \|\nabla u_t\|_{L^2} + \|\nabla (\sqrt{\rho}u_t)\|_{L^3} + \|\nabla^2 P\|_{L^q} \right) \]
\[ \leq C \left( 1 + \sigma \|\nabla u\|_{H^2} + \sigma^{\frac{1}{2}} (\sigma \|\nabla u_t\|_{H^1}^2)^{\frac{3(q-2)}{4q}} \right) \]
\[ \leq C + C \sigma^{\frac{1}{2}} (\sigma^{-1})^{\frac{3(q-2)}{4q}} \leq C, \]
(3.120)
together with (3.118) and (3.119) yields (3.112) and this completes the proof of Lemma 3.12.

---

### 4 Proof of Theorem 1.1

With all the a priori estimates in Section 3 at hand, we are going to prove the main result of the paper in this section.

**Proof of Theorem 1.1.** By Lemma 2.2, there exists a \( T_* > 0 \) such that the system (1.1)-(1.4) has a unique classical solution \((\rho, u, \Phi)\) on \( \Omega \times (0, T_*) \). One may use the a priori estimates, Proposition 3.1 and Lemmas 3.10-3.12 to extend the classical solution \((\rho, u, \Phi)\) globally in time.

First, by the definition of (3.1), the assumption of the initial data (1.14) and (3.69), one immediately checks that
\[ 0 \leq \rho_0 \leq \hat{\rho}, \quad A_1(0) = 0, \quad A_2(0) \leq C_0^\delta. \]
(4.1)
Therefore, there exists a \( T_1 \in (0, T_*) \) such that
\[ 0 \leq \rho_0 \leq 2\hat{\rho}, \quad A_1(T) \leq 2C_0^\frac{1}{2}, \quad A_2(\sigma(T)) \leq 2C_0^\delta, \]
(4.2)
hold for $T = T_1$. Next, we set

$$T^* = \sup \{T \mid (4.2) \text{ holds} \}. \quad (4.3)$$

Then $T^* \geq T_1 > 0$. Hence, for any $0 < \tau < T \leq T^*$ with $T$ finite, it follows from Lemmas 3.9, 3.10, and (4.5) that

$$\left\{ \begin{array}{l}
(\rho - \rho_\ast, \nabla (\Phi - \Phi_\ast)) \in C([0, T]; H^2 \cap W^{2,q}), \\
\nabla u \in C([\tau,T]; H^1), \quad \nabla u_t \in C([\tau,T]; L^q);
\end{array} \right. \quad (4.4)$$

where one has taken advantage of the standard embedding

$$L^\infty(\tau,T; H^1) \cap H^1(\tau,T; H^{-1}) \hookrightarrow C([\tau,T]; L^q), \quad \text{for any } q \in [2, 6].$$

Due to (3.91), (3.94), (3.112) and (1.1), we obtain

$$\int_\tau^T \left( \left( \int \rho |u_t|^2 \, dx \right) \right) \, dt \leq \int_\tau^T \left( \|\rho_t\|^2_{L^1} + 2\|\rho u_t \cdot u_t\|_{L^1} \right) \, dt \leq C \int_\tau^T \left( \|\sqrt{\rho}|u_t|^2\|_{L^2} \|\nabla u\|_{L^\infty} + \|u\|_{L^6} \|\nabla \rho\|_{L^2} \|u_t\|_{L^6} + \|\sqrt{\rho} u_t\|_{L^2} \right) \, dt \leq C,$$

which together with (4.4) yields

$$\sqrt{\rho} u_t, \quad \sqrt{\rho} \dot{u} \in C([\tau,T]; L^2). \quad (4.5)$$

Finally, we claim that

$$T^* = \infty. \quad (4.6)$$

Otherwise, $T^* < \infty$. Then by Proposition 3.1 it holds that

$$0 \leq \rho \leq \frac{7}{4} \beta, \quad A_1(T^*) \leq C_0^\frac{1}{3}, \quad A_2(\sigma(T^*)) \leq C_0^\frac{1}{3}, \quad (4.7)$$

It follows from Lemmas 3.9, 3.10, and (4.5) that $(\rho(x, T^*), u(x, T^*), H(x, T^*))$ satisfies the initial data condition (1.13) and (1.15), where $g(x) \triangleq \sqrt{\rho} \dot{u}(x, T^*), x \in \Omega$. Thus, Lemma 2.2 implies that there exists some $T^** > T^*$ such that (4.2) holds for $T = T^**$, which contradicts the definition of $T^*$. As a result, (4.6) holds. By Lemmas 2.2 and 3.9, 3.12, it indicates that $(\rho, u, \Phi)$ is in fact the unique classical solution defined on $\Omega \times (0, T]$ for any $0 < T < T^* = \infty$. The proof of Theorem 1.1 is finished. \qed

### A Some basic theories and lemmas

In this appendix, we review some elementary inequalities and important lemmas that are used extensively in this paper.

First, we recall the well-known Gagliardo-Nirenberg inequality (see [27]).

**Lemma A.1 (Gagliardo-Nirenberg)** Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$. For $p \in [2, 6], q \in (1, \infty)$, and $r \in (3, \infty)$, there exist two generic constants $C_1, C_2 > 0$ which may depend on $p, q$ and $r$ such that for any $f \in H^1(\Omega)$ and $g \in L^q(\Omega) \cap D^{1,r}(\Omega)$,

$$\|f\|_{L^p(\Omega)} \leq C_1 \|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}} + C_2 \|f\|_{L^2}, \quad (A.1)$$

$$\|g\|_{C(\overline{\Omega})} \leq C_1 \|g\|_{L^q(\Omega)}^{\frac{r(3-q)}{3r+q(3-3q)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(3-3q)}} + C_2 \|g\|_{L^2}. \quad (A.2)$$

Moreover, if $f \cdot n|_{\partial \Omega} = 0, g \cdot n|_{\partial \Omega} = 0$, then the constant $C_2 = 0$. 

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In order to get the uniform (in time) upper bound of the density $\rho$, we need the following Zlotnik inequality in [42].

**Lemma A.2** Suppose the function $y$ satisfy
\[ y'(t) = g(y) + b'(t) \text{ on } [0, T], \quad y(0) = y^0, \]
with $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and
\[ b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \quad \text{(A.3)} \]
for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then
\[ y(t) \leq \max\{y^0, \zeta\} + N_0 < \infty \text{ on } [0, T], \]
where $\zeta$ is a constant such that $g(\zeta) \leq -N_1$ for $\zeta \geq 0$.

The following two lemmas are given in Propositions 2.6-2.9 in [1].

**Lemma A.3** Let $k \geq 0$ be a integer, $1 < q < +\infty$, and assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ with $C^{k+1,1}$ boundary $\partial \Omega$. Then for $v \in W^{k+1,q}$ with $v \cdot n = 0$ on $\partial \Omega$, it holds that
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div}v\|_{W^{k,q}} + \|\text{curl}v\|_{W^{k,q}}). \quad \text{(A.5)} \]
In particular, for $k = 0$, we have
\[ \|\nabla v\|_{L^q} \leq C(\|\text{div}v\|_{L^q} + \|\text{curl}v\|_{L^q}). \quad \text{(A.6)} \]

**Lemma A.4** Let $k \geq 0$ be a integer, $1 < q < +\infty$. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^3$ and its $C^{k+1,1}$ boundary $\partial \Omega$ only has a finite number of 2-dimensional connected components. Then for $v \in W^{k+1,q}$ with $v \times n = 0$ on $\partial \Omega$, we have
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div}v\|_{W^{k,q}} + \|\text{curl}v\|_{W^{k,q}} + \|v\|_{L^q}). \quad \text{(A.7)} \]
In particular, if $\Omega$ has no holes, then
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div}v\|_{W^{k,q}} + \|\text{curl}v\|_{W^{k,q}}). \quad \text{(A.8)} \]

Finally, similar to [2][17], we need a Beale-Kato-Majda type inequality with respect to the slip boundary condition [13] which is given in [7].

**Lemma A.5** For $3 < q < \infty$, assume that $u \cdot n = 0$ and $\text{curl}u \times n = 0$ on $\partial \Omega$, $u \in W^{2,q}$, then there is a constant $C = C(q, \Omega)$ such that the following estimate holds
\[ \|\nabla u\|_{L^\infty} \leq C(\|\text{div}u\|_{L^\infty} + \|\text{curl}u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C. \quad \text{(A.9)} \]

**Acknowledgements**

This research was partially supported by National Natural Sciences Foundation of China No. 11671027, 11901025, 11971020, 11971217.
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