Entropy-driven phase transitions with influence of the field-dependent diffusion coefficient

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Abstract

We present a comprehensive study of the phase transitions in the single-field reaction-diffusion stochastic systems with field-dependent mobility of a power-low form and the internal fluctuations. Using variational principles and mean-field theory it was shown that the noise can sustain spatial patterns and leads to disordering phase transitions. We have shown that the phase transitions can be of critical or non-critical character.

Key words: Nonlinear diffusion; Internal noise; Phase transition
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1 Introduction

The ability of noise to induce spatial ordering has received a special attention in last two decades. It is well known that in many situations noise can actually play a constructive role, to name just a few one can consider: noise-induced transitions in zero-dimensional systems [1], noise induced phase transitions in extended systems [2], patterns formation [3], coupled Brownian motors [4,5], etc. Most of the noise-induced effects are caused by an external fluctuations and attributed to a short time instability. In particular, noise-induced patterns and the phase transitions usually have a dynamical origin. It is principally important that in the case of the internal noise, obeying fluctuation-dissipation relation, a spatio-temporal coherence in nonlinear systems can not be observed in a short-time scale. Such ordering processes follow an entropy-driven mechanism when a kinetic coefficient/mobility is a function of a stochastic field. It leads to the fact that a stationary distribution is described by a free energy

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functional reduced to a Lyapunov functional for deterministic dynamics and by an entropy contribution related to the field-dependent mobility \([6,7,8,9]\). It was shown by the most of works concerning such phenomena that quantitative change of the system behaviour is caused by the entropy variability.

It was shown that internal fluctuations can sustain the spatial coherence (patterns) in nonlinear reaction-diffusion stochastic systems \([8,10]\). In the simplest case such models have two essential features: the local dynamics and a transport phenomena. First of them is determined by the chemical reactions in the system and the second one relates to a diffusion of chemical species. Introducing a fluctuating source into the model, we arrive at the stochastic description of non-equilibrium system behaviour with possible spatial order. In this work we consider a prototype model of the reaction-diffusion systems where an internal fluctuations are of a multiplicative character (the noise intensity depends on a concentration field) (see Ref.[8] and citations therein). In the most of works devoting to study the system behaviour with a field-dependent mobility, the corresponding functional dependence is as follows: in the mixed state the kinetic coefficient is large, whereas in dense ones it has small values. A deterministic analysis of the systems with a generalized field-dependent mobility was performed in Ref.[11], where the form of the kinetic coefficient was proposed by authors solely. In our consideration we use a generalized approach basing on a nonlinear master equation proposed by G. Kaniadakis \([12]\) and derive the generalized form for the kinetic coefficient. Introducing the corresponding Fickian diffusion term into the model and a term to describe a local dynamics of the field variables, we arrive at a deterministic model for the reaction-diffusion systems. Considering a system in real conditions we take into account a fluctuation source related to an obtained mobility.

According to the well known approaches of entropy-driven phase transitions \([6,7]\) one can investigate a possibility of patterns formation and corresponding phase transitions. Our goal in this article is to perform a detailed study of spatial system’s coherent behaviour with a generalized kinetic coefficient in a bistable stochastic system. With help of variational principles we investigate a possibility of noise to sustain stationary spatial structures. A picture of noise-induced phase transitions will be studied with the mean-field approach and computer simulations.

Our paper is organized in the following manner. In the next section we present the general formalism to investigate the noise-induced phase transitions and patterns formation. In Section 3 we derive an expressions for the field-dependent diffusion coefficient using the formalism of nonlinear master equation and deformed Boltzmann-Gibbs statistics. According to the presented formalism the noise-sustained structures are considered in Section 4. The noise-induced phase transitions with the entropy-driven mechanism are studied in Section 5. Main results and prospects for the future are presented in the Conclusions.
Our starting point is the evolution equation for a particle density \( p = p(r, t) \) written in a most general form of the continuity equation \([12]\)

\[
\frac{\partial p}{\partial t} = \nabla \left[ D\gamma(p) \frac{\partial \ln \kappa(p)}{\partial p} \nabla p \right].
\] (1)

Here we take into account diffusion flux only with concentration-dependent effective diffusion coefficient assumed in the form

\[
D_{ef}(p) \equiv D\gamma(p) \frac{\partial \ln \kappa(p)}{\partial p}, \quad D = \text{const.}
\] (2)

From a formal viewpoint the nonlinear diffusion equation (1) can be derived from a master equation assuming that probabilities of the microscopic transitions explicitly depend on the concentrations of the initial and arrival states, described by functions \( \mu(p) \) and \( \nu(p) \), respectively. It was shown (see Refs.[13,14]) that functions \( \gamma(p) \) and \( \kappa(p) \) in Eq.(1) are defined as follows: \( \gamma(p) = \mu(p)\nu(p) \), \( \kappa(p) = \mu(p)/\nu(p) \). An explicit form for pairs \( (\mu, \nu) \) or \( (\gamma, \kappa) \) can be set due to the physics of the problem under consideration.

Introducing chemical reactions described by a rate \( f(p) \), a generalized continuity equation for the concentration of particles reads:

\[
\frac{\partial p}{\partial t} = f(p) + \nabla \left[ D_{ef}(p) \nabla p \right].
\] (3)

Formally, Eq.(3) can be written with help of the Lyapunov functional for the deterministic dynamics. Indeed, introducing notation for such a functional in the form

\[
F = \int dr \left\{ \frac{1}{2} \left[ D_{ef}(p) \nabla p \right]^2 + \varphi(p) \right\}
\] (4)

with

\[
\varphi = -\int_0^p f(p') D_{ef}(p') dp',
\] (5)

the deterministic evolution equation (3) takes a variational form \([8]\)

\[
\frac{\partial p}{\partial t} = -\frac{1}{D_{ef}(p)} \frac{\delta F[p]}{\delta p}.
\] (6)

Considering the system in realistic conditions one needs to introduce a fluctuation source \( \xi(p; r, t) \). In our stochastic analysis we assume that such fluctuating source obeys the fluctuation-dissipation relation \([15]\) and has following
\[ \langle \xi(p; r, t) \rangle = 0, \quad \langle \xi(p; r, t)\xi(p; r', t') \rangle = \frac{2\sigma^2}{D_{\text{ef}}(p)} \delta(r - r')\delta(t - t'), \quad (7) \]

where \( \sigma^2 \) is an intensity of the corresponding internal multiplicative noise. Formally, one can introduce an external noise related to fluctuations of a control parameter addressed to a local dynamics. In this paper we study influence of an internal fluctuations source only on the phase transitions picture with the concentration-dependent diffusion coefficient and the chemical reactions. In the following analysis we use the Stratonovich interpretation of the Langevin equation

\[ \frac{\partial p}{\partial t} = -\frac{1}{D_{\text{ef}}(p)} \frac{\delta F[p]}{\delta p} + \frac{1}{\sqrt{D_{\text{ef}}(p)}} \tilde{\xi}(r, t), \quad (8) \]

where \( \xi(p; r, t) = [D_{\text{ef}}(p)]^{-1/2} \tilde{\xi}(r, t) \).

Considering stationary properties of the system we exploit a stationary probability density functional, obtained as a solution of the corresponding Fokker-Planck equation [16]. In the framework of standard technique such stationary functional takes the form [2,6,17]

\[ P_{\text{st}} \propto \exp \left( -\frac{U_{\text{ef}}[p]}{\sigma^2} \right); \quad (9) \]

the effective energy functional

\[ U_{\text{ef}}[p] = F[p] - \frac{\sigma^2}{2} \int d\mathbf{r} \ln D_{\text{ef}}(p) \quad (10) \]

is defined through the free energy functional \( F[p] \) and the effective diffusion coefficient \( D_{\text{ef}}(p) \). Due to the stationary probability density functional (9) has an exact form, the effective potential (10) can be used to study possibility of the structure formation under the multiplicative noise influence. Moreover, we can apply the mean-field theory formalism to investigate the noise-induced phase transitions in systems of such a kind.

3 Model

The first problem we deal with is to set a generalized form for the effective diffusion coefficient \( D_{\text{ef}}(p) \), defined according to Eq.(2). The main criterion for the function \( D_{\text{ef}}(p) \) is a bell-shaped form in the interval \( p \in [0, 1] \). We assume that following properties are satisfied: \( D_{\text{ef}}(0) = D_{\text{ef}}(1) = 0; \) \( D_{\text{ef}}(1/2) = D_{\text{ef}}^{(\text{max})} \), where \( D_{\text{ef}}^{(\text{max})} \) is a maximal value. It means that fluctuations are large in a mixed state characterized by the value \( p = 1/2 \); in
dense states \((p = 1, \ p = 0)\) no fluctuations are realized. Considering a general problem, related to a complex systems investigation, let us assume that the function \(D_{eff}(p)\) has a power-law form. Indeed, as it was shown before the complex systems can be described by the nonlinear continuity equation and are characterized by a power-law form for main statistical quantities (see Refs.[18,19,20,21]). Most of complex statistical systems are described by deformed Boltzmann-Gibbs statistics or \(q\)-statistics exploiting deformed logarithm’s and exponential functions [21]. By now there are two well known kinds of deformations, proposing by G.Kaniadakis and C. Tsallis (see Refs.[13,21], respectively). In this paper we operate with a mathematical construction of the \(q\)-deformed logarithm \(\ln x \rightarrow \ln_q x = (x^{1-q} - 1)/(1 - q)\) [21], the exponent \(q\) is a non-additivity parameter; in the limit case \(q = 1\) one arrives at usual logarithm and the standard Boltzmann-Gibbs statistics. Thus, inserting \(q\)-deformed logarithm into Eq.(2), one has

\[
D_{eff}(p) = D\gamma(p)\kappa(p)^{-q}\frac{d\kappa(p)}{dp}.
\]  

(11)

In the most of works related to the nonlinear continuity equation (1) the functions \(\gamma(p)\) and \(\kappa(p)\) are assumed to be linear versus its argument. In the simplest case one has: \(\kappa(p) = p\), \(\gamma(p) = p\). It results to the construction of the form

\[
D_{eff}(p) = Dp^{1-q}.
\]  

(12)

It is seen that the linear approximation of \(\gamma(p)\) and \(\kappa(p)\) does not give the bell-shaped form for the effective diffusion coefficient. Evolution of the system with the effective diffusion coefficient (12) was described in Ref.[22]. It was shown that such complex systems manifest an anomalous diffusion.

To satisfy the main condition for the function \(D_{eff}(p)\) let us assume nonlinear constructions for \(\gamma(p)\) and \(\kappa(p)\). According to the fact that complex systems are self-similar, let us write \(\gamma(p)\) in the form

\[
\gamma(p) = p^\alpha;
\]  

(13)

where the exponent \(\alpha \in (0,1)\). To define the function \(\kappa(p)\) we suppose that the derivative \(d\kappa(p)/dp\) in Eq.(11) gives a power-law dependence, i.e. \(d\kappa(p)/dp = \kappa^\beta\), \(\beta > 0\). Such an assumption determines the power-low form for the function \(\kappa(p)\). Indeed, after some algebra one can obtain \(\kappa(p) = [C(\beta - 1) + (1 - \beta)p]^{1/\beta}\), where \(C > 0\) is a constant. Next, introducing a positive constant \(C_1 = C(\beta - 1)\) with \(\beta > 1\) one can represent the function \(\kappa(p)\) in the form of the Tsallis exponent [23]:

\[
\kappa(p) = [C_1 + (1 - \beta)p]^{1/\beta} \equiv C_1^{1/\beta} \exp_\beta \frac{p}{C_1}.
\]  

(14)

The exponent \(\beta\) plays a role of the non-additivity parameter. The Tsallis exponent in Eq.(14) becomes the usual one at \(\beta = 1\). Substituting expressions for the functions \(\gamma(p)\) and \(\kappa(p)\) from Eq.(13) and Eq.(14), respectively, into the
equation (11) and performing trivial calculations one arrives at the power-low form for the effective diffusion coefficient:

\[ D_{\text{ef}}(p) = D p^\alpha [C_1 - (\beta - 1)p]^{\delta}, \quad \delta = \frac{q - \beta}{\beta - 1}. \]  

(15)

The function \( D_{\text{ef}}(p) \) has a bell-shaped form only if \( \delta > 0 \) or if \( q > \beta \) with \( \beta > 1 \). Formally, the effective diffusion coefficient \( D_{\text{ef}}(p) \) defined by the exponent \( \alpha \neq \delta \) is a non-symmetrical function with respect to the point \( p = 1/2 \). In further investigation we are interested in studying the system properties when \( D_{\text{ef}}(p) \) is the symmetrical function. To this end we assume \( \delta = \alpha \). It allows to obtain the relation between the both non-additivity parameters \( q \) and \( \beta \) in the form

\[ \beta < q < 2\beta - 1. \]  

(16)

Therefore, we arrive at the symmetrical form for the effective diffusion coefficient

\[ D_{\text{ef}}(p) = \frac{\beta_0}{2} p^\alpha \left[ 1 - \frac{p}{p_s} \right]^\alpha, \]  

(17)

where \( p_s = C_1/(\beta - 1) \) is a saturation concentration, \( \beta_0 = 2DC_1^\alpha \) is a constant. The obtained formula (17) can be derived directly if probabilities \( \mu(p) \) and \( \nu(p) \) are known initially. Indeed, using the approach developed in Ref.[14] one can express functions \( \gamma(p) \) and \( \kappa(p) \) through \( \mu(p) \) and \( \nu(p) \), and after find the alternative construction for the effective diffusion coefficient:

\[ D_{\text{ef}}(p) = D \left( \frac{\mu(p)}{\nu(p)} \right)^{1-q} \left[ \frac{d}{dp} \mu - \frac{d}{dp} \nu \right]. \]  

(18)

As it follows from our considerations densities \( \mu(p) \) and \( \nu(p) \) can be obtained using relations between \( \gamma(p) \) and \( \kappa(p) \). After a trivial algebra one find:

\[ \mu(p) = [\gamma(p)\kappa(p)]^{1/2}, \quad \nu(p) = [\gamma(p)/\kappa(p)]^{1/2}. \]  

(19)

Therefore, the form for the effective diffusion coefficient is well defined.

Next, to derive a model for the function \( f(p) \) that describes possible chemical reactions in the system. Let us assume that in the deterministic regime there are three stationary concentration values \( p_0^{(1)}, p_0^{(2)}, p_0^{(3)} \) with \( p_0^{(i)} \neq p_0^{(j)}, i \neq j \). Thus, using the theory of dynamical systems one can suppose the deterministic force in the form \( f(p) = -\Pi_i (p - p_0^{(i)}) \). In the simplest case the following construction can be used:

\[ f(p) = \varepsilon(p - \lambda) - (p - \lambda)^3, \]  

(20)

where \( \varepsilon \) is a control parameter, \( \lambda \) determines the equilibrium concentration magnitude. In such a case stationary states are determined by the values: \( p_0^{(1)} = \lambda, p_0^{(2,3)} = \lambda \pm \sqrt{\varepsilon} \). In the following analysis we choose \( \lambda = 1/2 \).
Next, let us show that the corresponding internal multiplicative noise leads to a short-time instability of the mixed/disordered state $p = 1/2$. The linear stability analysis can be performed for an auxiliary field $y(r, t) = p(r, t) - \lambda$. The corresponding linearized Langevin equation takes the form

$$
\frac{\partial y(r, t)}{\partial t} = \varepsilon y + \kappa^2 \nabla^2 y + \bar{\sigma}^2 y + \kappa^{-1} \xi(r, t),
$$

(21)

where $\kappa^2 = 1/2\beta_0 \lambda^{2\alpha}$, $\bar{\sigma}^2 = (2\alpha \sigma^2)/(\beta_0 \lambda^{2(\alpha+1)})$. In the Fourier space Eq.(21) is

$$
\frac{\partial y(\pm k, t)}{\partial t} = \varepsilon y(\pm k, t) - \kappa^2 k^2 y(\pm k, t) + \bar{\sigma}^2 y(\pm k, t) + \kappa^{-1} \xi(\pm k, t).
$$

(22)

Writing the dynamical equation for the two point correlation function

$$
\frac{\partial}{\partial t} \langle y(k, t)y(-k, t) \rangle = \left\langle y(k, t) \frac{\partial}{\partial t} y(-k, t) \right\rangle + \left\langle y(-k, t) \frac{\partial}{\partial t} y(k, t) \right\rangle
$$

(23)

and calculating correlators $\langle \xi_{-k} y_k \rangle$, $\langle \xi_k y_{-k} \rangle$ with the help of Novikov’s theorem [24], one arrives at a dynamical equation for the structure function $S(k, t) = \langle y_k(t)y_{-k}(t) \rangle$ in the form

$$
\frac{\partial}{\partial t} S(k, t) = 2(\varepsilon - \kappa^2 k^2 + \bar{\sigma}^2) S(k, t) + 2\sigma^2 \kappa.
$$

(24)

It is seen that the internal multiplicative noise leads to a short-time instability of the mixed state $p = 1/2$. The stationary value of the structure function is

$$
S(k) = \frac{\sigma^2 / \kappa}{\kappa^2 k^2 - \varepsilon - \bar{\sigma}^2}.
$$

(25)

Hence, the homogeneous state $p(r, t) = 1/2$ is stable only if $\varepsilon < \kappa^2 k^2 - \bar{\sigma}^2$.

4 Noise sustained structures

As was shown in Ref.[8] the extrema of the effective potential $U_{ef}[p]$ correspond to the stationary noise-sustained structures $p_{st}$. Such structures can be computed as an solution of equation $\delta U_{ef}[p]/\delta p = 0$, where the effective functional (10) has the form

$$
U_{ef}[p] = \int dr \left[ \varphi(p) + \frac{1}{2} (D_{ef}(p) \nabla p)^2 \right] - \frac{\sigma^2}{2} \int dr \ln D_{ef}(p).
$$

(26)
Making the first variation of $U_{ef}[p]$ with respect to $p$ equal to zero, we arrive at the equation for the stationary structures

$$\Delta p = -\left[ \frac{\sigma^2}{2D_{ef}^2(p)} \frac{\partial D_{ef}(p)}{\partial p} + \frac{f(p)}{D_{ef}(p)} + \frac{1}{D_{ef}(p)} \frac{\partial D_{ef}(p)}{\partial p} (\nabla p)^2 \right]_{p=p_{st}}. \quad (27)$$

Considering homogeneous states, one can put $\nabla p = \Delta p = 0$ in Eq.(27). The reduced equation gives the most probable stationary states $p_e$, defining extrema positions of the function $U_{ef}(p)$. The corresponding solutions of such equation are shown in Fig.1a. It is seen that with an increase in the noise intensity two different values of the most probable concentrations degenerate at a bifurcation point $\sigma^2 = \sigma^2_b$. Despite the fixed point $p_e = 1/2$ exists always, the two additional fixed points (upper and lower curves in Fig.1a) are observed at $\sigma^2 < \sigma^2_b$. At $\sigma^2 > \sigma^2_b$ the stationary probability function has one extremum only located in $p_e = 1/2$. The phase diagram $\alpha(\sigma^2_b)$ illustrating the dependence of the bifurcation point position at various values of $\varepsilon$ is shown in Fig.1b. One can see that with an increase in the exponent $\alpha$ the bifurcation value $\sigma^2_b$ decreases. Therefore, if $D_{ef}$ decreases sharply at $p \simeq 0$ and $p \simeq 1$, then the stationary distribution becomes unimodal at large values of the noise intensity $\sigma^2$.

Next let us investigate stationary structures $p_{st}$. Firstly, we consider the vicinity of the homogeneous solution $p_e = 1/2$. In a linear approximation one can put $(\nabla p)^2 = 0$ in Eq.(27) and expand the left part of Eq.(27). Then, we obtain the second order differential equation in the form

$$\frac{\partial^2 p}{\partial \sigma^2} \simeq Ak_l^2 (p - p_{st}), \quad A = +1 \text{ if } \sigma^2 > \sigma^2_b,$$

$$A = -1 \text{ if } \sigma^2 < \sigma^2_b, \quad (28)$$

where $k_l = \sqrt{(\varepsilon - \bar{\sigma}^2)/x^2}$. So, one can conclude, that at $\sigma^2 < \sigma^2_b$ the station-
ary state $p_e = 1/2$ is locally stable and represents a center in the phase space $(p, \nabla p)$; in the opposite case ($\sigma^2 > \sigma^2_0$) it changes the stability and becomes a saddle. It is seen that the dependence $k_i(\tilde{\sigma}^2)$ has monotonically decreasing character. As it follows from our consideration the stationary periodic structures are formed in the vicinity of the point related to maximum of the function $U_{ef}(p)$.

In the numerical studying of the noise-induced spatial patterns we integrate Eq.(8) on a $d$-dimensional lattice of the mesh size $l$. In the discrete space Eq.(8) can be written as follows:

$$\frac{dp_i}{dt} = f(p_i) + \frac{1}{4l^2} \sum_{j \in nn^+(i)} (p_j - p_i)^2 \frac{dD_{ef}(p_i)}{dp_i} + D_{ef}(p_i) \sum_{j \in nn(i)} D_{ij} p_j$$

$$- \frac{\sigma^2}{2} \frac{dD_{ef}(p_i)}{dp_i} \frac{1}{D_{ef}(p_i)} + \frac{1}{\sqrt{D(p_i)}} \xi,$$

(29)

where $i = 1, \ldots, N^d$ enumerates the element of the square lattice of $N^d$ cells; periodic boundary conditions are used. The second term in Eq.(29) represents approximation of the gradient $|\nabla p|^2$, where $nn^+(i)$ indicates nearest neighbors in the positive direction of each axis, whereas the third term is related to a discrete Laplacian ($\nabla p \rightarrow \sum_{j \in nn(i)} D_{ij} p_j$, $nn(i)$ denotes nearest neighbors of the site $i$). To describe spatial patterns we use a spherical averaging of the structure function $S(k, t) = \int_{\Gamma_k} S(k, t) d\Gamma$, where $\Gamma_k$ is a spherical shell of a radius $k$. A convenient formula is

$$S(k, t) = \frac{1}{N_k} \sum_{k \leq k \leq k + \Delta k} S(k, t).$$

(30)

All calculations were performed in a two-dimension square lattice of $120 \times 120$ cells with lattice scaling $l = 1$, and integration time step $\tau = 5 \times 10^{-3}$. The stationary spherically averaged structure function is shown in Fig.2 at different values of the noise intensity.

It is principle, that an increase in the noise intensity $\sigma^2$ results to a shift of the peak of $S(k)$ to the small values of the wave vector $k$. This is different from what happens at the early stages of evolution. This fact indicates that when the system starts to evolve a multiplicative noise leads to instability of the state $p = 1/2$; the corresponding values of the wave vector $k$ increase with the noise intensity growth. For large times (in the stationary case) the situation is quite different. Here with an increase in the noise intensity the peak of $S(k)$ moves toward smaller values of $k$. This fact is related to an entropy mechanism of the phase transitions. Indeed, the effective energy functional (10) is defined through the initially known free energy functional $F[p]$ and a contribution $-\sigma^2/2 \int d\mathbf{r} \ln D_{ef}(p)$. The last one gives an effective entropy $S_{ef} = -1/2 \int d\mathbf{r} \ln D_{ef}(p)$ multiplied by the noise intensity $\sigma^2$. In such a
case we arrive at the standard thermodynamic definition of an internal energy

\[ U_{ef} = F + \sigma^2 S_{ef} \]

It is well known that in the theory of the entropy-driven phase transitions self-organization processes are not related to the short-time instability of the mixed/disordered state [6], its caused by entropy variations following from concentration-dependent mobility. In our case we have a quite similar situation. In particular, at early stages of the system evolution a noise destabilizes the disordered state \( p = 1/2 \), whereas at \( t \to \infty \) the entropy contribution plays a crucial role: it leads to patterning with small values of \( k \).

Obtained numerical results are in good corresponding with analytical predictions: in the stationary case the peak of \( S(k) \) moves toward small values of the wave vector when the noise intensity increases.

5 Mean Field Theory

In this section we will consider the noise-induced phase transitions with the above entropy mechanism. To this end we construct the Weiss’ mean-field (MF) approximation, based on the stationary distribution function. Let us approximate the gradient term in the functional (10) by the sum over nearest neighbors on the lattice as Eq.(29) shows. In the MF approximation we replace the exact value of the neighbors by a mean field \( \eta = \langle p \rangle \). It leads to the relation

\[ (\nabla p)^2 \to (p - \langle p \rangle)^2 = (p - \eta)^2. \]

In such a case the quantity \( \eta \) can be used as an order parameter. In this procedure we neglect fluctuations in neighboring sites. The value of the order parameter can be computed self-consistently as follows:

\[ \eta \equiv \langle p \rangle = \int P_{st}(p; \eta)pdp \equiv \Phi(\eta), \]

where \( P_{st}(p; \eta) \) is the stationary distribution in the MF approximation depending on the mean field \( \eta \). As usual, the parameter \( \eta \) allows to identify a phase transition from a disordered state with \( \eta = \eta_c \) to an ordered one characterized by the value \( \eta \neq \eta_c \); the symmetry that leads to \( \eta = \eta_c \) is embedded.
in the system. According to the discussion above, the effective potential has the form

\[ U_{ef}(p; \eta) = \frac{1}{2} D_{ef}^2(p)(p - \eta)^2 - \frac{\sigma^2}{2} \ln D_{ef}(p) - \int_{0}^{p} f(p') D_{ef}(p') dp' \]  (32)

and depends on the mean field \( \eta \). The stationary probability density function is

\[ P_{st}(p; \eta) = Z(\eta) \exp \left( \frac{-U_{ef}(p; \eta)}{\sigma^2} \right), \]  (33)

where \( Z(\eta) \) takes care of the normalization condition \( \int P_{st}(p; \eta) dp = 1 \). According to formalism proposed in Ref.[7] and quantitative analysis of the function \( \Phi(\eta) \) one can expect the second-order phase transitions in a system under consideration. Using Newton-Raphson condition, transitions between disordered and ordered phases occur if

\[ \left. \frac{d\Phi(\eta)}{d\eta} \right|_{\eta = \eta_c} = 1, \quad \eta_c = 1/2. \]  (34)

Introducing the susceptibility in the form

\[ \chi = \frac{d\Phi(\eta)}{d\eta}, \]  (35)

we relate the critical points to the values of \( \sigma^2 \) when \( \chi = 1 \). The corresponding phase diagram is shown in Fig.3.

![Fig. 3. Phase diagram at \( \varepsilon = 0.1 \) (shaded domain corresponds to the non-critical phase transitions)](image)

Here the surface plotting with the help of the solid lines corresponds to the critical values of the system parameters, when the second-order phase transition are occur. Shaded domain corresponds to the phase transitions which are of non-critical character (such situation is observed in magnetic systems when a weak external field is introduced into the system). Here the order parameter
decreases continuously and has not singularity in the vicinity of the transition point related to the shaded domain. The related susceptibility has the broad peak in the point corresponding to the system parameters values of the shaded domain shown in Fig.3. To understand a principle change in the system behaviour one can estimate contributions of all nonlinearities appeared in the effective potential (32). Here one has a competition between the noise characterized by $D_{ef}(p)$ and the driving force $f(p)$. Indeed, at $\alpha \simeq 1$ as the term related to the force $f(p)$ as the term related to $\ln D_{ef}(p)$ are essential and lead to phase transition of a critical character. At $\alpha \ll 1$ the main nonlinearity is caused by the force $f(p)$ only. Here the system can be approximately described by $D_{ef}(p) \approx \text{const}$. At the $\alpha$-values related to the shaded domain there are two above nonlinear terms having weak nonlinearities than the term related to $D_{ef}^2(p-\eta)^2$ gives. Therefore, at such intermediate values of $\alpha$ there are no nonlinearities suppressing the order in the system. A size of the domain for such transitions $\Delta \alpha = \alpha_0 - \alpha_m$ depends on control parameters $\beta_0$ and $\varepsilon$. Let us introduce an effective order parameter $\tilde{\eta} = |1/2 - \eta|$; in the “disordered” phase with $p = 1/2$ one has $\tilde{\eta} = 0$; in the “ordered” phase $\tilde{\eta} \neq 0$. The dependencies of the effective order parameter and the susceptibility versus the exponent $\alpha$ of the effective diffusion coefficient $D_{ef}$ and noise intensity $\sigma^2$ are shown in Fig.4. From the Fig.4a one can see that at $\alpha > \alpha_0$ and $\alpha < \alpha_m$ the effective order parameter $\tilde{\eta}$ decreases critically to zero with the noise intensity growth. In the domain $\alpha \in (\alpha_m, \alpha_0)$ the dependence $\tilde{\eta}(\sigma^2)$ has a smooth falling-down character. Therefore, such transitions are non-critical. Figure 4b illustrates peak of the susceptibility $\chi$ that corresponds to the value $\chi(\alpha, \sigma^2) = 1$ and determines the second-order phase transition point. In the interval of the exponent $\alpha \in (\alpha_m, \alpha_0)$ the susceptibility $\chi$ has the broad peak which is characterizes by value $\chi \simeq 1$ and relates to non-critical phase transitions. Thus, as it follows from our calculations to obtain the critical (second-order) noise-induced phase transitions the exponent $\alpha$ should lie in the interval $(0, \alpha_m) \cup (\alpha_0, 1)$. 

![Fig. 4. The character of changing of the order parameter and the susceptibility at $\varepsilon = 0.1$, $\beta_0 = 10.0$](image)
To verify our MF results we integrate numerically Langevin equation (8) on the two-dimensional lattice according to algorithm presented above (see Eq.(29)). For the order parameter we use the formula \( \tilde{\eta} = |1/2 - \langle p \rangle| \); the generalized susceptibility measures fluctuations of the field \( p \) in the vicinity of the critical points, \( \chi \sim \langle \Delta p \rangle^2 = \langle p^2 \rangle - \langle p \rangle^2 \), averaging over noise realizations, time, and the ensemble is taking into account. Solving the discrete Langevin equation Eq.(29) numerically on a two-dimension square lattice of \( 120 \times 120 \) cells with mesh size \( l = 1 \), and integration time step \( \tau = 5 \times 10^{-3} \), we obtain dependencies \( \tilde{\eta}(\sigma^2) \) and \( \chi(\sigma^2) \) shown in Fig.5. It is seen, that the order parameter decreases critically with the noise intensity growth. The generalized susceptibility has a well pronounced peak located at the critical transition point. With an increase in the exponent \( \alpha \) the peak location of the generalized susceptibility \( \chi \) is shifted toward small values of the noise intensity \( \sigma^2 \). Our numerical results are in a good correspondence with the MF theory.

Figure 6 illustrates dependence of the order parameter and the generalized susceptibility at the system parameters values related to the shaded domain in Fig.3. It is seen that the order parameter decreases smoothly than in Fig.5a;

\[ \eta \quad \chi \]

\[ \sigma^2 \]

Fig. 5. The dependence of the order parameter \( \tilde{\eta} \) and the susceptibility \( \chi \) vs. noise intensity \( \sigma^2 \) at \( \varepsilon = 0.1, \beta_0 = 3.0 \)

Fig. 6. The dependence of the order parameter \( \tilde{\eta} \) and the susceptibility \( \chi \) vs. noise intensity \( \sigma^2 \) at \( \varepsilon = 0.1, \alpha = 0.1, \beta_0 = 10.0 \)
the generalized susceptibility has no pronounced peak. Thus, one can conclude that the corresponding phase transition is of the non-critical character.

6 Conclusions

In this paper we have considered the physical system of the reaction-diffusion kind with a field-dependent mobility and internal fluctuations. It was shown that in generalized approach basing on the nonlinear kinetic equation the field-dependent mobility is of a power-low form proposed by C.L. Emmott and A.J. Bray [11]. Considering the local dynamics of the concentration field, it was found that internal fluctuations can sustain stable periodic patterns if the noise intensity does not exceed a critical value. The mechanisms of the phase transitions have been studied with the help of the mean field theory. It was shown that an increase in the noise intensity leads to disordering phase transitions. We have found that there is a special domain of the exponent determining the power-low form of the field-dependent diffusion coefficient when disordering phase transitions are not critical. It can be explained by the competition between noise and driving force. The corresponding bifurcation and phase diagrams are verified by computer simulations.

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