COMPOSITE S-BRANE SOLUTIONS RELATED TO TODA-TYPE SYSTEMS

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Abstract

Composite S-brane solutions in multidimensional gravity with scalar fields and fields of forms related to Toda-like systems are presented. These solutions are defined on a product manifold $\mathbb{R} \times M_1 \times \ldots \times M_n$, where $\mathbb{R}$ is a time manifold, $M_1$ is an Einstein manifold and $M_i (i > 1)$ are Ricci-flat manifolds. Certain examples of S-brane solutions related to $A_1 + \ldots + A_1$, $A_m$ Toda chains and those with "block-orthogonal" intersections (e.g. $SM$-brane solutions) are singled out. Under certain restrictions imposed a Kasner-like asymptotical behaviour of the solutions is shown.

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1 Introduction

Currently, there is a certain interest to so-called $S$-brane solutions [1], i.e. space-like analogues of $D$-branes, see also [2, 3, 4, 5] and references therein.

We remind that in perturbative string theory $D$-branes [6] are hypersurfaces where open strings end and the Dirichlet boundary conditions along transverse spacelike directions are imposed. Alternatively, $D$-branes may be also described as classical solutions in supergravity theories. Open strings can also obey Dirichlet boundary conditions along time-like directions that gives rise to space-like analogues of $D$-branes, i.e. $S$-branes. (For Euclidean brane solutions in type $II$ string theories see also [7, 8].)

It is well-known that $D$-branes play an important role in studying non-perturbative aspects of string/M theory and the AdS/CFT duality [10, 11, 12]. Analogously $S$-branes are expected to play the role of $D$-branes in realizing $dS/CFT$ duality [9] in string/M theory.

Nevertheless, from pure mathematical point of view $S$-brane solutions [1, 2, 3, 4, 5] are not new ones but were considered (mostly) in some earlier publications on string cosmology, see [13, 14, 15, 16, 24] etc. They are also special cases of more general exact solutions found in certain publications of Moscow group, see [17, 18, 19, 20, 21, 22] and references therein. These publications contain a lot of exact solutions defined on product of several Ricci-flat or Einstein spaces of arbitrary signatures. Among them generalized $S$-brane solutions governed by harmonic function with brane intersections corresponding to hyperbolic algebras were considered [26]. In [30] a general class of cosmological solutions with composite $S$-branes exhibiting an oscillating behaviour near the singularity was described, e.g. using the billiard representation near the singularity.

The aim of this paper is twofold. Firstly, we single out a family of cosmological solutions with composite $S$-branes from general Toda-like cosmological-type solutions suggested in [21]. Secondly, for special intersections (e.g. ”orthogonal” ones) we get a subfamily of $S$-brane solutions, containing the main part of solutions from [1, 2, 3, 4, 5] as special cases.

The paper is organized as follows. In Section 2 cosmological-type solutions related to Toda-type systems from [21] are presented (in a more or

\footnote{The solutions from [15] contain the spacelike brane solutions subsequently obtained by Strominger and Gutperle [1] and some other authors.}
less condensed manner). In Section 3 we single out composite \(S\)-brane solutions of general (Toda-like) form. Here we also consider special subclasses of \(S\)-brane solutions, e.g. with "orthogonal", "block-orthogonal" and \(A_m\) intersection rules. In Section 4 certain examples of solutions in \(D = 11\) supergravity (describing \(SM\)-branes) are singled out. Section 5 is devoted to asymptotical Kasner-like behaviour of \(S\)-brane solutions.

2 Cosmological-type solutions related to Toda-type systems

2.1 The model

We consider a model governed by the action

\[
S_g = \int d^Dx \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)] (F^a)^2 \right\}
\]

(2.1)

where \(g = g_{MN}(x) dx^M \otimes dx^N\) is a metric, \(\varphi = (\varphi^\alpha) \in \mathbb{R}^l\) is a vector of scalar fields, \((h_{\alpha\beta})\) is a constant symmetric non-degenerate \(l \times l\) matrix \((l \in \mathbb{N})\), \(\theta_a = \pm 1\),

\[
F^a = dA^a = \frac{1}{n_a!} F_{M_1...M_n}^a dz^{M_1} \wedge ... \wedge d z^{M_n}
\]

(2.2)

is a \(n_a\)-form \((n_a \geq 1)\), \(\lambda_a\) is a 1-form on \(\mathbb{R}^l\): \(\lambda_a(\varphi) = \lambda_{aa}\varphi^\alpha\), \(a \in \Delta\), \(\alpha = 1, ..., l\). In (2.1) we denote \(|g| = |\det(g_{MN})|\),

\[
(F^a)^2 = F_{M_1...M_{n_a}}^a F_{N_1...N_{n_a}}^a g^{M_1N_1} ... g^{M_{n_a}N_{n_a}}
\]

(2.3)

\(a \in \Delta\). Here \(\Delta\) is some finite set. For pseudo-Euclidean metric of signature \((-+, ..., +)\) all \(\theta_a = 1\).

2.2 Solutions with \(n\) Ricci-flat spaces

Let us consider a family of solutions to field equations corresponding to the action (2.1) and depending upon one variable \(u\) [21]. These solutions are defined on the manifold

\[
M = (u_-, u_+) \times M_1 \times M_2 \times ... \times M_n
\]

(2.4)
where \((u_-, u_+)\) is an interval belonging to \(\mathbb{R}\). The solutions read [21]

\[
g = \left(\prod_{s \in S} [f_s(u)]^{2d(I_s)h_s/(D-2)}\right) \left\{ \exp(2c^0u + 2\overline{c}^0)wdu \otimes du + \sum_{i=1}^{n} \left(\prod_{s \in S} [f_s(u)]^{-2h_s\delta_{idi}}\right) \exp(2c^i u + 2\overline{c}^i)g^i \right\},
\]

\[
\exp(\varphi^\alpha) = \left(\prod_{s \in S} f_s^{b_s}\chi_s^{\alpha_s}\right) \exp(c^\alpha u + \overline{c}^\alpha),
\]

\[
F^a = \sum_{s \in S} \delta^a_{as} F^s,
\]

\(\alpha = 1, \ldots, l\); In (2.5) \(w = \pm 1\), \(g^i = g^i_{m_i,n_i}(y_i)dy_{i}^{m_i} \otimes dy_{i}^{n_i}\) is a Ricci-flat metric on \(M_i, i = 1, \ldots, n\),

\[
\delta_{iI} = \sum_{j \in I} \delta_{ij}
\]

is the indicator of \(i\) belonging to \(I\): \(\delta_{iI} = 1\) for \(i \in I\) and \(\delta_{iI} = 0\) otherwise. The \(p\)-brane set \(S\) is by definition

\[
S = S_e \sqcup S_m, \quad S_e = \sqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v},
\]

\(v = e, m\) and \(\Omega_{a,e}, \Omega_{a,m} \subset \Omega\), where \(\Omega = \Omega(n)\) is the set of all non-empty subsets of \(\{1, \ldots, n\}\). Here and in what follows \(\sqcup\) means the union of non-intersecting sets. Any \(p\)-brane index \(s \in S\) has the form

\[
s = (a_s, v_s, I_s),
\]

where \(a_s \in \Delta\) is colour index, \(v_s = e, m\) is electro-magnetic index and the set \(I_s \in \Omega_{a_s,v_s}\) describes the location of \(p\)-brane worldvolume.

The sets \(S_e\) and \(S_m\) define electric and magnetic \(p\)-branes, correspondingly. In (2.6)

\[
\chi_s = +1, -1
\]

for \(s \in S_e, S_m\), respectively. In (2.7) forms

\[
F^s = Q_s \left(\prod_{s' \in S} f_{s'}^{-A_{s,s'}}\right) du \wedge \tau(I_s),
\]
\( s \in S_e \), correspond to electric \( p \)-branes and forms

\[
\mathcal{F}^s = Q_s \tau(I_s),
\]

(2.13)

\( s \in S_m \), correspond to magnetic \( p \)-branes; \( Q_s \neq 0, \ s \in S \). Here and in what follows

\[
\bar{I} \equiv I_0 \setminus I, \quad I_0 = \{1, \ldots, n\}.
\]

(2.14)

All manifolds \( M_i \) are assumed to be oriented and connected and the volume \( d_i \)-forms

\[
\tau_i \equiv \sqrt{|g_i(y_i)|} \ dy_1^i \wedge \ldots \wedge dy_{d_i}^i,
\]

(2.15)

and parameters

\[
\varepsilon(i) \equiv \text{sign}(\text{det}(g_{i_{1n_i}}^i)) = \pm 1
\]

(2.16)

are well-defined for all \( i = 1, \ldots, n \). Here \( d_i = \dim M_i, \ i = 1, \ldots, n \), \( D = 1 + \sum_{i=1}^n d_i \). For any set \( I = \{i_1, \ldots, i_k\} \in \Omega, \ i_1 < \ldots < i_k \), we denote

\[
\tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k},
\]

(2.17)

\[
M(I) \equiv M_{i_1} \times \ldots \times M_{i_k},
\]

(2.18)

\[
d(I) \equiv \dim(M(I)) = \sum_{i \in I} d_i,
\]

(2.19)

\[
\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k).
\]

(2.20)

\( M(I_s) \) is isomorphic to \( p \)-brane worldvolume manifold (see (2.10)). The parameters \( h_s \) appearing in the solution satisfy the relations

\[
h_s = (B_{ss})^{-1},
\]

(2.21)

where

\[
B_{ss'} \equiv d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2-D} + \chi_s \chi_{s'} \lambda_{\alpha_s \lambda_{\alpha_{s'}}} \lambda_{\beta_{s} \beta_{s'}} h_{\alpha \beta},
\]

(2.22)

\( s, s' \in S \), with \((h^{\alpha \beta}) = (h_{\alpha \beta})^{-1}\). Here we assume that

(i) \( B_{ss} \neq 0 \),

(2.23)

for all \( s \in S \), and

(ii) \( \det(B_{ss'}) \neq 0 \),

(2.24)
i.e. the matrix \( (B_{ss'}) \) is a non-degenerate one. In (2.12) another non-degenerate matrix (a so-called "quasi-Cartan" matrix) appears

\[
(A_{ss'}) = (2B_{ss'}/B_{s's'}) .
\] (2.25)

Here some ordering in the set \( S \) is assumed.

In (2.5), (2.6)

\[
f_s = \exp(-q^s),
\] (2.26)

where \((q^s) = (q^s(u))\) is a solution to Toda-type equations

\[
\ddot{q}^s = -\varepsilon_s B_{ss'} Q_s^2 \exp\left(\sum_{s' \in S} A_{ss'} q^{s'}\right),
\] (2.27)

\(s \in S\). Here

\[
\varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2}\varepsilon(I_s)\theta_{as},
\] (2.28)

where \(\varepsilon[g] \equiv \text{sign det}(g_{MN})\). More explicitly (2.28) reads: \(\varepsilon_s = \varepsilon(I_s)\theta_{as}\) for \(v_s = e\) and \(\varepsilon_s = -\varepsilon[g]\varepsilon(I_s)\theta_{as}\), for \(v_s = m\).

Vectors \(c = (c^A) = (c^i, c^\alpha)\) and \(\bar{c} = (\bar{c}^A)\) obey the following constraints

\[
\sum_{i \in I_s} d_i c^i - \chi_s \lambda_{a_s} c^\alpha = 0, \quad \sum_{i \in I_s} d_i \bar{c}^i - \chi_s \lambda_{a_s} \bar{c}^\alpha = 0, \quad s \in S,
\] (2.29)

\[
c^0 = \sum_{j=1}^n d_j c^j, \quad \bar{c}^0 = \sum_{j=1}^n d_j \bar{c}^j,
\] (2.30)

\[
2E = 2E_{TL} + h_{\alpha\beta} c^\alpha \bar{c}^\beta + \frac{1}{2} \sum_{i=1}^n d_i (c^i)^2 - \left( \sum_{i=1}^n d_i c^i \right)^2 = 0,
\] (2.31)

where

\[
E_{TL} = \frac{1}{4} \sum_{s, s' \in S} h_s A_{ss'} \dot{q}^s \dot{q}^{s'} + \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp\left(\sum_{s' \in S} A_{ss'} q^{s'}\right)
\] (2.32)

is an integration constant (energy) for the solutions from (2.27).

Eqs. (2.27) correspond to the Toda-type Lagrangian

\[
L_{TL} = \frac{1}{4} \sum_{s, s' \in S} h_s A_{ss'} q^s q^{s'} - \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp\left(\sum_{s' \in S} A_{ss'} q^{s'}\right).
\] (2.33)
The reduction of multidimensional cosmology with composite $p$-branes (and block-diagonal metric) to Toda-like system was performed earlier in [19].

Here we identify notations for $g^i$ and $\hat{g}^i$, where $\hat{g}^i = p^*_i g^i$ is the pullback of the metric $g^i$ to the manifold $M$ by the canonical projection: $p_i : M \to M_i$, $i = 1, \ldots, n$. An analogous agreement will be also kept for volume forms etc.

Due to (2.12) and (2.13), the dimension of $p$-brane worldvolume $d(I_s)$ is defined by

$$d(I_s) = n_{a_s} - 1, \quad d(I_s) = D - n_{a_s} - 1,$$

for $s \in S_c, S_m$, respectively. For a $p$-brane we have $p = p_s = d(I_s) - 1$.

### 2.3 Solutions with one curved Einstein space and $n-1$ Ricci-flat spaces

The cosmological solution with Ricci-flat spaces may be also modified to the following case:

$$\text{Ric}[g^1] = \xi_1 g^1, \quad \xi_1 \neq 0, \quad \text{Ric}[g^i] = 0, \quad i > 1,$$

i.e. the first space $(M_1, g^1)$ is Einstein space of non-zero scalar curvature and other spaces $(M_i, g^i)$ are Ricci-flat and

$$1 \notin I_s, \quad \forall s \in S,$$

i.e. all “brane” submanifolds do not contain $M_1$.

In this case the exact solution may be obtained by a little modifications of the solutions from the previous subsection [21].

The metric reads as follows

$$g = \left( \prod_{s \in S} [f_s(u)]^{2d(I_s)h_s/(D-2)} \right) \left\{ [f_1(u - u_1)]^{2d_1/(1-d_1)} \exp(2c_1^1 u + 2\bar{c}_1^1) \right\}$$

$$\times [wdu \otimes du + f_1^2(u - u_1)g^1] + \sum_{i=2}^n \left( \prod_{s \in S} [f_s(u)]^{-2h_s \delta I_s} \right) \exp(2c_i^i u + 2\bar{c}_i^i)g^i \right\}. $$

where

$$f_1(\tau) = R \sinh(\sqrt{C_1} \tau), \quad C_1 > 0, \quad \xi_1 w > 0,$$

(2.38)
\[ R \sin(\sqrt{|C_1|} \tau), \ C_1 < 0, \ \xi_1 w > 0; \quad (2.39) \]
\[ R \cosh(\sqrt{|C_1|} \tau), \ C_1 > 0, \ \xi_1 w < 0; \quad (2.40) \]
\[ |\xi_1(d_1 - 1)|^{1/2} \tau, \ C_1 = 0, \ \xi_1 w > 0, \quad (2.41) \]

\( u_1 \) and \( C_1 \) are constants, \( R = |\xi_1(d_1 - 1)/C_1|^{1/2} \), and
\[ C_1 \frac{d_1}{d_1 - 1} = 2E, \quad (2.42) \]
where \( E \) is defined in (2.31).

Now, vectors \( c = (c^A) \) and \( \bar{c} = (\bar{c}^A) \) satisfy also additional constraints
\[ c^1 = \sum_{j=1}^{n} d_j c^j = c^0, \quad \bar{c}^1 = \sum_{j=1}^{n} d_j \bar{c}^j = \bar{c}^0. \quad (2.43) \]
All other relations from the previous subsection are unchanged.

### 2.4 Restrictions on \( p \)-brane configurations.

The solutions presented above are valid if two restrictions on the sets of \( p \)-branes are satisfied [21]. These restrictions guarantee the block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints) [25].

Let us denote \( w_1 \equiv \{ i | i \in \{1,\ldots,n\}, \ d_i = 1 \} \), and \( n_1 = |w_1| \) (i.e. \( n_1 \) is the number of 1-dimensional spaces among \( M_i, \ i = 1,\ldots,n \)).

**Restriction 1.** For any \( a \in \triangle \) and \( v = e, m \) there are no \( I, J \in \Omega_{a,v} \) such that \( I = \{i\} \cup (I \cap J) \), and \( J = (I \cap J) \cup \{j\} \) for some \( i, j \in w_1 \), \( i \neq j \).

**Restriction 2.** For any \( a \in \triangle \) there are no \( I \in \Omega_{a,e} \) and \( J \in \Omega_{a,m} \) such that \( \bar{J} = \{i\} \cup I \).

Restriction 1 is satisfied for \( n_1 \leq 1 \) and also in the non-composite case: \( |\Omega_{a,e}| + |\Omega_{a,m}| = 1 \) for all \( a \in \triangle \). For \( n_1 \geq 2 \) it forbids the following pairs of two electric or two magnetic \( p \)-branes, corresponding to the same form \( F^a, a \in \triangle \):

- \( \begin{array}{c} \bullet \\ i \\ \end{array} \quad I \)
- \( \begin{array}{c} \bullet \bullet \\ j \\ \end{array} \quad J \)
Figure 1. A forbidden by Restriction 1 pair of two electric or two magnetic p-branes.

Here $d_i = d_j = 1, \ i \neq j, \ i, j = 1, \ldots, n$. Restriction 1 may be also rewritten in terms of intersections

$$(\text{R1}) \quad d(I \cap J) \leq d(I) - 2, \quad (2.44)$$

for any $I, J \in \Omega_{a,v}, \ a \in \triangle, \ v = e, m$ (here $d(I) = d(J)$).

Restriction 2 is satisfied for $n_1 = 0$. For $n_1 \geq 1$ it forbids the following electro-magnetic pairs, corresponding to the same form $F^a, a \in \triangle$:

\begin{tikzpicture}
  \node at (0,0) {\textbullet} node[anchor=west]{$i$};
  \node at (1,0) {\textbullet} node[anchor=east]{$\bar{j}$};
  \node at (0,-1) {\textbullet} node[anchor=west]{$I$};
  \node at (1,-1) {\textbullet} node[anchor=east]{$\bar{J}$};
\end{tikzpicture}

Figure 2. Forbidden by Restriction 2 electromagnetic pair of p-branes

Here $d_i = 1, \ i = 1, \ldots, n$. In terms of intersections Restriction 2 reads

$$(\text{R2}) \quad d(I \cap J) \neq 0, \quad (2.45)$$

for $I \in \Omega_{a,e}$ and $J \in \Omega_{a,m}, \ a \in \triangle$.

**Intersection rules.** From (2.21), (2.22) and (2.25) we get the $p$-brane intersection rules corresponding to the quasi-Cartan matrix $(A_{ss'}) [19]$

$$d(I_s \cap I_{s'}) = \frac{d(I_s)d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{a_s} \cdot \lambda_{a_{s'}} + \frac{1}{2} B_{s's'} A_{ss'}, \quad (2.46)$$

where $\lambda_{a_s} \cdot \lambda_{a_{s'}} = \lambda_{a a_s} \lambda_{b a_{s'}} h^{\alpha \beta} ; \ s, s' \in S$.

### 3 Composite S-brane solutions

In this section we single out special cosmological solutions called as $S$-branes.
3.1 Toda-type solutions

In what follows we suppose that all metrics \( g^i \) have Euclidean signatures \((+, \ldots, +)\) and \( w = -1 \). Thus the total metric \( g \) has the pseudo-Euclidean signature \((-+, \ldots, +)\). We put in action (2.1) \( \theta_a = 1 \).

Then, from definitions (2.20) and (2.28) we get

\[ \varepsilon_s = \varepsilon(I_s) = 1 \] (3.1)

for all \( s \).

We also put

\[ d_1 > 1 \] (3.2)

and \( 1 \notin I_s \), i.e. all branes do not contain \( M_1 \)-submanifold.

Let us assume that the (brane) matrix \((B_{ss'})\) is a positive definite and hence

\[ B_{ss} > 0 \] (3.3)

for all \( s \in S \). This suggestion and (3.1) imply

\[ E_{TL} > 0. \] (3.4)

Now we make the following choice of parameters in the solutions from subsections 2.2 and 2.3

\[ \hat{c}^i = 0, \quad i > 1, \quad c^\alpha = 0, \] (3.5)

for all \( \alpha \) and

\[ \bar{c}^i = \bar{c}^\alpha = 0, \] (3.6)

for all \( i, \alpha \). With this choice the brane constraints (2.29) are satisfied identically due to (2.36).

With the adopted choice of parameters both solutions from subsections 2.2 and 2.3 may be unified by the following relations

\[
\begin{aligned}
g &= \left( \prod_{s \in S} H_s^{2d(I_s)h_s/(D-2)} \right) \left\{ H^{2d_1/(1-d_1)}[-dt \otimes dt + H^2 g^1] \right\} \\
&\quad + \sum_{i=2}^n \left( \prod_{s \in S} H_s^{-2h_s \delta(I_s)} g^i \right),
\end{aligned}
\] (3.7)
\[ \exp(\varphi^\alpha) = \prod_{s \in S} H_s^{h_s \chi_s \lambda^a_{ss}}, \quad (3.8) \]

\[ F^a = \sum_{s \in S_c} \delta_{ss}^a Q_s \left( \prod_{s' \in S} H_{s'}^{A_{ss'}} \right) dt \wedge \tau(I_s) + \sum_{s \in S_m} \delta_{ss}^a Q_s \tau(I_s), \quad (3.9) \]

\( a \in \triangle; \quad \alpha = 1, \ldots, l; \) where

\[ H = |\xi_1(d_1 - 1)|^{1/2} \frac{\sinh(Mt)}{M}, \quad \xi_1 > 0; \quad (3.10) \]

\[ |\xi_1(d_1 - 1)|^{1/2} \frac{\cosh(Mt)}{M}, \quad \xi_1 < 0; \quad (3.11) \]

\[ \exp(Mt), \quad \xi_1 = 0; \quad (3.12) \]

\( M > 0, \) and

\[ \frac{M^2 d_1}{d_1 - 1} = 2E_{TL} = \frac{1}{2} \sum_{s, s' \in S} h_s A_{ss'} q^s q^{s'} + \sum_{s \in S} Q_s^2 \exp(\sum_{s' \in S} A_{ss'} q^{s'}). \quad (3.13) \]

Here

\[ H_s = \exp(-q^s(t)), \quad (3.14) \]

with \( q^s(t) \) obeying Toda-type equations

\[ \ddot{q}^s = -B_{ss} Q_s^2 \exp(\sum_{s' \in S} A_{ss'} q^{s'}), \quad s \in S. \quad (3.15) \]

The metric \( g^1 \) is an Einstein metric on \( M_1: \) \( \text{Ric}[g^1] = \xi_1 g^1, \) and all other metrics are Ricci-flat, i.e. \( \text{Ric}[g^i] = 0 \) for \( i > 1. \)

In combining of two subfamilies of solutions from the previous section we used the following definitions and identifications: \( u = t, \, f_1 = H, \, f_s = H_s, \)

\( M = \sqrt{C_1} \quad (C_1 > 0) \) for \( \xi_1 \neq 0 \) and \( (d_1 - 1)c^1 = M \) for \( \xi_1 = 0. \)

### 3.2 Solutions with "orthogonal" intersections

Let us consider "orthogonal" (or \( A_1 \oplus \ldots \oplus A_1 \)) intersection rules

\[ B_{ss'} = 0, \quad s \neq s', \quad (3.16) \]

(remind that \( B_{ss'} = (U^s, U^{s'}) \) are scalar products of brane vectors [25, 22]).
Then using relations from Appendix we get

\[ H_s = |Q_s||h_s|^{-1/2} \frac{\cosh(M_s(t - t_s))}{M_s}, \tag{3.17} \]

where all \( M_s > 0 \) and \( t_s \) are constants and

\[ 2E_{TL} = \sum_{s \in S} M_s^2 h_s. \tag{3.18} \]

Here we used the notations \( M_s = \sqrt{C_s}, \ C_s > 0 \) (see Appendix). For \( \xi_1 \neq 0 \) these ”orthogonal” \( S \)-brane solutions are special cases of more general ones from [19].

### 3.3 Solutions with ”block-orthogonal” intersections

Let us suppose that

\[ B_{ss'} = 0, \quad s \in S_i, \ s' \in S_j, \ i \neq j, \tag{3.19} \]

where

\[ S = S_1 \sqcup \ldots \sqcup S_k, \tag{3.20} \]

\( S_i \neq \emptyset, \ i, j = 1, \ldots, k \). Relation (3.20) means that the set \( S \) splits into \( k \) mutually non-intersecting subsets (blocks) \( S_1, \ldots, S_k \).

Then using relations from Appendix we get

\[ H_s = \left[ |Q_s||h_s|^{-1/2} \frac{\cosh(M_s(t - t_s))}{M_s} \right]^{b_s}, \tag{3.21} \]

where all \( M_s > 0 \) and \( t_s \) are constants coinciding inside ”blocks”:

\[ t_s = t_{s'}, \quad M_s = M_{s'}, \tag{3.22} \]

\( s, s' \in S_i, \ i = 1, \ldots, k \). The charges should be proportional to each other inside blocks

\[ \frac{Q_s^2}{b_s h_s} = \frac{Q_{s'}^2}{b_{s'} h_{s'}}, \tag{3.23} \]
\[ s, s' \in S_i, \ i = 1, \ldots, k. \] Here

\[ b_s = 2 \sum_{s' \in S} A^{ss'}, \quad (A^{ss'}) = (A_{ss})^{-1}. \]  \hfill (3.24)

For the Toda-like part of energy we get from Appendix

\[ 2E_{TL} = \sum_{s \in S} M_s^2 b_s h_s. \]  \hfill (3.25)

More general "block-orthogonal" solutions where considered in [27, 28, 29].

### 3.4 Solutions related to \( A_m \) Toda chain

Here we consider exact solutions to Toda-chain (3.15) equations [33] corresponding to the Lie algebra \( A_m = sl(m+1, \mathbb{C}) \) with the Cartan matrix

\[
(A_{ss'}) = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
& & & & & \cdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}
\] \hfill (3.26)

\( s, s' \in S = \{1, \ldots, m\}. \)

The Toda chain equations have the following solutions [34, 20, 21]

\[ H_s = C_s^{-1} \sum_{r_1 < \ldots < r_s} v_{r_1} \cdots v_{r_s} \Delta^2(w_{r_1}, \ldots, w_{r_s}) \exp[(w_{r_1} + \ldots + w_{r_s})t], \] \hfill (3.27)

\( s = 1, \ldots, m, \) where

\[ \Delta(w_{r_1}, \ldots, w_{r_s}) = \prod_{i<j} (w_{r_i} - w_{r_j}); \quad \Delta(w_{r_1}) \equiv 1, \] \hfill (3.28)

denotes the Vandermonde determinant. The real constants \( v_r \) and \( w_r, \ r = 1, \ldots, m+1, \) obey the relations

\[ \prod_{r=1}^{m+1} v_r = \Delta^{-2}(w_1, \ldots, w_{m+1}), \quad \sum_{r=1}^{m+1} w_r = 0. \] \hfill (3.29)
In (3.27)
\[ C_s = \prod_{s'=1}^{m} (B_{s's'}Q_{s'}^2)^{-A_{s's'}} \]  
(3.30)
where
\[ A_{s's'} = \frac{1}{m+1} \min(s, s') [m + 1 - \max(s, s')] \]  
(3.31)
s, s' = 1, \ldots, m, [35]. Here \( v_r \neq 0 \) and \( w_r \neq w_r', \) \( r \neq r'; \) \( r, r' = 1, \ldots, m + 1. \) Due to \( B_{ss} > 0, \) \( s \in S, \) all \( w_r, v_r \) are real, and, moreover, all \( v_r > 0, \) \( r = 1, \ldots, m + 1 \) [21].

For the (Toda) energy we get
\[ E_{TL} = \frac{h}{4} \sum_{r=1}^{m+1} w_r^2 \]  
(3.32)
where \( h = (B_{ss})^{-1}, s \in S \) (here all \( B_{ss} \) are equal).

It should be noted that pioneering cosmological solutions with \( p \)-branes related to Toda chains were considered earlier in [23, 15, 24].

4 Examples

Here we present well-known \( SM \)-brane solutions in \( D = 11 \) supergravity [32].

4.1 Solutions for algebra \( A_1 \)

We start with single \( S \)-branes.

4.1.1 \( SM2 \)-brane

Let \( n = 3, \) \( d_2 = 3. \) The \( SM2 \)-brane solution reads
\[ g = H_1^{1/3} \left\{ H^{2d_1/(1-d_1)} [-dt \otimes dt + H^2 g^1] + H_1^{-1} g^2 + g^3 \right\}, \]  
(4.1)
\[ F = Q_1 H_1^{-2} dt \wedge \tau_2, \]  
(4.2)
where \( H \) is defined in (3.10)-(3.12), \( H_1 \) is defined in (3.17) (\( h_1 = 1/2 \)) and \( 2M^2 d_1/(d_1 - 1) = M_1^2. \)
4.1.2 SM5-brane

Let us consider the magnetic solution dual to SM2. We put \( n = 3 \), \( d_2 = 6 \). The solution reads

\[
\begin{align*}
g & = H_1^{2/3} \left\{ H_2^{2d_1/(1-d_1)} [-dt \otimes dt + H^2 g^1] + H_1^{-1} g^2 + g^3 \right\}, \\
F & = Q_1 \tau_1 \wedge \tau_3,
\end{align*}
\]

where \( H \) and \( H_1 \) are defined in (3.10)-(3.12) and (3.17), respectively, \( h_1 = 1/2 \) and \( 2M^2 d_1/(d_1 - 1) = M_1^2 \).

4.2 SM2 \( \cap \) SM5-branes with \( A_1 \oplus A_1 \) intersection

Here we present a "superposition" of SM2 and SM5 solutions corresponding to "orthogonal intersection": \( d(I_1 \cap I_2) = 2 \). We put \( n = 5 \), \( d_1 = 2 \), \( d_2 = 1 \), \( d_3 = 2 \), \( d_4 = 4 \), \( d_5 = 1 \) and \( I_1 = \{2,3\} \), \( I_2 = \{3,4\} \). The solution reads

\[
\begin{align*}
g & = H_1^{1/3} H_2^{2/3} \left\{ H^{-4} [-dt \otimes dt + H^2 g^1] + H_1^{-1} g^2 + H_1^{-1} g^3 \right\}, \\
F & = Q_1 H_1^{-2} dt \wedge \tau_2 \wedge \tau_3 + Q_2 \tau_1 \wedge \tau_2 \wedge \tau_5,
\end{align*}
\]

where \( H \) and \( H_s \) are defined in (3.10)-(3.12) and (3.17), respectively (all \( h_s = 1/2 \)) and \( 4M^2 = M_1^2 + M_2^2 \).

4.3 SM2 \( \cap \) SM5-branes with \( A_2 \) intersection

Now we consider a (dyonic) solution consisting of SM2 and SM5 branes and with \( A_2 \) intersection: \( d(I_1 \cap I_2) = 1 \) [22]. We put \( n = 4 \), \( d_1 = d_2 = 2 \), \( d_3 = 1 \), \( d_4 = 5 \) and \( I_1 = \{2,3\} \), \( I_2 = \{3,4\} \). The solution reads (see subsection 3.3)

\[
\begin{align*}
g & = H_1 \left\{ H^{-4} [-dt \otimes dt + H^2 g^1] + H_1^{-1} g^2 + H_1^{-2} g^3 + H_1^{-1} g^4 \right\}, \\
F & = Q_1 H_1^{-1} dt \wedge \tau_2 \wedge \tau_3 \pm Q_1 \tau_1 \wedge \tau_2,
\end{align*}
\]
where $H$ is defined in (3.10)-(3.12) and
\[ H_1 = \left[ |Q_1| \frac{\cosh(M(t - t_1))}{M} \right]^2. \]  
(4.9)

This solution is also a special case of $A_2$ Toda chain solution (with $Q_1^2 = Q_2^2$) from subsection 3.4.

5 Kasner-like asymptotical behaviour

Here we consider asymptotical behaviour of $S$-brane solutions in the limit $t \to +\infty$, when
\[ H \sim \text{const} \exp(Mt), \quad H_s \sim \text{const} \exp(M_s t), \]  
(5.1)
$s \in S$.

It may be verified by a straightforward calculation for one-brane case that in the limit $t \to +\infty$ the metric and scalar fields have asymptotical Kasner-like behaviour
\[ g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^{n} A_i \tau^{2\alpha^i} g^i, \]  
(5.2)
\[ \varphi_{as} = \alpha^\beta \ln \tau + \text{const}, \]  
(5.3)
as $\tau \to +0$ ($\tau$ is synchronous time variable) with the set of Kasner parameters $\alpha = (\alpha^i, \alpha^\gamma)$, obeying
\[ \sum_{i=1}^{n} d_i \alpha^i = \sum_{i=1}^{n} d_i (\alpha^i)^2 + \alpha^\beta \alpha^\gamma h_{\beta\gamma} = 1. \]  
(5.4)

Here $\tau \sim \text{const} \exp(-\mathcal{M}t)$ for $t \to +\infty$, where $\mathcal{M} > 0$ is a linear combination of "masses" $M, M_s$. Analogous (though more tedious) calculations for several branes with orthogonal intersections give an analogous result.

Here we show that the Kasner-like asymptotical behaviour may be proved using the billiard representation for multidimensional cosmology with branes [30] under restrictions imposed in Section 3 and the following additional condition added: the matrix $(h_{\alpha\beta})$ is supposed to be positive definite. According
to results of refs. [30] for the proof of the Kasner-like asymptotical behaviour it is sufficient to verify that there exists a Kasner set obeying the inequalities

$$U_s(\alpha) = \sum_{i \in I_s} d_i \alpha^i - \chi_s \lambda_{a_s} \alpha^\gamma > 0,$$

where $\alpha^1 = (d_1 - \Delta)/[d_1(D - 1)]$, $\alpha^2 = ... = \alpha^n = (d + \Delta)/[d(D - 1)]$, and $\alpha^\gamma = 0$. Due to condition $1 \notin I_s$ this Kasner set does satisfy the relations (5.5).

When all $d_i > 1$ and $\alpha^\beta \alpha^\gamma h_{\beta\gamma} = 0$, then the Riemann tensor squared of the Kasner-type metric (5.2) diverges for $\tau \to +0$ [31] and, hence, the Riemann tensor squared of the original metric $g$ diverges as $t \to +\infty$. For $\alpha^\beta \alpha^\gamma h_{\beta\gamma} \neq 0$ the scalar curvature of (5.2) diverges as $\tau \to +0$ that implies $R[g] \to \infty$ for $t \to +\infty$.

Analogous consideration may be carried out for the solution with $H$ from (3.11) in the limit $t \to -\infty$.

**Remark. Conical singularity.** Let us consider the asymptotical behaviour of the solution with $H$ from (3.10) when $t \to +0$. In general case we get a conical singularity in this limit. This singularity is absent in special case when $(M_1, g_1)$ is unit Lobachevsky space of dimension $d_1$ (and $\xi_1 = -(d_1 - 1)$). This is a well-known Milne-type resolution of singularity.

**Remark. S-brane cosmology with oscillating behaviour near the singularity.** In the pioneering paper on billiard approach for multidimensional cosmology with $p$-branes [30] a large variety of composite $S$-brane solutions with never-ending oscillating behaviour near the singularity was described (and certain examples were considered). The open problem is to find examples of such type solutions (with block-orthogonal metrics) in string cosmology. We note that recent results of Damour and Henneaux on chaotic behaviour in $D = 10, 11$ supergravities [36, 37] (based on inequalities (5.5)) deal with 1-dimensional manifolds $M_1 = \ldots = M_n$ and sets of (composite) electric branes with "forbidden" intersections (see subsection 2.4) that lead to additional constraints [25]. The recent mathematical analysis carried out in the paper [38] gives certain arguments in favour of asymptotically
block-orthogonal (but not obviously block-orthogonal) form of metric in the vicinity of singularity for pure cosmological solutions in string cosmology with composite $p$-brane configurations from [36, 37].

6 Conclusions and discussions

In this paper a family of Toda-like composite $S$-brane solutions was presented. These ($S$-brane) solutions are special case of more general cosmological-type solutions from [21]. They generalize $S$-brane solutions from refs. [1, 2, 3, 4, 5] to a composite configurations with next to arbitrary intersecting rules.

Here several subclasses of $S$-brane solutions with "orthogonal", "block-orthogonal" and $A_m$ intersections were considered and certain examples of solutions in $D = 11$ supergravity (describing $SM$-branes with $A_1$, $A_1 \oplus A_1$ and $A_2$ intersections), were singled out.

The solutions under consideration have (in general position) an asymptotical Kasner-like behaviour near the singularity when $t \to +\infty$. This fact was shown here using the billiard approach for $p$-brane cosmology [30] under the assumption $1 / I_s$ imposed (i.e. when all branes do not contain $M_1$-submanifold). The relaxing of this assumption may lead to $S$-brane configurations with oscillating behaviour near the singularity (see [30]).

We remind that in refs. [19, 39] the Wheeler-DeWitt (WDW) equation for the quantum cosmology with composite electro-magnetic $p$-branes (e.g. $S$-branes) defined on product of Einstein spaces was obtained (for non-composite electric case see also [17]). In these papers the WDW equation was integrated for intersecting $p$-branes with orthogonal $U$-vectors, when $n - 1$ internal spaces are Ricci-flat and one is the Einstein space of a non-zero curvature (for non-composite electric case see [17]). A slightly different approach with classical field of forms (and a special brane setup) was suggested in [40]. An open problem is to generalize classical and quantum $S$-brane solutions from [19] (corresponding to the Lie algebra $A_1 \oplus \ldots \oplus A_1$) to other semisimple Lie algebras.
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Appendix

Solutions for "block-orthogonal" intersections. Here we consider "block-orthogonal" case (3.19), (3.20). In this case the moduli functions (2.26) read [29, 28, 22]

\[ f_s = (\bar{f}_s)^b_s, \quad (A.1) \]

with powers \( b_s \) defined in (3.24) and

\[ \bar{f}_s(u) = R_s \sinh(\sqrt{C_s}(u - u_s)), \quad C_s > 0, \quad \eta_s \varepsilon_s < 0; \quad (A.2) \]

\[ R_s \sin(\sqrt{|C_s|}(u - u_s)), \quad C_s < 0, \quad \eta_s \varepsilon_s < 0; \quad (A.3) \]

\[ R_s \cosh(\sqrt{C_s}(u - u_s)), \quad C_s > 0, \quad \eta_s \varepsilon_s > 0; \quad (A.4) \]

\[ |Q^s_b|b_s h_s|^{-1/2}(u - u_s), \quad C_s = 0, \quad \eta_s \varepsilon_s < 0, \quad (A.5) \]

where \( R_s = |Q^s_b|b_s h_s C_s|^{-1/2}, \quad \eta_s = \text{sign}(b_s h_s) = \pm 1, \quad C_s, \quad u_s - \) are constants, \( s \in S, \) coinciding inside "blocks": \( u_s = u_{s'}, \quad C_s = C_{s'}, \quad s, s' \in S_i; \quad i = 1, \ldots, k. \)

The charges should be proportional to each other inside blocks

\[ \varepsilon_s Q^2_s b_s h_s = \varepsilon_{s'} Q^2_{s'} b_{s'} h_{s'}, \quad (A.6) \]

\( s, s' \in S_i, \quad i = 1, \ldots, k. \)

The Toda part of the energy reads in this case

\[ E_{TL} = \frac{1}{2} \sum_{s \in S} C_s b_s h_s. \quad (A.7) \]

"Orthogonal" case. In the "orthogonal" case when all blocks consist of one element, i.e. \( |S_i| = \ldots = |S_k| = 1 \) [19], the quasy-Cartan matrix is diagonal \( A = \text{diag}(2, \ldots, 2), \) all \( b_s = 1 \) and

\[ f_s = \bar{f}_s, \quad (A.8) \]

\( s \in S. \)
References

[1] M. Gutperle and A. Strominger, Spacelike branes, *JHEP* **0204**, 018 (2002); hep-th/0202210.

[2] C.M. Chen, D.M. Gal’tsov and M. Gutperle, S-brane solutions in supergravity theories, hep-th/0204071.

[3] M. Kruczenski, R.C. Myers and A.W. Peet, Supergravity S-branes, *JHEP* **0205**, 039 (2002); hep-th/0204144.

[4] S. Roy, On supergravity solutions of space-like Dp-branes, hep-th/0205198.

[5] N.S. Degger and A. Kaya, Intersecting S-brane solutions of $D = 11$ supergravity, hep-th/0206057.

[6] J. Polchinski, Dirichlet-Branes and Ramond-Ramond charges, *Phys. Rev. Lett.* **75**, 4724 (1995); hep-th/9510017.

[7] C.M. Hull, Timelike T-Duality, de Sitter space, large $N$ gauge theories and topological field theory, *JHEP* **9807**, 021 (1998); hep-th/9806146.

[8] C.M. Hull, De Sitter space in supergravity and $M$ theory, *JHEP* **0111**, 012 (2001); hep-th/0109213.

[9] A. Strominger, The dS/CFT correspondence, *JHEP* **0110**, 034 (2001); hep-th/0106113.

[10] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2** (1998) 231, hep-th/9711200.

[11] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge theory correlators from non-Critical string theory, *Phys. Lett.* B **428**, 105 (1998); hep-th/9802109.

[12] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* **2**, 253 (1998); hep-th/9802150.

[13] K. Behrndt and S. Forste, String Kaluza-Klein cosmology, *Nucl. Phys.* B **430** 441 (1994) 441; hep-th/9403179 (see also hep-th/9312167, hep-th/9704013).

[14] A. Lukas, B.A. Ovrut and D. Waldram, Cosmological solutions in type II string theory, *Phys. Lett.* B **393**, 65 (1997); hep-th/9608195.

[15] H. Lü, S. Mukherji, C.N. Pope and K.-W. Xu, Cosmological solutions in string theories, *Phys. Rev.* D **55**, 7926-7935 (1997); hep-th/9610107.
[16] H. Lü, S. Mukherji and C.N. Pope, From p-branes to cosmology, Int. J. Mod. Phys. A 14 4121-4142 (1999); hep-th/9612224.

[17] M.A. Grebeniuk, V.D. Ivashchuk and V.N. Melnikov, Integrable multidimensional quantum cosmology for intersecting p-branes, Grav. and Cosmol. 3, No 3 (11), 243-249 (1997), gr-qc/9708031.

[18] K.A. Bronnikov, M.A. Grebeniuk, V.D. Ivashchuk and V.N. Melnikov, Integrable multidimensional cosmology for intersecting p-branes, Grav. Cosmol. 3, No 2(10), 105-112 (1997).

[19] V.D. Ivashchuk and V.N. Melnikov, Multidimensional classical and quantum cosmology with intersecting p-branes, hep-th/9708157; J. Math. Phys., 39, 2866-2889 (1998).

[20] V.R. Gavrilov and V.N. Melnikov, Toda chains associated with Lie algebras $A_m$ in multidimensional gravitation and cosmology with intersecting p-branes, Theor. Math. Phys. 123, No 3, 374-394 (2000) (in Russian).

[21] V.D. Ivashchuk and S.-W. Kim, Solutions with intersecting p-branes related to Toda chains, J. Math. Phys. 41, 444-460 (2000); hep-th/9907019.

[22] V.D. Ivashchuk and V.N. Melnikov, Exact solutions in multidimensional gravity with antisymmetric forms, topical review, Class. Quantum Grav., 18, R87-R152 (2001); hep-th/0110274.

[23] H. Lü, C.N. Pope and K-W. Xu, Liouville and Toda solutions of M-theory, Mod. Phys. Lett. A 11, 1785-1796 (1996); hep-th/9604058.

[24] H. Lü and C.N. Pope, SL(N+1,R) Toda solitons in supergravities, Int. J. Mod. Phys. A 12, 2061 -2074 (1997); hep-th/9607027.

[25] V.D. Ivashchuk and V.N. Melnikov, Sigma-model for the generalized composite p-branes, Class. Quantum Grav. 14, 3001 (1997); Corrigenda ibid. 15 (12), 3941 (1998); hep-th/9705036.

[26] V.D. Ivashchuk and V.N. Melnikov, Madjumdar-Papapetrou type solutions in sigma-model and intersecting p-branes, Class. Quantum Grav. 16, 849 (1999); hep-th/9802121.

[27] K.A. Bronnikov, Block-orthogonal brane systems, black holes and wormholes, Grav. Cosmol. 4, No 1 (13), 49 (1998); hep-th/9710207.
[28] V.D. Ivashchuk and V.N. Melnikov. Multidimensional cosmological and spherically symmetric solutions with intersecting p-branes. In Lecture Notes in Physics, Vol. 537, "Mathematical and Quantum Aspects of Relativity and Cosmology Proceedings of the Second Samos Meeting on Cosmology, Geometry and Relativity held at Pythagoreon, Samos, Greece, 1998, eds: S. Cotsakis, G.W. Gibbons., Berlin, Springer, 2000; gr-qc/9901001.

[29] V.D. Ivashchuk and V.N. Melnikov, Cosmological and spherically symmetric solutions with Intersecting p-branes, J. Math. Phys., 40, 6558-6576 (1999).

[30] V.D. Ivashchuk and V.N. Melnikov, Billiard representation for multidimensional cosmology with intersecting p-branes near the singularity, J. Math. Phys., 41, No 8, 6341-6363 (2000); hep-th/9904077.

[31] V.D. Ivashchuk and V.N. Melnikov, On singular solutions in multidimensional gravity, Grav. and Cosmol. 1, 204 (1996); hep-th/9612089.

[32] E. Cremmer, B. Julia, J. Scherk, Phys. Lett. B 76, 409 (1978).

[33] M. Toda, Progr. Theor. Phys. 45, 174 (1970).

[34] A. Anderson, J. Math. Phys. 37, 1349 (1996); hep-th/9507092.

[35] J. Fuchs and C. Schweigert, Symmetries, Lie algebras and Representations, A graduate course for physicists (Cambridge University Press, Cambridge, 1997).

[36] T. Damour and M. Henneaux, Chaos in superstring cosmology, Phys. Rev. Lett. 85, 920 (2000); hep-th/0003139.

[37] T. Damour and M. Henneaux, Oscillatory behaviour in homogeneous string cosmology models, Phys. Lett. B 488, 108 (2000); Erratum ibid. 491 377; hep-th/0006171.

[38] T. Damour, M. Henneaux, A.D. Rendall and M. Weaver, Kasner-like behaviour for subcritical Einstein-matter systems, models, gr-qc/0202069.

[39] V.D. Ivashchuk and V.N. Melnikov, Multidimensional quantum cosmology with intersecting p-branes, Hadronic J. 21, 319-335 (1998).

[40] H. Lü, J. Maharana, S. Mukherji and C.N. Pope, Cosmological solutions, p-branes and the Wheeler De Witt equation, Phys. Rev. D 57 , 2219-2229 (1997); hep-th/9707182.