A CHARACTERISTIC NUMBER OF HAMILTONIAN BUNDLES
OVER $S^2$

ANDRÉS VIÑA

Abstract. Each loop $\psi$ in the group $\text{Ham}(M)$ of Hamiltonian diffeomorphisms of a symplectic manifold $M$ determines a fibration $E$ on $S^2$, whose coupling class is denoted by $c$. If $VTE$ is the vertical tangent bundle of $E$, we relate the characteristic number $\int_E c_1(VTE)c^n$ with the Maslov index of the linearized flow $\psi^*_t$ and the Chern class $c_1(TM)$. We give the value of this characteristic number for loops of Hamiltonian symplectomorphisms of Hirzebruch surfaces.

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1. Introduction

A loop $\psi : S^1 \to \text{Ham}(M, \omega)$ in the group of Hamiltonian diffeomorphisms of a symplectic manifold $(M^{2n}, \omega)$ can be considered as a clutching function of a Hamiltonian fibration $E \xrightarrow{\pi} S^2$ with fibre $M$. The total space $E$ supports the coupling class $c \in H^2(E, \mathbb{R})$; this is the unique class such that $c^{n+1} = 0$, and $\iota_p^*(c)$ is the cohomology class of the symplectic structure on the fibre $\pi^{-1}(p)$, where $\iota_p$ is the inclusion of $\pi^{-1}(p)$ in $E$. Furthermore one can consider on $E$ the first Chern class $c_1(VTE)$ of the vertical tangent bundle of $E$. These canonical cohomology classes on $E$ determine the characteristic number (see (1.1))

$$I_\psi = \int_E c_1(VTE)c^n,$$

which depends only on the homotopy class of $\psi$. Since $I$ is an $\mathbb{R}$-valued group homomorphism on $\pi_1(\text{Ham}(M, \omega))$, the non vanishing of $I$ implies that the group $\pi_1(\text{Ham}(M, \omega))$ is infinite. That is, $I$ can be used to detect the infinitude of the corresponding homotopy group. Furthermore $I$ calibrates the Hofer’s norm $\nu$ on $\pi_1(\text{Ham}(M, \omega))$ in the sense that $\nu(\psi) \geq C|I_\psi|$, for all $\psi$, where $C$ is a positive constant.

$I$ is a generalization of the mixed action-Maslov homomorphism introduced by Polterovich for monotone manifolds, that is, when $[\omega] = ac_1(TM)$ and $a > 0$. The value of this mixed action-Maslov homomorphism on a loop $\psi$ is, in many cases, easy to calculate, since it is a linear combination of the symplectic action around any orbit $\{\psi_t(x_0)\}_t$ and the Maslov index of the linearized flow $(\psi_t)_*$ along this orbit. By contrast, $I$ is defined for Hamiltonian loops in general manifolds (non necessarily monotone), and its value is mostly not so easy to determine from the definition.

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Our purpose in this note is to obtain an explicit expression for $I_\psi$, which can be used to calculate its value. More precisely, when the bundle $TM$ admits local symplectic trivializations whose domains are fixed by the diffeomorphisms $\psi_t$, we deduce a formula for $I_\psi$ in which appear a contribution related to the Maslov indices of the linearized flow $\psi_1$, in the trivializations, and a second one in which are involved transition functions of the bundle $\det(TM)$. The second contribution is related with the Chern class $c_1(TM)$ in the following sense. Using the expression of $c_1(M)$ in terms of the transition functions of $TM$ determined by the trivializations, $\langle c_1(M)\omega^{n-1}, M \rangle$ can be written as a sum $\sum_j \int_{R_j} \sigma_j$, where $\sigma_j$ is a $2n-1$ form (see (3.10)). It turns out that the second contribution is equal to this sum “weighted” by a multiple of the Hamiltonian $f_t$ which generates $\psi$; more concretely, that contribution is $-n \sum_j \int_{R_j} (f_t \circ \psi_t) \sigma_j$.

Let $(M, \omega, f)$ be an integrable system such that the points where the integrals of motion are dependent form a set $P$ which is union of codimension 2 submanifolds of $M$, and such that $M \setminus P$ is invariant under $\psi_t$ and on it there exist action-angle coordinates. Furthermore we assume that there are $\psi_t$ invariant Darboux charts which cover $P$. Then the expression of $I_\psi$ in this atlas reduces to the aforesaid second contribution; that is, $I_\psi = -n \sum_j \int_{R_j} f \sigma_j$.

The paper is organized as follows. In Section 2 we recall the construction of the coupling class $c$ following [2]. Section 3 is concerned with the proof of the mentioned expression for $I_\psi$. First we express $\langle c_1(M)\omega^{n-1}, M \rangle$ as the sum $\sum_j \int_{R_j} \sigma_j$ of integrals of $2n-1$ forms, and next we use this result to prove the formula for the invariant $I_\psi$. In Section 4 we check and apply the formulae obtained in Section 3. Using these formulae, we calculate $I_\psi$, when $\psi$ is the loop in $\Ham(S^2)$ generated by the 1-turn rotation of $S^2$ around the z-axis. The result $I_\psi = 0$ agrees with the fact that $\pi_1(\Ham(S^2)) = \mathbb{Z}_2$ and $I$ is a group homomorphism on $\Ham(M)$. We also prove that $I$ on $\pi_1(\Ham(T^{2n}))$ vanishes identically. When $n = 1$ this result is consistent with the fact that $\pi_1(\Ham(T^2)) = 0$. Finally we determine the value of $I$ on the loops generated by action of $\mathbb{T}^2$ on a general symplectic Hirzebruch surface (see Theorem 3).

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2. The coupling class

Let $(M, \omega)$ be a compact connected symplectic $2n$-manifold. Let $\psi : S^1 = \mathbb{R}/\mathbb{Z} \to \Ham(M, \omega)$ be a loop in the group $\Ham(M, \omega)$ at id. By $X_t$ is denoted the time-dependent vector field generated by $\psi_t$ and $f_t$ is the normalized time-dependent Hamiltonian; that is,

$$\frac{d\psi_t}{dt} = X_t \circ \psi_t, \quad \iota_{X_t} \omega = -df_t, \quad \int_M f_t \omega^n = 0.$$

Given $\epsilon$, with $0 < \epsilon < \pi/2$, we set

$$D_\alpha^\circ := \{p \in S^2 \mid 0 \leq \theta(p) < \pi/2 + \epsilon\}$$

$$D_\alpha := \{p \in S^2 \mid \pi/2 - \epsilon < \theta(p) \leq \pi\},$$

where $\theta \in [0, \pi]$ is the polar angle from the z-axis.
Next we construct the Hamiltonian bundle $E$ over $S^2$ determined by $\psi$. First of all we extend $\psi$ to a map defined on $F := D^2_+ \cap D^2_-$ by putting $\psi(\theta, \phi) = \psi_t$, with $t = \phi/2\pi$, with $\phi$ the spherical azimuth angle. We set

\[ E = [(D^2_+ \times M) \cup (D^2_- \times M)]/ \sim, \quad \text{where} \]

\[ (\pm, p, x) \simeq (-, p', y) \iff \begin{cases} p = p' \in F, \\ y = \psi_t^{-1}(x), \ t = \phi(p)/2\pi. \end{cases} \]

In this way $M \hookrightarrow E \xrightarrow{\pi} S^2$ is a Hamiltonian bundle over $S^2$.

We assume that $D^2_\pm$ are endowed with the orientations induced by the usual one of $S^2$ (that is, the orientation of $S^2$ as border of the unit ball). We suppose that $S^1$ is oriented by $dt = d\phi/2\pi$, that is, $S^1$ is oriented as $\partial D_+$. In $E$ one considers the orientation induced by the one defined on $D^2_\pm \times M$ by $d\theta \wedge d\phi \wedge \omega^n$.

Let $\alpha$ be a monotone smooth map $\alpha : [\pi/2 - \epsilon, \pi] \to [0, 1]$, with $\alpha(\theta) = 1$ for $\theta \in [\pi/2 - \epsilon, \pi/2 + \epsilon]$ and $\alpha(\theta) = 0$ for $\theta$ near $\pi$. Now we consider the 2-form (see [1])

\[ \tau = \begin{cases} \omega, & \text{on } D^2_+ \times M \\ \omega + d(\alpha(f_t \circ \psi_t)) \wedge dt, & \text{on } D^2_- \times M. \end{cases} \]

As $\alpha$ vanishes near $\pi$, $\tau$ is well-defined on $D^2_\pm \times M$; moreover on $F \times M \subset D^2_\pm \times M$, $\tau$ reduces to $\omega + d(f_t \circ \psi_t) \wedge dt$. If we denote by $h$ the map

\[ h : F \times M \subset D^2_- \times M \to F \times M \subset D^2_+ \times M \]

given by $h(p, x) = (p, \psi_t(x))$, with $t = \phi(p)/2\pi$, then taking into account that $h_\ast \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + X_t \circ \psi_t$, it follows from $i_{X_t} \omega = -df_t$ that $h_\ast \omega = \omega + d(f_t \circ \psi_t) \wedge dt$.

So one has the following Proposition

**Proposition 1.** $\tau$ defines a closed 2-form on $E$.

Moreover the cohomology class $[\tau] \in H^2(E, \mathbb{R})$ restricted to each fibre coincides with $[\omega]$. On the other hand

\[ \int_E \tau^{n+1} = (n + 1) \int_{D^2_\pm \times M} (f_t \circ \psi_t) \alpha'(\theta) d\theta \wedge dt \wedge \omega^n. \]

From the normalization condition for $f_t$ it follows that $\int_E \tau^{n+1} = 0$. Hence $[\tau]$ is the coupling class $c$ of the fibration $E$ [6, 7].

**3.** **The characteristic number $I_\psi$.**

Denoting $TM = \{v_x \in T_x M \mid x \in M\}$, we put

\[ VTE = [(D^2_+ \times TM) \cup (D^2_- \times TM)]/ \sim, \]

with

\[ (+, p, v_x) \simeq (-, p', v_x') \iff p = p', \ x' = \psi_t^{-1}(x), \ v_x' = (\psi_t^{-1})_\ast(v_x) \]

where $t = \phi(p)/2\pi$. So $VTE$ is a vector bundle over $E$; by construction it is the vertical tangent bundle of $E$.

Let $(U; X_1, \ldots, X_{2n})$ be a symplectic trivialization of $TM$ on $U \subset M$, and $(V; Y_1, \ldots, Y_{2n})$ be a symplectic trivialization on $V \subset M$. We put

\[ U \pm := \{[\pm, p, x] \mid p \in D^2_\pm, x \in U\} \]
and similarly for $V_+$. Denoting $x_t := \psi_t^{-1}(x)$ one has
\[
U_+ \cap U_- = \{ [+p, x] \mid p \in F, x \in U, x_t \in U \}
\]
\[
V_+ \cap V_- = \{ [+p, x] \mid p \in F, x \in V, x_t \in V \}
\]
\[
V_- \cap U_- = \{ [-p, x] \mid p \in D^2_-, x \in V \cap U \}
\]
\[
U_+ \cap V_+ = \{ [+p, x] \mid p \in D^2_+, x \in V \cap U \}
\]
The corresponding transition functions of $VTE$ are
\[
g_{U_- U_+}( [+p, x]) = A(t, x) \in Sp(2n, \mathbb{R}) \text{, with } \psi_t^{-1}(X_i(x)) = \sum_k A^k_i(t, x) X_k(x_t)
\]
\[
g_{V_+ V_-}( [+p, x]) = B(t, x) \in Sp(2n, \mathbb{R}) \text{, with } \psi_t^{-1}(Y_i(x)) = \sum_k B^k_i(t, x) Y_k(x_t)
\]
\[
g_{V_- V_+}( [-p, x]) = R(x) = g_{U_- U_+}( [+p, x]) \text{, with } \psi_t(x)) = \sum_k R^k_i(x) X_k(x)
\]

We denote by $\rho$ the usual map $\rho : Sp(2n, \mathbb{R}) \to U(1)$ which restricts to the determinant map on $U(n)$ [10], then $l_{ab} := \rho \circ g_{ab}$ is a transition function for $\det(VTE)$. We also use the following notation, the matrices in $Sp(2n, \mathbb{R})$ are denoted with capital letters and its images by $\rho$ will be denoted by the corresponding small letters; that is,
\[
a(t, x) := \rho(A(t, x)), \quad b(t, x) := \rho(B(t, x)), \quad r_{UV}(x) := \rho(R(x)).
\]

If $\psi_t(U) \subset U$ for all $t$, given $x \in U$, the winding number of the map $t \in S^1 \to a^{-1}(t, x) \in U(1)$ is the integer
\[
\frac{i}{2\pi} \int_0^1 a^{-1}(t, x) \frac{\partial a}{\partial t}(t, x) dt.
\]
This integer is independent of the point $x \in U$, it will be denoted $J_U$. The number $J_U$ is the Maslov index in $U$ of the linearized flow $\psi_*$. Analogously, if $\psi_t(V) \subset V$ for all $t$ we have the integer
\[
J_V = \frac{i}{2\pi} \int_0^1 b^{-1}(t, x) \frac{\partial b}{\partial t}(t, x) dt,
\]
$x$ being any point of $V$; this is the Maslov index in $V$ of $\psi_*$. As a previous step to compute $I_{\psi}$ we shall prove the following Lemma, in which the value $\langle c_1(TM)[\omega]^{n-1}, [M] \rangle$ is expressed in terms of transition functions of $\det(TM)$.

**Lemma 2.** Let $\{B_1, \ldots, B_m\}$ be a set of trivializations of $TM$, such that its domains cover $M$. Then
\[
\langle c_1(TM)[\omega]^{n-1}, [M] \rangle = \frac{-i}{2\pi} \sum_{i<k} \int_{A_{ik}} d(\log s_{ik}) \wedge \omega^{n-1},
\]
s_{ik} being the corresponding transition function of $\det(TM)$ and
\[
A_{ik} = \left( \partial B_i \setminus \cup_{r<k} B_r \right) \cap B_k.
\]
Proof. $c_1(M)$ is represented on $B_a$ by the 2-form
$$\frac{-i}{2\pi} \sum_c d(\varphi_c d \log s_{ac}),$$
where $\{\varphi_c\}$ is a partition of unity subordinate to the covering $\{B_1, \ldots, B_m\}$.

If $m = 2$
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \frac{-i}{2\pi} \int_{B_1} d(\varphi_2 d \log s_{12}) \wedge \omega^{n-1} + \frac{-i}{2\pi} \int_{B_2 \setminus B_1} d(\varphi_1 d \log s_{21}) \wedge \omega^{n-1}.$$

By Stokes’ theorem
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{\partial B_1} \varphi_2 L_{12} + \int_{\partial(B_2 \setminus B_1)} \varphi_1 L_{21},$$
where
$$L_{jk} := (-i/2\pi)d \log s_{jk} \wedge \omega^{n-1}.$$

Since $\partial(B_2 \setminus B_1) \cap B_1 = \emptyset$, $\varphi_1$ vanishes on $\partial(B_2 \setminus B_1)$ and the last integral in (3.7) is zero.

As $\varphi_2$ is 1 on $\partial B_1$, we have
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{\partial B_1} L_{12}.$$

In this case $\partial B_1 \subset B_2$, so $\partial B_1 = A_{12}$, and the the Lemma is proved when $m = 2$.

If $m = 3$
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{\partial B_1} (\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{\partial(B_2 \setminus B_1)} (\varphi_1 L_{21} + \varphi_3 L_{23})$$
$$+ \int_{\partial(B_3 \setminus (B_1 \cup B_2))} (\varphi_1 L_{31} + \varphi_2 L_{32}).$$

As $\partial(B_3 \setminus (B_1 \cup B_2))$ and the interior of $B_1 \cup B_2$ are disjoint sets, $\varphi_1$ and $\varphi_2$ vanish on $\partial(B_3 \setminus (B_1 \cup B_2))$, and the integral in (3.9) is zero. Analogously $\partial(B_2 \setminus B_1)$ and support of $\varphi_1$ are disjoint so
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{\partial B_1} (\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{\partial(B_3 \setminus B_1)} \varphi_3 L_{23}.$$

On the other hand $\partial B_1 = A + D$, with $A := \partial B_1 \setminus B_2$ (oriented as $\partial B_1$) and $D := (\partial B_1 \setminus B_2) \cap B_2$ (see Figure 1). Moreover $\partial(B_2 \setminus B_1) = -A + C$ with $C := (\partial B_2 \setminus B_1) \cap B_3$ (oriented as $\partial B_2$).

Since $C \cap (B_1 \cup B_2) = \emptyset$, then $\varphi_3|C = 1$; thus
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{A + D} (\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{-A} \varphi_3 L_{23} + \int_{A_{23}} L_{23}.$$

The last integral in (3.11) is just the term in (3.8) with $i = 2, k = 3$.

Since $\varphi_3|D = 0$, for $j = 1, 2$, then $\varphi_3|D = 1$. As $A$ and support of $\varphi_1$ are disjoint sets, then $\langle \varphi_2 + \varphi_3 \rangle|_A = 1$. It follows from these facts together with the cocycle condition $L_{13} + L_{32} = L_{12}$ that
$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \int_{A + D} L_{12} + \int_{A_{13}} L_{13} + \int_{A_{23}} L_{23}.$$

On the other hand $A_{12} = (\partial B_1 \setminus B_1) \cap B_2 = A$. Similarly $A_{13} = D$. Therefore (3.12) is the formula given in the statement of Lemma when $m = 3$. 
The preceding arguments can be generalized to any $m$

\begin{equation}
\langle c_1(TM)[\omega]^{n-1}, [M] \rangle = \int_{\partial B_1} \sum_{j \neq 1} \varphi_j L_{1j} + \cdots + \int_{\partial (B_{m-1}\setminus \cup_{r<m} B_r)} \sum_{j \neq m-1} \varphi_j L_{m-1,j}
\end{equation}

\begin{equation}
+ \int_{\partial (B_m \setminus \cup_{r<m} B_r)} \sum_{j \neq m} \varphi_j L_{m-1,j}.
\end{equation}

For any $j = 1, \ldots, m-1$ support of $\varphi_j$ and $\partial (B_m \setminus \cup_{r<m} B_r)$ are disjoint sets. Thus the integral (3.14) is zero (as in the cases $m = 2, 3$). We decompose

$$\partial (B_{m-1}\setminus \cup_{r<m} B_r) = E + G,$$

with

$$E := \left(\partial B_{m-1}\setminus \cup_{r<m} B_r\right) \cap B_m.$$

Then $\varphi_j|_E = 0$ for all $j \neq m$ and $\varphi_m|_E = 1$; thus

\begin{equation}
\int_{\partial (B_{m-1}\setminus \cup_{r<m} B_r)} \sum_{j \neq m-1} \varphi_j L_{m-1,j} = \int_G + \int_{A_{m-1,m}} L_{m-1,m}.
\end{equation}

The last integral in (3.18) is the term in (3.10) which corresponds to $i = m-1, k = m$. An analogous, but more tedious, calculation to the one for the case $m = 3$ allows to identify in (3.13) the remainder terms of (3.5).

Lemma \ref{lemma1} gives a way for expressing $\langle c_1(TM)[\omega]^{n-1}, [M] \rangle$ as a sum of integrals of $2n-1$ differential forms on $2n-1$ chains. The righthand side of (3.10) can be written schematically

\begin{equation}
\sum_j \int_{R_j} \sigma_j.
\end{equation}

In next Theorem we use this expression to give an explicit formula for $I_\psi$ in terms of transition functions of $\det(TM)$ and Maslov indices of $\psi_t$.\qed
Theorem 3. If $\{B_1, \ldots, B_m\}$ is a set of symplectic trivializations for $TM$ which covers $M$, and such that $\psi_t(B_j) = B_j$, for all $t$ and all $j$, then

$$I_\psi = \sum_{i=1}^{m} J_i \int_{B_i \setminus \cup_{j<i} B_j} \omega^n + \sum_{i<k} N_{ik},$$

where

$$N_{ik} = n \frac{i}{2\pi} \int_{0}^{1} dt \int_{A_{ik}} (f_t \circ \psi_t)(d \log r_{ik}) \wedge \omega^{n-1},$$

$A_{ik} = (\partial B_i \setminus \cup_{r<k} B_r) \cap B_k$, $J_i$ is the Maslov index of $(\psi_t)_*$ in the trivialization $B_i$ and $r_{ik}$ the corresponding transition function of $\det(TM)$.

Proof. Using the notation (3.1) we put

$$O_{2a-1} := (B_a)_-, \quad O_{2a} := (B_a)_+.$$ 

Then $\{O_c | c = 1, \ldots, 2m\}$ is a covering for $E$. We shall denote by $l_{bc}$ the respective transition functions for $\det(VTE)$. If we set $U := B_1$, $V := B_2$, one has by (3.2)

$$l_{12} = a(t, x), \quad l_{13} = r_{UV}(x), \quad l_{34} = b(t, x).$$

We can determine $I_\psi = \langle c_1(VTE) c^a, [E] \rangle$ applying the result given in Lemma 2 to the set $\{O_c\}$ of trivializations of $VTE$. That is,

$$I_\psi = \sum_{a < b} T_{ab}, \text{ where } T_{ab} = \frac{i}{2\pi} \int_{A_{ab}} d \log l_{ab} \wedge \tau^n.$$ 

It follows from (3.18) and (3.21) that $\tau$ is equal to $\omega$ on $A_{ab}$ unless $a$ and $b$ are both odd; in this case $\tau = \omega + d(\alpha(f_t \circ \psi_t)) \wedge dt$.

We will calculate the summand $T_{12}$ in (3.19). The set $A_{12} = \partial O_1 \cap O_2 = \partial U_- \cap U_+$, and

$$\partial U_- = \{[+, p, x] | p \in \partial D^2, \ x \in U\} \cup \{[-, p, x] | p \in D^2, \ x \in \partial U\}. $$

So

$$A_{12} = \{[+, p, x] | p \in \partial D^2, \ x \in U\}.$$ 

Taking into account (3.21), (3.22) together with the fact that orientations of $S^1$ and $\partial D^2$ are opposite, we deduce

$$T_{12} = \frac{-i}{2\pi} \int_{U} \left( \int_{S^1} a^{-1}(t, x) \frac{\partial a(t, x)}{\partial t} dt \right) \omega^n = J_U \int_{U} \omega^n.$$ 

Next we consider the term $T_{34}$. The integration domain is

$$A_{34} = (\partial V_- \setminus (U_- \cup U_+)) \cap V_+ = \{[+, p, x] | p \in \partial D^2, \ x \in V \setminus U\}. $$

Hence

$$T_{34} = J_V \int_{V \setminus U} \omega^n.$$ 

In general,

$$A_{2j-1, 2j} = (\partial B_{j-1} \setminus \cup_{r<j} (B_{r+} \cup B_{r-})) \cap B_{j+}$$

$$= \{[+, p, x] | p \in \partial D^2, \ x \in B_j \setminus \cup_{r<j} B_r\}.$$
Hence the term in (3.19) with \( a = 2j - 1, \ b = 2j \) gives a contribution to \( I_\psi \) equal to

\[
J_{B_j} \int_{B_j \setminus U_{r < j} \setminus B_r} \omega^n
\]

Now we analyze \( T_{13} \).

\[
A_{13} = \{[-, p, x] \mid p \in D^2_-, \ x \in \partial U \cap V \}.
\]

\( D^2_\epsilon \) is oriented by the form \( d\theta \wedge dt \), and \( \partial U \cap V \) is oriented with the orientation of \( \partial U \). Hence

\[
T_{13} = -\frac{i}{2\pi} \int_{A_{13}} d \log r_{UV} \wedge (\omega + d(\alpha(f_t \circ \psi_t)) \wedge dt)^n
\]

\[
= -\frac{ni}{2\pi} \int_{A_{13}} d \log r_{UV}(f_t \circ \psi_t)\alpha'(\theta) d\theta \wedge dt \wedge \omega^{n-1}
\]

\[
= +\frac{ni}{2\pi} \int_0^1 dt \int_{\partial U \cap V} (f_t \circ \psi_t) d \log r_{UV} \wedge \omega^{n-1}.
\]

In general, if \( j < k \)

\[
T_{2j-1,2k-1} = \frac{ni}{2\pi} \int_0^1 dt \int_{A_{jk}} (f_t \circ \psi_t) d \log r_{jk} \wedge \omega^{n-1},
\]

where \( A_{jk} \) is the set defined in Lemma 2.

On the other hand

\[
A_{14} = (\partial U_+ \setminus (U_+ \cup V_-)) \cap V_+ = \{[-, p, x] \mid p \in D^2_-, \ x \in \partial U \setminus V \} \cap V_+ = \emptyset.
\]

Thus \( T_{14} = 0 \). In general, for \( j < k \) the integration domain \( A_{2j-1,2k} \) is of the form

\[
(\partial B_{j-1} \cup \cdot) \cap B_{k+}.
\]

In the union \( \cup \cdot \) appear the sets \( B_{k-} \) and \( B_{j+} \), hence

\[
A_{2j-1,2k} \subset (\partial B_{j-} \setminus (B_{j+} \cup B_{k-})) \cap B_{k+},
\]

and this set is empty by the same reason that \( A_{14} = \emptyset \). Therefore \( T_{2j-1,2k} = 0 \), for any \( j < k \).

The set \( A_{23} \) is

\[
A_{23} = (\partial U_+ \setminus U_-) \cap V_- = \{[+, p, x] \mid p \in F, \ x \in \partial U \setminus V \}.
\]

As \( d \log l_{23} \wedge \omega^n \) does not contain \( d\theta \), the term \( T_{23} \) vanishes. In general, if \( j < k \)

\[
A_{2j,2k-1} = (\partial B_{j+} \setminus \cdot) \cap B_{k-} \subset (\partial B_{j+} \setminus B_{j-}) \cap B_{k-}
\]

\[
= \{[+, p, x] \mid p \in F, \ x \in \partial B_j \setminus B_k \}.
\]

Then \( T_{2j,2k-1} \) vanishes by the same reason that \( T_{23} = 0 \).

Analogous arguments as the ones explained in the preceding paragraph show that \( T_{2j,2k} = 0 \), for any \( j < k \).

So, apart from the terms \( T_{ab} \) considered in (3.20) and in (3.22), the remainder summands in (3.19) are zero. The theorem follows from (3.20) and (3.22). \( \square \)
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From the definition of product in $\pi_1(\text{Ham}(M,\omega))$ by juxtaposition of paths and under the hypotheses of Theorem 4 it is obvious that

$$I : \pi_1(\text{Ham}(M,\omega)) \to \mathbb{R}$$

is a group homomorphism. This fact has been proved in [6] for the general case.

**Corollary 4.** If $U$ and $V$ are symplectic trivializations of $TM$, with $\psi_t(U) = U$, $\psi_t(V) = V$, for all $t$ and $U \cup V = M$ and $\int_{S^1}(f_t \circ \psi_t) dt$ is a constant $k$ on $\partial U \cap V$, then

$$I_{\psi} = J_U \int_U \omega^n + J_V \int_{V \setminus U} \omega^n - nk(c_1(TM)[\omega]^{n-1}, M).$$

**Corollary 5.** If $TM$ is trivial on $U := M \setminus \{q\}$, where $q$ is a point of $M$ fixed by $\psi_t$ for all $t$, then

$$I_{\psi} = J_U \int_M \omega^n - n \left( \int_{S^1} f_t(q) dt \right) (c_1(TM)[\omega]^{n-1}, M).$$

Now we analyze the expression for $I_{\psi}$ given in Theorem 3 in case of integrable systems. Let $f$ be the normalized Hamiltonian which generates the loop $\psi$. We assume that $(M,\omega,f)$ is completely integrable, with $f_1 = f, f_2, \ldots, f_n$ integrals of motion. We suppose that $df_1,\ldots,df_n$ are independent at the points of $M \setminus P := V$, where $P$ is a finite union of $2n-2$ dimensional submanifolds of $M$. We suppose that on $V$ are defined action-angle coordinates. We put

$$Q := \{ x \in P \mid \dim \text{Span} (df_1(x),\ldots,df_n(x)) = n-1 \}.$$ 

By $Q_1,\ldots,Q_k$ are denoted the connected components of $Q$, and let $V_j$ be a tubular neighborhood of $Q_j$ in $M$, invariant under $\psi_t$ for all $t$. We assume that on $V_j$ is defined a symplectic trivialization of $TM$. Then, for each $j$ one can choose a family of tubular neighborhoods $\{V_{jb} \subset V_j\}_{b=1,2,\ldots}$, such that

$$\lim_{b \to \infty} \int_{V_{jb}} \omega^n = 0.$$ 

Lemma 2 applied to the covering $\{V, V_{jb}\}_{j=1,\ldots,k}$ of $V \cup Q$ gives

$$\langle c_1(M)[\omega]^{n-1}, [M] \rangle = \frac{-i}{2\pi} \sum_{j=1}^k \int_{\partial V \cap V_{jb}} d \log r_{VV_{jb}} + \epsilon(b),$$

where $\epsilon(b)$ goes to 0 as $b \to \infty$.

Hence

$$(3.23) \quad \langle c_1(M)[\omega]^{n-1}, [M] \rangle = \sum_{j=1}^k z_j,$$

with

$$(3.24) \quad z_j := \frac{-i}{2\pi} \sum_{j=1}^k \int_{\partial V \cap V_{jb}} d \log r_{VV_{jb}} \wedge \omega^{n-1}.$$ 

**Proposition 6.** Let $(M,\omega,f,f_2,\ldots,f_n)$ be an integrable in which the preceding hypotheses hold, then

$$I_{\psi} = \sum_{j=1}^k z_j'.$$
where \( z'_j \) is obtained from the corresponding \( z_j \) by inserting the factor \( -nf \) in the integrand of (3.24).

Proof. The Maslov index \( J_U = 0 \) because of the particular form of the flow equations in action-angle coordinates. On the other hand

\[
\int_{V \setminus (V \cup \ldots)} \omega^n = 0.
\]

Thus the Proposition follows from Theorem 3, together with (3.23) and (3.24). □

Similar arguments to the ones involved in this Proposition are used in Section 4 for studying the invariant \( I \) in Hirzebruch surfaces.

4. Examples.

The invariant \( I \) when the manifold is the 2-sphere.

Let \( \psi_t \) be the rotation in \( \mathbb{R}^3 \) around \( \vec{e}_3 \) of angle \( 2\pi t \) with \( t \in [0, 1] \). Then \( \psi_t \) determines a Hamiltonian symplectomorphism of \( (S^2, \omega_{area}) \). In fact, the isotopy \( \{ \psi_t \} \) is generated by the vector field \( \frac{\partial}{\partial \phi} \), and the function \( f \) on \( S^2 \) defined by

\[
f(\theta, \phi) = -2\pi \cos \theta = -2\pi z.
\]

TS\(^2\) can be trivialized on \( U = D_2^+ \), and on \( V = D_2^- \). Moreover \( \partial U \cap V \) is the parallel \( \theta = \pi/2 + \varepsilon \). On \( \partial U \cap V \) the function \( f \circ \psi_t \) takes the value \( 2\pi \sin \varepsilon \).

\[
\int_U \omega = 2\pi(1 - k'), \quad \int_{V \setminus U} \omega = 2\pi(1 + k'),
\]

with \( k' := \cos(\pi/2 + \varepsilon) \).

Furthermore the north pole \( n \) and the south pole \( s \) are fixed points of the isotopy \( \psi_t \). The rotation \( \psi_t \) transforms the basis \( \vec{e}_1, \vec{e}_2 \) of \( T_nS^2 \) in

\[
(\cos 2\pi t \vec{e}_1 + \sin 2\pi t \vec{e}_2, -\sin 2\pi t \vec{e}_1 + \cos 2\pi t \vec{e}_2).
\]

So \( J_U \) is the winding number of the map

\[
t \in [0, 1] \rightarrow e^{2\pi i t} \in U(1);
\]

That is, \( J_U = +1 \).

Similarly, by considering the oriented basis \( \vec{e}_2, \vec{e}_1 \) of \( T_sS^2 \) it turns out that the Maslov index \( J_V \) of \( \psi_t \) is \(-1 \).

By Corollary 4

\[
I_\psi = 2\pi(1 - k') - 2\pi(1 + k') - (-2\pi k') \langle c_1(TS^2), S^2 \rangle = 0.
\]

Corollary 5 can also be applied to determine \( I_\psi \). One takes \( U := S^2 \setminus \{s\} \). As \( f(s) = -2\pi(-1) \), we obtain again

\[
I_\psi = +4\pi - 2\pi \langle c_1(TS^2), S^2 \rangle = 0.
\]

Using formula (??? we can determine \( I_\psi \) again. Now \( V \) is \( S^2 \setminus \{n, s\} \), \( U_1 \) is a small polar cap at \( n \) and \( U_2 \) the symmetric one at \( s \). By the symmetry

\[
\int_{\partial U_1 \cap V} d \log r_{U_1 \cap V} = \int_{\partial U_2 \cap V} d \log r_{U_2 \cap V},
\]

so \( y_1 = y_2 \). As \( f(n) = -f(s) \), then \( I_\psi = 0 \).

This result was expected, because \( \pi_1(\text{Ham}(S^2)) \) is isomorphic to \( \mathbb{Z}_2 \) (see [9]) and \( I \) is a group homomorphism.
The invariant \( I \) for Hamiltonian loops in \( \mathbb{T}^{2n} \).

We identify the torus \( \mathbb{T}^{2n} \) with \( \mathbb{R}^{2n}/\mathbb{Z}^{2n} \), and we suppose that \( \mathbb{T}^{2n} \) is equipped with the standard symplectic form \( \omega_0 \). If \( \psi_t \) is a Hamiltonian isotopy of \( \mathbb{T}^{2n} \), it can be written in the form

\[
\psi_t(x^1, \ldots, x^{2n}) = (x^1 + \alpha^1(t, x^i), \ldots, x^{2n} + \alpha^{2n}(t, x^i)),
\]

where the function \( \alpha^j \), for \( j = 1, \ldots, 2n \), is periodic of period 1 in each variable: \( t, x^1, \ldots, x^{2n} \). The vector fields \( \{ \frac{\partial}{\partial x^i} \} \) give a symplectic trivialization of the tangent bundle. In this case the right hand side of (3.17) has only one term. The matrix of \( (\psi_t)_* \) with respect to \( \{ \frac{\partial}{\partial x^i} \} \) is

\[
\left( \delta_t^j + \frac{\partial \alpha^j}{\partial x^i} \right) \in Sp(2n, \mathbb{R}).
\]

First, let us assume that each \( \alpha^i \) is a separate variables function; that is, \( \alpha^i(t, x^i) = f^i(t)w^i(x^i) \). Since \( \alpha^1 \) takes the same value at symmetric points on opposite faces of the cube \( I^{2n} \), there is a point \( p_1 \in I^{2n} \) such

\[
\frac{\partial u^1}{\partial x^i}(p_1) = 0,
\]

for all \( j \). Hence the first row of the matrix (4.1) at the point \( p_1 \) is \( (1, 0, \ldots, 0) \); that is, the matrix of \( (\psi_t)_*(p_1) \) is independent of \( f^1 \) and thus the Maslov index of \( \{ (\psi_t)_*(p_1) \}_t \) does not depend on \( f^1 \). From (3.17) it follows that \( I_{\psi} \) is independent of \( f^1 \). The independence of \( I_{\psi} \) with respect to \( f^1 \) is proved in a similar way. Thus in order to determine \( I_{\psi} \) we can assume that \( f^1 = 0 \) for all \( i \), but in this case \( I_{\psi} = 0 \) obviously.

If \( \alpha^j \) is sum of two separate variables functions

\[
\alpha^j(t, x^i) = f^j(t)w^j(x^i) + g^j(t)v^j(x^i),
\]

we take a point \( q_1 \in I^{2n} \), such that \( \frac{\partial \alpha^j}{\partial x^i}(q_1) = 0 \), for all \( j \). Then \( I_{\psi} \) is independent of \( g^1 \). The above reasoning gives \( I_{\psi} = 0 \) in this case as well.

By the Fourier theory, the original \( C^\infty \) periodic function \( \alpha^j \) can be approximated (in the uniform \( C^k \)-norm) by a sum of separated functions of the form \( \sum f_a(t)u_a(x^i) \), where \( f_a \) and \( u_a \) are 1-periodic. As \( I_{\psi} \) depends only on the homotopy class of \( \psi \), we conclude that \( I_{\psi} = 0 \) for a general Hamiltonian loop.

**Proposition 7.** The invariant \( I \) is identically zero on \( \pi_1(\text{Ham}(\mathbb{T}^{2n}, \omega_0)) \).

This result when \( n = 1 \) is consistent with the fact that \( \pi_1(\text{Ham}(\mathbb{T}^2)) = 0 \) (see [3]).

**Application to Hirzebruch surfaces.**

Given 3 numbers \( k, \tau, \mu \), with \( k \in \mathbb{Z}_{>0}, \tau, \mu \in \mathbb{R}_{>0} \) and \( k\mu < \tau \), the triple \( (k, \tau, \mu) \) determine a Hirzebruch surface \( M_{k, \tau, \mu} \) [3]. This manifold is the quotient

\[
\{ z \in \mathbb{C}^4 : k|z_1|^2 + |z_2|^2 + |z_4|^2 = \tau/\pi, |z_1|^2 + |z_3|^2 = \mu/\pi \}/\mathbb{T}^2,
\]

where the \( \mathbb{T}^2 \)-action is given by

\[
(a, b) \cdot (z_1, z_2, z_3, z_4) = (a^k z_1, az_2, bz_3, az_4),
\]

for \( (a, b) \in \mathbb{T}^2 \). The map

\[
[z_1, z_2, z_3, z_4] \mapsto \{ [z_2 : z_4], [z_2 z_3 : z_4 z_3 : z_1] \}
\]
allows us to represent $M_{k, \tau, \mu}$ as a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$. On the other hand the usual symplectic structures on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ induce a symplectic form $\omega$ on $M_{k, \tau, \mu}$, and the following $\mathbb{T}^2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^2$

$$(a, b)\left([u_0 : u_1], [x_0 : x_1 : x_2]\right) = \left([au_0 : u_1], [ax_0 : x_1 : bx_2]\right)$$

gives rise to a toric structure on $M_{k, \tau, \mu}$. In terms of the Delzant construction $(M_{k, \tau, \mu}, \omega)$ is associated to the trapezoid in $(\mathbb{R}^2)^*$ whose not oblique edges are $\tau, \mu, \lambda := \tau - k\mu$, [4] (see Figure 2). Moreover $\lambda$ is the value that the symplectic form $\omega$ takes on the exceptional divisor, $\{[z] \in M | z_3 = 0\}$, of $M := M_{k, \tau, \mu}$. And $\omega$ takes the value $\mu$ on the class of the fibre in the fibration $M \to \mathbb{C}P^1$.

![Figure 2. Delzant polytope associated to $M$.](image)

Since $M$ is a toric manifold, the $\mathbb{T}^2$-action define symplectomorphisms of $M$. More precisely, let $\psi_t$ the diffeomorphism of $M$ defined by

$$\psi_t[z_1, z_2, z_3, z_4] = [z_1 e^{2\pi it}, z_2, z_3, z_4].$$

$\psi = \{\psi_t : t \in [0, 1]\}$ is a loop of Hamiltonian symplectomorphisms of $(M, \omega)$. Similarly we have

$$\tilde{\psi}_t[z_1, z_2, z_3, z_4] = [z_1, z_2 e^{2\pi it}, z_3, z_4],$$

and the corresponding loop $\tilde{\psi}$ in $\text{Ham}(M, \omega)$.

Using Theorem 8 we shall calculate the values of $I_\psi$ and $I_{\tilde{\psi}}$ in terms of $\lambda$, $\tau$ and $k$. The result is stated in Theorem 8 below. The most laborious point in the proof of the following Theorem is to obtain Darboux charts for $M$ which give rise to simple transition functions for $\det(TM)$.

**Theorem 8.** Let $\psi$ and $\tilde{\psi}$ be the loops of symplectomorphisms of the Hirzebruch surface $(M_{k, \tau, \mu}, \omega)$, defined by (4.2) and (4.3) respectively, then

$$I_\psi = \frac{2k\mu^2}{3} \left(1 - \frac{\mu}{2\lambda + k\mu}\right), \text{ and } I_{\tilde{\psi}} = \frac{-k^2\mu^2}{3} \left(1 - \frac{\mu}{2\lambda + k\mu}\right).$$

$\lambda$ being $\tau - k\mu$.

**Proof.** We will define a Darboux atlas on $M$. First we consider the following covering for $M$

$$U_1 = \{[z] \in M : z_3 \neq 0 \neq z_4\}, \text{ } U_2 = \{[z] \in M : z_1 \neq 0 \neq z_4\}$$
the translation 

\[ \phi \] 

trivialization defined on 

\[ B \] 

endowed with the orientation given by 

Since 

\[ x = \frac{1}{2} \] 

by the formulae 

Then \( \omega \) on \( U_1 \) can be written 

Then \( \omega \) we consider the Darboux coordinates \((x_2, y_2, a_2, b_2)\), with 

On \( U_3 \) we put 

On \( U_3 \) we put 

On \( U_4 \) we set 

Finally, \( \omega = d\theta \) on subsets of the domains \( U \). 

The normalized Hamiltonian function for \( \psi_1 \) is 

It is not easy to determine the transition function of \( \det(TM) \) that corresponds to the coordinate transformation \((x_i, y_i, a_i, b_i) \rightarrow (x_j, y_j, a_j, b_j)\); that is why we will introduce polar coordinate on subsets of the domains \( U_j \). 

Given \( 0 < \epsilon < 1 \), for \( j = 1, 2, 3, 4 \) we put 

On \( B_0 \) are well-defined the coordinates \((\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2)\), and in this coordinates 

On \( B_0 \) we consider the Darboux coordinates \((x_j, y_j, a_j, b_j)\) defined above. Then \( B_0, B_1, B_2, B_3, B_4 \) are a Darboux atlas for \( M \). Assume that \( M \) is endowed with the orientation given by \( \omega^2 \). This orientation agrees on \( B_0 \) with the one defined by \( d\varphi_1 \wedge d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_2 \). 

It is evident that \( \psi_i(B_i) = B_i \), for \( i = 0, 1, 2, 3, 4 \). Since \( \psi \) on \( B_0 \) is simply the translation \( \varphi_1 \rightarrow \varphi_1 + 2\pi t \) of the variable \( \varphi_1 \), the Maslov index \( J_0 \) of \( \psi \) in the trivialization defined on \( B_0 \) vanishes. 

As \( B_j \) (for \( j = 1, 2, 3, 4 \)) has "infinitesimal size" and \( J_0 = 0 \), the expression for \( I_\psi \) of Theorem 3 can be written 

(4.5) 

\[ I_\psi = \sum_{i<k} N_{ik} + O(\epsilon) \] 

Since \( I_\psi \) is obviously independent of the coordinates, it follows from (4.3) that \( N_{ik} \) is independent, up to order \( \epsilon \), of the chosen Darboux coordinates in \( B_j \), for \( j = 1, 2, 3, 4 \). Moreover \( N_{ik} \) with \( 0 \neq i < k \) is also of order \( \epsilon \).
On the other hand, if we substitute $B_j$ by
\[ B'_j = \{ [z] \in B_j : |z_r| > \epsilon, \ r \neq j \} \]
in the definition of $N_{ik}$ (see Theorem 3) the new $N_{ik}$ differs from the old one in a quantity of order $\epsilon$. As on $B'_1$ the variable $\rho_2 \neq 0$, we can consider the Darboux coordinates
\[ (x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2) \]
on $B'_1$. Since $\rho_3 \neq 0$ on $B'_2$ we take the coordinates $(a_2, b_2, \frac{\rho_3^2}{2}, \xi_3)$ on $B'_2$. Similarly we will adopt the following coordinates: $(x_3, y_3, \frac{\rho_3^2}{2}, \chi_4)$ on $B'_3$ and $(\frac{\rho_3^2}{2}, \zeta_1, a_4, b_4)$ on $B'_4$.

Taking into account the preceding arguments
\[ I_\psi = \sum_{j=1}^{4} N'_{0j} + O(\epsilon), \]
where
\[ N'_{0j} = \frac{i}{\pi} \int_{A'_{0j}} f d \log r_{0j} \wedge \omega \]
and
\[ A'_{0j} = \{ [z] \in M : |z_r| > \epsilon, \text{ for all } r \neq j \text{ and } |z_j| = \epsilon \}. \]
The submanifold $A'_{0j}$ is oriented as a subset of $\partial B_0$; that is, with the orientation induced by the one of $B_0$.

Next we determine the value of $N'_{01}$. To know the transition function $r_{01}$ one needs the Jacobian matrix $R$ of the transformation
\[ (x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2) \rightarrow (\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2) \]
in the points of $A'_{01}$; with $\rho_1^2 = x_1^2 + y_1^2$, $\varphi_1 = \tan^{-1}(y_1/x_1)$. The non trivial block of $R$ is the diagonal one
\[ \begin{pmatrix} \frac{x_1}{r} & y_1 \\ r & s \end{pmatrix}, \]
with $r = -y_1(x_1^2 + y_1^2)^{-1}$ and $s = x_1(x_1^2 + y_1^2)^{-1}$. The non-real eigenvalues of $R$ are
\[ \lambda_{\pm} = \frac{x_1 + s \pm i \sqrt{4 - (s + x_1)^2}}{2}. \]
On $A'_{01}$ these non-real eigenvalues occur when $(s + x_1)^2 < 2$, that is, if $|\cos \varphi_1| < 2\epsilon(\epsilon^2 + 1)^{-1} =: \delta$. If $y_1 > 0$ then $\lambda_-$ of the first kind (see [10]) and $\lambda_+$ is of the first kind, if $y_1 < 0$.

Hence, on $A'_{01}$,
\[ \rho(R) = \begin{cases} \lambda_+ |\lambda_+|^{-1} = x + iy, & \text{if } |\cos \varphi_1| < \delta \text{ and } y_1 < 0; \\ \lambda_- |\lambda_-|^{-1} = x - iy, & \text{if } |\cos \varphi_1| < \delta \text{ and } y_1 > 0; \\ \pm 1, & \text{otherwise}. \end{cases} \]
where $x = \delta^{-1} \cos \varphi_1$, and $y = \sqrt{1 - x^2}$. 

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If we put $\rho(R) = e^{i\gamma}$, then $\cos \gamma = \delta^{-1} \cos \varphi_1$ (when $|\cos \varphi_1| < \delta$), and
\[
\begin{align*}
\sin \gamma &= -\sqrt{1 - \cos^2 \gamma}, & \text{if } \sin \varphi_1 > 0; \\
\sin \gamma &= \sqrt{1 - \cos^2 \gamma}, & \text{if } \sin \varphi_1 < 0.
\end{align*}
\]
So when $\varphi_1$ runs anticlockwise from 0 to $2\pi$, $\gamma$ goes round clockwise the circumference; that is, $\gamma = h(\varphi_1)$, where $h$ is a function such that
\[
h(0) = 2\pi, \text{ and } h(2\pi) = 0.
\]
As $r_{01} = \rho(R)$, then $d\log r_{01} = idh$.

On $A_{10}$ the form $\omega$ reduces to $(1/2)d\rho_2^2 \land d\varphi_2$. From (4.7) one deduces
\[
N'_{01} = \frac{i}{2\pi} \int_{A'_{01}} ifdh \land d\rho_2^2 \land d\varphi_2.
\]

On the other hand according to the convention about orientations, $\{[z] : |z| = \epsilon\}$ as subset of $\partial B_0$ is oriented by $-d\varphi_1 \land d\rho_2^2 \land d\varphi_2$. And on $A'_{01}$ the Hamiltonian function $f = -\kappa + O(\epsilon)$. Then it follows from (4.9) together with (4.8)
\[
N'_{01} = 2\tau\kappa + O(\epsilon).
\]

The contributions $N'_{02}, N'_{03}, N'_{04}$ to $I_\psi$ can be calculated in a similar way. One obtains the following results up to addends of order $\epsilon$:

\[
N'_{02} = 2\mu\kappa - \mu^2, \quad N'_{03} = 2\lambda(\kappa - \mu), \quad N'_{04} = \mu(2\kappa - \mu).
\]

As $I_\psi$ is independent of $\epsilon$, it follows from (4.10), (4.11), (4.12) and (4.13)
\[
I_\psi = \frac{2k\mu^2}{3} \left(1 - \frac{\mu}{2\lambda + k\mu}\right).
\]

Next we consider the loop $\tilde{\psi}$; the corresponding normalized Hamiltonian function is $\tilde{f} = \pi\rho_2^2 - \tilde{\kappa}$, where
\[
\tilde{\kappa} = \frac{3\lambda^2 + 3k\lambda\mu + k^2\mu^2}{3(2\lambda + k\mu)}.
\]

As in the preceding case
\[
I_{\tilde{\psi}} = \sum_{j=1}^{4} \tilde{N}'_{0j} + O(\epsilon),
\]

where
\[
\tilde{N}'_{0j} = \frac{i}{\pi} \int_{A'_{0j}} \tilde{f} d\log r_{0j} \land \omega.
\]

The expression for $\tilde{N}'_{01}$ can be obtained from (4.10) substituting $f$ for $\tilde{f}$; so
\[
\tilde{N}'_{01} = \tau(2\tilde{\kappa} - \tau) + O(\epsilon).
\]

Analogous calculations give the following values for the $\tilde{N}'_{0j}$’s, up to summands of order $\epsilon$
\[
\tilde{N}'_{02} = 2\mu\tilde{\kappa}, \quad \tilde{N}'_{03} = \lambda(2\tilde{\kappa} - \lambda), \quad \tilde{N}'_{04} = \mu(2\tilde{\kappa} - k\mu - 2\lambda).
\]

From (4.12), (4.13), (4.14) and (4.15) it follows the value for $I_{\tilde{\psi}}$ given in the statement of Theorem.
Remark. In [2] is proved that \( \pi_1(\text{Ham}(M)) = \mathbb{Z} \) when \( k = 1 \), therefore the quotient of \( I_\psi / I_\psi' \), for arbitrary Hamiltonian loops of symplectomorphisms, is a rational number. For the particular loops considered in Theorem 8 the quotient \( I_\psi / I_\psi \) equals \(-k/2\), so Theorem 8 is consistent with the result of Abreu and McDuff.

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Departamento de Física. Universidad de Oviedo. Avda Calvo Sotelo. 33007 Oviedo. Spain.

E-mail address: vina@uniovi.es