Proof Theory for Theories of Ordinals

III: $\Pi_N$-Reflection

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Abstract

This paper deals with a proof theory for a theory $T_N$ of $\Pi_N$-reflecting ordinals using a system $Od(\Pi_N)$ of ordinal diagrams. This is a sequel to the previous one [6] in which a theory for $\Pi_3$-reflecting ordinals is analysed proof-theoretically.

1 Prelude

This is a sequel to the previous ones [5] and [6]. Namely our aim here is to give finitary analyses of finite proof figures in a theory for $\Pi_N$-reflecting ordinals, via cut-eliminations as in Gentzen-Takeuti’s consistency proofs [10] and [15]. Throughout this paper $N$ denotes a positive integer such that $N \geq 4$.

Let $T$ be a theory of ordinals. Let $\Omega$ denote the (individual constant corresponding to the) ordinal $\omega^{CK}_1$. We say that $T$ is a $\Pi^2_2$-sound theory if

$$\forall \Pi_2 A(T \vdash A^\Omega \Rightarrow A^\Omega).$$

Definition 1.1 ($\Pi^2_2$-ordinal of a theory) Let $T$ be a $\Pi^2_2$-sound and recursive theory of ordinals. For a sentence $A$ let $A^\alpha$ denote the result of replacing unbounded quantifiers $Qx (Q \in \{\forall, \exists\})$ in $A$ by $Qx < \alpha$. Define the $\Pi^2_2$-ordinal $|T|_{\Pi^2_2}$ of $T$ by

$$|T|_{\Pi^2_2} := \inf\{\alpha \leq \omega^{CK}_1 : \forall \Pi_2 \text{ sentence } A(T \vdash A^\Omega \Rightarrow A^\alpha) \} < \omega^{CK}_1$$

Roughly speaking, the aim of proof theory for theories $T$ of ordinals is to describe the ordinal $|T|_{\Pi^2_2}$. This gives $\Pi^2_2$-ordinal of an equivalent theory of sets, cf. [4].
Let KP\(\Pi^N\) denote the set theory for \(\Pi^N\)-reflecting universes. KP\(\Pi^N\) is obtained from the Kripke-Platek set theory with the Axiom of Infinity by adding the axiom: for any \(\Pi^N\) formula \(A(u)\)

\[ A(u) \rightarrow \exists z (u \in z \& A^z(u)). \]

In [9] we introduced a recursive notation system \(Od(\Pi^N)\) of ordinals, which we studied first in [1]. An element of the notation system is called an ordinal diagram (henceforth abbreviated by o.d.). The system is designed for proof theoretic study of theories of \(\Pi^N\)-reflection. We [9] showed that for each \(\alpha < \Omega\) in \(Od(\Pi^N)\) KP\(\Pi^N\) proves that the initial segment of \(Od(\Pi^N)\) determined by \(\alpha\) is a well ordering.

Let \(T_N\) denote a theory of \(\Pi^N\)-reflecting ordinals. The aim of this paper is to show an upper bound theorem for the ordinal \(|T_N|_{\Pi^2}\):

**Theorem 1.1** \(\forall \Pi_2 A(T_N \vdash A^\Omega \Rightarrow \exists \alpha \in Od(\Pi_N) \mid \Omega A^\alpha)\)

Combining Theorem 1.1 with the result in [9] mentioned above yields the:

**Theorem 1.2** \(|KP\Pi^N|_{\Pi^2} = |T_N|_{\Pi^2} = \text{the order type of } Od(\Pi_N) | \Omega\)

Proof theoretic study for \(\Pi^N\)-reflecting ordinals via ordinal diagrams were first obtained in a handwritten note [2].

For an alternative approach to ordinal analyses of set theories, see M. Rathjen’s papers [11], [12] and [13].

Let us mention the contents of this paper.

In Section 2 a preview of our proof-theoretic analysis for \(\Pi^N\)-reflection is given. As in [6] inference rules \((c)_{\alpha_1}^\pi\) are added to analyze an inference rule (\(\Pi_N\)-rfl) saying the universe of the theory \(T_N\) is \(\Pi^N\)-reflecting. A chain is defined to be a consecutive sequence of rules \((c)\).

In Subsection 2.1 we expound that chains have to merge each other for a proof theoretic analysis of \(T_N\) for \(N \geq 4\). An ordinal diagram in the system \(Od(\Pi_N)\) defined in [9] may have its \(Q\) part, which has to obey complicated requirements. In Subsection 2.2 we explain what parts correspond to the \(Q\) part in proof figures.

In Section 3 the theory \(T_N\) for \(\Pi^N\)-reflecting ordinals is defined. In Section 4 let us recall briefly the system \(Od(\Pi_N)\) of ordinal diagrams (abbreviated by o.d.’s) in [9].

In Section 5 we extend \(T_N\) to a formal system \(T_{N_c}\). The language is expanded so that individual constants \(c_\alpha\) for o.d.’s \(\alpha \in Od(\Pi_N) \mid \pi\) are included. Inference rules \((c)_{\alpha_1}^\pi\) are added. Proofs in \(T_{N_c}\) defined in Definition 5.3 are proof figures enjoying some provisos and obtained from given proofs in \(T_N\) by operating rewriting steps. Some lemmata for proofs are established. These are needed to verify that rewritten proof figures enjoy these provisos. To each proof \(P\) in \(T_{N_c}\) an o.d. \(o(P) \in Od(\Pi_N) \mid \Omega\) is attached. Then the Main Lemma 5.1 is stated as follows: If \(P\) is a proof in \(T_{N_c}\), then the endsequent of \(P\) is true.
In Section 6 the Main Lemma 5.1 is shown by a transfinite induction on $o(P) \in Od(\Pi_N) \mid \Omega$.

This paper relies heavily on the previous ones [5] and [6].

**General Conventions.** Let $(X, <)$ be a quasiordering. Let $F$ be a function $F : X \ni \alpha \mapsto F(\alpha) \subseteq X$. For subsets $Y, Z \subseteq X$ and elements $\alpha, \beta \in X$, put

1. $\alpha \leq \beta \iff \alpha < \beta$ or $\alpha = \beta$
2. $Y \mid \alpha = \{ \beta \in Y : \beta < \alpha \}$
3. $Y < Z \iff \exists \beta \in Z \forall \alpha \in Y (\alpha < \beta)$
4. $Y < \beta \iff Y < \{ \beta \} \iff \forall \alpha \in Y (\alpha < \beta)$; $\alpha < Z \iff \{ \alpha \} < Z$
5. $Z \leq Y \iff \forall \beta \in Z \exists \alpha \in Y (\beta \leq \alpha)$
6. $\beta \leq Y \iff \{ \beta \} \leq Y \iff \exists \alpha \in Y (\beta \leq \alpha)$; $Z \leq \{ \alpha \}$
7. $F(Y) = \bigcup \{ F(\alpha) : \alpha \in Y \}$

## 2 A preview of proof-theoretic analysis

In this section a preview of our proof-theoretic analysis for $\Pi_N$-reflection is given.

Let us recall briefly the system $Od(\Pi_N)$ of o.d.'s in [9]. The main constructor in $Od(\Pi_N)$ is to form an o.d. $d^q_\sigma \alpha$ from a symbol $d$ and o.d.'s in $\{ \sigma, \alpha \} \cup q$, where $\sigma$ denotes a recursively Mahlo ordinal and $q = Q(d^q_\sigma \alpha)$ a finite sequence of quadruples of o.d.'s called $Q$ part of $d^q_\sigma \alpha$. By definition we set $d^q_\sigma \alpha < \sigma$. Let $\gamma \prec_2 \delta$ denote the transitive closure of the relation $\{ (\gamma, \delta) : \exists q, \alpha (\gamma = d^q_\sigma \alpha) \}$, and $\preceq_2$ its reflexive closure. Then the set $\{ \tau : \sigma \prec_2 \tau \}$ is finite and linearly ordered by $\prec_2$ for each $\sigma$.

An o.d. of the form $\rho = d^q_\sigma \alpha$ is introduced in proof figures only when an inference rule (\Pi_N-rfl) for $\Pi_N$-reflection is resolved by using an inference rule (c)\_.$\rho$.

$q$ in $\rho = d^q_\sigma \alpha$ includes some data $st_i(\rho), rg_i(\rho)$ for $2 \leq i < N$. $st_{N-1}(\rho)$ is an o.d. less than $\varepsilon_{i+1}$ and $rg_{N-1}(\rho) = \pi$, while $st_i(\rho), rg_i(\rho)$ for $i < N - 1$ may be undefined. If these are defined, then we write $rg_i(\rho) \downarrow$, etc. and $\kappa = rg_i(\rho)$ is an o.d. such that $\rho \prec_i \kappa$, where $\gamma \prec_i \delta$ is a transitive closure of the relation $pd_i(\gamma) = \delta$ on o.d.'s such that $\prec_i+1 \subseteq \prec_i$ and $\prec_2$ is the same as one mentioned above. $q$ also includes data $pd_i(\rho)$. $st_{N-1}(\rho)$ is defined so that

$$\gamma \prec_{N-1} \rho \Rightarrow st_{N-1}(\gamma) < st_{N-1}(\rho) \quad (1)$$

In Subsection 2.2 we explain what parts correspond to the $Q$ part in proof figures.
A theory $T_N$ for $\Pi_N$-reflection is formulated in Tait’s logic calculus, i.e., one-sided sequent calculus and $\Gamma, \Delta \ldots$ denote a sequent, i.e., a finite set of formulae. $T_N$ has the inference rule ($\Pi_N$-rfl):

$$
\Gamma, A \vdash \exists z A^\gamma, \Gamma
$$

where $A \equiv \forall x_N \exists x_{N-i} \cdots Qx_1 B$ with a bounded formula $B$.

So ($\Pi_N$-rfl) says $A \to \exists z A^\gamma$. 

To deal with the inference rule ($\Pi_N$-rfl) we introduce new inference rules ($c$)\, and ($\Sigma_i$)\, as in [6]:

$$
\frac{\Gamma, A^\sigma, \Lambda^\sigma}{\Gamma, A^\rho, (c)^\sigma} \quad \text{(c)}
$$

where $\Lambda$ is a set of $\Pi_N$-sentences as above, $\Lambda^\sigma = \{A^\sigma : A \in \Lambda\}$, the side formulae $\Gamma$ consists solely of $\Sigma_i^\sigma$-sentences and $\rho$ is of the form $d^\rho_i \alpha$.

$$
\frac{\Gamma, \neg A^\sigma, A^\sigma, \Lambda}{\Gamma, A} \quad \text{($\Sigma_i$)}
$$

where $A$ is a $\Sigma_i$ sentence. Although this rule ($\Sigma_i$)$^\sigma$ is essentially a (cut) inference, we need to distinguish between this and (cut) to remember that a ($\Pi_N$-rfl) was resolved.

When we apply the rule ($c$)$^\rho$ it must be the case:

- any instance term $\beta < \sigma$ for the existential quantifiers $\exists x_{N-i} < \sigma$ (i:odd)
- in $A^\sigma \equiv \forall x_N < \sigma \exists x_{N-1} < \sigma \cdots Qx_1 < \sigma B$ is less than $\rho$ (2)

As in [6] an inference rule ($\Pi_N$-rfl) is resolved by forming a succession of rules (c)'s, called a chain, which grows downwards in proof figures. We have to pinpoint, for each (c), the unique chain, which describes how to introduce the (c). To retain the uniqueness of the chain, i.e., not to branch or split a chain, we have to be careful in resolving rules with two uppersequents. Our guiding principles are:

(\textbf{ch1}) For any $\frac{A^\sigma}{A^\tau}$ (c)$^\sigma$ with $\tau = d^\rho_i \alpha$, if an o.d. $\beta$ is substituted for an existential quantifier $\exists y < \sigma$ in $A^\sigma$, i.e., $\beta$ is a realization for $\exists y < \sigma$, then $\beta < \tau$, cf. (2), and

(\textbf{ch2}) Resolving rules having several uppersequents must not branch a chain.

\footnote{For simplicity we suppress the parameter. Correctly $\forall u(A(u) \to \exists z(u < z \land A^z(u)))$.}
2.1 Merging chains

As contrasted with \[\Pi_3\]-reflection we have to merge chains here. Let us explain this phenomenon.

We omit side formulae in this subsection.

1) First resolve a \((\Pi_N\text{-rfl})\) in the left figure, and resolve the \((\Sigma_N)\)^\sigma\ 0 to the right figure with a \(\Sigma_{N-1}\) \(A_1\):

\[
\begin{array}{c}
\frac{A}{A^\sigma} (c)^\sigma_{\sigma} J_0 \\
\frac{A}{A^\sigma} \quad \frac{A}{A^\sigma} \\
\frac{-A^\sigma}{-A^\sigma} \\
\frac{A_1}{A_1^\sigma} \\
\frac{A_1^\sigma}{J_1 (\Sigma_{N-1})^\sigma}
\end{array}
\]

with \(A \equiv \forall x_N \exists x_{N-1} \forall x_{N-2} A_3\), \(\sigma = d^\sigma\alpha\), where \(A_3 \equiv \exists x_{N-3} A_4\) is a \(\Sigma_{N-3}\)-formula and \(\alpha\) denotes the o.d. attached to the uppersequent \(A\) of \((c)^\sigma_{\sigma}\).

2) Second resolve a \((\Pi_N\text{-rfl})\) above the \((c)^\sigma_{\sigma} I_0\) and a \((\Sigma_N)\) as in 1):

\[
\begin{array}{c}
\frac{-A^\sigma}{(\Sigma_{N-1})^\sigma} (\Sigma_{N-1})^\sigma \quad (\Sigma_{N-1})^\sigma \\
\frac{A^\sigma}{A^\sigma} B^\sigma \\
\frac{A^\sigma}{A^\sigma} B^\sigma \\
\frac{B^\sigma}{B^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A_1, B_1}{A_1^\sigma, B_1^\sigma} \quad (c)^\sigma_{\sigma} \\
\frac{B^\sigma}{B^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A_1, B_1}{A_1^\sigma, B_1^\sigma} \quad (c)^\sigma_{\sigma} \quad (\Sigma_{N-1})^\sigma
\end{array}
\]

with a \(\tau = d^\sigma\beta\) and a \(\Sigma_{N-1} B_1 \equiv \exists y_{N-1} \forall y_{N-2} B_3\), where \(\nu\) denotes the o.d. attached to the subproof \(P_1\) ending with the uppersequent \(A_1, B\) of \((c)^\sigma_{\sigma} I_0\).

After that resolve the \((\Sigma_{N-1})^\sigma\) \(J_1\):

\[
\begin{array}{c}
\frac{-A^\sigma}{(\Sigma_{N-1})^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A^\sigma}{A^\sigma} B^\sigma \\
\frac{A^\sigma}{A^\sigma} B^\sigma \\
\frac{B^\sigma}{B^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A_1, B_1, A_2}{A_1^\sigma, B_1^\sigma, A_2^\sigma} \\
\frac{B^\sigma}{B^\sigma} \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-1})^\sigma \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-1})^\sigma
\end{array}
\]

Then resolve the \((\Sigma_N)\)^\sigma\ 0:

\[
\begin{array}{c}
\frac{-A^\sigma}{\quad (\Pi_N\text{-rfl}) H} \\
\frac{P_3}{\quad \vdots} \\
\frac{-A^\sigma}{(\Sigma_{N-2})^\sigma \quad J_2} \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-2})^\sigma \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-2})^\sigma \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-2})^\sigma \\
\frac{A^\sigma}{A^\sigma} \quad (\Sigma_{N-2})^\sigma
\end{array}
\]

5
3) Thirdly resolve a \((\Pi_N\text{-rfl})H\) above the \((c)_{\rho}^\sigma I_0\). One cannot resolve the \((\Pi_N\text{-rfl})H\) by introducing a \((c)_{\rho}^\sigma\) with \(\rho < \tau\). Let me explain the reason.

Suppose that we introduce a new \((c)_{\rho}^\sigma I_1^1\) with \(\rho = d_{\sigma}^2\gamma\) immediately above the \((\Sigma_{N-2})^\tau J_2\) as in Fig. 3. Then the new \((c)_{\rho}^\sigma I_1^1\) is introduced after the \((c)_{\rho}^\sigma I_1\) and so \(\rho = d_{\sigma}^2\gamma < \tau\). Hence a new \((\Sigma_N)^\rho K'\) is introduced below the \((\Sigma_{N-1})^\tau K\):

\[
\begin{array}{c}
\dfrac{A_1, D}{\neg A_2^\sigma, \neg \hat{A}_1^\sigma, A_1^\rho, D^\sigma (c)_{\rho}^\sigma} \\
\dfrac{B_1^\sigma, A_2^\sigma}{\neg A_2^\sigma, D^\rho (c)_{\rho}^\sigma I_1^1} \\
\dfrac{\neg B_1^\tau, \neg B_1^\sigma}{B_1^\tau, D^\rho (c)_{\tau}^\gamma I_1} \\
\dfrac{\neg D^\rho}{-D^\rho K' \quad \text{Fig. 2}}
\end{array}
\]

with \(D \equiv \forall z_N \exists z_{N-1} \forall z_{N-2} D_3\). Nevertheless this does not work, because \(\neg A_2 \equiv \exists x_{N-3} \forall x_{N-4} - A_4\) is a \(\Sigma_{N-2}\) sentence with \(N - 2 \geq 2\). Namely the principle \((\text{ch1})\) may break down for the \((c)_{\rho}^\sigma I_1^1\) since any o.d. \(\delta < \sigma\), i.e., possibly \(\delta \geq \rho\) may be an instance term for the existential quantifier \(\exists x_{N-3}\) in \(A_2 \equiv \exists x_{N-2} \exists x_{N-3} A_4\) and may be substituted for the variable \(x_{N-3}\) in \(\neg A_2^\gamma\). Only we knows that such a \(\delta\) is less than \(\sigma\) and comes from the left upper part of \(J_2\).

4) Therefore the chain for \(H\) has to connect or merge with the chain \(I_0 - I_1\) for \(B:\)

\[
\begin{array}{c}
\dfrac{A_1, B}{\neg A_1^\tau, A_1^\rho, B^\sigma \hat{I}_0} \\
\dfrac{B^\sigma}{B^\tau, I_1^1} \\
\dfrac{\neg B_1^\sigma, \neg B_1^\tau}{B_1^\tau, D^\rho (c)_{\tau}^\gamma J_2} \\
\dfrac{\neg D^\rho}{\neg D^\rho (c)_{\rho}^\sigma I_2 \quad \text{Fig. 3}}
\end{array}
\]

with \(\rho = d_{\sigma}^2\gamma\) and a \((\Sigma_N)^\rho\) with the cut formula \(D^\rho\) follows this figure as in Fig. 2, where \(\eta\) denotes the o.d. attached to the uppersequent \(\hat{A}_1, D\) of \((c)_{\rho}^\sigma I_0^\nu\). \((\Sigma_{N-2})^\sigma J_2\) is a merging point for chains \(I_0 - I_1\) and \(I_0^\nu - I_1 - I_2\).

The principle \((\text{ch1})\) for the new \((c)_{\rho}^\sigma I_2\) will be retained for the simplest case \(N = 4\) as in Fig. 3. The problem is that the proviso \(\text{(II)}\) may break down: it may be the case \(\nu = st_{N-1}(\tau) \leq st_{N-1}(\rho) = \eta\) since we cannot expect the upper part of \((c)_{\rho}^\sigma I_0^\nu\) is simpler than the one of \((c)_{\rho}^\sigma I_0\).

In other words a new succession \(I_0^\nu - I_1 - I_2\) of collapsings starts. This is required to resolve \(\Sigma_{N-2}\) sentence \(\neg A_2^\gamma (N - 2 \geq 2)\) and hence \(\sigma\) has to be \(\Pi_{N-1}\)-reflecting.
If this chain $I'_0 - I_1 - I_2$ would grow downwards as in $\Pi_3$-reflection, i.e., in a chain $I'_0 - I_1 - I_2 - \cdots - I_n, I_n$ would come only from the upper part of $I'_0$, then the proviso (1) would suffice to kill this process. But the whole process may be iterated: in Fig.3 another succession $I''_0 - I_1 - I_2 - I_3$ may arise by resolving the $(\Sigma N)^\gamma J'_0$.

Nevertheless still we can find a reducing part, that is, the upper part of the $(c)^\gamma I_2$: the upper part of the $(c)^\gamma I_2$ becomes simpler in the step $I_2 - I_3$. Furthermore in the general case $N > 4$ merging processes could be iterated, viz. the merging point $(\Sigma N-2)^\rho J_2$ may be resolved into a $(\Sigma N-3)^\rho$, which becomes a new merging point to analyse a $\Sigma N-3$ sentence $\alpha^\sigma I_0$ where $\rho_1 < \rho$ is a $\Pi_{N-2}$-reflecting and so on. Therefore in $\text{Od}(\Pi_N)$ the $Q$ part of an o.d. may consist of several factors:

$$(\tau, \alpha, q = \{\nu_i, \kappa_i, \tau_i : i \in \text{In}(\rho)\}) \mapsto d^2_\alpha = \rho$$

with $\kappa_{N-1} = \text{rg}_{N-1}(\rho) = \pi$. $\text{In}(\rho)$ denotes a set such that

$$N - 1 \in \text{In}(\rho) \subseteq \{i : 2 \leq i \leq N - 1\}.$$  

We set for $i \in \text{In}(\rho)$:

$$st_i(\rho) = \nu_i, \text{rg}_i(\rho) = \kappa_i, pd_i(\rho) = \tau_i.$$  

If $i \notin \text{In}(\rho)$, set

$$pd_i(\rho) = pd_{i+1}(\rho), st_i(\rho) \simeq st_i(pd_i(\rho)), \text{rg}_i(\rho) \simeq \text{rg}_i(pd_i(\rho)).$$

Also these are defined so that $pd_2(\rho) = \tau$ for $\rho = d^2_\alpha$.

For the o.d. $\rho = d^2_\gamma$ in the Fig.3, $\text{In}(\rho) = \{N - 2, N - 1\}, st_{N-1}(\rho) = \eta, pd_{N-1}(\rho) = \sigma, \text{rg}_{N-2}(\rho) = \tau = pd_{N-2}(\rho), st_{N-2}(\rho) = \gamma = st_2(\gamma)$.

Thus $\nu_i = st_i(\rho)$ corresponds to the upper part of a $(c)^{\text{rg}_i(\rho)}$ while $\tau_{N-1} = pd_{N-1}(\rho)$ indicates that the first, i.e., uppermost merging point for a chain ending with a $(c)^\rho$ is a rule $(\Sigma N-2)^{\tau N-1}, e.g.,$ the rule in Fig.3. Note that $st_{N-1}(\rho) = \eta < st_{N-1}(pd_{N-1}(\rho))$, cf. (1). $\kappa_i = \text{rg}_i(\rho)$ is an o.d. such that there exists a $(c)^\kappa_i$ in the chain for $(c)^\rho$. We will explain how to determine the rule $(c)^{\text{rg}_i(\rho)}$, i.e., the point to which we direct our attention in Subsection 2.2.

The case $\text{In}(\rho) = \{N - 1\}$ corresponds to the case when a $(c)^{pd_{N-1}(\rho)}$ is introduced without merging points, i.e., as a resolvent of a $(\Pi N$-rfl) above the top of the chain whose bottom is a $(c)^{pd_{N-1}(\rho)}$. The case $\text{In}(\rho) = \{N - 2, N - 1\}$ corresponds to the case when a $(c)^{pd_{2}(\rho)}(pd_2(\rho) = pd_{N-2}(\rho))$ is introduced with a merging point $(c)^{pd_{N-1}(\rho)}$.

In Fig.3 a new succession with a merging point $(c)^\sigma I_2$ arises by resolving a $(\Sigma N)^\tau$ below the $(c)^\sigma I'_1, i.e., I_0 - I'_1 - I_2 - I_3 (c)^\sigma$ for a $\kappa$ with a $\lambda = st_{N-1}(\kappa)$. But in this case we have

$$\lambda = st_{N-1}(\kappa) < st_{N-1}(\tau) = \nu.$$  

$st_{N-1}(\kappa)$ corresponds to the upper part $P_1$ of a $(c)^\sigma I_0$ in Fig.1, when the $(c)^\sigma$ was originally introduced. This part $P_1$ is unchanged up to Fig.3:
2.2 The Q part of an ordinal diagram

In this subsection we explain how to determine the Q part \( q \) of \( \rho = \delta^2_\alpha \) from a proof figure when an inference rule \((c)^\rho\) is introduced.

In general such a \((c)^\rho\) is formed when we resolve an inference rule \((\Pi_N\text{-rfl}) H:\)

\[
\begin{array}{c}
\vdots \\
\vdots \\
\Gamma_{n+1}^{m+1} \\
\cdots \\
\Gamma_{m}^{m+1} (c)_{\sigma_{p+1}}^\sigma J_{p+1}^m \\
\cdots \\
\Gamma_{m}^{m+1} (c)_{\sigma_{n+1}}^\sigma J_{n+1}^m \\
\cdots \\
\Phi_m, \neg A_m \\
\end{array}
\quad
\begin{array}{c}
\vdots \\
\vdots \\
\Gamma_{n}^{n+1} \\
\cdots \\
\Gamma_{n}^{n+1} (c)_{\sigma_{n+1}}^\sigma J_{n+1}^n \\
\cdots \\
\Phi_m, \Psi_m \\
\end{array}
\quad
\begin{array}{c}
\Gamma_0 \\
\cdots \\
(c)_{\sigma_1}^\pi J_0 \\
\cdots \\
(c)_{\sigma_{p+1}}^\sigma J_p \\
\cdots \\
(c)_{\sigma_{n+1}}^\sigma J_{n+1} \\
\cdots \\
(S_i_{m})^\sigma_{n+1} K_m \\
\end{array}
\]

where \( \mathcal{R} = J_0, \ldots, J_{n-1} \) denotes a series of rules \((c)^\rho_{\sigma+1}, J_p\) with \( \pi = \sigma_0, \sigma = \sigma_n \).

\((\Pi_N\text{-rfl}) H\) is resolved into a \((c)^\rho J_p\) with \( \pi = \sigma_0, \sigma = \sigma_n \).

This series \( \mathcal{R} \) is devided into intervals \( \{ \mathcal{R}_m = J_{n_m-1+1}, \ldots, J_{n_m} : m \leq l \} \) with an increasing sequence \( n_{m-1} + 1 = 0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \) of numbers so that

1. \( \mathcal{R}_0 = J_0, \ldots, J_{n_0} \) is a chain \( C_{n_0} \) leading to \( J_{n_0} \).

2. For \( m < l \) \( \mathcal{R}_{m+1} = J_{n_m+1}, \ldots, J_{n_{m+1}} \) is a tail of a chain \( C_{n_{m+1}} = J_0^{m+1}, \ldots, J_{n_m+1}^{m+1}, J_{n_{m+1}}^{m+1}, \ldots, J_{n_m+1} \) leading to \( J_{n_{m+1}} \) such that the chain \( C_{n_{m+1}} \) passes through the left side of an inference rule \((\Sigma_{i_{m}})^{\sigma_{n_{m+1}}} K_m \) with \( 2 \leq \ i_m < N - 1 \), \( J_{n_m} \) is above the right uppersequent \( A_m, \Psi_m \) and \( J_{n_{m+1}} \) is above the left uppersequent \( \Phi_m, \neg A_m \) of \( K_m \), resp. \( A_m \) is a \( \Sigma_{i_{m}} \) sentence. Each rule \( J_{p+1}^m \) for \( p \leq n_m \) is again an inference rule \((c)^\rho_{\sigma+1}, J_m \) will be
a merging point of chains $C_{n_{m+1}}$ and a new chain $C_p = J_0, \ldots, J_{n-1}, J_n$ leading to $(c)_p J_n$.

3. There is no such a merging point below $J_{n-1}$, v.z. there is no $(\Sigma_k)^\sigma$ with $1 < k < N - 1$ such that $J_{n-1}$ is in the right upper part of the inference rule and there exists a chain passing through its left side.

Set $N - 1 \in In(\rho)$, $rg_{N-1}(\rho) = \pi$ and $st_{N-1}(\rho)$ is the o.d. attached to the upper part of $(c)^\sigma J_0$, where by the upper part we mean the part after resolving $(\Pi_N$-rfl) $H$.

First consider the case $l = 0$, i.e., there is no merging point for the new chain $C_\rho$ leading to the new $J_n$. Then set $In(\rho) = \{N - 1\}$ and $pd_{N-1}(\rho) = \sigma$.

Suppose $l > 0$ in what follows. Then set $pd_{N-1}(\rho) = \sigma_{n_l+1}$, i.e., $pd_{N-1}(\rho)$ is the superscript of the first uppermost merging point $(\Sigma_{i_0})^{\sigma_{q+1}} K_0$.

In any cases we have $st_{N-1}(\rho) < st_{N-1}(pd_{N-1}(\rho))$, cf. $st_i(\rho)$ always corresponds to the upper part of a $(c)^{\rho_2(i)}$ in the chain $C_\rho$ for $i \in In(\rho)$.

### 2.2.1 The simplest case $N = 4$

Here suppose $N = 4$ and we determine the Q part of $\rho$. First set $2 \in In(\rho)$, v.z. $In(\rho) = \{2, 3\}$ and $pd_2(\rho) = \sigma$. It remains to determine the o.d. $rg_2(\rho)$. In other words to specify a rule $(c)^{\rho_2} J_p$ with $rg_i(\rho) = \sigma_q$.

Note that $i_m = 2$ for any $m$ with $0 < m \leq l$ since $2 \leq i_m < N - 1 = 3$ in this case. There are two cases to consider. First suppose there is a $p < n$ such that

1. $p > n_0$, i.e., $\sigma_{p+1} \prec_2 \sigma_{n_0 + 1} = pd_3(\rho)$ and
2. $2 \in In(\sigma_{p+1})$, i.e., there was a merging point of the chain leading to $(c)^{\sigma_{p+1}} J_p$.

Then pick the minimal $q$ satisfying these two conditions, v.z. the uppermost rule $(c)^{\sigma_{q+1}} J_q$ below the first uppermost merging point $(\Sigma_{i_0})^{\sigma_{q+1}} K_0$ with $2 \in In(\sigma_{q+1})$. Then set

**Case 1** $rg_2(\rho) = rg_2(\sigma_{q+1})$.

Otherwise set

**Case 2** $rg_2(\rho) = \sigma = pd_2(\rho)$.

Consider the first case **Case 1** $rg_2(\rho) = rg_2(\sigma_{q+1}) \neq pd_2(\rho)$. From the definition we see $rg_2(\rho) = rg_2(\sigma_{q+1}) = pd_2(\sigma_{q+1}) = \sigma$. We have $\sigma_q = rg_2(\rho) \geq_3 pd_3(\rho) = \sigma_{n_0 + 1}$. This follows from the minimality of $q$, i.e., $\forall t (n_0 < t < q \rightarrow 2 \notin In(\sigma_{t+1}))$ and hence $\forall t (n_0 < t < q \rightarrow \sigma_t = pd_2(\sigma_{t+1}) = pd_3(\sigma_{t+1}))$.

Furthermore $q$ is minimal, i.e, $\sigma_q$ is maximal in the following sense:

$$
\forall t (n_0 < t < n (\leftrightarrow pd_2(\rho) = \sigma \geq_2 \sigma_{t+1} \prec_2 pd_3(\rho)) \& rg_2(\sigma_{t+1}) \leq_2 \sigma_q)
\rightarrow
rg_2(\sigma_{t+1}) \leq_2 \sigma_q
$$

(3)

In general we have the following fact.
Proposition 2.1  Let $C = J_0, \ldots, J_{n-1}$ be a chain leading to a $(c)^\sigma_{n-1} J_{n-1}$. Each $J_p$ is a rule $(c)^\sigma_{p+1}$, with $\sigma_0 = \pi$. Suppose that $2 \in In(\sigma_n)$ and the chain passes through the left side of a $(\Sigma_2)^\sigma_p K$ for a $p$ with $0 < p < n$ so that $J_{p-1}$ is in the left upper part of $K$ and $J_p$ is below $K$. Then $\sigma_q = rg_2(\sigma_n) \preceq_2 \sigma_p$, i.e., $q \geq p$.

\[
\begin{array}{c}
\vdots \vdots \\
\Phi, \Psi \\
\Phi, \Psi \Phi, \Psi \\
\vdots \vdots \vdots \\
\Gamma_{p-1} \quad \Gamma_{p-1} (c)^{\sigma_{p-1}} J_{p-1} \\
\vdots \vdots \vdots \\
\Phi, \Psi A^{\sigma_0} A^{\sigma_1}, \Psi (\Sigma_2)^{\sigma_p} K \\
\vdots \vdots \vdots \\
\Phi, \Psi \Phi, \Psi \\
\vdots \vdots \vdots \\
\Gamma_{p-1} (c)^{\sigma_p} J_p \\
\vdots \vdots \vdots \\
\Gamma_{n-1} (c)^{\sigma_{n-1}} J_{n-1}
\end{array}
\]

This means that when, in Fig.4 a $(\Sigma_3)^{\sigma_t} K^3 (0 < t \leq n)$ in the new chain $C_p = J_0, \ldots, J_{n-1}, J_n$ leading to $(c)^{\sigma_n} J_n$ is to be resolved into a $(\Sigma_2)^{\sigma_t} K^2$, then $t \leq q$, i.e., $rg_2(\rho) = \sigma_q \preceq_2 \sigma_t$. In other words any $(\Sigma_3)^{\sigma_t}$ with $q < t \leq n$, equivalently $(\Sigma_3)^{\sigma_t}$ which is below $(c)^{\sigma_n} J_q$ has to be resolved, until the chain $C_p$ will disappear by inversion.

For example consider, in Fig.4, an inference rule $(\Sigma_3)^{\sigma_t} K^3$ for $t = n_{m+1} + 1$. Its right cut formula is a $\Sigma_{m+2}$ sentence $C^{\sigma_t}$ and a descendent of a $\Sigma_3$ sentence $C$: a series of sentences from $C$ to $C^{\sigma_t}$ are in the chain $C_{n_{m+1}} = J_0^{m+1}, J_1^{m+1}, J_2^{m+1}, \ldots, J_{n_{m+1}}$ leading to $J_{n_{m+1}}$. Then the chain $C_{n_{m+1}}$ passes through the left side of the inference rule $(\Sigma_{m+1})^{\sigma_{n_{m+1}}} K_m$ and hence $K^3$ will not be resolved until $K_m$ will be resolved and its right upper part will disappear since we always perform rewritings of proof figure on the rightmost branch. But then the chain $C_p$ will disappear by inversion since it passes through the right side of $K_m$. In this way we see Proposition 2.1 cf. Lemma 5.7 in Section 3 for a full statement and a detailed proof.

It is seen from Proposition 2.1 and the minimality of $q$. Thus we have shown, cf. the conditions $(D^2.1)$ for $Od(\Pi_4)$ in 9 or Section 4.

\[rg_2(\sigma) = rg_2(pd_2(\rho)) \preceq_2 rg_2(\rho) \preceq_3 pd_3(\rho)\]

and

\[\forall t [rg_2(pd_2(\rho)) \preceq_2 \sigma_t \preceq_2 \sigma_q \Rightarrow rg_2(\sigma_t) \preceq_2 \sigma_q].\]

Furthermore we have

\[st_2(\rho) < st_2(\sigma_{p+1}) < \sigma_q^+ \quad (4)\]
for the maximal \( p \), vz. for the latest \((c) \sigma_{p+1} J_p\) with \( rg_2(\sigma_{p+1}) = \sigma_q \& 2 \in In(\sigma_{p+1})\).

Let \( m < l \) denote the number such that \( n_m < q \leq n_{m+1} \), i.e., \( J_q \) is a member of the tail \( R_{m+1} = J_{n_{m+1}}, \ldots, J_{n_m} \) of the chain \( C_{n_{m+1}} \). Then from Proposition 2.3 we see that \( J_p \) is also a member of \( R_{m+1} \) and further that \( J_q \) is a member of a chain \( C_p \) leading to \( J_p \). Thus the upper part of \((c)^{\sigma_q} J_q\) corresponding to \( st_2(p) \) is a result of performing several non-void rewritings to the upper part of a \((c)^{\sigma_q}\) which determined \( st_2(\sigma_{p+1}) \) when \((c) \sigma_{p+1} J_p\) was introduced originally. This yields (4).

Thus we have established the conditions (DQ).1 in \( \mathbb{I} \) or Section 4 for the newly introduced \( \rho \).

Why we choose such a \( \sigma_q \) as \( rg_2(\rho) \)? First introducing \( \sigma_q = rg_2(\rho) \) is meant to express the fact that \( \sigma_q \) is (iterated) \( \Pi_3 \)-reflecting and it is responsible to \( \Sigma_2^p \) sentences occurring above a \((c)^{\sigma_q}\). Therefore even if there exists a \( \sigma_{p+1} \) above \( pd_3(\rho) \), i.e., \( p \leq n_0 \) such that \( 2 \in In(\sigma_{p+1}) \), we ignore these in determining \( rg_2(\rho) \). Second in the Case 1 the reason why we chose \( \sigma_q \) as the uppermost one is explained by Proposition 2.3 any \((\Sigma_3)^{\rho}\) in the new chain \( C_p \) will not be resolved for \( q < t \leq n \) until the chain \( C_p \) will disapper by inversion. Hence any \( \sigma_q \) with \( rg_2(\sigma_{p+1}) = \sigma_q \& 2 \sigma_q \) for some \( p_1 \leq n \) will not be \( rg_2(\kappa) \) for \( \kappa \prec_2 \rho \) in the future. This means that a collapsing series \( \{(c) \kappa : rg_2(\kappa) = \sigma_q\} \) expressing the fact that \( \sigma_q \) is \( \Pi_3 \)-reflecting is killed by introducing \( \rho \) such that \( \rho \prec_2 \sigma_q \& 2 \sigma_q = rg_2(\rho) \). Therefore once we introduce such a \( \rho \), then we can ignore \( rg_2(\sigma_{p+1}) = \sigma_q \) between \( rg_2(\rho) \) and \( \rho \).

2.2.2 The general case \( N > 4 \)

Here suppose \( N > 4 \) and we determine the \( Q \) part of \( \rho \), i.e., determine the set \( In(\rho) \) and o.d.'s \( pd_i(\rho), rg_i(\rho) \) for \( i \in In(\rho) \) by referring Fig.4.

First set \( i_0 \in In(\rho) \) where \( i_0 \) denotes the number such that the first merging point is a \((\Sigma_{i_0})^{\sigma_{i_0+1}} K_0 \). Now let us assume inductively that for \( k_0 \geq 0 \) we have specified merging points \( \{K_{m_k} : k \leq k_0\} \) so that \( 0 = m_0 < \cdots < m_{k_0} \), \( N - 1 > i_{m_0} > \cdots > i_{m_{k_0}} \geq 2 \) and \( \forall m \forall k < k_0 [m_k < m < m_{k+1} \rightarrow i_m \geq i_{m_k}] \), and have setted \( \{i_{m_k} : k \leq k_0\} \subseteq In(\rho) \). Namely \( K_{m_0}, \ldots, K_{m_{k_0}} \) is a series of merging points going downwards with decreasing indices \( i_{m_k} \) and \( K_{m_k} \) is the uppermost merging point with \( i_{m_k} < i_{m_{k-1}} (i_{m_{k-1}} := N - 1) \).

If there exists an \( m < l \) such that \( m_{k_0} < m \& i_{m_{k_0}} > i_m \geq 2 \), then let \( m \) denote the minimal one, vz. the uppermost merging point \( K_m \) below the latest one \( K_{m_{k_0}} \) with \( i_{m_{k_0}} > i_m \), and set \( i_m \in In(\rho) \). Otherwise set

\[
In(\rho) = \{i_{m_k} : k \leq k_0\} \cup \{N - 1\}.
\]

This completes a description of the set

\[
In(\rho) = \{N - 1 = i_{m_{-1}}\} \cup \{i_{m_k} : 0 \leq k \leq k_1\}
= \{N - 1 = i_{m_{-1}} > i_{m_0} > \cdots > i_{m_{k_1}}\}.
\]

Observe that for \( i < N - 1 \)

\[
i \in In(\rho) \Leftrightarrow \exists m < l [i_m = i \& \forall p < m (i_p \geq i)].
\]
Now set \( pd_{i_{mk}}(p) = \sigma_{n_{mk+1}} \) for \(-1 \leq k \leq k_1\) with \( m_{k+1} := l\), viz. the merging point \( K_{m_k} \) chosen for \( i_{mk} \in In(p) \) is a \((\Sigma_{i_{mk}})^{pd_{i_{mk-1}}}(p)\) for \( 0 \leq k \leq k_1\) and \( pd_2(p) = pd_{i_{m_1}}(p) = \sigma_{n_1+1} = \sigma_1 = \sigma \). Observe that for any \( i\) with \( 2 \leq i \leq N - 1\) there exists an \( m(i) \leq l\) such that \( pd_i(p) = \sigma_{n_m(i)+1}\) and this \( m(i)\) is the minimal \( m\) for which \( i_m < i\).

It remains to determine the o.d.’s \( rg_i(p) \) for \( N - 1 \neq i = i_{m_k} \in In(p) \). As in the case \( N = 4 \) there are two cases to consider. First suppose there is a \( p < n\) such that

1. \( \rho \prec_i \sigma_{p+1} \prec_i \sigma_{n_{m_k+1}} = pd_{i_{m_k-1}}(p) = pd_{i+1}(p) \) and
2. \( i \in In(\sigma_{p+1}) \).

Then pick the minimal \( p\) satisfying these two conditions, viz. the uppermost rule \((c)\sigma_{p+1}, J_p\) below the merging point \((\Sigma_i)^{pd_{i+1}(p)} K_{m_k}\) with \( \sigma_{p+1} \prec_i pd_{i+1}(p) \& i \in In(\sigma_{p+1}) \). Then set

**Case 1** \( rg_i(p) := \sigma_q := rg_i(\sigma_{p+1}) \).

Otherwise set

**Case 2** \( rg_i(p) = pd_{i+1}(p) \).

In general we have the following fact.

**Proposition 2.2** Let \( C = J_0, \ldots, J_{n-1} \) be a chain leading to a \((c)^{\sigma_{n-1}} J_{n-1}\). Each \( J_p\) is a rule \((c)^{\sigma_{p+1}}\) with \( \sigma_0 = \pi\). Suppose that the chain passes through the left side of a \((\Sigma_p)^{\sigma_p} K\) for a \( p \) with \( 0 < p < n\) and a \( j \geq i \) so that \( J_{p-1} \) is in the left upper part of \( K\) and \( J_p\) is below \( K\). Then \( \sigma_n \prec_i \sigma_p\) and if further \( N - 1 \neq i \in In(\sigma_n)\), then \( \sigma_q = rg_i(\sigma_n) \prec_i \sigma_p\), cf. the figure in Proposition 2.1.

Let us explain this Proposition 2.2 using the new chain \( C_p = J_0, \ldots, J_{n-1} J_n\) leading to \((c)^{\sigma_{n+1}} J_n\), cf. Fig. 4. When a \((\Sigma_j)^{\sigma_j} K^{j+1}(0 < t \leq n)\) in the new chain \( C_p\) is to be resolved, a \((\Sigma_j)^{\sigma_j} K^{j+1}\) is introduced at a point below \( K^{j+1}\). The point and \( s \geq t\) is determined as the lowest position as far as we can lower a rule \((\Sigma_j)^{\sigma_j}\), cf. Definition 5.3 in Section 6. For example when \( K^{j+1}\) is the rule \((\Sigma_l)^{\sigma_{m+1}} K_m\) in Fig. 4, let \( m_1\) denote the minimal \( m\) such that \( i_{m_1} < i_m\) and we introduce a new \((\Sigma_{m_1-1})^{\sigma_{m_1}}\) \((s = n_{m+1})\) between the rules \((c)\sigma_{m+1} J_{m_1} \) and \((\Sigma_{m_1})^{\sigma_{m_1}} K_{m_1}\). Observe that the new \((\Sigma_{m_1})^{\sigma_{m_1+1}} \) together with \((\Sigma_{m_2}) K_{m_2}\) \((m < m_2 < m_{1})\) by inversion will be merging points for the next chain leading to a \((c)^{\sigma_{n}}\).

Let us consider the case when the \((\Sigma_{m_1-1})^{\sigma_{m_1+1}}\) is the rule \((\Sigma_j)^{\sigma_j} K\) in the Proposition 2.2 \( j = i_m - 1 \& p = n_{m_1+1}\). Also put \( pd_{i+1}(p) = \sigma_{n_{m_1+1}}\), where \( n(i)\) denotes the minimal \( n\) such that \( i_m < i\). Then \( i \leq j = i_m - 1\). By Proposition 2.3 below we see that \( i_{m} < i_{m_3}\) for any \( m_3 < m\), i.e., any merging point \((\Sigma_{i_{m_3}}) K_{m_3}\) above \((\Sigma_{i_{m}}) K^{j+1} = K_m\) has larger index since we are assuming that \( K_m\) is to be resolved. Therefore \( m(i) \geq m_1\), i.e., the merging point \((\Sigma_{i_{m(i)}})^{\sigma_{m(i)+1}} K_{m(i)}\) determining \( pd_{i+1}(p)\) is equal to or below the merging
point \((\Sigma_{i_{m_1}})^{\sigma_{n_{m_1}+1}} K_{m_1}\). In the former case we have \(pd_i(\rho) = \sigma_{n_{m(i)+1}} = \sigma_{n_{m_1}+1} = \sigma_p\) and hence \(\rho \prec_i \sigma_p\). In the latter case we have \(i_{m_3} \geq i\) for \(m_1 \leq m_3 < m(i)\). Thus we see \(\rho \prec_i \sigma_p\) inductively. This shows the first half of Proposition 2.2.

Now assume \(N - 1 \neq i \in In(\sigma_n)\) and show \(rg_i(\rho) \preceq_i \sigma_p\). Consider the

**Case 1**, vz. \(\sigma_q = rg_i(\rho) \neq pd_i(\rho)\). Let \(p_0\) denote the minimal \(p_0\) such that \(\rho \prec_i \sigma_{p_0+1} \prec_i pd_i(\rho)\) and \(i \in In(\sigma_{p_0+1})\). By the definition we have \(\sigma_q = r_{g_i}(\rho) = r_{g_i}(\sigma_{p_0+1})\).

Let \(m(i+1) < m(i)\) denote the number such that \(pd_{i+1}(\rho) = \sigma_{n_{m(i)+1}+1}\).

Then by \(i \in In(\rho)\) we have \(i_{m(i+1)} = i \leq i_{m_1}, m_1 \leq m(i+1) < m(i)\) and hence \(pd_{i+1}(\rho) \leq \sigma_{n_{m_1}+1} = \sigma_p\). On the other we have \(\rho \prec_i \sigma_q = r_{g_i}(\rho)\) by the definition and \(\rho \prec_i \sigma_p\) by the first half of the Proposition 2.2

Hence it suffices to show \(\sigma_q \leq \sigma_p\) since the set \(\{\tau : \rho \prec_i \tau\}\) is linearly ordered by \(\prec_i\) now. Now we see \(\sigma_q = r_{g_i}(\rho) = r_{g_i}(\sigma_{p_0+1}) \preceq_i pd_{i+1}(\rho)\) inductively, i.e., by using Proposition 2.2 for smaller parts. Thus we get \(\sigma_q \preceq_i pd_{i+1}(\rho) \leq \sigma_p\). This shows the second half of Proposition 2.2.

Further we have the following fact.

**Proposition 2.3** Let \(C = J_0, \ldots, J_{n-1}\) be a chain leading to a \((c)_{\sigma}^{\sigma_n-1} J_{n-1}\). Each \(J_k\) is a rule \((c)\sigma_{\tau_k+1}\) with \(\sigma_0 = \pi\). Suppose that the chain \(C\) passes through the left side of a \((\Sigma_j)^e K^l\) for a \(p < n\) so that \(J_{p-1}\) is in the left upper part of \(K^l\) and \(J_p\) is below \(K^l\).

Let \(D = I_0, \ldots, I_{m-1}\) be a chain leading to a \((c)\sigma_{\tau_k+1} I_{m-1}\). Each \(I_k\) is a rule \((c)\sigma_{\tau_k+1}\) such that \(\tau_k = \sigma_k\) for \(0 \leq k < \min\{n, m\}\). Suppose that the chain \(D\) passes through the left side of a \((\Sigma_j)^e K^u\) for a \(k \leq p\) so that \(I_{k-1}\) is in the left upper part of \(K^u\) and \(I_k\) is below \(K^u\). Further assume the rule \((c)\sigma_{\tau_k+1}\) is in the right upper part of \((\Sigma_j)^e K^l\) and \(i \leq j\).

Then the upper \(K^u\) foreruns the lower \(K^l\), i.e., analyses of \(K^u\) have to precede ones of \(K^l\).

Let us explain Proposition 2.3 by referring Fig. 4: \(C\) is the new chain \(C_\rho, K^l\) is the new \((\Sigma_{i_{m_1}-1})^e m_{m_1+1} K_{m_1}\) which is resulted from \((\Sigma_{i_{m_1}})^e m_{m_1+1} K_{m_1}\) with \(m = l - 1\), i.e., the resolved rule \(K_{l-1}\) is the lowest merging point. Then \(K^l\) is a \((\Sigma_{i_{m_1}})^e\) with \(m_1 = l\). Further \(D\) is the chain \(C_{n+1} = J_0, \ldots, J_{m+1}, J_{n+1}, \ldots, J_{m+n}\) leading to the last member \((c)\sigma_{J_{n+1}} (n - 1 = n_{m+1} = n_l)\) of the series \(R\). Then the last member \((c)\sigma_{J_{n-1}}\) is in the right upper part of \((\Sigma_{i_{m_1}})^e K^l\). Let \(I\) be a rule \((\Sigma_{i_{m_1}+1})^e\) such that the chain \(D\) passes through its right side. Suppose the rule \(J\) in the chain \(D\) is resolved and produces a \((\Sigma_{i_{m_1}})^e K^u\) for a \(k < n\) so that the chain \(D\) passes through the left side of \(K^u\).
\[ \Phi, \psi \] (\Sigma_{i_{m-1}})^{\sigma_{m+1}} K^{l_w} \]

where \( \neg A_m \equiv C_{m+1}^{\sigma_{m+1}} \equiv \forall x < \sigma_{m+1} C_0(x) \) and \( C_{m+1}^{\sigma_{m+1}} \equiv C_0(\alpha) \) for an \( \alpha < \sigma_{m+1} \).

\[ \Phi, \psi \] (\Sigma_{i_{m-1}})^{\sigma_{m+1}} K^{l_w} \]

We show, in Fig.6, no ancestor of the right cut formula \( C^\sigma \) of \( K^{l_w} \) is in the right upper part of \( K^{u_p} \) in order to see that \( K^{u_p} \) foreruns \( K^{l_w} \). It suffices to see that, in Fig.5, no ancestor of the right cut formula \( C^\sigma \) of \( K^{l_w} \) is in the left upper part of the resolved rule \( (\Sigma_{i_{m+1}})^{\sigma} I \). Any ancestor of the right cut formula \( C^\sigma \) of \( K^{l_w} \) comes from the left cut formula \( \neg A_m \equiv C_{m+1}^{\sigma_{m+1}} \) of \( (\Sigma_{i_{m-1}})^{\sigma_{m+1}} K_m \) and any ancestor of the latter is in the chain \( D \), which in turn passes through
the right side of $(\Sigma_{i+1})^T I$. Thus an ancestor of the right cut formula $C^\sigma$ of $K_{\text{up}}$ is in the right upper part of $I$ in Fig. 5, a fortiori, in the left upper part of $K_{\text{up}}$ in Fig. 6. This shows Proposition 2.3.

For full statements and proofs of Propositions 2.2, 2.3 see Lemmata 5.8, 7.4, the proviso (uplw) in Definition 5.8 in Section 3 and the case M7.2 in Section 6.

From Propositions 2.2, 2.3 we see that the conditions $(D^Q, 1)$ for Od$(\Pi_N)$ in [9] or Section 4 are enjoyed with respect to the $Q$ part of $\rho$ as for the case $N = 4$. A set-theoretic meaning and a wellfoundedness proof of Od$(\Pi_N)$ are derived from these conditions on o.d.’s as we saw in [8] and [9].

Consider a rule $(\Sigma_j)$ in the chain $\mathcal{C}_\rho$ for $j \geq i \in \text{In}(\rho)$ which is below $(\Sigma_{i_{m(i)}})^{p\rho}(p) K_{m(i)} (i_{m(i)} < i)$. Then from Proposition 2.3 we see that analyses of such a $(\Sigma_j)$ have to follow ones of the rule $(\Sigma_{i_{m(i)}})^{p\rho}(p) K_{m(i)}$. Thus when such a reversal happens, the lower rule with greater indices $(j > i_{m(i)})$ is dead and we can ignore it. The o.d. $p\rho(\rho)$ and the rule $(c)^{p\rho}(p) J_{n_{m(i)}}$ is the predecessor of the o.d. $\rho$ and the rule $(c)$ with respect to $i$: any member $(c)_\kappa$ of the chain $\mathcal{C}_\rho$ with $\rho < \kappa < p\rho(\rho)$ is irrelevant to the fact that $p\rho(\rho)$ and $r\rho_i(\rho)$ are iterated $\Pi_i$-reflecting. But the member may be relevant to $\Pi_j$-reflection for $j < i$. This motivates the definitions of $\text{In}(\rho)$ and $p\rho(\rho)$. A series $\kappa_i \prec \kappa_{i-1} \prec \cdots \prec \kappa_0$ expresses a possible stepping down for the fact that $\kappa_0$ is an iterated $\Pi_i$-reflecting ordinal. Degrees of iterations are measured by an ordinal $\nu < \kappa^+$ with $\kappa = r\rho_i(\kappa_0), \nu = st_i(\kappa_0)$ (and by predecessors of $r\rho_i(\kappa_0)$) as we saw in [8] and [9]. Therefore we search only for o.d.’s $\sigma_{p+1}$ with $\rho \prec \sigma_{p+1}$ in determining the o.d. $r\rho_i(\rho) = r\rho_i(\sigma_{p+1})$.

In the Case 1 the reason why we chose $\sigma_2$ as the uppermost one is explained by Propositions 2.2, 2.3 as in the case $N = 4$.

Now details follow.

3 The theory $T_N$ for $\Pi_N$-reflecting ordinals

In this section a theory $T_N$ of $\Pi_N$-reflecting ordinals is defined.

Let $T_0$ denote the base theory defined in [5]. $\mathcal{L}_1$ denotes the language of $T_0$. Recall that $\mathcal{L}_1 = \mathcal{L}_0 \cup \{ R^A, R^A_\prec : A \text{ is a } \Delta_0 \text{ formula in } \mathcal{L}_0 \cup \{ X \} \}$ with $\mathcal{L}_0 = \{ 0, 1, +, -, \cdot, q, r, \text{ max, } i, \langle \rangle, \langle \rangle =, < \}$. $R^A, R^A_\prec$ are predicate constants for inductively defined predicates. The axioms and inference rules in $T_0$ are designed for this language $\mathcal{L}_1$.

The language $\mathcal{L}(T_N)$ of the theory $T_N$ is defined to be $\mathcal{L}_1 \cup \{ \Omega \}$ with an individual constant $\Omega$.

The axioms of $T_N$ are the same as for the theory $T_3$ in [6], i.e., are obtained from those of $T_{22}$ in [5] by deleting the axiom $\Gamma$, $\text{Ad}(\Omega)$. Thus the axioms $\Gamma, \Lambda_f$ for the closure of $\Omega$ under the function $f$ in $\mathcal{L}_0$ are included as mathematical axiom in $T_N$.

The inference rules in $T_N$ are obtained from $T_0$ by adding the following rules...
(\Pi_N\text{-rfl}) and (\Pi_2^\Omega\text{-rfl}).

\[
\frac{\Gamma, A \neg \exists z (t_0 < z \land A^z), \Gamma}{\Gamma} \quad (\Pi_N\text{-rfl})
\]

where \( A \equiv \forall x_N \exists x_{N-1} \cdots Q x_1 B (x_N, x_{N-1}, \ldots, x_1, t_0) \) is a \( \Pi_N \) formula.

\[
\frac{\Gamma, A^\Omega \neg \exists z (t < z < \Omega \land A^z), \Gamma, t < \Omega}{\Gamma} \quad (\Pi_2^\Omega\text{-rfl})
\]

where \( A \equiv \forall y \exists x B(x, y, t) \) is a \( \Pi_2 \) formula.

Concepts related to proof figures, principal or auxiliary formulae, pure variable condition, branch, etc. are defined exactly as in Section 2 of [5].

4 The system \( Od(\Pi_N) \) of ordinal diagrams

In this section first let us recall briefly the system \( Od(\Pi_N) \) of ordinal diagrams (abbreviated by o.d.’s) in [9].

Let \( 0, \varphi, \Omega, +, \pi \) and \( d \) be distinct symbols. Each o.d. in \( Od(\Pi_N) \) is a finite sequence of these symbols. \( \varphi \) is the Veblen function. \( \Omega \) denotes the first recursively regular ordinal \( \omega^{CK}_1 \) and \( \pi \) the first \( \Pi_N \)-reflecting ordinal.

The set \( Od(\Pi_N) \) is classified into subsets \( R, SC, P \) according to the intended meanings of o.d.’s. \( P \) denotes the set of additive principal numbers, \( SC \) the set of strongly critical numbers and \( R \) the set of recursively regular ordinals (less than or equal to \( \pi \)). If \( \pi > \sigma \in R \), then \( \sigma^+ \) denotes the next recursively regular diagram to \( \sigma \).

Recall that \( K\alpha \) denotes the finite set of o.d.’s defined as follows.

1. \( K0 = \emptyset \).
2. \( K(\alpha_1 + \cdots + \alpha_n) = \bigcup \{ K\alpha_i : 1 \leq i \leq n \} \)
3. \( K\varphi\alpha\beta = K\alpha \cup K\beta \)
4. \( K\alpha = \{ \alpha \} \) otherwise, i.e., \( \alpha \in SC \).

Definition 4.1

1. \( D_\sigma (\alpha) \subseteq D_\sigma \).
   (a) \( D_\sigma (\alpha) = \emptyset \) if \( \alpha \in \{ 0, \Omega, \pi \} \).
   (b) \( D_\sigma (\alpha) = D_\sigma (K\alpha) \) if \( \alpha \notin SC \).
   (c) If \( \alpha \in D_\tau \),

   \[
   D_\sigma (\alpha) = \begin{cases} 
   D_\sigma ((\tau) \cup c(\alpha)) & \text{if } \tau > \sigma \\
   \{ \alpha \} \cup D_\sigma (c(\alpha)) & \text{if } \tau = \sigma \\
   D_\sigma (\tau) & \text{if } \tau < \sigma 
   \end{cases}
   \]

2. \( B_\sigma (\alpha) = \max \{ b(\beta) : \beta \in D_\sigma (\alpha) \} \).
3. \( B_{>\sigma}(\alpha) = \max \{ B_\tau(\alpha) : \tau > \sigma \} \).

For an o.d. \( \alpha \) set
\[
\alpha^+ = \min \{ \sigma \in R \cup \{ \infty \} : \alpha < \sigma \}.
\]

For \( \sigma \in R \), \( D_\sigma \subseteq SC \) denotes the set of o.d.’s of the form \( \rho = d_\sigma^2 \alpha \) with a (possibly empty) list \( q \), where the following condition has to be met:
\[
B_{>\sigma}(\{ \sigma, \alpha \} \cup q) < \alpha
\]
\( \alpha \) is the body of \( d_\sigma^2 \alpha \).

If \( q \) is not empty, then \( d_\sigma^2 \alpha \in D^{Q} \) by definition. Its \( Q \) part \( Q(d_\sigma^2 \alpha) = q = \nu \kappa \tau_j \) denotes a sequence of quadruples \( \nu_m \kappa_m \tau_m j_m \) of length \( l + 1 \) \((0 \leq l)\) such that

1. \( 2 \leq j_0 < j_1 < \cdots < j_l = N - 1 \),
2. \( \kappa_l = \pi, \kappa_m \in R \mid \pi (m < l) \& \sigma \preceq \kappa_m (m \leq l) \),
3. \( \nu_l \in Od(\Pi_N) \),
   \[
   \sigma = \pi \Rightarrow \nu_l \leq \alpha
   \] and
   \[
   m < l \Rightarrow \nu_m < \kappa_m^+
   \]
4. \( \tau_0 = \sigma, \tau_m \in \{ \pi \} \cup D^{Q}, \sigma \preceq \tau_m (m \leq l) \) and
   \[
   \tau_l = \pi \Rightarrow \sigma = \pi
   \]

From \( q = Q(\rho) \) define

1. \( in_j(\rho) = st_j(\rho)rg_j(\rho) \) (a pair) and \( pd_j(\rho) \): Given \( j \) with \( 2 \leq j < N \), put \( m = \min \{ m \leq l : j \leq j_m \} \).
2. \( pd_j(\rho) = \tau_m \).
3. \( \exists m \leq l (j = j_m) \): Then \( st_j(\rho) = \nu_m, rg_j(\rho) = \kappa_m \).
4. Otherwise: \( in_j(\rho) = in_j(pd_j(\rho)) = in_j(\tau_m) \). If \( in_j(\tau_m) = \emptyset \), then set \( st_j(\rho) \uparrow, rg_j(\rho) \uparrow \).
5. \( In(\rho) = \{ j_m : m \leq l \} \).

Observe that
\[
\pi < \beta \in q = Q(\rho) \Rightarrow \beta = \nu_l = st_{N-1}(\rho)
\]

The relation \( \alpha <_i \beta \) is the transitive closure of the relation \( pd_i(\alpha) = \beta \).

In [9] we impose several conditions on a diagram of the form \( \rho = d_\sigma^2 \alpha \) to be in \( Od(\Pi_N) \). For \( \alpha \in Od(\Pi_N), q \subseteq Od(\Pi_N) \& \sigma \in R \setminus \{ \Omega \}, \rho = d_\sigma^2 \alpha \in Od(\Pi_N) \) if the following conditions are fulfilled besides (5):
Proposition 4.1

Assume \( i \in \text{In}(\rho) \). Put \( \kappa = \text{rg}_i(\rho) \). Then

\[(DQ.1)\]
\[
\kappa = \text{rg}_i(\rho). \quad \text{Then}
\]
\[
(DQ.11) \quad \text{in}_i(\text{rg}_i(\rho)) = \text{in}_i(\text{pd}_{i+1}(\rho)), \quad \text{rg}_i(\rho) \preceq_i \text{pd}_{i+1}(\rho) \quad \text{and} \quad \text{pd}_{i}(\rho) \neq \text{pd}_{i+1}(\rho) \quad \text{if} \quad i < N - 1.
\]

Also \( \text{pd}_{i}(\rho) \preceq_i \text{rg}_i(\rho) \) for any \( i \).

Lemma 4.2

One of the following holds:

\[(DQ.12)\]
\[
\text{rg}_i(\rho) = \text{pd}_{i}(\rho) \& \mathbb{B}_{\kappa}(\text{st}_{i}(\rho)) < b(\alpha_1) \quad \text{with} \quad \rho \preceq \alpha_1 \in D_\kappa.
\]

\[(DQ.12.1)\]
\[
\text{rg}_i(\rho) = \text{pd}_{i}(\rho) \& \mathbb{B}_{\kappa}(\text{st}_{i}(\rho)) < b(\alpha_1) \quad \text{with} \quad \rho \preceq \alpha_1 \in D_\kappa.
\]

\[(DQ.12.2)\]
\[
\text{rg}_i(\rho) = \text{pd}_{i}(\rho) \& \text{st}_{i}(\rho) < \text{st}_{i}(\rho).
\]

\[(DQ.12.3)\]
\[
\forall \tau(\text{rg}_i(\text{pd}_{i}(\rho)) \preceq_i \tau \prec_i \kappa \rightarrow \text{rg}_i(\tau) \preceq_i \kappa) \& \text{st}_{i}(\rho) < \text{st}_{i}(\sigma_1) \quad \text{with}
\]
\[
\sigma_1 = \min\{\sigma_1 : \text{rg}_i(\sigma_1) = \kappa \& \text{pd}_{i}(\rho) \prec_i \sigma_1 \prec_i \kappa\}
\]

and such a \( \sigma_1 \) exists.

\[(DQ.2)\]
\[
\forall \kappa \preceq \text{rg}_i(\rho)(K_{\kappa}\text{st}_{i}(\rho) < \rho)
\]

\hspace{1cm}

\[(10)\]

for \( i \in \text{In}(\rho) \).

We set \( Q(d_\alpha) = \emptyset \), i.e., \( d_\alpha = d_\alpha \).

The order relation \( \alpha \prec \beta \) on \( D_\kappa \) is defined through finite sets \( K_\kappa \alpha \) for \( \tau \in R, \alpha \in \text{Od}(\Pi_\kappa) \), and the latter is defined through the relation \( \alpha \prec \beta \), which is the transitive closure of the relation \( \alpha \in D_\kappa \). Thus \( \alpha \prec \beta \iff \alpha \prec \beta \).

For \( \rho = d_\kappa \alpha \), \( c(\rho) = \{\tau, \alpha\} \cup q \) and

\[ K_\kappa \rho = \left\{ \begin{array}{ll}
K_\kappa(\{\tau\} \cup c(\rho)) = K_\kappa(\{\tau, \alpha\} \cup q). & \sigma < \tau \\
K_\kappa \tau, & \tau < \sigma \& \tau \not\in \sigma
\end{array} \right. \]

The following Proposition 4.1 is shown in \( \text{[5]} \).

**Proposition 4.1**

1. The finite set \( \{\tau : \sigma \prec_i \tau\} \) is linearly ordered by \( \prec_i \).

In the following assume \( \kappa = \text{rg}_i(\rho) \downarrow \).

2. \( \rho \prec_i \text{rg}_i(\rho) \).

3. \( \rho \prec_i \sigma \prec_i \tau \& \text{in}_i(\rho) = \text{in}_i(\tau) \Rightarrow \text{in}_i(\rho) = \text{in}_i(\sigma) \).

4. \( \rho \prec_i \tau \prec_i \text{rg}_i(\rho) \Rightarrow \text{rg}_i(\tau) \preceq_i \text{rg}_i(\rho) \).

**Definition 4.2**

For o.d.'s \( \alpha, \sigma \) with \( \sigma \in R \),

\[ K_\sigma(\alpha) := \max K_\sigma \alpha. \]

The following lemmata are seen as in \( \text{[5]} \).

**Lemma 4.1**

Suppose \( \mathbb{B}_{\kappa}(\alpha_i) < \alpha_i \) for \( i = 0, 1 \), and \( \alpha_0 < \alpha_1 \). Then

\[ \tau > \kappa \Rightarrow d_\tau \alpha_1 \in \text{Od}(\Pi_\kappa) \& d_\tau \alpha_0 < d_\tau \alpha_1. \]

**Lemma 4.2**

For \( \alpha, \beta, \sigma \in \text{Od}(\Pi_\kappa) \) with \( \sigma \in R|\pi \) assume \( \forall \tau < \pi[\mathbb{B}_{\tau}(\beta) \leq \mathbb{B}_{\tau}(\alpha)] \), and put \( \gamma = \max\{\mathbb{B}_{\tau}(\beta), \mathbb{B}_{\tau}(\{\sigma, \alpha\})\} + \omega^\beta \). Then \( \mathbb{B}_{\tau}(\{\sigma, \gamma, \gamma + K_\tau(\alpha)\}) < \gamma \), and hence \( \text{[6]} \) is fulfilled for \( d_\tau \gamma, d_\tau (\gamma + K_\tau(\alpha)) \in \text{Od}(\Pi_\kappa) \).
5 The system $T_{dc}$

In this section we extend $T_N$ to a formal system $T_{dc}$. The universe $\pi(T_N)$ of the theory $T_N$ is defined to be the o.d. $\pi \in Od(\Pi_N)$. The language is expanded so that individual constants $c_\alpha$ for o.d.'s $\alpha \in Od(\Pi_N)$ are included. Inference rules $(c)^\sigma$ are added. To each proof $P$ in $T_{dc}$ an o.d. $\alpha(P) \in Od(\Pi_N)$ is attached. Chains are defined to be a consecutive sequence of rules $(c)$. Proofs in $T_{dc}$ defined in Definition 5.8 are proof figures enjoying some provisos and obtained from given proofs in $T_N$ by operating rewriting steps. Some lemmata for proofs are established. These are needed to verify that re writed proof figures enjoy these provisos.

The language $L_{dc}$ of $T_{dc}$ is obtained from the language $L(N)$ by adding individual constants $c_\alpha$ for each o.d. $\alpha \in Od(\Pi_N)$ such that $1 < \alpha < \pi$ and $\alpha \neq \Omega$.

We identify the constant $c_\alpha$ with the o.d. $\alpha$.

In what follows $A, B, \ldots$ denote formulae in $L_{dc}$ and $\Gamma, \Delta, \ldots$ sequents in $L_{dc}$.

The axioms of $T_{dc}$ are obtained from those of $T_N$ as in [5].

Complexity measures $\deg(A)$, $\text{rk}(A)$ of formulae are defined as in [5] by replacing the universe $\pi(T_{22}) = \mu$ by $\pi(T_N) = \pi$.

Also the sets $\Delta^\sigma_0, \Sigma^\sigma_0$ of formulae are defined as in [5]. Recall that for a bounded formula $A$ and a multiplicative principal number $\alpha \leq \pi$, we have $A \in \Delta^\alpha \iff \deg(A) < \alpha$.

Definition 5.1

\[
\deg_N(A) := \begin{cases} 
\deg(A) + N - 1 & \text{if } A \text{ is a bounded formula} \\
\deg(A) & \text{otherwise}
\end{cases}
\]

Note that

\[
\deg_N(A) \not\in \{\alpha + i : i < N - 1, \alpha < \pi \text{ is a limit o.d.}\}
\]

The inference rules of $T_{dc}$ are obtained from those of $T_N$ by adding the following rules $(h)^\sigma (\sigma \in \{\alpha : \pi \leq \alpha < \pi + \omega\} \cup \{0, \Omega\})$, $(c\Pi_2)^{\alpha_1}_0$, $(c\Sigma_1)^{\alpha_1}_0$, $(c\Pi_N)^{\sigma}$, $(c\Sigma_{N-1})^{\sigma}$ for each $\sigma \in R \subseteq Od(\Pi_N) \& \sigma \neq \Omega$ and $(\Sigma_i)^\sigma$ for each $\sigma \in R \subseteq Od(\Pi_N) \& \sigma \notin \{\Omega, \pi\}$ and $i = 1, 2, \ldots, N$. The rule $(h)^\alpha$, $(c\Pi_2)^\Omega_{\alpha_1}$ and $(c\Sigma_1)^{\alpha_1}_0$ are the same as in [6]. We write $(w)$ for $(h)^0$.

1. 

\[
\Gamma, A^\sigma \frac{\Gamma, A^\sigma}{\Gamma, (c\Pi_N)^{\sigma}}
\]

where

(a) $A \equiv \forall x.N \exists x.N-1 \cdots Qx_1 B$ is a $\Pi_N$-sentence with a $\Delta^\tau$-matrix $B$,

(b) $\tau \in D_\sigma$ with the body $\alpha = b(\tau)$ of the rule and
(c) the formula \( A^\tau \) in the lowersequent is the principal formula of the rule and the formula \( A^\sigma \) in the uppersequent is the auxiliary formula of the rule, resp. Each formula in \( \Gamma \) is a side formula of the rule.

2.

\[
\frac{\Gamma, A^\sigma}{\Gamma, \Lambda} \ (c\Sigma_{N-1})^\tau
\]

where

(a) \( \Lambda \) is a nonempty set of unbounded \( \Pi_N \)-sentences with \( \Delta^\tau \)-matrices.
(b) \( \tau \in D_\sigma \) with the body \( \alpha = b(\tau) \) of the rule and
(c) each formula in \( \Gamma \) is a side formula of the rule.

3.

\[
\frac{\Gamma, \neg A^\sigma, A^\sigma, \Lambda}{\Gamma, \Lambda} \ (\Sigma_i)^\sigma
\]

where \( 1 \leq i \leq N \) and \( A^\sigma \) is a genuine \( \Sigma_i^\sigma \)-sentence, i.e., \( A^\sigma \in \Sigma_i^\sigma \) and \( A^\sigma \not\in \Pi_{i-1}^\sigma \cup \Sigma_{i-1}^\sigma \).

\( A^\sigma \ [\neg A^\sigma] \) is said to be the right [left] cut formula of the rule \( (\Sigma_i)^\sigma \), resp.

The rules \((c\Pi_2)^\Omega\) and \((c\Pi_N)\) are basic rules but not the rules \((h)^\sigma\), \((c\Sigma_1)^\Omega\), \((c\Sigma_{N-1})^\sigma\) and \((\Sigma_i)^\sigma\).

A preproof in \( T_{Nc} \) is a proof in \( T_{Nc} \) in the sense of [5], i.e., a proof tree built from axioms and inference rules in \( T_{Nc} \). The underlying tree \( \text{Tree}(P) \) of a preproof \( P \) is a tree of finite sequences of natural numbers such that each occurrence of a sequent or an inference rule receives a finite sequence. The root (empty sequence) \( (\ ) \) is attached to the endsequent, and in an inference rule

\[
a \ast (0, 0) : \Lambda_0 \ldots a \ast (0, n) : \Lambda_n \ (r) \ a \ast (0)
\]

where \((r)\) is the name of the inference rule. Finite sequences are denoted by Roman letters \( a, b, c, \ldots, I, J, K, \ldots \). Roman capitals \( I, J, K, \ldots \) denote exclusively inference nodes. We will identify the attached sequence \( a \) with the occurrence of a sequent or an inference rule.

Let \( P \) be a preproof and \( \gamma < \pi + \omega \) an o.d. in \( Od(\Pi_N) \). For each sequent \( a : \Gamma \) \( (a \in \text{Tree}(P)) \), we assign the height \( h_\gamma(a; P) < \pi + \omega \) of the node \( a \) with the base height \( \gamma \) in \( P \) as in [5] except we replace \( \pi(T_{22}) = \mu \) by \( \pi(T_{N}) = \pi \) and replace \( \text{deg}(A) \) by \( \text{deg}_N(A) \).

Then the height \( h(a; P) \) of \( a \) in \( P \) is defined to be the height with the base height \( \gamma = 0 \):

\[ h(a; P) := h_0(a; P). \]

A pair \((P, \gamma)\) of a preproof \( P \) and an o.d. \( \gamma \) is said to be height regulated if it enjoys the conditions in [5], or equivalently in [6], Definition 5.4. For the
rules \((\Sigma_i)\)^\sigma\), this requires the condition: If \(a : \Gamma\) is the lowersequent of a rule \((\Sigma_i)\)^\sigma \(a \ast (0)\) \((1 \leq i \leq N)\) in \(P\), then \(h_\gamma(a; P) \leq \sigma + i - 2\) if \(i = N - 1, N, 1\).

Otherwise \(h_\gamma(a; P) \leq \sigma + i - 1\).

Therefore for the uppersequent \(a \ast (0, k) : \Lambda\) of a \((\Sigma_i)\)^\sigma we have \(h_\gamma(a \ast (0, k); P) = \sigma + i - 1\). Note that this implies that there is no nested rules \((\Sigma_i)\)^\sigma, i.e., there is no \((\Sigma_i)\)^\sigma below any \((\Sigma_i)\)^\sigma for \(i \geq N - 1\).

A preproof is height regulated iff \((P, 0)\) is height regulated.

Let \(P\) be a preproof and \(\gamma < \pi + \omega\). Assume that \((P, \gamma)\) is height regulated. Then the o.d \(o_\gamma(a; P) \in O(\Pi_N)\) assigned to each node \(a\) in the underlying tree \(\text{Tree}(P)\) of \(P\) is defined exactly as in [6].

Furthermore for \(\tau \in \mathcal{R} \cap \text{Od}(\Pi_N)\), o.d.'s \(B_{\tau, \gamma}(a; P), B_{k\tau, \gamma}(a; P) \in O(\Pi_N)\) are assigned to each sequent node \(a\) such that \(h_\gamma(a; P) \leq \tau \in \mathcal{R}\) as in [6]. Namely

\[
B_{\tau, \gamma}(a; P) := \begin{cases} \pi \cdot o_\gamma(a; P) & \text{if } h_\gamma(a; P) = \tau = \pi \\ \max\{B_\tau(o_\gamma(a; P), B_{\gamma \tau}(\{\tau\} \cup (a; P)))\} + \omega o_\gamma(a; P) & \text{if } h_\gamma(a; P) < \pi \end{cases}
\]

\[
B_{k\tau, \gamma}(a; P) := B_{\tau, \gamma}(a; P) + K_{\tau}(a; P)
\]

\(B_\tau(a; P) [B_{k\tau, \gamma}(a; P)]\) denotes \(B_{\tau, 0}(a; P) [B_{k\tau, 0}(a; P)]\), resp.

Then propositions and lemmata (Rank Lemma 7.3, Inversion Lemma 7.9, etc.) in Section 9 of [5] and Replacement Lemma 5.15 in [6] hold also for \(T_{Nc}\).

Lemma [4.2] yields \(o_\gamma(a; P) \in \text{Od}(\Pi_N)\) for each node \(a \in \text{Tree}(P)\) if \((P, \gamma)\) is height regulated and \(\gamma < \pi + \omega\).

**Definition 5.2** Let \(T\) be a branch in a preproof \(P\) and \(J\) a rule \((\Sigma_i)\)^\sigma.

1. **Left branch:** \(T\) is a *left branch* of \(J\) if
   
   (a) \(T\) starts with a lowermost sequent \(\Gamma\) such that \(h(\Gamma) \geq \pi\),
   
   (b) each sequent in \(T\) contains an ancestor of the left cut formula of \(J\)
   
   (c) \(T\) ends with the left uppersequent of \(J\).

2. **Right branch:** \(T\) is a *right branch* of \(J\) if

   (a) \(T\) starts with a lowermost sequent \(\Gamma\) such that \(\Gamma\) is a lowersequent of a basic rule whose principal formula is an ancestor of the right cut formula of \(J\)

   (b) \(T\) ends with the right uppersequent of \(J\).

*Chains* in a preproof are defined as in Definition 6.1 of [6] when we replace \(((c\Pi_3), (\Sigma_3)), ((c\Sigma_2), (\Sigma_2))\) by \(((c\Pi_N), (\Sigma_N)), ((c\Sigma_{N-1}), (\Sigma_{N-1}))\). For definitions related to chains such as starting with, top, branch of a chain, passing through, see Definition 6.1 of [6]. Also *rope sequence* of a rule, the *end* of a rope sequence and the *bar* of a rule are defined as in Definition 6.2 of [6]. Moreover a *chain analysis* for a preproof together with the *bottom* of a rule is defined as in Definition 6.3 of [6].
Definition 5.3 \textit{Q} part of a chain and the \(i\)-origin.

1. Let \( C = J_0, J'_0, \ldots, J_n, J'_n \) be a chain starting with a \((c)\_\sigma J_n\). Put
   
   (a) \( In(C) := In(J_n) := In(\sigma) \).
   
   (b) \( in_i(C) := in_i(J_n) := in_i(\sigma) \) for \(2 \leq i < N\).
   
   (c) \( st_i(C) := st_i(J_n) := st_i(\sigma) \) for \(2 \leq i < N\).
   
   (d) \( J_k \) is the \(i\)-origin of the chain \( C \) or the rule \( J_n \) if \( J_k \) is a rule \((c)\_\kappa\) with \( \kappa = rg_i(\sigma) \downarrow \).
   
   (e) \( J_k \) is the \(i\)-predecessor of \( J_n \), denoted by \( J_k = pd_i(J_n) \) or \( i\)-predecessor of the chain \( C \), denoted by \( J_k = pd_i(C) \) if \( J_k \) is a rule \((c)\_\rho\) with \( \rho = pd_i(\sigma) \).

Definition 5.4 Knot and rope.

Assume that a chain analysis for a preproof \( P \) is given and by a chain we mean a chain in the chain analysis.

1. \(i\)-knot: Let \( K \) be a rule \((\Sigma_i)^\sigma \) (\(1 \leq i \leq N - 2\)). We say that \( K \) is an \(i\)-knot if there are an uppermost rule \((c)\_\sigma J_{lw} \) below \( K \) and a chain \( C \) such that \( J_{lw} \) is a member of \( C \) and \( C \) passes through the left side of \( K \).

The rule \( J_{lw} \) is said to be the lower rule of the \(i\)-knot \( K \). The member \((c)\_\sigma J_{ul} \) of the chain \( C \) is the upper left rule of \( K \) if such a rule \((c)\_\sigma J_{ur} \) exists.

\[
\begin{array}{ccc}
\vdots & C & \vdots \\
\Gamma, A^\sigma & A^\sigma, A & (\Sigma_i)^\sigma K \\
\Gamma, A & \vdots \\
\Delta & \text{uppermost} \ (c)^\sigma J_{lw} \in C \\
\vdots & \vdots \\
\end{array}
\]

2. A rule is a knot if it is an \(i\)-knot for some \(i > 1\).

\textbf{Remark.} Note that a 1-knot \((\Sigma_1)\) is not a knot by definition.

3. Let \( K \) be a knot, \( J_{lw} \) the lower rule of \( K \) and \( J_{ur} \) an upper right rule of \( K \). Then we say that \( K \) is a knot of \( J_{ur} \) and \( J_{lw} \).

4. Let \( C_n = J_0, \ldots, J_n \) be a chain starting with \( J_n \) and \( K \) a knot. \( K \) is a knot for the chain \( C_n \) or the rule \( J_n \) if

(a) the lower rule \( J_{lw} \) of \( K \) is a member \( J_k \) (\(k < n\)) of \( C_n \),

(b) \( C_n \) passes through the right side of \( K \), and
(c) for any $k < n$ the chain $C_k$ starting with $J_k$ does not pass through the right side of $K$.

The knot $K$ is a merging rule of the chain $C_n$ and the chain $C_k$ starting with the lower rule $J_{lw} = J_k$.

\[
\frac{\vdash C_k}{\vdash \Gamma, \neg A \sigma, A \sigma, \Lambda (\Sigma_i)^\sigma K} \quad \frac{\vdash \Delta_k}{\vdash \Delta_i K} \quad \frac{\vdash \Delta_n}{\vdash \Delta_n J_n}
\]

5. A series $\mathcal{R}_{J_0} = J_0, \ldots, J_{n-1} (n \geq 1)$ of rules (c) is said to be the rope starting with $J_0$ if there is an increasing sequence of numbers (uniquely determined)

\[0 \leq n_0 < n_1 < \cdots < n_l = n - 1 (l \geq 0)\]  \hspace{1cm} (11)

for which the following hold:

(a) each $J_{n_m}$ is the bottom of $J_{n_{m+1}}$ for $m \leq l (n_{-1} = -1)$,

(b) there is an uppermost knot $K_m$ such that $J_{n_m}$ is an upper right rule and $J_{n_{m+1}}$ is the lower rule of $K_m$ for $m < l$, and

(c) there is no knot whose upper right rule is $J_{n_l} = J_{n-1}$.

We say that the rule $J_{n-1}$ is the edge of the rope $\mathcal{R}_{J_0}$ or the edge of the rule $J_0$.

For a rope the increasing sequence of numbers (11) is called the knotting numbers of the rope.

Remark. These knots $K_m$ are uniquely determined for a proof defined below.

6. Let $K_{-1}$ be an $i_{-1}$-knot $(i_{-1} \geq 1)$ and $J_0$ the lower rule of $K_{-1}$. The left rope $K_{-1} \mathcal{R}$ of $K_{-1}$ is inductively defined as follows:

(a) Pick the lowermost rule (c) $J_{n_0}$ such that the chain $\mathcal{C}$ starting with $J_{n_0}$ passes through the left side of the $i_{-1}$-knot $K_{-1}$ and $J_0$ is a member of $\mathcal{C}$. Let $\mathcal{R} = I_0, \ldots, I_q$ be the part of the chain $\mathcal{C}$ with $J_0 = I_0$ & $J_{n_0} = I_q$.

(b) If there exists an uppermost knot $K_0$ such that $J_{n_0}$ is an upper right rule of $K_0$, then $K_{-1} \mathcal{R}$ is defined to be a concatenation:

\[K_{-1} \mathcal{R} = \mathcal{R} \bowtie K_0 \mathcal{R}\]

where $K_0 \mathcal{R}$ denotes the left rope of $K_0$.  

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(c) Otherwise. Set:

\[ \kappa_{-1} \mathcal{R} = \emptyset \mathcal{R} \]

Therefore for the left rope \( \kappa_{-1} \mathcal{R} = J_0, \ldots, J_{n-1} \) of \( K_{-1} \) there exists a uniquely determined increasing sequence of numbers \( (\text{11}) \) such that:

(a) each \( J_m \) is the lowermost rule (c) such that the chain \( \mathcal{C} \) starting with \( J_m \) passes through the left side of the \( i_{m-1} \)-knot \( K_{m-1} \) and \( J_{m-1} \) is a member of \( \mathcal{C} \) for \( m \leq l \),

(b) there is an \( i_m \)-knot \( K_m \) such that \( J_m \) is an upper right rule and \( J_{m+1} \) is the lower rule of \( K_m \) for \( m < l \), and

(c) there is no knot whose upper right rule is \( J_n \). ( \( K_{-1} \) is the \( i_{-1} \)-knot whose lower rule is \( J_0 \).

These numbers \( (\text{11}) \) is called the knotting numbers of the left rope and each knot \( K_m (m < l) \) a knot for the left rope.

By the left rope \( J_0 \mathcal{R} \) of the lower rule \( J_0 \) of \( K_{-1} \) we mean the left rope \( \kappa_{-1} \mathcal{R} \) of \( K_{-1} \).

When a rule \( (\Sigma_i + 1)^\sigma K \) is resolved, we introduce a new rule \( (\Sigma_i)^{\sigma_{n_{(i+1)}+1} \sigma} \) at a sequent \( \Phi \), which is defined to be the resolvent of \( K \) and a \( \sigma_{n_{(i+1)}+1} \leq \sigma \) defined as follows.

**Definition 5.5 Resolvent**

Let \( K \) be a rule \( (\Sigma_i + 1)^\sigma \) \( (0 < i < N) \). The resolvent of the rule \( K \) is a sequent \( a : \Phi \) defined as follows: let \( K' \) denote the lowermost rule \( (\Sigma_i + 1)^\sigma \) below or equal to \( K \) and \( b : \Psi \) the lowersequent of \( K' \).

**Case 1** The case when there exists an \( (i+1) \)-knot \( (\Sigma_i + 1)^\sigma \) which is between an uppersequent of \( K \) and \( b : \Psi \): Pick the uppermost such \( (\Sigma_i + 1)^\sigma \) \( K_{-1} \) and let \( \kappa_{-1} \mathcal{R} = J_0, \ldots, J_{n-1} \) denote the left rope of \( K_{-1} \). Each \( J_p \) is a knot for the left rope \( \kappa_{-1} \mathcal{R} \) of \( K_{-1} \).

Let

\[ 0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \quad (l \geq 0) \quad (\text{11}) \]

be the knotting numbers of the left rope \( \kappa_{-1} \mathcal{R} \) and \( K_m \) an \( i_m \)-knot \( (\Sigma_i + 1)^\sigma_{n_{m+1}} \) of \( J_m \) and \( J_{m+1} \) for \( m < l \). Put

\[ m(i + 1) = \max \{ m : 0 \leq m \leq l \& \forall p \in [0, m)(i + 1 \leq i_p) \} \quad (\text{12}) \]

Then the resolvent \( a : \Phi \) is defined to be the uppermost sequent \( a : \Phi \) below \( J_{n_{(i+1)}} \) such that \( h(a; P) < \sigma_{n_{(i+1)}+1} + i \).

**Case 2** Otherwise: Then the resolvent \( a \Phi \) is defined to be the sequent \( b : \Psi \).

**Definition 5.6** Let \( J \) and \( J' \) be rules in a preproof such that both \( J \) and \( J' \) are one of rules \( (\Sigma_i) \) \( (1 \leq i \leq N - 1) \) and \( J \) is above the right uppersequent of \( J' \). We say that \( J \) foreruns \( J' \) if any right branch \( \mathcal{T} \) of \( J' \) is left to \( J \), i.e., there
exists a merging rule $K$ such that $T$ passes through the left side of $K$ and the right uppersequent of $K$ is equal to or below the right uppersequent of $J$.

\[ \begin{array}{c}
\Gamma_0, \neg A^{\sigma_0} \\
\vdots
\end{array} \quad \begin{array}{c}
\Gamma_1, \neg B \\
\vdots
\end{array} \quad \begin{array}{c}
\Gamma_2, C, A_2 \\
(\Sigma_j) J
\end{array} \quad \begin{array}{c}
B_1, A_1 \\
K
\end{array} \quad \begin{array}{c}
\Gamma_1, A_1 \\
\vdots
\end{array} \quad \begin{array}{c}
\Gamma_0, A_0 \\
(\Sigma_i)^{\sigma_0} J
\end{array} \]

If $J$ foreruns $J'$, then resolving steps of $J$ precede ones of $J'$. In other words we have to resolve $J$ in advance in order to resolve $J'$.

**Definition 5.7** Let $R = J_0, \ldots, J_{n-1}$ denote a series of rules $(c)$. Each $J_p$ is a rule $(c)$. Assume that $J_0$ is above a rule $(\Sigma_i)^{\sigma} I$ and $\sigma = \sigma_p$ for some $p$ with $0 < p \leq n$. Then we say that the series $R$ reaches to the rule $I$.

In a proof defined in the next definition, if a series $R = J_0, \ldots, J_{n-1}$ reaches to the rule $(\Sigma_i)^{\sigma} I$, then either $R$ passes through $I$ in case $p < n$, or the subscript $\sigma_n$ of the last rule $(c)_{\sigma_{n-1}} J_{n-1}$ is equal to $\sigma$, i.e., $J_{n-1}$ is a lowermost rule $(c)$ above $I$.

**Definition 5.8** Proof

Let $P$ be a preproof. Assume a chain analysis for $P$ is given. The preproof $P$ together with the chain analysis is said to be a proof in $T_{NC}$ if it satisfies the following conditions besides the conditions (pure), (h-reg), (cs:side), (c:bound), (next), (h:bound), (ch:pass) (a chain passes through only rules $(c),(h), (\Sigma_i) (i < N)$), (ch:left), which are the same as in [6]:

**(st:bound)** Let $C$ be a chain, $i \in I_n(C)$ and $a : \Gamma$ be the uppersequent of the $i$-origin of the chain $C$.

**(st:bound1)** Let $i = N - 1$. Then

\[ o(a; P) \leq st_{N-1} (C). \]

**(st:bound2)** Let $i < N - 1$ and $\kappa = rg_i(C)$. Then for an $\alpha$

\[ st_i(C) = d_{\kappa + \alpha} \]

and

\[ B_{\kappa}(a; P) \leq \alpha. \]

**(ch:link)** Linking chains: Let $C = J_0, J_0', \ldots, J_{n'}, J_{n'}$ and $D = I_0, I_0', \ldots, I_m, I_m'$ be chains such that $J_i$ is a rule $(c)_{\tau_{i+1}}$ and $I_i$ a rule $(c)_{\sigma_{i+1}}$. Assume that branches of these chains intersect. Then one of the following three types must occur (Cf. [6] for Type1 (segment) and Type2 (jump)):
**Type1 (segment)**: One is a part of the other, i.e.,

\[ n \leq m & J_i = I_i \]

or vice versa.

Assume that there exists a merging rule \( K \) such that \( C \) passes through the left side of \( K \) and \( D \) the right side of \( K \). Then by (\textbf{ch:left}) the merging rule \( K \) is a \((\Sigma_j)^{\tau_j}\) for some \( j \leq n \) and some \( l \) with \( 1 \leq l \leq N - 2 \).

**Type2 (jump)**: The case when there is an \( i \leq m \) so that

1. \( J'_{j-1} \) is above \( K \) and \( J_j \) is below \( K \),
2. \( I_i \) is above \( K \),
3. \( I'_i \) is below \( J'_n \) and
4. \( \sigma_{i+1} < \tau_{n+1} \).

**Type3 (merge)**: The case when \( \tau_j = \sigma_j \). Then it must be the case:

1. \( l > 1 \),
2. \( I'_{j-1} \) and \( J'_{j-1} \) are rules \((\Sigma_{N-1})^{\tau_j}\) above \( K \), and
3. \( n < m \) & \( J_{j+k} = I_{j+k} \) & \( J'_{j+k} = I'_{j+k} \) for any \( k \) with
   \[ j \leq j + k \leq n. \]

That is to say, \( C \) and \( D \) share the part from \( J_j = I_j \) to \( J_n = I_n \), the right chain \( D \) has to be longer \( n < m \) than the left chain \( C \) and the merging rule \( K \) is not a rule \((\Sigma_1)\).

If **Type2 (jump)** or **Type3 (merge)** occurs for chains \( C \) and \( D \), then we say that \( D \) \textit{foreruns} \( C \), since the resolving of the chain \( D \) precedes the
resolving of the chain \( C \).

\[
\begin{array}{cccccc}
\Phi_{j-1}, \neg A_{j-1}^\tau & A_{j-1}^\tau, \Psi_{j-1} & (\Sigma_{N-1})^\tau_j \quad J_{j-1} \\
\Phi_{j-1}, \Psi_{j-1} & C & \Pi, \neg B^{\sigma_j}, B^{\sigma_j}, \Delta & (\Sigma_{N-1})^{\sigma_j} I_{j-1} \\
\Phi, \neg A^{\tau_i} & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Phi, \Psi & (\Sigma_i)^{\tau_j} K & \Gamma_j & (c\Sigma_{N-1})^{\tau_{j+1}} J_j = (c\Sigma_{N-1})^{\sigma_j+1} I_j \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& \Gamma_n & (c\Sigma_{N-1})^{\tau_{n+1}} J_n = (c\Sigma_{N-1})^{\sigma_{n+1}} I_n & \Phi_n, \neg A_n^{\sigma_{n+1}} & A_n^{\sigma_{n+1}}, \Psi_n & (\Sigma_{N-1})^{\tau_{n+1}} J_n' = (\Sigma_{N-1})^{\sigma_{n+1}} I_n' \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & \Gamma_m & (c)_{\sigma_{m+1}} I_m & Type 3 \\
\end{array}
\]

(ch:Qpt) Let \( C = J_0, \ldots, J_n \) be a chain with a \((c)_{\sigma_{p+1}}^\sigma, J_p (p \leq n)\) and put \( \rho = \sigma_{n+1} \). Then by (ch:link) there exists a uniquely determined increasing sequence of numbers

\[
0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \quad (l \geq 0)
\]

such that for each \( m < l \) there exists an \( i_m \)-knot \((\Sigma_{i_m})^{\sigma_{m+1}} K_m (2 \leq i_m \leq N - 2)\) for the chain \( C \). (The \( i_m \)-knot \( K_m \) is the merging rule of the chain \( C \) and the chain starting with the rule \( J_{n+1} \), cf. Type 3 (merge).) These numbers are called the knotting numbers of the chain \( C \).

Then \( pd_i(\rho), In(\rho), rg_i(\rho) \) have to be determined as follows:

1. For \( 2 \leq i < N \),

\[
pd_i(\rho) = \sigma_{n_{m(i)} + 1}
\]

with

\[
m(i) = \max\{m : 0 \leq m \leq l \land \forall p \in [0, m)(i \leq i_p)\}
\]

that is to say,

\[
J_{n_{m(i)}} = pd_i(J_n).
\]

2. For \( 2 \leq i < N - 1 \)

\[
i \in In(C) = In(\rho) \Leftrightarrow \exists p \in [0, m(i))(i_p = i)
\]

\[
\Leftrightarrow \exists p \in [0, l)(i_p = i \land \forall q < p(i_q > i))
\]

\[
\Leftrightarrow m(i) > m(i + 1) = \min\{m < l : i_m = i\}
\]

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And by the definition $N - 1 \in \text{In}(C) = \text{In}(\rho)$.

3. For $i \in \text{In}(C)$ & $i \neq N - 1$,
   (a) The case when there exists a $q$ such that
   \[
   \exists p [n_{m(i)} \geq p \geq q \geq n_{m(i+1)} & \rho \prec_i \sigma_{p+1} & \sigma_q = rg_i(\sigma_{p+1})] \tag{14}
   \]
   Then
   \[
r_{g_i}(\rho) = \sigma_q
   \]
   where $q$ denotes the minimal $q$ satisfying (14).
   (b) Otherwise.
   \[
r_{g_i}(\rho) = p_{d_i}(\rho) = \sigma_{n_{m(i)+1}}
   \]

(\textbf{Ibranch}) Any left branch of a $(\Sigma_i)$ is the rightmost one in the left upper part of the $(\Sigma_i)$.

(\textbf{forerun}) Let $J^{lw}$ be a rule $(\Sigma_j)^{\sigma}$. Let $R_{J_0} = J_0, \ldots, J_{n-1}$ denote the rope starting with a (c) $J_0$. Assume that $J_0$ is above the right uppersequent of $J^{lw}$ and the series $R_{J_0}$ reaches to the rule $J^{lw}$. Then there is no merging rule $K$, cf. the figure below, such that

1. the chain $C_0$ starting with $J_0$ passes through the right side of $K$, and
2. a right branch $T$ of $J^{lw}$ passes through the left side of $K$.

\[
\begin{array}{c}
\vdots \\
\Gamma, \neg B \quad B, \Lambda \\
\Gamma, \Lambda \\
\vdots \\
\Gamma_0 \\
\end{array}
\begin{array}{c}
\vcenter{\hbox{\vdots}} \\
\Phi, \neg A \\
\end{array}
\begin{array}{c}
\Phi, \Psi \\
\text{\vdots} \\
A, \Psi \\
\vdots \\
\end{array}
\begin{array}{c}
(\Sigma_i)^{\sigma} \\
J^{lw} \\
\end{array}
\end{array}
\]

(\textbf{uplw}) Let $J^{lw}$ be a rule $(\Sigma_j)^{\sigma}$ and $J^{up}$ an i-knot $(\Sigma_i)^{\sigma_0} (1 \leq i, j \leq N)$. Let $J_0$ denote the lower rule of $J^{up}$. Assume that the left rope $J_{up}R = J_0, \ldots, J_{n-1}$ of $J^{up}$ reaches to the rule $J^{lw}$. Then
(uplw) if $J^{up}$ is above the left uppersequent of $J^{lw}$, then $j < i < N$.

\[
\begin{array}{c}
\vdots C_0 \\
\Gamma, \neg B \quad B, A\\
\Gamma, A
\end{array}
\frac{(\Sigma_i)^{\sigma_0} J^{up}}{}
\]  
\begin{array}{c}
\vdots \\
\Gamma_0 \\
\Gamma_0 (c)^{\sigma_0} J_0
\end{array}
\frac{}{}
\begin{array}{c}
\vdots J^{up} \mathcal{R} \\
\Phi, \neg A \\
\Phi, \Psi
\end{array}
\frac{(\Sigma_j)^{\sigma} J^{lw}}{\Rightarrow j < i}
\]

where $C_0$ denotes the chain starting with $J_0$, and

(upwr) if $J^{up}$ is above the right uppersequent of $J^{lw}$ and $i \leq j \leq N$, then the rule $(\Sigma_i)^{\sigma_0} J^{up}$ foreruns the rule $(\Sigma_j)^{\sigma} J^{lw}$, cf. Proposition 2.3 in Subsection 2.2.

In other words if there exists a right branch $\mathcal{T}$ of $J^{lw}$ as shown in the following figure, then $j < i$.

\[
\begin{array}{c}
\vdots C_0 \\
\Gamma, \neg B \quad B, A\\
\Gamma, A
\end{array}
\frac{(\Sigma_i)^{\sigma_0} J^{up}}{}
\]  
\begin{array}{c}
\vdots \\
\Gamma_0 \\
\Gamma_0 (c)^{\sigma_0} J_0
\end{array}
\frac{}{}
\begin{array}{c}
\vdots J^{up} \mathcal{R} \\
\Pi, \neg C \\
\Pi, \Delta
\end{array}
\frac{\exists K}{\Rightarrow \Pi, \Delta)
\frac{\mathcal{T}}{\mathcal{T}}
\frac{\exists K}{\mathcal{T}}
\frac{\exists K}{\mathcal{T}}
\frac{}{\Phi, \Psi}
\frac{(\Sigma_j)^{\sigma} J^{lw}}{}
\]

Decipherment. These provisos for a preproof to be a proof are obtained by inspection to rewrited proof figures. We decipher only additional provisos from [6].

(ch:link) Now a new type of linking chains, Type3 (merge) enters, cf. Subsection 2.1.

For a chain $\mathcal{D} = I_0, I'_0, \ldots, I_m, I'_m$ and a member $I_n (n < m)$ of $\mathcal{D}$ let $\mathcal{C} = J_0, J'_0, \ldots, J_n, J'_n$ denote the chain starting with $J_n = I_n$. Then there are two possibilities:

Type1 (segment) $\mathcal{C}$ is a part $I_0, \ldots, I_n$ of $\mathcal{D}$ and hence the tops $I_0$ and $J_0$ are identical.
Type 3 (merge) The branch of $C$ is left to the branch of $D$.

(st:bound), (ch:Qpt) By these provisos we see that an o.d. $\rho$ is in $Od(\Pi_N)$ for a newly introduced rule $(\varepsilon)_{\rho}$, cf. Propositions 2.2, 2.3 in Subsection 2.2, Lemma 5.8 below and the case M5.2 in the next section 9.

(upwl) By the proviso we see that a preproof $P'$ which is resulted from a proof $P$ is again a proof with respect to the proviso (ch:Qpt), cf. Lemma 5.7.2.

(uplwr), (forerun), (lbranch) By these provisos we see that a preproof $P'$ which is resulted from a proof $P$ by resolving a rule $(\Sigma_{i+1})_{lw}$ is again a proof with respect to the provisos (forerun) and (uplw), cf. Proposition 2.2 in Subsection 2.2, the case M7.2 in the next section 9, Lemma 5.5 and Lemma 5.4.

In the following any sequent and any rule are in a fixed proof.

As in the previous paper [6] we have the following lemmata. Lemma 5.1 follows from the provisos (h-reg) and (ch:link) in Definition 5.8, Lemma 5.2 from (h-reg) and (c:bound1) and Lemma 5.3 from (h-reg).

**Lemma 5.1** Let $J$ be a rule $(\varepsilon)_{\rho}$ and $J'$ the trace $(\Sigma_{N-1})^\rho$ of $J$. Let $J_1$ be a rule $(\varepsilon)_{\rho}$ below $J'$. If there exists a chain $C$ to which both $J$ and $J_1$ belong, then $J_1$ is the uppermost rule $(\varepsilon)_{\rho}$ below $J$ and there is no rule $(\varepsilon)$ between $J'$ and $J_1$.

**Lemma 5.2** Let $J_{top}$ be a rule $(\varepsilon)_{\pi}$. Let $\Phi$ denote the bar of $J_{top}$. Assume that the branch $T$ from $J_{top}$ to $\Phi$ is the rightmost one in the upper part of $\Phi$. Then no chain passes through $\Phi$.

**Lemma 5.3** Let $J$ be a rule $(\varepsilon)$ and $b : \Phi$ the bar of the rule $J$. Then there is no (cut) $I$ with $b \subset I \subset J$ nor a right uppersequent of a $(\Sigma_N) I$ with $b \subset I \ast (1) \subset J$ between $J$ and $b : \Phi$.

The following lemma is used to show that a preproof $P'$ which results from a proof $P$ by resolving a rule $(\Sigma_j)_{lw}$ is again a proof with respect to the proviso (uplw), cf. the Claim 6.6 in the case M7 in the next subsection.

**Lemma 5.4** Let $J_{lw}$ be a rule $(\Sigma_j)$. Assume that there exists a right branch $T$ of $J_{lw}$ such that $T$ is the rightmost one in the upper part of $J_{lw}$. Then there is no i-knot $(\Sigma_i)_{up}$ above the right uppersequent of $J_{lw}$ such that $i \leq j$ and the left rope $j_{up} \mathcal{R}$ of $J_{up}$ and $J_{up}$ reaches to $J_{lw}$.

**Proof.** Suppose such a rule $J_{up}$ exists. By (uplwr) the rule $J_{up}$ foreruns $J_{lw}$. Thus the branch $T$ would not be the rightmost one.

\[
\begin{array}{c}
\vdots \\
\dfrac{\Psi, \neg B, B, \Phi}{\Psi, \Phi} (\Sigma_{i}) J_{up} \\
\vdots \\
\dfrac{\Gamma, \neg A, A, \Lambda}{\Gamma, \Lambda} (\Sigma_{j}) J_{lw}
\end{array}
\]
The following lemma is used to show that a preproof $P'$ which results from a proof $P$ by resolving a $(\Sigma_{i+1})^\sigma$ is a proof with respect to the proviso (uplwr), and to show a newly introduced rule $(\Sigma_i)$ in such a $P'$ does not split any chain, cf. the Claim 6.6 in the case M7.

**Lemma 5.5** Let $J$ be a rule $(\Sigma_{i+1})^\sigma_0$ ($0 < i < N$) and $b : \Phi$ the resolvent of $J$. Assume that the branch $T$ from $J$ to $b$ is the rightmost one in the upper part of $b$. Then every chain passing through $b$ passes through the right side of $J$.

**Proof.** Let $a*(0)$ denote the lowestmost rule $(\Sigma_{i+1})^\sigma_0$ below or equal to $J$, and $a : \Psi$ the lowersequent of $a*(0)$. The sequent $a : \Psi$ is the uppermost sequent below $J$ such that $h(a; P) < \sigma_0 + i$ by (h-reg).

**Case 2.** $b = a$: If a chain passes through $a$ and a left side of a $(\Sigma_{i+1})^\sigma_0 K_{-1}$ with $a \in K_{-1} \subseteq J$, then the chain would produce an $(i+1)$-knot $K_{-1}$.

**Case 1.** Otherwise: Then there exists an $(i+1)$-knot $(\Sigma_{i+1})^\sigma_0$ with $a \in K_{-1} \subseteq J$. Let $(\Sigma_{i+1})^\sigma_0 K_{-1}$ denote the uppermost such knot and $K_{-1} = J_0, \ldots, J_{n-1}$ the left rope of $K_{-1}$. Each $J_p$ is a rule $(c)^\sigma_{p+1}$. Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \ (l \geq 0)$$

be the knotting numbers of the left rope $K_{-1} \mathcal{R}$ and $K_m$ an $i_m$-knot $(\Sigma_{i_m})^\sigma_{m+1}$ of $J_{n_m}$ and $J_{n_m+1}$ for $m < l$. Put

$$m(i+1) = \max\{m : 0 \leq m \leq l \text{ and } \forall p \in [0, m)(i+1 \leq i_p)\}$$

Then the resolvent $b : \Phi$ is the uppermost sequent $b : \Phi$ below $J_{n_{m(i+1)}}$ such that

$$h(b; P) < \sigma_{n_{m(i+1)}+1} + i.$$ 

Then by (h-reg),

$$m = m(+1), \sigma = \sigma_{n_{m+1}}.$$ 

Assume that there is a chain $C$ passing through $b$. As in Case 2 it suffices to show that the chain $C$ passes through the right side of $K_{-1}$. Assume that this is not the case. Let $(c)^\rho K$ denote the lowermost member of $C$ which is above $b$.

**Claim 5.1** $K$ is on the branch $T$.

**Proof.** Assume that this is not the case. Then we see that there exists a merging rule $(\Sigma_j)^\rho I$ and a member $(c)^\rho K'$ of $C$ such that the chain $C$ passes through the left side of $I$. $K' \subset b \subset I$ and hence $h(K'; P) = \rho' \leq \sigma$. We see $\rho' = \sigma$ from (h-reg).

Suppose $m = l$. Then by the definition of the left rope $K_{-1} \mathcal{R}$, the rule $(\Sigma_j)^\rho I$ is not a knot, i.e., $j = 1$. But then $h(I*(1); P) = h(K'; P) = \sigma$, and hence $I \subset b$. A contradiction. Therefore $m < l$ and $i_m \leq i$. This means $K_{m*(1)} \subset b \subset I$. On the other hand we have $1 \leq i < j$ by $b \subset I$, and by Claim 5.1 $K'$ is the uppermost rule $(c)^\sigma$ below $(\Sigma_j)^\sigma I$. Therefore $(\Sigma_j)^\rho I$ would be a
knot below $J_{n_m}$. On the other side $K_m$ is the uppermost knot below $J_{n_m}$. This is a contradiction.

\[
\begin{array}{c}
\Pi \quad (c)_\sigma^p K \in C \\
\vdots \\
C \\
\Gamma_0, \neg A_0 \quad A_0, \Lambda_0 \\
\vdots \\
(\Sigma_j)^\sigma I \\
\end{array}
\]

Then as in the proof of Lemma 7.13 of [6] we see that $K = J_{n_m}$, i.e., $(c)_\sigma J_{n_m}$ and $(c)_\sigma^{n_m} J_{n_m}$ coincide. Consider the chain $C_m$ starting with $(c)_\sigma^{n_m} J_{n_m}$. Then by (ch:link) either $C_m$ is a segment of $C$ by Type1(segment), or $C$ foreruns $C_m$ by Type3(merge). Since $(c)_\sigma^{n_m} J_{n_m}$ is the lowest one such that $C_m$ passes through the left side of $K_{m-1}$ and $J_{n_m-1}+1$ is a member of $C_m$, Type1(segment) does not occur. In Type3(merge) $K_{m-1}$ has to be the merging rule of $C_m$ and $C$ since, again, $(c)_\sigma^{n_m} J_{n_m}$ is the lowest one, and the branch $T$ is the rightmost one. Therefore $C$ passes through the right side of $K_m-1$. If $m = 0$, then we are done. Otherwise we see the chain $C$ and the chain $C_{m-1}$ starting with $(c)_\sigma^{n_m-1} J_{n_m-1}$ has to share the rule $(c)_\sigma^{n_m-1} J_{n_m-1}$. As above we see that $C$ passes through the right side of $K_{m-2}$, and so forth.

\[\square\]

Lemma 5.6 Let $C = J_0, \ldots, J_n$ be a chain with rules $(c)_\sigma^{p+1} J_p$ for $p \leq n$, and $(\Sigma_j)^\sigma K (p < n)$ a rule such that $C$ passes through the right side of $K$ and the chain $C_p$ stating with $J_p$ passes through the left side of $K$. Further let $R = K R = J_p, \ldots, J_{q-1}$ denote the left rope of the $j$-knot $K$. Then the chain $C_q$ starting with $J_q$ is a part of the chain $C = C_n, C_q \subseteq C$. Therefore any knot for the chain $C$ is below $J_q$ and $q < n$, and in particular, if $K$ is a knot for the chain $C$, then $b = n$, cf. Definition 5.4.3

Proof. Suppose $q < n$. By the Definition 5.4.3 there is no knot of $J_{q-1}$ and $J_q$. Let $I_q$ denote a knot such that the chain $C_{q-1}$ starting with $J_{q-1}$ passes through the left side of $I_q$, $c \subseteq b$. From the definition of a left rope we see that the chain $C_q$ starting with $J_q$ does not pass through the left side of the knot $I_q$. Therefore by (ch:link) Type1(segment) the chain $C_q$ must be a part of the
chain \( C, C_q \subset C \), i.e., the top of the chain \( C_q \) is the top \( J_0 \) of \( C \).

\[
\begin{align*}
\Phi_p, \neg A_p & \quad \vdash \quad \Phi_p, \neg A_p, \neg A_{p+1}, \Psi_{p+1} \\
\Phi_p, \Psi_p & \quad \vdash \quad \Phi_p, \Psi_p, (\Sigma_j)^{\sigma_p} K \\
\vdots & \quad \vdash \quad \vdots \\
\Gamma_p & \quad \vdash \quad \Gamma_p, (c)^{\sigma_p} J_p \\
\vdots & \quad \vdash \quad \vdots \\
\Phi_q, \neg A_q & \quad \vdash \quad \Phi_q, \neg A_q, \neg A_{q+1}, \Psi_{q+1} \\
\Phi_{q-1}, \Psi_{q-1} & \quad \vdash \quad \Phi_{q-1}, \Psi_{q-1}, I_q \\
\vdots & \quad \vdash \quad \vdots \\
\Gamma_{q-1} & \quad \vdash \quad \Gamma_{q-1}, (c)^{\sigma_q} J_{q-1} \\
\vdots & \quad \vdash \quad \vdots \\
\Gamma_q & \quad \vdash \quad \Gamma_q, (c)^{\sigma_q} J_q \\
\vdots & \quad \vdash \quad \vdots \\
\Gamma_n & \quad \vdash \quad \Gamma_n, J_n
\end{align*}
\]

The following Lemma 5.7 is a preparation for Lemma 5.8. From the Lemma 5.8 we see that an o.d. \( \rho \) is in \( Od(\Pi_N) \) for a newly introduced rule \( (c)_\rho \), cf. the case M5.2 in the next subsection.

In the following Lemma 5.7, \( J \) denotes a rule \( (c)_\rho \) and \( C = J_0, \ldots, J_n \) the chain starting with \( J_n = J \). Each \( J_p \) is a rule \( (c)^{\sigma_{p+1}}_p \) for \( p \leq n \) with \( \sigma_{n+1} = \rho \).

\( K \) denotes a rule \( (\Sigma_j)^{\sigma_a} (j \leq N - 2, 0 < a \leq n) \) such that the chain \( C \) passes through \( K \). If \( C \) passes through the left side of \( K \), then \( j \leq N - 2 \) holds by (ch:left).

\( J_{a-1} \) denotes the lowermost member \( (c)_{\sigma_a} \) of \( C \) above \( K, K \subset J_{a-1} \).

Let

\[ 0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \quad (l \geq 0) \tag{11} \]

be the knotting numbers of the chain \( C \), cf. (ch:Qpt), and \( K_m \) an \( i_m \)-knot \( (\Sigma_{i_m})^{\sigma_{m+1}} \) of \( J_m \) and \( J_{m+1} \) for \( m < l \). Let \( m(i) \) denote the number

\[ m(i) = \max\{m : 0 \leq m \leq l \& \forall p \in [0, m)(i \leq i_p)\} \tag{13} \]

Lemma 5.7 (cf. Proposition 2.2 in Subsection 2.2.)

33
1. Let \( m \leq m(i) \). Then

\[
i \leq i_{m-1} \& \sigma_{n^{m+1}} \prec_i \sigma_{n^{m-1}+1}.
\]

2. Assume that \( C \) passes through the left side of the rule \( K \), i.e., \( K \ast (0) \subset J_{a-1} \). Then \( J_{a-1} \) is the upper left rule of \( K \). Let \( i \leq j \).

(a) \( \rho \prec_i \sigma_a \), and hence

(b) the \( i \)-predecessor of \( J \) is equal to or below \( J_{a-1} \), and

(c) if \( K_p \) is an \( i_p \)-knot \((\Sigma_{i_p})\) for the chain \( C \) above \( K \), then \( j < i_p \).

\[
\begin{array}{c}
\Phi_p, \neg A_p, \Phi_p, A_p, \Psi_p \\
\Phi_p, \Psi_p
\end{array}
\]

\[
(\Sigma_{i_p})^{\sigma_{n_p}+1} K_p
\]

\[
\begin{array}{c}
\Gamma_{a-1} \\
(\Sigma_{a-1})^{\sigma_{a-1}} J_{a-1}
\end{array}
\]

\[
\begin{array}{c}
\Phi, \neg A, \Phi, \Psi \\
\Phi, \Psi
\end{array}
\]

\[
(\Sigma_{j})^{\sigma_{j}} K
\]

\[
\begin{array}{c}
\Gamma_{n} \\
(\Sigma_{n})^{\sigma_{n}} J_{n} = J
\end{array}
\]

\[
\begin{array}{c}
\Gamma_{n} \\
(\Sigma_{n})^{\sigma_{n+1}} J_{n}
\end{array}
\]

\[
\neg \sigma_{n+1} \prec_i \sigma_{b} \Rightarrow i \leq j
\]

3. Let \( J_{b-1} \) be a member of \( C \) such that \( \rho \prec_i \sigma_b \) for an \( i \) with \( 2 \leq i \leq N - 2 \). Let \( C_{b-1} \) denote the chain starting with \( J_{b-1} \). Assume that the chain \( C_{b-1} \) intersects \( C \) of \textbf{Type3 (merge)} in \( \text{ch:link} \) and \( (\Sigma_{j}) K \) is the merging rule of \( C_{b-1} \) and \( C \).

Then \( i \leq j \).

\[
\begin{array}{c}
\Phi, \neg A, \Phi, A, \Psi \\
\Phi, \Psi
\end{array}
\]

\[
(\Sigma_{j})^{\sigma_{j}} K
\]

\[
\begin{array}{c}
\Gamma_{b-1} \\
(\Sigma_{b-1})^{\sigma_{b-1}} J_{b-1}
\end{array}
\]

\[
\begin{array}{c}
\Gamma_{b-1} \\
(\Sigma_{b-1})^{\sigma_{b-1}} J_{b-1}
\end{array}
\]

\[
\neg \sigma_{b} \prec_i \sigma_{b+1} \Rightarrow i \leq j
\]
4. Assume that \( C \) passes through the left side of the rule \( K \), i.e., \( K \ast (0) \subset J_{a-1} \). Let \( i \leq j \).

Assume that the \( i \)-origin \( J_q \) of \( C \) is not below \( K \), i.e., \( \sigma_q = rg_i(\rho) \downarrow q < a \). Then
\[
\forall b \in (a, n + 1) \{ \rho \preceq_i \sigma_b \prec_i \sigma_a \Rightarrow i \notin In(\sigma_b) \}
\]
and hence
\[
\forall b \in (a, n + 1) \{ \rho \preceq_i \sigma_b \prec_i \sigma_a \Rightarrow in_i(J) = in_i(J_{b-1}) = in_i(J_{a-1}) \}, \text{i.e.,}
\]
\[
in_i(\rho) = in_i(\sigma_b) = in_i(\sigma_a)\]

In particular by Lemma 5.7.2 we have
\[
\rho \prec_i \sigma_a \& in_i(J) = in_i(J_{a-1}) \}, \text{i.e., } in_i(\rho) = in_i(\sigma_a).
\]

5. Assume that \( C \) passes through the left side of the rule \( K \). Let \( J_{b-1} \) be a member of \( C \) such that \( J_{b-1} \) is below \( K \), i.e., \( a < b \), and assume that \( \sigma_{n+1} \preceq_i \sigma := \sigma_b \) for an \( i \leq j \). If \( \sigma_q = rg_i(\sigma) \downarrow q < a \), then
\[
\forall d \in (a, b) \{ \sigma \preceq_i \sigma_d \prec_i \sigma_a \Rightarrow i \notin In(\sigma_d) \}
\]
and
\[
\sigma \prec_i \sigma_a.
\]

Hence
\[
\forall d \in (a, b) \{ \sigma \preceq_i \sigma_d \prec_i \sigma_a \Rightarrow in_i(\sigma_d) = in_i(\sigma_a) \} \& in_i(\sigma) = in_i(\sigma_a).
\]

The following figure depicts the case \( \sigma_q = rg_i(\sigma) \downarrow \):

```
\begin{array}{c}
\vdots \\
\Gamma_q \\
\Gamma_q (c)^{rg_i(\sigma)} J_q \\
\vdots \\
\Phi, A \vdash C \Psi \\
\Phi, \neg A \vdash \Sigma_j \vdash \Sigma_j^{\sigma_a} K \\
\vdots \\
\Gamma_{b-1} \\
\Gamma_{b-1} (c)_\sigma J_{b-1} \\
\vdots \\
\Gamma_n \\
\Gamma_n (c)_{\sigma_{n+1}} J_n
\end{array}
```

6. Assume that the chain \( C \) passes through the left side of the rule \( K \). For an \( i \leq j \) assume that there exists a \( q \) such that

\[
\exists[n \geq p \geq q \geq a \& \rho \preceq_i \sigma_{p+1} \& \sigma_q = rg_i(\sigma_{p+1})] .
\]

Pick the minimal such \( q_0 \) and put \( \kappa = \sigma_{q_0} \). Then
\( (a) \forall d \in (a, q_0) \{ \sigma_{q_0} \preceq_i \sigma_d \preceq_i \sigma_a \rightarrow i \notin \text{In}(\sigma_d) \} \) and \( \text{in}_i(J_{m-1}) = \text{in}_i(J_{a-1}), \) i.e., \( \text{in}_i(\sigma_a) = \text{in}_i(\kappa) \) and \( \kappa \preceq_i \sigma_a. \)

\( (b) \forall \rho \preceq_i \sigma \preceq_i \kappa \Rightarrow \text{rg}_i(\sigma) \preceq_i \kappa \).

7. Assume that \( C \) passes through the left side of the rule \( K \). Let \( J_{b-1} \) be a member of \( C \) such that \( J_{b-1} \) is below \( K \), i.e., \( a < b \) and \( \rho \preceq_i \sigma \;:=\; \sigma_b \) for an \( i \leq j \). Suppose \( \text{rg}_i(\sigma) \downarrow \) and put \( \sigma_\ell = \text{rg}_i(\sigma) \). If the member \( (c)^\sigma J_q \) is below \( K \), i.e., \( a \leq q \), then for \( st_1(\sigma) = d_{\sigma_\ell} \alpha \), cf. \( (\text{st:bound}) \),

\[ B_{\sigma_\ell}(c; P) \leq \alpha \]

for the uppersequent \( c : \Gamma_q \) of the rule \( J_q \).

**Proof.** First we show Lemmata 5.7.1 and 5.7.2 simultaneously by induction on the number of sequents between \( K \) and \( J \).

**Proof of Lemma 5.7.1**

By the definition of the number \( m(i) \) we have \( i \leq i_{m-1} \). Since the chain \( C_{nm} \) starting with \( J_{nm} \) passes through the left side of the \( i_{m-1} \)-knot \( K_{m-1} \), we have the assertion \( \sigma_{n_{m+1}} \preceq_i \sigma_{n_{m-1}+1} \) by IH on Lemma 5.7.2.

This shows Lemma 5.7.1 \( \square \)

**Proof of Lemma 5.7.2**

By \( (\text{ch:Qpt}) \) we have

\[ pd_i(\rho) = \sigma_{n_{m(i)}+1} \] and \( J_{n_{m(i)}} = pd_i(J). \)

**Claim 5.2** \( a \leq n_{m(i)} + 1 \), i.e., \( J_{n_{m(i)}+1} \subset K \).

**Proof of Claim 5.2** If \( m(i) = l \), then \( a \leq n = n_l + 1. \) Assume

\[ m(i) < l \neq 0 \& a > n_{m(i)} + 1. \]

Then the \( i_{m(i)} \)-knot \( (\Sigma_{i_{m(i)}})^{a_{m(i)} + 1} K_{m(i)} \) is above the left uppersequent of \( K, K \ast (0) \subset K_{m(i)}, \) and \( j \geq i > i_{m(i)} \). Consider the left rope \( \kappa_{m(i)} \mathcal{R} = J_{n_{m(i)}+1}, \ldots, J_{b-1} \) of the knot \( K_{m(i)} \) for the chain \( C \). Then by Lemma 5.6 we have \( b = n \). Therefore \( \kappa_{m(i)} \mathcal{R} \) reaches to the rule \( K \). Thus by \( (\text{upwl}) \) we have \( i \leq j < i_{m(i)}. \) This is a contradiction. \( \square \)
By the Claim 5.2, we have Lemma 5.7.2.

**Case 1** \( a = n_{m(i)} + 1 \): This means that the \( i \)-predecessor \( J_{n_{m(i)}} \) of \( J \) is the rule \( J_{a-1} \), and \( pd_i(\rho) = \sigma_a \).

**Case 2** \( a < n_{m(i)} + 1 \): This means that \( J_{n_{m(i)}} \subset K \). Put
\[
m_1 = \min\{ m \leq m(i) : a < n_m + 1 \}
\]  
(15)

Then \( J_{n_m} \) is the uppermost rule \( J_n \) below \( K \). The chain \( C_{n_m} \) starting with \( J_{n_m} \) passes through the left side of the knot \( (\Sigma_{i_{m_1-1}})^{\sigma_{m_1-1}+1} K_{m_1-1} \). If \( K \subset K_{m_1-1} \), then \( C_{n_m} \) passes through the left side of \( K \).

And by the minimality of \( m_1 \), if \( K_{m_1-1} \subset K \), then \( J_{a-1} = J_{n_{m_1-1}} \), i.e., \( a = n_{m_1-1} + 1 \).

By Lemma 5.7.1 we have \( pd_i(\rho) = \sigma_{n_{m(i)}+1} \preceq \sigma_{n_m+1} \). Once again by IH we have \( \sigma_{n_m+1} \preceq \sigma_a \). Thus we have shown Lemma 5.7.2a, \( \rho \prec \sigma_a \).  
\[
\square
\]

**Proof of Lemma 5.7.2**  
\( j < i_p \): This is seen from (uplwl) as in the proof of the Claim 5.2 since in this case we have \( l \neq 0 \).

A proof of Lemma 5.7.2 is completed.  
\[
\square
\]

**Proof of Lemma 5.7.3**  
The chain \( C_{b-1} \) passes through the left side of \( K \) and \( C \) the right side of \( K \). By (ch:Qpt) we have
\[
pd_i(\sigma_{n+1}) = \sigma_{n_{m(i)}+1} \text{ and } J_{n_{m(i)}} = pd_i(J_n).
\]
If $K$ is a $K_m$ for an $m < l$, then the assertion $i \leq j = i_m$ follows from (13) since $b - 1 \geq n_{m(i)}$ by $\sigma_{n+1} \prec_i \sigma_b$, and hence $m < m(i)$.

Otherwise let $m \leq m(i)$ denote the number such that $n_{m} > b - 1 > n_{m-1}$, i.e., $J_{b-1}$ is between $K_{m-1}$ and $J_n$. Then $K$ is below $K_{m-1}$ and the rule $K$ is the merging rule of $C_{b-1}$ and the chain $C_{n_m}$ starting with $J_n$, i.e., $C_{n_m}$ passes through the right side of $K$.

By IH it suffices to show that $\sigma_{n_{m+1}} \prec_i \sigma_b$ and this follows from

$$\sigma_{n+1} \prec_i \sigma_{n_{m+1}} \tag{16}$$

since the set $\{ \tau : \sigma_{n+1} \prec_i \tau \}$ is linearly ordered by $\prec_i$, Proposition 4.11. Now (16) follows from (13) and Lemma 5.7.2a, i.e.,

$$\sigma_{n+1} \prec_i \rho_{d_i(\sigma_{n+1})} = \sigma_{n_{m(i)}+1} \prec_i \cdots \prec_i \sigma_{n_{m-1}+1} \prec_i \sigma_{n_m+1}.$$

This shows Lemma 5.7.3.

**Proof** of Lemma 5.7.4 by induction on the number of sequents between $K$ and $J$.

By the Claim 5.2 we have $a \leq n_{m(i)} + 1$.

**Case 1** $a = n_{m(i)} + 1$: This means that the $i$-predecessor $J_{n_{m(i)}}$ of $J$ is the rule $J_{a-1}$ and $pd_i(p) = \sigma_a$. By Lemma 5.7.2c we have $i_p > j \geq i$ for any $p < m(i)$. On the other side by (ch:Qpt)

$$i \in In(C) = In(\rho) \iff \exists p \in [0, m(i)) (i_p = i) \tag{17}$$

Hence $i \notin In(\rho)$. Thus $in_i(\sigma_a) = in_i(pd_i(\rho)) = in_i(\rho)$.

**Case 2** $a < n_{m(i)} + 1$: This means that $J_{n_{m(i)}}$ is below $K$. Let $m_1$ denote the number (15) defined in the proof of Lemma 5.7.2.
Claim 5.3 For each $m \in (m_1, m(i)]$ the $i$-origin of $J_{n_m}$ is not below $K_{m-1}$, $i < i_{m-1}$, $i \notin In(\rho)$, $\sigma_{n_m+1} \prec_i \sigma_{n_{m-1}+1}$ and $\forall b \in (n_{m-1}+1, n_m+1)\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \rightarrow \ i \notin In(\sigma_b)\}$ and 
\[
\forall b \in (n_{m-1}+1, n_m+1)\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \rightarrow \ i \notin In(\sigma_b)\}
\]
\[
\forall b \in (n_{m-1}+1, n_m+1)\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \rightarrow \ i \notin In(\sigma_b)\}
\]
\[
i_i(J_{n_m}) = i_i(J_{n_{m-1}}) \text{, i.e.,}
\]
\[
\sigma_{n_{m-1}+1} \prec_i \sigma_{n_{m-1}+1}
\]

Proof of the Claim 5.3 First we show $i < i_{m-1}$. By Lemma 5.7.1 we have 
i \leq i_{m-1}$. Assume $i = i_{m-1}$ for some $m \in (m_1, m(i)]$. Pick the minimal such $m_2$. Then by (ch:Qpt), 17 we have $i \in In(\rho)$ and hence $rg_i(\rho) \downarrow$. By Lemma 5.7.2c we have 
p < m_1 \Rightarrow i_p > j \geq i
(19)

Here $p < m_1$ means that $K_p$ is above $K$. Thus by (ch:Qpt)
$m(i + 1) = \max\{m : 0 \leq m \leq l \& \forall p \in [0, m)(i+1 \leq i_p)\} = m_2 - 1 \geq m_1$,
i.e.,
$J_{n_{m_2-1}} = pd_{i+1}(J) \& \sigma_{n_{m_2-1}+1} = pd_{i+1}(\rho) \& n_{m_2-1} \geq n_{m_1} \geq a$.
On the other hand by (ch:Qpt) we have for the $i$-origin $J_q$ of $C$, i.e., $\sigma_q = rg_i(\rho)$,
n_{m_2-1} = n_{m(i+1)} < q \leq n_{m(i)} + 1.
Thus $J_q$ is below $J_{n_{m_2-1}}$ and hence by $a \leq n_{m_2-1}$ the $i$-origin $J_q$ is below $K$.
This contradicts our hypothesis.

| $\Phi$ | $\neg A$ | $A, \Psi$ | $K$ |
|---------|--------|--------|-----|
| $\Phi, \Psi$ |        |        |     |
| $\Gamma_{n_{m_2-1}}$ | $(c)pd_{i+1}(\rho) J_{n_{m_2-1}}$ |        |     |
| $\Phi_{m_2-1}, \neg A_{m_2-1}$ | $A_{m_2-1}, \Psi_{m_2-1}$ | $(\Sigma_i) K_{m_2-1}$ |     |
| $\Phi_{m_2-1}, \Psi_{m_2-1}$ |        |        |     |
| $\Gamma_{q}$ | $(c)\sigma_i(\rho) J_q$ |        |     |
| $\Gamma_{n_{m(i)}}$ | $(c)pd_i(\rho) J_{n_{m(i)}}$ |        |     |
Thus we have shown $i < i_{m-1}$ for any $m \in (m_1, m(i)]$. From this, 19 and 17 we see $i \notin In(\rho)$ and hence

$$in_i(\rho) = in_i(pd_i(\rho)) = in_i(\sigma_{n_{m(i)+1}}).$$

In particular, if $rg_i(\rho) \downarrow$, then $\sigma_q = rg_i(\rho) = rg_i(\sigma_{n_{m(i)+1}})$, i.e., the $i$-origin of $J_{m(i)}$ equals to the $i$-origin $J_q$ of $J$. Therefore the $i$-origin of $J_{n_{m(i)}}$ is not below $K$ and a fortiori not below the $i_{m(i)-1}$-knot $K_{m(i)-1}$. Also the chain starting with $J_{n_{m(i)}}$ passes through the left side of $K_{m(i)-1}$ and $i \leq i_{m(i)-1}$. Thus by IH we have 18 for $m = m(i)$. We see similarly that for each $m \in (m_1, m(i)]$ the $i$-origin of $J_{n_m}$ is not below $K_{m-1}$ and 18.

Thus we have shown the Claim 5.3.

From Claim 5.3 we see

$$\forall b \in (n_{m_1} + 1, n + 1] \{ \rho \leq_i \sigma_b \prec_i \sigma_{n_{m_1}+1} \rightarrow i \notin In(\sigma_b) \}$$

and hence

$$\forall b \in (n_{m_1} + 1, n + 1] \{ \rho \leq_i \sigma_b \prec_i \sigma_{n_{m_1}+1} \rightarrow in_i(J) = in_i(J_{b-1}) = in_i(J_{n_{m_1}}), i.e., in_i(\rho) = in_i(\sigma_b) = in_i(\sigma_{n_{m_1}+1}) \}.$$ Further

$$\forall m \in (m_1, m(i)) \{ \rho \leq_i \sigma_{n_{m-1}+1} \prec_i \sigma_{n_{m-1}+1} \leq_i \sigma_{n_{m_1}+1} \}.$$ Once more by IH we have, cf. Figures in the proof of Lemma 5.7.2. Case 2.,

$$\forall b \in (a, n_{m_1} + 1] \{ \sigma_{n_{m_1}+1} \leq_i \sigma_b \prec_i \sigma_a \rightarrow i \notin In(\sigma_b) \}$$

and hence

$$\forall b \in (a, n_{m_1} + 1] \{ \sigma_{n_{m_1}+1} \leq_i \sigma_b \prec_i \sigma_a \rightarrow in_i(J_{n_m}) = in_i(J_{b-1}) = in_i(J_{a-1}), i.e., in_i(\sigma_{n_{m_1}+1}) = in_i(\sigma_b) = in_i(\sigma_a) \}.$$ Thus we have shown Lemma 5.7.4.

Proof of Lemma 5.7.4 by induction on the number of sequents between $K$ and $J_{b-1}$.

Let $C_b = I_0, \ldots, I_{b-1}$ denote the chain starting with $J_{b-1} = I_{b-1}$. Each rule $I_\rho$ is again a rule $(\Sigma)_{\sigma_{\rho+1}}$. Chains $C_b$ and $C$ intersect in a way described as Type1 (segment) or Type3 (merge) in (ch:link). If the chain $C_b$ passes through the left side of $K$, then the $i$-origin $I_q$ of $C_b$ is above $K$ if it exists, and hence the assertion follows from Lemma 5.7.4.

Otherwise there exists a merging rule $(\Sigma_i)^{\sigma_{\rho}} I$ below $K$ such that the chain
\(\phi, \neg A \ A, \psi \ K\)

\[\Gamma_{c-1} \ (c)_{\sigma_c} J_{c-1}\]

\(\Pi, \neg B \ B, \Delta \ \ (\Sigma)_{\sigma}\ I\)

\[\Gamma_{b-1} \ (c)_{\sigma} J_{b-1}\]

Then by Lemma 5.7.3 we have \(i \leq l\). The \(i\)-origin \(I_q\) of \(C_b\) is not below \(I\). Therefore by Lemma 5.7.4 we have

\[\forall d \in (c, b] \{ \sigma \preceq_i \sigma_d \prec_i \sigma_c \rightarrow i \notin \text{In}(\sigma_d) \}\]

and

\[\sigma \prec_i \sigma_c.\]

Hence

\[\forall d \in (c, b] \{ \sigma \preceq_i \sigma_d \prec_i \sigma_c \rightarrow \text{in}_i(\sigma_d) = \text{in}_i(\sigma_c) \}.\]

In particular

\[\text{in}_i(\sigma_c) = \text{in}_i(\sigma) \& \sigma \prec_i \sigma_c\]  \hspace{1cm} (20)

Now consider the member \(J_{c-1}\) of \(C\). \(J_{c-1}\) is again below \(K\), \(\sigma_{n+1} \preceq_i \sigma_c\) and \(rg_i(\sigma_c) \simeq rg_i(\sigma)\) by (20). Thus by IH we have

\[\forall d \in (a, c] \{ \sigma_c \preceq_i \sigma_d \prec_i \sigma_a \rightarrow i \notin \text{In}(\sigma_d) \}\]

and

\[\sigma_c \prec_i \sigma_a.\]

Therefore

\[\forall d \in (a, c] \{ \sigma_c \preceq_i \sigma_d \prec_i \sigma_a \rightarrow \text{in}_i(\sigma_d) = \text{in}_i(\sigma_a) \}.\]

This shows Lemma 5.7.5. \(\square\)

**Proof** of Lemma 5.7.6
Pick a $p_0$ so that $n ≥ p_0 ≥ q_0$, $ρ ≤_i σ_{p_0+1}$ and $rg_i(σ_{p_0+1}) = σ_{q_0} = κ$.

Lemma 5.7.6a First note that $ρ ≤_i σ_{p_0+1} ≺_i rg_i(σ_{p_0+1}) = κ$ by Proposition 4.1.2 (or by the proviso (ch:Qpt)) and hence $ρ ≺_i κ$. Thus the assertion follows from Lemma 5.7.5 and the minimality of $q_0$.

Lemma 5.7.6b Suppose $rg_i(σ_t) ≤_i κ$ for a $t$ with $ρ ≤_i σ_t ≺_i κ$. Put $σ_b = rg_i(σ_t)$. Then by Propositions 4.1.1 and 4.1.2 we have $κ = σ_{q_0} ≺_i σ_b$ and $b < q_0 < t$ & $q_0 ≥ a$. Hence by the minimality of $q_0$ we have $b < a$.

Thus by Lemma 5.7.5 we have

$$in_i(σ_a) = in_i(σ_t).$$
From this and Lemma 5.7.4 we have

\[ i_{n_i}(\sigma_a) = i_{n_i}(\sigma_t) = i_{n_i}(\kappa) \land \sigma_t \prec_i \kappa \preceq_i \sigma_a \]  

(21)

**Case 1** \( t \leq p_0 \): Then \( \sigma_{p_0+1} \prec_i \sigma_t \prec_i \kappa = r_{gi}(\sigma_{p_0+1}) \) by Proposition 4.1.1

By Proposition 4.1.1 we would have \( \sigma_b = r_{gi}(\sigma_t) \preceq_i \kappa \). Thus this is not the case.

Alternatively we can handle this case without appealing Proposition 4.1.1 as follows. Let \( p_0 \) denote the minimal \( p_0 \) such that

\[ n \geq p_0 \geq q_0 \land p \preceq_i \sigma_{p_0+1} \land r_{gi}(\sigma_{p_0+1}) = \sigma_{q_0} = \kappa. \]

Then by (ch:Qpt) we have \( \kappa = r_{gi}(\sigma_{p_0+1}) = p_{d_{i}}(\sigma_{p_0+1}) \) and hence this is not the case, i.e., \( p_0 < t \).

**Case 2** \( p_0 < t \): Then \( \sigma_t \preceq_i \sigma_{p_0+1} \prec_i \kappa \). By (21) and Proposition 4.1.1 or by Lemma 5.7.5 we would have \( i_{n_i}(\sigma_{p_0+1}) = i_{n_i}(\kappa) \). In particular \( \kappa = r_{gi}(\sigma_{p_0+1}) = r_{gi}(\kappa) \) but \( r_{gi}(\kappa) \) is a proper subdiagram of \( \kappa \). This is a contradiction.

This shows Lemma 5.7.6b. \( \Box \)

**Proof** of Lemma 5.7.7 by induction on the number of sequents between \( K \) and \( J_{b-1} \).

**Case 1** \( J_q \) is the \( i \)-origin of \( J_{b-1} \), i.e., \( J_q \) is a member of the chain starting with \( (c)_\sigma J_{b-1} \): By the proviso (st:bound) we can assume \( i \notin I_0(\sigma) \). Then \( i_{n_i}(\sigma) = i_{n_i}(p_{d_{i}}(\sigma)) = i_{n_i}(J_p) \) with \( J_p = p_{d_{i}}(J_{b-1}) \land a \leq b - 1 < p \) by Lemma 5.7.5. In particular \( r_{gi}(\sigma) = r_{gi}(p_{d_{i}}(\sigma)) \). IH and \( s_{t_{i}}(p_{d_{i}}(\sigma)) = s_{t_{i}}(\sigma) \) yields the lemma.

**Case 2** Otherwise: First note that \( \sigma_{n+1} \neq \sigma \) and \( \sigma_{n+1} \prec_i \sigma \). By (ch:Qpt) we have

\[ p_{d_{i}}(\sigma_{n+1}) = \sigma_{n_{m(i)}+1} \land J_{n_{m(i)}} = p_{d_{i}}(J_{n(i)}). \]

Also \( \sigma_{n_{m(i)}+1} \preceq_i \sigma \) and hence \( \sigma_{n_{m(i)}+1} \leq \sigma \). Let \( m_1 \) denote the number such that

\[ m_1 = \min\{m : \sigma_{n_{m+1}} \leq \sigma \} \leq m(i). \]

Then the rule \( (c)_\sigma J_{b-1} \) is a member of the chain \( C_{n_{m(i)}} \) starting with \( J_{n_{m(i)}} \) and \( J_{b-1} \) is below \( (\Sigma_{1_{m-1}}) K_{m_{1-1}} \). Also the chain \( C_{n_{m(i)}} \) passes through the left side of the knot \( K_{m_{1-1}} \). By \( m_1 \leq m(i) \) and Lemma 5.7.11 we have \( \sigma_{n_{m(i)}+1} \preceq_i \sigma_{n_{m}} \)

hence

\[ i \leq i_{m(i)} \land \sigma_{n_{m(i)}} \preceq_i \sigma. \]  

(22)

**Case 2.1** \( J_q \) is below \( K_{m_{1-1}} \), i.e., \( \sigma_q \leq \sigma_{n_{m_{1-1}}+1} \), i.e., \( n_{m_{1-1}} < q \): By IH and (22) we get the assertion.

**Case 2.2** Otherwise: By Lemma 5.7.5 and (22) we have

\[ i_{n_{i}}(\sigma) = i_{n_{i}}(\sigma_{n_{m(i)}+1}) \land \sigma \prec_i \sigma_{n_{m(i)}+1}. \]

Hence \( s_{t_{i}}(\sigma) = s_{t_{i}}(\sigma_{n_{m(i)}+1}) \land r_{gi}(\sigma) = r_{gi}(\sigma_{n_{m(i)}+1}). \) IH and \( s_{t_{i}}(\sigma_{n_{m(i)}+1}) = s_{t_{i}}(\sigma) \) yield the lemma. \( \Box \)
Lemma 5.8 Let $R = J_0, \ldots, J_{n-1}$ denote the rope starting with a top $(c)^n J_0$. Each $J_p$ is a rule $(c)^{\sigma_p+1}_p$. Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n - 1 (l \geq 0)$$

be the knotting numbers of the rope $R$, and $K_m$ an $i_m$-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of $J_{n_m}$: and $J_{n_m+1}$ for $m < l$. For $2 \leq i < N$ let $m(i)$ denote the number

$$m(i) = \max \{m : 0 \leq m \leq l \land \forall p \in [0, m)(i \leq i_p)\}$$

Note that $i_m \leq N - 2$ by (ch: left). Also put (cf. (ch: Qpt))

1. $pd_i = \sigma_{m(i)+1}$.

2. $i \in In \iff \exists p \in [0, m(i))(i_p = i)$.  

3. For $i \in In$ ($i \neq N - 1$),

**Case 1** The case when there exists a $q$ such that

$$\exists p[n_{m(i)} \geq p \geq q > n_{m(i+1)} \land pd_i \preceq \sigma_{p+1} \land \sigma_q = rg_i(\sigma_{p+1})] \quad (23)$$

Then

$$rg_i = \sigma_q$$

where $q$ denotes the minimal $q$ satisfying (23).

**Case 2** Otherwise.

$$rg_i = pd_i = \sigma_{m(i)+1}.$$
Proof.

Lemma 5.8.1

Let \( i \in \text{In} \), and put \( \sigma_q = \text{rg}_i \) and \( \sigma_p = \text{pd}_i \). By the definition we have \( p_0 = n_{m(i)} + 1 \) \& \( r = n_{m(i)+1} + 1 \), \( m(i) > m(i+1) \) \& \( i_{m(i)+1} = i \), \( p_0 \leq q_0 \leq r \) and \( \sigma_{p_0} \preceq_i \sigma_{q_0} \). Also

\[
\forall p \in [m(i+1), m(i))(i \leq i_p).
\]

From this and Lemma 5.7.1 we see

\[
\forall p \in [m(i+1), m(i))(\sigma_{n_p+1}+1 \prec_i \sigma_{n_p+1})
\]

On the other hand we have by the definition of \( \text{rg}_i \)

\[
\neg \exists q < q_0 \exists p[p_0 - 1 \leq p \leq q > r - 1 \& \text{pd}_i \preceq_i \sigma_{p+1} \& \sigma_q = \text{rg}_i(\sigma_p)]
\]

Case 2. Then \( \text{pd}_i = \text{rg}_i \), i.e., \( p_0 = q_0 \), and Lemma 5.8.1 vacuously holds. Lemma 5.8.1a \( \text{in}_i(\sigma_{q_0}) = \text{in}_i(\sigma_r) \& \sigma_{q_0} \preceq_i \sigma_r \) follows from (24) and Lemma 5.7.4 with (25).

Case 1. Let \( m \) denote the number such that

\[
m(i) \geq m > m(i+1) \& n_m \geq q_0 > n_{m-1}
\]
i.e., the rule \((c)^{\sigma_{q_0}} J_{q_0}\) is a member of the chain \( C_{n_m} \) starting with \( J_{n_m} \).

Claim 5.4 Let \( p_1 \) denote the minimal \( p_1 \) such that \( \sigma_{p_0} \preceq_i \sigma_{p_1+1} \) and \( \sigma_{q_0} = \text{rg}_i(\sigma_{p_1+1}) \). Then \( p_1 \leq n_m \& \sigma_{n_{m+1}} \preceq_i \sigma_{p_1+1} \).

Proof of Claim 5.4 Let \( m_1 \) denote the number such that

\[
m(i) \geq m_1 > m(i+1) \& n_{m_1} \geq p_1 > n_{m_1-1}.
\]
Then by (24), \( pd_i \preceq_i \sigma_{n_{m1}+1} \) and \( pd_i \preceq_i \sigma_{p1+1} \) we have \( \sigma_{n_{m1}+1} \preceq_i \sigma_{p1+1} \). It remains to show \( m = m_1 \). Assume \( m < m_1 \). Then by Lemma 5.7.5 and \( q_0 < n_{m1-1}+1 \) we would have \( in_i(\sigma_{n_{m1}-1}+1) = in_i(\sigma_{p1}+1) \) and hence \( rg_i(\sigma_{n_{m1}-1}+1) = rg_i(\sigma_{p1}+1) = \sigma_{q0} \). This contradicts the minimality of \( p_1 \) by (24).

\[
\begin{array}{c}
\Phi_{m-1}, \neg A_{m-1}, A_{m-1}, \Psi_{m-1} \to K_{m-1} \\
\Phi_{m-1}, \Psi_{m-1} \to \Gamma_{q0} \\
(c)rg_i(\sigma_{p1+1})J_{q0} \\
\Gamma_{q0} \to J_{n_{m1}} \end{array}
\]

This shows Claim 5.4. \( \Box \)

By the minimality of \( q_0 \) and the Claim 5.4 \( q_0 \) is the minimal \( q \) such that

\[
\exists p(n_m \geq p \geq q \geq n_{m-1} + 1 \& \sigma_{n_{m+1}} \preceq_i \sigma_{p+1} \& \sigma_q = rg_i(\sigma_{p+1})].
\]

Hence by Lemma 5.8.1a we have

\[
in_i(\sigma_{n_{m1}+1}) = in_i(\sigma_{q0}) \& \sigma_{q0} \preceq_i \sigma_{n_{m1}+1} \tag{27}
\]

and

\[
\forall t[\sigma_{n_{m1}} \preceq_i \sigma_t \prec_i \sigma_{q0} \Rightarrow rg_i(\sigma_t) \preceq_i \sigma_{q0}] \tag{28}
\]

Lemma 5.8.1a. By (27) it suffices to show that

\[
in_i(\sigma_{n_{m1}+1}) = in_i(pd_{i+1}) \& \sigma_{n_{m1}+1} \preceq_i \sigma_{pd_{i+1}}.
\]

This follows from (24) and Lemma 5.7.4 with (25).

Lemma 5.8.1b and 5.8.1c. In view of (28) it suffices to show the
Claim 5.5 \( \sigma_{p_0} \preceq_i \sigma_i \prec_i \sigma_{n_2+1} \Rightarrow rg_i(\sigma_i) \preceq_i \sigma_{q_0} \).

Proof of Claim 5.5 by induction on \( t \) with \( p_0 > t > n_2 + 1 \).

Let \( m_2 \geq m \) denote the number such that \( n_{m_2+1} \geq t > n_{m_2} \). Then the chain \( C_{n_2+1} \) starting with \( J_{n_2+1} \) passes through the left side of the rule \((\Sigma_j n_2+1) K_{m_2}\).

\[
\begin{array}{c}
\Phi_{m_2}, \neg A_{m_2} A_{m_2}, \Psi_{m_2} (\Sigma_j n_2+1) K_{m_2} \\
\Phi_{m_2}, \Psi_{m_2} \Gamma_{t-1} \sigma_1 \Gamma_{t-1} (c) J_{t-1} \\
\Gamma_{t-1} \Gamma_{t-1} \vdots \\
\Gamma_{n_{m_2+1}} \Gamma_{n_{m_2+1}} (c) \sigma_{n_{m_2+1}} \sigma_{n_{m_2+1}} J_{n_{m_2+1}}
\end{array}
\]

We have \( \sigma_{n_{m_2+1}} \preceq_i \sigma_i \prec_i \sigma_{n_2+1} \) (29) by (24). Put \( \sigma_b = rg_i(\sigma_i) \). It suffices to show \( b \geq q_0 \).

First consider the case when \( b \leq n_{m_2} \). Then by (29) and Lemma 5.7.6 we have \( \forall i \leq n_{m_2} \). Thus IH when \( m_2 > m \) and (28) when \( m_2 = m \) yield \( b \geq q_0 \).

Next suppose \( b > n_{m_2} \). Let \( q_1 \leq b \) denote the minimal \( q \leq b \) such that

\[ \exists p [n_{m_2+1} \geq p > n_{m_2} + 1 \& \sigma_{p+1} \leq_i \sigma_{p+1} \& \sigma_q = rg_i(\sigma_{p+1})] . \]

The pair \((p, q) = (t-1, b)\) enjoys this condition.

Then by Lemma 5.7.2 we have \( \sigma_{q_1} \preceq_i \sigma_{n_{m_2}+1} \). Thus \( \sigma_0 \preceq_i \sigma_{q_1} \preceq_i \sigma_{n_{m_2}+1} \preceq_i \sigma_{n_{m_2}} \preceq_i \sigma_{q_0} \). This shows Claim 5.5. \qed

Lemma 5.8.2

First observe that as in (22) in the proof of Lemma 5.7.1

\[ \forall m \leq m(i) [\sigma_{m(i)+1} \preceq_i \sigma_{m+1}] \] (30)

Put

\[ m_1 = \min \{m : p \leq n_m\} \]
\[ m_2 = \min \{m : q \leq n_m\}. \]

Then the rule \( J_p [J_q] \) is a member of the chain \( C_{n_{m_1}} [C_{n_{m_2}}] \) starting with \( J_{n_{m_1}} \) [starting with \( J_{n_{m_2}} \), resp. and \( m(i+1) < m_2 \leq m_1 \leq m(i) \).

Claim 5.6 (Cf. Claim 5.4) There exists a \( p_0 \) such that

\[ \forall i [\sigma_{p+1} = \sigma_{p_0+1} \& \sigma_{n_{m+1}} \leq_i \sigma_{p_0+1} \& n_{m_2} \geq p_0 > n_{m_2} - 1], \]

i.e., the rule \( J_{p_0} \) is a member of the chain \( C_{n_{m_2}} \) starting with \( J_{n_{m_2}} \).
Proof of Claim 5.6

1. The case \( m_1 = m_2 \): By (30) and \( \sigma_{n_m(i)+1} \preceq_i \sigma_{p+1} \) we have
   \[ \sigma_{n_1+1} \preceq_i \sigma_{p+1}. \tag{31} \]
   Set \( p_0 = p \).

2. The case \( m_2 < m_1 \): By (31) and Lemma 5.7.5 we have
   \[ \text{in}_{\sigma_{p+1}} = \text{in}_{\sigma_{n_m(i)-1+1}} = \cdots = \text{in}_{\sigma_{n_m(i)+1}} \]
   and
   \[ \sigma_{p+1} \prec_i \sigma_{n_m(i)-1+1} \prec_i \cdots \prec_i \sigma_{n_m(i)+1}. \]
   Set \( p_0 = n_{m_2} \).

This shows the Claim 5.6.

By the Claim 5.6 and Lemma 5.7.7 we conclude \( rg_i(\sigma_{p+1}) = rg_i(\sigma_{p_0+1}) \) and
\[ B_{\sigma_0}(c; P) \leq b(st_i(\sigma_{p_0+1})) = b(st_i(\sigma_{p+1})) = \alpha. \]

Main Lemma 5.1 If \( P \) is a proof, then the endsequent of \( P \) is true.

In the next section we prove the Main Lemma 5.1 by a transfinite induction on \( o(P) \in Od(\Pi_N) \setminus \Omega \).
Assuming the Main Lemma 5.1 we see Theorem 1.1 as in [6], i.e., attach \((h)^\pi, (c\Pi_J)^\Omega \) and \((h)^\Omega \) as last rules to a proof \( P_0 \) of \( A^\Omega \) in \( T_N \).

\[ P \]

6 Proof of Main Lemma

Throughout this section \( P \) denotes a proof with a chain analysis in \( T_{N_c} \) and \( r : \Gamma_{rdx} \) the redex of \( P \).

M1. The case when \( r : \Gamma_{rdx} \) is a lowersequent of an explicit basic rule \( J \).
M2. The case when \( r : \Gamma_{rdx} \) is a lowersequent of an \((ind) J \).
M3. The case when the redex \( r : \Gamma_{rdx} \) is an axiom.
These are treated as in [5], [6].
By virtue of M1-3 we can assume that the redex \( r : \Gamma_rdx \) of \( P \) is a lowersequent of a rule \( J = r \ast (0) \) such that \( J \) is one of the rules \((\Pi_2^2\text{-rfl}), (\Pi_N\text{-rfl})\) or an implicit basic rule.

M4. \( J \) is a \((\Pi_2^2\text{-rfl})\). As in \([5]\) introduce a \((c)_{\Pi_2^2\text{-rfl}}\) and a (cut).

M5. \( J \) is a \((\Pi_N\text{-rfl})\).

M5.1. There is no rule \((c)_{\Pi_N\text{-rfl}}\) below \( J \).

M5.2. There exists a rule \((c)_{\Pi_N\text{-rfl}}\) below \( J \).

Let \( P' \) be the following:

\[
\vdots 
\frac{\Gamma, A \quad \neg \exists z(t < z \land A^z), \Gamma}{r : \Gamma} \\
\vdots 
\frac{a : \Phi}{a_0 : \Lambda (h)\pi} \\
\frac{\vdots}{P}
\]

where \( a : \Phi \) denotes the uppermost sequent below \( J \) such that \( h(a; P) = \pi \). The sequent \( a_0 : \Lambda \) is the lowersequent of the lowermost \((h)\pi\).

Let \( P' \) be the following:

\[
\vdots 
\frac{\Gamma, A \quad (w)}{\vdots} \\
\vdots 
\frac{\neg A^\sigma, \Gamma}{\neg A^\sigma, \Gamma (w)} \\
\vdots 
\frac{\vdots}{\vdots} \\
\frac{a_1 : \Phi, A}{\Phi, A^\sigma (c\Pi_N)_{\sigma} J_0} \\
\frac{\Lambda, A^\sigma (h)\pi}{\neg A^\sigma, \Phi} \\
\frac{\vdots}{\vdots} \\
\frac{a_0 : \Lambda (\Sigma_{N-1})^\sigma J_0'}{P'}
\]

where the o.d. \( \sigma \) in the new \((c\Pi_N)_{\sigma} J_0 \) is defined to be

\[ \sigma = d_{q+1}^{\nu} \alpha \text{ with } q = \nu \pi \pi N - 1, \nu = o(a_1; P') \text{ and } \alpha = \pi \cdot o(a_1; P') + k_{\pi}(a; P) \]

Namely \( In(\sigma) = \{N - 1\}, st_{N-1}(\sigma) = \nu \text{ and } pd_{N-1}(\sigma) = rg_{N-1}(\sigma) = \pi \).

Then as in \([4]\) we see that \( \Phi \subseteq \Delta^\sigma, \alpha < Bk_{\pi}(a; P) \& \sigma < o(a_0; P), \sigma \in Od(\Pi_N) \) and \( o(a_0; P') < o(a_0; P) \). Hence \( o(P') < o(P) \). Moreover in \( P \), no chain passes through \( a_0 : \Lambda \), and the new \((\Sigma_N)^\sigma J_0' \) does not split any chain.

M5.2. There exists a rule \((c)_{\Pi_N\text{-rfl}}\) below \( J \).

Let \( \mathcal{R} = J_0, \ldots, J_{n-1} \) denote the rope starting with \( J_0 \). The rope \( \mathcal{R} \) need not to be a chain as contrasted with \([4]\). Each rule \( J_p \) is a \((c)_{\Pi_{n+1}}\). Put \( \sigma = \sigma_n \).
\[
\begin{array}{c}
\Gamma, A \quad \exists z (t < z \land A^z), \Gamma \\
\overline{r : \Gamma}
\end{array}
\]

\[
\begin{array}{c}
\Gamma_0 \quad \overline{a_0 \Gamma_0}
\end{array}
\]

\[
\begin{array}{c}
a_i : \Gamma_i \quad \overline{(c)_{\sigma_i}^\gamma J_i}
\end{array}
\]

\[
\begin{array}{c}
a_{n-1} : \Gamma_{n-1} \quad \overline{(c)_{\sigma_{n-1}}^\gamma J_{n-1}}
\end{array}
\]

\[
\begin{array}{c}
a_n : \Gamma_n \quad \overline{(\Sigma_{N-1})^\gamma J_{n-1}'}
\end{array}
\]

\[
\begin{array}{c}
a : \Phi \quad \overline{P}
\end{array}
\]

where \(a_n : \Gamma_n\) denotes the lowersequent of the trace \((\Sigma_{N-1})^\gamma J_{n-1}’\) of \(J_{n-1}\), and \(a : \Phi\) the bar of the rule \((c)_{\sigma} J_{i-1}\). Let \((\Sigma_{N-1})^{\sigma_{i+1}} J_i’\) denote the trace of \(J_i\) for \(0 \leq i < n\). Put

\[h := h(a; P)\]

By Lemma 5.2 there is no chain passing through the bar \(a : \Phi\).
Let $P'$ be the following:

\[
\begin{array}{c}
\cdots \\
\Gamma_n, A \\
\vdots \\
a_0^l : \Gamma_0, A \\
\Gamma_0', A^{\sigma_1}, J_0^l \\
\vdots \\
a_1^l : \Gamma_1, A^{\sigma_1}, J_1^l \\
\vdots \\
a_{n-1}^l : \Gamma_{n-1}, A^{\sigma_{n-1}}, J_{n-1}^l \\
\vdots \\
a_n^l : \Gamma_n, A^{\sigma_n}, (c\Pi_N)_\rho J_n
\end{array}
\quad
\begin{array}{c}
\cdots \\
\neg A^\rho, \Gamma \\
\vdots \\
a_0^l : \neg A^\rho, \Gamma_0 \\
\neg A^\rho, \Gamma_0' \\
\vdots \\
a_1^l : \neg A^\rho, \Gamma_1 \\
\neg A^\rho, \Gamma_1' \\
\vdots \\
a_{n-1}^l : \neg A^\rho, \Gamma_{n-1} \\
\neg A^\rho, \Gamma_{n-1}' \\
\vdots \\
a_n^l : \neg A^\rho, \Gamma_n \\
\neg A^\rho, \Phi
\end{array}
\]

\[\alpha : \Phi \quad (\Sigma_N)^\rho J_n^l \quad P'\]

For the proviso (lbranch) in $P'$, any ancestor of the left cut formula of the new $(\Sigma_N)^\rho J_n^l$ is a genuine $\Pi_N^\alpha$-formula $A^\tau$ for a $\tau$ with $\rho \leq \tau$. The formula $A^\tau$ is not in the branch $T$ from $r : \Gamma$ to $a : \Phi$ in $P$ since no genuine $\Pi_N^\alpha$-formula with $\tau > \Omega$ is on the rightmost branch $T$. Therefore any left branch of the new $(\Sigma_N)^\rho J_n^l$ is the rightmost one in the left upper part of the $J_n^l$ in $P'$.

In $P'$, a new chain $J_0^l, \ldots, J_{n-1}^l, J_n$ starting with the new $J_n$ in the chain analysis for $P'$ and $\rho = d^\rho_\alpha \alpha \in D_{\sigma}$ is determined as follows:

\[b(\rho) = \alpha = \max\{B_\sigma(o(a_0^l; P')), B_{\sigma}(\{\sigma\} \cup \langle a_n; P \rangle)\} + \omega(o(a_0^l; P')) + \max\{K_\sigma\langle a_n; P \rangle, K_\sigma(h)\},\]

\[\text{rg}_{N-1}(\rho) = \pi \text{ and st}_{N-1}(\rho) = o(a_0^l; P').\]

Let

\[0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \quad (l \geq 0)\]

be the knotting numbers of the rope $R$ and $K_m$ an $i_m$-knot $(\Sigma_{i_m})^{\sigma_{n+1}}$ of $J_{n_m}$ and $J_{n_{m+1}}$ for $m < l$. Let $m(i)$ denote the number

\[m(i) = \max\{m : 0 \leq m \leq l \text{ and } \forall p \in [0, m)(i \leq i_p)\}.\]

Then $pd_i(\rho), ln(\rho), rg_i(\rho), st_i(\rho)$ are determined so as to enjoy the provisos (ch:Qpt) and (st:bound).

1. $pd_i(\rho) = \sigma_{n(i)+1}$ for $2 \leq i < N$. Note that $pd_i(\rho) \neq \pi = \sigma_0$ since $n_0 \geq 0$, cf. the condition (\text{Qpt}) in Section 2 which says that $pd_{N-1}(\rho) = \pi \leftrightarrow \sigma = \pi$.  

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2. $N - 1 \in In(\rho)$ and $i \in In(\rho) \iff \exists p \in [0, m(i))(i_p = i)$ for $2 \leq i < N - 1$.

3. Let $i \in In(\rho) \& i \neq N - 1$. $q$ denotes a number determined as follows.

   **Case 1** The case when there exists a $q$ such that
   \[
   \exists p[n_m(i) \geq p \geq q > n_{m(i+1)} \& \rho < i] \sigma_{p+1} \& \sigma_q = rg_i(\sigma_{p+1}) \tag{13}
   \]
   Then $q$ denotes the minimal $q$ satisfying (13). Note that $\rho < i$ $\sigma_{p+1}$ is equivalent to $pd_i(\rho) = \sigma_{n_m(i)+1} \preceq_1 \sigma_{p+1}$.

   **Case 2** Otherwise. Then set $q = n_m(i) + 1$.
   In each case set $rg_i(\rho) = \sigma_q := \kappa$ for the number $q$, and $st_i(\rho) = d_\kappa + \alpha_i$ for
   \[
   \alpha_i = B_\kappa(a^l_\rho; P')
   \]
   where $a^l_\rho$ denotes the uppersequent $\Gamma$, $A^{\alpha_i}$ of $J^l_i$ in the left upper part of $(\Sigma_N)p J^l_i$ in $P'$.

   By Lemma 12 we have $B_{>\kappa}(\alpha_i) \subset B_{>\kappa}(\alpha_i) < \alpha_i$, and hence $st_i(\rho) \in Od(\Pi_N)$.

   Obviously the provisos (ch:Qpt) and (st:bound) are enjoyed for the new chain $J^l_0, \ldots, J^l_{n-1}, J_n$.

   Observe that, cf. [9] in Section 4
   \[
   \pi < \beta \in q = Q(\rho) \Rightarrow \beta = st_{N-1}(\rho).
   \]

**Claim 6.1** $\rho = d^l_\rho \alpha \in Od(\Pi_N)$.

**Proof of Claim 6.1**

**Case 1** $B_{>\sigma}(\sigma, \alpha) \cup q < \alpha$: By Lemma 12 we have $B_{>\sigma}(\{\sigma, \alpha\}) < \alpha$. It suffices to see $B_{>\sigma}(q) < \alpha$. By the definition we have $\{pd_i(\rho), rd_i(\rho) : i \in In(\rho)\} \subset \{\sigma_p, \sigma^+_p : p \leq n\}$. On the other hand we have $B_{>\sigma}(\{\sigma_p, \sigma^+_p : p \leq n\}) \subset B_{>\sigma}(\sigma)$.

   We have $B_{>\sigma}(st_{N-1}(\rho)) \subset B_{>\sigma}(\sigma)$. Finally for $st_i(\rho) = d_{\kappa+i}$ with $i < N - 1$, we have $B_{>\sigma}(st_i(\rho)) \subset B_{>\sigma}(\{\sigma, \alpha_i\}) \cup \{\alpha_i\}$, and $B_{>\sigma}(\{\sigma, \alpha_i\}) \subset B_{>\sigma}(\{\sigma, \alpha_i\})$ and $\alpha_i < \alpha$.

(DQ.12):

**Case 2** This corresponds to (DQ.12.1), $\kappa = rg_i(\rho) = pd_i(\rho)$. Let $\alpha_1$ denote the diagram such that $\rho \preceq_1 \alpha_1 \in D_\kappa$. Then
   \[
   \alpha_1 = \sigma_{n_m(i)+2}(pd_i(\rho) = \sigma_{n_m(i)+1} \& \sigma_{n+1} = \rho).
   \]
   We have by Lemma 12 and (c:bound2),
   \[
   B_{>\sigma}(B_\kappa(a_{n_m(i)+1}; P)) < B_\kappa(a_{n_m(i)+1}; P) \leq b(\alpha_1).
   \]

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On the other hand we have for \( st_i(\rho) = d_n + \alpha_i \)
\[
B_{>\kappa}(st_i(\rho)) \subset B_{>\kappa}(\{\kappa, \alpha_i\}) \leq B_{>\sigma}(B_{\kappa}(a_{n_m+1}; P)).
\]

Thus \( B_{>\kappa}(st_i(\rho)) < b(\alpha_1) \).

**Case 1** This corresponds to \((D^Q, 12.2), rg_i(\rho) = rg_i(pd_i(\rho))\) or to \((D^Q, 12.3), rg_i(pd_i(\rho)) < \kappa \) by Lemma 5.8.1c. Let \( p \) denote the maximal \( p \) such that
\[
rg_i(\sigma_{p+1}) = \sigma_q = rg_i(\rho) & pd_i(\rho) \leq \sigma_{p+1}.
\]

Thus \( st_i(\rho) < st_i(pd_i(\rho)) \) for the case \((D^Q, 12.2) \) and \( st_i(\rho) < st_i(\sigma_{p+1}) \) for the case \((D^Q, 12.3) \) follow from Lemma 5.8.2 since for \( rg_i(\sigma_{p+1}) = \sigma_q = rg_i(\rho) \)
\[
b(st_i(\rho)) = B_{\sigma_q}(a_q^l; P') < B_{\sigma_q}(a_q; P) \leq b(st_i(\sigma_{p+1}))
\]

and hence by Lemmata 4.1 and 4.2.

\[
st_i(\rho) < st_i(\sigma_{p+1}).
\]

\((D^Q, 11) \) and \((D^Q, 12.3) \): These follow from Lemma 5.8.1
\((D^Q, 2) \): \( \forall \tau \leq rg_i(\rho) (K_{r}(st_i(\rho) < \rho) \).

For \( \tau \leq \kappa = rg_i(\rho) \) and \( st_i(\rho) = d_n + \alpha_i \), we have \( K_{r}(st_i(\rho)) = K_{r}(\{\kappa, \alpha_i\}) \leq K_{r}(a_q^l; P') < \rho \) as in the case M6.2 in [6]. □

As in the case M6.2 in [6] we see that \( o(P') < o(P) \).

We have to verify that \( P' \) is a proof. The provisos other than \((\text{upwl})\) are seen to be satisfied as in the case M5.2 of [6]. For the proviso \((\text{forerun})\) see Claim 6.3 in the subcase M7.2 below. It suffices to see that \( P' \) enjoys the proviso \((\text{upwl})\) when the lower rule \( J^w \) is the new \((\Sigma_N)^p J'_n \). For example the left rope \( K_m R \) of the \( t_m \)-knot \( (\Sigma_{m+1})^p J_m \) of \( J_{n_m} \) and \( J_{n_m+1} \) ends with the rule \((c)_{\tau} J_{n-1} \). We show the following claim.

**Claim 6.2** Any left rope \( j_{\tau} R \) of a knot \( J^w \) in the left upper part of the new \((\Sigma_N)^p J'_n \) does not reach to \( J_n \).

**Proof** of Claim 6.2 Consider the original proof \( P \). By Lemma 5.2 there is no chain passing through the bar \( a: \Phi \) and hence it suffices to see that there is no rule \((c)^p \) above \( a: \Phi \). First observe that we have \( \rho < \tau \) for any rules \((c)_{\tau} \) and \((\Sigma_i)^p \) which are between (\( \Pi_N \)-rfl) \( J \) and \( a: \Phi \). Thus there is no rule \((c)^p \) on the branch \( T_0 \) from \((c)^p J_0 \) to \( a: \Phi \). Consider another branch \( T \) above \( a: \Phi \) and suppose that there is a rule \((c)^p \) \( I \) on \( T \). We can assume that the merging rule \( K \) of \( T \) and \( T_0 \) is below \( J_0 \) and hence the rule \( K \) is a \((\Sigma_i)^p \). By the proviso \((h-reg)\) (cf. Definition 5.4.4 in [6].) we have \( \tau \leq \sigma, \) i.e., \( K \) is between \( (c)^{p \tau_{n-1}} J_{n-1} \) and \( a: \Phi \). Then we have seen \( \rho < \tau \) and hence the trace \((\Sigma_{N-1})^p I_0 \) of \( (c)^p \) \( I \) is below \( K \) by the proviso \((h-reg)\). Therefore the chain stating with the trace \( I_0 \)
passes through the left side of \( K \). This is impossible by the proviso (ch:left).

\[
\begin{array}{c}
\frac{\Psi_2}{\Psi'_2} \frac{(c)^\sigma}{\rho} I \\
\frac{\Gamma_{n-1}}{\Gamma'_{n-1}} (c)^{\sigma_{n-1}} J_{n-1} \\
\vdots \quad \vdots \quad \vdots \\
\Psi_1, \neg C^\sigma \quad \Phi_1 \\
\vdots \quad \vdots \\
\Psi_0, \neg B^\rho \quad B^\rho, \Phi_0 \quad (\Sigma_{N-1})^\rho I_0 \\
\vdots \quad \vdots \\
a : \Phi
\end{array}
\]

\((\Sigma_i)^\tau K\)

In what follows we assume that \( r \ast (0) = J \) is a basic rule. Let \( v \ast (0) = I \) denote the vanishing cut of \( r \ast (0) = J \). \( v \ast (0) = I \) is either a \((\Sigma_i)\) or a \((\text{cut})\).

**M6.** \( I \) is a \((\Sigma_N)^\sigma\).

\[
\begin{array}{c}
\vdots \\
\alpha < \sigma, \Lambda_0 \quad \neg A^\sigma_{N-1}(\alpha), \Lambda_0 \quad (b\exists) J \\
\vdots \\
\exists x < \sigma \neg A^\sigma_{N-1}(x), \Lambda_0 \\
\vdots \\
\Gamma, A^\sigma \quad \neg A^\sigma, \Lambda \quad (\Sigma_N)^\sigma I \\
v : \Gamma, \Lambda
\end{array}
\]

where \( A \equiv \forall x A_{N-1}(x) \) is a \( \Pi_N \) formula.

Assuming \( \alpha < \sigma \) let \( P' \) be the following:

\[
\begin{array}{c}
\vdots \\
\neg A^\sigma_{N-1}(\alpha), \Lambda_0 \\
\neg A^\sigma, \Lambda_0, \neg A^\sigma_{N-1}(\alpha) \quad (w) \\
\vdots \\
\Gamma, A^\sigma \quad \neg A^\sigma, \Lambda \quad \neg A^\sigma_{N-1}(\alpha) \\
\vdots \\
\Gamma, \Lambda, \neg A^\sigma_{N-1}(\alpha) \quad \Gamma, A^\sigma_{N-1}(\alpha) \quad (\Sigma_{N-1})^\sigma I_{N-1} \\
v : \Gamma, \Lambda
\end{array}
\]

where, the preproof ending with \( \Gamma, A^\sigma_{N-1}(\alpha) \) is obtained from the left upper part of \( I \) in \( P \) by inversion.

As in the case M6 of [6] we see that \( o(v; P') < o(v; P) \).

For the proviso (lbranch) in \( P' \), cf. the case M5.2. We verify that \( P' \) is a proof with respect to the proviso (uplw).
The proviso (uplw) when the lower rule $J^l$ is the new $(\Sigma_{N-1})^\sigma I_{N-1}$: Consider the original proof $P$. By Lemma 5.3 no left rope in the right upper part of $(\Sigma N)^\sigma I$ reaches to $I$. Also by (uplw) with the lower rule $J^l = I$ there is no left rope of an $i$-knot $J^u$ reaching to $I$.

The proviso (uplw) when the lower rule $J^l$ is the new $(\Sigma_{N-1})^\sigma I_{N-1}$: As above there is no left rope of an $i$-knot $J^u$ reaching to $I$.

The proviso (uplw) when the upper rule $J^u$ is the (subcase) $(\Sigma_{N-1})^\sigma I_{N-1}$: $(\Sigma_{N-1})^\sigma I_{N-1}$ is not an $(N-1)$-knot since there is no chain passing through $(\Sigma N)^\sigma I$ by (ch:pass).

For the proviso (forerun) see Claim 6.3 in the subcase M7.2 below.

**M7.** $I$ is a $(\Sigma_{i+1})^\sigma$ with $1 \leq i < N-1$. Then $J$ is either an $(\exists)$ or a $(b\exists)$. Let $u_0 : \Psi$ denote the uppermost sequent below $I$ such that $h(u_0; P) < \sigma + i$. Also let $u : \Phi$ denote the resolvent of $I$, cf. Definition 5.5.  

**M7.1** $u_0 = u$.

\[
\frac{\alpha < \tau, \Lambda_0}{A^\tau_i(\alpha), \Lambda_0} \quad (x) \\
\frac{\Gamma, \neg A^\sigma_i+1}{A^\sigma_i+1, \Lambda} \\
\frac{\Gamma, \Lambda, (\Sigma_i)^\sigma I}{u : \Psi} \quad P
\]

where $A_{i+1} \equiv \exists y A_i(y)$ is a $\Sigma_{i+1}$ formula. Also if $x$ is an $(\exists)$, then $\tau = \pi$ and the left upper part of the true sequent $\alpha < \tau, \Lambda_0$ is absent. Anyway $\sigma \leq \tau$.

Assuming $\alpha < \tau$ and then $\alpha < \sigma$ by (c:bound), let $P'$ be the following:

\[
\frac{A^\tau_i(\alpha), \Lambda_0}{A^\tau_i+1, A^\tau_i(\alpha), \Lambda_0} \quad (w) \\
\frac{\Gamma, \neg A^\sigma_i+1}{A^\sigma_i+1, A^\sigma_i(\alpha), \Lambda} \\
\frac{\Gamma, \Lambda, \neg A^\sigma_i(\alpha)}{\Psi, A^\sigma_i(\alpha)} \quad (\neg A^\sigma_i(\alpha), \Psi) \quad (\Sigma_i)^\sigma \\
\frac{\Psi, A^\sigma_i(\alpha)}{u : \Psi} \quad P'
\]

It is easy to see that $o(u; P') < o(u; P)$. For the proviso (lbranch) in $P'$, cf. the case M5.2. To see that $P'$ is a proof with respect to the provisos (forerun), (uplw), cf. the subcase M7.2 below.
M7.2 Otherwise.
Let $K$ denote the lowermost rule $(\Sigma_{i+1})^\sigma$ below or equal to $I$. Then $u_0 : \Psi$ is the lowersequent of $K$ by (h-reg). There exists an $(i + 1)$-knot $(\Sigma_{i+1})^\sigma$ which is between an uppersequent of $I$ and $u_0 : \Psi$. Pick the uppermost such knot $(\Sigma_{j+1})^\sigma K_{-1}$ and let $K_{-1} = J_0, \ldots, J_{n-1}$ denote the left rope of $K_{-1}$. Each $J_p$ is a rule $(c)^{\sigma_p}$ with $\sigma = \sigma_0$. Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \quad (l \geq 0) \quad (11)$$

be the knotting numbers of the left rope $K_{-1} R$ and $K_m$ an $i_m$-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of $J_{n_m}$ and $J_{n_{m+1}}$ for $m < l$. Put

$$m(i + 1) = \max\{m : 0 \leq m \leq l \& \forall p \in [0, m)(i + 1 \leq i_p)\} \quad (12)$$

Then the resolvent $u : \Phi$ is the uppermost sequent $u : \Phi$ below $J_{n_m(i+1)}$ such that

$$h(u; P) < \sigma_{n_m(i+1)+1} + i.$$ 

In the following figure of $P$ the chain $C_{n_{m+1}}$ starting with $J_{n_{m+1}}$ passes through
the left side of $K_m$. 

\[
\begin{array}{c}
\frac{\alpha < \tau, \Lambda_0}{A_{i+1}^\sigma(a), \Lambda_0} J \\
\vdots
\Gamma, \neg A_{i+1}^\sigma \\
\frac{\nu : \Gamma, \Lambda}{(\Sigma_{i+1}^\sigma) I} \\
\vdots
\Gamma_0
\frac{(c)_{\sigma_1} J_0}{\Gamma_0} \\
\vdots
\Gamma_{nm}
\frac{(c)_{\sigma_{nm+1}} J_{nm}}{\Gamma_{nm}} \\
\vdots
\Pi_m, \neg B_m
\frac{B_m : \Delta_m}{(\Sigma_{nm}^\sigma)_{\sigma_{nm+1}} K_m} \\
\vdots
\frac{\Gamma_{nm+1}}{\Gamma_{nm+1}'}
\frac{(c)_{\sigma_{nm+1}} J_{nm+1}}{\Gamma_{nm+1}'} \\
\vdots
\frac{\Gamma_{nm+1}}{\Gamma_{nm+1}'}
\frac{(c)_{\sigma_{nm+1}} J_{nm+1}}{\Gamma_{nm+1}'} \\
\vdots
\frac{\Gamma_{nm(i+1)+1}}{\Gamma_{nm(i+1)+1}'}
\frac{(c)_{\sigma_{nm(i+1)+1}} J_{nm(i+1)+1}}{\Gamma_{nm(i+1)+1}'} \\
\vdots
\frac{\Gamma_1}{\Gamma_1} \\
u : \Phi
\end{array}
\]
Assuming \( \alpha < \tau \) and then \( \alpha < \sigma_n \leq \sigma_{n(m+i)+1} \), let \( P' \) be the following:

\[
\begin{align*}
&\frac{A_i^\tau(\alpha), \Lambda_0}{A_{i+1}^\tau(\alpha)} (w) \\
&\frac{\Gamma, \neg A_{i+1}^\tau}{A_i^\tau(\alpha)} J_i^l \quad \frac{\Gamma, \Lambda, A_i^\tau(\alpha)}{A_i^\tau(\alpha)} J_i^l \\
&\frac{\Gamma_0, A_i^\tau(\alpha)}{\Gamma_0, A_i^\tau(\alpha)} J_i^l \\
&\frac{\Gamma_{n_m+1}, A_i^{\sigma_{m+1}(\alpha)}}{\Gamma_{n_m+1}, A_i^{\sigma_{m+1}(\alpha)}} J_i^l \\
&\frac{\Gamma_{n_m+1}, A_i^{\sigma_{m+1}(\alpha)}}{\Gamma_{n_m+1}, A_i^{\sigma_{m+1}(\alpha)}} J_i^l \\
&\frac{\Gamma_{n_m(i+1)+1}, A_i^{\sigma_{m+1}(\alpha)}}{\Gamma_{n_m(i+1)+1}, A_i^{\sigma_{m+1}(\alpha)}} J_i^l \\
&\Phi, A_i^{\sigma_{m+1}(\alpha)} (\alpha) \quad u : \Phi \\
&\frac{\tau}{\Gamma, \neg A_i^\tau(\alpha)} (w) \\
&\frac{\Gamma, \neg A_i^\tau(\alpha)}{\Gamma, \neg A_i^\tau(\alpha)} \quad \frac{\Gamma, \Lambda, A_i^\tau(\alpha)}{\Gamma, \Lambda, A_i^\tau(\alpha)} \\
&\frac{\Gamma_0, A_i^\tau(\alpha)}{\Gamma_0, A_i^\tau(\alpha)} \\
&\frac{\Gamma_{n_m(i+1)+1}, A_i^{\sigma_{m+1}(\alpha)}}{\Gamma_{n_m(i+1)+1}, A_i^{\sigma_{m+1}(\alpha)}} \\
&\Phi, A_i^{\sigma_{m+1}(\alpha)} (\alpha) \quad u : \Phi
\end{align*}
\]

Here \( I_i \) denotes a \((\Sigma_i)^{\sigma_{n_m(i+1)+1}}\).

It is straightforward to see \( o(u; P') < o(u; P) \). We show \( P' \) is a proof.

First by Lemma \([5.5]\) in \( P \) every chain passing through the resolvent \( u : \Phi \) passes through the right side of \( I \) and, by inversion, the right upper part of \( I \) disappears in \( P' \). Hence the new \((\Sigma_i)^{\sigma_{n_m(i)+1}} \) \( I_i \) does not split any chain. For the proviso (\textit{branch}) in \( P' \), cf. the case \textbf{M5.2}.

\textbf{Claim 6.3} \textit{The proviso (forrun) holds for the lower rule} \( J^l = I_i \) \textit{in} \( P' \).
**Proof** of Claim 6.3. Consider a right branch $\mathcal{T}_r$ of $I_i$. We show that there is no rule $K$ such that $\mathcal{T}_r$ passes through the left side of $K$ and $h(a; P') < \pi$ with the lowersequent $a$ of $K$. The assertion follows from this and (h-reg). The ancestors of the right cut formula $\neg A_i^{\sigma_{m(i)+1}+1}(\alpha)$ of $I_i$ comes from the left cut formula $\neg A_i^{\sigma}+1$ of $I$ in $P$. Let $\mathcal{T}_r^i$ denote the branch in $P'$ from the lowersequent $v' : \neg A_i^{\sigma}(\alpha), \Gamma, \Lambda$ of the new $(w)$ to the right uppersequent $u \ast (1) : \neg A_i^{\sigma_{m(i)+1}+1}(\alpha), \Phi$ of $I_i$. Also let $\mathcal{T}_l$ denote a (the) left branch of $I$ in $P$. There exists a (possibly empty) branch $\mathcal{T}_0$ such that $\mathcal{T}_r = \mathcal{T}_l \neg \mathcal{T}_0 \neg \mathcal{T}_l^i$. By (branch) any left branch $\mathcal{T}_l$ of $I$ is the rightmost one in the left upper part of $I$. Therefore there is no rule $K$ such that $\mathcal{T}_r$ passes through the left side of $K$ and $h(a; P') < \pi$ with the lowersequent $a$ of $K$. □

**Claim 6.4** The proviso (uplwr) holds for the upper rule $J^{up} = I_i$ in $P'$.

**Proof** of Claim 6.3. Suppose that $I_i$ is a knot. Then there exists a chain $C_1$ starting with an $I_1$ such that $C_1$ passes through the left side of $I_i$. This chain comes from a chain in $P$ which passes through $u : \Phi$. Call the latter chain in $P$ $C_1$ again. Further assume that, in $P'$, the left rope $R_1$ of $I_i$ reaches to a rule $(\Sigma_j)^{\kappa}. J^{lw}$ with $i \leq j$. Let $I_2$ denote the lower rule of $I_i$. We have to show $I_i$ foreruns $J^{lw}$. It suffices to show that, in $P$, any right branch $\mathcal{T}$ of $J^{lw}$ passes through the right side of $I$ if the branch $\mathcal{T}$ passes through $u : \Phi$. Since, by inversion, the right upper part of $I$ disappears in $P'$, for such a branch $\mathcal{T}$ there exists a unique branch $\mathcal{T}'$ corresponding to it in $P'$ so that $\mathcal{T}'$ passes through the left side of $I_i$ and hence $\mathcal{T}'$ is left to $I_i$.

\[
\begin{array}{c}
\frac{\Gamma, \neg A_i^{\sigma}+1, \Lambda, A_i^{\sigma}}{\Gamma, \Lambda, (\Sigma_i^{i+1})^\kappa} \quad I
\\
\vdots
\\
\frac{\Gamma_{lw}, \neg C_{lw}}{\Gamma_{lw}, \Lambda_{lw}} \quad J^{lw}
\\
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma, \neg A_i^{\sigma}+1, \Lambda, A_i^{\sigma}(\alpha)}{\Gamma, \Lambda, A_i^{\sigma}(\alpha)} \quad I^l
\\
\vdots
\\
\frac{\Phi, A_i^{\sigma_{m(i)+1}}(\alpha), \neg A_i^{\sigma_{m(i)+1}}(\alpha), \Phi}{u : \Phi} \quad I_i
\\
\vdots
\\
\frac{\Gamma_{lw}, \neg C_{lw}}{\Gamma_{lw}, \Lambda_{lw}} \quad J^{lw}
\\
\end{array}
\]

\[
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\]
Case 1. The case when, in $P$, there exists a member $I_3$ of the chain $C_1$ such that $I_3$ is between $u : \Phi$ and $J_lu$, and the chain $C_3$ starting with $I_3$ passes through the resolvent $u : \Phi$ in $P$. Then by Lemma 5.7 the chain $C_3$ passes through the right side of $I$. The rope $R_{I_3}$ starting with $I_3$ in $P$ corresponds to a part (a tail) of the left rope $I, R$ in $P'$, Thus by the assumption the rope $R_{I_3}$ also reaches to $J_lu$ in $P$. Hence by (forerun) there is no merging rule $K$ such that

1. the chain $C_3$ starting with $I_3$ passes through the right side of $K$, and
2. the right branch $T$ of $J_lu$ passes through the left side of $K$.

Therefore the right branch $T$ of $J_lu$ passes through the right side of $I$ in $P$.

\[
\begin{array}{c}
\Gamma, \neg A_{i+1}^n \ A_{i+1}^n, \Lambda, A_i^0(\alpha) \\
\Gamma, \Lambda, A_i^0(\alpha) \\
\Phi, A_i^{n_{m(i)+1}}(\alpha) \\
\Phi \\
\vdots \Delta_2 \Delta_2 \\
\vdots \Delta_2 I_2 \\
\vdots \Delta_3 \Delta_3 \\
\vdots \Delta_3 I_3 \\
\vdots \Delta_4 \Delta_4 \\
\vdots \Delta_4 I_4 \\
\vdots \Delta_5 \Delta_5 \\
\vdots \Delta_5 I_5 \\
\vdots R_{I_3} \subset I, R \\
\vdots C_{I_{lw}} I_{lw} \\
\vdots C_{I_{lw}} I_{lw} \\
\vdots \neg C_{I_{lw}} I_{lw} \\
\vdots \Gamma_{I_{lw}} A_{I_{lw}} A_{I_{lw}} \\
\vdots \Gamma_{I_{lw}} A_{I_{lw}} A_{I_{lw}} \\
\vdots J_lu \ P' \\
\vdots J_lu \ P'
\end{array}
\]

Case 2. Otherwise: First we show the following claim:

Claim 6.5 In $P$, we have $m(i+1) < l$ for the number of knots $l$ in [1[1], and $I_2$ is the lower rule of the $i_{m(i+1)}$-knot $K_{m(i+1)}$. Let $K_{m(i+1)} R$ denote the left rope of $K_{m(i+1)}$ in $P$. Then $K_{m(i+1)} R$ reaches to $J_lu$.

Proof of Claim 6.5 In $P'$, the lower rule $I_2$ of the knot $I_i$ is a member of the chain $C_1$ starting with $I_1$ and passing through the left side of $I_i$. Further $I_2$ is above $J_lu$ since the left rope $I, R$ of $I_i$ is assumed to reach to $J_lu$ in $P'$, cf. Definition 5.7. Since we are considering when Case 1 is not the case, in $P$, $I_1$ is below $J_lu$ and the chain $C_2$ starting with $I_2$ does not pass through $u : \Phi$, and hence chains $C_1$ and $C_2$ intersect as Type3 (merge) in (ch:link). In other words there is a knot below $u : \Phi$ whose upper right rule is $(c)_{\sigma_{m(i+1)+1}} J_{m(i+1)}$. This means that the knot is the $i_{m(i+1)}$-knot $K_{m(i+1)}$. Thus we have shown that $m(i+1) < l$ and $I_2$ is the lower rule of the $i_{m(i+1)}$-knot $K_{m(i+1)}$.

Next we show that the left rope $K_{m(i+1)} R$ of $K_{m(i+1)}$ reaches to $J_lu$ in $P$. Suppose this is not the case. Let $(c)_{n_i} I_4$ denote the lowest (last) member of the
left rope \( K_{m(i+1)} R \). Then \( \kappa < \kappa_4 \) for the rule \((\Sigma_j)^\kappa J_{lw}\). By \( \kappa < \kappa_4 \), the next member \((c)^{\kappa_4} I_5\) of the chain \( C_1 \) is above \( J_{lw} \). Since we are considering when Case 1 is not the case, the chain \( C_5 \) starting with \( I_5 \) does not pass through \( u : \Phi \). By Definition 5.4.6 of left ropes and (ch:link) there would be a knot \( K' \) whose lower rule is \( I_5 \) and whose upper right rule is \( I_4 \). This is a contradiction since \( I_4 \) is assumed to be the last member of the left rope \( K_{m(i+1)} R \). This shows Claim 6.5.

In the following figure note that \( u : \Phi \) is above \( K_{m(i+1)} \) by (h-reg) and the definition of the resolvent \( u : \Phi \).

\[
\begin{array}{c}
C_2 = C_{m(i+1)+1} \\
\Pi_{m(i+1)}, \neg B_{m(i+1)} \\
B_{m(i+1)}, \Delta_{m(i+1)} \\
(\Sigma_{i_{m(i+1)}})^{\sigma_{m(i+1)}+1} K_{m(i+1)}
\end{array}
\]

\[
\begin{array}{c}
\Delta_2 \\
(c)^{\sigma_{m(i+1)}+1} I_2 \\
\Delta_4 \\
(c)^{\kappa_4} I_4 \\
\Pi, \neg B \quad B, \Delta \\
K'
\end{array}
\]

\[
\begin{array}{c}
\Delta_5 \\
(c)^{\kappa_4} I_5 \\
\Pi_{lw}, \neg C_{lw} \quad C_{lw}, \Lambda_{lw} \\
(\Sigma_{j})^{\kappa} J_{lw} \\
\Delta_1 \\
I_1
\end{array}
\]

By Claim 6.5 (uplwr) and \( i_{m(i+1)} \leq i \leq j, K_{m(i+1)} \) foreruns \( J_{lw} \) in \( P \). Therefore the right branch \( T \) of \( J_{lw} \) is left to \( K_{m(i+1)} \). Also by (h-reg) \( K_{m(i+1)} \) is below \( u : \Phi \). Hence \( T \) does not pass through \( u : \Phi \) in this case. This shows Claim 6.4. In the following figure \( C_2 \) denotes the chain starting with \( I_2 \).
Claim 6.6 The proviso $(\text{uplw})$ holds for the lower rule $J^{lw} = I_i$ in $P'$.

Proof of Claim 6.6 Let $J^{up}$ be a $j$-knot $(\Sigma_j)$ above $I_i$. Let $H_0$ denote the lower rule of $J^{up}$. Assume that the left rope $J^{up} \mathcal{R} = H_0, \ldots, H_{k-1}$ of $J^{up}$ reaches to the rule $I_i$. We show $i < j$
eq \text{even if } J^{up} \text{ is in the right upper part of } I_i. \text{ Consider the corresponding rule } J^{up} \text{ in } P.

Case 1 Either $J^{up}$ is $I$ or between $I$ and $u : \Phi$: If either $J^{up}$ is $I$ or an $i_m$-knot $K_m$ with $m < m(i+1)$, then $i < i + 1 = j$ or $i < i_m = j$ by (12), resp.

Otherwise $J^{up}$ is between $K_{m-1}$ and $J_{m}$ for some $m$ with $0 \leq m \leq m(i+1)$. Then the rule $J^{up}$ is the merging rule of the chain $\mathcal{C}_{n_m}$ starting with $J_{n_m}$ and the chain $\mathcal{C}_{H_0}$ starting with $H_0$ so that $\mathcal{C}_{n_m}$ passes through the right side of $J^{up}$ and $\mathcal{C}_{H_0}$ the left side of $J^{up}$. Hence by (ch:link) Type3 (merge) the rule $H_{k-1}$ is above $J_{n_m}$ and the left rope $H_0 \mathcal{R}$ does not reach to $I_i$. Thus this is not the case.
\[
\begin{align*}
\Pi_{m-1}, \Delta_{m-1} & \quad \Pi_{m-1}, -B_{m-1}, B_{m-1}, \Delta_{m-1} & \quad K_{m-1} \\
\vdots & \quad \vdots & \quad \vdots \\
\Delta, -C & \quad \Delta, C, \Psi & \quad J^\text{up} \\
\Lambda_q & \quad \Lambda_q & \quad H_q \\
\Lambda_{k-1} & \quad \Lambda_{k-1} & \quad H_{k-1} \\
\Gamma_{n_m} & \quad \Gamma_{n_m} & \quad J_{n_m}
\end{align*}
\]

where \(H_q\) denotes the lowermost member of \(H_0\mathcal{R}\) such that the chain \(C_{H_q}\) starting with \(H_q\) passes through the left side of \(J^\text{up}\). By (ch:link) Type3 (merge) the rule \(H_q\) is above \(J^\text{up}\) and so on.

**Case 2** \(J^\text{up}\) is in the right upper part of \(I\): Then the left rope \(H_0\mathcal{R}\) reaches to \(I\). Hence by Lemma 5.4, i.e., by (uplwr) we have \(i < i + 1 < j\).

**Case 3** \(J^\text{up}\) is in the left upper part of \(I\): Then the left rope \(H_0\mathcal{R}\) reaches to \(I\). Hence by (uplwr) we have \(i < i + 1 < j\).

**Case 4** Otherwise: Then there exists a rule \(K\) such that \(J^\text{up}\) is in the left upper part of \(K\) and \(K\) is between \(I\) and \(\Phi\). By (h-reg), (ch:pass) \(K\) is a rule \((\Sigma_\sigma)^\kappa\). The left rope \(H_0\mathcal{R} = H_0, \ldots, H_{k-1}\) reaches to \(K\). Hence by (uplwr) we have

\[p < j\]  \hspace{1cm} (32)

\[
\begin{align*}
\Delta, -C & \quad \Delta, C, \Psi & \quad J^\text{up} \\
\Gamma, \neg A_{i+1}^\sigma & \quad \Gamma, A_{i+1}^\sigma, \Lambda & \quad (\Sigma_{i+1}) I \\
\Gamma_K, \neg D & \quad D, \Lambda_K & \quad (\Sigma_\sigma)^\kappa K \\
\Gamma_K & \quad \Lambda_K & \quad \Phi
\end{align*}
\]

**Case 4.1** \(H_{k-1}\) is below \(K\): Let \(K'\) denote the uppermost knot such that \(K'\) is equal to or below \(K\), and there exists a member of \(H_0\mathcal{R}\) such that the chain
starting with the member passes through the left side of $K'$. Let $H_q$ be the lowermost member of $H_0\mathcal{R}$ such that the chain $C_{H_q}$ starting with $H_q$ passes through the left side of $K'$. If there exists a member of $H_0\mathcal{R}$ such that the chain starting with the member passes through the left side of $K$, then $K'$ is equal to $K$.

$$
\begin{array}{c}
J^{up} \\
\vdots \\
\Gamma_K, \neg D, D, \Lambda_K \\
\vdots \\
\Delta_q, \Lambda_K \\
\vdots \\
H_q
\end{array}
\quad (\Sigma_p)^{\kappa} K = K'
$$

Otherwise $K'$ is below $K$ and it is a knot for the left rope $H_0\mathcal{R}$. Let $H_{q-1}$ denote the lowermost member of $H_0\mathcal{R}$ above $K$. Then $H_{q-1}$ is an upper right rule of the knot $K'$ and $K'$ is a rule $(\Sigma_{p'})^{\kappa}$ with $p' \leq p$ by (h-reg).

$$
\begin{array}{c}
J^{up} \\
\vdots \\
\Delta_{q-1}, H_{q-1}, I \\
\vdots \\
\Gamma_K, \neg D, D, \Lambda_K \\
\vdots \\
\Delta_q, \Lambda_K \\
\vdots \\
H_q
\end{array}
\quad (\Sigma_p)^{\kappa} K
$$

By Lemma 5.1 the uppermost member of $C_{H_q}$ below $K'$ is the lower rule of the knot $K'$. By (h-reg) and Case 1 it suffices to show that the left rope $K'\mathcal{R} = G_0, \ldots, G_{k_0}$ of $K'$ reaches to $I_i$, i.e., to show the last member $G_{k_0}$ is equal to or below the rule $H_{k-1}$. Then we will have $i < p' \leq p < j$.

Let $G_0 = H_{q_0}$, denote the lower rule of $K'$ and $G_{k_1}$ the lowermost member of $K'\mathcal{R}$ such that the chain $C_{G_{k_1}}$ starting with $G_{k_1}$ passes through the left side of $K'$. Then by (ch:link) $G_{k_1}$ is equal to or below $H_q$.

**Case 4.1.1** $G_{k_1} = H_q$: Then $G_{k_0} = H_{k-1}$, i.e., $G_{q_1 - q_0} = H_{q_1}$ for any $q_1$ with
\(q_0 \leq q_1 < k\).

\[
\begin{array}{c}
\mathcal{C}_{H_{q_0}} \\
\Gamma_{K'}, \neg D' \quad D', \Lambda_{K'} \quad K'
\end{array}
\]

\[
\begin{array}{c}
\Gamma_{K'}, \Lambda_{K'} \\
\vdots \\
\Lambda_{q_0} \\
\Lambda_{q_0} \\
\vdots \\
\Lambda_q \\
\Lambda_q \\
\vdots \\
\Lambda_{q+1} \\
\Lambda_{q+1} \\
\vdots \\
\Lambda_{q_1} \\
\Lambda_{q_1}
\end{array}
\]

\[
\begin{array}{c}
H_{q_0} = G_0 \\
H_q = G_{k_1}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}_{H_{q_1}} \\
\Gamma_{K_1}, \neg D_1 \quad D_1, \Lambda_{K_1} \quad K_1
\end{array}
\]

\[
\begin{array}{c}
\Gamma_{K_1}, \Lambda_{K_1} \\
\vdots \\
\Lambda_{q+1} \\
\Lambda_{q+1} \\
\vdots \\
\Lambda_{q_1} \\
\Lambda_{q_1}
\end{array}
\]

\[
\begin{array}{c}
H_{q+1} = G_{q+1-q_0} \\
H_{q_1} = G_{q_1-q_0}
\end{array}
\]

where \(K_1\) is a knot of \(H_{q+1} = G_{q+1-q_0}\) and \(H_q = G_{k_1}\) with \(q + 1 - q_0 = k_1 + 1\).

**Case 4.1.2** Otherwise: Then by (ch:link) \(G_{k_1}\) is already below \(H_{k-1}\).
\[ \Gamma_{mg}, \neg D_{mg}, D_{mg}, \Lambda_{mg} \quad K_{mg} \]
\[ \Gamma_{K'}, \neg D', D', \Lambda_{K'} \quad K' \]
\[ \Lambda_{q_0} \quad \Lambda_{q_0}' \quad H_{q_0} = G_0 \]
\[ \Lambda_q \quad H_q = G_{q-q_0} \]
\[ \Gamma_{K_1}, \neg D_1, D_1, \Lambda_{K_1} \quad K_1 \]
\[ \Lambda_{q+1} \quad \Lambda_{q+1}' \quad H_{q+1} = G_{q+1-q_0} \]
\[ \Lambda_{q_1} \quad \Lambda_{q_1}' \quad H_{q_1} = G_{q_1-q_0} \]
\[ \Lambda_{k-1} \quad \Lambda_{k-1}' \quad H_{k-1} = G_{k-1-q_0} \]
\[ \Lambda_{k+q_0} \quad \Lambda_{k+q_0}' \quad G_{k_1} \]

where \( K_{mg} \) is a merging rule of the chain \( C_{H_q} \) starting with \( H_q \) and the chain \( C_{G_{k_1}} \) starting with \( G_{k_1} \). Since the chain \( C_{H_1} \) starting with the lower rule \( H_{q+1} = G_{q+1-q_0} \) of \( K_1 \) passes through the left side of \( K_1 \), \( G_{k_1} \) is not equal to \( H_{q+1} \) and hence is below \( H_{q+1} \) and so on.

**Case 4.2** \( H_{k-1} \) is above \( K \): Then \( H_{k-1} \) is a rule \((e)\sigma_{n_m(i+1)+1}\) and \( K \) is a rule \((\Sigma_p)^{\sigma_{n_m(i+1)+1}}\). Let \( d : \Gamma_K, \neg D \) denote an uppersequent of \( K \). By (h-reg) and the definition of the sequent \( u : \Phi \) we have \( \sigma_{n_m(i+1)+1} + i \leq h(d; P) \leq \sigma_{n_m(i+1)+1} + p - 1 \). Thus by (42) we get \( i \leq p - 1 < j \).
where the $i_{m(i+1)}$-knot $K_{m(i+1)}$ disappears when $m(i+1) = l$ in \textbf{[12]}. This shows Claim \textbf{6.6}.

\textbf{M8}. $I$ is a $(\Sigma_1)^\sigma$.

This is treated as in the case \textbf{M8} of \textbf{[6]}. Other cases are easy.

This completes a proof of the Main Lemma \textbf{5.1}.

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