SELF-SHRINKERS OF THE MEAN CURVATURE FLOW IN ARBITRARY CODIMENSION

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Abstract. In this paper we study self-similar solutions $M^m \subset \mathbb{R}^n$ of the mean curvature flow in arbitrary codimension. Self-similar curves $\Gamma \subset \mathbb{R}^2$ have been completely classified by Abresch & Langer [AL86] and this result can be applied to curves $\Gamma \subset \mathbb{R}^n$ equally well. A submanifold $M^m \subset \mathbb{R}^n$ is called spherical, if it is contained in a sphere. Obviously, spherical self-shrinkers of the mean curvature flow coincide with minimal submanifolds of the sphere. For hypersurfaces $M^m \subset \mathbb{R}^{m+1}, m \geq 2$, Huisken [Hui90] showed that compact self-shrinkers with positive scalar mean curvature are spheres. We will prove the following extension: A compact self-similar solution $M^m \subset \mathbb{R}^n, m \geq 2$, is spherical, if and only if the mean curvature vector $H$ is non-vanishing and the principal normal $\nu$ is parallel in the normal bundle. We also give a classification of complete noncompact self-shrinkers of that type.

1. Introduction

In this article we study an immersion $F : M \rightarrow \mathbb{R}^n$ of a smooth manifold $M$ of dimension $m$ and codimension $p = n - m$ into euclidean space that satisfies the quasilinear elliptic system

$$H = -F^\perp,$$

where $H$ denotes the mean curvature vector of the immersion and $^\perp$ is the projection onto the normal bundle of $M$.

Solutions of (1) give rise to homothetically shrinking solutions $M_t := F(M, t)$ of the mean curvature flow

$$F : M \times [0, T) \rightarrow \mathbb{R}^n,$$

$$\frac{d}{dt} F(x, t) = H(x, t).$$

Therefore they are often called ”self-shrinkers”. While until recently people were mostly interested in mean curvature flow of hypersurfaces, this non-linear evolution equation is getting more and more attention in higher codimension, see for instance [AS96], [Anc00], [BN99], [CLT02], [TW04], [SW02], [TY02].
Self-similar solutions play an important role in the formation of type-1 singularities of the mean curvature flow. They are characterized by their blow-up behaviour
\[
\sup_{M_t} |A| \leq \frac{c}{T-t},
\]
where \(A\) is the second fundamental form, \(c\) a constant and \(T\) denotes the blow-up time. It has been shown by Huisken [Hui90] that solutions of (2) forming a type-1 singularity can be homothetically rescaled so that any resulting limiting submanifold satisfies (1). Consequently, the classification of type-1 blow-ups is equivalent to the classification of self-shrinkers.

If \(M = \Gamma \subset \mathbb{R}^2\) is a curve, then all solutions of (1) have been classified by Abresch and Langer [AL86]. Except for the straight lines passing through the origin, the curvature \(k\) is positive for all them. In higher codimension the theorem of Abresch and Langer applies as well. To be precise, any self-shrinking curve \(\gamma \subset \mathbb{R}^n\) lies in a flat linear two-space \(E^2 \subset \mathbb{R}^n\) and coincides with one of the Abresch-Langer curves \(\Gamma\) in \(E^2\), because then equation (1) becomes an ODE of order two.

In [Hui90] it was proved that a compact hypersurface \(M^m \subset \mathbb{R}^{m+1}, m \geq 2\) satisfying (1) with positive scalar mean curvature \(H\) is \(S^m(\sqrt{m})\), i.e. a sphere of radius \(\sqrt{m}\). Later this was extended in [Hui93] to noncompact hypersurfaces with \(H > 0\) that appear as type-1 singularities of compact hypersurfaces. There it was shown that the blow-up limit belongs to one of the classes
\[
i) M = \Gamma \times \mathbb{R}^{m-1} \\
ii) M = S^{m+1-q}(\sqrt{m+1-q}) \times \mathbb{R}^{q-1}, q \in \{1, \ldots, m\},
\]
where \(\Gamma\) are the curves found by Abresch and Langer.

If one drops the condition \(H > 0\), then there are new examples, e.g. those found by Angenent [Ang92].

In higher codimension the situation becomes more complicated as the codimension increases. In particular, even under the condition \(|H| > 0\) we get new examples. Indeed, if \(\Gamma_1, \ldots, \Gamma_m\) are Abresch-Langer curves, then
\[
L := \Gamma_1 \times \cdots \times \Gamma_m \subset \mathbb{R}^{2m}
\]
is a self-shrinker with \(|H| > 0\). Moreover, this is a Lagrangian submanifold in \(\mathbb{C}^m\) and since the complex structure \(J\) on \(T\mathbb{C}^m\) induces an isometry between the tangent and the normal bundle of \(L\), these examples have a flat normal bundle. Except for the case \(\Gamma_1, \ldots, \Gamma_m = S^1\) they are not contained in a sphere. Let us also mention thatAnciaux [Anc00] classified equivariant Lagrangian self-shrinkers in \(\mathbb{C}^m\) and that all of them satisfy \(|H| > 0\) [GSSZ05].

On the other hand, one easily observes that spherical self-shrinkers coincide with minimal submanifolds of the sphere and that for spherical solutions the principal normal \(\nu := H/|H|\) is parallel in the normal bundle.

The purpose of this article is to show that the converse holds true as well. We state our main theorem.
Theorem 1.1. Let $M^m \subset \mathbb{R}^n$, $m \geq 2$ be a compact self-shrinker. Then $M$ is spherical, if and only if $H \neq 0$ and $\nabla^\perp \nu = 0$.

Remarks.

- It is trivial that the principal normal of a hypersurface is parallel. So this theorem forms a natural extension of Huisken’s result mentioned above.
- In codimension two we have a torsion 1-form $\tau$ defined by
  \[ \tau(X) := \langle \nabla^\perp X, b \rangle, \]
  where $b$ is the binormal vector $b = J \nu$ ($J$ denoting the complex structure on the normal bundle $NM$). In this case $\nabla^\perp \nu = 0 \iff \tau = 0$.

The idea in Huisken’s original paper [Hui90] was to show that the scaling invariant quantity $|A|^2/H^2$ is constant on a self-shrinking hypersurface. This will not work in our context and in fact the implications would be too strong. Instead, we look at the quantity $|P|^2/H^4$, where $P = \langle H, A \rangle$ is the second fundamental form w.r.t. the mean curvature vector itself. In codimension one this coincides with $|A|^2/H^2$.

One easily constructs many non-trivial examples of noncompact solutions of (1). In fact, any minimal cone $\Lambda \subset \mathbb{R}^k$ is a solution and if $M_1 \subset \mathbb{R}^k$, $M_2 \subset \mathbb{R}^l$ are solutions of (1), then so is $M_1 \times M_2 \subset \mathbb{R}^{k+l}$. In particular, a product of a minimal submanifold $M \subset S^{n-1}$ with a minimal cone $\Lambda \subset \mathbb{R}^k$ gives a noncompact solution of (1) in $\mathbb{R}^{n+k}$ with parallel principal normal $\nu$. Since minimal cones in higher codimension need not be analytic (see e.g. [HL82]), we find non-analytic examples of noncompact self-shrinkers with parallel principal normal. Therefore, the general picture in the noncompact case becomes more subtle. On the other side, minimal cones never form as type-1 singularities of compact submanifolds, since the blow-up must have uniformly bounded curvature. In the complete case we give the following classification.

Theorem 1.2. Let $M^m \subset \mathbb{R}^n$ be a complete connected self-shrinker with $H \neq 0$ and parallel principal normal. Suppose further that $M$ has uniformly bounded geometry, i.e. there exist constants $c_k$ such that $|\langle (\nabla)^k A \rangle| \leq c_k$ holds uniformly on $M$ for any $k \geq 0$. Then $M$ must belong to one of the following classes.

(i) $M^m = \Gamma \times \mathbb{R}^{m-1}$,  
(ii) $M^m = \tilde{M}^r \times \mathbb{R}^{m-r}$. 
Here, $\Gamma$ is one of the curves found by Abresch and Langer and $\tilde{M}^r$ is a complete minimal submanifold of the sphere $S^{n-m+r-1}(\sqrt{r}) \subset \mathbb{R}^{n-m+r}$, where $0 < r = \text{rank}(A') \leq m$ denotes the rank of the principal second fundamental form $A' = \langle \nu, A \rangle$.

Remarks.

- In codimension one, any self-shrinker is analytic and this was exploited in [Hui93] to classify the noncompact solutions with $H \neq 0$. In higher codimension, as pointed out above, this might not be the case. It turns out, analyticity is not needed in the proof of the splitting theorem [Jek]. On the other hand, theorem [Jek] implies that in codimension two any solution of (1) with $H \neq 0$ and parallel principal normal is analytic.
- The uniformly bounded geometry is essential only in the sense that we need to integrate by parts w.r.t. the Gauss kernel and this condition will prevent us from getting boundary terms at infinity. It may actually be replaced by a weaker condition which allows even a certain growth rate at infinity for $|\nabla^k A|$ and is only needed for $k \leq 2$. In addition, completeness and bounded geometry (here, bounded second fundamental form in $M \cap B(0, R)$ suffices for some large ball of radius $R$ centered at the origin) will exclude products of spherical self-shrinkers with minimal cones.
- Any blow-up (possibly noncompact) of a type-1 singularity forming on a compact submanifold will automatically be complete with uniformly bounded geometry, in particular theorem [Jek] may be applied to those blow-up limits.

The organization of the paper is as follows. After recalling the basic geometric structure equations for submanifolds in euclidean space in section two we exploit (1) to derive various elliptic equations for curvature quantities of self-similar solutions, in particular for those with parallel principal normal. This will be done in section three. In section four we give a proof of theorem [Jek] whereas in section five we proceed to classify noncompact solutions as stated in theorem [Jek].

This work has been initiated while I was supported by DFG as a Heisenberg fellow at the Albert Einstein Institute in Golm and the Max Planck Institute in Leipzig and I am indebted to my hosts, Gerhard Huisken and Jürgen Jost. Special thanks go to Guofang Wang for discussions on this subject.

2. Geometry in higher codimension

Let $F : M^m \to \mathbb{R}^n$ be a smooth immersion of a submanifold of codimension $p = n - m$. We let $(x^i)_{i=1,...,m}$ denote local coordinates on $M$ and we will always use cartesian coordinates $(y^a)_{a=1,...,n}$ on $\mathbb{R}^n$. Doubled greek and latin indices are summed from 1 to $m$ resp. from 1 to $n$. In local coordinates the differential $dF$ of $F$ is given by

$$dF = F^a_i \frac{\partial}{\partial y^a} \otimes dx^i,$$
where $F = y(F)$ and $F_i = \frac{\partial F}{\partial x^i}$. The coefficients of the induced metric $g_{ij} \, dx^i \otimes dx^j$ are

$$g_{ij} = \langle F_i, F_j \rangle = g_{\alpha\beta} F_{i}^{\alpha} F_{j}^{\beta},$$

where $g_{\alpha\beta} = \delta_{\alpha\beta}$ is the euclidean metric in cartesian coordinates. As usual, the Christoffel symbols are

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

The second fundamental form $A \in \Gamma(F^{-1}T\mathbb{R}^n \otimes T^*M \otimes T^*M)$ is defined by

$$A := \nabla dF =: A_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j.$$ 

Here and in the following all canonically induced full connections on bundles over $M$ will be denoted by $\nabla$. We will also use the connection on the normal bundle which will be denoted by $\nabla^\perp$.

It is easy to check that in cartesian coordinates on $\mathbb{R}^n$ we have

$$A_{ij}^\alpha = F_{ij} F_{k}^{\alpha} - \Gamma^k_{ij} F_{k}^{\alpha},$$

where $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$. By definition, $A$ is a section in the bundle $F^{-1}T\mathbb{R}^n \otimes T^*M \otimes T^*M$ and it is well known that $A$ is normal, i.e.

$$A \in \Gamma(\mathcal{N} M \otimes T^*M \otimes T^*M),$$

where $\mathcal{N} M$ denotes the normal bundle of $M$.

This means that

$$\langle F_k, A_{ij} \rangle := g_{\alpha\beta} F_{ij}^{\alpha} A_{ij}^{\beta} = 0, \quad \forall i,j,k. \quad (3)$$

The mean curvature vector field $H = H^\alpha \frac{\partial}{\partial y^\alpha}$ is defined by

$$H = g^{ij} A_{ij} = g^{ij} A_{ij}^\alpha \frac{\partial}{\partial y^\alpha}.$$ 

It will be convenient to rise and lower indices using the metric tensors $g_{ij}, g^{ij}$, for instance

$$A^i_j = g^{ki} A_{kj}.$$ 

The Riemannian curvature tensor on the tangent bundle will be denoted by $R_{ijkl}$, whereas the curvature tensor of the normal bundle, considered as a 2-form with values in $\mathcal{N} M \otimes \mathcal{N} M$, has components $R_{ij}^\perp$.

We summarize the equations of Gauß, Codazzi and Ricci in the following proposition.

**Proposition 2.1.** For an immersion $F : M^m \rightarrow \mathbb{R}^n$ holds true

$$R_{ijkl} = \langle A_{ik}, A_{jl} \rangle - \langle A_{il}, A_{jk} \rangle, \quad (\text{Gauß})$$

$$\nabla_i^\perp A_{jk} = \nabla_j^\perp A_{ik}, \quad (\text{Codazzi})$$

$$R_{ij}^\perp = A_{ik} \wedge A^k_j. \quad (\text{Ricci})$$
We define 
\[ P_{ij} := \langle H, A_{ij} \rangle, \quad Q_{ij} := \langle A^k_i, A^j_k \rangle, \quad S_{ijkl} := \langle A_{ij}, A_{kl} \rangle. \]

Then by Gauß' equation the Ricci curvature is given by 
\[ R_{ij} = \text{g}^{kl} R_{ikjl} = P_{ij} - Q_{ij}. \]

Moreover, we have Simons' identity

**Proposition 2.2.**

\[ \nabla_i \nabla_i H = \Delta_i A_{kl} + R_{klij} A^{ij} - R^i_k A_{il} + Q_{il} A^i_k - S_{kilj} A^{ij}. \]

We use the Simons identity to derive an expression for \( \langle A, (\nabla^\perp)^2 H \rangle \).

In a first step we compute

\[ 2\langle A, (\nabla^\perp)^2 H \rangle = \Delta_i |A|^2 - 2|\nabla^\perp A|^2 + 2R_{klij} S^{ijkl} - 2R_{ij} Q^{ij} + 2S_{ij}^i S^{ijkl} \tag{4} \]

For the normal curvature tensor \( R^i_j = A^i_k \wedge A^j_k \) one has

\[ |R^i_j|^2 = |A^i_{mn} A^j_{mn} - A^i_{mn} A^j_{nm}|^2 = 2|Q|^2 - 2S_{ikjl} S^{ijkl}, \]

so that by Gauß' equation (4) becomes

**Proposition 2.3.**

\[ 2\langle A, (\nabla^\perp)^2 H \rangle = \Delta_i |A|^2 - 2|\nabla^\perp A|^2 + 2|S|^2 - 2\langle P, Q \rangle + 2|R^\perp|^2. \]

If the immersion satisfies \( |H| > 0 \), we can define the principal normal

\[ \nu := \frac{H}{|H|}. \]

The principal normal is parallel in the normal bundle, iff

\[ \nabla^\perp_i \nu = 0. \]

This is equivalent to

\[ \nabla^\perp_i H = \nabla_i |H| \nu. \]

3. **Self-similar solutions in arbitrary codimension**

Suppose \( F : M^m \to \mathbb{R}^n \) is a self-shrinker, i.e.

\[ H = -F^\perp. \]

We define

\[ \theta := \frac{1}{2} d|F|^2 \]

and compute

\[ \nabla_i \theta = g_{ij} + \langle F^\perp, A_{ij} \rangle = g_{ij} - P_{ij}. \tag{5} \]
Moreover
\[ \nabla_i^\perp F^\perp = (\nabla_i (F - \theta^k F_k))^\perp = (F_i - \nabla_i \theta^k F_k - \theta^k A_{ik})^\perp , \]
so that
\[ \nabla_i^\perp F^\perp = -\theta^k A_{ik} \]
and
\[ \nabla_i^\perp H = \theta^k A_{ik}. \]

Taking another covariant derivative we derive
\[ \nabla_i^\perp \nabla_j^\perp F^\perp = -\nabla_i^\perp (\theta^k A_{jk}) \]
\[ = - (\nabla_i \theta^k A_{jk} + \theta^k \nabla_i A_{jk})^\perp \]
\[ = -\nabla_i \theta^k A_{jk} - \theta^k \nabla_i^\perp A_{jk} \]
\[ = -A_{ij} - (F^\perp, A^k_{ij}) A_{kj} - \theta^k \nabla_i^\perp A_{ij}, \]
where we have used (5) and the Codazzi equation in the last step. In particular, since \( H = -F^\perp \), we conclude
\[ \nabla_i^\perp \nabla_j^\perp H = A_{ij} - P^k_{ij} A_{kj} + \theta^k \nabla_i^\perp A_{ij} \]
and
\[ \Delta^\perp H - \theta^k \nabla^\perp_k H + P^{ij} A_{ij} - H = 0. \]

From this we derive
\[ \Delta |H|^2 - 2|\nabla^\perp H|^2 - \langle F^\perp, \nabla |H|^2 \rangle + 2|P|^2 - 2|H|^2 = 0. \]

For a self-similar solution we may exploit the Simons identity in proposition 2.2 and (7) to deduce
\[ \Delta^\perp A_{ij} - \theta^k \nabla^\perp_k A_{ij} + Q^k_{ij} A_{kj} + Q^k_{ji} A_{ki} + (R_{ikjl} - S_{ikjl}) A^{kl} - A_{ij} = 0 \]
and
\[ \Delta |A|^2 - 2|\nabla^\perp A|^2 - \langle F^\perp, \nabla |A|^2 \rangle + 2|S|^2 + 2|R^\perp|^2 - 2|A|^2 = 0. \]

We will need the following lemmata.

**Lemma 3.1.** Let \( M \subset \mathbb{R}^n \) be a self-shrinker with \( |H| > 0 \) and parallel principal normal, i.e. \( \nabla^\perp \nu = 0 \). Then we have
\[ P^{ij} A_{ij} = \frac{|P|^2}{|H|^2} \nu, \]
\[ S_{ijkl} P^{ij} P^{kl} = \frac{|P|^4}{|H|^4} \]
and
\[ P^k_{ij} A_{kj} = P^k_{ji} A_{ki}, \]
\[ S_{ikjl} P^{ij} P^{kl} = P^k_{i} P^l_{kj} Q^{ij}. \]
Proof. \( \Delta^\perp H = \Delta |H| \nu, \nabla^\perp_k H = \nabla_k |H| \nu \) and \( \Box \) imply that \( P^{ij} A_{ij} \) is a multiple of \( \nu \). But then
\[
P^{ij} A_{ij} = P^{ij} \langle \nu, A_{ij} \rangle \nu = \frac{|P|^2}{|H|^2} \nu \]
and
\[
S_{ijkl} P^{ij} P^{kl} = \langle A_{ij} P^{ij}, A_{kl} P^{kl} \rangle = \frac{|P|^4}{|H|^2} .
\]
This proves the first two equations. To prove the third we apply Ricci’s equation to \( H \)
\[
0 = \nabla^\perp_i \nabla^\perp_j H - \nabla^\perp_j \nabla^\perp_i H = \langle H, A^k_j \rangle A_{ki} - \langle H, A^k_i \rangle A_{kj} = P^k_i A_{ki} - P^k_i A_{kj} .
\]
It then follows
\[
S_{ijkl} P^{ij} P^{kl} = \langle A_{ik} P^{ij}, A_{jl} P^{kl} \rangle = \langle A^i_k P^i_k, A^j_l P^j_l \rangle = Q_{il} P^i_k P^j_l
\]
which is the last equation in the statement of the lemma. \( \square \)

Note, that the last two equations in lemma 3.1 hold for any submanifold with parallel principal normal since we did not use that \( M \) is a self-shrinker there.

Lemma 3.2. Let \( M \subset \mathbb{R}^n \) have parallel principal normal, then
\[
\frac{4}{|H|^4} (\nabla^\perp H, \nabla^\perp A_{ij}) P^{ij} = \frac{2}{|H|^2} \left( \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right) + \frac{4|P|^2}{|H|^6} \nabla |H|^{||^2} .
\]

Proof. From \( \nabla^\perp \nu = 0 \) we deduce
\[
\langle \nabla^\perp H, \nabla^\perp A_{ij} \rangle P^{ij} = \nabla^k |H| \langle \nu, \nabla^\perp_k A_{ij} \rangle P^{ij} = \nabla^k |H| \nabla^k \left( \langle \nu, A_{ij} \rangle \right) P^{ij} = \nabla^k |H| \nabla^k \left( \frac{P_{ij}}{|H|} \right) P^{ij} = \frac{1}{2|H|} \langle \nabla |H|, \nabla |P|^2 \rangle - \frac{|P|^2}{|H|^2} \nabla |H|^{||^2}
\]
and then the desired equation follows easily. \( \square \)

A straightforward computation shows
\[
\frac{2}{|H|^2} \left( \nabla_i |H| \frac{P_{jk}}{|H|} - |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right)^2 = \frac{2}{|H|^2} \left( \nabla \left( \frac{P}{|H|} \right) \right)^2 - 2 \frac{|P|^2}{|H|^6} \nabla |H|^{||^2} - \frac{2}{|H|} \left( \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right)
\]
and
\[
\frac{2}{|H|^2} \left( \nabla \left( \frac{P}{|H|} \right) \right)^2 = \frac{2|\nabla P|^2}{|H|^4} - 6 \frac{|P|^2}{|H|^6} \nabla |H|^{||^2} - \frac{2}{|H|} \left( \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right)
\]
so that
\[
\frac{2}{|H|^4} \left| \nabla_i |H| \frac{P_{jk}}{|H|} - |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right|^2 \\
= 2 \frac{\nabla P^2}{|H|^4} - 8 \frac{|P|^2}{|H|^6} |\nabla |H||^2 - \frac{4}{|H|} \left\langle \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle 
\]
(12)

Lemma 3.3. Let \( M \subset \mathbb{R}^n \) be a self-shrinker with \(|H| > 0 \) and parallel principal normal \( v \). Then the following equation holds
\[
\Delta \left( \frac{|P|^2}{|H|^4} \right) = \frac{2}{|H|^4} \left| \nabla_i |H| \frac{P_{jk}}{|H|} - |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right|^2 \\
+ \left\langle F^T, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle - \frac{2}{|H|} \left\langle \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle .
\]

Proof. First we use (8) and (10) to compute
\[
\Delta P_{ij} = \langle F^T, \nabla P_{ij} \rangle + 2 \langle \nabla^\perp H, \nabla^\perp A_{ij} \rangle \\
- Q^k_i P_{kj} - Q^k_j P_{ki} + 2(S_{kija} - S_{ijkl})P^{kl} + 2P_{ij}
\]
and then also
\[
\Delta |P|^2 = 2 \nabla P^2 + \langle F^T, \nabla |P|^2 \rangle + 4 \langle \nabla^\perp H, \nabla^\perp A_{ij} \rangle P^{ij} \\
- 4P^k_i P_{kj}Q^{ij} + 4(S_{kija} - S_{ijkl})P^{kl}P^{ij} + 4|P|^2.
\]
Since
\[
\Delta \left( \frac{|P|^2}{|H|^4} \right) = \frac{\Delta |P|^2}{|H|^4} - 2 \frac{|P|^2 \Delta |H|^2}{|H|^6} - \frac{8}{|H|} \left\langle \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle \\
- 8 \frac{|P|^2}{|H|^6} |\nabla |H||^2
\]
this and (11) implies
\[
\Delta \left( \frac{|P|^2}{|H|^4} \right) = \left\langle F^T, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle + 2 \frac{\nabla P^2}{|H|^4} + \frac{4}{|H|^4} \langle \nabla^\perp H, \nabla^\perp A_{ij} \rangle P^{ij} \\
- 4 \frac{|P|^2}{|H|^6} |\nabla^\perp H|^2 - \frac{8}{|H|} \left\langle \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle \\
- 8 \frac{|P|^2}{|H|^6} |\nabla |H||^2 \\
- \frac{4}{|H|^4} \left( P^k_i P_{kj}Q^{ij} + (S_{iija} - S_{ikjl})P^{kl}P^{ij} - \frac{|P|^4}{|H|^4} \right) .
\]
We apply lemma 3.1 to the last term and obtain
\[
\Delta \left( \frac{|P|^2}{|H|^4} \right) = \left\langle F^T, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle + 2 \frac{\nabla P^2}{|H|^4} + \frac{4}{|H|^4} \langle \nabla^\perp H, \nabla^\perp A_{ij} \rangle P^{ij} \\
- 12 \frac{|P|^2}{|H|^6} |\nabla |H||^2 - \frac{8}{|H|} \left\langle \nabla |H|, \nabla \left( \frac{|P|^2}{|H|^4} \right) \right\rangle ,
\]
where we have used $\nabla^\perp H = \nabla |H| \nu + |H| \nabla^\perp \nu = \nabla |H| \nu$ to replace $|\nabla^\perp H|^2$ by $|\nabla |H||^2$. In a next step we apply formula (12) and lemma 3.2 to continue

$$
\Delta \left( \frac{|P|^2}{|H|^2} \right) = 2 \frac{1}{|H|^4} \nabla_i |H| \left( \frac{P_{jk}}{|H|} \right)^2 - \frac{2}{|H|} \left( \nabla |H| \right. \left| \nabla \left( \frac{|P|^2}{|H|^2} \right) \right) \right)\right. \right) - \frac{1}{2} \left( \nabla_i |H| \left( \frac{P_{jk}}{|H|} \right)^2 + \nabla_j |H| \left( \frac{P_{ijk}}{|H|} \right)^2 \right) - \left. |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right) = 0.
$$

Consequently

$$
|P|^2 |\nabla |H||^2 - P_{ij} P^{jk} \nabla_i |H| \nabla_j |H| = 0.
$$

If $\nabla |H| \neq 0$ at some point $p \in M$, then at this point there is only one nonzero eigenvalue of the symmetric tensor $P_{ij}$ and the corresponding eigenvector is $\nabla |H| \nabla |H|$. At this point we have $|P|^2 = (\text{trace}(P))^2 = |H|^4$ which implies that the constant $c$ from above is 1. Since $\nu$ is parallel, equation (9) and $|P|^2 = |H|^4$ imply

$$
\Delta |H| - \langle F^T, \nabla |H| \rangle + |H|^3 - |H| = 0
$$

(13)
and as in [Hui90] partial integration gives

\[(m - 1) \int_M |H| \, d\mu = 0\]

which is impossible for \(m \geq 2\). Hence \(\nabla^\perp H = \nabla |H| \nu = 0\) everywhere on \(M\). From the equation \(\Delta F = H\) one deduces

\[\Delta |F|^2 = 2m + 2 (F^\perp, H) = 2(m - |H|^2),\]

where we used (1) in the last step. Since \(|H|^2\) is constant, the maximum principle implies

\[\Delta |F|^2 = 0 = m - |H|^2.\]  \hspace{1cm} (14)

So \(|F|^2\) is constant, \(\theta = 0\) and \(H = -F\). This proves \(M \subset S^{n-1}(\sqrt{m})\). \(\square\)

5. The complete case

In the complete case we cannot use the maximum principle equally well as in the last section. Instead we exploit integration by parts w.r.t. the Gauß kernel \(\rho(y) := e^{-|y|^2/2}\) on \(\mathbb{R}^n\).

Lemma 5.1. Let \(M \subset \mathbb{R}^n\) be a complete self-shrinker with \(|H| > 0\), parallel principle normal \(\nu\) and with bounded geometry, i.e. \(|(\nabla)^k A|^2 \leq c_k\) for suitable constants \(c_k\) and all \(k \geq 0\). Then the following integral expression holds

\[\int_M \left| \nabla \left( \frac{|P|^2}{|H|^4} \right) \right|^2 |H|^2 \rho \, d\mu + 2 \int_M \frac{|P|^2}{|H|^6} \nabla_i |H| \frac{P_{jk}}{|H|} - |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right|^2 \rho \, d\mu = 0. \]  \hspace{1cm} (15)

Proof. The bounded geometry of \(M\) guarantees that partial integration w.r.t. the Gauß kernel does not yield any boundary terms at infinity. Since \(\nabla_i \rho = -\theta_i \rho\), Lemma 3.3 and partial integration of \((|P|^2/|H|^2) \Delta(|P|^2/|H|^4) \rho \, d\mu\) gives (15). \(\square\)

Corollary 5.2. On any self-shrinker as in Lemma 5.1 we must have

\[\left| \nabla_i |H| \frac{P_{jk}}{|H|} - |H| \nabla_i \left( \frac{P_{jk}}{|H|} \right) \right| = 0,\]

\[\frac{|P|^2}{|H|^4} = \text{const.}\]

The next lemma on Abresch-Langer curves will become important in the proof of theorem 1.2 (see part (ii) below).
Lemma 5.3. Suppose $\Gamma \subset \mathbb{R}^2$ is one of the self-shrinking curves found by Abresch and Langer. There exists a constant $c_{\Gamma} > 0$ such that

$$k e^{-\frac{|F|^2}{2}} = c_{\Gamma},$$

(16)

holds true on all of $\Gamma$. Moreover, the critical values $k_0$ of the curvature function $k$ satisfy

$$k_0 e^{-k_0^2/2} = c_{\Gamma}.$$  
(17)

If $\Gamma_1, \Gamma_2$ are two such curves with $c_{\Gamma_1} = c_{\Gamma_2}$, then up to an euclidean motion $\Gamma_1 = \Gamma_2$. In particular, if $\Gamma_1, \Gamma_2$ are two different self-shrinking curves and $k_1, k_2$ are critical values of the curvature of $\Gamma_1$ resp. of $\Gamma_2$, then $k_1 \neq k_2$.

Proof. Let $\nu$ be the inner unit normal. Then (1) becomes $k = -\langle F, \nu \rangle$ and taking the gradient on both sides gives

$$\nabla_i k = k \theta_i.$$

This implies $\nabla(k e^{-\frac{|F|^2}{2}}) = 0$. It is clear that $c_{\Gamma}$ is positive and hence $k > 0$. Then at a critical point $k_0$ we have $\nabla_i k = \theta_i = 0$ and consequently $k_0 = -\langle F, \nu \rangle = -|F|$. Substituting $|F|$ in (16) gives (17). The remainder of the lemma follows from the fact that equation (16) is a second order ODE. □

Proof of theorem 1.2. Proceeding as in the proof of theorem 1.1 we first conclude that either $\nabla \perp H = 0$ everywhere, or $P$ admits only one non-zero eigenvalue which is $|H|^2$. Let us treat both cases separately.

(i) $\nabla \perp H = 0$ :

There exist several classification results for submanifolds in euclidean space with parallel mean curvature. In particular, under surprisingly similar assumptions on the relation between the mean curvature vector $H$ and the position vector $F$, Yano [Yan71] classified compact submanifolds with parallel mean curvature. It seems, an analogous classification to that of Yano for complete submanifolds with parallel mean curvature does not exist. However, in our special situation this can be established as follows. First observe, that in case $\nabla \perp H = 0$, equation (12) implies

$$\theta_i A_{ij} = 0.$$  
(18)

Moreover, from $\nabla \perp H = 0$ we deduce $\nabla P_{ij} = 0$ and then with equation (14)

$$P_{ij}^j = P_k^k P_{ij}^j.$$  
(19)

Hence, if $\lambda$ is an eigenvalue of $P_{ij}^j$, then either $\lambda = 0$ or $\lambda = 1$. Let us define

$$(P * A)_{ij} := P_k^k A_{kj}.$$  

By Lemma 3.1 $P * A$ is symmetric. We claim

$$\theta^k \nabla_k A_{ij} = 0 \quad \text{and} \quad A_{ij} = P_{ij}^k A_k.$$  
(20)
To prove (20), we exploit (7), (19) and \( \nabla P = 0 \) to get in a first step
\[
\theta^k \nabla_k (P^l A_{ij}) = \theta^k P^l \nabla_k (A_{ij})
\]
\[
= P^l P^m A_{mj} - P^l A_{ij}
\]
\[
= 0.
\] (21)

So it suffices to show \( |A|^2 - |P \ast A|^2 = 0 \). From (7), (19) and (21) we conclude
\[
\theta^k \nabla_k (|A|^2 - |P \ast A|^2) = 2 \theta^k (A_{ij}, \nabla_k A_{ij})
\]
\[
= 2\langle A_{ij}, P^k A_{kj} - A_{ij} \rangle
\]
\[
= 2\langle A_{ij}, P^l P^m A_{mj} - A_{ij} \rangle
\]
\[
= -2 (|A|^2 - |P \ast A|^2).
\] (22)

If \( \theta \) vanishes at a point \( p \in M \), then (22) implies \( A_{ij} = P^l A_{ij} \) and \( |A|^2 = |P \ast A|^2 \) at \( p \). So suppose \( \theta(p) \neq 0 \). Let \( \gamma \) be the integral curve of \( \theta \) with \( \gamma(0) = p \), i.e. \( d\gamma = \theta \). Then along \( \gamma \) we define the function
\[
f(t) := |\theta|^2(\gamma(t))
\]
and obtain
\[
\frac{d}{dt} f = d\gamma(\nabla |\theta|^2) = \theta^k \nabla_k |\theta|^2 = 2\theta^k \theta^l \nabla_k \theta_l.
\]
From (5), \( F^\perp = -H \) and (18) we see
\[
\nabla_i \theta_j = g_{ij} - P_{ij}, \quad \theta^i \nabla_i \theta_j = \theta_j,
\]
so that
\[
\frac{d}{dt} f = 2f.
\] (23)

Consequently
\[
|\theta|^2(\gamma(t)) = |\theta|^2(p)e^{2t} > 0, \forall t
\]
which by the completeness of \( M \) implies that the integral curve \( \gamma \) is regular and well-defined for all \( t \in \mathbb{R} \). Along the same curve \( \gamma \) we now define a new function
\[
\tilde{f}(t) := (|A|^2 - |P \ast A|^2)(\gamma(t))
\]
and from (22) we derive the evolution equation
\[
\frac{d}{dt} \tilde{f} = -2\tilde{f}
\]
and
\[
(|A|^2 - |P \ast A|^2)(\gamma(t)) = (|A|^2 - |P \ast A|^2)(p)e^{-2t}.
\]
This is the typical behaviour of a cone. If
\[
(|A|^2 - |P \ast A|^2)(p) \neq 0,
\]
then \( (|A|^2 - |P \ast A|^2)(\gamma(t)) \) becomes unbounded as \( t \to -\infty \). This contradicts the boundedness of the second fundamental form. So \( |A|^2 = |P \ast A|^2 \) and (20) is valid. The multiplicities of the two eigenvalues of \( P \) are constant on \( M \) since \( \nabla P_{ij} = 0 \). Let \( r \) denote the multiplicity of \( \lambda = 1 \), i.e. \( r \) is
the rank of the second fundamental form \( A^\nu = P/|H| \). We define the two distributions

\[
\mathcal{E}_p M := \{ V \in T_p M : P_i^i V^i = V^i \}, \\
\mathcal{F}_p M := \{ V \in T_p M : P_i^i V^i = 0 \},
\]

so that \( T_p M = \mathcal{E}_p M \oplus \mathcal{F}_p M \). In addition, for the nullspace \( \mathcal{E}_p M \) we have \( \mathcal{E}_p M \subset \ker(\theta) \) since \( (28) \) implies

\[
\theta(V) = \theta_j V^j = \theta_j P_i^j V^i = 0, \quad \forall V \in \mathcal{E}_p M.
\]

Let \( e_p \in \mathcal{E}_p M, f_p \in \mathcal{F}_p M \) and let \( \gamma(t) \) be a curve in \( M \) with \( \gamma(0) = p \). If \( e(t) \) with \( e(0) = e_p \) and \( f(t) \) with \( f(0) = f_p \) are varying along \( \gamma \) by parallel transport we obtain from \( \nabla P = 0 \)

\[
\frac{d}{dt} |Pe - \epsilon|^2 = 0 = \frac{d}{dt} |Pf|^2,
\]

so that \( e(t) \in \mathcal{E}_{\gamma(t)} M \) and \( f(t) \in \mathcal{F}_{\gamma(t)} M \). In particular for vector fields \( f_1, f_2 \in \mathcal{F} M \) and \( e_1, e_2 \in \mathcal{E} M \) we have

\[
\nabla e_1, e_2, [e_1, e_2] \in \mathcal{E} M, \\
\nabla f_1, f_2, [f_1, f_2] \in \mathcal{F} M,
\]

so that \( \mathcal{E} M, \mathcal{F} M \) are invariant under parallel transport and in particular involutive. This means that for each \( p \in M \) there exist two integral leaves \( \mathcal{E}_p, \mathcal{F}_p \) intersecting orthogonally in \( p \). By Lemma 3.1 we have for any normal vector \( V \) perpendicular to \( \nu \)

\[
\langle P, A^V \rangle = 0,
\]

where

\[
A^V := \langle V, A \rangle
\]

is the second fundamental form w.r.t. \( V \). At some point \( p \in M \) let us choose an orthonormal basis \( e_1, \ldots, e_r \) of \( \mathcal{E}_p M \). Then

\[
\text{tr}_\mathcal{E}(A^V) := \sum_{i=1}^r \langle V, A(e_i, e_i) \rangle = \langle P, A^V \rangle = 0,
\]

\( \forall V \in N_p M \) with \( \langle H, V \rangle = 0 \). (27)

From (24) we conclude that each integral leaf of \( \mathcal{E} M \) is an \( r \) - dimensional submanifold of the sphere \( S^{n-1}(\rho) \subset \mathbb{R}^n \) of radius \( \rho = |F| \), the radius depending on the leaf (see figure [1]). We claim that the leaves \( \mathcal{F}_p \) are \( (m-r) \) - dimensional affine subspaces of \( \mathbb{R}^n \) and that these affine subspaces are parallel for any \( p, p' \) in the same connected component of \( M \). To see this, let \( q \in \mathcal{F}_p \) be arbitrary and note that \( A_{ij} = P_i^k A_{kj} \) implies

\[
A(f, X) = 0, \quad \forall f \in \mathcal{F}_q M = T_q \mathcal{F}_p, \ X \in T_q M.
\]

The normal space of \( \mathcal{F}_p \) at \( q \) decomposes into

\[
N_q \mathcal{F}_p = N_q M \oplus \mathcal{E}_q M.
\]

By (28) the second fundamental form of \( \mathcal{F}_p \) vanishes w.r.t. any normal vector belonging to \( N_q M \) and by (26) also for all normal vectors belonging to \( \mathcal{E}_q M \). This shows that the integral leaves of \( \mathcal{F} M \) are affine subspaces of \( \mathbb{R}^n \) of dimension \( m-r \). Now we fix a point \( p \in M \) and suppose that \( p' \in \mathcal{E}_p \).
is arbitrary. Let $\gamma \subset \mathcal{E}_p$ be any smooth curve with $\gamma(0) = p$ and $\gamma(1) = p'$. Suppose $f_p \in \mathcal{F}_p$ is some vector and $f(t)$ is varying along $\gamma$ by parallel transport with $f(0) = f_p$. Considering $f(t)$ as a vector field along $\gamma$ in $\mathbb{R}^n$ we compute $\frac{d}{dt} f = \nabla_{\dot{\gamma}} f + (\frac{d}{dt} f)^\perp = (\frac{d}{dt} f)^\perp = A(\dot{\gamma}, f)$. But then by (28) $\frac{d}{dt} f = 0$. Since $\mathcal{F}M$ is invariant under parallel transport we conclude that $f_p \in \mathcal{F}_p M$ as well and since $f$ was arbitrary we must have $\mathcal{F}_p M = \mathcal{F}_{p'} M$. To see that this relation holds for all $p, p'$ in the same connected component of $M$ we proceed as follows. The leaf $\mathcal{E}_p$ is contained in a linear subspace of dimension $n - m + r$ which is orthogonal to $\mathcal{F}_p M$. Then we define an embedding

$$\tilde{F} : \mathcal{E}_p \times \mathbb{R}^{m-r} \to \mathbb{R}^n,$$

$$\tilde{F}(p', X) = (p', X).$$

Since the integral leaves of $\mathcal{F}M$ through $p' \in \mathcal{E}_p$ coincide with the affine subspace w.r.t. $\mathcal{F}_p M$ through $p'$, the $m$-dimensional image of $\mathcal{E}_p \times \mathbb{R}^{m-r}$ under $\tilde{F}$ is contained in the $m$-dimensional submanifold $M$ and by the completeness of both $\mathcal{E}_p$ and $M$ we conclude that $\tilde{F}(\mathcal{E}_p \times \mathbb{R}^{m-r})$ is a connected component of $M$. Since the integral leaves of $\mathcal{E}M$ are contained in spheres of radii $\rho = |F|$ and

$$|F|^2 = |F^\perp|^2 + |\theta|^2 = |H|^2 + |\theta|^2$$

we observe that $|\theta|^2$ must be constant along the leaves of $\mathcal{E}M$ since $H$ is parallel and hence $|H|$ is constant. Moreover, from (24) we conclude that the projection $\pi : M \to \mathcal{E}_p$ maps the points $q$ in the integral leaves of $\mathcal{E}M$ with $|q| = |F|$ to points $\tilde{q} \in \mathcal{E}_p$ with $|\tilde{q}|_{\mathcal{E}_p} = |F^\perp| = |H|$ and since $H = -F^\perp$ we see that $\tilde{M} := \mathcal{E}_p$ must be a minimal submanifold of the sphere $S^{n-m-1+r}(|H|)$. On the other hand $|H|^2 = g^{ij} P_{ij} = r$ since all eigenvalues of $P$ are either 0 or 1 and $r$ is the multiplicity of the latter. Thus $|H| = \sqrt{r}$ and each connected component of $M$ is isometric to $\tilde{M} \times \mathbb{R}^{m-r}$ with a minimal submanifold $\tilde{M} \subset S^{n-m-1+r}(|H|)$.

(ii) The remaining case is $\nabla^2 H(p_0) \neq 0$ at a point $p_0 \in M$. Then, as in the compact case, we find $|P|^2 = |H|^4$ globally. In addition, $P$ admits only one non-zero eigenvalue $\lambda = |H|^2$ and $\frac{\nabla |H|}{|\nabla |H||}$ spans the eigenspace. Then Lemma (3.3) implies the relations

$$P_i^k A_{kj} = |H| P_{ij} \nu, \quad P_i^j \nabla^i |H| = |H|^2 \nabla^j |H|. \quad (29)$$

Let us define

$$\tilde{M} := \{ p \in M : \nabla H(p) \neq 0 \}.$$

Following Huisken’s [Huis93] argument closely, we choose a connected component $U \subset \tilde{M}$ and consider the distributions

$$\mathcal{E}_{\tilde{M}} U := \{ X \in T_p U : p \in U, P X = |H|^2 X \}, \quad \mathcal{F}_{\tilde{M}} U := \{ X \in T_p U : p \in U, P X = 0 \}.$$

In contrast to part (i), these distributions are defined a-priori only on $\tilde{M}$. As in the first part, they are invariant under parallel transport and again (28), (29) are valid for them so that we may apply Frobenius’ theorem.
We claim that the integral leaves of $\mathcal{F}U$ are affine subspaces of dimension $m - 1$. This is equivalent to showing that

$$\hat{A}_{ij} := A_{ij} - \frac{1}{|H|} P_{ij} \nu$$

vanishes completely. From (6) we observe

$$|H| \nabla_i |H| = \theta^k P_{ki}$$

and then with (29)

$$\theta (\nabla |H|) = \frac{|\nabla |H||^2}{|H|}.$$

Let $\hat{\theta}$ be the projection of $\theta$ onto $\mathcal{F}U$, i.e.

$$\hat{\theta}_i = \theta_i - \frac{\theta (\nabla |H|)}{|\nabla |H||^2} \nabla_i |H| = \theta_i - \frac{1}{|H|} \nabla_i |H|.$$
A straightforward computation shows
\[
\tilde{\theta}^k A_{ki} = \left( \theta^k - \frac{1}{|H|} \nabla^k|H| \right) \left( A_{ki} - \frac{1}{|H|} P_{ki} \nu \right)
\]
\[\equiv \theta^k A_{ki} - \nabla_i|H|\nu - \frac{1}{|H|} \nabla^k|H|A_{ki} \]
\[+ \frac{1}{|H|^2} \nabla^k|H|P_{ki} \nu \]
\[\equiv \theta^k A_{ki} - \frac{1}{|H|^3} \nabla^j|H|P^k_j A_{ki} \]
\[\equiv \theta^k A_{ki} - \frac{1}{|H|^2} \nabla^j|H|P_{ij} \nu \]
\[\equiv \theta^k A_{ki} - \nabla_i|H|\nu \]
\[= 0. \tag{31}\]

Then
\[
0 = \nabla_i^\perp \left( \tilde{\theta}^k A_{ki} \right)
\]
\[\equiv \dot{\tilde{A}}_{ki} - P^k_i \dot{\tilde{A}}_{ki} + \frac{1}{|H|^2} \nabla_i|H|\nabla^k|H| \dot{\tilde{A}}_{ki} 
\]
\[\equiv \frac{1}{|H|} \nabla_i|H| \dot{A}_{ki} + \tilde{\theta}^k \nabla_i^\perp \dot{A}_{ki} \]
\[\equiv \dot{\tilde{A}}_{ki} - \frac{1}{|H|} \nabla_i|H| \dot{A}_{ki} + \tilde{\theta}^k \nabla_i^\perp \dot{A}_{ki}. \tag{32}\]

Since $\mathcal{F}U$ is invariant under parallel transport, we have
\[
\nabla f_1 \nabla f_2 |H| = f_1 \left( \langle \nabla |H|, f_2 \rangle \right) - \langle \nabla |H|, \nabla f_1, f_2 \rangle = 0
\]
for all $f_1, f_2 \in \mathcal{F}U$. Therefore we may write
\[
\nabla_i \nabla_j |H| = \frac{\Delta |H|}{|H|^2} P_{ij}
\]
and since (29) implies $P^k_i \dot{A}_{kj} = 0$, we get from the formula above
\[
\tilde{\theta}^k \nabla_i^\perp \dot{A}_{ki} = -\dot{\tilde{A}}_{ii}, \tag{32}\]
in particular both sides in (32) are symmetric in $i$ and $j$. From (25), $\nabla |H|(p) \in \mathcal{E}_p U$, $\tilde{\theta}(p) \in \mathcal{F}^*_p U$ and since $\mathcal{F}U$ is invariant under parallel transport we derive
\[
\eta \left( \nabla |\tilde{\theta}^2| \right) = 2\langle \eta, \tilde{\theta} \rangle, \quad \forall \eta \in \mathcal{F}^*_p U. \tag{33}\]
We may now proceed similarly as in part (i). If $\tilde{\theta}(p) = 0$, then (32) implies $\dot{\tilde{A}} = 0$. If $\tilde{\theta}(p) \neq 0$, we choose $\eta = \tilde{\theta}$ in (33) and consider the integral curve $\gamma$ w.r.t. $\tilde{\theta}$ starting at $p$ which as before proves that the integral curve is regular and contained in $U$ for all $t \in \mathbb{R}$. Applying formula (32) to $\gamma$ we get exponential growth in $t$ for the quantity $|\dot{\tilde{A}}|^2$ if $|\tilde{A}(p)| \neq 0$ which by the boundedness of the second fundamental form would be a contradiction. Therefore $\tilde{A} = 0$ everywhere and as in the first part we obtain for some fixed point $p \in U$ that the integral leaf $\mathcal{E}_p U$ is part of an
Abresch-Langer curve $\Gamma$ that extends smoothly up to the boundary of $U$ and that all integral leaves $\mathcal{F}_q U$ for $q \in U$ are parallel affine subspaces. So as above any connected component $U \subset \tilde{M}$ is a product of a part of an Abresch-Langer curve and some $R^{m-1}$. To extend this to all of $M$ we have to consider the points, where $\nabla |H| = 0$. We claim that the critical points of $|H|$ are transversally isolated along $\partial U$. By this we mean the following. The integral leaves $\mathcal{F}_p U$ of $U$ are geodesics in $M$ and by the completeness can be extended arbitrarily. So we obtain a complete geodesic $\gamma \subset M$ that coincides in $U$ with a portion of an Abresch-Langer curve $\Gamma = \mathcal{F}_p U$ and which intersects the boundary of $U$ orthogonally. Suppose the fraction of $\gamma \subset U$ is parametrized by $\gamma : (0, t_0) \to U$ and that $\lim_{t \to t_0} \nabla |H| (\gamma (t)) = 0$.

Then (8) implies
\[ \lim_{t \to t_0} \Delta |H| = |H| - |H|^3 \] (34)
On the other hand $|H| (\gamma (t))$ coincides in $U$ with the curvature $k$ of the Abresch-Langer curve $\Gamma$ and $\lim_{t \to t_0} k(t) = k(t_0) = k_0$ is a critical value of the curvature of $\Gamma$. Since $k_0 = 1$ is the critical value of the round circle, lemma 23 and 34 would imply that, if $\Delta |H| (t_0) = 0$, then $\Gamma$ is a fraction of $S^1$ and $\nabla k = 0 = \nabla |H|$ on all of $\Gamma$. Therefore $k_0 \neq 1$ and $\Delta |H| (t_0) \neq 0$. This implies that $\nabla |H| \neq 0$ immediately after passing $\gamma (t_0)$ in direction of $\gamma$. In particular, the boundary of each connected component $U \subset \tilde{M}$ consists of at most two affine subspaces of dimension $m-1$ that are parallel to each of the leaves $\mathcal{F}_q U$, $q \in U$. If $U_1, U_2$ are two connected components with $\tilde{U}_1 \cap \tilde{U}_2 \neq \emptyset$, then the integral leaves $\mathcal{F}_{q_1} U_1$, $\mathcal{F}_{q_2} U_2$ are parallel. We may thus join two such connected components $U_1, U_2$ along their common boundary and obtain two portions of Abresch-Langer curves $\Gamma_1, \Gamma_2$ that both intersect the common boundary of $U_1, U_2$ orthogonally and which meet at a boundary point, for instance at $\gamma (t_0)$. If $\Gamma_1$ and $\Gamma_2$ would be portions of different Abresch-Langer curves, then again by lemma 33 we get a contradiction, since then the critical values of the two curvature functions are different but both satisfy (34). By the completeness of $M$ we may extend this process until we get $M = \Gamma \times \mathbb{R}^{m-1}$.

□

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