A duality theorem for generalized Koszul algebras

Roberto Martínez Villa  
Instituto de Matemáticas, UNAM, AP 61-3  
58089 Morelia, Michoacán  
MEXICO  
mvilla@matmor.unam.mx

Manuel Saorín  
Departamento de Matemáticas  
Universidad de Murcia, Aptdo. 4021  
30100 Espinardo, Murcia  
SPAIN  
msaorinc@um.es

Abstract
We show that if \( \Lambda \) is a \( n \)-Koszul algebra and \( E = E(\Lambda) \) is its Yoneda algebra, then there is a full subcategory \( \mathcal{L}_E \) of the category \( \text{Gr}_E \) of graded \( E \)-modules, which contains all the graded \( E \)-modules presented in even degrees, that embeds fully faithfully in the category \( C(\text{Gr}_\Lambda) \) of cochain complexes of graded \( \Lambda \)-modules. That extends the known equivalence, for \( \Lambda \) Koszul (i.e. \( n = 2 \)), between \( \text{Gr}_E \) and the category of linear complexes of graded \( \Lambda \)-modules.

1 Introduction

From the classification of coherent sheaves over projective spaces by Bernstein, Gelfand and Gelfand (see [5] or [6]), Koszul algebras have deserved a lot of attention. A systematic treatment of them was given in [1], where the authors showed the existence of an equivalence of categories between large subcategories of the graded derived categories of a Koszul algebra \( \Lambda \) and its (quadratic) dual \( \Lambda^! \). They showed in addition that \( \Lambda^! \) is also Koszul and canonically isomorphic to the Yoneda algebra \( E = E(\Lambda) \) of \( \Lambda \). Recently (cf. [2]), the authors of

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the present paper showed that there is an abelian version of the mentioned equivalences, valid for more general graded algebras. Namely, after defining $A^!$ for an arbitrary positively graded algebra $A$, we showed that there is an equivalence between the category $Gr_A$ of graded $A$-modules and the category $LC_A$ (resp. $LC_A^!$) of linear complexes of projective (resp. almost injective) graded $A$-modules. In case $A$ is Koszul, the equivalence of [1] can be obtained from that by derivation.

As a generalization of Koszul algebras, Berger ([2]) introduced $n$-Koszul algebras, where $n > 1$ is an integer. They form a class of $n$-homogeneous algebras which includes all Artin-Schelter regular algebras of global dimension $\leq 3$ and, in case $n = 2$, coincides with the class of Koszul algebras. If $A$ is $n$-Koszul, then one also has a ($n$-homogeneous) dual algebra, still denoted $A^!$, from which the Yoneda algebra $E = E(A)$ can be obtained by killing supports and appropriate regrading on $A^!$ (see [7]). However, the existence, in the flavour of [9], of an equivalence of categories between reasonably large subcategories of $Gr_E$ and of the category $C(Gr_A)$ of cochain complexes of graded $A$-modules is still unknown. The same lack of knowledge can be applied to the existence, in the flavour of [1], of equivalences between reasonably large subcategories of the derived categories of graded modules over $E$ and $A$. Such an equivalence exists, however, in the context of $A_\infty$-algebras (cf. [8]).

The aim of this paper is to present, for every $n$-Koszul algebra $A$ and each integer $m$, an equivalence between a full subcategory $L_E$ of $Gr_E$ containing all the graded $E$-modules presented in even degrees and a full subcategory $\mathcal{Y}$ (depending on $m$) of $C(Gr_A)$ (cf. Corollary 4.6). Since $E = A^!_n$ as ungraded algebras, where $U = n\mathbb{Z} \cup (n\mathbb{Z} + 1)$, our strategy consists in showing an equivalence of categories between suitable subcategories of $Gr_{A^!_n}$ and $C(Gr_A)$ (cf. Theorem 4.3), which is valid for any graded algebra with relations of degree $\geq n$ and from which the desired result follows easily. The proof of last theorem is based on some consequences of the results in [9] (see Section 2), on equivalences between subcategories of $Gr_{A^!_n}$ and $Gr_{A^!_U}$ which were given in [10] for arbitrary group-graded algebras, and on the transport of a torsion theory from $Gr_{A^!_U}$ to the category $nLC_A$ of linear $n$-complexes of almost injective graded $A$-modules (see Section 3).

In this paper we borrow the terminology from [9], with some concrete adaptations. So the term positively graded algebra will stand for a $\mathbb{Z}$-graded algebra $A = \oplus_{i \in \mathbb{Z}} A_i$ such that $A_i = 0$, for all $i < 0$, $A_0$ is isomorphic to a direct product of copies of the ground field $K$ and $\dim_K A_1 < \infty$. The category of its right (resp. left) graded modules is denoted by $Gr_A$ (resp. $AGr$) and the full subcategory of locally finite graded modules is denoted $lfgr_A$ (resp. $Alfgr$). Unless explicitly said otherwise, ‘module’ will mean ‘right module’. Notice that, in our situation, the graded Jacobson radical of $A$ is $J^{gr}(A) = \oplus_{i > 0} A_i$ and an object $T \in Gr_A$ is semisimple if, and only if, $T J^{gr}(A) = 0$. That allows, in particular, to identify the category $Mod_{A^!_n}$ with that of semisimple graded $A$-modules concentrated in degree 0. If $M \in Gr_A$ then the graded socle $Soc^{gr}(M)$, which is by definition the largest graded semisimple submodule
of $M$, is given by $\text{Soc}^{gr}_n(M) = \{x \in M : xJ^n(A) = 0\}$. If $X \subseteq \mathbb{Z}$ then we shall say that $M$ is generated (resp. cogenerated) in degrees belonging to $X$ in case every nonzero factor (resp. subobject) of $M$ in $Gr_A$ has a support which intersects $X$ nontrivially. Also, we shall say that $M$ is presented in degrees belonging to $X$ in case it is the cokernel of a morphism in $Gr_A$ between projective objects generated in degrees belonging to $X$. Notice that if $j \in \mathbb{Z}$ and $M$ is cogenerated in degree $j$, then $\text{Soc}^{gr}_n(M) = M_j$. Recall also that the projective graded modules (i.e. the projective objects of $Gr_A$) are those in $\text{Add}(\oplus_{k \in \mathbb{Z}} A[k])$, while the almost injective graded modules are those in $\text{Add}(\oplus_{k \in \mathbb{Z}} D(A[k]))$, where $D = \text{Hom}_{A_0}(-, A_0) : Gr \rightarrow Gr_A$ is the canonical contravariant functor, which induces by restriction a duality $\text{Add} \xrightarrow{\simeq} \text{Add}$ (see \[9\] for details).

We have a finite quiver $Q$ associated to $A$, uniquely determined by the existence of isomorphisms $KQ_0 \cong A_0$ and $KQ_1 \cong A_1$ (of $K$-algebras and $KQ_0$-$KQ_0$-bimodules, respectively), together with a structural homomorphism of graded algebras $\pi_A : KQ \rightarrow A$ whose image is the (graded) subalgebra $\bar{A}$ of $A$ generated by $A_0 \oplus A_1$. When $\bar{A} = A$ (i.e. $\pi_A$ is surjective), we say that $A$ is a graded factor of a path algebra. We shall use the letter $\Lambda$ instead of $A$ when we want to emphasize that the algebra is a graded factor of a path algebras. All tensors $\otimes$ in the paper are tensor over $KQ_0 = A_0$. In general, if $I = \text{Ker}(\pi_A)$ then $I_k = \{x \in I : x$ is homogeneous of degree $k\}$ is an $A_0 - A_0$-sub-bimodule of $KQ$, for every $k \geq 0$. We can consider the orthogonal $I^\perp_k \subseteq KQ_n$ (for our fixed $n \geq 2$) with respect to the canonical duality $KQ_n^\vee \otimes KQ_n \rightarrow KQ_0 = A_0$ (see \[9\]). The graded algebra $A^I = KQ^\vee / \langle I^- \rangle$ will be called the n-homogeneous dual algebra of $A$ and we also put $A^I = (A^I)^\vee$. If $\Lambda = KQ/I$ is a graded factor of a path algebra, we shall say that it is n-homogeneous (resp. has relations of degree $\geq n$) when $I$ is generated by $I_n$ (resp. $\prod_{k \geq n} I_k$).

2. Graded modules versus n-complexes

In this section we extend the equivalences of \[9\] from (2)-complexes to $n$-complexes. Recall that we have a canonical $\mathbb{Z} \times \mathbb{Z}$-grading on $A[X]$ by putting $A[X]_{(i,j)} = A_iX^j$, in case $i,j \geq 0$, and $A[X]_{(i,j)} = 0$ otherwise. We refer to that paper to see the interpretation of the objects of $Gr_{A[X]}$ as pairs $(P, d)$, where $P \in Gr_{A^I}$ and $d : P \rightarrow P$ is a morphism in $Gr_{A^I}$ of degree +1. We then denote by $n\mathcal{LC}_A$ (resp. $n\mathcal{LC}^*_A$) the full subcategory of $Gr_{A[X]}$ consisting of those pair $(P, d)$ (resp. $(I, d)$) satisfying the following two conditions:

1. $(P, d)$ (resp. $(I, d)$) is a $n$-complex (i.e. $d^{k+n-1} \circ \ldots \circ d^{k+1} \circ d^k = 0$, for all $k \in \mathbb{Z}$)

2. $P^k$ (resp. $I^k$) is projective and generated in degree $-k$ (resp. almost injective and cogenerated in degree $-k$), for every $k \in \mathbb{Z}$

The objects of $n\mathcal{LC}_A$ (resp. $n\mathcal{LC}^*_A$) will be called linear $n$-complexes of projective (resp. almost injective) graded $A$-modules. We shall denote
by \( n \mathcal{L}_A \) (resp. \( n \mathcal{L}_c^A \)) the full subcategory of \( n \mathcal{L}_A \) (resp. \( n \mathcal{L}_c^A \)) consisting of those \((P, d)\) (resp. \((I, d)\)) such that \( P^k \) (resp. \( I^k \)) is finitely generated (resp. finitely cogenerated), for all \( k \in \mathbb{Z} \). We then consider the fully faithful embeddings \( \Psi_A, \nu_A : KQ Gr \rightarrow Gr_{A[X]} \) given in [9][Theorems 2.4 and 2.10]. We refer to that paper for their explicit definition, which we shall freely use here.

**Proposition 2.1.** Let \( A = \bigoplus_{i \geq 0} A_i \) be a positively graded algebra with quiver \( Q \) and \( A^! \) be its \( n \)-homogeneous dual. Then \( \Psi = \Psi_A : KQ Gr \rightarrow Gr_{A[X]} \) induces by restriction equivalences \( 1_A Gr = Gr_{A^!} \xrightarrow{\cong} n \mathcal{L}_A \) and \( \iota_A \text{fg} = \iota_{fgA^!} = \xrightarrow{\cong} n \mathcal{L}_A \).

**Proof.** By the proof of [9][Theorem 2.4], we know that \( \Psi \) establishes an equivalence of categories \( KQ Gr \xrightarrow{\cong} LG_A \), where \( LG_A \) is the full subcategory of \( Gr_{A[X]} \) consisting of those objects \((P, d)\) such that \( P^k \) is a projective object of \( Gr_A \) generated in degree \( -k \), for all \( k \in \mathbb{Z} \). We only need to prove that if \( M \in KQ Gr = Gr_{KQ^{op}} \) and \( \Psi(M) = (P, d) \), then \( M \cdot I_n^\perp = 0 \) if, and only if, \((P, d)\) is a \( n \)-complex.

We consider the canonical \( K \)-algebra homomorphism \( \pi_A : KQ \rightarrow A \) whose kernel is \( I \). It is convenient in the rest of this proof to view graded left \( KQ \)-modules as graded right \( KQ^{op} \)-modules. On one hand, \( \psi(M) \) is a cochain \( n \)-complex iff the composition \( M_k \otimes A[k] \xrightarrow{d^k} M_{k+1} \otimes A[k+1] \xrightarrow{d^{k+1}} \cdots \rightarrow \rightarrow M_{k+n} \otimes A[k+n] \) is zero, for each fixed \( k \in \mathbb{Z} \). But that is equivalent to say that \( d^k \circ \cdots \circ d^1 \) vanish on \( M_k \cong M_k \otimes A_0 \). Direct calculation shows that \((d^{k+n-1} \circ \cdots \circ d^1)(x) = \sum_{p \in Q_n} x p^0 \otimes \bar{p} \) for all \( x \in M_k \), where \( \bar{p} = \pi_A(p) \). Our goal is to show that this latter sum is zero for all \( x \in M_k \) iff \( M_k \cdot I_n^\perp = 0 \). To do that we consider an ordering of \( Q_n = \{p_1, \ldots, p_r, \ldots, p_{r+s+1}\} \), where i) \( \{p_1, \ldots, p_r\} \) is a basis of \( KQ_n \) modulo \( I \); ii) \( \{p_{r+1}, \ldots, p_{r+s}\} \) gathers the remaining \( p \in Q_n \) which do not belong to \( I \); iii) \( \{p_{r+s+1}, \ldots, p_{r+s+t}\} \) belongs to \( I \), for \( j = r+1, \ldots, r+s+1 \), which can be taken with the property that \( \lambda_{ij} \neq 0 \) implies that \( p_i \) and \( p_j \) share origin and terminus. Those linear combinations together with the \( p_j \), with \( j = r + s + 1, \ldots, r + s + t \), form a basis of \( I_n \).

By canonical methods of Linear Algebra, a basis of \( I_n^\perp \) is then given by the elements of the form \( h = p_i^0 + \lambda_{i+1, i} p_{i+1}^0 + \ldots + \lambda_{i+r+s, i} p_{i+r+s}^0 + \lambda_{i+1, i} p_{i+1}^1 + \ldots + \lambda_{i+r+s+1, i} p_{i+r+s+1}^1 \) (i = 1, \ldots, r). Now we have \((d^{k+n-1} \circ \cdots \circ d^1)(x) = \sum_{1 \leq i \leq r+s+t} x p_i^0 \otimes \bar{p}_i = \sum_{1 \leq i \leq r} x p_i^0 \otimes \bar{p}_i + \sum_{r+1 \leq i \leq r+s+t} x p_i^0 \otimes (\sum_{1 \leq i \leq r} \lambda_{ij} \bar{p}_i) \) (notice that, for \( r + s + 1 \leq i \leq r + s + t \), the summand \( x p_i^0 \otimes \bar{p}_i \) is zero because \( p_i \) belongs to \( I \)). We can write the last summatory as \( \sum_{1 \leq i \leq r} x p_i^0 \otimes \bar{p}_i = \sum_{1 \leq i \leq r} x h_i \otimes \bar{p}_i \). The fact that \( \{\bar{p}_i : \ i = 1, \ldots, r\} \) is a \( K \)-linearly independent subset of \( A_0 \) easily implies that the last summatory is zero in \( M_{k+n} \otimes A_n \) if \( x h_i = 0 \) for \( i = 1, \ldots, r \). This is equivalent to say that \( M_k \cdot I_n^\perp = 0 \) and we are done.

If the above result completes [9][Theorem 2.4], the following one completes [9][Theorem 2.10]:

**Proposition 2.2.** Let \( A = \bigoplus_{i \geq 0} A_i \) be a locally finite positively graded algebra with quiver \( Q \) and \( A^! \) be its \( n \)-homogeneous dual. Then \( v = v_A : KQ Gr \rightarrow \)
Gr}_{A[X]} induces by restriction equivalences \( \cdot \mathcal{L}C_A \) and \( \cdot \mathfrak{f}gr = lfg r_{A'} \Rightarrow_n \mathcal{L}C_A^* \).

**Proof.** Let \( \pi : KQ \to A \) be the canonical homomorphism of graded algebras whose kernel is \( I \). Again, we view the graded left \( KQ \)-module as right \( KQ^{op} \)-modules. We need to prove that if \( N \in \mathcal{G}r_{KQ^{op}} \) then \( \nu(N) \) is a cochain n-complex iff \( N_k \cdot I_n^+ = 0 \), for all \( k \in \mathbb{Z} \). We have that \( \nu(N) \) is a cochain n-complex iff the composition \( \mathcal{H}om_{A_0}(A, N_k)[k] \to \mathcal{H}om_{A_0}(A, N_{k+1})[k+1] \to \mathcal{H}om_{A_0}(A, N_{k+n})[k+n] \) is zero, for all \( k \in \mathbb{Z} \), iff its \(-(k+n)\)-component is a cochain \( n \)-complex iff \( \sum_{p \in Q_n} f(\bar{\nu})p_0^0 = 0 \), for all \( f \in \mathcal{H}om_{A_0}(A, N_k) \), where \( \bar{\nu} = \pi(p) \). Since \( A_i^n = A^i_1 \cdot A^i_1 \ldots A^i_1 \) is a direct summand of \( A_n \) in \( \text{Mod}_{A_0} \) and \( \bar{\nu} \in A^n_i \), for all \( p \in Q_n \), we get that \( \nu(N) \) is a cochain n-complex iff \( \sum_{p \in Q_n} f(\bar{\nu})p_0^0 = 0 \), for all \( f \in \mathcal{H}om_{A_0}(A^n_i, N_k) \)

Now we choose an ordering of \( Q_n = \{ p_1, \ldots, p_r, \ldots, p_{r+s}, \ldots, p_{r+s+t} \} \) with the same criterion as in the proof of Proposition 2.4. Then \( \{ p_1, \ldots, p_r \} \) is a basis of \( A^n_i \) and we have \( \sum_{p \in Q_n} f(\bar{\nu})p_0^0 = \sum_{1 \leq i < r} f(\bar{\nu}_i)p_i^0 + \sum_{r+1 \leq j < r+s} f(\sum_{1 \leq i < r} \lambda_{ij}\bar{\nu}_i)p_j^0 = \sum_{1 \leq i < r} f(\bar{\nu}_i)p_i^0 + \sum_{1 \leq i < r} \lambda_{ij}f(\bar{\nu}_i)h_i, \) for every \( f \in \mathcal{H}om_{A_0}(A^n_i, N_k) \), with the same terminology of the proof of Proposition 2.4.

On the other hand, the \( A_0 \)-homomorphisms \( A^n_i \to N_k \) of the form \( x\bar{\nu}_i^*(-) : a \to x\bar{\nu}_i^*(a) \), with \( i = 1, \ldots, r \) and \( x \in N_k \), generate \( \mathcal{H}om_{A_0}(A^n_i, N_k) \) (see [9][Remark 2.1]). But, for \( f = x\bar{\nu}_i^*(-) \), we have \( \sum_{1 \leq i < r} f(\bar{\nu}_i)h_i = xh_s \). Consequently, \( \nu(N) \) is a \( n \)-complex iff \( xh_s = 0 \), for all \( s = 1, \ldots, r \) and \( x \in N_k \), that is, if \( N_k \cdot I_n^+ = 0 \) for all \( k \in \mathbb{Z} \). That ends the proof. \( \square \)

The reader is invited to extend to general \( n \)-homogeneous algebras results like the equivalence of assertions 1, 2 and 5 in [9][Corollary 3.4] and of assertions 1 and 2 of [9][Corollary 3.5], which were given there for quadratic algebras.

## 3 Transport of a torsion theory

We know from [10] that if \( A = \oplus_{i \in \mathbb{Z}} A_i \) is a \( \mathbb{Z} \)-graded algebra and \( S \subseteq \mathbb{Z} \) is any subset, then \( T_S = \{ M \in \mathcal{G}r_A : M_S = 0 \} \) is a hereditary torsion class closed for products in \( \mathcal{G}r_A \). The following is a handy way of identifying its associated torsionfree class.

**Lemma 3.1.** Suppose that \( A = \oplus_{i \geq 0} A_i \) is positively graded, generated in degrees 0,1 and \( A_0 \) is semisimple. If \( S \subseteq \mathbb{Z} \) is not upper bounded then, for a graded \( A \)-module \( M = \oplus_{n \in \mathbb{Z}} M_n \), the following statements are equivalent:

1. \( M \) is \( T \)-torsionfree, where \( T = T_S \)

2. \( \mathcal{H}om_{\mathcal{G}r_A}(A_0[-m], M) = 0 \), for all \( m \in \mathbb{Z} \setminus S \)
Proof. 1) $\implies$ 2) is clear

2) $\implies$ 1) Notice that every morphism $A_0[-m] \to M$ in $Gr_A$ is given
by left multiplication by an element $x \in \text{ann}_{M_m}(A_1)$. Hence the hypothesis
is equivalent to say that $\text{ann}_{M_m}(A_1) = 0$, for all $m \in \mathbb{Z} \setminus S$. Recall that
$\text{Supp}(t(M)) \subseteq \mathbb{Z} \setminus S$. If $t(M) \neq 0$ (equivalently, $\text{Supp}(t(M)) \neq 0$) and $m \in
\text{Supp}(t(M))$, then we pick up $x \in t(M)_m \setminus \{0\}$. By assumption
$xA_1 \neq 0$ which implies that $t(M)_{m+1} \neq 0$ and, hence, that $m+1 \in \text{Supp}(t(M))$. By recurrence
we get that the interval $[m, +\infty)$ is contained in $\text{Supp}(t(M)) \subseteq \mathbb{Z} \setminus S$. But that
contradicts the fact that $S$ is not upper bounded.

Throughout the rest of the section $\Lambda = \oplus_{i \geq 0} \Lambda_i$ is a graded factor of a path
algebra (see section 1). We want to transfer the results of [10] (for $A = \Lambda^1$)
from $Gr_{\Lambda^1}$ to $n\mathcal{LC}_\Lambda$ and $n\mathcal{LC}_\Lambda$ via $v$ and $\Psi$. We refer the reader to [10]
for the definition and terminology about ring-supporting subsets and right modular
pairs of subsets of a group. Here we shall fix a modular pair $(S, U) = (m+U, U)$, with
$U = \bigcup_{k \in \mathbb{Z}} [kn, kn+r]$ and $m \in \mathbb{Z}$, where $n \geq 2$ and $0 \leq 2r \leq n$, with strict
inequality $2r < n$ in case $n > 2$. Then, in case $2r = n = 2$, we have $S = U = \mathbb{Z} = (S : U)$. In any other case, we have $(S : U) = m + (U : U) = m + n\mathbb{Z}$.

Lemma 3.2. Let $\Gamma = (\Gamma^i, d^i) \in_n \mathcal{LC}_\Lambda^*$ be a linear $n$-complex of almost injective
graded $\Lambda$-modules and let $M \in Gr_{\Lambda^1}$ be such that $\Gamma = \nu_{\Lambda}(M)$. The following
assertions hold:

1. $M \in T = T_S$ if, and only if, $I^j = 0$ for all $j \in S = \bigcup_{k \in \mathbb{Z}} [m+kn, m+kn+r]$

2. $M$ is $T$-torsionfree iff $\text{Hom}_{Gr_A}(D(\Lambda)[j], \text{Ker}(d^j)) = 0$ for all $j \notin S$

3. If $0 \leq 2r < n$ then the following conditions are equivalent:

   a) $M$ is generated in degrees belonging to $(S : U) = m + n\mathbb{Z}$

   b) The differential $d^{i-1} : I^{i-1} \to I^i$ satisfies that $(\text{Im}(d^{i-1}))_{-j} = (I^j)_{-j}$ for all $j \not\equiv m \pmod{n}$

   c) $\text{Soc}(I^j) \subseteq \text{Im}(d^{i-1})$, for all $j \not\equiv m \pmod{n}$.

Proof. By definition of $v$, we have $I^j = \text{Hom}_{\Lambda_0}(\Lambda, M_j)[j]$ and, hence, assertion
1 follows. The equivalence $v$ takes $A_0[-j]$ onto the stalk $n$-complex $D(\Lambda)[j]$ (at
the position $j$). Then, using Lemma 3.1, we have that $M$ is $T$-torsionfree if, and only if, there are no nonzero morphisms $f : D(\Lambda)[j] \to I^j$ in $n\mathcal{LC}_\Lambda^*$, for all
$j \notin S$. But such a morphism is completely determined by the induced morphism
in $Gr_A$ $f : D(\Lambda)[j] \to \text{Ker}(d^j)$. Therefore $M$ is $T$-torsionfree if, and only if,
$\text{Hom}_{Gr_A}(D(\Lambda)[j], \text{Ker}(d^j)) = 0$, for all $j \notin S$, which proves assertion 2.

On the other hand, since the algebra $\Lambda^1$ is generated in degrees $0, 1$, it is
easy to see that $M$ is generated in degrees belonging to $(S : U) = m + n\mathbb{Z}$
if, and only if, the multiplication map $M_{j-1} \otimes \Lambda_1^1 \to M_j$ is surjective, for
all $j \not\equiv m + n\mathbb{Z}$. By using adjunction, that is equivalent to say that the
$-j$-component $\text{Hom}_{\Lambda_0}(\Lambda_1, M_{j-1}) \to M_j$ of the ‘differential’ $d^{i-1} : I^{i-1} =
\( \text{Hom}_{\Lambda_0}(\Lambda, M_{j-1})[j-1] \rightarrow \text{Hom}_{\Lambda_0}(\Lambda, M_j)[j] = I^j \) is surjective for all \( j \neq m \) (mod \( n \)). Then the equivalence of conditions a) and b) in 3) follows. The equivalence of b) and c) is clear since \( \text{Soc}^\sigma(I) = (I)_{-j} \), for all \( j \in \mathbb{Z} \).

Recall from [10] that \( \mathcal{G}(\mathcal{S}, \mathcal{U}) \) is the full subcategory of \( \text{Gr}_A \) with objects those \( M \in \text{Gr}_A \) which are \( T \)-torsionfree and generated in degrees belonging to \( (S : U) \). We have the following:

**Proposition 3.3.** Let \( \mathcal{T} = T_S \) be the hereditary torsion class in \( \text{Gr}_{\Lambda^!} \) defined by \( S = \bigcup_{k \in \mathbb{Z}} [m + kn, m + kn + r] \). The following assertions hold:

1. \( \mathcal{T}^* = \nu_\Lambda(\mathcal{T}) \) is the hereditary torsion class of \( \mathcal{n}\mathcal{LC}_\Lambda^* \) consisting of those \( I \in \mathcal{n}\mathcal{LC}_\Lambda^* \) such that \( I^j = 0 \), for all \( j \in S \)

2. The functor \( \nu = \nu_\Lambda : \text{Gr}_{\Lambda^!} \rightarrow \text{Gr}_{\Lambda[X]} \) induces a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{G}(\mathcal{S}, \mathcal{U}) & \leftrightarrow & \text{Gr}_{\Lambda^!} \\
\cong & & \cong \\
\mathcal{G}^*(\mathcal{S}, \mathcal{U}) & \leftrightarrow & \mathcal{n}\mathcal{LC}_\Lambda^*
\end{array}
\]

where the vertical arrows are equivalences of categories and the compositions of horizontal arrows are fully faithful embeddings. Here \( \mathcal{G}^*(\mathcal{S}, \mathcal{U}) = \mathcal{n}\mathcal{LC}_\Lambda^* \), when \( 2r = n = 2 \), and is the full subcategory of \( \mathcal{n}\mathcal{LC}_\Lambda^* \) consisting of those \( I \in \mathcal{n}\mathcal{LC}_\Lambda^* \) satisfying conditions a) and b) below, when \( 0 \leq 2r < n \):

(a) \( \text{Hom}_{\text{Gr}\Lambda}(D(\Lambda)[j], \text{Ker}(d^j)) = 0 \), for all \( j \notin S \)

(b) \( \text{Soc}^\sigma(I) \subseteq \text{Im}(d^j-1) \), for all \( j \neq m \) (mod \( n \))

**Proof.** In case \( 2r = n = 2 \) we have \( \mathcal{G}(\mathcal{S}, \mathcal{U}) = \text{Gr}_A \) and the result follows from [9] [Theorem 2.10]. If \( 0 \leq 2r < n \) then the result follows from Proposition [2.2] Lemma 3.2 and [10] [Theorem 2.7].

## 4 From \( n \)-complexes to (2-)complexes

In this section we consider the case \( r = 1 \) of Section 3, i.e., the modular pair is \( (S, U) = (m + U, U) \), where \( U = n\mathbb{Z} \cup (n\mathbb{Z} + 1) \) and \( m \in \mathbb{Z} \). We want to pass from \( n \)-complexes to (2-)complexes via the appropriate contraction. We will consider parallels of the canonical contraction (see, e.g., [3] and [4]). Let \( C(\text{Gr}_\Lambda) \) be the category of (2-)complexes of graded \( \Lambda \)-modules. The unique strictly increasing function \( \delta_m := \delta_{(S, m)} : \mathbb{Z} \rightarrow \mathbb{Z} \) such that \( \delta_m(0) = m \) and \( \text{Im}(\delta_m) = S \) is given by \( \delta_m(2k) = m + kn \) and \( \delta_m(2k + 1) = m + kn + 1 \), for all \( k \in \mathbb{Z} \) (cf. [10] Lemma 4.9). Hence, \( \delta_m(j) = m + \delta(j) \) for all \( j \in \mathbb{Z} \), where \( \delta = \delta_{(U, 0)} \) is the map used,
for instance, in \[\mathcal{F}\]. We have an obvious additive functor \(H = H_m : \mathcal{L}_C^A \rightarrow C(Gr_A)\) defined as follows. We take \(H(f) = \bar{f}\) where \(\bar{f}^k = f \delta_{n,k}\), for all \(k \in \mathbb{Z}\) and, as differentials, \(\bar{d}^2 = d^m + jn : \bar{f}^{2j} = f^{m + jn} \rightarrow f^{m + jn + 1} = \bar{f}^{2j + 1}\) and \(\bar{d}^{2j + 1} = d^m + jn + 1 - 0 \cdots 0 d^m + jn + 1 : \bar{f}^{2j + 1} = f^{m + jn + 1} \rightarrow f^{m + jn + (j + 1)n} = \bar{f}^{2j + 2}\). The objects in the essential image of \(H\) will be called \(H\)-liftable.

The following observation is trivial, but very useful.

**Remark 4.1.** Let \(A = \bigoplus_{i \geq 0} A_i\) be a positively graded algebra generated in degrees 0, 1 and \((S, \mathcal{U})\) as above. If \(M, N \in \text{Gr}_{A_0}\) and \(f = (f_i : M_i \rightarrow N_i)_{i \in \mathbb{Z}}\) is a family of morphisms in \(\text{Mod}_{A_0}\), then the following assertions are equivalent:

1. \(f\) is a morphism in \(\text{Gr}_{A_0}\)
2. \(f(xa) = f(x)a\), for all \(x \in M_i\) and \(a \in A_1 \cup A_n\), where \(i \in \text{Supp}(M)\)

**Proof.** It is a straightforward consequence of the fact that, as an algebra, \(A_U\) is generated by \(A_0, A_1\) and \(A_n\) \(\Box\)

**Proposition 4.2.** The functor \(H = H_m : \mathcal{L}_C^A \rightarrow C(Gr_A)\) is an exact functor having the following properties:

1. \(\text{Ker}(H) = T^*_S =: T^*\) and \(H\) induces a faithful functor \(\mathcal{L}_C^A \rightarrow C(Gr_A)\)

2. The composition \(\mathcal{G}^*(S, \mathcal{U}) \hookrightarrow \mathcal{L}_C^A \xrightarrow{H} C(Gr_A)\) is a faithful functor whose essential image consists of those \(H\)-liftable complexes \((\bar{f}, \bar{d})\) satisfying condition a) below, in case \(n = 2\), or both conditions a) and b), in case \(n > 2\):

   (a) \(\bar{f}\) is almost injective and cogenerated in degree \(-\delta_m(j)\), for all \(j \in \mathbb{Z}\)

   (b) \(\text{Soc}^g(\bar{f}^{2j + 1}) \subseteq \text{Im}(\bar{d}^{2j})\), for all \(j \in \mathbb{Z}\)

3. When \(\Lambda\) has relations of degree \(\geq n\), the composition \(\mathcal{G}^*(S, \mathcal{U}) \hookrightarrow \mathcal{L}_C^A \xrightarrow{H} C(Gr_A)\) is also full

**Proof.** If \(n = 2\) then \(\mathcal{G}^* = 0, \mathcal{G}^*(S, \mathcal{U}) = \mathcal{L}_C^A\) and \(H\) is the composition \(\mathcal{L}_C^A \rightarrow C(Gr_A) \xrightarrow{?^m} C(Gr_A)\), where \(?^m\) is the canonical shifting of \((2-)\)complexes. Then the functor \(H\) is a fully faithful embedding and the three assertions trivially hold in this case.

We assume in the rest of the proof that \(n > 2\). The exactness of \(H\) and the fact that \(\text{Ker}(H) = T^*\) are clear. In order to prove property 1, we consider a morphism \(f : I \rightarrow J\) in \(\mathcal{L}_C^A\) and will prove that \(H(f) = 0\) iff \(f\) is the zero morphism in the quotient category \(\mathcal{L}_C^A\). To see that, using the equivalence of categories \(v = v_\Lambda : Gr_A \rightarrow \mathcal{L}_C^A\) (cf. Proposition 2.1), we have uniquely determined objects \(M, N\) and morphism \(g : M \rightarrow N\) in \(Gr_A\) such that \(\bar{I} = v(M), \bar{J} = v(N)\) and \(f = v(g)\). One readily sees that \(H(f) = 0\) iff the functor \((-)/\bar{g} : Gr_A \rightarrow Gr_K\) maps \(g\) onto zero. But, by [10] Proposition 2.1,
that happens iff $g$ is the zero morphism in the quotient category $\frac{\text{Gr}_{\Lambda}}{n\mathcal{L}^*_A}$, where $\mathcal{T} = \mathcal{T}_S$. But, by Proposition 3.3, we have that $g = 0$ in $\frac{\text{Gr}_{\Lambda}}{n\mathcal{L}^*_A}$ iff $f = 0$ in $\frac{\text{Gr}_{\Lambda}}{n\mathcal{L}^*_A}$.

On the other hand, also by Proposition 3.3, the composition $G^*(\mathcal{S}, \mathcal{U}) \hookrightarrow n\mathcal{L}^*_A \xrightarrow{pr} \frac{\text{Gr}_{\Lambda}}{n\mathcal{L}^*_A}$ is fully faithful. That together with the above proposition give that the composition $G^*(\mathcal{S}, \mathcal{U}) \hookrightarrow n\mathcal{L}^*_A \xrightarrow{H} C(\text{Gr}_{\Lambda})$ is a fully faithful functor. By definition of $G^*(\mathcal{S}, \mathcal{U})$, we readily see that if $(I, d)$ is the image of $(I, d') \in G^*(\mathcal{S}, \mathcal{U})$ by $H$, then $(\text{Im}(d^{(2)})_{m-jn-1} = (I^{(2)}_{m-jn-1})_{m-jn-1}$ for all $j \in \mathbb{Z}$ (i.e. $\text{Soc}_{2}(I^{(2)}_{m-1}) \subseteq \text{Im}(d^{(2)})$). Conversely, suppose that $(I, d')$ is an $H$-liftable (2-)complex satisfying conditions a) and b) of the statement of the proposition. We first choose $J' \in n\mathcal{L}^*_A$ such that $H(J') = I'$. Then we have a unique $M \in \text{Gr}_{\Lambda}$ such that $\nu(M) = J'$. Now condition b) translates into the fact that $(\text{Im}(d^{(m+jn)})_{m-jn-1} = (J^{m+jn+1}_{m-jn-1})_{m-jn-1}$, for all $j \in \mathbb{Z}$ (here $d'$ is the ‘differential’ of the $n$-complex $J'$). Bearing in mind [2] (Lemma 2.9) and the definition of $\nu$, that means that the multiplication map $\text{M}_{m+jn} \otimes \Lambda' \rightarrow M_{m+jn+1}$ is surjective, for all $j \in \mathbb{Z}$. From that we get that if $M'$ is the graded $\Lambda'$-submodule of $M$ generated by $M_{m+n\mathbb{Z}}$, then $\text{Supp}(\frac{M'}{M'}) \subseteq \mathbb{Z} \setminus \mathcal{S}$, so that $\frac{M'}{M'} \in \mathcal{T}$ and, hence, $M \cong M' \cong \frac{M'}{\text{Im}(M')}$. Then $J = \nu(M) \cong \nu(M') \in \frac{\text{Gr}_{\Lambda}}{n\mathcal{L}^*_A}$. We take $I = \nu\left(\frac{M'}{\text{Im}(M')}\right)$ and have that $H(I) = I'$. Since $\frac{M'}{\text{Im}(M')} \in \mathcal{G}(\mathcal{S}, \mathcal{U})$, we conclude that $I \in \mathcal{G}^*(\mathcal{S}, \mathcal{U})$ as desired.

We next prove assertion 3 assuming that the relations for $\Lambda$ have degree $\geq n$. Take $I, J \in \mathcal{G}^*(\mathcal{S}, \mathcal{U})$. Then we have uniquely determined $M, N \in \mathcal{G}(\mathcal{S}, \mathcal{U})$ such that $\nu(M) = I$ and $\nu(N) = J$. Let $f : H(I) \rightarrow H(J)$ be a morphism in $C(\text{Gr}_{\Lambda})$. Then we get morphisms $f^s : I^s = \text{Hom}_{\Lambda}(A, M_s)[s] \rightarrow \text{Hom}_{\Lambda}(A, N_s)[s] = J^s$ in $\text{Gr}_{\Lambda}$, for every $s \in \mathcal{S} = (m + n\mathbb{Z}) \cup (m + n + 1 \mathbb{Z})$, making commute the following diagrams:

\[
\begin{array}{ccc}
... & \cdots & \cdots & \cdots \\
... & \cdots & \cdots & \cdots \\
... & \cdots & \cdots & \cdots \\
\end{array}
\]

where $d^{n-1}$ means the composition of $n - 1$ consecutive ‘differentials’ of $I$.

By [2] (Lemma 2.8), that gives uniquely determined morphisms in $\text{Mod}_{\Lambda_0} g_s : M_s \rightarrow N_s$, for every $s \in (m + n\mathbb{Z}) \cup (m + n + 1 \mathbb{Z})$, such that $f^s = \text{Hom}(A, g_s)[s]$. The commutativity of the above diagram translates into the commutativity of
the following diagrams, for every \( k \in \mathbb{Z} \):

\[
\begin{align*}
\text{Hom}_{A_0}(\Lambda_{n-1}, M_{m+(k-1)n+1}) & \xrightarrow{d_{n-1}} M_{m+kn} \\
\text{Hom}_{A_0}(\Lambda_{n-1}, N_{m+(k-1)n+1}) & \xrightarrow{d_{n-1}} N_{m+kn} \\
\text{Hom}_{A_0}(\Lambda_1, M_{m+kn}) & \xrightarrow{d} M_{m+kn+1} \\
\text{Hom}_{A_0}(\Lambda_1, M_{m+kn}) & \xrightarrow{d} M_{m+kn+1}
\end{align*}
\]

where \( d_{n-1}(h) = \sum_{p \in Q_{n-1}} h(p)p^a \) and \( d(h) = \sum_{a \in Q_n} h(a)\alpha^a \), respectively.

Now we consider the obvious adaptation of [9] [Lemma 2.9], which is also true replacing \( Q_1 \) by \( Q_{n-1} \) due to the fact that \( \Lambda_i^1 = KQ^\text{op}_i \) for \( 0 \leq i < n \).

Then the commutativity of the last two diagrams is equivalent to say that \( \sum_{\alpha_i \in \Lambda_i^1} \beta_i \) and \( \sum_{\beta_i \in \Lambda_i^1} \alpha_i \) belong to \( S = (m + n\mathbb{Z}) \cup (m + 1 + n\mathbb{Z}) \). If now \( a \in \Lambda^1_1 \) and \( x \in M_{m+kn} \), then we have \( a = \sum_i \alpha_i \beta_i \), where \( \alpha_i = \Lambda_i^1 = KQ_1^\text{op} \) and \( \beta_i = \Lambda_i^1 = KQ_{n-1}^\text{op} \).

Then \( g(xa) = \sum_i g(x(\alpha_i \beta_i)) = \sum_i g((x\alpha_i)\beta_i) \). Now \( \deg(x\alpha_i) = m + kn + 1 \in S \) and the upper diagram above gives \( \sum_i g((x\alpha_i)\beta_i) = \sum_i g((x\alpha_i)\beta_i) \), while the lower diagram gives that the latter expression equals \( \sum_i g(x(\alpha_i \beta_i)) = \sum_i g(x(\alpha_i \beta_i)) = g(x) \sum_i \alpha_i \beta_i = g(x)a \), using the associative property of the \( A \)-module \( N \).

One can proceed in an analogous way when \( x \in M_{m+kn+1} \) and \( a \in \Lambda^1_1 \), but taking a decomposition \( a = \sum_i \gamma_i \beta_i \), where \( \beta_i = \Lambda_i^1 = KQ_{n-1}^\text{op} \) and \( \gamma_i = \Lambda_i^1 = KQ_1^\text{op} \).

According to Remark \( \ref{remark:lifting} \), we conclude that \( g : M_S \rightarrow N_S \) is a morphism in \( Gr_{A_0} \). Since, by definition, both \( M_S \) and \( N_S \) are liftable with respect to \( (-)_S \) and generated in degrees belonging to \( (S : U) \), Theorem 2.7 of [10] tells us that there exists a unique morphism \( \eta : M \rightarrow N \) in \( Gr_{\Lambda^1} \) such that \( \eta_S = g \). It is now a mere routine to check that \( \tilde{f} = \nu(\eta) : I \rightarrow J^* \) is a morphism in \( n\mathcal{L}_A^\text{op} \), such that \( H(\tilde{f}) = f \).

We can now put together all the pieces of the puzzle. Recall from [10] that if \( X \in Gr_{\Lambda_0} \), with \( \text{Supp}(X) \subseteq S \), and \( \mu_{m+kn,1} : X_{m+kn} \otimes \Lambda_1^1 \rightarrow X_{m+kn+1} \) and \( \mu_{m+kn,1n} : X_{m+kn} \otimes \Lambda_n^1 \rightarrow X_{m+(k+1)n} \) are the multiplication maps, then, working within the right \( \Lambda^1 \)-module \( X_{m+kn} \otimes \Lambda^1_1 \), it makes sense to consider \( \text{Ker}(\mu_{m+kn,1})\Lambda_{n-1}^1 \), which is a \( \Lambda_0 \)-submodule of \( X_{m+kn} \otimes \Lambda_1^1 \). We now have:
Theorem 4.3. Let $n \geq 2$ be a positive integer and consider the subsets $\mathcal{U} = n\mathbb{Z} \cup (n\mathbb{Z} + 1)$ and $\mathcal{S} = m + \mathcal{U}$, where $m \in \mathbb{Z}$. Let us assume that $\Lambda$ is a graded factor of a path algebra with relations of degree $\geq n$. There is an equivalence between the following categories:

1. The full subcategory $\mathcal{L}(\mathcal{S}, \mathcal{U})$ of $\text{Gr}_{\Lambda_i}$ with objects those $X$ which are generated in degrees belonging to $(\mathcal{S} : \mathcal{U})$ and satisfying that $\text{Ker}(\mu_{m+kn,1})\Lambda_{n-1} \subseteq \text{Ker}(\mu_{m+kn,n})$, for all $k \in \mathbb{Z}$

2. The full subcategory $\mathcal{Y}(\mathcal{S}, \mathcal{U})$ of $C(\text{Gr}_{\Lambda})$ whose objects are the $H_m$-liftable $(2)$-complexes $(\tilde{I}, d)$ satisfying condition a) below, in case $n = 2$, and both conditions a) and b), in case $n > 2$:
   
   (a) $\tilde{l}_i$ is almost injective and cogenerated in degree $-\delta_m(j)$, for every $j \in \mathbb{Z}$
   
   (b) $\text{Soc}^\mathcal{Y}(\tilde{l}_i)$ is almost injective and cogenerated in degree $-\delta_m(j)$, for all $j \in \mathbb{Z}$

Moreover, $\mathcal{L}(\mathcal{S}, \mathcal{U})$ contains all the graded $\Lambda_\mathcal{U}$-modules presented in degrees belonging to $(\mathcal{S} : \mathcal{U})$.

Proof. In case $n = 2$ everything is trivial, and is actually a shifting version of the equivalence $\text{Gr}_{\Lambda} \xrightarrow{\sim} \mathcal{L}_{\mathcal{U}}^*$ of [9] [Theorem 2.10]. Indeed instead of the functor $\nu = \nu_\Lambda$ there given, ours here is the composition $\text{Gr}_{\Lambda} \xrightarrow{\nu} C(\text{Gr}_{\Lambda}) \xrightarrow{\tau[m]} C(\text{Gr}_{\Lambda})$.

For the case $n > 2$, notice that $(\mathcal{S} : \mathcal{U}) = m + n\mathbb{Z}$. From Proposition 4.2 and [10] [Theorem 2.7] we get the following diagram, where all the thick arrows are equivalences of categories:

\[
\begin{array}{ccc}
\mathcal{G}(\mathcal{S}, \mathcal{U}) & \xrightarrow{\nu_\Lambda} & \mathcal{G}^*(\mathcal{S}, \mathcal{U}) \\
(-)_{\mathcal{S}} \downarrow & \cong & \cong \downarrow H \\
\mathcal{L}(\mathcal{S}, \mathcal{U}) & \sim & \mathcal{Y}(\mathcal{S}, \mathcal{U})
\end{array}
\]

Then the squig arrow making commute the diagram is also an equivalence.

\[\square\]

Remark 4.4. 1) We make explicit, in case $n > 2$, the definition of the equivalence $\mathcal{L}(\mathcal{S}, \mathcal{U}) \xrightarrow{\sim} \mathcal{Y}(\mathcal{S}, \mathcal{U})$. Assume that $\Lambda = KQ_I$, where $I$ is generated by $\rho \subseteq \prod_{i \geq n} \mathbb{K}Q_i$. Then $\Lambda^! = \frac{\mathbb{K}Q_{op}}{\leq_{<1>_n}}$. If $X \in \mathcal{L}(\mathcal{S}, \mathcal{U})$, then we have a morphism of $\Lambda_\mathcal{U}$-modules $\xi = \xi_k : X_{m+kn+1} \otimes \Lambda^!_{n-1} = X_{m+kn+1} \otimes \mathbb{K}Q^*_{n-1} \rightarrow X_{m+(k+1)n}$ defined as follows. Any element of $X_{m+kn+1} \otimes \mathbb{K}Q^*_{n-1}$ can be written as $\sum_{p \in Q_{n-1}} x_p \otimes \rho^i$, for uniquely determined $x_p \in X_{m+kn+1}e_{t(p)}$, where $e_i$ denotes the idempotent of $KQ_i$ corresponding to the vertex $i \in Q_0$. Since $X_{m+kn+1} = X_{m+kn} \Lambda^!_{1} = X_{m+kn} \mathbb{K}Q^*_{1}$, we can write $x_p$ as $x_p = \sum_{\alpha \in Q_1, a(\alpha) = t(p)} x_{\alpha,p} \alpha^*$. Then we put
\[ \xi(\sum_{p \in \mathbb{Q}_{n-1}} x_p \otimes p^\circ) = \sum_{\alpha \in \Lambda} x_{\alpha,p}(\alpha^\circ p^\circ) = \sum_{\alpha \in \Lambda} x_{\alpha,p}(\alpha^\circ p^\circ). \] That \( \xi \) is well-defined follows from the inclusions \( \text{Ker}(\mu_{m+kn,1})\Lambda_{n-1} \subseteq \text{Ker}(\mu_{m+kn,n}) \) given by the theorem. We put then \( \tilde{I}^2_j = \tilde{\text{Hom}}_{\Lambda_0}(\Lambda, X_{m+jn})[m + jn] \) and \( \tilde{I}^{2j+1} = \tilde{\text{Hom}}_{\Lambda_0}(\Lambda, X_{m+jn})[m + jn + 1], \) for all \( j \in \mathbb{Z}. \) The differential \( \tilde{d}^{2j} \) is completely identified by the map \( \text{Hom}_{\Lambda_0}(\Lambda_1, X_{m+jn}) \to X_{m+jn+1} \) which takes \( f \mapsto \sum_{\alpha \in \mathbb{Q}} f(\alpha)\alpha^\circ \) and the differential \( \tilde{d}^{2j+1} = \tilde{\text{Hom}}_{\Lambda_0}(\Lambda, X_{m+jn+1})[m + jn + 1] \to \text{Hom}_{\Lambda_0}(\Lambda, X_{m+jn+1})[m + jn + 1] \) which gives \( f \mapsto \sum_{p \in \mathbb{Q}_{n-1}} \xi(f(p) \otimes p^\circ). \) It is a mere routine to check that the equivalence given by last theorem takes \( X \) onto the here defined \((\tilde{I}, \tilde{d}).\)

2) If, instead of taking the functor \( \nu = \nu_\Lambda \) of Proposition \( \text{[2]} \) as basis of our arguments, one takes the functor \( \Psi = \Psi_\Lambda \) of Proposition \( \text{[3]}, \) then one gets results dual to those in sections \( \text{3 and 4.} \) In the case of locally finite graded modules, we can alternatively see that by using the canonical duality \( \mathcal{D}. \) Indeed, we have \( D(N \otimes \Lambda) \cong \text{Hom}_{\Lambda_0}(\Lambda, N), \) for all \( N \in \text{mod}_{\Lambda_0}, \) and then the commutativity of the following diagram follows, where \( \nu_\Lambda \) is taken for graded left \( \Lambda \)-modules:

\[
\begin{array}{ccc}
\Lambda \text{fg} & \cong & n\mathcal{L}_\Lambda \otimes p \\
\downarrow \cong & & \downarrow \cong \\
D & \cong & D \\
\downarrow \cong & & \downarrow \cong \\
l\text{fg} \mathcal{U}_\Lambda & \cong & n\mathcal{L}_\Lambda \\
\end{array}
\]

We make explicit, without proof, the dual of Theorem \( \text{[3]}, \) leaving the rest for the reader.

We have a canonical additive functor \( G = G_m : n\mathcal{L}_\Lambda \to C(\mathcal{G}_\Lambda) \) defined as follows. If \((P, d) \in n\mathcal{L}_\Lambda \) then its image by \( G \) is \((\tilde{P}, \tilde{d}), \) where \( \tilde{P} = P^{-\delta_m(-)} \) and the differentials are \( \tilde{d}^{2j-1} = d^{-m+2j-1} : P^{2j-1} = P^{-m+2j-1} \to P^{-m+2j} = \tilde{P}^{2j} \) and \( \tilde{d}^{2j} = d^{-m+2j+1} : P^{2j+1} = P^{-m+2j+1} \to P^{-m+2j+2} = \tilde{P}^{2j+1}, \) for all \( j \in \mathbb{Z}. \) To state the desired dual, for every \( X \in \text{Gr}_{\Lambda_0} \) with \( \text{Supp}(X) \subseteq -S = (-m + n\mathbb{Z}) \cup (-m - 1 + n\mathbb{Z}), \) we consider the comultiplications \( \Delta_{s,u} : X_{s-u} \to X_{s-u} \otimes KQ_u, \) for all \((s, u) \in (S, U), \) such that \( s + u \in S \) and \( u \geq 0. \) By definition, one has \( \Delta_{s,u}(x) = \sum_{p \in \mathbb{Q}} x \tilde{p}^\circ \otimes p, \) where \( \tilde{p}^\circ \) is the image of \( p^\circ \) by the canonical projection \( KQ_p^\circ \to \Lambda_u^\circ. \) We are now ready to state the dual of last theorem.

**Theorem 4.5.** Let \( n \geq 2 \) be a positive integer and and consider the subsets \( U = n\mathbb{Z} \cup (n\mathbb{Z} + 1) \) and \( S = m + U, \) where \( m \in \mathbb{Z}. \) Let us assume that \( \Lambda \) is a graded factor of a path algebra with relations of degree \( \geq n. \) There is an equivalence between the following categories:

1. The full subcategory \( \mathcal{L}_\circ(S, U) \) of \( \text{Gr}_{\Lambda_0} \) with objects those \( X \) which are
cogenerated in degrees belonging to \(-(S : U)\) and satisfy that the following
diagram in Mod\(_{\Lambda_0}\) can be completed for all \(k \in \mathbb{Z}\):

\[
\begin{array}{ccc}
X_{-m-(k+1)n} & \xrightarrow{\Delta} & X_{-m-kn} \otimes KQ_n \\
\downarrow & & \downarrow \cong \\
X_{-m-kn} \otimes KQ_{n-1} & \xrightarrow{\Delta \otimes 1} & X_{-m-kn} \otimes KQ_1 \otimes KQ_{n-1}
\end{array}
\]

2. The full subcategory \(Y^o(S, U)\) of \(C(Gr_{\Lambda})\) whose objects are the \(G_m\)-liftable
\((\mathcal{P}, \delta')\) satisfying condition a) below, in case \(n = 2\), and
conditions a) and b), in case \(n > 2\):

(a) \(\tilde{P}^j\) is a projective graded \(\Lambda\)-module generated in degree \(\delta_m(-j)\), for
every \(j \in \mathbb{Z}\)

(b) \((\text{Ker}\tilde{p}_{2k-1}) \subseteq \tilde{P}^{2k-1} \cdot J^g(\Lambda), \text{ for all } k \in \mathbb{Z}\)

In case \(\Lambda\) is a \(n\)-Koszul algebra with Yoneda algebra \(E = E(\Lambda)\) we know from [2] [Theorem 9.1] (see also [2] [Proposition 3.1]) that there is an algebra
isomorphism \(\varphi : E \cong \Lambda^1_{U}\), which we fix from now on, such that \(\varphi(E_j) = \Lambda^1_{\delta(j)}\),
for all \(j \in \mathbb{Z}\). We see it as an identification and, abusing of notation, we write
\(E_j = \Lambda^1_{\delta(j)}\), for all \(j \in \mathbb{Z}\). Hence, if \(V \in Gr_E\), we have multiplication maps
\(\tilde{\mu}_{2k,1} : V_{2k} \otimes E_1 = V_{2k} \otimes \Lambda^1 \rightarrow V_{2k+1}\) and \(\tilde{\mu}_{2k,2} : V_{2k} \otimes E_2 = V_{2k} \otimes \Lambda^1 \rightarrow V_{2k+2}\)
and we can take \(\text{Ker}(\tilde{\mu}_{2k,1})\Lambda^1_{n-1}\), which is an \(\Lambda_0 - \Lambda_0\)-sub-bimodule of
\(V_{2k} \otimes \Lambda^1 = V_{2k} \otimes E_2\). We then have the following consequence of Theorem 4.3

**Corollary 4.6.** Let \(n \geq 2\) be a positive integer and consider the subsets \(U = nZ \cup (n+1)Z\) and \(S = m+U\), where \(m \in \mathbb{Z}\). If \(\Lambda = \bigoplus_{i \geq 0} \Lambda_i\) is a \(n\)-Koszul
algebra and \(E\) is its Yoneda algebra, then there is an equivalence between:

1. The full subcategory \(L_E\) of \(Gr_E\), which, in case \(n = 2\), coincides with \(Gr_E\)
and, in case \(n > 2\), has as objects those \(V \in Gr_E\) which are generated in
even degrees and satisfy that \(\text{Ker}(\tilde{\mu}_{2k,1})\Lambda^1_{n-1} \subseteq \text{Ker}(\tilde{\mu}_{2k,2})\), for all \(k \in \mathbb{Z}\)

2. The full subcategory \(Y(S, U)\) of \(C(Gr_{\Lambda})\) defined in Theorem 4.3

Moreover, \(L_E\) contains all the graded \(E\)-modules presented in even degrees.

**Proof.** In case \(n = 2\), one has \(E = \Lambda^1\), \(L_E = Gr_E\) and \(Y(S, U)\) is the full
subcategory of \(C(Gr_{\Lambda})\) consisting of those complexes \((\tilde{I}, \delta')\) such that \(\tilde{I}^j\)
is almost injective and cogenerated in degree \(-m-j\), for every \(j \in \mathbb{Z}\). The
desired equivalence of categories is then the composition of the equivalences
\(\nu_{\Lambda} : Gr_E \cong Gr_{\Lambda^1} \xrightarrow{\approx} L\mathcal{C}_{\Lambda}\) and \(\approx [m] : L\mathcal{C}_{\Lambda} \xrightarrow{\approx} Y(S, U)\).
We next assume \( n > 2 \). Then, from [10][Theorem 2.7, Corollary 4.10 and Remark 4.11] we get equivalences of categories \( \mathcal{L}_E \cong \mathcal{G}(\mathcal{S}, \mathcal{U}) \cong \mathcal{L}(\mathcal{S}, \mathcal{U}) \) and, using Theorem 4.8, also an equivalence \( \mathcal{L}(\mathcal{S}, \mathcal{U}) \cong \mathcal{Y}(\mathcal{S}, \mathcal{U}) \).

**Example 4.7.** Let us take a positive integer \( n > 2 \) and consider the truncated algebra \( \Lambda = KQ/ < Q_n > \), so that \( \Lambda \) is \( n \)-Koszul and \( \Lambda' = KQ^{op} \). We give the equivalence of last corollary when \( m = 0 \) (i.e. \( \mathcal{S} = \mathcal{U} = n\mathbb{Z} \cup (n\mathbb{Z} + 1) \)). Then, as ungraded algebras, \( E = \Lambda_0^{op} = (KQ^{op})_\mathcal{U} \). The classical grading of \( E \) is given by assigning degree 1 to the \( \alpha^o \in Q_i^o \) and degree 2 to the \( p^o \in Q_n^p \). Then the subcategory \( \mathcal{L}_E \) consists of those \( V \in \mathcal{G}_E \) which are generated in even degrees and satisfy that if \( \sum_{\alpha \in Q} x_{\alpha}\alpha^o = 0 \), for a family of elements \( (x_{\alpha}) \) in \( V_2j \), then \( \sum_{\alpha \in Q} x_{\alpha}\alpha^o q^o = 0 \) for all \( q \in Q_{n-1} \). Using Remark 4.11(e) we then get that the equivalence \( \mathcal{L}_E \cong \mathcal{Y}(\mathcal{U}, \mathcal{U}) \) maps \( V \) onto the cochain complex \( (\hat{I}, \hat{d}) \) defined as follows: i) \( \hat{I}^j = \text{Hom}_\Lambda(\Lambda, V_2)[jn] \) and \( \hat{I}^{j+1} = \text{Hom}_\Lambda(\Lambda, V_{2j+1})[jn+1] \), for all \( j \in \mathbb{Z} \); ii) the differentials \( \hat{d} = \text{Hom}_\Lambda(\Lambda, V_2)[jn] \to \text{Hom}_\Lambda(\Lambda, V_{2j+1})[jn+1] = \hat{I}^{j+1} \) and \( \hat{d}^{j+1} = \text{Hom}_\Lambda(\Lambda, V_{2j+2})[(j+1)n] = \hat{I}^{2j+2} \) are defined by the formulas \( \hat{d}(f)(a) = \sum_{\alpha \in Q} f(\alpha^o)\alpha^o \) and \( \hat{d}(g)(a) = \sum_{\alpha \in Q} g(\alpha^o) \cdot p^o \), for all \( a \in \Lambda \), \( f \in \text{Hom}_\Lambda(\Lambda, V_2) \) and \( g \in \text{Hom}_\Lambda(\Lambda, V_{2j+1}) \). The last multiplication \( \cdot : V_{2j+1} \otimes \Lambda_{n-1} \to V_{2j+2} \) is given by \( \sum_{\alpha \in Q} x_{\alpha}\alpha^o \cdot p^o = \sum_{\alpha \in Q} x_{\alpha}(\alpha^o p^o) = \sum_{\alpha \in Q} x_{\alpha}(\alpha^o p^o) \), for all \( p \in Q_{n-1} \) and all \( x = \sum_{\alpha \in Q} x_{\alpha}\alpha^o \in X_{2j+1} \), where the \( x_{\alpha} \) belong to \( X_{2j} \).

If \( M \in \mathcal{L}_G \Lambda \), it shall be seen that \( M \) is \( \text{n-coKoszul} \) when it is cogenerated in degree 0 and its minimal (almost) injective graded resolution \( \hat{I}_M \) satisfies that \( \hat{I}^j \) is cogenerated in degree \( -\delta(j) \), for every \( j \geq 0 \), where \( \delta = \delta_0 \). In particular, \( \hat{I}_M \) satisfies conditions a) and b) of Theorem 4.8 for \( m = 0 \). It is easy to see that the assignment \( M \sim \hat{I}_M \) yields a fully faithful embedding of the category \( \mathcal{K}(\Lambda) \) of \( \text{n-coKoszul} \) \( \Lambda \)-modules into \( \mathcal{C}(\mathcal{L}_G \Lambda) \). We denote by \( \mathcal{K}_n(\Lambda) \) the full subcategory of \( \mathcal{L}_G \Lambda \) formed by the \( n \)-Koszul modules. It would be interesting to have an answer to the following question:

**Question and Remark 4.8.** Let \( \Lambda \) be a \( n \)-Koszul algebra \((n > 2)\). Which are the \( n \)-co-Koszul modules \( M \) such that \( \hat{I}_M \) is \( H_0 \)-lifting (and, hence, belongs to \( \mathcal{Y}(\mathcal{U}, \mathcal{U}) \)? If we denote the corresponding full subcategory of \( \mathcal{L}_G \Lambda \) by \( \mathcal{K}_\mathcal{U}(\Lambda) \), then its image by the canonical duality \( D : \mathcal{L}_G \Lambda \to \mathcal{L}_E \mathcal{U} \) which embeds \( \mathcal{K}_G(\Lambda) \) consists of those \((\text{locally finite})\) \( n \)-Koszul modules whose minimal graded projective resolution is \( G_0 \)-liftable (see Theorem 4.8). Now, going backward in Corollary 4.6 we get a contravariant fully faithful embedding \( \mathcal{K}_G(\Lambda) \cong \mathcal{K}_\mathcal{U}(\Lambda) \to \mathcal{L}_E \mathcal{U} \mathcal{U} \mathcal{U} \mathcal{U} \), which takes indecomposable projective graded \( \Lambda \)-modules onto simple graded \( E \)-modules and simple graded \( \Lambda \)-modules onto indecomposable projective graded \( E \)-modules. The last assertion follows from [4][Section 3], where the authors prove that the minimal projective resolution of \( \Lambda_0 \in \mathcal{L}_{\mathcal{G} \mathcal{U}} \) is obtained by contraction of the Koszul \( n \)-complex, and is thereby \( G \)-liftable.
References

[1] BEILINSON, A.; GINZBURG, V.; SOERGEL, W.: Koszul duality patterns in representation theory. J. Amer. Math. Soc. 9(2) (1996), 473-526

[2] BERGER, R.: Koszulity for nonquadratic algebras. J. Algebra 239 (2001), 705-734

[3] BERGER, R.; DUBOIS-VIOLETTE, M.; WAMBST, M.: Homogeneous algebras. J. Algebra 261 (2003), 172-185

[4] BERGER, R.; MARCONNET, N.: Koszul and Gorenstein properties for homogeneous algebras. Preprint, arXiv: math.QA/0310070 v2

[5] BERNSTEIN, J.; GELFAND, S.: Algebraic vector bundles and projective spaces. Appendix to Russian translation of M. Schneider, 'Holomorphic vector bundles on $\mathbb{P}^n$. Sem. Bourbaki 530 (1980), 80-102

[6] GELFAND, S.I.; MANIN, Y.I.: "Methods of homological algebra". Springer-Verlag (1996)

[7] GREEN, E.L.; MARCOS, E.N.; MARTINEZ-VILLA, R.; ZHANG, P.: D-Koszul algebras. J. Pure and Appl. Algebra 193 (2004), 359-378

[8] KELLER, B.: Introduction to $\mathbb{A}_\infty$-algebras and modules. Homology, Homotopy and Appl. 3 (2001), 1-35

[9] MARTINEZ VILLA, R.; SAORIN, M.: Koszul equivalences and dualities. Pacific J. Math. 204(2) (2004), 359-378

[10] MARTINEZ VILLA, R.; SAORIN, M.: Killing of supports on graded algebras. Preprint

[11] NASTASESCU, C.; VAN OYSTAEYEN, F.: "Graded Ring Theory". North-Holland (1982)