The heat flow for Kähler fibrations

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Abstract

We establish global existence of smooth solutions to heat flow for Yang-Mills-Higgs functional on Kähler fibrations. As an application, we give a new proof of the key inequality for Mundet’s Hitchin-Kobayashi correspondence theorem using the heat flow technique.

1 Introduction

Assume that $X$ is a compact Kähler manifold, $G$ is a compact connected Lie group, and $P \to X$ is a principle $G$-bundle on $X$. Now suppose that $(M, \omega, \mu)$ is a compact symplectic manifold which supports a Hamiltonian $G$-group action with a moment map $\mu : M \to \mathfrak{g}$. Let $\mathcal{A}$ be the space of connections on $P$, $\mathcal{A}^{1,1} \subset \mathcal{A}$ be the space of connection whose curvature belongs to $\Omega^{1,1}(P \times \text{Ad } \mathfrak{g})$. Let $\mathcal{F} = P \times_G M \to X$ be the associated fiber bundle (fibration) on $X$, $\mathcal{L}$ the space $\Gamma(\mathcal{F})$ of smooth sections of $\mathcal{F}$.

In 1999, Cieliebak-Gaio-Salamon [5] and Mundet [13] independently defined Gromov-Witten invariants of symplectic manifolds with Hamiltonian group actions using the vortex equation

$$\wedge F_A + \mu(u) = c,$$

where $F_A$ denotes the curvature of the connection $A$ on $P$, $\wedge$ the contraction with the Kähler form of $X$, $u \in \mathcal{L}$, and $c \in \mathfrak{g}$ is a fixed central element. They also introduced the Yang-Mills-Higgs functional (see section 2)

$$\mathcal{YMH}(A, u) = ||F_A||_{L^2}^2 + ||d_A u||_{L^2}^2 + ||\mu(u) - c||_{L^2}^2,$$

and proved its some fundamental properties. When $X = \mathbb{R}^2$, the functional appeared in the Ginzburg-Landau theory in particle physics, which is used to model superconductivity. When $X = \mathbb{R}^3, G = SU(2)$ and $P = \mathbb{R}^3 \times SU(2)$ is a trivial bundle, Taubes [18] studied the existence of critical points of Yang-Mills-Higgs functional. When $X = \Sigma$ is a compact Riemann surface and $M$ is a compact symplectic manifold, Song [17] proved the existence of critical points of Yang-Mills-Higgs functional above using Sacks-Uhlenbeck’s perturbation method.

On the other hand, the most interesting property of the Yang-Mills-Higgs functional is that the vortex equation (1) is just the minimum equation of
Yang-Mills-Higgs functional (2). And (1) has been studied in a large number of work, for example [3-5, 9, 13-15]. Especially, Bradlow in [4] established the relation between the $\tau$-stability of holomorphic bundles with a global section and the existence of a hermitian metric satisfying the $\tau$-vortex equations by minimizing the so-called Donaldson’s functional. In [13, 14], Mundet proved the Hitchin-Kobayashi correspondence between the $c$-stability and the existence of the vortex equation (1). Hong [8] gave another proof of Bradlow’s Hitchin-Kobayashi correspondence theorem by using heat flow approach, and his proof was considerably simpler than one in [4].

Motivated by Hong’s work, we investigate the Yang-Mills-Higgs functional (2) using heat flow approach. Recently, Venugopalan [21] studied the heat flow of a Yang-Mills-type functional similar to (2) on the space of gauged holomorphic maps using another approach.

Now let’s describe the outlines of this paper. First we will give the generalized covariant derivative $d_A$ (see section 2) introduced in [5]

$$d_A u = du + L_u A, \forall u \in \mathcal{L} = \Gamma(F),$$

and strictly define the Yang-Mills-Higgs functional

$$\mathcal{YMH}(A, u) = \left| \left| F_A \right| \right|_{L^2}^2 + \left| \left| d_A u \right| \right|_{L^2}^2 + \left| \left| \mu(u) - c \right| \right|_{L^2}^2.$$ 

Then we adopt Urakawa’s approach [20] and strictly deduce the Euler-Lagrange equations of Yang-Mills-Higgs functional as follows:

$$\nabla^* d_A u + (d\mu)^*(\mu(u) - c) = 0; \quad (3)$$

$$L^*_u d_A u + D^*_A F_A = 0. \quad (4)$$

We can see that the Euler-Lagrange equations above are common generalization of harmonic maps and Yang-Mills connections. Therefore, we are interested in studying the Euler-Lagrange equations and consider the following heat flow equations

$$\frac{\partial A}{\partial t} = -L^*_u d_A u - D^*_A F_A; \quad (5)$$

$$\frac{\partial u}{\partial t} = -\nabla^*_A d_A u - (d\mu)^*(\mu(u) - c). \quad (6)$$

And we can prove the global existence of (5), (6). Now we can state our main theorem as follows:

**Theorem 1.1.** Assume that both the base manifold $X$ and the fiber $M$ of the fiber bundle $F = P \times_G M \to X$ are compact Kähler manifolds (Kähler fibration), where $M$ supports a Hamiltonian $G$-group action, and $P$ is a holomorphic $G$-principal bundle over $X$. Let $H_0$ be a Hermitian metric on $P$, $A_0$ the canonical connection with respect to $H_0$, $u_0$ a holomorphic section
of the fiber bundle $P \times_G M$, i.e. $\partial_{\Lambda_0} u_0 = 0$. Then Yang-Mills-Higgs heat flow

$$\frac{\partial A}{\partial t} = -L^*_u d_A u - D^*_A F_A;$$

$$\frac{\partial u}{\partial t} = -\nabla^*_A d_A u - (d\mu)^*(\mu(u) - c),$$

exists an unique global smooth solution $(A, u)$ in $X \times [0, +\infty)$ with initial value

$$(A(\cdot, 0) = A_0, u(\cdot, 0) = u_0).$$

(7)

For the proof of theorem, we follow Donaldson’s approach [6, 8] to consider a flow of gauge transformation which is equivalent to (5), (6):

$$\frac{\partial h}{\partial t} = -\triangle h - i[h \hat{F}_{A_0} + \hat{F}_{A_0} h + 2(\mu(u_0) - c)h] + 2i \wedge (\partial_{A_0} hh^{-1} \partial_{A_0} h)$$

with $h(0) = I$. After that, we introduce the generalized Kähler identities and then prove some fundamental lemmas. Then modifying the idea of Donaldson, we obtain a local and global existence of the unique solution to (7) with initial value $h(0) = I$.

Next we recall the definition of $c$-stability of [14]. Then by studying the limiting behavior of the heat flow as $t \to \infty$, we can prove the key inequality [14, lemma 6.1] which is critical to the proof of the sufficiency part of Hitchin-Kobayashi correspondence theorem as follows:

**Theorem 1.2.** ([14]) Let $(A, u) \in \mathfrak{A}^{1,1} \times \mathcal{L}$ be a simple pair, and assume that $(A, u)$ is $c$-stable, then there exist positive constants $C_1, C_2$ such that for any $s \in \text{Met}_{2, B}^p$, one has

$$\sup |s| \leq C_1 \Psi^c(e^s) + C_2,$$

(8)

where the definitions of $\text{Met}_{2, B}^p$ and $\Psi^c$ can be seen in section 4.

This paper is organized as follows. In section 2, we define Yang-Mills-Higgs functional and strictly deduce its Euler-Lagrange equations. In section 3, we study the heat flow of Yang-Mills-Higgs functional and prove Theorem 1.1. As an application, we give another proof of Theorem 1.2 using the heat flow technique.

## 2 Preliminaries

### 2.1 Yang-Mills-Higgs functional

Suppose that $X$ is a compact Kähler manifold, $G$ is a compact connected Lie group, and $P \to X$ is a principle $G$-bundle on $X$. Let $(M, \omega)$ be a
compact symplectic manifold, and let $G$ acts on $M$ by symplectomorphisms.

Let $g = \text{Lie}(G)$ denote the Lie algebra of $G$. For every $\xi \in g$, denote by $X_\xi : M \to TM$ the vector field whose flow is given by the action of the 1-parameter subgroup generated by $\xi$. Suppose that the Lie algebra $g$ carries an invariant inner product $\langle \cdot, \cdot \rangle$. The action of $G$ is called Hamiltonian if there exists an equivariant map $\mu : M \to g$ such that, for every $\xi \in g$,

$$d < \mu, \xi > = \iota(X_\xi) \omega.$$  

This means that $X_\xi$ is the Hamiltonian vector field of the function $\langle \mu, \xi \rangle$. The map $\mu$ is called a moment map.

Now suppose that $(M, \omega, \mu)$ is a symplectic manifold with a Hamiltonian group $G$-action. Denote by $\mathcal{J}(M, \omega, \mu)$ the space of all almost complex structures $J$ on $TM$ which are invariant under the $G$-action and compatible with the symplectic structure $\omega$; that is, $\omega$ is $J$-invariant, and $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$ is a Riemannian metric on $M$. It follows from [12, Prop. 2.50] that the space $\mathcal{J}(M, \omega, \mu)$ is nonempty and contractible. Define an operator $L : M \to C^\infty(g, TM)$ as $L_x = L(x) : g \to T_x M, \forall x \in M$, where $L_x \xi := X_\xi(x), \forall \xi \in g$.

**Lemma 2.1.** Suppose that $(M, \omega, \mu)$ is a symplectic manifold with a Hamiltonian group $G$-action. Take a $\omega$-compatible almost complex structure $J$ on $M$, we have identities

$$L_x = -J(d \mu(x))^*, L_x^* = d \mu(x) J,$$

where $L_x^*$ is the adjoint operator of $L_x$ with respect to the invariant inner product $\langle \cdot, \cdot \rangle$ and the Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$.

**Proof.** First by the definition of moment map and $L_x$, we have

$$d < \mu(x), \xi > = \iota(X_\xi) \omega(x) = \omega(L_x \xi, \cdot), \forall \xi \in g. \quad (9)$$

By the definition of the Riemannian metric $g(\cdot, \cdot)$, we have

$$\omega(L_x \xi, \eta) = \omega(L_x \xi, -J(\eta)) = g(L_x \xi, -\eta) = -g(J \eta, L_x \xi), \forall \eta \in T_x M. \quad (10)$$

At the same time, $\forall \eta \in T_x M$ we have

$$d < \mu(x), \xi > (\eta) = g(d \mu(x) \eta, \xi) = g(\eta, (d \mu(x))^* \xi) = g(J \eta, J (d \mu(x))^* \xi). \quad (11)$$

By (9),(10),(11), we obtain $L_x = -J(d \mu(x))^*$. Note that $(L_x^*)^* = L_x, J^* = -J$, so $L_x^* = d \mu(x) J$.

Let $\mathcal{A}$ be the space of connections on $P$, $\mathcal{A}^{1,1} \subset \mathcal{A}$ be the space of connection whose curvature belongs to $\Omega^{1,1}(P \times \text{Ad}g)$. Let $\mathcal{F} = P \times_G M \to X$ be the associated bundle on $X$ with fiber $M$, $\mathcal{L}$ be the space $\Gamma(\mathcal{F})$ of smooth sections of $\mathcal{F}$.  


In order to define Yang-Mills-Higgs functional for pairs $A^{1,1} \times \mathcal{L}$, it will be necessary to extend the definition of covariant derivative on vector bundles to general fiber bundles.

According to [2], there is a representation of the sections of the associated fibration $\mathcal{F} = P \times_G M \to X$ as functions on the corresponding principle bundle $\pi : P \to X$. This representation is extremely useful in doing calculations in a coordinate free way.

**Theorem 2.2.** ([2]) Let $C^\infty_G(P, M)$ denote the space of equivariant maps from $P$ to $M$, that is, those maps $u : P \to M$ that satisfy $u(p \cdot g) = g^{-1}u(p)$.

There is a natural isomorphism between $\mathcal{L} = \Gamma(\mathcal{F})$ and $C^\infty_G(P, M)$, given by sending $u \in C^\infty_G(P, M)$ to $\tilde{u}$ defined by

$$\tilde{u}(x) = [p, u(p)],$$

where $x \in X$, $p$ is any element of $\pi^{-1}(x)$, and $[p, u(p)]$ is element of $\mathcal{F} = P \times_G M$ corresponding to $(p, u(p)) \in P \times M$.

By Theorem 2.2, from now on we can identify $\mathcal{L}$ as $C^\infty_G(P, M)$. We define the covariant derivative on fiber bundles $\mathcal{F} = P \times_G M \to X$ (see [5] for more details). $\forall u \in \mathcal{L} = C^\infty_G(P, M)$, the connection $A$ on $P$ determines a connection on the fiber bundle $\mathcal{F} = P \times_G M \to X$. More precisely, the tangent space of $\mathcal{F}$ at $\tilde{p}, x$ is the quotient

$$T_{\tilde{p}, x} \mathcal{F} = \frac{T_p P \times T_x M}{\{(p\xi, -X_p(x))|\xi \in g\}},$$

the vertical space consists of equivalence classes of the form $[0, w]$ with $w \in T_x M$, and the horizontal space consists of those equivalence classes $[v, w]$ where $v \in T_p P$ and $w \in T_x M$ satisfy $w + X_{A_p(v)}(x) = 0$. The covariant derivative of a section $u : P \to M$ with respect to the connection $A$ is the form $d_Au : TP \to u^*TM$ given by

$$d_Au(p)v = du(p)v + X_{A_p(v)}(u(p)).$$

It is easy to check that this 1-form is actually $G$-equivariant, and it satisfies $d_Au(p)p\xi = 0, \forall \xi \in g$. Hence it actually defines a covariant derivative on the fibration $\mathcal{F} = P \times_G M \to X$. Recall the definition of the operator $L$, then we can rewrite the covariant derivative as

$$d_Au = du + L_uA, \forall u \in \mathcal{L} = \Gamma(\mathcal{F}) = C^\infty_G(P, M). \tag{12}$$

We can rewrite identities of lemma 2.1. as follows

**Lemma 2.3.** $\forall u \in C^\infty_G(P, M)$, we have identities

$$L_u = -J(d\mu(u))^*, L_u^* = d\mu(u)J. \tag{13}$$
Now fix a central element $c \in g$. The Yang-Mills-Higgs functional $\mathcal{YMH} : A^{1,1} \times \mathcal{L} \to \mathbb{R}$ is defined as

$$\mathcal{YMH}(A, u) = \|F_A\|_{L^2}^2 + \|d_Au\|_{L^2}^2 + \|\mu(u) - c\|_{L^2}^2.$$ 

We can see that the functional is composed of the Yang-Mills functional $E_1 = \|F_A\|_{L^2}^2$, the energy functional $E_2 = \|d_Au\|_{L^2}^2$ and the Higgs functional $E_3 = \|\mu(u) - c\|_{L^2}^2$. Next we compute its critical points equations.

### 2.2 The Euler-Lagrange equations

Adopting [20]'s method, first fix the connection $A$ and take a variation $u_t$ of $u$, i.e., $u_t \in C_C^\infty (P, M)$, $u_0 = u$. So the Yang-Mills functional $E_1(A, u_t)$ is fixed, and we only need calculate $E_2$ and $E_3$. Choose a metric $h$ on $P$, $g = g(\cdot, \cdot) = \omega(\cdot, J \cdot)$ on $M$ and their corresponding Levi-civita connections $\nabla, \overline{\nabla}$. Further $\nabla, \overline{\nabla}$ induce connections $\nabla^* : T^*P \to \nabla^* P \otimes u^*TM$. Let $U \subset P, (x^i) \in U, u(U) \subset V \subset M, (y^\alpha) \in V$ be local coordinates. In the local coordinates, we can write

$$d_A(u) = u^i_\beta dx^i \otimes (\partial_{y^\beta} \cdot u),$$

where

$$u^i_\beta = d_Au(\frac{\partial}{\partial x^i} \otimes dy^\beta)$$

$$= (du + X_Au)(\frac{\partial}{\partial x^i} \otimes dy^\beta)$$

$$= \frac{\partial u^\beta}{\partial x^i} + X_A u^\beta$$

$$= \frac{\partial u^\beta}{\partial x^i} + X_A u^\beta,$$

and

$$< dx^i \otimes \frac{\partial}{\partial y^\alpha}, dx^j \otimes \frac{\partial}{\partial y^\beta} > = h^{ij} g_{\alpha\beta},$$

so

$$< d_A u, d_A u > = < \frac{\partial u^\beta}{\partial x^i} + X_A u^\beta dx^i \otimes \frac{\partial}{\partial y^\beta}, (\frac{\partial u^\alpha}{\partial x^j} + X_A u^\alpha) dx^j \otimes \frac{\partial}{\partial y^\alpha} >$$

$$= h^{ij} g_{\alpha\beta}(\frac{\partial u^\beta}{\partial x^i} + X_A u^\beta)(\frac{\partial u^\alpha}{\partial x^j} + X_A u^\alpha).$$

Therefore we have

$$E_2(A, u_t) = \int_X < d_A u_t, d_A u_t > = \int_X h^{ij} g_{\alpha\beta}(u_t)(\frac{\partial u^\beta_t}{\partial x^i} + X_A u^\beta_t)(\frac{\partial u^\alpha_t}{\partial x^j} + X_A u^\alpha_t).$$
By compatibility between metrics and connections above,
\[ \nabla_t (h^{ij} g_{\alpha \beta}) = 0, \nabla_k (h^{ij} g_{\alpha \beta}) = 0. \]

By symmetry,
\[ \frac{d}{dt} |_{t=0} E_2(A, u_t) = 2 \int_X h^{ij} g_{\alpha \beta}(u_t) \nabla_t |_{t=0} (\frac{\partial u_\beta^\alpha}{\partial x^j} + X_A^i u_\beta^\alpha). \]

Now we need compute \( \nabla_t |_{t=0} \frac{\partial u_\beta^\alpha}{\partial x^j} + X_A^i u_\beta^\alpha \). Since \( \nabla_t (\frac{\partial u_\beta^\alpha}{\partial x^j}) = \nabla_i (\frac{\partial u_\beta^\alpha}{\partial x^j}) \), we only need compute \( \nabla_t (X_A^i u_\beta^\alpha) \). Using the equivalent definition of connection by parallel translation, we get
\[ \nabla_t (X_A^i u_\beta^\alpha) = \bar{\nabla}_V (X_A^i). \]

Let \( V^\beta = \frac{\partial u_\beta^\alpha}{\partial t} |_{t=0} \), then
\[ \frac{d}{dt} |_{t=0} E_3(A, u_t) = 2 \int_X \nabla V^\beta (X_A^i u_\beta^\alpha) \bar{DA}_u + (d\mu)^* (\mu(u) - c >) \int_X \mu(u) - c > dvol. \]

As for \( E_3(A, u_t) = ||\mu(u_t) - c||^2_{L^2} = \int_X |\mu(u_t) - c|^2 dvol \), we have
\[ \frac{d}{dt} |_{t=0} E_3 = 2 \int_X < d\mu, (d\mu)^* (\mu(u) - c) > \int_X \mu(u) - c > dvol. \]

Thus the first variation formula of Yang-Mills-Higgs functional for \( u \) is:
\[ \frac{d}{dt} |_{t=0} \mathcal{YMH}(A, u_t) = \frac{d}{dt} |_{t=0} (E_2 + E_3) \]
\[ = 2 \int_X < \nabla_A V, dA u > + < d\mu(V), \mu(u) - c > \]
\[ = 2 \int_X < V, \nabla_A^* dA u > + < (d\mu)^* (\mu(u) - c) > \]
\[ = 2 \int_X < V, \nabla_A^* dA u > + (d\mu)^* (\mu(u) - c) >. \]
where $V = \frac{du}{dt}|_{t=0} \in \Gamma(u^*TM)$ is the variation vector field, $\nabla_A : \Gamma(u^*TM) \rightarrow \Omega^1(P, u^*TM)$ the covariant derivative, and $\nabla_A^*$ is the adjoint operator of $\nabla_A$.

In all, we get the first Euler-Lagrange equation with respect to $u$:

$$\frac{d}{dt}|_{t=0} \mathcal{YMH}(A, u_t) = 0 \iff \nabla^*_A d_A u + (d\mu)^*(\mu(u) - c) = 0.$$ 

Next we fix $u$ and study the first variation formula for $A \in \mathcal{A}$. It is obvious that we only need compute $E_1 = ||F_A||^2_{L^2}, E_2 = ||d_A u||^2_{L^2}$.

First we recall the variation of connection ([10]). Because space of connection on principle $G$-bundle $P \rightarrow X$ is an affine space: $d + \Omega^1(X, adP)$, where $adP$ is the associate bundle of $P$ with respect to the adjoint representation of $G$. Thus it suffices to vary $A$ along lines $A_t = A + tB, A_0 = A, \forall B \in d + \Omega^1(X, adP)$. It is easy to compute

$$F_{A_t} = F_A + tD_A B + t^2 B \wedge B,$$

$$d_A u = du + X_{A+tB}(u) = d_A u + tX_B(u).$$

So we have

$$E_1 + E_2 = \int_X \langle F_A + tD_A B + t^2 B \wedge B, F_A + tD_A B + t^2 B \wedge B \rangle + \langle d_A u + tX_B(u), d_A u + tX_B(u) \rangle$$

$$= 2t \int_X (\langle d_A(u), X_B(u) \rangle + \langle F_A, D_A B \rangle) + \cdots$$

Therefore

$$\frac{d \mathcal{YMH}(A_t, u)}{dt}|_{t=0} = 0 \iff \int_X \langle d_A u, X_B(u) \rangle + \langle F_A, D_A B \rangle$$

$$= \int_X \langle d_A u, L_u B \rangle + \langle F_A, D_A B \rangle$$

$$= \int_X \langle L^*_u d_A u + D_A^* F_A, B \rangle = 0,$$

where $L_u : g \rightarrow u^*TM$ defined by $\xi \in g \mapsto X_{\xi}(u)$, $L^*_u : u^*TM \rightarrow g$ is adjoint of $L_u$, $D_A^*$ is adjoint operator of $D_A$.

Therefore we obtain the second Euler-Lagrange equation of the Yang-Mills-Higgs functional for $A$:

$$L^*_u d_A u + D_A^* F_A = 0.$$ 

To sum up, we obtain
**Theorem 2.4.** The Euler-Lagrange equations of Yang-Mills-Higgs functional

\[ \mathcal{YMH}(A, u) = |F_A|^2_{L^2} + |d_A u|^2_{L^2} + |\mu(u) - c|^2_{L^2} \]

are as follows:

\[ \nabla^* d_A u + (d\mu)^*(\mu(u) - c) = 0; \]
\[ L^*_A d_A u + D^*_A F_A = 0. \]

**Remark 2.1.** When \( G = 1 \), then the Euler-Lagrange equations above can be reduced to the equation of harmonic maps

\[ \nabla^* du = -\text{Trace} \nabla du = 0. \]

When the fiber \( M = pt \), then the Euler-Lagrange equations above can be reduced to the equation of Yang-Mills connections [1, 6]:

\[ D^*_A F_A = 0. \]

Therefore, the Euler-Lagrange equations of Yang-Mills-Higgs functional are common generalization of harmonic maps and Yang-Mills connections.

In addition, there is the gauge group \( G = Aut(P) \) action on pair \((A, u)\). Similar to the Yang-Mills functional, the Yang-Mills-Higgs functional is invariant under group \( G \) action, therefore we have

**Proposition 2.5.** The Euler-Lagrange equations above are invariant under the gauge group \( G = Aut(P) \) action.

### 3 Heat flow

In this section, we will prove the long term existence of the heat flow of Yang-Mills-Higgs functional (2). The heat flow equations are

\[ \frac{\partial A}{\partial t} = -L^*_u d_A u - D^*_A F_A; \]
\[ \frac{\partial u}{\partial t} = -\nabla^*_A d_A u - (d\mu)^*(\mu(u) - c). \]

Modifying Hong [8]'s method, we first prove the energy inequality of Yang-Mills-Higgs functional.
3.1 Energy inequality

Consider the behavior of the Yang-Mills-Higgs functional $\mathcal{YMH}(A_t, u_t)$ along the heat flow equations

\[
\begin{align*}
\frac{\partial A}{\partial t} &= -L^*_u d_A u - D^*_A F_A; \\
\frac{\partial u}{\partial t} &= -\nabla^*_A d_A u - (d\mu)^*(\mu(u) - c).
\end{align*}
\]

Following [8], we can similarly obtain

**Lemma 3.1.** (Energy inequality) If the pair $(A_t, u_t)$ is a solution to Yang-Mills-Higgs heat flow (5), (6) on $X \times [0, \infty)$, then for $t < \infty$

\[
\mathcal{YMH}(A_t, u_t) + 2 \int_0^t \int_X (|\frac{\partial u}{\partial t}|^2 + |L^*_u d_A u + D^*_A F_A|^2) dx d\tau = \mathcal{YMH}(A_0, u_0).
\]

(14)

**Proof.** On one side, by (5), (6) we have

\[
\int_X |\frac{\partial u}{\partial t}|^2 dx = \left(\frac{\partial u}{\partial t}, -\nabla^*_A d_A u - (d\mu)^*(\mu(u) - c)\right)(6)
\]

\[
= -(\frac{\partial u}{\partial t}, \nabla^*_A d_A u) - (\frac{\partial u}{\partial t}, (d\mu)^*(\mu(u) - c))
\]

\[
= -(\nabla^*_A \frac{\partial u}{\partial t}, d_A u) - (\frac{\partial u}{\partial t}, (d\mu)^*(\mu(u) - c))
\]

\[
= -(\frac{\partial (d_A u)}{\partial t}, d_A u) + (\frac{\partial A}{\partial t}, d_A u) - (\frac{\partial}{\partial t}, (d\mu)^*(\mu(u) - c))
\]

\[
= -(\frac{\partial (d_A u)}{\partial t}, d_A u) - (L^*_u d_A u + D^*_A F_A)(u), d_A u
\]

\[
-((d\mu)(\frac{\partial}{\partial t}), \mu(u) - c)(5)
\]

\[
= -\frac{1}{2} \frac{d}{dt} \int_X (|d_A u|^2 + |\mu(u) - c|^2) - (L^*_u d_A u + D^*_A F_A), L^*_u d_A u).
\]

On the other side,

\[
\frac{1}{2} \frac{d}{dt} \int_X |F_A|^2
\]

\[
= \left(\frac{\partial F_A}{\partial t}, F_A\right)
\]

\[
= \left(D_A \frac{\partial A}{\partial t}, F_A\right)
\]

\[
= -(D_A (L^*_u d_A u + D^*_A F_A), F_A)(5)
\]

\[
= -(L^*_u d_A u + D^*_A F_A, D^*_A F_A).
\]

Therefore, we have

\[
\frac{1}{2} \frac{d}{dt} \int_X (|F_A|^2 + |d_A u|^2 + |\mu(u) - c|^2) dx = -\int_X (|\frac{\partial u}{\partial t}|^2 + |L^*_u d_A u + D^*_A F_A|^2) dx.
\]

Integration on both sides we complete the proof. \qed
Corollary 3.2. Yang-Mills-Higgs functional decreases along the heat flow.

3.2 Local existence

Following Donaldson’s approach [6, 8], consider the complex gauge group \( \mathcal{G}^\mathbb{C} \), which acts on \( A_{1,1} \) by

\[
\bar{\partial}_{g(A)} = g \cdot \bar{\partial}_A \cdot g^{-1}, \quad \partial_{g(A)} = g^{* -1} \cdot \partial_A \cdot g^*
\]

where \( g^* \) denotes the conjugate transpose of \( g \). Extending the action of the unitary gauge group,

\[
\mathcal{G} = \{ g \in \mathcal{G}^\mathbb{C} | h(g) = g^* g = I \},
\]

which means

\[
g^{-1} \cdot d_{g(A)} \cdot g = \bar{\partial}_A + h^{-1} \partial_A h, \quad g^{-1} F_{g(A)} g = F_A + \bar{\partial}_A (h^{-1} \partial_A h),
\]

(16)

where \( h = g^* g \).

Consider following heat equation for a one-parameter family \( H_t \) of metrics on some holomorphic bundle over \( X \):

\[
\frac{\partial H}{\partial t} = -2i H [\hat{F}_t + \mu(u) - c],
\]

(17)

where \( \hat{F}_t = \wedge F_{H_t} \). This heat equation is completely equivalent to the equation:

\[
\frac{\partial h}{\partial t} = -\triangle_{A_0} h - i [h \hat{F}_{A_0} + \hat{F}_{A_0} h + 2(\mu(u_0) - c)h] + 2i (\bar{\partial}_{A_0} hh^{-1} \partial_{A_0} h),
\]

(18)

with \( h(0) = I \), where \( \triangle_{A_0} \) is the Laplacian defined through an integrable connection \( A_0 (\bar{\partial}^2_{A_0} = 0) \), \( h \) is a positive self-adjoint endomorphism of a unitary bundle such that \( H(t) = H_0 h(t) \), and \( H_0 \) is the initial Hermitian metric on \( P \). As pointed out in [6, 8], (18) is a nonlinear parabolic equation, so we obtain a short time solution to (18) by standard parabolic PDE theory. Therefore we obtain a short-time solution to (18).

Then we need do some necessary calculations. First we need generalize the Kähler identities ([6, 7, 11]) from vector bundle-values case to fiber bundle-values case where the fibre is a manifold not a vector space.

Fix the Kähler metric on \( X \) as

\[
h_0 = h_{kj} dz_k \otimes d\bar{z}_j,
\]

and define the Kähler form \( \omega_X \) as

\[
\omega_X = \frac{i}{2} h_{kj} dz_k \wedge d\bar{z}_j, \quad i = \sqrt{-1}.
\]
Let $A$ be a connection on the principal bundle $P$ over the Kähler manifold $X$, so we have
\[ d_A : C^\infty_G(P,M) \to \Omega^1_G(P,u^*TM), d_Au := du + LuA. \]
Then we have
\[ \partial_A : C^\infty_G(P,M) \to \Omega^{1,0}_G(P,u^*TM) \]
\[ \partial_Au = \frac{1}{2}(d_Au - J \cdot d_Au \cdot j), \]
and
\[ \bar{\partial}_A : C^\infty_G(P,M) \to \Omega^{0,1}_G(P,u^*TM) \]
\[ \bar{\partial}_Au = \frac{1}{2}(d_Au + J \cdot d_Au \cdot j), \]
making up
\[ d_A = \partial_A + \bar{\partial}_A, \]
where $j$ is the fixed complex structure of the Kähler manifold $X$, and $J$ is a $\omega$-compatible almost complex structure of the symplectic manifold $M$.

We denote $\Omega^k$ the $k$-form and recall the decomposition
\[ \Omega^k = \sum_{p+q=k} \Omega^{p,q}, \]
of $k$-forms into $(p,q)$ forms. On the Kähler manifold $X$ with Kähler form $\omega_X$ we define
\[ L : \Omega^{p,q} \to \Omega^{p+1,q+1}, L(\eta) = \eta \wedge \omega_X. \]
The we can define an algebraic trace operator $\wedge$ by
\[ \wedge = L^* : \Omega^{p,q} \to \Omega^{p-1,q-1}. \]
Through the definition of $\wedge$, we have

**Lemma 3.3.** *(Generalized Kähler identities)* When the fiber $M$ of fiber bundle $F = P \times_G M \to X$ is a compact Kähler manifold, or equivalently $J$ is a complex structure of $M$, we have
\[ \bar{\partial}_A^* = i[\partial_A, \wedge]; \quad \eqref{eq:19} \]
\[ \partial_A^* = -i[\bar{\partial}_A, \wedge]. \quad \eqref{eq:20} \]

**Proof.** Our case is essentially same as vector bundle-value case [6, 7, 11]. □

Then we will do some calculations.
Lemma 3.4. Let $A_0 \in \mathcal{A}^{1,1}$ be a connection on $P$. Assume that $u_0 \in C_G^\infty(P, M)$ is holomorphic with respect to the connection $A_0$ and $J$ ($A_0$ and $J$ determine a holomorphic structure on $P \times_G M$), i.e., $\bar{\partial}_{A_0} u_0 = 0$. Assume that $A(t) = g(t)(A_0), u(t) = g(t)(u_0), g(t) \in G^C$. Then we have

$$\bar{\partial}_{A(t)} u(t) = 0, F_{A(t)} \in \Omega^{1,1}.$$  \hfill (21)

Proof. By (15) and (16), we have

$$\bar{\partial}_{g(A_0)} = g \cdot \bar{\partial}_{A_0} \cdot g^{-1};$$

$$g^{-1}F_{g(A_0)} g = F_{A_0} + \bar{\partial}_{A_0}(h^{-1} \partial h),$$

where $h = g^* g$. Thus by the transformation formulas and conditions above we have

$$F_{A(t)} = F_{g^*(A_0)} = g(F_{A_0} + \bar{\partial}_{A_0}(h^{-1} \partial h))g^{-1} \in \Omega^{1,1}.$$  

And

$$\bar{\partial}_{A(t)} u(t) = g \cdot \bar{\partial}_{A_0} \cdot g^{-1} g u_0 = g \bar{\partial}_{A_0} u_0 = 0.$$  

This completes the proof. \hfill \Box

Lemma 3.5. Assume the same conditions as lemma 3.4. and write

$$A = A(t) = g(t)(A_0), u = u(t) = g(t)(u_0),$$

we have

$$(\bar{\partial}_A - \partial_A)(\mu(u)) = i \cdot L^*_u (d_A u).$$  \hfill (22)

Proof. By previous definitions of $\bar{\partial}_A, \partial_A$, we have

$$(\bar{\partial}_A - \partial_A)(\mu(u)) = i \cdot d_A(\mu(u)) \cdot j = i \cdot d\mu(u)(d_A u \cdot j),$$  \hfill (23)

where $j$ is the fixed complex structure of $X, i = \sqrt{-1}$.

On the other hand, by lemma 2.3, we have

$$L^*_u (d_A u) = d\mu(u) J(d_A u) = d\mu(u)(J \cdot d_A u).$$  \hfill (24)

By lemma 3.4, we know

$$0 = \bar{\partial}_{A(t)} u(t) = \bar{\partial}_A u = d_A u + J \cdot d_A u \cdot j$$

$$\Leftrightarrow J \cdot d_A u = d_A u \cdot j.$$  \hfill (25)

Thus combining (23), (24) and (25) we complete the proof. \hfill \Box
Lemma 3.6. Suppose that $M$ is a compact Kähler manifold. Let $A_0 \in \mathcal{A}^{1,1}$, and $u_0 \in C^\infty_G (P, M)$ such that $\bar{\partial}_{A_0} u_0 = 0$. Assume that $A = A(t) = g(t)(A_0), u = u(t) = g(t)(u_0), g(t) \in G_C$. We have

$$d_A^* F_A = i(\partial_A - \bar{\partial}_A) \wedge F_A. \quad (26)$$

Proof. First by Bianchi identity $d_A F_A = 0$ we have

$$\partial_A F_A = - \bar{\partial}_A F_A. \quad (27)$$

By lemma 3.4 we know $F_A \in \Omega^{1,1}$, so $\partial_A F_A \in \Omega^{2,1}, \bar{\partial}_A F_A \in \Omega^{1,2}$, we have

$$\partial_A F_A = \bar{\partial}_A F_A = 0. \quad (28)$$

Thus by generalized Kähler identities (lemma 3.3) and (28) we have

$$d_A^* F_A = (\bar{\partial}_A^* + \partial_A^*) F_A = i(\partial_A - \bar{\partial}_A) \wedge F_A - i \wedge (\partial_A F_A - \bar{\partial}_A F_A) = i(\partial_A - \bar{\partial}_A) \wedge F_A.$$

This completes the proof.

Lemma 3.7. Assume the same conditions as lemma 3.6, we have

$$d_A^* d_A = i \wedge (\bar{\partial}_A \partial_A - \partial_A \bar{\partial}_A)$$

and

$$d_A^* d_A - i \wedge F_A = 2 \bar{\partial}_A^* \bar{\partial}_A.$$

Proof. By generalized Kähler identities and lemma 3.4,

$$d_A^* d_A = (\bar{\partial}_A^* + \partial_A^*)(\bar{\partial}_A + \partial_A) = i \wedge (\partial_A \partial_A - \partial_A \bar{\partial}_A),$$

and we also have

$$d_A^* d_A = \partial_A^* \partial_A + \bar{\partial}_A^* \bar{\partial}_A.$$

Then

$$\partial_A^* \partial_A - \bar{\partial}_A^* \bar{\partial}_A = i \wedge \partial_A \partial_A + i \wedge \partial_A \bar{\partial}_A = i \wedge (d_A^2) = i \wedge F_A.$$

Therefore we have

$$d_A^* d_A - i \wedge F_A = 2 \bar{\partial}_A^* \bar{\partial}_A.$$

This completes the proof of lemma 3.7.
Now take any $g \in G^C$ such that $g^*g = h$ (for example $g = h^{1/2}$). Since $h$ solve (18), we have

$$\frac{\partial g}{\partial t}g^{-1} + g^*\frac{\partial g^*}{\partial t} = -2i[\wedge F_g(A_0) + \mu(u) - c].$$  

(29)

Here we use a fact that the moment map $\mu$ is equivariant, i.e., $g\mu(u_0)g^{-1} = \mu(g(u_0)) = \mu(u)$. Then as [8], at the time $t$,

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}(\partial_{(g+\epsilon\partial g)}A_0 + \tilde{\partial}_{(g+\epsilon\partial g)}A_0)
= \partial_A(g^{-1}\partial g) + \tilde{\partial}_A(g\partial g^{-1}).$$

So we have

$$\frac{\partial A}{\partial t} = \partial_A(g^*\partial g^*) - \tilde{\partial}_A(\partial_{gg^{-1}})
= -\partial_A i[\wedge F_A + \mu(u) - c] - \frac{1}{2}\partial_A(\partial_{gg^{-1}}) + \frac{1}{2}\partial_A(g^*\partial g^*) + 
\tilde{\partial}_A i[\wedge F_A + \mu(u) - c] + \frac{1}{2}\tilde{\partial}_A(g^*\partial g^*) - \frac{1}{2}\tilde{\partial}_A(\partial_{gg^{-1}})
= i(\tilde{\partial}_A - \partial_A)[\wedge F_A + \mu(u) - c] + d_A(\alpha(t))
= -i(\tilde{\partial}_A - \partial_A)\wedge F_A + i(\tilde{\partial}_A - \partial_A)(\mu(u) - c) + d_A(\alpha(t)).$$

Due to lemma 3.5 and lemma 3.6, we have

$$\frac{\partial A}{\partial t} = -d_A^*F_A - L_u^*(d_Au) + d_A(\alpha(t)), \quad (30)$$

where $\alpha(t)$ is defined by $\alpha(t) = \frac{1}{2}(g^*\frac{\partial g^*}{\partial t} - \partial_{gg^{-1}})$.

Using (29) we obtain

$$\frac{\partial u}{\partial t} = \partial_{gg^{-1}}u
= -i[\wedge F_A + \mu(u) - c]u - \frac{1}{2}(g^*\frac{\partial g^*}{\partial t} - \partial_{gg^{-1}})u
= -i \wedge F_A u - iL_u(\mu(u) - c) - \alpha(t)u.$$ 

By lemma 2.3, lemma 3.4 and lemma 3.7, we have

$$\frac{\partial u}{\partial t} = -d_A^*d_Au - (d\mu)^*(\mu(u) - c) - \alpha(t)u. \quad (31)$$

Assume $h$ is the solution of (18), let $g = h^{\frac{1}{2}}$. The corresponding pair $(A(t), \tilde{u}(t)) = (g(A_0), g(u_0))$ thus is a solution to (30) and (31). Through a
gauge transformation of the equivalent flow (30), (31), we prove the local existence of Yang-Mills-Higgs heat flow.

Now let $S(t)$ be the unique smooth solution to the following initial value problem:

$$\frac{dS}{dt} \cdot S^{-1} = -\alpha(t), S(0) = I, \quad (32)$$

where

$$\alpha(t) = \frac{1}{2}(g \cdot \partial g^* - \partial_t gg^{-1}).$$

Let

$$d_A = S^{-1} \cdot d_{\tilde{A}} \cdot S, \quad u = S^{-1} \tilde{u},$$

we have

$$S^{-1} \cdot d^{\ast}_{\tilde{A}} F_{\tilde{A}} \cdot S = d^{\ast}_A F_A;$$
$$S^{-1} \cdot L_u(d_{\tilde{A}} \tilde{u}) \cdot S = L_u^u(d_A u),$$

and

$$d_{\tilde{A}}(\alpha) = d_{\tilde{A}} \cdot \alpha - \alpha \cdot d_{\tilde{A}}.$$

Combining these equalities with (32), we have

$$\frac{\partial A}{\partial t} = -d^u_A F_A - L_u^u(d_A u);$$
$$\frac{\partial u}{\partial t} = -d^u_A d_A u - (d\mu)^u(\mu(u) - c).$$

We ultimately get a smooth solution on $X \times [0, \epsilon)$. This completes the proof of the following local existence theorem:

**Theorem 3.8 (Local existence).** Assume that both the base manifold $X$ and the fiber $M$ are compact Kähler manifolds, where $M$ supports a Hamiltonian $G$-group action, and $P$ is a holomorphic $G$-principal bundle over $X$. Let $A_0 \in A^{1,1}$ be a given smooth connection on $P$ with curvature $F_{A_0}$ of type $(1,1)$ and let $u_0 \in \mathcal{L}$ be holomorphic, i.e. $\tilde{\partial}_{A_0} u_0 = 0$. Then there exist a positive constant $\epsilon > 0$ and a smooth solution $(A(x,t), u(x,t))$ such that $(A, u)$ solve the Yang-Mills-Higgs heat flow (5), (6) in $X \times [0, \epsilon)$ with initial values $A(x,0) = A_0$ and $u(x,0) = u_0$. 

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3.3 Global existence

Recall the equivalent heat equation (17)

\[
\frac{\partial H_t}{\partial t} = -2iH_t[\hat{F}_t + \mu(u) - c].
\]

In order to study the uniqueness of heat flow, we also introduce a distance between metrics as in [6].

**Definition 3.1.** For any two metrics \(H, K\) on the principle \(G\)-bundle \(P\), set

\[
\tau(H, K) = Tr(H^{-1}K);
\]

\[
\sigma(H, K) = \tau(H, K) + \tau(K, H) - 2\text{rank}(G).
\]

It follows immediately from that the inequality

\[
\lambda + \lambda^{-1} \geq 2, \forall \lambda \geq 0,
\]

that \(\sigma(H, K) \geq 0\) with equality if and only if \(H = K\). Following [6], we can prove a lemma which is exactly the same as [6, Prop. 12]

**Lemma 3.9.** If \(H_t, K_t\) are two solutions of the evolution equation (17), then writing \(\sigma = \sigma(H, K)\), we have

\[
\frac{\partial \sigma}{\partial t} + \Delta \sigma \leq 0.
\]

Using lemma 3.9, we can prove local uniqueness of the heat equation (17)

**Theorem 3.10.** (Local uniqueness) Any two smooth solutions \(H_t, K_t\), which are defined for \(0 \leq t < \epsilon\), are continuous at \(t = 0\), and we have the same initial condition \(H_0 = K_0\), agree for all \(t \in [0, \epsilon)\).

**Proof.** Using lemma 3.9, apply the maximum principle to \(\sigma(H_t, K_t)\).

Now we prepare to prove global existence of the heat flow (5),(6). Recall that we take an invariant inner product on \(\mathfrak{g}\), which induces a norm \(\| \cdot \|\). Then set

\[
\hat{e} = | \wedge F_A + \mu(u) - c|^2.
\]

By Theorem 3.8, assume that \((A, u)\) is a solution of the Yang-Mills-Higgs heat flow (5),(6) on \(X \times [0, T)\):

\[
\frac{\partial A}{\partial t} = -L^*_udA - D^*_AF_A;
\]

\[
\frac{\partial u}{\partial t} = -\nabla^*_AdAu - (d\mu)^*(\mu(u) - c),
\]

with curvature \(F_A \in \Omega^{1,1}\) and \(\bar{\partial}u = 0\) on \(X \times [0, T)\), then we have
Theorem 3.11. Assume that both the base manifold $X$ and the fiber $M$ are compact Kähler manifolds, where $M$ supports a Hamiltonian $G$-group action, and $P$ is a holomorphic $G$-principal bundle over $X$. Let $A_0 \in \mathcal{A}^{1,1}$ be a given smooth connection on $P$ with curvature $F_{A_0}$ of type $(1,1)$ and let $u_0 \in \mathcal{L}$ be holomorphic, i.e. $\partial A_0 u_0 = 0$. Suppose that $(A,u)$ is local solution of Yang-Mills-Higgs heat flow (5),(6) in $X \times [0,\varepsilon)$ with initial values $A_0$ and $u_0$, then we have

$$\left( \frac{\partial}{\partial t} + \triangle \right) \hat{\dot{e}} \leq 0,$$

where $\triangle = \nabla^*_A \nabla_A$ is the Laplacian.

To prove this theorem, it suffices to prove an equality

Lemma 3.12. Assume the same conditions as in Theorem 3.11, we have

$$\left( \frac{\partial}{\partial t} + \triangle \right) \hat{\dot{e}} = -2|\nabla_A(\wedge F_A + \mu(u) - c)|^2 - 2|L_u(\wedge F_A + \mu(u) - c)|^2.$$

Proof. By direct computation, we have

$$\frac{1}{2} \frac{\partial}{\partial t} |\wedge F_A + \mu(u) - c|^2 = Re \left\langle \frac{\partial}{\partial t} (\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= Re \left\langle \frac{\partial}{\partial t} (\wedge F_A + \mu(u) - c), \wedge F_A + \mu(u) - c \right\rangle$$

$$= Re \left\langle \frac{\partial}{\partial t} (\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$+ Re \left\langle \frac{\partial}{\partial t} (\mu(u) - c), \wedge F_A + \mu(u) - c \right\rangle.$$

And we also have

$$Re \left\langle \frac{\partial}{\partial t} (\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= Re \left\langle \wedge d_A(\frac{\partial}{\partial t} A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= Re \left\langle - \wedge d_A(L_u^* d_A u + d_A^* F_A), \wedge F_A + \mu(u) - c \right\rangle \quad (\text{5})$$

$$= Re \left\langle i \wedge d_A(\tilde{A} - \partial A) \wedge F_A - \wedge d_A(L_u^* d_A u), \wedge F_A + \mu(u) - c \right\rangle \quad (\text{lemma 3.6})$$

$$= Re \left\langle i \wedge \partial A \tilde{\partial} A - \tilde{\partial} A \partial A \wedge F_A, \wedge F_A + \mu(u) - c \right\rangle$$

$$= Re \left\langle \wedge F_A(L_u^* d_A u), \wedge F_A + \mu(u) - c \right\rangle \quad (\tilde{\partial} A = 0)$$

$$= -Re \left\langle \nabla_A \nabla_A(\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= -Re \left\langle \wedge d_A(\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= -Re \left\langle \wedge d_A(d_A u), L_u(\wedge F_A) > -Re \left\langle \wedge d_A(d_A u), L_u(\mu(u) - c) \right\rangle \quad (\text{lemma 3.7})$$

$$= -Re \left\langle \nabla_A \nabla_A(\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle$$

$$= -Re \left\langle \wedge F_A(L_u^* d_A u), L_u(\wedge F_A) \right\rangle$$

$$= -Re \left\langle \wedge F_A(L_u^* d_A u), L_u(\mu(u) - c) \right\rangle$$

$$= -Re \left\langle \nabla_A(\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle - |L_u(\wedge F_A)|^2$$

$$= -Re \left\langle L_u(\wedge F_A), \wedge F_A + \mu(u) - c \right\rangle \quad (\tilde{d}^2 = F_A),$$

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and
\[
Re < \frac{\partial}{\partial t}(\mu(u)), \land F_A + \mu(u) - c >
\]
\[
= -Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c >
\]
\[
- Re < d\mu((d\mu)^*(\mu(u) - c)), \land F_A + \mu(u) - c > \quad ((6))
\]
\[
= - Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c >
\]
\[
- Re < L^*_u L_u (\mu(u) - c), \land F_A + \mu(u) - c > \quad (\text{lemma } 2.3)
\]
\[
= - Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c >
\]
\[
- Re < L_u (\mu(u) - c), L_u (\land F_A) > |L_u (\mu(u) - c)|^2.
\]

Then add equalities above, we have
\[
\frac{1}{2} \frac{\partial}{\partial t} \hat{e} = - Re < \nabla_A^* \nabla_A (\land F_A), \land F_A + \mu(u) - c >
\]
\[
- Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c > - |L_u (\mu(u) - c)|^2
\]
\[
- |L_u (\land F_A)|^2 - 2 Re < L_u (\land F_A), L_u (\mu(u) - c) >
\]
\[
= - Re < \nabla_A^* \nabla_A (\land F_A), \land F_A + \mu(u) - c >
\]
\[
- Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c > - |L_u (\land F_A + \mu(u) - c)|^2.
\]

On the other hand, we have
\[
\frac{1}{2} \Delta \hat{e} = \nabla_A Re < \nabla_A (\land F_A + \mu(u) - c), \land F_A + \mu(u) - c >
\]
\[
= Re < \nabla_A^* \nabla_A (\land F_A + \mu(u) - c), \land F_A + \mu(u) - c >
\]
\[
- |\nabla_A (\land F_A + \mu(u) - c)|^2
\]
\[
= Re < \nabla_A^* \nabla_A (\land F_A), \land F_A + \mu(u) - c >
\]
\[
+ Re < \nabla_A^* \nabla_A (\mu(u)), \land F_A + \mu(u) - c >
\]
\[
- |\nabla_A (\land F_A + \mu(u) - c)|^2
\]
\[
= Re < \nabla_A^* \nabla_A (\land F_A), \land F_A + \mu(u) - c >
\]
\[
+ Re < d\mu(\nabla_A^* dA u), \land F_A + \mu(u) - c >
\]
\[
- |\nabla_A (\land F_A + \mu(u) - c)|^2.
\]

Combining equalities above we complete the proof of lemma 3.12. \(\square\)

**Lemma 3.13.** Assume the same conditions as in Theorem 3.11, then \(\sup_X \hat{e}\) is a decreasing function of \(t\).

**Proof.** This follows from Theorem 3.11 and the maximum principal for the heat operator \((\frac{d}{dt}) + \Delta\) on \(X\). \(\square\)

Then using this lemma, we can prove the following lemma similar to [8, Lemma 8].
Corollary 3.14. Let $H(t) = H_0 h(t)$ where $b(t)$ is smooth solution of (18) in $X \times [0, T)$ with the initial value $H_0$ for a finite time $T > 0$. Then $H(t)$ converges in $C^0$ to a nondegenerate continuous metric $H(T)$ as $t \to T$.

Using almost the same procedure as [6] (or [8]), we can further prove

Lemma 3.15. Let $H(t), 0 \leq t < T$, be any one-parameter family of Hermitian metrics on a holomorphic principal bundle $P$ over a compact Kähler manifold $X$ such that

(i) $H(t)$ converges in $C^0$ to some continuous metric $H(T)$ as $t \to T$,

(ii) $\sup_X |\hat F_H|$ is uniformly bounded for $t < T$.

Then $H(t)$ is bounded in $C^{1,\alpha}$ independently of $t \in [0, T)$ for $0 < \alpha < 1$.

Now we can prove the global existence of (17) in the following theorem.

Theorem 3.16. Assume that both the base manifold $X$ and the fiber $M$ are compact Kähler manifolds, where $M$ supports a Hamiltonian $G$-group action, and $P$ is a holomorphic $G$-principal bundle over $X$. Let $H_0$ be a Hermitian metric on $P$, $A_0$ the canonical connection with respect to $H_0$ (it is well-known that $A_0 \in \mathcal{A}^{1,1}$), $u_0$ a holomorphic section of the fiber bundle $P \times_G M$, i.e. $\bar \partial_{A_0} u_0 = 0$. Then the equation

$$\frac{\partial H}{\partial t} = -2iH[\wedge F_H + \mu(u) - c]$$

has a unique solution $H(t)$ which exists on $X$ for $0 \leq t < \infty$.

Proof. The proof is totally similar to [8], so here we only give a sketch. The local existence and uniqueness of this equation has been proved by previous result. Suppose that the solution exists for $0 \leq t < T$. By lemma 3.13, $\sup_X |\hat F_H|^2$ is bounded independently of $t \in [0, T)$ and $\hat F_H$ is bounded independently of $t \in [0, T)$. By corollary 3.14, $H(t)$ converges in $C^0$ to a nondegenerate continuous limit metric $H(T)$ as $t \to T$. Thus by lemma 3.15, $H(t)$ is bounded in $C^{1,\alpha}$ independently of $t \in [0, T)$. Let $H(t) = H_0 h(t)$, where $h(t)$ solves

$$\frac{\partial h}{\partial t} = -\Delta_{A_0} h - i[h\hat F_{A_0} + \hat F_{A_0} h + 2(\mu(u_0) - c)h] + 2i(\bar \partial_{A_0} hh^{-1}\partial_{A_0} h).$$

Then it is totally the same as the proof of [8, Theorem 11] to prove that $h$ is $C^{2,\alpha}$ and $\frac{\partial h}{\partial t}$ is $C^{\alpha}$ with bounds independent of $t \in [0, T)$. Thus $H(t) \to H(T)$ in $C^{2,\alpha}$, hence in $C^\infty$. For the initial value $H(T)$ we use local existence theorem (Theorem 3.8) again. Therefore the solution continues to exist for $t < T + \epsilon$, for some $\epsilon$. This completes the proof.

Proof of Theorem 1.1. By Theorem 3.8, Theorem 3.10 and Theorem 3.16, we can immediately obtain Theorem 1.1.
4 Application

In this section we will use the heat flow of Yang-Mills-Higgs functional to give a new proof of Theorem 1.2 which is critical to the Hitchin-Kobayashi correspondence theorem [13, 14]. The idea is the same as [8]’s approach to combine Donaldson’s method [6] with Uhlenbeck and Yau’s method [19].

Due to the property of the heat flow, we apply a well-known PDE result to prove an inequality which relates $C^0$ and $L^1$ norms of $s \in Met_{p,B}^0$, that is

$$\sup |s| \leq C_B \|s\|_{L^1}. \quad (37)$$

Then we prove

$$\|s\|_{L^1} \leq C_1' \Psi^c(e^s) + C_2'. \quad (38)$$

After combining the two inequalities, we can prove Theorem 1.2.

4.1 c-stability

First let’s recall the definition of c-stability [14]. Suppose that $(M, \omega, \mu)$ is a compact symplectic manifold with a Hamiltonian group $G$-action and $\mu : M \to g$ is the corresponding moment map, where the Lie algebra $g$ carries an invariant inner product $\langle \cdot, \cdot \rangle$.

Definition 4.1. $\forall x \in M$ and $s \in g$. Let $\lambda_t(x; s) = \mu_s(e^{its}x) = \langle \mu(e^{its}x), s \rangle$.

Define the maximal weight $\lambda(x; s)$ of the action of $s$ on $x$ to be

$$\lambda(x; s) = \lim_{t \to \infty} \lambda_t(x; s) \in \mathbb{R} \cup \{\infty\}.$$ 

Let $g = \text{Lie}(G)$, we have $g = l \oplus g^*$, where $l$ is the center of $g$, and $g^* = [g, g]$. Suppose that $h \subset g^*$ is a Cartan subalgebra, and $R \subset h^*$ is the set of roots, then we have $g = l \oplus h \oplus \bigoplus_{\alpha \in R} g_{\alpha}$, where $g_{\alpha} \subset g^*$ is the subspace on which $h$ acts through the character $\alpha \in h^*$. We can decompose the set $R$ of roots in positive and negative roots as: $R = R^+ \cup R^-$. Write the set of simple roots $\Delta = (\alpha_1, \ldots, \alpha_r) \subset R^+$. Take a maximal compact subgroup $K \subset G$, and let $\mathfrak{k} = \text{Lie}(K)$. Then take any subset $A = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Delta$, and let

$$D = D_A = \{\alpha \in R| \alpha = \Sigma_{j=1}^r m_j \alpha_j, \text{ where } m_{i_t} \geq 0 \text{ for } 1 \leq t \leq s\}.$$ 

Definition 4.2. We call the subalgebra $q = l \oplus h \oplus \bigoplus_{\alpha \in D} g_{\alpha}$ parabolic subalgebra of $g$ with respect to $A \subset \Delta$. The connected subgroup $Q$ of $G$ whose subalgebra is $q$ is called the parabolic subgroup of $G$ with respect to $A$. Let $\lambda_1, \ldots, \lambda_r$ be the set of fundamental weights, then we call any positive (resp. negative) linear combination of the fundamental weights $\lambda_{i_1}, \ldots, \lambda_{i_s}$ plus an element of the dual of $i(l \cap \mathfrak{k})$ a dominant (resp. antidominant)
character on \(q\) (or on \(Q\)).

**Definition 4.3.** Let \(P \to X\) be principal \(G\)-bundle, \(P_G = P \times_K G\), and \(P(G/Q) = P_G \times_G (G/Q)\). We call the space \(\Gamma(P(G/Q))\) of sections of the bundle \(P(G/Q)\) as the space of *reductions* of the structure group of \(P_G\) from \(G\) to \(Q\). A reduction \(\sigma\) is holomorphic if the map \(\sigma : X \to P(G/Q)\) is holomorphic.

According to [14], there is a section \(g_{\sigma, \chi} \in \Omega^0(P \times \text{Ad}\mathfrak{k})\) which is fiberwise the dual of \(\chi\) for a parabolic subgroup \(Q \subset G\), a reduction \(\sigma \in \Gamma(P(G/Q))\) and an antidominant character \(\chi\). [14] has also proved that \(L\) is a Kähler manifold with a Hamiltonian \(G_K(= \Gamma(P \times \text{Ad}\mathfrak{k})\)-group action, its corresponding moment map is

\[\mu_L(u) = \mu(u) = \mu \cdot u.\]

Therefore, we can define the maximal weight of \(s \in \text{Lie}(G_K) = \Omega^0(P \times \text{Ad}\mathfrak{k})\) acting on a section \(u \in L\) as

\[\int_{x \in X} \lambda(u(x); s(x)).\]

Fixing any central element \(c \in \mathfrak{l} \cap \mathfrak{k}, \forall u \in L\), we define the *c–total degree* of the pair \((\sigma, \chi)\) as follows:

\[T^c_u(\sigma, \chi) = \text{deg}(\sigma, \chi) + \int_{x \in X} \lambda(u(x); -ig_{\sigma, \chi}(x)) + < i\chi, c > \text{Vol}(X),\]

where the definition of \(\text{deg}(\sigma, \chi)\) can be found in [14].

**Definition 4.4.** A pair \((A, u) \in \mathcal{A}^{1,1} \times L\) is called *c-stable*, if for any \(X_0 \subset X\) whose complement \(X \setminus X_0\) is a complex codimension 2 submanifold of \(X\), for any parabolic subgroup \(Q \subset G\), for any antidominant character \(\chi\) of \(Q\), and for any holomorphic reduction \(\sigma \in \Gamma(X_0; P(G/Q))\), we have

\[T^c_u(\sigma, \chi) > 0.\]

**Definition 4.5.** A pair \((A, u)\) is called *simple* if there is no semisimple element in \(\text{Lie}(G_G)\) which leaves \((A, u)\) fixed, where \(G_G = \Gamma(P \times \text{Ad}\ G)\).

Now we can state the main theorem of [14] as follows:

**Theorem 4.1** (Hitchin-Kobayashi correspondence). Assume that \((A, u) \in \mathcal{A}^{1,1} \times L\) is a simple pair. Then there exists a gauge transformation \(g \in G_G\) such that

\[\wedge F_g(A) + \mu(g(u)) = c,\]

if and only if \((A, u)\) is c-stable.
4.2 Heat flow proof of Mundet’s theorem

According to [14], both $\mathcal{A}^{1,1}$ and $\mathcal{L}$ are Kähler manifolds. Hence $\mathcal{A}^{1,1} \times \mathcal{L}$ is also a Kähler manifold with a Hamiltonian $\mathcal{G}_K$-action, and the moment map is $\mu^c(A, u) = \wedge F_A + \mu(u) - c$. Then we have the integral of the moment map $\Psi^c$ (for more details see [13, 14]) defined on $\mathcal{A}^{1,1} \times \mathcal{L} \times \mathcal{G}_G$. We will see that if the pair $(A, u) \in \mathcal{A}^{1,1} \times \mathcal{L}$ is $c$-stable, then the map $\Psi^c$ satisfies an inequality like that in [4, 16]. This method is almost the same as that appears in [8]. Therefore, we only give a sketch in some steps of the proof, referring to [8] (or [4]) for details.

Recall that there is an invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ which induces a norm $|\cdot|$. On $\Omega^0(P \times \Ad \mathfrak{g})$ we can define the $L^p$ norm:

$$||s||_{L^p} = \left( \int_X |s(x)|^p \right)^{\frac{1}{p}},$$

and the $L^2_p$ norm:

$$||s||_{L^2_p} = ||s||_{L^p} + ||d_A s||_{L^p} + ||\nabla d_A s||_{L^p},$$

where $d_A$ is the covariant derivative with respect to the connection $A$, and

$$\nabla = \nabla^{LC} \otimes d_A : \Omega^0(T^*X \otimes P \times \Ad \mathfrak{g}) \rightarrow \Omega^1(T^*X \otimes P \times \Ad \mathfrak{g}),$$

where $\nabla^{LC}$ is the Levi-Civita connection. Now we can define

$$\mathcal{M}et^p_2 = L^2_p(P \times \Ad \mathfrak{k})$$

as the closure of $\Omega^0(P \times \Ad \mathfrak{k})$ with respect to the norm $||\cdot||_{L^2_p}$. Then take a subset of $\mathcal{M}et^p_2$ as follows:

$$\mathcal{M}et^p_{2, B} = \{ s \in \mathcal{M}et^p_2 ||\mu^c(e^s(A, u))||^p_{L^p} \leq B \},$$

where $B \in \mathbb{R}^+$. 

Next using previous theorems on the global existence of heat flow in section 3, we can prove a lemma totally similar to [8, lemma 12]:

**Lemma 4.2.** Let $H(t) = H_0 h(t) = H_0 e^{u(t)}$ be Hermitian metrics which $h$ is the global solution of (18) on $X \times [0, +\infty)$. Then for any sequence of $t_j \rightarrow \infty$, $\Tr(\wedge F_H(t_j) + \mu(e^{u(t_j)} u_0))$ converges to some constant $\alpha$ in weakly $L^1_1$. Moreover, $\int_X <\mu(e^{u(t_j)} u_0), s(t_j)>$ also converges a constant as $t_j \rightarrow \infty$.

**Proof.** By lemma 3.1 (energy inequality), we know

$$\int |L^* \alpha| d_A^* u(t_j) + D^*_{A(t_j)} F_A(t_j)^2 \rightarrow 0,$$  \hspace{1cm} (39)

as $t_j \rightarrow \infty$. By lemma 3.5 and lemma 3.6 we have

$$i(-\partial_A + \bar{\partial}_{A^*}) (\wedge F_A + \mu(u)) = -L^*_u d_A u - D^*_{A^*} F_A.$$  \hspace{1cm} (40)
Therefore
\[ \int |\nabla A(\wedge F_{A(t_j)} + \mu(u(t_j)))|^2 = \int |L^*_{u(t_j)}dA_{A(t_j)}u(t_j) + D^*_{A(t_j)}F_{A(t_j)}|^2 \to 0. \]

Since \( \nabla_A Tr = Tr \nabla_A \), we have
\[ \int |\nabla_A Tr(\wedge F_{A(t_j)} + \mu(u(t_j)))|^2 \to 0. \tag{41} \]

By Theorem 3.11, we know that \( |\wedge F_{A(t_j)} + \mu(u(t_j))| \) is bounded and then
\[ Tr(\wedge F_H(t_j) + \mu(e^{(t_j})u_0)) = Tr(\wedge F_{A(t_j)} + \mu(u(t_j))) \to \alpha, \]
weakly in \( L^2_1 \) as \( t_j \to \infty \) for some function \( \alpha \) with \( \nabla \alpha = 0 \). This means that \( \alpha \) is a constant. By Chern-Weil theory, we have
\[ 2i \int Tr(\wedge F_H + \mu(e^s u_0)) = 4\pi C_1(P) + 2i \int < \mu(e^s u_0), s >. \]

Then
\[ \lim_{t_j \to \infty} \int < \mu(e^{s(t_j)}u_0), s(t_j) > = aVol(X) + 2i\pi C_1(P). \tag{42} \]

This completes the proof. \( \square \)

**Lemma 4.3.** Let \( H(t) = H_0h(t) = H_0 e^{s(t)} \) be Hermitian metrics where \( h(t) \) is the solution of (18), then there exists a constant \( C_B \) such that
\[ \sup |s| \leq C_B ||s||_{L^1}. \tag{43} \]

**Proof.** The proof is totally similar to the proof of [8, lemma 13]. \( \square \)

**Proof of Theorem 1.2.** Using lemma 4.3, and following Simpson’s method [8, 16], we can prove Theorem 1.2.

For \( H = H_0h = H_0 e^{s} \), define the integral of the moment map (see [14] for more details) as follows:
\[ \Psi^c(e^s) = M_{u_0,c}(H(t)) \]
\[ = 2i \int_X Tr(s \wedge F_{H_0}) dX + 2 \int_X < \phi(s) \bar{\partial}A_0 s, \bar{\partial}A_0 s > dX \]
\[ + 2i \int_X < \mu(e^s u_0) - \mu(u_0), s > dX - 2i \int_X \log \det (e^s \cdot c) dX, \]

where \( \phi \) is constructed from the function
\[ \phi(\lambda_1, \lambda_2) = \frac{e^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1) - 1}{(\lambda_2 - \lambda_1)^2} \]

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The functional can be seen as a modified Donaldson functional. In fact, when \( M = \{ pt \} \), it coincides with the Donaldson functional [6].

Then we have

\[
\frac{d}{dt} \Psi_c(e^s) = \frac{d}{dt} M_{u_0,c}(H(t)) = -4 \int_X | F_H + \mu(u) - c |^2 dX \leq 0. \tag{44}
\]

Since \( M_{u_0,c}(H(0)) = 0 \), it yields that

\[
\Psi_c(e^s) = M_{u_0,c}(H(t)) = -4 \int_0^t \int_X | F_H + \mu(u) - c |^2 dXd\tau \leq 0. \tag{45}
\]

By lemma 4.3, to prove Theorem 1.2, we only need prove

\[
||s||_{L^1} \leq C_1' \Psi_c(e^s) + C_2'. \tag{46}
\]

We can suppose the contrary and then deduce that the pair \((A, u)\) cannot be \(c\)-stable in this case. If there exists not such constants, then we can find a sequence of \(C_j \to \infty\) and \(s_j \in \mathcal{M} \) with \(||s_j||_{L^1} \to \infty\) such that \(||s_j||_{L^1} \geq C_j \Psi_c(e^s)\). Let \( l_j = ||s_j||_{L^1} \), \( u_j = l_j^{-1} s_j \), so \(||u_j||_{L^1} = 1\) and \(sup|u_j| \leq C\). Then we have

**Lemma 4.4.** ([14]) After passing to a subsequence, there exists \(u_\infty \in L_1^2(P \times \text{Ad} \ g)\) such that \(u_j \to u_\infty\) in \(L_1^2(P \times \text{Ad} \ g)\) and

\[
\lambda((A, u); -iu_\infty) \leq 0.
\]

**Proof.** Using the proposition (2) of the integral of the moment map in [14, Proposition 3.3], we have

\[
\frac{d}{dt} \Psi_c(e^s) = \frac{d}{dt} M_{u_0,c}(H(t)) = \lambda_t((A, u); -is).
\]

By (44) we can obtain

\[
\lambda_t((A, u); -iu_j) \leq 0.
\]

In particular \(\lambda_t((A, u); -iu_\infty) \leq 0, \forall t > 0\). By Definition 4.1, we have

\[
\lambda((A, u); -iu_\infty) \leq 0.
\]

This completes the proof of lemma 4.4. \(\square\)

Following [6] or [8], then using lemma 4.2, the same argument in [16, lemma 5.5] yields that all eigenvalues \(\lambda_i\) of \(u_\infty\) are constants for almost \(x \in X\), and \(u_\infty\) defines a filtration of \(V\) by holomorphic subbundles in the complement of a complex codimension 2 subvariety \(X_0\) of \(X\). The filtration of \(V\) on \(X_0\) and \(u_\infty\) lead to a holomorphic reduction \(\sigma \in \Gamma(X_0; P(G/Q))\) for some parabolic subgroup \(Q\) of \(G\), and an antidominant character \(\chi\) of
Q. By [14, lemma 4.3], the c-total degree $T^c_u(\sigma, \chi)$ of the pair $(\sigma, \chi)$ equals $\lambda((A, u); -iu_\infty)$. Then by lemma 4.4 we have

$$T^c_u(\sigma, \chi) \leq 0.$$ 

But this contradicts the c-stability condition. Thus we complete the proof of Theorem 1.2.

\[\square\]

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