On the total and strong version for Roman dominating functions in graphs

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Abstract. The total and strong version of the Roman domination number (for graphs) is introduced in this research, and the study of its mathematical properties is therefore initiated. We establish upper bounds for such a parameter, and relate it with several parameters concerning vertex domination in graphs. In addition, among other results, we show that for any tree T of order n(T) \geq 3, maximum degree \( \Delta(T) \) and \( s(T) \) support vertices, the total strong Roman domination number is bounded below by \( \left\lceil \frac{n(T)+s(T)}{\Delta(T)} \right\rceil + 1 \).

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1. Introduction

From now on we assume \( G \) is a simple graph with vertex set \( V(G) \) and edge set \( E(G) \) (\( V \) and \( E \) for short) having no isolated vertex. The order of \( G \) (or number of vertices of \( G \)) is \( |V| \), and represented by \( n = n(G) \). For a vertex \( v \in V(G) \), its open neighborhood is the set \( N_G(v) = N(v) = \{ u \in V(G) \mid uv \in E(G) \} \). The closed neighborhood of \( v \) is the set \( N_G[v] = N[v] = N(v) \cup \{ v \} \). The degree of a vertex \( v \in V \) in \( G \) is \( \deg_G(v) = d(v) = |N(v)| \). The maximum and minimum degrees of \( G \) are \( \Delta = \Delta(G) \) and \( \delta = \delta(G) \), respectively. For a fixed set \( S \subset V(G) \), its open neighborhood is \( N_G(S) = N(S) = \cup_{v \in S} N(v) \), and its closed neighborhood is \( N_G[S] = N[S] = N(S) \cup S \). The diameter of \( G \), denoted by \( \text{diam}(G) \), represents the maximum possible value among all the minimum distances between pairs of vertices of \( G \). The girth \( g(G) \) of \( G \) is taken as the length of a shortest cycle in \( G \). We make the assumption that graphs with no cycle (trees) have girth equal to \( \infty \).
A vertex of degree one in $G$ is a leaf. A vertex being adjacent to a leaf is a support vertex. For a given support vertex $v \in V(G)$, by $L_v$ we represent the set of all leaves adjacent to $v$. The special case of a tree obtained from two stars $K_{1,p}$ and $K_{1,q}$, ($p \geq q \geq 1$) by adding an edge between their central vertices is the double star $DS_{p,q}$. For a rooted tree $T$: $C(v)$ represents the set of all children of a vertex $v \in V(T)$; $D(v)$ is taken as the set of descendants of $v$; and we write $D[v] = D(v) \cup \{v\}$. The depth of the vertex $v$, written depth$(v)$, is the largest distance between $v$ and any vertex of $D(v)$. Finally, by a subtree $T_v$ of $T$, we mean a maximal subtree induced by $v$ and the descendants $D(v)$ of $v$ (that is the subtree induced by $D[v]$). For a graph $G$, the corona of $G$, denoted cor$(G)$, is a graph obtained from $G$ by adding a pendant vertex to every vertex of $G$.

To see some background on domination in graphs we suggest the books [26–28]. One common research line nowadays is related to the study of different variants of the standard domination concept. Maybe the most common variations are the total domination (see [29,30]), the independent domination (see [25]) and the Roman domination (see [22]). Each one of these three mentioned variants has its own variants. In this work, we make a contribution to some combination of two of these variants, namely the total and the Roman domination.

A set $S \subset V(G)$ is a dominating set of $G$ if every vertex not in $S$ has a neighbor in $S$, or equivalently, if $N[S] = V$. The domination number of $G$, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of $G$. By a $\gamma(G)$-set, we mean a dominating set of cardinality $\gamma(G)$. A set $D \subset V(G)$ is an efficient dominating set for $G$ if for every vertex $v \in V(G)$, there is exactly one $u \in D$ dominating $v$ (a vertex of $D$ dominates itself). The set $S$ is a total dominating set, abbreviated as TD-set, if $N(S) = V$ (every vertex of $G$ has a neighbor in $S$, including the vertices of $S$). The total domination number, denoted by $\gamma_t(G)$, is the cardinality of a smallest total dominating set of $G$.

Roman dominating functions in graphs were first formally defined by Cockayne et al. [22], and this was partly motivated by a work of Stewart [34]. The idea comes from the ancient Roman Empire and a strategy of securely protecting the empire from external attacks. In recent years, there has been an “explosion” of research works concerning Roman dominating functions, and by now, this topic is very well studied. The literature on this topic has been surveyed here [16–20]. There are still a lot of ongoing work and open problems that are of high interest. Variations of the standard Roman domination give more insight into the classical problem, and new strategies of “protecting a hypothetical Roman Empire” are of interest. We now aim to continue contributing to the topic of Roman dominating functions in graphs.

A Roman dominating function defined over the graph $G$ (RD-function for short) is a map $f: V(G) \rightarrow \{0, 1, 2\}$ such that if $u$ satisfies $f(u) = 0$, then $u$ is adjacent to a vertex $v$ with $f(v) = 2$. The weight of $f$, written $\omega(f)$, is
The Roman domination number, denoted by $\gamma_R(G)$, is the minimum weight among all RD-functions of $G$. An RD-function with the minimum possible weight $\gamma_R(G)$ in $G$ is a $\gamma_R(G)$-function. For an RD-function $f$, we assume $V_i^f = \{v \in V(G): f(v) = i\}$ for $i = 0, 1, 2$. These three sets uniquely determine $f$, and so, it can be written as $f = (V_0^f, V_1^f, V_2^f)$. Note that $\omega(f) = |V_1^f| + 2|V_2^f|$.

One recent variation of Roman dominating functions in graphs, introduced by Liu and Chang [31] (although in a more general setting), is the total Roman domination concept. A total Roman dominating function of a graph $G$ without isolated vertices (or TRD-function for short), is a Roman dominating function $f$ on $G$, such that the subgraph induced by the vertices labeled 1 or 2 under $f$, has minimum degree at least one. The total Roman domination number, denoted by $\gamma_{tR}(G)$, is the minimum possible weight among all TRD-functions on $G$. A $\gamma_{tR}(G)$-function is a TRD-function with weight $\gamma_{tR}(G)$. The total Roman domination parameter was introduced by Liu and Chang [31], but studied in a more general setting. Further on, specific studies on such parameters were developed for instance in the articles [1–4,6–14].

The defensive strategy of Roman domination states that a vertex labeled with 0 (unsecured place) must have at least a neighbor labeled with 2 (secured place). Thus, if an unsecured position is attacked by one neighbor, then a secured (or stronger) neighbor can send one of the two legions it possesses, in order to defend this neighbor vertex from the attack. However, if there is a secured place which has many neighbors labeled with 0, and a “kind of simultaneous” attack occurs, then this secured place cannot properly proceed. In order to deal with such a situation, the idea of strong Roman dominating functions in graphs was introduced in [5].

Consider a map $f : V(G) \rightarrow \{0, 1, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ over $V(G)$. Let $B_j = \{v \in V : f(v) = j\}$ for $j = 0, 1$ and let $B_2 = V \setminus (B_0 \cup B_1) = \{v \in V : f(v) \geq 2\}$ Then, $f$ is a strong Roman dominating function on $G$, abbreviated StRD-function, if every $v \in B_0$ has a neighbour $u$ with $u \in B_2$ and $f(u) \geq 1 + \left\lceil \frac{1}{2}|N(u) \cap B_0|\right\rceil$.

Total Roman domination can be also seen as a strategy of protection in which isolated elements are avoided. For the strong Roman domination, the “weakness” of protecting elements is dealt with. However, isolated elements can still occur. It is then our goal to develop a protection strategy in which isolated elements are avoided as well as strong protection is considered.

A total strong Roman dominating function, abbreviated TStRD-function, represents a strong Roman dominating function such that the set of vertices labeled positive induces a graph with minimum degree at least one. The minimum possible weight $\omega(f) = f(V) = \sum_{v \in V} f(v)$, among all total strong Roman dominating functions $f$ on $G$, is called the total strong Roman domination number of $G$, denoted by $\gamma_{tStR}(G)$. A TStRD-function $f$ is called a
Figure 1. A total strong Roman dominating function with minimum weight

$\gamma_{tStR}(G)$-function if $\omega(f) = \gamma_{tStR}(G)$. Figure 1 shows a graph with a labeling corresponding to a TStRD-function of minimum weight.

We next initiate the study of the total strong Roman domination number of graphs. We begin by establishing several tight upper bounds on the total strong Roman domination number of graphs concerning other related graph parameters and invariants. In addition, we study the behavior of such a parameter over the structure of a tree graph, and for instance prove that, for any given tree $T$ of order $n(T) \geq 3$, having maximum degree $\Delta(T)$ and $s(T)$ support vertices, $\gamma_{tStR}(T) \geq \left\lceil \frac{n(T)+s(T)}{\Delta(T)} \right\rceil + 1$.

The following basic or known results shall be used in this paper. The lower bound in the first observation follows from the fact that assigning 2 to every vertex with label at least 2 in any TStRD-function induces a TRD-function, and the upper bound, is deduced by defining a TStRD-function in which every vertex of a total dominating $S$ of $G$ is labeled with $\left\lceil \frac{\Delta+1}{2} \right\rceil$ and the remaining vertices with 0.

Observation 1. For every graph $G$ with no isolated vertex, $\gamma_{tR}(G) \leq \gamma_{tStR}(G) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil \gamma_{t}(G)$.

Observation 2. Let $G$ be a graph of order $n \geq 3$ with no isolated vertex. Then $3 \leq \gamma_{tStR}(G) \leq n$.

Observation 3. Let $G$ be a connected graph of order at least three and let $f = (B_0, B_1, B_2)$ be a $\gamma_{tStR}(G)$-function. Then,

1. $|B_2| \leq |B_0|$.
2. If $x$ is a leaf and $y$ a support vertex in $G$, then $x \notin B_2$ and $y \notin B_0$.

Proof. Every vertex of $B_0$ is adjacent to a vertex of $B_2$. If $|B_2| > |B_0|$, then there must be a vertex $v \in B_2$ such that every vertex $u \in N(v) \cap B_0$ has another neighbor $v' \in B_2$ with $v' \neq v$, or $N(v) \cap B_0 = \emptyset$. Since $v$ must have a neighbor in $B_1 \cup B_2$, both possibilities above lead to contradiction by changing the label of $v$ to 1. This proves item 1.
For item 2., if a support vertex $y$ has label 0, then all leaves adjacent to $y$ must have positive labels, and thus, they are isolated in the subgraph induced by vertices with positive labels, which is not possible in any TStRD-function. If a leaf $x$ belongs to $B_2$, then its adjacent support must belong to $B_1 \cup B_2$. Thus, by redefining the label of $x$ to 1, we obtain a new TStRD-function with smaller weight, which is a contradiction with $f$ being a $\gamma_{\text{StR}}(G)$-function.

**Theorem A.** [5] For any connected graph $G$ with $\Delta \leq 2$, $\gamma_{\text{StR}}(G) = \gamma_R(G)$.

**Observation 4.** For any connected graph $G$ with $\Delta \leq 3$, $\gamma_{\text{tStR}}(G) = \gamma_{\text{tR}}(G)$.

**Proof.** From Observation 1, $\gamma_{\text{tR}}(G) \leq \gamma_{\text{StR}}(G)$. Since every vertex $v$ labeled positive in any TStRD-function $f$ must have a neighbor also labeled positive under $f$, we deduce that such $v$ can have at most two neighbors labeled 0 under $f$, because $\Delta \leq 3$. Consequently, any TRD-function on $G$ is indeed a TStRD-function, and so, $\gamma_{\text{tR}}(G) \geq \gamma_{\text{StR}}(G)$.

**Theorem B.** [15] For any path $P_n$ and any cycle $C_n$, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.

**Theorem C.** [1] If $G$ is a nontrivial path or a cycle on $n$ vertices, then $\gamma_{\text{tR}}(G) = n$.

Let $G$ be the following family of graphs. We begin with a 4-cycle $(v_1v_2v_3v_4)$. We next add $k_1 + k_2 \geq 1$ different paths $P_2$ (that is with vertex disjoint sets), and join $v_1$ to the end of $k_1$ paths of them, and also join $v_2$ to the end of the remaining $k_2$ paths (it is possible that $k_1 = 0$ or $k_2 = 0$). Moreover, we consider the family $\mathcal{H}$ containing all the graphs that can be constructed from a double star by making one subdivision of each pendant edge of the double star, and also making a subdivision of the non-pendant edge with $r \geq 0$ vertices. These families of graphs were introduced in [1].

**Theorem D.** [1] Let $G$ be a connected graph of order $n$. Then $\gamma_{\text{tR}}(G) = n$, if and only if one of the following holds.

1. $G$ is a path or a cycle.
2. $G$ is the corona of some graph $F$.
3. $G$ is a subdivided star.
4. $G \in G \cup \mathcal{H}$.

**Theorem E.** [1] If $G$ is a graph with no isolated vertex, then $\gamma_{\text{t}}(G) = \gamma_{\text{tR}}(G)$ if and only if $G$ is the disjoint union of copies of $K_2$.

**Theorem F.** [1] Let $G$ be a connected graph of order $n \geq 3$. Then, $\gamma_{\text{tR}}(G) = \gamma_{\text{t}}(G) + 1$ if and only if $\Delta(G) = n - 1$.

**Theorem G.** Let $G$ be a graph of minimum degree at least one. Then, (a): [32] $\gamma(G) \leq \frac{|V(G)|}{2}$ and, (b): [23,33] $\gamma(G) = \frac{|V(G)|}{2}$ if and only if the components of $G$ are a cycle $C_4$ or the corona of any connected graph $H$. 
2. Bounds on the total strong Roman domination number

The main goal of this section concerns finding a few interesting tight bounds for the total strong Roman domination number of graphs, in which we relate it to other parameters or invariants of the graph. To this end, we need the following concepts. A matching in a graph $G$ is formed by a set of edges having no vertices in common. The matching number of the graph $G$ is the maximum cardinality among all possible matchings in $G$, which we denote by $\alpha'(G)$.

**Theorem 5.** Let $G$ be a graph of order $n \geq 4$, maximum degree $\Delta$, and without isolated vertices and different from a star. Then,

$$\gamma_{StR}^t(G) \leq n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil.$$  

**Proof.** We note that $\alpha'(G) \geq 2$ because $G$ is not a star. Let $v$ be a vertex of maximum degree $\Delta$ and let $X = V(G) \setminus N_G[v]$. If $X = \emptyset$, then clearly $v$ has degree $\Delta = n - 1$ and also, it is satisfied $\gamma_{StR}^t(G) \leq \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1$. Hence,

$$\gamma_{StR}^t(G) \leq 1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1 = n - \Delta + \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1$$

$$\leq n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil,$$

which gives the desired bound.

Assume now that $X \neq \emptyset$ and that $S$ is the set consisting of all isolated vertices of the subgraph induced by $X$, from now on denoted $G[X]$. If $S = \emptyset$, then let $u \in N(v)$ and define $f : V(G) \rightarrow \{0, 1, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ by $f(v) = \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1, f(u) = 1, f(x) = 1$ for every $x \in X$, and $f(y) = 0$ otherwise. Since $S$ is empty, we notice that $f$ is a TStRD-function of $G$ of weight $n - \Delta + \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1$ yielding

$$\gamma_{StR}^t(G) \leq n - \Delta + \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1 \leq n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil.$$  

Let $S \neq \emptyset$. Since $\delta \geq 1$, every vertex $s \in S$ is adjacent to at least one vertex of $N(v)$. Let $S'$ be the smallest subset of $N(v)$ such that every vertex in $S$ is adjacent to a vertex of $S'$. By the choice of $S'$, each vertex $u' \in S'$ has a private neighbor $u \in S$ with respect to $S'$, and so $|S'| \leq |S|$. Let $M = \{uu' : u' \in S' \text{ and } u \in S \text{ is a private neighbor of } u'\}$. Obviously, $M$ is a matching in $G$ yielding $|S'| \leq \alpha'(G)$. We now consider four cases.

**Case 1.** $S' = N(v) \text{ and } S = X$.

The function $f : V(G) \rightarrow \{0, 1, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ defined as $f(v) = 1, f(x) = \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1$ for every $x \in S'$ and $f(x) = 0$ otherwise, is a TStRD-function of $G$.  

and so,
\[
\gamma_{StR}^t(G) \leq \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil + 1\right)|S'| + 1 = 1 + |S'| + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil\right)|S'|
\leq 1 + |S| + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil\right)|S'| \leq n - \Delta + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil\right)|S'|
\leq n - \Delta + \alpha'(G)\left\lceil\frac{\Delta - 1}{2}\right\rceil.
\]

(1)

**Case 2.** \(S' = N(v)\) and \(S \not\subseteq X\).

Assume that \(ww' \in E(G[X - S])\). Then \(M \cup \{ww'\}\) is a matching of \(G\) and so, \(|S'| \leq \alpha'(G) - 1\). Define the function \(f : V(G) \rightarrow \{0, 1, \ldots, \left\lceil\frac{\Delta}{2}\right\rceil + 1\}\) as \(f(v) = 1\), \(f(x) = \left\lceil\frac{\Delta - 1}{2}\right\rceil + 1\) for any \(x \in S'\), \(f(x) = 1\) for every \(x \in X - S\), and \(f(x) = 0\) otherwise. Clearly, \(f\) is a TStRD-function of \(G\), and this implies that
\[
\gamma_{StR}^t(G) \leq \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil + 1\right)|S'| + 1 + n - 1 - \Delta - |S|
\leq n - \Delta + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil\right)|S'|
\leq n - \Delta + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil\right)(\alpha'(G) - 1)
< n - \Delta + \alpha'(G)\left\lceil\frac{\Delta - 1}{2}\right\rceil.
\]

**Case 3.** \(S' \not\subseteq N(v)\) and \(S = X\).

Let \(z \in N(v) - S'\). Hence, \(M \cup \{vz\}\) is a matching of \(G\) and so, \(|S'| \leq \alpha'(G) - 1\). We consider the function \(f : V(G) \rightarrow \{0, 1, \ldots, \left\lceil\frac{\Delta}{2}\right\rceil + 1\}\) defined as \(f(v) = 1 + \left\lceil\frac{\Delta - |S'|}{2}\right\rceil\), \(f(x) = \left\lceil\frac{\Delta - 1}{2}\right\rceil + 1\) for every \(x \in S'\), and \(f(x) = 0\) otherwise. Note that \(f\) is a TStRD-function of \(G\) and so,
\[
\gamma_{StR}^t(G) \leq 1 + \left\lceil\frac{\Delta - |S'|}{2}\right\rceil + \left(\left\lceil\frac{\Delta - 1}{2}\right\rceil + 1\right)|S'|
= 1 + |S'| + \left\lceil\frac{\Delta - |S'|}{2}\right\rceil + \left\lceil\frac{\Delta - 1}{2}\right\rceil|S'|
\leq 1 + |S| + \left\lceil\frac{\Delta - |S'|}{2}\right\rceil + \left\lceil\frac{\Delta - 1}{2}\right\rceil(\alpha'(G) - 1)
\leq n - \Delta + \left\lceil\frac{\Delta - 1}{2}\right\rceil\alpha'(G).
\]

(2)

**Case 4.** \(S' \not\subseteq N(v)\) and \(S \not\subseteq X\).

Suppose that \(z \in N(v) - S'\) and that \(uu' \in E(G[X - S])\). Then clearly \(M \cup \{vz, uu'\}\) is a matching of \(G\) and so, we have \(|S'| \leq \alpha'(G) - 2\). Define
Figure 2. The graphs $F_1, F_2, F_3, F_4$ and $F_5$

the function $f : V(G) \to \{0, 1, \ldots, \lceil \Delta \rceil + 1\}$ by $f(v) = \lceil \Delta - |S'| \rceil + 1$, $f(x) = \lceil \Delta - 1 \rceil + 1$ if $x \in S'$, $f(x) = 1$ if $x \in X - S$ and $f(x) = 0$ otherwise. It is easy to see that $f$ is a TStRD-function of $G$. Thus,

$$
\gamma_{t StR}^t(G) \leq n - \Delta + \left\lceil \Delta - |S'| \right\rceil + \left(\left\lceil \Delta - \frac{1}{2} \right\rceil + 1\right) |S'|
$$

$$
\leq n - \Delta + \left\lceil \frac{\Delta - |S'|}{2} \right\rceil + \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 1\right) |S'|
$$

$$
\leq n - \Delta + \left\lceil \frac{\Delta - |S'|}{2} \right\rceil + \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil \right) (\alpha'(G) - 2)
$$

$$
< n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil,
$$

and the proof is complete. $\square$

We next characterize the graphs attaining the bound from Theorem 5, among graphs having girth at least four. To this end, we need the set of graphs appearing in Fig. 2 and Observation 6. Also, for a given graph $G$, by $S(G)$ we mean a graph obtained from $G$ by subdividing all its edges.

Observation 6. For any star graph $S_t$ on $t$ leaves, $\gamma_{t StR}^t(S_t) = \left\lceil \frac{t-1}{2} \right\rceil + 2$.

Proof. It can be noted that a function $f$ on $S_t$ that assigns $\left\lceil \frac{t-1}{2} \right\rceil + 1$ to the center of the star, 1 to exactly one leaf, and 0 otherwise, is a $\gamma_{t StR}^t(S_t)$-function. $\square$

Theorem 7. Let $G$ be a connected graph of order $n$, maximum degree $\Delta$, and such that $g(G) \geq 4$. Then, $\gamma_{t StR}^t(G) = n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil$ if and only if $G$ is one of the graphs in the set $\{P_4, P_5, C_4, C_5, DS_{1,2}, S(K_{1,3}), F_1, F_2, F_3, F_4, F_5\}$.

Proof. Assume that $\gamma_{t StR}^t(G) = n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil$. By Observation 2, we have $n - \Delta + \alpha'(G) \left\lceil \frac{\Delta - 1}{2} \right\rceil \leq n$. First observe that if $\Delta = 2$ or $\Delta \geq 4$, then
\( \alpha'(G) \leq 2 \). Also, if \( \Delta = 3 \), then \( \alpha'(G) \leq 3 \). If \( \alpha'(G) = 1 \), then \( G \) is a star and we get a contradiction, by using Observation 6.

Hence, we may assume that \( \alpha'(G) = 2 \) or \( \alpha'(G) = \Delta = 3 \). Let \( v \) be a vertex of maximum degree \( \Delta \), and let \( X \), \( S \) and \( S' \) denote the sets defined in the proof of Theorem 5. Since \( g(G) \geq 4 \), we have \( X \neq \emptyset \). First let \( S = \emptyset \). Then the function \( f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1 \} \) defined by \( f(v) = \lceil \frac{\Delta-1}{2} \rceil + 1 \), \( f(u) = 1 \), \( f(x) = 1 \) for \( x \in X \) and \( f(x) = 0 \) otherwise, is a TStRD-function of \( G \) of weight \( n - \Delta + \lceil \frac{\Delta-1}{2} \rceil + 1 \). Then \( n - \Delta + \alpha'(G) \lceil \frac{\Delta-1}{2} \rceil \leq n - \Delta + \lceil \frac{\Delta-1}{2} \rceil + 1 \).

If \( \alpha'(G) = \Delta = 3 \), then

\[
n = n - 3 + 3 \left[ \frac{3 - 1}{2} \right] \leq n - 3 + \left[ \frac{3 - 1}{2} \right] + 1 = n - 1
\]

which is a contradiction. Assume that \( \alpha'(G) = 2 \). Then it then follows that \( n - \Delta + 2 \lceil \frac{\Delta-1}{2} \rceil \leq n - \Delta + \lceil \frac{\Delta-1}{2} \rceil + 1 \), yielding \( \Delta = 2 \) or \( \Delta = 3 \). If \( \Delta = 2 \), then, from Observation 4 and Theorem C, we deduce that \( G \in \{ P_5, C_5 \} \). Let \( \Delta = 3 \).

We deduce from \( \alpha'(G) = 2 \) and \( S = \emptyset \), that \( G[X] \) is a star \( K_{1, t} \) whose central vertex, say \( z \), is adjacent to a neighbor of \( v \), say \( u \). Since \( \Delta = 3 \), we have \( t \in \{1, 2\} \). If \( t = 2 \), then the function \( g : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1 \} \) defined by \( g(v) = g(z) = 2 \), \( g(u) = 1 \) and \( g(y) = 0 \) otherwise, is a TStRD-function of \( G \) of weight less than \( n - \Delta(G) + \alpha'(G) \lceil \frac{\Delta-1}{2} \rceil = 6 \), which is a contradiction.

Thus \( t = 1 \). If \( g(G) = \infty \), then we observe that \( G \) is a graph obtained from a double star \( DS_{1,2} \), by subdividing its central edge once, and so \( G = F_1 \). Let \( g(G) < \infty \). Since \( \Delta(G) = 3 \) and \( \alpha'(G) = 2 \), we have \( g(G) = 4 \). Hence, \( z \) must be adjacent to some other neighbors of \( v \). It is easy to see that \( G \in \{ F_2, F_3 \} \) in this case.

Now suppose that \( S \neq \emptyset \). By the proof of Theorem 5, we only consider the following cases.

**Case 1.** \( S' = N(v) \) and \( S = X \).

By (1), we deduce that \( \alpha'(G) = |S'| = |S| = \Delta \). Since \( n - \Delta + \Delta \lceil \frac{\Delta-1}{2} \rceil \leq n \), we have \( \Delta \leq 3 \). If \( \Delta = 2 \), then clearly \( G \in \{ P_5, C_5 \} \). Let \( \Delta = 3 \). Since \( G \) is triangle-free, \( N(v) \) is independent and since each vertex in \( S' \) has a private neighborhood in \( S \) with respect to \( S' \), we conclude that each vertex in \( S \) is of degree one. Thus \( G \) is obtained from \( K_{1,3} \) by subdividing its edges, i.e., \( G = S(K_{1,3}) \).

**Case 2.** \( S' \not\subseteq N(v) \) and \( S = X \).

In this case all inequalities occurring in (2) must be equalities, and so \( \alpha'(G) - 1 = |S'| = |S| = 1 \). Let \( S' = \{u\} \). Hence, the function \( g : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1 \} \) defined by \( g(v) = \lceil \frac{\Delta-1}{2} \rceil + 1 \), \( g(u) = 2 \), and \( g(y) = 0 \) otherwise, is a TStRD-function of \( G \) of weight \( \lceil \frac{\Delta-1}{2} \rceil + 3 \). We must have \( \lceil \frac{\Delta-1}{2} \rceil + 3 \geq n - \Delta + 2 \lceil \frac{\Delta-1}{2} \rceil \), and so, \( n - \Delta + \lceil \frac{\Delta-1}{2} \rceil \leq 3 \). Since \( n = \Delta + 2 \), we obtain \( \lceil \frac{\Delta-1}{2} \rceil \leq 1 \), which leads to \( \Delta \leq 3 \). If \( \Delta(G) = 2 \), then clearly \( G \in \{ P_4, C_4 \} \). If \( \Delta(G) = 3 \), then we note that \( G \in \{ F_4, F_5, DS_{1,2} \} \). This completes the proof. \( \Box \)
We now continue with some other bounds in which we also involve some other invariants of the graph, like the minimum degree, the diameter, and the girth.

**Proposition 1.** Let $G$ be a connected graph of order $n \geq 2$ and minimum degree $\delta$. Then,

$$\gamma_{StR}^t(G) \leq n - \left\lfloor \frac{\delta - 1}{2} \right\rfloor.$$  

**Proof.** Let $v \in V(G)$ be a vertex of minimum degree $\delta$ and let $u \in N(v)$. Define the function $f : V(G) \to \{0, 1, 2, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ such that $f(v) = \left\lceil \frac{\delta - 1}{2} \right\rceil + 1$, $f(u) = 1$, $f(x) = 0$ for any $x \in N(v) - \{u\}$, and $f(y) = 1$ for every $y \notin N[v]$. Note that $f$ is a total strong Roman dominating function of $G$. Namely, if there is a vertex $w \notin N[v]$ that has no neighbor outside of $N[v]$, then $w$ must be adjacent to every neighbor of $v$, since otherwise $w$ is a vertex of degree smaller than $v$, which is not possible. Thus, $\gamma_{StR}^t(G) \leq \left\lceil \frac{\delta - 1}{2} \right\rceil + 2 + n - 1 - \delta = n - \left\lfloor \frac{\delta - 1}{2} \right\rfloor$, as desired. □

**Proposition 2.** Let $G$ be a graph of order $n$ with $\text{diam}(G) = 2$, minimum degree $\delta$, and maximum degree $\Delta$. Then,

$$\gamma_{StR}^t(G) \leq \delta \left(1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil \right) + 1.$$  

**Proof.** Suppose $v \in V(G)$ is a vertex of minimum degree $\delta$. Clearly, $N(v)$ dominates all vertices of $G$ since $G$ has diameter two. Define the function $f : V(G) \to \{0, 1, \ldots, 1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil\}$ such that $f(v) = 1, f(x) = 1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil$ for every $x \in N(v)$ and $f(x) = 0$ otherwise. It is clear that $f$ is a total strong Roman dominating function yielding $\gamma_{StR}^t(G) \leq \left(1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil \right) + 1$. □

Next we establish upper bounds in terms of the order, diameter and girth of the graph.

**Proposition 3.** Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq 3$. Then,

$$\gamma_{StR}^t(G) \leq n - \left\lfloor \frac{\text{diam}(G) + 1}{3} \right\rfloor.$$  

**Proof.** Assume $P = v_1v_2 \ldots v_{\text{diam}(G)+1}$ is a diametral path in $G$ and let $f$ be a $\gamma_{StR}(P)$-function. By Theorems A and B, we have $\omega(f) = \left\lceil \frac{2\text{diam}(G)+2}{3} \right\rceil$. Hence, the function $g : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ defined as $g(u) = f(u)$ for every $u \in V(P)$ and $g(u) = 1$ for every $u \in V(G) \setminus V(P)$, is a TStRD-function of $G$. Therefore,

$$\gamma_{StR}^t(G) \leq (n - \text{diam}(G) - 1) + \left\lceil \frac{2\text{diam}(G) + 2}{3} \right\rceil = n - \left\lceil \frac{\text{diam}(G) + 1}{3} \right\rceil,$$  

which is our desired bound. □
Proposition 4. Let $G$ be a connected graph of order $n$ with $g(G) \geq 4$ and $\delta \geq 3$. Then
\[
\gamma_{\text{Str}}^{t}(G) \leq n - \left\lfloor \frac{g(G)}{3} \right\rfloor.
\]

Proof. Let $C$ be a cycle of $G$ with $g(G)$ edges. For any $\gamma_{\text{Str}}^{t}(C)$-function, the function $g : V(G) \to \{0, 1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor + 1\}$ defined by $g(u) = f(u)$ for $u \in V(C)$ and $g(u) = 1$ for $u \in V(G) \setminus V(C)$, is a TStrD-function of $G$. By Theorems A and B, we have
\[
\gamma_{\text{Str}}^{t}(G) \leq \omega(f) + (n - g(G)) = n - \left\lfloor \frac{g(G)}{3} \right\rfloor,
\]
and the proof is complete. \qed

We conclude this section by characterizing all the graphs attaining the largest possible value in the total strong Roman domination number, and giving some Norhauss–Gaddum result for such a parameter.

Theorem 8. Let $G$ be a connected graph of order $n$. Then $\gamma_{\text{Str}}^{t}(G) = n$, if and only if one of the next items is satisfied.

(i) $G$ is a cycle or a path.
(ii) $G$ is the corona of some graph $F$.
(iii) $G$ is a subdivided star graph.
(iv) $G \in \mathcal{G} \cup \mathcal{H}$, with $\mathcal{G}, \mathcal{H}$ as defined in the Introduction.

Proof. Suppose that the graph $G$ satisfies at least one of the (four) conditions given in the statement of the theorem. Then, by using Observations 1 and 2 and Theorem D, we deduce that $\gamma_{\text{Str}}^{t}(G) = n$.

Conversely, let $\gamma_{\text{Str}}^{t}(G) = n$. We claim that $\gamma_{\text{Str}}^{t}(G) = \gamma_{tR}(G)$. Let $f = (B_0, B_1, B_2)$ be a $\gamma_{\text{Str}}^{t}(G)$-function. If there is a vertex, say $v \in B_2$, for which $f(v) \geq x$ where $x \geq 3$, then $|N(v) \cap B_0| \geq 2x - 3 \geq x$. Thus, $\gamma_{\text{Str}}^{t}(G) < n$, which is a contradiction. Consequently, every vertex in $B_2$ has value 2. Therefore, $\gamma_{\text{Str}}^{t}(G) = \gamma_{tR}(G) = n$ and by Theorem D, the graph $G$ satisfies one of the conditions appearing in the statement of our result. \qed

Proposition 5. Let $G$ and $\overline{G}$ be connected graphs of order $n \geq 4$. Then $8 \leq \gamma_{\text{Str}}^{t}(G) + \gamma_{\text{Str}}^{t}(\overline{G}) \leq 2n$. Moreover, $\gamma_{\text{Str}}^{t}(G) + \gamma_{\text{Str}}^{t}(\overline{G}) = 2n$ if and only if $G = P_4$.

Proof. From the condition, $\gamma_{\text{Str}}^{t}(G)$ and $\gamma_{\text{Str}}^{t}(\overline{G})$ exist. Let without loss of generality, $\gamma_{\text{Str}}^{t}(G) \geq \gamma_{\text{Str}}^{t}(\overline{G})$. First we will prove the right hand side inequality. By Observation 2, $\gamma_{\text{Str}}^{t}(G) + \gamma_{\text{Str}}^{t}(\overline{G}) \leq 2n$ with equality if and only if $\gamma_{\text{Str}}^{t}(G) = \gamma_{\text{Str}}^{t}(\overline{G}) = n$. Now by Theorem 8, $G$ and $\overline{G}$ are $P_4$.

Now we will prove the left hand side inequality. Without loss of generality assume $\gamma_{\text{Str}}^{t}(G) \leq \gamma_{\text{Str}}^{t}(\overline{G})$. By Observation 2, $\gamma_{\text{Str}}^{t}(G) \geq 3$. If $\gamma_{\text{Str}}^{t}(G) = 3$, then clearly $3 = \gamma_{\text{Str}}^{t}(G) = \gamma_{tR}(G) = \gamma_{t}(G)+1$ and we conclude from Theorem...
F that $G$ has order 4 and $\Delta = 3$, but then $\overline{G}$ has isolated vertices, which is a contradiction. Therefore, $\gamma_{StR}(G) \geq 4$ and so $\gamma_{StR}(\overline{G}) \geq 4$, which means $\gamma_{StR}^t(G) + \gamma_{StR}^t(\overline{G}) \geq 8$.

**3. Comparing $\gamma_{StR}^t(G)$ with $\gamma_{stR}(G)$, $\gamma_t(G)$ and $\gamma(G)$**

We next present a bound for the total strong Roman domination number which is related to the strong Roman domination parameter.

**Theorem 9.** If $G$ is a graph of order $n \geq 4$ and minimum degree at least one, then

$$\gamma_{StR}^t(G) \leq 2(\gamma_{StR}(G) - 1).$$

This bound is sharp for $G \in \{P_4, C_4, P_6, C_6\}$.

**Proof.** Let $f = (B_0^f, B_1^f, B_2^f)$ be a $\gamma_{StR}(G)$-function such that $|B_2^f| = 0$. Suppose firstly that $|B_1^f| \neq 0$. Let $B_{12}^f$ be the set formed by vertices belonging to $B_1^f$ for which there is a neighbor in $B_2^f$. Consider $|B_{12}^f| \neq 0$, and let $u \in B_{12}^f$ and $v \in N(u) \cap B_2^f$. If $N(v) \cap B_0^f$ is odd, then the function $g : V(G) \rightarrow \{0, 1, \ldots, \left\lceil \frac{\Delta}{2} \right\rceil + 1\}$ defined by $g(u) = 0$ and $g(x) = f(x)$ for $x \in V(G) - \{u\}$ is an StRD-function of weight less than $f(V(G))$, a contradiction. Hence, $|N(v) \cap B_0^f|$ is even, for every vertex $v \in N(B_{12}^f) \cap B_2^f$. Let $B_{11}^f$ be the subset of $B_1^f$ such that $N(B_{11}^f) \subseteq B_0^f$. Note that all neighbors of a vertex in $B_{11}^f$ belong to $B_0^f$ and every vertex in $B_1^f - B_{11}^f$ has a neighbor with positive label. For every vertex $w \in B_{11}^f$, we chose any of its neighbors, say $w'$, and we let $W = \bigcup_{w \in B_{11}^f} w'$.

Notice that $|W| \leq |B_{11}^f| \leq |B_1^f|$, and that $W \subseteq B_0^f$. Since every vertex in $W \cup B_{12}^f$ is adjacent to at least one vertex from the set $B_2^f$, we deduce $|B_2^f| \geq 1$ and $\sum_{v \in B_1^f} \left\lceil \frac{|N(v) \cap B_0^f|}{2} \right\rceil \geq 1$. If $G[B_1^f \cup B_2^f]$ has no isolated vertex, then $f$ is a $\gamma_{StR}^t$-function of $G$, which implies that $\gamma_{StR}^t(G) = \gamma_{StR}(G) < 2(\gamma_{StR}(G) - 1)$.

Assume now that $G[B_1^f \cup B_2^f]$ has at least one isolated vertex. Consider the function $h = (B_0^h, B_1^h, B_2^h) = (V_0 - W, B_1 \cup W, B_2)$. If $G[B_1^h \cup B_2^h]$ has no isolated vertex, then $h$ is a TStRD-function on $G$ such that

$$\gamma_{StR}^t(G) \leq h(V(G)) \leq 2|B_1^h| + |B_2^h| + \sum_{y \in B_2^h} \left\lceil \frac{|N(y) \cap B_0^h|}{2} \right\rceil$$

$$= 2\gamma_{StR}(G) - |B_2^h| - \sum_{y \in B_2^h} \left\lceil \frac{|N(y) \cap B_0^h|}{2} \right\rceil \leq 2\gamma_{StR}(G) - 2$$

$$= 2(\gamma_{StR}(G) - 1).$$

If the subgraph induced by $G[B_1^h \cup B_2^h]$ has a vertex of degree zero, then we consider $U$ as the set of such vertices of degree zero in $G[B_1^h \cup B_2^h]$. Since
the induced subgraph $G[B_1^h]$ has no vertex of degree zero, we can check that $U \subseteq B_2^h$. As we noted before, every vertex from the set $W$ has at least one neighbor vertex in the set $B_2$. This implies that $U \subseteq B_2^h$. Now, for any vertex $u \in U$, we chose one of its neighbors, say $u'$, and we take $M = \bigcup_{u \in U} \{u\}$. Note that $|M| \leq |U| < |B_2^h|$ and $M \subseteq B_0^h$. Then the function $h' = (B_0^h, B_1^h, B_2') = (V_0^h - M, B_1^h \cup M, B_2)$ is a TStrRD-function of $G$, implying that

$$
\gamma_{StR}^t(G) \leq h'(V(G)) \leq 2|B_1^h| + 2|B_2'| + \sum_{z \in B_2'} \left\lfloor \frac{|N(z) \cap B_0^h|}{2} \right\rfloor - 1
= 2\gamma_{StR}^t(G) - \sum_{z \in B_2'} \left\lfloor \frac{|N(z) \cap B_0^h|}{2} \right\rfloor - 1 \leq 2\gamma_{StR}^t(G) - 2.
$$

We now consider $|B_1^h| = 0$, and assume $U$ is formed by the set of vertices of degree zero in the induced subgraph $G[B_2^h]$. If $|U| = 0$, then we obtain $\gamma_{StR}^t(G) = \gamma_{StR}(G)$. Hence, we may consider that $|U| \neq 0$. For every vertex $u \in U$, we take any one of its neighbors, say $u'$, and we set $U' = \bigcup_{u \in U} \{u\}$. Notice that we have $|U'| \leq |U| \leq |B_2^h|$. Then the function $h'' = (B_0^{h''}, B_1^{h''}, B_2^{h''}) = (B_0^h - U', U', B_2^h)$ is a TStrRD-function of $G$, implying that

$$
\gamma_{StR}^t(G) \leq h''(V(G)) \leq 2|B_2^{h''}| + \sum_{w \in B_2^{h''}} \left\lfloor \frac{|N(w) \cap B_0^{h''}|}{2} \right\rfloor - 1 \leq 2\gamma_{StR}^t(G) - 1.
$$

If $\gamma_{StR}^t(G) = 2\gamma_{StR}(G) - 1$, then $\gamma_{StR}(G) = 2$, implying that $n \leq 3$, which is a contradiction. Therefore $\gamma_{StR}^t(G) \leq 2(\gamma_{StR}(G) - 1)$ and this completes the proof.

Since the vertices labeled with positive numbers in any TStrRD-function of a graph $G$ form a total dominating set of $G$, it is clear that $\gamma_{StR}^t(G) \geq \gamma_t(G)$. We next characterize the class of graphs attaining equality in such a bound.

**Proposition 6.** Let $G$ be a graph of order $n$. Then $\gamma_{StR}^t(G) = \gamma_t(G)$ if and only if $G$ is the disjoint union of copies of $K_2$.

**Proof.** Assume $\gamma_{StR}^t(G) = \gamma_t(G)$, and let $f = (B_0, B_1, B_2)$ be a $\gamma_{StR}^t(G)$-function. Then $B_1 \cup B_2$ is a total dominating set of $G$. Thus,

$$
\gamma_t(G) \leq |B_1| + |B_2| \leq |B_1| + |B_2| + \sum_{w \in B_2} \left\lfloor \frac{1}{2} |N(w) \cap B_0| \right\rfloor = \gamma_{StR}^t(G).
$$
Then there must occur an equality situation in this inequality chain. In particular, it must happen that \( \sum_{w \in B_2} \frac{1}{2} |N(w) \cap B_0| = 0 \), implying that \( |B_2| = 0 \), and consequently, \( V(G) = |B_1| \). Since \( f \) is an arbitrary \( \gamma^t_{StR}(G) \)-function, \( (\emptyset, V(G), \emptyset) \) is the only \( \gamma^t_{StR}(G) \)-function. By Theorem E, \( G \) is the disjoint union of copies of \( K_2 \), which completes this implication. The reverse implication is straightforward and the proof is complete. □

Based on the relatively simple deduction of the result above, we next discuss graphs \( G \) for which \( \gamma^t_{StR}(G) = \gamma_t(G) + 1 \).

**Proposition 7.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then, \( \gamma^t_{StR}(G) = \gamma_t(G) + 1 \) if and only if \( G \) is \( P_3 \) or \( C_3 \).

**Proof.** If \( G \) is \( P_3 \) or \( C_3 \), then clearly \( \gamma^t_{StR}(G) = \gamma_t(G) + 1 \).

Conversely, let \( \gamma^t_{StR}(G) = \gamma_t(G) + 1 \) and let \( f = (B_0, B_1, B_2) \) be a \( \gamma^t_{StR}(G) \)-function. If \( B_2 = \emptyset \), then \( \gamma^t_{StR}(G) = n \), and by Theorem 8, \( G \) is one of the graphs in Theorem 8, but clearly in such cases \( \gamma_t(G) < n - 1 \) which is a contradiction. Thus, \( B_2 \neq \emptyset \). Since \( B_1 \cup B_2 \) is a TD-set of \( G \), we have

\[
\gamma^t_{StR}(G) - 1 = \gamma_t(G) \leq |B_1| + |B_2| \leq |B_1| + |B_2| + \sum_{w \in B_2} \left[ \frac{1}{2} |N(w) \cap B_0| \right] - 1 = \gamma^t_{StR}(G) - 1.
\]

Consequently, this inequality chain must become a chain of equal quantities. In particular, \( \sum_{w \in B_2} \left[ \frac{1}{2} |N(w) \cap B_0| \right] = 1 \), which leads to \( |B_2| = 1 \) and \( \left[ \frac{1}{2} |N(w) \cap B_0| \right] = 1 \) where \( B_2 = \{w\} \). Therefore, \( \gamma^t_{StR}(G) = \gamma_t(G) \) and by Theorem F, \( \Delta(G) = n - 1 \), which leads to \( G = P_3 \) or \( G = C_3 \). □

Another relationship between \( \gamma^t_{StR}(G) \) and \( \gamma_t(G) \) was already noted in Observation 1. We next characterize the limit case of such a bound.

**Proposition 8.** Let \( G \) be a graph of order \( n \) and \( \Delta > 1 \). Then \( \gamma^t_{StR}(G) = \left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G) \) if and only if there exists a \( \gamma^t_{StR}(G) \)-function \( f = (B_0, B_1, B_2) \) such that \( |B_1| = 0 \) and \( |N(w) \cap B_0| = \Delta - 1 \) for each \( w \in B_2 \).

**Proof.** Let \( \gamma^t_{StR}(G) = \left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G) \) and \( S \) be an arbitrary \( \gamma_t(G) \)-set. The function \( f = (B_0, B_1, B_2) \) that assigns the weight \( 1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil \) to each vertex of \( S \), and the weight \( 0 \) to all remaining vertices of \( G \) is a TStRD-function on \( G \). Thus,

\[
\left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G) = \gamma^t_{StR}(G) \leq f(V(G)) = |B_2| + \sum_{w \in B_2} \left[ \frac{1}{2} |N(w) \cap B_0| \right] = \left( 1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil \right) |S| = \left\lceil \frac{\Delta + 1}{2} \right\rceil |S| \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G).
\]
We notice that the last inequality must be equality, since $S$ is a $\gamma_t(G)$-set. Also, observe that $|B_2| = |S|$. Thus, this inequality chain must become a chain of equal quantities. Particularly,

$$\gamma_{StR}^t(G) = f(V(G)) = \left\lceil \frac{\Delta + 1}{2} \right\rceil |B_2| = |B_1| + |B_2| + \sum_{w \in B_2} \frac{1}{2} |N(w) \cap B_0|.$$ 

As a consequence, $|B_1| = 0$ and $\left\lceil \frac{\Delta - 1}{2} \right\rceil |B_2| = \sum_{w \in B_2} \left\lfloor \frac{1}{2} |N(w) \cap B_0| \right\rfloor$ hold, which leads to that $f$ is a $\gamma_{StR}(G)$-function for which $|B_1| = 0$ and also $|N(w) \cap B_0| = \Delta - 1$ for each $w \in B_2$.

Conversely, assume there exists a $\gamma_{StR}^t(G)$-function $f = (B_0, B_1, B_2)$ such that $|B_1| = 0$ and that $|N(w) \cap B_0| = \Delta - 1$ for each $w \in B_2$. Since $B_1 \cup B_2 = B_2$ is a total dominating set of $G$, we have

$$\gamma_t(G) \leq |B_2| \leq \frac{1}{1 + \left\lceil \frac{\Delta - 1}{2} \right\rceil} \gamma_{StR}^t(G) = \frac{1}{\left\lceil \frac{\Delta + 1}{2} \right\rceil} \gamma_{StR}^t(G).$$

Thus, it follows $\gamma_{StR}^t(G) \geq \left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G)$. Therefore, by using Observation 1, we obtain the equality $\gamma_{StR}^t(G) = \left\lceil \frac{\Delta + 1}{2} \right\rceil \gamma_t(G)$. \hfill $\square$

Our final results in this section relate the total strong Roman domination number and the (standard) domination number of graphs.

**Theorem 10.** Let $G$ be a graph without isolated vertices. Then,

$$\gamma_{StR}^t(G) \leq \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2\right) \gamma(G).$$

Moreover, if equality holds, then every $\gamma(G)$-set $S$ is an efficient dominating set, and every vertex in $S$ has degree $\Delta$.

**Proof.** Let $S$ be a $\gamma(G)$-set, and let $S'$ be the set of vertices in $S$ that have degree zero in $G[S]$ (it can happen that $S' = \emptyset$). For every vertex $v \in S'$ (if it exists), we chose a vertex adjacent to $v$, and denote this vertex by $v'$. Let $S'' = \cup_{v \in S'} \{v'\}$. Let $f$ be a TStRD-function on $G$ defined as follows: (a) for each vertex $v \in S$, let $f(v) = \left\lceil \frac{\Delta - 1}{2} \right\rceil + 1$, (b) for each vertex $v \in S''$, let $f(v) = 1$, and (c) for each vertex $v \in V(G) \setminus (S \cup S'')$, let $f(v) = 0$. Thus, it follows $\gamma_{StR}^t(G) \leq f(V(G)) \leq \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 1\right)|S| + |S''| \leq \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2\right)|S| = \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2\right)\gamma(G)$, as desired.

Assume next $\gamma_{StR}^t(G) = \left(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2\right)\gamma(G)$. Let $S$ be any $\gamma(G)$-set, and let $S'$ and $S''$ be two sets defined as previously described. It is true that $(\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2)\gamma(G) = \gamma_{StR}^t(G) \leq (\left\lceil \frac{\Delta - 1}{2} \right\rceil + 1)|S| + |S''| \leq (\left\lceil \frac{\Delta - 1}{2} \right\rceil + 1)|S| + |S'| \leq (\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2)|S| = (\left\lceil \frac{\Delta - 1}{2} \right\rceil + 2)\gamma(G)$. As a consequence, an equality relation must occur in this last inequality chain. So, $|S''| = |S'| = |S|$, which leads to the claim that $S$ is an independent set. Moreover, every vertex belonging to the set $S$ is of degree $\Delta$. But then $d_G(u, v) \geq 2$ (the distance between $u$ and $v$) for any two distinct vertices $u$ and $v$ in $S$. We shall show that $d_G(u, v) \geq 3$. For
a contradiction, we suppose \( d_G(u, v) = 2 \). Let \( w \) be a vertex adjacent to both vertices \( u \) and \( v \), and select \( u' = u' = w \) where, as above, the vertices \( u' \) and \( v' \) are the vertices chosen to be neighbors of \( u \) and \( v \), respectively. According to this, we note that \( |S''| < |S'| \), which produces a contradiction. Therefore, we must have \( d_G(u, v) \geq 3 \) for any pair of distinct vertices \( u \) and \( v \) from the set \( S \). Consequently, the set \( S \) is an efficient dominating set in the graph \( G \). Therefore, we have obtained that every \( \gamma(G) \)-set \( S \) is an efficient dominating set in \( G \), and that every vertex in \( S \) has degree \( \Delta \). □

**Proposition 9.** Let \( G \) be a connected graph of order \( n \) and without isolated vertices. Then \( \gamma(G) + \gamma^t_{StR}(G) \leq \frac{3n}{2} \). We have equality if and only if \( G = C_4 \) or \( G = cor(F) \) for some graph \( F \).

**Proof.** By Theorem G(a), \( \gamma(G) \leq \frac{n}{2} \), and by Observation 2, \( \gamma^t_{StR}(G) \leq n \). Hence, \( \gamma(G) + \gamma^t_{StR}(G) \leq \frac{3n}{2} \). We have equality if and only if both \( \gamma(G) = \frac{n}{2} \) and \( \gamma^t_{StR}(G) = n \) are valid. The required result follows by combining Theorem G(b) and Theorem 8. □

4. Trees

In this section, we present two bounds for the total strong Roman domination number of trees. First, we note that by Proposition 7, for any tree \( T \) of order \( n \geq 4 \), it follows \( \gamma^t_{StR}(T) \geq \gamma_t(T) + 2 \). We shall improve this bound if the maximum degree is larger than five. To this end, we need the following lemma.

**Lemma 1.** Let \( T \) be a tree different from a star. Then, there exists a \( \gamma^t_{StR}(T) \)-function \( f = (B_0, B_1, B_2) \) such that every leaf of \( T \) belongs to \( B_0 \).

**Proof.** Let \( f = (B_0, B_1, B_2) \) be a \( \gamma^t_{StR}(T) \)-function. Suppose there exists a leaf \( x \) of \( T \) such that \( f(x) \geq 1 \) and let \( x' \) be the support vertex adjacent to \( x \). By definition \( f(x') \geq 1 \). If \( f(x) \geq 2 \), then one can easily construct a TStRD-function with weight smaller than \( f \), which is not possible. Thus, we must have \( f(x) = 1 \). If the support vertex \( x' \) has \( t \geq 2 \) adjacent leaves labeled with 1, then we can construct a new TStRD-function of \( T \) of weight smaller than or equal to \( \gamma^t_{StR}(T) \), by relabeling \( t - 1 \) of such leaves with 0, and the vertex \( x' \) with \( f(x') + \lfloor (t - 1)/2 \rfloor \). In consequence, we may assume that every support vertex has at most one leaf labeled with 1 under \( f \), say our \( x \), for the support vertex \( x' \). If there exists a non leaf vertex \( y \in N(x') \setminus \{x\} \) for which \( f(y) = 0 \), then one can “interchange” the labels of \( x \) and \( y \) to construct a new \( \gamma^t_{StR}(T) \)-function satisfying \( f(x) = 0 \). Hence, we may assume \( f(y) \neq 0 \) for every non leaf \( y \in N(x') \setminus \{x\} \). This also leads to the conclusion that \( f(x') = 1 + \left\lfloor \frac{|L_{x'}|-1}{2} \right\rfloor \), otherwise we could decrease the weight of \( f \), which is not possible. But, then we can construct a new \( \gamma^t_{StR}(T) \)-function by changing the labels of \( x \) and \( x' \).
This implies that $n$ to zero and $1 + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil$, respectively, and this is either not possible or satisfies our requirement. 

**Theorem 11.** For any nontrivial tree $T$ with maximum degree $\Delta(T)$,

$$\gamma_{StR}^t(T) \geq \gamma_t(T) + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil.$$ 

This bound is sharp for stars.

**Proof.** By Proposition 7, for any tree $T$ of order $n \geq 4$, it follows $\gamma_{StR}^t(T) \geq \gamma_t(T) + 2$. Thus, if $\Delta(T) \leq 5$, then we deduce that $\gamma_{StR}^t(T) \geq \gamma_t(T) + 2 \geq \gamma_t(T) + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil$. In consequence, from now on we may assume $\Delta(T) \geq 6$. This implies that $n \geq 7$, and also that $\text{diam}(T) \geq 2$.

If $\text{diam}(T) = 2$, then $T$ is a star with $\left\lceil \frac{\Delta(T) - 1}{2} \right\rceil + 2 = \gamma_{StR}^t(T) = \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil + \gamma_t(T)$ (this also shows the sharpness of the bound). If $\text{diam}(T) = 3$, then $T$ is a double star $DS_{p,q}$ $(p \geq 1, q \geq 5)$, where $\Delta(T) = q + 1$, $\gamma_t(T) = 2$ and $\gamma_{StR}^t(T) = \lceil \frac{q}{2} \rceil + 1 + \lceil \frac{q}{2} \rceil + 1$. By the fact that $\lceil \frac{q}{2} \rceil \geq 1$, we have $\lceil \frac{q}{2} \rceil + \lceil \frac{q}{2} \rceil + 2 > \lceil \frac{q}{2} \rceil + 2$. Hence $\gamma_{StR}^t(T) > \gamma_t(T) + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil$. The remaining part of the proof shall be done by induction on the order $n$ of $T$.

Let $T$ be a tree of order $n$ and $\text{diam}(T) \geq 4$, and assume that any tree $T'$ of order $n' < n$ and $\Delta(T') \geq 6$ satisfies $\gamma_{StR}^t(T') \geq \gamma_t(T') + \left\lceil \frac{\Delta(T') - 1}{2} \right\rceil$. Suppose $\text{diam}(T) = k - 1$, and let $P := v_1, v_2, \ldots, v_k$ be a diametral path of $T$ such that $v_2$ has the smallest possible degree. Root $T$ at $v_k$ and let $T' = T - \{v_1\}$.

Note that by the choice of $P$, removing vertex $v_1$ does not change the maximum degree of $T'$ with respect to that of $T$. Thus $\Delta(T') = \Delta(T) \geq 6$. On the other hand, it clearly happens that $\gamma_t(T) - 1 \leq \gamma_t(T') \leq \gamma_t(T)$. Also, by Lemma 1, there exists a $\gamma_{StR}^t(T)$-function $f = (B_0, B_1, B_2)$ such that for every leaf $x$ of $T$, it follows $f(x) = 0$. So, we can claim that $\gamma_{StR}^t(T) \geq \gamma_{StR}^t(T')$.

Now, if $\gamma_t(T') = \gamma_t(T)$, then by using the induction hypothesis, we deduce

$$\gamma_{StR}^t(T) \geq \gamma_{StR}^t(T') \geq \gamma_t(T') + \left\lceil \frac{\Delta(T') - 1}{2} \right\rceil = \gamma_t(T) + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil.$$

We next consider $\gamma_t(T') = \gamma_t(T) - 1$. We then observe that $v_2$ must have degree 2 in $T$, since otherwise the removal of $v_1$ to obtain $T'$ would not give $\gamma_t(T')$ strictly smaller than $\gamma_t(T)$. Since $f(v_1) = 0$, it must happen that $f(v_2) \geq 2$. Moreover, the neighbor of $v_2$ (which is indeed $v_3 \in P$) other than $v_1$ satisfies $f(v_3) \geq 1$. It is clear now that $f(v_2) = 2$, for otherwise we could construct a TStRD-function of $T$ of weight smaller than that of $f$, which is not possible.
Thus, from $f$, we construct a TStRD-function $f'$ in $T'$ by taking the restriction of $f$ to $T'$ and only changing the label of $v_2$, that is, taking $f'(v_2) = 1$. So, we obtain that $\gamma_{stR}^t(T') \leq \gamma_{stR}^t(T) - 1$. Therefore, by using the induction hypothesis, we deduce

$$\gamma_{stR}^t(T) \geq \gamma_{stR}^t(T') + 1 \geq \gamma_t(T') + \left\lceil \frac{\Delta(T') - 1}{2} \right\rceil + 1 = \gamma_t(T) - 1 + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil + 1 = \gamma_t(T) + \left\lceil \frac{\Delta(T) - 1}{2} \right\rceil,$$

which completes the proof. $\square$

**Theorem 12.** For any tree $T$ of order $n(T) \geq 3$ with maximum degree $\Delta(T)$ and $s(T)$ support vertices,

$$\gamma_{stR}^t(T) \geq \left\lceil \frac{n(T) + s(T)}{\Delta(T)} \right\rceil + 1.$$

Furthermore, this bound is sharp for $T \in \{P_3, P_4, P_5, DS_{1,2}, DS_{2,2}, K_{1,3}\}$.

**Proof.** The proof will be done by induction on the order $n(T)$ of the tree $T$. One can verify that the statement is true for $P_3$. Hence, from now on in this proof we assume $n(T) \geq 4$, and that every tree $T'$ of order $n(T') < n(T)$ with $s(T')$ support vertices satisfies the bound $\gamma_{stR}^t(T') \geq \left\lceil \frac{n(T') + s(T')}{\Delta(T')} \right\rceil + 1$.

If $T = K_{1,n-1}$ is a star, then $\gamma_{stR}^t(T) = \left\lceil \frac{n+2}{2} \right\rceil \geq \left\lceil \frac{n+1}{n-1} \right\rceil + 1$ and we have equality for $n = 4$. Likewise, assume that $T$ is a double star. If $p = q = 1$, then $T = P_4$ and $\gamma_{stR}^t(T) = 4 = \left\lceil \frac{6}{2} \right\rceil + 1$. If $T = DS_{1,q}$, with $q \geq 2$, then we have $\gamma_{stR}^t(T) = \left\lceil \frac{q}{2} \right\rceil + 3 \geq \left\lceil \frac{q+5}{q+1} \right\rceil + 1$ and we have equality for $q = 2$. If $T = DS_{p,q}$, with $q \geq p \geq 2$, then $\gamma_{stR}^t(T) = \left\lceil \frac{q}{2} \right\rceil + \left\lceil \frac{p}{2} \right\rceil + 2 \geq \left\lceil \frac{p+5}{q+1} \right\rceil + 1$ and we have equality for $p = q = 2$. If $T$ is a path $P_n$, then by using Observation 4 and Theorem C, we have $\gamma_{stR}^t(T) = n \geq \left\lceil \frac{n+2}{2} \right\rceil + 1$, and there is equality if $n = 4$ or $n = 5$.

Consequently, we may assume that $\text{diam}(T) \geq 4$ and that $\Delta(T) \geq 3$. Let $v_1v_2 \ldots v_k$ be a diametral path in $T$ such that $v_2$ has the smallest possible degree. Note that all the descendants of $v_2$ are leaves adjacent to $v_2$. We consider the tree $T$ rooted at the vertex $v_k$, and analyze the following situations.

**Case 1:** $v_2$ has degree two. Let $T' = T - \{v_1\}$. As in the proof of Theorem 11, by the choice of $P$, removing vertex $v_1$ does not change the maximum degree of $T'$ with respect to that of $T$. Thus $\Delta(T') = \Delta(T) \geq 3$. Also, $v_2$ is not a support vertex of $T'$, and by using the same idea as in the proof of Theorem 11, we
deduce that $\gamma_{\text{Str}}(T) \geq \gamma_{\text{Str}}(T') + 1$. Thus, by using the induction hypothesis and taking into account that $s(T) = s(T') + 1$ and $n(T) = n(T') + 1$, it follows,

$$\gamma_{\text{Str}}(T) \geq \gamma_{\text{Str}}(T') + 1$$

$$\geq \left\lceil \frac{n(T') + s(T')}{\Delta(T')} \right\rceil + 2$$

$$\geq \left\lceil \frac{n(T) - 1 + s(T) - 1}{\Delta(T)} \right\rceil + 2$$

$$= \left\lceil \frac{n(T) + s(T) + \Delta(T) - 2}{\Delta(T)} \right\rceil + 1$$

$$\geq \left\lceil \frac{n(T) + s(T)}{\Delta(T)} \right\rceil + 1.$$

**Case 2:** $v_2$ has degree larger than 3. This means $v_2$ has at least three adjacent leaves. Let $v'_1$ be a leaf adjacent to $v_2$ other than $v_1$ and let $T'' = T - \{v_1, v'_1\}$. Again, removing vertices $v_1, v'_1$ does not change the maximum degree, and so $\Delta(T'') = \Delta(T) \geq 3$. Moreover, by Lemma 1, there exists a $\gamma_{\text{Str}}^t(T)$-function $f = (B_0, B_1, B_2)$ such that for every leaf $x$ of $T$, it follows $f(x) = 0$. In this sense, since $f(v_1) = f(v'_1) = 0$, we can construct a TStrD-function for $T''$ by decreasing the label of $v_2$ by one, and leaving the remaining labels unchanged. This leads to the claim $\gamma_{\text{Str}}^t(T) \geq \gamma_{\text{Str}}^t(T'') + 1$. Thus, since $v_2$ continues to be a support vertex in $T''$, by using the induction hypothesis and the equalities $s(T) = s(T'')$ and $n(T) = n(T'') + 2$, we deduce,

$$\gamma_{\text{Str}}^t(T) \geq \gamma_{\text{Str}}^t(T'') + 1$$

$$\geq \left\lceil \frac{n(T'') + s(T'')}{\Delta(T'')} \right\rceil + 2$$

$$\geq \left\lceil \frac{n(T) - 2 + s(T) - 1}{\Delta(T)} \right\rceil + 2$$

$$= \left\lceil \frac{n(T) + s(T) + \Delta(T) - 2}{\Delta(T)} \right\rceil + 1$$

$$\geq \left\lceil \frac{n(T) + s(T)}{\Delta(T)} \right\rceil + 1.$$

**Case 3:** $v_2$ has degree 3. In order to simplify the proof, we shall adapt Case 2 to this situation. We again define $T'' = T - \{v_1, v'_1\}$. Clearly, now $v_2$ is not a support vertex, but a leaf in $T''$. Since $v_2$ has degree 3 in $T'$ and is adjacent to two leaves $v_1, v'_1$ such that $f(v_1) = f(v'_1) = 0$, we must have $f(v_3) \geq 1$. Consequently, $f(v_2) = 2$. Thus, we construct a new TStrD-function on $T''$ by relabeling $v_2$ with 1, which means $\gamma_{\text{Str}}^t(T) \geq \gamma_{\text{Str}}^t(T'') + 1$. Now, since $v_2$ is not a support vertex in $T''$, by using the induction hypothesis and the equalities $s(T) = s(T'') + 1$ and $n(T) = n(T'') + 2$, we deduce that (the last
inequality follows since $\Delta(T) \geq 3$,
\[
\gamma_{StR}^4(T) \geq \gamma_{StR}^4(T'') + 1
\]
\[
\geq \left\lceil \frac{n(T'') + s(T'')}{\Delta(T'')} \right\rceil + 2
\]
\[
\geq \left\lceil \frac{n(T) - 2 + s(T) - 1}{\Delta(T)} \right\rceil + 2
\]
\[
= \left\lceil \frac{n(T) + s(T) + \Delta(T) - 3}{\Delta(T)} \right\rceil + 1
\]
\[
\geq \left\lceil \frac{n(T) + s(T)}{\Delta(T)} \right\rceil + 1,
\]
and this completes the proof. □

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