Decentralized Complete Dictionary Learning via $\ell^4$-Norm Maximization

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Abstract—With the rapid development of information technologies, centralized data processing is subject to many limitations, such as computational overheads, communication delays, and data privacy leakage. Decentralized data processing over networked terminal nodes becomes an important technology in the era of big data. Dictionary learning is a powerful representation learning method to exploit the low-dimensional structure from the high-dimensional data. By exploiting the low-dimensional structure, the storage and the processing overhead of data can be effectively reduced. In this paper, we propose a novel decentralized complete dictionary learning algorithm, which is based on $\ell^4$-norm maximization. Compared with existing decentralized dictionary learning algorithms, comprehensive numerical experiments show that the novel algorithm has significant advantages in terms of per-iteration computational complexity, communication cost, and convergence rate in many scenarios. Moreover, a rigorous theoretical analysis shows that the dictionaries learned by the proposed algorithm can converge to the one learned by a centralized dictionary learning algorithm at a linear rate with high probability under certain conditions.

Index Terms—Consensus averaging, dictionary learning, decentralized algorithms, non-convex optimization.

I. INTRODUCTION

With the development of the Internet of Things (IoT), sensors and various data acquisition devices are ubiquitous and widely used in intelligent transportation, smart grids, and other fields. The centralized data processing method requires network terminal nodes to transmit locally generated data to a central node for data analysis and processing. However, transmitting a large amount of local data can cause network congestion, communication delay, data privacy leakage, and other problems, which limit the real-time decision in the network. Moreover, since the centralized node needs to process all the data, it has too much computational overhead, which limits the application of this processing method in large-scale scenarios. Since a lot of data is generated at network terminals, decentralized real-time data processing at network terminal nodes becomes an important technology to solve the above problems during data transmission.

In the era of big data, it has been observed that many high-dimensional real-world data present a low-dimensional structure under the projection of proper bases [1]. The low-dimensional structure of big data can be effectively exploited to complete various signal processing tasks with reduced resource consumption, such as image denoising [2], face recognition [3], and matrix completion [4]. Therefore, it is critical to learn the low-dimensional structure underlying big data. Dictionary learning (DL) is a representation learning method that can efficiently find low-dimensional structures in high-dimensional data. Given the observed data $Y$, DL aims to decompose the observation matrix $Y$ into a dictionary matrix $D$ and a sparse coefficient matrix $X$ such that $Y = DX$ or $Y \approx DX$. The concept of DL was firstly introduced by Olshausen and Field [5], [6], after which, many DL algorithms have been proposed including Method of Optimal Directions (MOD) [7], K-SVD [8], and Matching, Stretching, and Projection (MSP) [9]. However, these algorithms are centralized algorithms, which require the collection of data over the network and thus cause overhead problems or privacy issues.

In this paper, we focus on the problem of decentralized DL, where the decentralized nodes collaboratively learn the dictionary from the local data by exchanging information among neighboring nodes. The decentralized DL shares the advantages of low latency, low transmission overheads, and data privacy protection, and at the same time, the compressibility of high-dimensional data is effectively utilized through dictionary learning to efficiently process high-dimensional data.

A. Previous Work

In the past decades, several decentralized DL algorithms have been proposed [10]–[20]. Those algorithms can be divided into two main categories, the first one is based on centralized DL algorithms and the second one is based on decentralized frameworks. [11], [16], [19] belong to the first category, where [11], [16] are based on K-SVD [8] and [19] is based on Analysis SimCO [21]. However, these algorithms are not attractive in terms of convergence rate, communication cost, per-iteration computational complexity, etc. Specifically, the per-iteration computational complexity of all three algorithms is very high because they are all based on computational-intensive centralized algorithms. Meanwhile, [11], [19] need thousands of iterations to achieve a good convergence result, thus introducing excessive communication overheads. [10], [12]–[15], [17], [18], [20] belong to the second category, and they are based on some novel or existing decentralized frameworks, such as EXTRA [22], that can be applied directly or with minor modifications to DL. Similarly, these algorithms are also not satisfactory in terms of convergence performance, convergence rate, communication cost, etc. Specifically, some algorithms cannot converge. For example, numerical experiments in [18] showed that the decentralized algorithm proposed in [10] failed to reach consensus.
among local dictionaries. To the best of our knowledge, the fastest convergence rate proven among these algorithms can only achieve a sub-linear rate [18], [20], which leads to this category of algorithms all requiring thousands of iterations to achieve a good convergence result. The difficulties in improving the performance of these two categories of existing decentralized DL algorithms are as follows:

- The key to proposing an effective and efficient decentralized DL algorithm based on existing centralized DL algorithms is to find a centralized algorithm that is effective, efficient and easy to be decentralized. However, although many novel centralized DL formulations and algorithms, such as [23]–[25], proposed in recent years have elaborate designs and rigorous convergence analyses, they are not efficient and effective enough in practice, especially in large-scale scenarios.

- Most of the existing decentralized frameworks and optimization algorithms that can be applied to DL are based on $\ell^4$-norm penalization. However, solving a non-convex optimization problem whose objective function is non-smooth is NP-hard in general [26], which means that it is very difficult or even impossible to design an efficient and effective decentralized DL algorithm based on such formulation.

In the past few years, the development of centralized DL algorithms can inspire us to design more advanced decentralized DL algorithms. It was shown in [24] that if the sparse matrix $X$ obeys the Bernoulli-Gaussian distribution, the complete DL problem can be converted to an orthogonal complete DL problem without loss of generality by preconditioning $Y$. [27] showed that the orthogonal complete DL algorithm can achieve comparable learning performance in terms of the per-iteration computational complexity, communication cost, convergence rate, etc.

- We prove that the dictionaries learned by DMSP converge to the one learned by MSP at a linear rate with high probability under certain conditions.

- We present extensive numerical experiments to show that DMSP can converge to MSP under rather mild conditions and by fully utilizing the advantages of $\ell^4$-based formulation, the proposed DMSP achieves unmatched performance in terms of the per-iteration computational complexity, communication cost, convergence rate, compared with various baselines.

Notations

In this paper, lowercase regular letters denote scalars, lowercase bold letters denote vectors, and uppercase bold letters denote matrices. $x_i$ denotes the $i$-th entry of vector $x$, $x_{i,j}$ denotes the $i$-th row or matrix $X$, $x_{s,i}$ denotes the $i$-th column of matrix $X$, and $x_{i,j}$ denotes the entry in the $i$-th row and $j$-th column of matrix $X$. $X_i$ denotes a local matrix at node $i$, $X^{(i)}$ denotes a matrix at the $i$-th iteration, and $X^{(i,s)}$ denotes a matrix at the $s$-th inner iteration in the $t$-th outer iteration. $\| \cdot \|_4$ denotes the element-wise $\ell^4$-norm. $\| \cdot \|_F$ denotes the Frobenius norm. $[N]$ denotes a set $\{1, 2, \ldots, N\}$. $(A_i)_{i \in \mathbb{N}}$ denotes a tuple $(A_1, \ldots, A_N)$. $\circ$ denotes the Hadamard product and $(\cdot)^{\text{ht}}$ denotes the $j$-th Hadamard power. $\mathcal{P}_G(\cdot)$ denotes a projection onto group $G$. $O(n)$ denotes the $n$-dimensional orthogonal group.

II. PROBLEM FORMULATION

Consider a decentralized network with $N$ nodes and this network can be abstracted as a time-varying directed graph sequence $\{G^{(t,s)}\}_{t,s}$, where $G^{(t,s)} = (\mathcal{V}, \mathcal{E}^{(t,s)})$ is the graph at the $s$-th inner iteration in the $t$-th outer iteration, $\mathcal{V} \subseteq [N]$, and $(i,j) \in \mathcal{E}^{(t,s)}$ if and only if a node $i$ can send messages to another node $j$ at the $s$-th inner iteration in the $t$-th outer iteration. Define the in-neighborhood of node $i$ at the $s$-th inner iteration in the $t$-th outer iteration as $\mathcal{N}_{i,:,:}^{(t,s)} = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}^{(t,s)} \} \cup \{i\}$ and define the out-neighborhood of node $i$ at the $s$-th inner iteration in the $t$-th outer iteration as $\mathcal{N}_{i,:out}^{(t,s)} = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}^{(t,s)} \} \cup \{i\}$. We assume that the graph sequence $\{G^{(t,s)}\}_{t,s}$ is uniformly
strongly connected, i.e., there exists a positive integer \( B \), such that \( G(t) = \left( V, \{U_{i,j}^{(k+1)}B^{-1}E(t)\}_{i,j,k=0}^{B-1} \right) \) is strongly connected for every \( k > 0 \) and every \( t > 0 \). Let \( W(t,s) \in \mathbb{R}^{n \times n} \) denote the weighted adjacency matrix at the \( s \) inner iteration in the \( t \) outer iteration. The weighted adjacency matrix \( W(t,s) \) needs to satisfy the following assumptions: \( W(t,s) \) is a column stochastic matrix and \( w_{i,j}^{(t,s)} > 0 \) if and only if \( j \in \mathcal{N}_{i,in} \).

Suppose that each node \( i \) generates its local observation matrix \( Y_i \in \mathbb{R}^{n \times p_i} \), which can be expressed as \( Y_i = D_o X_i \), where \( D_o \in \mathbb{R}^{n \times n} \) is the ground truth dictionary and \( X_i \in \mathbb{R}^{n \times p_i} \) is a Bernoulli-Gaussian random matrix with sparsity \( \theta \), i.e., \( \{X_i\}_{j,k} \sim i.i.d. BG(\theta) \). Since the complete DL problem can be converted to an orthogonal complete DL problem without loss of generality under the Bernoulli-Gaussian model, we consider orthogonal complete DL in this paper, which means \( D_o \) is a square matrix satisfying the orthogonality constraint.

Let \( Y = [Y_1, \ldots, Y_N] \in \mathbb{R}^{n \times p} \) denote the collection of local data, where \( p = \sum_{i \in [N]} p_i \). If all local data are available at a fusion center, an \( \ell^4 \)-based orthogonal complete DL problem can be formulated as

\[
\begin{align*}
\max_{D,X} & \quad \|X\|_4^4 \\
\text{s.t.} & \quad Y = DX, \\
& \quad D \in O(n).
\end{align*}
\]

Let \( A = D^T \), (1) can be reformulated as

\[
\begin{align*}
\max_{A} & \quad \|AY\|_4^4 \\
\text{s.t.} & \quad A \in O(n).
\end{align*}
\]

However, data collection in a centralized fusion center can induce huge communication overheads and leak data privacy. In this paper, we aim to design a decentralized orthogonal complete DL algorithm based on \( \ell^4 \)-norm maximization, so that the nodes in the network can collaboratively learn the dictionary without exchanging the raw data.

Suppose that \( P \in \mathbb{R}^{n \times n} \) is a signed permutation matrix. Since \( D_o \) and \( D_o P^T \) are both orthogonal, \( X \) and \( PX \) have the same sparsity level, i.e., \( \|X\|_0 = \|PX\|_0 \), the orthogonal complete DL problem is signed permutation ambiguous. Therefore, it is considered to be perfectly recovery if any signed permutation of the ground truth dictionary is found.

### III. \( \ell^4 \)-Based Orthogonal Complete Dictionary Learning Algorithm

#### A. Centralized \( \ell^4 \)-Based Dictionary Learning Algorithm

Most of existing DL algorithms are based on \( \ell^1 \)-norm minimization to promote sparsity. An \( \ell^1 \)-based orthogonal complete DL problem can be formulated as

\[
\begin{align*}
\min_{A} & \quad \|AY\|_1 \\
\text{s.t.} & \quad A \in O(n).
\end{align*}
\]

However, due to the non-smoothness of \( \ell^1 \)-norm and the non-convexity of orthogonality constraint, (3) is NP-hard in general [26]. On the contrary, the smoothness of \( \ell^4 \)-norm and the sparsity imposition of \( \ell^4 \)-norm maximization enable a more efficient DL algorithm design.

Designing an efficient and effective optimization algorithm to solve (2) is tough because of the non-convex objective function and the orthogonality constraint of (2). Based on Algorithm 1 in [29], [9] proposed MSP (see Algorithm 1) to solve (2). Each iteration of MSP performs a projected gradient ascent with infinite step size, and MSP is thus able to achieve a super-linear convergence rate. Numerical experiments in [9], [30] verified that MSP outperforms existing DL algorithms in many scenarios (including noise, outliers, and sparse corruptions).

#### Algorithm 1 MSP (Algorithm 2 in [9])

1: Initialize \( A^{(0)} \in O(n) \)
2: for \( t = 0, 1, \ldots, T - 1 \) do
3: \( \partial A^{(t)} \leftarrow 4 (A^{(t)} Y^\top)^{\odot 3} Y^\top \)
4: \( U \Sigma V^\top \leftarrow \text{SVD} (\partial A^{(t)}) \)
5: \( A^{(t+1)} \leftarrow UV^\top \)
6: end for
7: Output \( A^{(T)} \)

#### B. Decentralized \( \ell^4 \)-Based Dictionary Learning Algorithm

The key to proposing an efficient and effective \( \ell^4 \)-based decentralized orthogonal complete DL algorithm is to find out which part of MSP can be implemented in a decentralized manner. From observation, Step 3 and Step 5 of MSP have the potential for decentralized framework design.

If the decentralized variant of MSP is realized by designing the decentralized framework for Step 5 of MSP, consensus averaging can be employed to exchange the dictionaries learned at the individual node \( A_i^{(t)} \), \( i \in [N] \) after each iteration. However, this decentralized framework cannot guarantee to satisfy the orthogonality constraint in (2). Specifically, the orthogonal group is non-convex, so locally estimated dictionaries after consensus averaging might not be orthogonal, which violates the orthogonality constraint in (2). If each node projects the linearly combined estimated matrix onto the orthogonal group again, the convergence of this decentralized framework cannot be guaranteed even if the perfect consensus is achieved in every iteration because projection onto a non-convex set is not non-expansive.

This paper proposes the algorithm DMSP (see Algorithm 2) by designing the decentralized framework for Step 3 in MSP. DMSP performs consensus averaging to exchange the local gradient matrix of each node \( \partial A_1^{(t)}, \ldots, \partial A_N^{(t)} \) in each iteration. By exchanging the gradient instead of raw data, the proposed algorithm DMSP can protect data privacy. This design is effective because MSP has two useful properties:

- Separable: \( (AY)^{\odot 3} Y^\top = \sum_{i \in [N]} (AY_i)^{\odot 3} Y_i^\top \).
- Scale-Invariant: \( \mathcal{P}_{O(n)}(X) = \mathcal{P}_{O(n)}(\alpha X), \forall X \in \mathbb{R}^{n \times n}, \forall \alpha > 0 \).

Benefitting from two aforementioned properties, we can know if each node has the same initial local dictionary and consensus averaging achieves perfect consensus in each iteration, DMSP will output the same estimated dictionary as MSP.
It is worth mentioning that the scale-invariant property of projection onto the orthogonal group \( O(n) \) is a quite benign property not only because the estimated average gradient matrices obtained by consensus averaging can be used directly, but also because the auxiliary variable used in the Push-Sum protocol does not need to be maintained when the graph is directed, which makes this algorithm less computational and communication intensive. To be more specific, because it is no longer needed to be maintained when the graph is directed, the estimated average gradient matrices are only needed to be maintained when the auxiliary variable is not used in the Push-Sum protocol, where the estimated average gradient matrices obtained by consensus averaging in a directed graph is usually through the Push-Sum protocol, where the estimated average gradient matrices only need to be divided by an auxiliary variable to get the correct estimated average variable. However, benefit from the scale-invariant property, dividing or not dividing by the auxiliary variable has no effect on the result after projection, so we can dispense the auxiliary variable.

**Algorithm 2 DMSP (Simplified Version)**

1: **Initialize** \( A_1^{(0)} = A_2^{(0)} = \cdots = A_N^{(0)} \in O(n) \)
2: for \( t = 0, 1, \ldots, T - 1 \) do
3: \( \partial A_i^{(t)} \leftarrow 4 \left( A_i^{(t)} Y_i \right) \cdot Y_i^T, \forall i \in [N] \)
4: \( \left( \partial \bar{A}_i^{(t)} \right)_{i \in [N]} \leftarrow \text{ConsensusAveraging} \left( \left( \partial A_i^{(t)} \right)_{i \in [N]} \right) \)
5: \( U_i \Sigma_i V_i^T \leftarrow \text{SVD} \left( \partial \bar{A}_i^{(t)} \right), \forall i \in [N] \)
6: \( A_i^{(t+1)} \leftarrow U_i V_i^T, \forall i \in [N] \)
7: end for
8: **Output** \( A_1^{(T)}, A_2^{(T)}, \ldots, A_N^{(T)} \)

Since the auxiliary variable can be dispensed in the Push-Sum protocol, DMSP can use the same algorithmic framework (see Algorithm 3) for both directed and undirected graphs. For undirected graphs, the in-neighborhood and the out-neighborhood are the same, define the neighborhood of node \( i \) at the \( s \) th inner iteration in the \( t \) th outer iteration as \( N_i^{(t,s)} = N_{i,in}^{(t,s)} \). The weighted adjacency matrix \( W^{(t,s)} \) for undirected graphs can be constructed according to the Metropolis weights [33, 34]

\[
w_{i,j}^{(t,s)} = \begin{cases} 
1 & \text{if } (i, j) \in E^{(t,s)}, \\
\frac{1}{\max\{N_{i,in}^{(t,s)}, N_{j,in}^{(t,s)}\}} & \text{if } i = j, \\
1 - \sum_{k \in N_{i,in}^{(t,s)} \setminus \{i\}} w_{i,k}^{(t,s)} & \text{if } i \neq j, \\
0, & \text{otherwise},
\end{cases}
\]

while for directed graphs, the weighted adjacency matrix \( W^{(t,s)} \) can be constructed according to

\[
w_{i,j}^{(t,s)} = \begin{cases} 
\frac{1}{N_{i,in}^{(t,s)}}, & \text{if } j \in N_{i,in}^{(t,s)}, \\
0, & \text{otherwise}.
\end{cases}
\]

**IV. CONVERGENCE ANALYSIS**

In practical scenarios, the consensus averaging iteration usually cannot achieve perfect consensus due to the running time limitation. After each iteration, the locally estimated dictionary obtained by each node will have a deviation from the one obtained by MSP because perfect consensus usually cannot be achieved, and this deviation will accumulate over the iteration process and affect the DL performance in subsequent iterations. To analyze whether the dictionaries learned by DMSP can approach the dictionary learned by MSP, it is important to quantify the deviation between the dictionaries learned by DMSP and the dictionary learned by MSP.

Let \( \delta^{(t)} = \max_{i \in [N]} \| A_i^{(t)} - A_i^{(0)} \|_F \) denote the deviation of the dictionaries learned by DMSP from the dictionary learned by MSP after \( t \) iterations, and let \( \delta^{(t)} (t_{c}^{(t)}) = \max_{i \in [N]} \| A_i^{(t_{c}^{(t)})} (t_{c}^{(t)}) - \bar{A}_i^{(t)} \|_F \) denote the deviation of dictionaries learned by DMSP from the dictionary learned by MSP if the perfect consensus is achieved at the \( t_{c}^{(t)} \) th iteration, where \( t_{c}^{(t)} \) is the number of consensus averaging iterations in the \( t \) th iteration of DMSP. We have the following convergence result of DMSP.

**Theorem 1** (Convergence of DMSP, Informal Version of Theorem 6). Suppose that there exists a signed permutation matrix \( P \) and a non-negative real number \( \epsilon \in \left[ 0, \frac{2\alpha(1-\theta)+3\sqrt{2}(1+\alpha)(1-\theta+\epsilon)}{\alpha-\theta} \right] \) such that

\[
\| A^{(t)} - PD_i^T \|_F \leq \epsilon, \quad (6)
\]

\[
\| A_i^{(t)} - PD_i^T \|_F \leq \epsilon, \quad \forall i \in [N], \quad (7)
\]

\[
\| A^{(t)} D_i^T \|_F \leq n - \frac{1}{2} (\epsilon - \delta^{(t)})^2, \quad (8)
\]

where constant \( \alpha \in (0, 1) \). If

\[
\delta^{(k)} = \frac{2 \| Z^{(k)} - \sum_{i \in [N]} Z_i^{(k)} \|_F}{\sigma_n (Z^{(k)}) + \sigma_n (\sum_{i \in [N]} Z_i^{(k)})}
\]

and \( \delta^{(k)} > 0 \) for every \( k \geq t \), where \( Z^{(k)} = (A^{(k)} Y)^{\odot 3} Y^T \) and \( Z_i^{(k)} = (A_i^{(k)} Y_i)^{\odot 3} Y_i^T \), then

\[
\delta^{(k+1)} < \alpha \delta^{(k)}, \forall k \geq t
\]
holds with high probability, when \( p \) is large enough.

Theorem 1 shows that the deviation of the dictionaries learned by DMSP from the dictionary learned by MSP will converge to 0 at a linear rate with high probability if the estimated dictionaries are within a neighborhood of a global optimum and the number of consensus averaging iterations is regulated properly in each iteration. Later in this section, we will present the formal version of Theorem 1 and prove it rigorously.

Theorem 1 takes into account the \( t \)th and all subsequent iterations. To help our analysis, we start by analyzing the relationship between \( \delta^{(t+1)} \) and \( \delta^{(t)} \). As mentioned before, the deviation between the locally estimated dictionary obtained by each node and the estimated dictionary obtained by MSP at the \( t+1 \)th iteration can be decomposed into two parts:

\[
\delta^{(t+1)} = \max_{i \in [N]} \| A_i^{(t+1)} - A^{(t+1)} \|_F
\]

\[
\leq \max_{i \in [N]} \| A_i^{(t+1)} - \bar{A}^{(t+1)} \|_F + \| \bar{A}^{(t+1)} - A^{(t+1)} \|_F, \tag{11} \]

where \( \bar{A}^{(t+1)} \) is the estimated dictionary learned by DMSP if the perfect consensus is achieved at the \( t+1 \)th iteration.

(11) gives an upper bound of \( \delta^{(t+1)} \), so we only need to analyze the relationship between \( \delta^{(t+1)} \) and \( \delta^{(t)} \), when the estimated dictionaries are both within a neighborhood of a global optimum. Since consensus averaging usually cannot achieve perfect consensus in practical scenarios, we only consider the case \( \delta^{(t)} > 0 \) without loss of generality. By the Theorem 1 in [35], the orthogonal invariance of the Frobenius norm, and the orthogonal invariance of singular values, we know that

\[
\delta_a^{(t+1)} \leq \frac{2 \| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \|_F}{\sigma_n(B^{(t)}) + \sigma_n(\sum_{i \in [N]} B_i^{(t)})}, \tag{12} \]

where \( B^{(t)} = (U^{(t)}X)^{\frac{3}{2}}X^T \), \( U^{(t)} = A^{(t)}D_\alpha \), \( B_i^{(t)} = (U_i^{(t)}X_i)^{\frac{3}{2}}X_i^T \), \( U_i^{(t)} = A_i^{(t)}D_\alpha \), \( \forall i \in [N] \).

(12) implies that we only need to analyze the relationship between \( \frac{2 \| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \|_F}{\sigma_n(B^{(t)}) + \sigma_n(\sum_{i \in [N]} B_i^{(t)})} \) and \( \delta^{(t)} \). There are two main difficulties in analyzing this relationship: \( X \) is a random matrix making we can only express their relationship in terms of probability, and it is also tough to obtain a lower bound of \( \sigma_n(\cdot) \). To solve the first difficulty, we will first consider the deterministic case that all random variables are equal to their expectation, and then concentration bounds can be used. We leverage a gersgorin-type lower bound for the smallest singular value to solve the second difficulty [36].

**Theorem 2.** If there exists a signed permutation matrix \( P \in \mathbb{R}^{n \times n} \) and a non-negative real number \( \epsilon \in [0, \frac{\alpha - \theta}{2 \alpha n(1-\theta)+3\sqrt{2(1-\alpha)(1+\theta)+\alpha \theta}}) \), such that

\[
\| U^{(t)} - P \|_F \leq \epsilon, \tag{13} \]

\[
\| U_i^{(t)} - P \|_F \leq \epsilon, \forall i \in [N], \tag{14} \]

where constant \( \alpha \in (\theta, 1) \), then

\[
\frac{2 \| E[B^{(t)}] - E[\sum_{i \in [N]} B_i^{(t)}] \|_F}{\sigma_n(E[B^{(t)]) + \sigma_n(E[\sum_{i \in [N]} B_i^{(t)})]} < \alpha \delta^{(t)}. \tag{15} \]

Proof of Theorem 2 is given in Appendix B. We will next use Theorem 2 and concentration bounds to derive a lower bound of the probability of \( \delta_a^{(t+1)} < \alpha \delta^{(t)} \) under some assumptions in a stochastic scenario.

**Theorem 3.** On the basis of satisfying all the assumptions in Theorem 2, if \( \frac{p}{(\ln p)^4} = \omega(\frac{\delta^{(t)+2}}{\delta^{(t)} C n}) \), then

\[
\begin{align*}
\mathbb{P}\left( \frac{2 \| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \|_F}{\sigma_n(B^{(t)}) + \sigma_n(\sum_{i \in [N]} B_i^{(t)})} < \alpha \delta^{(t)} \right) \geq & 1 - 8np\theta \exp\left(-\frac{(\ln p)^2}{2}\right) \\
&-8n^2\exp\left(-\frac{3p(\delta^{(t)})^2C^2}{c_2(\delta(t) + 2)^2n^2\theta + 16n^2(\ln p)^4\delta(t)(\delta(t) + 2)C}\right),
\end{align*} \tag{16} \]

where \( C = 6\alpha(1 - \theta)(1 - 2\alpha) - 6(1 - \alpha)^2 - 6\alpha\theta^2 - 18\sqrt{2}(\alpha + 1)(1 - \theta)\epsilon \) and a constant \( c_2 > 6.8 \times 10^4 \).

Proof of Theorem 3 is given in Appendix C. Now we know the relationship between \( \delta_a^{(t+1)} \) and \( \delta^{(t)} \), we need to take \( \delta_a^{(t+1)} \) into account to obtain the relationship between \( \delta_c^{(t+1)} + \delta_a^{(t+1)} \) and \( \delta^{(t)} \).

**Theorem 4.** On the basis of satisfying all the assumptions in Theorem 3, if

\[
\delta_c^{(t+1)} \in \left[ 0, \alpha \delta^{(t)} - \frac{2 \| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \|_F}{\sigma_n(B^{(t)}) + \sigma_n(\sum_{i \in [N]} B_i^{(t)})} \right], \tag{17} \]

then

\[
\begin{align*}
\mathbb{P}\left( \delta^{(t+1)} < \alpha \delta^{(t)} \right) \geq & 1 - 8np\theta \exp\left(-\frac{(\ln p)^2}{2}\right) \\
&-8n^2\exp\left(-\frac{3p(\delta(t) + 2)^2n^2\theta + 16n^2(\ln p)^4\delta(t)(\delta(t) + 2)C}{c_2(\delta(t) + 2)^2n^2\theta + 16n^2(\ln p)^4\delta(t)(\delta(t) + 2)C}\right),
\end{align*} \tag{18} \]

where \( C = 6\alpha(1 - \theta)(1 - 2\alpha) - 6(1 - \alpha)^2 - 6\alpha\theta^2 - 18\sqrt{2}(\alpha + 1)(1 - \theta)\epsilon \) and a constant \( c_2 > 6.8 \times 10^4 \).

Proof of Theorem 4 is given in Appendix D. On the basis of satisfying all the assumptions in Theorem 4, if
$p/(ln p)^3 = \omega\left(\frac{(\delta^{(t)+2})^2n^{2\theta}}{(\delta^{(t)})^2C^2} + \frac{(\delta^{(t)+2})n^{5/2}}{\delta^{(t)}C}\right)$, we can know that $\delta^{(t+1)} < \alpha \delta^{(t)}$ with high probability. Based on this conclusion, the relationship between $\delta^{(t+2)}$ and $\delta^{(t+1)}$ will be discussed below.

**Theorem 5.** On the basis of satisfying all the assumptions in Theorem 3, if

$$\|U^{(t)}\|_F^4 \geq n - \frac{1}{2} \left(\epsilon - \delta^{(t)}\right)^2,$$

then

$$\|U^{(t+1)} - P\|_F^p \leq \epsilon - \delta^{(t+1)},$$

$$\|U_i^{(t+1)} - P\|_F^p \leq \epsilon, \quad \forall i \in [N],$$

$$\|U^{(t+1)}\|_F^4 \geq n - \frac{1}{2} \left(\epsilon - \delta^{(t+1)}\right)^2.$$  

Proof of Theorem 5 is given in Appendix E. By Theorem 5, we know that if $\frac{p}{(ln p)^3} = \omega\left(\frac{(\delta^{(t+1)+2})^2n^{2\theta}}{(\delta^{(t+1)})^2C^2} + \frac{(\delta^{(t+1)+2})n^{5/2}}{\delta^{(t+1)}C}\right)$ and $\delta^{(t+2)} \in \left[0, \alpha \delta^{(t+1)} - \frac{2}{\sigma_n(B^{(t+1)} + \sum_{i \in [N]} B^{(t+1)}_i)_F}\right]$ also hold, then $\delta^{(t+2)} < \alpha \delta^{(t+1)}$ with high probability and assumptions in Theorem 5 still hold. Therefore, we can obtain the following theorem.

**Theorem 6 (Convergence of DMSP).** Suppose that there exists a signed permutation matrix $P$ and a non-negative real number $\epsilon \in \left[0, \frac{\theta - \theta^2}{2\alpha(1-\theta)(1-\theta) + \alpha \theta}\right]$ such that

$$\|A^{(t)} - PD^T\|_F \leq \epsilon,$$

$$\|A_i^{(t)} - PD^T\|_F \leq \epsilon, \quad \forall i \in [N],$$

$$\|A^{(t)}D\|_F^4 \geq n - \frac{1}{2} \left(\epsilon - \delta^{(t)}\right)^2,$$

where constant $\alpha \in (\theta, 1)$. If

$$\delta^{(k+1)} \in \left[0, \alpha \delta^{(k)} - \frac{2}{\sigma_n(B^{(k)} + \sum_{i \in [N]} B^{(k)}_i)_F}\right],$$

and $\delta^{(k)} > 0$ for every $k \geq t$, then

$$\delta^{(k+1)} < \alpha \delta^{(k)}, \quad \forall k \geq t$$

holds with high probability, when $\frac{p}{(ln p)^3} = \omega\left(\frac{(\delta^{(t)+2})^2n^{2\theta}}{(\delta^{(t)})^2C^2} + \frac{(\delta^{(t)+2})n^{5/2}}{\delta^{(t)}C}\right)$.

**V. NUMERICAL RESULTS**

In this section, the results of comprehensive numerical experiments will be presented. In Subsection V-A, DMSP will be compared with MSP to show the effectiveness of DMSP. In Subsection V-B, DMSP will be compared with three existing decentralized DL algorithms Cloud K-SVD [16], Linearized D^4L [18], and DESTINY [20] to further corroborate the efficiency and effectiveness of DMSP. All experiments are conducted on a server with two Intel Xeon Gold 6354 CPUs. All codes are implemented in python and mpi4py [37] is used for node-to-node communications. All experimental results in this section are averaged among 5 trials except for denoised images.

**A. Comparison with MSP**

We first consider a time-varying directed network comprising $N = 36$ nodes and the edges of this network are randomly generated by an Erdos-Renyi model with probability parameter $P$. Observation matrix $Y \in \mathbb{R}^{n \times p}$ is generated by $Y = D_oX$, where $D_o \in \mathbb{R}^{n \times n}$ is a random orthogonal matrix and $X \in \mathbb{R}^{n \times p}$ is a Bernoulli-Gaussian random matrix with sparsity $\theta$. Local observation matrices $Y_i$ are obtained by slicing the observation matrix $Y$ approximately evenly. Decentralized nodes collaboratively recover the dictionary by exchanging the information among neighboring nodes and the recovery error in each node is measured by $\|1 - \frac{n}{\|A_iD_o\|_2^4/2n}\|_2/\|n\|$ because a perfect recovery gives a $0\%$ error. The number of outermost iterations for all experiments in this subsection is 15 and the weighted adjacency matrices $W^{(t,s)}$ are constructed according to (5).

Table I compares the recovery error of MSP and DMSP under different choices of $P$ and $T_c$. Figure 1 shows the convergence performance of MSP and DMSP under different choices of $T_c$. Table I and Figure 1 show that DMSP converges to MSP within a small number of consensus averaging iterations per outermost iteration even in a sparse network. From these experimental results, it can be seen that DMSP can converge to MSP under rather mild conditions and the effectiveness of DMSP is verified.

![Fig. 1. Convergence plots under different choices of $T_c$ when $n = 25$, $p = 10,000$, $\theta = 0.1$, and $P = 0.2$.](image)

**B. Comparison with Existing Decentralized DL Algorithms**

In this subsection, DMSP will be compared with three existing decentralized DL algorithms Cloud K-SVD [16], Linearized D^4L [18], and DESTINY [20], using both synthetic and real data. Cloud K-SVD, Linearized D^4L, and DESTINY
are chosen as baselines because they are representatives of $\ell^0$-based, $\ell^1$-based, and $\ell^2$-based decentralized DL algorithms, respectively. To meet the requirements of all the baselines being compared, we consider a time-invariant undirected network comprising $N = 36$ nodes and the edges of this network are randomly generated by an Erdos-Renyi model with probability parameter $P = 0.5$. To ensure that the communication cost of each outermost iteration of these four algorithms is approximately the same, let $T_c = 2$ in DMSP and $T_p = 2$, $T_c = 1$ in Cloud K-SVD. The weighted adjacency matrices $\mathbf{W}^{(t,s)}$ are constructed according to (4).

1) Experiments Using Bernoulli-Gaussian Data: The observation matrix $\mathbf{Y} \in \mathbb{R}^{n \times p}$ is generated by $\mathbf{Y} = \mathbf{D}_o \mathbf{X}$, where $\mathbf{D}_o \in \mathbb{R}^{n \times n}$ is a random orthogonal matrix and $\mathbf{X} \in \mathbb{R}^{n \times p}$ is a Bernoulli-Gaussian random matrix with sparsity $\theta$. Local observation matrices $\mathbf{Y}_i \in \mathbb{R}^{n \times p_i}$ are obtained by slicing the observation matrix $\mathbf{Y}$ approximately evenly. The metric for recovery error is the same as the metric in Subsection V-A.

Table II compares DMSP with three baseline algorithms in terms of the running time and the recovery errors under different choices of $n$, $p$, and $\theta$. The experimental results in Table II show that DMSP has a lower per-iteration computational complexity and faster convergence rate than existing decentralized DL algorithms, which also results in a lower communication cost required to achieve a desired convergence result. Moreover, the low per-iteration computational complexity and fast convergence rate of DMSP indicate that it is more suitable for large-scale scenarios than existing decentralized DL algorithms. Figure 2 presents the convergence performance of DMSP and three existing decentralized DL algorithms mentioned above when $n = 25$, $p = 10,000$, and $\theta = 0.1$. Figure 2 shows that DMSP converges faster than existing decentralized DL algorithms.

| $n$ | $p$ | $\theta$ | $T_p = 0$ | $T_p = 1$ | $T_p = 2$ | $T_p = 3$ | $T_p = 4$ | $T_p = 5$ | MSP |
|-----|-----|---------|----------|----------|----------|----------|----------|----------|-----|
| 25  | 10,000 | 0.1 | 18.50% | 1.03% | 0.27% | 0.20% | 0.19% | 0.19% | 0.19% |
| 25  | 10,000 | 0.1 | 18.50% | 0.37% | 0.20% | 0.19% | 0.19% | 0.19% | 0.19% |
| 25  | 10,000 | 0.1 | 18.50% | 0.24% | 0.19% | 0.19% | 0.19% | 0.19% | 0.19% |
| 25  | 10,000 | 0.1 | 18.50% | 0.20% | 0.19% | 0.19% | 0.19% | 0.19% | 0.19% |

VI. CONCLUSION

In this paper, we proposed a novel decentralized DL algorithm DMSP, which allows decentralized nodes collaboratively

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

Fig. 3. Original image, corrupted image, and denoised images after 30 outermost iterations of Boat. (a): original image; (b): corrupted image (PSNR = 26.04 dB); (c): denoised image by DMSP (PSNR = 30.03 dB); (d): denoised image by Cloud K-SVD (PSNR = 28.30 dB); (e): denoised image by DESTINY (PSNR = 24.28 dB); (f): denoised image by Linearized D^4L (PSNR = 27.27 dB).

2) Experiments on Image Denoising: The theoretical analysis of DMSP heavily relies on the Bernoulli-Gaussian model. Experiments in this part aim to illustrate the performance of DMSP beyond the Bernoulli-Gaussian model by comparing the effectiveness and efficiency of DMSP with existing decentralized DL algorithms on image denoising.

The original image chosen for experiments on image denoising are two $512 \times 512$ grayscale images Boat and Barbara. The corrupted images are obtained by adding Gaussian white noise to the original images with different noise levels. For each experiment, the observation matrix $\mathbf{Y}$ is generated by extracting all $8 \times 8$ patches from a corrupted image. Local observation matrices $\mathbf{Y}_i$ are obtained by slicing the observation matrix $\mathbf{Y}$ approximately evenly. Peak signal-to-noise ratio (PSNR) is used to measure the effectiveness of denoising and the number of outermost iteration for all experiments in this part is 30.

Table III compares DMSP with three baseline algorithms in terms of the running time and the effectiveness of denoising under different choices of noise level. Figure 3 and Figure 4 presents denoised images generated by DMSP and three baseline algorithms when the noise level is 0.0025. The experimental results in Table III, Figure 3, and Figure 4 show that DMSP outperforms existing decentralized DL algorithms on image denoising in terms of the effectiveness of denoising, per-iteration computational complexity, convergence rate, and communication cost.

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

![Image](314x280 to 385x352)

Fig. 3. Original image, corrupted image, and denoised images after 30 outermost iterations of Boat. (a): original image; (b): corrupted image (PSNR = 26.04 dB); (c): denoised image by DMSP (PSNR = 30.03 dB); (d): denoised image by Cloud K-SVD (PSNR = 28.30 dB); (e): denoised image by DESTINY (PSNR = 24.28 dB); (f): denoised image by Linearized D^4L (PSNR = 27.27 dB).
TABLE II
RECOVERY PERFORMANCE UNDER DIFFERENT CHOICES OF n, p, and \( \theta \).

| n   | p    | \( \theta \) | iter. | DMSP (Ours) | Time | Error | Cloud K-SVD | Time | Error | DESTINY | Time | Error | Linearized D^L | Time | Error |
|-----|-----|-------|------|-----------|------|-------|------------|------|-------|---------|------|-------|----------------|------|-------|
| 25  | 10,000 | 0.1  | 15   | 0.06 s  | 0.26% | 1.18 s | 13.71%     | 0.06 s | 57.89% | 0.07 s  | 94.61% |
| 25  | 10,000 | 0.3  | 15   | 0.05 s  | 0.38% | 1.30 s | 13.45%     | 0.05 s | 60.20% | 0.07 s  | 99.73% |
| 50  | 20,000 | 0.1  | 20   | 2.42 s  | 0.27% | 107.48 s| 15.84%     | 3.23 s | 63.49% | 4.78 s  | 88.30% |
| 50  | 20,000 | 0.3  | 20   | 2.98 s  | 0.39% | 79.38 s| 15.24%     | 3.73 s | 72.23% | 3.69 s  | 97.24% |
| 100 | 40,000 | 0.1  | 25   | 8.69 s  | 0.25% | 485.30 s| 18.49%     | 10.33 s | 72.86% | 10.21 s | 92.36% |
| 100 | 40,000 | 0.3  | 25   | 8.54 s  | 0.40% | 499.42 s| 16.28%     | 9.58 s | 81.44% | 12.25 s | 99.60% |

TABLE III
DENOSING PERFORMANCE UNDER DIFFERENT CHOICES OF IMAGE AND NOISE LEVEL.

| Variance | PSNR | Time | Error | Corrupted Image | PSNR | Time | Error | Cloud K-SVD | PSNR | Time | Error | DESTINY | PSNR | Time | Error | Linearized D^L | PSNR | Time |
|---------|------|------|-------|-----------------|------|------|-------|------------|------|------|-------|---------|------|------|-------|----------------|------|------|
| Boat    |      |      |       |                 |      |      |       |             |      |      |       |         |      |      |       |               |      |      |
| 0.005   | 23.07 dB | 20.31 s | 28.99 dB | 3049.90 s | 28.20 dB | 36.70 s | 23.75 dB | 46.85 s | 26.79 dB |
| 0.01    | 20.11 dB | 19.75 s | 27.39 dB | 2833.23 s | 27.35 dB | 37.81 s | 22.94 dB | 46.72 s | 26.57 dB |
| Barbara |      |      |       |                 |      |      |       |             |      |      |       |         |      |      |       |               |      |      |
| 0.005   | 23.05 dB | 20.52 s | 29.25 dB | 3058.77 s | 27.68 dB | 38.16 s | 20.63 dB | 45.69 s | 25.40 dB |
| 0.01    | 20.14 dB | 19.70 s | 27.77 dB | 2917.84 s | 27.12 dB | 36.85 s | 19.82 dB | 45.59 s | 24.73 dB |

Fig. 4. Original image, corrupted image, and denoised images after 30 outermost iterations of Barbara. (a): original image; (b): corrupted image (PSNR = 26.04 dB); (c): denoised image by DMSP (PSNR = 30.15 dB); (d): denoised image by Cloud K-SVD (PSNR = 28.37 dB); (e): denoised image by DESTINY (PSNR = 23.22 dB); (f): denoised image by Linearized D^L (PSNR = 25.56 dB).

learn the dictionary from the local data. Convergence analysis showed that DMSP can converge to MSP at a linear rate with high probability under certain conditions, which indicates that DMSP can effectively leverage the advantages of \( \ell^q \)-norm maximization to achieve excellent performance in terms of per-iteration computational complexity, convergence rate, etc. Extensive experiments further corroborated the effectiveness and efficiency of DMSP.

APPENDIX A
LEMmAS FOR THE CONVERGENCE ANALYSIS

Lemma 7. If \( A \in \mathbb{R}^{n \times n} \), then

\[
\sigma_n(A) \geq \min_{i \in [n]} \left\{ \left| a_{i,i} \right| - \frac{1}{2} \left( \sum_{j \neq i} \left| a_{i,j} \right| + \sum_{j \neq i} \left| a_{j,i} \right| \right) \right\}. \quad (29)
\]

Proof: Please refer to Theorem 3 in [36].

Lemma 8. If \( U \in O(n), X \in \mathbb{R}^{n \times p} \), \( x_{i,j} \sim_{i.i.d.} \text{BG}(\theta) \), then

\[
\{ E \left[ (UX)^{o_3} (UX)^T \right] \}_{i,j} = \begin{cases} 3p\theta(1-\theta) \sum_{k \in [n]} u_{i,k}^3 u_{j,k} & \text{if } i \neq j, \\ 3p\theta(1-\theta) \sum_{k \in [n]} u_{i,k}^4 + 3p\theta^2 & \text{if } i = j. \end{cases} \quad (30)
\]

Proof: By (98) in [9], we know that

\[
E \left[ (UX)^{o_3} X^T \right] = 3p\theta(1-\theta)U^{o_3} + 3p\theta^2 U. \quad (31)
\]

Since \( U \) is deterministic and orthogonal,

\[
E \left[ (UX)^{o_3} (UX)^T \right] = 3p\theta(1-\theta)U^{o_3} U^T + 3p\theta^2 I. \quad (32)
\]

Therefore,

\[
\{ E \left[ (UX)^{o_3} (UX)^T \right] \}_{i,j} = \begin{cases} 3p\theta(1-\theta) \sum_{k \in [n]} u_{i,k}^3 u_{j,k} & \text{if } i \neq j, \\ 3p\theta(1-\theta) \sum_{k \in [n]} u_{i,k}^4 + 3p\theta^2 & \text{if } i = j. \end{cases} \quad (33)
\]

Lemma 9. Let \( P \in \mathbb{R}^{n \times n} \) be a signed permutation matrix and \( U \in O(n) \). If \( \| U - P \|_F = \epsilon \) and \( p_{i,j} \neq 0 \), then

\[
| u_{i,j} | \geq 1 - \frac{1}{2} \epsilon^2. \quad (34)
\]

Proof: We can assume that \( P \) is a diagonal matrix without loss of generality.
If $p_{i,i} = 1$, let $u_{i,*} = [\delta_1, \ldots, \delta_{i-1}, 1 - \delta_i, \delta_{i+1}, \ldots, \delta_n]$ and $c_i^2 = \sum_{i \in [n]} \delta_i^2$. Since $U$ is orthogonal, we know that

$$\delta_1^2 + \cdots + \delta_{i-1}^2 + (1 - \delta_i)^2 + \delta_{i+1}^2 + \cdots + \delta_n^2 = 1. \quad (35)$$

Substitute $\sum_{j \neq i} \delta_j^2$ by $c_i^2 - \delta_i^2$ in (35), we obtain that

$$\sum_{i \in [n]} \delta_i^2 = 1, \quad (36)$$

which implies that

$$\delta_i = \frac{1}{2} \epsilon_i^2 \leq \frac{1}{2} \epsilon_i. \quad (37)$$

which indicates that

$$|u_{i,i}| \geq 1 - \frac{1}{2} \epsilon_i^2. \quad (38)$$

If $p_{i,i} = -1$, we can get the same result by the same approach. Therefore, we can conclude that if $\|U - P\|_F = \epsilon$ and $p_{i,j} \neq 0$, then

$$|u_{i,j}| \geq 1 - \frac{1}{2} \epsilon_i^2. \quad (39)$$

**Lemma 10.** Let $P \in \mathbb{R}^{n \times n}$ be a signed permutation matrix and $U \in O(n)$. If $\|U - P\|_F = \epsilon \leq 1$ and $i \neq j$, then

$$|u_{i,i}^* u_{j,j}^T| \leq 2 \epsilon. \quad (40)$$

**Proof:** Let $\theta_{u_i, u_j^3}$, $\theta_{u_i, u_j}$, $\theta_{u_i, u_j^3, u_j}$ be the angles between vector $u_i$ and vector $u_j^3$, vector $u_i$ and vector $u_j$, and vector $u_i^3$ and vector $u_j$, respectively. The orthogonality of $U$ implies that

$$|u_{i,i}^* u_{j,j}^T| \leq \|u_{i,i}^3\|_2 \|u_{j,j}\|_2 \cos \theta_{u_i^3, u_j} \leq \cos \theta_{u_i^3, u_j}. \quad (41)$$

If $\theta_{u_i^3, u_j} \in [0, \frac{\pi}{2}]$, we know that

$$\theta_{u_i^3, u_j} \geq \theta_{u_i, u_j} - \theta_{u_i, u_j^3} = \frac{\pi}{2} - \theta_{u_i, u_j^3}, \quad (42)$$

which implies that

$$\left| \cos \theta_{u_i^3, u_j} \right| \leq \left| \cos \left( \frac{\pi}{2} - \theta_{u_i, u_j^3} \right) \right| = \sin \theta_{u_i, u_j^3}. \quad (43)$$

If $\theta_{u_i^3, u_j} \in (\frac{\pi}{2}, \pi]$, we know that

$$\theta_{u_i^3, u_j} \leq \theta_{u_i, u_j} + \theta_{u_i, u_j^3} = \frac{\pi}{2} + \theta_{u_i, u_j^3}, \quad (44)$$

which implies that

$$\left| \cos \theta_{u_i^3, u_j} \right| \leq \left| \cos \left( \frac{\pi}{2} + \theta_{u_i, u_j^3} \right) \right| = \sin \theta_{u_i, u_j^3}. \quad (45)$$

By (41), (43), and (45), we know that

$$|u_{i,i}^* u_{j,j}^T| \leq \cos \theta_{u_i^3, u_j} \leq \sin \theta_{u_i, u_j^3}. \quad (46)$$

We can further obtain that

$$|u_{i,i}^* u_{j,j}^T| \leq \sin \theta_{u_i, u_j^3}.$$
\[ \|U^{03} - V^{03}\|_F \leq 3\sqrt{2\epsilon}\|U - V\|_F. \] (53)

**Proof:** Let \( \delta = \|U - V\|_F \). If \( \delta = 0 \), (53) is trivial, so we only need to consider the case \( \delta > 0 \).

We can assume that \( P \) is a diagonal matrix without loss of generality. Let \( U - V = D + N \) where \( D \) is the diagonal part of \( U - V \) and \( N \) is the off-diagonal part of \( U - V \). Let \( \|D\|_F = \delta_D \) and \( \|N\|_F = \delta_N \), so \( \delta_D^2 + \delta_N^2 = \delta^2 \). Hence,

\[
\|U^{03} - V^{03}\|_F^2 \leq \sum_{i,j \in [n]} (u_{i,j}^3 - v_{i,j}^3)^2 \\
\leq \sum_{i,j \in [n]} (u_{i,j} - v_{i,j})^2 (u_{i,j}^2 + u_{i,j}v_{i,j} + v_{i,j}^2)^2 \\
= \sum_{i \in [n]} (u_{i,i} - v_{i,i})^2 (u_{i,i}^2 + u_{i,i}v_{i,i} + v_{i,i}^2)^2 \\
+ \sum_{i \in [n]} (u_{i,j} - v_{i,j})^2 (u_{i,j}^2 + u_{i,j}v_{i,j} + v_{i,j}^2)^2 \\
\leq 9(\delta_D^2 + \epsilon^4 \delta_N^2) \\
\leq 9\left(\frac{\delta_D^2}{\delta_D^2 + \epsilon^4}\right) \delta^2,
\] (54)

which implies that

\[
\|U^{03} - V^{03}\|_F \leq 3\sqrt{\frac{\delta_D^2}{\delta_D^2 + \epsilon^4}} \delta.
\] (55)

Since \( \delta \) and \( \epsilon \) are fixed, we need to find an upper bound of \( \delta_D^2 \), or equivalently, a lower bound of \( \delta_N^2 \). We will accomplish this goal in two steps. In the first step we will decouple each row of \( U \) and \( V \) into \( n \) subproblems, and after that we will solve each subproblem. In the second step we will derive the conclusion of the original problem from the conclusion obtained by solving the subproblems.

1. **Decouple and Solve Subproblems**

Let \( \delta_i = \|u_{i,*} - v_{i,*}\|_2, \delta_D,i = \|d_{i,*}\|_2, \delta_N,i = \|n_{i,*}\|_2, \forall i \in [n] \). If

\[
f(\delta, \epsilon) = \max_{\|U - V\|_F = \delta_D, \|U - P\|_F \leq \epsilon} \delta_D^2,
\] (56)

\[
g_i(\delta_i, \epsilon) = \max_{\|u_{i,*} - v_{i,*}\|_2 = \delta_i, \|v_{i,*} - p_{i,*}\|_2 \leq \epsilon} \delta_D,i, \forall i \in [n].
\] (57)

we know that

\[
f(\delta, \epsilon) \leq \max_{\sum_i \delta_i^2 = \delta^2} \sum_i g_i(\delta_i, \epsilon).
\] (58)

Therefore, we can decouple each row of \( U \) and \( V \) into \( n \) subproblems, solving the \( n \) easier subproblems first. For the \( i^{th} \) subproblem, since \( \delta_i \) and \( \epsilon \) are known, we need to find an upper bound of \( \delta_D,i^2 \), or equivalently, a lower bound of \( \delta_N,i^2 \).

Next we will consider the impact of the choice of diagonal and non-diagonal entries of \( U \) and \( V \) on \( \delta_N,i^2 \) in turn.

1.1. **The Choice of Non-diagonal Entries**

Since

\[
\delta_N,i^2 = \sum_{j \neq i} (u_{i,j} - v_{i,j})^2 \\
= \sum_{j \neq i} u_{i,j}^2 + \sum_{j \neq i} v_{i,j}^2 - 2 \sum_{j \neq i} u_{i,j}v_{i,j} \\
= 1 - u_{i,i}^2 + 1 - v_{i,i}^2 - 2 \sum_{j \neq i} u_{i,j}v_{i,j},
\] (59)

we know that \( \delta_N,i^2 \) is minimized, no matter what the values of \( u_{i,i} \) and \( v_{i,i} \), if

\[
u_{i,j} = \begin{cases} \sqrt{1 - u_{i,j}^2}, & \text{if } j = (i + 1) \mod n, \\ 0, & \text{if } j \neq i \text{ and } j \neq (i + 1) \mod n. \end{cases}
\] (60)

\[
v_{i,j} = \begin{cases} \sqrt{1 - v_{i,j}^2}, & \text{if } j = (i + 1) \mod n, \\ 0, & \text{if } j \neq i \text{ and } j \neq (i + 1) \mod n. \end{cases}
\] (61)

1.2. **The Choice of Diagonal Entries**

We will further consider the choice of diagonal entries of \( U \) and \( V \) on the basis of the subsection 1.1. We can assume that \( p_{i,i} = 1 \) and \( u_{i,i} \geq v_{i,i} \) without loss of generality. Lemma 9 implies that

\[
u_{i,i} \geq v_{i,i} \geq 1 - \frac{1}{2}\epsilon^2.
\] (62)

Assume that \( v_{i,i} = C + x \) and \( u_{i,i} = C + \delta_D,i + x \) where \( C = 1 - \frac{1}{2}\epsilon^2 \), hence,

\[
\delta_D,i^2 = \left(\sqrt{1 - (C + x)^2} - \sqrt{1 - (C + \delta_D,i + x)^2}\right)^2 \\
= 1 - (C + x)^2 + 1 - (C + \delta_D,i + x)^2 \\
- 2\sqrt{1 - (C + x)^2}\sqrt{1 - (C + \delta_D,i + x)^2}.
\] (63)

Since

\[
\frac{\partial \delta_D,i^2}{\partial x} = -2C - 2x - 2C - 2\delta_D,i - 2x \\
+ 2\sqrt{1 - (C + \delta_D,i + x)^2}(C + x) \\
+ 2\sqrt{1 - (C + x)^2}(C + \delta_D,i + x)
\] (64)

we know that \( x = 0 \), i.e., \( v_{i,i} = C \) and \( u_{i,i} = C + \delta_D,i \) can minimize \( \delta_N,i^2 \).

2. **Solve the Original Problem**

In this section, we will unify \( n \) subproblems again to find a lower bound of \( \delta_N^2 \). Consider the optimization problem (65)

\[
\min_{\delta_D,1, \ldots, \delta_D,n} \sum_{i \in [n]} \delta_N,i^2
\]
Apply the Karush-Kuhn-Tucker conditions to (66), we have

$$f_i(x) = \sqrt{1 - C^2 x(C + x)^2} + C (1 - (C + x)^2)^{\frac{3}{2}} - C\sqrt{1 - C^2} (1 - (C + x)^2).$$

The derivative of $f_i(x)$ is

$$f'_i(x) = 3(C + x) \left( \sqrt{1 - C^2 (C + x)} - C \sqrt{1 - (C + x)^2} \right).$$

Since $f_4(0) = 0$, we only need to show that $f'_4(x) > 0$ in $(0, 1 - C)$. Note that

$$(C + x)^2 > C^2 \Rightarrow (C + x)^2 - C^2(C + x)^2 > C^2(C + x)^2 \Rightarrow (1 - C^2)(C + x)^2 > C^2(1 - (C + x)^2) \Rightarrow \sqrt{1 - C^2}(C + x) > C\sqrt{1 - (C + x)^2}.$$ (73)

we know that $f'_4(x) > 0$ in $(0, 1 - C)$. Therefore, $\Delta_{D,i}$ are equal for every $i \in S$ which implies that $\delta_{D,i}$ are equal for every $i \in S$.

Now, we need to consider the cardinality of $S$. Suppose that $|S| = m \in [1, n]$, then $\delta_{D,i} = \frac{\mu_i}{m}$ and the objective of (66) is

$$g_1(m) = m \left( 1 - C^2 + 1 - \left( C + \frac{\delta_D}{\sqrt{m}} \right)^2 \right) - 2m \sqrt{1 - C^2} \sqrt{1 - \left( C + \frac{\delta_D}{\sqrt{m}} \right)^2}.$$ (74)

Since $g_2(x) = \delta_{D,i}^2$ is strictly monotonic decreasing in $(0, 1 - C)$, we only need to show that $g_3(x) = \frac{1}{n^3} (g_1 \circ g_2)(x)$ is strictly monotonic increasing in $(0, 1 - C)$ if we want to show that $g_1(m)$ is strictly monotonic decreasing in $(\frac{\delta_{D,i}^2}{(1-C)^2} + \infty)$. The derivative of $g_3(x)$ is

$$g'_3(x) = \frac{\sqrt{1 - C^2 x^2} - 2\sqrt{1 - (C + x)^2}(1 - C^2)}{x^3\sqrt{1 - (C + x)^2} (1 - C^2)} + \frac{2\sqrt{1 - C^2 (1 - (C + x)^2)}}{x^3\sqrt{1 - (C + x)^2}} \left( \frac{(1 - (C + x)^2) - \sqrt{1 - C^2} Cx}{x^3\sqrt{1 - (C + x)^2}} \right).$$ (75)

Note that denominator of (75) is positive, i.e., $x^3\sqrt{1 - (C + x)^2} > 0$, in $(0, 1 - C)$, so we only need to show that the numerator of (75) is also positive in $(0, 1 - C)$. Let

$$g_4(x) = \frac{\sqrt{1 - C^2 x^2} - 2\sqrt{1 - (C + x)^2}(1 - C^2)}{x^3\sqrt{1 - (C + x)^2} (1 - C^2)} + 2\sqrt{1 - C^2 (1 - (C + x)^2)} + \frac{(\sqrt{1 - (C + x)^2} + \sqrt{1 - C^2}) Cx}. (76)$$
The derivative of $g_4(x)$ is
\[ g'_4(x) = (3C + 2x) \frac{1 - Cx - C^2 - \sqrt{1 - C^2} \sqrt{1 - (C + x)^2}}{\sqrt{1 - (C + x)^2}}. \] (77)

Since $g_4(0) = 0$, we only need to show that $g'_4(x) > 0$ in $(0, 1 - C)$. Note that
\begin{align*}
C^4 + 2C^3x + C^2x^2 - 2C^2 - 2Cx - x^2 + 1 \\
< C^4 + 2C^3x + C^2x^2 - 2C^2 - 2Cx + 1 \\
\Rightarrow (1 - C^2)(1 - (C + x)^2) < (1 - Cx + C^2)^2 \\
\Rightarrow \sqrt{1 - C^2\sqrt{1 - (C + x)^2}} < 1 - Cx + C^2, \tag{78}
\end{align*}
we know that $g'_4(x) > 0$ in $(0, 1 - C)$ and $|S| = n$. Hence, a lower bound of $\delta_2^2$ is
\[ \delta_2^2 \geq n \left(1 - C^2 + 1 - \left(1 + \frac{\delta_D}{\sqrt{n}}\right)^2\right) \\
- 2n \sqrt{1 - C^2} \sqrt{1 - \left(1 + \frac{\delta_D}{\sqrt{n}}\right)^2}. \tag{79} \]

Let $C_1 = \sqrt{1 - C^2}$, $C_2 = \sqrt{1 - \left(1 + \frac{\delta_D}{\sqrt{n}}\right)^2}$, we have
\[ \delta_2^2 \geq n \left(\frac{C_1^2 - C_2^2}{C_1 - C_2}\right)^2 \\
= n \frac{2C_2^2}{C_1 - C_2} \geq \frac{n}{4} \left(1 - C^2\right) \geq \frac{C_1^2}{1 - C^2} \delta_2^2, \tag{80} \]
which implies that
\[ \delta_2^2 \leq \frac{\delta_2^2}{\delta_D^2 + \frac{C_1^2}{1 - C^2} \delta_D^2} = 1 - \left(1 - \frac{C_1^2}{2}\right)^2 \leq 2e^2 - e^4. \tag{81} \]

Therefore, we can conclude that
\[ \|U^o - V^o\|_F \leq 3\sqrt{\frac{\delta_2^2}{\delta_D^2}} + e^4 \delta \leq 3\sqrt{2e}\|U - V\|_F. \tag{82} \]

**Lemma 13.** Let $U, U_1, \ldots, U_N \in O(n)$ such that $\|U - U_i\|_F \leq \delta$, $\forall i \in [N]$, and let $X = [X_1, \ldots, X_N] \in \mathbb{R}^{n \times p}$, $x_{i,j} \sim_{i.i.d.} \text{BG}(\theta)$ where $X_i \in \mathbb{R}^{n \times p}$, $\forall i \in [N]$. If there exists a signed permutation matrix $P$, such that $\|U - P\|_F \leq \epsilon$, $\forall i \in [N]$, and $\|U_i - P\|_F \leq \epsilon$, $\forall i \in [N]$, then
\[ \|U^o - V^o\|_F \leq 3\sqrt{2e}\|U - V\|_F. \tag{82} \]

**Proof:** (85), (86), and (87) in [9] implies that
\[ \|U^o - V^o\|_F \leq 3n^2\|U - V\|_F. \tag{87} \]

**Lemma 14.** Let $U, U_1, \ldots, U_N \in O(n)$ such that $\|U - U_i\|_F \leq \delta$, $\forall i \in [N]$, and let $X = [X_1, \ldots, X_N] \in \mathbb{R}^{n \times p}$, $x_{i,j} \sim_{i.i.d.} \text{BG}(\theta)$ where $X_i \in \mathbb{R}^{n \times p}$, $\forall i \in [N]$. If $X$ is the truncation of $X$ by bound $B$
\[ \bar{x}_{i,j} = \begin{cases} x_{i,j}, & \text{if } |x_{i,j}| \leq B, \\ 0, & \text{otherwise}, \end{cases} \tag{85} \]
then
\[ \|U^o - \sum_{i \in [N]} (U_i^o X_i^o) X_i^T\|_F \leq 3n^2\|B\|_F \delta. \tag{86} \]

**Proof:** (85), (86), and (87) in [9] implies that
\[ \|U^o - \sum_{i \in [N]} (U_i^o X_i^o) X_i^T\|_F \leq 3n^2\|B\|_F \delta. \tag{87} \]
\[ \mathbb{P} \left( \| \sum_{i \in [N]} (U_i X_i)^3 X_i^T - E \left[ (U_i X_i)^3 X_i^T \right] \|_F \geq np\xi \right) \leq 2n^2 \exp \left( \frac{-3p\xi^2}{c_1 \theta + 8n^2 (\ln p)^4 \xi} \right) + 2np \theta \exp \left( \frac{-(\ln p)^2}{2} \right). \] (88)

for a constant \( c_1 > 1.7 \times 10^4 \).

**Proof:** Let \( \tilde{X} \) be the truncation of \( X \) by bound \( \ln p \)

\[ \tilde{x}_{i,j} = \begin{cases} x_{i,j}, & \text{if } |x_{i,j}| \leq \ln p, \\ 0, & \text{otherwise.} \end{cases} \] (89)

Let \( U = U_1 \). Since \( \|U_i - U_j\| \leq \delta_c, \forall i, j \in [N] \), we know that

\[ \|U - U_i\|_F \leq \delta_c, \forall i \in [N]. \] (90)

Lemma 34 in [9] implies that

\[ \mathbb{P} \left( \| \sum_{i \in [N]} (U_i X_i)^3 X_i^T - E \left[ (U_i X_i)^3 X_i^T \right] \|_F \geq np\xi \right) \leq \mathbb{P} \left( \mathbb{P} \left( \| \sum_{i \in [N]} \exp \left( \| -3p\xi^2 \right) + 2np \theta \exp \left( \frac{-(\ln p)^2}{2} \right). \right) \right) \] (91)

Now we only need to obtain an upper bound of \( \Gamma \). By Lemma 13, Lemma 14, we have

\[ \Gamma \leq \mathbb{P} \left( \| \sum_{i \in [N]} \left( (U_i X_i)^3 X_i^T - (U X)^3 X_i^T \right) \|_F \right) \leq \mathbb{P} \left( \| \sum_{i \in [N]} \left( (U_i X_i)^3 X_i^T \right) \|_F \right) \] (92)

Since \( \xi \geq 8n(\ln p)^4 \delta_c \), we have

\[ \Gamma \leq \mathbb{P} \left( \| (U X)^3 X_i^T - E \left[ (U X)^3 X_i^T \right] \|_F \geq np\xi \right). \] (93)

By (232), (233), and (239) in [9], we have

\[ \Gamma \leq \mathbb{P} \left( \| (U X)^3 X_i^T - E \left[ (U X)^3 X_i^T \right] \|_F \geq \frac{np\xi}{4} \right). \] (94)

for a constant \( c_1 > 1.7 \times 10^4 \).

Combine (91) and (94), we have

\[ \mathbb{P} \left( \| \sum_{i \in [N]} (U_i X_i)^3 X_i^T - E \left[ (U_i X_i)^3 X_i^T \right] \|_F \geq np\xi \right) \leq 2n^2 \exp \left( \frac{-3p\xi^2}{c_1 \theta + 8n^2 (\ln p)^4 \xi} \right) + 2np \theta \exp \left( \frac{-(\ln p)^2}{2} \right). \] (95)

for a constant \( c_1 > 1.7 \times 10^4 \).

**Lemma 16.** Let \( U_1, \ldots, U_N \in O(n) \) such that \( \|U - U_i\|_F \leq \delta_c, \|U_i - U_j\|_F \leq \delta_c, \forall i, j \in [N] \) and let \( X = [X_1, \ldots, X_N] \in \mathbb{R}^{n \times p}, x_{i,j} \sim_{\text{i.i.d.}} \text{BG}(\theta) \) where \( X_i \in \mathbb{R}^{n \times p}, \forall i \in [N] \). If there exists a signed permutation matrix \( P \in \mathbb{R}^{n \times n} \), such that \( \|U - P\|_F = \epsilon \leq 1, \|U_i - P\|_F \leq \epsilon \leq 1, \forall i \in [N], \) and \( \xi \geq 16n(\ln p)^4 \delta_c \), then

\[ \mathbb{P} \left( \left\| \frac{(U X)^3 X_i^T - \sum_{i \in [N]} (U_i X_i)^3 X_i^T \right\|_F \geq \left( 9\sqrt{2p\theta} \right) \right. \right) \leq 4n^2 \exp \left( \frac{-3p\xi^2}{c_2 \theta + 16n^2 (\ln p)^4 \xi} \right) + 4np \theta \exp \left( \frac{-(\ln p)^2}{2} \right). \] (96)

for a constant \( c_2 > 6.8 \times 10^4 \).

**Proof:** The triangle inequality, Lemma 13, and Lemma 15 implies that

\[ \mathbb{P} \left( \left\| \left( U X \right)^3 X_i^T - \sum_{i \in [N]} (U_i X_i)^3 X_i^T \right\|_F \right) \] (97)

\[ < \left( 9\sqrt{2p\theta} \right) \] (98)
Let \( i \in [N] \). If there exists a signed permutation matrix \( P \in \mathbb{R}^{n \times n} \), such that \( \| U - P \|_F \leq \epsilon \leq 1 \), \( \| U_i - P \|_F \leq \epsilon \leq 1 \), \( \forall i \in [N], \) and \( \xi \geq 16n(\ln p)^4 \delta \), then

\[
\mathbb{P} \left( \sigma_n \left( \left( UX \right)^{\odot 3} X^T \right) + \sigma_n \left( \sum_{i \in [N]} (U_i X_i)^{\odot 3} X_i^T \right) \leq 6p\delta^2 \right) + 6p\delta(1 - \theta)(1 - 2\epsilon) - \left( 9\sqrt{2p\theta}(1 - \theta)\epsilon + 3p^2 \right) \delta - np\xi 
\]

\[
\leq 4n^2 \exp \left( \frac{-3p\xi^2}{c_2 \theta + 16n^2(\ln p)^4 \xi} \right) + 4np\exp \left( \frac{-(\ln p)^2}{2} \right), 
\]

for a constant \( c_2 > 6.8 \times 10^4 \).

**Proof:** Lemma 11 and Lemma 17 implies that

\[
\mathbb{E} \left[ (UX)^{\odot 3} X^T \right] + \mathbb{E} \left[ \sum_{i \in [N]} (U_i X_i)^{\odot 3} X_i^T \right] > 6p\delta^2 
\]

By Lemma 15, we can further know that

\[
\mathbb{P} \left( \sigma_n \left( \left( UX \right)^{\odot 3} X^T \right) + \sigma_n \left( \sum_{i \in [N]} (U_i X_i)^{\odot 3} X_i^T \right) > 6p\delta^2 \right) 
\]

\[
\leq 1 - \mathbb{P} \left( \left\| \left( UX \right)^{\odot 3} X^T \right\|_F - \mathbb{E} \left[ (UX)^{\odot 3} X^T \right] \right\|_F \geq \frac{n\xi}{2} \right) \]

\[
\geq 1 - 4n^2 \exp \left( \frac{-3p\xi^2}{c_2 \theta + 16n^2(\ln p)^4 \xi} \right) - 4np\exp \left( \frac{-(\ln p)^2}{2} \right), 
\]

for a constant \( c_2 > 6.8 \times 10^4 \). \( \blacksquare \)

**Lemma 17.** Let \( U_1, U_2, \ldots, U_N \in O(n) \) such that \( \| U - U_i \|_F \leq \delta \), \( \| U_i - U_j \|_F \leq \delta \), \( \forall i, j \in [N] \) and let \( X = [X_1, \ldots, X_N] \in \mathbb{R}^{n \times p} \), \( x_{i,j} \sim_{i.i.d.} \) BG(\( \theta \)) where \( \theta \) is a constant.

\[
\sigma_n \left( \mathbb{E} \left[ \sum_{i \in [N]} (U_i X_i)^{\odot 3} X_i^T \right] \right) \geq 3p\theta(1 - \theta)(1 - 2\epsilon) + 3p^2 - \left( 9\sqrt{2p\theta}(1 - \theta)\epsilon + 3p^2 \right) \delta. 
\]

**Proof:** (98) can obtained through Lemma 11, Lemma 13, and Theorem 7.3.5 in [38]. \( \blacksquare \)

**Appendix B**

**Proof of Theorem 2**

**Lemma 18.** Let \( U_1, U_2, \ldots, U_N \in O(n) \) such that \( \| U - U_i \|_F \leq \delta \), \( \| U_i - U_j \|_F \leq \delta \), \( \forall i, j \in [N] \) and let \( X = [X_1, \ldots, X_N] \in \mathbb{R}^{n \times p} \), \( x_{i,j} \sim_{i.i.d.} \) BG(\( \theta \)) where

\[
\| X_i \|_\infty \leq \xi, \forall i \in [N]. \]
Hence, we only need to show that
\[
6\alpha \rho \theta (1 - \theta) (1 - 2 \eta c) + 6\alpha \rho \theta^2
- \left(9 \sqrt{2} \alpha \rho \theta (1 - \theta) \varepsilon + 3 \alpha \rho \theta^2 \right) \delta(t)
> 18\sqrt{2} \rho \theta (1 - \theta) \varepsilon + 6 \rho \theta^2.
\]
\[\tag{103}\]
Since \(\delta(t) \leq \max_{i \in [N]} \left\| U^{(i-1)} - P \right\|_F + \left\| U^{(i-1)} - P \right\|_F \leq 2 \varepsilon \) and \(\varepsilon \leq 1\), we only need to show that
\[
6\alpha \rho \theta (1 - \theta) (1 - 2 \eta c) + 6\alpha \rho \theta^2
- \left(18\sqrt{2} \rho \theta (1 - \theta) + 6 \rho \theta^2 \right) \varepsilon
> 18\sqrt{2} \rho \theta (1 - \theta) \varepsilon + 6 \rho \theta^2.
\]
\[\tag{104}\]
Since \(\varepsilon \in \left[0, \frac{\alpha - \theta}{2\alpha (1 - \theta) + 3 \sqrt{2} (1 - \alpha) (1 + \theta) + \alpha \theta} \right]\), we can know that
the inequality (15) holds. 

**APPENDIX C**

**PROOF OF THEOREM 3**

**Proof:** Let \(\left\| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \right\|_F = (9 \sqrt{2} \rho \theta (1 - \theta) \varepsilon + 3 \rho \theta^2) \delta(t) + n p \xi_1\), \(\sigma_n (B^{(t)}) + \sigma_n \left(\sum_{i \in [N]} B_i^{(t)}\right) = 6 \rho \theta (1 - \theta) (1 - 2 \eta c) + 6 \rho \theta^2 - (9 \sqrt{2} \rho \theta (1 - \theta) \varepsilon + 3 \rho \theta^2) \delta(t) - n p \xi_2\).

Theorem 2 implies that
\[
\mathbb{P} \left( \frac{2 \left\| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \right\|_F}{\sigma_n (B^{(t)}) + \sigma_n \left(\sum_{i \in [N]} B_i^{(t)}\right)} < \alpha \delta(t) \right)
\geq \mathbb{P} \left( \frac{2 n p \xi_1}{\delta(t)} + n p \xi_2 < p C \right)
\geq \mathbb{P} \left( \frac{2 n p \xi_1}{\delta(t)} < \frac{2}{\delta(t) + 2} p C, n p \xi_2 < \frac{\delta(t)}{\delta(t) + 2} p C \right)
\geq 1 - \mathbb{P} \left( n p \xi_1 \geq \frac{\delta(t)}{\delta(t) + 2} p C \right) - \mathbb{P} \left( n p \xi_2 \geq \frac{\delta(t)}{\delta(t) + 2} p C \right),
\]
where \(C = 6 \alpha \theta (1 - \theta) (1 - 2 \eta c) - 6(1 - \alpha) \theta^2 - 6 \alpha \theta^2 \varepsilon - 18 \sqrt{2} (\alpha + 1) \theta (1 - \theta) \varepsilon\).

By Lemma 16 and Lemma 18, we can know that the inequality (16) holds.

**APPENDIX D**

**PROOF OF THEOREM 4**

**Proof:** Since
\[
\delta(t+1) = \begin{cases} 0, & \alpha \delta(t), \\ \frac{2 \left\| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \right\|_F}{\sigma_n (B^{(t)}) + \sigma_n \left(\sum_{i \in [N]} B_i^{(t)}\right)} \end{cases},
\]
we know that
\[
\mathbb{P} \left( \delta(t+1) < \alpha \delta(t) \right)
\geq \mathbb{P} \left( \delta(t+1) + \alpha \delta(t) < \alpha \delta(t) \right)
\geq \mathbb{P} \left( \frac{2 \left\| B^{(t)} - \sum_{i \in [N]} B_i^{(t)} \right\|_F}{\sigma_n (B^{(t)}) + \sigma_n \left(\sum_{i \in [N]} B_i^{(t)}\right)} < \alpha \delta(t) \right)
\geq 1 - 8 n p \theta \exp \left( - \frac{(\ln p)^2}{2} \right)
- 8 n^2 \exp \left( - \frac{3 p (\delta(t))^2 \gamma^2}{4 \alpha^2 (\ln p)^4 (\delta(t) + 2) C} \right),
\]
\[\tag{107}\]
where \(C = 6 \alpha \theta (1 - \theta) (1 - 2 \eta c) - 6(1 - \alpha) \theta^2 - 6 \alpha \theta^2 \varepsilon - 18 \sqrt{2} (\alpha + 1) \theta (1 - \theta) \varepsilon\) and a constant \(c_2 > 6.8 \times 10^4\). 

**APPENDIX E**

**PROOF OF THEOREM 5**

**Proof:** Lemma 6 in [9], (19), and (20) implies that
\[
\left\| U^{(t+1)} - P \right\|_F \leq \varepsilon - \delta(t) \leq \varepsilon - \delta(t+1).
\]
\[\tag{108}\]
Since \(\delta(t+1) = \max_{i \in [N]} \left\| U^{(t+1)} - U^{(t+1)} \right\|_F\), we know that
\[
\max_{i \in [N]} \left\| U_i - P \right\|_F \leq \varepsilon.
\]
\[\tag{109}\]
Since \(\left\| U^{(t+1)} \right\|_4 \geq \left\| U^{(t)} \right\|_4\), we know that
\[
\left\| U^{(t+1)} \right\|_4 \geq \left\| U^{(t)} \right\|_4 \geq \varepsilon \geq n - \frac{1}{2} \left( \varepsilon - \delta(t) \right)^2.
\]
\[\tag{110}\]

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