Modified newtonian dynamics and non-relativistic ChSAS gravity

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Abstract

In the context of the non-relativistic theories, a generalization of the Chern–Weil-theorem allows us to show that extended Chern–Simons actions for gravity in \( d = 4 \) invariant under some specific non-relativistic groups lead to modified Poisson equations. In some particular cases, these modified equations have the form of the so-called MOND approach to gravity. The modifications could be understood as due to the effects of dark matter. This result could leads us to think that dark matter can be interpreted as a non-relativistic limit of dark energy.
I. INTRODUCTION

In Ref. [1] it was shown that: (i) it is possible to obtain non-relativistic versions of both generalized Poincaré algebras $\mathfrak{B}_n$ [2–4] and generalized AdS-Lorentz algebras [3, 5–7]. (ii) Using an analogous procedure to that used in Ref. [8], it is possible to find the non-relativistic limit of the five-dimensional Einstein–Chern–Simons gravity which leads to a modified version of the Poisson equation.

On the other hand, in the context of the so-called extended gauge theory, Antoniadis, Konitopoulos and Savvidy introduced background-free gauge invariants including gauge potentials described by higher degree differential forms [9–12]. This construction has allowed to study in Refs. [13–15] a generalization of the Chern–Weil theorem, which made possible to construct generalized $2n$ and $(2n + 2)$-dimensional transgression forms, to reproduce the $(2n + 2)$-dimensional Chern–Simons forms obtained in Refs. [11, 12] and to show that the $2n$-dimensional Chamseddine’s topological gravity [16–18] corresponds to a Chern–Simons–Antoniadis–Savvidy (ChSAS) form. These mathematical results were then used to study the construction of an off-shell invariant ChSAS action for gravity in $d = 4$ which is gauge quasi-invariant under the generalized gauge transformations for the Maxwell algebra. In Ref. [19] it was shown that the extended invariants found by Antoniadis, Konitopoulos and Savvidy can also be obtained by gauging free differential algebras.

It is the purpose of this paper to consider the non-relativistic versions of the generalized Poincaré algebra $\mathfrak{B}_4$ (Maxwell algebra [20, 21]) denoted by $\mathcal{G}\mathfrak{B}_4$ and the AdS-Lorentz algebra $AdS\mathfrak{L}_4$ denoted by $\mathcal{G}\mathfrak{L}_4$ to find the non-relativistic limit of the four dimensional ChSAS action for gravity.

This paper is organized as follows. In Section 2 we present a short review: (i) on the non-relativistic versions of the Maxwell algebra $\mathfrak{B}_4$ and the AdS-Lorentz, known as galilean algebra type I $\mathcal{G}\mathfrak{B}_4$ and galilean algebra type II $\mathcal{G}\mathfrak{L}_4$ respectively; (ii) on ChSAS gravity. In Sections 3 and 4, using an analogous procedure to that used in Ref. [8], generalizations of the Newtonian gravity are found by gauging the $\mathcal{G}\mathfrak{B}_4$ and $\mathcal{G}\mathfrak{L}_4$ algebras. In Section 5 we study the possible relations between the Newtonian gravities found in the previous sections and the so called Modified Newtonian Dynamics (MOND) models. Finally our conclusions are presented in Section 6.
II. NON-RELATIVISTIC $\mathfrak{B}_4$ AND $\mathfrak{L}_4$ ALGEBRAS AND CHSAS GRAVITY

In [1] were found the non-relativistic versions of the generalized Poincaré algebras $\mathfrak{B}_n$ [2, 4] and the generalized AdS-Lorentz algebras $AdS_Ln$ [3, 5–7, 22] using a generalized Inönü–Wigner contraction. These non-relativistic algebras were called generalized Galilean algebras type I and type II and denoted by $\mathfrak{G}\mathfrak{B}_n$ and $\mathfrak{G}\mathfrak{L}_n$ respectively.

A. Galilean algebra type I $\mathfrak{G}\mathfrak{B}_4$

In Ref. [1] was found in that, separating the spatial and temporal components in the generators $\{P_a, J_{ab}, Z_{ab}\}$ of Maxwell algebra $\mathfrak{B}_4$, performing the rescaling $K_i \to c^{-1}J_0$, $P_i \to R^{-1}P_i$, $H \to cR^{-1}P_0 - c^2M$, $Z_i \to c^{-1}Z_0$ and taking the limits $c, R \to \infty$ (with $c$ the speed of light and $R$ is the cosmic radius), the generators of the $\mathfrak{G}\mathfrak{B}_4$ algebra satisfy the following non-vanishing commutation relations

$$[J_{ij}, J_{kl}] = \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{ij}J_{lk} - \delta_{ik}J_{jl},$$
$$[J_{ij}, K_k] = \delta_{kj}K_i - \delta_{ki}K_j,$$
$$[K_i, P_j] = -\delta_{ij}M,$$
$$[J_{ij}, P_k] = \delta_{kj}P_i - \delta_{ki}P_j,$$
$$[K_i, H] = -P_i,$$
$$[J_{ij}, Z_{kl}] = \delta_{jk}Z_{il} + \delta_{il}Z_{jk} - \delta_{ij}Z_{lk} - \delta_{ik}Z_{jl},$$
$$[J_{ij}, Z_k] = \delta_{kj}Z_i - \delta_{ki}Z_j,$$
$$[P_i, H] = Z_i,$$
$$[Z_{ij}, K_k] = \delta_{kj}Z_i - \delta_{ki}Z_j,$$

Others $= 0,$

where $i, j, k, l = 1, 2, 3$. This algebra correspond to a S-expansion of the Newton–Hooke algebra with central extension [23–25], whose representation can be obtained from the SO(3, 2) algebra using the gamma matrices

$$\Gamma_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu],$$
where $\gamma_\mu$ satisfied the Clifford algebra $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$ with $\mu, \nu = 0, 1, \ldots, 4$ and $\eta_{\mu\nu} = \text{diag} (-c^2, 1, 1, 1, -R^2)$. The identification $J_{ij} = \Gamma_{ij}$, $\Gamma_{i0} = cK_i$, $\Gamma_{i4} = RP_i$, $\Gamma_{04} = RP_0 = Rc^{-1}(H + c^2M)$ and $\Gamma_* = 2M$ with $\Gamma_* = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$ gives us the commutation relations of Newton–Hooke algebra with central extension and allows to know the invariant tensors of generalized Galilean algebras by means of the S-expansion procedure \[23, 24\].

**B. Galilean algebra type II $\mathcal{GL}_4$**

The $\mathcal{GL}_4$ algebra corresponds to the nonrelativistic limit of the $\text{AdSL}_4 \equiv so(D - 1, 1) \oplus so(D - 1, 2)$ algebra \[1\]. This algebra was introduced in Refs. \[5–7\], reobtained from the Maxwell algebra in Ref. \[31\] using a method known as deformation of Lie algebras and later from de $\text{AdS}$ algebra in Ref. \[3\] using the so called S-expansion procedure.

In Ref. \[1\] it was found that separating the spatial and temporal components in the generators $\{P_a, J_{ab}, Z_{ab}\}$ of $\text{AdSL}_4$ algebra, performing the rescaling $K_i \rightarrow c^{-1}J_{i0}$, $P_i \rightarrow R^{-1}P_i$, $H \rightarrow cR^{-1}P_0 - c^2M$, $Z_i \rightarrow c^{-1}Z_{i0}$ and taking the limit $c, R \rightarrow \infty$, the generators of the $\mathcal{GL}_4$ algebra satisfy the following non-vanishing commutation relations

\[
\begin{align*}
[J_{ij}, J_{kl}] &= \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{jl}J_{ik} - \delta_{ik}J_{jl}, \\
[J_{ij}, K_k] &= \delta_{kj}K_i - \delta_{ki}K_j, \\
[K_i, P_j] &= -\delta_{ij}M, \\
[J_{ij}, P_k] &= \delta_{kj}P_i - \delta_{ki}P_j, \\
[K_i, H] &= -P_i, \\
[J_{ij}, Z_{kl}] &= \delta_{jk}Z_{il} + \delta_{il}Z_{jk} - \delta_{jl}Z_{ik} - \delta_{ik}Z_{jl}, \\
[J_{ij}, Z_k] &= \delta_{kj}Z_i - \delta_{ki}Z_j,
\end{align*}
\]
\[ [P_i, H] = Z_i, \]
\[ [Z_{ij}, K_k] = \delta_{kj} Z_i - \delta_{ki} Z_j, \]
\[ [Z_{ij}, Z_{kl}] = \delta_{jk} Z_{il} + \delta_{il} Z_{jk} - \delta_{jl} Z_{ik} - \delta_{ik} Z_{jl}, \]
\[ [Z_{ij}, Z_k] = \delta_{kj} Z_i - \delta_{ki} Z_j, \]
\[ [Z_{ij}, P_k] = \delta_{kj} P_i - \delta_{ki} P_j, \]
\[ [Z_i, P_j] = -\delta_{ij} M, \]
\[ [Z_i, H] = -P_i. \]  
(3)

This algebra can be also written as the direct sum \( \mathcal{G} \Sigma_4 = NH \oplus E(3) \), where \( NH \) is the Newton-Hooke with central extension and \( E(3) \) is the Euclidean algebra in three dimensions. In fact, carrying out the base change

\[ \tilde{K}_i = K_i - Z_i, \quad \tilde{Z}_{ij} = J_{ij} - Z_{ij}, \]

in the \( \mathcal{G} \Sigma_4 \) algebra (3), we find that the only non vanishing commutator are:

(a) the Newton-Hooke algebra with central extension

\[ [J_{ij}, J_{kl}] = \delta_{jk} J_{il} + \delta_{il} J_{jk} - \delta_{jl} J_{ik} - \delta_{ik} J_{jl}, \]
\[ [J_{ij}, P_k] = \delta_{kj} P_i - \delta_{ki} P_j, \]
\[ [J_{ij}, Z_k] = \delta_{kj} Z_i - \delta_{ki} Z_j, \]
\[ [Z_i, P_j] = -\delta_{ij} M, \]
\[ [Z_i, H] = -P_i, \]
\[ [P_i, H] = Z_i. \]  
(4)

(b) the Euclidian algebra in three dimensions \( E(3) \)

\[ [\tilde{Z}_{ij}, \tilde{Z}_{kl}] = \delta_{jk} \tilde{Z}_{il} + \delta_{il} \tilde{Z}_{jk} - \delta_{jl} \tilde{Z}_{ik} - \delta_{ik} \tilde{Z}_{jl}, \]
\[ [\tilde{Z}_{ij}, \tilde{K}_k] = \delta_{kj} \tilde{K}_i - \delta_{ki} \tilde{K}_j, \]
\[ [\tilde{K}_i, \tilde{K}_j] = 0. \]  
(5)
(c) $J_{ij}$ commutators are given by

$$
\begin{align*}
[J_{ij}, P_k] &= \delta_{kj} P_i - \delta_{ki} P_j, \\
[J_{ij}, Z_k] &= \delta_{kj} Z_i - \delta_{ki} Z_j, \\
[J_{ij}, \tilde{Z}_{kl}] &= \delta_{jk} \tilde{Z}_{il} + \delta_{il} \tilde{Z}_{jk} - \delta_{jl} \tilde{Z}_{ik} - \delta_{ik} \tilde{Z}_{jl}, \\
[J_{ij}, \tilde{K}_k] &= \delta_{kj} \tilde{K}_i - \delta_{ki} \tilde{K}_j,
\end{align*}
$$

Others = 0. \quad \text{(6)}

From where we can see that the Newton Hooke algebra with central extension is subalgebra of $\mathcal{G} \mathcal{L}_4$ which correspond to the non-relativistic limit of $AdS$ algebra. It is interesting to note that in (5) the generator $\tilde{Z}_{ij}$ corresponds to a rotation in $so(3)$ and $\tilde{K}_i$ is a translation in $R^3$. On the other hand, in (6) $J_{ij}$ corresponds to a rotation in $so(3)$, $\tilde{Z}_i$ is a boost, $M$ corresponds to the center of the algebra and $P_i$, $H$ are space and time translation operators respectively.

C. Pontryagin–Chern and ChSAS forms

The idea of extending the Yang–Mills fields to higher rank tensor gauge fields was used in Refs. [10–12] to construct gauge invariant and metric independent forms in higher dimensions. These forms are analogous to the Pontryagin–Chern forms in Yang–Mills gauge theory and are given by,

$$
\begin{align*}
\Gamma_{2n+3} &= \langle F^n, F_3 \rangle = dC^{(2n+2)}_{\text{ChSAS}}, \\
\Gamma_{2n+4} &= \langle F^n, F_4 \rangle = dC^{(2n+3)}_{\text{ChSAS}}, \\
\Xi_{2n+6} &= \langle F^n, F_6 \rangle + n\langle F^{n-1}, F_4^2 \rangle = dC^{(2n+5)}_{\text{ChSAS}}, \\
\Upsilon_{2n+8} &= \langle F^n, F_8 \rangle + 3n\langle F^{n-1}, F_4, F_6 \rangle + n(n-1)\langle F^{n-2}, F_4^3 \rangle = dC^{(2n+7)}_{\text{ChSAS}}, \quad \text{(7)}
\end{align*}
$$

where $F_3, F_4, F_6, F_8$ are the field-strength tensors for the gauge fields $A_2$, $A_3$, $A_5$, and $A_7$ respectively and where $C^{(2n+2)}_{\text{ChSAS}}, C^{(2n+3)}_{\text{ChSAS}}, C^{(2n+5)}_{\text{ChSAS}}, C^{(2n+7)}_{\text{ChSAS}}$ are the corresponding Chern–Simons-
Antoniads Savvidy forms which are given by

\[ C_{\text{ChSAS}}^{(2n+2)} = \langle F^n, A_2 \rangle + d\varphi_{2n+1}, \]
\[ C_{\text{ChSAS}}^{(2n+3)} = \langle F^n, A_3 \rangle + d\varphi_{2n+2}, \]
\[ C_{\text{ChSAS}}^{(2n+5)} = \langle F^n, A_5 \rangle + n \langle F^{n-1}, F_4, A_3 \rangle, \]
\[ C_{\text{ChSAS}}^{(2n+7)} = \langle F^n, A_7 \rangle + n(n - 1) \langle F_4, F_4, A_3, F^{n-2} \rangle + n \langle F_6, A_3, F^{n-1} \rangle + 2n \langle F_4, A_5, F^{n-1} \rangle. \]  

(8)

In Ref. [19] were shown that the ChSAS invariants found in Refs. [10–12] can be constructed from an algebraic structure known as free differential algebra (FDA).

D. Four-dimensional ChSAS gravity

In Ref. [13] was constructed generalized \((2n+2)\)-dimensional trangression forms and reproduced the so called \((2n+2)\)-dimensional ChSAS forms obtained in Refs. [11, 12]. These mathematical results were used to construct a four-dimensional action for gravity in \(d = 4\)

\[ S_{\text{ChSAS}} = \int_{M^4} \mathcal{L}_{\text{ChSAS}}, \]

whose Lagrangian \(\mathcal{L}_{\text{ChSAS}} = \langle F, A_2 \rangle \equiv \langle F, B \rangle\) is a ChSAS form.

III. NEWTONIAN CHSAS GRAVITY FOR THE \(\mathcal{G}B_4\) ALGEBRA

In Ref. [8] was shown how the Newton–Cartan formulation of Newtonian gravity can be obtained from gauging the Bargmann algebra. In Refs. [13] it was shown that the gauging of \(B_4\) allows us to construct a four-dimensional ChSAS gravity which leads general relativity in a certain limit. On the other hand, we have seen that the non-relativistic version of the \(B_4\) algebra is given by the \(G\mathcal{B}_4\) algebra. In this Section we show that, using an analogous procedure to that used in Ref. [8], it is possible to find a generalization of the Newtonian gravity.
A. Gauging the \( \mathcal{GB}_4 \) algebra

The one-form gauge connection \( A \) valued in the \( \mathcal{GB}_4 \) algebra is given by

\[
A = \frac{v}{l} \tau H + \frac{1}{l} e^i P_i + \frac{1}{vl} m M + \frac{1}{v} \omega^i K_i + \frac{1}{2} \omega^i j_{ij} + \frac{1}{2} k^i j_{ij} + \frac{1}{v} k^i Z_i, \tag{10}
\]

where \( \nu, l \) are parameters of dimensions velocity and length respectively. The corresponding 2-form curvature \( F = dA + \frac{1}{2} [A, A] \) is given by

\[
F = \frac{v}{l} \tau H + \frac{1}{l} (T^i - \omega^i \tau) P_i + \frac{1}{vl} (m - \omega^i e_i) M + \frac{1}{v} D \omega^i K_i + \frac{1}{2} R^i j_{ij} + \frac{1}{v} \tau G^i + \frac{1}{l} e^i B_{ij} e^j - \frac{1}{v} \omega^i B^0 + \frac{1}{2} B^i j_{ij} + \frac{1}{2} \omega^i k^i Z_i + \frac{1}{2} \omega^i k^i B_{ij} + \frac{1}{2} \omega^i k^i B_{ij} + \frac{1}{2} C \omega^i B^i, \tag{11}
\]

where \( T^i = D e^i \) and \( R^i j_{ij} = d \omega^i j_{ij} + \omega^i k^i \omega^j \). Here \( D \) is the covariant derivative with respect to the \( SO(3) \) transformations. For the two-form gauge potential \( B \) we can write

\[
B = B^i P_i + B^0 H + B^{(m)} M + \frac{1}{2} B^i j_{ij} + G^i K_i + \frac{1}{2} \beta^i j_{ij} + \beta^i Z_i, \tag{12}
\]

whose associated 3-form curvature is given by

\[
F_3 = D B = d B + [A, B]
= H^0 H + H^i P_i + H^{(m)} M + \frac{1}{2} H^i j_{ij} + L^i K_i + \frac{1}{2} \Theta^i j_{ij} + \Theta^i Z_i, \tag{13}
\]

where

\[
\begin{align*}
H^0 &= d B^0, \\
H^i &= D B^i + \frac{v}{l} \tau G^i - \frac{1}{l} B^i j_{ij} - \frac{1}{v} \omega^i B^0, \\
H^{(m)} &= d B^{(m)} + \frac{1}{l} e^i G_i - \frac{1}{v} \omega^i B^i, \\
H^i j_{ij} &= D B^i j_{ij}, \\
L^i &= D G^i - \frac{1}{v} B^i j_{ij} \omega_j, \\
\Theta^i j_{ij} &= D \beta^i - k^i j_{ij} B^{kj} + k^i j_{ij} B^{ki}, \\
\Theta^i &= D \beta^i - \frac{1}{l} \tau B^i + \frac{1}{l} e^i B^0 - \frac{1}{l} B^i j_{ij} + k^i j_{ij} G_j - \frac{1}{v} \beta^i j_{ij} \omega_j. \tag{14}
\end{align*}
\]

These equations are analogous to Eq. (2.13) of Ref. [26] and Eq. (III.6.47) of Ref. [27] and therefore they are not a FDA. However, when the condition \( H^0 = H^i = H^{(m)} = H^i j_{ij} = L^i = \Theta^i j_{ij} = \Theta^i = 0 \) is imposed we get the FDA for the fields \( B^i, B^0, B^{(m)}, B^i j_{ij}, G^i, \beta^i j_{ij}, \beta^i \). The
problem now is to express the form $B$ in terms of the one-forms $\tau, e^i, m, \omega^{ij}, \omega^i, k^{ij}, k^i$ of the non-relativistic Maxwell algebra.

To express the 2-forms as the wedge product of the 1-forms, we follow a procedure developed in Refs. [26, 28]. We impose the ansatz

$$B^i = \frac{a_1}{2l} \omega^{ij} e_j + \frac{a_2}{2l} \omega^i \tau + \frac{a_3}{2l} k^{ij} e_j + \frac{a_4}{l} k^i \tau + \frac{a_5}{2v^2} \omega^i m + \frac{a_6}{2v^2} k^i m,$$

$$B^0 = \frac{b_1}{2vl} \omega^i e_i + \frac{b_2}{2vl} k^i e_i,$$

$$B^{(m)} = \frac{c_1}{2vl} \omega^i e_i + \frac{c_2}{2vl} k^i e_i,$$

$$B^{ij} = \frac{d_1}{2l^2} e^i e^j + \frac{d_2}{2v^2} \omega^i k^j + \frac{d_3}{2v^2} \omega^j k^i + \frac{d_4}{2} k^i k^j + \frac{d_5}{2v^2} k^i k^j + \frac{d_6}{2v^2} \omega^j k^i + \frac{d_7}{2v^2} \omega^i k^j + \frac{d_8}{2v^2} \omega^i \omega^j + \frac{d_9}{2v^2} \omega^j \omega^i,$$

$$G^i = \frac{f_1}{2l^2} e^i \tau + \frac{f_2}{2vl} \omega^i e^m + \frac{f_3}{2v} \omega^i \omega^j + \frac{f_4}{2v} k^i j^j + \frac{f_5}{2v} \omega^j k^i + \frac{f_6}{2v} k^i j^j,$$

$$\beta^{ij} = \frac{g_1}{2l^2} e^i e^j + \frac{g_2}{2v^2} \omega^i k^j + \frac{g_3}{2v^2} \omega^j k^i + \frac{g_4}{2v} k^i k^j + \frac{g_5}{2v^2} k^i k^j + \frac{g_6}{2v^2} \omega^j k^i + \frac{g_7}{2v^2} \omega^i k^j + \frac{g_8}{2v^2} \omega^i \omega^j + \frac{g_9}{2v^2} \omega^j \omega^i,$$

$$\beta^i = \frac{h_1}{2l^2} e^i \tau + \frac{h_2}{2vl} \omega^i e^m + \frac{h_3}{2v} \omega^i \omega^j + \frac{h_4}{2v} k^i j^j + \frac{h_5}{2v} \omega^j k^i + \frac{h_6}{2v} k^i j^j,$$  \hspace{2cm} (15)

where $a_1, \ldots, a_6, b_1, b_2, c_1, c_2, d_1, \ldots, d_9, f_1, \ldots, f_6, g_1, \ldots, g_9$ and $h_1, \ldots, h_6$ are arbitrary constants. Introducing (15) in the corresponding FDA for the fields $B^i, B^0, B^{(m)}, B^{ij}, G^i, \beta^{ij}, \beta^i$, we find relations between these constants. These relations lead to the following form to (15)

$$B^i = \frac{a_1}{2l} \omega^i \tau + \frac{a_2}{2l} k^{ij} e_j + \frac{a_3}{2l} k^i \tau + \frac{a_5}{2v^2} \omega^i m,$$

$$B^0 = -\frac{a_5 + d_5}{2vl} \omega^i e_i, \quad B^{(m)} = \frac{c_1}{2vl} \omega^i e_i + \frac{c_2}{2vl} k^i e_i,$$

$$B^{ij} = \frac{d_1}{2l^2} e^i e^j + \frac{d_2}{2v^2} \omega^i k^j + \frac{d_3}{2v^2} \omega^j k^i + \frac{d_4}{2} k^i k^j + \frac{d_5}{2v^2} k^i k^j + \frac{d_6}{2v^2} \omega^j k^i + \frac{d_7}{2v^2} \omega^i k^j + \frac{d_8}{2v^2} \omega^i \omega^j + \frac{d_9}{2v^2} \omega^j \omega^i,$$

$$\beta^{ij} = \frac{g_1}{2l^2} e^i e^j + \frac{g_2}{2v^2} \omega^i k^j + \frac{g_3}{2v^2} \omega^j k^i + \frac{g_4}{2v} k^i k^j + \frac{g_5}{2v^2} k^i k^j + \frac{g_6}{2v^2} \omega^j k^i + \frac{g_7}{2v^2} \omega^i k^j + \frac{g_8}{2v^2} \omega^i \omega^j + \frac{g_9}{2v^2} \omega^j \omega^i,$$

$$\beta^i = \frac{h_1}{2l^2} e^i \tau - \frac{a_5}{2vl} \omega^i e^m + \frac{g_2}{2v} \omega^i \omega^j + \frac{h_4}{2v} k^i j^j + \frac{h_6}{2v} k^i j^j.$$  \hspace{2cm} (16)

There are 14 arbitrary constants in the FDA expansion in terms of 1-forms; the fields given by Eqs. (16) represent the most general solution that can be built with the fields $\tau, e^i, m, \omega^{ij}, \omega^i, k^{ij}, k^i$. Any choice of the constants represent a solution to the FDA.
B. Non-relativistic ChSAS Lagrangian

Using the theorem VII.2 of Ref. [23], it is possible to show that the invariant tensors for \( G_B^4 \) are given by

\[
\langle J_{ij} K_k \rangle = \alpha_0 v \varepsilon_{ijk}, \quad \langle J_{ij} Z_k \rangle = \alpha_2 v \varepsilon_{ijk}, \quad \langle K_i Z_{kl} \rangle = \alpha_2 v \varepsilon_{ijkl},
\]

(17)

being \( \alpha_0 \) and \( \alpha_2 \) arbitrary constants. The ChSAS Lagrangian is given then by the following Chern-Simons form

\[
\mathcal{C}^{(4)}_{\text{ChSAS}} = \langle F, B \rangle
\]

(18)

whose explicit form is

\[
\mathcal{C}^{(4)}_{\text{ChSAS}} = \frac{\alpha_0}{2} \varepsilon_{ijkl} D^j \omega^i B^{kl} + \frac{\alpha_2}{2} \varepsilon_{ijkl} \omega^i \omega^j + \frac{\alpha_0}{2} v \varepsilon_{ijkl} R^{ij} G^k + \frac{\alpha_2}{2} v^2 \varepsilon_{ijkl} \omega^i \omega^j + \frac{\alpha_2}{2} v \varepsilon_{ijkl} R^{ij} \beta^k
\]

(19)

Introducing the FDA expansion given by Eqs. (16) in (19), we find that when \( v, l \to \infty \) we find that the non-relativistic ChSAS Lagrangian for the \( G_B^4 \) algebra takes the form

\[
\mathcal{L}^{(4)}_{\text{NR-ChSAS}} = \frac{\alpha_2}{2} \varepsilon_{ijkl} D^j \omega^i B^{kl} + \frac{\alpha_2}{2} \varepsilon_{ijkl} \omega^i \omega^j + \frac{\alpha_0}{2} v \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2}{2} v^2 \varepsilon_{ijkl} \omega^i \omega^j + \frac{\alpha_2}{2} \varepsilon_{ijkl} D^j \omega^i \omega^j + \frac{\alpha_2}{2} \varepsilon_{ijkl} R^{ij} \omega^k.
\]

(20)

In presence of matter, the complete Lagrangian of the theory is given by

\[
\mathcal{L} = \mathcal{L}^{(4)}_{\text{NR-ChSAS}} + \kappa \mathcal{L}_M.
\]

(21)

The variation of \( \mathcal{L} \) leads to the following equations of motion

\[
\frac{\alpha_2 h_1 + \alpha_0 (a_3 - h_6)}{4} \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2 (a_3 - h_6)}{4} \varepsilon_{ijkl} D^j \omega^i \omega^j + \frac{\alpha_2 g_2}{2} \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2 h_4 + \alpha_0 (a_3 + a_4)}{4} \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2 g_2}{2} \varepsilon_{ijkl} D^j \omega^i \omega^j + \frac{\alpha_2}{2} \varepsilon_{ijkl} R^{ij} \omega^k = 0,
\]

(22)

\[
\frac{\alpha_2 g_2}{4} \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2 h_4 + \alpha_0 (a_3 + a_4)}{4} \varepsilon_{ijkl} D^j \omega^i \omega^j - \frac{\alpha_2 g_2}{4} \varepsilon_{ijkl} D^j \omega^i \omega^j + \frac{\alpha_2}{2} \varepsilon_{ijkl} R^{ij} \omega^k + \frac{\alpha_2}{2} \varepsilon_{ijkl} D^j \omega^i \omega^j = 0,
\]

(23)
\[
\frac{\alpha_2 h_6}{4} \varepsilon_{ijk} R_k^i k_j^i + \kappa \frac{\delta L_M}{\delta k^i} = 0, \tag{24}
\]
\[
\frac{\alpha_2}{4} \varepsilon_{ijk} \left( \frac{h_1 v^2}{l^2} T^k \tau + g_2 D \omega_1 \omega^l - g_2 \omega_2 k_1 D \omega^l + h_4 D k^i \omega^l - h_4 k_1 D \omega^l \right) + \frac{\alpha_2 g_2}{2} \varepsilon_{ijk} \left( a_3 - h_6 \right) \frac{v^2}{l^2} T^k \tau + (a_3 + a_4) D k^i \omega^l - (a_3 + a_4) k_1 D \omega^l \right) + \frac{\alpha_2 a_4}{8} \varepsilon_{ijk} k^i \left( a_3 - h_6 \right) \frac{v^2}{l^2} e^l \tau + (a_3 + a_4) k_1 \omega^l \right) + \alpha_2 g_4 \left( \frac{\delta L_M}{\delta \omega^l} \varepsilon_{ijkl} k^i \omega^l \right) = 0, \tag{25}
\]
where we have used
\[
T_a = \left( \frac{\delta L_M}{\delta e^a} \right), \quad T_0 = \left( \frac{\delta L_M}{\delta \tau} \right). \tag{28}
\]
Taking into account that
\[
*(T_0) \delta \tau = - \det(g) \delta^a \tau^0 \delta \tau^a dx^4,
\]
\[
\varepsilon_{ij} k^i \omega^j \delta \tau = 2 \det(g) \left( \delta^a \gamma^0 - \delta^0 \gamma^a - D a_k \omega^k + D a_k \omega^k \right) \delta \tau^a dx^4, \tag{29}
\]
we find that the field equation (27) can be written as
\[
\left[ \left( \frac{\alpha_2 h_1}{4} + \alpha_0 \frac{a_3 - h_6}{4} \right) \delta_{\sigma \delta} R - 2 R_{\sigma \delta} \right] + \alpha_2 \frac{a_3 - h_6}{2} \left( \delta_{\alpha \delta} D a_k^{\alpha \beta} - D a_k^{\alpha \beta} + D a_k^{\alpha \beta} \right) - \frac{\tau^2}{v^2} \kappa \delta_{\alpha \delta} T_0^0 \right] \det(g) \delta \tau^a dx^4 = 0. \tag{30}
\]
The contraction of this equation with $g^{\alpha \delta}$ leads to
\[
R = \alpha_2 \frac{a_3 - h_6}{2} D a_k^{\alpha \beta} - \frac{2 l^2}{v^2} \kappa T_0. \tag{31}
\]
Taking the components 00 of (30) and using (31) we find
\[
R_{00} - \left( \frac{\alpha_2 h_1}{2} + \alpha_0 \frac{a_3 - h_6}{2} \right) R_{00} + \alpha_2 \frac{a_3 - h_6}{2} \left( 2 D a_k^{\alpha \beta} - D a_k^{\alpha \beta} + D a_k^{\alpha \beta} \right) - \frac{3 l^2}{v^2} \kappa T_0 = 0. \tag{32}
\]
Following the procedure of Ref. [8], we find

\[ R_{00} = \nabla^2 \phi, \quad g_{00} = \tau_0 \tau_0 = 1, \quad T_{00} = \rho. \tag{33} \]

From (33) and (32) we finally obtain

\[ \nabla^2 \phi = 6 \alpha l^2 v^2 \kappa \rho - \alpha \beta \left( 2D_\alpha k^\alpha \beta - D_0 k_\beta \beta + D_\beta k_\alpha \beta \right), \tag{34} \]

where

\[ \alpha = -[\alpha_2 h_1 + \alpha_0 (a_3 - h_6)]^{-1}, \quad \beta = a_2 (a_3 - h_6). \tag{35} \]

IV. MOND THEORY CONNECTION

The modified form of Poisson equation (34) suggests a possible connection with the so-called MOND approach to gravity interactions. In fact, the first complete MOND theory was constructed by Milgrom and Bekenstein in Ref. [30]. It involves a modification of the Poisson equation and can be derived from the following Lagrangian

\[ \mathcal{L}_{\text{Mond}} = -\int d^3r \left\{ \rho \varphi + \frac{a_0^2}{8\pi G} \tilde{\mathcal{F}} \left[ \frac{\nabla \varphi}{a_0} \right]^2 \right\}, \]

where \( \varphi \) is the gravitational potential (meaning that for a test particle \( \vec{a} = -\vec{\nabla} \varphi \)), \( \rho \) denotes the matter mass density, and \( \tilde{\mathcal{F}}(x^2) \) is an arbitrary function. The variation of \( \mathcal{L}_{\text{Mond}} \) with respect to \( \varphi \) leads to the following field equations

\[ \nabla \cdot \left[ \mu \left( \frac{[\nabla \varphi]}{a_0} \right) \nabla \varphi \right] = 4\pi G \rho, \]

with \( \mu(x) = \tilde{\mathcal{F}}'(x^2) \). A little bit of algebra allows us to see that this equation takes the form

\[ \mu(x) \nabla^2 \varphi(x) = 4\pi G \rho - \nabla \mu(x) \cdot \nabla \varphi(x). \tag{36} \]

Comparing this last equation with (34) we can see that in some particular cases the MOND approach to gravity could coincide with such modified Poisson equation. In fact, if we consider the case where the \( k^\alpha \beta \) field does not depend on time and its non zero components are given by \( k^{i0}_0 = -k^{0i}_0 = \sigma(x) \delta^{ij} \partial_j \phi(x) \), we find

\[ 2D_\alpha k^\alpha \beta - D_0 k_\beta \beta + D_\beta k_\alpha \beta = D_i k^{i0}_0, \tag{37} \]
where $D_\alpha$ is the covariant derivative. Following Ref. [8] with $\Gamma^i_{00} = \delta^{ij}\partial_j\phi(x)$ we can see that

$$D_i k^{i0}_0 = \vec{\nabla} \sigma(x) \cdot \vec{\nabla} \phi(x) + \sigma(x) \nabla^2 \phi(x). \tag{38}$$

Introducing (37,38) in (34) we find

$$[1 + \alpha \beta \sigma(x)] \nabla^2 \phi(x) = 6 \alpha \frac{l^2}{v^2} \kappa \rho - \alpha \beta \vec{\nabla} \sigma(x) \cdot \vec{\nabla} \phi(x). \tag{39}$$

Comparing (39) with (36), we can see that if $\phi(x) = \varphi(x)$ and $\mu(x) = 1 + \alpha \beta \sigma(x)$, we can see that Eq. (39) matches with the MOND equation (36) for the following values of the $\alpha$ and $\beta$

$$\alpha = \frac{4\pi G v^2}{6l^2 \kappa} = \frac{v^2}{12l^2}, \quad \beta = 1. \tag{40}$$

V. NEWTONIAN CHSAS GRAVITY FOR $\mathcal{L}_4$ ALGEBRA

Let us now consider the one and two forms gauge fields $A, B$ valued in the $\mathcal{L}_4$ algebra

$$A = \frac{v}{l} \tau H + \frac{1}{l} e^i P_i + \frac{1}{vl} m M + \frac{1}{v} \omega^i K_i + \frac{1}{2} \omega^{ij} J_{ij} + \frac{1}{2} k^{ij} Z_{ij} + \frac{1}{v} k^i Z_i,$$

$$B = B^i P_i + B^0 H + B^{(m)} M + \frac{1}{2} B^{ij} J_{ij} + G^i K_i + \frac{1}{2} \beta^{ij} Z_{ij} + \beta^i Z_i. \tag{41}$$

Following the same procedure used in the previous section, it is found that the corresponding non-relativistic ChSAS Lagrangian leads to the following generalized Poisson equation

$$\nabla^2 \phi = -\frac{6}{\alpha_0 (a_4 - a_2 + a_3)} \frac{l^2}{v^2} \kappa \rho + \frac{\alpha_2}{\alpha_0} \left(2D_\alpha k^{\alpha\beta}_\beta - D_0 k^{\alpha\beta}_\beta + D_\beta k^{\alpha\beta}_0 \right) + \frac{2\alpha_2}{\alpha_0} \left(1 + \frac{a_3}{(a_4 - a_2 + a_3)} \right) \left(k^{\alpha\gamma}_\alpha k^{\beta}_\gamma \beta - k^{\alpha\gamma}_\beta k^{\beta}_\gamma \alpha \right) - \left(k^{0\gamma}_0 k^{\beta}_\gamma \beta - k^{0\gamma}_\beta k^{\beta}_0 \right) + \left(k^{0\alpha}_\alpha k^{0\beta}_\beta - k^{0\alpha}_\beta k^{0\beta}_\alpha \right). \tag{42}$$

Comparing (42) with (36) we can see that in some particular cases, the MOND approach to gravity could coincide with such modified Poisson equation (42). If we consider again the case where the $k^{\alpha\beta}_\beta$ field does not depend on time and its non-zero components are given by

$$k^{i0}_0 = -k^{0i}_0 = \sigma(x) \delta^{ij} \partial_j \phi(x), \tag{43}$$

13
we find \( k^{\alpha \gamma} k_{\gamma \beta} - k^{\alpha \gamma} k_{\gamma \beta} = 0, \ k^{0 \gamma} k_{\gamma \beta} - k^{0 \gamma} k_{\gamma \beta} = 0, \ k^{\alpha \gamma} k_{\gamma 0} - k^{\alpha \gamma} k_{\gamma 0} = 0. \) So that, Eq. (42) takes the form

\[
[1 + \beta \sigma(x)] \nabla^2 \phi(x) = 6 \alpha \frac{l^2}{v^2} \kappa \rho - \beta \vec{\nabla} \sigma(x) \cdot \vec{\nabla} \phi(x)
\]  

(44)

where

\[
\alpha = - \frac{1}{\alpha_0 (a_4 - a_2 + a_3)}, \quad \beta = - \frac{\alpha_2}{\alpha_0}.
\]  

(45)

Comparing (44) with (36) we can see that if \( \phi(x) = \varphi(x) \) and \( \mu(x) = 1 + \beta \sigma(x) \), then Eq. (44) matches with the Mond equation (36) if

\[
\alpha = \frac{4 \pi G v^2}{6 l^2 \kappa} = \frac{v^2}{12l^2},
\]

\[
\beta = 1.
\]

VI. COMMENTS

In the present work we have studied the non-relativistic versions of the generalized Poincaré algebra \( \mathfrak{B}_4 \) denoted by \( \mathcal{G} \mathfrak{B}_4 \) and the AdS-Lorentz algebra \( \text{AdS} \mathfrak{L}_4 \) denoted by \( \mathcal{G} \mathfrak{L}_4 \) to find the non-relativistic limit of the four dimensional ChSAS action for gravity.

We have shown that the gauging of non-relativistic algebras \( \mathcal{G} \mathfrak{B}_4 \) and \( \mathcal{G} \mathfrak{L}_4 \) permits to construct generalizations of the Newtonian gravity which leads to modified versions of the Poisson equation. In some particular cases, it is possible to find relations between the generalized Newtonian gravities and the so called MOND model. These modifications to the Poisson equation, could be compatible with dark matter and would allow us to conjecture that dark matter could be interpreted as the non-relativistic limit of dark energy.

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[1] N. González, G. Rubio, P. Salgado and S. Salgado, Generalized Galilean algebras and Newtonian gravity, Phys. Lett. B 755 (2016) 433.
[2] F. Izaurieta, P. Minning, E. Rodríguez, A. Pérez and P. Salgado, *Standard General Relativity from Chern-Simons Gravity*, Phys. Lett. B 678 (2009) 213.

[3] P. Salgado and S. Salgado, *so(D-1,1)⊕so(D-1,2) algebras and gravity*, Phys. Lett. B 728, (2014) 5.

[4] P. K. Concha, D. M. Peñafiel, E. K. Rodríguez and P. Salgado, *Even-dimensional General Relativity from Born–Infeld gravity*, Phys. Lett. B 725 (2013) 419.

[5] D. V. Soroka and V. A. Soroka, *Tensor extension of the Poincaré algebra*, Phys. Lett. B 607 (2005) 302.

[6] D. V. Soroka and V. A. Soroka, *Semi-simple extension of the (super)Poincaré algebra*, Adv. High Energy Phys. 2009 (2009) 234147.

[7] D. V. Soroka and V. A. Soroka, *Semi-simple o(N)-extended super-Poincaré algebra*, arXiv:1004.3194 [hep-th].

[8] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, *Newtonian Gravity and the Bargmann Algebra*, Class. & Quant. Grav. 28 (2011) 105011.

[9] G. Savvidy, *Topological mass generation in four-dimensional gauge theory*, Phys. Lett. B 694 (2010) 65.

[10] S. Konitopoulos and G. Savvidy, *Extension of Chern–Simons forms and new gauge anomalies*, J. Math. Phys. 55 (2014) 06234.

[11] I. Antoniadis and G. Savvidy, *New gauge anomalies and topological invariants in various dimensions*, Eur. Phys. J. C 72 (2012) 2140.

[12] I. Antoniadis and G. Savvidy, *Extension of Chern–Simons forms and new gauge anomalies*, Int. J. Mod. Phys. A 29 (2014) 1450027.

[13] F. Izaurieta, I. Muñoz, P. Salgado, *A Chern–Simons gravity action in d=4*, Phys. Lett. B 750 (2015) 39.

[14] P. Catalán, F. Izaurieta, P. Salgado and S. Salgado, *Topological gravity and Chern–Simons forms in d=4*, Phys. Lett. B 751 (2015) 205.

[15] F. Izaurieta, P. Salgado, S. Salgado, *Chern-Simons-Antoniadis-Savvidy forms and standard supergravity*, Phys. Lett. B 767 (2017) 360.

[16] A. H. Chamseddine, *Topological gravity and supergravity in various dimensions*, Nucl. Phys. B 346 (1990) 213.

[17] A. H. Chamseddine, *Topological Gauge Theory of Gravity in Five-dimensions and All Odd
Dimensions, Phys. Lett. B 233 (1989) 291.

[18] A. H. Chamseddine, Nucl. Phys. B 340 (1990) 595.

[19] P. Salgado, S. Salgado, Extended gauge theory and gauged free differential algebras, Nucl. Phys. B 926 (2018) 179.

[20] H. Bacry, P. Combe, J.L. Richard, Group-theoretical analysis of elementary particles in an external electromagnetic field. I. the relativistic particle in a constant and uniform field, Nuovo Cimento. A 67 (1970) 267.

[21] R. Schröder, The Maxwell Group and the Quantum Theory of Particles in Classical Homogeneous Electromagnetic Fields, Fortschr. Phys. 20 (1972) 701.

[22] J. Díaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodríguez, P. Salgado and O. Valdivia, A generalized action for (2+1)-dimensional Chern–Simons gravity, J. Phys. A: Math. Theor. 45, (2012) 255207.

[23] F. Izaurieta, E. Rodríguez and P. Salgado, Expanding Lie (super) algebras through abelian semigroups, J. Math. Phys. 47 (2006) 123512.

[24] F. Izaurieta, E. Rodríguez and P. Salgado, The Extended Cartan Homotopy Formula and a Subspace Separation Method for Chern-Simons Theory, Lett. Math. Phys. 80 (2007) 127.

[25] Y. Tian, H. Y. Guo, C. G. Huang, Z. Xu and B. Zhou, Mechanics and Newton-Cartan-like gravity on the Newton-Hooke space-time, Phys. Rev. D 71 (2005) 044030.

[26] R. D’Auria and P. Fré, Geometric Supergravity in d = 11 and Its Hidden Supergroup, Nucl. Phys. B 201 (1982) 101.

[27] L. Castellani, R. D’Auria and P. Fré, Supergravity and Superstring. A Geometric Perspective, World Scientific 1991.

[28] I. Bandos, J.A. de Azcárraga, M. Picón and O. Varela, On the formulation of D = 11 supergravity and the composite nature of its three-form gauge field, Ann. Phys. 317 (2005) 238.

[29] A. Pérez, P. Minning and P. Salgado, Eleven-dimensional CJS supergravity and the D’Auria-Fré group, Phys. Lett. B 660 (2008) 407.

[30] J. D. Bekenstein and M. Milgrom, Does the missing mass problem signal the breakdown of Newtonian gravity? Astrophys. J. 286 (1984) 7.

[31] J. Gomis, K. Kamimura, J. Lukierski, Deformations of Maxwell algebra and their Dynamical Realizations, JHEP 08 (2009) 039, [arXiv:0906.4464] [hep-th].