An inverse problem of the elasticity of $n$ elastic inclusions embedded into an elastic half-plane is analysed. The boundary of the half-plane is free of traction. The half-plane and the inclusions are subjected to antiplane shear, and the conditions of ideal contact hold in the interfaces between the inclusions and the half-plane. The shapes of the inclusions are not prescribed and have to be determined by enforcing uniform stresses inside the inclusions. The method of conformal mappings from a slit domain onto the $(n+1)$-connected physical domain is worked out. It is shown that to recover the map and the shapes of the inclusions, one needs to solve a vector Riemann–Hilbert problem on a genus-$n$ hyperelliptic surface. In a particular case of loading, the vector problem reduces to two scalar Riemann–Hilbert problems on $n+1$ slits on a hyperelliptic surface. In the elliptic case, in addition to three parameters of the model, the conformal map possesses a free geometric parameter. The results of numerical tests in the elliptic case show the impact of these parameters on the inclusion shape.

1. Introduction

Methods of conformal mappings have numerous applications in model problems of continuum mechanics. This particularly concerns inverse problems for multiply connected domains arising in fluid mechanics and elasticity. The former includes free boundary problems of Hele-Shaw and Muscat flow [1,2], supercavitating
flow [3] and vortex dynamics [4, 5]. The study of inverse elastic problems for the determination of the profiles of the cavities and inclusions was initiated in [6]. One of the first such problems, the Cherepanov model [7], concerns an elastic plane with $n$ cavities subjected to constant normal and tangential traction on the boundary when the shapes of the holes are to be determined by enforcing constant tangential normal stresses on the boundary. In the symmetric case of two cavities their shapes were recovered [7] by applying a conformal mapping and solving two Schwarz problems on two slits. Methods of integral equations [8], the Riemann–Hilbert problems on a hyperelliptic surface of genus $n - 1$ [9] and the Schottky–Klein prime function [10] were developed to generalize the solution [7] to the case of any number of holes and to analyse the properties of the conformal mappings employed and the solution derived. The method of the Riemann–Hilbert problem on a hyperelliptic surface was worked out [11] to construct a meromorphic solution to the elastic–plastic antiplane model for a multiply connected domain [12].

The design of inclusions with a uniform internal field is important in the micromechanical analysis of composite materials [13]. That is why inverse elastic problems of antiplane strain in multiply connected domains have been attracting the attention of many studies. The case of two symmetric inclusions was analysed in [14] by using the Weierstrass zeta function and the Schwarz–Christoffel formula. Another approach to the problem of two finite inclusions that are not necessarily symmetric was proposed in [15]. The method was designed for a specific case of uniform stress distribution inside the inclusions and was based on a Laurent series representation of a conformal map from an annulus to the exterior of two inclusions. This approach was recently applied [16] to the case when one of the uniformly stressed inclusions was finite, while the second one was a semi-infinite body. Numerical solutions for multiply connected domains were obtained by the method of finite elements in [17] and by the Faber series with the coefficients determined by nonlinear systems in [13].

Two methods of conformal mappings from a canonical domain onto the physical multiply connected domain were proposed [18, 19] for the inverse problem of antiplane strain of a plane with $n$ uniformly stressed inclusions. In this model, the stresses $\tau_{13}$ and $\tau_{23}$ inside all of the inclusions are equal to constants $\tau_1$ and $\tau_2$, respectively, and are independent of the stresses prescribed at infinity. In [18] the canonical domain was chosen to be a parametric plane with $n$ slits lying in the real axis, while in [19] the canonical domain was a plane with $n$ circular holes. In the former case, the method of the Riemann–Hilbert problem on a hyperelliptic surface was applied and the conformal map was reconstructed in terms of singular integrals. In the case of a circular domain, the method of the Riemann–Hilbert problem of the theory of automorphic functions generated by a Schottky symmetry group was used. Since for any doubly and triply connected domain $D_0$ there exists a conformal map from a parametric slit domain $D$ with slits lying in the same line onto the domain $D_0$, the method of slit conformal maps was able to recover the whole family of inclusions by quadratures when $n = 2$ or $n = 3$ and only a particular family in the case $n \geq 4$. The circular map gave a series representation of the solution for any finite number of inclusions.

The goal of this paper is to develop a method of conformal mappings for the inverse antiplane problem with inclusions uniformly stressed and embedded into a half-plane. Its aim is to construct a conformal map that is capable of recovering the shapes of inclusions and that does not change the straight boundary of the surrounding semi-infinity body. We state the problem and reduce it to a boundary value problem for a single analytic function in §2. In §3, we show that in two cases, (i) $\tau_{13} = \tau_1$, $\tau_{23} = \tau_2$ and (ii) $\tau_{13} = \mu_j \nu_1$, $\tau_{23} = \mu_j \nu_2$, the problem is equivalent to a vector Riemann–Hilbert problem on $n + 1$ contours on a hyperelliptic surface. Here, $\tau_{13}$, $\tau_{23}$ and $\mu_j$ are the shear stresses and shear moduli in the $j$th inclusion, respectively. When $\tau_{23} = 0$, the vector problem is decoupled, and two scalar Riemann–Hilbert problems on $n$ finite and one semi-infinite contours on a genus-$n$ Riemann surface need to be solved. This case is considered in §4. To solve these problems, we propose a new analogue of the Cauchy kernel for a hyperelliptic surface that is convenient when the density is not decaying at infinity, while the infinite point lies on the
contour of the problem. In §5, we derive the conformal map needed through the solution to the two Riemann–Hilbert problems solved in the previous section and present numerical results in the elliptic case (a single inclusion in a half-plane).

2. Setting

Consider a semi-infinite elastic body \( \mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0, x_2 > 0, |x_3| < \infty \} \) and \( n \) elastic inclusions \( \{(x_1, x_2) \in D_j, |x_3| < \infty \} (j = 1, 2, \ldots, n) \) embedded into the body. Denote a cross-section of the body orthogonal to the axis \( x_3 \) and external to the inclusions by \( D_0 = \mathbb{R}_+^2 \setminus \bigcup_{j=1}^n D_j \). The shear moduli of the domains \( D_0 \) and \( D_j \) are \( \mu_0 \) and \( \mu_j \), respectively, and the inclusions are in ideal contact with the external body. Suppose the body is subjected to antiplane shear \( \tau_1 = \tau_1^\infty \) as \( x_1 \to \pm \infty \), \( 0 < x_2 < \infty \). The boundary of the body \( L_0 = \{(x_1, x_2) = 0 \} \) is free of traction, \( \tau_2 = 0 \), and at infinity as \( x_2 \to \infty \) and \( |x_1| < \infty \), \( \tau_3 = 0 \). We aim to recover the whole family of possible uniformly stressed inclusions \( D_j \) such that \( \tau_1 = \tau_1^j \) and \( \tau_3 = \tau_3^j \), \( (x_1, x_2) \in D_j \), where \( \tau_1^j \) and \( \tau_3^j \) are prescribed constants.

Denote by \( w_j(x_1, x_2) \), \( (x_1, x_2) \in D_j \) \((j = 0, 1, \ldots, n)\), the \( x_3 \)-component of the displacement vector. The function \( w_j(x_1, x_2) \) is harmonic in \( D_j \), and the shear stresses are expressed through the displacement by

\[
\sigma_{13} = \mu_j \frac{\partial w_j}{\partial x_1}, \quad \sigma_{23} = \mu_j \frac{\partial w_j}{\partial x_2}, \quad (x_1, x_2) \in D_j, \quad j = 0, 1, \ldots, n. 
\]

(2.1)

On the interfaces \( L_j \), the boundary conditions of ideal contact read

\[
w_0 = w_j, \quad \mu_0 \frac{\partial w_0}{\partial \nu} = \mu_j \frac{\partial w_j}{\partial \nu}, \quad x \in L_j, \quad j = 1, \ldots, n,
\]

(2.2)

where \( L_j \) are the boundaries of the inclusions, and \( \frac{\partial}{\partial \nu} \) is the normal derivative. Since

\[
\mu_j \frac{\partial w_j}{\partial x_1} = \tau_1^j, \quad \mu_j \frac{\partial w_j}{\partial x_2} = \tau_2^j, \quad (x_1, x_2) \in D_j, \quad j = 1, \ldots, n,
\]

(2.3)

we can recover the function \( w_j \) and its harmonic conjugate \( w_j^* \) up to arbitrary constants \( a_j \) and \( a_j' \),

\[
w_j = \frac{\tau_1^j x_1 + \tau_2^j x_2}{\mu_j} + a_j \quad \text{and} \quad w_j^* = \frac{-\tau_2^j x_1 + \tau_1^j x_2}{\mu_j} + a_j', \quad (x_1, x_2) \in D_j.
\]

(2.4)

Denote next a harmonic conjugate \( w_j^*(x_1, x_2) \) of the function \( w_0 \) and introduce a function \( f(z) \)

\[
f(z) = w_0(x_1, x_2) + iw_0^*(x_1, x_2) - A z, \quad z = x_1 + ix_2 \in D_0,
\]

(2.5)

analytic in the domain \( D_0 \). Here, \( A \) is a complex constant. Then \([19]\), the real boundary conditions (2.2) are equivalent to the following complex conditions on the contours \( L_j \) for the analytic function \( f(z) \):

\[
f(z) + Az + ib_j' = \frac{\kappa_j}{2\mu_j} (\bar{\tau}^j z + d_j \mu_j) - \frac{\kappa_j}{2\mu_j} (\bar{\tau}^j \bar{z} + d_j' \mu_j), \quad z \in L_j, \quad j = 1, \ldots, n,
\]

(2.6)

where \( \kappa_j = \mu_j/\mu_0 \), \( \bar{\tau}^j = \tau_1^j + i\tau_2^j \), \( d_j = a_j + i a_j' \) and \( b_j' \) are real constants. In two particular cases of loading these boundary conditions can be simplified. These cases are (i) \( \tau_1^j = \tau_1 \), \( \tau_2^j = \tau_2 \) and (ii) \( \tau_1^j/\mu_j = v_1 \), \( \tau_2^j/\mu_j = v_2 \), \( j = 1, 2, \ldots, n \), where \( \tau_1 \), \( \tau_2 \), \( v_1 \) and \( v_2 \) are some parameters. In the former case, \( A = \bar{\tau}/\mu_0 \), while in the second case, \( A = \bar{v} \), where \( \tau = \tau_1 + i\tau_2 \) and \( v = v_1 + iv_2 \).
In case (i), the boundary conditions (2.6) have the form
\[ f(z) = \frac{1}{\lambda_j} \text{Re} \left( \frac{\tilde{r} z}{\mu_0} + a_j + ib_j \right), \quad z \in L_j, \ j = 1, 2, \ldots, n, \]  
where \( \lambda_j = \kappa_j/(1 - \kappa_j) \) and \( b_j = \kappa_0 a_j - b_j' \). By using the asymptotics of \( w_0(z) \) as \( z \to \infty \) and the Cauchy–Riemann conditions, we find from (2.5) that
\[ f(z) \sim \frac{r^\infty - \tilde{r}}{\mu_0} z + \text{const}, \quad z \to \infty. \]  
In case (ii), the function \( f(z) \) given by (2.5) with \( A = \tilde{\nu} \) is analytic in the domain \( D_0 \), grows at infinity as
\[ f(z) \sim \left( \frac{r^\infty}{\mu_0} - \tilde{\nu} \right) z + \text{const}, \quad z \to \infty, \]  
and satisfies the interface conditions
\[ f(z) = i(\kappa_j - 1) \text{Im}(\tilde{\nu}z) + a_j + ib_j, \quad z \in L_j, \ j = 1, 2, \ldots, n. \]  

3. Vector Riemann–Hilbert problem on a hyperelliptic surface

Note that if the contours \( L_j \) had been prescribed, then the boundary conditions would constitute an ill-posed problem since (2.7) and (2.10) specify the real and the imaginary parts of an analytic function on the contours. In this section, we shall apply the method of conformal mappings to recover the function \( f(z) \) and the contours \( L_j \). We shall show that in cases (i) and (ii) the problem of determination of the conformal mapping is equivalent to a vector Riemann–Hilbert problem on a genus-\( n \) two-sheeted Riemann surface.

(a) Case (i)

Let \( z = \omega(\xi) \) be a conformal map \( \mathcal{D} \to D_0 \) from a parametric \( \xi \)-plane cut along segments \( l_0 = [m, \infty) \) and \( l_j = [k_{2j-1}, k_{2j}], \ j = 1, 2, \ldots, n, \) where
\[ k_{2n-1} < k_{2n} < k_{2n-3} < k_{2n-2} < \cdots < k_3 < k_1 < k_2 < m, \quad k_1 = 0, k_2 = 1. \]  
The function \( \omega(\xi) \) maps the two-sided finite segments \( l_j \) onto the contours \( L_j \) \( (j = 1, \ldots, n) \) and the two-sided semi-infinite contour \( l_0 \) onto the \( x_1 \)-axis of the physical plane. When there are one or two inclusions in the half-plane and the domain is doubly or triply connected, such a map always exists. In the case \( n \geq 3 \), we assume that the inclusions are arranged such that all pre-images \( l_j \) of their boundaries \( L_j \) lie in the real axis.

Introduce a new function \( F(\xi) = f(\omega(\xi)) \) analytic in the domain \( \mathcal{D} \). Then the complex boundary conditions (2.7) can be equivalently written in the form
\[ \text{Im} F(\xi) = b_j, \quad \text{Re} \frac{\tilde{r} \omega(\xi)}{\mu_0} = \lambda_j [\text{Re} F(\xi) - a_j], \quad \xi \in l_j, \ j = 1, \ldots, n. \]  
Derive next two boundary conditions on the two-sided contour \( l_0 \). The first condition is obvious. Since \( x_2 = 0 \) on the contour \( L_0 \), we immediately have
\[ \text{Im} \omega(\xi) = 0, \quad \xi \in l_0. \]  
To determine the second relation, we use the condition \( \partial w_0 / \partial x_2 = 0 \) as \( x_2 \to 0^+ \), \( -\infty < x_1 < \infty \), and the Cauchy–Riemann condition to connect the partial derivatives of the function \( w_0 \) and its harmonic conjugate \( w_0^* \). This results in \( w_0^* = b_0 \) on the line \( L_0 \), where \( b_0 \) is an arbitrary real constant.
Therefore,

$$\Im f(z) = b_0 - \Im \frac{\bar{\tau}z}{\mu_0}, \quad z \in L_0, \quad (3.4)$$

and the second condition on the slit \(l_0\) reads

$$\Im F(\xi) = b_0 - \Im \frac{\bar{\tau}\omega(\xi)}{\mu_0}, \quad \xi \in l_0. \quad (3.5)$$

The boundary conditions (3.2), (3.3) and (3.5) have to be complemented by the conditions at infinity. We have

$$F(\xi) \sim \frac{\tau^{i0} - \bar{\tau}}{\mu_0} \omega(\xi), \quad \omega(\xi) \sim c_{\pm} \xi^{1/2}, \quad \xi = \xi \pm 0, \quad \xi \to \infty, \quad (3.6)$$

where \(c_{\pm}\) are constants.

We wish now to show that the boundary conditions derived can be transformed into a vector Riemann–Hilbert on a Riemann surface. Let \(R\) be a genus-\(n\) hyperelliptic surface of the algebraic function

$$u^2 = p(\xi), \quad p(\xi) = \xi(1 - \xi)(\xi - m)\prod_{j=3}^{2n}(\xi - k_j). \quad (3.7)$$

Fix a single branch of the function \(p^{1/2}(\xi)\) in the \(\xi\)-plane cut along the contours \(l_j\) (\(j = 0, 1, \ldots, n\)) by the condition \(p^{1/2}(\xi + i0) = \sqrt{|p(\xi)|}, \quad \xi > m\). Take two replicas \(D^+\) and \(D^-\) of the parametric domain \(D\) and attach the ‘+’ sides of the slits \(l_j\) on the upper sheet \(D^+\) to the ‘−’ sides of the corresponding slits on the lower sheet \(D^-\). Then we attach the sides \(l_j^- \subset D^+\) to the sides \(l_j^+ \subset D^-\), \(j = 0, 1, \ldots, n\). Points \((\zeta, u(\zeta))\) of the upper sheet and \((\zeta, -u(\zeta))\) of the lower sheet of the resulting genus-\(n\) Riemann surface \(R\) are symmetric with respect to the contour \(L = \bigcup_{j=0}^{n} l_j\).

Denote \((\zeta, -u(\zeta)) = (\zeta_s, u_s)\). Assume that a point \((\zeta, p^{1/2}(\zeta))\) lies in the upper sheet \(D^+\). Then the point \((\zeta_s, u_s)\) is symmetric to it and lies in the lower sheet \(D^-\). On this surface, we introduce two functions

$$\Phi_1(\xi, u) = \begin{cases} F(\xi), & (\xi, u) \in D^+, \\ F(\bar{\xi}), & (\xi, u) \in D^-, \end{cases} \quad \text{and} \quad \Phi_2(\xi, u) = \begin{cases} i\mu_0^{-1} \bar{\tau} \omega(\xi), & (\xi, u) \in D^+, \\ -i\mu_0^{-1} \bar{\tau} \omega(\bar{\xi}), & (\xi, u) \in D^- \end{cases}. \quad (3.8)$$

These functions are symmetric with respect to the contour \(L\),

$$\Phi_j(\xi, u) = \Phi_j(\zeta_s, u_s), \quad (\xi, v) \in L, \quad (3.9)$$

and their limit values on the contour \(L\) from the upper and lower sheets are expressed through the functions \(F(\xi)\) and \(\omega(\xi)\) as

$$\Phi_1^+(\xi, v) = F(\xi), \quad \Phi_2^+(\xi, v) = i\mu_0^{-1} \bar{\tau} \omega(\xi), \quad \Phi_1^-(\xi, v) = F(\bar{\xi}), \quad \Phi_2^-(\xi, v) = -i\mu_0^{-1} \bar{\tau} \omega(\bar{\xi}), \quad (\xi, v) \in L \subset R, \quad (3.10)$$

where \(v = u(\xi)\).

It remains to write the boundary conditions (3.2), (3.3) and (3.5) in terms of the functions (3.10). As a result, we arrive at a vector Riemann–Hilbert problem on the Riemann surface \(R\) for the vector \(\Phi(\xi, u) = (\Phi_1(\xi, u), \Phi_2(\xi, u))\) analytic everywhere on the surface \(R \setminus L\), Hölder-continuous up to the contour \(L\) and whose limit values satisfy the boundary condition

$$\Phi^+(\xi, v) = G(\xi, v)\Phi^-(\xi, v) + g(\xi, v), \quad (\xi, v) \in L \subset R, \quad (3.11)$$

where \(G(\xi, v)\) is a piecewise constant matrix

$$G(\xi, v) = \begin{pmatrix} 1 & 0 \\ 2i\lambda_j & 1 \end{pmatrix}, \quad (\xi, v) \in l_j, \quad G(\xi, v) = \begin{pmatrix} 1 & i(1 - \bar{\tau}/\tau) \\ 0 & -\bar{\tau}/\tau \end{pmatrix}, \quad (\xi, v) \in l_0. \quad (3.12)$$
and \( g(\xi, v) \) is a piecewise constant vector

\[
g(\xi, v) = \begin{pmatrix} 2ib_j \\
-2i\lambda_j(a_j - ib_j) \end{pmatrix}, \quad (\xi, v) \in l_j, \quad g(\xi, v) = \begin{pmatrix} 2ib_0 \\
0 \end{pmatrix}, \quad (\xi, v) \in l_0. \tag{3.13}
\]

The vector \( \Phi(\xi, u) \) is symmetric with respect to the contour \( \mathcal{L}, \Phi(\xi, u) = \overline{\Phi(\xi, u_*)} \), and its components satisfy the conditions at infinity

\[
\Phi_1^+(\xi, u) \sim \frac{\tau_1^\infty - \tau}{i\tau} \Phi_2^+(\xi, u) \quad \text{and} \quad \Phi_2(\xi, u) = O(\tau^{1/2}), \quad \tau \to \infty. \tag{3.14}
\]

When \( \tau_j^i = 0, j = 1, \ldots, n \), that is, when \( \tau = \tau_1 \), the vector Riemann–Hilbert problem is equivalent to two scalar problems to be solved consequently,

\[
\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2ib_j, \quad (\xi, v) \in l_j, \quad j = 0, 1, \ldots, n \tag{3.15}
\]

and

\[
\Phi_2^+(\xi, v) + \Phi_2^-(\xi, v) = 0, \quad (\xi, v) \in l_0,
\]

\[
\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 2ib_j[\Re \Phi_1^+(\xi, v) - a_j], \quad (\xi, v) \in l_j, \quad j = 1, \ldots, n. \tag{3.16}
\]

(b) Case (ii)

In the case when \( \tau_j^i / \mu_j = v_1, \tau_j^j / \mu_j = v_2, j = 1, 2, \ldots, n \), and \( v_1 \) and \( v_2 \) are constants, the counterpart of the boundary conditions (3.2) on the contours \( l_j \) is

\[
\begin{align*}
\Re F(\xi) &= a_j, \quad (\kappa_j - 1) \Im[\bar{v}_j(c_j)] = \Im F(\xi) - b_j, \quad \xi \in l_j, \quad j = 1, \ldots, n. \tag{3.17}
\end{align*}
\]

The two conditions (3.3) and (3.5) remain the same. To reduce this problem to a vector Riemann–Hilbert problem on the surface \( \mathcal{R} \), we define the functions \( \Phi_1(\xi, u) \) and \( \Phi_2(\xi, u) \) in a way different from case (i). We put

\[
\Phi_1(\xi, u) = \begin{cases} 
\frac{iF(\xi)}{2}, & (\xi, u) \in \mathcal{D}^+, \\
-\frac{i\bar{F}(\xi)}{2}, & (\xi, u) \in \mathcal{D}^-,
\end{cases} \quad \text{and} \quad \Phi_2(\xi, u) = \begin{cases} 
\bar{v}_j(\xi), & (\xi, u) \in \mathcal{D}^+, \\
v_j(\xi), & (\xi, u) \in \mathcal{D}^-.
\end{cases} \tag{3.18}
\]

These functions solve the vector Riemann–Hilbert problem (3.11) on the contour \( \mathcal{L} \) with the matrix coefficient \( G(\xi, v) \) and the right-hand side \( g(\xi, v) \) having the form

\[
G(\xi, v) = \begin{pmatrix} 1 & 0 \\
-\frac{2iv}{\kappa_j - 1} & 1 \end{pmatrix}, \quad (\xi, v) \in l_j, \quad G(\xi, v) = \begin{pmatrix} -1 & i\left(1 - \frac{\bar{v}}{v}\right) \\
0 & \frac{\bar{v}}{v} \end{pmatrix}, \quad (\xi, v) \in l_0 \tag{3.19}
\]

and

\[
g(\xi, v) = \begin{pmatrix} 2ib_j \\
-\frac{2i\kappa_j}{\kappa_j - 1}(b_j + ia_j) \end{pmatrix}, \quad (\xi, v) \in l_j, \quad g(\xi, v) = \begin{pmatrix} -2ib_0 \\
0 \end{pmatrix}, \quad (\xi, v) \in l_0.
\]

The vector \( \Phi(\xi, u) \) satisfies the symmetry condition \( \Phi(\xi, u) = \overline{\Phi(\xi, u_*)} \), \( (\xi, u) \in \mathcal{R} \), and the condition at infinity

\[
\Phi_1^+(\xi, u) \sim \frac{i(\tau_1^\infty - \bar{\tau})}{\mu_0\bar{v}} \Phi_2^+(\xi, u) \quad \text{and} \quad \Phi_2(\xi, u) = O(\tau^{1/2}), \quad \tau \to \infty. \tag{3.20}
\]

Again, the vector problem is decoupled when \( \tau_2^j = 0 \), that is, when \( v = \bar{v} = v_1 = \tau_1^i / \mu_j, j = 1, \ldots, n \). In this case, we deduce the following two scalar Riemann–Hilbert problems on the
weierstrass kernel (\(\xi\)) of the problems (3.15) and (3.21) are bounded at infinity. Therefore, the direct use of the

\[ V^+(\xi, v) - V^-(\xi, v) = 2iu_j, \quad (\xi, v) \in l_j, \quad j = 1, \ldots, n, \]

\[ V^+(\xi, v) + V^-(\xi, v) = -2b_0, \quad (\xi, v) \in l_0 \]  \hspace{1cm} (3.21)

and

\[ \Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = -\frac{2i}{\kappa_j - 1} [\text{Re} \Phi_1^+(\xi, v) + b_j], \quad (\xi, v) \in l_j, \quad j = 1, \ldots, n, \]

\[ \Phi_2^+(\xi, v) + \Phi_2^-(\xi, v) = 0, \quad (\xi, v) \in l_0. \]  \hspace{1cm} (3.22)

Since the scalar problems in cases (i) and (ii) are similar, we confine ourselves to considering case (i) only.

4. Two scalar Riemann–Hilbert problems on a hyperelliptic surface

(a) Analogue of the Cauchy kernel on a hyperelliptic surface

One of the contours of both Riemann–Hilbert problems, \(l_0\), is semi-infinite, while the right-hand

sides of the problems (3.15) and (3.21) are bounded at infinity. Therefore, the direct use of the

Weierstrass kernel \((u + v)/2\text{v}(d\xi/\xi - \zeta)\) leads to divergent integrals. This kernel has order-\(n\) poles at both infinite points on the surface. We propose another analogue of the Cauchy kernel that does not have poles at infinity. Instead, it has \(n\) simple poles at finite points on both sheets of the surface \(\mathcal{R}\). It has the form

\[ V(\xi, v; \zeta, u) d\xi = \frac{1}{2} \left( \frac{\zeta - \xi_0}{\xi - \xi_0} + \frac{u}{v} \prod_{j=1}^{n} \frac{\xi - \xi_j}{\xi - \xi_j} \right) \frac{d\xi}{\xi - \zeta}, \]  \hspace{1cm} (4.1)

where \(u = u(\xi), \quad v = u(\xi), \quad \xi_j, \quad (j = 0, 1, \ldots, n)\) are arbitrary real fixed points not lying on the contour \(\mathcal{L}\). We select the points \(\xi_1, \ldots, \xi_n\) to be distinct, while the point \(\xi_0 \notin \mathcal{L}\) may coincide with one of the points \(\xi_1, \ldots, \xi_n\). In the elliptic case, we shall take \(\xi_1 = \xi_0\).

Since all the points \(\xi_j \quad (j = 0, \ldots, n)\) are chosen to be real, the kernel \(V d\xi\) is symmetric with respect to the contour \(\mathcal{L}\),

\[ V(\xi, v; \zeta, u) d\xi = V(\xi, v; \bar{\zeta}, -u(\bar{\xi})) d\bar{\xi}. \]

Singular integrals with this kernel satisfy the Sokhotski–Plemelj formulæ on the contour \(\mathcal{L}\). The kernel \(V d\xi\) also satisfies the following two properties.

(a) With respect to the variable \((\xi, v), V d\xi\) is an Abelian differential of the third kind. It has

simple poles at the points \((\zeta, u), (\xi_0, u(\xi_0))\) and \((\xi_0, -u(\xi_0))\). As \(\xi \to \infty, \quad V = O(\xi^{-3/2})\).

(b) With respect to the variable \((\zeta, u), V d\xi\) is a meromorphic function on the surface \(\mathcal{R}\). It has

simple poles at the points \((\xi, v), (\xi_j, u(\xi_j))\) and \((\xi_j, -u(\xi_j))\), \(j = 1, \ldots, n\). The residues of the kernel \(V d\xi\)

\[ \text{res}_{(\zeta, u)=(\xi_j, \pm u(\xi_j))} V(\xi, v; \zeta, u) d\xi = \pm \frac{u(\xi_j)}{2} \varphi_j(\xi_j) d\xi \]  \hspace{1cm} (4.2)

generate a basis of Abelian differentials of the first kind

\[ \varphi_j(\xi_j) d\xi = \left( \prod_{s=1, s \neq j}^{n} \frac{\xi - \xi_s}{\xi_j - \xi_s} \right) \frac{d\xi}{v}, \quad j = 1, 2, \ldots, n. \]  \hspace{1cm} (4.3)

In particular, we have

\[ n = 1: \quad \varphi_1(\xi) d\xi = \frac{d\xi}{v}, \]

\[ n = 2: \quad \varphi_1(\xi) d\xi = \frac{\xi - \xi_2}{\xi_1 - \xi_2} \frac{d\xi}{v}, \quad \varphi_2(\xi) d\xi = \frac{\xi - \xi_1}{\xi_2 - \xi_1} \frac{d\xi}{v}. \]  \hspace{1cm} (4.4)

At infinity, the kernel \(V d\xi\) is bounded, \(V d\xi = O(1), \quad \xi \to \infty\).
(b) The first Riemann–Hilbert problem

Now equipped with the kernel $V d\xi$ we shall proceed to solve the first Riemann–Hilbert problem (3.15). Owing to the simple poles of the kernel and the asymptotics of the function $\Phi_1(\zeta)$ at infinity, the general solution has the form

$$\Phi_1(\zeta, u) = N_0 + in(\zeta) \sum_{j=1}^{n} \frac{N_j}{\zeta - \xi_j} + \frac{1}{\pi} \sum_{j=0}^{n} b_j \int_{l_j} V(\xi, v; \zeta, u) d\xi. \quad (4.5)$$

Here, $N_j (j = 0, 1, \ldots, n)$ are real constants. At this stage, they are arbitrary. It is directly verified that the function (4.5) satisfies the symmetry condition (3.9). The points $(\xi_j, \pm u(\xi_j))$ are removable singularities of the function $\Phi_1(\zeta, u)$ if and only if the constants $N_j$ are expressed through the constants $b_s$ as

$$N_j = -\frac{1}{2\pi i} \sum_{s=0}^{n} b_s \int_{l_s} \psi_j(\xi) d\xi, \quad j = 1, 2, \ldots, n. \quad (4.6)$$

In general, the solution is growing at infinity as $\zeta^{n-1/2}$. To satisfy the conditions at infinity (3.14) for $n \geq 2$, we require

$$\lim_{\zeta \to \infty} \zeta^{-1/2-s} u(\zeta) \sum_{j=1}^{n} \frac{N_j}{\zeta - \xi_j} = 0, \quad s = 1, 2, \ldots, n - 1. \quad (4.7)$$

The solution (4.5) possesses $2n + 2$ real constants $N_j$ and $b_j (j = 0, 1, \ldots, n)$. The constants $N_1, \ldots, N_n$ and $b_0, \ldots, b_n$ are connected by $2n - 1$ linear relations ((4.6) and (4.7)). Therefore, three constants $N_0$ and, say, $N_1$ and, $b_0$ are free.

In the elliptic case, $n = 1$, there are no conditions (4.7) and one condition (4.6). Compute the two integrals in (4.6)

$$\int_{l_0} \frac{d\xi}{v} = \frac{2}{i} \int_{m}^{\infty} \frac{d\xi}{\sqrt{|p(\xi)|}} = -4ikK$$
$$\int_{l_1} \frac{d\xi}{v} = \frac{2}{i} \int_{0}^{1} \frac{d\xi}{\sqrt{|p(\xi)|}} = 4ikK, \quad k = m^{-1/2}, \quad (4.8)$$

where $K = K(k)$ is the complete elliptic integral of the first kind. On substituting the integrals in (4.6), we express $b_1$ through the free constants $b_0$ and $N_1$

$$b_1 = b_0 - \frac{\pi N_1}{2ikK}. \quad (4.9)$$

In the case of the genus-2 surface, from (4.7), we immediately deduce that $N_2 = -N_1$, while the relations (4.6) yield two equations,

$$\frac{1}{2\pi i} \sum_{s=0}^{2} b_s \int_{l_s} \frac{(\xi - \xi_j)}{v} d\xi = (\xi_2 - \xi_1)N_1, \quad j = 1, 2. \quad (4.10)$$

The constants $N_0, N_1$ and $b_0$ are free, while the constants $b_1$ and $b_2$ are determined through $b_0$ and $N_1$ by the two linear equations (4.10).

(c) The second Riemann–Hilbert problem

Rewrite equations (3.16) in the form

$$\Phi_2^+(\xi, v) = G_2(\xi, v)\Phi_2^-(\xi, v) + g_2(\xi, v), \quad (\xi, v) \in L, \quad (4.11)$$

where

$$G_2(\xi, v) = \begin{cases} -1, & (\xi, v) \in l_0, \\ 1, & (\xi, v) \in L', \end{cases} \quad g_2(\xi, v) = \begin{cases} 0, & (\xi, v) \in l_0, \\ 2i\lambda_j[\Re \Phi_1^-(\xi, v) - a_j], & (\xi, v) \in l_j, j \neq 0. \end{cases} \quad (4.12)$$
Here, $L' = \bigcup_{j=1}^{n} l_j$. The function $\Phi_2(\zeta, u)$ has to satisfy the symmetry condition (3.9) and the condition at infinity (3.14).

First, we factorize the coefficient $G_2(\xi, v)$, that is, find a function $X(\zeta, u)$ that meets the symmetry condition $X(\zeta, u) = \overline{X(\zeta, u)}$, $(\zeta, u) \in \mathcal{R}$, meromorphic in $\mathcal{R} \setminus L$, and its limit values satisfy the relation

$$G_2(\xi, v) = X^+(\xi, v) [X^- (\xi, v)]^{-1}, \quad (\xi, v) \in L.$$  

(4.13)

To find such a function, we cut the surface $\mathcal{R}$ along canonical cross-sections $a_j$ and $b_j$ ($j = 1, 2, \ldots, n$) to form a simply connected domain $\mathcal{R}$. Choose $a_j$ as two-sided slits $l_{j-1}$ that belong to both sheets of the surface (figure 1). The contour $b_1$ consists of the segment joining the point $m$ and $k_{2n}$ along the upper sheet $D^+$ and the segment $[k_{2n}, m] \subset D^-$. The other cross-sections $b_j$ combine the segments $[k_{2j-3} k_{2n}] \subset D^+$ and $[k_{2n} k_{2j-3}] \subset D^-$, $j = 2, 3, \ldots, n, k_1 = 0$. The positive direction on the boundary $\partial \mathcal{R}$ of the surface $\mathcal{R}$,

$$\partial \mathcal{R} = a_1^+ b_1^+ a_1^- b_1^- , \ldots, a_n^+ b_n^+ a_n^- b_n^- ,$$  

(4.14)

is chosen in the standard way, that is, when a point traverses the boundary, the surface $\mathcal{R}$ is on the left.

Show that the function

$$X(\zeta, u) = \chi(\zeta, u) \overline{X(\zeta, u)} \exp \left\{ \frac{1}{2\pi i} \int_{l_0} \log(-1)V(\xi, v; \zeta, u) \, d\xi \right\}$$  

(4.15)

solves the factorization problem. Here,

$$\chi(\zeta, u) = \exp \left\{ - \sum_{j=1}^{n} \left( \int_{\gamma_j} + n_j \int_{a_j} + m_j \int_{b_j} \right) V(\xi, v; \zeta, u) \, d\xi \right\},$$  

(4.16)

where $n_j$ and $m_j$ are integers and $\gamma_j$ are contours lying in the surface $\mathcal{R}$, not crossing the canonical cross-sections and passing through the branch point $k_{2n-1}$. The starting points of the contours $\gamma_j$, $p_j = (\eta_j, u_j)$ are arbitrary fixed distinct points. We select them on the upper sheet $D^+$, $u_j = \sqrt[p]{\eta_j}$, $j = 1, \ldots, n$. The terminal points $q_j = (\zeta_j, u_j)$, $u_j = u(\zeta_j)$ cannot be chosen a priori. They may fall on either sheet and are to be determined.

We assert that the function $X(\zeta, u)$ is symmetric with respect to the contour $L$, $X(\zeta, u) = \overline{X(\zeta, u)}$, $(\zeta, u) \in \mathcal{R}$. By the Sokhotski–Plemelj formulae, the function $X(\zeta, u)$ satisfies the relation (4.13). The integrals in (4.16) have jumps across the contours of integration. However, these jumps are multiples of $2\pi i$, and the functions $X(\zeta, u)$ and $\overline{X(\zeta, u)}$ are meromorphic on the surface $\mathcal{R}$.

The integrals over the contours $\gamma_j$ in (4.16) have logarithmic singularities at the endpoints of the contours. Therefore, the function $\chi(\zeta, u)$ has simple zeros at the points $p_j \in D^+$ and simple poles at the points $q_j \in \mathcal{R}$. Because of the symmetry, the factorizing function $X(\zeta, u)$ has $2n$
simple zeros at the points \( p_j \) and \( p_{js} = (\tilde{\eta}_j, -\sqrt{p(\tilde{\eta}_j)}) \) and 2\( n \) simple poles at the points \( q_j \) and \( q_{js} = (\tilde{\xi}_j, -u(\tilde{\xi}_j)) \).

Since the kernel \( V(\xi, \zeta; u, v) \) has simple poles at the points \( (\xi_j, u(\xi_j)) \) and \( (\tilde{\xi}_j, -u(\tilde{\xi}_j)) \) \( (j = 1, 2, \ldots, n) \), the function \( X(\xi, u) \) has essential singularities at these points. Owing to formula (4.2), to convert them into removable singularities, it is necessary and sufficient to require that

\[
\frac{1}{2} \sum_{j=1}^{n} \left( \int_{0}^{\eta_j} \psi_1(\xi) \, d\xi - \int_{\eta_j}^{1} \psi_1(\xi) \, d\xi + 2n \int_{a_j} \psi_1(\xi) \, d\xi \right) = 0, \quad s = 1, 2, \ldots, n. \tag{4.17}
\]

These \( n \) relations constitute the imaginary equations of the following genus-\( n \) Jacobi inversion problem.

Find \( n \) points \( q_j = (\zeta_j, u_j') \) on the surface \( R \) and 2\( n \) integers \( n_j \) and \( m_j \) \( (j = 1, 2, \ldots, n) \) such that

\[
\sum_{j=1}^{n} \left( \int_{\eta_j}^{\eta_j} \psi_1(\xi) \, d\xi + n_j A_{js} + m_j B_{js} \right) = h_{s}, \quad s = 1, 2, \ldots, n. \tag{4.18}
\]

Here,

\[
A_{js} = \int_{a_j} \psi_1(\xi) \, d\xi \quad \text{and} \quad B_{js} = \int_{b_j} \psi_1(\xi) \, d\xi
\]

are the cyclic \( A \)- and \( B \)-periods of the Abelian integrals of the first kind

\[
J_s(q) = \int_{k_{s-n-1}}^{n} \psi_1(\xi) \, d\xi, \quad q \in R
\]

and

\[
h_s = \frac{1}{4} A_{1s} + \sum_{j=1}^{n} \int_{k_{s-n-1}}^{n} \psi_1(\xi) \, d\xi.
\]

In the elliptic case, the Jacobi inversion problem is solvable [20] in terms of elliptic functions. We have \( k_1 = 0, \psi_1(\xi) = \frac{1}{2}, \)

\[
A_{11} = -2i \int_{m}^{\infty} \frac{d\xi}{\sqrt{|p(\xi)|}} = -4ikK, \quad \text{and} \quad B_{11} = 2 \int_{m}^{1} \frac{d\xi}{\sqrt{|p(\xi)|}} = 4ikK',
\]

where \( K' = K\left(\sqrt{1-k^2}\right) \) and \( h_1 \) is given by

\[
h_1 = -ikK + \int_{0}^{m} \frac{d\xi}{p^{1/2}(\xi)}.
\]

By inverting the elliptic integral we find \( \zeta_1 \) in terms of the elliptic sine \( \zeta_1 = sn^2(\eta_1/2k) \). Evaluate next the following two integrals:

\[
J_{\pm} = \int_{0}^{m} \frac{d\xi}{p^{1/2}(\xi)} \pm \int_{0}^{\xi_1} \frac{d\xi}{p^{1/2}(\xi)} - ikK
\]

and the constants \( n_1 \) and \( m_1 \)

\[
n_1 = -\text{Im} \frac{J_+}{4kK} \quad \text{and} \quad m_1 = \text{Re} \frac{J_+}{4kK'}.
\]

If both constants \( n_1 \) and \( m_1 \) are integers, then the point \( q_1 = (\zeta_1, \sqrt{p(\zeta_1)}) \in D^+ \). Otherwise, \( q_1 = (\zeta_1, -\sqrt{p(\zeta_1)}) \in D^- \), and the constants

\[
n_1 = -\text{Im} \frac{J_+}{4kK} \quad \text{and} \quad m_1 = \text{Re} \frac{J_+}{4kK'}
\]

have to be integers.

In the hyperelliptic case, \( n \geq 2 \), the solution procedure requires transforming the basis of the Abelian differentials \( \{\psi_1(\xi) \, d\xi\}_{j=1}^{n} \) into the canonical basis, computing the Riemann constants.
constructing the associated genus-\(n\) Riemann theta function and determining its zeros [21]. The last step reduces the problem to a certain degree-\(n\) algebraic equation [22].

It turns out that, owing to the symmetry of the Riemann surface, the factorizing function \(X(\xi, u)\) is independent of the integers \(m_j\), and formula (4.15) can be simplified and written in the form

\[
X(\xi, u) = \exp \left\{ \frac{u}{2} \sum_{j=1}^{n} \frac{d\xi_j}{(\xi - \xi_j) v} - \sum_{j=1}^{n} \int_{\gamma_j} V(\xi, v; \xi, u) \, d\xi \right\} + \int_{\gamma_j} V(\xi, v; \xi, -u) \, d\xi + 2nju \int_{a_j^+} \left( \prod_{j=1}^{n} \frac{d\xi_j}{(\xi - \xi_j)(\xi - \xi_j)v} \right) \, d\xi. \tag{4.27}
\]

Here, \(a_j^+\) is the upper side of the cross-section \(a_j\) (the upper side of the slit \(l_j-1\)).

Having factorized the function \(G_2(\xi, v)\) we can proceed with the solution of the Riemann–Hilbert problem (4.11). On replacing the function \(G_2(\xi, v)\) in (4.11) by \(X^+(\xi, v)[X^-(\xi, v)]^{-1}\) we arrive at

\[
\frac{\Phi_2^+(\xi, v)}{X^+(\xi, v)} = \frac{\Phi_2^-(\xi, v)}{X^-(\xi, v)} + \frac{g_2(\xi, v)}{X^+(\xi, v)}, \quad (\xi, v) \in \mathcal{L}. \tag{4.28}
\]

Introduce next a singular integral over the contour \(\mathcal{L}' = \bigcup_{j=1}^{n} l_j\)

\[
\Psi(\xi, u) = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{g_2(\xi, v)}{X(\xi, v)} V(\xi, v; \xi, u) \, d\xi, \quad (\xi, u) \in \mathcal{R}. \tag{4.29}
\]

In the elliptic case (we selected \(\xi_0 = \xi_1\)), this integral can be written as follows:

\[
\Psi(\xi, u) = \frac{1}{4\pi i} \int_0^1 \left[ \frac{g_2(\xi^+, u^+)}{X^+(\xi^+, u^+)} \left( \frac{\xi - \xi_0}{\xi - \xi_0} + \frac{u \xi - \xi_0}{v^+ \zeta - \xi_0} \right) - \frac{g_2(\xi^-, u^-)}{X^+(\xi^-, u^-)} \left( \frac{\xi - \xi_0}{\xi - \xi_0} + \frac{u \xi - \xi_0}{v^- \xi - \xi_0} \right) \right] \frac{d\xi}{\xi - \xi}. \tag{4.30}
\]

Here, \((\xi^\pm, v^\pm) = (\xi \pm i0, \sqrt{p(\xi \pm i0)})\) are points on the sides \(l_j^+ \subset D^+, u = u(\xi)\) and

\[
g_2(\xi^\pm, v^\pm) = 2i \left[ N_0^+ \pm \frac{\sqrt{p(\xi)}}{\xi - \xi_0} g_0(\xi) \right], \quad 0 < \xi < 1, \tag{4.31}
\]

where \(N_0^+ = N_0 - a_1\) and

\[
g_0(\xi) = 2N_1 - \frac{b_0}{\pi} \int_m^\infty \frac{(\tau - \xi_0) \, d\tau}{\sqrt{p(\tau)(\tau - \xi)}}, \quad 0 < \xi < 1. \tag{4.32}
\]

The Cauchy principal value for the last singular integral is assigned.

We now return to the hyperelliptic case, use the Sokhotski–Plemelj formulæ to represent the second term in the right-hand side of equation (4.28) by the difference of the limit values of the function \(\Psi(\xi, u), \Psi^+(\xi, v) - \Psi^-(\xi, v)\), apply the continuity principle and the generalized Liouville theorem on the surface \(\mathcal{R}\) and deduce the general solution to the Riemann–Hilbert problem (4.11)

\[
\Phi_2(\xi, u) = X(\xi, u)[\Psi(\xi, u) + \Omega(\xi, u)], \quad (\xi, u) \in \mathcal{R}. \tag{4.33}
\]

Here, \(\Omega(\xi, u)\) is a rational function of the surface \(\mathcal{R}\). Since the functions \(\Phi_2(\xi, u), X(\xi, u)\) and \(\Psi(\xi, u)\) are symmetric with respect to the contour \(\mathcal{L}\), the function \(\Omega(\xi, u)\) has to be symmetric as well.

The function \(\Psi(\xi, u)\) has simple poles at the points \((\xi_j, \pm u(\xi_j))\), the poles of the kernel \(V(\xi, v; \xi, u)\). At the same time, the solution \(\Phi_2(\xi, u)\) cannot have poles at these points. Therefore, in order to make them removable singularities, we admit simple poles of the function \(\Omega(\xi, u)\) at these points and require that

\[
\operatorname{res}_{\xi = \xi_j} [\Psi(\xi, u) + \Omega(\xi, u)] = 0, \quad j = 1, 2, \ldots, n. \tag{4.34}
\]

Next, the function \(X(\xi, u)\) has simple poles at the points \(q_j = (\xi_j, u_j)\) \((j = 1, 2, \ldots, n)\) that lie on either the upper or lower sheet of the surface (this is determined by the solution of the Jacobi
problem). To make them removable singularities of the solution, we enforce the following $n$
complex conditions:

$$
\Psi(\zeta_j, u_j^\prime) + \Omega(\zeta_j, u_j^\prime) = 0, \quad j = 1, 2, \ldots, n.
$$  \tag{4.35}

At the points $p_j = (\eta_j, u_j) \in D^+$, the function $X(\zeta, u)$ has simple zeros. Therefore, the function
$\Omega(\zeta, u)$ may have simple poles at these points. Finally, owing to (3.14), $\Omega(\zeta, u) = O(\zeta^{1/2})$, $\zeta \to \infty$, and the principal term of its asymptotics at infinity has to be chosen such that the first condition
in (3.14) holds.

The most general form of the rational function $\Omega(\zeta, u)$ is

$$
\Omega(\zeta, u) = M_0 + \sum_{j=1}^{n} \left[ (M_{1j} + iM_{2j}) \frac{u(\zeta) + u(\eta_j)}{\zeta - \eta_j} - (M_{1j} - iM_{2j}) \frac{u(\zeta) - u(\eta_j)}{\zeta - \eta_j} \right].
$$  \tag{4.36}

where $M_0, M_{1j}, M_{2j}$ and $M_{3j}$ ($j = 1, 2, \ldots, n$) are arbitrary real constants. From the conditions (4.34),
we can fix the constants $M_{3j}$

$$
M_{3j} = \frac{1}{4\pi} \int_{\eta_j}^{\infty} \frac{g_2(\xi, v)\psi_j(\xi) \, d\xi}{X^+(\xi, v)}.
$$  \tag{4.37}

The conditions at infinity bring us $n$ relations for the other constants. The first $n - 1$ equations have the form

$$
\lim_{\zeta \to \infty} \frac{u}{\xi^s} \sum_{j=1}^{n} \left( \frac{M_{1j} + iM_{2j}}{\zeta - \eta_j} - \frac{M_{1j} - iM_{2j}}{\zeta - \eta_j} + \frac{iM_{3j}}{\zeta - \xi_j} \right) = 0, \quad s = 1, 2, \ldots, n - 1,
$$  \tag{4.38}

and the $n$th equation is

$$
\Phi_1^+(\zeta, u) - \frac{\tau_1}{i\tau_1} \Phi_2^+(\zeta, u) \sim 0, \quad \zeta \to \infty.
$$  \tag{4.39}

In addition to these $n$ real conditions, we have $n$ complex equations (4.35). Thus, in total we have $3n$
real equations for $5n$ free real constants, $M_0, M_{1j}, M_{2j}, a_j (j = 1, 2, \ldots, n)$, $m, k_3, k_4, \ldots, k_{2n}$.
Note that the $3n$ real equations (4.35), (4.38) and (4.39) constitute a linear system with respect to the $3n + 1$
constants $M_0, M_{1j}, M_{2j}$ and $a_j$. Thus, there are $2n$ free real parameters—say, $a_1, m, k_3, k_4, \ldots, k_{2n}$—in the solution of the second Riemann–Hilbert problem.

In the elliptic case, we have only one complex condition (4.35) and two real conditions ((4.37) and (4.39)). On evaluating the limit

$$
\lim_{\zeta \to \infty} X(\zeta, u) = iX_\infty \quad \text{and} \quad X_\infty = \left| \frac{\zeta_1 - \xi_0}{\eta_1 - \xi_0} \right|,
$$  \tag{4.40}

we find from (4.39)

$$
M_{21} = \frac{M_{31}}{2} + \frac{\tau_1 N_1}{2(\tau_1^\infty - \tau_1) X_\infty}.
$$  \tag{4.41}

The constant $M_{31}$ is fixed by (4.37), which that can be written in the form

$$
M_{31} = -\frac{1}{4\pi i} \int_{\eta_1}^{\infty} \left[ \frac{g_2(\xi^+, v^+)}{X^+(\xi^+, v^+)} + \frac{g_2(\xi^-, v^-)}{X^+(\xi^-, v^-)} \right] \, d\xi / |p(\xi)|,
$$  \tag{4.42}

The constants $M_{11}$ and $M_0$ are determined by the complex equation (4.35)

$$
M_{11} = \frac{(P^+ + P^-)M_{21} + P_0M_{31} + Q}{Q^- - Q^+}
$$  \tag{4.43}

and

$$
M_0 = (P^- - P^+)M_{11} + (Q^+ + Q^-)M_{21} + Q_0M_{31} - P,
$$

where $P, Q, P_0, Q_0, P^\pm$ and $Q^\pm$ are real constants given by

$$
P + iQ = \Psi(\zeta_1, u_1), \quad P_0 + iQ_0 = \frac{u(\zeta_1)}{\zeta_1 - \xi_0},
$$

$$
p^+ + iQ^+ = \frac{u(\zeta_1) + u(\eta_1)}{\zeta_1 - \eta_1}, \quad p^- + iQ^- = \frac{u(\zeta_1) - u(\eta_1)}{\zeta_1 - \eta_1}.
$$  \tag{4.44}
5. Family of conformal maps: numerical results in the elliptic case

The family of conformal mappings \( z = \omega(\zeta) \) from the \((n + 1)-\)connected \( \zeta \)-domain \( \mathcal{D} \) onto the \((n + 1)-\)connected domain \( D_0 \) is described by the formula

\[
\omega(\zeta) = -\frac{i\mu_0}{\tau_1} \Phi^+_2(\zeta, u), \quad (\zeta, u) \in \mathcal{D}^+.
\]  

(5.1)

In the hyperelliptic case, in addition to the three dimensionless parameters of the elasticity model, \( \hat{\tau}_1 = \tau_1/\mu_0, \hat{\tau}_1^\infty = \tau_1^\infty/\mu_0 \) and \( \kappa = \mu_1/\mu_0 \), the map possesses \( 2n + 2 \) other real parameters. They are \( N_0^s = N_0 - n_1, N_1 \neq 0, b_0, m, k_3, \ldots, k_{2n} \). The non-zero real parameter \( N_1 \) is a scaling parameter, and the map \( \hat{\omega}(\zeta) = N_0^{-1} \omega(\zeta) \) has \( 2n + 1 \) real parameters, \( \hat{N}_0^s = N_0^s/N_1, \hat{b}_0 = b_0/N_1 \) and the \( n \) geometric parameters. Our numerical tests in the elliptic case show that the map is invariant of the second parameter \( \hat{b}_0 \). As for the first parameter \( \hat{N}_0^s \), it is a translation parameter. Variation of this parameter leads to translation of the inclusion along the real axis and does not change the inclusion profile and the distance of the inclusion points to the \( x_1 \)-axis. This means that, given the model three parameters \( \hat{\tau}_1, \hat{\tau}_1^\infty \) and \( \kappa \), the parameter \( m \in (1, \infty) \) generates a one-parametric family of maps \( \hat{\omega}(\zeta) \) and therefore a one-parametric family of scaled uniformly stressed inclusions embedded into a half-plane. The parameter \( m > 1 \) has to be chosen such that the inclusion boundary does not intersect the \( x_1 \)-axis, the boundary of the external elastic body. We conjecture that the same is true in the genus-\( n \) hyperelliptic case, and the family of conformal maps has exactly \( 2n - 1 \) free geometric parameters, \( m, k_3, k_4, \ldots, k_{2n} \), and is invariant of \( \hat{b}_0 \) and \( \hat{N}_0^s \).

We wish next to analyse the solution in the elliptic case and transform the formulae to the form convenient for numerical tests. To verify that the function \( z = \omega(\zeta) \) maps the two-sided slits \( l_0^1 \) and \( l_1^1 \) onto the boundary of the \( z \)-half-plane and the inclusion \( D_1 \), respectively, indeed, we write down the function (5.1) on the contours \( l_0^1 \) and \( l_1^1 \). We have

\[
\omega(\xi^\pm) = -\frac{i\mu_0}{\hat{\tau}_1} X^+(\xi^\pm, u^\pm)[\Psi(\xi^\pm, u^\pm) + \Omega(\xi^\pm, u^\pm)], \quad (\xi^\pm, u^\pm) \in [l_0^1 \cup l_1^1] \subset \mathcal{D}^+,
\]  

(5.2)

where \( \xi^\pm = \xi \pm i0, u^\pm = p^{1/2}(\xi^\pm) \). On the two sides of the contour \( l_0^1 \), the functions \( \Psi(\xi^\pm, u^\pm) \) and \( \Omega(\xi^\pm, u^\pm) \) are real. By the Sokhotski–Plemelj formulae, we discover that Re \( X^+(\xi^\pm, u^\pm) = 0 \). Therefore, the imaginary part of the function \( \omega(\xi^\pm) \) is equal to 0, while Re \( \omega(\xi^\pm) \in (-\infty, \infty) \). This means that the contour \( l_0^1 \) is mapped to the real axis of the physical plane.

Consider now the contour \( l_1 \). On applying the Sokhotski–Plemelj formulae, we obtain from (4.5) and (4.33) that

\[
\left\{ \begin{array}{l}
\Phi^+_1(\xi, v) - \Phi^-_1(\xi, v) = 2i b_1, \quad (\xi, v) \in l_1 \\
\Phi^+_2(\xi, v) - \Phi^-_2(\xi, v) = 2i \lambda \left[ \text{Re} \Phi^+_1(\xi, v) - a_1 \right], \quad (\xi, v) \in l_1.
\end{array} \right.
\]  

(5.3)

In view of (3.8), we conclude that the complex boundary condition (2.7) and the interface conditions (2.2) hold. It is directly verified that the solution determined satisfies the condition at infinity (2.8).

To recover the contour \( L_1 \), the boundary of the inclusion, we need to let a point \( \zeta \) traverse the contour \( l_1 \) along the positive and negative sides. Since all our numerical tests show that the point \( \zeta_1 \) falls on the upper sheet of the surface \( \mathcal{R} \), we simplify the formula for the function \( X^+(\xi^\pm, u^\pm) \) for this case

\[
X^+(\xi^\pm, u^\pm) = \exp \left\{ \mp \frac{1}{2} - 2n_1 \right\} \frac{1 - \tau \xi_0}{\sqrt{\tau(1 - \tau)}(1 - m \tau)(1 - \tau \xi)}\left[ \int_0^{1/m} \left( \frac{1 - \tau \xi_0}{\sqrt{\tau(1 - \tau)}(1 - m \tau)(1 - \tau \xi)} \right) \frac{d\tau}{\tau - \xi} \right], \quad (\xi^\pm, v^\pm) \in [l_1^1 \subset \mathcal{D}^+].
\]  

(5.4)

A similar formula can be written when \( \zeta_1 \in \mathcal{D}^- \). In this case, a part of the contour \( \gamma_1 \), a contour \( (\eta_1, 0) \), lies on the upper sheet \( \mathcal{D}^+ \) and the second part \( (0, \zeta_1) \) lies on the lower sheet \( \mathcal{D}^- \). Both of
the integrals in (5.4) are non-singular, and the Gauss quadrature formulae give a good accuracy of computations.

Write next the limit values of the function \( \Psi(\xi, u) \) on the sides \( l_1^\pm \) of the contour \( l_1 \). By the Sokhotski–Plemelj formulae, we find from (4.27)

\[
\Psi(\xi^\pm, u^\pm) = \frac{g_2(\xi^\pm, u^\pm)}{2X^+(\xi^\pm, u^\pm)} + \frac{1}{4\pi i} \left[ \frac{g_2(\tau^+, v^+)}{X^+(\tau^+, v^+)} \left( \frac{\xi - \xi_0}{\tau - \xi_0} + \frac{u^\pm \tau - \xi_0}{v^+ - \xi - \xi_0} \right) \right]
\]

On replacing the function \( g_2 \) by its expression (4.28), we split the integral into three other integrals. We write

\[
\Psi(\xi^\pm, u^\pm) = \frac{i}{X^+(\xi^\pm, u^\pm)} \left[ N^0 \pm \sqrt{\frac{p(\xi)}{\xi - \xi_0}} g_0(\xi) \right] + \mathcal{I}_1(\xi) \pm \mathcal{I}_2(\xi) + \mathcal{I}_3^\pm(\xi), \quad 0 < \xi < 1,
\]

where

\[
\mathcal{I}_1(\xi) = \frac{\xi - \xi_0}{2\pi i} \int_0^1 \frac{\sqrt{\tau(1 - \tau)f_1(\tau, \xi)}}{\tau - \xi} d\tau,
\]

\[
\mathcal{I}_2(\xi) = \frac{N^0 \sqrt{\frac{p(\xi)}{\xi - \xi_0}}}{2\pi (\xi - \xi_0)} \int_0^1 \frac{\tau - \xi_0}{\sqrt{\tau(1 - \tau)}} Y_+(\tau) \frac{d\tau}{\tau - \xi},
\]

and

\[
\mathcal{I}_3^\pm(\xi) = \frac{1}{2\pi} \int_0^1 \frac{N^0(\xi - \xi_0)}{\tau - \tau_0} \pm \frac{\sqrt{\frac{p(\xi)}{\xi - \xi_0}} g_0(\tau)}{\sqrt{\tau(1 - \tau)}} Y_-(\tau) \frac{d\tau}{\tau - \xi},
\]

and

\[
Y_\pm(\tau) = \frac{1}{X^+(\tau^+, v^+)} \pm \frac{1}{X^+(\tau^-, v^-)}.
\]

We assert that all three of these integrals are singular. The density of the first integral vanishes at the endpoints, and the integral can be represented in the form

\[
\mathcal{I}_1(\xi) = \frac{\xi - \xi_0}{2\pi} \int_0^1 \frac{\sqrt{\tau(1 - \tau)f_1(\tau, \xi)}}{\tau - \xi} d\tau \quad \text{and} \quad f_1(\tau, \xi) = \frac{\sqrt{m - \tau} g_0(\tau) Y_+(\tau)}{(\tau - \xi)^2}.
\]

To compute this integral, we expand the function \( f_0 \) in terms of the Chebyshev polynomials of the second kind

\[
f_1(\tau, \xi) = \sum_{i=1} d_i(\xi) U_{i-1}(2\tau - 1), \quad 0 < \xi < 1,
\]

where the expansion coefficients

\[
d_i(\xi) = \frac{2}{\pi} \int_{-1}^{1} f_1 \left( \frac{\tau + 1}{2}, \xi \right) U_{i-1}(\tau) \sqrt{1 - \tau^2} d\tau
\]

are evaluated by the order-\( N \) Gaussian quadrature formula

\[
d_i(\xi) = \frac{2}{N + 1} \sum_{j=1}^{N} \sin \frac{j\pi}{N + 1} \sin \frac{j\pi}{N + 1} f_1 \left( \frac{x_j^0 + 1}{2}, \xi \right), \quad x_j^0 = \cos \frac{j\pi}{N + 1}.
\]

On substituting the expansion (5.10) into formula (5.9) and expressing the integral in terms of the Chebyshev polynomials of the first kind \( T_i \) we deduce the series expansion for the integral (5.9)

\[
I_1(\xi) = -\frac{\xi - \xi_0}{4} \sum_{i=1}^{\infty} d_i(\xi) T_i(2\xi - 1), \quad 0 < \xi < 1.
\]

The density of the second integral \( I_2(\xi) \) in (5.7) has the square root singularity at the endpoints. The integral can be computed in a similar fashion [18]

\[
I_2(\xi) = \frac{N^0 \sqrt{\frac{p(\xi)}{\xi - \xi_0}}}{\xi - \xi_0} \sum_{i=1}^{\infty} c_i U_{i-1}(2\xi - 1),
\]
Figure 2. Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \geq 0$ for different values of $m$ when $\nu = 0.5$, $N_0^\nu = 0$, $\tau_1/\mu_0 = -1$ and $\tau_1^\infty/\mu_0 = -2$.

\[
\begin{align*}
\tilde{F}_0(\tau, \xi) &= \frac{1 - \tau \xi_0}{\sqrt{1 - \tau(1 - \tau \xi)}}, \\
\tilde{F}_1(\tau) &= \frac{\tau - \xi_0}{\sqrt{\tau}}.
\end{align*}
\]}

The first integral in (5.16) is evaluated by the Gauss quadrature rule, while the second one is computed by expanding it in terms of the Chebyshev polynomials of the second kind, as was done for the integral $I_2(\xi)$ in (5.14).

The results of computations are presented in figures 2–6. In figure 2, for the model parameters selected to be $\kappa = \mu_1/\mu_0 = 0.5$, $\tau_1/\mu_0 = -1$ and $\tau_1^\infty/\mu_0 = -2$, we plot the inclusion profile for some values of the mapping parameter $m$. By decreasing the parameter $m$, we increase the inclusion size and eventually intersect the boundary of the external body. At the same time, given a set of the problem parameters $\kappa \neq 1$ and $\tau_1 \neq \tau_1^\infty$, it is always possible to increase the parameter $m$ such that the inclusion is completely embedded into a half-plane, that is, its boundary does not intersect the $x_1$-axis.
Figure 3. Normalized inclusion in the half-plane \(|x_1| < \infty, x_2 \geq 0\) for different values of \(\kappa \in (0, 1)\) when \(m = 1.6, N_0^\kappa = 0, \tau_1/\mu_0 = -1\) and \(\tau_1^\infty/\mu_0 = -2\).

Figure 4. Normalized inclusion in the half-plane \(|x_1| < \infty, x_2 \geq 0\) for different values of \(\kappa > 1\) when \(m = 2, N_0^\kappa = 0, \tau_1/\mu_0 = -1\) and \(\tau_1^\infty/\mu_0 = -2\).

In figures 3 and 4, \(\tau_1/\mu_0 = -1\) and \(\tau_1^\infty = -2\), the parameter \(m\) is fixed, while the parameter \(\kappa\) has different values. In figure 3, \(\kappa \in (0, 1)\) and \(\kappa > 1\) in figure 4. If \(\kappa \to 1^\pm\), then the dimensionless size of the inclusion is growing, and in order to prevent the intersection of the inclusion boundary with the \(x_1\)-axis, it is necessary to increase the parameter \(m\).

In addition to the singular case \(\kappa = 1\), there is another singular case, \(\tau_1 = \tau_1^\infty\), when the solution to the problem does not exist. In figures 5 and 6, we show the inclusion profile when \(\kappa = 0.5\),...
Figure 5. Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \leq 0$ for different values of $\tau_1/\mu_0$ when $m = 2$, $N_0^\infty = 0$, $\kappa = 0.5$ and $\tau_1^\infty/\mu_0 = -2$.

Figure 6. Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \geq 0$ for different values of $\tau_1/\mu_0$ when $m = 5$, $N_0^\infty = 0$, $\kappa = 0.5$ and $\tau_1^\infty/\mu_0 = -2$. 
When the values of the stress $τ_1$ inside the inclusion are close to the values of the stress at infinity $τ_1^∞$, then the inclusion size decreases. What is different between the cases $τ_1/τ_1^∞ < 1$ and $τ_1/τ_1^∞ > 1$ is that the function $ω(ζ)$ maps the parametric $ζ$-plane onto the lower half-plane in figure 5 and the upper half-plane in figure 6. Another difference is that when $τ_1 → τ_1^∞$ and $|τ_1/τ_1^∞| > 1$ the inclusion approaches the half-plane boundary, while when $|τ_1/τ_1^∞| < 1$ it drifts away from the boundary of the external body. At the same time, when $|τ_1/τ_1^∞| → 0$, the inclusion is growing, becomes closer to the $x_1$-axis and we need to increase the parameter $m$ to keep the inclusion inside the external body.

### 6. Conclusion

In the previous sections, a closed-form solution has been derived to the inverse problem of antiplane shear of an elastic half-plane with $n$ finite elastic inclusions $D_j$ ($j = 1, \ldots, n$). In this model, the boundary of the half-plane $\{|x_1| < ∞, x_2 = 0\}$ is kept free of traction $τ_{23}$, the conditions of ideal contact on the interfaces between the inclusions and the half-plane hold and the stress field inside the inclusions is uniform, while the shape of the inclusions is to be recovered. This free boundary harmonic problem is well suited for the method of conformal mappings. It has been shown that a map from a parametric plane cut along $\mathbb{R}^2 \setminus \cup_{j=1}^n D_j$ can be recovered by solving a vector Riemann–Hilbert problem with a piecewise constant matrix coefficient on a hyperelliptic genus-$n$ surface $\mathcal{R}$. Under the assumption that $τ_{23} = 0$ inside of all the inclusions, the vector problem has been decoupled into two scalar Riemann–Hilbert problems on $n+1$ slits on the surface $\mathcal{R}$. The solution has been determined by employing a new analogue of the Cauchy kernel on a hyperelliptic surface and solving a factorization problem. The associated Jacobi inversion problem has been solved explicitly in terms of elliptic functions in the genus-1 case.

In the elliptic case, not counting the scaling parameter, the model and the solutions to the Riemann–Hilbert problems generate a four-parametric family of conformal mappings that possesses three model parameters, $κ = μ_1/μ_0$, $τ_1/μ_0$ and $τ_1^∞/μ_0$, and one geometric parameter, $m$. Numerical implementation of this method has shown that, although the inclusion shape resembles an ellipse, it is not an actual ellipse and is not symmetric with respect to any line parallel to the $x_1$-axis. In the hyperelliptic case, in addition to the three model parameters, the family of conformal mappings possesses $2n - 1$ free geometric parameters.

Finally, we note that the map constructed recovers the whole family of inclusions embedded into a half-plane when $n = 1$ and 2. In the case $n ≥ 3$, the map has been found for the case when the inclusions are arranged in a way that their pre-images lie in the real axis. This restricts the applicability of the method. For the general case, another technique may be developed. It includes employing a circular $(n+1)$-connected domain as a parametric domain and solving the associated Riemann–Hilbert problems of the theory of automorphic functions [23,24]. This is planned as a part of future research.

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