Abstract

We study the multi-armed bandit problem where the rewards are realizations of general non-stationary stochastic processes, a setting that generalizes many existing lines of work and analyses. In particular, we present a theoretical analysis and derive regret guarantees for rested bandits in which the reward distribution of each arm changes only when we pull that arm. Remarkably, our regret bounds are logarithmic in the number of rounds under several natural conditions. We introduce a new algorithm based on classical UCB ideas combined with the notion of weighted discrepancy, a useful tool for measuring the non-stationarity of a stochastic process. We show that the notion of discrepancy can be used to design very general algorithms and a unified framework for the analysis of multi-armed rested bandit problems with non-stationary rewards. In particular, we show that we can recover the regret guarantees of many specific instances of bandit problems with non-stationary rewards that have been studied in the literature. We also provide experiments demonstrating that our algorithms can enjoy a significant improvement in practice compared to standard benchmarks.

1. Introduction

We consider the classical multi-armed bandit (MAB) problem where, at each round, an agent selects an arm out of $K$ available arms and receives the reward of the selected arm while remaining unaware of the rewards of the other arms. This problem has been studied extensively in the stochastic scenario with i.i.d. rewards (Lai & Robbins, 1985; Auer et al., 2002a) as well as in the adversarial scenario (Auer et al., 2002b), where there is no assumption on the reward distribution.

Recently, there have been several studies of the MAB setting where the rewards are stochastic, but not necessarily independent and identically distributed. This is a natural framework for many real-world problems, which may contain reward sequences that exhibit certain levels of non-stationarity but are also not entirely adversarial.

For example, in the actual multi-armed bandit scenario, where a learner is faced with the decision of pulling an arm from a number of slot machines, it’s possible that the machines may get serviced or replaced and that the payout distribution of each arm may change over time. As another example, consider the online advertising problem where an agent must select an ad to present to a user. Here, each ad is an arm of the bandit and presenting an ad to a user equates to pulling that arm. Since showing the same ad to a user may result in varying degrees of fatigue or excitement, it is reasonable to expect that a user’s reaction to an ad is dependent on their previous responses. This example readily extends to any scenario where there is a need to customize user recommendations, and where the user’s response can change over time depending on past responses, e.g. product recommendations, price setting, and content display. A third illustration is the medical diagnosis setting, where a doctor must prescribe treatments to a patient. Repeated application of the same treatment may yield progressively different results, so it is unreasonable to assume that the outcome of a treatment is i.i.d. over time and does not depends on the previously made decisions.

There have been two predominant ways of modeling the evolution of non-stationary rewards in the stochastic multi-armed bandit problem: the restless setting and the rested setting. In the former, the rewards of the arms are stochastic processes that evolve continually regardless of which arm was chosen by the learner at each round (Whittle, 1981). In the latter, the stochastic process for each arm evolves only when it is chosen, so that the dynamics of the rewards depend on the choices of the learner. Note that the first example above corresponds to the restless setting, and the second and third examples correspond to the rested setting.

Both the restless and rested bandit scenarios have been studied in the literature, and in each analysis, the authors impose different and specific assumptions about the stochastic process. For instance, a common assumption for the restless setting is that the non-stationary rewards evolve according
We introduce a new notion of regret, the path-dependent dynamic pseudo-regret that is a natural performance measure of a bandit algorithm in the presence of non-stationary rewards, and, at the same time, allows us to recover the results from the prior work referenced above.

Our approach towards this problem and our analysis is based on the notion of weighted discrepancy, a tool that was shown to be useful for measuring the degree of non-stationarity in a stochastic process (Kuznetsov & Mohri, 2015). By adapting weighted discrepancy to the multi-armed bandit scenario and incorporating it into state-of-the-art techniques, we are able to design new UCB-style algorithms that can achieve compelling guarantees in the non-stationary rested bandit scenario.

In particular, we first show that knowledge of the discrepancies of each arm in this setting is sufficient to design algorithms that guarantee at most logarithmic path-dependent dynamic pseudo-regret. This indicates that the notion of discrepancy is a key quantity for this problem. We then consider the more realistic scenario where the discrepancy is not known to the learner. In this case, we show that when the discrepancy is bounded, there exist algorithms which can again guarantee at most logarithmic path-dependent dynamic pseudo-regret. Furthermore, we show that this assumption of bounded discrepancy is satisfied by many of the examples referenced above (as well as by other new processes). Thus, we present a unifying framework for analyzing problems in this scenario, and we introduce a single family of algorithms that can achieve logarithmic path-dependent dynamic pseudo-regret for many types of stochastic processes, matching the best-known results for many of the specific cases studied in prior work. We also present a lower bound which implies that in certain non-stationary settings, the guarantees provided by our algorithms are optimal.

Our paper is organized as follows. In Section 2, we formally introduce the rested non-stationary multi-armed bandit (NSMAB) problem, including some new and natural notions of regret that are specific to this more general setting. In Section 3, we define the notion of weighted discrepancy, present key properties of this quantity, and introduce an algorithm for the rested NSMAB problem when the discrepancy is known to the learner. In Section 4, we consider the more realistic scenario where the discrepancy is unknown but bounded. For this setting, we present an algorithm which admits at most logarithmic regret, and we show that many of the stochastic processes studied in the literature adhere to the bounded discrepancy assumption. We also present a lower bound demonstrating that in certain non-stationary settings, logarithmic regret is provably optimal. In Section 5, we present experimental results that support our theoretical treatment of this problem. Our algorithms and analyses are new and based on the adaptation of discrepancy to the rested non-stationary multi-armed bandit setting.

2. Learning Scenario

We consider the classical multi-armed bandit problem (Lai & Robbins, 1985; Auer et al., 2002a) with stochastic rewards.
At each round \( t \), the learner selects an arm \( I_t \) out of a finite set of \( K \) arms and receives a random reward \( X_{t,I_t} \) for the selected arm, without receiving any information about the rewards of the other arms. The objective of the learner is to achieve the largest expected cumulative reward over the course of \( T \) rounds.

Unlike the common scenario adopted in the literature, we do not assume that the rewards are drawn i.i.d. from a fixed distribution. Instead, we allow the reward sequences to be realizations of an arbitrary stochastic processes, potentially non-stationary. Furthermore, we focus on the rested setting, where the reward distribution of each arm evolves only when that arm is pulled. More precisely, if \( X_{1,i}, \ldots, X_{t,i} \) is the realization of the reward process for arm \( i \) and arm \( i \) is pulled at time \( t \), then the learner observes \( X_{T_i(t),i} \) as its reward, where \( T_i(t) \) is the number of times arm \( i \) was pulled by time \( t \). In this work, we consider an oblivious adversary. That is, an adversary that selects a stochastic process for each arm before the game begins and then a particular realization of the reward sequence is drawn according to this stochastic process. This is a natural assumption that is commonly adopted in the prior work. In fact, in the absence of this assumption, even notion of standard pseudo-regret (defined below) does not admit a straightforward interpretation (Arora et al., 2012). Nevertheless, the assumption is not necessary for our proofs and results. The only other assumptions that we make is that reward sequences for different arms are independent and that \( X_{t,i} \in [0,1] \) for all \( i \) and \( t \).

The standard performance measure of an algorithm in the multi-armed bandit problem is the notion of pseudo-regret, which compares the cumulative expected reward of the algorithm to the cumulative expected reward of the best static arm chosen in hindsight:

\[
\text{Reg}_T = \max_{i \in [K]} \sum_{t=1}^{T} \mathbb{E}[X_{t,i}] - \mathbb{E}[X_{T_i(t),i}].
\] (1)

To simplify the notation, in what follows, we will write \( X_{t,I_t} \) instead of \( X_{T_i(t),i} \).

However, since in our setting future reward distribution may be vastly different depending on a particular realization of the past rewards and the best arm to pull is changing over time, we consider two other extensions of pseudo-regret defined below: path-dependent pseudo-regret and path-dependent dynamic pseudo-regret.

Let \( X_1^{-1} \) be the sequence of random reward observed up to time \( t \): \( X_{1,1}, X_{2,1}, \ldots, X_{t,1} \). We define the path-dependent pseudo-regret:

\[
\text{reg}_T = \max_{i \in [K]} \sum_{t=1}^{T} \mathbb{E}[X_{t,i} | X_1^{t-1}] - \mathbb{E}[X_{t,I_t} | X_1^{t-1}].
\] (2)

Figure 1. The path-dependent dynamic pseudo-regret, \( \delta \text{reg}_T \), upper bounds all other definitions of regret. Dynamic regret bounds non-dynamic regret by sub-additivity of the maximum. Path-dependent regret bounds non-path dependent regret by taking expectations and Jensen’s inequality.

We note that by the independence of arms \( \mathbb{E}[X_{t,I_t} | X_1^{t-1}] = \mathbb{E}[X_{t,I_t}] \), where \( s_1 < \ldots < s_k < t \) are times at which arm \( I_t \) was pulled.

We define the path-dependent dynamic pseudo-regret:

\[
\delta \text{reg}_T = \sum_{t=1}^{T} \max_{i \in [K]} \mathbb{E}[X_{t,i} | X_1^{t-1}] - \mathbb{E}[X_{t,I_t} | X_1^{t-1}],
\] (3)

and we let \( i^*(t) = \arg \max_{i \in [K]} \mathbb{E}[X_{t,i} | X_1^{t-1}] \) denote the best arm to pull after \( t \) rounds. Observe, that sub-additivity of maximum, \( \delta \text{reg}_T \geq \text{reg}_T \).

In the i.i.d. setting, the path-dependent dynamic pseudo-regret, the path-dependent pseudo-regret, and the classical pseudo-regret coincide. However, in non-stationary environments \( \text{reg}_T \) and \( \delta \text{reg}_T \) are finer measure of the algorithm’s performance than \( \text{Reg}_T \), since they take into account past realizations of the stochastic process. Indeed, by Jensen’s inequality, the expectation of the path-dependent pseudo-regret upper-bounds the standard pseudo-regret:

\[
\mathbb{E} \left[ \max_{i \in [K]} \sum_{t=1}^{T} \mathbb{E}[X_{t,i} | X_1^{t-1}] - \mathbb{E}[X_{t,I_t} | X_1^{t-1}] \right]
\geq \max_{i \in [K]} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}[X_{t,i} | X_1^{t-1}] - \mathbb{E}[X_{t,I_t} | X_1^{t-1}] \right]
= \max_{i \in [K]} \sum_{t=1}^{T} \mathbb{E}[X_{t,i}] - \mathbb{E}[X_{t,I_t}] = \text{Reg}_T.
\]

Therefore, any regret guarantee for \( \delta \text{reg}_T \) also imply regret guarantees for \( \text{reg}_T \) and \( \text{Reg}_T \). We summarize relations between different notions of regret in Figure 1. For completeness, we also include the notion of non-path-dependent dynamic pseudo-regret, however, this notion is not used in this paper.

Note that that even though \( \delta \text{reg}_T \) is mathematically well-defined, it is difficult to interpret, since the reward observations in this expression depend on the arms chosen by
the algorithm and imply that the benchmark we compare to depends on the algorithm. Nevertheless, we will use this definition of regret in our analysis, since providing guarantees for this notion of regret allows us to immediately give bounds for $\text{Reg}_s$ and $\text{Reg}_T$.

Finally, in the i.i.d. setting, regret guarantees are often given in terms of gaps $\Delta_t$ between mean reward of the best arm $i^*$ and arm $i$. Our bounds on the dynamic pseudo-regret will depend on the natural extension of this notion that measures gap between the conditional rewards of arm $i$ and of the best arm $i^*(t)$ at time $t$:

$$\Delta_{t,i} = \mathbb{E}[X_{t,i^*(t)}|X_{t-1}^i] - \mathbb{E}[X_{t,i}|X_{t-1}^i].$$

### 3. Discrepancy

The upper confidence bounds used for the reward of each arm in UCB strategies (Lai & Robbins, 1985; Agrawal, 1995; Auer et al., 2002a) are obtained by using concentration inequalities such as Hoeffding’s bound (Hoeffding, 1963). In a general non-stationary scenario, Hoeffding’s inequality is no longer applicable. Instead, we adopt an analysis similar to that of Kuznetsov & Mohri (2015) in the study of time series forecasting.

#### 3.1. Concentration Bound

We start with the observation that here, unlike the i.i.d. scenario, a simple average of the past rewards for arm $i$ may no longer be an informative estimate of its future performance. Instead, we will consider a weighted average $\sum_{s=1}^{t} q_{t,i}(s)X_{s,i}$, where the weights $q_{t,i}(s)$ can help emphasize more relevant past observations and de-emphasize less relevant ones. Since we assume a rested setting, at each time $t$, $q_{t,i}$ is a distribution over $[t]$ supported only by the set of times $s \in [t]$ at which arm $i$ was selected by the algorithm ($I_s = i$) and $q_{t,i}(s)$ is the weight assigned to time $s$. We will denote by $u_{t,i}$ the special case of the uniform distribution over $\{s \in [t] : I_s = i\}$.

At any time $t$, a crucial problem for the learner is then to come up with weights $q_{t,i}(s)$ that can help accurately estimate the rewards of arm $i$ at time $t+1$. The quality of these weights can be measured by the following notion of $q_{t,i}$-weighted discrepancy $D_{t,i}(q_{t,i})$:

$$D_{t,i}(q_{t,i}) = \mathbb{E}[X_{t+1,i}|X_{t}^i] - \sum_{s=1}^{t} q_{t,i}(s) \mathbb{E}[X_{s,i}|X_{t-1}^i],$$

for any $i \in [K]$, $t \in [T]$, and distribution $q_{t,i}$. Figure 2 provides an illustration of this notion of discrepancy and the weights $q_{t,i}$. Discrepancy is the difference between the conditional expectation of the reward of arm $i$ at time $t+1$ and the weighted average of the conditional expectations at all past times $s \leq t$. In our setting, discrepancy can be interpreted as the expected (signed) error of our estimate $\sum_{s=1}^{t} q_{t,i}(s)X_{s,i}$ due to the non-stationarity in the process. In other words, weighted discrepancy measures how well $\sum_{s=1}^{t} q_{t,i}(s)X_{s,i}$ approximates the current reward at time $t+1$ in (conditional) expectation. In the special case where the rewards are drawn i.i.d., the discrepancy values vanish, $D_{t,i}(q_{t,i}) = 0$, and we recover the standard case where a simple average of past observations is an accurate estimate of the current reward. We refer the reader to (Kuznetsov & Mohri, 2015; Zimin & Lampert, 2017) for further examples.

The following is a concentration inequality for the weighted rewards in terms of the weighted discrepancies and the weights $q_{t,i}$, in the general case of non-stationary stochastic processes.

**Proposition 1.** Fix $t \in [T]$ and $i \in [K]$ and a distribution $q_{t,i}$ over $[t]$. Then, for any $\delta > 0$, each of the following inequalities holds with probability at least $1 - \delta$:

$$\mathbb{E}[X_{t+1,i}|X_{t}^i] - \sum_{s=1}^{t} q_{t,i}(s)X_{s,i} - D_{t,i}(q_{t,i}) \leq \|q_{t,i}\| \sqrt{\frac{\log \frac{1}{\delta}}{2}},$$

$$\sum_{s=1}^{t} q_{t,i}(s)X_{s,i} - \mathbb{E}[X_{t+1,i}|X_{t}^i] + D_{t,i}(q_{t,i}) \leq \|q_{t,i}\| \sqrt{\frac{\log \frac{1}{\delta}}{2}}.$$
observed. Thus, we will always assume that the weights \( q_{t,i} \) are supported on time steps \( s \in [t] \) for which \( I_s = i \).

Choosing appropriate distributions \( q_{t,i} \) is crucial for obtaining a useful confidence bound. For instance, assigning all the probability mass to the single most relevant observation can minimize the discrepancy but would lead to a significantly larger slack term \( \|q_{t,i}\| \sqrt{\frac{1}{2} \log \frac{1}{\Delta_i}} \) with \( \|q_{t,i}\| = 1 \). At the other extreme, choosing uniform weights to minimize \( \|q_{t,i}\| \) can result in an unfavorable discrepancy.

### 3.2. Algorithm for Known Discrepancy

In this section, we show that the knowledge of the discrepancies is sufficient for achieving logarithmic regret. In general, the algorithms we consider admit two stages: at each round \( t \), the algorithm first chooses appropriately a distribution \( q_{t,i} \) for each arm \( i \in [K] \), next it uses these distributions to select an arm. Here, we will make the ideal assumption that, for the distribution weights \( q_{t,i} \), the algorithm is choosing in the first stage, it can compute the discrepancies \( D_{t,i}(q_{t,i}) \). Of course, this assumption in general does not hold in practice and we will later omit it.

In light of the concentration bound of Proposition 1, we can then design a Discrepancy-based UCB (DIsc-UCB) algorithm that, in the second stage, selects the arm with the highest upper confidence bound at each round \( t \):

\[
I_{t+1} = \arg\max_{i \in [K]} \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} + D_{t,i}(q_{t,i}) + \|q_{t,i}\| \sqrt{\frac{1}{2} \log t}.
\]

The following results shows that, if in the first stage, the weights are chosen so that \( \|q_{t,i}\|^2 T_{i}(t) \) is bounded by a constant, then DIsc-UCB algorithm admits a logarithmic path-dependent dynamic pseudo-regret guarantee.

**Theorem 1.** Assume that the weights \( q_{t,i} \) are selected for any \( t \in [T] \) and \( i \in [K] \) so that \( \|q_{t,i}\|^2 T_{i}(t) \leq A^2 \), for some positive constant \( A^2 \). Then, the dynamic pseudo-regret of DIsc-UCB, \( \delta_{\text{reg}}(\text{DIsc-UCB}) \), is upper-bounded by

\[
\sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \left[ \max_{t \in [T]} \frac{8A^2 \log(t)}{\Delta^2_{t,i}} + 2 \right].
\]

This bound can be viewed as the counterpart in the non-stationary case of the gap-dependent pseudo-regret bounds for UCB in the i.i.d. scenario. Here, the upper bound is somewhat more complex since the gap between the conditional expectation of the rewards of arm \( i \) and that of the best varies with time. Notice, however, that in the particular case where the gaps are fixed over time, \( \Delta_{t,i} \equiv \Delta_i \) for all \( t \in [T] \), that is the conditional expected rewards of the arms vary in the same way over time, then the regret expression greatly simplifies and each term becomes inversely proportional to a static gap, as in the standard UCB regret bound:

\[
\delta_{\text{reg}}(\text{DISC-UCB}) \leq \sum_{\Delta_i > 0} \left[ \frac{8A^2 \log(T)}{\Delta_i} + 2 \right].
\]

The static gap assumption holds trivially in the i.i.d. scenario but it may also hold in other natural scenarios where the reward distributions shift concomitantly.

Observe also that the theorem suggests that if the learner can compute the discrepancies in the case of the uniform weights, then the choice \( q_{t,i} = u_{t,i} \) is optimal since the upper bound \( A \) can be chosen to be \( A = 1 \) in that case. More generally, this analysis shows that the knowledge of the discrepancies is sufficient for achieving logarithmic regret without any assumption about the family of stochastic processes. In practice, however, the learner is not provided with the true discrepancy measures and needs to either estimate them or exploit some prior knowledge about the family of stochastic processes.

### 4. Bounded Discrepancy

In this section, we no longer assume that the algorithm receives the weighted discrepancies for the weight distributions it has chosen. Instead, we will only assume that the discrepancies are bounded. We give a general algorithm, \( \text{WEn} \text{TUCB} \), for the non-stationary rested bandit problem under that boundedness assumption. We show that when the weights are chosen appropriately and the upper bound on discrepancies is sufficiently small, then \( \text{WEn} \text{TUCB} \) admits logarithmic regret guarantees. We prove a lower bound in the rested setting showing that the dependency of this bound on the gaps cannot be improved in the static gap scenario. We then review a series of applications of our algorithm to scenarios where the family of stochastic processes (not the actual process of course) is known. We show how in all of those cases the weights can be chosen appropriately so that the discrepancies are bounded and so that our algorithm achieves logarithmic regret guarantees.

#### 4.1. Algorithm for Bounded Discrepancy

Recall that the standard UCB algorithm first finds an upper confidence bound on the expected reward of each arm based on a concentration inequality such as Hoeffding’s inequality and then chooses the arm with the largest upper confidence bound. In our setting, Proposition 1 cannot be directly applied since the discrepancy term is unknown. However, suppose for now that the weights \( q_{t,i} \) are chosen to ensure that the discrepancy term is bounded. More precisely, we assume that \( D_{t,i}(q_{t,i}) \leq C\|q_{t,i}\| \sqrt{2 \log(T)} \) (we show in the next section that this assumption holds for many stochastic processes if the weights are chosen carefully). Then, Proposition 1 can be used to define a UCB-type algorithm.
that we will refer to as \textsc{WeightedUCB} and that is defined by the following selection of an arm $I_{t+1}$ at round $(t + 1)$:

$$I_{t+1} = \arg\max_{i \in [K]} \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} + (C + 1)\|q_{t,i}\|\sqrt{2\log t}.$$ 

\textsc{WeightedUCB} admits the following logarithmic path-dependent dynamic pseudo-regret guarantee.

**Theorem 2.** Assume that the weights $q_{t,i}$ are selected for any $t \in [T]$ and $i \in [K]$ so that $\|q_{t,i}\|^2 T_i(t) \leq A^2$ and $|D_{t,i}(q_{t,i})| \leq C\sqrt{\log(t)/T_i(t)}$, for some positive constants $A^2$ and $C$. Then, the dynamic pseudo-regret of \textsc{WeightedUCB}, $\delta_{\text{reg}}(T)(\text{Weighted-UCB})$, is upper-bounded by

$$\sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \left[ \max_{t \in [T]} \frac{8(A+C)^2 \log(t)}{\Delta_{t,i}^2} + 2 \right].$$

Thus, while the discrepancies are no longer assumed to be known, remarkably, the regret bound of the theorem matches (modulo constants) that of Theorem 1. In the next section, we will show that the assumptions of the theorem are satisfied for a number of scenarios examined in the past. Thus, in each case, \textsc{Weighted-UCB} admits a favorable regret guarantee that is always at least as favorable as the state-of-the-art, thereby providing a unified solution.

### 4.2. Lower bound

It is natural to ask if the bound of Theorem 2 is tight and whether it is possible to design algorithms with even more favorable the path-dependent dynamic regret guarantees. In Appendix B, we present a lower bound that showing that, when the discrepancy is bounded, using weights that are not too concentrated as in the assumption of Theorem 2, and when the optimality gaps are static (i.e. $\Delta_{t,i} \equiv \Delta_i$, for all $t \in [T]$), then the regret guarantee of the \textsc{Weighted-UCB} algorithm is optimal. Note that the lower bounds of Besbes et al. (2014b) does not apply to our setting since they analyze the restless scenario.

### 4.3. Applications

In this section, we analyze our algorithm under the assumption that the reward sequences of each arm are realizations of a stochastic process coming from a family that is known to the learner. This setup is adopted in all of the prior work on non-stationary bandits. As mentioned in Section 1, a variety of different processes have been considered in the restless case. For the rested setting, the prior work is significantly more limited. Rotting arms have been studied in (Levine et al., 2017) and Markov arms in (Tekin & Liu, 2012). We show that the discrepancy-based analysis introduced in previous sections allows us not only to straightforwardly recover previous work on rested bandits for Markov and rotting arms but also to easily study many new cases such as changing means, drifting distributions, known trend function, periodic means, rotting arms with jumps and more.

We show that having prior knowledge about the family of stochastic processes (but not the process itself) is enough to help the learner derive the functional form of the discrepancy. This subsequently enables the learner to select weights $q_{t,i}$ that can bound the discrepancy in a way that provides compelling guarantees through Theorem 2. More precisely, we choose weights for each example such that the discrepancy is bounded by $C\sqrt{\log(t)/T_i(t)}$ and such that $\|q_{t,i}\|^2_2 \leq \mathcal{O}(1/\sqrt{T_i(t)})$. If both of these conditions hold, then Theorem 2 implies a logarithmic bound on the path-dependent dynamic pseudo-regret. Remarkably, for some of these examples, these conditions can be fulfilled with uniform weights. This implies that running standard UCB in these non-stationary settings is theoretically sound and admits logarithmic regret.

In what follows, recall that $T_i(t)$ denotes the number of times we chose an arm $i$ up to time $t$ and that in the rested scenario, the stochastic process of an arm remains static unless that arm is chosen.

**I.I.D. process.** If the stochastic process governing the dynamics of each arm is i.i.d. and the $q_{t,i}$s form probability distributions, then the discrepancy term is zero. Thus, we can choose $q_{t,i}(s) = \frac{1}{T_i(t)}$ such that $\|q_{t,i}\|^2_2 = \frac{1}{\sqrt{T_i(t)}}$. Thus, by Theorem 2, the path-dependent dynamic pseudo-regret is in $\mathcal{O}(\log(T))$.

**Complete dependence.** Another extreme scenario is when the rewards are completely correlated, that is $X_{s,i} = X_{s+1,i}$ for all $i$ and all $t$. Then setting $q_{t,i}(s) = \frac{1}{T_i(t)} \rightarrow \delta_{\text{reg}}(T)(\text{UCB})$ leads to $\Delta_{t,i}(q_{t,i}) = 0$ and $\|q_{t,i}\|^2_2 T_i(t) = 1$ and we conclude that the regret of our algorithm is in $\mathcal{O}(\log(T))$.

**Rotting arms.** Levine et al. (2017) analyzed the setting of sub-Gaussian random rewards with mean $\mu_i + r^{-\theta^*_i}$ at time $t$ for $\theta^*_i \in \Theta = \{\theta_1, \theta_2, \ldots\}$. Recalling that $q_{t,i}$ is supported on $T_i(t)$ points, the discrepancy is given by $\Delta_{t,i}(q_{t,i}) = (T_i(t) + 1)^{-\theta^*_i} - \sum_{s=1}^{T_i(t)} q_{t,i}(s)s^{-\theta^*_i}$. We then consider the following parametric form $q_{t,i}(s) = \frac{1}{s^{-\alpha}} \mu_i + r^{-\theta^*_i}$. The discrepancy term can be bounded as follows $\Delta_{t,i}(q_{t,i}) \leq \mathcal{O}(T_i(t)^{-\alpha} - T_i(t)^{1-\alpha}T_i(t)^{\alpha})$. Thus, choosing $\alpha = -1$ results in a favorable discrepancy while at the same time admitting $\|q_{t,i}\|^2_2 = \frac{1}{\sqrt{T_i(t)}}$. By Theorem 2, the path-dependent dynamic pseudo-regret is $\mathcal{O}(\log(T))$.

**Markov chain.** As in (Tekin & Liu, 2012), we assume that the stochastic process for each arm $i$ is a Markov chain with a state space $S_i$. We only consider rewards from the distribution we are trying to predict and evenly distribute mass over them, that is, $q_{t,i}(s) = \sum_{j=1}^{S_i} \frac{1}{S_i} x_{i,j}$, where $X_s$ is the
reward observation at time $s$ of arm $i$. The discrepancy term is zero ($D_{t,i}(q_{t,i}) = 0$) and $\|q_{t,i}\|_2 = \sqrt{\sum_{j=1}^{t-1} 1_{s,j,i} - x_{j,i}}$. If the number of times we visit a state is $\Omega(T_i(t))$, then we attain $O(\log(T))$. We remark that since $q_{t,i}$s are chosen in a data-dependent fashion this result requires a slight extension of Proposition 1 that holds uniformly over all choices of $q_{t,i}$. This extension can be proven following the same arguments as in Theorem 8 of (Kuznetsov & Mohri, 2015).

**Rarely changing means.** For each arm $i$, fix a sequence of means $\mu_{1,i}, \ldots, \mu_{k,i}$ and change points $1 = c_{1,i} < \ldots < c_{M,i} < A$. The rewards of arm $i$ are drawn independently from distributions with means $\mu_{j,i}$ from times $[c_{j,i}, c_{j+1,i})$. This setup has not been previously studied in the setting of non-stationary rested bandits. The most natural form of $q_{t,i}$ would be to distribute the weight uniformly over the samples since the last change. That is, if the last change was at $c_{j,i}$, then $q_{t,i}(s) = \frac{1}{c_{j,i} - c_{j,i} + 1}$ for $c_{j,i} \leq s \leq T_i(t)$ and $q_{t,i}(s) = 0$ for $1 \leq s \leq c_{j,i} - 1$. The discrepancy term is zero: $D_{t,i}(q_{t,i}) = 0$, and $\|q_{t,i}\|_2 = \frac{1}{c_{j,i} - c_{j,i} + 1}$. In the worst case, the last change $c_{j,i}$ is close to $T_i(t)$, meaning $T_i(t) \sim c_{j,i}$. Then $\|q_{t,i}\|_2^2 T_i(t) = \frac{T_i(t)}{c_{j,i} - c_{j,i} + 1} = O(c_{j,i})$, which implies that the regret is $O(A \log(T))$. In the case where the change points are unknown, change point detection algorithms can be used to determine these times.

**Rotting arms with jumps.** We consider an entirely novel setup in which rotting arms can be restarted via a jump. For each arm $i$, fix a sequence of means $\mu_{1,i}, \ldots, \mu_{k,i}$ and change points $1 = c_{1,i} < \ldots < c_{M,i} < A$. The rewards of arm $i$ are sub-Gaussian random variables with mean $\mu_{j,i} + (t - c_{j,i} + 1)^{-\gamma}$ at time $t \in [c_{j,i}, c_{j+1,i})$ for $\Theta^1_i \in \Theta = \{\theta_1, \theta_2, \ldots\}$. Set $q_{t,i}(s) = \frac{T_i(t)}{c_{j,i} - c_{j,i} + 1}$ for $c_{j,i} \leq s \leq T_i(t)$ and $q_{t,i}(s) = 0$ for $1 \leq s \leq c_{j,i} - 1$. As in the previous example, the discrepancy is negligible and the regret of our algorithm is in $O(A \log(T))$.

**Periodic means.** For each arm $i$, let $\mu_{1,i}, \ldots, \mu_{p_i,i}$ be a sequence of reward means. We assume that for each $s$, $X_{p_i,t+j,i}$ is drawn from a distribution with mean $\mu_{j,i}$ for $0 \leq j < p_i$. In other words, in this setting, the means are changing rapidly as opposed to the rarely changing means scenario considered earlier. This scenario has not been studied before in the rested setting. The natural choice of weights is $q_{0,t+j,i}(s) = 1/t$ if $s = p_i k + j$ for some $k$ and zero otherwise. In this case, $D_{t,i}(q_{t,i}) = 0$ and $\|q_{t,i}\|_2^2 T_i(t) \leq 2p_i$. Thus, the regret is $O(\log(T) \max p_i)$.

**Known trend function.** Consider the following setup that has been analyzed before in (Bouneffouf & Féraud, 2016) for the restless setting, but has not been analyzed in the rested setting. Let $R: \mathbb{N} \to \{r_1, \ldots, r_N\}$ be a known trend function. We assume that each $X_{t,i} = X'_{t,i} R(T_i(t))$ where $X'_{t,i}$ are drawn i.i.d. from some distribution with unknown mean $\mu_i$. We choose $q_{t,i}(s) = \frac{1}{R(T_i(s) = R(T_i(t))/\sum_{j=1}^{t-1} 1_{R(T_i(j)) = R(T_i(t))}}$ which achieves zero discrepancy and guarantees $O(\log(T))$ regret provided that $j, R$ take values of $r, \Omega(T_i(t))$ times.

**Drifting means.** To simplify notation, we write $t$ instead of $T_i(t)$ for this example, and summation over $[t]$ is always meant to indicate a summation over the rounds in which arm $i$ is selected. Consider the scenario where $|E[X_{s+1,i}] - E[X_{s,i}]| \leq \delta_t$ for every $s \leq t$, that is, the conditional path-dependent rewards drift over time. We may consider the setting where $\delta_t = \frac{1}{t^\gamma}$ for some $\gamma > 0$. In this case, we can bound the discrepancy as follows:

$$|D_{t,i}(q_{t,i})| = |E[X_{t+1,i}] - \sum_{s=1}^{t} q_{t,i}(s) E[X_{s,i}]|$$

$$\leq \sum_{s=1}^{t} q_{t,i}(s)(t+1-s)\delta_t.$$

We show that choosing $q_{t,i}$ to be the uniform distribution over the most recent $n$ rounds in which arm $i$ was selected and choosing an appropriate $n$ results in a favorable regret bound. For this choice of weights, it follows that

$$|D_{t,i}(q_{t,i})| \leq \sum_{s=1}^{t} q_{t,i}(s)(t+1-s)\delta_t$$

$$\leq \frac{1}{n} \sum_{s=t-n+1}^{t} (t+1-s)\delta_t \leq \delta t n = \frac{n}{t^\gamma}.$$

For our condition on the discrepancy in Theorem 2 to hold, that is for $|D_{t,i}(q_{t,i})| \leq O\left(\sqrt{1/t}\right) \leq O\left(\sqrt{1/n}\right)$ to be satisfied, we must require $n$ to be selected such that $\frac{n}{t^\gamma} = O\left(\sqrt{1/n}\right)$, i.e. $n \leq t^{\frac{2}{\gamma}}$. Thus, we can set $n = t^{\frac{2}{\gamma}}$, which implies that $|q_{t,i}|^2 \leq n^{-1} = t^{-\frac{2}{\gamma}}$. Now, since $|q_{t,i}|^2 T_i(t) = t^{-\frac{2}{\gamma}} T_i(t) \leq t^{-\frac{2}{\gamma} + 1}$, then $|q_{t,i}|^2 T_i(t) \leq O(1)$ for $\gamma \geq \frac{2}{3}$. Since this condition holds and the discrepancy is bounded, the path-dependent dynamic pseudo-regret is in $O(\log(T))$ by Theorem 2. The drifting distributions scenario has not been studied before in either the rested or restless setting.

**Mixed arms.** In this setup, we allow a subset of arms $I$ to follow any of the processes defined above (e.g., Markov) and its complement $I'$ to follow any of the other processes (e.g., rotting arms). Then, for the Markov arms, we can use weights $q$ as described in the Markov example and for the rotting arms, we can use the weights from the rotting example to achieve regret that scales $O(\log(T))$. This approach straightforwardly generalizes to an arbitrary number of subsets. The mixed arms scenario has not been studied previously in either the rested or restless case.
5. Experiments

We report the results of experiments with WEIGHTEDUCB for several of the stochastic processes described in Section 4.3. We compared our algorithm with EXP3 (Auer et al., 2002b), a standard algorithm with provable guarantees for the general adversarial setting. The algorithms were tested for $T = 5,000$ rounds using $K = 150$ arms.

The stochastic processes were generated as follows. For the i.i.d. scenario, the rewards of the $K$ arms were drawn according to the binomial distribution with means in $[1/K, 2/K, \ldots, 1]$. In the rarely changing means setting, for each arm $i$, we first chose the number of times $N_i$ that the mean changes by sampling uniformly in the set $\{1, \ldots, 10\}$. The changing times $c_j$ were then evenly spaced in $\{1 + T/N_i, 1 + 2T/N_i, \ldots, T\}$, and the means $\{\mu_1, \ldots, \mu_{N_i}\}$ chosen from the uniform distribution in $(0, 1)$. Depending on the time $t$ and the time change $c_j$, we drew from the normal distribution $\mathcal{N}(\mu_j, 0.1)$ for each arm $j$. In the rotting arms setting, for each arm $i$, we chose the parameter $\theta_i$ in the set $\{0.1, 0.1 + 10/K, \ldots, 10\}$ and drew the rewards of arm $i$ from the binomial distribution where the mean decays as $t^{-\theta_i}$. For the drifting scenario, the reward of arm $i$ at time $t$ was drawn from the binomial distribution with mean $\mu_{t,i}$. The initial mean $\mu_{0,i}$ of each arm $i$ was drawn from the uniform distribution in $(1-1/\sqrt{T}, 1+1/\sqrt{T})$, where this interval was chosen so that the means cannot drift outside $(0, 1)$. Then for each time $t$, the mean changes by $\mu_{t,i} = \mu_{t-1,i} + b(t, i)/T^{2/3}$ where $b(t, i)$ is a Bernoulli random variable in $\{-1, 1\}$ drawn at each time $t$ for each arm $i$. For known trends, recall that $X_{t,i} = X_{t,i} R(T_i(t))$ for a known trend function $R$. For each arm $i$, the $X_{t,i}$ are drawn from the normal distribution $\mathcal{N}(\mu_i, 0.3)$ with means $\mu_i$ in the set $[0.1, 0.1 + 6/K, \ldots, 6]$. The trend function was defined as follows: $R(T_i(t)) = 0.1$ if $T_i(t)$ mod $3 = 0$, $R(T_i(t)) = 1$ if $T_i(t)$ mod $3 = 1$ and $R(T_i(t)) = 3$ if $T_i(t)$ mod $3 = 2$. For the periodic means, the length of each period is $50$ time steps with a total of three repeating periods, $p_i = 3$, for all arms. The rewards of each arm are drawn from the normal distribution $\mathcal{N}(\mu_{t,i}, 0.3)$ with means $\mu_{1,i} \in [10, 10 + 20/K, \ldots, 20]$ for the first period, $\mu_{2,i} \in [5, 5 + 9/K, \ldots, 9]$ for the second period, and $\mu_{3,i} \in [1, 1 + 4/K, \ldots, 4]$ for the third period.

We tested WEIGHTEDUCB and EXP3 on these stochastic processes. We calculated at each round their reward per round, that is the ratio of the cumulative reward and the number of rounds, and averaged that quantity over ten trials. Figure 3 shows plots of these averages as a function of the number of rounds for each family of stochastic process. WEIGHTEDUCB outperforms EXP3 in all of these tasks. For the i.i.d., drifting and rarely changing means settings, the average cumulative reward of WEIGHTEDUCB is strictly one standard deviation above that of EXP3 after about 2,000 rounds and it appears that its average rewards are steadily increasing. For the rotting arms, the rewards steadily decrease as expected but WEIGHTEDUCB achieves higher average cumulative rewards for a longer period. For periodic means, WEIGHTEDUCB exploits the higher means for all arms in the first period before moving on to the next period and for known trends, WEIGHTEDUCB quickly finds a winning strategy with a high reward.

6. Conclusion

We presented an analysis of the non-stationary multi-armed bandit problem in the rested setting. We introduced algorithms leveraging the notion of discrepancy, for which we proved regret guarantees in different scenarios: known discrepancies, bounded discrepancies, and known families of stochastic processes. For the first scenario, we show that it is always possible to logarithmic regret, and for the second scenario, we designed an algorithm that, under certain assumptions, guarantees logarithmic regret, and we further illustrated situations in which the guarantees of the algorithm are provably optimal. For the third scenario, we studied several instances for which we showed that WEIGHTEDUCB achieves a logarithmic regret. We also showed empirically that this algorithm outperforms EXP3 for those instances.
References

Agrawal, Rajeev. Sample mean based index policies with $\log(n)$ regret for the multi-armed bandit problem. In *Applied Mathematics*, 1995.

Arora, Raman, Dekel, Ofer, and Tewari, Ambuj. Online bandit learning against an adaptive adversary: from regret to policy regret. In *ICML*, 2012.

Audiffren, Julien and Ralaivola, Liva. Cornering stationary and restless mixing bandits with remix-UCB. In *NIPS*, 2015.

Auer, Peter, Cesa-Bianchi, Nicolò, and Fischer, Paul. Finite-time analysis of the multi-armed bandit problem. In *MLJ*, 2002a.

Auer, Peter, Cesa-Bianchi, Nicolò, Freund, Yoav, and Schapire, Robert. The non-stochastic multi-armed bandit problem. In *SIAM Journal on Computing*, 2002b.

Besbes, Omar, Gur, Yonatan, and Zeevi, Assaf. Optimal exploration-exploitation in a multi-armed bandit problem with non-stationary rewards. Technical report, Stanford University, Graduate School of Business, 2014a.

Besbes, Omar, Gur, Yonatan, and Zeevi, Assaf. Stochastic multi-armed bandit problem with non-stationary rewards. In *NIPS*, 2014b.

Bouneffouf, Djallel and Féraud, Raphaël. Multi-armed bandit problem with known trend. In *Neurocomputing*, 2016.

Garivier, Aurelien and Moulines, Eric. On upper-confidence bound policies for non-stationary bandit problems. In *ALT*, 2011.

Hoeffding, Wassily. Probability inequalities for sums of bounded random variables. In *Journal of the American Statistical Association*, 1963.

Kuznetsov, Vitaly and Mohri, Mehryar. Learning theory and algorithms for forecasting non-stationary time series. In *NIPS*, 2015.

Lai, T.L. and Robbins, Herbert. Asymptotically efficient adaptive allocation rules. In *Applied Mathematics*, 1985.

Levine, Nir, Crammer, Koby, and Mannor, Shie. Rotting bandits. In *ArXiv*, 2017.

Ortner, Ronald, Daniil, Ryabko, Peter, Auer, and Munos, Rémi. Regret bounds for restless Markov bandits. In *Theoretical Computer Science*, 2014.

Pandey, Sandeep, Chakrabarti, Deepayan, and Agarwal, Deepak. Multi-armed bandit problems with dependent arms. In *ICML*, 2007.

Raj, Vishnu and Kalyani, Sheetal. Taming non-stationary bandits: A Bayesian approach. In *Arxiv*, 2017.

Slivkins, Aleksandrs and Upfal, Eli. Adapting to a changing environment: the Brownian restless bandits. In *COLT*, 2008.

Tekin, Cem and Liu, Mingyan. Adaptive learning of uncontrolled restless bandits with logarithmic regret. In *Annual Allerton Conference, IEEE*, 2011.

Tekin, Cem and Liu, Mingyan. Online learning of rested and restless bandits. In *IEEE Transactions on Information Theory*, 2012.

Whittle, Peter. Arm-acquiring bandits. In *Annals of Probability*, 1981.

Zimin, Alexander and Lampert, Christoph. Learning theory for conditional risk minimization. In *NIPS*, 2017.
A. Path-dependent dynamic pseudo-regret analysis

In this section, we provide the proofs for our results concerning path-dependent dynamic pseudo-regret. We begin with the following concentration bound.

**Proposition 1.** Fix \( t \in [T] \) and \( i \in [K] \). Let \( q_{t,i} \) be a fixed distribution over \([t]\). Then, for any \( \delta > 0 \), each of the following inequalities holds with probability at least \( 1 - \delta \):

\[
\mathbb{E}[X_{t+1,i} | X_1^i] - \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} - D_{t,i}(q_{t,i}) \leq \|q_{t,i}\| \sqrt{\frac{1}{2} \log \frac{1}{\delta}}
\]

\[
- \mathbb{E}[X_{t+1,i} | X_1^i] + \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} + D_{t,i}(q_{t,i}) \leq \|q_{t,i}\| \sqrt{\frac{1}{2} \log \frac{1}{\delta}}.
\]

**Proof.** Define \( V_s = q_{t,i}(s)(\mathbb{E}[X_{s,i} | X_1^{s-1}] - X_{s,i}) \) for \( s \in [t] \). Since \( q_{t,i} \) is a fixed distribution, \( \mathbb{E}[q_{t,i}(s)|X_1^{s-1}] = q_{t,i}(s) \), and so \( \mathbb{E}[V_s | X_1^{s-1}] = 0 \). Thus, \( \{V_s\}_{s=1}^{t} \) is a martingale difference sequence with respect to the filtration induced by the random variables \( \{X_{s,i}\}_{s=1}^{t} \). Since the rewards \( X_{s,i} \) are bounded in \([0, 1] \), the martingale difference sequence satisfies:

\[
0 \leq V_s \leq q_{t,i}(s)
\]

for each \( s \in [t] \). Thus, by Azuma’s inequality,

\[
\sum_{s=1}^{t} q_{t,i}(s) \mathbb{E}[X_{s,i} | X_1^{s-1}] - \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} \leq \|q_{t,i}\| \sqrt{\frac{1}{2} \log \frac{1}{\delta}}.
\]

By introducing the term \( \mathbb{E}[X_{t+1,i} | X_1^i] \) and using the definition of discrepancy,

\[
\mathbb{E}[X_{t+1,i} | X_1^i] - \sum_{s=1}^{t} q_{t,i}(s) X_{s,i} - \mathbb{E}[X_{t+1,i} | X_1^i] + \sum_{s=1}^{t} q_{t,i}(s) \mathbb{E}[X_{s,i} | X_1^{s-1}] \leq \|q_{t,i}\| \sqrt{\frac{1}{2} \log \frac{1}{\delta}},
\]

and we attain the first bound in the theorem. The second inequality follows by a symmetric argument. \( \square \)

For ease of notation in the arguments that follow, we define the following quantities:

\[
\mu_t^i = \mathbb{E}[X_{t,i} | X_1^{t-1}], \quad \tilde{\mu}_t^i = \sum_{s=1}^{t} q_{t,i}(s) X_{s,i}, \quad D_t^i = D_{t,i}(q_{t,i}), \quad S_t^i = \|q_{t,i}\| \sqrt{2 \log (t)}.
\]

We also remind the reader that \( X_{t,i} \) is actually the \( T_i(t) \)-th reward of arm \( i \) and that \( q_{t,i} \) is only supported on \( T_i(t) \) time steps in \([t]\). These facts will be important for understanding the proofs of the following theorems. An easy extension of the following theorems is to derive regret bounds for the non-path dependent dynamic pseudo-regret.

**Theorem 1.** Assume the weights \( q_{t,i} \) are selected for any \( t \in [T] \) and \( i \in [K] \) so that \( \|q_{t,i}\| T_i(t) \leq A^2 \), for positive constant \( A^2 \). Then, the dynamic pseudo-regret of \( \text{DISC-UCB}_{T}(\delta_{reg,T}(\text{DISC-UCB})) \), is upper-bounded by:

\[
\sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \left[ \max_{t \in [T]} \frac{8 A^2 \log(t)}{A^2_{t,i}} + 2 \right].
\]

**Proof.** We begin by decomposing the path-dependent dynamic pseudo-regret as follows:

\[
\sum_{t=1}^{T} \mathbb{E}[X_{t,i^*(t)} | X_1^{t-1}] - \mathbb{E}[X_{t,i_t} | X_1^{t-1}] = \sum_{t=1}^{T} \sum_{i=1}^{K} \left( \mathbb{E}[X_{t,i^*(t)} | X_1^{t-1}] - \mathbb{E}[X_{t,i} | X_1^{t-1}] \right) \mathbb{E}[1_{i_t=i} | X_1^{t-1}]
\]

\[
\leq \sum_{i=1}^{K} \max_{t \in [T]} \left( \mathbb{E}[X_{t,i^*(t)} | X_1^{t-1}] - \mathbb{E}[X_{t,i} | X_1^{t-1}] \right) \sum_{t=1}^{T} \mathbb{E}[1_{i_t=i} | X_1^{t-1}].
\]

This last expression allows us to work with each arm \( i \) separately, so that we can focus on bounding \( \sum_{t=1}^{T} \mathbb{E}[1_{i_t=i} | X_1^{t-1}] \) for each arm \( i \).
In order to bound this sum, we split the expectation according to the events $T_i(t-1) \leq s_i$ and $T_i(t-1) > s_i$ where $s_i$ is a quantity to be determined.

$$\sum_{t=1}^{T} \mathbb{E}[1_{I_t=i}|X_t^{t-1}] = \sum_{t=1}^{T} \mathbb{E}[1_{I_t=i,T_i(t-1)\leq s_i} + 1_{I_t=i,T_i(t-1)\geq s_i}|X_t^{t-1}] \leq s_i + \sum_{t=s_i+1}^{T} \mathbb{E}[1_{I_t=i,T_i(t-1)\geq s_i}|X_t^{t-1}] .$$

We will show that for $s_i$ sufficiently large, the expression $\sum_{t=s_i+1}^{T} \mathbb{E}[1_{I_t=i,T_i(t-1)\geq s_i}|X_t^{t-1}]$ will be bounded by a mild constant independent of $T$.

Note first that if $I_t = i$, then by the design of the DISC-UCB algorithm, it must be the case that:

$$\hat{\mu}_i^{t-1} + D_i^{t-1} + S_i^{t-1} \geq \hat{\mu}_i^{t-1} + D_i^{t-1} + S_i^{t-1} ,$$

where for simplicity, we have written $i^*$ instead of $i^*(t-1)$.

By rearranging this inequality, and introducing the true conditional rewards $\mu_i^{t-1}$ and $\mu_i^{t-1}$, this implies that

$$\hat{\mu}_i^{t-1} + D_i^{t-1} + S_i^{t-1} - \mu_i^{t-1} + \mu_i^{t-1} + S_i^{t-1} - S_i^{t-1} - [\hat{\mu}_i^{t-1} + D_i^{t-1} + S_i^{t-1}] + \mu_i^{t-1} - \mu_i^{t-1} \geq 0 .$$

We can split this last inequality into three components:

$$\left[\hat{\mu}_i^{t-1} - \mu_i^{t-1} + D_i^{t-1} - S_i^{t-1}\right] + \left[2S_i^{t-1} - \Delta_{t-1,i}\right] + \left[\mu_i^{t-1} - \mu_i^{t-1} - D_i^{t-1} - S_i^{t-1}\right] \geq 0 .$$

Now, if we can show that the middle term in the above inequality is negative, then the first and third terms must be positive, which we will show can only occur with low probability. Intuitively, the middle term will be negative when the slack term $S_i^{t-1}$ is small enough, i.e. when we pull arm $i$ enough times. This amounts to choosing $s_i$, the threshold for $T_i(t-1)$ in our decomposition, such that the middle term is negative.

By assumption, $\|q_{t-1,i}\|^2 T_i(t-1) \leq A^2$. Thus, by choosing

$$s_i = \max_{t \in [T]} \frac{8A^2 \log(t)}{\Delta_{t-1,i}^2} ,$$

we find that

$$T_i(t-1) \geq s_i$$

$$\Rightarrow T_i(t-1) \geq \frac{8 \log(t-1) \|q_{t-1,i}\|^2 T_i(t-1)}{\Delta_{t-1,i}^2}$$

$$\Rightarrow \Delta_{t-1,i} - 2S_i^{t-1} > 0 ,$$

where we used the fact that $A^2 \geq \|q_{t-1,i}\|^2 T_i(t-1)$.

Thus, we can bound the number of times expert $i$ has been chosen by

$$s_i + \sum_{t=s_i+1}^{T} \mathbb{E}[1_{I_t=i,T_i(t-1)\geq s_i}|X_t^{t-1}] \leq s_i + \sum_{t=s_i+1}^{T} \mathbb{P}\left[-\mu_i^{t-1} + \mu_i^{t-1} + D_i^{t-1} - S_i^{t-1} \geq 0\|X_t^{t-1}\| + \mathbb{P}\left[\mu_i^{t-1} - \mu_i^{t-1} - D_i^{t-1} - S_i^{t-1} \geq 0\|X_t^{t-1}\| \right]$$

Now recall that $q_{t-1,i}$ is supported on $T_i(t-1)$ time steps in $[t-1]$. Since we do not know $T_i(t-1)$ and $q_{t-1,i}$ can change depending on its value, we apply a union bound over all possible realizations of $T_i(t-1)$. For the following bound, we will write $\hat{\mu}_i^{t-1} = \hat{\mu}_i^{t-1,T_i(t-1)}$, $s_i^{t-1} = S_i^{t-1,T_i(t-1)}$, and $D_i^{t-1} = D_i^{t-1,T_i(t-1)}$ to reflect the dependency of each of these
terms on \( q_{t-1,i} \) and hence on \( T_i(t-1) \). Now, by choosing \( \delta = (t-1)^{-4} \) in Proposition 1, we can bound the first of the two probabilities above in the following way:

\[
\mathbb{P}[-\mu_i^{t-1} + \mu_i^{t-1,T_i(t-1)} + D_i^{t-1,T_i(t-1)} - S_i^{t-1,T_i(t-1)} \geq 0 | X_i^{t-1}] \\
\leq \mathbb{P}\{s \in [1, t-1] : -\mu_i^{t-1} + \mu_i^{t-1,s} + D_i^{t-1,s} - S_i^{t-1,s} \geq 0 | X_i^{t-1}\} \\
\leq \sum_{s=1}^{t-1} \frac{1}{(t-1)^2}.
\]

By a similar reasoning, we can bound \( \mathbb{P}[\mu_i^{t-1} - \mu_i^{t-1} - D_i^{t-1} - S_i^{t-1} \geq 0 | X_i^{t-1}] \leq \sum_{s=1}^{t-1} \frac{1}{(t-1)^2}. \)

Putting the above bounds together, we can write

\[
s_i + \sum_{t=s_i+1}^{T} \mathbb{E}[1_{I_t=i|1_{T_i(t-1)>s_i}|X_i^{t-1}] \leq s_i + \sum_{t=s_i+1}^{T} \frac{1}{(t-1)^2} \leq s_i + 2
\]

By summing over all the arms \( i \), the path-dependent dynamic pseudo-regret can be bounded as follows:

\[
\delta_{\text{reg}} \leq \sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \sum_{t=1}^{T} \mathbb{E}[1_{I_t=i}|X_i^{t-1}] \leq \sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] |s_i + 2|
\]

\[
= \sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \left[ \max_{t \in [T]} \frac{8 A^2 \log(t)}{\Delta_{t,i}^2} + 2 \right].
\]

We next present a proof of Theorem 2, which bounds the path-dependent dynamic pseudo-regret incurred by the Weighted-UCB algorithm.

**Theorem 2.** Assume the weights \( q_{t,i} \) are selected for any \( t \in [T] \) and \( i \in [K] \) so that \( \|q_{t,i}\|^2 T_i(t) \leq A^2 \) and \( |D_{t,i}(q_{t,i})| \leq C \sqrt{\log(t)/T_i(t)} \), for positive constants \( A^2 \) and \( C \). Then, the dynamic pseudo-regret of Weighted-UCB, \( \delta_{\text{reg}}(\text{Weighted-UCB}) \), is upper-bounded by:

\[
\delta_{\text{reg}} \leq \sum_{i=1}^{K} \left[ \max_{t \in [T]} \Delta_{t,i} \right] \left[ \max_{t \in [T]} \frac{8 (A + C)^2 \log(t)}{\Delta_{t,i}^2} + 2 \right].
\]

**Proof.** By following similar arguments as in the beginning of the proof of Theorem 1, we can focus on bounding \( \mathbb{E}[1_{I_t=i|T(t-1)>s_i}|X_i^{t-1}] \).

If \( I_{t+1} = i \), then by construction of the Weighted-UCB algorithm, it must be the case that

\[
\hat{\mu}_i^{t-1} + (1 + C) S_i^{t-1} \geq \hat{\mu}_i^{t-1} + (1 + C) S_i^{t-1}
\]

By introducing the true conditional rewards \( \mu_i^{t} \) and \( \mu_i^{t} \), and the discrepancies \( D_i^{t-1} \) and \( D_i^{t-1} \), this implies that

\[
\hat{\mu}_i^{t-1} + (1 + C) S_i^{t-1} - \left( \hat{\mu}_i^{t-1} + (1 + C) S_i^{t-1} \right) \\
- \mu_i^{t-1} + \mu_i^{t-1} + \mu_i^{t-1} - S_i^{t-1} + S_i^{t-1} - D_i^{t-1} + D_i^{t-1} - D_i^{t-1} + D_i^{t-1} \geq 0.
\]

We can split this last inequality into the following three components:

\[
(\hat{\mu}_i^{t-1} - \mu_i^{t-1} + D_i^{t-1} - S_i^{t-1}) + (\mu_i^{t-1} + \mu_i^{t-1} + (2 + C) S_i^{t-1} - C S_i^{t-1} - D_i^{t-1} + D_i^{t-1}) \\
+ (\mu_i^{t-1} - \mu_i^{t-1} - D_i^{t-1} - S_i^{t-1}) \geq 0.
\]
By our assumption on the discrepancies \(|D_{t-1,i}(q_{t-1,i})|\) ≤ \(C \sqrt{\log(t-1)/T_i(t-1)}\) and the fact that \(||q_{t-1,i}||^2 T_i(t-1)\leq A^2\) implies \(S_{i}^{t-1} \leq A \sqrt{2 \log(t-1)/T_i(t-1)}\), it follows that
\[
\left(\hat{\mu}_i^{t-1} - \mu_i^{t-1} + D_i^{t-1} - S_i^{t-1}\right) + \left(-\mu_i^t + \mu_i^{t-1} + 2(A + C) \sqrt{\frac{2 \log(t-1)}{T_i(t-1)}}\right) + \left(-\hat{\mu}_i^t + \mu_i^t - D_i^t - S_i^t\right) \geq 0.
\]
As in the proof of Theorem 1, the first and third term can be shown to be negative with high probability. Thus, it remains to show that if we choose arm \(i\) enough times, then the middle term is negative as well.

Now define
\[
s_i = \max_{t \in [T]} \frac{8(A + C)^2 \log(t)}{\Delta_{i,t}^2}.
\]
Then, if \(T_i(t-1) \geq s_i\), it follows that
\[
T_i(t-1) \geq \max_{t \in [T]} \frac{8(A + C)^2 \log(t)}{\Delta_{i,t}^2}
\]
\[
\Rightarrow T_i(t-1) \geq \frac{8(A + C)^2 \log(t-1)}{\Delta_{i,t-1}^2}
\]
\[
\Rightarrow 0 \geq -\mu_i^{t-1} + \mu_i^{t-1} + 2(A + C) \sqrt{\frac{2 \log(t-1)}{T_i(t-1)}}.
\]

The remainder of the proof follows by the same reasoning as in the proof of Theorem 1.

**B. Lower bound**

In this section, we present a lower bound on the path-dependent dynamic regret of the non-stationary multi-armed bandit problem in the rested setting. In particular, we show that for every algorithm which guarantees sub-polynomial path-dependent dynamic regret, there exists an environment with unit-variance Gaussian rewards for which the algorithm must incur at least logarithmic path-dependent dynamic regret.

We first introduce some definitions which will formalize the notions of algorithm and environment in a functional and probabilistic sense.

**Definition 1** (bandit environment). Given \(K \in \mathbb{N}\), a \(K\)-arm bandit environment \(\nu\) is a distribution over sequences of random vectors (i.e. rewards) of dimension \(K\): \((X_1, X_2, \ldots)\). We denote the set of all \(K\)-arm bandit environments by \(\mathcal{E}_K\), and we denote the set of all \(K\)-arm bandit environments with unit-variance Gaussian rewards by \(\mathcal{E}_{K,N}^\star\).

**Definition 2** (algorithm). An algorithm \(A\) is a function that maps sequences of arms and rewards to (potentially trivial) distributions over arms: \(([K] \times \mathbb{R})^* \rightarrow \Delta_K\).

We denote the distribution output by the algorithm at time \(t\) by \(\pi_t = \pi_t([I_1, X_1, \ldots, I_{t-1}, X_{t-1}])\).

Thus, the path-dependent dynamic regret \(\delta_{\text{reg}}(\nu, A)\) will always refer to a specific bandit environment \(\nu\) and an algorithm \(A\), i.e. \(\delta_{\text{reg}}(\nu, A)\).

**Definition 3** (algorithmic consistency). An algorithm \(A\) is consistent over the set of bandit environments \(\mathcal{E}\) if for all non-stationary bandit environments \(\nu \in \mathcal{E}\) and for all \(p > 0\), it holds that \(\delta_{\text{reg}}(\nu, A) = O(n^p)\) as \(T \rightarrow \infty\).

**Theorem 3.** For any algorithm \(A\) consistent over \(K\)-action unit-variance Gaussian environments, \(\mathcal{E}_{K,N}\) and any \(\nu \in \mathcal{E}_{K,N}\), it holds that
\[
\liminf_{T \rightarrow \infty} \frac{\delta_{\text{reg}}(\nu, A)}{\log(T)} \geq \sum_{i : \text{max}_{t \in [T]} \Delta_{i,t} > 0} \frac{2}{\text{max}_{t \in [T]} \Delta_{i,t}}.
\]

**Proof.** Let \(A\) be a consistent algorithm \(A\), and let \(\nu \in \mathcal{E}_{K,N}\) be an arbitrary unit-variance Gaussian environment \(\nu \in \mathcal{E}_{K,N}\) such that at each time \(t \in [T]\), the rewards are drawn from the product distribution \((P_{t,1}, \ldots, P_{t,K})\) with means \(\mu_{t,i}\) for each arm \(i \in [K]\) and where \(P_{t,i} = P_{t}([X_{t,i}])\).
We can decompose the regret as

\[ \delta_{\text{reg}}(\nu, A) = \sum_{i \in [K]} \sum_{t=1}^{T} \mathbb{E}[1_{I_t = i} | X_{t-1}^t] \Delta_{t,i}, \]

and so we focus on lower bounding

\[ \sum_{t=1}^{T} \mathbb{E}[1_{I_t = i} | X_{t-1}^t] \Delta_{t,i} \]

for each arm \( i \in [K] \) that is suboptimal at least at some time steps \( t \in [T] \).

Now, let \( N_t \) be the set of time steps in which arm \( i \) is suboptimal. Consider an environment \( \nu' \in \mathcal{E}_{K,N} \) where the distribution at time \( t \) is given by \( (P_{t,i}^1, \ldots, P_{t,K}^t) \), where \( P_{t,i}^t = N(\mu_{t,i} + \lambda_{t,i}, 1) \) with \( \lambda_{t,i} \geq \Delta_{t,i} \) to be selected later for each time \( t \in [N_t] \).

Let \( \mathcal{P}_{\nu,A} \) be the joint distribution of the actions and rewards in the game determined by environment \( \nu \) and algorithm \( A \).

Now for each \( t \in [T] \), define the random variable \( T_t(t) = \sum_{i=1}^{t} 1_{I_t = i, \Delta_{t,i} > 0} = \sum_{s \in N_t \cap [t]} 1_{I_s = i} \), and consider the event \( A_t = \{ T_t(T) \geq T/2 \} \). Then by Pinsker’s inequality, it follows that

\[ D(\mathcal{P}_{\nu,A}, \mathcal{P}_{\nu',A}) \geq \frac{1}{2} \log \left( \frac{\mathbb{P}_{\nu,A}(A_t)}{\mathbb{P}_{\nu',A}(A_t)} \right). \]

Notice that arm \( i \) is suboptimal in \( \nu \) but optimal in \( \nu' \) at each of the times \( t \in N_t \). Moreover, for any \( j \neq i \) and \( t \in [N_t] \), \( \Delta_{t,j} \geq \lambda_{t,i} - \Delta_{t,i} \). Thus, it follows that \( \delta_{\text{reg}}(\nu, A) \geq \frac{T}{2} \min_{t \in N_t} \Delta_{t,i} \mathcal{P}_{\nu,A}(A_t) \) and \( \delta_{\text{reg}}(\nu', A) \geq \frac{T}{2} \min_{t \in N_t} (\lambda_{t,i} - \Delta_{t,i}) \mathcal{P}_{\nu',A}(A_t) \).

Let \( k_i = \frac{\min_{t \in [N_t]} \Delta_{t,i}}{\min_{t \in [N_t]} (\lambda_{t,i} - \Delta_{t,i})} \). Then, \( \frac{\delta_{\text{reg}}(\nu, A) + \delta_{\text{reg}}(\nu', A)}{T k_i} \geq \mathcal{P}_{\nu,A}(A_t) + \mathcal{P}_{\nu',A}(A_t) \).

Now, we claim that \( D(\mathcal{P}_{\nu,A}, \mathcal{P}_{\nu',A}) = \sum_{i=1}^{K} \sum_{t=1}^{T} \mathbb{E}_{\nu,A}[1_{I_t = i} | X_{t-1}^t] D(P_{t,i}, P_{t,i}') \).

This follows because

\[ D(\mathcal{P}_{\nu,A}, \mathcal{P}_{\nu',A}) = \mathbb{E}_{\nu,A} \left[ \log \frac{\mathcal{P}_{\nu,A}}{\mathcal{P}_{\nu',A}} \right] \]

\[ = \mathbb{E}_{\nu,A} \left[ \log \sum_{t=1}^{T} P_{t,i}(X_{t-1}^t) P_{t,i}'(X_{t-1}^t) \right] \]

\[ = \sum_{i=1}^{K} \mathbb{E}_{\nu,A} \left[ \sum_{t=1}^{T} \log \frac{P_{t,i}(X_{t-1}^t)}{P_{t,i}'(X_{t-1}^t)} \right] \]

by definition of relative entropy once we condition on \( I_t \). Furthermore, by decomposing this last expression over arms, we can write

\[ \sum_{i=1}^{K} \sum_{t=1}^{T} \mathbb{E}_{\nu,A}[1_{I_t = i} D(P_{t,i}, P_{t,i}')] = \sum_{i=1}^{K} \sum_{t=1}^{T} \mathbb{E}_{\nu,A} \left[ 1_{I_t = i} D(P_{t,i}, P_{t,i}') | X_{t-1}^t \right] \]

\[ = \sum_{i=1}^{K} \sum_{t=1}^{T} \mathbb{E}_{\nu,A} \left[ 1_{I_t = i} | X_{t-1}^t \right]. \]
Since the claim holds, the fact that $D(P_{t,i}, P_{t,i}') = \frac{\lambda_{t,i}^2}{2}$ and $D(P_{t,j}, P_{t,j}') = 0$ for $j \neq i$ imply that
\[
\sum_{t=1}^{T} \frac{\lambda_{t,i}^2}{2} \mathbb{E}_{\nu,A}[1_{I_t=i}] \geq \text{log} \left( \frac{k_i}{2} \right) \frac{T}{\text{reg}_T(\nu, A) + \text{reg}_T(\nu', A)}.
\]
\[
= \text{log} \left( \frac{k_i}{2} \right) + \text{log}(T) - \text{log}(\text{reg}_T(\nu, A) + \text{reg}_T(\nu', A)).
\]
Dividing by $\log(T)$, it follows that
\[
\sum_{t=1}^{T} \frac{\lambda_{t,i}^2}{2} \mathbb{E}_{\nu,A}[1_{I_t=i}] \geq 1 - \frac{\text{log}(\text{reg}_T(\nu, A) + \text{reg}_T(\nu', A))}{\log(T)}.
\]
Now, by assumption, for any $p > 0$, there exists $A > 0$ such that $\text{reg}_T(\nu, A) + \text{reg}_T(\nu', A) \leq A_n^p$. Thus,
\[
\limsup_{T \to \infty} \frac{\text{log}(\text{reg}_T(\nu, A) + \text{reg}_T(\nu', A))}{\log(T)} \leq p.
\]
Letting $p \to 0$, it follows that
\[
\liminf_{T \to \infty} \max_{t \in N} \frac{\lambda_{t,i}^2}{2} \mathbb{E}_{\nu,A}[1_{I_t=i}] \geq 1.
\]
By taking the infimum over both sides over $\lambda_{t,i} \geq \Delta_{t,i}$,
\[
\liminf_{T \to \infty} \max_{t \in N} \frac{\lambda_{t,i} \sum_{t=1}^{T} \mathbb{E}_{\nu,A}[1_{I_t=i}]}{\log(T)} \geq 1.
\]
Thus, by repeating the same argument for all $i \in [K]$ and summing,
\[
\liminf_{T \to \infty} \frac{\text{reg}_T(\nu, A)}{\log(T)} \geq \sum_{i \in [K]} \frac{1}{\max_{t: \Delta_{t,i} > 0} \Delta_{t,i}}.
\]
Notice that Theorem 2 implies that when a bandit problem is such that the discrepancy is bounded using weights that are not too concentrated as in the assumption of the theorem, then the WeightedUCB algorithm guarantees sub-polynomial regret. Thus, the lower bound in Theorem 3 applies to the algorithm. If the optimality gaps $\Delta_{t,i}$ are static, then the upper and lower bounds are of the same order, from which we can assert that the WeightedUCB algorithm is optimal.