Comparison between test-particle simulations and test-particle theories for cosmic ray transport: III. Dynamical turbulence

To cite this article: M Gammon et al 2019 J. Phys. Commun. 3 015016

View the article online for updates and enhancements.
Comparison between test-particle simulations and test-particle theories for cosmic ray transport: III. Dynamical turbulence

M Gammon, M Heusen and A Shalchi
Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
E-mail: andreasm4@yahoo.com

Keywords: energetic particles, quasi-linear theory, isotropic turbulence

Abstract
We explore analytically and numerically the motion of energetic particles such as electrons and protons through space. The electrically charged particles interact with a large scale or mean field $\vec{B}_0$ and a turbulent component $\delta \vec{B}$ leading to a complicated stochastic motion. This type of physical scenario is important in plasma physics as well as particle astrophysics. Years ago a quasi-linear theory for particle transport was developed and applied in hundreds of research papers. Whereas it became clear that quasi-linear theory does not work for the transport of energetic particles across a large scale field, it is still unclear for which parameter regimes the theory works if diffusion along that field is explored. In the current paper, we therefore combine quasi-linear theory with dynamical isotropic turbulence and different turbulence spectra. The obtained results are then compared with test-particle simulations. We show that in the general case, quasi-linear theory does not provide an accurate description of parallel transport for isotropic turbulence even if dynamical turbulence effects are included.

1. Introduction
A well-known problem in theoretical physics is the transport of energetic particles through a plasma. Examples are plasmas in fusion devices such as tokamaks, the solar wind plasma, and the interstellar medium. Particle transport in the interplanetary space is of particular interest since the properties of energetic particles such as solar energetic particles and cosmic rays can be measured in situ by using space probes.

To understand the motion of energetic particles through a plasma theoretically is challenging because simple models such as so-called Bohm diffusion do not work except in extreme parameter regimes (see, e.g. Hussein and Shalchi 2014). The standard approach for investigating particle transport is the application of quasi-linear theory (QLT, see Jokipii 1966). In the comprehensive work of Bieber et al (1994), for instance, the latter theory was used to reproduce interplanetary measurements of solar energetic particles and cosmic rays. However, those investigations focused on diffusion of particles along the solar magnetic field and they were based on a specific turbulence configuration (e.g. two component turbulence).

We explore the transport of energetic particles in a systems which consists of a mean magnetic field $\vec{B}_0$ and a turbulent component $\delta \vec{B}$. The former field deflects the particles and forces them to follow a helical motion. The turbulent field scatters the particles leading to a stochastic motion. Furthermore, the mean field breaks the symmetry of our system. Therefore, we need to distinguish between transport along and across that field. The associated diffusion parameters are usually called the parallel and perpendicular diffusion coefficients, respectively.

In more recent years it was shown that quasi-linear theory is incomplete if used to study perpendicular diffusion (see, e.g. Shalchi 2009 for a review). Perpendicular transport is a complicated non-linear process which is based on several effects such as particle trajectories being influenced by random walking magnetic field lines, a strong influence of parallel diffusion itself, and particles being scattered away from field lines. Furthermore, the field lines themselves need to be described non-linearly in the general case (see, e.g. Matthaeus et al 1995).
Therefore, perpendicular transport needs to be described by non-linear theories (see, e.g. Matthaeus et al. 2003, Shalchi 2010, 2015, 2017). Quasi-linear theory is only valid for small values of the so-called Kubo number and for very long parallel mean free paths, usually corresponding to high particle energies.

The question remains whether quasi-linear theory can be used to compute the diffusion parameter along the mean magnetic field. Well-known is that there are some limitations (see again Shalchi 2009 for a review). Quasi-linear theory fails, for instance, if a turbulence model with dominant two-dimensional character is considered (see, e.g. Shalchi et al. 2004 and Qin et al. 2006). Furthermore, quasi-linear theory does not work for pitch-angle scattering if the pitch-angle is close to 90° (see, e.g. Völkl 1973, Jones et al. 1973, Owens 1974, Völkl 1975, Goldstein 1976, Jones et al. 1978, and Kaiser et al. 1978). Therefore, a so-called second order quasi-linear theory was proposed in Shalchi (2005). The latter theory takes into account orbit fluctuations rather than unperturbed orbits as used in quasi-linear theory. As a consequence, one finds a broadened resonance condition compared to the Dirac delta found within quasi-linear theory. Second order QLT, however, was derived for magnetostatic turbulence. In realistic physical scenarios such as the solar wind, one expects to find dynamical effects. Time-dependent magnetic fields can be related to the propagation of magnetohydrodynamic waves (e.g. shear Alfvén waves) or different damping effects (see, e.g. Schlickeiser 2002 for more details). The latter effects also broaden the resonance similar compared to resonance broadening due to non-linearities.

Up to date it remains unclear whether quasi-linear theory works for parallel diffusion in realistic time-dependent turbulence. It is the purpose of the current article to combine quasi-linear theory with isotropic dynamical turbulence and to compute the parallel spatial diffusion coefficient and the corresponding mean free path. We compare our analytical findings with test-particle simulations performed for the same turbulence model. This type of comparison was not performed before for isotropic dynamical turbulence and this will, therefore, complement previous work where quasi-linear theory was tested for pure static turbulence and undamped magnetohydrodynamic waves (see Tautz et al. 2006a, 2006b).

The paper is organized as follows: In section 2 we present the basic formulas of quasi-linear theory for isotropic dynamical turbulence. In section 3 we discuss test-particle simulations performed for the same turbulence configuration and we compare the simulations with analytical theory. In section 4 we summarize and conclude.

2. Quasi-linear parallel transport in isotropic turbulence

The parallel mean free path $\lambda_\|_p$ is controlled by the Fokker-Planck coefficient of pitch-angle scattering $D_{\mu \mu}$ via the relation (see, e.g. Earl 1974)

$$\lambda_\|_p = \frac{3v}{8} \int_{-1}^{+1} d\mu \frac{(1 - \mu^2)^2}{D_{\mu \mu}(\mu)}$$

where we have used the pitch-angle cosine $\mu$ and the particle speed $v$. Alternatively, one could compute the parallel spatial diffusion coefficient via $\kappa_\parallel = v \lambda_\|_p / 3$. For certain values of $\mu$ one could find $D_{\mu \mu} = 0$ so that the integral in equation (1) is not convergent. This problem is known as the 90° problem.

The general form of the quasi-linear pitch-angle Fokker-Planck coefficient is (see, e.g. equation (25) of Teufel and Schlickeiser 2002)

$$D_{\mu \mu}(\mu) = \frac{\Omega^2 (1 - \mu^2)}{2B_0^2} \int d^3k \sum_{n=-\infty}^{+\infty} \mathcal{R}_n(\vec{k}, \mu) \times \left\{ P_{RR}(\vec{k}) I_{n+1}(W) + P_{LL}(\vec{k}) I_{n-1}(W) - I_{n+1}(W) I_{n-1}(W) \{ P_{RL}(\vec{k}) e^{2i\theta} + P_{LR}(\vec{k}) e^{-2i\theta} \} \right\}$$

with the helical components of the magnetic correlation tensor $P_{RR}, P_{LL}, P_{RL}$, and $P_{LR}$ which are discussed below. Furthermore, we have used the parameter

$$W = \frac{v}{\Omega} k_\perp \sqrt{1 - \mu^2},$$

where $\Omega$ is the unperturbed gyro-frequency given by $\Omega = (q B_0) / (m c \gamma)$ where we have used the electric charge of the particle $q$, the rest mass $m$, the speed of light $c$, and the Lorentz factor $\gamma$. In equations (2) and (3) we have
also used the wave vector \( \vec{k} \) which has the following Cartesian components

\[
\begin{align*}
    k_x &= k_x \cos \psi = k \sin \theta \cos \psi, \\
    k_y &= k_y \sin \psi = k \sin \theta \sin \psi, \\
    k_z &= k_z = k \cos \theta.
\end{align*}
\]

Furthermore, we have used the resonance function

\[
R_{\alpha}(\vec{k}, \mu) = \Re \int_0^{\infty} dt \, e^{-i (\nu k_3 + \alpha \Omega) t - \alpha \nu_A k^2} \\
= \frac{\alpha \nu_A k}{(\nu k_3 + \alpha \Omega)^2 + (\alpha \nu_A k)^2}
\]

with the absolute value of the wave vector \( k = |\vec{k}| \). Here we have employed the damping model of dynamical turbulence proposed in Bieber et al. (1994). In the latter model the Cartesian components of the spectral tensor, which is related to the magnetic correlation tensor via a Fourier transform, are given by

\[
P_{ij}(\vec{k}, t) = P_{ij}(\vec{k}) e^{-\alpha \nu_A k t}
\]

where we have used the static tensor components \( P_{ij}(\vec{k}) \) which are discussed below. The time \( t \) in this equation is really the time-difference between the considered time and the initial time. However, in our investigations we set the initial time equal to zero. In the damping model of dynamical turbulence we assume an exponential form of the temporal correlations of the magnetic fields. The parameter \( \nu_A \) is the usual Alfvén speed. In Gaussian units it is given by

\[
\nu_A \approx \frac{B_0}{\sqrt{4\pi \rho_0}}
\]

where we have used the background density \( \rho_0 \). Note that the latter speed is indeed the propagation speed of the Alfvén wave as long as that speed is much smaller than the speed of light in vacuum. The parameter \( \alpha \) used in the damping model of dynamical turbulence is not known exactly but it can be used to adjust the strength of dynamical turbulence effects. Usually we set \( \alpha = 1 \) as originally done in the work of Bieber et al. (1994). Note that the damping model used here is justified in part in closures, as in Edwards (1964). However, more modern perspectives recognize that the speed \( \nu_A \) in this model is better treated as the fluctuation amplitude \( \delta v \). This is in agreement with Bieber et al. (1994) if we set \( \alpha = \bar{b} v / \nu_A \). Therefore, our choice \( \alpha = 1 \) is equivalent to the assumption that \( \bar{b} v \) is in the same order as \( \nu_A \). The time scales of turbulence were discussed in detail in the literature (see, e.g. Chen and Kraichnan (1989) and Zhou et al. (2004)). Furthermore, the time relaxation for isotropic magnetohydrodynamic turbulence without mean magnetic field was explored numerically by Servidio et al. (2001) who found a decorrelation time that scales almost linearly with wavenumber \( k \). This supports the functional form of the time scale employed in the current paper, even though the present paper assumes that a guide field is present. There is some tension between the assumption of isotropy and the presence of a mean magnetic field which is known to produce anisotropic spectra in magnetohydrodynamics (see, e.g. Oughton et al. 1994). More advanced dynamical turbulence models were discussed and used in particle diffusion theory (see Shalchi et al. 2006). Those advanced models, however, would make the analytical calculations presented in the current article even more difficult.

We also need to specify the static correlation tensor components \( P_{ij}(\vec{k}) \). Those are controlled by the turbulence geometry indicating how the magnetic fields depend upon direction. In the current paper we use a three-dimensional model rather than models with reduced dimensionality as used in some previous work such as Bieber et al. (1994). The simplest 3D model for magnetic turbulence is, of course, the isotropic model in which the magnetic correlation tensor is given by

\[
P_{ij} = \frac{G(k)}{8\pi k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right)
\]

where the magnetic helicity is assumed to be zero. The spectrum \( G(k) \) is discussed below. The conversion to helical coordinates, defined via

\[
\delta B_{L,R} = \frac{1}{\sqrt{2}} (\delta B_x \pm i \delta B_y),
\]
yields

\[ P_{LL} = P_{RR} = \frac{1}{2} (P_{xx} + P_{yy}) \]
\[ = \frac{1}{2} \frac{G(k)}{8\pi k^2} \left( 2 - \frac{k_x^2 + k_y^2}{k^2} \right) \]
\[ = \frac{G(k)}{16\pi k^2} \left( 2 - \frac{k_x^2}{k^2} \right), \]

\[ P_{LR} = \frac{1}{2} [P_{xx} - P_{yy} + i(P_{xx} + P_{yy})] \]
\[ = \frac{G(k)}{16\pi k^2} \left( \frac{k_x^2 - k_y^2}{k^2} - 2i k_x k_y \right), \]

\[ P_{RL} = \frac{1}{2} [P_{xx} - P_{yy} - i(P_{xx} + P_{yy})] \]
\[ = \frac{G(k)}{16\pi k^2} \left( \frac{k_x^2 - k_y^2}{k^2} + 2i k_x k_y \right). \]

(10)

To evaluate equation (2) we follow Tautz et al. (2006a, 2006b). Using spherical coordinates for the wave vector (see equation (4) of the current paper) provides

\[ D_{\mu\nu} = \frac{\Omega^2 (1 - \mu^2)}{2B_0} \int d^3k \sum_{n=-\infty}^{\infty} R_n(\vec{k}, \mu) \]
\[ \times \frac{G(k)}{16\pi k^2} \left[ (2 - \sin^2 \theta) [J_{n+1}^2(W) + J_{n-1}^2(W)] - J_{n+1}(W) J_{n-1}(W) \right] \]
\[ \times \left[ (-\sin^2 \theta \cos^2 \psi + \sin^2 \theta \sin^2 \psi)e^{2i\psi} + 2i \sin^2 \theta \sin \psi \cos \psi e^{2i\psi} \right. \]
\[ \left. + (-\sin^2 \theta \cos^2 \psi + \sin^2 \theta \sin^2 \psi)e^{-2i\psi} - 2i \sin^2 \theta \sin \psi \cos \psi e^{-2i\psi} \right]. \]

(11)

By using Euler’s relation, the second squared bracket can be simplified to give \([\ldots] = -2 \sin^2 \theta\). With \(\eta = \cos \theta\) we obtain

\[ D_{\mu\nu} = \frac{\Omega^2 (1 - \mu^2)}{16B_0} \frac{1}{\Omega} \int_{-1}^{1} d\eta \int_{0}^{\infty} dk \frac{G(k)}{k} \sum_{n=-\infty}^{\infty} R_n(\vec{k}, \mu) \]
\[ \times \left\{ (1 + \eta^2) [J_{n+1}^2(W) + J_{n-1}^2(W)] \right. \]
\[ + 2(1 - \eta^2) J_{n+1}(W) J_{n-1}(W) \left\} \right. \]
\[ \right. \}

(12)

where now

\[ W = k \nu \sqrt{(1 - \mu^2)(1 - \eta^2)} \Omega. \]

(13)

With the addition theorems for Bessel functions (see, e.g. Watson 1966) we find

\[ \{\ldots\} = \eta^2 (J_{n+1} - J_{n-1})^2 + (J_{n+1} + J_{n-1})^2 = 4\eta^2 J_n^2 + \frac{4n^2}{W^2} J_n^2. \]

(14)

It is straightforward to show that the \(n = 0\) contribution in the infinite sum in equation (12) is equal to zero. Therefore we find

\[ D_{\mu\nu} = \frac{\Omega^2 (1 - \mu^2)}{4B_0} \int_{-1}^{1} d\eta \int_{0}^{\infty} \frac{G(k)}{k} \]
\[ \times \sum_{n=-\infty}^{\infty} \frac{\alpha_n \nu_k}{(\nu_n \nu_k + n\Omega)^2 + (\alpha_n \nu_k)^2} \]
\[ \times \left[ \eta \nu_n^2 (W) + \frac{n^2}{W^2} J_n^2 (W) \right]. \]

(15)
Due to symmetry in \( n \), we can write this as

\[
D_{\mu\nu} = \frac{\Omega^2(1 - \mu^2)}{2B_0^2} \int_{-1}^{1} d\eta \int_{0}^{\infty} dk \ G(k)
\times \sum_{n=1}^{\infty} \frac{(\eta n k + n!)^2 + (\alpha \nu A k)^2}{(\alpha \nu A k)^2} \times \left[ \eta^2 J_n^2(W) + \frac{n^2}{W^2} J_n^2(W) \right].
\]

(16)

To evaluate the latter formula, we need to specify the spectrum \( G(k) \). In the current paper we consider two different model spectra.

### 2.1. A spectrum containing only inertial and energy ranges

We start our investigations with a spectrum without dissipation range. The latter effect will be included in section 2.2. Based on the work presented in Shalchi and Weinhorst (2009) we use the model

\[
G(k) = 4D(s, q)l_0 \delta B^2 \frac{(kl_0)^q}{[1 + (kl_0)^2]^{(s+q)/2}} \quad (17)
\]

with the normalization function

\[
D(s, q) = \frac{\Gamma\left(\frac{s+q}{2}\right)}{2\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s+q+1}{2}\right)} \quad (18)
\]

where we have used gamma functions. The model spectrum used here contains the characteristic scale \( l_0 \) which is usually called the bendover scale. This scale separates the energy range of the spectrum \((kl_0 < 1)\) from the inertial range \((kl_0 > 1)\). In the former range the spectrum scales like \( \propto k^q \) and, therefore, the parameter \( q \) is called the energy range spectral index. In the inertial range, on the other hand, the spectrum behaves like \( \propto k^{-s} \) and consequently \( s \) is called the inertial range spectral index. The constants in the model spectrum, and the function defined in equation (18) in particular, were chosen so that we satisfy the normalization condition

\[
\int_{0}^{\infty} dk \ G(k) = \delta B^2.
\]

(19)

In order to compare our analytical results with previous results and simulations, it is convenient to use the dimensionless pitch-angle Fokker-Planck coefficient

\[
\tilde{D}_{\mu\nu} = \frac{l_0}{v} D_{\mu\nu}.
\]

(20)

Therewith, equation (1) can be rewritten as

\[
\lambda_{\parallel} = \frac{3}{8} \int_{-1}^{+1} d\mu \frac{(1 - \mu^2)^2}{\tilde{D}_{\mu\nu}(\mu)}.
\]

(21)

In the following we rewrite equation (16). In order to do this, we use the spectrum (17) and the integral transformation \( x = kl_0 \) to derive

\[
\tilde{D}_{\mu\nu} = 2D(s, q)(1 - \mu^2) \delta B^2 \frac{B_0}{B_0^2} \int_{0}^{\infty} dx \frac{x^q}{(1 + x^2)^{s+q/2}}
\times \int_{-1}^{+1} d\eta \sum_{n=1}^{\infty} \frac{\alpha \epsilon x}{(\mu R \epsilon x + n)^2 + (\alpha \epsilon R x)^2}
\times \left[ \eta^2 J_n^2(W) + \frac{n^2}{W^2} J_n^2(W) \right]
\]

(22)

Here we have used the parameters \( \epsilon = \nu A/v \) and \( R = R_L/l_0 \). Furthermore, we now have

\[
W = x R \sqrt{(1 - \mu^2)(1 - \eta^2)}.
\]

(23)

In section 3 we will solve equation (22) numerically. Before we do this we consider an alternative spectrum.

### 2.2. A spectrum with dissipation range

In the following we extend the spectrum given by equation (17) by including a dissipation range so that (see, e.g. Bieber et al 1994 for more details)
$G(k) = 4D(s,q)l_0^q B^2$
\[
x = \begin{cases} 
\frac{(k_0 l_0)^q}{(1 + k_0^2 l_0^2)^{\eta + q/2}} & \text{if } k \leq k_d, \\
\frac{(k_d l_0)^q}{(1 + k_d^2 l_0^2)^{\eta + q/2}} \left(\frac{k_d}{k}\right)^p & \text{if } k \geq k_d. 
\end{cases}
\] (24)

Please note that this spectrum is only correctly normalized if $k_d l_0 \gg 1$. The additional parameter $k_d$ is usually called the dissipation wavenumber and the parameter $p$ the dissipation range spectral index. Note that the parameter $k_d$ separates the dissipation range from the inertial range. In the former range the spectrum scales like $\propto k^{-p}$. With this form of the spectrum, the Fokker-Planck coefficient of pitch-angle scattering becomes

\[
\tilde{D}_{\mu
u} = 2D(s,q)(1 - \mu^2) \frac{\delta B^2}{B_0^2} \int_0^\infty dx \frac{\alpha \epsilon x}{(\mu Rx + n)^2 + (\alpha \epsilon Rx)^2} 
\times \left[ \eta J_n^2(W) + \frac{n^2}{W^2} J_n^2(W) \right] 
\] (25)

instead of equation (22). Here we have used the function

\[
h(x) = \begin{cases} 
\frac{x^q}{(1 + x^2)^{\eta + q/2}} & \text{if } x \leq x_d, \\
\frac{x_d^q}{(1 + x_d^2)^{\eta + q/2}} \left(\frac{x_d}{x}\right)^p & \text{if } x \geq x_d.
\end{cases}
\] (26)

with $x_d = k_d l_0$. The spectrum used here is very similar compared to the one used by Bieber et al. (1994) for slab and two-dimensional turbulence but our spectrum is more general in the energy range. Furthermore, we assume isotropic turbulence.

3. Comparison with test-particle simulations

3.1. Test-particle simulations

An alternative way of computing particle orbits and diffusion coefficients is to perform test-particle simulations (see, e.g. Michalek and Ostrowski 1996, Giacalone and Jokipii 1999, Qin et al. 2002a, 2002b, and Tautz and Shalchi 2013). Within this approach one still needs to specify the used turbulence model via the components of the magnetic correlation tensor in the way described above. Particle orbits are then computed numerically by solving the Newton-Lorentz equation of a charged particle interacting with a turbulent magnetic field. This is done by employing standard methods of computational physics. For instance, one uses an adaptive higher-order method such as a Runge-Kutta solver (see, e.g. Press et al. 2007). By considering thousands of test-particles, one can compute the mean square displacements of particle orbits and then the corresponding diffusion coefficient via

\[
\kappa_\parallel = \lim_{t \to \infty} \frac{\langle (\Delta z)^2 \rangle}{2t}.
\] (27)

We have performed test-particle simulations for dynamical turbulence as described in Hussein and Shalchi (2016). Furthermore, we have employed the isotropic model and used the model spectrum with dissipation range (see equation (24) of the current paper). Our results are listed in tables 1 and 2 for protons and electrons. The data is plotted in figure 1. Between rigidities of $R = 0.01$ and $R = 1$ the results shown here are very similar compared to the simulations shown in Tautz et al. (2008) performed for magnetostatic turbulence. In the latter paper, however, slightly different turbulence parameters were used. For rigidities much smaller than $R = 0.01$ we find the usual split between the electron and proton results as obtained in the pioneering work of Bieber et al. (1994).

3.2. Quasi-linear results for the spectrum without dissipation range

Above we have shown the results from simulations obtained for the parallel mean free path. Before we compare them with the quasi-linear diffusion parameters, we computed the pitch-angle Fokker-Planck coefficient as given by equation (22). The two integrals therein as well as the sum were evaluated numerically. We performed our calculations for protons as well as electrons and we have considered three different values of the magnetic rigidity, namely $R = 0.01, R = 1$, and $R = 10$. For our numerical solution of equation (22), we use the following parameter values: $\delta B/B_0 = 0.5, \alpha = 1, s = 5/3,$ and $q = 3$. All formulas depend upon the parameter...
Figure 1. The parallel mean free path $\lambda_\parallel /b_0$ versus the magnetic rigidity $R = R/\lambda_0$. The results shown here were obtained from test-particle simulations performed for the damping model of dynamical turbulence and isotropic turbulence. Furthermore, a spectrum with dissipation range was employed (see equation (24) of the current paper).

Table 1. Transport parameters obtained from test-particle simulations using isotropic turbulence and the damping model of dynamical turbulence. The listed values were obtained for protons ($R_0 = 0.169$). Furthermore, we have used $\delta B/b_0 = 0.5, \alpha = 1, v_\perp = 33.5 \, \text{km} \, \text{s}^{-1}, s = 5/3, q = 3, p = 3$, and $k_B t_0 = 9 \times 10^3$.

| $R = R_1/b_0$ | Rigidity in MV | $\lambda_\parallel /b_0$ | $\lambda_\perp$ in AU | $\lambda_\perp /b_0$ | $\lambda_\parallel /\lambda_\perp$ |
|----------------|----------------|-------------------------|-----------------------|-----------------------|-------------------------------|
| $10^{-4}$      | 0.554 3        | 0.28                    | $8.4 \times 10^{-3}$  | 0.01                  | 0.035                         |
| $5 \times 10^{-4}$ | 2.772        | 0.7                     | 0.021                 | 0.011                 | 0.016                         |
| $10^{-3}$      | 5.543          | 1.08                    | 0.03                  | 0.012                 | 0.011                         |
| $5 \times 10^{-3}$ | 27.72         | 2.39                    | 0.071                 | 0.013                 | $5.44 \times 10^{-3}$        |
| $10^{-2}$      | 55.43          | 2.7                     | 0.081                 | 0.014                 | $5.19 \times 10^{-3}$        |
| $5 \times 10^{-2}$ | 277.2         | 5.55                    | 0.17                  | 0.02                  | $3.6 \times 10^{-3}$         |
| $10^{-1}$      | 554.3          | 6.74                    | 0.2                   | 0.023                 | $3.4 \times 10^{-3}$         |
| $5 \times 10^{-1}$ | 2772          | 19.8                    | 0.594                 | 0.066                 | $3.33 \times 10^{-3}$        |
| $10^0$         | 5543           | 42.5                    | 1.28                  | 0.08                  | $1.88 \times 10^{-3}$        |
| $5 \times 10^0$ | $27.72 \times 10^3$ | 420                     | 12.6                  | 0.09                  | $2.14 \times 10^{-4}$        |
| $10^1$         | $55.43 \times 10^3$ | 1280                   | 38.4                  | 0.105                 | $8.2 \times 10^{-5}$          |

Table 2. Transport parameters obtained from test-particle simulations using isotropic turbulence and the damping model of dynamical turbulence. The listed values were obtained for electrons ($R_0 = 9.2 \times 10^3$). Furthermore, we have used $\delta B/b_0 = 0.5, \alpha = 1, v_\perp = 33.5 \, \text{km} \, \text{s}^{-1}, s = 5/3, q = 3, p = 3$, and $k_B t_0 = 9 \times 10^3$.

| $R = R_1/b_0$ | Rigidity in MV | $\lambda_\parallel /b_0$ | $\lambda_\perp$ in AU | $\lambda_\perp /b_0$ | $\lambda_\parallel /\lambda_\perp$ |
|----------------|----------------|-------------------------|-----------------------|-----------------------|-------------------------------|
| $10^{-4}$      | 0.554 3        | 3.43                    | 0.103                 | 0.025                 | $7.3 \times 10^{-3}$          |
| $5 \times 10^{-4}$ | 2.772        | 2.91                    | 0.087                 | 0.021                 | $7.2 \times 10^{-3}$          |
| $10^{-3}$      | 5.543          | 2.93                    | 0.088                 | 0.021                 | $7.17 \times 10^{-3}$         |
| $5 \times 10^{-3}$ | 27.72         | 2.72                    | 0.082                 | 0.014                 | $5.15 \times 10^{-3}$         |
| $10^{-2}$      | 55.43          | 3.3                     | 0.099                 | 0.015                 | $4.54 \times 10^{-3}$         |
| $5 \times 10^{-2}$ | 277.2         | 5.2                     | 0.16                  | 0.021                 | $4.0 \times 10^{-3}$          |
| $10^{-1}$      | 554.3          | 6.8                     | 0.204                 | 0.027                 | $3.97 \times 10^{-3}$         |
| $5 \times 10^{-1}$ | 2772          | 18.9                    | 0.57                  | 0.063                 | $3.33 \times 10^{-3}$         |
| $10^0$         | 5543           | 40.5                    | 1.22                  | 0.082                 | $2.02 \times 10^{-3}$         |
| $5 \times 10^0$ | $27.72 \times 10^3$ | 430                     | 12.9                  | 0.083                 | $1.93 \times 10^{-4}$         |
| $10^1$         | $55.43 \times 10^3$ | 1270                   | 38.1                  | 0.105                 | $8.27 \times 10^{-5}$         |
\[ \epsilon = \frac{v_A}{c} \sqrt{\frac{R_0^2 + R^2}{R}} \]

with

\[ R_0 = \frac{1}{l_{dab} B_0} \begin{cases} 0.511 \text{MV} & \text{for } e^- \\ 938 \text{MV} & \text{for } p^+ \end{cases} \]

For the heliospheric parameters considered in the current paper we have for electrons \( R_0(e^-) \approx 9.2 \times 10^{-3} \) and protons \( R_0(p^+) \approx 0.169 \). For the Alfvén speed we set \( v_A = 33.5 \) km s\(^{-1}\). We have computed the pitch-angle Fokker–Planck coefficient \( D_{\mu \mu} \) and all our results are visualized in figures 2–7. Furthermore, we computed the quasi-linear parallel mean free path by using equation (1). The integral therein was evaluated numerically as well. The obtained parallel mean free paths are visualized in figure 8 versus the magnetic rigidity for protons and electrons, respectively. As in the simulations we find a split between the electron and proton mean free paths at lower rigidities. This is also what was found in the work of Bieber et al (1994).

### 3.3. Quasi-linear results for the spectrum with dissipation range

In section 3.2 we have obtained the quasi-linear pitch-angle Fokker-Planck coefficient and the corresponding parallel mean free path for a spectrum without dissipation range. In the following we repeat our computations for a spectrum with dissipation range. In this case the pitch-angle Fokker–Planck coefficient is given by equation (25). For the additional parameters we have used \( x_d = k_d l_0 = 9 \times 10^3 \) and \( p = 3 \).

Our results obtained for the pitch-angle Fokker-Planck coefficient \( D_{\mu \mu} \) are visualized in figures 9–14. Furthermore, the quasi-linear parallel mean free path is plotted versus magnetic rigidity in figure 15. Although we now included dissipation effects, it seems that the results are not fundamentally different compared to the results obtained for the spectrum without dissipation range (see figure 8).

### 3.4. Quasi-linear results versus test-particle simulations

In order to explore the validity of quasi-linear theory for isotropic dynamical turbulence, we compare the quasi-linear results with the simulations described above. In both cases the results are based on a spectrum with dissipation range and they are visualized in figure 16. As we can see, quasi-linear results do not agree well with the simulations regardless of the value of magnetic rigidity and the particle species. For intermediate rigidities the quasi-linear mean free paths are much larger than the values obtained from simulations. The reason is that nonlinear effects are neglected in quasi-linear theory and, thus, one obtains a too small pitch-angle Fokker-Planck coefficient resulting in a too large parallel spatial diffusion coefficient. For small rigidities, however, the situation...
is slightly different. Whereas we still find too large electron mean free paths, the quasi-linear proton mean free paths are now much smaller than the values obtained from simulations.

4. Summary and conclusion

In the current paper we explored quasi-linear transport of energetic particles in isotropic dynamical turbulence. The main motivation of this work was to determine whether quasi-linear theory is valid for parallel diffusion in the considered turbulence model. Therefore, we computed parallel diffusion parameters and compared our results with test-particle simulations.

It is known that quasi-linear theory (QLT) cannot describe pitch-angle scattering for pitch-angles close to $90^\circ$ (corresponding to $\mu = 0$) accurately. Within magnetostatic QLT we find zero scattering at $\mu = 0$. It was shown in the past that this is a consequence of the assumption of unperturbed orbits. As soon as fluctuations of the particle trajectories are taken into account, one obtains a broadened resonance and a pitch-angle scattering...
coefficient which is no longer equal to zero at small values of $\mu$ (see Shalchi 2005 and Shalchi et al 2009). Furthermore, there are indications that quasi-linear theory can also not describe pitch-angle scattering and parallel diffusion for two-dimensional turbulence correctly (see Shalchi et al 2004 and Qin et al 2006).

On the other hand, QLT is able to reproduce interplanetary observations of particle diffusion coefficients as shown in the comprehensive work of Bieber et al (1994). Therefore, it is important to present a detailed comparison of diffusion parameters based on QLT and test-particle simulations which provide an independent tool for computing diffusion coefficients. In previous work such a detailed comparison was presented for magnetostatic turbulence as well as undamped magnetohydrodynamic waves (see Tautz et al 2006a, 2006b). In the current work we complement such previous investigations by studying parallel diffusion in dynamical turbulence.

By comparing QLT results with test-particle simulations we have shown that the quasi-linear parallel diffusion coefficients are usually much larger than the simulations. We believe that this is a consequence of quasi-linear theory not being able to describe pitch-angle scattering at $\mu \approx 0$. Therefore, the scattering coefficient which is no longer equal to zero at small values of $\mu$ (see Shalchi 2005 and Shalchi et al 2009).

**Figure 5.** The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum without dissipation range, electrons, and a magnetic rigidity of $R = 0.01$.

**Figure 6.** The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum without dissipation range, electrons, and a magnetic rigidity of $R = 1$. 
coefficients $D_{\mu \nu}$ obtained from QLT are too small and, thus, the parallel mean free paths computed by employing equation (1) are too large. Therefore, one has to be careful if quasi-linear theory is applied. It seems that the theory is not correct in the general case regardless whether the turbulence is static, consists of undamped waves, or is fully dynamical. However, there might be some exceptions where quasi-linear theory is correct. Examples for such exceptions could be pure slab turbulence for certain spectra or a slab + 2D model with certain magnetic field values (see again Bieber et al 1994).

The results of the current paper also motivate future investigations. Clearly QLT does not agree with simulations in most cases. However, previous non-linear treatments of the transport have been performed for magnetostatic turbulence. Therefore, it has to be subject of future work to combine dynamical turbulence models with non-linear tools. Candidates for such theories would be the second-order theory of Shalchi (2005) or the theory developed by Subedi et al (2017) for scattering in isotropic turbulence without mean field.
Figure 9. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, protons, and a magnetic rigidity of $R = 0.01$.

Figure 10. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, protons, and a magnetic rigidity of $R = 1$. 

Figure 11. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, protons, and a magnetic rigidity of \( R = 10 \).

Figure 12. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, electrons, and a magnetic rigidity of \( R = 0.01 \).
Figure 13. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, electrons, and a magnetic rigidity of $R = 1$.

Figure 14. The quasi-linear pitch-angle Fokker-Planck coefficient obtained for a spectrum with dissipation range, electrons, and a magnetic rigidity of $R = 10$. 
Acknowledgments

Support by the Natural Sciences and Engineering Research Council of Canada (NSERC) is acknowledged.

ORCID iDs

A Shalchi  © https://orcid.org/0000-0002-2923-0489

References

Bieber J W, Matthaeus W H, Smith C W, Wanner W, Kallenrode M-B and Wibberenz G 1994 Astrophys. J. 420 294
Chen S and Kraichnan R H 1989 Physics of Fluids A: Fluid Dynamics 1 2019
Earl J A 1974 Astrophys. J. 193 231
Edwards S F 1964 J. Fluid Mech. 18 239
Giacalone J and Jokipii J R 1999 Astrophys. J. 520 204
Goldstein M I. 1976 Astrophys. J. 204 900

Figure 15. The parallel mean free path $\lambda_P/l_0$ versus the magnetic rigidity $R = R_L/l_0$ for a spectrum with dissipation range. The results shown here were obtained by employing quasi-linear theory for the damping model of dynamical turbulence and isotropic turbulence.

Figure 16. The parallel mean free path $\lambda_P/l_0$ versus the magnetic rigidity $R = R_L/l_0$ for a spectrum with dissipation range. The results shown here were obtained by employing quasi-linear theory as well as test-particle simulations. In all cases we employed the the damping model of dynamical turbulence and assumed isotropic turbulence.
Hussein M and Shalchi A 2014 Astrophys. J. 785 31
Hussein M and Shalchi A 2016 Astrophys. J. 817 136
Jokipii J R 1966 Astrophys. J. 146 480
Jones F C, Birmingham T J and Kaiser T B 1973 Astrophys. J. 180 L139
Jones F C, Birmingham T J and Kaiser T B 1978 Phys. Fluids 21 347
Kaiser T B, Birmingham T J and Jones FC 1978 Phys. Fluids 21 361
Matthaeus W H, Gray P C, Pontius D H Jr. and Bieber J W 1995 Phys. Rev. Lett. 75 2136
Matthaeus W H, Qin G, Bieber JW and Zank GP 2003 Astrophys. J. 590 L53
Michalek G and Ostrowski M 1996 Nonlinear Processes Geophys 3 66
Oughton S, Priest E R and Matthaeus W H 1994 J. Fluid Mech. 280 95
Owens A J 1974 Astrophys. J. 191 235
Press W H, Teukolsky S A, Vetterling W T and Flannery B P 2007 Numerical Recipes (University Press: Cambridge)
Qin G, Matthaeus W H and Bieber JW 2002a GeRL 29 1048
Qin G, Matthaeus W H and Bieber JW 2002b Astrophys. J. 578 L117
Qin G, Matthaeus W H and Bieber JW 2006 Astrophys. J. 640 L103
Schlickeiser R 2002 Cosmic Ray Astrophysics (Springer: Berlin)
Servidio S, Carbonell V, Dmitruk P and Matthaeus W H 2001 Europhys. Lett. 96 55003
Shalchi A, Bieber JW, Matthaeus WH and Qin G 2004 Astrophys. J. 616 617
Shalchi A 2005 Phys. Plasmas 12 052905
Shalchi A, Bieber JW, Matthaeus WH and Schlickeiser R 2006 Astrophys. J. 642 230
Shalchi A 2009 Nonlinear Cosmic Ray Diffusion Theories. Astrophysics and Space Science Library vol 362 (Berlin: Springer)
Shalchi A, Škoda T, Tautz RC and Schlickeiser R 2009 Astron. Astrophys. 507 589
Shalchi A and Weinhorst B 2009 Adv. Space Res. 43 1429
Shalchi A 2010 Astrophys. J. 720 L127
Shalchi A 2015 Phys. Plasmas 22 015001
Shalchi A 2017 Phys. Plasmas 24 055010
Subedi P et al 2017 Astrophys. J. 837 140
Tautz R C, Shalchi A and Schlickeiser R 2006a J. Phys. G: Nucl. Part. Phys. 32 809
Tautz R C, Shalchi A and Schlickeiser R 2006b J. Phys. G: Nucl. Part. Phys. 32 1045
Tautz R C, Shalchi A and Schlickeiser R 2008 Astrophys. J. 685 L165
Tautz R C and Shalchi A 2013 J. Geophys. Res. 118 642
Teufel A and Schlickeiser R 2002 Astron. Astrophys. 393 703
Völk H J 1973 Rev. Geophys. Space Sci. 13 547
Watson G N 1966 A Treatise on the Theory of Bessel Functions (Cambridge: University Press)
Zhou Y, Matthaeus WH and Dmitruk P 2004 Rev. Mod. Phys. 76 1015