Chaotic maps and flows: exact Riemann–Siegel lookalike for spectral fluctuations

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Abstract

To treat the spectral statistics of quantum maps and flows that are fully chaotic classically, we use the rigorous Riemann–Siegel lookalike available for the spectral determinant of unitary time evolution operators \(F\). Concentrating on dynamics without time reversal invariance, we get the exact two-point correlator of the spectral density for finite dimension \(N\) of the matrix representative of \(F\), as phenomenologically given by random matrix theory. In the limit \(N \to \infty\), the correlator of the Gaussian unitary ensemble is recovered. Previously conjectured cancellations of contributions of pseudo-orbits with periods beyond half the Heisenberg time are shown to be implied by the Riemann–Siegel lookalike.

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1. Introduction

Universal fluctuations in quantum energy spectra under conditions of full classical chaos are well understood in terms of Gutzwiller’s semiclassical periodic-orbit theory [1, 2]. The phenomenological description previously given by random matrix theory has been fully recovered for individual (rather than ensemble averages of) systems from the unitary, orthogonal [1, 2] and symplectic [3] symmetry classes.

The analogous problem for periodically driven dynamics will be taken up in this paper. A unitary Floquet operator \(F\) describing the evolution of such systems over one period of the driving and its quasi-energy spectrum will be investigated. Limiting ourselves to dynamics without time reversal invariance, we shall aim at recovering the spectral fluctuations of the circular unitary ensemble of random matrix theory. Surprisingly, the semiclassical analysis will not only produce the two-point correlator of the density of quasi-energy levels in the limit of infinite dimension \(N\) of the Floquet matrix (which coincides with the correlator of energy levels of autonomous dynamics), but even the finite-dimension variant (a periodic function of the spacing variable, with \(N\) times the mean level spacing as the period).
Floquet operators turn out easier to analyze than Hamiltonians. It is the so-called self-

inverseness of the secular polynomial of unitary matrices which comes to our help; it provides

a variant of the secular polynomial, a ‘zeta’ function \( \zeta(\varphi) = \sqrt{\text{det}(-F^\dagger)} e^{-iN\varphi/2} \text{det}(1-e^{i\varphi}F) \),

which is real for real values of the phase variable \( \varphi \) and enjoys a rigorous finite-N Riemann–

Siegel lookalike. Four zetas will be combined as

\[
Z = (\zeta(\varphi_C)\zeta(\varphi_D)\zeta(\varphi_A)^{-1}\zeta(\varphi_B)^{-1})
\]

to a generating function \( Z \), the latter having the two-point correlator of interest as a suitable

derivative. Due to the fixed order \( N \) of \( \text{det}(1-e^{i\varphi}F) \), the generating function is represented

by finite Fourier series with respect to the variables \( \varphi_C, \varphi_D \); the Fourier coefficients are related

to the traces of powers of the Floquet operator, \( t_n = \text{Tr} F^n \), by the venerable relations given

by Isaac Newton nearly 350 years ago. In contrast, the inverse zetas in \( Z \) have infinite Fourier

series where the Fourier coefficients can again be expressed in terms of the traces \( t_n \). Inasmuch

as those \( t_n \) are determined by a Gutzwiller trace formula we get semiclassical access to \( Z \). The

trace formula for \( t_n \) is a well-behaved one, without divergence necessitating regularization.

When scrutinizing the semiclassical limit of \( Z \) according to the strategy just sketched, we shall meet with a remarkable property, long hoped for but hitherto elusive. It is well to

highlight that property right away in intuitive semiclassical parlance, before pointing to the

rigorous algebraic origin. The finite Fourier series representing \( Z \) with respect to the variables

\( \varphi_C, \varphi_D \) have the welcome consequence, due to the Riemann–Siegel lookalike, that periodic

orbits with periods only up to half the Heisenberg time determine the constituent zetas \( \zeta(\varphi_C) \)

and \( \zeta(\varphi_D) \). On the other hand, orbits with periods up to infinity enter the infinite Fourier series

of the inverse zetas in \( Z \). That ‘asymmetry’ in the pairs \( \varphi_A, \varphi_B \) and \( \varphi_C, \varphi_D \) creates a vexatious

inconvenience in evaluating \( Z \). In previous work on autonomous flows, the analogous difficulty

was fought with the help of imaginary parts of the energy variables large enough to cut off the

contributions of orbits with periods beyond half the Heisenberg time. In our present case, no

trick at all is needed. Orbits with periods larger than half the Heisenberg time make exactly

zero contribution to \( \zeta(\varphi_C)\zeta(\varphi_D) \) in \( Z \). The algebraic basis for the property under discussion

is the rigorous Riemann–Siegel lookalike for the zeta function. That powerful identity

forces the coefficients of the secular polynomial \( \text{det}(1-e^{i\varphi}F) = \sum_{n=0}^{N} A_n e^{i\varphi n} \)

with \( n > N \) (and thus the coefficients of \( e^{i\varphi n} \) in the finite Fourier series for \( \zeta(\varphi) \) with \( |n| = \frac{N}{2} \)) to vanish—

even though Gutzwiller’s periodic-orbit sum for the traces \( t_n \) contains no information on the

finite dimension of the Floquet matrix.

The rigorous Riemann–Siegel lookalike can be carried over to the treatment of the energy

level fluctuations for autonomous flows. One has to resort to a stroboscopic description and

work with the Floquet operator for a suitable strobe period which secures a one-to-one relation

between \( N \) energy levels and eigenphases. Again, finite Fourier series arise for the generating

function \( Z \) for the phases \( \varphi_C, \varphi_D \). For the unitary symmetry class, the diagonal approximation

yields, in the limit \( N \to \infty \), the exact \( Z \) known from the Gaussian unitary ensemble. As

regards corrections, we show that the present approach reproduces, for \( N \to \infty \), the previous results

for all Wigner–Dyson symmetry classes.

The rest of the paper is organized as follows. In section 2, we present the Riemann–Siegel

lookalike for the zeta function of finite-dimensional unitary matrices \( F \). Newton’s relations are

employed to express zeta in terms of traces of powers of \( F \). In section 3, the generating function

is defined and decomposed in additive pieces according to the Riemann–Siegel lookalike. In

section 4, we invoke the Gutzwiller trace formula for powers of Floquet operators and show

the diagonal approximation for the unitary symmetry class to produce the exact generating

function known from random matrix theory. Some hints are given on why we are certain that

the vanishing of corrections can be shown in a generalization of previous work on autonomous

flows. In section 5, we argue that autonomous flows can be described stroboscopically in terms

of unitary Floquet operators. In the final section 6, we summarize the progress made through
the rigorous Riemann–Siegel lookalike, comment on related previous work and speculate on future ramifications.

2. Riemann–Siegel lookalike for unitary matrices

Toward capturing spectral fluctuations we look at the secular determinant

\[ A(\varphi) = \det(1 - e^{i\varphi} F) = \sum_{n=0}^{N} A_n e^{i\varphi_n}, \]  

(1)

where the coefficients \( A_n \) include \( A_0 = 1 \) and \( A_N = \det(-F) \). The determinant \( A(\varphi) \) is defined in the whole complex \( \varphi \)-plane. We may think of the determinant as a discrete one-sided Fourier transform of the coefficients with the inverse

\[ A_n = \int_{0}^{2\pi} \frac{d\varphi}{2\pi} A(\varphi) e^{-i\varphi_n}. \]  

(2)

The information stored in \( A(\varphi) \) is also encoded in the (trace of the) resolvent

\[ t(\varphi) = \text{Tr} \left( \frac{1}{1 - e^{i\varphi} F} \right). \]  

(3)

To decode, a geometric series for the resolvent is useful, and that series takes different forms in the upper and lower halves of the complex \( \varphi \)-plane. Defining the ‘traces’ (of powers of the Floquet operator),

\[ t_n = \text{Tr} F^n, \quad n = 0, \pm 1, \pm 2, \ldots, \]  

(4)

we have

\[ t(\varphi) = \begin{cases} 
\sum_{n=0}^{\infty} e^{i\varphi_n} t_n & \text{Im} \varphi > 0 \\
-\sum_{n=1}^{\infty} e^{-i\varphi_n} t_n^* & \text{Im} \varphi < 0
\end{cases} \]  

(5)

In both cases, imaginary parts of the phase may be seen as convergence ensurers. As above for the secular coefficients, we recover the traces \( t_n \) from the resolvent by inverting the one-sided Fourier transform,

\[ t_n = \int_{0}^{2\pi} \frac{d\varphi}{2\pi} t(\varphi) e^{-i\varphi_n}. \]

Inasmuch as the determinant of any matrix equals the exponentiated trace of its logarithm, we can express the resolvent in terms of the determinant as

\[ t(\varphi) = N + i \frac{\partial}{\partial \varphi} \ln A(\varphi) \quad \leftrightarrow \quad A(\varphi) = \exp \left( -i \int_{0}^{\varphi} \text{d}\varphi' \left( t(\varphi') - N \right) \right). \]  

(6)

The foregoing relation leads us to different expressions for the secular determinant for the two possible signs of \( \text{Im} \varphi \),

\[ A(\varphi) = \begin{cases} 
\exp \left( -\sum_{n=1}^{\infty} \frac{t_n e^{i\varphi_n}}{n} \right) & \text{Im} \varphi > 0 \\
A_N \exp \left( iN\varphi - \sum_{n=1}^{\infty} \frac{t_n^* e^{-i\varphi_n}}{n} \right) & \text{Im} \varphi < 0.
\end{cases} \]  

(7)

We have fixed integration constants by invoking \( A(\varphi) \to 1 \) for \( \text{Im} \varphi \to +\infty \) in the first case and \( A(\varphi) \to A_N e^{iN\varphi} \) for \( \text{Im} \varphi \to -\infty \) in the second case. By expanding the right members of (7) in powers of \( e^{i\varphi} \), we formally obtain infinite series. However, the traces \( t_n \), \( t_n^* \) conspire to secure \( A_n = 0 \) for \( n > N \). Even on the real \( \varphi \)-axis, the two expressions exist and coincide, provided \( \varphi \) does not equal any of the eigenphases of \( F \).
The traces $t_n$ and the secular coefficients $A_n$ are directly related as well. The Taylor expansion of (7) gives $A(\varphi) = \sum_{\nu=0}^{\infty} (-1)^\nu \left( \sum_{l=1}^{\infty} t_l e^{i\nu\varphi} \right)^l$. The secular coefficients are obtained by employing the multinomial expansion for the powers $(-1)^\nu$ and collecting the coefficients of $e^{i\nu\varphi}$ with $n = 0, 1, \ldots$. To formulate the result, we consider all partitions of the positive integer $n$ into sums of smaller positive integers $l$ with multiplicities $v_l$, namely $n = \sum_{l=0,1,\ldots} v_l$. For the ‘vector’ $\vec{v}$, we define two properties, $L(\vec{v}) = \sum_{l=0,1,\ldots} l v_l$ and $V(\vec{v}) = \sum_{l=0,1,\ldots} v_l$. We then have [4–6]

$$A_n = \sum_{\vec{v}} (-1)^{V(\vec{v})} \prod_{l \geq 1} \frac{t_{l v_l}}{l v_l!}.$$  

(8)

It is remarkable that expression (8) holds independently of the degree $N$ of the secular polynomial. Nevertheless, if the traces $t_n$ are evaluated for fixed $N$ we obtain $A_n = 0$ for $n > N$. Expression (8) can be regarded as the explicit solution of Isaac Newton’s recursive relation $(N - n)A_n = \sum_{m=0}^{n} t_m A_{n-m}$ for the secular coefficients; for that reason we shall refer to it as the Newton relation.

The unitarity of the Floquet operator entails what is known as the ‘self-inversiveness’ of the secular coefficients [6–9],

$$A_n = A_{N-n}^* A_N.$$  

(9)

That property makes sure that the set of the secular coefficients contains exactly $N$ independent real parameters (as which one may see the eigenphases of $F$). For $N$ odd, the $A_n$ with $n = 1, 2, \ldots, \frac{N-1}{2}$ together with $A_N$ fully represent the secular polynomial since the remaining coefficients are given by self-inversiveness according to (9). Similarly, for even $N$, the $A_n$ with $n = 1, 2, \ldots, \frac{N}{2}$ carry the full information. As a further consequence of self-inversiveness, the following variant of the secular determinant, to be called ‘zeta function’,

$$\zeta(\varphi) = \sqrt{A_N^*} e^{-i\varphi N/2} A(\varphi) - (10)$$

is real valued for real $\varphi$. For the case of even $N$, which we shall stick to from here on, it is convenient to split the zeta function into three additive pieces:

$$\zeta(\varphi) = \sqrt{A_N^*} e^{-i\varphi N/2} A_N e^{i\varphi/2} + \sqrt{A_N^*} A_N^* e^{-i\varphi/2} \sum_{n=0}^{\frac{N}{2}-1} A_n e^{i\nu\varphi} + \sqrt{A_N^*} A_N^* e^{-i\varphi/2} \sum_{n=0}^{\frac{N}{2}-1} A_n^* e^{-i\nu\varphi}$$

$$\equiv \zeta_-(\varphi) + \zeta_0(\varphi) + \zeta_+(\varphi).$$  

(11)

The terms $\zeta_-$ and $\zeta_+$ contain Fourier components $e^{i\nu\varphi}$ with, respectively, $\nu = -1, -2, \ldots, -\frac{N}{2}$ and $\nu = 1, 2, \ldots, \frac{N}{2}$, while $\zeta_0$ is the zeroth Fourier component, $\zeta_0 = \sqrt{A_N^*} A_N^* = \sqrt{\frac{1}{2} A_N^* A_N^*} = \zeta_0^*$. The decomposition (11) of zeta can be seen as a rigorous Riemann–Siegel lookalike [5, 6, 10–13].

An alternative form for the zeta function is useful. It is obtained by importing representation (7) of $A(\varphi)$ to definition (10),

$$\zeta(\varphi) = \begin{cases} \sqrt{A_N^*} \exp \left\{ -i \frac{N}{2} \varphi - \sum_{n=1}^{\infty} t_n e^{i\nu\varphi} \right\} & \text{Im } \varphi > 0 \\ \sqrt{A_N^*} \exp \left\{ +i \frac{N}{2} \varphi - \sum_{n=1}^{\infty} t_n^* e^{-i\nu\varphi} \right\} & \text{Im } \varphi < 0. \end{cases}$$  

(12)
That representation of zeta by an exponentiated infinite Fourier sum allows us to write the reciprocal $1/\zeta$ as

$$\frac{1}{\zeta(\varphi)} = \begin{cases} \sqrt{\mathcal{A}_N} \exp \left\{ \frac{N}{2} \varphi + \sum_{n=1}^{\infty} \frac{t_n e^{i n \varphi}}{n} \right\} & \text{Im } \varphi > 0 \\ \sqrt{\mathcal{A}_N^*} \exp \left\{ -\frac{N}{2} \varphi + \sum_{n=1}^{\infty} \frac{t_n^* e^{-i n \varphi}}{n} \right\} & \text{Im } \varphi < 0 \end{cases} \quad (13)$$

The innocent looking sign changes relative to (12) have in fact drastic consequences. Clearly, the Fourier series expansion of $1/\zeta$ cannot be a finite-order polynomial in $e^{\pm i \varphi}$ but rather is an infinite series,

$$\frac{1}{\zeta(\varphi)} = \begin{cases} \sqrt{\mathcal{A}_N} e^{i N \varphi/2} \sum_{n=0}^{\infty} \tilde{A}_n e^{i n \varphi} & \text{Im } \varphi > 0 \\ \sqrt{\mathcal{A}_N^*} e^{-i N \varphi/2} \sum_{n=0}^{\infty} \tilde{A}_n^* e^{-i n \varphi} & \text{Im } \varphi < 0 \end{cases} \quad (14)$$

wherein the Fourier coefficients $\tilde{A}_n$ differ from the secular coefficients $A_n$ only by the absence of the sign factor in (8),

$$\tilde{A}_n = \sum_{\alpha} \prod_{l \geq 1} \frac{1}{t_{\alpha l}!} t_{\alpha l}.$$

3. Generating function

In analogy to standard practice for energy spectra, we introduce the multiplicative combination of four spectral determinants,

$$\mathcal{Z} = \left\{ \frac{\zeta(\varphi_C)\zeta(\varphi_D)}{\zeta(\varphi_A)\zeta(\varphi_B)} \right\} \quad (16)$$

The angular brackets denote two averages. One is over a $2\pi$-interval of a real center phase $\phi$ defined through

$$\varphi_{A/C} = \phi + \frac{e_{A/C}}{N}, \quad \varphi_{B/D} = \phi - \frac{e_{B/D}}{N}. \quad (17)$$

A second average is, in principle, necessary over a small offset window [6]. The generating function $\mathcal{Z}$ deserves interest since it yields the two-point function of the level density through two derivatives, see [6, 14].

To make the average $\langle \cdot \rangle = (2\pi)^{-1} \int_0^{2\pi} d\varphi \langle \cdot \rangle$ well defined, we must assign nonvanishing imaginary parts of opposite sign to the phase arguments in the denominator on the rhs of (16). Therefore, we endow the offset variables $e_{A/B}$ with positive imaginary parts. On the other hand, the phase arguments of the numerator zetas are unrestricted by the definition of $\mathcal{Z}$.

To prepare for the semiclassical analysis of the generating function $\mathcal{Z}$, we represent the two inverse zetas in definition (16) by the infinite series (14) but employ the Riemann–Siegel lookalike (11) in the numerator. The generating function thus becomes a sum of nine terms,

$$\mathcal{Z} = \sum_{i,j=+,-,0} Z_{ij}, \quad Z_{ij} = \langle \zeta(\varphi_A)^{-1} \zeta(\varphi_B)^{-1} \zeta(\varphi_C)\zeta(\varphi_D) \rangle. \quad (18)$$
The symmetries

\[ Z_+ (e_A, e_B, e_C, e_D) = Z_+ (e_A, e_B, -e_D, -e_C) \]
\[ Z_+ (e_A, e_B, e_C, e_D) = Z_- (e_A, e_B, -e_D, -e_C) \]
\[ Z_0 (e_A, e_B, e_C, e_D) = Z_0 (e_A, e_B, -e_D, -e_C) \]
\[ Z_0 (e_A, e_B, e_C, e_D) = Z_0 (e_A, e_B, -e_D, -e_C) \]

show that only five components, say, \( Z_{++}, Z_{−−}, Z_{00}, Z_{0+}, Z_{00}, \) are independent. The explicit expression for the \( Z_{ij} \) will be written as need arises. For now, we just note the example

\[ Z_{++} = e^{i (e_a + e_b - e_c - e_d)} \sum_{a,b=0}^{\infty} \sum_{c,d=0}^{\infty} \tilde{A}_{\alpha} \tilde{A}_{\beta}^* \tilde{A}_{\gamma}^* \tilde{A}_{\delta} \left( e^{i (a - b + c - d) \phi} \right) e^{i (a e_A + b e_B + c e_C + d e_D)} \]

(20)

each of the center-phase averages yields the restriction \( a + c = b + d \) such that only the offset phases remain as variables. The summations over the integers \( c, d \) go from 0 to \( \frac{N}{2} - 1 \), due to the definitions of \( \zeta_\alpha \). Clearly then, \( Z_{++} \) is a polynomial of order \( \frac{N}{2} \) both with respect to \( e^{-i e_C / N} \) and \( e^{-i e_D / N} \) without zero-order terms in either variable. It is now conventional to introduce projectors \( P_P \) and \( P_D \) leaving the powers \( e^{i e_C / N} \) and \( e^{i e_D / N} \) unchanged and killing all others. Under the protection of the product \( P_P P_D \), the sums over the integers \( c, d \) can be extended to go from 0 to infinity, such as the sums over \( a, b \). With the help of the expansion coefficients (8), (15) we obtain \( Z_{++} \) represented by

\[ Z_{++} = P_P P_D \left( e^{i (e_a + e_b - e_c - e_d)} \sum_{a,b,c,d} \Phi_{a,b,c,d} \phi^{(a+b-c-d)} \prod_{n=1}^{\infty} \right. \]
\[ \times \left. \left( \begin{array}{c} L(d) + L(c) + L(b) + L(a) \\ n \end{array} \right) \left( \begin{array}{c} \frac{\alpha}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\beta}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\gamma}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\delta}{n} \phi \end{array} \right) \prod_{n=1}^{\infty} \right) \right) \]

Similarly, we define projectors \( P_P, P_D \) which leave unchanged the phase-independent term in the respective zetas while killing all \( e^{i e_C / N} \) and \( e^{i e_D / N} \) with \( \nu \neq 0 \). We then have \( Z_{00} = P_P P_D (\cdots), Z_{+} = P_P P_D (\cdots) \) and \( Z_{-} = P_P P_D (\cdots) \) with the parentheses being the same as that for \( Z_{++} \) above. The analogous series for \( Z_{--} \) reads

\[ Z_{--} = P_P P_D \left( e^{i (e_a + e_b - e_c - e_d)} \sum_{a,b,c,d} \Phi_{a,b,c,d} \phi^{(a+b-c-d)} \prod_{n=1}^{\infty} \right. \]
\[ \times \left. \left( \begin{array}{c} L(d) + L(c) + L(b) + L(a) \\ n \end{array} \right) \left( \begin{array}{c} \frac{\alpha}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\beta}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\gamma}{n} \phi \end{array} \right) \left( \begin{array}{c} \frac{\delta}{n} \phi \end{array} \right) \prod_{n=1}^{\infty} \right) \right) \]

where the projector \( P_P \) leaves unchanged the powers \( e^{i e_C / N} \) with \( 0 < \nu \leq \frac{N}{2} \) and kills all others.

There is a price to pay for the introduction of the infinite sums over the vectors \( \tilde{c}, \tilde{d} \) in (21), (22). To ensure convergence, we must allow for infinitesimal positive imaginary parts of \( e_{C/D} \), as already done for \( e_{A/B} \).

4. Semiclassics for maps

4.1. Trace formula

We now foray into asymptotics by invoking the Gutzwiller trace formula. For the traces \( t_n \) of Floquet maps of periodically driven systems we have [6]

\[ t_n \sim \frac{1}{\sqrt{\det(M - I)}} e^{i S / \hbar} \]

(23)
a sum over period-$n$ orbits. The contribution of each orbit involves a phase factor where the action $S$ (defined to include the Maslov phase and measured in units of Planck’s constant) appears as the phase. That $S$ is the ‘time-dependent’ action which generates the periodic orbit in $n$ steps of the classical map. The prefactor involves the primitive period $n$ and accounts for the instability of the orbit in terms of the so-called monodromy matrix $M$. For the sake of simplicity, we shall treat periodically driven single-freedom systems. We thus have a single positive Lyapounov rate $\lambda$ to reckon with for each periodic orbit. For beauty, we refrain from putting subscripts distinguishing different period-$n$ orbits on the action and the monodromy matrix.

The number $\mathcal{N}(n)$ of period-$n$ orbits is known to asymptotically grow exponentially with the period $n$ [15–17]. On average, over a small window $\Delta n$ of periods centered at $\bar{n}$, such that $1 \ll \Delta n \ll \bar{n}$, one has

$$\langle \mathcal{N}(n) \rangle = e^{nt}/n.$$  \hspace{1cm} (24)

The finiteness of the number of period-$n$ orbits makes the trace formula (23) a well-defined object not in need of any regularization, quite in contrast to the more widely known one for the density of energy levels of autonomous dynamics (see the following section).

Long orbits dominate the semiclassical asymptotics, and therefore we have written the trace formula (23) without regard for $r$-fold traversals of orbits with the primitive period $n_0 = n/r$. Assuming ergodicity, we encounter one and the same Lyapounov rate for all long periodic orbits.

### 4.2. Diagonal approximation, unitary class

We now focus on the diagonal approximation for the unitary symmetry class: only those terms from the multinomial expansion of the powers of periodic-orbit sums in $(t_0)^{\alpha_1+\alpha_2} (t_0^*)^{b_1+d_1}$ are kept in which each phase factor $e^{S/n}$ pairs up with its complex conjugate such that all dependence on the actions cancels. We thus immediately obtain $[t_n]^m (t_n^*)^m]_{\text{diag}} \propto \delta_{mn}$. For fixed $m$, any choice of $m$ action-phase factors $e^{S/n}$ comes with $m!$ possibilities to pick the matching inverses. With any of these choices the $m$-fold summation over periodic orbits, considering the HOdA sum rule for ergodic maps,

$$\sum_{\text{period-} n \text{ orbits}} \frac{1}{|\det(M-1)|} \sim \frac{1}{n}, \quad n \gg 1,$$  \hspace{1cm} (25)

leads to the $n$th power of the diagonal approximation for the product $t_n t_n^*$ of just two traces

$$\begin{bmatrix} t_n^m & (t_n^*)^m \end{bmatrix}_{\text{diag}} = \delta_{nm} m! [t_n t_n^*]_{\text{diag}} = \delta_{nm} m!,$$  \hspace{1cm} (26)

a behavior reminiscent of Gaussian statistics.

Using the result (26) for $Z_{\text{diag}}^{\mathcal{C}}$ we face

$$Z_{\text{diag}}^{\mathcal{C}} = P_{\mathcal{C}}^C P_{\mathcal{D}}^D (\ldots)$$  \hspace{1cm} (27)

with the building block

$$\begin{bmatrix} \ldots \end{bmatrix} = e^{\frac{i}{\hbar} (a_1 + a_2 - \cdots - c \pi)} \prod_{n=1}^{\infty} \sum_{\alpha_1, \alpha_2 = b_1, d_1} (-1)^{\alpha_1 + \alpha_2} \frac{1}{m!} m^n \times \binom{m}{b_n} \frac{m}{a_n} e^{\frac{i}{\hbar} (a_1 + a_2 + \cdots + c \pi)}.$$  \hspace{1cm} (28)

We here do two binomial sums and sum over $m$,

$$\begin{bmatrix} \ldots \end{bmatrix} = e^{\frac{i}{\hbar} (a_1 + a_2 - \cdots - c \pi)} \exp \sum_{n=1}^{\infty} \frac{1}{n} \left( e^{i \frac{\pi}{\hbar} a_1} - e^{i \frac{\pi}{\hbar} a_2} \right) \left( e^{i \frac{\pi}{\hbar} c_1} - e^{i \frac{\pi}{\hbar} c_2} \right).$$  \hspace{1cm} (29)
It should be noted that the sum rule (25) and its consequence (26) are true when averaged over a narrow window $\Delta n$ of $n$; their use in $Z_{-+}^{\text{diag}}$ is justified inasmuch as the cofactor depending on the offset variables $e_{A/B/C/D}$ varies smoothly with $n$ on the scale $\Delta n$. By this ‘self-averaging’ in the semiclassical limit, the generating function becomes smooth without any average beyond the one over the center phase $\phi$.

Finally, using $\sum_{n}^\infty e^{i\nu n} = -\ln(1 - e^{i\nu})$, we obtain

\begin{equation}
(\cdots) = e^{i(e_{A} + e_{B} - e_{C} - e_{D})} 1 - e^{i(e_{A} + e_{B})} 1 - e^{i(e_{A} + e_{C})} 1 - e^{i(e_{A} + e_{D})}, \tag{29}
\end{equation}

Let us first throw a glance at $Z_{-+}$ as given by (22) and (26). The phase matching required by the diagonal approximation yields the restriction $a_{n} + c_{n} + d_{n} = b_{n}$, which contradicts the one obtained from the center-phase average: $\sum_{n} a_{n} + c_{n} + d_{n} - b_{n} = N$. We conclude that $Z_{-+}^{\text{diag}}$ vanishes. Due to the symmetry (19), we have $Z_{+}^{\text{diag}} = 0$ as well. The full generating function thus comes out as

\begin{equation}
Z_{\text{diag}} = Z_{00}^{\text{diag}} + (Z_{0+}^{\text{diag}} + Z_{0-}^{\text{diag}} + Z_{-+}^{\text{diag}}) + (Z_{-+}^{\text{diag}} + Z_{+0}^{\text{diag}} + Z_{+}^{\text{diag}}). \tag{30}
\end{equation}

The first four terms are obtained from $(\cdots)$ by applying respectively the projectors $P_{C}^{C}P_{D}^{D}P_{0}^{0}P_{0}^{0}$, $P_{C}^{C}P_{D}^{D}P_{-}^{0}P_{-}^{0}$, $P_{C}^{C}P_{D}^{D}P_{-}^{0}P_{0}^{0}$; the remaining ones follow from the symmetry conditions (19). To let these projectors do their job we expand $(1 - e^{i(e_{A} + e_{D})})^{-1} = \sum_{n=0}^\infty e^{i(e_{A} + e_{D})}$. After elementary calculations given in [14], we obtain a polynomial of order $N^2$ in $e_{A}^{\pm}$, $e_{B}^{\pm}$ and $e_{C}^{\pm}$. Amazingly, the final expression coincides with the sum of $(\cdots)$ and its Riemann–Siegel complement (obtained by the replacement $e_{A}, e_{D} \to -e_{D}, -e_{A}$).

\begin{equation}
Z_{\text{diag}} = e^{i(e_{A} + e_{B} - e_{C} - e_{D})} \left(1 - e^{i(e_{A} + e_{B})} 1 - e^{i(e_{A} + e_{C})} 1 - e^{i(e_{A} + e_{D})} \right) + e^{i(e_{A} + e_{B} + e_{C} + e_{D})} \left(1 - e^{i(e_{A} - e_{C})} 1 - e^{i(e_{A} - e_{D})} \right).
\end{equation}

Indeed, the denominator $1 - e^{i(e_{A} + e_{D})}$ cancels from the sum of the two terms in (31) such that the polynomial character mentioned arises.

Most remarkably, we have arrived at the exact generating function for the circular unitary ensemble [18] for finite $N$. In particular, the periodicity in the offset phases with period $\alpha N$ is a nice finite-size effect for Floquet operators of periodically driven dynamics.

Of course, in the limit $N \to \infty$, we obtain the generating function describing the correlation decay at finite offsets

\begin{equation}
\lim_{N \to \infty} Z_{\text{diag}} = e^{i(e_{A} + e_{B} - e_{C} - e_{D})/2} \left(\frac{(e_{A} + e_{D})(e_{B} + e_{C})}{(e_{A} + e_{B})(e_{C} + e_{D})} - e^{i(e_{A} + e_{B} + e_{C} + e_{D})/2} \left(\frac{(e_{A} - e_{C})(e_{B} - e_{D})}{(e_{A} + e_{B})(e_{C} + e_{D})} \right) \right), \tag{32}
\end{equation}

as for the Gaussian unitary ensemble.

The case of odd dimension $N$ leads to the same result. The pertinent calculation differs in technical details not worth being spelled out here.

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3 Strictly speaking, we were not allowed to use the sum rule (25) for small periods $n$. However, the error thus committed is of order $(n_{0}/N)^{2}$ with $n_{0}$ being some period above which the HOdA sum rule is reliable. For a rough argument, assume the offsets of order unity and consider $\sum_{n=1}^{n_{0}} (e^{i\bar{e}_{A}^{n_{0}} - e^{i\bar{e}_{C}^{n_{0}}}}(e^{i\bar{e}_{B}^{n_{0}} - e^{i\bar{e}_{D}^{n_{0}}}}/n \sim (e_{A} - e_{C})(e_{B} - e_{D}) \sum_{n=1}^{n_{0}} \frac{1}{n} \approx (\frac{n_{0}}{n_{0}})^{2}$; the argument is easily modified to allow for the period $2\pi N$ of the offset variables.
4.3. Off-diagonal corrections

Inasmuch as the foregoing diagonal approximation already gives the exact result for the unitary symmetry class, all off-diagonal contributions must cancel. There is nothing much to add to the previous work for flows of autonomous dynamics as far as the limit $N \to \infty$ is concerned. Some nontrivial modifications are needed for finite $N$ and the periodicity of $Z$. While a full proof of the cancellation of off-diagonal corrections will be reserved for a separate paper, we would like to at least give some hints here.

As previously, quadruples of non-ordered sets of periodic orbits (pseudo-orbits) come into play such that the pseudo-orbits within each quadruple have action differences of the order of Planck’s constant (see the following section for some more details). The (Feynman-type) diagrams characterizing those quadruples remain unchanged, and so is the ordering of the diagrams in terms of the number of vertices ($V$) and links ($L$) by increasing values of $L - V$. The rules for associating analytic expressions with diagrams are modified. Straightforward computer-based summations yield order-by-order cancellation of the corrections with $L - V = 1, 2, \ldots, 6$ which altogether involve 7404 non-equivalent diagrams.

5. Stroboscopic maps for autonomous flows

The rigorous Riemann–Siegel lookalike for finite-dimensional unitary matrices can also be made available for autonomous flows, by choosing a suitable stroboscopic description. We propose to show that the status of previous work benefits from that modification. In particular, the diagonal approximation for the unitary symmetry class will again lead us to the exact finite-$N$ result (31), without needing large imaginary parts for the offset variables $e_C/D$. The periodicity in the offset variables in this case is an artifact of the stroboscopic description; only the limit $N \to \infty$ is of interest here.

5.1. Classical strobe period

We consider a small stretch of the spectrum of a time-independent Hamiltonian containing $N \gg 1$ levels $\{E_1 \leq E_2 \leq \cdots \leq E_N\}$ and the pertinent time evolution operator over a certain strobe period $\tau$, to be called the Floquet operator, $F = \sum_{\mu=1}^{N} e^{-iE_\mu \tau / \hbar} |\mu\rangle \langle \mu|$. Choosing the strobe period such that the eigenphases $\phi_\mu = E_\mu \tau / \hbar$ of $F$ just fill the interval $[0, 2\pi]$ once, namely $\tau = 2\pi \hbar / (E_N - E_1)$, the increasing order of the eigenenergies carries over to the eigenphases. For large $N$, the spectral fluctuations of the Hermitian Hamiltonian and the unitary Floquet operator become identical, since the neighborship of $\phi_1$ and $\phi_N$ loses any significance.

Three timescales are of importance for us: the Heisenberg time $T_H = N\tau$, the Ehrenfest time $T_E = \lambda^{-1} \ln(S/\hbar)$ with $\lambda$ being the Lyapounov rate and $S$ some classical action scale, and the strobe period $\tau$. These times are ordered as $\tau \ll T_E \ll T_H$. In fact, we may take the strobe period to be a classical time, independent of Planck’s constant; in view of $T_H \sim \hbar^{-1/4 + 1}$ and classical $\tau$ means $N \sim \hbar^{-1/4 + 1}$. On the other hand, classical $\tau$ means an overall energy span $E_N - E_1 = O(\hbar)$, and therefore the mean level density therein, $\bar{\rho} = N / (E_N - E_1)$, needs no unfolding.
5.2. Trace formula

We adopt the Gutzwiller formula for the density of energy levels [6, 19],

$$\rho(E) \sim \frac{1}{2\pi\hbar} \sum T \frac{1}{\sqrt{\det(M - 1)}} \left( e^{iS/E} + e^{-iS/E} \right). \quad (33)$$

Here the sum is over periodic orbits on the energy shell \(E\) and \(S^0\) is the energy-dependent action (sometimes called reduced action), which as a generating function generates periodic orbits for fixed energy \(E\); finally, \(T\) denotes the primitive period. A Legendre transformation connects the energy-dependent action and its time-dependent counterpart as \(S = S^0 - ET\).

The traces of the \(N \times N\) Floquet matrix \(t_n = \sum_{i=1}^{N} e^{-i\epsilon/E_i/\hbar}\) can be written as

$$t_n = \int dE' \left[ \theta(E' - E + \Delta E/2) - \theta(E' - E - \Delta E/2) \right] \rho(E') e^{-iE\tau/E/\hbar}. \quad (34)$$

The step functions select the energy interval \([E - \Delta E/2, E + \Delta E/2]\) filled by the spectrum, such that \(\tau \Delta E / \hbar = 2\pi\). Semiclassically for \(n > 0\) we have

$$t_n \sim \frac{1}{2\pi\hbar} \sum_{E - \Delta E/2}^{E + \Delta E/2} T \frac{1}{\sqrt{\det(M - 1)}} e^{-iE\tau/E/\hbar} \left( e^{iS(E)/E} + e^{-iS(E)/E} \right). \quad (35)$$

The reference energy at the mid-point of the spectrum will henceforth be set to zero, \(E = 0\). To check how strongly the phase of the exponentials within the integrand varies across that interval, we expand \([S'(E') - E\tau/E]/\hbar = \left[ S'(0) + (T - \tau)E' + \frac{1}{2} S''(0)(E')^2 + \cdots \right]/\hbar\). The third term in that expansion defines a width \(\propto \sqrt{\hbar}\) larger than the span of the spectrum such that the term can be dropped. The second term of the expansion, on the other hand, varies by \(\Delta\phi \equiv 2\pi(T - \tau)/\tau\) across the spectral range. We can conclude that the term \(e^{-iS/E}/\hbar\) is negligible for \(n > 0\), while \(e^{iS/E}/\hbar\) plays no role for \(n < 0\). Inasmuch as we need the traces \(t_n\) only for positive \(n\) we have

$$t_n \sim \sum \frac{T}{\tau \sqrt{\det(M - 1)}} e^{iS/E/\hbar} \frac{1}{\Delta E} \int_{-\Delta E/2}^{\Delta E/2} dE' e^{i(T - \tau)E'/E/\hbar} \quad (36)$$

The function \(\text{sinc} x = \sin \pi x / \pi x\) favors orbits whose periods \(T\) do not differ from \(n\tau\) by more than a few \(\tau\) (see following subsection). The sharpness of the energy and the uncertainty of the orbit periods of the order \(\tau\) are the important differences of the present trace formula from the one valid for periodically driven systems, see equation (23). As a consequence of that difference it will turn out necessary to introduce a certain regularization below.

5.3. Diagonal approximation, unitary symmetry

We begin with calculating the product of \(t_t t_n^*\) in the diagonal approximation following from the periodic-orbit expansion (36). Denoting by \(T_0\) the smallest period above which the average number of periodic orbits with periods in \(T, T + dT\) is given by the exponential proliferation law \(dT e^{\epsilon T}/T\) and the HOdA sum rule is applicable, we have

$$[t_t t_n^*]_{\text{diag}} = \sum \left( \frac{T}{\tau \sqrt{\det(M - 1)}} \right)^2 \text{sinc}^2 \left( \frac{T}{\tau} - n \right) \quad (37)$$

This expression formally diverges at the upper limit due to the slow decay of the sinc function at large values of the argument. It is here that the regularization announced just above becomes...
necessary. To that end, suppose that in (34) we smooth the cutoffs at $E' = \pm \Delta E/2$ such that transitions from 0 to 1 and back take place through intervals of the order of the mean level spacing $\delta E = \Delta E/N$. That smoothing cannot notably change traces of the Floquet matrix. However, in the periodic-orbit expansion the contributions with periods significantly larger than the Heisenberg time will be suppressed. The integral (37) then becomes finite with the main contribution from the region where the argument of sinc is close to zero such that $T$ can be replaced by $nt \tau$. After that the integration interval can be safely extended to $[-\infty, \infty]$ with the result

$$[t_n t_n^*</{\text{diag}}'] = n \int_{-\infty}^{\infty} \frac{dT}{\tau} \sin^2 \left( \frac{T}{\tau} - n \right) = n. \quad (38)$$

The diagonal approximation for products of higher powers of traces $t_n^m(t_n^*)^k$ can be expressed in terms of $[t_n t_n^*]_{\text{diag}}$ in the same way as for periodically driven systems, and we recover equation (26).

Likewise, all subsequent steps of the diagonal approximation in section 4.2 are unchanged such that the generating function (31) is rederived. In the present case of autonomous flows, only the limit of small offsets, $e_{A/B/C/D}/N \to 0$ yielding (32), is of physical interest. Two significant advantages over the derivation in [1, 2, 6] are worth noting. First, we now need only infinitely small imaginary parts of the offset phases $e_{C/D}$. Second, as long as $N$ is finite, the finiteness of the Fourier series in the variables $e_{C/D}$ is rigorously preserved; while the Riemann–Siegel components of $Z$ individually have non-vanishing contributions from pseudo-orbits with periods $T > T_H/2$, their sum enjoys cancellation of these contributions.

5.4. Quadruples of pseudo-orbits for $Z$

We recall the Riemann–Siegel lookalike (11) and the ensuing decomposition (18) of the generating function. It is now convenient to write the component $Z_{\pm \pm}$ in the form

$$Z_{\pm \pm} = \left\{ \mathcal{P}^C \mathcal{P}^D e^{iN(\varphi_4 - \varphi_2 + \varphi_1 - \varphi_3)} \exp \sum_{n=1}^{\infty} \left\{ \frac{t_n}{n} e^{i\varphi_1} - e^{i\varphi_3} + \frac{t_n^*}{n} e^{-i\varphi_2} - e^{-i\varphi_4} \right\} \right\}, \quad (39)$$

which is easily seen to be correct when using the projectors $\mathcal{P}^C, \mathcal{P}^D$ in (12).

After invoking the trace formula (36) the sums over the traces in (39) are expressed in terms of the periodic orbits,

$$\sum_{n=1}^{\infty} \frac{t_n}{n} e^{i\varphi_1} \sim \sum e^{iS^p/\hbar} \frac{T}{\sqrt{\det(M - 1)}} \sum_{n=1}^{\infty} \frac{1}{n \tau} e^{i\varphi_1} \sin \left( \frac{T}{\tau} - n \right). \quad (40)$$

Inasmuch as terms with $n \sim T/\tau$ dominate in the inner sum, we may replace the factor $T_{\tau}$ with unity, accepting a relative error of the order $1/N$. The lower limit of the sum over $n$ can then be shifted to $-\infty$.

Next, we employ the discrete Fourier transform

$$\sum_{n=-\infty}^{\infty} e^{i\varphi_1} \sin(x - n) = e^{i\varphi_1} \sin(x), \quad (41)$$

where $\varphi_1$ is the $(2\pi)$-periodic sawtooth function equaling $\varphi$ in the interval $[-\pi, \pi]$. For using further below, we note right away that restricting the sum over $n$ to the range $|n| \leq \frac{N}{2}$

4 We can, e.g., replace the $\theta$s in (34) by the error functions $\frac{1}{\sqrt{\pi}} \left[ 1 + \text{erf} \left( \frac{E-E_0}{\Delta} \right) \right]$. The periodic-orbit expansion of traces (35) will then acquire the Gaussian factor $e^{-T^2/\tau_H^2}$, $\tau_H = 2\pi \hbar/\delta E$, and the result (38) will be correct to within corrections of order of $\ln N$, which may be neglected compared with the relevant values of $n$. 

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means putting a soft cutoff on the parameter $x$ to $|x| < \frac{N}{2}$, because for fixed $n$ and varying $x$ the function $\text{sinc}(x - n)$ is weakly localized near $x = n$. At any rate, the sum in (40) can be approximated as

$$
\sum_{n=1}^{N} \frac{t_n}{n} e^{i n \omega_T} \sim \sum_{\pm} \frac{1}{\sqrt{|\det(M - 1)|}} e^{i \theta/\hbar + i |\theta|/\tau},
$$

and herein the truncation $n < \frac{N}{2}$ would entail the soft cutoff $T < \frac{N}{2}$.

Four such sums appear exponentiated in expression (39) for $Z_1$. We Taylor expand all four pertinent exponentials and so obtain a fourfold sum over pseudo-orbits. Let us denote for brevity by $F_A$ the product of the stability coefficients $1/\sqrt{\det(M - 1)}$ of periodic orbits comprising the pseudo-orbit $A$ and $F_{B/C/D}$ denote similar properties for the pseudo-orbits $B, C$ and $D$. Neglecting orbit repetitions and again representing the four phases $\varphi_{A/B/C/D}$ as in (17) we find [1, 2, 6]

$$
Z_{-+} \sim \mathcal{P}^C \mathcal{P}^D e^{(\epsilon_1 + r - \epsilon_e - r_0)/2} \sum_{A, B, C, D} (-1)^{\nu_A + \nu_B} F_A F_B F_C F_D \times \exp \left[ \frac{i}{\hbar} \left( S_A + S_C - S_B - S_D \right) \right]
$$

$$
\times \left\{ \exp \left[ \frac{i}{\tau} \left( T_A (\phi + \frac{2\pi}{N}) + T_C (\phi + \frac{2\pi}{N}) - T_B (\phi - \frac{2\pi}{N}) - T_D (\phi - \frac{2\pi}{N}) \right) \right] \right\},
$$

Here $A, B, C, D$ are the pseudo-orbits associated with the denominator (numerator) of the generating function; $v_C$ and $v_B$ stand for the number of the orbits in $C, D$.

Next we turn to the center-phase average over the interval $-\pi < \phi < \pi$. We write the sawtooth functions as $[\phi + \delta] = \phi + \delta - 2\pi \theta (\phi - \delta)$ and $[\phi - \delta] = \phi - \delta - 2\pi \theta (-\pi - \phi + \delta)$, for $\delta \geq 0$. The $\phi$ dependence outside the step functions in the exponent is given by $i\phi \Delta T/\tau$, where $\Delta T = T_A + T_C - T_B - T_D$ is the period mismatch between the pseudo-orbits $A \cup C$ and $B \cup D$. For partner pseudo-orbits, the period and action mismatches are both of order $\hbar$, and therefore $i\phi \Delta T/\tau$ can be dropped in the semiclassical limit. The remaining elementary but cumbersome integral over $\phi$ is periodic in $\frac{2\pi}{N}$. The projectors $\mathcal{P}^C \mathcal{P}^D$ serve to truncate the respective Fourier expansions at $-\frac{N}{2}$, the cumulative periods $T_{C/D}$ of the pseudo-orbits $C, D$ are thus restricted to be smaller than half the Heisenberg time. For autonomous flows, the remaining periodicity in the offset variables is an artifact of the stroboscopic description. Of physical interest is only the limit $e_{A/B/C/D}/N \to 0$. In that limit, the step functions in the exponent can be dropped since they make only for $O(1/N)$ corrections. The projectors $\mathcal{P}^C \mathcal{P}^D$ lose their meaning and are to be removed, but the soft cutoff for the periods $T_{C/D}$ at half the Heisenberg time must be kept. The final result thus reached,

$$
Z_{-+} \sim e^{(\epsilon_1 + r - \epsilon_e - r_0)/2} \sum_{A, B, C, D} (-1)^{\nu_A + \nu_B} F_A F_B F_C F_D
$$

$$
\times e^{i(T_A + T_C + T_B + T_D)/\hbar} \left[ \exp \left( i(S_A + S_C - S_B - S_D) \right) \right],
$$

is identical to the one previously obtained in [1, 2, 6]. In a quite analogous manner, we recover, in our present context, the previous result $Z_{-\oplus} \sim 0$. Equally ignorable is then the distinction of the zeroth Fourier component $\zeta_0$ and we can simply write $Z = Z_{-+} + Z_{+-}$.

We would like to note that the reasoning in the present subsection is not restricted to the unitary symmetry class.

6. Summary and discussion

We employ the rigorous Riemann–Siegel lookalike available for unitary $N \times N$ matrices to treat spectral fluctuations both for periodically driven and autonomous dynamics.
For periodically driven systems, a finite Gutzwiller trace formula arises for the traces of powers of the Floquet matrix, \( t_n = \text{Tr} F^n \). Only period-\( n \) orbits are involved in \( t_n \). The Riemann–Siegel lookalike then represents the spectral determinant by pseudo-orbits with cumulative periods up to \( n = N/2 \), half the dimensionless Heisenberg time \( N \). Restricting ourselves to the unitary symmetry class (no time reversal invariance), we find the diagonal approximation to yield the exact generating function of the two-point correlator of the density of quasi-energies provided by the circular unitary ensemble of random matrices [18], for finite matrix dimension \( N \). That generating function reduces to the one for the Gaussian unitary ensemble in the limit \( N \to \infty \). All corrections, due to partnerships of periodic (pseudo-)orbits generated in close self-encounters, must therefore cancel. Using computer-based summation of the contributing partner pseudo-orbits we ascertain order-by-order cancellation in the lowest six orders; the order is given by the difference \( L - V \), with \( V \) being the overall number of relevant self-encounters and \( L \) their overall number of encounter stretches.

For autonomous flows, we define a suitable strobe period and the pertinent time evolution operator to obtain a Floquet matrix capturing a finite number \( N \) of energy levels as quasi-energies. The pertinent trace formula, derived from Gutzwiller’s one for the density of energy levels, needs regularization. Again, for the unitary symmetry class the diagonal approximation yields the exact finite-\( N \) CUE generating function and, with \( N \to \infty \), the exact GUE result. We recover the previously obtained pseudo-orbit expansions [1–3] for higher order corrections in the limit \( N \to \infty \), for all Wigner–Dyson symmetry classes.

A certain charm and, in fact, progress over the previous treatment of autonomous flows can be seen in two facts. First, the polynomial character of the secular determinant \( \text{det}(1 - e^{i\phi} F) \) carries over to the generating function; the latter retains its polynomial nature under the pseudo-orbit expansion, even when the restriction imposed by the Riemann–Siegel lookalike, \( n \leq N/2 \), is lifted. Second, only infinitesimal imaginary parts are needed for the quasi-energy variables. That progress is owed to the rigorous Riemann–Siegel lookalike.

We would like to pay respect to related work. The explicit solution of Newton’s relations giving the coefficients \( A_n \) of the secular polynomial in terms of the traces \( t_n \) has a long history. Some recent references are [4–6]. The rigorous finite-\( N \) Riemann–Siegel lookalike has been noted before [5, 6, 10–13, 20]. Already more than a decade ago, the possibility of semiclassically accessing spectral fluctuations using Newton’s relations and Riemann–Siegel for unitary matrices was suggested [6, 20]. First estimates of the oscillatory part of the spectral correlator based on Riemann–Siegel were made by Bogomolny and Keating [21]. Reasons for expecting cancellation of contributions from pseudo-orbits with periods beyond half the Heisenberg time in correlators of spectral determinants were put forth in [22]. More recently, such cancellation has been demonstrated for spectral fluctuations in graphs [23]. Of course, the exactness of the diagonal approximation for the small-time form factor of dynamics without time reversal invariance was already found by Berry in the 1985 paper [24] which marked the beginning of the semiclassical analysis of spectral statistics.

Postponed to future work is a demonstration of the cancellation of corrections to all orders of the ‘diagrammatic’ expansion for Floquet matrices of the unitary symmetry class. An adaption of the strategy of [2] should not be overly difficult to come by, even for the orthogonal and symplectic symmetry classes.

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