ON FRACTIONAL REGULARITY OF DISTRIBUTIONS OF FUNCTIONS IN GAUSSIAN RANDOM VARIABLES

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Abstract

We study fractional smoothness of measures on $\mathbb{R}^k$, that are images of a Gaussian measure under mappings from Gaussian Sobolev classes. As a consequence we obtain Nikolskii–Besov fractional regularity of these distributions under some weak nondegeneracy assumption.

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1. Introduction

Let $\gamma$ be a Gaussian measure on a locally convex space $E$ and let $f: E \to \mathbb{R}^k$ be a polynomial mapping. It was shown in [5] and [9] that the density of the image measure $\gamma \circ f^{-1}$ belongs to a certain Nikolskii–Besov class. Here we consider a general Sobolev mapping $f \in W^{p,2}(\gamma)$ and provide an estimate of the total variation norm $\| (\gamma \circ f^{-1})_h - \gamma \circ f^{-1} \|_{TV}$ in terms of the behavior of $\gamma(\Delta f \leq t)$ (see Theorems 4.1, 5.1 and Corollaries 4.1, 5.1), where $\mu_h(A) := \mu(A - h)$ is the shift of the measure $\mu$ to the vector $h$, and where $\Delta f$ is the determinant of the Malliavin matrix $M_f$ of the mapping $f$ (all the necessary definitions are given in the first section).

This result provides a quantitative estimate of smoothness of $\gamma \circ f^{-1}$ and complements the classical theorem (see [4, Theorem 9.2.4]) which asserts that such a distribution possesses a density with respect to the standard Lebesgue measure if $\Delta f(x) \neq 0$ for $\gamma$-almost every point $x$. However, it
should be mentioned that in this classical result only the inclusion of $f$ to the first Sobolev class is assumed. We also note that in [11, Theorem 2.11] the lower semi-continuity of densities of such distributions was established.

The obtained results also provide a quantitative estimate in the following qualitative theorem (see [7] and [5], which generalizes [11, Theorem 2.14]). Let $f_n = (f_{n,1}, \ldots, f_{n,k}) : E \to \mathbb{R}^k$ be a sequence of functions such that $f_{n,i} \in W^{4k,2}(\gamma)$. Set

$$\delta(\varepsilon) := \sup_n \gamma(\Delta f_n \leq \varepsilon)$$

and assume that

$$\sup_n \|f_n\|_{W^{4k,2}(\gamma)} = a < \infty \quad \text{and} \quad \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0.$$ 

If the sequence of measures $\gamma \circ f_n^{-1}$ converges in distribution, it also converges in variation. Corollary 5.2 of the present paper asserts that under the same assumptions one has

$$\|\gamma \circ f_n^{-1} - \nu\|_{TV} \leq C(k,a) \left[ \delta(\|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/8}) \right]^{1/(4k)} + \|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/(32k)},$$

where $\nu$ is the limiting distribution and $\|\cdot\|_{KR}$ is the Kantorovich–Rubinstein norm, which metrizes weak convergence of probability measures. A similar bound is also valid for mappings from $W^{p,2}(\gamma)$ for any $p > 4k - 1$, which is also an improvement of the above result.

The approach in this work is similar to the classical Malliavin method developed in [11] (see also [4] and [16]). The main idea of the method is to obtain bounds of the form

$$\int \varphi^{(n)}(f) d\gamma \leq C_n \sup_t |\varphi(t)|, \quad \forall \varphi \in C_0^\infty(\mathbb{R})$$

which yields that the density of $\gamma \circ f^{-1}$ is infinitely differentiable. In works [5], [9], the Malliavin condition was modified to treat the case of Nikolskii–Besov fractional smoothness of distributions. In this work we similarly employ the results of [10] which estimate the quantity $\|\mu_h - \mu\|_{TV}$ in terms of the function

$$\sigma(\mu, t) := \sup \left\{ \int \partial_v \varphi \, d\mu : \|\varphi\|_\infty \leq t, \|\partial_v \varphi\|_\infty \leq 1 \right\},$$

where the supremum is taken over all functions $\varphi \in C_0^\infty(\mathbb{R}^k)$ and unit vectors $e$.

To apply the classical Malliavin method one should assume some non-degeneracy of mapping $f$, for example in the form of integrability of $\Delta f^{-1}$ to some power $p > 1$. Such condition is sometimes very restrictive and difficult for verification. For example, the required integrability is not valid for polynomial mappings. Nevertheless, for polynomials on Gaussian space,
the following weak nondegeneracy condition holds: $\Delta_f^{-1}$ is integrable to every power $\theta < \frac{1}{2d(k-1)}$ (this follows from the Carbery–Wright inequality [8], [12]). Thus, a natural question is to investigate the smoothness properties of distributions $\gamma \circ f^{-1}$ for Sobolev mappings $f$ under the weak nondegeneracy assumption of the integrability of $\Delta_f^{-1}$ to some power $\theta \in (0, 1)$. Corollaries 4.3 and 5.3 give the Nikolskii–Besov fractional smoothness of distributions $\gamma \circ f^{-1}$ for Sobolev mappings $f$ under such weak assumption which generalizes the results of [5] about the polynomial mappings. Our results also give an estimate of the total variation distance between two such distributions under a common weak nondegeneracy assumption in terms of the Kantorovich–Rubinstein distance between these distributions.

2. Definitions and notations

In this section we introduce the definitions and notation used throughout the paper.

Let $C_0^\infty(\mathbb{R}^n)$ denote the space of all infinitely smooth functions with compact support and let $C_b^\infty(\mathbb{R}^n)$ denote the space of all bounded smooth functions with bounded derivatives of every order. The standard Euclidian inner product on $\mathbb{R}^k$ is denoted by $\langle \cdot, \cdot \rangle$, and the standard norm is denoted by $|\cdot|$. For the standard Lebesgue measure on $\mathbb{R}^k$ we will use the symbol $\lambda^k$.

Let $\mu$ be a bounded measure on a measurable space. Recall that $\mu \circ f^{-1}$ denotes the image of the measure $\mu$ under a $\mu$-measurable mapping $f$, i.e., the following equality holds:

$$\mu \circ f^{-1}(A) = \mu(f^{-1}(A)).$$

For a Borel measure $\mu$ on $\mathbb{R}^k$, its shift to the vector $h$ is the measure $\mu_h$ defined by the equality

$$\mu_h(A) = \mu(A - h) \quad \text{for every Borel set } A.$$

The total variation norm of a Borel measure $\mu$ on $\mathbb{R}^k$ (possibly signed) is defined by the equality

$$\|\mu\|_{TV} := \sup \left\{ \int \phi \, d\mu, \, \phi \in C_0^\infty(\mathbb{R}^k), \, \|\phi\|_{\infty} \leq 1 \right\},$$

where

$$\|\phi\|_{\infty} := \sup_{x \in \mathbb{R}^k} |\phi(x)|.$$

The Kantorovich–Rubinstein norm (which is sometimes called the Fortet–Mourier norm) of a Borel measure $\mu$ on $\mathbb{R}^k$ is defined by the formula

$$\|\mu\|_{KR} := \sup \left\{ \int \phi \, d\mu : \phi \in C_0^\infty(\mathbb{R}^k), \, \|\phi\|_{\infty} \leq 1, \, \|\nabla \phi\|_{\infty} \leq 1 \right\}.$$
We note here that, for probability measures, convergence in the Kantorovich–Rubinstein norm is equivalent to weak convergence (convergence in distribution for random variables). We also introduce the Kantorovich norm of a measure \( \mu \) on \( \mathbb{R}^k \) with finite first moment (\( \int |x| |\mu|(dx) < \infty \)) and with \( \mu(\mathbb{R}^k) = 0 \):

\[
\|\mu\|_K := \sup \left\{ \int \varphi \, d\mu, \varphi \in C_0^\infty(\mathbb{R}^k), \|\nabla \varphi\|_\infty \leq 1 \right\}.
\]

We recall (see [2], [13], and [17]) that the Nikolskii–Besov space \( B^\alpha(\mathbb{R}^k) := B^\alpha_1,\infty(\mathbb{R}^k) \) with \( \alpha \in (0, 1) \) consists of all functions \( \rho \in L^1(\mathbb{R}^k) \) for which there is a constant \( C \) such that for every \( h \in \mathbb{R}^k \) one has

\[
\int_{\mathbb{R}^k} |\rho(x + h) - \rho(x)| \, dx \leq C|h|^{\alpha}.
\]

When the function \( \rho \) is the density (with respect to \( \lambda^k \)) of the measure \( \mu \) the above condition can be represented in the following form:

\[
\|\mu_h - \mu\|_{TV} \leq C|h|^{\alpha}.
\]

We now recall several facts about Gaussian measures on locally convex spaces.

Let \( E \) be a locally convex space with the topological dual \( E^* \). Let \( \gamma \) be a centered Gaussian measure on \( E \), i.e. it is a Radon measure such that every functional \( \ell \in E^* \) is a normally distributed random variable with zero mean (its distribution is either the Dirac measure at zero or has a centered Gaussian density). Let \( H \subset E \) be the Cameron–Martin space of the measure \( \gamma \) consisting of all vectors \( h \) with finite Cameron–Martin norm \( |h|_H < \infty \), where

\[
|h|_H = \sup \left\{ \ell(h): \int_E \ell^2 \, d\gamma \leq 1, \ell \in E^* \right\}.
\]

For the standard Gaussian measure on \( \mathbb{R}^n \), the Cameron–Martin space is \( \mathbb{R}^n \) itself. For a general Radon Gaussian measure, the Cameron–Martin space is a separable Hilbert space (see [3, Theorem 3.2.7 and Proposition 2.4.6]) with the inner product \( \langle \cdot, \cdot \rangle_H \) generated by \( |\cdot|_H \).

It is known (see, for example, [3, Section 2.10]) that for an arbitrary orthonormal family \( \{\ell_i\}_{i=1}^\infty \subset E^* \) in \( L^2(\gamma) \) there is an orthonormal family \( \{e_i\}_{i=1}^\infty \) in \( H \) such that \( \ell_i(e_j) = \delta_{i,j} \). Let \( \gamma_n \) be the distribution of the vector \( (\ell_1, \ldots, \ell_n) \) on \( \mathbb{R}^n \). This distribution is the standard Gaussian measure on \( \mathbb{R}^n \) with density \( (2\pi)^{-n/2} \exp(-|x|^2/2) \).

For a function \( f \in L^p(\gamma) \) we set

\[
\|f\|_p := \|f\|_{L^p(\gamma)} := \left( \int |f(x)|^p \, \gamma(dx) \right)^{1/p}, \quad p \in [1, \infty).
\]

Let \( FC^\infty(E) \) be the set of all functions \( \varphi \) of the form

\[
\varphi(x) = \psi(\ell_1(x), \ldots, \ell_n(x)),
\]
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where \( \psi \in C_0^\infty(\mathbb{R}^n) \) and \( n \in \mathbb{N} \).

For a function \( \varphi \in \mathcal{F}C^\infty(E) \) of the form \( \varphi(x) = \psi(\ell_1(x), \ldots, \ell_n(x)) \) set

\[
D^1\varphi(x) = \nabla \varphi(x) = \sum_{j=1}^n (\partial_j \psi)(\ell_1(x), \ldots, \ell_n(x)) e_j,
\]

\[
(D^2\varphi)_{i,j}(x) = (\partial_i \partial_j \psi)(\ell_1(x), \ldots, \ell_n(x)).
\]

The Sobolev space \( W^{p,m}(\gamma) \), \( m \in \{1, 2\} \), is the closure of the class \( \mathcal{F}C^\infty(E) \) with respect to the norm

\[
\|\varphi\|_{W^{p,m}(\gamma)} := \|\varphi\|_p + \sum_{i=1}^m \|D^i\varphi\|_p,
\]

where \( \|D^1\varphi\|_p := \|\nabla \varphi\|_{L^p}, \|D^2\varphi\|_p := \||D^2\varphi|_{HS}\|_p, \) and \( |\cdot|_{HS} \) is the Hilbert–Schmidt norm.

Let \( L \) be the Ornstein–Uhlenbeck operator defined by

\[
L\varphi(x) = \Delta \varphi(x) - \langle x, \nabla \varphi(x) \rangle
\]

for \( \varphi \in C_0^\infty(\mathbb{R}^n) \), where \( \Delta \) is the Laplace operator. We note that

\[
\|L\varphi\|_{L^p(\gamma)} \leq c_1(p) \|\varphi\|_{W^{p,2}(\gamma)}
\]

for \( p > 1 \) with some constant \( c_1(p) \) depending only on \( p \) (see [3, Theorem 5.7.1]).

Let \( f : E \to \mathbb{R}^k \) be a mapping such that its components \( f_1, \ldots, f_k \) belongs to \( W^{1,1}(\gamma) \). Let us define the Malliavin matrix \( M_f \) of the mapping \( f \) by

\[
M_f(x) = (m_{i,j}(x))_{i,j \leq k}, \quad m_{i,j}(x) := \langle \nabla f_i(x), \nabla f_j(x) \rangle_H.
\]

Let

\[
A_f := \{a_{i,j}\}
\]

be the adjugate matrix of \( M_f \), i.e., \( a_{i,j} = M_{ji} \), where \( M_{ji} \) is the cofactor of \( m_{ji} \) in the matrix \( M_f \). Set

\[
\Delta_f := \det M_f.
\]

Note that

\[
\Delta_f : M_f^{-1} = A_f. \tag{2.1}
\]

For a function \( g \geq 0 \) we set

\[
u_\gamma(g, \varepsilon) := \int_0^\infty (s + 1)^{-2\gamma} (g \leq \varepsilon s) \, ds.
\]

We need the following simple lemma.

**Lemma 2.1.** For an arbitrary function \( g \geq 0 \) and arbitrary numbers \( r \geq 1, \varepsilon > 0 \) one has

\[
\int (g + \varepsilon)^{-r} d\gamma \leq \tau \varepsilon^{-r} u_\gamma(g, \varepsilon).
\]
Proof. By Fubini’s theorem and Chebyshev’s inequality one has
\[
\int (g + \varepsilon)^{-r} \, d\gamma = r \int_0^{\varepsilon^{-1}} t^{r-1} \gamma((g + \varepsilon)^{-1} \geq t) \, dt
\]
\[
= r \int_0^{\infty} (s + \varepsilon)^{-r-1} \gamma(g \leq s) \, ds \leq r \varepsilon^{-r} \int_0^{\infty} (s + 1)^{-r-1} \gamma(g \leq \varepsilon s) \, ds
\]
\[
\leq r \varepsilon^{-r} \int_0^{\infty} (s + 1)^{-2} \gamma(g \leq \varepsilon s) \, ds.
\]
The lemma is proved.

3. Smoothness properties of measures on \( \mathbb{R}^k \)

The following modulus of continuity plays a crucial role below.

**Definition 3.1.** For a measure \( \mu \) on \( \mathbb{R}^k \) and \( t > 0 \) we set
\[
\sigma(\mu, t) := \sup \left\{ \int \partial_e \varphi \, d\mu : \|\varphi\|_{\infty} \leq t, \|\partial_e \varphi\|_{\infty} \leq 1 \right\},
\]
where the supremum is taken over all functions \( \varphi \in C_0^\infty(\mathbb{R}^k) \) and over all unit vectors \( e \).

The following theorem is proved in [10] (see also [6]).

**Theorem 3.1.** For any measure \( \mu \) on \( \mathbb{R}^k \) one has
\[
\|\mu_h - \mu\|_{TV} \leq 2\sigma(\mu, |h|/2), \quad \sigma(\mu, t) \leq 6k \sup_{|h| \leq t} \|\mu_h - \mu\|_{TV}.
\]

This theorem implies that the measure \( \mu \) is absolutely continuous with respect to Lebesgue measure if (and only if) \( \sigma(\mu, t) \to 0 \) as \( t \to 0 \).

The modulus of continuity \( \sigma(\mu, \cdot) \) can be used to compare different distances on the space of probability measures. In the following theorem we estimate the total variation distance between two probability measures \( \mu \) and \( \nu \) in terms of the Kantorovich–Rubinstein distance and the quantity \( \sigma(\mu - \nu, \cdot) \). This result generalizes some estimates from [5] and [9].

**Lemma 3.1.** Let \( \mu \) and \( \nu \) be two probability measures on \( \mathbb{R}^k \). Then for any \( \varepsilon \in (0, 1) \) one has
\[
\|\mu - \nu\|_{TV} \leq 3\sqrt{k} \sigma(\mu - \nu, \varepsilon) + \sqrt{k} \varepsilon^{-1} \|\mu - \nu\|_{KR}.
\]
In particular, since \( \sigma(\mu - \nu, \varepsilon) \leq \sigma(\mu, \varepsilon) + \sigma(\nu, \varepsilon) \), we have
\[
\|\mu - \nu\|_{TV} \leq 6\sqrt{k} \max\{\sigma(\mu, \varepsilon), \sigma(\nu, \varepsilon)\} + \sqrt{k} \varepsilon^{-1} \|\mu - \nu\|_{KR}.
\]
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Proof. Set

\[ \rho(x) = (2\pi)^{-k/2} \exp(-|x|^2/2) \]

and

\[ \rho_\varepsilon(x) = \varepsilon^{-k} \rho(t/\varepsilon). \]

For the measure \( \omega := \mu - \nu \) we have

\[ \|\mu - \nu\|_{TV} = \|\omega\|_{TV} \leq \|\omega - \omega \ast \rho_\varepsilon\|_{TV} + \|\omega \ast \rho_\varepsilon\|_{TV}, \]

where \( \omega \ast \rho_\varepsilon \) is the convolution of the measures \( \omega \) and \( \rho_\varepsilon \) \( dx \). For the first term above, we have

\[ \|\omega - \omega \ast \rho_\varepsilon\|_{TV} \leq \int_{\mathbb{R}^k} \|\omega - \omega_{\varepsilon y}\|_{TV} \rho(y) \, dy \leq 2 \int_{\mathbb{R}^k} \sigma(\omega, \varepsilon |y|/2) \rho(y) \, dy. \]

For the second term, we have

\[ \|\omega \ast \rho_\varepsilon\|_{TV} = \sup_{\varphi \in C^\infty_0(\mathbb{R}^k)} \int_{\mathbb{R}^k} \varphi(x) \int_{\mathbb{R}^k} \rho_\varepsilon(x - y) \omega(dy) \, dx = \sup_{\varphi \in C^\infty_0(\mathbb{R}^k)} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \varphi(x) \rho_\varepsilon(x - y) \, dx \omega(dy). \]

Note that

\[ \nabla \int_{\mathbb{R}^k} \varphi(x) \rho_\varepsilon(x - y) \, dx = \varepsilon^{-k} \int_{\mathbb{R}^k} \varphi(x) \varepsilon^{-1} (\nabla \rho)((x - y)/\varepsilon) \, dx. \]

Thus, the Lipschitz constant of the function

\[ y \mapsto \int_{\mathbb{R}^n} \varphi(x) \rho_\varepsilon(x - y) \, dx \]

can be estimated from above by \( \varepsilon^{-1} \int_{\mathbb{R}^n} |\nabla \rho(x)| \, dx \leq \varepsilon^{-1} \sqrt{k} \). Moreover,

\[ \left| \int_{\mathbb{R}^n} \varphi(x) \rho_\varepsilon(x - y) \, dx \right| \leq \varepsilon^{-1} \sqrt{k} \]

for \( \varepsilon \in (0,1) \). So,

\[ \|\omega\|_{TV} \leq 2 \int_{\mathbb{R}^k} \sigma(\omega, \varepsilon |y|/2) \rho(y) \, dy + \sqrt{k} \varepsilon^{-1} \|\omega\|_{KR} \]

\[ = 2 \int_{|y| \leq 2} \sigma(\omega, \varepsilon |y|/2) \rho(y) \, dy + 2 \int_{|y| > 2} \sigma(\omega, \varepsilon |y|/2) \rho(y) \, dy + \sqrt{k} \varepsilon^{-1} \|f\|_{KR}. \]

In the first integral \( \sigma(\omega, \varepsilon |y|/2) \leq \sigma(\omega, \varepsilon) \) by monotonicity of the function \( \sigma(\omega, \cdot) \) and in the second integral \( \sigma(\omega, \varepsilon |y|/2) \leq |y|/2 \sigma(\omega, \varepsilon) \), since \( \sigma(\mu, \varepsilon t) \leq t \sigma(\mu, \varepsilon) \) for \( t \geq 1 \). Thus,

\[ \|\omega\|_{TV} \leq c_k \sigma(\omega, \varepsilon) + \sqrt{k} \varepsilon^{-1} \|f\|_{KR}, \]
where $c_k = 2\int_{|y|\leq 2} \rho(y)dy + \int_{|y|>2} |y|\rho(y)dy \leq 2 + \sqrt{k} \leq 3\sqrt{k}$. The lemma is proved.

Remark 3.1. By a similar reasoning, one can prove that, for an arbitrary pair of probability measures $\mu$ and $\nu$ on $\mathbb{R}^k$ and any $\varepsilon > 0$, one has

$$\|\mu - \nu\|_{TV} \leq 3\sqrt{k}\nu(\mu - \nu, \varepsilon) + \sqrt{k}\varepsilon^{-1}\|\mu - \nu\|_K.$$ 

4. One-dimensional case

In this section we study smoothness properties of the distribution $\gamma \circ f^{-1}$ on the real line generated by a Sobolev smooth function $f$ on a locally convex space equipped with a centered Gaussian measure $\gamma$.

We start with the following technical lemma.

Lemma 4.1. Let $p > 1$, $r \geq 1$, $a > 0$. Then there is a constant $c(p)$ depending only on $p$ such that for every function $f \in W^{p,2}(\gamma)$ with

$$\|f\|_{W^{p,2}(\gamma)} \leq a,$$

and for every function $g \in W^{r,1}(\gamma) \cap L^\infty(\gamma)$ one has

$$\int \frac{\langle \nabla g, \nabla f \rangle_H}{\|\nabla f\|_H^2 + \varepsilon^2} d\gamma \leq (c(p)a\|g\|_\infty \varepsilon^{-2}u(\|\nabla f\|_H, \varepsilon)^{1-1/p})$$

for any $\varepsilon > 0$.

Proof. We first assume that the functions $g, f$ belong to $FC^\infty(E)$ and are of the form $g = g(\ell_1, \ldots, \ell_n)$, $f = f(\ell_1, \ldots, \ell_n)$. Integrating by parts, we have

$$\int_E \frac{\langle \nabla g, \nabla f \rangle_H}{\|\nabla f\|_H^2 + \varepsilon^2} d\gamma_n = \int_{\mathbb{R}^n} \frac{\langle \nabla g, \nabla f \rangle}{\langle \nabla f, \nabla f \rangle + \varepsilon^2} d\gamma_n$$

$$= -\int_{\mathbb{R}^n} g \frac{L_f}{\langle \nabla f, \nabla f \rangle + \varepsilon^2} - 2\frac{\langle D^2 f \cdot \nabla f, \nabla f \rangle}{\langle \nabla f, \nabla f \rangle + \varepsilon^2} d\gamma_n$$

$$\leq \|g\|_\infty \int_{\mathbb{R}^n} \frac{|L_f|}{\langle \nabla f, \nabla f \rangle + \varepsilon^2} d\gamma_n + 2 \int_{\mathbb{R}^n} \frac{\|D^2 f\|_H}{\langle \nabla f, \nabla f \rangle + \varepsilon^2} d\gamma_n$$

$$\leq \|g\|_\infty (\|L_f\| + 2\|D^2 f\|_p) \|\|\nabla f\|_H^2 + \varepsilon^2\|^{-1/p-1})$$

$$\leq (c_1(p) + 2)\|g\|_\infty \|f\|_{W^{p,2}(\gamma)} \|\|\nabla f\|_H^2 + \varepsilon^2\|^{-1/p-1}), \quad (4.1)$$

where $L$ is the Ornstein–Uhlenbeck operator associated with the standard Gaussian measure $\gamma_n$. 
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Thus, we have a function $f$ such that $f^n \to f$ in $W^{p,2}(\gamma)$ which also converges almost everywhere along with first and second derivatives. Passing to the limit in the above inequality we obtain the same inequality for a general function $f \in W^{p,2}(\gamma)$ and a function $g \in FC^\infty(E)$. Now, for a function $g \in W^{r,1}(\gamma) \cap L^\infty(\gamma)$ we can take functions $g^n \in FC^\infty(E)$ such that $g^n \to g$ in $W^{r,1}(\gamma)$ and almost everywhere. Let us consider function $\varphi \in C^\infty_0(\mathbb{R})$ such that $\varphi(t) = 1$ for $t \in [-\|g\|_\infty, \|g\|_\infty]$ and $|\varphi(t)| \leq 2\|g\|_\infty$. Then the sequence $\{\varphi(g^n)\}$ also converges to the function $g$ in $W^{r,1}(\gamma)$ and almost everywhere, $\|\varphi(g^n)\|_\infty \leq 2\|g\|_\infty$. We can pass to the limit in the above inequality and obtain a similar estimate for general functions $f \in W^{p,2}(\gamma)$ and $g \in W^{r,1}(\gamma) \cap L^\infty(\gamma)$:

$$
\int \frac{\langle \nabla g, \nabla f \rangle}{|\nabla f|_H^2 + \varepsilon^2} d\gamma \leq c_2(p)\|g\|_\infty \|f\|_{W^{p,2}(\gamma)} \langle |\nabla f|_H^2 + \varepsilon^2 \rangle^{-1/p}.
$$

By Lemma 2.1 we have

$$
\langle |\nabla f|_H^2 + \varepsilon^2 \rangle^{-1/p} \leq 2\langle |\nabla f|_H + \varepsilon \rangle^{-2/p} \langle |\nabla f|_H, \varepsilon \rangle^{1-1/p}.
$$

Thus,

$$
\int \frac{\langle \nabla u, \nabla f \rangle}{|\nabla f|_H^2 + \varepsilon^2} d\gamma \leq c(p) a \|g\|_\infty \varepsilon^{-2} u_\gamma \langle |\nabla f|_H, \varepsilon \rangle^{1-1/p}.
$$

The lemma is proved. \hfill \Box

**Theorem 4.1.** Let $p > 1$, $a > 0$. Then there is a constant $c(p)$, depending only on $p$, such that for every function $f \in W^{p,2}(\gamma)$ with

$$
\|f\|_{W^{p,2}(\gamma)} \leq a
$$

one has

$$
\sigma(\gamma \circ f^{-1}, t) \leq c(p) a t \varepsilon^{-2} u_\gamma \langle |\nabla f|_H, \varepsilon \rangle^{1-1/p} + 4u_\gamma \langle |\nabla f|_H, \varepsilon \rangle
$$

for every number $\varepsilon > 0$.

**Proof.** For all $\varphi \in C^\infty_0(\mathbb{R})$ and $\varepsilon > 0$, we can write

$$
\int \varphi'(f) d\gamma = \int \varphi'(f) \frac{\langle \nabla f, \nabla f \rangle}{|\nabla f|_H^2 + \varepsilon^2} d\gamma + \varepsilon^2 \int \varphi'(f) \frac{\langle \nabla f, \nabla f \rangle}{|\nabla f|_H^2 + \varepsilon^2} d\gamma.
$$

For the first term, by Lemma 2.1 we have

$$
\int \frac{\langle \nabla (\varphi \circ f), \nabla f \rangle}{|\nabla f|_H^2 + \varepsilon^2} d\gamma \leq c(p) a \|\varphi\|_\infty \varepsilon^{-2} u_\gamma \langle |\nabla f|_H, \varepsilon \rangle^{1-1/p}.
$$

The second term, by Lemma 2.1 does not exceed

$$
4\|\varphi\|_\infty u_\gamma \langle |\nabla f|_H, \varepsilon \rangle.
$$
Therefore,
\[
\int \varphi'(f) \, d\gamma \leq c(p)a \|\varphi\|_{\infty} \epsilon^{-2} u_{\gamma}(|\nabla f|_H, \epsilon) \left( |\nabla f|_H, \epsilon \right)^{1-1/p} + 4 \|\varphi'\|_{\infty} u_{\gamma}(|\nabla f|_H, \epsilon).
\]

The theorem is proved. \(\square\)

Since \(u_{\gamma}(|\nabla f|_H, \epsilon) \leq 1\), taking \(\epsilon = \sqrt{t}\) in the previous theorem, we obtain the following result.

**Corollary 4.1.** Let \(p > 1\), \(a > 0\). Then there is a constant \(c(p)\), depending only on \(p\), such that for every function \(f \in W^{p,2}(\gamma)\) with
\[
\|f\|_{W^{p,2}(\gamma)} \leq a
\]
one has
\[
\sigma(\gamma \circ f^{-1}, t) \leq (c(p)a + 4)u_{\gamma}(|\nabla f|_H, \sqrt{t})^{1-1/p}.
\]

The following corollary provides a quantitative bound in the following result from [7]: convergence in distribution of random variables \(f_n\) from a certain Sobolev class implies convergence in variation under some uniform nondegeneracy assumption and uniform boundedness of their Sobolev norms.

**Corollary 4.2.** Let \(p > 1\) and let \(f_n \in W^{p,2}(\gamma)\) be a sequence such that
\[
\sup_n \|f_n\|_{W^{p,2}(\gamma)} = a < \infty, \quad \delta(\epsilon) := \sup_n \gamma(|\nabla f_n|_H \leq \epsilon) \to 0.
\]
Assume that the sequence of distributions \(\gamma \circ f_n^{-1}\) converges weakly to the measure \(\nu\) (equivalently \(\|\gamma \circ f_n^{-1} - \nu\|_{KR} \to 0\)). Then \(\|\gamma \circ f_n^{-1} - \nu\|_{TV} \to 0\) and there is a constant \(C(p, a)\) such that
\[
\|\gamma \circ f_n^{-1} - \nu\|_{TV} \leq C(p, a) \left( \delta \left( \|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/8} \right) \right)^{1-1/p} + \|\gamma \circ f_n^{-1} - \nu\|_{KR}^{(p-1)/(8p)}.
\]

**Proof.** By Lemma 3.1 and Corollary 4.1 we have
\[
\|\gamma \circ f_n^{-1} - \gamma \circ f_m^{-1}\|_{TV} \leq 6(c(p)a + 4) \left( \int_{0}^{\infty} \delta(s \sqrt{\epsilon}) \, ds \right)^{1-1/p} + \epsilon^{-1} \|\gamma \circ f_n - \gamma \circ f_m^{-1}\|_{KR}.
\]
Passing to the limit as \(m \to \infty\), we obtain a similar estimate with \(\nu\) in place of \(\gamma \circ f_m^{-1}\). We now note that
\[
\int_0^\infty (s + 1)^{-2} \delta(s \sqrt{\varepsilon}) \, ds
= \int_0^{\varepsilon^{-1/4}} (s + 1)^{-2} \delta(s \sqrt{\varepsilon}) \, ds + \int_{\varepsilon^{-1/4}}^{\infty} (s + 1)^{-2} \delta(s \sqrt{\varepsilon}) \, ds
\leq \delta(\varepsilon^{1/4}) + \frac{\varepsilon^{1/4}}{\varepsilon^{1/4} + 1} \leq \delta(\varepsilon^{1/4}) + \varepsilon^{1/4}.
\]
Taking \(\varepsilon = \|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/2}\), we get
\[
\|\gamma \circ f_n^{-1} - \nu\|_{TV}
\leq 12(c(p)a + 4) \left( \delta(\|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/8})^{1-1/p} + \|\gamma \circ f_n^{-1} - \nu\|_{KR}^{1/8-1/8p} \right).
\]
The corollary is proved. \(\square\)

The following corollary gives the Nikolskii–Besov smoothness of \(\gamma \circ f_n^{-1}\) under the assumption of \(\gamma\)-integrability of \(|\nabla f|_H^{-\theta}\) to some power \(\theta \in (0,1)\). This result generalizes [5, Theorem 5.1] to the case of general Sobolev functions.

**Corollary 4.3.** Let \(p > 1\), \(a, b > 0\), \(\theta \in (0,1)\). Set \(\alpha := \frac{p\theta}{2p+\theta}\). There is a constant \(C := C(p,a,b,\theta)\) such that for every function \(f \in W^{p,2}(\gamma)\) with
\[
\|f\|_{W^{p,2}(\gamma)} \leq a, \quad \int |\nabla f|^{-\theta}_H \, d\gamma \leq b
\]
one has
\[
\|\gamma \circ f^{-1}_n - \gamma \circ f^{-1}\|_{TV} \leq C|h|^\alpha, \quad \forall h \in \mathbb{R}.
\]
Equivalently, the measure \(\gamma \circ f^{-1}\) possesses a density from the Nikolskii–Besov class \(B^\alpha(\mathbb{R})\).

**Proof.** Under our assumptions, we have
\[
u_\gamma(|\nabla f|_H, \varepsilon) := \int_0^\infty (s + 1)^{-2} \gamma(|\nabla f|_H \leq \varepsilon s) \, ds
\leq \varepsilon^\theta b \int_0^\infty s^\theta (s + 1)^{-2} \, ds = c(b, \theta) \varepsilon^\theta.
\]
By Theorem 4.1 for every \(\varepsilon > 0\), one has
\[
\sigma(\gamma \circ f^{-1}, t) \leq c(p) a t \varepsilon^{-2} u_\gamma(|\nabla f|_H, \varepsilon)^{1-1/p} + 4 u_\gamma(|\nabla f|_H, \varepsilon)
\leq c(p) a (c(b, \theta))^{1-1/p} t \varepsilon^{-2} \varepsilon^\theta (1-1/p) + 4 c(b, \theta) \varepsilon^\theta.
\]
Taking \(\varepsilon = t^{2p+\theta}\) and applying Theorem 3.1 we get the desired bound. \(\square\)
The following corollary generalizes [5, Theorem 5.2] and [15, Theorem 3.1].

**Corollary 4.4.** Let \( p > 1, a, b > 0, \theta \in (0,1) \). Set
\[
\beta := \frac{p\theta}{(2+\theta)(p+\theta)}.
\]
There is a constant \( C_1 := C_1(p,a,b,\theta) \) such that such that, for every pair of functions \( f, g \in W^{p,2}(\gamma) \) with
\[
\|f\|_{W^{p,2}(\gamma)} \leq a, \quad \|g\|_{W^{p,2}(\gamma)} \leq a, \quad \int |\nabla f|^{-\theta}_{H} \, d\gamma \leq b, \quad \int |\nabla g|^{-\theta}_{H} \, d\gamma \leq b,
\]
one has
\[
\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV} \leq C_1(p,a,b,\theta)\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR}.
\]

**Proof.** By Lemma 3.1 for each \( \varepsilon \in (0,1) \) one has
\[
\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV} \leq 6\max\{\sigma(\gamma \circ f^{-1}, \varepsilon), \sigma(\gamma \circ g^{-1}, \varepsilon)\} + \varepsilon^{-1}\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR}
\]
\[
\leq 6C(p,a,b,\theta)\varepsilon^{\alpha} + \varepsilon^{-1}\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR},
\]
where \( \alpha = \frac{p\theta}{2p+\theta} \). Taking \( \varepsilon = \|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR} \) we get the desired bound. \( \square \)

5. **Multidimensional case**

We now proceed to the case of multidimensional mappings
\[
f = (f_1, \ldots, f_k) : E \to \mathbb{R}^k
\]
and the properties of their distributions \( \gamma \circ f^{-1} \) on \( \mathbb{R}^k \).

We start with the following analog of Lemma 4.1.

**Lemma 5.1.** Let \( k \in \mathbb{N}, p > 1, q > 1, r \geq 1, a > 0 \). Then there exists a number \( C_0 := C_0(k,p,q,a) > 0 \) such that, for every mapping \( f = (f_1, \ldots, f_k) : E \to \mathbb{R}^k \), where \( f_i \in W^{p,2}(\gamma) \) and
\[
\|f\|_{W^{p,2}(\gamma)} := \max_{i=1,\ldots,k} (\|f_i\|_{W^{p,2}(\gamma)}) \leq a,
\]
for every pair of functions \( u \in W^{r,1}(\gamma) \cap L^{\infty}(\gamma), v \in W^{q,1}(\gamma) \) with
\[
1/q + 1/p + 1/r = 1, \quad 1 - 1/q - (2k+1)/p > 0
\]
and for every number \( \varepsilon \in (0,1) \), one has
\[
\int_{E} \frac{\langle \nabla u, \nabla f_j \rangle}{\Delta f + \varepsilon} \, d\gamma \leq C_0\|u\|_{\infty}\|v\|_{W^{r,1}(\gamma)}\varepsilon^{-2}u_{\gamma}(\Delta f, \varepsilon)^{1-1/q-(2k+1)/p}.
\]
So, the second term in (5.1) is estimated by the determinant. We note that for an arbitrary matrix $A$, the third term in (5.1) can be estimated by $\sum_i G_{\Delta f} L_i$, where $L_i$ is the Ornstein–Uhlenbeck operator associated with the standard Gaussian measure $\gamma_n$. We now estimate each of these three terms. The first term in (5.1) can be estimated from above by

$$||u||_\infty ||v||_q ||L f_j||_p ||(\Delta f + \varepsilon)^{-1}||_{1-1/p-1/q}$$

$$\leq c_1(p) ||u||_\infty ||v||_{W^{q,1}(\gamma)} ||f_j||_{W^{p,2}(\gamma)} ||(\Delta f + \varepsilon)^{-1}||_{1-1/p-1/q}.$$

The third term in (5.1) can be estimated by

$$||u||_\infty \|\nabla f_j\|_p \|\nabla v\|_q \|(\Delta f + \varepsilon)^{-1}\|_{1-1/p-1/q}$$

$$\leq ||u||_\infty \|f_j\|_{W^{p,2}(\gamma)} \|v\|_{W^{q,1}(\gamma)} \|(\Delta f + \varepsilon)^{-1}\|_{1-1/p-1/q}.$$

To estimate the second term in (5.1) we need to estimate the gradient of the determinant. We note that for an arbitrary matrix $C$, one has $|\det C| \leq \prod_i |c_i^r|$, where $\{c_i^r\}$ are columns of the matrix $C$. We have $\langle \nabla f_j, \nabla \Delta f \rangle = \sum_i \det C_i$, where $C_i = \{c_i^{m,r}\}$ is the matrix such that $c_i^{m,r} = \langle \nabla f_m, \nabla f_r \rangle$ for $r \neq i$ and $c_i^{m,i} = \langle D^2 f_m \cdot \nabla f_i, \nabla f_j \rangle + \langle D^2 f_i \cdot \nabla f_m, \nabla f_j \rangle$. Thus,

$$\langle \nabla f_j, \nabla \Delta f \rangle \leq \sum_i |\det C_i| \leq \prod_i |c_i^r|$$

$$\leq (\sum_m |\nabla f_m|)^{2(k-1)} \sum_i \sum_m |\langle D^2 f_m \cdot \nabla f_i, \nabla f_j \rangle| + |\langle D^2 f_i \cdot \nabla f_m, \nabla f_j \rangle|$$

$$\leq 2|\nabla f_j| (\sum_m |\nabla f_m|)^{2k-1} \sum_i \|D^2 f_i\|_{HS} \leq 2 (\sum_m |\nabla f_m|)^{2k} \sum_i \|D^2 f_i\|_{HS}.$$
\[2\|u\|_\infty \int_{\mathbb{R}^n} |v| \left( \sum_m |\nabla f_m| \right)^{2k} \left( \sum_i \|D^2 f_i\|_{HS} \right) (\Delta f + \varepsilon)^{-2} \, d\gamma_n \]

\[\leq 2\|u\|_\infty \|v\|_q \left( \sum_m |\nabla f_m| \right)^{2k} \left( \sum_i \|D^2 f_i\|_{HS} \right) \| (\Delta f + \varepsilon)^{-2} \| \frac{1}{1-\frac{1}{q}-\frac{1}{2k+1}} \]

\[\leq c_2(k)\|u\|_\infty \|v\|_{W^{q,1}(\gamma)} \|f\|_{W^{p,2}(\gamma)}^{2k+1} (\Delta f + \varepsilon)^{-2} \left( \frac{1}{1-\frac{1}{q}-\frac{1}{2k+1}} \right) \]

for some constant \( c_2(k) \), which depends only on \( k \).

Therefore, we have

\[\int \frac{\langle \nabla u, \nabla f_j \rangle}{\Delta f + \varepsilon} \, d\gamma \]

\[\leq \left( c_1(p) + 1 \right) \|u\|_\infty \|v\|_{W^{q,1}(\gamma)} \|f_j\|_{W^{p,2}(\gamma)} (\Delta f + \varepsilon)^{-1} \left( \frac{1}{1-\frac{1}{p-1}} \right) + c_2(k)\|u\|_\infty \|v\|_{W^{q,1}(\gamma)} \|f\|_{W^{p,2}(\gamma)}^{2k+1} (\Delta f + \varepsilon)^{-2} \left( \frac{1}{1-\frac{1}{q}-\frac{1}{2k+1}} \right) \]

for functions \( u, v, f_j \in FC^\infty(E) \), \( i = 1, 2, \ldots, k \). For functions \( f_j \in W^{p,2}(\gamma) \), \( v \in W^{q,1}(\gamma) \), we take sequences \( f^n_j \in FC^\infty(E) \), \( v^n \in FC^\infty(E) \) such that \( f^n_j \rightarrow f_j \) in \( W^{p,2}(\gamma) \), \( v^n \rightarrow v \) in \( W^{q,1}(\gamma) \) and both sequences (along with the sequences of their derivatives) also converge almost everywhere. Passing to the limit in the above inequality we obtain the same inequality for general functions \( f_j \in W^{p,2}(\gamma) \), \( v \in W^{q,1}(\gamma) \), \( u \in FC^\infty(E) \).

Now, for a function \( u \in W^{r,1}(\gamma) \cap L^\infty(\gamma) \), we take functions \( u^n \in FC^\infty(E) \) such that \( u^n \rightarrow u \) in \( W^{r,1}(\gamma) \) and almost everywhere. Let us consider a function \( \varphi \in C_0^\infty(\mathbb{R}) \) such that \( \varphi(t) = t \) for \( t \in [-\|u\|_\infty, \|u\|_\infty] \) and \( |\varphi(t)| \leq 2\|u\|_\infty \). Then, the sequence \( \{\varphi(u^n)\} \) also converges to the function \( u \) in \( W^{r,1}(\gamma) \) and almost everywhere, \( \|\varphi(u^n)\|_\infty \leq 2\|u\|_\infty \). We can pass to the limit in the above inequality and obtain a similar estimate for general functions \( f_j \in W^{p,2}(\gamma) \), \( i = 1, 2, \ldots, k \), \( v \in W^{q,1}(\gamma) \), \( u \in W^{r,1}(\gamma) \cap L^\infty(\gamma) \):

\[\int_E \frac{\langle \nabla u, \nabla f_j \rangle}{\Delta f + \varepsilon} \, d\gamma \]

\[\leq c_3(k, p)\|u\|_\infty \|v\|_{W^{q,1}(\gamma)} \|f_j\|_{W^{p,2}(\gamma)} (\Delta f + \varepsilon)^{-1} \left( \frac{1}{1-\frac{1}{p-1}} \right) + \|f\|_{W^{p,2}(\gamma)}^{2k+1} (\Delta f + \varepsilon)^{-2} \left( \frac{1}{1-\frac{1}{q}-\frac{1}{2k+1}} \right), \]

with \( c_3(k, p) = 2(c_1(p) + 1) + 2c_2(k) \).

By Lemma 2.1, we have

\[\|f\|_{W^{p,2}(\gamma)}^{2k+1} (\Delta f + \varepsilon)^{-2} \left( \frac{1}{1-\frac{1}{q}-\frac{1}{2k+1}} \right) \leq 2\varepsilon^{-1} u_\gamma (\Delta f + \varepsilon)^{1-1/p-1/q} \]

and
\[
\| (\Delta_f + \varepsilon)^{-2} \|_{1-1/q-(2k+1)/p} \leq 3\varepsilon^{-2}u_\gamma(\Delta_f, \varepsilon)^{1-1/q-(2k+1)/p}.
\]
Since \( \varepsilon \leq 1 \) and \( u_\gamma(\Delta_f, \varepsilon) \leq 1 \) we have
\[
\varepsilon^{-1}u_\gamma(\Delta_f, \varepsilon)^{1-1/p-1/q} \leq \varepsilon^{-2}u_\gamma(\Delta_f, \varepsilon)^{1-1/q-(2k+1)/p}.
\]
Thus,
\[
\int_E \frac{\langle \nabla u, \nabla f_j \rangle_H v}{\Delta_f + \varepsilon} \, d\gamma \\
\leq C_0(k, p, q, a)\| u \|_{\infty} \| v \|_{W^{4,1}(\gamma)} \varepsilon^{-2}u_\gamma(\Delta_f, \varepsilon)^{1-1/q-(2k+1)/p}
\]
with \( C_0(k, p, q, a) = c_3(k, p)(2a + 3a^{2k+1}) \). The lemma is proved.

**Theorem 5.1.** Let \( k \in \mathbb{N}, \ a > 0, \) and \( p > 4k - 1 \). Then there exists a number \( C_1 := C_1(p, k, a) > 0 \) such that, for every mapping \( f = (f_1, \ldots, f_k) : E \to \mathbb{R}^k \), where \( f_i \in W^{p,2}(\gamma) \) and
\[
\| f \|_{W^{p,2}(\gamma)} := \max_{i=1, \ldots, k} (\| f_i \|_{W^{p,2}(\gamma)}) \leq a,
\]
for every \( \varepsilon \in (0, 1) \), one has
\[
\sigma(\gamma \circ f^{-1}, t) \leq C_1 t \varepsilon^{-2}u_\gamma(\Delta_f, \varepsilon)^{1-(4k-1)/p} + u_\gamma(\Delta_f, \varepsilon).
\]

**Proof.** Fix an arbitrary function \( \varphi \in C_0^\infty(\mathbb{R}^k) \) with \( \| \varphi \|_\infty \leq t, \| \partial e \varphi \|_\infty \leq 1 \), and an arbitrary unit vector \( e \in \mathbb{R}^k \). It can be easily verified that
\[
M_f(\partial_1 \varphi(f), \ldots, \partial_k \varphi(f)) = (\langle \nabla (\varphi \circ f), \nabla f_1 \rangle_H, \ldots, \langle \nabla (\varphi \circ f), \nabla f_k \rangle_H).
\]
Here the left-hand side is interpreted as the standard product of a matrix and a column vector. Then by (2.1) we have
\[
(\partial_e \varphi)(f) \Delta_f = \langle v, A_{f^e} \rangle, \quad v = (\langle \nabla (\varphi \circ f), \nabla f_1 \rangle_H, \ldots, \langle \nabla (\varphi \circ f), \nabla f_k \rangle_H)
\]
which yields the following equality:
\[
\Delta_f(\partial_e \varphi)(f) = \sum_{i,j} \langle \nabla (\varphi \circ f), \nabla f_j \rangle_H a_f^{ij} e_i.
\]
For any fixed number \( \varepsilon \in (0, 1) \) we can write
\[
\int \partial_e \varphi(f) \, d\gamma = \int \partial_e \varphi(f) \frac{\Delta_f}{\Delta_f + \varepsilon} \, d\gamma + \varepsilon \int \partial_e \varphi(f) (\Delta_f + \varepsilon)^{-1} \, d\gamma. \quad (5.2)
\]
For the first term by the above reasoning we have
\[
\int \partial_e \varphi \frac{\Delta_f}{\Delta_f + \varepsilon} \, d\gamma = \sum_{i,j} \int \frac{\langle \nabla (\varphi \circ f), \nabla f_j \rangle_H a_f^{ij} e_i}{\Delta_f + \varepsilon} \, d\gamma.
\]
We note that $a^{i,j} e_i \in W^{p/(2k-2),1}(\gamma)$ and there is a constant $c_4(k)$ such that \[
\|a^{i,j} e_i\|_{W^{p/(2k-2),1}(\gamma)} \leq c_4(k)\|f\|^{2k-2}_{W^{p,2}(\gamma)} \leq c_4(k)a^{2k-2}.
\]
We also note that $\varphi \circ f \in W^{p,1}(\gamma)$ and $(2k-2)/p + 1/p + 1/p \leq 1$. Hence we have obtained the estimate \[
\varphi \circ f \in W^{1-(2k-1)/p,1}(\gamma)
\]
and $\|\varphi \circ f\|_{W^{1-(2k-1)/p,1}(\gamma)} \leq \|\varphi \circ f\|_{W^{p,1}(\gamma)}$. Moreover, we have \[
1 - (2k-2)/p - (2k+1)/p = 1 - (4k-1)/p > 0.
\]
Applying now Lemma 5.1 with $r = (1 - (2k-1)/p)^{-1}$ and $q = p/(2k-2)$ we obtain

\[
\int \partial_x \varphi \frac{\Delta f}{\Delta f + \varepsilon} \ d\gamma \leq C_1(k,p,a)\|\varphi\|_{\infty} \varepsilon^{-2} u_\gamma(\Delta f, \varepsilon)^{1-(4k-1)/p},
\]

with $C_1(k,p,a) = k^2 c_4(k) C_0(k,p,p/(2k-2),a) a^{2k-2}.$

Using Lemma 2.1 we can estimate the second term in (5.2) in the following way:

\[
\varepsilon \int \partial_x \varphi(f)(\Delta f + \varepsilon)^{-1} \ d\gamma \leq \|\partial_x \varphi\|_{\infty} u_\gamma(\Delta f, \varepsilon) \leq u_\gamma(\Delta f, \varepsilon).
\]

Hence we have obtained the estimate \[
\int \partial_x \varphi \ d\gamma \leq C_1(k,p,a)\|\varphi\|_{\infty} \varepsilon^{-2} u_\gamma(\Delta f, \varepsilon)^{1-(4k-1)/p} + u_\gamma(\Delta f, \varepsilon).
\]

Since $\|\varphi\|_{\infty} \leq t$, the theorem is proved.

Taking $\varepsilon = \sqrt{t}$ we get the following result.

**Corollary 5.1.** Let $k \in \mathbb{N}$, $a > 0$, and $p > 4k - 1$. Then there exists a constant $C := C(p,k,a) > 0$ such that, for every mapping $f = (f_1, \ldots, f_k) : E \rightarrow \mathbb{R}^k$, where $f_i \in W^{p,2}(\gamma)$ and \[
\|f\|_{W^{p,2}(\gamma)} := \max_{i=1,\ldots,k} \left(\|f_i\|_{W^{p,2}(\gamma)}\right) \leq a,
\]
for every $t \in (0,1)$, one has \[
\sigma(\gamma \circ f^{-1}, t) \leq C(p,k,a) u_\gamma(\Delta f, \sqrt{t})^{1-(4k-1)/p}.
\]

The following corollary is a multidimensional analog of Corollary 1.2. It asserts that convergence in distribution of random vectors $f_n$ from a Sobolev class implies convergence in variation provided they are uniformly nondegenerate and uniformly bounded in the Sobolev norm.
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Corollary 5.2. Let \( k \in \mathbb{N}, a > 0, \) and \( p > 4k-1. \) Let \( f_n = (f_{n,1}, \ldots, f_{n,k}) \in W^{p,2}(\gamma) \) be a sequence of mappings such that
\[
\sup_n \| f_n \|_{W^{p,2}(\gamma)} = a < \infty, \quad \delta(\varepsilon) := \sup_n \gamma(\Delta f_n \leq \varepsilon) \to 0.
\]
Assume also that the sequence of distributions \( \gamma \circ f_n^{-1} \) converges weakly to some measure \( \nu \) (equivalently, \( \| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}} \to 0 \)). Then
\[
\| \gamma \circ f_n^{-1} - \nu\|_{\text{TV}} \to 0
\]
and
\[
\| \gamma \circ f_n^{-1} - \nu\|_{\text{TV}} \leq C_2(p, k, a) \left( \delta(\| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}}^8) \right)^{1-(4k-1)/p} + \| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}} 1-(4k-1)/p.
\]

Proof. By Lemma 3.1 and Corollary 5.1 we have
\[
\| \gamma \circ f_n^{-1} - \gamma \circ f_m^{-1}\|_{\text{TV}} \leq 6C(p, k, a) \left( \int_0^\infty (s + 1)^{-2} \delta(s\sqrt{x}) \, ds \right)^{1-(4k-1)/p} + \sqrt{k}\varepsilon^{-1}\| \gamma \circ f_n - \gamma \circ f_m^{-1}\|_{\text{KR}}.
\]
Passing to the limit as \( m \to \infty, \) we obtain a similar estimate with \( \nu \) in place of \( \gamma \circ f_m^{-1}. \) Now we proceed as in Corollary 4.2
\[
\int_0^\infty (s + 1)^{-2} \delta(s\sqrt{x}) \, ds = \int_0^{2^{1/8}\varepsilon^{-1/4}} (s + 1)^{-2} \delta(s\sqrt{x}) \, ds + \int_{2^{1/8}\varepsilon^{-1/4}}^\infty (s + 1)^{-2} \delta(s\sqrt{x}) \, ds
\]
\[
\leq \delta(2^{1/8} \varepsilon^{1/4}) + \frac{\varepsilon^{1/4}}{\varepsilon^{1/4} + 2^{1/8}} \leq \delta(2^{1/8} \varepsilon^{1/4}) + 2^{-1/8} \varepsilon^{1/4}.
\]
Taking \( \varepsilon = 2^{-1/2}\| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}}^{1/2} \leq 1 \) we get
\[
\| \gamma \circ f_n^{-1} - \nu\|_{\text{TV}} \leq C_2(p, k, a) \left( \delta(\| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}}^{1/8}) \right)^{1-(4k-1)/p} + \| \gamma \circ f_n^{-1} - \nu\|_{\text{KR}}^{1-(4k-1)/8p}.
\]
The corollary is proved. \( \square \)

We now apply Theorem 5.1 to show the Nikolskii–Besov smoothness of \( \gamma \circ f^{-1} \) under our weak nondegeneracy condition: \( \Delta f^{-1} \) is \( \gamma \)-integrable to some power \( \theta \in (0, 1). \) The following corollary generalizes [5, Theorem 4.1].
Corollary 5.3. Let $k \in \mathbb{N}$, $a > 0$, $b > 0$, $\theta \in (0,1)$, $p > 4k - 1$. Set $\alpha := \frac{p \theta}{2p + (4k - 1)\theta}$. Then there exists a number $C := C(p, k, a, b, \theta) > 0$ such that, for every mapping $f = (f_1, \ldots, f_k): E \to \mathbb{R}^k$, where $f_i \in W^{p, 2}(\gamma)$ and

$$\|f\|_{W^{p, 2}(\gamma)} := \max_{i = 1, \ldots, k} (\|f_i\|_{W^{p, 2}(\gamma)}) \leq a, \quad \int \Delta_f^{-\theta} \, d\gamma \leq b,$$

one has

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV} \leq C \|f^{-1} - g^{-1}\|_{TV}^{\alpha} \quad \forall h \in \mathbb{R}^k.$$

In other words, the density of $\gamma \circ f^{-1}$ belongs to the Nikolskii–Besov space $B^\alpha(\mathbb{R}^k)$.

Proof. Let us estimate $u_\gamma(\Delta_f, \varepsilon)$:

$$u_\gamma(\Delta_f, \varepsilon) := \int_0^{\infty} (s + 1)^{-2} \gamma(\Delta_f \leq \varepsilon s) \, ds \leq \varepsilon \theta \int_0^{\infty} s^\theta (s + 1)^{-2} \, ds = c_1(b, \theta) \varepsilon^\theta.$$

By Theorem 5.1 for $\varepsilon \in (0, 1)$ one has

$$\sigma(\gamma \circ f^{-1}, t) \leq C_1(p, k, a) t \varepsilon^{-2} u_\gamma(\Delta_f, \varepsilon)^{1 - (4k - 1)/p} + u_\gamma(\Delta_f, \varepsilon) \leq C_2(p, k, a, b, \theta) (t \varepsilon^{-2 + (1 - (4k - 1)/p)} + \varepsilon^\theta).$$

Taking $\varepsilon = t^{\frac{p}{2p + (4k - 1)\theta}}$ for $t < 1$ and noting that $\sigma(\gamma \circ f^{-1}, t) \leq 1 \leq t$ for $t \geq 1$, by Theorem 3.1 we get the desired bound. \(\square\)

The next corollary is a generalization of [5, Theorem 4.2] and of [14, Theorem 4.1] to the case of Sobolev mappings in place of polynomials.

Corollary 5.4. Let $k \in \mathbb{N}$, $a > 0$, $b > 0$, $\theta \in (0,1)$, $p > 4k - 1$. Set $\alpha := \frac{p \theta}{2p + (4k - 1)\theta}$. Then there exists a number $C := C(p, k, a, b, \theta) > 0$ such that for every pair of mappings $f = (f_1, \ldots, f_k), g = (g_1, \ldots, g_k): E \to \mathbb{R}^k$, where $f_i, g_i \in W^{p, 2}(\gamma)$ and

$$\|f\|_{W^{p, 2}(\gamma)} \leq a, \quad \|g\|_{W^{p, 2}(\gamma)} \leq a, \quad \int \Delta_f^{-\theta} \, d\gamma \leq b, \quad \int \Delta_g^{-\theta} \, d\gamma \leq b,$$

one has

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV} \leq C(p, k, a, b, \theta) \|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV}^{\alpha} \quad \forall h \in \mathbb{R}^k.$$

Proof. By Lemma 3.1, for an arbitrary $\varepsilon \in (0, 1)$, one has
\[ \|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{TV} \leq 6\sqrt{k} \max\{\sigma(\gamma \circ f^{-1}, \epsilon), \sigma(\gamma \circ g^{-1}, \epsilon)\} + \sqrt{k} \epsilon^{-1}\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR} \leq C_1(p, k, a, b, \theta)(\epsilon^\alpha + \epsilon^{-1}\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{KR}). \]

Taking \( \epsilon = 2^{-1}\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\frac{1}{\alpha}KR} \) we get the desired bound. \(\square\)

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