AUTOMATA, REDUCED WORDS AND GARSIDE SHADOWS IN COXETER GROUPS

CHRISTOPHE HOHLWEG, PHILIPPE NADEAU, AND NATHAN WILLIAMS

ABSTRACT. In this article, we introduce and investigate a class of finite deterministic automata that all recognize the language of reduced words of a finitely generated Coxeter system \((W,S)\). The definition of these automata is straightforward as it only requires the notion of weak order \(\leq_R\) on \((W,S)\) and the related notion of Garside shadows in \((W,S)\), an analog of the notion of a Garside family. Then we discuss the relations between this class of automata and the canonical automaton built from Brink and Howlett’s small roots. We end this article by providing partial positive answers to two conjectures: (1) the automata associated to the smallest Garside shadow is minimal; (2) the canonical automaton is minimal if and only if the support of all small roots is spherical, i.e., the corresponding root system is finite.

1. Introduction

In this article, we introduce and investigate a class of finite deterministic automata that recognize the language \(\text{Red}(W,S)\) of reduced words of a finitely generated Coxeter system \((W,S)\). The definition of these automata is straightforward, requiring only the notion of (right) weak order \(\leq_R\) on \((W,S)\) [1, 2] and the related notion of Garside shadows, introduced by M. Dyer and the first author in [13] as an analog of the notion of a Garside family in a monoid; see [11, 10] and the references therein. For general definitions and properties, we refer the reader to [25] regarding automata and to [2, 22] regarding Coxeter groups.

A Garside shadow in \((W,S)\) is a subset \(B \subseteq W\) that contains \(S\) and is closed under join (for the right weak order) and by taking suffixes. In [13], the authors show that finite Garside shadows exist in any Coxeter system \((W,S)\). Let \(B\) be a finite Garside shadow in \((W,S)\). So \(\bigvee X \in B\) for any bounded subset \(X\) of \(B\), i.e., a subset that has an upper bound. Therefore, the following projection from \(W\) to \(B\) is well-defined:

\[
\pi_B : W \to B \\
 w \mapsto \bigvee \{ g \in B \mid g \leq_R w \}
\]

We denote by \(\ell : W \to \mathbb{N}\) the length function of the Coxeter system \((W,S)\).

**Definition 1.1.** We define a finite deterministic automaton \(A_B(W,S)\) over the alphabet \(S\) as follows:

Date: April 18, 2016.

2010 Mathematics Subject Classification. Primary 20F55; secondary 20F10; 05E15; 06F99.

Key words and phrases. Coxeter groups, Garside shadows, low elements, weak order, dominance order, small roots, automaton and reduced words.

\*supported by NSERC Discovery grant Coxeter groups and related structures.
the set of states is $B$;
• the initial state is the identity $e$ of $W$, and all states are final;
• the transitions are: $x \rightarrow \pi_B(sx)$ whenever $\ell(sx) > \ell(x)$.

Since the intersection of Garside shadows is again a Garside shadow, there is a smallest Garside shadow $\tilde{S}$ in $(W,S)$. As a first example, the finite automaton built out of the smallest Garside shadow $\tilde{S}$ for the infinite dihedral group is shown in Figure 1. Further examples are given in §3.6 and in Figures 5 and 6.

Our main result is that $A_B(W,S)$ recognizes the language of reduced words of $(W,S)$.

**Theorem 1.2.** If $B$ is a finite Garside shadow in $(W,S)$, then the finite deterministic automaton $A_B(W,S)$ recognizes the language $\text{Red}(W,S)$.

Theorem 1.2 is proved in §2. In §3, we show that an inclusion $B \subseteq C$ of Garside shadows induces a surjective morphism $A_C(W,S) \rightarrow A_B(W,S)$ between their associated automata. The smallest Garside shadow being finite [13, Corollary 1.2], we are led to the following conjecture.

**Conjecture 1.** The automaton $A_{\tilde{S}}(W,S)$ is the minimal automaton recognizing $\text{Red}(W,S)$.

Using Sage [23, S+09], we checked that Conjecture 1 holds for all Coxeter groups $W$ of rank at most 4 whose corresponding Coxeter graph $\Gamma_W$ has edge labels less than 10; see Remark 3.15 for more details.

Our initial motivation for this work was to provide a purely combinatorial definition for an automaton that recognizes the language of reduced words. Indeed, as we now recall, all previously-defined automata recognizing $\text{Red}(W,S)$ require the introduction of an auxiliary geometric representation and root system.

In 1993, B. Brink and R. Howlett [4] showed that finitely-generated Coxeter groups are automatic, in the sense of [16], thereby filling a gap in the proof of the “Parallel Wall Theorem” of [9]. For each Coxeter system $(W,S)$, they provided a word-acceptor—that is, a finite automaton that recognizes the language of lexicographically minimal reduced words in $W$. This particular automaton is built using their notion of small roots, and therefore requires a geometric representation of $(W,S)$ and its associated root system. In a series of articles [7, 8, 5, 6], Casselman explains how to perform practical computations in Coxeter groups using Brink and Howlett’s word-acceptor.

We are often interested in all reduced words, not only those that are lexicographically-ordered; see for instance [26]. In his thesis [17], H. Eriksson studied a finite deterministic automaton $A_0(W,S)$ over $S$ that recognizes the language $\text{Red}(W,S)$. The automaton $A_0(W,S)$ is called the canonical automaton in [2 §4.8], and is built using B. Brink and R. Howlett’s technology of small roots. An immediate consequence is that the language $\text{Red}(W,S)$ is regular, a result we recover in Theorem 1.2.

In particular, the generating function for the number of reduced words in $(W,S)$ with respect to their length is a rational function.

For $n \in \mathbb{N}$, the canonical automaton was extended, replacing small roots with $n$-small roots, in [15] and [13] to the $n$-canonical automaton $A_n(W,S)$. We recall these notions in §8 and discuss morphisms between $A_n(W,S)$ and the automata $A_B(W,S)$ arising from certain finite Garside shadows $B$. In particular, we
show in Corollary 3.13 that any \( n \)-canonical automaton surjects into the automaton \( \mathcal{A}_S(W, S) \), providing evidence for Conjecture 1.

Both H. Eriksson [17, Theorem 80] and P. Headley [19, Theorem V.8] prove that in type \( \tilde{A}_n \), the canonical automaton \( \mathcal{A}_0(\tilde{A}_n, S) \) is minimal. Furthermore, they note that \( \mathcal{A}_0(W, S) \) is not minimal for general affine groups \( W \).

We conjecture a necessary condition for the canonical automaton to be minimal.

The sufficient condition is shown in Proposition 3.14.

**Conjecture 2.** Let \( W \) be irreducible. Then \( \mathcal{A}_0(W, S) \) is minimal if and only if \( \Sigma = \Phi^+_{sp} \), where \( \Phi^+_{sp} \) denotes the set of roots whose support is a finite standard parabolic subgroup.

Since \( \mathcal{A}_0(W, S) \) surjects onto \( \mathcal{A}_S(W, S) \) (Corollary 3.13), Conjecture 2 implies Conjecture 1 for Coxeter systems for which \( \Sigma = \Phi^+_{sp} \). In §3.5, we prove Conjecture 2 in the following cases.

**Theorem 1.3.** Conjecture 2 holds in each of the following cases:

1. \( W \) is finite.
2. \( W \) is right-angled, i.e. \( m_{st} = 2 \) or \( \infty \) for all \( s \neq t \).
3. \( W \) is a complete graph, i.e. \( m_{st} > 2 \) for all \( s \neq t \).
4. \( W \) is of type \( \tilde{A}_{n-1} \).
5. \( W \) has rank 3.

In the first four cases, \( \Sigma = \Phi^+_{sp} \) and \( \mathcal{A}_0(W, S) \) is minimal.

We also checked that Conjecture 2 holds if \( W \) has rank 4 and \( m_{st} < 10 \) for all \( s \neq t \); see Remark 3.15 for more details.

When \( (W, S) \) is an affine Coxeter system, P. Headley described a remarkable connection between the canonical automaton and the Shi arrangement [24]: the states of \( \mathcal{A}_0(W, S) \) are in bijection with the (minimal elements in the) connected regions of the complement of the Shi arrangement for \( (W, S) \) [19]. The same relationship holds for the states of \( \mathcal{A}_n(W, S) \) and the regions of the \( n \)-Shi arrangement, as we outline in §3.6.

## 2. Garside shadow automata

Fix \( (W, S) \) a Coxeter system with length function \( \ell : W \to \mathbb{N} \). The rank of \( W \) is the cardinality of the set of simple reflections \( S \). A word \( s_1 \cdots s_k \) on the alphabet \( S \) is a reduced word for \( w \in W \) if \( w = s_1 \cdots s_k \) and \( k = \ell(w) \). For \( u, v, w \in W \), we say that:

- \( w = uv \) is reduced if \( \ell(w) = \ell(u) + \ell(v) \), i.e., the concatenation of any reduced word for \( u \) with any reduced word for \( v \) is a reduced word for \( w \);
- \( u \) is a prefix of \( w \) if a reduced word for \( u \) is a prefix of a reduced word for \( w \);
- \( v \) is a suffix of \( w \) if a reduced word for \( v \) is a suffix of a reduced word for \( w \).

Observe that if \( w = uv \) is reduced, then \( u \) is a prefix of \( w \) and \( v \) is a suffix of \( w \). The subset \( D_L(w) = \{ s \in S | \ell(sw) < \ell(w) \} \) of \( S \) is called the left descent set of \( w \in W \). The descent set plays an important role in the study of reduced words since it coincides with the set of the possible first letters of reduced words of an element \( w \in W \); see [2].

The standard parabolic subgroup \( W_I \) is the subgroup of \( W \) generated by \( I \subseteq S \). It is well-known that \( (W_I, I) \) is itself a Coxeter system and that the length function
\(\ell_I : W_I \to \mathbb{N}\) is the restriction of \(\ell\) to \(W_I\). Moreover, \(W_I\) is finite if and only if it contains a longest element, which is then unique and is denoted by \(w_{o,I}\).

The set \(X_I := \{x \in W | \ell(sx) > \ell(x), \forall s \in I\}\) is the set of minimal-length coset representatives for the coset \(W_I \setminus W\). For any \(w \in W\), there is a unique decomposition \(w = w_Iw'\), with \(w_Iw'\) reduced; see \cite[Proposition 2.4.4]{2}. See \cite{22} for more details.

### 2.1. Weak order and Garside shadows.

The (right) weak order is the order on \(W\) defined by \(u \leq_R v\) if \(u\) is a prefix of \(v\). Since we only consider the right weak order in this article, we only use from now on the term weak order. The weak order gives a natural orientation of the Cayley graph of \((W,S)\): for \(w \in W\) and \(s \in S\), we orient an edge \(w \to ws\) if \(w \leq_R ws\). We recall the following well-known useful properties linking descent sets and weak order, which is a rephrasing of part of \cite[Proposition 3.1.2]{2}.

**Lemma 2.1.** Let \(u, v \in W\) and \(s \in S\).

(a) \(s \in D_L(u)\) if and only if \(s \leq_R u\).

(b) If \(s \in D_L(u) \cap D_L(v)\), then \(u \leq_R v\) if and only if \(su \leq_R sv\).

(c) If \(s \notin D_L(u)\) and \(s \notin D_L(v)\), then \(u \leq_R v\) if and only if \(su \leq_R sv\).

A. Björner \cite[Theorem 8]{1} proved that the weak order \((W,\leq_R)\) is a complete meet semilattice: for any \(A \subseteq W\), there exists an infimum \(\bigwedge A \in W\), also called the meet of \(A\); see \cite[Chapter 3]{2}.

A subset \(X \subseteq W\) is bounded in \(W\) if there exists a \(g \in W\) such that \(x \leq_R g\) for any \(x \in X\). Therefore, any bounded subset \(X \subseteq W\) admits a least upper bound \(\bigvee X\) called the join of \(X\):

\[
\bigvee X = \bigwedge \{g \in W | x \leq_R g, \forall x \in X\}.
\]

When \(W\) is finite, any element \(w \in W\) is a prefix of the longest element \(w_o\), so that \(W\) itself is bounded. In fact, \((W,\leq_R)\) turns out to be a complete ortholattice; see \cite[Corollary 3.2.2]{2}.

**Definition 2.2** (\cite{13}). A subset \(B \subseteq W\) is a Garside shadow in \((W,S)\) if \(B\) contains \(S\) and:

(i) \(B\) is closed under join in the weak order: if \(X \subseteq B\) is bounded, then \(\bigvee X \in B\);

(ii) \(B\) is closed under taking suffixes: if \(w \in B\), then any suffix of \(w\) is also in \(B\).

Since a standard parabolic subgroup \(W_I\) with its canonical set of generators \(I \subseteq S\) forms a Coxeter system, it is natural to say that a subset \(B \subseteq W_I\) is a Garside shadow of \((W_I,I)\) if \(B\) contains \(I\) and verifies Conditions (i)–(ii) of Definition \ref{Definition 2.2}.

Note that if \(B\) is a Garside shadow in \((W,S)\), then \(B \cap W_I\) is a Garside shadow in \((W_I,I)\) \cite[Remark 2.5(c)]{13}. Since the intersection of Garside shadows is again a Garside shadow, there exists a smallest Garside shadow of \((W,S)\) containing \(X \subseteq W\), which we denote by \(\text{Gar}_S(X)\). In \cite[Corollary 1.2]{13}, Dyer and the first author show that the smallest Garside shadow

\[
\tilde{S} := \text{Gar}_S(S)
\]

is finite. The automaton constructed from the smallest Garside shadow \(\tilde{S}\) of the infinite dihedral group is illustrated in Figure \ref{Figure}.
Remark 2.3. The finiteness of $\tilde{S}$ is shown in [13] using the geometry of the root system. A direct computational proof is still open. The problem of computing $\tilde{S}$ relies on finding an efficient criterion for a subset of $W$ to be bounded.

Figure 1. The weak order on the infinite dihedral group $D_\infty$ and the automaton associated to the smallest Garside shadow $\{e,s,t\}$, which is represented by red vertices.

2.2. Garside Shadow Projections.

Definition 2.4. Let $B$ be a Garside shadow in $(W,S)$. We call the surjection

$$\pi_B : W \to B$$
$$w \mapsto \bigvee \{g \in B \mid g \leq_R w\},$$

the $B$-projection.

Since $S \subseteq B$, the set $\{g \in B \mid g \leq_R w\}$ is non-empty and bounded for any $w \in W$. Together with Condition (i) of Definition 2.2, this implies that the $B$-projection is well-defined. Note that $\pi_B(w)$ can be characterized as the unique longest prefix of $w$ which belongs to $B$.

Proposition 2.5. Let $B$ be a Garside shadow in $(W,S)$ and $u, w \in W$, then:

(a) $\pi_B \circ \pi_B = \pi_B$;
(b) $\pi_B(w) \leq_R w$, with equality holding if and only if $w \in B$;
(c) If $u \leq_R w$, then $\pi_B(u) \leq_R \pi_B(w)$.

Proof. Properties (a) and (b) are clear from the definition. For (c), if $u \leq_R w$ then $x \leq_R u$ implies $x \leq_R w$ for any $x \in B$, from which we conclude the proposition.

The next proposition states that left descent sets are invariant under Garside shadow projections.

Proposition 2.6. Let $B$ be a Garside shadow in $(W,S)$ and $w \in W$. Then $D_L(w) = D_L(\pi_B(w))$, and $s \pi_B(w) \leq_R sw$ for any $s \in S$.

Proof. We first show $D_L(w) = D_L(\pi_B(w))$. Let $r \in D_L(\pi_B(w))$. By Lemma 2.1(a) and Proposition 2.5(b), we have $r \leq_R \pi_B(w) \leq_R w$. So $D_L(\pi_B(w)) \subseteq D_L(w)$.

Conversely, let $s \leq_R w$. Since $S \subseteq B$, we have $s = \pi_B(s)$ by Proposition 2.5(b). Then $s \leq_R \pi_B(w)$ by Proposition 2.5(c). So $D_L(w) \subseteq D_L(\pi_B(w))$. 
We now show $s\pi_B(w) \leq_R sw$ for any $s \in S$. We write $D := D_L(w) = D_L(\pi_B(w))$. If $s \notin D$, then the result follows from Proposition 2.5(b) and Lemma 2.1(c). If $s \in D$, the statement follows from Proposition 2.5(b) and Lemma 2.1(b).

**Remark 2.7.** Definition 2.4 of $\pi_B$ and the proof of Proposition 2.6 require only the conditions $S \subseteq B$ and Condition (i) from Definition 2.2. The proof of Proposition 2.8 requires all the conditions from Definition 2.2.

**Proposition 2.8.** Let $B$ be a Garside shadow in $(W,S)$. Let $w \in W$ and $s \in S$ such that $s \notin D_L(w)$. Then $\pi_B(sw) = \pi_B(s\pi_B(w))$.

**Proof.** We have $s\pi_B(w) \leq_R sw$ by Proposition 2.6. Therefore, by Proposition 2.5(c), $\pi_B(s\pi_B(w)) \leq_R \pi_B(sw)$. To complete the proof, we will show that $\pi_B(sw) \leq_R \pi_B(s\pi_B(w))$, which is equivalent by Proposition 2.5(a), (b), and (c) to the statement that $\pi_B(sw) \leq_R s\pi_B(w)$. We prove this last relation as follows. Using Proposition 2.6, we see that $s \in D_L(sw) = D_L(\pi_B(sw))$, so that since $\pi_B(sw) \leq_R sw$ by Proposition 2.5(b), Lemma 2.1(b) allows us to conclude that $s\pi_B(sw) \leq_R w$. Now $s\pi_B(sw) \in B$ because $B$ is closed under taking suffixes by Definition 2.2(ii), so that $\pi_B(s\pi_B(sw)) = s\pi_B(sw) \leq_R \pi_B(w)$ by Proposition 2.5(b,c). Multiplying both sides by $s$ and using Lemma 2.1(c) gives $\pi_B(sw) \leq_R s\pi_B(w)$.

**Remark 2.9.** The proof of Proposition 2.8 requires all the conditions from Definition 2.2.

**Corollary 2.10.** Let $v \in W$ and $s_1, \ldots, s_k \in S$ such that $s_k \cdots s_1 v$ is reduced. Then

$$\pi_B(s_k \cdots s_1 v) = \pi_B(s_k \pi_B(s_k-1 \pi_B(\cdots s_2 \pi_B(s_1 \pi_B(v))))) = \pi_B(s_k \cdots s_1 \pi_B(v)).$$

In particular:

(a) If $w = s_k \cdots s_1$ is a reduced word, then

$$\pi_B(w) = \pi_B(s_k \cdots s_1) = \pi_B(s_k \pi_B(s_k-1 \pi_B(\cdots s_2 \pi_B(s_1))));$$

(b) If $uv$ is reduced, $u \in W$, then $\pi_B(u \pi_B(v)) = \pi_B(uv)$.

**Proof.** We prove the first equality by induction on $k > 0$. The case $k = 1$ is Proposition 2.8. Now assume the property for $k-1 > 0$. Then since $s_k s_{k-1} \cdots s_1 v$ is reduced, we have $s_k \notin D_L(s_k-1 \cdots s_1 v)$. By Proposition 2.8 we obtain

$$\pi_B(s_k \cdots s_1 v) = \pi_B(s_k \pi_B(s_k-1 \cdots s_1 v)).$$

By induction, $\pi_B(s_k-1 \cdots s_1 v) = \pi_B(s_k-1 \pi_B(\cdots s_2 \pi_B(s_1 \pi_B(v))))$.

Now for the second equality, observe that $\pi_B(v)$ is a prefix of $v$ by Proposition 2.5(b). Therefore $s_k \cdots s_1 \pi_B(v)$ must be reduced, since $s_k \cdots s_1 v$ is reduced.

We conclude by applying the first equality to $\pi_B(v)$, recalling that $\pi_B$ is a projection. In particular (a) is obtained by taking $v = e$ and (b) by considering a reduced word $s_k \cdots s_1$ for $u$. □
2.3. Garside Shadow Automata and Proof of Theorem 1.2. Before proving Theorem 1.2, we recall some terminology about automata theory; see [25]. A finite deterministic automaton $A$ over the alphabet $S$ is a quadruple $(Q, q_0, F, \delta)$ where $Q$ is a finite set of states, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta$ is a partial function $Q \times S \rightarrow Q$. If $\delta(q, s) = q'$ then $q \xrightarrow{s} q'$ is a transition. An automaton $A$ can thus be seen as a directed graph on the vertex set $Q$ with edges labeled by elements of $S$ such that for any $q, s$ there is at most one edge with source $q$ and label $s$.

For an automaton $A$, one naturally extends $\delta$ to a partial function $Q \times S^* \rightarrow Q$. A word $s_1 \cdots s_k \in S^*$ is accepted by $A$ if $\delta(q_0, s_1 \cdots s_k)$ is defined and is in $F$. The set of all words accepted by $A$ is the language recognized by $A$ and denoted by $L(A)$. Languages $L \subseteq S^*$ occurring in this way are called regular, and it is a fundamental theorem of Kleene that the class of such languages coincides with the class of rational languages; see [25, Theorem 2.1].

Let $B$ be a Garside shadow. Recall from Definition 1.1 in the introduction that the automaton $A_B(W, S)$ is defined by:

- the set of states is $B$;
- the initial state is the identity $e$ of $W$, and all states are final;
- the transitions are: $x \xrightarrow{s} \pi_B(sx)$ whenever $s \notin D_L(x)$.

We denote $A_B := A_B(W, S)$ if there is no possible confusion. We prove now that $A_B(W, S)$ recognizes the language $\text{Red}(W, S)$ of reduced words in $(W, S)$.

**Proof of Theorem 1.2** We prove the theorem by induction. Let $P(k)$ $(k \in \mathbb{N})$ be the following property:

For any sequence $s_1, \ldots, s_k$ of simple reflections, $s_k \cdots s_1$ is reduced if and only if there is a path in $A_B$ starting at the initial state $e$ with edges labeled successively by $s_1, \ldots, s_k$. The final state of such a path is $\pi_B(s_k \cdots s_1)$.

By definition of $A_B$, properties $P(0)$ and $P(1)$ are easily seen to be true. Now let $k > 1$ be such that $P(i)$ holds for all $i < k$, and consider any sequence $s_1, \ldots, s_k \in S$. Let $w_j := s_j \cdots s_1$, and let $x_j := \pi_B(s_j x_{j-1})$.

We first show that the sequence of edge labels for a path in $A_B$ is reduced. Suppose there is a path in $A_B$

$$e \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \cdots \xrightarrow{s_{k-1}} x_{k-1} \xrightarrow{s_k} x_k$$

from the state $e$ to the state $x_k$, with edges labeled by $s_1, \ldots, s_k$. By induction, $s_k \cdots s_1$ is reduced, so that by Corollary 2.10, $\pi_B(w_{k-1}) = x_{k-1}$. Since $x_{k-1} \xrightarrow{s_1} x_k$ is an edge in the automaton $A_B$, $s_k \notin D_L(x_{k-1})$ by definition. Therefore $s_k \notin D_L(w_{k-1})$, since $D_L(x_{k-1}) = D_L(w_{k-1})$ by Proposition 2.6. In particular, since $s_k \cdots s_1$ is reduced, $s_k w_{k-1} = s_k s_{k-1} \cdots s_1$ is also reduced.

We now show that any reduced word $s_k s_{k-1} \cdots s_1$ gives a path in $A_B$ from $e$, with the desired edge labels and ending state. For the sake of contradiction, suppose that the sequence $s_1, \ldots, s_k$ does not define a path in $A_B$.

This gives rise to two cases, both of which lead to a contradiction of our initial assumption that $s_k s_{k-1} \cdots s_1$ is reduced. If the initial sequence $s_1, \ldots, s_{k-1}$ does not define a path in $A_B$, then, by induction $s_{k-1} \cdots s_1$ is not reduced, contradicting our assumption. Otherwise, the sequence $s_1, \ldots, s_{k-1}$ ends at the state $x_{k-1}$ and, by induction, $s_{k-1} \cdots s_1$ is reduced. In particular, $\pi_B(w_{k-1}) = x_{k-1}$ by Corollary 2.10. Since $s_1, \ldots, s_k$ does not define a path, $s_k \notin D_L(x_{k-1})$, so that $s_k \notin D_L(w_{k-1})$ by
Proposition 2.6. But then $s_ks_{k-1}\cdots s_1 = s_kw_{k-1}$ is not reduced, which again contradicts our initial assumption. 

Remark 2.11. Neither the definition of $A_B(W,S)$, the definition of $\pi_B$ nor the proof of Theorem 1.2 requires $B$ to be finite. However we chose to state the result for finite Garside shadows in Theorem 1.2, since those produce finite automata.

2.4. Root systems and inversion sets. Before studying the relation between Garside shadow automata and standard parabolic subgroups, we need to introduce a geometric representation and a root system for $(W,S)$.

Recall that a quadratic space $(V,B)$ is a data of a real vector space $V$ with a symmetric bilinear form $B$. The group $O_B(V)$ is the group consisting of all linear maps that preserve $B$. For any non-isotropic vector $\alpha \in V$, i.e., $B(\alpha,\alpha) \neq 0$, we associate a $B$-reflection $s_\alpha$ given by the formula $s_\alpha(v) = v - 2\frac{B(\alpha,v)}{B(\alpha,\alpha)}\alpha$, for all $v \in V$.

We consider now a geometric representation of $(W,S)$, i.e., a faithful representation of $W$ as a subgroup of $O_B(V)$, where $S$ is mapped into the set of $B$-reflections associated to a simple system $\Delta = \{\alpha_s \mid s \in S\}$ ($s = s_\alpha$). Then the $W$-orbit $\Phi = W(\Delta)$ is a root system with positive roots $\Phi^+ = \text{cone}_B(\Delta)$ and negative roots $\Phi^- = -\Phi^+$, where $\text{cone}(X)$ is the set of nonnegative linear combination of vectors in $X \subseteq V$ and $\text{cone}_B(X) = \text{cone}(X) \cap \Phi$; see [21 §1] for more details.

We recall now some useful well-known results linking roots and reduced words in $(W,S)$.

The left inversion set of $w \in W$ of $w \in W$ is defined by $N(w) := \Phi^+ \cap w(\Phi^-)$. The following proposition may be found in [20 §2.3-§2.5]; part (b) is due to M Dyer [12].

Proposition 2.12. The map $N : (W,\leq_R) \to (P(\Phi^+),\subseteq)$ is a poset monomorphism. Furthermore:

(a) For any $u,w \in W$, $u \leq_R w$ if and only if $N(u) \subseteq N(w)$;
(b) For any bounded $X \subseteq W$, $N(\bigvee X) = \text{cone}_B(\bigcup_{x \in X} N(x))$.

If $I \subseteq S$, then $\Delta_I := \{\alpha_s \mid s \in I\}$ is a simple system with root system $\Phi_I := W_I(\Delta_I)$ and positive root system $\Phi_I^+ := \Phi_I \cap \Phi^+$ for the standard parabolic subgroup $W_I$. The following statement is well-known; we include a proof here for completeness.

Corollary 2.13. Let $I \subseteq S$ and $w = w_I w^I$ with $w_I \in W_I$ and $w^I \in X_I$, then $N(w_I) = N(w) \cap \Phi_I^+.$

Proof. The left-to-right inclusion follows from Proposition 2.12 (a) since $w_I$ is a prefix of $w$. Now let $\alpha \in N(w) \cap \Phi_I^+$. So $w_I^{-1}(\alpha) \in \Phi_I$ and $w^{-1}(\alpha) = (w_I)^{-1}w_I^{-1}(\alpha)$ is an element of $\Phi^-$. Now $(w_I)^{-1}(\Phi_I^+) \subseteq \Phi^+$, since $w_I^I \in X_I$. Thus $w_I^{-1}(\alpha)$ must be in $\Phi_I^+$, and so $\alpha \in N(w_I).$ 

2.5. Parabolic Subgroups. We now discuss the behaviour of Garside shadows with respect to standard parabolic subgroups.

Let $B$ be a Garside shadow in $(W,S)$ and $W_I$ be the standard parabolic subgroup generated by $I \subseteq S$. Then $B \cap W_I$ is a Garside shadow [13, Remark 2.5(c)]. Let $A_B(I)(W,S)$ be the restriction of the automaton $A_B(W,S)$ to the states corresponding to $B \cap W_I$ and to the transitions corresponding to $s \in I$.

Proposition 2.14. Let $B$ be a Garside shadow and $I \subseteq S$. 

(a) The restriction of \( \pi_B \) to \( W_I \) is the \((B \cap W_I)\)-projection \( \pi_{B \cap W_I} : W_I \to B \cap W_I \).

(b) \( A_B^{(I)}(W, S) = A_{B \cap W_I}(W_I, I) \).

**Proof.** (a) By definition of Garside shadow projections, we need to show that for any \( w \in W_I \) we have \( \{ g \in B \mid g \leq_R w \} = \{ g \in B \cap W_I \mid g \leq_R w \} \).

The right-to-left inclusion is obvious. Now let \( w \in W_I \) and \( g \leq_R w \). Since \( w \in W_I \), any reduced word for \( w \) uses only letters from \( I \), by [2 Corollary 1.4.8(ii)]. In particular any prefix of \( w \) is in \( W_I \). Since \( g \leq_R w \), \( g \) is a prefix of \( W_I \). Therefore \( g \in W_I \), which concludes the proof of (a).

(b) By definition, the states of \( A_B^{(I)}(W, S) \) and of \( A_{B \cap W_I}(W_I, I) \) are the same. The fact that the transitions are the same follows by (a). \( \square \)

**Remark 2.15** (Minimal automata and restriction to standard parabolic subgroups). We do not know if the restriction of a Garside shadow to \( W_I \) is compatible with Garside closure [13 Remark 2.5(c)]. In other words, if \( B \) is a Garside shadow in \((W_I, I)\), is \( \text{Gars}(B) \cap W_I = B \)? In particular, we do not know if \( A_B^{(I)}(W, S) = A_I(W_I, I) \).

Another way to restrict a Garside shadow to a standard parabolic subgroup is by the mean of the minimal coset representatives decomposition; the associated automaton structure is discussed in Proposition 3.2. Recall that any element \( w \in W \) has a unique decomposition \( w = w_1 w^I \) with \( w_1 \in W_I \) and \( w^I \in X_I \). We denote by \( p_I : W \to W_I \) the projection defined by \( p_I(w) := w_I \).

**Proposition 2.16.** Let \( B \) be a Garside shadow in \((W, S)\) and \( I \subseteq S \).

(a) The set \( p_I(B) \) is a Garside shadow in \((W_I, I)\).

(b) We have \( p_I \circ \pi_B = \pi_{p_I(B)} \circ p_I \).

**Remark 2.17.** We have \( B \cap W_I \subseteq p_I(B) \), but equality does not hold in general. Indeed, let \( S = \{ s, t, u \} \) and \( W = \{ s \mid s^2 = t^2 = u^2 = 1, su = us \} \). One checks that \( B := \{ 1, s, t, u, su, tu, stu \} \) is a Garside shadow for \((W, S)\). Now pick \( I = \{ s, t \} \). Then we have \( p_I(stu) = st \notin B \) while \( stu \in B \), so \( st \in p_I(B) \setminus (B \cap W_I) \).

To prove the proposition, we need the following lemma.

**Lemma 2.18.** Let \( X \) be a bounded set in \( W \) and \( I \subseteq S \), then \( p_I(\bigvee X) = \bigvee p_I(X) \).

**Proof.** By Corollary 2.13 and Proposition 2.12(b) we have

\[
N \left( p_I \left( \bigvee X \right) \right) = N \left( \bigvee X \right) \cap \Phi_I = \text{cone}_\Phi \left( \bigcup_{x \in X} N(x) \right) \cap \Phi_I.
\]

Since our statement is about combinatorics of reduced words, we consider without loss of generality the simple system to be a basis of \( V \). So in particular \( \text{span}(\Phi_I) \) is a supporting hyperplane of \( \text{cone}(\Delta) \). Therefore,

\[
\text{cone}_\Phi \left( \bigcup_{x \in X} N(x) \right) \cap \Phi_I = \text{cone}_\Phi \left( \bigcup_{x \in X} N(x) \cap \Phi_I \right),
\]

since there are only finitely many generators for each cone. So by Corollary 2.13 we obtain

\[
N \left( p_I \left( \bigvee X \right) \right) = \text{cone}_\Phi \left( \bigcup_{x \in X} N(p_I(x)) \right).
\]
Finally, by Proposition 2.12(b) and Corollary 2.13 again we have:

\[ N \left( \bigvee p_I(X) \right) = \text{cone}_{\Phi_I} \left( \bigcup_{x \in X} N(p_I(x)) \right) = N \left( p_I \left( \bigvee X \right) \right). \]

Proof of Proposition 2.16. (a) We verify the conditions in Definition 2.2. It is clear that \( p_I(B) \subseteq W_I \). Now, since \( p_I(s) = s \) for any \( s \in I \) and \( I \subseteq S \subseteq B \) we have \( I \subseteq p_I(B) \). For Condition (i), consider \( X \subseteq B \) bounded in \( W \). So \( p_I(X) \) is bounded in \( W \), so its join \( \bigvee p_I(X) \) exists. We have to show that \( \bigvee p_I(X) \in p_I(B) \).

By Lemma 2.18 we have \( \bigvee p_I(X) = p_I(\bigvee X) \), which is an element of \( p_I(B) \) since \( \bigvee X \in B \). For Condition (ii), consider \( w \in B \) and a suffix \( v \) of \( w \). Since \( w' \in X_I \), the expression \( vw' \) is reduced. Therefore, \( vw' \) is a suffix of \( w \in B \). Since \( B \) is a Garside shadow, \( vw' \in B \). Furthermore, \( p_I(vw') = v \), so \( v \in p_I(B) \).

(b) It is enough to show that for \( w \in W \), we have

\[ \{ p_I(g) \mid g \in B, g \leq_R w \} = \{ g' \in p_I(B) \mid g' \leq_R p_I(w) \}. \]

But this follows easily from Proposition 2.12(a) together with Corollary 2.13.

\[ \square \]

3. Morphisms and Garside Shadow Automata

In this section, we discuss morphisms between Garside shadow automata, then we compare the automata of a particular family of Garside shadows, the set of n-low elements with the family of n-canonical automata. We first recall the definitions of morphisms of automata, minimal automata, and the concept of minimal roots for \( (W, S) \).

3.1. Morphisms of automata. We refer the reader to [25, Chapter II(3)] for additional details on morphisms of automata. Here we shall only use this notion in a particular case suited to our various automata.

Definition 3.1 (see [25, Chapter II(3)]). Let \( \mathcal{A} = (Q, q_0, F, \delta) \) and \( \mathcal{A}' = (Q', q'_0, F', \delta') \) be two finite deterministic automata over the same alphabet \( S \). A function \( f : Q \rightarrow Q' \) is a morphism of automata between \( \mathcal{A} \) and \( \mathcal{A}' \) if

(i) \( f(q_0) = q'_0 \);
(ii) \( f(F) \subseteq F' \);
(iii) If \( q_1 \xrightarrow{s} q_2 \) is a transition in \( \mathcal{A} \), then \( f(q_1) \xrightarrow{s} f(q_2) \) is a transition in \( \mathcal{A}' \).

A morphism of automata \( f \) is totally surjective if \( f \) is surjective, satisfies \( f^{-1}(F') = F \) and if, for any transition \( q'_1 \xrightarrow{s} q'_2 \) in \( \mathcal{A}' \), there exists \( q_1, q_2 \) such that \( f(q_1) = q'_1, f(q_2) = q'_2 \) and \( q_1 \xrightarrow{s} q_2 \) in \( \mathcal{A} \). In this case \( \mathcal{A}' \) is called a quotient of \( \mathcal{A} \).

If \( f \) is a morphism between \( \mathcal{A} \) and \( \mathcal{A}' \) then \( L(\mathcal{A}) \subseteq L(\mathcal{A}') \). If \( f \) is totally surjective then \( L(\mathcal{A}) = L(\mathcal{A}') \).

The following proposition gives a first example of a totally surjective morphism related to Garside shadow automata and arising from the surjection \( p_I \) from Proposition 2.16.

Proposition 3.2. Let \( I \subseteq S \) and \( B \) be a Garside shadow in \( (W, S) \). The automaton \( \mathcal{A}_B(W, I) \) is defined by taking the same states \( B \), initial state \( e \), and final states as \( \mathcal{A}_B(W, S) \), but with only the transitions of \( \mathcal{A}_B(W, S) \) corresponding to letters in \( I \). Then the surjection \( p_I : B \rightarrow p_I(B) \) induces a totally surjective morphism from \( \mathcal{A}_B(W, I) \) to \( \mathcal{A}_{p_I(B)}(W_I, I) \).
Proof. Let us first show that \( p_I \) verifies the conditions in Definition 3.1. We have \( p_I(e) = e \), and all states are final in both \( A_B(W, I) \) and \( A_{p_I(B)}(W_I, I) \), so (i) and (ii) hold. To prove (iii), let \( w \xrightarrow{\delta} \pi_B(sw) \) be a transition in \( A_B(W, I) \) with \( s \in I \setminus D_L(w) \). We have to show that \( p_I(w) \xrightarrow{\delta} p_I(\pi_B(sw)) \) is a transition in \( A_{p_I(B)}(W_I, I) \).

Since \( D_L(w) \cap I = D_L(p_I(w)) \), there is a transition \( p_I(w) \xrightarrow{\delta} \pi_{p_I(B)}(sp_I(w)) \). The equality \( p_I(\pi_B(sw)) = \pi_{p_I(B)}(sp_I(w)) \) is then guaranteed by Proposition 2.16(b) since \( p_I(sw) = sp_I(w) \) for any \( s \in I \setminus D_L(w) \).

To prove that \( p_I \) is totally surjective, let \( p_I(w) \xrightarrow{\delta} \pi_{p_I(B)}(sp_I(w)) \) be a transition in \( A_{p_I(B)}(W_I, I) \), with \( w \in B \) and \( s \in I \setminus D_L(p_I(w)) \). Then \( w \xrightarrow{\delta} \pi_B(sw) \) is a transition in \( A_B(W, I) \) since \( D_L(w) \cap I = D_L(p_I(w)) \), and we conclude as in the previous paragraph.

Minimal automata. Given a regular language \( L \in S^* \), there exists an automaton \( R(L) \) which recognizes \( L \) and is a quotient of all automata that recognize \( L \), called the minimal automaton of \( L \).

It can be constructed as follows: given \( u \in S^* \), define \( u^{-1}L \) to be the set of \( v \in S^* \) such that \( uv \in L \), and let \( Q_L = \{ u^{-1}L \mid u \in S^* \} \). We then define \( R(L) = (Q_L, q_I, F_L, \delta_L) \) with \( q_I = L \), \( F_L = \{ u^{-1}L \mid u \in L \} \) and transitions \( \delta_L(u^{-1}L, a) = (ua)^{-1}L \). This automaton clearly recognizes \( L \).

Now pick any deterministic, complete automaton \( A \) such that \( L = L(A) \). Given \( q \in Q \), let \( L_q(A) \) be the language recognized by the automaton \( A \) with \( q \) replacing \( q_0 \) as initial state. If \( L_q(A) = L_{q'}(A) \) then \( q \) and \( q' \) are called equivalent states. Then \( q \mapsto L_q(A) \) is a totally surjective morphism from \( A \) to \( R(L) \). Therefore in order to prove that an automaton is minimal, one must show that distinct states are never equivalent.

Remark 3.3. Denote by \( A_{min}(W, S) \) the minimal automaton that recognizes the language \( \text{Red}(W, S) \). For \( I \subseteq S \), we define the automaton \( A_{min}^{(I)}(W, S) \) to be the restriction of \( A_{min}(W, S) \) to the transitions in \( I \) and the states that can be reached from the initial state using these transitions. We now show that \( A_{min}^{(I)}(W, S) = A_{min}(W_I, I) \), so that minimal automata remain minimal upon restriction to a parabolic subgroup.

Let \( q_1, q_2 \) be distinct states in \( A_{min}^{(I)}(W, S) \); \( q_1, q_2 \) can be reached by reading (reduced words for elements) \( w_1, w_2 \in W_I \), respectively. Since \( q_1, q_2 \) are non-equivalent states in \( A_{min}(W, S) \) by minimality, there exists \( w \in W \) such that \( w_1w \) is reduced while \( w_2w \) is not reduced. Now use the decomposition \( w = w_1w' \) with \( w_I \in W_I \) and \( w' \in X_I \), then \( w_1w_I \) is reduced while \( w_2w_I \) is not reduced. So the states are not equivalent in \( A_{min}^{(I)}(W, S) \), which is therefore the minimal automaton \( A_{min}(W_I, I) \) that recognizes \( \text{Red}(W_I, I) \).

Therefore—assuming Conjecture 1—we conclude that \( A_{S}^{(I)}(W, S) = A_I(W_I, I) \), resolving our question in Remark 2.16.

3.2. Inclusion of Garside shadows and morphisms of automata.

Proposition 3.4. If \( C \subseteq B \) are two Garside shadows, then \( \pi_C \circ \pi_B = \pi_C \).

\(^1\)This means that \( \delta \) is defined everywhere. Any automaton can be transformed into a complete one by adding a non-final sink state † and transitions \( b(q, s) := \dagger \) whenever \( \delta \) is not previously defined.
Proof. It is enough to show that for \( w \in W \), we have
\[
\{ g \in C \mid g \leq_R \pi_B(w) \} = \{ g \in C \mid g \leq_R w \}.
\]
The left-to-right inclusion follows from Proposition 2.5(b). Now let \( g \in C \) such that \( g \leq_R w \). Since \( C \subseteq B \) we have by Proposition 2.5(b,c) that
\[
g = \pi_B(g) \leq_R \pi_B(w),
\]
which concludes the proof. \( \square \)

**Corollary 3.5.** If \( C \subseteq B \) are two Garside shadows, then the \( C \)-projection \( \pi_C \) induces a totally surjective morphism from \( A_B \) to \( A_C \). In particular, \( A_\mathcal{S} \) is a quotient of any Garside shadow automaton.

**Proof.** The \( C \)-projection \( \pi_C : B \rightarrow C \) is surjective, since if \( w \in C \), then \( w \in B \) and \( \pi_C(w) = w \). We now show that \( \pi_C \) verifies the conditions in Definition 3.1.

(i) Since \( e \in C \), \( \pi_C(e) = e \).

(ii) Let \( w \rightarrow \pi_B(sw) \) be a transition in \( A_B \) with \( s \notin D_L(w) \). We have to show that \( \pi_C(w) \rightarrow \pi_C(sw) = \pi_C(sw) \) is a transition in \( A_C \), using Proposition 3.4. Since \( D_L(w) = D_L(\pi_C(w)) \) by Proposition 2.6, we have \( s \notin D_L(\pi_C(w)) \). So \( \pi_C(w) \rightarrow \pi_C(sw) \) is a transition in \( A_C \) by definition, and we conclude by Proposition 2.8.

(iii) This holds since all states are final in both automata and \( \pi_C \) is surjective.

To prove that \( \pi_C \) is totally surjective, it remains to show that, if \( v \rightarrow v' \) is a transition in \( A_C \), there is a transition \( u \rightarrow u' \) in \( A_B \) with \( \pi_C(u) = v \) and \( \pi_C(u') = v' \). But this is guaranteed by taking \( u := v \) and \( u' := v \) since \( C \subseteq B \). \( \square \)

To conclude this discussion, we show Conjecture 1 in the finite case.

**Proposition 3.6.** Assume that \( W \) is finite. Then \( \tilde{S} = W \) and \( A_\mathcal{S} \) is the minimal automaton that recognizes \( \text{Red}(W, S) \).

**Proof.** The fact that \( W \) is finite implies that \( \tilde{S} = W \) [18, Proposition 2.2(3)]. Therefore \( \pi_\mathcal{S} = \pi_W = \text{Id}_W \), the identity map on \( W \). Thus \( A_\mathcal{S} \) has states indexed by \( W \) and transitions \( w \rightarrow sw \) if \( s \notin D_L(w) \).

Let \( u, v \in W \) be two equivalent states, i.e., for any \( s_1, s_2, \ldots, s_k \), we have that \( s_k \cdots s_1u \) is reduced if and only if \( s_k \cdots s_1v \) is reduced. We must prove that \( u = v \). Let \( k \geq 0 \) be maximal so that there exist \( s_1, s_2, \ldots, s_k \) with \( s_k \cdots s_1u \) reduced; note that \( k \) exists since \( W \) is finite. By hypothesis, \( s_k \cdots s_1v \) is reduced and \( k \) is necessarily also maximal for that property. But there is a unique element \( w \) satisfying \( D_L(w) = S \), namely the longest element \( w_o \) [2, Proposition 2.3.1(ii)]. This shows that \( s_k \cdots s_1u = w_o = s_k \cdots s_1v \) and thus \( u = v \). \( \square \)

### 3.3. Small Inversion Sets and Low Elements

In [4], the authors introduced a partial order \( \preceq \) on \( \Phi^+ \) called the dominance order defined by:
\[
\alpha \preceq \beta \iff (\forall w \in W, \beta \in N(w) \implies \alpha \in N(w)).
\]
The \( \infty \)-depth of \( \beta \in \Phi^+ \) is the number of positive roots strictly dominated by \( \beta \):
\[
dp_\infty(\beta) := |\{ \alpha \in \Phi^+ \mid \alpha \prec \beta \}|.
\]

**Definition 3.7.** Let \( n \in \mathbb{N} \), we say that a root \( \beta \in \Phi^+ \) is \( n \)-small if \( \dp_\infty(\beta) \leq n \) and set \( \Sigma_n(W) \) to be the set of \( n \)-small roots.
A 0-small root is called a small root and we write $\Sigma(W) := \Sigma_0(W)$. B. Brink and R. Howlett showed in [1] that $\Sigma_n$ is finite for $n = 0$. This result was later extended by X. Fu [18] to all $n \geq 0$. The (left) $n$-small inversion set of $w \in W$ is

$$\Sigma_n(w) := N(w) \cap \Sigma_n,$$

and we denote by $\Lambda_n(W) \subseteq \mathcal{P}(\Sigma_n(W))$ the set of all $n$-inversion sets. Since $\Sigma_n(W)$ is finite, $\Lambda_n(W)$ is also finite. We write $\Sigma(w) := \Sigma_0(w)$.

**Definition 3.8.** An element $w \in W$ is $n$-low if $N(w) = \text{cone}_\Phi(\Sigma_n(w))$. We denote by $L_n(W)$ the set of $n$-low elements in $W$.

A 0-low element is called a low element and we write $L(W) := L_0(W)$. Low elements were introduced by P. Dehornoy, M. Dyer, and the first author in [11], and extended for any $n \in \mathbb{N}$ by M. Dyer and the first author in [13]. We refer the reader to [13, §3.1-§3.3] for more details and examples; examples of low elements are also given in Figure 4. We summarize here some results concerning $n$-small inversion sets and $n$-low elements.

**Theorem 3.9** ([13]). Let $n \in \mathbb{N}$.

(a) The map $\Sigma_n : L_n(W) \to \Lambda_n(W)$ is injective.

(b) The set $L_n(W)$ of $n$-low elements is finite and closed under join in $(W, \leq_R)$.

(c) The set of low elements $L(W)$ is a finite Garside shadow in $(W, S)$.

(d) If $(W, S)$ is finite, affine or with Coxeter graph with edges labelled by $3, \infty$, then the set $L_n(W)$ of $n$-low elements is a finite Garside shadow in $(W, S)$.

The statements (a) and (b) are [13, Proposition 3.26], the statement (c) is [13, Theorem 1.1] and (d) is [13, Theorem 1.3 and Theorem 4.17]. We end this discussion by recalling two conjectures from [13]:

13 Conjecture 1: The set $L_n(W)$ is a finite Garside shadow in $(W, S)$.

13 Conjecture 2: The map $\Sigma_n : L_n(W) \to \Lambda_n(W)$ is a bijection.

### 3.4. Low Element Automata and Canonical Automata.

Let $n \in \mathbb{N}^*$, the $n$-canonical automaton is the finite automaton $\mathcal{A}_n(W, S)$ over $S$ defined as follows:

- the (finite) set of states is $\Lambda_n(W)$;
- the initial state is $\emptyset (= \Sigma_n(e))$ and all states are final;
- the transitions are: $A \to \{\alpha_s\} \cup (s(A) \cap \Sigma_n)$ whenever $\alpha_s \notin A$.

As shown in [13], if $A = \Sigma_n(w)$ then $s \notin D_L(w)$ if and only if $\alpha_s \notin A$, and in this case $\{\alpha_s\} \cup (s(A) \cap \Sigma_n) = \Sigma_n(sw)$. The transitions are thus well defined. Also, one has immediately that if $w = s_1 \cdots s_k$ is reduced, then the path from $\emptyset$ with labels $s_1, \ldots, s_k$ ends in the state $\Sigma_n(w)$.

Therefore the $n$-canonical automaton $\mathcal{A}_n(W, S)$ recognizes $\text{Red}(W, S)$, for any $n \in \mathbb{N}$.

The 0-canonical automaton, or simply the canonical automaton, was studied by H. Eriksson in his thesis [17] and named in [2, §4.8].

When $L_n(W)$ is a Garside shadow in $(W, S)$—which we suspect is always the case [13, Conjecture 1]— we may consider the associated finite Garside shadow projection and automaton.

**Proposition 3.10.** Let $n \in \mathbb{N}$.

(a) The Garside shadow projection $\pi_{L_n(W)}$ is well-defined.
(b) The map $\pi_n : \Lambda_n(W) \to L_n(W)$ defined by $\pi_n(\Sigma_n(w)) := \pi_{L_n(W)}(w)$ is a well-defined surjection.

(c) The map $\pi_n$ induces a totally surjective morphism from the canonical automaton $A_0(W,S)$ to the automaton $A_{L_n(W)}(W,S)$.

(d) If $L_n(W,S)$ is a Garside shadow in $(W,S)$, then $\pi_n$ induces a totally surjective morphism from the $n$-canonical automaton $A_n(W,S)$ to the automaton $A_{L_n(W)}(W,S)$.

Proof. (a) As observed in Remark 2.7, the definition of the Garside shadow projection $\pi_{L_n(W)}$ only requires $L_n(W)$ to contain $S$ and be closed under taking joins, which is guaranteed by Theorem 3.9(b).

(b) The fact that $\pi_n$ is surjective follows from the definition of $\Lambda_n(W)$. To prove that $\pi_n$ is well-defined, let $u, v \in W$ such that $\Sigma_n(u) = \Sigma_n(v)$. We have to show that $\pi_{L_n(W)}(u) = \pi_{L_n(W)}(v)$. By Definition 2.4 it is enough to show that
\[ \{ g \in L_n(W) \mid g \leq_R u \} = \{ g \in L_n(W) \mid g \leq_R v \}. \]

Let $g \in L_n(W)$ such that $g \leq_R u$. By Proposition 2.12 we have $N(g) \subseteq N(u)$, and therefore $\Sigma_n(g) \subseteq \Sigma_n(u) = \Sigma_n(v)$. Since $g$ is $n$-low we have by definition $N(g) = \text{cone}_\Phi(\Sigma_n(g)) \subseteq \text{cone}_\Phi(\Sigma_n(v)) \subseteq N(v)$.

Therefore, again by Proposition 2.12 $g \leq_R v$. This shows the left-to-right inclusion, and we conclude the other inclusion by symmetry.

(c) This follows from (d) and Theorem 3.9(c).

(d) We must check the three conditions of Definition 3.1:

(i) This follows from the fact that $\pi_n(\emptyset) = \pi_n(\Sigma_n(e)) = \pi_{L_n(W)}(e) = e$.

(ii) This follows since every state in both automata is final and $\pi_n$ is surjective.

(iii) By definition of the transitions in $A_{L_n(W)}(W,S)$ and $A_n(W,S)$, one must check that if $s \notin D_L(w)$, then $\pi_{L_n(W)}(s \pi_{L_n(W)}(w)) = \pi_{L_n(W)}(sw)$. This follows immediately from Proposition 2.8.

To prove that $\pi_n$ is totally surjective, it remains to show that, if $w \xrightarrow{s} \pi_{L_n(W)}(sw)$ is a transition in $A_{L_n(W)}(W,S)$, then $\Sigma_n(w) \xrightarrow{\alpha_s} \Sigma_n(sw)$ is a transition in $A_n(W,S)$. But $s \notin D_L(w)$, therefore $\alpha_s \notin \Sigma_n(w)$, and so $\Sigma_n(w) \xrightarrow{\alpha_s} \Sigma_n(sw)$ is a transition in $A_n(W,S)$.

\[ \square \]

Remark 3.11. Let $n \in \mathbb{N}$. If the map $\Sigma_n : L_n(W) \to \Lambda_n(W)$ is a bijection, i.e. [13] Conjecture 2] has a positive answer, then $\pi_n$ would induce an isomorphism from the $n$-canonical automaton $A_n(W,S)$ to $A_{L_n(W)}(W,S)$.

Proposition 3.12. If $W$ is finite, then the canonical automaton $A_0(W,S)$ is minimal.

Proof. Since $W$ is finite, we have $\Phi^+ = \Sigma(W)$. Then in particular $\Sigma(w) = N(w)$ for any $w \in W$ and therefore $L(W) = W = \tilde{S}$, by Proposition 3.6. So $N = \Sigma : L(W) = W = \tilde{S} \to \Lambda(W)$ is a bijection and therefore $A_0(W,S)$ and $A_{\tilde{S}}(W,S)$ are isomorphic. The result follows therefore by Proposition 3.6.

The next corollary of Proposition 3.10 together with Corollary 3.5 strengthens the evidence for Conjecture 3.

Corollary 3.13. The automaton $A_{\tilde{S}}(W,S)$ associated to the smallest Garside shadow $\tilde{S}$ in $(W,S)$ is a quotient of all canonical automata $A_n(W,S)$. 

Figure 2 illustrates all of our automata recognizing Red($W, S$), and the maps between them.

3.5. Minimality of the canonical automaton. A positive root $\beta \in \Phi^+ = \text{cone}_\Phi(\Delta)$ has a unique expression with nonnegative linear combination of vectors in $\Delta$: $\beta = \sum_{s \in S} a_s \alpha_s$, with $a_s \geq 0$; we define the support of $\beta$ to be the set $\text{supp}(\beta) := \{ s \in S | a_s > 0 \}$.

We say that a positive root $\beta$ is spherical if the standard parabolic subgroup $W_{\text{supp}(\beta)}$ is finite, and we write $\Phi^\oplus_{\text{sph}}$ for the set of spherical roots.

Spherical roots are always small. Now if the reverse inclusion holds, the following proposition shows that the canonical automaton is minimal, so that one implication in Conjecture 2 is true.

**Proposition 3.14.** Let $W$ be irreducible. If $\Sigma = \Phi^\oplus_{\text{sph}}$, then $A_0(W, S)$ is minimal.

The following proof is inspired by Theorem V.8 in P. Headley’s thesis [19].

**Proof.** Let $\Sigma(u)$ and $\Sigma(v)$ be two equivalent states of $A_0(W, S)$. This means that for any $s_1, s_2, \cdots, s_k$ in $S$, we have that $s_k \cdots s_1 u$ is reduced if and only if $s_k \cdots s_1 v$ is reduced. We have to show that $\Sigma(u) = \Sigma(v)$.

Assume that they are distinct, so that, up to exchanging the role of $u$ and $v$, there is $\alpha \in \Sigma(u) \setminus \Sigma(v)$. By assumption, $\Sigma = \Phi^\oplus_{\text{sph}}$, so there is $I \subseteq S$ such that $W_I$ is finite and $\alpha \in \Phi^\oplus_I$.

Now we use the decompositions $u = u_I u_I'$ and $v = v_I v_I'$ in $W_I \times X_I$. The expression $wu$ is reduced if and only if $wu_I$ is reduced, since $gu'$ is reduced for any $g \in W_I$. So we have that for any $s_1, \ldots, s_k \in I$, $s_k \cdots s_1 u_I$ is reduced if and only if $s_k \cdots s_1 v_I$ is reduced.

Since $W_I$ is finite, the automaton $A_0(W_I, I)$ is minimal by Proposition 3.12. Therefore $\Sigma(u_I) = \Sigma(v_I)$. Note that $\Sigma(u_I) := \Sigma(u) \cap \Phi_I$ and $\Sigma(v_I) := \Sigma(v) \cap \Phi_I$. 

**Figure 2.** Commutative diagram relating the various automata described in this article. The projections $\pi_i$ are conjectured to be bijections [13, Conjecture 2], and so is $p_{\min}$ (Conjecture 1). The sets $L_i \subseteq W$ are conjectured to be Garside shadows [13, Conjecture 1]; only $L_0$ is known to be. Finally Conjecture 2 characterizes when the bottom row $p_{\min} \circ \pi_{\tilde{S}} \circ \pi_0$ is an isomorphism.
are small inversion sets for \((W_I, I)\), by Corollary 2.13 and the definition of small inversion sets. But \(\alpha\) was chosen to be in \(\Phi_I \cap \Sigma(u)\), which contradicts that \(\alpha \in \Sigma(u) \setminus \Sigma(v)\). Therefore, \(\Sigma(u) = \Sigma(v)\).

We conclude by proving Theorem 1.3.

Proof of Theorem 1.3.

(1) This is Proposition 3.12, since \(A_0(W, S)\) and \(A_H(W, S)\) are isomorphic in this case.

(2) Here \(\Sigma = \Phi_{sph}^+ = S\); see for instance \([11, \text{Proposition 5.1}( iii)]\). So we are in the case of Proposition 3.14.

(3) In this case \(\Phi_{sph}^+\) consists of the union of all \(\Phi_{s,t}^+\) where \(m_{st} < \infty\). This is equal to the whole of \(\Sigma\) since the support of a small root is a tree with no \(\infty\)-edge \([3]\). We conclude again by Proposition 3.14. Note that one can actually give an explicit description of the canonical automaton in this case and prove its minimality directly.

(4) The fact that the automaton is minimal in this case is due to Eriksson \([17, \text{Theorem 80}]\). Now recall that the Coxeter graph is a simply-laced cycle. Since the support of a small root is a tree \([3]\), we have \(\Sigma = \Phi_{sph}^+\) here and the conjecture holds by Proposition 3.14.

(5) The case of complete graphs was already checked, so one may assume that we have generators \(s, t, u\) with \(m_{su} = 2\) and \(3 \leq m_{st} \leq m_{tu}\). Denote \(m = m_{st}\) and \(p = m_{tu}\). If \(p = \infty\), or if \(m = 3\) and \(p < 6\), we have \(\Sigma = \Phi_{sph}^+\), so Proposition 3.14 gives us the result.

We may now assume \((m = 3\) and \(p \geq 6)\) or \((m, p \geq 4)\); in particular \(W\) is not finite. Write \(c_i = 2 \cos(\pi/i)\). Then \(\alpha := us\alpha_t = c_m\alpha_s + \alpha_t + c_p\alpha_u\) is a small root which is not spherical, so that \(\Sigma \neq \Phi_{sph}^+\). To show that the conjecture holds we thus need to find two distinct equivalent states in \(A_0(W, S)\).

Now \(su\) and \(tsu\) are reduced words, with distinct final states in \(A_0(W, S)\) given by \(\Sigma(su) = \{\alpha_s, \alpha_u\}\) and \(\Sigma(sut) = \{\alpha_s, \alpha_u, \alpha_t\}\). We have \(D_L(su) = D_L(tsu) = \{s, u\}\), so only \(t\) can be read from any of these states. Now a quick computation shows \(t(\alpha) = \alpha + (c_m^2 + c_p^2 - 1)\alpha_t\). Since \(c_m^2 + c_p^2 - 1 \geq 2\) for all considered values of \(m\) and \(p\), we have \(t(\alpha) \notin \Sigma\) and therefore \(\Sigma(tsu) = \Sigma(tsut)\). This shows that \(\Sigma(su)\) and \(\Sigma(tsu)\) are equivalent states, and thus that \(A_0(W, S)\) is not minimal.

Remark 3.15 (On Conjectures 1 and 2). Using Sage \([23, 5^+09]\), we wrote code to compute the set of small roots. We used these to compute the canonical automaton, from which we determined the minimal automaton. It is simple to test if a given small roots is spherical by examining the simple roots that occur in its support, from which we are able to check Conjecture 2. This code is sufficiently fast to compute examples in rank 5—for example, we determined that the minimal automaton for \(\tilde{D}_5\) has size 58965.

We also wrote a naive implementation to determine the minimal Garside shadow using Definition 2.2 to check Conjecture 1. This code finishes in a few minutes on standard hardware in rank four (and below), but already takes longer than several hours in rank five.
Our software confirms that Conjectures 1 and 2 hold for all Coxeter groups of rank 4 with edge labels less than 10. Figure 3 includes data for a few selected Coxeter groups of low rank.

| Name | Coxeter Diagram | $|A_0(W,S)|$ | $|A_{\tilde{S}}(W,S)|$ | $|A_{\text{min}}(W,S)|$ | $|\Sigma|$ | $\Phi^{+}_{sph}$ |
|------|----------------|-------------|----------------|----------------|-------------|---------------|
| $\tilde{A}_2$ | ![Diagram](image) | 16 | 16 | 16 | 6 | 6 |
| $\tilde{C}_2$ | ![Diagram](image) | 25 | 24 | 24 | 8 | 7 |
| $\tilde{G}_2$ | ![Diagram](image) | 49 | 41 | 41 | 12 | 8 |
| $\tilde{A}_3$ | ![Diagram](image) | 125 | 125 | 125 | 12 | 12 |
| $\tilde{C}_3$ | ![Diagram](image) | 343 | 317 | 317 | 18 | 15 |
| $\tilde{B}_3$ | ![Diagram](image) | 343 | 315 | 315 | 18 | 15 |
| $\tilde{C}_3$ | ![Diagram](image) | 92 | 92 | 92 | 15 | 15 |
| $\tilde{G}_3$ | ![Diagram](image) | 164 | 164 | 164 | 21 | 21 |
| $\tilde{A}_4$ | ![Diagram](image) | 91 | 80 | 80 | 18 | 14 |
| $\tilde{C}_4$ | ![Diagram](image) | 100 | 90 | 90 | 30 | 25 |

Figure 3. Numerical data for selected Coxeter groups. Note that in affine type $\tilde{W}_n$, $|A_0(W,S)| = (h + 1)^n$ and $|\Sigma| = nh$, where $h$ is the Coxeter number of the corresponding finite Weyl group.

3.6. **Canonical automata and Shi arrangements.** We end this article by describing some rank 3 examples of automata. It turns out these examples can be drawn in a very nice way: their states form a convex set in the (dual of the) geometric representation of $(W,S)$. The reason in the affine case is related to a property of the Shi arrangement, which leads us to discuss a generalization of the Shi arrangement for any Coxeter system.

Let $\Phi_0$ be a reduced, irreducible, crystallographic root system of rank $r$ for a finite Weyl group $W_0$ in a real vector space $V_0$ with $W_0$-invariant positive definite scalar product $\langle \cdot, \cdot \rangle$. Let $\Phi^+_0$ be a choice of positive roots, let $\Delta_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$
be the corresponding simple roots. The height of a positive root $\alpha = \sum_{i=0}^{n} c_i \alpha_i$ is $\sum_{i=0}^{n} c_i$; for example, the highest root $\alpha_h$ in $\Phi_0^+$ has height $h - 1$, where $h$ is the Coxeter number of $W_0$. Define $V := V_0 \oplus \mathbb{R}\delta$ and define the set of affine roots to be

$$\Phi := \{ \alpha + k\delta | \alpha \in \Phi \text{ and } k \in \mathbb{Z} \}.$$

The positive affine roots are $\Phi^+ := \{ \alpha + k\delta | \alpha \in \Phi^+_0 \text{ and } k \geq 0 \in \mathbb{Z} \} \cup \{ \alpha + k\delta | \alpha \in -\Phi^+_0 \text{ and } k > 0 \in \mathbb{Z} \}$, and the simple affine roots are $\Delta := \Delta_0 \cup \{ \alpha_0 \}$, where $\alpha_0 := -\alpha_h + \delta$.

For $\alpha \in \Phi_0$ and $k \in \mathbb{Z}$, we consider in $V_0$, seen as an affine space, the affine hyperplane

$$H_{\alpha,k} := \{ x \in V_0 | \langle x, \alpha \rangle = k \}.$$

The affine Weyl group $W$ is the group generated by affine reflections in the simple affine hyperplanes: $H_{\alpha,0}$ for $\alpha \in \Delta_0$ and $H_{\alpha_h,1}$. The fundamental alcove $K$ is the (interior of the) compact region bounded by the simple affine hyperplanes. The closure of $K$ is a fundamental domain for the action of $W$.

The $n$-Shi arrangement is the collection of hyperplanes

$$\text{Shi}_n(W) := \{ H_{\alpha,k} | \alpha \in \Phi^+, -n + 1 \leq k \leq n \}.$$

We abbreviate $\text{Shi}(W) := \text{Shi}_1(W)$, and call it the Shi arrangement\footnote{This was called the sandwich arrangement in [19]}. The roots corresponding to the hyperplanes in $\text{Shi}_n(W)$ coincide with the $n$-small roots, so that $\Sigma_n$ can be thought of as a generalization of the $n$-Shi arrangement to any Coxeter group. The crystallographic affine root system $\Phi$ is easily and bijectively convertible to a root system; so the dominance order and $n$-small roots are well-defined in a crystallographic root system. It particular, the only relations in the dominance order on $\Phi^+$ are $\alpha + k\delta \preceq \alpha + l\delta$ for $\alpha \in \Phi, k \leq \ell \in \mathbb{Z}$; see [13, Example 3.9]. We obtain therefore the following proposition.

**Proposition 3.16.** If $(W, S)$ is of affine type, then

$$\text{Shi}_n(W) = \{ H_{\alpha} | \alpha \in \Sigma_n(W) \}.$$  

The Shi arrangements for types $\tilde{A}_2$ and $\tilde{C}_2$ are drawn in Figure 4.

In affine type, the small inversion sets have previously been studied under the guise of the minimal alcoves of the $n$-Shi arrangement. More precisely, for $W$ of affine type, $\Lambda_n(W) = \{ w \in W | \{ w(s) | s \in D_L(w) \} \subseteq \Sigma_n \}$. The corresponding statement for $n$-low elements and general type is given as [13, Conjecture 2], restated above in §3.

It turns out that there are $(nh + 1)^r$ $n$-low elements in affine type. The reason for this is that the inverses of such elements coalesce into an $(nh + 1)$-fold dilation of the fundamental alcove.

**Theorem 3.17** (J. Y. Shi). Let $K$ be the fundamental alcove for an affine Weyl group $W$, and let $h$ be the Coxeter number of the corresponding finite Weyl group $W_0$. Then

$$\{ w^{-1}K | w \text{ an } n\text{-low element} \} \cong (nh + 1)K.$$
In particular, the alcoves corresponding to the inverses of \( n \)-low elements form a convex set. This theorem is illustrated for the infinite dihedral group \( \tilde{A}_1 = I_2(\infty) \) in Figures 1 and 5, which show the automata built from the 1- and 2-low elements, respectively. Figure 6 illustrates this theorem for types \( \tilde{A}_2 \) and \( \tilde{C}_2 \), simultaneously drawing the automaton.

We note that convexity does not necessarily hold for the subset of alcoves coming from the inverses of elements in \( \tilde{S} \), as seen for example in Figure 6—for \( \tilde{C}_2 \), \( \tilde{S} = \Sigma \setminus \{s_1 s_3 s_2\} \).

On the basis of the affine rank three examples, it is tempting to conjecture that equivalent states are given by intersecting intervals in the weak order with \( L_n(W) \). The (non-affine) triangle group \((5, 3, 5)\) is a counterexample to this claim.

When the Coxeter system \((W, S)\) is of indefinite type, i.e., \( W \) is not finite nor affine, the isotropic cone \( Q = \{x \in V \mid B(x, x) = 0\} \), and the region where \( x \in V \) verifies \( Q(x, x) < 0 \), are nonempty. In this case, following \[21\] \[14\], we consider the projective representation for \((W, S)\) associated to the geometric representation of
(W, S), with roots system Φ and simple system Δ in §2.4. More precisely, since Φ = Φ⁺ ⊔ Φ⁻ is encoded by the set of positive roots Φ⁺, we represent Φ by an ‘affine cut’: there is an affine hyperplane V₁ in V transverse to Φ⁺, i.e., for any β ∈ Φ⁺, the ray R⁺β intersects V₁ in a unique nonzero point ˜β. So Rβ ∩ V₁ = {˜β} for any β ∈ Φ. The set of normalized roots ˜Φ = {˜β | β ∈ Φ} is contained in the compact set conv( ˜∆) and therefore admits a set E of accumulation points called the set of limit roots, which verifies E ∈ ˜Q. The group W acts on ˜Φ ⊔ E ∪ conv(E) componentwise: w · x = ˜w(x).

Now, the role of the affine space V₀ for an affine Coxeter system (W, S) with the tiling obtained by the action of W on the fundamental alcove K is replaced for indefinite Coxeter systems by a tiling of the imaginary convex body conv(E) by the projective action of W on the non-empty fundamental region

\[ K = \{ x ∈ \text{conv}(\tilde{\Delta}) | B(x, \alpha_s) ≥ 0, \forall s ∈ S \}. \]

Denote \( H_α = \{ x \in V | B(x, α) = 0 \} \), then K is the region of conv( ˜Δ) bounded by the hyperplanes \( H_α \). This is illustrated in Figure 7 and Figure 8 see also [14, Figures 2 and 14]. We refer the reader to [14] for more details.

In view of Proposition 3.16 it is natural to give the following definition.

**Definition 3.18.** Let (W, S) be an indefinite Coxeter system. The \( n \)-Shi arrangement of (W, S) is the collection of hyperplanes

\[ \text{Shi}_n(W, S) := \{ H_α = \{ x ∈ V | B(x, α) = 0 \} | α ∈ \Sigma_n(W) \}. \]

If [13] Conjecture 2, restated above in §3 is true, it would mean that the set \( L_n(W) \) of n-low elements parameterized the region of the \( n \)-Shi arrangement. Furthermore, each region Shiₙ(W, S) would be have a unique minimal-length region of the form \( w · K \) with \( w ∈ L_n(W) \). Moreover, we observed in numerous cases in rank 3 and 4 the following statement.

**Figure 6.** The automata \( A_0(\tilde{A}_2, S) \) and \( A_0(\tilde{C}_2, S) \), drawn using Theorem 3.17. A [green, red, blue] edge represents multiplication by \([s_1, s_2, s_3]\). There is one omitted (red) edge between 132 and 213 in \( A_0(\tilde{C}_2, S) \).
Conjecture 3. Let $n \in \mathbb{N}$, then the subset $\bigcup_{w \in L_n(W)} w^{-1} \cdot K$ of the imaginary convex body is convex.

Reasonable evidence for Conjecture 3 is supplied by the fact that $L_n(W)$ is closed under taking suffixes. Figure 8 illustrates Conjecture 3 for several rank three examples.

Figure 8. The regions $\bigcup_{w \in L_n(W)} w^{-1} \cdot K$ for the triangle groups $(3, 3, 6), (3, 4, 4)$, and $(4, 7, 2)$. 
Acknowledgments. This work was initiated in LaCIM (Montreal) while Philippe Nadeau was visiting thanks to a research travel grant from the Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO). The first author (CH) thanks Christophe Reutenauer for interesting conversations regarding minimality of automata and restriction to standard parabolic subgroups. We thank an anonymous referee for suggesting that we provide more evidence for Conjectures 1 and 2.

References

[1] A. Björner. Orderings of Coxeter groups. In Combinatorics and algebra (Boulder, Colo., 1983), volume 34 of Contemp. Math., pages 175–195. Amer. Math. Soc., Providence, RI, 1984.

[2] A. Björner and F. Brenti. Combinatorics of Coxeter Groups, volume 231 of GTM. Springer, New York, 2005.

[3] B. Brink. The set of dominance-minimal roots. J. Algebra, 206(2):371–412, 1998.

[4] B. Brink and R. Howlett. A finiteness property and an automatic structure for coxeter groups. Math. Ann., 296:179–190, 1993.

[5] B. Casselman. Computation in Coxeter groups. I. Multiplication. Electron. J. Combin., 9(1):Research Paper 25, 22 pp. (electronic), 2002.

[6] B. Casselman. Computation in Coxeter groups. II. Constructing minimal roots. Represent. Theory, 12:260–293, 2008.

[7] W. A. Casselman. Machine calculations in Weyl groups. Invent. Math., 116(1-3):95–108, 1994.

[8] W. A. Casselman. Automata to perform basic calculations in Coxeter groups. In Representations of groups (Banff, AB, 1994), volume 16 of CMS Conf. Proc., pages 35–58. Amer. Math. Soc., Providence, RI, 1995.

[9] M. Davis, M. Shapiro. Coxeter groups are automatic. Preprint, 1991.

[10] P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, and J. Michel. Fundataions of Garside theory, volume 22 of EMS Tracts in Mathematics European Mathematical Society (EMS), Zürich, 2015.

[11] P. Dehornoy, M. Dyer, and C. Hohlweg. Garside families in Artin-Tits monoids and low elements in Coxeter groups. Comptes Rendus Mathematique, 353:403–408., 2015.

[12] M. J. Dyer. On the weak order of Coxeter groups. 2011. http://arxiv.org/abs/1108.5557.

[13] M. Dyer and C. Hohlweg. Small roots, low elements, and the weak order in Coxeter groups. ArXiv e-prints, May 2015.

[14] M. Dyer, C. Hohlweg, and V. Ripoll. Imaginary cones and limit roots of infinite Coxeter groups. http://arxiv.org/abs/1303.6710, Mar. 2013.

[15] T. Edgar. Dominance and regularity in coxeter groups. PhD Thesis, University of Notre Dame, 2009. http://etd.nd.edu/ETD-db/.

[16] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. Word processing in groups. Jones and Bartlett Publishers, Boston, MA, 1992.

[17] H. Eriksson. Computational and combinatorial aspects of Coxeter groups. ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Takn.dr)–Kungliga Tekniska Hogskolan (Sweden).

[18] X. Fu. The dominance hierarchy in root systems of Coxeter groups. J. Algebra, 366:187–204, 2012.

[19] P. Headley. Reduced expressions in infinite Coxeter groups. UMI, Ann Arbor, MI, 1994. Thesis–University of Michigan.

[20] C. Hohlweg and J.-P. Labbé. On Inversion Sets and the Weak Order in Coxeter Groups. http://arxiv.org/abs/1502.06926, 2015.

[21] C. Hohlweg, J.-P. Labbé, and V. Ripoll. Asymptotical behaviour of roots of infinite Coxeter groups. Canad. J. Math., 66:323–353, 2014.

[22] J. E. Humphreys. Reflection groups and Coxeter groups, volume 29. Cambridge University Press, Cambridge, 1990.

[8+] SageMath, the Sage Mathematics Software System (Version x.y.z), The Sage Developers, YYYY, http://www.sagemath.org.
[23] The Sage-Combinat community, *Sage-Combinat*: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, http://combinat.sagemath.org, 2008.

[24] J. Y. Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, volume 1179 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[25] J. Sakarovitch. *Elements of automata theory*, Translated from the 2003 French original by Reuben Thomas. Cambridge University Press, Cambridge, 2009.

[26] J.-Y. Shi. The reduced expressions in a Coxeter system with a strictly complete Coxeter graph. *Adv. Math.*, 272:579–597, 2015.

(Christophe Hohlweg) Université du Québec à Montréal, LaCIM et Département de Mathématiques, CP 8888 Succ. Centre-Ville, Montréal, Québec, H3C 3P8, Canada

E-mail address: hohlweg.christophe@uqam.ca
URL: http://hohlweg.math.uqam.ca

(Philippe Nadeau) CNRS & Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cedex, France

E-mail address: nadeau@math.univ-lyon1.fr
URL: http://math.univ-lyon1.fr/~nadeau

(Nathan Williams) Université du Québec à Montréal, LaCIM, CP 8888 Succ. Centre-Ville, Montréal, Québec, H3C 3P8, Canada

E-mail address: nathan.f.williams@gmail.com
URL: http://thales.math.uqam.ca/~nwilliams/