Irreducibility of SPDEs driven by pure jump noise

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Abstract: The irreducibility is fundamental for the study of ergodicity of stochastic dynamical systems. In the literature, there are very few results on the irreducibility of stochastic partial differential equations (SPDEs) and stochastic differential equations (SDEs) driven by pure jump noise. The existing methods on this topic are basically along the same lines as that for the Gaussian case. They heavily rely on the fact that the driving noises are additive type and more or less in the class of stable processes. The use of such methods to deal with the case of other types of additive pure jump noises appears to be unclear, let alone the case of multiplicative noises.

In this paper, we develop a new, effective method to obtain the irreducibility of SPDEs and SDEs driven by multiplicative pure jump noise. The conditions placed on the coefficients and the driving noise are very mild, and in some sense they are necessary and sufficient. This leads to not only significantly improving all of the results in the literature, but also to new irreducibility results of a much larger class of equations driven by pure jump noise with much weaker requirements than those treatable by the known methods. As a result, we are able to apply the main results to SPDEs with locally monotone coefficients, SPDEs/SDEs with singular coefficients, nonlinear Schrödinger equations, Euler equations etc. We emphasize that under our setting the driving noises could be compound Poisson processes, even allowed to be infinite dimensional. It is somehow surprising.

Keywords: Irreducibility; pure jump noise; stochastic partial differential equations; ergodicity; locally monotone coefficients; singular coefficients; Schrödinger equations; Euler equations;

AMS Subject Classification (2020): 60H15; 60G51; 37A25; 60H17.
1 Introduction and motivation

Let $H$ be a topological space with Borel $\sigma$-field $\mathcal{B}(H)$, and let $X := \{X^x(t), t \geq 0; x \in H\}$ be an $H$-valued Markov process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $X$ is said to be irreducible in $H$ if for each $t > 0$ and $x \in H$

$$\mathbb{P}(X^x(t) \in B) > 0 \quad \text{for any non-empty open set } B.$$ 

In this paper, we are concerned with the irreducibility of stochastic partial differential equations (SPDEs) and stochastic differential equations (SDEs) driven by pure jump noise.

The irreducibility is a fundamental property of stochastic dynamic systems. More precisely, the importance of the study of the irreducibility lies in its relevance in the analysis of the ergodicity of Markov processes. The uniqueness of the invariant measures/ergodicity is usually obtained by proving irreducibility and the strong Feller property, or the asymptotic strong Feller property, or the $e$-property; see [7, 10, 11, 18, 23, 24, 27]. These methods are proved to be powerful tools to establish the uniqueness of invariant measures/ergodicity for various dynamical systems driven by Gaussian noise, and there is a huge amount of literature on this topic; see e.g., [15, 16, 17, 33] and the references therein. Irreducibility plays an indispensable role in establishing large deviations of the occupation measures of Markov processes, we refer the reader to [20, 21, 22, 25, 26, 38]; and it also plays an important role in the study of the recurrence of Markov processes; see [7].

The study of the irreducibility for stochastic dynamical systems driven by Gaussian noise has a long history; see, for instance, the classical works [10, 27] and the books [7, 8]. There is a large amount of literature devoted to this topic. To obtain the irreducibility for stochastic equations driven by Gaussian noise, one usually needs to solve a control problem. In doing so, three ingredients play a very important role: the (approximate) controllability of the associated PDEs, Girsanov’s transformation of Wiener processes, and the support of Wiener processes/stochastic convolutions on path spaces. See also [12, 14, 15, 27, 41] and the references therein.

However, things become quite different when the driving noises are pure jump processes. It seems that no results on the irreducibility of stochastic dynamical systems driven by pure jump noise had been available before the paper [29]. Compared with the case of the Gaussian driving noise, there are very few results on the irreducibility of the case of the pure jump driving noise, because the systems behave drastically differently due to the appearance of jumps. The existing methods on the irreducibility are basically along the same lines as that of the Gaussian case. They heavily rely on the fact that the driving noises are additive type and more or less in the class of stable processes. The use of such methods to deal with the
case of other types of additive pure jump noises appears to be unclear, let alone the case of multiplicative noises. The methods and techniques available for dynamical systems driven by Gaussian noise are not well suited for investigating the irreducibility of systems driven by jump type noise for two main reasons. One is that there exist very few results on the support of the pure jump Lévy processes/the stochastic convolutions on path spaces. Due to the discontinuity of trajectories, the characterization of the support of the pure jump processes is much harder than that of the Gaussian case. The other is that Girsanov’s transformation of the pure jump Lévy process is much less effective than that of Gaussian case, because the density of the Girsanov transform of a Poisson random measure is expressed in terms of nonlinear invertible and predictable transformations, and is to censore jumps or thin the size of jumps. So far, there is a lack of effective methods to obtain the irreducibility of stochastic equations driven by pure jump noise. This strongly motivates the current paper.

Now we mention the results on the irreducibility of SPDEs driven by pure jump noise. To do this, we first introduce the so-called cylindrical pure jump Lévy processes defined by the orthogonal expansion

$$L(t) = \sum \beta_i L_i(t) e_i, \quad t \geq 0,$$

(1.1)

where \(\{e_i\}\) is an orthonormal basis of a separable Hilbert space \(H\), \(\{L_i\}\) are real valued i.i.d. pure jump Lévy processes, and \(\{\beta_i\}\) is a given sequence of non zero real numbers.

The first paper dealing with the irreducibility of stochastic equations driven by pure jump noise was published in [29]. The authors obtained the irreducibility of semilinear SPDEs with Lipschitz coefficients; see [29, Theorem 5.4]. The driving noises they considered are the so-called cylindrical symmetric \(\alpha\)-stable processes, \(\alpha \in (0, 2)\), which have the form (1.1) with \(\{L_i\}\) replaced by real valued i.i.d. symmetric \(\alpha\)-stable processes. A key point in their analysis is to study the support of \(L_i\) and the stochastic convolutions with respect to \(L_i\) and \(L\) on some path spaces. Subsequently, the authors solved a control problem to obtain the irreducibility. The proofs of the support are based on carefully controlling moments and sizes of jumps of the driving noises, and require a strong condition that the support of the intensity measure of \(L_i\) contains interval \([-R, R]\) for some \(R > 0\), which is naturally satisfied by real valued symmetric \(\alpha\)-stable processes. If one follows their ideas, the strong condition seems very difficult to be improved/removed. Their approach must be modified to study the irreducibility of stochastic equations with highly nonlinear terms. There are several papers doing so (see [9, 13, 36, 37]), which we describe below.

Following the spirit in [8] for the Gaussian cases and [29], the authors in [36] proved the irreducibility of stochastic real Ginzburg-Landau equation on torus \(T = \mathbb{R} \setminus \mathbb{Z}\) in \(H := \{h \in \mathbb{R}\} \setminus \mathbb{Z}\)
$L^2(\mathbb{T}) \setminus \{ \int_{\mathbb{T}} h(y) dy = 0 \}$ driven by cylindrical symmetric $\alpha$-stable processes with $\alpha \in (1, 2)$; see [36, Theorem 2.3]. Due to the polynomial nonlinearity, they established a new support result for stochastic convolutions on some suitable path space; see [8, Lemma 3.2]. And then, they solved a control problem with polynomial term to obtain the irreducibility. Although their method could deal with the non-Lipschitz term, some technical restrictions are placed on the driving noises. For example, (ii) on page 1182 of [36], i.e.,

$$\alpha \in (1, 2)$$

(1.2)

and

$$C_1 \gamma_i^{-\beta} \leq |\beta_i| \leq C_2 \gamma_i^{-\beta} \text{ with } \beta > \frac{1}{2} + \frac{1}{2\alpha} \text{ for some positive constants } C_1 \text{ and } C_2,$$

(1.3)

here $\{\gamma_i = 4\pi^2 |i|^2\}$ are the eigenvalues of the Laplace operator on $H$. By improving the methods in [36] and [29], the authors in [37] and [9] established the irreducibility of stochastic reaction-diffusion equation and stochastic Burgers equation driven by the subordinated cylindrical Wiener process with a $\alpha/2$-stable subordinator, $\alpha \in (1, 2)$, respectively. However, since the main ideas of [37] [9] are similar to that of [36], technical restrictions as (1.2) and (1.3) on the driving noises are required in [37] [9].

In [13] the authors studied the irreducibility of some stochastic Hydrodynamical systems with bilinear term; see [13, Theorem 3.5]. Their framework can cover the 2D Navier-Stokes equation, the GOY and Sabra shell models etc. To obtain the irreducibility, the driving noises $L$ they considered are of the form (1.4) satisfying

(A1) The intensity measure $\mu$ of each component process $L_i$ satisfies that there exists a strictly monotone and $C^1$ function $q : (0, \infty) \to (0, \infty)$ such that

$$\lim_{r \to \infty} q(r) = 0, \lim_{r \to 0^+} q(r) = 1, \text{ and } \mu(dz) = q(|z|)|z|^{-1-\theta} dz, \theta \in (0, 2);$$

(1.4)

$$\int_{\mathbb{R}} (1 - q^{1/2}(|z|))^2 \mu(dz) < \infty.$$  

(1.5)

(A2) There exist a certain $\epsilon \in (0, 2)$ and $\vartheta \in [0, 1/2)$ such that

$$\sum_i (|\beta_i| + \beta_i^2 \lambda_i^{-2\vartheta} + \beta_i^2 \lambda_i^{-1} + \beta_i^4 \lambda_i^\epsilon) < \infty.$$  

(1.6)

Here $0 < \lambda_1 < \lambda_2 < \ldots$ are the eigenvalues associated with a positive self-adjoint operator appearing in the equations they studied.

The driving noises $L$ could not cover cylindrical $\alpha$-stable processes. In the sense of distribution, they are in the class of cylindrical $\alpha$-stable processes. Indeed, Assumption (A1) implies
that the distribution of each component process $L_i$ on path spaces is equivalent to that of $\alpha$-stable processes; see [13, Lemma 4.2]. Therefore, the support of stochastic convolutions with respect to $L_i$ is the same as that of the cases with respect to $\alpha$-stable process, which was studied in [29]. Combining this with the idea of proving [29, Theorem 4.10], under the assumptions (A1) and (A2), the support for stochastic convolutions with respect to $L$ on a path space is established. After solving a control problem with bilinear term, they obtained the irreducibility. The assumption (A1) plays an indispensable role in their methods and, at the same time, this limits the applications of their results. As mentioned in [13], their results could not be applied to the family of truncated Lévy flights which requires that $q$ appearing in (A1) satisfies $q(r) \equiv 0$, $r \not\in (0, R]$ for some $R > 0$. The authors also pointed out that the irreducibility in their framework with the driving noises replaced by stable processes does not follow from their results and is still an open problem.

Although, the irreducibility of several interesting SPDEs driven by pure jump noise is obtained, there are always some very restrictive assumptions on the driving Lévy noise: The driving noises are additive type and more or less in the class of stable processes, and other technical assumptions such as (1.2)-(1.6) are required. Actually, the conditions on the driving noise to obtain the irreducibility are much stronger than that to obtain the well posedness. Using the existing methods, it seems very hard to deal with the case of other types of additive pure jump noises, let alone the case of multiplicative noises; and also seems very difficult to solve the irreducibility of many other physical models with highly nonlinear terms, such as porous medium equation, $p$-Laplace equation, fast diffusion equation, in particular, SPDEs with singular coefficients, e.g., nonlinear Schrödinger equations, Euler equations etc.

Our main results are Theorem 2.1 and Theorem 2.2. As an application of the main results of this paper, Proposition 4.1 in Section 4 not only covers all of the results obtained in [36, 37, 9, 13] but also requires much weaker conditions on the coefficients and the driving Lévy noise, and also covers the setting of multiplicative driving noise. In a few words, to get the irreducibility, we just impose the conditions under which the well posedness can be guaranteed and a nondegenerate condition on the intensity measure of the driving Lévy noise, i.e., Assumptions 2.3 and 2.4 in Section 2 respectively for the case of the multiplicative noise and the case of the additive noise. The framework of Proposition 4.1 in Section 4 covers SPDEs such as stochastic reaction–diffusion equations, stochastic semilinear evolution equation, stochastic porous medium equation, stochastic $p$-Laplace equation, stochastic fast diffusion equation, stochastic Burgers type equations, stochastic 2D Navier-Stokes equation, stochastic magneto-hydrodynamic equations, stochastic Boussinesq model for the Bénard convection, stochastic 2D magnetic Bénard problem, stochastic 3D Leray-$\alpha$ model, stochastic
equations of non-Newtonian fluids, several stochastic Shell Models of turbulence, and many other stochastic 2D Hydrodynamical systems, most of which can not be covered by the existing results. An example of the driving noises required in Proposition 4.1 is the form (1.1) with

\[(A3)\] the intensity measure \(\mu\) of each component process \(L_i\) satisfies that there exists a Borel measurable function \(q : (-\infty, \infty) \to [0, \infty)\) such that

\[
\mu(dz) = q(z)|z|^{-1-\theta}dz, \quad \theta \in (0, 2),
\]

with the Lebesgue measures of \(\{r > 0 : q(r) > 0\}\) and \(\{r < 0 : q(r) > 0\}\) are strictly positive;

\[(A4)\] \(\{\beta_i\}\) is a given sequence of non zero real numbers satisfying the conditions under which the well posedness can be proven.

The assumptions (A3) and (A4) remove technical assumptions appeared in \([36, 13]\), such as (1.2)-(1.6), and in particular, cover the cylindrical symmetric/non-symmetric \(\alpha\)-stable processes with \(\alpha \in (0, 2)\), the family of truncated Lévy flights, etc. The driving noises considered in Proposition 4.1 could cover a very large class of the so-called subordination of Lévy processes, including the subordinated cylindrical Wiener process with a \(\alpha/2\)-stable subordinator, \(\alpha \in (0, 2)\), which removes restrictive assumptions appeared in \([37, 9]\), e.g., (1.2) and (1.3). For more details, see Subsection 4.1.

In addition, we solved the open problem raised in \([13]\) which we mentioned above. As an application of Proposition 4.1 of our paper, combining with the \(e\)-property, we can obtain the uniqueness of invariant measures of stochastic evolution equations with weakly dissipative drifts such as stochastic fast diffusion equations and singular stochastic \(p\)-Laplace equations. It seems quite difficult to get these results with other means due to the lack of strong dissipativity of the equations. See Proposition 4.2 and Examples 4.9 and 4.10 below. In \([13]\) and \([2]\), the authors studied irreducibility and exponential mixing of some stochastic Hydrodynamical systems driven by cylindrical pure jump noise. As an another application of Proposition 4.1 one is able to significantly improve the main results (including exponential mixing) in \([13]\) and \([2]\). Our main results are also applicable to SPDEs with singular coefficients, including the nonlinear Schrödinger equations, Euler equations etc; see Subsections 4.4 and 4.5. They are new in the pure jump cases. And, to the best of our knowledge, the corresponding results on the irreducibility in the case of Gaussian driving noise are not known either. We emphasize that under our setting the driving noises could be compound Poisson processes, even allowed to be infinite dimensional. It is somehow surprising.
Now we introduce the results on the irreducibility of SDEs driven by pure jump noise. In [39], the authors studied the irreducibility of SDEs with singular coefficients driven by symmetric and rotationally invariant $\alpha$-stable processes with $\alpha \in (1, 2)$. They obtained a two sided estimate of the transition density function of the solution, which implies the irreducibility. This is the only paper to get the irreducibility of stochastic equations driven by multiplicative pure jump noise. In [1] the authors obtained the irreducibility of a class of multidimensional Ornstein-Uhlenbeck processes driven by additive pure jump noise $L$. $L$ is of the form $L(t) = L_1(t) + L_2(t)$, where $L_1$ and $L_2$ are independent $d$-dimensional pure jump Lévy processes, such that one of the following conditions is satisfied

(1) $L_1$ is a subordinate Brownian motion, and $L_2$ can be any pure jump Lévy process or vanish;

(2) $L_1$ is an anisotropic Lévy process with independent symmetric one dimensional $\alpha$-stable components for $\alpha \in (0, 2)$, and $L_2$ is a compound Poisson process.

Their proofs are based on studying the corresponding infinitesimal generator of the Ornstein-Uhlenbeck processes. The proofs of [39] and [1] rely on the special driving noises. Their methods seem very difficult to deal with other types of pure jump Lévy noises, and, in particular, it is impossible to apply the method of [39] to deal with compound Poisson noises. Their methods could not be used to deal with the case of SPDEs. The main results in this paper could be applied to the cases considered in [39] and [1] under the same assumptions. Indeed, Proposition 4.1 covers the case of [1] with much weaker conditions on the coefficients and the driving Lévy noises. In the present paper, instead of applying our main results to give a new proof of the irreducibility obtained in [39], we establish the irreducibility for a class of SDE with singular coefficients driven by non-degenerate $\alpha$-stable-like Lévy process with $\alpha \in (0, 2)$, the well posedness of which was obtained in a recent paper [6]. We notice that the study of the supercritical case $\alpha \in (0, 1)$ is much harder and attracts a lot of attention. Our results for the supercritical case are new. We stress that our results for the case of the additive noise are sharp. See Subsection 4.6.

In this paper, we have found an effective, general method to obtain the irreducibility of SDEs and SPDEs driven by multiplicative pure jump Lévy noise. Our method is so general, which places very mild conditions on the coefficients and the driving Lévy noise. This leads to not only significantly improving all of the results in the literature, but also to new irreducibility results of a much larger class of equations driven by pure jump noise with much weaker requirements than those treatable by the known methods. As a result, new applications include SPDEs with locally monotone coefficients, SPDEs/SDEs with singular coefficients, nonlinear Schrödinger equations, Euler equations etc.
We now describe the main idea of this paper. Let $H$ be a separable Hilbert space, and let $X := \{X_x(t), t \geq 0; x \in H\}$ be an $H$-valued càdlàg strong Markov process on some probability space $(\Omega, \mathcal{F}, P)$. For example, $X^x = (X^x(t), t \geq 0)$ could be the unique solutions to SPDEs/SDEs driven by pure jump Lévy noise with initial data $x \in H$. For any $x, y \in H$, $T > 0$ and $\kappa > 0$, our aim is to prove that

$$P\left(X^x(T) \in B(y, \kappa)\right) > 0. \quad (1.7)$$

Here, for any $h \in H$ and $l > 0$, denote $B(h, l) = \{\overline{h} \in H : \|\overline{h} - h\|_H < l\}$.

To do this, we impose two main assumptions: Assumptions 2.2 and 2.3. Intuitively speaking, the first one is a weakly continuous assumption on $X$ uniformly in the initial data. The second one is a nondegenerate condition on the intensity measure of the driving Lévy noise, which basically says that for any $\overline{h}, \overline{\overline{h}} \in H$, the neighbourhoods of $\overline{\overline{h}}$ can be reached with positive probability from $\overline{h}$ through a finite number of choosing jumps.

Applying Assumption 2.2 to the given $y$ and $\kappa$, there exist $\epsilon_0 := \epsilon(y, \frac{\kappa}{2}) \in (0, \frac{\kappa}{2})$ and $t_0 := t(y, \frac{\kappa}{2}) > 0$ such that for any $\overline{h} \in B(y, \epsilon_0)$,

$$P\left(\left\{X^\overline{h}(t) \in B(y, \frac{\kappa}{2}), \forall t \in [0, t_0]\right\}\right) > 0. \quad (1.8)$$

Therefore, set $T_0 = T - \frac{t_0}{2}$, once we prove that there exists $\overline{T} \in (T_0, T)$ such that

$$P\left(X^x(\overline{T}) \in B(y, \epsilon_0)\right) > 0, \quad (1.9)$$

by the Markov property of $X$, (1.7) follow from (1.8) and (1.9), completing the proof.

We now explain the ideas of proving (1.9). First, notice that there exists $\zeta \in H$ such that for any $\rho > 0$

$$P\left(\left\{X^x(T_0) \in B(\zeta, \rho)\right\}\right) > 0. \quad (1.10)$$

By Assumption 2.3, $B(y, \epsilon_0)$ can be reached with positive probability from $\zeta$ through a finite number of choosing jumps. Set $\sigma_i$ be the $i$-th jump time. One key step to obtain (1.9) is to prove that there exist $\rho_0 > 0$, $\rho_1 > 0$, $q_1 \in H$, and $T_1 \in (T_0, T)$ such that

$$P\left(\left\{X^x(T_0) \in B(\zeta, \rho_0)\right\} \cap \{X^x(t) \in B(\zeta, 2\rho_0), \forall t \in [T_0, \sigma_1]\}\right. \left.\cap \{X^x(\sigma_1) \in B(q_1, \frac{\rho_1}{2})\} \cap \{X^x(t) \in B(q_1, \rho_1), \forall t \in (\sigma_1, T_1]\}\right) > 0, \quad (1.11)$$

which implies that

$$P\left(\left\{X^x(T_0) \in B(\zeta, \rho_0)\right\} \cap \{X^x(T_1) \in B(q_1, \rho_1)\}\right) > 0. \quad (1.12)$$

To get (1.11), the following claims will be used:
(C1) Assumption 2.2, (1.10) and the Markov property of \( X \) imply that

\[
\Pr \left( \{X^x(T_0) \in B(\zeta, \rho_0)\} \cap \{X^x(t) \in B(\zeta, 2\rho_0), \forall t \in [T_0, \sigma_1]\} \right) > 0.
\]

(C2) The first choosing jump ensures that the neighbourhood of \( q_1, B(q_1, \frac{\rho}{2}) \), can be reached with positive probability from \( B(\zeta, 2\rho_0) \).

To complete the proof of (1.11), a further delicate argument is carried out, which requires an intricate cutoff procedure and employs stopping time techniques, etc. The argument exploits the strong Markov property of \( X \), the fact that the jumps of the Poisson random measure on disjoint subsets are mutually independent, and the fact that with probability one, two independent Lévy processes can not jump simultaneously at any given moment, etc. It also relies on carefully choosing moments and sizes of jumps of the driving noises.

After getting (1.11) and (1.12), following a recursive procedure we are able to prove that there exist \( \{q_i, i = 1, 2, ..., n\} \subseteq H, \{\rho_i, i = 1, 2, ..., n\} \subseteq (0, \infty) \) and \( T_0 < T_1 < T_2 < ... < T_n < T \) such that

\[
\Pr \left( \{X^x(T_0) \in B(\zeta, \rho_0)\} \cap \bigcap_{i=1}^{n} \{X^x(T_i) \in B(q_i, \rho_i)\} \right) > 0. \tag{1.13}
\]

Carefully choosing \( B(q_n, \rho_n) \subset B(y, \epsilon_0) \), the above inequality implies that

\[
\Pr \left( \{X^x(T_n) \in B(y, \epsilon_0)\} \right) \geq \Pr \left( \{X^x(T_n) \in B(q_n, \rho_n)\} \right) > 0.
\]

Therefore, (1.9) holds, completing the proof.

The approach to prove the irreducibility we are proposing here is completely different with the existing approaches. Our approach gets rid of solving the (approximate) controllability for the associated PDEs, proving the support of Lévy processes/stochastic convolutions on path spaces, and relying on Girsanov’s transformation of Lévy processes. An important novelty of this article is that, for the first time, we find a nondegenerate condition on the intensity measure of the driving Lévy noises to prove the irreducibility. A further novelty is that the main assumptions, Assumptions 2.2 and 2.3, are imposed separately on the process \( X \) and the intensity measure of the driving Lévy noises. These two assumptions are basically independent of each other. Both of them are held for most of the applications. Therefore, the approach we are proposing here is quite robust, and covers much more general types of nonlinear stochastic PDEs, including SPDEs/SDEs with singular coefficients, than that of the existing approaches.

The paper is organized as follows. In Section 2, we will give the main framework and main results: Theorems 2.1 and 2.2. Section 3 is devoted to the proof of the main results. In Section 4, we provide applications to SDEs and SPDEs including many interesting physical
models. Since Assumptions 2.2 and 2.3 are basically independent of each other, Section 4 is divided into three parts: Subsection 4.1 presents examples of the additive driving noises satisfying Assumption 2.4 (the corresponding Assumption 2.3 in the setting of the additive noise). Subsection 4.2 gives examples of the multiplicative driving noises satisfying Assumption 2.3. Subsections 4.3-4.6 are to provide examples of physical models satisfying Assumption 2.2. The irreducibility of many interesting physical models driven by pure jump Lévy noise is established in Subsections 4.3-4.6.

2 Preliminaries and statements of the main results

In this section, we will introduce the framework and state the main results. Let

\[ V \subset H \simeq H^* \subset V^* \]

be a Gelfand triple, i.e., \((H, \langle \cdot, \cdot \rangle_H)\) is a separable Hilbert space and identified with its dual space \(H^*\) by the Riesz isomorphism, \(V\) is a reflexive Banach space that is continuously and densely embedded into \(H\). If \(V, \langle \cdot, \cdot \rangle_V\) denotes the dualization between \(V\) and its dual space \(V^*\), then it follows that

\[ \langle u, v \rangle_V = \langle u, v \rangle_H, \quad u \in H, \; v \in V. \]

Let \((\Omega, \mathcal{F}, F, \mathbb{P})\), where \(F = \{\mathcal{F}_t\}_{t \geq 0}\), be a filtered probability space satisfying the usual conditions.

For a metric space \((X, d_X)\), the Borel \(\sigma\)-field on \(X\) will be written as \(\mathcal{B}(X)\). For any \(x \in X\) and \(l \geq 0\), denote \(B(x, l) = \{y \in X : d_X(x, y) < l\}\) and \(B(x, l) = \{y \in X : d_X(x, y) \leq l\}\). If \(I \subset \mathbb{R}\) is a time interval, we denote by \(D(I, X)\) the space of all càdlàg paths from \(I\) to \(X\).

Let \((Z, \mathcal{B}(Z))\) be a metric space, and \(\nu\) a given \(\sigma\)-finite measure \(\nu\) on it, that is, there exists \(Z_n \in \mathcal{B}(Z), n \in \mathbb{N}\) such that \(Z_n \uparrow Z\) and \(\nu(Z_n) < \infty, \forall n \in \mathbb{N}\). Let \(N : \mathcal{B}(Z) \times \mathbb{R}^+ \times \Omega \to \bar{\mathbb{N}} := \mathbb{N} \cup \{0, \infty\}\) be a time homogeneous Poisson random measure on \((Z, \mathcal{B}(Z))\) with intensity measure \(\nu\). For the existence of such Poisson random measure, we refer the reader to [19]. We denote by \(\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt\) the compensated Poisson random measure associated to \(N\).

Now we consider the following SPDEs driven by pure jump noise:

\[
\begin{align*}
\frac{dX(t)}{dt} &= \mathcal{A}(X(t))dt + \int_{Z_1^c} \sigma(X(t^-), z)\tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t^-), z)N(dz, dt), \\
X(0) &= x,
\end{align*}
\]

(2.1)

where \(\mathcal{A} : V \to V^*\) and \(\sigma : H \times Z \to H\) are Borel measurable mappings, and, for any \(m \in \mathbb{N}\), \(Z_m^c\) denotes the complement of \(Z_m\) relative to \(Z\).
Definition 2.1 An $H$-valued càdlàg $\mathbb{F}$-adapted process $X$ is called a solution of (2.1) if the following conditions are satisfied:

(I) $X(t, \omega) \in V$ for $dt \otimes \mathbb{P}$-almost all $(t, \omega) \in [0, \infty) \times \Omega$, where $dt$ stands for the Lebesgue measures on $[0, \infty)$;

(II) $\int_{0}^{t} |A(X(s))|_{V} ds + \int_{0}^{t} \int_{Z_{1}} |\sigma(X(s), z)|_{H} \nu(dz) ds + \int_{0}^{t} \int_{Z_{1}} |\sigma(X(s), z)|_{H} N(dz, ds) < \infty$, $\forall t \geq 0$, $\mathbb{P}$-a.s.,

(III) $\mathbb{P}$-a.s.

$$X(t) = x + \int_{0}^{t} A(X(s)) ds + \int_{0}^{t} \int_{Z_{1}} \sigma(X(s), z) \tilde{N}(dz, ds)$$

$$+ \int_{0}^{t} \int_{Z_{1}} \sigma(X(s), z) N(dz, ds), \quad t \geq 0,$$

(2.2) as an equation in $V^*$. For notational convenience, we use the notation $X^x$ to indicate the solution of (2.1) starting from $x$.

Our starting point is the following assumption.

**Assumption 2.1** For any $x \in H$, there exists a unique global solution $X^x = (X^x(t))_{t \geq 0}$ to (2.1) and $\{X^x, x \in H\}$ forms a strong Markov process.

**Remark 2.1** Our primary concern in this paper is the irreducibility of the solutions of SPDEs. We simply impose the assumption 2.1 which of course holds under many variants of standard assumptions on the coefficients. See the examples in Section 4.

For any $m \in \mathbb{N}$, since $\nu(Z_m) < \infty$, (2.2) can be rewritten as follows:

$$X^x(t) = x + \int_{0}^{t} A(X^x(s)) ds + \int_{0}^{t} \int_{Z_m \setminus Z_1} \sigma(X^x(s), z) \tilde{N}(dz, ds)$$

$$- \int_{0}^{t} \int_{Z_m \setminus Z_1} \sigma(X^x(s), z) \nu(dz) ds + \int_{0}^{t} \int_{Z_m} \sigma(X^x(s), z) N(dz, ds), \quad t \geq 0.$$  

Removing the big jumps in the above equation, consider

$$dX_m(t) = A(X_m(t)) dt + \int_{Z_m \setminus Z_1} \sigma(X_m(t), z) \tilde{N}(dz, dt) - \int_{Z_m \setminus Z_1} \sigma(X_m(t), z) \nu(dz) dt,$$

$$X_m(0) = x.$$  

(2.3)

Set

$$N(Z_m, t) := \int_{0}^{t} \int_{Z_m} N(dz, ds), \quad t \geq 0,$$

and

$$\tau^i_m = \inf\{t \geq 0 : N(Z_m, t) = i\}, \quad i \in \mathbb{N}. \quad (2.4)$$

It is clear that $\{X^x(t), t \in [0, \tau^i_m]\}$ is a solution to (2.3) on $t \in [0, \tau^i_m)$. The next result shows that the existence and uniqueness of the solution of equation (2.3) follows from that of equation (2.1).
Proposition 2.1  Fix an arbitrary $m \in \mathbb{N}$. For any $x \in H$, there exists a unique global solution $X^x_m = (X^x_m(t))_{t \geq 0}$ to equation (2.3), which also has the strong Markov property.

Proof  Note that equation (2.3) does not involve the jumps of the Poisson random measure $N$ in the set $Z_m$. Therefore, the solution $X^x_m$ is continuous at the jumping times $\tau^i_m$, $i \in \mathbb{N}$. $X^x_m$ can be constructed as follows:

For $t \in [0, \tau^1_m)$, set $X^x_m(t) = X^x(t)$. Define $X^x_m(\tau^1_m) = \lim_{t \uparrow \tau^1_m} X^x(t)$. Let $Y^1(t), t \geq \tau^1_m$ be the solution of the following SPDE:

$$Y^1(t) = X^x_m(\tau^1_m) + \int_{\tau^1_m}^t \mathcal{A}(Y^1(s))ds + \int_{\tau^1_m}^t \int Z_m \sigma(Y^1(s), z)N(dz, ds) - \int_{\tau^1_m}^t \int Z_m \sigma(Y^1(s), z)\nu(dz)ds + \int_{\tau^1_m}^t \int Z_m \sigma(Y^1(s), z)N(dz, ds), \quad t \geq \tau^1_m.$$  

For $t \in [\tau^1_m, \tau^2_m)$, set $X^x_m(t) = Y^1(t)$, and define $X^x_m(\tau^2_m) = \lim_{t \uparrow \tau^2_m} Y^1(t)$. Recursively, let $Y^i(t), t \geq \tau^i_m$, for $i \geq 2$, denote the solution of the following SPDE:

$$Y^i(t) = X^x_m(\tau^i_m) + \int_{\tau^i_m}^t \mathcal{A}(Y^i(s))ds + \int_{\tau^i_m}^t \int Z_m \sigma(Y^i(s), z)N(dz, ds) - \int_{\tau^i_m}^t \int Z_m \sigma(Y^i(s), z)\nu(dz)ds + \int_{\tau^i_m}^t \int Z_m \sigma(Y^i(s), z)N(dz, ds), \quad t \geq \tau^i_m.$$  

Set $X^x_m(t) = Y(t)$ for $t \in [\tau^i_m, \tau^{i+1}_m)$, and define $X^x_m(\tau^{i+1}_m) = \lim_{t \uparrow \tau^{i+1}_m} Y(t)$. Since $\nu(Z_m) < \infty$, we have $\lim_{i \to \infty} \tau^i_m = \infty$, $\mathbb{P}$-a.s. The solution $X^x_m = (X^x_m(t))_{t \geq 0}$ is uniquely determined.

The strong Markov property of $\{X^x_m, x \in H\}$ is implied by that of $\{X^x, x \in H\}$.  

We denote by $\mathcal{G}_m$ the $\mathbb{P}$-completion of $\sigma\{N(U \cap Z_m, t), U \in \mathcal{B}(Z), t \geq 0\}$, and $\mathcal{G}^c_m$ the $\mathbb{P}$-completion of $\sigma\{N(U \cap Z^c_m, t), U \in \mathcal{B}(Z), t \geq 0\}$. Then $\mathcal{G}_m$ and $\mathcal{G}^c_m$ are independent. Since $X^x_m \in \mathcal{G}^c_m$, and $\tau^1_m \in \mathcal{G}_m$, we have the following result.

Proposition 2.2  $\sigma\{X^x_m\}$ and $\sigma\{\tau^1_m\}$ are independent.

For any $x, y \in H, \eta > 0$ and $m \in \mathbb{N}$, define $\mathbb{F}$-stopping times

$$\tau^{\eta}_{x,y} = \inf\{t \geq 0 : X^x(t) \not\in B(y, \eta)\} \quad \text{and} \quad \tau^{\eta}_{x,y,m} = \inf\{t \geq 0 : X^x_m(t) \not\in B(y, \eta)\}.$$  

(2.5)

Since $X^x \in D([0, \infty), H)$ and $X^x_m \in D([0, \infty), H)$, $\mathbb{P}$-a.s., we have $\mathbb{P}(\tau^{\eta}_{x,x} > 0) = 1$ and $\mathbb{P}(\tau^{\eta}_{x,x,m} > 0) = 1$.

Remark 2.2  Note that $X^x(\tau^{\eta}_{x,y})$ may not belong to $\overline{B(y, \eta)}$, in general the following statement is not true:

$$\sup_{s \in [0, \tau^{\eta}_{x,y}]} \|X^x(s) - y\|_H \leq \eta \quad \text{on} \ \{\tau^{\eta}_{x,y} > 0\}, \quad \mathbb{P}$-a.s.$$

However, we have

$$\sup_{s \in [0, \tau^{\eta}_{x,y}]} \|X^x(s) - y\|_H \leq \eta \quad \text{and} \quad \sup_{s \in [0, \tau^{\eta}_{x,y,m}]} \|X^x(s) - y\|_H \leq \eta \quad \text{on} \ \{\tau^{\eta}_{x,y} > 0\}, \quad \mathbb{P}$-a.s.$$

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We need the following assumption, which is held for most of the applications.

**Assumption 2.2** For any $h \in H$, there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exist $(\epsilon, t) = (\epsilon(h, \eta), t(h, \eta)) \in (0, \frac{\eta}{2}] \times (0, \infty)$ satisfying,

$$\inf_{\hat{h} \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{\hat{h}, h} \geq t) > 0.$$

**Proposition 2.3** Assume that Assumptions 2.1 and 2.2 hold. For any $m \in \mathbb{N}$ and $h \in H$, there exists $\eta^m_0 > 0$ such that, for any $\eta \in (0, \eta^m_0]$, there exist $(\epsilon, t) = (\epsilon(m, h, \eta), t(m, h, \eta)) \in (0, \frac{\eta}{2}] \times (0, \infty)$ satisfying,

$$\inf_{\hat{h} \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{\hat{h}, h, m} \geq t) > 0. \quad (2.6)$$

**Proof** Recall $\tau^1_m$ defined by (2.4). For any $\hat{h}, h \in H$, $\eta, s > 0$, we have

$$\mathbb{P}(\tau^\eta_{\hat{h}, h} \geq s) = \mathbb{P}(\{\tau^\eta_{\hat{h}, h} \geq s\} \cap \{\tau^1_m \leq s\}) + \mathbb{P}(\{\tau^\eta_{\hat{h}, h} \geq s\} \cap \{\tau^1_m > s\}). \quad (2.7)$$

Note that $\{X^x(t), t \in [0, \tau^1_m]\}$ coincides with $\{X^x_m(t), t \in [0, \tau^1_m]\}$ on $t \in [0, \tau^1_m)$. We have

$$\mathbb{P}(\{\tau^\eta_{\hat{h}, h} \geq s\} \cap \{\tau^1_m > s\}) = \mathbb{P}(\tau^\eta_{\hat{h}, h, m} \geq s) \mathbb{P}(\tau^1_m > s).$$

Applying Proposition 2.2, we have

$$\mathbb{P}(\{\tau^\eta_{\hat{h}, h, m} \geq s\} \cap \{\tau^1_m > s\}) = \mathbb{P}(\tau^\eta_{\hat{h}, h, m} \geq s) \mathbb{P}(\tau^1_m > s).$$

Combining this equality with (2.7), one concludes that

$$\mathbb{P}(\tau^\eta_{\hat{h}, h, m} \geq s) \geq \frac{\mathbb{P}(\tau^\eta_{\hat{h}, h} \geq s) - \mathbb{P}(\tau^1_m \leq s)}{\mathbb{P}(\tau^1_m > s)}. \quad (2.8)$$

By Assumption 2.2, there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exist $(\epsilon, t) = (\epsilon(h, \eta), t(h, \eta)) \in (0, \frac{\eta}{2}] \times (0, \infty)$ satisfying

$$\beta := \inf_{\hat{h} \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{\hat{h}, h} \geq t) > 0. \quad (2.9)$$

Then, for any $s \in (0, t]$,

$$\inf_{\hat{h} \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{\hat{h}, h} \geq s) \geq \beta. \quad (2.10)$$

Recall that $\tau^1_m$ has the exponential distribution with parameter $\nu(Z_m) < \infty$, that is,

$$\mathbb{P}(\tau^1_m > s) = e^{-\nu(Z_m)s}, \quad \mathbb{P}(\tau^1_m \leq s) = 1 - e^{-\nu(Z_m)s}. \quad (2.11)$$

Putting together (2.8)–(2.11) we see that there exists $s_0 > 0$ small enough such that

$$\inf_{\hat{h} \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{\hat{h}, h, m} \geq s_0) > 0.$$
This completes the proof. \(\square\)

Now we introduce the conditions on the jumping measure of the driving noise of the equation (2.1), which basically says that for any \(h, y \in H\), the neighbourhoods of \(y\) can be reached from \(h\) through a finite number of choosing jumps.

**Assumption 2.3** For any \(h, y \in H\) with \(h \neq y\) and any \(\eta > 0\), there exist \(n, m \in \mathbb{N}\), and \(\{l_i, i = 1, 2, ..., n\} \subset Z_m\) such that, for any \(\eta \in (0, \frac{n}{2})\), there exist \(\{\epsilon_i, i = 1, 2, ..., n\} \subset (0, \infty)\) and \(\{\eta_i, i = 0, 1, ..., n\} \subset (0, \infty)\) such that, denoting

\[
q_0 = h, \quad q_i = q_{i-1} + \sigma(q_{i-1}, l_i), \quad i = 1, 2, ..., n,
\]

\[
\begin{align*}
&\bullet \quad 0 < \eta_0 \leq \eta_1 \leq \ldots \leq \eta_{n-1} \leq \eta_n \leq \eta; \\
&\bullet \quad \text{for any } i = 0, 1, ..., n-1, \{\tilde{q} + \sigma(\tilde{q}, l_i), \tilde{q} \in B(q_i, \eta_i), l_i \in B(l_{i+1}, \epsilon_{i+1})\} \subset B(q_{i+1}, \eta_{i+1}); \\
&\bullet \quad B(q_n, \eta_n) \subset B(y, \frac{n}{2}); \\
&\bullet \quad \text{for any } i = 1, 2, ..., n, \nu(B(l_i, \epsilon_i)) > 0; \\
&\bullet \quad \text{there exists } m_0 \geq m \text{ such that } \bigcup_{i=1}^n B(l_i, \epsilon_i) \subset Z_m.
\end{align*}
\]

**Remark 2.3** Note that

\[
\lim_{\eta \searrow 0} \eta_i = 0, \quad \forall i = 0, 1, ... n.
\]

Moreover, in the above assumption, without loss of generality, it is not difficult to see that we can require that \(\epsilon_i, i = 1, 2, ..n\) is non-increasing as \(\eta \searrow 0\).

Now we are in a position to state the main result of the paper.

**Theorem 2.1** Suppose Assumptions 2.1, 2.2 and 2.3 hold. Then the Markov process formed by the solution \(\{X^x, x \in H\}\) of equation (2.1) is irreducible in \(H\).

In the rest of this section, let us consider the particular case of the additive noise. Let now \(Z = H\), and \(\nu\) a given \(\sigma\)-finite intensity measure of a Lévy process on \(H\). Recall that \(\nu(\{0\}) = 0\) and \(\int_H(\|z\|^2_H \wedge 1)\nu(dz) < \infty\). Let \(N : \mathcal{B}(H \times \mathbb{R}^+) \times \Omega \rightarrow \bar{N}\) be the time homogeneous Poisson random measure with intensity measure \(\nu\). Again \(\bar{N}(dz, dt) = N(dz, dt) - \nu(dz)dt\) denotes the compensated Poisson random measure associated to \(N\).

Let us point out that, see e.g., [28] Theorems 4.23 and 6.8, in this case

\[
L(t) = \int_0^t \int_0 \int_{\|z\|_H \leq 1} z\tilde{N}(dz, ds) + \int_0^t \int_{\|z\|_H > 1} zN(dz, ds), t \geq 0
\]

defines an \(H\)-valued Lévy process.

Let \(Z_m = \{\chi \in H : \|\chi\|_H > \frac{1}{m}\}, m \in \mathbb{N}\). Now consider the SPDE (2.1) with the additive noise \(dL(t)\), that is,

\[
\begin{align*}
&dX(t) = A(X(t))dt + dL(t), \quad (2.12)
&\text{for any } t \geq 0.
\end{align*}
\]
\[ X(0) = x. \]

Let us now formulate the condition on the jumping measure in this setting.

**Assumption 2.4** For any \( h \in H \) and \( \eta_h > 0 \), there exist \( n \in \mathbb{N} \), a sequence of strict positive numbers \( \eta_1, \eta_2, \ldots, \eta_n \), and \( a_1, a_2, \ldots, a_n \in H \setminus \{0\} \), such that \( 0 \not\in B(a_i, \eta_i), \nu(B(a_i, \eta_i)) > 0 \), \( i = 1, \ldots, n \), and that \( \sum_{i=1}^{n} B(a_i, \eta_i) := \{ \sum_{i=1}^{n} h_i : h_i \in B(a_i, \eta_i), 1 \leq i \leq n \} \subset B(h, \eta_h) \).

As an application of Theorem 2.1, we have

**Theorem 2.2** Under Assumptions 2.1, 2.2 and 2.4, the Markov process formed by the solution \( \{X^x, x \in H\} \) of equation (2.12) is irreducible in \( H \).

Considering (2.12) with \( A = 0 \) and \( x = 0 \), then we have

**Corollary 2.1** Assume that Assumptions 2.4 holds. For any \( s > 0 \), \( \epsilon > 0 \) and \( h \in H \), \( \mathbb{P}(L(s) \in B(h, \epsilon)) > 0 \).

**Remark 2.4** The result in Theorem 2.2 is optimal in the sense that it is false when Assumption 2.4 fails. Here is an example. Consider the following stochastic differential equation on the real line \( \mathbb{R} \):

\[ dX(t) = b(X(t))dt + dL(t), \quad (2.13) \]

where the function \( b : \mathbb{R} \to \mathbb{R}^+ \) is Borel measurable, and \( L = \{ L(t) = \int_0^t \int_{z>0} zN(dz, ds), t \geq 0 \} \) is a Lévy process on \( \mathbb{R} \) with the intensity measure \( \nu \) satisfying \( \nu(\{0\}) = 0 \), \( \nu(\{z < 0\}) = 0 \) and \( \int_{z>0} z\nu(dz) < \infty \), which implies that \( L(t) \geq 0 \), \( \mathbb{P} \)-a.s. In this case, Assumption 2.4 obviously fails. It is easy to see that any solution \( X^x \) to (2.13) (if it exists) satisfies \( \mathbb{P}(X^x(t) \in (-\infty, x)) = 0 \), \( \forall t \geq 0 \), which in particular means that \( \{X^x, x \in \mathbb{R}\} \) is not irreducible in \( \mathbb{R} \).

We also like to stress that Corollary 2.1 covers both finite and infinite dimensional Lévy processes, and is even new for \( \mathbb{R}^n \)-valued Lévy processes. We refer the reader to Chapter 5 in [31] for the study of the support of \( \mathbb{R}^n \)-valued Lévy processes.

### 3 Proofs of the main results

In this section, we will give the proof of Theorem 2.1. To this end, we need to prepare a number of results. Recall \( \tau_{x,y}^\eta \) introduced in (2.5).

**Proposition 3.1** Assume that Assumptions 2.1 and 2.2 hold. For any \( h \in H \), there exists \( \eta_h > 0 \) such that, for any \( \eta \in (0, \eta_h] \), there exist \( s > 0 \), \( \varpi_0 \in (0, \frac{s}{2}] \) satisfying for any \( h \in B(h, \varpi_0) \),

\[ \mathbb{P}(\sup_{l \in [0,s]} \|X^h(l) - h\| \leq \eta) > 0. \quad (3.1) \]
Proof  By Assumption 2.2, for any \( h \in H \), there exists \( \eta_h > 0 \) such that, for any \( \eta \in (0, \eta_h] \), there exists \( (\epsilon, t) = (\epsilon(h, \eta), t(h, \eta)) \in (0, \frac{n}{2}] \times (0, \infty) \) such that, for any \( \tilde{h} \in B(h, \epsilon) \),

\[
\mathbb{P}(\tau_{\tilde{h}, h}^n \geq t) > 0,
\]

which implies that for any \( \tilde{h} \in B(h, \frac{\epsilon}{2}) \),

\[
\mathbb{P}( \sup_{l \in [0, \frac{\epsilon}{2}]} \|X^h(l) - h\|_H \leq \eta) > 0.
\]

We simply choose \( s = \frac{\epsilon}{2} \) and \( \varpi_0 = \frac{\epsilon}{2} \) to complete the proof. \( \square \)

By Proposition 2.3 a similar argument to that used for the proof of Proposition 3.1 leads to the following statement.

Proposition 3.2  Assume that Assumptions 2.1 and 2.2 hold. For any \( m \in \mathbb{N} \) and \( h \in H \), there exists \( \eta^m_h > 0 \) such that, for any \( \eta^m \in (0, \eta^m_h] \), there exist \( s^m > 0 \), \( \varpi^m_0 \in (0, \frac{n}{2}] \) such that for any \( \tilde{h} \in B(h, \varpi^m_0) \),

\[
\mathbb{P}( \sup_{l \in [0, s^m]} \|X^h_m(l) - h\|_H \leq \eta^m) > 0.
\] (3.2)

Proposition 3.3  Assume that Assumption 2.3 holds. For any \( h, y \in H \) with \( h \neq y \) and any \( \bar{\eta} > 0 \), there exist \( n, m \in \mathbb{N} \), and \( \{l_i, i = 1, 2, \ldots, n\} \subset Z_m \) such that, for any \( \eta \in (0, \frac{\bar{\eta}}{2}) \), there exist \( \{\epsilon_i, i = 1, 2, \ldots, n\} \subset (0, \infty) \) and

\[
0 < \eta_0 < \eta_1 < 2\eta_1 < \eta_2 < 2\eta_2 < \eta_3 < 2\eta_3 < \ldots < \eta_{n-1} < 2\eta_{n-1} < \eta'_{n-1} < \eta_n < 2\eta_n \leq \eta'_{n} < \eta
\]

such that, denoting \( q_0 = h, q_i = q_{i-1} + \sigma(q_{i-1}, l_i), i = 1, 2, \ldots, n, \)

- \( \{\tilde{q} + \sigma(\tilde{q}, l), \tilde{q} \in B(q_0, \eta_0), l \in B(l_1, \epsilon_1)\} \subset B(q_1, \eta_1), \)
- for any \( i = 1, \ldots, n - 1 \), \( \{\tilde{q} + \sigma(\tilde{q}, l), \tilde{q} \in B(q_i, \eta_i), l \in B(l_{i+1}, \epsilon_{i+1})\} \subset B(q_{i+1}, \eta_{i+1}) \);
- \( B(q_n, 2\eta_n) \subset B(q_n, \eta''_n) \subset B(y, \frac{n}{2}) \);
- for any \( i = 1, 2, \ldots, n \), \( \nu(B(l_i, \epsilon_i)) > 0 \);
- there exists \( m_0 \geq m \) such that \( \bigcup_{i=1}^{m} B(l_i, \epsilon_i) \subset Z_{m_0} \);
- for any \( i = 1, 2, \ldots, n \), \( \epsilon_i \) is non-increasing as \( \eta \searrow 0 \).

Proof  By Assumption 2.3 and Remark 2.3, for any \( h, y \in H \) with \( h \neq y \) and any \( \bar{\eta} > 0 \), there exist \( n, m \in \mathbb{N} \), and \( \{l_i, i = 1, 2, \ldots, n\} \subset Z_m \), denoting \( q_0 = h, q_i = q_{i-1} + \sigma(q_{i-1}, l_i), i = 1, 2, \ldots, n, \)

such that, for any \( \eta \in (0, \frac{\bar{\eta}}{2}) \), setting \( \eta_k = \frac{\eta}{2^k}, k \in \mathbb{N} \), there exist \( \{\epsilon^k_i, i = 1, 2, \ldots, n\} \subset (0, \infty) \) and \( \{\eta^k_i, i = 0, 1, \ldots, n\} \subset (0, \infty) \) satisfying
\[ 0 < \eta_0^k \leq \eta_1^k \leq \ldots \leq \eta_{n-1}^k \leq \eta_n^k; \]

- for any \( i = 0, 1, \ldots, n - 1 \), \( \{ \hat{q} + \sigma(\hat{q}, l), \ \hat{q} \in B(q, \eta_i^k), l \in B(l_{i+1}, \epsilon_{i+1}) \} \subset B(q_{i+1}, \eta_{i+1}^k); \]

- \( B(q_n, \eta_n^k) \subset B(y, \frac{\tilde{y}}{2}); \)

- for any \( i = 1, 2, \ldots, n \), \( \nu(B(l_i, \epsilon_i^k)) > 0; \)

- for any \( i = 0, 1, 2, \ldots, n \), \( \lim_{k \to \infty} \eta_i^k = 0; \)

- for any \( i = 1, 2, \ldots, n \), \( \epsilon_i^k \) is non-increasing as \( k \to \infty; \)

- there exists \( m_0 \geq m \) such that \( \bigcup_{i=1}^n B(l_i, \epsilon_i^k) \subset Z_m. \)

Noting that, for any \( q \in H, z \in Z, \omega_1 > \omega_2 \geq 0, \beta_1 \geq \beta_2 \geq 0, \)

\[ \overline{B(q, \omega_2)} \subset B(q, \omega_1) \]

and

\[ \{ \hat{q} + \sigma(\hat{q}, l), \ \hat{q} \in \overline{B(q, \omega_2)}, l \in B(z, \beta_2) \} \subset \{ \hat{q} + \sigma(\hat{q}, l), \ \hat{q} \in B(q, \omega_1), l \in B(z, \beta_1) \}, \]

one can easily choose appropriate integers \( k, \eta_i^k \) and \( \epsilon_i^k \) above to get the positive numbers \( \eta_i, \eta_i' \) and \( \epsilon_i \) required in the statement of the proposition. \( \square \)

Combining Propositions 3.1, 3.2 and 3.3 together, we arrive at the following.

**Proposition 3.4** Assume that Assumptions 2.1, 2.2 and 2.3 hold. For any \( h, y \in H \) with \( h \neq y \) and any \( \bar{h} > 0 \), there exist \( n, m \in \mathbb{N} \), and \( \{ l_i, i = 1, 2, \ldots, n \} \subset Z_m \) such that, denoting

\[ q_0 = h, \ q_i = q_{i-1} + \sigma(q_{i-1}, l_i), \ i = 1, 2, \ldots, n, \]

for any \( \eta \in (0, \frac{\bar{h}}{2}), \) there exist \( s > 0, \eta_i \in (0, \eta), \omega_i^1 \in (0, \frac{\omega_i}{2}] \) and \( \omega_i^2 \in (0, \frac{\omega_i}{2}], \ i = 0, 1, 2, \ldots, n, \) and \( \{ \epsilon_i, i = 1, 2, \ldots, n \} \subset (0, \infty), \) and \( m_0 \geq m, \) such that, for any \( s' \in (0, s], \)

- for any \( i = 0, 1, 2, \ldots, n - 1, \) \( \{ \hat{q} + \sigma(\hat{q}, l), \ \hat{q} \in \overline{B(q_i, \eta_i)}, l \in B(l_{i+1}, \epsilon_{i+1}) \} \subset \overline{B(q_{i+1}, \omega_{i+1}^2)}; \)

- \( \overline{B(q_n, \omega_n^1)} \subset \overline{B(q_n, \eta_n)} \subset B(y, \frac{\tilde{y}}{2}); \)

- for any \( i = 1, 2, \ldots, n, \) \( \nu(B(l_i, \epsilon_i)) > 0; \)

- for any \( i = 1, 2, \ldots, n, \) \( \epsilon_i \) is non-increasing as \( \eta \searrow 0; \)

- \( \bigcup_{i=1}^n B(l_i, \epsilon_i) \subset Z_{m_0}; \)

- for \( i = 0, 1, 2, \ldots, n, \)

\[ \mathbb{P} \left( \sup_{l \in [0, s']} \| X^h l - q_i \|_H \leq \omega_i^1 \right) > 0, \ \forall \ h \in \overline{B(q_i, \omega_i^2)}; \]
\[
\mathbb{P}\left(\sup_{t \in [0,s]} \|X^{\tilde{h}}(t) - q_i\|_H \leq \eta_i\right) > 0, \quad \forall \tilde{h} \in B(q_i, \omega_i^1); \]

and
\[
\mathbb{P}\left(\sup_{t \in [0,s']} \|X^{\hat{h}}_{z_0}(t) - q_i\|_H \leq \omega_i^1\right) > 0, \quad \forall \hat{h} \in B(q_i, \omega_i^1); \]
\[
\mathbb{P}\left(\sup_{t \in [0,s']} \|X^{\hat{h}}_{z_0}(t) - q_i\|_H \leq \eta_i\right) > 0, \quad \forall \hat{h} \in B(q_i, \omega_i^1).
\]

**Proof of Theorem 2.1**

**Proof** We will prove that, for any \(x, y \in H\), \(T > 0\) and \(\kappa > 0\),
\[
\mathbb{P}(X^x(T) \in B(y, \kappa)) > 0. \tag{3.3}
\]

Fix now \(x, y \in H\), \(T > 0\) and \(\kappa > 0\). Recall
\[
\tau^{\eta}_{h,y} = \inf\{t \geq 0 : X^{\tilde{h}}(t) \not\in B(y, \eta)\}.
\]

By Assumption 2.2, there exists \(\eta_y > 0\) such that, for any \(\eta \in (0, \eta_y]\), there exists \((\epsilon_y^\eta, t_y^\eta) \in (0, \eta_y^2] \times (0, \infty)\) satisfying, for any \(\tilde{h} \in B(y, \epsilon_y^\eta),
\[
\mathbb{P}(\tau^{\eta}_{h,y} \geq t_y^\eta) > 0.
\]

Without loss of generality, we may assume
\[
\kappa < \eta_y.
\]

Now, let in particular \(\eta = \frac{\kappa}{4}\) to get a pair \((\epsilon_y, t_y) \in (0, \frac{\kappa}{4}] \times (0, \infty)\) such that for any \(\tilde{h} \in B(y, \epsilon_y),
\[
\mathbb{P}(\tau^{\frac{\kappa}{4}}_{h,y} \geq t_y) > 0.
\]

This implies that, for any \(\hat{h} \in B(y, \epsilon_y),
\[
\mathbb{P}\left(\sup_{s \in [0,t_y^\eta]} \|X^{\hat{h}}(s) - y\|_H \leq \frac{\kappa}{4}\right) > 0. \tag{3.4}
\]

We may also assume \(t_y \leq T\). Set \(T_0 = T - \frac{t_y}{2}\). In the rest of the proof, we distinguish the following two cases:

**Case 1:** for any \(\epsilon > 0\), \(\mathbb{P}(X^x(T_0) \in B(y, \epsilon)) > 0;\)

**Case 2:** there exists \(\hat{\epsilon} > 0\) such that \(\mathbb{P}(X^x(T_0) \in B(y, \hat{\epsilon})) = 0.

For an \(H\)-valued measurable mapping \(S\) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we will denote by \(\mathbb{P}^S\) the measure induced by \(S\) on \((H, \mathcal{B}(H))\).
We first consider the Case 1, namely, assume that for any \( \epsilon > 0 \), \( \mathbb{P}(X^z(T_0) \in B(y, \epsilon)) > 0 \). In particular, we have
\[
\mathbb{P}(X^z(T_0) \in B(y, \frac{\epsilon y}{2})) > 0. \tag{3.5}
\]

By the Markov property of \( X := \{X^z, x \in H\} \) and (3.5), we have
\[
\mathbb{P}(X^z(T) \in B(y, \kappa)) \\
\geq \mathbb{P}(\{X^z(T_0) \in B(y, \frac{\epsilon y}{2})\} \cap \{X^z(t) \in B(y, \kappa), \forall t \in [T_0, T]\}) \\
= \int_{B(y, \frac{\epsilon y}{2})} \mathbb{P}(X^\tilde{h}(t) \in B(y, \kappa), \forall t \in [0, T - T_0]) \mathbb{P}^{X^z(T_0)}(d\tilde{h}) \\
= \int_{B(y, \frac{\epsilon y}{2})} \mathbb{P}(X^\tilde{h}(t) \in B(y, \kappa), \forall t \in [0, \frac{T_0}{2}]) \mathbb{P}^{X^z(T_0)}(d\tilde{h}). \tag{3.6}
\]

In view of (3.4), we have
\[
\mathbb{P}(X^\tilde{h}(t) \in B(y, \kappa), \forall t \in [0, \frac{T_0}{2}]) > 0 \quad \text{for all} \quad \tilde{h} \in B(y, \frac{\epsilon y}{2}).
\]

It follows from (3.5) and (3.6) that \( \mathbb{P}(X^z(T) \in B(y, \kappa)) > 0 \), completing the proof of the theorem in the Case 1.

Next we consider the Case 2, namely, assume that there exists \( \tilde{\epsilon} > 0 \) such that
\[
\mathbb{P}(X^z(T_0) \in B(y, \tilde{\epsilon})) = 0, \tag{3.7}
\]
which implies that there exists \( \zeta \not\in B(y, \tilde{\epsilon}) \) such that for any \( \rho > 0 \)
\[
\mathbb{P}(X^z(T_0) \in B(\zeta, \rho)) > 0. \tag{3.8}
\]

By Proposition 3.4, we have the following statements.

For the point \( \zeta, y \in H \) and \( \tilde{y} = \kappa_0 := \frac{\epsilon y}{2} \),

(C) there exist \( n^\zeta, m^\zeta \in \mathbb{N} \), and \( \{l^\zeta_i, i = 1, 2, ..., n^\zeta\} \subset Z_{m^\zeta} \), denoting
\[
q^\zeta_0 = \zeta, \quad q^\zeta_i = q^\zeta_{i-1} + \sigma(q^\zeta_{i-1}, l^\zeta_i), \quad i = 1, 2, ..., n^\zeta,
\]
such that, for \( \eta = \frac{\kappa_0}{8} \), there exist \( s^\zeta > 0 \), \( \eta^\zeta_i \in (0, \eta) \), \( \varpi^1^\zeta_i \in (0, \frac{\eta^\zeta_i}{2}) \) and \( \varpi^2^\zeta_i \in (0, \frac{\eta^1^\zeta_i}{2}) \), \( i = 0, 1, 2, ..., n^\zeta \), and \( \{\epsilon^\zeta_i, i = 1, 2, ..., n^\zeta\} \subset (0, \infty) \), and \( m^\zeta \geq m^\zeta \), such that, for
\[
s' = s_0 := \frac{t_0}{4m^\zeta} \wedge \frac{\epsilon^\zeta}{2},
\]

(C-1) for any \( i = 0, 1, 2, ..., n^\zeta - 1 \), \( \{\tilde{q} + \sigma(\tilde{q}, l), \tilde{q} \in \overline{B(q^\zeta_i, \eta^\zeta_i)}, l \in B(l^\zeta_i, \epsilon^\zeta_i)\} \subset B(q^\zeta_{i+1}, \varpi^2_{i+1});
\]

(C-2) \( B(q^\zeta_{n^\zeta}, \varpi^1_{n^\zeta}) \subset B(q^\zeta_{n^\zeta}, \eta^\zeta_{n^\zeta}) \subset B(y, \frac{\eta}{2}) = B(y, \frac{\epsilon y}{4});
\]

(C-3) for any \( i = 1, 2, ..., n^\zeta \), \( \nu(B(l^\zeta_i, \epsilon^\zeta_i)) > 0; \)
(C-4) for any $i = 1, 2, ..., n^c$, $\epsilon^c_i$ is non-increasing as $\eta \searrow 0$;

(C-5) $\bigcup_{i=1}^{n^c} B(l^c_i, \epsilon^c_i) \subset \mathcal{Z}_{m_0}$;

(C-6) for any $i = 0, 1, 2, ..., n^c$,

\[
\mathbb{P}(\sup_{t \in [0, s_0]} \|X^h(l) - q^c_i\|_H \leq \omega^1_i) > 0, \quad \forall h \in B(q^c_i, \omega^2_i);
\]

\[
\mathbb{P}(\sup_{t \in [0, s_0]} \|X^h(l) - q^c_i\|_H \leq \eta^c_i) > 0, \quad \forall h \in B(q^c_i, \omega^1_i);
\]

and

\[
\mathbb{P}(\sup_{t \in [0, s_0]} \|X^{h}_{m_0}(l) - q^c_i\|_H \leq \omega^1_i) > 0, \quad \forall h \in B(q^c_i, \omega^2_i);
\]

\[
\mathbb{P}(\sup_{t \in [0, s_0]} \|X^{h}_{m_0}(l) - q^c_i\|_H \leq \eta^c_i) > 0, \quad \forall h \in B(q^c_i, \omega^1_i).
\]

By (3.8) and $\omega^2_i > 0$, we have

\[
\mathbb{P}(X^x(T_0) \in B(\zeta, \omega^2_i)) > 0. \tag{3.9}
\]

Set $T_i := T_0 + is_0$. We will prove that, for any $i = 1, 2, ..., n^c$,

\[
\mathbb{P}(\{X^x(T_0) \in B(\zeta, \omega^2_i)\} \bigcap \{X^x(T_j) \in B(q^c_i, \omega^1_i)\}) > 0. \tag{3.10}
\]

By the Markov property of $X$,

\[
\mathbb{P}(\{X^x(T_0) \in B(\zeta, \omega^2_i)\} \bigcap \{X^x(T_1) \in B(q^c_i, \omega^1_i)\})
\]

\[
= \int_{B(\zeta, \omega^2_i)} \mathbb{P}(X^x(s_0) \in B(q^c_i, \omega^1_i)) \mathbb{P}^{X^x(T_0)}(dh).
\]

In view of (3.9), to prove (3.10) with $i = 1$, it is sufficient to show that for any $h \in B(\zeta, \omega^2_i)$,

\[
\mathbb{P}(X^x(s_0) \in B(q^c_i, \omega^1_i)) > 0. \tag{3.12}
\]

Let $\sigma_1 = \inf\{t \geq 0 : \int_0^t \int_{B(l^c_i, \epsilon^c_i)} N(dz, ds) = 1\}$ be the first jumping time of the Poisson process $N(B(l^c_i, \epsilon^c_i), [0, t])$. $\sigma_1$ has the exponential distribution with parameter $0 < \nu(B(l^c_i, \epsilon^c_i)) < \nu(Z_{m_0}) < \infty$. In particular, we have

\[
\mathbb{P}(\sigma_1 \in (0, s_0]) > 0. \tag{3.13}
\]

By the strong Markov property of $X$, for any $h \in B(\zeta, \omega^2_i)$,

\[
\mathbb{P}(X^h(s_0) \in B(q^c_i, \omega^1_i))
\]

\[
\geq \mathbb{P}(\{X^h(s_0) \in B(q^c_i, \omega^1_i)\} \cap \{\sigma_1 \in (0, s_0]\}) \cap \{X^h(\sigma_1) \in B(q^c_i, \omega^2_i)\}) \tag{3.14}
\]
\[ \geq \mathbb{P}\{\sigma_1 \in (0, s_0] \cap \{X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1)\} \cap \{\sup_{s \in [\sigma_1, \sigma_1 + s_0]} \|X^h(s) - q^{i_1}_1\|_H \leq \omega^{1\zeta}_1\} \] \\
= \mathbb{E}\left( \mathbb{E}\left( 1_{(0, s_0]}(\sigma_1) \frac{1}{B(q^{i_1}_1, \omega^{2\zeta}_1)}(X^h(\sigma_1))1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [\sigma_1, \sigma_1 + s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \mathcal{F}_{\sigma_1} \right) \right) \\
= \mathbb{E}\left( 1_{(0, s_0]}(\sigma_1) \frac{1}{B(q^{i_1}_1, \omega^{2\zeta}_1)}(X^h(\sigma_1)) \mathbb{E}\left( 1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [0, s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \right) \right).
\\
If we set \( f(\tilde{h}) = \mathbb{E}\left( 1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [0, s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \right) \), then
\[ \mathbb{E}\left( 1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [0, s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \right) = f(X^h(\sigma_1)). \quad (3.15) \]

Using (C-6) for \( i = 1 \), we have, for any \( \tilde{h} \in B(q^{i_1}_1, \omega^{2\zeta}_1) \),
\[ \mathbb{E}\left( 1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [0, s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \right) > 0. \]

So \( f(X^h(\sigma_1)) > 0 \) for \( X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1) \). Hence, to prove (3.12), in view of (3.14) we only need to show, for any \( \tilde{h} \in B(\zeta, \omega^{2\zeta}_0) \),
\[ \mathbb{E}\left( 1_{[0, \omega^{1\zeta}_1]} \left( \sup_{s \in [0, s_0]} \|X^h(s) - q^{i_1}_1\|_H \right) \right) = \mathbb{P}\{\sigma_1 \in (0, s_0] \cap \{X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1)\} \} > 0. \quad (3.16) \]

Set \( \tau_1 = \text{inf}\{t \geq 0 : \int_0^t \int_{Z_{m_0}} \mathcal{L}(B(l^{i_1}_1, \epsilon^{i_1}_1), X^h(\sigma_1)) \right\} N(dz, ds) = 1 \}. \) Since \( \nu(Z_{m_0}^{i_1}) < \infty \),
\[ \mathbb{P}(\tau_1 > s_0) > 0. \quad (3.17) \]

For any \( h \in B(\zeta, \omega^{2\zeta}_0) \),
\[ \mathbb{P}\{\sigma_1 \in (0, s_0] \cap \{X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1)\} \} \]
\[ \geq \mathbb{P}\{\sigma_1 \in (0, s_0] \cap \{\tau_1 > s_0\} \cap \{X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1)\} \} \]
\[ \geq \mathbb{P}\{\sigma_1 \in (0, s_0] \cap \{\sup_{s \in [0, \sigma_1]} \|X^h(s) - \zeta\|_H \leq \omega^{1\zeta}_1\} \cap \{\tau_1 > s_0\} \cap \{X^h(\sigma_1) \in B(q^{i_1}_1, \omega^{2\zeta}_1)\}. \quad (3.18) \]

Since \( \nu(Z_{m_0}^{i_1}) < \infty \), \( \nu(B(l^{i_1}_1, \epsilon^{i_1}_1)) < \infty \) and (C-5), the solution to (2.2) with \( x = h \) satisfies the following equation:
\[ X^h(t) = h + \int_0^t A(X^h(s)) ds + \int_0^t \int_{Z_{m_0}^{i_1}} \sigma(X^h(s^-), z) \mathcal{N}(dz, ds) \]
\[ - \int_0^t \int_{Z_{m_0}^{i_1} \setminus Z_1} \sigma(X^h(s^-), z) \mathcal{U}(dz) ds + \int_0^t \int_{B(l^{i_1}_1, \epsilon^{i_1}_1)} \sigma(X^h(s^-), z) \mathcal{N}(dz, ds) \]
\[ + \int_0^t \int_{Z_{m_0}^{i_1} \setminus B(l^{i_1}_1, \epsilon^{i_1}_1)} \sigma(X^h(s^-), z) \mathcal{N}(dz, ds), \quad t \geq 0. \]

Notice that
\[ \int_0^t \int_{B(l^{i_1}_1, \epsilon^{i_1}_1)} \sigma(X^h(s^-), z) \mathcal{N}(dz, ds) = 0, \text{ on } \{t \in [0, \sigma_1]\}, \quad \mathbb{P}\text{-a.s.;} \]
Recalling the solution $X_m$ introduced in (2.3) we conclude that for $s < \sigma_1 \leq s_0 < \tau_1$, $X^h(s) = X^h_{m_0^\zeta}(s)$. Moreover, by (C-1) with $i = 0$, we see that

$$
\{ \sup_{s \in [0, \sigma_1)} \| X^h(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} 
\subseteq
\{ X^h(\sigma_1-) := \lim_{s \to \sigma_1} X^h(s) \in B(\zeta, \varpi_0^{1, \zeta}) \}
\subseteq
\{ X^h(\sigma_1) = X^h(\sigma_1-) + (X^h(\sigma_1) - X^h(\sigma_1-)) \in B(q^\zeta, \varpi_1^{1, \zeta}) \}. \tag{3.19}
$$

Here we have used the fact that

$$
X^h(\sigma_1) - X^h(\sigma_1-)
= \int_0^{\sigma_1} \int_{Z^c_{m_0^\zeta}} \sigma(X^h(s), z) \tilde{N}(dz, ds) + \int_0^{\sigma_1} \int_{B(l_1^\xi, \xi_1)} \sigma(X^h(s), z) N(dz, ds)
+ \int_0^{\sigma_1} \int_{Z^c_{m_0^\zeta} \setminus B(l_1^\xi, \xi_1)} \sigma(X^h(s), z) N(dz, ds)
\subseteq \sup_{s \in [0, \sigma_1]} \| X^h(s) - \zeta \|_H \leq \varpi_0^{1, \zeta}
\subseteq \{ \sigma(X^h(\sigma_1-), l), \ l \in B(l_1^\xi, \xi_1) \}.
$$

We therefore arrive at

$$
\{ \sigma_1 \in (0, s_0] \} \cap \{ \sup_{s \in [0, \sigma_1)} \| X^h(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} \subseteq \{ \tau_1 > s_0 \} \cap \{ X^h(\sigma_1) \in B(q^\zeta, \varpi_1^{1, \zeta}) \}
= \{ \sigma_1 \in (0, s_0] \} \cap \{ \sup_{s \in [0, \sigma_1)} \| X^h(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} \cap \{ \tau_1 > s_0 \}
= \{ \sigma_1 \in (0, s_0] \} \cap \{ \sup_{s \in [0, \sigma_1)} \| X^h_{m_0^\zeta}(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} \cap \{ \tau_1 > s_0 \}
\supseteq \{ \sigma_1 \in (0, s_0] \} \cap \{ \sup_{s \in [0, s_0]} \| X^h_{m_0^\zeta}(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} \cap \{ \tau_1 > s_0 \}. \tag{3.20}
$$

Similar to the proof of Proposition 2.2, because the following events are determined by the jumps of the Poisson random measure on disjoint subsets,

$$
\{ \sigma_1 \in (0, s_0] \}, \ \{ \sup_{s \in [0, s_0]} \| X^h_{m_0^\zeta}(s) - \zeta \|_H \leq \varpi_0^{1, \zeta} \} \text{ and } \{ \tau_1 > s_0 \}. \tag{3.21}
$$
are mutually independent. Combining (3.13), (3.17), (3.18), (3.20), and (3.21) together, we obtain that, for any \(h \in B(\zeta, \varpi_0^2 \zeta)\),

\[
P\{\{\sigma_1 \in (0, s]\cap \{X^h(\sigma_1) \in B(q_1^2, \varpi_1^2 \zeta)\}\}
\geq P(\sigma_1 \in (0, s])P(\sup_{s \in [0, s]} \|X^h(\sigma_1) - \zeta\|_H \leq \varpi_0^2 \zeta))P(\tau_1 > s)
> 0,
\]

(3.22)

which proves (3.16). (C-6) has been used in the last inequality. We have proved (3.10) for \(i = 1\).

Now we prove (3.10) with \(i = 2\). By the Markov property of \(X\),

\[
P(\{X^x(T_0) \in B(\zeta, \varpi_0^{2 \zeta})\} \cap \{X^x(T_j) \in B(q_j^2, \varpi_j^{1 \zeta})\})
= E\left(E\left(1_{B(\zeta, \varpi_0^{2 \zeta})}(X^x(T_0)) \cdot \frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(T_1)) \cdot \frac{1}{B(q_2^2, \varpi_2^{1 \zeta})}(X^x(T_2))|F_{T_1}\right)\right)
= E\left(E\left(1_{B(\zeta, \varpi_0^{2 \zeta})}(X^x(T_0)) \cdot \frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(T_1)) \cdot \frac{1}{B(q_2^2, \varpi_2^{1 \zeta})}(X^x(T_2))|F_{T_1}\right)\right)
= E\left(E\left(1_{B(\zeta, \varpi_0^{2 \zeta})}(X^x(T_0)) \cdot \frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(T_1)) \cdot \frac{1}{B(q_2^2, \varpi_2^{1 \zeta})}(X^x(T_2))|h = X^x(T_1)\right)\right).
\]

Here,

\[
E\left(\frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(s_0))\right) = E\left(\frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(s_0))\right)_{h = X^x(T_1)}
\]

(3.24)

(3.10) with \(i = 1\) says that

\[
E\left(1_{B(\zeta, \varpi_0^{2 \zeta})}(X^x(T_0)) \cdot \frac{1}{B(q_1^2, \varpi_1^{1 \zeta})}(X^x(T_1))\right) = P(\{X^x(T_0) \in B(\zeta, \varpi_0^{2 \zeta})\} \cap \{X^x(T_1) \in B(q_1^2, \varpi_1^{1 \zeta})\}) > 0.
\]

In view of (3.23), to prove (3.10) for \(i = 2\), we only need to show that, for any \(h \in B(q_1^2, \varpi_1^{1 \zeta})\),

\[
E\left(\frac{1}{B(q_2^2, \varpi_2^{1 \zeta})}(X^x(s_0))\right) = P(\{X^x(s_0) \in B(q_2^2, \varpi_2^{1 \zeta})\}) > 0.
\]

This can be proved similarly as the proof of (3.12). Thus we have proved (3.10) also for \(i = 2\).

Following a recursive procedure we are able to prove (3.10) for any \(i = 1, 2, \ldots, n^\zeta\).

Now we will prove that

\[
P\left(\left\{X^x(T_0) \in B(\zeta, \varpi_0^{2 \zeta})\right\} \cap \left\{X^x(T_j) \in B(q_j^2, \varpi_j^{1 \zeta})\right\} \cap \left\{X^x(T_{n^\zeta}) \in B(y, \frac{\varpi_0}{2})\right\} \cap \left\{X^x(T) \in B(y, \kappa_0)\right\}\right) > 0.
\]

(3.25)

Recall \(s_0 = \frac{t_0}{4\varpi_0} \land \frac{t_0^2}{2} \land \kappa_0 = \frac{\varpi_0}{2} \land T_0 = T - \frac{t_0}{2}\). By (C-2) and (3.10), we have

\[
P\left(\left\{X^x(T_0) \in B(\zeta, \varpi_0^{2 \zeta})\right\} \cap \left\{X^x(T_j) \in B(q_j^2, \varpi_j^{1 \zeta})\right\} \cap \left\{X^x(T_{n^\zeta}) \in B(y, \frac{\varpi_0}{2})\right\}\right)
\geq P\left(\left\{X^x(T_0) \in B(\zeta, \varpi_0^{2 \zeta})\right\} \cap \left\{X^x(T_j) \in B(q_j^2, \varpi_j^{1 \zeta})\right\}\right)
\]

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Noticing that $T_{n\zeta} = T_0 + n\zeta s_0 \in (T_0, T)$ and applying the Markov property of $X$ again, we have

$$
P\left(\bigcap_{j=1}^{n\zeta-1} \{X^x(T_j) \in B\left(q_j, \frac{\zeta}{2}\right)\} \cap \{X^x(T_{n\zeta}) \in B(y, \frac{\zeta}{2})\} \cap \{X^x(T) \in B(y, \kappa)\} \right) = E\left(1_{B(\zeta, \frac{\zeta}{2})}(X^x(T_0)) \prod_{j=1}^{n\zeta-1} 1_{B(q_j, \frac{\zeta}{2})}(X^x(T_j)) 1_{B(y, \frac{\zeta}{2})}(X^x(T_{n\zeta})) \cdot E\left(1_{B(y, \kappa)}(X^h(T-T_{n\zeta})) \mid h = X^x(T_{n\zeta})\right)\right).$$

(3.27)

In view of (3.26) and (3.27), to prove (3.25), we only need to prove that, for any $h \in B(y, \frac{\zeta}{2})$,

$$E\left(1_{B(y, \kappa)}(X^h(T-T_{n\zeta})) \right) = P\left(X^h(T-T_{n\zeta}) \in B(y, \kappa)\right) > 0.$$  

(3.28)

As $T-T_{n\zeta} \in (0, \frac{\zeta}{2})$, (3.28) follows from the choice of $t_y$, see (3.4). Hence (3.25) is established, which in particular yields

$$P(X^x(T) \in B(y, \kappa)) > 0.$$  

The proof is finished in the Case 2, completing the whole proof of Theorem 2.1. □

4 Applications

In this section, we provide applications of our main results to SDEs and SPDEs including many interesting physical models. Since Assumptions 2.2 and 2.3 are basically independent of each other, this section is divided into three parts: Subsection 4.1 presents examples of the additive driving noises satisfying Assumption 2.4 (the corresponding Assumption 2.3 in the case of the additive noise). Subsection 4.2 gives examples of multiplicative driving noises satisfying Assumption 2.3. Subsections 4.3-4.6 are to provide examples of physical models satisfying Assumption 2.2. Combination of Subsections 4.1-4.6 produces plenty of examples for the irreducibility of SDEs and SPDEs driven by pure jump Lévy noise, including many interesting physical models.

4.1 Sufficient conditions and examples for Assumption 2.4

For any measure $\rho$ on $H$, its support $S_\rho = S(\rho)$ is defined to be the set of $x \in H$ such that $\rho(G) > 0$ for any open set $G$ containing $x$. Set

$$H_0 := \left\{ \sum_{i=1}^{n} m_i a_i, \ n, m_1, \ldots, m_n \in \mathbb{N}, \ a_i \in S_\nu \right\}.$$

(4.1)

It is not difficult to see that Assumption 2.4 holds if and only if $H_0$ is dense in $H$.  

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Assumption 2.4 actually places very mild conditions on the intensity measures $\nu$ of the Lévy processes. The examples we can include are much more general than that considered in the literature where the irreducibility of SPDEs/SDEs driven by pure jump noise were studied. In this subsection, we give several explanatory examples that satisfy Assumption 2.4.

Example 4.1 Let $H = \mathbb{R}$. The intensity measure $\nu$ of the Lévy process satisfies Assumption 2.4, namely, $H_0$ defined in (4.1) is dense in $\mathbb{R}$, if one of the following conditions is satisfied:

1. There exist $a < 0$, $b > 0$ and $c_n \neq 0$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} c_n = 0$, $\{a, b, c_n, n \in \mathbb{N}\} \subseteq S_\nu$.

2. $\nu(\mathbb{R}) = \infty$, and there exist $a > 0$ and $b < 0$ such that $\{a, b\} \subseteq S_\nu$.

3. There exist $a \neq 0$ and $b_n \neq -a$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} b_n = -a$, $\{a, b_n, n \in \mathbb{N}\} \subseteq S_\nu$.

4. There exist $a \neq 0$ and $b_n \neq 0$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} b_n = \infty$ and there exists a subsequence of $\{na+b_n, n \in \mathbb{N}\}$ strictly increase (or decrease) to $0$, $\{a, b_n, n \in \mathbb{N}\} \subseteq S_\nu$.

5. Set $S_\nu^+ = \{a \in S_\nu : a > 0\}$ and $S_\nu^- = \{a \in S_\nu : a < 0\}$. Let $\text{Leb}(S_\nu^+) > 0$ and $\text{Leb}(S_\nu^-) > 0$. Here $\text{Leb}$ is the Lebesgue measure on $\mathbb{R}$.

6. Let $S_\nu^+$ and $S_\nu^-$ defined as in (5). There exist $a \in S_\nu^+$ and $b \in S_\nu^-$ such that $a/b$ is an irrational number.

Recall that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} |x|^2 \wedge 1\nu(dx) < \infty$. $\nu(\mathbb{R}) = \infty$ implies that there exist $\{c_n \neq 0, n \in \mathbb{N}\} \subseteq S_\nu$ such that $\lim_{n \to \infty} c_n = 0$. Therefore (2) follows from (1).

We point out that $H_0$ defined in (4.1) is dense in $\mathbb{R}$ if and only if there exist $\{a, b, a_n, n \in \mathbb{N}\} \subseteq H_0 \setminus \{0\}$ such that $a < 0$, $b > 0$ and $\lim_{n \to \infty} a_n = 0$.

The proofs of statements are elementary and omitted here. We stress that it is easy to find many examples with $\nu(\mathbb{R}) < \infty$ that satisfy Assumption 2.4. This means that the driving Lévy process $L$ could be a compound Poisson process on $\mathbb{R}$.

Remark 4.1 The so-called tempered stable processes on $\mathbb{R}$ have the intensity measure $\nu$ given by

$$\nu(dx) = q^+(x)1_{(0,\infty)}(x)\frac{1}{x^{1+\beta^+}}dx + q^-(x)1_{(-\infty,0)}(x)\frac{1}{|x|^{1+\beta^-}}dx,$$

(4.2)

here $q^+: (0,\infty) \to [0,\infty)$ and $q^- : (0,\infty) \to [0,\infty)$ are called tempering functions, $\beta^+$ and $\beta^-$ are positive constants. One can easily see that under mild conditions on $q^+$ and $q^-$, the intensity measure $\nu$ satisfies one of the conditions in Example 4.1.

Example 4.2 Let $H = \mathbb{R}^d$, $d \in \mathbb{N} \cup \{+\infty\}$. Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis of $H$. Let $\{L_i(t), t \geq 0\}_{i \in \mathbb{N}}$ be mutually independent one dimensional pure jump Lévy processes with
their intensity measures $\nu_i$ satisfying one of the conditions listed in Example 4.1. Choosing $\beta_i \in \mathbb{R} \setminus \{0\}, i \in \mathbb{N}$ such that

$$\sum_{i=1}^{d} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \nu_i(dx_i) < \infty. \quad (4.3)$$

$L(t) = \sum_{i=1}^{d} \beta_i L_i(t)e_i, t \geq 0$ defines an $H$-valued Lévy process, and its intensity measure $\nu$ satisfies Assumption 2.4.

**Remark 4.2** (4.3) is a natural condition, because it implies that

$$\int_{\mathbb{H}} \|z\|^2_\mathbb{H} \wedge 1 \nu(dz) = \sum_{i=1}^{d} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \nu_i(dx_i) < \infty.$$ 

There are many examples such that $\sum_{i=1}^{d} \nu_i(\mathbb{R}) < \infty$, even for the infinite dimensional setting, and then, the driving Lévy process $L$ is a compound Poisson process on $H$.

The following example is concerned with the subordination of Lévy processes, which is an important idea to obtain new Lévy processes.

**Example 4.3** Let $H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\}$. Let $Z = \{Z_t, t \geq 0\}$ be a subordinator (an increasing Lévy process on $\mathbb{R}$) with Lévy measure $\rho$, drift $\beta_0$, which satisfy

$$\beta_0 \geq 0 \text{ and } \int_{(0,\infty)} (1 \wedge s) \rho(ds) < \infty.$$ 

Let $X = \{X_t, t \geq 0\}$ be a Lévy process on $H$ with intensity measure $\nu_X$. $Z$ and $X$ are independent. Define

$$L_t(\omega) = X_{Z_t(\omega)}(\omega), \ t \geq 0, \ \omega \in \Omega. \quad (4.4)$$

Then $\{L_t, t \geq 0\}$ is a Lévy process on $H$, and its intensity measure $\nu$ satisfies that

$$\nu(B) = \beta_0 \nu_X(B) + \int_{(0,\infty)} \mu^*_X(B) \rho(ds), \ B \in \mathcal{B}(H).$$

Here $\mu^*_X$ is the law of $X_s$. See [31, Theorem 30.1] for details.

If one of the following conditions holds, then the intensity measure $\nu$ of (4.4) satisfies Assumption 2.4.

- one of $\int_{(0,\infty)} (1 \wedge s) \rho(ds)$ and $\beta_0$ is not equal to 0, and Assumption 2.4 holds with $\nu$ replaced by $\nu_X$.

- $\int_{(0,\infty)} (1 \wedge s) \rho(ds) > 0$ and $S_{\mu^*_X}$ is dense in $H$.

Now let $\{Z_t, t \geq 0\}$ be a subordinator with Lévy measure $\rho$ satisfying $\int_{(0,\infty)} (1\wedge s) \rho(ds) > 0$. We have the following two concrete examples.

(1) Let $\{X_t, t \geq 0\} = \{W_t, t \geq 0\}$ be a $Q$-Wiener process on $H, Q \in L(H)$ is nonnegative, symmetric, with finite trace and $\text{Ker}Q = \{0\}$; here $L(H)$ denotes the set of all bounded linear
operators on $H$. As it is well-known that, for any $h \in H$, $t > 0$ and $r > 0$, $\mathbb{P}(W_t \in B(h, r)) > 0$, we see that $L_t = W_{Z_t}$, $t \geq 0$ is a Lévy process on $H$ whose intensity measure $\nu$ satisfies $S_{\nu} = H$, hence $\nu$ satisfies Assumption 2.4.

(2) Let $\{X_t, t \geq 0\}$ be a Lévy process introduced in Example 4.2. Corollary 2.1 implies that $S_{\mu_X}$ is dense in $H$. Then $L_t = X_{Z_t}$, $t \geq 0$ is a Lévy process on $H$, and its intensity measure $\nu$ satisfies Assumption 2.4.

**Example 4.4** Let $H = \mathbb{R}^d$, $d \in \mathbb{N}$. Assume that the intensity measure $\nu$ of the Lévy process is absolutely continuous with respect to the Lebesgue measure $dz$ on $\mathbb{R}^d$, that is, $\nu(dz) = q(z)dz$, for some measurable function $q : \mathbb{R}^d \to [0, \infty)$. Let $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Assume that $q$ is a continuous function and that $q(x) > 0$ for $x = e_i$, $i = 1, 2, ..., d$ and $x = -\sum_{j=1}^d e_i$. Then we can easily see that the intensity measure $\nu$ satisfies Assumption 2.4.

One can find other mild conditions on $q$ such that the intensity measure $\nu$ satisfies Assumption 2.4, even for the case that $q$ is not a continuous function.

**Example 4.5** Let $H = \mathbb{R}^d$, $d \in \mathbb{N}$. Assume that the intensity measure $\nu$ of the Lévy process is defined in polar coordinates by

$$
\nu(B) = \int_{(0, \infty)} \int_{S^{d-1}} 1_B(ru)M(dr, du), \quad B \in \mathcal{B}(\mathbb{R}^d).
$$

Here $M$ is of the form

$$
M(dr, du) = \frac{q(r, u)}{r^{1+\alpha}}drdu, \quad (r, u) \in (0, \infty) \times S^{d-1},
$$

where $\alpha \geq 0$, $\sigma$ is a finite measure on the unit sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ and $q : (0, \infty) \times S^{d-1} \to [0, \infty)$ is a Borel function satisfying

$$
\int_{\mathbb{R}^d} |x|^2 \land 1 \nu(dx) = \int_{(0, \infty)} \int_{S^{d-1}} |ru|^2 \land 1 M(dr, du) < \infty.
$$

When $\alpha \in (0, 2)$, the corresponding Lévy processes are called tempered $\alpha$-stable processes; see, e.g., [30, 33]. A special case is the $d$-dimensional $\alpha$-stable-like Lévy processes; see, e.g., [6], in which the authors established the well-posedness of singular SDEs driven by $d$-dimensional non-degenerate $\alpha$-stable-like process with $\alpha \in (0, 2)$.

Now we give a sufficient condition to guarantee that $\nu$ satisfies Assumption 2.4.

Let $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Assume that

- there exist $n \in \mathbb{N}$, $f_1, f_2, ..., f_n \in S^{d-1}$ such that for any $i = 1, 2, ..., d$ and $j = 1, 2, ..., n$, there exist $a_{ij} \geq 0$ and $b_{ij} \geq 0$ such that $e_i = \sum_{j=1}^n a_{ij}f_j$ and $-e_i = \sum_{j=1}^n b_{ij}f_j$;
- $\{f_1, f_2, ..., f_n\} \subseteq \mathcal{S}_{\sigma}$;
- for any $f_i, i = 1, 2, ..., n$, $q(r, f_i) : (0, \infty) \to [0, \infty)$ is continuous with respect to $r$;
- for any $i = 1, 2, ..., d$ and $j = 1, 2, ..., n$, if $a_{ij} > 0$, then $q(a_{ij}^2, f_j) > 0$;
Then \( \nu \) satisfies Assumption 2.4.

This can be seen as follows: under the assumptions above, there exists \( \epsilon > 0 \) such that, for any \( i = 1, 2, \ldots, d \) and \( j = 1, 2, \ldots, n \), if \( a_j^i > 0 \), then \( \{l_{f_j}, 0 < a_j^i - \epsilon \leq l \leq a_j^i + \epsilon\} \subseteq S_\nu; \) if \( b_j^i > 0 \), then \( \{l_{f_j}, 0 < b_j^i - \epsilon \leq l \leq b_j^i + \epsilon\} \subseteq S_\nu. \)

An interested reader would have little difficulties to find other conditions on \( q, \) in particular, without the continuity of \( q \) required as above, such that \( \nu \) satisfies satisfying Assumption 2.4.

**Remark 4.3** Let \( H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\} \). The driving Lévy processes on \( H \) satisfying Assumption 2.4 include not only a large class of compound Poisson processes, but also a large class of heavy-tailed Lévy processes, even the processes who’s intensity measures \( \nu \) satisfy for any small \( \alpha > 0 \), \( \int_{\|z\|_H > 1} \|z\|_H^\alpha \nu(dz) = \infty. \)

### 4.2 Examples for Assumption 2.3

A simple example is as follows.

**Example 4.6** Let \( H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\} \). Assume that the noise term in (2.1) has the form:

\[
\int_{Z_1} \sigma(X(t-), z)\tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t-), z)N(dz, dt) = \int_{Z_1} \sigma_1(X(t-), z)\tilde{N}_1(dz, dt) + \int_{Z_1} \sigma_1(X(t-), z)N_1(dz, dt) + dL(t),
\]

here \( L \) is a Lévy process on \( H \) satisfying Assumption 2.4 (see the examples in Subsection 4.1), \( N_1 \) can be any Poisson random measure on \( Z \), and \( L \) and \( N_1 \) are independent. Then Assumption 2.3 holds.

**Example 4.7** Let \( H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\} \). Recall \( L = \{L_t = W_{Z_t}, t \geq 0\} \) in Example 4.3, where \( W = \{W_t, t \geq 0\} \) is a Q-Wiener process on \( H, Q \in L(H) \) is nonnegative, symmetric, with finite trace and \( \text{Ker}Q = \{0\} \), \( Z = \{Z_t, t \geq 0\} \) is a subordinator with Lévy measure \( \rho \) satisfying \( \int_{(0,\infty)} (1 \wedge s)\rho(ds) > 0 \), and \( W \) and \( Z \) are independent. The intensity measure of \( L \) is denoted by \( \nu \), which satisfies \( S_\nu = H \). Denote by \( N \) the Poisson random measure corresponding to \( L \), and \( \tilde{N} \) the associated compensated Poisson random measure. Then

\[
L_t = \int_0^t \int_{0 < \|z\|_H \leq 1} zd\tilde{N}(dz, ds) + \int_0^t \int_{\|z\|_H > 1} zdN(dz, ds), t \geq 0.
\]

Denote by \( L_2(Q^{1/2}(H), H) \) the space of all Hilbert-Schmidt operators from \( Q^{1/2}(H) \) to \( H \) equipped with the Hilbert-Schmidt norm. Assume that \( \sigma : H \to L_2(Q^{1/2}(H), H) \) is continuous, and, for any \( h \in H, \text{Ker}\sigma(h) = 0. \) Then the driving noise

\[
\int_0^t \sigma(X(s-))dL_s
\]
there exists

Let \( s \) satisfies Assumption 2.3. Here, \( \lambda \) of \( \mu \) with intensity measure \( \nu \). Let \( x \) of \( \beta \) for any \( (D2) \sigma \) \( \mathbb{P} \). Assume that

\[
\|h - P_N h\|_H \leq \frac{n}{8} \text{ and } \|y - P_N y\|_H \leq \frac{n}{8}.
\]

(4.5)

Here, \( \{e_1, e_2, ..., e_d\} \) is an orthonormal basis of \( H \), and for any \( x \in H \), \( P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle_H e_i \).

Let \( x_N := P_N(y - h) \), \( l = \sigma(h)^{-1} x_N \) and \( q = h + \sigma(h) l = h - P_N h + P_N y \). Then \( \|q - y\|_H \leq \frac{n}{4} \). By the continuity of \( \sigma \) and \( S_\nu \), it is easy to see that Assumption 2.3 holds with \( n = 1 \).

Example 4.8 Let \( H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\} \), and \( \{e_i, i = 1, 2, ..., d\} \) be an orthonormal basis of \( H \). Let \( \{L_i = \{L_i(t), t \geq 0\}, i \in \mathbb{N}\} \) be a sequence of i.i.d. one dimensional Lévy processes with intensity measure \( \mu \) on some filtered probability space \( (\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P}) \). Assume that

\((C1)\) there exists \( c^+_n > 0, c^-_n < 0, n \in \mathbb{N} \) such that \( \lim_{n \to \infty} c^-_n = \lim_{n \to \infty} c^+_n = 0 \) and \( \{c^+_n, c^-_n, n \in \mathbb{N}\} \subseteq S_\mu \).

\((C2)\) there exists \( \kappa_1 > \kappa_2 > 0 \) and \( \sigma_i : H \to \mathbb{R}, i \in \mathbb{N} \) such that, for any \( i \in \mathbb{N} \),

\((D1)\) \( \sigma_i : H \to \mathbb{R} \) is continuous,

\((D2)\) for any \( h \in H \), \( \kappa_2 < |\sigma_i(h)| < \kappa_1 \).

\((C3)\) Let \( \beta_i \in \mathbb{R} \setminus \{0\}, i \in \mathbb{N} \) be given constants such that

\[
\sum_{i=1}^{d} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \mu(dx_i) < \infty.
\]

Denote by \( N_i \) the Poisson random measure corresponding to \( L_i \), and \( \tilde{N}_i \) the associated compensated Poisson random measure. Suppose that the noise term in (2.4) is of the form:

\[
\int_{0}^{t} \int_{z_i} \sigma(X(s-), z) \tilde{N}(dz, ds) + \int_{0}^{t} \int_{z_i} \sigma(X(s-), z) N(dz, ds)
\]

\[
= \sum_{i=1}^{d} \int_{0}^{t} \beta_i \sigma_i(X(s-)) dL_i(s) e_i
\]

\[
= \sum_{i=1}^{d} \left( \int_{0}^{t} \int_{0<|z_i| \leq 1} \beta_i \sigma_i(X(s-)) z_i \tilde{N}_i(dz_i, ds) e_i + \int_{0}^{t} \int_{|z_i| > 1} \beta_i \sigma_i(X(s-)) z_i N_i(dz_i, ds) e_i \right).
\]

Then Assumption 2.3 holds.

Proof For any \( h, y \in H \) with \( h \neq y \) and \( \eta > 0 \), there exists \( N \in \mathbb{N} \) such that

\[
\|h - P_N h\|_H \leq \frac{n}{8} \text{ and } \|y - P_N y\|_H \leq \frac{n}{8}.
\]

(4.6)
Here, for any \( x \in H \), \( P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle H e_i \).

Let \( \delta = \frac{\eta}{4N} \). There exists \( n_0 \in \mathbb{N} \) such that

\[
\max\{ |\kappa_2 \beta_1 c_{n_0}^+ |, |\kappa_1 \beta_1 c_{n_0}^+ |, |\kappa_2 \beta_1 c_{n_0}^- |, |\kappa_1 \beta_1 c_{n_0}^- |, i = 1, 2, ..., N \} \leq \frac{\delta}{2}.
\]

(4.7)

It is easy to see that

\[
\min\{ |\kappa_2 \beta_1 c_{n_0}^+ |, |\kappa_1 \beta_1 c_{n_0}^+ |, |\kappa_2 \beta_1 c_{n_0}^- |, |\kappa_1 \beta_1 c_{n_0}^- |, i = 1, 2, ..., N \} > 0.
\]

(4.8)

Set \( q_0 = h \), and for \( i = 1, 2, ..., N \), \( x_i = \langle y, e_i \rangle H - \langle h, e_i \rangle H \).

If \( x_1 = 0 \), then turn to \( x_2 \). If \( x_1 \neq 0 \), for \( j \in \mathbb{N} \), let \( sgn_j : = sgn(\sigma_1(q_{j-1}) \beta_1) sgn(x_1) \), \( q_j = q_{j-1} + \sigma_1(q_{j-1}) \beta_1 c_{n_0}^{sgn_j-1} e_1 \). By (4.7) and (4.8), there exists \( m_1 \in \mathbb{N} \) such that \( \| y, e_1 \rangle H - \langle q_{m_1}, e_1 \rangle H | \leq \frac{\delta}{2} \). Now we have obtained \( \{ q_0, q_1, ..., q_{m_1} \} \) and \( \{ l_j = c_{n_0}^{sgn_j-1} e_1, j = 1, 2, ..., m_1 \} \).

Turn to \( x_2 \). If \( x_2 = 0 \), then turn to \( x_3 \). If \( x_2 \neq 0 \), for \( j \in \mathbb{N} \), let \( sgn_{m_1+j-1} = sgn(\sigma_2(q_{m_1+j-1}) \beta_2) sgn(x_2) \), \( q_{m_1+j} = q_{m_1+j-1} + \sigma_2(q_{m_1+j-1}) \beta_2 c_{n_0}^{sgn_{m_1+j-1}} e_2 \). By (4.7) and (4.8), there exists \( m_2 \in \mathbb{N} \) such that \( \| y, e_2 \rangle H - \langle q_{m_1+m_2}, e_2 \rangle H | \leq \frac{\delta}{2} \). Now we have obtained

\[
\{ q_0, q_1, ..., q_{m_1}, q_{m_1+1}, ..., q_{m_1+m_2} \}
\]

and

\[
\{ l_j = c_{n_0}^{sgn_j-1} e_1, j = 1, 2, ..., m_1 \} \cup \{ l_{m_1+j} = c_{n_0}^{sgn_{m_1+j-1}} e_2, j = 1, 2, ..., m_2 \}.
\]

Recursively, we obtain \( \{ q_0, q_1, ..., q_{m_1}, q_{m_1+1}, ..., q_{m_1+m_2}, q_{m_1+m_2+1}, ..., q_{m_1+m_2+...+m_N} \} \) and

\[
\{ l_j = c_{n_0}^{sgn_j-1} e_1, j = 1, 2, ..., m_1 \} \cup \{ l_{m_1+j} = c_{n_0}^{sgn_{m_1+j-1}} e_2, j = 1, 2, ..., m_2 \} \cup ... \{ l_{m_1+m_2+...+m_N, j} = c_{n_0}^{sgn_{m_1+m_2+...+m_N-1+j-1}} e_N, j = 1, 2, ..., m_N \}.
\]

The construction of \( q_{m_1+m_2+...+m_N} \) implies that

\[
\begin{align*}
\| q_{m_1+m_2+...+m_N} - y \|_H & \leq \| q_{m_1+m_2+...+m_N} - P_N y - (h - P_N h) \|_H + \| P_N y - y \|_H + \| P_N h - h \|_H \\
& \leq \sum_{i=1}^{N} \| \langle y, e_i \rangle H - \langle q_{m_1+m_2+...+m_N}, e_i \rangle H \| + \frac{\eta}{4} \\
& = \sum_{i=1}^{N} \| \langle y, e_i \rangle H - \langle q_{m_1+m_2+...+m_i}, e_i \rangle H \| + \frac{\eta}{4} \\
& \leq N \cdot \frac{\delta}{2} + \frac{\eta}{4} \\
& = \frac{3\eta}{8}.
\end{align*}
\]

Combining with the fact that \( \sigma_1 \) is continuous, we see that Assumption 2.3 holds.

Using similar (but more involved) arguments as above, we have the following claim.

Assume that (C1)–(C3) hold with (D2) replaced by

\[
(D2') \text{ for any } h \in H, \kappa_2 |\sigma_1(h)| < \kappa_1 (1 + \| h \|_H).
\]

Then Assumption 2.3 holds.
4.3 Locally Monotone SPDEs

Under the general framework as in [40], we will obtain the irreducibility for a large class of coercive and local monotone SPDEs driven by pure jump noise.

Our results in this subsection are applicable to SPDEs such as stochastic reaction–diffusion equations, stochastic semilinear evolution equation, stochastic porous medium equation, stochastic $p$-Laplace equation, stochastic Burgers type equations, stochastic 2D Navier-Stokes equation, stochastic magneto-hydrodynamic equations, stochastic Boussinesq model for the Bénard convection, stochastic 2D magnetic Bénard problem, stochastic 3D Leray-$\alpha$ model, stochastic equations of non-Newtonian fluids, several stochastic Shell Models of turbulence, and many other stochastic 2D Hydrodynamical systems.

Recall that we consider the following SPDEs.

\[ dX(t) = A(X(t))dt + \int_{Z_1} \sigma(X(t^-), z) \tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t^-), z) N(dz, dt), \quad (4.9) \]
\[ X(0) = x. \]

Let us formulate the assumptions on the coefficients $A$ and $\sigma$. Suppose that there exist constants $\alpha > 1$, $\beta \geq 0$, $\theta > 0$, $C > 0$, $F > 0$ and a measurable (bounded on balls) function $\rho : V \to [0, +\infty)$ such that the following conditions hold for all $v, v_1, v_2 \in V$:

(H1) (Hemicontinuity) The map $s \mapsto \langle A(v_1 + sv_2), v \rangle_V$ is continuous on $\mathbb{R}$,

(H2) (Local monotonicity)

\[
2\langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V + \int_{Z_1} \|\sigma(v_1, z) - \sigma(v_2, z)\|^2 H\nu(dz) \\
\leq (C + \rho(v_2))\|v_1 - v_2\|^2_H.
\]

(H3) (Coercivity)

\[
2\langle A(v), v \rangle_V + \theta\|v\|_V^2 \leq F + C\|v\|_H^2,
\]

(H4) (Growth)

\[
\|A(v)\|_{\frac{1}{\alpha} V^*} \leq (F + C\|v\|_V^2)\left(1 + \|v\|_H^2\right).
\]

The following well-posedness was proved in [40] Theorem 1.2.

**Lemma 4.1** Suppose that conditions (H1)–(H4) hold, and there exists a constant $\gamma < \frac{\theta}{2\beta}$ such that for all $v \in V$

\[
\int_{Z_1} \|\sigma(v, z)\|^2_H \nu(dz) \leq F + C\|v\|_H^2 + \gamma\|v\|_V^2;
\]

\[
\int_{Z_1} \|\sigma(v, z)\|^2 H\nu(dz) \leq F\frac{\beta + 2}{\beta} + C\|v\|_H^{\beta + 2};
\]

\[X(0) = x.\]
\[
\rho(v) \leq C(1 + \|v\|^\alpha_V)(1 + \|v\|^\beta_H).
\]

Then for any \( x \in H \), \((4.9)\) has a unique solution \( X^x = (X^x(t), t \geq 0) \).

Now, we state the main result in this subsection.

**Proposition 4.1** Under the same assumptions of Lemma 4.1, assume that for any fixed \( z \in Z_1 \), \( \sigma(\cdot, z) : H \rightarrow H \) is continuous, and that the driving noise term satisfies Assumption 2.3 then the solution \( \{X^x, x \in H\} \) to equation \((4.9)\) is irreducible in \( H \).

**Proof** By Lemma 4.1 and the fact that for any fixed \( z \in Z_1 \), \( \sigma(\cdot, z) : H \rightarrow H \) is continuous, it is classical that \( \{X^x, x \in H\} \) forms a strong Markov process on \( H \). Therefore, Assumption 2.1 is satisfied.

Applying Theorem 2.1 we see that the proof of this proposition will be complete once we prove that Assumption 2.2 is satisfied, which we will do in the rest of the proof.

The proof is divided into two steps.

**Step 1.** Consider (2.3) with \( m = 1 \), that is
\[
dX_1(t) = A(X_1(t))dt + \int_{Z_1} \sigma(X_1(t-), z) \tilde{N}(dz, dt),
\]
\[
X_1(0) = x.
\]

By [10, Theorem 1.2], for any \( x \in H \), \((4.10)\) has a unique solution \( X^x_1 = (X^x_1(t), t \geq 0) \).

For any \( h, \tilde{h} \in H \), applying the Itô formula, we have
\[
e^{-\int_0^t (C + \rho(X^x_1(s)))ds} \|X_1^\tilde{h}(t) - X_1^h(t)\|^2_H - \|\tilde{h} - h\|^2_H \\
\leq \int_0^t e^{-\int_s^t (C + \rho(X^x_1(r)))dr} \left( 2V \cdot (A(X_1^\tilde{h}(s)) - A(X_1^h(s)), X_1^{\tilde{h}}(s) - X_1^h(s))_V \\
- (C + \rho(X_1^h(s)))\|X_1^\tilde{h}(s) - X_1^h(s)\|^2_H \right)ds \\
+ 2 \int_0^t \int_{Z_1} e^{-\int_s^t (C + \rho(X^x_1(r)))dr} \langle \sigma(X_1^{\tilde{h}}(s-), z) - \sigma(X_1^h(s-), z), X_1^{\tilde{h}}(s-) - X_1^h(s-) \rangle_H \tilde{N}(dz, ds) \\
+ \int_0^t \int_{Z_1} e^{-\int_s^t (C + \rho(X^x_1(r)))dr} \|\sigma(X_1^{\tilde{h}}(s-), z) - \sigma(X_1^h(s-), z)\|^2_H \tilde{N}(dz, ds) \\
\leq 2 \int_0^t \int_{Z_1} e^{-\int_s^t (C + \rho(X^x_1(r)))dr} \langle \sigma(X_1^{\tilde{h}}(s-), z) - \sigma(X_1^h(s-), z), X_1^{\tilde{h}}(s-) - X_1^h(s-) \rangle_H \tilde{N}(dz, ds) \\
+ \int_0^t \int_{Z_1} e^{-\int_s^t (C + \rho(X^x_1(r)))dr} \|\sigma(X_1^{\tilde{h}}(s-), z) - \sigma(X_1^h(s-), z)\|^2_H \tilde{N}(dz, ds).
\]

Assumption (H2) has been used for the last inequality. Applying stochastic Gronwall’s inequality, see [39, Lemma 3.7], we deduce that for any \( 0 < q < p < 1 \) and any \( T > 0 \),
\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{-\int_0^t (C + \rho(X^x_1(s)))ds} \|X_1^\tilde{h}(t) - X_1^h(t)\|^2_H \right)^q \right] \leq \left( \frac{p}{p - q} \right) \|\tilde{h} - h\|^{2q}_H.
\]

Define the stopping time
\[
r_1^{\tilde{h}} := \inf\{t \geq 0 : e^{-\int_0^t (C + \rho(X^x_1(s)))ds} \leq 1/2\}.
\]

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Therefore,

\[
\lim_{T \to 0} \mathbb{P}(\tau_1^h < T) = 0.
\] (4.12)

By Chebyshev’s inequality, for any \( \eta > 0 \),

\[
\mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_1^h(t) - X_1^h(t)\|_H > \frac{\eta}{2}\right)
\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_1^h(t) - X_1^h(t)\|_H > \frac{\eta}{2}, \tau_1^h \geq T\right) + \mathbb{P}(\tau_1^h < T)
\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{-\int_0^t (C + \rho(X_1^h(s)))ds}\|X_1^h(t) - X_1^h(t)\|_H^2 > \frac{\eta^2}{8}, \tau_1^h \geq T\right) + \mathbb{P}(\tau_1^h < T)
\leq \left(\frac{p}{p - q}\right)\|\hat{h} - h\|^2 H_\rho^2 \left(\frac{8}{\eta^2}\right)^q + \mathbb{P}(\tau_1^h < T).
\] (4.13)

Notice that

\[
\mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_1^h(t) - h\|_H > \eta\right)
\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_1^h(t) - h\|_H > \frac{\eta}{2}\right) + \mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_1^h(t) - X_1^h(t)\|_H > \frac{\eta}{2}\right),
\] (4.14)

and \( X_1^h \in D([0, \infty), H) \), \( \mathbb{P} \)-a.s. implies that

\[
\lim_{T \to 0} \mathbb{P}(\sup_{0 \leq t \leq T} \|X_1^h(t) - h\|_H > \frac{\eta}{2}) = 0.
\] (4.15)

Combining (4.12)-(4.15) together we see that there exist \( T_0 = T_0(h, \eta) > 0 \) and \( \epsilon_0 = \epsilon_0(h, \eta) > 0 \) small enough such that

\[
\sup_{\hat{h} \in B(h, \epsilon_0)} \mathbb{P}(\sup_{0 \leq t \leq T_0} \|X_1^h(t) - h\|_H > \eta) \leq \frac{1}{4}.
\] (4.16)

Therefore,

\[
\inf_{\hat{h} \in B(h, \epsilon_0)} \mathbb{P}(\sup_{0 \leq t \leq T_0} \|X_1^h(t) - h\|_H \leq \eta) \geq \frac{3}{4}.
\] (4.17)

**Step 2.** Recall \( \tau_1^1 = \inf\{t \geq 0 : N(Z_1, t) = 1\} \) as defined in (2.4). Notice that \( \{X^x(t), t \in [0, \tau_1^1]\} \) coincides with \( \{X_1^x(t), t \in [0, \tau_1^1]\} \). By the independence of \( X_1^x \) and \( \tau_1^1 \) (see Proposition 2.2), we have

\[
\inf_{\hat{h} \in B(h, \epsilon_0)} \mathbb{P}(\sup_{0 \leq t \leq T_0} \|X_1^h(t) - h\|_H \leq \eta)
\geq \inf_{\hat{h} \in B(h, \epsilon_0)} \mathbb{P}(\sup_{0 \leq t \leq T_0} \|X_1^h(t) - h\|_H \leq \eta, \tau_1^1 > 2T_0)
\geq \inf_{\hat{h} \in B(h, \epsilon_0)} \mathbb{P}(\sup_{0 \leq t \leq T_0} \|X_1^h(t) - h\|_H \leq \eta) \mathbb{P}(\tau_1^1 > 2T_0).
\]
\begin{align}
\geq \frac{3}{4} P(\tau_1 > 2T_0) > 0. \tag{4.18}
\end{align}

For the last inequality, we have used the fact that \( \tau_1 \) has the exponential distribution with parameter \( \nu(Z_1) < \infty \). (4.18) implies Assumption 2.2, completing the proof. \( \square \)

Now we turn to SPDEs driven by additive noise.

**Proposition 4.2** Consider the case of additive noise, i.e., \( \sigma(\cdot, z) \equiv z \); see (2.12). Under the same assumptions of Lemma 4.1 with (H2) replaced by

\[ 2V_* \langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V \leq 0, \tag{4.19} \]

and assume that the driving noise term satisfies Assumption 2.4. Then there exists at most one invariant measure to the solution of the equation (2.12).

**Proof** For any \( x, y \in H \), by (4.19) apply the Itô formula to \( \|X^x(t) - X^y(t)\|_H^2 \) to get

\[ E(\|X^x(t) - X^y(t)\|_H^2) \leq \|x - y\|_H^2, \ \forall t \geq 0. \]

Hence \( \{X^x, x \in H\} \) satisfies the so-called \( e \)-property; see [23, 24]. This together with the irreducibility given in Proposition 4.1 implies the uniqueness of the invariant measure (if it exists); see Theorem 2 in [23]. \( \square \)

We would like to point out again that the driving Lévy processes on \( H \) satisfying Assumption 2.4 could include a large class of compound Poisson processes and heavy-tailed Lévy processes etc.; see Remark 4.3. These driving Lévy processes could be used in the application of Proposition 4.2.

As the application of Proposition 4.2 we can obtain the uniqueness of invariant measures of stochastic evolution equations with weakly dissipative drifts such as stochastic fast diffusion equations and singular stochastic \( p \)-Laplace equations. It seems quite difficult to get these results with other means due to the lack of strong dissipativity of the equations. Here are the examples.

**Example 4.9** Let \( \Lambda \) be an open (possibly unbounded) domain in \( \mathbb{R}^d, d \in \mathbb{N} \), with smooth boundary. Consider the following Gelfand triple

\[ V := W^{1,p}_0(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W^{1,p}_0(\Lambda))^* \]

and the stochastic \( p \)-Laplace equation

\[ dX(t) = [\text{div}(|\nabla X(t)|^{p-2}\nabla X(t))]dt + dL(t), \quad X(0) = x \in H, \tag{4.20} \]

where \( p \in (1, \infty) \). If the intensity measure of \( L(t) \) satisfies Assumption 2.7, applying Proposition 4.3 we conclude that there exists at most one invariant measure to (4.20). We stress that this example covers the singular case, i.e., \( p \in (1, 2) \), and the case that \( \Lambda \) is an unbounded domain.
Example 4.10 Let Λ be an open (possibly unbounded) domain in \( \mathbb{R}^d \), \( d \in \mathbb{N} \), with smooth boundary. Consider the following Gelfand triple

\[ V := L^{r+1}(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq (L^{r+1}(\Lambda))^* \]

We consider the stochastic fast diffusion equation

\[
\begin{aligned}
dX(t) &= \Delta(|X(t)|^{r-1}X(t))dt + dL(t), \quad X(0) = x \in H, \\
X(t, \xi) &= 0, \quad \forall \xi \in \partial \Lambda,
\end{aligned}
\]

where \( r \in (0, \infty) \). If the intensity measure of \( L(t) \) satisfies Assumption 2.4, applying Proposition 4.2 we again conclude that there exists at most one invariant measure to (4.21).

Notice that when \( r \in (0, 1) \), (4.21) is called stochastic fast diffusion equation, and when \( r \geq 1 \), (4.21) is the stochastic version of classical porous medium equation.

In [13] and [2], the authors studied irreducibility and exponential mixing of some stochastic Hydrodynamical systems driven by cylindrical pure jump process. Applying Proposition 4.1 in this paper, one is able to significantly improve the main results (including exponential mixing) in [13] and [2].

4.4 Nonlinear Schrödinger equations

The nonlinear Schrödinger equation (NLS) is a fundamental model describing wave propagation that appears in various fields such as nonlinear optics, nonlinear water propagation, quantum physics, Bose-Einstein condensate, plasma physics and molecular biology.

In this subsection, we study the irreducibility of stochastic NLS driven by additive Lévy noise. Without further notice, all the \( L^p \) spaces in this subsection are referred as spaces of complex-values functions.

Consider (2.12) with \( H = L^2(\mathbb{R}^d) \), \( d \in \mathbb{N} \), \( \mathcal{A}(u) = i[\Delta u - \lambda|u|^{\alpha-1}u] \), where \( \lambda \in \{-1, 1\} \), and \( 1 < \alpha < 1 + \frac{4}{d} \). Now consider NLS with additive noise, that is,

\[
\begin{aligned}
dX(t) &= \mathcal{A}(X(t))dt + \int_{0<\|z\|_H\leq 1} z\tilde{N}(dz, dt) + \int_{\|z\|_H>1} zN(dz, dt), \\
X(0) &= x.
\end{aligned}
\]

We say a pair \((p, r)\) is admissible if \( p, r \in [2, \infty) \) and \((p, r, d) \neq (2, \infty, 2)\) satisfying \( \frac{2}{p} + \frac{d}{r} = \frac{d}{2} \). The following result provides the existence and uniqueness of the solution of the stochastic NLS (4.22) whose proof was given in [35].

**Proposition 4.3** Let \( p \geq 2, \ 1 < \alpha < 1 + \frac{4}{d}, \ r = \alpha + 1 \) such that \((p, r)\) is an admissible pair. For any \( h \in H \), there exists a unique global mild solution \( X^h = (X^h(t), t \geq 0) \) of (4.22) satisfying

\[
X^h \in D([0, \infty); H) \cap L^p_{\text{loc}}(0, \infty; L^r(\mathbb{R}^d)), \ \mathbb{P}-\text{a.s.}
\]
Proposition 4.4 If the driving noise satisfies Assumption 2.4, then the solution \{X^h, x \in H\} of (4.22) is irreducible in H.

Proof It is classical that the solution \{X^h, h \in H\} of (4.22) forms a strong Markov process on H. Therefore, Assumptions 2.1 holds. To apply Theorem 2.2 to conclude the proof, it only remains to show that Assumption 2.2 holds. From the proof of Proposition 4.1 we see that Assumption 2.2 is implied by the following results.

For any \(h \in H\) and \(\eta > 0\), there exist \(T_0 = T_0(h, \eta) > 0\) and \(\epsilon_0 = \epsilon_0(h, \eta) > 0\) small enough such that

\[
\sup_{\eta > 0} \mathbb{P} \left( \sup_{0 \leq t \leq T_0} \|X^h_1(t) - X^{\tilde{h}}_1(t)\|_H > \frac{\eta}{2} \right) < \frac{1}{2},
\]

(4.23)

where \(X^h_1\) is the mild solution of the following equation:

\[
dX^h_1(t) = A(X^h_1(t))dt + \int_{0 < \|z\|_H \leq 1} z\tilde{N}(dz, dt),
\]

(4.24)

\[X^h_1(0) = \tilde{h}.\]

We now prove (4.23). For \(t > 0\), set

\[Y_t := L^\infty(0, t; H) \cap L^p(0, t; L^r(\mathbb{R}^d)),\]

(4.25)

and for \(u \in Y_t\),

\[\|u\|_{Y_t} := \sup_{s \in [0, t]} \|u(s)\|_H + \left( \int_0^t \|u(s)\|_{L^r(\mathbb{R}^d)}^p ds \right)^{\frac{1}{p}}.\]

(4.26)

Let \(\theta : \mathbb{R}_+ \to [0, 1]\) be a non-increasing \(C_0^\infty\) function such that \(1_{[0, 1]} \leq \theta \leq 1_{[0, 2]}\) and \(\inf_{x \in \mathbb{R}_+} \theta'(x) \geq -2\). For the fixed \(h \in H\), set \(R = \|h\|_H + 2\) and \(\theta_R(\cdot) = \theta(\frac{\cdot}{R})\).

For any \(h \in H\), let \(Z^h_R\) be the solution of the truncated stochastic Schrödinger equation:

\[
dZ^h_R(t) = i[\Delta Z^h_R(t) - \lambda \theta_R(\|Z^h_R\|_H)]Z^h_R(t)|^{a-1}Z^h_R(t)dt + \int_{0 < \|z\|_H \leq 1} z\tilde{N}(dz, dt),
\]

(4.27)

\[Z^h_R(0) = \tilde{h}.\]

By Propositions 2.2 and 3.1, for any \(T > 0\), we have

\[
\|Z^h_R - Z^\tilde{h}_R\|_{Y_T} \leq C\|h - \tilde{h}\|_H + CT^{1-\frac{(a-1)d}{4}} R^{a-1} \|Z^h_R - Z^\tilde{h}_R\|_{Y_T}.
\]

(4.28)

For any \(\eta > 0\), we can choose \(\tilde{\epsilon} \in (0, \frac{\eta}{4} \land 1]\) and \(\tilde{T} > 0\) small enough such that

\[2C\tilde{\epsilon} \leq \frac{1}{2} \land \frac{\eta}{4}\]

and

\[CT^{1-\frac{(a-1)d}{4}} R^{a-1} \leq \frac{1}{2}.\]

Then, for any \(\tilde{h} \in B(h, \tilde{\epsilon})\),

\[
\|Z^h_R - Z^\tilde{h}_R\|_{Y_{\tilde{T}}} \leq 2C\|h - \tilde{h}\|_H \leq \frac{1}{2} \land \frac{\eta}{4}.
\]

(4.29)
here, for any $\epsilon > 0$, $B(h, \epsilon) = \{ h \in H : \| h - h \|_H < \epsilon \}.$

Define $\tau = \inf \{ s \geq 0 : \| Z^h(t) \|_Y > \| h \|_H + 1/2 \}$. Then $\mathbb{P}(\tau > 0) = 1$, which implies that there exists $T_0 \in (0, \tilde{T}]$ such that

$$\mathbb{P}(\tau > T_0) \geq 11/12.$$  \hfill (4.30)

Combining this with (4.29), for any $\tilde{h} \in B(h, \tilde{\epsilon})$,

$$||Z^h_R||_{Y_{T_0}} \leq ||h||_H + 1 < R \text{ on } \{ \tau > T_0 \} \text{ P-a.s..}$$  \hfill (4.31)

Let us define $\tau^h_R = \inf \{ s \geq 0 : \| Z^h(t) \|_Y > R \}$. Then, for any $\tilde{h} \in B(h, \tilde{\epsilon})$,

$$\mathbb{P}(\tau^h_R > 0) = 1 \text{ and } Z^h_R(t) = X^h(t) \text{ on } t \in [0, \tau^h_R) \text{ P-a.s..}$$  \hfill (4.32)

Note that (4.31) implies that, for any $\tilde{h} \in B(h, \tilde{\epsilon})$,

$$\mathbb{P}(\tau \land \tau^h_R > T_0) = \mathbb{P}(\tau > T_0).$$  \hfill (4.33)

Combining (4.29)–(4.33) together, we deduce that, for any $\tilde{h} \in B(h, \tilde{\epsilon})$,

$$\mathbb{P} \left( \sup_{0 \leq t \leq T_0} \| X^h(t) - X^h(t) \|_H > \frac{\eta}{2} \right) \leq \mathbb{P} \left( \sup_{0 \leq t \leq T_0} \| Z^h_R(t) - Z^h_R(t) \|_H > \frac{\eta}{2} \land \tau^h_R > T_0 \right) + \mathbb{P}(\tau \land \tau^h_R \leq T_0)$$

$$= \mathbb{P}(\tau \leq T_0) \leq \frac{1}{12}.$$  \hfill (4.34)

This completes the proof. \hfill \square

4.5 Euler equations

The Euler equations are the classical model for the motion of an inviscid, incompressible, homogeneous fluid. In this subsection, we assume that the intensity measure of the driving noise has finite mass, i.e., $\nu(H) < \infty$, which includes important impulsive random processes often found in neural and financial engineering models.

We consider the stochastic Euler equation on $\mathbb{R}^2$:

$$du^h(t) + (u^h(t) \cdot \nabla)u^h(t) = -\nabla p^h(t) + \int_H zN(dz, dt),$$

$$\text{div } u^h(t) = 0, \quad u^h(0) = h,$$  \hfill (4.35)

where $p^h(t)$ is the scalar pressure. Let $k > 2$ be an integer and $H = H^k(\mathbb{R}^2)$.

**Proposition 4.5** Let $h \in H$. Then, there exists a unique solution $u^h \in D(0, \infty); H)$ of the equation (4.35), and $\{u^h, h \in H\}$ forms a strong Markov process on $H$. If the driving noise satisfies Assumption 2.4, then $\{u^h, h \in H\}$ is irreducible in $H$. 37
Proof: It is known from [4,5] that the deterministic Euler equation with no noise admits a unique $H$-continuous solution for every initial data in $H$. As we assume $\nu(H) < \infty$, the driving Lévy process has a finite number of jumps on every finite time interval. The solution of the Euler equation (4.35) can be uniquely constructed by piecing together the solutions of the deterministic Euler equations between jumping times. It is a classical fact that the solutions $\{u^\epsilon, h \in H\}$ forms a strong Markov process on $H$. Therefore, Assumptions 2.1 is satisfied.

Applying Theorem 2.2, we see that the proof is complete once we verify Assumption 2.2.

Define $\tau = \inf\{t \geq 0 : N(H,t) = 1\}$. We have $\mathbb{P}(\tau \leq T) = 1 - e^{-\nu(H)T}, T \geq 0$. As in the proof of Proposition 4.1 to show that Assumption 2.2 holds, we only need to prove the following result.

For any $h \in H$ and $\eta \in (0,1)$, there exist $T_0 = T_0(h,\eta) > 0$ and $\epsilon_0 = \epsilon_0(h,\eta) > 0$ small enough such that

$$\sup_{\tilde{h} \in B(h,\epsilon_0)} \sup_{0 \leq t \leq T_0} \|u^\epsilon_t(t) - u^\tilde{h}_t(t)\|_H < \frac{\eta}{2}, \quad (4.36)$$

where for any $h \in H$, $u^\epsilon_t(t) = u^\epsilon_t(t)1_{[0,\tau)}(t)$.

Now we prove (4.36).

Consider the Naiver-Stokes equation with viscosity coefficients $\epsilon > 0$,

$$du^\epsilon_t(t) + (u^\epsilon_t(t) \cdot \nabla)u^\epsilon_t(t) = \epsilon \Delta u^\epsilon_t(t) - \nabla p^\epsilon_t(t),$$

$$\text{div } u^\epsilon_t(t) = 0, \quad u^\epsilon_t(0) = h. \quad (4.37)$$

From the proof of [4, Theorem 1.2] or [5], we see that

$$\sup_{\tilde{h} \in B(h,1)} \sup_{0 < \epsilon < 1} \sup_{0 \leq t \leq 1} \|u^\epsilon_t(t) - u^\tilde{h}_t(t)\|_{H^k} \leq C(k,h) < \infty, \quad (4.38)$$

$$\sup_{0 \leq t \leq 1} \|u^\epsilon_t(t) - u^h_t(t)\|_{H^s} \leq C(h,s,k)\epsilon \frac{k}{s}, \quad \text{for } 0 \leq s < k. \quad (4.39)$$

Therefore, we have

$$\sup_{0 \leq t \leq 1} \|u^h_t(t) - u^\tilde{h}_t(t)\|_{H^k} \leq \liminf_{\epsilon \to 0} \sup_{0 \leq t \leq 1} \|u^\epsilon_t(t) - u^\tilde{h}_t(t)\|_{H^k}. \quad (4.40)$$

As a result, we need to estimate the right side of the above inequality. By the chain rule,

$$\|u^\epsilon_t(t) - u^\tilde{h}_t(t)\|_{H^k}^2 + \epsilon \int_0^t \|u^\epsilon_l(l) - u^\tilde{h}_l(l)\|_{H^{k+1}}^2 dl \leq \|	ilde{h} - h\|_{H^k}^2 + \int_0^t \langle (u^\epsilon_l(l) \cdot \nabla)u^\epsilon_l(l) - (u^\tilde{h}_l(l) \cdot \nabla)u^\tilde{h}_l(l), u^\epsilon_l(l) - u^\tilde{h}_l(l) \rangle_{H^k} dl. \quad (4.41)$$

Recall the following estimates in [32] (see Lemmas 1.2 and 1.3 there).
(i) For $s > 2$, $H^s(\mathbb{R}^2)$ is an algebra for the pointwise multiplication and
\[
\|uv\|_{H^s} \leq c(s)(\|u\|_{H^s}\|v\|_{H^{s-1}} + \|u\|_{H^{s-1}}\|v\|_{H^s}),
\] (4.42)
for any $u, v \in H^s(\mathbb{R}^2)$.

(ii) Let $s > 2$ be an integer. Then for any $u, v \in H^s(\mathbb{R}^2)$
\[
\|A^s(uv) - uA^s v\|_{L^2} \leq c(s)\|u\|_{H^s}\|v\|_{H^{s-1}},
\] (4.43)
where $A^s = \Delta^\frac{s}{2}$.

Thus, for any $u, v \in H^{k+1}$ with $\text{div}\, u = \text{div}\, v = 0$, applying (4.42) and (4.43) we get
\[
|\langle A^k((u \cdot \nabla)u - (v \cdot \nabla)v), A^k(u - v)\rangle_{L^2}|
\leq |\langle A^k((u \cdot \nabla)(u - v)) - (u \cdot \nabla)A^k(u - v), A^k(u - v)\rangle_{L^2}|
+ |\langle A^k(((u - v) \cdot \nabla)v) - ((u - v) \cdot \nabla)A^k v, A^k u\rangle_{L^2}|
\leq c(k)(\|u - v\|_{H^k}^2 + \|u - v\|_{H^k}\|v\|_{H^{k+1}}^2)
+c(k)(\|u\|_{H^k}\|u - v\|_{H^k}\|v\|_{H^{k+1}} + \|u - v\|_{H^{k+1}}\|v\|_{H^{k+1}})
\leq c(k)(\|u - v\|_{H^k}^2(\|u\|_{H^k} + 1) + c(k)(\|u\|_{H^k}\|v\|_{H^{k+1}}^2 + \|v\|_{H^{k+1}}^4),
\] (4.44)
and it is well-known that
\[
|\langle (u \cdot \nabla)u - (v \cdot \nabla)v, u - v \rangle_{L^2}| \leq \|u - v\|_{H^1}\|u - v\|_{L^2}\|u\|_{H^1} \leq \|u - v\|_{H^k}\|u\|_{H^k}.
\] (4.45)

Take $h' \in B(h, \frac{\eta}{16})$ such that $h' \in H^{k+1}$. From the proof of [4, Theorem 1.2], we see that
\[
\|u_i^{e,h'}(t)\|_{H^{k+1}} \leq C(k, h') < \infty.
\] For any $\epsilon \in (0, 1)$ and $\tilde{h} \in B(h, \epsilon)$, replace $h$ by $h'$ in (4.41) and combine with (4.38), (4.44)-(4.45) to obtain, for any $T \in [0, 1],
\[
\sup_{0 \leq t \leq T} \|u_i^{e,h}(t) - u_i^{e,h'}(t)\|_{H^k}^2
\leq e^{\epsilon^2} \int_0^T (\|u_i^{e,h'}(t)\|_{H^{k+1}}^2 + \|u_i^{e,h'}(t)\|_{H^{k+1}}^4) dt
\leq e^{\epsilon^2} \left(\epsilon + \frac{\eta}{16}\right)^2 + C(k, h, h') T
\leq \frac{\eta^2}{16},
\] (4.46)
if we choose $\epsilon = \epsilon_0 = \frac{\eta}{16}$ and $T = T_0$ small enough. Thus,
\[
\sup_{0 \leq t \leq T_0} \|u_i^{e,h}(t) - u_i^{e,h'}(t)\|_{H^k}^2
\leq 2 \sup_{0 \leq t \leq T_0} \|u_i^{e,h}(t) - u_i^{e,h'}(t)\|_{H^k}^2 + 2 \sup_{h \in B(h, \epsilon_0)} \sup_{0 \leq t \leq T_0} \|u_i^{e,h}(t) - u_i^{e,h'}(t)\|_{H^k}^2 \leq \frac{\eta^2}{4}.
\]
This together with (4.40) gives the desired inequality (4.36).
4.6 Singular SDEs

Let $L = (L_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$, $d \in \mathbb{N}$, and denote its intensity measure by $\nu$. To state the condition on $\nu$, for $\alpha \in (0, 2)$, denote by $\mathbb{L}_\alpha^\text{non}$ the space of all non-degenerate $\alpha$-stable Lévy measure $\nu^{(\alpha)}$; that is,

$$
\nu^{(\alpha)}(A) = \int_0^\infty \left( \int_{S^{d-1}} \frac{1_A(r\theta) \vartheta(d\theta)}{r^{1+\alpha}} \right) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),
$$

where $\vartheta$ is a finite measure over the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ with

$$
\inf_{\theta_0 \in S^{d-1}} \int_{S^{d-1}} |\theta_0 \cdot \theta| \vartheta(d\theta) > 0. \tag{4.47}
$$

For $R > 0$, denote by $B_R$ the closed ball in $\mathbb{R}^d$ centered at the origin with radius $R$. We assume that there are $\nu_1, \nu_2 \in \mathbb{L}_\alpha^\text{non}$, so that

$$
\nu_1(A) \leq \nu(A) \leq \nu_2(A) \quad \text{for} \quad A \in \mathcal{B}(B_1). \tag{4.48}
$$

In [6], the authors call Lévy processes with intensity measure satisfying (4.48) non-degenerate $\alpha$-stable-like Lévy process. The Lévy measure $\nu$ could be singular with respect to the Lebesgue measure on $\mathbb{R}^d$ and its support could be a proper subset of $\mathbb{R}^d$.

For a Borel measurable drift $b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and diffusion matrix $\sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, consider the following SDE

$$
dX_t = b(X_t)dt + \sigma(X_t)dL_t = b(X_t)dt + \int_{0<|z|\leq 1} \sigma(X_{t-})z\tilde{N}(dz, dt) + \int_{|z|>1} \sigma(X_{t-})zN(dz, dt). \tag{4.49}
$$

Here $N$ and $\tilde{N}$ are the Poisson random measure and compensated Poisson random measure associated with $L$, respectively. In a recent paper [6], the authors established the following well posedness of the SDE (4.49).

**Lemma 4.2** Assume that $\nu$ satisfies (4.48) with $\alpha \in (0, 2)$. Assume that there are constants $\beta \in (1 - \alpha/2, 1]$ and $\Lambda > 0$ so that for all $x, y, \xi \in \mathbb{R}^d$,

$$
|b(x)| \leq \Lambda and |b(x) - b(y)| \leq \Lambda|x - y|^{\beta}, \tag{4.50}
$$

$$
\Lambda^{-1}|\xi| \leq |\sigma(x)\xi| \leq \Lambda|\xi| and ||\sigma(x) - \sigma(y)|| \leq \Lambda|x - y|. \tag{4.51}
$$

Then, there is a unique strong solution $X^x = (X^x(t), t \geq 0)$ to (4.49) for any initial data $x \in \mathbb{R}^d$.

We are concerned with the irreducibility of the solutions $\{X^x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$. To obtain the irreducibility, we introduce the following conditions. Let $\{e_i\}_{i=1,2,\ldots,d}$ be an orthonormal basis of $\mathbb{R}^d$. We are concerned with the irreducibility of the solutions $\{X^x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$.
Proof of Proposition 4.6

We will apply Theorem 2.1 to get the irreducibility. First we verify Assumption 2.2.

Removing the big jumps in (4.49), consider the following SDE:

\[ dX_1(t) = b(X_1(t))dt + \int_{0<|z|\leq 1} \sigma(X_1(t-))z\tilde{N}(dz, dt), \quad X_1(0) = x. \]  

(4.52)

We will prove that for any \( h \in \mathbb{R}^d \) and \( \eta > 0 \), there exist \( T_0 = T_0(h, \eta) > 0 \) and \( \epsilon_0 = \epsilon_0(h, \eta) > 0 \) small enough such that

\[ \sup_{h \in B(h, \epsilon_0)} \mathbb{P} \left( \sup_{0 \leq t \leq T_0} |X_1^h(t) - X_1^h(t)| > \frac{\eta}{2} \right) < \frac{1}{2}. \]  

(4.53)

We now fix \( h \in \mathbb{R}^d \) and set \( R = |h| + 2 \). Let \( \theta : \mathbb{R}^+ \rightarrow [0, 1] \) be a non-increasing \( C_0^\infty \) function such that \( 1_{[0, R]} \leq \theta \leq 1_{[0, R+2]} \). Let \( b(x) = b(x)\theta(x) \). For any \( \zeta \in (0, 1] \), define

\[ \sigma_\zeta(x) = \begin{cases} 
\frac{\sigma(x + h)}{\theta(x)} & |x| \leq \zeta/2, \\
\frac{2(|x| - \zeta)}{\zeta} \sigma(x) + h & \zeta/2 < |x| \leq \zeta, \\
\sigma(h) & |x| > \zeta.
\end{cases} \]

For \( \zeta_0 \in (0, 1] \) small enough, by the proof of [6, Theorem 1.1] and applying [6, Theorem 4.1], for any \( h \in \mathbb{R}^d \), the following SDE admits a unique strong solution

\[ Y_t^h = h + \int_0^t b_\theta(Y_{s-}^h + h)ds + \int_0^t \int_{0<|z|\leq 1} \sigma_\zeta_0(Y_{s-}^h)z\tilde{N}(dz, dt), \quad t \geq 0. \]
Moreover, from the proof of [6, Theorem 4.1] we see that for any $T > 0$ there exists a constant $C_T > 0$ such that for any $h_1, h_2 \in \mathbb{R}^d$

$$
E\left( \sup_{t \in [0,T]} |Y_t^{h_1} - Y_t^{h_2}|^2 \right) \leq C_T |h_1 - h_2|^2.
$$

(4.54)

Define

$$
\tau^h = \inf\left\{ t > 0 : |Y_t^h| \geq \zeta_0/2 \right\}.
$$

Then for any $\tilde{h} \in B(h, \zeta_0/8)$,

$$
P(\tau^{\tilde{h}} > 0) = 1,
$$

(4.55)

$$
X_1^h(t) = h + Y_t^{\tilde{h}} - h \quad \text{on } t \in [0, \tau^{\tilde{h}}) \text{ P-a.s.},
$$

(4.56)

and

$$
X_1^h(t) - X_1^{\tilde{h}}(t) = Y_t^{0} - Y_t^{\tilde{h}} \quad \text{on } t \in [0, \tau^0 \wedge \tau^{\tilde{h}}) \text{ P-a.s.}
$$

(4.57)

Define $\tilde{\tau}^0 = \inf\{ t > 0 : |Y_t^0| \geq \zeta_0/16 \}$. Then $P(\tilde{\tau}^0 > 0) = 1$, which implies that there exists $T_0 \in (0, 1]$ such that

$$
P(\tilde{\tau}^0 > T_0) \geq 11/12.
$$

(4.58)

Combining the inequality above with (4.54) and the Chebyshev inequality,

$$
P\left( \sup_{t \in [0,T_0]} |Y_t^{\tilde{h}} - h| \leq \zeta_0/8 \right)
\geq P\left( \{ \sup_{t \in [0,T_0]} |Y_t^{\tilde{h}} - Y_t^0| \leq \zeta_0/16 \} \cap \{ \sup_{t \in [0,T_0]} |Y_t^0| \leq \zeta_0/16 \} \right)
\geq 1 - P\left( \sup_{t \in [0,T_0]} |Y_t^{\tilde{h}} - Y_t^0| > \zeta_0/16 \right) - P\left( \sup_{t \in [0,T_0]} |Y_t^0| > \zeta_0/16 \right)
\geq 1 - \frac{16^2}{\zeta_0^2} C_{T_0} |\tilde{h} - h|^2 - P(\tilde{\tau}^0 \leq T_0)
\geq 11/12 - \frac{16^2}{\zeta_0^2} C_{T_0} |\tilde{h} - h|^2.
$$

Hence, there exists $\epsilon_1 > 0$ such that

$$
\inf_{\tilde{h} \in B(h, \epsilon_1)} P\left( \sup_{t \in [0,T_0]} |Y_t^{\tilde{h}} - h| \leq \zeta_0/8 \right) \geq 10/12,
$$

which implies that

$$
\inf_{\tilde{h} \in B(h, \epsilon_1)} P(\tilde{\tau}^{\tilde{h}} > T_0) \geq 10/12.
$$
Combining this inequality with (4.58), we further have that for any \( \tilde{h} \in B(h, \epsilon) \)

\[
\mathbb{P}(\tau^0 \wedge \tau^{\tilde{h}-}\!> T_0) \geq 1 - \mathbb{P}(\tau^0 \leq T_0) - \mathbb{P}(\tau^{\tilde{h}-}\! \leq T_0) \\
\geq \mathbb{P}(\tau^{\tilde{h}-}\! > T_0) - \mathbb{P}(\tau^0 \leq T_0) \\
\geq \frac{3}{4}.
\]

(4.59)

For the second inequality, we have used \( \{ \tau^0 \leq T_0 \} \subseteq \{ \tilde{\tau}^0 \leq T_0 \} \).

For any \( \eta > 0 \), let \( \epsilon_0 = \epsilon_1 \cap \sqrt{\frac{\eta^2}{32C_{\epsilon_0}}} \wedge 2^{-45} \). Then, by (4.57), the Chebyshev inequality, and (4.59), for any \( \tilde{h} \in B(h, \epsilon_0) \),

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T_0} |X^h_i(t) - X^{\tilde{h}}_i(t)| > \frac{\eta}{2} \right) \\
\leq \mathbb{P}\left( \{ \sup_{0 \leq t \leq T_0} |X^h_i(t) - X^{\tilde{h}}_i(t)| > \frac{\eta}{2} \} \cap \{ \tau^0 \wedge \tau^{\tilde{h}-}\!> T_0 \} \right) + \mathbb{P}(\tau^0 \wedge \tau^{\tilde{h}-}\! \leq T_0) \\
\leq \mathbb{P}\left( \sup_{0 \leq t \leq T_0} |Y^0_l - Y^{\tilde{h}-}\!| > \frac{\eta}{2} \right) + \frac{1}{4} \\
\leq \frac{4}{\eta^2} C_{T_0} |\tilde{h} - h| + \frac{1}{4} \\
\leq 3/8.
\]

The proof of (4.53) is complete. Now following the similar arguments as that in the proof of Proposition 4.1, we see that Assumption 2.2 holds.

Assumption 2.2 follows from Lemma 4.2. We now verify Assumption 2.3.

Note that \( \{ t f_i, l \in (0, 1], i = 1, 2, \ldots, n \} \subseteq S \).

For any \( h \neq y \in \mathbb{R}^d \) and \( \eta > 0 \), set \( q_0 = h \). Choose \( i_0 \in \{ 1, 2, \ldots, n \} \) such that

\[
\varepsilon_0 := \frac{\langle \sigma(q_0)f_{i_0}, y - q_0 \rangle}{|\sigma(q_0)f_{i_0}|} = \sup_{i = 1, 2, \ldots, n} \frac{\langle \sigma(q_0)f_i, y - q_0 \rangle}{|\sigma(q_0)f_i|}.
\]

Then \( \varepsilon_0 \in [\kappa, 1] \). Let \( \theta = |y - q_0| \) and \( \theta \) be such that \( \cos \theta = \varepsilon_0 \). Define

\[
g(r) = (\theta - r \cos \theta)^2 + (r \sin \theta)^2 = \theta^2 - 2r \theta \varepsilon_0 + r^2 \leq \theta^2 - 2r \theta \kappa + r^2, \quad r \geq 0.
\]

Take \( r_0 \in (0, |\sigma(q_0)f_{i_0}|) \supset (0, \Lambda^{-1}] \) such that \( g(r_0) = \inf_{r \in (0, |\sigma(q_0)f_{i_0}|]} g(r) \leq \inf_{r \in (0, \Lambda^{-1}]} g(r) \).

Since \( \varepsilon_0 \in [\kappa, 1] \), if \( \theta \kappa > \Lambda^{-1} \), \( g(r_0) \leq g(\Lambda^{-1}) \leq \theta^2 - \Lambda^{-2} \); if \( \theta \kappa \in (0, \Lambda^{-1}] \), \( g(r_0) = g(\theta \kappa) = \theta^2 (1 - \kappa^2) \).

Now let \( q_1 = q_0 + \frac{\sigma(q_0)f_{i_0}}{|\sigma(q_0)f_{i_0}|} r_0 \) and \( l_1 = \frac{f_{i_0}}{|\sigma(q_0)f_{i_0}|} r_0 \). Then \( |q_1 - y|^2 = g(r_0) \). Recursively, we can construct \( q_m, l_m, m \geq 2 \) until that \( |q_m - y| \leq \frac{\eta}{8} \). Since \( \sigma \) is continuous and \( \{ t f_i, l \in (0, 1], i = 1, 2, \ldots, n \} \subseteq S \), then Assumption 2.3 holds.

The proof of the proposition is complete.

\[ \Box \]

**Acknowledgement.** This work is partially supported by NSFC (No. 12131019, 11971456, 11721101). Jianliang Zhai’s research is also supported by the School Start-up Fund (USTC)
KY0010000036 and the Fundamental Research Funds for the Central Universities (No. WK3470000016).

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