ALMOST VOLUME CONE IMPLIES ALMOST METRIC CONE FOR ANNULUSES
CENTERED AT A COMPACT SET IN RCD(\(K,N\))-SPACES

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ABSTRACT. In [10], Cheeger-Colding considered manifolds with lower Ricci curvature bound and gave some almost rigidity results about warped products including almost metric cone rigidity and quantitative splitting theorem. As a generalization of manifolds with lower Ricci curvature bound, for metric measure spaces in RCD(\(K,N\)), 1 < N < \(\infty\), splitting theorem [10] and “volume cone implies metric cone” rigidity for balls and annuluses of a point [30] have been proved. In this paper we will generalize Cheeger-Colding’s [10] result about “almost volume cone implies almost metric cone for annuluses of a compact subset” to RCD(\(K,N\))-spaces. More precisely, consider a RCD(\(K,N\))-space \((X,d,m)\) and a Borel subset \(\Omega \subset X\). If the closed subset \(S = \partial \Omega\) has finite outer curvature, the diameter \(\text{diam}(S) \leq D\) and the mean curvature of \(S\) satisfies

\[ m(x) \leq m, \forall x \in S, \]

and

\[ m(A_{a,b}(S)) \geq (1 - \epsilon) \int_a^b \left( \frac{\sin_H(r)}{r} + \frac{m}{N-1} \frac{\sin_H(r)}{r} \right)^{n-1} dr \text{vol}(S), \]

then \(A_{a',b'}(S)\) is measured Gromov-Hausdorff close to a warped product \((a', b') \times \frac{\sin_H(r)}{r} + \frac{m}{N-1} \frac{\sin_H(r)}{r} Y\), \(A_{a,b}(S) = \{x \in X \setminus \Omega, a < d(x,S) < b\}\), \(a < a' < b\), \(Y\) is a metric space with finite components with each component a RCD(\(0, N-1\))-space when \(m = 0, K = 0\) or a RCD(\(N - 2, N - 1\))-space for other cases and \(H = \frac{N}{N-1}\). Note that when \(m = 0, K = 0\), our result is a kind of quantitative splitting theorem and in other cases it is an almost metric cone rigidity.

To prove this result, different from [16, 30], we will use [22]’s second order differentiation formula and a method similar as [10].

1. INTRODUCTION

In [10], Cheeger-Colding gave the following “almost volume cone implies almost metric cone” rigidity.

**Theorem 1.1** ([10]). If a complete \(N\)-manifold \(M\) with \(\text{Ric}_M \geq (N - 1)H\) and a compact subset \(\Omega \subset M\) satisfies that the mean curvature

\[ m(x) \leq m, \forall x \in S = \partial \Omega, \]

and

\[ \text{diam}(S) \leq D, \]

then

\[ m(A_{a,b}(S)) \geq (1 - \epsilon) \int_a^b \left( \frac{\sin_H(r)}{r} + \frac{m}{N-1} \frac{\sin_H(r)}{r} \right)^{n-1} dr \text{vol}(S), \]

where \(A_{a,b}(S) = \{x \in X \setminus \Omega, a < d(x,S) < b\}\), \(Y\) is a length metric space with at most \(C(N,H,a,b,D)\) components \(Y_i\) such that \(\text{diam}(Y_i) \leq c(N,H,m,a,b,\alpha,D)\) and

\[ \sin_H(r) = \begin{cases} \frac{\sin \sqrt{Hr}}{r}, & H > 0; \\ \frac{\sin \sqrt{Hr}}{Hr}, & H = 0; \\ \frac{\sinh \sqrt{-Hr}}{\sqrt{-Hr}} r, & H < 0. \end{cases} \]

If \(S\) is a hypersurface of \(M\), by Heinze-Karcher [24, (1.1)] implies that

\[ \text{vol}(A_{a,b}(S)) \leq \int_a^b \left( \frac{\sin_H(r)}{r} + \frac{m}{N-1} \frac{\sin_H(r)}{r} \right)^{N-1} dr \text{vol}(S). \]
In particular, if $M$ is compact with $\tilde{D} = \text{diam}(X)$,

$$
\text{vol}(M) \leq \int_{[-D, D]} \left( \frac{m}{N-1} \text{sn}_H(r) \right)^{N-1} dr \text{vol}(S).
$$

(1.4)

And when $H > 0$, the equality holds in (1.4) iff $M$ and $N$ have constant curvature (see [24]).

In [27], Ketterer extended Heintze-Karcher [24]'s results about the volume comparison (1.3) and the rigidity result for $K > 0$ in RCD($K, N$)-spaces with $S = \partial \Omega$; $\Omega$ is Borel and $H = \frac{K}{N-1}$.

In this note, we will generalize Theorem 1.1 to RCD($K, N$)-spaces which can also be treated as a quantitative version and a generalization of Ketterer [27]'s work (see Theorem 2.19) to arbitrary $K$. In the following, we will use the same definitions of mean curvature, finite outer curvature and measure on $S$, $m_S$ as in [27] (see Section 2.7 for these definitions).

**Theorem 1.2.** If a metric measure space $(X, d, m) \in \text{RCD}(K, N)$, $3 \leq N < \infty$, $\text{supp}(m) = X$ and a Borel subset $\Omega \subset X$ satisfies that $S = \partial \Omega$ is closed, diam$(S) \leq D$, $m(S) = 0$, $S$ has finite outer curvature, the mean curvature

$$
m(x) \leq m, \forall x \in S,
$$

and

$$
\text{m}(A_{a, b}(S)) \geq (1 - \epsilon) \int_{a}^{b} \left( \frac{m}{N-1} \text{sn}_H(r) \right)^{n-1} dr m_S(S),
$$

(1.6)

then

$$
d_{mGH}(A_{a', b'}(S), (a', b') \times \text{sn}_H(r) + \frac{m}{N-1} \text{sn}_H(r)) \leq \Psi(\epsilon |N, K, m, a, b, D),
$$

where $H = \frac{K}{N-1}$, $a' = a + \frac{(b-a)}{3}$, $b' = b - \frac{(b-a)}{3}$ and $(Y, d_Y, m_Y)$ has at most $C(N, K, a, b, D)$ components $Y_i$ with each $Y_i \in \text{RCD}(0, N - 1)$ for $m = 0$, $K = 0$, $Y_i \in \text{RCD}(N - 2, N - 1)$ for $m \neq 0$ or $K \neq 0$.

**Remark 1.3.** (i) When $K = 0, m = 0$, (1.6) becomes

$$
m(A_{a, b}(S)) \geq (1 - \epsilon)(b-a)m(S)
$$

and Theorem 1.2 is a kind of quantitative splitting theorem.

In [25], Huang gave a quantitative splitting rigidity under (1.7) and a measure-decreasing-along-distance-function (MDADF) condition. The assumption that $S$ has finite outer measure and mean curvature upper bound (1.5) implies MDADF condition (see Lemma 3.3 and the definition of finite outer curvature). A better result we have is that $Y$ can be chosen as each component in RCD$(0, N - 1)$.

(ii) When $K \neq 0$ or $m \neq 0$, assume

$$
m = (N - 1) \frac{\text{sn}_H(r_0)}{\text{sn}_H(r_0)},
$$

then (1.6) becomes

$$
\frac{m(A_{a, b}(S))}{m_S(S)} \geq (1 - \epsilon) \frac{\text{vol}(A_{r_0, b+r_0})}{\text{vol}(\partial B^H_{r_0})}.
$$

(1.8)

In [30], Philippis-Gigli pointed out that “volume cone implies metric cone” holds for annulus centered at a point where they assume

$$
\frac{m(A_{a, b}(x))}{m(\partial B_\delta(x))} \geq (1 - \epsilon) \frac{\text{vol}(A^H_{a, b})}{\text{vol}(\partial B^H_{\delta})},
$$

and derived a quantitative rigidity as in Theorem 1.2. Here

$$
m(\partial B_\delta(x)) = \limsup_{\delta \to 0} \frac{m(A_{a, a+\delta}(x))}{\delta}.
$$

Our result is a generalization of [30] in some sense.

(iii) When the equality holds in (1.6), we have that $A_{a', b'}(S)$ has a warped product structure and $\partial B_{a'}(S)$ has constant mean curvature.
Now we give a sketch of the proof of Theorem 1.2. Consider a sequence of RCD\((K, N)\)-spaces, \((X_i, d_i, \nu_i)\) which is measured Gromov-Hausdorff convergent to a RCD\((K, N)\)-space \((X, d, \nu)\). Assume \(S_i = \partial \Omega_i, \Omega_i \subset X_i\) is Borel with \(\nu_i(S_i) = 0\) and \(\text{diam}(S_i) \leq D\). Define a signed distance function associated to \(\Omega\).

\[
d_{s,i}(x) = \begin{cases} d_i(x, S_i), & x \in X_i \setminus \Omega_i; \\
-d_i(x, S_i), & x \in \Omega_i.
\end{cases}
\]

Obviously, \(d_{s,i}\) is 1-Lipschitz. Then by \cite{36} Proposition 2.70 or \cite{29} Proposition 2.12, there is a 1-Lipschitz function \(d_s : X \to \mathbb{R}\) such that \(d_{s,i}\) converges uniformly to \(d_s\) on any compact set. Let \(S = \{x \in X, d_s(x) = 0\}\), \(\Omega = \{x \in X, d_s(x) \leq 0\}\). Then as in \cite{25} Lemma 3.25, we know that \(d_s\) is a signed distance function associated to \(\Omega\).

Assume \(S_i\) has finite outer curvature and the mean curvature

\[
m(x) \leq m, \forall x \in S_i,
\]

\[(1.10)\]

\[m(A_{a,b}(S_i)) \geq (1 - \epsilon_i) \int_a^b \left( s_{n_H}(r) + \frac{m}{n-1} s_{n_H}(r) \right)^{n-1} dr m_{s_i}(S_i), \epsilon_i \to 0.\]

To prove Theorem 1.2, we only need to show that \(A_{a',b'}(S)\) is isometric to \((a', b') \times s_{n_H}(r) + \frac{m}{n-1} s_{n_H}(r) Y\) where \((Y, d_Y, m_Y) \in \text{RCD}(0, N - 1)\) for \(m = 0, K = 0\), \((Y, d_Y, m_Y) \in \text{RCD}(N - 2, N - 1)\) for the other cases.

To obtain this result, first we have that:

\((*)\) With intrinsic metric \(A_{a', b'}(S)\) is isometric to a warped product \((a', b') \times s_{n_H}(r) + \frac{m}{n-1} s_{n_H}(r) Y\) (for the definition see Section 2.6).

We will follow the process as in \cite{10}. The assumption \(S\) has finite outer curvature and \(m(X) \leq m\), together with the laplacian formula derived by \cite{44}, we will derive the laplacian comparison of \(d_s\) and relative volume comparison (see Lemma 3.3 and Lemma 3.6). Then by the volume condition \((1.10)\), we will get a laplacian estimate of \(d_s\) in Theorem 3.1. These laplacian estimates and improved Bochner’s inequality in RCD-spaces \((K, N)\)-spaces, see also Theorem 2.11, give the Hessian estimates Theorem 4.11. Then using the second order differentiation in RCD\((K, N)\)-space \cite{22}, we can show that the metric in the path connected component of \(A_{a', b'}(S)\) satisfies the Pythagoras theorem when \(m = 0, K = 0\) and Cosine law for the other cases. This gives the warped product structure of \(A_{a', b'}(S)\). And the relative volume comparison gives that \(Y\) has at most \(C(N, K, D, b, a)\) components.

Assume \(Y\) has one component. Then \((*)\) and that \(A_{a', b'}(S) \subset (X, d, \nu) \in \text{RCD}(K, N)\) enable us to derive that:

\((***)\) \((Y, d_Y, m_Y) \in \text{RCD}(0, N - 1)\) for \(m = 0, K = 0\) and \((Y, d_Y, m_Y) \in \text{RCD}(N - 2, N - 1)\) for the other cases (see Section 6).

Endow \(Y\) with an admissible metric and an admissible measure from the warped product structure \((*)\). A similar argument as in \cite{5} shows that \((Y, d_Y, m_Y)\) is infinitesimally Hilbertian and satisfies Sobolev to Lipschitz property (see Theorem 6.1). Now by local to global property we can see that \(\mathbb{R} \times Y\) (when \(m = 0, K = 0\) and the Euclidean cone \(C(Y)\) (when \(m \neq 0, K = 0\) are \(\text{RCD}(0, N)\)-spaces. Then \cite{14} and \cite{27} gives \((***)\) for \(K = 0\). For \(K \geq 0\), we will follow the argument in the proof of \cite{27} Theorem 1.2 where Ketterer showed that if the \((K, N)\)-cone \((C(Y), d_K, m_N)\) is a \(\text{RCD}(K, N)\)-space, then \(Y\) is \(\text{RCD}(N - 2, N - 1)\)-space.

The paper is organized as follows. In Section 2, we will present some basic definitions and facts we need in the proof of Theorem 1.2. Then by studying the relative volume comparison for annuluses centered at a compact subset, we give the Laplacian estimates of the distance function from the compact subset in Section 3. Then in Section 4, we will give the corresponding Hessian estimates in \(A_{a', b'}(S)\). In Section 5, by the second differential formula in \(\text{RCD}(K, N)\)-spaces \cite{22}, using the Hessian estimates and a methods as in \cite{10} we will derive that with the intrinsic metric the annulus \(A_{a', b'}(S)\) satisfies Pythagoras theorem or Cosine law. In Section 6, we will give the warped product structure of \(A_{a', b'}(S)\) and by studying the properties of the section \(Y\) of the warped product, we will prove that \(Y\) is a \(\text{RCD}\)-space.

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2. Preliminary

In this section, we recall some basic definitions and properties that we need in the proof of Theorem 1.2.

Let \((X, d, \nu)\) be a metric measure space satisfying that \((X, d)\) is a complete, separable and locally compact geodesic metric space endowed with a nonnegative Radon measure \(\nu\) which is supported on \(X\) and is finite.
on any bounded sets. We refer readers to the survey [1] for an overview of the topic and bibliography about curvature-dimension bounds in metric measure spaces.

2.1. **Calculus in metric measure spaces.** For the details of this subsection one can confer [16].

Consider a metric measure space \((X, d, m)\) as above. Let \(C([0, 1], X)\) be the space of continuous curves with weak convergence topology and let \(\mathcal{P}(C([0, 1], X))\) be the space of Borel probability measures of \(C([0, 1], X)\). A measure \(\pi \in \mathcal{P}(C([0, 1], X))\) is called a **test plan** if for some \(c > 0\),

\[
(e_t)_2(\pi) \leq cm, \forall t \in [0, 1], \quad \int_0^1 |\gamma(t)|dt \pi(\gamma) < \infty,
\]

where \(|\gamma(t)| = \lim_{h \to 0} d(\gamma(t + h), \gamma(t))/h|\) and \(e_t: C([0, 1], X) \to X, e_t(\gamma) = \gamma(t)\) is the evaluation map. Sobolev class \(S^2(X, d, m)\) is defined as the space of \(f: X \to \mathbb{R}\), such that there exists \(G \in L^2(X, m)\),

\[
\int |f(\gamma(1)) - f(\gamma(0))|d\pi(\gamma) \leq \int_0^1 G(\gamma(t))|\gamma(t)|dt \pi(\gamma), \forall \text{ test plan } \pi,
\]

where \(G\) is called a weak upper gradient of \(f\). Let \(|\nabla f|_w\) be the minimal (in \(m\)-a.e. sense) weak upper gradient of \(f\).

The space \(W^{1, 2}(X, d, m) = L^2(X, m) \cap S^2(X, d, m)\) endowed with the norm

\[
\|f\|_{W^{1, 2}}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2,
\]

is a Banach space.

Define the **Cheeger energy** as \(\text{Ch}: L^2(X, m) \to [0, \infty]\)

\[
\text{Ch}(f) = \begin{cases} \frac{1}{2} \int \|\nabla f\|_{L^2}^2 dm, & f \in W^{1, 2}(X, d, m) \\ +\infty, & \text{otherwise.} \end{cases}
\]

We say \((X, d, m)\) is **infinitesimally Hilbertian** if \(W^{1, 2}(X, d, m)\) is a Hilbert space, i.e., the Cheeger energy is a quadratic form.

In the following of this section we always assume that \((X, d, m)\) is infinitesimally Hilbertian.

For an open subset \(\Omega \subset X\), let \(W^{1, 2}_{\text{loc}}(\Omega)\) be the space of function \(f: \Omega \to \mathbb{R}\) that locally equal to some function in \(W^{1, 2}(X, d, m)\). For \(f, g \in W^{1, 2}_{\text{loc}}(\Omega)\), define

\[
\Gamma(f, g) = \langle \nabla f, \nabla g \rangle = \liminf_{\epsilon \to 0} \frac{\|g + \epsilon f\|_w^2 - \|g\|_w^2}{2\epsilon}.
\]

In fact \(\langle \nabla f, \nabla g \rangle\) can be achieved by taking limit directly \(m\)-a.e. \((\text{cf. } [17]). By [17], the map \(\Gamma: W^{1, 2}_{\text{loc}}(\Omega) \times W^{1, 2}_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Omega)\) is symmetric, bilinear and \(\Gamma(f, f) = \|\nabla f\|_w^2\).

**Definition 2.1.** For \(f \in W^{1, 2}_{\text{loc}}(\Omega)\), if there exists a Radon measure \(\mu\) on \(\Omega\) such that

\[
-\int \langle \nabla f, \nabla g \rangle = \int g d\mu
\]

holds for any Lipschitz function \(g: \Omega \to \mathbb{R}\), \(\text{supp } g \subset \subset \Omega\), then \(\mu\) is called the **distributional Laplacian** or **measure valued Laplacian** of \(f\) and denote it by \(\Delta f|_\Omega\).

Let \(D(\Delta, \Omega)\) be the space of \(f\) which has a distribution Laplacian. By the property of \(\Gamma\), we know that \(D(\Delta, \Omega)\) is a vector space and the Laplacian is linear. For \(f \in W^{1, 2}(X, d, m) \cap D(\Delta, X)\), if \(\Delta f = hm, h \in L^2(X, m)\), we denote \(\Delta f = h\).

2.2. **Tangent and cotangent modules.** The details of this subsection can be found in [13].

Consider a measured space \((X, A, m)\) where \(A\) is its \(\sigma\)-algebra. Let \(\mathcal{B}(X) = A/\sim\), where \(A, B \in A, A \sim B\) if \(m((A \setminus B) \cup (B \setminus A)) = 0\). A Banach space \((M, \|\cdot\|)\) is called a \(L^\infty(X, m)\)-premodule if there is a bilinear map

\[
L^\infty(X, m) \times M \to M, \quad (f, v) \mapsto f \cdot v,
\]

such that for each \(v \in M, f, g \in L^\infty(X, m)\),

\[
(fg) \cdot v = f \cdot (g \cdot v), \quad 1 \cdot v = v, \quad \|f \cdot v\| \leq \|f\|_{L^\infty(X, m)}\|v\|.
\]

An \(L^\infty(X, m)\)-premodule \((M, \|\cdot\|)\) is called a \(L^\infty(X, m)\)-**module** if

(1) Locality: for each \(x \in M, A_n \in \mathcal{B}(X)\),

\[
\forall n, \chi_{A_n} \cdot v = 0 \Rightarrow \chi_{\cup_n A_n} \cdot v = 0;
\]

(2) Positivity: \(\|f\|_{L^\infty(X, m)} > 0 \Rightarrow \|f \cdot v\| > 0, \quad \forall v \
eq 0\).
(2) Gluing: for every sequence \(\{v_n\} \subset M, \{A_n\} \subset B(X),\) if
\[
\chi_{A_i \cap A_j} \cdot v_i = \chi_{A_i \cap A_j} \cdot v_j, \forall i, j, \limsup_{n \to \infty} \left\| \sum_{i=1}^{n} \chi_{A_i} \cdot v_i \right\| < \infty,
\]
then there is \(v \in M,
\[
\chi_{A_i} \cdot v = \chi_{A_i} \cdot v_i, \forall i, \quad \|v\| \leq \liminf_{n \to \infty} \left\| \sum_{i=1}^{n} \chi_{A_i} \cdot v_i \right\|.
\]

A module morphism is a map \(T : M_1 \to M_2\) which is bounded and linear by viewing \(M_1\) and \(M_2\) as Banach spaces and satisfies the locality condition
\[
T(f \cdot v) = f \cdot T(v), \forall v \in M_1, f \in L^{\infty}(X, m).
\]
Denote all module morphism from \(M_1\) to \(M_2\) by \(\text{Hom}(M_1, M_2)\). The dual module \(M^* = \text{Hom}(M, L^1(X, m))\).

If there is a non-negative map \(\| \cdot \| : M \to L^p(X, m), p \in [0, \infty]\) satisfying that
\[
\|v\|_{L^p(X, m)} = \|v\|, \quad |f| \cdot v = |f| \cdot \|v\|, m - a.e., \forall v \in M, f \in L^{\infty}(X, m),
\]
then \(M\) is called a \(L^p(X, m)\)-normed \(L^{\infty}(X, m)\)-premodule (resp. module) when \(M\) is a \(L^{\infty}(X, m)\)-premodule (resp. module). \(\| \cdot \|\) is called the pointwise \(L^p(X, m)\)-norm. And locally, for \(A \in B(X)\), we can define \(M|_A = \{v \in M, \|v\| = 0\} - a.e. \) on \(A^2\) which is a \(L^p(X, m)\)-normed \(L^{\infty}(X, m)\)-module. A \(L^2(X, m)\)-normed \(L^{\infty}(X, m)\)-module which is a Hilbert space under \(\| \cdot \|\) is called a Hilbert module.

Consider a \(L^2(X, m)\)-normed \(L^{\infty}(X, m)\)-module \(M, V \subset M\). Let \(\text{Span}(V)\) be the collection of \(v \in M\) such that there is a Borel decomposition \(\{X_n\}\) of \(X\), and for each \(n\), there are \(v_{1,n}, \ldots, v_{k_n,n} \in V, f_{1,n}, \ldots, f_{k_n,n} \in L^{\infty}(X, m),\)
\[
\chi_{X_n}v = \sum_{i=1}^{k_n} f_{i,n}v_{i,n}, \forall n.
\]
And we say \(V\) generates \(M\) if \(\text{Span}(V) = M\).

Definition 2.2 ([19]). There is a unique, up to isomorphism, Hilbert module \(L^2(T^*X)\) endowed with a linear map \(d : W^{1,2}(X, d, m) \to L^2(T^*X)\) satisfying
\[
|df| = |\nabla f|_w, m - a.e., \forall f \in W^{1,2}(X, d, m); \quad d(W^{1,2}(X, d, m)) \text{ generates } L^2(T^*X).
\]
We call \(L^2(T^*X)\) the cotangent module of \((X, d, m)\). The dual of \(L^2(T^*X)\) is called the tangent module of \((X, d, m)\) and denoted by \(L^2(TX)\). Elements of \(L^2(TX)\) is called vector fields. And denote by \(\nabla f\) the dual of \(df\).

Let \(D(\text{div}) \subset L^2(TX)\) be the space of vector fields \(v\) satisfying that there is \(f \in L^2(X, m)\) such that for any \(g \in W^{1,2}(X, d, m),\)
\[
\int fgdm = - \int dg(v)dm.
\]
f is called the divergence of \(v\) and denoted by \(\text{div}(v)\). If \(f \in D(\Delta)\), then \(\nabla f \in D(\text{div})\) and \(\text{div}(\nabla f) = \Delta f\) (see [19 Proposition 2.3.14]).

For two Hilbert module \(\mathcal{H}_1, \mathcal{H}_2\), we can define the tensor product \(\mathcal{H}_1 \otimes \mathcal{H}_2\) and the exterior product \(\bigwedge\mathcal{H}_1 \otimes \mathcal{H}_2\) (see Section 1.5 in [19]). And denote the pointwise \(L^2(X, m)\)-normal of the tensor product \(L^2(\bigwedge^2T^*X)\) by \(\| | \cdot | \|\).
For $N \geq 1$, $K$, let $\sigma_{K,N} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ be as
\[
\sigma_{K,N}(\theta) = \begin{cases} 
\frac{+\infty}{\sin(\theta \sqrt{K/N})}, & K \theta^2 \geq N \pi^2, \\
\frac{\sin(\theta \sqrt{K/N})}{\sin(\theta \sqrt{K/N})}, & 0 < K \theta^2 < N \pi^2, \\
\frac{\sinh(\theta \sqrt{K/N})}{\sinh(\theta \sqrt{K/N})}, & K \theta^2 = 0, \\
\frac{\sinh(\theta \sqrt{K/N})}{\sinh(\theta \sqrt{K/N})}, & K \theta^2 < 0.
\end{cases}
\]
and let
\[
\tau_{K,N}^t(\theta) = t^\frac{1}{K-1} \sigma_{K,N-1}^t(\theta)^{\frac{1}{N-1}}.
\]

**Definition 2.3** ([28, 34, 35]). Given $K \in \mathbb{R}, N \geq 1$, we say a metric measure space $(X, d, m)$ is a CD$(K,N)$-space if for any two measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support which contains in $m$’s support, there exists $\pi \in \text{OpGeo}(\mu_0, \mu_1)$ such that for each $t \in [0,1]$
\[
-\int \rho_t^{-1} \frac{\Delta}{\rho_t} \log \rho_t dm \leq -\int \tau_{K,N}^t d\pi - \Delta \tau_{K,N}^t d\mu_0 + \tau_{K,N}^t d\tau_{K,N}^t d\mu_1 - \Delta \tau_{K,N}^t d\mu_1,
\]
where $(e_t)_{2} = \rho_t m + \mu_t, \mu_t \perp m$. We call $(X, d, m)$ is a CD$(K,N)$-space if the above inequality holds for $\sigma_{K,N}$ instead of $\tau_{K,N}$.

**Definition 2.4** ([2, 18]). A metric measure space $(X, d, m)$ is a RCD$(K, N)$-space (resp. RCD$^*$$(K,N)$-space) if it is an infinitesimally Hilbertian CD$(K,N)$-space (resp. CD$^*$$(K,N)$-space).

And we say a metric measure space $(X, d, m)$ is a CD$_{\text{loc}}$(K,N) space if for a cover $\{A_i\}$ of $X$, $A_i \subset X$, $\cup_i A_i = X$, CD$(K,N)$ holds in each $A_i$. At the local level $\int_{K' \leq K} \text{CD}_{\text{loc}}(K',N)$ coincide with $\int_{K' \leq K} \text{CD}_{\text{loc}}(K',N)$. We call $(X, d, m)$ is essentially non-branching if for any $\mu, \nu \in \mathcal{P}_2(X)$ with bounded support, each $\pi \in \text{OpGeo}(\mu, \nu)$ is concentrated on a Borel set of non-branching geodesics. It was proved in [12], an essentially non-branching metric measure space $(X, d, m)$ is CD$(K,N)$ if and only if it is CD$^*$$(K,N)$ if and only if it is CD$_{\text{loc}}$(K,N). By [2, 32], a RCD$(K, \infty)$-space is essentially non-branching.

**Theorem 2.5** ([15]). Assume a metric measure space $(X, d, m)$ with supp$(m) = X$ satisfies the infinitesimally Hilbertian and Sobolev to Lipschitz property, i.e. any $f \in W^{1,2}(X, d, m)$ with $|\nabla f|_w \leq 1$ m.a.e. admits a 1-Lipschitz representative. Then the followings are equivalent:

(i) $(X, d, m) \in \text{CD}^*(K,N)$;

(ii) The Bakry-Ledoux pointwise gradient estimate $\text{BL}(K,N)$ holds: for $f$ of finite Cheeger energy,
\[
|\nabla H_t(f)|_w^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta H_t(f)|_w^2 \leq e^{-2Kt} H_t(|\nabla f|_w^2), m - a.e.
\]

where
\[
\frac{d}{dt} H_t(f) = \Delta H_t(f), \quad H_0(f) = f.
\]

(ii) The Bochner/Bakry-Emery inequality $\text{BK}(K,N)$ holds: for $f \in D(\Delta), \Delta f \in W^{1,2}(X, d, m)$, $g \in D(\Delta), g \geq 0$, $\Delta g \in L^\infty(X, m)$,
\[
\frac{1}{2} \int \Delta g |\nabla f|_w^2 dm - \int g \langle \nabla (\Delta f), \nabla f \rangle dm \geq K \int g |\nabla f|_w^2 dm + \frac{1}{N} \int g |\Delta f|^2 dm.
\]

Last, let’s recall the following existence of good cut-off functions.

**Lemma 2.6** ([29]). Let $(X, d, m)$ be an RCD$(K, N)$-space for $N < \infty$ and let compact subset $S = \partial \Omega$ with $\Omega$ Borel. For each $R > 0$, $0 < 10r_1 < r_2 < R$, there exists a Lipschitz function $\phi : X \to [0,1]$ such that

(i) $\phi = 1$ on $A_{3r_1, \frac{R}{2}}(S)$, $\phi = 0$ on $X \setminus A_{2r_1, \frac{R}{2}}(S)$;

(ii) $r_2^2 |\Delta \phi| + |r_1 |\nabla \phi| \leq C(K,N,R)$ a.e. on $A_{2r_1,3r_1}(S)$;

(iii) $r_2^2 |\Delta \phi| + r_2 |\nabla \phi| \leq C(K,N,R)$ a.e. on $A_{2r_1,3r_1}(S)$.

2.4. **Regular Lagrangian flow.** In this subsection, we recall the definition and some facts about Regular Lagrangian flow (see [5, 21]).

**Definition 2.7.** Consider a metric measure space $(X, d, m) \in \text{RCD}(K, N)$ and a time-dependent vector field $V_t \in L^2([0,1], L^2_{\text{loc}}(TX))$. We say that

\[
F : [0,1] \times X \to X
\]
is a Regular Lagrangian flow (RLF for brief) of $V_t$ if

(i) $(F_t)_{t \geq 0}$ is a $C^1$-flow for some $C > 0$;

(ii) For $m$-a.e. $x \in X$, the curve $s \mapsto F_s(x)$, $s \in [0,1]$ is continuous and $F_0(x) = x$;

(iii) For each $f \in W^{1,2}(X, d, m)$, for $m$-a.e. $x \in X$, the function $s \mapsto f(F_s(x))$ belongs to $W^{1,1}(0,1)$ and

$$\frac{d}{ds} f(F_s(x)) = df(V_s)(F_s(x)), m \times L^1 - \text{a.e.}(x, s).$$

**Theorem 2.8** ([5]). For a time-dependent vector $V_t \in L^1([0,T], L^2(TX))$ with $V_t \in D(div)$ for a.e. $t$, if

$$\text{div}(V_t) \in L^1([0,T], L^2(X, m), \max\{-\text{div}(V_0), 0\}) \in L^1([0,T], L^\infty(X, m)), \nabla V_t \in L^1([0,T], L^2(T^{\otimes 2}X))$$

then there exists a unique, up to $m$-a.e. equality, RLF $(F_t)_{t \in [0,T]}$ for $V_t$ and

$$(F_t)_* m \leq \exp \left( \int_0^t \|\max\{-\text{div}(V_s), 0\}\|_{L^\infty} ds \right) dm.$$

The RLFS are closely related with the continuity equation. A $W_2$-continuous curve $(\mu_t)_{t \in [0,T]} \subset \mathcal{P}(X)$ with $\mu_t \leq Cm$ and a vector $V_t \in L^2([0,T], L^2(TX))$ is said a solution of the continuity equation

$$\frac{d}{dt} \mu_t + \text{div}(V_t \mu_t) = 0$$

iff for each $f \in W^{1,2}(X, d, m)$, the map $t \mapsto \int f d\mu_t$ is absolutely continuous and

$$\frac{d}{dt} \int f d\mu_t = \int df(V_t) d\mu_t, \text{a.e. } t \in [0,T].$$

For $V_t$ as in Theorem 2.8 and $F_t$ the unique RLF of $V_t$, let $\mu_t = (F_t)_* \mu_0$, $\mu_0 \in \mathcal{P}_2(X)$. By [5], $(\mu_t, V_t)$ is a solution of the continuity equation.

2.5. **The differential formula in RCD($K, N$)-spaces.** In this subsection, we always assume $(X, d, m)$ is a RCD($K, N$)-space.

Define the class of test functions as

$$\text{Test}(X) = \{f \in D(\Delta) \cap L^\infty(X, m), |\nabla f|_w \in L^\infty(X, m), \Delta f \in W^{1,2}(X, d, m)\}.$$

It was shown in [33, 23] that if $f \in \text{Test}(X)$, then $|\nabla f|^2 \in D(\Delta)$ and one may define

$$\Gamma_2(f) = \frac{1}{2} \Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle.$$

**Definition 2.9** ([12]). Let $W^{2,2}(X, d, m)$ be the set of $f \in W^{1,2}(X, d, m)$ satisfying that there is $A \in L^2((T^*)_\otimes 2)$ such that for any $g_1, g_2, h \in \text{Test}(X)$

$$2 \int hA(\nabla g_1, \nabla g_2)dm = -\int \langle \nabla f, \nabla g_1 \rangle \text{div}(hg_2) + \langle \nabla f, \nabla g_2 \rangle \text{div}(hg_1) + h \langle \nabla f, \nabla \langle \nabla g_1, \nabla g_2 \rangle \rangle dm.$$

$A$ is called the Hessian of $f$, denoted by $\text{Hess}(f)$.

It was proved in [19] that $D(\Delta) \subset W^{2,2}(X, d, m)$ and let $H^{2,2}(X)$ be the closure of $D(\Delta)$ in $W^{2,2}(X, d, m)$.

**Theorem 2.10** (Improved Bochner inequality, [33, 23]). For $(X, d, m) \in \text{RCD}(K, N)$, $K \in \mathbb{R}$, $N \in [1, \infty)$ and $f \in \text{Test}(X)$, we have that $f \in W^{2,2}(X, d, m)$ and

$$\Gamma_2(f) \geq (K|\nabla f|^2 + |\text{Hess}(f)|_{H^2}) \mu.$$

Given a function $\phi : X \to \mathbb{R} \cup \{-\infty\}$ not identically $-\infty$, its $c$-transform $\phi^c : X \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$\phi^c(x) = \inf_{y \in X} \frac{d^2(x, y)}{2} - \phi(y).$$

We call $\phi$ is $c$-concave if $\phi^c = \phi$. Let $\partial^c \phi \subset X^2$ be the set of $(x, y) \in X^2$ such that

$$\phi(z) - \phi(x) \leq \frac{d^2(z, y)}{2} - \frac{d^2(x, y)}{2}, \forall z \in X.$$

A test plan $\pi \in \text{OpGeo}(\mu, \nu)$ if and only if there is a $c$-concave function $\phi$ such that $\text{supp}((e_0, e_1)_2 \pi) \subset \partial^c \phi$ and such $\phi$ is called Kantorovich potential from $\mu$ to $\nu$. And for any $t \in (0,1)$, $t\phi$ is a Kantorovich potential from $\mu$ to $(e_t)_{2}(\pi)$.
3.15, we have that \((\mu, \gamma)\)

\[ (2.2) \]

Theorem 2.12

And the measure \(m \mapsto \sum t \in [0, 1] \]

\[ (2.3) \]

A corollary of Theorem 2.11 is the following first order differential estimates along geodesics (see [14, Corollary 3.14]):

**Corollary 2.13** ([14]). Let \((X, d, m)\) be an RCD\((K, N)\)-space and let \(p \in X, f \in W^{1,2}(X, d, m)\). For \(m\)-a.e. \(x \in X\), the map \(t \mapsto f(\gamma_{x,p}(t))\) is in \(W_{loc}^{1,1}([0, d(p, x)])\) and

\[ (2.4) \]

where \(\gamma_{x,p}\) is a unit speed geodesic from \(x\) to \(p\).

For a fixed point \(p \in X\) and a measure \(\mu \in \mathcal{P}_2(X)\) with \(\mu \leq Cm\), let \(\gamma_{x,p}(t), t \in [0, 1]\) be a constant speed geodesic from \(x \in \text{supp}(\mu)\) to \(p\), let \(D = \sup_{x \in \text{supp}(\mu)} d(x, p)\) and let \(\mu_t = (\gamma_{x,p}(t))^\sharp(\mu)\). By [14, Theorem 3.15], we have that \((\mu_t)\) solves the continuity equation

\[ \frac{d}{dt} \mu_t + \text{div}(-\nabla_{d_p}\mu_t) = 0. \]

And we have the second order differential formula, as [14, Proposition 3.21],

**Corollary 2.14.** Let the assumption be as in Corollary 2.13, let \(\Pi \leq C(m \times m)\) be a nonnegative, compactly supported measure on \(X \times X\) and let \(f \in H^{2,2}(X)\). Then for each \(t, s \in (0, 1]\)

\[ (2.5) \]

where \(\gamma_{x,y} : [0, 1] \to X\) is a constant speed geodesic from \(x\) to \(y\).

2.6. **Warped product and \((K, N)\)-cone over metric measure spaces.** Consider two metric measure spaces \((Z, d_Z, m_Z)\) and \((Y, d_Y, m_Y)\) and two continuous maps \(w_d, w_m : Z \to [0, \infty)\) with \(\{w_d = 0\} \subset \{w_m = 0\}\). A **warped product** \(Z \times_w Y\) is the space \(Z \times Y\) admitting the metric

\[ d_w(p, q) = \inf \{l_w(\gamma) \mid \gamma\text{ is a absolutely continuous curve between } p, q\}, \]

where \(\gamma = (\gamma^Z, \gamma^Y)\),

\[ l_w(\gamma) = \int_0^1 \sqrt{\dot{\gamma}^Z_1^2 + w_d^2(\gamma^Z) \dot{\gamma}^Y_1^2} \, dt. \]

And the measure \(m_w\) on \(Z \times_w Y\) is defined as

\[ dm_w = w_m dm_Z \otimes dm_Y. \]

**Definition 2.15.** The \((K, N)\)-cone \((C(Y), d_K, m_N)\) over a metric measure space \((Y, d, m_Y)\) is defined as follows: let \(H = K/(N - 1)\) and \((t, y_1), (t, y_2) \in C(Y)\),

\[ C(Y) = \begin{cases} [0, \pi / \sqrt{H}] / \{0, \pi / \sqrt{H}\} \times Y, & K > 0; \\ [0, \infty) \times Y / \{0 \times Y\}, & K \leq 0; \end{cases} \]

\[ d_K((t, y_1), (s, y_2)) = \begin{cases} \sqrt{t^2 + s^2 - 2st \cos \min\{\pi, d(y_1, y_2)\}} & K = 0; \\ (\sin_H' t)^{-1} (\sin_H' (s) + H \sin_H(s) \sin_H(t) \cos \min\{\pi, d(y_1, y_2)\}) , & K \neq 0; \end{cases} \]

\[ m_N = \sin_H^{-1}(t) dt \otimes m_Y. \]
Almost volume cone implies almost metric cone

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It is obvious that if \(\text{diam}(Y) \leq \pi\), then \(C(Y) = I_K \times \frac{N}{\text{sn}_H} Y\), where \(I_K = [0, \frac{1}{\sqrt{H}}]\) for \(K > 0\) and \(I_K = [0, \infty)\) for \(K \leq 0\) and \(w_m = \text{sn}_H^{-1} w_d = \text{sn}_H\). And in [20], Ketterer showed that for \(N \geq 2\), \(K \geq 0\), a \((K, N)\)-cone over \((Y, d, m)\) is a RCD\((K, N)\)-space if and only if \((Y, d, m)\) is a RCD\((N - 2, N - 1)\)-space. More precisely,

**Theorem 2.16** ([20]). (i) Assume a metric measure space \((Y, d, m)\) \(\in\) RCD\((N - 2, N - 1)\), \(N \geq 2\), \(K \geq 0\) and \(\text{diam}(Y) \leq \pi\). Then the \((K, N)\)-cone over \((Y, d, m)\), \((C(Y), d_K, m_N)\) \(\in\) RCD\((K, N)\);

(ii) Assume the \((K, N)\)-cone over a metric measure space \((Y, d, m)\), \((C(Y), d_K, m_N)\) \(\in\) RCD\((K, N)\) with \(N \geq 2\), then \((Y, d, m)\) \(\in\) RCD\((N - 2, N - 1)\) with \(\text{diam}(Y) \leq \pi\).

2.7. Measure decomposition and Heintze-Karcher inequality in RCD\((K, N)\)-spaces. In this subsection, we briefly recall the following measure decomposition in RCD\((K, N)\)-spaces by [11, 13] and the Heintze-Karcher inequality in RCD\((K, N)\)-spaces given by [27]. Note that in [11], to derive the measure decomposition result they also assume the condition that \(m(X) < \infty\). In [13], they showed that this assumption is unnecessary.

Consider a measurable space \((R, \mathcal{R}, m)\) and a map \(\Omega : R \to Q\). One can equip \(Q\) with a \(\sigma\)-algebra \(\mathcal{Q}\) if \(\Omega^{-1}(B) \in \mathcal{R}\). Let \(\mathfrak{Q}m = q\) which is a probability measure on \(Q\).

A **disintegration** of \(m\) which is consistent with \(\Omega\) is a map \(\mathcal{R} \times Q \to [0, 1]\), \((A, \alpha) \mapsto m_{\alpha}(A)\) satisfying that:

(i) \(m_{\alpha}\) is a probability measure on \((R, \mathcal{R})\) for each \(\alpha \in Q\);

(ii) \(\alpha \mapsto m_{\alpha}(A)\) is \(q\)-measurable for each \(A \in \mathcal{R}\);

(iii) For \(A \in \mathcal{R}, B \in \mathcal{Q}\),

\[
m(A \cap \Omega^{-1}(B)) = \int_B m_{\alpha}(A)dq(\alpha).
\]

We call \(\{m_{\alpha}\}_{\alpha \in Q}\) a disintegration of \(m\) and call \(m_{\alpha}\) the conditional probability measures.

A disintegration \(\{m_{\alpha}\}_{\alpha \in Q}\) is strongly consistent with \(\Omega\) if for q-a.e. \(\alpha\), \(m_{\alpha}(\Omega^{-1}(\alpha)) = 1\).

Let \((X, d, m)\) be as in the beginning of this section and be essentially non-branching. Let \(u : X \to \mathbb{R}\) be a 1-Lipschitz map. A set \(A \subset X \times X\) is \(d\)-cyclically monotone if for any finite set of points \((x_1, y_1), \ldots, (x_n, y_n) \in A\)

\[
\sum_{i=1}^n d(x_i, y_i) \leq \sum_{i=1}^n d(x_i, y_{i+1}),\ y_{n+1} = y_1.
\]

A \(d\)-cyclically monotone set associated with \(u\) is defined as

\[
\Gamma = \{ (x, y) \in X \times X, u(x) - u(y) = d(x, y) \}.
\]

It is obvious that \((x, y) \in \Gamma\) implies that for any minimizing geodesic \(\gamma\) from \(x\) to \(y\), \((\gamma_t, \gamma_s) \in \Gamma\) for \(0 \leq s \leq t \leq 1\).

Let the transport rays \(R = \Gamma \sqcup \Gamma^{-1}\), where \(\Gamma^{-1} = \{ (x, y) \in X \times X, (y, x) \in \Gamma \}\). Let \(T = P_1(R \setminus \{ (x, y), x = y \in X \}) \subset X\) where \(P_1(x, y) = x\). And the sets

\[
A_+ = \{ x \in T, \exists z, w \in \Gamma(x), (z, w) \notin R \},
\]

\[
A_- = \{ x \in T, \exists z, w \in \Gamma^{-1}(x), (z, w) \notin R \},
\]

are called forward and backward branching points respectively, where \(\Gamma(x) = \{ y \in X, (x, y) \in \Gamma \}\), and similar define \(\Gamma^{-1}(x), R(x)\).

The initial and final points are

\[
a = \{ x, \Gamma^{-1}(x) = \{ x \} \}, \ b = \{ x, \Gamma(x) = \{ x \} \}.
\]

Let \(\mathcal{T}_u = T \setminus (A_+ \cup A_-)\). It was proved that

\[
x \sim y \Leftrightarrow (x, y) \in R
\]

is an equivalence relation on \(\mathcal{T}_u\) ([11, 13]). Denote this relation by \(R_u\) and the equivalence classes by \(\{X_{\alpha}\}_{\alpha \in Q}\).

Each \(X_{\alpha}\) is isometric to an interval \(I_{\alpha} \subset \mathbb{R}\) via an isometry \(\gamma_{\alpha} : I_{\alpha} \to X_{\alpha}\) such that \(d(\gamma_{\alpha}(t), \gamma_{\alpha}(s)) = |t - s|\) for \(t, s \in I_{\alpha}\). The map \(\gamma_{\alpha}\) extends to a geodesic in \(X\) which is also denoted by \(\gamma_{\alpha}\). Denote the closure \(\bar{I}_{\alpha}\) of \(I_{\alpha}\) by \([a(X_{\alpha}), b(X_{\alpha})]\).

**Theorem 2.17** ([11, 13]). Let \((X, d, m)\) be an essentially non-branching CD\((K, N)\)-space with \(\text{supp}(m) = X\).

Let \(u : X \to \mathbb{R}\) be a 1-Lipschitz function. Then

(i) there exists a disintegration \(\{m_{\alpha}\}_{\alpha \in Q}\) of \(m_{\mathcal{T}_u}\) that is strongly consistent;
Lemma 2.18. Let the assumption be as in Theorem 2.17. Then for each \(0 \leq a < b \leq 1\),
\[
\frac{h_{n}(b)}{h_{n}(a)} \leq \left(\sinh(b-a) + \frac{(\ln h_{n})'_{+}(a)}{N-1} \sinh(b-a)\right)^{N+1},
\]
where
\[
(\ln h_{n})'_{+}(a) = \lim_{h \downarrow 0} \frac{\ln h_{n}(a+h) - \ln h_{n}(a)}{h},
\]
h_{n}(t) = h_{n}(\gamma(t)) and \(f_{+} = \max\{f, 0\}\).

Consider \((X, d, m)\) be as above lemma. Let \(\Omega \subset X\) be a Borel subset and let \(S = \partial \Omega\) with \(m(S) = 0\). Let
\[
d_{s} = \begin{cases}
  d(x, S), & x \in X \setminus \Omega; \\
  -d(x, S), & x \in \Omega.
\end{cases}
\]
Then \(d_{s}\) is 1-Lipschitz. By Theorem 2.17 there is a disintegration of \(d_{s}\), \(\{m_{a}\}_{a \in Q}\) and a partition \(\{X_{a}\}_{a \in Q}\) of \(X\) up a measure zero set (By [23] Lemma 3.4), \(m(\Omega \setminus \mathcal{T}_{d_{s}}) = 0\). And \(\mathcal{T} \supset X \setminus S\). Thus \(m(S) = 0\) implies \(m(X \setminus \mathcal{T}_{d_{s}}) = 0\).

Let \(A = \mathcal{Q}^{-1}(\Omega(S \cap \mathcal{T}_{d_{s}}))\). Then for each \(a \in \Omega(A)\), there is a unique \(t_{a} \in (a(X_{a}), b(X_{a}))\) such that \(X_{a} \cap S = \{\gamma(t_{a})\} \neq \emptyset\). Identify the measurable set \(\Omega(A) \subset \mathcal{Q}\) with \(A \cap S\) and one can assume \(t_{a} = t_{0}\).

Let \(Q = \{a \in Q, X_{a} \cap \mathcal{T}_{d_{s}} \subset a \cup \Omega\}\). By [11] Theorem 7.10, \(q(\mathcal{Q} \setminus \mathcal{Q}) = 0\). Let \(\mathcal{T}_{d_{s}}^{*} = \mathcal{Q}^{-1}(\mathcal{Q} \setminus \mathcal{Q})\). The sets \(B_{lin} = \Omega^{2} \cap \mathcal{T}_{d_{s}}^{*} \setminus (A \cap \mathcal{T}_{d_{s}}^{*})\) and \(B_{out} = \Omega^{2} \cap \mathcal{T}_{d_{s}}^{*} \setminus (A \cap \mathcal{T}_{d_{s}}^{*})\) are measurable (see [27] Remark 5.1).

We say that \(S\) has \textbf{finite outer curvature} if \(m(B_{out}) = 0\), \(S\) has \textbf{finite inner curvature} if \(m(B_{lin}) = 0\), and \(S\) has \textbf{finite curvature} if \(m(B_{lin} \cup B_{out}) = 0\).

If \(S\) has finite outer curvature, we can define its \textbf{outer mean curvature} as
\[
p \in S \mapsto H^{+}(p) = \begin{cases}
  \left. \frac{d}{dt} \ln h_{n}(\gamma_{a}(0)) \right|_{t=0}, & p = \gamma_{a}(0) \in S \cap A \cap \mathcal{T}_{d_{s}}, \\
  -\infty, & p \in R_{d_{s}}(B_{lin}) \cap S, \\
  c \text{ for some } c \in \mathbb{R}, & \text{otherwise}.
\end{cases}
\]
Switch the roles of \(\Omega\) and \(\mathcal{T}_{d_{s}}^{*}\) and assume \(S\) has finite inner curvature, we call the corresponding outer curvature the \textbf{inner mean curvature} and write as \(H^{-}\).

If \(S\) has finite curvature, the \textbf{mean curvature} defined as \(\max\{H^{+}, -H^{-}\} = m\).

The \textbf{surface measure} of \(S\) is defined as
\[
\int_{S} \phi(x) dm_{S}(x) = \int_{\Omega(A \cap \mathcal{T}_{d_{s}}^{*})} \phi(\gamma_{a}(0)) h_{n}(0) dq(a),
\]
for any continuous function \(\phi : X \rightarrow \mathbb{R}\).

In [27], Ketterer generalized the Heintze-Karcher inequality to CD(K, N)-spaces.

Theorem 2.19. Assume \((X, d, m) \in \text{CD}(K, N)\) is an essentially non-branching metric measure space, \(N > 1\).
Let \(\Omega \subset X\) be a closed Borel subset and let \(S = \partial \Omega\) such that \(m(S) = 0\) and \(S\) has finite outer curvature. Then
\[
m(B_{lin}(t) \setminus \Omega) \leq \int_{S} \int_{0}^{t} J_{K,H^{+}(p),N}(r) dr dm_{S}(p), \forall t \in (0, \bar{D}],
\]
where \(\bar{D} = \text{diam}(X)\), \(J_{K,H^{+}(p),N}(r) = \left(\sinh_{K/N-1}(r) + \frac{H^{-}}{N-1} \sinh_{K/N-1}(r)\right)^{N-1}
\).

If \(S\) has finite curvature, then
\[
m(X) \leq \int_{S} \int_{-\bar{D}}^{\bar{D}} J_{K,m(p),N}(r) dr dm_{S}(p).
\]
In particular, if \((X, d, m)\) is a RCD(K, N)-space, \(K > 0\), then the equality holds in (2.7) if there is a RCD(N - 2, N - 1)-space \((Y, d_{Y}, m_{Y})\) such that \((X, d, m) = (C(Y), d_{K}, m_{N})\) and \(S\) is a constant mean curvature surface in \(X\).
3. LAPLACIAN ESTIMATES

Let \((X_i, d_i, m_i)\) be a sequence of \(RCD(K, N)\)-spaces and let \(\Omega_i \subset X_i\) be a Borel subset with \(S_i = \partial \Omega_i\) closed and \(\text{diam}(S_i) \leq D\). Define a signed distance function associated to \(\Omega_i\).

\[
d_{s,i}(x) = \begin{cases} 
  d_i(x, S_i), & x \in X_i \setminus \Omega_i; \\
  -d_i(x, S_i), & x \in \Omega_i.
\end{cases}
\]

Obviously, \(d_{s,i}\) is 1-Lipschitz.

Assume \((X_i, d_i, m_i)\) is measured Gromov-Hausdorff convergent to a metric measure space \((X, d, m) \in RCD(K, N)\). Then by [76] Proposition 2.70 or [24] Proposition 2.12, there is a 1-Lipschitz function \(d_s: X \to \mathbb{R}\) such that \(d_{s,i}\) converges uniformly to \(d_s\) on any compact set. Let \(S = \{x \in X, d_s(x) = 0\}\), \(\Omega = \{x \in X, d_s(x) \leq 0\}\). Then as in [24] Lemma 3.25, we know that \(d_s\) is a signed distance function associated to \(\Omega\). Let \(H = \frac{K}{N} \), and for \(m \neq 0\) or \(K \neq 0\), let

\[
m = (N - 1)\frac{\text{sn}'_H(r_0)}{\text{sn}_H(r_0)}.
\]

In this section, we will show that

**Theorem 3.1** (Laplaceian estimates). Let \((X, d, m)\), \(d_s\) be as above. Assume \(S_i\) has finite outer curvature, \(\text{diam}(S_i) \leq D\) and the mean curvature \(m_i(x_i) \leq m\) for each \(i\) and any \(x_i \in S_i\). Then if \(1.10\)

\[
\frac{m(A_{a,b}(S_i))}{m(S_i)} \geq (1 - \epsilon_i) \int_a^b \left( \frac{\text{sn}'_H(r)}{\text{sn}_H(r)} + \frac{m}{N - 1} \right)^{n-1} dr,
\]

holds with \(\epsilon_i \to 0\), \(d_s \in \Delta(A_{a,b}(S))\). In particular, for \(x \in A_{a,b}(S)\),

(i) For \(m = 0\) and \(K = 0\), we have that \(\Delta d_s = 0\);

(ii) For \(m \neq 0\) or \(K \neq 0\),

\[
\Delta d_s = (N - 1)\frac{\text{sn}'_H(d_s + r_0)}{\text{sn}_H(d_s + r_0)}.
\]

To prove the above Laplacian estimates, we first recall the following Laplacian formula in [13] Corollary 4.16.

**Theorem 3.2** [13]. Let \((X, d, m)\) be a RCD\((K, N)\)-space. Consider the signed distance function \(d_s\) associated with a Borel subset \(\Omega \subset X\) and a compact boundary \(S = \partial \Omega\) with \(m(S) = 0\). And assume the associated disintegration of \(d_s\), \(m = \int_Q \int_{X, a} h_\alpha(r) dr d\alpha\). Then \(d_s \in D(\Delta, X \setminus S)\) and

\[
\Delta d_s = \int_S h_a(r) m_{X \setminus S} - \int_Q h_a (\delta_{a(X_a) \cap \Omega}) m_{X \setminus S} d\alpha.
\]

where \((\ln h_\alpha)'\) is roughly the directional derivative of \(\ln h_\alpha\) in the direction of \(\nabla d_s\). In particular

\[
[\Delta d_s]_{X \setminus S} = (\ln h_\alpha)' m_{X \setminus S}
\]

and

\[
[\Delta d_s]_{X \setminus S} \text{ sing} = - \int_Q h_a (\delta_{a(X_a) \cap \Omega}) m_{X \setminus S} d\alpha.
\]

Note that in [13] Corollary 4.16, there is a negative sign in front of the derivative \((\ln h_\alpha)'\), where they defined \(h_\alpha'\) as

\[
h_\alpha'(x) = \lim_{t \to 0} \frac{h_\alpha(g_t(x)) - h_\alpha(x)}{t},
\]

and \(g_t(y) = y\) such that \(d_s(x) - d_s(y) = t\). Roughly speaking the derivative \(h_\alpha'\) there is the directional derivative in the direction of \(-\nabla d_s\). Compared the one in Lemma 2.13 in this paper, we always denote

\[
h_\alpha'(x) = \lim_{t \to 0} \frac{h_\alpha(\gamma(t_0 + t)) - h_\alpha(\gamma(t_0))}{t},
\]

where \(\gamma(t_0) = x\) and \(\gamma\) is a unit speed geodesic in \(X_\alpha\) such that \(d_s(\gamma(l)) - d_s(\gamma(t)) = t - l\).

Now using [24], as the discussion of [27] Lemma 4.1 and the proof of Laplacian comparison in manifolds with lower Ricci curvature bound, we have the following Laplacian comparison.
Lemma 3.3. Let the assumption be as in Theorem 3.2 and assume $S$ has finite outer curvature. Assume that for each $x \in S$, the mean curvature $m(x) \leq m$.

(i) If $m = 0$ and $K = 0$, we have

$$[\Delta d_{sLX}(\Omega \cup S)]_{\text{reg}}(x) \leq 0;$$

(ii) If $m \neq 0$ or $K \neq 0$, then

$$[\Delta d_{sLX}(\Omega \cup S)]_{\text{reg}}(x) \leq (N-1) \frac{\sn_H(d_s(x) + r_0)}{\sn_H(d_s(x) + r_0)}.$$

Proof. Let $u(t) = h_\alpha^{1/2}(\gamma(t))$ where $h_\alpha$ satisfies (2.6) as in Theorem 2.17. Then $u$ is semi-concave and satisfies

$$u'' + Hu \leq 0$$

in the distributional sense. And the limits

$$u'_+(r) = \lim_{\delta \downarrow 0} \frac{u(r + \delta) - u(r)}{\delta}, \quad u'_-(r) = \lim_{\delta \downarrow 0} \frac{u(r - \delta) - u(r)}{-\delta}$$

exist.

Take $\phi \in C^\infty_0((-1, 1))$, $\int_{-1}^1 \phi = 1$, $\phi_c(t) = \frac{1}{c} \phi\left(\frac{t}{c}\right)$ and let

$$\tilde{u}(s) = \int_{-\epsilon}^\epsilon \phi_c(-r)u(s - r)dr.$$

Then

$$\tilde{u}''(s) \leq -Hu,$$

and

$$\left(\frac{\tilde{u}'}{u}\right)' = \frac{\tilde{u}''}{u} - \left(\frac{\tilde{u}'}{u}\right)^2 \leq -H - \left(\frac{\tilde{u}'}{u}\right)^2.$$

Let $f = \frac{\tilde{u}'}{u}$ and let $f_H = \frac{\sn_H'}{\sn_H}$. Then $f'_H = -H - f^2_H$ and

$$(\sn^2_H(f'(s) - f_H(s)))' = 2\sn_H(s)\sn_H'(s)(f'(s) - f_H(s)) + \sn^2_H(s)(f''(s) - f'_H(s)) \leq 2\sn^2_H(f'(s) - f_H(s)) - \sn^2_H(s)(f''(s) - f'_H(s)) = -\sn^2_H(s)(f'(s) - f^2_H(s))^2 \leq 0.$$

Thus for $s \geq 0$,

$$\sn^2_H(s + r_0)(f'(s + r_0) - f_H(s + r_0)) - \sn^2_H(r_0)(f(r_0) - f_H(r_0)) \leq 0.$$

If $f(r_0) \leq f_H(r_0)$, then

$$f(s + r_0) \leq f_H(s + r_0).$$

Now note that as $\epsilon \to 0$, $\tilde{u} \to u$ and $\tilde{u}'_+ \to u'_+$ and $\frac{u'}{u} = \frac{1}{N-1}(\ln h_\alpha)'$. And by the Laplacian formula Theorem 3.2, we have (ii).

For (i), as above we have that $f' \leq 0$. Thus for $s \geq 0$,

$$f(s + r_0) \leq f(r_0) \leq 0.$$

Remark 3.4. The above laplacian comparison can also be seen in [7].

Now by Lemma 3.3 and Lemma 2.18, we have the following volume element comparison.

Lemma 3.5 (Volume element comparison). Let the assumption be as in Lemma 3.3. Then for any $r_1, r_2 \in (0, b(X_\alpha))$, $r_1 \leq r_2$,

(i) for $m = 0$ and $K = 0$,

$$h_\alpha(r_2) \leq h_\alpha(r_1);$$

(ii) for $m \neq 0$ or $K \neq 0$,

$$\frac{h_\alpha(r_2)}{\sn^{N-1}_H(r_2 + r_0)} \leq \frac{h_\alpha(r_1)}{\sn^{N-1}_H(r_1 + r_0)}.$$
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\textbf{Proof.} By Lemma 2.18 for \( r_1 \leq r_2 \) as above,
\[
    h_\alpha(r_2) \leq J_{K,H+\gamma_\alpha(r_1))},N(r_2-r_1)h_\alpha(r_1) = \left( \text{sn}_H^N(r_2-r_1) + \frac{(\ln h_\alpha)'_+(r_1)}{N-1}\text{sn}_H^N(r_2-r_1) \right) + h_\alpha(r_1).
\]

Thus for \( m = 0 \) and \( K = 0 \), by Lemma 3.3 \((\ln h_\alpha)' \leq 0\), we have
\[
    h_\alpha(r_2) \leq h_\alpha(r_1);
\]

For \( m \neq 0 \) or \( K \neq 0 \), by Lemma 3.3
\[
    h_\alpha(r_2) \leq \frac{\text{sn}_H^{N-1}(r_2 + r_0)}{\text{sn}_H^{N-1}(r_1 + r_0)}h_\alpha(r_1).
\]

In the following, let \( h_\alpha(r) = 0 \) when \( r \) increases and \( h_\alpha(r) \) becomes undefined. If \( S \) has finite outer curvature, then for almost all \( a \geq 0 \),
\[
    m(\partial B_a(S)) = \limsup_{\delta \to 0} \frac{m(A_{a,a+\delta}(S))}{\delta} = \int_{Q(\partial B_a(S) \cap \mathcal{T}^2_a)} h_\alpha(a)d\alpha.
\]

Using above lemma, we have the following relative volume comparison.

\textbf{Lemma 3.6} (Relative volume comparison). Let the assumption be as in Lemma 3.3. Then for \( 0 \leq a < b \),

(i) if \( m = 0 \) and \( K = 0 \), then
\[
    (3.1) \quad m(\partial B_b(S)) \leq m(\partial B_a(S));
\]

(ii) if \( m \neq 0 \) or \( K \neq 0 \),
\[
    (3.2) \quad m(A_{a,b}(S)) \leq (b-a)m(\partial B_a(S));
\]

\[
    (3.3) \quad m(\partial B_b(S)) \leq m(\partial B_a(S)) \frac{\text{vol}(\partial B_{b+r_0})}{\text{vol}(\partial B_{a+r_0})};
\]

\[
    (3.4) \quad m(A_{a,b}(S)) \leq m(\partial B_a(S)) \frac{\text{vol}(A_{a+r_0,b+r_0})}{\text{vol}(\partial B_{a+r_0})}.
\]

\textbf{Proof.} The proof is similar as the the one of [27] Theorem 1.1,
\[
    m(A_{a,b}(S)) = m(A_{a,b}(S) \cap \mathcal{T}^2_{a} \setminus B_{out}) = \int_{\Omega(A_{a,b}(S) \cap \mathcal{T}^2_{a})} \int_{A_{a,b}(S) \cap \mathcal{T}^2_{a}} h_\alpha(r)drd\alpha.
\]

By Lemma 3.3 for \( m \neq 0 \) or \( K \neq 0 \),
\[
    m(A_{a,b}(S)) \leq \int_{\Omega(A_{a,b}(S) \cap \mathcal{T}^2_{a})} \int_{a}^{b} \left( \frac{\text{sn}_H^N(r + r_0)}{\text{sn}_H^N(a + r_0)} \right) d\alpha \frac{dr}{n-1} h_\alpha(a)d\alpha
\]
\[
    \leq m(\partial B_a(S)) \int_{a}^{b} \left( \frac{\text{sn}_H^N(r + r_0)}{\text{sn}_H^N(a + r_0)} \right) \frac{dr}{n-1}
\]
\[
    = m(\partial B_a(S)) \frac{\text{vol}(A_{a+r_0,b+r_0})}{\text{vol}(\partial B_{a+r_0})}.
\]

For \( m = 0 \) and \( K = 0 \),
\[
    m(A_{a,b}(S)) \leq (b-a)m(\partial B_a(S)).
\]

By above relative volume comparison, as in [11], we have that
Lemma 3.7. Let the assumption be as in Lemma 3.6 and assume (1.6). Then for each $d \in (a, b)$,

(i) if $m = 0$ and $K = 0$,

\[
(3.5) \quad \frac{m(\partial B_d(S))}{m(\partial B_a(S))} \geq 1 - \frac{b - a}{b - d};
\]

(ii) if $m \neq 0$ or $K \neq 0$,

\[
(3.6) \quad \frac{m(\partial B_d(S))}{m(\partial B_a(S))} \geq \left(1 - \epsilon \frac{\text{vol}(A_{d+r_0,a+b+r_0})}{\text{vol}(A_{d+r_0,b+r_0})}\right) \frac{\text{vol}(\partial B_{d+r_0}(S))}{\text{vol}(\partial B_{a+r_0})}.
\]

Proof. By

\[
\frac{m(A_{a,b}(S))}{m(S)} \geq (1 - \epsilon) \frac{\text{vol}(A_{d+r_0,b+r_0})}{\text{vol}(A_{d+r_0,b+r_0})},
\]

we have that,

\[
\frac{m(A_{a,b}(S))}{m(S)} \frac{\text{vol}(\partial B_{d+r_0}(S))}{\text{vol}(A_{d+r_0,b+r_0})} - \text{vol}(A_{d+r_0,b+r_0}) \geq -\epsilon \text{vol}(A_{d+r_0,b+r_0}).
\]

Thus for all $a < d < b$,

\[
1 - \epsilon \frac{\text{vol}(A_{d+r_0,b+r_0})}{\text{vol}(A_{d+r_0,b+r_0})} \leq 1 + \frac{m(A_{a,b}(S))}{m(S)} \frac{\text{vol}(\partial B_{d+r_0}(S))}{\text{vol}(A_{d+r_0,b+r_0})} - \frac{\text{vol}(A_{d+r_0,b+r_0})}{\text{vol}(A_{d+r_0,b+r_0})} = \frac{m(A_{a,b}(S))}{m(S)} \frac{\text{vol}(B_{d+r_0})}{\text{vol}(A_{d+r_0,b+r_0})}.
\]

By relative volume comparison Lemma 3.6

\[
\text{vol}(A_{d+r_0,b+r_0}) \geq m(A_{a,d}(S)) \cdot \frac{\text{vol}(B_{d+r_0})}{m(S)}.
\]

So by (3.6),

\[
1 - \epsilon \frac{\text{vol}(A_{d+r_0,b+r_0})}{\text{vol}(A_{d+r_0,b+r_0})} \leq \frac{m(A_{a,b}(S))}{m(S)} \frac{\text{vol}(\partial B_{d+r_0}(S))}{\text{vol}(\partial B_{r_0})} \leq \frac{m(A_{a,b}(S))}{m(S)} \frac{\text{vol}(\partial B_{d+r_0}(S))}{\text{vol}(\partial B_{r_0})}.
\]

Note that by (3.1) and (1.6) implies that for $0 \leq a < b$,

\[
m(A_{a,b}(S)) \geq (1 - \epsilon)(b - a)m(\partial B_a(S)), \text{ for } m = 0 \text{ and } K = 0;
\]

\[
m(A_{a,b}(S)) \geq (1 - \epsilon) \frac{\text{vol}(A_{d+r_0,b+r_0})}{\text{vol}(\partial B_{d+r_0})} m(\partial B_a(S)), \text{ for } m \neq 0 \text{ or } K \neq 0.
\]

Then same argument as above gives the results. \(\square\)

Now we will use above volume estimates Lemma 3.6 Lemma 3.7 and Laplacian comparison Lemma 3.3 to prove the Laplacian estimates Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.3 we have that for $m = 0$, $K = 0$,

\[
\Delta d_{s,i} \leq 0;
\]

for $m \neq 0$ or $K \neq 0$,

\[
\Delta d_{s,i} \leq (N - 1) \frac{s_n H^i (r + r_0)}{s_n H (r + r_0)}.
\]

And for $a < d < b$ (see also [5] Proposition 3.6)

\[
\int_{A_{a,d}(S_i)} \Delta d_{s,i} = \frac{1}{m(A_{a,d}(S_i))} \int_{\Omega(A_{a,d}(S_i)) \cap \eta_{d}^{2}} \int_{A_{a,d}(S_i)} \Delta d_{s,i} h_{a}(r) dr dq(\alpha)
\]

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\[ \Delta d_{s,i} \geq \frac{1}{\operatorname{vol}(A_\alpha)} \int_{\Omega(A_\alpha \cap \mathcal{F}_{d,i})} \frac{h'_\alpha(r)}{h_\alpha(r)} d\alpha \]

Thus for \( m = 0 \) and \( K = 0 \), by (3.5) and (3.1)

\[ \int_{A_\alpha \cap \mathcal{F}_{d,i}} \Delta d_{s,i} \geq \frac{-\varepsilon_i(b - a)}{b - d - \varepsilon_i(b - a)} \frac{1}{\operatorname{vol}(A_\alpha)} \]

And for \( m \neq 0 \) or \( K \neq 0 \), if \( \operatorname{vol}(\partial B^H_{d + r_0}) > \operatorname{vol}(\partial B^H_{a + r_0}) \) which is always holds for \( K \leq 0 \), by (3.4) and (3.6)

\[ \int_{A_\alpha \cap \mathcal{F}_{d,i}} \Delta d_{s,i} \geq \frac{1}{\operatorname{vol}(\partial B^H_{d + r_0})} \left( (1 - \varepsilon_i C(N, H, a, b, d, r_0)) \operatorname{vol}(\partial B^H_{d + r_0}) - \operatorname{vol}(\partial B^H_{a + r_0}) \right) \frac{\operatorname{vol}(B^H_{d + r_0})}{\operatorname{vol}(A_\alpha)} \]

If \( \operatorname{vol}(\partial B^H_{d + r_0}) \leq \operatorname{vol}(\partial B^H_{a + r_0}) \) for \( K > 0 \), as the \( m = 0 \), \( K = 0 \) case,

\[ \int_{A_\alpha \cap \mathcal{F}_{d,i}} \Delta d_{s,i} \geq \frac{1}{\operatorname{vol}(\partial B^H_{d + r_0})} \left( (1 + \varepsilon_i C(N, H, a, b, d, r_0)) \operatorname{vol}(\partial B^H_{d + r_0}) - \operatorname{vol}(\partial B^H_{a + r_0}) \right) \frac{\operatorname{vol}(B^H_{d + r_0})}{\operatorname{vol}(A_\alpha)} \]

And by Lemma 2.18

\[ \int_{A_\alpha \cap \mathcal{F}_{d,i}} (N - 1) \frac{\operatorname{sn}'_H(r + r_0)}{\operatorname{sn}_H(r + r_0)} dm \]

\[ \leq \frac{1}{\operatorname{vol}(\partial B^H_{d + r_0})} \left( (1 + \varepsilon_i) \operatorname{vol}(\partial B^H_{d + r_0}) - \operatorname{vol}(\partial B^H_{a + r_0}) \right) \frac{\operatorname{vol}(B^H_{d + r_0})}{\operatorname{vol}(A_\alpha \cap \mathcal{F}_{d,i})} \]

We have that

\[ (3.7) \quad \int_{A_\alpha \cap \mathcal{F}_{d,i}} \Delta d_{s,i} \geq (1 - \Psi(\varepsilon_i|N, K, a, d, b, m)) \int_{A_\alpha \cap \mathcal{F}_{d,i}} (N - 1) \frac{\operatorname{sn}'_H(r + r_0)}{\operatorname{sn}_H(r + r_0)} \]

Now as the proof of Claim 1 in [9, Proposition 4.3], passing to the limit, we have that on \( A_{a,d}(S) \)

(i) for \( m = 0 \) and \( K = 0 \),

\[ \Delta d_s = 0; \]
(ii) for $m \neq 0$ or $K \neq 0$,
\[ \Delta d_s = (N-1)\frac{sn'(d_s + r_0)}{sn(d_s + r_0)}. \]

\[ \square \]

**Corollary 3.8.** Let the assumption be as in Theorem 3.1. For $m \neq 0$ or $K \neq 0$, let
\[ f_H(x) = f_H(d_s(x)), \quad f_H(r) = \int sn_H(r + r_0)dr. \]

Then in $A_{a,b}(S)$
\[ \nabla f_H = sn_H(d_s + r_0)\nabla d_s, \]
and
\[ \Delta f_H = N sn_H(d_s + r_0). \]

\section{4. Hessian estimates}

In this section we will use the Laplacian estimates Theorem 3.1 to give an estimate of $\text{Hess}(d_s)$ for $m = 0$ and $K = 0$ and estimates of $\text{Hess}(f_H)$ for $m \neq 0$ or $K \neq 0$.

**Theorem 4.1** (Hessian estimates). Let the assumption be as in Theorem 3.1 and let $a < d < b$.

(i) For $m = 0, K = 0$, in $A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$ m.a.e.
\[ \text{Hess}(d_s) = 0. \]

(ii) For $m \neq 0$ or $K \neq 0$, in $A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$ m.a.e.
\[ \text{Hess}(f_H) = sn'_H(d_s + r_0). \]

**Proof.** Take a cut-off function $\phi : X \to [0, 1]$ as in Lemma 2.10 such that

(1) $\phi = 1$ on $A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$, $\phi = 0$ on $X \setminus A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$;

(2) $|\Delta\phi| + |\nabla\phi| \leq C(K, N, a, b)$ a.e. on $A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$.

For $m = 0, K = 0$, by the improved Bochner inequality Theorem 2.10 and Theorem 3.1,
\[ 0 = -\frac{1}{2} \int X \Gamma(\phi, |\nabla d_s|^2) = \int X \phi \frac{1}{2} |\nabla d_s|^2 \geq \int X \phi (|\text{Hess}(d_s)|_{HS}^2 + \Gamma(d_s, \Delta d_s)) = \int X \phi |\text{Hess}(d_s)|_{HS}^2. \]

And thus in $A_{a+\frac{m}{b-a}, b-\frac{m}{b-a}}(S)$ m.a.e.
\[ \text{Hess}(d_s) = 0. \]

For $m \neq 0, K = 0$, $f_0 = \frac{1}{2}(r + r_0)^2$,
\[ 0 = -\int X \Gamma(\phi, \frac{1}{2} |\nabla f_0|^2 - f_0) = \int X \phi \left( \frac{1}{2} |\nabla f_0|^2 - \Delta f_0 \right) \geq \int X \phi (|\text{Hess}(f_0)|_{HS}^2 + \Gamma(f_0, \Delta f_0) - N) \geq \int X \phi |\text{Hess}(f_0)|_{HS}^2. \]

For $K = N - 1$, $f_1 = -\cos(r + r_0)$,
\[ 0 = -\frac{1}{2} \int X \Gamma(\phi, |\nabla f_1|^2 + f_1^2 - 1) = \frac{1}{2} \int X \phi (\Delta |\nabla f_1|^2 + 2 |\nabla f_1|^2 + 2 f_1 \Delta f_1) \geq \int X \phi (|\text{Hess}(f_1)|_{HS}^2 + (N-1)|\nabla f_1|^2 + \Gamma(f_1, \Delta f_1) + |\nabla f_1|^2 + f_1 \Delta f_1) \]
\[ = \int X \phi (|\text{Hess}(f_1)|_{HS}^2 + (N-1) \sin^2(r + r_0) - N \sin^2(r + r_0) + \sin^2(r + r_0) - N \cos^2(r + r_0)) \geq \int X \phi |\text{Hess}(f_1)|_{HS}^2. \]

For $K = -(N-1)$, $f_{-1} = \cos(r + r_0)$,
\[ 0 = -\frac{1}{2} \int X \Gamma(\phi, |\nabla f_{-1}|^2 - f_{-1}^2 + 1) = \frac{1}{2} \int X \phi (\Delta |\nabla f_{-1}|^2 - 2 |\nabla f_{-1}|^2 - 2 f_{-1} \Delta f_{-1}) \]
\[ \geq \int X \phi (|\text{Hess}(f_{-1})|_{HS}^2 - 2 |\text{Hess}(f_{-1})|_{HS}^2 - 2 f_{-1} \Delta f_{-1}) \]
\[ \geq \int X \phi |\text{Hess}(f_{-1})|_{HS}^2. \]
\[ \geq \int_X \phi \left( \| \text{Hess}(f-1) \|_{\text{HS}}^2 - (N-1)\|\nabla f-1\|^2 + \Gamma(f-1, \Delta f-1) - \|\nabla f-1\|^2 - f-1\Delta f-1 \right) \]

\[ = \int_X \phi \left( \| \text{Hess}(f-1) \|_{\text{HS}}^2 - (N-1)\sinh^2(r + r_0) + N\sinh^2(r + r_0) - \sin^2(r + r_0) - N\cosh^2(r + r_0) \right) \]

\[ \geq \int_X \phi \| \text{Hess}(f-1) - \cosh(r + r_0) \|_{\text{HS}}^2. \]

\[ \square \]

5. Pythagoras theorem and cosine law

Let \((X, d, m)\) be as in Theorem 3.1. In this section, we will use a method as in [10] to show that the metric in \(A_{a+\delta, b-\delta}(S)\) (some \(\delta > 0\)) satisfies Pythagoras theorem for \(m = 0\) and \(K = 0\) or Cosine law for \(m \neq 0\) or \(K \neq 0\).

First note that for \(t \in \left[\frac{-(b-a)}{3}, \frac{(b-a)}{3}\right], td_s\) is c-concave, thus we can apply the differential formula in Section 2.5 to \(d_s\).

**Lemma 5.1.** Let the assumption be as in Theorem 3.1. Then \(td_s\) is c-concave for \(t \in \left[\frac{-(b-a)}{3}, \frac{(b-a)}{3}\right],\) and

\[ (td_s)^c(y) = -td_s - \frac{t^2}{2}. \]

**Proof.** The proof is the same as in [10]. First take small \(\tau > 0\) and consider

\[ T_{a,b}^i = \{ T_{a,b} \cap A_{a,b-\tau}(S), T_{d,s} \cap A_{b-\tau,b}(S) \neq \emptyset \}. \]

Then for any \(x \in T_{a,b}^i\), there is \(y \in A_{b-\tau,b}(S)\), such that

\[ d_{s,i}(y) - d_{s,i}(x) = d_i(x, y). \]

Since \(m_i(X_i \setminus T_{d,s,i}) = 0\),

\[ m_i(T_{d,s} \cap A_{b-\tau,b}(S_i)) = m_i(A_{b-\tau,b}(S_i)). \]

And by volume element comparison Lemma 3.5 and [10],

\[ \frac{m_i(T_{a,b}^i)}{m_i(A_{a,b}(S_i))} = \frac{m_i(T_{a,b}^i)}{m_i(A_{a,b}(S_i))} \frac{\text{vol}(A_{a,b}(S_i))}{\text{vol}(A_{a,b}(S_i))} \]

\[ \geq \frac{A_{b-\tau,b}(S_i)}{A_{a,b}(S_i)} \frac{\text{vol}(A_{a,b}(S_i))}{\text{vol}(A_{a,b}(S_i))} \]

\[ \geq (1 - \epsilon_i) \frac{\text{vol}(A_{a,b}(S_i))}{\text{vol}(A_{a,b}(S_i))} \]

\[ \geq (1 - \epsilon_i) \frac{\text{vol}(A_{a,b}(S_i))}{\text{vol}(A_{a,b}(S_i))}. \]

Since \(\tau\) can be taken arbitrary small, without loss of generality, we may let \(\tau \to 0\) and derive that for each \(x \in A_{a,b}(S_i)\), there is \(y \in \partial B_b(S_i)\) such that

\[ d_i(x, y) \geq b - d_{s,i}(x) \geq (1 - \epsilon_i)d_i(x, y). \]

Since \(d_s\) is 1-Lipschitz, as in [10], for \(x, y \in X,\)

\[ td_s(x) - td_s(y) \leq |t|d(x, y) \leq \frac{t^2}{2} + \frac{d^2(x, y)}{2}. \]

And by the definition of c-transform,

\[ (td_s)^c(y) = \inf_{x \in X} \frac{d^2(x, y)}{2} - td_s(x) \geq -td_s(y) - \frac{t^2}{2}. \]

For the opposite inequality, note that for \(y \in A_{a,b}(S),\) there is \(y_i \in A_{a,b}(S_i)\) and \(y_i^- \in \partial B_b(S_i), y_i^+ \in \partial B_b(S_i)\) such that \(y_i \to y\)

\[ d_i(y_i, y_i^-) = d_{s,i}(y_i) - a, d_{s,i}(y_i^+) - d_{s,i}(y_i) = d_i(y_i^+, y_i) \geq -\epsilon_i. \]

Then

\[ b - a \leq d_i(y_i^+, y_i^-) \leq d_i(y_i, y_i^-) + d_i(y_i, y_i^+) \leq b - a + \epsilon_i. \]
Let $\gamma_i$ be a unit speed minimal geodesic from $y_i^-$ to $y$ and let $\gamma_i^+$ be a unit speed minimal geodesic from $y$ to $y_i^+$. Assume $\gamma_i^- \cup \gamma_i^+ \rightarrow \gamma \in X$.

For $y \in A_{b-a, b+a}(S)$, for each $t \in (-\frac{b-a}{b+a}, \frac{b-a}{b+a})$, we can take $\gamma_t = \gamma(d_s(y) + t)$ such that

$$d_s(\gamma_t) - d_s(y) = t = \text{sign}(t)d(y, \gamma_t).$$

Thus

$$(td_s)^c(y) \leq \frac{d^2(\gamma_t, y)}{2} - td_s(\gamma_t) = -td_s(y) - \frac{t^2}{2}.$$

Take a cut-off function $\phi : X \rightarrow [0, 1]$ as in the proof of Theorem 4.1. Consider the vector field $\phi \nabla d_s$. Then as the discussion in section 3.4 of [8], by [5] (see Theorem 2.8), we know that the Regular Lagrangian flow $F_t$ for $\phi \nabla d_s$ exists and is unique.

Using Hessian estimates Theorem 4.1, Lemma 5.1 and the differential formula Theorem 2.11 Corollary 2.13 and Corollary 2.14 we will derive Pythagoras theorem for $m = 0$, $K = 0$, and Cosine law for $m \neq 0$ or $K \neq 0$.

Note that by Corollary 2.13 we have that for a Lipschitz map $f$, we have

$$f(p) = f(x) - \lim_{\delta \rightarrow 0} \int_{|x - y| < \delta} d(x, p) \langle \nabla f, \nabla d_p \rangle (\gamma_{p,x}(t))dt,$$

where $\gamma_{p,x}$ is a constant geodesic from $p$ to $x$ and $d_p$ is the distance function from $p$. In the following for simplicity we will write (5.2) just as

$$f(x) - f(p) = \int_0^1 d(x, p) \langle \nabla f, \nabla d_p \rangle (\gamma_{p,x}(t))dt.$$

**Theorem 5.2** (Pythagoras theorem and Cosine law). *Let the assumption be as in Theorem 3.1. Let $F_t$ be the regular Lagrangian flow of $\phi \nabla d_s$. Denote $A = A_{a-b, a+b}(S)$. Let $B = B_r(x) \subset A$ be a geodesic ball in $A$.\n
(i) For $m = 0$ and $K = 0$,

$$\int_{B \times B} |d^2(x, y) - (d_s(x) - d_s(y))^2 - d^2(x, F_{-d_s(y) + d_s(\gamma_s(x))}(y))| \, dm(x) m(y) = 0;$$

(ii) For $m \neq 0$ or $K \neq 0$,

(i-1) if $K = 0$, we have that

$$\int_{B \times B} \frac{(d_s(x) + r_0)^2 + (d_s(y) + r_0)^2 - d^2(x, y)}{2(d_s(x) + r_0)(d_s(y) + r_0)} - \frac{2(r_0 + d_s(x))^2 - d^2(x, F_{-d_s(y) + d_s(\gamma_s(x))}(y))}{2(r_0 + d_s(x))^2} \, dm(x)m(y) = 0;$$

(ii-2) if $K = \pm(N - 1)$, and thus $H = \pm 1$,

$$\int_{B \times B} \frac{|\nabla_H'(d_s(x)) - \nabla_H'(d_s(x) + r_0)\nabla_H'(d_s(y) + r_0) - \nabla_H'(d_s(x) + r_0)\nabla_H'(d_s(y) + r_0)|}{\nabla_H'(d_s(x)) \nabla_H'(d_s(y) + r_0)} = 0;$$

Proof. Assume for $x, y \in B$, $d_s(x) = t_0$, $d_s(y) = t$. Let $\gamma_t : [0, 1] \rightarrow B$ be a constant speed geodesic from $x$ to $F_{-\gamma_t}(y)$, $\tau \in [0, t - t_0]$.

For $m = 0$, $K = 0$,

$$\int_{B \times B} \frac{d^2(x, y) - (t - t_0)^2 - d^2(x, F_{-\gamma_t}(y))}{2} \, dm(x)dm(y)$$

$$= \int_{B \times B} \frac{1}{2} \left| d^2(x, F_{-\gamma_t}(y)) - (t - t_0 - \tau)^2 \right|_{\tau = 0} ^{t - t_0} \, dm(x) dm(y)$$

$$\leq \int_{B \times B} \int_0^{t - t_0} |d(x, F_{-\gamma_t}(y)) \nabla d_s(F_{-\gamma_t}(y)) - (t - t_0 - \tau)\, d\tau dm(x) dm(y)$$

$$= \int_{B \times B} \int_0^{t - t_0} |d(x, F_{-\gamma_t}(y)) \nabla d_s(F_{-\gamma_t}(y)) - d_s(\gamma_t(l))| \, d\tau dm(x) dm(y)$$

$$\leq \int_{B \times B} \int_0^{t - t_0} |d(x, F_{-\gamma_t}(y)) (\nabla d_s(F_{-\gamma_t}(y)) - (\nabla d_s(F_{-\gamma_t}(y)) \nabla d_s(\gamma_t(l))| \, dl d\tau dm(x) dm(y)$$

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\[ \leq \int_0^1 \int_0^1 \int_{B \times B} \int_{l-t_0}^{l-t_0} d^2(x, F_{-\tau}(y)) \left| \text{Hess}_{d_\gamma}(\gamma_{\tau}(\xi), \gamma_{\tau}(\xi)) \right| d\tau dm(x)dm(y) d\xi dl \]

\[ \leq \int_0^1 \int_0^1 \int_{B \times B} \int_{l-t_0}^{l-t_0} d^2(x, F_{-\tau}(y)) \left| \text{Hess}_{d_\gamma}(\gamma_{\tau}(\xi), \gamma_{\tau}(\xi)) \right| dm(x)dm(y) d\xi d\tau \]

\[ = 0. \]

Now prove the cosine law for \( K = 0 \) and \( m \neq 0 \) where \( f_0(x) = \frac{1}{2}(d_s + r_0)^2 \).

\[ \int_{B \times B} \left| \frac{(t_0 + r_0)^2 + (t + r_0)^2 - d^2(x, y)}{2(t_0 + r_0)(t + r_0)} - \frac{2 (r_0 + t_0)^2 - d^2(x, F_{t_0-t}(y))}{2 (r_0 + t_0)^2} \right| dm(x)m(y) \]

\[ = \int_{B \times B} \left| \frac{(t_0 + r_0)^2 + (t + r_0 - \tau)^2 - d^2(x, F_{-\tau}(y))}{2(t_0 + r_0)(t + r_0 - \tau)} \right| dm(x)m(y) \]

\[ \leq \frac{1}{2(a + r_0)^3} \int_{B \times B} \int_0^{l-t_0} \left| (r_0 + t - \tau)^2 - (t_0 + r_0)^2 + d^2(x, F_{-\tau}(y)) \right| d\tau \]

\[ = \frac{1}{(a + r_0)^3} \int_{B \times B} \int_0^{l-t_0} \left| f_0(\gamma_{\tau}(l))^{1/3} + \frac{1}{2} d^2(x, F_{-\tau}(y)) - d(x, F_{-\tau}(y)) \left( \nabla d_x, \nabla d_s + d^2(x, F_{-\tau}(y)) \right) \right| d\tau \]

\[ \leq \frac{1}{(a + r_0)^3} \int_0^1 \int_{B \times B} \int_0^{l-t_0} d^2(x, F_{t_0}(y)) \left| \text{Hess}_{f_0}(\nabla d_x, \nabla d_s)(\gamma_{\tau}(\xi)) \right| dm(x)dm(y) d\xi dl \]

\[ = 0 \]

For \( K = (N - 1) \), consider \( f_1 = -\cos(d_s + r_0) \), we can see that

\[ \int_{B \times B} \left| \frac{\cos(d(x, y)) - \cos(t_0 + r_0) \cos(t + r_0)}{\sin(t_0 + r_0) \sin(t + r_0)} - \frac{\cos(d(x, F_{t_0-t}(y))) - \cos^2(r_0 + t_0)}{\sin^2(r_0 + t_0)} \right| dm(x)m(y) \]

\[ \int_{B \times B} \left| \frac{\cos d(x, F_{-\tau}(y)) - \cos(t_0 + r_0) \cos(t + r_0 - \tau)}{\sin(t_0 + r_0) \sin(t + r_0 - \tau)} \right|^{l-t_0} dm(x)m(y) \]

\[ \leq c \int_{B \times B} \int_0^{l-t_0} \left| \sin d(x, F_{-\tau}(y)) \left( \nabla d_x, \nabla f_1 \right)(F_{-\tau}(y)) - \cos(t_0 + r_0) + \cos d(x, F_{-\tau}(y)) \cos(t + r_0 - \tau) \right| d\tau \]

\[ = c \int_{B \times B} \int_0^{l-t_0} \left| \sin d(x, F_{-\tau}(y)) \left( \nabla d_x, \nabla f_1 \right)(F_{-\tau}(y)) + \cos d(x, \gamma_{\tau}(0))f_1(\gamma_{\tau}(0)) - \cos d(x, \gamma_{\tau}(1))f_1(\gamma_{\tau}(1)) \right| d\tau \]

\[ = c \int_{B \times B} \int_0^{l-t_0} \left| \left( \sin d(x, \gamma_{\tau}(l)) \left( \nabla d_x, \nabla f_1 \right)(F_{-\tau}(y)) - \cos d(x, \gamma_{\tau}(l))f_1(\gamma_{\tau}(l)) \right) \right|^{l-t_0} d\tau \]

\[ \leq c \int_0^1 \int_{B \times B} \int_0^1 \left| \frac{d(x, F_{-\tau}(y)) \cos d(x, \gamma_{\tau}(l)) \left( \nabla f_1, \nabla d_x \right)(F_{-\tau}(y))}{\cos d(x, \gamma_{\tau}(l)) \left( \nabla f_1, \nabla d_x \right)(F_{-\tau}(y))} \right| d\tau \]

\[ = c \int_0^1 \int_{B \times B} \left| \nabla f_1 \left( \nabla f_1, \nabla d_x \right)(\gamma_{\tau}(\xi)) \right|^{l-t_0} d\tau \]

\[ \leq c \int_0^1 \int_{B \times B} \int_0^1 \left| \nabla f_1 \left( \nabla f_1, \nabla d_x \right)(\gamma_{\tau}(\xi)) \right|^{l-t_0} d\tau \]

\[ = c \int_0^1 \int_{B \times B} \int_0^1 \left| \nabla f_1 \left( \nabla f_1, \nabla d_x \right)(\gamma_{\tau}(\xi)) \right|^{l-t_0} \]
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\[ + \int_0^1 \int_0^l \cos d(x, \gamma_r(\xi)) d^2(x, F_{-r}(y)) f_1(\gamma_r(l)) d\xi dl \]

\[ = c \int_0^1 \int_{B \times B} \left| \int_0^1 d^2(x, F_{-r}(y)) \cos d(x, \gamma_r(l)) \int_0^l \text{Hess}_1(\gamma'_r(\xi), \gamma'_r(\xi)) d\xi dl \right| \]

\[ + \int_0^1 \int_0^l \cos d(x, \gamma_r(l)) d^2(x, F_{-r}(y)) f_1(\gamma_r(l)) d\xi dl \]

\[ = c \int_0^1 \int_0^1 \int_{B \times B} d^2(x, F_{-r}(y)) |\cos d(x, \gamma_r(l))| |\text{Hess}_1(\nabla d_x, \nabla d_x)(\gamma'_r(\xi)) + f_1(\gamma_r(\xi))| \]

\[ = 0. \]

For \( K = -(N - 1) \), the same argument as \( K = (N - 1) \) gives the result. \( \square \)

By the Pythagoras theorem and Cosine law above, we have that

**Theorem 5.3.** The regular Lagrangian flow \( F : (-\frac{b-a}{3}, \frac{b-a}{3}) \times A \rightarrow A_{a,b}(S) \) admits a continuous representation with respect to the measure \( \mathcal{L}^1 \times \mathfrak{m} \) which we still denote by \( F \). In particular the Pythagoras theorem or Cosine law in Theorem 5.2 holds pointwise for \( F \).

**Proof.** By the definition of regular Lagrangian flow, for \( \mathfrak{m} \)-a.e. \( x, t \mapsto F_t(x) \) is continuous. And since

\[ d(F_t(x), F_t(y)) = d(F_0(x), F_0(y)) + d(F_t(x), F_t(x)) + d(F_t(x), F_t(y)) \leq d(F_0(y), F_0(x)) + d(F_0(x), F_0(y)), \]

it sufficient to show that for \( \mathfrak{m} \times \mathfrak{m} \)-a.e. \( x, y \in A \) with \( d_s(x) = d_s(y) = l \) and \( d(x, y) < \delta < 1 \), for

\[ t \in (-\frac{b-a}{3}, \frac{b-a}{3}), \]

(5.6) \[ d(F_t(x), F_t(y)) \leq \Psi(\delta|K, a, b, r_0). \]

If \( m = 0, K = 0 \), by Pythagoras theorem, we see that for \( \mathfrak{m} \times \mathfrak{m} \)-a.e. \( x, y \in A \) with \( d_s(x) = d_s(y) = l \), for

\[ t \in (-\frac{b-a}{3}, \frac{b-a}{3}), \]

\[ d(x, y) = d(F_t(x), F_t(y)); \]

If \( K = 0, m \neq 0 \), by Cosine law,

\[ \frac{2(l+ r_0)^2 - d^2(x, y)}{2(l + r_0)^2} = \frac{2(l + r_0 + t)^2 - d^2(F_t(x), F_t(y))}{2(l + r_0 + t)^2}, \]

that is

\[ d^2(F_t(x), F_t(y)) = \left( \frac{l + r_0 + t}{l + r_0} \right)^2 d^2(x, y) \leq \frac{(b + r_0)^2}{(a + r_0)^2} d^2(x, y). \]

If \( K = (N - 1) \), by Cosine law,

\[ \frac{\cos d(x, y) - \cos^2(l + r_0)}{\sin^2(l + r_0)} = \frac{\cos d(F_t(x), F_t(y)) - \cos^2(l + r_0)}{\sin^2(l + r_0)}, \]

that is

\[ \frac{\cos d(F_t(x), F_t(y)) - 1}{\sin^2(b + r_0)} \leq \frac{\cos d(F_t(x), F_t(y)) - 1}{\sin^2(l + r_0)} = \frac{\cos d(x, y) - 1}{\sin^2(l + r_0)} \leq \Psi(\delta|a, b, r_0). \]

Thus (5.6) holds.

For \( K = -(N - 1) \), it is similar as the \( K = (N - 1) \) case. \( \square \)

6. WARPED PRODUCT STRUCTURE

In this section, we will finish the proof of Theorem 1.2. First by the Pythagoras theorem or Cosine law, we derive a warped product structure \( (a', b') \times_{\sin_{H}(r)} + \sin_{H}(r) \times_{\alpha} Y \) of \( A_{a', b'}(S) \) and see that \( Y \) has finite components. Then using this warped product and the RCD-condition of \( A_{a', b'}(S) \) we prove that each component of \( Y \) is infinitesimally Hilbertian and satisfies the Sobolev to Lipschitz property (see Theorem 6.1). And last, by a methods as in [26], we show that each component of \( Y \) is a RCD-space (see Theorem 6.8).
6.1. Warped product structure. Let \( a' = a + (b - a)/3, b' = b - (b - a)/3 \) and let \( A = A_{a', b'}(S) \) be as in Theorem 3.2. Let \( F : [-\frac{b - a}{3}, \frac{b - a}{3}] \times A \to A_{a,b}(S) \) be a continuous map as in Theorem 5.3. Let \( Y = \partial B_{a'}(S) \) and let \( \iota : Y \to A_{a,b}(S) \) be the inclusion map. Assume \( Y \) has one component. For \( y_1, y_2 \in Y \), define \( d_Y(y_1, y_2) \) and the measure \( m_Y \) as follows:

(i) for \( m = 0, K = 0, \)

\[
d_Y(y_1, y_2) = d(y_1, y_2);
\]

And for \( E \subset Y, \)

\[
m_Y(E) = \frac{m(\{x \in A, F_{-d_s(x) + a'}(x) \in E\})}{b' - a'};
\]

(ii) for \( m \neq 0 \) or \( K \neq 0, \)

\[
d_Y(y_1, y_2) = \frac{1}{\sin H(r_0 + a')} \inf \{L(\tilde{\gamma}), \tilde{\gamma} \in \iota(Y), \tilde{\gamma}_0 = y_1, \tilde{\gamma}_1 = y_2\},
\]

where

\[
L(\tilde{\gamma}) = \lim_{\delta \to 0} \left\{ \sum d(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i+1})), \frac{d(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i+1}))}{\delta}, 0 \leq t_1 \leq \cdots \leq t_n \leq b \text{ is a division of } \tilde{\gamma} \right\}
\]

For \( E \subset Y \), define the measure \( m_Y(E) \) as

\[
m_Y(E) = \frac{m(\{x \in A, F_{-d_s(x) + a'}(x) \in E\})}{\int_{a'}^{b'} \sin H^{-1}(t + r_0) dt}.
\]

Define a map from \( A \) to a warped product space as the following:

For \( m = 0 \) and \( K = 0, \)

\[
\Phi : A \to (a', b') \times Y, x \mapsto \Phi(x) = (d_s(x), F_{-d_s(x) + a'}(x)).
\]

Then by Pythagoras theorem, for \( x, x' \in A, \)

\[
d^2(x, x') = (d_s(x) - d_s(x'))^2 + d^2(F_{-d_s(x) + a'}(x), F_{-d_s(x') + a'}(x)) = d^2(\Phi(x), \Phi(x')).
\]

For \( m \neq 0 \) or \( K \neq 0, \)

\[
\Phi : A \to (a' + r_0, b' - r_0) \times_{\sin H(r)} Y, x \mapsto \Phi(x) = (d_s(x) + r_0, F_{-d_s(x) + a'}(x)).
\]

And by Cosine law and the definition of \( d_Y \), we have that for \( x, x' \in A \)

\[
d(x, x') = d_w(\Phi(x), \Phi(x')).
\]

In fact, assume \( \gamma \) is a minimal geodesic from \( x \) to \( x' \) with \( d(x, x') \) sufficient small such that \( \gamma \subset A \). Let \( F_{-d_s(\gamma(t)) + a'}(\gamma(t)) = \tilde{\gamma}(t) \). Dived \( \gamma \) with \( 0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1 \) such that \( d(\gamma(t_i), \gamma(t_{i+1})) \leq \delta \). Then

\[
d(x, x') = \lim_{\delta \to 0} \sum d(\gamma(t_i), \gamma(t_{i+1})) = \lim_{\delta \to 0} \sum \frac{\sin H(d_s(\gamma(\xi_i)) + r_0)}{\sin H(a' + r_0)} d(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i+1})) = \lim_{\delta \to 0} \sum \sin H(d_s(\gamma(\xi_i)) + r_0) d_Y(\gamma(t_i), \gamma(t_{i+1})) = d_w(\Phi(x), \Phi(x')).
\]

And more precisely, we have that for \( d_Y(y_1, y_2) \leq \delta, \) some \( \delta > 0, \)

(ii-1) if \( K = 0 \) and \( m \neq 0, \)

\[
d_Y(y_1, y_2) = \arccos \left( 1 - \frac{d^2(y_1, y_2)}{2(r_0 + a + (b - a)/3)^2} \right);
\]

(ii-2) if \( K = N - 1, \)

\[
d_Y(y_1, y_2) = \arccos \left( \frac{\cos d(y_1, y_2) - 1}{\sin^2(r_0 + a + (b - a)/3)^2} + 1 \right);
\]

(ii-3) if \( K = -(N - 1), \)

\[
d_Y(y_1, y_2) = \arccos \left( \frac{1 - \cosh d(y_1, y_2)}{\sinh^2(r_0 + a + (b - a)/3)^2} + 1 \right).
\]
If $Y$ has more than one components, by above discussion, restricted to each component of $A$, $\Phi$ is an isometry. Now as the discussion in [25 Claim 5.8], by relative volume comparison, we know that the number of $Y$’s components $\leq C(N,H,D,b,a)$. In fact, for each component $Y_k$, there is $y_k \in Y_k$ such that $x_k = F_{(b'-a')/2}(y_k) \in A$ and thus $B_{(b'-a')/3}(x_k) \subset A$. And for $l \neq k$, $B_{(b'-a')/3}(x_k) \cap B_{(b'-a')/3}(x_l) = \emptyset$. By relative volume comparison and that $A \subset B_{D+b}(p)$ for any $p \in S$, we know that $A$ contains at most $C(N,H,D,b,a)$ points which are $(b' - a')/3$ separated.

In the following, we will always assume $Y$ has one component.

By the uniqueness of solutions of the continuity equation (2.1) (more precisely the local uniqueness, see [8 Lemma 3.14]), we have that for any $E \subset A$, $t \in \left[\frac{1}{2} - (b-a)/3, (b-a)/3\right]$,

(i) for $m = 0$, $K = 0$,

$$\frac{d}{dt}(F_t)_* \mu(E) = \mu(E);$$

(ii) for $m \neq 0$ or $K \neq 0$, for a.e. $x \in A$, $\mu_t = (F_t)_* \mu$ satisfies

$$\frac{d}{dt} \mu_t(x) + \Delta d_s(F_{-t}(x)) \mu_t(x) = 0$$

which implies that

$$\frac{d}{dt} \mu_t(x) = \frac{sn_{H}^{-1}\left(d_s(x) - t + r_0\right)}{sn_{H}^{-1}\left(d_s(x) + r_0\right)} \mu_0(x).$$

By the definition of $m_Y$, (6.1), (6.2), and the property (6.3), (6.4) and the Laplacian estimates Theorem 3.1 as the proof of [16 Proposition 5.28] (see also [8 Lemma 5.11]), we have that for $a' \leq c < d \leq b'$ and a Borel subset $E \subset Y$,

$$m(E^d_c) = \begin{cases} m_Y(\overline{E}(d-c)), & m = 0 \text{ and } K = 0; \\ m_Y(\overline{E}\left(\int_c^d sn_{H}^{-1}(t + r_0)dt\right)), & m \neq 0 \text{ or } K \neq 0, \end{cases}$$

where

$$E^d_c = \{ x \in A, c \leq d_s(x) \leq d, F_{-d_s(x)+a'} \in \overline{E} \}.$$  

In fact, for $m \neq 0$ or $K \neq 0$, (6.0) can also be derived by (6.5) and the fact that

$$\int_{a' - d}^{a' - c} \mu_t(\overline{E})dt = m(E^d_c).$$

Since $\Phi$ is a isometry, we know that $A$ has a warped product structure. And thus $(a' + r_0, b' + r_0) \times sn_{H} Y$ is a $CD_{\text{loc}}(K,N)$-space for $m \neq 0$ or $K \neq 0$, $(a', b') \times Y$ is a $CD_{\text{loc}}(0,N)$-space for $m = 0$ and $K = 0$.

In the following, we denote

$$(Y_w, d_w, m_w) = \begin{cases} (\mathbb{R} \times Y, d, \mathcal{L}^1 \otimes m_Y), & m = 0, K = 0; \\ (C(Y), d_K, m_N), & m \neq 0 \text{ or } K \neq 0. \end{cases}$$

6.2. Properties of the metric measure space $(Y, d_Y, m_Y)$. In this subsection, we consider the metric measure space $(Y, d_Y, m_Y)$ as above and as in [8] we will show that

**Theorem 6.1.** Consider the metric measure space $(Y, d_Y, m_Y)$ defined as in above subsection. We have that $(Y, d_Y, m_Y)$ is infinitesimally Hilbertian, satisfies the almost everywhere locally doubling property, supports a local Poincaré inequality and is a measured-length space.

For any $[a'', b''] \subset [a', b']$, let

$$T : [a'', b''] \times Y \to A, \quad (t, y) \mapsto T(t, y) = F_t(y),$$

$$T : C([0, 1], Y) \times [a'', b''] \to C([0, 1], X), \quad (\gamma_s, t) \mapsto T(\gamma, t)_s = F_t(\gamma_s).$$

And let

$$P : A_{a,b}(S) \to Y, \quad x \mapsto P(x) = F_{-d_s(x)+a'}(x),$$

$$P : C([0, 1], A_{a,b}(S)) \to C([0, 1], Y), \quad \gamma_s \mapsto \tilde{P}(\gamma)_s = F_{-d_s(\gamma_s)+a'}(\gamma_s).$$

Consider the inclusion map $\iota : Y \to A_{a', b'}(S)$. By the definition of $d_Y$, for $m = 0, K = 0$, $\iota$ is an isometric embedding; for $m \neq 0$ or $K \neq 0$, $\iota$ is a Lipschitz map.

Since Poincaré inequality is invariant under a bi-Lipschitz map (cf. [8] Section 4.3, more precisely the proof of [8 Proposition 4.16]) and $\iota : Y \to A_{a', b'}(S)$ is Lipschitz, the local Poincaré inequality in $A_{a', b'}(S)$
implies the local Poincaré inequality in \((Y, d_Y, m_Y)\). And since \(A_{a', b'}(S) \subset X\) which is a RCD-space, by [31], \(A_{a', b'}(S)\) supports a local Poincaré inequality.

To see \((Y, d_Y, m_Y)\) is infinitesimally Hilbertian we only need to show that

**Lemma 6.2.** Given a nonnegative function \(h \in \text{Lip}(\mathbb{R})\) with \(h(t) = 0\) for \(t \in \mathbb{R} \setminus [a + \frac{b-a}{4}, b - \frac{b-a}{4}]\) and 
\(h(t) = 1\) for \(t \in [a', b']\), for each \(g \in L^2(Y, m_Y)\), define \(f(x) = g(P(x))h(d_x(x))\). Then \(g \in W^{1,2}(Y, d_Y, m_Y)\) if and only if \(f \in W^{1,2}(X, d, m)\) and for \(x \in A\),

(i) for \(m = 0, K = 0\),
\[
|\nabla f(x)| = |\nabla g|(F_{-a}(x)) = |\nabla Y g|(F_{-a}(x));
\]
(ii) for \(m \neq 0\) or \(K \neq 0\),
\[
|\nabla f(x)| = \frac{|\nabla g|(F_{-a}(x))}{|\nabla Y g|(F_{-a}(x))} = \frac{1}{|\nabla Y g|(F_{-a}(x))} = \frac{1}{\text{sn}_H(d_x(x) + r_0)}|\nabla g|(F_{-a}(x)).
\]

**Proof.** The proof is similar as in [3], Proposition 5.12, Theorem 5.13.

Consider test plans \(\Pi\) on \(Y\). Let \(\Pi = T_{\bar{\gamma}}\left(\text{Lip}(b' - a' - 1 L^1_{[a', b']}\right)\), where \([a'', b''] \subset [a', b']\) as above.

Claim 1: \(\Pi\) is a test plan of \(X\).

First note that
\[
\int_{C([0,1], X)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) = \int_{C([0,1], Y)} (b'' - a'' - 1) \int_0^{b''} \int_0^1 |\bar{\Pi}(\bar{\gamma}, t)|^2 ds d\Pi(\bar{\gamma}),
\]
and
\[
|\bar{\Pi}(\bar{\gamma}, t)|^2 = \lim_{h \to 0} \frac{d(\bar{\Pi}(\bar{\gamma}, t), h)}{|h|} = \lim_{h \to 0} \frac{d(F_t(\bar{\gamma} + h), F_t(\bar{\gamma}))}{|h|}.
\]
By Theorem 5.3 for \(m = 0, K = 0\),
\[
|\bar{\Pi}(\bar{\gamma}, t)|^2 = |\bar{\gamma}|,
\]
and thus
\[
\int_{C([0,1], X)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) = \int_{C([0,1], Y)} \int_0^1 |\gamma_s|^2 ds d\Pi(\bar{\gamma}) < \infty;
\]
For \(m \neq 0, K \neq 0\),
\[
|\bar{\Pi}(\bar{\gamma}, t)|^2 = \lim_{h \to 0} \frac{r_0 + a'}{r_0 + a'} ds d\Pi(\bar{\gamma}) = \frac{t + r_0 + a'}{|h|} |\bar{\gamma}_s|
\]
and thus
\[
\int_{C([0,1], X)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) = \int_{C([0,1], Y)} \int_0^1 |\gamma_s|^2 ds d\Pi(\bar{\gamma}) < \infty.
\]
For \(K = (N - 1)\),
\[
|\bar{\Pi}(\bar{\gamma}, t)|^2 = \lim_{h \to 0} \frac{\sin(r_0 + a') |d(\bar{\gamma}_s, \bar{\gamma})|}{|h|} = \frac{\sin(r_0 + a') |d(\bar{\gamma}_s, \bar{\gamma})|}{|h|}
\]
and thus
\[
\int_{C([0,1], X)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) \leq \frac{1}{\sin(r_0 + a')^2} \int_{C([0,1], Y)} \int_0^1 |\gamma_s|^2 ds d\Pi(\bar{\gamma}) < \infty.
\]
For \(K = -(N - 1)\), it is similar as the case \(K = N - 1\).

And for the set \(E_c^d = \{x \in A_{a', b}(S)\} \setminus d \leq d' \leq d', x \in E\}, \) where \(E\) is a Borel set in \(Y\),
\[
\int_{C([0,1], Y)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) = |\bar{\Pi} \times (b'' - a'' - 1 L^1_{[a', b']}((e_t \circ \bar{\Pi})^{-1} E_c^d) |\leq C m_Y(E_c^d) \leq C m(E_c^d).
\]

Claim 2: If \(f \in W^{1,2}(X, d, m)\), then \(g \in W^{1,2}(Y, d_Y, m_Y)\).

Assume \(f \in W^{1,2}(X, d, m)\), then
\[
\int_{C([0,1], Y)} \int_0^1 |\gamma_s|^2 ds d\Pi(\gamma) = \int_{C([0,1], Y)} (b'' - a'' - 1) \int_0^{b''} g(\gamma) h(\gamma) d\Pi(\gamma)
\]

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Thus, if for any such that for \( A \) implies almost metric cone Lina Chen.

Assume the pieces By the definition of \( d \), \( K \in \mathbb{N} \). For \( t \neq 0 \), \( t_{i+1} \in A_{a,b} \). By taking cut points for each \( i \), then

\[
|f(\gamma_i) - f(\gamma_0)| \leq |f(\gamma^1_i) - f(\gamma^0_i)| + \cdots + |f(\gamma^k_i) - f(\gamma^k_0)|.
\]

If for any \( \gamma \in C([0,1], A_{a,b}(S)) \), we have

\[
|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t) |\dot{\gamma}_t| dt,
\]

then we can take

\[
\tilde{G}(x) = \begin{cases} G(x), & x \in A_{a,b}(S), \\ 0, & x \in X \setminus A_{a,b}(S), \end{cases}
\]

such that

\[
|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 \tilde{G}(\gamma_t) |\dot{\gamma}_t| dt.
\]

Let \( \tilde{\Pi} = \tilde{P}_t(\Pi) \) where \( \Pi \) is supported in \( C([0,1], A_{a,b}(S)) \). As claim 1, we have that \( \tilde{\Pi} \) is a test plan of \( Y \). To see this as above

\[
\int_{C([0,1],Y)} \int_0^1 |\dot{\gamma}_t|^2 dt d\tilde{\Pi} = \int_{C([0,1],X)} \int_0^1 \left| \frac{d}{dt} F_{-d_s(\gamma_t) + \alpha'(\gamma_t)} \right|^2 dt d\tilde{\Pi}.
\]

For \( m = 0, K = 0 \),

\[
d^2(F_{-d_s(\gamma_{t+h}) + \alpha'(\gamma_{t+h})}, F_{-d_s(\gamma_{t}) + \alpha'(\gamma_{t})}) = d^2(\gamma_{t+h}, \gamma_t) - |d_s(\gamma_{t+h}) - d_s(\gamma_t)|^2 \leq d^2(\gamma_{t+h}, \gamma_t),
\]

Thus

\[
\int_{C([0,1],Y)} \int_0^1 |\dot{\gamma}_t|^2 dt d\tilde{\Pi} \leq \int_{C([0,1],X)} \int_0^1 |\dot{\gamma}_t|^2 dt d\tilde{\Pi} < \infty.
\]

For \( m \neq 0, K = 0 \), by

\[
\frac{d^2(\gamma_{t+h}, \gamma_t) - (d_s(\gamma_{t+h}) + r_0)^2 - (d_s(\gamma_t) + r_0)^2}{2(d_s(\gamma_{t+h}) + r_0)(d_s(\gamma_t) + r_0)} = \frac{d^2(F_{-d_s(\gamma_{t+h}) + \alpha'(\gamma_{t+h})}, F_{-d_s(\gamma_{t}) + \alpha'(\gamma_{t})}) - 2(\alpha' + r_0)^2}{2(\alpha' + r_0)^2},
\]

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we have
\[ d^2(F_{-d_s(\gamma_t+h)}+a'(\gamma_t+h), F_{-d_s(\gamma_t)+a'(\gamma_t)}) \leq \frac{(a'+r_0)^2}{(d_s(\gamma_t)+r_0)(d_s(\gamma_t)+r_0)} d^2(\gamma_t+h, \gamma_t). \]
Thus
\[ \int_{C([0,1],Y)} |\gamma_t'|^2 dt d\Pi \leq \int_{C([0,1],X)} \left( \frac{a'+r_0}{d_s(\gamma_t)+r_0} \right)^2 |\gamma_t'|^2 dt d\Pi < \infty. \]
For \( K = (N-1) \), by
\[
\frac{\cos d(\gamma_t+h, \gamma_t) - \cos(d_s(\gamma_t+h) + r_0) \cos(d_s(\gamma_t) + r_0)}{\sin(d_s(\gamma_t+h) + r_0) \sin(d_s(\gamma_t) + r_0)} = \frac{\cos d(F_{-d_s(\gamma_t+h)}+a'(\gamma_t+h), F_{-d_s(\gamma_t)+a'(\gamma_t)}) - \cos^2(a'+r_0)}{\sin^2(a'+r_0)},
\]
i.e.
\[
\frac{\cos d(\gamma_t+h, \gamma_t) - \cos(d_s(\gamma_t+h) - d_s(\gamma_t))}{\sin(d_s(\gamma_t+h) + r_0) \sin(d_s(\gamma_t) + r_0)} = \frac{\cos d(F_{-d_s(\gamma_t+h)}+a'(\gamma_t+h), F_{-d_s(\gamma_t)+a'(\gamma_t)}) - 1}{\sin^2(a'+r_0)},
\]
cos \( d(\gamma_t+h, \gamma_t) - 1 \leq \frac{\sin(d_s(\gamma_t+h) + r_0) \sin(d_s(\gamma_t) + r_0)}{\sin^2(a'+r_0)} (\cos d(F_{-d_s(\gamma_t+h)}+a'(\gamma_t+h), F_{-d_s(\gamma_t)+a'(\gamma_t)}) - 1). \]
thus
\[ \int_{C([0,1],Y)} |\gamma_t'|^2 dt d\Pi \leq \int_{C([0,1],X)} \left( \frac{\sin(a'+r_0)}{\sin(d_s(\gamma_t) + r_0)} \right)^2 |\gamma_t'|^2 dt d\Pi < \infty. \]
For \( K = -(N-1) \), similarly as the \( K = (N-1) \) case we have
\[ \int_{C([0,1],Y)} |\gamma_t'|^2 dt d\Pi \leq \int_{C([0,1],X)} \left( \frac{\sin(a'+r_0)}{\sin(d_s(\gamma_t) + r_0)} \right)^2 |\gamma_t'|^2 dt d\Pi < \infty. \]
And for \( E \subset Y \),
\[
(e_t)_\sharp \Pi(E) = \Pi((e_t \circ \hat{P})^{-1}(E)) = \Pi(e^{-1}_t(E_0^b)) = (e_t)_\sharp \Pi(E_0^b) \leq c m(E_0^b) \leq c'm_y(E).
\]
Now, since \( g \in W^{1,2}(Y, d_Y, m_Y) \) and
\[ \int_{C([0,1],X)} f(\gamma_1) - f(\gamma_0) d\Pi(\gamma) \]
\[ = \int_{C([0,1],X)} g(F_{-d_s(\gamma_1)+a'(\gamma_1)}) h(d_s(\gamma_1)) - g(F_{-d_s(\gamma_0)+a'(\gamma_0)}) h(d_s(\gamma_0)) d\Pi(\gamma) \]
\[ \leq \int_{C([0,1],Y)} g(\gamma_1) - g(\gamma_0) d\Pi(\gamma) + \int_{C([0,1],X)} g(F_{-d_s(\gamma_0)+a'(\gamma_0)}) \int_0^1 h'(d_s(\gamma_0)) |\gamma_1'| dt d\Pi(\gamma) \]
\[ \leq \int_{C([0,1],Y)} |\nabla g(\gamma_t)| |\gamma_t'| dt d\Pi(\gamma) + \int_{C([0,1],X)} g(F_{-d_s(\gamma_0)+a'(\gamma_0)}) \int_0^1 h'(d_s(\gamma_0)) |\gamma_1'| dt d\Pi(\gamma) \]
\[ \leq \int_{C([0,1],X)} \int_0^1 |\nabla g(F_{-d_s(\gamma_0)+a'(\gamma_0)}))| P(\gamma_1') + g(F_{-d_s(\gamma_0)+a'(\gamma_0)}) h'(d_s(\gamma_0)) |\gamma_1'| dt d\Pi(\gamma) \]
Thus for \( x \in A \),
\[ |\nabla f| (x) \leq |\nabla g|(F_{-d_s(x)+a'(x)}) \text{ Lip}(\hat{P}). \]
For \( m = 0, K = 0 \),
\[ |\nabla f| (x) \leq |\nabla g|(F_{-d_s(x)+a'(x)}); \]
For \( m \neq 0, K = 0 \),
\[ |\nabla f| (x) \leq \frac{a'+r_0}{d_s(x) + r_0} |\nabla g|(F_{-d_s(x)+a'(x)}); \]
For \( K \neq 0 \),
\[ |\nabla f| (x) \leq \frac{\text{sn}_H(a'+r_0)}{\text{sn}_H(d_s(x) + r_0)} |\nabla g|(F_{-d_s(x)+a'(x)}). \]

Recall the definitions of almost everywhere locally doubling property and measured-length property.
**Definition 6.3.** A metric measure space \((X, d, m)\) is almost everywhere locally doubling if there is a full measure Borel subset \(X \subset X\) satisfying that for each \(x \in X\), there exists an open set \(U \ni x\), constants \(C, R > 0\), such that for \(r \in (0, R)\), \(y \in U\),

\[
m(B_{2r}(y)) \leq Cm(B_r(y)).
\]

**Definition 6.4.** A metric measure space \((X, d, m)\) is measured-length if there is a full measure subset \(X \subset X\) satisfying the following: For \(x_0, x_1 \in X\), there exist \(\epsilon > 0\), a map

\[
(0, \epsilon)^2 \to \mathcal{P}(C([0, 1], X)), (t_0, t_1) \mapsto \Pi_{t_0, t_1}^{x_0, x_1},
\]

such that

(i) For \(\phi \in C_0(C([0, 1], X))\), the map

\[
(0, \epsilon)^2 \to \mathbb{R}, (t_0, t_1) \mapsto \int \phi d\Pi_{t_0, t_1}^{x_0, x_1}
\]

is Borel;

(ii) For \(i = 0, 1\),

\[
(c_i)_t \Pi_{t_0, t_1}^{x_0, x_1} = \frac{1}{m(B_t(x_i))}m;
\]

(iii)

\[
\limsup_{t_0, t_1 \to 0} \int \int_0^1 |\gamma| dtd\Pi_{t_0, t_1}^{x_0, x_1}(\gamma) \leq d^2(x_0, x_1).
\]

Now we prove \((Y, d_Y, m_Y)\) is almost everywhere locally doubling and measured-length.

**Lemma 6.5.** The metric measure space \((Y, d_Y, m_Y)\) is almost everywhere locally doubling.

**Proof.** For \(E \subset Y\), let \(E_r = \{x \in A, 0 \leq d_a(x) - a' \leq r, F_{-d_a(x) + a'} \in E\}\).

For each \(x \in Y\), there is open set \(Y \supset U \ni x, R > 0\), that for each \(y \in U\), \(r < R\),

(i) for \(m = 0, K = 0\),

\[
m_Y(B_{2r}(y)) = \frac{m(B_{2r}(y))}{2r} \leq \frac{m(B_{2r}(F_r(y)))}{2r} \leq \frac{\text{vol}(B_r^H)}{\text{vol}(B_r^H)} m_Y(B_r(y)).
\]

(ii) for \(m \neq 0\) or \(K \neq 0\), note that for any \(r < R\), there are \(0 < c(K, R) < C(K, R)\) such that

\[
B_{c(K, R)r}(F_r(y)) \subset B_r(y) \subset B_{2r}(y) \subset B_{C(K, R)r}(F_r(y)).
\]

Then by relative volume comparison, a similar argument as the \(m = 0, K = 0\) case gives the almost everywhere locally doubling property.

**Lemma 6.6.** The metric measure space \((Y, d_Y, m_Y)\) is a measured-length space.

The proof of this lemma is the same as the one [3 Proposition 5.14]. Here we omit it. And by the following theorem, we can derive a series of properties about \((Y_w, d_w, m_w)\).

**Theorem 6.7** (\cite{20,3}). Consider a warped product space \(Y_w = I \times_w Y\) where \(I\) is a bounded interval in \(\mathbb{R}\) and \(w, w_m : I \to [0, \infty)\) with \(w_m > 0\) for points in the interior of \(I\). Assume \((Y, d_Y, m_Y)\) is a.e. locally doubling, measured length, infinitesimally Hilbertian, then \((Y_w, d_w, m_w)\) is a.e. locally doubling, measured length, infinitesimally Hilbertian and it has the Sobolev to Lipschitz property.
6.3. \((Y, d_Y, m_Y)\) is a RCD-space. For \(m = 0, K = 0\), we will show that \((\mathbb{R} \times Y, d, \mathcal{L}^1 \otimes m_Y)\) satisfies the CD_{\text{loc}}(0, N) condition. Then by the local-to-global property [15, Theorem 3.14] and [32], we know that \(\mathbb{R} \times Y\) is essentially non-branching and is a CD(0, N)-space. Finally an argument as in [16] gives that 
\((Y, d_Y, m_Y) \in \text{RCD}(0, N - 1)\).

For \(K \neq 0\) or \(m \neq 0\), Ketterer [20, Theorem 1.2] (see Theorem [2.10]) proved that if \((C(Y), d_K, m_N) \in \text{RCD}^*(K, N)\), then \((Y, d_Y, m_Y) \in \text{RCD}^*(N - 2, N - 1)\) and diam\((Y) \leq \pi\). We have known that: \((a' + r_0, b' + r_0) \times_{\mathbb{R}_{t'}} Y\) is isometric to \(A_{d_{t'}}(S) \subset X\) which is a CD\((K, N)\)-space. We will show that Ketterer’s result (ii) of Theorem [2.10] holds under this weaker condition.

First recall some basic definitions about Dirichlet forms. See [2] or [26] for more details.

Let \((X, d, m)\) be a locally compact, separable Hausdorff metric measure space. A symmetric Dirichlet form \(\mathcal{E}^X\) defined in \(D(\mathcal{E}^X) \subset L^2(X, m)\) is a \(L^2(X, m)\)-lower semi-continuous, quadratic form that satisfies the Markov property. The domain \(D(\mathcal{E}^X)\) is a Hilbert space with respect to
\[
(u, u)_{D(\mathcal{E}^X)} = (u, u)_{L^2(X, m)} + \mathcal{E}^X(u, u).
\]

There is a self-adjoint, negative-definite operator \((L^X, D_2(L^X))\) on \(L^2(X, m_X)\) where
\[
D_2(L^X) = \{u \in D(\mathcal{E}^X), \exists v \in L^2(X, m), -(v, u)_{L^2(X, m)} = \mathcal{E}^X(u, w), \forall w \in D(\mathcal{E}^X)\}.
\]

Let \(v = L^X u\).

Denote \(D_{\infty}(\mathcal{E}^X) = D(\mathcal{E}^X) \cap L^\infty(X, m)\). For \(u, \phi \in D_{\infty}(\mathcal{E}^X)\), define
\[
\Gamma^X(u; \phi) = \mathcal{E}^X(u, u\phi) - \frac{1}{2} \mathcal{E}^X(u^2, \phi),
\]

which can be extended by continuity to \(u \in D(\mathcal{E}^X)\).

For \(u, v \in D(\mathcal{E}^X), \phi \in D(\mathcal{E}^X)\), define
\[
\Gamma^X(u, v; \phi) = \frac{1}{2} \left(\Gamma^X(u; \phi) + \Gamma^X(v; \phi) - \Gamma^X(u - v; \phi)\right).
\]

And let
\[
2\Gamma^X_2(u, v; \phi) = \Gamma^X(u, v; L^X \phi + L^X_v u; \phi) - 2\Gamma^X(u, L^X u; \phi), \quad \Gamma^X_2(u; \phi) = \Gamma^X_2(u, u; \phi),
\]

where \(u, v \in D(\Gamma^X_2) = \{u \in D_2(L^X), L^X u \in D(\mathcal{E}^X)\}\), test function \(\phi \in D_{\infty}(L^X) = \{\phi \in D_2(L^X), \phi, L^X \phi \in L^\infty(X, m), \phi > 0\}\).

Let \(D'\) be the set of \(u\) such that the map \(\phi \mapsto \Gamma^X(u; \phi)\) is an absolutely continuous measure w.r.t. \(m\) which is denoted by \(\Gamma^X(u)m\). If \(D' = D(\mathcal{E}^X)\), we call \(\mathcal{E}^X\) admits a “carré du champ” operator.

If \(\mathcal{E}^X\) is strongly local and admits a “carré du champ” operator (see [26, Section 2.1]), one can define \(D_{\text{loc}}(\mathcal{E}^X) \subset L^2_{\text{loc}}(X, m)\) and thus there is an intrinsic distance of \(\mathcal{E}^X\),
\[
d_{\mathcal{E}^X}(x, y) = \sup\{u(x) - u(y), u \in D_{\text{loc}}(\mathcal{E}^X) \cap C(X), \Gamma^X(u) \leq 1, m - a.e.\}.
\]

Note that if \((X, d, m)\) is infinitesimally Hilbertian, the Cheeger energy of \(X\), \(\text{Ch}^X\) is a symmetric Dirichlet form. And the corresponding \(L^X u = \Delta_X u\), the Laplacian of \(u\), \(\Gamma^X\) is the same as the one defined in Subsection 2.1. Compared with \(\Gamma^2\) defined in RCD\((K, N)\) (Subsection 2.5), \(\Gamma^X_2\) here is in a weak sense.

**Theorem 6.8.** Let \((Y, d_Y, m_Y)\) be as one in the beginning of this section.

(i) For \(m = 0, K = 0\), \((\mathbb{R} \times Y, d, \mathcal{L}^1 \otimes m_Y)\) is a RCD\((0, N)\)-space and thus \((Y, d_Y, m_Y)\) is a RCD\((0, N - 1)\)-space;

(ii) For \(m \neq 0, K = 0\), \((C(Y), d_0, m_N)\) is a RCD\((0, N)\)-space and \((Y, d_Y, m_Y)\) is a RCD\((0, N - 2, N - 1)\)-space;

(iii) For \(K \neq 0\), \((Y, d_Y, m_Y)\) is a RCD\((N - 2, N - 1)\)-space. Especially for \(K > 0\), \((C(Y), d_K, m_N)\) is a RCD\((K, N)\)-space.

**Proof.** For \(m = 0\) and \(K = 0\), note that for \(\delta > 0\), \((a' + \delta, b' - \delta) \times Y\) is locally a CD\((0, N)\)-space. And for any \(t \in \mathbb{R}, (t \times Y, t + b' - a' - 2\delta) \times Y\) is isometric to \((a' + \delta, b' - \delta) \times Y\) and thus is locally a CD\((0, N)\)-space. Since \(\mathbb{R} \times Y\) can be covered by \((t \times Y, t + b' - a' - 2\delta) \times Y, t \in \mathbb{R}\), we know \((\mathbb{R} \times Y, d, \mathcal{L}^1 \otimes m_Y)\) is in CD_{\text{loc}}\((0, N)\).

By the discussion above Theorem 3.14 in [15] and the infinitesimally Hilbertian property of \(\mathbb{R} \times Y\) (Theorem 6.4 and Theorem 6.7), we know that \(\mathbb{R} \times Y\) is essentially non-branching (see [32]). And then by local to global property [12], \((\mathbb{R} \times Y, d, \mathcal{L}^1 \otimes m_Y)\) is RCD\((0, N)\)-space.

Now \((Y, d_Y, m_Y)\) is a RCD\((0, N - 1)\)-space by the argument in [16].
For $m \neq 0$ or $K \neq 0$, first endow $Y$ with a new metric $d'_Y$ such that $\text{diam}(Y) \leq \pi$:

$$d'_Y(y_1, y_2) = \begin{cases} 
  d_Y(y_1, y_2), & \text{if } d_Y(y_1, y_2) \leq \pi; \\
  \pi, & \text{if } d_Y(y_1, y_2) > \pi.
\end{cases}$$

Then $(Y, d_Y)$ is locally isometric to $(Y, d'_Y)$ and the $(K, N)$-cone $C(Y, d_Y) = C(Y, d'_Y)$. In the following we will assume $Y$ endowed with the metric $d'_Y$.

For $m \neq 0$, $K = 0$, note that for any $r > 0$, $(C(Y), r^{-1}d_0, c_r m_N)$ is isometric to $(C(Y), d_0, m_N)$, where

$$c_r = \left( \int_{B_r(x)} 1 - \frac{d_0(x, y)}{r} dm_N(y) \right)^{-1}.$$ 

Then for any $t > 0$, $(a' + r_0 + \delta, b' + r_0 - \delta) \times_r Y \subset (C(Y), (a' + r_0 + \delta)/td_0, c_t(a' + r_0 + \delta)m_N)$ is isometric to the one in $(C(Y), d_0, m_N)$ which is locally a CD(0, $N$)-space. Rescaling back, we can see that

$$\left( t \cdot \frac{b' + r_0 - \delta}{a' + r_0 + \delta} \right) \times_r Y \in (C(Y), d_0, m_N)$$

is in $\text{CD}_{\text{loc}}(0, N)$ which implies $(C(Y), d_0, m_N) \in \text{CD}_{\text{loc}}(0, N)$. As the above discussion we know that $(C(Y), d_0, m_N)$ is a RCD$(0, N)$-space. Then (ii) of Theorem 2.10 implies that $(Y, d'_Y, m_Y)$ is a RCD$(0, N - 1)$-space. And so is $(Y, d_Y, m_Y)$.

For (iii), we follow the argument in the proof of [20, Theorem 1.2].

By Theorem 6.1 we know that $(Y, d_Y, m_Y)$ is infinitesimally Hilbertian and satisfies the Sobolev to Lipschitz property. Thus by [15] to prove $(Y, d_Y, m_Y)$ is a RCD$(N - 2, N - 1)$-space, one only need to show it satisfying the $(N - 2, N - 1)$ Bakry-Emery inequality: for any $f \in W^{1,2}(Y, d_Y, m_Y)$, $t > 0$,

$$(6.7) \quad |\nabla(H_t(f))|^2 + \frac{4(N - 2)t^2}{(N - 1)(e^{2(N-2)t} - 1)}|\Delta H_t(f)|^2 \leq e^{-2(N-2)t}H_t(|\nabla f|^2) m_Y - a.e.$$ 

As in the proof of [20, Theorem 1.2], we can derive (6.7) by the following kind of Bakry-Emery inequality: for any $u \in D(\Gamma^Y_Y)$,

$$(6.8) \quad \frac{1}{2} \int_Y L^Y \phi \Gamma^Y(u) dm_Y - \int_Y \Gamma^Y(u, L^Y u) \phi dm_Y \geq (N - 1) \int_Y \Gamma^Y(u) \phi dm_Y + \frac{1}{N} \int_Y (L^Y u)^2 \phi dm_Y - \frac{1}{(N + 1)N} \int_F (L^Y u + Nu)^2 \phi dm_Y.$$ 

Namely the methods and calculations from [6.3] to [6.7] are similar as the proof of Bakry-Emery inequality and Bakry-Ledoux estimate in [13] (see [15] Proposition 4.7, 4.9, see also the proof of [20, Theorem 1.2]). Here we omit it and only point out that in the proof one needs the regularity of $\text{Ch}^Y$: The intrinsic distance of $\text{Ch}^Y$, $d_{\text{Ch}^Y} = d_Y$ and $\text{Ch}^Y$ is strongly regular [20, Lemma 5.14].

In [20, Lemma 5.14], Ketterer assumed that $(C(Y), d_K, m_N)$ is a RCD$^*$-space. By examining the proof there, we can see that it only needs the Sobolev to Lipschitz property of $(C(Y), d_K, m_N)$ which can be seen from Theorem 6.1.

In the following, we will see how to derive (6.8) from the relations between the Cheeger energies $\text{Ch}^Y$, $\text{Ch}^{C(Y)}$ and the warped product $C^C$, where $C^C$ is a symmetric form on $L^2(C(Y), \text{sn}^{-1}N(t)dt \otimes m_Y)$ defined as

$$C^C(u) = \int_Y \int_{I_K} |u_y'(t)|^2 dt dm_Y + \int_{I_K} \text{Ch}^Y(u_p) \text{sn}^{-3}N(t) dt$$

where $u \in C_0^\infty(I_K) \otimes D(\text{Ch}^Y)$ and $u_p = u(p, \cdot)$, $u_y = u(\cdot, y)$.

First note that:

- Claim 1: As in [20, Lemma 5.11], we have that the intrinsic distance $d_{C^C}$ of $C^C$ satisfies that, for $x, y \in A_{a', b'}(S)$,

$$d_{C^C}(x, y) = d_K(x, y).$$

- Claim 2: As [20, Corollary 5.12], for $u \in D_{\text{loc}}(C^C) \cap L^\infty(A_{a, b}(S))$, we have that

$$C^C(u) = \text{Ch}^{C(Y)}(u).$$
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By the definition of $\mathcal{E}^C$, for $u_1 \otimes u \in C^0_0((a', b')) \otimes D(G^Y_0)$, $1 \otimes \phi \in 1 \otimes D^{h,2}(L^Y)$, a careful calculation as in [26] gives that (see also (33) in [26]):

$$\Gamma^C_2(u_1 \otimes u; 1 \otimes \phi) = \int_{C(Y)} I_{2}^{K, sn_H^{-1}}(u_1)u^2 \phi + \frac{u^2}{sn_H} \Gamma_2^Y(u)\phi + \frac{1}{2} \Gamma^K(u_1) \frac{1}{sn_H} L^Y(u)\phi$$

$$(6.9) + \int_{C(Y)} \left( \frac{1}{2} L^{I_K, sn_H^{-1}} - \frac{u^2}{sn_H} \right) \Gamma^Y(u) = \frac{u_1}{sn_H} L^{I_K, sn_H^{-1}}(u_1) \Gamma^Y(u) - \Gamma^K \left( u_1, \frac{u_1}{sn_H} \right) u L^Y(u) \phi,$$

where $L^{I_K} = d^2/dt^2$ and

$$\mathcal{E}^{I_K, sn_H^{-1}}(u) = \int I_K (u')^2 sn_H^{-1} dt,$$

thus

$$L^{I_K, sn_H^{-1}}(u) = u'' + \frac{N - 1}{sn_H} sn_H u'.$$

By Claim 2, $\mathcal{E}^C = Ch(C^Y)$. And the Bakry-Émery inequality holds in $A_{a', b'}(S)$:

$$\Gamma^C_2(u_1 \otimes u; 1 \otimes \phi) \geq K \Gamma^C(u_1 \otimes u; \phi) + \frac{1}{N} \int (L^C u_1 \otimes u)^2 \phi.$$

Then as in [26] Theorem 3.10, by taking $u_1 = \sin t$, for $K = N - 1$, and $u_1 = \sinh t$, for $K = -(N - 1)$, $t \in (a', b')$ in (6.3) and using the above Bakry-Émery inequality, we have (6.3).

To see Claim 1, note that in the proof of [26] Lemma 5.11, the only place where Ketterer used the RCD$^*$-condition of $C(Y)$ is to derive the inequality (see (47) in [26])

$$(6.10)$$

Here we want to show (6.10) holds locally in $A_{a', b'}(S)$. This can be seen by restricting all Ketterer’s estimates in $A_{a', b'}(S)$ and then the curvature-dimension condition can be applied similarly.

In the proof of [27] Corollary 5.12, one only needs the local doubling property and local Poincaré inequality of $Y$ and $C(Y)$, and Claim 1. Thus Claim 2 is derived.

REFERENCES

[1] L. Ambrosio, Calculus heat flow and curvature-dimension bounds in metric measure spaces. Proceedings of ICM 2018, (2018)
[2] L. Ambrosio, N. Gigli, G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, Duke Math. J., 163(7), (2014), 1406-1490
[3] L. Ambrosio, N. Gigli, A. Mondino, T. Rajala, Riemannian Ricci curvature lower bounds in metric measure space with $\sigma$-finite measure. Trans. AMS, 367, (2015), 4661-4701.
[4] L. Ambrosio, A. Mondino, G. Savaré, Nonlinear diffusions and curvature conditions in metric measure spaces, Mem. Amer. Math. Soc., 262 (2019), no. 1270
[5] L. Ambrosio, D. Trevisan . Well-posedness of Lagrangian flows and continuity equations in metric measure spaces[J]. Analysis and PDE, 7(5), (2014),1179-1234
[6] A. Björn, J. Björn, Nonlinear potential theory on metric spaces, volume 17 of EMS Tracts in Math. European Mathematical Society (EMS), Zürich, (2011).
[7] A. Bartscher, C. Ketterer, R. McCann, E. Woolgar, Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 131, 29. MR 4185085.
[8] C. Connell, X. Dai, J. Núñez-Zimbrón, R. Perales, P. Suárez-Serrato, G. Wei, Maximal volume entropy rigidity for RCD$^*$(−(N − 1), N) spaces, https://arxiv.org/abs/1809.06909
[9] L. Chen, Quantitative maximal volume entropy rigidity on Alexandrov spaces, to appear in Proceeding of AMS.
[10] J. Cheeger; T.H. Colding, Almost rigidity of warped products and the structure of spaces with Ricci curvature bounded below, Ann. of Math. (2) 144 (1996), 189 – 237. MR 1405949.
[11] F. Cavalletti, A. Mondino, Sharp and rigidity isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds, Invent. Math., 208(2017), no. 3, 803-849.
[12] F. Cavalletti, E. Milman, The globalization theorem for the curvature-dimension condition. Invent. math. (2021) 226: 1-137
[13] F. Cavalletti, A. Mondino, New formulas for the Laplacian of distance functions and applications, Anal. PDE 13(7) (2020), 2091-2147.
[14] Q. Deng, Hölder continuity of tangent cones in RCD(K, N) spaces and applications to non-branching, arXiv: 2009.07956
[15] M. Erbar, K. Kuwada, K. Strumr, On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure space. Invent. Math., 201(3), (2015), 993-1071.
[16] N. Gigli, The splitting theorem in non-smooth context[J]. Mathematics, (2013).
[17] N. Gigli, An overview on the proof of the splitting theorem in non-smooth context, Anal. Geom. Metric Spaces, 2 (2014)
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[18] N. Gigli, On the differential structure of metric measure spaces and applications. Memoirs of the American Mathematical Society, 236(1113), (2015)
[19] N. Gigli, Nonsmooth differential geometry-an approach tailored for spaces with Ricci curvature bounded from below, Mem. Amer. Math. Soc. 251 (2018), v-161
[20] N. Gigli, B. Han, Sobolev spaces on warped products, J. Funct. Anal. 275 (2018), no. 8, 2059-2095
[21] N. Gigli, C. Rigoni, Recognizing the flat torus among RCD*(0, N) spaces via the study of the first cohomology group, Calc. Var. PDE, 57(2018), no. 4
[22] N. Gigli, L. Tamanini, Second order differentiation formula on RCD*(K, N) spaces, to appear in J. Eur. Math. Soc. (JEMS) (2018).
[23] B. Han, Ricci tensor on RCD*(K, N) spaces, J. Geom. Anal., 28 (2018), no. 2, 1295-1314.
[24] E. Heintze, H. Karcher, A general comparison theorem with application to volume estimates for submanifolds, Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 451-470
[25] X. Huang, An almost rigidity theorem and its applications to noncompact RCD(0, N) spaces with linear volume growth, arXiv: 1710.05830
[26] C. Ketterer, Cones over metric measure spaces and the maximal diameter theorem, J. Math. Pures Appl. 103, (2015) 1228-1275.
[27] C. Ketterer, The Heintze-Karcher inequality for metric measure spaces, arXiv: 1908.06146
[28] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport. Ann. Of Math., (2), 169(3), (2009), 903-991
[29] A. Mondino and A. Naber, Structure theory of metric-measure spaces with lower Ricci curvature bounds, arxiv:1405.222 (to appear in Journ. European Math Soc.)
[30] G. Philippis, N. Gigli, From volume cone to metric cone in the nonsmooth setting, Geom. Funct. Anal., Vol. 26, (2016),1526-1587.
[31] T. Rajala, Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Strum. J. Funct. Anal., 263(4), (2012), 896-924.
[32] T. Rajala, K. Sturm, Non-branching geodesics and optimal maps in strong CD(K,∞) spaces, Calc. Var. PDE, 50 (2014), 831-846.
[33] G. Savaré, Self-improvement of the Bakry-Émery condition and Wassertein contraction of the heat flow in RCD(K,∞) metric measure spaces, Disc. Cont. Dyn. Sist. 34 (2014), 1641-1661.
[34] K. Sturm, On the geometry of metric measure spaces. I. Acta Math. 196(1), (2006), 65-131
[35] K. Sturm, On the geometry of metric measure spaces. II. Acta Math. 196(1), (2006),133-177
[36] C. Villani, Optimal transport. Old and New, Springer-Verlag, (2009), Berlin

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