Optimal local expansion of W states using linear optics and Fock states

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We derive the maximum success probability of circuits with passive linear optics for post-selectively expanding an N-photon W state to an \((N + n)\)-photon W state, by accessing only one photon of the initial W state and adding \(n\) photons in a Fock state. We show that the maximum success probability is achieved by a polarization-dependent beamsplitter and \(n - 1\) polarization-independent beamsplitters.

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I. INTRODUCTION

Multipartite entanglement has rich structure and diverse properties, which can be exploited for many kinds of applications. With an increasing number of parties sharing entanglement, the fact that the structure becomes more complex also means that preparation of such an entanglement may become more difficult, especially if one tries to generate the whole state in a single step. Instead, one may start with an initial entangled state among a small number of subsystems, and then expand it by adding ancillary subsystems. In fact, many interesting classes of multipartite entangled states, such as Greenberger-Horne-Zeilinger (GHZ) states [1], cluster states [2] and W states [3], are defined for an arbitrary number of qubits, which makes the expansion strategy look attractive. There are several proposals of state expansion for multipartite entangled states by local manipulation on a single site without accessing other qubits [4–6]. For example, in the case of GHZ states, a deterministic local expansion of an N-qubit state to an \((N + n)\)-qubit state is possible in principle. In optical systems, several experimental demonstrations have been performed [7–10] by using quantum parity checking gates [11].

W states are an interesting class of multipartite entangled states in that they have a web-like entanglement structure. An N-qubit W state is represented by \(|W_N\rangle = (|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |0\cdots1\rangle)/\sqrt{N}\). Every qubit in \(|W_N\rangle\) has bonds with every other qubit, and the pairwise entanglement survives even if all the other \((N - 2)\)-qubits are discarded [3, 12, 13]. In recent years, there have been a number of theoretical proposals and experimental demonstrations using W states in multiparty protocols such as quantum key distribution [14], leader election [15] as well as preparation of W states in optical systems [16–26]. In the case of W states, a deterministic local expansion is impossible even in principle because the marginal state of the remaining untouched \(N - 1\) qubits is different for \(|W_N\rangle\) and \(|W_{N+n}\rangle\), so it is worth seeking nontrivial ways of expanding W states probabilistically. Since the qualitative difference from GHZ states in the expandability may arise from the difference in the nature of multipartite entanglement, study of efficient local expansions of W states is interesting theoretically as well as practically.

For photonic polarization-based qubits, recent proposals and a demonstration include the expansion of \(|W_N\rangle\) to \(|W_{N+1}\rangle\) [4, 5] and to \(|W_{N+2}\rangle\) [6, 27]. These expansion schemes are composed of passive linear optics and one or two photons in a Fock state. In this paper, we address the question of what is the best way of expanding \(|W_N\rangle\) to \(|W_{N+n}\rangle\) for photonic polarization-based qubits. We discuss the maximum success probability of local expansion methods composed of passive linear optics and an ancilla mode in an \(n\)-photon Fock state. We derive the maximum success probability, and also show that it is achieved by a polarization-dependent beam splitter (PDBS) and \(n - 1\) polarization-independent beamsplitters (BSs). In the case of \(n = 2\), the optimal success probability is higher than that of the expanding gate proposed so far [6].

This paper is organized as follows: In Sec. II, we describe the expansion schemes considered in this paper. In Sec. III, we derive an upper bound on the success probability for expanding W states. In Sec. IV, we explicitly construct a linear optical circuit that achieves the derived upper bound on the success probability. Finally, in Sec. V, we give a brief summary and conclusions.

II. EXPANSION METHODS OF W STATES

The optical circuit considered in this paper for expanding W states is composed of only passive linear optics, that is, of PDBSs, BSs, phase shifters, wave plates, and arbitrary linear losses. As shown in Fig. 1 (a), the circuit has two input spacial modes 1 and 2. One photon from an \(N\)-photon polarization-entangled W state, which is represented by \(|W_N\rangle = (|VH\cdotsH\rangle + |HVH\cdotsH\rangle + \cdots + |H\cdotsHV\rangle)/\sqrt{N}\) where \(|H(V)\rangle\) is the state of an horizontally (H-) and vertically (V-) polarized photon, is fed to mode 1, and an H-polarized \(n\)-photon Fock state (we denote it by \(|nH\rangle\)) is fed to mode 2. We define the successful operation of the circuit to be the events where the photons come out from the output spacial modes 1, . . . , \(n+1\), one by one, namely, exactly one photon from every output mode. We require that the output state is exactly the \((N + n)\)-photon W state for the success events.

As is well known, linear optical losses can be equiva-
lently described by lossless optical circuits with auxiliary spacial modes. This is because a linear loss with transmission $T$ is equivalently realized by using a BS with transmission $T$. Since a BS is a lossless component with two input modes and two output modes, we can simulate losses with lossless components by introducing auxiliary input modes initially in the vacuum and the same number of auxiliary output modes from which the ‘lost’ photons escape. The circuit in Fig. 1 (a) is thus described equivalently by a lossless circuit with $L(\geq n + 1)$ input and output spacial modes. The family of schemes considered in this paper is described as follows. The expansion circuit has $2L$ input and output modes composed of $L$ spatial modes with each having two polarizing (H- and V-polarized) modes. One photon from an $N$-photon polarization-entangled W state enters into input spacial mode 1 and an H-polarized $n$-photon Fock state enters into input spacial mode 2. All of the other input spacial modes receive vacuum states. Arbitrary lossless linear optical operations are applied to the 2$L$ input modes. We show this expansion circuit in Fig. 1 (b).

Let $|\text{vac}\rangle_{\text{in(out)}}$ be the state of all the 2$L$ input(output) modes in the vacuum. The unitary operation $U$ for any passive linear optical operations satisfies $U|\text{vac}\rangle_{\text{in}} = |\text{vac}\rangle_{\text{out}}$, and relevant actions of $U$ are represented by

$$Ua_{1H}^\dagger U^\dagger = \sum_{j=1}^L (\beta_j H b_{jH}^\dagger + \beta_j V b_{jV}^\dagger),$$

$$Ua_{1V}^\dagger U^\dagger = \sum_{j=1}^L (\gamma_j H b_{jH}^\dagger + \gamma_j V b_{jV}^\dagger),$$

and

$$Ua_{2H}^\dagger U^\dagger = \sum_{j=1}^L (\alpha_j H b_{jH}^\dagger + \alpha_j V b_{jV}^\dagger),$$

where $a_{1H(V)}^\dagger$ and $a_{2H(V)}^\dagger$ are the H(V)-polarized photon creation operators for input spacial mode 1 and 2, $b_{jH(V)}^\dagger$ is the H(V)-polarized photon creation operator for output spacial mode $j$, and $\alpha_j, \beta_j, \gamma_j$ are complex numbers. Using the commutation relation $[a_{js}, a_{js'}^\dagger] = [b_{js}, b_{js'}^\dagger] = \delta_{jj'}\delta_{ss'}(s, s' = H, V)$ and Eqs. (1) – (3), we obtain

$$\sum_{j=1}^L (\Omega_j H \Omega_j^* + \Omega_j V \Omega_j^* V) = \delta_{HH'},$$

for $\Omega, \Omega' = H, V$. By denoting $|H_N\rangle \equiv |H\cdots H\rangle$, the input state to the expansion circuit in Fig. 1 can be written as

$$|W_N\rangle |n\rangle = \frac{1}{\sqrt{N}} \left( \sqrt{N-1} |W_{N-1}\rangle \otimes \frac{a_{1H}^\dagger (a_{2H}^\dagger)^n}{\sqrt{n!}} |\text{vac}\rangle_{\text{in}} |W_{N-1}\rangle \otimes \frac{a_{1V}^\dagger (a_{2V}^\dagger)^n}{\sqrt{n!}} |\text{vac}\rangle_{\text{in}} \right),$$

while state $|W_{N+n}\rangle$ which we desire as an output is

$$|W_{N+n}\rangle = \frac{1}{\sqrt{N+n}} \left( \sqrt{N-1} |W_{N-1}\rangle \otimes \prod_{j=1}^{n+1} b_{jH}^\dagger |\text{vac}\rangle_{\text{out}} + |H_{N-1}\rangle \otimes \sum_{j=1}^{n+1} \prod_{i\neq j}^{n+1} b_{ij}^\dagger |\text{vac}\rangle_{\text{out}} \right).$$

The post-selected events where the photons come out from output spacial mode 1, . . . , $n + 1$ one by one are described by a projector written as

$$\Pi_{\text{post}} = \sum_{s_1 = H, V} \cdots \sum_{s_{n+1} = H, V} \left( \prod_{i=1}^{n+1} b_{is_i}^\dagger \right) |\text{vac}\rangle_{\text{out}} |\text{vac}\rangle_{\text{out}} \prod_{j=1}^{n+1} b_{js_j}^\dagger,$$

By applying $U$ and $\Pi_{\text{post}}$ to Eq. (5), we should obtain Eq. (6). Hence

$$\Pi_{\text{post}} U |W_N\rangle |n\rangle = \sqrt{P_{\text{suc}}} |W_{N+n}\rangle,$$
where $P_{\text{suc}}$ is the probability of success. By operating $(W_{N-1} \otimes \langle \text{vac} \rangle_{\text{out}}(\prod_{i=1}^{n+1} b_{iH}))$ on Eq. (8) from the left, we have

$$
\sqrt{P_{\text{suc}}} = \sqrt{\frac{n!(N+n)}{N}} \eta_0 ,
$$

(9)

where

$$
\eta_0 \equiv \langle \text{vac} \rangle_{\text{out}} \left( \prod_{i=1}^{n+1} b_{iH} \right) \frac{U a_{iH}^\dagger (a_{2H}^\dagger)^n}{n!} \langle \text{vac} \rangle_{\text{in}},
$$

(10)

$$
= \sum_{i=1}^{n+1} \beta_{iH} \prod_{j \neq i}^{n+1} \alpha_{jH}.
$$

(11)

Here we have used Eqs. (1)–(3), (5) and (6). Similarly, by operating $(H_{N-1} \otimes \langle \text{vac} \rangle_{\text{out}}(b_{iv} \prod_{i=1}^{n+1} b_{iH}))$ on Eq. (8) from the left, we have

$$
\sqrt{P_{\text{suc}}} = \sqrt{\frac{n!(N+n)}{N}} \eta_i ,
$$

(12)

where

$$
\eta_i \equiv \langle \text{vac} \rangle_{\text{out}} \left( b_{iv} \prod_{l \neq i}^{n+1} b_{lH} \right) \frac{U a_{iV}(a_{2H}^\dagger)^n}{n!} \langle \text{vac} \rangle_{\text{in}},
$$

(13)

$$
= \gamma_{iV} \prod_{j \neq i}^{n+1} \alpha_{jH} + \alpha_{iV} \sum_{j \neq i}^{n+1} \gamma_{jH} \prod_{k \neq i,j}^{n+1} \alpha_{kH} ,
$$

(14)

for $1 \leq i \leq n+1$. Here we have used Eqs. (1)–(3), (5) and (6). From Eqs. (9) and (12), we obtain

$$
P_{\text{suc}} = \frac{n!(N+n)}{N} |\eta_0|^2 ,
$$

(15)

and

$$
\eta_0 = \cdots = \eta_{n+1}.
$$

(16)

III. BOUND ON THE SUCCESS PROBABILITY

In this section, we derive an upper bound on success probability $P_{\text{suc}}$. Since the cases with $P_{\text{suc}} = 0$ are irrelevant for the upper bound, we focus on the cases with $P_{\text{suc}} > 0$ here. The requirement of producing the exact W state, Eq. (8), narrows down the choices of the unitary operator $U$ significantly. As shown in Appendix A, in order to satisfy $P_{\text{suc}} > 0$, it is necessary that $\alpha_{iV} = \beta_{iV} = 0$ for $1 \leq i \leq n+1$ and $\prod_{j=1}^{n+1} \alpha_{jH} \neq 0$. Thus, from Eqs. (11) and (14), we are allowed to express $\eta_0$ and $\eta_i$, for later use, as

$$
\eta_0 = \prod_{i=1}^{n+1} \alpha_{iH} \sum_{j=1}^{n+1} \beta_{jH} \alpha_{jH} ,
$$

(17)

$$
\eta_i = \gamma_{iV} \prod_{j \neq i}^{n+1} \alpha_{jH} ,
$$

(18)

for $1 \leq i \leq n+1$. Define $P_i \equiv |\alpha_{iH}|^2$ ($1 \leq i \leq n+1$), $P \equiv (P_1, \ldots, P_{n+1})$,

$$
S(P) \equiv \sum_{j=1}^{n+1} P_j ,
$$

(19)

and

$$
\Pi(P) \equiv \prod_{j=1}^{n+1} P_j .
$$

(20)

Because $\prod_{j=1}^{n+1} \alpha_{jH} \neq 0$ and $S(P) \leq 1$ from Eq. (4), we obtain

$$
P \in \mathcal{R} \equiv \{ P | P_i > 0 (1 \leq i \leq n+1), 0 < S(P) \leq 1 \}. 
$$

(21)

Combining $\alpha_{iV} = \beta_{iV} = 0$ for $1 \leq i \leq n+1$ with $\sum_{j=1}^{L} (\alpha_{jH}^* \beta_{jH} + \alpha_{iV}^* \beta_{iV}) = 0$ and $\sum_{j=1}^{L} (|\alpha_{jH}|^2 + |\alpha_{jV}|^2) = \sum_{j=1}^{L} (|\beta_{jH}|^2 + |\beta_{jV}|^2) = 1$ from Eq. (4), we have

$$
\left| \sum_{j=1}^{n+1} \frac{\beta_{jH}}{\alpha_{jH}} \right|^2 = \left| \sum_{j=1}^{L} \frac{\zeta_{js} - \alpha_{jH}^* \alpha_{jH}}{\alpha_{jH}} \beta_{jH} \right|^2 
$$

$$
\leq \sum_{j=1}^{L} \frac{1}{P_j} - (n+1)^2 ,
$$

(22)

where $\zeta_{iH} \equiv \frac{1}{\alpha_{iH}^*}$ and $\zeta_{iV} \equiv 0$ for $1 \leq i \leq n+1$, and $\zeta_{iH} = \zeta_{iV} \equiv 0$ for $n+2 \leq i \leq L$. Here we have used the Cauchy-Schwarz inequality. From Eqs. (17), (20) and (22), we obtain a bound on $|\eta_0|^2$ written as

$$
|\eta_0|^2 \leq F(P) \equiv \Pi(P) \left( \sum_{k=1}^{n+1} \frac{1}{P_k} - (n+1)^2 \right) .
$$

(23)

Note that $F(P) > 0$ since we are focusing on the cases with $P_{\text{suc}} > 0$. Using Eqs. (16), (18), (19) and $\sum_{j=1}^{n+1} |\gamma_{jV}|^2 \leq 1$ from Eq. (4), we have

$$
|\eta_0|^2 S(P) = \sum_{j=1}^{n+1} P_j |\eta_j|^2 = \sum_{j=1}^{n+1} P_j |\gamma_{jV}|^2 \sum_{k \neq j} P_k \leq \sum_{j=1}^{n+1} |\gamma_{jV}|^2 \Pi(P) \leq \Pi(P) .
$$

Then we obtain another bound on $|\eta_0|^2$ written as

$$
|\eta_0|^2 \leq G(P) \equiv \frac{\Pi(P)}{S(P)} .
$$

(24)

From Eqs. (15), (21), (23) and (24), we obtain

$$
P_{\text{suc}} \leq \frac{n!(N+n)}{N} \max_{P \in \mathcal{R}} H(P) ,
$$

(25)

where $H(P) \equiv \min \{F(P), G(P)\}$.

Before conducting the optimization over $P$ in Eq. (25), let us discuss physical intuition behind the bound $H(P)$. 
Recall that \( P_i = |\alpha_{ih}|^2 \) is the probability of a photon in the input mode 2 in Fig. 1 (a) to appear at the output mode \( i \). One of the bounds on \(|\eta_0|^2\), \( G(P) \), was derived through constraints on \(|\eta_i|^2\), which is proportional to the probability of having a V-polarized photon at the output mode \( i \) and \( n \) H-polarized photons in the other \( n \) output modes, one in each [See Eq. (13)]. From the definition of \( G(P) \) in Eq. (24), we see that

\[
S(P) = P_1 + \cdots + P_{n+1} = 1 \quad (26)
\]

and

\[
P_1 = \cdots = P_{n+1} \quad (27)
\]

give the maximum of \( G(P) \). This means lossless and equal distribution of the \( n \) H-polarized photons incident on the input mode 2 is the best for maximizing the amplitude of the terms including one V-polarized photon. This result does not change even if we regard photons as classical distinguishable particles, since the origin of the V photon (input mode 1) and that of an H photon (input mode 2) are uniquely determined and no interference occurs.

On the other hand, the bound \( F(P) \) stems directly from a constraint on \(|\eta_0|^2\), which is proportional to the probability of \((n+1)\) H-polarized photons to appear at the \((n+1)\) output modes, one by one. In this case, there are \((n+1)\) indistinguishable paths, depending on which of the \((n+1)\) output photons is traced back to the input photon in mode 1. As a result, the total amplitude \( \eta_0 \) can be given by the sum over \((n+1)\) terms as in Eq. (11), which can be rewritten as

\[
\eta_0 = \left( \prod_{j=1}^{n+1} \alpha_{jH} \right) \sum_{i=1}^{n+1} \frac{1}{P_i} \alpha_{ih}^* \beta_{ih} \quad (28)
\]

Let us see the interference among these terms at the choice of \( P \) satisfying Eqs. (26) and (27). The parameters \( \beta_{ih}(i = 1, \ldots, L) \) describe how the input photon in mode 1 is distributed. They must satisfy the unitarity condition of Eq. (4), which gives, under the lossless condition of Eq. (26), a constraint

\[
\sum_{i=1}^{n+1} \alpha_{ih}^* \beta_{ih} = 0 \quad (29)
\]

Together with Eq. (27), we see that the total amplitude \( \eta_0 \) always vanishes regardless of the choice of the parameters \( \beta_{ih}(i = 1, \ldots, n+1) \). This is the reason why we have \( F(P) \to 0 \) for \( P \to ((n+1)^{-1}, \ldots, (n+1)^{-1}) \). Incidentally, the case with \( n = 1 \) is equivalent to the well-known two-photon interference effect at the symmetric beamsplitter [28], in which the two photons never leave separated. The above result with general \( n \) can thus be regarded as an extension of the \((1,1)\)-photon case to the \((1,n)\)-photon case. What is interesting here is that the symmetry is required only for the \( n \)-photon input, and not for the one-photon input, to achieve the complete destructive interference.

In order to obtain a nonzero value \( H(P) \to 0 \), one must go away from the point \( P_1 = \cdots = P_{n+1} = (n+1)^{-1} \) by dropping either the lossless condition Eq. (26) and/or the symmetry condition Eq. (27). If one breaks the symmetry, the orthogonality condition Eq. (29) no longer implies \( \eta_0 = 0 \) in Eq. (28). On the other hand, introduction of loss relaxes the orthogonality condition Eq. (29) itself, since it is equivalent to introducing auxiliary modes in Fig. 1 (b), namely, \( L > n + 1 \). The condition Eq. (29) then changes to

\[
\sum_{i=1}^{n+1} \alpha_{ih}^* \beta_{ih} = - \sum_{i=n+2}^{L} \alpha_{ih}^* \beta_{ih} - \sum_{i=n+2}^{L} \alpha_{iv}^* \beta_{iv} \quad , \quad (30)
\]

which allows more freedom in the choice of parameters \( \beta_{ih}(i = 1, \ldots, n+1) \).

In the following, we optimize over \( P \) in Eq. (25) by deriving necessary conditions for \( P \) to achieve the maximum of \( H(P) \). We assume \( P_1 \leq \cdots \leq P_{n+1} \) without loss of generality, and we consider the two cases, \( S(P) < 1 \) and \( S(P) = 1 \), separately.

In the case of \( S(P) < 1 \), for \( P \) to be a local maximum, there exists \( \epsilon > 0 \) such that \( H(P + \Delta P) \leq H(P) \) for any \( \Delta P \equiv \alpha \mu_0 + \beta \nu_0 \) with \( \alpha^2 + \beta^2 < \epsilon \), where \( \mu_0 \equiv (1, 0, \ldots, 0) \) and \( \nu_0 \equiv (0, 0, \ldots, 1) \). As shown in Appendix B, this leads to

\[
\langle u_0, \nabla F \rangle \langle v_0, \nabla G \rangle = \langle v_0, \nabla F \rangle \langle u_0, \nabla G \rangle \quad , \quad (31)
\]

where \( \langle X, Y \rangle \) means the inner product between \( X \) and \( Y \). From \( \nabla_i F(P) = \partial F(P)/\partial P_i = P_i^{-1}(F(P) - \Pi(P)P_i^{-1}) \), \( \partial G(P)/\partial P_i = G(P)(P_i^{-1} - S(P)^{-1}) \) and Eq. (31), we obtain

\[
\left( \frac{1}{P_{n+1}} - \frac{1}{P_1} \right) \left( \frac{F(P)}{\Pi(P)} + \frac{1}{P_{n+1}} \sum_{j=2}^{n+1} P_j \right) = 0 \quad . \quad (32)
\]

Since \( P_1 \leq \cdots \leq P_{n+1} \), Eq. (32) means \( P_1 = \cdots = P_{n+1} = S(P)(n+1)^{-1} \). At this point, \( F(P) \) and \( G(P) \) are regarded as functions of single parameter \( S(P) \). We show in Appendix C that \( S(P) = 1 - (n+1)^{-2} \) gives the local maximum of \( H(P) \), whose value is

\[
H_{\text{loc max}} = \frac{n^n(n+2)^n}{(n+1)^{3n+1}} \quad . \quad (33)
\]

In the case of \( S(P) = 1 \), since \( P_i > 0 \) for all \( i \) for \( P \) to be a local maximum under the constraint \( S(P) = 1 \), there exists \( \epsilon > 0 \) such that \( H(P + \Delta P_i) \leq H(P) \) for any \( \Delta P_i \equiv \alpha \mu_i + \beta \nu_i \) with \( \alpha^2 + \beta^2 < \epsilon \), where \( \mu_1 \equiv u_0 - \langle u_0, \nabla S(P) \rangle \langle \nabla S(P), \nabla S(P)^{-1} \rangle \nabla S(P) = (n(n+1)^{-1}, -(n+1)^{-1}, \ldots, -(n+1)^{-1}) \) and \( \nu_1 \equiv v_0 - \langle v_0, \nabla S(P) \rangle \langle \nabla S(P), \nabla S(P)^{-1} \rangle \nabla S(P) = (-n+1)^{-1}, \ldots, -(n+1)^{-1} \). This leads to

\[
\langle u_1, \nabla F \rangle \langle v_1, \nabla G \rangle = \langle v_1, \nabla F \rangle \langle u_1, \nabla G \rangle \quad , \quad (34)
\]
from Appendix B. From Eq. (34), we obtain
\[
\left( \frac{1}{P_j} - \frac{1}{P_{n+1}} \right)^{n+1} \sum_{j=1}^{n+1} \left( \frac{1}{P_j} - \frac{1}{P_j} \right) \left( \frac{1}{P_{n+1}} - \frac{1}{P_j} \right) = 0. \tag{35}
\]

Here the first factor is nonzero since \( P_1 < P_{n+1} \) from \( F(P) > 0 \). In Eq. (35), since \( (P_1^{-1} - P_j^{-1}) \geq 0 \) and \( (P_{n+1}^{-1} - P_j^{-1}) \leq 0 \) for all \( j \), we obtain either \( P_j = P_1 \) or \( P_j = P_{n+1} \) for every \( j \), which implies \( P_1 = \cdots = P_m < P_{m+1} = \cdots = P_{n+1} (1 \leq m \leq n) \). We thus find that, at the local maximum,
\[
F(P) = P_m^{n-1} P_{n+1}^{n-m} \left[ m P_{n+1} + (n + 1 - m) P_1 - (n + 1)^2 P_1 P_{n+1} \right], \tag{36}
\]
and
\[
G(P) = P_1^m P_{n+1}^{n+1-m}, \tag{37}
\]
with
\[
m P_1 + (n + 1 - m) P_{n+1} = 1. \tag{38}
\]
In Appendix D, we derive the local maximum of \( H(P) \) under the constraint \( S(P) = 1 \), whose value is \( H_m \) in Eq. (D8). Furthermore, in Appendix E, we show the maximum of \( H_m \) is given by \( m = 1 \), and its value is
\[
H_1 = \left( \frac{1}{n} \right)^n P_1^{opt}(1 - P_1^{opt})^n, \tag{39}
\]
where
\[
P_1^{opt} = \frac{2n + 3 - \sqrt{4n + 1}}{2(n^2 + 2n + 2)}. \tag{40}
\]

From Eqs. (33) and (39), \( \max_{P \in \mathbb{R}} H(P) \) is equal to \( \max \{ H_{lossy}, H_1 \} \). In Appendix E, we show \( H_1 > H_{lossy} \), and thus we obtain
\[
P_{suc} \leq \frac{n!(N + n)}{N} \left( \frac{1}{n} \right)^n P_1^{opt}(1 - P_1^{opt})^n. \tag{41}
\]

IV. EXPLICIT CONSTRUCTION OF AN OPTIMAL CIRCUIT

Here we construct an optical circuit which achieves the right hand side of Eq. (41). The circuit in Fig. 2 is composed of a PDBS and \( n - 1 \) BSs, and this circuit has \( n + 1 \) input and output spacial modes with no auxiliary spacial modes. We post-select the events where exactly one photon comes out from every output mode. We denote the transmittance and reflectance for \( H(V) \)-polarized photons of the PDBS by \( T_{H(V)} \) and \( R_{H(V)} \), and the transmittance and reflectance of the \( k \)-th BS (BSk) by \( T_k \) and \( R_k \). The parameters of the PDBSs are set as \( T_{H} = R_{V} = P_1^{opt} \) and \( R_{H} = T_{V} = 1 - P_1^{opt} \). The parameters of the BSs are set to output \( n \) photons from \( n \) BSs one by one with equal probability, that is, they are set as \( T_k = (n - k)(n + 1 - k)^{-1} \) and \( R_k = (n + 1 - k)^{-1} \). In this case, \( \alpha_{1H} = \sqrt{T_1} \), \( \alpha_{2H} = \cdots = \alpha_{(n+1)H} = \sqrt{n^{-1}R_1} \), \( \beta_{H} = \alpha_{1H}^{-1} - \alpha_{jH}(n + 1) \) and \( \gamma_{jV} = \alpha_{jH} \). All of the other variables of \( \alpha_{jH(V)}, \beta_{jH(V)} \) and \( \gamma_{jH(V)} \) are equal to zero. We have \( \eta_0 = \cdots = \eta_{n+1} \) in this case, and the success probability of this circuit is equal to the right hand side of Eq. (41). Hence we conclude that the maximum of \( P_{suc} \) is
\[
P_{max} = \frac{n!(N + n)}{N} \left( \frac{1}{n} \right)^n P_1^{opt}(1 - P_1^{opt})^n. \tag{42}
\]

The dependence of \( P_{max} \) on \( n \) and \( N \) is shown in Fig. 3.

In the case of \( n = 1 \), we see \( P_{max} = (N + 1)/(5N) \) for \( T_{V(H)} = (5 \pm \sqrt{5})/10 \) and \( T_{V(H)} = (5 \pm \sqrt{5})/10 \). This value is the same as that in Ref. [4, 5]. In the case of \( n = 2 \), we have \( P_{max} = 8(N + 2)/(125N) \) for \( T_{V(H)} = 4/5 \), \( T_1 = 1/2 \) and \( R_1 = 1/2 \). This value is higher than \( (N + 2)/(16N) \) given by the circuit composed of only half BSs in Ref. [6].

We also construct optical circuits which achieve the success probability of the other local maximum, \( H_m \), in Eq. (D8) of Appendix D and \( H_{lossy} \) in Eq. (33). A circuit which achieves \( P_{suc} = n!(N + n)N^{-1}H_m \) is constructed with a PDBS and \( n - 1 \) BSs. (BS1, ..., BS\( n-1 \)) are placed at each output of the PDBS in series, as shown in Fig. 4 (a). This circuit has \( n + 1 \)
input and output spacial modes without auxiliary spacial modes. Parameters of the PDBS and the BSs are set as 
\[ T_H = R_V = m \xi_m, \quad T_H = H_V = 1 - m \xi_m \]
where \( \xi_m \) is in Eq. (D4), and 
\[ T_k = (m - k)(m + 1 - k)^{-1} \quad \text{and} \quad R_k = (m + 1 - k)^{-1} \]
for \( 1 \leq k \leq m - 1 \). In this case, 
\[ \alpha_H = \cdots = \alpha_{mH} = \sqrt{m^{-1}T_H}, \quad \alpha_{m+1H} = \cdots = \alpha_{(m+1)H} = \frac{\sqrt{n+1-m}}{\sqrt{n+2-m}}R_H, \]
\[ \beta_H = \alpha_H - \alpha_{H(n+1)} \]
and 
\[ \gamma_H = \alpha_H \]
for all \( j \). All of the other variables of \( \alpha_{H(V)}, \beta_{H(V)}, \gamma_{H(V)} \) are equal to zero.

A circuit which achieves \( P_{\text{suc}} = P_{\text{lossy}} \equiv n!(N + n)N^{-1}H_{\text{lossy}} \) is constructed with a PDBS and \( n \) BSs. The PDBS and BS1, \ldots, BSn are placed in series, as shown in Fig. 4 (b). We post-select the events where the photons come out from output spacial modes of \( n \) BSs, one by one. This circuit can be regarded as a lossless circuit with \( n+2 \) input and output modes, but if we regard it as a circuit with 2 inputs and \( n+1 \) output, as in Fig. 1 (a), it is a lossy circuit. Parameters of the PDBS and the BSs are set as 
\[ R_H = 1 - (n + 1)^{-2}, \quad T_H = 1 - (n + 1)^{-2}, \quad R_V = 0, T_V = 1, \]
\[ T_k = (n + 1 - k)(n + 2 - k)^{-1} \quad \text{and} \quad R_k = (n + 2 - k)^{-1} \]
for \( 1 \leq k \leq n \). In this case, 
\[ \alpha_H = \cdots = \alpha_{(n+1)H} = \sqrt{R_H(n + 1)^{-T}}, \quad \alpha_{(n+2)H} = \sqrt{T_H \cdot \beta_H = \sqrt{R_H(n + 1)^{-T}},} \]
\[ \beta_{(n+2)H} = -\sqrt{R_H} \alpha_{(n+2)H}, \gamma_{(n+2)H} = \sqrt{(n + 1)^{-T} \quad \text{for} \quad 1 \leq j \leq n + 1} \]
for all \( j \). All of the other variables of \( \alpha_{H(V)}, \beta_{H(V)}, \gamma_{H(V)} \) are equal to zero.

Let us compare the success probability \( P_{\text{lossy}} \) for the lossy circuit with the global maximum \( P_{\text{max}} \) for a few examples. In the case of \( n = 1 \), we see \( P_{\text{lossy}} = 3(N + 1)/(16N) \) for \( T_H = 1/4 \) and \( T_1 = 1/2 \), whereas \( P_{\text{max}} = (N + 1)/(5N) \). In the case of \( n = 2 \), we have \( P_{\text{lossy}} = 128(N + 2)/(2187N) \approx 0.059(N + 2)/N \) for \( T_H = 1/9, T_1 = 2/3, T_2 = 1/2 \), while \( P_{\text{max}} = 0.064(N + 2)/N \).

\section{Conclusion}

We have derived the maximum success probability of the circuits composed of passive linear optics and an ancilla mode in an \( n \)-photon Fock state for post-selectively expanding an \( N \)-photon polarization-entangled W state to an \( (N + n) \)-photon polarization-entangled W state, by accessing only one photon of the initial W state. Whereas the symmetry in W states suggests that photons from beamsplitters should be equally distributed among \( n + 1 \) output modes, bosonic nature of photons requires us to introduce either optical losses or to break symmetry in order to reduce interference effects between one photon from the \( N \)-photon W state and the photons from the Fock state. In fact, both cases possess local maximums at which the success probability does not increase by infinitesimal changes in variables. We showed that the overall maximum success probability is achieved by a PDBS and \( n - 1 \) BSs. In the case of \( n = 2 \), the maximum success probability is higher than that of the expanding gate proposed in Ref. [6].

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\section*{Appendix A}

In the following, we prove that when \( P_{\text{suc}} > 0 \) is satisfied, we have \( \alpha_{iV} = 0 \) and \( \beta_{iV} = 0 \) for \( 1 \leq i \leq n + 1 \), and \( \prod_{i=1}^{n+1} \alpha_{iH} \neq 0 \). For nonzero success probability \( P_{\text{suc}} > 0 \), it is necessary to have \( \eta_h, \eta_i \neq 0 \). From Eqs. (5) and (8), the term in \( |W_N\rangle \) including \( a_{iH} \) must not be transformed into the terms including \( b_{iV} \) by the unitary operator \( U \).
Then we obtain

\[ \Gamma_i \equiv \langle \text{vac}_{\text{out}} \left( b_i Y \prod_{j \neq i}^{n+1} b_{jH} \right) U_{iH}^a (a_{PH})^n \rangle_{\text{vac}_{\text{in}}} = \beta_i \prod_{j \neq i}^{n+1} \alpha_{jH} + \alpha_i \sum_{j \neq i}^{n+1} \beta_j \prod_{k \neq i, j}^{n+1} \alpha_{kH} = 0, \]  

(A1)

for \( 1 \leq i \leq n+1 \) and

\[ \langle \text{vac}_{\text{out}} \left( \prod_{j=1}^{n+1} b_j Y \right) U_{iH}^a (a_{PH})^n \rangle_{\text{vac}_{\text{in}}} = \sum_{j=1}^{n+1} \beta_j \prod_{k \neq j}^{n+1} \alpha_k \nu = 0. \]  

(A2)

Because \( \sum_{i=1}^{n+1} \Gamma_i \alpha_{iH} \prod_{j \neq i}^{n+1} \alpha_{jV} = m_n \prod_{j=1}^{n+1} \alpha_{jH} = 0 \) is obtained from Eqs. (A1) and (A2), we have \( \prod_{j=1}^{n+1} \alpha_{jH} = 0 \). Assuming that \( \alpha_{iV} = 0 \), we find three facts: (i) \( \prod_{j \neq \ell}^{n+1} \alpha_{jH} \neq 0 \) because \( \eta_i = \gamma_i \prod_{j \neq \ell}^{n+1} \alpha_{jH} \neq 0 \), (ii) \( \gamma_i \nu = 0 \) for \( 1 \leq j \leq n+1 \) because \( \eta_i = \gamma_i \prod_{j \neq \ell}^{n+1} \alpha_{jH} \neq 0 \) and

\[ \langle \text{vac}_{\text{out}} \left( b_{\ell V} b_{j V} \prod_{k \neq \ell, j}^{n+1} b_{kH} \right) U_{iH}^a (a_{PH})^n \rangle_{\text{vac}_{\text{in}}} = \gamma_i \nu \gamma_{jV} \prod_{k \neq j}^{n+1} \alpha_k \alpha_{iH} = 0, \]  

(A3)

for \( 1 \leq j(\neq \ell) \leq n+1 \) from Eq. (8), and (iii) \( \beta_i \nu = 0 \) from \( \Gamma_i = \beta_i \nu \prod_{j \neq \ell}^{n+1} \alpha_{jH} = 0 \). Therefore, from \( \prod_{j=1}^{n+1} \alpha_{jH} = 0 \) and recursive use of (i)-(iii), we have \( \alpha_{iV} = 0 \) and \( \beta_i \nu = 0 \) for \( 1 \leq i \leq n+1 \), and \( \prod_{j=1}^{n+1} \alpha_{jH} \neq 0 \).

**Appendix B**

Here we prove the following statement. Suppose \( H(P) \equiv \min \{ F(P), G(P) \} \). Let \( \epsilon > 0 \) be a constant and \( u \) and \( v \) be arbitrary vectors. If \( H(P + \Delta P) \leq H(P) \) for any \( \Delta P \equiv \alpha u + \beta v \) with \( \alpha^2 + \beta^2 < \epsilon \), then

\[ \langle u, \nabla F \rangle \langle v, \nabla G \rangle = \langle v, \nabla F \rangle \langle u, \nabla G \rangle, \]  

(B1)

is satisfied where \( \langle X, Y \rangle \) means the inner product between \( X \) and \( Y \).

We define two vectors \( a \) and \( b \) as

\[ a \equiv \langle u, \nabla F \rangle v - \langle v, \nabla F \rangle u, \]  

(B2)

and

\[ b \equiv \langle v, \nabla G \rangle u - \langle u, \nabla G \rangle v. \]  

(B3)

Suppose \( \langle u, \nabla F \rangle \langle v, \nabla G \rangle > \langle v, \nabla F \rangle \langle u, \nabla G \rangle \). From Eqs. (B2) and (B3), we obtain \( \langle a, \nabla F \rangle = \langle b, \nabla G \rangle = 0 \) and \( \langle a, \nabla G \rangle = \langle b, \nabla F \rangle > 0 \). Hence both \( \langle a + b, \nabla F \rangle > 0 \) and \( \langle a + b, \nabla G \rangle > 0 \), which implies the direction of \( a + b \) increases \( F \) and \( G \) at the same time. This fact contradicts with \( H(P + \Delta P) \leq H(P) \) for any \( \Delta P \). In the case of \( \langle u, \nabla F \rangle \langle v, \nabla G \rangle < \langle v, \nabla F \rangle \langle u, \nabla G \rangle \), both \( \langle -a - b, \nabla F \rangle > 0 \) and \( \langle -a - b, \nabla G \rangle > 0 \) are satisfied, which also contradicts with \( H(P + \Delta P) \leq H(P) \) for any \( \Delta P \). Therefore we obtain Eq. (B1).

**Appendix C**

We derive the maximum of \( H(P) = \min \{ F(P), G(P) \} \) in the case of \( P_1 = \cdots = P_{n+1} = S(P)(n+1)^{-1} \), namely,

\[ F(P) = \frac{S(P)^n}{(n+1)^{-1}} (1 - S(P)), \]  

(C1)

and

\[ G(P) = \frac{1}{S(P)^{n+1}} (S(P) - n+1 - 1). \]  

(C2)

\( F(P) \) and \( G(P) \) are regarded as functions of single parameter \( S(P) \). From Eq. (C1), we have

\[ \frac{dF(P)}{dS(P)} = \left( \frac{S(P)}{n+1} \right)^{-1} (n - S(P)(n+1)) . \]  

(C3)

We find that \( F(P) \) decreases monotonously for \( n(n+1)^{-1} \leq S(P) \leq 1 \) from Eq. (C3), and \( G(P) \) increases monotonously for \( 0 < S(P) \leq 1 \) from (C2). From Eqs. (C1) and (C2), the solution of \( F(P) = G(P) \) is given by \( S(P) = 1 - (n+1)^{-2} \). Since \( (n+1)^{-1} < 1 - (n+1)^{-2} \) is satisfied, \( S(P) = 1 - (n+1)^{-2} \) gives the maximum of \( H(P) \), whose value is

\[ H_{\text{max}} = \frac{n^n (n+2)^n}{(n+1)^{2n+1}}. \]  

(C4)

**Appendix D**

We derive the maximum of \( H(P) = \min \{ F(P), G(P) \} \) under Eqs. (36) – (38), namely,

\[ F(P) = F(\xi) \equiv \xi^{m-1} \zeta^{n-1} (m \zeta + (n+1) - m) \xi^{m-1} - (n+1)\xi - 1, \]  

(D1)

\[ G(P) = G(\xi) \equiv \xi^m \zeta^{n+1-m}, \]  

(D2)

with

\[ \zeta = \frac{1 - m \xi}{n+1 - m}. \]  

(D3)

Because \( \xi < \zeta \), \( 0 < \xi < (n+1)^{-1} \) is satisfied. From Eqs. (D1) and (D2), \( F(\xi) = G(\xi) \) has four roots, which are given by \( \xi = 0, m^{-1} \) and the two roots of \( I(\xi) \equiv m \zeta + (n+1 - m) \xi - (n+1)^2 \xi - \xi = 0 \). Since \( I(0) > 0 \)
and \( f((n+1)^{-1}) < 0 \), there is only one root satisfying \( F(\xi) = G(\xi) \) for \( 0 < \xi < (n+1)^{-1} \), which is

\[
\xi = \xi_m \equiv \frac{2m(n+1) + 1 - \sqrt{4m(n+1) - m + 1}}{2m(n+1)^2 + 1}.
\] (D4)

From Eqs. (D1), (D2) and (D3), we obtain

\[
\frac{dF(\xi)}{d\xi} = m\xi^{m-2} \xi^{n-m-1}(\xi - \xi) f(\xi),
\] (D5)

where

\[
f(\xi) \equiv \frac{m(n+1)^2}{n+1-m} \xi^2 - \frac{n+1+2nm}{n+1-m} \xi + \frac{m-1}{n+1-m},
\] (D6)

and

\[
\frac{dG(\xi)}{d\xi} = m\xi^{m-1} \xi^{n-m}(\xi - \xi).
\] (D7)

From Eq. (D7), we find that \( G(\xi) \) increases monotonously for \( 0 < \xi < (n+1)^{-1} \). Since \( f(\xi) \) is a convex function, \( f(\xi_m) = (\xi-1)(\xi-1) - 1 < 0 \) and \( f((n+1)^{-1}) = -2(n+1)^{-1} - 1 < 0 \) assure that \( F(\xi) \) decreases monotonously for \( \xi_m \leq \xi < (n+1)^{-1} \). Therefore \( \xi = \xi_m \) gives the maximum of \( H(P) \), and the value is

\[
H_m \equiv F(\xi_m) = \xi_m^n \left( \frac{1-m\xi_m}{n+1-m} \right)^{n+1-m}.
\] (D8)

**Appendix E**

We show that \( H_1 > H_m(2 \leq m \leq n) \) and \( H_1 > H_{lossy} \), where \( H_m \) is defined in Eq. (D8) and \( H_{lossy} \) is in Eq. (33).

Since \( H_m > 0 \), let us analyze the property of \( \log H_m \) as a continuous function of \( m \) for \( 1 \leq m \leq n \). Let \( h(m) \equiv H_m^{-1} dH_m/dm \) be its derivative. Because we have

\[
\frac{dh(m)}{dm} = \frac{(n+1) - 2m}{m(n+1-m)(4m(n+1-m)+1)^{3/2}},
\] (E1)

we find that \( m = (n+1)/2 \) gives the maximum of \( h(m) \). From \( dh((n+1)/2)/dm = 2(n+1)^{-1}((n+1)^2+1)^{-3/2} > 0 \) for all \( n \) and \( h((n+1)/2) \to 0 \) \( (n \to \infty) \), we obtain \( h(m) < 0 \) for all \( m \). Thus, because \( dh_m/dm < 0 \) holds, \( H_m \) takes its maximum for \( m = 1 \), whose value is

\[
H_1 = \left( \frac{1}{n} \right)^n \xi_1(1-\xi_1)^n,
\] (E2)

from Eq. (D8), where \( \xi_1 \) is in Eq. (D4).

Next we show \( H_1 > H_{lossy} \). Since \( H_1 > H_{(n+1)/2} \) is satisfied, we show \( H_{(n+1)/2} > H_{lossy} \). From Eqs. (33) and (D8),

\[
\frac{H_{(n+1)/2}}{H_{lossy}} = \frac{(n+1)^4}{(n+1)^4 - 1} \times \frac{(n+1)^2}{(n+1)^2 - 1} > 1,
\] (E3)

is satisfied. Hence we obtain \( H_1 > H_{lossy} \).

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