Continuous time mean-variance portfolio selection with nonlinear wealth equations and random coefficients

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Abstract. This paper concerns the continuous time mean-variance portfolio selection problem with a special nonlinear wealth equation. This nonlinear wealth equation has nonsmooth random coefficients and the dual method developed in [7] does not work. To apply the completion of squares technique, we introduce two Riccati equations to cope with the positive and negative part of the wealth process separately. We obtain the efficient portfolio strategy and efficient frontier for this problem. Finally, we find the appropriate sub-derivative claimed in [7] using convex duality method.

Key words. mean-variance portfolio selection; nonlinear wealth equation; Riccati equation; convex duality

Mathematics Subject Classification (2010) 60H10 93E20

1 Introduction

A mean-variance portfolio selection problem is to find the optimal portfolio strategy which minimizes the variance of its terminal wealth while its expected terminal wealth equals a prescribed level. Markowitz [17], [18] first studied this problem in the single-period setting. Its multi-period and continuous time counterparts have been studied extensively in the literature; see, e.g. [1], [8], [14], [15], [21] and the references therein. Most of the literature on mean-variance portfolio selection focuses on an investor with linear wealth equation. But in some cases, one need to consider nonlinear wealth equations. For example, a large investor’s portfolio selection may affect the return of the stock’s price which leads to a nonlinear wealth equation. When some taxes must be paid on the gains made on the stocks, we also have to deal with a nonlinear wealth equation.

As for the continuous time mean-variance portfolio selection problem with nonlinear wealth equation, Ji [7] obtained a necessary condition for the optimal terminal wealth when the coefficient of the wealth equation is smooth. Jin studied the continuous time mean-variance portfolio selection problem with higher borrowing rate in which the wealth equation is nonlinear and the coefficient is not smooth. They employed the viscosity solution of the HJB equation to characterize the optimal portfolio strategy.

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In this paper, the continuous time mean-variance portfolio selection problem with a special kind of nonlinear wealth equation is studied. This kind of nonlinear wealth equation has a nonsmooth coefficient. When the coefficients are all deterministic continuous functions, [4] solves this problem via explicit viscosity solution of the corresponding HJB equation. We generalize the result of [4] to cover random coefficients. Our main idea are inspired by [6]. In [6], the authors study continuous time mean-variance portfolio selection problem with cone constraints. Their methods inspire us to handle the positive and negative parts of the wealth process separately. In fact, the optimal wealth process will remain positive (negative) if the initial investment is positive (negative).

The paper is organized as follows. In section 2, we formulate the problem. Our main results are given in section 3. In section 4, we find the appropriate sub-derivative claimed in Corollary 4.4 of [7] using convex duality method.

2 Formulation of the problem

Let \( W = (W^1, \ldots, W^d)' \) be a standard \( d \)-dimensional Brownian motion defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \), where \( \{\mathcal{F}_t\}_{t \geq 0} \) denotes the natural filtration associated with the \( d \)-dimensional Brownian motion \( W \) and augmented.

We introduce the following spaces:

\[
L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) = \left\{ \xi : \Omega \to \mathbb{R} | \xi \text{ is } \mathcal{F}_T\text{-measurable, and } E|\xi|^2 < \infty \right\},
\]

\[
M^2(0, T; \mathbb{R}^d) = \left\{ \phi : [0, T] \times \Omega \to \mathbb{R}^d | (\phi_t)_{0 \leq t \leq T} \text{ is } \mathcal{F}_t\text{-adapted process, and } ||\phi||^2 = E \int_0^T |\phi_t|^2 dt < \infty \right\},
\]

\[
L^\infty(\Omega; C(0, T; \mathbb{R})) = \left\{ \phi : [0, T] \times \Omega \to \mathbb{R} | (\phi_t)_{0 \leq t \leq T} \text{ is } \mathcal{F}_t\text{-adapted essentially bounded process with continuous sample paths.} \right\}
\]

We consider a financial market consisting of a riskless asset (the money market instrument or bond) whose price is \( S^0 \) and \( d \) risky securities (the stocks) whose prices are \( S^1, \ldots, S^d \). An investor decides at time \( t \in [0, T] \) what amount \( \pi_t^i \) of his wealth \( X_t \) to invest in the \( i \)th stock, \( i = 1, \ldots, d \). The portfolio \( \pi_t = (\pi_t^1, \ldots, \pi_t^d)' \) and \( \pi_0^i := X_t - \sum_{i=1}^d \pi_t^i \) are \( \mathcal{F}_t \)-adapted.

Throughout this paper, we take the following notations.

For any \( x \in \mathbb{R}^d \),

\[
x^+ := (x_1^+, \ldots, x_d^+)' := (x_1, \ldots, x_d)' ,
\]

and the functions \( x_i^+ := \begin{cases} x_i, & \text{if } x_i \geq 0; \\ 0, & \text{if } x_i < 0, \end{cases} \)

Consider the following kind of nonlinear wealth equation:

\[
\begin{align*}
    dX_t &= (r_t X_t + (\pi^1_t)' \sigma^1_t \bar{\omega} - (\pi^-_t)' \sigma^1_t \bar{\theta}_t) dt + \pi^1_t \sigma_t dW_t, \\
    X_0 &= x_0, \quad t \in [0, T] 
\end{align*}
\]

(2.1)
where the interest rate $r_t$ is a deterministic uniformly bounded scalar-valued function. All processes $\tilde{\theta}_t = (\tilde{\theta}_1^t, ..., \tilde{\theta}_d^t)$, $\bar{\theta}_t = (\bar{\theta}_1^t, ..., \bar{\theta}_d^t)$, $\sigma_t = \{\sigma_{ij}^t\}_{1 \leq i,j \leq d}$ are assumed to be $\mathcal{F}_t$-adapted and bounded uniformly in $(t, \omega) \in [0,T] \times \Omega$. We assume throughout that $\sigma_t$ is uniformly nondegenerate:

$$\exists \varepsilon > 0, \quad \rho^t \sigma(t) \sigma'(t) \rho \geq \varepsilon ||\rho||^2, \quad \forall \rho \in \mathbb{R}^d, \ t \in [0,T], \ a.s. \quad (2.2)$$

and

$$\bar{\theta}_t^i \leq \tilde{\theta}_t^i, \ t \in [0,T], \ a.s. \ i = 1, ..., d. \quad (2.3)$$

This kind of nonlinear wealth equation has nonsmooth coefficients and can cover the following three important models: the first model is proposed by Jouini and Kallal [10] and El Karoui et al [3] in which an investor has different expected returns for long and short position of the stock; the second one is given in section 4 of [2] for a large investor; the third one is introduced in [4] to study the wealth equation with taxes paid on the gains. Please refer [8] for a synthetic reference.

**Remark 2.1** When $\tilde{\theta}_t = \tilde{\theta}_t$, $t \in [0,T], \ a.s.$, the wealth equation (2.1) reduces to the classical linear wealth equation.

**Definition 2.2** A portfolio $\pi$ is said to be admissible if $\pi \in M^2(0,T; \mathbb{R}^d)$ and $(X, \pi)$ satisfies Eq. (2.1).

Denote by $A(x_0)$ the set of portfolio $\pi$ admissible for the initial investment $x_0$.

For a given expectation level $K \geq x_0 e^{\int_0^T r_s ds}$, consider the following continuous time mean-variance portfolio selection problem:

Minimize $\text{Var}X_T = E(X_T - K)^2$,

s.t.

$$\begin{align*}
EX_T &= K, \\
(X, \pi) &\text{ is admissible for Eq. (2.1).}
\end{align*} \quad (2.4)$$

The above problem is called feasible if there is at least one portfolio satisfying the constraints of (2.4). The optimal strategy $\pi^*$ to (2.4) is called an efficient strategy corresponding to $K$. Denote the optimal terminal value by $X^*_T$. Then, $(\text{Var}X^*_T, K)$ is called an efficient point. The set of all efficient points $\{(\text{Var}X^*_T, K) \mid K \in [x_0 e^{\int_0^T r_s ds}, +\infty)\}$ is called the efficient frontier.

### 3 Main results

**Theorem 3.1** The mean-variance problem (2.4) is feasible for every $K \in [x_0 e^{\int_0^T r_s ds}, +\infty)$ if and only if

$$\sum_{i=1}^d E \int_0^T (\mu_i^t)^+ dt > 0, \quad \text{or} \quad \sum_{i=1}^d E \int_0^T (\bar{\mu}_i^t)^- dt > 0, \quad (3.1)$$

where $\mu := \sigma \tilde{\theta}$, $\bar{\mu} := \sigma \bar{\theta}_t$, $t \in [0,T], \ a.s.$
Remark 3.2: The link between problem (2.4) and (3.2) is provided by the Lagrange duality theorem (see Luenberger [16]).

The auxiliary optimal stochastic control problem:

$$\pi$$ of the form

$$\pi_t := \begin{cases} \mu_t^0, & \text{if } (t, \omega) \in M_{i_0}; \\ 0, & \text{if } (t, \omega) \notin M_{i_0}. \end{cases}$$

The wealth corresponding to $$\pi^\beta$$ at time $$T$$ is

$$X_T^\beta = x_0 e^{\int_0^T r_s ds} + \beta \int_0^T e^{\int_0^T r_s ds}((\pi^+_t)'\mu_t - (\pi^-_t)'\bar{\mu}) dt + \beta \int_0^T e^{\int_0^T r_s ds} \pi_t' \sigma_t dW_t,$$

thanks to the positive homogeneity of $$x^+$$ and $$x^-$$.

Thus problem (2.4) is feasible if there exists $$\beta \geq 0$$ such that

$$K = EX_T^\beta = x_0 e^{\int_0^T r_s ds} + \beta E \int_0^T e^{\int_0^T r_s ds}((\pi^+_t)'\mu_t - (\pi^-_t)'\bar{\mu}) dt.$$

This is equivalent to $$E \int_0^T e^{\int_0^T r_s ds}((\pi^+_t)'\mu_t - (\pi^-_t)'\bar{\mu}) dt > 0$$ which can be easily verified from the construction of $$\pi^\beta$$.

For the case $$\sum_{i=1}^d E \int_0^T (\mu_t^-)^- dt > 0$$, the proof is similar.

Conversely, if problem (2.4) is feasible for $$K \geq x_0 e^{\int_0^T r_s ds}$$, then for $$K \geq x_0 e^{\int_0^T r_s ds}$$, there exists an admissible portfolio $$\pi$$ such that $$K = EX_T = x_0 e^{\int_0^T r_s ds} + E \int_0^T e^{\int_0^T r_s ds}((\pi^+_t)'\mu_t - (\pi^-_t)'\bar{\mu}) dt$$. Then

$$E \int_0^T e^{\int_0^T r_s ds}((\pi^+_t)'\mu_t - (\pi^-_t)'\bar{\mu}) dt > 0,$$

for some admissible portfolio $$\pi$$, this leads to (3.1).

This completes the proof. \(\square\)

Throughout this paper, we shall assume (3.1) holding.

To deal with the constraint $$EX_T = K$$, we introduce a Lagrange multiplier $$-2\lambda \in \mathbb{R}$$ and get the following auxiliary optimal stochastic control problem:

Minimize $$E(X_T - K)^2 - 2\lambda(EX_T - K) = E(X_T - d)^2 - (d - K)^2 =: \hat{J}(\pi, d),$$

s.t.

$$\pi \in M^2(0, T),$$

$$(X, \pi) \text{ satisfies Eq. (2.1)},$$

where $$d := K + \lambda$$.

Remark 3.2: The link between problem (2.4) and (3.2) is provided by the Lagrange duality theorem (see Luenberger [16]).

$$\min_{\pi \in \mathcal{A}(x_0), EX_T = K} \max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} \hat{J}(\pi, d).$$
There exists a solution to the Riccati equation (3.6) if it satisfies (3.5). We first address the existence of these equations via truncation technique and the results of [13]. But to our knowledge, there are no results which can cover the Riccati equations (3.5) and (3.6).

\[ J(\pi, d) = \max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} J(\pi, d). \]

Introduce the following two Riccati equations:

\[
\begin{cases}
    dP_1(t) = -[2r_1P_1(t) + H_1(P_1(t), \Lambda_1(t))]dt + \Lambda_1(t)dW_t, \\
    P_1(T) = 1, \\
    P_1(t) > 0;
\end{cases}
\]

\[
\begin{cases}
    dP_2(t) = -[2r_1P_2(t) + H_2(P_2(t), \Lambda_2(t))]dt + \Lambda_2(t)dW_t, \\
    P_2(T) = 1, \\
    P_2(t) > 0,
\end{cases}
\]

where

\[
H_1(P, \Lambda) := \inf_{\pi \in \mathbb{R}^d} [P\pi^t\sigma_1^t\pi + 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda],
\]

\[
H_2(P, \Lambda) := \inf_{\pi \in \mathbb{R}^d} [P\pi^t\sigma_1^t\pi - 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda].
\]

For \( P > 0 \), denote

\[
\pi_1(t, \omega, P, \Lambda) := \arg\min_{\pi \in \mathbb{R}^d} [P\pi^t\sigma_1^t\pi + 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda],
\]

\[
\pi_2(t, \omega, P, \Lambda) := \arg\min_{\pi \in \mathbb{R}^d} [P\pi^t\sigma_1^t\pi - 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda].
\]

Functions \( P\pi^t\sigma_1^t\pi + 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda \) and \( P\pi^t\sigma_1^t\pi - 2P((\pi^+)^t\sigma_1\theta_1 - (\pi^-)^t\sigma_1\theta_1) + \pi^t\sigma_1\Lambda \) are strictly convex with respect to \( \pi \), so \( \pi_1(t, \omega, P, \Lambda) \) and \( \pi_2(t, \omega, P, \Lambda) \) are uniquely defined.

**Definition 3.3** A pair of processes \( (P_1, \Lambda_1) \) \( \in L^\infty(\Omega; C(0, T); \mathbb{R}) \times L^2(0, T; \mathbb{R}^d) \) is called a solution to the Riccati equation (3.5) if it satisfies (3.5), and there exists constants \( 0 < m < M < \infty \), such that \( m \leq P_1(\cdot) \leq M \). The solution to the Riccati equation (3.6) is similar.

The Riccati equations (3.5) and (3.6) are highly nonlinear BSDEs which violate the standard Lipschitz assumptions for existence. There are lots of results on the solvability of Riccati equations, see for example [6.12.13]. But to our knowledge, there are no results which can cover the Riccati equations (3.5) and (3.6). We first address the existence of these equations via truncation technique and the results of [13].

**Theorem 3.4** There exists a solution \( (P_1, \Lambda_1) \) (respectively, \( (P_2, \Lambda_2) \)) to the BSDE (3.5) (respectively, (3.6)).
Proof: We proof only the claim for BSDE \((3.5)\), the existence for BSDE \((3.6)\) is similar.

Step 1: For any subset \(I\) of the index set \(\{1,\ldots,d\}\), denote \(\mu^I := (\mu^1,\ldots,\mu^d)^t\), where

\[
\mu^i = \begin{cases} 
\mu^i, & \text{if } i \in I; \\
\bar{\mu}, & \text{if } i \notin I,
\end{cases}
\]

and denote \(\mathbb{R}^I := \{\pi \in \mathbb{R}^d \mid \pi^i \geq 0, \text{ if } i \in I; \pi^i < 0, \text{ if } i \notin I\}\). Then we have

\[
H_1(P, \Lambda) = \inf_{\pi \in \mathbb{R}^d} \{ P\pi' \sigma_\pi \sigma' \pi + 2[P((\pi^i)' \sigma_t \bar{\theta}_t) - (\pi^i)' \sigma_t \bar{\theta}_t] + \pi_1 \sigma_1 \Lambda \}
\]

\[
= \inf_{\pi \in \mathbb{R}^d} \{ P\pi' \sigma_\pi \sigma' \pi + 2[P((\pi^i)' \mu_t) - (\pi^i)' \mu_t] + \pi_1 \sigma_1 \Lambda \}
\]

\[
= \min_{\pi \in \mathbb{R}^d} \inf_{I \subseteq \{1,\ldots,d\}} \{ P\pi' \sigma_\pi \sigma' \pi + 2[P\pi_1 \mu_t + \pi_1 \sigma_1 \Lambda] \}
\]

\[
\geq \min_{\pi \in \mathbb{R}^d} \inf_{I \subseteq \{1,\ldots,d\}} \{ P\pi' \sigma_\pi \sigma' \pi + 2[\pi_1 \mu_t + \pi_1 \sigma_1 \Lambda] \}
\]

\[
= \min_{\pi \in \mathbb{R}^d} \inf_{I \subseteq \{1,\ldots,d\}} \{ P[(\pi_t + (\sigma_t \sigma_t')^{-1}(\mu_t + \sigma_t \Lambda)^t)(\pi_t + (\sigma_t \sigma_t')^{-1}(\mu_t + \sigma_t \Lambda))] - P(\mu_t + \sigma_t \Lambda)^t(\mu_t + \sigma_t \Lambda) \}
\]

\[
= \min_{\pi \in \mathbb{R}^d} \inf_{I \subseteq \{1,\ldots,d\}} \{ -P(\mu_t + \sigma_t \Lambda)^t(\mu_t + \sigma_t \Lambda) \}
\]

\[
\geq \min_{\pi \in \mathbb{R}^d} \inf_{I \subseteq \{1,\ldots,d\}} \{ -P(\mu_t)^t(\mu_t) - P(\sigma_t \Lambda)^t(\sigma_t \Lambda) \} - \frac{\Lambda \Lambda}{P} \]

\[
= : f(P, \Lambda).
\] (3.7)

Step 2: We claim that the following BSDE (the argument \(t\) is suppressed) has a solution in terms of Definition 3.3

\[
\begin{align*}
\begin{cases}
dP_3 = -[2rP_3 + f(P_3, A_3)]dt + \Lambda_3 dW;
\end{cases}
\end{align*}
\]

\[
P_3(T) = 1,
\]

\[
P_3 > 0.
\] (3.8)

Note that we have assumed that \(r, \bar{\theta}, \bar{\theta}\) and \(\sigma\) are uniformly bounded, then there exists a nonnegative constant \(c \) independent of \(I\), such that \(2r \leq c\) and \(-\mu^t(\sigma_t \sigma_t')^{-1}\mu_t \geq -c\) simultaneously. Set \(c_1 := \epsilon \int_0^\alpha (2r - c) ds\).

Consider the following BSDE:

\[
\begin{align*}
\begin{cases}
dP_3 = -[2rP_3^t + \min_{\pi \in \mathbb{R}^d} \{ -P(\mu_t)^t(\mu_t) - 2P(\sigma_t \Lambda)^t(\sigma_t \Lambda) \} - \frac{\Lambda \Lambda}{P}] dW;
\end{cases}
\end{align*}
\]

\[
P_3(T) = 1,
\] (3.9)

where \(g : \mathbb{R}^+ \to [0, 1]\) is a smooth truncation function satisfying \(g(x) = 0\) for \(x \in [0, \alpha]\), and \(g(x) = 1\), for \(x > c_1\). According to Theorem 1 in [13], there exists a bounded maximal solution (see the precise definition in [13]) to this BSDE denoted as \((P_3, A_3)\).
And the following BSDE
\[
\begin{aligned}
    dP_b &= -\left[2rP_b^+ + \min_{\alpha \in \mathcal{A}(\pi)} \left\{ -cP_b^+ - 2(\mu^t)^t(\sigma^t)^{-1}\Lambda_5 - \frac{\sigma^t \Lambda_5}{\rho \sigma^t} g(P_b^+) \right\} \right] dt + \Lambda_5^t dW_t, \\
    P_b(T) &= 1,
\end{aligned}
\] 
(3.10)
has a solution \((e^{j(t)}(2r_{\alpha} - c)ds, 0)\). Thus from the comparison theorem (Corollary 2 in [13]), we get
\[
P_4(t) \geq e^{j(t)} \geq e^{j(t)}(2r_{\alpha} - c)ds = c_1.
\]
This shows that \((P_4, \Lambda_4)\) is actually a solution of BSDE \((3.8)\).

Step 3: Complete the proof. Consider the following BSDE:
\[
\begin{aligned}
    dP_b &= -\left[2rP_b + H_1(P_b, \Lambda_b)g_2(P_b^+)\right] dt + \Lambda_5^t dW_t, \\
    P_b(0) &= 1,
\end{aligned}
\] 
(3.11)
where \(g_2\) is the truncation function in Step 2. From the inequality (3.7) and Theorem 1 in [13], there exists a bounded, maximal solution of BSDE (3.11), denoted as \((P_0, \Lambda_0)\). Inequality (3.7) gives also \(P_1(t) \geq P_2(t) \geq c_1, \forall t \in [0, T]\). Therefore \((P_0, \Lambda_0)\) is actually a solution of BSDE (3.8). This completes the proof. \(\Box\)

The following corollary is useful in determining the Lagrange multiple.

**Corollary 3.5** Let \((P_1(t), \Lambda_1(t))\) and \((P_2(t), \Lambda_2(t))\) be the unique solutions of (3.8) and (3.6) respectively, then we have
\[
P_1(0)e^{-2\int_0^T r_{\alpha} ds} < 1, \quad P_2(0)e^{-2\int_0^T r_{\alpha} ds} < 1.
\]
**Proof**: We prove \(P_1(0)e^{-2\int_0^T r_{\alpha} ds} < 1\) only. From the definition of \(H_1(P, \Lambda)\), we deduce \(H_1(P, \Lambda) \leq 0\). Thus we have \(P_1(t) \leq e^{2\int_0^T r_{\alpha} ds}\), moreover \(P_1(0)e^{-2\int_0^T r_{\alpha} ds} \leq 1\). If \(P_1(0)e^{-2\int_0^T r_{\alpha} ds} = 1\), then \(H_1(P_1(t), \Lambda_1(t)) = 0\). That is \((P_1(t), \Lambda_1(t)) = (e^{2\int_0^T r_{\alpha} ds}, 0)\), therefore
\[
H_1(P_1(t), 0) = P_1(t) \inf_{\pi \in \mathcal{A}(\mu)} \left( \sigma^t \pi^2 + 2(\pi^+)^t(\mu^t)^t(\pi^-)^t(\bar{\mu})_t \right) = 0.
\]
Thus \(\mu = \bar{\mu} = 0, \forall t \in [0, T]\), a.s., but this contradicts with (3.11). This completes the proof. \(\Box\)

**Theorem 3.6** The state feedback control
\[
\pi^*_t = \pi_1(t, \omega, P_1(t), \Lambda_1(t))(X_t - de^{-\int_0^t r_{\alpha} ds})^+ + \pi_2(t, \omega, P_2(t), \Lambda_2(t))(X_t - de^{-\int_0^t r_{\alpha} ds})^-
\] 
(3.12)
is optimal for problem (3.3). Moreover, in this case the optimal cost is
\[
\inf_{\pi \in \mathcal{A}(\xi_0)} E(X_T - d)^2 = \begin{cases} 
    P_1(0)(x_0 - de^{-\int_0^t r_{\alpha} ds})^2, & \text{if } x_0 \geq de^{-\int_0^t r_{\alpha} ds}, \\
    P_2(0)(x_0 - de^{-\int_0^t r_{\alpha} ds})^2, & \text{if } x_0 \leq de^{-\int_0^t r_{\alpha} ds}.
\end{cases}
\] 
(3.13)
**Proof**: Denote \(Y^*_t := X_t - de^{-\int_0^t r_{\alpha} ds}\). Applying Tanaka’s formula to \(Y^*_t\), we get
\[
dY^*_t = I_{\{Y_t > 0\}}(r_t Y_t + (\pi^*_t)^t(\bar{\mu})_t - (\pi^-_t)^t(\sigma^t)^{-1}) dt + I_{\{Y_t > 0\}}(\pi^*_t)^t dW_t + \frac{1}{2} dL_t,
\]
where $L_t$ is the local time of $Y_t$ at $0$.

Applying Ito’s formula to $(Y_t^+)^2$, we get

$$d(Y_t^+)^2$$

$$= 2Y_t^+ \left\{ I_{\{Y_t^+>0\}}(r_t Y_t + (\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t)dt + I_{\{Y_t^+>0\}}\pi_t^+ \sigma_t \pi_t dt \right\} + \frac{1}{2} dL_t$$

$$= \left\{ 2r_t (Y_t^+)^2 + 2Y_t^+ ((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + I_{\{Y_t^+>0\}}\pi_t^+ \sigma_t \pi_t \right\} dt + 2Y_t^+ \pi_t^+ \sigma_t dW_t,$$

where we have used the fact \( \int_0^t |Y_t| dL_t = 0 \), a.s., please refer Proposition 1.3 in Chapter VI of \[19\] for the proof.

Applying Ito’s formula to \( P_1(t)(Y_t^+)^2 \), we get

$$dP_1(t)(Y_t^+)^2$$

$$= \left\{ I_{\{Y_t^+\leq 0\}} P_1(t)\pi_t^+ \sigma_t \pi_t - 2(Y_t^-) [P_1(t)((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_1] - (Y_t^-)^2 H_1(P_1(t), \Lambda_1(t)) \right\} dt$$

$$+ \left\{ -2P_1(t)Y_t^- \pi_t^+ \sigma_t + (Y_t^-)^2 \Lambda_2(t) \right\} dW_t.$$

Similarly, we can get

$$dP_2(t)(Y_t^-)^2$$

$$= \left\{ I_{\{Y_t^+\leq 0\}} P_2(t)\pi_t^+ \sigma_t \pi_t - 2(Y_t^-) [P_2(t)((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_2] - (Y_t^-)^2 H_2(P_2(t), \Lambda_2(t)) \right\} dt$$

$$+ \left\{ -2P_2(t)Y_t^- \pi_t^+ \sigma_t + (Y_t^-)^2 \Lambda_2(t) \right\} dW_t.$$

We define an increasing sequence of stopping times $\tau_n$, $n \geq 1$ which converging to $T$ almost surely as follows:

$$\tau_n := \inf \{ t > 0 \} \int_0^t \left| 2P_1(t)Y_t^+ \pi_t^+ \sigma_t + (Y_t^+)^2 \Lambda_2 ds \right| + \int_0^t \left| -2P_2(t)Y_t^- \pi_t^+ \sigma_t + (Y_t^-)^2 \Lambda_2 ds \right| \geq n \} \land T,$$

where $\inf \emptyset := +\infty$.

Integrating (3.14) and (3.15) from 0 to $\tau_n$, summing them and taking expectation, we get

$$E[P_1(\tau_n)(Y_{\tau_n}^+)^2 + P_2(\tau_n)(Y_{\tau_n}^-)^2]$$

$$= P_1(0)(Y_0^+)^2 + P_2(0)(Y_0^-)^2$$

$$+ E \int_0^{\tau_n} \left\{ I_{\{Y_t^+>0\}} P_1(t)\pi_t^+ \sigma_t \pi_t + 2(Y_t^+)[P_1(t)((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_1] - (Y_t^+)^2 H_1(P_1(t), \Lambda_1(t)) \right\} dt$$

$$+ I_{\{Y_t^+\leq 0\}} P_2(t)\pi_t^+ \sigma_t \pi_t - 2(Y_t^-) [P_2(t)((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_2] - (Y_t^-)^2 H_2(P_2(t), \Lambda_2(t)) \right\} dt.$$

From the definition of $H_1(P, \Lambda)$, we know the integrand on the right-hand side in the above equation, denoted as $\phi(Y_t, \pi_t)$, is nonnegative. For instance, when $Y_t > 0$, set $\pi_t := Y_t u_t$, for some $u_t \in \mathbb{R}^d$, then

$$P_1(t)\pi_t^+ \sigma_t \pi_t + 2Y_t [P_1(t)((\pi_t^+)^\prime \sigma_t \theta_t - (\pi_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_1] - (Y_t)^2 H_1(P_1(t), \Lambda_1(t))$$

$$= (Y_t)^2 \left\{ P_1(t)u_t^\prime \sigma_t \pi_t u_t + 2[P_1(t)((u_t^+)^\prime \sigma_t \theta_t - (u_t^-)^\prime \sigma_t \bar{\theta}_t) + \pi_t^+ \sigma_t \Lambda_1] - H_1(P_1(t), \Lambda_1(t)) \right\}$$

$$\geq 0.$$
For any $\pi \in \mathcal{A}(x_0)$, it’s easy to verify $E \sup_{t \in [0, T]} |Y_t|^2 < \infty$. Let $n \to \infty$, and by the dominated convergence theorem, we have

$$E(X_T - d)^2 = E(Y_T)^2 = E\left[ P_1(T)(Y_T^+)^2 + P_2(T)(Y_T^-)^2 \right]$$
$$= P_1(0)(Y_0^+)^2 + P_2(0)(Y_0^-)^2 + E \int_0^T \phi(Y_t, \pi_t) dt$$
$$\geq P_1(0)(Y_0^+)^2 + P_2(0)(Y_0^-)^2,$$

(3.16)

where the equality holds at

$$\pi_t^* = \pi_1(t, \omega, P_1(t), \Lambda_1(t))(X_t - de^{-f_t^r \cdot ds})^+ + \pi_2(t, \omega, P_2(t), \Lambda_2(t))(X_t - de^{-f_t^r \cdot ds})^-,$$

which is (3.12). As a consequence, (3.13) is verified.

Now we need to prove $\pi^* \in M^2(0, T; \mathbb{R}^d)$. Note that $(t, \omega)$ is suppressed

$$(\pi^+)^* = \pi_1^+ Y^+ + \pi_2^+ Y^-, \quad \text{and} \quad (\pi^-)^* = \pi_1^- Y^+ + \pi_2^- Y^-.$$%

We claim the following equation has a unique continuous $\mathcal{F}_t$-adapted solution.

$$\begin{cases}
    dY_t = (r_t Y_t + ((\pi_t^+)^' \sigma_t^x_1 - ((\pi_t^-)^' \sigma_t^x_1) \, dt + (\pi_t^+)' \sigma_t dW_t
    
    \end{cases}$$

(3.17)

Consider the following two equations:

$$\begin{cases}
    d\tilde{Y}_t = (r_t \tilde{Y}_t + (\pi_t^+)' \sigma_t^x_1 \tilde{Y}_t - (\pi_t^-)' \sigma_t \hat{Y}_t) dt + \tilde{Y}_t \pi_t' \sigma_t dW_t
    
    \tilde{Y}_0 = (x_0 - de^{-f_0^r \cdot ds})^+, \quad t \in [0, T],
\end{cases}$$

(3.18)

and

$$\begin{cases}
    d\tilde{Y}_t = (r_t \tilde{Y}_t - (\pi_t^+)' \sigma_t^x_1 \tilde{Y}_t + (\pi_t^-)' \sigma_t \hat{Y}_t) dt - \tilde{Y}_t \pi_t' \sigma_t dW_t
    
    \tilde{Y}_0 = (x_0 - de^{-f_0^r \cdot ds})^-, \quad t \in [0, T],
\end{cases}$$

(3.19)

Then

$$\tilde{Y}_t = (x_0 - de^{-f_0^r \cdot ds})^+ \exp \left\{ \int_0^t (r_s + (\pi_s^+)' \sigma_s^x_1 - (\pi_s^-)' \sigma_s \tilde{Y}_t - \frac{1}{2} \pi_s^2 \sigma_s^2 \pi_s^2) dt + \int_0^t \pi_s' \sigma_s dW_t \right\},$$

and

$$\tilde{Y}_t = (x_0 - de^{-f_0^r \cdot ds})^- \exp \left\{ \int_0^t (r_s - (\pi_s^+)' \sigma_s^x_1 + (\pi_s^-)' \sigma_s \tilde{Y}_t - \frac{1}{2} \pi_s^2 \sigma_s^2 \pi_s^2) dt + \int_0^t \pi_s' \sigma_s dW_t \right\}.$$%

(And it’s easy to verify that $Y_t := \tilde{Y}_t - \hat{Y}_t$ solves Eq. (3.17). The proof of uniqueness is similar. From (3.10), we know the solution of (3.11) satisfies $E(Y_T)^2 = P_1(0)(Y_0^+)^2 + P_2(0)(Y_0^-)^2 < \infty$. Eq. (3.17) can also be regarded as a BSDE, from the classical BSDE theory, we conclude $\pi^* \in M^2(0, T; \mathbb{R}^d).$ This completes the proof. □
Remark 3.7 For any \( s \in [0, T] \) and \( x_s \in L^2(\Omega, \mathcal{F}_s, P; \mathbb{R}) \), consider a dynamic version of problem (3.3) on \([s, T]\):

\[
\text{ess inf } E[(X_T - d)^2 | \mathcal{F}_s], \text{ s.t. } \pi \in \mathcal{A}(x_s).
\] (3.20)

By similar procedure as in the proof of Theorem 3.6, we can get

\[
\text{ess inf } E[(X_T - d)^2 | \mathcal{F}_s] = \begin{cases}
    P_1(s)(x_s - de^{-\int_s^T r(t) dt})^2, & \text{if } x_s \geq de^{-\int_s^T r(t) dt}, \\
    P_2(s)(x_s - de^{-\int_s^T r(t) dt})^2, & \text{if } x_s \leq de^{-\int_s^T r(t) dt},
\end{cases}
\]

where \((P_1, \Lambda_1)\) and \((P_2, \Lambda_2)\) are any solutions of the Riccati equations (3.5) and (3.6) respectively. This shows that the solution of Riccati equation (3.5) is unique and so do (3.6).

Now we determine the Lagrange multiple \(d^*\) which attains \(\max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} \tilde{J}(\pi, d)\).

From (3.22),

\[
\min_{\pi \in \mathcal{A}(x_0)} \tilde{J}(\pi, d) = \begin{cases}
    \text{inf}_{\pi \in \mathcal{A}(x_0)} \mathbb{E}(X_T - d)^2 - (d - K)^2, & \text{if } x_0 \geq de^{-\int_0^T r(t) dt}, \\
    (P_1(0)(e^{-\int_0^T r(t) dt} - 1))^2 d^2 - (2x_0P_1(0)(e^{-\int_0^T r(t) dt} - 2K)d + P_1(0)x_0^2 - K^2), & \text{if } d \leq x_0 e^{-\int_0^T r(t) dt}, \\
    (P_2(0)(e^{-\int_0^T r(t) dt} - 1))^2 d^2 - (2x_0P_2(0)(e^{-\int_0^T r(t) dt} - 2K)d + P_2(0)x_0^2 - K^2), & \text{if } d \geq x_0 e^{-\int_0^T r(t) dt}.
\end{cases}
\]

Denote

\[
f(d) := (P_1(0)e^{-\int_0^T r(t) dt} - 1)^2 d^2 - (2x_0P_1(0)e^{-\int_0^T r(t) dt} - 2K)d + P_1(0)x_0^2 - K^2,
\]

and

\[
h(d) := (P_2(0)e^{-\int_0^T r(t) dt} - 1)^2 d^2 - (2x_0P_2(0)e^{-\int_0^T r(t) dt} - 2K)d + P_2(0)x_0^2 - K^2.
\]

Noting that \(K \geq x_0 e^{-\int_0^T r(t) dt}\), by simple calculation, we get

\[
\max_{d \leq x_0 e^{-\int_0^T r(t) dt}} f(d) = f(x_0 e^{-\int_0^T r(t) dt}) = -(x_0 e^{-\int_0^T r(t) dt} - K)^2.
\]

And

\[
\max_{d \geq x_0 e^{-\int_0^T r(t) dt}} h(d) = h(x_0 e^{-\int_0^T r(t) dt} - K) = h(x_0 e^{-\int_0^T r(t) dt} - K)^2 = f(x_0 e^{-\int_0^T r(t) dt}).
\]

Then \(d^* := \frac{x_0P_2(0)e^{-\int_0^T r(t) dt} - K}{P_2(0)e^{-\int_0^T r(t) dt} - 1} \geq x_0 e^{-\int_0^T r(t) dt}\) is the maximum point of \(\max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} \tilde{J}(\pi, d)\).

Under the optimal \(\pi^*\) defined in (3.12) with \(d^*\), we know \(Y_t = \hat{Y}_t - \hat{Y}_t = -\hat{Y}_t \leq 0\) from (3.18) and (3.19).

The above analysis boils down to the following theorem.
Theorem 3.8 The efficient strategy of the problem (2.4) can be written as a function of time $t$ and wealth $X$:

$$\pi^*(t, X) = -\pi_2(t, \omega, P_2(t), \Lambda_2(t)) (X_t - d^* e^{-\int_t^T r_s ds})$$

Moreover, the efficient frontier is

$$\text{Var} X_T = \frac{P_2(0) e^{-\int_0^T r_s ds}}{1 - P_2(0) e^{-\int_0^T r_s ds}} (EX_T - x_0 e^{\int_0^T r_s ds})^2.$$  

Remark 3.9 When the dimension $d = 1$ and $\sigma_t > 0$, $t \in [0,T]$, a.s., we have

$$H_2(P, \Lambda) := \inf_{\pi \in \mathbb{R}} [P \sigma_t^2 \pi^2 - 2P(\pi^+ \sigma_t \bar{\theta}_t - \pi^- \sigma_t \bar{\theta}_t) + \pi \sigma_t \Lambda]$$

$$= \begin{cases} 
-\frac{(P \sigma_t + \Lambda)^2}{P^2 \sigma_t^2}, & \text{if } \frac{\Lambda}{P} \geq -\frac{\bar{\theta}}{\sigma_t}, \\
0, & \text{if } -\frac{\bar{\theta}}{\sigma_t} \leq \frac{\Lambda}{P} \leq -\frac{\bar{\theta}}{\sigma_t}, \\
-\frac{(P \sigma_t + \Lambda)^2}{P^2 \sigma_t^2}, & \text{if } \frac{\Lambda}{P} \leq -\frac{\bar{\theta}}{\sigma_t}, 
\end{cases}$$

and

$$\pi_2(t, \omega, P_2(t), \Lambda_2(t)) = \begin{cases} 
P_2(t) \frac{\sigma_t + \Lambda_2(t)}{P_2(t) \sigma_t}, & \text{if } \frac{\Lambda_2(t)}{P_2(t) \sigma_t} \geq -\frac{\bar{\theta}}{\sigma_t}, \\
0, & \text{if } -\frac{\bar{\theta}}{\sigma_t} \leq \frac{\Lambda_2(t)}{P_2(t) \sigma_t} \leq -\frac{\bar{\theta}}{\sigma_t}, \\
P_2(t) \frac{\sigma_t - \Lambda_2(t)}{P_2(t) \sigma_t}, & \text{if } \frac{\Lambda_2(t)}{P_2(t) \sigma_t} \leq -\frac{\bar{\theta}}{\sigma_t}, 
\end{cases}$$  

(3.22)

Note that we have assumed that $\bar{\theta} \leq \bar{\theta}_t$, $t \in [0,T]$, a.s.

Remark 3.10 When $d = 1$, $\bar{\theta}_t, \bar{\theta}_t, \sigma_t$ are deterministic continuous functions on $[0,T]$, the unique solutions of Riccati equations (3.3) and (3.3) are given by

$$(P_1(t), \Lambda_1(t)) = (e^{\int_0^T (2\bar{r}_s - \bar{\theta}_s) ds}, 0); \ (P_2(t), \Lambda_2(t)) = (e^{\int_0^T (2\bar{r}_s - \bar{\theta}_s) ds}, 0),$$

respectively, then we recover the results of [3].

4 Convex duality

In [1], the terminal perturbation method depends heavily on the differentiability of wealth equation with respect to $\pi$. So for problem (2.4), [1] have only sufficient condition (Corollary 4.4) with derivative replaced by sub-derivatives. But for wealth equation (3.21), the sub-derivatives are $[\mu, \bar{\mu}]$. Now we try to find an appropriate sub-derivative by convex duality method.

For notation simplicity, we set the dimension $d = 1$.

By the conclusion in Section 3, we know problem (2.4) is equivalent to the following problem

Minimize $E(X_T - d^+)^2$, s.t. $\pi \in \mathcal{A}(x_0)$,  

(4.1)
where \( d^* = \frac{\sigma_0 P_0(0) e^{-\int_0^T r_s ds}}{P_0(0) r_0^{-1}} \). Notice that \( x_0 \leq d^* e^{-\int_0^T r_s ds} \), the optimal wealth process \( X_t^* \leq d^* e^{-\int_0^T r_s ds}, t \in [0,T], a.s. \). Especially, we have \( X_T^* \leq d^* \). Thus problem (4.1) is equivalent to

\[
\text{Minimize } E((X_T - d^*)^-)^2, \quad \text{s.t. } \pi \in A(x_0),
\]

(4.2)

Besides (2.2) and (2.3), we assume \( \sigma_t > 0, t \in [0,T], a.s. \). For problem (4.2), selecting \( \pi \) is equivalent to selecting the terminal wealth \( X_T \) by the existence and uniqueness result of BSDEs. Set \( \bar{\sigma}_t := \sigma_t \pi_t, t \in [0,T] \), then problem (4.2) is equivalent to

\[
\text{Minimize } E((\xi - d^*)^-)^2,
\]

s.t.

\[
\begin{cases}
\xi \in L^2, \\
V^\xi_0 \leq x_0,
\end{cases}
\]

(4.3)

where \( (V^\xi_t, q_t) \) is the unique solution of the following BSDE,

\[
V^\xi_t = \xi - \int_t^T (r_s V^\xi_s + q_s^+ \bar{\theta}_s - q_s^- \bar{\theta}_s) ds - \int_t^T q_s dW_s
\]

(4.4)

The generator of the above BSDE is convex, thus we have the following variational representation of \( (V^\xi_t, q_t) \):

\[
V^\xi_t = \text{ess sup}_{v \in B} E[N^{\xi,v}_{t,T}], \quad \text{a.s.},
\]

where,

\[
N^{\xi,v}_{t,s} = e^{-\int_t^s (r_\alpha + \frac{1}{2} v_\alpha^2) d\alpha - \int_t^s v_\alpha dW_\alpha},
\]

and

\[
B = \{ v \mid v \text{ is } \{F_t\}_{t \geq 0} \text{-adapted and } \theta_{t,1} \leq v_t \leq \bar{\theta}_t, t \in [0,T], a.s. \}.
\]

Especially, we have

\[
V^\xi_0 = \sup_{v \in B} E[N^{\xi,v}_{0,T}].
\]

Set \( \bar{u}(\zeta) := \inf_{x \leq d^*} [(x - d)^- + \zeta x] = d^* - \frac{\zeta^2}{4}, \zeta > 0. \)

Then \( \forall \zeta > 0, v \in B, \xi \in L^2, \xi \leq d, \) we have

\[
E((\xi - d^*)^-)^2 \geq E[d^* \zeta N^{\xi,v}_{0,T} - \frac{\zeta^2}{4} (N^{\xi,v}_{0,T})^2 - \zeta N^{\xi,v}_{0,T}] \\
\geq d^* \zeta e^{-\int_0^T r_s ds} - \frac{\zeta^2}{2} E(N^{\xi,v}_{0,T})^2 - x_0 \zeta,
\]

and the equality holds if and only if there exists \( \bar{\xi} \in L^2, \zeta > 0, \bar{v} \in B \) such that

\[
\bar{\xi} = d^* - \frac{\zeta}{2} N^{\bar{\xi},\bar{v}}_{0,T},
\]

(4.5)

and

\[
E[\bar{\xi} N^{\bar{\xi},\bar{v}}_{0,T}] = x_0
\]

(4.6)
hold simultaneously. So we introduce the dual problem
\[
\sup_{\zeta > 0, v \in B} \left[ -\frac{\zeta^2}{4} E(N_{0,T}^{r,v})^2 + d^* \zeta e^{-\int_0^T r_s ds} - x_0 \zeta \right] 
\]
\[
= - \inf_{\zeta > 0, v \in B} \left[ \frac{\zeta^2}{4} E(N_{0,T}^{r,v})^2 - d^* \zeta e^{-\int_0^T r_s ds} + x_0 \zeta \right] 
\]
\[
= - \inf_{\zeta > 0} \left[ \frac{\zeta^2}{4} \inf_{v \in B} E(N_{0,T}^{r,v})^2 + \zeta (x_0 - d^* e^{-\int_0^T r_s ds}) \right] 
\]
(4.7)

Let’s first deal with \( \inf_{v \in B} E(N_{0,T}^{r,v})^2 \) which is apparently independent of \( \zeta \). Consider its more general dynamic counterpart
\[
\hat{V}(t, v) := \text{essinf}_{\hat{v} \in B_t(v)} E[(N_{0,T}^{r,v})^2 | F_t],
\]
where \( B_t(v) = \{ \hat{v} \in B \mid \hat{v}_s = v_s, \ s \in [0, t] \} \).

We conjecture that \( \hat{V}(t, v) \) has the following form
\[
\hat{V}(t, v) = (N_{0,t}^{r,v})^2 e^{\hat{Y}_t},
\]
where \((\hat{Y}, \hat{Z})\) is the unique solution of the following BSDE
\[
\hat{Y}_t = \int_t^T g(s, \hat{Z}_s) ds - \int_t^T \hat{Z}_s dW_s.
\]
(4.8)

For any \( v \in B \), \( (N_{0,t}^{r,v})^2 e^{\hat{Y}_t} \) is submartingale, \( \hat{v} \in B \) is the solution of the above dynamic problem if and only if \( (N_{0,t}^{r,v})^2 e^{\hat{Y}_t} \) becomes a martingale. From the expression of \( N_{t,s}^{r,v} \) and \( \hat{Y}_t \), we know
\[
(N_{0,t}^{r,v})^2 e^{\hat{Y}_t} = e^{-\int_0^T (2r_s + v_s^2) ds - \int_0^T 2v_s dW_s + \int_0^T (\hat{Y}_s - \int_0^s \hat{Z}_r ds) ds + \int_0^T \hat{Z}_s dW_s}
\]
\[
= e^{\hat{Y}_0 - \int_0^T (\hat{Z}_s - 2v_s) ds - \int_0^T (\frac{1}{2} \hat{Z}_s^2 - 2Z_s v_s + v_s^2 - 2r_s - g(s, \hat{Z}_s)) ds},
\]
Thus for any \( v \in B \), we have \( \frac{1}{2} \hat{Z}_s^2 - 2Z_s v_s + v_s^2 - 2r_s - g(s, \hat{Z}_s) \geq 0. \)

Set
\[
g(s, Z) := \inf_{\theta_s \leq v_s \leq \hat{\theta}_s} (v_s^2 - 2Z_s v_s + \frac{1}{2} Z_s^2 - 2r_s)
\]
\[
= \begin{cases} 
(\hat{\theta}_s - Z)^2 - \frac{1}{2} Z_s^2 - 2r_s, & \text{if } Z > \hat{\theta}_s, \\
- \frac{1}{2} Z_s^2 - 2r_s, & \text{if } \hat{\theta}_s < Z \leq \hat{\theta}_s, \\
(\hat{\theta}_s - Z)^2 - \frac{1}{2} Z_s^2 - 2r_s, & \text{if } Z < \hat{\theta}_s.
\end{cases}
\]

By the result of Kobylanski [11], the quadratic BSDE (4.8) has a unique solution \((\hat{Y}, \hat{Z})\).
Then the internal infimum in (4.7) is attained at
\[
\hat{v}_t := \text{argmin}_{\hat{\theta}_t \leq v_t \leq \hat{\theta}_t} (v_t^2 - 2\hat{Z}_t v_t + \frac{1}{2} \hat{Z}_t^2 - 2r_t)
\]
\[
= \begin{cases} 
\hat{\theta}_t, & \text{if } \hat{Z}_t > \hat{\theta}_t, \\
\hat{Z}_t, & \text{if } \hat{\theta}_t \leq \hat{Z}_t \leq \hat{\theta}_t, \\
\hat{\theta}_t, & \text{if } \hat{Z}_t < \hat{\theta}_t, \ t \in [0, T], a.s.,
\end{cases}
\]
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and \( \inf_{v \in \mathcal{G}} E(N_{t,v}^{r,\psi})^2 = E(N_{0,T}^{r,\psi}) = e^{\hat{Y}_0} \). The external infimum in (4.7) is attained at 
\[
\hat{\zeta} = -2e^{-\hat{Y}_0}(x_0 - d^*e^{-\int_0^T r_s ds}) > 0.
\]

Thus the optimal solution of problem (4.3) is
\[
\hat{\zeta} = d^* - \frac{\hat{\zeta}}{2} N_{0,T}^{r,\psi}.
\]

And it's easy to verify (4.6). Actually, \( \hat{v} \) is the appropriate sub-derivative that we are looking for. And problem (2.4) is equivalent to

\[
\text{Minimize } E(X_T - d^*)^2,
\]
\[
\begin{align*}
\pi \in M^2(0, T), \\
dX_t &= (r_t X_t + \pi_t \sigma_t \hat{v}_t)dt + \pi_t \sigma_t dW_t, \\
x &\leq x_0.
\end{align*}
\]

(4.9)

So far, the problem (2.4) has been solved by two method: the completion of squares method and the convex duality method. Next we will explain that the optimal solutions obtained by these two method are the same. First the solutions of BSDE (3.6) and BSDE (4.8) have the following relationship.

Proposition 4.1 \( \frac{1}{p_2(t)} = e^{\hat{Y}_t}; \left( \frac{\Lambda_2(t)}{p_2(t)} \right) = -\hat{Z}_t, \; t \in [0, T], \; \text{a.s.} \)

Put the optimal feedback control \( \pi^* \) (3.21) into the wealth equation (2.1), and noticing that the optimal wealth process \( X_{t}^* < d^*e^{-\int_0^T r_s ds} \), we can get the optimal terminal wealth \( X_{T}^* \). We claim that \( X_{T}^* = \hat{\zeta} \). Here is the reason.

Set \( A_t = \{ \frac{\Lambda_2(t)}{p_2(t)} \geq -\theta_t \} = \{ \hat{Z}_t \leq \theta_t \} \), \( B_t = \{ \frac{\Lambda_2(t)}{p_2(t)} \leq -\theta_t \} = \{ \hat{Z}_t \geq \theta_t \} \).

Then we have
\[
X_{T}^* = d^* + (x_0 - d^*e^{-\int_0^T r_s ds})e^{\hat{Y}_0} - \hat{\zeta}
\]
\[
\begin{align*}
&= e^{-\int_0^T (r_t - \frac{\hat{\zeta}}{2} r_t) dt} \left( \frac{\Lambda_2(t)}{p_2(t)} \right) I_{A_t} - \hat{\zeta}_t \theta_t + \frac{\Lambda_2(t)}{p_2(t)} \frac{\hat{\zeta}_t}{\theta_t} I_{B_t},
\end{align*}
\]

We need to show \( X_{T}^* = d^* - \frac{\hat{\zeta}}{2} N_{0,T}^{r,\psi} \). Notice that
\[
\hat{\zeta} = -2(x_0 - d^*e^{-\int_0^T r_s ds})e^{-\hat{Y}_0}
\]
\[
\begin{align*}
&= -2(x_0 - d^*e^{-\int_0^T r_s dt})e^{-\hat{Y}_0} - \frac{\hat{\zeta}}{2} \left( (\theta_t - \hat{\zeta}_t)^2 - \frac{1}{4} \hat{\zeta}_t^2 - 2 r_t \theta_t + \frac{\Lambda_2(t)}{p_2(t)} \frac{r_t}{\theta_t} + \frac{\Lambda_2(t)}{p_2(t)} \frac{\hat{\zeta}_t}{\theta_t} \right) dt
\end{align*}
\]

and
\[
N_{0,T}^{r,\psi} = e^{-\int_0^T (r_t + \frac{\hat{\zeta}_t}{2} r_t) dt - \int_0^T \hat{v}_t dW_t}
\]
\[
\begin{align*}
&= e^{-\int_0^T (r_t + \frac{\Lambda_2(t)}{2} r_t) \frac{r_t}{\theta_t} I_{A_t} + \frac{\Lambda_2(t)}{2} I_{A_t} + \frac{\Lambda_2(t)}{2} \frac{r_t}{\theta_t} I_{A_t}^c \cap B_t) dt - \int_0^T (\theta_t I_{A_t} + \frac{\Lambda_2(t)}{2} I_{A_t} + \frac{\Lambda_2(t)}{2} \frac{r_t}{\theta_t} I_{A_t}^c \cap B_t) dW_t.
\end{align*}
\]
Compare the expressions of $X_t^\ast$, $\hat{\zeta}$, $N_{0,T}$, it’s sufficient to show
\[
\begin{align*}
    r_t - \theta_t (\theta_t + \frac{\Lambda_2(t)}{P_2(t)})I_{A_t} - \tilde{\theta}_t (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)})I_{B_t} - \frac{1}{2} (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)})^2 I_{A_t} - \frac{1}{2} (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)})^2 I_{B_t} \\
    = -((\tilde{\theta}_t - \tilde{Z}_t)^2 - \frac{1}{2} \tilde{Z}_t^2 - 2r_t) I_{A_t} + \frac{1}{2} \tilde{Z}_t^2 I_{A_t \cap B_t} - ((\tilde{\theta}_t - \tilde{Z}_t)^2 - \frac{1}{2} \tilde{Z}_t^2 - 2r_t) I_{A_t} \\
    - (r_t + \frac{1}{2} \theta_t^2 I_{B_t} + \frac{1}{2} \theta_t^2 I_{A_t} + \frac{1}{2} \tilde{Z}_t^2 I_{A_t \cap B_t})
\end{align*}
\]
and
\[
\begin{align*}
    -((\theta_t + \frac{\Lambda_2(t)}{P_2(t)}) I_{A_t} + (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)}) I_{B_t}) = \tilde{Z}_t - (\tilde{\theta}_t I_{B_t} + \tilde{\theta}_t I_{A_t} + \tilde{Z}_t I_{A_t \cap B_t})
\end{align*}
\]
Take note of proposition 4.1, equality (4.11) is obvious. And equality (4.10) is equivalent to
\[
\begin{align*}
    -\frac{1}{2} \theta_t^2 I_{B_t} - \frac{1}{2} \theta_t^2 I_{A_t} + \frac{1}{2} \tilde{Z}_t^2 I_{A_t} + \frac{1}{2} \tilde{Z}_t^2 I_{B_t} - (\tilde{\theta}_t - \tilde{Z}_t)^2 I_{B_t} - (\tilde{\theta}_t - \tilde{Z}_t)^2 I_{A_t} \\
    = -\theta_t (\theta_t + \frac{\Lambda_2(t)}{P_2(t)}) I_{A_t} - \tilde{\theta}_t (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)}) I_{B_t} - \frac{1}{2} (\theta_t + \frac{\Lambda_2(t)}{P_2(t)})^2 I_{A_t} - \frac{1}{2} (\theta_t + \frac{\Lambda_2(t)}{P_2(t)})^2 I_{B_t}
\end{align*}
\]
Compare the coefficients before $I_{A_t}$ and $I_{B_t}$, the above equality is equivalent to
\[
\begin{align*}
    \frac{1}{2} \theta_t^2 + \frac{1}{2} \tilde{Z}_t^2 - (\tilde{\theta}_t - \tilde{Z}_t)^2 = -\theta_t (\theta_t + \frac{\Lambda_2(t)}{P_2(t)}) - \frac{1}{2} (\theta_t + \frac{\Lambda_2(t)}{P_2(t)})^2
\end{align*}
\]
and
\[
\begin{align*}
    \frac{1}{2} \tilde{\theta}_t^2 + \frac{1}{2} \tilde{Z}_t^2 - (\tilde{\theta}_t - \tilde{Z}_t)^2 = -\tilde{\theta}_t (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)}) - \frac{1}{2} (\tilde{\theta}_t + \frac{\Lambda_2(t)}{P_2(t)})^2.
\end{align*}
\]
Notice proposition 4.1, these two equalities are easy to check.

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