Bounding Inefficiency of Equilibria in Continuous Actions Games using Submodularity and Curvature

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Abstract

Games with continuous strategy sets arise in several machine learning problems (e.g. adversarial learning). For such games, simple no-regret learning algorithms exist in several cases and ensure convergence to coarse correlated equilibria (CCEs). The efficiency of such equilibria with respect to a social function, however, is not well understood. In this paper, we define the class of valid utility games with continuous strategies and provide efficiency bounds for their CCEs. Our bounds rely on the social function being a monotone DR-submodular function. We further refine our bounds based on the curvature of the social function. Furthermore, we extend our efficiency bounds to a class of non-submodular functions that satisfy approximate submodularity properties. Finally, we show that valid utility games with continuous strategies can be designed to maximize monotone DR-submodular functions subject to disjoint constraints with approximation guarantees.

1 Introduction

Game theory is a powerful tool for modelling many real-world multi-agent decision making problems [7]. In machine learning, game theory has received substantial interest in the area of adversarial learning (e.g. generative adversarial networks [14]) where models are trained via games played by competing modules [2]. Apart from modelling interactions among agents, game theory is also used in the context of distributed optimization. In fact, specific games can be designed so that multiple entities can contribute to optimizing a common objective function [24, 22].

A game is described by a set of players aiming to maximize their individual payoffs which depend on each others’ strategies. The efficiency of a joint strategy profile is measured with respect to a social function, which depends on the strategies of all the players. When the strategies for each player are uncountably infinite, the game is said to be continuous.

Continuous games describe a broad range of problems where integer or binary strategies may have limited expressiveness. In market sharing games [13], for instance, competing firms may invest continuous amounts in each market, or may produce an infinitely divisible product. Also, several integer problems can be generalized to continuous domains. For example, in budget allocation problems continuous amounts can be allocated to each media channel [4]. In machine learning, many games are naturally continuous [21].

1.1 Related work

Although continuous games are finding increasing applicability, from a theoretical viewpoint they are less understood than games with finitely many strategies. Recently, no-regret learning algorithms [7] have been proposed for continuous games under different set-ups [32, 30, 29]. Similarly to finite games [7], these no-regret dynamics converge to coarse correlated equilibria (CCEs) [30, 2], the weakest class of equilibria which includes pure Nash equilibria, mixed Nash equilibria and correlated equilibria. However, CCEs may

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be highly suboptimal for the social function. A central open question is to understand the (in)efficiency of such equilibria. Differently from the finite case, where bounds on such inefficiency are known for a large variety of games [28], in continuous games this question is not well understood.

To measure the inefficiency of CCEs arising from no-regret dynamics, [6] introduces the price of total anarchy. This notion generalizes the well-established price of anarchy (PoA) of [19] which instead measures the inefficiency of the worst pure Nash equilibria of the game. There are numerous reasons why players may not reach a pure Nash equilibrium [6, 29, 28]. In contrast, regret minimization can be done by each player via simple and efficient algorithms [6]. Recently, [28] generalizes the price of total anarchy defining the robust PoA which measures the inefficiency of any CCE (including the ones arising from regret minimization), and provides examples of games for which it can be bounded.

In the context of distributed optimization, where a game is designed to optimize a given objective [24], bounds on the robust price of anarchy find a similar importance. In this setting, a distributed scheme to optimize the social function is to let each player implement a no-regret learning algorithm based only on its payoff information. A bound on the robust PoA provides an approximation guarantee to such optimization scheme.

Bounds on the robust PoA provided by [28] mostly concern games with finitely many actions. A class of such games are the valid utility games introduced by [31]. In such games, the social function is a submodular set function and, using this property, [28] showed that the PoA bound derived in [31] indeed extends to all CCEs of the game. This class of games covers numerous applications including market sharing, facility location, and routing problems, and were used by [24] for distributed optimization. Strategies consist of selecting subsets of a ground set, and can be equivalently represented as binary decisions. Recently, authors in [23] extend the notion of valid utility games to integer domains. By leveraging properties of submodular functions over integer lattices, they show that the robust PoA bound of [28] extends to the integer case. The notion of submodularity has recently been extended to continuous domains, mainly in order to design efficient optimization algorithms [4, 10, 11]. To the best of author’s knowledge, such notion has not been utilized for analyzing efficiency of equilibria of games over continuous domains.

1.2 Our contributions

We bound the robust price of anarchy for a subclass of continuous games, which we denote as valid utility games with continuous strategies. They are the continuous counterpart of the valid utility games introduced by [31] and [23] for binary and integer strategies, respectively. Our bounds rely on a particular game structure and on the social function being a monotone DR-submodular function [4, Definition 1]. Hence, we define the curvature of a monotone DR-submodular function on continuous domains, analyze its properties, and use it to refine our bounds. We also show that our bounds can be extended to non-submodular functions which have ‘approximate’ submodularity properties. This is in contrast with [31, 23] where only submodular social functions were considered. Finally, employing the machinery of [24], we show that valid utility games with continuous strategies can be designed to maximize non convex/non concave functions in a distributed fashion with approximation guarantees. Depending on the curvature of the function, the obtained guarantees can improve the ones available in the literature.

1.3 Notation

We denote by $e_i$, $0$, and $I$, the $i^{th}$ unit vector, null vector, and vector of all ones of appropriate dimensions, respectively. Given $n \in \mathbb{N}$, with $n \geq 1$, we define $[n] := \{1, \ldots, n\}$. Given vectors $x, y$, we use $|x|_i$ and $x_i$ interchangeably to indicate the $i^{th}$ coordinate of $x$, and $(x, y)$ to denote the vector obtained from their concatenation, i.e., $(x, y) := [x^\top, y^\top]^\top$. Moreover, for vectors of equal dimension, $x \leq y$ means $x_i \leq y_i$ for all $i$. Given $x \in \mathbb{R}^n$ and $j \in \{0, \ldots, n\}$, we define $|x|_j := (x_1, \ldots, x_j, 0, \ldots, 0) \in \mathbb{R}^n$ with $|x|_0 = 0$. A function $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is monotone if, for all $x \leq y \in \mathcal{X}$, $f(x) \leq f(y)$. Moreover, $f$ is affine if for all $x, y \in \mathcal{X}$, $f(x + y) = f(x) + f(y)$. For simplicity, we assume each strategy $s_i$ is $d$-dimensional, although different dimensions could exist for different players. Each player aims to maximize her payoff function $\pi_i : \mathcal{S} \to \mathbb{R}$, which in general depends on the strategies of all the players. We let the social function be $\gamma : \mathbb{R}^{Nd} \to \mathbb{R}_+$. For the rest of the paper we assume $\gamma(0) = 0$. We denote such games with the tuple $G = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma)$. Given an outcome $s$ we use
the standard notation \((s_i, s_{-i})\) to denote the outcome
where player \(i\) chooses strategy \(s_i\) and the other players select strategies \(s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)\).

A pure Nash equilibrium is an outcome \(s \in S\) such that
\[
\pi_i(s) \geq \pi_i(s_i', s_{-i}),
\]
for every player \(i\) and for every strategy \(s_i' \in S_i\). A coarse correlated equilibrium (CCE) is a probability distribution \(\sigma\) over the outcomes \(S\) that satisfies
\[
E_{s \sim \sigma} [\pi_i(s)] \geq E_{s \sim \sigma} [\pi_i(s_i', s_{-i})],
\]
for every player \(i\) and for every strategy \(s_i' \in S_i\). CCE’s are the weakest class of equilibria and they include pure Nash, mixed Nash, and correlated equilibria [23].

Since each player selfishly maximizes her payoff, the outcome \(s \in S\) of the game is typically suboptimal for the social function \(\gamma\). To measure such suboptimality, [23] introduced the robust price of anarchy (robust PoA) which measures the inefficiency of any CCE. Given \(\mathcal{G}\), we let \(\Delta\) be the set of all the CCEs of \(\mathcal{G}\) and define the robust PoA as the quantity
\[
\text{PoA}_{CCE} := \max_{s \in S} \frac{\gamma(s)}{\min_{s \in \Delta} E_{s \sim \sigma}[\gamma(s)]},
\]
It can be easily seen that \(\text{PoA}_{CCE} \geq 1\). As discussed in the introduction, \(\text{PoA}_{CCE}\) has two important implications. In multi-agent systems, \(\text{PoA}_{CCE}\) bounds the efficiency of no-regret learning dynamics followed by the selfish agents. In fact, these dynamics converge to a CCE of the game [20]. In the context of distributed optimization, no-regret learning algorithms can be implemented distributively to optimize a given function and \(\text{PoA}_{CCE}\) certifies the overall approximation guarantee. Bounds for \(\text{PoA}_{CCE}\), however, were obtained mostly for games with finitely many actions [23].

In this paper, we are interested in upper bounding \(\text{PoA}_{CCE}\) for continuous games \(\mathcal{G}\) defined above. To motivate our results, we present two relevant examples of such games. The first one is a budget allocation game, while in the second example a continuous game can be designed for distributed maximization in the spirit of [24]. We will come back to these examples in Section 3 and derive upper bounds for their respective \(\text{PoA}_{CCE}\)’s.

**Example 1 (Continuous budget allocation game).** A set of \(N\) advertisers enters a market consisting of a set of \(d\) media channels. By allocating (or investing) part of their budget in each advertising channel, the goal of each advertiser is to maximize the expected number of activated customers, i.e., customers who purchase her product. The market is described by a bipartite graph \(\mathcal{G} = (\mathcal{R} \cup \mathcal{T}, \mathcal{E})\), where the left vertices \(\mathcal{R}\) denote channels and the right vertices \(\mathcal{T}\) denote customers, with \(d = |\mathcal{R}|\). For each advertiser \(i\) and edge \((r, t) \in \mathcal{E}\), \(p_i(r, t) \in [0, 1]\) is the probability that advertiser \(i\) activates customer \(t\) via channel \(r\). Each advertiser chooses a strategy \(s_i \in \mathbb{R}_d^+\), which represents the amounts allocated (or invested) to each channel, subject to budget constraints \(S_i = \{s_i \in \mathbb{R}_d^+ : c_i^\top s_i \leq b_i, 0 \leq s_i \leq \bar{s}_i\}\). This generalizes the set-up in [23], where strategies \(s_i\) are integer. Hence, we consider the continuous version of the game modeled by [23]. For every customer \(t \in \mathcal{T}\) and advertiser \(i \in [N]\), we define \(\Gamma(t) = \{r \in \mathcal{R} : (r, t) \in \mathcal{E}\}\) and the quantity
\[
P_i(s_i, t) = 1 - \prod_{r \in \Gamma(t)} (1 - p_i(r, t))^{[s_i]_r},
\]
which is the probability that \(i\) activates \(t\) when the other advertisers are ignored. For each customer \(t\), a permutation \(\rho \in \mathcal{P}_N\) is drawn uniformly at random, where \(\mathcal{P}_N\) is the set of all permutations of \([N]\). Then, according to \(\rho\) each advertiser sequentially attempts to activate customer \(t\). Hence, for a given allocation \(s = (s_1, \ldots, s_N) \in S = \prod_{i=1}^N S_i\), the payoff of each advertiser can be written in closed form as [23]:
\[
\pi_i(s) = \frac{1}{N!} \sum_{t \in \mathcal{T}} \sum_{\rho \in \mathcal{P}_N} P_i(s_i, t) \prod_{j<i} (1 - P_j(s_j, t))\]
where \(j \prec_i\) indicates that \(j\) precedes \(i\) in \(\rho\). The term \(\pi_i(s)\) represents the expected number of customers activated by advertiser \(i\) in allocation \(s\). The goal of the market analyst, which assumes the role of the game planner, is to maximize the expected number of customers activated. Hence, for any \(s\), the social function \(\gamma\) is
\[
\gamma(s) = \sum_{i=1}^N \pi_i(s) = \sum_{t \in \mathcal{T}} \left(1 - \prod_{i=1}^N (1 - P_i(s_i, t))\right)
\]

**Example 2 (Sensor coverage with continuous assignments).** Given a set of \(N\) autonomous sensors, we seek to monitor a finite set of \(d\) locations in order to maximize the probability of detecting an event. For each sensor, a continuous variable \(x_i \in \mathbb{R}_d^+\) indicates the energy assigned (or time spent) to each location, subject to budget constraints \(\mathcal{X}_i := \{x_i \in \mathbb{R}_d^+ : c_i^\top x_i \leq b_i, 0 \leq x_i \leq \bar{x}_i\}\). This generalizes the well-known sensor coverage problem studied in [24] (and previous works), where \(x_i\)’s are binary and indicate the locations sensor \(i\) is assigned to. The probability that sensor \(i\) detects an event in location \(r\) is \(1 - (1 - p_i^r)^{[x_i]_r}\), with \(0 \leq p_i^r \leq 1\), and it increases as more energy is assigned to the location. Hence, given a strategy \(x = (x_1, \ldots, x_N)\), the joint probability of detecting an event in location \(r\) is
\[
P(r, x) = 1 - \prod_{i \in [N]} (1 - p_i^r)^{[x_i]_r}.
\]
The goal of the planner is to maximize the probability of detecting an event
\[
\gamma(x) = \sum_{r \in [d]} w_r P(r, x),
\]
where \( w_i \)'s represent the a priori probability that an event occurs in location \( r \). As in [24], we can set up a continuous game \( G = (N, \{ S_i \}_{i=1}^n, \{ \pi_i \}_{i=1}^N, \gamma) \) where \( S_i = X_i \) for each \( i \), and \( \pi_i \)'s are designed so that good monitoring solutions can be obtained when each player selfishly maximizes her payoff.

**Proofs of the upcoming propositions and remarks are presented in Appendix A.**

### 3 Main results

We derive PoACCE bounds for a subclass of continuous games \( \mathcal{G} \) by extending the valid utility games considered in [31] and [23] to continuous strategy sets. Hence, in Definition 3 we will define the class of valid utility games with continuous strategies. As will be seen, the two problems described above and several other examples fall into this class. At the end of the section, we will show that valid utility games can be designed to maximize non-convex/non-concave objectives in a distributed fashion with approximation guarantees.

#### 3.1 Robust PoA bounds

As in [31, 23], the PoACCE bounds obtained rely on the social function \( \gamma \) experiencing diminishing returns (DR). Differently from set functions, in continuous (and integer) domains, different notions of DR exist. Similarly to [23], our first main result relies on \( \gamma \) satisfying the strongest notion of DR, also known as DR property [4], which we define in Definition 1. Moreover, as in [31] our bound can be refined depending on the curvature of \( \gamma \). While DR properties have been recently studied also in continuous domains, notions of curvature of a submodular function were only explored for set functions [10, 17] (see [3, Appendix C] for a comparison of the existing notions). Hence, in Definition 2 we define the curvature of a monotone DR-submodular function on continuous domains.

**Definition 1 (DR property).** A function \( f : \mathcal{X} = \prod_{i=1}^n X_i \rightarrow \mathbb{R} \) with \( X_i \subseteq \mathbb{R} \) is DR-submodular if for all \( x \leq y \in \mathcal{X}, \forall i \in [n], \forall k \in \mathbb{R}^{+} \) such that \( x + k e_i \) and \( y + k e_i \) are in \( \mathcal{X} \),

\[
 f(x + k e_i) - f(x) \geq f(y + k e_i) - f(y). 
\]

When restricted to binary sets \( Z = \{0, 1\}^n \), Definition 1 coincides with the standard notion of submodularity for set functions. An equivalent characterization of the DR property for a twice-differentiable function is that all the entries of its Hessian are non-positive [4]:

\[
 \forall x \in \mathcal{X}, \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, \quad \forall i, j.
\]

**Definition 2 (curvature).** Given a monotone DR-submodular function \( f : \mathcal{X} \subseteq \mathbb{R}_n^+ \rightarrow \mathbb{R} \), and a set \( Z \subseteq \mathcal{X} \) with \( 0 \in Z \), we define the curvature of \( f \) with respect to \( Z \) by

\[
 \alpha(Z) = 1 - \inf_{x \in Z, \forall i \in [n]} \lim_{k \to +\infty} \frac{f(x + k e_i) - f(x)}{f(k e_i) - f(0)}. 
\]

**Remark 1.** For any monotone function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( \forall Z \subseteq \mathbb{R}^n \) with \( 0 \in Z \), \( \alpha(Z) \in [0, 1] \).

When restricted to binary sets \( Z = \{0, 1\}^n \), Definition 2 coincides with the total curvature defined in [10]. Moreover, if \( f \) is monotone DR-submodular and differentiable, its curvature with respect to a set \( Z \) can be computed as:

\[
 \alpha(Z) = 1 - \inf_{x \in Z} \left[ \frac{\nabla f(x)}{\nabla f(0)} \right].
\]

Based on the previous definitions, we define the class of valid utility games with continuous strategies.

**Definition 3.** A game \( G = (N, \{ S_i \}_{i=1}^n, \{ \pi_i \}_{i=1}^N, \gamma) \) is a valid utility game with continuous strategies if:

i) The function \( \gamma \) is monotone DR-submodular.

ii) For each player \( i \) and for every outcome \( s, \pi_i(s) \geq \gamma(s) - \gamma(0, s_{-i}) \).

iii) For every outcome \( s \), \( \gamma(s) \geq \sum_{i=1}^{N} \pi_i(s) \).

Intuitively, the conditions above ensure that the payoff for each player is at least their contribution to the social function and that optimizing \( \gamma \) is somehow bound to the goals of the players. Defining the set \( S := \{ x \in \mathbb{R}_{+}^{Nd} | 0 \leq x \leq s_{\text{max}} \} \), with \( s_{\text{max}} \) such that \( \forall s, s' \in S, s + s' \leq s_{\text{max}} \), we can establish the following main theorem.

**Theorem 1.** Let \( G = (N, \{ S_i \}_{i=1}^n, \{ \pi_i \}_{i=1}^N, \gamma) \) be a valid utility game with continuous strategies with social function \( \gamma : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+} \) having curvature \( \alpha(S) \leq \alpha \). Then, \( PoACCE \leq (1 + \alpha) \).

We will prove Theorem 1 in Section 4.2.

**Remark 2.** If \( G \) is a valid utility game with continuous strategies, then \( PoACCE \leq 2 \).

**Remark 3.** The notion of valid utility games above is an exact generalization of the one by [23] for integer strategy sets. Leveraging recent advances in ‘approximate’ submodular functions, in Section 4 we relax condition i) and derive \( PoACCE \) bounds for a strictly larger class of games.

Using Theorem 1, the following proposition upper bounds \( PoACCE \) of Example 1.

**Proposition 1.** The budget allocation game defined in Example 1 is a valid utility game with continuous
strategies. Moreover, $PoA_{CCF} \leq 1 + \alpha < 2$ with 
\[ \alpha := 1 - \frac{\sum_{t \in [T], r \in [d]} \ln(1 - p_i(t)) \prod_{j \in [N]} (1 - P_j(2\delta_j(t)))}{\sum_{t \in [T], r \in [d]} \ln(1 - p_i(t))} \].

In our more general continuous actions framework, the obtained bound strictly improves the bound of 2 by 
\[ [23], \] since the curvature of the social function was not 
considered in \[ [23] \]. We will visualize our bound in the experiments of Section \[ 6 \]

Using Theorem \[ 1 \] we now generalize Example \[ 2 \] and 
show that valid utility games with continuous strategies 
can be designed to maximize monotone DR-
submodular functions subject to decoupled constraints 
with approximation guarantees. The proposed optimization scheme will be used in Section \[ 6 \] to solve an 
instance of the sensor coverage problem defined in Ex-
ample \[ 2 \].

### 3.2 Game-based monotone DR-submodular maximization

Consider the general problem of maximizing a mono-
tone DR-submodular function $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ subject 
to decoupled constraints $\mathcal{X} = \prod_{i=1}^N \mathcal{X}_i \subseteq \mathbb{R}^n$. We can 
assume $\mathcal{X}_i \subseteq \mathbb{R}^+$ without loss of generality \[ [2] \], since 
otherwise one could optimize $\gamma$ over a shifted version 
of its constraints. Moreover, we assume $\gamma(0) = 0$ for ease 
of exposition. Note that the class of monotone DR-
submodular functions includes non concave functions.

To find approximate solutions, we set up a game 
\[ \hat{G} := (\mathcal{N}, \{\hat{\mathcal{S}}_i\}_{i=1}^N, \{\hat{\pi}_i\}_{i=1}^N, \gamma) \],

where for each player $i$, $\hat{\mathcal{S}}_i := \mathcal{X}_i$, and $\hat{\pi}_i(s) := \gamma(s) - \gamma(0, s_{-i})$ for every outcome $s \in \mathcal{S} = \mathcal{X}$. By using 
DR-submodularity of $\gamma$, we can affirm the following.

**Fact 1.** $\hat{G}$ is a valid utility game with continuous 
strategies.

Assume there exists $x_{max} \in \mathbb{R}_+^n$ such that $\forall x, x' \in \mathcal{X}, x + x' \leq x_{max}$. Then, we denote with $\alpha(\mathcal{X})$ 
the curvature of $\gamma$ with respect to $\mathcal{X} := \{x \in \mathbb{R}^n_+ \mid 0 \leq x \leq x_{max}\}$ and let $\alpha \in [0, 1]$ be an upper bound for 
$\alpha(\mathcal{X})$. If such $x_{max}$ does not exist, we let $\alpha = 1$. Moreover, 
assume that for each player $i \in [N]$ there exists a no-regret algorithm \[ [27] \] Sec. \[ 3 \] to play $\hat{G}$. That is, 
when $\hat{G}$ is repeated over time, player $i$ can ensure that 
\[ \max_{x \in \mathcal{S}_i} \sum_{t=1}^T \hat{\pi}_i(s_t, s_{-i}) - \frac{1}{T} \sum_{t=1}^T \hat{\pi}_i(s_t, s_{-i}) \rightarrow 0 \] 
as $T \rightarrow \infty$, for any sequence $\{s_{-i}^T\}_{t=1}^T$. We let D-
noREGRET be the distributed algorithm where such 
no-regret algorithms are simultaneously implemented 
for each player. We can establish the following corol-
lary of Theorem \[ 1 \].

**Corollary 1.** Let $x^* = \arg\max_{x \in \mathcal{X}} \gamma(x)$. Then, 
D-noREGRET converges to a distribution $\sigma$ over $\mathcal{X}$ 
such that $E_{x \sim \sigma}[\gamma(x)] \geq 1/(1 + \alpha) \gamma(x^*)$.

Note that the FRANK-WOLFE variant of \[ 4 \] can also be 
used to maximize $\gamma$ with $(1 - e^{-1})$ approximations, un-
der the additional assumption that $\mathcal{X}$ is down-closed.

For small $\alpha$’s, however, our guarantee can strictly 
prove the one by \[ 4 \].

If $\mathcal{X}$ is convex compact and $\gamma$ is concave in each $\mathcal{X}_i$, then 
$\hat{\pi}_i$’s are concave in each $x_i$, and the online gradient 
 ascent algorithm by \[ 32 \] ensures no-regret for each 
player \[ 4 \]. Using Corollary \[ 1 \] we show that the sen-
so r coverage problem of Example \[ 2 \] falls into this class 
and D-noREGRET has approximation guarantees that 
depend on the sensing probabilities $P(r, \cdot)$’s.

**Proposition 2.** Consider the sensor coverage prob-
lem of Example \[ 2 \] and assume we set up the game $\hat{G}$. Then, 
online gradient ascent \[ 32 \] is a no-regret algo-
rithm for each player. Moreover, D-noREGRET has 
an expected approximation ratio of $1/(1 + \alpha)$, where 
$\alpha := \max_{r \in [d]} P(r, 2X)$ and $\mathfrak{x} = (x_1, \ldots, x_N)$.

Note that the obtained approximation ratio is strictly 
larger than $\frac{1}{2}$ and it increases when the number of sen-
ors $N$ or the detection probabilities decrease, a fact 
also noted in \[ 24 \] for the binary setting. We compare 
the performance of D-noREGRET and the FRANK-
WOLFE variant of \[ 4 \] in Section \[ 6 \].

A decentralized maximization scheme for submodular 
functions is also proposed in \[ 26 \], albeit in a different 
setting. In \[ 26 \], $\gamma$ consists of a sum of local functions 
such as a common down-closed convex constraint set, 
while we considered a generic objective $\gamma$ subject 
to local constraints.

### 4 Analysis

In order to prove Theorem \[ 1 \] and its extension to non-
submodular functions (Section \[ 5 \]), we first review the 
main properties of submodularity in continuous do-
 mains. We will introduce two straightforward interpre-
tations of the DR properties defined in the literature, 
and show a fundamental property of the curvature of the 
curve of a monotone DR-submodular function.

#### 4.1 Submodularity and curvature on continuous domains

Submodularity in continuous domains has received 
recent attention for approximate maximization and 
minimization of non convex/non concave functions 
\[ 2 \] A similar version of the corollary can be obtained when 
no-o-regret \[ 18 \] Definition \[ 4 \] exist for each 
player, such as the ones by \[ 9,8 \] for online submodular 
maximization.
Submodular continuous functions are defined on subsets of \( \mathbb{R}^n \) of the form \( \mathcal{X} = \bigcap_{i=1}^n \mathcal{X}_i \), where each \( \mathcal{X}_i \) is a compact subset of \( \mathbb{R} \). A function \( f : \mathcal{X} \to \mathbb{R} \) is submodular if for all \( x, y \in \mathcal{X} \), \( i, j \in \{1, \ldots, n\} \) and \( a, b > 0 \) s.t. \( x_i + a \in \mathcal{X}_i \) and \( x_j + a \in \mathcal{X}_j \),

\[
f(x + a, e_i) - f(x) \geq f(x + a, e_i + a, e_j) - f(x + a, e_j).
\]

The above property also includes submodularity of set functions, by restricting \( \mathcal{X}_i \)'s to \( \{0, 1\} \), and over integer lattices, by restricting \( \mathcal{X}_i \)'s to \( \mathbb{Z}_+ \). We are interested, however, in submodular continuous functions, where \( \mathcal{X}_i \)'s are compact subsets of \( \mathbb{R} \). As thoroughly studied for set functions, submodularity is related to diminishing return properties of \( f \). However, differences exist when considering functions over continuous (or integer) domains. In particular, submodularity is equivalent to the following weak DR property [5].

**Definition 4 (weak DR property).** A function \( f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R} \) is weakly DR-submodular if, for all \( x \leq y \in \mathcal{X} \), \( i \), and \( y + k e_i \) are in \( \mathcal{X} \),

\[
f(x + ke_i) - f(x) \geq f(y + ke_i) - f(y).
\]

The DR property, which we defined in Definition 1 [6] characterizes the full notion of diminishing returns and indentifies a subclass of submodular continuous functions. While weak DR and DR properties coincide for set functions, this is not the case for functions on integer or continuous lattices. As the next section reveals, the weak DR property of \( \gamma \) is indeed not sufficient to prove Theorem 1 [6]. However, it will be useful in Section 5 when considering results to non-submodular functions. In Appendix A.7 we discuss submodularity for differentiable functions.

To prove the main results of the paper, the following two propositions provide equivalent characterizations of weak DR and DR properties, respectively [6].

**Proposition 3.** A function \( f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R} \) is weakly DR-submodular (Definition 4) if and only if for all \( x \leq y \in \mathcal{X} \), \( i \), with \( z_i = 0 \) \( \forall i \in [n] \) \( : y_i > x_i \),

\[
f(x + z) - f(x) \geq f(y + z) - f(y).
\]

**Proposition 4.** A function \( f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R} \) is DR-submodular (Definition 1) if and only if for all \( x \leq y \in \mathcal{X} \), \( i \), and \( y + z \) are in \( \mathcal{X} \),

\[
f(x + z) - f(x) \geq f(y + z) - f(y).
\]

Finally, the following proposition clarifies the role of the curvature of a DR-submodular function and is key for the proof of Theorem 1 [6].

**Proposition 5.** Consider a monotone DR-submodular function \( f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R} \), and a set \( \mathcal{Z} := \{ x \in \mathbb{R}^n : 0 \leq x \leq z_{\text{max}} \} \subseteq \mathcal{X} \). Then, for any \( x, y \in \mathcal{Z} \) such that \( x + y \in \mathcal{Z} \),

\[
f(x + y) - f(x) \geq (1 - \alpha(\mathcal{Z}))[f(y) - f(0)],
\]

where \( \alpha(\mathcal{Z}) \) is the curvature of \( f \) with respect to \( \mathcal{Z} \).

### 4.2 Proof of Theorem 1

The proof uses submodularity of the social function similarly to [28, Example 2.6] and [23, Proposition 4]. However, it allows us to consider the curvature of \( \gamma \). Differently from [23, Proposition 4], our proof does not rely on the structure of the strategy sets \( S_i \). The weak DR and DR properties are used separately in the proof, to show that the weak DR property of \( \gamma \) is not sufficient to obtain the results. This fact was similarly noted in [23] for the integer case.

To upper bound \( Pa_{\text{ACCE}} \), we first prove that for any pair of outcomes \( s, s^* \in \mathcal{S} \),

\[
\sum_{i=1}^N \pi_i(s^*_i, s_{-i}) \geq \gamma(s^*) - \alpha(\gamma)(s).
\]

In the framework of [28], this means that \( \mathcal{G} \) is a \((1, \alpha)\)-smooth game. Then, few inequalities from [28] show that \( Pa_{\text{ACCE}} \leq (1 + \alpha) \).

The smoothness proof is obtained as follows. Consider any pair of outcomes \( s, s^* \in \mathcal{S} \). For \( i \in \{0, \ldots, N\} \) with a slight abuse of notation we define \( [s^*]_i = (s^*_1, \ldots, s^*_i, 0, \ldots, 0) \) with \( [s^*]_i = 0 \), where \( s^*_i \) is the strategy of player \( j \) in the outcome \( s^* \). We have:

\[
\sum_{i=1}^N \pi_i(s_i^*, s_{-i}) \geq \sum_{i=1}^N \pi_i(\gamma(s_i^*, s_{-i}) - \gamma(0, s_{-i})
\geq \sum_{i=1}^N \gamma(s_i^* + s_i, s_{-i}) - \gamma(s)
\geq \sum_{i=1}^N \gamma(s_i^* + s_i, s_{-i}) - \gamma(s + [s^*]_i^{-1})
\geq \gamma(s + s^*) - \gamma(s)
\geq (1 - \alpha)\gamma(s) + \gamma(s^*) - \gamma(s) = \gamma(s^*) - \alpha(\gamma)(s)
\]

The first inequality follows from condition ii) of valid utility games as per Definition 8. The second inequality from \( \gamma \) being DR-submodular (and using Proposition 3). The third inequality from \( \gamma \) being weakly DR-submodular (and using Proposition 3). The last inequality follows since, by Proposition 5

\[
\gamma(s + s^*) - \gamma(s^*) \geq (1 - \alpha(\mathcal{S}))[\gamma(s) - \gamma(0)],
\]

and \( \alpha(\mathcal{S}) \leq \alpha \).

For completeness we report the steps of [28] to prove that \( Pa_{\text{ACCE}} \leq (1 + \alpha) \). Let \( s^* = \arg \max_{s \in \mathcal{S}} \gamma(s) \). Then, for any CCE \( \sigma \) of \( \mathcal{G} \) we have

\[
E_{s \sim \sigma}[\gamma(s)] \geq \sum_{i=1}^N E_{s \sim \sigma}[\pi_i(s)] \geq \sum_{i=1}^N E_{s \sim \sigma}[\pi_i(s^*_i, s_{-i})]
\geq \gamma(s^*) - \alpha E_{s \sim \sigma}[\gamma(s)],
\]
where the first inequality is due to condition iii) of valid utility games as per Definition 3 the second inequality holds from σ being a CCE, and the last one since G is (1, α)-smooth. Moreover, linearity of expectation was used throughout. From the inequalities above it holds that for any CCE σ of G, γ(s*)/E_{σ→σ}[γ(s)] ≤ 1 + α. Hence PoA_{CCE} ≤ 1 + α.

Remark 4. Although Theorem 1 requires DR-submodularity of γ over \( \mathbb{R}^d_+ \) (for simplicity), only DR-submodularity over \( \mathcal{S} \) was used. In case γ is DR-submodular only over \( \mathcal{S} \), one could consider \( \tilde{\gamma} : \mathbb{R}^N_+ \rightarrow \mathbb{R}_+ \), defined as \( \tilde{\gamma}(s) = \gamma(\min(s, s_{\max})) \) which is DR-submodular over \( \mathbb{R}^N_+ \). This can be proved using DR-submodularity and monotonicity of γ over \( \mathcal{S} \). The same smoothness proof is obtained with \( \tilde{\gamma} \) in place of γ since the two functions are equal over \( \mathcal{S} \). However, the curvature of \( \tilde{\gamma} \) with respect to \( \mathcal{S} \) is 1 and therefore a bound of 2 for PoA_{CCE} is obtained.

5 Extension to the non-submodular case

In many applications, functions are close to being submodular, where this closeness has been measured in terms of submodularity ratio \( \eta \) (for set functions) and weak-submodularity \( \eta \) (on continuous domains). Accordingly, in this section we relax condition i) of valid utility games (Definition 3) and provide bounds for PoA_{CCE} when the social function γ is not necessarily DR-submodular. This case was never considered for the valid utility games of [31, 23]. We relax the weak DR property of γ with the following definition.

Definition 5. Given a game \( G = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma) \) with γ monotone, we define generalized submodularity ratio of γ as the largest scalar \( \eta \) such that for any pair of outcomes \( s, s' \in \mathcal{S} \),

\[
\sum_{i=1}^N \gamma(s_i + s'_i, s_{-i}) - \gamma(s) \geq \eta[\gamma(s + s') - \gamma(s)].
\]

It is straightforward to show that \( \eta \in [0, 1] \). Moreover, as stated in Appendix B (Proposition 3), if γ is weakly DR-submodular then γ has generalized submodularity ratio \( \eta = 1 \). When strategies \( s_i \) are scalar (i.e., \( d = 1 \)), Definition 5 generalizes the submodularity ratio by [11] to continuous domains.\(^4\)

In addition, we relax the DR property of γ as follows.

Definition 6. Given a game \( G = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma) \), we say that γ is playerwise DR-submodular if for every player \( i \) and vector of strategies \( s_{-i} \), \( \gamma(\cdot, s_{-i}) \) is DR-submodular.

Analogously to Definition 1 if γ is twice-differentiable, it is playerwise DR-submodular iff for every \( i \in [N] \)

\[
\forall s \in \mathcal{S}, \quad \frac{\partial^2 \gamma(s)}{\partial [s_i]_i \partial [s_i]_m} \leq 0, \quad \forall i, m \in [d].
\]

While Definition 5 concerns the interactions among different players, Definition 6 requires that γ is DR-submodular with respect to each individual player. When the social function γ is DR-submodular, then it is also playerwise DR-submodular. Moreover, since the DR property γ is DR-submodular, then it is also playerwise DR-submodular. Moreover, since the DR property is stronger than weak DR, γ has generalized submodularity ratio \( \eta = 1 \). If γ is not DR-submodular, however, the notions of Definition 5 and Definition 6 are not related. We visualize their differences in the following example.

Example 3. Consider a game with \( N = 2, d = 2 \), and γ twice-differentiable. Let \( \eta \) be the generalized submodularity ratio of γ. Assume the Hessian of γ satisfies one of the three cases below, where with ‘+’ or ‘−’ we indicate the sign of its elements:

\[
\begin{align*}
&1. \quad (-) \quad (-) \quad (+) \quad (+) \\
&2. \quad (+) \quad (-) \quad (-) \quad (+) \\
&3. \quad (+) \quad (+) \quad (-) \quad (-)
\end{align*}
\]

From the previous definitions, the function γ is playerwise DR-submodular iff all the entries highlighted in red are non-positive, while \( \eta \) depends on all the off-diagonal entries. In case 1., all the entries are negative, hence γ is DR-submodular. Thus, it is playerwise DR-submodular and has generalized submodularity ratio \( \eta = 1 \). In case 2., all off-diagonal entries are negative, hence γ is weakly DR-submodular (see Appendix 3). and thus \( \eta = 1 \). However, γ is not playerwise DR-submodular since some highlighted entries are positive. In case 3., γ is playerwise DR-submodular and its generalized submodularity ratio depends on its parameters.

Note that only case 1. of the previous example satisfies the conditions of Theorem 1. However, the following Theorem 2 is applicable also to a subset of functions which fall in case 3. The proof can be found in Appendix D.

Theorem 2. Let \( G = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma) \) be a game where γ is monotone, playerwise DR-submodular and has generalized submodularity ratio \( \eta > 0 \). Then, if conditions ii) and iii) of Definition 5 are satisfied, PoA_{CCE} \leq (1 + \eta)/\eta.

In light of the previous comments, when γ is DR-submodular Theorem 2 yields a bound of 2 which is always higher than \((1 + \alpha)\) from Theorem 1. This is because the notion of curvature in Definition 2 cannot be used in the more general setting of Theorem 2 since γ may not be DR-submodular.

In Appendix E we show that examples of functions with generalized submodularity ratio \( 1 > \eta > 0 \) are

\[\text{In Appendix B we define an exact generalization of the submodularity ratio by [11] to continuous domains. We relate it to Definition 5 and compare it to the ratio by [10].}\]
products of monotone weakly DR-submodular functions and monotone affine functions. As a consequence, the following generalization of Example 2 falls into the set-up of Theorem 2.

Sensor coverage problem with non-submodular objective. Consider the sensor coverage problem defined in Example 2 where the weights \( w_r \)'s are monotone affine functions \( w_r : \mathbb{R}^{N \times d} \rightarrow \mathbb{R}_+ \) rather than constants. For instance, the probability that an event occurs in location \( r \) can increase with the average amount of energy allocated to that location. That is, \( \gamma(x) = \sum_{r \in [d]} w_r(x) P(r, x) \) with \( w_r(x) = a_r \frac{\sum_{i=1}^{n} |x_i|}{N} + b_r \). To maximize \( \gamma \) one could set up a game \( G \) where condition ii) of Definition 3 is satisfied with equality, as shown in Section 3.2. In Appendix 13 we show that \( \gamma \) has generalized submodularity ratio \( 1 > \eta > 0 \), it is playerwise DR-submodular, and that \( \gamma(x) = \gamma(x) \geq \gamma(x) \), for every \( x \), which is a weaker version of condition iii). Nevertheless, using Theorem 3 and the last proof steps of Section 12 we prove that \( \text{PoACCE}(1 + 0.5\eta) \). We also show that \( \gamma \) is concave in each \( x_i \). Therefore a distributed implementation of online gradient ascent maximizes \( \gamma \) up to \( 0.5\eta/(1 + 0.5\eta) \) approximations.

We remark that our definitions of curvature, submodularity ratio, and Theorems 12 can also be applied to games and optimizations over integer domains, i.e., when \( S_i \subseteq \mathbb{Z}_+^d \) and \( \gamma \) is defined on integer lattices.

6 Experimental results

In this section we analyze the examples defined in Section 2 using the developed framework.

6.1 Continuous budget allocation game

We consider \( N = 10 \) advertisers in a market with \( d = 100 \) channels and \( |T| = 10^4 \) customers. For the budget constraints we select \( b_i = 1 \), \( s_i = 1 \) and each entry of \( c_i \) is sampled uniformly at random from \([0, 1]\).

For each \( i \in [N], r \in \mathcal{R}, t \in T \), \( p_{i}(r, t) \) is drawn uniformly at random in \([0.8, 1]\) \( p_{\text{max}} \). In Figure 1a we visualize the bound for \( \text{PoACCE} \) obtained in Proposition 1 for different values of \( p_{\text{max}} \) and the number of random edges connected to each customer. The chosen ranges ensure that a sufficient fraction of customers will be activated. For instance, for \( p_{\text{max}} = 0.01 \) and drawing 20 random edges for each customer, we obtained \( \frac{\text{PoACCE}}{\text{D-noRegret}} = \frac{3}{2} \( \gamma(x_k) \) as a function of the number of iterations \( K \) (left plot). Moreover, for \( K = 3000 \), we compare the performance of the algorithms when we enlarge the constraints \( \mathcal{X}_i \) by choosing \( b_i = b \) for each \( i \), with \( b \in \{1, 1.1, 1.2, \ldots, 2\} \) (right plot). As visible, \( \text{D-noRegret} \) shows faster convergence than \( \text{Frank-Wolfe} \) variant. However, for \( K = 3000 \) the two algorithms return the same values. Average computation times per iteration are 0.019 s and 0.009 s for \( \text{Frank-Wolfe} \) and \( \text{D-noRegret} \), respectively, on a 16 Gb machine at 3.1 GHz using Matlab.

7 Conclusions and future work

We bounded the robust price of anarchy for a subclass of continuous games, denoted as valid utility games with continuous strategies. Our bound relies on a particular structure of the game and on the social function being monotone DR-submodular. We introduced the notion of curvature of a monotone DR-submodular function and refined the bound using this notion. In addition, we extended the obtained bounds to a class of non-submodular functions. We showed that valid utility games can be designed to maximize monotone DR-submodular functions subject to disjoint constraints. For a subclass of such functions, our approximation guarantees improve the ones in the literature. We demonstrated our results numerically via a continuous budget allocation game and a sensor coverage problem. In light of the obtained approximation guarantees, we believe that the introduced notion of curvature of a monotone DR-submodular function can be used.

Footnote: Since \( \gamma \) is monotone DR-submodular and \( S \) is downclosed, we used the \( \text{FRANK-WOLFE} \) variant by 4 to maximize \( \gamma \) up to \((1 - e^{-1}) \) approximations.
to tighten existing guarantees for constrained maximization. Currently, we are studying the tightness of the obtained bounds and their applicability to several continuous games such as auctions.

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Bounding Inefficiency of Equilibria in Continuous Actions Games using Submodularity and Curvature

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A Supplementary material for Sections 3-4

A.1 Proof of Remark 1

Since $f$ is monotone, $f(x + ke_i) \geq f(x)$ and $f(ke_i) \geq f(0)$ for any $x \in \mathcal{Z}$, $i \in [n]$, and $k \in \mathbb{R}_+$. Hence, $\alpha(\mathcal{Z}) \leq 1$. Moreover, $\inf_{x \in \mathcal{Z}} \lim_{k \to 0^+} \frac{f(x + ke_i) - f(x)}{f(ke_i) - f(0)} \leq 1$, since the considered ratio equals 1 when $x = 0$. Hence, $\alpha(\mathcal{Z}) \geq 0$.

A.2 Proof of Remark 2

The proof is obtained simply noting that the curvature $\alpha(S)$ of $\gamma$ is always upper bounded by 1.

A.3 Proof of Proposition 1

We first show that the budget allocation game of Example 1 is a valid utility game with continuous strategies. In fact, for any $l \in [Nd]$

$$|v_j(\gamma(s))| = \sum_{t \in T \cap m \in \Gamma(t)} -\ln(1-p_l(m,t)) \prod_{i=1}^N (1-P_i(s_i,t)),$$

where $j \in [N]$ and $m \in [d]$ are the indexes of advertiser and channel corresponding to coordinate $l \in [Nd]$, respectively. Hence, $\gamma$ is monotone since $|v_j(\gamma(s))| \geq 0$ for any $l \in [Nd]$ and $s \in \mathbb{R}_+^N$. Moreover, $\gamma$ is DR-submodular since $\gamma(s) = \sum_{t \in T \cap m \in \Gamma(t)} \gamma_t(s) - 1 - \prod_{i=1}^N (1 - P_i(s_i,t))$ is such that for any $j, l \in [N], m, n \in [d]$, $\frac{\partial^2 \gamma_t(s)}{\partial s_j \partial s_i} = -\ln(1 - p_l(m,t)) \ln(1 - p_l(n,t)) \prod_{i=1}^N (1 - P_i(s_i,t)) \leq 0$ for any $s \in \mathbb{R}_+^N$. Finally, condition ii) can be verified equivalently as in [23] Proposition of Proof of Proposition 5 and condition iii) holds with equality.

Then the set $\mathcal{S} := \{s \in \mathbb{R}_+^N \mid 0 \leq s \leq s_{max}\}$ is such that $s + s' \leq s_{max}$ for any pair $s, s' \in \mathcal{S}$. Moreover, using the expression of $\nabla \gamma(s)$, the curvature of $\gamma$ with respect to $\mathcal{S}$ is

$$1 - \alpha(S) = \inf_{s \in \mathcal{S}} \frac{|v_j(s)|}{|v_j(0)|} = \sum_{t \in T \cap m \in \Gamma(t)} \min_{i \in [N], r \in [d]} \frac{\ln(1-p_l(r,t)) \prod_{i=1}^N (1 - P_i(2s_j, t))}{\ln(1-p_l(r,t))} = 1 - \alpha > 0.$$

Hence, using Theorem 1 we conclude that $P_{\alpha CCE} \leq 1 + \alpha$.

A.4 Proof of Fact 1

Condition i) holds since $\gamma$ is monotone DR-submodular by definition. Also, condition ii) holds with equality. Moreover, defining (with abuse of notation) $|s|^1 = (s_1, \ldots, s_0, \ldots, 0)$ for $i \in [N]$ with $|s|^0 = 0$, condition iii) holds since by DR-submodularity one can verify that $\frac{\sum_{i=1}^N \hat{v}_i(s) - \sum_{i=1}^N \hat{v}_i(0, s_{-i})}{\gamma(|s|^1) - \gamma(|s|^0)} = \gamma(x) - \gamma(0) = \gamma(x)$.

A.5 Proof of Corollary 1

By definition of $\alpha$, and according to Theorem 1, $\hat{\gamma}$ is such that $P_{\alpha CCE} \leq (1 + \alpha)$. In other words, letting $s^* = \arg \max_{x \in \mathcal{S}} \gamma(x)$, any CCE $\sigma$ of $\hat{\gamma}$ satisfies $\mathbb{E}_{x \sim \sigma} [\gamma(x)] \geq 1/(1 + \alpha) \gamma(s^*)$. Moreover, since players simultaneously use no-regret algorithms DNOREGRET converges to one of such CCE [13, 28]. Hence, the statement of the remark follows.

A.6 Proof of Proposition 2

Consider the sensor coverage problem with continuous assignments defined in Example 2. We first show that $\gamma$ is a monotone DR-submodular function. In fact, for any $i \in [Nd], |

$$\nabla \gamma(x)|_{i} = -\ln(1 - \frac{p^m(m, t)}{m}) \prod_{i=1}^N (1 - P_i(s_i, t)), \text{ and }$$

hence $\nabla \gamma(x)|_{i} \geq 0$ for any $i \in [Nd]$ and $s \in \mathbb{R}_+^N$. Moreover, $\gamma$ is DR-submodular since $\gamma(s) = \sum_{t \in T \cap m \in \Gamma(t)} \gamma_t(s) - 1 - \prod_{i=1}^N (1 - P_i(s_i, t))$ is such that for any $j, l \in [N], m, n \in [d]$, $\frac{\partial^2 \gamma_t(s)}{\partial s_j \partial s_i} = -\ln(1 - p_l(m,t)) \ln(1 - p_l(n,t)) \prod_{i=1}^N (1 - P_i(s_i, t)) \leq 0$ for any $s \in \mathbb{R}_+^N$. Finally, condition ii) can be verified equivalently as in [23] Proposition of Proof of Proposition 5 and condition iii) holds with equality.

The set $\mathcal{S} := \{x \in \mathbb{R}_+^N \mid 0 \leq x \leq x_{max}\}$ is such that $x + x' \leq x_{max}$ and $x + x' \leq x_{max}$ for any pair $x, x' \in \mathcal{S}$. Moreover, using the expression of $\nabla \gamma(x)$, the curvature of $\gamma$ with respect to $\mathcal{S}$ is

$$1 - \alpha(S) = \inf_{s \in \mathcal{S}} \frac{|v_j(s)|}{|v_j(0)|} = \sum_{t \in T \cap m \in \Gamma(t)} \min_{i \in [N], r \in [d]} \frac{\ln(1-p_l(r,t)) \prod_{i=1}^N (1 - P_i(2s_j, t))}{\ln(1-p_l(r,t))} = 1 - \alpha > 0.$$

Hence, using Theorem 1 we conclude that $P_{\alpha CCE} \leq 1 + \alpha$.

A.7 Properties of (twice) differentiable submodular functions

As mentioned in Section 1, submodular continuous functions are defined on subsets of $\mathbb{R}^n$ of the form
\( X = \prod_{i=1}^{n} X_i \), where each \( X_i \) is a compact subset of \( \mathbb{R} \). From the weak DR property (Definition 4) it follows that, when \( f \) is differentiable, it is submodular iff
\[
\forall x, y \in X : x \leq y, \forall i \text{ s.t. } x_i = y_i, \nabla_i f(x) \geq \nabla_i f(y).
\]
That is, the gradient of \( f \) is a weak antitone mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

Moreover, we saw that a function \( f : X \rightarrow \mathbb{R} \) is submodular iff for all \( x \in X, \forall i \neq j \) and \( a_i, a_j > 0 \) s.t. \( x_i + a_i \in X_i, x_j + a_j \in X_j \), we have [1]
\[
f(x + a_i e_i) - f(x) \geq f(x + a_i e_i + a_j e_j) - f(x + a_j e_j).
\]
As visible from the latter condition, when \( f \) is twice-differentiable, it is submodular iff all the off-diagonal entries of its Hessian are non-positive [1]:
\[
\forall x \in X, \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, \forall i \neq j.
\]
Hence, the class of submodular continuous functions contains a subset of both convex and concave functions.

Similarly, from the DR property (Definition 4) it follows that for a differentiable continuous function DR-submodularity is equivalent to
\[
\forall x \leq y, \nabla f(x) \geq \nabla f(y).
\]
That is, the gradient of \( f \) is an antitone mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). More precisely, [4] Proposition 2] showed that a function \( f \) is DR-submodular iff it is submodular (weakly DR-submodular) and coordinate-wise concave. A function \( f : X \rightarrow \mathbb{R} \) is coordinate-wise concave if, for all \( x \in X, \forall i \in [n], \forall k \in \mathbb{R}_+ \), and \((x + ke_i)\) and \((x + ke_i)\) are in \( X \), we have
\[
f(x + ke_i) - f(x) \geq f(x + (k + l)e_i) - f(x + le_i),
\]
or equivalently, if twice differentiable, \( \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0 \) \( \forall i \in [n] \). Hence, as stated in Section [3] a twice-differentiable function is DR-submodular iff all the entries of its Hessian are non-positive:
\[
\forall x \in X, \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, \forall i, j.
\]

A.8 Proof of Proposition 3

(property of Proposition [3] → weak DR)
We want to prove that for all \( x \leq y \in X, \forall i \text{ s.t. } x_i = y_i, \forall k \in \mathbb{R}_+ \text{ s.t. } (x + ke_i) \) and \((y + ke_i)\) are in \( X \),
\[
f(x + ke_i) - f(x) \geq f(y + ke_i) - f(y).
\]
This is trivially done choosing \( z = ke_i \). Note that \( z \) is such that \( z_i = 0, \forall i \in \{i | y_i > x_i\} \), so the property of Proposition [3] can indeed be applied.

(weak DR → property of Proposition 4]
For all \( x \leq y \in X, \forall z \in \mathbb{R}_+^n \text{ s.t. } (x + z) \) and \((y + z)\) are in \( X \), with \( z_i = 0, \forall i \in [n] : y_i > x_i \), we have
\[
f(x + z) - f(x) = \sum_{i=1}^{n} f(x + [z]_i) - f(x + [z]_i^{-1})
\]
\[
= \sum_{i : x_i = y_i} f(x + [z]_i^{-1} + z_ie_i) - f(x + [z]_i^{-1})
\]
\[
\geq \sum_{i : x_i = y_i} f(y + [z]_i^{-1} + z_ie_i) - f(y + [z]_i^{-1})
\]
\[
= \sum_{i=1}^{n} f(y + [z]_i) - f(y + [z]_i^{-1})
\]
\[
= f(y + z) - f(y).
\]
The first equality is obtained from a telescoping sum, the second equality follows since when \( y_i > x_i, z_i = 0 \). The inequality follows from weak DR property of \( f \) and the last two equalities are similar to the first two.

A.9 Proof of Proposition 4

(property of Proposition [4] → DR)
We want to prove that for all \( x \leq y \in X, \forall i \in [n], \forall k \in \mathbb{R}_+ \text{ s.t. } (x + ke_i) \) and \((y + ke_i)\) are in \( X \),
\[
f(x + ke_i) - f(x) \geq f(y + ke_i) - f(y).
\]
This is trivially done choosing \( z = ke_i \) and applying the property of Proposition [4]

(\( DR \rightarrow \) property of Proposition 4]
For all \( x \leq y \in X, \forall z \in \mathbb{R}_+^n \text{ s.t. } (x + z) \) and \((y + z)\) are in \( X \), we have
\[
f(x + z) - f(x) = \sum_{i=1}^{n} f(x + [z]_i) - f(x + [z]_i^{-1})
\]
\[
= \sum_{i=1}^{n} f(x + [z]_i^{-1} + z_ie_i) - f(x + [z]_i^{-1})
\]
\[
\geq \sum_{i=1}^{n} f(y + [z]_i^{-1} + z_ie_i) - f(y + [z]_i^{-1})
\]
\[
= \sum_{i=1}^{n} f(y + [z]_i) - f(y + [z]_i^{-1})
\]
\[
= f(y + z) - f(y).
\]
The first and last equalities are telescoping sums and the inequality follows from the DR property of \( f \).
A.10 Proof of Proposition \textbf{5}

By Definition \textbf{2} the curvature $\alpha(Z)$ of $f$ w.r.t. $Z$ satisfies

$$f(x + ke_i) - f(x) \geq (1 - \alpha(Z))[f(k e_i) - f(0)] \quad (1)$$

for any $x \in Z, i \in [n]$ s.t. $x + ke_i \in Z$ with $k \to 0_+$. We firstly show that condition \textbf{11} indeed holds for any $x \in Z, i \in [n], k \in \mathbb{R}_+$ s.t. $x + ke_i \in Z$, by using monotonicity and coordinate-wise concavity of $f$. As seen in Appendix \textbf{A.7}, DR-submodularity implies coordinate-wise concavity. To this end, we define

$$\alpha^k_i(Z) = 1 - \inf_{x \in Z; x + ke_i \in Z} \frac{f(x + ke_i) - f(x)}{f(ke_i) - f(0)}.$$ 

Hence, it suffices to prove that, for any $i \in [n]$, $\alpha^k_i(Z)$ is non-increasing in $k$. Note that by DR-submodularity,

$$\alpha^k_i(Z) = 1 - \frac{f(z_{max}) - f(z_{max} - ke_i)}{f(ke_i) - f(0)}.$$ 

Hence, for any pair $l, m \in \mathbb{R}_+$ with $l < m$, $\alpha^m_i(Z) \geq \alpha^l_i(Z)$ is true whenever

$$\frac{f(z_{max}) - f(z_{max} - ke_i)}{f(ke_i) - f(0)} \geq \frac{f(z_{max}) - f(z_{max} - le_i)}{f(le_i) - f(0)}.$$ 

The last inequality is satisfied since, by coordinate-wise concavity, $[f(z_{max}) - f(z_{max} - me_i)]/m \geq [f(z_{max}) - f(z_{max} - le_i)]/l$ and $[f(me_i) - f(0)]/m \leq [f(le_i) - f(0)]/l$. This is because, given a concave function $g : \mathbb{R} \to \mathbb{R}$, the quantity

$$R(x_1, x_2) := \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

is non-increasing in $x_1$ for fixed $x_2$, and vice versa. Moreover, monotonicity ensures that all of the above ratios are non-negative.

To conclude the proof of Proposition \textbf{5} we show that if condition \textbf{11} holds for any $x \in Z, i \in [n], k \in \mathbb{R}_+$ s.t. $x + ke_i \in Z$, then the result of the proposition follows. Indeed, for any $x, y$ s.t. $x + y \in Z$ we have

$$f(x + y) - f(x) = \sum_{i=1}^n f(x + [y]^i_1) - f(x + [y]^{i-1}_1)$$

$$= \sum_{i=1}^n f(x + [y]^{i-1}_1 + y_i e_i) - f(x + [y]^{i-1}_1)$$

$$\geq (1 - \alpha(Z)) \sum_{i=1}^n f(y_i e_i) - f(0)$$

$$\geq (1 - \alpha(Z)) \sum_{i=1}^n f([y]^i_1) - f([y]^{i-1}_1)$$

$$= (1 - \alpha(Z))(f(y) - f(0)),$$
B Supplementary material for Section 5

In the first part of this appendix we generalize the submodularity ratio defined in [11] for set functions to continuous domains and discuss its main properties. We compare it to the ratio by [16] and we relate it to the generalized submodularity ratio defined in Definition 5. Then, we provide a class of social functions with generalized submodularity ratio 0 < η < 1 and we report the proof of Theorem 2. Finally, we analyze the sensor coverage problem with the non-submodular objective defined in Section 6.

B.1 Submodularity ratio of a monotone function on continuous domains

We generalize the class of submodular continuous functions, defining the submodularity ratio η ∈ [0, 1] of a monotone function defined on a continuous domain.

Definition 7 (submodularity ratio). The submodularity ratio of a monotone function f : X ⊆ ℝ^n → ℝ is the largest scalar η such that for all x, y ∈ X such that x + y ∈ X,

\[ \sum_{i=1}^{n} [f(x + y_i e_i) - f(x)] \geq \eta [f(x + y) - f(x)]. \]

It is straightforward to show that η ∈ [0, 1] and, when restricted to binary sets X = \{0, 1\}^n, Definition 7 coincides with the submodularity ratio defined in [11] for set functions. A set function is submodular iff it has submodularity ratio η = 1 [11]. However, functions with submodularity ratio 0 < η < 1 still preserve ‘nice’ properties in term of maximization guarantees. Similarly to [11], we can affirm the following.

Proposition 6. A function f : X ⊆ ℝ^n → ℝ is weakly DR-submodular (Definition 4) iff it has submodularity ratio η = 1.

Proof. If f is weakly DR-submodular (Definition 4), then for any x, y ∈ X,

\[ \sum_{i=1}^{d} f(x + y_i e_i) - f(x) \]

\[ \geq \sum_{i=1}^{d} f(x + \lceil y_i \rceil_1) - f(x + \lceil y_i \rceil_1^{-1}) = f(x + y) - f(x). \]

Assume now f has submodularity ratio η = 1. We prove that f is weakly DR-submodular by proving that it is submodular. Hence, we want to prove that for all x ∈ X, ∀i ≠ j and a_i, a_j > 0 s.t. x_i + a_i ∈ X_i, x_j + a_j ∈ X_j,

\[ f(x + a_i e_i) - f(x) \geq f(x + a_i e_i + a_j e_j) - f(x + a_j e_j). \]

Consider y = a_i e_i + a_j e_j ∈ X. Since f has submodularity ratio η = 1, we have

\[ f(x + a_i e_i) - f(x) + f(x + a_j e_j) - f(x) \geq f(x + a_i e_i + a_j e_j) - f(x), \]

which is equivalent to the submodularity condition [2].

An example of functions with submodularity ratio η > 0 is the product between an affine and a weakly DR-submodular function, as stated in the following proposition.

Proposition 7. Let f, ρ : X ⊆ ℝ^n → ℝ be two monotone functions, with f weakly DR-submodular, and g affine such that ρ(x) = a^T x + b with a ≥ 0 and b > 0. Then, provided that X is bounded, the product g(x) := f(x)ρ(x) has submodularity ratio η = \inf_{x,y\in X} \sum_{i,j} a_i a_j x_i y_j > 0.

Proof. Note that since ρ is affine, for any x, y ∈ X we have that g(x + y) = f(x + y)ρ(x + y) = f(x)ρ(x) + f(y)ρ(y) = f(x)ρ(x) + f(y)ρ(y). For any pair x, y ∈ X we have:

\[ \sum_{i=1}^{n} [g(x + y_i e_i) - g(x)] \]

\[ = \sum_{i=1}^{n} \rho(x + y_i e_i) [f(x + y_i e_i) - f(x)] + f(x) (y_i a^T e_i) \]

\[ \geq \min_{i\in[n]} \rho(x + y_i e_i) \sum_{i=1}^{n} f(x + y_i e_i) - f(x) + f(x) (a^T y) \]

\[ \geq \min_{i\in[n]} \rho(x + y_i e_i) (f(x) - f(x)) \]

\[ \eta(x, y) \cdot [g(x + y) - g(x)]. \]

The first inequality follows since ρ is affine non-negative and f is non-negative. The second inequality is due to f being weakly DR-submodular (f has submodularity ratio η = 1) and 0 < η(x, y) ≤ 1, which holds because b > 0 and a ≥ 0. Hence, it follows that γ has submodularity ratio

\[ \eta := \inf_{x,y\in X} \eta(x, y) = \inf_{i\in[n], x,y\in X} \frac{b}{b + \sum_{j\neq i} a_j y_j} > 0. \]
B.1.1 Related notion by [16]

A generalization of submodular continuous functions was also provided in [16] together with provable maximization guarantees. However, it has different implications than the submodularity ratio defined above. In fact, [16] considered the class of differentiable functions \( f : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) with parameter \( \eta \) defined as

\[
\eta = \inf_{x, y \in \mathcal{X}, x \leq y} \inf_{i \in [n]} \frac{\nabla f(x)_i}{\nabla f(y)_i}.
\]

For monotone functions \( \eta \in [0, 1] \), and a differentiable function is DR-submodular iff \( \eta = 1 \) [16]. Note that the parameter \( \eta \) of [16] generalizes the DR property of \( f \), while our submodularity ratio \( \gamma \) generalizes the weak DR property.

B.2 Relations with the generalized submodularity ratio of Definition 5

In Proposition 8 we saw that submodularity ratio \( \eta = 1 \) is a necessary and sufficient condition for weak DR-submodularity. In contrast, a generalized submodularity ratio (Definition 5) \( \eta = 1 \) is only necessary for the social function \( \gamma \) to be weakly DR-submodular. This is stated in the following proposition. For non submodular \( \gamma \), no relation can be established between submodularity ratio of Definition 7 and generalized submodularity ratio of Definition 5.

Proposition 8. Given a game \( \mathcal{G} = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma) \). If \( \gamma \) is weakly DR-submodular, then \( \gamma \) has generalized submodularity ratio \( \eta = 1 \).

Proof. Consider any pair of outcomes \( s, s' \in \mathcal{S} \). For \( i \in \{0, \ldots, N\} \), with abuse of notation we define \( [s]_i^0 := (s^i_0, \ldots, s^i_0, 0, \ldots, 0) \) with \( [s]_i^0 = 0 \). We have,

\[
\sum_{i=1}^N \gamma(s_i + s'_i, s_{-i}) - \gamma(s) = \sum_{i=1}^N \gamma(s + [s']_i^0 - [s]_i^0) - \gamma(s + [s']_i^0)
\]

where the inequality follows since \( \gamma \) is weakly DR-submodular and the equality is a telescoping sum.

Similarly to Proposition 7 in the previous section, in the following proposition we show that social functions \( \gamma \) defined as product of weakly DR-submodular functions and affine functions have generalized submodularity ratio \( \eta > 0 \).

Proposition 9. Given a game \( \mathcal{G} = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma) \). Let \( \gamma \) be defined as \( \gamma(s) := f(x)\rho(x) \) with \( f, \rho : \mathbb{R}^d \rightarrow \mathbb{R}^n \) be two monotone functions, with \( f \) weakly DR-submodular, and \( g \) affine such that \( \rho(x) = a^\top x + b \) with \( a = (a_1, \ldots, a_N) \geq 0 \) and \( b > 0 \). Then, \( \gamma \) has generalized submodularity ratio \( \eta = \inf_{i \in [N], x \in S} \frac{b}{b + \sum_{j \neq i} |a_j| \gamma_j} > 0 \).

Proof. The proof is equivalent to the proof of Proposition 7 with the only difference that \( s_i \) belong to \( \mathbb{R}^n \) instead of \( \mathbb{R}^d \).

Note that for the game considered in the previous proposition, using Proposition 7 one could also affirm that \( \gamma \) has submodularity ratio \( \eta = \inf_{i \in [N], x \in S} \frac{b}{b + \sum_{j \neq i} |a_j| \gamma_j} > 0 \) which, unless \( d = 1 \), is strictly smaller than its generalized submodularity ratio.

B.3 Proof of Theorem 2

The proof is equivalent to the proof of Theorem 1 with the only difference that here we prove that \( \mathcal{G} \) is a \( (\eta, \eta) \)-smooth game in the framework of [28]. Then, it follows that \( \text{PoA}_{\text{CCE}} \leq (1 + \eta)/\eta \).

For the smoothness proof, consider any pair of outcomes \( s, s^* \in \mathcal{S} \). We have:

\[
\sum_{i=1}^N \pi_i(s_i^*, s_{-i}) \geq \sum_{i=1}^N \gamma(s_i^*, s_{-i}) - \gamma(0, s_{-i})
\]

\[
\geq \sum_{i=1}^N \gamma(s_i^* + s_i, s_{-i}) - \gamma(s)
\]

where the inequality follows since \( \gamma \) is weakly DR-submodular and the equality is a telescoping sum. The first inequality is due to condition ii) of Definition 8. The second inequality follows since \( \gamma \) is player-wise DR-submodular (applying Proposition 4 for each player \( i \)) and the second inequality from \( \gamma \) having generalized submodularity ratio \( \eta \).

B.4 Analysis of the sensor coverage problem with non-submodular objective

We analyze the sensor coverage problem with non-submodular objective defined in Section 3 where \( \gamma(x) = \sum_{r \in [d]} w_r(x) P(r, x) \) with \( w_r(x) = a_r \sum_{i=1}^n [x_i]_i + b_r \). Note that by Proposition 9 the function \( \gamma_r(x) := w_r(x) P(r, x) \) has generalized submodularity ratio \( \eta > 0 \), hence it is not hard to show that \( \gamma(x) = \sum_{r \in [d]} \gamma_r(x) \) shares the same property. Moreover, there exist parameters \( a_r, b_r \) for which \( \gamma \) is not submodular. Interestingly, \( \gamma \) is convex in each \( X_r \).
In fact, $\gamma_i$’s are concave in each $X_i$ since $P(r, x)$’s are concave in each $X_i$ and $w_r$’s are positive affine functions. Moreover, $\gamma$ is playerwise DR-submodular since the $(d \times d)$ blocks on the diagonal of its Hessian are diagonal (and their entries are non-positive, by concavity of $\gamma$ in each $X_i$).

To maximize $\gamma$, as outlined in Section 3.2, we can set up a game $G = (N, \{S_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma)$ where for each player $i$, $S_i = X_i$, and $\pi_i(s) = \gamma(s) - \gamma(0, s_{-i})$ for every outcome $s \in S = X$. Hence, condition ii) of Definition 3 is satisfied with equality. Following the proof of Theorem 2, we have that:

$$\sum_{i=1}^N \pi_i(s^*_i, s_{-i}) \geq \eta \gamma(s + s^*) - \eta \gamma(s)$$

In order to bound PoA$_{CCE}$, the last proof steps of Section 4.2 still ought to be used. Such steps rely on condition iii), which in Section 3.2 was proved using submodularity of $\gamma$. Although $\gamma$ is not submodular, we prove a weaker version of condition iii) as follows. By definition of $\gamma_r$ and for every outcome $x$ we have $\sum_{i=1}^N \gamma_r(s) - \gamma_r(0, s_{-i}) = \sum_{i=1}^N w_r(x)[P(r, s) - P(r, 0, s_{-i})] + [w_r(s_i, 0) - w_r(0)]P(r, 0, s_{-i}) \leq w_r(x)P(r, s) + P(r, 0, s) \sum_{i=1}^N [w_r(s_i, 0) - w_r(0)] = (1 + \frac{w_r(s) - w_r(0)}{w_r(x)}) \gamma_r(x) \leq 2 \gamma_r(x)$. The equalities are due to $w_r$ being affine, the first inequality is due to $P(r, \cdot)$ being submodular and monotone, and the last inequality holds since $w_r$ is positive and monotone. Hence, from the inequalities above we have $\sum_{i=1}^N \pi_i(x) = \sum_{i=1}^N \gamma(x) - \gamma(0, x_{-i}) \leq 2 \gamma(x)$. Note that a tighter condition can also be derived depending on the functions $w_r$’s, using $(1 + \max_{s \in X, r \in [d]} \frac{w_r(s) - w_r(0)}{w_r(x)})$ instead of 2. We will now use such condition in the same manner condition iii) was used in Section 4.2.

Let $s^* = \arg\max_{s \in S} \gamma(s)$. Then, for any CCE $\sigma$ of $G$ we have

$$E_{s \sim \sigma}[\gamma(s)] \geq \frac{1}{2} \sum_{i=1}^N E_{s \sim \sigma}[\pi_i(s)] \geq \frac{1}{2} \sum_{i=1}^N E_{s \sim \sigma}[\pi_i(s^*_i, s_{-i})]$$

$$\geq \frac{\eta}{2} \gamma(s^*) - \frac{\eta}{2} E_{s \sim \sigma}[\gamma(s)].$$

Hence, PoA$_{CCE} \leq (1 + 0.5\eta)/0.5\eta$. 

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