KAZHDAN-LUSZTIG LEFT CELLS IN TYPE $B_n$ FOR INTERMEDIATE PARAMETERS

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Abstract. Using a characterization of a generalized $\tau$-invariant for intermediate parameter Hecke algebras in type $B_n$ obtained in [8], we verify a conjectural description of Kazhdan-Lusztig cells in this setting due to C. Bonnafé, L. Iancu, M. Geck, and T. Lam.

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1. Introduction

Hecke algebras play a central role in the representation theory of reductive algebraic groups over finite and $p$-adic fields. Each can be constructed as a specialization of an Iwahori-Hecke algebra $H$ which itself can be defined via generators and relations from a Coxeter group $W$ without explicit dependence on the underlying algebraic group.

Defined in [12] and broadened by G. Lusztig to the setting of weighted Iwahori-Hecke algebras in [14] and [15], Kazhdan-Lusztig cells describe partitions of $W$ known as left, right, and two-sided cells. Their well-known classification in type $A$ motivates our work. We briefly recount the results. The Robinson-Schensted map

$$RS : S_n \to SYT(n) \times SYT(n)$$

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defines a bijection between the symmetric group and same-shape pairs of standard Young tableaux. If we write \( RS(w) = (P(w), Q(w)) \), then \( P(w) \) and \( Q(w) \) are known as the left and right tableaux of \( w \). It follows from the work of A. Joseph in [10] and [11] on primitive ideals for complex Lie algebras of type \( A \) that left cells in this setting consist of those permutations whose right tableaux agree, right cells of those permutations whose left tableaux agree, and two-sided cells of all permutations whose image tableaux have the same shape. See [1] for a comprehensive exposition.

For weighted Coxeter groups in type \( B_n \), the focus of this paper, partitions into cells depend on a single positive parameter \( s \). When \( s = 1 \), known as the equal-parameter case, cell partitions can again be derived from the classification of primitive ideals for complex Lie algebras of type \( B_n \). Obtained by D. Garfinkle in [5], [6], and [7], the theory mimics type \( A \) results, but depends on a correspondence between the Coxeter group \( W_n \) of type \( B_n \) and same-shape pairs of standard domino tableaux. This time, left and right cells in \( W_n \) are the pre-images of certain equivalence classes of right and left tableaux, respectively; and two-sided cells are described by equivalence classes of partitions.

Kazhdan-Lusztig cells have been determined for a few other values of the parameter \( s \) in type \( B_n \). When \( s = \frac{1}{2} \) and \( s = \frac{3}{2} \), their classification appears already in [14] and for \( s > n - 1 \), known as the asymptotic case, the problem was resolved by C. Bonnafé and L. Iancu in [4]. Reconciling these descriptions, C. Bonnafé, L. Iancu, M. Geck, and T. Lam formulated a set of conjectures relating cells for all values of the parameter \( s \) to a one-parameter family \( G_r \) of domino-insertion algorithms in [2] modeled after Garfinkle’s original map defined in [5].

In the present paper, we address the case \( s = n - 1 \). Using results obtained in [8], we verify the above conjectures. The key is an enhancement of D. Vogan’s generalized \( \tau \)-invariant [19] proposed in [3]. We show that as is true in type \( A \) and the equal-parameter case of type \( B_n \), the left-cell, generalized \( \tau \)-invariant, and certain tableaux-based partitions of \( W \) coincide.

Our paper has the following structure. In Section 2, we describe preliminaries about the Coxeter groups of type \( B_n \), domino tableaux, and the domino insertion maps \( G_r \). Section 3 recounts the definitions of Kazhdan-Lusztig cells, a conjecture of C. Bonnafé, L. Iancu, M. Geck, and T. Lam, and results on the \( \tau \)-invariant and descent sets. The final Section 4 contains the proofs of the main results as well as motivating examples.

2. Preliminaries

2.1. Hyperoctahedral groups. Let \( W_n = W(B_n) \) be the Weyl group of type \( B_n \). We realize it as the set of signed permutations on \( n \) letters, writing an element in one-line notation as \( w = (w_1 w_2 \ldots w_n) \). It is a Coxeter group \( (W_n, S) \) described by the Coxeter diagram:

\[
\begin{array}{ccccccc}
& & & & & & s_{n-1} \\
t & s_1 & s_2 & \cdots & \cdots &
\end{array}
\]

where we identify the generating reflections with signed permutations as

\[
t = (1 2 \ldots n) \quad \text{and} \quad s_i = (1 2 \ldots i + 1 i \ldots n).
\]
We will use a bar to denote a negative entry. Write \( t = t_1 \) and for \( 2 \leq k \leq n \), let 
\[ t_k := s_{k-1} \cdots s_1 t s_1 \cdots s_{k-1}, \]
so that 
\[ t_k = (1 \ 2 \ \cdots \ k \ \cdots \ n). \]

2.2. Domino tableaux. Let \( \lambda \) be a partition of \( n \in \mathbb{N} \) and identify it with a Young diagram \( Y(\lambda) \), a left-justified array of squares whose row lengths decrease weakly. A staircase partition takes the form \( \lambda_r = [r, r-1, \ldots, 1] \) for some \( r \in \mathbb{N} \). We extend this notion, letting \( \lambda_0 \) denote the empty partition.

Write \( s_{ij} \) for the square lying in row \( i \) and column \( j \) of \( Y(\lambda) \). A domino is a pair of squares in \( Y(\lambda) \) of the form \( \{s_{ij}, s_{i+1,j}\} \) or \( \{s_{ij}, s_{i,j+1}\} \); it is removable from \( Y(\lambda) \) if deleting its underlying squares leaves either the empty diagram, or another Young diagram which contains the square \( s_{11} \). Starting with a Young diagram \( Y(\lambda) \) we can iteratively delete removable dominos. This process always terminates in a diagram \( Y(\lambda_r) \) for some \( r \geq 0 \). In fact, \( r \) is determined entirely by \( \lambda \) and does not depend on the precise sequence of removable domino deletions. We will say that \( \lambda_r \) is the \( 2 \)-core, or simply the core of \( \lambda \) and that \( \lambda \) is a partition of rank \( r \). Let \( \delta_k \) denote the set of all squares \( s_{ij} \) which satisfy \( i + j = k + 1 \).

Example 1. The partition \( \lambda = [7, 6, 1^3] \) is of rank three. Its Young diagram \( Y(\lambda) \) as well as a domino tiling representing deletions of removable dominos are given below.

![Diagram](attachment:image.png)

**Definition 2.1.** A standard domino tableau of rank \( r \) and shape \( \lambda \) is a tiling of the non-core squares of \( Y(\lambda) \) by dominos, each labeled by a unique integer from the set \( \{1, \ldots, n\} \) in such a way that the labels increase along the rows and columns of \( Y(\lambda) \). We will write \( \text{SDT}_r(\lambda) \) for the set of standard domino tableaux of rank \( r \) of shape \( \lambda \) and \( \text{SDT}_r(n) \) for the set of standard domino tableaux of rank \( r \) which contain exactly \( n \) dominos.

For convenience, we will label the squares in the core of \( T \) with the integer 0. The set of ordered pairs of same-shape domino tableaux in \( \text{SDT}_r(n) \) will play an important role in what follows; we will denote it as \( \text{SSDT}_r(n) \). We will also have occasion to refer to domino tableaux that satisfy exactly the conditions above, but the set of whose squares with label 0 does not form a staircase partition. We will call these simply standard domino tableaux.

2.3. Cycles. The notion of a cycle within a domino tableau was first introduced in [5] and extended to domino tableaux of rank \( r \) in [18]. We refer the reader to these references for details. Identifying a set of fixed squares in a tableau allows us to define

- a partition of its domino labels into disjoint cycles, and
- the operation of moving through a cycle to produce another domino tableau.

There are two natural choices for the set of fixed squares for a rank \( r \) tableau, each producing a distinct cycle partition.

**Definition 2.2.** Consider a tableau \( T \in \text{SDT}_r(n) \).
(a) If fixed squares in $T$ are defined as those $s_{ij}$ for which $i + j$ has the opposite parity from $r$, we will call the resulting cycles of $T$ regular cycles, or simply cycles.

(b) On the other hand, if fixed squares in $T$ are those $s_{ij}$ for which $i + j$ has the same parity as $r$, we will call the resulting cycles of $T$ opposite cycles.

In prior work, for instance [18] and [16], it has only been necessary to address regular cycles for a rank $r$ tableau, but opposite cycles will play an important role herein.

**Example 2.** Consider the following standard domino tableaux of rank 2:

\[
S = \begin{array}{ccc}
0 & 1 & 2 \\
4 & 3 & 1
\end{array} \quad T = \begin{array}{ccc}
0 & 1 & 2 \\
4 & 3 & 1
\end{array}
\]

Each domino label forms its own regular cycle in both $S$ as well as in $T$. The family of opposite cycles in $S$ is $\{\{1\}, \{2\}, \{3, 4\}\}$, while in $T$ it is $\{\{1\}, \{2, 4\}, \{3\}\}$.

Given a cycle $c$ in $T$, the moving through map constructs another domino tableau $MT(T, c)$ that differs from $T$ only in the labels of the variable squares of $c$. Disjoint cycles can be moved though independently, and if $U$ is a set of cycles, we will write $MT(T, U)$ for the domino tableau obtained by simultaneously moving through all of them. When discussing tableau pairs, define

\[
MT((S, T), (U, V)) := (MT(S, U), MT(T, V)).
\]

When $c$ is an opposite cycle, we will write $MT_{op}(T, c)$ to emphasize the opposite choice of fixed squares is being used to define the operation. We refer the reader to [18, §2.2] for the detailed definitions of the moving through map.

Cycles in a tableau come in a few distinct flavors depending on whether or not the moving through operation changes the underlying Young diagram and which squares are affected. A cycle for which moving through changes the shape of the underlying tableau is called an open cycle, otherwise it is closed. An open cycle is a core cycle if moving through it changes the total number of squares in the underlying Young diagram; it is non-core otherwise. Respectively, we will write

\[
CC(T), OC(T), KC(T), \text{ and } NC(T)
\]

for the sets of closed, open, core, and non-core cycles in a domino tableau $T$. Note that $OC(T)$ is a disjoint union of $KC(T)$ and $NC(T)$. Our notation for the corresponding sets of opposite cycles will include the superscript $op$.

**Example 2 (Continued).** First we consider the cycle partition for both $S$ and $T$.

Each of $S$ and $T$ has three core cycles and one non-core cycle:

\[
KC(S) = KC(T) = \{\{1\}, \{2\}, \{3\}\}, \text{ while } NC(S) = NC(T) = \{4\}.
\]

In the opposite cycle partition, each tableau has two core cycles and one closed cycle:

\[
KC^{op}(S) = \{\{1\}, \{2\}\}, CC^{op}(S) = \{\{3, 4\}\},
\]

\[
KC^{op}(T) = \{\{1\}, \{3\}\}, CC^{op}(T) = \{\{2, 4\}\}.
\]
Moving through all the core cycles of a standard domino tableau of rank \( r \) results in a standard domino tableau of rank \( r + 1 \). Similarly, moving through all of the opposite core cycles reduces the rank of a standard domino tableau by one. For instance, with \( S \) and \( T \) as above,

\[
\text{MT}(S, KC(S)) = \begin{array}{c}
1 & 2 \\
3 & 4 \\
5 & 6 \\
7 & 8 \\
\end{array}
\quad \text{and} \quad \text{MT}(T, KC(T)) = \begin{array}{c}
1 & 2 \\
3 & 4 \\
5 & 6 \\
7 & 8 \\
\end{array}
\]

In general, for \((S, T) \in \text{SSDT}_r(n)\) the shape of \(\text{MT}(S, KC(S))\) will be different from the shape of \(\text{MT}(T, KC(T))\), as occurs above. However, it is possible to amend this by including additional non-core open cycles in the moving though map. Again looking at regular cycles in the tableaux above, let

\[
\gamma(S, T) = KC(S) \cup \{4\} \quad \text{and} \quad \gamma(T, S) = KC(T) \cup \{4\},
\]

then moving through produces a shape-shape pair of standard domino tableaux:

\[
\text{MT}(S, \gamma(S, T)) = \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\quad \text{and} \quad \text{MT}(T, \gamma(T, S)) = \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\]

**Definition 2.3.** Consider a pair \((S, T) \in \text{SSDT}_r(n)\). Let \(\gamma(S, T)\) and \(\gamma(T, S)\) be the minimal sets of open cycles in \(S\) and \(T\), respectively, that satisfy:

(a) \(KC(S) \subset \gamma(S, T)\), \(KC(T) \subset \gamma(T, S)\), and

(b) the tableaux \(\text{MT}(S, \gamma(S, T))\) and \(\text{MT}(T, \gamma(T, S))\) have the same shape.

As in \[6, 2.3.1\], we refer to \(\gamma(S, T)\) as the set of extended open cycles in \(S\) relative to \(T\) through \(KC(S)\) and define \(\gamma(T, S)\) similarly. Finally, write \(\gamma_{ST}\) for the pair \((\gamma(S, T), \gamma(T, S))\).

As detailed in \[18\], the above discussion allows us to define a bijective map on same-shape domino tableau pairs \(\Gamma_r : \text{SSDT}_r(n) \to \text{SSDT}_{r+1}(n)\) by letting

\[\Gamma_r((S, T)) = \text{MT}((S, T), \gamma_{ST}).\]

### 2.4. Domino insertion.

Modeled on the Robinson-Schensted algorithm, \[5\] and \[13\] introduced a one-parameter family of bijections

\[G_r : W_n \to \text{SSDT}_r(n)\].

In its original form, these maps are defined by using a domino insertion algorithm where a pair of standard domino tableaux is constructed simultaneously, the left tableaux by inserting and bumping dominos starting with a Young diagram of the staircase partition \(\lambda_r\), while the right tracking the shape of the insertion tableau. We will write \(G_r(w) = (P_r(w), Q_r(w))\) for the image of an element of \(W_n\) under this map and extend this notation writing \(G^k_r(w) = (P^k_r(w), Q^k_r(w))\) for the tableaux obtained after \(k\) insertion steps.

**Example 3.** Consider the signed permutation \(w = (4 \ 1 \ 3 \ 2) \in W_4\). The following sequence of tableau pairs represents its images under \(G_r\) for increasing values of \(r\).

\[
P_0(w) = \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array} \quad \quad Q_0(w) = \begin{array}{c}
1 & 2 \\
3 & 4 \\
\end{array}
\]
When $r \geq n-1$, $G_r$ recovers a well-known bijection between signed permutations and same-shape pairs of standard Young bitableaux, see [4]. Positive values are inserted as horizontal dominos at the top of the tableau and negative values as vertical dominos at the bottom. If $r \geq n-1$, the two sets of labels do not interact during insertion as is true in the bitableaux algorithm. Tableaux with this latter property will play a role in what follows, so we define:

**Definition 2.4.** A standard domino tableau $T$ of rank $r$ is split iff its diagonal $\delta_{r+2}$ contains at least one empty square. We extend this notion in the natural way to same-shape tableau pairs.

There is a simple relationship among the tableau pairs in $G_r(w)$ for the different values of $r$. It can be described completely in terms of extended cycles. With $\Gamma_r$ defined as in Section 2.3, we have:

**Theorem 2.5** ([18]). $\Gamma_r(G_r(w)) = G_{r+1}(w)$.

The inverse of $\Gamma_r$, which we write as $\Upsilon_r$, is defined on a tableau pair $(S, T)$ by moving through the extended cycles $\upsilon_{ST} = (\upsilon(S, T), \upsilon(T, S))$, where $\upsilon(S, T)$ is the extended opposite open cycle in $S$ relative to $T$ through $KC^{op}(S)$ and $\upsilon(T, S)$ is the extended opposite open cycle in $T$ relative to $S$ through $KC^{op}(T)$.

We will be particularly interested in equivalence classes on $W_n$ defined by domino tableaux. The simplest are those defined by having the same right tableaux.

**Definition 2.6.** For $T \in SDT_r(n)$, let

$$C(T) = \{ w \in W_n \mid Q_r(w) = T \}.$$  

For each value of $r$, these classes define a partition of $W_n$. As $r$ varies, there is an intricate, but tractable, relationship among them. We describe it in the following proposition.

**Proposition 2.7.** Let $T \in SDT_r(n)$ and define $T' := MT(T, KC(T)) \in SDT_{r+1}(n)$. If $U$ is a set of regular cycles in $T$, then it is also a set of opposite cycles in $T'$. Write $T_U := MT(T, U)$ and $T'_U := MT_{op}(T', U)$. Then

$$\bigcup_{U \subseteq NC(T)} C(T_U) = \bigcup_{U \subseteq NC(T)} C(T'_U).$$
Proof. First note the following two sets of simple relationships between sets of open cycles among the above tableaux: \( KC(T_U) = KC(T) = KC^{op}(T') \) and \( NC(T_U) = NC(T) = NC^{op}(T') \). We will work with sets of same-shape pairs of standard domino tableaux instead of subsets of \( W_n \). Write \( D(T) := G_r(C(T)) \) for the set of same-shape rank \( r \) standard domino tableau pairs whose right tableaux is \( T \).

Let

\[
X(T) := \bigsqcup_{U \subset NC(T)} D(T_U) \quad \text{and} \quad Y(T) := \bigsqcup_{U \subset NC(T)} D(T'_U).
\]

We will show that \( \Gamma_r(X(T)) = Y(T) \). Hence consider \((S, T_U) \in SSDT_r(n)\). By definition of \( \Gamma_r \), the right tableau of \( \Gamma_r(S, T_U) \) equals \( MT(T_U, \gamma(T_U, S)) \). The extended cycle \( \gamma(T_U, S) \) in \( T_U \) consists of all the open cycles in \( KC(T_U) = KC(T) \) together with a subset of non-core open cycles \( U' \subset NC(T_U) = NC(T) \) which depends on \( S \).

If we write \( \Delta \) for the symmetric-difference relation on sets, our observation means that \( \Gamma_r(S, T_U) \in D(T_U \setminus U') \). Thus we have \( \Gamma_r(X(T)) \subset Y(T) \).

Now consider \((S', T'_U) \in SSDT_{r+1}(n)\). Then the right tableau of \( \Upsilon_r(S', T'_U) \) equals \( MT(T'_U, v(T'_U, S')) \) where \( v(T'_U, S') \) consists of all the opposite core open cycles in \( KC^{op}(T'_U) = KC(T) \) together with a subset of opposite non-core open cycles \( U' \subset NC^{op}(T') = NC(T) \) which depends on \( S' \). This time, \( \Upsilon_r(S', T_U) \in D(T_U \cup U') \) and we have \( X(T) \supset \Upsilon_r(Y(T)) \). Together, the two inclusions give us the desired equality.

We will use the above proposition in two especially tractable cases, when the set \( NC(T) \) of non-core open cycles is either empty, or contains exactly one cycle.

3. Cells

Following G. Lusztig’s construction in [15] and the combinatorics of standard domino tableaux detailed in [2] and [16], we describe two one-parameter families of partitions of \( W_n \), as well as a conjecture relating them.

3.1. Kazhdan-Lusztig Cells. The first partition is defined in terms of the algebraic structure of a two-parameter algebra derived from \( W_n \). Let \( S \) be the set of simple roots of \( W_n \) as defined in Section 2.1. Write \( \ell : W_n \to \mathbb{Z}_{\geq 0} \) for the standard length function computed using the generators in \( S \). A weight function \( \mathcal{L} : W_n \to \mathbb{Z} \) is a map satisfying \( \mathcal{L}(wy) = \mathcal{L}(w) + \mathcal{L}(y) \) whenever \( \ell(wy) = \ell(w) + \ell(y) \).

It is uniquely determined by its values on the simple roots \( S \) and is subject to the condition that whenever \( s, s' \in S \) and \( ss' \) is of odd order, \( \mathcal{L}(s) = \mathcal{L}(s') \). We will let \( \mathcal{L}(t) = b \) and \( \mathcal{L}(s) = a \) for all \( i \) with \( a, b \in \mathbb{N} \) and write \( \mathcal{L} = \mathcal{L}(a, b) \).

Let \( \mathcal{H}_n \) be the generic Iwahori-Hecke algebra over \( \mathbb{A} = \mathbb{Z}[v, v^{-1}] \) with parameters \( \{v_s \mid s \in S\} \), where \( v_w = v^{\mathcal{L}(w)} \) for all \( w \in W_n \). The algebra \( \mathcal{H}_n \) is free over \( \mathbb{A} \) and has a basis \( \{T_w \mid w \in W\} \) in terms of which multiplication takes the form

\[
T_sT_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w), \\
T_{sw} + (v_s - v_s^{-1})T_w & \text{if } \ell(sw) < \ell(w),
\end{cases}
\]

for \( s \in S \) and \( w \in W_n \). As detailed in [15], each choice of weight function \( \mathcal{L} \), or equivalently, each pair \( a, b \in \mathbb{N} \), defines partitions of \( W_n \) into left, right, and two-sided cells. In fact, these partitions depend only on the ratio \( \frac{b}{a} \) of the parameters. In what follows, we will only consider integer values of this ratio.
Notation: We will restrict our attention to \( \frac{k}{a} \in \mathbb{N} \), write \( r = \frac{k}{a} - 1 \), and call components of the corresponding partitions left, right, and two-sided \( r \)-cells. Denote the resulting equivalence relations by \( \approx_L^r \), \( \approx_R^r \), and \( \approx_{LR}^r \), respectively, and let \( K_L^r(w) \), \( K_R^r(w) \), and \( K_{LR}^r(w) \) be the cells containing the element \( w \in W_n \). We will focus almost exclusively on left cells and will omit the superscript in our notation unless there is a potential for confusion.

3.2. Combinatorial cells. Inspired by the classification of Kazhdan-Lusztig cells for classical Weyl groups in terms of standard Young and domino tableaux, we have the following definition of a family of partitions of \( W_n \) based on the images of the maps \( G_r \).

Definition 3.1. Consider an non-negative integer \( r \) and let \( S, T \in \text{SDT}_r(n) \). We will write \( S \sim_r T \) if and only if there is a set of non-core open cycles \( U \subset \mathcal{NC}(T) \) such that \( S = \text{MT}(T, U) \). If \( w, y \in W_n \), we will say:

(a) \( w \sim_L^r y \) if and only if \( Q_r(w) \sim_r Q_r(y) \),
(b) \( w \sim_R^r y \) if and only if \( P_r(w) \sim_r P_r(y) \),
(c) \( w \sim_{LR}^r y \) if and only if there is a sequence \( w = w_0, w_1, \ldots, w_k = y \) where for all \( i < k \), either \( w_i \sim_{LR}^r w_{i+1} \) or \( w_i \sim_L^r w_{i+1} \) or \( w_i \sim_R^r w_{i+1} \).

Notation: We will call the equivalence classes on \( W_n \) defined by the relations \( \sim_L^r \), \( \sim_R^r \), and \( \sim_{LR}^r \) left, right, and two-sided combinatorial \( r \)-cells writing \( C_L^r(w) \), \( C_R^r(w) \), and \( C_{LR}^r(w) \) for the cells containing the element \( w \in W_n \). Again, we will omit the superscript for left cells in our notation unless there is a potential for confusion.

When \( r \geq n - 1 \) it is easy to see from the definition of the insertion algorithm \( G_r \) that all combinatorial left \( r \)-cells and \( (r + 1) \)-cells coincide. The same is true for combinatorial right and two-sided cells. We will call these cells asymptotic and write \( C_{aL}^r(w) \), \( C_{aR}^r(w) \), and \( C_{aLR}^r(w) \), usually omitting the superscript for left cells.

Left combinatorial \( r \)-cells can be written as unions of sets of the form \( C(T) \) for rank \( r \) tableaux. Using notation from Section 2.4 we have

\[
C_r(w) = \bigcup_{U \in \mathcal{NC}(T)} C(T_U)
\]

where \( T = Q_r(w) \) and \( T_U := \text{MT}(T, U) \). At the heart of this paper is the conjecture of C. Bonnafé, L. Iancu, M. Geck, and T. Lam which, when interpreted in light of the results from [15], states that combinatorial \( r \)-cells agree with Kazhdan-Lusztig \( r \)-cells on \( W_n \):

Conjecture 3.2 ([2]). For \( w \in W_n \) and \( r \in \mathbb{Z}_{\geq 0} \), \( C_r(w) = K_r(w) \).

Among integral values of \( r \), this conjecture has been verified for \( r = 0 \) in [7] and for \( r \geq n - 1 \) in [4], the latter in particular implying that \( C_a(w) = K_a(w) \) for all \( w \in W_n \). Our work concerns \( r = n - 2 \), which we follow [8] in calling the intermediate parameter case and will refer to \( r \)-cells as intermediate cells.
3.3. **Descent set and an enhanced \(\tau\)-invariant.** For a Coxeter group \(W\) with generating reflections \(S\), the right descent set, or the right \(\tau\)-invariant, of an element \(w \in W\) is the set

\[
\tau(w) = \{ s \in S \mid \ell(ws) < \ell(w) \}.
\]

We avoid handedness and refer to it simply as the \(\tau\)-invariant of \(w\). In the type \(B_n\) setting where the weight function has unequal parameters, \([17]\) and \([8]\) extended the \(\tau\)-invariant to draw from not only simple reflections but also some of the elements of the form \(t_j\) defined in Section 2.1. More precisely,

**Definition 3.3** (\([17]\) and \([8]\)). For \(w \in W_n\) and weight function \(L = L(a, b)\), the enhanced \(\tau\)-invariant is the set

\[
\tau^L(w) = \tau(w) \cup \{ t_j \mid \frac{b}{a} > j - 1 \text{ and } \ell(w t_j) < \ell(w) \}.
\]

When the parameters \(a\) and \(b\) of the weight function \(L(a, b)\) are equal, then this definition recovers the usual \(\tau\)-invariant and \(\tau^L(w) = \tau(w)\).

It is possible read-off the enhanced \(\tau\)-invariant both from the signed-permutation representation of \(w\) as well as from the domino tableaux \(G_r(w)\), although in the latter case this is only straightforward for weight functions with certain parameters. The following instructs us how to proceed in the case of signed permutations:

**Proposition 3.4.** Let \(w \in W_n\). Then

(a) \(\ell(ws_i) < \ell(w) \iff w(i + 1) < w(i)\), and
(b) \(\ell(w t_j) < \ell(w) \iff w(j) < 0\).

To compute \(\tau^L(w)\) from the tableaux \(G_r(w)\), we first imitate \([8]\) Section 2.1 and define the notion of an enhanced \(\tau\)-invariant for a domino tableau \(Q\) of rank \(r\). Inclusion in this set will be determined by the relative positions of the dominoes in \(Q\). We restrict the definition to the setting where the weight function \(L = L(a, b)\) satisfies \(r \geq \frac{b}{a} - 1\). Note that in the equal-parameter case, this inequality is satisfied by all values of \(r\).

**Definition 3.5.** Consider \(Q \in \text{SDT}_r(n)\) and let \(L = L(a, b)\) with \(r \geq \frac{b}{a} - 1\). Then

(a) \(t_j \in \tau^L(Q)\) if and only if \(D(j, Q)\) is vertical for \(1 \leq j \leq r + 1\), and
(b) \(s_i \in \tau^L(Q)\) if and only if \(D(i, Q)\) lies strictly above \(D(i+1, Q)\) for \(1 \leq i \leq n\).

Extending this definition to pairs of domino tableaux, we let \(\tau^L((P, Q)) := \tau^L(Q)\). The ordinary \(\tau\)-invariant \(\tau(Q)\) and \(\tau(P, Q)\) is defined by taking the intersection of this set with \(S\).

The following proposition relating the enhanced \(\tau\)-invariant for signed permutations and domino tableaux generalizes \([8]\) 2.1.18 where it is stated for the regular \(\tau\)-invariant when \(r = 0\). It is possible to verify it by carefully extending the original proof to domino tableaux of rank \(r\). Instead we will take a more streamlined approach using the maps \(\Gamma_r\) which relate the domino insertion maps for tableaux of different ranks.

**Proposition 3.6.** Let \(w \in W_n\) and suppose that \(L = L(a, b)\) where \(\frac{b}{a}\) is an integer and \(r \geq \frac{b}{a} - 1\). Then

\[
\tau^L(w) = \tau^L(G_r(w)).
\]

This proposition has the following immediate corollary for the regular \(\tau\)-invariant:
Corollary 3.7. Let \( w \in W_n \). For all \( r \geq 0 \), we have
\[
\tau(w) = \tau(G_r(w)).
\]

Our proof relies on two intermediate results. The first will help us understand the effect of the maps \( \Gamma_r \) on the ordinary \( \tau \)-invariant.

Lemma 3.8. Suppose that \( T \) is a standard domino tableau and \( c \) is either an open cycle in \( T \) or a closed cycle not of the form \( c = \{i, i+1\} \). Then \( \tau(T) = \tau(\text{MT}(T, c)) \). The same holds for corresponding opposite cycles in \( T \) as long as \( T \) is not a standard domino tableau of rank 0.

Proof. This is a generalization of [2, 3.1.4], allowing both general rank tableaux as well as certain core open cycles, which the original result omits. Thus consider a cycle or opposite cycle \( c \) in \( T \) which satisfies the conditions of the proposition. Let \( T' = \text{MT}(T, c) \) or \( T' = \text{MT}_\text{op}(T, c) \). We will show that \( \tau(T) \subset \tau(T') \). Since moving through is an involution, this will suffice.

First suppose that \( t \in \tau(T) \). Then \( D(1, T) \) is vertical, and unless \( s_{11} \in D(1, T) \) and \( c \) is an opposite cycle, so will be \( D(1, T') \). Thus \( t \in \tau(T') \). If \( s_i \in \tau(T) \), then we have to examine the relative positions of \( D(i, T) \) and \( D(i+1, T) \). If we assume that in fact \( s_i \not\in \tau(T') \), then \( D(i, T') \) and \( D(i+1, T') \) must be adjacent and their relative positions must be one of the following:

where we have written \( i' := i + 1 \). We eliminate each possibility successively from left to right, writing \( s_{kl} \) for the top, leftmost square of \( D(i, T') \). The first possibility implies that the fixed squares with labels \( i \) and \( i + 1 \) lie in the same row in both \( T \) and \( T' \), which contradicts our original assumption that \( s_i \in \tau(T) \). The second and third imply impossible labellings of a standard domino tableau: \( s_{k+1, l+1} \) in the second and \( s_{k, l+1} \) in the third would have to be labeled with an integer between \( i \) and \( i + 1 \). For the final diagram to have arisen from a tableau \( T \) with \( s_i \in \tau(T) \), \( s_{kl} \) must be a fixed square. But in this case, \( c = \{i, i + 1\} \) is a closed cycle, which we have excluded from consideration.

The following statement resolves the inclusion of the \( t_j \) in the enhanced \( \tau \)-invariant for domino tableaux of sufficiently large rank. It follows directly from the definition of the domino insertion maps \( G_r \).

Lemma 3.9. Let \( w \in W_n \) and suppose that \( \mathcal{L} = \mathcal{L}(a, b) \) where \( \frac{b}{a} \) is an integer and \( r \geq \frac{b}{a} - 1 \). When \( k \leq r + 1 \), the following statements are equivalent for all \( 1 \leq j \leq k \):

(a) \( D(w(j), P^k(w)) \) is horizontal,
(b) \( D(w(j), Q^k(w)) \) is horizontal,
(c) \( w(j) > 0 \), and
(d) \( t_j \not\in \tau\mathcal{L}(w) \).

Proof of Proposition [3, 2.1.4]. The original version of this result [2, 2.1.9] gives us the desired statement for the ordinary \( \tau \)-invariant and the map \( G_0 \). Since for a domino tableau pair \( (P, Q) \) of rank \( r \), the rank \( r + 1 \) tableaux \( \Gamma_r(P, Q) \) are defined by moving through a set of open cycles in \( P \) and \( Q \), and by Proposition [2, 3] \( G_{r+1} = \Gamma_r \circ G_r \)
for all \( r \), Lemma 3.8 implies that our result holds for the ordinary \( \tau \)-invariant for all maps \( G_r \).

To verify the proposition for the enhanced \( \tau \)-invariant, we need only compare membership of \( t_j \) for \( 2 \leq j \leq r+1 \) in \( \tau^C(w) \) and \( \tau^C(G_r(w)) \). Note that the restriction of the recording tableau \( Q_r(w) \) to its core of rank \( r \) and the dominos with labels in the set \( \{1, 2, \ldots, r+1\} \) is a split domino tableau. We may now apply Lemma 3.9 to conclude the proof.

\[ \square \]

4. Intermediate cells

We are ready to address the conjecture of Bonnafé, Geck, Iancu, and Lam. Using the inverse of the rank-increasing map \( \Gamma_r \) on tableaux pairs, we first describe the relationship between Kazhdan-Lusztig asymptotic left cells and combinatorial left intermediate cells, that is, cells for the parameter \( r = n-2 \). We show that each combinatorial left intermediate cell is either an asymptotic left cell itself or a union of two such cells. When reconciled with the results on Kazhdan-Lusztig left intermediate cells in [8], this allows us to deduce Conjecture 3.2 in this setting.

Throughout most of this section we will work in a more general environment obtaining results for certain combinatorial left cells for all values of \( r \). We begin with two examples that illustrate the issues involved.

4.1. Examples. The first example concerns a set of tableaux for which rank-\( r \) and rank-(\( r+1 \)) left combinatorial cells coincide. In the second example, rank-\( r \) left combinatorial cells are unions of rank-(\( r+1 \)) left combinatorial cells.

Example 4. Consider the following tableaux pair \( (S, T) \in \text{SSDT}_2(5) \):

\[
S = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
5 & 1
\end{array} \quad T = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
3 & 5 \\
4 & 2 \\
5 & 1
\end{array}
\]

For both tableaux, each row of horizontal dominos and each column of vertical dominos forms a core open cycle. Each also forms its own extended cycle. In fact, recalling notation from Section 2.4 for extended cycles, \( \gamma(S, T) = \{\{1, 4\}, \{2\}, \{3, 5\}\} \) and \( \gamma(T, S) = \{\{1\}, \{2, 5\}, \{3, 4\}\} \). Consequently, \( \Gamma_2(S, T) = (S', T') \) where:

\[
S' = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
5 & 1
\end{array} \quad \text{and} \quad T' = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
3 & 5 \\
4 & 2 \\
5 & 1
\end{array}
\]

It is easy to see that the extended cycles \( \gamma(T, S) \) in \( T \) do not depend on \( S \). The tableaux \( T \) and \( T' \) contain no non-core open cycles, thus \( C(T) \) is a combinatorial left 2-cell and \( C(T') \) is a combinatorial left 3-cell. Proposition 2.7 in fact implies that we have equality \( C(T) = C(T') \).

Example 5. Next, consider the tableaux pair \( (S, T) \in \text{SSDT}_2(4) \):

\[
S = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
5 & 1
\end{array} \quad T = \begin{array}{cc}
1 & 3 \\
2 & 4 \\
3 & 5 \\
4 & 2 \\
5 & 1
\end{array}
\]
In both tableaux, each domino forms its own open cycle. Extended cycles are more intricate with \( \gamma(S,T) = \{(1,2,3), (3,4)\}\) and \( \gamma(T,S) = \{(1,2,4), (2,3)\}\). Consequently \( \Gamma_2(S,T) = (S',T')\) where

\[
S' = \begin{array}{ccc}
2 & 3 & 1 \\
4 & & \\
\end{array} \quad \text{and} \quad T' = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
\end{array}
\]

This time, the extended cycles in \( \gamma(T,S) \) in \( T \) do depend on the tableau \( S \). Note that since \( \{4\} \) is an open cycle in \( T \), then it is also an opposite open cycle in \( T' \) and write \( \overline{T} = MT(T,\{4\}) \) as well as \( \overline{T'} = MT_{op}(T',\{4\}) \). Since \( \{4\} \) is the only non-core open cycle in \( T \), we know that \( C(T) \cup C(\overline{T}) \) is a combinatorial left 2-cell. Neither \( T' \) nor \( \overline{T'} \) have non-core open cycles, thus \( C(T') \) and \( C(\overline{T'}) \) each form a combinatorial left 3-cell. By Proposition 2.7,

\[
C(T) \cup C(\overline{T}) = C(T') \cup C(\overline{T'})
\]

which we read as a decomposition of a combinatorial left 2-cell into two combinatorial left 3-cells.

4.2. A partition of \( W_n \). Domino tableaux that are split carry a particularly simple cycle structure. A split domino tableau \( T \in SDT_r(n) \) consists of rows of horizontal and columns of vertical dominos. The labels of the dominos in each such row and column form the complete set of its cycles. Each is a core cycle and consequently open. Since \( T \) has no non-core cycles, the set \( C(T) \) is in fact a left combinatorial \( r \)-cell. Further, if we let \( T' = MT(T,KC(T)) \), then \( T' \in SDT_{r+1}(n) \) is also split and by Proposition 2.7, \( C(T) = C(T') \). This process can be repeated with \( T' \) and iterated, eventually showing that:

**Proposition 4.1.** \( C(T) \) is a left asymptotic cell whenever \( T \in SDT_r(n) \) is split.

For any \( w \in W_n \) and \( r \geq n-1 \), the tableau pair \( G_r(w) \) is always split. Let \( s(w) \) be the smallest integer such that \( G_{s(w)}(w) \) is split and define \( W_n^k \) to be the set of \( w \) for which \( s(w) = k \). Then:

\[
W_n = \bigcup_{k=0}^{n-1} W_n^k.
\]

**Definition 4.2.** We will say that \( w \) is non-split if \( w \in W_n^{n-1} \) and split otherwise. Extending this notion slightly, we will say \( w \in W_n^k \) is \( k \)-split for \( k < n - 1 \).

The set of non-split elements in \( W_n \) admits a simple description as detailed in [8]. We reproduce it as part of the following proposition, whose proof follows directly from the definition of the map \( G_{n-1} \).

**Proposition 4.3.** Consider the signed permutation \( w = w(1)w(2)\ldots w(n) \in W_n \) as a sequence, writing \( x_1, \ldots, x_q \) for the subsequence of negative integers as well as \( y_1, \ldots, y_{n-q} \) for the subsequence of positive integers contained therein. We will say \( w \in Z_n \) if and only if \( |\{x_i\}| \) and \( \{y_i\} \) form decreasing sequences. Then

\[
W_n^{n-1} = Z_n.
\]
4.3. Unions of asymptotic cells. We will now show that the Kazhdan-Lusztig and combinatorial left intermediate cells coincide. The proof is different for split and non-split elements of $W_n$. We summarize the salient results on Kazhdan-Lusztig cells in the following theorem, which is an amalgam of Corollaries 5.9 and 8.11; Lemmas 5.10(iii), 8.3, and 8.7; and Theorem 8.8 of [5]:

Theorem 4.4 ([5]). The sets of split and non-split elements of $W_n$ are both unions of left intermediate cells. Further:

(a) If $w$ is split, then its left intermediate cell is also a left asymptotic cell.

(b) If $w$ is not split, then its left intermediate cell is the union of two left asymptotic cells and coincides with the set of all non-split elements with the same $\tau$-invariant as $w$.

We can show that the combinatorial left intermediate cells for split $w$ are simply left asymptotic cells. In fact, a little more can be said. The following is inspired by Example 4.

Proposition 4.5. Suppose that $w \in W_n^r$. Then for all $r \geq k$, $C_r(w) = C_a(w)$. In particular, if $w$ is split, then $C_{n-2}(w) = C_a(w)$.

Proof. If $r \geq k$, the tableaux pair $(S, T) := G_r(w)$ is split. Since split tableaux have no non-core cycles, $C(T)$ is a combinatorial left $r$-cell and equals $C_r(w)$. By Proposition 4.4, $C(T)$ is a left asymptotic cell and so it equals $C_a(w)$.

Lemma 4.6. If $w$ is split, then $C_{n-2}(w) = K_{n-2}(w)$.

Proof. This is a direct consequence of Proposition 4.5 which gives us the equality $C_{n-2}(w) = C_a(w)$, and Theorem 4.4(a) gives us $K_{n-2}(w) = C_a(w)$.

Immediately, we obtain the following analogue of the first statement of Theorem 4.4 for combinatorial cells. It also admits a straightforward independent proof.

Proposition 4.7. The sets of split and non-split elements of $W_n$ are both unions of combinatorial left intermediate cells.

The proof of Lemma 4.6 is straightforward partially because the extended cycles in $G_r(w)$ for $k$-split $w$ take on a particularly simple form whenever $r \geq k$. In general, if $r < k$ this cycle structure can be quite complex, see Section 2 of [16]. There is one instance when it is still tractable, namely when $k = r + 1$. Analyzing this case will let us show that on Kazhdan-Lusztig and combinatorial left cells agree for non-split elements. The proof proceeds via a sequence of propositions. The first details the cycle structure of the tableau pair $G_r(w)$ for $(r + 1)$-split signed permutations and is modeled after Example 5.

Proposition 4.8. Suppose $w \in W_n^{r+1}$, $T := Q_r(w)$, and $T' := Q_{r+1}(w)$. Then:

(a) There is a unique non-core open cycle $c$ in $T$. With this exception, every other cycle in $T$ is a core open cycle and consists of the labels of all horizontal dominos in a row or all vertical dominos in a column of $T$.

(b) With cycle $c$ as above, define tableaux $\overline{T} = MT(T, c)$ and $\overline{T'} = MT_{op}(T', c)$. Then $C(T) \cup C(\overline{T})$ is a combinatorial left $r$-cell, and

$$C(T) \cup C(\overline{T}) = C(T') \cup C(\overline{T'}).$$

When $c$ is a cycle consisting of the label of a single domino, then $C(T')$ as well as $C(\overline{T})$ are left asymptotic cells and the above equation is a decomposition of a combinatorial left $r$-cell into two left asymptotic cells.
Proof. As sets of labels, cycles in $T$ coincide with opposite cycles in $T'$, a split tableau. The non-core cycles of $T$ are the non-core opposite cycles in $T'$. There are exactly $r + 2$ opposite cycles in $T'$, all open, and exactly one core opposite open cycle for each square in $\delta_{r+1}(T')$. This leaves one opposite non-core open cycle $c$ in $T'$, proving part (a).

The first part of (b) is clear from the definition of combinatorial left cells since $c$ is the unique non-core open cycle in $T$. The second part is just a restatement of Proposition \ref{prop:core_cycles} adapted to the present setting, keeping in mind that here $MT(T, KC(T))$ may be either $T'$ or $\overline{T'}$. For the last part, we have to show that when $c$ consists of the label of a single domino, then $NC(T') = NC(\overline{T'}) = \emptyset$. For the tableau $T'$ this is straightforward: $T'$ is split and Proposition \ref{prop:core_cycles} applies. This is generally not true for $\overline{T'}$ unless $c$ is a singleton. So assume that $c = \{m\}$ and that $D(m, T') = \{s_{ij}, s_{i+1,j}\}$ is vertical. Then $s_{i,j+1}$ is an empty square in $\delta_{r+3}(T')$ and since $c$ is a non-core cycle, $D(m, \overline{T'}) = \{s_{ij}, s_{i+1,j}\}$ and $s_{i+1,j}$ is an empty square in $\delta_{r+3}(T')$. In particular, this means $\overline{T'}$ is split and $NC(\overline{T'}) = \emptyset$. If on the other hand we assume that $D(m, T')$ is horizontal, a similar argument applies. This gives us the desired decomposition of a combinatorial left $r$-cell in terms of two left asymptotic cells.

Proposition 4.9. Suppose $w$ and $y$ are not split and let $S := Q_{n-1}(w)$, and $T := Q_{n-1}(y)$. The set $c = \{n\}$ is an opposite non-core open cycle in both $S$ and $T$. If $\tau(S) = \tau(T)$, then either $T = S$ or $T = MT_{op}(S, c)$.

Proof. For the tableaux $S$ and $T$, the label of each of its $n$ dominoes forms both a cycle and an opposite cycle. Only $n - 1$ of these are opposite core cycles, one for every square in on the diagonal $\delta_{n-1}$. Because both tableaux are standard, the sole opposite non-core cycle $c$ must consist of the largest label which in this case is $n$. Let $S' = MT_{op}(S, c)$. By Lemma \ref{lem:op_loc}, $\tau(S) = \tau(S')$. By Proposition \ref{prop:core_cycles} (b), $C(S)$ and $C(S')$ are left asymptotic cells. Further, it is easy to see that $C(S)$ and $C(S')$ consist of non-split elements. Similarly, we see that all elements of $C(T)$ are not split and that it forms a left asymptotic cell.

By Theorem \ref{thm:op_loc} (b), there are exactly two left asymptotic cells within $W_n$ whose elements share the same $\tau$-invariant $\tau(S)$. Since $\tau(S) = \tau(S') = \tau(T)$ and $S \neq S'$, the proposition follows.

Proposition 4.10. Suppose that $w$ and $y$ are not split. If $\tau(w) = \tau(y)$, then $C_{n-2}(w) = C_{n-2}(y)$.

Proof. Let $(S, T) := G_{n-2}(w)$, $(\overline{S}, \overline{T}) := G_{n-2}(y)$, $(S', T') := G_{n-1}(w)$, and $(\overline{S'}, \overline{T'}) := G_{n-1}(y)$. We need to show that $T = MT(T, U)$ for some, perhaps empty, set $U$ of non-core open cycles in $\overline{T}$. Recall that $G_{n-2} = \gamma_{n-2} \circ G_{n-1}$. From the definition of $\gamma_{n-2}$ we obtain $T = MT_{op}(T', KC^{op}(T') \cup V_1)$ and $\overline{T} = MT_{op}(\overline{T'}, KC^{op}(\overline{T'}) \cup V_2)$ for some subsets $V_1 \subset NC^{op}(T')$ and $V_2 \subset NC^{op}(\overline{T'})$. By Proposition \ref{prop:core_cycles} we know that $V_1$ and $V_2$ are not complicated as $NC^{op}(T') = NC^{op}(\overline{T'}) = \{n\}$.

Since $\tau(w) = \tau(y)$, Corollary \ref{cor:op_loc} implies that $\tau(T') = \tau(\overline{T'})$ and again by Proposition \ref{prop:core_cycles} we know that either $T' = \overline{T'}$ or $T' = MT_{op}(\overline{T'}, \{n\})$. But this means that either $T = \overline{T}$ or $T = MT(\overline{T}, \{n\})$, as desired.

Lemma 4.11. If $w$ is not split, then $C_{n-2}(w) = K_{n-2}(w)$.
Proof. Restricting our attention to the set of non-split elements of \( W_n \),

- Proposition 4.10 implies that the \( \tau \)-invariant defines a partition finer than the one given by combinatorial left intermediate cells,
- Corollary 3.7 implies that the partition into combinatorial left \( r \)-cells is finer than the \( \tau \)-invariant partition for all \( r \geq 0 \), and
- Theorem 4.4(b) implies that the \( \tau \)-invariant partition agrees with the one given by left intermediate cells.

Taken together with Proposition 4.7, the lemma follows. \( \square \)

From Lemmas 4.6 and 4.11 we can now obtain the following explicit formulation of Conjecture 3.2:

**Theorem 4.12.** Consider \( w,y \in W_n \), write \( r = n-2 \), and let

\[
G_r(w) = (S_1, T_1) \quad \text{while} \quad G_r(y) = (S_2, T_2).
\]

Then \( w \approx_L^r y \) if and only if \( T_2 = MT(T_1, U) \) for some \( U \subset NC(T_1) \).

In the intermediate parameter setting, Theorem 4.4 shows that all left cells either coincide with, or are unions of, asymptotic left cells. Together with Conjecture 3.2, the combinatorics of tableaux in Propositions 4.5, 4.8, and 2.7 suggest that this is still the case for specific Kazhdan-Lusztig left \( r \)-cells with \( r < n-2 \).

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