Self-averaging characteristics of spectral fluctuations

Petr Braun\textsuperscript{1,2} and Fritz Haake\textsuperscript{1}

\textsuperscript{1}Fakultät für Physik, Universität Duisburg-Essen, D-47048 Duisburg, Germany
\textsuperscript{2}Institute of Physics, Saint-Petersburg University, 198504 Saint Petersburg, Russia

E-mail: petr.braun@uni-due.de

Received 12 January 2015
Accepted for publication 12 February 2015
Published 10 March 2015

Abstract

The spectral form factor as well as the two-point correlator of the density of (quasi-)energy levels of individual quantum dynamics are not self-averaging. Only suitable smoothing turns them into useful characteristics of spectra. We present numerical data for a fully chaotic kicked top, employing two types of smoothing: one involves primitives of the spectral correlator, the second, a small imaginary part of the quasi-energy. Self-averaging universal (like the circular unitary ensemble (CUE) average) behavior is found for the smoothed correlator, apart from noise which shrinks like $1/\sqrt{N}$ as the dimension $N$ of the quantum Hilbert space grows. There are periodically repeated quasi-energy windows of correlation decay and revival wherein the smoothed correlation remains finite as $N \to \infty$ such that the noise is negligible. In between those windows (where the CUE averaged correlator takes on values of the order $N^{-1/2}$) the noise becomes dominant and self-averaging is lost. We conclude that the noise forbids distinction of CUE and GUE-type behavior. Surprisingly, the underlying smoothed generating function does not enjoy any self-averaging outside the range of its variables relevant for determining the two-point correlator (and certain higher-order ones). We corroborate our numerical findings for the noise by analytically determining the CUE variance of the smoothed single-matrix correlator.

Keywords: spectral fluctuations, quantum chaos, correlation functions

(Some figures may appear in colour only in the online journal)
1. Introduction

Recent semiclassical work based on Gutzwiller’s periodic-orbit theory has revealed universal spectral fluctuations for quantum dynamics with a fully chaotic classical limit [1–6]. However, the present semiclassical theory leaves not satisfactorily answered the question of whether universal behavior prevails only under the protection of suitable averages over ensembles of quantum systems which all share the same classical limit (‘$\hbar$-averages’).

We have therefore thought it desirable to do a thorough investigation of spectral fluctuations of individual quantum dynamics and have chosen a kicked top without time reversal invariance for a case study. As is well known, the spectral form factor $K(n) = 1/N | \text{tr} U^n |^2$ (where $n = 1, 2, \ldots$ is a dimensionless discrete time and $N$ the dimension of the Floquet matrix $U$) as well as the two-point correlator $C(e)$ of the density of levels (the Fourier transform of $K(n)$) need smoothing in order to become self-averaging indicators of universal spectral fluctuations (or absence thereof) [7–9]. We have checked a certain second primitive of the form factor to be self-averaging and faithful to the average over the circular unitary ensemble (CUE) for times $n$ not negligibly small compared to the Heisenberg time $N$. The correlator, a periodic function of a quasi-energy variable $e$ conjugate to the time $n$, has self-averaging first and second primitives inside certain quasi-energy windows within which correlations have not yet subsided to ‘noise’; the CUE correlator of periodically driven dynamics is practically identical to the Gaussian unitary ensemble (GUE) correlator of autonomous flows, the periodicity for the CUE apart. Outside those windows of correlation decay and revival (which are tiny in width compared to the period of $C(e)$), the CUE and GUE correlators are different but cannot be distinguished (again, apart from the periodicity for the CUE) for times $n$ not negligibly small compared to the Heisenberg time $N$. The noise can only be removed by some ensemble average; the latter fact was not appreciated previously. We are led to the same result when smoothing the correlator by allowing for an imaginary part $\text{Im} e > 0$, the latter sufficiently large (see below) but smaller than the mean spacing of the eigenphases of $U$.

We represent the correlator as a descendant of a generating function $Z(e, \delta_x, \delta_z)$, a periodic function of three variables. Only the behavior near $\delta_z = 0$ determines the correlator $C(e)$. A primitive of $Z$ w.r.t. $e$ turns out smooth, self-averaging, and faithful to the CUE average for an individual kicked top, for $\delta_z$ near zero and as long as $e$ remains in the windows of correlation decay and revival. Outside, however, there is no self-averaging. Inasmuch as such absence of universality is irrelevant for correlator and form factor (and certain higher-order ones, see below) one might dispatch it as physically uninteresting. Nonetheless, the question arises as to why previous semiclassical work has yielded the random-matrix theory (RMT) generating function without manifest necessity for ensemble averaging. We confirm our numerical findings for the role of noise in section 6 by analytically determining the CUE variance of the smoothed single-$U$ correlator, building on results of Conrey et al for higher-order generating functions [10]. We thus generalize the previously known ‘ergodicity’ of the correlator within the CUE.

The Floquet operator

\[
U = \exp\left(-i\frac{\tau_z J_z^2}{2j + 1} - i \alpha_z J_z\right) \exp\left(-i\frac{\tau_y J_y^2}{2j + 1} - i \alpha_y J_y\right) \exp\left(-i\frac{\tau_x J_x^2}{2j + 1} - i \alpha_x J_x\right)
\] (1)
of our kicked top lacks time reversal invariance. The rotation angles were chosen as
\( \alpha_x = \alpha_y = 1, \alpha_z = 1.1 \) and the torsion strengths as \( \tau_x = 10, \tau_y = 0, \tau_z = 4 \), to bring about
predominance of classical chaos; islands of regular motion cumulatively cover less than a
single Planck cell. The angular momenta \( J_{xyz} \) obey \( [J_x, J_y] = i J_z \) etc. The quantum number \( j \) fixes the
dimension of the quantum Hilbert space as \( N = 2j + 1 \); our calculations involve
values of \( j \) between \( 10^2 \) and \( 10^4 \).

2. Generating function, generalized correlator and form factor

The analysis of spectral fluctuations of unitary quantum maps is conveniently based on the
generating function

\[
Z(a, b, c, d) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\det\left(1 - c e^{i\phi} U\right) \det\left(1 - de^{-i\phi} U^d\right)}{\det\left(1 - a e^{i\phi} U\right) \det\left(1 - be^{-i\phi} U^d\right)} \, d\phi,
\]

\( \{a, b\} < 1, \ c, d \in \mathbb{C} \). \hspace{1cm} (2)

That function \([6, 11]\) generates the two-point correlator of the eigenphase density of
\( U \) as

\[
C(e) = \frac{2ab}{N^2} \left[ a + \right]_{a = b = c = d = e^{iN/2}}.
\]

The center-phase average, i.e., the integral \( \int_0^{2\pi} \frac{d\phi}{2\pi} (\cdot) \) in definition (2), entails interesting
properties. If we set \( a = c \), two of the spectral determinants in (2) cancel. The \( \phi \)-integral then
simply yields \( Z(a, b, a, d) = 1 \), as is easily checked by going to a complex \( \phi \)-plane and
complementing the integral to one over a suitable closed loop not encircling the poles pro-
vided by the eigenphases of \( U \). Similarly, for \( b = d \) we have \( Z(a, b, c, b) = 1 \) and thus
\( Z - 1 = (a - c)(b - d)[\ldots] \) with \( \{\ldots\} \) a polynomial of order \( N - 1 \) in both \( c \) and \( d \). In fact,
the quantity \( \frac{Z - 1}{(a - c)(b - d)} = \{\ldots\} \) yields the complex two-point correlator as

\[
C(e) = \frac{2ab}{N^2} \frac{Z - 1}{(a - c)(b - d)} \left|_{a = b = c = d = e^{iN/2}} \right.
s.
\]

The foregoing result suggests an algebraic kinship of the generating function to be called the
generalized correlator,

\[
C = \left( \frac{ab}{cd} \right)^{N-1} \frac{2ab}{N^2} \frac{Z - 1}{(a - c)(b - d)}
\]

\hspace{1cm} (4)

which yields the correlator as \( C(e) = C \left|_{a = b = c = d = e^{iN/2}} \right. \), without differentiation. Note that we
have sneaked in the factor \( (ab/cd)(N-1)/2 \) which becomes unity for the correlator \( C \); our
motivation for that import will be revealed below.

As a further consequence of the center-phase average, the generating function and the
generalized correlator \( C \) depend on the four complex variables \( a, b, c, d \) only through three
independent combinations which may be chosen as

\[
ab = e^{i\delta a/2}, \quad \frac{c}{a} = e^{i\delta c/2}, \quad \frac{d}{b} = e^{i\delta d/2}.
\]

While \( \delta \) can be arbitrary complex, the variable \( e \) is restricted by \( \text{Im} \ e > 0 \). Since we lose no
physically important information by restricting ourselves to real \( \delta \), that choice will be made
without exception. Moreover, unless noted otherwise, we shall take the quasi-energy \( e \) as real
in the sense that \( \text{Im} \ e = 0 \).
We note in passing that the generating function $Z$ defined above allows access to higher-order correlation functions through higher derivatives w.r.t. to $\epsilon$, $d$ or to $\delta_z$ evaluated at $\delta_z = 0$. All those functions then appear, of course, with the single quasi-energy $\epsilon$ as the exclusive argument. There is thus physics in the generalized correlator $C$ for $\delta_z$ near zero for this reason we shall check for self-averaging there as well, not just at $\delta_z = 0$.

We can proceed to the Fourier transform of $C(\epsilon, \delta_\pm, \delta_-)$ with respect to $\epsilon$,

$$K(n, \delta_+, \delta_-) = \int_0^{\pi N} \frac{d\epsilon}{2\pi} C(\epsilon, \delta_+, \delta_-) e^{-i2\pi n/N},$$

(6)

which we shall refer to as the generalized form factor since it reduces to the physical form factor for $\delta_+ = \delta_- = 0$.

For a given spectrum $\{e^{i\mu e}, \mu = 1, 2, \ldots N\}$ of the Floquet operator $U$, the generating function $Z$ can be further evaluated by doing the center-phase average explicitly with the help of Cauchy’s theorem. We may write the result as

$$C(\epsilon, \delta_+, \delta_-) = \frac{2}{N^2} \sum_{\mu, \nu=1}^{N} e^{-i\mu \epsilon} f_\mu(\delta_+ f\nu(\delta_-), \text{and)}$$

$$K(n, \delta_+, \delta_-) = \frac{1}{N} \sum_{\mu, \nu} e^{-i\mu \epsilon} f_\mu(\delta_+ f\nu(\delta_-)$$

(7)

with

$$z = e^{i\mu e/N}, \Delta_{\mu \nu} = \delta_{\mu} - \delta_{\nu}, \text{and } f_\mu(\delta_\pm) = \prod_{\nu(\neq \mu)} \frac{\sin \frac{\Delta_{\mu \nu} + \delta_\pm/N}{2}}{\sin \frac{\Delta_{\mu \nu}}{2}}. \text{ (8)}$$

We read off the period $\pi N$ for the $e$-dependence of $C$ which allows us to limit the range of $\epsilon$ to $[-\pi N/2, \pi N/2]$. Five properties of the auxiliary function $f_\mu(\delta_\pm)$ and their consequences for the generalized correlator and form factor are worth noting:

- The functions $f_\mu$ obey the sum rule $\sum_{\mu=1}^{N} f_\mu(\delta) = \frac{\sin \frac{\delta}{2}}{\sin \frac{\delta}{2N}}$.
- Since $f_\mu(0) = 1$, the generalized correlator and form factor reduce to the physical ones for $\delta_z = 0$.
- For real $\delta_\pm$, the functions $f_\mu(\delta_\pm)$ are real. Consequently, for $\text{Im } \epsilon \downarrow 0$ we obtain $\text{Re } C$ as a weighted sum of periodic delta-functions,

$$\text{Re } C(\epsilon, \delta_+, \delta_-) = -\frac{\sin \frac{\delta_+}{2}}{N \sin \frac{\delta_+}{2N}} \frac{\sin \frac{\delta_-}{2}}{N \sin \frac{\delta_-}{2N}} + \frac{2\pi}{N^2} \sum_{\mu, \nu=1}^{N} f_\mu(\delta_+ f\nu(\delta_-) \delta_{2}\varepsilon \left( \frac{2e}{N} - \Delta_{\mu \nu} \right). \text{ (9)}$$

we took into account the sum rule for $f_\mu$ and the relation

$$\text{Re } \left[ e^{i\epsilon \Delta_{\mu \nu}/N} \right] = -\frac{1}{2} + \pi \delta_{2}\varepsilon \left( \frac{2e}{N} - \Delta_{\mu \nu} \right) \text{ valid when } \text{Im } \epsilon \downarrow 0. \text{ In particular, for } \delta_z = 0 \text{ we arrive at the ‘physical correlator’ } R(\epsilon) = \text{Re } C(\epsilon, 0, 0) \text{ in the well known form }$$

$$R(\epsilon) = -1 + \frac{2\pi}{N^2} \sum_{\mu, \nu=1}^{N} \delta_{2}\varepsilon \left( \frac{2e}{N} - \Delta_{\mu \nu} \right). \text{ (10)}$$

Analogously, $\text{Im } C$ engenders $\text{Im } C(\epsilon)$ which contains principal-value $1/\epsilon$ singularities. That property is owed to the factor $\left( \frac{\text{ab}}{\text{cd}} \right)^{N-1/2}$ in definition (4) of $C$. 

4
The product structure of $f_\mu$ reveals Fourier components $e^{i2\delta_\mu/2N}$ for $C$ and $K$ with $\nu = \pm(N-1), \pm(N-3), \ldots$. Therefore, $C$ and $K$ are periodic in $\delta_\mu$ with period $2\pi N$ for $N$ odd and $4\pi N$ for $N$ even; moreover, the finest scale of variation in $\delta_\mu$ is $2\pi$.

For real $\delta_\mu = -\delta_-, K$ becomes non-negative while $C$ acquires a real (imaginary) part even (odd) in $\epsilon$. In particular, $R(\epsilon) = R(-\epsilon)$, and therefore we need to look at the real correlator $R(\epsilon)$ only in the interval $[0, \pi N/2]$.

### 3. CUE averages

Checking for self-averaging and universality means comparing spectral characteristics with their CUE averages. The generating function $[6, 10, 12]$

$$Z_{\text{CUE}}(1) = (a-c)(b-d) = \left(1 - \langle cd\rangle^N\right) / [(1 - ab)(1 - cd)],$$

yields,

$$C_{\text{CUE}}(\epsilon, \delta_+, \delta_-) = \frac{ie^{i\epsilon}}{N^2 \sin \frac{\epsilon}{N}} \frac{\sin \left(\frac{e + \delta_+ + \delta_-}{2}\right)}{\sin \left[\frac{1}{N} \left(\frac{e + \delta_+ + \delta_-}{2}\right)\right]}$$

$$\Rightarrow C_{\text{CUE}}(\epsilon, \delta, \delta) = C_{\text{CUE}}(\epsilon) = \frac{e^{2\epsilon} - 1}{2N^2 \sin^2(\epsilon/N)}. \quad (11)$$

The generalized correlator depends on the phases $\delta_\mu$ only through their sum $\delta_\mu + \delta_\nu$; that property is shared by the generalized form factor (6). The limit $N \to \infty$ with $N/e \to 0$ gives $C_{\text{CUE}}(\epsilon) \to C_{\text{GUE}}(\epsilon) = (e^{2\epsilon} - 1)/2e^2$ and correspondingly $Z_{\text{GUE}}(\epsilon, \delta_+, \delta_-)$ and $C_{\text{GUE}}(\epsilon, \delta_+ \delta_-)$. The two determinants in the denominator of (2) have their zeros along the $\phi$-axis shifted by $2\epsilon$ whereas those in the numerator have relative shift $2\epsilon + \delta_+ + \delta_-$. It is these two shifts which are the arguments in the factors of the CUE generalized correlator.

### 4. Smoothing by integration

The explicit form of the generalized correlator $C$ given in (7) is a good starting point for integrating over $\epsilon$ (recall $z = e^{i2\epsilon/N}$), inasmuch as we are facing a slightly compacted form of the partial-fraction decomposition in terms of $(1 - ze^{-i2\epsilon})^{-1}$. The first primitive $\int e^{i2\epsilon} C(e', \delta_+ \delta_-)$ varies with the phase $e/N$ on the scale $1/N^2$, the latter measuring the typical distance between neighboring quasi-energy differences $\Delta_{\mu \nu}$. The real part of the first primitive is thus piecewise constant in $\epsilon$, with steps of height and width $\propto N^{-1}$. For $N \gg 1$, the real part of the first primitive will appear smooth on the scales $\epsilon \sim 1$ and $\int \left|e^{i2\epsilon} C(e', \delta_+ \delta_-)\right| \sim 1$ with the steps not being resolved. The imaginary part of $\int e^{i2\epsilon} C(e', \delta_+ \delta_-) \sim 1$ then also appears smooth: its inconspicuous logarithmic singularities are not resolved either. The first primitive of the physical correlator can be defined as

$$R^{(1)}(\epsilon) = \int_{\epsilon}^{N\pi/2} de' R(e') \quad (12)$$

with the reference point at the right border of the principal interval of $\epsilon$. It is close to zero for all $\epsilon$ comparable to $N$. When $\epsilon$ tends to $+0$ the first primitive of the physical correlator tends to the universal limit $R^{(1)}(0^+) = -\pi/2$ for any spectrum; this is easily seen when doing the
\[ e - \text{integral and using the explicit form (10). At } e = 0 \text{ the first primitive experiences a discontinuity due to the diagonal terms in the double sum in (10), namely,} \]
\[ R^{(0)}(0) = \pi / 2 \text{ and } R^{(1)}(1) = -R^{(1)}(-1). \]

It follows that the first primitive defined in (12) is odd, \[ R^{(1)}(e) = -R^{(1)}(-e), \]
and evenness of the correlator \( R(e) \) being taken into account.

Having discussed at length general properties of the first primitive \( R^{(1)}(e) \) we now turn to numerical results. The first primitive has long been known to be self-averaging and faithful to RMT for fully chaotic dynamics [5], as illustrated for the kicked top in figure 1; the phase \( e/N \) is reckoned there in a unit proportional to the mean level spacing \( ms \), which is such that the mean level spacing in terms of the quasi-energy \( e \) is \( \pi/2 \). The staircase character of \( R^{(1)}(e) \) (step width and height \( \propto 1/N \)) is not resolved in figure 1. Self-averaging and universality

\[ e/N \]

Figure 1. Kicked top, \( j = 9600 \). First primitive \( R^{(1)}(e) \) closely agrees with RMT average (dashed); \( ms = \text{mean spacing, } 1ms = 2\pi/N \). Staircase character unresolved. Inset shows symmetry \( R^{(1)}(e) = -R^{(1)}(-e) \) and jump at \( e = 0 \).

Figure 2. Same for \( e \) up to \( \pi N/2 \). Window of correlation decay compressed such that \( R^{(1)}(e) \) becomes the near-vertical line therein. Only noise remains once \( e \) exceeds a few \( ms \). Blowup of noise stretch in inset.
reign, apart from tiny ‘noise’ for $e$ in a range of a few ms, in the region of noticeable level-level correlations. Outside that region, in particular for $e \sim N$, nothing but noise remains: $R^{(1)}(e)$ fluctuates around zero without systematic $e$-dependence, see figure 2; the black ribbon of constant width consists of irregular fluctuations; the inset shows an enlargement. The noise amplitude decays as $1/\sqrt{N}$ with $N \to \infty$ (accuracy better than 2% for absolute deviation from mean over two decades of $N$). Both the finite-$N$ CUE average, $R^{(1)}_{\text{CUE}}(e) \propto N^{-1}$, and the difference between the CUE and GUE are overwhelmed by the noise. It follows that a single spectrum does not distinguish the infinite-$N$ GUE behavior from the finite-$N$ CUE one, the periodicity of the latter apart. The overall behavior of $R^{(1)}(e)$ for individual kicked tops is consistent with the ‘ergodicity’ of the correlator within the CUE [5, 7]. This is to be looked at more closely in section 6.

As mentioned above, the first primitive appears smooth on the physically most interesting scales where the steps in the real part (and the logarithmic singularities in the imaginary part) of $C^{(1)}(e)$ are not resolved. On the other hand, if one wants to strictly ban all singularities one may focus on the second primitive of $C$ which is continuous in the phase $e$. In that vein, we define

$$C^{(2)}(e, \delta_+, \delta_-) = \int_0^e de' \int_{-N/2}^{N/2} de'' C(e'', \delta_+, \delta_-)$$

and

$$K^{(2)}(n, \delta_+, \delta_-) = \sum_{n'=1}^n \sum_{\delta=1}^N K(n', \delta_+, \delta_-).$$

Our numerical studies of the second primitives reveal self-averaging and fidelity to RMT only for $|\delta_{\pm}|$ at most of order unity and for $e$ within the windows of correlation decay (around $|e| = 0 \mod N\pi$) already met with in $R^{(1)}(e)$. The self-averaging of $R^{(2)}$, $K^{(2)}$, together with the mentioned limitations, is revealed in figures 3–5.

In particular, figure 3 depicts a narrow bundle of three curves for the second primitive of the real correlator $R^{(2)}(e)$: one representing the CUE average and the other two pertaining to the kicked top with $j = 9600$ and 9601 in the range of $e$ between 0 and 10 ms. Throughout that

![Figure 3. Self-averaging Re $C^{(2)}$ of kicked top for $e \ll N, \delta = 0$. Violation of universality for $\delta = 10$ ms.](image-url)
range, the three curves differ from one another by negligible amounts. It is to be noted that for $e$ equaling 5 ms the CUE correlator $R_{\text{CUE}}(e)$ has decayed to a practically negligible level $\sim 1/(10e)^2$. A fourth curve in figure 3 pertains to the second primitive of the real part of the generalized correlator $C(e, \delta)$ with $\delta_+ = -\delta_- = \delta$ equaling 10 mean spacings. That curve strongly deviates from the CUE prediction (which, as already mentioned, does not depend on $\delta$) and signals non-self-averaging behavior.

For larger values of $e$ the relative differences grow and signal non-universality, see figure 4. However, that growth is of no importance for two reasons. First, outside the phase window of correlation decay (and revival; note the periodicity with period $\pi N$) only weak

Figure 4. $R^{(2)}$ Noisy and non-universal for large $e$.

Figure 5. (a) Self-averaging $K^{(2)}$ of kicked top for $\delta = 0$ and $\delta = 1$ ms. (b, c) No universality for $\delta = 10$ ms.
noise remains for $R^{(1)}(e)$. Second, the second primitive grossly exaggerates all large-$e$ structures turning the $1/e^2$ decay of $R_{\text{CUE}}(e)$ into a logarithmic one.

The same salient message is signaled in figure 5 for the second primitive of the form factor as a function of the scaled discrete time $n/N$. Three curves (a), unresolved from one another, refer to the CUE and the kicked top for $j = 9600$ with $\delta_+ = -\delta_+ = \delta$ equal to zero and a single mean spacing indicate excellent self-averaging and universality. Two further curves pertain to the kicked top with $\delta = 10$ ms, one (b) for $j = 9600$ and the other (c) for $j = 9601$. Universality would require both curves (b,c) to coincide with the triple (a) and is grossly violated. The noisy small-time behavior of $\kappa^{(2)}$, invisible on the scale of figure 5, corresponds to the large-$e$ noise in $C^{(2)}$.

5. Smoothing by complex quasi-energy

In figure 6 we show the real correlator $\text{Re } C(e + i\eta)$ where $\eta$ is a small imaginary addition to the real quasi-energy $e$. To smooth away the delta peaks arising for vanishing $\eta$ that imaginary part must be larger than the finest scale of variation in $e$, seen to be $\sim 1/N$ at the end of section 4. On the other hand, we want to keep $2\eta/N$ smaller than the mean spacing $2\pi/N$ of the eigenvalues of $U$ in order not to tamper with the $e$ dependence on that latter scale. These restrictions are respected in figure 6. In order to better check on deviations from the CUE average, we have taken out the contribution of the diagonal terms of the double sum over the eigenphases in (7): that contribution, a delta function broadened to width $\eta$, is universal, i.e. the same for all spectra. Just as for smoothing by integration we see excellent agreement with the CUE average within the principal window of correlation decay but noise dominance outside: for $e/N$ not small, the CUE average scales as $1/N^2$ while the standard deviation is found numerically to decay only like $1/\sqrt{N}$ as $N$ grows at fixed $\eta$ (evidence not shown here).
6. CUE fluctuations

In order to better understand the noise in single-top correlators and the systematic non-self-averaging in the generalized correlator, we here look at fluctuations of the single-$U$ correlator throughout the CUE. We start with some numerical findings and shall then proceed to discuss an explicit analytic result for the CUE variance.

6.1. Numerical results

Figure 7 further conveys the absence of self-averaging and universality in the generalized correlator (and thus the generating function) outside the physically relevant range of its variables. The 3D plot in figure 7 shows the mean absolute deviation $\langle |\text{Re} \ C(2)_{\text{CUE}}(\delta) - \text{Re} \ C(2)_{\text{CUE}}(0)| \rangle$, the mean $\langle \cdot \rangle$ taken over 13 937 spectra of 201 $\times$ 201 matrices randomly drawn from the CUE (with the algorithm described in [13]), as a function of $\delta_+$ and $\delta_-$. With $e = 1$ ms; the range captured for $\delta_\pm$ is chosen as 20 ms, symmetric about zero. The line $\delta_+ = \delta_- = 0$ is obviously distinguished: thereon, the mean absolute deviation is small only in a narrow interval near $\delta_+ = \delta_- = 0$ while outside a high ridge arises and signals large fluctuations of $R(2)$ within the CUE; a single unitary matrix, either for a kicked top or drawn at random from the CUE, would entail even stronger and noisier absolute deviation and thus reveal non-self-averaging. Perpendicular to the line $\delta_+ = \delta_- = 0$ there is less drama: the mean absolute deviation undergoes oscillations, roughly with the finest scale of variation allowed by the Fourier series for $f_\mu(\delta_\pm)$, and decay with increasing $|\delta_+ - \delta_-|$. Even though less flagrant and not obvious from figure 7, non-self-averaging is also incurred along the line $\delta_+ - \delta_- = 0$ with increasing distance from the center $\delta_+ = \delta_- = 0$; already a few mean spacings beyond the center, relative mean absolute deviations of order unity become typical.

In figure 8 we concentrate on the behavior of $\langle |\text{Re} \ C(2)_{\text{CUE}}(\delta) - \text{Re} \ C(2)_{\text{CUE}}(0)| \rangle$ along the ridge $\delta_+ + \delta_- = 0$. Three curves pertain to ensembles of CUE matrices for different values of $N$; we face a tendency of the deviation from the full CUE average to shrink with increasing $N$. On the other hand, there is no distinct tendency to less erratic behavior as either $N$ or the sample size increases. The figure also contains two single-spectrum curves, one pertaining to our kicked top with $j = 9601$ and the other to a single matrix drawn at random from the CUE with the same dimension; again, we see gross violations of self-averaging for large $\delta_+$ with no qualitative difference between the random matrix and the kicked top.

While accumulating the many spectra making up figures 7, 8 we observed slow convergence, the slowness due to ‘rare extreme events’; a single odd spectrum can significantly alter the partial average previously accumulated. We suspect that such extreme events are spectra with exceptionally close nearest-neighbor pairs of levels; for such spectra the coefficients $f_\mu$ in the spectral representation (7) become almost singular and thus give rise to huge deviations from the universal limit and flagrant absence of self-averaging as soon as $|\delta_+ - \delta_-|$ exceeds a few mean spacings. That suspicion is nourished by the investigation of the circular orthogonal and symplectic ensembles (COE and CSE). For the COE, the weaker (linear rather than quadratic) level repulsion allows for more frequent closely neighboring levels and we did find the expected slower convergence and orders-of-magnitude larger variance of the generalized correlator with the same $\delta_\pm \neq 0$ than for the CUE. Conversely, the CSE with strongest (quartic) degree of level repulsion shows quicker convergence and smaller variance. These findings are depicted in figure 9 showing mean deviation of the generalized correlator $R(e, \delta) = \text{Re} \ C(e, \delta, -\delta)$ as function of $\delta$ at fixed complex $e = (1 + 0.05i)$ ms.
6.2. Analytic results

All of the foregoing numerical results are supported and, in part, generalized by the CUE variance of the correlator smoothed by either an imaginary part of the quasi-energy \( e \) or by integration. That variance can be found in the following way. We start by defining the covariance of two generating functions \( \tilde{Z}(a_i, b_i, c_i, d_i) \equiv Z(i), \ i = 1, 2, \) as \( \text{Cov}(Z(1), Z(2)) = \langle Z(1)Z(2) \rangle - \langle Z(1) \rangle \langle Z(2) \rangle \) where \( \langle \ldots \rangle \) stands for the CUE average; the second summand in the right-hand side is a product of the CUE generating functions written out in the beginning of section 3. The first summand is formally a double integral over the central phases \( \phi_1, 2 \) of \( Z(1) \) and \( Z(2) \), but in view of the CUE averaging only integration over the relative central phase is needed.
\[ \langle Z(1)Z(2) \rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} \times \left\{ \frac{\det\left(1 - c_1e^{i\phi}U\right)\det\left(1 - d_1e^{-i\phi}U^*\right)}{\det\left(1 + a_1e^{i\phi}U\right)\det\left(1 + b_1e^{-i\phi}U^*\right)} \right\}. \] (14)

The CUE average of the ratio of 4/4 spectral determinants in the integrand can be imported from [10]; the \( \phi \)-integral then is elementary. The resulting covariance can be reformulated as the covariance of two generalized correlators (4); see the appendix for the somewhat bulky expression and the web version [14] for details. We here confine ourselves to discussing the special cases of interest in our context. We shall thus be led to a generalization of Pandey’s ergodicity of the two-point correlator within the CUE [5, 7].

**Complex quasi-energy:** we first turn to the generalized correlator smoothed by an imaginary part \( \eta \) added to the real quasi-energy \( e \). The real pair \( \delta_{\pm} \) will be restricted as \( \delta_+ = -\delta_- = \delta \). That case deserves special attention due to the strong fluctuations at large \( \delta \) seen in figure 7 while the CUE average is \( \delta \)-independent. The variance \( \text{Var}_{\text{CUE}}(\text{Re } C(e + i\eta, \delta)) \) is given by a somewhat cumbersome expression; we show only the leading term of its expansion in powers of \( \eta \).

\[ \text{Var}_{\text{CUE}}(\text{Re } C(e + i\eta, \delta)) \sim \frac{1}{2N\eta} \left( 1 - \frac{\sin^2 \frac{\delta}{2N}}{\sin^2 \frac{e}{N}} \right)^2 \left( 1 - \frac{\sin^2 e}{N^2 \sin^2 \frac{e}{N}} \right) \times (1 + \eta f(e, N) + \ldots) \] (15)

with \( f(e, N) \) finite for all \( e \) and \( N \to \infty \). In particular, the variance \( \text{Var}_{\text{CUE}} \text{Re}(C(e + i\eta)) \) of the physical correlator arises for \( \delta = 0 \) and is seen to be \( \propto 1/N\eta \). For it to become small for large \( N \), the smoothing imaginary part \( \eta \) must, as already argued in section 5, be larger than the minimum scale of variation in \( e \), that is \( 1/N \), and small compared to the mean spacing, that
is $2\pi$ in the units used for $\epsilon$. These restrictions are with respect to $\eta$, a small number independent of $N$. The ensuing rms deviation is $\propto N^{-1/2}$ and explains the numerically found $1/\sqrt{N}$ noise in the correlator. That noise exceeds the mean once $\epsilon \gtrsim \sqrt{\eta N}$, as is easily checked using the CUE correlator given in (11).

Rather drastic non-self-averaging of the generalized correlator arises when $\epsilon$ ranges within the windows of correlation decay and revival while $\delta \sim N^{\beta} \sim N^{1/2}$; the variance then becomes $\sim N^{\beta-1}$. But the region $\delta \gg 1$ is devoid of physical interest.

**Primitives:** proceeding to smoothing by integration we propose to discuss the variance of the first primitive $R^{(1)}(\epsilon)$ at $\text{Im} \epsilon \downarrow 0$. That variance is readily obtained from the covariance of the complex correlator given in the appendix as

$$\text{Var}_{\text{CUE}} R^{(1)}(\epsilon) = \int_{0}^{\epsilon} \text{d}y \left[ \frac{\sin 4y}{N^3 \sin^2 \frac{2y}{N}} \ln \frac{\sin^2 \frac{\epsilon - 2y}{N}}{\sin^2 \frac{\epsilon}{N}} + \frac{\pi}{N} \left( 1 - \frac{\sin^2 y}{N^2 \sin^2 \frac{2y}{N}} \right) \right] + \frac{(\epsilon - y)}{N^4 \sin^2 \frac{2y}{N}} \left( 4N \cos 4y - 2 \cot \frac{2y}{N} \sin 4y \right) - \frac{1}{N^2} \ln \frac{(\epsilon - 2y)^2}{\epsilon^2}.$$  

The variance is exactly zero at $\epsilon = 0$ and $\epsilon = N\pi/2$ which is understandable since $R^{(1)}(\epsilon)$ of any spectrum has the same value at these points. Away from the end points, the variance steeply rises to approximately $1/N$. It is not difficult to extract the large-$N$ asymptotics within the principal window of correlation decay and revival where $\epsilon/N \ll 1$ and $\text{Var}_{\text{CUE}} R^{(1)}(\epsilon) \sim 1/N$,

$$\text{Var}_{\text{CUE}} R^{(1)}(\epsilon) \sim \frac{1}{N} \left[ 1 - \frac{1}{2} \cos 4\epsilon - \frac{\sin 4\epsilon}{8\epsilon} - 2\epsilon \text{ Si } 4\epsilon \right]$$

$$+ \pi \left( e + \frac{\sin^2 \frac{\epsilon}{e}}{e} - \text{ Si } 2\epsilon \right) + \int_{0}^{\epsilon} \frac{(\sin 4y - 4y) \log \frac{1 - 2y}{e}}{2y} \text{d}y.$$  

![Figure 10. Variance of the first primitive of the physical correlator with $N = 11$, exact (full line) and large-$N$ asymptotics (dashed).](image-url)
Here $x = \int_0^1 \sin \frac{t}{r} \, dt$; the expression in the square brackets is close to $e^{x^2/2}$ for $e \ll 1$ and 1 for $e \gg 1$. In figure 10, both the exact and the asymptotic form of $\text{Var} R^{(1)}(e)$ are shown for $N = 11$. Both curves practically agree, except for $e$ in the immediate neighborhood of the half period, where the large-$N$ asymptotics is practically constant instead of falling back to 0. The close agreement of the two curves in the figure is in fact amazing since the large-$N$ asymptotics is derived only for small $e/N$. The scaling with $N$, $\text{Var}_{\text{CUE}} R^{(1)}(e) \sim 1/N$, is manifest in the figure. Once again we see confirmed the $(\text{rms}) 1/\sqrt{N}$ noise numerically found, now for the correlator smoothed by integration.

The overwhelming of the mean by the noise for $e$ outside the windows of correlation decay and revival is also found again, here for the first primitive $R^{(1)}(e)$, such that the ratio of standard deviation and mean is of the order $\sqrt{N}$.

To characterize the CUE fluctuations of the generalized correlator, we finally comment on the variance of $\text{Re} C^{(1)}(e, \delta)$ with $\delta = \delta_+ = -\delta_-$. We sketch the explicit result only for the $e$-window of correlation decay and revival, $\frac{\pi}{N} \ll 1$,

$$\text{Var} \text{Re} C^{(1)}(e, \delta) \approx \left( \text{Var} R^{(1)}(e) \right) \left[ 1 + N^2 \sin^2 \frac{\delta}{2N} F_1(e) + N^4 \sin^4 \frac{\delta}{2N} F_2(e) \right], \quad (18)$$

where $F_1, F_2$ are some $N$-independent functions of $e$. The most important conclusion is negligibility of fluctuations in the same sense as for $\delta = 0$ as long as $\delta$ remains of order unity. The extreme case of $\delta \sim N$ has $\text{Var} \text{Re} C^{(1)}(e, \delta) \sim N^3$ and thus no self-averaging. Even the less excessive growth $\delta \sim N^2$ with $\beta < 1$ gives $\text{Var} \text{Re} C^{(1)}(e, \delta) \sim N^{4(\beta - 1)}$ and thus loss of self-averaging for $\delta \gg 1/N$. Indeed, in our numerical calculations the largest $N$ was of the order $10^4$ with $N^{1/4} \sim 10$; the break-up of self-averaging could thus be expected (and was observed) at $\delta \gtrsim 2\pi = 1 \text{ ms}$.

7. Discussion and outlook

We have found fidelity to RMT of the first and second primitives of spectral form factor $K(n)$ and real two-point correlator $R(e)$ for an individual kicked top in periodically (period $\propto N$) repeated windows of correlation decay and revival for the quasi-energy variable $e$. In between these windows correlations are so weak as to be negligible for large $N$. System-specific noise in the $e$-dependence is found to have the order $1/\sqrt{N}$ and thus to be negligible as well. Interestingly, however, the noise overwhelms the CUE average of the correlator in between the windows of correlation decay and revival. In fact the noise is strong enough to preclude distinguishability of the CUE and GUE prediction for a single spectrum. The same behavior is found when the correlator is smoothed by an imaginary addition in the real quasi-energy $e$, provided $\eta$ is (i) large enough to iron out the singularities in the $\eta \downarrow 0$ correlator and (ii) small enough to not noticeably attenuate $R(e)$. On the other hand, we find the underlying generating function $Z(e, \delta_+, \delta_-)$ self-averaging and universal only under the additional restriction of $|\delta_{\perp}|$ no larger than a few mean spacings.

Our results for the CUE variance of the correlator $R(e)$, the latter smoothed by suitable $\eta$ or integration, mean that in the large-$N$ limit any CUE matrix can, with overwhelming probability (for $N \to \infty$ with probability one), be expected to have a smoothed correlator equaling the CUE average. Likewise, by working out the smoothed correlator for a single top (or any other fully chaotic map) with large $N$, one has overwhelming probability to get the CUE average. Exceptions are possible but will in practice not be met with.

We would like to comment on the status of previous semiclassical work based on Gutzwiller’s periodic-orbit theory. That periodic-orbit approach involves a sum over certain
combinations of classical periodic orbits which can be seen as a ‘perturbation series’ for \( Z \). Such work has resulted in universality of \( Z(e, \delta_+, \delta_-) \), without restriction for the independent variables and without manifest necessity of any ensemble average [4, 6].

The original work on autonomous flows [4] relied on an imaginary part \( \eta \) of the quasi-energy variable much larger than a mean spacing—return to real quasi-energy was possible only after summing up the series. The later extension to Floquet maps [6] could make do with small \( \eta \). Our present investigation reveals limits within which \( \eta \) must lie for the semiclassical periodic-orbit expansions to be applicable to individual Floquet maps, at least for the correlator \( R(e) \) in the limit \( N \to \infty \).

On the other hand, we have found here that one may stick to real quasi-energies (in the sense \( \eta \downarrow 0 \)) if one smoothes by going to the first primitive \( R^{(1)}(e) \); self-averaging then takes place for \( N \to \infty \) and even for large finite \( N \) apart from negligible noise. The semiclassical periodic-orbit expansions can be done under the protection of that smoothing and the resulting single-dynamics \( R^{(1)}(e) \) also has the right to be equal to the CUE average in the limit \( N \to \infty \).

Inasmuch as periodic orbits have yielded the CUE average of the full generating function \( Z(e, \delta_+, \delta_-) \) it appears that too much is proven since we now know that a single spectrum comes with gross violations of self-averaging for large \( \delta_- \). One does not need to worry too much since large \( \delta_- \) do not harbor any physically relevant information. With the correlator (or its primitive) satisfactorily treated, one can live with subjecting the periodic-orbit expansion to a suitable further average to justify the RMT result for \( Z \). We have numerically checked (but not shown here) that an average over \( N \propto 1/\hbar \) leaving unchanged the classical limit does the job for our kicked top. Averages over small intervals of classical control parameters (small in the sense of shrinking to zero length for \( \hbar \to 0 \)) have been shown to work as well [11].

We conclude with a speculative outlook. As already stated above, and illustrated in figure 4, the second primitive enhances large-\( e \) structures. Exploiting that property, we have averaged the primitive \( R^{(2)}(e) \) for single kicked-top spectra over a range \([j, j + \Delta j]\) with \( 1 \ll \Delta j \ll j \). That average did away with noise at large \( e \propto N = 2j + 1 \) but left oscillations in \( e \), around the CUE average \( R^{(2)}_{\text{CUE}}(e) \). On the other hand, when doing the same with matrices randomly drawn from the CUE we found no oscillations. Further investigation must reveal whether the oscillations found for the top are due to Ehrenfest-time effects [15–17], short orbits, or some other effect.

Acknowledgments

We thank Sven Gnutzmann for discussions and gratefully acknowledge support by the Sonderforschungsbereich SFBTR12 ‘Symmetries and universality in mesoscopic systems’ of the Deutsche Forschungsgemeinschaft.

Appendix. CUE covariance of complex correlator

The covariance of the generalized correlator can be extracted from the results of [10] for the underlying generating functions, see equation (14). Allowing for complex quasi-energies \( e_1, e_2 \) and limiting ourselves with the case of real \( \delta_1 = \delta_2 = -\delta_- = -\delta_- = \delta \) we find

\[
\text{Cov}\left(C(e_1, \delta), C(e_2, \delta) \right) = \frac{4}{N^4} A(z_1, z_2, \delta) B(z_1, z_2),
\]  

(A.1)
with $z_l = e^{i2\pi l/N}$ and

$$A(z_1, z_2, \delta) = \frac{z_1 z_2}{(z_1 z_2 - 1)^2} \prod_{k=1,2} \left( \frac{z_k e^{i\delta/N} - 1}{z_k - 1} \right)^2,$$

$$B(z_1, z_2) = -\frac{z_1 z_2 + 1}{z_1 - 1} + \frac{z_2}{z_1 - z_2} \left[ \frac{z_2^N (z_1 + 1)}{z_1 - 1} - \frac{z_1^N (z_2 + 1)}{z_2 - 1} \right]$$

$$+ z_2^N \left( \frac{2}{z_1 - 1} + \frac{2}{z_2 - 1} + \frac{6}{z_1 z_2 - 1} + 3 - 2N \right).$$

Observing $C^R(e, \delta) = C(e^h, \delta)$ we get the covariance of the real part $\text{Re} C(e, \delta)$.

References

[1] Berry M V 1987 The Bakerian lecture, 1987: quantum chaosology Proc. R. Soc. A 413 183–98
[2] Sieber M and Richter K 2001 Correlations between periodic orbits and their role in spectral statistics Phys. Scr. 2001 128
[3] Heusler S, Müller S, Altland A, Braun P and Haake F 2007 Periodic-orbit theory of level correlations Phys. Rev. Lett. 98 044103
[4] Müller S, Heusler S, Altland A, Braun P and Haake F 2009 Periodic-orbit theory of universal level correlations in quantum chaos New J. Phys. 11 103025
[5] Haake F 2010 Quantum Signatures of Chaos (Springer Series in Synergetics) (Berlin: Springer)
[6] Braun P and Haake F 2012 Chaotic maps and flows: exact Riemann–Siegel lookalike for spectral fluctuations J. Phys. A: Math. Theor. 45 425101
[7] Pandey A 1979 Statistical properties of many-particle spectra: III. Ergodic behavior in random-matrix ensembles Ann. Phys., NY 119 170–91
[8] Prange R E 1997 The spectral form factor is not self-averaging Phys. Rev. Lett. 78 2280–3
[9] Smilansky U 1999 Semiclassical quantization of maps and spectral correlations Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 173–92
[10] Conrey J B, Farmer D W and Zimbauer M R 2007 Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $u(n)$ (arXiv:math-ph/0511024v2)
[11] Zyczkowski K and Kuś M 1994 Random unitary matrices J. Phys. A: Math. Gen. 27 4235
[12] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[13] Mehta M L 2004 Random Matrices (Pure and Applied Mathematics vol 142) 3rd edn (New York: Academic)
[14] Conrey J B, Farmer D W and Zimbauer M R 2007 Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $u(n)$ (arXiv:math-ph/0511024v2)
[15] Zyczkowski K and Kuś M 1994 Random unitary matrices J. Phys. A: Math. Gen. 27 4235
[16] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[17] Mehta M L 2004 Random Matrices (Pure and Applied Mathematics vol 142) 3rd edn (New York: Academic)
[18] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[19] Conrey J B, Farmer D W and Zimbauer M R 2007 Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $u(n)$ (arXiv:math-ph/0511024v2)
[20] Zyczkowski K and Kuś M 1994 Random unitary matrices J. Phys. A: Math. Gen. 27 4235
[21] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[22] Mehta M L 2004 Random Matrices (Pure and Applied Mathematics vol 142) 3rd edn (New York: Academic)
[23] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[24] Conrey J B, Farmer D W and Zimbauer M R 2007 Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $u(n)$ (arXiv:math-ph/0511024v2)
[25] Zyczkowski K and Kuś M 1994 Random unitary matrices J. Phys. A: Math. Gen. 27 4235
[26] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[27] Mehta M L 2004 Random Matrices (Pure and Applied Mathematics vol 142) 3rd edn (New York: Academic)
[28] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[29] Conrey J B, Farmer D W and Zimbauer M R 2007 Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $u(n)$ (arXiv:math-ph/0511024v2)
[30] Zyczkowski K and Kuś M 1994 Random unitary matrices J. Phys. A: Math. Gen. 27 4235
[31] Zyczkowski K and Kuś M 1999 Pair correlations of quantum chaotic maps from supersymmetry Supersymmetry and Trace Formulae: Chaos and Disorder Proc. NATO ASI (Cambridge, UK, 8–20 September 1997) ed I V Lerner et al (New York: Plenum) 153–72
[32] Mehta M L 2004 Random Matrices (Pure and Applied Mathematics vol 142) 3rd edn (New York: Academic)