COXETER GROUPS, IMAGINARY CONES AND DOMINANCE

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Abstract. Brink and Howlett have introduced a partial ordering, called dominance, on the root systems of Coxeter groups in their proof that all finitely generated Coxeter groups are automatic (Math. Ann. 296 (1993), 179–190). Recently a function called $\infty$-height is defined on the reflections of Coxeter groups in an investigation of various regularity properties of Coxeter groups (Edgar, Dominance and regularity in Coxeter groups, PhD thesis, 2009). In this paper, we show that these two concepts are closely related to each other. We also give applications of dominance to the study of imaginary cones of Coxeter groups.

1. Introduction

In this paper we attempt to extend the understanding of a partial ordering (called dominance) defined on the root system of an arbitrary Coxeter group. The dominance ordering was introduced by Brink and Howlett in their paper [3] (where it was used to prove the automaticity of all finitely generated Coxeter groups). Dominance ordering has been further studied in the 1990’s by Brink [5] and Krammer [22], and later reproduced in [23], and it has only been recently examined again (Dyer [10], in connection with the representation theory of Coxeter groups; the PhD thesis of Edgar [11]; and a recent paper by the author [13]). The present paper is a short addition to both [11] and [13], and it could serve as a building block in the general knowledge on dominance ordering and on the combinatorics and geometry of Coxeter groups in general.

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More specifically, this paper has the following two objectives: (1) investigating the connection between the dominance ordering on the root system of an arbitrary Coxeter groups $W$ and a specific function (called $\infty$-height) defined on the set of reflections of $W$; (2) exploring the applications of the dominance ordering to the imaginary cone of $W$ (as defined by Kac).

The paper is organized into three sections. In the first section, background material is introduced: root basis, Coxeter datum, and root systems are defined in the context of the paper, and some basic properties of Coxeter groups are recalled for later use in the paper (most of them can be found in Howlett’s lectures [18]). Here we follow the definition used in [22], which gives a slight variant of the classical notion of root system, particularly adapted when working with arbitrary (not necessarily crystallographic) Coxeter groups. Furthermore, this framework allows easy passing to reflection subgroups. Indeed, we recall the fundamental property ([7, Theorem 1.8]) that the reflection subgroups of a Coxeter group are themselves Coxeter groups, and this particular framework allows us to apply all the definitions and properties to the reflection subgroups and not only to the over-group.

In the second section, the first main theorem (giving the connection between $\infty$-height and dominance order) is stated and proved. All results are related to an arbitrary Coxeter datum, implying the data of a root system $\Phi$, its associated Coxeter group $W$, and the set $T$ of all reflections of $W$ (consisting of all the $W$-conjugates of the Coxeter generators). The main objects of study are:

- the dominance order on $\Phi$ (Definition 3.1): given $x, y \in \Phi$, we say $x$ dominates $y$ if whenever $w \in W$ such that $wx \in \Phi^-\Phi$ then $wy \in \Phi^-\Phi$ too (where $\Phi^-$ denotes the set of negative roots);
- the function $\infty$-height on $T$. It is a variant of the usual (standard) height function of a reflection $t \in T$, namely, the minimal length of an element of $W$ that maps $\alpha_t$ (the unique positive root associated to $t$) to an element of the root basis. Adhering to the general framework of this paper, our definition of the height function applies to all reflection subgroups of $W$. It is easy to check (Lemma 3.13) that the height of $t$ is equal to the sum of the heights of $t$ relative to each maximal (with respect to inclusion) dihedral reflection subgroup containing $t$. The $\infty$-height of $t$ is then defined as a sub-sum of this sum, taking into account only those subgroups which are infinite (Definition 3.8).

We then show that these two concepts are closely related in the following way. The canonical bijection $t \leftrightarrow \alpha_t$, between $T$ and $\Phi^+$ (the set of positive roots), restricts to a bijection between (for any $n \in \mathbb{N}$):

- the set $T_n$ of all reflections whose $\infty$-height is $n$; and
• the set $D_n$ of all positive roots which strictly dominate exactly $n$ other positive roots.

The proof of this fact (Theorem 3.15) relies on a study of dihedral reflection subgroups. We have previously studied the partition $(D_n)_{n \in \mathbb{N}}$ of $\Phi^+$ in [13]; in particular, we showed there that each $D_n$ is finite and we gave an upper bound for its cardinality. Together with Theorem 3.15, this allows us to deduce here some information on the combinatorics of the $T_n$’s (Corollary 3.23).

The final section explores the relation between the dominance order and the imaginary cone of a Coxeter group. The concept of imaginary cone was introduced by Kac in [21] to study the imaginary roots of Kac-Moody Lie algebras, and was later generalized to Coxeter groups by Hée [14, 15] and Dyer [10]. It is defined as the subset of the dual of the Tits cone (denoted as $U^*$ here) consisting of elements $v \in U^*$ such that $(v, \alpha) > 0$ for only finitely-many $\alpha \in \Phi^+$ (where $(\ , \ )$ denotes the bilinear form associated to the Coxeter datum). The main results (Theorem 4.13 and Corollary 4.15) of this section state the following property: whenever $x, y \in \Phi$, then $x$ dominates $y$ if and only if $x - y$ lies in the imaginary cone. One direction of this property was first suggested to us by Howlett (private communications), and it is a special case of a result obtained independently (but earlier) by Dyer. We are deeply indebted to both of them for helpful discussions inspiring us to study the imaginary cone. We would also like to thank the referee of this paper for many valuable suggestions, especially those resulting in Corollary 4.15. To close this section, we include an alternative definition for the imaginary cone in the case where $W$ is finitely generated.

2. Background Material

Definition 2.1. (Krammer [22]) Suppose that $V$ is a vector space over $\mathbb{R}$ and let $(\ , \ )$ be a bilinear form on $V$ and let $\Delta$ be a subset of $V$. Then $\Delta$ is called a root basis if the following conditions are satisfied:

(C1) $(a, a) = 1$ for all $a \in \Delta$, and for distinct elements $a, b \in \Delta$ either

$(a, b) = -\cos(\pi/m_{ab})$ for some integer $m_{ab} = m_{ba} \geq 2$, or else

$(a, b) \leq -1$ (in which case we define $m_{ab} = m_{ba} = \infty$);

(C2) $0 \notin \text{PLC}(\Delta)$, where PLC($A$), the positive linear cone of a set $A$, denotes the set

$\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a \in A \text{ and } \lambda_{a'} > 0 \text{ for some } a' \in A \}.$

If $\Delta$ is a root basis, then we call the triple $\mathcal{C} = (V, \Delta, (\ , \ ))$ a Coxeter datum. Throughout this paper we fix a particular Coxeter datum $\mathcal{C}$. We stress that our definition of a root basis is not the most classical one of [2] or even [20]: the root system (see Definition 2.5) arising from our definition of a root basis is not necessarily crystallographic (indeed, the bilinear form can take values less than $-1$), and the root basis is
not assumed to be linearly independent (this allows us to transmit easily the definitions and properties of a Coxeter group to its reflection subgroups, indeed the requirements in our definition of a root basis of a Coxeter group are identical to those in the characterization of the equivalent of a root basis in any reflection subgroup). Observe that (C1) implies that for each \( a \in \Delta \), \( a \notin \text{PLC}(\Delta \setminus \{a\}) \), and furthermore, (C1) together with (C2) yield that \( \{a, b, c\} \) is linearly independent for all distinct \( a, b, c \in \Delta \). Note also that (C2) is equivalent to the requirement that 0 does not lie in the convex hull of \( \Delta \).

For each \( a \in \Delta \), define \( \rho_a \in \text{GL}(V) \) by the rule:

\[
\rho_a x = x - 2(x, a)a,
\]

for all \( x \in V \). Observe that \( \rho_a \) is a reflection, and \( \rho_a \rho_a = -a \).

The following proposition summarizes a few useful results:

**Proposition 2.2.** [18, Lecture 1] (i) Suppose that \( a, b \in \Delta \) are distinct such that \( m_{ab} \neq \infty \). Set \( \theta = \frac{\pi}{m_{ab}} \). Then

\[
(\rho_a \rho_b)^i a = \sin(2i + 1)\theta a + \frac{\sin 2i \theta}{\sin \theta} b,
\]

for each integer \( i \), and in particular, \( \rho_a \rho_b \) has order \( m_{ab} \) in \( \text{GL}(V) \).

(ii) Suppose that \( a, b \in \Delta \) are distinct such that \( m_{ab} = \infty \). Set \( \theta = \cosh^{-1}(-(a, b)) \). Then

\[
(\rho_a \rho_b)^i a = \begin{cases} 
\sinh(2i + 1)\theta a + \frac{\sinh 2i \theta}{\sinh \theta} b, & \text{if } \theta \neq 0 \\
(2i + 1)a + 2ib, & \text{if } \theta = 0,
\end{cases}
\]

for each integer \( i \), and in particular, \( \rho_a \rho_b \) has infinite order in \( \text{GL}(V) \). \( \square \)

Let \( G_\varepsilon \) be the subgroup of \( \text{GL}(V) \) generated by \( \{ \rho_a \mid a \in \Delta \} \). Suppose that \( (W, S) \) is a Coxeter system in the sense of [16] or [20] with \( S = \{ r_a \mid a \in \Delta \} \) being a set of involutions generating \( W \) subject only to the condition that the order of \( r_a r_b \) is \( m_{ab} \) for all \( a, b \in \Delta \) with \( m_{ab} \neq \infty \). Then Proposition 2.2 yields that there exists a group homomorphism \( \phi_\varepsilon : W \to G_\varepsilon \) satisfying \( \phi_\varepsilon(r_a) = \rho_a \) for all \( a \in \Delta \). This homomorphism together with the \( G_\varepsilon \)-action on \( V \) give rise to a \( W \)-action on \( V \): for each \( w \in W \) and \( x \in V \), define \( wx \in V \) by \( wx = \phi_\varepsilon(w)x \). It can be easily checked that this \( W \)-action preserves \((\ ,\ )\). Denote the length function of \( W \) with respect to \( S \) by \( \ell \), and call an expression \( w = r_1r_2\cdots r_n \) (where \( w \in W \) and \( r_i \in S \)) reduced if \( \ell(w) = n \). The following is a useful result:

**Proposition 2.3.** [18, Lecture 1, Theorem, Page 4] Let \( G_\varepsilon, W, S \) and \( \ell \) be as above, and let \( w \in W \) and \( a \in \Delta \). If \( \ell(wa) \geq \ell(w) \) then \( wa \in \text{PLC}(\Delta) \). \( \square \)

An immediate consequence of the above proposition is the following important fact:
Corollary 2.4. [18 Lecture 1, Corollary, Page 5] Let $G_{\mathcal{C}}, W, S$ and $\phi_{\mathcal{C}}$ be as above. Then $\phi_{\mathcal{C}} : W \rightarrow G_{\mathcal{C}}$ is an isomorphism. □

In particular, the above corollary yields that $(G_{\mathcal{C}}, \{\rho_a | a \in \Delta\})$ is a Coxeter system isomorphic to $(W, S)$. We call $(W, S)$ the abstract Coxeter system associated to the Coxeter datum $\mathcal{C}$, and we call $W$ a Coxeter group of rank $\#S$ (where $\#$ denotes cardinality).

Definition 2.5. The root system of $W$ in $V$ is the set

$$\Phi = \{wa | w \in W \text{ and } a \in \Delta\}.$$ 

The set $\Phi^+ = \Phi \cap \text{PLC}(\Delta)$ is called the set of positive roots, and the set $\Phi^- = -\Phi^+$ is called the set of negative roots.

From Proposition 2.3 we may readily deduce that:

Proposition 2.6. ([18 Lecture 3]) (i) Let $w \in W$ and $a \in \Delta$. Then

$$\ell(wr_a) = \begin{cases} \ell(w) - 1, & \text{if } wa \in \Phi^-, \\ \ell(w) + 1, & \text{if } wa \in \Phi^+. \end{cases}$$

(ii) $\Phi = \Phi^+ \uplus \Phi^-$, where $\uplus$ denotes disjoint union.

(iii) $W$ is finite if and only if $\Phi$ is finite. □

Define $T = \bigcup_{w \in W} wSw^{-1}$. We call $T$ the set of reflections in $W$. For each $x \in \Phi$, let $\rho_x \in \text{GL}(V)$ be defined by the rule: $\rho_x(v) = v - 2(v, x)x$, for all $v \in V$. Since $x \in \Phi$, it follows that $x = wa$ for some $w \in W$ and $a \in \Delta$. Direct calculations yield that $\rho_x = (\phi_{\mathcal{C}}(w))\rho_a(\phi_{\mathcal{C}}(w))^{-1} \in G_{\mathcal{C}}$. Now let $r_x \in W$ be such that $\phi_{\mathcal{C}}(r_x) = \rho_x$. Then $r_x = wr_aw^{-1} \in T$ and we call it the reflection corresponding to $x$. It is readily checked that $r_x = r_{-x}$ for all $x \in \Phi$ and $T = \{r_x | x \in \Phi\}$. For each $t \in T$ we let $\alpha_t$ be the unique positive root with the property that $r_{\alpha_t} = t$. It is also easily checked that there is a bijection $\psi : T \rightarrow \Phi^+$ given by $\psi(t) = \alpha_t$, and we call $\psi$ the canonical bijection.

For each $x \in \Phi^+$, as in [3], we define the depth of $x$ relative to $S$ to be $\min\{\ell(w) | w \in W \text{ and } wx \in \Phi^-\}$, and we denote it by $dp(x)$. The following lemma gives some basic properties of depth:

Lemma 2.7. ([3] [4] [24]).

(i) Let $\alpha \in \Phi^+$. Then $dp(\alpha) = \frac{1}{2}(\ell(r_{\alpha}) + 1)$.

(ii) Let $r \in S$ and $\alpha \in \Phi^+ \setminus \{\alpha_r\}$. Then

$$dp(r\alpha) = \begin{cases} dp(\alpha) - 1 & \text{if } (\alpha, \alpha_r) > 0, \\ dp(\alpha) & \text{if } (\alpha, \alpha_r) = 0, \\ dp(\alpha) + 1 & \text{if } (\alpha, \alpha_r) < 0. \end{cases}$$

Proof. (ii): [4] Corollary 2.7.

(ii): [3] Lemma 1.7. □
Remark 2.8. Part (i) of the above Lemma is equivalent to the property that any reflection in a Coxeter group has a palindromic expression which is reduced, and this was indeed noted in [24, Proposition 4.3].

Define functions \( N: W \to \mathcal{P}(\Phi^+) \) and \( \overline{N}: W \to \mathcal{P}(T) \) (where \( \mathcal{P} \) denotes power set) by setting \( N(w) = \{ x \in \Phi^+ | wx \in \Phi^- \} \) and \( \overline{N}(w) = \{ t \in T | \ell(wt) < \ell(w) \} \) for all \( w \in W \). We call \( \overline{N} \) the reflection cocycle of \( W \) (sometimes \( \overline{N}(w) \) is also called the right descent set of \( w \)). Standard arguments as those in [20, § 5.6] yield that for each \( w \in W \),

\[
\ell(w) = \#N(w), \tag{2.1}
\]

and

\[
\overline{N}(w) = \{ r_x | x \in N(w) \}. \tag{2.2}
\]

In particular, \( N(r_a) = \{ a \} \) for \( a \in \Delta \). Moreover, \( \ell(wv^{-1}) + \ell(v) = \ell(w) \), for some \( w, v \in W \) if and only if \( N(v) \subseteq N(w) \).

A subgroup \( W' \) of \( W \) is a reflection subgroup of \( W \) if \( W' = \langle W' \cap T \rangle \) (\( W' \) is generated by the reflections contained in it). For any reflection subgroup \( W' \) of \( W \), let

\[
S(W') = \{ t \in T | \overline{N}(t) \cap W' = \{ t \} \}
\]

and

\[
\Delta(W') = \{ x \in \Phi^+ | r_x \in S(W') \}.
\]

It was shown by Dyer ([8]) and Deodhar ([4]) that \( (W', S(W')) \) forms a Coxeter system:

**Theorem 2.9.** (Dyer) (i) Suppose that \( W' \) is an arbitrary reflection subgroup of \( W \). Then \( (W', S(W')) \) forms a Coxeter system. Moreover, \( W' \cap T = \bigcup_{w \in W'} wS(W')w^{-1} \).

(ii) Suppose that \( W' \) is a reflection subgroup of \( W \), and suppose that \( a, b \in \Delta(W') \) are distinct. Then

\[
(a, b) \in \{- \cos(\pi/n) | n \in \mathbb{N} \text{ and } n \geq 2\} \cup (-\infty, -1].
\]

And conversely if \( \Delta' \) is a subset of \( \Phi^+ \) satisfying the condition that

\[
(a, b) \in \{- \cos(\pi/n) | n \in \mathbb{N} \text{ and } n \geq 2\} \cup (-\infty, -1]
\]

for all \( a, b \in \Delta' \) with \( a \neq b \), then \( \Delta' = \Delta(W') \) for some reflection subgroup \( W' \) of \( W \). In fact, \( W' = \langle \{ r_a | a \in \Delta' \} \rangle \).

**Proof.** (i) [8, Theorem 3.3].

(ii) [8, Theorem 4.4]. \( \square \)

Let \( (\cdot, \cdot)' \) be the restriction of \( (\cdot, \cdot) \) on the subspace \( \text{span}(\Delta(W')) \). Then \( \mathcal{C}' = (\text{span}(\Delta(W'))), \Delta(W'), (\cdot, \cdot)' \) is a Coxeter datum with \( (W', S(W')) \) being the associated abstract Coxeter system. Thus the notion of a root system applies to \( \mathcal{C}' \). We let \( \Phi(W'), \Phi^+(W') \) and \( \Phi^-(W') \) be, respectively, the set of roots, positive roots and negative
roots for the datum $\mathcal{C}'$. Then $\Phi(W') = W'\Delta(W')$ and Theorem 2.3 (i) yields that $\Phi(W') = \{ x \in \Phi | r_x \in W' \}$. Furthermore, we have $\Phi^+(W') = \Phi(W') \cap \text{PLC}(\Delta(W'))$ and $\Phi^-(W') = -\Phi^+(W')$. We call $S(W')$ the set of canonical generators of $W'$, and we call $\Delta(W')$ the set of canonical roots of $\Phi(W')$. In this paper a reflection subgroup $W'$ is called a dihedral reflection subgroup if $\#S(W') = 2$.

A subset $\Phi'$ of $\Phi$ is called a root subsystem if $r_x, y \in \Phi'$ whenever $x, y$ are both in $\Phi'$. It is easily seen that there is a bijective correspondence between the set of reflection subgroups $W'$ of $W$ and the set of root subsystems $\Phi'$ of $\Phi$: $W'$ uniquely determines the root subsystem $\Phi(W')$, and $\Phi'$ uniquely determines the reflection subgroup $\{ \{ r_x | x \in \Phi' \} \}$. The notion of a length function also applies to the Coxeter system $(W', S(W'))$, and we let $\ell_{(W', s(W'))}: W' \to \mathbb{N}$ be the length function for $(W', S(W'))$. If $w \in W'$ and $a \in \Delta(W')$ then applying Proposition 2.6 to the Coxeter datum $\mathcal{C}' = (\text{span}(\Delta(W')))$ yields

$$
\ell_{(W', s(W'))}(w)a = \begin{cases} 
\ell_{(W', s(W'))}(w) - 1, & \text{if } wa \in \Phi^-(W'), \\
\ell_{(W', s(W'))}(w) + 1, & \text{if } wa \in \Phi^+(W').
\end{cases} \quad (2.3)
$$

Similarly the notion of a reflection cocycle also applies to the Coxeter system $(W', S(W'))$. Let $\overline{N}_{(W', s(W'))}: W \to \mathcal{P}(W' \cap T)$ denote the reflection cocycle for $(W', S(W'))$. Then for each $w \in W'$,

$$
\overline{N}_{(W', s(W'))}(w) = \{ t \in W' \cap T \mid \ell_{(W', s(W'))}(wt) < \ell_{(W', s(W'))}(w) \}. \quad (2.4)
$$

And we define $N_{(W', s(W'))}(w) = \{ x \in \Phi^+(W') \mid wx \in \Phi^-(W') \}$, for each $w \in W'$. It is shown in [7] that $\overline{N}_{(W', s(W'))}(w) = \overline{N}(w) \cap W'$ for arbitrary reflection subgroup $W'$ of $W$. Furthermore, it is readily seen that the canonical bijection $\psi$ restricts to a bijection $\psi': T \cap W' \to \Phi^+(W')$ given by $\psi'(t) = a_t$. For $w \in W'$, applying (2.4) to the Coxeter datum $\mathcal{C}' = (\text{span}(\Delta(W'))$, $\Delta(W')$, $\langle \cdot, \cdot \rangle$) yields

$$
\ell_{(W', s(W'))}(w) = \#N_{(W', s(W'))}(w). \quad (2.5)
$$

Furthermore, $\ell_{(W', s(W'))}(wv^{-1}) + \ell_{(W', s(W'))}(w) = \ell_{(w', s(w'))}(w)$, for some $w, v \in W'$, precisely when $N_{(W', s(W'))}(w) \subseteq N_{(W', s(W'))}(w)$. For a Coxeter datum $\mathcal{C} = (V, \Delta, \langle \cdot, \cdot \rangle)$, since $\Delta$ may be linearly dependent, the expression of a root in $\Phi$ as a linear combination of elements of $\Delta$ may not be unique. Thus the concept of the coefficient of an element of $\Delta$ in any given root in $\Phi$ is potentially ambiguous. We close this section by specifying a canonical way of expressing a root in $\Phi$ as a linear combination of elements from $\Delta$. This canonical expression follows from a standard construction similar to that considered in [10, Proposition 2.9]. Given a Coxeter datum $\mathcal{C} = (V, \Delta, \langle \cdot, \cdot \rangle)$, let $E$ be a vector space over $\mathbb{R}$ with basis $\Delta_E = \{ e_a | a \in \Delta \}$ in bijective correspondence with $\Delta$, and let $\langle \cdot, \cdot \rangle_E$ be the unique bilinear form on $E$ satisfying

$$
\langle e_a, e_b \rangle_E = (a, b) \text{ for all } a, b \in \Delta.
$$
Then $\mathcal{C}_E = (E, \Delta_E, (, )_E)$ is a Coxeter datum. Moreover, $\mathcal{C}_E$ and $\mathcal{C}$ are associated to the same abstract Coxeter system $(W, S)$; indeed Corollary 2.4 yields that the abstract Coxeter group $W$ is isomorphic to both $G_E = \langle \{ \rho_a \mid a \in \Delta \} \rangle$ and $G_{\mathcal{C}_E} = \langle \{ \rho_{a_n} \mid a \in \Delta \} \rangle$. Furthermore, $W$ acts faithfully on $E$ via $r_a y = \rho_{e_a} y$ for all $a \in \Delta$ and $y \in E$.

Let $f: E \to V$ be the unique linear map satisfying $f(e_a) = a$, for all $a \in \Delta$. It is readily checked that $(f(x), f(y)) = (x, y)_E$, for all $x, y \in E$. Now for all $a \in \Delta$ and $y \in E$,

$$r_a(f(y)) = \rho_a(f(y)) = f(y) - 2(f(y), a)a = f(y) - 2(f(y), e_a)f(e_a) = f(y - 2(y, e_a)_E e_a) = f(r_a y).$$

Then it follows that $w(f(y)) = f(wy)$, for all $w \in W$ and all $y \in E$, since $W$ is generated by $\{ r_a \mid a \in \Delta \}$. Let $\Phi_E$ denote the root system associated to the datum $\mathcal{C}_E$. Standard arguments yield that:

**Proposition 2.10.** [13, Proposition 2.1] The restriction of $f$ defines a $W$-equivariant bijection $\Phi_E \leftrightarrow \Phi$. \hfill \Box

Since $\Delta_E$ is linearly independent, it follows that each root $y \in \Phi_E$ can be written uniquely as $y = \sum_{e_a \in \Delta_E} \lambda_a e_a$; we say that $\lambda_a$ is the coefficient of $e_a$ in $y$, and it is denoted by $\text{coeff}_{e_a}(y)$. We use this fact together with the $W$-equivariant bijection $f: \Phi_E \leftrightarrow \Phi$ to give a canonical expression of a root in $\Phi$ in terms of $\Delta$:

**Definition 2.11.** Suppose that $x \in \Phi$. For each $a \in \Delta$, define the canonical coefficient of $a$ in $x$, written $\text{coeff}_a(x)$, by requiring that $\text{coeff}_a(x) = \text{coeff}_{e_a}(f^{-1}(x))$. The support, written $\text{supp}(x)$, is the set of $a \in \Delta$ with $\text{coeff}_a(x) \neq 0$.

3. **Dominance, Maximal Dihedral Reflection Subgroups and Infinity Height**

Throughout this section, let $W$ be the abstract Coxeter group associated to the Coxeter datum $\mathcal{C} = (V, \Delta, (, ))$, and let $\Phi$ and $T$ be the corresponding root system and the set of reflections respectively. Recently in [11], a uniquely determined non-negative integer, called $\infty$-height, is assigned to each reflection in $W$. Naturally, the set $T$ is then the disjoint union of the sets $T_0, T_1, T_2, \ldots$, where the set $T_n$ consists of all the reflections with $\infty$-height equal to $n$.

These $T_n$’s were utilized to demonstrate nice regularity properties of $W$ ([11, Ch. 5]). Furthermore, they gave rise to a family of modules in the generic Iwahori-Hecke algebra associated to $W$, and in turn, these modules were used to prove a weak form of Lusztig’s conjecture on the boundedness of the $a$-function (Dyer, unpublished). It is also known (Dyer, unpublished) that if $W$ is of finite rank, then there are finitely many reflections in $T_n$ for each $n$. 
In this section we prove that for an arbitrary reflection \( t \in T \) whose \( \infty \)-height equals \( n \), the corresponding positive root \( \alpha_t \) dominates precisely \( n \) other positive roots. This observation will then establish a bijection between the set of all reflections in \( W \) with \( \infty \)-height equal to \( n \) and the set of all positive roots each dominates precisely \( n \) other positive roots. Recent results on dominance obtained in [13] may then be immediately applied to the \( T_n \)'s, answering a number of basic questions about these \( T_n \)'s.

Following [13] and [12 § 4.7], we generalize the definition of dominance to the whole of \( \Phi \) (whereas in [3] and [5], dominance was only defined on \( \Phi^+ \)), and we stress that all the notations are the same as in the previous section.

Definition 3.1. (i) Let \( W' \) be a reflection subgroup of \( W \), and let \( x, y \in \Phi(W') \). Then we say that \( x \) dominates \( y \) with respect to \( W' \) if

\[
\{ w \in W' \mid wx \in \Phi^-(w') \} \subseteq \{ w \in W' \mid wy \in \Phi^-(w') \}.
\]

If \( x \) dominates \( y \) with respect to \( W' \) then we write \( x \text{ dom} W' y \).

(ii) Let \( W' \) be a reflection subgroup of \( W \) and let \( x \in \Phi^+(W') \). Define

\[
D_{W'}(x) = \{ y \in \Phi^+(W') \mid y \neq x \text{ and } x \text{ dom}_{W'} y \}.
\]

If \( D_{W'}(x) = \emptyset \) then we call \( x \) elementary with respect to \( W' \). For each non-negative integer \( n \), define

\[
D_{W',n} = \{ x \in \Phi^+(W') \mid \#D_{W'}(x) = n \}.
\]

In the case that \( W' = W \), we write \( D(x) \) and \( D_n \) in place of \( D_{W'}(x) \) and \( D_{W',n} \), respectively. If \( D(x) = \emptyset \) then we call \( x \) elementary.

It is readily checked that dominance with respect to any reflection subgroup \( W' \) of a Coxeter group \( W \) is a partial ordering on \( \Phi(W') \).

The following lemma summarizes some basic properties of dominance:

Lemma 3.2. ([13 Lemma 2.2]) (i) Let \( x, y \in \Phi^+ \) be arbitrary. Then \( x \text{ dom}_W y \) if and only if \( (x, y) \geq 1 \) and \( dp(x) \geq dp(y) \).

(ii) Dominance is \( W \)-invariant, that is, if \( x \text{ dom}_W y \) then \( wx \text{ dom}_W wy \) for all \( w \in W \).

(iii) Let \( x, y \in \Phi \) be such that \( x \text{ dom}_W y \). Then \( -y \text{ dom}_W -x \).

(iv) Let \( x, y \in \Phi \). Then there is dominance between \( x \) and \( y \) if and only if \( (x, y) \geq 1 \).

Corollary 3.3. Let \( x, y \in \Phi \), and let \( W' \) be an arbitrary reflection subgroup containing both \( r_x \) and \( r_y \).

(i) There is dominance with respect to \( W' \) between \( x \) and \( y \) if and only if \( (x, y)' \geq 1 \), where \( (, )' \) is the restriction of \((, )\) to the subspace \( \text{span}(\Delta(W')) \).

(ii) \( x \text{ dom}_W y \) if and only if \( x \text{ dom}_{W'} y \).

Proof. (i) Follows from Lemma 3.2(iv) applied to the Coxeter group \( W' \) and the datum \( G' = (\text{span}(\Delta(W'))), \Delta(W'), (, )' \).

(ii) The desired result is trivially true if \( x = y \), so we may assume that \( x \neq y \). It is clear that \( x \text{ dom}_W y \) implies that \( x \text{ dom}_{W'} y \). Conversely,
suppose that $x \text{ dom}_{W'} y$. Then part (i) yields that $(x, y) = (x, y)' \geq 1$. Thus Lemma 3.2 (iv) yields that either $x \text{ dom}_{W} y$, or else $y \text{ dom}_{W} x$. If the latter is the case, then by the first part of the current proof, $y \text{ dom}_{W'} x$, and hence it follows that $x = y$ (since dominance with respect to $W'$ is a partial ordering), contradicting our choice of $x$ and $y$.

Next is a well-known result whose proof can be found in the remarks immediately before Lemma 2.3 of [3]:

**Lemma 3.4.** ([3]) There is no non-trivial dominance between positive roots in the root system of a finite Coxeter group. ⊓⊔

Then we have a technical result which is going to be used repeatedly in the rest of this paper.

**Proposition 3.5.** Let $\alpha, \beta \in \Phi^+$ with $(\alpha, \beta) \leq -1$, and let $W'$ be the dihedral reflection subgroup generated by $r_\alpha$ and $r_\beta$. Further, if we set $\theta = \cosh^{-1}(-(\alpha, \beta))$, and for each $i \in \mathbb{Z}$ adopt the notation

$$c_i = \begin{cases} 
\frac{\sinh i \theta}{\sinh \theta}, & \text{if } \theta \neq 0 \\
i, & \text{if } \theta = 0.
\end{cases}$$

Then

(i) $W'$ is infinite, and $\Phi(W') = \{ c_{i+1} \alpha + c_i \beta \mid i \in \mathbb{Z} \}$.

(ii) Suppose that $x, y \in \Phi(W')$. Then $(x, y) \in (-\infty, -1] \cup [1, \infty)$, and in particular, if $x \neq \pm y$ then $\langle \{ r_x, r_y \} \rangle$ is an infinite dihedral reflection subgroup. More specifically,

(a) If $x = c_{n+1} \alpha + c_n \beta$ and $y = c_{m+1} \alpha + c_m \beta$, then either $(x, y) = \cosh((n - m)\theta) \geq 1$ if $\theta \neq 0$, or $(x, y) = 1$ if $\theta = 0$.

(b) If $x = c_{n+1} \alpha + c_n \beta$ and $y = c_{m} \alpha - c_m \beta$, then either

$(x, y) = -\cosh((n + m)\theta) \leq -1$ if $\theta \neq 0$, or $(x, y) = -1$ if $\theta = 0$.

(c) If $x = c_{n} \alpha + c_{n} \beta$ and $y = c_{m+1} \alpha + c_{m} \beta$, then either

$(x, y) = -\cosh((n + m)\theta) \leq -1$ if $\theta \neq 0$, or $(x, y) = -1$ if $\theta = 0$.

(d) If $x = c_{n} \alpha + c_{n} \beta$ and $y = c_{m-1} \alpha + c_{m} \beta$, then either

$(x, y) = \cosh((n - m)\theta) \geq 1$ if $\theta \neq 0$, or $(x, y) = 1$ if $\theta = 0$.

(iii) If $x \in \Phi^+(W') \setminus \{ \alpha, \beta \}$ then $D_{W'}(x) \neq \emptyset$.

**Proof.** (i) Proposition 4.5.4 (ii) of [11] implies that $W'$ is infinite, and the rest of statement follows from direct calculations similar to those in Proposition 2.2.

(ii) follows from Part (i) above and a direct calculation.

(iii) If $x \in \Phi^+(W') \setminus \{ \alpha, \beta \}$ then Part (i) above yields that either $x = c_{n+1} \alpha + c_n \beta$ (for some $n \neq 0$), or else $x = c_{n} \alpha + c_{n} \beta$ (for some $n \neq 1$). Then Part (ii) above and Corollary 3.3 (i) imply that we can find some $y \in \Phi^+(W') \setminus \{ x \}$ such that $x \text{ dom}_{W'} y$. 


Proof. Suppose for a contradiction that $W$ is a subgroup of $W_{\infty}$, with the following well-known one, and for completeness, we include a few technical results on infinite dihedral reflection subgroups. We begin with the following well-known one, and for completeness, we include a proof here.

**Proposition 3.6.** (Dyer [9]) Suppose that $\alpha, \beta \in \Phi^+$ are distinct. Let $W' = \langle \{ r_\gamma \mid \gamma \in (\mathbb{R}\alpha + \mathbb{R}\beta) \cap \Phi^+ \} \rangle$. Then $W'$ is a dihedral reflection subgroup of $W$.

**Proof.** Suppose for a contradiction that $W'$ is not dihedral. Then $\#S(W') \geq 3$, and let $x_1, x_2, x_3 \in \Delta(W')$ be distinct. Then $\Delta(W')$ is contained in a unique maximal dihedral reflection subgroup, namely $\langle \{ r_\alpha, r_\beta \} \rangle$. Theorem 2.9 (ii) then yields that $(x_i, x_j) \leq 0$ whenever $i, j \in \{1, 2, 3\}$ are different. Clearly $x_1, x_2, x_3$ are all in the two dimensional subspace $\mathbb{R}\alpha + \mathbb{R}\beta$, and thus a contradiction arises if we could show that $x_1, x_2, x_3$ are linearly independent. Let $c_1, c_2, c_3 \in \mathbb{R}$ be such that $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$. Since $x_1, x_2, x_3 \in \Phi^+$, and $0 \notin \text{PLC}(\Delta)$, it follows that $c_1, c_2, c_3$ cannot be all positive or all negative. Rename $x_1, x_2, x_3$ if necessary, we have the following three possibilities:

$$c_1, c_2 \geq 0 \quad \text{and} \quad c_3 < 0,$$

or

$$c_1, c_2 \leq 0 \quad \text{and} \quad c_3 > 0,$$

or

$$c_1, c_2, c_3 = 0.$$

If (3.2) is the case then $0 = (c_1 x_1 + c_2 x_2 + c_3 x_3, x_3) < 0$, and if (3.3) is the case then $0 = (c_1 x_1 + c_2 x_2 + c_3 x_3, x_3) > 0$, both are clearly absurd. Hence (3.4) must be the case and $x_1, x_2, x_3$ are linearly independent, a contradiction as required. □

Let $\alpha, \beta \in \Phi^+$ be distinct. Let $W''$ be an arbitrary dihedral reflection subgroup of $W$ containing the dihedral reflection subgroup $\langle \{ r_\alpha, r_\beta \} \rangle$. Let $x, y$ be the canonical roots for $W''$. It can be readily checked that $\mathbb{R}x + \mathbb{R}y = \mathbb{R}\alpha + \mathbb{R}\beta$, and hence $x, y \in (\mathbb{R}\alpha + \mathbb{R}\beta) \cap \Phi^+$. It then follows that $W'' \subseteq \langle \{ r_\gamma \mid \gamma \in (\mathbb{R}\alpha + \mathbb{R}\beta) \cap \Phi^+ \} \rangle$. This observation together with Proposition 3.6 readily yield the following well-known result:

**Proposition 3.7.** Every dihedral reflection subgroup $\langle \{ r_\alpha, r_\beta \} \rangle$ of $W$ (where $\alpha, \beta \in \Phi^+$ are distinct), is contained in a unique maximal dihedral reflection subgroup, namely $\langle \{ r_\gamma \mid \gamma \in \Phi^+ \cap (\mathbb{R}\alpha + \mathbb{R}\beta) \} \rangle$. □

**Definition 3.8.** (i) Define $\mathcal{M}$ to be the set of all maximal dihedral reflection subgroups of $W$.

(ii) Define $\mathcal{M}_\infty$ to be the set $\{ W' \in \mathcal{M} \mid \#W' = \infty \}$. 
(iii) For each \( t \in T \), define \( \mathcal{M}_t \) to be the set \( \{ W' \in \mathcal{M} \mid t \in W' \} \).

(iv) Let \( W' \) be a reflection subgroup of \( W \), and let \( t \in W' \cap T \). Define the standard height, \( h_{(W', S_{W'})}(t) \), of \( t \) with respect to the Coxeter system \((W', S(W'))\) to be

\[
\min \{ \ell_{(W', S_{W'})}(w) \mid w \in W', \, w\alpha_t \in \Delta(W') \}.
\]

For the standard height of \( t \) with respect to the Coxeter system \((W, S)\), we simply write \( h(t) \) in place of \( h_{(W, S)}(t) \).

Remark 3.9. For arbitrary reflection subgroup \( W' \) of \( W \), the depth function naturally applies to \( \Phi^+(W') \): if \( x \in \Phi^+(W') \), then the depth of \( x \) relative to \( S(W') \) (written \( dp_{(W', S_{W'})}(x) \)) is defined to be

\[
\min \{ \ell_{(W', S_{W'})}(w) \mid w \in W', \, wx \in \Phi^-(W') \}.
\]

Now for each \( t \in W' \cap T \), it is easily checked that

\[
dp_{(W', S_{W'})}(\alpha_t) = h_{(W', S_{W'})}(t) + 1,
\]

and hence applying Lemma 2.7 (i) to the Coxeter system \((W', S(W'))\) yields that

\[
h_{(W', S_{W'})}(t) = \frac{\ell_{(W', S_{W'})}(t) - 1}{2}. \tag{3.5}
\]

The following appears in [11], and for completeness we give a proof here:

Lemma 3.10. For each \( t \in T \), we have \( T \setminus \{ t \} = \bigcup_{W' \in \mathcal{M}_t} ((W' \cap T) \setminus \{ t \}) \).

Proof. It is readily checked that \( T \setminus \{ t \} = \bigcup_{W' \in \mathcal{M}_t} ((W' \cap T) \setminus \{ t \}) \), and hence we only need to check that this union is indeed disjoint. Suppose for a contradiction that there are distinct \( W_1, W_2 \in \mathcal{M}_t \) with \( r \in W_1 \cap W_2 \) for some \( r \in T \setminus \{ t \} \). Then clearly \( \{r, t\} \subseteq W_1 \) and \( \{r, t\} \subseteq W_2 \), contradicting Proposition 3.7.

The canonical bijection \( \psi : T \leftrightarrow \Phi^+ \) and the above immediately yield that:

Corollary 3.11. \( \Phi^+ \setminus \{ \alpha \} = \bigcup_{W' \in \mathcal{M}_\alpha} (\Phi^+(W') \setminus \{ \alpha \}) \), for each \( \alpha \in \Phi^+ \).

Remark 3.12. In particular, the above corollary yields implies that for \( t \in T \), if \( W_1, W_2 \in \mathcal{M}_t \) are distinct then \( \Phi^+(W_1) \cap \Phi^+(W_2) = \{ \alpha_t \} \).

Lemma 3.13. ([11]) Let \( t \in T \) be arbitrary. Then

\[
h(t) = \sum_{W' \in \mathcal{M}_t} h_{(W', S_{W'})}(t).
\]
Proof. For any reflection \( t \in T \), Corollary 3.11 yields that
\[
\{ \alpha \in \Phi^+ \mid t\alpha \in \Phi^- \} = \{ \alpha_i \} \cup \bigcup_{W' \in \mathcal{K}} \{ \alpha \in \Phi^+(W') \setminus \{ \alpha_i \} \mid t\alpha \in \Phi^-(W') \}.
\]  (3.6)
Since \( h(t) = \frac{1}{2}(\ell(t) - 1) = \frac{1}{2}(\#N(t) - 1) \), it follows from (3.6) that
\[
h(t) = \frac{1}{2} \left( \sum_{W' \in \mathcal{K}} \#\{ \alpha \in \Phi^+(W') \setminus \{ \alpha_i \} \mid t\alpha \in \Phi^-(W') \} \right)
= \sum_{W' \in \mathcal{K}} \frac{1}{2}(\ell(W', S_{W'})(t) - 1) \quad ( \text{by (2.4)} )
= \sum_{W' \in \mathcal{K}} h(W', S_{W'})(t) \quad ( \text{by (3.5)} ).
\]
\[\Box\]

**Definition 3.14.** \([11]\) For \( t \in T \), define the \( \infty \)-height of \( t \) to be
\[
h^\infty(t) = \sum_{W' \in \mathcal{K} \cap \mathcal{K}_\infty} h(W', S_{W'})(t),
\]
and for each non-negative integer \( n \), we define
\[T_n = \{ t \in T \mid h^\infty(t) = n \}.\]

Observe that from the above definition, it is not clear whether, for a specific non-negative integer \( n \), there is any reflection \( t \in T \) with \( h^\infty(t) = n \). It turns out that a number of basic questions like this can in fact be resolved with the aid of the results obtained in \([13]\) once we prove the following:

**Theorem 3.15.** For each non-negative integer \( n \), there is a bijection \( T_n \leftrightarrow D_n \) given by \( t \leftrightarrow \alpha_i \).

The proof of the above theorem will be deferred until we have all the necessary tools.

**Proposition 3.16.** Suppose that \( t \in T \), and let \( W' \) be an infinite dihedral reflection subgroup containing \( t \). If \( h(W', S_{W'})(t) \geq 1 \) then there exists some \( x \in \Phi^+(W') \) with \( \alpha_i \text{ dom}_W x \).

**Proof.** Observe that the condition \( h(W', S_{W'})(t) \geq 1 \) is equivalent to \( \alpha_i \notin \Delta(W') \), and hence the required result follows immediately from Proposition 3.5 (iii). \[\Box\]

The following proposition will be a key step to prove Theorem 3.15:

**Proposition 3.17.** Let \( W' \) be an infinite dihedral reflection subgroup, and let \( \Delta(W') = \{ \alpha, \beta \} \).
(i) There are two disjoint dominance chains in $\Phi(W')$, namely:

$$\ldots \text{dom}_W r_\alpha r_\beta r_\alpha(\beta) \text{ dom}_W r_\alpha(\beta) \text{ dom}_W r_\alpha(\beta) \text{ dom}_W \alpha$$
$$\text{dom}_W (-\beta) \text{ dom}_W r_\beta(-\alpha) \text{ dom}_W r_\beta r_\alpha(-\beta) \text{ dom}_W \cdots (3.7)$$

and

$$\ldots \text{dom}_W r_\beta r_\alpha r_\beta(\alpha) \text{ dom}_W r_\beta r_\alpha(\beta) \text{ dom}_W r_\beta(\alpha) \text{ dom}_W \beta$$
$$\text{dom}_W (-\alpha) \text{ dom}_W r_\alpha(-\beta) \text{ dom}_W r_\alpha r_\beta(-\alpha) \text{ dom}_W \cdots . (3.8)$$

In particular, each root in $\Phi(W')$ lies in exactly one of the above two chains, and the negative of any element of one chain lies in the other. Furthermore, the roots in $\Phi(W')$ dominated by either $\alpha$ or $\beta$ are all negative.

(ii) If $x \in \Phi(W')$ then $\#D_{W'}(x) = h_{(W', s(W'))}(r_x)$. $\square$

Proof. (i) Theorem 2.9 (ii) and [1] Proposition 4.5.4 (ii) yield that $(\alpha, \beta) \leq -1$. Hence it follows from Lemma 3.2 (iv) that $\alpha \text{ dom}_W -\beta$ and $\beta \text{ dom}_W -\alpha$. Then we can immediately verify the existence of the two dominance chains (3.7) and (3.8), and from these two chains the remaining statements in part (i) follow readily.

(ii) Follows immediately from the definition of $h_{(W', s(W'))}(r_x)$ and the two dominance chains (3.7) and (3.8). $\square$

**Proposition 3.18.** Suppose that $x, y \in \Phi^+$ are distinct with $x \text{ dom}_W y$, and let $W'$ be a dihedral reflection subgroup containing $r_x$ and $r_y$. Then $h_{(W', s(W'))}(r_x) \geq 1$.

Proof. It follows from Corollary 3.3 (ii) that $x \text{ dom}_{W'} y$, so Lemma 3.4 above yields that $W'$ is an infinite dihedral reflection subgroup. Let $\{\alpha, \beta\} = \Delta(W')$. We know from Proposition 3.17 (i) that the roots in $\Phi(W')$ dominated by either $\alpha$ or $\beta$ are all negative, and since $x \text{ dom}_W y \in \Phi^+$, it follows that $x \notin \{\alpha, \beta\}$. Hence by definition $h_{(W', s(W'))}(r_x) \geq 1$. $\square$

From the last two propositions we may deduce the following special case of Theorem 3.15.

**Lemma 3.19.** There is a bijection $T_0 \leftrightarrow D_0$ given by $t \leftrightarrow \alpha_t$.

Proof. Let $t \in T_0$, and suppose for a contradiction that $\alpha_t \notin D_0$. Then there exists $s \in T \setminus \{t\}$ such that $\alpha_t \text{ dom}_W \alpha_s$. Let $W'$ be the unique maximal dihedral reflection subgroup of $W$ containing $\{(s, t)\}$. Proposition 3.18 yields that $h_{(W', s(W'))}(t) \geq 1$. Since $\alpha_t \text{ dom}_W \alpha_s$, it follows from Lemma 3.4 that $W' \in \mathcal{M}_\infty$, and consequently $h_\infty(t) \geq 1$, contradicting the assumption that $t \in T_0$.

Conversely, suppose that $\alpha_t \in D_0$, and suppose for a contradiction that $t \notin T_0$. Then there exists some $W' \in \mathcal{M}_t \cap \mathcal{M}_\infty$ with
h(W, s(W′))(t) ≥ 1. But then Proposition 3.16 yields that αt /∈ D0, producing a contradiction as required.

Observe that Proposition 3.17 (ii) can be equivalently stated as:

**Proposition 3.20.** Suppose that t ∈ T, and suppose that W′ is an infinite dihedral reflection subgroup containing t. Then

\[ \#D_{W′}(\alpha_t) = h(W′, s(W′))(t). \]

**Proposition 3.21.** Suppose that t ∈ T is arbitrary. Then

\[ \bigcup_{W′ ∈ M_t ∩ M_∞} D_{W′}(\alpha_t) = D(\alpha_t). \]

**Proof.** First we observe that Remark 3.12 yields that the union of the sets D_{W′}(\alpha_t) over all W′ in M_t ∩ M_∞ is indeed disjoint.

It is clear that \[ \bigcup_{W′ ∈ M_t ∩ M_∞} D_{W′}(\alpha_t) \subseteq D(\alpha_t). \]

Conversely, suppose that x ∈ D(\alpha_t). Let W′ be the unique maximal dihedral reflection subgroup of W containing \{t, r_x\}. Then Corollary 3.3 (ii) yields that αt dom_{W′} x. Finally since there is no non-trivial dominance in any finite Coxeter group, it follows that W′ ∈ M_∞, as required.

Now we prove that for any reflection t ∈ W, its ∞-height \( h^\infty(t) \) equals the number of positive roots strictly dominated by αt:

**Theorem 3.22.** Let t ∈ T be arbitrary. Then \( h^\infty(t) = \#D(\alpha_t). \)

**Proof.** It follows from Proposition 3.20 and Proposition 3.21 that

\[ h^\infty(t) = \sum_{W′ ∈ M_t ∩ M_∞} h(W′, s(W′))(t) = \sum_{W′ ∈ M_t ∩ M_∞} \#D_{W′}(\alpha_t) = \#D(\alpha_t). \]

Finally we are in a position to prove Theorem 3.15:

**Proof of Theorem 3.15** The desired result follows immediately from Theorem 3.22.

Now combining Theorem 3.8 of [13], Corollary 3.9 of [13], Corollary 3.21 of [13] and Theorem 3.15 above we may deduce:

**Corollary 3.23.** (i) For each positive integer n,

\[ T_n \subseteq \{ tt' t | t ∈ T_0 and t' ∈ T_m for some m ≤ n − 1 \}. \]

(ii) Suppose that W is an infinite Coxeter group with \#S < ∞. Then \( 0 < \#T_n ≤ (\#T_0)^{n+1} − (\#T_0)^n \) for each positive integer n.

□
Remark 3.24. An upper bound for $\#T_0(= \#D_0)$ is given in [3], Furthermore, for any fixed finitely generated Coxeter group, this number can be explicitly calculated following the methods presented in [5].

4. Dominance and Imaginary Cone

Kac introduced the concept of an imaginary cone in the study of the imaginary roots of Kac-Moody Lie algebras. In [21, Ch. 5] the imaginary cone of a Kac-Moody Lie algebra was defined to be the positive cone on the positive imaginary roots. The generalization of imaginary cones to arbitrary Coxeter groups was first introduced by Hée in [14], and subsequently reproduced in [15]. This generalization has also been studied by Dyer ([10]) and Edgar ([11]). In this section we investigate the connections between this generalized imaginary cone and dominance in Coxeter groups, in particular, we show that whenever $x$ and $y$ are roots of a Coxeter group, then $x \ dom_W y$ if and only if $x - y$ lies in the imaginary cone of that Coxeter group.

Let $(W, S)$ be the abstract Coxeter system associated to the Coxeter datum $\mathcal{G} = (V, \Delta, ( , ))$ and let $\Phi$ be the corresponding root system. For any real vector space $X$ we write $X^\ast = \text{Hom}(X, \mathbb{R})$. In this section we take $X$ to be some suitable subspace of $V$. Also in this paper all cones are assumed to be convex cones. For any cone $C$ in $X$, we define $C^\ast = \{ f \in X^\ast \mid f(v) \geq 0 \text{ for all } v \in C \}$ and call it the dual of $C$; and for any cone $F$ in $X^\ast$, we define $F^\ast = \{ v \in X^\ast \mid f(v) \geq 0 \text{ for all } f \in F \}$ and call it the dual of $F$. If $W$ acts on $X$, then $X^\ast$ bears the contragredient representation of $W$ in the following way: if $w \in W$ and $f \in X^\ast$ then $wf \in X^\ast$ is given by the rule $(wf)(v) = f(w^{-1}v)$ for all $v \in X$. It is readily checked that for a cone $C$ in $X$ we have $C \subseteq C^\ast$, and also for any $w \in W$, we have $(wC)^\ast = wC^\ast$.

The following is a well-known result whose proof can be found in [18, Notes (c), Lecture 1]:

**Lemma 4.1.** Suppose that $X$ is a real vector space of finite dimension, and let $C$ be a cone in $X$. Then $(C^\ast)^\ast = C$, where $C$ is the topological closure of $C$ in $X$ (with respect to the standard topology on $X$). □

Set $P = \text{PLC}(\Delta) \cup \{0\}$. It is clear that $P$ is a cone in $V$. We define the Tits cone of $W$ in the same way as in 5.13 of [20]:

**Definition 4.2.** The Tits cone of the Coxeter group $W$ is the $W$-invariant set $U = \bigcup_{w \in W} wP^\ast$.

It is not obvious from the above definition that the Tits cone is indeed a cone, however, this can be made clear by the following result:

**Proposition 4.3.**

$$U = \{ f \in \text{span}(\Delta)^\ast \mid f(x) \geq 0 \text{ for all but finitely many } x \in \Phi^+ \}, \quad (4.1)$$
Suppose that Lemma 4.5.

Proof. Use induction on \(v\). If \(\ell\) then \(f \in P^*\) and it is readily checked that \(\text{Neg}(f) \subseteq N(w^{-1})\). Since \(N(w^{-1})\) is a finite set, it follows that \(f \in Y\), and hence \(U \subseteq Y\). Conversely, suppose that \(f \in Y\). If \(\text{Neg}(f) = \emptyset\) then \(f \in P^*\). Thus we may assume that \(#\text{Neg}(f) > 0\), and proceed with an induction. Observe that then there exists some \(\alpha \in \Delta\) such that \(f(\alpha) < 0\). It is then readily checked that \(#\text{Neg}(r_{\alpha} f) = #\text{Neg}(f) - 1\), and hence it follows from the inductive hypothesis that \(r_{\alpha} f \in U\). Since \(U\) is \(W\)-invariant, it follows that \(f \in U\), and hence \(Y \subseteq U\). \(\square\)

Lemma 4.4. \(U^* = \bigcap_{w \in W} wP^*\). Furthermore, \(U^* = \bigcap_{w \in W} wP\), whenever \(\Delta\) is a finite set.

Proof.

\[
U^* = \{ v \in V \mid f(v) \geq 0, \text{ for all } f \in U \}
= \{ v \in V \mid (w\phi)(v) \geq 0, \text{ for all } \phi \in P^* \text{, and for all } w \in W \}
= \{ v \in V \mid \phi(w^{-1}v) \geq 0, \text{ for all } \phi \in P^*, \text{ and for all } w \in W \}
= \bigcap_{w \in W} \{ v \in V \mid \phi(w^{-1}v) \geq 0, \text{ for all } \phi \in P^* \}
= \bigcap_{w \in W} \{ wv \in V \mid \phi(v) \geq 0, \text{ for all } \phi \in P^* \}
= \bigcap_{w \in W} \{ wv \in V \mid v \in (P^*)^* \}.
\tag{4.2}
\]

Let \(X = \text{span}(\Delta)\). If \(#\Delta\) is finite then it follows from Lemma 4.1 that \((P^*)^* = \overline{P}\). It is clear that \(P\) is topologically closed, hence (4.2) yields that \(U^* = \bigcap_{w \in W} wP\) when \(\Delta\) is a finite set. \(\square\)

Lemma 4.5. Suppose that \(v \in V\) has the property that \((a, v) \leq 0\) for all \(a \in \Delta\). Then \(wv - v \in P\) for all \(w \in W\). Moreover, if \(v \in P\) then \(v \in U^*\).

Proof. Use induction on \(\ell(w)\). Note that if \(\ell(w) = 0\) then there is nothing to prove. If \(\ell(w) \geq 1\) then we may write \(w = w'r_a\) where \(w' \in W\) and \(a \in \Delta\) with \(\ell(w) = \ell(w') + 1\). Then Proposition 2.3 yields that \(w'a \in \Phi^+ \subseteq P\), and we have

\[
wv - v = (w'r_a)v - v = w'(v - 2(v, a)a) - v = (w'v - v) - 2(a, v)w'a.
\]

Note that by the inductive hypothesis \(w'v - v \in P\). Since \((a, v) \leq 0\), it follows from the above that \(wv - v \in P\).

If \(v \in P\) then \(wv = (wv - v) + v \in P\) for all \(w \in W\), and hence \(v \in \bigcap_{w \in W} w^{-1} P \subseteq U^*\). \(\square\)
The following is a useful result from [13]:

**Proposition 4.6.** ([13], Proposition 3.4) Suppose that \( x, y \in \Phi \) are distinct with \( x \ \text{dom}_W y \). Let \( W' \) be the dihedral reflection subgroup generated by \( r_x \) and \( r_y \), and let \( \Delta(W') = \{ \alpha, \beta \} \). Then there exists some \( w \in W' \) such that either

\[
\begin{align*}
wx &= \alpha \\
wy &= -\beta
\end{align*}
\]

or else

\[
\begin{align*}
wx &= \beta \\
wy &= -\alpha.
\end{align*}
\]

In particular, \((x, y) = -(a, b)\). \(\Box\)

**Proposition 4.7.** Suppose that \( x, y \in \Phi \) such that \( x \ \text{dom}_W y \). Then \( w(x - y) \in \text{PLC}(\Delta) \) for all \( w \in W \), that is, \( x - y \in U^* \).

**Proof.** The assertion is trivially true if \( x = y \), so we may assume that \( x \neq y \). Since \( x \ \text{dom}_W y \), Lemma 3.2 (iv) yields that \( (x, y) \geq 1 \). Let \( W' \) be the (infinite) dihedral subgroup of \( W \) generated by \( r_x \) and \( r_y \). Let \( S(W') = \{ s, t \} \) and \( \Delta(W') = \{ \alpha, \alpha_t \} \). Proposition 4.6 yields that \((\alpha, \alpha_t) = -(x, y) \leq -1 \). Set \( c_i \) as in Proposition 3.5 for each \( i \in \mathbb{Z} \).

Since \( x \ \text{dom}_W y \), it follows that \((x, y) \geq 1 \), and Proposition 3.5 (ii) then yields that either

\[
\begin{align*}
x &= c_{n+1} \alpha_s + c_n \alpha_t \\
y &= c_{m+1} \alpha_s + c_m \alpha_t \\
&\quad \text{or else} \quad \begin{align*}
x &= c_{n-1} \alpha_s + c_n \alpha_t \\
y &= c_{m-1} \alpha_s + c_m \alpha_t.
\end{align*}
\end{align*}
\]

Next we shall show that \( n > m \). Suppose for a contradiction that \( m \geq n \). Then either \( x = y \) (when \( n = m \)) or else there will be a \( w \in W' \) such that \( wx \in \Phi(W') \cap \Phi^- \) and yet \( wy \in \Phi(W') \cap \Phi^+ \) (when \( n < m \)), both contradicting the fact that \( x \ \text{dom}_W y \). Since \( c_n > c_m \) whenever \( n > m \), it follows that \( x - y \in \text{PLC}(\Delta) \). Given the \( W \)-invariance of dominance, for any \( w \in W \), repeat the above argument with \( x \) replaced by \( wx \) and \( y \) replaced by \( wy \), we may conclude that \( w(x - y) \in \text{PLC}(\Delta) \subseteq (P^*)^* \). It then follows from Lemma 4.4 that \( x - y \in U^* \). \(\Box\)

When \( \# \Delta \) is finite, it can be checked that Lemma 4.4 yields that whenever \( x, y \in \Phi \) such that \( x - y \in U^* \), then \( x \ \text{dom}_W y \). In fact we can remove this finiteness condition and still prove the same result, and to do so we need some special notations and few extra elementary results.

We thank the referee of this paper for prompting us to look into this direction.

**Notations 4.8.** For a subset \( I \) of \( S \) we set \( \Delta_I = \{ x \in \Delta \mid r_x \in I \} \); \( V_I = \text{span}(\Delta_I) \); \( W_I = \langle I \rangle \); and \( P_I = \text{PLC}(\Delta_I) \cup \{0\} \). Furthermore, we set

\[
P_I^* = \{ f \in \text{Hom}(V_I, \mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \in P_I \};
\]

and

\[
P_I^{**} = \{ x \in V_I \mid f(x) \geq 0 \text{ for all } f \in P_I^* \}.
\]
Then $G_I = (V_I, \Delta_I, (,)_I)$ (where $(,)_I$ is the restriction of $(,)$ on $V_I$) is a Coxeter datum with corresponding Coxeter system $(W_I, I)$, and we call $W_I$ the standard parabolic subgroup of $W$ corresponding to $I$. Clearly $W_I$ preserves $V_I$.

**Lemma 4.9.** Suppose that $I$ is a subset of $S$. Then $P^{**} \cap V_I \subseteq P_I^{**}$.

*Proof.* Write $V = V_I \oplus V_I'$, where $V_I'$ is a vector space complement of $V_I$. Consequently, every $v \in V$ is uniquely written as $v = v_I + v_I'$, where $v_I \in V_I$ and $v_I' \in V_I'$. Then we observe that every $g \in P_I^*$ gives rise to a $g' \in P^*$ as follows: for any $v \in V$, simply set $g'(v) = g(v_I)$. Now let $x \in P^{**} \cap V_I$ and $f \in P_I^*$ be arbitrary. Then $f(x) = f'(x) \geq 0$, since $f' \in P^*$ and $x \in P^{**}$. Hence $x \in P_I^{**}$, and so $P^{**} \cap V_I \subseteq P_I^{**}$. □

**Proposition 4.10.** Let $x, y \in \Phi$. Then $x - y \in U^*$ if and only if $x \, \text{dom}_W \, y$.

*Proof.* By Proposition 4.7 we only need to prove that when $x$ and $y$ are both roots then $x - y \in U^*$ implies that $x \, \text{dom}_W \, y$. The assertion certainly holds if $x = y$, thus we only need to check the case when $x \neq y$.

Since dominance and $U^*$ are both $W$-invariant, it follows that we only need to prove the following statement: if $x \in \Phi^-$ then $y \in \Phi^-$ too.

Take $I = \{ r_\alpha | \alpha \in \text{supp}(x) \cup \text{supp}(y) \}$, and note that in particular, $I$ is a finite set. Now in view of Lemma 4.4, Lemma 4.9 and the fact that $W_I$ preserves $V_I$ we have

$$x - y \in (\bigcap_{w \in W} wP^{**}) \cap V_I \subseteq (\bigcap_{w \in W_I} wP^{**}) \cap V_I \subseteq \bigcap_{w \in W_I} w(P^{**} \cap V_I) \subseteq \bigcap_{w \in W_I} wP_I^{**} = \bigcap_{w \in W_I} wP_I,$$

where the equality follows from Lemma 4.11 since $I$ is a finite set. Thus $x - y \in P_I$, and this implies, precisely, that $y \in \Phi^-$ whenever $x \in \Phi^-$. □

Next we have a technical result which is a key component of the main theorem of this section.

**Proposition 4.11.** Suppose that $x, y \in \Phi$ are distinct with $x \, \text{dom}_W \, y$. Then there exists some $w \in W$ such that $wx \in \Phi^+$, $wy \in \Phi^-$ and $(w(x - y), z) \leq 0$ for all $z \in \Phi^+$.

*Proof.* Clearly it is enough to show that under such assumptions there exists some $w \in W$ with $wx \in \Phi^+$, $wy \in \Phi^-$ and $(w(x - y), z) \leq 0$ for all $z \in \Delta$.

Let $W'$ be the (infinite) dihedral reflection subgroup of $W$ generated by $r_x$ and $r_y$, and let $\Delta(W') = \{a_0, b_0\}$. Clearly $a_0, b_0 \in \Phi^+$, and
Proposition 4.6 yields that \((a_0, b_0) = -(x, y) \leq -1\), furthermore, there is some \(u \in \{r_x, r_y\}\) such that either
\[
\begin{align*}
\{ u(x) = a_0 \\
u(y) = -b_0 \\
\}
\begin{align*}
\{ u(x) = b_0 \\
u(y) = -a_0. \\
\}
\end{align*}
\]
At any rate, \(u(x - y) = a_0 + b_0\). Since the \(W\)-action preserves \((, ,\), it follows that \((a_0, a_0) = 1 = (b_0, b_0)\), and hence \((a_0 + b_0, a_0) \leq 0\) and \((a_0 + b_0, b_0) \leq 0\). However there may exist some \(c_1 \in \Delta\) with \((a_0 + b_0, c_1) > 0\). If this is the case, set \(a_1 = r_{c_1} a_0\) and \(b_1 = r_{c_1} b_0\). Recall that \((d, c_1) \leq 0\) for all \(d \in \Delta \setminus \{c_1\}\), so it follows that
\[
c_1 \in \text{supp}(a_0) \cup \text{supp}(b_0). (4.4)
\]
Since \((a_0 + b_0, c_1) > 0\), whereas \((a_0 + b_0, a_0) \leq 0\) and \((a_0 + b_0, b_0) \leq 0\), it follows that \(a_0 \neq c_1\) and \(b_0 \neq c_1\). Therefore we see that \(a_1, b_1 \in \Phi^+\), and \((a_1, b_1) = (a_0, b_0) \leq -1\). Consequently Theorem 2.9 (ii) yields that \(a_1, b_1\) are the canonical roots for the root subsystem \(\Phi(\{r_{a_1}, r_{b_1}\})\). Since \(r_{c_1}(a_0 + b_0) = a_0 + b_0 - 2(a_0 + b_0, c_1)c_1\) and \((a_0 + b_0, c_1) > 0\), it follows that
\[
\text{supp}(a_1) \cup \text{supp}(b_1) \subseteq \text{supp}(a_0) \cup \text{supp}(b_0),
\]
and
\[
\sum_{a \in \Delta} \text{coeff}_a(a_1) + \sum_{a \in \Delta} \text{coeff}_a(b_1) < \sum_{a \in \Delta} \text{coeff}_a(a_0) + \sum_{a \in \Delta} \text{coeff}_a(b_0).
\]
Moreover, since \((a_0 + b_0, c_1) > 0\), it follows that at least one of \((a_0, c_1)\) or \((b_0, c_1)\) must be strictly positive. Hence Lemma 2.7 yields that
\[
dp(a_1) + dp(b_1) \leq dp(a_0) + dp(b_0).
\]
Repeat this process and we can obtain new pairs of positive roots \(\{a_2, b_2\}, \ldots, \{a_{m-1}, b_{m-1}\}, \{a_m, b_m\}\) with
\[
\text{supp}(a_m) \cup \text{supp}(b_m) \subseteq \text{supp}(a_{m-1}) \cup \text{supp}(b_{m-1}) \subseteq \cdots \subseteq \text{supp}(a_0) \cup \text{supp}(b_0)
\]
and \(\text{dp}(a_m) + \text{dp}(b_m) \leq \text{dp}(a_{m-1}) + \text{dp}(b_{m-1}) \leq \cdots \leq \text{dp}(a_0) + \text{dp}(b_0)\), so long as we can find a \(c_m \in \Delta\) such that \((a_{m-1} + b_{m-1}, c_m) > 0\). Note that this process only terminates at a pair \(\{a_n, b_n\}\) for some \(n\), if \((a_n + b_n, z) \leq 0\) for all \(z \in \Delta\). Now if we could show that this process terminates at some such \(\{a_n, b_n\}\) after a finite number of iterations, then we have in fact found a \(w \in W\) given by
\[
w = r_{c_n} r_{c_{n-1}} \cdots r_{c_1} u, \text{ where } u \text{ is as in } (4.3), (4.5)
\]
satisfying
\[
(w(x - y), z) = (r_{c_n} \cdots r_{c_1} (a_0 + b_0), z) = (a_n + b_n, z) \leq 0
\]
for all \(z \in \Delta\).

Observe that the set of positive roots having depth less than the specific bound \(\text{dp}(a_0) + \text{dp}(b_0)\) and support in a fixed finite subset
supp($a_0$) $\cup$ supp($b_0$) of $\Delta$ is finite, indeed, Lemma 2.7 (ii) implies that there are at most $\#(\text{supp}(a_0) \cup \text{supp}(b_0))^{dp(a_0) + dp(b_0)}$ many such positive roots. Hence it follows that the possible pairs of positive roots $\{a_i, b_i\}$ obtainable in the above process must be finite too. Finally since

$$\sum_{a \in \Delta} \text{coeff}_a(a_j) + \sum_{a \in \Delta} \text{coeff}_a(b_j) < \sum_{a \in \Delta} \text{coeff}_a(a_i) + \sum_{a \in \Delta} \text{coeff}_a(b_i)$$

for all $j > i$, it follows that the sequence $\{a_0, b_0\}, \{a_1, b_1\}, \ldots$ must terminate at $\{a_n, b_n\}$ for some finite $n$, as required.

Finally, keep $w$ as in (4.5), we see from the above construction that either $wx = a_n \in \Phi^+$ and $wy = -b_n \in \Phi^-$, or else $wx = b_n \in \Phi^+$ and $wy = -a_n \in \Phi^-$. □

**Definition 4.12.** We define the imaginary cone $Q$ of $W$ by

$$Q = \{ v \in U^* \mid (v, a) \leq 0 \text{ for all but finitely many } a \in \Phi^+ \}.$$

The following result was obtained independently by Dyer as a consequence of [10, Theorem 6.3], stating that the imaginary cone of a reflection subgroup is contained in that of the over-group.

**Theorem 4.13.** Suppose that $x, y \in \Phi$ such that $x \text{ dom}_W y$. Then $x - y \in Q$.

**Proof.** By Proposition 4.7 we know that $x - y \in U^*$, thus to prove the desired result, we only need to show that $(x - y, z) \leq 0$ for all but finitely many $z \in \Phi^+$. Suppose that $z \in \Phi^+$ such that $(x - y, z) > 0$. Let $w \in W$ be as in Proposition 4.11. Then $(w(x - y), wz) > 0$, and by Proposition 4.11 this is possible only if $z \in N(w)$. Since $\#N(w)$ is clearly finite (of size $\ell(w)$), it follows that indeed $(x - y, z) \leq 0$ for all but finitely many $z \in \Phi^+$. □

**Remark 4.14.** The above theorem is a special case of Dyer’s result when the subgroup is dihedral. In fact, Dyer’s result, when applied to dihedral reflection subgroups, implies that if $x$ and $y$ are roots with $x \text{ dom}_W y$ then $x - cy \in Q$ for an explicit range of $c \in \mathbb{R}$ depending on the value of $(x, y)$. Our formulation was first suggested to us by Howlett and Dyer, and we gratefully acknowledge their help.

Theorem 4.13 combined with Proposition 4.10 immediately imply the following:

**Corollary 4.15.** Let $x, y \in \Phi$. Then $x - y \in Q$ if and only if $x \text{ dom}_W y$.

**Remark 4.16.** Incidentally, we observe from Proposition 4.10 and Corollary 4.15 that when $x, y \in \Phi$, it is impossible for $x - y$ to be in $U^* \setminus Q$.

**Corollary 4.17.** Suppose that $x, y \in \Phi$ are distinct. Then the following are equivalent:
(i) whenever $x \text{ dom}_W z \text{ dom}_W y$ for some $z \in \Phi$, then either $z = x$ or $z = y$ (thus forming a cover of dominance);

(ii) there exists a $w \in W$ such that $wx \in D_0$ and $wy \in -D_0$.

Proof. Suppose that (i) is the case. Let $w$ be as in Proposition 4.11 above. First we show that then $wx \in D_0$. Suppose for a contradiction that $wx \notin D_0$, and let $z \in D(wx)$. Then Proposition 4.11 yields that $wy \in \Phi^-$ and $(wy, z) \geq (wx, z) \geq 1$. Hence it is clear that $z \text{ dom}_W wy$. But this implies that $x \text{ dom}_W w - 1 z \text{ dom}_W y$ with $x \neq w - 1 z \neq y$, contradicting (i). Therefore $wx \in D_0$, as required. Exchanging the roles of $x$ and $-y$ we may deduce that $wy \in -D_0$.

Suppose that (ii) is the case and suppose for a contradiction that there exists some $z \in \Phi \setminus \{x, y\}$ such that $x \text{ dom}_W z \text{ dom}_W y$. Let $w \in W$ with $wx \in D_0$ and $wy \in -D_0$. If $wz \in \Phi^+$ then Lemma 3.2 (ii) yields that $wx \text{ dom}_W wz$, contradicting the fact that $wx \in D_0$. On the other hand, if $wz \in \Phi^-$, then Lemma 3.2 (ii) and (iii) yield that $-wy \text{ dom}_W -wz \in \Phi^+$, contradicting the fact that $-wy \in D_0$.

$\blacksquare$

Observe that applying Corollary 4.17 to arbitrary reflection subgroup $W'$ of $W$ yields the following:

Corollary 4.18. Suppose that $W'$ is a reflection subgroup of $W$ with $x$ and $y \in \Phi(W')$ being distinct. Then the following are equivalent:

(i) whenever $x \text{ dom}_{W'} z \text{ dom}_{W'} y$ for some $z \in \Phi(W')$, then either $z = x$ or $z = y$;

(ii) there exists a $w \in W'$ such that $wx \in D_{W', 0}$ and $wy \in -D_{W', 0}$.

$\blacksquare$

Definition 4.19. Suppose that $W'$ is a reflection subgroup of $W$ and $x, y \in \Phi(W')$ satisfy both (i) and (ii) of Corollary 4.18. Then we say that the dominance between $x$ and $y$ is minimal with respect to $W'$.

Proposition 4.20. Suppose that $x, y \in \Phi$ are distinct with $x \text{ dom}_W y$, and let $W'$ be the dihedral reflection subgroup generated by $r_x$ and $r_y$. Then the dominance between $x$ and $y$ with respect to $W'$ is minimal.

Proof. It follows from Corollary 3.3 (ii) that $x \text{ dom}_{W'} y$, and hence Lemma 3.4 yields that $W'$ is infinite dihedral. Let $\Delta(W') = \{\alpha, \beta\}$. Then Proposition 3.17 (i) yields that $D_{W', 0} = \{\alpha, \beta\}$.

On the other hand, it follows from Proposition 4.6 that there is some $w \in W'$ such that either

$$\begin{cases} wx = a \\ wy = -b \end{cases} \quad \text{or else} \quad \begin{cases} wx = b \\ wy = -a, \end{cases}$$

consequently Corollary 4.18 yields that the dominance between $x$ and $y$ with respect to $\langle\{r_x, r_y\}\rangle$ is minimal.

$\blacksquare$

From the above proposition we may deduce:
Proposition 4.21. Suppose that \( x \in \Phi^+ \) with \( D(x) = \{ x_1, x_2, \ldots, x_m \} \). For each \( i \in \{ 1, 2, \ldots, m \} \), set \( W_i = \langle \{ r_x, r_x \} \rangle \). Then \( W_i \neq W_j \) whenever \( i \neq j \).

Proof. For each \( i \in \{ 1, 2, \ldots, m \} \), set \( \{ s_i, t_i \} = S(W_i) \). Suppose for a contradiction that \( W' = W_i = W_j \) for some \( i \neq j \). Then we may write \( \{ s, t \} = \{ s_i, t_i \} = \{ s_j, t_j \} \). Corollary 3.3(ii) yields that \( x \, \text{dom}_{W_k} \, x_k \) for all \( k \in \{ 1, 2, \ldots, m \} \), and since there is no non-trivial dominance in finite Coxeter groups, it follows that \( W_1, W_2, \ldots, W_m \) are all infinite dihedral reflection subgroups. Hence it follows from Proposition 4.5.4 of [1] that \( (\alpha_s, \alpha_t) \leq -1 \). Set \( c_n \) as in Proposition 3.5 for each \( n \in \mathbb{Z} \).

Since \( x \, \text{dom}_{W} \, x_i \) and \( x \, \text{dom}_{W} \, x_j \), Proposition 3.5(ii) yields that either

\[
\begin{cases}
  x = c_m \alpha_s + c_{m+1} \alpha_t \\
  x_i = c_m \alpha_s + c_{m+1} \alpha_t \\
  x_j = c_{m'} \alpha_s + c_{m' + 1} \alpha_t
\end{cases}
\]

or else

\[
\begin{cases}
  x = c_m \alpha_s + c_{m-1} \alpha_t \\
  x_i = c_m \alpha_s + c_{m-1} \alpha_t \\
  x_j = c_{m'} \alpha_s + c_{m'-1} \alpha_t
\end{cases}
\]

for some distinct integers \( m, m' \) and \( m'' \). Observe that in either case \( (x_i, x_j) \geq 1 \), and therefore there will be (non-trivial) dominance between \( x_i \) and \( x_j \). Without loss of any generality, we may assume that \( x \, \text{dom}_{W} \, x_i \) and \( x \, \text{dom}_{W} \, x_j \). Then \( x \, \text{dom}_{W} \, x_i \, \text{dom}_{W} \, x_j \) by Corollary 3.3(ii), contradicting Proposition 4.20.

We close this paper with an alternative characterization for the imaginary cone \( Q \) when \( \# \Delta < \infty \).

Proposition 4.22. If \( \# \Delta < \infty \) then

\[
Q = \{ \, vw \mid w \in W \text{ and } v \in P \text{ such that } (v, a) \leq 0 \text{ for all } a \in \Phi^+ \}.
\]  

(4.6)

Proof. First we denote the set on the right hand side of (4.6) by \( Z \), and for each \( b \in P \), define \( \text{Pos}(b) = \{ c \in \Phi^+ \mid (b, c) > 0 \} \). Recall that under the assumption that \( \# \Delta < \infty \), Lemma 4.4 yields that

\[
Q = \{ \, v \in \bigcap_{w \in W} wP \mid (v, a) \leq 0 \text{ for all but finitely many } a \in \Phi^+ \}.
\]

Let \( u \in Q \) be arbitrary. Since \( \# \Delta < \infty \), it follows from Lemma 4.4 that \( u \in P \). If \( \text{Pos}(u) = \emptyset \), then trivially \( u \in Z \). Therefore we may assume that \( \text{Pos}(u) \neq \emptyset \), and proceed by an induction on \( \# \text{Pos}(u) \) (this is only possible because \( u \in Q \), and so \( \# \text{Pos}(u) < \infty \)). Let \( a \in \Delta \) be chosen such that \( (u, a) > 0 \). Then it can be readily checked that \( \text{Pos}(r_au) = r_a(\text{Pos}(u) \setminus \{a\}) \). Thus the inductive hypothesis yields that \( r_au \in Z \). Clearly \( Z \) is \( W \)-invariant, and so \( u \in Z \), and hence \( Q \subseteq Z \).

Conversely, if \( x \in Z \), then \( x = vw \) for some \( w \in W \) and \( v \in P \) such that \( (v, a) \leq 0 \) for all \( a \in \Delta \). Lemma 4.5 yields that \( v \in U^* \), and since \( U^* \) is clearly \( W \)-invariant, it follows that \( x \in U^* \). Suppose that \( y \in \Phi^+ \) with \( (x, y) > 0 \). Since \( (x, y) = (vw, y) = (v, w^{-1}y) \), and
since \((v, a) \leq 0\) for all \(a \in \Phi^+\), it follows that \(w^{-1}y \in \Phi^-\) and thus \(y \in N(w^{-1})\). The finiteness of the set \(N(w^{-1})\) then implies that \(x \in Q\), and hence \(Z \subseteq Q\).

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