Hotelling Model Analysis Based on Product Quality Difference with the Network Externality

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Abstract. A network externality was introduced into the two-stage dynamic Hotelling model with the quadratic function of transportation cost on the basis of the quality difference between two products. In consideration of three different position relations among the consumers' indifference curve, quality preferences for the products and population distribution density, the existence condition of equilibrium price between two manufacturers about the Hotelling model were obtained. Attention was paid to traffic-flow distribution technology and infinite dimensional variational inequality. Finally, the influences of network externality, transportation cost, cost difference and quality difference on equilibrium prices between two manufactures were analyzed.

Keywords: Hotelling model; Product quality differentiation; Network externality; Dynamic game.

1. Introduction
The purpose of proposing the Hotelling model is to resolve the Bertrand paradox by introducing product differences [1]. The core idea of the Hotelling model lies in its recognition of the relationship between product differentiation and geographical location, which has become a key factor of market competition between manufacturers [2].

The scholars considered a problem reflecting the difference degree of product in linear space about traditional model of the Hotelling. Based on this model, Aspremont discussed the optimal location problem and price-equilibrium existence conditions of the Hotelling model with linear transportation costs [3]. In the discussion of location in the supply chain about Hotelling model, XIE focused on competitive strategies at the same supply price [4]. Guo [5] and Huo [6] analyzed the price competition problem when internal members share information. From the perspective of decentralized structure and the entry mode of multinational corporations, Yang [7] and Li [8] studied the related decisions and the influence of supply chain. Based on Bass diffusion model, Lee analyzed the influence of different sales strategies on equilibrium prices [9]. Based on Lee's conclusion, Shivendu further introduced network externality about the model [10]. Combining the Hotelling model with new decision variables, Yang, Guo and Wang evolved the game process of duopoly manufacturers' price competition [11-14]. The aforementioned literature assumed that the effect of product difference (manufacturer's location is different) on competitive equilibrium in the same situation as the quality of the products.

In fact, consumer’s choice is affected by product brand, quality, performance and other factors. All consumers using products with the same quality and brand are not in line with realistic conditions or economic laws. For example, fabrics are different in quality and price, but consumers have different preferences and, therefore, the fabrics will have their own sales volumes in the same market. Although
Xu B\textsuperscript{[15]}, to a certain degree, studied the Hotelling model with product quality difference; these studies did not involve the product differentiation with network externality. Therefore, network externality was introduced into the Hotelling model in this work. The complete information dynamic game problems of two-stage involved of two products with network externality made by two different manufacturers were discussed when consumers' quality preferences obeyed uniform distribution on the interval $[d, e]$. There are three different position relations among the consumers' indifference curve of product selection, quality preferences and consumer distribution areas. Therefore, the existence conditions of the equilibrium prices of two products made by two manufacturers respectively were obtained. Finally, the effect of network externality characteristics and unit transportation cost of products and differences in quality of two kinds of products on the equilibrium prices of two manufacturers were analyzed.

2. Model Establishment and Related Assumptions
Suppose the distribution of the city is linear, and the positions of manufacturer A (Producing product 1) and manufacturer B (Producing product 2) are $a$ and $b$ in the city respectively. Moreover, $\Delta = 1 - b - a$ is position difference between two the manufacturers. The length of the city, as well as the total number of consumers, is normalized to one. The transportation costs of products 1 and 2 are the quadratic functions $t(x - a)^2$ and $t(1 - b - x)^2$ respectively, where $t$ is unit transportation cost. Consumer preference $\theta$ for product quality obeys uniform distribution in the interval $[d, e]$, and consumer location $x$ in linear city and its preference $\theta$ are independent of each other. $p_i$ and $q_i$ are the price and market share of product $i$, respectively. $c_i$ means the production cost of the product $i$. $T_1$ is the quality of the product $i$, with $i = 1, 2$.

To simplify the analysis, assume that product $i$ has the same network coefficient $u$, and $\Delta T_1 - T_2 > 0$. The two manufacturers conducted the dynamic game of two-stage complete information. They determined operating position (meeting $a \geq 0, b \geq 0, a + b \leq 1$) at the same time in the first stage, and priced their products at the same time to maximize the profits in the second stage. Assume that all consumers have the same, large enough reservation utility $R_0$ for product $i$ and that the market can be completely covered. The indifference curve of consumer choice can be obtained through the condition:

$$U_1(x, \theta) = U_2(x, \theta)$$

$x$ is the position of consumers, $U_i(x, \theta)$ is the utility function for product $i$, with $i = 1, 2$. Then, the utility $U_i(x, \theta)$ is:

$$U_1(x, \theta) = R_0 + \theta T_1 - P_1 - (x - a)^2 t + uq_1$$
$$U_2(x, \theta) = R_0 + \theta T_2 - P_2 - (1 - b - x)^2 t + uq_2$$

(1)

Assume that other consumers do not change product selection in network effect product market. Then, every consumer's change in strategy cannot increase the utility when the market reaches equilibrium. The method described by Marcotte\textsuperscript{[16]} was used to make the market share of network effect product relative to the balancing flow in traffic problem. With the condition that the distribution of consumers is uneven, the selection problem of products with network effect in a linearly distributed city was characterized through infinite dimensional variational inequality:

$$\langle P - R - \theta T + G(x) - uQ, Q - y \rangle \leq 0 \quad \forall y \in \Omega(x, \theta) = \{q / q \in Q\}.$$

The problem is a lateral difference problem with network externality, providing a new way of thought for the Hotelling model solution.
\( P = (p_1, p_2) \) is product price vector in the inequality, 
\( R = (R_0, R_0) \) is the reservation utility vector of product (a constant vector with equal component), 
\( T = (T_1, T_2) \) is quality vector for each product with network effect, \( Q = (q_1, q_2) \) is market share vector for each product, and 
\[ q_i = \int \int_{V \cup d \theta_i} \frac{1}{e^{-d}} dx d\theta, \quad q_2 = 1 - q_1. \]
\( G(x) \) is the vector of transportation cost function.
The variational inequality proposed by Marcotte can be used to solve the related problems of the Hotelling model. The market share of network product is equivalent to the solution of \( Q = (q_1, q_2) \), satisfying infinite dimensional variational inequality:
\[ \langle P - R - \partial T + G(x) - uQ, Q - y \rangle \leq 0 \quad \forall y \in \Omega(x, \theta) \quad (2) \]

### 3. Existence Conditions of Equilibrium Price under the Product Market with Network Effect and Price Competition Analysis

Customer choice theory shows that the consumer will choose a product with the largest utility. Therefore, \( r(x, \theta) = U_1(x, \theta) - U_2(x, \theta) \) and \( \Delta p = p_1 - p_2 \) are given. The analysis shows that there are three categories of market share.
The necessary and sufficient condition for product 1 to completely occupy the market is as follows, with \( q_1 = 1 \):
\[ r(x, \theta) > 0, \forall x \in [0,1], \theta \in [d,e] \Rightarrow \Delta p < d\Delta T - (1 + b - a)\Delta t + u(q_1 - q_2) \quad (3) \]
It's worth noting that \( \Delta = 1 - b - a \) in the inequality.
The necessary and sufficient condition for product 2 to completely occupy the market is as follows, with \( q_1 = 0 \):
\[ r(x, \theta) < 0, \forall x \in [0,1], \theta \in [d,e] \Rightarrow \Delta p > e\Delta T + (1 - b + a)\Delta t + u(q_1 - q_2) \quad (4) \]
Therefore, the necessary and sufficient condition for two products to simultaneously occupy the market is:
\[ d\Delta T - (1 + b - a)\Delta t + u(2q_1 - 1) \leq \Delta p \leq e\Delta T + (1 - b + a)\Delta t + u(2q_1 - 1) \quad (5) \]

### 3.1. Model Analysis When There is no Difference in Site Selection between Two Manufacturers
The indiffERENCE curve of consumers is obtained by Equation (1) and Equation (2), with no location difference between the two products \( a + b = 1 \):
\[ \theta(x) = \frac{1}{\Delta T} \left[ \Delta P - u(q_1 - q_2) \right] \]
Equation (1) shows that space \( U_1 \geq U_2 \) is a rectangular region formed by \( 0 \leq x \leq 1 \) and \( \frac{\Delta p - u(q_1 - q_2)}{\Delta T} \leq \theta \leq e \), shown in Figure 1:
Therefore, the following equations can be obtained by Equation (5), with \( a + b = 1 \) and \( \Delta = 0 - d\Delta T + u(2q_1 - 1) \leq \Delta p \leq e\Delta T + u(2q_1 - 1) \) when two products occupy market together.

\[
q_1 = \int_{\Delta T}^{\Delta T - (q_2 - q_1)w} \left( \int_0^1 \frac{1}{e - d} dx \right) d\theta = \frac{\Delta T e - u - \Delta p}{\Delta T (e - d) - 2u}
\]

\[
q_2 = \int_d^{\Delta T - (q_2 - q_1)w} \left( \int_0^1 \frac{1}{e - d} dx \right) d\theta = \frac{\Delta p - u - \Delta Td}{\Delta T (e - d) - 2u}
\]

\[
\pi_1(p_1, p_2) = \frac{(p_1 - c_1)(\Delta T e - u - \Delta p)}{\Delta T (e - d) - 2u}
\]

\[
\pi_2(p_1, p_2) = \frac{(p_2 - c_2)(\Delta p - u - \Delta Td)}{\Delta T (e - d) - 2u}
\]

\( \pi_i (i = 1,2) \) is the continuous differentiable concave function with \( p_i \). The following proposition can be obtained through first-order condition of the equilibrium price with this condition \( p_1^* \geq c_1, p_2^* \geq c_2 \).  

**Proposition 1:** With the condition \( a + b = 1 \), the necessary and sufficient condition for the existence of equilibrium prices of two manufacturers meeting the model conditions and conducting price competition in the linear city is

\[
(2d - e)\Delta T + 3u \leq \Delta c \leq (2e - d)\Delta T - 3u
\]

Therefore, the equilibrium prices and profits for each manufacturer are:

\[
\begin{align*}
P_1^* &= \frac{(2e - d)\Delta T + 2c_1 + c_2 - u}{3} \\
P_2^* &= \frac{(e - 2d)\Delta T + c_1 + 2c_2 - u}{3}
\end{align*}
\]

\[
\begin{align*}
\pi_1(P_1^*, P_2^*) &= \frac{[(2e - d)\Delta T - c_1 + c_2 - 3u]^2}{9([\Delta T (e - d) - 2u]}
\pi_2(P_1^*, P_2^*) &= \frac{[(e - 2d)\Delta T + c_1 - c_2 - 3u]^2}{9([\Delta T (e - d) - 2u]}
\end{align*}
\]

3.2. Model Analysis of Price Competition between Two Firms with Location Difference
The indifference curve of consumers’ product selection was obtained by Equation (1) and Equation (2), with no level difference between two products when \( a + b < 1 \).
\[ \theta(x) = \frac{1}{\Delta T} \left[ \Delta P - (1 - b + a - 2x)\Delta t - (2uq_1 - u) \right] \]  

(9)

There exist the following three different position relations among indifference curve, quality preferences and population distribution areas, as shown in Figure 2.

![Figure 2](image)

**Figure 2.** The geometric relationship among indifference curves, population distribution and product preferences with \( a + b < 1 \)

(a) The following equation was obtained by Equation (5) and Equation (9) when \( \theta(0) < d \leq \theta(1) < e \).

\[ \theta(x_0)d\Delta T - (1 + b - a)\Delta t + u(2q_1 - 1) \leq \Delta p \leq d\Delta T + (1 - b + a)\Delta t + u(2q_1 - 1) \]

According to equation (9), \( \theta(x_0) = d \) was given, then

\[ x_0 = \frac{d\Delta T - \Delta p}{2\Delta t} + \frac{1 - b + a}{2} + \frac{2uq_1 - u}{2\Delta t} \]

which was substituted into expression for \( q_1 \) to obtain:

\[ q_1 = \int_0^{x_0} dx \int_{\theta(x)}^e \frac{1}{e - d} d\theta + \int_{\theta(x)}^{x_0} dx \int_0^{\theta(x)} \frac{1}{e - d} d\theta \]

\[ q_1 = x_0 + \int_0^{x_0} dx \int_{\theta(x)}^e \frac{1}{e - d} d\theta \]

\[ = \frac{1}{4(e - d)\Delta T \Delta t} \left[ (1 + b - a)\Delta t + \Delta p - d\Delta t \right]^2 + \frac{2uq_1 - u}{e - d} + \frac{2uq_1 - u}{\Delta T(e - d)} + \frac{(2uq_1 - u)^2}{4(e - d)\Delta t \Delta T} \]

The first-order Taylor series expansion of \( \frac{2uq_1 - u}{4(e - d)\Delta T \Delta t} \) to \( (2uq_1 - u) \) was solved:

\[ q_1 = 1 - \frac{\left[ (1 + b - a)\Delta t + \Delta p - d\Delta t \right]^2}{4(e - d)\Delta T \Delta t} + \frac{2uq_1 - u}{\Delta T(e - d)} \]

\[ q_1 = 1 - \frac{\left[ (1 + b - a)\Delta t + \Delta p - d\Delta T \right]^2}{4\Delta T(e - d)\Delta t - u \left[ -d\Delta T + \Delta p + (1 + b - a)\Delta t \right]} \]
(b) Similarly, the following equations were obtained by Equation (5) and Equation (9) when $d \leq \theta(0) \leq \theta(1) \leq e$:

$$d\Delta T + (1 + b - a)\Delta t + u(2q_1 - 1) \leq \Delta p \leq e\Delta T - (1 + b - a)\Delta t + u(2q_1 - 1)$$

$$q_1 = \int_0^e dx \int_{\phi(x)}^{e} \frac{1}{e - d} d\theta = \frac{e\Delta T - \Delta p + (a - b)\Delta t}{(e - d)\Delta T} + \frac{1}{(e - d)\Delta T}(2uq_1 - u)$$

$$q_1 = \frac{e\Delta T - \Delta p + (a - b)\Delta t - u}{(e - d)\Delta T - 2u}$$

(c) One can get the following equations when $d \leq \theta(0) \leq e < \theta(1)$:

$$e\Delta T - (1 + b - a)\Delta t + u(2q_1 - 1) \leq \Delta p \leq e\Delta T + (1 - b + a)\Delta t + u(2q_1 - 1)$$

$$q_1 = \int_0^e dx \int_{\phi(x)}^{e} \frac{1}{e - d} d\theta$$

$$= \frac{1}{e - d} \left[ e - \Delta p + \frac{(1 - b + a)\Delta t}{\Delta T} \right] - \frac{2uq_1 - u + (2uq_1 - u)^2 + \left[ e\Delta T + (1 - b + a)\Delta t - \Delta p \right]^2}{2\Delta t + 4\Delta T(e - d)\Delta t}$$

The following equation was obtained with the same analysis as situation (a):

$$q_1 = \frac{\left[ e\Delta T + (1 - b + a)\Delta t - \Delta p \right]^2 - 2u \left[ e\Delta T - \Delta p + (1 - b + a)\Delta t \right]}{4\left[ (e - d)\Delta T \Delta t - \left[ e\Delta T - \Delta p + (1 - b + a)\Delta t \right]u \right]}$$

$q_1$ is smallest when $d \leq \theta(0) \leq e < \theta(1)$ in the three cases for manufacturer A. Further analysis shows that $q_1$ is largest when $d \leq \theta(0) \leq \theta(1) \leq e$. Similarly, the yield analysis of manufacturer B shows that $q_2 = 1 - q_1$ is largest when $d \leq \theta(0) \leq \theta(1) \leq e$.

The equilibrium solution $(P_1^*, P_2^*)$ can only occur in the second case. Therefore:

$$\pi_1(p_1, p_2) = (p_1 - c_1) \cdot \frac{e\Delta T - \Delta p + (a - b)\Delta t - u}{(e - d)\Delta T - 2u}$$

$$\pi_2(p_1, p_2) = (p_2 - c_2) \cdot \frac{-d\Delta T + \Delta p - (a - b)\Delta t - u}{(e - d)\Delta T - 2u}$$

In conclusion, the following proposition can be obtained through first-order conditions and relations $P_1^* \geq c_1, P_2^* \geq c_2$ between equilibrium price and cost.

**Proposition 2:** When $a + b < 1$, the necessary and sufficient condition for the existence of the equilibrium prices of two manufacturers meeting the model conditions and conducting price competition in the linear city is:
\[(2d - e)\Delta T + (a - b)\Delta t + 3u \leq \Delta c \leq (2e - d)\Delta T + (a - b)\Delta t - 3u\]

Therefore, the equilibrium prices and profits of the manufacturers are:

\[
\begin{align*}
P_1^* &= \frac{(2e - d)\Delta T + (a - b)\Delta t + 2c_1 + c_2 - u}{3} \\
P_2^* &= \frac{(e - 2d)\Delta T + (b - a)\Delta t + c_1 + 2c_2 - u}{3}
\end{align*}
\]

\[
\begin{align*}
\pi_1(P_1^*, P_2^*) &= \left(\frac{(2e - d)\Delta T + (a - b)\Delta t - c_1 + c_2 - 3u}{9(e - d)\Delta T - 2u}\right)^{2} \\
\pi_2(P_1^*, P_2^*) &= \left(\frac{(e - 2d)\Delta T + (b - a)\Delta t + c_1 - c_2 - 3u}{9(e - d)\Delta T - 2u}\right)^{2}
\end{align*}
\]

4. Location Decision of Manufacturers and the Comparative Static Analysis for the Model

The location decision problem of manufacturers was analyzed with \(a + b < 1\):

\[
\begin{align*}
\frac{\partial \pi_1(P_1^*, P_2^*)}{\partial a} &= \frac{2(1 - 2a)\left[\frac{(2e - d)\Delta T + (a - b)\Delta t - c_1 + c_2 - 3u}{9(e - d)\Delta T - 2u}\right]}{9(e - d)\Delta T - 2u} \\
\frac{\partial \pi_2(P_1^*, P_2^*)}{\partial b} &= \frac{2(1 - 2b)\left[\frac{(e - 2d)\Delta T + (b - a)\Delta t + c_1 - c_2 - 3u}{9(e - d)\Delta T - 2u}\right]}{9(e - d)\Delta T - 2u}
\end{align*}
\]

It can be observed that manufacturers have a tendency to concentrate in the urban centre. If two manufacturers minimize the product difference (including the differences caused by the location of manufacturers), decision of the equilibrium location for manufacturers is \(a^* = b^* = \frac{1}{2}\). This shows that two manufacturers meeting the above model conditions with the difference of production cost meeting \((2d - e)\Delta T + 3u \leq \Delta c \leq (2e - d)\Delta T - 3u\) should select the midpoint of a linearly distributed city in the first stage. Then, they conduct price competition in the case of no positional difference. Equilibrium prices and profits of products of two manufacturers were given by Equation (10) and Equation (11) on the basis of first-order conditions and relations \(P_1^* \geq c_1, P_2^* \geq c_2\) between equilibrium prices and costs, respectively.

\[
\frac{\partial P_1^*}{\partial u} = -1, \frac{\partial P_2^*}{\partial u} = -1,
\]

so the equilibrium prices of two products decrease with increase of the network externality coefficient \(u\).

Meanwhile, \(\frac{\partial P_1^*}{\partial \Delta T} = \frac{2e - d}{3}, \frac{\partial P_2^*}{\partial \Delta T} = \frac{e - 2d}{3}\). Therefore, the equilibrium prices of two products increase with the increase of quality difference \(T\) when \(e > 2d\). The equilibrium prices of two products decrease with the increase in quality difference \(T\), when \(e < d / 2\).
\[ \frac{\partial P_1'}{\partial t} = \frac{a-b}{3} - \frac{\partial P_2'}{\partial t} = \frac{b-a}{3} \]

shows that the equilibrium price of product 1 increases with an increase in unit transportation cost \( t \) when \( a > b \), but the equilibrium price of product 2 decreases. Similarly, the equilibrium price of product 1 decreases with an increase in unit transportation cost \( t \), but the equilibrium price of product 2 increases when \( a < b \).

Assuming that \( \Delta c \) is the cost difference between the products, the existence conditions of manufacturers' equilibrium prices show:

\[
\frac{\partial \pi_1}{\partial \Delta c} = -\frac{2[(2e-d)\Delta t + (a-b)\Delta t - c_1 + c_2 - 3u]}{9[e-d]\Delta T - 2u} < 0
\]

\[
\frac{\partial \pi_2}{\partial \Delta c} = \frac{2[(e-2d)\Delta T + (b-a)\Delta t + c_1 - c_2 - 3u]}{9[e-d]\Delta T - 2u} > 0
\]

Therefore, the optimal profit of manufacturer A decreases, and the optimal profit of manufacturer B increases with increase of the cost difference between the products.

5. Conclusion

Network externality was introduced into the Hotelling model with quadratic transport cost. The dynamic game problems of two-stage complete information of two manufacturers were studied. There are three different position relations among the consumers' indifference curve of product selection, quality preferences and population distribution areas. Therefore, the existence conditions of the optimal prices were obtained by traffic-flow distribution technology and infinite dimensional variational inequality. Finally, in addition to the influence of network externality on the equilibrium prices, the effect of differences in quality and cost between two products and unit transportation cost of two manufacturers on the equilibrium prices was analyzed. Therefore, the general conclusions about the Hotelling model are extended.

This issue can be further studied with consumers' uneven distribution in a linearly distributed city and variable network externality coefficient on the basis of this model.

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