The Bijectivity of Mirror Functors on Tori

Kazushi Kobayashi∗

Abstract

By the SYZ construction, a mirror pair (X, ˇX) of a complex torus X and a mirror partner ˇX of the complex torus X is described as the special Lagrangian torus fibrations X → B and ˇX → B on the same base space B. Then, by the SYZ transform, we can construct a simple projectively flat bundle on X from each affine Lagrangian multi section of ˇX → B with a unitary local system along it. However, there are non-unique choices of transition functions of it, and this fact actually causes difficulties when we try to construct a functor between the symplectic geometric category and the complex geometric category. In the present paper, by solving this problem, we prove that there exists a bijection between the set of the isomorphism classes of their objects.

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1 Introduction

Let X be an n-dimensional complex torus, and we denote by ˇX a mirror partner of the complex torus X. For this mirror pair (X, ˇX), the homological mirror

∗Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Yayoicho 1-33, Inage, Chiba, 263-8522 Japan. E-mail : afka9031@chiba-u.jp. 2010 Mathematics Subject Classification : 14J33, 14F05, 53D37. Keywords : torus, homological mirror symmetry, SYZ transform.
symmetry conjecture [11], which is proposed by Kontsevich in 1994, states that there exists an equivalence

$$D^b(Coh(X)) \cong Tr(Fuk(\hat{X}))$$

of triangulated categories. Here, $D^b(Coh(X))$ is the bounded derived category of coherent sheaves on $X$, and $Tr(Fuk(\hat{X}))$ is the derived category of the Fukaya category $Fuk(\hat{X})$ on $\hat{X}$ [4] which is obtained by the Bondal-Kapranov-Kontsevich construction [3], [11]. Historically, first, this conjecture has been studied when $(X, \hat{X})$ is a pair of elliptic curves (see [18], [17], [1] etc.), and after that, the case of abelian varieties has been studied in [5] as a generalization of the case of elliptic curves to the higher dimensional case (see also [12]).

On the other hand, the SYZ construction [19], which is proposed by Strominger, Yau, and Zaslow in 1996, proposes a way of constructing mirror pairs geometrically. By this construction, the mirror pair $(X, \hat{X})$ is realized as the trivial special Lagrangian torus fibrations $\pi : X \to B$ and $\hat{\pi} : \hat{X} \to B$ on the same base space $B$ which is homeomorphic to an $n$-dimensional real torus. Here, for each point $b \in B$, the special Lagrangian torus fibers $\pi^{-1}(b)$ and $\hat{\pi}^{-1}(b)$ are related by the T-duality. In particular, it is expected that the homological mirror symmetry on the mirror pair $(X, \hat{X})$ is realized by the SYZ transform (an analogue of the Fourier-Mukai transform) along the special Lagrangian torus fibers of $\pi : X \to B$ and $\hat{\pi} : \hat{X} \to B$.

Considering the above discussions, we explain the purpose of this paper. For a given mirror pair $(X, \hat{X})$, we regard it as the trivial special Lagrangian torus fibrations $\pi : X \to B$ and $\hat{\pi} : \hat{X} \to B$ in the sense of the SYZ construction. First, in the symplectic geometry side, we consider the Fukaya category $Fuk(\hat{X})$ consisting of affine Lagrangian multi sections of $\hat{\pi} : \hat{X} \to B$ with unitary local systems along them. Then, according to the discussions in [13] and [2], we can obtain a holomorphic vector bundle on $X$ which admits a constant curvature connection from each object of $Fuk(\hat{X})$. This is called the SYZ transform. More precisely, the above constant curvature is expressed locally as

$$dz^t R d\bar{z} \cdot \text{id},$$

(1)

where $z = (z_1, \cdots, z_n)^t$ is the local complex coordinates of $X$, and $R$ is a constant matrix of order $n$ (actually, $R$ is a Hermitian matrix of order $n$). On the other hand, for a holomorphic vector bundle on $X$ with the Hermitian connection, if its curvature form is expressed locally as the form (1), such a holomorphic vector bundle admits a projectively flat structure (for example, see [10]). Therefore, we see that each object of $Fuk(\hat{X})$ is transformed to a projectively flat bundle on $X$, which in particular becomes simple. However, there are non-unique choices of transition functions of it. In this paper, we consider the DG-category $DG_X$ consisting of such simple projectively flat bundles with any compatible transition functions. We expect that this $DG_X$ generates $D^b(Coh(X))$ though we do not discuss it in this paper. At least, it is known that it split-generates $D^b(Coh(X))$ when $X$ is an abelian variety (cf. [16], [1]). In this setting, when we fix a choice of transition functions of holomorphic vector
bundles in $DG_X$, we can obtain a map

$$\iota : \text{Ob}(\text{Fuk}(\check{X})) \rightarrow \text{Ob}(DG_X)$$

by the SYZ transform. Then, for example, it is shown in [5, Proposition 13.2] that the map $\iota$ induces an injection

$$\iota_{\text{isom}} : \text{Ob}_{\text{isom}}(\text{Fuk}(\check{X})) \rightarrow \text{Ob}_{\text{isom}}(DG_X),$$

where $\text{Ob}_{\text{isom}}(DG_X)$ and $\text{Ob}_{\text{isom}}(\text{Fuk}(\check{X}))$ denote the set of the isomorphism classes of holomorphic vector bundles in $DG_X$ and the set of the isomorphism classes of objects of $\text{Fuk}(\check{X})$, respectively. Thus, in the present paper, we prove that the map $\iota_{\text{isom}}$ is actually a bijection by constructing a natural map

$$\text{Ob}(DG_X) \rightarrow \text{Ob}(\text{Fuk}(\check{X}))$$

whose direction is opposite to the direction of the map $\iota$.

This paper is organized as follows. In section 2, we take a complex torus $X$, and explain the definition of a mirror partner $\check{X}$ of the complex torus $X$. In section 3, we define a class of a certain kind of simple projectively flat bundles on $X$, and construct the DG-category $DG_X$ consisting of those holomorphic vector bundles. We also study the isomorphism classes of them in section 3. In section 4, we consider the Fukaya category $\text{Fuk}(\check{X})$ consisting of affine Lagrangian multi sections of $\hat{\pi} : \check{X} \rightarrow B$ with unitary local systems along them, and study the isomorphism classes of objects of $\text{Fuk}(\check{X})$. In section 5, we explicitly construct a bijection $\text{Ob}_{\text{isom}}(DG_X) \rightarrow \text{Ob}_{\text{isom}}(\text{Fuk}(\check{X}))$. This result is given in Theorem 5.1.

2 Preparations

In this section, we define a complex torus $T_{j=T}^{2n}$ and a mirror partner $\check{T}_{j=T}^{2n}$ of the complex torus $T_{j=T}^{2n}$.

First, we define an $n$-dimensional complex torus $T_{j=T}^{2n}$ as follows. Let $T$ be a complex matrix of order $n$ such that $\text{Im}T$ is positive definite. We consider the lattice $2\pi(Z^n \oplus T\mathbb{Z}^n)$ in $\mathbb{C}^n$ and define

$$T_{j=T}^{2n} := \mathbb{C}^n / 2\pi(Z^n \oplus T\mathbb{Z}^n).$$

Sometimes we regard the $n$-dimensional complex torus $T_{j=T}^{2n}$ as a $2n$-dimensional real torus $\mathbb{R}^{2n} / 2\pi\mathbb{Z}^{2n}$. In this paper, we further assume that $T$ is a non-singular matrix. Actually, in our setting described below, it turns out that the mirror partner of the complex torus $T_{j=T}^{2n}$ does not exist if $\text{det}T = 0$. However, we can avoid this problem and discuss the homological mirror symmetry even if $\text{det}T = 0$ by modifying the definition of the mirror partner of the complex torus $T_{j=T}^{2n}$ and a class of holomorphic vector bundles which we treat. This fact will
be discussed in [9]. Here, we fix an \( \varepsilon > 0 \) small enough and let

\[
O_{m_1,\ldots,m_n}^{I_{1},\ldots,I_{n}} := \left\{ \begin{array}{l}
\left( \frac{x}{y} \right) \in T_{j=\tau}^{2n} | \frac{2}{3} \pi (l_j - 1) - \varepsilon < x_j < \frac{2}{3} \pi l_j + \varepsilon, \\
\frac{2}{3} \pi (m_k - 1) - \varepsilon < y_k < \frac{2}{3} \pi m_k + \varepsilon, \quad j, k = 1, \ldots, n
\end{array} \right. \}
\]

be a subset of \( T_{j=\tau}^{2n} \), where \( l_j, m_k = 1, 2, 3, \)

\[\begin{align*}
x := (x_1, \ldots, x_n)^t, \quad y := (y_1, \ldots, y_n)^t,
\end{align*}\]

and we identify \( x_i \sim x_i + 2\pi, \ y_i \sim y_i + 2\pi \) for each \( i = 1, \ldots, n \). Sometimes we denote \( O_{m_1=1,\ldots,m_n}^{I_{1},\ldots,I_{n}} \) instead of \( O_{m_1,\ldots,m_n}^{I_{1},\ldots,I_{n}} \) in order to specify the values \( l_j = l, \ m_k = m \). Then, \( \{O_{m_1=1,\ldots,m_n}^{I_{1},\ldots,I_{n}}\}_{l_j,m_k=1,2,3} \) is an open cover of \( T_{j=\tau}^{2n} \), and we define the local coordinates of \( O_{m_1=1,\ldots,m_n}^{I_{1},\ldots,I_{n}} \) by

\[\begin{align*}
(x_1, \ldots, x_n, y_1, \ldots, y_n)^t \in \mathbb{R}^{2n}.
\end{align*}\]

Furthermore, we locally express the complex coordinates \( z := (z_1, \ldots, z_n)^t \) of \( T_{j=\tau}^{2n} \) by \( z = x + Ty \).

Next, we define a mirror partner of \( T_{j=\tau}^{2n} \). We consider a \( 2n \)-dimensional real torus \( T^{2n} = \mathbb{R}^{2n} / 2\pi \mathbb{Z}^{2n} \), and of course, for each point \( (x^1, \ldots, x^n, y^1, \ldots, y^n)^t \in T^{2n} \), we identify \( x^i \sim x^i + 2\pi, \ y^i \sim y^i + 2\pi \), where \( i = 1, \ldots, n \). We also denote by \( (x^1, \ldots, x^n, y^1, \ldots, y^n)^t \) the local coordinates in the neighborhood of an arbitrary point \( (x^1, \ldots, x^n, y^1, \ldots, y^n)^t \in T^{2n} \). Furthermore, we use the same notation \( (x^1, \ldots, x^n, y^1, \ldots, y^n)^t \) when we denote the coordinates of the covering space \( \mathbb{R}^{2n} \) of \( T^{2n} \). Here, for simplicity, we set

\[\begin{align*}
\tilde{x} := (x^1, \ldots, x^n)^t, \quad \tilde{y} := (y^1, \ldots, y^n)^t.
\end{align*}\]

We define a complexified symplectic form \( \tilde{\omega} \) on \( T^{2n} \) by

\[\begin{align*}
\tilde{\omega} := d\tilde{x}^t (-T^{-1})^t d\tilde{y}.
\end{align*}\]

where \( d\tilde{x} := (dx^1, \ldots, dx^n)^t \) and \( d\tilde{y} := (dy^1, \ldots, dy^n)^t \). We decompose \( \tilde{\omega} \) into

\[\begin{align*}
\tilde{\omega} = d\tilde{x}^t \text{Re}(-T^{-1})^t d\tilde{y} + i d\tilde{x}^t \text{Im}(-T^{-1})^t d\tilde{y},
\end{align*}\]

and define

\[\begin{align*}
\omega := \text{Im}(-T^{-1})^t, \quad B := \text{Re}(-T^{-1})^t.
\end{align*}\]

Here, \( i = \sqrt{-1} \). Sometimes we identify the matrices \( \omega \) and \( B \) with the 2-forms \( d\tilde{x}^t \omega d\tilde{y} \) and \( d\tilde{x}^t Bd\tilde{y} \), respectively. Then, \( \omega \) defines a symplectic form on \( T^{2n} \). The closed 2-form \( B \) is often called the B-field. This complexified symplectic torus \( (T^{2n}, \tilde{\omega} = d\tilde{x}^t (-T^{-1})^td\tilde{y}) \) is a mirror partner of the complex torus \( T_{j=\tau}^{2n} \), so hereafter, for simplicity, we denote

\[\begin{align*}
T_{j=\tau}^{2n} := (T^{2n}, \tilde{\omega} = d\tilde{x}^t (-T^{-1})^t d\tilde{y}).
\end{align*}\]
3 Complex geometry side

The purpose of this section is to explain the complex geometry side in the homological mirror symmetry setting on \((T^2_{J=\mathbb{T}}, T^2_{\bar{J}=\mathbb{T}})}\). In subsection 3.1, we define a class of holomorphic vector bundles

\[ E_{(r, A, r', \mathcal{U}, p, q)} \to T^2_{J=\mathbb{T}}, \]

and construct the DG-category

\[ \text{DG}_{T^2_{J=\mathbb{T}}} \]

consisting of these holomorphic vector bundles \(E_{(r, A, r', \mathcal{U}, p, q)}\). In particular, we first construct \(E_{(r, A, r', \mathcal{U}, p, q)}\) as a complex vector bundle, and then discuss when it becomes a holomorphic vector bundle later in Proposition 3.1. In subsection 3.2, we study the isomorphism classes of holomorphic vector bundles \(E_{(r, A, r', \mathcal{U}, p, q)}\) by using the classification result of simple projectively flat bundles on complex tori by Matsushima [14] and Mukai [15].

3.1 The definition of \(E_{(r, A, r', \mathcal{U}, p, q)}\)

We assume \(r \in \mathbb{N}, A = (a_{ij}) \in M(n; \mathbb{Z})\), and \(p, q \in \mathbb{R}^n\). First, we define \(r' \in \mathbb{N}\) by using a given pair \((r, A) \in \mathbb{N} \times M(n; \mathbb{Z})\) as follows. By the theory of elementary divisors, there exist two matrices \(A, B \in GL(n; \mathbb{Z})\) such that

\[
AA^tB = \begin{pmatrix}
\tilde{a}_1 & \cdots & \tilde{a}_s & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \tilde{a}_s \\
\end{pmatrix},
\]

(2)

where \(\tilde{a}_i \in \mathbb{N} (i = 1, \ldots, s, 1 \leq s \leq n)\) and \(\tilde{a}_i|a_{i+1} (i = 1, \ldots, s - 1)\). Then, we define \(r'_i \in \mathbb{N}\) and \(a'_i \in \mathbb{Z} (i = 1, \ldots, s)\) by

\[
\frac{\tilde{a}_i}{r} = \frac{a'_i}{r'_i}, \quad \gcd(r'_i, a'_i) = 1,
\]

where \(\gcd(m, n) > 0\) denotes the greatest common divisor of \(m, n \in \mathbb{Z}\). By using these, we set

\[
r' := r'_1 \cdots r'_s \in \mathbb{N}.
\]

(3)

This \(r' \in \mathbb{N}\) is uniquely defined by a given pair \((r, A) \in \mathbb{N} \times M(n; \mathbb{Z})\), and it is actually the rank of \(E_{(r, A, r', \mathcal{U}, p, q)}\). Now, we define the transition functions of \(E_{(r, A, r', \mathcal{U}, p, q)}\) as follows (although the following notations are complicated, roughly speaking, the transition functions of \(E_{(r, A, r', \mathcal{U}, p, q)}\) in the cases of \(x_j \mapsto \))
\( x_j + 2\pi, y_k \mapsto y_k + 2\pi \) are given by \( e^{\frac{\pm x_j}{y}V_j, U_k}, \) respectively, where \( j, k = 1, \ldots, n \). Let

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} : O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \to \mathbb{C}', \quad l_j, m_k = 1, 2, 3
\]

be a smooth section of \( E_{(r, A, r', U, p, q)} \). The transition functions of \( E_{(r, A, r', U, p, q)} \) are non-trivial on

\[
O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}, \ldots, O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n},
\]

and otherwise are trivial. We define the transition function on \( O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \) by

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}} = e^{\frac{a_j}{y}V_j - \frac{a_j}{y}V_j} \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}}
\]

where \( a_j := (a_{j1}, \ldots, a_{jn}) \in \mathbb{Z}^n \) and \( V_j \in U(r') \). Similarly, we define the transition function on \( O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \) by

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}} = U_k \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}}
\]

where \( U_k \in U(r') \). In the definitions of these transition functions, actually, we only treat \( V_j, U_k \in U(r') \) which satisfy the cocycle condition, so we explain the cocycle condition below. When we define

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}} = U_k \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}} = \left( U_k \right) \cdot e^{\frac{a_j}{y}V_j} \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \bigg| _{O^{l_1 \cdots l_n}_{m_1 \cdots m_n} \cap O^{l_1 \cdots l_n}_{m_1 \cdots m_n}}
\]

the cocycle condition is expressed as

\[
V_j V_k = V_k V_j, \quad U_j U_k = U_k U_j, \quad \zeta^{-a_j} U_k V_j = V_j U_k,
\]

where \( \zeta \) is the \( r \)-th root of 1 and \( j, k = 1, \ldots, n \). We define a set \( \mathcal{U} \) of unitary matrices by

\[
\mathcal{U} := \left\{ V_j, U_k \in U(r') \mid V_j V_k = V_k V_j, \quad U_j U_k = U_k U_j, \quad \zeta^{-a_j} U_k V_j = V_j U_k, \quad \right. \quad j, k = 1, \ldots, n \right\}.
\]
Of course, how to define the set $\mathcal{U}$ relates closely to (in)decomposability of $E_{(r,A,r',\mathcal{U},p,q)}$. Here, we only treat the set $\mathcal{U}$ such that $E_{(r,A,r',\mathcal{U},p,q)}$ is simple. Actually, we can take such a set $\mathcal{U} \neq \emptyset$ for any $(r,A,r') \in \mathbb{N} \times M(n;\mathbb{Z}) \times \mathbb{N}$, and this fact is discussed in Proposition 3.2. Furthermore, we define a connection $\nabla_{(r,A,r',\mathcal{U},p,q)}$ on $E_{(r,A,r',\mathcal{U},p,q)}$ locally as

$$\nabla_{(r,A,r',\mathcal{U},p,q)} := d - \frac{i}{2\pi} \left( \left( \frac{1}{r} p^t A^t + \frac{1}{r} q^t A^t \right) \right) dy \cdot I_{r'},$$

where $dy := (dy_1, \ldots, dy_n)^t$ and $d$ denotes the exterior derivative. In fact, $\nabla_{(r,A,r',\mathcal{U},p,q)}$ is compatible with the transition functions and so defines a global connection. Then, its curvature form $\Omega_{(r,A,r',\mathcal{U},p,q)}$ is expressed locally as

$$\Omega_{(r,A,r',\mathcal{U},p,q)} = -\frac{i}{2\pi} dx^t A^t dy \cdot I_{r'}, \quad (4)$$

where $dx := (dx_1, \ldots, dx_n)^t$. In particular, this local expression (4) implies that holomorphic vector bundles $E_{(r,A,r',\mathcal{U},p,q)}$ are simple projectively flat bundles (for example, the definition of projectively flat bundles is written in [10]). Moreover, the interpretation of these simple projectively flat bundles $E_{(r,A,r',\mathcal{U},p,q)}$ by using the factors of automorphy is given in section 3 of [8]. Now, we consider the condition such that $E_{(r,A,r',\mathcal{U},p,q)}$ is holomorphic. We see that the following proposition holds.

**Proposition 3.1.** For a given quadruple $(r,A,p,q) \in \mathbb{N} \times M(n;\mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n$, the complex vector bundle $E_{(r,A,r',\mathcal{U},p,q)} \to T^{2n}_T$ is holomorphic if and only if $AT = (AT)^t$ holds.

**Proof.** In general, a complex vector bundle is holomorphic if and only if the $(0,2)$-part of its curvature form vanishes, so we calculate the $(0,2)$-part of $\Omega_{(r,A,\mu,\mathcal{U})}$. It turns out to be

$$\Omega^{(0,2)}_{(r,A,r',\mathcal{U},p,q)} = \frac{i}{2\pi} d\bar{z}^t \{ T(T - T)^{-1} \}^t A^t (T - T)^{-1} d\bar{z} \cdot I_{r'},$$

where $d\bar{z} := (d\bar{z}_1, \ldots, d\bar{z}_n)^t$. Thus, $\Omega^{(0,2)}_{(r,A,r',\mathcal{U},p,q)} = 0$ is equivalent to that $\{ T(T - T)^{-1} \}^t A^t (T - T)^{-1}$ is a symmetric matrix, i.e., $AT = (AT)^t$. \hfill $\square$

Considering the above discussions, here, we mention the simplicity of holomorphic vector bundles $E_{(r,A,r',\mathcal{U},p,q)}$. In general, the following proposition holds.

**Proposition 3.2.** For each quadruple $(r,A,p,q) \in \mathbb{N} \times M(n;\mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n$, we can take a set $\mathcal{U} \neq \emptyset$ such that $E_{(r,A,r',\mathcal{U},p,q)}$ is simple.

**Proof.** For a given pair $(r,A) \in \mathbb{N} \times M(n;\mathbb{Z})$, we can take two matrices $A$, $B \in GL(n;\mathbb{Z})$ which satisfy the relation (2). Then, note that $r' := r'_1 \cdots r'_s \in \mathbb{N}$ is uniquely defined in the sense of the relation (3). We fix such matrices $A$, $B \in GL(n;\mathbb{Z})$, and set

$$T' := B^{-1} T A^t.$$
By using this $T'$, we can consider the complex torus $T_{j=T}^{2n} = \mathbb{C}^n / 2\pi (\mathbb{Z}^n \oplus T' \mathbb{Z}^n)$, and we locally express the complex coordinates $Z := (Z_1, \cdots, Z_n)^t$ of $T_{j=T}^{2n}$ by $Z = X + TY$, where $X := (X_1, \cdots, X_n)^t$, $Y := (Y_1, \cdots, Y_n)^t$. In particular, the complex torus $T_{j=T}^{2n}$ is biholomorphic to the complex torus $T_{j=T}^{2n}$, and the biholomorphic map
\[ \varphi : T_{j=T}^{2n} \rightarrow T_{j=T}^{2n} \]
is actually given by
\[ \varphi(z) = B^{-1}z. \]
Furthermore, when we regard the complex manifolds $T_{j=T}^{2n}$ and $T_{j=T}^{2n}$ as the real differentiable manifolds $\mathbb{R}^{2n} / 2\pi \mathbb{Z}^{2n}$, the biholomorphic map $\varphi$ is regarded as the diffeomorphism
\[ \varphi \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} B^{-1} & O \\ O & (A^{-1})^t \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right). \]
Now, we define a set $U' \neq \emptyset$ as follows. First, we set
\[ V_i := \left( \begin{array}{ccc} 0 & 1 & \cdots \\ \cdots & \cdots & \cdots \\ 1 & 0 & \cdots \end{array} \right) \in U'(r'_i), \quad U_i := \left( \begin{array}{ccc} 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array} \right) \in U'(r'_i), \]
where $\zeta_i := e^{\frac{2\pi i}{r'_i}}$, $i = 1, \cdots, s$. By using these matrices $V_i, U_i$, we define
\[ V'_i := e^{\frac{2\pi i}{r'_i} Y_i} V_i \otimes I_{r'_i} \otimes \cdots \otimes I_{r'_i} \in U'(r'_i), \]
\[ V'_2 := I_{r'_1} \otimes e^{\frac{2\pi i}{r'_2} Y_2} V_i \otimes \cdots \otimes I_{r'_i} \in U'(r'_i), \]
\[ \vdots \]
\[ V'_s := e^{\frac{2\pi i}{r'_s} Y_s} V_i \otimes \cdots \otimes I_{r'_s} \in U'(r'_i), \]
\[ V'_{s+1} := I_{r'_1} \otimes \cdots \otimes I_{r'_s} \in U'(r'_i), \]
\[ \vdots \]
\[ V'_{s+1} := \cdots U_i \in U'(r'_i), \]
\[ U'_1 := \bar{U}_1^{-a'_1} I_{r'_1} \otimes \cdots \otimes I_{r'_s} \in U'(r'_i), \]
\[ U'_2 := I_{r'_1} \otimes \bar{U}_2^{-a'_2} I_{r'_1} \otimes \cdots \otimes I_{r'_s} \in U'(r'_i), \]
\[ \vdots \]
\[ U'_{s+1} := \cdots U_i \in U'(r'_i), \]
\[ \vdots \]
\[ U'_{s+1} := I_{r'_1} \otimes \cdots \otimes I_{r'_s} \in U'(r'_i), \]
\[ \vdots \]

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and set
\[ U' := \{ V'_1, V'_2, \ldots, V'_s, V'_{s+1}, \ldots, V'_n, U'_1, U'_2, \ldots, U'_s, U'_{s+1}, \ldots, U'_n \in U(r') \} \neq \emptyset. \]

Then, we can construct the holomorphic vector bundle
\[ E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \rightarrow T^{2n}_{J=T'}; \]
and in particular, \( V'_j \) and \( U'_k \) are used in the definition of the transition functions of \( E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \) in the \( X_j \) and \( Y_k \) directions, respectively \((j, k = 1, \ldots, n)\). For this holomorphic vector bundle \( E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \), we can consider the pullback bundle \( \varphi^* E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \) by the biholomorphic map \( \varphi \), and we can regard the pullback bundle \( \varphi^* E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \) as the holomorphic vector bundle
\[ E_{(r, A, r', U, p, q)} \rightarrow T^{2n}_{J=T} \]
by using a suitable set \( U \) which is defined by employing the data \((U', A, B)\). In particular, since we can check \( U \neq \emptyset \) easily, in order to prove the statement of this proposition, it is enough to prove that \( E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \) is simple.

Let
\[ \varphi^{r_1 \cdots r'_s} \]
be an element in \( \text{End}(E_{(r, A^{AB}, r', U', A^{P}, B^{q})}) \). Since the rank of \( E_{(r, A^{AB}, r', U', A^{P}, B^{q})} \) is \( r' = r'_1 \cdots r'_s \), we can treat \( \varphi^{r_1 \cdots r'_s} \) as a matrix of order \( r' \). Then, we can divide \( \varphi^{r_1 \cdots r'_s} \) as follows.

\[
\varphi^{r_1 \cdots r'_s} = \begin{pmatrix}
\varphi_{11}^{r_1} & \cdots & \varphi_{1r'_1}^{r_1} \\
\vdots & \ddots & \vdots \\
\varphi_{r'_1}^{r_1} & \cdots & \varphi_{r'_1 r'_1}^{r_1}
\end{pmatrix} = \left( \varphi_{k_1 l_1}^{r_1} \right)_{1 \leq k_1, l_1 \leq r'_1}.
\]

Here, each \( \varphi_{k_1 l_1}^{r_1} \) is a matrix of order \( r'_2 \cdots r'_s \). Similarly, we can divide each \( \varphi_{k_1 l_1}^{r_1} \) as follows.

\[
\varphi_{k_1 l_1}^{r_1} = \begin{pmatrix}
\varphi_{k_1 l_1}^{r_1} & \cdots & \varphi_{k_1 l_1}^{r_1} \\
\vdots & \ddots & \vdots \\
\varphi_{k_2 l_2}^{r_1} & \cdots & \varphi_{k_2 l_2}^{r_1}
\end{pmatrix} = \left( \varphi_{k_2 l_2}^{r_1} \right)_{1 \leq k_2, l_2 \leq r'_2}.
\]

Here, each \( \varphi_{k_2 l_2}^{r_1} \) is a matrix of order \( r'_3 \cdots r'_s \). By repeating the above steps, as a result, we can express \( \varphi^{r_1 \cdots r'_s} \) as
\[
\left( \left( \left( \cdots \left( \varphi_{k_{s-1} l_{s-1}}^{r_1} \right)_{k_{s-1} l_{s-1}} \right)_{k_{s-2} l_{s-2}} \right)_{k_{s-3} l_{s-3}} \cdots \right)_{k_1 l_1}^{r_s}.
\]
where $1 \leq k_i, l_i \leq r'_i \ (i = 1, \cdots, s)$. Hereafter, we consider the local expression of each component

$$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{2,l_2}} \right) \cdots \right)^{r'_s}_{k_{s-1,l_{s-1}}}_{k,l_s}$$

(5)

of $\phi^{r'_1 \cdots r'_s}$. First, for each $k = 1, \cdots, n$, by considering the transition functions of $E_{(r, AAB, r', \partial U', A, B, q)}$ in the $Y_k$ direction, we see that the morphism $\phi^{r'_1 \cdots r'_s}$ must satisfy

$$U'_k \cdot \phi^{r'_1 \cdots r'_s}(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = \phi^{r'_1 \cdots r'_s}(X_1, \cdots, X_n, Y_1, \cdots, Y_k + 2\pi, \cdots, Y_n) \cdot U'_k.$$  

(6)

Furthermore, the morphism $\phi^{r'_1 \cdots r'_s}$ need to satisfy not only the relation (6) but also the Cauchy-Riemann equation

$$\bar{\partial}(\phi^{r'_1 \cdots r'_s}) = 0.$$  

(7)

Therefore, by the relations (6) and (7), we can give a local expression of the component (5) of $\phi^{r'_1 \cdots r'_s}$ as follows.

$$\sum_{l_{k_1 l_1}, \cdots, l_{k_s l_s}, I_{s+1}, \cdots, I^n} \lambda^{l_{k_1 l_1}, \cdots, l_{k_s l_s}, I_{s+1}, \cdots, I^n}_{k_{1,l_1}, \cdots, k_{s,l_s}} \cdot i^{\frac{\pi}{2\pi}(l_k - k)} T^{-1}(X + TY).$$

Here,

$$\lambda^{l_{k_1 l_1}, \cdots, l_{k_s l_s}, I_{s+1}, \cdots, I^n}_{k_{1,l_1}, \cdots, k_{s,l_s}} \in \mathbb{C}$$

is an arbitrary constant. Finally, for each $j = 1, \cdots, n$, we consider the conditions on the transition functions of $E_{(r, AAB, r', \partial U', A, B, q)}$ in the $X_j$ direction. By a direct calculation, if $j = 1, \cdots, s$, we obtain

$$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{j,l_j}} \right) \cdots \right)^{r'_j + 1 \cdots r'_s}_{k_{j+1,l_{j+1}}} (X_1, \cdots, X_j + 2\pi, \cdots, X_n, Y_1, \cdots, Y_n)$$

= $$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{j,l_j}} \right) \cdots \right)^{r'_j + 1 \cdots r'_s}_{(k_j+1)(l_j+1)} (X_1, \cdots, X_n, Y_1, \cdots, Y_n),$$

(8)

and if $j = s + 1, \cdots, n$, we obtain

$$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{s,l_s}} \right) \cdots \right)_{k_{s+1,l_{s+1}}} (X_1, \cdots, X_j + 2\pi, \cdots, X_n, Y_1, \cdots, Y_n)$$

= $$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{s+1,l_{s+1}}}} \right) \cdots \right)_{k_{s+1,l_{s+1}}} (X_1, \cdots, X_n, Y_1, \cdots, Y_n).$$

(9)

In particular, the relation (8) implies

$$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{s,l_s}} \right) \cdots \right)_{k_{s+1,l_{s+1}}} (X_1, \cdots, X_j + 2\pi r'_j, \cdots, X_n, Y_1, \cdots, Y_n)$$

= $$\left( \cdots \left( \frac{\phi_{k_{1,l_1}}^{r'_1 \cdots r'_s}}{k_{s+1,l_{s+1}}}} \right) \cdots \right)_{k_{s+1,l_{s+1}}} (X_1, \cdots, X_n, Y_1, \cdots, Y_n),$$

(10)
and by using the relations (9) and (10), we have the condition
\[
\left( \frac{a'_l}{r'_1} (l_1 - k_1) + I^1_{k_1 t_1}, \ldots, \frac{a'_s}{r'_{s}} (l_s - k_s) + I^s_{k_s t_s}, I^{s+1}, \ldots, I^n \right) T^{l-1}
\in \mathbb{Z} / r'_1 \times \cdots \times \mathbb{Z} / r'_{s} \times \mathbb{Z} \times \cdots \mathbb{Z} \subset \mathbb{R}^n.
\] (11)

Now, since \(\text{Im} T^{l-1}\) is positive definite, the condition (11) turns out to be
\[
\left( \frac{a'_l}{r'_1} (l_1 - k_1) + I^1_{k_1 t_1}, \ldots, \frac{a'_s}{r'_{s}} (l_s - k_s) + I^s_{k_s t_s}, I^{s+1}, \ldots, I^n \right) = 0. \] (12)

Here, recall the relation (8). By the relation (8), it is enough to consider the condition (12) in the case \(k_1 = \cdots = k_s = 1\). We focus on the first component in the condition (12), i.e.,
\[
\frac{a'_l}{r'_1} (l_1 - 1) + I^1_{t_1} = 0. \] (13)

In the case \(l_1 = 1\), the condition (13) turns out to be \(I^1_{t_1} = 0\), so by the relation (8), we see
\[
\varphi_{11}^{r_2^{r_2} \cdots r'_s} = \varphi_{22}^{r_3^{r_3} \cdots r'_s} = \cdots = \varphi_{r'_1 t'_1}^{r'_2 \cdots r'_s} \in M(r_2^{r_2} \cdots r'_s; \mathbb{C}).
\]

We consider the cases \(l_1 = 2, \cdots, r'_1\). Note that \(\gcd(r'_1, a'_1) = 1\) holds by the assumption. Therefore, we have
\[
l_1 - 1 \in r'_1 \mathbb{Z}
\]
by the condition (13). However, this fact contradicts the assumption \(l_1 = 2, \cdots, r'_1\). Thus, for each \(l_1 = 2, \cdots, r'_1\), we obtain
\[
\varphi_{r'_1 t'_1}^{r'_2 \cdots r'_s} = O,
\]
and by using the relation (8) again, we also obtain
\[
\varphi_{2(1+1)}^{r'_2 \cdots r'_s} = \varphi_{3(1+2)}^{r'_3 \cdots r'_s} = \cdots = \varphi_{r'_1 t'_1(1-l)}^{r'_2 \cdots r'_s} = O.
\]

Similarly, by focusing on the second component in the condition (12), we see
\[
\left( \varphi_{11}^{r'_2 \cdots r'_s} \right)_{11}^{r'_3 \cdots r'_s} = \left( \varphi_{11}^{r'_2 \cdots r'_s} \right)_{22}^{r'_3 \cdots r'_s} = \cdots = \left( \varphi_{11}^{r'_2 \cdots r'_s} \right)_{r'_2 r'_2}^{r'_3 \cdots r'_s} \in M(r_3, \cdots, r'_s; \mathbb{C}),
\]
and the other components \(\varphi_{11}^{r'_2 \cdots r'_s} k_{2 l_2}^{r'_3 \cdots r'_s}\) of the matrix \(\varphi_{11}^{r'_2 \cdots r'_s}\) vanish. By repeating the above discussions, as a result, we have
\[
\varphi^{r'_1 \cdots r'_s} = \left( \cdots \left( \left( \varphi_{11}^{r'_1 \cdots r'_s} \right)_{11}^{r'_3 \cdots r'_s} \right)_{11} \cdots \right)_{11} \cdot I'_{r'},
\]
We decompose \( \text{DG} \) together with the wedge product for the anti-holomorphic differential forms. We can check that this linear map is a differential. Furthermore, the product part is denoted \( \nabla \). In particular, the degree \( r \) where the grading is defined as the degree of the anti-holomorphic differential forms is the space of homomorphisms from \( E_\rho \) where \( \Omega \) is the space of anti-holomorphic differential forms, and \( \text{End}(E_{(r,A,B',U',Ap,B')}) \cong \mathbb{C} \).

We define the DG-category \( \text{DG}_{T^2_{j=T}} \) consisting of holomorphic vector bundles \( (E_{(r,A,B',U',Ap,B')}, \nabla_{(r,A,B',U',Ap,B')}) \). This definition is an extension of the case of \( (T^2_{j=T}, T^2_{j=T}) \) to the higher dimensional case in [7] (see section 3), and it is also written in section 2 of [8]. The objects of \( \text{DG}_{T^2_{j=T}} \) are holomorphic vector bundles \( E_{(r,A,B',U',Ap,B')} \) with \( U(r') \)-connections \( \nabla_{(r,A,B',U',Ap,B')} \). Of course, we assume \( AT = (AT)^\ell \). Sometimes we simply denote \( (E_{(r,A,B',U',Ap,B')}, \nabla_{(r,A,B',U',Ap,B')}) \) by \( E_{(r,A,B',U',Ap,B')} \).

For any two objects
\[
\begin{align*}
E_{(s,B,s',V,u,v)} &= (E_{(r,A,r',U,p,q)}, \nabla_{(r,A,r',U,p,q)}), \\
E_{(s,B,s',V,u,v)} &= (E_{(s,B,s',V,u,v)}, \nabla_{(s,B,s',V,u,v)}),
\end{align*}
\]
the space of morphisms is defined by
\[
\begin{align*}
\text{Hom}_{\text{DG}_{T^2_{j=T}}}(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)}) := \Gamma(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)}) \otimes \Omega^{0,\ast}(T^2_{j=T}),
\end{align*}
\]
where \( \Omega^{0,\ast}(T^2_{j=T}) \) is the space of anti-holomorphic differential forms, and
\[
\Gamma(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)})
\]
is the space of homomorphisms from \( E_{(r,A,r',U,p,q)} \) to \( E_{(s,B,s',V,u,v)} \). The space of morphisms \( \text{Hom}_{\text{DG}_{T^2_{j=T}}}(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)}) \) is a \( \mathbb{Z} \)-graded vector space, where the grading is defined as the degree of the anti-holomorphic differential forms. In particular, the degree \( r \) part is denoted
\[
\text{Hom}_{\text{DG}_{T^2_{j=T}}}(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)}).
\]

We decompose \( \nabla_{(r,A,r',U,p,q)} \) into its holomorphic part and anti-holomorphic part \( \nabla_{(r,A,r',U,p,q)} = \nabla^{(1,0)}_{(r,A,r',U,p,q)} + \nabla^{(0,1)}_{(r,A,r',U,p,q)} \), and define a linear map
\[
\begin{align*}
\text{Hom}_{r}^{+1}(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)}) \to \text{Hom}_{r+1}^{+1}(E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)})
\end{align*}
\]
by
\[
\psi \mapsto (2\nabla^{(0,1)}_{(s,B,s',V,u,v)}(\psi) - (-1)^{r}\nabla^{(0,1)}_{(r,A,r',U,p,q)}(\psi)).
\]
We can check that this linear map is a differential. Furthermore, the product structure is defined by the composition of homomorphisms of vector bundles together with the wedge product for the anti-holomorphic differential forms. Then, these differential and product structure satisfy the Leibniz rule. Thus, \( \text{DG}_{T^2_{j=T}} \) forms a DG-category.
3.2 The isomorphism classes of \( E_{(r,A,r',t,p,q)} \)

In this subsection, we fix \( r \in \mathbb{N}, A \in M(n; \mathbb{Z}) \), and consider the condition such that \( E_{(r,A,r',t,p,q)} \cong E_{(r,A,r',t',p',q')} \) holds. Here,

\[
p, q, p', q' \in \mathbb{R}^n
\]

and

\[
\mathcal{U} := \left\{ V_j, U_k \in U(r') \mid V_j V_k = V_k V_j, \quad U_j U_k = U_k U_j, \quad \zeta^{-a_{ij}} U_j V_j = V_j U_k, \right\}
\]

\[
\mathcal{U}' := \left\{ V'_j, U'_k \in U(r') \mid V'_j V'_k = V'_k V'_j, \quad U'_j U'_k = U'_k U'_j, \quad \zeta^{-a_{ij}} U'_j V'_j = V'_j U'_k, \right\}
\]

Furthermore, for each \( j = 1, \ldots, n \), we define \( \xi_j, \theta_j, \xi'_j, \theta'_j \in \mathbb{R} \) by

\[
e^{i\xi_j} = \det V_j, \quad e^{i\theta_j} = \det U_j, \quad e^{i\xi'_j} = \det V'_j, \quad e^{i\theta'_j} = \det U'_j,
\]

and set

\[
\xi := (\xi_1, \ldots, \xi_n)^t, \quad \theta := (\theta_1, \ldots, \theta_n)^t, \quad \xi' := (\xi'_1, \ldots, \xi'_n)^t, \quad \theta' := (\theta'_1, \ldots, \theta'_n)^t \in \mathbb{R}^n.
\]

(14)

Now, in order to consider the condition such that \( E_{(r,A,r',t,p,q)} \cong E_{(r,A,r',t',p',q')} \) holds, we recall the following classification result of simple projectively flat bundles on complex tori by Matsushima and Mukai (see [14, Theorem 6.1], [15, Proposition 6.17 (1)], and note that the notion of semi-homogeneous vector bundles in [15] is equivalent to the notion of projectively flat bundles).

**Theorem 3.3** (Matsushima [14], Mukai [15]). For two simple projectively flat bundles \( E, E' \) over a complex torus \( \mathbb{C}^n/\Gamma \) (\( \Gamma \subset \mathbb{C}^n \) is a lattice) which satisfy \((\text{rank} E, c_1(E)) = (\text{rank} E', c_1(E'))\), there exists a line bundle \( L \in \text{Pic}^0(\mathbb{C}^n/\Gamma) \) such that

\[
E' \cong E \otimes L.
\]

Furthermore, since we are going to use in the proof of Theorem 3.4 (this theorem gives the condition such that \( E_{(r,A,r',t,p,q)} \cong E_{(r,A,r',t',p',q')} \) holds), we again take the biholomorphic map

\[
\varphi : T^{2n}_{\mathbb{Z}} \cong T^{2n}_{\mathbb{Z}}
\]

in the proof of Proposition 3.2. Of course, we can also regard this biholomorphic map \( \varphi \) as the diffeomorphism \( \mathbb{R}^{2n}/2\pi \mathbb{Z}^{2n} \cong \mathbb{R}^{2n}/2\pi \mathbb{Z}^{2n} \) which is expressed locally as

\[
\varphi \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} B^{-1} & O \\ O & (A^{-1})^t \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right).
\]

Then, the following theorem holds.
Theorem 3.4. Two holomorphic vector bundles $E_{(r, A, r', \mathcal{U}, p, q)}$, $E_{(r, A, r', \mathcal{U}', p', q')}$ are isomorphic to each other, $E_{(r, A, r', \mathcal{U}, p, q)} \cong E_{(r, A, r', \mathcal{U}', p', q')}$, if and only if

$$(p + T^t q) - (p' + T^t q') \equiv \frac{r}{r'} (\theta - \theta') + T^t \frac{p}{r'} (\xi' - \xi) \equiv 0 \pmod{2\pi r (A^{-1} \begin{pmatrix} \frac{Z}{r} \\ \vdots \\ \frac{Z}{r} \end{pmatrix} \otimes T^t (B^{-1})^t \begin{pmatrix} \frac{Z}{r} \\ \vdots \\ \frac{Z}{r} \end{pmatrix})}$$

holds.

Proof. First, we consider the condition such that

$$(\varphi^{-1})^* E_{(r, A, r', \mathcal{U}, p, q)} \cong (\varphi^{-1})^* E_{(r, A, r', \mathcal{U}', p', q')},$$

holds. Now, by using the suitable sets $\tilde{U}$ and $\tilde{U}'$ (the definitions of $\tilde{U}$ and $\tilde{U}'$ depend on the data $(\mathcal{U}, A, B)$ and $(\mathcal{U}', A, B)$, respectively), we can regard $(\varphi^{-1})^* E_{(r, A, r', \mathcal{U}, p, q)}$ and $(\varphi^{-1})^* E_{(r, A, r', \mathcal{U}', p', q')}$ as $E_{(r, \tilde{A}, r', \tilde{U}, \tilde{\beta}, \tilde{q})}$ and $E_{(r, \tilde{A}, r', \tilde{U}', \tilde{\beta}', \tilde{q}')}$, respectively, where

$$\tilde{A} := AAB = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_s \\ 0 \\ \vdots \\ \tilde{a}_s \\ 0 \end{pmatrix}, \quad \tilde{p} := Ap, \quad \tilde{q} := B^t q, \quad \tilde{p}' := Ap', \quad \tilde{q}' := B^t q'.$$

Then, constant vectors $\xi, \theta, \xi', \theta' \in \mathbb{R}^n$ are also transformed to constant vectors

$$\tilde{\xi} := B^t \xi, \quad \tilde{\theta} := A\theta, \quad \tilde{\xi}' := B^t \xi', \quad \tilde{\theta}' := A\theta' \in \mathbb{R}^n,$$

respectively. For these holomorphic vector bundles $E_{(r, \tilde{A}, r', \tilde{U}, \tilde{\beta}, \tilde{q})}$ and $E_{(r, \tilde{A}, r', \tilde{U}', \tilde{\beta}', \tilde{q}')}$, by Theorem 3.3, we see that there exists a holomorphic line bundle $E_{(1, O, 1, V, u, v)}$ such that

$$E_{(r, \tilde{A}, r', \tilde{U}', \tilde{\beta}', \tilde{q}') \cong E_{(r, \tilde{A}, r', \tilde{U}, \tilde{\beta}, \tilde{q})} \otimes E_{(1, O, 1, V, u, v)}.$$

Here,

$$\mathcal{V} := \left\{ e^{i\tau_1}, \ldots, e^{i\tau_n}, e^{i\sigma_1}, \ldots, e^{i\sigma_n} \in U(1) \right\}, \quad u, v \in \mathbb{R}^n,$$

and for simplicity, we set

$$\tau := (\tau_1, \ldots, \tau_n)^t, \quad \sigma := (\sigma_1, \ldots, \sigma_n)^t \in \mathbb{R}^n.$$
In particular, for each $j = 1, \cdots, n$, we assume that $e^{i\tau_j}$ and $e^{i\sigma_j}$ are the transition function of $E_{(1,0,1,\nu',u,v)}$ in the $x_j$ direction and the transition function of $E_{(1,0,1,\nu,u,v)}$ in the $y_j$ direction, respectively. Therefore, since

$$E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})} \cong E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})}$$

(15)

holds if and only if

$$E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})} \cong E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})} \otimes E_{(1,0,1,\nu,u,v)}$$

(16)

holds, our first goal is to find the relation on the parameters $u$, $v$, $\tau$, $\sigma \in \mathbb{R}^n$ such that the relation (16) holds. By using Theorem 3.3 again, we see that there exists a holomorphic line bundle $E_{(1,0,1,\nu',u',v')}$ such that

$$E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})} \cong E_{(r,A,r',\tilde{U},\tilde{p},\tilde{q})} \otimes E_{(1,0,1,\nu',u',v')},$$

where

$$\tilde{U}_0 := \left\{ V_1', \cdots, V_n', U_1', \cdots, U_n' \in U(r') \right\}, \quad p_0, q_0 \in \mathbb{R}^n.$$

Here, note that the definitions of $V_1', \cdots, V_n', U_1', \cdots, U_n' \in U(r')$ are given in the proof of Proposition 3.2. Thus, since we can rewrite the relation (16) to

$$E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \otimes E_{(1,0,1,\nu',u',v')} \cong \left( E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \otimes E_{(1,0,1,\nu',u',v')} \right) \otimes E_{(1,0,1,\nu,u,v)},$$

as a result, it is enough to consider the condition such that

$$E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \cong E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \otimes E_{(1,0,1,\nu,u,v)}$$

holds. By a direct calculation, we can actually check that

$$E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \cong E_{(r,A,r',\tilde{U}_0,p_0,q_0)} \otimes E_{(1,0,1,\nu,u,v)}$$

holds if and only if

$$u + T'^{u}v \equiv \sigma - T'^{u} \tau \text{ (mod 2}\pi(\begin{pmatrix} \frac{z}{r_1} \\ \vdots \\ \frac{z}{r_n} \end{pmatrix} \oplus T'^{u} (\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} )) \right) \right)$$

(17)

holds, and in particular, we can regard the relation (17) as

$$(\bar{p} + T'^{u} \tilde{q}) - \{ (\bar{p} + T'^{u} \tilde{q}) + r(u + T'^{u}v) \} \equiv \frac{r}{r'} \{ \bar{\theta} - (\bar{\theta} + r'\sigma) \} + T'^{u} \frac{r}{r'} \{(\bar{\xi} + r'\tau) - \bar{\xi} \}$$

$$\left( \begin{pmatrix} \frac{z}{r_1} \\ \vdots \\ \frac{z}{r_n} \end{pmatrix} \oplus T'^{u} (\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} ) \right).$$

(18)
we can give the condition such that the relation (15) holds as follows.

\[ (\tilde{p} + T^u \tilde{q}) - \{(\tilde{p} + T^u \tilde{q}) + r(u + T^u v)\} \equiv \frac{r}{\nu'}\{\tilde{\theta} - (\tilde{\theta} + r' \sigma)\} + T^u \frac{r}{\nu'}\{\tilde{\xi} + r' \tau - \tilde{\xi}\} \pmod{\frac{r}{\nu'} 2\pi(Z^n + T^u Z^n)} \]

Hence, we see that the relation (16) holds if and only if the relation (18) holds. Here, note that there exists an isomorphism

\[ \det E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \cong \det (E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \otimes E_{(1, O, 1, V, u, v)}) \]

with

\[ (\tilde{p} + T^u \tilde{q}) - \{(\tilde{p} + T^u \tilde{q}) + r(u + T^u v)\} \equiv \frac{r}{\nu'}\{\tilde{\theta} - (\tilde{\theta} + r' \sigma)\} + T^u \frac{r}{\nu'}\{\tilde{\xi} + r' \tau - \tilde{\xi}\} \pmod{\frac{r}{\nu'} 2\pi(Z^n + T^u Z^n)} \]

Now, we consider the condition such that the relation (15) holds. As a necessary condition for the relation (15),

\[ \det E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \cong \det E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q}')} \]

need to hold, and by a direct calculation, we can rewrite the relation (19) to the following.

\[ (\tilde{p} + T^u \tilde{q}) - (\tilde{p}' + T^u \tilde{q}') \equiv \frac{r}{\nu'}\{\tilde{\theta} - \tilde{\theta}'\} + T^u \frac{r}{\nu'}\{\tilde{\xi}' - \tilde{\xi}\} \pmod{\frac{r}{\nu'} 2\pi(Z^n + T^u Z^n)} \]

Therefore, since the definition of \( E_{(1, O, 1, V, u, v)} \) is given by

\[ E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \cong E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q}')} \otimes E_{(1, O, 1, V, u, v)}, \]

and the relations (16) and (18) are equivalent, by considering the relation (20), we can give the condition such that the relation (15) holds as follows.

\[ \begin{pmatrix} \frac{Z}{\nu'} & \frac{Z}{\nu'} \\ \vdots & \vdots \\ \frac{Z}{\nu'} & \frac{Z}{\nu'} \end{pmatrix} \oplus T^u \begin{pmatrix} \frac{Z}{\nu'} & \frac{Z}{\nu'} \\ \vdots & \vdots \\ \frac{Z}{\nu'} & \frac{Z}{\nu'} \end{pmatrix} \pmod{2\pi r A^{-1} (B^{-1})^t} \]

Thus, by considering the pullback bundles \( \varphi^* E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \) and \( \varphi^* E_{(r, A, r', \tilde{A}, \tilde{p}', \tilde{q}')} \), we can conclude that

\[ E_{(r, A, r', \tilde{A}, \tilde{p}, \tilde{q})} \cong E_{(r, A, r', \tilde{A}, \tilde{p}', \tilde{q}')} \]

holds if and only if

\[ \begin{pmatrix} \frac{Z}{\nu'} & \frac{Z}{\nu'} \\ \vdots & \vdots \\ \frac{Z}{\nu'} & \frac{Z}{\nu'} \end{pmatrix} \oplus T^u (B^{-1})^t \begin{pmatrix} \frac{Z}{\nu'} & \frac{Z}{\nu'} \\ \vdots & \vdots \\ \frac{Z}{\nu'} & \frac{Z}{\nu'} \end{pmatrix} \pmod{2\pi r A^{-1} (B^{-1})^t} \]

holds. \( \square \)
Remark 3.5. When we work over a pair $(T_{j=\tau}^2, \check{T}_{j=\tau}^2)$ of elliptic curves, Theorem 3.4 implies that there exists a one-to-one correspondence between the set of the isomorphism classes of holomorphic vector bundles $E_{(r, A, r', s, p, q)}$ and the set of the isomorphism classes of holomorphic line bundles $\det E_{(r, A, r', s, p, q)}$.

4 Symplectic geometry side

In this section, we define the objects of the Fukaya category $\text{Fuk}(\check{T}_{j=\tau}^2)$ corresponding to holomorphic vector bundles $E_{(r, A, r', s, p, q)} \rightarrow T_{j=\tau}^2$, and study the isomorphism classes of them. The discussions in this section are based on the SYZ construction [19] (see also [13], [2]). Note that objects of the Fukaya category are pairs of Lagrangian submanifolds and unitary local systems on them.

4.1 The definition of $L_{(r, A, p, q)}$

In this subsection, we define a class of pairs of affine Lagrangian submanifolds

$$L_{(r, A, p)}$$

in $\check{T}_{j=\tau}^2$ and unitary local systems

$$L_{(r, A, p, q)} \rightarrow L_{(r, A, p)}.$$ 

First, let us consider the following $n$-dimensional submanifold $\check{L}_{(r, A, p)}$ in $\mathbb{R}^{2n}$.

$$\check{L}_{(r, A, p)} := \left\{ \left( \check{x}, \check{y} \right) \in \mathbb{R}^{2n} \mid \check{y} = \frac{1}{r} A \check{x} + \frac{1}{r} p \right\}.$$ 

We see that this $n$-dimensional submanifold $\check{L}_{(r, A, p)}$ becomes a Lagrangian submanifold in $\mathbb{R}^{2n}$ if and only if $\omega A = (\omega A)^t$ holds. Then, for the covering map $\check{\pi} : \check{T}_{j=\tau}^2 \rightarrow \mathbb{R}^n/2\pi \mathbb{Z}^n$, $L_{(r, A, p)} := \check{\pi}(\check{L}_{(r, A, p)})$ defines a Lagrangian submanifold in $\check{T}_{j=\tau}^2$. On the other hand, we can also regard the complexified symplectic torus $\check{T}_{j=\tau}^2$ as the trivial special Lagrangian torus fibration $\check{\pi} : \check{T}_{j=\tau}^2 \rightarrow \mathbb{R}^n/2\pi \mathbb{Z}^n$, where $\check{x}$ is the local coordinates of the base space $\mathbb{R}^n/2\pi \mathbb{Z}^n$ and $\check{y}$ is the local coordinates of the fiber of $\check{\pi} : \check{T}_{j=\tau}^2 \rightarrow \mathbb{R}^n/2\pi \mathbb{Z}^n$. Then, we can regard each affine Lagrangian submanifold $L_{(r, A, p)}$ in $\check{T}_{j=\tau}^2$ as the affine Lagrangian multi section

$$s(\check{x}) = \frac{1}{r} A \check{x} + \frac{1}{r} p$$ 

of $\check{\pi} : \check{T}_{j=\tau}^2 \rightarrow \mathbb{R}^n/2\pi \mathbb{Z}^n$. 

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Remark 4.1. As explained in section 3, although \( r' := r'_1 \cdots r'_s \in \mathbb{N} \) is the rank of \( E_{(r,A',r'',p,q)} \to T^{2n}_{\mathbb{Z}^n} \) (see the relations (2) and (3)), in the symplectic geometry side, this \( r' \in \mathbb{N} \) is interpreted as follows. For the affine Lagrangian submanifold \( L_{(r,A,p)} \) in \( T^{2n}_{\mathbb{Z}^n} \), which is defined by a given data \((r, A, p) \in \mathbb{N} \times M(n; \mathbb{Z}) \times \mathbb{R}^n \), we regard it as the affine Lagrangian multi section \( s(\tilde{x}) = \frac{1}{r} A\tilde{x} + \frac{1}{r} p \) of \( \tilde{\pi} : \tilde{T}^{2n}_{\mathbb{Z}^n} \to \mathbb{R}^n/2\pi \mathbb{Z}^n \). Then, for each point \( \tilde{x} \in \mathbb{R}^n/2\pi \mathbb{Z}^n \), we see

\[
\begin{align*}
 s(\tilde{x}) &= \left\{ \left( \frac{1}{r} A\tilde{x} + \frac{1}{r} p + \frac{2\pi}{r} ABM_s \right) \in \tilde{\pi}^{-1}(\tilde{x}) \approx \mathbb{R}^n/2\pi \mathbb{Z}^n \right\},
 M_s = (m_1, \cdots, m_s, 0, \cdots, 0)^t \in \mathbb{Z}^n, \ 0 \leq m_i \leq r'_i - 1, \ i = 1, \cdots, s
\end{align*}
\]

and this fact indicates that \( s(\tilde{x}) \) consists of \( r' \) points. Thus, we can regard \( r' \in \mathbb{N} \) as the multiplicity of \( s(\tilde{x}) = \frac{1}{r} A\tilde{x} + \frac{1}{r} p \).

Furthermore, we consider the trivial complex line bundle

\[
\mathcal{L}_{(r,A,p,q)} \to L_{(r,A,p)}
\]

with the flat connection

\[
\nabla_{\mathcal{L}_{(r,A,p,q)}} := d - \frac{1}{2\pi r} q^t d\tilde{x}.
\]

Note that \( q \in \mathbb{R}^n \) is the holonomy of \( \mathcal{L}_{(r,A,p,q)} \) along \( L_{(r,A,p)} \approx T^n \). By the definition of the Fukaya category, the relation

\[
\Omega_{\mathcal{L}_{(r,A,p,q)}} = d\tilde{x}^t B d\tilde{y}_{|L_{(r,A,p)}}
\]

need to hold, where \( \Omega_{\mathcal{L}_{(r,A,p,q)}} \) is the curvature form of the flat connection \( \nabla_{\mathcal{L}_{(r,A,p,q)}} \), i.e., \( \Omega_{\mathcal{L}_{(r,A,p,q)}} = 0 \). Hence, we see

\[
d\tilde{x}^t B d\tilde{y}_{|L_{(r,A,p)}} = \frac{1}{r} d\tilde{x}^t BAd\tilde{x} = 0,
\]

so one has \( BA = (BA)^t \). Note that \( \omega A = (\omega A)^t \) and \( BA = (BA)^t \) hold if and only if \( AT = (AT)^t \) holds. Hereafter, for simplicity, we set

\[
\mathcal{L}_{(r,A,p,q)} := (L_{(r,A,p)}, \mathcal{L}_{(r,A,p,q)}).
\]

By summarizing the above discussions, we obtain the following proposition.

**Proposition 4.2.** For a given quadruple \((r,A,p,q) \in \mathbb{N} \times M(n; \mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n\), \( \mathcal{L}_{(r,A,p,q)} \) is an object of the Fukaya category \( \text{Fuk}(T^{2n}_{\mathbb{Z}^n}) \) if and only if \( AT = (AT)^t \) holds.

Here, note that the condition \( AT = (AT)^t \) in Proposition 4.2 is also the condition such that a complex vector bundle \( E_{(r,A',r'',p,q)} \to T^{2n}_{\mathbb{Z}^n} \) becomes a holomorphic vector bundle in Proposition 3.1.
4.2 The isomorphism classes of $\mathcal{L}_{(r,A,p,q)}$

The discussions in this subsection correspond to the discussions in subsection 3.2, so throughout this subsection, we fix $r \in \mathbb{N}$, $A \in M(n; \mathbb{Z})$, and consider the condition such that $\mathcal{L}_{(r,A,p,q)} \cong \mathcal{L}_{(r,A,p',q')}$. holds, where

$$p, q, p', q' \in \mathbb{R}^n.$$

Here, we explain the definition of the equivalency of objects of the Fukaya categories. Let $(M, \omega)$ be a symplectic manifold. We consider two objects $\mathcal{L} := (L, L), \mathcal{L}' := (L', L')$ of the Fukaya category $\text{Fuk}(M, \omega)$, where $L, L'$ are Lagrangian submanifolds in $(M, \omega)$, and $L \to L, L' \to L'$ are unitary local systems. Then, if there exists a symplectic automorphism $\Phi : (M, \omega) \sim \to \mathcal{L}'$ such that

$$\Phi^{-1}(L') = L,$$

$$\Phi^* \mathcal{L}' \cong \mathcal{L},$$

we say that $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, and write $\mathcal{L} \cong \mathcal{L}'$.

Since we considered the complex torus $T^{2n}_{2n}$, which is biholomorphic to the complex torus $T^{2n}_{2n}$ in subsection 3.2, first, we take a mirror partner of the complex torus $T^{2n}_{2n}$. Here, we consider the complexified symplectic torus $\tilde{T}^{2n}_{2n} := T^{2n}_{2n}$ as a mirror partner of the complex torus $T^{2n}_{2n}$. We denote the local coordinates of $\tilde{T}^{2n}_{2n}$ by $(X^1, \cdots, X^n, Y^1, \cdots, Y^n)^t$, and set

$$\tilde{X} := (X^1, \cdots, X^n)^t, \tilde{Y} := (Y^1, \cdots, Y^n)^t.$$

Let us consider a diffeomorphism $\phi : \tilde{T}^{2n}_{2n} \sim \to \tilde{T}^{2n}_{2n}$ which is expressed locally as

$$\phi \left( \begin{array}{c} \tilde{x} \\ \tilde{y} \end{array} \right) = \left( \begin{array}{cc} B^{-1} & O \\ O & A \end{array} \right) \left( \begin{array}{c} \tilde{x} \\ \tilde{y} \end{array} \right).$$

By a direct calculation, we see

$$\phi^*(d\tilde{X}^t(-T^{-1})^td\tilde{Y}) = d\tilde{x}^t(-T^{-1})^td\tilde{y},$$

where $d\tilde{X} := (dX^1, \cdots, dX^n)^t$, $d\tilde{Y} := (dY^1, \cdots, dY^n)^t$, so this diffeomorphism $\phi$ is a symplectomorphism.

Now, we consider the isomorphism classes of $\mathcal{L}_{(r,A,p,q)}$, namely, we consider the condition such that $\mathcal{L}_{(r,A,p,q)} \cong \mathcal{L}_{(r,A,p',q')}$ holds as an analogue of Theorem 3.4. Actually, the following theorem holds.

**Theorem 4.3.** Two objects $\mathcal{L}_{(r,A,p,q)}, \mathcal{L}_{(r,A,p',q')}$ are isomorphic to each other,

$$\mathcal{L}_{(r,A,p,q)} \cong \mathcal{L}_{(r,A,p',q')}.$$
if and only if

\[ p \equiv p' \pmod{2\pi rA^{-1}} \quad \left( \begin{array}{c} \frac{Z}{r_1} \\ \vdots \\ \frac{Z}{r_n} \end{array} \right), \quad q \equiv q' \pmod{2\pi r(B^{-1})^t} \quad \left( \begin{array}{c} \frac{Z}{r_1} \\ \vdots \\ \frac{Z}{r_n} \end{array} \right) \]

hold.

Proof. We define

\[ (\phi^{-1})^*L_{(r,A,p,q)} := ((\phi^{-1})^{-1}(L_{(r,A,p)}), (\phi^{-1})^*L_{(r,A,p,q)}), \]

\[ (\phi^{-1})^*L_{(r,A,p',q')} := ((\phi^{-1})^{-1}(L_{(r,A,p')}), (\phi^{-1})^*L_{(r,A,p',q')}). \]

First, we consider the condition such that

\[ (\phi^{-1})^*L_{(r,A,p,q)} \cong (\phi^{-1})^*L_{(r,A,p',q')} \]

holds, namely, our first goal is to consider when it is possible to construct a symplectic automorphism \( \Phi : \tilde{T}_{2n}^{2n} \to \tilde{T}_{2n}^{2n} \) such that

\[ \Phi^{-1}((\phi^{-1})^{-1}(L_{(r,A,p')})) = (\phi^{-1})^{-1}(L_{(r,A,p)}), \quad (21) \]

\[ \Phi^*(\phi^{-1})^*L_{(r,A,p',q')} \cong (\phi^{-1})^*L_{(r,A,p,q)}. \]

Now, we consider the condition (21). Since \( r \in \mathbb{N}, A \in M(n;\mathbb{Z}) \) are fixed, in the case \( (\phi^{-1})^{-1}(L_{(r,A,p)} = (\phi^{-1})^{-1}(L_{(r,A,p')}) \) only, we can take the map

\[ \Phi = \text{id}_{\tilde{T}_{2n}^{2n}} \]

as a symplectic automorphism \( \Phi : \tilde{T}_{2n}^{2n} \to \tilde{T}_{2n}^{2n} \) which satisfies the condition (21), and

\[ (\phi^{-1})^{-1}(L_{(r,A,p)}) = (\phi^{-1})^{-1}(L_{(r,A,p')}) \]

holds if and only if

\[ \mathcal{A}p \equiv \mathcal{A}p' \pmod{2\pi r} \quad \left( \begin{array}{c} \frac{Z}{r_1} \\ \vdots \\ \frac{Z}{r_n} \end{array} \right) \]

holds. Then, the condition (22) simply becomes

\[ (\phi^{-1})^*L_{(r,A,p,q)} \cong (\phi^{-1})^*L_{(r,A,p',q')}, \]

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so hereafter, we consider when \((\phi^{-1})^*L_{(r,A,p,q)} \cong (\phi^{-1})^*L_{(r,A,p',q')}\) holds on \((\phi^{-1})^{-1}(L_{(r,A,p)}) = (\phi^{-1})^{-1}(L_{(r,A,p')})\) by computing an isomorphism

\[ \psi : (\phi^{-1})^*L_{(r,A,p,q)} \cong (\phi^{-1})^*L_{(r,A,p',q')} \]

explicitly. The morphism \(\psi\) need to satisfy the differential equation

\[ \nabla_{(\phi^{-1})^*L_{(r,A,p',q')}}\psi = \psi \nabla_{(\phi^{-1})^*L_{(r,A,p,q)}}, \tag{24} \]

In particular, since the differential equation (24) turns out to be

\[
\left( \frac{\partial \psi}{\partial X^1}, \cdots, \frac{\partial \psi}{\partial X^n} \right) - \frac{i}{2\pi \tau} ((q'^1B)_1 - (q'B)_1, \cdots, (q'^nB)_n - (q'B)_n) \psi = 0, \tag{25}
\]

by solving the differential equation (25), we obtain a solution

\[ \psi(\tilde{X}) = \lambda e^{\frac{i}{\pi \tau}(q'-q)'B\tilde{X}}, \tag{26} \]

where \(\lambda \in \mathbb{C}\) is an arbitrary constant. Furthermore, since \((\phi^{-1})^*L_{(r,A,p,q)}\) and \((\phi^{-1})^*L_{(r,A,p',q')}\) are trivial, this morphism \(\psi\) must satisfy

\[ \psi(X_1, \cdots, X^i + 2\pi r'_i, \cdots, X^n) = \psi(X_1, \cdots, X^i, \cdots, X^n) \quad (i = 1, \cdots, s), \]
\[ \psi(X_1, \cdots, X^i + 2\pi, \cdots, X^n) = \psi(X_1, \cdots, X^i, \cdots, X^n) \quad (i = s + 1, \cdots, n). \]

By a direct calculation, we see that

\[ \psi(X_1, \cdots, X^i + 2\pi r'_i, \cdots, X^n) = e^{\frac{i}{\pi \tau}(q'-q)'B_i}\psi(X_1, \cdots, X^i, \cdots, X^n) \quad (i = 1, \cdots, s), \]
\[ \psi(X_1, \cdots, X^i + 2\pi, \cdots, X^n) = e^{\frac{i}{\pi \tau}(q'-q)'B_i}\psi(X_1, \cdots, X^i, \cdots, X^n) \quad (i = s + 1, \cdots, n) \]

hold, where \(B_i := (B_{i1}, \cdots, B_{in})^t\), so we obtain

\[ e^{\frac{i}{\pi \tau}(q'-q)'B_i} = 1 \quad (i = 1, \cdots, s), \]
\[ e^{\frac{i}{\pi \tau}(q'-q)'B_i} = 1 \quad (i = s + 1, \cdots, n). \]

Clearly, these relations are equivalent to

\[ B_i(q' - q) = 2\pi \tau N_{r'}, \]

where

\[ N_{r'} := \left( \frac{N_1}{r'_1}, \cdots, \frac{N_s}{r'_s}, N_{s+1}, \cdots, N_n \right)^t, \quad (N_1, \cdots, N_s, N_{s+1}, \cdots, N_n) \in \mathbb{Z}^n, \]

and then, by the formula (26), the isomorphism \(\psi\) is expressed locally as

\[ \psi(\tilde{X}) = \lambda e^{IN_{r'}\tilde{X}} \]
with \( \lambda \neq 0 \in \mathbb{C} \). Hence,
\[
(\phi^{-1})^* \mathcal{L}_{(r,A,p,q)} \cong (\phi^{-1})^* \mathcal{L}_{(r,A,p',q')}
\]
holds on \((\phi^{-1})^{-1}(L_{(r,A,p)}) = (\phi^{-1})^{-1}(L_{(r,A,p')})\) if and only if
\[
B'tq \equiv B'tq' \pmod{2\pi r^{-1}}
\]
holds. Thus, by the relations (23) and (27), we can conclude that
\[
\mathcal{L}_{(r,A,p,q)} \cong \mathcal{L}_{(r,A,p',q')}
\]
holds if and only if
\[
p \equiv p' \pmod{2\pi rA^{-1}}, \quad q \equiv q' \pmod{2\pi r(B^{-1})t'}
\]
hold.

Hence, by comparing Theorem 3.4 with Theorem 4.3, we can expect that the isomorphism classes of holomorphic vector bundles \( E_{(r,A,r',U,p,q)} \rightarrow T_{J=\bar{T}}^{2n} \) correspond to the isomorphism classes of objects \( \mathcal{L}_{(r,A,p,q)} \) of the Fukaya category \( \text{Fuk}(T_{J=\bar{T}}^{2n}) \). Actually, by a direct calculation, we can check that this correspondence is correct. We discuss this fact in section 5.

5 Main result

In this section, we prove that there exists a bijection between the set of the isomorphism classes of holomorphic vector bundles \( E_{(r,A,r',U,p,q)} \rightarrow T_{J=\bar{T}}^{2n} \) and the set of the isomorphism classes of objects \( \mathcal{L}_{(r,A,p,q)} \) of the Fukaya category \( \text{Fuk}(T_{J=\bar{T}}^{2n}) \).

First, we prepare two notations. We denote the set of the isomorphism classes of objects of the DG-category \( \text{DG}_{T_{J=\bar{T}}^{2n}} \) (i.e., the set of the isomorphism classes of holomorphic vector bundles \( E_{(r,A,r',U,p,q)} \)) by
\[
\text{Ob}^{\text{isom}}(\text{DG}_{T_{J=\bar{T}}^{2n}}).
\]
Similarly, we denote the set of the isomorphism classes of objects \( \mathcal{L}_{(r,A,p,q)} \) of the Fukaya category \( \text{Fuk}(T_{j=T}^{2n}) \) by

\[
\text{Ob}^{\text{isom}}(\text{Fuk}(\tilde{T}_{j=T}^{2n})).
\]

Now, in order to state the main theorem, we define a map \( F : \text{Ob}(DG_{T_{j=T}^{2n}}) \rightarrow \text{Ob}(\text{Fuk}(\tilde{T}_{j=T}^{2n})) \) as follows. Clearly, we need four parameters \( r, A, p, q \) when we define objects \( \mathcal{L}_{(r,A,p,q)} \) of \( \text{Fuk}(\tilde{T}_{j=T}^{2n}) \). On the other hand, we need five parameters \( r, A, p, q, \mathcal{U} \) when we define objects \( E_{(r,A,r',\mathcal{U},p,q)} \) of \( DG_{T_{j=T}^{2n}} \). Hence, when we define a map \( \text{Ob}(DG_{T_{j=T}^{2n}}) \rightarrow \text{Ob}(\text{Fuk}(\tilde{T}_{j=T}^{2n})) \), we must transform not only the information about four parameters \( r, A, p, q \) but also the information about \( \mathcal{U} \). For example, let us consider a map \( \text{Ob}(DG_{T_{j=T}^{2n}}) \rightarrow \text{Ob}(\text{Fuk}(\tilde{T}_{j=T}^{2n})) \) which is simply defined by

\[
E_{(r,A,r',\mathcal{U},p,q)} \mapsto \mathcal{L}_{(r,A,p,q)}.
\]

For a quadruple \( (r, A, p, q) \in \mathbb{N} \times M(n; \mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n \) with mutually distinct \( \mathcal{U} \) and \( \mathcal{U}' \), \( E_{(r,A,r',\mathcal{U},p,q)} \) and \( E_{(r,A,r',\mathcal{U}'},p,q) \) are not isomorphic to each other in general. However, the above map sends both \( E_{(r,A,r',\mathcal{U},p,q)} \) and \( E_{(r,A,r',\mathcal{U}',p,q)} \) to the same object \( \mathcal{L}_{(r,A,p,q)} \) unfortunately. Thus, by considering these facts, here, we define a map

\[
F : \text{Ob}(DG_{T_{j=T}^{2n}}) \rightarrow \text{Ob}(\text{Fuk}(\tilde{T}_{j=T}^{2n}))
\]

by

\[
F(E_{(r,A,r',\mathcal{U},p,q)}) = \mathcal{L}_{(r,A,p,r' \varphi + q \xi)},
\]

where \( \xi, \theta \in \mathbb{R}^n \) denote the vectors associated to \( \mathcal{U} \) in the sense of the definition (14).

In the above setting, we present the following theorem which is the main theorem in this paper.

**Theorem 5.1.** The map \( F \) induces a bijection between \( \text{Ob}^{\text{isom}}(DG_{T_{j=T}^{2n}}) \) and \( \text{Ob}^{\text{isom}}(\text{Fuk}(\tilde{T}_{j=T}^{2n})) \).

**Proof.** In this proof, for a given object \( E_{(r,A,r',\mathcal{U},p,q)} \in \text{Ob}(DG_{T_{j=T}^{2n}}) \), we denote by \( \xi, \theta \in \mathbb{R}^n \) the vectors associated to \( \mathcal{U} \) in the sense of the definition (14). Similarly, for a given object \( E_{(s,B,s',\mathcal{V},u,v)} \in \text{Ob}(DG_{T_{j=T}^{2n}}) \), we denote by \( \tau, \sigma \in \mathbb{R}^n \) the vectors associated to \( \mathcal{V} \) in the sense of the definition (14). We denote the induced map from the map \( F \) by

\[
F^{\text{isom}} : \text{Ob}^{\text{isom}}(DG_{T_{j=T}^{2n}}) \rightarrow \text{Ob}^{\text{isom}}(\text{Fuk}(\tilde{T}_{j=T}^{2n})).
\]

Explicitly, it is defined by

\[
F^{\text{isom}}([E_{(r,A,r',\mathcal{U},p,q)}]) = [F(E_{(r,A,r',\mathcal{U},p,q)})],
\]

where, of course, \([E_{(r,A,r',\mathcal{U},p,q)}] \) and \([F(E_{(r,A,r',\mathcal{U},p,q)}]) \) denote the isomorphism class of \( E_{(r,A,r',\mathcal{U},p,q)} \) and the isomorphism class of \( F(E_{(r,A,r',\mathcal{U},p,q)}) \), respectively.
First, we check the well-definedness of the map \( F_{\text{isom}} \). We take two arbitrary objects \( E_{(r,A,r',\mathcal{U},p,q)} \), \( E_{(s,B,s',\mathcal{V},u,v)} \) ∈ \( \text{Ob}(DG_{T_{2g}^2}) \) and assume
\[
E_{(r,A,r',\mathcal{U},p,q)} \cong E_{(s,B,s',\mathcal{V},u,v)}.
\]
By considering the \( i \)-th Chern characters \( \chi_i(E_{(r,A,r',\mathcal{U},p,q)}) \), \( \chi_i(E_{(s,B,s',\mathcal{V},u,v)}) \) of the holomorphic vector bundles \( E_{(r,A,r',\mathcal{U},p,q)} \), \( E_{(s,B,s',\mathcal{V},u,v)} \) for each \( i \in \mathbb{N} \), we see
\[
\chi_i(E_{(r,A,r',\mathcal{U},p,q)}) = \chi_i(E_{(s,B,s',\mathcal{V},u,v)}).
\]
(28)
We consider the equality (28) in the cases \( i = 0, 1 \). Then, we obtain
\[
r' = s',
\]
\[
\frac{r'}{r}A = \frac{s'}{s}B,
\]
so one has
\[
\frac{1}{r}A = \frac{1}{s}B.
\]
Hence, we see that there exists a \( k \in \mathbb{N} \) such that
\[
s = kr, \tag{29}
\]
\[
B = kA. \tag{30}
\]
Therefore, since we can regard \( E_{(s,B,s',\mathcal{V},u,v)} \) as \( E_{(kr,kA,r',\mathcal{V},u,v)} = E_{(r,A,r',\mathcal{V},\frac{r}{r'}u,\frac{r}{r'}v)}, \)
by Theorem 3.4, we see
\[
p - \frac{r}{r'}\theta \equiv \frac{1}{k} \left( u - \frac{s}{s'} \sigma \right) \pmod{2\pi r A^{-1}} \begin{pmatrix} \frac{r}{r'} \mathbb{Z} \\ \vdots \\ \mathbb{Z} \end{pmatrix}, \tag{31}
\]
\[
q + \frac{r}{r'}\xi \equiv \frac{1}{k} \left( v + \frac{s}{s'} \tau \right) \pmod{2\pi r (B^{-1})^t} \begin{pmatrix} \frac{r}{r'} \mathbb{Z} \\ \vdots \\ \mathbb{Z} \end{pmatrix}. \tag{32}
\]
Thus, by Theorem 4.3 and the relations (29), (30), (31), (32), we can conclude
\[
\mathcal{L}(r,A, p - \frac{r}{r'}\theta, q + \frac{r}{r'}\xi) \cong \mathcal{L}(s,B, u - \frac{s}{s'} \sigma, v + \frac{s}{s'} \tau),
\]
namely,
\[
F(E_{(r,A,r',\mathcal{U},p,q)}) \cong F(E_{(s,B,s',\mathcal{V},u,v)}).
\]
Next, we prove that $F_{\text{isom}}$ is injective. We take two arbitrary objects $E_{(r,A,r',U,p,q)}, E_{(s,B,s',V,u,v)} \in \text{Ob}(DG_{T_{J=\tau}^{\mathbb{Z}}})$ and assume

$$F(E_{(r,A,r',U,p,q)}) \cong F(E_{(s,B,s',V,u,v)}),$$

namely,

$$\mathcal{L}_{(r,A,\frac{r-p}{\tau},\frac{s-q}{\tau}+\xi)} \cong \mathcal{L}_{(s,B,\frac{u-s}{\tau},\frac{v-u}{\tau}+\tau)}.$$

Then, we see that there exists a $k \in \mathbb{N}$ which satisfies the relations (29) and (30). Here, we take two matrices $A, B \in GL(n; \mathbb{Z})$ such that

$$AAB = \begin{pmatrix}
\tilde{a}_1 & & \\
& \ddots & \\
& & \tilde{a}_t
\end{pmatrix}, \quad \tilde{a}_t \neq 0 \quad (33)$$

where $\tilde{a}_i \in \mathbb{N}$ ($i = 1, \cdots, t, 1 \leq t \leq n$) and $\tilde{a}_i|a_{i+1}$ ($i = 1, \cdots, t-1$). Therefore, since the relation (30) holds, we obtain

$$ABB = A(kA)B = \begin{pmatrix}
k\tilde{a}_1 & & \\
& \ddots & \\
& & k\tilde{a}_t
\end{pmatrix} \quad (34)$$

In particular, the relations (29), (30), (33), (34) imply

$$r' = s'. \quad (35)$$

Hence, by Theorem 4.3, the relations (31) and (32) hold. Now, note that we can regard $E_{(s,B,s',V,u,v)}$ as $E_{(kr,kA,r',V,u,v)} = E_{(r,A,r',V,\frac{1}{k}u,\frac{1}{k}v)}$ by the relations (29), (30), (35). Thus, by Theorem 3.4 and the relations (31) and (32), we see that

$$E_{(r,A,r',U,p,q)} \cong E_{(r,A,r',V,\frac{1}{k}u,\frac{1}{k}v)}$$

holds, and this relation indicates

$$E_{(r,A,r',U,p,q)} \cong E_{(s,B,s',V,u,v)}.$$  

Finally, we prove that $F_{\text{isom}}$ is surjective. We take an arbitrary quadruple $(r, A, p, q) \in \mathbb{N} \times M(n; \mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n$, and consider the element

$$[\mathcal{L}_{(r,A,p,q)}] \in \text{Ob}_{\text{isom}}(Fuk(T_{J=\tau}^{\mathbb{Z}})).$$
In particular, a representative of \([L_{(r,A,p,q)}]\) is expressed as

\[L_{(r,A,p+2\pi rA^{-1}M,q+2\pi r(B^{-1})^tN)}\]

by using a pair

\[
(M, N) \in \left(\begin{array}{c}
\frac{Z}{r_1} \\
\vdots \\
\frac{Z}{r_2} \\
\frac{Z}{Z}
\end{array}\right) \times \left(\begin{array}{c}
\frac{Z}{r_1} \\
\vdots \\
\frac{Z}{r_2} \\
\frac{Z}{Z}
\end{array}\right)
\]

(see Theorem 4.3). For the element \([L_{(r,A,p,q)}]\), we consider the element

\[E_{(r,A,r',M',p+\frac{\pi}{r}q-\frac{\pi}{r}\xi)} \in \text{Ob}_\text{isom}(DG_{T_{3k}T}),\]

where \(\xi, \theta \in \mathbb{R}^n\) are the vectors associated to \(U\) in the sense of the definition (14). Here, note that how to choose a set \(U\) is not unique even if we fix a quadruple \((r, A, p, q) \in \mathbb{N} \times M(n; \mathbb{Z}) \times \mathbb{R}^n \times \mathbb{R}^n\). Therefore, a representative of \([E_{(r,A,r',M',p+\frac{\pi}{r}q-\frac{\pi}{r}\xi)}]\) is expressed as

\[E_{(r,A,r',M',p+\frac{\pi}{r}q+2\pi rA^{-1}M,q-\frac{\pi}{r}\xi+2\pi r(B^{-1})^tN)}\]

by using a set \(U'\) with the associated vectors \(\xi', \theta' \in \mathbb{R}^n\) and a pair

\[
(M, N) \in \left(\begin{array}{c}
\frac{Z}{r_1} \\
\vdots \\
\frac{Z}{r_2} \\
\frac{Z}{Z}
\end{array}\right) \times \left(\begin{array}{c}
\frac{Z}{r_1} \\
\vdots \\
\frac{Z}{r_2} \\
\frac{Z}{Z}
\end{array}\right)
\]

(see Theorem 3.4). Then, by a direct calculation, we see

\[
F_{\text{isom}}([E_{(r,A,r',M',p+\frac{\pi}{r}q+\frac{\pi}{r}\xi)}])
= F_{\text{isom}}([E_{(r,A,r',M',p+\frac{\pi}{r}q+2\pi rA^{-1}M,q-\frac{\pi}{r}\xi+2\pi r(B^{-1})^tN)}])
= [F(E_{(r,A,r',M',p+\frac{\pi}{r}q+2\pi rA^{-1}M,q-\frac{\pi}{r}\xi+2\pi r(B^{-1})^tN))]
= [L_{(r,A,(p+\frac{\pi}{r}q+2\pi rA^{-1}M)-\frac{\pi}{r}q,(q-\frac{\pi}{r}\xi+2\pi r(B^{-1})^tN))}
= [L_{(r,A,p+2\pi rA^{-1}M,q+2\pi r(B^{-1})^tN)}]
= [L_{(r,A,p,q)}].
\]

This completes the proof.
Acknowledgment

I would like to thank Hiroshige Kajiura for various advices in writing this paper. I am also grateful to Masahiro Futaki and Atsushi Takahashi for helpful comments. This work is supported by Grant-in-Aid for JSPS Research Fellow 18J10909.

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