The Boltzmann equation for colorless plasmons in hot QCD plasma. Semiclassical approximation

Yu.A. Markov * and M.A. Markova *

Institute of System Dynamics and Control Theory Siberian Branch of Academy of Sciences of Russia, P.O. Box 1233, 664033 Irkutsk, Russia

Abstract

Within the framework of the semiclassical approximation, we derive the Boltzmann equation describing the dynamics of colorless plasmons in a hot QCD plasma. The probability of the plasmon-plasmon scattering at the leading order in the coupling constant is obtained. This probability is gauge-independent at least in the class of the covariant and temporal gauges. It is noted that the structure of the scattering kernel possesses important qualitative difference from the corresponding one in the Abelian plasma, in spite of the fact that we focused our study on the colorless soft excitations. It is shown that four-plasmon decay is suppressed by the power of $g$ relative to the process of nonlinear scattering of plasmons by thermal particles at the soft momentum scale. It is stated that the former process becomes important in going to the ultrasoft region of the momentum scale.

PACS: 12.38.Mh, 24.85.+p, 11.15.Kc

Keywords: Quark-gluon plasma; Boltzmann equation; Four-plasmon decay

*e-mail: markov@icc.ru
1 Introduction

For about three decades there has been an increasing interest in theoretical research into various dynamical properties of (ultra)relativistic many-particle systems. It is connected with manifold applications to various problems in astrophysical systems, modern cosmology, in multiparton processes in experiment with high energy heavy ion collisions etc. The kinetic phenomenons, having a purely collective character, are one of the more important aspects of complicated dynamics of many-particle systems under extreme conditions. Here, the basic element in the description of transport phenomena is the derivation of the corresponding kinetic equations which would take into account (depending on the character of the problem being studied) the presence of mean fields in the system, two- (and more) body collisions, the possible renormalization effects, the effects of quantum fluctuations (stochastic), pair production, etc. Here, we restrict our consideration to a brief review of the results of the derivation of relativistic kinetic equations essentially based on two-body collisions in hot gauge theories.

At present there are a few methods of construction of relativistic collision integrals. In particular, we mention the Zubarev’s method of the non-equilibrium statistical operator [1] (use of this method on a relativistic systems of quark-gluon plasma (QGP) type can be found in Ref. [2]), and the method developed by Klimontovich [3] for an ordinary nonrelativistic plasma (the so-called second momentum or polarization approximation) and expanded to relativistic (semi)classical systems - in [4]. It should be stressed that the above-mentioned methods are particularly effective in the construction of collision integrals for relativistic (semi)classical systems, whose evaluation is described by so-called exact ”microscopic” dynamical equations arising from the classical equations of motion. However, the extension of these methods to the relativistic quantum systems encounters some difficulties and therefore is uneffective. For the latter systems, mention may be made of the method based on the use of the reduction formulae of quantum field theory given by de Groot et al [5]. A more powerful and more convenient tool to derive the approximate relativistic kinetic equations from exact field Shwinger-Dyson equations, is the so-called closed-time-path (CTP) formalism [6]. Examples of its relativistic field-theoretical generalization can be found in [7, 8].

As is known, the cornerstone of derivation of the kinetic equations for a hot non-Abelian plasma is a fundamental separation of the momentum scale. The physical justification for such a separation is the fact that the collective excitations which develop at a particular energy scale $gT$ (or $g^2T$, where $T$ is the temperature, and $g$ is the coupling constant), are well separated, when $g \ll 1$, from the typical energies of plasma particles $\sim T$ [9]. Generally speaking one can define two types of kinetic equations: the equations for hard particles - hard quarks($q$), antiquarks($\bar{q}$) and hard transverse gluons($g$), and

---

1For the connection with our previous papers [10] here, we shall follow the definitions of the momentum scales, accepted in [18]: the hard scale, corresponding to momentum of order $T$, the soft scale $\sim gT$, and the ultrasoft scale $\sim g^2T$. 
equation for soft (ultrasoft) collective modes (in the case of Bose excitations - transverse and longitudinal modes, the latter are called by plasmons). Most efforts were directed at the derivation of the first type of equations. However for considerably excited states, when a characteristic time of relaxation of the hard particle distributions $f_s, s = q, \bar{q}, g$ is commensurable with a characteristic time of relaxation of the soft oscillations or even significantly exceeds it, along with kinetic equations for hard particles it is necessary to use the kinetic equation for the soft modes.

The calculation of the collision term for the quark-gluon plasma was apparently first made in [11]. Within the framework of the concepts developed in the theory of electron-ion plasma, the scattering probability of (anti)quarks among themselves through longitudinal and transverse virtual oscillations which account for dynamical screening, is derived. We note that in this paper the Boltzmann equation for hard quarks and antiquarks was supplemented by an equation characterized the relaxation of field excitations and was presented as the second type of kinetic equation mentioned above. In paper [12] a similar Boltzmann equation for hard gluons was obtained. Here, the scattering probability was deduced within the framework of usual diagrammatic perturbation theory with inclusion of screening effects in the random-phase approximation (one-loop order). Although the relativistic Boltzmann equations constructed in these papers take into account such an important QGP property as screening, the range of their validity restricts their use to just colorless deviations from equilibrium distribution functions.

In [4] within the framework of the (semi)classical representation of QGP [13] the Balescu-Lenard-type collision terms for small color and singlet deviations of the distributions from the initial colorless equilibrium, were derived by the Klimontovich method. However, in these papers, the organizing role of the various momentum scales was not recognized, resulting in some inconsistent and complicated transport equations for hard particles.

Only in recent years it been possible to derive the Boltzmann equation for hard modes of hot non-Abelian plasma by a rigorous and self-consistent way using an expansion in the coupling constant, and clearly clarify the nature of the approximation involved, and thus fix its range of applicability. It was be shown that for longer wavelengths ($\lambda \sim 1/g^2T$) of color excitations in the non-Abelian plasma, not only was consideration of interaction of hard particles with the soft degrees of freedom represented by mean fields essential, but also taking into account the collisions of hard particles among themselves. A similar Vlasov-Boltzmann equation reproduces exactly (at leading order in $g$) a large variety of the thermal results obtained by a more fundamental analysis of the diagrammatic perturbation theory [14] and provides in some cases a letter description of phenomena do not yield to a perturbative analysis.

Here, one can extract three approaches to constructing an effective kinetic equation for hard particles with collision term. The first of them is connected with Bödeker’s effective theory for the ultrarelativistic field modes [15]. Starting from the collisionless non-Abelian Vlasov equation, which is the result of integrating out the scale $T$ [9], Bödeker has shown
how one can integrate out the scale $gT$ in an expansion in the gauge coupling $g$. At leading order in $g$, he has obtained the linearized Vlasov-Boltzmann equation for the hard field modes, which besides a collision term also contains a Gaussian noise. Subsequently, this equation was also proposed by Arnold et al [16] who derive the relevant collision term on phenomenological grounds – by analyzing the scattering processes between hard particles in the plasma. The kinetic equation derived in [15, 16] has a non-trivial matrix structure, since, the distribution function that describes color fluctuations is not diagonal in color space.

Afterwards, an alternative derivation of the collision term of Balescu-Lenard-type was proposed by Litim and Manuel, and by Valle Basagoiti [17]. The former authors used a classical transport theory, whereas the latter used the set of ”microscopic” dynamical equations coming from the HTL effective action describing the evolution of the collisionless plasma. In both cases the collision terms were derived by averaging the statistical fluctuations in the plasma on the basis of the method developed by Klimontovich [3].

Blaizot and Iancu [18] suggested a detailed derivation of the Vlasov-Boltzmann equation, starting from the Kadanoff-Baym equations. The derivation is based on the method of gauge covariant gradient expansion, which was first proposed by them for the collective dynamics at the scale $gT$ [9]. By using the given equation they obtain [19] the effective amplitudes for the ultrasoft color fields, which generalize the HTL’s by including the effects of the collisions (see also Guerin [20]).

The paper of Bezzerides and DuBois [21] devoted to the non-thermal QED plasma, is one of the first papers, in which the relativistic kinetic equation for the soft correlation function was considered. Over 60-70 years in connection with application to thermonuclear fusion in the theory of the nonrelativistic electron-ion plasma, a powerful perturbative method (the so-called weak turbulent approximation) was developed [22-24] for research into various nonlinear plasma processes of the following types: on-shell scattering of soft modes on hard particles (the nonlinear Landau damping), three- and four-wave decays, etc. In spite of the fact that this method is not able to describe the phenomena connected with strong turbulence; nevertheless, it enables one within the framework of a unified scheme to encompass a wide class of plasma phenomena. The paper [21] is the attempt at the extending of the above-mentioned weak turbulent theory to the electron-positron-photon plasma governed by the quantum electrodynamics. As a basic tool for such an extension, the CTP-formalism was used. However, since the main effort here was directed into investigating the collision integrals for hard electron and positrons, the authors have restricted their derivation to the plasmon kinetic equation, only taking into account pair production and the (linear) Landau damping process. Properly high-order processes, which we are interested in and are responsible for the nonlinear interaction mechanisms of the plasma waves, were not considered at all (see also the discussion of one-loop computations below).

A similar kinetic equation for soft Bose-modes in a QGP was apparently first made by Heinz et al [25]. In the context of the imaginary time formalism in the one-loop approx-
imation the imaginary part of the complete color linear response function was deduced and it is shown that it can be expressed in the form of the Boltzmann-Nordheim collision term. On the basis of such a derived rate of decay \( \Gamma_d \) and the rate for regeneration of the perturbations \( \Gamma_i \), the kinetic equation defining the evolution of a phase space distribution \( N(x, t; k, \omega) \) of soft electric perturbations of the momentum \( k = (\omega, k) \) in the form proposed by Weldon [26], was written as

\[
\frac{dN}{dt} = -N \Gamma_d + (1 + N) \Gamma_i.
\]

(1.1)

As in a previous case, the higher orders, when \( \Gamma_d \) and \( \Gamma_i \) itself can functionally (in the general case, nonlinearly) depend on \( N \) here, were not considered. However the derivation of such dependence becomes important if we take into account that all computations in [25] ([21]) were performed with the ”rigid” one-loop approximation, with bare propagators of massless gluons and quarks (electrons). However, as is known, quarks (electrons) and gluons inside the loop are not massless, they acquire the effective temperature-induced masses. The consequence of this fact is a kinematic prohibition of a decay of the soft perturbations into physical states. Furthermore by virtue of the fact that the phase velocities of both transverse and longitudinal eigenmodes of the plasma exceed velocity of light, the linear Landau damping is also absent. By virtue of the above-mentioned, the rates of decay and regeneration are just zero in this approximation.

This paper is devoted to further study of the kinetic equation for soft modes of the non-Abelian plasma. The theoretical framework of this paper is derived from synthesis of two formal developments. The first one is the development of a nonlinear theory of plasma wave interactions in ordinary plasma - more exactly the weak turbulent approximation [22-24]. The second is the development of effective theory of hot QCD originally proposed by Braaten and Pisarski [27], Frenkel and Taylor [28], Jackiw and Nair [29], Blaizot and Iancu [9] and then developed in the papers [15-20]. In our previous papers [10] without resorting to a complicated diagrammatic technique within the framework of the semiclassical representations, the following term in the expansions of \( \Gamma_d \) and \( \Gamma_i \), linear on a phase space distribution of soft perturbations, was derived. However, as was shown in [10] (see also section 2), this approximation is not sufficient for a complete definition of the relaxation process of the soft modes in QGP. Here, we consider the next terms in the expansions of \( \Gamma_d \) and \( \Gamma_i \), and show that the corresponding nonlinear equation (1.1) is of purely Boltzmann type, i.e. the collision term on the r.h.s. of this equation has a standard Boltzmann structure, with a gain term and a loss term.

The outline of the paper is as follows. In section 2 the preliminary comments, with regard to derivation of the Boltzmann equation, describing the plasmon-plasmon scattering are explained. In section 3 the essential features of the scheme, which we used previously in [10] to derive the kinetic equation with allowance for the nonlinear Landau damping, are summarized. In section 4 we discuss the consistency with gauge symmetry of the approximation scheme used. Section 5 is devoted to the determination of the interacting fields in the form of the expansion in free fields with the necessary accuracy for further
research. In section 6 we select all terms in the expansion of the color random current responsible for the four-plasmon decay and derive the intermediate kinetic equation which then in section 7 will be rewritten in the terms of HTL-amplitudes. Section 8 is devoted to deriving the probability of plasmon-plasmon scattering, which is the main result of this work. In the next section on the basis of the explicit form of the obtained collision integral, the expression for lifetimes of colorless plasmons is defined and an estimate for the leading order in the coupling at the soft momentum scale is deduced. Finally in section 10 we present our conclusions and future avenues of study.

2 Preliminary comments

We denote the localized number density of the plasmons by $N_l(k, x) \equiv N^l_k$, and the distribution function of hard thermal gluons by $f_p(p, x) \equiv f_p$. In this paper we consider processes with longitudinal oscillations only, propagating in a purely gluonic plasma, with no quarks. Besides, we suppose that there is no external color current and/or mean color field in the system, and the system is in the global equilibrium state, i.e.

$$f_{pa}^b = \delta^a_b f_p \equiv \delta^a_b 2 \frac{1}{e^{E_p/T} - 1}. \quad (2.1)$$

Here, $E_p \equiv |p|$ for a massless hard gluon, the coefficient 2 takes into account that the hard gluon has two helicity states and $a, b = 1, \ldots, N_c^2 - 1$ for the $SU(N_c)$ gauge group. The triviality of the color structure of the plasmon number density is a consequence of these restrictions (see section 4)

$$N_{k}^{l \, \alpha \beta} = \delta^{\alpha \beta} N^l_k.$$  

The dispersion relation for plasmons $\omega = \omega^l(k) \equiv \omega_k^l$ is defined from

$$\text{Re} \, \varepsilon_l^l(\omega, k) = 0, \quad (2.2)$$

where

$$\varepsilon_l^l(\omega, k) = 1 + \frac{3\omega_{pl}^2}{k^2} \left[1 - F\left(\frac{\omega}{|k|}\right)\right], \quad F(x) \equiv \frac{x}{2} \left[\ln \left|\frac{1 + x}{1 - x}\right| - i\pi \theta(1 - |x|)\right] \quad (2.3)$$

is longitudinal color permeability and $\omega_{pl}^2 = g^2 N_c T^2 / 9$ is a plasma frequency.

We expect the time-space evolution of $N^l_k$ to be described by

$$\frac{\partial N^l_k}{\partial t} + V^l_k \frac{\partial N^l_k}{\partial x} = -N^l_k \Gamma_d[N^l_k] + (1 + N^l_k) \Gamma_i N^l_k, \quad (2.4)$$

where $V^l_k = \partial \omega^l_k / \partial k$ is the group velocity of the longitudinal oscillations. For a generalized decay rate $\Gamma_d$ and inverse decay rate $\Gamma_i$ it is shown in an explicit form that in general

\[\text{Notice, that in the general case, when the distribution function of hard gluons is a slowly varying function in time and space, equation (2.2) is replaced by Re} \, \varepsilon_l^l(\omega, k; t, x) = 0. \text{In this case the l.h.s. of equation (2.4) should be supplemented by} (\partial \omega^l(k, x) / \partial x)(\partial N^l(k, x) / \partial k).\]
case they are functionals dependent on the plasmon number density. Although the approach we shall use in the subsequent discussion is correct only within the framework of semiclassical approximation, it is convenient to interpret the terms entering into $\Gamma_d$ and $\Gamma_i$, using a quantum language.

The Eq. (2.4) in general, describes two principal processes of the nonlinear wave-interaction. The first of them represents the process of the stimulated emission and absorption of the collective wave quanta by hard particles of plasma. In this case the more general expression for the decay rate $\Gamma_i^{(S)}$ and the regeneration rate $\Gamma_i^{(S)}$ can be written in the following forms, respectively:

$$\Gamma_d^{(S)}[N^l_k] = \sum_{n,m} \int \frac{dp}{(2\pi)^3} \int dT_{nm}^{(S)} w(p|k_1, \ldots, k_n; k'_1, \ldots, k'_m) N^l_{k_1} \ldots N^l_{k_n} \times (1 + N^l_{k'_1}) \ldots (1 + N^l_{k'_m}) f_p [1 + f_{p'}],$$

(2.5)

and

$$\Gamma_i^{(S)}[N^l_k] = \sum_{n,m} \int \frac{dp}{(2\pi)^3} \int dT_{nm}^{(S)} w(p'|k'_1, \ldots, k'_m; k, k_1, \ldots, k_n) N^l_{k'_1} \ldots N^l_{k'_m} \times (1 + N^l_{k_1}) \ldots (1 + N^l_{k_n}) f_{p'} [1 + f_p].$$

Here, the phase-space integration is

$$\int dT_{nm}^{(S)} = \int \frac{dk_1}{(2\pi)^3} \ldots \frac{dk_n}{(2\pi)^3} \frac{dk'_1}{(2\pi)^3} \ldots \frac{dk'_m}{(2\pi)^3}$$

(2.7)

$$\times (2\pi) \delta(E_p + \omega^{k_1}_1 + \ldots + \omega^{k_n}_n - E_{p'} - \omega^{k'_1}_1 - \ldots - \omega^{k'_m}_m),$$

with the delta function expressing the energy conservation of the processes of stimulated emission and absorption of the plasmons. The function $w(p|k_1, \ldots, k_n; k'_1, \ldots, k'_m)$ is the probability of absorption of $n + 1$ plasmons with the frequencies $\omega^{k_1}_1, \ldots, \omega^{k_n}_n$ and the wavevectors $k_1, \ldots, k_n$ by a hard gluon carrying of momentum $p$ with consequent radiation of $m$ plasmons with frequencies $\omega^{k'_1}_1, \ldots, \omega^{k'_m}_m$ and the wavevectors $k'_1, \ldots, k'_m$. The function $w(p'|k'_1, \ldots, k'_m; k, k_1, \ldots, k_n)$ is the probability of inverse process - the absorption of $m$ plasmons by a hard gluon with the momentum $p' \equiv p + k_1 + \ldots + k_n - k'_1 - \ldots - k'_m$ with consequent radiation of $n + 1$ plasmons. Diagrammatically this corresponds to a Feynman graph with two hard external lines and an arbitrary number of $n + m + 1$ soft external lines. By using the fact that $|p| \gg |k_1|, |k_1|, \ldots, |k_n|, |k'_1|, \ldots, |k'_m|$, the energy conservation law can be represented in the form of following "generalized" resonance condition:

$$\omega^{k_1}_1 + \ldots + \omega^{k_n}_n - \omega^{k'_1}_1 - \ldots - \omega^{k'_m}_m - v(k + k_1 + \ldots + k_n - k'_1 - \ldots - k'_m) = 0,$$

(2.8)

where $v \equiv p/|p|$. Furthermore, one can approximate the distribution function of hard gluons on the r.h.s. of equations (2.5) and (2.6),

$$f_{p'} \simeq f_p + (k + k_1 + \ldots + k_n - k'_1 - \ldots - k'_m) \frac{\partial f_p}{\partial p},$$
and set $1 + f_p \simeq 1 + f_{p'} \simeq 1$ by virtue of $f_p, f_{p'} \ll 1$.

The r.h.s. of (2.3) and (2.6) can be formally considered as an expansions of $\Gamma_d^{(S)}$ and $\Gamma_i^{(S)}$ in the functional series in powers of the plasmon number density. The actual dimensionless parameter of expansion here, is (for classical statistic) the ratio of the energy of longitudinal plasma excitations to the averaged thermal energy per particle, i.e.

$$\varepsilon = \left( \int \frac{d k}{(2\pi)^3} \omega_k \rho_k \right) / \left( \bar{n} \int \frac{d p}{(2\pi)^3} E_p f_p \right),$$

where $\bar{n}$ is the mean density. In conditions, when the excitations energy is a small quantity compared with the thermal energy of hard particles, we have

$$\varepsilon \ll 1. \quad (2.9)$$

The last inequality means that the fields of longitudinal oscillations are sufficiently small and they cannot essentially change such “crude” equilibrium parameters of a plasma as particles density, temperature and thermal energy (this, in particular, justifies the choice of the distribution function of thermal gluons in the form of (2.1)).

On the other hand, however, we shall consider the energy of the plasma oscillations to be sufficiently large, i.e. greatly exceeding the energy of thermal fluctuations of the the color field in the plasma. The consequence of the last requirement is the inequality

$$\varepsilon \gg \delta, \quad (2.10)$$

where $\delta$ is the plasma parameter

$$\delta = \frac{\bar{r}^3}{r_D^2} \ll 1.$$

Here, $\bar{r}$ is the inter-particle distance ($\sim \bar{n}^{-1/3}$), and $r_D$ is the Debye length

$$r_D^2 = \frac{T}{4\pi \bar{n} g^2 N_c}.$$

The condition (2.10) demonstrates the validity of ignoring the hard gluon collisions among themselves relative to their interactions with soft plasma modes.

Inequalities (2.9) and (2.10) correspond to the weak turbulent approximation, within the framework of which one can restrict the consideration to several first terms in a functional expansions of $\Gamma_d^{(S)}$ and $\Gamma_i^{(S)}$. We note however, that when the energy level of the plasma excitations becomes comparable with the thermal energy of particles, e.g. as the result of development of a strong instability (strong turbulence), the perturbation theory here, is no longer applicable and the problem of summation of all the relevant contributions thus appears. The last situation can be really take place in the processes proceed in QGP emerging from the heavy ion collisions at higher energies. In this work we do not consider this very complicated problem, assuming that the inequality (2.9) is always fulfilled.
Let us discuss in more detail the first terms in the expansions of \( \Gamma_d^{(S)} \) and \( \Gamma_i^{(S)} \). For \( n = m = 0 \), the equation (2.8) results in the relation
\[
\omega_k^I - v k = 0,
\]
which is well-known as the Cherenkov resonance condition, which does not hold in the gluon plasma. Therefore \( \Gamma_\alpha^{(S)} = \Gamma_i^{(S)} = 0 \), when only \( O(\varepsilon^0) \) terms on the r.h.s. of (2.5) and (2.6) are retained.

For \( n = 1, m = 0 \) we have
\[
\omega_k^I + \omega_{k_1}^I - v (k + k_1) = 0,
\]
and for \( n = 0, m = 1 \), respectively (here, we replace \( k'_1 \) by \( k_1 \))
\[
\omega_k^I - \omega_{k_1}^I - v (k - k_1) = 0.
\]
The first resonance condition (2.11) describes the simultaneous radiation (or absorption) of two plasmons with frequencies \( \omega_k^I, \omega_{k_1}^I \) and wavevectors \( k, k_1 \). By virtue of the fact that the phase velocity of the longitudinal oscillations exceeds the velocity of light, this process is kinematically forbidden. The first non-trivial terms in the expansion of the functionals \( \Gamma_d^{(S)} \) and \( \Gamma_i^{(S)} \) are defined by the second resonance condition (2.12). It is associated with the absorption of the plasmon by a hard gluon with frequency \( \omega_k^I \) and wavevector \( k \) with its consequent radiation with frequency \( \omega_{k_1}^I \) and wavevector \( k_1 \) (and vice versa). Schematically this process can be represented as follows:
\[
g^* + g \leftrightarrow g^* + g,
\]
where \( g^* \) are the plasmon collective excitations and \( g \) are excitations with characteristic momenta of order \( T \). In the theory of the ordinary plasma [22-24] this process is known as the nonlinear Landau damping. In the case of QGP it was studied in detail in [10]. We have shown, that the nonlinear Landau damping rate
\[
\gamma_l^I (k) \equiv (\Gamma_d^{(S)} [N_k^I] - \Gamma_i^{(S)} [N_k^I])|_{n=0, m=1}
\]
defines two processes: the effective pumping of energy across the spectrum towards small wavenumbers with the conservation of excitation energy and properly nonlinear dissipation (damping) of the longitudinal plasma waves by hard particles, where the first process is crucial. The consequence of this fact is the inequality: \( \gamma_l^I (0) < 0 \), i.e. \( k = 0 \) - mode is increased. The main conclusion, which we drew in [10] is that the only process of the nonlinear Landau damping does not lead to the total relaxation of soft excitations in the homogogeneous isotropic plasma. At the scale of a small \( |k| \) (\( |k| \ll gT \)) it is necessary to consider the processes of higher-order in \( \varepsilon \), than (2.13), which lead to the suppression of increase of the \( k = 0 \)-mode.

The following terms in the expansion of the decay rate (2.5) and the inverse decay rate (2.6), corresponding to \( n, m = 1, 2 \), are defined by
\[
\omega_k^I \pm \omega_{k_1}^I \pm \omega_{k_2}^I - v (k \pm k_1 \pm k_2) = 0,
\]
\[ \omega_k^l \pm \omega_{k_1}^l \mp \omega_{k_2}^l - \mathbf{v}(\mathbf{k} \pm \mathbf{k}_1 \mp \mathbf{k}_2) = 0. \]

Physically this corresponds to simultaneous absorption (radiation) of three plasmons by a thermal gluon, or simultaneous absorption (radiation) of two plasmons with consequent radiation (absorption) of one plasmon. At the long-wavelength range these processes are kinematically forbidden and therefore in this approximation \( \Gamma_d^{(S)} \) and \( \Gamma_i^{(S)} \) vanish.

However there are contributions different from zero in the expansions of the generalized decay rate \( \Gamma_d \) and the inverse \( \Gamma_i \) in the equation (2.4) which are of the same order as the processes (2.15), i.e. of order \( O(\varepsilon^2) \). These contributions are concerned with the second type of the nonlinear processes defining the time-space evolution of \( N_k^l \) and going without exchange of energy between hard thermal gluons and plasmons. They represent the processes of decays, fusions of plasmons and their scattering off each other. Diagrammatically, this corresponds to Feynman graph, where all external lines are soft. The relevant decay rate \( \Gamma_d^{(P)} \) and regenerating rate \( \Gamma_i^{(P)} \) can be formally represented in the form

\[
\Gamma_d^{(P)}[N_k^l] = \sum_{n,m} \int d\mathcal{T}_{nm}^{(P)} w(k, k_1, \ldots, k_n; k_1', \ldots, k_m') N_{k_1}^l \cdots N_{k_n}^l (1 + N_{k_1}^l) \cdots (1 + N_{k_m}^l),
\]

and

\[
\Gamma_i^{(P)}[N_k^l] = \sum_{n,m} \int d\mathcal{T}_{nm}^{(P)} w(k_1', \ldots, k_m'; k, k_1, \ldots, k_n) N_{k_1'}^l \cdots N_{k_m'}^l (1 + N_{k_1}^l) \cdots (1 + N_{k_n}^l).
\]

Here, the phase-space measure is

\[
\int d\mathcal{T}_{nm}^{(P)} = \int \frac{dk_1}{(2\pi)^3} \cdots \frac{dk_n}{(2\pi)^3} \frac{dk_1'}{2\pi} \cdots \frac{dk_m'}{2\pi} (2\pi)^4 \delta^{(4)}(k + k_1 + \ldots + k_n - k_1' - \ldots - k_m'),
\]

with the delta functions expressing the energy and the momentum conservation of the decay processes, the fusion and the plasmon scattering among themselves. The decay rate (2.16) and the regenerating rate (2.17) are different from zero if the following ”resonance” conditions are obeyed

\[
\omega_k^l + \omega_{k_1}^l + \ldots + \omega_{k_n}^l - \omega_{k_1'}^l - \ldots - \omega_{k_m'}^l = 0,
\]

\[
\mathbf{k} + \mathbf{k}_1 + \ldots + \mathbf{k}_n - \mathbf{k}_1' - \ldots - \mathbf{k}_m' = 0.
\]

The first contribution is different from zero in \( \Gamma_d^{(P)} \) and \( \Gamma_i^{(P)} \), which would arise in the case of \( n = m = 1, 2 \) as defined by the system of equations

\[
\omega_k^l \pm \omega_{k_1}^l = 0, \quad \mathbf{k} \pm \mathbf{k}_1 = 0.
\]

These conservation laws describe a decay of one plasmon into two plasmons and the reverse process of fusion of two plasmons into one plasmon,

\[
\mathbf{g}^* \rightleftharpoons \mathbf{g}_1^* + \mathbf{g}_2^*.
\]
Three-plasmon decay (2.20) is as important as the process of nonlinear Landau damping (2.13). However, the specific peculiarity of a dispersion law of the longitudinal oscillations in hot non-Abelian plasma is that resonance equations (2.19) have no solutions no matter what the values of wavevectors $k, k_1$ and $k_2$ may be. The processes (2.20) are kinematically forbidden by the conservation laws, and it makes a contribution only in the second order over parameter $\varepsilon$ of perturbation theory, for $n, m = 1, 2, 3$.

In general case, with four plasmons, two different processes occur

$$g^* \leftrightarrow g_1^* + g_2^* + g_3^*, \quad (2.21)$$

$$g^* + g_1^* \leftrightarrow g_2^* + g_3^*. \quad (2.22)$$

The first of them corresponds to the process of the decay of one plasmon $g^*$ into three plasmons $g_1^*, g_2^*, g_3^*$, and the reverse process of fusion of three plasmons into one plasmon $g^*$. The second process presents the plasmon scattering by plasmon. The last one is considered as the decay process and is interpreted as the process of the decay (fusion) of two plasmons $g^*$ and $g_1^*$ into two plasmons $g_2^*$ and $g_3^*$. For four-plasmon decay process (2.21), the following resonance conditions are obeyed:

$$\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \omega^l_{\mathbf{k}_3} = 0, \quad (2.23)$$

$$\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0,$$

and for the scattering process (2.22) we have, respectively,

$$\omega^l_{\mathbf{k}} + \omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \omega^l_{\mathbf{k}_3} = 0, \quad (2.24)$$

$$\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0.$$

It is not difficult to show that the conservation laws (2.23) and (2.24) kinematically forbid the processes of the direct decay of the plasmon into three (and vice versa) (2.21) and admit only the processes of (2.22) type. As shown in section 9, at the soft scale the latter process is suppressed by a power of $g$ relative to the process of nonlinear Landau damping (2.13). However at the ultrasoft scale, one would expect that the process of the elastic scattering of the plasmon by a plasmon may be as larger as the process (2.13) and thus plays an important role in the kinetics of the plasmons at larger wavelength (see the end of section 9).

Putting the expressions (2.16) and (2.17) into equation (2.4), where only $O(\varepsilon^2)$ relevant terms in the expansions are retained, we result in the Boltzmann equation, describing four-plasmon decays of (2.22) type

$$\frac{\partial N^l_{\mathbf{k}}}{\partial t} + V^l_{\mathbf{k}} \frac{\partial N^l_{\mathbf{k}}}{\partial \mathbf{x}} =$$

$$= \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{d\mathbf{k}_3}{(2\pi)^3} (2\pi)^4 \delta(\omega^l_{\mathbf{k}} + \omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \omega^l_{\mathbf{k}_3}) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$$

11
\[ \times w(k, k_1; k_2, k_3) \{ N_{k_2}^l N_{k_3}^l (1 + N_{k_2}^l) (1 + N_{k_1}^l) - N_{k_1}^l N_{k_1}^l (1 + N_{k_2}^l) (1 + N_{k_3}^l) \} \]

\[ \approx \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} (2\pi)^4 \delta(\omega_k^l + \omega_{k_1}^l - \omega_{k_2}^l - \omega_{k_3}^l) \delta(k + k_1 - k_2 - k_3) \times w(k, k_1; k_2, k_3) \{ N_{k_1}^l N_{k_2}^l N_{k_3}^l + N_{k_1}^l N_{k_2}^l N_{k_3}^l - N_{k_1}^l N_{k_1}^l N_{k_2}^l - N_{k_1}^l N_{k_1}^l N_{k_3}^l \}. \]

In writing this equation, we have used the fact that the probabilities of direct and reverse processes are equal and besides in the last line in the semiclassical regime we consider the soft modes to be strongly populated, i.e. \((1 + N_{k}^l)(1 + N_{k_1}^l) \approx N_{k}^l N_{k_1}^l + N_k^l + N_{k_1}^l\) etc. A similar Boltzmann equation for plasmons was studied intensively for the ordinary plasma [30, 31] (in [24, 30] the explicit expression for the function \(w(k, k_1; k_2, k_3)\) can be found).

The main purpose of this work is the derivation in the explicit form of the probability of plasmon-plasmon scattering for hot non-Abelian plasma. The function \(w(k, k_1; k_2, k_3)\) must satisfy the symmetry relations over permutation of arguments

\[ w(k, k_1; k_2, k_3) = w(k_2, k_3; k_1) = w(k, k_1; k_3, k_2) = w(k_1, k; k_2, k_3), \quad (2.26) \]

which are the consequence of the indistinguishable of the plasmons (recall that here, we discuss only colorless plasmons, i.e. those for which the occupation number does not carry adjoint color indices).

In conclusion of this section we note some general properties of the processes of four-plasmon decays. Multiplying the r.h.s. of Eq. (2.25) in turn by \(\omega_k^l\) and \(k\), integrating with respect to the wavevector \(k\), and taking into account (2.26), it is easily checked that

\[ E \equiv \int \frac{dk}{(2\pi)^3} \omega_k^l N_k^l = \text{const}, \quad K \equiv \int \frac{dk}{(2\pi)^3} k N_k^l = \text{const}. \]

These relations are evident a consequence of the conservation laws of energy and momentum in the plasmon-plasmon scattering.

On the other hand, the total plasmon numbers in such decays should also be conserved, because the only process in which two plasmons decay into two others is permitted. Really, by integrating (2.25) over all \(k\)-space and taking into account the relations (2.26), it is easy to verify that

\[ N \equiv \int \frac{dk}{(2\pi)^3} N_k^l = \text{const}, \]

i.e. four-plasmon decay not changes the total number of plasmons.

### 3 The random-phase approximation

In this section let us briefly recall the main methodological notion used in study of the processes of nonlinear wave-interaction in a non-Abelian plasma within semiclassical approximation in [10].
We use the metric $g_{\mu\nu} = diag(1, -1, -1, -1)$ and choose units such that $c = k_B = 1$. The gauge field potentials are $N_c \times N_c$-matrices in a color space defined by $A_\mu = A_\mu^a t^a$ with $N_c^2 - 1$ Hermitian generators of the $SU(N_c)$ group in the fundamental representation. The field strength tensor $F_{\mu\nu} = F_{\mu\nu}^a t^a$ with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

obeys the Yang-Mills (YM) equation in a covariant gauge

$$\partial_\mu F^{\mu\nu}(X) - ig[A_\mu(X), F^{\mu\nu}(X)] - \xi^{-1} \partial^\nu \partial^\mu A_\mu(X) = -j^\nu(X),$$

where $\xi$ is a gauge parameter. $j^\nu$ is a color current

$$j^\nu = gt^a \int d^4p \, p^\nu \text{Tr}(T^a f),$$

where $T^a$ are Hermitian generators of $SU(N_c)$ group in the adjoint representation ($(T^a)^{bc} = -if^{abc}$, $\text{Tr}(T^a T^b) = N_c \delta^{ab}$). The distribution function of gluons $f$ satisfies the dynamical equation which in the semiclassical limit (when polarization effects are neglected [32]) is [13]

$$p^\mu \tilde{D}_\mu f + \frac{1}{2} gp^\mu \{F_{\mu\nu}, \frac{\partial f}{\partial p_\nu}\} = 0,$$

where $\tilde{D}_\mu$ is a covariant derivative acting as

$$\tilde{D}_\mu = \partial_\mu - ig[A_\mu(X), \cdot],$$

with $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denoting the commutator and anticommutator, respectively, and $A_\mu$, $F_{\mu\nu}$ are defined as $A_\mu = A_\mu^a T^a$, $F_{\mu\nu} = F_{\mu\nu}^a T^a$.

The distribution function $f$ can be decomposed into two parts: regular and random, where the latter is generated by spontaneous fluctuations in the plasma

$$f = f^R + f^T,$$

so that

$$\langle f \rangle = \langle f^R \rangle, \quad \langle f^T \rangle = 0.$$

Here, angular brackets $\langle \cdot \rangle$ indicate a statistical ensemble of averaging. The initial values of parameters which characterize the collective degree of plasma freedom is a such statistical ensemble. For almost linear collective motion to be considered below this may be initial values of oscillation phases.

We also use the definition

$$A_\mu = A_\mu^R + A_\mu^T, \quad \langle A_\mu^T \rangle = 0.$$
By averaging equation (3.4) over statistical ensemble, we obtain the kinetic equation for the regular part of the distribution function of hard gluons $f^R$

$$p^\mu \partial_\mu f^R = i g p^\mu \langle [A^T_\mu, f^T] \rangle - \frac{1}{2} g p^\mu \langle (\mathcal{F}^T_{\mu\nu})_L, \frac{\partial f^T}{\partial p_\nu} \rangle - \frac{1}{2} g p^\mu \langle (\mathcal{F}^T_{\mu\nu})_{NL}, \frac{\partial f^R}{\partial p_\nu} \rangle - \frac{1}{2} g p^\mu \langle \{ (\mathcal{F}^T_{\mu\nu})_{NL}, \partial f^R \partial p_\nu \} \rangle. \tag{3.7}$$

Here, the indices $L$ and $NL$ denote the linear and nonlinear parts with respect to field $A^a_\mu$ of the strength tensor (3.1). The correlation functions on the r.h.s. of this equation are collision terms due to particle-wave interactions and describe the backreaction of the background state from the plasma waves.

We assume that the typical time the nonlinear relaxation for the oscillations is a small quantity relative to the time scale over which the distribution of hard transverse gluons $f^R$ vary substantially. Therefore we neglect by change of the regular part of the distribution function with space and time, assuming that this function is specified and describes the global equilibrium in the gluon plasma

$$f^R \equiv f^0 = 2 \frac{2\theta(p_0)}{(2\pi)^3} \frac{1}{(e^{(p_0)/T} - 1)^2}, \tag{3.8}$$

where $u_\mu$ is the 4-velocity of the plasma. (Here, for convenience, we somewhat over-determine the equilibrium distribution function of thermal gluons (2.1).)

We use the expansion in powers of the oscillations amplitude of the random function $f^T$ to investigate non-equilibrium processes in QGP, such that the excitation energy of waves is small quantity in relation to the total energy of the particles

$$f^T = \sum_{n=1}^{\infty} f^{T(n)}, \tag{3.9}$$

where $f^{T(n)}$ collects the contributions of the $n$-th power in $A^T_\mu$. The expansion of a color current, corresponding to (3.9) has the form

$$j_\mu = j^R_\mu + j^T_\mu, \quad \langle j_\mu \rangle = j^R_\mu, \quad j^T_\mu = \sum_{n=1}^{\infty} j_{T(n)}^\mu, \tag{3.10}$$

where by the definition (3.3), we have

$$j_{T(n)}^\mu = g t^a \int d^4 p p_\mu \text{Tr} (T^a f^{T(n)}). \tag{3.11}$$

The regular part of a current vanishes for the global equilibrium gluon plasma.

Substituting the expansion (3.9) into (3.4), and collecting the terms of the same order in $A^T_\mu$, we derive the system of equations

$$p^\mu \partial_\mu f^{T(1)} = -\frac{1}{2} g p^\mu \langle (\mathcal{F}^T_{\mu\nu})_L, \frac{\partial f^R}{\partial p_\nu} \rangle, \tag{3.12}$$
\[ p^\mu \partial_\mu f^{T(2)} = igp^\mu ([A^T_\mu, f^{T(1)}] - \langle [A^T_\mu, f^{T(1)}] \rangle) \]  
\[ - \frac{1}{2}gp^\mu((F^T_{\mu
u})_{L, \nu} - \langle (F^T_{\mu
u})_{L, \nu} \rangle) - \frac{1}{2}gp^\mu((F^T_{\mu
u})_{NL} - \langle (F^T_{\mu
u})_{NL} \rangle, \frac{\partial f^R}{\partial p_\nu}), \]  
\[ p^\mu \partial_\mu f^{T(3)} = igp^\mu ([A^T_\mu, f^{T(2)}] - \langle [A^T_\mu, f^{T(2)}] \rangle) - \frac{1}{2}gp^\mu((F^T_{\mu
u})_{L, \nu} - \langle (F^T_{\mu
u})_{L, \nu} \rangle) \]  
\[ - \frac{1}{2}gp^\mu((F^T_{\mu
u})_{NL, \nu} - \langle (F^T_{\mu
u})_{NL, \nu} \rangle, \frac{\partial f^{T(1)}}{\partial p_\nu}). \]  

We rewrite the Yang-Mills equation (3.2), connecting a gauge field with a color current, in the following form

\[ \partial_\mu (F^{T\mu\nu})_L - \xi^{-1} \partial_\nu \partial^\mu A^T_\mu + j^{T(1)}_\mu = \]  
\[ = - j^{T^0}_L + ig \partial_\mu ([A^{T_\mu}, A^{T_\nu}] - \langle [A^{T_\mu}, A^{T_\nu}] \rangle) \]  
\[ + ig ([A^T_\mu, (F^{T\mu\nu})_L] - \langle [A^T_\mu, (F^{T\mu\nu})_L] \rangle) + g^2 ([A^T_\mu, [A^{T_\mu}, A^{T_\nu}] - \langle [A^T_\mu, [A^{T_\mu}, A^{T_\nu}] \rangle]). \]

Here, on the l.h.s. we collect all linear terms with respect to $A^T_\mu$ and we denote: \( j^{T^0}_L \equiv j^{T(2)_L} + j^{T(3)_L} + \ldots \). It will be shown that at leading order in $g$ only, the first two terms in \( j^{T^0}_L \) need to be kept.

It is not difficult to obtain the explicit form of the terms in the color current expansion (3.10) from the system of equations (3.12)-(3.14) and the relation (3.11). In the momentum representation and to the leading order in coupling constant up to the third-order terms, we have [10]

\[ j^{T(1)}_\mu (k) = \Pi^{\mu\nu} (k) A^T_\nu (k), \]

where

\[ \Pi^{\mu\nu} (k) = g^2 N_c \int d^4 p \frac{p^\mu (p^\nu (k^\mu p) - (k^\mu )^0 p^0^\nu ) f^0}{p k + i p^0 \epsilon}, \epsilon \rightarrow +0 \]

is the high temperature polarization tensor;

\[ j^{T(2)_{ab}} (k) = f^{abc} \int S^{(II)_{\mu\nu\lambda}}_{k,k_1,k_2} (A^b_\nu (k_1) A^c_\lambda (k_2) - A^b_\nu (k_1) A^c_\lambda (k_2)) \delta (k - k_1 - k_2) dk_1 dk_2, \]

where

\[ S^{(II)_{\mu\nu\lambda}}_{k,k_1,k_2} = -ig^3 N_c \int d^4 p \frac{p^\mu p^\nu p^\lambda}{p k + i p^0 \epsilon} \frac{(k_2 \partial_\rho f^0)}{p k_2 + i p^0 \epsilon}; \]

\[ j^{T(3)_{ab}} (k) = f^{abf} \int f^{fde} \int S^{(II)_{\mu\nu\lambda}}_{k,k_1,k_2} (A^b_\nu (k_3) A^c_\lambda (k_1) A^e_\sigma (k_2) - A^b_\nu (k_3) A^c_\lambda (k_1) A^e_\sigma (k_2)) \delta (k - k_1 - k_2 - k_3) dk_1 dk_2 dk_3, \]

and

\[ \Sigma^{(II)_{\mu\nu\lambda}}_{k,k_1,k_2,k_3} = -g^4 N_c \int d^4 p \frac{p^\mu p^\nu p^\lambda p^\rho}{p k + i p^0 \epsilon} \frac{1}{p (k_1 + k_2) + i p^0 \epsilon} \frac{(k_2 \partial_\rho f^0)}{p k_2 + i p^0 \epsilon}. \]

For simplicity, hereafter we drop the superscript $T$ of a gauge field.
Furthermore, we rewrite the equation (3.13) in the momentum representation. By inserting the linear part of the random current (3.16) and nonlinear corrections (3.18) into Eq. (3.15), one finds

\[ [k^2g^{\mu\nu} - (1 + \xi^{-1})k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu^a(k) \]

\[ + \int_{bcf} S_{k,k_1,k_2}(A_\nu^c(k_1)A_\lambda^d(k_2) - \langle A_\nu^c(k_1)A_\lambda^d(k_2) \rangle)\delta(k - k_1 - k_2)dk_1dk_2 \]

\[ + \int_{bdf} \Sigma_{k,k_1,k_2,k_3}(A_\zeta^c(k_3)A_\lambda^d(k_1)A_\nu^e(k_2) - A_\zeta^c(k_3)\langle A_\lambda^d(k_1)A_\nu^e(k_2) \rangle) \]

\[ \times \delta(k - k_1 - k_2 - k_3)dk_1dk_2dk_3. \]

Here, \( S_{k,k_1,k_2} = S_{k,k_1,k_2}^{(I)\mu\nu\lambda} + S_{k,k_1,k_2}^{(II)\mu\nu\lambda} \), \( \Sigma_{k,k_1,k_2,k_3} = \Sigma_{k,k_1,k_2,k_3}^{(I)\mu\nu\lambda\sigma} + \Sigma_{k,k_1,k_2,k_3}^{(II)\mu\nu\lambda\sigma} \). The functions \( S^{(I)} \) and \( \Sigma^{(II)} \) are defined by expressions (3.19) and (3.24), respectively, and

\[ S_{k,k_1,k_2}^{(I)\mu\nu\lambda} = -ig(k'^\nu g^{\mu\lambda} + k_2^\nu g^{\mu\lambda} - k_2^\mu g^{\nu\lambda}), \Sigma_{k,k_1,k_2,k_3}^{(II)\mu\nu\lambda\sigma} = g^2 g^{\nu\lambda} g^{\mu\sigma}. \]

These tensor structures are caused by self-action of a gauge field. They are defined by nonlinear terms on the r.h.s. of equation (3.13), which are not associated with a color current in QGP. In section 5 equation (3.22) will be formally solved by iteration.

In [10] we introduce the correlation function of random oscillations

\[ I_{\mu\nu}^{ab}(k', k) = \langle A_\mu^a(k')A_\nu^b(k) \rangle \].

(3.24)

In order not to over burden equations by the symbol "*" we use a dagger "†" to denote complex conjugation. In thermal equilibrium, when the correlation function (3.24) in the coordinate representation depends only on the relative coordinates and time \( \Delta X = X' - X \), we have

\[ I_{\mu\nu}^{ab}(k', k) = I_{\mu\nu}^{ab}(k')\delta(k' - k). \]

(3.25)

Off-equilibrium perturbations which are slowly varying in space and time lead to a delta function broadening, and \( I_{\mu\nu}^{ab} \) depends on both arguments \( k \) and \( k' \).

Let us introduce \( I_{\mu\nu}^{ab}(k', k) = I_{\mu\nu}^{ab}(k, \Delta k), \Delta k = k' - k \) with \( | \Delta k/k | \ll 1 \) and insert the spectral intensity function in the Wigner form

\[ I_{\mu\nu}^{ab}(k, x) = \int I_{\mu\nu}^{ab}(k, \Delta k)e^{-i\Delta kx}d\Delta k, \]

depending slowly on \( x \).

Now we multiple equation (3.22) by \( A_\mu^a(k') \), subtract to it the complex-conjugated equation (with the replacement \( k \leftrightarrow k' \), \( a \leftrightarrow b \)) and average the equation using the formula (3.24). Furthermore we expand the polarization tensor into Hermitian and anti-Hermitian parts

\[ \Pi^{\mu\sigma}(k) = \Pi^{H\mu\sigma}(k) + \Pi^{A\mu\sigma}(k), \Pi^{H\mu\sigma}(k) = \Pi^{H\mu\sigma}(k), \Pi^{A\mu\sigma}(k) = -\Pi^{A\mu\sigma}(k). \]
The term with $\Pi^A$ corresponds to linear Landau damping. As was shown by Heinz and Siemens [33], that linear Landau damping for waves in QGP is absent and hence, this term vanishes. We expand the remaining terms on the l.h.s. in powers of $\Delta k$ and keep only linear ones. This corresponds to a gradient expansion procedure, usually used in the derivation of kinetic equations. Multiplying the difference equation by $e^{-i\Delta k x}$ and integrating over $\Delta k$ with regard to

$$\int \Delta k \lambda I_{ab}^{\mu\nu}(k, \Delta k) e^{-i\Delta k x} d\Delta k = i \frac{\partial I_{ab}^{\mu\nu}(k, x)}{\partial x^\lambda},$$

we finally obtain the equation, which is a starting point for our further research

$$\frac{\partial}{\partial k^\lambda} [k^2 g^{\mu\nu} - (1 + \xi^{-1})k^\mu k^\nu - \Pi^{\mu\nu}(k)] \frac{\partial I_{ab}^{\mu\nu}(k, x)}{\partial x^\lambda} = \int \sum_{k, k_1, k_2, k_3} \{ f_{abc} f_{def} \langle A_{[a}^{\mu}(k') \rangle \langle A_{c]}^{\nu}(k_1) \rangle \langle A_{d]}^{\sigma}(k_2) \rangle \delta(k - k_1 - k_2 - k_3) d k_1 d k_2 d k_3 - \{ a \leftrightarrow b, k \leftrightarrow k', \text{compl. conj.} \}. \tag{3.26}$$

4 Consistency with gauge symmetry

In this section we shall discuss the consistency of the approximation scheme which we use with the requirement of the non-Abelian gauge symmetry.

The initial dynamical equation (3.4) and the Yang-Mills equation (3.2) (without the gauge-fixing condition) transform covariantly under local transformations

$$\tilde{A}_{\mu}(X) = h(X) A_{\mu}(X) + \frac{i}{g} \partial_{\mu} h^\dagger(X), \quad h(X) = \exp (i \theta^a(X) t^a)$$

with the parameter $\theta^a(X)$. We also have transformation of gluon distribution function [13]

$$\tilde{f}(p, X) = H(X) f(p, X) H^\dagger(X),$$

where $H^{ab}(X) = \text{Sp}[t^a h(X) t^b h^\dagger(X)]$.

As is known (see, e.g. [17]), after the splitting (3.5), (3.6) the resulting equations left two symmetries: the background gauge symmetry,

$$\tilde{A}_{\mu}^R(X) = h(X) A_{\mu}^R(X) + \frac{i}{g} \partial_{\mu} h^\dagger(X), \quad \tilde{A}_{\mu}^T(X) = h(X) A_{\mu}^T(X) h^\dagger(X), \tag{4.1}$$

and the fluctuation gauge symmetry,

$$\tilde{A}_{\mu}^R(X) = 0, \quad \tilde{A}_{\mu}^T(X) = h(X) A_{\mu}^R(X) + A_{\mu}^T(X) + \frac{i}{g} \partial_{\mu} h^\dagger(X). \tag{4.2}$$
The condition which we impose on a regular part of the gauge field $A_{\mu}^R$ in the preceding section and the requirement that the statistical average of the fluctuation vanishes $\langle A_{\mu}^T \rangle = 0$, break down both types of symmetry (4.1) and (4.2). Thus in the case of a gauge transformation (4.1) we obtain $\bar{A}_{\mu}^R \neq 0$, and in the case of (4.2) we arrive at non-invariance of the constraint $\langle A_{\mu}^T \rangle = 0$. Moreover, the introduced correlation function (3.24) also has an explicitly gauge non-covariant character. This leads to the fact that calculations in the previous section are gauge non-covariant, and therefore the value of these manipulations is doubtful.

Nevertheless, there is a special case, when the preceding (and following) conclusions are justified. This is a case of colorless fluctuation, where $I_{ab}^{\mu\nu}(k, x) = \delta_{ab}I_{\mu\nu}^{\mu
u}(k, x)$. We can obtain a gauge-invariant equation for $I_{\mu\nu}^{\mu
u}(k, x)$ only in this restriction, in spite of the fact that the intermediate calculations spoil non-Abelian gauge symmetry of the initial equations (3.2)-(3.4).

In principle, we shall be able to maintain an explicit background gauge symmetry (4.1) at each step of our calculations, as has been done, for example, by Blaizot and Iancu [18] for derivation of the Boltzmann equation describing the relaxation of ultrasoft color excitations. First of all we assume that $A_{\mu}^R \neq 0$. Then as the gauge-fixing condition for the random field $A_{\mu}^T$, we choose the background field gauge

$$D_{\mu}^R(X)A_{\mu}^T(X) = 0, \quad D_{\mu}^R(X) \equiv \partial_{\mu} - igA_{\mu}^R(X),$$

(4.3)

which is manifestly covariant with respect to gauge transformations of the background gauge field $A_{\mu}^R(X)$. Lastly we define a gauge-covariant Wigner function as in [13, 18]

$$I_{ab}^{\mu\nu}(k, x) = \int \bar{I}_{ab}^{\mu\nu}(s, x) e^{iks} ds, \quad s \equiv X_1 - X_2, \quad x \equiv \frac{1}{2}(X_1 + X_2),$$

where

$$\bar{I}_{ab}^{\mu\nu}(s, x) \equiv U^{aa'}(x, x + \frac{s}{2}) I_{a'b'}^{a'b'}(x + \frac{s}{2}, x - \frac{s}{2}) U^{b'b}(x - \frac{s}{2}, x),$$

instead of the usual Wigner function $I_{ab}^{\mu\nu}(k, x)$, whose ‘poor’ transformation properties follow from the initial definition $I_{ab}^{\mu\nu}(X_1, X_2) = \langle A_{\mu}^{T a}(X_1)A_{\nu}^{T b}(X_2) \rangle$. The function $U(x, y)$ is the non-Abelian parallel transporter

$$U(x, y) = \text{P exp} \left\{ -ig \int_{\gamma} dz^\mu A_{\mu}^R(z) \right\}.$$

The path $\gamma$ is the straight line joining $x$ and $y$.

The derivation of the kinetic equation for plasmons in this approach becomes quite cumbersome and non-trivial. For example, on the l.h.s. of the equations for a random part of the distribution (3.12)-(3.14), the covariant derivative $D_{\mu}^R$ will be used instead of the ordinary one $\partial_{\mu}$. Besides, we cannot assume that the regular part of the distribution function is specified and equal to the Bose-Einstein distribution (3.8). It is necessary to also take into account their change using the kinetic equation (3.7) with the collision terms.
on the r.h.s. of (3.7), which describes the backreaction of the background distributions from the soft fluctuations. The correlators on the r.h.s. of equation (3.7) can be expressed in terms of the function \( f^{ab}_{\mu} \) and the distribution of hard transverse gluons \( f^{R}(p, X) \) only.

However, if we restrict our consideration to the study of colourless excitations and replace the distribution function of hard gluons by its equilibrium value (3.8), then this leads to an effective vanishing of terms with mean field \( A^{R}_{\mu} \). This follows, for example, from the analysis of the derivation of the Boltzmann equation by Blaizot and Iancu [18]. Therefore, the simplest way to derive the kinetic equations for soft colorless QGP excitations is to assume \( A^{R}_{\mu} = 0 \) and to use the prime gauge non-covariant correlator (3.24). In this case the background field gauge (4.3) is reduced to a covariant one. The resulting Boltzmann equation for colorless plasmons will be gauge invariant if all contributions to the probability of plasmon-plasmon scattering at the leading order in \( g \) are taken into account (see also discussion in conclusion).

5 The interacting fields as the functions of free fields

Let us define approximate solution of equation (3.22) accurate up to third-order in the oscillations amplitude of the free field. For this purpose, it is convenient to write this equation in a more compact form.

We introduce the following notation

\[
J^{(2)\mu\nu}(k) \equiv f^{abc} \int S_{k,k_{1},k_{2}}^{\mu\nu\lambda}(A_{b}^{\lambda}(k_{1})A_{c}^{\lambda}(k_{2}) - \langle A_{b}^{\lambda}(k_{1})A_{c}^{\lambda}(k_{2}) \rangle)\delta(k-k_{1}-k_{2})dk_{1}dk_{2}, \quad (5.1)
\]

\[
J^{(3)\mu\nu}(k) \equiv f^{abf}f^{def} \int \sum_{k,k_{1},k_{2},k_{3}}A_{b}^{\mu}(k_{3})A_{c}^{\lambda}(k_{1})A_{a}^{\nu}(k_{2}) - A_{b}^{\mu}(k_{3})A_{c}^{\lambda}(k_{1})A_{a}^{\nu}(k_{2})\delta(k-k_{1}-k_{2}-k_{3})dk_{1}dk_{2}dk_{3}. \quad (5.2)
\]

The expressions (5.1) and (5.2) present the nonlinear color currents, including the self-action effects of gauge fields, in contrast to (3.18) and (3.20).

Using \( *D_{\mu\nu}(k) \) we denote the medium modified (retarded) gluon propagator, which in a covariant gauge has a form

\[
*D_{\mu\nu}(k) = -P_{\mu\nu}(k) *\Delta^{l}(k) - Q_{\mu\nu}(k) *\Delta^{l}(k) + \xi D_{\mu\nu}(k) \Delta^{0}(k), \quad (5.3)
\]

where \( *\Delta^{l}(k) = 1/(k^{2} - \Pi^{l}(k)), \Pi^{l}(k) = \frac{1}{2}\Pi^{\mu\nu}(k)P_{\mu\nu}(k), \Pi^{l}(k) = \Pi^{\mu\nu}(k)Q_{\mu\nu}(k); \Delta^{0}(k) = 1/k^{2} \). The Lorentz matrices in (5.3) are members of the basis

\[
P_{\mu\nu}(k) = g_{\mu\nu} - D_{\mu\nu}(k) - Q_{\mu\nu}(k), \quad Q_{\mu\nu}(k) = \frac{\bar{u}_{\mu}(k)\bar{u}_{\nu}(k)}{\bar{u}^{2}(k)}, \quad C_{\mu\nu}(k) = -\frac{(\bar{u}_{\mu}(k)k_{\nu} + \bar{u}_{\nu}(k)k_{\mu})}{\sqrt{-2k^{2}\bar{u}^{2}(k)}},
\]

\[
D_{\mu\nu} = k_{\mu}k_{\nu}/k^{2}, \quad \bar{u}_{\mu}(k) = k^{2}u_{\mu} - k_{\mu}(ku). \quad (5.4)
\]
Let us assume that we are in the rest frame of a heat bath, so that \( u_\mu = (1, 0, 0, 0) \).

Using the above introduced functions, the equation (3.22) can be rewritten in the form

\[
* \mathcal{D}^{-1 \mu \nu}(k) A^a_\nu(k) = - J^{T(2)\mu\nu}(A, A) - J^{T(3)\mu\nu}(A, A, A). \tag{5.5}
\]

The nonlinear integral equation (5.5) is solved by the approximation scheme method - the weak free field expansion (small perturbations). Discarding the nonlinear terms in \( A \) on the r.h.s. of equation (5.5), we obtain in the first approximation

\[
* \mathcal{D}^{-1 \mu \nu}(k) A^a_\nu(k) = 0.
\]

The solution of this equation, which we denote by \( A^{(0)a}_\mu(k) \) is the solution for free fields.

Further keeping the term, quadratic in the field on the r.h.s. of Eq. (5.5), we derive the equation

\[
* \mathcal{D}^{-1 \mu \nu}(k) A^a_\nu(k) = - J^{T(2)\mu\nu}(A^{(0)}, A^{(0)}),
\]

where on the r.h.s. we substitute free fields instead of interacting ones. The general solution of the last equation can be given in the form

\[
A^a_\mu(k) = A^{(0)a}_\mu(k) - * \mathcal{D}_{\mu\nu}(k) J^{T(2)\nu\lambda}(A^{(0)}, A^{(0)}).
\]

This approximate solution was used in research of nonlinear plasmon damping in QGP [10].

The following term in the expansion of interacting fields is defined from equation

\[
* \mathcal{D}^{-1 \mu \nu}(k) A^a_\nu(k) = - J^{T(2)\mu\nu}(- * \mathcal{D} J^{T(2)}(A^{(0)}, A^{(0)}), A^{(0)}) - J^{T(3)\mu\nu}(A^{(0)}, A^{(0)}, A^{(0)}) - * \mathcal{D}_{\mu\nu}(k) J^{T(3)\nu\lambda}(A^{(0)}, A^{(0)}, A^{(0)}).
\]

Using the explicit expressions for the currents (5.7) and (5.8), after cumbersome algebraic transformations, we obtain the form of the interacting field from the equation (5.6) with the accuracy required for our further calculations

\[
A^a_\mu(k) = A^{(0)a}_\mu(k) - * \mathcal{D}_{\mu\nu}(k) J^{T(2)\nu\lambda}(A^{(0)}, A^{(0)}), A^{(0)}), A^{(0)}) - * \mathcal{D}_{\mu\nu}(k) J^{T(3)\nu\lambda}(A^{(0)}, A^{(0)}, A^{(0)}).
\]

Here, the third-order color current on the r.h.s. is defined by the expression

\[
J^{T(3)\nu\lambda}(A^{(0)}, A^{(0)}, A^{(0)}) \equiv f^{abf} f^{fde} \int \tilde{\Sigma}^{\nu\nu\lambda\rho}_{k_1, k_2, k_3} (A^{(0)b}_\lambda(k_3) A^{(0)d}_\rho(k_2)) - A^{(0)b}_\lambda(k_3) \langle A^{(0)d}_\rho(k_2) \rangle \delta(k - k_1 - k_2 - k_3) dk_1dk_2dk_3,
\]

where

\[
\tilde{\Sigma}^{\nu\nu\lambda\rho}_{k_1, k_2, k_3} \equiv \Sigma^{\nu\nu\lambda\rho}_{k_1, k_2, k_3} - \frac{1}{2} * \mathcal{D}_{\nu\lambda}(k_1 + k_2)(S^{\nu\lambda\rho}_{k_1, k_2, k_3} - S^{\nu\rho\lambda}_{k_1, k_2, k_3})(S^{\rho\lambda\nu}_{k_1 + k_2, k_2, k_1} - S^{\rho\nu\lambda}_{k_1, k_2, k_1}),
\]

and we take into account, that the third-order correlation function \( \langle A^{(0)}(0) A^{(0)}(0) A^{(0)}(0) \rangle \) vanishes by virtue of the fact that \( A^{(0)} \) represents the amplitude fully non-correlative gauge fields. The factor of \( \frac{1}{2} \), in front of the second term on the r.h.s. of (5.9) arises from symmetrization with respect to permutation of the potentials \( A^{(0)b}_\lambda(k_3) \) and \( A^{(0)d}_\rho(k_2) \) in the expression (5.8). The current (5.8) may be interpreted as a certain third-order effective color current, in contrast to the initial ‘bare’ expression (5.2).
6 The correspondence principle

For the determination of the probability of plasmon-plasmon scattering in a gluon plasma the method developed in the theory of the nonlinear processes in electron-ion plasma and known as the correspondence principle [24, 30], is usable. For the non-Abelian plasma this approach is especially effective in the temporal gauge, when we have closer correspondence with the electrodynamics of an ordinary plasma. The gist of this method is as follows.

The change in the plasmon numbers, caused by spontaneous processes of four-plasmon decays only, is

\[ \left( \frac{\partial N_k^l}{\partial t} + V_k^l \frac{\partial N_k^l}{\partial x} \right)^{sp} = \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \times (2\pi)^4 \delta(\omega_k^l + \omega_{k_1}^l - \omega_{k_2}^l - \omega_{k_3}^l) \delta(k + k_1 - k_2 - k_3) w(k, k_1, k_2, k_3) N_{k_1}^l N_{k_2}^l N_{k_3}^l. \]

This equation follows from equation (2.25) in the limit of small intensity \(N_k^l \to 0\). In this case the change of energy of the longitudinal excitations is

\[ \left( \frac{d\mathcal{E}}{dt} \right)^{sp} = \int \frac{dk}{(2\pi)^3} \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \times (2\pi)^4 \delta(\omega_k^l + \omega_{k_1}^l - \omega_{k_2}^l - \omega_{k_3}^l) \omega_k^l w(k, k_1, k_2, k_3) N_{k_1}^l N_{k_2}^l N_{k_3}^l. \]

On the other hand the value \(\left( \frac{d\mathcal{E}}{dt} \right)^{sp}_e\) represents the emitted radiant power of the longitudinal waves \(\mathcal{I}^l\), which in turn is equal to the work done by the radiation field with the color current, creating it, in unit time

\[ \mathcal{I}^l = \int dx \left( E^a(x, t) J^a(x, t) \right) = \int \frac{d\omega d\omega'}{2\pi} \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{dk}{(2\pi)^3} \langle E_{k,\omega'}^a J_{-k,\omega}^a \rangle e^{-i(\omega' - \omega)t} = \]

\[ = \frac{1}{2} \int \frac{d\omega d\omega'}{2\pi} \frac{d\omega}{2\pi} \frac{d\omega}{2\pi} \frac{dk}{(2\pi)^3} \left( \frac{1}{\omega'\epsilon'(\omega', k)} - \frac{1}{\omega\epsilon(\omega, k)} \right) k^i k^j \langle J_{k,\omega'}^{ai} J_{k,\omega}^{aj} \rangle e^{-i(\omega' - \omega)t}. \]

Here, \(E^a(x, t) = -\partial A^a(x, t)/\partial t\) is chromoelectric field in the temporal gauge. The sign on the r.h.s. of (5.3) corresponds to the choice of sign in front of the current in the Yang-Mills equation (3.2). In conclusion of the last line (6.3) we take into account that the Fourier-component of a field \(E^a_{k,\omega} = k E^a_{k,\omega}/|k|\) is associated with \(J^a_{k,\omega}\) by the Yang-Mills equation

\[ E^a_{k,\omega} = \frac{i}{\omega\epsilon(\omega, k)} \frac{(k \cdot J^a_{k,\omega})}{|k|}. \]

In order to define the probability of the four-plasmon decays, the correlation function on the r.h.s. of equation (6.3) has to contain terms of sixth-order in the free field \(A^{(0)}\). The required sixth-order correlator yields the color current \(J^{T(3)ai}(5.2)\) (more precisely, its expression in the temporal gauge). However here, it is also necessary to take into account the effects, that arise from iteration of the current \(J^{T(2)ai}(5.4)\). Defining in this way all necessary contributions, making the correlation decoupling of the sixth-order correlators
in terms of pairs and next expressing next \( \langle A^{(0)} A^{(0)} \rangle \) in terms of \( N^l \), we obtain an emitted radiant power \( \mathcal{I}^l \). Comparing \( \mathcal{I}^l \) with (1.2), one identifies the required probability \( w(k, k_1; k_2, k_3) \).

However this method encountered certain difficulties in deciding on the other gauges, e.g. the covariant gauge. Here, it is convenient for the definition of the probability of plasmon-plasmon scattering to start immediately from equation (3.26). Using the above-obtained expression (5.7) for the potentials of interacting fields, by a simple search we extract all the sixth-order correlators responsible for four-plasmon decays of the type \( \text{plasmon-plasmon scattering} \). The rule used to choose the relevant terms is defined by the simple fact that when we make the correlation decoupling of the sixth-order correlators in the terms of the pairs (in order to define the product \( N_{k_1}^l N_{k_2}^l N_{k_3}^l \)) as a factor, the \( \delta \)-functions arise in the form

\[
\delta(\omega_k^l + \omega_{k_1}^l - \omega_{k_2}^l - \omega_{k_3}^l) \delta(k + k_1 - k_2 - k_3).
\]

Also the coefficient function preceding \( N_{k_1}^l N_{k_2}^l N_{k_3}^l \) must satisfy properties (2.26). As will be shown below, these conditions are sufficient to calculate the plasmon-plasmon scattering probability in the covariant gauge (this rather cumbersome and physically not quite transparent approach is suited for other gauges).

At first, we consider the contribution to the r.h.s. of the initial equation (3.26), associated with \( \Sigma \)-functions. Here, for convenience of further reference we write it separately

\[
-i \int dk' d f^{bcf} d f^{de} \sum_{k, k_1, k_2, k_3} \left( \langle A^a_{\mu}(k') A^c_{\nu}(k_3) A^d_{\lambda}(k_1) A^e_{\sigma}(k_2) \rangle - \langle A^a_{\mu}(k') A^c_{\nu}(k_3) \rangle \langle A^d_{\lambda}(k_1) A^e_{\sigma}(k_2) \rangle \right) \delta(k - k_1 - k_2 - k_3) dk_1 dk_2 dk_3 - (a \leftrightarrow b, k \leftrightarrow k', \text{compl. conj.}) \right). \tag{6.5}
\]

One can obtain the sixth-order terms in the free field by two waves. The first one is as follows. We substitute in turn the expression from the r.h.s. of (5.7), which contains only cubic terms in the potentials of free fields, instead of each potential of the interacting ones, i.e.

\[
A^a_{\mu}(k) \rightarrow - \ast \mathcal{D}_{\mu\nu'}(k) \mathcal{J}^{T(3)\mu\nu'}(A^{(0)}, A^{(0)}, A^{(0)}).
\]

We replace the remaining potentials by the rule \( A^c_{\nu}(k_3) \rightarrow A^{(0)\nu}(k_3) \), etc. The second way is to substitute the quadratic term in \( A^{(0)} \) from the r.h.s. of (5.7) instead of any two potentials of the interacting fields, i.e.

\[
A^a_{\mu}(k) \rightarrow - \ast \mathcal{D}_{\mu\nu'}(k) \mathcal{J}^{T(2)\mu\nu'}(A^{(0)}, A^{(0)}).
\]

We replace the remaining potentials by free ones. It is necessary to look at all possible substitutions in both first and second ways.

The number of terms appearing can be cut if we note, that it is need to keep only such terms in intermediate expressions which contain the propagators \( \ast \mathcal{D}_{\mu\nu'}(k) \) and \( \ast \mathcal{D}_{\mu\nu'}(k') \). These propagators give the terms, proportional to

\[
\delta(Re \varepsilon^l(k)) = \left( \frac{\partial Re \varepsilon^l(k)}{\partial \omega} \right)^{-1}_{\omega=\omega_k^l} [\delta(\omega - \omega_k^l) + \delta(\omega + \omega_k^l)], \tag{6.6}
\]

where \( \varepsilon^l(k) \) is the propagator associated with \( \Sigma \)-functions.
i.e. the factor taken into account the existence of plasmons with wavevector \( \mathbf{k} \) and energy \( \omega'_k \), in spite of the fact that the number density of the plasmon \( N'_k \) is explicitly absent.

Hence it follows that for the first way only the replacement in (6.3)

\[
A'^{i\alpha}_{\mu}(k') \rightarrow - S^{\dagger}_{\mu\nu'}(k') J^{jT(3)\mu\nu'}(A^{(0)}_{\mu}, A^{(0)}, A^{(0)}), \quad A^{\nu}_{\alpha}(k_3) \rightarrow A^{\nu\sigma}_{\alpha}(k_3), \ldots
\]
gives desired contribution (similar for the conjugate term). This leads (6.3) to the expression

\[
i \int dk' \{ \int b e f f d f d e f a e^* f g d^* e^* * S^{\dagger}_{\mu\nu'}(k') \sum_{k,k_1,k_2,k_3} \delta(k',k'_1,k'_2,k'_3) \langle (A^{(0)}_{\mu})^{i\alpha'}(k'_3) A^{(0)}_{\nu'}(k'_1) A^{(0)}_{\nu'}(k'_2) \rangle \delta(k_3,k_1,k_2) - \langle A^{(0)}_{\mu} A^{(0)}_{\nu'}(k'_3) A^{(0)}_{\nu'}(k'_1) A^{(0)}_{\nu'}(k'_2) \rangle \delta(k_3,k_1,k_2) \} \delta(k' - k_1 - k_2 - k_3) \prod_{i=1}^{3} dk_i \delta k'_i - (a \leftrightarrow b, k \leftrightarrow k', \text{compl. conj.).}
\]

In the second case, at first step it should be replaced by

\[
A'^{i\alpha}_{\mu}(k') \rightarrow - S^{\dagger}_{\mu\nu'}(k') J^{jT(2)\mu\nu'}(A^{(0)}_{\mu}, A^{(0)}).
\]

This gives

\[
i \int dk' \{ \int b e f f d f d e f a b e' * S^{\dagger}_{\mu\nu'}(k') \sum_{k,k_1,k_2,k_3} \delta(k',k'_1,k'_2,k'_3) \langle (A^{(0)}_{\mu}^{i\alpha'})^{(0)}(k'_3) A^{(0)}_{\nu'}(k'_1) A^{(0)}_{\nu'}(k'_2) \rangle \delta(k_3,k_1,k_2) - \langle A^{(0)}_{\mu}^{i\alpha'}(k'_3) A^{(0)}_{\nu'}(k'_1) A^{(0)}_{\nu'}(k'_2) \rangle \delta(k_3,k_1,k_2) \} \delta(k' - k_1 - k_2 - k_3) \prod_{i=1}^{3} dk_i \delta k'_i + (k \leftrightarrow k', a \leftrightarrow b, \text{compl. conj.).}
\]

By virtue of the stochasticity of gauge fields, the last term inside the parentheses vanishes. We replace the remaining potentials of the interacting fields by the rules

\[
A^{a}_{\nu}(k_3) \rightarrow - S^{\dagger}_{\nu\nu'}(k_3) J^{jT(2)\nu\nu'}(A^{(0)}_{\nu}, A^{(0)}), \quad A^{d}_{\alpha}(k_1) \rightarrow A^{d}_{\alpha}(k_1), \quad A^{d}_{\nu}(k_2) \rightarrow A^{d}_{\nu}(k_2), \quad \text{etc.}
\]

By inspecting the expression (6.8), one sees that, complex-conjugate and non-conjugate amplitudes of free fields enter by a non-symmetric fashion. As will be shown below, in this case \( \delta \)-functions (6.3) type expressing the energy and momentum conservation laws in the plasmon-plasmon scattering have not arisen.

By virtue of the fact that we restrict our consideration to the derivation of the kinetic equation for colorless excitations, i.e.

\[
\langle A^{(0)}_{\mu} A^{(0)}_{\nu} \rangle \sim \delta^{de}, \quad \langle A^{(0)}_{\mu} A^{(0)}_{\nu} \rangle \sim \delta^{de},
\]

23
all terms inside the parentheses of the expression (3.24), excepting the first term (the sixth-order correlator) vanish because \(\delta^{de}, \delta^{dc}, \ldots\), are contracted with antisymmetric structure constants. Further, it is necessary to decouple the averaging of six potentials into pairs. We define below, what the correlations decoupling is responsible for in the four-plasmon decay processes. By using the definition of the correlation function (3.24), (3.25), we have

\[
\langle A^{(0)\dagger\epsilon'}_{\nu'}(k_3')A^{(0)\dagger\mu'}_{\alpha'}(k_1')A^{(0)\dagger\epsilon}(k_2')A^{(0)\dagger\epsilon}(k_3)A^{(0)d}_{\lambda}(k_1)A^{(0)d}_{\alpha}(k_2) \rangle
\]

(6.10)

\[
= \langle A^{(0)\dagger\epsilon'}_{\nu'}(k_3')A^{(0)\dagger\mu'}_{\alpha'}(k_1') \rangle \langle A^{(0)\dagger\epsilon}(k_2')A^{(0)d}_{\lambda}(k_1) \rangle \langle A^{(0)d}_{\alpha}(k_2) \rangle + \ldots
\]

\[= I_{\nu'\lambda}(k_3')\delta^{d\epsilon'}d(k_3' - k_1')I_{\alpha'\lambda}(k_2')\delta^{d\epsilon}d(k_2' - k_1)I_{\alpha\sigma}(k_3)\delta^{\epsilon\epsilon} \delta(k_3 + k_2) + \ldots.\]

After substitution of the first term on the r.h.s. (6.10) into (6.7) and performing integration over \(dk_1dk_2dk_3dk'\) and elementary color algebra, we obtain

\[
\delta^{ab}_{\nu\sigma} N^2_n (-i) \int \{ \mathcal{D}_{\mu'k}^\dagger (k) \mathcal{D}_{\mu'k'}^\dagger (k') \mathcal{D}_{\mu\lambda}^\dagger (k) \mathcal{D}_{\mu\sigma}^\dagger (k) \}
\]

As can be seen from the last expression, the required \(\delta\)-functions (6.4) are not appeared and therefore this expression is not associated with the plasmon-plasmon scattering and it should be dropped. This is a general rule. The decomposition of averaging of free field amplitudes into the correlators containing the pair of complex-conjugate potentials or one of the nonconjugate potentials between the inside of the angular brackets (statistical averaging), does not give a contribution to the process of interest to us. For this reason, it is necessary to fully drop all contribution in the process of the plasmon-plasmon scattering defined by the expression (5.8), since making the correlation decoupling, the pair with complex-conjugate potentials or without conjugate necessarily arises. We write out decoupling of the sixth-order correlator, which gives a contribution to equation (5.4). Suppressing color and Lorentz indices and employing a condensed notion, \(A_1 \equiv A^{(0)}_{\mu}(k_1)\), we have

\[
\langle A^{\dagger}_1, A^{\dagger}_1, A^{\dagger}_1, A_2, A_3, A_2, A_2 \rangle = 3 \langle A^{\dagger}_1, A_3 \rangle \langle A^{\dagger}_1, A_1 \rangle \langle A^{\dagger}_1, A_2 \rangle + \langle A^{\dagger}_1, A_3 \rangle \langle A^{\dagger}_1, A_2 \rangle \langle A^{\dagger}_1, A_1 \rangle + \langle A^{\dagger}_1, A_1 \rangle \langle A^{\dagger}_1, A_3 \rangle \langle A^{\dagger}_1, A_2 \rangle + \langle A^{\dagger}_1, A_1 \rangle \langle A^{\dagger}_1, A_2 \rangle \langle A^{\dagger}_1, A_3 \rangle + \langle A^{\dagger}_1, A_2 \rangle \langle A^{\dagger}_1, A_1 \rangle \langle A^{\dagger}_1, A_3 \rangle + \langle A^{\dagger}_1, A_2 \rangle \langle A^{\dagger}_1, A_3 \rangle \langle A^{\dagger}_1, A_2 \rangle.
\]

(6.11)

Now we consider the terms with \(S\)-functions on the r.h.s. of equation (3.26) and here, we also write them separately

\[
-i \int dk' \{ f^{bcd}_{\nu'k,k_1,k_2} \langle A^{(0)}_{\mu}(k')A^{(0)}_{\nu'}(k_1)A^{(0)}_{\lambda}(k_2) \rangle \delta(k - k_1 - k_2)dk_1dk_kd2
\]

\[
- f^{acd}_{\nu'k,k_1,k_2} \langle A^{(0)}_{\mu}(k)A^{(0)}_{\nu'}(k_1)A^{(0)}_{\lambda}(k_2) \rangle \delta(k' - k_1 - k_2)dk_1dk_2 \}.
\]

(6.12)
According to the previous discussion, at a first step it is necessary to perform the replacement

\[ A_\mu^a(k') \rightarrow -*D_{\mu\nu}^a(k') \tilde{\Gamma}^{(3)\mu\nu}(A^{(0)}, A^{(0)}, A^{(0)}), \]

\[ A_\mu^b(k) \rightarrow -*D_{\mu\nu}^b(k) \tilde{\Gamma}^{(3)\mu\nu}(A^{(0)}, A^{(0)}, A^{(0)}). \]

Furthermore we consequently replace the remaining two potentials of the interacting fields in the correlators by \( A_\nu^a(k_1) \rightarrow -*D_{\nu\rho}^a(k_1) \tilde{\Gamma}^{(2)\rho}(A^{(0)}, A^{(0)}) \), \( A_\lambda^a(k_2) \rightarrow A_\lambda^a(k_2) \), etc. This automatically leads to symmetry of contribution in \( A^{(0)} \) and \( A^{(0)} \). As result we obtain

\[
\frac{i}{2} \int \left\{ *D_{\mu\nu}^a(k') f_{\beta c d} f_{e g} f_{h i j} g \right\}
\times \left( S_{k,k_1+k_2,k_3} - S_{k,k_1+k_2,k_3} \right) \tilde{\Sigma}_{k',k'_1,k'_2,k'_3} *D_{\rho\sigma}(k_1 + k_2)
\times (S_{k_1+k_2,k_1,k_2} - S_{k_1+k_2,k_1,k_2}) \left( (A^{(0)}_\nu^c(k_3)) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) \right)
\times \delta(k-k_1-k_2-k_3) \delta(k'-k_1'-k_2'-k_3') \prod_{i=1}^{3} dk_i dk_i' - (a \leftrightarrow b, k \leftrightarrow k', \text{compl. conj.}) \}.
\]

Because of (6.11), in the last expression it is necessary to retain only the sixth-order correlator. Finally adding (6.7) and (6.13), we obtain the following equation, instead of (6.20)

\[
\frac{\partial}{\partial k_\lambda} [k^2 g_{\mu\nu} - (1 + \xi^{-1}) k_\mu k_\nu - \Pi^{H\mu\nu}(k)] \frac{\partial I_{ab}^{\mu\nu}}{\partial x^\lambda} =
= i \int \left( f_{\beta c d} f_{e g} f_{h i j} g \right) *D_{\mu\nu}^a(k') \tilde{\Sigma}_{k,k_1,k_2,k_3} \tilde{\Sigma}_{k',k'_1,k'_2,k'_3} *D_{\rho\sigma}(k_1 + k_2)
\times (A^{(0)}_\nu^c(k_3) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) \right)
\times \delta(k-k_1-k_2-k_3) \delta(k'-k_1'-k_2'-k_3') \prod_{i=1}^{3} dk_i dk_i' - (a \leftrightarrow b, k \leftrightarrow k', \text{compl. conj.}) \}.
\]

7 HTL-amplitudes

Let us transform equation (6.14) into a suitable form for our further research. For this purpose we perform the symmetrization of the coefficient functions in the integrand on the r.h.s. of equation (6.14) over possible permutations of color and Lorentz indices, and arguments of potentials of gauge fields within of the two groups \( (A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) \) and \( (A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) \) inside the statistical averaging angular brackets.

For example, for the first group this symmetrization leads to the expression

\[
f_{\beta c d} f_{e g} f_{h i j} \tilde{\Sigma}_{k,k_1,k_2,k_3} A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) A^{(0)\sigma}_{\lambda}(k_1) = \frac{1}{3!} \left( f_{\beta c d} f_{e g} f_{h i j} \tilde{\Sigma}_{k,k_1,k_2,k_3} - \tilde{\Sigma}_{k,k_1,k_2,k_3} \right)
\]
Furthermore, we use initial definitions (3.23), (3.19) and (3.21). We present the integration of the equilibrium distributions (3.8) (for [9, 27, 28]. Actually, by the definition of the \( \tilde{\Sigma} \)-function (5.9) the coefficient in front of the second

\[
\text{ing the correlators on the r.h.s. of the kinetic equation (6.14) in term s of HTL-amplitudes}
\]

\[\frac{f_{bcf} f_{fde}}{f_{fde} f_{fde}} f_{fde} f_{fde} (7.1)\]

In a similar way we transform the coefficient \( f_{a'e} f_{g'd'e' \Sigma^a' \Sigma^b' \Sigma^c' \Sigma^d'} \) in front of the second group of potentials of gauge fields.

The expression (7.1) is convenient because it enables us to rewrite the functions preceding the correlators on the r.h.s. of the kinetic equation (6.14) in terms of HTL-amplitudes [9, 27, 28]. Actually, by the definition of the \( \Sigma \)-function (5.9) the coefficient in front of \( f_{bcf} f_{fde} \) equals

\[
\sum_{k,k_1,k_2,k_3} - \sum_{k,k_2,k_1,k_3} - \mathcal{D}_{\rho a}(k_1 + k_2)(S_{k_1,k_2,k_3} + S_{k_1,k_2,k_3}) (S_{k_1,k_2,k_3} - S_{k_1,k_2,k_3})
\]

\[\] + (\nu \leftrightarrow \sigma, k_2 \leftrightarrow k_3) \quad (7.2)

Furthermore, we use initial definitions (3.23), (3.19) and (3.21). We present the integration measure \( d^4p \) as \( dp^0 |p|^2 dp |d\Omega, \) where \( d\Omega \) is the angular measure. Using the definition of the equilibrium distributions (3.8) (for \( \mu = 0 \)) and taking into account

\[
N_c \int_{-\infty}^{+\infty} |p|^2 dp \int_{-\infty}^{+\infty} p_0 dp_0 \frac{df^0(p_0)}{dp_0} = -\frac{3}{4\pi} \left( \frac{\omega_{pl}}{g} \right)^2,
\]

we perform the integral over \( dp_0 \) and the radial integral over \( dp \) in the expressions for \( S^{(I)} \)-function (3.19) and \( \Sigma^{(I)} \)-function (3.21).

This enables us to present the expression (7.2) in the following form

\[
-g^2 \{ *\Gamma^{\mu\sigma\lambda}(k, k_1, k_2, k_3) - \mathcal{D}_{\rho a}(k_1 + k_2) *\Gamma^{\mu\nu\rho}(k, k_1, k_2) *\Gamma^{\alpha\lambda\sigma}(k_1 + k_2, k_1, k_2) \]

\[\]

\[
- *\mathcal{D}_{\rho a}(k_1 + k_3) *\Gamma^{\mu\sigma\rho}(k_1 + k_3, k_1) *\Gamma^{\alpha\lambda\nu}(k_1 + k_3, k_1, k_3) \}
\]

\[\] \equiv -g^2 *\Gamma^{\mu\sigma\lambda}(k, k_1, k_2, k_3) \quad (7.3)

where

\[
*\Gamma^{\mu\nu\lambda}(k, k_1, k_2, k_3) \equiv \Gamma^{\mu\nu\lambda} + \delta \Gamma^{\mu\nu\lambda}(k, k_3, k_1, k_2) \quad (7.4)
\]

is the effective four-gluon vertex, which represents a sum of a bare four-gluon vertex

\[
\Gamma^{\mu\nu\lambda} = 2 g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}.
\]
and a corresponding HTL-correction

\[ \delta \Gamma_{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^{\mu\nu}v^{\lambda\sigma}}{vk + i\epsilon} \left[ \frac{1}{v(k + k_1)} + i\epsilon \left( \frac{\omega_2}{vk_2 - i\epsilon} - \frac{\omega_3}{vk_3 - i\epsilon} \right) \right], \]

\[ (v^\mu = (1, v), v^2 = 0); \]

\[ \ast \Gamma^{\mu\nu\rho}(k, k_1, k_2) \equiv \Gamma^{\mu\nu\rho}(k, k_1, k_2) + \delta \Gamma^{\mu\nu\rho}(k, k_1, k_2) \] (7.5)

is the effective three-gluon vertex. It also represents a sum of the bare three-gluon vertex and corresponding HTL-correction

\[ \Gamma^{\mu\nu\rho}(k, k_1, k_2) = g^{\mu\nu}(k - k_1)^\rho + g^{\nu\rho}(k_1 - k_2)^\mu + g^{\mu\rho}(k_2 - k)^\nu \] (7.6)

and corresponding HTL-correction

\[ \delta \Gamma^{\mu\nu\rho}(k, k_1, k_2) = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^{\mu\nu}v^{\rho}}{vk + i\epsilon} \left( \frac{\omega_2}{vk_2 - i\epsilon} - \frac{\omega_1}{vk_1 - i\epsilon} \right), \] (7.7)

\[ m_D^2 = 3 \omega_p^2 \] is the Debye screening mass.

The polarization tensor (3.17) in this notation takes the form

\[ \Pi^{\mu\nu}(k) = m_D^2 \left( g^{\mu0}g^{\nu0} - \omega \int \frac{d\Omega}{4\pi} \frac{v^{\mu\nu}}{vk + i\epsilon} \right). \]

In a similar way, the coefficient in front of the product of structure constants \( f_{bdf} f_{fde} \) in (7.1) may be presented as

\[-g^2 \{ \ast \Gamma^{\mu\nu\sigma}(k, -k_1, -k_2, -k_3) - \ast D_{\rho\alpha}(k_2 + k_3) \ast \Gamma^{\mu\nu\rho}(k, -k_1, -k_2 - k_3) \ast \Gamma^{\alpha\nu\sigma}(k_2 + k_3, -k_3, -k_2) \]

\[ + \ast D_{\rho\alpha}(k_1 + k_3) \ast \Gamma^{\mu\nu\rho}(k, -k_2, -k_1 - k_3) \ast \Gamma^{\alpha\lambda\nu}(k_1 + k_3, -k_1, -k_3) \}\n
\[ \equiv -g^2 \hat{\Gamma}^{\mu\nu\sigma}(k, -k_1, -k_3, -k_2). \] (7.8)

Performing transformation of the coefficient preceding \((A_{\nu'}^{(0)\ell d'}(k_1)A_{\sigma'}^{(0)\ell e'}(k_2)A_{\nu}^{(0)\ell e}(k_3))\) in a similar manner, we can cast the equation (6.14) in the following form

\[ \frac{\partial}{\partial k_\lambda} [k^2 g^{\mu\nu} - (1 + \xi^{-1})k^\mu k^\nu] - \Pi^{H\mu\nu}(k) \frac{\partial f_{ab}^{\mu\nu}}{\partial x^\lambda} \] (7.9)

\[ = i \frac{g^4}{(3l)^2} \int dk' [\{ f_{bdf}^{\ell de} \ast \hat{\Gamma}^{\ell\mu\nu\lambda\sigma}(k, -k_2, -k_1, -k_3) + f_{bdf} f_{fde} \ast \hat{\Gamma}^{\mu\nu\lambda\sigma}(k, -k_1, -k_3, -k_2) \]

\[ \times \ast D_{\mu\nu'}^{\ell d'}(k') \{ f_{\alpha\beta}^{d'e} g f_{\beta\gamma}^{d'\epsilon} \ast \hat{\Gamma}^{\ell\mu'\nu'\lambda\gamma\nu'}(k', -k_2', -k_1', -k_3') \}

\[ \ast D_{\mu\nu'}^{\ell d'}(k') \{ f_{\alpha\beta}^{d'e} g f_{\beta\gamma}^{d'\epsilon} \ast \hat{\Gamma}^{\ell\mu'\nu'\lambda\gamma\nu'}(k', -k_2', -k_1', -k_3') \}

\[ \times \delta(k - k_1 - k_2 - k_3)\delta(k' - k_1' - k_2' - k_3') \prod_{i=1}^{3} \int dk_i dk'_i - (a \leftrightarrow b, k \leftrightarrow k', \text{compl.conj.})]. \]
The r.h.s. of equation (7.9) has a non-trivial color structure that actually is well represented by non-trivial color structure of the initial dynamical equation (3.4). As will be shown below, this leads to a qualitative distinction between the elastic scattering probability \( w(k, k_1; k_2, k_3) \) of colorless plasmons in a hot QCD plasma and a similar one of plasmons in a hot QED plasma [24, 30].

At the end of this Section we present the identities analogous to the effective Ward one in hot gauge theories [27, 28, 9]. It can be shown that the following equalities hold

\[
k_\mu *\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) = *\Gamma^{\nu\lambda\sigma}(k_1, k, k_1, k_2, k_3),
\]

(7.10)

\[
k_{1\nu} *\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) = *\Gamma^{\mu\lambda\sigma}(k + k_1, k_2, k_3) - *\Gamma^{\mu\lambda\sigma}(k, k_1 + k_2, k_3)
\]

(similar contractions with \( k_{2\lambda}, k_{3\sigma} \)),

\[
k_\mu *\Gamma^{\mu\nu\rho}(k, k_1, k_2) = *D^{-1\nu\rho}(-k_1) - *D^{-1\nu\rho}(-k_2)
\]

(7.11)

(similar contractions with \( k_{1\nu}, k_{2\rho} \)). Here, \(*D^{-1\mu\nu}(k) = P^{\mu\nu}(k) *\Delta^{-1}\mu\nu(k) + Q^{\mu\nu}(k) *\Delta^{-1}\mu\nu(k)\)

is the inverse propagator without the gauge fixing term.

8 The kinetic equation for plasmons

We are now in a position to explicitly compute the probability of plasmon-plasmon scattering. First of all, we make the correlation decoupling of the sixth-order correlators on the r.h.s. of equation (7.9) in terms of the pair ones by the rule (6.11). After cumbersome calculations, the r.h.s. of equation (7.9) can be written as

\[
g^4 \text{Im} \left( *\Gamma^{\mu\nu\rho\lambda\sigma}(k) \right) \int \left\{ f^{abc} f^{def} *\tilde{\Gamma}^{\mu\rho\lambda\sigma}(k, -k_1, -k_1, -k_3) + f^{bde} f^{fcd} *\tilde{\Gamma}^{\mu\rho\lambda\sigma}(k, -k_1, -k_3, -k_2) \right\}
\]

\[
\times \left\{ f^{ace} f^{bde} *\tilde{\Gamma}^{\nu\lambda\rho\sigma'}(k, -k_1, -k_1, -k_3) + f^{ade} f^{bcd} *\tilde{\Gamma}^{\nu\lambda\rho\sigma'}(k, -k_1, -k_3, -k_2) \right\}
\]

\[
\times I_{\mu\nu}(k_3) I_{\lambda\lambda'}(k_1) I_{\rho\sigma'}(k_2) \delta(k - k_1 - k_2 - k_3) dk_1 dk_2 dk_3.
\]

In deriving of (8.1) we have used two relations, which satisfy \(*\tilde{\Gamma}^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3)\)

\[
\ast\tilde{\Gamma}^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) + \ast\tilde{\Gamma}^{\mu\lambda\nu\sigma}(k, k_2, k_1, k_3) + \ast\tilde{\Gamma}^{\mu\nu\sigma\lambda}(k, k_1, k_3, k_2) = 0,
\]

\[
\ast\tilde{\Gamma}^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) = \ast\tilde{\Gamma}^{\mu\nu\lambda\sigma}(k, k_3, k_2, k_1).
\]

(8.2)

Their correctness may be verified by a direct calculation using known properties of HTL-amplitudes [27, 28], entering into the definition of \(*\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3)\) (7.3). The first of the relations in (8.2) supplements, in the same sense, a similar relation for the HTL-correction \( \delta\Gamma \) to the bare four-gluon vertex, originally proposed by Frenkel and Taylor [28]. The second one is a property of invariance of the function \(*\tilde{\Gamma} \) when the momenta order is reversed. Notice that order of the space-time indices in equation (8.2) is important.
Furthermore, if we restrict our consideration to the study of plasmons scattering by plasmons, then in the spectral decomposition of $I_{\mu\nu'}(k_3)$, we determine the imaginary part of

$$\epsilon^\prime(k),$$

satisfy the equations

$$\text{Re} \epsilon^\prime(k) = \frac{1}{(k^2 - \Pi^l(k))}$$

In the propagator $^*\mathcal{D}_{\mu\nu'}(k)$ we also keep only the longitudinal part $-Q_{\mu\nu'}(k)^*\Delta^l(k)$. To determine the imaginary part of $^*\Delta^l(k) = 1/(k^2 - \Pi^l(k))$ we use the approximation (see e.g., Pustovalov and Silin in [22])

$$\frac{1}{k^2 - \Pi^l(k)} \approx \frac{P}{\text{Re}(k^2 - \Pi^l(k))} - \frac{i\pi \text{sign}(\text{Im}(k^2 - \Pi^l(k))) \delta(\text{Re}(k^2 - \Pi^l(k)))}{k^2 \text{Re} \epsilon^l(k)} - \frac{i\pi \text{sign} \omega \delta(\text{Re} \epsilon^l(k))}{k^2},$$

which holds for a small $\text{Im} \epsilon^l(k)$. Here, we consider that because of (2.3), $\text{sign}(\text{Im} \epsilon^l(k)) = \text{sign} \omega$. The symbol $P$ denotes a principal value. We perceive the $\delta$-function of the real part of the longitudinal permeability, which appears in (8.4) in the ordinary sense (6.6).

Substituting (6.6) into (8.4) and choosing $\omega = \omega^l_k > 0$ for definiteness, we obtain the required relation

$$\text{Im} ^*\mathcal{D}_{\mu\nu'}(k) \approx \frac{\pi}{k^2} Q_{\mu\nu'}(k) \left( \frac{\partial \text{Re} \epsilon^l(k)}{\partial \omega} \right)^{-1}_{\omega = \omega^l_k} \delta(\omega - \omega^l_k).$$

With allowing (8.3), (8.5) and performing the color algebra with the following identities

$$f^{bcf} f^{fde} f^{acg} f^{gde} = N_c^2 \delta^{ab},$$

$$f^{bcf} f^{fde} f^{adg} f^{gce} = \frac{1}{2} N_c^2 \delta^{ab},$$

we rewrite (8.3) in the form

$\delta^{ab} \pi g^4 N_c^2 \frac{1}{k^2} \left( \frac{\partial \text{Re} \epsilon^l(k)}{\partial \omega} \right)^{-1}_{\omega = \omega^l_k} \delta(\omega - \omega^l_k) \int \{|^*\Gamma(k, -k_2, -k_1, -k_3)|^2 + |^*\Gamma(k, -k_1, -k_3, -k_2)|^2 + \text{Re} \left( ^*\Gamma(k, -k_2, -k_1, -k_3) \times \Gamma^l(k_1) \Gamma^l(k_2) \Gamma^l(k_3) \delta(k - k_1 - k_2 - k_3) \right| \}

\int \frac{1}{u^2(k_1)u^2(k_2)u^2(k_3)}$$

Here, we denote

$$^*\Gamma(k, k_1, k_2, k_3) \equiv ^*\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) u_\mu(k) u_\nu(k_1) u_\lambda(k_2) u_\sigma(k_3).$$

By virtue of the fact, that the spectral intensity functions $I^l(k_i), i = 1, 2, 3$, entering into (8.7), satisfy the equations $\text{Re} \epsilon^l(k_i) I^l(k_i) = 0$ [10], they have the following structure

$$I^l(k_i) = I^l_{k_i} \delta(\omega_i - \omega^l_{k_i}) + I^l_{-k_i} \delta(\omega_i + \omega^l_{k_i}), i = 1, 2, 3,$$
where \( I^i_{k_i} \) are certain functions of the wavevectors \( k_i \).

We substitute (8.9) into (8.7) and perform the integration over the frequency \( d\omega \) with the help of the \( \delta \)-functions. The function \( I^i_{k_i} \) in the l.h.s. of the kinetic equation (7.9) also has a structure of form (8.9). Here, we keep only a positive branch \( \omega > 0 \) in agreement with (8.3). Furthermore performing integration (7.9) over \( d\omega \), where the r.h.s. of (7.9) has the form (8.7), as result we obtain

\[
\pi g^4 N_c^2 \frac{1}{k^2} \left( \frac{\partial \text{Re} \varepsilon^i(k)}{\partial \omega} \right)_{\omega=\omega_k}^{-1} \frac{\partial I^i_{k_i}}{\partial t} + (-k^2) \left( \frac{\partial \text{Re} \varepsilon^i(k)}{\partial k} \right)_{\omega=\omega_k} \frac{\partial I^i_{k_i}}{\partial k} \]  

\[
= \pi g^4 N_c^2 \frac{1}{k^2} \left( \frac{\partial \text{Re} \varepsilon^i(k)}{\partial \omega} \right)_{\omega=\omega_k}^{-1} \int [Q(k, k_1, k_2, k_3)\delta(k + k_1 + k_2 + k_3) \\
+ Q(k, -k_1, -k_2, -k_3)\delta(k - k_1 - k_2 - k_3) + Q(k, k_1, k_2, -k_3)\delta(k + k_1 + k_2 - k_3) \\
+ Q(k, k_1, -k_2, k_3)\delta(k + k_1 - k_2 + k_3) + Q(k, -k_1, k_2, k_3)\delta(k - k_1 + k_2 + k_3) \\
+ Q(k, k_1, -k_2, -k_3)\delta(k + k_1 - k_2 - k_3) + Q(k, -k_1, k_2, -k_3)\delta(k - k_1 + k_2 - k_3) \\
+ Q(k, -k_1, -k_2, k_3)\delta(k - k_1 - k_2 + k_3)]_{\text{on-shell}} I^i_{k_1} I^j_{k_2} I^l_{k_3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,
\]

where we have defined

\[
Q(k, k_1, k_2, k_3) \equiv \frac{1}{\bar{u}^2(k)\bar{u}^2(k_1)\bar{u}^2(k_2)\bar{u}^2(k_3)} \times \{|^*\bar{\Gamma}(k, k_2, k_1, k_3)|^2 + |^*\bar{\Gamma}(k, k_1, k_2, k_3)|^2 + \text{Re} (|^*\bar{\Gamma}(k, k_2, k_1, k_3)^*\bar{\Gamma}^i(k, k_1, k_3, k_2))\}. \tag{8.11}
\]

In equation (8.10) the \( x \)-dependence of \( I^i_{k_i}, I^j_{k_i}, \) etc is understood, although not written explicitly.

The first term inside the square brackets on the r.h.s. of (8.10) describes the process of simultaneous fusion or emission of four plasmons in the plasma. Considering that the \( \delta \)-function has no support on the plasmon mass shell, its contribution to the kinetic equation vanishes. The second, third and fourth terms describe the decay of one plasmon into three or the fusion of three plasmons into one. As was mentioned in section 2, these processes are forbidden by the conservation law (2.23). Therefore these terms also vanish. The remaining terms describe the scattering of two plasmons off two plasmons. Three terms imply the existence of three possible channels of a given process, which change the plasmon numbers density \( N_{k_i}^i \)

\[
g^* + g_1^* \leftrightarrow g_2^* + g_3^*, \quad g^* + g_2^* \leftrightarrow g_1^* + g_3^*, \quad g^* + g_3^* \leftrightarrow g_1^* + g_2^*.
\]

If we perform replacements \( k_1 \leftrightarrow k_2 \) in the next to last term on the r.h.s. of (8.10), and \( k_1 \leftrightarrow k_3 \) in the last term, then the r.h.s. of (8.10) can be presented as

\[
\pi g^4 N_c^2 \frac{1}{k^2} \left( \frac{\partial \text{Re} \varepsilon^i(k)}{\partial \omega} \right)_{\omega=\omega_k}^{-1} \int [Q(k, k_1, -k_2, -k_3) + Q(k, -k_2, k_1, -k_3) + Q(k, -k_3, -k_2, k_1)]_{\text{on-shell}} I^i_{k_1} I^j_{k_2} I^l_{k_3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.
\]
× δ(ω_k + ω_{k_1} - ω_{k_2} - ω_{k_3}) \delta(\mathbf{k} + \mathbf{k_1} - \mathbf{k_2} - \mathbf{k_3}) I_k^l \, I_{k_1}^l \, I_{k_2}^l \, \mathop{d\mathbf{k}_1} \, \mathop{d\mathbf{k}_2} \, \mathop{d\mathbf{k}_3}.

By using the definition of kernel $Q$ (8.11) and the properties (8.2), it can be shown, that

$$Q(k, -k_2, k_1, -k_3) + Q(k, -k_3, -k_2, k_1) = 2Q(k, k_1, -k_2, -k_3).$$

Taking into account the last relation, it goes over from the function $I_k^l$ to the number density of plasmons

$$N_k^l = -\left(k^2 \frac{\partial \text{Re} \varepsilon^l(k)}{\partial \omega} \right)_{\omega = \omega_k^l} I_k^l$$

and recovering the complete form for finite values $N_k^l$, we obtain the required Boltzmann equation (2.25), where the probability of plasmon-plasmon scattering is defined by the following expression

$$w(k, k_1; k_2, k_3) = 3\pi (2\pi)^5 g^4 N_c^2 \left[ \frac{1}{k^2 k_1^2 k_2^2 k_3^2} \left( \frac{\partial \text{Re} \varepsilon^l(k)}{\partial \omega} \right)^{-1} \left( \frac{\partial \text{Re} \varepsilon^l(k_1)}{\partial \omega_1} \right)^{-1} \right. \left\{ | \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) |^2 + | \ast \tilde{\Gamma}(k, k_1, -k_3, -k_2) |^2 
\right.
\left. + \text{Re} \left( \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) \ast \tilde{\Gamma}^\dagger(k, k_1, -k_3, -k_2) \right) \right\} \frac{1}{\bar{u}^2(k) \bar{u}^2(k_1) \bar{u}^2(k_2) \bar{u}^2(k_3)}. \right. \tag{8.12}$$

The result (8.12) is rather unexpected. As we can see from the expression (8.12), this probability does not reduce to the squared module of one scalar function, as this occurs in the Abelian plasma. Here, the scattering probability is defined by the squared module of two independent scalar functions and their interference\(^3\), in spite of the fact that in this paper we restrict our consideration to a study of the nonlinear interaction of only colorless excitations. This radically distinguishes the Boltzmann equation (2.25), describing the effects of the collisions among colorless soft excitations from the corresponding Boltzmann equation including the effects of the collisions among colorless hard excitations [34, 18, 19]. In the last case, the Boltzmann equation, corrected to color factors, fully coincides with the corresponding one in the Abelian plasma. This is a point which we find difficult to interpret and therefore additional analysis of this problem is required (see, also discussion in conclusion).

The scattering probability can be written in the form, which is manifestly symmetric under permutations of the external momenta $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ and $\mathbf{k}_3$

$$w(k, k_1; k_2, k_3) = \pi (2\pi)^5 g^4 N_c^2 \left[ \frac{1}{k^2 k_1^2 k_2^2 k_3^2} \left( \frac{\partial \text{Re} \varepsilon^l(k)}{\partial \omega} \right)^{-1} \left( \frac{\partial \text{Re} \varepsilon^l(k_1)}{\partial \omega_1} \right)^{-1} \left( \frac{\partial \text{Re} \varepsilon^l(k_2)}{\partial \omega_2} \right)^{-1} \right. \left\{ | \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) |^2 + | \ast \tilde{\Gamma}(k, k_1, -k_3, -k_2) |^2 + | \ast \tilde{\Gamma}(k, k_1, -k_3, k_2) |^2 + \right. \left. \frac{1}{\bar{u}^2(k) \bar{u}^2(k_1) \bar{u}^2(k_2) \bar{u}^2(k_3)} \right\}. \right.$$
Re \left( \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) \ast \tilde{\Gamma}^\dagger(k, k_1, -k_3, -k_2) \right) + \Re \left( \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) \ast \tilde{\Gamma}^\dagger(k, k_1, -k_2, -k_3) \right) \\
+ \Re \left( \ast \tilde{\Gamma}(k, k_2, -k_1, -k_3) \ast \tilde{\Gamma}^\dagger(k, k_1, -k_2, -k_3) \right) \frac{1}{u^2(k)u^2(k_1)u^2(k_2)u^2(k_3)} \right)_{\text{on-shell}}.

This expression is suitable for checking the symmetry conditions (2.26), which is imposed on the plasmon-plasmon scattering probability.

Recall that the function \( \ast \tilde{\Gamma} \), which appears in the expression for probability (8.12), is defined by expressions (8.8) and (7.3). As we have shown in [10], the expression (8.8) (exactly, its part, independent on a gauge-parameter) is gauge invariant at least in a class of covariant and temporal gauges. With regard to the reasoning in section 4 this problem of the dependence of the plasmon-plasmon scattering probability (8.12) on a gauge parameter coming from the gauge fixing term in the gluon propagator, is discussed below.

Let us separate the terms with a gauge parameter from the expression \( \ast \tilde{\Gamma}(k, -k_2, k_1, -k_3) \). By using the Ward identities (7.10) and (7.11), we have

\[
\xi \{ D_{\rho\alpha}(-k_1 + k_2)\Delta^0(-k_1 + k_2) \ast \Gamma^{\mu\rho}(k, -k_3, k_1 - k_2) \ast \Gamma^{\alpha\lambda\sigma}(-k_1 + k_2, k_1, -k_2) \\
+ D_{\rho\alpha}(-k_1 + k_3)\Delta^0(-k_1 + k_3) \ast \Gamma^{\mu\rho}(k, -k_2, k_1 - k_3) \ast \Gamma^{\alpha\lambda\nu}(-k_1 + k_3, k_1, -k_3) \}
\times \tilde{u}_\mu(k)\tilde{u}_\lambda(k_1)\tilde{u}_\sigma(k_2)\tilde{u}_\nu(k_3).
\]

(8.13)

It is easily shown that expression on the r.h.s. of (8.13) vanish either because \( \ast D^{-1}\mu\nu(k) \) is transverse, or by the definition of the mass-shell condition, i.e.

\[
k_\mu \ast D^{-1}\mu\nu(k) = 0, \ \ast D^{-1}\mu\nu(k)|_{\omega = \omega_k} = 0.
\]

(8.14)

If we carry out the following replacements on the r.h.s. of the expressions (8.8), (7.3)

\[
\bar{u}_\mu(k) \rightarrow \bar{u}_\mu(k) \equiv \frac{k^2}{ku}(k_\mu - u_\mu(ku)),
\]

and

\[
\ast D_{\rho\alpha}(k) \rightarrow \ast \tilde{D}_{\rho\alpha}(k) = \ast D_{\rho\alpha}(k) - \left( \frac{\sqrt{-2k^2\bar{u}^2}}{k^2(ku)} C_{\rho\alpha}(k) + \frac{\bar{u}^2(k)}{k^2(ku)^2} D_{\rho\alpha}(k) \right) \ast \Delta^1(k)
\]

\[
- \xi D_{\rho\alpha}(k)\Delta^0(k) - \xi_0 \frac{k^2}{(ku)^2} D_{\rho\alpha}(k),
\]

where \( \xi_0 \) is a gauge parameter in the temporal gauge, then it can be proved that a similar statement holds in the temporal gauge also.
Thus, the gauge-dependent parts of \( w(k, k_1; k_2, k_3) \) drop out, since they are multiplied by the mass-shell factor. These factors are proportional to \( (\omega - \omega'_{k}) \). However, in the quantum case Baier et al [35] observed that a naive calculation of the gluon damping rate in a covariant gauge appears to violate this consideration. The mass-shell factor is multiplied by the integral involving a power infrared divergence which is cut-off exactly on the scale \( (\omega - \omega'_{k}) \sim g^2 T \). This problem was careful discussed in consideration of the nonlinear Landau damping [10], when in (7.3) and consequently (8.13) it is necessary to set
\[
k_1 = -k, \quad k_2 = -k_1, \quad k_3 = k_1.
\] (8.15)

We have shown, that the gauge-dependent part of the nonlinear Landau damping rate vanishes on a mass-shell at least for the plasmons with zero momentum. In the case of the process of the plasmon-plasmon scattering considered here, the problem on the dependence of \( w(k, k_1; k_2, k_3) \) on a gauge parameter is more subtle, since instead of (8.15), we have condition: \( k + k_1 = k_2 + k_3 \) only. Research into this nontrivial question goes beyond our present goal.

9 Lifetimes of plasmons

To calculate the lifetimes of colorless plasmons we first linearize the Boltzmann equation (2.25) (here, we assume that off-equilibrium fluctuation is perturbative small), writing the number density of plasmons as
\[
N^l_k = N^l_{eq}(k) + \delta N^l_k
\]
where \( N^l_{eq}(k) = \left( e^{\omega^l_k/T^*} - 1 \right)^{-1} \) is the Planck distribution function and \( T^* \) is a certain constant, which can be interpreted as a plasmon gas temperature in the statistical equilibrium state. Then we find
\[
\frac{\partial \delta N^l_k}{\partial t} + V^l_k \frac{\partial \delta N^l_k}{\partial x} = (9.1)
\]
\[
= \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} (2\pi)^4 \delta(\omega^l_{k_1} + \omega^l_{k_2} - \omega^l_{k_3}) \delta(k + k_1 - k_2 - k_3) w(k, k_1; k_2, k_3)
\times \{ \delta N^l_k [N^l_{eq}(k_2)N^l_{eq}(k_3)(N^l_{eq}(k_1) + 1)] - N^l_{eq}(k_1)(N^l_{eq}(k_2) + 1)(N^l_{eq}(k_3) + 1)] + \\
\delta N^l_{k_2} [N^l_{eq}(k_3)(N^l_{eq}(k_2) + 1)(N^l_{eq}(k_1) + 1)] - N^l_{eq}(k_1)(N^l_{eq}(k_2) + 1)(N^l_{eq}(k_3) + 1)] + (k \leftrightarrow k_1, \quad k_2 \leftrightarrow k_3) \}\right) .
\]
This equation can be further simplified if we use the following parametrization for off-equilibrium fluctuation of the occupation number \( \delta N^l_k \) [12, 18]
\[
\delta N^l_k \equiv -\frac{dN^l_{eq}(k)}{d\omega^l_k} W^l_k = (1/T^*) N^l_{eq}(k)(N^l_{eq}(k) + 1) W^l_k.
\] (9.2)
Let us introduce the momentum transfers, which carries the soft gluon exchanged in the collision of two plasmons, setting

\[ \mathbf{k}_2 = \mathbf{k} - \mathbf{q}, \quad \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{q}. \] (9.3)

Performing the integration over \( d\mathbf{k}_3 \), replacing the integration over \( d\mathbf{k}_2 \) by one with respect to momentum transfer and taking into account (9.2) and (9.3), finally we derive a linearized Boltzmann equation for \( \mathcal{W}_k \) function

\[ \frac{\partial \mathcal{W}_k^l}{\partial t} + \mathbf{V}_k \cdot \frac{\partial \mathcal{W}_k^l}{\partial \mathbf{x}} = - \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{N_{eq}^l(\mathbf{k}_1)(N_{eq}^l(\mathbf{k} - \mathbf{q}) + 1)(N_{eq}^l(\mathbf{k}_1 + \mathbf{q}) + 1)}{(N_{eq}^l(\mathbf{k}) + 1)} \times w(\mathbf{k}, \mathbf{k}_1; \mathbf{q}) (2\pi)^3 \delta(\omega_\mathbf{k} - \omega_\mathbf{k}_1 - \omega_{\mathbf{k}_1+\mathbf{q}} + \omega_{\mathbf{k}_1+\mathbf{q}}). \] (9.4)

Here, we goes over from the function \( w(\mathbf{k}, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3) \) (8.12) to a new function \( w(\mathbf{k}, \mathbf{k}_1; \mathbf{q}) \)

\[ w(\mathbf{k}, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3)|_{\mathbf{k}_2 = \mathbf{k} - \mathbf{q}, \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{q}} \equiv w(\mathbf{k}, \mathbf{k}_1; \mathbf{q}). \]

Based on the exact form of the r.h.s. of equation (9.4), we define the lifetimes of the plasmon of momentum \( \mathbf{k} \) as follows

\[ \frac{1}{\tau_{pl}(\mathbf{k})} = \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{N_{eq}^l(\mathbf{k}_1)(N_{eq}^l(\mathbf{k} - \mathbf{q}) + 1)(N_{eq}^l(\mathbf{k}_1 + \mathbf{q}) + 1)}{(N_{eq}^l(\mathbf{k}) + 1)} \times w(\mathbf{k}, \mathbf{k}_1; \mathbf{q}) (2\pi)^3 \delta(\omega_\mathbf{k} - \omega_\mathbf{k}_1 - \omega_{\mathbf{k}_1+\mathbf{q}} + \omega_{\mathbf{k}_1+\mathbf{q}}). \] (9.5)

Here, the integrand has a more involved structure in comparison with a similar expression for the case of the lifetimes of the hard transverse gluon [34]. The reason of this fact is that the momentum of the soft quasiparticles (plasmons) becomes of the same order as momentum transfers and under these conditions one must take into account the non-trivial character of the frequency dependence \( \omega \) of the collective gluon modes on momentum \( \mathbf{k} \) and vertex corrections.

It is not difficult to estimate the order of the expression (9.5) at the momentum scale \( gT \). Considering the plasmon gas in thermal equilibrium with hard particles from the heat bath, i.e. \( T^* \simeq T \), and using the definition of \( w \) function, we obtain

\[ \frac{1}{\tau_{pl}} \sim g^3 N_c^2 T. \] (9.6)

However, obtaining a numerical factor of proportionality in (9.6) is a complicated problem even for limiting case of \( \mathbf{k} = 0 \)-mode. Here, we restrict our consideration to following general remark, which somewhat simplifies the matter.

In proving the gauge invariance of the nonlinear Landau damping rate, we have shown [10] that the \( \ast \Gamma \) function (8.3), (7.3) entering into the definition of the scattering probability (8.12) can be introduced in its simplest form

\[ \ast \Gamma(k, -k_2, k_1, -k_3) = \]
\[ k^2 k_1^2 k_2^2 ln \{ * \Gamma^{0000}(k, -k_2, k_1, -k_3) - * D_{\rho \rho}(k - k_3)^* \Gamma^{000}(k, -k_3, k_1 - k_2)^* \Gamma^{000}(k_1 + k_2, k_1, -k_2) \] 
\[ - * D_{\rho \rho}(k - k_2)^* \Gamma^{000}(k, -k_2, k_1 - k_3)^* \Gamma^{000}(k_1 + k_3, k_1, -k_3) \} \bigg|_{\text{on-shell}}. \]

However, even if the last expression is accounted for, after performing the integration over solid angles in \( \delta \Gamma^{0000} \) and \( \delta \Gamma^{000} \) HTL-amplitudes (see, e.g., [28]), we obtain expressions which are very cumbersome. For this reason passage to the limit \( |k| \rightarrow 0 \) is non-trivial.

The estimate (9.6) shows, that at the momentum scale \( gT \) a ”collision” damping rate of the soft longitudinal mode is suppressed by a power of \( g \) relative to a value of the nonlinear Landau damping rate (\( \sim g^2 T \)) [10]. Therefore at the soft momentum scale it can be neglected by the influence of the plasmon interactions among themselves on the relaxation dynamics of a plasma excitations.

As was mentioned in Sec. 2, the process of the nonlinear Landau damping leads to an effective pumping of energy across the spectrum towards small wavenumbers. By the effect of pumping, all plasmons will tend to concentrate near small \( |k| = |k_0| \rightarrow 0 \). However, phase space, within which the plasmons are occupied, proportional to \( |k_0|^3 \), will also be very small (a similar state, when a great many plasmons with small wavenumbers are mainly concentrated in a small phase volume, in the theory of strong turbulent electron-ion plasma this is sometimes called a plasmon condensate [36]). By virtue of this fact intensive collisions between plasmons arise and this scattering process is described by the Boltzmann equation (2.23). This leads to a ”throw out” of plasmons from region of small \( |k| \) and thus a suppression of the increase of the \( k = 0 \)-mode. In mathematical language this denotes that for definite values of the momentum, the magnitude of the off-equilibrium fluctuation \( \delta N^I_k \) becomes as large as \( N^{eq}_k(k) \), and therefore a linearization of the Boltzmann equation (2.23) is no longer valid, like the estimation (9.6), corrected within the framework of this approximation. In the region \( |k| \ll gT \), where collisions among plasmons start to play a role, it is necessary to solve the exact nonlinear integro-differential equation, whose r.h.s. is considered as both the process of the scattering plasmons off thermal particles, and the process of four-plasmon decays, i.e.

\[
\frac{\partial N^I_{k_1}}{\partial t} + V^I_{k_1} \frac{\partial N^I_{k_1}}{\partial x} = -3(\omega_{pl}/g)^2 \int \frac{d\mathbf{k}_1}{(2\pi)^3} \left( \omega_{k_1} - \omega_{k_1}^I \right) Q(k, k_1) N^I_{k_1} N^I_{k_1} \\
+ \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{d\mathbf{k}_3}{(2\pi)^3} (2\pi)^4 \delta(\omega_{k_1}^I + \omega_{k_1}^I - \omega_{k_2}^I - \omega_{k_3}^I) \delta(k + k_1 - k_2 - k_3) \\
n \times w(k, k_1; k_2, k_3) (N^I_{k_1} N^I_{k_2} N^I_{k_3} + N^I_{k_1} N^I_{k_2} N^I_{k_3} - N^I_{k_1} N^I_{k_1} N^I_{k_2} - N^I_{k_1} N^I_{k_1} N^I_{k_3})
\]

where

\[
Q(k, k_1) = (2\pi)^4 N_c \left( \frac{\partial \text{Re} \varepsilon^I(k)}{\partial \omega} \right)^{-1} \left( \frac{\partial \text{Re} \varepsilon^I(k_1)}{\partial \omega} \right)^{-1} \omega = \omega_{k_1}^I \omega_1 = \omega_{k_1}^I \\
\times \int \frac{d\Omega}{4\pi} \delta(\omega_{k_1}^I - \omega_{k_1}^I - \mathbf{v}(k - k_1)) \left| \mathcal{M}^c(k, k_1) + \mathcal{M}^I(k, k_1) + \mathcal{M}^d(k, k_1) \right|^2.
\]
The amplitudes $\mathcal{M}^c$, $\mathcal{M}^∥$ and $\mathcal{M}^⊥$ have the following forms [10]

$$\mathcal{M}^c(k, k_1) = \frac{g^2}{|k||k_1|} \frac{1}{\omega_k^2 \omega_{k_1}^2} \frac{(kv)(k_1v)}{\omega_k - (kv)},$$

$$\mathcal{M}^∥(k, k_1) = \frac{g^2}{|k||k_1|} \frac{(k-k_1)v}{(k-k_1)^2} \times \left( \frac{(k-k_1)^2 \Delta^l(k-k_1)}{\omega_1(\omega - \omega_1)^2} \right) \delta \Gamma^{ijl}(k,-k_1,-k+k_1) k^i k_1^j (k-k_1)^l, \quad \omega = \omega_k, \omega_1 = \omega_{k_1}$$

$$\mathcal{M}^⊥(k, k_1) = \frac{g^2}{|k||k_1|} \frac{([n(k-k_1)]v)}{n^2(k-k_1)^2} \times \left( \frac{\Delta^l(k-k_1)}{\omega_1} \right) \Gamma^{ijl}(k,-k_1,-k+k_1) k^i k_1^j [n(k-k_1)]^l, \quad \omega = \omega_k, \omega_1 = \omega_{k_1}.$$

Here, $n \equiv [kk_1]$. We present the amplitudes $\mathcal{M}^c$, $\mathcal{M}^∥$ and $\mathcal{M}^⊥$ in their simplest form, defined by temporal gauge.

A qualitative analysis of the solution behavior of a similar equation in the case of electron-ion plasma with a rather crude approximation of the kernel $Q(k, k_1)$ and the probability of plasmon-plasmon scattering $w(k, k_1; k_2, k_3)$, was performed by Kovrizhnykh [30]. He has shown, in particular, that really in the kinematic regime of small wavenumbers $|k|$, the process of plasmon-plasmon scattering starts to play a dominant role, that prevents an increase of the $k = 0$-mode. In the case of the non-Abelian plasma, an analogous qualitative investigation of the solution of equation (9.7) presents difficulties because of the above-mentioned complexity of the integrands on the r.h.s. of equation (9.7), and therefore it requires some additional approximations, allowing these functions to be made more visible and suitable for numerical calculations.

### 10 Conclusion

Within the framework of the semiclassical kinetic theory of a hot gluon plasma we have obtained the transport equation for the plasmons, taking into account four-plasmon decay. The probability of plasmon-plasmon scattering at the leading order in the coupling constant is derived. This is defined with the help of three-gluon and four-gluon effective vertices, and the effective propagator only, as the probability of the process of scattering of the plasmon by hard QGP particles [10]. It is proved that the scattering probability is gauge-independent at least within a class of covariant and temporal gauges.

In this paper we have restricted our consideration to a derivation of the Boltzmann equation for the simplest collective Bose-modes of hot gluon plasma, colorless plasmons. This fact was reflected in deciding on a trivial color structure of the plasmon occupation...
number: $N_{k}^{l,ab} \sim \delta^{ab}$. It is clear that such a choice of trivial color structure is completely neglected by the fundamental property of plasmons in the non-Abelian plasma, such as the availability of color charge. This is a very important difference from the Abelian case, where the plasmons do not carry electric charge. While the initial state of the problem can have a trivial color structure, the dynamics of these collective modes can change the color of the plasmons. In light of these remarks, here we are dealing with an Abelianized version of more complicated colored plasmon dynamics. It is a first step towards a full description of plasmon transport properties to derive the Boltzmann equation which takes into account the precession of the color charge of the plasmon. It is probably only in the construction of such a complete equation that we gain a reasonable explanation of the structure of the scattering probability (8.12).

The scheme of the derivation of the Boltzmann equation for colorless plasmons, proposed in this paper, admits the straightforward generalization to the case, when the number density of the plasmon $N_{k}^{l,ab}$ has a non-trivial color structure, e.g. such as $N_{k}^{l,ab} = (T_{c})^{ab} N_{k}^{l,c}$. In this case, as was discussed in Sec. 4, it is necessary to assume that the regular (background) part of the field $A^{R}_{\mu}(X)$ is different from zero. The density matrix, effective propagator, three-gluon, four-gluon vertices become the functionals of the background field $A^{R}_{\mu}(X)$ [9, 18], which assumed vanishing at $X_{0} \to -\infty$. This significantly complicates the problem of construction of the required kinetic equation. It can somewhat simplify the problem if it is considered that the fluctuation $\delta N_{k}^{l,ab}$ is a small perturbation in relation to the equilibrium value $\delta^{ab} N_{eq}^{l}(k)$, where $N_{eq}^{l}(k)$ is the Planck distribution, and restrict the consideration to the linearized Boltzmann equation for colored plasmons. The exact consideration of a given question will be subject of separate research.

There is an independent test of the validity of the derived Boltzmann equation (2.25) (exactly, its linearized version (9.4)). As was mentioned in introduction in [18] the fundamental derivation of the Boltzmann equation for a high temperature Yang-Mills plasma was proposed by Blaizot and Iancu within the CTP formalism framework. Their derivation relies on a gauge-covariant gradient expansion of the Kadanoff-Baym equations for the gluon two-point function. The Boltzmann equation has emerged as the quantum transport equation at leading order in $g$ for the gauge-covariant fluctuation $\delta G$ of a hard gluon propagator.

Besides the above-mentioned paper, the Kadanoff-Baym equations for the off-equilibrium propagator of the soft gluon $D_{\mu\nu}(X,Y)$, which are formally identical to those for a hard gluon propagator $G_{\mu\nu}(X,Y)$, are written out. These equations are used in [18] only to deduce the relation between the off-equilibrium gauge-covariant fluctuation $\delta D^{<}(k, x)$ and the gauge-covariant fluctuation of the leading-order soft gluon polarization tensor $\delta \Pi^{<}(k, x)$, and the problem of self-interactions of the soft fields is not considered here. However, in principal there is nothing to forbid the use of these equations for research in the dynamics of the soft fields and construction of the relevant transport equation within the framework of the scheme, suggested by Blaizot and Iancu [18]. Here, by $\delta \Pi^{<}(k, x)$
we mean the fluctuation of the next-to-leading order of the soft gluon self-energy, involving three- and four-gluon off-equilibrium vertices with soft external lines. The relevant effects of self-interaction of the soft fields are encoded in these functions. Here, we note that in the real-time formalism, technical complications resulting from the doubling of degrees of freedom are arisen. However in recent years a suitable technique allowing one effectively to work not only with the single-particle propagator, but also with three- and four-point functions, has been developed [8]. This greatly simplifies calculations in real time and enables us to derive the transport equation for the soft gluons, in particular, the plasmons, directly from the underlying quantum field theory and compare it with the equation, obtained in this paper in the context of the semiclassical approximation. This formal scheme is also needed to specify the limits of validity of the semiclassical kinetic approach to the research of the processes of nonlinear interaction of the soft fields in a hot QCD plasma.

**Acknowledgement**

We are very grateful to Stanislaw Mrówczyński for useful correspondence. This work was supported in part by Grant INTAS (No. 2000-15) and Grant for Young Scientist of Russian Academy of Sciences (No. 1999-80).
References

[1] D.N. Zubarev, *Nonequilibrium Statistical Thermodynamics* (Consultant Bureau, New York, 1974).

[2] S.V. Erokhin, A.V. Prozorkevich, S.A. Smoljansky, and V.D. Toneev, Theor. Math. Phys. 94 (1993) 416; M.V. Tokarchuk, T. Arimitsu, and A.E. Kobrun, Condensed Matter Phys. 1 (1998) 605.

[3] Yu.L. Klimontovich, *The Statistical Theory of Nonequilibrium Processes in a Plasma* (Pergamon Press, Oxford, 1967); Yu.L. Klimontovich, *Kinetic Theory of Nonideal Gases and Nonideal Plasmas* (Pergamon Press, Oxford, 1982); Yu.L. Klimontovich, *Statistical Physics* (Harwood Academic Publishers, 1986).

[4] A.V. Selikhov, Phys. Lett. 268 (1991) 263; Erratum Phys. Lett. B 285 (1992) 398; A.V. Selikhov and M. Gyulasssy, Phys. Rev. C 49 (1994) 1726; Yu.A. Markov and M.A. Markova, Theor. Math. Phys. 103 (1995) 444; ibid. 108 (1996) 977.

[5] S.R. de Groot, W.A. van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).

[6] J. Schwinger, J. Math. Phys. 2 (1961) 407; P.M. Bakshi and K.T. Mahanthappa, J. Math. Phys. 4 (1963) 1; *ibid.* 4 (1963) 12; L.V. Keldysh, Sov. Phys. JETP 20 (1965) 1018. See, also, E.M. Lifshitz and L.P. Pitaevskii, *Physical Kinetic* (Pergamon, Oxford, 1981).

[7] A. Niemi and G. Semenoff, Ann. Phys. (N.Y.) 152 (1984) 105; G.Z. Zhou, Z.B. Su, B.L. Hao, and L. Yu, Phys. Rep. 118 (1985) 1; R.D. Jordan, Phys. Rev. D 33 (1986) 444; E. Calzetta and B.L. Hu, Phys. Rev. D 37 (1988) 2878; T.S. Evans, Phys. Rev. D 47 (1993) 4196; P. Aurencche and T. Becherrawy, ENSLAPP-A-452/93, NSF-ITP-93-155 (unpublished); M.A. van Eijk, R. Kobes, and Ch.G. van Weert, Phys. Rev. D 50 (1994) 4097; F. Gelis, Nucl. Phys. B 508 (1997) 483.

[8] M.E. Carrington and U. Heinz, Eur. Phys. J. C 1 (1998) 619; Hou Defu and U. Heinz, Eur. Phys. J. C 4 (1998) 129; Hou Defu, E. Wang and U. Heinz, J. Phys. G 24 (1998) 1861; M.E. Carrington *et al*., Phys. Rev. D 61 (2000) 25011; Hou Defu, M.E. Carrington, R. Kobes, and U. Heinz, Phys. Rev. D 61 (2000) 085013.

[9] J.-P. Blaizot and E. Iancu, Phys. Rev. Lett. 70 (1993) 3376; Nucl. Phys. B 417 (1994) 608.

[10] Yu.A. Markov and M.A. Markova, Transp. Theor. Stat. Phys. 28 (1999) 645; J. Phys. G 26 (2000) 1581.

[11] V.P. Silin and V.N. Ursov, Sov. Phys. Dokl. 30 (1985) 594.
[12] G. Baym, H. Monien, C.J. Pethick, and D.G. Ravenhall, Phys. Rev. Lett. 64 (1990) 1867.

[13] U. Heinz, Phys. Rev. Lett. 51 (1983) 351; Ann. Phys. (N.Y.) 161 (1985) 48; ibid. 168 (1986) 148; J. Winter, J. Physique 45 (1984) C6-53; H.-Th. Elze, M. Gyulassy, and D. Vasak, Phys. Lett. B 177 (1986) 402; Nucl. Phys. B 276 (1986) 706; H.-Th. Elze and U. Heinz, Phys. Rep. 183 (1989) 81; St. Mrówczyński, Phys. Rev. D 39 (1989) 1940; Yu.A. Markov and M.A. Markova, Theor. Math. Phys. 111 (1997) 601.

[14] V.V. Lebedev and A.V. Smilga, Physica A 181 (1992) 187; S. Jeon and L.G. Yaffe, Phys. Rev. D 53 (1996) 5799; M.E. Carrington, R. Kobes, and E. Petitgirard, Phys. Rev. D 57 (1998) 2631; M.E. Carrington and R. Kobes, Phys. Rev. D 57 (1998) 6372; E. Petitgirard, Phys. Rev. D 59 (1999) 045004; D. Bödeker, Nucl. Phys. B 566 (2000) 402.

[15] D. Bödeker, Phys. Lett. B 426 (1998) 351; Nucl. Phys. B 559 (1999) 502.

[16] P. Arnold, D.T. Son, and L.G. Yaffe, Phys. Rev. D 59 (1999) 105020.

[17] D.F. Litim and C. Manuel, Nucl. Phys. B 562 (1999) 237; M.A. Valle Basagoiti, EHU-FT/9905, hep-ph/9903462.

[18] J.-P. Blaizot and E. Iancu, Nucl. Phys. B 557 (1999) 183.

[19] J.-P. Blaizot and E. Iancu, Nucl. Phys. B 570 (2000) 326.

[20] F. Guerin, Phys. Rev. D 63 (2001) 045017.

[21] B. Bezzerides and D.F. DuBois, Ann. Phys. (N.Y.) 70 (1972) 10.

[22] B.B. Kadomtsev, Plasma Turbulence (Academic Press, New York, 1965); V.N. Tsytovich, Non-linear Effects in Plasma (Pergamon, Oxford, 1970); V.V. Pustovalov and V.P. Silin, Proc. P.H. Lebedev Inst. 61 (1972) 42; V.N. Tsytovich, Phys. Rep. 178 (1989) 261.

[23] A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko, and K.N. Stepanov, Plasma Electrodynamics (Pergamon, Oxford, 1975).

[24] V.N. Tsytovich, Theory of a Turbulent Plasma (Consultant Bureau, New York, 1977).

[25] U. Heinz, K. Kajantie, and T. Toimela, Ann. Phys. (N.Y.) 176 (1987) 218.

[26] H.A. Weldon, Phys. Rev. D 28 (1983) 2007.

[27] E. Braaten and R.D. Pisarski, Nucl. Phys. B 337 (1990) 569; ibid. B 339 (1990) 310.

[28] J. Frenkel and J.C. Taylor, Nucl. Phys. B 334 (1990) 199.

[29] R. Jackiw and V.P. Nair, Phys. Rev. D 48 (1993) 4991.
[30] L.M. Kovrizhnykh, Sov. Phys. JETP 22 (1966) 168.

[31] V.E. Zakharov, Zh. Eksp. Theor. Fiz. 51 (1966) 688 (in Russian); V.A. Liperovsky and V.N. Tsytovich, Izv. Vusov Radiofiz. IX (1966) 469 (in Russian); ibid. XII (1969) 823; S.B. Pikelner and V.N. Tsytovich, Sov. Phys. JETP b 28 (1969) 514; A.A. Galeev and R.Z. Sagdeev, Nonlinear Theory of Plasma in: Problems of Plasma Theory, Vol. 7 (Atomizdat, Moscow, 1973); Rev. Plasma Phys. 7 (1979) 1.

[32] In our paper (Markov and Markova in Ref. [13]) it was shown that existence of the spin degree of freedom for gluons in the semiclassical approximation results in two covariant coupled dynamical equations for the distribution function $f$ and the function related to circular polarization. Therefore if required, more subtle effects associated with gluons polarization can be without difficulty taken into account in the derivation scheme of kinetic equation that outlined at present work.

[33] U. Heinz and P.J. Siemens, Phys. Lett. B 158 (1985) 11.

[34] H. Heiselberg and C.J. Pethick, Phys. Rev. D 47 (1993) R769.

[35] R. Baier, G. Kunstatter, and D. Schiff, Nucl. Phys. B 388 (1992) 287.

[36] V.E. Zakharov, Sov. Phys. JETP 35, 908 (1972); K. Komilov, L. Stenflo, F.Kh. Khakimov, and V.N. Tsytovich, Zh. Techn. Fiz. XLVI, 680 (1976) (in Russian).