A group field theory for 3d quantum gravity coupled to a scalar field

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ABSTRACT

We present a new group field theory model, generalising the Boulatov model, which incorporates both 3-dimensional gravity and matter coupled to gravity. We show that the Feynman diagram amplitudes of this model are given by Riemannian quantum gravity spin foam amplitudes coupled to a scalar matter field. We briefly discuss the features of this model and its possible generalisations.

I. INTRODUCTION

Spin foam models [1, 2] represent a purely combinatorial and algebraic implementation of the sum-over-histories approach to quantum gravity, in any signature and spacetime dimension, with an abstract 2-complex playing the role of a discrete spacetime, and algebraic data from the representation theory of the Lorentz group playing the role of geometric data assigned to it. Indeed, the first model of quantum gravity to be ever proposed, the Ponzano-Regge model, was a spin foam model for Euclidean quantum gravity without cosmological constant [3]. This approach has recently been developed to a great extent in the 3-dimensional case. It is now clear that it provides a full quantisation of pure gravity [4], whose relation with the one obtained by other approaches is well understood [5, 6]. Moreover, matter can be consistently included in the picture [4, 7], providing a link between spin foam models and effective field theory [8] living on a non-commutative geometry. This picture allows us to naturally address the semi-classical limit of spin foam models and shows that quantum gravity in dimension 3 effectively follows the principle of the so-called deformed (or doubly) special relativity [9].

The group field theory formalism [10] represents a generalisation of matrix models of 2-dimensional quantum gravity [11]. It is a universal structure lying behind any spin foam model for quantum gravity [12, 13], providing a third quantisation point of view on gravity [10] and allowing us to sum over pure quantum gravity amplitudes associated with different topologies [14]. In this picture, spin foams, and thus spacetime itself, appear as (higher-dimensional analogues of) Feynman diagrams of a field theory defined on a group manifold and spin foam amplitudes are simply the Feynman amplitudes weighting the different graphs in the perturbative expansion of the quantum field theory. On the other hand, we can construct a non commutative field theory whose Feynman diagram amplitudes reproduce the coupling of matter fields to 3d quantum gravity for a trivial topology of spacetime [8].Remarkably, the momenta of the fields are labelled also by group elements. Moreover, in three dimensions there is a duality between matter and geometry, and the insertion of matter can be understood as the insertion of a topological defect charged under the Poincaré group [4]. This suggests that one should be able to treat the third quantisation of gravity and the second quantisation of matter fields in one stroke (see [15] for an early attempt). The purpose of this paper is to study how the coupling of matter to quantum gravity is realised in the group field theory, and whether it is possible to write down a group field theory for gravity and particles that reproduces the amplitudes derived in [4] coupling quantum matter to quantum geometry.

This is what we achieve in the present work. The way the correct amplitudes are generated as Feynman amplitudes of the group field theory is highly non-trivial. It requires an extension of the usual group field theory (gft) formalism to a higher number of field variables, and produces an interesting intertwining of gravity and matter degrees of freedom, as we are going to discuss in the following. The formalism we present is still based on the classical SU(2), it bears strong similarity with a recent work of Krasnov [16] who considered gft based on the quantum group DSU(2). This should not be a surprise since it is well understood that the particle spin foam amplitudes have a quantum group structure hidden in them [4, 13]. The gft model we propose here is, however, very different from the ones considered in Krasnov’s work since our Feynman graphs reproduce explicitly the insertion of particles coupled to gravity.

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II. THE PONZANO-REGGE SPIN FOAM MODEL COUPLED TO POINT PARTICLES

The general form of Feynman graph amplitudes for spinning particles coupled to 3 dimensional quantum gravity - eg the Ponzano-Regge model- has been written in [4]. In this paper we focus on the case of spinless particles and we recall in this section the definition of these amplitudes before deriving them from a gft. We start from a triangulation $\Delta$ of our spacetime $M$ and consider also the dual $\Delta^*$: dual vertices, edges and faces correspond respectively to tetrahedra, faces and edges of $\Delta$. We choose our Feynman graph, $\gamma$, to be embedded in the triangulation $\Delta$ such that edges of $\gamma$ are edges of the triangulation. Each edge of $\gamma$ is labelled by an angle $\theta \in [0, \pi]$

$$\theta = \kappa m, \quad \kappa = 4\pi G_N,$$

where $G_N$ is Newton’s constant, $\kappa$ is the inverse Planck mass and $m$ is the mass of the particle. To each angle $\theta$ we associate an element of the Cartan subgroup $H$ of SU(2)

$$h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

which corresponds to a rotation of angle $2\theta$ around a given axis. Given a group $G$, here SU(2), we assign group elements $g_e$ to all dual edges $e^*$ of the triangulation. We constrain the holonomies around dual faces $f^* \sim e$ to be flat if there is no particle and we constrain it to be in the conjugacy class $\theta$ if $e$ is an edge of $\gamma$. More precisely, let us denote by $G_e (= G_{f^*})$ the product of the group elements around a dual face (or plaquette) $f^* \sim e$:

$$G_e = G_{f^*} = \prod_{e^* \in \partial f^*} g_{e^*}^{e^*},$$

where $e^*(e^*) = \pm 1$ records the orientation of the edge $e^*$ in the boundary of the (dual) face $f^*$. The amplitude is given by

$$Z_M(\gamma_\theta) = \Delta(\theta)^{|E_\gamma|} \int \prod_{e^*} dg_{e^*} \int \prod_{e^* \in \gamma} du_e \prod_{e^* \notin \gamma} \delta(G_e) \prod_{e \in \gamma} \delta(G_e u_e h_\theta u_e^{-1}),$$

(1)

where $dg$ is the normalised Haar measure and $\delta(g)$ the corresponding delta function on $G$, $\Delta(\theta) \equiv \sin(\theta)$ and $|E_\gamma|$ is the number of edges in the particle graph $\gamma$. We see that two types of group elements arise in the construction of this amplitude, the $g_{e^*}$ describe pure gravity excitations and the $u_e$ variables are associated with the particle degrees of freedom. They arise because the insertion of a particle locally breaks the Lorentz and translational symmetries of the gravity model and the former gauge transformation becomes dynamical at the location of the particle [4]. The $u_e$ are then interpreted as giving the direction of the particle momenta propagating along the edge $e$. The fact that we are talking about spinless particles implies that $u_e$ should not be thought of as an element of $G$ but as an element of $G/H$ (with $H = U(1)$), that is momentum space. The insertion of spinning particles can be achieved by taking into account a non trivial value under the $H$ part of $u_e$. The main lesson which we learn from this amplitude, and which gives the key idea leading to a gft construction of such an amplitude, is the fact that we need both $G$ variables describing the gravity excitation and $G/H$ variables describing the propagation of particles. We can expand the $\delta$ functions in terms of characters

$$\delta(g) = \sum_j d_j \chi_j(g),$$

$d_j = 2j + 1$ being the dimension of the spin $j$ representation and perform the integration over $g_{e^*}$, $u_e$ in order to obtain a state sum model

$$Z_M(\gamma_\theta) = \Delta(\theta)^{|E_\gamma|} \sum_{\{j_x\}} \prod_{e^* \notin \gamma} d_{j_e} \prod_{e^* \in \gamma} \chi_{j_e}(h_\theta) \prod_t \{ j_{e_1}, j_{e_2}, j_{e_3} \},$$

(2)

where the summation is over all edges of $\Delta$ and the product of normalised $6j$ symbols is over all tetrahedra $t$. For each tetrahedron, the admissible triples of edges, e.g. $(j_{e_1}, j_{e_2}, j_{e_3})$, corresponds to faces of this tetrahedron. Boulatov [17] was the first to show that the amplitude [17] can be obtained as a Feynman graph evaluation of a group field theory. It is important to note however that this amplitude is generically divergent. It is now understood [18] that this divergence is due to a translational gauge symmetry (equivalent on-shell to diffeomorphism symmetry) acting on the Ponzano-Regge model. This symmetry should be gauge-fixed in order to obtain well defined and triangulation
III. A GFT MODEL FOR 3D QUANTUM GRAVITY COUPLED TO SCALAR MATTER

A. Action and Feynman rules

We shall now define a field theory on a group manifold, whose Feynman expansion gives the above modified Ponzano-Regge model. We consider a generic real field $\phi_j$ on a 6-argument field, this chunk of quantum geometry carries also additional degrees of freedom, labelled by the interpretation of a '3rd quantised' chunk of quantum geometry \[10, 12\]. However, in this extended formulation based on a 6-argument field, this chunk of quantum geometry carries also additional degrees of freedom, that acquire the physical meaning of particle degrees of freedom (more precisely particle momenta) when a mass parameter is inserted in a suitable way, as we are going to show in the following. Let us now list the symmetries that this field is required to satisfy.

- We require that $\phi$ is invariant under (even) elements $\sigma$ of the permutation group of three elements $S_3$, acting on pairs of field variables $(g_1, u_1)$:

$$\phi(g_1, g_2, g_3; u_1, u_2, u_3) = \phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}; u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}).$$

If we require the field to be invariant under even permutations of the three pairs of arguments, then this is equivalent to dealing with a complex field instead, with the odd permutations mapping the field to its complex conjugate \[11\], and the Feynman amplitudes produced by the corresponding group field theory are in one-to-one correspondence with orientable 2-complexes, as explained in \[12, 19\]. We can more generally require the field to transform under an arbitrary representation (not necessarily reducible) of $S_3$. This will affect the type of 2-complexes generated by the perturbative expansion of the theory. We stress, however, that this would not imply any change for the amplitudes of the Feynman diagrams.

- Furthermore, we pick a $U(1)$ subgroup $H$, of $SU(2)$, with the interpretation of the invariance subgroup for the particle momenta, and project three of the arguments into $SU(2)/U(1)$ equivalence classes

$$P_\delta \phi(g_1, g_2, g_3; u_1, u_2, u_3) \equiv \int_{H^3} \prod_{i=1}^3 db_i \, \phi(g_1, g_2, g_3; u_1 b_1, u_2 b_2, u_3 b_3),$$

so that the field becomes in fact a function over three copies each of $SU(2)$ and $SU(2)/U(1)$.

- Finally, we project the first half of the field, i.e. the part dependent on the first three arguments, into its $SU(2)$ invariant part, by imposing invariance under simultaneous right action of $SU(2)$ on the first three arguments:

$$P_\alpha \phi(g_1, g_2, g_3; u_1, u_2, u_3) \equiv \int_{SU(2)} d\alpha \, \phi(g_1 \alpha, g_2 \alpha, g_3 \alpha; u_1, u_2, u_3).$$

This last symmetry has a geometric interpretation, as in the usual Boulatov model. It imposes the closure of the triangle of which the field $\phi$ represents the 2nd quantisation, by constraining the spin variables dual to the $g_i$ variables associated to its three edges.

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1 A connected set of edges of $\Delta \backslash \gamma$ passing through every vertex of $\Delta \backslash \gamma$
Given such a field, we can write down a Boulatov-like action, with the extra \(u\) variables simply mimicking the relations among the gravity degrees of freedom \(g\):

\[
S[\phi] = \frac{1}{2} \int \prod_{i=1}^{3} dg_i du_i [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)] [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)]
+ \frac{\lambda}{4!} \int \prod_{i=1}^{6} dg_i du_i [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)] [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)]
\times [P_{\alpha} P_{b} \phi(g_4, g_5, g_6; u_4, u_5, u_6)] [P_{\alpha} P_{b} \phi(g_4, g_5, g_6; u_4, u_5, u_6)].
\]

As we will see the Feynman amplitudes obtained from this model are proportional to those obtained by the Boulatov model, i.e. the usual Ponzano-Regge spin foam amplitudes describing pure 3d Riemannian quantum gravity. This shows that the \(u\) variables are completely redundant at this stage and do not have any real physical meaning. They are going to acquire it soon, however. Now we introduce a mass parameter in the theory, turning this redundant description of pure quantum gravity into a model for gravity coupled to scalar matter. We define a mass insertion operator \(P_{b}\), acting on the field \(\phi\) as follows:

\[
P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3) \equiv \phi(u_1 h_{\theta} u_1^{-1}, g_1, g_2, g_3; u_1, u_2, u_3),
\]

where \(h_{\theta} = \exp(\theta J_0) \in H\); \(\theta\) is half the deficit angle created by the presence of a mass \(m\), \(\theta = 4\pi G m^2\), being the Planck mass; \(J_0\) is the generator of the U(1) subgroup \(H\), the same subgroup under which the \(u_i\) variables of the field are invariant and \(\epsilon\) is the non-trivial Weyl group element, given in the fundamental representation by:

\[
\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

We define then a new group field theory model, representing 3d Riemannian quantum gravity coupled to scalar matter, whose dynamics are given by the action:

\[
S[\phi] = \frac{1}{2} \int \prod_{i=1}^{3} dg_i du_i [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)] [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)]
- a[P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)] [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)]
+ \frac{\lambda}{4!} \int \prod_{i=1}^{6} dg_i du_i [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)] [P_{\alpha} P_{b} \phi(g_1, g_2, g_3; u_1, u_2, u_3)]
\times [P_{\alpha} P_{b} \phi(g_4, g_5, g_6; u_4, u_5, u_6)] [P_{\alpha} P_{b} \phi(g_4, g_5, g_6; u_4, u_5, u_6)],
\]

where \(a\) is a free parameter and all integrals are with respect to the normalised Haar measure. Let us write the Feynman rules of the theory in coordinate space. We have to identify kinetic and vertex operators. For this purpose, we write the action as

\[
S[\phi] = \frac{1}{2} \int \prod_{i=1}^{3} dg_i du_i \prod_{j=1}^{3} dB_j dB_j \phi(g_i, u_i) \sum K(g_i, g_j, u_i, \tilde{u}_j) \phi(g_j, \tilde{u}_j)
+ \int \prod_{i,j} dg_i du_i \sum \mathcal{V}(g_{ij}, u_{ij}) \phi(g_{ij}, u_{ij}) \phi(g_{ij}, u_{ij}).
\]

where in this integral, \(i \neq j\), and \(\phi(g_{ij}, u_{ij}) = \phi(g_{i1}, g_{i2}, g_{i3}; u_{i1}, u_{i2}, u_{i3})\), and so forth. The kinetic and vertex operators are

\[
K(g_i, g_j, u_i, \tilde{u}_j) = \int \prod_{i=1}^{3} dB_i \delta(g_i g_j^{-1}) \delta(u_i b_i \tilde{u}_i^{-1}) - a \int \prod_{i=1}^{3} dB_i \delta(u_i h_{\theta} u_i^{-1} g_i g_j^{-1}) \delta(u_i b_i \tilde{u}_i^{-1}) \prod_{j=2}^{3} \delta(g_j g_j^{-1}) \delta(u_j b_j \tilde{u}_j^{-1}),
\]

\[
\mathcal{V}(g_{ij}, u_{ij}) = \frac{\lambda}{4!} \int \prod_{i=1}^{4} d\alpha_i \prod_{j>i} dB_{ij} \delta(\alpha_i g_{ij}^{-1} g_{ij} \alpha_i^{-1}) \delta(b_{ij} u_{ij}^{-1} u_{ij} b_{ij}^{-1}).
\]
We have purposefully discarded the $\alpha$ variables in the $\delta$-functions of the kinetic term. This does not change, up to an overall factor, the computation of the amplitudes since $P_0$ commutes with $P_g$. We have also sidelined the sum over permutations, for ease of notation. Care should be taken in inverting the kinetic term, however, on the subspace of symmetric fields only. On this subspace, the kinetic term is indeed diagonal and we can proceed as follows. We now define the operators $I$ and $K_\theta$:

$$K(g, \bar{g}, u, \bar{u}) = I - a K_\theta.$$  

The propagator is the inverse of the kinetic operator. Furthermore, the operator $K_\theta$ satisfies $(K_\theta)^2 = I$, as laid out below:

$$(K_\theta)^2 = \int d^3 \bar{g} d^3 \bar{u} \delta(\bar{g}_1 h_\theta^{-1} \bar{u}_1 u_1 \bar{g}_3^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1})$$

$$\times \delta(\bar{u}_1 h_\theta \bar{u}_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1}) \delta(\bar{u}_1 h_\theta \bar{u}_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1})$$

We integrate with respect to the $\bar{g}$ variables

$$(K_\theta)^2 = \int d^3 \bar{u} d^3 \bar{u} \delta(\bar{u}_1 h_\theta \bar{u}_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1})$$

and then the $\bar{u}$ variables

$$(K_\theta)^2 = \int d^3 \bar{b} d^3 \bar{b} \delta(\bar{u}_1 h_\theta \bar{u}_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1})$$

But $b_1 h_\theta^{-1} b_1^{-1} = b_1 \bar{h}_\theta^{-1} b_1^{-1} = h_\theta^{-1}$ since $ch_\theta^{-1} = h_\theta^{-1}$ and since $b_1$ and $\bar{h}_\theta^{-1}$ are in the same commutative $U(1)$ subgroup. Furthermore, $\tilde{b}_1 \epsilon = -\bar{b}_1^{-1}$. Finally, redefining $-\bar{b}_1^{-1}$ as $\bar{b}_1$, $b_2 \bar{b}_2 \rightarrow b_2$ and $b_3 \bar{b}_3 \rightarrow b_3$ gives us

$$(K_\theta)^2 = \int d^3 \bar{u} d^3 \bar{u} \delta(g_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1}) \delta(\bar{u}_1 h_\theta \bar{u}_1 \bar{g}_1^{-1}) \delta(g_2 \bar{g}_2^{-1}) \delta(g_3 \bar{g}_3^{-1})$$

This leads to a nice closed form for the propagator

$$\mathcal{P}(g_1, \bar{g}_j, u_1, \bar{u}_j) = \frac{I + aK_\theta}{1 - a^2}.$$  

### B. Feynman amplitudes and spin foam formulation

To construct a generic Feynman amplitude, we will analyse the structure and gluing properties of the propagator and vertex operator:

#### 1. Vertex Operator

We scrutinise in two parts:

$$\mathcal{V}(g_{ij}, u_{ij}) = \frac{1}{4!} \sum_{i=1}^4 \int \prod_{j>i} \alpha_i \prod_{j>i} db_{ij} \delta(a_j g_{ji}^{-1} g_{ij} \alpha_i^{-1}) \delta(b_{ij} u_{ij}^{-1} u_{ij} b_{ij}^{-1}).$$

The $\delta$-functions over the $g$ variables are the usual holonomies around the six wedges dual to the edges $e$, of a tetrahedron. So the model already has the structure of a 2-complex dual to a triangulation. The $u$ variables represent the momenta of the particles and as such are identified with the edges of the tetrahedron. Each edge of the tetrahedron is shared by two triangles. The $\delta$-functions above ensure that the momentum associated to an edge is the same when viewed from either of these triangles. This extra structure is not present in the Boulantov model.
2. Propagator

The operator \( P \) glues two tetrahedra at a triangular interface. From the analysis of the vertex operator above, we know that each triangle of a tetrahedron has three wedges (dual to each of its three edges) and three momenta associated to it. The propagator has two terms with different gluing properties.

\[
P(g_i, g_j, u_i, u_j) = \frac{1}{1 - a^2} I + \frac{a}{1 - a^2} K_\theta.
\]

For \( P_{\text{massless}} \), the \( \delta \)-functions over the \( g \) variables glue three wedges of one tetrahedron to three wedges of another tetrahedron pairwise, giving the holonomies around three composite wedges. The \( \delta \)-functions over the \( u \) variables ensure that the momenta on the edges of the two triangles match. For \( P_{\text{massive}} \), two of the \( \delta \)-functions over the \( g \)-variables act as for \( P_{\text{massless}} \), similarly for the \( u \) variables. One of the \( \delta \)-functions, however, couples the \( g_1 \) and \( u_1 \) variables, effectively placing a massive particle at the point where the edge of the tetrahedron intersects the composite wedge. The final \( \delta \)-function to consider, \( \delta(u_1 b_1 e_i u_1^{-1}) \), places a tag on the edge \( e_i \) of the tetrahedron with the mass insertion. The tag ensures that if another propagator further around the sequence of Feynman graph edges \( e^* \), forming the boundary of a (dual) face \( f^* \), inserts a mass along the edge \( e \), it will cancel with the first according to the property \((K_\theta)^2 = I \). We find in general that we can only have two possibilities when a face is fully assembled: no particle on the edge \( e \), or one particle on the edge, according to whether there have been an even or an odd number of insertions of \( K_\theta \) on the boundary of the face. To be clearer, we will calculate more explicitly the amplitude for a generic dual face in the next subsection. In the end the partition function for the field theory, when expanded in Feynman graphs (i.e. in a power series for \( \lambda \)) takes the form:

\[
Z = \int D\phi \, e^{-S[\phi]} = \sum_\Gamma \frac{\lambda^{v[\Gamma]}}{sym[\Gamma]} Z[\Gamma],
\]

where \( v[\Gamma] \) is the number of vertices in the Feynman graph; \( sym[\Gamma] \) is its symmetry factor; and \( Z[\Gamma] \) is the amplitude for each Feynman graph, being given explicitly by:

\[
Z[\Gamma] = N[\Gamma] \int e^{ \sum e^* d\alpha_e \prod e \, du_e \prod e \in \partial f \, \delta(G_e) \prod e \in \partial f \, \delta(G_e h_\theta u_e^{-1})},
\]

where \( N[\Gamma] \) is a normalisation factor, discussed later, arising from the (partial) redundancy of the \( u \) variables in each diagram and the \( a \) dependence; \( \alpha_e \) is the holonomy along an edge of the Feynman graph; \( G_e = \prod_{e' \in \partial f} \alpha_{e'}^{-1} \) is the holonomy around a face \( f^* \) of the Feynman graph, which is dual to the edge of \( e \) of the triangulation; and \( \partial f \) is the set of edges of the triangulation that have a particle present. We recognise in \( Z[\Gamma] \) the Ponzano-Regge amplitude \( \mathbb{P} \). We see that the group field theory we have defined gives, in addition to a sum over all possible quantum gravity spin foams arising as usual as Feynman graphs of the theory, a sum over all possible massive spinless particle insertions in the spin foam, interpreted as a sum over all possible Feynman diagrams for a scalar field theory. Each gravity + particle configuration is weighted exactly by the amplitude of the Ponzano-Regge model coupled with massive spinless particles given in \( \mathbb{P} \), and provides us also with a definite normalisation factor for each of these amplitudes.

C. Amplitude for a generic face of the Feynman graph

Consider a face \( f^* \) of a Feynman graph, the boundary of which is formed by \( N \) contiguous edges, \( e^* \), labelled \( e_1^*, \ldots, e_N^* \). The amplitude for this face is

\[
A(f^*) = \int (d \ldots) P_{f^*}^{1} V_{12}^{f^*} \ldots P_{v_N}^{f^*} V_{v_N v_1}^{f^*},
\]

where \( P_i^{f^*} \) are the \( \delta \)-functions from the propagator along \( e_i^* \) relating to the face \( f^* \); \( V_{ij}^{f^*} \) are the \( \delta \)-functions from the vertex where \( e_i^* \) and \( e_j^* \) meet, pertaining to the face \( f^* \). If we contract all the \( g \) variable \( \delta \)-functions we get a final \( \delta \)-function of the form

\[
A(f^*) = \sum_{n=0}^{N} \int (d \ldots) \delta(G_{f^*} u_{a_1} h_\theta u_{a_1}^{-1} \times \cdots \times u_{a_n} h_\theta u_{a_n}^{-1}) \times (\delta \text{-functions over the } u \text{ variables}),
\]
the sum is over different combinations of mass insertions; \( n \leq N \) is the number of particle insertions in that specific term and \( G_f \) is the holonomy around the face built up from a product of \( \alpha \). If there are \( n \) particle insertions in the \( \eta \) variable part then there will be \( n \) \( \delta \)-functions in the \( u \) variable part with \( \epsilon \) inserted. Once we contract the \( u \) variables we get that the masses cancel just as in the calculation of \((K_\theta)^2 = I\). In the end, for each term in the sum we have two possibilities: For \( n \) even we get

\[
\delta(G_f)\delta(b_f),
\]

where \( b_f \) is a product of the \( b \) variables around the face. Therefore we get no particle insertion and a pure gravity face modulo an extra factor which we take into the normalisation. For \( n \) odd we get

\[
\delta(G_f u_\theta u^{-1})\delta(b_f \epsilon),
\]

thus a single particle insertion and another factor which we take into the normalisation.

### D. Overall normalisation of the Feynman graph

For a kinetic term with the structure \( I - aK_\theta \), the propagator takes the form:

\[
\frac{1}{1 - a^2} (I + aK_\theta),
\]

as we have shown. This produces \( a \)-dependent amplitudes when the expansion in Feynman graphs is performed. We have not specified what this \( a \) is and how it depends on the physical parameters of gravity or matter; indeed there is quite some freedom involved in choosing a specific expression for \( a \) and only further analysis of the model we proposed can narrow the range of possibilities down to a restricted one. This parameter will enter the normalisation coefficients controlling the relative strength of Feynman diagrams with and without particles. The normalisation factor will clearly contain an overall factor \((1 - a^2)^{-|e^*|}\) where \(|e^*|\) denotes the number of dual edges of the two complex. There will be an additional factor \( a \) each time \( K_\theta \) is inserted along a dual edge. If an even number of \( K_\theta \) are inserted along a face no particle circulates along that face. A useful formula in order to get the right normalisation factor in a given example is

\[
(I + aK_\theta)^n = \frac{1}{2} ((1 + a)^n + (1 - a)^n) I + \frac{1}{2} ((1 + a)^n - (1 - a)^n) K_\theta.
\]

Along with this numerical factor, there is a singular factor coming from a redundant \( \delta \)-function for each face as shown in (23, 24). For each face not carrying a particle we have a factor

\[
\int_{U(1)} db \delta(b) = \sum_j (2j + 1),
\]

and for each face carrying a particle we have a factor

\[
\int_{U(1)} db \delta(b\epsilon) = \sum_j (2j + 1)(-1)^j,
\]

the sum, being over integer \( j \), is obtained from the character expansion of the delta function. These expressions are unfortunately ill defined and they need a regularisation. A proper regularisation that can preserve all the symmetries of the theory is to replace the usual SU(2) group by a quantum group \( U_q(SU(2)) \). It is expected that with this choice the key features of the model can be preserved and that the corresponding normalisation coefficients are given by

\[
\sum_{j=0}^{N} [2j + 1] q^j = \frac{1 + t + (q + q^{-1})t^{N+1}}{(1 - q^2t)(1 - q^2t)},
\]

\[
2 \text{ a natural possibility in view of } [1] \text{ is to consider } a = \Delta(\theta)
\]
with \( q = \exp(i \pi/27) \), \([n]_q = (q^n - q^{-n})/(q - q^{-1})\) and \( t = -1\) for a face with particle and \( t = +1\) otherwise. Let us recall that if we consider the original Boulatov model such factors do not arise since they come from a redundant summation over spins dual to the \( u \) variables. The Feynman graph amplitudes of the Boulatov model are, however, not equal to the physical quantum gravity amplitudes. In order to get the quantum gravity amplitudes one has to divide out the infinite volume of a gauge symmetry. This is conveniently done by restricting the summation over spins as presented in the first section. This gauge-fixing procedure which is well defined at the level of the quantum gravity amplitudes is, however, not fully understood at the level of the gft. When we extend the gft to include the momenta as presented in the first section. This gauge-fixing procedure which is well defined at the level of the quantum gravity amplitude. This is conveniently done by restricting the summation over spins not equal to the physical quantum gravity amplitudes. In order to get the quantum gravity amplitudes one has to be understood already at the gft level both for the Boulatov model and for our extension including particles; that is, whether we can already for the Boulatov model identify at the level of the gft the translational (or diffeomorphism) symmetry responsible for the divergences of the naive gravity amplitude. A similar identification should also be implemented for our particle model. We do not resolve this issue in the present work.

IV. DISCUSSION

A. Features of the model

We have seen that the model correctly generates spin foam configurations with some dual faces carrying particle data, i.e. a mass label, indicating that a particle of the given mass is propagating along the edge of the triangulation dual to that face. This means that the group field theory produces all possible Feynman graphs for a scalar field embedded in the triangulation dual to the quantum gravity 2-complex, specifying the field propagator on each line of the Feynman graph, and this only. Interestingly, this is enough, in this 3-dimensional setting, for specifying fully the dynamics of the particles, i.e. their interaction. In fact, this is dictated by the Bianchi identity constraining the sum of curvatures in the boundary of any 3-cell of the dual complex around any given vertex of the triangulation. When one or more particles are meeting at that vertex, thus interacting there, this implies momentum conservation for their interaction, which is the only content of any \( \phi^n \) theory. Because any number of particles can be incident to any given vertex of the triangulation in the model we propose, this means that this corresponds to a scalar field theory with a potential given by a sum over any power of the field operators: \( \phi^j(x) + \phi^k(x) + \ldots \).

We have seen that the crucial property of the modified kinetic term we propose for producing mass insertions in the spin foam amplitudes is, besides the extension of the field to 6 arguments, the property \((K_\phi)^2 = I\) for the operator \( K_\phi\) inserting the mass of the particles in the group field theory action. It is interesting to note that the presence of this operator which inserts particles also breaks a symmetry that the pure gravity model possesses. This bears some similarity with the fact that particles in 3d can be understood as defects breaking the translational symmetry of the theory without matter \[4\]. The symmetry is the following: Let’s consider the transformation

\[
\phi(g_1, g_2, g_3; u_1, u_2, u_3) \rightarrow \phi(v_1g_1, g_2, g_3; w_1u_1, u_2, u_3),
\]

where \( v_1, w_1 \) are arbitrary fixed group elements. This transformation is clearly a symmetry of the pure gravity action \[7\]. This symmetry is, however, broken by the insertion of a mass term and only the transformation

\[
\phi(g_1, g_2, g_3; u_1, u_2, u_3) \rightarrow \phi(v_1g_1, g_2, g_3; v_1u_1, u_2, u_3),
\]

preserves the action \[4\].

B. A direct generalisation

The model we presented in section \[11\] produces mass insertions in different faces of the spin foam 2-complex by application of the operator \( K_\phi \) in the 1st argument of the field, carrying the gravity variable \( g_1 \). Because of permutation symmetry, the fact that one has chosen the 1st argument of the field for inserting a mass parameter and not, say, the 2nd is irrelevant, as one can easily convince oneself. Still, one may find the fact that a mass parameter is inserted in only -one- of the arguments of the field a bit unsatisfactory, for symmetry reasons. Here we want to discuss briefly what happens when one relaxes this condition. The result is that one can write down a generalised version of the model presented above, that is, however, basically equivalent to it, and leads to the same type of graphs being generated. One can consider defining a generalised kinetic term with matter insertions, defined using a sum of operators \( K_\phi(1), K_\phi(2), K_\phi(3)\), each \( K_\phi(i)\) inserting a mass parameter in the i-th argument of field, thus having
as kinetic term an operator with the structure $I - a(K_\theta(1) + K_\theta(2) + K_\theta(3))$. It is obvious that a model like this would generate exactly the same type of graphs and amplitudes as the one we have defined above. It is also easy to realise that, once such an operator is introduced, there will be graphs in which the insertion of a $K_\theta(1)$ and a $K_\theta(2)$, say, in different propagators would produce the same amplitude that would have been generated by the use in the kinetic term of an operator of the form $K_\theta(1, 2)$, i.e., an operator inserting a mass parameter in both the 1st and 2nd arguments of the field at once; and indeed one could generalise further the kinetic term to an operator of the form: 

$$I - a(K_\theta(1) + K_\theta(2) + K_\theta(3)) - b(K_\theta(1, 2) + K_\theta(2, 3) + K_\theta(1, 3)).$$

Carrying this line of reasoning even further, one is led to the kinetic term:

$$\mathcal{K} = I - a(K_\theta(1) + K_\theta(2) + K_\theta(3)) - b(K_\theta(1, 2) + K_\theta(2, 3) + K_\theta(1, 3)) - cK_\theta(1, 2, 3) \equiv I - K_{total}. \quad (31)$$

where the $K_\theta(1, 2, 3)$ is defined as the operator inserting a mass parameter in all the first three arguments of the field. Again, this generalised kinetic term leads to the same kind of Feynman graphs and amplitudes, as it is easy to verify. In fact the structure of the amplitudes is determined by the property $K_\theta(i)^2 = 1$, as we have explained, so it is enough for each mass insertion produced by the operators $K_\theta(i, j)$ and $K_\theta(i, j, k)$ to satisfy that property in order for the resulting amplitude to be of the form we have described. Of course, the normalisation factors for the amplitudes are going to be different from those of the simpler model presented in section III and it will depend in general on three different coupling constants $a$, $b$ and $c$. This gives additional freedom that may well turn out to be useful in some situation.

It is interesting to note that there are several choices of coupling constant that lead to further simplifications. For example, one can show that in order for the added terms to satisfy $(K_{total})^2 \propto I$, then one needs to choose

$$a = 0, \quad b = 0, \quad \text{or} \quad b = 0, \quad a + c = 0, \quad (32)$$

that is

$$\mathcal{K} = I - cK_\theta(1, 2, 3), \quad \text{or} \quad \mathcal{K} = I - a(K_\theta(1) + K_\theta(2) + K_\theta(3) - K_\theta(1, 2, 3)). \quad (33)$$

Finally, a simple and highly symmetrical choice is

$$\mathcal{K} = (I - aK_\theta(1))(I - aK_\theta(2))(I - aK_\theta(3))$$

$$= I - a(K_\theta(1) + K_\theta(2) + K_\theta(3)) + a^2(K_\theta(1, 2) + K_\theta(2, 3) + K_\theta(1, 3)) - a^3K_\theta(1, 2, 3). \quad (34)$$

Since the $(I - aK_\theta(i))$ commute, we can easily compute the propagator

$$\mathcal{P} = \frac{(I + aK_\theta(1))(I + aK_\theta(2))(I + aK_\theta(3))}{(1 - a^2)^3}. \quad (35)$$

It is not clear at the present stage, however, which specific properties one should ask the gft propagator to fulfill.

C. Alternatives

After having discussed some possible generalisations of the proposed model leading to very similar structures, we would like to discuss two genuine alternatives to it. One based on a much simpler action constructed inserting a mass parameter in the simplest way in the usual 3-argument field, leading however to a problematic structure for the Feynman amplitudes, and thus showing the need for the 6-argument extension on which we have based our model. The other keeps the same structure of the model presented in section III but inserts the mass parameter by means of a modification of the interaction term in the gft action, instead of the kinetic one. As we will show, this modification is completely harmless. Consider first a three argument field, the same on which the Boulatov model is based, and insert the mass parameter $h_\theta$ and the velocities for the relevant particles $u_i$ in the 1st argument of the field. The model we obtain is therefore realised by forgetting about the presence of the $u_i$ variables in the extra slots of the generalised field used in the model presented in section III. The action is then:

$$S[\phi] = \frac{1}{2} \int \prod_{i=1}^{3} dg_i du_1 \left( \phi(g_1, g_2, g_3)\phi(g_1, g_2, g_3) - a \phi(g_1, g_2, g_3)\phi(u_1 h_\theta u_1^{-1}g_1, g_2, g_3) \right)$$

$$+ \frac{\lambda}{4!} \int \prod_{i=1}^{6} dg_i \phi(g_1, g_2, g_3)\phi(g_4, g_5, g_6)\phi(g_4, g_2, g_6)\phi(g_1, g_5, g_6). \quad (36)$$
The Feynman rules for this action can be read out easily, and the amplitudes constructed in the usual way. The kinetic operator is:

$$K(g_i, \bar{g}_i) = \prod_{i=1}^{3} \delta(g_i \bar{g}_i^{-1}) - a \int du_1 \delta(u_1 h_\theta u_1^{-1} g_1 \bar{g}_1^{-1}) \prod_{i=2}^{3} \delta(g_i \bar{g}_i^{-1}) \equiv I - aK_\theta,$$

while the interaction operator is the usual Boulatov one. It is easy to see from the expression for the kinetic operator, that the mass-inserting operator does not satisfy any property like $(K_\theta)^2 = I$ anymore. Instead, its square gives:

$$(K_\theta)^2 = \int du_1 d\bar{u}_1 \delta(\bar{u}_1 h_\theta \bar{u}_1^{-1} u_1 h_\theta u_1^{-1} g_1 \bar{g}_1^{-1}) \prod_{i=2}^{3} \delta(g_i \bar{g}_i^{-1}).$$

This makes the propagator much more complicated, being given by:

$$\mathcal{P}(g_i, \bar{g}_i) = I + \sum_{n>0} (aK_\theta)^n,$$

resulting in considerably more arduous Feynman diagrammatics. An example of an amplitude of this model, for a typical dual face is:

$$A(f^*) = \delta(G_f, u_{a_1} h_\theta u_{a_1}^{-1} \times \cdots \times u_{a_n} h_\theta u_{a_n}^{-1}),$$

for $n$ less than the number of edges bounding the dual face $f^*$. We see that there is no ‘multiple mass cancellation’ anymore, and we end up having multiple mass insertions on each face, in the typical Feynman diagram, each of which is associated to a different $SU(2)/U(1)$ velocity element. There is a possible physical interpretation of the resulting configuration, which is in terms of multi-particle states. In other words, we would have more than one particle with a given mass associated to a dual face and so to an edge of the triangulation. We are not in the position of being able to exclude this interpretation, or definitely reject the model that generates these configurations altogether; however, we feel that such an interpretation is problematic for at least two reasons: 1) it would imply that more than one particle is propagating along the same link of the triangulation, so they would be located at the same point in the manifold and interact together with other (multi-)particles at the vertices of the triangulation; most important, 2) interpreting the particle configurations in the triangulation as Feynman graphs of some effective field theory would become much less straightforward than for the model proposed in section \textbf{III} and applying any procedure to extract this effective field theory, like that used in \textbf{8} to the amplitudes generated by this ‘simplified’ model would be quite cumbersome, if at all possible. We note that the same problem of ‘multiple mass insertions’ is generated by most other modifications of the structure of the model presented in section \textbf{III} affecting the 3 extra arguments of the field, i.e. the $u_i$ variables. Such modifications lead to losing the property $(K_\theta)^2 = I$ which is responsible for ‘mass cancellation’ in the dual faces when the operator is inserted more than once, and that leads to the presence of only one mass in each dual face, and furthermore, to consistency with the interpretation of the mass-labelled graphs in the triangulation as Feynman graphs of a field theory.

We conclude by mentioning instead a harmless modification of the model, that may even turn out to be useful in future studies. One can keep the structure of the field to be a function of 6 arguments, and one can keep the same form for the pure gravity action based on this 6 argument field, but choose to insert a mass parameter not in the kinetic term but in the interaction term of the group field theory action. The group field theory action would then be:

$$S[\phi] = \frac{1}{2} \int \prod_{i=1}^{3} dg_i du_i[P_\alpha P_\beta \phi(g_1, g_2, g_3; u_1, u_2, u_3)][P_\alpha P_\beta \phi(g_1, g_2, g_3; u_1, u_2, u_3)]$$

$$+ \frac{\lambda}{4!} \int \prod_{i=1}^{6} dg_i du_i[P_{\alpha_1} P_{\beta_1} \phi(g_1, g_2, g_3; u_1, u_2, u_3)][P_{\alpha_2} P_{\beta_2} \phi(g_4, g_5, g_6; u_4, u_5, u_6)]$$

$$\times [P_{\alpha_3} P_{\beta_3} \phi(g_4, g_5, g_6; u_4, u_5, u_6)][P_{\alpha_4} P_{\beta_4} \phi(g_1, g_5, g_6; u_1, u_5, u_6)]$$

$$+ \frac{\lambda}{4!} \int \prod_{i=1}^{6} dg_i du_i[aP_\theta P_{\alpha_1} P_{\beta_1} \phi(g_1, g_2, g_3; u_1, u_2, u_3)][P_{\alpha_2} P_{\beta_2} \phi(g_4, g_5, g_6; u_4, u_5, u_3)]$$

$$\times [P_{\alpha_3} P_{\beta_3} \phi(g_4, g_5, g_6; u_4, u_5, u_6)][P_{\alpha_4} P_{\beta_4} \phi(g_1, g_5, g_6; u_1, u_5, u_6)].$$

As we have anticipated, however, it is straightforward to verify that this action leads to the same type of Feynman graphs and amplitudes that one gets instead by modifying the kinetic term, the only difference being that one would get a different normalisation factor, in front of each amplitude.
V. CONCLUSIONS

We have defined a new group field theory for 3-dimensional Riemannian quantum gravity, and constructed its perturbative expansion in Feynman diagrams. These diagrams correspond to spin foam 2-complexes describing quantum gravitational degrees of freedom dual to 3d triangulations, as in the Boulatov model, but they also carry additional labels which describe massive spinless particles propagating in the spacetime one reconstructs from the gravity degrees of freedom. They have the interpretation of Feynman graphs for a scalar field theory embedded in the triangulation representing spacetime. The amplitudes for these diagrams have exactly the form obtained in [4] from classical considerations, i.e. are given by the Ponzano-Regge model coupled to massive spinless particles, and are shown in [8] to admit an effective (non-commutative) scalar field theory description, confirming the above physical interpretation.

The model presented possesses some quite non-trivial and interesting features, that we highlighted in the paper, and that deserve further analysis. This represents a first step in a program of analysing the coupling of matter and gauge fields to quantum gravity in the group field theory approach. The next steps would be first of all the inclusion of spin degrees of freedom and the construction of a group field theory reproducing the amplitudes given in [4] for massive spinning particles. Second, one should develop the study of the gft observables in the presence of particles and its relation with spin networks with open ends. Next, one would like to show how the effective field theory picture for the particle degrees of freedom can be obtained directly, and possibly in a simpler way, from the group field theory formulation. Finally, the problem of the coupling of gauge fields and the description of their interaction with both gravity and matter fields, again in the group field theory formalism, should be tackled.

To conclude, as we mention in the text, one of the most pressing issues for our model and the Boulatov model is to understand whether there are symmetries at the gft level that justify the gauge fixing needed at the level of the Feynman graphs to reproduce physical 3d gravity amplitudes. This is necessary in order to promote this type of gft to a fundamental model of three dimensional gravity coupled to matter.

Acknowledgments: We would like to thank K. Krasnov for keeping us informed of his progress and PI for an invitation which initiated this collaboration. J.R. would like to thank the occupants of room B0.10 for correcting innumerable typos.

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