Positive definite functions on Coxeter groups with applications to operator spaces and noncommutative probability

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A new class of positive definite functions related to colour-length function on arbitrary Coxeter group is introduced. Extensions of positive definite functions, called the Riesz-Coxeter product, from the Riesz product on the Rademacher (Abelian Coxeter) group to arbitrary Coxeter group is obtained. Applications to harmonic analysis, operator spaces and noncommutative probability is presented. Characterization of radial and colour-radial functions on dihedral groups and infinite permutation group are shown.

Introduction

In 1979 Uffe Haagerup in his seminal paper [HAA79], essentially proved the positive definiteness, for $0 \leq q \leq 1$, of the function $P_q(x) = q^{|x|} = \exp(-t|x|)$, where $|\cdot|$ is the word length on a free Coxeter group $W = \mathbb{Z}/2 \ast \cdots \ast \mathbb{Z}/2$. From this he deduced also Khinchine type inequalities. He has shown that the regular C*-algebra of W has bounded approximation property and later [DCH85] the completely bounded approximation property (CBAP). These results of Uffe Haagerup have had significant impact on harmonic analysis on free groups and, more generally, on Coxeter groups; they also influenced free probability theory and other noncommutative probability theories.

In the paper [BJS88] it was shown that the function $P_q(x) = q^{|x|}$ is positive definite for $q \in [-1, 1]$ and all Coxeter groups, where the length $|\cdot|$ is the natural word length function on a Coxeter group with respect to the set of its Coxeter generators. This fact implies that infinite Coxeter groups have the Haagerup property and do not have Kazhdan’s property (T).

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Later, Januszkiewicz [JAN02] and Fendler [FENO2B] showed, in the spirit of Haagerup proof, that $w \mapsto z^{|w|}$ is a coefficient of a uniformly bounded Hilbert representation of $W$ for all $z \in \mathbb{C}$ such that $|z| < 1$. As shown in a very short paper of Valette [VAL93], this implies CBAP. See the book [BOO8] for further extension of Uffe Haagerup results for a big class of groups.

In the paper [BS96] Bożejko and Speicher considered the free product (convolution) of classic normal distribution $N(0,1)$ and the new length function on the permutation group $S_n$ (i.e. the Coxeter group of type $A$) was introduced, which we shall call the colour-length function $\| \cdot \|$. It is defined as follows: for $w \in S_n$ in the minimal (reduced) representations $w = s_1 \ldots s_k$, where each $s_j$ belong to the set $S$ of transpositions of the form $(i, i+1)$, we put $\|w\| = \#\{s_1, s_2, \ldots , s_k\}$.

For our study one of the most important results of this paper is that the function called Riesz-Coxeter product $R_q$ defined on all Coxeter groups $(W, S)$ as

$$R_q(s) = q_s, \text{ for } s \in S, \text{ and } R_q(xy) = R_q(x)R_q(y), \text{ if } \|xy\| = \|x\| + \|y\|$$

is positive definite for $0 \leq q \leq 1$.

This implies, in particular, that in an arbitrary Coxeter group the set of its Coxeter generators is a weak Sidon set and also it is completely bounded $\Lambda_{cb}^p$-set, see Theorems 8.1 and 8.2. Equivalently, the span of the linear operators $\{\lambda(s) | s \in S\}$ in the noncommutative $L^p$-space $L^p(W)$ is completely boundedly isomorphic to row and column operator Hilbert space (see Theorem 8.2).

Another interesting connection between the two length functions $\| \cdot \|$ and $\| \cdot \|$ appeared in [BS96] in the formula for the moments of free additive convolution power of the Bernoulli law $\mu_{-1} = (\delta_{-1} + \delta_1) / 2$ (cf. Corollary 6 in cited paper):

$$m_{2n}(\mu_{-1}^{\boxplus q}) = q^n \sum_{\pi \in \mathcal{P}_2(n)} (-1)^{\text{sym} \pi} q^{-\|\pi\|},$$

for $q \in \mathbb{N}$. (See also Section 9 of the present paper.)

Also, in [BBLS11] the colour-length function on the permutation group $S_n$ was studied. Some of its extensions to pairpartitions appeared in the presentation of the proof that classical normal law $N(0,1)$ is free infinitely divisible under free additive convolution $\boxplus$.

Since we have recent extensions of the free probability (which is related to type $A$ Coxeter groups) to the free probability of type $B$ Coxeter groups (see [BEH15]), it seems to be interesting to determine the role of the colour-length functions for the Coxeter groups of type $B$ and $D$. 
The plan of the paper is as follows.

In Section 1 we recall definitions of Coxeter groups and of the length and the colour-length functions.

In Section 2 we recall the definition of positive definite funcions and discuss various classes of those, namely radial, colour-radial, and colour-dependant.

In Section 3 we discuss Abelian Coxeter groups.

In Section 4 we show the following formula characterizing the radial normalised positive definite functions on these Coxeter groups which contain the infinite Rademacher group $\bigoplus_{i=1}^{\infty} \mathbb{Z}/2$ as a parabolic subgroup (these include the infinite permutation group $S_\infty$):

Every radial positive definite function $\varphi$ is of the form

$$\varphi(w) = \int_{-1}^{1} q^{\|w\|} \mu(dq)$$

for a probability measure $\mu$.

That characterisation is a variation on the classical de Finetti theorem. A noncommutative version was shown by Köstler ans Speicher [KS09] (see also [LEHO4]).

We also show in Theorem 4.3, that the function $\exp(-t\|w\|^p)$ is positive definite for all $t \geq 0$ if and only if $p \in [0, 1]$.

In Section 5 we give a short proof of the equivalence of the two known results concerning positive definite functions on finite Coxeter groups.

In Section 6 we present the main properties of the colour-dependent positive definite functions on Coxeter groups, in particular we show in Proposition 4.4. that on $S_\infty$ and some other Coxeter groups, the function $w \mapsto r^{\|w\|}$ is positive definite if and only if $r \in [0, 1]$.

The Section 7 gives characterization of all colour-length functions on finite and infinite dihedral groups $D_m$, for $m = 1, 2, \ldots, \infty$.

In Section 8 we prove that the set $S$ of Coxeter generators is a weak Sidon set in arbitrary Coxeter groups $(W, S)$ with constant 2 and that it is also a completely bounded $\Lambda(p)$ set with contants as $C \sqrt{p}$, for $p > 2$.

In Section 9 we prove for arbitrary finitely generated Coxeter group an identity involving both lengths $\cdot$ and $\| \cdot \|$ (see Proposition 9.1). We apply it to give a proof of Corollary 7 from [BS96], (see Equation (9.1)) where the proof, involving probabilistic considerations, was not presented in [BS96].
1. Coxeter groups

In this part we recall the basic facts regarding Coxeter groups and introduce notation which will be used throughout the rest of the paper. For more details we refer to [BOU68, HUM90].

A group $W$ is called a Coxeter group if it admits the following presentation:

$$W = \left\{ \langle S \mid (s_1 s_2)^{m(s_1, s_2)} = 1 : s_1, s_2 \in S, m(s_1, s_2) \neq \infty \rangle \right\},$$

where $S \subset W$ is a set and $m$ is a function $m : S \times S \to \{1, 2, 3, \ldots, \infty\}$ such that $m(s_1, s_2) = m(s_2, s_1)$ for all $s_1, s_2 \in S$ and $m(s_1, s_2) = 1$ if and only if $s_1 = s_2$. The pair $(W, S)$ is called a Coxeter system. In particular, every generator $s \in S$ has order two and every element $w \in W$ can be represented as

\[(1.1) \quad w = s_1 s_2 \ldots s_m\]

for some $s_1, s_2, \ldots, s_m \in S$. If the sequence $(s_1, \ldots, s_m) \in S^m$ is chosen in such a way that $m$ is minimal then we write $|w| = m$ and call it the length of $w$. In such a case the right hand side of (1.1) is called a reduced representation or reduced word of $w$. This is not unique in general, but the set of involved generators is unique [BOU68, Ch. IV, §1, Prop. 7], i.e. if $w = s_1 s_2 \ldots s_m = t_1 t_2 \ldots t_m$ are two reduced representations of $w \in W$ then $\{s_1, s_2, \ldots, s_m\} = \{t_1, t_2, \ldots, t_m\}$. This set $\{s_1, s_2, \ldots, s_m\} \subseteq S$ will be denoted $S_w$ and called the colour of $w$.

Given a subset $T \subset S$ by $W_T$ we denote the subgroup generated by $T$ and call it the parabolic subgroup associated with $T$. To see that $S_w$ is independent of the reduced representation of $w$ notice that

\[(1.2) \quad s \in S_w \iff w \notin W_{S \setminus \{s\}}.\]

We define the colour-length of $w$ putting $\|w\| = |S_w|$ (the cardinality of $S_w$). Both lengths satisfy the triangle inequality and we have $\|w\| \leq |w|$. In the case of the permutation group the colour-length has the following pictorial interpretation. If $\sigma$ is a permutation in $S_n$ then $\|\sigma\|$ equals $n$ minus the number of connected components of the diagram representing $\sigma$. Notice, that $|\sigma|$ equals to the number of crossings in the diagram (the number of pairs of chords that cross).
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\[ \sigma \quad e \quad (12) \quad (12)(23) \quad (12)(23)(12) \]

\[
\begin{array}{cccc}
|\sigma| & 0 & 1 & 2 \\
\|\sigma\| & 0 & 1 & 2 & 2
\end{array}
\]

It would be convenient to define

\[
(1.3) \quad \|w\|_s = \begin{cases} 
0 & \text{if } s \notin S_w, \\
1 & \text{if } s \in S_w,
\end{cases}
\]

then, clearly, \(\|w\| = \sum_{s \in S} \|w\|_s\).

2. Positive defined functions

A complex function \(\varphi\) on a group \(\Gamma\) is called positive definite if we have

\[
\sum_{x, y \in \Gamma} \varphi(y^{-1}x)\alpha(x)\overline{\alpha(y)} \geq 0
\]

for every finitely supported function \(\alpha : \Gamma \to \mathbb{C}\).

A positive definite \(\varphi\) function is Hermitian and satisfies \(|\varphi(x)| \leq \varphi(e)\) for all \(x \in \Gamma\). Usually it is assumed, that \(\varphi\) is normalised, i.e. that \(\varphi(e) = 1\).

In this and the following sections we discuss the radial functions on Coxeter groups. These are functions which depend on \(|w|\) rather than \(w\).

We call a function \(\varphi\) on \((W, S)\) colour-dependent if \(\varphi(w)\) depends only on \(S_w\). We call it colour-radial if it depends only on \(\|w\|\).

An Abelian Coxeter group generated by \(S\) is isomorphic to the direct product \(\oplus_{s \in S} \mathbb{Z}/2\). On these groups the lengths \(|\cdot|\) and \(\|\cdot\|\) coincide and all functions are colour dependent.

The main example of a positive definite function will be the Riesz–Coxeter function. Given a sequence \(q = (q_s)_{s \in S}\) we define \(R_q(w) = \prod_{s \in S} q_\|w\|_s = \prod_{s \in S_w} q_s\). We will abuse notation and denote by \(R_q\) also the associated operator \(\sum_{w \in W} R_q(w)w\). That is

\[
R_q = 1 + \sum_{s \in S} q_s s + \sum_{w : S_w = \{s, s_2\}} q_s q_{s_2} w + \sum_{w : S_w = \{s, s_2, s_3\}} q_s q_{s_2} q_{s_3} w + \cdots
\]
In the case all $q_s = q$ we get $R_q = \sum q\|w\|w$.

This generalises the classical case of Rademacher–Walsh functions in the Rademacher group $\text{Rad}_n$. If we denote the generator of the $i$-th factor $\mathbb{Z}/2$ of the latter by the symbol $r_i$ then, by definition, $r_i^2 = 1$ and $r_ir_j = r_jr_i$ and

$$R_q = \prod_{i=1}^{n}(1 + q_ir_i).$$

### 3. Rademacher groups

In this section we are going to study positive definite radial functions on the Abelian Coxeter groups, $(W,S) = \text{Rad}_S$. Since positive definiteness is tested on functions with finite support, we can assume that $S$ is countable. If $\#S = n$ we will write $\text{Rad}_n$ instead of $\text{Rad}_S$. Given $n \in \mathbb{N} \cup \{\infty\}$, we denote by $P_n$ the class of all functions $f : \{0,1,\ldots,n\} \to \mathbb{R}$ for $n$ finite and $f : \mathbb{N} \to \mathbb{R}$ if $n = \infty$ such that $\varphi(w) = f(|w|)$ is a normalised positive definite on $\text{Rad}_n$.

The following observation is straightforward.

**Proposition 3.1.** Assume that $1 \leq m < n \leq \infty$ and $f \in P_n$. Then the restriction of $f$ to $\{0,\ldots,m\}$ belongs to $P_m$. A function $f$ belongs to $P_\infty$ if and only if all its restrictions to $\{0,\ldots,m\}$ for any $m \in \mathbb{N}$ belong to $P_m$.

**Theorem 3.2.** Assume $n$ is finite. The set $P_n$ form a simplex whose vertices (extreme points) are $f^n_l(k) = \binom{n}{l}^{-1}\sum_{i=0}^{l}(-1)^i\binom{k}{i}\binom{n-k}{l-i}$, where $0 \leq l \leq n$. Equivalently, every normalised radial positive definite function on the group $\text{Rad}_n$ is of the form

$$\varphi(x) = \sum_{l=0}^{n}\lambda_ff^n_l(|x|),$$

where the sequence of nonnegative numbers $(\lambda_l)_{l=0}^{n}$ is unique and satisfies $\sum_{l=0}^{n}\lambda_l = 1$.

**Proof.** We can indentify the dual $\hat{\text{Rad}}_n$ group of $\text{Rad}_n$ with $\text{Rad}_n$ via the paring $(x,y) = (-1)^{\sum_{i=1}^{n}x_iy_i}$. By Bochner’s theorem every normalised positive definite function $\varphi$ on $\text{Rad}_n$ is of the form

$$\varphi(x) = \int_{\text{Rad}_\infty}(x,y)\mu(dy),$$

for some probability measure $\mu$. Clearly, such a function is radial if and only if $\mu$ is invariant under the action of $\mathbb{S}_n$. 
Among such measures extreme ones are measures $\mu_l$ for $0 \leq l \leq n$, where $\mu_l$ is equally distributed among elements of length $l$. Moreover,

$$\varphi(x) = \int_{\text{Rad}_\infty} (x, y) \mu_l(dy) = f_l^n(|x|)$$

as claimed. \hfill \Box

The following theorem is a version of the classical de Finetti Theorem (see [FEL71, p. 223]) for the infinite Rademacher group.

**Theorem 3.3.** Assume that $\varphi$ is a radial function on the Rademacher group $\text{Rad}_\infty$. Then $\varphi$ is a normalised positive definite if and only if there exists a probability measure $\mu$ on $[-1, 1]$ such that

$$\varphi(x) = \int_{-1}^{1} q^{(x)} \mu(dq).$$

This measure $\mu$ is unique.

**Proof.** Since the function $q^{(x)}$ is normalised positive definite for $q \in [-1, 1]$, the “if” implication is obvious.

Assume that $\varphi$ is normalised positive definite. The group $\text{Rad}_\infty$ is discrete and Abelian and its dual is the compact group $\text{Rad}_\infty = \prod_{n=1}^{\infty} \mathbb{Z}/2$.

By Bochner’s theorem, there exists a probability measure $\eta$ on $\text{Rad}_\infty$ such that

$$\varphi(x) = \int_{\text{Rad}_\infty} (x, y) d\eta(y),$$

where for $x = (x_1, x_2, \ldots) \in \text{Rad}_\infty$, $y = (y_1, y_2, \ldots) \in \text{Rad}_\infty$, we put $(x, y) = (-1)^{\sum_{i=1}^{\infty} x_i y_i}$. The radiality of $\varphi$ is equivalent to the fact that for every permutation $\sigma \in \mathcal{S}_\infty$, we have $\varphi(x) = \varphi(\sigma(x))$, where $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. This, in turn, implies that $\eta$ is $\sigma$-invariant for every $\sigma \in \mathcal{S}_\infty$, i.e. we have $\eta(A) = \eta(\sigma(A))$ for every Borel subset $A \subset \text{Rad}_\infty$.

For a sequence $\varepsilon = (\varepsilon_i)_{i=1}^{n} \in \{0, 1\}^n$ we define $C_n(\varepsilon) \subseteq \text{Rad}_\infty$ by

$$C_n(\varepsilon) = \{ y \in \text{Rad}_\infty | y_i = \varepsilon_i : 1 \leq i \leq n \},$$

in particular $C_\emptyset(\emptyset) = \text{Rad}_\infty$. Then we have $\eta(C_n(\varepsilon)) = \eta(C_n(\varepsilon'))$ if $\varepsilon'_i = \varepsilon_{\sigma(i)}$ for some $\sigma \in \mathcal{S}_n$ and every $1 \leq i \leq n$. For $\varepsilon \in \mathbb{R}$ we put

$$\varepsilon^n = (\varepsilon, \varepsilon, \ldots, \varepsilon)_{1 \leq i \leq n}$$

and $a_n = \eta(C_n(1^n))$. Moreover, for $n, k \geq 0$ we define the difference operators $\Delta^k a_n$ by induction: $\Delta^0 a_n = a_n$ and $\Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n$.

We claim that

$$(-1)^k \Delta^k a_n = \eta\left(C_{n+k}(1^n 0^k)\right).$$
Denoting the right hand side of (3.1) by \( c(n,k) \) we note that \( c(n,0) = a_n \) and
\[
C_{n+k+1}(1^n o^k o) \cup C_{n+k+1}(1^n o^k 1) = C_{n+k}(1^n o^k),
\]
is a disjoint union. This implies
\[
c(n, k + 1) = c(n, k) - c(n + 1, k).
\]
This formula, by induction on \( k \), leads to (3.1).

From (3.1) we see that the sequence \( (a_n) \) is completely monotone, i.e. that \( (-1)^k \Delta^k a_n \geq 0 \) for all \( n, k \geq 0 \). By the celebrated theorem of Hausdorff (see [HAU21, Sätze II und III]), there exists a unique probability measure \( \rho \) on \([0,1]\) such that
\[
(3.2) \quad (-1)^k \Delta^k a_n = \int_0^1 u^n (1-u)^k \, d \rho(u).
\]
(Note that Equation (3.2) for arbitrary \( k \geq 0 \) follows from the case \( k = 0 \).)

For \( x = (1^n o^\infty) \in \text{Rad}_\infty \) so that \( |x| = n \), we have
\[
\phi(x) = \int_{\text{Rad}_\infty} (x, y) \, d \eta(y) = \int_{\text{Rad}_\infty} (-1)^{\sum_{i=1}^n y_i} \, d \eta(y)
\]
\[
= \sum_{\epsilon \in \{0,1\}^n} (-1)^{\sum_{i=1}^n \epsilon_i} \eta(C_n(\epsilon)) = \sum_{k=0}^n \binom{n}{k} (-1)^k \eta(C_n(1^k o^{n-k}))
\]
\[
= \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^1 u^k (1-u)^{n-k} \, d \rho(u) = \int_0^1 (1-2u)^n \, d \rho(u)
\]
\[
= \int_{-1}^1 q^n \, d \mu(q),
\]
where \( \mu \) is defined by \( \mu(A) = \rho(\frac{1}{2} + \frac{1}{2} A) \) for a Borel set \( A \subseteq [-1,1] \). \( \square \)

4. Remarks on radial positive definite functions on some infinitely generated Coxeter groups

In this Section we extend the last theorem of the previous section to a certain class of Coxeter groups.

**Theorem 4.1.** Assume that \((W,S)\) is a Coxeter system and that there is an infinite subset \(S_0 \subseteq S\) such that \( st = ts \) for \( s, t \in S_0\). Assume that \( \phi \) is a radial function on \( W \) with \( \phi(e) = 1 \). Then \( \phi \) is positive definite if and only if there exists a probability measure \( \mu \) on \([-1,1]\) such that
\[
\phi(\sigma) = \int_{-1}^1 q^{\mid \sigma \mid} \, d \mu(q).
\]
This measure \( \mu \) is unique.
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**Proof.** It is sufficient to note that the group generated by $S_0$ is a parabolic subgroup isomorphic with $\text{Rad}_\infty$. □

**Example.** For $W = S_\infty$ we have $S = \{(n, n + 1) : n \in \mathbb{N}\}$. Then we can take $S_0 = \{(2n - 1, 2n) : n \in \mathbb{N}\}$. Similar $S_0$ can be found in infinitely generated groups of type B and D.

**Problem 4.2.** When $-1 \leq q \leq 1$, $q \neq 0$ is the positive definite function $q^{[x]}$ on $\mathbb{S}_\infty$ an extreme point in the set of normalised positive definite functions?

**Theorem 4.3.** The function $q_p(\sigma) = e^{-t|\sigma|^p}$ is positive definite on $\mathbb{S}_\infty$ if and only if $0 \leq p \leq 1$.

**Proof.** A contrario. Assume that for some $p > 1$ and $t_0 > 0$ the function $\psi_p(\sigma) = e^{-t_0|\sigma|^p}$ is positive definite on $\mathbb{S}_\infty$.

For $q_0 = e^{-t_0}$, choosing $\sigma$ such that $|\sigma| = 2n$ we have $q_0^{(2n)^p} = \int_{-1}^{1} q^{2n} d\mu_0(q)$ for some probability measure $\mu_0$ on $[-1, 1]$. Since $(\int_{-1}^{1} q^{2n} d\mu_0(q))^{1/n}$ tends to $\max\{q^2 | q \in \text{supp}\mu_0\}$ while $(q_0^{(2n)^p})^{1/n} \to 0$, we conclude that $\mu_0$ is the Dirac measure at 0, which is a contradiction.

The “if” part is standard. We need to show that $f(x) = e^{-tx^p}$ is the Laplace transform of some probability measure supported on $[0, \infty)$, so $f$ is a convex combination of functions of the form $e^{-sx}$.

By characterisation of Laplace transforms (see [HAU21, Satz III]) this is equivalent to complete monotonicity, that is $(-1)^n f^{(n)} > 0$ for all $n = 0, 1, \ldots$. And indeed, by induction, $(-1)^n f^{(n)}$ is a positive linear combination of positive functions of the form $x^{pj-n} f(x)$ for $1 \leq j \leq n$. □

The measures with Laplace transforms $e^{-tx^p}$ for $t \geq 0$ and $0 \leq p \leq 1$ are studied in detail in [YOS80, Ch. IX.11] (see Propositions 1 and 2 there).

Let us note that for such groups like $\mathbb{Z}^k$ or $\mathbb{R}^k$ with the Euclidean distance $d$ the functions $\exp(-td^p)$ are positive definite for all $t \geq 0$ and $0 \leq p \leq 2$ (the case $p = 2$ corresponds to the Gaussian Law).

5. **The longest element**

If a Coxeter group $W$ is finite, then it contains the unique element $\omega_0$ which has the maximal length with respect to $|\cdot|$. 
From the definition it is clear, that a function \( \varphi \) on a group \( \Gamma \) with values in the field of complex numbers \( \mathbb{C} \) is positive definite if and only \( \sum_{g \in \Gamma} \varphi(g)g \) is a nonnegative (bounded if the group is finite) operator on \( \ell^2 \Gamma \). (We will identify \( g \in \Gamma \) with \( \lambda(g) \in B(\ell^2 \Gamma) \), where \( \lambda \) is the left regular representation, for short.)

Let \( W \) be a finite Coxeter group. The following two statements are well known.

(A) The function \( q^{\mid w \mid} \) is positive definite for any \( 0 \leq q \leq 1 \).

(B) The function \( \Delta(w) = \mid \omega_0 \mid / 2 - \mid w \mid \) is positive definite.

The first one was proven in \([bjs88]\) (even for infinite Coxeter groups and also for \(-1 \leq q \leq 1 \)) while the second — in \([bso3, \text{Proposition 6}]\). Here we give a short direct prove of the following.

**Proposition 5.1.** The above statements (A) and (B) are equivalent.

**Proof.** Let \( q = e^{-t} \) (with \( t \geq 0 \), as we assume \( q \leq 1 \)). The case (A) is equivalent to \( \Phi_t = \sum_{w \in W} e^{t \Delta(w)} w = e^{t \mid \omega_0 \mid / 2} \sum_{w \in W} q^{\mid w \mid} w \) being nonnegative.

Assume (A). Recall first, that \( \mid \omega_0 w \mid = \mid \omega_0 \mid - \mid w \mid = \mid w \omega_0 \mid \). Therefore \( \mid \omega_0 \mid / 2 - \mid \omega_0 w \mid = -(\mid \omega_0 \mid / 2 - \mid w \mid) \), i.e. \( \Delta(\omega_0 w) = -\Delta(w) \) and similarly, \( \Delta(\omega_0 \omega_0) = -\Delta(w) \).

The equality \( \Delta(\omega_0 w) = \Delta(w \omega_0) \) implies that \( \omega_0 \) (and thus \( Q = (1 - \omega_0) / 2 \)) commutes with \( \Delta \) (and thus \( \Phi_t \)). Since \( Q = Q^2 \) is nonnegative we conclude that

\[
\Phi_t Q = \sum_{w \in W} e^{t (\mid \omega_0 \mid / 2 - \mid w \mid)} - e^{t (\mid \omega_0 \mid / 2 - \mid w \omega_0 \mid)} \frac{w}{2t} \sum_{w \in W} \frac{\sinh(t \Delta(w))}{t} w.
\]

is nonnegative. Therefore, taking the limit as \( t \to 0 \), we obtain that \( \sum_{w \in W} \Delta(w) w \) is nonnegative. Thus (B).

Assuming (B) and using the Schur lemma, which says that the (point-wise) product of positive definite functions is positive definite, we get that

\[
\Phi_t = \sum_{w \in W} e^{t \Delta(w)} w = \sum_{n \geq 0} \frac{t^n}{n!} \left( \sum_{w \in W} \Delta(w)^n w \right)
\]

is nonnegative. Thus (A).

\( \square \)

6. Colour-dependent positive definite functions on Coxeter groups

The question which colour-dependant or colour-radial functions are positive functions on Coxeter groups is wide open. In this section
we provide some sufficient conditions. In the next section we will examine the dihedral groups in full details.

**Lemma 6.1.** Let \( H \) be a subgroup of a group \( \Gamma \) of index \( d \). Then the function \( \varphi_r \) defined by \( \varphi_r(x) = 1 \) if \( x \in H \) and \( \varphi_r(x) = r \) otherwise is positive definite on \( \Gamma \) if and only if \( r \in [-1/(d-1), 1] \), with natural convention that if \( d = \infty \) then \(-1/(d-1) = 0\).

Note, that if \( H = \{1\} \) then \( d = |\Gamma|\).

**Proof.** First assume that \( d \) is finite and let us enumerate the left cosets:
\[
\{gH : g \in \Gamma\} = \{H_1, H_2, \ldots, H_d\}.
\]

Note, that for \( x \in H_i, y \in H_j \) we have \( y^{-1}x \in H \) if and only if \( i = j \). Therefore, for \( r_0 = -1/(d-1) \) and for a finitely supported complex function \( f \) on \( \Gamma \) we have
\[
\sum_{x,y \in \Gamma} \varphi_{r_0}(y^{-1}x)f(x)\overline{f(y)} = \frac{1}{d-1} \sum_{1 \leq i < j \leq d} \left| \sum_{x \in H_i} f(x) - \sum_{y \in H_j} f(y) \right|^2,
\]
which proves that \( \varphi_{r_0} \) is positive definite. For \( r \in [-1/(d-1), 1] \) the function \( \varphi_r \) is positive definite as a convex combination of \( \varphi_{r_0} \) and the constant function \( \varphi_1 \).

On the other hand, if we choose \( x_i \in H_i \) for each \( i \leq d \) and define \( f \) as the characteristic function of the set \( \{x_1, \ldots, x_d\} \) then
\[
\sum_{x,y \in \Gamma} \varphi_r(y^{-1}x)f(x)\overline{f(y)} = d + (d^2 - d)r,
\]
which proves that \( r \geq -1/(d-1) \) is a necessary condition for positive definiteness of \( \varphi_r \).

If \( d = \infty \) then \( r_0 = 0 \) and the function \( \varphi_0 \) is positive definite as the characteristic function of the subgroup \( H \). For “only if” part we chose an arbitrarily long sequence \( x_1, \ldots, x_{d'} \) of elements from different left cosets and use (6.1) with \( d' \) instead of \( d \). \( \Box \)

**Theorem 6.2.** Assume that for every \( s \in S \) we are given a number \( q_s \),
\[
-1 \leq q_s \leq 1,
\]
where \( d_s \) denotes the index if the parabolic subgroup generated by \( S \setminus \{s\} \) in \( W \) : \( d_s = [W : W_{S \setminus \{s\}}] \). Then the Riesz–Coxeter \( R_q \) is positive definite on \( W \).

**Proof.** From Lemma 6.1 the function \( w \mapsto q_s^{|w|_s} \) is positive definite for \( s \in S \) and \(-1/(d_s - 1) \leq q_s \leq 1 \). Since the pointwise product of positive definite functions is positive definite, the statement holds. \( \Box \)
Example. Take $W = S_n$, the permutation group on the set $\{1,2,\ldots,n\}$. It is generated by the transpositions $S = \{s_i = (i,i+1), 1 \leq i \leq n-1\}$. For $1 \leq i \leq n-1$ the parabolic subgroup generated by $S \setminus \{s_i\}$ is isomorphic with $S_{i-1} \times S_{n-i-1}$, so its index is $i^i(n-1)$. It would be interesting to determine for which $r$ the function $w \mapsto r^{\|w\|}$ is positive definite. By Proposition 6.2 this holds for $r \in [-1/(d-1),1]$, where $d$ is the maximal index of the parabolic subgroups of the form $W_{S \setminus \{s\}}$. We note a necessary condition.

Proposition 6.3. Assume that we have distinct generators $s_o,s_1,\ldots,s_n \in S$ such that $s_os_k \neq sks_o$ (i.e. $m(s_o,s_k) > 2$) for $1 \leq k \leq n$. If the function $w \mapsto r^{\|w\|}$ is positive definite on $W$, then $-\frac{1}{(n-1)} \leq r^3 \leq 1$.

If there is an element $s_o \in S$ for which there are infinitely many $s \in S$ such that $s_os \neq ss_o$ then $r^{\|w\|}$ is positive definite on $W$ if and only if $0 \leq r \leq 1$.

Proof. Consider elements $w_k = s_os_k s_o$. Note, that for $k \neq l$ we have $\|w_l^{-1}w_k\| = 3$. If $q_r$ is positive definite on $W$ then we have

$$0 \leq \sum_{k,l=1}^{n} q_r(x_l^{-1}x_k) = n + (n^2 - n)r^3,$$

which implies $r^3 \geq -\frac{1}{(n-1)}$. □

Corollary 6.4. The function $w \mapsto q^{\|w\|}$ on $S_\infty$ is positive definite if and only if $0 \leq q \leq 1$.

Problem 6.5. Thus, it is valid to ask the following. Is it true that every normalised positive definite colour-lenght-radial function $\phi: S_\infty \to \mathbb{R}$ is of the form $\phi(\sigma) = \int_0^1 q^{\|w\|} d\mu(q)$ for some probability measure $\mu$ on $[0,1]$?

7. Dihedral groups

In this part we are going to examine the class of colour-dependent positive definite functions on the case the simplest nontrivial Coxeter groups. Assume that $W = D_{2n} = \langle s,t\rangle$ (i.e. the group of symmetries of a regular $n$-gon), and define a colour-dependent function on $W$:

$$\phi(w) = \begin{cases} 1 & \text{if } w = e, \\ p & \text{if } w = s, \\ q & \text{if } w = t, \\ r & \text{otherwise.} \end{cases}$$

If $p = q$ then $\phi$ is colour radial. We are going to determine for which parameters $p,q,r$ the function $\phi$ is positive definite on $W$. It is easy
to observe necessary conditions: \( p, q, r \in [-1, 1] \). Moreover, since \( \langle st \rangle \) is a cyclic subgroup of order \( n \), Lemma 6.1, implies a necessary condition: \(-1/(n-1) \leq r \leq 1\).

**Finite dihedral groups.** Assume that \( W \) is a finite dihedral group, \( W = D_{2n} \), so that \( (st)^n = 1 \). We will use the following version of Bochner's theorem: A function \( f \) on a compact group \( G \) is positive definite if and only if its Fourier transform:

\[
\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx
\]

is a positive operator for every \( \pi \in \hat{G} \), where \( \hat{G} \) denotes the dual object of \( G \), i.e. the family of all equivalency classes of unitary irreducible representations of \( G \), see [SIM96]. Then we have

\[
f(x) = \sum_{\pi \in \hat{G}} d_\pi \text{tr} \left[ \hat{f}(\pi)\pi(x) \right].
\]

Therefore, for every irreducible representation \( \pi \) of \( D_{2n} \) we are going to find

\[
\hat{\phi}(\pi) = \frac{1}{2n} \sum_{g \in G} \phi(g)\pi(g^{-1}).
\]

We will identify \( s \) with \((0, -1)\) and \( t \) with \((1, -1)\). If \( n \) is odd then \( D_{2n} \) possesses two characters: \( \chi_{+,+} \) such that \( \chi_{+,+}(w) = 1 \) for every \( w \in D_{2n} \) and \( \chi_{-,+} \) such that \( \chi_{-,+}(s) = \chi_{-,+}(t) = -1 \). If \( n \) is even then we have two additional characters \( \chi_{+,+} \) and \( \chi_{-,+} \) such that \( \chi_{+,+}(s) = \chi_{-,+}(t) = 1 \) and \( \chi_{-,+}(t) = \chi_{-,+}(s) = -1 \). It is easy to check that

\[
2n\hat{\phi}(\chi_{+,+}) = 1 + p + q + (2n - 3)r,
\]

\[
2n\hat{\phi}(\chi_{-,+}) = 1 - p - q + r,
\]

which gives

\[-1 - (2n - 3)r \leq p + q \leq 1 + r\]

and, for \( n \) even,

\[
2n\hat{\phi}(\chi_{+,+}) = 1 + p - q - r,
\]

\[
2n\hat{\phi}(\chi_{-,+}) = 1 - p + q - r,
\]

which implies

\[|p - q| \leq 1 - r.\]

We have also the family of two dimensional representations \( U_a \):

\[U_a(k, 1) = \begin{pmatrix} e^{2\pi ika/n} & 0 \\ 0 & e^{-2\pi ika/n} \end{pmatrix},\]

\[U_a(k, -1) = \begin{pmatrix} 0 & e^{2\pi ika/n} \\ e^{-2\pi ika/n} & 0 \end{pmatrix}.,\]
where $a = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$. Then for the function given by (7.1) we have

$$2n\hat{\phi}(U_a) = (1-r)\text{Id} + (p-r)U_a(o,-1) + (q-r)U_a(1,-1)$$

$$= \begin{pmatrix} 1-r & p-r + (q-r)e^{2\pi ia/n} \\ p-r + (q-r)e^{-2\pi ia/n} & 1-r \end{pmatrix}.$$ 

This matrix is positive definite if and only if $r \leq 1$ and

$$|p-r + (q-r)e^{2\pi ia/n}| \leq 1 - r.$$ 

Therefore we have

**Proposition 7.1.** The function $\phi$ given by (7.1) is positive definite on $D_{2n}$ if and only if

$$1 + p + q + (2n-3)r \geq 0, \quad 1 - p - q + r \geq 0$$

(plus

$$1 + p - q - r \geq 0, \quad 1 - p + q - r \geq 0$$

whenever $n$ is even) and

$$|p-r + (q-r)e^{2\pi ia/n}| \leq 1 - r.$$ 

for $a = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$.

Let us confine ourselves to colour-radial functions.

**Corollary 7.2.** Assuming that $p = q$, the function $\phi$ defined by (7.1) is positive definite on $W = D_{2n}$ if and only if

$$\max \left\{ \frac{-2p-1}{2n-3}, 2p-1 \right\} \leq r \leq \frac{1 + 2p \cos(\pi/n)}{1 + 2 \cos(\pi/n)},$$

i.e. if and only if the point $(p, r)$ belongs to the triangle whose vertices are

$$\left( \frac{1 - n \cos(\pi/n)}{1 + (2n-1)\cos(\pi/n)}, \frac{1 - \cos(\pi/n)}{1 + (2n-1)\cos(\pi/n)}, \frac{n-2}{2n-2}, \frac{-1}{n-1} \right), \quad (1,1).$$

**Proof.** For $p = q$ the conditions from Proposition 7.1 reduce to

$$2p - 1 \leq r, \quad -1 - 2p \leq (2n-3)r, \quad 2 \cos(\pi/n)|p-r| \leq 1 - r.$$ 

It is sufficient to note that $2p - 1 \leq r$ implies $2 \cos(\pi/n)(p-r) \leq 1 - r$ for $p \leq 1$.

**Example.** For $D_4$ we have the positive definiteness of $\phi$ is equivalent to

$$-1 + |p+q| \leq r \leq 1 - |p-q|,$$

which means that the set of all possible $(p,q,r)$ forms a tetrahedron with vertices $(-1,1,-1), (1,-1,-1), (-1,-1,1), (1,1,1)$. For $p = q$ the condition reduces to $2|p| - 1 \leq r \leq 1$. 

\[\square\]
In the case of $D_6$ Proposition 7.1 leads to the following conditions:

$$1 - p - q + r \geq 0, \quad 1 + p + q + 3r \geq 0,$$

$$1 - r \geq \sqrt{p^2 + q^2 + r^2 - pq - pr - qr},$$

which can be expressed as

$$\max \left\{ \frac{-1 - p - q}{3}, p + q - 1 \right\} \leq r \leq \frac{1 - p^2 - q^2 + pq}{2 - p - q}.$$}

**Proposition 7.3.** The function $\phi$ given by (7.1) is positive definite on $W = D_\infty$ if and only if $0 \leq r$ and $|p - r| + |q - r| \leq 1 - r$, i.e.

$$(7.2) \quad \max \{0, p + q - 1\} \leq r \leq \min \left\{ 1 - |p - q|, \frac{1 + p + q}{3} \right\}.$$}

**Proof.** First we note that the set of $(p, q, r) \in \mathbb{R}^3$ satisfying (7.2) constitutes a pyramid which is the convex hull of the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(1, 1, 1)$ (apex). For these particular parameters it is easy to see that $\phi$ is positive definite: $(1, 1, 1)$ corresponds to the constant function $1$, $(1, 0, 0)$ to the characteristic function of the subgroup $\langle s \rangle = \langle s \rangle$, and $(-1, 0, 0)$ to the characteristic function of $\langle s \rangle$. Similarly for $(0, \pm 1, 0)$. This, by convexity, proves that (7.2) is a sufficient condition.

On the other hand, we know already that $r \geq 0$ is a necessary condition. Let us fix $n$ and define $W^+(n) = \{ x \in W : |sx| < |x| \leq 2n \}$, $W^-(n) = \{ x \in W : |tx| < |x| \leq 2n \}$ and

$$f(x) = \begin{cases} \pm 1 & \text{if } x \in W^\pm(n), \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in W^+(n)$ we have $S_{y^{-1}x} = \emptyset$ in $2n - 2$ cases (namely, if $x = y$) $S_{y^{-1}x} = \{ s \}$ in $2n - 2$ cases (namely if $|x| = 2k, |y| = 2k + 1$ or vice-versa, $k = 1, \ldots, n - 1$) $S_{y^{-1}x} = \{ t \}$ in $2n$ cases (namely if $|x| = 2k, |y| = 2k - 1$ or vice-versa, $k = 1, \ldots, n$) and $S_{y^{-1}x} = \{ s, t \}$ in all the other $(2n - 1)(2n - 2)$ cases. Similarly, for $x, y \in W^-(n)$ we have $S_{y^{-1}x} = \emptyset$ in $2n$ cases, $S_{y^{-1}x} = \{ s \}$ in $2n$ cases, $S_{y^{-1}x} = \{ t \}$ in $2n - 2$ cases and $S_{y^{-1}x} = \{ s, t \}$ in $2n - 1)(2n - 2)$ cases. If $x \in W^+(n), y \in W^-(n)$ or vice-versa then $S_{y^{-1}x} = \{ s, t \}$. Summing up, we get

$$\sum_{x, y \in W} \phi(y^{-1}x)f_n(x)f_n(y)$$

$$= 4n + (4n - 2)p + (4n - 2)q + (4n - 2)(2n - 2)r - 8n^2r$$

$$= 4n + (4n - 2)p + (4n - 2)q - (12n - 4)r.$$
Therefore for every \( n \in \mathbb{N} \) we have a necessary condition
\[
1 + \left(1 - \frac{1}{2n}\right)p + \left(1 - \frac{1}{2n}\right)q - \left(3 - \frac{1}{n}\right)r \geq 0.
\]
Letting \( n \to \infty \) we get \( 1 + p + q \geq 3r \).

Put \( x_k = \text{stst} \ldots, |x_k| = k \). Fix \( n \) and define
\[
g(x) = \begin{cases} x_{-+}(x) & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n, \\ 0 & \text{otherwise}, \end{cases}
\]
where, as before, \( x_{-+} \) is the character on \( W \) for which \( x_{-+}(s) = -1 \), \( x_{-+}(t) = 1 \). Then
\[
\sum_{x,y \in W} \phi(y^{-1}x)g(x)g(y) = \sum_{k,l=1}^{4n} \phi(x_l^{-1}x_k)g(x_k)g(x_l).
\]
Denote \( c_{k,l} = \phi(x_l^{-1}x_k)g(x_k)g(x_l) \). Then we have \( c_{k,k} = 1 \), \( 1 \leq k \leq 4n \), \( c_{k,k-1} = q \) if \( k \) is even, \( c_{k,k-1} = -p \) if \( k \) is odd, \( 2 \leq k \leq 4n \) and \( c_{k,l} = c_{l,k} \) for all \( 1 \leq k, l \leq 4n \). If \( 1 \leq k, l \leq 4n \) and \( |k - l| \geq 2 \) then \( c_{k,l} = (-1)^j/r \), where \( j \) is the total number of \( s \) appearing in \( x_k \) and \( x_l \). Now it is not difficult to check that
\[
\sum_{l=1}^{4n} c_{k,l} = \begin{cases} 1 + q - 2r & \text{if } k = 1 \text{ or } k = 4n, \\ 1 - p + q - r & \text{if } 1 < k < 4n, \end{cases}
\]
which implies
\[
\sum_{x,y \in W} \phi(y^{-1}x)g(x)g(y) = 4n - (4n - 2)p + 4nq - (4n + 2)r
\]
and leads to necessary condition \( r \leq 1 - p + q \). In a similar manner we get \( r \leq 1 + p - q \).

Finally, define a function \( h \) similarly like \( g \), but now we use the character \( x_{-+} \):
\[
h(x) = \begin{cases} x_{-+}(x) = (-1)^k & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n, \\ 0 & \text{otherwise}. \end{cases}
\]
Putting \( d_{k,l} = \phi(x_l^{-1}x_k)h(x_k)h(x_l) \) we have \( d_{k,k} = 1 \), \( d_{k,k-1} = -p \) if \( 2 \leq k \leq 4n \) is even and \( d_{k,k-1} = -q \) if \( k \) is odd. Moreover, if \( |k - l| \geq 2 \), \( 1 \leq k, l \leq 4n \) then \( d_{k,l} = (-1)^{k+l}/r \). Now one can check that
\[
\sum_{l=1}^{4n} d_{k,l} = \begin{cases} 1 - q & \text{if } k = 1 \text{ or } k = 4n, \\ 1 - p - q + r & \text{if } 1 < k < 4n, \end{cases}
\]
which yields \( 1 - p - q + r \geq 0 \) and completes the proof that the conditions (7.2) are necessary. \( \square \)
8. Weak Sidon sets and operator Khinchin inequality

The aim of this section is to show that the set of Coxeter generators $S$ in an arbitrary Coxeter group $W$ is a weak Sidon set, i.e. an interpolation set for the Fourier–Stieltjes algebra $B(W)$.

Given a group $\Gamma$, the Fourier–Stieltjes algebra consists of linear combinations of positive definite functions on $\Gamma$, i.e. every element of $B(\Gamma)$ is of the form $f = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ for some positive definite functions $\varphi_i$ ($1 \leq i \leq 4$) on $\Gamma$. The norm on $B(\Gamma)$ is defined as

$$\|f\|_{B(\Gamma)} = \inf \left\{ \sum \varphi_i(e) \middle| \text{where } f \text{ decomposes as above} \right\}$$

**Theorem 8.1.** The set of Coxeter generators $S$ in an arbitrary Coxeter group $W$ is a weak Sidon set, i.e. for every bounded function $f : S \to [-1, 1]$ there exists positive definite functions $\varphi_{\pm}$, such that $f(s) = \varphi_{\pm}(s)$ for any $s \in S$. One can take $\varphi_{\pm} = R_{q_{\pm}}$ for a suitable choice of $q_{\pm}$.

Moreover

$$\|\varphi_{\pm} \|_{B(W)} \leq 2$$

**Proof.** Put $S_{\pm}(f) = \{ s \in S | \pm f(s) > 0 \}$. Set

$$q_{\pm}^s = \begin{cases} \pm f(s) & \text{for } s \in S_{\pm}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(s) = R_{q_{\pm}}(s) - R_{q_{\pm}}(s)$ as claimed. The rest of the statement hold as the Riesz-Coxeter function at the identity element equals to one.

Given a matrix $A \in M_n(\mathbb{C})$ and $p \geq 1$ the Schatten $p$-class norm $\|A\|_{S_p}$ is defined as $\|A\|_{S_p}^p = (\text{tr}|A|)^{1/p}$, where $|A| = (A^*A)^{1/2}$.

Let $\lambda$ denote the left regular representation of a group $\Gamma$. Given a finite sum $f = \sum c_g \lambda(g) \in \mathbb{C}[\Gamma]$ we define noncommutative $L^p$-norm

$$\|f\|_{L^p(\Gamma)}^p = \left( \tau \left( (f* f)^{1/2} \right) \right)^{1/p}$$

where $\tau(f) = c_e$ is the von Neumann trace and $L^p(\Gamma)$ is a completion of $\mathbb{C}[\Gamma]$ with respect to the above norm.

We recall, that a scalar-valued map $\varphi$ on a group $\Gamma$ is called a completely bounded Fourier multiplier on $L^p(\Gamma)$ if the associated operator

$$M_{\varphi}(\lambda(g)) = \varphi(g)\lambda(g), \quad g \in \Gamma$$

extends to a completely bounded operator on $L^p(\Gamma)$.

We let $M_{cb}(L^p(\Gamma))$ to be an algebra of completely bounded Fourier multipliers equipped with the norm

$$\|\varphi\|_{M_{cb}(L^p(\Gamma))} = \|M_{\varphi} \otimes \text{id}_{S^p} \|.$$
Following Pisier [PISO3], for \( a_s \in M_n(C) \), where \( s \in S \), we define

\[
\| (a_s)_{s \in S} \|_{R \cap C} = \max \left\{ \left\| \sum_{s \in S} (a_s a_s^*)^{1/2} \right\|_{SP}, \left\| \sum_{s \in S} (a_s^* a_s)^{1/2} \right\|_{SP} \right\}.
\]

For a set \( E \in \Gamma \) we define the completely bounded constant \( \Lambda_p^{cb}(E) \) as infimum of \( C \) such that

\[
\left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq C \| (a_s)_{s \in S} \|_{R \cap C}
\]

for all matrices \( a_s \in M_n(C) \) and \( n \in \mathbb{N} \).

**Theorem 8.2.** If \( a_s \in M_n(C) \), then for all \( p \geq 2 \) and any Coxeter system \((W, S)\) we have

\[
\| (a_s)_{s \in S} \|_{R \cap C} \leq \left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq 2 A' \sqrt{p} \| (a_s)_{s \in S} \|_{R \cap C}.
\]

**Proof.** It was shown by Harcharras [LP86, Prop. 1.8] that \( \Lambda_p^{cb}(E) \) if finite if and only if \( E \) is an interpolation set for \( M_p(L^p(\Gamma)) \), i.e. every bounded function on \( E \) can be extended to a multiplier, and

\[
\Lambda_p^{cb}(E) \leq \Lambda_p^{cb}(R) \mu_p^{cb}(E),
\]

where \( R \) is the generating set in the Rademacher group \( \text{Rad}_\infty \) and \( \mu_p^{cb}(E) \) is the interpolation constant.

As shown by Buchholz [BUco5, Thm. 5] for \( p = 2n \), and \( S \) the standard generating set in \( \text{Rad}_\infty \), \( \Lambda_{2n}^{cb}(R) = ((2n - 1)!!)^{1/2n} \leq A \sqrt{p} \) for some absolute \( A \). This was extended by Pisier [PISO3, Thm. 9.8.2] for any \( p \geq 2 \), i.e.

\[
\Lambda_p^{cb}(R) \leq A' \sqrt{p},
\]

for an absolute constant \( A' \).

We have shown in Theorem 8.1 that in an arbitrary Coxeter group \( W \) its Coxeter generating set \( S \) is a weak Sidon set, i.e. it is interpolation set for the Fourier–Stieltjes algebra \( B(W) \). Since for \( p \geq 1 \), \( B(\Gamma) \) is a subalgebra of \( M_p(L^p(\Gamma)) \) and

\[
\| \varphi \|_{M_p(L^p(\Gamma))} \leq \| \varphi \|_{B(\Gamma)},
\]

we see that \( \mu_p^{cb}(S) \leq 2 \). Thus \( \Lambda_p^{cb}(S) \leq 2 A' \sqrt{p} \). This finishes the proof of the right inequality.

The left inequality holds for any group \( \Gamma \) and any \( S \subset \Gamma \) (see [LP86]).
Remark 8.3. Fendler [FENO2A] has shown that if for all \( s, t \in S \), \( s \neq t \), we have \( m_{s,t} \geq 3 \), then
\[
\Lambda^c_b(S) \leq 2 \sqrt{2}.
\]
See also [BOZ75] and [BUC99] for related results in the case of free Coxeter groups. Also Haagerup and Pisier have shown that \( \Lambda^c_b(\infty)(S) = 2 \), where \( \Lambda^c_b(E) = \sup_{p \geq 2} \Lambda^c_p(E) [HP93] \). See the paper of Haagerup [HAA81] where the best constant was calculated for the set of Coxeter generators of the Rademacher group in case when \( a \) are scalars.

9. Chromatic length function for Coxeter groups and pairpartitions

Let \([2n] = \{1, \ldots, 2n\}\). Let \(2^{[2n]}\) denote the set of subsets of \([2n]\). By a partition of \([2n]\) we mean \( \pi \subset 2^{[2n]} \) such that \( \bigcup \pi = [2n] \) and if \( \pi', \pi'' \in \pi \) then \( \pi' = \pi'' \) or \( \pi' \cap \pi'' = \emptyset \). We say, that partition \( \rho \) is a coarsening of a partition \( \pi \) if for any \( \pi' \in \pi \) there exists \( \rho' \in \rho \) such that \( \pi' \subset \rho' \).

A partition is called crossing if there exist \( 1 \leq a < b < c < d \leq 2n \) and \( \pi_1, \pi_2 \in \pi \) with \( a, c \in \pi_1 \neq \pi_2 \ni b, d \); otherwise it is called noncrossing. For any partition \( \pi \) there exists th the smallest noncrossing coarsening \( \Phi(\pi) \) of \( \pi \) (ie. if \( \rho \) is a noncrossing coarsening of \( \pi \) then it is a coarsening of \( \Phi(\pi) \)). We define \( \|\pi\| = n - \#\Phi(\pi) \). The notion for the map \( \Phi \) was introduced in [BY06].

We say that \( \pi \) is a pairpartition if every member of \( \pi \) has cardinality two. The set of pairpartitions of \([2n]\) is denoted by \( P_2(2n) \). A partition \( \pi \in P_2(2n) \) we write \( |\pi| \) to denote the number of ordered quadruples \( 1 \leq a < b < c < d \leq 2n \) such that both \( \{a, c\} \) and \( \{b, d\} \) belong to \( \pi \). Note, that \( |\pi| = 0 \) precisely when \( \pi \) is noncrossing. The set of noncrossing pairpartitions is denoted \( NC_2(2n) \).

Given a noncrossing pairpartition \( \omega \) we call \( \{b, c\} \in \omega \) an inner block if there exists \( \{a, d\} \in \omega \) with \( a < b < c < d \). The number of inner blocks of \( \omega \) we denote as \( \text{inn}(\omega) \).

In [BS96, Cor. 7] Bożejko and Speicher observed the following identity.

\[
(9.1) \quad \sum_{\pi \in P_2(2n)} (-1)^{|\pi|} q^{|\pi|} = \sum_{\omega \in NC_2(2n)} (1 - q)^{\text{inn}(\omega)}.
\]

Let \( f_n(q) = \sum_{\omega \in NC_2(2n)} (1 - q)^{\text{inn}(\omega)} \). It is elementary to derive
\[
f_n(q) = C_n \, {}_2F_1 \left( \frac{n, 1 - n}{n + 2} | q \right),
\]
where $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \# NC_2(2n)$ denote the $n$-th Catalan number and $2F_1$ is the classical hypergeometric function. If we write $f(q) = \sum_{j=0}^{n} t_j q^j$, then the triangle $(t_j^n)_{0 \leq j \leq n}$ appears in [SLO01] as “sequence” A062991. Since we are not going to use this formula, we leave it as an exercise to the reader. For the expansion of $f_n(1-t)$ and the Delanoy triangle appearing there the reader may consult [BW01, Prop 6.1].

In what follows, we prove a result about an arbitrary finitely generated Coxeter group, which for permutation groups implies the above one. Given a permutation $\sigma \in S_n$, we construct a pairpartition $\sigma = \{(i, 2n + 1 - \sigma(i)) | 1 \leq i \leq n\}$. Note, that $|\sigma|$ is equal to the length of $\sigma$ with respect to the Coxeter generators $(1, 2), \ldots, (n-1, n)$ of $S_n$. Therefore, denoting by $|w|$ the Coxeter length of an element $w$ of some Coxeter group $W$ will not lead into any confusion.

It is also clear that, with respect to the identification of permutations with a subset of pairpartitions, the two definitions of $\|\cdot\|$ agree (see Equations (1.3) and (1.2)).

By $W(t)$ we denote a growth series of a finitely generated Coxeter group $W$ (length function). That is, a power series $W(t) = \sum_{w \in W} t^{|w|}$. (Note, that the coefficient at $t$ equals to $\#S$. This explains why here and in the rest of this section we consider only finitely generated Coxeter groups. We will not repeat this assumption for short.) Moreover, for $X \subset W$ we write $X(t) = \sum_{w \in X} t^{|w|}$.

Let us define a multivariable formal power series (chromatic length function). For any $X \subset W$ define

$$X(t, q) = \sum_{w \in X} t^{|w|} \prod_{s \in S_w} q_s.$$

In particular $X(t) = X(t, 1)$, where $1 = (1)_{s \in S}$.

**Proposition 9.1.** The polynomial (or formal power series, if $W$ is infinite) $W(t, q)$ satisfies

$$W(t, q) = \sum_{T \subset S} W_T(t) \prod_{r \in T} r_s \prod_{s \in S \setminus T} (1 - q_s).$$

**Proof.** Let $W_{R}^o$ denote the set of all elements of $W_R$ not contained in any proper parabolic subgroup of $W_R$, ie. $W_{R}^o = W_R - \bigcup_{T \subset R} W_T$. Then, by inclusion-exclusion principle, $W_R^o(t) = \sum_{T \subset R} (-1)^{\#(R-T)} W_T(t)$. 


Therefore,

\[
W(t, q) = \sum_{w \in W} \lambda^{\|w\|} q_{\|w\|} = \sum_{R \in \mathcal{C} \sigma} W_{R}^0(t) \prod_{r \in R} q_{r} = \sum_{R \in \mathcal{C} \sigma} W_{R}^0(t) (-1)^{(R-\mathcal{T})} \prod_{r \in R} q_{r} = \sum_{T \in \mathcal{C} \sigma} W_{T}(t) \prod_{r \in T} q_{r} \sum_{T \in \mathcal{C} \sigma} (-q_{s}) \prod_{s \in R \setminus T} (1 - q_{s}).
\]

\[
\Box
\]

**Corollary 9.2.** If \( W \) is a finite Coxeter group then

\[
(9.2) \quad W(-1, q) = \prod_{s \in S}(1 - q_{s}).
\]

**Proof.** Choose \( s \in T \) and put \( W_{T}^{[s]} = \{ w \in W : |w| < |w_s| \} \). Clearly, \( W_{T} = W_{T}^{[s]} W_{T} \) therefore \( W_{T}(t) = W_{T}^{[s]}(t) W_{T}(t) \). Since \( W \) is a finite group, \( W_{T}^{[s]} \) is a polynomial. Thus \( W_{T}(-1) = 0 \) if \( T \) is nonempty (and \( W_{T}(-1) = 1 \)). \( \square \)

In order to prove Equation (9.1) we define the Wick map \( P_{2}(2n) \ni \pi \mapsto \pi : \in \mathcal{NC}_{2}(2n) \) (related to the normal order in quantum field theory). Given a pairpartition \( \pi \) we define \( \pi \) by repetitive resolving crossings. That is, we replace repetitively every crossing pair \( \{a, c\} \) and \( \{b, d\} \) with \( a < b < c < d \) by \( \{a, d\} \) and \( \{b, c\} \). In order to see that the result is independent of the order of resolution we describe \( \pi \) in an equivalent way.

Let \( \Phi(\pi) \) be the smallest noncrossing coarsening of \( \pi \). For each block \( \beta \) of \( \Phi(\pi) \) define \( \beta^{+} = \{ y \in (\exists x) x \in \beta, y > x, \{ x, y \} \in \pi \} \) and \( \beta^{-} = \{ x \in (\exists y) y \in \beta, y > x, \{ x, y \} \in \pi \} \). Order \( \beta^{+} = \{ y_{1}, \ldots, y_{k} \} \) in increasing way and \( \beta^{-} = \{ x_{1}, \ldots, x_{k} \} \) in decreasing way. Then all pairs \( \{x_{i}, y_{j}\} \) will be parts of \( \pi \).

Equation (9.1) will follow from a more refined statement.

**Proposition 9.3.** For every \( \omega \in \mathcal{NC}_{2}(2n) \)

\[
(9.3) \quad \sum_{\pi \in \mathcal{P}_{2}(2n)} (-1)^{|\pi|} q^{\|\pi\|} = (1 - q)^{\text{inn}(\omega)}.
\]

**Proof.** Let us first consider the case of \( \omega = \mathcal{T} = \{(i, 2n+1-i) | 1 \leq i \leq n \} \). Clearly, \( [\pi : \pi = \mathcal{T}] = [\mathcal{S}_{\pi} : \pi \in \mathcal{S}_{n}] \). And Equation (9.3) is equivalent to Equation (9.2) (as all \( q_{s} \) are set to \( q \)) for \( W = \mathcal{S}_{n} \).
In a general case observe, that $\Phi(\pi)$ is a coarsening of $\omega = \pi c$. Yet, not every coarsening may appear. The obvious condition is that for each block $\beta$ of $\pi c$ the pair $[\min \beta, \max \beta]$ belong to $\omega$. For the purpose of this proof we will call such a coarsening admissible. Its clear, that admissible coarsenings $\rho$ are in one to one correspondence with subsets of $\omega$ containing all outer (not inner) parts of $\omega$ of the form $\{(\min \rho', \max \rho')|\rho' \in \rho\}$.

Let us refine Equation (9.3) further. For every $\omega \in NC_2(2n)$ and any admissible coarsening $\eta$ of $\omega$ we have

\[
\sum_{\pi \in P_2(2n) \atop \pi = \omega, \Phi(\pi) = \rho} (-1)^{|\pi|} = (-1)^{|\rho|}.
\]

Equation (9.3) follows from (9.4) by multiplying by $q^{n-|\eta|}$ and summing over all admissible coarsenings $\eta$ of $\omega$.

Equation (9.4) is again equivalent to to Equation (9.2) (for all apermutation groups and all $q_s$ set to $q$) as both sides factor as a product over blocks of $\rho$.

**Question 9.4.** We have proven Equation 9.1 with the help of an embedding $S_n \ni \sigma \mapsto \sigma \in NC_2(2n)$ (or several such embeddings, one for each outer block of $\omega$). Corollary 9.2 holds for any Coxeter group. Is there a corresponding formula concerning some generalization of pairpartitions?

In the proof of Proposition 9.1 we have not assumed that $W$ was finite. Let us finish this section with a discussion of infinite Coxeter groups. Recall, that $-1$ does not lie in the radius of convergence on $W(t)$ if $W$ is not finite. Nevertheless, $W(t)$ represents a rational function as follows from the following result.
Proposition 9.5. ([STE68], [SER71, Prop. 26]) Let \((W, S)\) be an an infinite Coxeter system. Then

\[
\frac{1}{W(t)} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|} W_T(1/t)}{W_T(1/t)}.
\]

Where \(\mathcal{F}\) denote the family of subsets \(T \subset S\), such that the group \(W_T\) generated by \(T\) is finite. In particular, \(W(t)\) is a series of a rational function (i.e. a quotient of polynomials).

One may ask a question what is the class of (infinite) Coxeter groups such that \(W_T(-1) = 0\) for any nonempty subset \(T\) of generators. A na"ıve argument that

\[W(t) = W_\{s\}(t)W^{\{s\}}(t) = (1 + t)W^{\{s\}}(t)\]

shows, that the question if \(W(-1) \neq 0\) is equivalent to whether \(W^{\{s\}}(t)\) can have a pole at \(t = -1\). On the other hand note, that if \(W\) is of type \(A_2\), i.e. \(W\) is given by a presentation \(\langle s_i : 1 \leq i \leq 3 \mid s_i^2, (s_i s_j)^3 : 1 \leq i < j \leq 3 \rangle\) then, by Equation (9.5), \(W(t) = \frac{1+t+t^2}{(1-t)^2}\) and \(W(-1) = 1/4\).

More generally, it is known ([BOU68]) that in each coset of \(W_T\) there exists the unique shortest element. Let \(W^T\) denote the set of those shortest representatives. Moreover if \(w = w_T w_T^\prime\) with \(w_T \in W_T^\prime \subset W_T\) then \(|w| = |w_T^\prime| + |w_T|\). Therefore \(W^T(t)W^T(t) = W(t)\). In particular, \(W^T(t)\) represents a rational function, and it is legitimate to ask about the value of \(W^T(-1)\).

In the case of finite Coxeter group \(W\), Eng [ENG91] observed that

\[W^T(-1) = \# \{w \in W^T|ww_0w \in W_T\},\]

where \(w_0\) is the longest element in \(W\). (Eng’s proof was case-by-case. Later, a general classification-free proof of Eng’s theorem was given in [RWS04]).

Subsequently, Reiner [REI02] has shown that if \(W\) is crystallographic (i.e. the Weyl group in a compact Lie group \(G\)), then both sides of the above equality compute the signature of the corresponding flag variety \(G/Q_T\), where \(Q_T\) is a parabolic subgroup associated to \(T\).

What is the meaning of \(W(-1)\) or \(W^T(-1)\) for infinite \(W\)?

We do not know if it possible for \(W(-1)\) to be negative. If one takes \(W = \langle s_i : 1 \leq i \leq 4 \mid s_i^2, (s_i s_j)^3 : 1 \leq i < j \leq 4 \rangle\). Then, by Equation (9.5), \(W(t) = \frac{(1+t)(1+t+t^2)}{3^2-2t^2-2t+1}\) and \(W^{s_1}(-1) = -1/2\).
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