The Absence of Ultralocal
Ginsparg-Wilson Fermions

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It was shown recently by I. Horváth that lattice fermions obeying the standard form of the Ginsparg-Wilson relation cannot be ultralocal. However, there are more general forms of the Ginsparg-Wilson relation, which also guarantee the physical properties related to chirality, but which are not covered by Horváth’s consideration. Here we present a proof which applies to all Ginsparg-Wilson fermions, demonstrating that they can only be local in the sense of an exponential decay of their couplings, but not ultralocal.
A formulation of lattice fermions is characterized by some lattice Dirac operator $D$. The famous Nielsen-Ninomiya No-Go theorem \[1\] excludes — based on mild assumptions, namely Hermiticity and discrete translation invariance — the existence of undoubled lattice fermions, which are chiral (in the sense that $\{D, \gamma_5\} = 0$) and local (in the sense that the couplings in $D$ decay at least exponentially).

Recently, much attention has been attracted to an old idea by P. Ginsparg and K. Wilson \[2\], who suggested to break the chiral symmetry on the lattice in a particularly smooth way, so that

$$\{D_{x,y}, \gamma_5\} = 2(DR\gamma_5D)_{x,y}, \quad (1)$$

where the term $R$ is non-trivial and local. (Here we refer to a lattice of unit spacing in the $d$ dimensional Euclidean space, $x, y \in \mathbb{Z}^d$.)

In addition $R$ commutes with $\gamma_5$. This can be explained for instance by starting from the prescription $\{D^{-1}, \gamma_5\} = 2\gamma_5R = 2R\gamma_5$ (note that a local term $R$ doesn’t shift the poles in $D^{-1}$). Alternatively, we can start off by requiring invariance of the Lagrangian to $O(\epsilon)$ for a lattice modified chiral transformation in the spirit of Ref. \[3\], $\bar{\psi} \to \bar{\psi}(1 + \epsilon[1 - D]R)\gamma_5$, $\psi \to \psi(1 + \epsilon\gamma_5[1 - RD])$ ($R$ local; $DR, RD$ convolutions in c-space), which leads to $\{D, \gamma_5\} = D\{R, \gamma_5\}D$. Then we see that only the part of $R$, which commutes with $\gamma_5$, contributes. This is the term that we call $R$, i.e. $\{R, \gamma_5\} = 2R\gamma_5$.

There are three types of local lattice fermion formulations in the literature, the perfect as well as the classically perfect fermions \[2, 4, 5\] and another formulation by H. Neuberger \[6\] (based on the so-called overlap formalism \[7\]), which obey the Ginsparg-Wilson relation (GWR), eq. (1), \[8, 9, 10\]. In fact, this relation preserves the essential physical properties related to chirality \[2, 8, 9, 3, 11\], even for chiral gauge theory \[12\].

As a virtue of the slight relaxation of the chiral symmetry condition for $D$, fermions obeying eq. (1) (GW fermions) can be local in sense of an exponential decay of the couplings in $D$. This is a great progress, but it does not mean that GW fermions can even be ultralocal, i.e. that their couplings drop to zero beyond a finite number of lattice spacings. The absence of ultralocal GW fermions has first been conjectured intuitively \[14\]. In fact, it has been shown by I. Horváth \[16\] that ultralocality is excluded for the standard form of the GWR, which is given by $R_{x,y} = \frac{1}{2} \delta_{x,y}$.

However, the question if this is still true for any choice of the Ginsparg-Wilson kernel $R$ has not been answered yet, and the answer is not obvious at all from Horváth’s consideration. Here we are going to prove the absence of ultralocal solutions $D$ for all local kernels $R$, i.e. for all Ginsparg-Wilson fermions.

\footnote{For Neuberger fermions in QCD, locality has been discussed in detail in Ref. \[13\], and it holds at least up to moderate coupling strength. Other types of overlap fermions have a still higher degree of locality \[14, 15\].}
We start from the following observations: (i) It is sufficient to show the absence of free ultralocal GW fermions. (ii) If we can show this property in $d = 2$, then ultralocal GW fermions in all dimensions $d > 2$ are ruled out as well, because they could always be mapped on a 2d solution of the GWR. In momentum space, such a mapping corresponds to the restriction $D(p_1, p_2, 0, \ldots, 0)$.

We assume Hermiticity, discrete translation invariance, as well as invariance under reflections and exchange of the axes. Then a general ansatz for $D$ in $d = 2$ reads

$$D(p) = \rho_\mu(p)\gamma_\mu + \lambda(p),$$

where $\lambda(p)$ is a real Dirac scalar, whereas $\rho_\mu(p)$ is imaginary. Here $\rho_\mu$ is odd in the $\mu$-direction and even in the other direction, while $\lambda$ is even in both directions. Furthermore exchange symmetry of the axes implies $\rho_1(p_1, p_2) = \rho_2(p_2, p_1)$ and $\lambda(p_1, p_2) = \lambda(p_2, p_1)$. As a consequence, the GW kernel $R$ is a Dirac scalar, $R$ is even, and $R(p_1, p_2) = R(p_2, p_1)$.

The fermion has to be massless, and the operator $D$ must have the correct continuum limit, which implies

$$\rho_\mu(p) = ip_\mu + O(\epsilon^3), \quad \lambda(p) \leq O(\epsilon^2),$$

if $p_1, p_2 = O(\epsilon)$.

We assume $D$ — and therefore $\rho_\mu$ and $\lambda$ — to be ultralocal, and we are going to demonstrate that such a GW fermion does not exist.

To capture all local kernels $R \neq 0$ we proceed in two steps.

**STEP 1**

In a first step we assume $R$ to be *ultralocal*. Then the modified operator $D'$

$$D'(p) := 2R(p)D(p) = \rho_\mu'(p)\gamma_\mu + \lambda'(p)$$

$$\rho_\mu'(p) = 2r_0 ip_\mu + O(\epsilon^3), \quad \lambda'(p) \leq O(\epsilon^2),$$

is ultralocal as well (where $r_0 := R(p = 0)$).

Now the free GWR can be written as

$$-\rho_1^2(p) - \rho_2^2(p) + \lambda'(p) = 1,$$

where $\lambda'(p) := 1 - \lambda'(p)$.

A free GW fermion has to satisfy eq. (5) at any momentum $p$. We first consider this condition only for the special case $p_1 = p_2 := q$ and look at the quantities

$$\rho_n^{(\text{dia})} := \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \rho_1'(q, q) \exp(iqn) \quad \text{and}$$

$$\tilde{\rho}_n^{(\text{dia})} := \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \tilde{\lambda}'(q, q) \exp(iqn) \quad (n \in \mathbb{Z})$$
(note that $\rho'_1(q, q) = \rho'_2(q, q)$). They have to be ultralocal, i.e. confined to some finite interval $|n| \leq L_{\text{dia}}$. We choose $L_{\text{dia}}$ so that it is the maximal distance over which a non-trivial coupling occurs. According to the Lemma in Ref. [16], only the “extreme” couplings with $n = \pm L_{\text{dia}}$ can contribute to $\rho_n^{(\text{dia})}$, $\ell_n^{(\text{dia})}$ [17]. From the low momentum expansion (4) we obtain

\begin{align*}
\rho^{(\text{dia})}(q) &= \frac{2r_0 i \sin(L_{\text{dia}}q)}{L_{\text{dia}}}, \\
\ell^{(\text{dia})}(q) &= \cos(L_{\text{dia}}q),
\end{align*}

(6)

so that only discrete values

\begin{equation}
2r_0 = \pm \frac{L_{\text{dia}}}{2^{3/2}}
\end{equation}

(7)

lead to a solution of the free GWR (3) restricted to $p_1 = p_2$. I. Horváth considered the case of a constant $R(p) = 1/2 = r_0$ (standard GW kernel), and he observed that there is no solution for that. However, we see now that the diagonal case $p_1 = p_2$ is not sufficient to rule out ultralocal GW fermions in general. All the cases where $2^{3/2}r_0$ is an integer are not covered by this consideration.

Of course we have exploited only a small part of condition (3) so far. We now take into consideration another special case by setting $p_2 = 0$. For this “mapping to $d = 1$” eq. (3) simplifies to

\begin{equation}
- \rho'^2(p_1, 0) + \tilde{\lambda}'^2(p_1, 0) = 1.
\end{equation}

(8)

We repeat exactly the same procedure as in the diagonal case, based on the Lemma in Ref. [16]. In this case, we denote the maximal (and only) coupling distance of

\begin{align*}
\rho_n^{(1d)} &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} dp_1 \rho'_1(p_1, 0) \exp(ip_1n) \quad \text{and} \\
\ell_n^{(1d)} &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} dp_1 \tilde{\lambda}'(p_1, 0) \exp(ip_1n)
\end{align*}

as $L_{1d}$, and eqs. (4) and (8) now yield the condition

\begin{equation}
2r_0 = \pm L_{1d}.
\end{equation}

(9)

We see that a number of ultralocal solutions for eq. (8) exist. For instance, the 1d Wilson fermion solves the 1d mapping of the standard GWR.

We now combine the two conditions which arise from our two special cases of eq. (3). Eqs. (7) and (9) lead to the requirement

\begin{equation}
L_{\text{dia}}^2 = 2L_{1d}^2
\end{equation}

(10)
with the only solution $L_{dia} = L_{1d} = 0$. Since $\rho'_\mu$ is odd, $L_{dia} = 0$ further implies

$$\rho'_\mu(q, q) = 0, \quad q \in [-\pi, \pi].$$ (11)

Since $R$ is ultralocal and even with respect to both axes, $R(p)$ can be written as

$$\sum_{x_1=0}^{N} \sum_{x_2=0}^{N} \Gamma_{x_1, x_2} \cos(x_1 p_1) \cos(x_2 p_2)$$ (12)

($N$ finite). Exchange symmetry of the axes implies $\Gamma_{x_1, x_2} = \Gamma_{x_2, x_1}$, and from property (13) we infer $0 = \Gamma_{x_1, x_2} + \Gamma_{x_2, x_1} = 2\Gamma_{x_1, x_2}$ for all $x_1, x_2$. However, this means $R = 0$, which contradicts the assumptions.

This completes STEP 1, i.e. the proof of the absence of ultralocal solutions for the class of ultralocal GW kernels $R$. As a side-remark we add that the result of STEP 1 even holds if we allow for fermion doubling.

STEP 2

In the second step we consider the case where $R$ decays exponentially, i.e. we assume it to be local but not ultralocal. The question is, if such a kernel $R$ exists, that is: is it possible that

$$2R = \frac{\lambda}{-\rho^2 + \lambda^2}$$ (13)

is local (but not ultralocal), when $\rho_\mu$ and $\lambda$ are ultralocal?

In the ratio on the right-hand side of eq. (13) both, the numerator and the denominator are even and symmetric under exchange of the axes. Hence both of them take the form (12), where $N$ is finite again (because $D$ is ultralocal). Now we factorize all the terms with $x_\mu > 1$ so that only $\cos p_1$ and $\cos p_2$ occur. Furthermore we define

$$c_\mu := 1 - \cos p_\mu \quad (\mu = 1, 2), \quad c_\mu = O(\epsilon^2).$$ (14)

After this factorization we can obviously express numerator and denominator of eq. (13) as polynomials (of a finite degree) in $c_1$, $c_2$. From eq. (8) we know

$$-\rho^2(c_1, c_2) = 2(c_1 + c_2) + O(\epsilon^4), \quad \lambda = O(\epsilon^2).$$ (15)

As a next step, we assume that the polynomials in ratio (13) are simplified maximally. This means that the maximal common factor of numerator and denominator is divided off, with the condition that both preserve their form as polynomials in $c_1, c_2$ (or in other words: these polynomials are reduced to their minimal degree).

After this simplification, we consider the denominator and distinguish three cases:

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(a) The simplified denominator reduces to a constant.

Of course, after this simplification also the numerator is still a finite polynomial and therefore ultralocal, hence in this case \( R \) would be ultralocal. This means that we are actually not in the class of GW kernels that we want to consider in STEP 2; this case has already been discussed (and ruled out) in STEP 1.

(b) The simplified denominator vanishes at \( c_1 = c_2 = 0 \).

Then the same must be true for the numerator, since \( R(p) \) must be regular. For small momenta, we call the order of the denominator \( O(\epsilon^2k) \), and that of the numerator \( O(\epsilon^{2k}) \) (where \( k, \bar{k} \) are natural numbers and \( k \geq k \geq 1 \) so that \( R(c_1, c_2) = O(\epsilon^{2(k-k)}) \). Now we take \((k-k+1)\) derivatives of \( R \) with respect to \( c_1 \) or \( c_2 \). The result will diverge at \( p = (0,0) \) (the situation, where such a derivative vanishes does not belong to case (b)). Therefore \( R \) is not analytical in momentum space, and hence it is non-local. So in this case we are not dealing with a GW fermion.

(c) The simplified denominator is momentum dependent, and it does not vanish at \( c_1 = c_2 = 0 \).

Without loss of generality we can assume the denominator to take the form \( 1 + O(\epsilon^2) \).

At first glance, this case seems to allow for many ultralocal solutions. As a simple example, one could set \( \lambda = -\rho^2 \). However, such a term \( \lambda \) does not avoid doubling, hence in this case — and only in this case — the condition that our GW fermion is \textit{free of doubling} is crucial. Technically it means that \( D \) has no zeros in the Brillouin zone \([−\pi, \pi]^2\), except for the physical one at \( p_1 = p_2 = 0 \).

In case (c), in the above simplification a momentum dependent factor was divided off. We call this factor \( K \), and \( -\rho^2 = KS, \lambda = KT \), where \( K, S, T \) are all polynomials in \( c_1, c_2 \).

We now consider the possible forms of \( K \) and \( S \). Let us first go back to the terms \( \rho_\mu(p_1, p_2) \). They can be factorized in the same way as we treated eq. (13), and we arrive at the form

\[
\rho_1(p_1, p_2) = i \sin p_1 F(c_1, c_2),
\rho_2(p_1, p_2) = i \sin p_2 F(c_2, c_1),
\]

(16)

where \( F(c_1, c_2) = 1 + O(\epsilon^2) \) is once more a polynomial. This implies

\[
-\rho^2 = KS = c_1(2 - c_1)F(c_1, c_2)^2 + c_2(2 - c_2)F(c_2, c_1)^2.
\]

(17)

Since \( S = 1 + O(\epsilon^2) \), we see from eq. (15) that \( K \) can be written as

\[
K = c_1 X(c_1, c_2) + c_2 X(c_2, c_1),
\]

(18)
where \( X(c_1, c_2) = 2 + O(\epsilon^2) \). \( X \) is again a polynomial in \( c_1, c_2 \), and it is strictly forbidden that it contains any factor \((2 - c_1)\) or \((2 - c_2)\). (Otherwise this would also be a factor of \( \lambda \), and then doubling occurs at momentum \((p_1, p_2) = (\pi, 0)\) resp. \((0, \pi)\).)

On the other hand, such factors are allowed in \( S \). We finally decompose \( S \) as
\[
S(c_1, c_2) = (2 - c_1)^{n_1} (2 - c_2)^{n_2} Y(c_1, c_2). \tag{19}
\]
where \( n_1, n_2 \in \mathbb{N}_0 \). This decomposition is done such that \( Y \) does not contain any factors \((2 - c_1)\) or \((2 - c_2); all\) these factors are extracted, hence \( n_1, n_2 \) are maximal. Of course, due to exchange symmetry in the axes we know that \( n_1 = n_2 \). \tag{20}

Combining eqs. (18) and (19) we arrive at
\[
K S = c_1 (2 - c_1)^{n_1} (2 - c_2)^{n_2} X(c_1, c_2) Y(c_1, c_2) + c_2 (2 - c_1)^{n_1} (2 - c_2)^{n_2} X(c_2, c_1) Y(c_1, c_2). \tag{21}
\]
We recall that \( X \) and \( Y \) do not contain any factors \( c_\mu \) or \((2 - c_\mu)\), due to the above decompositions. Together with eqs. (16), (17) we obtain
\[
F(c_1, c_2)^2 = (2 - c_1)^{n_1 - 1} (2 - c_2)^{n_2 - 1} X(c_1, c_2) Y(c_1, c_2). \tag{22}
\]
Since \( F(c_1, c_2) \) is a polynomial itself, we conclude that \( n_1 \) must be odd, whereas \( n_2 \) must be even.

This contradicts eq. (20), and therefore case \((c)\) is excluded as well. \( \square \)

Now we have completed the general proof that GW fermions cannot be ultralocal. \( \square \)

In view of practical applications, this result means that we cannot simulate fermions obeying the GWR as formulated in the infinite volume. In finite volume with certain boundary conditions, the GWR — with these boundary conditions implemented — may hold, but this requires the coupling over all distances in the given volume, which is inconvenient. What one can work on is a very fast exponential decay of the couplings \( \hat{1}, \hat{1}, \hat{5} \). In order to construct an overlap fermion with a high level of locality, it turned out to be useful to start from a short-ranged approximate GW fermion, which is then inserted into the overlap formula.

As a criterion for the quality of a short-ranged approximate free GW fermion, we can insert it into the GWR and solve for \( R \). This term is only a pseudo-GW kernel, since it has got to be non-local, according to our result. Indeed, if we insert for instance the Wilson fermion, the resulting term decays

\( \text{\footnote{W. Kerler suggests that the GWR should generally take the form } } \{ D, \gamma_5 \} = 2RD\gamma_5D \text{ instead of eq. } \{ D, \gamma_5 \}. \text{ Of course, our proof applies to that formulation too.} \)
as $R_{x,y} \sim 1/(4|x-y|^4)$ in $d = 2$, and $R_{x,y} \sim 1/(1.6|x-y|^6)$ in $d = 4$.

For comparison, a truncated perfect free hypercube fermion (with couplings inside a unit hypercube on the lattice) provides a better approximation to a GW fermion, and the corresponding pseudo-GW kernel decays much faster [19]: $R_{x,y} \sim 1/(290|x-y|^4)$ in $d = 2$, resp. $R_{x,y} \sim 1/(120|x-y|^6)$ in $d = 4$.

To summarize, we repeat that we are dealing with a new variant of a No-Go theorem for lattice fermions. The well-known Nielsen-Ninomiya theorem excludes locality if the fermion obeys $\{D, \gamma_5\} = 0$. If we relax this condition to the GWR, then locality is possible, but ultralocality still not. We have demonstrated this for all GW kernels in any dimension $d \geq 2$, and therefore for all GW fermions.

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Consider two functions $A_n, B_n$ ($n \in \mathbb{Z}$), which obey in momentum space $A(k)^2 + cB(k)^2 = 1$ (for some $c \neq 0$). If $L$ is the maximal range of $A_n, B_n$ (i.e. $L$ is the minimal integer so that $A_n = B_n = 0$ for all $|n| > L$), then the Lemma states that $A_n = B_n = 0$ also if $|n| < L$. This can be shown by induction in coordinate space \[16\].

This Lemma applies to $-2\rho^{(\text{dia})}(q)^2 + \ell^{(\text{dia})}(q)^2 = 1$, as well as eq. (8).

\[18\] W. Kerler, hep-lat/9905010.

\[19\] W. Bietenholz, hep-lat/0001001.

\[20\] I. Horváth, Phys. Rev. D60 (1999) 034510.