Magnetotransport of a periodically modulated graphene monolayer

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(Date textdate; Received textdate; Revised textdate; Accepted textdate; Published textdate)

Abstract

We have performed a detailed investigation of the electrical properties of a graphene monolayer which is modulated by a weak one dimensional periodic potential in the presence of a perpendicular magnetic field. The periodic modulation broadens the Landau Levels into bands which oscillate with $B$. The electronic conduction in this system can take place through either diffusive scattering or collisional scattering off impurities. Both these contributions to electronic transport are taken into account in this work. In addition to the appearance of commensurability oscillations in both the collisional and diffusive contributions, we find that Hall resistance also exhibits commensurability oscillations. Furthermore, the period and amplitude of these commensurability oscillations in the transport parameters and how they are affected by temperature are also discussed in this work.
I. INTRODUCTION

Recent successful preparation of a single layer of graphene has generated a lot of interest in this system as experimental and theoretical studies have shown that the nature of quasiparticles in this two-dimensional system is very different from those of conventional two-dimensional electron gas (2DEG) systems realized in semiconductor heterostructures. Graphene has a honeycomb lattice of carbon atoms. The quasiparticles in graphene have a band structure in which electron and hole bands touch at two points in the Brillouin zone. At these Dirac points the quasiparticles obey the massless Dirac equation. In other words, they behave as massless, chiral Dirac Fermions leading to a linear dispersion relation $E_k = \hbar v_F k$ (with the characteristic velocity $v_F \simeq 10^6 m/s$). This difference in the nature of the quasiparticles in graphene from a conventional 2DEG has given rise to a host of new and unusual phenomena such as the anomalous quantum Hall effect[1, 2] with profound effects on transport in these systems. The transport properties of graphene are currently being explored in the presence of nonuniform potentials, such as in $p - n$ junctions[3], as well as in periodic potentials. Effects of periodic potential on electron transport in 2D electron systems has been the subject of continued interest, where electrical modulation of the 2D system can be carried out by depositing an array of parallel metallic strips on the surface or through two interfering laser beams[4]. More recently in graphene, electrostatic[5] and magnetic[6] periodic potentials have been shown to modulate its electronic structure in unique ways leading to fascinating physics and possible applications. Periodic potentials are induced in graphene by interaction with a substrate[7] or controlled adatom deposition[8]. In addition, it was shown that periodic ripples in suspended graphene also induces a periodic potential in a perpendicular electric field[9]. Epitaxial growth of graphene on top of a prepatterned substrate is also a possible route to modulation of the potential seen by the electrons. In this work, we complement these recent studies to discuss the effects of a weak electric modulation on the electrical conductivity in a graphene monolayer subjected to an external magnetic field perpendicular to the graphene plane. Electric modulation introduces a new length scale, period of modulation, in the system giving rise to interesting physical effects on the transport response. Commensurability (Weiss) oscillations, in addition to Shubnikov de Hass (SdH) oscillations, are found to occur as a result of commensurability of the electron cyclotron diameter at the Fermi energy and the period of the electric
modulation. In [10], on the same subject, diffusive contribution to magnetoconductivity was considered whereas in the present work we determine collisional and Hall contributions as well. This makes this paper a complete study of electric modulation induced effects on electrical conductivities/resistivities in a graphene monolayer in the presence of a magnetic field.

In the next section, we present the formulation of the problem and derive expressions for electrical conductivities in a graphene monolayer. In section III, results of numerical work are presented and discussed, followed by the conclusions in section IV.

II. FORMULATION

We consider a graphene sheet in the $x-y$ plane. The magnetic field $\mathbf{B}$ is applied along the $z-$ direction. The system is also subjected to a 1D weak periodic modulation $U(x)$ in the $x-$ direction. The one electron Hamiltonian reads

$$H = v_F \sigma \cdot (p + e\mathbf{A}) + U(x)$$

where $p$ is the momentum operator, $\sigma = \{\sigma_x, \sigma_y\}$ are Pauli matrices and $v_F (\sim 10^6 \text{m/s})$ characterizes the electron velocity in graphene. In the absence of modulation, i.e. for $U(x) = 0$ and for the vector potential chosen in the Landau guage $A = (0, Bx, 0)$, the normalized eigenfunctions of Eq. (1) are given by

$$\phi_n(x), \phi_{n-1}(x)$$

are the harmonic oscillator wavefunctions centred at $x_o = l^2 k_y$. $n$ is the Landau level index, $l = \sqrt{\frac{\hbar}{eB}}$ the magnetic length and $L_y$ the length of 2D graphene system in the $y$ direction. The corresponding eigenvalue is

$$E_n = \hbar \omega_g \sqrt{n} \quad \text{where} \quad \omega_g = v_F \sqrt{2eB/\hbar} = v_F \sqrt{2}/l.$$ 

The modulation potential is approximated by the first Fourier component of the periodic potential

$$U(x) = V_o \cos Kx$$

where $K = 2\pi/a$, $a$ is the period of modulation and $V_o$ is the constant modulation amplitude. This potential lifts the degeneracy of Landau Levels (LLs) and the energy becomes dependent on the position $x_o$ of the guiding centre. Thus energy eigenvalues for weak modulation ($V_o \ll E_F$), using first order perturbation theory, are

$$E_{n,k_y} = E_n + V_{n,B} \cos Kx_o$$

where $V_{n,B} = \frac{V_o}{2} e^{-u/2} \left[ L_n(u) + L_{n-1}(u) \right]$ with $L_n(u), L_{n-1}(u)$ the Laguerre polynomials and $u = K^2 l^2/2$. We note that the electric modulation induced broadening of the energy
spectrum is nonuniform. The Landau bandwidth \( \sim V_{n,B} \) oscillates as a function of \( n \) since \( L_n(u) \) are oscillatory functions of index \( n \). \( V_{n,B} \) at the Fermi energy can be approximated, using an asymptotic expression for \( n \gg 1 \) appropriate for low magnetic-field range relevant to the present study, as

\[
V_B = V_0 \sqrt{\frac{2}{\pi K R_c}} \cos(K R_c - \frac{\pi}{4})
\]  

where \( R_c = k_F l^2 \) is the classical cyclotron orbit, \( k_F = \sqrt{2\pi n_e} \) and \( n_e \) is the electron number density. The above expression shows that \( V_B \) oscillates with \( B \), through \( R_c \), and the width of Landau bands \( 2 |V_B| \) becomes maximum at

\[
\frac{2R_c}{a} = i + \frac{1}{4} \quad (i = 1, 2, 3, ...)
\]

and vanishes at

\[
\frac{2R_c}{a} = i - \frac{1}{4} \quad (i = 1, 2, 3, ...).
\]

which is termed the flat band condition. The oscillations of the Landau bandwidth is the origin of the commensurability (Weiss) oscillations and, at the same time, are responsible for the modulation of the amplitude and the phase of the Shubnikov-de Hass (SdH) oscillations.

To calculate the electrical conductivity in the presence of weak modulation we use Kubo formula [11]. The diffusive contribution to conductivity which arises due to the scattering induced migration of the Larmor circle center has already been determined for a graphene monolayer in [10]. Our focus, in this work, will be the calculation of the collisional contribution to the conductivity and the the Hall conductivity.

1. **COLLISIONAL CONDUCTIVITY:**

To obtain collisional contribution to conductivity, we assume that electrons are elastically scattered by randomly distributed charge impurities as it has been shown that charged impurities play a key role in the transport properties of graphene near the Dirac point [17, 18]. This type of scattering is dominant at low temperature. The collisional conductivity when spin degeneracy is considered is given by

\[
\sigma_{col}^{xx} = \frac{\beta e^2}{\Omega} \sum_{\xi, \xi'} f_\xi (1 - f_{\xi'}) W_{\xi\xi'} (\alpha_\xi^x - \alpha_{\xi'}^x)^2
\]  

where \( W_{\xi\xi'} \) is the scattering rate and \( f_\xi \) is the Fermi-Dirac distribution function with \( \xi \) denoting the Landau level index. The collisional conductivity is proportional to the square of the Fermi energy, \( E_F \), and inversely proportional to the density of states at the Fermi level, \( N(E_F) \), and the scattering rate, \( W_{\xi\xi'} \). The effect of spin degeneracy is included by considering the spin-dependent scattering rates, \( W_{\xi\xi'}^{\uparrow\downarrow} \) and \( W_{\xi\xi'}^{\downarrow\uparrow} \), which account for scattering between spin-up and spin-down states.

The electronic contribution to the conductivity is given by the Kubo formula [11] and is

\[
\sigma_{elec}^{xx} = \frac{e^2}{\Omega} \sum_{\xi, \xi'} f_\xi (1 - f_{\xi'}) W_{\xi\xi'} (\alpha_\xi^x - \alpha_{\xi'}^x)^2
\]  

where \( \alpha_\xi^x \) is the spin component of the momentum transfer to the electron at the Landau level \( \xi \). The spin-dependent scattering rates, \( W_{\xi\xi'}^{\uparrow\downarrow} \) and \( W_{\xi\xi'}^{\downarrow\uparrow} \), are determined by the spin susceptibility of the system. The collisional contribution to the conductivity is significant at low magnetic fields and higher temperatures, while the electronic contribution is dominant at high magnetic fields and lower temperatures. The total conductivity is the sum of the collisional and electronic contributions.

The Hall conductivity, \( \sigma_{xy} \), is given by

\[
\sigma_{xy} = \frac{\pi e^2}{4h}N_{\xi\xi'}(\alpha_\xi^y - \alpha_{\xi'}^y)^2
\]

where \( N_{\xi\xi'} \) is the density of states at the Landau level \( \xi \) and \( \xi' \). The Hall conductivity is proportional to the square of the Fermi energy and inversely proportional to the density of states at the Fermi level, \( N(E_F) \), and the scattering rate, \( W_{\xi\xi'} \). The effect of spin degeneracy is included by considering the spin-dependent scattering rates, \( W_{\xi\xi'}^{\uparrow\downarrow} \) and \( W_{\xi\xi'}^{\downarrow\uparrow} \), which account for scattering between spin-up and spin-down states.

The electronic contribution to the Hall conductivity is given by the Kubo formula [11] and is

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where \( \alpha_\xi^y \) is the spin component of the momentum transfer to the electron at the Landau level \( \xi \). The spin-dependent scattering rates, \( W_{\xi\xi'}^{\uparrow\downarrow} \) and \( W_{\xi\xi'}^{\downarrow\uparrow} \), are determined by the spin susceptibility of the system. The collisional contribution to the Hall conductivity is significant at low magnetic fields and higher temperatures, while the electronic contribution is dominant at high magnetic fields and lower temperatures. The total Hall conductivity is the sum of the collisional and electronic contributions.

The total conductivity tensor is given by

\[
\sigma_{ij} = \sigma_{col}^{xx} \delta_{ij} + \sigma_{elec}^{xx} \delta_{ij} + \sigma_{col}^{xy} \delta_{ij} + \sigma_{elec}^{xy} \delta_{ij}
\]

where \( \delta_{ij} \) is the Kronecker delta function. The total conductivity tensor describes the transport properties of graphene in a magnetic field, taking into account both the collisional and electronic contributions.

The total Hall conductivity tensor is given by

\[
\sigma_{ij}^{xy} = \sigma_{elec}^{xy} + \sigma_{col}^{xy} \delta_{ij}
\]

where \( \delta_{ij} \) is the Kronecker delta function. The total Hall conductivity tensor describes the transport properties of graphene in a magnetic field, taking into account both the collisional and electronic contributions.

The total conductivity tensor and the total Hall conductivity tensor provide a comprehensive description of the transport properties of graphene in a magnetic field, taking into account both the collisional and electronic contributions. The collisional contribution is significant at low magnetic fields and higher temperatures, while the electronic contribution is dominant at high magnetic fields and lower temperatures. The total conductivity and the total Hall conductivity are given by the sum of the collisional and electronic contributions.
where \( f_\xi = [\exp(\frac{E_\xi - \mu}{k_B T}) + 1]^{-1} \) is the Fermi Dirac distribution function with \( f_\xi = f_{\xi'} \) for elastic scattering, \( k_B \) is the Boltzmann constant and \( \mu \) the chemical potential. \( W_{\xi\xi'} \) is the transmission rate between the one-electron states \( |\xi\rangle \) and \( |\xi'\rangle \), \( \Omega \) the volume of the system, \( e \) the electron charge, \( \tau(E) \) the relaxation time and \( \alpha_\xi = \langle \xi | r_x | \xi \rangle \) the mean value of the \( x \) component of the position operator when the electron is in state \( |\xi\rangle \).

Collisional conductivity arises as a result of migration of the cyclotron orbit due to scattering by charge impurities. The scattering rate \( W_{\xi\xi'} \) is given by

\[
W_{\xi\xi'} = \sum_q \left| U_q \right|^2 \left| \langle \xi | e^{i q \cdot (r - R)} | \xi' \rangle \right|^2 \delta(E_\xi - E_{\xi'}). \tag{7}
\]

The Fourier transform of the screened impurity potential is \( U_q = 2\pi e^2/\varepsilon \sqrt{q^2 + k_s^2} \), where \( r \) and \( R \) are the position of electron and of impurity respectively; \( k_s \) is the screening wave vector, \( \varepsilon \) is the dielectric constant of the material. By performing an average over random distribution of impurities, \( (N_I \equiv \text{impurity density}) \), the contribution of the unperturbed part of the wavefunction, \( |\xi\rangle \equiv |n, k_y\rangle \), to the scattering rate is

\[
W_{\xi\xi'}^{(\text{c})} = \frac{2\pi N_I}{A_o \hbar} \sum_q \left| U_q \right|^2 \left| \langle n, k_y | e^{i q \cdot (r - R)} | n', k_y' \rangle \right|^2 \delta(E_{n, k_y} - E_{n', k_y'}) \tag{8}
\]

with

\[
\left| \langle n, k_y | e^{i q \cdot (r - R)} | n', k_y' \rangle \right|^2 = \frac{1}{4} \left[ J_{n,n'}(\gamma) + J_{n-1,n'-1}(\gamma) \right] \delta_{k_y-k_y',q_y} \tag{9}
\]

and

\[
|J_{n,n'}(\gamma)|^2 = \frac{n!}{n'!} e^{-\gamma} \gamma^{n-n'} \left[ L_{n'}^{n-1}(\gamma) \right]^2 ; n' \leq n. \tag{10}
\]

Here \( A_o = L_x L_y \) is the area of the graphene monolayer and \( \gamma = l^2(q_x^2 + q_y^2)/2 = \frac{q^2}{2} \) with \( q^2 = (q_x^2 + q_y^2) \). Inserting Eq. (8) in Eq. (6) we obtain

\[
\sigma_{xx}^{\text{col}} = \frac{e^2 \beta A^4}{A_o \hbar} \sum_{n,n',k_y} \sum_{k_y} \sum_q \left| U_q \right|^2 \frac{1}{4} \left[ J_{n,n'}(\gamma) + J_{n-1,n'-1}(\gamma) \right] q_y f_{n, k_y} (1 - f_{n, k_y}) \delta(E_{n, k_y} - E_{n', k_y}) \tag{11}
\]

with \( f_{n, k_y} \equiv f(E_{n, k_y}) \), the Fermi Dirac distribution function. Taking \( \sum_q \rightarrow \frac{A_o}{4\pi^2} \int d\varphi \int d\gamma \) and \( q_y = q_{y \perp} \sin \varphi, |U_q|^2 \sim |U_o|^2 \) in Eq. (11), we obtain

\[
\sigma_{xx}^{\text{col}} = \frac{e^2 \beta N_I}{A_o \hbar} |U_o|^2 \sum_{n,n',k_y} f_{n, k_y} (1 - f_{n, k_y}) \int_{0}^{\infty} \gamma \left[ J_{n,n'}(\gamma) + J_{n-1,n'-1}(\gamma) \right]^2 d\gamma \delta(E_{n, k_y} - E_{n', k_y}). \tag{12}
\]
Using the following integral identity \[11, 16\]:

\[
\int_0^\infty \gamma [J_{n,n'}(\gamma)]^2 \, d\gamma = \int_0^\infty \gamma e^{-\gamma} [L_n(\gamma)]^2 \, d\gamma = (2n + 1) \tag{13}
\]

where for \(n = n'\), \([J_{n,n'}(\gamma)]^2 = e^{-\gamma} [L_n(\gamma)]^2\) with the result

\[
\int_0^\infty \gamma [J_{n-1,n'-1}(\gamma)]^2 \, d\gamma = \int_0^\infty \gamma e^{-\gamma} [L_{n-1}(\gamma)]^2 \, d\gamma = (2n - 1) \tag{14}
\]

\[
\int_0^\infty \gamma J_{n,n}(\gamma) J_{n-1,n'-1}(\gamma) \, d\gamma = \int_0^\infty \gamma e^{-\gamma} [L_n(\gamma)] [L_{n-1}(\gamma)] \, d\gamma = 0. \tag{15}
\]

Finally, replacing the \(\delta\) function by a Lorentzian of zero shift and constant width \(\Gamma\), \(\sum_{k_y} \rightarrow \int dk_y\), \(A_o \rightarrow L_x L_y\), and performing the sum on \(n'\), keeping only the dominant term \(n' = n\) in Eq. (12), we obtain the following result

\[
\sigma_{xx}^{\text{col}} \approx \frac{e^2}{h} \frac{N_I U_o^2}{\pi a \Gamma} \sum_{n=0}^\infty \int_0^{a/\ell^2} dk_y \beta f_n(k_y) (1 - f_{n,k_y}). \tag{16}
\]

2. DIFFUSIVE CONDUCTIVITY:

For completeness, we also present the result for diffusive conductivity which was determined in \([10]\),

\[
\sigma_{yy}^{\text{diff}} = 2\pi e^2 \frac{V_2^2 \tau \beta}{h} \sum_{n=0}^\infty \frac{\partial f(E)}{\partial E} \left[ L_n(u) - L_{n-1}(u) \right]^2 \tag{17}
\]

where \(\tau\) is the constant scattering time and \(\frac{\partial f(E)}{\partial E} = \exp(\beta(E - E_F)) / [\exp(\beta(E - E_F)) + 1]^2\).

Now \(\sigma_{yy} = \sigma_{xx}^{\text{col}} + \sigma_{yy}^{\text{diff}}\).

3. HALL CONDUCTIVITY:

The nondiagonal contribution to conductivity \([11]\) is given by

\[
\sigma_{yx} = \frac{2i\hbar e^2}{\Omega} \sum_{\xi \neq \xi'} f_\xi (1 - f_{\xi'}) \langle \xi | v_y | \xi' \rangle \langle \xi' | v_x | \xi \rangle \frac{1 - e^{\beta(E_\xi - E_{\xi'})}}{(E_\xi - E_{\xi'})^2}. \tag{18}
\]
Since \( f_\xi (1 - f_\xi') (1 - e^{\beta (E_\xi - E_\xi')}) = f_\xi' (1 - f_\xi) \) and \( \Omega \rightarrow A_0 \equiv L_x L_y \), we obtain
\[
\sigma_{yx} = \frac{2ihe^2}{\Omega} \sum_{\xi \neq \xi'} f_\xi' (1 - f_\xi) \frac{\langle \xi' | v_y | \xi' \rangle \langle \xi | v_x | \xi \rangle}{(E_\xi - E_\xi')^2}.
\]  
(19)

Since the \( x \) and \( y \) components of velocity operator are \( v_x = \frac{\partial H}{\partial p_x} \) and \( v_y = \frac{\partial H}{\partial p_y} \) when \( H = v_F \sigma. (p + eA) \). Therefore, \( v_x = v_F \sigma_x \) and \( v_y = v_F \sigma_y \). Hence
\[
\langle \xi' | v_x | \xi \rangle = \langle n', k_y | v_x | n, k_y \rangle = -i v_F
\]  
and
\[
\langle \xi | v_y | \xi' \rangle = \langle n, k_y | v_y | n', k_y \rangle = v_F.
\]  
(20)

Substituting the values of the matrix elements of velocity in Eq. (19) yields
\[
\sigma_{yx} = \frac{2he^2 v_F^2}{L_x L_y} \sum_{\xi \neq \xi'} f_\xi' (1 - f_\xi) \langle \xi' | v_x | \xi \rangle \langle \xi | v_y | \xi' \rangle \langle \xi | v_x | \xi \rangle \langle \xi' | v_y | \xi' \rangle. 
\]  
(22)

Since \( E_\xi \equiv E_{n,k_y} = E_n + V_{n,B} \cos Kx_o \) where \( E_n = \hbar \omega_g \sqrt{n} \) and \( V_{n,B} = \frac{V_o}{2} e^{-u/2} [L_n(u) + L_{n-1}(u)] \) we obtain
\[
(E_\xi - E_{\xi'})^2 = \hbar^2 \omega_g^2 \left[ \sqrt{n + 1} - \sqrt{n + \lambda_n \cos Kx_o} \right]^2
\]  
(23)

where
\[
\lambda_n = \frac{V_o}{2\hbar \omega_g} e^{-u/2} (L_{n+1}(u) - L_{n-1}(u)).
\]  
(24)

Substituting Eq. (23) in Eq. (22) we obtain the Hall conductivity in graphene as
\[
\sigma_{yx} = \frac{e^2 L_y^2}{h} \sum_{n=0}^{\infty} \int_0^{a/2} dk_y \frac{f_{n,k_y} - f_{n+1,k_y}}{\left[ \sqrt{n + 1} - \sqrt{n + \lambda_n \cos Kx_o} \right]^2}.
\]  
(25)

Elements of the resistivity tensor \( \rho_{\mu\nu}(\mu,\nu=x,y) \) can be determined from those of the conductance tensor \( \sigma_{\mu\nu} \), obtained above, using the expressions: \( \rho_{xx} = \sigma_{yy} / S, \rho_{yy} = \sigma_{xx} / S \) and \( \rho_{xy} = -\sigma_{yx} / S \) where \( S = \sigma_{xx} \sigma_{yy} - \sigma_{xy} \sigma_{yx} \) with \( S \approx \sigma_{xy}^2 = n_c e^2 / B^2 \).

### III. RESULTS AND DISCUSSION

The above expressions for the (collisional, diffusive and Hall) conductivities, Eqs. (16), (17) and (25) are the principal results of this work. The integrals appearing in these equations are evaluated numerically and the results are presented in Figure (1a) at temperature...
$T = 2 \, K$ for a graphene monolayer with electron density $n_e = 3.0 \times 10^{11} \text{cm}^{-2}$, electric modulation strength $V_o = 0.5 \text{meV}$ with period $a = 350 \text{nm}$. In addition, the following parameters were employed \cite{18, 19, 20}: $\tau = 4 \times 10^{-13} \text{s}$, $\Gamma = 0.4 \text{meV}$, impurity density $N_I = 2.5 \times 10^{11} \text{cm}^{-2}$ and $\varepsilon = 3.9$ (using SiO$_2$ as the substrate material). We observe that SdH oscillations are visible in collisional conductivity $\sigma_{xx}$ whereas the Hall conductivity $\sigma_{yx}$ decreases with increasing magnetic field, $B$. Furthermore, Weiss oscillations superimposed on SdH oscillations are seen in $\sigma_{yy}$. To highlight the effects of modulation, we also calculate the correction to the conductivity (change in conductivity) as a result of modulation which is expressed as $\Delta \sigma_{\mu\nu} = \sigma_{\mu\nu}(V_o) - \sigma_{\mu\nu}(V_o = 0)$ and is shown in Figure (1b). Electric modulation acting on the system results in a positive contribution to $\Delta \sigma_{yy}$ and a negative contribution to $\Delta \sigma_{xx}$ whereas $\Delta \sigma_{yx}$ oscillates around zero. We find that $\Delta \sigma_{yy} \gg \Delta \sigma_{xx}$, which is a consequence of the fact that $\Delta \sigma_{xx}$ has only collisional contribution, while $\Delta \sigma_{yy}$, in addition to the collisional part, has contributions due to band conduction which are much larger. It is also seen that the oscillations in $\Delta \sigma_{xx}$ and $\Delta \sigma_{yy}$ are $180^\circ$ out of phase. To determine the effects of temperature on magnetoconductivities, comparison of conductivities and corrections to the conductivities at two different temperatures $T = 2K$ (solid curve) and $T = 6K$ (broken curve) are presented in Figures (2) and (3) respectively. $\Delta \sigma_{xx}$ shows strong temperature dependence which is a clear signature that SdH oscillations are dominant here. Oscillations in $\Delta \sigma_{yy}$ show comparatively weaker dependence on temperature as Weiss oscillations, that are weakly dependent on temperature, play a more significant role in $\sigma_{yy}$. Furthermore, Weiss oscillations are also seen in $\Delta \sigma_{yx}$ and they are weakly sensitive to temperature a low magnetic fields (that is when $B < 0.188T$). In graphene system, the value of $B$ defining the boundary between SdH and Weiss oscillations is quite low (it lies between 0.1 and 0.15Tesla). For smaller values of $B$, the amplitude of Weiss oscillations remain essentially the same at various temperatures. When $B$ is large, SdH oscillations dominate and the amplitude of oscillations gets reduced considerably at comparatively higher temperatures. However, oscillatory phenomenon still persists.

It can be seen from Figure (1a), (2a) and (2b) that amplitude of SdH oscillations remains large at those values of the magnetic field where the flat band condition is satisfied i.e at $B(\text{Tesla}) = 0.6897, 0.2956, 0.1881, 0.1379, 0.1089...$ when $i = 1, 2, 3, 4...$ in Eq.(5) while is supressed at the maximum bandwidth/broad band condition, i.e at $B(\text{Tesla}) = 0.4138, 0.2299, 0.1592, 0.1217, 0.0985, ...$ for $i = 1, 2, 3, 4...$ in Eq.(4). Furthermore, zeros in
\( \Delta \sigma_{\mu\nu} \) appear in close agreement with values predicted from the flat band condition. The amplitude of \( \Delta \sigma_{xx} \) and \( \Delta \sigma_{yy} \) becomes maximum at the broad band condition (as seen in Figure (3)), whereas the amplitude of \( \Delta \sigma_{yx} \) crosses the zero level at the broad band condition and then a phase change of amplitude occurs.

Components of the resistivity tensor \( \rho_{\mu\nu} \) have also been computed and shown in Figure (4a) as a function of \( B \) for \( T = 2K \) (solid curve) and \( 6K \) (broken curve) respectively. The correction (change) in \( \rho_{\mu\nu} \) due to the modulation is shown in Figure (4b). To verify our results, we compare them in the absence of modulation with the unmodulated experimental results presented in [14]. In order to carry this out, we note that the number density \( n_e \) is related to the gate voltage \( (V_g) \) through the relationship [13] \( n_e = \epsilon_o \epsilon V_g / t e \), where \( \epsilon_o \) and \( \epsilon \) are the permittivities for free space and the dielectric constant of graphene, respectively. \( e \) is the electron charge and \( t \) the thickness of the sample. It yields \( V_g = 4.8V \) for \( n_e = 3.0 \times 10^{11} cm^{-2} \). We find that the results for magnetoresistivities obtained in this work are in good agreement with the values given in reference [14] for the unmodulated case at \( V_g = 4.8V \).

We observe in Figure (4), that the dominant effect of Weiss oscillations appears in \( \rho_{xx} \) as it is proportional to \( \sigma_{yy} \), whereas the amplitude of oscillations in \( \rho_{yy} \) show a monotonic increase in amplitude with magnetic field signifying dominance of SdH in \( \rho_{yy} \). In Figure (5), we observe that the oscillations in \( \Delta \rho_{xx} \) and \( \Delta \rho_{yy} \) are out of phase and the amplitude of the oscillation in \( \Delta \rho_{xx} \) is greater than the amplitude of oscillation in \( \Delta \rho_{yy} \). The out of phase character of the oscillations can be understood by realizing that the conduction along the modulation direction, which contributes to \( \rho_{yy} \), occurs due to hopping between Landau states and it is minimum when the density of states at the Fermi level is minimum. Oscillations in \( \rho_{xx} \) are much larger than those in \( \rho_{yy} \) as a new mechanism of conduction due to modulation contributes to \( \rho_{xx} \). To highlight temperature effects on the modulated system, we present in Figure (6), corrections to magnetoresistivities at two different temperatures (2K, solid curve and 6K, broken curve). These results exhibit SdH oscillation when \( B \) becomes greater than \( 0.188T \) as seen in Figures (5) and (6). The Weiss oscillations in \( \Delta \rho_{xx} \) are in phase with those of \( \Delta \rho_{xy} \). From Figure (5c) one might infer that Hall resistivity is not affected by modulation. This is not so, as even Hall resistivity carries modulation effects and that is seen if we draw the slope of \( \rho_{xy} \) as a function of magnetic field (Figure (7)).

In order to quantitatively analyze the results presented in the figures we consider the density of states (DOS) of this system. At finite temperature, the oscillatory part of resistivities
\((\Delta \rho/\rho_o)\) are proportional to the oscillatory part of the density of states (DOS) at the Fermi energy, \(A(T/T_c)\Delta D(E_F)/D_o\) where \(A(T/T_c) = (T/\tau)/\sinh(T/\tau)\), \(D_o\) is the DOS and \(\rho_o\) is resistivity in the absence of magnetic field, respectively [15]. For not too small magnetic fields \(B \gtrsim 0.05T\), \(\Delta \rho/\rho_o \simeq (\omega_\gamma \tau)^2 \Delta \sigma/\sigma_o\) to a good approximation, where \(\sigma_o = \frac{e^2 v_F^2}{2} \tau D_o\) represents conductivity at zero magnetic field and \(\tau\) is the relaxation time. The analytic expression for the density of states (DOS) of a graphene monolayer in the presence of a magnetic field is [15]

\[ D(E, V_B) = D_o \left[ 1 + 2 \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \cos[2\pi k(\varepsilon - v_B \cos t)] dt \exp(-2\pi k\eta) \right] \]

\[ = D_o \left[ 1 + 2 \sum_{k=1}^{\infty} \cos(2\pi k\varepsilon) J_o(2\pi k\varepsilon) \exp(-2\pi k\eta) \right] \]

where \(D_o = \frac{2E}{\hbar \omega_g^2 \pi^2} = \frac{E}{\pi (\hbar \omega_g)^2}\), \(\varepsilon = (\frac{E}{\hbar \omega_g})^2\), \(\eta = \frac{1}{\hbar \omega_g}\) and \(v_B = \frac{2v_F E}{\hbar \omega_g}\). \(J_o(x)\) is the Bessel function of order zero. Since \(\exp(-2\pi k\eta) \ll 1\) for weak magnetic fields, it is usually a good approximation to keep only the \(k = 1\) term in the sum: \(D(E) \simeq D_o + \Delta D_1(E)\) with

\[ \frac{\Delta D_1(E)}{D_o} = 2 \cos(2\pi \varepsilon) J_o(2\pi v_B) \exp(-2\pi \eta) \]

To determine the effects of an external magnetic field on the conductivities/resistivities of the system we consider Eq.(26). With a decrease in \(B\), \(v_B\) oscillates periodically with respect to \(1/B\) around \(v_B = 0\), increasing its amplitude proportionally to \(1/\sqrt{B}\) [Eq. (3)]. The function \(J_o(2\pi v_B)\) decreases from 1 with an increase of \(|v_B| = 0.3827 \simeq 3/8\) and than changes its sign. Therefore the oscillations of \(\Delta D_1(E)\) takes a minimum amplitude at the maximum bandwidth conditions while \(|v_B|\) stays less than \(3/8\); it disappears when a maximum of \(|v_B|\) touches at \(\sim 3/8\); it reappears with an inverted sign for \(|v_B|\) larger than \(3/8\). Therefore, if we assume that \(\Delta \rho/[\rho_o A(T/T_c)] \propto \Delta D(E_F)/D_o\) holds, we can find the position where oscillations of \(\Delta \rho/[\rho_o A(T/T_c)]\) vanish. That occurs at \(|V_B| = 0.19135(\hbar \omega_g)^2/E_F\).

We can also find the period of oscillations in conductivities/resistivities from Eq. (26) as follows. We have \(D(E, V_B) \simeq D_o \{1 + 2 \cos(2\pi \varepsilon) J_o(2\pi v_B) \exp(-2\pi k\eta)\} \simeq D_o \{1 + 2 \cos(2\pi \varepsilon)(1 - \pi^2 v_B^2) \exp(-2\pi k\eta)\}\). Since \(v_B^2 \propto \cos^2(K R_c - \frac{\pi}{4})\). The period of oscillation can be estimated by equating the increment of the cosine argument with \(\pi\),

\[ K \Delta(R_c) = \pi, \]

(28)
which leads to

$$\Delta \left( \frac{1}{B} \right) = \left( \frac{e}{2\sqrt{2\pi} \hbar} \right) \frac{a}{\sqrt{n_e}}. \quad (29)$$

In our work \((n_e = 3.0 \times 10^{11} \text{cm}^{-2}\) and \(a = 350\text{nm}\), therefore the period of oscillations comes out to be \(1.933T^{-1}\) which is in good agreement with the results shown in the figures.

Damping of these oscillations with temperature can also be discussed. In Ref. \[10\], the temperature scale for damping of Weiss oscillations is given by

$$K_B T_c^{\text{Weiss}} = \frac{b \hbar v_F}{4\pi^2 a} \quad \text{where } b = \left(\frac{a}{l}\right)^2 \text{ and } v_F = \frac{\omega}{\sqrt{2}}.$$  

To determine the damping temperature for SdH oscillations we, following Ref.\[10\] and [11], use asymptotic expression for magnetoconductivity. For this, we use DOS (Eq. 26),

$$D(E) = \frac{2E}{(\hbar \omega_g)^2 \pi l^2} \left[ 1 + 2 \exp(-2\pi \eta) \cos\left(2\pi \frac{E}{(\hbar \omega_g)^2} \right) \right]. \quad (31)$$

In the asymptotic limit of weak magnetic fields when many filled Landau levels occur, we take \(L_n \approx L_{n-1}\) and replace \(e^{-u/2}L_n\) by \(1/\sqrt{\pi \sqrt{n}u} \cos(2\sqrt{n}u - \pi/4)\) and inserting the continuum approximation \(\sum_{n=0}^{\infty} \rightarrow \int_0^{\infty} dE D(E)\) in Eq. 17, we obtain the following result

$$\frac{\sigma_{yy}^{\text{diff}}}{\sigma_o} = \frac{4\sqrt{2\pi^2 l} V_o^2}{aE_F (\hbar \omega_g)} \left[ F + 2 \exp(-2\pi \eta) A(T/T_c^{\text{SdH}}) \cos\left(2\pi \frac{E_F}{(\hbar \omega_g)^2} \right) \cos^2 \left(\sqrt{2K} l \frac{E_F}{\hbar \omega_g} - \frac{\pi}{4}\right) \right] \quad (32)$$

where \(F = \frac{1}{2} \left[ 1 - A(T/T_c^{\text{Weiss}}) + 2A(T/T_c^{\text{Weiss}}) \cos^2 \left(\sqrt{2K} l \frac{E_F}{\hbar \omega_g} - \frac{\pi}{4}\right) \right] \) is the contribution of Weiss oscillations and \(A(T/T_c^{\text{SdH}}) = [4\pi^2 E_F K_B T/(\hbar \omega_g)^2]/\sinh[4\pi^2 E_F K_B T/(\hbar \omega_g)^2] \) is the amplitude of the SdH oscillations. Therefore, the characteristic temperature of SdH oscillations is given by

$$K_B T_c^{\text{SdH}} = \frac{(\hbar \omega_g)^2}{4\pi^2 E_F} \frac{\hbar \omega_g}{2\pi^2} \left( \frac{1}{\sqrt{2k_F l}} \right). \quad (33)$$

The amplitude of oscillations is given by \(A = \frac{x}{\sinh(x)}\), where \(x = \frac{T}{T_c}\). The amplitude of Weiss oscillations at \(B = 0.3T\) are 0.9993 and 0.9938 at \(T = 2K\) and \(T = 6K\), respectively. The corresponding amplitudes for SdH oscillations are 0.6878 and 0.0882. The SdH amplitude decreases by \(\sim 87\%\) whereas the amplitude of Weiss oscillations decreases by \(\sim 0.55\%\) for \(4K\) change in temperature. In Figures (2), (3), (5) and (6); the SdH amplitude decreases by \(\sim 77\%\) when temperature is changed from \(T = 2K\) to \(T = 6K\), and it
is in good agreement with the results obtained from Eqs. (30) and (33). It is due to the fact $K_BT_{c,Weiss} \gg K_BT_{c,SdH}$ that the Weiss oscillations are more robust against temperature changes.

Finally, we compare the results obtained for the conductivity/ resistivity of graphene with those of a 2DEG given in [11]. The characteristic damping temperatures for Weiss and SdH oscillations in 2DEG are $K_BT_{c,2DEG}^{Weiss} = \frac{\hbar \omega_c}{2\pi^2} \left( \frac{ak_F}{2} \right)$ and $K_BT_{c,2DEG}^{SdH} = \frac{\hbar \omega_c}{2\pi^2}$, respectively. In contrast, the corresponding damping temperatures in graphene are given by Eqs (30) and (33). On comparing the two temperature scales, we find that the damping temperatures of both oscillations in graphene are higher than that of a 2DEG. The ratio is found to be $\frac{T_{c,e}}{T_{c,e}} = \frac{m^* v_F}{\hbar k_F} \approx 4.2$; where $m^*$ is the electron mass in a 2DEG and $T_{c,e}$ is the critical temperature of a 2DEG; which implies that a comparatively higher temperature is required for damping of oscillations in graphene. This is due to the higher Fermi velocity of Dirac electrons in graphene compared to standard electrons in a 2DEG systems. It is evident from the numerical results that both, Sdh and Weiss-type oscillations, are more enhanced and more robust against temperature in graphene.

To conclude, we have investigated the effects of a weak periodic electric modulation on the conductivity of a graphene monolayer subjected to a perpendicular magnetic field. As a result of modulation a new length scale, period of modulation, enters the system leading to commensurability oscillations in the diffusive, collisional and Hall contributions to conductivities/resistivities. These modulation induced effects on graphene magnetotransport are discussed in detail in this work.

IV. APPENDIX

Here we derive the expression for the density of states, Eq. (26) in the text. We consider monolayer graphene subjected to a uniform quantizing magnetic field $B = B\hat{z}$ in the presence of an additional weak periodic modulation potential. The energy spectrum in the quasi classical approximation, i.e. when many Landau bands are filled may be written as

$$E_{n,x} = \sqrt{n} \hbar \omega_g + V_{n,B} \cos Kx_0$$

where $V_{n,B} = \frac{V_o}{2} e^{-u/2} [L_n (u) + L_{n-1} (u)]$ with $L_n (u), L_{n-1} (u)$ the Laguerre polynomials and $u = K^2l^2/2$. For large $n$; $L_n (u) \approx L_{n-1} (u)$ and $V_{n,B} = V_o e^{-u/2} L_n (u)$. Using the asymptotic
expression for the Laguerre polynomials \[ e^{-u/2}L_n(u) \to \frac{1}{\sqrt{\pi}\sqrt{n}} \cos(2\sqrt{n}u - \frac{\pi}{4}) \] and taking the continuum limit \( n \to \frac{1}{2}(\frac{\ell E}{v_F})^2 \), where \( v_F = \omega_g/\sqrt{2} \) we get

\[
V_{n,B} = V_0 \pi^{-1/2} \left( \frac{1}{2} K^2 l^2 E}{h \omega_g} \right)^{-1/4} \cos \left( \sqrt{2} K l \frac{E}{h \omega_g} - \frac{\pi}{4} \right)
\]

(35)

\[
V_{n,B} = V_0 \pi^{-1/2} \left( \frac{1}{2} K^2 l^2 E}{h \omega_g} \right)^{-1/4} \cos \left( \sqrt{2} K l \frac{E}{h \omega_g} - \frac{\pi}{4} \right).
\]

(36)

To obtain a more general result which will lead to the result that we require as a limiting case we consider impurity broadened Landau levels. The self energy may be expressed as

\[
\Sigma^-(E) = \Gamma_o^2 \sum_n \int_0^a \frac{dx_o}{a} \frac{1}{E - E_{n,x_0} - \Sigma^-(E)}
\]

(37)

which yields

\[
\Sigma^-(E) = \int_0^a \frac{dx_o}{a} \sum_{-\infty}^\infty \frac{\Gamma_o^2}{E - \Sigma^-(E) - V_{n,B} \cos K x_o - \sqrt{n}h \omega_g}.
\]

(38)

\( \Gamma_o \) is the broadening of the levels due to the presence of impurities. The density of states is related to the self energy through

\[
D(E) = \text{Im} \left[ \frac{\Sigma^-(E)}{\pi^2 l^2 \Gamma_o^2} \right].
\]

(39)

The residue theorem has been used to sum the series \( \sum_{n} f(n) = -\{\text{Sum of residues of } \pi(\cot \pi n) f(n) \text{ at all poles of } f(n)\} \). Here \( f(n) = \sum_{-\infty}^\infty \frac{b}{E - \Sigma^-(E) - V_{n,B} \cos K x_o} \) with \( b = \Gamma_o^2 \), \( c = E - \Sigma^-(E) - V_{n,B} \cos K x_o \) and \( d = h \omega_g \). The function \( f(n) \) has a pole at \( c^2/d^2 \) and the residue of \( (\pi(\cot \pi n) f(n)) \) at the pole is \( \frac{2b \pi}{d^2} \pi \cot(\frac{\pi^2}{d^2}) \). Hence \( \sum_{-\infty}^\infty f(n) = \frac{2b \pi}{d^2} \pi \cot(\frac{\pi^2}{d^2}) \) and we obtain

\[
\Sigma^-(E) = \int_0^a \frac{dx_o}{a} \frac{2\pi \Gamma_o^2 (E - \Sigma^-(E) - V_{n,B} \cos K x_o)}{(h \omega_g)^2} \cot \left( \frac{\pi(E - \Sigma^-(E) - V_{n,B} \cos K x_o)}{(h \omega_g)^2} \right)
\]

(40)

\[
\approx \frac{2\pi \Gamma_o^2 E}{(h \omega_g)^2} \int_0^a \frac{dx_o}{a} \cot \left( \frac{\pi E}{(h \omega_g)^2} [E - 2\{\Sigma^-(E) + V_{n,B} \cos(K x_o)\}] \right).
\]

Separating \( \Sigma^-(E) \) into real and imaginary parts

\[
\Sigma^-(E) = \Delta(E) + i \frac{\Gamma(E)}{2}.
\]

(41)
Eq. (40) takes the form

\[ \Delta(E) + i \frac{\Gamma(E)}{2} = \frac{2 \pi \Gamma_0^2 E}{(\hbar \omega_g)^2} \int_0^a \frac{dx_o \sin 2u + i \sinh 2v}{a \cosh 2v - \cos 2u} \quad (42) \]

where

\[ u = \frac{\pi E}{(\hbar \omega_g)^2} \left[ \varepsilon - 2 \{ \Delta(E) + V_{n,B} \cos(Kx_o) \} \right] \quad (43) \]

\[ v = \frac{\pi \Gamma(E) E}{(\hbar \omega_g)^2} \quad (44) \]

\[ \text{Im} [\Sigma^-(E)] = \frac{2 \pi \Gamma_0^2 E}{(\hbar \omega_g)^2} \int_0^a \frac{dx_o \sinh 2v}{a \cosh 2v - \cos 2u} = \frac{2 \pi \Gamma_0^2 E}{(\hbar \omega_g)^2} \int_0^a \frac{dx_o}{a} \left( 1 + 2 \sum_{k=1}^\infty \cos(2k u) \exp(-2k v) \right). \quad (45) \]

If we define dimensionless variables \( \varepsilon = \frac{E}{(\hbar \omega_g)^2} \), \( \eta = \frac{\Gamma(E) E}{(\hbar \omega_g)^2} \) and \( v_B = \frac{2V_B E}{(\hbar \omega_g)^2} \) the density of states is obtained as

\[ D(E, V_B) = D_0(E) \left\{ 1 + 2 \sum_{k=1}^\infty \int_0^a \frac{dx_o}{a} \cos[2\pi k(\varepsilon - v_B \cos Kx_o)] \exp(-2\pi k\eta) \right\} \quad (46) \]

where \( D_0(E) = \frac{2E}{(\hbar \omega_g)^2 \pi} \). Let \( Kx_o = t \) in the above expression results in

\[ D(E, V_B) = D_0(E) \left\{ 1 + 2 \sum_{k=1}^\infty \frac{1}{2\pi} \int_0^{2\pi} \cos[2\pi k(\varepsilon - v_B \cos t)] dt \exp(-2\pi k\eta) \right\}. \quad (47) \]

Solving the integral yields

\[ D(E, V_B) = D_0(E) \left\{ 1 + 2 \sum_{k=1}^\infty \cos(2\pi k\varepsilon) J_0(2\pi kv_B) \exp(-2\pi k\eta) \right\}. \quad (48) \]

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