Abstract. We give a conceptual proof of the fact that if \( M \) is a complete submanifold of a space form, then the maximal integral manifolds of the nullity distribution of its second fundamental form through points of minimal index of nullity are complete.

1. Introduction

Let \( M \) be a submanifold of a space form and let \( \mathcal{N} \) be the nullity distribution of its second fundamental form. The index of nullity of \( M \) at \( p \) is the dimension of \( \mathcal{N}_p \). It is well known, from the Codazzi equation, that \( \mathcal{N} \) is an autoparallel distribution restricted to the open and dense subset of \( M \) where the index of nullity is locally constant.

If \( M \) is complete and one restricts to the open subset \( U \) of points of \( M \) where the index of nullity is minimal, then the integral manifolds of \( \mathcal{N} \) through points of \( U \) are also complete, from a result of Ferus \([1]\). We will give a conceptual proof of this result, as a corollary of a general theorem, whose proof involves very simple geometric ideas.

2. Main results

Lemma 2.1. Let \( M \) be a Riemannian manifold and let \( f : M \to N \) be a differentiable function of constant rank such that \( f(M) \) is an embedded submanifold of \( N \) (this can always be assumed locally). Assume that the distribution \( \ker(df) \) is autoparallel and let \( \Sigma \) be an integral manifold of \( \ker(df) \). Let \( \gamma(t) \) be a geodesic in \( \Sigma \), \( v \in T_{f(\gamma(0))}f(M) \) and let \( J(t) \) be the horizontal lift of \( v \) along \( \gamma \), i.e., \( J(t) \in \ker(df_{\gamma(t)})^\perp \) and \( df J(t) = v \). Then \( J(t) \) is a Jacobi vector field along \( \gamma \).

Proof. Let \( c(s) \) be a (short) curve in \( f(M) \) such that \( c'(0) = v \). The horizontal lift of \( c(s) \) through points of \( \Sigma \) gives rise to a perpendicular variation \( \tilde{c}_\gamma(s) \) by totally geodesic submanifolds, which must be by isometries. Therefore \( \tilde{c}_\gamma(t)(s) \) is a variation by geodesics whose associated variation field is \( J(t) \). □

Theorem 2.2. Let \( M \) be a complete Riemannian manifold, \( f : M \to N \) be a differentiable function and let \( U \) be the open subset of \( M \) where the rank of \( f \) is maximal. Assume that \( \ker(df)|_U \) is autoparallel. Then its integral manifolds are complete.
Proof. Let $\Sigma$ be a totally geodesic integral manifold of $\text{ker}(df)$ through a point $p \in U$ and let $\gamma : [0, b) \to \Sigma$ be a maximal geodesic in $\Sigma$.

Observe that $f$ has maximal rank in a neighborhood of each point of $\Sigma$. From the local form of maps of constant rank it is not difficult to see that given $t_1, t_2 \in [0, b)$ there are open neighborhoods $V_1$ and $V_2$ of $\gamma(t_1), \gamma(t_2)$ such that $f(V_1)$ and $f(V_2)$ are embedded submanifolds of $N$ and $f(V_1) \cap f(V_2)$ contains an open neighborhood of $f(\gamma(t_1)) = f(\gamma(t_2))$ in both $f(V_1)$ and $f(V_2)$. In particular, $T_{f(\gamma(t_1))}f(V_1) = T_{f(\gamma(t_2))}f(V_2) =: V$.

Let $v \in V$ and apply the previous lemma to define a Jacobi field $J$ along $\gamma$ that projects down to $v$. Since $M$ is complete $\gamma(b)$ and $J(b)$ are well defined and $J(b)$, by the continuity of $df$, also projects down to $v$. Then $df_{\gamma(b)}(T_{\gamma(b)}M)$ contains $V$. So, $\text{rank}(df_{\gamma(b)}) = \text{rank}(df_{\gamma(0)})$ and therefore $\gamma(b) \in \Sigma$. □

If $M^n$ is a submanifold of the Euclidean space $\mathbb{R}^{n+k}$, the Gauss map of $M$ is the map $G : M \to G_k(\mathbb{R}^{n+k})$ defined by $p \mapsto \nu_p M$, where $\nu_p M$ denotes the normal space of $M$ at $p$. If $M^n$ is a submanifold of the sphere $S^{n+k} \subset \mathbb{R}^{n+k+1}$, then the Gauss map of $M$ is defined to be the map $G : M \to G_k(\mathbb{R}^{n+k+1})$ that sends each point to its normal space in the sphere, regarded as a subspace of $\mathbb{R}^{n+k+1}$ (see the remark below). A similar construction can be made for a submanifold $M^n$ of the hyperbolic space $H^{n+k}$, regarded as a submanifold of the Lorentz space $\mathbb{R}^{n+k,1}$.

It is well known that in the three cases, the nullity distribution of $M$ coincides with the kernel of its Gauss map. Therefore we get:

Corollary 2.3. Let $M$ be a complete submanifold of a space form. Then any maximal integral manifold of the nullity distribution of $M$ through a point of minimal index of nullity is complete.

Remark 2.4. Let $M$ be a submanifold of a space form and let $\nu_1$ be a parallel sub-bundle of the normal bundle. One can regard to the nullity of the second fundamental form projected to this sub-bundle. This is equivalent to regard the common kernel of all shape operators of vectors in this sub-bundle. This generalized nullity space coincides with the kernel of the generalized Gauss map $p \mapsto \nu_1(p)$. So, as in the corollary, if $M$ is complete one has completeness of the integral manifolds where the kernel has minimal dimension.

References

[1] D. Ferus. On the completeness of nullity foliations. *Michigan Math. J.*, 18:61–64, 1971. 89