1-rigidity of CR submanifolds in spheres

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Abstract We propose a unified computational framework for the problem of deformation and rigidity of submanifolds in a homogeneous space under geometric constraint. A notion of 1-rigidity of a submanifold under admissible deformations is introduced. It means every admissible deformation of the submanifold osculates a one parameter family of motions up to 1st order.

We implement this idea to the question of rigidity of CR submanifolds in spheres. A class of submanifolds called Bochner rigid submanifolds are shown to be 1-rigid under type preserving CR deformations. 1-rigidity is then extended to a rigid neighborhood theorem, which roughly states that if a CR submanifold $M$ is Bochner rigid, then any pair of mutually CR equivalent CR submanifolds that are sufficiently close to $M$ are congruent by an automorphism of the sphere.

A local characterization of Whitney submanifold is obtained, which is an example of a CR submanifold that is not 1-rigid. As a by product, we give a simple characterization of the proper holomorphic maps from the unit ball $\mathbb{B}^{n+1}$ to $\mathbb{B}^{2n+1}$.

Key words: Moving frames, Maurer-Cartan equation, 1-rigidity, CR submanifold, Bochner rigid, Whitney submanifold
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Introduction

In the study of submanifolds in a homogeneous space $X = G/P$ of a Lie group $G$, the method of moving frames is both a unifying concept and an effective tool, which is the version of the method of equivalence applied to submanifold geometry. Let $\phi$ be the left invariant, Lie algebra $\mathfrak{g}$-valued Maurer-Cartan form of $G$. The local equivalence problem for a submanifold $f : M \hookrightarrow X$ in many interesting cases is solved on a canonical adapted subbundle $E_f : B_f \hookrightarrow G$ together with $\pi = E_f^* \phi$ in such a way that the complete set of local invariants is generated by the coefficients of $\pi$ [Ga]. The pair $(B_f, \pi)$ measures in a sense how the submanifold $M$ deviates from a flat model submanifold. The method was systematically exploited and applied by Cartan himself, and by Chern in various geometric problems, [Ch] and the references therein.
A submanifold $M$ often inherits a geometric structure from the ambient space $X$, e.g., metric, conformal structure, CR structure. A general problem of interest is the deformation of a submanifold preserving the induced geometric structure and the rigidity phenomena thereof. There exist a wealth of works on the subject ranging from local differential algebraic analysis of the prolonged Gauß equations in the isometric embedding problem [BBG], to symbol analysis via Lie algebra cohomology in Griffiths-Harris rigidity of Hermitian symmetric spaces [La][HY]. So far, however, there appears to be little work which may serve as a common conceptual ground for these problems.

The purpose of this paper is to propose a unified conceptual perspective and, more importantly, a computational framework for the problem of deformation and rigidity of submanifolds in a homogeneous space under geometric constraint. We introduce a notion of 1-rigidity of a submanifold under admissible deformations, which can be considered as a geometric definition of infinitesimal rigidity in classical differential geometry. The idea is to apply the method of equivalence to deformation of submanifolds. 1-rigidity means every admissible deformation is equivalent to a one parameter family of motions up to 1st order. In essence, we differentiate the method of moving frames once. This approach to rigidity problems may provide an alternative basis for the existing works [KT][BBG][La][HY][Ha][ChH].

We implement this idea to the question of rigidity of CR submanifolds in a sphere $\Sigma^m = \partial \mathbb{B}^{m+1}$, where $\mathbb{B}^{m+1}$ is the unit ball in $\mathbb{C}^{m+1}$. The main result is that a class of submanifolds called Bochner rigid submanifolds are 1-rigid under type preserving CR deformations, Theorem 4.1. A CR submanifold in $\Sigma^m$ is Bochner rigid if its fundamental forms are rigid in an algebraic sense, Definition 3.5. 1-rigidity is then extended to the following local rigidity or rigid neighborhood theorem, Theorem 4.2. Let $M \subset \Sigma^m$ be a Bochner rigid CR submanifold. Then any pair of mutually CR equivalent CR submanifolds that are sufficiently close to $M$ are congruent by an automorphism of $\Sigma^m$. This generalizes the recent work [EHZ] where it was proved, among other things, that a
nondegenerate CR submanifold \( M^n \subset \Sigma^{n+r} \) of CR dimension \( n \) is CR-rigid when \( r \leq \frac{n}{2} \).

The proof of Theorem 4.1 is essentially equivalent to showing that the linearized Maurer-Cartan equation

\[-d(\delta \pi) = \pi \wedge \delta \pi + \delta \pi \wedge \pi\]  

(1)

has \( \delta \pi = 0 \) as its only solution up to certain group action \( (5) \). The computation involved is algorithmic as in the moving frame method, and it proceeds as follows. The condition of admissible deformation, type preserving CR deformation, imposes a set of initial conditions on \( \delta \pi \). Repeated applications of the linearized Maurer-Cartan equation \( (1) \) then give rise to a sequence of compatibility conditions on \( \delta \pi \). The Bochner rigid assumption on the CR submanifold \( M \) implies that any two adjacent sequence of compatibility conditions are tightly related, thus ensuring the propagation of the sequence of compatibility conditions to force \( \delta \pi = 0 \).

In Section 1, the equation of deformation under geometric constraint is introduced by expanding the Maurer-Cartan equation of one parameter family of immersions, for which \( (1) \) is the first set of compatibility conditions. This sequence of equations lead to the notion of 1-rigidity under admissible deformations in Section 2. Certain group action naturally arises, which plays an important role analogous to absorption in the method of equivalence. In Section 3, we set up the basic structure equations for CR submanifolds in spheres, which are then refined for the class of Bochner rigid submanifolds. The main theorem Theorem 4.1 is proved in Section 4, and then extended to local rigidity theorem Theorem 4.2. The algorithmic computation for CR 1-rigidity is carried out in detail in the proof of Theorem 4.1. In Section 5, we present a local characterization that a CR flat submanifold \( \Sigma^n \hookrightarrow \Sigma^{2n}, n \geq 2 \), is a part of either a linear embedding or a Whitney submanifold, Theorem 5.1. The structure equation of a Whitney submanifold together with Cartan’s generalization of Lie’s third fundamental theorem implies that a Whitney submanifold is CR deformable in exactly one direction, thus providing an example of a CR submanifold which is not 1-rigid. Theorem 5.1 also gives a simple
characterization of the proper holomorphic maps from the unit ball \( \mathbb{B}^{n+1} \) to \( \mathbb{B}^{2n+1} \),

**Corollary 5.1.** Section 6 is devoted to an algebraic proof that the canonical \( S^1 \)-bundles over Plücker embeddings of Grassmann manifolds are Bochner rigid.

Our original motivation for the present work is to provide a geometric description of the complete system for CR maps in [Ha], the analysis of prolonged Gauss equations for isometric embeddings in [BBG], and the Griffiths-Harris rigidity of Hermitian symmetric spaces in [La][HY]. In order to apply our idea to isometric embedding problem, the higher order terms in the deformation equation (4) should be taken into account. Application to jet rigidity of certain homogeneous varieties will be reported elsewhere.

We shall agree that the expression "differentiating A mod B" would mean "differentiating A and considering mod B". CR dimension of a CR manifold is by definition the dimension of the contact hyperplane fields as a complex vector space, **Definition 3.1**.

We assume throughout this paper the CR submanifolds under consideration are of CR dimension \( \geq 2 \).

## 1 Deformation

Let \( X = G / P \) be a homogeneous space of a Lie group \( G \), and consider a submanifold \( M \subset X \) defined by an immersion

\[
f : M \hookrightarrow X.
\]  \hspace{1cm} (2)

In many geometric situations, the standard reduction process of moving frame method gives rise to an adapted subbundle \( E_f : B_f \hookrightarrow G \) with a structure group \( H \subset P \) in a canonical way [Ga][Ch].

\[
f^*G \supset B_f \quad \begin{array}{c} 
E_f \\
\downarrow H \\
M \end{array} \begin{array}{c} 
\hookrightarrow \\
\downarrow P \\
X
\end{array} \begin{array}{c} 
G \\
\hookrightarrow \\
X
\end{array}
\]
Such $B_f$ captures the geometry of $f$ in the following sense. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\phi$ be the $\mathfrak{g}$-valued left invariant Maurer-Cartan form of $G$: Let $f_1, f_2$ be two immersions with the induced $H$-bundles $B_{f_1}, B_{f_2}$ respectively. $f_1$ and $f_2$ are congruent up to a left motion by an element of $G$ iff there exists a $H$-bundle isomorphism of the pair $(B_{f_1}, E^*_{f_1}\phi)$ and $(B_{f_2}, E^*_{f_2}\phi)$ [Br1][Gr].

We wish to describe in this section the geometry of the deformation of a submanifold in this setting. Assume for definiteness $g : G \hookrightarrow \text{GL}(N, \mathbb{R})$ or $\text{GL}(N, \mathbb{C})$ for an integer $N$. Then in particular $\phi = g^{-1}dg$, and $\phi$ satisfies Maurer-Cartan structure equation

$$-d\phi = \phi \wedge \phi.$$ 

Consider a deformation $f_t$ of $f_0 = f$ for $t \in (-\epsilon, \epsilon)$, and let $E_t : B_t \hookrightarrow G$ be the associated adapted $H$-bundle.

$$f^*_t G \supset B_t \xrightarrow{E_t} G \xrightarrow{\downarrow H} P \xrightarrow{\downarrow f_t} M \xrightarrow{\downarrow P} X$$

A general element $E_t = (e_1(t), e_2(t), ..., e_N(t))$ may be written as

$$e_A(t) = e_B(0) g^B_A(t), \; A, B = 1, ..., N.$$ 

Equivalently in matrix form

$$E_t = E_0 g_t,$$

where $g_t = (g^B_A(t))$ is the unique $G$-valued function on $B_f \times (-\epsilon, \epsilon)$ that describes the deformation of the bundle $B_f$. Let us expand $g_t$ (formally) with respect to $t$

$$g_t = \exp \left( \sum_{k=1}^\infty \frac{t^k}{k!} U_k \right)$$
for a sequence of $g$-valued functions $\{U_k\}$ on $B_f$. Then

$$dE_t = dE_0 g_t + E_0 dg_t$$

$$= E_0 \pi_0 g_t + E_0 dg_t$$

$$= E_t (g_t^{-1} \pi_0 g_t + g_t^{-1} dg_t)$$

$$= E_t \pi_t$$

where $\pi_t = E_t^{-1} dE_t = E_t^* \phi$. The expression for $\pi_t = \sum_{k=1}^{\infty} \frac{t^k}{k!} \pi_k$ can be computed as follows.

$$\pi_1 = dU_1 + \pi_0 U_1 - U_1 \pi_0$$

$$\pi_2 = dU_2 + \pi_0 U_2 - U_2 \pi_0 + \pi_1 U_1 - U_1 \pi_1$$

$$\vdots$$

$$\pi_k = dU_k + \pi_0 U_k - U_k \pi_0 \mod \{\pi_{k-1}, \pi_{k-2}, \ldots, \pi_1\}$$

$$\vdots$$

On the other hand, the deformations we are interested in are not arbitrary ones but those deformations that satisfy certain geometric constraint. Since the induced Maurer-Cartan form $\pi_t$ captures the geometry of the immersion $f_t$, it is reasonable to define the condition of admissible deformation in terms of a finite set of ordinary differential relations among the coefficients of $\pi_t$. These relations are in turn expressed as a sequence of (algebraic) relations among the coefficients of $\{\pi_1, \pi_2, \ldots\}$.

Remark. There are situations where the condition of admissible deformation is given in a form which is more general than a sequence of algebraic relations among the coefficients of $\{\pi_1, \pi_2, \ldots\}$, e.g., the jet rigidity of homogeneous varieties in projective spaces [LM]. The condition of admissible deformation in this case may involve a sequence of differential relations among the coefficients of $\{\pi_1, \pi_2, \ldots\}$.

Definition 1.1 Let $f : M \hookrightarrow X$ be a submanifold in a homogeneous space $X = G/P$. 
Suppose a condition of admissible deformation is given as a finite set of ordinary differential relations among the coefficients of \( \pi_t \). The equation of deformation at \( f_0 = f \) is the sequence of first order equations \( (3) \) for a sequence of \( g \)-valued functions \( \{ U_1, U_2, \ldots \} \) on the adapted \( H \)-bundle \( B_f \) with \( \{ \pi_1, \pi_2, \ldots \} \) satisfying the sequence of relations that correspond to the given set of differential relations on \( \pi_t \).

The idea of this set up is to lift the problem of deformation to Lie group \( G \) where a uniform treatment is available through Maurer-Cartan form and its structure equation.

Note by expanding the structure equation

\[ -d\pi_t = \pi_t \wedge \pi_t \]

with respect to \( t \), we obtain a sequence of differential equations for \( \{ \pi_1, \pi_2, \ldots \} \).

\[ -d\pi_1 = \pi_0 \wedge \pi_1 + \pi_1 \wedge \pi_0, \quad (4) \]
\[ -d\pi_2 = \pi_0 \wedge \pi_2 + \pi_2 \wedge \pi_0 + 2 \pi_1 \wedge \pi_1, \]
\[ \vdots \]
\[ -d\pi_k = \sum_{i+j=k} \frac{k!}{i!j!} \pi_i \wedge \pi_j, \]
\[ \vdots \]

These are in fact the set of compatibility conditions obtained by differentiating \( (3) \) once.

We now introduce a certain group action on the space of deformations that is tangent to Cauchy characteristics of the deformation equation, which plays an important role in our treatment of rigidity questions.

Assume a deformation \( E_t \) is given. Suppose \( h_t \) is a \( H \)-valued function on \( B_f \times (-\epsilon, \epsilon) \), and consider a new deformation

\[ \hat{E}_t = E_t h_t. \]
Let \( \Pi : G \rightarrow X = G/P \) be the projection map. Since \( H \subset P \), \( \Pi(E_t) = \Pi(\hat{E}_t) = f_t \) describes the same deformation of the immersion \( f \). This action of \( C^\infty(B_f \times (-\epsilon, \epsilon), H) \) simply rotates along the fiber of \( B_t \rightarrow M \).

The nature of this ambiguity is similar to the one in the reduction process of moving frame method. In order to see this group action explicitly, let us write

\[
\eta_t = \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k!} V_k\right)
\]

for a sequence of \( \mathfrak{h} \)-valued functions \( \{V_1, V_2, \ldots\} \) on \( B_f \), where \( \mathfrak{h} \) is the Lie algebra of \( H \). Then \( V_k \)'s acts on \( \pi_k \)'s by translation as follows.

\[
\Delta \pi_1 = \pi_0 V_1 - V_1 \pi_0 \quad (5)
\]
\[
\Delta \pi_2 \equiv \pi_0 V_2 - V_2 \pi_0 \mod \{V_1\}
\]
\[
\vdots
\]
\[
\Delta \pi_k \equiv \pi_0 V_k - V_k \pi_0 \mod \{V_{k-1}, V_{k-2}, \ldots V_1\}
\]
\[
\vdots
\]

Here \( \Delta \pi_k \) stands for the contribution from \( \{V_1, V_2, \ldots\} \). We will utilize this extra degree of freedom to normalize certain coefficients in the rigidity related computations. This is analogous to the process of absorption in the method of equivalence [Ga].

2 1-rigidity

The notion of 1-rigidity under admissible deformations of a submanifold in a homogenous space is introduced in this section. 1-rigidity can be considered as a geometric and unifying definition of infinitesimal rigidity in classical differential geometry. We continue to use the notations adopted in Section 1.

Suppose in the deformation equation (3) \( \pi_i = 0 \) for \( 1 \leq i \leq k \),

or equivalently

\[
dU_i + \pi_0 U_i - U_i \pi_0 = 0 \quad \text{for} \ 1 \leq i \leq k.
\]
Then it is easy to check $U_i = E_0^{-1} a_i E_0$ for a constant $a_i \in \mathfrak{g}$, $1 \leq i \leq k$, and

$$E_t = E_0 g_t = E_0 \exp\left(\frac{t^i}{i!} U_i\right)$$

$$\equiv \exp\left(\sum_{i=1}^{k} \frac{t^i}{i!} a_i\right) E_0 \mod t^{k+1}.$$ 

The vanishing of the first $k \pi_i$’s thus means that the deformation osculates a one parameter family of motions by $G$ up to order $k$.

**Definition 2.1** Suppose a condition of admissible deformations is given as a finite set of ordinary differential relations among the coefficients of $\pi_t$. A submanifold in a homogeneous space is 1-rigid under admissible deformations if every solution to the deformation equation (3) necessarily has $\pi_1 = 0$ modulo the group action (5).

The original deformation equation (3) is however rather difficult to use in practice to verify 1-rigidity of a submanifold. Instead, note $\pi_1$ satisfies the compatibility equation (4),

$$-d\pi_1 = \pi_0 \wedge \pi_1 + \pi_1 \wedge \pi_0.$$ 

For all practical purposes, it is this equation modulo the group action (5) that we will use to test for 1-rigidity. The process of simplification of this sort, that is removing nongeometric terms by differentiating once, has been capitalized in [BG] in their study of characteristic cohomology.

As it is the case with the method of moving frames, this definition is best explained and understood in practice. We consider in this section as an example a submanifold in the homogeneous space of conformal sphere $S^m$, and its conformal deformation. In the course of computation, a set of criteria for conformal 1-rigidity of a submanifold naturally emerges. This example also serves as a guide through the computation for CR 1-rigidity in Section 4. For general reference for conformal geometry, [Br2].

Let $\mathbb{R}^{m+1,1}$ be given a metric of signature $(m + 1, 1).$ Let $G = SO_0(m + 1, 1)$ be the identity component of the group of linear transformations that preserve the metric.
Let $X = S^m$ be the set of positive null rays through the origin. It is well known that $SO_0(m + 1, 1)$ acts transitively on $S^m$, and that $S^m$ inherits an $SO_0(m + 1, 1)$ invariant conformal structure.

Let $f : M \hookrightarrow X$ be an $n$-dimensional submanifold. Upon an appropriate choice of basis for $\mathbb{R}^{m+1,1}$, we may arrange so that the induced Maurer-Cartan form $\pi$ on a $1$-adapted $H$-bundle $B_f$ takes the following form, [Br2] for details on $1$-adapted bundle $B_f$.

$$\pi = \begin{pmatrix} \pi_0^0 & \pi_j^0 & \pi_b^0 & 0 \\
\pi_i^0 & \pi_j^i & \pi_b^i & \pi_{m+1}^i \\
0 & \pi_j^a & \pi_b^a & \pi_{m+1}^a \\
0 & \pi_j^{m+1} & 0 & \pi_{m+1}^{m+1} \end{pmatrix} \tag{6}$$

where $\pi_j^i = -\pi_i^j$, $\pi_b^a = -\pi_a^b$, $\pi_j^{m+1} = -\pi_0^j$, $\pi_j^0 = -\pi_{m+1}^j$, $\pi_b^0 = -\pi_{m+1}^b$, $\pi_{m+1}^m = -\pi_0^m$, and $1 \leq i, j \leq n$, $n + 1 \leq a, b \leq m$. $\pi_j^i$'s are semibasic 1-forms, and the quadratic form $\pi_j^i \circ \pi_0^i$ represents the induced conformal structure on $M$.

$\pi$ satisfies the structure equations $-d\pi = \pi \wedge \pi$. Differentiating $\pi_0^a = 0$, we get $\pi_i^a \wedge \pi_0^i = 0$, and by Cartan’s Lemma

$$\pi_i^a = h_{ij}^a \pi_j^0$$

for a set of coefficients $h_{ij}^a = h_{ji}^a$. Using the group action by $\pi_{m+1}^a$, we normalize so that $\text{tr} \ h_{ij}^a = 0$. The trace free parts of the second fundamental forms $H^a = h_{ij}^a \pi_0^i \circ \pi_0^j$ are conformal invariant of the submanifold $M$. Note that an element of the Lie algebra $\mathfrak{h}$ of $H$ is of the following form.

$$\begin{pmatrix} V_0^0 & V_j^0 & 0 & 0 \\
0 & V_j^i & 0 & V_{m+1}^i \\
0 & 0 & V_b^a & 0 \\
0 & 0 & 0 & V_{m+1}^{m+1} \end{pmatrix}$$

10
Suppose we are interested in 1-rigidity under the deformations that preserve the induced conformal structure on $M$. From (6), this is written as an ordinary differential equation

$$\frac{d}{dt} \pi^i_0(t) \circ \pi^i_0(t)|_{t=0} = \lambda \pi^i_0(0) \circ \pi^i_0(0)$$

for a scaling factor $\lambda$. Using the group action by $V^0_j$, $V^i_j$ components, however, we may in fact normalize so that

$$\frac{d}{dt} \pi^i_0(t)|_{t=0} = 0.$$

Hence $\pi_1 = \delta \pi$ is of the following form.

$$\delta \pi = \begin{pmatrix} \delta \pi^0_0 & \delta \pi^0_j & \delta \pi^0_b & 0 \\ 0 & \delta \pi^j_j & \delta \pi^j_b & \delta \pi^j_{m+1} \\ 0 & \delta \pi^a_j & \delta \pi^a_b & \delta \pi^a_{m+1} \\ 0 & 0 & 0 & \delta \pi^m_{m+1} \end{pmatrix}$$

(7)

The remaining *group variables* at this stage are $V^a_b$ and $V^0_i = -V^i_{m+1}$.

Differentiating $\delta \pi^0_0 = 0, \delta \pi^0_i = 0$ in (7) using (3), and by Cartan’s lemma, it is easy to check there exists a set of coefficients $p_i, p_{ij}^0 = p_{ji}^0$ such that

$$\delta \pi^0_0 = p_i \pi^i_0,$$

$$\delta \pi^0_i = \delta \pi^0_j = p_i \pi^0_0 - p_j \pi^0_j,$$

$$\delta \pi^a_i = p_{ij}^a \pi^j_0.$$

Using the group action by $V^0_j$ component as above, we may translate $p^j = 0$. Now differentiating $\delta \pi^0_0 = 0, \delta \pi^0_j = 0$ with these relations, we obtain the following compatibility conditions for deformation.

$$\delta \pi^0_i = p_{ij} \pi^j_0, \quad p_{ij} = p_{ji},$$

$$\pi^a_i \wedge \delta \pi^a_j + \delta \pi^a_i \wedge \pi^a_j - \pi^0_i \wedge \delta \pi^0_j - \delta \pi^0_i \wedge \pi^0_j = 0.$$  

(8)
Fix a point \( p \in M \), and let \( V^n = T_pM \), \( W^{m-n} = N_pM \) represent the tangent space and the normal space respectively. Take a conformal basis \( \{ x^i \} \) for \( V^* \) and \( \{ w_a \} \) for \( W \). Set

\[
H = h_i^a x^i x^j \otimes w_a \in S^2 V^* \otimes W,
\]
\[
P = p_i^a x^i x^j \otimes w_a \in S^2 V^* \otimes W.
\]

Let \( K(V) \subset \wedge^2 V^* \otimes \wedge^2 V^* \) be the space of curvature like tensors, which decomposes into \( K(V) = W \oplus \text{Ric} \), Weyl curvature tensor and Ricci tensor. Then the equation (8) is equivalent to

\[
\gamma(H, P)^W = 0, \tag{9}
\]
\[
\gamma(H, P)^{\text{Ric}} = p_{ij} x^i x^j \tag{10}
\]

where \( \gamma(H, P) \) is the linearized Gauss map [BBG]. An element \( H \in S^2 V^* \otimes W \) is called Weyl rigid when the space of solutions to (9) is \( \{ P = v^a b \, h_{ij} x^i x^j \otimes w_a \mid v^a b = -v^a b \} \).

Note for any \( Q \in S^2 V^* \otimes W \), \( \gamma(Q, P)^W = \gamma(Q_0, P)^W \), where \( Q_0 \) is the trace free part of \( Q \). We mention in passing that when \( \dim W = 1 \), a traceless quadratic form \( H \) is Weyl rigid if it has 3 nonzero eigenvalues.

Assume the trace free part of the second fundamental form \( H \) of \( M \) is Weyl rigid. Then using the group action by \( V^a b \) component, we may translate \( \delta \pi^a_i = 0 \). It follows \( \delta \pi^0_i = 0 \) by (10), and the only remaining nonzero elements of \( \delta \pi \) are \( \delta \pi^0_b \) and \( \delta \pi^a_b \). Differentiating \( \delta \pi^a_i = 0, \delta \pi^0_i = 0 \), we finally get

\[
\delta \pi^0_b \wedge \pi^a_i = 0, \tag{11}
\]
\[
\delta \pi^a_b \wedge \pi^0_i = 0. \tag{12}
\]

An element \( H \in S^2 V^* \otimes W \) is nondegenerate if the associated map \( H : S^2 V \to W \) is surjective. We first show that for a nondegenerate \( H \), (12) implies \( \delta \pi^a_b = 0 \). Let \( \delta \pi^a_b = q^a_i \pi^i, q^a_i = -q^b_i \). Then (12) is equivalent to \( q^a_i h^b_{ji} = q^a_i h^b_{ji} = q^a_{ijk} \) with \( q^a_{ijk} \) fully
symmetric in lower indices. Multiplying both sides by $h_{sp}^a$ and summing over $a$,

$q_{ijk}^a h_{sp}^a = q_{bi}^a h_{jk}^b h_{sp}^a$

$= -q_{ai}^b h_{jk}^b h_{sp}^a$

$= -q_{isp}^b h_{jk}^b$.

But $q_{ijk}^a$ is symmetric in lower indices and $h_{ij}^a$ is nondegenerate, hence $q_{ijk}^a = 0$ and consequently $q_{bi}^a = 0$.

We next show if $H$ is both nondegenerate and Weyl rigid, then (11) implies $\delta\pi^0_b = 0$. Let $\delta\pi^0_b = q_{bi}^a \pi^i$. Then (11) is equivalent to $q_{ai}^b h_{jk}^b = q_{bk}^b h_{ji}^b = q_{ijk}^b$ with $q_{ijk}^b$ fully symmetric in lower indices. For an arbitrary set of coefficients $\{f_i\}$, set

$Q = Q^a \otimes e_a = (q_{ai}^f j + q_{aj}^f i) x^i x^j \otimes e_a \in S^2 V^* \otimes W$.

Then one easily sees $\gamma(H, Q)^{WV} = 0$. Since $H$ is Weyl rigid, $q_{ai}^f j + q_{aj}^f i = v^a_b h_{ij}^b$ for skew symmetric $v^a_b = -v^b_a$. Multiplying both sides by $h_{sp}^a x^i x^j x^s x^p$ and summing over $a, i, j, s, p$, we get

$(f_i^i x^i) (q_{jsp} x^j x^s x^p) = 0$.

Since $\{f_i\}$ is arbitrary, $q_{ijk} = 0$, and consequently $q_{bi} = 0$ for $h_{ij}^a$ is nondegenerate.

Thus a submanifold $M \hookrightarrow S^m$ is conformally 1-rigid if it has a nondegenerate Weyl rigid second fundamental form. One can apply the results of [KT] and show that the canonical isometric embeddings of irreducible compact Hermitian symmetric spaces into spheres satisfy this criteria, and hence they are conformally 1-rigid.

3 CR submanifolds

3.1 Fundamental forms

In this section, we set up the basic structure equations for CR submanifolds in spheres. A sequence of invariants called fundamental forms are derived from the structure equations.
For general reference in CR geometry, [ChM][Ja].

Let $\mathbb{C}^{m+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{m+1})$, $1 \leq A \leq m$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + i(z^0 \bar{z}^m - z^m \bar{z}^0).$$

Let $\Sigma^m$ be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\text{SU}(m+1,1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. Then $\text{SU}(m+1,1)$ acts transitively on $\Sigma^m$, and

$$p : \text{SU}(m+1,1) \to \Sigma^m = \text{SU}(m+1,1)/P$$

for an appropriate subgroup $P$ [ChM].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{m+1}) \in \text{SU}(m+1,1)$ as an ordered set of $(m+2)$-column vectors in $\mathbb{C}^{m+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{m+1} \rangle = -\langle Z_{m+1}, \bar{Z}_0 \rangle = i,$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by $Z_0$. The left invariant Maurer-Cartan form $\phi$ of $\text{SU}(m+1,1)$ is defined by the equation

$$d Z = Z \phi,$$

which is in coordinates

$$d(Z_0, Z_A, Z_{m+1}) = (Z_0, Z_B, Z_{m+1}) \begin{pmatrix}
\phi_0^0 & \phi_0^A & \phi_0^{m+1} \\
\phi_A^0 & \phi_A^B & \phi_A^{m+1} \\
\phi_{m+1}^0 & \phi_{m+1}^A & \phi_{m+1}^{m+1}
\end{pmatrix}. \quad (14)$$
Coefficients of $\phi$ are subject to the relations obtained from differentiating (13) which are

$$
\phi^0_0 + \phi^m_{m+1} = 0 \\
\phi^{m+1}_0 = \phi^m_0, \quad \phi^0_{m+1} = \phi^0_{m+1} \\
\phi^m_A = -i \phi^A_0, \quad \phi^A_{m+1} = i \phi^0_A \\
\phi^A_B + \phi^B_A = 0 \\
\text{tr} \phi = 0,
$$

and $\phi$ satisfies the structure equation

$$
-d\phi = \phi \wedge \phi.
$$

(15)

It is well known that the $\text{SU}(m + 1, 1)$-invariant CR structure on $\Sigma^m \subset \mathbb{C}P^{m+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2m+1} = \partial B^{m+1}$, where $B^{m+1} \subset \mathbb{C}^{m+1}$ is the unit ball. The structure equation (14) shows that for any local section $s : \Sigma^m \to \text{SU}(m + 1, 1)$, this CR structure is defined by the hyperplane fields $(s^*\phi^m_0 + 1) = \mathcal{H}$ and the set of (1,0)-forms $\{ s^*\phi_A^0 \}$.

**Definition 3.1** Let $M$ be a manifold of dimension $2n + 1$. A submanifold defined by an immersion $f : M \hookrightarrow \Sigma^m$ is a CR submanifold if $f_* T_p M \cap \mathcal{H}_{f(p)}$ is a complex subspace of $\mathcal{H}_{f(p)}$ of dimension $n$ for each $p \in M$.

Note the induced hyperplane fields $f^{-1}_*(f_* (TM) \cap \mathcal{H})$ is necessarily a contact structure on $M$, and thus $M$ has an induced nondegenerate CR structure.

Let $f : M \hookrightarrow \Sigma^m$ be a CR submanifold. For each $p \in M$, we wish to define an associated decomposition of $\mathbb{C}^{m+1,1}$, which is a part of the reduction process of moving frame method applied to CR submanifold. Let $\Pi : \mathbb{C}^{m+1,1} - \{0\} \to \mathbb{C}P^{m+1}$ denote the projection map.

Set $V_0 = \langle \Pi^{-1}(f(p)) \rangle$ and $V_0 + V_1 = \langle \Pi^{-1}(f(p)), \Pi^{-1}(\partial f) \rangle$, where $\partial f$ stands for the holomorphic or (1,0) derivatives of $f$. Successively define the sequence of subspaces...
$W_2, W_3, \ldots, W_\tau$ with $\dim W_l = r_l$ so that

$$(V_0 + V_1) \oplus W_2 \oplus \ldots W_l = \langle \Pi^{-1}(f(p)), \Pi^{-1}(\partial f), \Pi^{-1}(\partial^2 f), \ldots \Pi^{-1}(\partial^l f) \rangle$$

and $n + r_2 + \ldots r_\tau = m$, where $\partial^l f$ stands for the $l$-th order $(1,0)$ derivatives of $f$. The set of numbers $(r_2, r_3, \ldots r_\tau)$ are called the type numbers of $f$ at $p$, and $\tau$ is called the height.

**Remark.** The structure equation (14), (19), and $\pi_0^A = i \bar{\pi}_m^{A+1}$ show this orthogonal decomposition, rather than a filtration, is well defined.

**Definition 3.2** A CR submanifold is of constant type if the type numbers are constant.

We assume $M \hookrightarrow \Sigma^m$ is a CR submanifold of constant type from now on.

In terms of the structure equation (14), we may arrange so that

$$V_0 = \langle Z_0 \rangle$$

$$(V_0 + V_1) = \langle Z_0 \rangle + \langle Z_i \rangle_{i=1}^n$$

$$(V_0 + V_1) \oplus W_2 = (\langle Z_0 \rangle + \langle Z_i \rangle_{i=1}^n) \oplus \langle Z_{i_2} \rangle_{i_2=n+1}^{n_2}$$

$$\vdots$$

$$(V_0 + V_1) \oplus W_2 \oplus \ldots W_l = (\langle Z_0 \rangle + \langle Z_i \rangle_{i=1}^n) \oplus \langle Z_{i_2} \rangle_{i_2=n+1}^{n_2} \oplus \ldots \langle Z_{i\tau} \rangle_{i_{\tau}=n_{\tau-1}+1}^{n_{\tau}}$$

where $n_l = n_{l-1} + r_l$ and $n_1 = n$. This decomposition then induces an associated adapted subbundle $B$

$$E : f^*(\text{SU}(m+1,1)) \supset B \hookrightarrow \text{SU}(m+1,1)$$

on which the following weak structure equations hold. Let $\pi = E^*\phi$.

$$\pi_0^A = 0 \quad \text{for} \quad n + 1 \leq A \leq m,$$

$$\pi_i^A \equiv 0 \mod \theta, \pi^{0,1} \quad \text{for} \quad n_2 + 1 \leq A \leq m,$$

$$\pi_{i_l}^A \equiv 0 \mod \theta, \pi^{0,1} \quad \text{for} \quad n_{l+1} + 1 \leq A \leq m, \quad 2 \leq l \leq \tau - 2,$$
where \( n_{l-1} + 1 \leq i_l \leq n_l \) for \( 2 \leq l \leq \tau \), and mod \( \pi^{0,1} \) means mod \( \bar{\pi}_0, \bar{\pi}_1, ... \bar{\pi}_n \). Implies the following equation in matrix form.

\[
\pi \equiv \begin{pmatrix}
\pi^i \\
\pi^i_0 \\
0 \pi^i_{i_1} \\
0 0 \pi^i_{i_2} \\
\vdots \\
0 0 0 0 \\
\end{pmatrix} \mod \pi^{m+1}_0, \pi^{0,1}.
\]

Denote \( \pi^{m+1}_0 = \theta, \pi^i = \pi^i_0 \). Differentiating \( \pi^A_0 = 0 \) for \( n + 1 \leq A \leq m \), we get \( \pi^A_i \wedge \pi^A_0 + \pi^{A+1}_0 \wedge \theta = 0 \). By Cartan’s lemma, there exists a set of coefficients \( h^A_{ij} = h^A_{ji}, h^A_i, h^A \) such that

\[
\begin{pmatrix}
\pi^A_i \\
\pi^{A+1}_0 \\
\end{pmatrix} = \begin{pmatrix} h^A_{ij} & h^A_i \\
\end{pmatrix} \begin{pmatrix} \pi^j \\
\theta \\
\end{pmatrix}.
\] (19)

Note by definition of \( W_2 \), \( h^A_{ij} = 0 \) for \( A \geq n_2 + 1 \).

Since the CR structure on \( \Sigma^m \) is integrable, the differential ideal generated by \( \{ \theta, \pi^{0,1} \} \) is closed. By successively differentiating \( \pi^{i+1}_{i-1} \equiv 0 \mod \theta, \pi^{0,1} \) for \( l = 2, ... \tau - 1 \), we obtain the following structure equations.

\[
\pi^{i+1}_{i+1} \wedge \pi^i_{i-1} \equiv 0 \mod \theta, \pi^{0,1}, \text{ for } 2 \leq l \leq \tau - 1.
\] (20)

Set \( \pi^{i+1}_{i+1} \equiv h^{i+1}_{ij} \pi^j \mod \theta, \pi^{0,1} \). The sequence of equations (20) then gives

\[
h^{i+1}_{ij} h^i_{ik} = h^{i+1}_{ij} h^i_{kj}.
\]

and thus

\[
h^i_{i_1} h^i_{i_2} ... h^i_{k_1} = h^i_{ij} \ldots h^i_{ks}
\]

is fully symmetric in lower indices.
Definition 3.3 Let $M \rightarrow \Sigma^m$ be a CR submanifold of constant type with the associated decomposition (17) and the Maurer Cartan form (18). The $l$-th fundamental form $F^l$ is defined by

$$F^l = h_{i_1}^{i_1} h_{i_2}^{i_2} \cdots h_{i_l}^{i_l} \pi^{i_1} \otimes \pi^{i_2} \otimes \cdots \otimes \pi^{i_l} \otimes Z_{i_l}$$

$$\equiv \pi_{i_{l-1}}^{i_{l-1}} \otimes \pi_{i_{l-2}}^{i_{l-2}} \cdots \pi_i^{i_2} \otimes \pi_i^{i_1} \otimes Z_i \text{ mod } \theta, \pi_0^1.$$

A computation with the structure equations (14) shows $F^l$ is up to scale a well defined section of the bundle $S^{l,0}(V^*) \otimes W_l$ over $M$, where $V$ is the holomorphic or $(1,0)$ tangent space of $M$. $F^l$ is one of the simplest $l$-th order extrinsic invariant of the CR submanifold $M$.

3.2 Bochner rigid submanifolds

The weak structure equations (18) is refined in this section for a class of CR submanifolds whose fundamental forms are algebraically rigid.

We start with an algebraic definition. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^r$ with the standard Hermitian metric. Let $S^{k,0}(V^*)$ ($S^{0,k}(V^*)$) be the space of holomorphic(anti-holomorphic) homogeneous polynomials of degree $k$, and denote $S^{k,l} = S^{k,0}(V^*) \otimes S^{0,l}(V^*)$. Let $\{x^i\}$ be a unitary basis of $S^{1,0}(V^*)$. For $k, l \geq 1$, we define the subspace

$$S^{k-1,l-1}_1 = \{x^k \bar{x}^k f \mid f \in S^{k-1,l-1}\} \subset S^{k,l}.$$

Let $\{e_a\}$ be a unitary basis of $W$. For $H = H^a \otimes e_a \in S^{k,0}(V^*) \otimes W$ and $P = P^a \otimes e_a \in S^{l,0}(V^*) \otimes W$, we define

$$\langle H, P \rangle = H^a \otimes \bar{P}^a \in S^{k,l}(V^*).$$

Definition 3.4 An element $H \in S^{k,0}(V^*) \otimes W$ is Bochner rigid if for any $P \in S^{k,0}(V^*) \otimes W$, the equation

$$\gamma(H, P) = \langle H, \bar{P} \rangle + \langle P, \bar{H} \rangle \in S^{k-1,k-1}_1 \subset S^{k,k}$$
implies
\[ \gamma(H, P) = 0. \]  

(22)

Suppose a Bochner rigid \( H = H^a \otimes e_a \in S^{k,0}(V^*) \otimes W \) is nondegenerate, or equivalently the associated map \( H : S^{k,0}(V) \to W \) is surjective. Then by Cartan’s Lemma, (22) implies
\[ P^a = u^a_b H^b \]  

(23)

for a skew Hermitian matrix \( u^a_b = -\bar{u}_b^a \).

It is known that when \( \dim W \leq \frac{1}{2} \dim V \), every \( H \in S^{k,0}(V^*) \otimes W \) is Bochner rigid [Hu]. We record the following for later application.

**Lemma 3.1** Let \( H \in S^{k,0}(V^*) \otimes W \) be a Bochner rigid polynomial. Suppose for \( B \in S^{1,0}(V^*) \otimes W \),
\[ \langle H, \bar{B} \rangle \in S^{k-1,0}_1 \subset S^{k,1}. \]

Then we necessarily have
\[ \langle H, \bar{B} \rangle = 0. \]

If \( H \) is moreover nondegenerate, then \( B = 0 \).

**Proof.** Let \( \{ x^i \} \) be a unitary (1,0)-basis of \( V^* \), and denote \( \langle H, \bar{B} \rangle = (x^i \bar{x}^i) g_B \) for \( g_B \in S^{k-1,0} \). Take any \( g \in S^{k-1,0} \). Then \( \gamma(H, g B) = (x^i \bar{x}^i) (g_B \bar{g} + g \bar{g}_B) = 0 \) by Bochner rigidity of \( H \). Since \( g \) is arbitrary, \( g_B = 0 \). The rest follows from the definition of a nondegenerate form. \( \Box \)

**Definition 3.5** Let \( M \hookrightarrow \Sigma^m \) be a CR submanifold of constant type with the associated decomposition (17) and the fundamental forms (21). \( M \) is a Bochner rigid submanifold if each of its fundamental forms is Bochner rigid.

**Example 3.1** Let \( M_{n,p} \hookrightarrow \mathbb{C}P^N = P(\bigwedge^n \mathbb{C}^{n+p}) \) be the Plücker embedding of the Grassmannian \( Gr(n, \mathbb{C}^{n+p}) \), \( p \leq n \). Let \( \tilde{M}_{n,p} \hookrightarrow S^{2N+1} \) be the inverse image of \( M_{n,p} \) under
Hopf map, which is an $S^1$-bundle over $M_{n,p}$. When $p = 2, 3$, $\hat{M}_{n,p}$ is a Bochner rigid CR submanifold of $S^{2N+1}$.

We postpone the computation of this example to **Section 6**. It is likely the case that $\hat{M}_{n,p} \hookrightarrow S^{2N+1}$ is Bochner rigid for all $p$.

The main result of this section is the following refined structure equations for Bochner rigid submanifolds. We continue to use the notations of **Section 3.1**

**Theorem 3.1** Let $M \hookrightarrow \Sigma^m$ be a Bochner rigid submanifold. Then the weak structure equations (18) can be refined as follows.

\[
\begin{align*}
\pi^A_0 &= 0 & \text{for } n + 1 \leq A \leq m & \quad (24) \\
\pi^A_i &= 0 & \text{for } n_2 + 1 \leq A \leq m \\
\pi^A_{il} &= 0 & \text{for } n_{l+1} + 1 \leq A \leq m, & \quad 2 \leq l \leq \tau - 2
\end{align*}
\]

These equations furthermore imply

\[
\begin{align*}
\pi^{il+1}_{il} &= 0 \mod \theta, \pi^{1,0} & \text{for } 1 \leq l \leq \tau - 1, & \quad (25) \\
\pi^A_{m+1} &= 0 & \text{for } n_2 + 1 \leq A \leq m.
\end{align*}
\]

**Proof.** By definition of $W_2$ and the weak structure equations (18), we in fact have

\[
\begin{align*}
\pi^A_i &= h^A_i \theta, & \quad (26) \\
\pi^A_{m+1} &= h^A_i \pi^i \mod \theta & \text{for } n_2 + 1 \leq A \leq m
\end{align*}
\]

for a set of coefficients $h^A_i$. Set $\pi^A_{i_2} \equiv B^A_{i_2j} \pi^j \mod \theta, \pi^{1,0}$ for $n_2 + 1 \leq A \leq m$. Differentiating (26) mod $\theta$,

\[
i h^A_i \varpi \equiv \pi^A_{i_2} \pi^i + \pi^A_{m+1} \wedge \pi^A_i \mod \theta, \quad \text{for } n_2 + 1 \leq A \leq m,
\]

where $\varpi = \pi^k \wedge \pi^k$, and this gives

\[
B^A_{i_2p} h^A_{ij} = -i(h^A_i \delta_{jp} + h^A_j \delta_{ip}).
\]
But $h_{ij}^2$ represents the second fundamental form $F^2$ which is Bochner rigid. By Lemma 3.1 $B_{i_2p}^A = 0$, $h_j^A = 0$ for $n_2 + 1 \leq A \leq m$. We thus have, since $\pi_{i_2}^A \equiv 0 \mod \theta, \pi^{0,1}$ for $n_3 + 1 \leq A \leq m$ from (18),

\[
\pi_i^A = 0 \quad \text{for} \quad n_2 + 1 \leq A \leq m,
\]

\[
\pi_{i_2}^A \equiv 0 \mod \theta, \pi^{1,0} \quad \text{for} \quad n_2 + 1 \leq A \leq n_3,
\]

\[
\pi_{i_2}^A = h_{i_2}^A \theta \quad \text{for} \quad n_3 + 1 \leq A \leq m.
\]

(27)

Set $\pi_{i_3}^A \equiv B_{i_3j}^A \pi^j \mod \theta, \pi^{1,0}$ for $n_3 + 1 \leq A \leq m$. Differentiating (27) mod $\theta$,

\[
h_{i_2}^A \varpi \equiv \pi_{i_3}^A \wedge \pi_{i_2}^i \mod \theta \quad \text{for} \quad n_3 + 1 \leq A \leq m,
\]

for $\pi_{m+1}^A \equiv 0 \mod \theta$ for $n_2 + 1 \leq A \leq m$ from (26), and this gives

\[
B_{i_3p}^A h_{i_2}^i = -i h_{i_2}^A \delta_{kp}.
\]

Multiplying $h_{ij}^2$ and summing over $i_2$ we get

\[
B_{i_3p}^A h_{i_2}^i h_{i_2}^j = -i h_{i_2}^A h_{ij}^2 \delta_{kp} \quad \text{for} \quad n_3 + 1 \leq A \leq m.
\]

But $h_{i_2}^2 h_{ij}^2 = h_{i_2}^i h_{ij}^2$ represents the third fundamental form $F^3$ which is Bochner rigid, and both $F^2$, $F^3$ are nondegenerate by definition. By Lemma 3.1 $B_{i_3p}^A = 0$, $h_{i_2}^A = 0$ for $n_3 + 1 \leq A \leq m$. We thus have by similar argument as before

\[
\pi_{i_2}^A = 0 \quad \text{for} \quad n_3 + 1 \leq A \leq m,
\]

\[
\pi_{i_3}^A \equiv 0 \mod \theta, \pi^{1,0} \quad \text{for} \quad n_3 + 1 \leq A \leq n_4,
\]

\[
\pi_{i_3}^A = h_{i_3}^A \theta \quad \text{for} \quad n_4 + 1 \leq A \leq m.
\]

Continuing in this manner, it is straightforward to verify (24).

The first equation of (25) is already proved. Once (24) is true, we have from (19)

\[
\pi_{m+1}^A = p^A \theta \quad \text{for} \quad n_2 + 1 \leq A \leq m.
\]

Differentiating this equation mod $\theta$ and collecting (1,1)-terms, we get $p^A \varpi \equiv 0$ for $n_2 + 1 \leq A \leq m$. □
4 Rigidity of CR submanifolds

4.1 CR 1-rigidity

Let \( f : M \hookrightarrow \Sigma^m \) be a CR submanifold of constant type. A \textit{type preserving CR deformation} of \( f \) is a deformation of \( f \) through CR immersions with the same type numbers and the equivalent induced CR structures as \( f \).

**Theorem 4.1** Let \( f : M \hookrightarrow \Sigma^m \) be a Bochner rigid CR submanifold of CR dimension \( \geq 2 \). Then \( f \) is 1-rigid under type preserving CR deformations.

The idea is to explore the structure of the compatibility equation (4) in the presence of the associated decomposition (17) and the corresponding adapted structure equation (24), (25). The proof is carried out by a sequence of over-determined pde computations. Simply put, Bochner rigidity of the fundamental forms implies that each adjacent sequence of compatibility conditions of the over-determined pde’s are tightly related. It thus plays the role of connecting the propagation of the associated sequence of compatibility conditions.

It would be interesting and more desirable to remove the type preserving hypothesis in the theorem, which is the case when the height \( \tau = 2 \). The computation involved without the hypothesis is, however, more differential algebraic than algebraic which we are not able to resolve.

**Corollary 4.1** [Hu][EHZ] Let \( f : M^n \hookrightarrow \Sigma^m \) be a CR submanifold of constant type and of CR dimension \( n \geq 2 \). Suppose each of the type numbers \( (r_2, r_3, \ldots, r_{\tau}) \) are bounded by \( r_l \leq \frac{n}{2}, \ 2 \leq l \leq \tau \). Then \( f \) is 1-rigid under type preserving CR deformations.

**Corollary 4.2** Let \( f_{n,p} : \hat{M}_{n,p} \hookrightarrow \Sigma^m \) be the canonical \( S^1 \)-bundle over the Plücker embedding of the Grassmannian \( Gr(n, \mathbb{C}^{n+p}) \), \( p \leq n \). When \( p = 2, 3 \), \( f_{n,p} \) is 1-rigid under type preserving CR deformations.
We present the proof of the theorem for the case $f$ has height $\tau = 3$, as the proof for the general case can be easily deduced from this. We shall agree on the index range

$$1 \leq i, j, k, l, p, s \leq n$$

$$n + 1 \leq a, b \leq n_2$$

$$n_2 + 1 \leq \nu, \mu \leq n_3,$$

and we continue to use the notations in Section 3.

**Proof of Theorem 4.1.** Since $M$ is a Bochner rigid submanifold, the induced Maurer-Cartan form $\pi$ on the associated $H$-bundle $B$ takes the following form.

$$\pi = \begin{pmatrix}
\pi^0_j & 0 & 0 & 0 & 0 \\
0 & \pi^0_i & 0 & 0 & 0 \\
0 & 0 & \pi^a_j & 0 & 0 \\
0 & 0 & 0 & \pi^\nu_i & 0 \\
\theta & \pi^{m+1}_j & 0 & 0 & \pi^{m+1}_i
\end{pmatrix}$$

(28)

with

$$\pi^a_j = h^a_{jk} \pi^k + h^a_j \theta,$$

$$\pi^a_{m+1} \equiv h^a_j \pi^j \mod \theta,$$

$$\pi^\nu_i \equiv h^\nu_i \pi^k \mod \theta.$$ 

Here $h^a_{jk}$ and $h^\nu_i h^\nu_j = h^\nu_{ijk}$ are fully symmetric in lower indices representing the second and the third fundamental forms. The Lie algebra of $H$-valued group variable $V$ in the group action is of the following form at this stage.

$$V = \begin{pmatrix}
V^0_0 & V^0_j & 0 & 0 & V^{m+1}_m \\
0 & V^i_j & 0 & 0 & V^{i}_{m+1} \\
0 & 0 & V^a_j & 0 & 0 \\
0 & 0 & 0 & V^\nu_i & 0 \\
0 & 0 & 0 & 0 & V^{m+1}_{m+1}
\end{pmatrix}.$$
For notational purpose, let $\delta \pi$ denote $\pi_1$ in the deformation equation (4). Since the type preserving CR deformation is considered, it is clear that $\delta \pi$ should take the following form after some absorption by group actions (5).

$$
\delta \pi = \begin{pmatrix}
\delta \pi^0_0 & \delta \pi^0_j & \delta \pi^0_b & \delta \pi^0_\mu & \delta \pi^0_{m+1} \\
0 & \delta \pi^j_j & \delta \pi^i_b & \delta \pi^i_\mu & \delta \pi^i_{m+1} \\
0 & \delta \pi^a_j & \delta \pi^a_b & \delta \pi^a_\mu & \delta \pi^a_{m+1} \\
0 & \delta \pi^\nu_j & \delta \pi^\nu_b & \delta \pi^\nu_\mu & \delta \pi^\nu_{m+1} \\
0 & 0 & 0 & 0 & \delta \pi^{m+1}_{m+1}
\end{pmatrix}
$$

(29)

Here $\delta \pi^i_0$ is translated to 0 by $V^i_j - \delta_{ij} V^0_0$ and $V^i_m + 1$ component, and $\delta \pi^m_{m+1}$ is translated to 0 by Re $V^0_0$ component for it is a CR deformation, and $\delta \pi^\nu_j \equiv 0 \mod \theta$, $\pi^{0,1}$ for it is type preserving. The remaining group variables at this stage are $V^0_0$, $V^a_b$, $V^0_{m+1}$, and $V^\nu_m$ with $V^0_0 = -\overline{V^0_0}$.

**Step 1.** Differentiating $\delta \pi^0_0 = 0$, $\delta \pi^a_0 = 0$, $\delta \pi^\nu_0 = 0$, $\delta \pi^m_{m+1} = 0$, we get

$$
\Delta^i_j \wedge \pi^j + \delta \pi^i_{m+1} \wedge \theta = 0
$$

(30)

$$
\delta \pi^a_i \wedge \pi^i + \delta \pi^a_{m+1} \wedge \theta = 0
$$

$$
\delta \pi^\nu_i \wedge \pi^i + \delta \pi^\nu_{m+1} \wedge \theta = 0
$$

Re $\delta \pi^0_0 \wedge \theta = 0,$

where $\Delta^i_j = \delta \pi^i_j - \delta \pi^0_0 \delta_{ij}$. Using the group action (5) by $V^0_{m+1}$ component, we can absorb Re $\delta \pi^0_0$ to 0. Then $(\Delta^i_j)$ is a skew Hermitian matrix valued 1-form such that

$$
\Delta^i_j \wedge \pi^j \equiv 0 \mod \theta,
$$
and Cartan’s lemma implies $\Delta^i_j \equiv 0 \mod \theta$, $\delta \pi^\nu_j \equiv 0 \mod \theta$, $\pi^{0,1}$ and \cite{31} implies $\delta \pi^\nu_j \equiv 0 \mod \theta$. We thus have

\[
\begin{pmatrix}
\Delta^i_j \\
\delta \pi^i_j \\
\delta \pi^a_j \\
\delta \pi^a_j \\
\delta \pi^\nu_j \\
\delta \pi^\nu_j
\end{pmatrix} = \begin{pmatrix}
0 & q^i_j \\
q^i_j & q^i_j \\
p^a_j & p^a_j \\
p^a_j & p^a_j \\
0 & p^\nu_j \\
p^\nu_j & p^\nu_j
\end{pmatrix} \begin{pmatrix}
\pi^j \\
\theta \\
\pi^j \\
\theta \\
\pi^j \\
\theta
\end{pmatrix},
\]

Re $\delta \pi^0_0 = 0$,

where $q^i_j = -\bar{q}^i_j$, $p^a_j = p^a_j$. Differentiating Re $\delta \pi^0_0 = 0$ with these relations, we get

\[
\delta \pi^0_{m+1} = \frac{i}{2}(\bar{q}^i \pi^i - q^i \bar{\pi}^i) + q^0 \theta,
\]

for a real coefficient $q^0$. Note $\delta \pi^{m+1}_{m+1} = -\delta \pi^0_0 = \delta \pi^0_0$. The remaining group variables at this stage are $V^0_0$, $V^a_0$, and $V^\nu_\mu$.

**Step 2.** Differentiating $\Delta^i_j = q^i_j \theta \mod \theta$ and collecting terms, we obtain an equation which is equivalent to

\[
h^a_{jk} \bar{p}^a_{il} + p^a_{jk} \bar{h}^a_{il} = i(q^i_j \delta_{kl} + q^i_k \delta_{lj} + q^i_j \delta_{ik} + q^i_k \delta_{ij}).
\]

(31)

Since $h^a_{ij}$ is Bochner rigid, this equation implies

\[
q^i_j = 0,
\]

\[
h^a_{jk} \bar{p}^a_{il} + p^a_{jk} \bar{h}^a_{il} = 0.
\]

From \cite{23}, $p^a_{ij} = u^a_b h^b_{ij}$ for a skew Hermitian matrix ($u^a_b$), and thus can be absorbed to 0 by the group action by $V^a_b - \delta_{ab} V^0_0$ component. Differentiating $\delta \pi^i_{m+1} = q^i \theta \mod \theta$,

\[
q^i \pi^i \equiv \pi^i \wedge \delta \pi^0_{m+1} + \pi^i \wedge \delta \pi^a_{m+1} \mod \theta
\]

\[
\equiv \pi^i \wedge (\frac{i}{2}(\bar{q}^i \pi^i - q^i \bar{\pi}^i)) - \bar{h}^a_{jk} p^a_j \bar{\pi}^k \wedge \pi^i \mod \theta,
\]

25
where \( \varpi = \pi^k \land \bar{\pi}^k \). Collecting \((2, 0)\)-terms, we get \( q^i = 0 \). Since \( h^a_{ij} \) represents the nondegenerate second fundamental form, the remaining equation \( h^a_{ik} p^k_j = 0 \) implies \( p^a_i = 0 \). Now differentiating \( \pi^a_{m+1} = p^a \theta \mod \theta \), we get \( p^a = 0 \). \( \delta \pi \) is now reduced to the following form.

\[
\delta \pi = \begin{pmatrix}
\delta \pi^0_0 & 0 & 0 & \delta \pi^0_\mu & q^0 \theta \\
0 & \delta \pi^0_0 & \delta_{ij} & 0 & \delta \pi^i_\mu & 0 \\
0 & 0 & \delta \pi^a_\nu & \delta \pi^a_\mu & 0 \\
0 & \delta \pi^\nu_j & \delta \pi^\nu_0 & \delta \pi^\nu_\mu & \delta \pi^\nu_{m+1} \\
0 & 0 & 0 & 0 & \delta \pi^0_0
\end{pmatrix}
\]

The remaining group variables at this stage are \( V^0_0 \) and \( V^\nu_\mu \).

**Step 3.** Differentiating \( \delta \pi^\nu_i = p^\nu_i \theta \mod \theta \), we get

\[
p^\nu_i \varpi \equiv \delta \pi^\nu_i \land \pi^a_i \mod \theta.
\]

Set \( \delta \pi^\nu_i \equiv p^\nu_i \pi^i + C^a_{\nu i} \bar{\pi}^i \mod \theta \). Collecting \((2, 0)\), and \((1, 1)\) terms in \((32)\) we get

\[
\begin{align*}
p^\nu_{ai} h^a_{jk} &= p^\nu_{aj} h^a_{ik} = p^\nu_{ijk} \quad \text{fully symmetric in lower indices} \\
i p^\nu_i \delta_{jk} &= -C^a_{\nu k} h^a_{ij}.
\end{align*}
\]

By Bochner rigidity of \( h^a_{ij} \), \((33)\) implies \( C^a_{\nu k} = 0, p^\nu_i = 0 \). Differentiating \( \delta \pi^\nu_{m+1} = p^\nu \theta \mod \theta \) with these relations gives \( p^\nu = 0 \). Differentiating \( \delta \pi^0_{m+1} = q^0 \theta \mod \theta \) at this stage gives \( q^0 = 0 \). \( \delta \pi \) is now reduced to the following form.

\[
\delta \pi = \begin{pmatrix}
\delta \pi^0_0 & 0 & 0 & 0 & 0 \\
0 & \delta \pi^0_0 & \delta_{ij} & 0 & 0 \\
0 & 0 & \delta \pi^a_\nu & \delta \pi^a_\mu & 0 \\
0 & \delta \pi^\nu_b & \delta \pi^\nu_\mu & 0 \\
0 & 0 & 0 & 0 & \delta \pi^0_0
\end{pmatrix}
\]

**Step 4.** Differentiating \( \delta \pi^a_i = 0 \mod \theta \),

\[
\Delta^a_b \land \pi^b_i \equiv 0 \mod \theta
\]
where $\Delta_a^b = \delta \pi_b^a - \delta \pi_0^a \delta_{ab}$. Since the coefficients of $\pi_i^a \equiv h_{ij}^a \pi^j \mod \theta$ represent the second fundamental form which is nondegenerate by definition, a variant of Cartan’s lemma, [GH], implies $\Delta_a^b \mod \theta$ is in the span of $\{ \pi_i^a \}$. But $\pi_i^a$ consists of $(1,0)$-forms mod $\theta$ and $\Delta_a^b$ is skew Hermitian. Hence $\Delta_a^b \equiv 0 \mod \theta$, and we write

$$\Delta_a^b = q_a^b \theta$$

where $q_a^b = -\bar{q}_b^a$. Differentiating this equation, we get $-i q_a^b \varpi \equiv \bar{\pi}_a^b \wedge \delta \pi_b^a + \delta \bar{\pi}_a^b \wedge \pi_b^a \mod \theta$, or equivalently

$$-i q_a^b \delta_{sk} = \bar{h}_{as}^\nu \bar{p}_{bk}^\nu + \bar{p}_{as}^\nu h_{bk}^\nu.$$

Multiplying both sides by $\bar{h}_{ip}^a h_{ij}^b$ and summing with respect to $a, b$,

$$-i q_a^b \bar{h}_{ip}^a h_{ij}^b \delta_{sk} = \bar{h}_{ips}^\nu \bar{p}_{ijk}^\nu + \bar{p}_{ips}^\nu h_{ijk}^\nu.$$

By Bochner rigidity of $h_{ijk}^\nu$ representing the third fundamental form, this implies

$$q_a^b = 0,$$

$$p_{ijk}^\nu = p_{ai}^\nu h_{jk}^a = u_{ai}^\nu h_{jk}^a = u_{ai}^\nu h_{jk}^a$$

for a skew Hermitian $u_{ai}^\nu = -\bar{u}_{ai}^\nu$. Since $h_{ijk}^a$ is nondegenerate, this implies

$$p_{ai}^\nu = u_{ai}^\nu h_{ai}^\nu,$$

and we may absorb $p_{ai}^\nu$ to 0 by group action by $V_{\mu}^\nu - \delta_{\nu \mu} V_0^0$ component as before. Put $\delta \pi_a^\nu = p_a^\nu \theta$. Differentiating $\Delta_a^b = 0$ and collecting $\theta \wedge \pi^{1,0}$-terms, we get $h_{ai}^\nu \bar{p}_b^0 = 0$. Multiplying $h_{ijk}^a$ and summing with respect to $a$, we get

$$h_{ijk}^\nu \bar{p}_b^0 = 0.$$
Since \( h_{ijk}^\nu \) represents the third fundamental form which is nondegenerate by definition, this equation implies \( p_\nu^\nu = 0 \). \( \delta \pi \) is now reduced to the following form.

\[
\delta \pi = \begin{pmatrix}
\delta \pi^0_0 & 0 & 0 & 0 & 0 \\
0 & \delta \pi^0_0 \delta_{ij} & 0 & 0 & 0 \\
0 & 0 & \delta \pi^0_0 \delta_{ab} & 0 & 0 \\
0 & 0 & 0 & \delta \pi^\nu_\mu & 0 \\
0 & 0 & 0 & 0 & \delta \pi^0_0 \\
\end{pmatrix}
\]

Note there are no remaining group variables at this stage.

**Step 5.** Differentiating \( \delta \pi^\nu_a = 0 \mod \theta \),

\[
\Delta^\nu_\mu \wedge \pi^\mu_a \equiv 0 \mod \theta
\]

where \( \Delta^\nu_\mu = \delta\pi^\nu_\mu - \delta\pi^0_0 \delta_{\nu\mu} \). Since the coefficients of \( \pi^\mu_a \equiv h^\mu_{aj} \pi^j \mod \theta \) represents the nondegenerate third fundamental form, a variant of Cartan’s lemma as before implies

\[
\Delta^\nu_\mu = \delta\pi^\nu_\mu - \delta\pi^0_0 \delta_{\nu\mu} \equiv 0 \mod \theta.
\]

Now \( \delta \pi \) takes values in \( \mathfrak{su}(m + 1, 1) \), and in particular \( \text{tr} \delta \pi = 0 \). Hence \( \delta\pi^0_0 \) is a multiple of \( \theta \), and consequently

\[
\delta \pi = P \theta
\]

for a scalar \( \mathfrak{su}(m + 1, 1) \)-valued coefficient \( P \). But differentiating this equation mod \( \theta \) gives \( P \varpi \equiv 0 \), hence \( P = 0 \). \( \square \)

Here we summarize the process of computation for 1-rigidity in analogy with the game of dominoes. Imagine \( \delta \pi \) is a \((\tau + 2)\)-by\(- (\tau + 2)\) standing blocks of dominoes. Our objective is to lay down all of these domino blocks. There are three main ingredients in the process.

**A.** The initial push: This corresponds to imposing the condition of admissible deformations as in [24].
B. When a block gets hit, it falls toward the nearby standing blocks: This is expressed by the compatibility conditions obtained by differentiation using the deformation equation \[4\].

C. The nearby standing blocks in process B are within hitting distance, and gets hit by at least one domino block, and falls: This in CR case is guaranteed by the Bochner rigid conditions.

Once there was the initial push A, the processes B and C alternate just as it is the case with real dominoes. We finally mention that in process C, it is conceivable that a standing block gets hit but the force is not enough to lay it down, or it is equally conceivable that a standing block gets hit by more than one blocks and the hitting forces cancel. In these cases, one has to analyze the prolongation of the deformation equation. This phenomena occurs when trying to remove the type preserving hypothesis from Theorem 4.1.

4.2 Local rigidity

Although it is formulated based on a natural geometric consideration, 1-rigidity introduced in Section 2 is an abstract definition. It is not clear whether 1-rigidity has any bearing on the actual rigidity property of submanifolds under admissible deformation. The main result of this section is that 1-rigidity can be extended to actual local rigidity at least for Bochner rigid CR submanifolds in spheres. The idea of proof is a certain averaging trick, which might be of use in rigidity problems in other homogeneous spaces as well.

**Theorem 4.2** Let \( f_0 : M \hookrightarrow \Sigma^m \) be a Bochner rigid CR submanifold of height \( \tau \). Then there exists a \( C^{\tau} \) neighborhood \( \mathcal{U} \subset C^{\tau}(M, \Sigma^m) \) of \( f_0 \) with the following property. Suppose \( f, g \in \mathcal{U} \) are CR immersions such that their induced CR structures on \( M \) are mutually equivalent and that they are of the same type as \( f_0 \). Then \( f \) and \( g \) are
congruent up to an automorphism of $\Sigma^m$.

Note when $\tau = 2$, the type preserving hypothesis can be dropped.

We present the proof for the case $\tau = 2$, for the proof for the general case is essentially the same. We follow the notations in Section 4.1.

**Proof of Theorem 4.2** Given two CR immersions $f, g$, let $F : B_f \hookrightarrow f^*\text{SU}(m+1,1)$, $G : B_g \hookrightarrow g^*\text{SU}(m+1,1)$ be the associated adapted $H$-bundles over $M$. Let

$$s_f = (F_0, F_1, \ldots F_{m+1}),$$
$$s_g = (G_0, G_1, \ldots G_{m+1})$$

be any sections of these bundles, and denote

$$s_f^*(F^*\phi) = \alpha,$$
$$s_g^*(G^*\phi) = \beta$$

where $\phi$ is the Maurer-Cartan form of $\text{SU}(m+1,1)$. Set

$$\pi = \frac{1}{2}(\alpha + \beta),$$
$$\delta\pi = \alpha - \beta;$$

and observe

$$-d(\delta\pi) = \alpha \wedge \alpha - \beta \wedge \beta$$
$$= \delta\pi \wedge \pi + \pi \wedge \delta\pi.$$  \hspace{1cm} (34)

For a fixed section $s_g$, we wish to show there exists a section $s_f$ such that $\delta\pi = 0$.

Since $f$ and $g$ induce the equivalent CR structures on $M$, we may take a section $s_f$ such that

$$\alpha^i_0 = \beta^i_0 = \pi^i,$$
$$\alpha^{m+1}_0 = \beta^{m+1}_0 = \theta$$

30
and \( \delta \pi \) takes the following form.

\[
\delta \pi = \begin{pmatrix}
\delta \pi_0^0 & * & * \\
0 & * & * \\
0 & * & * \\
0 & 0 & 0 & * \\
\end{pmatrix}.
\]

Differentiating \( \delta \pi_0^{m+1} = 0 \) using (34), we get Re \( \delta \pi_0^0 \land \theta = 0 \). Modifying \( F_{m+1} \) by a multiple of \( F_0 \) if necessary, which is still a legitimate section of \( B_f \) preserving the relations (35), we may in fact have Re \( \delta \pi_0^0 = 0 \) (this requires a computation, which is rather similar to the absorption by group action as in CR 1-rigidity and we omit). Now differentiating \( \delta \pi_0^i = 0 \) we get

\[
\begin{pmatrix}
\Delta_i^j \\
\delta \pi^i_{m+1}
\end{pmatrix} = \begin{pmatrix}
0 & q^j_i \\
q^j_i & q^i_j
\end{pmatrix} \begin{pmatrix}
\pi^i \\
\theta
\end{pmatrix}
\]

(36)

where \( \Delta_i^j = \delta \pi_i^j - \delta \pi_0^0 \delta_{ij}, \ q^j_i = -q^i_j \).

Let \( f^a = f^a_{ij} \pi^i \circ \pi^j \) and \( g^a = g^a_{ij} \pi^i \circ \pi^j \) represent the second fundamental forms of \( f \) and \( g \). If \( f \) and \( g \) are sufficiently \( C^2 \) close to \( f_0 \) which is Bochner rigid, then \( f \) and \( g \) would be Bochner rigid. Thus the average \( \frac{1}{2} (f^a + g^a) \) would be Bochner rigid too for a section \( s_f \) with appropriate \( (F_{n+1}, F_{n+2}, ... F_{n+r}) \)-part. Differentiating (36) and proceeding as in (31), we get

\[
q^i_j = 0, \\
f^a - g^a = u^a_b (f^a + g^a)
\]

(37)

for a skew Hermitian matrix \((u^a_b) = u = -u^*\).

Set \( \vec{f} = (f^{n+1}, f^{n+2}, ... f^{n+r})^t, \vec{g} = (g^{n+1}, g^{n+2}, ... g^{n+r})^t \), and write (37) in matrix form

\[
(I - u) \vec{f} = (I + u) \vec{g}.
\]
When \( f \) and \( g \) are sufficiently \( C^2 \) close to \( f_0 \), \( u \) is small and \( I - u \) would be invertible.

Thus we may write

\[
\vec{f} = (I - u)^{-1} (I + u) \vec{g}.
\]

But

\[
(I - u)^{-1} (I + u)((I - u)^{-1} (I + u))^* = (I - u)^{-1} (I + u)(I - u) (I + u)^{-1}
\]
\[
= (I - u)^{-1} (I - u)(I + u) (I + u)^{-1}
\]
\[
= I.
\]

Thus we may modify \( (F_{n+1}, F_{n+2}, ... F_{n+r}) \) part of the section \( s_f \) by unitary matrices close to identity to have

\[
\delta \pi^a_i = 0.
\]

\( \delta \pi \) now becomes

\[
\delta \pi = \begin{pmatrix}
\delta \pi^0_0 & * & * & \delta \pi^{0}_{m+1} \\
0 & \delta \pi^0_0 \delta_{ij} & 0 & *
\end{pmatrix}.
\]

The rest of the computation proceeds the same as in the proof of Theorem 4.11 and we conclude for this section \( s_f \),

\[
\delta \pi = \alpha - \beta = 0.
\]

By the fundamental theorem [Gr, p780], the sections \( s_f \) and \( s_g \) are congruent by an element of \( SU(m+1,1) \), and hence \( f \) and \( g \) are congruent by an automorphism of \( \Sigma^m \).

\( \square \)

5 Whitney submanifold

The main result of this section is the local characterization that every nonlinear CR-flat submanifold \( M^n \subset \Sigma^{2n}, n \geq 2 \), is a part of a Whitney submanifold. Whitney
submanifold is an example of a CR submanifold which is not 1-rigid. The result of this section in fact implies it is CR deformable in exactly 1 direction, see the end of this section.

Let $V^{n+2} = \mathbb{C}^{n+1,1}$ be the complex vector space with coordinates $\xi = (\xi^0, \xi^i, \xi^{n+1}), \ 1 \leq i \leq n$, and a Hermitian scalar product

\[ Q_n(\xi, \bar{\xi}) = \xi^i \bar{\xi}^i + i (\xi^0 \bar{\xi}^{n+1} - \xi^{n+1} \bar{\xi}^0). \]

Let $\Sigma^n \simeq S^{2n+1}$ be the set of equivalence classes up to scale of null vectors.

Let $\mu = (\mu^0, \mu^i, \mu^{n+i}, \mu^{2n+1}), \ 1 \leq i \leq n$, be the coordinates of $V^{2n+2}$. Whitney submanifold $\Gamma_n : \Sigma^n \to \Sigma^{2n}$ is an immersion induced by the quadratic map $\hat{\Gamma}_n : V^{n+2} \to V^{2n+2}$ defined as

\[ \begin{align*}
\mu^0 &= 2 \xi^0 \xi^{n+1}, \\
\mu^{2n+1} &= (\xi^{n+1})^2 - (\xi^0)^2, \\
\mu^i &= \xi^i (i \xi^0 + \xi^{n+1}), \\
\mu^{n+i} &= \xi^i (-i \xi^0 + \xi^{n+1}).
\end{align*} \]

$\hat{\Gamma}_n Q_{2n} = 2 (\xi^0 \bar{\xi}^0 + \xi^{n+1} \bar{\xi}^{n+1}) Q_n$, and the induced map $\Gamma_n$ on $\Sigma^n$ is well defined. It is easy to check $\Gamma_n$ is CR-equivalent to the boundary map $\partial W_n : S^{2n+1} \to S^{4n+1}$ of the following Whitney map $W_n : \mathbb{B}^{n+1} \to \mathbb{B}^{2n+1}$, where $\mathbb{B}^* \subset \mathbb{C}^*$ is the unit ball and $(z^0, z^i), \ 1 \leq i \leq n$, is a coordinate of $\mathbb{C}^{n+1}$.

\[ W_n(z^0, z^i) = ( (z^0)^2, z^0 z^i, z^i). \quad (38) \]

This equivalence is via the isomorphism $\Sigma^n \simeq S^{2n+1}$ given in coordinates

\[ \begin{align*}
z^0 &= \frac{i \xi^0 + \xi^{n+1}}{-i \xi^0 + \xi^{n+1}}, \\
z^i &= \frac{\sqrt{2} \xi^i}{-i \xi^0 + \xi^{n+1}}.
\end{align*} \]
Set $\Sigma_0^n = \{ [\xi] \in \Sigma^n | \xi^i = 0, \forall i \}$ and $\Sigma_s^n = \{ [\xi] \in \Sigma^n | i\xi^0 + \xi^{n+1} = 0 \}$. Then $\Gamma_n$ is an immersion which is 1 to 1 on $\Sigma^n - \Sigma_0^n$, 2 to 1 on $\Sigma_0^n$, and the second fundamental form vanishes along $\Sigma_s^n$.

**Theorem 5.1** Let $M^n \hookrightarrow \Sigma^{2n}$ be a $C^3$ nonlinear CR-flat submanifold of CR dimension and codimension $n \geq 2$. Then $M$ is congruent to a part of the Whitney submanifold up to an automorphism of $\Sigma^{2n}$.

CR-flat submanifold $M^1 \hookrightarrow \Sigma^2$ has been classified by Faran [Fa]. In contrast to $n \geq 2$ cases, there are four inequivalent CR flat submanifolds when $n = 1$.

As a corollary, we have a simple characterization of the proper holomorphic maps from a unit ball $\mathbb{B}^{n+1}$ to $\mathbb{B}^{2n+1}$ [HJ].

**Corollary 5.1** Let $F : \mathbb{B}^{n+1} \to \mathbb{B}^{2n+1}, n \geq 2$, be a nonlinear proper holomorphic map which is $C^{n+1}$ up to the boundary. Then $F$ is equivalent to the Whitney map [38] up to automorphisms of the unit balls.

**Theorem 5.1** is based on the following algebraic lemma due to Iwatani on the normal form of the second fundamental form of a Bochner-Kähler submanifold [Iw][Br3]. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^n$ with the standard Hermitian scalar product. Let $\{ z^i \}$ be a unitary $(1,0)$-basis for $V^*$, and let $\{ w_a \}$ be a unitary basis for $W$.

**Lemma 5.1** [Iw] Suppose $H = h_{ij}^a z^i z^j \otimes w_a \in S^{2,0}(V^*) \otimes W$ satisfies

$$\gamma(H, H) = h_{ij}^a h_{kl}^b z^i z^j \otimes z^k z^l \in S_1^{1,1} \subset S^{2,2},$$

or simply $\gamma(H, H)$ is Bochner-flat [Br3]. Then up to a unitary transformation on $V$,

$$H = h_{in}^a z^i z^n \otimes w_a.$$ 

Set $\nu_i = h_{im}^a w_a \in W$. A computation shows $\gamma(H, H)$ is Bochner-flat whenever

$$\langle \nu_i, \nu_j \rangle = 0 \quad \text{for} \ i \neq j,$$

$$\langle \nu_n, \nu_n \rangle = 4 < \nu_q, \nu_q > \quad \text{for} \ q < n.$$
Thus up to a unitary transformation on $W$, we may assume

$$\nu_q = r w_q \quad \text{for } q < n,$$

$$\nu_n = 2 r w_n,$$

for some $r \geq 0$.

Let $M^n \hookrightarrow \Sigma^{2n}$ be a CR-flat submanifold. The second fundamental form of $M$ is Bochner-flat [EHZ]. In the notation of Section 3, Lemma 5.1 then implies we may write

$$\pi^{n+i}_q \equiv r \delta_{iq} \omega^n \mod \theta, \quad \text{for } q < n,$$

$$\pi^{n+i}_n \equiv r(1 + \delta_{in}) \omega^i \mod \theta,$$

where $\theta$ is the dual to the contact hyperplane fields. Assume $M$ is not linear, $H \neq 0$, and we scale $r = 1$ using the group action by $\text{Re} \pi^0$. We thus obtain the following local normal form of the second fundamental form of a nonlinear CR-flat submanifold $M^n \hookrightarrow \Sigma^{2n}$.

$$\pi^{n+i}_q = \delta_{iq} \omega^n + h^i_q \theta \quad \text{for } q < n,$$

$$\pi^{n+i}_n = (1 + \delta_{in}) \omega^i + h^i_n \theta$$

for some coefficients $h^i_j$.

**Theorem 5.1** is now obtained by successive differentiation of this normalized structure equation. We assume $n \geq 3$ for simplicity for the rest of this section, as $n = 2$ case can be treated in a similar way. We shall agree on the index range $1 \leq p, q, s, t \leq n-1$, and denote $p' = n+p$, $n' = n+n$. Recall our convention $-d\theta \equiv i \pi^k \wedge \bar{\pi}^k \equiv i \varpi \mod \theta$, and we denote $\pi^i = \omega^i$ for the sake of notation.

**Step 1.** Differentiating $\pi^{n+n}_s = h^n_s \theta \mod \theta$, we get

$$i h^n_s \varpi \equiv (\pi^{n'}_{s'} - 2\pi^n_s) \wedge \omega^n + \pi^{n'}_s \wedge (-i \bar{\omega}^s) \mod \theta.$$
Since \( n - 1 \geq 2 \), this implies \( h^s_n = 0 \), and by Cartan’s lemma

\[
\begin{pmatrix}
\pi^t_{s'} - 2\pi^s_n \\
-\pi^t_{s} \\
\pi^t_{2n+1}
\end{pmatrix} \equiv
\begin{pmatrix}
c_s & u \\
u & 0
\end{pmatrix}
\begin{pmatrix}
\omega^n \\
-i\bar{\omega}^s
\end{pmatrix} \mod \theta
\]

for coefficients \( c_s, u \).

**Step 2.** Differentiating \( \pi^t_{s'} = h^t_s \theta \mod \theta \) for \( t \neq s \), we get

\[
i h^t_s \varpi \equiv (\pi^t_{s'} - \pi^t_s) \wedge \omega^n - \pi^t_{s} \wedge \omega^t + \pi^t_{2n+1} \wedge (i\bar{\omega}^s) \mod \theta.
\]

Since \( n - 1 \geq 2 \), this implies \( h^t_s = 0 \) for \( t \neq s \), and by Cartan’s lemma

\[
\begin{pmatrix}
\pi^t_{s'} - \pi^t_s \\
-\pi^t_{s} \\
\pi^t_{2n+1}
\end{pmatrix} \equiv
\begin{pmatrix}
0 & b_s & -i\bar{b}_t \\
b_s & 0 & a \\
-i\bar{b}_t & a & 0
\end{pmatrix}
\begin{pmatrix}
\omega^n \\
\omega^t \\
-i\bar{\omega}^s
\end{pmatrix} \mod \theta
\]

for coefficients \( c_s, u \). Since \( \pi^t_{s'} - \pi^t_s \) is skew Hermitian, it cannot have any \( \omega^n \)-term.

**Step 3.** Differentiating \( \pi^t_{s'} = \omega^n + i h^t_s \theta \mod \theta \) and collecting terms, we get

\[
h^t_s \varpi \equiv (\pi^t_{s'} - \pi^t_s + \pi^0_0 - \pi^0_n) \wedge \omega^n + (b_p \omega^n - i a \bar{\omega}^p) \wedge \omega^p + (-b_t \omega^t + \bar{b}_t \bar{\omega}^t) \wedge \omega^n \mod \theta.
\]

Since \( n - 1 \geq 2 \), this implies \( h^t_s = a \), and

\[
\Delta_t = \pi^t_{s'} - \pi^t_s + \pi^0_0 - \pi^0_n
= a_t \omega^n - i a \bar{\omega}^n + (b_t \omega^t - \bar{b}_t \bar{\omega}^t) + \sum_p b_p \omega^p - A_t \theta
\]

for coefficients \( a_t, A_t \).

**Step 4.** From **Step 2**, we may use the group action by \( \pi^0_{2n+1} \) to translate \( a = 0 \), which we assume from no on. We also translate \( h^t_n = 0 \) by \( \pi^t_{2n+1} \). Differentiating \( \pi^t_{s'} = \omega^t \mod \theta \) with these relations, we get

\[
0 \equiv \bar{b}_t \left( \sum_p \omega^p \wedge \bar{\omega}^p \right) + (2\bar{c}_t - 3\bar{b}_t) \omega^n \wedge \bar{\omega}^n + \omega^n \wedge (a_t + 2i \bar{u}) \wedge \omega^t \mod \theta.
\]

Thus \( b_t = c_t = 0, \ a_t = -2i \bar{u} \).
Step 5. Differentiating $\pi^n' = 2\omega^n + h^n \theta \mod \theta$ and collecting terms, we get $A_t = A$ for a single variable, and

$$i h^n \tilde{\omega} \equiv 2(\pi^n' - \pi^n + \pi^n_0 - \pi^n) \land \omega^n + i u \omega^p \land \bar{\omega}^p - i u \omega^n \land \bar{\omega}^n \mod \theta.$$  

This implies $h^n = u$, and

$$\Delta_n = \pi^n' - \pi^n + \pi^n_0 - \pi^n = a_n \omega^n - i u \bar{\omega}^n - A_n \theta$$

for coefficients $a_n, A_n$. But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with Step 3, $a_n = -3 i \bar{u}$.

Step 6. Now by considering $\theta$-terms in Step 1, 2, 3, 4, 5 and the fact $\pi^n_i \land \omega^i + \pi^n_{2n+1} \land \theta = 0$, we obtain the following simple structure equations. We omit the details of computations.

\[
\begin{pmatrix}
\pi^n_s' - 2\pi^n_s \\
\pi^n_{2n+1}' \\
\pi^n_i' - \pi^n_i \\
-\pi^n_i \\
\pi^n_{2n+1}'
\end{pmatrix} = \begin{pmatrix}
0 & u & 0 \\
u & 0 & -i \bar{\omega}^s \\
\end{pmatrix} \begin{pmatrix}
\omega^n \\
-\bar{\omega}^s \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
\pi^n_s' - \pi^n_s \\
-\pi^n_i \\
\pi^n_{2n+1}'
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]

$$\pi^n_{2n+1} = (A - i u \bar{u}) \omega^n + B_t \theta$$

$$\pi^n_{2n+1} = A \omega^n + B_n \theta$$

Step 7. Differentiating $\pi^n_s' - \pi^n_s = 0, \pi^n_s = 0$, we get first $B_s = 0, B = 0$, and

$$A - \bar{A} = i(u \bar{u} - 1). \quad (40)$$

Step 8. Differentiating $\pi^n_n' = 2\omega^n + u \theta$ and $\pi^n_{2n+1}' = u \omega^n$,

$$du = u(-\pi^n_n + \pi^n_{2n+1}) + 2(A - A_n) \omega^n + i u (3 \bar{u} \omega^n + u \bar{\omega}^n) + u (2A - A_n) \theta. \quad (41)$$

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**Step 9.** Differentiating $\pi_n' = -iu \bar{\omega}^s$ using (41) and collecting terms in $\theta \wedge \bar{\omega}^s$,

$$A_n = 2A - \bar{A}.$$ 

**Step 10.** Differentiating $\pi_{2n+1}^t = (\bar{A} - i) \omega^t$, $\pi_{2n+1}^n = A \omega^n$, we get

$$dA = \pi_{2n+1}^0 + (A + i) (\pi_{2n+1}^{2n+1} - \pi_0^0) - (A + i)^2 \theta.$$  (42)

We normalize $A = i\alpha$ for a real number $\alpha$ using group action by $\pi_{2n+1}^0$. Since $\pi_i^i + \bar{\pi}_i^i = 0$, $\Delta_t + \bar{\Delta}_t = \pi_0^0 + \bar{\pi}_0^0$ and (42) is now reduced to

$$d\alpha = 2i(\alpha + 1) (\bar{u} \omega^n - u \bar{\omega}^n)$$  (43)

$$\pi_{2n+1}^0 = - (\alpha + 1)^2 \theta.$$  

When $u \neq 0$, we may also rotate $u$ to be a positive number, in which case it is determined by (40)

$$2\alpha + 1 = u \bar{u}.$$  (44)

At this stage, note that the only independent coefficients in the structure equations are $\alpha$, $u$, and that the expression for their derivatives does not involve any new variables. The structure equations for the CR-flat submanifold $M^n \subset \Sigma^{2n}$ thus close up as follows.

\[
\begin{pmatrix}
\pi_q' & \pi_p' \\
\pi_q^n' & \pi_n^n'
\end{pmatrix} = \begin{pmatrix}
\delta_{pq} \omega^n & \omega^p \\
0 & 2 \omega^n + u \theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\pi^n_s' - 2\pi^n_s \\
\pi_{2n+1}'
\end{pmatrix} = \begin{pmatrix}
0 & u \\
u & 0
\end{pmatrix} \begin{pmatrix}
\omega^n \\
-i \bar{\omega}^s
\end{pmatrix}
\]

\[
\begin{pmatrix}
\pi_t^n' - \pi_t^n \\
-\pi_s^n \\
\pi_{2n+1}'
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
\[ \pi_{2n+1}^t = (A - i u \bar{u}) \omega^t \]
\[ \pi_{2n+1}^n = A \omega^n \]
\[ \Delta_t = -2 i \bar{u} \omega^n - A \theta \]
\[ \Delta_n = -3 i \bar{u} \omega^n - i u \bar{\omega}^n - A_n \theta \]
\[ du = u \left( -\pi_n^n + \pi_{2n+1}^{2n+1} \right) + i u \left( 3 \bar{u} \omega^n + u \bar{\omega}^n \right) \]
\[ + 2 \left( A - A_n \right) \omega^n + u \left( 2 A - A_n \right) \theta \]
\[ d\alpha = 2 i (\alpha + 1) \left( \bar{u} \omega^n - u \bar{\omega}^n \right) \]
\[ \pi^{0}_{2n+1} = - (\alpha + 1)^2 \theta \]

where \( A = i \alpha, A_n = 3 i \alpha, \) and \( \alpha, u \) satisfy the relation (41). Moreover, a long but direct computation shows that these structure equations are compatible, that is \( d^2 = 0 \) is a formal identity of the structure equations.

Remark. The structure equation closes up at order 3 [Ha]. Since both \( \Sigma^n \) and \( \Sigma^{2n} \) are real analytic, this implies a \( C^3 \) nonlinear CR-flat submanifold \( f : \Sigma^n \hookrightarrow \Sigma^{2n} \) is real analytic.

Let \( \Sigma^* = \Sigma^n - \Sigma^n_s \), and note that it is a connected set. Note also the structure equation (45) implies that the set of points where \( u = 0 \) or equivalently \( \alpha = -\frac{1}{2} \) cannot have any interior on \( \Sigma^* \). We claim the invariant \( \alpha \) takes any value \( > -\frac{1}{2} \) on \( \Sigma^* \).

Suppose \( \alpha_+ = \sup_{\Sigma^*} \alpha > -\frac{1}{2} \) is finite. Applying the existence part of Cartan’s generalization of Lie’s third fundamental theorem on closed structure equations, [Br3], there exists for any \( p_0 \in \Sigma^* \) a neighborhood \( U \subset \Sigma^* \) and a CR immersion \( g : U \to \Sigma^{2n} \) with invariant \( \alpha|_{p_0} = \alpha_+ \), hence necessarily \( u_{p_0} = \sqrt{2 \alpha_0 + 1} \). From (40), \( d\alpha|_{p_0} \neq 0 \) and let \( p_- \in U \) be a point with \(-\frac{1}{2} < \alpha|_{p_-} < \alpha_+ \). Then by uniqueness part of Cartan’s theorem, there exists a neighborhood \( U' \subset U \) of \( p_- \) on which \( g \) agrees with the Whitney map \( \Gamma_n \) up to automorphisms of \( \Sigma^n \) and \( \Sigma^{2n} \). Since \( g \) and \( \Gamma_n \) satisfy the closed set of structure equations, they are real analytic. Thus \( g \) is a part of \( \Gamma_n \). But \( du|_{p_0} \neq 0 \),
and there exists a point $p_+ \in U \subset \Sigma^*$ such that $\alpha|_{p_+} > \alpha_+$, a contradiction. By similar argument, $\inf_{\Sigma^*} \alpha = -\frac{1}{2}$, and the claim follows for $\Sigma^*$ is connected.

Proof of Theorem 5.1. Since the set of points $\alpha = -\frac{1}{2}$ cannot have any interior, let $p \in M$ be a point with $\alpha|_p > -\frac{1}{2}$. Then from the results above, there exists a point $q \in \Sigma^n$ such that $\alpha|_q = \alpha|_p$. By similar argument as above after identifying $p \in M$ with $q \in \Sigma^n$, there exists an automorphisms $\tau_{2n}$ of $\Sigma^{2n}$ such that $f = \tau_{2n} \circ \Gamma_n$ on a neighborhood $U$ of $p$. The theorem follows for both $f$ and $\Gamma_n$ are real analytic. □

Proof of Corollary 5.1. By the regularity theorem [Mi], $F$ is real analytic up to $\partial F$. Since the CR structure on $S^{2n+1} = \Sigma^n$ is definite, the set of points where $\partial F$ has holomorphic rank $n$ is a dense open subset. Since $F$ is not linear, there exists a point $p \in \Sigma^n$ where the second fundamental form does not vanish either. By Theorem 5.1, $\partial F$ agrees with the Whitney map $\Gamma_n$ in a neighborhood of $p$ up to automorphisms of the unit balls. The real analyticity then implies $\partial F = \Gamma_n$ on $\Sigma^n$, and hence $F = W_n$ on $B^{n+1}$. □

We may apply Cartan’s generalization of Lie’s third fundamental theorem and show that the Whitney submanifold provides an example of a deformable CR-submanifold, and in particular that it is not 1-rigid. Take a point $p \in \Sigma^n$ and an analytic one parameter family of real numbers $\alpha_t > -\frac{1}{2}$, and set $u_t = \sqrt{2 \alpha_t + 1}$. Then by the existence part of Cartan’s theorem, there exists a neighborhood $U$ of $p$ and a one parameter family of CR immersions $f_t : U \rightarrow \Sigma^{2n}$ with the induced structure equations (45) such that the invariants $\alpha, u$ have the prescribed values $\alpha_t, u_t$ at $p$. Of course this deformation is tangential and does not actually deform the submanifold. It is due to an intrinsic CR symmetry of $\Sigma^n$ that cannot be extended to a symmetry of the ambient $\Sigma^{2n}$ along the Whitney submanifold.
6 Proof of Example 3.1

Example 3.1 Let $M_{n,p} \hookrightarrow \mathbb{C}P^N = P(\bigwedge^n \mathbb{C}^{n+p})$ be the Plücker embedding of the Grassmannian $Gr(n, \mathbb{C}^{n+p})$, $p \leq n$. Let $\hat{M}_{n,p} \hookrightarrow S^{2N+1}$ be the inverse image of $M_{n,p}$ under Hopf map, which is an $S^1$-bundle over $M_{n,p}$. When $p = 2, 3$, $\hat{M}_{n,p}$ is a Bochner rigid CR submanifold of $S^{2N+1}$.

It is easy to check that the canonical $S^1$-bundle $\hat{M} \hookrightarrow S^{2N+1}$ over any complex submanifold $M \hookrightarrow \mathbb{C}P^N$ is a CR submanifold. Moreover, the fundamental forms of $\hat{M}$ as a CR submanifold are simply the pull back, in an appropriate sense, of the usual fundamental forms of $M$ as a projective subvariety [HY]. It thus suffices to show that the fundamental forms of the Plücker embeddings of the complex Grassmannian manifolds are Bochner rigid when $p = 2, 3$. Note the height $\tau$ of $\hat{M}_{n,p}$ is $p$.

Let $V$ denote the tangent space of $M_{n,p}$, and we follow the notations of Section 3.

Case $p = 2$. There exists a unitary basis $\{x^i, y^i\}, i = 1, \ldots, n$, of $V^*$ such that the second fundamental form is given by $\frac{n(n-1)}{2}$ quadratic forms

$$F^{ij} = x^i y^j - x^j y^i, \quad i < j.$$ 

Let $P^{ij} \in S^{2,0}(V^*)$, $i < j$, be such that

$$F^{ij} P^{ij} + P^{ij} F^{ij} = (x^i \bar{x}^i + y^i \bar{y}^i) Q$$

for some $Q \in S^{1,1}(V^*)$. Since $F^{ij}$ has only $x, y$ terms, $Q$ cannot have $x \bar{x}$ terms nor $y \bar{y}$ terms, and thus $Q$ only has $x \bar{y}$, $y \bar{x}$ terms. Consider $x^1 \bar{x}^1 Q$ term. Since it contains three $x$’s, the only possible contribution comes from $x^1 y^j \bar{P}^{1j}_{xx} + P^{1j}_{xx} x^1 \bar{y}^j$, where $P^{1j}_{xx}$ denotes the $xx$ component of $P^{1j}$. But

$$\frac{\partial}{\partial y^1} (x^1 y^j \bar{P}^{1j}_{xx} + P^{1j}_{xx} x^1 \bar{y}^j) = 0.$$ 

Hence $\frac{\partial}{\partial y^1} Q = 0$. By permutation symmetry in $x, y$, and in $i$, and since $Q = \bar{Q}$, $Q = 0$. 

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Case \( p = 3 \). There exists a unitary basis \( \{ x^i, y^i, z^i \}, i = 1, \ldots, n \), of \( V^* \) such that the second fundamental form is given by the quadratic forms

\[
F_{ij}^1 = y^i z^j - y^j z^i, \quad i < j \\
F_{ij}^2 = z^i x^j - z^j x^i, \quad i < j \\
F_{ij}^3 = x^i y^j - x^j y^i, \quad i < j.
\]

Let \( P_{ij}^1, P_{ij}^2, P_{ij}^3 \in S^{2,0}(V^*), i < j, \) be such that

\[
F_{ij}^a \bar{P}_{ij}^a + P_{ij}^a \bar{F}_{ij}^a = (x^i \bar{x}^i + y^i \bar{y}^i + z^i \bar{z}^i) Q
\]

for some \( Q \in S^{1,1}(V^*) \). Consider this equation mod \( x, y, \) and \( z \) in turn. Since \( Q \) is quadratic, the result for the case \( p = 2 \) implies \( Q = 0 \).

The third fundamental form is given by the cubic forms

\[
F_{ijk} = x^i y^j z^k + x^j y^k z^i + x^k y^i z^j - x^i y^j z^k - x^j y^i z^k - x^k y^j z^i, \quad i < j < k.
\]

Let \( P_{ijk} \in S^{3,0}(V^*), i < j < k, \) be such that

\[
F_{ijk} P_{ijk} + P_{ijk} F_{ijk} = (x^i \bar{x}^i + y^i \bar{y}^i + z^i \bar{z}^i) Q
\]

for some \( Q \in S^{2,2}(V^*) \). Considering this equation mod \( x, y, \) and \( z \) in turn, every monomials of \( Q \) is either a \( xxyz, yxyz, \) or \( zxyz \) term (ignoring the conjugation). Consider \( xxyz \) component \( Q_{xxyz} \) and the \( x^1 \bar{x}^1 Q_{xxyz} \) term in the above equation. Since there are 4 \( x \)-terms, the only possible contribution comes from \( x^1 y^j z^k P_{xxx}^{1jk} + P_{xxx}^{1jk} x^1 \bar{y}^j \bar{z}^k \), where \( P_{xxx}^{1jk} \) denotes the \( xxx \) component of \( P_{ijk}^{1jk} \). But

\[
\frac{\partial}{\partial y^1} (x^1 y^j z^k P_{xxx}^{1jk} + P_{xxx}^{1jk} x^1 \bar{y}^j \bar{z}^k) = 0.
\]

Hence \( \frac{\partial}{\partial y^1} Q_{xxyz} = 0 \). By permutation symmetry in \( x, y, z, \) and in \( i \), and since \( Q = \bar{Q} \), \( Q = 0 \).
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