A tale of two probability distributions

Reply to Comment on "Troublesome aspects of the Renyi-MaxEnt treatment by Thomas Oikonomou and G. Baris Bagci

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Abstract
This Reply is intended as a refutation of the preceding Comment [Oikonomou and Bagci, Phys. Rev. E 96, 056101 (2017)] on our paper [Plastino et al., Phys. Rev. E 94, 012145 (2016)]. We show that the Tsallis probability distribution of our paper does not coincide with the Tsallis distribution studied by Oikonomou and Bagci. Consequently, their findings do not apply to our paper.

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1 Introduction

The authors, OB for short, are here criticizing our PRE paper cited in their reference [1]. Let us start with three statements regarding the essence of the
A1: It is shown in [1] that the MaxEnt variational approach used in conjunction with Renyi’s entropy leads to inconsistencies.

A2: These inconsistencies are due to a hidden relation between the concomitant Legendre multipliers ($\lambda_1$ and $\lambda_2$) discovered while dissecting the variational process that leads to the appropriate MaxEnt probability distribution.

A3: Renyi’s entropy is not trace form, while Tsallis’ one is of such nature. Thus, we can expect differences to arise in a MaxEnt treatment.

B1: OB do not question the first two above points. Their claim is that they apply also to Tsallis’ entropy. More precisely, they purport to discover that a hidden relation emerges in the Tsallis’ MaxEnt treatment as well.

2 The two-distributions’ problem

OB work with the pseudo-Tsallis probability distribution (PD)

$$P_{OB} = Z^{-1} [1 + (1 - q)\beta Z^{q-1}(U - <U>)]^{\frac{1}{q-1}}$$

$$Z = \left( \int P_{OB}^q d\mu \right)^{\frac{1}{q-1}},$$

which does not coincide with Tsallis’ pioneer PD of 1988 [2], which reads

$$P_T = \frac{[1 + \beta(1 - q)U]^{\frac{1}{q-1}}}{Z_T},$$

$$Z_T = \int [1 + \beta(1 - q)U]^{\frac{1}{q-1}} d\mu.$$

Thus, we can rephrase B1 as stating:

B1: OB purport to discover that a hidden relation emerges in the Tsallis’ MaxEnt treatment for their PD $P_{OB}$, which is not Tsallis’ PD. We could finish our Reply right here. However, let us delve deeper into the issue in order to
gain some insight into why OB get their peculiar PD distribution \([1]\). Such is the subject of next Section.

3 Boltzmann-Gibbs à la Oikonomou-Bagci

3.1 Normal procedure

In order to better illustrate the OB procedure, we apply it here to the Boltzmann-Gibbs (BG) exponential distribution. One maximizes in such an instance

$$F_{SB}(P) = -\int P \ln P d\mu + \lambda_1 \left( \int P U d\mu - < U > \right) + \lambda_2 \left( \int P d\mu - 1 \right).$$

(5)

The first variation becomes

$$F_{SB}(P + h) - F_{SB}(P) = -\int \left( \ln P - \lambda_1 U - \lambda_2 + 1 \right) h d\mu + O(h^2).$$

(6)

Accordingly,

$$\ln P - \lambda_1 U - \lambda_2 + 1 = 0.$$  

(7)

Here, as most people do, but not OB, one immediately deduces \( P \) and is immediately led to

$$P = e^{\lambda_1 U} e^{\lambda_2 - 1}.$$  

(8)

\( P \)–normalization entails

$$e^{\lambda_2 - 1} \int e^{\lambda_1 U} d\mu = 1,$$

(9)

and then

$$e^{\lambda_2 - 1} = \frac{1}{\int e^{\lambda_1 U} d\mu}.$$  

(10)

In other words,

$$P = \frac{e^{\lambda_1 U}}{\int e^{\lambda_1 U} d\mu}.$$  

(11)

Further, one finds

$$< U > = \frac{\int U e^{\lambda_1 U} d\mu}{\int e^{\lambda_1 U} d\mu}.$$  

(12)

Well known physical arguments, as shown first by Gibbs himself [3], allow one to identify \( \lambda_1 \)

$$\lambda_1 = -\beta = -\frac{1}{kT}.$$  

(13)
3.2 OB procedure

Starting with Eq. (7), OB follow a different trajectory so as to ascertain which is proper $P$. They first multiply (7) by $P$ and integrate, finding

$$
\int P \ln P d\mu - \lambda_1 < U > - \lambda_2 + 1 = 0, \tag{14}
$$

so that

$$
\lambda_2 = \int P \ln P d\mu + \beta < U > + 1, \tag{15}
$$

which OB would call a hidden relation between $\lambda_2$ and $\beta$. Replacing (15) into (7) OB obtain

$$
\ln P + \beta (U - < U >) - \int P \ln P d\mu = 0, \tag{16}
$$

or

$$
P = e^{-\beta(U-<U>)+\int P \ln P d\mu}. \tag{17}
$$

Integrating once again OB are led to

$$
e^{\int P \ln P d\mu} \int e^{-\beta(U-<U>)} d\mu = 1. \tag{18}
$$

This is a critical stage. OB choose to write $Z$ not as

$$
Z^{-1} = e^{\int P \ln P d\mu+\beta<U>}, \tag{19}
$$

but as

$$
Z_{OB}^{-1} = e^{\int P_{OB} \ln P_{OB} d\mu}, \tag{20}
$$

leading to

$$
P_{OB} = \frac{e^{\beta(U-<U>)}}{Z_{OB}}, \tag{21}
$$

and then it follows that

$$
S = \ln Z_{OB}, \tag{22}
$$

which is obviously an incorrect result. This happens because of the OB-choice (20). With the selection (19) they would have reached instead

$$
P = \frac{e^{-\beta U}}{Z}, \tag{23}
$$
so that
\[ e^{-\int P \ln P d\mu} = Z e^{\beta <U>}, \]  
(24)
and
\[ S = \ln Z + \beta <U>, \]  
(25)
the correct result. We have clearly identified, with reference to the BG
distribution, the origin of OB’s troubles.

4 Tsallis’ PD

4.1 Normal procedure

The first variation’s equation is [1]
\[ \frac{q}{1-q} P^{q-1} + \lambda_1 U + \lambda_2 = 0 \]  
(26)
\[ \lambda_1 = -\beta q Z_q^{1-q} \quad \lambda_2 = \frac{q}{q-1} Z_q^{1-q}, \]  
(27)
leading to
\[ P = \frac{[1 + (1-q)\beta U]^{1/q}}{Z_q}, \]  
(28)
and, for \( S_q \),
\[ S_q = \ln_q Z_q + Z_q^{1-q} \beta <U>, \]  
(29)
the correct result.

4.2 Tsallis’ PD à la Oikonomou-Bagci

OB multiply [20] by \( P \) and integrate:
\[ e^{-\int P d\mu} + \lambda_1 <U> + \lambda_2 = 0, \]  
(30)
which they call a hidden relation between two Lagrange multipliers entering
the MaxEnt treatment. This is equation (1) in the Comment, the hard core
of their present contribution. They now choose
\[ \lambda_1 = -\frac{\beta q \int P d\mu}{1 + (1-q)\beta <U>}, \]  
(31)
and obtain
\[ \lambda_2 = \frac{q}{q-1} \int P^q d\mu + \frac{\beta q \int P^q d\mu}{1+(1-q)\beta <U>}, \]  
(32)
so that
\[ P^{q-1} = \frac{\beta q \int P^q d\mu}{1+(1-q)\beta <U>}[1+(1-q)\beta U], \] 
(33)
or
\[ P = \left( \frac{\beta q \int P^q d\mu}{1+(1-q)\beta <U>} \right)^{\frac{1}{q-1}}[1+(1-q)\beta U]^{\frac{1}{q-1}}. \]  
(34)

OB are here at a critical stance. Had they selected
\[ \lambda_1 = -\beta q, \]  
(36)
they would have found for \( P \) the expression (28), the right Tsallis' result, obtained the hidden relation (30) notwithstanding. We see that, contrary to OB’s claim, the hidden relation impedes nothing. However, at this crucial stage OB chose to write
\[ \lambda_1 = -\beta q. \]  
(36)
leading to
\[ P = \left( \int P^q d\mu \right)^{\frac{1}{q-1}} \left[ 1 + \frac{1+(1-q)\beta (U-<U>)}{\int P^q d\mu} \right]^{\frac{1}{q-1}}, \] 
(37)
which is Eq. (1) above. According to the OB-choice (36) above we have now
\[ Z_q = \left( \int P^q d\mu \right)^{\frac{1}{1-q}}, \] 
(38)
and
\[ S_q = \ln_q Z_q, \]  
(39)
an incorrect result, arising because \( \lambda_1 \) was incorrectly chosen.
5 MaxEnt reciprocity relations

A word of caution. We have above used words like "choosing" or "selection". This is speaking in a rather loose fashion. In fact, MaxEnt prescribes a definite recipe in order to find the Lagrange multipliers $\lambda$. MaxEnt asserts that, if the a priori known information concerns $N$ expectation values $<A_k>$, and then $N+1$ (accounting for normalization) Lagrange multipliers $\lambda_k$, then the entropy $S$ acquires the form (MaxEnt version of $S$)

$$S = \lambda_0 + \sum_{k=1}^{N} \lambda_k <A_k> \ .$$

(40)

The $\lambda_k$'s are obtained via so-called reciprocity relations (see, for example, [4])

$$\lambda_k = \frac{\partial S}{\partial <A_k>} .$$

(41)

In practice, however, instead of solving equations (41) one often makes educated guesses for the $\lambda_k$'s, as reported above in this paper.

How to make such an educated guess in the Tsallis’ instance? A main criterion is to choose Tsallis’ $\lambda_1$ in such a manner that, in the limit $q \to 1$, it should coincide with the Boltzmann-Gibbs’ $\lambda_1$. In that case, from such correct $\lambda_1$ one immediately derives a Tsallis’ $\lambda_2$, that yields then the usual Tsallis’ distribution. This $\lambda_1$-criterion is satisfied by both OB’s guess (31) and Tsallis’ one (27). One can appeal then to Ockham’s razor to select (27). A word of caution seems appropriate. Guessing is an art, not science. If inspiration fails in guessing the Lagrange parameter, one can always appeal to Eq. (41), that never fails.

6 Conclusion

The Comment to which we have replied here serves the useful purpose of highlighting issues related to Tsallis’ statistics, but does not invalidate our paper [1]. OB are not using Tsallis’ PD distribution but one concocted by them. The Comment’s main result is its Eq. (1), which we showed does not impede one to arrive at the correct Tsallis’ PD. The Comment’s error lies in a not judicious choice of Lagrange multipliers. This invalidates its conclusions, fruits of the poisoned tree.
References

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