Decomposition of enzyme kinetics equations

V Sobolev

1Samara National Research University, Moskovskoye Shosse 34, Samara, Russia, 443086
e-mail: hsablem@gmail.com

Abstract. The problem of the splitting of singularly perturbed differential systems is invesigated and the application to the problem of suicide substrate kinetics is given.

1. Introduction
The object of our investigation is the following differential system:

\[ \dot{x} = f(x, y, t, \varepsilon), \quad \varepsilon \dot{y} = g(x, y, t, \varepsilon), \]

where \( x \in R^m, \ y \in R^n, \ t \in R, \) and \( \varepsilon \) is a small positive parameter.

The paper is devoted to the construction of a transformation reducing (1)-(2) to the system

\[ \dot{v} = \varphi(v, t, \varepsilon), \quad \varepsilon \dot{z} = \eta(v, z, t, \varepsilon). \]

This approach was firstly suggested in [6] and it has been used to solve different control problems [2, 3, 5, 8]. The splitting transformation uses functions that describe fast and slow integral manifolds and the construction of a splitting transformation is not a simple task in the general case. Some classes of singularly perurbed systems for which this splitting transformation can be effectively constructed are described in the paper, see also [7].

2. Slow integral manifold
Suppose that system (1), (2) agrees with the following hypotheses.

I. Equation \( g(x, y, t, 0) = 0 \) has an isolated solution \( y = h_0(x, t) \) for \( t \in R, \ x \in R^m \).

II. The functions \( f, g, \) and \( h_0 \) are sufficiently smooth in the \( \rho \)-neighbourhood of \( h_0(x, t) \) for \( 0 \leq \varepsilon \leq \varepsilon_0 \) and some positive \( \rho \) and \( \varepsilon_0 \).

III. All eigenvalues \( \lambda_i(x, t) \) \( (i = 1, \ldots, n) \) of the matrix \( B = B(x, t) \),

\[ B = \frac{\partial g}{\partial y}(x, h_0(x, t), t, 0), \]

have negative real parts, i.e.

\[ Re \lambda_i(x, t) \leq -2\gamma < 0. \]

Then (1), (2) possesses the slow integral manifold \( y = h(x, t, \varepsilon) \) [6] the flow on which is described by the system

\[ \dot{x} = f(x, h(x, t, \varepsilon), t, \varepsilon). \]
Note that an exact calculations of $h$ are impossible in the general case. Usually, the asymptotic expansions

$$h(x, t, \varepsilon) = h_0(x, t) + \varepsilon h_1(x, t) + \varepsilon^2 h_2(x, t) + \ldots$$

are used.

3. Fast integral manifold

Setting $z = y - h(x, t, \varepsilon)$ we consider the differential system in some neighbourhood of the slow integral manifold. Additionally, let us introduce a new variable $v$, which satisfies the differential equation on the slow integral manifold

$$\dot{v} = f(v, h(v, t, \varepsilon), t, \varepsilon),$$

and else one new variable $w = x - v$. For $z$, $v$ and $w$ we obtain the differential system

$$\dot{v} = \varphi(v, t, \varepsilon),$$

$$\dot{w} = W(v, w, z, t, \varepsilon),$$

$$\varepsilon \dot{z} = B(v, t)z + Z(v, w, z, t, \varepsilon),$$

with

$$\varphi(v, t, \varepsilon) = f(v, h(v, t, \varepsilon), t, \varepsilon),$$

$$Z(v, w, z, t, \varepsilon) = g(v + w, z + h(v + w, t, \varepsilon), t, \varepsilon) - B(v, t)z - \varepsilon \frac{\partial h}{\partial t}(v + w, t, \varepsilon)$$

$$- \varepsilon \frac{\partial h}{\partial x}(v + w, t, \varepsilon)f(v + w, z + h(v + w, t, \varepsilon), t, \varepsilon),$$

$$W(v, w, z, t, \varepsilon) = f(v + w, z + h(v, w, t, \varepsilon), t, \varepsilon) - \varphi(v, t, \varepsilon).$$

Let all assumptions I–III hold, then for some $\varepsilon_1$, $0 < \varepsilon_1 \leq \varepsilon_0$ and all $\varepsilon \in (0, \varepsilon_1]$ the differential system (3)–(5) has the integral manifold (fast integral manifold) $w = \varepsilon H(v, z, t, \varepsilon)$, and the flow on this fast manifold is described by the system

$$\dot{v} = \varphi(v, t, \varepsilon),$$

$$\varepsilon \dot{z} = \eta(v, z, t, \varepsilon),$$

where

$$\eta(v, z, t, \varepsilon) = B(v, t)z + Z(v, \varepsilon H(v, z, t, \varepsilon), z, t, \varepsilon).$$

It is possible in some cases to find $H$ as an asymptotic expansion

$$H(v, z, t, \varepsilon) = H_0(v, z, t) + \varepsilon H_1(v, z, t) + \varepsilon^2 H_2(v, z, t) + \ldots$$

from the invariance equation

$$\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial v} \varphi(v, t, \varepsilon) + \frac{\partial H}{\partial z} [B(v, t)z + Z(v, \varepsilon H, z, t, \varepsilon)] = W(v, \varepsilon H, z, t, \varepsilon).$$

(6)
4. Main transformation

We use now the transformation

\[ x = v + \varepsilon H(v, z, t, \varepsilon), \]
\[ y = z + h(x, t, \varepsilon) \]

(7)
(8)

to reduce the system (1), (2) to the form

\[ \dot{v} = \varphi(v, t, \varepsilon), \]
\[ \varepsilon \dot{z} = \eta(v, z, t, \varepsilon). \]

(9)
(10)

Let \((x(t), y(t))\) be a solution to (1), (2) with an initial condition \(x(t_0) = x_0, y(t_0) = y_0\). There exists a solution \((v(t), z(t))\) of (9), (10) with the initial condition \(v(t_0) = v_0, z(t_0) = z_0\), such that

\[ x(t) = v(t) + \varepsilon H(v(t), z(t), t, \varepsilon), \quad y(t) = z(t) + h(x(t), t, \varepsilon). \]

(11)

It is sufficient to show that (11) holds for \(t = t_0\). Setting \(t = t_0\) in (11) we obtain

\[ x_0 = v_0 + \varepsilon H(v_0, z_0, t_0, \varepsilon), \quad y_0 = z_0 + h(x_0, t_0, \varepsilon) \]

and, therefore, \(z_0 = y_0 - h(x_0, t_0, \varepsilon)\).

For \(v_0\) we have the equation

\[ v_0 = x_0 - H(v_0, z_0, t_0, \varepsilon) = V(v_0) \]

(12)

which has a unique solution for any \(x_0 \in R^m\) and fixed \(z_0\) and \(t_0\), where

\[ \|z_0\| = \|y_0 - h(x_0, t_0, \varepsilon)\| \leq \rho_1 \]

for some \(\rho_1\).

The following statement is true.

Let all assumptions I–III hold. Then there exist positive numbers \(\varepsilon_2\) and \(\rho_2\) such that for all \(\varepsilon \in (0, \varepsilon_2]\) any solution \(x = x(t, \varepsilon)\), \(y = y(t, \varepsilon)\) of system (1), (2) with the initial condition \(x(t_0, \varepsilon) = x_0, y(t_0, \varepsilon) = y_0\), where \(||y_0 - h(x_0, t_0, \varepsilon)\|| \leq \rho_2\), can be represented in form of (11).

This statement means that in the \(\rho_2\)-neighbourhood of the slow integral manifold \(y = h(x, t, \varepsilon)\) of system (1), (2) can be reduced to the form (9), (10) by the splitting transformation (7), (8). Thus, system (1), (2) was splitted into two subsystems, the first of which is independent and contains a small parameter in a regular manner. Note that the initial value \(v_0\) can be calculated from (12) in the form of an asymptotic expansion:

\[ v_0 = v_{00} + \varepsilon v_{01} + \varepsilon^2 v_{02} + \ldots \]

For example, \(v_{00} = x_0, v_{01} = -H(x_0, z_{00}, t_0, 0)\), where \(z_{00} = y_0 - h(x_0, t_0)\).

Note that there exists number \(K, \quad K > 1\) such that

\[ ||z(t, \varepsilon)|| \leq K \exp(-\gamma t/\varepsilon) ||z_0||, \quad t \geq 0. \]

This means that the solution \(x = x(t, \varepsilon), y = y(t, \varepsilon)\) of system (1)-(2) with the initial condition \(x(0, \varepsilon) = x_0, y(t_0, \varepsilon) = y_0\) can be represented of form

\[ x(t, \varepsilon) = v(t, \varepsilon) + \varepsilon \varphi_1(t, \varepsilon), \]
\[ y(t, \varepsilon) = y(t, \varepsilon) + \varphi_2(t, \varepsilon). \]

(13)

Thus, this solution is represented as a sum of solution which lies on the slow integral manifold, i.e.

\[ x = x(t, \varepsilon) = v(t, \varepsilon), \quad y(t, \varepsilon) = h(v(t, \varepsilon), t, \varepsilon), \]

and exponentially decreasing functions

\[ \varepsilon \varphi_1(t, \varepsilon) = \varepsilon H(v(t, \varepsilon), z(t, \varepsilon), t, \varepsilon), \]
\[ \varphi_2(t, \varepsilon) = z(t, \varepsilon) + h(v(t, \varepsilon) + \varepsilon H(v(t, \varepsilon), z(t, \varepsilon), t, \varepsilon), t, \varepsilon) - h(v(t, \varepsilon), t, \varepsilon). \]
5. Linear with respect to fast variables systems

5.1. General case

Consider the following system of differential equations

\[
\dot{x} = \zeta(x, t, \varepsilon) + F(x, t, \varepsilon)y, \tag{14}
\]

\[
\varepsilon \dot{y} = \xi(x, t, \varepsilon) + G(x, t, \varepsilon)y, \tag{15}
\]

with vector variables \(x \in \mathbb{R}^m, y \in \mathbb{R}^n,\) and \(t \in \mathbb{R}\).

This system is a typical model of enzyme kinetics [1].

Assuming that all the eigenvalues \(G(x, t, 0)\) have negative real parts and \(\zeta, \xi, F\) and \(G\) are sufficiently smooth we obtain that (14)-(15) possesses the slow integral manifold

\[
y = h(x, t, \varepsilon) = h_0(x, t) + \varepsilon h_1(x, t) + \ldots,
\]

and the functions \(h_i\) can be found from the invariance equation

\[
\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x}(\zeta + Fh) = \xi + Gh.
\]

Using the representations

\[
F(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j F_j(x, t), \quad G(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j G_j(x, t),
\]

\[
\zeta(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j \zeta_j(x, t), \quad \xi(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j \xi_j(x, t)
\]

where \(G_0 = G_0(x, t)\) plays the role of the matrix \(B(x, t)\), the formulae for the coefficients \(h_i = h_i(x, t)\) take the form

\[
h_0 = G_0^{-1} \zeta_0,
\]

\[
h_k = G_0^{-1} \left[ \frac{\partial h_{k-1}}{\partial t} + \sum_{p=0}^{k-1} \frac{\partial h_p}{\partial x} (\zeta_{k-1-p} + \sum_{j=0}^{k-1-p} F_j h_{k-p-1-j}) \right] - \xi_k - \sum_{j=1}^{k} G_j h_{k-j}], \quad k \geq 1. \tag{16}
\]

The invariance equation (6) for \(H(v, z, t, \varepsilon)\) takes the following form

\[
\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial v} [\zeta(v, t, \varepsilon) + F(v, t, \varepsilon)h(v, t, \varepsilon)] + \frac{\partial H}{\partial z} [G(v + \varepsilon H, t, \varepsilon)
\]

\[
-\varepsilon \frac{\partial h}{\partial x} (v + \varepsilon H, t, \varepsilon)F(v + \varepsilon H, t, \varepsilon)]z = \zeta(v + \varepsilon H, t, \varepsilon) - \zeta(v, t, \varepsilon)
\]

\[
+ F(v + \varepsilon H, t, \varepsilon)(z + h(v + \varepsilon H, t, \varepsilon)) - F(v, t, \varepsilon)h(v, t, \varepsilon).
\]

For \(\varepsilon = 0\) we obtain

\[
\frac{\partial H_0}{\partial z} G_0(v, t)z = F_0(v, t)z.
\]

This means that we can represent \(H_0(v, z, t)\) of form \(H_0(v, z, t) = D_0(v, t)z\). Here \(D_0(v, t)\) satisfies the equation

\[
D_0(v, t) G_0(v, t) = F_0(v, t).
\]
Thus, we obtain
\[ H_0(v, z, t) = F_0(v, t)G_0^{-1}(v, t)z. \]

Note that the transformation
\[ x = v + \varepsilon H_0(v, z, t), \quad y = z + h_0(x, t) + \varepsilon h_1(x, t) \] (17)
reduces (14) to the following form:
\[ \dot{v} = \zeta_0(v, t) + F_0(v, t)h_0(v, t) + \varepsilon[\zeta_1(v, t) + F_0(v, t)h_1(v, t) + F_1(v, t)] + O(\varepsilon^2), \]
\[ \varepsilon \dot{z} = [G_0(v, t) + \varepsilon G_1(v, t) + \frac{\partial G_0}{\partial x}(v, t)H_0(v, z, t) - \frac{\partial h_0}{\partial x}(v, t)F_0(v, t)]z + O(\varepsilon^2). \] (18)

### 5.2. Partial case
If (14)-(15) is the autonomous differential system then (16) take the forms
\[ h_k = G_0^{-1}\zeta_0, \]
\[ h_k = G_0^{-1}\sum_{p=0}^{k-1} \frac{\partial h_p}{\partial x}(\zeta_{k-1-p} + \sum_{j=0}^{k-1-p} F_j h_{k-1-p-j}) - \xi_k - \sum_{j=1}^{k} G_j h_{k-j}, \quad k \geq 1. \]

The invariance equation (6) for \( H = H(v, z, \varepsilon) \) in this case takes the form
\[ \varepsilon \frac{\partial H}{\partial v}[\zeta(v, \varepsilon) + F(v, \varepsilon)h(v, \varepsilon)] + \frac{\partial H}{\partial z}[G(v + \varepsilon H, \varepsilon) - \zeta(v, \varepsilon)] + F(v + \varepsilon H, \varepsilon)z = \zeta(v + \varepsilon H, \varepsilon) - \zeta(v, \varepsilon) + F(v + \varepsilon H, \varepsilon)(z + h(v + \varepsilon H, \varepsilon)) - F(v, \varepsilon)h(v, \varepsilon). \]

In the limiting case \( \varepsilon = 0 \) we have
\[ \frac{\partial H_0}{\partial z}G_0(v)z = F_0(v)z. \]

Thus,
\[ H_0(v, z) = F_0(v)G_0^{-1}(v)z \]
and (17) takes the form
\[ x = v + \varepsilon H_0(v, z), \quad y = z + h_0(x) + \varepsilon h_1(x). \]

Finally, the system (18) takes the form
\[ \dot{v} = \zeta_0(v) + F_0(v)h_0(v) + \varepsilon[\zeta_1(v) + F_0(v)h_1(v) + F_1(v)] + O(\varepsilon^2), \]
\[ \varepsilon \dot{z} = [G_0(v) + \varepsilon G_1(v) + \frac{\partial G_0}{\partial x}(v)H_0(v, z) - \frac{\partial h_0}{\partial x}(v)F_0(v)]z + O(\varepsilon^2). \]
5.3. Suicide Substrate Kinetics

Consider the dimensionless equations of suicide substrate kinetics [4]

$$\dot{s} = -s[(\sigma + 1) - \sigma x - (\sigma + 1)y - (\sigma + 1)e_i] + \frac{\rho}{1 + \rho} x,$$

(19)

$$\varepsilon \dot{x} = s[(\sigma + 1) - \sigma x - (\sigma + 1)y - (\sigma + 1)e_i] - x,$$

(20)

$$\varepsilon \dot{y} = \left( \frac{\sigma}{(1 + \sigma)(1 + \rho)} \right) x - \psi y,$$

(21)

$$\varepsilon \dot{e}_i = \phi y$$

(22)

with the initial conditions

$$s(0) = 1, x(0) = 0, y(0) = 0, e_i(0) = 0.$$  

(23)

Following [4] we introduce $\sigma = \varepsilon p$, where $p$ is of order $O(1)$. Now we introduce a new variable

$$\eta = \phi y + \psi e_i,$$

i.e. $e_i = (\eta - \phi y)/\psi$. Then the differential system (19)-(22) with four variables $s$, $x$, $y$ and $e_i$ become two equations for slow variables $s$ and $\eta$, and two equations for fast variables $x$ and $y$

$$\dot{s} = -s[(\varepsilon p + 1) - \varepsilon px - (\varepsilon p + 1)y - (\varepsilon p + 1)(\eta - \phi y)/\psi] + \frac{\rho}{1 + \rho} x,$$

(24)

$$\dot{\eta} = \frac{p\phi}{(1 + \varepsilon p)(1 + \rho)} x.$$

(25)

$$\varepsilon \dot{x} = s[(\varepsilon p + 1) - \varepsilon px - (\varepsilon p + 1)y - (\varepsilon p + 1)(\eta - \phi y)/\psi] - x,$$

(26)

$$\varepsilon \dot{y} = \left( \frac{\varepsilon p}{(1 + \varepsilon p)(1 + \rho)} \right) x - \psi y,$$

(27)

with the initial conditions

$$s(0) = 1, \eta(0) = 0, x(0) = 0, y(0) = 0.$$  

(28)

To calculate the approximation of slow invariant manifold and the equation on this manifold we assume that $0 < \varepsilon \ll 1$.

In this case we have

$$\zeta = \begin{pmatrix} -s(\varepsilon p + 1) + \frac{1 + \varepsilon p}{\psi} s \eta s \\ 0 \end{pmatrix} ; \quad F = \begin{pmatrix} \frac{\rho}{1 + \rho} + \frac{\varepsilon p s}{\rho \phi} & s(\varepsilon p + 1) - \frac{1 + \varepsilon p}{\psi} \phi \\ (1 + \varepsilon p)(1 + \rho) & 0 \end{pmatrix} ;$$

$$\xi = \begin{pmatrix} s(\varepsilon p + 1) - \frac{1 + \varepsilon p}{\psi} s \eta s \\ 0 \end{pmatrix} ; \quad G = \begin{pmatrix} -s(\varepsilon p + 1) - \frac{1 + \varepsilon p}{\psi} \phi \\ (1 + \varepsilon p)(1 + \rho) & -\psi \end{pmatrix}.$$  

The corresponding vector and matrix coefficients of asymptotic expansions

$$\zeta = \zeta_0 + \varepsilon \zeta_1 + O(\varepsilon^2), \quad \xi = \xi_0 + \varepsilon \xi_1 + O(\varepsilon^2),$$

$$F = F_0 + \varepsilon F_1 + O(\varepsilon^2), \quad G = G_0 + \varepsilon G_1 + O(\varepsilon^2).$$
take the form

\[ \xi_0 = \begin{pmatrix} s(1 - \eta/\psi) \\ 0 \end{pmatrix}; \]
\[ \zeta_0 = -\xi_0, \; \zeta_1 = -p\xi_0, \; \xi_1 = p\xi_0, \]
\[ F_0 = \begin{pmatrix} \frac{p\phi}{1+p} - sp s(1 - \frac{\phi}{\psi}) \\ 0 \end{pmatrix}; \quad F_1 = \begin{pmatrix} ps \quad ps(1 - \frac{\phi}{\psi}) \\ -p^2\phi \quad 0 \end{pmatrix}; \]
\[ G_0 = \begin{pmatrix} -1 & -ps(1 - \frac{\phi}{\psi}) \\ 0 & -\psi \end{pmatrix} \quad G_1 = \begin{pmatrix} -sp & -s(1 - \frac{\phi}{\psi}) \\ \frac{p}{1+p} & 0 \end{pmatrix}. \]

The degenerate system is

\[ \dot{s} = -s[1 - y - (\eta - \phi y)/\psi] + \frac{\rho}{1+\rho} x, \quad (29) \]
\[ \dot{\eta} = \frac{p\phi}{1+\rho} x, \quad (30) \]
\[ 0 = s[1 - y - (\eta - \phi y)/\psi] - x, \quad (31) \]
\[ 0 = -\psi y. \quad (32) \]

It is clear that the slow surface is stable.

Two algebraic equations (31)-(32) have the unique solution

\[ x = s(1 - \eta/\psi), \]
\[ y = 0, \]

and the inverse matrix \( G_0^{-1} \) is given by the relationship

\[ G_0^{-1} = \begin{pmatrix} -1 & s(1 - \phi/\psi)/\psi \\ 0 & -\psi^{-1} \end{pmatrix}. \]

As a result, we obtain the first order approximation to the system on the slow invariant manifold

\[ \dot{s} = -s(1 - \eta/\psi) \frac{1}{1+\rho} + \varepsilon S(s, \eta) + O(\varepsilon^2), \]
\[ \dot{\eta} = p\phi s(1 - \eta/\psi) \frac{1}{1+\rho} + \varepsilon E(s, \eta) + O(\varepsilon^2), \]

where the slow invariant manifold is

\[ \begin{pmatrix} x \\ y \end{pmatrix} = h_0(s, \eta) + \varepsilon h_1(s, \eta) + O(\varepsilon^2). \]

Here

\[ h_0(s, \eta) = -G_0^{-1}(s)\xi_0(s, \eta) = \xi_0(s, \eta), \]
\[ h_1(s, \eta) = G_0^{-1}(s)[\xi_0'(s, \eta)(F_0(s) - I) - pI - G_1(s)]\xi_0(s, \eta), \]
\[ \xi_0'(s, \eta) = \begin{pmatrix} 1 - \eta/\psi & -s/\psi \\ 0 & 0 \end{pmatrix}. \]
\[
\left( \begin{array}{c}
S(s, \eta) \\
E(s, \eta)
\end{array} \right) = F_1(s) h_0(s, \eta) + F_0(s) h_1(s, \eta).
\]

We omit expressions for \(S, E\) due to their tremendous.

Using \(H_0 = F_0(v) G^{-1}_0(v) z\) we obtain now

\[
H_0(v_1, v_2, z_1, z_2) = \left( \begin{array}{c}
-\frac{\rho}{1+p} + v_1 p \\
-\frac{\rho_0}{1+p} v_1 (1 - \frac{\phi}{\psi})/\psi
\end{array} \right) \left( \begin{array}{c}
z_1 \\
z_2
\end{array} \right)
\]

and therefore,

\[
\left( \begin{array}{c}
v_1(0) \\
v_2(0)
\end{array} \right) = \left( \begin{array}{c}
s(0) \\
\eta(0)
\end{array} \right) - \varepsilon H_0(s(0), \eta(0), x(0) - s(0)(1 - \eta(0)/\psi), y(0)) + O(\varepsilon^2)
\]

\[
= \left( 1 + \varepsilon \left( \frac{\rho}{1+p} - p \right) \right) + O(\varepsilon^2).
\]

Finally, we obtain the reduced model of form

\[
v_1 = -v_1(1 - \frac{\psi}{v_2}) \frac{1}{1+p} + \varepsilon S(v_1, v_2) + O(\varepsilon^2),
\]

\[
v_2 = p\phi v_1(1 - \frac{\psi}{v_2}) \frac{1}{1+p} + \varepsilon E(v_1, v_2) + O(\varepsilon^2),
\]

with the initial condition

\[
v_1(0, \varepsilon) = 1 + \varepsilon \left( \frac{\rho}{1+p} - p \right) + O(\varepsilon^2),
\]

\[
v_2(0, \varepsilon) = 0 + O(\varepsilon^2).
\]

6. Conclusion

The effective separation of slow and fast dynamics is the main goal of the paper. As the result the independent subsystem for the slow dynamics is obtained.

7. References

[1] Fehrst A 1985 Enzyme Structure and Mechanisms (New York: W.F. Freeman & Co)
[2] Fridman E 1999 Exact slow-fast decomposition of a class of nonlinear singularly perturbed optimal control problems vi ainvariant manifolds Int. J. Control 72(17) 1609
[3] Fridman E 2000 Exact slow-fast decomposition of nonlinear singularly perturbed optimal control problem Syst. Control Lett. 40(2) 121
[4] Murray J D 2001 Mathematical Biology I. An Introduction (New York: Springer)
[5] Prasov A A and Khalil K 2015 Tracking performance of a highgain observer in the presence of measurement noise Int. J. Adapt. Control Signal Proc. 30(8-10) 1228
[6] Sobolev V A 1984 System and Control Letters 5 169
[7] Sobolev V 2019 Efficient decomposition of singularly perturbed systems Math. Model. Nat. Phenom 14(4) DOI 10.1051/mnpp/2019023
[8] Voropaeva N V 2008 Decomposition of problems of optimal control and estimation for discrete systems with fast and slow variables Autom. Remote Control 69(6) 920
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