VARIETIES CONNECTED BY CHAINS OF LINES

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ABSTRACT. In this paper we give for any integer \( l \geq 2 \) a numerical criterion ensuring the existence of a chain of length \( l \) of lines through two general points of an irreducible variety \( X \subset \mathbb{P}^N \), involving the degrees and the number of homogeneous polynomials defining \( X \). We show that our criterion is sharp.

CONTENTS

Introduction 1
1. Notation and Preliminaries 2
2. Chains of Lines 3
2.1. Sharpness of Theorem 2.1 6
References 8

INTRODUCTION

Bonavero and Höring, in [BH], consider a smooth scheme theoretical complete intersection \( X \subset \mathbb{P}^N \) and give a bound involving the degrees of the polynomials defining the variety that grants the conic-connectedness. However their result ensures the existence of a smooth conic that in general is weaker than the existence of a singular conic. Indeed from classical arguments of deformations of chains of rational curves we have that a singular conic through two general points on a smooth variety can be deformed into a smooth conic. The existence of a smooth conic \( f : \mathbb{P}^1 \to X \) through two general points on a projective variety does not imply the existence of a singular conic. This is true if \( \dim_{\text{H}}(\text{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}})) \geq 2 \) by Mori’s Bend-and-Break lemma [De, Proposition 3.2].

We start from this result and consider the more general case of a non necessarily smooth variety \( X \subset \mathbb{P}^N \), set theoretically defined by homogeneous polynomials; such varieties do not need to be a complete intersection. In [MMT, Theorem 4.4] the authors and Saeed Tafazolian give a numerical criterion ensuring the existence of a chain of length two of lines through two general points of a variety \( X \subset \mathbb{P}^N \). In section 2 we generalize [MMT, Theorem 4.4] and we obtain the following result (Theorem 2.1).

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Theorem. Let \( X \subset \mathbb{P}^N \) be a variety set theoretically defined by homogeneous polynomials \( G_i \) of degree \( d_i \), for \( i = 1, \ldots, m \), and let \( l \geq 2 \) be an integer. If
\[
\sum_{i=1}^{m} d_i \leq \frac{N(l - 1) + m}{l}
\]
then \( X \) is rationally chain connected by chains of lines of length at most \( l \).
In particular if \( X \) is smooth and the above inequality is satisfied then \( X \) is rationally connected by rational curves of degree at most \( l \).

Finally we prove the sharpness of this result considering a hypersurface \( X_{l+1} \) of degree \( l+1 \) in \( \mathbb{P}^{l+2} \).

1. Notation and Preliminaries

We work over the complex field. Throughout this paper we denote by \( X \subset \mathbb{P}^N \) an irreducible variety of dimension \( n \geq 1 \). We assume \( X \) to be non-degenerate of codimension \( c \), so that \( N = n + c \).

Prime Fano, covered by lines and conic connected varieties. Let \( x \in X \subset \mathbb{P}^N \) be a general point. We denote by \( \mathcal{L}_x \) the (possibly empty) variety of lines through \( x \), contained in \( X \). Note that \( \mathcal{L}_x \) is embedded in the space of tangent directions at \( x \), that is \( \mathcal{L}_x \subseteq \mathbb{P}(t_x X^*) = \mathbb{P}^{n-1} \), where \( t_x X \) denotes the affine embedded Zariski tangent space at \( x \).

We denote by \( a := \dim(\mathcal{L}_x) \) the dimension of \( \mathcal{L}_x \). We say that \( X \) is covered by lines if \( L_x \neq \emptyset \) for \( x \in X \) general. When \( \mathcal{L}_x \) is irreducible, it can proved that \( a = \deg(\mathcal{N}_{l/X}) \), where \( l \) is a line in \( X \) through \( x \), and \( \mathcal{N}_{l/X} \) is its normal bundle. When \( a \geq \frac{n-1}{2} \), \( \mathcal{L}_x \subset \mathbb{P}^{n-1} \) is smooth and irreducible; if, moreover, \( \text{Pic}(X) \) is cyclic, it is also non-degenerate, see [Hw].

Recall that \( X \subset \mathbb{P}^N \) is a prime Fano variety of index \( i(X) \) if its Picard group is generated by the class \( H \) of a hyperplane section and \( -K_X = i(X)H \) for some positive integer \( i(X) \). By the work of Mori, see [Mo], if \( i(X) > \frac{n+1}{2} \) then \( X \) is covered by lines.

A variety \( X \subset \mathbb{P}^N \) is called a conic-connected (CC) variety if for \( x, y \in X \) general points there is a conic \( C_{x,y} \) passing through \( x, y \) and contained in \( X \).

Loci of Chains. Let \( X \) be a variety covered by lines and let \( x \in X \) be a general point. We define the loci determined on \( X \) by chains of lines through \( x \) as follows.

Definition 1.1. The locus of lines in \( X \) though \( x \) is defined as
\[
\mathcal{L}_{\mathcal{O}_1}(x) = \bigcup_{[L] \in \mathcal{L}_x} L
\]
and the locus of chains of lines of length \( l \) in \( X \) though \( x \) is defined recursively as
\[
\mathcal{L}_{\mathcal{O}_l}(x) = \bigcup_{[L] \in \mathcal{L}_y \mid y \in \mathcal{L}_{\mathcal{O}_{l-1}}(x) \text{ is a general point}} L.
\]

We denote by \( d_l \) the maximal dimension of the irreducible components of \( \mathcal{L}_{\mathcal{O}_l}(x) \). If there exists an integer \( l \) such that \( d_l = \dim(X) \) but \( d_{l-1} < \dim(X) \) we say that \( X \) has length \( l \). In particular a variety of length 2 is conic-connected.
In [HK] Hwang and Kebekus give a lower bound on $d_l$ under the irreducibility assumption on $\mathcal{L}_x$ for $x \in X$ general point. However their proof work even without this irreducibility assumption. The following theorem can be found in [W4, Theorem 4.6].

**Theorem 1.2.** Let $X$ be a prime Fano variety of dimension $n \geq 3$. Then we have
\[ d_1 = \dim(\mathcal{L}_x) + 1 \text{ and } d_l \leq l(\dim(\mathcal{L}_x) + 1) \text{ for each } l \geq 1. \]

In the proof of Theorem 2.1 it will be necessary to perform intersections in a product of projective spaces, so we recall briefly the structure of the Chow ring of a product, for more details see [Fu].

**Chow ring of a product.** Let us recall, in order to fix the notation, that for a cartesian product of projective spaces the Chow ring is given by
\[ A^*(\mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_k}) \cong \mathbb{Z}[h_1, \ldots, h_k]/(h_1^{N_1}, \ldots, h_k^{N_k}) \]
where $h_i$ is the hyperplane class of $\mathbb{P}^{N_i}$. It follows that the class of subvariety $Z \subset \mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_k}$ can be written in the form
\[ [Z] = \sum_{i_1+\ldots+i_k=\dim(Z)} \lambda_{i_1, \ldots, i_k} h_1^{N_1-i_1} \ldots h_k^{N_k-i_k}, \]
where the $\lambda_{i_1, \ldots, i_k}$ are the multidegrees of $Z$.

## 2. Chains of Lines

We want to generalize [MMT, Theorem 4.4] giving conditions on the equations defining the variety that ensure the existence of a chain of lines of a prescribed length through two general points of $X$. In this case we have to perform a intersection in a product of projective spaces.

**Theorem 2.1.** Let $X \subset \mathbb{P}^N$ be a variety set theoretically defined by homogeneous polynomials $G_i$ of degree $d_i$, for $i = 1, \ldots, m$, and let $l \geq 2$ be an integer. If
\[ \sum_{i=1}^{m} d_i \leq \frac{N(l-1) + m}{l} \]
then $X$ is rationally chain connected by chains of lines of length at most $l$.

*In particular if $X$ is smooth and the above inequality is satisfied then $X$ is rationally connected by rational curves of degree at most $l$.*

**Proof.** Let $x, y \in X$ be two general points. We can assume that $x = [1 : 0 : \ldots : 0]$ and $y = [0 : \ldots : 0 : 1]$. Moreover let us consider other $l-1$ points in $\mathbb{P}^N$, that we denote by $p^i = [p^i_1 : \ldots : p^i_N]$, for $i = 1, \ldots, l-1$. Our goal is to find the set of $l-1$ points that will represent the intersections of the lines we are seeking to build the chain and we would like to see such set as a point of the cartesian product $\mathbb{P}^N \times \ldots \times \mathbb{P}^N_{l-1}$.

Let us consider the line that join the points $x$ and $p^1$, that we will denote by $ux + vp^1 = [u + vp^1_0 : \ldots : vp^1_N]$. We may observe that $G_i(ux + vp^1)$ is a polynomial of degree $d_i$ in the variables $u$ and $v$; it has $d_i + 1$ coefficients, but in this case there is no monomial of the type $u^{d_i}$ because of our assumption $x \in X$. So imposing $G_i(ux + vp^1) \equiv 0$ gives us $d_i$ conditions, involving only on the coordinates of the point $p^1$.

Let us now consider the line $up^1 + vp^2 = [up^1_0 + vp^2_0 : \ldots : up^1_N + vp^2_N]$. Again
\( G_i(up^1 + vp^2) \) is a homogeneous polynomial of degree \( d_i \) in the variables \( u \), \( v \) and imposing \( G_i(up^1 + vp^2) \equiv 0 \) gives us \( d_i + 1 \) equations. Let us give a closer look to such equations; \( d_i - 1 \) of them will be homogeneous polynomials in the coordinates of the points \( p^1 \), \( p^2 \), one of them will be a polynomial only in coordinates of the point \( p^1 \), which is exactly the equation \( G_i = 0 \) written in the coordinates of \( p^1 \) that we have already found in the previous step, and one of them will be a polynomial only in coordinates of the point \( p^2 \).

In the same way for every line \( up^{j-1} + vp^j = [up_{l-1}^j + vp_{l-1}^j : \ldots : up_{N}^j + vp_{N}^j] \), for \( i \) from 2 to \( l - 1 \), imposing \( G_i(up^{j-1} + vp^j) \equiv 0 \), we get \( d_i + 1 \) equations; \( d_i - 1 \) of them will be homogeneous polynomials in the coordinates of the points \( p^{j-1} \), \( p^j \), one of them will be a polynomial only in the coordinates of the point \( p^{j-1} \), which is the equation \( G_i = 0 \) written in the coordinates of \( p^{j-1} \) that we have already found in the previous step, and one of them will be a polynomial only in coordinates of the point \( p^j \).

We now consider the line \( up^{j-1} + vy = [up_{l-1}^{j-1} : \ldots : up_{N}^{j-1} + vy] \); we notice that \( G_i(up^{j-1} + vy) \) is a polynomial of degree \( d_i \) in the variables \( u \) and \( v \), it has \( d_i + 1 \) coefficients, but in this case there is no monomial of the type \( v^d \) because of our assumption \( y \in X \). So imposing \( G_i(up^{j-1} + vy) \equiv 0 \) gives us \( d_i \) conditions, only on the coordinates of the point \( p^{j-1} \).

Summarizing, we get the following conditions:

- \( \sum_{i=1}^{m} d_i \) homogeneous equations only in the \( p_j \)'s and \( \sum_{i=1}^{m} d_i \) equations only in the \( p_j^{i-1} \)’s,
- \( m \) homogeneous equations only in the \( p_j \)’s, for every \( k = 2, \ldots, l - 2 \),
- \( \sum_{i=1}^{m} d_i - m \) bihomogeneous equations in the variables \( p_j^{i-1}, p_j^k \), for every \( k = 2, \ldots, l - 1 \).

We want to perform the intersection of these hypersurfaces in \( \mathbb{P}^N_1 \times \ldots \times \mathbb{P}^N_{l-1} \). If \( h_i \) is the hyperplane class of \( \mathbb{P}^N_1 \), the intersection is given by an expression of the form

\[
\sum_{i=1}^{m} d_i h_i \equiv (h_1 + \ldots + h_l) \sum_{i=1}^{m} d_i - m
\]

and each summand of the expression above is of the form

\[
\sum_{i=1}^{m} d_i h_i \equiv (h_1 + \ldots + h_l) \sum_{i=1}^{m} d_i - m
\]

Our aim is to prove that under the numerical hypothesis of the theorem at least one of these summands does not vanish. Take

\[ \bar{J}_k = \left\lfloor \frac{k}{l - 1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor, \quad \text{for } k = 1, \ldots, l - 2, \]

where \( \lfloor p \rfloor \) is the greatest integer smaller or equal than \( p \).

Note that \( 0 \leq \bar{J}_k \leq \sum_{i=1}^{m} d_i - m \) for \( k = 1, \ldots, l - 2 \). Consider the term

\[
\sum_{i=1}^{m} d_i h_i \sum_{i=1}^{m} d_i \bar{J}_k - \bar{J}_k
\]

For any \( k = 2, \ldots, l - 2 \) the exponent of \( h_k \) is \( \sum_{i=1}^{m} d_i - \bar{J}_{k-1} + \bar{J}_k \). In order to ensure the intersection in \( \mathbb{P}^N \) to be not empty we impose

\[ N - \sum_{i=1}^{m} d_i + \bar{J}_{k-1} - \bar{J}_k \geq 0 \quad \text{for any } k = 2, \ldots, l - 2. \]
Substituting we have

$$\left\lfloor \frac{k-1}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor \geq \left\lfloor \frac{k}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor - N + \sum_{i=1}^{m} d_i.$$  

Since the number on the right is an integer it is enough to prove that

$$\left\lfloor \frac{k}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor \geq \left\lfloor \frac{k-1}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor - N + \sum_{i=1}^{m} d_i.$$  

That is

$$\frac{k-1}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) + N - \sum_{i=1}^{m} d_i \geq \left\lfloor \frac{k}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor.$$

Consider now the term on the left, from our hypothesis we get $N \geq \frac{t \sum_{i=1}^{m} d_i - m}{l-1}$. So

$$\frac{k-1}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) + N - \sum_{i=1}^{m} d_i \geq \frac{k}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) + \frac{t \sum_{i=1}^{m} d_i - m}{l-1} - \sum_{i=1}^{m} d_i$$

This prove that the intersection in $\mathbb{P}^N$ is not empty for any $k = 2, \ldots, l - 2$.  

Consider now the exponent of $h_1$. We have to impose

$$N - \sum_{i=1}^{m} d_i - \overline{f}_1 = N - \sum_{i=1}^{m} d_i - \left\lfloor \frac{\sum_{i=1}^{m} d_i - m}{l-1} \right\rfloor \geq 0.$$  

But $N - \sum_{i=1}^{m} d_i \geq \frac{\sum_{i=1}^{m} d_i - m}{l-1}$ implies the last inequality and is equivalent to the numerical hypothesis of the theorem. This show that the intersection in $\mathbb{P}^N_1$ is also not empty. Finally consider the exponent of $h_{l-1}$ and impose

$$N - 2 \sum_{i=1}^{m} d_i + m + \overline{f}_{l-2} \geq 0.$$  

We have $\left\lfloor \frac{l-2}{l-1} \left( \sum_{i=1}^{m} d_i - m \right) \right\rfloor \geq 2 \sum_{i=1}^{m} d_i - m - N$, since the number on the right is an integer it is enough to prove that $(l-2)(\sum_{i=1}^{m} d_i - m) \geq (l-1)(2 \sum_{i=1}^{m} d_i - m - N)$ and again this is exactly our numerical hypothesis. We conclude that the intersection in $\mathbb{P}^N_{l-1}$ is not empty.

At this point we know that the equations define non-empty subvarieties of $\mathbb{P}^N_j$ for any $j = 1, \ldots, l - 1$. To ensure that these subvarieties lift to subvarieties of $\mathbb{P}^N_1 \times \ldots \times \mathbb{P}^N_{l-1}$ rationally equivalent to cycles having non-empty intersection we force

$$N(l-1) - (l-2) \sum_{i=1}^{m} d_i - 2 \sum_{i=1}^{m} d_i + m \geq 0,$$

which once again is our initial hypothesis.

So our system of equations on the product $\mathbb{P}^N_1 \times \ldots \times \mathbb{P}^N_{l-1}$ has at least a solution, which represents the sequence of connection points in the chain of lines we were looking for. Finally, if $X$ is smooth, by general smoothing arguments ([De], Proposition 4.24) we can deform our chain in a rational curve of degree at most $l$ connecting $x$ and $y$.  

□
2.1. **Sharpness of Theorem 2.1.** The inequality in Theorem 2.1 is sharp. Consider a smooth hypersurface $X_{l+1}$ of degree $l + 1$ in $\mathbb{P}^{l+2}$. Then $d = l + 1$, $N = l + 2$ and $m = 1$, so we have $\frac{N(l(l-1)+m)}{l} = \frac{l^2 + l - 1}{l}$. Since

$$l \leq \frac{l^2 + l - 1}{l} < l + 1$$

the hypersurface $X_{l+1}$ is a good candidate to prove sharpness. The equalities imply $d = l + 1 = l + 2 - 1 = N - 1$ we have $\dim(L_x) = N - 1$. Now by Theorem 1.2 the dimension $d_l = \dim(L_{o_l}(x))$ is bounded by $d_l \leq l(\dim(L_x) + 1) = l$. Since $\dim(X_{l+1}) = l + 1 > l$ we have

$$\dim(L_{o_l}(x)) < \dim(X)$$

and $X$ is not connected by chains of length $l$ of lines.

From Theorem 2.1 we get the following Corollary which can also be found in [Ko, Lemma 4.8.1].

**Corollary 2.2.** Let $X \subset \mathbb{P}^N$ be a hypersurface of degree $d \leq N - 1$. Then $X$ is rationally connected by a chain of lines of length at most $N - 1$.

**Proof.** If $l = N - 1$ the inequality $d \leq \frac{N(N-2)+1}{N-1} = \frac{N^2 - 2N + 1}{N-1} = N - 1$ of Theorem 2.1 is satisfied by hypothesis.

**Corollary 2.3.** Let $X \subset \mathbb{P}^N$ be a scheme theoretical complete intersection. If $\deg(X) \leq \frac{N(l(l-1)+c)}{l}$ then $X$ is rationally chain connected by chains of lines of length at most $l$. If, in addition, $X$ is smooth and Fano of index $i_X \geq \frac{n+4}{n+1}$ then $X$ is rationally connected by rational curves of degree at most $l$.

**Proof.** The first assertion follows from the inequality $\sum_{i=1}^c d_i \leq \prod_{i=1}^c d_i = \deg(X)$ combined with Theorem 2.1. If $X$ is a smooth, Fano, complete intersection, Theorem 2.1 and the equality $i_X = N + 1 - \sum_{i=1}^c d_i$ imply the second assertion.

**Proposition 2.4.** Let $X \subseteq \mathbb{P}^N$ be a smooth complete intersection defined by homogeneous polynomials $G_i$, of degree $d_i$, for $i = 1, ..., c$, such that $\sum_{i=1}^c d_i \leq N - 1$. Then

$$\text{length}(X) = \left\lfloor \frac{N - c}{N - \sum_{i=1}^c d_i} \right\rfloor,$$

where $\lfloor k \rfloor$ is the smallest integer greater or equal than $k$.

**Proof.** Since the integer $l_{\text{min}} := \left\lfloor \frac{N - c}{N - \sum_{i=1}^c d_i} \right\rfloor$ satisfies the inequality of Theorem 2.1 we have that $X$ is $l_{\text{min}}$-chain connected. We have to prove that $X$ is not $(l_{\text{min}} - 1)$-chain connected. Now $l_{\text{min}} - 1 = \left\lfloor \frac{N - c}{N - \sum_{i=1}^c d_i} - 1 \right\rfloor$. Note that $\dim(L_x) = N - \sum_{i=1}^c d_i - 1 \geq 0$. By Theorem 1.2 we have $d_{l_{\text{min}} - 1} \leq (l_{\text{min}} - 1)(\dim(L_x) + 1)$, we distinguish two cases

- If $\frac{N - c}{N - \sum_{i=1}^c d_i} - 1$ is an integer then

$$d_{l_{\text{min}} - 1} \leq \left(\frac{N - c}{N - \sum_{i=1}^c d_i} - 1\right)(N - \sum_{i=1}^c d_i) = \sum_{i=1}^c d_i - c < N - c = n.$$
If \( \frac{N-c}{\sum_{i=1}^{c} d_i} \) is not an integer then

\[
d_{\min} - 1 \leq (l_{\min} - 1)(N - \sum_{i=1}^{c} d_i) < \frac{N-c}{\sum_{i=1}^{c} d_i}(N - \sum_{i=1}^{c} d_i) = n.
\]

Then \( d_{\min} - 1 = \dim(\mathcal{O}_{\min}(x)) < \dim(X) \). \( \square \)

**Remark 2.5.** In the case \( l = 2 \) we find again [MMT] Theorem 4.4], in fact, the inequality in \([2.1]\) simply becomes \( \sum_{i=1}^{m} d_i \leq \frac{N+m}{2} \).

**Remark 2.6.** In the range of Theorem \([2.1]\) \( X \) is covered by lines. Indeed under the numerical hypothesis of Theorem \([2.1]\) we have \( m < \sum_{i=1}^{c} d_i \leq \frac{N-1}{l} \) which gives \( m < N \). So we get the inequality

\[
\sum_{i=1}^{m} d_i \leq \frac{N(l-1)+m}{l} < N,
\]

which forces \( X \) to be covered by lines.

**Counting chains of lines.** We discuss now an example that shows how it is possible to count the number of possible chains of lines when the equality in Theorem \([2.1]\) holds.

Let us consider a cubic threefold \( X \subset \mathbb{P}^4 \). In this case the equality holds when \( l = 3 \), so we are looking for all the possible 3-chains of lines connecting two general points \( x, y \) of \( X \). Following the proof of the theorem, we have to perform intersection in \( \mathbb{P}_1^4 \times \mathbb{P}_2^4 \), we are looking for two points \( p^1, p^2 \). We have 3 conditions on the coordinates of \( p^1 \) namely \( h_1, 2h_1, 3h_1 \), describing the cone of lines in \( X \) through \( x \). Furthermore we have 3 other conditions \( h_2, 2h_2, 3h_2 \) on the coordinates of \( p^2 \), describing the cone of lines in \( X \) through \( y \). Finally we have 2 conditions involving the coordinates of both points \( p^1, p^2 \), namely \( 2h_1 + h_2 \) and \( h_1 + 2h_2 \). Their intersection product is given by

\[
h_12h_13h_1h_22h_23h_2(2h_1 + h_2)(h_1 + 2h_2) = 36h_1^3h_2^3(2h_1^2 + 5h_1h_2 + 2h_2^2) = 180h_1^4h_2^4,
\]

and we conclude that we have 180 possibilities.

A geometrical description of this fact is the following: there are exactly 6 = \( h_12h_13h_1 \) lines in \( X \) through \( x \) and 6 = \( h_22h_23h_2 \) lines in \( X \) through \( y \). Take a line \( L_x \) of the first family and a line \( L_y \) of the second. These lines are skew otherwise \( X \) would be connected by singular conics and we know this is not possible by classical arguments of projective geometry, so \( L_x \) and \( L_y \) generate a 3-plane \( H \). The linear section \( H \cap X := S \) is a smooth cubic surface in \( \mathbb{P}^3 \) and we can consider the lines \( L_x, L_y \) as two exceptional divisors of a proper blow-up of the projective plane in 6 points; we denote by \( p, q \) the points in the plane which are blown-up in \( L_x \) and \( L_y \). There are exactly 5 lines joining \( L_x \) and \( L_y \) namely the strict transform of the line \( \langle p, q \rangle \) and of the conics passing through \( p, q \) and 3 of the 4 remaining points. In conclusion we have \( 6 \cdot 6 \cdot 5 = 180 \) possibilities, that double-checks the counting made before.

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