RELATIVE GROUP HOMOLOGY THEORIES WITH COEFFICIENTS AND THE COMPARISON HOMOMORPHISM

JOSÉ ANTONIO ARICINIEGA-NEVÁREZ, JOSÉ LUIS CISNEROS-MOLINA, AND LUIS JORGE SÁNCHEZ SALDAÑA

Abstract. Let $G$ be a group, let $H$ be a subgroup of $G$ and let $Or(G)$ be the orbit category. In this paper we extend the definition of the relative group homology theories of the pair $(G,H)$ defined by Adamson and Takasu to have coefficients in an $Or(G)$-module. There is a canonical comparison homomorphism defined by Cisneros-Molina and Ariciniega-Navarez from Takasu’s theory to Adamson’s theory. We give a necessary and sufficient condition on the subgroup $H$ for which the comparison homomorphism is an isomorphism for all coefficients. We also use the Lück-Wiermann construction to introduce a long exact sequence for Adamson homology. Finally, we provide some examples of explicit computations for the comparison homomorphism.

1. Introduction

Let $G$ be a group, let $H$ be a subgroup of $G$, and let $M$ be an arbitrary $G$-module. In the literature there are two relative group (co)homology theories associated to the pair $(G,H)$ with coefficients in $M$. Both reduce to the classical group (co)homology of $G$ when $H$ is the trivial subgroup. The first relative group (co)homology theory, denoted by $H_*([G : H]; M)$, was defined by Adamson [Ada54] and later Hochschild Hoc56 interpreted Adamson’s theory in terms of relative homological algebra. The second relative group (co)homology theory, denoted by $H_*(G, H; M)$, was introduced by Massey [Mas55, Problem 22] and later studied by Takasu in [Tak57] and [Tak59]. The reader can look in [ANCM17] for more details about the history and references. From now on, we call the former Adamson relative group (co)homology theory and the latter Takasu relative group (co)homology theory.

To the best of our knowledge Adamson relative group homology theory and Takasu relative homology theory were compared for the first time in [ANCM17]. In fact, in [ANCM17] Section 7 Arciniega-Navarez and Cisneros-Molina defined a canonical homomorphism from Takasu homology theory to Adamson homology theory $\varphi : H_*(G, H; \mathbb{Z}) \rightarrow H_*([G : H]; \mathbb{Z})$, which from now on we call the comparison homomorphism. They gave a sufficient condition for $\varphi$ to be an isomorphism: if $H$ is a malnormal subgroup of $G$ then, for all $i \geq 1$, the comparison homomorphism $\varphi : H_i([G : H]; \mathbb{Z}) \rightarrow H_i(G, H; \mathbb{Z})$ is an isomorphism.

Recall that classical group (co)homology can be defined topologically as the (co)homology groups of the classifying space $BG$ of $G$ with local coefficients associated to $M$. In analogy with this, taking $M$ as a trivial $G$-module, topological definitions of Adamson and Takasu relative (co)homology theories were given [Bio77, Tak59, ANCM17]. The topological definition of Takasu relative group (co)homology extends without problem to consider coefficients in an arbitrary $G$-module $M$ taking local coefficients associated to $M$, but the topological definition of Adamson relative group (co)homology only works with trivial coefficients. So, a natural question is if it is possible to give a topological definition for Adamson relative group (co)homology with “more general coefficients”, such that, when taking trivial coefficients it coincides with the topological definition given in [ANCM17], and when taking coefficients in an arbitrary $G$-module, it coincides with Adamson’s original algebraic definition.

The solution to the questions above mentioned is to define both Adamson and Takasu relative homology theories with coefficients in an $Or(G)$-module, using Bredon (co)homology, where $Or(G)$ is the orbit category. Also, once we consider more general coefficients, we prove the converse of Arciniega-Navarez and Cisneros-Molina’s theorem (see Theorem 4.10): the comparison homomorphism is an isomorphism for all $Or(G)$-module coefficients if and only if $H$ is a malnormal subgroup of $G$. It is worth saying that the comparison of both relative (co)homology theories is not only motivated by curiosity. Recently, Adamson and Takasu theories have been used to construct invariants of complete hyperbolic 3-manifolds.
Throughout this section \cite{IK14} we can see a natural application of the comparison homomorphism.

The main feature of Adamson relative (co)homology is that, provided \( H \) is a normal subgroup of \( G \), it is isomorphic to the classical group (co)homology of \( G/H \) (see \cite{Ada54} Theorem 3.2), \cite{Hoc56} Section 6] and \cite{ANCMI17} Corollary 4.29). For Takasu relative (co)homology this is not true in general. The main property of Takasu relative (co)homology is a long exact sequence, which (via the topological definition) corresponds to that of a pair of topological spaces in singular homology \cite{Ta59} Proposition 2.3). Such a sequence is not available in Adamson homology. We show that the extension of Adamson and Takasu relative homologies presented here still satisfy their corresponding property.

As in the classical group homology theory, there are descriptions of \( H_i([G : H]; M) \) and \( H_i(G, H; M) \) using derived functors once we are able to define exact sequences and projective resolutions in a suitable context. This was done in Takasu’s and Hochshild’s original papers. In \cite{ANCMI17} Section 7.1, there is also a description of the comparison homomorphism in a purely algebraic setting using derived functors. In the present paper we also describe Adamson and Takasu homology with coefficients in an \( \text{Or}(G) \)-module using the language of derived functors.

The paper is organized as follows. In Section 2 we recall the definitions of Takasu and Adamson relative group homology theories, and we show why the topological definition for Adamson’s theory given in \cite{ANCMI17} does not work for coefficients in an arbitrary \( G \)-module. In Section 3 we recall the definition of the (restricted) orbit category \( \text{Or}(G, F) \) as well as the definition of \( \text{Or}(G, F) \)-modules and \( \text{Or}(G, F) \)-spaces. Section 4 is devoted to define Adamson and Takasu homology for a pair \((G, H)\) with coefficients in a \( \text{Or}(G) \)-module \( M \), as far as we know these homology theories have not been compared in this context; later on, we define the comparison homomorphism and prove in Theorem 4.16 a converse of Cisneros-Molina and Archiniega-Nevárez theorem \cite{ANCMI17} Theorem 7.13 for the comparison homomorphism. In Section 5 we provide descriptions of Adamson and Takasu homology using derived functors in suitable categories, as well as an algebraic description of the comparison homomorphism. Section 6 has as a main goal to describe a long exact sequence for Adamson homology, analogue to that of Takasu homology, using the Lück–Weiermann construction from \cite{LW12}. Finally, in Section 7 we provide explicit computations of the comparison homomorphism; in particular, we show that the comparison homomorphism associated to \((C_4, C_2)\) (with constant coefficients) is not an isomorphism, therefore, at least to our knowledge, in this concrete case, the isomorphism between Adamson and Takasu homology seems to be a coincidence.

Every result and every construction in this paper has its cohomological version.

2. Adamson and Takasu Relative Group Homology Theories

In this article, we always consider groups as topological spaces endowed with the discrete topology. Throughout this section \( G \) will denote a group and \( H \) a subgroup of \( G \).

In this section we recall the definitions of Takasu and Adamson relative group homology theories for the pair \((G, H)\). The topological definition for Takasu relative group homology given in \cite{ANCMI17} §5.1 for trivial coefficients, also works with coefficients in an arbitrary \( G \)-module \( M \) taking homology with local coefficients associated to \( M \). We also recall the topological definition of Adamson relative group homology given in \cite{ANCMI17} §4.3 for trivial coefficients, and we see why it cannot be extended to use coefficients in an arbitrary \( G \)-module \( M \) taking homology with local coefficients associated to \( M \).

2.1. \( G \)-spaces. Let \( X \) and \( X' \) be two \( G \)-spaces. We denote by \( \text{map}_G(X, X') \) the set of \( G \)-maps from \( X \) to \( X' \). Let \( \pi : K \to G \) be a homomorphism of groups, we denote \( \text{res}_\pi X \) the space \( X \) with the natural \( K \)-action induced by \( \pi \). If \( \pi \) is the inclusion we denote the respective space by \( \text{res}_K^G X \). If \( Y \) is an \( H \)-space, the ind\( \text{uction} \ \text{ind}_G^H Y \) is the \( G \)-space \( G \times Y \) divided by the \( H \)-action \( (g, x) \cdot h = (g, h^{-1}x) \).

2.2. Classifying Spaces for Families. Let \( G \) be a group, a family of subgroups of \( G \) is a nonempty collection \( \mathcal{F} \) of subgroups of \( G \) which is closed under conjugation and taking subgroups. Examples of families are \( \text{Tr} \) the family containing only the trivial subgroup, and \( \text{Fin}, \text{VCyc} \), ALL the families of finite, virtually-cyclic and all subgroups respectively.

In the present work, we are interested in the family \( \mathcal{F}(H) = \{ K \leq G | g^{-1}Kg \leq H \ \text{for some} \ g \in G \} \) of subgroups of \( G \) that we call the family generated by \( H \). Given a family \( \mathcal{F} \) of subgroups of \( G \), define \( \mathcal{F} \cap H = \{ L \cap H \mid L \in \mathcal{F} \} \).

Given a group \( G \) and a family of subgroups \( \mathcal{F} \), a model for the classifying space \( E_{\mathcal{F}} G \) is a \( G \)-CW–complex \( X \) satisfying:

- The isotropy group \( G_x \) belongs to \( \mathcal{F} \), for all \( x \in X \), and
- the fixed point set \( X^H \) is contractible for every \( H \in \mathcal{F} \).
Equivalently, a model for \( E_F G \) is a terminal object in the \( G \)-homotopy category of \( G \)-CW-complexes with isotropy in \( F \), sometimes called \( G-F \)-CW-complexes. Given a group \( G \) and a family of subgroups \( \mathcal{F} \), there always exists a model for \( E_F G \) and it is unique up to \( G \)-homotopy equivalence [Lic00, Theorem 1.9].

**Remark 2.1.** When \( \mathcal{F} = \text{Tr} \), we have that \( E_F G \) corresponds to the universal bundle \( E G \) of \( G \). The \( G \)-orbit space of \( E G \) is the classical classifying space \( BG \) of \( G \). In analogy with \( BG \), we denote by \( B_F G \) the \( G \)-orbit space of \( E_F G \).

### 2.3. Takasu relative group homology

Denote by \( G - \text{mod} \) (resp. \( \text{mod} - G \)) the category of left (resp. right) \( G \)-modules, and given two \( G \)-modules \( M \) and \( M' \) we denote by \( \text{map}_G(M, M') \) the set of \( G \)-module homomorphisms from \( M \) to \( M' \).

In [Tak59, Ch. I §1] Takasu defines the \( G \)-module \( I_{(G,H)}(\mathbb{Z}) \) to be the kernel of the augmentation homomorphism \( \mathbb{Z}[G/H] \to \mathbb{Z} \).

Let \( M \) be an arbitrary \( G \)-module. The **Takasu relative homology of the pair \( (G, H) \) with coefficients in \( M \)** is given by

\[
H_n(G, H; M) = \text{Tor}^G_{n-1}(I_{(G,H)}(\mathbb{Z}), M).
\]

The classifying space \( BH \) can be regarded as a subspace of the classifying space \( BG \). In fact, there is a map \( i : BH \to BG \) induced by the inclusion of \( H \) in \( G \); the **mapping cylinder** \( \text{Cyl}(i) \) of \( i \) is a model for \( BG \) since it is homotopically equivalent to \( BG \) and it clearly contains \( BH \) as a subspace. Then, an alternative definition for the **Takasu relative homology with coefficients in \( M \)** is defined as the homology of the pair of spaces \((BG, BH)\) with local coefficients in \( M \). That is

\[
H_n(G, H ; M) = \hat{H}_n(BG, BH ; M).
\]

It is clear that when \( H \) is the identity subgroup we recover the (reduced) homology of the group \( G \) with coefficients in \( M \).

**Problem 2.2.** When \( M \) is a trivial \( G \)-module one can consider the **mapping cone** \( \text{Cone}(i) = \text{Cyl}(i)/BH \) of \( \iota \), so taking reduced homology we have that

\[
\hat{H}_n(G, H; M) = H_n(\text{Cyl}(i), BH; M) = \hat{H}_n(\text{Cone}(\iota); M).
\]

But when \( M \) is an arbitrary \( G \)-module we cannot use \( \text{Cone}(\iota) \). Using Seifert–van Kampen Theorem one can compute the fundamental group of \( \text{Cone}(\iota) \) we have that \( \pi_1(\text{Cone}(\iota)) = G/N \) with \( N \) the normal subgroup generated by \( H \). In order to compute \( \hat{H}_n(\text{Cone}(\iota); M) \) the homology of \( \text{Cone}(\iota) \) with local coefficients associated to \( M \), the module \( M \) has to be a \( G/N \)-module, but in defining \( \hat{H}_n(\text{BG}, BH; M) \) we can use any \( G \)-module!

### 2.4. Adamson relative group homology

Consider the following chain complex \((C_\ast(G/H), \partial_\ast)\) of \( G \)-modules: let \( C_\ast(G/H) \) be the free abelian group generated by the ordered \((n + 1)\)-tuples of elements of \( G/H \); define the \( i \)-th face homomorphism \( \partial_i : C_\ast(G/H) \to C_{\ast-1}(G/H) \) by \( \partial_0(g_0, \ldots, g_n) = (g_0H, \ldots, g_iH, \ldots, g_nH) \), where \( g_iH \) denotes deletion, and the boundary homomorphism \( \partial_n : C_n(G/H) \to C_{n-1}(G/H) \) by \( \partial_n = \sum_{i=0}^n (-1)^i d_i \). We have that the augmented complex

\[
\cdots \to C_n(G/H) \to C_{n-1}(G/H) \to \cdots \to C_1(G/H) \xrightarrow{\partial_1} C_0(G/H) \xrightarrow{\partial_0} \mathbb{Z} \to 0,
\]

is acyclic [ANC97, Proposition 3.2]. Hence given an arbitrary \( G \)-module \( M \) define

\[
B_\ast(G/H; M) = C_\ast(G/H) \otimes_{\mathbb{Z}[G]} M.
\]

The **Adamson relative homology with coefficients in \( M \)** is given by

\[
\hat{H}_n([G : H]; M) = H_n(B_\ast(G/H; M)).
\]

This is the definition given by Adamson in [Ada54 §3]. It is clear that when \( H \) is the identity subgroup the complex \( C_\ast(G/H) \) is the canonical free \( G \)-resolution of \( \mathbb{Z} \) and we recover the classical group homology. Instead of \( G/H \), we can use any other isomorphic \( G \)-set \( X_{(H)} \), in this case, we call \( C_\ast(X_{(H)}) \) the **standard complex of \( (G, H) \)**, then we can see Adamson relative group homology as a particular case of the homology of a permutation representation defined by Snapper [Sna64]. Hochschild [Hoc65] interpreted Adamson’s theory in terms of relative homological algebra by proving that the complex (3) is a relative projective resolution of \( \mathbb{Z} \) [ANC97, Proposition 4.11].

There is also a topological definition, given in [ANC97], when \( M \) is a trivial \( G \)-module. Consider the family of subgroups \( \mathcal{F}(H) \) generated by \( H \). Then

\[
\hat{H}_n([G : H]; M) = H_n(B_{\mathcal{F}(H)}G; M).
\]
Problem 2.3. To take coefficients in an arbitrary $G$–module $M$ as in the algebraic definition, we cannot simply take homology with local coefficients associated to the module $M$. There is a problem analogous to Problem 2.2 in the case of Takasu’s theory; to take homology with local coefficients associated to a module $M$, the module has to be a $\pi_1(B_{F(H)}G)$–module. Since $\pi_1(B_{F(H)}G) = G/N$ where $N$ is the normal subgroup of $G$ generated by $H$ [ANCM17 Proposition 4.23], given a $G$–module $M$ we need to modify it in order to use it as coefficient. But in the algebraic definition we can take any $G$–module $M$!

In the present article, we give a suitable topological definition using Bredon homology with coefficients in any module over the orbit category to solve (in a more general setting) Problems 2.2 and 2.3.

3.2. Modules over the orbit category

In this section we introduce the (restricted) orbit category, as well as some objects over the orbit category. All the material of this section can be found in great detail in [MV03, CDS7] and [Lie89]. Throughout this section $G$ will denote a group and $F$ a family of subgroups of $G$.

The restricted orbit category $\text{Or}(G, F)$ is the category whose objects are homogeneous spaces (also called orbits) $G/H$ with $H \in F$, and whose morphisms are $G$–maps. The set of $G$–maps between the orbits $G/H$ and $G/K$ is denoted by $\text{map}_G(G/H, G/K)$. We denote $\text{Or}(G, \text{ALL})$ simply by $\text{Or}(G)$. Note that, for every family $F$, we have a canonical inclusion $\text{Or}(G, F) \to \text{Or}(G)$.

It is easy to see that every element in $\text{map}_G(G/H, G/K)$ is of the form $R_a: G/H \to G/K$, $gH \mapsto ga^{-1}K$, provided $aHa^{-1} \subseteq K$. Also $R_a = R_b$ if and only if $ab^{-1} \in K$, and $R_b \circ R_a = R_{ba}$, whenever the composition makes sense.

3.1. Modules over the orbit category. A covariant (resp. contravariant) $\text{Or}(G, F)$–module is a covariant (resp. contravariant) functor from $\text{Or}(G, F)$ to the category of abelian groups. A morphism $M \to N$ of $\text{Or}(G, F)$–modules of the same variance is a natural transformation between the underlying functors. We denote by $\text{Hom}_{\text{Or}(G, F)}(M, N)$ the set of all morphisms $M \to N$. We denote by $\text{Mod} \circ \text{Or}(G, F)$ (resp. $\text{Mod} \to \text{Or}(G, F)$) the category of covariant (resp. contravariant) $\text{Or}(G, F)$–modules. These are abelian categories [ML95 Proposition IX 3.1] with enough projectives [MV03] p. 10.

Example 3.1. Denote the free abelian group with basis $\text{map}_G(G/H, G/K)$, by $\mathbb{Z}[G/H, G/K]$. So, $\mathbb{Z}[G/H, -]$ and $\mathbb{Z}[-, G/K]$ respectively define a covariant and a contravariant $\text{Or}(G, F)$–modules. The module $\mathbb{Z}[-, G/K]$ happens to be a free contravariant $\text{Or}(G, F)$–module.

Remark 3.2. The categories $\text{Mod} \circ \text{Or}(G, \text{Tr})$ (resp. $\text{Mod} \to \text{Or}(G, \text{Tr})$–mod) and $\text{mod} \circ G$ (resp. $\text{mod} \to G$–mod) are canonically isomorphic. To the $\text{Or}(G, \text{Tr})$–module $M$ corresponds the $G$–module $M(G/I)$, where $I$ denotes the trivial subgroup of $G$.

Given a contravariant $\text{Or}(G, F)$–module $M$ and a covariant $\text{Or}(G, F)$–module $N$, there is a tensor product $M \otimes_{\text{Or}(G, F)} N$, which is a suitable abelian group (see [MV03 p. 14]). Hence we have a tensor product functor $- \otimes_{\text{Or}(G, F)} N$ from the category of contravariant $\text{Or}(G, F)$–modules to the category of abelian groups. For every $\text{Or}(G, F)$–module $M$ with a suitable variance and for all $H \in F$, we have a Yoneda-type isomorphism (see [MV03] p. 9, p. 14])

$$\mathbb{Z}[-, G/H] \otimes_{\text{Or}(G, F)} M = M(G/H).$$

3.2. Restriction, induction and coinduction for modules over the orbit category. Let $\varphi: H \to G$ be a homomorphism of groups. Denote $\varphi^*F$ the family of subgroups of $H$ that are mapped by $\varphi$ to a group in $F$. We have a natural functor

$$\overline{\varphi}: \text{Or}(H, \varphi^*F) \to \text{Or}(G, F)$$

Given an $\text{Or}(G, F)$–module $M$, the restriction functor $\text{res}_{\varphi} (M)$ is the $\text{Or}(H, F \cap H)$–module of the same variance as $M$, defined by $\text{res}_{\varphi} (M) = M \circ \overline{\varphi}$. It is no difficult to see that $\text{res}_{\varphi}$ defines covariant functors

$$\text{res}_{\varphi}: \text{Mod} \circ \text{Or}(G, F) \to \text{Mod} \to \text{Or}(H, \varphi^*F),$$

$$\text{res}_{\varphi}: \text{Or}(G, F) \to \text{Mod} \circ \text{Or}(H, \varphi^*F) \to \text{Mod}.$$
We also have induction and coinduction functors (see [Lue89] for the definitions)

\[
\text{ind}_\varphi : \text{Or}(H, \varphi^* F) \rightarrow \text{mod} \rightarrow \text{Or}(G, F) \rightarrow \text{mod},
\]

\[
\text{coind}_\varphi : \text{mod} \rightarrow \text{Or}(H, \varphi^* F) \rightarrow \text{mod} \rightarrow \text{Or}(G, F).
\]

such that all the usual adjoint properties hold. If \( \varphi \) is an inclusion, then we use the notation \( \text{res}_H^G, \text{ind}_H^G \), and \( \text{coind}_H^G \).

### 3.3. Coinvariants

Given a \( G \)-module \( M \), we define the **coinvariants functor** \( M = \mathbb{Z}[-] \otimes_{\mathbb{Z}G} M \). We have that \( M(G/K) \) are the \( K \)-coinvariants \( M_K \) of \( M \). This defines a covariant functor from the category of \( G \)-modules to the category of coinvariant \( \text{Or}(G, F) \)-modules.

### 3.4. Spaces over the orbit category

A covariant (resp. contravariant) \( \text{Or}(G, F) \)-space is a covariant (resp. contravariant) functor \( \text{Or}(G, F) \rightarrow \text{Top} \), where \( \text{Top} \) is the category of topological spaces.

If \( X \) is a \( G \)-space we can define the fixed point contravariant \( \text{Or}(G, F) \)-space \( \overline{X} = \text{map}_G(-, X) \). We have that \( \overline{X}(G/K) = X^K \). This defines a functor from the category of \( G \)-spaces to the category of \( \text{Or}(G, F) \)-spaces.

### 4. Adamson and Takasu theories using Bredon homology

In this section we introduce Bredon homology for \( G \)-CW–complexes. The material of this section can be found in great detail in [MV03], [SG05]. In this section we define the relative group homology theories of Adamson and Takasu with coefficients in an \( \text{Or}(G) \)-module, as well as the comparison homomorphism between them. Such definitions generalize those existing in the literature.

#### 4.1. Bredon homology

If \( X \) is a \( G \)-CW–complex then we have associated the fixed point functor, i.e. a contravariant \( \text{Or}(G) \)-space \( \overline{X} : \text{Or}(G) \rightarrow \text{Top} \), and the contravariant functor \( \text{C}_*(X)(-) : \text{Or}(G) \rightarrow \text{Ch}(\mathbb{A}b) \), which is the composition of \( \overline{X} \) with the cellular chain complex functor (Subsection 3.4). Let \( M \) be a covariant \( \text{Or}(G) \)-module, we can define a chain complex of abelian groups by considering the tensor product

\[
\text{C}_*(X; M) := \text{C}_*(X) \otimes_{\text{Or}(G)} M.
\]

We describe \( \text{C}_*(X; M) \) in more detail, for further details about the boundary morphisms see [Bred7].

Let \( \Delta_i \) be the set of \( i \)-cells of \( X \), since \( G \) acts cellularly on \( X \), the set \( \Delta_i \) is a \( G \)-set. Let \( K_\sigma \) be the isotropy group of \( \sigma \in \Delta_i \). Let \( \Sigma_i \) be a set of representatives for the \( G \)-orbits in \( \Delta_i \). Then, using the distributive property of the tensor product and the Yoneda-type isomorphism

\[
\text{C}_*(X) \otimes_{\text{Or}(G)} M = \bigoplus_{\sigma \in \Sigma_i} (\mathbb{Z}[-, G/K_\sigma] \otimes_{\text{Or}(G)} M)
\]

By definition, the **Bredon homology** of the \( G \)-CW–complex \( X \) with coefficients in the \( \text{Or}(G) \)-module \( M \) (with suitable variance) is

\[
H_*^G(X; M) = H_*(\text{C}_*(X) \otimes_{\text{Or}(G)} M).
\]

In a complete analogous way, for a \( G \)-CW-pair \( (X, Y) \).

#### Example 4.1

If \( M \) is a \( G \)-module, we have the coinvariants functor \( M \). Hence, we can define the **homology of the \( G \)-CW–complex \( X \) with coefficients in the \( G \)-module \( M \) by**

\[
H_*^G(X; M) = H_*^G(X; M).
\]

#### Example 4.2

Let \( X \) be a \( G \)-CW–complex and \( M \) a \( G \)-module. Then the chain complex of abelian groups \( \text{C}_*(X; M) := \text{C}_*(X) \otimes_{\text{Or}(G)} M \) is isomorphic to the tensor product \( S_*(X) \otimes_{\mathbb{Z}[G]} M \), where \( S_*(X) \) is the classical cellular chain complex of \( X \) with the induced action of \( G \). In fact, this can be easily seen by decomposing \( S_*(X) \) as a direct sum of \( G \)-modules of the form \( \mathbb{Z}[G/K] \), the Yoneda isomorphism [5].
and the isomorphism $\mathbb{Z}[G/K] \otimes_{\mathbb{Z}[G]} M = M_K$:

$$S_\ast(X) \otimes_{\mathbb{Z}[G]} M = \bigoplus_{\sigma \in \Sigma_\ast} (\mathbb{Z}[G/K_\sigma] \otimes_{\mathbb{Z}[G]} M)$$

$$= \bigoplus_{\sigma \in \Sigma_\ast} M_{G/K_\sigma}$$

$$= \bigoplus_{\sigma \in \Sigma_\ast} (\mathbb{Z}[-, G/K_\sigma] \otimes_{\text{Or}(G)} \mathbb{M})$$

where $\Sigma_\ast$ is, as before, a set of representatives for the $G$–orbits in $\Delta_\ast$. The computation of the boundary homomorphism is straightforward.

**Example 4.3.** If $A$ is a trivial $G$–module (i.e. $ga = a$ for all $g \in G$ and $a \in A$) the functor of coinvariants $\Delta$ associated to $A$ is the constant Or($G$)–module given by $\Delta(G/K) = A$ for all subgroups $K$, and $\Delta(A) = \text{Id}$ for every morphism in Or($G$). By Example 4.4 Bredon homology of $X$ with coefficients in $\Delta$ recovers the cellular homology of $X/G$ with coefficients in the trivial $G$–module $A$.

$$H^G_\ast(X; A) = H^G_\ast(X; \Delta) = H_\ast(X/G; A).$$

**Example 4.4.** Let $(X, Y)$ be a pair of $G$-CW–complexes with $X$ simply-connected and with free $G$–action, and let $M$ be a covariant Or($G$)–module. Then $H^G_\ast(X, Y; M)$ is isomorphic to the homology of the pair $(X/G, Y/G)$ with local coefficients associated to the $G$–module $M(G/I)$. In fact, since $X$ is simply-connected we know that the canonical projection $X \to X/G$ is the universal covering projection so that $G \cong \pi_1(X/G)$.

The following theorem will be useful in the next section in order to establish the main properties of Adamson and Takasu relative homology theories.

**Lemma 4.5.** Let $G$ be a group and $H$ a subgroup of $G$. Consider a covariant Or($G$)–module $M$. Then, for any $H$-CW–complex $X$, there are natural isomorphisms

$$H^H_\ast(X; \text{res}^G_H(M)) \cong H^G_\ast(\text{ind}^G_H X; M).$$

**Proof.** The proof follows by applying the definition of Bredon homology and the classical adjunction properties of $\text{res}^G_H$ and $\text{ind}^G_H$ (see [DL98, Lemma 1.9]).

4.2. **Adamson relative group homology.** Consider a discrete group $G$, a subgroup $H$ of $G$, and an Or($G$)–module $M$ (with suitable variance). Recall that $\mathcal{F}(H)$ is the family of subgroups of $G$ generated by $H$. Then we define the **Adamson relative group homology** of the pair $(G, H)$ with coefficients in $M$ by

$$H_\ast([G : H]; M) := H^G_\ast(E_{\mathcal{F}(H)}G; M).$$

**Remark 4.6.** Note that:

- By Example 4.3 our definition reduces to the definition of Adamson relative group homology with coefficients in a trivial $G$–module given in [ANCM17].
- For $H$ trivial, we have $\mathcal{F}(H) = \text{Tr}$ and, by Remark 2.1 we get the universal covering $EG$ of the classical classifying space $BG$ of $G$, we recover the classical homology of $G$ with coefficients in the $G$–module $M(G/I)$.
- If $M$ is a $G$–module, then we recover the definition of Adamson homology with coefficients in a $G$–module given in (4) (see Example 4.1 and Example 4.2).

Now we state the main property of Adamson relative group homology. Roughly speaking, it describes an excision phenomenon. As a consequence the Adamson homology of $(G, H)$ is the group homology of the quotient $G/H$ if $H$ is a normal subgroup of $G$.

**Proposition 4.7.** Let $N$ be a normal subgroup of $G$ contained in $H$, and let $\pi: G \to G/N$ be the quotient projection. Let $M$ be an Or($G$)–module. Then, for all $n \geq 0$, we have the following isomorphisms

$$H_\ast([G : H]; M) \cong H_\ast([G/N : H/N]; \text{ind}_\pi(M)).$$

**Proof.** Consider $X$ a model for $E_{\mathcal{F}(H)/N}G/N$, then $\text{res}_\pi X$ is a model for $E_{\mathcal{F}(H)}G$. It is not difficult to see that $C_\ast(\text{res}_\pi X) = \text{res}_\pi C_\ast(X)$ as chain complexes in $\text{mod}$–Or($G$). Now, using [MP02, Lemma 3.1
and Proposition 3.2, we have the following isomorphisms
\[
\text{res}_\pi C_\ast(X) \otimes_{\text{Or}(G)} M \cong (C_\ast(X) \otimes_{\text{Or}(G/N)} \mathbb{Z}[\pi(-), -]) \otimes_{\text{Or}(G)} M \\
\cong C_\ast(X) \otimes_{\text{Or}(G/N)} (\mathbb{Z}[\pi(-), -] \otimes_{\text{Or}(G)} M) \\
\cong C_\ast(X) \otimes_{\text{Or}(G/N)} \text{ind}_\pi M.
\]
Applying homology to both sides we get the conclusion for Adamson homology. □

Using Example [13] and a straightforward computation of the coefficients, we get the following corollary.

**Corollary 4.8.** Let \( N \) be a normal subgroup of \( G \) contained in \( H \). Let \( M \) be a \( G \)-module. Then, for all \( n \geq 0 \), we have the following isomorphisms
\[
H_n([G:H]; M) \cong H_n([G/N:N]/M)
\]
where \( M_N \) is the group of coinvariants of \( M \).

### 4.3. Takasu relative group homology

Let \( G \) be a discrete group and let \( H \) be a subgroup of \( G \). Regarding \( EG \) as an \( H \)-CW–complex, we have a map \( EH \rightarrow EG \) unique up to \( H \)–homotopy, which, finally, leads to a \( G \)-map \( \iota_H^G : \text{ind}_H^G EH \rightarrow EG \). From now on, we will always assume that the map \( \iota_H^G \) is an inclusion, by replacing \( EG \) with the mapping cylinder of \( \iota_H^G \). This assumption is due to the fact that we want to consider the \( CW \)–pair \( (EG, \text{ind}_H^G EH) \), which does not always make sense. A second reason is that we will make use of some push-out constructions, (see equation (6) and Theorem 6.1), hence our assumption will turn them into homotopy \( G \)-push-outs.

Consider an \( \text{Or}(G) \)-module \( M \). Define the **Takasu relative group homology of \( (G, H) \)** to be
\[
H_\ast(G, H; M) := H_\ast^G(EG, \text{ind}_H^G EH; M).
\]

**Remark 4.9.** Note that:

- By Example [13] our definition reduces to the definition of Takasu relative group homology with coefficients in a trivial \( G \)-module given in [16].
- If \( H = I \) is the trivial subgroup, then we recover the classical homology of \( G \) with coefficients in the \( G \)-module \( M(G/I) \).
- If \( M \) is a \( G \)-module, then we recover the definition of Takasu homology with coefficients in a \( G \)-module given in [13], (see Example [13] and Example [13]).

Now we state the main property of Takasu relative group homology, this is, a long exact sequence that relates the homology of \( G \) and \( H \) to the homology of the pair \( (G, H) \). This long exact sequence can be interpreted as the fact that Takasu’s theory resembles the quotient of the homologies of \( G \) and \( H \). In fact, in Subsection [4.4] we show an example where this phenomenon is more evident.

**Theorem 4.10.** Let \( G \) be a group and \( H \) a subgroup. Let \( M \) be a covariant \( \text{Or}(G) \)-module. For \( n \geq 0 \), there exists a long exact sequence of the form,
\[
\cdots \rightarrow H_{n+1}(G, H; M) \rightarrow H_n(H; \text{res}_H^G M(H/I)) \rightarrow H_n(G; M(G/I)) \rightarrow H_n(G, H; M) \rightarrow \cdots.
\]

**Proof.** It follows from the induction structure and the long exact sequence of the pair \( (EG, \text{ind}_H^G EH) \). In fact, we have the following commutative diagram where every vertical arrow is an isomorphism using Lemma [4.5] and Example [4.4].

\[
\begin{array}{ccccccc}
\cdots & \rightarrow & H_n^G(\text{ind}_H^G EH; M) & \rightarrow & H_n^G(EG; M) & \rightarrow & H_n^G(EG, \text{ind}_H^G EH; M) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & H_n^H(EG; \text{res}_H^G M) & \rightarrow & H_n^G(EG; M) & \rightarrow & H_n(G, H; M) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & H_n(G; \text{res}_H^G M(H/I)) & \rightarrow & H_n(G; M(G/I)) & \rightarrow & H_n(G, H; M) & \rightarrow & \cdots \\
\end{array}
\]

□

**Remark 4.11.** By Example [13] this exact sequence reduces to the one in [16] for \( M \) the constant functor \( \mathbb{Z} \).
Let us give an equivalent definition of Takasu relative homology, at least in degrees greater than or equal to 2. This addresses Problem 2.2.

The Takasu space $T(G, H)$ of $(G, H)$ is the $G$-CW–complex given by the following $G$–pushout

(6) \[
\begin{array}{ccc}
\text{ind}^G_H \mathbb{E}H & \xrightarrow{\phi} & \mathbb{E}G \\
\downarrow & & \downarrow \\
G/H & \longrightarrow & T(G, H),
\end{array}
\]

where the left map is induced by collapsing each connected component of $\text{ind}^G_H \mathbb{E}H$ to a point.

**Remark 4.12.** Note that the cone points form a $G$–orbit of 0–cells of $T(G, H)$ that can be identified with $G/H$, hence we can make sense to the pair $(T(G, H), G/H)$. Note that the points in this orbit are the only ones with non-trivial isotropy.

**Theorem 4.13.** Let $G$ be a group and $H$ a subgroup. Then, for all $n \geq 0$, the quotient map $(\mathbb{E}G, \text{ind}^G_H \mathbb{E}H) \to (T(G, H), G/H)$ induces an isomorphism

$$H_n(G; H; M) \to H_n^G(T(G, H), G/H; M),$$

for every $G$–module $M$.

**Proof.** The proof is essentially the same as that of [Hat02, Proposition 2.22], using the excision and homotopy invariance axioms of Bredon homology.

The following corollary gives an answer to Problem 2.2.

**Corollary 4.14.** For all $n \geq 2$ and every $G$–module $M$ (in particular for every $G$–module), We have the following isomorphism

$$H_n(G; H; M) \cong H_n^G(T(G, H); M)$$

**Proof.** It follows from the long exact sequence of the pair $(T(G, H), G/H)$ and the fact that $H_n^G(G/H; M)$ is zero for all $n \geq 1$ and for every $M$.

**4.4. The comparison homomorphism.** Since $T(G, H)$ is defined via the $G$–pushout (6) and both $\text{ind}^G_H \mathbb{E}H$ and $\mathbb{E}G$ are free $G$-CW–complexes, while for the points of $G/H$ all the isotropy groups are conjugated to $H$, we conclude that all the isotropy groups belong to the family $F(H)$. Therefore there is a $G$–map, unique up to $G$–homotopy

(7) \[
T(G, H) \to E_F(H)G,
\]

which leads to the $G$–map of pairs

$$(T(G, H), G/H) \to (E_F(H)G, G/H).$$

Applying the long exact sequence to this map of pairs, and using the fact that $H_i^G(G/H; M) = 0$ for all $i \geq 1$, we get (see Corollary 4.14), for $n \geq 2$ the following commutative square

(8) \[
\begin{array}{ccc}
H_n(T(G, H); M) & \xrightarrow{\cong} & H_n(G, H; M) \\
\downarrow & & \downarrow \\
H_n([G : H]; M) & \xrightarrow{\cong} & H_n(E_F(H)G, G/H; M).
\end{array}
\]

Hence we have a homomorphism

$$\phi_n : H_n(G, H; M) \to H_n([G : H]; M),$$

for all $n \geq 2$ and for every covariant $G$–module $M$, and analogously for cohomology. We call $\phi$ the comparison homomorphism.

**Remark 4.15.** Using the construction above, we still have the following commutative diagram with exact rows, that might help us to compare Takasu and Adamson relative homology theory in dimensions 0 and 1:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & H_0(T(G, H); M) & \longrightarrow & H_1(T(G, H); M) & \longrightarrow & H_0(G/H; M) & \longrightarrow & H_0(T(G, H); M) & \longrightarrow & H_0(G, H; M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_0([G : H]; M) & \longrightarrow & H_1(E_F(H)G, G/H; M) & \longrightarrow & H_0(G/H; M) & \longrightarrow & H_0([G : H]; M) & \longrightarrow & H_0(E_F(H)G, G/H; M) & \longrightarrow & 0
\end{array}
\]
Note that, in case $M$ is a constant Or($G$)-module, then the maps $H^*_G(G/H; M) \rightarrow H^*_G(T(G, H); M)$ and $H^*_G(G/H; M) \rightarrow H^*_G(E_{\mathcal{F}H}G; M)$ are split injective since they can be identified with singular homology maps (see Example 4.3), therefore, at least in this case, we also have defined the comparison homomorphism in dimensions 0 and 1.

Recall that a subgroup $H$ of $G$ is said to be malnormal if $g^{-1}Hg \cap H = I$ for all $g \in G \setminus H$. The following theorem addresses the first question in Problem 2.4.

**Theorem 4.16.** The following are equivalent

1. $H$ is a malnormal subgroup of $G$;
2. the Takasu space $T(G, H)$ of $(G, H)$ is a model for the classifying space $E_{\mathcal{F}H}G$.
3. The comparison homomorphism

$$\varphi: H_*(G, H; M) \rightarrow H_*([G : H]; M)$$

is an isomorphism for every covariant Or($G$)-module $M$.

**Proof.** By [ANCM17] Proposition 7.11, the Takasu space $T(G, H)$ is a model for $E_{\mathcal{F}H}G$ if and only if $H$ is a malnormal subgroup of $G$. Hence (1) is equivalent to (2).

Assume that $T(G, H)$ is a model for $E_{\mathcal{F}H}G$. Since $E_{\mathcal{F}H}G$ is unique up to $G$ homotopy, the (unique up to $G$ homotopy) $G$–map $T(G, H) \rightarrow E_{\mathcal{F}H}G$ in (7) induces an isomorphism in homology for every Or($G$)-module $M$, therefore the left map in (8) is an isomorphism and we conclude that the comparison homomorphism is an isomorphism.

Now suppose that $\varphi$ is an isomorphism for all Or($G$)–modules $M$, we shall verify that $T(G, H)$ is a model for $E_{\mathcal{F}H}G$.

From the definition, we can conclude that $T(G, H)$ is contractible since we are collapsing contractible subcomplexes of the contractible space $E_{\mathcal{F}}G$ to obtain $T(G, H)$. Also, we know that, for a non-trivial subgroup $K \in \mathcal{F}(H)$, the space $T(G, H)^K$ coincides with $(G/H)^K$, in particular, it is a discrete subspace of $T(G, H)$. Hence $T(G, H)^K$ is contractible if and only if it consist of exactly one point. Now we shall prove that $T(G, H)^K$ is a discrete space with the (singular) cohomology of the one-point space. From [MS95] Pag. 252, eq. (5), there exists an Or($G$)–module $N$, such that, for any $G$–space $X$, we have the following natural isomorphism

$$H^*_G(X; N) \cong H^*(X^K; Z),$$

where the right hand side is singular cohomology. Hence $\varphi$ induces the following commutative diagram

$$
\begin{array}{ccc}
H^*_G(T(G, H); N) & \xrightarrow{\cong} & H^*(T(G, H)^K; Z) \\
\downarrow{\varphi^*} & & \downarrow{\varphi^*} \\
H^*_G(E_{\mathcal{F}H}G; N) & \xrightarrow{\cong} & H^*((E_{\mathcal{F}H}G)^K; Z).
\end{array}
$$

Hence, we conclude that $T(G, H)^K$ is a one-point space. This finishes the proof.

**Remark 4.17.** In [ANCM17] Theorem 7.13, the if part of the Theorem 4.16 is proved using a constant Or($G$)-module as coefficients, while for the only if part we strongly use the fact that the comparison homomorphism is an isomorphisms for any coefficients.

## 5. ADAMSON AND TAKASU THEORIES VIA DERIVED FUNCTORS

Let $G$ be a group and let $\mathcal{F}$ be a family of subgroups of $G$. In this section we recall the definition of the derived functor Tor in the category of Or($G$, $\mathcal{F}$)–modules. Then, taking $H \leq G$, we use them to give algebraic definitions of Adamson and Takasu relative homology theories for the pair $(G, H)$. We then give a description of the comparison homomorphism using derived functors.

Recall there is an inclusion $\text{mod- Or}(G, \mathcal{F}) \rightarrow \text{mod- Or}(G)$, which sends a contravariant Or($G$, $\mathcal{F}$)–module $M$ to a contravariant Or($G$)–module, which we denote by $\widehat{M}$, by setting $\widehat{M}(G/K) = M(G/K)$ for $K \in \mathcal{F}$ and $\widehat{M}(G/K) = 0$ for $K \notin \mathcal{F}$.

### 5.1. Tor functors.

A sequence of $\text{Or}(G, \mathcal{F})$-modules is exact if, for all $K \in \mathcal{F}$, when evaluated at $G/K$, the resulting sequence of abelian groups is exact.

Let $N$ be a contravariant Or($G$, $\mathcal{F}$)–module and let $M$ be a covariant Or($G$, $\mathcal{F}$)–module. Consider a projective resolution $P_N$ of $N$. As in the classical setting we have a Tor functor given by

$$\text{Tor}_i^{\text{Or}(G, \mathcal{F})}(N, M) = H_i(P_N \otimes_{\text{Or}(G, \mathcal{F})} M).$$
5.2. Adamson relative group homology. Now we are interested in defining Adamson homology of a pair \((G, H)\) using the Tor and Ext functors. For this, we take \(F = F(H)\). One can obtain \(Or(G, F(H))\)-projective resolutions from a model of \(E_{F(H)} G\) (see [MIV03, p. 11]). We include this construction for completeness. First, define the augmentation homomorphism \(\varepsilon : C_0(X)(-) \to \mathbb{Z}\) as the usual augmentation homomorphism \(\varepsilon : C_0(X^K) \to \mathbb{Z}\) for all \(K \in F(H)\).

**Proposition 5.1.** Let \(X\) be a model for \(E_{F(H)} G\). Then the augmented chain complex \(C_\ast(X)\) is a free, and therefore projective, \(Or(G, F(H))\)-resolution of the constant \(Or(G, F(H))\)-module \(\mathbb{Z}\).

**Proof.** Since \(X^K\) is contractible, the augmented chain complex \(C_\ast(X)(G/K) = C_\ast(X^K)\) is acyclic. Therefore the augmented chain complex functor \(C_\ast(X)\) is exact.

On the other hand, since \(X\) has isotropy groups in \(F(H)\), the \(Or(G, F)\)-modules \(C_\ast(X)\) are all free modules. Therefore the augmented chain complex \(C_\ast(X)\) is a free resolution of \(\mathbb{Z}\).

Let \(P_*\) be an \(Or(G, F(H))\)-projective resolution of the constant \(Or(G, F(H))\)-module \(\mathbb{Z}\). Hence the induced sequence \(\hat{P}_*\) of \(Or(G)\)-modules, is an \(Or(G)\)-projective resolution of the \(Or(G)\)-module \(\mathbb{Z}\).

**Theorem 5.2.** Let \(G\) be a group and \(H\) a subgroup. For all \(i \geq 0\) and for all \(Or(G, F)\)-module \(M\), we have the following isomorphisms

\[
H_i([G : H]; M) \cong \text{Tor}_i^{Or(G)}(\mathbb{Z}, M)
\]

**Proof.** It follows from Proposition 5.1.

**Remark 5.3.** In [Hoc56], Adamson relative group homology is described as derived functors, using the language of relative homological algebra. On the other hand, in the above description, Adamson homology is described using derived functors within an abelian category with enough projectives. This gives an algebraic approach completely analogue to the classical group homology.

5.3. Takasu relative group homology. Now we want to define Takasu homology of a pair \((G, H)\) using derived functors. In Takasu’s original paper [Tak59], there is already a very nice approach for \(H_*(G, H; M)\) via derived functors (see also [ANCM17, Section 5.3]). Recall that the category of contravariant \(Or(G, Tr)\)-modules is canonically isomorphic to both, the category of right \(G\)-modules and the category of left \(G\)-modules, hence there is no substantial difference between the three of them (see Remark 3.2). Consider the \(Or(G, Tr)\)-module \(I_{(G, H)}(\mathbb{Z}) := \text{ker}(\mathbb{Z}[G/H] \to \mathbb{Z})\) and the corresponding \(Or(G)\)-module \(I_{(G, H)}(\mathbb{Z})\). Hence we have that \(I_{(G, H)}(\mathbb{Z})(G/I) = I_{(G, H)}(\mathbb{Z})\) and \(I_{(G, H)}(\mathbb{Z})(G/K) = 0\) for all \(K \neq I\). Hence, a \(G\)-projective resolution of the \(G\)-module \(I_{(G, H)}(\mathbb{Z})\) corresponds to a \(Or(G)\)-projective resolution of the \(Or(G)\)-module \(I_{(G, H)}(\mathbb{Z})\).

**Theorem 5.4.** For every \(Or(G)\)-module \(M\) and every \(i \geq 1\), We have the following isomorphisms

\[
H_i(G, H; M) \cong \text{Tor}_{i-1}^{Or(G)}(I_{(G, H)}(\mathbb{Z}), M(G/I)) \cong \text{Tor}_{i-1}^{Or(G)}(I_{(G, H)}(\mathbb{Z}), M)\]

**Proof.** Consider the standard resolution \(C_\ast(G)\) (resp. \(C_\ast(H)\)) of the trivial \(G\)-module (resp. \(H\)-module) \(\mathbb{Z}\). By [Tak59, Proposition 3.3] the exact sequence

\[
\cdots \to C_2(G)/\text{ind}_H^G C_2(H) \to C_1(G)/\text{ind}_H^G C_1(H) \to I_{(G, H)}(\mathbb{Z}) \to 0
\]

is a \(G\)-projective resolution of \(I_{(G, H)}(\mathbb{Z})\), so it can be seen as an \(Or(G)\)-projective resolution of \(I_{(G, H)}(\mathbb{Z})\). On the other hand, by Remark 3.2 the sequence (9) corresponds to the cellular chain complex of the Takasu space \(T(G, H)\). Thus for any \(Or(G)\)-moduel \(M\), (9) can be used to compute the Takasu relative homology of \((G, H)\).

5.4. The algebraic version of the comparison homomorphism. As in Subsection 5.3, consider an \(Or(G)\)-projective resolution \(P_* \to I_{(G, H)}(\mathbb{Z})\).

On the other hand, since the family \(F(H)\) is generated by \(H\), we have that the augmentation map \(\varepsilon : \mathbb{Z}[-, G/H] \to \mathbb{Z}\) is surjective. Next complete this surjection to a projective \(Or(G, F(H))\)-resolution \(Q_* \to \mathbb{Z}\) so that \(Q_0 = \mathbb{Z}[-, G/H]\). Notice that \(I_{(G, H)}(\mathbb{Z}) = \ker \varepsilon(G/I)\). Consider an \(Or(G)\)-projective resolution \(\hat{Q}_* \to \hat{\mathbb{Z}}\). Since \(I_{(G, H)}(\mathbb{Z}) = \ker \varepsilon(G/I)\), the inclusion \(I_{(G, H)}(\mathbb{Z}) \to \ker \varepsilon\), can be extended to a morphism of resolutions

\[
\varphi_* : P_* \to \hat{Q}_*.
\]
Given an $\text{Or}(G)$–module $M$, this induces homomorphisms
\begin{equation}
\varphi_i : H_i(G, H; M) \to H_i([G : H]; M), \quad \text{for all } i \geq 2.
\end{equation}

Finally, we construct a long exact sequence on which the comparison homomorphism (for each dimension) fits. This addresses the second question in Problem 2.3.

Let $J(G, H)$ be the quotient $\ker \varepsilon / I(G, H)(\mathbb{Z})$, so that we have the following long exact sequence
\[ 0 \to I(G, H)(\mathbb{Z}) \to \ker \varepsilon \to J(G, H) \to 0 \]

Note that $\text{Tor}^\text{Or}(G)(\ker \varepsilon, M) = H_1([G : H]; M)$. Given an $\text{Or}(G)$–module $M$, we get a long exact sequence of $\text{Tor}^\text{Or}(G)(\cdot, M)$ groups
\[ \cdots \to H_1(G, H; M) \xrightarrow{\varphi_1} H_1([G : H]; M) \to \text{Tor}^\text{Or}(G)(\widehat{G/H}, M) \to H_1(G, H; M) \to \cdots \]
\[ \to H_2(G, H; M) \xrightarrow{\varphi_2} H_2([G : H]; M). \]

Essentially $\text{Tor}^\text{Or}(G)(\widehat{G/H}, M)$ is measuring how far is the comparison homomorphism from being an isomorphism.

We describe $J(G, H) = \ker \varepsilon / I(G, H)(\mathbb{Z})$ in more detail. Recall that $\varepsilon : \mathbb{Z}[-, G/H] \to \mathbb{Z}$ is the augmentation homomorphism. Then we have
\[ J(G, H)(G/K) = \begin{cases} 
\ker(\mathbb{Z}[G/H]^K \to \mathbb{Z}) & \text{if } K \neq I \text{ and } K \in \mathcal{F}(H), \\
0 & \text{otherwise}.
\end{cases} \]

The fixed point set $(G/H)^K$ is the set of all cosets $gH$ such that $K \leq g^{-1}Hg$. In particular, if $K \in \mathcal{F}(H)$ is contained in only one conjugate $g^{-1}Hg$, then $\ker(\mathbb{Z}[G/H]^K \to \mathbb{Z}) = \ker \mathbb{Z} \to \mathbb{Z} = 0$.

Hence, we can conclude that $J(G, H)$, and therefore $\text{Tor}^\text{Or}(G)(\widehat{G/H}, M)$, only depends on the following family
\[ \mathcal{G}(H) := \{ K \leq G : K \leq g_i^{-1}Hg_i \cap g_j^{-1}Hg_j \text{ for some } g_i, g_j \text{ such that } g_iH \neq g_jH \} \subset \mathcal{F}(H) \]

Explicitly, for $K \in \mathcal{F}(H)$,
\[ J(G, H)(G/K) = \begin{cases} 
\ker(\mathbb{Z}[G/H]^K \to \mathbb{Z}) & \text{if } K \neq I \text{ and } K \in \mathcal{G}(H), \\
0 & \text{otherwise}.
\end{cases} \]

As a very particular case, assume $H$ is a malnormal subgroup of $G$. Then $\mathcal{G}(H)$ is the trivial family, hence $J(G, H)(G/K) = 0$ for every $K \in \mathcal{F}(H)$. Thus we partially recover Theorem 4.10 that is, we have that the comparison homomorphism $\varphi_i$ is an isomorphism, for $i \geq 2$, provided $H$ is a malnormal subgroup of $G$.

6. The Lück–Weiermann construction and some exact sequences

In this section, we obtain in Corollary 4.3 a Takasu-type long exact sequence for Adamson homology. To get such a sequence we describe a construction of the classifying space $E_{\mathcal{F}(H)\Gamma}$ similar to [8], using [LW12] Theorem 2.3.

First we need some notation. Let $G$ be a group. Consider two families $\mathcal{F} \subseteq \mathcal{G}$ of subgroups of $G$. Suppose we have an equivalence relation $\sim$ on $\mathcal{G} \setminus \mathcal{F}$ satisfying:

- **Closed under taking subgroups**: For $H, K \in \mathcal{G} \setminus \mathcal{F}$, with $H \subseteq K$ then must be $H \sim K$, and
- **Invariant under conjugation**: For $H, K \in \mathcal{G} \setminus \mathcal{F}$ and $g \in G$, then must be $H \sim K$ if and only if $g^{-1}Hg \sim g^{-1}Kg$.

For $H \in \mathcal{G} \setminus \mathcal{F}$, define the subgroup of $G$
\[ N_G[H] = \{ g \in G \mid g^{-1}Hg \sim H \}. \]

Also define the following family of subgroups of $N_G[H]$
\[ \mathcal{G}[H] = \{ K \subseteq N_G[H] \mid K \in \mathcal{G} \setminus \mathcal{F}, K \sim H \} \cup \{ K \subseteq N_G[H] \mid K \in \mathcal{F} \}. \]

As an immediate consequence of [LW12] Theorem 2.3 we have the following,
Theorem 6.1. With the same notation as above, let $H$ be a subgroup of $G$, let $\mathcal{G}$ the family of subgroups of $G$ generated by $H$, and let $\mathcal{F}$ be a subfamily of $\mathcal{G}$, such that $E_\mathcal{F}N_G[H]$ and $E_{\mathcal{G}/H}N_G[H], and a model for $E_\mathcal{F}G$. Now consider $X$ defined by the $G$-pushout:

$$G \times N_G[H] E_{\mathcal{F}}N_G[H] \xrightarrow{i} E_{\mathcal{F}}G$$

$$\downarrow \quad \downarrow$$

$$G \times N_G[H] E_{\mathcal{G}/H}N_G[H] \xrightarrow{\text{Id}_G \times N_G[H]} X$$

where the maps starting from the left upper corner are cellular and one of them is an inclusion of $G$-CW-complexes. Then $X$ is a model for $E_{\mathcal{G}}G$.

Remark 6.2. The $G$-pushout from the previous theorem, can be seen as a kind of analogue of the $G$-pushout used to define the Takasu’s space $T(G, H)$. This point of view might be helpful since this pushout construction gives us a more explicit description of the difference between the classifying space $E_{\mathcal{F}(G)H}G$ and Takasu’s space $T(G, H)$.

Corollary 6.3. Let $G$ be a group, let $\mathcal{G}$ be the family generated by a subgroup $H$ of $G$, let $\mathcal{F}$ be a subfamily of $\mathcal{G}$ and let $M$ be an $Or(G, \mathcal{G})$–module. Then we have the following Mayer–Vietoris exact sequence

$$\cdots \rightarrow H_n^{N_G[H]}(E_\mathcal{F}N_G[H]; \text{res}^G_{N_G[H]} M) \rightarrow H^n(G; M) \oplus H_n^{N_G[H]}(E_{\mathcal{G}/H}N_G[H]; \text{res}^G_{N_G[H]} M) \rightarrow H_n([G : H]; M) \rightarrow \cdots$$

In particular, if $\mathcal{F}$ is the trivial family, then we have the following exact sequence

$$\cdots \rightarrow H_n(N_G[H]; \text{res}^G_{N_G[H]} (M(N_G[H]/I))) \rightarrow H_n(G; M(I)) \oplus H_n^{N_G[H]}(E_{\mathcal{G}/H}N_G[H]; \text{res}^G_{N_G[H]} M(N_G[H]/I)) \rightarrow H_n([G : H]; M) \rightarrow \cdots$$

6.1. Malnormal subgroups. Let $H$ be a malnormal subgroup of $G$. We consider $\text{Tr} \subset \mathcal{F}(H)$. Define the equivalence relation $\sim$ on $\mathcal{F}(H) \setminus \text{Tr}$ as follows: Let $K_1$ and $K_2$ be groups in $\mathcal{F}(H) \setminus \text{Tr}$, then there exists $g_1, g_2 \in G$ such that $K_1 \subseteq g_1Hg_1^{-1}$ and $K_2 \subseteq g_2Hg_2^{-1}$. Hence we say that $K_1 \sim K_2$ if and only if $g_1Hg_1^{-1} \cap g_2Hg_2^{-1} \neq \emptyset$. Then $N_G[H] = \{g \in G \mid gHg^{-1} \cap H \neq \emptyset\} = H$ and $\mathcal{G}[H]$ is the family of all subgroups of $H$.

Consider an $Or(G)$–module $M$, then applying Corollary 6.3 we have the following long exact sequence

$$\cdots \rightarrow H^1_{\text{Tr}H}(E_\mathcal{F}H; M) \rightarrow H_n(G; M) \oplus H^1_{\text{ALL}H}(E_{\mathcal{G}H}H; M) \rightarrow H_n([G : H]; M) \rightarrow \cdots$$

since $E_{\text{ALL}B}$ has as a model the one-point space. Hence the above sequence translates to

$$\cdots \rightarrow H_{n+1}([G : H]; M) \rightarrow H_n(H; M) \rightarrow H_n(G; M) \rightarrow H_n([G : H]; M) \rightarrow \cdots$$

6.2. The long exact sequence for good triples. In this subsection we will show that, for certain triples $K \leq H \leq G$, we have a long exact sequence in Adamson homology similar to the sequence of a triple of spaces in singular homology.

Consider a triple of groups $K \leq H \leq G$, and consider the families $\mathcal{F}(K)$ and $\mathcal{F}(H)$ generated by $K$ and $H$ respectively. Define the following equivalence relation in $\mathcal{F}(H) \setminus \mathcal{F}(K)$: for any $H_1, H_2$ in $\mathcal{F}(H) \setminus \mathcal{F}(K)$, there exist $g_1, g_2 \in G$ such that $H_1 \leq g_1Hg_1^{-1}$ and $H_2 \leq g_2Hg_2^{-1}$, then we set

$$H_1 \sim H_2 \iff g_1Hg_1^{-1} \cap g_2Hg_2^{-1} \in \mathcal{F}(H) \setminus \mathcal{F}(K),$$

i.e. $g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$ is not subconjugate to $K$.

It is straightforward that this equivalence relation satisfies the required properties in order to use the Lück–Weiermann construction.

We say that $K \leq H \leq G$ is a good triple, if $H = N_G[H] = \{g \in G \mid H \sim gHg^{-1}\}$, i.e. if, for all $g \in G \setminus H$, $H \cap gHg^{-1}$ is subconjugated to $K$.

Theorem 6.4. Using the above notation. Suppose that $K \leq H \leq G$ is a good triple. Then, for all $Or(G)$–module $M$, we have a long exact sequence such that, for all $n > 0$, looks like

$$\cdots \rightarrow H_n([H : K]; \text{res}^G_{H} M) \rightarrow H_n([G : K]; M) \rightarrow H_n([G : H]; M) \rightarrow H_{n-1}([H : K]; \text{res}^G_{H} M) \rightarrow \cdots,$$

while for $n = 0$ we have

$$\cdots \rightarrow H_0([H : K]; \text{res}^G_{H} M) \rightarrow H_0([G : K]; M) \oplus M(G/H) \rightarrow H_0([G : H]; M) \rightarrow 0.$$
Proof. Note that for a good triple $N_G[H] = H$ and $G[H]$ is the family of all subgroups of $N_G[H] = H$, since $H$ always belongs to $G[H]$ by definition. Hence $E_{G[H]}N_G[H]$ has as a model the one-point space.

The conclusion follows from Corollary \ref{cor:iso} by setting $F = F(K)$, $G = F(H)$. \hfill \Box

Example 6.5. If $H$ is a malnormal subgroup of $G$, then it follows that for every $K \leq H$, the triple $K \leq H \leq G$ is a good triple.

Example 6.6. Consider the triple $T \leq B \leq G$, where $G = SL_2(\mathbb{C})$,

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}, \quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \right\}.$$

We also consider this groups endowed with the discrete topology. Then, we claim that they are a good triple. In fact, since $B$ is a proper maximal subgroup of $G$ (see \text{[Land2]} Proposition XIII.8.2), we conclude that $N_G[B]$ is either $G$ or $B$. Therefore it suffices to exhibit an element in $G$ that does not belong to $N_G[B]$, i.e. an element $g \in G$ such that $gBg^{-1} \cap B$ is subconjugate to $T$. A direct computation shows that we can take $g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

7. Examples

7.1. Normal subgroups. Let $G$ be a group and $H$ a normal subgroup of $G$. Let $M$ be a $G$–module. Then by Corollary \ref{cor:iso, cor:iso2}

$$H_\ast([G:H]; M) \cong H_\ast(G/H; M_H)$$

Recall the Lyndon–Hochshild–Serre spectral sequence

$$E^2_{p,q} = H_p(G/H; H_q(H; \text{res}^G_H M)) \implies H_{p+q}(G; M)$$

Proposition 7.1. Suppose that we have a normal subgroup $H$ of $G$, and a $G$–module $M$ such that $E^2_{p,q} = 0$, for all $p, q > 0$. Then we have the commutative diagram

$$\cdots \longrightarrow H_n(H; \text{res}^G_H M)_{G/H} \longrightarrow H_n(G; M) \longrightarrow H_n(G/H; M_H) \longrightarrow H_{n-1}(H; \text{res}^G_H M)_{G/H} \longrightarrow \cdots$$

where the first vertical arrow is the quotient map, the second is the identity, and the third is the comparison map composed with the isomorphism above.

Proof. The top row is straightforward from the spectral sequence and the hypothesis $E^2_{p,q} = 0$, for all $p, q > 0$. The commutativity of the diagram follows from the following commutative diagram (up to $G$–homotopy)

$$\begin{array}{ccc}
G \times H EH & \longrightarrow & EG \\
\downarrow & & \downarrow \\
G \times H EH & \longrightarrow & E_{G(H)}G = E(G/H)
\end{array}$$

and the definition of the transgression (see \text{[McC01]} p. 185) homomorphism in the Lyndon–Hochshild–Serre spectral sequence. \hfill \Box

Corollary 7.2. Consider $G$, $H$, and $M$ as in the previous proposition. If the action of $G$ on $H$ is by inner automorphisms, then the comparison homomorphism

$$\varphi: H_\ast(G, H; M) \to H_\ast([G : H]; M)$$

is an isomorphism, for all $G$–modules $M$.

Proof. It follows from the previous proposition and the five lemma, since the left vertical arrow in the Proposition \ref{prop:iso} is then the identity. \hfill \Box

Remark 7.3. Note that the hypothesis from Proposition \ref{prop:iso} and its corollary depend on the coefficients $M$. Therefore this does not lead to a contradiction with Theorem \ref{thm:iso}. \hfill \Box
Example 7.4. As a straightforward example we have the cyclic groups \((C_n, C_m)\) with \(m\) and \(n\) relative primes and \(M\) equal to the trivial \(G\)-module \(\mathbb{Z}\). In fact, using the universal coefficient theorem and the fact that \(H_i(C_m; \mathbb{Z})\) vanishes for \(i \neq 0\), we have that
\[
E^2_{p,q} = H_p(C_n/C_m; H_q(C_m; M)) \cong 0
\]
for all \(p, q > 0\). Moreover, since both \(H_i([C_n : C_m]; \mathbb{Z})\) and \(H_i(C_n, C_m; \mathbb{Z})\) are finite groups for every \(i > 0\), we can conclude that the comparison homomorphism is actually an isomorphism. We will show in Subsection below that, the hypothesis of \(m\) and \(n\) being relative prime numbers, is necessary.

7.2. The pair \((K \times H, H)\). Throughout this section we assume the coefficients module is \(\mathbb{Z}\). Consider the pair \((K \times H, H)\), where we are making an abuse of notation by denoting \(I \times H\) by \(I\). Then we have the split short exact sequence
\[
I \rightarrow H \xrightarrow{i} K \times H \rightarrow K \rightarrow I,
\]
and the projection homomorphism \(j: K \times H \rightarrow H\) such that \(j \circ i\) is the identity homomorphism on \(H\).

Now, from Proposition we have the following long exact sequence for Takasu’s relative homology
\[
\cdots \rightarrow H_{n+1}(K \times H, H) \rightarrow H_n(H) \xrightarrow{j_*} H_n(K \times H) \rightarrow H_n(K \times H, H) \rightarrow \cdots
\]
Thus, the homomorphism induced by \(j\) leads to short split exact sequences
\[
I \rightarrow H_n(H) \xrightarrow{i_*} H_n(K \times H) \rightarrow H_n(K \times H, H) \rightarrow I
\]
for all \(n \geq 0\). Therefore
\[
H_n(K \times H, H) \cong H_n(K \times H)/H_n(H)
\]
for all \(n \geq 1\).

On the other hand, \(H_0(G) = H_n(K \times H)\) can be calculated using the Künneth formula
\[
H_n(K \times H) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(H) \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(K), H_j(H))
\]
While for Adamson’s group homology, from Proposition we have
\[
H_i([K \times H, H]) \cong H_i(K \times H/H) = H_i(K)
\]
In this case the comparison homomorphism \(\varphi_i: H_i(K \times H, H) \rightarrow H_i([K \times H : H])\) can be completely described. In fact, it is not difficult to see that this homomorphism comes from projecting \(H_n(G, H)\) onto the copy of \(H_n(K)\) contained as a summand (described above as the column in the spectral sequence-type array). Hence the comparison homomorphism is surjective and the kernel is given by
\[
\bigoplus_{i+j=n, i \geq 0, j > 0} H_i(K, H_j(H)) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(H) \oplus \bigoplus_{i+j=n-1, i > 0, j > 0} \operatorname{Tor}(H_i(K), H_j(H))
\]
For the case of more general coefficients an explicit description of the comparison homomorphism, seems to be more complicated.

7.3. Finite cyclic groups. Let \(C_i\) denote the cyclic group of order \(i\). In Example 7.3, the authors exhibit the pair \((C_4, C_2)\) as an example where the Adamson’s and Takasu’s homology groups are isomorphic, which might sound as a contradiction with Theorem Nevertheless, they do not mention anything about this isomorphism being related to the comparison homomorphism. In this example we show that the comparison homomorphism \(\varphi_i: H_i(C_4, C_2; \mathbb{Z}) \rightarrow H_i([C_4 : C_2]; \mathbb{Z})\) vanishes for all \(i \geq 2\), in particular, it is not an isomorphism.

Let \(G = C_n\) be the cyclic group of order \(n\) generated by the element \(t\). Consider \(N = 1+t+t^2+\cdots+t^{n-1}\), then
\[
\nu : \cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0
\]
is a \(G\)-projective resolution of the trivial \(G\)-module \(\mathbb{Z}\), where the homomorphisms are multiplication by \(t-1\) and \(N\), and \(\varepsilon\) is the augmentation homomorphism (see for instance [Bro82, Pag. 20]).

On the other hand, consider a subgroup \(C_m = H\) of \(G\). Hence, \(H\) is normal and the quotient group \(G/H\) is also cyclic generated by \(tH\), so
\[
\nu' : \cdots \xrightarrow{N'} \mathbb{Z}[G/H] \xrightarrow{(t-1)H} \mathbb{Z}[G/H] \xrightarrow{N'} \mathbb{Z}[G/H] \xrightarrow{(t-1)H} \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0
\]
where $N' = H + tH + \cdots + t^{n/m-1}H$, is a $G/H$–projective resolution of $Z$.

From Section 5.3 and [ANCM17, Lemma 5.3], we have the following facts,

- If $P$ is a projective $G$–module, then $I_{G,H}(P)$ is also a projective $G$–module;
- The $G$–homomorphism $\eta: ZG \otimes_Z H \to Z[G/H]$, given by $\eta(g \otimes n) = ngH$, is an isomorphism;
- Moreover $\eta$ restricts to an isomorphism $I_{G,H}(Z) \cong \ker(\varepsilon : Z[G/H] \to Z)$;
- $I_{G,H}(M)$ is generated (as $Z$–module) by the elements $x \otimes n - 1 \otimes xn$ for all $x \notin H$ and $n \in M$.

From now on we will focus on the case $(G,H) = (C_4,C_2)$. We can apply the functor $I_{G,H}(-)$ to the resolution $\nu$ in order to obtain a $G$–projective resolution of $I_{G,H}(Z)$. Since $\nu'$ is an exact sequence, the isomorphism $\eta: I_{G,H}(Z) \to \ker(\varepsilon)$ can be extended to a morphism between the following exact sequences.

$$
\begin{array}{c}
I_{G,H}(\nu) : \\
1 \longrightarrow \lambda_0 \\
\vdots \longrightarrow \lambda_1 \longrightarrow \lambda_2 \longrightarrow \eta \\
\nu' \longrightarrow \cdots \\
\longrightarrow Z[G/H] \\
\longrightarrow Z[G/H] \\
\longrightarrow Z[G/H] \\
\longrightarrow \ker(\varepsilon) \longrightarrow 0.
\end{array}
$$

Our following task is to compute $\lambda_i$ for all $i \geq 0$. First we should note that $I_{G,H}(ZG)$ is a rank-one free $G$–module with basis $t \otimes 1 - 1 \otimes t$. In fact, by definition of $I_{G,H}(ZG)$ we have the short exact sequence

$$
0 \to I_{G,H}(ZG) \to Z[G] \otimes_Z H \to Z[G] \to 0,
$$

which splits since $Z[G]$ is a free $G$–module. Therefore $I_{G,H}(Z[G])$ is a free abelian group of rank 4. Now, a direct calculation shows that $t^i(t \otimes 1 - 1 \otimes t)$, for $i = 0, 1, 2, 3$ form a $Z$–basis of $I_{G,H}(ZG)$, and we conclude that it is a free $G$–module.

In order to calculate $\lambda_0$, we use the right hand side square of the above diagram:

$$
\eta \circ (\text{Id} \otimes \varepsilon)(t \otimes 1 - 1 \otimes t) = \eta(t \otimes 1 - 1 \otimes 1) = tH - H = (t - 1)H.
$$

hence we can define $\lambda_0(t \otimes 1 - 1 \otimes t) = H$ and extend to a $G$–equivariant map. Now we compute $\lambda_1$ using the second square from right to left.

$$
\lambda_0 \circ (\text{Id} \otimes (t - 1))(t \otimes 1 - 1 \otimes t) = \lambda_0(t \otimes (t - 1) - 1 \otimes (t^2 - t)) = \lambda_0(t \otimes t - t \otimes 1 - 1 \otimes t^2 + 1 \otimes t) = \lambda_0(-(t \otimes 1 - 1 \otimes t) - t(t \otimes 1 - 1 \otimes t)) = \lambda_0(-(t + 1)(t \otimes 1 - 1 \otimes t)) = -(t + 1)H,
$$

therefore we can define $\lambda_1(t \otimes 1 - 1 \otimes t) = -H$. For $\lambda_2$ we have

$$
\lambda_1 \circ (\text{Id} \otimes N)(t \otimes 1 - 1 \otimes t) = \lambda_1(t \otimes (1 + t + t^2 + t^3) - 1 \otimes (1 + t + t^2 + t^3)) = \lambda_1((t \otimes 1 - 1 \otimes t) - t(t \otimes 1 - 1 \otimes t) + t^2(t \otimes 1 - 1 \otimes t) - t^3(t \otimes 1 - 1 \otimes t)) = \lambda_1(0) = 2(t - 1)H
$$

and we define $\lambda_2(t \otimes 1 - 1 \otimes t) = 2H$. Proceeding in a similar way we finally obtain the following formulas

$$
\begin{align*}
\lambda_{2i}(t \otimes 1 - 1 \otimes t) & = 2^iH, \\
\lambda_{2i+1}(t \otimes 1 - 1 \otimes t) & = -2^iH.
\end{align*}
$$

And we can conclude that after tensoring with the trivial $G$–module $Z$, we obtain the homomorphisms of abelian groups $\lambda_i \otimes \text{Id}: I_{G,H}(Z) \otimes ZG \to Z[G/H] \otimes ZG$ given by the following formulas

$$
\begin{align*}
\lambda_{2i} \otimes \text{Id}(t \otimes 1 - 1 \otimes t) & = 2^i, \\
\lambda_{2i+1}(t \otimes 1 - 1 \otimes t) & = -2^i.
\end{align*}
$$
Finally the comparison homomorphisms at the level of chain complexes looks as follows

\[
\begin{array}{ccccccccc}
\cdots & Z & 0 & Z & 2 & Z & 0 & Z & 2 & Z & 0 \\
\downarrow & 4 & \downarrow & 0 & \downarrow & 2 & \downarrow & 2 & \downarrow & 2 & \downarrow \\
\cdots & Z & 0 & Z & 2 & Z & 0 & Z & 2 & Z & 0 \\
\end{array}
\]

Now we can explicitly compute the comparison homomorphism \( \varphi_1 : H_i(C_4, C_2; \mathbb{Z}) \rightarrow H_i([C_4 : C_2]; \mathbb{Z}) \). In fact \( \varphi_1 \) is the identity while \( \varphi_i = 0 \) for all \( i \geq 2 \).

**Remark 7.5.** In general, the algebraic version of the comparison homomorphism only gives the comparison homomorphisms \( \varphi_i : H_i(G, H; M) \rightarrow H_i([G : H]; M) \) for \( i \geq 2 \) (see Subsection 5.4 or [ANCM17, §7.1]) because when we remove ker \( \varepsilon \) in the projective resolution to get the reduced resolution, the kernel becomes \( C_1([G : H]) \) but in general it is smaller, so this truncated sequence, does not compute \( H_1([G : H]; M) \). But in the particular case of this example, that does not happen because taking the reduced resolution tensored with the trivial coefficients \( \mathbb{Z} \), we get that the homomorphism induced by the homomorphism before the augmentation is zero (see for instance [Bro82, Pag. 35]), so the lower sequence in diagram (13) at the first place on the right, indeed computes \( H_1([C_4 : C_2]; \mathbb{Z}) \), so we also get the comparison homomorphism \( \varphi_1 \).

**References**

[Ada54] Iain T. Adamson. Cohomology theory for non-normal subgroups and non-normal fields. *Proc. Glasgow Math. Assoc.*, 2:66–76, 1954.

[ANCM17] José Antonio Arciniega-Nevárez and José Luis Cisneros-Molina. Comparison of relative group (co)homologies. *Bol. Soc. Mat. Mex. (3)*, 23(1):41–74, 2017.

[ANCM18] José Antonio Arciniega-Nevárez and José Luis Cisneros-Molina. Invariants of hyperbolic 3-manifolds in relative group homology. arXiv:1303.2986v3 [math.GT], November 2018.

[Blo77] James V. Blowers. The classifying space of a permutation representation. *Trans. Amer. Math. Soc.*, 227:345–355, 1977.

[Bre67] Glen E. Bredon. *Equivariant cohomology theories*. Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin-New York, 1967.

[Bro82] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in \( K \)- and \( L \)-theory. *K-Theory*, 15(3):201–252, 1998.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Hoc56] G. Hochschild. Relative homological algebra. *Trans. Amer. Math. Soc.*, 82:246–269, 1956.

[IK14] Ayumu Inoue and Yuichi Kabaya. Quandle homology and complex volume. *Geom. Dedica.*, 171:265–292, 2014.

[Lam02] Serge Lang. Algebra. Volume 211 of Graduate Texts in Mathematics. Springer-Verlag, New York, third edition, 2002.

[Lüc89] Wolfgang Lück. *Transformation groups and algebraic \( K \)-theory*, volume 1408 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1989. Mathematische Grundlehren.

[Lüc05] Wolfgang Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[LW12] Wolfgang Lück and Michael Weiermann. On the classifying space of the family of virtually cyclic subgroups. *Pure Appl. Math. Q.*, 8(2):497–555, 2012.

[Mas55] W. S. Massey. Some problems in algebraic topology and the theory of fibre bundles. *Ann. of Math. (2)*, 62:327–359, 1955.

[McC01] John McCleary. *A user’s guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[ML95] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[MP02] Conchita Martínez-Pérez. A spectral sequence in Bredon (co)homology. *J. Pure Appl. Algebra*, 176(2-3):161–173, 2002.

[MS95] I. Moerdijk and J.-A. Svensson. A Shapiro lemma for diagrams of spaces with applications to equivariant topology. *Compositio Math.*, 96(3):249–282, 1995.

[MV03] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[Nos17] Takefumi Nosaka. *Quandles and topological pairs*. SpringerBriefs in Mathematics. Springer, Singapore, 2017. Symmetry, knots, and cohomology.

[SG05] Ruben José Sánchez-García. *Equivariant \( K \)-homology of the classifying space for proper actions*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–University of Southampton (United Kingdom).

[Sna64] Ernst Snapper. Cohomology of permutation representations. *I. Spectral sequences*, 13:133–161, 1964.

[Tak57] Satoru Takasu. On the change of rings in the homological algebra. *J. Math. Soc. Japan*, 9:315–329, 1957.

[Tak59] Satoru Takasu. Relative homology and relative cohomology theory of groups. *J. Fac. Sci. Univ. Tokyo, Sect. I*, 8:75–110, 1959.
[tD87] Tammo tom Dieck. *Transformation Groups*. Number 8 in Studies in Mathematics. Walter de Gruyter, Berlin, 1987.

[Zic09] Christian K. Zickert. The volume and Chern-Simons invariant of a representation. *Duke Math. J.*, 150(3):489–532, 2009.

División de Ingenierías, Universidad de Guanajuato

E-mail address: ja.arciniega@ugto.mx

Unidad Cuernavaca del Instituto de Matemáticas, National University of Mexico, Mexico 62210

E-mail address: jlcisneros@im.unam.mx

Unidad Cuernavaca del Instituto de Matemáticas, National University of Mexico, Mexico 62210

E-mail address: luisjorge@im.unam.mx