Adjacency and Tensor Representation in General Hypergraphs.

Part 2: Multisets, Hb-graphs and Related $\epsilon$-adjacency Tensors

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Abstract

HyperBagGraphs (hb-graphs as short) extend hypergraphs by allowing the hyperedges to be multisets. Multisets are composed of elements that have a multiplicity. When this multiplicity has positive integer values, it corresponds to non ordered lists of potentially duplicated elements. We define hb-graphs as family of multisets over a vertex set; natural hb-graphs correspond to hb-graphs that have multiplicity functions with positive integer values. Extending the definition of $\epsilon$-adjacency to natural hb-graphs, we define different way of building an $\epsilon$-adjacency tensor, that we compare before having a final choice of the tensor. This hb-graph $\epsilon$-adjacency tensor is used with hypergraphs.

1 Introduction

Hypergraphs were introduced in Berge and Minieka [1973]. Hypergraphs are defined as a family of nonempty subsets - called hyperedges - of the set of vertices. Elements of a set are unique. Hence elements of a given hyperedge are also unique in a hypergraph.

Multisets extend sets by allowing duplication of elements. As mentioned in Singh et al. [2007], N.G. de Bruijn proposed to Knuth the terminology multi-set in replacement of a variety of existing terms, such as bag or weighted set. Multisets are used in database modelling: in Albert [1991] relational algebra extension basements were introduced to manipulate bags - see also Klug [1982] - by studying bags algebraic properties. Queries for such bags have been largely studied in a series of articles - see references in Grumbach et al. [1996]; bags are intensively used in database queries as duplicate search is a costly operation. In Hernich and Kolaitis [2017] information integration under bag semantics is
studied as well as the tractability of some algorithmic problems over bag semantics: they showed that the GLAV (Global-And-Local-As-View) mapping of two databases problem becomes untractable over such semantic. In Radoaca [2015a] and Radoaca [2015b], the author extensively study multisets and propose to represent them by two kind of Venn diagrams. Multisets are also used in P-computing in the form of labelled multiset, called membrane - see Păun [2006] for more details.

Taking advantage of this duplication allowance, we construct in this article an extension of hypergraphs called hyper-bag-graphs (shortcut as hb-graphs). There are two main reasons to get such an extension. The first reason is that multisets are extensively used in databases as they allow presence of duplicates Lamperti et al. [2000] - removing duplicates (and thus obtaining sets and hypergraphs) being an expensive operation. The second reason is that natural hb-graphs - hb-graphs based on multisets with non-negative integer multiplicity values - allow results on the hb-graph adjacency tensor: hypergraph being particular case of hb-graph, other hypergraph e-adjacency tensors than the ones proposed in Banerjee et al. [2017], Ouvrard et al. [2017] can be built by giving meaningful interpretation to the steps taken during its construction via hb-graph.

Section 2 gives the mathematical background, including main definitions on hypergraphs and multisets. Section 3 gives mathematical construction of the Hyper-Bag-Graphs (or hb-graphs). Section 4 gives algebraic description of hb-graphs and consequences for the adjacency tensor of hypergraphs. Section 5 gives results on the constructed tensors. Section 6 evaluates the constructed tensors and proceed to a final choice on the hypergraph e-adjacency tensor. Section 7 gives future work.

2 Mathematical background

2.1 Hypergraphs

As mentioned in Ouvrard et al. [2017], hypergraphs fit collaboration networks modelling - Newman [2001a,b] -, co-author networks - Grossman and Ion [1995], Taramasco et al. [2010] -, chemical reactions - Temkin et al. [1996] -, genome - Chauve et al. [2013] -, VLSI design - Karypis et al. [1999] - and other applications. More generally hypergraphs fit perfectly to keep entities grouping information. Hypergraphs succeed in capturing p-adic relationships. In Berge and Minieka [1973], Stell [2012] and Bretto [2013] hypergraphs are defined in different ways. In this article, the definition of Bretto [2013] - as it doesn’t impose the union of the hyperedges to cover the vertex set - is used:

Definition 2.1. An (undirected) hypergraph $\mathcal{H} = (V, E)$ on a finite set of $n$ vertices (or vertices) $V = \{v_j : j \in [n]\}$ is defined as a family of $p$ hyperedges $E = \{e_j : j \in [p]\}$ where each hyperedge is a non-empty subset of $V$.

A weighted hypergraph is a triple: $\mathcal{H}_w = (V, E, w)$ where $\mathcal{H} = (V, E)$ is a hypergraph and $w$ a mapping where each hyperedge $e \in E$ is associated to a real
number \( w(e) \).

The \( 2 \)-section of a hypergraph \( \mathcal{H} = (V, E) \) is the graph \( [\mathcal{H}]_2 = (V, E') \) such that:

\[
\forall u \in V, \forall v \in V : (u, v) \in E' \iff \exists e \in E : u \in e \land v \in e
\]

Let \( k \in \mathbb{N}^* \). A hypergraph is \( k \)-uniform if all its hyperedges have the same cardinality \( k \).

A directed hypergraph \( \mathcal{H} = (V, E) \) is a hypergraph where each hyperedge \( e_i \in E \) accepts a partition in two non-empty subsets, called the source - written \( e_{si} \) - and the target - written \( e_{ti} \) - with \( e_{si} \cap e_{ti} = \emptyset \).

**Definition 2.2.** Let \( \mathcal{H} = (V, E) \) be a hypergraph.

The degree of a vertex is the number of hyperedges it belongs to. For a vertex \( v_i \), it is written \( d_i \) or \( \text{deg}(v_i) \). It holds: \( d_i = |\{ e : v_i \in e \}| \)

In this article only undirected hypergraphs will be considered. Hyperedges link one or more vertices together. Broadly speaking, the role of the hyperedges in hypergraphs is playing the role of edges in graphs.

### 2.2 Multisets

#### 2.2.1 Generalities

Basic on multisets are given in this section, based mainly on Singh et al. [2007].

**Definition 2.3.** Let \( A \) be a set of distinct objects. Let \( \mathcal{W} \subseteq \mathbb{R}^+ \)

Let \( m \) be an application from \( A \) to \( \mathcal{W} \).

Then \( A_m = (A, m) \) is called a multiset - or mset or bag - on \( A \).

\( A \) is called the ground or the universe of the multiset \( A_m \), \( m \) is called the multiplicity function of the multiset \( A_m \).

\( A_m^* = \{ x \in A : m(x) \neq 0 \} \) is called the support - or root or carrier - of \( A_m \).

The elements of the support of a mset are called its generators.

A multiset where \( \mathcal{W} \subseteq \mathbb{N} \) is called a natural multiset.

We write \( M(A) \) the set of all multisets of universe \( A \).

Some extensions of multisets exist where the multiplicity function can have its range in \( \mathbb{Z} \) - called hybrid set in Loeb [1992]. Some other extensions exist like fuzzy multisets Syropoulos [2000].

Several notations of mssets exist. One common notation which we will use, if \( A = \{ x_j : j \in [n] \} \) is the ground of a mset \( A_m \) is to write:

\[
A_m = \{ x_i^{m_i} : i \in [n] \}
\]

where \( m_i = m(x_i) \).

An other notation is:

\[
\{ x_1, \ldots, x_n \}_{m_1, \ldots, m_n}
\]

or even:

\[
m_1 \{ x_1 \} + \ldots + m_n \{ x_n \}.
\]
If \( A_m \) is a natural multiset an other notation is:

\[
\left\{ \frac{x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n}{m_1 \text{ times}, \ldots, m_n \text{ times}} \right\}
\]

which is similar to have an unordered list.

**Remark 2.1.**
1. Two msets can have same support and same support objects multiplicities but can differ by their universe. Also to be equal two msets must have same universe, same support and same multiplicity function.
2. The multiplicity function corresponds to a weight that is associated to objects of the universe.
3. Multiplicity in natural multisets can also be interpreted as a duplication of support elements. In this case, a mset can be viewed as a non ordered list with repetition. In a natural multiset the copies of a generator \( a \) of the support in \( m(a) \) instances are called **elements** of the multiset.
4. Some definitions of multisets also consider \( W = \mathbb{R} \) which could lead to interesting applications. We don’t develop such case here.

**Definition 2.4.** Let \( A_m \) be a mset.

The **m-cardinality** of \( A_m \) written \( \#_m A_m \) is defined as:

\[
\#_m A_m = \sum_{x \in A} m(x).
\]

The **cardinality** of \( A_m \) - written \( \# A_m \) is defined as:

\[
\# A_m = |A^*_m|.
\]

**Remark 2.2.** In general multisets, m-cardinality and cardinality are two separated notions as for instance: \( A = \{a^{1.2}, b^{0.8}\} \), \( B = \{a^{0.2}, b^{1.8}\} \) and, \( C = \{a^{0.5}, b^{0.5}\} \) have all same cardinality with different m-cardinalities for \( C \) compared to \( A \) and \( B \).

In natural multisets, m-cardinality and cardinality are equal if and only if the multiplicity of each element in the support is 1, ie if the natural multiset is a set. It doesn’t generalize to general multisets - see \( A \) and \( B \) of the former example.

**Definition 2.5.** Two msets \( A_{m_1} \) and \( B_{m_2} \) are said to be **cognate** if they have same support.

They are not necessarily equal: for instance, \( \{a^1, b^2\} \) and \( \{a^2, b^1\} \) are cognate but different.
Definition 2.6. Let $A = U_{m_A}$ and $B = U_{m_B}$ be two msets on the same universe $U$.

If $A^* = \emptyset$ $A$ is called the empty mset and written $\emptyset$.

$A$ is said to be included in $B$ - written $A \subseteq B$ - if for all $x \in U$: $m_A(x) \leq m_B(x)$. In this case, $A$ is called a submset of $B$.

The union of $A$ and $B$ is the mset $C = A \cup B$ of universe $U$ and of multiplicity function $m_C$ such that for all $x \in U$:

$$m_C(x) = \max (m_A(x), m_B(x)).$$

The intersection of $A$ and $B$ is the mset $D = A \cap B$ of universe $U$ and of multiplicity function $m_D$ such that for all $x \in U$:

$$m_D(x) = \min (m_A(x), m_B(x)).$$

The sum of $A$ and $B$ is the mset $E = A \oplus B$ of universe $U$ and of multiplicity function $m_E$ such that for all $x \in U$:

$$m_E(x) = m_A(x) + m_B(x).$$

Proposition 2.1. $\cup$, $\cap$ and $\oplus$ are commutative and associative laws on msets of same universe. They have the empty mset of same universe as identity law.

$\oplus$ is distributive for $\cup$ and $\cap$.

$\cup$ and $\cap$ are distributive one for the other.

$\cup$ and $\cap$ are idempotent.

Definition 2.7. Let $A$ be a mset.

The power set of $A$, written $\bar{P}(A)$, is the multiset of all submsets of $A$.

2.2.2 Copy-set of a multiset

Let consider a multiset: $A_m = (A, m)$ where the range of the multiplicity function is a subset of $\mathbb{N}$. Equivalent definition - see Syropoulos [2000] - is to give a couple $< A_0, \rho >$ where $A_0$ is the set of all instances (including copies) of $A_m$ with an equivalency relation $\rho$ where:

$$\forall x \in A_0, \forall x' \in A_0 : x \rho x' \leftrightarrow \exists ! c \in A : x \rho c \land x' \rho c.$$ 

Definition 2.8. Two elements of $A_0$ such that: $x \rho x'$ are said copies one of the other. The unique $c \in A$ is called the original element. $x$ and $x'$ are said copies of $c$.

Also $A_0/\rho$ is isomorphic to $A$ and:

$$\forall \overline{x} \in A_0/\rho, \exists ! c \in A : \{|x : x \in \overline{x}\}| = m(c) \land \forall x \in \overline{x} : x \rho c.$$ 

Definition 2.9. The set $A_0$ is called a copy-set of the multiset $A_m$.

Remark 2.3. A copy-set for a given multiset is not unique. Sets of equivalency classes of two couples $< A_0, \rho >$ and $< A'_0, \rho' >$ of a given multiset are isomorphic.
2.2.3 Algebraic representation of a multiset

We suppose given a natural multiset \( A_m = (A, m) \) of universe \( A = \{\alpha_i : i \in [n]\} \) and multiplicity function \( m \). It yields:

\[
A_m = \left\{ \alpha_{ij}^m : \alpha_{ij} \in A_m^* \right\}.
\]

**Vector representation:** A multiset can be conveniently represented by a vector of length the cardinality of the universe and where the coefficients of the vector represent the multiplicity of the corresponding element.

**Definition 2.10.** The vector representation of the multiset \( A_m \) is the vector \( \vec{A} = (m(\alpha))_{\alpha \in A} \).

This representation requires \(|A|\) space and has \(|A| - |A_m^*|\) null elements.

The sum of the elements of \( \vec{A} \) is \( \sum_m A_m \).

This representation will be useful later when considering family of multisets in order to build the incident matrix.

**Hypermatrix representation:** An alternative representation is built by using a symmetric hypermatrix. This approach is needed to reach our goal of constructing an \( e \)-adjacency tensor for general hypergraphs.

**Definition 2.11.** The unnormalized hypermatrix representation of the multiset \( A_m \) is the symmetric hypermatrix \( A_u = (a_{i_1...i_r})_{(i_1,...,i_r) \in [n]^r} \) of order \( r = \sum_m A_m \) and dimension \( n \) such that:

\[
a_{i_1...i_r} = 1 \quad \text{if} \quad \forall j \in [r] : i_j \in [n] \wedge \alpha_{ij} \in A_m^*.
\]

Hence the number of non-zero elements in \( A_u \) is \( \frac{r!}{\prod_{\alpha \in A_m^*} m(\alpha)} \) out of the \( n^r \) elements of the representation.

The sum of the elements of \( A_u \) is then:

\[
\frac{r!}{\prod_{\alpha \in A_m^*} m(\alpha)}.
\]

To achieve a normalisation, we enforce the sum of the elements of the hypermatrix to be the \( m \)-rank of the multiset it encodes. It yields:

**Definition 2.12.** The normalized hypermatrix representation of the multiset \( A_m \) is the symmetric hypermatrix \( A = (a_{i_1...i_r})_{(i_1,...,i_r) \in [n]^r} \) of order:

\[
\prod_{\alpha \in A_m^*} m(\alpha) \frac{1}{(r - 1)!} \quad \text{if} \quad \forall j \in [r] : i_j \in [n] \wedge \alpha_{ij} \in A_m^*.
\]

The other elements are null.
3 Hb-graphs

Hyper-bag-graphs - hb-graphs for short - are introduced in this section. Hb-graphs extend hypergraphs by allowing hyperedges to be msets. The goal of this section is to revisit some of the definitions and results found in Bretto [2013] for hypergraphs and extend them to hb-graphs.

3.1 Generalities

3.1.1 First definitions

Definition 3.1. Let \( V = \{v_i : i \in [n]\} \) be a nonempty finite set.

A hyper-bag-graph - or hb-graph - is a family of msets with universe \( V \) and support a subset of \( V \). The msets are called the hb-edges and the elements of \( V \) the vertices.

We write \( E = (e_i)_{i \in [p]} \) the family of hb-edges and \( \mathcal{H} = (V, E) \) such a hb-graph.

We consider for the remainder of the article a hb-graph \( \mathcal{H} = (V, E) \), with \( V = \{v_i : i \in [n]\} \) and \( E = (e_i)_{i \in [p]} \) the family of its hb-edges.

Each hb-edge \( e_i \in E \) is of universe \( V \) and has a multiplicity function associated to it: \( m_{e_i} : V \rightarrow \mathbb{W} \) where \( \mathbb{W} \subseteq \mathbb{R}^+ \). When the context make it clear the notation \( m_i \) is used for \( m_{e_i} \) and \( m_{ij} \) for \( m_{e_i}(v_j) \).

Definition 3.2. A hb-graph is said with no repeated hb-edges if:

\[ \forall i_1 \in [p], \forall i_2 \in [p] : e_{i_1} = e_{i_2} \Rightarrow i_1 = i_2. \]

Definition 3.3. A hb-graph where each hb-edge is a natural mset is called a natural hb-graph.

Remark 3.1. For a general hb-graph each hb-edge has to be seen as a weighted system of vertices, where the weights of each vertex are hb-edge dependent.

In a natural hb-graph the multiplicity function can be viewed as a duplication of the vertices.

Definition 3.4. The order of a hb-graph \( \mathcal{H} \) - written \( O(\mathcal{H}) \) - is:

\[ O(\mathcal{H}) = \sum_{j \in [n]} \max_{e \in E} (m_e (v_j)). \]

Its size is the cardinality of \( E \).

Definition 3.5. The empty hb-graph is the hb-graph with an empty set of vertices.

The trivial hb-graph is the hb-graph with a non empty set of vertices and an empty family of hb-edges.
If: \( \bigcup_{i \in [p]} e_i^* = V \) then the hb-graph is said with no isolated vertices. Otherwise, the elements of \( V \setminus \bigcup_{i \in [p]} e_i^* \) are called the isolated vertices. They correspond to elements of hyperedges which have zero-multiplicity for all hb-edges.

**Remark 3.2.** A hypergraph is a natural hb-graph where the vertices of the hb-edges have multiplicity one for any vertex of their support and zero otherwise.

### 3.1.2 Support hypergraph

**Definition 3.6.** The support hypergraph of a hb-graph \( \mathcal{H} = (V, E) \) is the hypergraph whose vertices are the ones of the hb-graph and whose hyperedges are the support of the hb-edges in a one-to-one way. We write it \( \mathcal{H} = (V, E^*) \), where \( E^* = \{ e^* : e \in E \} \).

**Remark 3.3.** Given a hypergraph, an infinite set of hb-graphs can be generated that all have this hypergraph as support. To each of these hb-graphs corresponds a hb-edge family: to each support of these hb-edges corresponds at least a hyperedge in the hypergraph and reciprocally to each hyperedge corresponds at least a hb-edge in each hb-graph of the infinite set.

To have unicity, the considered hypergraph and hb-graphs should be respectively with no repeated hyperedge or with no repeated hb-edge.

### 3.1.3 m-uniform hb-graphs

**Definition 3.7.** The m-range of a hb-graph \( \mathcal{H} = (V, E) \) is by definition:

\[
\text{r}_m(\mathcal{H}) = \max_{e \in E} \#_m e.
\]

The range of a hb-graph \( \mathcal{H} \) - written \( r(\mathcal{H}) \) - is the range of its support hypergraph \( \mathcal{H} \).

The m-co-range of a hb-graph - written \( \text{cr}_m(\mathcal{H}) \) - is by definition:

\[
\text{cr}_m(\mathcal{H}) = \min_{e \in E} \#_m e.
\]

The co-range of a hb-graph \( \mathcal{H} \) - written \( c(\mathcal{H}) \) - is the range of its support hypergraph \( \mathcal{H} \).

**Definition 3.8.** A hb-graph is said k-m-uniform if all its hb-edges have same m-cardinality \( k \).

A hb-graph is said k-uniform if its support hypergraph is k-uniform.

**Proposition 3.1.** A hb-graph \( \mathcal{H} \) is k-m-uniform if and only if:

\[
\text{r}_m(\mathcal{H}) = \text{cr}_m(\mathcal{H}) = k.
\]

**Proof.** Immediate. \( \square \)
3.1.4 HB-star and m-degree

**Definition 3.9.** The **HB-star** of a vertex \( x \in V \) is the multiset - written \( H(x) \) - defined as:

\[
H(x) = \left\{ e^{m_e(x)} : e \in E \land x \in e^* \right\}.
\]

**Remark 3.4.** The support of the HB-star \( H^*(x) \) of a vertex \( x \in V \) of a hb-graph \( \mathcal{H} \) is exactly the star of this vertex in the support hypergraph \( \mathcal{H} \).

**Definition 3.10.** The **m-degree** of a vertex \( x \in V \) of a hb-graph \( \mathcal{H} \) - written \( \deg_m(x) = d_m(x) \) - is defined as:

\[
\deg_m(x) = \#_m H(x).
\]

The **maximal m-degree** of a hb-graph \( \mathcal{H} \) is written \( \Delta_m = \max_{x \in V} \deg_m(x) \).

The degree of a vertex \( x \in V \) of a hb-graph \( \mathcal{H} \) - written \( \deg(x) = d(x) \) - corresponds to the degree of this vertex in the support hypergraph \( \mathcal{H} \).

The maximal degree of a hb-graph \( \mathcal{H} \) is written \( \Delta \) and corresponds to the maximal degree of the support hypergraph \( \mathcal{H} \).

**Definition 3.11.** A hb-graph having all of its hb-edges of same m-degree \( k \) is said **m-regular** or **k-m-regular**.

A hb-graph is said **regular** if its support hypergraph is regular.

3.1.5 Dual of a hb-graph

**Definition 3.12.** Considering a hb-graph \( \mathcal{H} \), its dual is the hb-graph \( \tilde{\mathcal{H}} \) with a set of vertices \( \tilde{V} = \{ \tilde{x}_i : i \in [p] \} \) which is in bijection \( f \) with the set of hb-edges \( E \) of \( \mathcal{H} \):

\[
\forall \tilde{x}_i \in \tilde{V}, \exists e_i \in E : \tilde{x}_i = f(e_i).
\]

And the set of hb-edges \( \tilde{E} = \{ \tilde{e}_j : j \in [n] \} \) is in bijection \( g : \tilde{x}_j \mapsto \tilde{e}_j \) - where \( \tilde{e}_j = \left\{ \tilde{x}_i^{m_{\tilde{e}_j}(\tilde{x}_i)} : i \in [p] \land \tilde{x}_i = f(e_i) \land \tilde{x}_j \in e_i^* \right\} \) - with the set of vertices of \( \mathcal{H} \).

Switching from the hb-graph to its dual:

| \( \mathcal{H} \) | \( \tilde{\mathcal{H}} \) |
|-------------------|-------------------|
| Vertices \( x_i \), \( i \in [n] \) | \( \tilde{x}_j = f(e_j), j \in [p] \) |
| Edges \( e_j \), \( j \in [p] \) | \( \tilde{e}_i = g(x_i), i \in [n] \) |
|Multiplicity \( x_i \in e_j \) with \( m_{e_j}(x_i) \) | \( \tilde{x}_j \in \tilde{e}_i \) with \( m_{\tilde{e}_i}(\tilde{x}_j) \) |
| \( d_m(x_i) \) | \( \#_m \tilde{e}_i \) |
| \( \# m e_i \) | \( d_m(\tilde{x}_j) \) |
| \( k \)-m-uniform | \( k \)-m-regular |
| \( k \)-m-regular | \( k \)-m-uniform |
3.2 Additional concepts for natural hb-graphs

3.2.1 Numbered copy hypergraph of a natural hb-graph

In natural hb-graphs the hb-edge multiplicity functions have their range in the natural number set. The vertices in a hb-edge with multiplicities strictly greater than 1 can be seen as copies of the original vertex.

Deepening this approach copies have to be understood as “numbered” copies. Let $A$ and $B$ be two hb-edges. Let $v_i$ be a vertex of multiplicity $m_A$ in $A$ and $m_B$ in $B$. $A \cap B$ will hold $\min(m_A, m_B)$ copies: the ones “numbered” from 1 to $\min(m_A, m_B)$. The remaining copies will be held either in $A$ xor $B$ depending which set has the highest multiplicity of $v_i$.

More generally, we define the numbered-copy set of a multiset:

Definition 3.13. Let $A_m = \{x_i^{m_i} : i \in [n]\}$.

The numbered copy-set of $A_m$ is the copy-set $\hat{A}_m = \{[x_{ij}]_{m_i} : i \in [n]\}$ where: $[x_{ij}]_{m_i}$ is a shortcut to indicate the numbered copies of the original element $x_i$: $x_{i1}$ to $x_{im_i}$ and $j$ is designated as the copy number of the element $x_i$.

Definition 3.14. Let $\mathcal{H} = (V, E)$ be a natural hb-graph.

Let $V = \{v_j : j \in [n]\}$ be the vertices of the hb-graph. Let $E = \{e_k : k \in [p]\}$ be the hb-edges of the hb-graph and for $k \in [p]$, $m_{e_k}$ the multiplicity function of $e_k \in E$.

The maximum multiplicity function of $\mathcal{H}$ is the function $m : V \to \mathbb{N}$ defined for all $v \in V$ by: $m(v) = \max_{e \in E} m_{e}(v)$.

Definition 3.15. Let $\mathcal{H} = (V, E)$ be a natural hb-graph where $V = \{v_i : i \in [n]\}$ is the vertex set and $E = (e_k)_{k \in [p]}$ is the hb-edge family of the hb-graph.

Let $m$ be the maximum multiplicity function.

Let consider the numbered-copy-set of the multiset $\{v_i^{m(v_i)} : i \in [n]\}:

\[ V = \{[v_{ij}]_{m(v_i)} : i \in [n]\}. \]

Then each hb-edge $e_k = \{v_{ij}^{m_k(i)} : j \in [k] \land i_j \in [n]\}$ is associated to a copy-set / equivalency relation $< e_{k0}, \rho_k >$ which elements are in $V$ with copy number as small as possible for each vertex in $e_k$.

Then $\mathcal{H}_0 = (V, E_0)$ where $E_0 = \{e_{k0} : k \in [p]\}$ is a hypergraph called the numbered-copy-hypergraph of $\mathcal{H}$.

Proposition 3.2. A numbered-copy-hypergraph is unique for a given hb-graph.

Proof. It is immediate by the way the numbered-copy-hypergraph is built from the hb-graph.
Allowing the duplicates to be numbered prevent ambiguities; nonetheless it has to be seen as a conceptual approach as duplicates are entities that are not discernible.

3.2.2 Paths, distance and connected components

Defining a path in a hb-graph is not straightforward as vertices are duplicated in a hb-graph. The duplicate of a vertex strictly inside a path must be at the intersection of two consecutive hb-edges.

**Definition 3.16.** A strict m-path \( x_0 e_1 x_1 \ldots e_s x_s \) in a hb-graph from a vertex \( x \) to a vertex \( y \) is a vertex / hb-edge alternation with hb-edges \( e_1 \) to \( e_s \) and vertices \( x_0 \) to \( x_s \) such that \( x_0 = x \), \( x_s = y \), \( x \in e_1 \) and \( y \in e_s \) and that for all \( i \in [s-1] \), \( x_i \in e_i \cap e_{i+1} \).

A large m-path \( x_0 e_1 x_1 \ldots e_s x_s \) from a vertex \( x \) to a vertex \( y \) is a vertex / hb-edge alternation with hb-edges \( e_1 \) to \( e_s \) and vertices \( x_0 \) to \( x_s \) such that \( x_0 = x \), \( x_s = y \), \( x \in e_1 \) and \( y \in e_s \) and that for all \( i \in [s-1] \), \( x_i \in e_i \cup e_{i+1} \).

\( s \) is called in both cases the length \( l(x, y) \) of the m-path from \( x \) to \( y \).

Vertices from \( x_1 \) to \( x_{s-1} \) are called interior vertices of the m-path.

\( x_0 \) and \( x_s \) are called extremities of the m-path.

If the extremities are different copies of the same object, then the m-path is said to be an almost cycle.

If the extremities designate exactly the same copy of one object, the m-path is said to be a cycle.

**Remark 3.5.**

1. For a strict m-path, there are:

\[
\prod_{i \in [s-1]} m_{e_i \cap e_{i+1}} (x_i)
\]

possibilities of choosing the interior vertices along a given m-path \( x_0 e_1 x_1 \ldots e_s x_s \) and:

\[
m_{e_1} (x_0) \prod_{i \in [s-1]} m_{e_i \cap e_{i+1}} (x_i) m_{e_s} (x_s)
\]

possible strict m-paths in between the extremities.

2. For a large m-path, there are:

\[
\prod_{i \in [s-1]} m_{e_i \cup e_{i+1}} (x_i)
\]

possibilities of choosing the interior vertices along a given m-path \( x_0 e_1 x_1 \ldots e_s x_s \) and:

\[
m_{e_1} (x_0) \prod_{i \in [s-1]} m_{e_i \cup e_{i+1}} (x_i) m_{e_s} (x_s)
\]

possible large m-paths in between the extremities.
3. As large m-paths between two extremities by a given sequence of interior vertices and hb-edges include strict m-paths, we often refer as m-paths for large m-paths.

4. If an m-path exists from \( x \) to \( y \) then an m-path also exists from \( y \) to \( x \).

**Definition 3.17.** An m-path \( x_0 e_1 x_1 \ldots e_s x_s \) in a hb-graph corresponds to a unique path in the hb-graph support hypergraph called the **support path**.

**Proposition 3.3.** Every m-path \( x_0 e_1 x_1 \ldots e_s x_s \) traversing same hyperedges and having similar copy vertices as intermediate and extremity vertices share the same support path.

The notion of distance is similar to the one defined for hypergraphs.

**Definition 3.18.** Let \( x \) and \( y \) be two vertices of a hb-graph. The distance \( d(x, y) \) from \( x \) to \( y \) is the minimal length of an m-path from \( x \) to \( y \) if such an m-path exists. If no m-path exist, \( x \) and \( y \) are said disconnected and \( d(x, y) = +\infty \).

**Definition 3.19.** A hb-graph is said **connected** if its support hypergraph is connected, disconnected otherwise.

**Definition 3.20.** A **connected component** of a hb-graph is a maximal set of vertices such that every pair of vertices of the component has an m-path in between them.

**Remark 3.6.** A connected component of a hb-graph is a connected component of one of its copy hypergraph.

**Definition 3.21.** The **diameter** of a hb-graph \( H \) - written \( \text{diam}(H) \) - is defined as:

\[
\text{diam}(H) = \max_{x, y \in V} d(x, y).
\]

### 3.2.3 Adjacency

**Definition 3.22.** Let \( k \) be a positive integer.

Let consider \( k \) vertices not necessarily distinct belonging to \( V \).

Let write \( V_{k,m} \) the mset consisting of these \( k \) vertices with multiplicity function \( m \).

The \( k \) vertices are said **\( k \)-adjacent** in \( H \) if it exists \( e \in E \) such that \( V_{k,m} \subseteq e \).

Considering a hb-graph \( H \) of m-range \( \overline{k} = r_H \), the hb-graph can’t handle more than \( \overline{k} \)-adjacency in it. This maximal \( k \)-adjacency is called the **\( k \)-adjacency** of \( H \).

**Definition 3.23.** Let consider a hb-edge \( e \) in \( H \).

Vertices in the support of \( e \) are said **\( e^* \)-adjacent**.

Vertices in the hb-edge \( e \) with nonzero multiplicity are said **\( e \)-adjacent**.
Remark 3.7.  
- $e^*$-adjacency doesn’t support redundancy of vertices.
- $e$-adjacency allows the redundancy of vertices.
- The only case of equality is where the hb-edge has all its nodes of multiplicity 1 at the most.

Definition 3.24. Two hb-edges are said incident if their support intersection is not empty.

3.2.4 Sum of two hb-graphs
Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hb-graphs.

The sum of two hb-graphs $H_1$ and $H_2$ is the hb-graph written $H_1 + H_2$ defined as the hb-graph that has:

- $V_1 \cup V_2$ as vertex set and where the hb-edges are obtained from the hb-edges of $E_1$ and $E_2$ with same multiplicity for vertices of $V_1$ (respectively $V_2$) but such that for each hyperedge in $E_1$ (respectively $E_2$) the universe is extended to $V_1 \cup V_2$ and the multiplicity function is extended such that $\forall v \in V_2 \setminus V_1 : m(v) = 0$ (respectively $\forall v \in V_1 \setminus V_2 : m(v) = 0$)

- $E_1 + E_2$ as hb-edge family, i.e. the family constituted of the elements of $E_1$ and of the elements of $E_2$.

$H_1 + H_2 = (V_1 \cup V_2, E_1 + E_2)$

This sum is said direct if $E_1 + E_2$ doesn’t contain any new pair of repeated hb-edge than the ones already existing in $E_1$ and those already existing in $E_2$. In this case the sum is written $H_1 \oplus H_2$.

3.3 An example

Example 3.1. Considering $H = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{e_1, e_2, e_3, e_4\}$ with: $e_1 = \{v_1^2, v_4^2, v_5^1\}$, $e_2 = \{v_2^3, v_3^1\}$, $e_3 = \{v_3^1, v_5^3\}$, $e_4 = \{v_6\}$.

It holds:

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $\# m e_j$ |
|-------|-------|-------|-------|-------|-------|-------|-------------|
| 2     | 3     | 0     | 0     | 2     | 0     | 1     | 5           |
| 0     | 3     | 0     | 0     | 2     | 0     | 0     | 4           |
| 0     | 0     | 1     | 0     | 2     | 0     | 0     | 3           |
| 2     | 0     | 0     | 0     | 3     | 1     | 0     | 1           |
| 1     | 0     | 2     | 0     | 1     | 0     | 0     | 1           |
| 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0           |
| 2     | 1     | 3     | 1     |       |       |       |             |
Therefore the order of $\mathcal{H}$ is $O(\mathcal{H}) = 2 + 3 + 1 + 2 + 1 + 0 = 10$ and its size is $|E| = 4$.

$v_7$ is an isolated vertex.

$e_1$ and $e_3$ are incident as well as $e_4$ and $e_2$. $e_4$ is not incident to any hb-edge.

$v_1$, $v_4$ and $v_5$ are $e^*$-adjacent as they hold in $e_1^*$.

$v_1^2$, $v_4^1$ and $v_3^1$ are $e$-adjacent as they hold in $e_1$.

The dual of $\mathcal{H}$ is the hb-graph: $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{E})$ with:

- $\tilde{V} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$ with $f(\tilde{x}_i) = e_i$ for $1 \leq i \leq 4$
- $\tilde{E} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7\}$ with:
  - $\tilde{e}_1 = \tilde{e}_4 = \{\tilde{x}_1^2\}$
  - $\tilde{e}_2 = \{\tilde{x}_2^3\}$
  - $\tilde{e}_3 = \{\tilde{x}_2^1, \tilde{x}_3^1\}$
  - $\tilde{e}_5 = \{\tilde{x}_1^1, \tilde{x}_3^2\}$
  - $\tilde{e}_6 = \{\tilde{x}_4^1\}$
  - $\tilde{e}_7 = \emptyset$

$\tilde{\mathcal{H}}$ has duplicated hb-edges and one empty hb-edge.

4 Algebraic representation of a hb-graph

4.1 Incidence matrix of a hb-graph

A mset is well defined by giving itself its universe, its support and its function of multiplicity. We have seen that a mset can be represented by a vector called the vector representation of the m-set.

Hb-edges of a given hb-graph have all the same universe.

**Definition 4.1.** Let $n$ and $p$ be two positive integers.

Let $\mathcal{H} = (V, E)$ be a non-empty hb-graph, with vertex set $V = \{v_i : i \in \llbracket n \rrbracket\}$ and $E = \{e_j : j \in \llbracket p \rrbracket\}$.

The matrix $\tilde{H} = [m_{ij}(v_i)]_{i \in \llbracket n \rrbracket}^{j \in \llbracket p \rrbracket}$ is called the **incidence matrix** of the hb-graph $\mathcal{H}$.

This incidence matrix is intensively used in Ouvrard et al. [2018b] for diffusion by exchanges in hb-graphs.

4.2 $e$-adjacency tensor of a natural hb-graph

To build the $e$-adjacency tensor $A(\mathcal{H})$ of a natural hb-graph $\mathcal{H} = (V, E)$ without repeated hb-edge - with vertex set $V = \{v_i : i \in \llbracket n \rrbracket\}$ and hb-edge set $E = \{e_j : j \in \llbracket p \rrbracket\}$ - we use a similar approach that was used in Ouvrard et al. [2017] using the strong link between cubical symmetric tensors and homogeneous polynomials.
Definition 4.2. An elementary hb-graph is a hb-graph that has only one non repeated hb-edge in its hb-edge family.

Claim 4.1. Let $\mathcal{H} = (V, E)$ be a hb-graph with no repeated hb-edge.

Then:

$$\mathcal{H} = \bigoplus_{e \in E} \mathcal{H}_e$$

where $\mathcal{H}_e = (V, (e))$ is the elementary hb-graph associated to the hb-edge $e$.

Proof. Let $e_1 \in E$ and $e_2 \in E$. As $\mathcal{H}$ is with no repeated hb-edge, $e_1 + e_2$ doesn’t contain new pairs of repeated elements. Thus $\mathcal{H}_{e_1} + \mathcal{H}_{e_2}$ is a direct sum.

A straightforward iteration over elements of $e \in E$ leads trivially to the result.

We need first to define hypermatrices for the $k$-adjacency of an elementary hb-graph and of a $m$-uniform hb-graph.

4.2.1 Normalised $k$-adjacency tensor of an elementary hb-graph

We consider an elementary hb-graph $\mathcal{H}_e = (V, (e))$ where $V = \{v_i : i \in [n]\}$ and $e$ is a multiset of universe $V$ and multiplicity function $m$. The support of $e$ is $e^* = \{v_{j_1}, \ldots, v_{j_k}\}$ by considering, without loss of generality: $1 \leq j_1 < \ldots < j_k \leq n$.

$e$ is the multiset: $e = \{v_{j_1}^{m_{j_1}}, \ldots, v_{j_k}^{m_{j_k}}\}$ where $m_{j} = m(v_{j})$.

The normalised hypermatrix representation of $e$, written $Q_e$, describes uniquely the $m$-set $e$. Thus the elementary hb-graph $\mathcal{H}_e$ is also uniquely described by $Q_e$ as $e$ is the unique hb-edge. $Q_e$ is of rank $r = \#_m e = \sum_{j=1}^k m_{j}$ and dimension $n$.

Hence, the definition:

Definition 4.3. Let $\mathcal{H} = (V, (e))$ be an elementary hb-graph with $V = \{v_i : i \in [n]\}$ and $e$ the multiset $\{v_{j_1}^{m_{j_1}}, \ldots, v_{j_k}^{m_{j_k}}\}$ of m-rank $r$, universe $V$ and multiplicity function $m$.

The normalised $k$-adjacency hypermatrix of an elementary hb-graph $\mathcal{H}_e$ is the normalised representation of the multiset $e$: it is the symmetric hypermatrix $Q_e = (q_{j_1 \ldots j_r})$ of rank $r$ and dimension $n$ where the only nonzero elements are:

$$q_{\sigma(j_1)^{m_{\sigma(j_1)}} \ldots \sigma(j_k)^{m_{\sigma(j_k)}}} = \frac{m_{j_1}! \ldots m_{j_k}!}{(r-1)!}$$

where $\sigma \in S_r$.

In a elementary hb-graph the $k$-adjacency corresponds to $\#_m e$-adjacency. This hypermatrix encodes the $k$-adjacency of the elementary hb-graph; as the $k$-adjacency corresponds to $e$-adjacency in such a hb-graph is encodes also the $e$-adjacency of the elementary hb-graph.
4.2.2 hb-graph polynomial

Homogeneous polynomial associated to a hypermatrix: With a similar approach than in Ouvrard et al. [2017] where full details are given, let write $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$.

$(e_{i_1} \otimes \ldots \otimes e_{i_k})_{i_1, \ldots, i_k \in [n]}$ is a basis of $\mathcal{L}_k^0(\mathbb{K}^n)$, where $\otimes$ is the Segre outer-product.

A tensor $Q \in \mathcal{L}_k^0(\mathbb{K}^n)$ is associated to a hypermatrix $Q = (q_{i_1 \ldots i_r})_{i_1, \ldots, i_r \in [n]}$ by writing $Q$ as:

$$Q = \sum_{i_1, \ldots, i_r \in [n]} q_{i_1 \ldots i_r} e_{i_1} \otimes \ldots \otimes e_{i_r}$$

Considering $n$ variables $z_i$ attached to the $n$ vertices $v_i$ and $z = \sum_{i \in [n]} z_i e_i$, the multilinear matrix product $(z, \ldots, z) . Q = (z)_{[r]} . Q$ is a polynomial $P(z_0)$ of degree $r$.

**Elementary hb-graph polynomial:**

Considering a hb-graph $\mathcal{H}_e = (V, (e))$ with $V = \{v_i : i \in [n]\}$ and $e$ the multiset $\{v_{j_1}^{m_{j_1}}, \ldots, v_{j_k}^{m_{j_k}}\}$ of m-rank $r$, universe $V$ and multiplicity function $m$.

Using the normalised $\mathbb{K}$-adjacency hypermatrix $Q_e = (q_{i_1 \ldots i_r})_{i_1, \ldots, i_r \in [n]}$, which is symmetric, we can write the reduced version of its attached homogeneous polynomial $P_e$:

$$P_e(z_0) = \sum_{i_1, \ldots, i_r \in [n]} c_{e_i} P_{e_i}(z_0)$$

where $c_{e_i}$ is a technical coefficient. $P(z_0)$ is called the **hb-graph polynomial**. The choice of $c_{e_i}$ is made in order to retrieve the m-degree of the vertices from the $e$-adjacency tensor.

---

1As a reminder: $z_0 = (z_1, \ldots, z_n)$
4.2.3 $k$-adjacency hypermatrix of a m-uniform natural hb-graph

We now extend to m-uniform hb-graph the $k$-adjacency hypermatrix obtained in the case of an elementary hb-graph.

In the case of a $r$-m-uniform natural hb-graph with no repeated hb-edge, each hb-edge has the same $m$-cardinality $r$. Hence the $k$-adjacency of a $r$-m-uniform hb-graph corresponds to $r$-adjacency where $r$ is the m-rank of the hb-graph. The $k$-adjacency tensor of the hb-graph has rank $r$ and dimension $n$. The elements of the $k$-adjacency hypermatrix are:

$$a_{i_1...i_r}$$

with $i_1,\ldots,i_r \in [n]$.

The associated hb-graph polynomial is homogeneous of degree $r$.

We obtain the definition of the $k$-adjacency tensor of a $r$-m-uniform hb-graph by summing the $k$-adjacency tensor attached to each hyperedge with a coefficient $c_i$ equals to 1 for each hyperedge.

**Definition 4.4.** Let $H = (V,E)$ be a hb-graph. $V = \{v_i : i \in [n]\}$.

The $k$-adjacency hypermatrix of a $r$-m-uniform hb-graph $H = (V,E)$ is the hypermatrix $A_H = (a_{i_1...i_r})_{i_1,...,i_r,[n]}$ defined by:

$$A_H = \sum_{i \in [r]} Q_e_i$$

where $Q_{e_i}$ is the $k$-adjacency hypermatrix of the elementary hb-graph associated to the hb-edge $e_i = \{v_{j_1}^{m_{i1}},\ldots,v_{j_k}^{m_{ik}}\} \in E$.

The only non-zero elements of $Q_{e_i}$ are the elements of indices obtained by permutation of the multiset $\{j_1^{m_{i1}},\ldots,j_k^{m_{ik}}\}$ and are all equals to $\frac{m_{i1}!\ldots m_{ik}!}{(r-1)!}$.

**Remark 4.1.** When a $r$-m-uniform hb-graph has 1 as vertex multiplicity for any vertices in each hb-edge support of all hb-edges, then this hb-graph is a $r$-uniform hypergraph: in this case, we retrieve the result of the degree-normalized tensor defined in Cooper and Dutle [2012].

**Claim 4.2.** The $m$-degree of a vertex $v_j$ in a $r$-m-uniform hb-graph $H$ of $k$-adjacency hypermatrix is:

$$\deg_m(v_j) = \sum_{j_2,...,j_r \in [n]} a_{jj_2...j_r}.$$  

**Proof.** $\sum_{j_2,...,j_r \in [n]} a_{jj_2...j_r}$ has non-zero terms only for corresponding hb-edges $e_i$ that have $v_j$ in it. For such a hb-edge containing $v_j$, it is described by $e_i = \{v_j^{m_{ij}},v_{j_2}^{m_{i2}},\ldots,v_{j_k}^{m_{ik}}\}$. It means that the multiset $\{j_2,\ldots,j_r\}$ corresponds exactly to the multiset $\{j_1^{m_{i1}},j_2^{m_{i2}},\ldots,j_k^{m_{ik}}\}$. For each $e_i$ such that $v_j \in e_i$,
4.2.4 Elementary operations on hb-graphs

In Ouvrard et al. [2017], we describe two elementary operations that are used in the hypergraph uniformisation process. We describe here two similar operations and some additional operations for hb-graphs.

Operation 4.1. Let \( \mathcal{H} = (V, E) \) be a hb-graph. Let \( w_1 \) be a constant weighted function on hb-edges with constant value 1. The weighted hb-graph \( \mathcal{H}_1 = (V, E, w_1) \) is called the canonical weighted hb-graph of \( \mathcal{H} \).

The application \( \phi_{cw} : \mathcal{H} \mapsto \mathcal{H}_1 \) is called the canonical weighting operation.

Operation 4.2. Let \( \mathcal{H} = (V, E, w_1) \) be a canonical weighted hb-graph. Let \( c \in \mathbb{R}^{++} \). Let \( w_c \) be a constant weighted function on hb-edges with constant value \( c \).

The weighted hb-graph \( \mathcal{H}_c = (V, E, w_c) \) is called the \( c \)-dilatated hb-graph of \( \mathcal{H} \).

The application \( \phi_{c-d} : \mathcal{H}_1 \mapsto \mathcal{H}_c \) is called the \( c \)-dilatation operation.

Operation 4.3. Let \( \mathcal{H}_w = (V, E, w) \) be a weighted hb-graph. Let \( y \notin V \) be a new vertex.

The y-complemented hb-graph of \( \mathcal{H}_w \) is the hbgraph \( \mathcal{H}_w = (\tilde{V}, \tilde{E}, \tilde{w}) \) where

\[
\tilde{V} = V \cup \{y\}, \quad \tilde{E} = (\xi(e))_{e \in E} - \text{with the map } \xi : E \to \mathcal{M}(\tilde{V}) \text{ such that for all } e \in E, \xi(e) \in \mathcal{M}(\tilde{V}) \text{ and is the multiset } \left\{ x^{m_{\xi(e)}(x)} : x \in \tilde{V} \right\}, \text{ with } \]

\[m_{\xi(e)}(x) = \begin{cases} m_e(x) & \text{if } x \in e^* \\ r_{\mathcal{H}} - \# m_e & \text{if } x = y \end{cases}\]

- and, the weight function is \( \tilde{w} \) is such that \( \forall e \in E: \tilde{w}(\xi(e)) = w(e) \).

The application \( \phi_{y-c} : \mathcal{H}_w \mapsto \mathcal{H}_w \) is called the y-complemented operation.
Definition 4.5. Let $\mathcal{H} = (V, E)$ and $\mathcal{H}' = (V', E')$ be two hb-graphs. Let $\phi : \mathcal{H} \rightarrow \mathcal{H}'$.

$\phi$ is said preserving e-adjacency if vertices of $V'$ that are e-adjacent in $\mathcal{H}'$ are either e-adjacent vertices in $\mathcal{H}$ or the maximal subset of these vertices that are in $V$ are e-adjacent in $\mathcal{H}$.

$\phi$ is said preserving exactly e-adjacency if vertices that are e-adjacent in $\mathcal{H}'$ are e-adjacent in $\mathcal{H}$ and reciprocally.

We can extend these definitions to $\psi : (\mathcal{H}_i)_{i \in I} \rightarrow \mathcal{H}'$.

Definition 4.6. Let $(\mathcal{H}_i)_{i \in I}$ be a family of hb-graphs with $\forall i \in I$, $\mathcal{H}_i = (V_i, E_i)$ and $\mathcal{H}' = (V', E')$ a hb-graph.
Let consider \( \psi : (\mathcal{H}_i)_{i \in I} \mapsto \mathcal{H}' \).

\( \psi \) is said preserving e-adjacency if vertices that are e-adjacent in \( \mathcal{H}' \) are either e-adjacent vertices in exactly one of the \( \mathcal{H}_i, i \in I \) or the maximal subset of these vertices that is in \( V = \bigcup_{i \in I} V_i \) is e-adjacent in exactly one of the \( \mathcal{H}_i \).

\( \phi \) is said preserving exactly e-adjacency if vertices that are e-adjacent in \( \mathcal{H}' \) are e-adjacent in exactly one of the \( \mathcal{H}_i, i \in I \) and reciprocally.

We can extend these definitions to \( \nu : \mathcal{H} \mapsto (\mathcal{H}_i)_{i \in I} \).

**Definition 4.7.** Let \((\mathcal{H}_i)_{i \in I}\) be a family of hb-graphs with \( \forall i \in I, \mathcal{H}_i = (V_i, E_i) \) and \( \mathcal{H} = (V, E) \) a hb-graph.

Let consider \( \nu : \mathcal{H} \mapsto (\mathcal{H}_i)_{i \in I} \).

\( \psi \) is said preserving e-adjacency if vertices that are e-adjacent in one of the \( \mathcal{H}_i, i \in I \) are either e-adjacent vertices in \( \mathcal{H} \) or the maximal subset of these vertices that is in \( V \) is e-adjacent in \( \mathcal{H} \).

\( \phi \) is said preserving exactly e-adjacency if vertices that are e-adjacent in one of the \( \mathcal{H}_i, i \in I \) are e-adjacent in \( \mathcal{H} \) and reciprocally.

**Claim 4.3.** Let \( \mathcal{H} = (V, E) \) be a hb-graph.

The canonical weighting operation, the \( \epsilon \)-dilatation operation, the merging operation and, the decomposition operation preserve exactly e-adjacency.

The \( \gamma \)-complemented operation and the \( \gamma^\alpha \)-vertex-increasing operation preserve e-adjacency.

**Proof.** Immediate.

---

**Claim 4.4.** The composition of two operations which preserve (respectively exactly) e-adjacency preserves (respectively exactly) e-adjacency.

The composition of two operations where one preserves exactly e-adjacency and the other preserves e-adjacency preserves e-adjacency.

**Proof.** Immediate.

---

### 4.2.5 Processes involved for building the e-adjacency tensor

In a general natural hb-graph \( \mathcal{H} \), hb-edges are not forced to have same m-cardinality: the rank of the \( \mathcal{E} \)-adjacency tensor of the elementary hb-graph associated to each hb-edge depends on the m-cardinality of the hb-edge. As a consequence the hb-graph polynomial is no more homogeneous. Nonetheless techniques to homogenize such a polynomial are well known.

The hb-graph m-uniformisation process (Hm-UP) transform a given hb-graph of m-range \( r_\mathcal{H} \) into a \( r_\mathcal{H} \)-m-uniform hb-graph written \( \mathcal{H} \): this uniformisation can be mapped to the homogenization of the attached polynomial of the original hb-graph, called the polynomial homogenization process (PHP).

The Hm-UP can be achieved by different means:
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- **straightforward m-uniformisation** levels directly all hb-edges by adding a Null vertex $y_0$ with multiplicity the difference between the hb-graph m-rank and the hb-edge m-cardinality. It is achieved by considering the $y_0$-complemented hb-graph of $H$.

- **silo m-uniformisation** processes the hb-edges of given m-cardinality $r$, regrouped in the dilatated hb-graph $H_{c_r}$ obtained by canonical weighting and dilatation of the $r$-m-uniform sub-hb-graph $H_r$ of $H$ containing all its hb-edges of m-cardinality $r$ - by adding a $r$-dependent null vertex $y_r$ in multiplicity $r_H - r$ so it levels the hb-edges to a constant m-cardinality of $r_H$; it uses the $y_r^{r_H-r}$-vertex-increasing operation on each $H_{c_r}$. A merging operation is then used to gather all the $y_r^{r_H-r}$-vertex-increased hb-graph into a single $r_H$-m-uniform.

- **layered m-uniformisation** processes m-uniform hb-subgraphs of increasing m-cardinality by successively adding a vertex and merging it to the hb-subgraph of the above layer. The layered homogenisation process applied to hypergraphs was explained with full details in Ouvrard et al. [2017]; it involves a two-phase step iterations based on successive $\{y_k\}$-vertex-increased hb-graphs and merging with the dilatated weighted hb-graph of the next layer.

4.2.6 On the choice of the technical coefficient $c_{e_i}$

The technical coefficient $c_{e_i}$ has to be chosen such that by using the elements of the $e$-adjacency hypermatrix $A = (a_{i_1,...,i_r})_{i_1,...,i_r \in [n]}$, the hypermatrix allows to retrieve:

1. the m-degree of the vertices: $\sum_{i_2,...,i_r \in [n]} a_{i_2...i_r} = \deg_m(v_i)$.
2. the number of hb-edges $|E|$.

The approach is similar to the one used in Part 1: we consider a hb-graph $H$ that we decompose in a family of $r$-m-uniform hb-graphs $(H_r)_{r \in [r_H]}$.

To achieve it, let $R$ be the equivalency relation defined on $E$ the family of hb-edges of $H$: $e E e' \iff \#_m e = \#_m e'$.

$E/R$ is the set of classes of hb-edges of same m-cardinality. The elements of $E/R$ are the sets: $E_r = \{ e \in E : \#_m e = r \}$.

Considering $R = \{ r : E_r \in E/R \}$, it is set $E_r = \emptyset$ for all $r \in [r_H] \setminus R$.

Let consider the hb-graphs: $H_k = (V, E_r)$ for all $r \in [r_H]$ which are all $r$-m-uniform.

It holds: $E = \bigcup_{r \in [r_H]} E_r$ and $E_{r_1} \cap E_{r_2} = \emptyset$ for all $r_1 \neq r_2$, hence $(E_r)_{r \in [r_H]}$ constitutes a partition of $E$ which is unique from the way it has been defined.

Hence:

$H = \bigoplus_{r \in [r_H]} H_r$.

Each of these $r$-m-uniform hb-graph $H_r$ can be associated to a $k$-adjacency tensor $A_r$ viewed as a hypermatrix $A_{H_r} = (a_{i(r)i_2...i_r})$ of order $r$, hypercubic.
and, symmetric of dimension $|V| = n$.

We write $\left(a_{i_1,\ldots,i_r}^{c}\right)_{i_1,\ldots,i_r\in[n]}$ the $c$-adjacency hypermatrix associated to $\mathcal{H}$: it is going to be built hereafter and $n_1$ depends on the way the hypermatrix is built.

The number of hb-edges in $\mathcal{H}_r$ is given by summing the elements of $A_r$:

$$
\sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{(r)} = \sum_{i_1,\ldots,i_r\in[n]} a_{i_2,\ldots,i_r}^{(r)} = \sum_{i_1,\ldots,i_r\in[n]} \deg_m(v_i) = r |E_r|
$$

In the whole hb-graph $\mathcal{H}$ can also be calculated as:

$$
\sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{(r)} = r \sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{(r)} = r |E_r|
$$

As

$$
|E| = \sum_{r=1}^{n} |E_r| = \sum_{r=1}^{n} \frac{1}{r} \sum_{i=1}^{n} \deg_m(v_i) = \sum_{r=1}^{n} \frac{1}{r} \sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{(r)}
$$

Hence, it follows:

$$
\sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{H} = \sum_{r=1}^{n} \frac{r}{r} \sum_{i_1,\ldots,i_r\in[n]} a_{i_1,\ldots,i_r}^{(r)}
$$

Also, choosing for all $i \in [p]$ such that $\#e_i = r$, $c_{e_i} = \frac{r \mathcal{H}}{r}$. We write for all $r \in [r_H]$: $c_r = \frac{r \mathcal{H}}{r}$. It is the technical coefficient for the corresponding layer of level $r$ of the hb-graph $\mathcal{H}$.

Hence, the HUP is initiated by applying the canonical weighting to each $m$-uniform hb-graph $\mathcal{H}_r$ that transforms it into $\mathcal{H}_{r,1}$. Then the $c_r$-dilatation operation is applied to each weighted $m$-uniform hb-graph $\mathcal{H}_{r,1}$ to obtain its $c_r$-dilatated hb-graph $\mathcal{H}_{r,c_r}$. 
4.2.7 Straightforward approach

**Straightforward m-uniformisation:** We first decompose $\mathcal{H} = \bigoplus_{r \in [r_H]} \mathcal{H}_r$ as viewed in sub-section 4.2.6.

We transform each $\mathcal{H}_r, r \in [r_H]$ into a canonical weighted hb-graph $\mathcal{H}_{r,1}$ that we dilate using the dilatation coefficient obtaining the $c_r$-dilatated hb-graph $\mathcal{H}_{r,c_r}$.

This family $(\mathcal{H}_{r,c_r})$ is then merged into the hb-graph: $\mathcal{H}_{w,d} = \bigoplus_{r \in [r_H]} \mathcal{H}_{r,c_r}$.

To get a m-uniform hb-graph at last we generate a vertex $y_1 \not\in V$ and apply to $\mathcal{H}_{w,d}$ the $y_1$-complemented operation to obtain $\tilde{\mathcal{H}}_{w,d}$ the $y_1$-complemented hb-graph of $\mathcal{H}_{w,d}$.

The different steps are summarized in Figure 1.

**Claim 4.5.** The transformation $\phi_s : \mathcal{H} \mapsto \tilde{\mathcal{H}}_{w,d}$ preserves the $e$-adjacency.

**Proof.** $\phi_s = \phi_{y_1} \circ \phi_{m} \circ \left( \phi_{c-d} \circ \phi_{cw} \right) \circ \phi_d$.  

The operations involved either preserve $e$-adjacency or preserve exactly $e$-adjacency, also by composition $\phi_s$ preserve $e$-adjacency.

**Straightforward homogenization:** To homogenize the hb-graph, we add an additional vertex $N$ into the universe of the hb-graph, ie the vertex set, corresponding to one additional variable $y_1$.

The $k$-adjacency hypermatrix of the hb-edge $e_i = \left\{ v_{j_1}^{m_{i,j_1}}, \ldots, v_{j_k}^{m_{i,j_k}} \right\}$ is $\mathcal{Q}_e_i$ of rank $\rho_i = \#m_{e_i}$ and dimension $n$. The corresponding reduced polynomial is $P_{e_i}(z_0) = \rho_i z_{j_1}^{m_{i,j_1}} \cdots z_{j_k}^{m_{i,j_k}}$.

To get it of degree $r_\mathcal{H}$ we add an additional variable $y_1$ with multiplicity $m_{i,n+1} = r_\mathcal{H} - \rho_i$.

---

2 \( \vdots \) \( \vdots \) \( \vdots \) \( \vdots \) indicates parallel operations on each member of the family indicated in index of the right parenthesis.
The term \( P_{e_i}(z_0) \) with attached tensor \( P_{e_i} \) of rank \( \rho_i \) and dimension \( n \) is transformed in:

\[
R_{e_i}(z_1) = P_{e_i}(z_0) y_1^{m_{i+n+1}} = r_i z_{j_1}^{m_{ij1}} \cdots z_{j_{k_i}}^{m_{ijk_i}} y_1^{m_{i+n+1}}
\]

with attached tensor \( R_{e_i} \) of rank \( r_H \) and dimension \( n+1 \).

The only non-zero elements of \( R_{e_i} \) are:

\[
r_{j_i}^{m_{ij1} \cdots m_{ijk_i}(n+1)m_{i+n+1}} = \frac{\rho_i m_{ij1}! \cdots m_{ijk_i}! m_{i+n+1}!}{r_H!}
\]

and all the elements of \( R_{e_i} \) obtained by permutation of the indices and with same value. The number of permutation is:

\[
\frac{r_H!}{m_{ij1}! \cdots m_{ijk_i}! m_{i+n+1}!}
\]

The hb-graph polynomial \( P(z_0) = \sum_{i \in [p]} c_i P_{e_i}(z_0) \) is transformed into a homogeneous polynomial:

\[
R(z_1) = \sum_{i \in [p]} c_i R_{e_i}(z_1) = \sum_{i \in [p]} c_i z_{j_1}^{m_{ij1}} \cdots z_{j_{k_i}}^{m_{ijk_i}} y_1^{m_{i+n+1}}
\]

representing the homogenized hb-graph \( H \) with attached tensor \( R = \sum_{i=1}^{p} c_e_i R_{e_i} \), where \( c_{e_i} = \frac{r_H}{\# m e_i} \).

**Definition 4.8.** The straightforward \( e \)-adjacency hypermatrix of a hb-graph \( H = (V, E) \) is the hypermatrix \( A_{str,H} \) defined by:

\[
A_{str,H} = \sum_{i \in [p]} c_{e_i} R_{e_i},
\]

where for \( e_i = \{ v_{j_1}^{m_{ij1}}, \ldots, v_{j_{k_i}}^{m_{ijk_i}} \} \in E \) the associated hypermatrix is: \( R_{e_i} = (r_{i_1} \ldots r_{i_H}) \), which only non-zero elements are:

\[
r_{j_i}^{m_{ij1} \cdots m_{ijk_i}(n+1)m_{i+n+1}} = \frac{m_{ij1}! \cdots m_{ijk_i}! m_{i+n+1}!}{r_H!}
\]

- with \( m_{i+n+1} = r_H - \# m e_i \) - and the ones with same value and obtained by permutation of the indices and where \( c_{e_i} = \frac{r_H}{\# m e_i} \).

**4.2.8 Silo approach**

**Silo m-uniformisation:** The first steps are similar to the straightforward approach.
The hb-graph $\mathcal{H}$ is decomposed in layers $\mathcal{H} = \bigoplus_{r \in [r_H]} \mathcal{H}_r$ as described in subsection 4.2.6. Each $\mathcal{H}_r, r \in [r_H]$ is canonically weighted and $c_r$-dilatated to obtain $\mathcal{H}_{r,c}$. We generate $r_H - 1$ new vertices $y_i \notin V, i \in [r_H - 1]$. We then apply to each $\mathcal{H}_{r,c}, r \in [r_H - 1]$ the $y_r^{r_H - r}$-vertex-increasing operation to obtain $\mathcal{H}^+_{r,c}$ the $y_r^{r_H - r}$-complemented hb-graph of each $\mathcal{H}_{r,c}, r \in [r_H - 1]$. The family $(\mathcal{H}^+_{r,c})_{r \in r_H}$ is then merged using the merging operation to obtain the $r_H$-m-uniform hb-graph $\hat{\mathcal{H}}_{\hat{\omega}}$.

The different steps are summarized in Figure 2.

\begin{claim}
The transformation $\phi_s : \mathcal{H} \mapsto \hat{\mathcal{H}}_{\hat{\omega}}$ preserves the $e$-adjacency.
\end{claim}

\begin{proof}
$\phi_s = \phi_m \circ \left( \phi_{y_r^{r_H - r} \circ \phi_{c,d} \circ \phi_{cw}} \right) \circ \phi_d$.

The operations involved either preserve $e$-adjacency or preserve exactly $e$-adjacency, also by composition $\phi_s$ preserve $e$-adjacency.
\end{proof}

\section*{Silo homogenization:}

In this homogenization process we suppose that the $r_H$-edges are sorted by $m$-cardinality.

We add $r_H - 1$ vertices $N_1$ to $N_{r_H - 1}$ into the universe, i.e., the vertex set, corresponding to $r_H - 1$ additional variables respectively $y_1$ to $y_{r_H - 1}$.

The term $P_{e_i}(z_0) = z_j^{m_{ij_1}} \ldots z_{j_k}^{m_{ij_k}}$ of $P$ has degree the $m$-cardinality of the $r_H$-edge $e_i$, i.e. $\# m e_i$. To get it of degree $r_H$, we use the additional variable $y_{\# m e_i}$ with multiplicity $m_{i \# m e_i} = r_H - \# m e_i$.

The term $P_{e_i}(z_0)$ with attached tensor $\mathcal{T}_{e_i}$ of rank $\# m e_i$ and dimension $n$ is transformed in $R_{e_i}(z_{\# m e_i}) = P_{e_i}(z_0) y_{\# m e_i}$ with attached tensor $R_{e_i}$ of rank $r_H$ and dimension $n + 1$.

The only non-zero elements of $R_{e_i}$ are:

$$\begin{bmatrix}
\begin{array}{c}
m_{ij_1}! \ldots m_{ij_k}! m_{i n + \# m e_i}! \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\frac{r_H!}{c_{i j_1} \ldots c_{i j_k} (n + \# m e_i)_{m_{i n + \# m e_i}}} \\
\end{array}
\end{bmatrix}
$$

and the same value elements which indices are obtained by permutation of this first element.
$P$ is transformed into a homogeneous polynomial

$$R(z_{rH^{-1}}) = \sum_{i \in [p]} c_i R_{e_i}(z_{\# m e_i}) = \sum_{i \in [p]} c_i z_{ji_{i_1}}^{m_{i_1}} \ldots z_{ji_{i_k}}^{m_{i_k}} y_{\# m e_i}^{m_{i_{n+\# m e_i}}}$$

representing the homogenized hb-graph $\overline{H}$ with attached tensor $R = \sum_{i \in [p]} c_i R_{e_i}$, where: $c_{e_i} = \frac{r_H}{\# m e_i}$.

**Definition 4.9.** The silo $e$-adjacency hypermatrix of a hb-graph $H = (V, E)$ is the hypermatrix $A_{sil, H} = (a_{i_1 \ldots i_{rH}})_{i_1, \ldots, i_{rH} \in [n]}$ defined by:

$$A_{sil, H} = \sum_{i \in [p]} c_{e_i} R_{e_i}$$

and where for $e_i = \{v_{j_{i_1}}^{m_{i_1}}, \ldots, v_{j_{i_k}}^{m_{i_k}}\} \in E$ the associated hypermatrix is:

$$R_{e_i} = (r_{i_1 \ldots i_{rH}}),$$

which only non-zero elements are:

$$r_{j_{i_1}^{m_{i_1}} \ldots j_{i_k}^{m_{i_k}} (n+\# m e_i)^{m_{i_{n+\# m e_i}}}} = \frac{m_{i_1}! \ldots m_{i_{k}}! m_{i_{n+\# m e_i}}!}{r_H!}$$

and all elements of $R_{e_i}$ obtained by permuting

$$j_{1}^{m_{i_1}} \ldots j_{k_{i}}^{m_{i_k}} (n+\# m e_i)^{m_{i_{n+\# m e_i}}},$$

with:

$$m_{i_{n+\# m e_i}} = r_H - \sum_{l \in [k_i]} m_{i_{j_l}},$$

and where:

$$c_{e_i} = \frac{r_H}{\# m e_i}.$$

**Remark 4.3.** In this case,

$$A_{sil, H} = \sum_{r \in [r_H]} c_r \sum_{e_i \in \{e: \# m e = r\}} R_{e_i}$$

where $c_r = \frac{r_H}{r}$.

### 4.2.9 Layered approach

**Layered uniformisation:** The first steps are similar to the straightforward approach.

The hb-graph $H$ is decomposed in layers $H = \bigoplus_{r \in [r_H]} H_r$ as described in subsection 4.2.6. Each $H_r, r \in [r_H]$ is canonically weighted and $c_r$-dilatated to obtain $H_{r,c_r}$.
We generate $r_H - 1$ new vertices $y_i \notin V$, $i \in [r_H - 1]$ and write $V_s = \{y_i : i \in [r_H - 1]\}$.

A two-phase steps iteration as it has been done with hypergraphs in Ouvrard et al. [2017] and Ouvrard et al. [2018a] is considered: the inflation phase (IP) and the merging phase (MP). At step $k = 0$, $K_0 = H_{1,c_1}$ and no further action is made but increasing $k$ of 1 and going to the next step. At step $k > 0$, the input is the $k$-m-uniform weighted hb-graph $K_k$ obtained from the previous iteration. In the IP, $K_k$ is transformed into $K_k^+$ the $y_k^1$-vertex-increased hb-graph, which is ($k + 1$)-m-uniform.

The MP is merging the hypergraphs $K_k^+$ and $H_{k+1,c_{k+1}}$ into a single ($k + 1$)-m-uniform hb-graph $\tilde{K}$. We iterate while $k < r_H$, increasing in between each step $k$ of 1. When $k$ reaches $r_H$, we stop iterating and the last $K_{\tilde{r}}$ obtained, written $\tilde{H}_{\tilde{r}}$ is called the $V_S$-layered m-uniform hb-graph of $H$. The different steps are summarized in Figure 3.

Claim 4.7. The transformation $\phi_s : H \mapsto \tilde{H}_{\tilde{r}}$ preserves the e-adjacency.

Proof. $\phi_s = \psi \circ \left( \phi_{c-d} \circ \phi_{cw} \right) \circ \phi_d$, where $\psi$ is called the iterative layered operation that converts the family obtained by $\left( \phi_{c-d} \circ \phi_{cw} \right) \circ \phi_d$ and transform...
it into the $V_S$-layered m-uniform hb-graph of $\mathcal{H}$.

The operations involved in the operations $\phi_{c-d}$, $\phi_{c-w}$ and $\phi_d$ either preserve $e$-adjacency or preserve exactly $e$-adjacency, and so forth by composition.

The iterative layered operation preserve $e$-adjacency as the operations involved are preserving $e$-adjacency and that the family of hb-graphs at the input has hb-edges family that are totally distinct.

Also by composition $\phi_n$ preserve $e$-adjacency.

Layered homogenization: This solution was first developed in Ouvrard et al. [2017] for general hypergraphs. The idea is to sort the hb-edges as in the silo homogenization and considering as well $r_{\mathcal{H}} - 1$ additional vertices $L_1$ to $L_{r_{\mathcal{H}} - 1}$ into the universe, corresponding to $r_{\mathcal{H}} - 1$ additional variables respectively $y_1$ to $y_{r_{\mathcal{H}} - 1}$.

But these vertices are added successively to each hb-edge to fill the hb-edges to a $r_{\mathcal{H}}$ value of the $m$-cardinality: a hb-edge of initial cardinality $#_{m} e_i$ will be filled with elements $L_{#_{m} e_i}$ to $L_{r_{\mathcal{H}} - 1}$. It matches to add the $k$-m-uniform hb-subgraph $\mathcal{H}_k$ with the $k + 1$-m-uniform hb-subgraph $\mathcal{H}_{k+1}$ by filling the hb-edge of $\mathcal{H}_k$ with the additional vertex $L_k$ to get a homogenised $k + 1$-m-uniform hb-subgraph of the homogenised hb-graph $\overline{\mathcal{H}}$.

A hb-edge of $m$-cardinality $#_{m} e_i$ is represented by the polynomial

$$P_{e_i} (z_0) = z_{m_{j_1}}^{m_{j_1}} \cdots z_{m_{j_k}}^{m_{j_k}}$$

of degree $#_{m} e_i$.

All the hb-edges of same m-cardinality $m$ belongs to the same layer of level $m$. To transform the hb-edge of m-cardinality $#_{m} e_i + 1$ we fill it with the element $L_{#_{m} e_i}$.

In this case, the polynomial $P_{e_i} (z_0)$ is transformed into:

$$R_{(1)e_i} (z_{#_{m} e_i}) = P_{e_i} (z_0) y_{#_{m} e_i}^{1}$$

of degree $#_{m} e_i + 1$.

Iterating over the layers the polynomial:

$$P_{e_i} (z_0) = z_{m_{j_1}}^{m_{j_1}} \cdots z_{m_{j_k}}^{m_{j_k}}$$

is transformed in:

$$R_{(r_{\mathcal{H}} - #_{m} e_i)e_i} (z_{r_{\mathcal{H}} - 1}) = P_{e_i} (z_0) y_{#_{m} e_i}^{1} \cdots y_{r_{\mathcal{H}} - 1}^{1}$$

of degree $r_{\mathcal{H}}$.

The polynomial $P_{e_i} (z_0)$ with attached tensor $\mathcal{P}_{e_i}$ of rank $#_{m} e_i$ and dimension $n$ is transformed in:

$$R_{(r_{\mathcal{H}} - #_{m} e_i)e_i} (z_{r_{\mathcal{H}} - 1}) = R_{e_i} (z_0) y_{#_{m} e_i}^{1} \cdots y_{r_{\mathcal{H}} - 1}^{1}$$

with attached tensor $\mathcal{R}_{(r_{\mathcal{H}} - #_{m} e_i)e_i}$ of rank $r_{\mathcal{H}}$ and dimension $n + r_{\mathcal{H}} - 1$. 
The only non-zero elements of $R_{(r_H - \# m_{e_i})e_i}$ are:

$$r_{(r_H - \# m_{e_i})j_k}^{m_{ij1} \ldots m_{ijk1}} [n + \# m_{e_i}] \ldots [n + r_H - 1]^1 = \frac{m_{ij1}! \ldots m_{ijk1}!}{r_H!}$$

and all the elements of $R_{(r - \# m_{e_i})e_i}$ obtained by permuting:

$$j_1^{m_{ij1}} \ldots j_{k_i}^{m_{ijk1}} [n + \# m_{e_i}] \ldots [n + r_H - 1]^1$$

And $P$ is transformed in a homogeneous polynomial

$$R(z_{r_H-1}) = \sum_{i \in [p]} c_{e_i} R_{(r_H - \# m_{e_i})e_i} (z_{r_H-1})$$

representing the homogenized hb-graph $\overline{H}$ with attached tensor

$$R = \sum_{i \in [p]} c_{e_i} R_{(r_H - \# m_{e_i})e_i},$$

where:

$$c_{e_i} = \frac{r_H}{\# m_{e_i}}.$$

**Definition 4.10.** The layered $e$-adjacency tensor of a hb-graph $H = (V, E)$ is the tensor

$$A_{lay}(H) = (a_{i_1 \ldots i_{r_H}})_{1 \leq i_1, \ldots, i_{r_H} \leq n}$$

defined by:

$$A_{lay}(H) = \sum_{i \in [p]} c_{e_i} R_{(r_H - \# m_{e_i})e_i}$$

where for $e_i = \{v_{j_{11}}^{m_{ij1}}, \ldots, v_{j_{k_i}}^{m_{ijk1}}\} \in E$ the associated tensor is:

$$R_{(r_H - \# m_{e_i})e_i} = \left(r_{(r_H - \# m_{e_i})i_1 \ldots i_{r_H}}\right),$$

which only non-zero elements are:

$$r_{(r_H - \# m_{e_i})j_k}^{m_{ij1} \ldots m_{ijk1}} [n + \# m_{e_i}] \ldots [n + r_H - 1]^1 = \frac{m_{ij1}! \ldots m_{ijk1}!}{r_H!}$$

and all elements of $R_{e_i}$ with same value obtained by permuting:

$$j_1^{m_{ij1}} \ldots j_{k_i}^{m_{ijk1}} [n + \# m_{e_i}] \ldots [n + r_H - 1]^1,$$

and where: $c_{e_i} = \frac{r_H}{\# m_{e_i}}$. 
Remark 4.4. $A_{lay}(H)$ can also be written:

$$A_{lay}(H) = \sum_{r \in [r_H]} c_r \sum_{e_i \in \{e : \#_m e = r\}} R_{e_i},$$

where $c_r = \frac{r_H}{r}$.

5 Results on the constructed tensors

Each of the tensor built is of rank $r_H$ and of dimension $n + n_A$ where $n_A$ is:

- in the straightforward approach: $n_A = 1$.
- in the silo approach and the layered approach: $n_A = r_H - 1$.

5.1 Information on hb-graph

5.1.1 m-degree of vertices

We built the different tensors in a way that the retrieval of the vertex m-degree is possible; the null vertex(-ices) added give(s) additional information on the structure of the hb-graph.

Claim 5.1. Let consider for $j \in [n]$ a vertex $v_j \in V$.

Then in each of the e-adjacency tensors built, it holds:

$$\sum_{j_2, \ldots, j_{r_H} \in [n + n_A]} a_{jj_2 \ldots j_{r_H}} = \sum_{i : v_j \in e_i} m_{ij} = \deg_m(v_j)$$

Proof. For $j \in [n]$: $\sum_{j_2, \ldots, j_{r_H} \in [n + n_A]} a_{jj_2 \ldots j_{r_H}}$ has non-zero terms only for corresponding hb-edges of original hb-graph $e_i$ that have $v_j$ in it. For such a hb-edge containing $v_j$, it is described by $e_i = \{v_j^{m_{i,j}}, l_{i_2}^{m_{i,l_2}}, \ldots, l_{i_k}^{m_{i,l_k}}\}$. It means that the multiset $\{\{j_2, \ldots, j_{r_H}\}\}$ corresponds exactly to the multiset $\{j_{m_{i,j} - 1}, l_{i_2}^{m_{i,l_2}}, \ldots, l_{i_k}^{m_{i,l_k}}\}$.

In the straightforward approach, for each $e_i$ such that $v_j \in e_i$, there are:

$$\frac{(r_H - 1)!}{(m_{i,j} - 1)! m_{i,l_2}! \ldots m_{i,l_k}! m_{i,n+1}!}$$

possible permutations of the indices $j_2$ to $j_{r_H}$ and

$$a_{jj_2 \ldots j_{r_H}} = m_{i,j}! m_{i,l_2}! \ldots m_{i,l_k}! m_{i,n+1}! \frac{1}{(r_H - 1)!}.$$  

In the silo approach, for each $e_i$ such that $v_j \in e_i$, there are:

$$\frac{(r_H - 1)!}{(m_{i,j} - 1)! m_{i,l_2}! \ldots m_{i,l_k}! m_{i,n+1+\#_e e_i}!}.$$
possible permutations of the indices $j_2$ to $j_{r_H}$ and
\[
a_{j_2 \ldots j_{r_H}} = \frac{m_{i}! m_{i_2}! \ldots m_{i_k}! m_{n+\#e_i}!}{(r_H - 1)!}.
\]

In the layered approach, for each $e_i$ such that $v_j \in e_i$, there are:
\[
\frac{(r_H - 1)!}{(m_{i_j} - 1)! m_{i_2}! \ldots m_{i_k}!}
\]
possible permutations of the indices $j_2$ to $j_{r_H}$ which have all the same value equals to:
\[
a_{j_2 \ldots j_{r_H}} = \frac{m_{i_j}! m_{i_2}! \ldots m_{i_k}!}{(r_H - 1)!}.
\]

Also, whatever the approach taken:
\[
\sum_{j_2 \ldots j_{r_H} \in [n]} a_{j_2 \ldots j_{r_H}} = \sum_{i: v_j \in e_i} m_{i_j} = \deg_{m_i}(v_j).
\]

\[\square\]

5.1.2 Additional vertex information

The additional vertices carry information on the $hb$-edges of the $hb$-graph: the information carried depends on the approach taken.

Claim 5.2. The layered $e$-adjacency tensor allows the retrieval of the distribution of the $hb$-edges.

Proof. For $j \in [n_A]$:
\[
\sum_{j_2 \ldots j_{r_H} \in [n+n_A]} a_{n+j_2 \ldots j_{r_H}}
\]
has non-zero terms only for corresponding $hb$-edges of the uniformized $hb$-graph $\tau_i$ that have $v_j$ in it. Such a $hb$-edge is described by:
\[
\tau_i = \{v_k^{m_{i_k}} : 1 \leq k \leq n + n_A\}.
\]

It means that the multiset:
\[
\{\{j_2, \ldots, j_{r_H}\}\}
\]
corresponds exactly to the multiset:
\[
\{(n + j)^{m_{i_n+j-1}}\} + \{k^{m_{i_k}} : 1 \leq k \leq n + n_A, k \neq j\}.
\]

The number of possible permutations of elements in this multiset is:
\[
\frac{(r_H - 1)!}{(m_{i_{n+j}} - 1)! \prod_{k \in [n]} m_{i_k}! \prod_{k \in [n+n_A]} \prod_{k \neq j} m_{i_k}!}.
\]
and the elements corresponding to one hb-edge are all equals to:

$$\prod_{k \in [n_A]} m_k k! \over (r_H - 1)!.$$  

Thus:

$$\sum_{j_2 \ldots j_{r_H} \in [n+n_A]} a_{n+j_2 \ldots j_{r_H}} = \sum_{j_2 \ldots j_{r_H} \in [n]} m_{n+j} = \text{deg}_m (N_j)$$

The interpretation differs between the different approaches.

**For the silo approach:**

There is one added vertex in each hb-edge. The silo of hb-edges of m-cardinality $m_s$ ($m_s \in [r_H - 1]$) is associated to the null vertex $N_{m_s}$. The multiplicity of $N_{m_s}$ in each hb-edge of the silo is $r_H - m_s$.

Hence:

$$\text{deg}_m (N_j) \over r_H - m_s = |\{e : #m e = m_s\}|.$$  

The number of hb-edges in the silo $m_s$ is then deduced by the following formula:

$$|\{e : #m e = m_s\}| = |E| - \sum_{m_s \in [r_H - 1]} \text{deg}_m (N_j) \over r_H - m_s.$$  

**For the layered approach:**

The vertex $N_j$ corresponds to the layer of level $j$ and added to each hb-edge that has m-cardinality less or equal to $j$ with a multiplicity of 1.

Also:

$$\text{deg}_m (N_j) = |\{e : #m e \leq j\}|.$$  

Hence, for $j \in [2; r_H - 1]$:

$$|\{e : #m e = j\}| = \text{deg}_m (N_j) - \text{deg}_m (N_{j-1})$$

and:

$$|\{e : #m e = 1\}| = \text{deg}_m (N_1)$$

$$|\{e : #m e = r_H\}| = |E| - \text{deg}_m (N_{r_H-1})$$

**For the straightforward approach:**

In a hb-edge of m-cardinality $j \in [r_H - 1]$, the vertex $N_1$ is added in multiplicity $r_H - j$. The number of hb-edge of m-cardinality $j$ can be retrieved by considering the elements of $A_{\text{str}} (H)$ of index $(n+1)i_1 \ldots i_{r_H-1}$ where $1 \leq i_1 \leq \ldots \leq i_j \leq n$ and $i_{j+1} = \ldots = i_{r_H-1} = n+1$ and thus of indices obtained by permutation.

It follows for $j \in [r_H - 1]$:
\[ |\{e : #_m e = j\}| = |\{e : N_1 \in e \land m_e (N_1) = r_H - j\}| \]
\[ = \sum_{i_1, \ldots, i_{r_H - 1} \in [n + 1]}^{a_{n+1} \ldots i_{r_H - 1}} |\{i_k = n + 1\}| = r_H - j - 1 \]

The terms of this sum \(a_{n+1} \ldots i_{r_H - 1}\) are non-zero only for corresponding hb-edges \(\tau\) of the uniformized hb-graph that have \(N_1\) in multiplicity \(r_H - j\) in it. Such a hb-edge is described by:
\[ \tau = \{u_k^{m_k} : 1 \leq k \leq n\} + \{N_1^{r_H - j}\}. \]

It means that the multiset:
\[ \{\{i_1, \ldots, i_{r_H - 1}\}\} \]
corresponds exactly to the multiset:
\[ \{k^{m_k} : k \in [n]\} + \{n + 1^{r_H - j - 1}\}. \]

The number of possible permutations of elements in this multiset is:
\[ \frac{(r_H - 1)!}{\prod_{k \in [n]} m_k! (r_H - j - 1)!} \]
and the elements corresponding to one hb-edge all equal:
\[ \frac{\prod_{k \in [n]} m_k! \times (r_H - j)!}{(r_H - 1)!}. \]

Hence:
\[ \frac{1}{r_H - j} \sum_{i_2, \ldots, i_{r_H} \in [n + 1]}^{a_{n+1} \ldots i_{r_H}} = |\{e : #_m e = j\}| \]
\[ |\{i_k = n + 1 : 2 \leq k \leq r_H\}| = r_H - j - 1 \]

The number of hb-edges of m-cardinality \(r_H\) can be retrieved by:
\[ |\{e : #_m e = r_H\}| = |E| - \sum_{j \in [r_H - 1]} |\{e : #_m e = j\}|. \]
\[ \square \]
5.2 Initial results on spectral analysis

Let $\mathcal{H} = (V, E)$ be a general hb-graph of $e$-adjacency tensor $A_{\mathcal{H}} = (a_{i_1 \ldots i_{k_{\text{max}}}})$ of order $k_{\text{max}}$ and dimension $n + n_A$.

In the $e$-adjacency tensor $A_{\mathcal{H}}$ built, the diagonal entries are no longer equal to zero. As all elements of $A_{\mathcal{H}}$ are all non-negative real numbers and as we have shown that:

$$\sum_{i_2, \ldots, i_m \in [n+n_A]} a_{i_2 \ldots i_m} = 0$$

It follows:

Claim 5.3. The $e$-adjacency tensor $A_{\mathcal{H}} = (a_{i_1 \ldots i_{k_{\text{max}}}})$ of a general hypergraph $\mathcal{H} = (V, E)$ has its eigenvalues $\lambda$ such that:

$$|\lambda| \leq \max(\Delta, \Delta^*) + r_{\mathcal{H}}$$

where $\Delta = \max_{i \in [n]} (d_i)$ and $\Delta^* = \max_{i \in [n_A]} (d_{n+i})$

Proof. From

$$\forall i \in [1, n], (Ax^{m-1})_i = \lambda x_i^{m-1}$$

we can write as $a_{i_2 \ldots i_m}$ are non-negative real numbers, that for all $\lambda$ it holds:

$$|\lambda - a_{i_2 \ldots i_m}| \leq \sum_{i_2, \ldots, i_m \in [n+n_A]} a_{i_2 \ldots i_m}$$

Considering the triangular inequality:

$$|\lambda| \leq |\lambda - a_{i_2 \ldots i_m}| + |a_{i_2 \ldots i_m}|$$

Combining 3 and 4 yield:

$$|\lambda| \leq \sum_{i_2, \ldots, i_m \in [n+n_A]} a_{i_2 \ldots i_m} + |a_{i_1 \ldots i_m}|$$

But, whatever the approach taken, if $\{i^{r_n}\}$ is an hb-edge of the hb-graph

$$|a_{i_1 \ldots i_m}| = r_{\mathcal{H}}$$

otherwise:

$$|a_{i_1 \ldots i_m}| = 0$$

and thus writing $\Delta = \max_{i \in [n]} (\deg_m(v_i))$ and $\Delta^* = \max_{i \in [n_A]} (\deg_m(N_i))$ and using 5 yield:

$$|\lambda| \leq \max(\Delta, \Delta^*) + r_{\mathcal{H}}.$$

□
Remark 5.1. In the straightforward approach:
\[ \Delta^* = \deg_{m}(N_1) \]
\[ = \sum_{j \in [r_H-1]}(r_H - j) |\{e : \#_m e = j\}| \]

In the silo approach:
\[ \Delta^* = \max_{j \in [r_H-1]}(\deg_m(N_j)) \]
\[ = \max_{j \in [r_H-1]}(|\{e : \#_m e = j\}|) \]

In the layered approach:
\[ \Delta^* = \max_{j \in [r_H-1]}(\deg_m(N_j)) \]
\[ = \max_{j \in [r_H-1]}(|\{e : \#_m e \leq j\}|) \]
\[ = |\{e : \#_m e \leq r_H - 1\}| \]

The values of \( \Delta \) don’t change whatever the approach taken is.

6 Evaluation and final choice

6.1 Evaluation

We have put together some key features of the e-adjacency tensors proposed in this article: the one of the straightforward approach \( A_{\text{str}}(H) \), the one of the silo approach \( A_{\text{sil}}(H) \) and the one of the layered approach \( A_{\text{lay}}(H) \).

The constructed tensors have all same order \( r_H \). \( A_{\text{sil}}(H) \) and \( A_{\text{lay}}(H) \) dimensions are \( r_H - 2 \) bigger than \( A_{\text{str}}(H) \) (\( n - 2 \) in the worst case). \( A_{\text{str}}(H) \) has a total number of elements \( \frac{(n + 1)^n}{(n + r_H - 1)^n} \) times smaller than the two other tensors.

Elements of \( A_{\text{str}}(H) \) - respectively \( A_{\text{sil}}(H) \) - are repeated \( \frac{1}{n_j!} \) - respectively \( \frac{1}{n_j k!} \) - times less than elements of \( A_{\text{lay}}(H) \). The total number of non null elements filled for a given hb-graph in \( A_{\text{str}}(H) \) and \( A_{\text{sil}}(H) \) are the same and is smaller than the total number of non null elements in \( A_{\text{lay}}(H) \).

The number of elements to be filled before permutation to have full description of a hb-edge is constant and equals to 1 whatever the approach taken and the value depends only on the hb-edge composition.

All tensors are symmetric and allow reconstructivity of the hb-graph from the elements.

Nodes degree can be retrieved as it has been shown previously, but it is easier with the silo and layered approach.
6.2 Final choice

The approach by silo seems to be a good compromise between the easiness of calculating the m-degree of vertices and the shape of the hb-graph by the Null vertices added and the number of elements to be filled in the tensor: in other words \( A_{\text{sil}}(H) \) is a good compromise between \( A_{\text{str}}(H) \) and \( A_{\text{lay}}(H) \). We take \( A_{\text{sil}}(H) \) as definition of the e-adjacency tensor of the hb-graph. The preservation of the information on the shape of the hb-edges through the null vertices added allow to keep the diversity of the m-cardinality of the hb-edges.

6.3 Hypergraphs and hb-graphs

Hypergraphs are particular case of hb-graphs and hence the e-adjacency tensor defined for e-adjacency tensor can be used for hypergraphs. As the multiplicity function for vertices of a hyperedge seen as hb-edge has its values in \{0, 1\}, the elements of the e-adjacency tensor that differs only by a factorial due to the cardinality of the hyperedge.

The definition that is retained for the hypergraph e-adjacency tensor is:

**Definition 6.1.** The e-adjacency tensor of a hypergraph \( H = (V, E) \) having maximal cardinality of its hyperedges \( k_{\text{max}} \) is the tensor \( A(H) = (a_{i_1...i_r})_{1 \leq i_1, ..., i_r \leq n} \) defined by:

\[
A(H) = \sum_{i \in [p]} c_{e_i} R_{e_i}
\]

and where for \( e_i = \{v_{j_1}, \ldots, v_{j_k}\} \in E \) the associated tensor is: \( R_{e_i} = (r_{i_1...i_r}) \), which only non-zero elements are:

\[
r_{j_1...j_{k_i}(n+k_i)^{k_{\text{max}}-k_i}} = \frac{(k_{\text{max}}-k_i)!}{k_{\text{max}}!}
\]

and all elements of \( R_{e_i} \) obtained by permuting

\[
(j_1 \ldots j_{k_i}(n + k_i)^{k_{\text{max}}-k_i}),
\]

and where:

\[
c_{e_i} = \frac{k_{\text{max}}}{k_i}.
\]

As in Ouvrard et al. [2017] we compare the e-adjacency tensor obtained by Banerjee et al. [2017] and the chosen e-adjacency tensor. The results are presented in Figure
Table 1: Evaluation of the \( e \)-adjacency tensor depending on construction

\( A_{str} (\mathcal{H}) \) refers to the \( e \)-adjacency tensor built by the straightforward approach;

\( A_{sil} (\mathcal{H}) \) refers to the \( e \)-adjacency tensor built by the silo approach;

\( A_{lay} (\mathcal{H}) \) refers to the \( e \)-adjacency tensor built by the layered approach.
| Order $k_{\text{max}}$ | $A(H)$ $k_{\text{max}}$ |
|---|---|
| Dimension | $n$ | $n + k_{\text{max}} - 1$ |
| Total number of elements | $n^{k_{\text{max}}}$ | $(n + k_{\text{max}} - 1)^{k_{\text{max}}}$ |
| Total number of elements potentially used by the way the tensor is build | $n^{k_{\text{max}}}$ | $(n + k_{\text{max}} - 1)^{k_{\text{max}}}$ |
| Number of non-nul elements for a given hypergraph | $\sum_{s=1}^{k_{\text{max}}} \alpha_s |E_s|$ with $\alpha_s = p_s(k_{\text{max}}) \frac{k_{\text{max}}!}{k_1!...k_s!}$ | $\sum_{s=1}^{k_{\text{max}}} \alpha_s |E_s|$ with $\alpha_s = \frac{k_{\text{max}}!}{k_1!...k_{\text{s}}!}$ with $n_s = k_{\text{max}} - s$ |
| Number of repeated elements per hyperedge of size $s$ | $\frac{k_{\text{max}}!}{k_1!...k_s!}$ | $\frac{k_{\text{max}}!}{k_1!...k_{s}!}$ |
| Number of elements to be filled per hyperedge of size $s$ before permutation | Varying $p_s(k_{\text{max}})$ | Constant 1 |
| Number of elements to be described to derived the tensor by permutation of indices | $\sum_{s=1}^{k_{\text{max}}} p_s(k_{\text{max}})|E_s|$ | $|E|$ |
| Value of elements of a hyperedge of hyperedge composition $s$ | Dependent of $\frac{s}{\alpha_s}$ | Dependent of hyperedge size $\frac{(k_{\text{max}} - s)!}{s(k_{\text{max}} - 1)!}$ |
| Symmetric | Yes | Yes |
| Reconstructivity | Need computation of duplicated vertices | Straightforward: delete special vertices |
| Nodes degree | Yes | Yes |
| Spectral analysis | Yes | Special vertices increase the amplitude of the bounds |
| Interpretability of the tensor in term of hypergraph / hb-graph | No / No | No / Yes |

Table 2: Evaluation of the hypergraph $e$-adjacency tensor

$B_H$ designates the adjacency tensor defined in Banerjee et al. [2017]

$A(H)$ designates the $e$-adjacency tensor as defined in this article.
7 Conclusion

In this article, extending the concept of hypergraphs to support multisets to hb-graphs has allowed us to define a systematic approach to build the $e$-adjacency tensor of a hb-graph. This systematic approach has allowed us to apply it to hypergraphs.

The tensor constructed in Banerjee et al. [2017] appears as a transformation of the hypergraph $H = (V, E)$ into a weighted hb-graph $H_B = (V, E', w_e)$: the hb-graph has the same vertex set but the hb-edges are obtained from the hyperedges of the original hypergraph by transforming them in a way that for a given hyperedge all the hb-edges having this hyperedge as support are considered with multiplicities of vertices such that it reaches $k_{\text{max}}$.

We intend to use our new tensor in building a spectral analysis of hypergraphs.

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