Sets of iterated Partitions and the Bell iterated Exponential Integers

Ivar Henning Skau  
University of South-Eastern Norway  
3800 Bø, Telemark  
ivar.skau@usn.no

Kai Forsberg Kristensen  
University of South-Eastern Norway  
3918 Porsgrunn, Telemark  
kai.f.kristensen@usn.no

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Abstract

It is well known that the Bell numbers represent the total number of partitions of an n-set. Similarly, the Stirling numbers of the second kind, represent the number of k-partitions of an n-set. In this paper we introduce a certain partitioning process that gives rise to a sequence of sets of “nested” partitions. We prove that at stage m, the cardinality of the resulting set will equal the m-th order Bell number. This set-theoretic interpretation enables us to make a natural definition of higher order Stirling numbers and to study the combinatorics of these entities. The cardinality of the elements of the constructed ”hyper partition” sets are explored.

1 A partitioning process.

Consider the 3-set $S = \{a, b, c\}$. The partition set $\wp_3^{(1)}$ of $S$, where the elements of $S$ are put into boxes, contains the five partitions shown in the second column of Figure

Now we proceed, putting boxes into boxes. This means that we create a second order partition set to each first order partition in $\wp_3^{(1)}$. The union of all the second order partition sets is denoted by $\wp_3^{(2)}$, and appears in the third column of Figure
\[ S = \wp_3^{(0)} \]

| \{a, b, c\} | \{abc, abc, abc\} | \{abc, abc, abc\} |
|-------------|-----------------|-----------------|
| \{abc, abc\} | \{abc, abc, abc\} | \{abc, abc, abc\} |
| \{abc, abc\} | \{abc, abc, abc\} | \{abc, abc, abc\} |
| \{abc, abc\} | \{abc, abc, abc\} | \{abc, abc, abc\} |
| \{abc, abc\} | \{abc, abc, abc\} | \{abc, abc, abc\} |

Figure 1: The basic set \( S \) together with the partition sets \( \wp_3^{(1)} \) and \( \wp_3^{(2)} \)

**Definition.** \( \wp_n^{(1)} \) is the set of all partitions of a given \( n \)-set. For \( m > 1 \), \( \wp_n^{(m)} \), called the \( m \)-th order partition set, is the union of the complete collection of sets, each being the partition set of an element in \( \wp_n^{(m-1)} \).

We observe that the number of partitions in \( \wp_3^{(2)} \) is \( |\wp_3^{(2)}| = 12 \), which also, and not by coincidence, turns out to be the second order Bell number \( B_3^{(2)} \).

### 2 Connecting higher order Bell numbers and hyper partitions.

The \( m \)-th order Bell numbers \( B_n^{(m)} \) \((n = 0, 1, \ldots)\), studied by E. T. Bell in [1], are given by the exponential generating functions

\[
E_m(x) = \sum_{n=0}^{\infty} B_n^{(m)} \frac{x^n}{n!} \quad (m \geq 1),
\]

where \( E_1(x) = \exp(\exp(x) - 1) \) and \( E_{m+1}(x) = \exp(E_m(x) - 1) \).

In Table 1, \( B_n^{(m)} \) is computed for a few values of \( m \) and \( n \).

**Theorem 1.** The number of \( m \)-th order partitions of an \( n \)-set is \( B_n^{(m)} \) \((m, n \geq 1)\), i.e.

\[
|\wp_n^{(m)}| = B_n^{(m)}. \quad (1)
\]
Table 1: Higher order Bell numbers $B^{(m)}_n$ when $1 \leq n \leq 8$ and $1 \leq m \leq 5$

| $m \setminus n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1              | 1   | 2   | 5   | 15  | 52  | 203 | 877 | 4140|
| 2              | 1   | 3   | 12  | 60  | 358 | 2471| 19302|167894|
| 3              | 1   | 4   | 22  | 154 | 1304| 12915|146115|185570|
| 4              | 1   | 5   | 35  | 315 | 3455| 44590|660665|11035095|
| 5              | 1   | 6   | 51  | 561 | 7556| 120196|2201856|45592666|

Proof. The proof makes use of generating functions. Recall that there are altogether $S(n, k)$ (Stirling number of the second kind) distinct $k$-partitions of the given $n$-set, i.e. there are $S(n, k)$ elements in $\wp^{(1)}_n$ which are $k$-sets. Each of these $k$-sets gives rise to $|\wp^{(m)}_k|$ distinct partitions of order $m+1$ of the $n$-set we started with, i.e. elements in $\wp^{(m+1)}_n$. Furthermore, different elements in $\wp^{(1)}_n$ of course give different elements in $\wp^{(m+1)}_n$, since they are already different at the "ground level". This means that we have the recurrence formula

$$|\wp^{(m+1)}_n| = \sum_{k=1}^{n} |\wp^{(m)}_k| S(n, k), \quad |\wp^{(0)}_k| = 1.$$  \hspace{1cm} (2)

Now, let $P^{(m)}(x) = \sum_{n=1}^{\infty} |\wp^{(m)}_n| \cdot x^n/n!$ denote the exponential generating function of $\{|\wp^{(m)}_n|\}_{n=1}^{\infty}$. Multiplication with $x^n/n!$, summation over $n$ and changing the order of summation in (2), leads to

$$P^{(m+1)}(x) = \sum_{k=1}^{\infty} |\wp^{(m)}_k| \sum_{n=1}^{\infty} S(n, k) \cdot \frac{x^n}{n!}.$$  \hspace{1cm} (3)

It is well known that $\sum_{n=1}^{\infty} S(n, k) \cdot x^n/n! = (e^x - 1)^k/k!$ is the exponential generating function of the Stirling numbers of the second kind. (A proof of this fact is included in Example 4). From (3) we therefore get the recurrence formula

$$P^{(m+1)}(x) = \sum_{k=1}^{\infty} |\wp^{(m)}_k| \cdot \frac{(e^x - 1)^k}{k!} = P^{(m)}(e^x - 1).$$  \hspace{1cm} (4)

Now, since $P^{(0)}(x) = e^x - 1$, a straightforward induction argument yields $P^{(m)}(x) = E_m(x) - 1$, which proves (1) as well as the relation

$$B^{(m+1)}_n = \sum_{k=1}^{n} B^{(m)}_k S(n, k), \quad B^{(0)}_k = 1,$$  \hspace{1cm} (5)

which now follows from (2).
Example 1. To demonstrate the power of hyper partition thinking, we give a combinatorial proof of the relation

\[ B_n^{(m)} = \sum_{s=0}^{n-1} \binom{n-1}{s} B_s^{(m)} B_{n-s}^{(m-1)}, \quad B_0^{(m)} = 1, \quad (6) \]

that appeared in [1, p. 545, (2.11)].

Each element (i.e. a hyper partition) in \( \wp_n^{(m)} \) consists of nested sets where the sets at the ground level are subsets of the basic n-set \( S \). For a partition \( p \in \wp_n^{(m)} \) we call elements of \( S \) related if they "reside" in the same outer set (box) in \( p \).

Now, fix an arbitrary \( a \in S \). We count the number of partitions in \( \wp_n^{(m)} \) according to which elements \( a \) is related: Let \( A \) be a basic \((n-s)\)-set, including \( a \), of related elements. Now for each of these \( \binom{n-1}{n-s-1} = \binom{n-1}{s} \) sets there are \( B_{n-s}^{(m-1)} \) inner structures. For the complementary \( s \)-set \( C = S \setminus A \), the partitioning process will generate \( B_s^{(m)} \) partitions of order \( m \). By combining the possibilities, (6) follows.

Observe that if we in the same manner as above fix two (or more) elements in \( S \), new formulas emerge.

By putting \( B_n^{(0)} = 1 \), we note that (6) is a generalized version of the well known formula (see [3, p. 210])

\[ B_n = \sum_{s=0}^{n-1} \binom{n-1}{s} B_s. \]

3 Higher order Stirling numbers.

Having established the relationship between partitions of order \( m \) and higher order Bell numbers, defining higher order Stirling numbers seems like a natural thing to do.

Definition. The \( m \)-th order Stirling number \( S^{(m)}(n, k) \) (of the second kind) is the number of \( k \)-sets in \( \wp_n^{(m)} \).

In [2] E. T. Bell gave an analytical definition of what he called generalized Stirling numbers \( \zeta_n^{(k,m)} \) by means of generating functions. In Theorem [2] we prove that \( S^{(m)}(n, k) = \zeta_n^{(k,m)} \).

We note that the higher order Stirling numbers \( S^{(m)}(n, k) \) are the entries of the matrix \( S^m \), where \( S = (S(n, k)) \). This can be seen by induction from the relation (7) in the proof of Theorem[2] Table[2] is computed with the aid of such matrices.
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
m & S^{(m)}(5,1) & S^{(m)}(5,2) & S^{(m)}(5,3) & S^{(m)}(5,4) & S^{(m)}(5,5) & B_5^{(m)} \\
\hline
5 & 3455 & 3325 & 725 & 50 & 1 & 7556 \\
20 & 1115320 & 233050 & 11900 & 200 & 1 & 1360471 \\
50 & 45533300 & 3706375 & 74750 & 500 & 1 & 49314926 \\
\hline
\end{array}
\]

Table 2: Some examples of higher order Stirling and Bell numbers

**Theorem 2.** Let \( S^{(m)}(n, k) \) be the \( m \)-th order Stirling numbers of the second kind. Then we have
\[
S^{(m)}(n, k) = \zeta^{(k,m)}_n,
\]
where \( \zeta^{(k,m)}_n \) are the generalized Stirling numbers of the second kind, defined by E.T. Bell in [2, p. 91] by the generating functions
\[
\frac{(E_{m-1}(t) - 1)^k}{k!} = \sum_n \zeta^{(k,m)}_n \cdot t^n \cdot \frac{n!}{n!}.
\]

**Proof.** Let \( F_k^{(m)} \) denote the generating function of \( \{S^{(m)}(n, k)\}_{n=k}^\infty \). Then we have
\[
F_k^{(m)}(x) = \sum_{n=k}^\infty S^{(m)}(n, k) \frac{x^n}{n!}.
\]
The idea is to come up with an analogous formula to (4), in order to obtain an analogous formula to (4). We claim that
\[
S^{(m+1)}(n, k) = \sum_{i=k}^n S^{(m)}(i, k) S(n, i).
\]
This is true because we know that each of the \( S(n, i) \) first order partitions will generate \( S^{(m)}(i, k) \) \( k \)-partitions of order \( m+1 \). Multiplication by \( x^n/n! \) followed by summation over \( n \) in (7) gives
\[
F_k^{(m+1)}(x) = \sum_{n,i} S^{(m)}(i, k) S(n, i) \frac{x^n}{n!} = \sum_i S^{(m)}(i, k) \left( \frac{e^x - 1}{i!} \right) = F_k^{(m)}(e^x - 1),
\]
because \( \sum_n S(n, i) x^n/n! = (e^x - 1)^i/i! \). We have thus deduced the recurrence formula
\[
F_k^{(m+1)}(x) = F_k^{(m)}(e^x - 1).
\]
Since \( F_k^{(1)}(x) = (e^x - 1)^k/k! \), induction yields
\[
F_k^{(m)}(x) = \frac{(E_{m-1}(x) - 1)^k}{k!}, \quad (E_0(x) = e^x),
\]
which completes the proof. \qed
We notice that Theorem 1 is proved once more since we have
\[ |\mathcal{P}(m)_n| = \sum_{k=1}^{n} S^{(m)}(n, k) \]
and
\[ P^{(m)}(x) = \sum_{n=1}^{\infty} |\mathcal{P}(m)_n| x^n/n! = \sum_{k=1}^{\infty} P_k^{(m)}(x) = E_m(x) - 1. \]

How the introduction of the higher order Stirling numbers opens up the scope for hyper partition thinking, is illustrated in the next examples.

**Example 2.** If we, in our construction process of \( \mathcal{P}(m)_n \), stop at the \( r \)-th intermediate stage, i.e. in \( \mathcal{P}(r)_n \), making up status so far by grouping the elements according to their cardinality before advancing further on, we get the following generalized version of (7):
\[ S^{(m)}(n, k) = \sum_{i=k}^{n} S^{(m-r)}(i, k) S^{(r)}(n, i), \]
since there are \( S^{(r)}(n, i) \), \( i \)-sets in \( \mathcal{P}(r)_n \). This also follows from the matrix representation \( S^m = S^{m-r} S^r \), as well as from (7) by induction.

Summing from \( k = 1 \) to \( n \) yields
\[ B_n^{(m)} = \sum_{i=1}^{n} B_i^{(m-r)} S^{(r)}(n, i), \]
which generalizes (5).

**Example 3.** The formula
\[ S^{(m)}(n, k) = \sum_{s=k-1}^{n-1} \binom{n-1}{s} B_{n-s}^{(m-1)} S^{(m)}(s, k-1) \]
may be proved combinatorially in just the same manner as (6) in Example 1. Note that (8) yields (7) by summation over \( k \).

**Example 4.** Counting the \( k \)-sets in \( \mathcal{P}(m)_n \) by first forming the \( k \) "families" (outer sets) of related elements, we get
\[ S^{(m)}(n, k) = \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} B_{i_1}^{(m-1)} \cdots B_{i_k}^{(m-1)}, \]
where in this case \( B_{i_k}^{(0)} = 1 \), \( k \geq 1 \) and \( B_{0}^{(m)} = 0 \), because a family with \( i \) relatives yields \( B_{i}^{(m-1)} \) elements in \( \mathcal{P}(m-1)_i \), i.e. there are exactly \( B_{i}^{(m-1)} \) possible "inner" structures for an "i-family".

Summing over \( k \) yields
\[ B_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} B_{i_1}^{(m-1)} \cdots B_{i_k}^{(m-1)}. \]
Note that we have used nothing but the set-theoretic hyper partition interpretation/definition of $B_n^{(m)}$ and $S^{(m)}(n,k)$ in establishing (11) and (10). Now, therefore, let $f_{m-1}(t) = f_{m-1}$ be the exponential generating function (e.g.f.) to \( \{B_n^{(m-1)}\}_{n=0}^{\infty} \). And observe then that $f_{m-1}^k$ is the e.g.f. of the sequence \( \{ \sum_{i_1 + \cdots + i_k = n} B_{i_1}^{(m-1)} \cdots B_{i_k}^{(m-1)} \}_{n=0}^{\infty} \).

Now we multiply (9) and (10) with $x^n/n!$, sum over $n$ and change the order of summation, to obtain

$$F_k^{(m)}(x) = \frac{1}{k!} f_{m-1}^k(x) \quad \text{and} \quad f_{m}(x) = \sum_{k=1}^{\infty} \frac{f_{m-1}^k(x)}{k!} = \exp(f_{m-1}(x)) - 1.$$ 

And since $f_0(x) = e^x - 1$, we have $f_{m}(x) = E_m(x) - 1$. So by this reasoning "from outside in", we have an alternative proof of Theorem 1 as well as of Theorem 2.

### 4 An asymptotic consideration

In [1, p. 545] E.T. Bell proved the interesting formula

$$B_n^{(m)} = c_{n-1} m^{n-1} + c_{n-2} m^{n-2} + \cdots + c_0,$$

where $c_{n-1}, \ldots, c_0$ are rational numbers, independent of $m$. When $n$ is fixed, this implies that

$$\lim_{m \to \infty} \frac{B_n^{(m)}}{B_n^{(m-1)}} = 1,$$

enabling us to say something more about the cardinality of the members of $\wp_n^{(m)}$.

**Remark.** Via a slightly different proof of (11) than in [1] one might show that $c_{n-1} = n!/2^{n-1}$, see [4].

When looking at the "children" of a $p \in \wp_n^{(m)}$, i.e. the elements in $\wp_n^{(m+1)}$ that $p$ gives rise to, we see that all but one of them have lower cardinality than their "father" $p$. Following the next generations in the partitioning process, it appears that the great majority of the descendants are 1-element sets (see Table 2). It is therefore easy to conjecture that the average value $A_n^{(m)}$ of the cardinality of the sets in $\wp_n^{(m)}$ approaches 1 as $m \to \infty$, i.e.

$$A_n^{(m)} = \frac{1}{B_n^{(m)}} \sum_{k=1}^{n} k S^{(m)}(n,k) \rightarrow 1.$$
Let us see why (13) is true. (12) yields, in conjunction with the observation $S^{(m)}(n, 1) = B^{(m-1)}_n$, that $S^{(m)}(n, 1) \sim B^{(m)}_n \quad (m \to \infty)$. Since $B^{(m)}_n = \sum_{k=1}^{n} S^{(m)}(n, k)$, we consequently have

$$S^{(m)}(n, k) = o(B^{(m)}_n), \quad k \geq 2 \quad (m \to \infty),$$

and (13) follows immediately.

5 A summary comment

The set-theoretic interpretation of the higher order Bell numbers places them in a natural and fundamental context. Together with the higher order Stirling numbers, they take the same central position in the combinatorics of the higher order partition sets as their predecessors $B_n$ and $S(n, k)$ have at the ground level.

References

[1] Bell, E.T., The Iterated Exponential Integers, *Annals of mathematics*, Vol. 39, July 1938, 539 - 557.

[2] Bell, E.T., Generalized Stirling transforms of sequences, *Amer. Journal of Math*, Vol. 61, 1939, 89 - 101.

[3] Comtet, L., *Advanced Combinatorics*, Reidel, 1974.

[4] Skau, I.H., Kristensen, K.F., An asymptotic Formula for the iterated exponential Bell Numbers, [http://arxiv.org/abs/1903.07979](http://arxiv.org/abs/1903.07979), 2019.