Abstract

The aim of this paper is to show that there exists a deterministic algorithm that can be applied to compute the factors of a polynomial of degree 2, defined over a finite field, given certain conditions.
## CONTENTS

6 The Theorem
  6.1 Statement of the Theorem
    6.1.1 The problem
    6.1.2 The hypothesis
  6.2 The Proof

7 The original statement of the Theorem
  7.1 Polynomials of higher degree
  7.2 Advanced topics
    7.2.1 Curves of Genus $g$
    7.2.2 The Zeta function

8 Running Time
  8.1 Introduction to running times
    8.1.1 Definition
  8.2 Running time of our algorithm

9 Acknowledgements
0.1 Preface

The study of prime numbers has been puzzling Number Theorists for several centuries. Certainly since the remarkable results found by Pierre de Fermat in the 17th century, a new wave of motivation has triggered some of the most talented mathematicians to research this field in more depth.

The field is vast, and possibly one of the most challenging ones: whilst the statement of a problem in this field may at first sound like a lunchtime brainteaser for a hobby-number-cruncher, its solution will in general be extremely complex and in many cases has taken centuries to find, if this has been achieved at all yet!

But this field is not only of high importance to theoretical research. The most recent developments in this field have been concentrated on computational number theory. The fact that still so little is known about prime numbers, and that it is such a difficult field to make much progress in, has been exploited by the computer industry during the last century.

Secure transmission of data is made possible by prime numbers, and hence research in Number Theory is nowadays mainly revolved around finding ways to ensure that this level of security is maintained.

At the heart of this lies the problem of finding roots of polynomials modulo prime numbers.

Whilst it is in theory possible to do this, the procedures that we know about so far are not very efficient and would in general take far too long to be of any practical use.

This dissertation (unfortunately) does not provide us with a magic key to cracking such codes. I will show that there exists a deterministic algorithm that can, under certain circumstances, find the factors of polynomials modulo a prime number. However the running time of this algorithm is still much higher than some probabilistic (and fairly reliable!) algorithms that are in use already.

The result that we will obtain here is hence rather of interest to mathematicians working in Algorithmic Number Theory than of practical use. Perhaps, however, similar techniques will eventually be developed that might be put to more use in practice. Perhaps the purely theoretical side of mathematics will find its applications in practice, and Albert Einstein will be proved wrong for his remark “As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.”
Chapter 1

Introduction

The idea for this project originates from a claim made by Dr Neeraj Kayal in 2005, together with some further refinements added by Prof Bjorn Poonen (University of California, Berkeley).

A well-known open problem in Algorithmic Number Theory is the efficient calculation of roots of polynomials modulo a prime number \( q \) in deterministic polynomial-time.

A very basic example of this is the following: let \( q \) be prime and \( a \) and number between 0 and \( q - 1 \). The study of Elementary Number Theory provides us with easy tools to check whether there exists a number \( b \) between 0 and \( q - 1 \) such that \( b^2 = a \pmod{q} \) - in that case, \( a \) is called a \textit{quadratic residue}.

For example, we could apply what is called “Euler’s Criterion”. It says that if \( q \) is an odd prime, then for all \( a \in \mathbb{N} \), we have

\[
\left( \frac{a}{q} \right) \equiv a^{(q-1)/2} \pmod{q},
\]

where the fraction on the left hand side denotes the Legendre Symbol, defined by

\[
\left( \frac{a}{q} \right) := \begin{cases} 
1 & \text{if such a number } b \text{ exists} \\
-1 & \text{if no such } b \text{ exists} \\
0 & \text{if } q \text{ divides } a
\end{cases}
\]

But how can we calculate this number \( b \), if it exists? We would need to solve the equation \( h(z) := z^2 - a \equiv 0 \pmod{q} \). This is a much harder problem, if it is to be solved efficiently. Of course we could try substituting
every value in \{0, 1, \ldots, q - 1\} for \(z\) to check whether the equation is satisfied; however as we are in practical applications more concerned with large primes, this could take quite a while.

1.1 The Claim

Kayal claimed that we can factorise such a polynomial \(h(z)\) defined over a finite field \(\mathbb{F}_q\) using a deterministic algorithm with running time bounded by a universal polynomial in \((\log q)\), given certain circumstances (Poonen’s input to this claim will be discussed later). Roughly speaking, the underlying condition is that we can construct an algebraic family of bivariate polynomials \(C_z(X, Y)\), each member of which has a different number of solutions modulo \(q\).

I will restrict the detailed proof to the case \(\deg(h) = 2\). The case for higher degree polynomials will be discussed briefly afterwards. I will then also give a brief discussion about the running time of the algorithm.

1.2 The idea of the proof

The idea of the proof can be outlined as follows:

We have a deterministic algorithm, known as “Schoof’s Algorithm”, which is used to compute the number of rational points on an elliptic curve given in Weierstrass Form and defined over a finite field.

Let \(h(z)\) denote the polynomial that we wish to factorise, and \(\mathbb{F}_q\) the finite field over which \(h\) is defined. We consider the ring \(R := \mathbb{F}_q[z]/(h(z))\).

If we are given an elliptic curve \(C\) over \(R\) that satisfies the underlying condition, and we attempt to apply Schoof’s Algorithm to count the number of rational points on \(C\), the algorithm will at some point break down, and thereby reveal the factors of \(h(z)\).

Now why does this happen?

Consider the difference between a ring and a field: a field contains all its inverses, which is not necessarily true for a ring. This is precisely why
the algorithm will not work when it is working with a ring: whilst trying to compute the inverse of an element (which the algorithm can easily do when in a field), it will at some point not be able to find that inverse and will therefore stop running.

At this point we know that it must have found an element of the ring that has no inverse. But by inspecting this ring \( R \) more closely, we can see which elements in the ring do not have an inverse: it is precisely the set of elements in \( F_q[z] \) spanned by the factors of \( h(z) \).

So all we need to do is compute the greatest common divisor of this element that made the algorithm stop, and \( h(z) \) (since this element may be a multiple of a factor of a factor), to obtain a non-trivial factor of \( h(z)! \)

To present a detailed proof however requires a lot more careful explanation; this is what will follow now.

1.3  Structure of this Paper

In Chapter (2) of this dissertation, I will define elliptic curves and explain some of their elementary properties that we will need to be aware of in order to understand Schoof’s Algorithm.

Chapter (3) contains a brief discussion about counting rational points on elliptic curves, which is followed by a rather technical section explaining the essential tools that underlie Schoof’s Algorithm.

I will give a full description of Schoof’s Algorithm for elliptic curves over a finite field in Chapter (4).

For the purpose of a clear and thorough understanding of the theorem and its proof, I will then include a short chapter on elementary Ring Theory; it will be a collection of standard results that should only serve as a reference to the following chapter.

A slightly simplified version of the actual assertion will finally be explained in Chapter (5), together with a detailed proof. The next chapter will then explain how this simplified version differs from the original claim made by Kayal & Poonen, and what changes might be made to the proof in the previous chapter in order to adapt it to the “full version”.
Finally I will, in chapter 8, provide the reader with some background about algorithms and computations, and give a brief discussion about the running time of the algorithm.
Chapter 2

Elliptic Curves

I will first of all state a few definitions and standard results from the study of elliptic curves. As some of the proofs require a few technical lemmas that are not directly relevant to this dissertation I will omit most of them; they are standard bookwork and can be found e.g. in [20] and [5] (N.B. those sources also provide the interested reader with a thorough insight into Elliptic Curves).

2.1 Preliminary Definitions

Throughout these definitions, we shall denote by $K$ some field.

**Definition 2.1.1** $A_n(K) = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in K\}$, is called affine $n$-space.

**Definition 2.1.2** When $P \in A_n(K)$, we say that $P$ is $K$-rational or defined over $K$.

**Definition 2.1.3** Let $P_n(K) := \{(x_0, \ldots, x_n) : x_0, \ldots, x_n \in K, not all 0\}$, subject to the relation that $(x_0, \ldots, x_n) = (y_0, \ldots, y_n) \in P_n(K)$ if there exists $r \in K, r \neq 0$, such that $(y_0, \ldots, y_n) = (rx_0, \ldots, rx_n)$. $P_n(K)$ is called projective $n$-space over $K$.

**Definition 2.1.4** A polynomial in $n$ projective variables is an $(n + 1)$-variable homogeneous polynomial.

**Definition 2.1.5** A projective curve in $P_2$ is defined by a homogeneous polynomial in 3 variables $F(X, Y, Z) = 0$. 
Definition 2.1.6 Let \( C : f(x, y) = 0 \) be an affine curve and let \( P = (x_0, y_0) \) be a point on \( C \). We say that \( P \) is a singular point on \( C \) if
\[
\frac{d}{dx}(f)|_P = \frac{d}{dy}(f)|_P = 0.
\]
A curve is called non-singular if it does not contain any singular points.

Finally, we are in a position to unambiguously define elliptic curves:

Definition 2.1.7 (Elliptic Curves) An elliptic curve over a field \( K \) is a non-singular, projective cubic curve, defined over \( K \), with a \( K \)-rational point.

Definition 2.1.8 Let \( C : f(x, y) = 0 \) and \( C' : f(x, y) = 0 \) be curves over \( K \). A rational map \( \phi \) over \( K \) from \( C \) to \( C' \) is a map given by a pair \( \phi_1, \phi_2 \) of rational functions in \( (x, y) \), defined over \( K \), with the property that given any point \( P = (x_0, y_0) \) on \( C \), then \( (\phi_1(x_0, y_0), \phi_2(x_0, y_0)) \) lies on \( C' \).

If there also exists a rational map \( \psi \) from \( C' \) to \( C \) such that \( \psi \cdot \phi \) is the identity on \( C \) and \( \phi \cdot \psi \) is the identity on \( C' \) then we say that \( \phi \) is a birational transformation over \( K \) from \( C \) to \( C' \), and that \( C \) and \( C' \) are birationally equivalent over \( K \).

Remark about the terminology An elliptic curve is not to be confused with an ellipse, which is a plane algebraic curve usually given in the form
\[
\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1
\]
for some non-zero constants \( a, b \) in some field. There is however an explanation for the terminology.

Consider the relationship between the trigonometric functions sine, cosine and tangent, and the arc lengths of a circle. The further study of elliptic curves shows that there is a similar relationship between elliptic curves and arc lengths on ellipses. These give rise to so-called elliptic integrals of the form
\[
\int \frac{dx}{4x^3 + Ax + B} \tag{2.1}
\]
Integrals like (2.1) are multi-valued and only well-defined modulo a period lattice \( L \). The "inverse" function of those integrals is a doubly periodic function called an elliptic function.

In fact every such function \( P \) with periods independent over \( L \) satisfies an equation of the form

\[
P'^2 = 4P^3 + AP + B
\]  

(2.2)

If we consider \((P, P')\) as a point in space then we can define a mapping from the solutions of this equation to the curve

\[
Y^2 = X^3 + AX + B
\]  

(2.3)

This is the standard form for an elliptic curve that we shall be concerned with throughout this dissertation.

2.2 Arithmetic on Elliptic Curves

**Definition 2.2.1 (Addition and Inverses)** Let \( C \) be an elliptic curve over a field \( K \). Let \( o \) be its \( K \)-rational point. For any two points \( a, b \) on \( C \), denote by \( l_{a,b} \) the line through \( a \) and \( b \); if \( a = b \) then \( l_{a,b} \) is defined to be the tangent to \( C \) at \( a = b \). Let \( d \) be the third point of intersection of \( l_{a,b} \) with \( C \). Define \( c \) to be the third point of intersection between \( C \) and \( l_{o,d} \), the line through \( o \) and \( d \). We then define \( a + b := c \).

Let \( k \) be the third point of intersection between \( C \) and \( l_{o,o} \), the tangent to \( C \) at \( o \). Let \( a' \) be the third point of intersection between \( C \) and \( l_{a,k} \). Define \(-a := a' \).

**Comment**

It is often convenient to write elliptic curves in affine form, although it should be understood that we always mean a projective curve. For example, \( C : y^2 = x^3 + 1 \) will be used as the shorthand notation for the projective curve \( C : ZY^2 = X^3 + Z^3 \).

It can be shown that any elliptic curve over \( K \) can be birationally transformed over \( K \) to the Weierstrass form

\[
C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]  

(2.4)

To further simplify the equation we can use the following theorem.
**Theorem 2.2.2** Let $K$ be a field with $\text{char}(K) \neq 2$. Then any elliptic curve over $K$ is birationally equivalent over $K$ to a curve of the form $Y^2 = \text{cubic in } x$. If $\text{char}(K) \neq 2$ or $3$ then we can further reduce (2.4) to the form

$$Y^2 = X^3 + AX + B.$$  

(2.5)

For the purpose of this dissertation, we shall only be concerned with elliptic curves over fields of characteristic $q > 3$, hence (2.5) will be treated as our standard equation for an elliptic curve. We will also adapt the convention to choose $o = (0, 1, 0)$, the point at infinity.

Note that $Z = 0$ meets $C$ at $o$ three times. Given $a = (x, y, z)$, the third point of intersection between the curve and the line through $a$ and $o$ is $(x, -y, z)$, which must then be $-a$. This leads to the following result.

**Lemma 2.2.3** For an elliptic curve $C$ written in our standard form (2.5), we can simplify the formulae for addition and inverses of points on $C$ as follows:

- $-(x, y) := (x, -y)$

- If $d := (x_3, y_3)$, the third point of intersection of $C$ and $l_{a,b}$, then $a + b = (x_3, -y_3)$.

### 2.3 The Group Structure

Now let us have a closer look at the rational points on an elliptic curve $C$. With the above definitions of addition and $o$ we can show the following:

After a few computations it is easy to see that for all $a$ and $b$ on $C$ we have

- $a + b = b + a$,

- $a + o = o + a = a$, and

- $a + (-a) = (-a) + a = o$.

Moreover, further computations that involve a few technical lemmas, will reveal that for $c$ on $C$ we also have
• \((a + b) + c = a + (b + c)\).

From this we can deduce the following theorem:

**Theorem 2.3.1** Let \(C\) be an elliptic curve over \(K\). The points on \(C\), together with the operation \(a + b\) as defined in Lemma 2.2.3, form a group. The point \(o\) acts as the identity in this group, and inverses are given by \(-a\) as in the definition above.

For a natural number \(m \in \mathbb{N}\) we will from now on adapt the notation 
\([m]P := P + P + \cdots + P\) \((m\) times). This map is also known as the “multiplication-by-
\(m\)-map” from the curve to itself. We can extend the definition of this to \(m \in \mathbb{Z}\) by defining \([0]P := 0\) and \([-m]P := -[m]P\).

So for example, if we have \(P = (0, 1)\) on \(C : Y^2 = X^3 + 1\), then 
\(-[2]P = -(P + P)\). Computing the tangent at \(P\), we obtain the line \(L : Y = 1\), and the “third point of intersection” of \(C\) and \(L\) being again \(P\). So \([2]P = -P = (0, -1)\), and hence \(-[2]P = P\).

This map plays a central part in elliptic curve cryptography; its applications will later on be extremely useful in this dissertation.

### 2.4 Elliptic Curves over Finite Fields

Now let us consider an elliptic curve \(C\) over a finite field \(\mathbb{F}_q\). Recall that we are only considering fields of characteristic \(q > 3\) here. The cases for \(q = 2\) or 3 are similar, and some of our computations and notations could be adapted to include those cases, too. However, for the entire purpose of this dissertation, those two cases will be irrelevant and we will therefore exclude them in all our computations.

Consider the group of rational points on \(C\) over \(\mathbb{F}_q\). The following result should be immediately obvious:

**Theorem 2.4.1** Over a finite field \(\mathbb{F}_q\), the number of rational points on an elliptic curve \(C\) is finite.

We shall denote this number by \(\#C(\mathbb{F}_q)\). It may be asked whether we can find out anything about this quantity. The answer to this is that we can indeed, and in fact the computation of this number lies right at the heart of the proof of the theorem.
Before I give an in-depth discussion of how to compute the actual value of $\#C(\mathbb{F}_q)$, I will give an upper and lower bound on it, and define a few tools that we will later on need in our computations.

Discussion

Consider a straight line $L$ over $\mathbb{F}_q$, given by $L : Y = aX + b$. What do we know about the number of points on $L$?

For every possible value $x \in \mathbb{F}_q$, i.e. $x \in 0, 1, \ldots, q - 1$, there exists exactly one solution in $\mathbb{F}_q$ for $y$, so we obtain $q$ rational points. Also, the point at infinity is always a rational point; in total we therefore have exactly $q + 1$ rational points on $L$.

Now we can consider the number of points on a curve $C$ of the form $Y^2 = f(X)$ in a similar way: for each of the $q$ possible values for $X$ we have one of the three cases:

- If $f(x)$ is a quadratic residue modulo $q$, we obtain two solutions for $Y$, namely $y = \pm \sqrt{f(x)}$;
- If $f(x)$ is a quadratic non-residue modulo $q$, we will have no solutions for $Y$;
- If $f(x) = 0$ we have precisely one solution for $Y$, namely $y = 0$.

From elementary Number Theory we know that in $\mathbb{F}_q$, exactly half the values in $\{1, \ldots, q-1\}$ are quadratic residues, so we would expect the number of rational points on $C$ to be roughly $q$ to represent the “fifty-fifty chance of $f(x)$ being a quadratic residue”, hence yielding 2 solutions. We then add the point at infinity, and obtain as a rough estimate $q + 1$ rational points.

**Definition 2.4.2 (Trace of Frobenius)** For a given curve $C$ over $\mathbb{F}_q$, the **trace of Frobenius $t$** is the quantity defined by the relation $\#C(\mathbb{F}_q) = q + 1 - t$.

$t$ can therefore be regarded as the “error term” in our estimate of $\#C(\mathbb{F}_q)$. The following theorem gives a bound on this error term:

**Theorem 2.4.3 (Hasse’s Theorem)**

$$|t| \leq 2\sqrt{q} \quad (2.6)$$
A detailed proof of this can be found in [18]. In Subsection 2.2.2 of Chapter 7 in this paper we will see an alternative argument to deduce this.

A map that should be well-known to anyone who has studied basic algebra and number theory is the Frobenius map. It has a very interesting property that has important applications in the study of elliptic curves, as we shall soon see.

**Definition 2.4.4** The $q^{th}$-power Frobenius map $\phi$ defined on an elliptic curve $C$ over $\mathbb{F}_q$ maps points on $C$ to points on $C$ as follows:

$$\phi = \begin{cases} C(\mathbb{F}_q) & \to & C(\mathbb{F}_q) \\ (x, y) & \mapsto & (x^q, y^q) \\ o & \mapsto & o \end{cases}$$

It is easily verified that $\phi$ is a group endomorphism for the group of rational points on $C$ over $\mathbb{F}_q$ and is therefore most commonly referred to as the Frobenius endomorphism.

As mentioned above, a deeper study of $\phi$ will reveal several interesting results; the following property of $\phi$ is crucial to this dissertation, and deserves particular attention.

**Lemma 2.4.5** The Frobenius endomorphism has characteristic polynomial

$$\chi(X) : X^2 - tX + q.$$  

**Outline Proof**

A full proof of this involves a lot of technical Lemmas; I will therefore only state the main idea of the proof. It relies on the fact that

$$\# \text{Ker}(\phi - 1) = \deg(\phi - 1) = q + 1 - t.$$  

From this we can deduce that if we take an integer $m \geq 1$ with $\gcd(m, q) = 1$, then we have

$$\det(\phi_m) \equiv q \pmod{m}, \quad \text{tr}(\phi_m) \equiv t \pmod{m}.$$  

The details of this proof can be found in [18].
Clearly this is the same as writing

\[ \phi^2 - [t]\phi + [q] = [o]. \] (2.7)

**Corollary 2.4.6** Hence we have that for a point \( P := (x, y) \) on \( C \):

\[ (x^q, y^q) - [t](x^q, y^q) + [q](x, y) = o. \] (2.8)
Chapter 3

Counting Rational Points on Elliptic Curves

As mentioned in Chapter \( \text{(1)} \), the key to the proof of our assertion is part of an algorithm that reveals the factors of \( h(z) \).

Although this dissertation is not about the efficient computation of the number of rational points on elliptic curves, the algorithm that we will later on adapt is in its original form a point-counting algorithm for elliptic curves over finite fields. I will therefore give a brief introduction to such algorithms in general.

As mentioned earlier, over a finite field the number of rational points on an elliptic curve is clearly finite. In Chapter \( \text{(2)} \) we have seen an upper and lower bound for the number of such points.

Now it may be asked if we can actually compute the precise number of rational points on a given curve. The answer is that we can indeed, and there are several methods that can be applied to do this.

An explicit formula for the number of rational points on an elliptic curve \( C : Y^2 = X^3 + AX + B \) over \( \mathbb{F}_q \) is given by the following sum:

\[
\#C(\mathbb{F}_q) = 1 + \sum_{x \mod q} \left( \left( \frac{X^3 + AX + B}{p} \right) + 1 \right)
\]

where \( (\frac{a}{p}) \) denotes the Legendre Symbol.

Computing \( \#C(\mathbb{F}_q) \) this way takes \( O(q^{1+\epsilon}) \) bit operations\(^1\); this is clearly

\(^1\)For a definition of “bit operations”, see Chapter \( \text{(8)} \).
not very practical when \( q \) is a large prime number (which, in practical applications of our theorem, it usually will be!).

Due to Rene Schoof however, we have a more efficient way of computing \( \# C(\mathbb{F}_q) \). In his paper [17], Schoof gives an explicit deterministic algorithm to compute the exact number of points on any given curve over a finite field.

In the next chapter I will go into detail about this particular algorithm, but first we will need yet more technical tools in order to understand the algorithm better.

As Corollary (2.4) suggests, the algorithm involves calculating coordinates of rational points of the form \([m]P\) where \( P \) is a rational point and \( m \) a positive integer. We will therefore need techniques to efficiently compute those coordinates.

It should be clear that the coordinates of \( P_1 + P_2 \) are rational functions of the coordinates \((x_1, y_1)\) of \( P_1 \) and \((x_2, y_2)\) of \( P_2 \). By repetition of this calculation we can see that multiplication by \([m]\) given by

\[
(x, y) \mapsto [m](x, y)
\]

can also be expressed in terms of rational functions in \( x \) and \( y \). Explicitly, we have the formulae given in the following section.

### 3.1 The Division Polynomials

**Lemma 3.1.1** Let \( C \) be an elliptic curve defined over a field \( K \) and let \( m \) be a positive integer. There exist polynomials \( \psi_m, \theta_m, \omega_m \in K[x, y] \) such that for \( P = (x, y) \in C(K) \) with \([m]P \neq o\) we have

\[
[m]P = \left( \frac{\theta_m(x, y)}{\psi_m(x, y)^2}, \frac{\omega_m(x, y)}{\psi_m(x, y)^3} \right)
\]  

(3.1)

The polynomial \( \psi_m \) is generally referred to as the \( m \)th Division Polynomial of \( C \). \( \theta_m \) and \( \omega_m \) can both be expressed in terms of \( \psi_m \) as shown in the explicit recursive expressions for \( \psi_m \) below.

**Remark**

The expressions for \( \psi_m \) that I will give are simplified for the case where we can write our curve in the form \( C : Y^2 = X^3 + AX + B \). They are given in a more general form in [2] for the general curve...
CHAPTER 3. COUNTING RATIONAL POINTS ON ELLIPTIC CURVES

\[ C : Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6, \]

which could also be defined over fields of characteristic 2 or 3.

3.1.1 Explicit Expressions for $\psi$

Let \( C : Y^2 = X^3 + AX + B \) be defined over \( K \).

Then \( \psi_m \) can be computed as follows:

\[
\begin{align*}
\psi_0 &= 0, \psi_1 = 1, \psi_2 = 2y, \\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2, \\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3), \\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3, \text{ } m \geq 2, \\
\psi_{2m} &= \frac{(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1})\psi_m}{2y}, \text{ } m > 2
\end{align*}
\]

We can now in turn define \( \theta_m \) and \( \omega_m \) in terms of the division polynomials:

\[
\begin{align*}
\theta_m &= x\psi_m^2 - \psi_{m-1}\psi_{m+1} \\
\omega_m &= \frac{\psi_{2m}}{2\psi_m}
\end{align*}
\]

Finally, we define

\[
f_m := \begin{cases} 
\psi_m & \text{if } m \text{ odd} \\
\psi_m/(2y) & \text{if } m \text{ even}
\end{cases}
\]

The proof of these formulae involves straightforward but lengthy calculations and will therefore be omitted; some more detail is included in [12]. It is however important to note the following two facts:

**Corollary 3.1.2** Let \( f_m \) be defined as in (3.3). Then

1. \( f_m \) is a polynomial in \( x \) only.

2. The degree of \( f_m \) is at most \((m^2 - 1)/2\) if \( m \) is odd, and at most \((m^2 - 4)/2\) if \( m \) is even.

The latter fact will be relevant in Chapter 8 when calculating the running time of the algorithm.
3.2 The $m$-Torsion Subgroup

Clearly, when $K$ is a finite field, $\mathbb{F}_q$ for some prime $q$, then $C(K)$ is a torsion group, i.e. every point on the curve has finite order (since $C(K)$ itself is finite). For a non-negative integer $m$, the set of $m$-Torsion points on $C(K)$ is defined by

$$C[m] := \{ P \in C(K) | [m]P = O \}.$$

We can now also express Corollary 2.4 in terms of elements of this subgroup:

**Corollary 3.2.1** For points of order $m$ on $C$, i.e. $P = (x, y) \in C[m]$, we have

$$\phi^2_m(P) - [\tau]\phi_m(P) + [k](P) = [o](P) = o,$$

where we define $\tau \equiv t \pmod{m}$ and $k \equiv q \pmod{m}$.

It is easily verified that this is a subgroup of $C(K)$. By definition, $o \in C[m]$. The $m^{th}$ division polynomial $\psi_m$ characterises the $m$-Torsion subgroup as stated in the following theorem.

**Theorem 3.2.2** Let $P \in C(K) \setminus o$, and let $m \geq 1$. Then

$$P \in C[m] \iff \psi_m(P) = 0.$$

Clearly, this condition is equivalent to the following corollary, which is more useful for our computations later:

**Corollary 3.2.3** Let $P = (x, y) \in C(K) \setminus \{o\}$ be such that $[2]P \neq o$ and let $m \geq 2$. Then

$$P \in C[m] \iff f_m(x) = 0.$$  \hspace{1cm} (3.5)

The 2-torsion points are excluded in this, since they satisfy $\psi_2P = 0$, which we need to divide $\psi_m$ by in order to obtain $f_m$ if $m$ is even. However, we can immediately recognise points of order 2 due to the fact that their $y$-coordinate is always equal to zero (the reader may check this as an easy exercise to become familiar with the arithmetic on elliptic curves).

To finish this rather technical section off, I will give an explicit expression for $[m]P$, which is again just a straightforward transformation of (3.1):

$$[m]P = \left( x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2}, \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y\psi_m^3} \right).$$  \hspace{1cm} (3.6)
In the actual application of this result, \( \psi_m \) will be replaced by \( f_m \) so that \([m]P\) is a rational function of \( x \) only. We will show this explicitly later.

In the following chapter I will give a detailed explanation of the deterministic algorithm that provides us with an efficient method to count the number of rational points on a given elliptic curve over a finite field.
Chapter 4

Schoof’s Algorithm

Schoof’s Algorithm (published in April 1985) provides us with a tool to compute \( \#C(K) \), where \( C \) is an elliptic curve given in Weierstrass form, and \( K = \mathbb{F}_q \) is a finite field. The algorithm takes \( O((\log q)^8) \) elementary operations and is deterministic; it does not depend on any unproved hypotheses. As usual, we will restrict ourselves to the case where \( \text{char}(K) \neq 2 \) or 3; those cases again need separate treatment, however as mentioned before, they are irrelevant for our purposes.

I will first of all list the main steps of the algorithm, so that the reader can refer to them when working through the following section. Note that the purpose of some of those steps may not seem immediately obvious, and some notation may be unfamiliar, but the details will of course be filled in afterwards.

### 4.1 Outline

**INPUT:** An elliptic curve \( C : Y^2 = X^3 + AX + B \) defined over \( \mathbb{F}_q \) where \( q \) is a prime \( \neq 2 \) or 3.

1. Compute the quantity \( l_{\text{max}} \) defined in (4.1) in Section 4.2
2. Set \( l = 3 \). Compute \( \tau = t \pmod{l} \) as follows:
   (a) Set \( \tau \equiv 0 \pmod{l} \).

\footnote{Note that in his paper, Schoof shows that the running time of his algorithm is \( O((\log q)^9) \); it can however be shown that one can make improvements on this bound. This will be discussed in more depth in Chapter 8.}
(b) To test whether there exists a point $P$ on $C$ such that $\phi^l_P + [k]P = \pm[\tau]\phi_P$, where $k \equiv q \pmod{l}$:
Compute $H_{k,\tau}$ as defined in (4.6) and (4.10) in Section 4.2.

(c) Compute $\gcd(H_{k,\tau}, f_l)$, where $f_l$ is as defined in (3.3) in Section 3.1.1.
   i. If $\gcd(H_{k,\tau}, f_l) = 1$, go to Step (2d).
   ii. If $\gcd(H_{k,\tau}, f_l) \neq 1$, determine the correct sign of $\tau$ using the methods explained in Section 4.2. Set $\tau = \pm\tau$ accordingly. Go to step (2e).

(d) Set $\tau = \tau + 1$. Go to step (2b).

(e) OUTPUT: $(\tau, l)$. Set $l =$ next prime $\leq l_{\text{max}}$, go to step (2a). If the next prime $l > l_{\text{max}}$, go to step (3).

3. Compute $t$ using the Chinese Remainder Theorem applied to $(\tau, l)$ for all $l$.

4. Compute $\#C(\mathbb{F}_q)$.

5. FINAL OUTPUT: $\#C(\mathbb{F}_q)$.

4.2 Explanation

Now the above looks very abstract and clearly requires explanation.

Note that only in the last step we are concerned with $\#C(\mathbb{F}_q)$. The algorithm actually computes the trace of Frobenius; this is clearly equivalent to computing $\#C(\mathbb{F}_q)$ due to the one-to-one correspondence between the two quantities, $\#C(\mathbb{F}_q) = q + 1 - t$.

Since Hasse’s Theorem 2.4.3 provides us with a bound on $t$, it will be sufficient to compute $t$ modulo a sufficiently large number of primes and then recover the value of $t$ by an application of the Chinese Remainder Theorem.
I will explain the algorithm for $\tau \equiv 0$ in thorough detail first; the case for $\tau \in \{1, \ldots, (l - 1)/2\}$ will then only be outlined. It follows the same idea, but involves computing slightly more complicated polynomials.

Since the algorithm is in its abstract form very technical and hence somewhat difficult to follow, I will, as an example of its application, demonstrate each step by performing it on the elliptic curve $C : Y^2 = X^3 + 1$ over the finite field $\mathbb{F}_5$.

Note that this example is almost trivial, since over $\mathbb{F}_5$ we can find the number of points by inspection rather easily. The example will however also show that the computations over such small fields already involve very complicated looking polynomials. In practice, we would apply Schoof’s Algorithm to fields of characteristic a large prime.\footnote{In order to avoid confusion I will print the example in blue so that the reader can easily distinguish more easily between “theory” and “practice”, since I will often skip between the two.}

We begin by defining $l_{\text{max}}$ to be the smallest prime such that

$$
\prod_{\substack{l \text{ prime} \\ 2 \leq l \leq l_{\text{max}}} \ l > 4 \cdot \sqrt{q}.}
$$

(4.1)

This bound is sufficient for us to obtain enough values of $\tau \pmod{l}$ to recover $t$ using the Chinese Remainder Theorem.

We know from (2.7) that the trace of Frobenius $t$ satisfies

$$
\phi^2 + q = t\phi,
$$

so if we reduce this equation modulo $l$ we have

$$
\phi_l^2 + k = \tau\phi_l , \text{ where } \tau \equiv t \pmod{l}, \ k \equiv q \pmod{l}
$$

(4.2)

for all points on $C$ of order $l$, i.e. for $P = (x, y) \in C[l]$.\footnote{In order to avoid confusion I will print the example in blue so that the reader can easily distinguish more easily between “theory” and “practice”, since I will often skip between the two.}
In order to find \( \tau \) we need to check for which \( \tau \in \{0, 1, \ldots, l\} \) the relation (4.2) holds. To do this, we will test for \( \tau \in \{0, 1, \ldots, (l - 1)/2\} \) whether a point \( P = (x, y) \) exists in \( C[l] \) such that

\[
\phi^2_l P + [k] P = \pm [\tau] \phi P.
\]

By applying (3.6), we can see that this is equivalent to testing for which \( \tau \) we have

\[
(x^{q^2}, y^{q^2}) + \left( x - \frac{\psi_{q-1}\psi_{q+1}}{\psi_q^2}, \frac{\psi_{q+2}\psi_{q-1} - \psi_{q-2}\psi_{q+1}}{4y\psi_q^3} \right)
= \begin{cases} 
0 & \text{if } \tau \equiv 0 \pmod{l} \\
\left( x^q - \left( \frac{\psi_{q-1}\psi_{q+1}}{\psi_q^2} \right)^q, \left( \frac{\psi_{q+2}\psi_{q-1} - \psi_{q-2}\psi_{q+1}}{4y\psi_q^3} \right)^q \right) & \text{otherwise.}
\end{cases}
\]

So let us now run through the algorithm to see what happens at each step. First of all we compute \( l_{\text{max}} \).

### 4.2.1 Step (2)

We set \( l = 3, \tau = 0 \). We then test whether there exists a point in \( C[l] \) such that \( \phi^2_l P = \pm [k] P \).

Comparing the x-coordinates of both sides in (4.4), we can see that this holds if and only if

\[
x^{q^2} = x - \frac{\psi_{k-1}\psi_{k+1}}{\psi_k^2}
\]

In order to obtain a univariate polynomial in \( x \) only, we replace the \( \psi_n \) by \( f_n \) and multiply through by the denominator. Let us now define \( H_{k,0}(x) \) as follows:

\[
H_{k,0}(x) := \begin{cases} 
(x^{q^2} - x)f_k^2(x)(x^3 + Ax + B) + f_{k-1}(x)f_{k+1}(x) & (k \text{ even}) \\
(x^{q^2} - x)f_k^2(x) + f_{k-1}(x)f_{k+1}(x)(x^3 + Ax + B) & (k \text{ odd})
\end{cases}
\]

### 4.2.2 Step (2c)

We have now reduced the problem of testing whether relation (1.2) holds for \( \tau \equiv 0 \pmod{l} \) to checking whether there exists a \( P = (x, y) \in C[l] \) such that \( H_{k,0}(x) = 0 \).
Let us consider our example: Since we have $q = 5$, we can take $l_{\text{max}} = 5$. We will now compute $f_n$ now for $n = 0, \ldots, 4$:

\[
\begin{align*}
  f_0(x) &= 0, f_1(x) = 1, f_2(x) = 1, \\
  f_3(x) &= 3x^4 + 12x, f_4 = x^6 + 20x^3 - 8.
\end{align*}
\]

Since we have set $l = 3$, $\tau = 0$, now need to compute $H_{2,0}$.

\[
H_{2,0} = (x^{25} - x)(x^3 + 1)f_2(x)^2 + f_1(x)f_3(x)
\]

\[
= (x^{25} - x)(x^3 + 1) + f_3(x)
\]

Although it may not seem immediately obvious, why this is any easier than the original problem, it is indeed a simplification: We recall from (3.5) that

\[
P = (x, y) \in C[l] \iff \ell P = 0 \iff f_1(x) = 0.
\]

On the other hand we know that if our chosen $\tau$ is indeed the trace of Frobenius, then for all such $x$, we have $H_{k,\tau}(x) = 0$.

From this we deduce that all roots of $f_l$ are also roots of $H_{k,\tau}$, and the two polynomials therefore have a non-trivial greatest common divisor.

So rather than attempting to solve the equation $H_{k,0}(x) = 0$, we only need to compute the greatest common divisor of $H_{k,\tau}$ and $f_l$; this explains step (2c) of the algorithm. Note here that in order to compute the greatest common divisor we use the Euclidean Algorithm, which the reader should be familiar with; it is briefly outlined in the next chapter.

Now consider the case where $\gcd(H_{k,0}, f_l) = 1$. This happens if and only if $H_{k,0}$ and $f_l$ have no roots in common. In that case we have $H_{k,0}(x) \neq 0$ for any $P = (x, y) \in C[l]$ and so we conclude that there exists no $P \in C[l]$ such that (4.2) holds. Clearly this means that $t \neq \tau \pmod{l}$, and we go to step (2a) to set $\tau = \tau + 1$ and try again for this new value of $\tau$. I will return to this case later.

\[3\text{Note here that we never hit } \tau + 1 > (l - 1)/2 \text{ as we know that exactly one } \tau \in \{0, \ldots, (l - 1)/2\} \text{ will satisfy (4.2) and so } (l - 1)/2 \text{ is the largest value that } |\tau| \text{ can take. Once we have hit this value we know it is the correct solution and we find ourselves in Step (2a) ii, from which we proceed to (2a) straight away.} \]
If, on the other hand, the greatest common divisor is non-trivial, then we know that if we have a point $P$ in $C[l]$, it will necessarily satisfy the desired property (4.2).

In our example, this step boils down to finding the greatest common divisor of $(x^{25} - x)(x^3 + 1) + f_3(x)$ and $f_3(x)$. This in turn is just
\[
\gcd((x^{25} - x)(x^3 + 1), f_3(x)) = \gcd((x^{25} - x)(x^3 + 1), 3x^4 + 12x).
\]

We find that this greatest common divisor turns out to be $x$ and hence proceed to sub-step ii, which is explained below:

\[
\gcd \neq 1, \tau = 0
\]

Now there are two “subcases” to be considered: Namely when $\phi_l^2 P = [\pm q]P$ and when $\phi_l^2 P = [\mp q]P$. We now run through a “sub-algorithm” for the case $\tau = 0$.

- Test whether $\phi_l^2 P = [\pm k]P$ by checking the $y$-coordinate in a similar way.

  - If $\phi_l^2 P = [\pm k]P$, go to step (2a) of the main algorithm.

  - If $\phi_l^2 P = [\pm k]P$, test whether $q$ is a square modulo $l$

    * If $(\frac{q}{l}) = -1$, go to step (2a) of the main algorithm.

    * If $(\frac{q}{l}) = 1$, let $\omega^2 = q \pmod{l}$ and test whether $\phi_l P = [\omega]P$ or $\phi_l P = [-\omega]P$. Set $\omega_0 = \pm \omega$ accordingly. Set $\tau = 2\omega_0$, go to step (2a) of the main algorithm.

Now why are we doing all this?

**Case 1** First assume that $\phi_l^2 P = [\pm k]P$. So we know that $\tau\phi_l P = 0$, and since $\phi_l P \neq 0$, we can conclude that $t \equiv 0 \pmod{l}$. So we proceed to Step (2a) in the main algorithm and then run the algorithm for the next prime $l$. 
Case 2  On the other hand, if \( \phi_2 P = [+k]P \), then
\[
(2q - \tau \phi_{\ell})P = 0 \quad \text{and so} \quad \phi_{\ell}P = \frac{2q}{t} P.
\]
Let us apply \( \phi_{\ell} \) to both sides and use the equality satisfied by \( P \); so we get
\[
qP = \phi_2^3 P = \phi_{\ell} \left( \frac{2q}{t} P \right) = \left( \frac{2q}{t} \right)^2 P,
\]
and hence that \( t^2 \equiv 4q \pmod{l} \). Again, we must split this into two subcases: When \( q \) is a quadratic residue modulo \( l \) and when it is not.

- \( \left( \begin{array}{c} q \end{array} \right)_l = -1: \) In this case we can conclude that \( \tau \equiv 0 \pmod{l} \) and go to Step (2e).
- \( \left( \begin{array}{c} q \end{array} \right)_l = 1: \) Let \( 0 < \omega < q - 1 \) denote a square root of \( q \) modulo \( l \). Since we have \( (2q - \tau \phi_{\ell})P = 0 \), we can see that \( 2q/t \) is an eigenvalue of \( \phi_{\ell} \); but \( t/2 = \pm \sqrt{q} \), so either \( \sqrt{q} \) or \( -\sqrt{q} \) is an eigenvalue of \( \phi_{\ell} \). To test this, we proceed exactly as before in checking whether \( \phi_{\ell}P = \left[ \pm \sqrt{q} \right] P \).

In our example, we check the \( y \)-coordinates of (4.4) and proceed as above: assuming that \( \phi_2^3 P = -[k]P \), we turn this into a polynomial in \( x \) that depends on \( k \) and \( \tau \), and compute its greatest common divisor with \( f_3 \). We find that this greatest common divisor is non-trivial and hence \( \phi_3^2 P = -[2]P \) is indeed the correct solution. So \( t \equiv 0 \pmod{3} \) in our example.

Applying the same methods, we compute \( \tau \pmod{5} \) and find that \( t \equiv 0 \pmod{5} \), too.

Let us return to the algorithm to see what would have happened if the greatest common divisor had been trivial.

\( \text{gcd} = 1 \)

Here we have that for no point \( P \in C[l] \), relation (4.3) is satisfied. From this we conclude that \( t \neq \tau \pmod{l} \) and so we need to check whether the next value of \( \tau \) is the trace of Frobenius mod \( l \), i.e. whether \( \phi_2^3 P + [k]P = \pm [\tau] \phi_{\ell}P \) for \( P \in C[l] \). So we go back to step (2b) and compute \( H_{k,\tau} \) for this new value of \( \tau \). Referring to (4.4) again, we know that
\[
(\phi_2^3 P + kP)X = x^3 + x + \frac{f_{k-1}f_{k+1}}{f_k^2} + \lambda^2 + \lambda,
\]
(4.8)
where
\[
\lambda = \frac{(y^q + y + x)xf_k^3 + f_{k-2}f_{k+1}^2 + (x^2 + x + y)(f_{k-1}f_kf_{k+1})}{x f_k^2(x + x^q) + xf_{k-1}f_kf_{k+1}}.
\]

On the other side we have that
\[
(\pm \tau \phi l P)_X = x^q + \left(\frac{f_{\tau+1}f_{\tau-1}}{f^2_\tau}\right)^q.
\]
(4.9)

Now we can, in a similar way as above, transform the equation by reducing modulo the curve equation so that we have polynomials of degree at most one in \(y\), since we can substitute \((x^3 + Ax + B)^m\) for any \(y^{2m}\).

We then obtain an equation of the form \(a(x) - yb(x) = 0\), hence \(y = a(x)/b(x)\) for some \(a(x), b(x) \in \mathbb{F}_q(x)\). Again substituting for \(y\) in the curve equation we therefore finally get
\[
y^2 = (x^3 + Ax + B) = \left(\frac{a(x)}{b(x)}\right)^2.
\]

So we define
\[
H_{k,\tau} := a(x)^2 - (x^3 + Ax + B)b(x)^2,
\]
(4.10)
a polynomial in \(x\) only.

Now we can proceed precisely as before: We want to check whether for \(x\) such that \(f_l(x) = 0\) we also have \(H_{k,\tau} = 0\), i.e. whether the roots of \(f_l\) are also roots of \(H_{k,\tau}\).

So in Step (2c) we compute the greatest common divisor of \(H_{k,\tau}\) and \(f_l\).

If the points on \(C[l]\) do not satisfy the Frobenius relation and hence the greatest common divisor is 1, we conclude that this value of \(\tau\) is also not the correct value.
We are therefore sent to Step (2d) to proceed to the next possible value of \(\tau\) and then return to Step (2b), where we run the same test for that new value.

Otherwise we have that \(t \equiv \pm \tau(l)\) for our chosen \(\tau\). In this case we need to check which sign is correct: We refer to (4.4) again, this time comparing the \(y\)-coordinates of both sides, and check in a similar manner which is the correct sign.
4.2.3 The final steps

This way we eventually obtain enough values for $t \pmod{l}$ so that we can finally proceed to Step 4 and apply the Chinese Remainder Theorem to the pairs $(\tau, l)$. Finally we can calculate $\#C(\mathbb{F}_q)$, which completes the algorithm.

Applying the Chinese Remainder Theorem to our example $C : Y^2 = X^3 + 1$ over $\mathbb{F}_5$, where we had that $t \equiv 0 \pmod{3}$ and $t \equiv 0 \pmod{5}$, we can deduce that $t = 0$. So the number of rational points on $C$ over $\mathbb{F}_5$ is $5 + 1 - 0 = 6$.

Since we have chosen such a simple example, we can verify this result by inspection, i.e. by trying each value of $x \in \mathbb{F}_5$:

On $C : Y^2 = X^3 + 1$ over $\mathbb{F}_q$ we have the following rational points:

$$o, (0, \pm1), (2, \pm2), (4, 0).$$

So we obtain the same result; there are 6 rational points on $C$ over $\mathbb{F}_5$.

The topic of point-counting algorithms, improvements of Schoof’s Algorithm and its applications is an extremely interesting and wide-ranging one; I refer the interested reader to [2] and [9] for the further study of this subject. Any deeper discussion about this subject is however irrelevant to this dissertation.
Chapter 5

Some Ring Theory

For the purpose of a clearer understanding of the proof that will follow in the next chapter, we will need to recall some elementary theory about rings and fields.

The following results should be known to the reader. I will therefore omit proofs to the assertions made; they should be regarded as a list of results that the reader may refer to in some steps of the proof of the theorem.

Details of proofs can be found in e.g. [3], [4] or [7].

5.1 Elliptic Curves defined over a Ring

Let $h(z)$ in $\mathbb{F}_q[z]$ be a nonzero polynomial of degree 2 with distinct roots in $\mathbb{F}_q$, and consider a curve $C$ over the ring $R := \mathbb{F}_q[z]/(h(z))$. We may view $C$ as a pair of curves over $\mathbb{F}_q$ as follows:

$C$ has coefficients of the form $(rz + s)$, where $r, s \in \mathbb{F}_q$.

Since $R \cong \mathbb{F}_q \times \mathbb{F}_q$, we can apply the isomorphic map $(rz + s) \mapsto (ra + s, -ra + s)$, where $\pm a$ are the roots of $h(z)$ in $\mathbb{F}_q$, to the curve to obtain a pair of curves, both defined over $\mathbb{F}_q$. I.e.:

$$C : Y^2 = X^3 + (\alpha z + \beta)X + (\gamma z + \delta) \mapsto \begin{cases} C_+ : & Y^2 = X^3 + (\alpha a + \beta)X + (\gamma a + \delta) \\ C_- : & Y^2 = X^3 + (-\alpha a + \beta)X + (-\gamma a + \delta) \end{cases}$$

(5.1)
5.2 Rings and Fields

The fundamental difference between a ring and a field is that in a ring we may have non-units. That is, we may have elements $r \in R$, such that there exists no $s \in R$ with $r \cdot s = 1_R$.

A zero divisor is an element $r \in R, r \neq 0$ such that there exists $s \in R \setminus \{0\}$ with $r \cdot s = 0_R$.

**Lemma 5.2.1** If $r \in R$ is a zero divisor, then it is a non-unit.

**Example** For instance, in the ring $A = \mathbb{Z}/4\mathbb{Z}$, we have that $2 \neq 0$, but $2 \cdot 2 = 4 = 0_A$. $2$ is therefore a zero divisor. It is also a non-unit: there is no element $s \in A$ such that $s \cdot 2 = 1_A$.

Although this should be obvious, the following result is worth some particular attention:

**Corollary 5.2.2** In $R \setminus \{0\} = (\mathbb{F}_q[z]/(z^2-a^2)) \setminus \{0\}$, the non-units are $(z-a)$ and $(z + a)$.

5.3 Euclid’s Polynomial Division Algorithm

Let us consider two univariate polynomials $f(x), g(x)$ defined over some field $F$. Euclid’s Polynomial Division Algorithm provides us with an efficient tool to compute the greatest common divisor of $f$ and $g$.

5.3.1 Long Division of Polynomials

Recall from school how we divide polynomials: First we divide the leading term of the higher degree polynomial by the leading term of the lower degree polynomial. Now think about what “dividing” means: we try to find an element $a$ such that $a \cdot b = c$, where $b$ is the leading coefficient of the lower degree polynomial and $c$ that of the higher degree polynomial. All this should of course be clear, but as it will be crucial later on, it is again worth noting down the following result:

**Lemma 5.3.1** We have $a = c \cdot b^{-1}$.

Hence in order to find $a$, we compute the inverse of $b$ and premultiply it by $c$. 
5.3.2 Euclid’s Algorithm

This is just a brief outline of the algorithm. Details can be found in any undergraduate book on linear algebra, e.g. [3], [4] or [7].

Proposition

For \( f(x), g(x) \in F[X] \) with \( 0 < \deg(g) < \deg(f) \), there exist \( r_1(x), q_0(x) \in F[X] \) such that we can write

\[
f(x) = g(x)q_0(x) + r_1(x), \text{ with } \deg(r_1) < \deg(f) \text{ or } r_1 \equiv 0.
\]

The polynomials \( r_1(x) \) and \( q_0(x) \) are computed by long division of polynomials. As the next step in the Algorithm, we define a sequence \( r_i(x) \) as follows:

\[
r_0(x) = g(x)
\]

\[
r_i(x) = r_{i+1}(x)q_{i+1}(x) + r_{i+2}(x) \text{ with } \deg(r_{i+2}) < \deg(r_{i+1}) \text{ or } r_{i+2} = 0.
\]

We eventually obtain

\[
r_{n-1}(x) = r_n(x)q_n(x) + r_{n+1}(x), \quad r_{n+1}(x) = 0, r_n(x) \neq 0.
\]

At this point the algorithm ends, and returns \( r_n(x) \) as the greatest common divisor.

Now we are finally ready to tackle the actual problem we are aiming to solve.
Chapter 6

The Theorem

In this chapter I will discuss the theorem to be proved. First of all I will give the already simplified version of the theorem and prove it. The original statement of it is somewhat more complicated and requires a few more definitions; this will be discussed in Chapter 7.

6.1 Statement of the Theorem

6.1.1 The problem

The problem to be solved here is:

Find a deterministic algorithm with

- **INPUT:**
  - A finite field \( F_q \) and
  - a nonzero polynomial \( h(z) \) of degree 2 in \( F_q[z] \).

(6.1)

- **OUTPUT:**
  - The factors of \( h(z) \) over \( F_q \).

- Running time: polynomial in the size of the input, i.e., bounded by a universal polynomial in \( (1 + \deg h)(\log q) \).
6.1.2 The hypothesis

We are given a polynomial \( h(z) \) of degree 2 with roots in the finite field \( \mathbb{F}_q \), where \( q \) is a prime.

Assume that there exists an elliptic curve \( C : Y^2 = X^3 + AX + B \) over the ring \( R := \mathbb{F}_q[z]/(h(z)) \) for which there exists a prime \( l \), such that we have \( \#C_+(\mathbb{F}_q) \neq \#C_-(\mathbb{F}_q) \) (mod \( l \)).

(6.2)

Theorem 6.1.1 (Kayal) Given (6.2), there exists an algorithm as in (6.1).

6.2 The Proof

Let us define \( h(z) := (z^2 - a^2) \in \mathbb{F}_q[z] \) (where we do not know \( a \) but only \( a^2 \)) and let

\[ C : Y^2 = X^3 + (\alpha z + \beta)X + (\gamma z + \delta) \]

over the ring \( R \) as above. Let \( l \) be the prime number that satisfies the hypothesis of the theorem, i.e. such that \( \#C_+(\mathbb{F}_q) \neq \#C_-(\mathbb{F}_q) \) (mod \( l \)).

Let \( t_+, t_- \) be the respective traces of Frobenius of \( C_+ \) and \( C_- \). Then we have that \( t_+ \neq t_- \) (mod \( l \)).

The idea of the proof is that Schoof’s point counting algorithm is an algorithm that solves our problem of factorising \( h(z) \). I claimed earlier that when we apply it to \( C \), it will at some point reveal the factors of \( h \).

Schoof’s Algorithm is defined for elliptic curves over finite fields, whereas \( C \) is defined over a ring. Note that if the underlying hypothesis for the theorem were not fulfilled, we could in general run the algorithm over curves defined over a ring without any problems.

Running Schoof’s Algorithm over \( C \) is equivalent to running it over \( C_+ \) and \( C_- \) simultaneously. Every operation that we are performing on \( C \) can be thought of as performing the same operations on \( C_+ \) and \( C_- \) if we map \( z \) to \( \pm a \) accordingly.

Assume that we are in Step (2a) of the algorithm with \( l \), the prime number with the desired property. Checking every value of \( \tau \in \{0, \ldots, (l-1)/2\} \) to see whether it satisfies \( \phi^2 - \tau\phi + q = 0 \) is hence the same as checking whether there exists a point \( P^+ = (x^+, y^+) \) in \( C_+[l] \) such that the relation...
\[ \phi_2 - \tau \phi + q = 0 \] is satisfied, and whether for a point \( P^- = (x^-, y^-) \) in \( C_-[l] \), this equation holds.

Let \( f^+_l \) and \( f^-_l \) denote the \( l \)th division polynomial on \( C_+ \) and \( C_- \) respectively, and let \( H^+_k,\tau \) and \( H^-_k,\tau \) be as defined in (4.10) for the two curves accordingly.

Now, without loss of generality, we assume that \( t_+ < t_- \). Consider Step 2c in the algorithm with \( \tau = \tau_+ \equiv t_+ \pmod{l} \). We compute \( H^+_k,\tau \) and \( H^-_k,\tau \). Since \( \tau = \tau_+ \), we will find that for all points \( P^+ = (x^+, y^+) \) in \( C_+[l] \), we have \( H^+_k,\tau (x^+) = 0 \).

On the other hand however, since \( \tau \not\equiv \tau_- \equiv t_- \pmod{l} \), we know that no point in \( C_-[l] \) satisfies \( H^-_k,\tau = 0 \).

Now consider this step of the algorithm over \( C \) itself. So we attempt to compute \( \gcd(H_k,\tau, f_l) \), as usual, using the Euclidean Algorithm.

Suppose we are trying to divide some polynomial \( r_i(x) \) by \( r_{i+1}(x) \) where \( \deg(r_i) > \deg(r_{i+1}) \). I will now make the following claim:

**Proposition 6.2.1** The leading coefficient \( c(z) \) of \( r_{i+1}(x) \) is a non-unit in \( R \), for some \( i \).

**Corollary 6.2.2** Proposition 6.2.1 completes the proof.

**Proof of Proposition 6.2.1** Imagine that the lower degree polynomial \( r_{i+1}(x) \) never has leading coefficient a non-unit in \( R \). The Euclidean Algorithm will just run smoothly over the ring as if it were a field.
Now recall that everything we are doing with the curve $C$ is equivalent to performing the same operations on the pair of curves $C_+$ and $C_-$ simultaneously.

Let us once more consider in detail the relationship between the polynomials $r_i$ for $C$ and the $r_i^+$ and $r_i^-$ for $C_+$ and $C_-$. The latter two are just evaluations of the coefficients of $r_i$ at $z = a$ and $z = -a$ respectively.

So if we assume that the leading coefficient of $r_i$ is a unit for every $i = 0, \ldots, n$ (where we have that $r_{n+1} = 0$), then the leading coefficient of $r_i$ never vanishes on $C_+$ and $C_-$. This implies that the degree of the polynomials $r_i$ is the same as the degree of the $r_i^+$ and $r_i^-$. In particular, we note that for all $i = 0, \ldots, n$, the degrees of $r_i^+$ and $r_i^-$ are the same.

So finally, we conclude that the degree of $r_n^+$ is the same as the degree of $r_n^-$. But recall that $r_n^+$ and $r_n^-$ are the greatest common divisors of $H_{k,\tau}$ and $f_l^+$, and $H_{k,\tau}$ and $f_l^-$ respectively.

By assumption however, the greatest common divisor of $H_{k,\tau}$ and $f_l^-$ is 1, since $\tau \neq \tau_- (\mod l)$, whereas that of $H_{k,\tau}$ and $f_l^+$ is strictly non-trivial!

This is clearly a contradiction.

We can now see that at some point we must encounter a non-unit as the leading coefficient of some $r_i$.

By Corollary 6.2.2, this completes the proof.
Chapter 7

The original statement of the Theorem

7.1 Polynomials of higher degree

As mentioned earlier, the full assertion made by Dr Kayal is slightly more advanced. Instead of restricting himself to polynomials of degree 2, he claimed that the assertion would hold for any polynomial with distinct roots in a finite field.

On closer inspection, one can see that this is plausible, and that in fact the proof will be very similar to the one given above. One needs to think of an elliptic curve over the ring

\[ R := \mathbb{F}_q[z]/(h(z)) = \mathbb{F}_q[z]/((z - \alpha_1)(z - \alpha_2)(\ldots)(z - \alpha_n)) \]

as a family of curves over \( \mathbb{F}_q \) in the same way as above, i.e. with \( z \) evaluated at \( \alpha_i \) for each “subcurve” \( C_i \).

If we are then given a curve \( C \) over this ring, such that for some prime \( l \) the number of rational points on \( C_i \) is not congruent to the number of rational points on \( C_j \) modulo \( l \), for some \( i < j \), the same problem as discussed above will arise in Schoof’s Algorithm.

Let \( \tau_i \) be the trace of Frobenius of \( C_i \) modulo \( l \), which is hence not equivalent to the trace of Frobenius of \( C_j \). Adapting a similar notation as before, and using the same arguments, we can deduce that \( \text{gcd}(H_{k,\tau_i}^{(i)}, f_l^{(i)}) \) is strictly non-trivial, whereas \( \text{gcd}(H_{k,\tau_i}^{(j)}, f_l^{(j)}) = 1 \).
CHAPTER 7. THE ORIGINAL STATEMENT OF THE THEOREM

When computing $gcd(H_{k,\tau_i}, f_l)$, on $C$, the Euclidean Algorithm will again break down in an attempt to compute the inverse of a non-unit in $R$, which we will inevitably encounter as the leading coefficient of some $r_j$. The reason for this is precisely the same as in the case for $deg(h) = 2$: if this never happened, then we would be able to conclude from this fact that the greatest common divisors $gcd(H_{k,\tau_i}^{(i)}, f_l^{(i)})$ and $gcd(H_{k,\tau_i}^{(i)}, f_l^{(j)})$ have the same degree.

The above very brief outline of the proof already shows that a detailed proof of this version of the assertion would have involved a lot of careful, possibly confusing, notation (just imagine a detailed account of Schoof’s Algorithm with this notation!). It should be clear however, that the proof follows the same string of arguments.

7.2 Advanced topics

There are some further simplifications of the assertion that I have made. Some of the topics underlying the full claim made by Dr Kayal, and Prof Poonen’s addition to this, are somewhat too advanced to give a “brief” explanation of them before being able to prove the theorem. Details of such topics however can be found e.g. in [14], [18] and [9].

In its original form, the assertion has the following underlying hypothesis:

Let $h(t)$ in $\mathbb{Z}[t]$ be a nonzero polynomial, and let $C$ be a smooth projective curve of genus $g$ over $\mathbb{Z}[t]/(h(t))$.

The hypothesis is: There exists a $C$ as above such that for each sufficiently large primes $p$, the zeta functions of the curves $C_{p,o_i}$ are distinct.

7.2.1 Curves of Genus $g$

In this dissertation I have restricted myself to the case of very “simple” curves, i.e. elliptic curves, which are also often defined as non-singular curves of genus 1.

Giving a detailed discussion about curves of higher genus would take us too far afield in this dissertation. As a brief description of “genus” however, I will just say that any curve $F(x,y) = 0$ has a non-negative integer $g$ associated with it; $g$ is referred to as the genus. In general (for example if the curve is non-singular), the genus increases as the degree of $F$ increases.
For curves of the form $Y^2 = F(X)$, we have that $\deg(F) = 2g + 1$. Hence in our case the genus is 1, and a curve of the form $Y^2 = X^5 + a_4X^4 + \cdots + a_0$ has genus 2.

The addition that Prof Poonen made to Dr Kayal’s initial claim is in fact to do with such curves of genus greater or equal to 2.

In trying to find “applicable” curves for this problem, he remarked that if one used a curve of higher genus, the probability of the hypothesis being fulfilled would be much higher than for elliptic curves.

In fact, he claimed that the probability of finding an elliptic curve over a ring $\mathbb{F}_q[z]/(h(z))$, for which the fibres at each root $\alpha_i$ (that is, the “sub-curves” $C_i$ for all $i$) have the same number of rational points modulo some prime number $l$, is of order $1/q$.

For curves of higher genus, that probability is, according to Prof Poonen, much smaller - in fact it is of order $1/q^2$. This is of course an important result if one tries to find curves to apply this theorem to.

We can however find suitable elliptic curves, too, that fulfil our hypothesis. For example, consider the curve

$$C : Y^2 = X^3 + zX \text{ over } R = \mathbb{F}_5[z]/(z^2 - 1).$$

The subcurves, i.e. $C$ evaluated at the roots of $(z^2 - 1) = (z + 1)(z - 1)$ are given by

$$C_+: Y^2 = X^3 + X \text{ and } C_- : Y^2 = X^3 - X,$$

both defined over $\mathbb{F}_5$.

By inspection, we can see that the points in $C_+(\mathbb{F}_5)$ are

$$\{o, (0, 0), (2, 0), (3, 0)\},$$

so there are 4 of them.

On the other hand, $C_- (\mathbb{F}_5)$ has the points

$$\{o, (0, 0), (1, 0), (2, 1), (2, -1), (3, 2), (3, -2), (4, 0)\}$$

- a set of 8 rational points!
Clearly, \(4 \neq 8 \pmod{3}\). So this curve satisfies our hypothesis and could be used in applications of the theorem (although of course it would be a fairly trivial and pointless example).

Now in order to show that what we have proved above is (almost) the same as the original assertion by Dr Kayal, we will just need to understand what the “Zeta function of \(C\)” is.

### 7.2.2 The Zeta function

Let \(C\) be a curve defined over \(\mathbb{F}_q\). Clearly if \(C\) is defined over \(\mathbb{F}_q\) then it is also defined over \(\mathbb{F}_{q^n}\) for all \(n \geq 1\). It may therefore be interesting to consider

\[
N_n = \#C(\mathbb{F}_{q^n})
\]

for \(n \geq 1\), i.e. the number of rational points on \(C\) over \(\mathbb{F}_{q^n}\).

**Definition 7.2.1 (The Zeta Function)** Define the series

\[
Z(E; T) = \exp \left( \sum_{n \geq 1} \frac{N_n}{n} \cdot T^n \right)
\]

(7.1)

for an indeterminate \(T\). This is called the Zeta function of \(C\) over \(\mathbb{F}_q\).

Due to work by Hasse - and for a more general case extending to curves of genus higher than 1, by Weil - we can show that the Zeta function has a simpler form:

**Theorem 7.2.2 (Weil conjectures for an elliptic curve)** Let \(C\) be a curve defined over \(\mathbb{F}_q\). Denote by \(c_n\) the trace of Frobenius of \(C\) over \(\mathbb{F}_{q^n}\), i.e. \(c_n = \#C(\mathbb{F}_{q^n}) - q^n - 1\). The Zeta function is a rational function of \(T\) and takes the form

\[
Z(C; T) = \frac{P(T)}{(1 - T)(1 - qT)}
\]

(7.2)

where \(P(T) = 1 - c_1 T + qT^2 = (1 - \alpha)(1 - \bar{\alpha})\). Furthermore, the discriminant of \(P(T)\) is non-positive and the magnitude of \(\alpha\) is \(\sqrt{q}\).

A proof of this theorem can be found e.g. in [10], for the case \(g = 1\), or [19], for curves of higher genus. Note in particular that the last line of the theorem implies that \(c_1^2 < 4q\), which is Hasse’s Theorem.
More importantly however, if we take the derivative of the logarithm of both sides in (7.2), substituting in (7.1) for the left hand side, we can show after some straightforward series manipulations and partial fraction expansions that this implies

\[ N_n = \#C(F_{q^n}) = q^n + 1 - \alpha^n - \bar{\alpha}^n = |1 - \alpha^n|^2. \]

Since both \( c_1 \) and \( \alpha \) can be immediately derived from knowledge of \( N_1 \), we can uniquely determine \( N_n \) for all \( n \geq 1 \) once we know the number of rational points of \( C \) over the base field \( F_q \).

It should be clear that this result is an extremely important one which has many useful applications not only in attempting to prove the above theorem. However, if we return to Kayal/Poonen’s claim, we can now simplify the proof of the theorem as follows:

The underlying hypothesis for our factorisation is a different Zeta function for the fibres at \( \alpha_i \) (mod \( l \)) and \( \alpha_j \) (mod \( l \)) for some prime \( l \). Now that we know that we can determine \( N_n \) unambiguously from computing \( N_1 \), and since \( Z(C; T) \) only depends on the \( N_n \), it will be sufficient to use a curve for which the fibres at \( \alpha_i \) and \( \alpha_j \) have a different number of rational points over \( F_q \), modulo \( l \). This is clearly what we have done above.
Chapter 8

Running Time

To finish this dissertation off, I will now give an account of the running time of the algorithm.

Since algorithms and computations thereof is a rather broad mathematical subject on its own, and one which I assume the reader to be unfamiliar with, I will restrict this discussion to the key points. I recommend in particular [1] to the interested reader; it contains a comprehensive introduction to this subject, and will also fill in some details about the running time of the individual steps in our algorithm that I will omit.

8.1 Introduction to running times

In order to calculate the running time of an algorithm, we count the number of basic operations performed by the algorithm on the “worst-case input”. The worst-case input is the input for which the most basic operations are required. We count the basic operations as follows:

8.1.1 Definition

Let $n \in \mathbb{Z}$. Define

$$lg n := \begin{cases} 
1, & \text{if } n = 0; \\
1 + \lfloor \log_2 |n| \rfloor, & \text{if } n \neq 0.
\end{cases}$$

Then $lg n$ counts the number of bits in the binary representation of $n$.

A step is the fundamental unit of computation. Now, different situations require different units. For example, analysing a sorting algorithm would
require counting the number of comparison steps, whereas in the case of an algorithm that computes the evaluation of a polynomial at a certain point we may want to count each addition, subtraction and multiplication as a single step.

In general we therefore adapt the convention to equate “step” with “bit operation”: We write all integers in binary code, so we are only working with variables that take the values 0 or 1. We then perform logical operations on these variables: conjunction ($\land$), disjunction ($\lor$) and negation ($\neg$). Each of those operations takes 1 bit.

The running time, or cost of computation, is then the total number of such logical operations performed in an algorithm. It depends on the size of the input.

For example, the operation $a + b$ takes $\lg a + \lg b$ bit operations. However we usually just aim to find an upper bound of the running time, rather than an exact number; we therefore only note that the running time is $O(\lg a + \lg b)$ (where the $O$ is the “Big-Oh-notation”, which should be well-known to the reader).

**Lemma 8.1.1** If we have a sequence of operations in an algorithm, say $P$ and $Q$, then we have that the running time of the algorithm “operation $P$ followed by operation $Q$” is

$$\text{Time}(P; Q) = \text{Time}(P) + \text{Time}(Q).$$

**8.2 Running time of our algorithm**

Lemma 8.1.1 tells us that in order to compute the precise running time of the algorithm, we need to add up the running times required for each individual part of the algorithm.

However, to find an upper bound of the running times, it will be sufficient to find an upper bound for the part of the algorithm that has the largest running time, as stated in the following Lemma:

**Lemma 8.2.1** If $f(x) = O(g(x))$ then $f(x) + g(x) = O(g(x))$.

Let us now go through the steps of Schoof’s Algorithm, and analyse the amount of computations involved in each step. I will use several standard
results for running times without proof; details can be found e.g. in [1].

It is a well-known result that there exists a universal constant \( C \) such that

\[
\prod_{\text{l prime } 2 \leq l \leq l_{\text{max}}} l > C \cdot e^L.
\]

for every \( L > 0 \). For a proof of this, see e.g. [15].

So we can take \( l_{\text{max}} := O(\lg q) \). The number of primes occurring in the product is \( O(\lg q) \) and the primes \( l \) themselves are clearly also \( O(\lg q) \).

Now consider the running time involved in Step (2) of the algorithm. This step clearly requires the largest amount of computation, so that its running time will in the end dominate over the others.

We now need to state some more results from complexity theory.

**Definition 8.2.2** Let \( f(x) \in R[X] \) for some ring \( R \) with \( |R| = p^m \). Define

\[
\lg f := \begin{cases} 
1, & \text{if } f = 0; \\
(1 + \deg(f))\lg |R|, & \text{if } f \neq 0.
\end{cases}
\]

**Lemma 8.2.3** Let \( f, g \) be polynomials in \( R[X] \). Then

1. \( f \pm g \) can be computed with \( O(\lg f + \lg g) \) bit operations.
2. \( f \cdot g \) can be computed using \( O((\lg f)(\lg g)) \) bit operations.
3. Computing the greatest common divisor of \( f \) and \( g \) also requires \( O((\lg f)(\lg g)) \) bit operations.

From Corollary 3.1.1, Part 2 we have that \( \deg(f_l) = O(l^2) \), and from above we know that \( l_{\text{max}} = O(\lg q) \).

Computing the \( H_{k,\tau} \) will involve computing \( x^q, y^q, x^{q^2} \) and \( y^{q^2} \) (reduced modulo the curve equation) modulo \( f_l \).

For \( x^q \) and \( x^{q^2} \), this will require \( O(\lg q) \) multiplications in the ring each - hence \( O((\lg q)^2) \) together. Reducing modulo \( f_l \) takes \( O(l^2) = O((\lg q)^2) \) bit operations, so the computation of \( x^q \) and \( x^{q^2} \) will require \( O((\lg q)^4) \) multiplications in the ring.
Since the order of the ring in our case is $|\mathbb{F}_q| = O(lgq)$, multiplication of any two elements in $R$ takes $O((lgq)^2)$ bit operations.

We therefore need $O((lgq)^6)$ bit operations in total to compute $x^q$ and $x^{q^2}$. For $y^q$ and $y^{q^2}$, the computations are similar and hence their complexity will not affect the asymptotic upper bound.

Now the $x^q$, $y^q$, $x^{q^2}$ and $y^{q^2}$ are computed once for each prime $l$, so $O(lgq)$ times, and then stay the same for each $\tau$. Now $\tau$ is also $O(lgq)$, so we have $O((lgq)^7)$ bit operations for each prime $l$.

Finally, we have $O(lgq)$ primes $l$, so the complexity in the entire Step (2) of Schoof’s Algorithm amounts to $O((lgq)^8)$ bit operations.

This does indeed dominate the computations of both $l_{\text{max}}$ and the Chinese Remainder Theorem in the last step, so we will not have to compute the complexities involved in those (we are not concerned with the latter anyway though, as in our algorithm, we will never get as far as computing the group order!).

In fact, one can make improvements to find a slightly lower “upper bound” for the complexity, but let us finish this dissertation with the conclusion that our algorithm computes the factors of $h(z)$ over $\mathbb{F}_q$ using at most $O((lgq)^8)$ bit operations.

If we convert our “Big-Oh-notation” to a polynomial, we can say that, indeed, the running time of the algorithm is bounded by a polynomial in $logq$, as asserted at the beginning of this dissertation.
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Chapter 9

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L.D.S.

49