HOW TO DISTINGUISH IDENTICAL PARTICLES. THE GENERAL CASE

FEDOR HERBUT

Serbian Academy of Sciences and Arts, Knez Mihajlova 35,
11000 Belgrade, Serbia

fedorh@infosky.net and fedorh@mi.sanu.ac.yu

The many-identical-particle quantum correlations are revisited utilizing the machinery of basic group theory, especially that of the group of permutations. It is done with the purpose to obtain precise definitions of effective distinct particles, and of the limitations involved. Namely, certain restrictions allow one to distinguish identical particles in the general case of $N$ of them, and of $J$ clusters of effectively distinct identical particles, where $N$ and $J$ are arbitrary integers (but $2 \leq J \leq N$). Mutually orthogonal, single-particle distinguishing projectors (events or properties), $J$ of them, are the backbone of the construction. The general results are exemplified by local quantum mechanics, and by the case of nucleons. The former example suits laboratory experiments, and a critical view of it is presented.

Keywords: Identical fermions, identical bosons, clusters of effective distinct particles.

1 Introduction

The inventor of the exclusion principle, Pauli, is reported to have said (private communication by the late R. E. Peierls) that if two electrons are apart, then they are distinct particles by this very fact. His principle applies to those that are not in this relation. De Muynck has pointed to Schiff’s unsuccessful formalization of Pauli’s statement.

Generalizing Pauli, Schiff stipulates that two identical particles are distinguishable when the two-particle probability amplitude $a(1, 2)$ of some
dynamical variable is different from zero only when the two particles have their values in *disjoint* ranges of the spectrum of the variable. But, as de Muynck remarks,¹ this actually *cannot ever occur* when the wave function is (anti)symmetric, for then \[ a(2, 1) = -a(1, 2) \] (for identical bosons and identical fermions respectively).

In the next section it will be shown that the present author’s previous result³ for two identical particles in the form of a suitable theorem resolves this difficulty.

It will also be explained how this theorem incorporates Mirman’s important claim⁴ that distinguishability of identical particles is essentially an experimental notion. Namely, following Jauch,⁵ one can distinguish *intrinsic* and *extrinsic properties* of particles. According to him, identical are those particles that have equal intrinsic properties. But, as de Muynck remarks,¹ ”an intrinsic property may show up dynamical behavior”, and turn out to be extrinsic like the proton and neutron states (cf subsections 5.C and 6.C). It all depends on the experimental conditions.

In Section 3 some mathematical notions required for the intended generalization to \( N \) particles are presented. In Section 4 the mentioned theorem from previous work,³ which makes it clear how one can distinguish two identical particles turning formally extrinsic properties into intrinsic ones, is generalized to an arbitrary number of particles and an arbitrary number of effectively distinct clusters of particles.

The analysis is set against quotations from two standard textbooks on quantum mechanics, that of Messiah,⁶ and that of Cohen-Tannoudji et al.⁷ An attempt is made to show where these textbooks are right and where they lack detail and precision in an important way.

In Section 5 some illustrations are given. Finally, in Section 6 concluding
In *first-quantization* quantum mechanics one has $N$ single-particle state spaces $\{\mathcal{H}_n : n = 1, \ldots, N\}$. The identicalness of the particles is expressed in terms of isomorphisms $\{I_{i\rightarrow j} : i, j = 1, \ldots, N; i \neq j\}$ connecting pairs of single-particle spaces: $\mathcal{H}_j = I_{i\rightarrow j}\mathcal{H}_i$, $i, j = 1, \ldots, N; i \neq j$. Naturally, $I_{i\rightarrow j}I_{j\rightarrow i} = I_j$, $I_j$ being the identity operator in $\mathcal{H}_j$.

As an illustration for the action of the operators $I_{i\rightarrow j}$ we mention that the second-particle radius-vector operator is: $\vec{r}_2 = I_{1\rightarrow 2}\vec{r}_1I_{2\rightarrow 1}$.

The *effective* $N$-distinct-particle space, on which the description of identical particles in first-quantization quantum mechanics is based, is $\mathcal{H}_{1\ldots N} \equiv \prod_{n=1}^{\otimes N} \mathcal{H}_n$, where $\otimes$ denotes the tensor (or direct) product of Hilbert spaces. (We shall use this symbol also for the tensor product of vectors and of operators.)

The basic mathematical tool in the investigation that follows is elementary group theory, in particular, the use of the group $\{p : p \in \mathcal{S}_N\}$ of all permutations $p$ of $N$ objects. (This group is usually called ”the symmetric group”. We will use the usual symbol $\mathcal{S}_N$, but not the term to avoid confusion because it is not the group that is symmetric, but the group is the natural tool to express the identical-particle symmetries of the state of the system.)

To make the reading easier for those theoretical physicists who are not quite familiar with group theory, not even in its elementary form, the requisite group-theoretic machinery is systematized, and to some extent derived, in Appendix A. Proof of Theorems 1 and 2, which are hopefully new and are the main result (but their proof is somewhat more intricate) is relegated to Appendices B and C respectively. Finally, in Appendix D the concept of pos-
session of a property by a system in a state and by an observable is explained.

2 Distinguishability of two identical particles

As it was mentioned, the identical-particle idea rests on distinguishing intrinsic and extrinsic properties of a particle. The former do not enter the quantum-mechanical formalism; they form the physical basis of the single-particle state space (e.g., of that of the electron, which is characterized by mass, charge, spin, gyromagnetic factor etc, as unique intrinsic properties). The extrinsic properties play an important role in the formalism in terms of projectors.

Returning to Schiff’s attempt to formalize a generalization of Pauli’s distinguishing identical particles\(^2\) (cf the Introduction), we imagine that the single-particle observable at issue, if incomplete, is completed by some suitable compatible observables into a complete set (in principle, this is always possible), and that the two-particle amplitude \(a(1,2)\) is the two-particle wave function in the representation of this complete set. Then, we learn from textbooks that \(a(2,1) = \mp a(1,2)\) for identical bosons and identical fermions respectively.

Let the projectors \(E\) and \(F\) correspond to two disjoint regions in the spectrum of the observable at issue. Introducing the orthocomplementary projector \(E^\perp (\equiv 1 - E)\) (which, in this case, projects onto the set-theoretical complementary region in the spectrum), the relation \(E_1 \phi(1) = \phi(1)\), is equivalent to \(E_1^\perp \phi(1) = 0\). Hence, the former relation (as also the latter) means that for particle 1 the observable at issue has positive probability values only in the corresponding region. Analogous statements
hold true for $F$.

Let, further, the index value in $E_i$, $F_i$, $i = 1, 2$ show to which of the particles the extrinsic property applies. Then, the correct way to express Pauli’s criterion of distinguishability is to say that the two-particle system possesses the property $(E_1 F_2 + F_1 E_2)$ (it must be symmetrized): $(E_1 F_2 + F_1 E_2)a(1, 2) = a(1, 2)$. (More on the concept of possession of a property in Appendix D.)

The mathematical results of previous work, in terms of isomorphism and equivalence of relevant operators, then show that the extrinsic properties $E$ and $F$ can be transformed effectively into intrinsic ones by isomorphic transition from the subspace $(E_1 F_2 + F_1 E_2)S_{12}^{s,a}(H_1 \otimes H_2)$ ($S_{12}^{s,a}$ denotes the symmetrizer or the antisymmetrizer) to the effective distinct-particle state space $(E_1 H_1 \otimes F_2 H_2)$. Schiff’s mentioned criterion is actually valid in the latter, distinct-particle space.

Mirman’s claim of the essential role played by experiments shows up in the fact that the mentioned transformation of extrinsic properties into effective intrinsic ones is restricted to experiments in which the possession of the property $(E_1 F_2 + F_1 E_2)$ is preserved.

Thus, a generalized Pauli criterion of distinguishing identical particles can be expressed in the quantum-mechanical formalism quite satisfactorily as far as two identical particles are concerned. The motivation for this article is the belief that generalization to any number of particles and any number of distinguishing properties is desirable. The more so because there are two important examples of effective distinguishing identical particles: that of non-overlapping spatial domains (see subsection 5.B), and the case of nucleons in nuclei (subsections 5.C and 6.C).
3 Identical particles and the maximal-symmetry projectors

Definition 1. One speaks of identical particles if the particles have identical complete sets of intrinsic properties.

This condition has the prerequisite that long experience suggests that one is unable to convert any of the intrinsic properties by dynamical means into extrinsic ones, and that one is unable to extend the set of such properties. These are impotency stipulations analogous to those of thermodynamics on which the thermodynamical principles are based.

Explanation is in order. Some time ago the electron neutrino and the muon neutrino were believed to be identical particles because they had their, up-to-then known, intrinsic properties in common. Later it was discovered that they differ; the former has the electronic leptonic quantum number, and the latter the muonic one. Thus, their other common properties were incomplete; after completion it turned out that they no longer have all intrinsic properties equal.

An illustration for converting an intrinsic property into an extrinsic one is the case of parity and weak interaction. Until the advent of the famous parity-non-conserving weak interaction experiments, parity could be considered an intrinsic property of the elementary particles. These experiments converted it into an extrinsic one, and nowadays we must work with the parity observable with its parity-plus and parity-minus eigen-projectors.

Now, a few remarks of mathematical nature on how one associates the unitary operator representative $P_{1\ldots N}$ in $H_{1\ldots N}$ with a given permutation
If \( p \) is a transposition, transposing, e. g., the state vectors of the first and the second particle, then the corresponding permutation, a so-called *exchange operator*, acts on uncorrelated \( N \)-particles vectors as the operator tensor product

\[
P^{1 \leftrightarrow 2}_{1 \ldots N} \equiv I_{1 \rightarrow 2} \otimes I_{2 \rightarrow 1} \otimes I_3 \otimes \ldots \otimes I_N
\]

(cf A.III.1 in Appendix A). Its action on correlated vectors follows then uniquely as an immediate consequence of requiring linearity and continuity. It is easily seen that this operator takes an uncorrelated basis (induced from the factor spaces) into itself because it amounts to a permutation of its vectors. Hence the operator is unitary.

All transpositions (special permutations) in \( \mathcal{H}_{1 \ldots N} \) are analogously connected with the corresponding isomorphisms \( I_{i \rightarrow j} \), and, as it is well known, all permutations factorize into transpositions. In this manner, all permutations in \( \mathcal{H}_{1 \ldots N} \) can be defined in terms of the isomorphisms \( \{I_{i \rightarrow j} : i, j = 1, \ldots, N; i \neq j\} \) connecting pairs of single-particle state spaces. All permutation operators are unitary because any product of unitary operators is unitary.

It is well known that bosons and fermions differ sharply in some properties (e. g., Bose condensation). But, as it was shown in previous work,\(^3\) as far as distinguishing identical ones of them goes, they behave equally. In this study we extend this result to any number of particles.

To treat bosons and fermions together, we will write \( \text{sign}(p) \), which is, by definition, 1 if one treats bosons, and it equals \((-)^p\), the parity of the permutation (cf A.II.1) if one deals with fermions.

Now we write the *symmetry theorem* in a concise and practical form. For every permutation \( p \in \mathcal{S}_N \) and every state vector \( |\psi\rangle_{1 \ldots N} \) of an \( N \)-identical-particle system the action of the former on the latter amounts to

\[
p \in \mathcal{S}_N.
\]
no more than a possible change of sign as follows:

\[ P_{1...N} |\psi\rangle_{1...N} = \text{sign}(p) |\psi\rangle_{1...N} \]  

(1a).

It is known that the identical-particle symmetry correlations are expressible in terms of maximal-symmetry operators. We define them as projectors. They are \( S^{s,a}_{1...N} \equiv (S^{s}_{1...N} \text{ or } S^{a}_{1...N}) \). The symmetrizer (for identical bosons) is \( S^{s}_{1...N} \equiv (N!)^{-1} \sum_{p \in S_{N}} P_{1...N} \), and the antisymmetrizer (for identical fermions) is \( S^{a}_{1...N} \equiv (N!)^{-1} \sum_{p \in S_{N}} (-1)^{p} P_{1...N} \). Thus, \( S^{s,a}_{1...N} \) is the N-identical-particle maximal-symmetry projector operator. We write

\[ S^{s,a}_{1...N} = (N!)^{-1} \sum_{p \in S_{N}} \text{sign}(p) P_{1...N}. \]  

(2)

Maximal symmetry (boson symmetry or fermion antisymmetry) of a state vector can be expressed, besides by (1a), also (equivalently) by

\[ S^{s,a}_{1...N} \Psi_{1...N} = \Psi_{1...N} \]  

(1b).

It is straightforward to prove this claim.

The geometrical meaning of (1b) is that every physically meaningful state vector is within the subspace \( S^{s,a}_{1...N} \mathcal{H}_{1...N} \) of \( \mathcal{H}_{1...N} \). We call the former the first-principle state space of identical particles.

Obviously, each mixed-or-pure state (density operator) \( \rho_{1...N} \) has its range within the former subspace, or, equivalently, any decomposition of \( \rho_{1...N} \) into pure states results in state vectors from the subspace \( S^{s,a}_{1...N} \mathcal{H}_{1...N} \). In standard language, one speaks of Bose-Einstein statistics if one has bosons (if \( S^{s,a}_{1...N} = S^{s}_{1...N} \)), and of Fermi-Dirac statistics in the case of fermions (when \( S^{s,a}_{1...N} = S^{a}_{1...N} \)).
4 How to obtain distinct particles

In both textbooks Cohen-Tannoudji et al. and Messiah the way how to distinguish identical particles is presented in some detail and fairly correctly (cf pp. 1406-1408 in the former and pp. 600-603 in the latter). For instance, in Messiah (pp. 600-601) one can find the following passage.

"In practice, the electrons of a system are all inside a certain spatial domain $D$, and the dynamical properties in which we are interested all correspond to measurements to be made inside this domain. It turns out that the other electrons may simply be ignored so long as they remain outside $D$ and so long as their interaction with the electrons of the system remain negligible. This is a general result and applies to bosons as well as to fermions. We shall prove it here for the special case of a system of two fermions."

The exposition is restricted to spatial and spin-projector distinctions. But the general procedure of distinguishing is not given, and the precise restrictions involved are not clear. It is the purpose of the two theorems that follow to make up for these deficiencies.

We utilize the powerful tool of projectors. Thus, $Q_D \equiv \int \int \int _{D} \langle \vec{r} | \vec{r} \rangle d^3 \vec{r}$, and $Q_{\text{out}} \equiv \int \int \int _{D^c} \langle \vec{r} | \vec{r} \rangle d^3 \vec{r}$, are the projectors corresponding to the mentioned domain $D$ and to the complementary (in the set-theoretical sense) domain $D^c$ in $\mathbb{R}^3$, which means "outside $D". One should note that $Q_D$ and $Q_{\text{out}}$ are orthogonal: $Q_D Q_{\text{out}} = 0$, and that $Q_{\text{out}} = Q_D^\perp$ ( $Q_D^\perp$ being the projector orthocomplementary to $Q_D$ ). Thus, in the quoted passage, these two projectors distinguish between the electrons that one is interested in and those that one is not.

In the general case, which we are now going to investigate, let the distinguishing properties or events be given by arbitrary $J$ orthogonal single-particle projectors: $\{Q_n^j : j = 1, \ldots, J \} : n = 1, \ldots, N$, $\forall j, \forall n$:
(Q_n^j)^\dagger = Q_n^j \quad \text{(Hermitian operators)}, \quad \forall n : Q_n^j Q_n^{j'} = \delta_{j,j'} Q_n^j \quad \text{(orthogonal projectors)}, \quad \text{and finally,} \quad \forall j : Q_n^j = I_{1\rightarrow n} Q_1^j I_{n\rightarrow 1}, \quad n = 2, \ldots, N \quad \text{(mathematically, equivalent projectors; physically, same properties or events)}.

We have in mind \(J\) clusters of effectively-distinct particles, \(2 \leq J \leq N\). We write them in an ordered way according to the (arbitrarily fixed) values of \(j\): \(j = 1, \ldots, J\). The \(j\)-th cluster contains a certain number of particles, which we denote by \(N_j\), \(\sum_{j=1}^J N_j = N\). It will prove useful to introduce also the sum of particles up to the beginning of the \(j\)-th cluster: \(M_j \equiv \sum_{j'=1}^{(j-1)} N_j'\) for \(j \geq 2\), and \(M_1 \equiv 0\).

The distinguishing projectors appear in \(H_{1\ldots N}\) through the tensor product of distinguishing projectors:

\[
Q_{1\ldots N} = \prod_{j=1}^J \left( \prod_{n=(M_j+1)}^{(M_j+N_j)} Q_n^j \right).
\]

One should note that the last product (in the brackets) applies to the \(j\)-th cluster, and that it multiplies tensorically physically equal (mathematically equivalent via transpositions) single-particle projectors.

We introduce the corresponding effective distinct-cluster space \(H_{1\ldots N}^D\), which is the state space of \(J\) ordered distinct-particle clusters, each consisting of identical particles:

\[
H_{1\ldots N}^D = \left\{ \prod_{j=1}^J \left[ S_{(M_j+1)\ldots(M_j+N_j)}^a \left( \prod_{n=(M_j+1)}^{(M_j+N_j)} Q_n^j \right) \right] \right\} H_{1\ldots N} = \prod_{j=1}^J \left[ S_{(M_j+1)\ldots(M_j+N_j)}^a \left( \prod_{n=(M_j+1)}^{(M_j+N_j)} (Q_n^j H_n) \right) \right] = Q_{1\ldots N} \left( \prod_{j=1}^J S_{(M_j+1)\ldots(M_j+N_j)}^a \right) H_{1\ldots N},
\]

\((3a)\)

\((4a, b, c)\)
where \(a, b, c\) refer to the three expressions of \(\mathcal{H}_{1,...,N}^D\). (and the two operator factors in (4c) commute).

Note that the distinct cluster spaces (factors in the tensor product \(\prod_{j=1}^{\otimes J}j\) in (4a) or (4b)) are decoupled from each other (in the sense of identical-particle symmetry correlations), i.e., one has the tensor product \(\prod_{j=1}^{\otimes J}\), but the factor spaces within each cluster are coupled by the corresponding maximal-symmetry projectors.

On the other hand, there is the symmetrized tensor product of distinguishing projectors in \(\mathcal{H}_{1,...,N}\) determined by (3a) and the permutation operators:

\[
Q_{1,...,N}^{sym} \equiv \left( \sum_{p \in S_N} (P_{1...N}Q_{1...N}P_{1...N}^{-1}) \right) / \prod_{j=1}^{J} (N_j!).
\]

We call it the distinguishing property.

For each term in (3b) there exist \(\left[\prod_{j=1}^{J}(N_j!)\right] - 1\) other terms equal to it (cf (3a)). There are \(\left((N!) / \prod_{j=1}^{J}(N_j!))\right)\) distinct terms in (3b), and they are orthogonal projectors in \(\mathcal{H}_{1,...,N}\). The operator \(Q_{1,...,N}^{sym}\) is a symmetric projector, i.e., one that commutes with every permutation operator \(P_{1...N}\). (Proof of these claims see in A.III.4 and A.III.5.)

The corresponding \(N\)-identical-particle subspace \(\mathcal{H}_{1,...,N}^{Id}\) of \(\mathcal{H}_{1,...,N}\) is defined as the range of \(Q_{1,...,N}^{sym}\) in the first-principle state space:

\[
\mathcal{H}_{1,...,N}^{Id} \equiv Q_{1,...,N}^{sym}S_{1,...,N}^{s,a}\mathcal{H}_{1,...,N} = S_{1,...,N}^{s,a}Q_{1,...,N}^{sym}\mathcal{H}_{1,...,N}.
\]

**Theorem 1.** The subspaces \(\mathcal{H}_{1,...,N}^{Id}\) and \(\mathcal{H}_{1,...,N}^D\) are isomorphic, and the maps

\[
I_{1,...,N}^{Id \rightarrow D} \equiv \left( (N!) / \prod_{j=1}^{J}(N_j!)) \right)^{1/2} Q_{1,...,N}\mathcal{H}_{1,...,N}^{Id},
\]

\[
I_{1,...,N}^{D \rightarrow Id} \equiv \left( (N!) / \prod_{j=1}^{J}(N_j!)) \right)^{1/2} S_{1,...,N}^{s,a}\mathcal{H}_{1,...,N}^{D}.
\]
where $|\ldots\rangle$ denotes the restriction to the corresponding subspace, are mutually inverse unitary isomorphisms mapping $\mathcal{H}^{Id}_{1\ldots N}$ onto $\mathcal{H}^{D}_{1\ldots N}$ and vice versa:

$$\mathcal{H}^{D}_{1\ldots N} = I^{D\rightarrow Id}_{1\ldots N}\mathcal{H}^{Id}_{1\ldots N} \quad \text{and} \quad \mathcal{H}^{Id}_{1\ldots N} = I^{Id\rightarrow D}_{1\ldots N}\mathcal{H}^{D}_{1\ldots N}.$$  

Theorem 1 is proved in Appendix B.

**Definition 2.** In case the state $\rho^{Id}_{1\ldots N}$ of an $N$-identical-particle system satisfies the relation

$$Q^{sym}_{1\ldots N}\rho^{Id}_{1\ldots N} = \rho^{Id}_{1\ldots N}, \quad (8)$$

we say that the system possesses the *distinguishing property* $Q^{sym}_{1\ldots N}$ in the state in question (cf relations (D.2) and (D.3) in Appendix D). In this case, and only in this case, it is amenable to Theorem 1. Since (8) is actually a restriction on the choice of state, we refer to $Q^{sym}_{1\ldots N}$ also as the *restricting property*.

The distinguishing (single-particle) properties $\{Q^{j}: j = 1, \ldots, J\}$ determine the restricting property $Q^{sym}_{1\ldots N}$. This is the backbone of the presented answer to the question "How to distinguish identical particles?".

The physical meaning of the *decoupling* and the *coupling isomorphisms* $I^{Id\rightarrow D}_{1\ldots N}$ and $I^{D\rightarrow Id}_{1\ldots N}$ respectively given in the theorem shows up, of course, in the *observables* that are defined in $\mathcal{H}^{Id}_{1\ldots N}$ and $\mathcal{H}^{D}_{1\ldots N}$. The corresponding or equivalent operators (obtained by the similarity transformation) are of the same kind: Hermitian, unitary, projectors etc. because all these notions are defined in terms of the Hilbert-space structure, which is preserved by the (unitary) isomorphisms. In Theorem 2 (on observables) below, the restricting role of $Q^{sym}_{1\ldots N}$ will be additionally clarified.

It is seen that a prerequisite for describing an evolution or a measurement
in the subspaces \( H^{Id}_{1...N} \) and \( H^{D}_{1...N} \) is the possession of the restricting properties (occurrence of the events) \( Q^{sym}_{1...N} \) and \( Q_{1...N} \) respectively, and their preservation.

As it is clear from (5), a relevant observable for the decoupling, i.e., a Hermitian operator that reduces in \( H^{Id}_{1...N} \), is one that \textit{commutes} with the restricting projector \( Q^{sym}_{1...N} \), and \textit{one confines oneself to its reducee} in \( H^{Id}_{1...N} \) (cf (5)). In physical terms, the observable must be \textit{compatible} with the restricting property \( Q^{sym}_{1...N} \) and one must assume that the property is \textit{possessed} (cf (D.2) and (D.3) in Appendix D), and that this is preserved if some process is at issue.

\textbf{Theorem 2. A)} Let \( A^{D}_{1...N} \) be any Hermitian operator (observable) in \( H_{1...N} \) that commutes (is compatible) both with every permutation associated with the distinct-cluster representation \( \forall p \in G_{D} : [A^{D}_{1...N}, P_{1...N}] = 0 \) (cf property A.III.4) and with \( Q_{1...N} \) (cf (3a)). (Hence, \( A^{D}_{1...N} \) reduces in \( H^{D}_{1...N} \).) Let, further,

\[
A^{D,Q,sym}_{1...N} \equiv \left( \prod_{j=1}^{J} (N_{j}!) \right)^{-1} \sum_{p \in S_{N}} P_{1...N} A^{D}_{1...N} Q_{1...N} P_{1...N}^{-1}
\]

be the symmetrized product \( A^{D}_{1...N} Q_{1...N} \). Then (the symmetric operator) \( A^{D,Q,sym}_{1...N} \) commutes with \( Q^{sym}_{1...N} \) (cf (3b)), and the reducee of \( A^{D}_{1...N} \) in \( H^{d}_{1...N} \) and that of \( A^{D,Q,sym}_{1...N} \) in \( H^{Id}_{1...N} \) respectively are equivalent (physically the same observables) with respect to the isomorphisms in Theorem 1:

\[
A^{D,Q,sym}_{1...N}|_{H^{Id}_{1...N}} = \left( I^{D\rightarrow Id}_{1...N} \right) A^{D}_{1...N} \left( I^{Id\rightarrow D}_{1...N} \right).
\]

(One could write pedantically \( (A^{D}_{1...N}|_{H^{D}_{1...N}}) \) instead of simply \( A^{D}_{1...N} \) in (10a).)
B) Conversely, let $B_{1 \ldots N}^{Id}$ be any (completely) symmetric Hermitian operator (identical-particle observable) in $\mathcal{H}_{1 \ldots N}$ that commutes (is compatible with) $Q_{1 \ldots N}^{\text{sym}}$ (cf (3b)). Then the Hermitian operator (in $\mathcal{H}_{1 \ldots N}$)

$$B_{1 \ldots N}^{D} \equiv \left[ \frac{(N)!}{\left( \prod_{j=1}^{J} (N_j !) \right)} \right] Q_{1 \ldots N} B_{1 \ldots N}^{Id} S_{1 \ldots N}^{s,a} Q_{1 \ldots N}$$

(10b)

commutes with every distinct-cluster permutation $\forall p \in G_{D} : [P_{1 \ldots N}, B_{1 \ldots N}^{D}] = 0$, and with $Q_{1 \ldots N}$. The reducee of $B_{1 \ldots N}^{D}$ in $\mathcal{H}_{1 \ldots N}^{D}$ is equivalent with (physically the same observable as) the reducee of $B_{1 \ldots N}^{Id}$ in $\mathcal{H}_{1 \ldots N}^{Id}$:

$$B_{1 \ldots N}^{D}|_{\mathcal{H}_{1 \ldots N}^{D}} = \left( I_{1 \ldots N}^{Id \rightarrow D} \right) B_{1 \ldots N}^{Id} \left( I_{1 \ldots N}^{D \rightarrow Id} \right).$$

(10c)

Proof of Theorem 2 is given in Appendix C.

The following result is an immediate consequence of Theorem 1 and the two parts of Theorem 2.

**Corollary 1.** If one considers an a priori given operator $B_{1 \ldots N}^{Id}$ as specified in Theorem 2.B, and if one utilizes (10b) to derive $B_{1 \ldots N}^{D}$ from it, and then one takes the symmetrized form

$$B_{1 \ldots N}^{D,\text{sym}} \equiv \left( \prod_{j=1}^{J} (N_j !) \right)^{-1} \sum_{p \in S_{n}} P_{1 \ldots N} B_{1 \ldots N}^{D} P_{1 \ldots N}^{-1}$$

of the latter, then, though, in general, $B_{1 \ldots N}^{Id}$ and $B_{1 \ldots N}^{D,\text{sym}}$ are distinct operators in $\mathcal{H}_{1 \ldots N}$, they have one and the same reducee in $\mathcal{H}_{1 \ldots N}^{Id}$. In this sense, one can consider the latter operator as rewriting the former operator in suitable form (having $\mathcal{H}_{1 \ldots N}^{Id}$ in mind).
Corollary 2. Let \( \rho^{I_d}_{1...N} \) and \( \rho^{D}_{1...N} \equiv I^{I_d\rightarrow D}_{1...N} \rho^{I_d}_{1...N} I^{D\rightarrow I_d}_{1...N} \), its equivalent distinct-cluster state be given. Then

\[
\langle B^{I_d}_{1...N} \rangle_{\rho^{I_d}_{1...N}} \equiv \text{tr}(\rho^{I_d}_{1...N} B^{I_d}_{1...N}) = \text{tr}(\rho^{D}_{1...N} B^{D}_{1...N})
\]  

(11)

(cf Theorem 2.B), i.e., the expectation values are equal.

Proof follows immediately from (10c) and the fact that the maps \( I^{D\rightarrow I_d}_{1...N} \) and \( I^{I_d\rightarrow D}_{1...N} \) are mutually inverse unitary isomorphisms (cf Theorem 1). \( \square \)

One can speak of identical-particle representation (description in \( \mathcal{H}^{I_d}_{1...N} \)) and of distinct-cluster representation (treatment in \( \mathcal{H}^{D}_{1...N} \)) in the sense of Theorems 1, 2 and Corollary 2.

Corollary 3. Let \( U^{I_d}_{1...N} \) be the unitary evolution operator in \( \mathcal{H}_{1...N} \) for some time interval for the \( N \)-identical-particle system under consideration such that \( [U^{I_d}_{1...N} , Q^{\text{sym}}_{1...N}] = 0 \) (cf (3b)), \( \forall p \in \mathcal{S}_N : [U^{I_d}_{1...N}, P_{1...N}] = 0 \), and hence \( [U^{I_d}_{1...N}, S^{s,a}_{1...N}] = 0 \). Let, further, \( \rho^{I_d,i}_{1...N} \) and \( \rho^{I_d,f}_{1...N} \) be the initial and the final \( N \)-identical-particle density operators (physically, states) such that \( Q^{\text{sym}}_{1...N} \rho^{I_d,k}_{1...N} = \rho^{I_d,k}_{1...N}, k = i,f \). Then the evolution \( \rho^{I_d,f}_{1...N} = U^{I_d}_{1...N} \rho^{I_d,i}_{1...N} (U^{I_d}_{1...N})^{-1} \) can be transferred from \( \mathcal{H}^{I_d}_{1...N} \) to \( \mathcal{H}^{D}_{1...N} \) (cf (5) and (4a-c)), i.e., the identical-particle description can be replaced by the distinct-cluster one, to obtain \( \rho^{D,f}_{1...N} = U^{D}_{1...N} \rho^{D,i}_{1...N} (U^{D}_{1...N})^{-1} \). (Naturally, this is due to the unitary nature of the isomorphisms \( I^{I_d\rightarrow D}_{1...N} \) and \( I^{D\rightarrow I_d}_{1...N} \) - cf (6) and (7)).

Finally, as the last layer of quantum mechanical description, we discuss measurement. We confine ourselves to ideal measurement, the one to which
most textbooks of quantum mechanics confine themselves, and where the change of state is given by the Lüders formula.\(^8\)

Let \(A_{1\ldots N}^{id}\) be an \(N\)-identical-particle Hermitian operator (observable) in \(\mathcal{H}_{1\ldots N}\) for which \([A_{1\ldots N}^{id},Q_{1\ldots N}^{sym}] = 0\) (cf (3b)) is valid. Let, further, \(A_{1\ldots N}^{id} = \sum_i a_i E_{1\ldots N}^i\), \(i \neq i' \Rightarrow a_i \neq a_i'\) be its (unique) spectral form. As it is well known, the commutation of \(A_{1\ldots N}^{id}\) both with \(P_{1\ldots N}\), \(\forall p \in S_N\) and with \(Q_{1\ldots N}^{sym}\) is necessarily valid also for each eigen-projector \(E_{1\ldots N}^i\).

**Nonselective** ideal measurement converts any \(N\)-identical-particle state \(\rho_{1\ldots N}\) into the (nonselective) Lüders state

\[
\sum_i (E_{1\ldots N}^i \rho_{1\ldots N} E_{1\ldots N}^i).
\] (12a)

On the other hand, **selective** ideal measurement, in which, e. g., the fixed result \(a_i\) that is detectable, i. e., such that \(\text{tr}(E_{1\ldots N}^i \rho_{1\ldots N}) > 0\), is selected, converts \(\rho_{1\ldots N}\) into the selective Lüders state

\[
E_{1\ldots N}^i \rho_{1\ldots N} E_{1\ldots N}^i / \left(\text{tr}(E_{1\ldots N}^i \rho_{1\ldots N})\right).
\] (13a)

After this elementary introduction, we can reduce the given state changes to the subspace \(\mathcal{H}_{1\ldots N}^{id}\) (cf (5)). To this purpose, we make the assumption that \(Q_{1\ldots N}^{sym} A_{1\ldots N}^{id} \neq 0\), i. e., that the operator \(A_{1\ldots N}^{id}\) has a nonzero reducee in \(\mathcal{H}_{1\ldots N}^{id}\) (physically, a relevant component). Further, we take a state possessing (cf (D.2)) the distinguishing property: \(Q_{1\ldots N}^{sym} \rho_{1\ldots N} = \rho_{1\ldots N}^{id}\). Then the **nonselective Lüders change of state** gives\(^8\)

\[
\sum_i' \left(E_{1\ldots N}^i |_{\mathcal{H}_{1\ldots N}^{id}}\right) \rho_{1\ldots N}^{id} \left(E_{1\ldots N}^i |_{\mathcal{H}_{1\ldots N}^{id}}\right),
\] (12b)

where the prim on the sum denotes that the zero terms are omitted. The **selective Lüders change of state** reads\(^8\)

\[
\forall i, \text{tr}(E_{1\ldots N}^i \rho_{1\ldots N}^{id}) > 0 : \rho_{1\ldots N}^{id} \rightarrow \rho_{1\ldots N}^{id} \rho_{1\ldots N}^{id} / \left(\text{tr}(E_{1\ldots N}^i \rho_{1\ldots N})\right).
\]
The isomorphism \( I_{1\ldots N}^{d \to D} \), mapping \( \mathcal{H}_{1\ldots N}^{Id} \) onto \( \mathcal{H}_{1\ldots N}^{D} \), further converts (12b) and (13b) into

\[
\sum_i E_{i1\ldots N}^{i,D} \rho_{1\ldots N}^{D} E_{i1\ldots N}^{i,D},
\]

(12c)

and

\[
\forall i, \text{tr}(E_{i1\ldots N}^{i,D} \rho_{1\ldots N}^{D}) > 0 : \quad \rho_{1\ldots N}^{D} \xrightarrow{\forall i} E_{i1\ldots N}^{i,D} \rho_{1\ldots N}^{D} E_{i1\ldots N}^{i,D}/(\text{tr}(E_{i1\ldots N}^{i,D} \rho_{1\ldots N}^{D})).
\]

(13c)

respectively. Here

\[
E_{i1\ldots N}^{i,D} = (I_{1\ldots N}^{D \to Id}) E_{i1\ldots N}^{i,D} \mathcal{H}_{1\ldots N}^{Id} (I_{1\ldots N}^{D \to Id}).
\]

(14)

One should note that the isomorphism \( I_{1\ldots N}^{d \to D} \), which enables one to decouple the clusters from each other, applies only to a restricted set of observables. This set is determined by the requirement of compatibility with the distinguishing property \( Q_{1\ldots N}^{sym} \). Overmore, only the reducees in \( \mathcal{H}_{1\ldots N}^{Id} \) (physically, the relevant components) are taken into account for transfer into \( \mathcal{H}_{1\ldots N}^{D} \) (physically, for conversion into the distinct-cluster description).
5 Illustrations

A. Valence electrons
It is well known in quantum molecular physics (also called quantum chemistry) that only the outermost so-called valence electrons, on which the attractive action of the nucleus is relatively weakest, partake in forming the bonds between the atoms to make molecules. Hence, to treat the bonds it is practical to consider the core electrons and the valence ones as distinct particles. The distinguishing properties are defined in terms of the relevant shell-model single-particle states.

B. Non-overlapping spatial domains
Let \( D_e \) be a spatial domain comprising a laboratory on earth, and \( D_m \) an analogous domain on the moon. Let, further, \( D_3 \) be any third spatial domain disjoint with both preceding ones (and these two are, of course, disjoint from each other). The distinguishing projectors are \( Q_i \equiv \int \int \int_{D_i} |\vec{r}\rangle \langle \vec{r}| \) \( d^3\vec{r}, \ i = e, m, 3 \). They are orthogonal due to the disjointness of the domains.

Since all experiments are done in the laboratories (on earth and on moon), the relevant observables satisfy the required restrictions of compatibility with and preservation of the corresponding distinguishing property. Hence, the easier thing, and the thing that is done, is to work in the decoupled space \( \mathcal{H}^D_{1...N} \) and not in the coupled space \( \mathcal{H}^{1d}_{1...N} \). More on this in the critical discussion in subsection 6.D.

C. Nucleons
The single-nucleon state space has three tensor-factor spaces: the orbital (or spatial) one, the spin one, and the isospin one. The single-nucleon distinct-
guishing projectors are the eigen-projectors of $t_z$, the z-projection of isospin, which is completely analogous to the spin-1/2 case. Protons correspond to the eigenvalue $t_z = +1/2$ and neutron to $t_z = -1/2$ respectively. (The projectors are multiplied tensorically by the identity operators in the orbital and in the spin factor space).

When weak interaction does not play a role, i.e., when no $\beta$-radioactivity (converting protons into neutrons and vice versa) is taking place, then the distinguishing property $Q^{sym}_{1...N}$ is possessed by the state $\rho^{Id}_{1...N}$ of the nucleus. Namely, this property physically simply says that there are $N_p$ protons and $N_n$ neutrons in the $N$-nucleonic nuclear state $\rho^{Id}_{1...N}$ ($N = N_p + N_n$; in nuclear physics the notation is $N_p = Z$, $N_n = N$, and $N = A$).

Hence, one can transfer the quantum-mechanical description from the first-principle completely antisymmetric space $S_{1...N}^{sa} \mathcal{H}_{1...N}$ (in which the so-called extended Pauli principle is valid) to the effective distinct-cluster space $\mathcal{H}^{D}_{1...N}$. We have two clusters here, that of protons and that of neutrons. (Some formulae are given in the preceding article for the two-nucleon case, where the deuteron is discussed in some detail.) More about nucleons in subsection 6.C.

6 Concluding remarks

In this final section the essential features of the expounded theory are summed up, and some important special cases are critically discussed.

A. The effective distinct-cluster subspace and its role
The theory is based on $J \left( 2 \leq J \leq N \right)$ orthogonal single-particle pro-
jectors \( \{Q^j_1 : j = 1, 2, \ldots, J\} \), called "distinguishing projectors". They determine the distinct-cluster subspace

\[
\mathcal{H}^D_N \equiv \bigotimes_j \prod_{j=1}^J \left(S^{s,a}_{(M_j+1)\ldots(M_j+N_j)} \left(\prod_{n=(M_j+1)}^{(M_j+N_j)} (Q^j_n \mathcal{H}_n)\right)\right)
\]

(cf (4b)) of the (formal) distinct-particle (most encompassing) space \( \mathcal{H}_{1\ldots N} \equiv \bigotimes_{n=1}^N \mathcal{H}_n \). The subspace \( \mathcal{H}^D_N \) is isomorphic (cf Theorem 1) to the corresponding \( N \)-identical-particle subspace \( \mathcal{H}^{Id}_{1\ldots N} \equiv Q^{sym}_{1\ldots N} \left(S^{s,a}_{1\ldots N} \mathcal{H}_{1\ldots N}\right) \) of the first-principle identical-particle state space \( S^{s,a}_{1\ldots N} \mathcal{H}_{1\ldots N} \) (cf (5)). Here \( Q^{sym}_{1\ldots N} \) is the symmetrized tensor product of the distinguishing projectors (cf (3b) and (3a)), called the distinguishing or the restricting property.

The distinct-cluster subspace \( \mathcal{H}^D_{1\ldots N} \) is relevant only for \( N \)-identical-particle states \( \rho^{Id}_{1\ldots N} \) that possess the distinguishing property \( Q^{sym}_{1\ldots N} \), i.e., that satisfy \( Q^{sym}_{1\ldots N} \rho^{Id}_{1\ldots N} = \rho^{Id}_{1\ldots N} \). This key relation between subspace \( \mathcal{H}^{Id}_{1\ldots N} \) and state \( \rho^{Id}_{1\ldots N} \) can, naturally, be inverted: If an \( N \)-identical-particle state \( \rho_{1\ldots N} \) is given, and one can find \( J \) distinguishing (single-particle) projectors making up an \( (N \)-identical-particle) distinguishing property \( Q^{sym}_{1\ldots N} \) possessed by the state, then the mentioned isomorphism (Theorem 1) becomes relevant. ("Possession" of a given property is explained in Appendix D.)

In case of possession of the distinguishing property by the state \( \rho^{Id}_{1\ldots N} \), the mentioned isomorphism \( \mathcal{H}^{Id}_{1\ldots N} \rightarrow \mathcal{H}^D_{1\ldots N} \) enables one to make a transition from \( \rho^{Id}_{1\ldots N} \) to a corresponding effective \( J \)-distinct-cluster state, which is denoted by \( \rho^{D}_{1\ldots N} \). Theorem 2 endows this transition with the physical meaning of genuine (though only effective) distinct-particle clusters at the price of a serious restriction: it is valid only for some \( N \)-identical-particle Hermitian operators \( A_{1\ldots N} \) in \( \mathcal{H}_{1\ldots N} \), those that possess the distinguishing property \( Q^{sym}_{1\ldots N} \) (cf Appendix D): they are Hermitian operators that com-
mute with the distinguishing projector $Q_{1\ldots N}^{\text{sym}}$ (physically, that are compatible with the distinguishing property), and of which subsequently the reducee $A_{1\ldots N}|_{\mathcal{H}_{1\ldots N}^{\text{d}}}$ in $\mathcal{H}_{1\ldots N}^{\text{id}}$ (physically, the relevant component) is taken.

It is important to emphasize that the effective distinct-cluster description (in the effective state space $\mathcal{H}_{1\ldots N}^{\text{d}}$, cf (4a-c)) is not an approximation (as effective particles often are); for the observables that possess the distinguishing property the description is exact, and for those that do not possess it, it does not make sense.

B) Converting extrinsic properties into intrinsic ones

As it was stated in Definition 1, the notion of identical particles rests on the idea of equal intrinsic properties of the particles. One can view the theory expounded as the general framework how to convert some extrinsic properties, represented by nontrivial projectors in the single-particle state space, into intrinsic ones. The converted extrinsic properties are the distinguishing projectors $\{Q_j^l : j = 1, 2, \ldots, J\}$. In the effective distinct-cluster space $\mathcal{H}_{1\ldots N}^{\text{d}}$ these properties become intrinsic (cf (4b)).

C) The reverse algorithm: converting intrinsic properties into extrinsic ones

Sometimes the reverse conversion of intrinsic properties into extrinsic ones takes place. For this algorithm the same conceptual framework can be used as for the direct conversion (see the preceding section). The theory presented in this article covers also this case.

The best example is that of protons and neutrons (cf subsection 5.C and
If being a proton or a neutron is considered as an intrinsic property of the particle, then the state \( \rho_{D_{1\ldots N}} \) of the nucleus is a density operator in the two-distinct-cluster space \( \mathcal{H}_{1\ldots N}^{D} \equiv \left( S_{1\ldots N_p}^{a} \prod_{n=1}^{N_p} \mathcal{H}_{n}^{p} \right) \otimes \left( S_{1\ldots N_n}^{a} \prod_{n=1}^{N_n} \mathcal{H}_{n}^{n} \right) \), where \( \mathcal{H}_{n}^{p} \) and \( \mathcal{H}_{n}^{n} \) are the space of the \( n \)-th single proton and the \( n \)-th single neutron. (The suffix \( n \), which denotes the neutron, should not be confused with the index \( n \).

When weak interaction (or \( \beta \)-radioactivity) is taken into account, the single-particle spaces \( \mathcal{H}_{n}^{p} \) and \( \mathcal{H}_{n}^{n} \) have to be replaced by a doubly dimensional nucleonic space of the \( n \)-th particle \( \mathcal{H}_{n} \equiv Q_{n}^{p} \mathcal{H}_{n} \oplus Q_{n}^{n} \mathcal{H}_{n} \), where \( Q_{n}^{p} \) and \( Q_{n}^{n} \) are the proton and the neutron projectors respectively (cf subsection 5.C), and \( \oplus \) denotes the orthogonal sum of subspaces. The first-principle \( N \)-identical-nucleon space is then \( S_{1\ldots N}^{s,a} \mathcal{H}_{1\ldots N} \). \( Q_{n}^{p} \) and \( Q_{n}^{n} \) are the distinguishing projectors, and the symmetrized \( N \)-identical-nucleon projector \( Q_{1\ldots N} \equiv \left( \prod_{n=1}^{N_p} Q_{n}^{p} \right) \otimes \left( \prod_{n=(N_p+1)}^{(N_p+N_n)} Q_{n}^{n} \right) \) is the distinguishing property (cf (3a) and (3b)). The corresponding two-distinct-cluster space is now rewritten as

\[
\mathcal{H}_{1\ldots N}^{D} \equiv \left( S_{1\ldots N_p}^{a} \prod_{n=1}^{N_p} Q_{n}^{p} \mathcal{H}_{n} \right) \otimes \left( S_{(N_p+1)\ldots N}^{a} \prod_{n=(N_p+1)}^{N_n} Q_{n}^{n} \mathcal{H}_{n} \right),
\]

where \( Q_{n}^{p} \mathcal{H}_{n} = \mathcal{H}_{n}^{p} \) and \( Q_{n}^{n} \mathcal{H}_{n} = \mathcal{H}_{n}^{n} \) are the \( n \)-th single-proton and single-neutron spaces respectively (cf subsection 5.C).

The reverse process at issue consists in transferring the quantum-mechanical description from \( \mathcal{H}_{1\ldots N}^{D} \) to the subspace \( \mathcal{H}_{1\ldots N}^{id} \) of the first-principle space \( S_{1\ldots N}^{a} \mathcal{H}_{1\ldots N} \). Inclusion of \( \beta \)-radioactivity requires the use of the latter space because that of the former does not suffice.

Perhaps additional light is shed on the reverse application if the ex-
pounded theory by discussing a fictitious case. Suppose we want to treat the proton (p) and the electron (e) as two states of a single particle (like the proton and the neutron). Can we do this? The answer is affirmative, and the way to do it is to use the theory of this article in the, above explained, reverse direction.

The new first-particle space would be \( \mathcal{H}_1 \equiv Q_p \mathcal{H}_1 \oplus Q_e \mathcal{H}_1 \), where \( Q_p \) and \( Q_e \) project \( \mathcal{H}_1 \) onto the proton and the electron subspace respectively. The rest is analogous as in case of the nucleon above with the important difference that there is no counterpart of the proton-state- or neutron-state-property non-conserving weak interaction. This means that every \( N \)-particle state \( \rho_{1...N}^{ld} \) possesses the distinguishing property, and can never lose it. Hence, the corresponding distinct-cluster space \( \mathcal{H}_1^{D_{1...N}} \) will always do for description, and we are back to permanently distinct particles.

D) A critical view of local quantum mechanics

It was stated in subsection 6.A that the expounded theory is not an approximate one; in some states and some processes it is valid exactly, and in others not at all. In some cases approximation is nevertheless present (in a different sense). Local quantum mechanics is one of them.

For the description of, e. g., electrons in an earth laboratory (cf subsection 5.B), the unsymmetrized distinguishing property (cf (3a)) is

\[
Q_{1...N} \equiv \left( \prod_{n=1}^{N_e} Q_n^e \right) \otimes \left( \prod_{n=(N_e+1)}^{N} Q_n^{c \perp} \right) \tag{16}
\]

(cf (3a)), and \( Q_{1...N}^{sym} \) is its symmetrized form (cf (3b)). In (16) \( N \) and \( N_e \) are the number of all electrons in the universe and that of all electrons in the earth domain \( \mathcal{D}_e \). The projectors \( Q_n^e \) and its orthocomplementary
$Q^e_n$ map the space $\mathcal{H}_n$ onto the domain $\mathcal{D}_e$ and out of it respectively.

The corresponding distinct-cluster space is

$$\mathcal{H}^D_{1\ldots N} \equiv \left( S^a_{1\ldots N_e} \prod_{n=1}^{N_e} Q^e_n \mathcal{H}_n \right) \otimes \left( S^a_{(N_e+1)\ldots N} \prod_{n=(N_e+1)}^{N} Q^e_n \mathcal{H}_n \right).$$

(17)

Since the two tensor factors in (17) are ipso facto decoupled, one can restrict the description to the first factor $\mathcal{H}^{1d}_{1\ldots N_e} \equiv S^a_{1\ldots N_e} \left( \prod_{n=1}^{N_e} Q^e_n \mathcal{H}_n \right)$ as far as Hermitian operators (observables) acting in this space are concerned. Quantum-mechanical description in this space we call local quantum mechanics).

One wonders if there is anything wrong with this. The answer is "yes". We give two arguments against the exactness of local quantum mechanics.

(i) We can imagine classically that every electron is either on earth (in $\mathcal{D}_e$), or outside it. But quantum-mechanically this is not so. In any realistic state there are delocalized electrons, which, put in a simplified way, are in a state of superposition of being on earth and being outside it. Thus, the above distinguishing property is not possessed in an exact way.

This is where approximation enters the scene. We approximate the above realistic state by a state $\rho^{1d}_{1\ldots N}$ that possesses the distinguishing property $Q^{sym}_{1\ldots N}$ determined by (16) (cf (3a) and (3b)), and we apply the presented theory to it. All electrons that are involved in laboratory experiments are certainly on earth; the delocalized ones do not participate. Hence the replacement of the exact $\rho_{1\ldots N}$ by the approximate $\rho^{1d}_{1\ldots N}$ is believed to be a good approximation.

(ii) When the orbital (or spatial) tensor-factor space of a single particle is determined by the basic set of observables, which are the position, the linear
momentum, and their functions, one obtains an irreducible space, i.e., a space that has no non-trivial subspace invariant simultaneously for all the basic observables (for position and linear momentum; cf sections 5 and 6 in chapter VIII of Messiah’s book[^6]). Hence, the above used subspace \( \mathcal{Q}_1^e \mathcal{H}_1 \) (for the local, earth quantum-mechanical description) is not invariant either. It is, of course, invariant for position, but linear momentum has to be replaced by another Hermitian operator approximating it.

**Appendix A.**

**The mathematics required**

**A.I General properties of groups and group representations used in the article.**

**General property A.I.1** Multiplication of all elements \( g \) of a group \( \mathcal{G} \) from the left (or from the right) maps \( \mathcal{G} \) in a one-to-one way onto itself. (The claim is easily proved.)

**General property A.I.2** Conjugation of all elements \( g \) of a group \( \mathcal{G} \) by any fixed element \( g' \) from the group maps \( \mathcal{G} \) in a one-to-one way onto itself. Hence, \( \{ g' g (g')^{-1} : g \in \mathcal{G} \} = \mathcal{G} \). (An immediate consequence of Property A.I.1.)

**General property A.I.3** If \( \mathcal{G}' \) is a proper finite subgroup of a group \( \mathcal{G} \), \( g' \in \mathcal{G} \) is an element outside \( \mathcal{G}' \) that leaves \( \mathcal{G}' \) by conjugation invariant, then the conjugation maps \( \mathcal{G}' \) in a one-to-one way onto itself. (Immediate
consequence of the easily proved one-to-one mapping and the finiteness of \( G' \).

**General property A.I.4** Let \( \{ x : x \in X \} \) be a set, and let \( G \) be a group of transformations acting in \( X \). Further, let \( \{ f[x] : \forall f \} \) be the set of all single-valued mappings \( f \) of \( X \) into some set \( Y \) (\( f[x] \in Y \)). Then \( \hat{g}f[x] \equiv f[g^{-1}(x)] \) induces a representation of \( G \) in the set of functions. One should note that this means that the map \( \forall g \in G : g \Rightarrow \hat{g} \) is a homomorphism, i.e., it preserves the product of two factors due to
\[
\hat{g}_1\hat{g}_2f[x] = \hat{g}_1[f[g_2^{-1}(x)] = f[g_2^{-1}(g_1^{-1}(x))] = f[(g_1g_2)^{-1}(x)] = (\hat{g}_1\hat{g}_2)f[x],
\]
and it maps the unit element \( e \) into the unit element: \( \hat{e}f[x] \equiv f[ex] = f[x] \).

**General property A.I.5** Let \( G \) be a finite group of \( D \) elements and \( G' \) its subgroup of \( d \) elements. Subsets of the form \( gG' \equiv \{ gg' : g' \in G' \} \) (\( G'g \), \( g \in G \), are called left (right) cosets of the subgroup. Let \( \{g_k : k = 1, 2, \ldots, (D/d)\} \) be elements of \( G \), one from each left coset, and symmetrically \( \{g'_k : k = 1, 2, \ldots, (D/d)\} \) from the right cosets. Then
\[
\sum_{k=1}^{D/d} g_k G' = G = \sum_{k=1}^{D/d} G' g'_k \tag{A.1a, b}
\]
are the left-coset and the right-coset (set-theoretical) decompositions of \( G \) into non-overlapping sets or classes. (The symbol \( \sum \) denotes the union of disjoint sets. The left and the right cosets coincide if and only if the subgroup is an invariant one.) Each coset contains \( d \) elements, and one can take \( g_{k=1} = g'_{k=1} = e \). The first cosets equal \( G' \). (Property A.I.5 is a direct consequence of A.I.1.)
General property A.I.6 Taking the inverse element is an anti-isomorphism (isomorphism with transposing the factors) of any group $\mathcal{G}$ onto itself. (Proof is straightforward.)

A.II Properties of the group $S_N$ of all permutations $p$ of $N$ objects used in the article.

Property A.II.1 Each permutation $p \in S_N$ can be factorized into transpositions, in general, non-uniquely, but the number of factors is either even in all factorizations or odd. This property of a permutation $p$ is unique, it is written as $(-)^p$, and it is called ”parity” of the permutation. It is by definition $+1$ if the number of factors is even, and it is $-1$ in case the number is odd. (See basic ideas about groups in Hamermesh or in Messiah.6)

Property A.II.2 Parity $(-)^p$ of the permutation is a homomorphism of $S_N$ into the multiplicative group $\{+1, -1\}$: $\forall p, p' \in S_N : (-)^{pp'} = (-)^p(-)^{p'}$. The unit element $e$ has parity $+1$. $\forall p \in S_N : (-)^p = (-)^{p^{-1}}$ (as follows from $pp^{-1} = e$).

Property A.II.3 The group $S_N$ has only two one-dimensional representations: the so-called identical (or symmetric) one ($\forall p \Rightarrow 1$) and the antisymmetric one ($\forall p \Rightarrow (-)^p$) (p. 1117 there).

Property A.II.4 The group $S_N$ has one and only one invariant subgroup $\mathcal{G}'$ : the group of even permutations $\mathcal{G}' \equiv \{p : p \in S_N, (-)^p = 1\}$ (see Messiah’s textbook, p. 1111 there).
Property A.II.5 Let us define for all $p \in S_N$ $\text{sign}(p) \equiv +1$ if one deals with bosons, and $\text{sign}(p) \equiv (-)^p$ in case of fermions. Further, let $G' \subseteq S_N$ be any (proper or improper) subgroup of $S_N$. Let $G'$ have $d$ elements. Finally, we define $S_{1\ldots N}^{G',s,a} \equiv d^{-1} \sum_{p \in G} \text{sign}(p) P_{1\ldots N}$ (cf A.II.6). One has

$$\forall p \in G': \quad \left( \text{sign}(p) P_{1\ldots N} \right) S_{1\ldots N}^{G',s,a} = S_{1\ldots N}^{G',s,a} = S_{1\ldots N}^{G',s,a} \left( \text{sign}(p) P_{1\ldots N} \right).$$

(A.2a, b)

(It is a direct consequence of A.I.1 and A.II.2.)

Property A.II.6 The operator $S_{1\ldots N}^{G',s,a}$ (cf A.II.5) is a projector.

(This is so because

$$(S_{1\ldots N}^{G',s,a})^\dagger = d^{-1} \sum_{p \in G} \text{sign}(p) P_{1\ldots N}^\dagger = d^{-1} \sum_{p \in G} \text{sign}(p) P_{1\ldots N}^{-1} =$$

$$d^{-1} \sum_{p^{-1} \in G} \text{sign}(p^{-1}) P_{1\ldots N}^{-1} = S_{1\ldots N}^{G',s,a}$$

- cf I.6 and II.2. Further,

$$(S_{1\ldots N}^{G',s,a})^2 = d^{-1} \sum_{p \in G} \text{sign}(p) P_{1\ldots N} S_{1\ldots N}^{G',s,a} = d^{-1} d S_{1\ldots N}^{G',s,a} = S_{1\ldots N}^{G',s,a}$$

- cf (A.2a).)

Property A.II.7 The symmetry projector $S_{1\ldots N}^{G',s,a}$ of any subgroup $G'$ of $S_N$ (cf A.II.5 and A.II.6)) satisfies

$$S_{1\ldots N}^{G',s,a} S_{1\ldots N}^{s,a} = S_{1\ldots N}^{s,a} = S_{1\ldots N}^{s,a} S_{1\ldots N}^{G',s,a},$$

(A.3a, b)

where

$$S_{1\ldots N}^{s,a} \equiv S_{1\ldots N}^{S_N,s,a} = (N!)^{-1} \sum_{p \in S_N} \text{sign}(p) P_{1\ldots N}. \quad (A.4)$$
(Namely, on account of (A.2.a), one can write
\[ S_{1...N}^{G',s,a}S_{1...N}^{s,a} = d^{-1} \sum_{p \in G'} \text{sign}(p)P_{1...N}^{s,a} = d^{-1} \sum_{p \in G'} S_{1...N}^{s,a} = (d^{-1})dS_{1...N}^{s,a} = S_{1...N}^{s,a}, \]
where \( d \) is the order of (the number of elements in) \( G' \). The symmetrical argument utilizing (A.2.b) leads to (A.3.b).)

A.III Basic properties of the representations of \( S_N \) in \( \mathcal{H}_{1...N} \) that are used in the article.

**Property A.III.1** An uncorrelated vector \( \prod_{n=1}^{\otimes} |\psi_n\rangle \) in \( \mathcal{H}_{1...N} \) consists of the choice of the \( N \) vectors \( |\psi_n\rangle \) from the single-particle space (the \( N \) spaces \( \mathcal{H}_n \) can be considered as one thanks to the identicalness of the particles - see the Introduction), and of tensor multiplication (the second index enumerates the factors). The choice can be understood as a map of the set \( \{1, \ldots, N\} \) into the single-particle space. Hence we are dealing with \( f[x] \) (cf A.I.4), and we can apply the procedure of induction specified in A.I.4 to obtain a representation. Thus
\[ \forall p \in S_N : \quad P_{1...N}^{\otimes}(\prod_{n=1}^{\otimes} |\psi_n\rangle) = \prod_{n=1}^{\otimes} |\psi_{p^{-1}(n)}\rangle. \quad (A.5) \]
(A.5) is a definition of the operators \( P_{1...N} \), it is easily seen that it coincides with that given in Section 3.

**Property A.III.2** Let \( \prod_{n=1}^{\otimes} O_n^n \) be the tensor product of any \( N \) single-particle operators \( \{O^n : n = 1, \ldots, N\} \). Then
\[ \forall p \in S_N : \quad P_{1...N}^{\otimes}(\prod_{n=1}^{\otimes} O_n^n)P_{1...N}^{-1} = (\prod_{n=1}^{\otimes} O_{p^{-1}(n)}^n). \quad (A.6) \]
Property A.III.3 Let $\bigotimes_{n=1}^{N} O_n$ be the tensor product of $N$ single-particle operators (with some possibly equal) $\{O^n : n = 1, \ldots, N\}$. Then (cf (A.4)):

$$S_{s,a}^{s,a}(\bigotimes_{n=1}^{N} O_n) S_{1\ldots N}^{s,a} = (N!)^{-1} S_{1\ldots N}^{s,a} \left( \sum_{p \in S_N} (\bigotimes_{n=1}^{N} O_n) P_{1\ldots N}^{-1} \right) S_{1\ldots N}^{s,a}. \quad (A.7)$$

(This is so because, making use of II.5 and II.2, one can write

$$\forall p \in S_N : \text{lhs} = S_{1\ldots N}^{s,a} P_{1\ldots N} (\bigotimes_{n=1}^{N} O_n) P_{1\ldots N}^{-1} S_{1\ldots N}^{s,a}.$$ 

Adding this up for all $N!$ permutations, (A.7) ensues.)

Property A.III.4 We now write down a set-theoretical decomposition that is relevant for the distinct-cluster space $\mathcal{H}_{1\ldots N}^{D}$ (cf (4a-c)).

Let $N = \sum_{j=1}^{J} N_j$, $\forall j : 1 \leq N_j \leq N$ be a decomposition of the natural $N$ into $J$ naturals. Further, let $\forall j, j \geq 2 : M_j \equiv \sum_{j'=1}^{j-1} N_{j'}$; and $M_{j=1} \equiv 0$. Finally, let

$$\{1, 2, \ldots, N\} = \sum_{j=1}^{J} \{(M_j + 1), (M_j + 2), \ldots, (M_j + N_j)\} \quad (A.8)$$

be a set-theoretical decomposition into classes (non-overlapping subsets - hence the union is replaced by $\sum$) each containing $N_j$ successive naturals. (The natural $M_j$ is the number of naturals that precede the $j$-th class.)

Let, further, $j(n)$ denote the class to which $n$ belongs. We assume that $N$ first-particle operators $\{O_1^n : n = 1, \ldots, N\}$ are given, and that
those and only those that belong to one and the same class in the decomposition (A.8) are equal. Hence, the index \( j \) enumerates also the distinct operators, and one can rewrite the set of operators as \( \{O_j^{i(n)} : n = 1, \ldots, N\} \).

The \( N \)-particle operator (in \( \mathcal{H}_{1...N} \))

\[
O^{\text{sym}}_{1...N} \equiv \left( \prod_{j=1}^{J} (N_j!) \right)^{-1} \sum_{p \in S_N} P_{1...N} \left( \prod_{n=1}^{\otimes N} O_n^{i(n)} \right) P_{1...N}^{-1}
\]  

(A.9)

is the symmetrized form of \( \prod_{n=1}^{\otimes N} O_n^{i(n)} \). It is symmetric, i.e. \( \forall p \in S_N : [O^{\text{sym}}_{1...N}, P_{1...N}] = 0 \iff P_{1...N}O^{\text{sym}}_{1...N}P_{1...N}^{-1} = O^{\text{sym}}_{1...N} \). (The latter relation for \( O^{\text{sym}}_{1...N} \) follows from I.1.)

The operator \( O^{\text{sym}}_{1...N} \) consists of \( (N!)/\prod_{j=1}^{J} (N_j!) \) distinct tensor-product terms. To prove this claim, we define \( G_D \) as the subgroup of \( S_N \) containing all permutations that leave each class in decomposition (A.8) invariant:

\[
\forall p \in G_D, \forall n : \text{ if } n \in \{(M_j + 1), \ldots, (M_j + N_j)\} \Rightarrow p(n) \in \{(M_j + 1), \ldots, (M_j + N_j)\}.
\]

Taking into account (A.1.a) (\( G \equiv S_N, \ G' \equiv G_D \)), (A.6), and using the notations \( D \equiv (N!), \ d \equiv \prod_{j=1}^{J} (N_j!) \), the definition (A.9) leads to

\[
O^{\text{sym}}_{1...N} = d^{-1} \sum_{p \in G_D} \sum_{k=1}^{D/d} \prod_{n=1}^{\otimes N} O_n^{j[p_k^{-1}]}(n) =
\]

\[
d^{-1} \sum_{p \in G_D} \sum_{k=1}^{D/d} \prod_{n=1}^{\otimes N} O_n^{j[p_k^{-1}(p_k^{-1})]}.
\]

Further, since the permutation \( p^{-1} \) (\( p \in G_D \)), does not change the \( j \) value of \( p_k^{-1}(n) \), one further has

\[
O^{\text{sym}}_{1...N} = (d^{-1}d) \sum_{k=1}^{D/d} \prod_{n=1}^{\otimes N} O_n^{j(p_k^{-1}(n))} = \sum_{k=1}^{D/d} \prod_{n=1}^{\otimes N} O_n^{j(p_k^{-1}(n))}.
\]
Finally, arguing *ab contrario*, we take $k \neq k'$ and assume that the corresponding terms are equal: $\forall n : O_n^{j[p_k^{-1}(n)]} = O_n^{j[p_{k'}^{-1}(n)]}$. Then, $\forall n : j[p_k^{-1}(n)] = j[p_{k'}^{-1}(n)]$. Hence, $p_k p_k^{-1} \equiv p \in \mathcal{G}_D \iff p_k' = p p_k$ in contradiction with (A.1.a).

**Property A.III.5** Let the distinct operators $O^j$ in the preceding property be *orthogonal single-particle projectors* $\{Q^j : j = 1, \ldots, J\}$. Then $Q_{1\ldots N}^{sym}$ (cf (A.9)) is a symmetric projector in $\mathcal{H}_{1\ldots N}$, which consists of $(\frac{(N!)^J}{\prod_{j=1}^J(N_j!)}$ orthogonal projector terms.

(This is due to the fact that now “distinct” means orthogonal on the single-particle level, hence the product of any two distinct tensor-product projector terms in $Q_{1\ldots N}^{sym}$ multiplies for some value of $n$ orthogonal single-particle projectors giving zero.)

**Property A.III.6** Let $O^D_{1\ldots N}$ be an operator in $\mathcal{H}_{1\ldots N}$ that commutes with all distinct-cluster permutations $\forall p \in \mathcal{G}_D : [O^D_{1\ldots N}, P_{1\ldots N}] = 0$ (cf property A.III.4). Then:

$$S_{1\ldots N}^{s,a} O^D_{1\ldots N} S_{1\ldots N}^{s,a} = \left[\left(\prod_{j=1}^J (N_j!)/((N!)^J)\right)\right] S_{1\ldots N}^{s,a} O_{1\ldots N}^{D, sym} S_{1\ldots N}^{s,a}, \quad (A.10)$$

where

$$O_{1\ldots N}^{D, sym} \equiv \left(\prod_{j=1}^J (N_j!)^{-1}\right) \sum_{p \in \mathcal{S}_N} P_{1\ldots N} (O_{1\ldots N}^D) P_{1\ldots N}^{-1} \quad (A.11)$$

is the symmetrized form of $O^D_{1\ldots N}$.

(This is so because, making use of A.II.5 and A.II.2, one can write

$$\forall p \in \mathcal{S}_N : \text{lhs}(A.10) = S_{1\ldots N}^{s,a} P_{1\ldots N} (O_{1\ldots N}^D) P_{1\ldots N}^{-1} S_{1\ldots N}^{s,a}.$$ 

Adding this up for all $(N!)$ permutations, (A.10) is obtained.) The operator $O_{1\ldots N}^{D, sym}$ is symmetric. (This is so because $\forall p' \in \mathcal{S}_N : P'_{1\ldots N} O_{1\ldots N}^{D, sym} (P'_{1\ldots N})^{-1} =$
Appendix B

Now a proof of Theorem 1 is presented.

To begin with, we omit the restriction sign \( | \) wherever the restriction is anyway fulfilled, and we prove that \( I_{1...N}^{d} \mathcal{H}_{1...N}^{d} \subset \mathcal{H}_{1...N} \). One has (cf (3a) and (3b)):

\[
Q_{1...N} Q_{1...N}^{sym} = Q_{1...N} = Q_{1...N} Q_{1...N}, \quad (B.1a, b)
\]
due to orthogonality of \( Q_{1...N} \) to all the \([N!] / \left( \prod_{j=1}^{J} (N_j) \right)\] distinct projector terms in \( Q_{1...N}^{sym} \) except to the term \( Q_{1...N} \) itself (cf A.III.5).

Also, one has

\[
\left( \prod_{j=1}^{J} S_{s,a}^{(M_j+1)...(M_j+N_j)} \right) S_{1...N}^{s,a} = S_{1...N}^{s,a} \quad (B.1c)
\]
(cf (A.3.a)).

Hence, in view of (6), (5), (B.1a), and (B.1c), \( I_{1...N}^{d} \mathcal{H}_{1...N}^{d} = Q_{1...N} \mathcal{H}_{1...N}^{d} = \left[ Q_{1...N} \left( \prod_{j=1}^{J} S_{s,a}^{(M_j+1)...(M_j+N_j)} \right) \right] S_{1...N}^{s,a} \mathcal{H}_{1...N} \). In view of (4c), this is a subspace of the space \( \mathcal{H}_{1...N}^{d} \) because \( (S_{1...N}^{s,a} \mathcal{H}_{1...N}) \) is a subspace of \( \mathcal{H}_{1...N} \).

Next, we show the reverse claim that the operator \( I_{1...N}^{d} \) (given by (7)) takes the subspace \( \mathcal{H}_{1...N}^{d} \) into \( \mathcal{H}_{1...N}^{d} \). Utilizing (4c) and (B.1b), one has

\[
I_{1...N}^{d} \mathcal{H}_{1...N}^{d} = S_{1...N}^{s,a} Q_{1...N} \left( \prod_{j=1}^{J} S_{s,a}^{(M_j+1)...(M_j+N_j)} \right) \mathcal{H}_{1...N} = \]

\[
S_{1...N}^{s,a} Q_{1...N}^{sym} \left( \prod_{j=1}^{J} S_{s,a}^{(M_j+1)...(M_j+N_j)} \right) \mathcal{H}_{1...N}.
\]

Since \( \left[ Q_{1...N} \left( \prod_{j=1}^{J} S_{s,a}^{(M_j+1)...(M_j+N_j)} \right) \right] \mathcal{H}_{1...N} \) is a subspace of \( \mathcal{H}_{1...N} \), we have ended up with a subspace of \( \mathcal{H}_{1...N}^{d} \) (cf (5)).
Next, we show that the maps $I_{1\ldots N}^{D\to Id}$ and $I_{1\ldots N}^{D\to Id}$ in application to the subspaces $\mathcal{H}_{1\ldots N}^{Id}$ and to $\mathcal{H}_{1\ldots N}^{D}$ respectively are each other's inverse.

Owing to the definitions (7) and (6), and to the definition (A.4) of $S_{1\ldots N}^{s,a}$, one has the following equality of maps:

$$I_{1\ldots N}^{D\to Id} I_{1\ldots N}^{D\to Id} = \left\{ \left( (N!) / \prod_{j=1}^{J} (N_j !) \right)^{1/2} Q_{1\ldots N} \left( (N!) / \prod_{j=1}^{J} (N_j !) \right)^{1/2} S_{1\ldots N}^{s,a} \right\} |_{\mathcal{H}_{1\ldots N}^{D}} =$$

$$\left\{ \left( \prod_{j=1}^{J} (N_j !) \right)^{-1} \sum_{p \in S_N} \text{sign}(p) Q_{1\ldots N} P_{1\ldots N} \right\} |_{\mathcal{H}_{1\ldots N}^{D}}.$$

On account of (4c), we can further write

$$\text{lhs} = \left\{ \left( \prod_{j=1}^{J} (N_j !) \right)^{-1} \sum_{p \in S_N} \text{sign}(p) Q_{1\ldots N} P_{1\ldots N} Q_{1\ldots N} \right\} |_{\mathcal{H}_{1\ldots N}^{D}} =$$

$$\left\{ \left( \prod_{j=1}^{J} (N_j !) \right)^{-1} \sum_{p \in S_N} \text{sign}(p) Q_{1\ldots N} P_{1\ldots N}^{-1} Q_{1\ldots N} P_{1\ldots N} \right\} |_{\mathcal{H}_{1\ldots N}^{D}}.$$

All $\left( P_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1} \right)$ multiply with $Q_{1\ldots N}$ into zero except when $p \in \mathcal{G}_{D}$ (cf A.III.4). For the permutation operators from this subgroup one has, $P_{1\ldots N} = \prod_{j=1}^{J} P_{(M_j+1)\ldots(M_j+N_j)}$, corresponding (in the sense of homomorphism) to $p \in \mathcal{G}_{D}: p = \prod_{j=1}^{J} p_j$, where $p_j$ permutes possibly non-trivially only in the $j$-th class, in the rest of the classes it permutes trivially (cf (A.8)). Utilizing this, the idempotency of $Q_{1\ldots N}$, and the fact that $Q_{1\ldots N} \left( \prod_{j=1}^{J} S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} \right)$ acts on $\mathcal{H}_{1\ldots N}^{D}$ as the identity operator, we can, further, write

$$\text{lhs} = \left\{ \left( \prod_{j=1}^{J} (N_j !) \right)^{-1} \times \right.$$

$$\sum_{p \in \mathcal{G}_{D}} \left[ Q_{1\ldots N} \left( \text{sign}(p) \prod_{j=1}^{J} P_{(M_j+1)\ldots(M_j+N_j)} \right) Q_{1\ldots N} \left( \prod_{j=1}^{J} S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} \right) \right] \right\} |_{\mathcal{H}_{1\ldots N}^{D}}.$$

$$(B.2)$$
Further, one has \( \text{sign}(p) = \prod_{j=1}^{J} \text{sign}(p_j) \) for \( p \in \mathcal{G}_D \) (cf A.II.2), and, in accordance with (A.2.a), for each of the \( J \) class factors

\[
\text{sign}(p_j)P_{(M_j+1)\ldots(M_j+N_j)} \left( \otimes_{n=(M_j+1)}^{(M_j+N_j)} Q_n^j \right) S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} = \\
\left( \otimes_{n=(M_j+1)}^{(M_j+N_j)} Q_n^j \right) S_{(M_j+1)\ldots(M_j+N_j)}^{s,a}
\]

is valid. Further, using (3a) and (A.2a) once more, one obtains

\[
\left( \otimes_{j=1}^{J} \text{sign}(p_j)P_{(M_j+1)\ldots(M_j+N_j)} \right) Q_{1\ldots N} \left( \otimes_{j=1}^{J} S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} \right) = \\
Q_{1\ldots N} \left( \otimes_{j=1}^{J} S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} \right).
\]

Substituting (B.3) in (B.2), and recognizing that the sum \( \sum_{p \in \mathcal{G}_D} \) adds up precisely \( \left( \prod_{j=1}^{J} (N_j!) \right) \) equal terms, we finally have

\[
\text{lhs} = \left\{ Q_{1\ldots N} \left( \otimes_{j=1}^{J} S_{(M_j+1)\ldots(M_j+N_j)}^{s,a} \right) \right\} |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}}^{|\mathcal{H}_1^{d\rightarrow Id}} = 1 |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}}.
\]

This establishes the claim that \( I_{1\ldots N}^{Id\rightarrow Id} \) is the inverse of \( I_{1\ldots N}^{Id\rightarrow Id} \).

Analogously, in view of (7), (6), and (A.4), we have the following equality of maps:

\[
I_{1\ldots N}^{Id\rightarrow Id} I_{1\ldots N}^{Id\rightarrow Id} = \left\{ \left( (N!) / \prod_{j=1}^{J} (N_j!) \right)^{1/2} S_{1\ldots N}^{s,a} \right\} |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}} = \\
\left\{ \left( \prod_{j=1}^{J} (N_j!) \right)^{-1} \sum_{p \in S_N} \left( P_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1} \right) \text{sign}(p) P_{1\ldots N} \right\} |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}}.
\]

In view of (5), in \( \mathcal{H}_1^{Id \ldots N} \) \( S_{1\ldots N}^{s,a} \) acts as the identity operator. Therefore, taking into account (A.2.a) and (3b), one further obtains

\[
\text{lhs} = \left\{ \left( \prod_{j=1}^{J} (N_j!) \right)^{-1} \sum_{p \in S_N} \left( P_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1} \right) \text{sign}(p) P_{1\ldots N} S_{1\ldots N}^{s,a} \right\} |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}} = \\
\sum_{p \in S_N} \left( P_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1} \right) \text{sign}(p) P_{1\ldots N} S_{1\ldots N}^{s,a} |_{\mathcal{H}_1^{d\rightarrow Id \ldots N}}.
\]
\[ \{Q_{1\cdots N}^{sym}S_{1\cdots N}^{s,a}\}_{H_{1\cdots N}^{d}} = 1_{H_{1\cdots N}^{d}}. \]

Thus, the claim that the two maps are the inverse of each other is proved.

Since the maps are the inverse of each other, it is easily seen that they are necessarily surjections and injections, i.e., bijections as claimed.

Next, we prove that \( I_{1\cdots N}^{D \to Id} \) preserves the scalar product, which we write as \((\ldots,\ldots)\). Let \( \Psi_{1\cdots N} \) and \( \Phi_{1\cdots N} \) be two arbitrary elements of \( H_{1\cdots N}^{D} \). On account of (7) and the Hermiticity and the idempotency of \( S_{s,a}^{1\cdots N} \) (cf (A.4) and A.II.6), one has

\[ \left( I_{1\cdots N}^{D \to Id} \Psi_{1\cdots N}, I_{1\cdots N}^{D \to Id} \Phi_{1\cdots N} \right) = \\
\left( \left( (N)! \prod_{j=1}^{J} (N_{j}!) \right)^{1/2} \Psi_{1\cdots N}, \left( (N)! \prod_{j=1}^{J} (N_{j}!) \right)^{1/2} S_{1\cdots N}^{s,a} \Phi_{1\cdots N} \right) = \\
\left[ (N)! \prod_{j=1}^{J} (N_{j}!) \right] \left( \Psi_{1\cdots N}, S_{1\cdots N}^{s,a} \Phi_{1\cdots N} \right). \]

Further, on account of (A.4) and the fact that \( Q_{1\cdots N} \) acts as the identity operator on \( \Psi_{1\cdots N} \) and \( \Phi_{1\cdots N} \), one can apply it to both and write

\[ \text{lhs} = \left( \prod_{j=1}^{J} (N_{j}!) \right)^{-1} \sum_{p \in \mathcal{G}_{D}} \text{sign}(p) \left( \Psi_{1\cdots N}, Q_{1\cdots N}(P_{1\cdots N}Q_{1\cdots N}P_{1\cdots N}^{-1})P_{1\cdots N} \Phi_{1\cdots N} \right). \]

One has \( Q_{1\cdots N}(P_{1\cdots N}Q_{1\cdots N}P_{1\cdots N}^{-1}) = 0 \), except if \( p \in \mathcal{G}_{D} \), when it is equal to \( Q_{1\cdots N} \). If \( p \in \mathcal{G}_{D} \), then \( p \) permutes only within the classes, hence the corresponding permutation operator \( P_{1\cdots N} \) commutes with \( Q_{1\cdots N} \) (cf (3a)). Also \( \Phi_{1\cdots N} = \left( \prod_{j=1}^{J} (S_{(M_{j}+1)\cdots(M_{j}+N_{j})}) \right) \Phi_{1\cdots N} \) (cf (4c)) is valid, and for \( p \in \mathcal{G}_{D} \), \( \text{sign}(p)P_{1\cdots N} \left( \prod_{j=1}^{J} (S_{(M_{j}+1)\cdots(M_{j}+N_{j})}) \right) = \prod_{j=1}^{J} (S_{(M_{j}+1)\cdots(M_{j}+N_{j})}) \) (cf A.II.5). Therefore,

\[ \text{lhs} = \left( \prod_{j=1}^{J} (N_{j}!) \right)^{-1} \sum_{p \in \mathcal{G}_{D}} \left( \Psi_{1\cdots N}, \Phi_{1\cdots N} \right) = \left( \Psi_{1\cdots N}, \Phi_{1\cdots N} \right). \]
It is easy to see that also the inverse of a scalar-product preserving bijection must be scalar-product preserving. This completes the proof of Theorem 1.

Appendix C

We prove now Theorem 2.

A) To prove that the operator

$$A^{D,Q,\text{sym}}_{1\ldots N} \equiv \left( \prod_{j=1}^{J} (N_j!) \right)^{-1} \sum_{p \in S_N} P_{1\ldots N} A^{D}_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1}$$

(cf (9)) commutes with \( Q^{\text{sym}}_{1\ldots N} \), we utilize the idempotency of \( Q_{1\ldots N} \) and its commutation with \( A^{D}_{1\ldots N} \), and we rewrite \( A^{D,Q,\text{sym}}_{1\ldots N} \) in the form

$$A^{D,Q,\text{sym}}_{1\ldots N} = \left( \prod_{j=1}^{J} (N_j!) \right)^{-1} \sum_{p \in S_N} P_{1\ldots N} Q_{1\ldots N} A^{D}_{1\ldots N} Q_{1\ldots N} P_{1\ldots N}^{-1}.$$ Then, on account of the facts that \( Q^{\text{sym}}_{1\ldots N} \) is symmetric and that \( Q^{\text{sym}}_{1\ldots N} Q_{1\ldots N} = Q_{1\ldots N} Q^{\text{sym}}_{1\ldots N} = Q_{1\ldots N} \) (cf (B.1a,b)), the claimed commutation becomes obvious.

Next, to prove (10a), we start with its rhs and we utilize (7) and (6):

$$\text{rhs}(10a) \equiv \left( I^{D\rightarrow Id}_{1\ldots N} \right) A^{D}_{1\ldots N} \left( I^{Id\rightarrow D}_{1\ldots N} \right) =$$

$$\left[ (N!) / \left( \prod_{j=1}^{J} (N_j!) \right) \right] \left[ S^{s,a}_{1\ldots N} A^{D}_{1\ldots N} Q_{1\ldots N} S^{s,a}_{1\ldots N} \right] |_{\mathcal{H}^{Id}_{1\ldots N}}.$$ Since \( S^{s,a}_{1\ldots N} \) acts as the identity operator in \( \mathcal{H}^{Id}_{1\ldots N} \), one further has

$$\text{rhs}(10a) = \left[ (N!) / \left( \prod_{j=1}^{J} (N_j!) \right) \right] \left[ S^{s,a}_{1\ldots N} A^{D}_{1\ldots N} Q_{1\ldots N} S^{s,a}_{1\ldots N} \right] |_{\mathcal{H}^{Id}_{1\ldots N}}.$$ Finally, on account of (A.10),

$$\text{rhs}(10a) = \left[ S^{s,a}_{1\ldots N} A^{D,Q,\text{sym}}_{1\ldots N} S^{s,a}_{1\ldots N} \right] |_{\mathcal{H}^{Id}_{1\ldots N}} = A^{D,Q,\text{sym}}_{1\ldots N} |_{\mathcal{H}^{Id}_{1\ldots N}} \equiv \text{lhs}(10a).$$
B) It is obvious that the operator

\[ B_{1\ldots N}^D \equiv \left[ (N!) \left/ \left( \prod_{j=1}^{J} (N_j !) \right) \right. \right] Q_{1\ldots N} B_{1\ldots N}^{Id} S_{1\ldots N}^s a Q_{1\ldots N} \]

(cf (10b)) commutes both with every distinct-cluster permutation and with \( Q_{1\ldots N} \).

Proof of (10c) is straightforward: Utilizing the definitions (6) and (7) of the isomorphisms, one obtains

\[ \text{rhs}(10c) = \left[ (N!) \left/ \left( \prod_{j=1}^{J} (N_j !) \right) \right. \right] Q_{1\ldots N} B_{1\ldots N}^{Id} S_{1\ldots N}^s a \left| H_{D} \right. \]

Since in \( H_{D} \) \( Q_{1\ldots N} \) acts as the identity operator, a glance at (10b) establishes the claim.

Appendix D

The mathematical relation

\[ A\rho = a\rho, \quad (D.1) \]

where \( A \) is a Hermitian operator, \( \rho \) a density operator, and \( a \) a real number, implies that \( a \) belongs to the eigenvalues of \( A \), and that \( \rho \) is a state in which the system has the eigenvalue \( a \) of \( A \). This can be seen by substituting a spectral form \( \rho = \sum_i r_i |\psi_i\rangle \langle \psi_i| \) in (D.1), and by multiplying scalarly the obtained relation from the right by one of the eigenvectors of \( \rho \), e. g., by \( |\psi_i\rangle \), that corresponds to a positive eigenvalue \( r_i \). Then \( A |\psi_i\rangle = a |\psi_i\rangle \) ensues. (It is easy to extend the argument to any decomposition of \( \rho \) into pure states by expanding these in the complete eigenbasis, and
by taking into account that expansion coefficients along eigenvectors of \( \rho \) corresponding to zero are zero.)

If the observable \( A \) is a property (projector) \( F \), then one has the following special case of (D.1):

\[
F \rho = \rho. \tag{D.2}
\]

It has the physical meaning that the system in the state \( \rho \) possesses the property \( F \). Namely, the probability of \( F \) in \( \rho \) is 1, i.e., in suitable measurement, the event \( F \) necessarily occurs (the property \( F \) is necessarily obtained) in the state \( \rho \).

It is immediately seen (by adjoining) that (D.2) implies

\[
[\rho, F] = 0 \tag{D.3}
\]

with the physical meaning of compatibility of state and property.

Relation (D.3) expresses a weaker property of the state \( \rho \) than relation (D.2). If only the former is valid, then \( F \rho F = F \rho \) satisfies also the latter relation.

Besides (D.1), there is another mathematical generalization of (D.2) relevant for the investigation in this article:

\[
FA = A, \tag{D.4}
\]

where \( F \) is a projector, and \( A \) is a Hermitian operator (an observable). Like above, (D.4) implies the (weaker) compatibility relation

\[
[F, A] = 0. \tag{D.5}
\]

Conversely, if a Hermitian operator \( B \) satisfies only (D.5) mutatis mutandis, i.e., if \( [F, B] = 0 \), then for the Hermitian operator (observable) \( FB \)
the stronger relation (D.4) is valid mutatis mutandis:

\[ F(FB) = (FB) \]  \hspace{1cm} (D.6)

(as obvious due to the idempotency of \( F \)).

If \( A = \sum_i a_i P_i, \ i \neq i' \Rightarrow a_i \neq a_{i'} \) is the (unique) spectral decomposition of \( A \), then (D.4) implies

\[ \forall i, a_i \neq 0 : FP_i = P_i. \]  \hspace{1cm} (D.7a)

(This is seen by substituting the spectral form of \( A \) on both sides of (D.4), and by multiplying subsequently the relation by \( P_i \) - with a fixed value of \( i \). Taking into account that (D.5) is, as well known, equivalent to \( \forall i : [F, P_i] = 0 \), the relation \( a_i FP_i = a_i P_i \) ensues.)

As well known, (D.7a) is symbolically written as

\[ \forall i, a_i \neq 0 : P_i \leq F \]  \hspace{1cm} (D.7b)

with the physical meaning that if the event \( P_i \) occurs, i.e., the result \( a_i \neq 0 \) is obtained in a measurement of the observable \( A \), then necessarily also the event \( F \) occurs (or the property \( F \) is valid).

For the null projector of a Hermitian operator \( A \) that satisfies (D.4), i.e., for the eigen-projector of the latter corresponding to the eigenvalue zero, only the weaker condition of compatibility with \( F \) is valid. Hence, as it is easy to see, instead of (D.7b) one has

\[ a_{i_0} = 0 \Rightarrow P_{i_0} = FP_{i_0} + F^\perp, \]  \hspace{1cm} (D.7c)

where \( F^\perp \equiv 1 - F \).

In the sense of (D.7b) and (D.7c), one can give (D.4) the physical interpretation that the observable \( A \) possesses the property \( F \).
It is perhaps interesting to realize that geometrically (D.4) means that the (topologically closed) range \( \bar{\mathcal{R}}(A) \) of \( A \) is entirely within that of \( F : \bar{\mathcal{R}}(A) \subseteq \mathcal{R}(F) \). In more detail, \( \forall i, a_i \neq 0 : \mathcal{R}(P_i) \subseteq \mathcal{R}(F) \), and for \( a_{i_0} = 0, \mathcal{R}(FP_{i_0}) \subseteq \mathcal{R}(F), \mathcal{R}(F^\perp P_{i_0}) = \mathcal{R}(F^\perp) \) (cf (D.7c)).

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