A curious model of the Higgs field: a complex scalar finite field meets the Monster symmetry

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Abstract

As a small step toward understanding the nature of the Higgs field, a curious observation is reported that calculates the mass of the Higgs boson with a 99.7 \% agreement to its experimentally measured value. Particularly, motivated by the desire to avoid infinities, a complex scalar finite field that aims to describe the Higgs field is proposed as the underlying field of an extremely huge finite group of Lie type. Then, the occurrence of a critical point of the phase transition in that finite field is explained, and the mass of its bosons is computed employing the Klein-Gordon equation. Next, in a backward analysis, to curiously estimate magnificence of that finite group of Lie type, the mass of Higgs boson is inputted to the obtained mass relation, and a huge order close to the order of the Monster group is outputted. Supported by such a clue as well as the fact that the Monster group is critical among the sporadic groups, it is therefore assumed that the critical order of the corresponding symmetry group (of Lie type), at which its underlying finite field reaches the critical point of a phase transition, is the order of the Monster group. Based on this assumption, the main observation is reported as \( \frac{m_H}{m_P} \simeq D \frac{m_e^*}{m_2} \), where \( M_H, M_P, m_q, m_2 \), and \( D \), are respectively, the mass of Higgs boson, the reduced Planck mass, the mass of bosons of the finite field, the mass of field with two elements, and the number of spatial dimensions which perfectly match the central charge \( (i.e., \text{number of degrees of freedom}) \) in the Monster conformal field theory.

\textbf{Keywords:} Finite field, Higgs field, Klein-Gordon equation, Phase transition, Monster group

1. Motivation

In the progress of the quantum field theory, the continuum assumption of the infinite number of elements (that is also referred to as degrees of freedom, microstates, or points) for a physical field, was a source of undesirable divergences and unphysical results. “These problems have stopped the development of quantum field theory for almost twenty years and were ‘apparently’ resolved with the introduction of renormalization. The interpretation given to this procedure has evolved over the decades. In recent years, it has become customary to regard continuum field theories as approximations to more fundamental theories. This justifies the use of a cut-off: a lattice spacing, or some other kind of regularization that effectively suppresses the degrees of freedom associated with very small distances” \cite{1}.

Such a cut-off, if exists, is profoundly entangled into the structure of the universe. The current letter is, then, an attempt to explore such a fundamental feature of nature via assuming a field with an extremely large but finite number of elements that play the role of infinity. Like the traditional assumption, which considers the field as a limit of a system with \( q \) elements constrained in a lattice where the continuum obtains as \( q \) approaches ‘infinity’, let us assume that the real physics obtains as \( q \) approaches such an enormously large (cut-off) number of elements. A very property of such a huge \( q \) is its criticality. It is critical since it is a maximum, acting like infinity in the finite realm. Hence, let denote it by \( q_c \). Notably, these elements are associated with the symmetries of a system. During the emergence of a system, due to several symmetry-breaking processes, \( q \) evolves accordingly. Excluding the accidental symmetries, the emergence of \( q \) is often increasing in time. As an instance, let imagine that \( q \) emerges

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like 2, 3, ..., $q_\epsilon - 1$, $q_\epsilon$, $q_\epsilon$, ... in a certain system. In such a system, two different eras are distinguishable; one era before $q$ reaches $q_\epsilon$, and one afterward. If the behavior of the system differs in these two eras, they may be two separate phases of the system and the point of $q = q_\epsilon$ may consequently indicate a critical point of the phase transition. In this case, the “order parameter”, that is typically zero above the transition point (here $q_\epsilon$) and becomes nonzero below it, can be defined proportional to a power of $|q_\epsilon - q|$ [2][3].

Now, let assess three consequences of assuming a finite number of elements for a field:

I. The symmetries of the system would be discrete instead of continuous.

Discreteness is the straightforward consequence of a finite number of symmetries. However, the existence of $q_\epsilon$ implies more interesting consequences. Remarkably, if such a critical point exists in a system, as it plays the role of infinity, one can assume approximately continuous symmetries for the post-transition epoch. Particularly, at the critical point and afterward, numerous points produce an extremely dense distribution implying the existence of an element in the infinitesimal vicinity of each element, i.e., an (approximate) continuous symmetry. It is also a direct requirement of the lattice renormalization group (RG) theory of the phase transition, that a continuum limit of a lattice theory must occur at its phase transition point. Meaning that the study of a continuum limit of a lattice theory is the analysis of critical phenomena [4]. As a result, the theorems addressing the continuous symmetries, like Goldstone’s theorem, would be applicable in the transition and post-transition period wherein the numerous points approximately produce the continuum.

II. The corresponding infinite symmetry group would be replaced by a very huge finite group.

As the starting point of our interest, let consider the Higgs field that in the Standard Model of elementary particles is defined as a complex scalar infinite field of $SU(2)$ symmetry transformation [5]. The Lie group $SU(2)$, which is an infinite group, therefore, should be replaced by its finite version, the Steinberg group $SU_2(q^2)$ of Lie type. In this way, the underlying finite field of $SU_2(q^2)$ plays the role of the finite complex scalar (Higgs) field.

Notably, the order (i.e., number of elements) of $SU_2(q^2)$ is $q^3 - q$, where $q$ is a prime power representing the order of its underlying finite field $F_q$. The elements of $F_q$, that are the roots of unity and are all lying on the unit circle, are given by $F_q = \exp \left( \frac{2\pi i}{q} n \right)$, where $n = 0, 1, 2, ..., q - 1$ [6]. This form is comparable with the continuous spatial form $\exp(i\theta)$ in the infinite fields. E.g., recall the Higgs field that is believed to have an infinite number of degenerate minima (vacuum states) given by $\phi = \phi_0 \exp(i\theta)$, for any real $\theta$ between 0 and $2\pi$, where $\phi_0$ is a real constant [5]. Such a comparison between a finite field and an infinite one is shown in table 1.

| Field            | $\phi(\chi)$                          | $F(n)$                          |
|------------------|---------------------------------------|---------------------------------|
| No. of microstates | $\infty$                             | $q$                             |
| Spatial Form     | $\phi_0 e^{i\theta}$ (x)             | $F_0 e^{\frac{2\pi i n}{q}}$   |
|                  | $0 \leq \theta < 2\pi$              | $n = 0, 1, 2, ..., q - 1$       |
| Symmetry group   | $SU(2)$                               | $SU_2(q^2)$                     |
| Order of the symmetry group | $\infty$ | $q^3 - q$ |

Table 1
III. A phase transition can be imagined due to an extremely dense distribution of the elements

Inspired by the ‘mechanism’ of phase transition of the Lee-Yang type, one can also imagine a similar mechanism for the abovementioned phase transition in a finite field. According to the Lee-Yang theory, in the infinite-size limit of a finite-size system, when the complex zeros of the partition function become numerous and may become dense or condense along a certain arc, a phase transition can be triggered. Particularly, in the original Ising model considered by Lee and Yang, with the change of variable \( \rho = e^{\pi i} \), the Lee-Yang theorem implies that all complex zeros, \( \rho \), lie on the unit circle [7]. In the same manner, when the elements of \( F_q \) become numerous, they become highly dense along the unit circle and a phase transition is conceivable.

2. Method

A. Calculating the mass of quanta of a complex scalar finite field

Let consider a discrete complex scalar finite field whose periodic spatial distribution of elements is given by

\[
F(n) = F_0 \exp \left( \frac{2\pi i}{q} n \right), \quad n = 0, 1, 2, ..., q - 1, \quad \text{and} \quad q \quad \text{a prime power, and a real number.}
\]

As \( F \) is scalar and its quanta (bosons) are spinless, it obeys the Klein-Gordon equation. In the reduced Planck units where \( c = \hbar = 1 \), the Klein-Gordon equation for an infinite continuous scalar field \( \varphi \) is \((\partial^2 - \nabla^2)\varphi = m^2 \varphi \), where \( m \) is the mass of bosons of \( \varphi \), and \( \nabla^2 \) is the continuous Laplace operator. So, for a discrete complex scalar field \( F \), the Klein-Gordon equation becomes

\[
(\partial_t^2 - \Delta)F = m_q^2 F
\]

Where, \( m_q \) is the mass of bosons of \( F \) and \( \Delta \) is the discrete Laplace operator. The one-dimensional plane-wave Ansatz \( F(n, t) = F_0 \exp \left( \frac{2\pi i}{q} n - \omega t \right) \) is a solution of the equation (1) provided the dispersion (i.e., the energy-momentum) relation [5]

\[
\omega^2 - \left( \frac{2\pi i}{q} \right)^2 = m_q^2
\]

The existence of a critical point of the phase transition at \( q = q_c \) divide the history of the system into two distinguished eras, i.e., an emerging pre-transition era and a stable post-transition era. To the best of our knowledge, in the emerging era, a series of symmetry-breakings, in which \( q \) symmetries of the initial state of the system gradually become hidden, increases the entropy of the system and brings it to such a critical point. The equilibrium state (of maximum entropy) is then reached, and the system becomes stable. In such a stable epoch, where \( q = q_c \), equation (1) is time-independent, and equation (2) becomes

\[
- \left( \frac{2\pi i}{q_c} \right)^2 = m_{q_c}^2
\]

This result could also be inferred from the assumption of \( \omega \ll \frac{2\pi}{q} \) in equation (2). Taking the square roots of the equation (3) leads to

\[
|m_{q_c}| = \frac{2\pi}{q_c}
\]

The inverse relationship between \( |m_{q_c}| \) and \( q_c \) was imaginable. It is well-known that, near a critical point of a phase transition, the (thermodynamic) properties of the system can be described by a few parameters which in the present model is, currently, the number of microstates (further explanation is provided in the first part of the
A minimum amount of information is needed to determine the state of the system. That is, a microcanonical ensemble wherein the probability distribution of the microstates is uniform. In the uniform distribution, the probability of each state, and its expected value, is the inverse of the total number of microstates confirming the inverse relationship between $q_c$ and the expected value of mass-energy.

**B. Calculating an estimation of the critical orders**

The second consequence of assuming a finite number of elements addresses the last couple of rows in table 1. It is stated that the corresponding symmetry group of the field would be a huge finite group instead of an infinite one. In this part, the aim is to curiously estimate how huge that finite group would be if its underlying finite field describes the Higgs field. To this aim, the estimation of $q_c$ is required. $q_c$ can be calculated from equation (5) if the value of $m_{q_c}$ is known. Thus, temporarily, let assume that $F$ describes the Higgs field to borrow the known value of the mass of Higgs boson for $m_{q_c}$. That is, let assume $m_{q_c} = m_H$, where $m_H$ is the mass of Higgs boson in the reduced Planck units (i.e., $\frac{M_H}{M_P}$ where $M_P$ is the reduced Planck mass and $M_H$ is the mass of Higgs boson). According to the 2012 discovery of the Higgs boson at LHC, the mass of Higgs boson is reported to be about 125.1 Gev/$c^2$ [8]. Noting that the reduced Planck mass is $2.435 \times 10^{-17}$ Gev/$c^2$,

$$m_{q_c} = m_H = \frac{M_H}{M_P} = \frac{125.1 \text{ Gev}/c^2}{2.435 \times 10^{18} \text{ Gev}/c^2} \approx 5.137 \times 10^{-17}$$

(5)

Then, equation (4) implies

$$q_c = \frac{2\pi}{m_{q_c}} \approx 1.223 \times 10^{17}$$

(6)

As it was expected, intriguingly, it is a huge number of microstates implicating an extremely dense distribution, that is capable of a phase transition. Having an estimation of $q_c$, now, the order of the corresponding symmetry group $SU_2(q_c^2)$ can be estimated.

$$|SU_2(q_c^2)| = q_c^3 - q_c \approx 1.83 \times 10^{51}$$

(7)

Strikingly, this order is close to the order of the Monster group ($\approx 8 \times 10^{53}$) [9]. This clue leads us to the main observation in the next section.

| Sporadic group | $M_{11}$ | $M_{12}$ | ... | Monster |
|---------------|---------|---------|-----|---------|
| $|\text{Sporadic group}|$ | $\approx 8 \times 10^3$ | $\approx 9.5 \times 10^4$ | ... | $\approx 8 \times 10^{53}$ |

Table 2

| $q$ | 2 | 3 | ... | $q_c$ |
|-----|---|---|-----|------|
| $|SU_2(q^2)|$ | 6 | 24 | ... | $1.83 \times 10^{51}$ |

Table 3
3. Main observation

In this section, the main observation is presented. On one hand, the huge order of $|SU_2(q^\ast)|$ estimated in (7) is close to the order of the Monster group (i.e., $\approx 8 \times 10^{53}$). On the other hand, the Monster group is critical as it is the largest among 26 sporadic groups [9]. Comparing such a criticality (maximality), which is depicted in table 2, with that of $q_\ast$, which is depicted in table 3, as well as the above-mentioned closeness of the orders, one may naturally conjecture that there may be a hidden ‘relation’ between the order of the Monster group and the order of the proposed finite group of the Lie type. Let, curiously, propose that such a relation exists, and it is simply equality. In other words, let propose that the critical order of $SU_2(q^\ast)$, at which its underlying finite field reaches the critical point of a phase transition, is the order of the Monster group. That is,

$$|SU_2(q^\ast)| = |\text{Monster group}|$$ (8)

Where, $q^\ast$ indicates the critical $q$ associated with the order of the Monster group.

Next, (8) implies,

$$q^2 \sim q^\ast \approx 8.08 \times 10^{53}$$ (9)

Then, $q^\ast$ is estimated as $q^\ast \approx 9.314 \times 10^{17}$. Since $q^\ast$ should be a prime power, let select $q^\ast = 9650949592 \approx 9.314 \times 10^{17}$, which is the closest prime power to the solution of equation (9). Now, having $q^\ast$, the mass of bosons of the underlying finite field with $q^\ast$ elements can similarly be computed from the equation (4)

$$|m_q| = \frac{2\pi}{q^\ast} \approx 6.746 \times 10^{-18}$$ (10)

Then, the first form of our main observation is

$$m_H \approx 24 \frac{m_q}{\pi}$$ (11)

As already mentioned, $m_H = M_H/M_p$, where $M_p$ is the reduced Planck mass and $M_H$ is the mass of Higgs boson. Such a mass ratio on the LHS of (11) delicately suggests an existence of a mass ratio on the RHS. Interestingly, extrapolating (3) to $q = 2$, yields $\lim_{q \rightarrow 2} m_q = \pi$, which is nicely appeared in the denominator of the RHS, confirming such a suggestion. More precisely, employing the momentum-energy relation of the equation (3) for the field with 2 elements, $2 \ll q_e$ implies $\pi \gg \frac{2\pi}{q_e} \gg \omega$, that results in $m_2 = \pi$. Replacing $\pi$ by $m_2$, modifies the equation (11) to

$$\frac{M_H}{M_p} \approx 24 \frac{m_q}{m_2}$$ (12)

This observation recommends two correspondences. First, between $m_q$ and $M_H$. Second, between $M_p$ and $m_2$. The first one is obvious, as the finite field with $q^\ast$ elements aims to describe the Higgs field. The latter also makes sense because both $M_p$ and $m_2$ belong to the blocks or units of their realm: $m_2$ is the mass of the smallest irreducible finite field, and $M_p$ is the unit of mass in the reduced Planck units (additional explanation is provided in the last part of the discussion section).

The equation (12) can be modified too. To understand the role of 24, let explore the physical models with the Monster symmetry in the literature. The only reputed physical model with the Monster symmetry is the FLM (Frenkel, Lepowsky, and Meurman) model, dubbed the Monster conformal field theory (CFT). In 1984, FLM constructed a model of a holomorphic CFT with the Monster symmetry and the central charge 24, whose partition
function is the modular invariant J-function [10]. In the context of the CFT, the central charge counts the number of degrees of freedom (DOF) [1]. Not wishing to look a gift horse in the mouth, the Monster CFT suggests that 24 appeared in the equation (12) equals the number of DOF. It perfectly matches our calculations, because the Laplacian in the Klein-Gordon equation (1) was written for only one spatial DOF. If one rewrites the equation (3) for $D$ dimensions of space with the same wavenumbers, then 

$$-D \left( \frac{2\pi c}{q} \right)^2 = m_q^2,$$

that results in $m_{q_c} = D \frac{2\pi c}{q_c}$, where $D$ is the number of spatial DOF. Hence, representing 24 as the number of spatial DOF (or the central charge in the CFT context) modifies the equation (12) to the main observation of this letter

$$\frac{M_H}{M_p} \cong D \frac{m_{q^*}}{m_2}$$

(13)

Furthermore, the general relation $m = q \Phi$ in the quantum field theory [5], where $\Phi$ and $q$ are the amplitude and the charge respectively, also confirms that 24 in the equation (12) can be a charge parameter.

Finally, before starting the discussions, it is worthwhile to compute the accuracy of the model in predicting the mass of the Higgs boson using the equation (13).

$$M_H \cong 24 \frac{m_{q^*}}{m_2} M_p \cong 125.47 \text{ GeV/c}^2$$

(14)

This value is almost identical, with a 99.7% match, to the measured mass of the Higgs boson (i.e., 125.1 GeV/c$^2$).

4. Discussions

4.1 Phase transition, universality classes, and CFT

The phase transitions are often associated with order and symmetry. As a measure of the degree of order, the order parameter is already proposed to be proportional to a power of $|q^* - q|$. According to the Landau theory, the behavior of physical quantities around the phase transition can be described by a power law, like $|q^* - q|^\alpha$, where $\alpha$ is called the critical exponent. E.g., the heat capacity near a phase transition is proportional to $|T^* - T|^\alpha$, where $T$ and $T^*$ are the temperature and the critical temperature respectively [2][3].

Since $q$ takes the prime power values, there is no discontinuity or a big jump in the proposed order parameter near the critical point. As a result, the phase transition is seemingly a second-order phase transition. An impressive observation related to the second-order phase transitions is the universality. It claims that there exist classes of apparently different systems that have the same value of critical exponent(s) in their phase transitions. The universality is indeed a prediction of the RG theory of phase transition that states the (thermodynamic) properties of a system near a phase transition depend only on a few general features, namely the number of spatial dimensions and the symmetry [2][3]. Such a prediction accords with our observation, which introduces the mass (of quanta of the system) in terms of symmetry and the number of spatial dimensions. These couple of parameters used in our observation is in perfect agreement with those of the Monster CFT. While such a match is impressive, the appearance of the CFT is not a surprise, because the statistical and thermodynamic systems are often conformally invariant at their critical points of the phase transition [1].

Thus, if one accepts that the proposed critical finite field describes the Higgs field, then, the Monster CFT seems a suitable candidate to model the transition point and the post-transition era of the electroweak symmetry breaking (EWSB). Therefore, it is worth investigating and checks if the pre-transition era can be modeled by a set of theories with the symmetry structure of other sporadic groups.
4.2 Extension field

Some authors denote the Steinberg groups by $A_n(q^2)$, while others denote them by $A_n(q)$. It is because there are two fields involved: one quadratic of order $q^2$, and its fixed field of order $q$. Specifically, if $F_q$ is a (unique up to isomorphism) finite field of size $q$, there is a unique quadratic separable extension of $F_q$, such that the extension field is a finite field of order $q^2$ [6].

Noteworthy, in the calculations, only the number of elements of the fixed field is considered and discussed. However, one may curiously follow the same procedure and compute a mass corresponding to the extension field. Such a mass would be extremely small ($\approx 10^{-6} \text{eV}/c^2$) as it is proportional to $q^{-2}$. However, the probable existence of such a tiny mass is speculative and needs further assessment.

4.3 $SL_2(q)$ vs $SU_2(q^2)$

Since the Steinberg group $SU_2(q^2)$ and the Chavely group $SL_2(q)$ of Lie type are isomorphic [5], from the beginning, one could replace $SU_2(q^2)$ by $SL_2(q)$ and obtain the same results. In this case, $SL_2(q)$ that directly relates to the modular group, and therefore the monstrous, moonshine may illuminate the role of the Monster group in the proposed observation from a different point of view.

4.4 The singularity of the field with one element and the Big Bang

The well-known “field with one element” denoted by $F_1$ (or $F_{un}$) is a single element field-like entity that attracted much attention despite its simplicity [11]. Not only the field loses its meaning at $q = 1$ (since there is no field with a single element based on the definition of the field), but also if one curiously extrapolates the order of $SU_2(q^2)$ to the point $q = 1$, it becomes zero implying the existence of a singularity. So far, it is proposed that the finite field $F$ (with $q^2$ elements) is an alternative to model the Higgs field, where its phase transition indicates the EWSB. Moving backward in time, the number of elements decreases, and we first reach to $F_2$ and finally $F_1$. We already proposed the correspondence between $m_2$ and $M_P$ which connotes the correspondence between the epoch of $F_2$ and the Planck epoch. Now, the existence of a singularity at $F_1$ strongly reinforces such a correspondence, because the backward extrapolation of the expansion of the universe in time yields the Big Bang singularity shortly before the Planck epoch.

Such a correspondence can also be strengthened by another hint. In the early stages of the universe (i.e., the moment immediately following the Big Bang), all the forces were unified, and fundamental interactions were not distinct. That unity is nicely observable in the index of $F_1$. Further, by the introduction of $F_1$, various expressions of objects and features in the arithmetic geometry become straightforwardly coherent [11], reminding the mentioned coherency of the early moments.

5. Conclusion

Since this letter is structured to report an observation, conclusive judgments are left to the readers. However, some comments are provided below.

What can be conveyed as the first message of this letter is that nature, in its basic levels, is finite and discrete. Second, the observation reported seems to be a manifestation of profound connections between the pure mathematics, namely the largest sporadic group, and the observable physical quantities, namely the Higgs field. Third, the observation possibly addresses the hierarchy problem, as it proposes an answer to why the mass of the Higgs boson is so much lighter than the Planck mass.

Moreover, to the best of our knowledge, what brings the system to the critical point of the phase transition is a series of symmetry-breakings in which $q$ symmetries of the initial state of the system gradually become hidden.
Indeed, the entropy of the system increases toward its maximum value at the equilibrium state. At this critical point, the entropy of the system needs to be increased, but the underlying symmetry structure of the system (here, the Monster symmetry) is saturated and does not allow more entropy increment in that structure. Therefore, to allow more entropy increment, a phase transition is required. A helpful metaphor is blowing up a balloon until it reaches its maximum surface tension and spontaneously explodes. The blowing is analogous to the entropy increment (symmetry-breakings), the maximum surface tension is analogous to the maximum order (here, the order of Monster group), and the explosion is like the phase transition with spontaneously symmetry-breaking. In short, the phase transitions in systems with a certain underlying symmetry structure perhaps occurs as the system reaches its maximum entropy allowed by its underlying symmetry structure.

Finally, in the future, further studies are not only needed to shed more light on what was reported in this letter and improve it, but also to investigate among other sporadic groups via a physical perspective.

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