\[ \tau^- \to (\pi\pi\pi)^- \nu_{\tau} : \text{Theory versus Experiment} \]

D. Gómez Dumm \textsuperscript{a}, A. Pich \textsuperscript{b}, J. Portolés \textsuperscript{b}

\textsuperscript{a}IFLP, Depto. de Física, Universidad Nacional de la Plata, C.C. 67, 1900 La Plata, Argentina

\textsuperscript{b}Departament de Física Teòrica, IFIC, CSIC-Universitat de València, Apt. Correus 22085, E-46071 València, Spain

We analyse \( \tau^- \to \pi\pi\pi\nu_{\tau} \) decays within the framework of the resonance chiral theory of QCD. We have worked out the relevant Lagrangian that describes the axial–vector current hadronization contributing to these processes, and the new coupling constants that arise have been constrained by imposing the asymptotic behaviour of the corresponding spectral function within QCD. Hence we compare the theoretical framework with the experimental data, obtaining a good quality fit from the ALEPH spectral function and branching ratio. We also get values for the mass and on–shell width of the \( a_1(1260) \) resonance, and provide the \( \tau^- \to \pi\pi\pi\nu_{\tau} \) structure functions that have been measured by OPAL and CLEO-II finding an excellent agreement.

1. Introduction

The hadronization of currents in exclusive processes provide a detailed knowledge on the strong interacting mechanisms driven by non-perturbative QCD. Within this framework \( \tau^- \) decays into hadrons allow to study the properties of the vector and axial–vector QCD currents, and yield relevant information on the dynamics of the resonances entering into the processes. At very low energies, typically \( E \ll M_\rho \) (being \( M_\rho \) the mass of the \( \rho(770) \)), chiral perturbation theory (\( \chi \)PT) \textsuperscript{11} is the corresponding effective theory of QCD. However the decays \( \tau^- \to \pi\pi\pi\nu_{\tau} \), through their full energy spectrum, happen to be driven by the \( \rho(770) \) and \( a_1(1260) \) resonances, mainly, in an energy region where the invariant hadron momentum approaches the masses of the resonances (\( \sqrt{Q^2} \sim M_\rho \)). Consequently \( \chi \)PT is no longer applicable to the study of the whole spectrum but only to the very low energy domain \textsuperscript{2}. Until now the standard way of dealing with these decays has been to use \( O(p^2) \) \( \chi \)PT to fix the normalization of the amplitudes in the low energy region and, accordingly, to include the effects of vector and axial–vector meson resonances by modulating the amplitudes with ad hoc Breit–Wigner functions \textsuperscript{34}. However we have seen \textsuperscript{50} that this modelization is not consistent with \( O(p^4) \) \( \chi \)PT, a fact that could spoil any outcome provided by the analysis of experimental data using this procedure.

Lately several experiments have collected good quality data on \( \tau^- \to \pi\pi\pi\nu_{\tau} \), such as branching ratios and spectra \textsuperscript{2} or structure functions \textsuperscript{8}. Their analysis within a model–independent framework is highly desirable if one wishes to collect information on the hadronization of the relevant QCD currents. At energies \( E \sim M_\rho \) the resonance mesons are active degrees of freedom that cannot be integrated out, as in \( \chi \)PT, and they have to be properly included into the relevant Lagrangian. The procedure is ruled by the approximated chiral symmetry of QCD under \( SU(3)_L \otimes SU(3)_R \), that drives the interaction of Goldstone bosons (the lightest octet of pseudoscalar mesons), and the \( SU(3)_V \) assignments of the resonance multiplets. Its systematic arrangement has been put forward in Refs. \textsuperscript{9,10} as the Resonance Chiral Theory (R\( \chi \)T). A complementary tool is the large number of colours (\( N_C \)) limit
of QCD. It has been pointed out [11] that the inverse of the number of colours of the gauge group $SU(N_\text{C})$ could provide the expansion parameter involved in the perturbative treatment of the amplitudes. Indeed large–$N_\text{C}$ QCD shows features that resemble, both qualitatively and quantitatively, the $N_\text{C} = 3$ case.

In Ref. [5] we have performed an analysis of the $\tau \to \pi\pi\pi\nu_\tau$ decays using de above–mentioned tools, and we present in the following its main results.

2. The Resonance Effective Theory of QCD

The final hadron system in the $\tau \to \pi\pi\pi\nu_\tau$ decays spans a wide energy region $3m_\pi \lesssim E \lesssim M_\tau$ that is heavily populated by resonances. As a consequence an effective theory description of the full energy spectrum requires to include the resonances as active degrees of freedom. RχT is the appropriate framework to work with and, accordingly, we consider the Lagrangian

$$L_{\text{RχT}} = \frac{F^2}{4} (u_\mu u^\mu + \lambda_+ ) + \frac{F_\pi}{2\sqrt{2}} (V_{\mu\nu} f_{\mu\nu}^\pi )$$

$$+ i \frac{G_V}{\sqrt{2}} (V_{\mu\nu} u_\mu u^\nu + \frac{F_A}{2\sqrt{2}} (A_{\mu\nu} f_{\mu\nu}^A )$$

$$+ \mathcal{L}_{\text{kin}}^V + \mathcal{L}_{\text{kin}}^A + \sum_{i=1}^{5} \lambda_i \mathcal{O}_\text{VAP}^i ,$$

with $\lambda_i$ unknown real adimensional couplings, and the operators $\mathcal{O}_\text{VAP}^i$ are given by

$$\mathcal{O}_\text{VAP}^1 = \langle [V^\mu\nu, A_{\mu\nu}] \chi^- \rangle ,$$

$$\mathcal{O}_\text{VAP}^2 = i \langle [V^\mu\nu, A_{\nu\alpha}] h_\alpha^\mu \rangle ,$$

$$\mathcal{O}_\text{VAP}^3 = i \langle [\nabla^\mu V_{\mu\nu}, A^{\nu\alpha}] h_\alpha^\mu \rangle ,$$

$$\mathcal{O}_\text{VAP}^4 = i \langle [\nabla^\alpha V_{\mu\nu}, A_\nu^\mu] u_\alpha \rangle ,$$

$$\mathcal{O}_\text{VAP}^5 = i \langle [\nabla^\alpha V_{\mu\nu}, A^{\nu\mu}] u_\alpha \rangle ,$$

where $h_\mu^\nu = \nabla_\mu u_\nu + \nabla_\nu u_\mu$ and the notation is that of Ref. [9]. Notice that we use the antisymmetric tensor formulation to describe the spin 1 resonances and, consequently, we do not consider the $\mathcal{O}(p^4)$ chiral Lagrangian of pseudoscalars [10].

3. Axial–vector current form factors in $\tau \to \pi\pi\pi\nu_\tau$

The decay amplitude for the $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ and $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ processes can be written as

$$M_{\pm} = - \frac{G_F}{\sqrt{2}} V_{ud} \bar{u}_\nu \gamma^\mu (1 - \gamma_5) u_\tau T_{\pm \mu} ,$$

where $T_{\pm \mu}$ is the hadronic matrix element of the participating $V_{\mu} - A_{\mu}$ QCD currents. In the isospin limit there is no contribution of the vector current to these processes and, therefore, only the axial–vector current $A_\mu$ appears:

$$T_{\pm \mu}(p_1, p_2, p_3) = \langle \pi_1(p_1)\pi_2(p_2)\pi_3^+(p_3)|A_\mu|0 \rangle ,$$

being $\pi^+$ the one in $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ and $\pi^-$ that in $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$. The hadronic tensor can be written in terms of three form factors, $F_1, F_2$ and $F_P$, as [12]

$$T^\mu = V^\mu_1 F_1 + V^\mu_2 F_2 + V^\mu_P F_P ,$$

where

$$V^\mu_1 = \left( g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_1 - p_3)_\nu ,$$

$$V^\mu_2 = \left( g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_2 - p_3)_\nu ,$$

$$V^\mu_P = Q^\mu = p_{1\mu}^\nu + p_{2\mu}^\nu + p_{3\mu}^\nu .$$

The form factors $F_1$ and $F_2$ have a transverse structure in the total hadron momenta $Q$, and drive a $J^P = 1^-$ transition. Bose symmetry under interchange of the two identical pions in the final state demands that $F_1(Q^2, s, t) = F_2(Q^2, t, s)$ where $s = (p_1 + p_3)^2$ and $t = (p_2 + p_3)^2$. Meanwhile $F_P$ accounts for a $J^P = 0^-$ transition that carries pseudoscalar degrees of freedom and vanishes with the square of the pion mass. Hence its contribution to the decay processes will be very much suppressed and we will not consider it in the following.

In the low $Q^2$ region, the matrix element in Eq. (4) can be calculated using χPT. At $O(p^2)$ one has two contributions, arising from the diagrams in Fig. 1. The sum of both graphs yields

$$T^\chi_{\pm \mu} = \frac{2\sqrt{2}}{3F} \{ V_{1\mu} + V_{2\mu} \} .$$

(7)
Figure 1. Diagrams contributing to the hadronic amplitude $T_{\pm \mu}$ in $\mathcal{O}(p^2) \chi$PT.

Figure 2. Resonance–mediated diagrams contributing to $T_{\pm \mu}$.

We now include the resonance–mediated contributions to the amplitude, to be evaluated through the interacting terms in $\mathcal{L}_{R\chi T}$ Eq. (1). The relevant diagrams to be taken into account are those shown in Fig. 2. We get

$$T_{\pm \mu}^R = \mp \frac{\sqrt{2} F_V G_V}{3 F^3} \times \left[ \alpha(Q^2, s, t) V_{1\mu} + \alpha(Q^2, t, s) V_{2\mu} \right]$$

$$\pm \frac{4 F_A G_V}{3 F^3} \frac{Q^2}{Q^2 - M_A^2} \times \left[ \beta(Q^2, s, t) V_{1\mu} + \beta(Q^2, t, s) V_{2\mu} \right].$$

The functions $\alpha(Q^2, s, t)$ and $\beta(Q^2, s, t)$ are given explicitly in Ref. [5] and depend on three combinations of the $\lambda_i$ couplings in $\mathcal{L}_{R\chi T}$ in Eq. (1), that we call $\lambda_0$, $\lambda'$ and $\lambda''$.

The form factors in Eq. (8) include zero–width $\rho(770)$ and $a_1(1260)$ propagator poles, leading to divergent phase–space integrals in the calculation of the $\tau \rightarrow \pi \pi \pi \nu_\tau$ decay width as the kinematical variables go along the full energy spectrum. The result can be regularized through the inclusion of resonance widths, which means to go beyond the leading order in the $1/N_C$ expansion, and implies the introduction of some additional theoretical inputs. This issue has been analysed in detail within the resonance chiral effective theory in Ref. [13] and, accordingly, we include off–shell widths for both resonances [5].

![Figure 3. Fit to the ALEPH data for the normalized $\tau^{-} \rightarrow \pi^{+} \pi^{-} \pi^{-} \nu_\tau$.](image)

4. QCD constraints on the coupling constants

A look to our results above shows that we have six unknown couplings (or combination of couplings), namely : $F_V$, $G_V$, $F_A$, $\lambda_0$, $\lambda'$ and $\lambda''$. The QCD ruled short–distance behaviour of the vector and axial–vector form factors in the large–$N_C$ limit (approximated with only one octet of vector resonances) constrains the couplings of $\mathcal{L}_{R\chi T}$ in Eq. (1), which must satisfy [10] :

$$1 - \frac{F_V G_V}{F^2} = 0 ,$$

$$2 F_V G_V - F_A^2 = 0 .$$

(9)

In addition, the first Weinberg sum rule, in the limit where only the lowest narrow resonances contribute to the vector and axial–vector spectral functions, leads to

$$F_V^2 - F_A^2 = F^2 .$$

(10)

In this way all three couplings $F_V$, $G_V$ and $F_A$ can be written in terms of the pion decay constant : $F_V = \sqrt{2} F$, $G_V = F/\sqrt{2}$ and $F_A = F$. These results are well satisfied phenomenologically and we have adopted them [2]. The proper QCD driven behaviour of the $T_{\pm \mu}$ form factor imposes in addition the constraints :

$$2 \lambda' - 1 = 0 ,$$

$$\lambda'' = 0 .$$

(11)

[2] A more thorough study of this procedure is given in Ref. [3].
Figure 4. Theoretical values for the $w_A, w_C, w_D$ and $w_E$ integrated structure functions in comparison with the experimental data from CLEO-II (solid) and OPAL (dashed) [8].

5. Phenomenology of $\tau \to \pi \pi \pi \nu_\tau$ processes

To analyse the experimental data we will only consider the dominating $J^P = 1^+$ driven axial–vector form factors, that satisfy $T_{+\mu} = -T_{-\mu}$ hence providing the same predictions for both $\tau^- \to \pi^+ \pi^- \pi^- \nu_\tau$ and $\tau^- \to \pi^- \pi^0 \pi^0 \nu_\tau$ processes in the isospin limit.

We have fitted the experimental values for the $\tau^- \to \pi^+ \pi^- \pi^- \nu_\tau$ branching ratio and normalized spectral function obtained by ALEPH [7] and we get a reasonable $\chi^2/d.o.f. = 64.5/52$ shown in Fig. 3. Hence we get the axial–vector $a_1(1260)$ parameters $M_A = (1.204 \pm 0.007)$ GeV and $\Gamma_{a_1}(M^2_A) = (0.48 \pm 0.02)$ GeV, where the errors are only statistical. We predict, accordingly, the structure functions [12] that we compare with the experimental results for $\tau^- \to \pi^- \pi^0 \pi^0 \nu_\tau$ in Fig. 3.

Acknowledgements

We wish to thank S. Narison and his team for the organization of the QCD03 Conference. This work has been supported in part by TMR EURIDICE, EC Contract No. HPRN-CT-2002-00311, by MCYT (Spain) under grant FPA2001-3031, by Generalitat Valenciana under grant GRUPOS03/013, by the Agencia Española de Cooperación Internacional (AECI) and by ERDF funds from the EU Commission.

REFERENCES

1. S. Weinberg, Physica A 96 (1979) 327;
   J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142.
2. G. Colangelo, M. Finkemeier and R. Urech, Phys. Rev. D 54 (1996) 4403.
3. J. H. Kühn and A. Santamaria, Z. Phys. C 48 (1990) 445.
4. R. Fischer, J. Wess and F. Wagner, Z. Phys. C 3 (1980) 313;
A. Pich, Phys. Lett. B 196 (1987) 561;
A. Pich, “QCD Tests From Tau Decay Data”, SLAC Rep.-343 (1989) 416;
J. J. Gomez- Cadenas, M. C. Gonzalez-Garcia and A. Pich, Phys. Rev. D 42 (1990) 3093.
5. D. Gómez Dumm, A. Pich and J. Portolés, “τ → πππντ decays in the Resonance Effective Theory”, [arXiv:hep-ph/0312183].
6. J. Portolés, Nucl. Phys. Proc. Suppl. 98 (2001) 210.
7. R. Barate et al. [ALEPH Collaboration], Eur. Phys. J. C 4 (1998) 409.
8. K. Ackerstaff et al. [OPAL Collaboration], Z. Phys. C 75 (1997) 593;
T. E. Browder et al. [CLEO Collaboration], Phys. Rev. D 61 (2000) 052004.
9. G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B 321 (1989) 311.
10. G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B 223 (1989) 425.
11. G. ’t Hooft, Nucl. Phys. B 72 (1974) 461;
E. Witten, Nucl. Phys. B 160 (1979) 57.
12. J. H. Kühn and E. Mirkes, Z. Phys. C 56 (1992) 661 [Erratum-ibid. C 67 (1995) 364].
13. D. Gómez Dumm, A. Pich and J. Portolés, Phys. Rev. D 62 (2000) 054014.