On metastability in nearly-elastic systems

Wenqing Hu

Abstract

We consider a nearly-elastic model system with one degree of freedom. In each collision with the "wall", the system can either lose or gain a small amount of energy due to stochastic perturbation. The weak limit of the corresponding slow motion, which is a stochastic process on a graph, is calculated. A large deviation type asymptotics and the metastability of the system is also considered.

Keywords: Averaging, large deviations, metastability, Markov processes on graphs, random walk.

2010 Mathematics Subject Classification Numbers: 70K65, 34C28, 37D99, 60J25, 60F10, 60G50.

1 Introduction

Consider a model of a one-dimensional system with several potential wells (Fig.1). A particle of unit mass moves freely in an interval \([q_1, q_n]\) with elastic reflection at the ends of the interval if the initial velocity is large enough. Let a finite number of points \(q_2, q_3, \ldots, q_{n-1} \in (q_1, q_n)\) be given. Suppose at each \(q_i\) there is a "wall" of certain height which gives the particle instantaneous reflection once the particle hits it from either side. The "height" coordinate \(H\) is the energy of the particle. The potential wells are numbered by 1, 2, ..., \(N\) (see Fig.1, where \(N = 7\)). Note that some of the wells are the combination of "smaller" wells. For example, in Fig.1 well 5 consists of wells 1 and 2, well 6 consists of wells 5 and 3, and well 7 consists of wells 6 and 4. The speed of the particle at energy level \(H\) is \(\sqrt{2H}\). In the following, we always make the convention that the bigger wells, like well 5 which consists of wells 1 and 2, are of energy level between the top of that well and the one that separates the two smaller wells. For example, in Fig.1 well 5 is supposed to be of energy level between \(H_6\) and \(H_5\); well 6 is supposed to be of energy level between \(H_7\) and \(H_6\), etc. Under this convention each well with number \(i\) has a minimum energy level \(H_i\) (see Fig.1). We assume that all \(H_i\)'s are bounded away from 0. Within well \(i\), at energy level \(H\), the particle moves between the

\*Department of Mathematics, University of Maryland at College Park, huwenqing@math.umd.edu
walls of that well and has speed \( v = \sqrt{2H} \). At each collision with the wall, the particle is instantaneously reflected and the speed of the particle remains the same. The energy \( H \) is preserved in the system.

Assume now that the collisions with the walls are not absolutely elastic. If the particle is in well \( i \) with energy \( H \), then it hits the left (right) wall of that well and was reflected, while at the same time its energy becomes \( H - \varepsilon \xi_k^{(i)} \) ( \( H - \varepsilon \eta_k^{(i)} \) ) (if at the bottom of the well \( i \) there is no smaller wells the energy decays to \( H \lor (H - \varepsilon \xi_k^{(i)}) \) or \( H \lor (H - \varepsilon \eta_k^{(i)}) \), respectively, and \( a \lor b = \max(a,b) \)). Here \( 0 < \varepsilon << 1 \) is a small parameter and \( k \) denotes the number of collisions with the left (right) wall (when the particle is at some energy level which is the bottom of a "big" well, i.e., one which contains two smaller wells we take \( \xi_k^{(i)} \) and \( \eta_k^{(i)} \) to be those corresponding to the big well). The sequences of random variables \( \{\xi_k^{(i)}\}_{k \geq 1}, \{\eta_k^{(i)}\}_{k \geq 1} \) are i.i.d. and mutually independent with \( \mathbb{E} (\xi_k^{(i)} + \eta_k^{(i)}) > 0 \). We assume that these random variables are bounded \( \mathbb{P}\{|\xi_k^{(i)}| \leq M\} = \mathbb{P}\{|\eta_k^{(i)}| \leq M\} = 1 \) for some \( M > 0 \) and they all have continuous densities. In all the following, when we use random variables such as \( \xi, \eta \) without subscript, they are understood as independent random variables and having the same distribution as corresponding \( \xi_k \) and \( \eta_k \)’s. Also, later in this paper we will always denote \( \zeta_k = -(\xi_k + \eta_k) \) and \( \zeta = -(\xi + \eta) \).

The position of the particle in our perturbed system can now be described by a stochastic process \( \tilde{X}_t^\varepsilon = (\tilde{H}_t^\varepsilon, \tilde{q}_t^\varepsilon) \) where \( \tilde{H}_t^\varepsilon \) is the energy of the particle at time \( t \) and \( \tilde{q}_t^\varepsilon \) is the horizontal position of the particle (see Fig.1). We denote the width of the \( i \)-th well by \( D_i \). In Fig.1 we have \( D_5 = D_1 + D_2, D_6 = D_5 + D_3 \) and \( D_7 = D_6 + D_4 \).

The perturbed system \( \tilde{X}_t^\varepsilon \) has, for \( 0 < \varepsilon << 1 \), fast and slow components. The
fast component consists of the motion along the non-perturbed trajectory. To describe the slow component, consider the graph $\Gamma$ obtained after identification of points of each well with a given energy level $H$. Denote by $\sqcap$ the phase space of our system: $\sqcap$ is the union of all wells and it is assumed that each interior well consists of two sides, left and right. Denote by $Y: \sqcap \to \Gamma$ the identification map of the phase space $\sqcap$ to $\Gamma$. The slow component of the motion is $\tilde{Y}^\varepsilon_t = Y(\tilde{H}^\varepsilon_t, \tilde{q}^\varepsilon_t)$ (compare with [5, Ch.8], [4]). We rescale time $t \mapsto t/\varepsilon$. Define $X^\varepsilon_t = \tilde{X}^\varepsilon_{t/\varepsilon}$, $H^\varepsilon_t = \tilde{H}^\varepsilon_{t/\varepsilon}$, $q^\varepsilon_t = \tilde{q}^\varepsilon_{t/\varepsilon}$, $Y^\varepsilon_t = \tilde{Y}^\varepsilon_{t/\varepsilon}$.

We make a convention here: in the following processes with a tilde on it are original processes with natural time parameter $t$; processes without such a tilde on it are time-rescaled process with time $t/\varepsilon$; processes with a hat on it are piecewise linear modifications of the one under the hat. For example, $H^\varepsilon_t = \hat{H}^\varepsilon_t$ and $\hat{\tilde{H}}^\varepsilon_t$ is a piecewise linear modification of $\tilde{H}^\varepsilon_t$. Number the edges of the graph: $\Gamma = \{I_1, I_2, ..., I_N\}$ where $N$ is the number of the wells (in Fig.1 $N = 7$). The $i$-th well corresponds to edge $I_i$. Exterior vertex $V_k$ corresponds to the bottom of the $k$-th well. Interior vertex $O_l$ corresponds to the lowest energy level (as was in the convention made before) of the $l$-th well ("big" well). Then $Y(H, q) = (H, K(H, q))$ where $K(H, q)$ is the number of the edge containing $Y(H, q)$ and $H$ is the energy. We see that after time rescaling, the slow component is the process $Y^\varepsilon_t = (H^\varepsilon_t, K(H^\varepsilon_t, q^\varepsilon_t))$.

We will show that the process $Y^\varepsilon_t$ converges, as $\varepsilon \downarrow 0$, to a stochastic process $Y_t$ on $\Gamma$. The process $Y_t$ is a deterministic motion within each edge of $\Gamma$ and has stochasticity only at the interior vertices $O_l$ of $\Gamma$.

Since we allow random variables $\xi^{(i)}_{k_l}$, $\eta^{(i)}_{k_l}$ to be less than 0, it can happen that the particle enters certain well and sooner or later it jumps out of that well. Since we assumed that $E(\xi^{(i)}_{k_l} + \eta^{(i)}_{k_l}) > 0$, this is a large deviation type event. We will calculate the "quasi-potential" describing how difficult it is to switch from one well to another. For the system with many wells, metastability and asymptotic behavior of the system will be considered in Section 4.

2 The limiting process

In this section we first consider the two well case. Let us assume that our system has two wells 1 and 2 and their combination is well 3. Interior vertex is $O_3$ and exterior vertices are $V_1$ and $V_2$. The edges are $I_1$, $I_2$ and $I_3$. We assume that $P(\{|\xi^{(i)}_{k_l}| \leq M\} = P(\{|\eta^{(i)}_{k_l}| \leq M\} = 1$ for some constant $M > 0$. Similar to [4], by using the standard averaging principle, we get
Lemma 2.1. Let $H_0^0 = H_0 > H(O_3)$. Within each edge of the graph $\Gamma$, as $\varepsilon \downarrow 0$, the process $H_\varepsilon = H_{t_\varepsilon}$, converges uniformly in probability on $0 < t < T < \infty$, to a deterministic motion $H(t)$ which is defined by the equations

$$H(t) = \left( \sqrt{H_0 - t} - \frac{\mathbb{E}\xi(3) + \mathbb{E}\eta(3)}{2\sqrt{2D_3}} \right)^2, 0 < t \leq t_0 \text{ on } I_3;$$

(2.1)

and

$$H(t) = \left( \sqrt{H(O_3)} - (t - t_0) \frac{\mathbb{E}\xi(1) + \mathbb{E}\eta(1)}{2\sqrt{2D_1}} \right)^2, t > t_0 \text{ on } I_1;$$

(2.2)

$$H(t) = \left( \sqrt{H(O_3)} - (t - t_0) \frac{\mathbb{E}\xi(2) + \mathbb{E}\eta(2)}{2\sqrt{2D_2}} \right)^2, t > t_0 \text{ on } I_2$$

(2.3)

respectively. Here $H(O_3)$ is the energy corresponding to the interior vertex $O_3$ and $t_0 = 2\sqrt{2D_3(\sqrt{H_0} - \sqrt{H(O_3)})}$ is the time for $H(t)$ to come to the interior vertex $O_3$.

Similarly as was done in [4], we consider a piecewise linear modification $\tilde{H}_\varepsilon$ of $\tilde{H}_t$. Under the convention made in the introduction we put $\hat{H}_\varepsilon = \tilde{H}_\varepsilon / \varepsilon$ and $\hat{X}_\varepsilon = (\hat{H}_\varepsilon, q_\varepsilon)$. Let $\hat{Y}_\varepsilon = (\hat{H}_\varepsilon, K(\hat{X}_\varepsilon))$. It is clear that for fixed $\varepsilon > 0$,

$$\mathbb{P}\{ |\tilde{H}_t - H_t| < C \varepsilon \} = 1$$

(2.4)

for some positive constant $C > 0$ and $0 < t < T < \infty$. We have, as in [4],

Lemma 2.2. For each $T > 0$, the family $\{\hat{Y}_\varepsilon\}_{\varepsilon > 0}$ is tight in $C_{0T}(\Gamma)$.

We now turn to the problem of determining the asymptotic branching probability for the process $\hat{Y}_\varepsilon$ as $\varepsilon \downarrow 0$, at the interior vertex $O_3$. Let us first present an auxiliary lemma about certain properties of random walk (compare with [4]).

Let $\{\xi_k\}_{k \geq 1}, \{\eta_k\}_{k \geq 1}$ be i.i.d, mutually independent sequences of random variables. Assume that the random variables have continuous densities and $\mathbb{P}\{-\infty < -\alpha < \xi_k < \alpha < \infty\} = 1, \mathbb{P}\{0 < \eta_k < \alpha < \infty\} = 1$ for some positive constant $\alpha > 0$. Notice that we allow $\xi_k$ to be negative but we assume that $\mathbb{E}(\xi_k + \eta_k) > 0$. Put, for $m \geq 0$, that

$$S_0 = 0, \quad S_{2m} = \sum_{k=1}^{m} (\xi_k + \eta_k), \quad S_{2m+1} = S_{2m} + \xi_{m+1}.$$

Define $\tau^\lambda_n = \min\{m : S_m > n\lambda\}$ for $\lambda > 0$.

Since $\mathbb{E}(\xi_k + \eta_k) > 0$, the law of large numbers implies that $\mathbb{P}\{\tau^\lambda_n < \infty\} = 1$ for any $\lambda > 0, n \in \mathbb{Z}$.
Let $\zeta_k = \xi_k + \eta_k$. Let $T_n = S_{2n} = \sum_{k=1}^{n} \zeta_k$. Sample trajectories of $S_n$ and $T_n$ are shown in Fig.2. Let $N = \min_{k \geq 1} \{T_n > 0\}$. Put

$$E_n(I) = P\{N = n, T_n \in I\}$$

for $I \subset (0, +\infty)$. In other words, $E_n(I)$ is the probability of the event

$$\{T_1 \leq 0, T_2 \leq 0, ..., T_{n-1} \leq 0, T_n > 0, T_n \in I\}.$$

Consider random variables $a = T_{N-1}$, $b = S_{2N-1}$, $c = T_N$. We are now ready to state

**Lemma 2.3.** Under mentioned above conditions,

$$\lim_{n \to \infty} P\{\tau_n^\lambda \text{ is odd}\} = \frac{Eb_{b>0}}{Ec},$$

$$\lim_{n \to \infty} P\{\tau_n^\lambda \text{ is even}\} = 1 - \frac{Eb_{b>0}}{Ec}.$$

**Proof.** We say a strong ascending ladder point (see [3, Ch.12]) for $\{T_n\}_{n \geq 1}$ (respectively, $\{S_n\}_{n \geq 1}$) occurs at step $k$ if

$$T_k > \max\{T_r : 0 \leq r \leq k-1\}$$

(respectively, $S_k > \max\{S_r : 0 \leq r \leq k-1\}$).
If the successive strong ascending ladder points for $T_n$ are $W_1, W_1 + W_2, \ldots$, we write $T_{W_1 + W_2 + \ldots + W_k} - T_{W_1 + W_2 + \ldots + W_{k-1}}$ (suppose $W_0 = 0$) as $Z_k$, the $k$-th strong ascending ladder step for $\{T_n\}_{n \geq 1}$.

The random variables $W_k, k \geq 1$ are i.i.d with common distribution the same as that of $N$. The random variables $Z_k, k \geq 1$ are i.i.d with common distribution the same as that of $c = T_N$.

Since we assumed that $\mathbb{P}\{0 < \eta_k < \alpha < \infty\} = 1$, the occurrence of a strong ascending ladder point for $\{T_n\}_{n \geq 1}$ at step $k$ implies that a strong ascending ladder point for $\{S_n\}_{n \geq 1}$ happens either at step $2k$ or at step $2k - 1$ (see Fig.2). Define $R_k = S_{2(W_1 + W_2 + \ldots + W_k) - 1} - S_{2(W_1 + W_2 + \ldots + W_{k-1})}$. Since each piece of the random walk between steps $W_1 + \ldots + W_{k-1}$ and $W_1 + \ldots + W_k$ are i.i.d, the random variables $R_k$ are i.i.d with common distribution the same as that of $b = S_{2N-1}$. By using the same local limit argument as that in [4, Lemma 3.3], one can see that

$$\lim_{n \to \infty} \mathbb{P}\{\tau_n^\lambda \text{ is odd}\} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} R_k 1_{R_k > 0}}{\sum_{k=1}^{n} Z_k} = \frac{\mathbb{E}b 1_{b > 0}}{\mathbb{E}c},$$

and the result follows. □

By using the Wiener-Hopf theory (see [3, Ch.12, Ch.18]), one can sometimes determine the distribution of $c$ by $E(dx) = \sum_{n=0}^{\infty} E_n(dx)$ and thus $\mathbb{E}c = \int_{0}^{\infty} xE(dx)$. The distribution of $b$ can be determined by the convolution relation

$$E(dx) = \int_{-\infty}^{x} b(dy)F_\eta(dx - y).$$

Here $b(dy) = \mathbb{P}\{b \in dy\}$ and $F_\eta$ is the common distribution function of $\eta_k$. We refer the reader to [3, Ch.12, Ch.18].

Now let us turn back to our system. Assume that our system always loses energy on the right walls, i.e. $\mathbb{P}\{0 < \eta_k^{(i)} < M < \infty\} = 1$. On the left walls the system can either gain or lose energy - we only assume that $\mathbb{P}\{-\infty < -M < \xi_k^{(i)} < M < \infty\} = 1$. Let $a^{(i)}, b^{(i)}, c^{(i)}$ be defined in the same way as $a, b, c$ in Lemma 2.3 for random walks $S_k^{(i)}$ constructed from $\xi_k^{(i)}, \eta_k^{(i)}$:

$$S_0^{(i)} = 0, \quad S_{2m}^{(i)} = \sum_{k=1}^{m} (\xi_k^{(i)} + \eta_k^{(i)}), \quad S_{2m+1}^{(i)} = S_{2m}^{(i)} + \xi_{m+1}^{(i)}.$$

Let $\lambda = H_0 - H(O_3), n = \left\lfloor \frac{1}{\lambda} \right\rfloor$, and apply Lemma 2.3 directly, we get
Lemma 2.4.
\[
\lim_{\varepsilon \downarrow 0} \mathbb{P}\{X_\varepsilon^t \text{ finally falls into well } 1\} = \frac{\mathbb{E}b^{(3)}1_{b^{(3)}>0}}{\mathbb{E}c^{(3)}} = \rho_1^{(3)},
\]
\[
\lim_{\varepsilon \downarrow 0} \mathbb{P}\{X_\varepsilon^t \text{ finally falls into well } 2\} = 1 - \frac{\mathbb{E}b^{(3)}1_{b^{(3)}>0}}{\mathbb{E}c^{(3)}} = \rho_2^{(3)}.
\]

Define a process \(Y_t\) on \(\Gamma\): \(Y_t = (H(t), K(t))\); on each edge \(H_t\) satisfies equations (2.1), (2.2), (2.3), respectively. The process \(Y_t\), when arriving at the interior vertex \(O_3\), immediately leaves that vertex and goes into edge \(I_1\) or \(I_2\) with probabilities \(\rho_1^{(3)}\) and \(\rho_2^{(3)}\), respectively. We have,

Theorem 2.1. Under the same assumption mentioned before Lemma 2.4, as \(\varepsilon \downarrow 0\), process \(\hat{Y}_\varepsilon^t\) converges weakly, for \(0 < T < \infty\), in \(C_0^T(\Gamma)\) with uniform topology, to \(Y_t\).

The above result can be easily generalized to the case when the system has more than two wells. The averaging principle is the same as before: within each edge \(I_i\), as \(\varepsilon \downarrow 0\), \(H_\varepsilon^t\) converges to a deterministic motion \(H(t)\) which satisfies the differential equation

\[
\frac{dH}{dt} = -\frac{\mathbb{E}\xi^{(i)} + \mathbb{E}\eta^{(i)}}{T_i(H)},
\]

where \(i\) is the number of the well, \(T_i(H) = \frac{2D_i}{\sqrt{2H}}\) is the period of the elastic motion within well \(i\).

We assume that the system always loses energy on the right walls, i.e. \(\mathbb{P}\{0 < \xi_k^{(i)} < M < \infty\} = 1\); and on the left walls the system can either gain or lose energy - we only assume that \(\mathbb{P}\{-\infty < -M < \xi_k^{(i)} < M < \infty\} = 1\). The branching probabilities for the limiting motion \(Y_t\) at the bottom of well \(i\) can be given by \(\rho_1^{(i)} = \frac{\mathbb{E}b^{(i)}1_{b^{(i)}>0}}{\mathbb{E}c^{(i)}}\) (for entering the left well) and \(\rho_2^{(i)} = 1 - \frac{\mathbb{E}b^{(i)}1_{b^{(i)}>0}}{\mathbb{E}c^{(i)}}\) (for entering the right well). The branching at each interior vertex is independent of the others.

Finally we briefly consider the case when we throw away the artificial restriction that \(\mathbb{P}\{0 < \eta_k < \alpha < \infty\} = 1\). Suppose \(\xi_k\) and \(\eta_k\) are two i.i.d. series and mutually independent. Let \(\mathbb{P}\{-\alpha < \xi_k < \alpha\} = \mathbb{P}\{-\alpha < \eta_k < \alpha\} = 1\) for some \(\alpha > 0\) and \(k = 1, 2, 3, \ldots\). Suppose all \(\xi_k\)’s and \(\eta_k\)’s have continuous densities. We also assume that \(\mathbb{E}(\xi_k + \eta_k) > 0\). Let us add one more assumption that \(\mathbb{P}\{\xi_k > 0\} > 0, \mathbb{P}\{\eta_k > 0\} > 0\).

Let us consider the strong ascending ladder points for the random walk

\[
S_0 = 0, \quad S_{2m} = \sum_{k=1}^{m} (\xi_k + \eta_k), \quad S_{2m+1} = S_{2m} + \xi_{m+1}.
\]
We define these strong ascending ladder points to be $J_1, J_1 + J_2, \ldots$. Let $M_k = 1$ if $J_1 + \ldots + J_k$ is odd and $M_k = 0$ if $J_1 + \ldots + J_k$ is even. Let $M_0 = 0$.

Let us consider another random walk $S_0' = 0, S_{2m}' = \sum_{k=1}^{m} \eta_k + \xi_k, S_{2m+1}' = S_{2m} + \eta_{m+1}$ and the corresponding strong ascending ladder points $J_1', J_1' + J_2', \ldots$. We consider the first strong ascending ladder steps $\gamma(0) = S_J$ for $\{S_n\}_{n \geq 0}$ and $\gamma(1) = S'_J$ for $\{S'_n\}_{n \geq 0}$.

By strong Markov property of the random walk $S_n$ and our assumptions on $\xi_k$ and $\eta_k$, it is easy to see that $M_k$ is an ergodic Markov chain with two states $\{0, 1\}$ and an invariant measure $\mu(\{0\}) = \mu_0$ and $\mu(\{1\}) = \mu_1$ for some $0 < \mu_i < 1$ and $\sum \mu_i = 1$, $i = 0, 1$. The coupling chain $(M_k - 1, M_k)$ is also an ergodic Markov chain with four states $\{(0,0), (0,1), (1,0), (1,1)\}$ and an invariant measure $\mu(\{(0,0)\}) = \mu_{00}, \mu(\{(0,1)\}) = \mu_{01}, \mu(\{(1,0)\}) = \mu_{10}, \mu(\{(1,1)\}) = \mu_{11}$. Here $0 < \mu_{ij} < 1$ and $\sum \mu_{ij} = 1$ for $i, j = 0, 1$. It is clear that $\mu_{11} + \mu_{10} = \mu_1, \mu_{01} + \mu_{00} = \mu_0$.

Let $\gamma_k^{(0)}$ be a sequence of i.i.d random variables which has common distribution same as $\gamma(0)$. Let $\gamma_k^{(1)}$ be a sequence of i.i.d random variables which has common distribution same as $\gamma(1)$. The random variables $\gamma_k^{(0)}$ and $\gamma_k^{(1)}$ are bounded and have continuous densities. We choose these random variables such that they are mutually independent and also independent of the $M_k$'s.

Define $\tau_n^\lambda = \min \{m : S_m > n\lambda\}$ for $\lambda > 0$.

We claim the

**Lemma 2.5.** Under mentioned above conditions,

$$
\lim_{n \to \infty} \mathbb{P}\{\tau_n^\lambda \text{ is odd}\} = \frac{\mu_{11} E \gamma^{(1)} + \mu_{01} E \gamma^{(0)}}{\mu_1 E \gamma^{(1)} + \mu_0 E \gamma^{(0)}},
$$

$$
\lim_{n \to \infty} \mathbb{P}\{\tau_n^\lambda \text{ is even}\} = \frac{\mu_{10} E \gamma^{(1)} + \mu_{00} E \gamma^{(0)}}{\mu_1 E \gamma^{(1)} + \mu_0 E \gamma^{(0)}}.
$$

**Proof.** We use the same local limit theorem argument as in [4, Lemma 3.3]. We first apply the local limit theorem to sequence $T_n$ (as defined in the proof of Lemma 2.3). Then we use the fact that
\[ \lim_{n \to \infty} \mathbb{P}\{ \tau_{\lambda}^n \text{ is odd} \} = \lim_{n \to \infty} \sum_{k=1}^{n} M_k \gamma_k^{(M_k-1)} \]

\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \gamma_k^{(M_k-1)} \]

\[ = \frac{\nu_{11}(n)}{n} \frac{1}{\nu_1(n)} \sum_{k=1}^{\nu_1(n)} \gamma_k^{(1)} + \frac{\nu_{01}(n)}{n} \frac{1}{\nu_0(n)} \sum_{k=1}^{\nu_0(n)} \gamma_k^{(0)} \]

and the Lemma follows. Here for \( i, j = 0, 1 \) we set

\[ \nu_{ij}(n) = \text{number of } k \text{'s such that } (M_k-1, M_k) = (i, j), 1 \leq k \leq n \]

and for \( i = 0, 1 \) we set

\[ \nu_i(n) = \text{number of } k \text{'s such that } M_k-1 = i, 1 \leq k \leq n . \]

\[ \square \]

But in this case the process \( T_n \) loses its ability to "detect" a strong ascending ladder point for the process \( S_n \). Actually it might happen that \( S_1, ..., S_{2n} \) have a strong ascending ladder point at \( S_{2n-1} \), yet \( T_1, ..., T_n \) have no strong ascending ladder point. Therefore one might not get explicit formulas as in Lemma 2.3. This problem of explicitly calculating the asymptotic branching probability still remains open.

Now we turn back to our original system. By the same arguments that we use to prove Theorem 2.1 we assert that under the assumptions made in Section 1 and an additional assumption \( \mathbb{P}\{ \xi_k^{(i)} > 0 \} > 0, \mathbb{P}\{ \eta_k^{(i)} > 0 \} > 0 \), we have

**Theorem 2.2.** As \( \varepsilon \downarrow 0 \) the process \( \hat{Y}_t^\varepsilon \) converges weakly for \( 0 < T < \infty \) in \( C_0T(\Gamma) \) with uniform topology to a process \( Y_t \) on \( \Gamma \) which is a Markov process on \( \Gamma \). It is deterministic inside the edges and only has stochasticity (i.e. certain branching probabilities) at the interior vertices.

### 3 Large deviations

We now calculate large deviation type asymptotics. We consider the simplest case when there is only one well. The general case follows from our result for one well case and will be discussed in the next section.
Suppose our well has width $D$. The perturbation for the collision at the walls is given by i.i.d and mutually independent sequences $\{\xi_k\}_{k \geq 1}$ and $\{\eta_k\}_{k \geq 1}$. We assume that $\mathbb{P}\{-M \leq \xi_k \leq M\} = \mathbb{P}\{-M \leq \eta_k \leq M\} = 1$ for some $M > 0$. We assume that $\mathbb{E}(\xi_k + \eta_k) > 0$. Both $\xi$ and $\eta$ have continuous density. This implies that the process $H_\varepsilon^t$ is bounded for time $0 \leq t \leq T < \infty$. Let us assume that for the time $0 \leq t \leq T$ we have $0 < H_0 \leq H_\varepsilon^t \leq \overline{H} < \infty$. Let $q_0^\varepsilon = q_0$. Let

$$Q_\varepsilon^t = q_0 + \text{the total horizontal distance that } q_\varepsilon^t \text{ traveled up to time } t.$$ 

The system $(H_\varepsilon^t, Q_\varepsilon^t)$ satisfies the equations:

$$\begin{aligned}
\dot{H}_\varepsilon^t & = -f(Q_\varepsilon^t), \\
\dot{Q}_\varepsilon^t & = \frac{1}{\varepsilon} \sqrt{2} H_\varepsilon^t.
\end{aligned} \tag{3.1}$$

Here random function $f(Q) = \sum_{k=1}^{\infty} (\xi_k \delta(Q - (2k - 1)D) + \eta_k \delta(Q - 2kD))$ where $\delta(\cdot)$ is the Dirac $\delta$-function.

Consider a piecewise linear modification $\tilde{H}_\varepsilon^t$ of the step function $\tilde{H}_\varepsilon^t$, as defined at the beginning of Section 2. We see that by (2.4) $\tilde{H}_\varepsilon^t$ is a good approximation of $H_\varepsilon^t$.

System (3.1) has fast component $Q$ and slow component $H$ and they depend on each other. Let $H_0 \leq h \leq \overline{H}$. Let $Q^h(t) = q_0 + t\sqrt{2h}$. Let $\beta \in \mathbb{R}$. Define

$$\mathcal{H}(h, \beta) = \lim_{T \to \infty} \frac{1}{T} \ln \mathbb{E} \left( \exp \left( -\beta \int_0^T f(Q^h(t)) dt \right) \right)$$

$$= \frac{\sqrt{2h}}{2D} \ln \mathbb{E} \left( \exp \left( -\beta (\xi + \eta) \right) \right). \tag{3.2}$$

Let $\mathcal{L}$ be the Legendre transform of $\mathcal{H}$:

$$\mathcal{L}(h, \alpha) = \sup_{\beta} (\alpha \beta - \mathcal{H}(h, \beta)).$$

Let $\varphi \in C_{[0,T]}([H_0, \overline{H}])$. Let

$$S_{0T}(\varphi) = \begin{cases} 
\int_0^T \mathcal{L}(\varphi_s, \dot{\varphi}_s) ds, & \text{for } \varphi \text{ absolutely continuous}, \\
+\infty, & \text{otherwise}. \tag{3.3}
\end{cases}$$

We have:

**Theorem 3.1.** The family $\tilde{H}_\varepsilon^t$, $0 < t < T$ satisfies the large deviation principle as $\varepsilon \downarrow 0$ in the space $C_{[0,T]}([H_0, \overline{H}])$ with normalizing factor $\varepsilon^{-1}$ and an action functional $S_{0T}(\varphi)$.
To be precise, Theorem 3.1 means the following (see [5, Ch.3]):

(0) The set $\Phi(s) = \{ \varphi \in C_{[0,T]}([H_0,H]) : S_{0T}(\varphi) \leq s \}$ is compact for every $s \geq 0$.

(I) For any $\nu > 0$, any $\delta > 0$, any $\varphi \in C_{[0,T]}([H_0,H])$, there exist $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we have

$$
P\{\rho_{0T}(\hat{H}^\varepsilon, \varphi) < \delta \} \geq \exp(-\varepsilon^{-1}(S_{0T}(\varphi) + \nu)).$$

(II) For any $\delta > 0$, any $\nu > 0$ and any $s > 0$ there exist an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we have

$$
P\{\rho_{0T}(\hat{H}^\varepsilon, \Phi(s)) \geq \delta \} \leq \exp(-\varepsilon^{-1}(s - \nu)).$$

Here for $\varphi, \psi \in C_{0T}([H_0,H])$ we denote $\rho_{0T}(\varphi, \psi) = \max_{0 \leq t \leq T} |\varphi(t) - \psi(t)|$ and $\rho_{0T}(\varphi, \Phi(s)) = \max_{\psi \in \Phi(s)} \max_{0 \leq t \leq T} |\varphi(t) - \psi(t)|$.

Let us consider an example.

**Example.** Let function $H_0(\beta) = \ln \mathbb{E} \exp(-\beta(\xi + \eta))$. We have $\mathcal{H}(h, \beta) = \frac{\sqrt{2h}}{2D} H_0(\beta)$. Using the convexity of exponential function, we get $H_0(\beta) \geq \ln \mathbb{E} \exp(-\beta(\xi + \eta)) = -\beta \mathbb{E}(\xi + \eta)$, i.e. $H_0(\beta) + \beta \mathbb{E}(\xi + \eta) \geq 0$. The minimum is achieved at $\beta = 0$.

Now we let $\varphi_t = H(t)$. Here $H(t)$ is the limiting motion of $H^\varepsilon_t$ as $\varepsilon \downarrow 0$. Standard averaging principle gives us

$$
\frac{dH(t)}{dt} = -\frac{\sqrt{2H(t)}}{2D} (\mathbb{E} \xi + \mathbb{E} \eta) , \quad H(0) = H_0^0.
$$

Now $\mathcal{H}(H(t), \beta) = \frac{\sqrt{2H(t)}}{2D} H_0(\beta)$, and

$$
\mathcal{L}(H(t), \hat{H}(t)) = \sup_{\beta} (\hat{H}(t)\beta - \mathcal{H}(H(t), \beta))
$$

$$
= \sup_{\beta} (\hat{H}(t)\beta - \frac{\sqrt{2H(t)}}{2D} H_0(\beta))
$$

$$
= -\frac{\sqrt{2H(t)}}{2D} \inf_{\beta} (\mathbb{E} \xi + \mathbb{E} \eta) \beta + H_0(\beta) = 0.
$$

This means that $S_{0T}(H(t)) = 0$, which is not surprising since $H(t)$ is the averaged motion of the system.

On the other hand, for any absolutely continuous trajectory $\varphi_t$ such that $\dot{\varphi}_t > 0$ for $0 \leq t \leq T$ we have $\mathcal{L}(\varphi_t, \dot{\varphi}_t) = \sup_{\beta} (\dot{\varphi}_t \beta - \mathcal{H}(\varphi_t, \beta)) > 0$ since $\mathcal{H}(\varphi_t, 0) = 0$ and
\[ \frac{\partial}{\partial \beta} \mathcal{H}(\varphi, \beta) \big|_{\beta=0} = -\frac{\sqrt{2\pi t}}{2D} \mathbb{E}(\xi + \eta) < 0 \] (see Lemma 3.2.1). This gives \( S_{0T}(\varphi) > 0 \) which means that there is a "difficulty" for the system to gain some energy. \( \Box \)

The **Proof** of Theorem 3.1 is based on a combination of Cramér’s large deviation principle for i.i.d. sums and the technique to calculate large deviations from an averaged system with full dependence, which was developed in [6], [7].

Let us first formulate an analogue of the classical Cramér’s large deviation principle for i.i.d. sums and the technique to calculate large deviations from an averaged system with full dependence, which was developed in [6], [7].

**Lemma 3.1.** Let \( \zeta_1, \ldots, \zeta_n, \ldots \) be a sequence of bounded i.i.d random variables, which have continuous densities and let

\[ \mathcal{H}_0(\beta) = \ln \mathbb{E} \exp(\beta \zeta_1) . \]

Let \( \mathcal{L}_0(\alpha) = \sup_{\beta \in \mathbb{R}} (\alpha \beta - \mathcal{H}_0(\beta)) \). Let \( \varepsilon > 0 \). Suppose integer \( n(\varepsilon) \to \infty \) as \( \varepsilon \downarrow 0 \). Suppose for a bounded \( -\infty < \underline{x} < x(\varepsilon) < \overline{x} < \infty \) we have \( |\text{argmax}_\beta (x(\varepsilon) \beta - \mathcal{H}_0(\beta))| \leq b < \infty \) is uniformly bounded. Then for any \( \nu > 0 \), there exist \( \delta > 0 \) such that, for any \( 0 < \delta < \delta(\varepsilon) < \overline{\delta} < \infty \), there exist \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) we have

\[ \exp(-n(\varepsilon)(\mathcal{L}_0(x(\varepsilon)) + \nu)) \leq \mathbb{P} \left\{ \left| \frac{\sum \zeta_{n(\varepsilon)}}{n(\varepsilon)} - x(\varepsilon) \right| < \delta(\varepsilon) \right\} \leq \exp(-n(\varepsilon)(\mathcal{L}_0(x(\varepsilon)) + \nu)) . \]

**Proof.** Let \( A(n) = \frac{1}{n}(\zeta_1 + \ldots + \zeta_n) \). Let \( A = \mathbb{E} A(n) \). We are estimating

\[ \mathbb{P} \{ x(\varepsilon) - \delta(\varepsilon) < A(n(\varepsilon)) < x(\varepsilon) + \delta(\varepsilon) \} . \]

About the upper bound. Consider first the case \( x(\varepsilon) - \delta(\varepsilon) > A \). We have, using Chebyshev inequality, for \( \beta \geq 0 \), that

\[ \mathbb{P} \{ x(\varepsilon) - \delta(\varepsilon) < A(n(\varepsilon)) \} \leq \exp(-\beta n(\varepsilon)(x(\varepsilon) - \delta(\varepsilon))) \mathbb{E}(\beta n(\varepsilon) A(n(\varepsilon))) \]

\[ = \exp(-\beta n(\varepsilon)(x(\varepsilon) - \delta(\varepsilon))) \mathbb{E} \prod_{k=1}^{n(\varepsilon)} \exp(\beta \zeta_k) \]

\[ = \exp(-n(\varepsilon)((x(\varepsilon) - \delta(\varepsilon))\beta - \mathcal{H}_0(\beta))) . \]

Since for \( x(\varepsilon) - \delta(\varepsilon) > A \) and \( \beta \geq 0 \) we have \( \mathcal{L}_0(x(\varepsilon) - \delta(\varepsilon)) = \sup_{\beta \geq 0} ((x(\varepsilon) - \delta(\varepsilon))\beta - \mathcal{H}_0(\beta)) \), we optimize the above inequality and we get

\[ \mathbb{P} \{ x(\varepsilon) - \delta(\varepsilon) < A(n(\varepsilon)) \} \leq \exp(-n(\varepsilon)\mathcal{L}_0(x(\varepsilon) - \delta(\varepsilon))) . \]
Since our choice of $x(\varepsilon)$ makes $|\text{argmax}_\beta (x(\varepsilon) \beta - \mathcal{H}_0(\beta))|$ uniformly bounded, the uniform continuity of $\mathcal{L}_0$ gives the upper bound in this case. That is, we can choose $\delta > 0$ small enough such that for $0 < \delta < \delta(\varepsilon) < \delta$ we have $\mathcal{L}_0(x(\varepsilon) - \delta(\varepsilon)) \geq \mathcal{L}_0(x(\varepsilon)) - \nu$.

In the case when $x(\varepsilon) + \delta(\varepsilon) < A$, we estimate, for $\beta \geq 0$, that

$$
\begin{align*}
\mathbb{P}\{- (x(\varepsilon) + \delta(\varepsilon)) < - A(n(\varepsilon))\} & \\
& \leq \exp(\beta n(\varepsilon)(x(\varepsilon) + \delta(\varepsilon))) \mathbb{E}(\beta n(\varepsilon) A(n(\varepsilon))) \\
& = \exp(\beta n(\varepsilon)(x(\varepsilon) + \delta(\varepsilon))) \prod_{k=1}^{n(\varepsilon)} \exp(-\beta \zeta_k) \\
& = \exp(-n(\varepsilon)((x(\varepsilon) + \delta(\varepsilon))(-\beta) - \mathcal{H}_0(-\beta))).
\end{align*}
$$

Now we use the fact that for $x(\varepsilon) + \delta(\varepsilon) < A$ we have $\mathcal{L}_0(x(\varepsilon) + \delta(\varepsilon)) = \sup_{\beta \geq 0}((x(\varepsilon) + \delta(\varepsilon))(-\beta) - \mathcal{H}_0(-\beta))$ and we apply a similar argument.

Now in the case of $x(\varepsilon) - \delta(\varepsilon) \leq A \leq x(\varepsilon) + \delta(\varepsilon)$, we choose $\delta > 0$ small enough such that $|\mathcal{L}_0(x(\varepsilon)) - \mathcal{L}_0(A)| < \nu/2$ and we notice that $\mathcal{L}_0(A) = 0$. This gives the trivial upper bound as $\varepsilon \downarrow 0$.

Now we prove the lower bound. Consider the unique solution of the equation

$$
\mathcal{H}_0'(\eta) = x(\varepsilon).
$$

By our assumptions on the uniform boundedness of $|\text{argmax}_\beta (x(\varepsilon) \beta - \mathcal{H}_0(\beta))|$ and about the boundedness an having density of $\zeta$’s it is easy to check that the solution of this equation exists and is unique. Now define a new measure $\hat{d}\mathbb{P}^\varepsilon$ in terms of $\mathbb{P}$ as

$$
\frac{d\hat{d}\mathbb{P}^\varepsilon}{d\mathbb{P}}(x) = \exp(\eta x - \mathcal{H}_0(\eta)).
$$

This $\hat{d}\mathbb{P}^\varepsilon$ is a probability measure since

$$
\int_{\mathbb{R}} d\hat{d}\mathbb{P}^\varepsilon = \frac{1}{\mathbb{E}\exp(\eta \zeta)} \int_{\mathbb{R}} \exp(\eta x) d\mathbb{P} = 1.
$$

Also under $\hat{d}\mathbb{P}^\varepsilon$ the expected value of $\zeta$ is $\hat{d}\mathbb{E}^\varepsilon \zeta = \frac{1}{\mathbb{E}\exp(\eta \zeta)} \int_{\mathbb{R}} x \exp(\eta x) d\mathbb{P} = \mathcal{H}_0'(\eta) = x(\varepsilon)$. Now we have

$$
\begin{align*}
\mathbb{P}\{|A(n(\varepsilon)) - x(\varepsilon)| < \delta(\varepsilon)\} & \\
& = \int_{\bigcup_{k=1}^{n(\varepsilon)} \{x_k - x(\varepsilon)\} < n(\varepsilon) \delta(\varepsilon)} d\mathbb{P}(dx_1) \ldots d\mathbb{P}(dx_n) \\
& \geq \exp(-n(\varepsilon)\delta(\varepsilon)\eta) \exp(-n(\varepsilon)\eta) \int_{\bigcup_{k=1}^{n(\varepsilon)} \{x_k - x(\varepsilon)\} < n(\varepsilon) \delta(\varepsilon)} \exp(\eta \sum_{k=1}^{n(\varepsilon)} x_k) d\mathbb{P}(dx_1) \ldots d\mathbb{P}(dx_n) \\
& = \exp(-n(\varepsilon)\delta(\varepsilon)\eta) \exp(-n(\varepsilon)(x(\varepsilon)\eta - \mathcal{H}_0(\eta))) \hat{d}\mathbb{P}^\varepsilon\{|A(n(\varepsilon)) - x(\varepsilon)| < \delta(\varepsilon)\} \\
& \geq \exp(-n(\varepsilon)\delta(\varepsilon)b) \exp(-n(\varepsilon)\mathcal{L}_0(x(\varepsilon))) \hat{d}\mathbb{P}^\varepsilon\{|A(n(\varepsilon)) - x(\varepsilon)| < \delta(\varepsilon)\}.
\end{align*}
$$

As we have, in this case $\hat{d}\mathbb{P}^\varepsilon\{|A(n(\varepsilon)) - x(\varepsilon)| \geq \delta(\varepsilon)\} \leq \frac{\hat{d}\mathbb{E}^\varepsilon|\zeta - x(\varepsilon)|^2}{n(\varepsilon)\delta^2} \to 0$ as $\varepsilon \downarrow 0$, we have $\hat{d}\mathbb{P}^\varepsilon\{|A(n(\varepsilon)) - x(\varepsilon)| < \delta(\varepsilon)\} \to 1$ as $\varepsilon \downarrow 0$. We choose $\varepsilon_0 = \varepsilon_0(\nu, \delta)$ small enough
such that for $0 < \varepsilon < \varepsilon_0$ we have $\hat{P}_x \{ |A(n(\varepsilon)) - x(\varepsilon)| < \delta(\varepsilon) \} \geq \exp(-n(\varepsilon)\nu/2)$. We then choose $\delta$ small enough such that $\delta b \leq \nu/2$. This then gives the lower bound. \hfill \Box

The next lemma gives some simple but important properties of the functions $\mathcal{H}(h, \beta)$ and $\mathcal{L}(h, \alpha)$, which will be used later.

Let us denote $\beta[h, \alpha] = \arg\max_\beta (\alpha \beta - \mathcal{H}(h, \beta))$. Let $h \in [H_0, \mathcal{H}]$.

**Lemma 3.2.** We have

1. $\mathcal{H}(h, 0) = 0$ and $\frac{\partial}{\partial \beta} \mathcal{H}(h, \beta)|_{\beta=0} < 0$.

2. For any $b > 0$ the functions $\mathcal{H}$ and $\frac{\partial}{\partial \beta} \mathcal{H}$ are uniformly continuous in $(h, \beta)$, $|\beta| < b$. The function $\mathcal{H}(h, \beta)$ is $C^\infty$ in the variables $h$ and $\beta$.

3. The function $\mathcal{H}(h, \beta)$ is strictly convex in $\beta$.

4. We have $\left| \frac{\partial}{\partial \beta} \mathcal{H}(h, \beta) \right| \leq U \sqrt{2h}$ for some constant $U > 0$. When $|\alpha| > U \sqrt{2h}$ we have $\mathcal{L}(h, \alpha) = +\infty$.

5. The set $A(h) = \{ \alpha : \mathcal{L}(h, \alpha) < \infty \}$ has nonempty interior.

6. Let $\hat{\alpha}$ be such that $\mathcal{L}(h, \hat{\alpha}) = 0$, then $\hat{\alpha}$ is in the interior of the set $A(h)$.

7. Let $|\beta[h, \alpha]| \leq b < \infty$. Then for any small $\kappa > 0$ and any $|\alpha' - \alpha| < \kappa$, $|h' - h| < \kappa$ we have $|\beta[h, \alpha] - \beta[h', \alpha']| < C(b, \kappa)$ and $|\mathcal{L}(h, \alpha) - \mathcal{L}(h', \alpha')| < C(b, \kappa)$ for a constant $C(b, \kappa) \downarrow 0$ as $\kappa \downarrow 0$.

**Proof.** For notational convenience let $\zeta = - (\xi + \eta)$.

1. Let $\mathcal{H}_0(\beta) = \ln \mathbb{E} \exp(\beta \zeta)$. We have $\mathcal{H}(h, \beta) = \frac{\sqrt{2h}}{2D} \mathcal{H}_0(\beta)$. It is obvious that $\mathcal{H}(h, 0) = 0$. Also we have $\left. \frac{\partial}{\partial \beta} \mathcal{H}(h, \beta) \right|_{\beta=0} = \frac{\sqrt{2h}}{2D} \mathbb{E} \zeta < 0$.

2. We have

$$\mathcal{H}(h, \beta) = \frac{\sqrt{2h}}{2D} \ln \mathbb{E} \exp(\beta \zeta)$$

and

$$\frac{\partial}{\partial \beta} \mathcal{H}(h, \beta) = \frac{\sqrt{2h} \mathbb{E} \zeta \exp(\beta \zeta)}{2D \mathbb{E} \exp(\beta \zeta)}$$

so that they are uniformly continuous in $(h, \beta)$ for $|\beta| < b$. One can take higher derivatives also so that the function $\mathcal{H}(h, \beta)$ is $C^\infty$ in both variables $h$ and $\beta$.

3. We can calculate

$$\frac{\partial^2}{\partial \beta^2} \mathcal{H}(h, \beta) = \frac{\sqrt{2h} (\mathbb{E} \zeta^2 \exp(\beta \zeta) \mathbb{E} \exp(\beta \zeta) - (\mathbb{E} \zeta \exp(\beta \zeta))^2)}{2D (\mathbb{E} \exp(\beta \zeta))^2} > 0$$

since the Cauchy-Schwarz inequality is now a strict one. This means that the function $\mathcal{H}(h, \beta)$ is strictly convex in $\beta$.  

14
4. From 2 we can have \( \left| \frac{\partial}{\partial \beta} H(h, \beta) \right| \leq U \sqrt{2h} \). This gives, when \(|\alpha| > U \sqrt{2h}\), that \( \mathcal{L}(h, \alpha) = +\infty \).

5. Since we assumed that \( \mathbb{P}\{|\xi| \leq M\} = \mathbb{P}\{|\eta| \leq M\} = 1 \) and both have density, we can assume that there exist \( c < 0, C > 0 \) such that \( \mathbb{P}(c < \xi + \eta < C) = 1 \) and there exist \( \kappa > 0 \) and \( \mu > 0 \) such that \( \mathbb{P}(\xi + \eta > C - \mu) \geq \kappa \) and \( \mathbb{P}(\xi + \eta < c + \mu) \geq \kappa \), also \( C - \mu > c + \mu \). From here we get, that for \( \beta > 0 \), \( H(h, \beta) \geq -\frac{\sqrt{2h}}{2D} (c + \mu)\kappa \beta \) and for \( \beta < 0 \), \( H(h, \beta) \geq -\frac{\sqrt{2h}}{2D} (c - \mu)\kappa \beta \). This fact helps us to conclude that \( \{\alpha \in \mathbb{R}, -\frac{\sqrt{2h}}{2D} (C - \mu)\kappa \beta < \alpha < -\frac{\sqrt{2h}}{2D} (c + \mu)\kappa \} \subset A^\alpha(h) \).

6. We are proving that the number \( \hat{\alpha} \) which makes \( \mathcal{L}(h, \hat{\alpha}) = 0 \) is in the interior of the set \( A(h) \). Since \( \hat{\alpha} = \frac{\partial}{\partial \beta} H(h, [\beta, \hat{\alpha}]) \) and \( \hat{\alpha}[h, \hat{\alpha}] = H(h, [\beta, \hat{\alpha}]) \). By strict convexity of \( H \) in \( \beta \) this means that \( \hat{\alpha} = \frac{\partial}{\partial \beta} H(h, \beta)|_{\beta = 0} \). The statement reduces to proving that \( -\frac{\sqrt{2h}}{2D} (C - \mu)\kappa \beta < \frac{\partial}{\partial \beta} H(h, \beta)|_{\beta = 0} < -\frac{\sqrt{2h}}{2D} (c + \mu)\kappa \), which is straightforward.

7. Suppose \(|\beta[h, \alpha]| \leq b < \infty \). Then by strict convexity of \( H(h, \beta) \) in \( \beta \) we see that \( \beta[h, \alpha] \) is the unique solution of the equation \( H_0(\beta) = \frac{2D\alpha}{\sqrt{2h}} \). This also gives \(|\alpha| \leq K(b) \) for some constant \( K(b) > 0 \). For any \(|\alpha' - \alpha| < \kappa \) and \(|h' - h| < \kappa \) we have \( \frac{2D\alpha}{\sqrt{2h}} - \frac{2D\alpha'}{\sqrt{2h'}} < V \kappa \) for some \( V > 0 \). Therefore from the smoothness of the function \( H_0(\beta) \) and the strict monotonicity of \( H_0'(\beta) \) we conclude that the unique solution \( \beta[h', \alpha'] \) of the equation \( H_0'(\beta) = \frac{2D\alpha'}{\sqrt{2h'}} \) is close to \( \beta[h, \alpha] \): \(|\beta[h, \alpha] - \beta[h', \alpha']| < C(b, \kappa) \). This gives also the fact that

\[
|\mathcal{L}(h, \alpha) - \mathcal{L}(h', \alpha')| \\
\leq |\alpha|\beta[h, \alpha] - \alpha'\beta[h', \alpha']| + |H(h, \beta[h, \alpha]) - H(h', \beta[h', \alpha'])| \\
\leq |\alpha|\beta[h, \alpha] - \beta[h', \alpha']| + |\alpha - \alpha'||\beta[h', \alpha']| + \\
|H(h, \beta[h, \alpha]) - H(h', \beta[h, \alpha])| + |H(h', \beta[h, \alpha]) - H(h', \beta[h', \alpha'])| \\
< C(b, \kappa)
\]

for some \( C(b, \kappa) > 0 \), and we have \( C(b, \kappa) \downarrow 0 \) as \( \kappa \downarrow 0 \). \( \square \)

The next lemma is an analogue of Lemma 5 in [7].

**Lemma 3.3.** For any \( \nu > 0 \) there exist some \( \Delta(\nu) > 0 \), \( \delta_0(\nu) > 0 \) such that for any fixed \( 0 < \delta_0 < \delta_0(\nu) \) and fixed \( 0 < \Delta < \Delta(\nu) \), there exist \( \delta_1(\nu, \Delta) > 0 \) such that for any \( 0 < \delta_1 < \delta_1(\nu, \Delta) \) on the set \( \tilde{H}_{t_0}^\varepsilon - h_0 \) < \( \delta_0 \), uniformly with respect to \( t_0, h_0, \tilde{H}_{t_0}^\varepsilon, q_0 \),
under the condition \(\left|\beta[h_0, \frac{h_1 - h_0}{\Delta}]\right| \leq b < \infty\), as \(\varepsilon \downarrow 0\), we have
\[
\exp(-\varepsilon^{-1}(\Delta(L(h_0, \frac{h_1 - h_0}{\Delta})) - C(b)\nu\Delta - \tilde{C}(b, \delta_0))) \geq \mathbb{P}\{|\hat{H}_t - h_1| < \delta_1|\mathcal{F}_t\} \geq \exp(-\varepsilon^{-1}(\Delta(L(h_0, \frac{h_1 - h_0}{\Delta})) + C(b)\nu\Delta + \tilde{C}(b, \delta_0)))
\]
where \(C(b) > 0\) is a constant and \(\tilde{C}(b, \delta) \downarrow 0\) as \(\delta \downarrow 0\).

**Proof.** Let \(\zeta = -(\xi + \eta)\). Let \(\mathcal{H}_0(\beta) = \ln \mathbb{E}\exp(\beta\zeta)\). Let \(\mathcal{L}_0(\alpha) = \sup(\alpha(\beta - \mathcal{H}_0(\beta))\).

Let \(\Delta > 0\). Let \(N^\varepsilon(t_0, \Delta)\) be the number of crossings that the process \(\mathcal{Q}_t\) make with the set \(\{Q = kD, k \in \mathbb{N}\}\) during time \([t_0, t_0 + \Delta]\). Let \(n^\varepsilon(t_0, \Delta) = N^\varepsilon(t_0, \Delta)/2\) if \(N^\varepsilon(t_0, \Delta)\) is even and \(n^\varepsilon(t_0, \Delta) = (N^\varepsilon(t_0, \Delta) - 1)/2\) if \(N^\varepsilon(t_0, \Delta)\) is odd. Since
\[
\frac{1}{\varepsilon}\sqrt{2H_0} \leq \dot{Q}_t^\varepsilon = \frac{1}{\varepsilon}\sqrt{2H_t^\varepsilon} \leq \frac{1}{\varepsilon}\sqrt{\frac{2}{\varepsilon^2}}
\]
we have
\[
\frac{1}{\varepsilon}\Delta\sqrt{2H_0} \geq \int_{t_0}^{t_0 + \Delta} \dot{Q}_t^\varepsilon dt \geq \frac{1}{\varepsilon}\Delta\sqrt{2H_0}.
\]
This together with the fact that
\[
(N^\varepsilon(t_0, \Delta) - 1)D \leq \int_{t_0}^{t_0 + \Delta} \dot{Q}_t^\varepsilon dt \leq N^\varepsilon(t_0, \Delta)D.
\]
gives
\[
C_1\frac{\Delta}{\varepsilon} \leq N^\varepsilon(t_0, \Delta) \leq C_2\frac{\Delta}{\varepsilon}
\]
for some \(C_1 > 0, C_2 > 0\). Since we assumed that \(\mathbb{P}\{|\xi| \leq M\} = \mathbb{P}\{|\eta| \leq M\} = 1\) we see that \(|H_t^\varepsilon - H_{t_0}\| \leq C_3\Delta\) for \(t \in [t_0, t_0 + \Delta]\) and some \(C_3 > 0\). As we have \(|H_{t_0}^\varepsilon - h_0| < \delta_0\) we have \(|H_t^\varepsilon - h_0| < \delta_0 + C_3\Delta\) for \(t \in [t_0, t_0 + \Delta]\). This gives
\[
|\sqrt{2H_t^\varepsilon} - \sqrt{2h_0}| \leq C_4(\delta_0 + \Delta)
\]
for some \(C_4 > 0\) and \(t \in [t_0, t_0 + \Delta]\). Therefore
\[
\left|N^\varepsilon(t_0, \Delta)D - \frac{1}{\varepsilon}\sqrt{2h_0}\Delta\right| \leq \int_{t_0}^{t_0 + \Delta} (\dot{Q}_t^\varepsilon - \frac{1}{\varepsilon}\sqrt{2h_0})dt + D \leq \frac{C_4(\delta_0 + \Delta)}{\varepsilon}\Delta + D.
\]
This gives
\[
\left|\varepsilon n^\varepsilon(t_0, \Delta) - \frac{\sqrt{2h}\Delta}{2D}\right| \leq C_5(\varepsilon + (\delta_0 + \Delta)\Delta)
\]
for some \(C_5 > 0\). This implies that for \(\varepsilon + (\delta_0 + \Delta)\Delta << \Delta\) we have \(n^\varepsilon(t_0, \Delta) \rightarrow \infty\) as \(\varepsilon \downarrow 0\). Also, in this case \(C_6\Delta \leq \varepsilon n^\varepsilon(t_0, \Delta) \leq C_7\Delta\) for some \(C_6, C_7 > 0\).
Let \( \zeta_k = -(\xi_k + \eta_k) \). Now we have, for \( \varepsilon > 0 \) small enough,
\[
\mathbb{P}\{ |H_{t_0}^{\varepsilon} + h_1| < \delta_1 | F_{t_0} \} \\
= \mathbb{P}\{ |H_{t_0}^{\varepsilon} + H_{t_0}^{\varepsilon} - h_1| < \delta_1 | F_{t_0} \} \\
\geq \mathbb{P}\left\{ |\varepsilon(\zeta_1 + \ldots + \zeta_n(t_0, \Delta)) + H_{t_0}^{\varepsilon} - h_1| < \delta_1/2 | F_{t_0} \right\}.
\]

Fix some \( \lambda > 0 \) such that \( C_5 \lambda < \frac{\sqrt{2h}}{8D} \). We then choose \( \delta_0(\nu) \) and \( \Delta(\nu) \) such that \( \delta_0(\nu) + \Delta(\nu) < \lambda \) and we fix some \( 0 < \delta_0 < \delta_0(\nu) \) and \( 0 < \Delta < \Delta(\nu) \). We then choose \( \varepsilon \) small enough such that \( C_5 \varepsilon/\Delta < \frac{\sqrt{2h}}{8D} \). We see that for a chosen \( \delta_1(\nu, \Delta) > 0 \) such that \( \delta_1(\nu, \Delta)/\Delta \) is small, for any \( 0 < \delta_1 < \delta_1(\nu, \Delta) \), we can make \( \frac{\delta_1/2}{\varepsilon_n(\varepsilon)(t_0, \Delta)} \) to be smaller than the \( \bar{\eta} \) in Lemma 3.1. Also, we notice that \( \frac{\delta_1/2}{\varepsilon_n(\varepsilon)(t_0, \Delta)} \) is bounded away from 0 as \( \varepsilon \downarrow 0 \), for fixed \( \delta_1 \) and \( \Delta \). On the other hand, since we have
\[
\arg\max_{\beta} \left( \frac{h_1 - H_{t_0}^{\varepsilon}}{\varepsilon_n(\varepsilon)(t_0, \Delta)} \beta - H_0(\beta) \right) = \arg\max_{\beta} \left( (h_1 - H_{t_0}^{\varepsilon})\beta - \varepsilon_n(\varepsilon)(t_0, \Delta)H_0(\beta) \right)
\]
which, by Lemma 3.2.7, is close to
\[
\beta[h_0, \frac{h_1 - h_0}{\Delta}] = \arg\max_{\beta} \left( (h_1 - h_0)\beta - \frac{\sqrt{2h_0}}{2D} \Delta H_0(\beta) \right),
\]
say, within a distance of \( \kappa(\delta_0, (\delta_0 + \Delta)\Delta) \), as \( \varepsilon \) is small. And this \( \kappa(\delta_0, (\delta_0 + \Delta)\Delta) \to 0 \) as \( (\delta_0, \Delta) \to (0, 0) \). We shall choose our \( \delta_0(\nu) \) and \( \Delta(\nu) \) to be small such that \( |\kappa(\delta_0(\nu), (\delta_0(\nu) + \Delta(\nu))\Delta(\nu))| < 1 \). By our assumption \( |\beta[h_0, \frac{h_1 - h_0}{\Delta}]| \leq b < \infty \).

Now we see that Lemma 3.1 applies. We then get, on the set \( \{ |H_{t_0}^{\varepsilon} + h_0| < \delta_0 \} \), as \( \varepsilon \) is small, we have
\[
\mathbb{P}\{ |H_{t_0}^{\varepsilon} + h_1| < \delta_1 | F_{t_0} \} \\
\geq \mathbb{P}\left\{ |\zeta_1 + \ldots + \zeta_n(t_0, \Delta) - \frac{h_1 - H_{t_0}^{\varepsilon}}{\varepsilon_n(\varepsilon)(t_0, \Delta)}| < \frac{\delta_1/2}{\varepsilon_n(\varepsilon)(t_0, \Delta)} | F_{t_0} \right\} \\
\geq \exp(-n\varepsilon(t_0, \Delta)(\sup_{\beta \in \mathbb{R}} \left( \frac{h_1 - H_{t_0}^{\varepsilon}}{\varepsilon_n(\varepsilon)(t_0, \Delta)} \beta - H_0(\beta) \right) + \nu)) .
\]
Now we use Lemma 3.2.7 to get the bound
\[
\mathbb{P}\{ |H_{t_0}^{\varepsilon} + h_1| < \delta_1 | F_{t_0} \} \\
\geq \exp(-\varepsilon^{-1} \left( \varepsilon_n(t_0, \Delta) \sup_{\beta \in \mathbb{R}} \left( \frac{h_1 - h_0}{\varepsilon_n(\varepsilon)(t_0, \Delta)} \beta - H_0(\beta) \right) + C_9(\nu) \Delta + C_9(b, \delta_0) \right) \) \\
= \exp(-\varepsilon^{-1} (\Delta \sup_{\beta \in \mathbb{R}} \left( \frac{h_1 - h_0}{\Delta} \beta - \frac{\varepsilon_n(t_0, \Delta)}{\Delta} H_0(\beta) \right) + C_9 \nu \Delta + C_9(b, \delta_0)) \) \\
\geq \exp(-\varepsilon^{-1} (\Delta(\nu) \frac{h_1 - h_0}{\Delta}) + C_9 \nu \Delta + C_9(b, \delta_0)).
\]
Here the auxiliary constant $C_8 > 0$ and positive functions $C_{10}(b, \delta_0, \Delta) \to 0$ as $(\delta_0, \Delta) \to (0, 0)$ and $C_9(b, \delta_0) \to 0$ as $\delta_0 \to 0$. We choose $\delta_0(\nu)$ and $\Delta(\nu)$ small enough such that $C_8\nu + C_{10}(b, \delta_0(\nu), \Delta(\nu)) \leq C_{11}(b)\nu$ for $C_{11} > 0$. This gives, for some $C(b) > 0$ and $C(b, \delta_0) \downarrow 0$ as $\delta_0 \downarrow 0$, the bound

$$\mathbb{P}\{|H_{t_0+\Delta} - h_1| < \delta_1|F_{t_0}\} \geq \exp(-\varepsilon^{-1}(\Delta(L(h_0, \frac{h_1-h_0}{\Delta}))) + C(b)\nu\Delta + \bar{C}(b, \delta_0)) \right)$$.

As we have (2.4), this also gives, as $\varepsilon$ is small, that on the set $\{|\hat{H}_{t_0}^\varepsilon - h_0| < \delta_0\}$ we have

$$\mathbb{P}\{|\hat{H}_{t_0+\Delta}^\varepsilon - h_1| < \delta_1|F_{t_0}\} \geq \exp(-\varepsilon^{-1}(\Delta(L(h_0, \frac{h_1-h_0}{\Delta}))) + C(b)\nu\Delta + \bar{C}(b, \delta_0)) \right)$$.

Similarly one can estimate

$$\mathbb{P}\{|\hat{H}_{t_0+\Delta}^\varepsilon - h_1| < \delta_1|F_{t_0}\} \leq \exp(-\varepsilon^{-1}(\Delta(L(h_0, \frac{h_1-h_0}{\Delta}))) - C(b)\nu\Delta - \bar{C}(b, \delta_0)) \right)$$.

□

Remark. In the proof of Theorem 3.1 we will iteratively use this Lemma and we emphasize that the choice of $\delta_1$ does not depend on the choice of $\delta_0$ (of course, provided that $\delta_0 < \delta_0(\nu)$). Also, the choice of small $\varepsilon$ may depend on $\nu$, $\Delta$, $\delta_1$, $\delta_1/\Delta$ (coming from the dependence of $\varepsilon$ on $\Delta$ in Lemma 3.1).

Proof of Theorem 3.1.

1. Set-up. Let $L$ be the Legendre transform of $\mathcal{H}$:

$$L(h, \alpha) = \sup_{\beta} (\alpha \beta - \mathcal{H}(h, \beta))$$.

And the action functional is defined as

$$S_{0T}(\varphi) = \left\{ \begin{array}{ll}
\int_0^T L(\varphi_s, \dot{\varphi}_s)ds, & \text{for } \varphi \in C_{0T}([H_0, H]) \text{ absolutely continuous}, \\
+\infty, & \text{otherwise}. \end{array} \right.$$.

Part (0) of the large deviation principle can be shown as Lemma 7.4.2 of [5].

2. First part of the proof. The lower bound (I).

Let $S(\varphi) < \infty$. We show that, given any $\nu > 0$, any $\delta > 0$, we have for $\varepsilon$ small enough, that
\[ \varepsilon \ln \mathbb{P}\{\rho_{0T}(\hat{H}^c, \varphi) < \delta\} \geq -S(\varphi) - \nu. \]

Assume that for any \( s \), \( L(\varphi_s, \dot{\varphi}_s) < \infty \) for any \( s \). The reason is the same as in [7], Section 4, Step 1. By Lemma 3.2.4 we can assume that \( \sup_{0 \leq s \leq T} |\dot{\varphi}_s| \leq U \sqrt{2H} \) (\( U \) is the constant coming from Lemma 3.2.4).

3. By Lemma 3.2.5 for each \( s \in [0, T] \) the set \( \{\alpha : L(\varphi_s, \alpha) < \infty\} \) has non-empty interior \( L^o[\varphi_s] \). Since \( L(\varphi_s, \dot{\varphi}_s) < \infty \) we have, as in Section 4, Step 5 of [7] that \( L(\varphi_s, \dot{\varphi}_s) = \liminf_{\alpha \to \varphi_s, \alpha \in L^o[\varphi_s]} L(\varphi_s, \alpha) \). For each such \( \alpha \) there exists a (unique in our case since \( \mathcal{H}(h, \beta) \) is strictly convex in \( \beta \)) finite adjoint \( \beta[\varphi_s, \alpha] \). We have \( \mathcal{H}(\varphi_s, \beta) = \alpha \beta[\varphi_s, \alpha] - L(\varphi_s, \alpha) \) and \( L(\varphi_s, \alpha) = \alpha \beta[\varphi_s, \alpha] - \mathcal{H}(\varphi_s, \beta) \). We then choose \( \dot{\varphi}_s = \alpha \in L^o[\varphi_s] \) so that the value \( L(\varphi_s, \dot{\varphi}_s) \) is close to \( L(\varphi_s, \dot{\varphi}_s) \).

Put \( \tilde{\varphi}_t = \hat{H}_0^t + \int_0^t \tilde{\varphi}_s ds \). We can choose this new curve to be as close to \( \varphi_t \) as we like, therefore we can make

\[ \left| \int_0^T L(\varphi_s, \dot{\varphi}_s) ds - S_{0T}(\varphi) \right| \leq \nu/3. \]

4. For any \( s \) we choose a measurable \( \tilde{\alpha}_s \) such that \( L(\varphi_s, \tilde{\alpha}_s) = 0 \). This is the same as in Section 4, Step 6 of [7]. We see that \( |\tilde{\alpha}_s| \leq U \sqrt{2H} \) for the constant \( U \) in Lemma 3.2.4. Also, \( \tilde{\alpha}_s \in L^o[\varphi_s] \) by Lemma 3.2.6. For this \( \tilde{\alpha}_s \) the corresponding adjoint \( \beta[\varphi_s, \tilde{\alpha}_s] = \arg\max_\beta (\beta \tilde{\alpha}_s - \mathcal{H}(\varphi_s, \beta)) \) exists, is unique and finite.

5. As is the same in Section 4, Step 7 of [7], we take for given \( b \) that

\[ \varphi_t^b = \hat{H}_0^t + \int_0^t \left( \tilde{\varphi}_s 1(|\beta[\varphi_s, \tilde{\varphi}_s]| \leq b) + \tilde{\alpha}_s 1(|\beta[\varphi_s, \tilde{\varphi}_s]| > b) \right) ds. \]

Since we choose \( |\tilde{\alpha}_s| \leq U \sqrt{2H} \) in Step 4 of our proof we can find a \( b \) such that the curve \( \varphi_t^b \) is still close to \( \varphi \) in \( \rho_{0T} \) norm, and the values \( \int_0^T L(\varphi_s, \varphi_s^b) ds \) and \( \int_0^T L(\varphi_s, \dot{\varphi}_s) ds \) are close to each other, say

\[ \left| \int_0^T L(\varphi_s, \varphi_s^b) ds - S_{0T}(\varphi) \right| \leq 2\nu/3. \]

At the same time, we make \( |\beta[\varphi_s, \varphi_s^b]| \leq b \). And for \( b \) large enough we make \( \{\rho_{0T}(\hat{H}^c, \varphi) < \delta\} \supset \{\rho_{0T}(\hat{H}^c, \varphi^b) < \delta/2\} \).

6. Similarly as in Section 4, Step 9 of [7], we change our functions \( \varphi \) and \( \varphi_t^b \) into a step function \( \psi \) and a piecewise linear function \( \chi \) on \([0, T]\) so that first \( \psi_s = \varphi_{[s/|\Delta|]} \) and \( \chi_s = \varphi_{[s/|\Delta|]}^b \) and the steplength \( \Delta \) of \( \psi, \chi \) satisfies \( \Delta < \Delta(\nu) \) (the value from Lemma 3.3). Secondly,

\[ \left| \int_0^T L(\psi_s, \dot{\chi}_s) - S_{0T}(\varphi) \right| \leq \nu. \]
Thirdly,
\[ \{ \rho_{0T}(\hat{H}^\varepsilon, \varphi) < \delta \} \subset \{ \rho_{0T}(\hat{H}^\varepsilon, \chi) < \delta' \} \]
if \( \delta' \) is small w.r.t. \( \delta \). At last, our choice of \( \chi \) can make all the Fenchel-Legendre adjoint to the \( \chi \) variable \( \beta[\psi(m-1)\Delta, \hat{x}(m-1)\Delta+] = \arg\max_\beta(\beta\hat{x}(m-1)\Delta + \mathcal{H}(\psi(m-1)\Delta, \beta)) \) uniformly bounded, \( |\beta[\psi(m-1)\Delta, \hat{x}(m-1)\Delta+]| \leq b \).

To achieve these goals, we need to choose \( \delta' << \delta \) and \( \Delta \) small enough such that \( \rho_{0T}(\varphi, \chi) < \delta' \) is small and \( \rho_{0T}(\chi, \psi) < \delta' \). \( \psi \) is not in the space \( C_{0T}([H_0, H]) \) but we can still use the distance \( \rho_{0T} \) is small. This \( \Delta \) is chosen based on given small \( \nu, \delta, \delta' \) and the value \( b \) found in Step 5.

7. The next step is the same as in Section 4, Step 10 of [7]. Let \( N\Delta = T \). Let \( \varphi^\Delta = (\varphi_\Delta, \ldots, \varphi_N\Delta) \) and \( (\hat{H}^\varepsilon)^\Delta = (\hat{H}_\Delta, \ldots, \hat{H}_N^\varepsilon) \). First of all we have

\[ \{ \rho_{0T}(\hat{H}^\varepsilon, \chi) < \delta' \} \supset \{ \rho_{0T}^{\text{discrete}}((\hat{H}^\varepsilon)^\Delta, \chi^\Delta) < \delta'' \} \]
if ever \( \delta'' \) and \( \Delta \) are small. The \( \delta'' \) and \( \Delta \) are chosen based on given small \( \delta' \). Here \( \rho_{0T}^{\text{discrete}}((\psi^\Delta, \chi^\Delta) = \sup_m |\psi_m\Delta - \varphi_m\Delta| \) and the inclusion comes from the fact that as \( \Delta \) is small we have (regarding \( \chi^\Delta \) and \( (\hat{H}^\varepsilon)^\Delta \) as step functions also)

\[ \rho_{0T}(\hat{H}^\varepsilon, \chi) \leq \rho_{0T}^{\text{discrete}}((\hat{H}^\varepsilon)^\Delta, \chi^\Delta) + \rho_{0T}((\hat{H}^\varepsilon)^\Delta, \hat{H}^\varepsilon) + \rho_{0T}(\chi^\Delta, \chi) \leq \delta'' + C\Delta < \delta' \]
if ever \( \delta'' < \delta''(\delta') \) and \( \Delta \leq \Delta(\delta') \).

Then we estimate

\[ P\{ \rho_{0T}^{\text{discrete}}(\hat{H}_\Delta, \chi^\Delta) < \delta'' \} \geq E \prod_{m=1}^N 1(|\hat{H}_m\Delta - \chi_m\Delta| < \delta'''_m) \]
with \( \delta'''_1 < \delta'''_2 < \ldots < \delta'''_m < \delta''' \wedge \delta_0(\nu) \wedge \delta_1(\nu, \Delta) \) to be chosen later (\( \delta_0(\nu) \) and \( \delta_1(\nu, \Delta) \) are from Lemma 3.3).

8. Let us estimate the conditional expectation \( E(1(|\hat{H}_m\Delta - \chi_m\Delta| < \delta'''_m)|\mathcal{F}_{(m-1)\Delta}) \) on the set \( \{|\hat{H}^\varepsilon_{(m-1)\Delta} - \chi_{(m-1)\Delta}| < \delta'''_{m-1}\} \).

We apply Lemma 3.3 and Lemma 3.2.7 on the set \( \{|\hat{H}^\varepsilon_{(m-1)\Delta} - \chi_{(m-1)\Delta}| < \delta'''_{m-1}\} \) to get the estimate

\[ E(1(|\hat{H}^\varepsilon_{m\Delta} - \chi_{m\Delta}| < \delta'''_m)|\mathcal{F}_{(m-1)\Delta}) \]
\[ \geq \exp(-e^{-1}(\Delta\mathcal{L}(\chi_{(m-1)\Delta}, \hat{x}_{(m-1)\Delta}+) + C(b + \kappa(\delta'), \delta'''_{m-1})\nu \Delta + \tilde{C}(b + \kappa(\delta'), \delta'''_{m-1}))) \]
\[ \geq \exp(-e^{-1}(\Delta\mathcal{L}(\psi_{(m-1)\Delta}, \hat{x}_{(m-1)\Delta}+) + A(b, \delta')\Delta + KC(b, \delta'''_{m-1})\nu \Delta + \tilde{K}C(b, \delta'''_{m-1}))) \]
\[ \geq \exp(-e^{-1}(\Delta(\mathcal{L}(\psi_{(m-1)\Delta}, \hat{x}_{(m-1)\Delta}+) + A(b)\nu \Delta + K\tilde{C}(b, \delta'''_{m-1}))) \].

Here \( \kappa(\delta') \to 0 \) as \( \delta' \downarrow 0 \). We have used the fact that \( \rho_{0T}(\chi, \psi) < \delta' \) and Lemma 3.2.7, as well as the fact that \( |\beta[\psi_{(m-1)\Delta}, \hat{x}_{(m-1)\Delta}+]| \leq b \). We are choosing \( \delta' \) small such
Consider two vertices $E$, see [5, Ch.6] and compare with [1]). We will explain below what this is through an example.

Due to the stochasticity of the limiting process at interior vertices of the graph $\Gamma$, the metastability phenomenon in our case will be metastability of probability distributions rather than metastability of single states (as in the classical Freidlin-Wentzell theory, see [5, Ch.6] and compare with [1]). We will explain below what this is through an example.

We consider generic case when all the width and depth of the wells are different. Consider two vertices $E_1$ and $E_2$ of our graph $\Gamma$ (see Fig.1). The vertices $E_1$ and $E_2$ might be exterior vertices (like $V_1, V_2, V_3, V_4$ in Fig.1) or interior vertices (like $O_b, O_6, O_7$ in Fig.1). We suppose that the energy levels corresponding to $E_1$ and $E_2$ are $H_{E_1}$ and $H_{E_2}$, and $H_{E_2} > H_{E_1}$. Let us first assume that $E_1$ and $E_2$ can be joined by one edge $I_{N(E_1, E_2)}$. Here $N(E_1, E_2)$ is the number of the well that has energy level between $H_{E_1}$ and $H_{E_2}$ (recall that under our convention every well has a highest and lowest energy level). Recall that the well $N(E_1, E_2)$ has width $D_{N(E_1, E_2)}$. Let
\[ V(E_1, E_2) = \inf \{ S_{0T}^{N(E_1, E_2)}(\varphi) : H_{E_1} \leq \varphi_t \leq H_{E_2}, 0 \leq t \leq T < \infty, \varphi_0 = H_{E_1}, \varphi_T = H_{E_2} \}, \]

and

\[ V(E_2, E_1) = 0. \]  

(4.1)

Here the functional \( S_{0T}^{N(E_1, E_2)} \) is defined by

\[ S_{0T}^{N(E_1, E_2)}(\varphi) = \int_0^T \mathcal{L}^{N(E_1, E_2)}(\varphi_t, \dot{\varphi}_t) dt \]

if \( \varphi \) is absolutely continuous and \( +\infty \) otherwise.

The function

\[ \mathcal{L}^{N(E_1, E_2)}(h, \alpha) = \sup_{\beta} (\alpha \beta - \mathcal{H}^{N(E_1, E_2)}(h, \beta)) , \]

(4.3)

where \( \alpha, \beta \in \mathbb{R} \) and \( h \geq 0 \) is the Legendre transform of the function

\[ \mathcal{H}^{N(E_1, E_2)}(h, \beta) = \frac{\sqrt{2h}}{2D_{N(E_1, E_2)}} \ln \mathbb{E} \exp(-\beta (\xi^{N(E_1, E_2)} + \eta^{N(E_1, E_2)})) . \]  

(4.4)

(If we are at some ”small” well, i.e., it contains no smaller wells we make \( \mathcal{H}^{N(E_1, E_2)}(H_{E_1}, \beta) = \frac{\sqrt{2H_{E_1}}}{2D_{N(E_1, E_2)}} \ln \mathbb{E} \exp(-\beta (\xi^{N(E_1, E_2)} + \eta^{N(E_1, E_2)})) \mathbb{1}(\xi^{N(E_1, E_2)} < 0, \eta^{N(E_1, E_2)} < 0) \).

In particular, we see that our function \( V(E_1, E_2) \) depends on the width \( D_{N(E_1, E_2)} \) of the \( N(E_1, E_2) \)-th well, the energy levels \( H_{E_1} \) and \( H_{E_2} \) of the \( N(E_1, E_2) \)-th well and properties of the random variables \( \xi^{N(E_1, E_2)} \) and \( \eta^{N(E_1, E_2)} \) which give perturbations at the left and right walls when the particle is in the \( N(E_1, E_2) \)-th well.

One can verify that \( V(E_1, E_2) \) and \( V(E_2, E_1) \) define the ”quasi-potential” for all adjacent vertices \( E_1 \) and \( E_2 \) (with \( H_{E_2} > H_{E_1} \)) on our graph \( \Gamma \). To do this, we shall notice that by similar arguments as we did in Section 3, the action functional for the perturbed dynamical system \( \hat{Y}_{t}^{\beta} = (\hat{H}_{t}^{\beta}, K(\hat{H}_{t}^{\beta}, \hat{q}_{t}^{\beta})) \) on the graph \( \Gamma \) shall be defined by

\[ S_{0T}(\varphi, K) = \int_0^T \mathcal{L}(\varphi_s, K(s), \varphi_s) ds \]

where \( \mathcal{L}(\varphi_s, K(s), \varphi_s) = \sup_{\beta} (\varphi_s \beta - \mathcal{H}(\varphi_s, K(s), \beta)) \).

Here \( \varphi : [0, T] \rightarrow [H_0, H] \) is absolutely continuous (otherwise the action functional is \(+\infty \) and \( H > \hat{H}_{t}^{\beta} \) for \( 0 \leq t \leq T \)). The function \( K(s) : [0, T] \rightarrow \{1, 2, ..., N\} \) where \( N \) is the number of edges of graph \( \Gamma \). The function

\[ \mathcal{H}(h, K, \beta) = \frac{\sqrt{2h}}{2D_K} \ln \mathbb{E} \exp(-\beta (\xi^{(K)} + \eta^{(K)})) \]

whenever \( (h, K) \) does not correspond to the bottom of a ”small” well and it is \( \mathcal{H}(h_0, K_0, \beta) = \frac{\sqrt{2h_0}}{2D_{K_0}} \ln \mathbb{E} \exp(-\beta (\xi^{(K_0)} + \eta^{(K_0)})) \mathbb{1}(\xi^{(K_0)} < 0, \eta^{(K_0)} < 0) \) when \( (h_0, K_0) \) corresponds to
the bottom of a "small" well. Since we assume that $\mathbb{E}(\xi + \eta) > 0$, we find (compare with the example given in Section 3) that the minimum in the definition of the quasi-potential between $E_1$ and $E_2$ is achieved within the class of functions that satisfy $H_{E_1} \leq \varphi_t \leq H_{E_2}$ for $0 \leq t \leq T$, as defined in (4.1).

Now for any two vertices $F_1$ and $F_2$ on the graph $\Gamma$, let

$$V(F_1, F_2) = \min_{(E_1, \ldots, E_m)} \sum_{i=1}^{m-1} V(E_i, E_{i+1}).$$

(4.5)

Here $(E_1, \ldots, E_m)$ is a path of $\Gamma$ for which $E_1 = F_1, E_m = F_2$ and each pair $E_i, E_{i+1}$ can be joined by an edge of the graph $\Gamma$.

One can verify that the function $V(F_1, F_2)$ defines the "quasi-potential" between $F_1$ and $F_2$, as was defined in $[5, \text{Ch.6}]$.

In particular, one can easily check that for any interior vertex $O_i$, there is an exterior vertex $V_k$ such that $V(O_i, V_k) = 0$. Therefore interior vertices are unstable (compare with $[5, \text{Ch.6, Lemma 6.4.3}]$).

Now let us consider the example given in Fig.1. We suppose that, after using (4.1)- (4.4), we have the following: $V(V_1, O_5) = 2, V(V_2, O_5) = 1, V(O_5, O_6) = 1, V(O_6, O_7) = 1, V(V_3, O_6) = 6, V(V_4, O_7) = 5$ and $V(O_7, O_6) = V(O_6, O_5) = V(O_6, V_3) = V(O_5, V_1) = V(O_5, V_2) = V(O_7, V_4) = 0$.

Suppose our process $\hat{Y}_t^{\varepsilon} = (\hat{H}_t^{\varepsilon}, K_t^{\varepsilon})$ starts from a point $(H_0, 7)$ with $H_0$ large enough. Here the process $\hat{H}_t^{\varepsilon}$ is the piecewise linear modification defined at the beginning of Section 2 and $\hat{Y}_t^{\varepsilon} = Y(\hat{H}_t^{\varepsilon}, \hat{q}_t^{\varepsilon})$ is the identification map introduced in Section 1.

Let $Y_t$ be the (weak) limiting process of $\hat{Y}_t^{\varepsilon}$ as $\varepsilon \downarrow 0$ on the graph $\Gamma$. It is a Markov process on $\Gamma$ which is a deterministic motion within each edge and only has stochasticity at the interior vertices (see Theorem 2.1 and Theorem 2.2). In particular, let the branching probabilities at vertex $O_7$ be given by $p_6$ (for entering $I_6$) and $p_4 = 1 - p_6$ (for entering $I_4$); the branching probabilities at vertex $O_6$ be given by $p_5$ (for entering $I_5$) and $p_3 = 1 - p_5$ (for entering $I_3$); and the branching probabilities at vertex $O_5$ be given by $p_1$ (for entering $I_1$) and $p_2 = 1 - p_1$ (for entering $I_2$).

After long enough finite time, as $\varepsilon \downarrow 0$, the position of the process $\hat{Y}_t^{\varepsilon}$ will be given by a probability distribution which is approximately $(p_1 p_5 p_6, p_2 p_5 p_6, p_3 p_6, p_4) = (p_1 p_5 p_6, (1 - p_1)p_5 p_6, (1 - p_5) p_6, 1 - p_6)$ among the exterior vertices $(V_1, V_2, V_3, V_4)$. Let us denote the distribution $(p_1 p_5 p_6, p_2 p_5 p_6, p_3 p_6, p_4)$ by $U_0$.

Let us now consider behavior of the process $\hat{Y}_t^{\varepsilon}$ at exponentially long time scale $t = t(\varepsilon) \propto \exp(C \varepsilon^{-1})$. To this end we first remind the reader of some classical results in $[5, \text{Ch.6}]$. Consider a set of $K_i$’s, $i = 1, \ldots, l$, which are equilibriums of a deterministic dynamical system, say $Z_{t}$. Suppose the corresponding stochastic dynamical system $Z_{t}^{\varepsilon}$, which is a small random perturbation of $Z_{t}$, satisfies a large deviation principle with normalizing factor $\frac{1}{\varepsilon}$ and the quasi-potentials between $K_i$ and $K_j$ are given by $V(K_i, K_j)$. 23
We decompose the set of $K_i$'s into hierarchy of cycles $\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(s)}$, unified into the last cycle of maximal rank. For any cycle $\pi^{(k)}$, $0 \leq k \leq s$, we define
\[
C(\pi^{(k)}) = A(\pi^{(k)}) - \min_{t \in \pi^{(k)}} \min_{g \in G_{\pi^{(k)}}(t)} \sum_{(m \rightarrow n) \in g} V(K_m, K_n),
\]
where
\[
A(\pi^{(k)}) = \min_{g \in G(L \setminus \pi^{(k)})} \sum_{(m \rightarrow n) \in g} V(K_m, K_n).
\]

Here $L$ is the set of indices for the points $K_1, \ldots, K_i$. The set $G(L \setminus \pi^{(k)})$ is the collection of all $L \setminus \pi^{(k)}$-graphs and the set $G_{\pi^{(k)}}(t)$ is the collection of all $t$-graphs restricted to $\pi^{(k)}$ (see [5, Ch.6, Section 6]).

Then for sufficiently small $\rho > 0$ we have,
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}_x \tau_{\pi^{(k)}} = C(\pi^{(k)}),
\]
uniformly in $x$ belonging to some $\rho$-neighborhood of the set $\bigcup_{i \in \pi^{(k)}} K_i$, where $\tau_{\pi^{(k)}}$ is the first exit time for the system $Z_t^{\pi}$ to exit from $\pi^{(k)}$ (see Theorem 6.6.2 of [5, Ch.6]).

Also, the asymptotic as $\varepsilon \downarrow 0$ exit position $Z_{\tau_{\pi}}^{\varepsilon}$ in $L \setminus \pi^{(k)}$ for the system to exit from $\pi^{(k)}$ is given by one of the $K_i$'s which is the end of the chains of arrows in an $L \setminus \pi^{(k)}$ graph that minimizes the sum in (4.7) (see Theorem 6.6.1 of [5, Ch.6]).

Now let us turn back to our example. We start from the distribution $U_0 = (p_1p_5p_6, p_2p_5p_6, p_3p_6, p_4)$. The cycles of rank 0 are just the vertices $V_1, V_2, V_3, V_4$ and we call them $\pi^{(i-1)} = \{V_i\}$, $i = 1, \ldots, 4$. We calculate $C(\pi^{(0)}) = 2$, $C(\pi^{(1)}) = 1$, $C(\pi^{(2)}) = 6$, $C(\pi^{(3)}) = 5$. Therefore by using (4.8), we see that at time scale $t = t(\varepsilon) \asymp \exp(\varepsilon^{-1})$, our system will be jumping out from $V_2$ first. By determining the $\{V_1, V_3, V_4, O_5, O_6, O_7\}$-graph minimizing the sum in $A(\pi^{(1)})$, the first vertex that it approaches will be $V_1$ (to be precise, it will be $O_5$ but $O_3$ is unstable). Taking into account that there is a branching probability at vertex $O_5$, we see that one such transition will make the distribution be $(p_1p_5p_6 + p_1p_2p_5p_6, p_2^2p_5p_6, p_3p_6, p_4)$ and $n$ such transitions will make the distribution be $(p_1p_5p_6 + p_1p_2p_5p_6 + p_1p_2^2p_5p_6 + \ldots + p_1p_2^n p_5p_6, p_2^{n+1} p_5p_6, p_3p_6, p_4)$. Therefore after many such transitions, when $n$ is very large, the distribution will be approximately $U_1 = (p_5p_6, 0, p_3p_6, p_4)$. The distribution $U_1$ will be the "metastable distribution" over time scale $t = t(\varepsilon) \asymp \exp(\varepsilon^{-1})$ (compare with [1, Theorems 4.1 and 4.2]).

We increase our time scale. Since $C(\pi^{(0)}) = 2$, $C(\pi^{(2)}) = 6$, $C(\pi^{(3)}) = 5$, at time scale $t = t(\varepsilon) \asymp \exp(2\varepsilon^{-1})$, the system begins to jump out from $V_1$ and transit to $V_2$, which makes the distribution $U_1 = (p_5p_6, 0, p_3p_6, p_4)$ be $(p_1p_5p_6, p_2p_5p_6, p_3p_6, p_4)$, $(p_2^2p_5p_6, p_2p_5p_6 + p_2p_1p_5p_6, p_3p_6, p_4)$, ..., $(p_1^{n+1}p_5p_6, p_2p_5p_6 + p_2p_1p_5p_6 + \ldots + p_2^n p_5p_6, p_3p_6, p_4)$, and so on. But notice that one such transition happens at time scale $t = t(\varepsilon) \asymp \exp(2\varepsilon^{-1})$, within which transitions from $V_2$ to $V_1$, as described in the above para-
over time scale $t = t(\varepsilon) \approx \exp(2\varepsilon - 1)$, $U_1 = (p_5p_6, 0, p_3p_6, p_4)$ will still be the metastable distribution of our system.

Over time scale $t = t(\varepsilon) \approx \exp(2\varepsilon - 1)$, our system has already formed a cycle \{V_1, O_5, V_2\}, which we call $\pi^{(4)}$. We calculate $C(\pi^{(4)}) = 3 < C(\pi^{(3)}) \land C(\pi^{(2)})$. That means, at time scale $t = t(\varepsilon) \approx \exp(2\varepsilon - 1)$, jumping out from cycle $\pi^{(4)}$ happens first. By determining the \{V_3, V_4, O_6, O_7\}-graph minimizing the sum in $A(\pi^{(4)})$, we will first jump to $V_3$ (again, it is actually $O_6$ but $O_6$ is unstable). Taking into account of the branching probabilities, this will make the distribution be finally $U_2 = (0, 0, p_6, p_4)$. The distribution $U_2$ is the metastable distribution over time scale $t = t(\varepsilon) \approx \exp(3\varepsilon - 1)$.

We now consider the cycle \{V_1, O_5, V_2, O_6, V_3\} and we call it $\pi^{(5)}$. We calculate $C(\pi^{(5)}) = 7$. Since $C(\pi^{(2)}) = 6$, $C(\pi^{(3)}) = 5$, by the same reasoning above, over time scale $t = t(\varepsilon) \approx \exp(5\varepsilon - 1)$, transition from $V_4$ to $V_3$ happens first and that leads to a new metastable distribution $U_3 = (0, 0, 1, 0)$.

Over time scale $t = t(\varepsilon) \approx \exp(6\varepsilon - 1)$, transition from $V_3$ to $V_4$ happens. By the same reasoning above, we see that this leads to the fact that the metastable distribution over time scale $t = t(\varepsilon) \approx \exp(6\varepsilon - 1)$ is still $U_3$. After that time scale, although new transition might still happen, the metastable distribution will remain to be $U_3$.

Acknowledgements

I thank my advisor M.Freidlin for posing this problem to me and for pointing out the references [1], [3], [6], [7], as well as many useful discussions.

References

[1] A.Athreya, M.Freidlin, Metastability for random perturbations of nearly-Hamiltonian systems, Stochastics and Dynamics, Vol. 8, No.1 (2008) 1-21.

[2] A.Dembo, O.Zeitouni, Large Deviations techniques and applications, Springer, 1998.

[3] W.Feller, An Introduction to Probability Theory and Its Applications, Vol.2, Second Edition, John Wiley and Sons, 1971.

[4] M.Freidlin, W.Hu, On stochasticity in nearly-elastic systems, Stochastics and Dynamics, online, DOI:10.1142/S0219493711500201.

[5] M.Freidlin, A.Wentzell, Random perturbations of dynamical systems, Springer, 1998.

[6] A.Yu.Veretennikov, On large deviations in the averaging principle for SDE’s with a “full dependence”, Annals of Probability, 27, 1999, No.1, 284-296.

[7] A.Yu.Veretennikov, On large deviations in the averaging principle for SDE’s with a “full dependence”, correction, arxiv. MATH.PR.0502098.