The Kardar–Parisi–Zhang model of a random kinetic growth: effects of a randomly moving medium

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Abstract
The effects of a randomly moving environment on a randomly growing interface are studied by the field theoretic renormalization group. The kinetic roughening of an interface is described by the Kardar–Parisi–Zhang (KPZ) stochastic differential equation while the velocity field of the moving medium is modelled by the Navier–Stokes equation with an external random force. It is found that the large-scale, long-time (infrared) asymptotic behavior of the system is described by four non-equilibrium universality classes associated with possible fixed points of the renormalization group equations. In addition to the previously known regimes of asymptotic behavior (ordinary diffusion, kinetic growth, and passively advected scalar field), a new nontrivial regime (non-equilibrium universality class) is found. That regime corresponds to a process in which the motion of the environment and the nonlinearity of the KPZ equation are important simultaneously. The fixed points coordinates, their regions of stability and the critical exponents are calculated to the first order of the expansion in $\varepsilon = 2 - d$ (one-loop approximation). However, the new fixed point is either infrared repulsive ($d > 2$) or corresponds to imaginary coupling constant ($d < 2$). Possible physical interpretation in terms of mapping to certain reaction-diffusion models and Bose systems is discussed.

Keywords: statistical mechanics, nonequilibrium systems, turbulence, renormalization group
1. Introduction

An importance of kinetic growth as a study subject takes its root in how wide-spread the phenomenon of kinetic roughening of a surface is: it is observed in flame and smoke propagation, in the growth of colloid aggregates and tumours, in a deposition of a substance on a substrate, and so on [1–9]. As a surface or a phase boundary (interface) evolves with time, it becomes rougher and rougher and develops coarser features. In particular, the nth order structure function of a kinetic growth behaves as [1–5]:

\[ S_n(t, r) \approx r^\chi F_n(r/t^{1/\zeta}) \]  

(1.1)

Here \( F_n(\cdot) \) is a scaling function, \( \chi \) is the roughness exponent, and \( \zeta \) is the dynamical exponent. The structure function \( S_n(t, r) \) is obtained by averaging the difference between the heights of the interface profile over the statistical ensemble:

\[ S_n(t, r) \equiv \langle (h(t, x) - h(0, 0))^n \rangle, \]

(1.2)

where the brackets \( \langle \ldots \rangle \) denote the averaging, \( h \) is the interface height, and \( |x| = r \).

The power law (1.1) describes the asymptotic behavior in the infrared (IR) range (the time \( t \) and space \( r \) differences are large in comparison with characteristic microscopic scales). Self-similar (scaling) behavior with universal exponents in the IR range is one of the features of equilibrium nearly-critical systems, thus, universality classes (types of critical or scaling behavior) of kinetic roughening can be established by using the methods and approaches developed for the study of critical phenomena.

While there is a number of microscopic models for kinetic growth [6–9] (most notably, the Edwards–Wilkinson model [7]), a simplified model for a smoothed (coarse-grained) height field may be sufficient to describe universal properties of kinetic roughening (see similar point for nearly-critical systems in, e.g. [10, 11]). The Kardar–Parisi–Zhang (KPZ) nonlinear stochastic differential equation [12] is one such model:

\[ \partial_t h = \varepsilon_0 \partial^2 h + \lambda_0 (\partial h)^2/2 + f. \]

(1.3)

Here the height field \( h(x) = h(t, x) \) depends on the \( d \)-dimensional coordinate \( x \), \( \partial_t = \partial/\partial t \), \( \partial = \{\partial_i\} = \{\partial/\partial x_i\} \), \( \partial^2 = (\partial \cdot \partial) = \partial_i \partial_i \) is the Laplace operator and \( (\partial h)^2 = (\partial h \cdot \partial h) = \partial_i h \partial_i h \); the summations over repeated tensor indices are always implied. The first term in the right-hand side of the equation (1.3) stands for the surface tension (i.e. \( \varepsilon_0 > 0 \)) while the second term models growth along the interface local normal. The sign of the parameter \( \lambda_0 \) determines whether the growth is positive or negative. The term \( f = f(x) \) is the Gaussian random noise with a zero mean and a pair correlation function

\[ \langle f(x)f(x') \rangle = C \delta(t - t') \delta^{(d)}(x - x'), \]

(1.4)

where \( C > 0 \). It is sufficient to consider only the case \( C = 1 \); indeed, any nontrivial amplitude \( C \) can be scaled out (absorbed by the fields and other parameters of the model). Thus, we set \( C = 1 \) in the following. (The case \( C < 0 \) or, equivalently, that of complex \( \lambda_0 \), also has a meaningful interpretation and will be discussed later.)

When \( \lambda_0 = -1 \) and \( d = 1 \), the KPZ equation is, in fact, the Burgers equation. Thus, it actually appeared for the first time in [13] in terms of the vector field \( \mathbf{u} = \partial h \). The KPZ model is also connected via mapping to a model of directed polymers in random media and to a model of Bose many-particle system with attraction; see e.g. [14].

Moreover, the first two terms on the right-hand side of the equation (1.3) are, in fact, the simplest local ones that respect the symmetries \( h \to h + \text{const} \) and \( O(d) \). In this sense, the KPZ
model can be expected to describe a variety of numerous, rather different in their physical
nature, non-equilibrium, disordered and driven diffusive systems. For example, in [15] the
KPZ model was used in the study of large-scale matter distribution in the Universe.

Numerous modifications of the original KPZ model were introduced: random noise with
finite correlation time [16], vector or matrix field $h$ [17], modified form of the nonlinearity
[18], higher-order diffusion term [19], inclusion of the turbulent advection [20], spatially cor-
related noise [21], $1/f$-noise [22], conserved form [23], coupling with directed percolation
[24], anisotropic modifications [25–28]. In connection to the latter, it is also worth mentioning
continuous anisotropic models of self-organized criticality [29]. Some mathematical aspects
of the KPZ model were studied, e.g. in [30].

The field theoretic renormalization group (RG) approach is often used to great effect in the
study of critical phenomena [10, 11]. The RG approach allows one to find IR attractive fixed
points of renormalizable field theoretic models. The fixed points correspond to universality
classes and provide their critical exponents.

The standard perturbative RG analysis of the KPZ model [12, 13, 31, 32] proved that the
field theory related to the stochastic problem (1.3) and (1.4) is multiplicatively renormal-
able. There is a nontrivial fixed point of the RG equations that corresponds to a universality
class described by the exponents $\chi = 0, z = 2$. The fixed point becomes IR attractive when
$\varepsilon > 0$, however, in this case the coordinates of the fixed point do not lie inside the physical
range of the model parameters ($C > 0, \sigma_0 > 0, \lambda_0$ is a real number). All these results are ‘per-
turbatively exact’, i.e. exact in all orders of the expansion in $\varepsilon \equiv 2 - d$.

The KPZ model, nevertheless, could possess a hypothetical ‘essentially non-perturbative’
IR attractive fixed point that is not ‘visible’ within any kind of perturbation theory. Under
this assumption one can obtain the exact values $\chi = 1/2, z = 3/2$ for $d = 1$ (dictated by the
fluctuation-dissipation theorem along with the Galilean symmetry) [12, 13]. The exact values
for $d = 2$ and $d = 3$ can be derived after making further, rather nontrivial, assumptions [33].
Functional (‘exact’ or ‘non-perturbative’) RG, indeed, provided the evidence of the existence
of this fixed point [34–36] which turned out to be stable even in the anisotropic case [27].
Although convincing, this analysis is based on a certain appropriate Ansatz and does not
involve (at least formal) small parameter, which complicates comparison to other approaches
and systematic improvement of the results. Thus, further investigation is still desirable.

The behavior of real systems near their critical points can be very sensitive to external dis-
turbances, presence of impurities and so on; see, e.g. [37, 38] for discussion and references. In
particular, deterministic or chaotic flows can drastically affect the critical behavior [39–45],
thus, it is important to consider their effects. In this connection, the paper [46] should be men-
tioned. There, the Edwards–Wilkinson model with a constant drift was studied. It was shown
that, in contrast to naive dimensional analysis, both the diffusion and convection terms are
relevant for the resulting scaling behaviour.

The aim of this paper is to study the influence of the random motion of the fluid environ-
ment on the IR behavior of kinetic growth. The advection by the velocity field $\nu(x) \equiv \{\nu_i(x)\}$
is introduced by the usual ‘minimal’ replacement in the equation (1.3)

$$\partial_t h \to \nabla_i h \equiv \partial_i h + (\nu \cdot \partial) h = \partial_i h + \nu_i \partial_i h,$$

(1.5)

where $\nabla_i$ is the Galilean covariant (Lagrangean) derivative that also preserves the symmetry
$h \to h+\text{const}$ of the original KPZ model. The field $h(x)$ is assumed to have no effect on the
dynamics of the velocity field $\nu(x)$ (‘passive’ advection). This approximation is sufficient
for a preliminary qualitative understanding of what can happen if the fluid motion is taken
into account. Dynamics of the velocity field are described by the microscopic model of an
incompressible viscous fluid near thermal equilibrium, namely, by the Navier–Stokes (NS) equation with a thermal noise as an external random force [13, 47]:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \nabla P + \mathbf{F}. \quad (1.6)$$

Here $\mathbf{v}$ is the velocity field vector (it is transverse due to incompressibility: $(\partial \cdot \mathbf{v}) = \partial_i v_i = 0$) with a zero mean, $P$ is the pressure, $\mathbf{F}$ is the transverse external random force per unit mass (all of these quantities depend on $x$), $\nu_0$ is the kinematic coefficient of viscosity. The both equations (1.3) and (1.6) are studied on the entire $t$ axis and are supplemented by the retardation condition and by the condition that the fields $h, v$ vanish asymptotically for $t \to -\infty$. The random force $\mathbf{F}$ represents a thermal noise that has a Gaussian statistics with zero mean and the following correlation function:

$$\langle F_i(t, \mathbf{x}) F_j(t', \mathbf{x}') \rangle = \delta(t-t') D_0 \int_{|k| > m} \frac{dk}{(2\pi)^d} \mathbb{P}_y(k) k^2 \exp(i(k \cdot \mathbf{r})). \quad (1.7)$$

Here $\mathbb{P}_y(k) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |k|$ is a wave number, $D_0 > 0$ is an amplitude factor and $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. The integral cutoff at $k = m$ provides IR regularization; here $m \equiv 1/L$ while $L$ is an analogue of an integral turbulence scale. The precise form of the cutoff is unimportant, thus, the sharp cutoff is chosen for simplicity.

The equation (1.6) with the random force $\mathbf{F}$ (1.7) describes spontaneous velocity fluctuations (relaxation to equilibrium of sufficiently small externally induced fluctuations) and was studied in relation to the problem of long tails in Green functions [13, 47]. In the framework of our investigation this choice of the velocity statistics is the most instructive one. Indeed, the both nonlinearities in the scalar equations (1.3) and (1.6) (the KPZ interaction and the advection term) become logarithmic at $d = 2$. This means that they become IR relevant simultaneously and, thus, they are equally important for the analysis of the IR asymptotic behavior. If it were not so, one of them would be IR irrelevant for some values of $d$ and would give but corrections to the leading IR asymptotic behavior.

Previously, the KPZ model coupled with the Kraichnan’s rapid-change velocity ensemble was considered in [20]. The Kraichnan’s model is described by a ‘synthetic’ random Gaussian field, white in time and self-similar in space [48]. Some time ago it attracted enormous attention among the turbulent community because of the insight it offers into the origin of intermittency and anomalous scaling in fluid turbulence; see [49] and references therein. Owing to its relative simplicity it allows for some exact results and accurate simulations.

However, it is desirable to employ more realistic models, in particular, to include finite correlation time. Unfortunately, synthetic Gaussian ensembles with finite correlation time (like, e.g. the Ornstein–Uhlenbeck process) suffer from the lack of Galilean symmetry, which may lead to some pathologies; see, e.g. [50].

In this paper, we employ the stochastic NS equation, which implies finite correlation time, non-Gaussianity and, at the same time, is manifestly Galilean covariant.

Moreover, now the velocity field has its own dynamics, which opens the possibility to study the feedback of the advected fields on the fluid dynamics itself (‘active scalar field’). Recent study of two coupled generalized stochastic KPZ equations [24] shows that a highly nontrivial pattern of types of critical behaviour can arise.

A possible step in the study of all those interesting effects is to consider the KPZ model coupled with the velocity field described by the NS equation. It should also be noted that the calculations in the problem containing the NS equations are, naturally, a somewhat more involved than in the case of the problem with the simple synthetic ensemble, and the field theoretic RG is a most suitable tool to deal with such models.
The paper consists of six sections. The field theory equivalent to the full stochastic problem (1.3), (1.4), (1.7) and its diagrammatic technique are described in section 2. Analysis of the ultraviolet (UV) divergences of the model and discussion of its multiplicative renormalizability are given in section 3. Derivation of the RG equations is given in section 4. Fixed points of the RG equations, possible universality classes and corresponding critical exponents are discussed in section 5. Analysis of the composite fields $h^n(x)$ required to derive expressions like (1.1) is also discussed in this section.

It is found out that the RG equations possess a new nontrivial fixed point, in addition to the line of the Gaussian fixed points (trivial regime of critical behavior—a free field theory that describes ordinary diffusion), purely ‘kinematic’ fixed point (the KPZ nonlinearity is IR irrelevant—a passively advected scalar field universality class), and the curve of the fixed points related to the pure KPZ model (random motion of the medium is irrelevant—universality class of kinetic growth). The fixed points coordinates and their regions of IR stability are derived to the first order of the expansion in $\varepsilon = 2 - d$ (one-loop approximation), while the corresponding critical exponents are found in one-loop approximation or exactly.

A new nontrivial fixed point corresponds to a process in which the motion of the medium and the KPZ nonlinearity are relevant simultaneously. It turns out, that for the values of $\varepsilon > 0$ the IR asymptotic behavior is governed by this new fixed point. However, the coordinates of the fixed point lie in the unphysical region. Thus, careful physical interpretation of these results requires special attention.

The in-depth discussion of the possible solutions to the physical interpretation of the obtained results, non-perturbative fixed points, and application of the functional RG to the problem is given in section 6.

The one-loop structure of the Green functions containing divergences to be eliminated by renormalization, and calculation of corresponding diagrams are presented in the appendix.

2. The field theory of the model

According to a general statement (see, e.g. monographs [10], section 17.2 and [11], section 5.3 and references therein), the stochastic problem (1.3) and (1.4) without the motion of the medium taken into account is equivalent to the field theory with the set of fields $\Phi = \{h, h'\}$ and the action functional

$$S(\Phi) = \frac{1}{2} h'^2 h + h' \left\{ -\partial_t h + z_0 \nabla^2 h + \frac{1}{2} \lambda_0 (\partial h)^2 \right\}.$$  \hspace{1cm} (2.1)

Here and below, the integrations over $x = (t, x)$ are always implied, e.g.

$$\frac{1}{2} h'^2 h = \frac{1}{2} \int dt \int dx \ h'(t, x) h'(t, x).$$  \hspace{1cm} (2.2)

Correlation functions and response functions of the stochastic problem (1.3) and (1.4) are identified with Green functions of the field theory (2.1), i.e. they are represented by the functional averages over the full set of fields $\Phi = \{h, h'\}$ with the weight $\exp S(\Phi)$.

The bare propagators are determined by the free part of the action (2.1) and have the following form in the frequency–momentum ($\omega-k$) representation:
The model includes the interaction vertex $\lambda_0 h'(\partial h)^2/2 = h' V h h/2$ with the vertex factor $V = -i k_i (-ip_j) \lambda_0$, where $k$ and $p$ are the momenta flowing out of the vertex via the fields $h$.

Coupling with the velocity field $v(x)$ is introduced by the substitution (1.5) in equations (1.3) and (2.1). The full problem is then equivalent to the field theory with the four fields $\Phi = \{h, h', v, v'\}$ and the action functional

$$S(\Phi) = \frac{1}{2} D_0 (\partial v')^2 + \left( v' \cdot \left\{ -\partial_i v - (v \cdot \partial) v + \nu_0 \partial^2 v \right\} \right)$$

$$+ \frac{1}{2} h' h' + h' \left\{ -\partial_i h - (v \cdot \partial) h + \nu_0 \partial^2 h + \frac{1}{2} \lambda_0 (\partial h)^2 \right\}. \quad (2.4)$$

The first term contains the factor $(\partial v')^2 = \partial_i v_i' \partial_j v_j'$ (the summations over repeated tensor indices are implied) because the correlation function (1.7) contains the factor $k^2$ in frequency–momentum representation.

The transversality of the auxiliary field $v'$ makes it possible to drop the purely longitudinal pressure contribution $\partial P$ from equation (1.6) in the action functional (2.4).

Thus, another four propagators emerge for the full model (2.4):

$$\langle v v' \rangle_0 = \langle v' v \rangle_0^* = \frac{P_{\phi}(k)}{-i\omega + \nu_0 k^2},$$

$$\langle v_v' \rangle_0 = \frac{D_0 k^2 P_{\phi}(k)}{\omega^2 + \nu_0^2 k^2}, \quad \langle v' v \rangle_0 = 0. \quad (2.5)$$

There are also the two new vertices: firstly, $(v' \cdot (v \cdot \partial) v) = v' V_{ij} v_i v_j / 2$ with the vertex factor $V_{ij} = i (k_i \delta_{ij} + k_j \delta_{ij})$ where $k$ is the momentum flowing into the vertex via the field $v'$; secondly, $h'(v \cdot \partial) h = h' V_j v_j / h$ with the vertex factor $V_j = -i k_j = ip_j$ where $k$ is the momentum flowing into the vertex via the field $h$ and $p$ is the momentum flowing into the vertex via the field $h'$.

There are three coupling constants:

$$g_0 = D_0 / \nu_0^3 \sim \Lambda^5, \quad \lambda_0 = \lambda_0 / \nu_0^3 \sim \Lambda^{3/2},$$

$$w_0 = \nu_0 / \nu_0. \quad (2.6)$$

The first two relations are obtained from the dimension analysis (see the next section) and define the typical UV momentum scale $\Lambda$. The constant $w_0$ is completely dimensionless and as such must be considered alongside the other coupling constants.

3. UV divergences and renormalization

The analysis of UV divergences is based on the canonical dimensions analysis (‘power counting’); see, e.g. monographs [10, 11]. There are two independent scales to be considered in the dynamic models of the type (2.4): the time scale $T$ and the length scale $L$; see, e.g. [11].
and the momentum dimension $d^\kappa$ are the numbers of corresponding fields entering into the function $v - \varepsilon \lambda$ and in some function $m h + \cdots$. Substituting the data from table 1 to (3.1) one obtains:

\[ [F] \sim [T]^{-d^F} [L]^{-d^\kappa}. \]

Normalization conditions

\[ d^k = -d^\kappa = 1, \quad d^c = d^\omega = 0, \quad d^\nu = d^\rho = 0, \quad d^\nu = -d^\rho = 1 \]

and the requirement that each term of the action functional is dimensionless further determine $d^F$ and $d^\kappa$. The total canonical dimension is defined as $d_F = d^F + 2d^\kappa$ (in the free theory, $\partial \propto \partial^2$).

Canonical dimensions of the fields and the parameters of the theory (2.4) are presented in the table 1. The table also includes renormalized parameters (the ones without the subscript ‘0’ and the reference mass scale $\mu$, an additional parameter of the renormalized theory. It is needed for a correct renormalization of the coupling constants and will be defined later.

Thus, the model is logarithmic at $d = 2$ when all of the coupling constants simultaneously become dimensionless. The UV divergences in the Green functions manifest themselves as poles in $\varepsilon = 2 - d$.

The total canonical dimension of an arbitrary one-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_1^{\cdots \cdots \cdots}$ with $\Phi = \{h, h', v, v'\}$ in the frequency–momentum representation is given by the relation:

\[ d^\Gamma = d + 2 - d_h N_h - d_{h'} N_{h'} - d_v N_v - d_{v'} N_{v'}, \tag{3.1} \]

where $N_h, N_{h'}, N_v, N_{v'}$ are the numbers of corresponding fields entering into the function $\Gamma$; see, e.g. [11], section 5.15, equation (5.88) and [51], section 1.4, equation (1.26).

The total dimension $d^\Gamma$ in the logarithmic theory (i.e. at $\varepsilon = 0$) is the formal index of the UV divergence: $d^\Gamma = d^\Gamma |_{\varepsilon = 0}$. Substituting the data from table 1 to (3.1) one obtains:

\[ d^\Gamma \sim 4 - 2N_{h'} - N_v - N_{v'}. \tag{3.2} \]

However, if a number of external momenta occurs as an overall factor in all diagrams of a certain Green function, the index of divergence should be modified. In the present case the fields $h$ and $v'$ do, indeed, enter the vertices $h'(\partial h) \wedge h' (v \cdot \partial) v$ and $(v' \cdot (v \cdot \partial) v) = v' v \partial_v = - (\partial_v v') v \partial_v$ only in the form of spatial derivatives. Thus, any appearance of $h$ or $v'$ in some function $\Gamma$ gives an external momentum, and the real index of divergence is given by the expression $d^\Gamma_{\text{ext}} = d^\Gamma - N_h - N_{v'}$, hence

\[ d^\Gamma_{\text{ext}} = 4 - N_h - 2N_{h'} - N_v - 2N_{v'}. \tag{3.3} \]

Superficial UV divergences can only be present in the one-irreducible functions that correspond to the non-negative index of divergence $d^\Gamma_{\text{ext}}$; see; e.g. [32], sections 3.3 and 5.15, and [51], section 1.4.

| $F$  | $h$  | $h'$ | $v$  | $v'$ | $\kappa$, $\nu$, $\omega$, $\nu'$ | $\lambda^2$ | $\rho_0$, $\lambda^2_0$ | $w_0$, $w$, $\tilde{\lambda}$, $\mu$, $\Lambda$ |
|------|------|------|------|------|----------------------------------|-------------|-----------------|------------------|
| $d^F_F$ | -1/2 | 1/2  | 1    | -1   | 3                                | 0           | 0               | 0                |
| $d^F_\kappa$ | d/2  | d/2  | 1    | -1   | $d + 1$                         | -2          | $-d - 4$        | 2 - d $\equiv \varepsilon$ 0 | 1          |
| $d_F$ | d/2 - 1 | d/2 + 1 | 1 | $d - 1$ | 0                             | 2 - $d$    | $\varepsilon$ | 0               | 1          |

### Table 1. Canonical dimensions of the fields and the parameters of the theory (2.4).
Canonical dimensions analysis should be augmented by the following considerations. As a result of causality, all the one-irreducible diagrams without external ‘tails’ of the auxiliary fields $v'$, $h'$ involve closed circuits of retarded propagators and, therefore, vanish. Thus, it is sufficient to consider only the functions with $N_v + N_{v'} > 0$.

The field $h$ is passive in the sense that it does not affect the dynamics of the velocity field. This means that the full Green functions with $N_h = 0$ and $N_{v'} > 0$ and the one-irreducible Green functions with $N_h > 0$ and $N_{v'} = 0$ vanish identically (for any given value of the numbers $N_v$, $N_{v'}$). In particular, this forbids the divergence in the one-irreducible function $\langle v' hh \rangle_{1-ir}$ with the counterterm $(\partial \cdot h) \nabla^2 h$.

The counterterms that have the form of total derivatives (or can be reduced to such form using the integration by parts) vanish after the integration over $x = \{t, \mathbf{x}\}$ and should be ignored; consequently, the counterterms that differ by a total derivative should be identified with each other.

Lastly, the transversality conditions $(\partial \cdot v) = (\partial \cdot v') = 0$ for the vector fields should not be forgotten.

Taking all of the above into account, one can ascertain that superficial UV divergences can be present only in the following one-irreducible functions:

\[
\begin{align*}
\langle h'h' \rangle_{1-ir} &\quad (\delta_\Gamma = 0, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } h'h', \\
\langle h'h \rangle_{1-ir} &\quad (\delta_\Gamma = 2, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } h'(\partial h)^2, \\
\langle h'h \rangle_{1-ir} &\quad (\delta_\Gamma = 2, \delta_{\Gamma'} = 1) \quad \text{with the counterterm } h'\nabla^2 h, \\
\langle vv' \rangle_{1-ir} &\quad (\delta_\Gamma = 2, \delta_{\Gamma'} = 1) \quad \text{with the counterterm } (v' \cdot \nabla^2 v), \\
\langle v'vv \rangle_{1-ir} &\quad (\delta_\Gamma = 1, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } (v' \cdot (v \cdot \partial)v), \\
\langle v'v' \rangle_{1-ir} &\quad (\delta_\Gamma = 2, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } (\partial v')^2, \\
\langle h'hv \rangle_{1-ir} &\quad (\delta_\Gamma = 1, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } h'(v \cdot \partial) h, \\
\langle h'vv \rangle_{1-ir} &\quad (\delta_\Gamma = 0, \delta_{\Gamma'} = 0) \quad \text{with the counterterm } h'v^2. 
\end{align*}
\]

Additional considerations related to the symmetry of the model further reduce the number of the counterterms. The action functional of the KPZ model is invariant with respect to the transformation

\[
h(t, \mathbf{x}) \rightarrow h(t, \mathbf{x} + u \mathbf{t}) - \frac{(u \cdot \mathbf{x})}{\lambda_0} + \frac{u^2 t}{2 \lambda_0}, \quad h'(t, \mathbf{x}) \rightarrow h'(t, \mathbf{x} + u \mathbf{t})
\]

with an arbitrary constant parameter $u$. This invariance is the Galilean symmetry in terms of the vector field $\partial h$; it is violated in the full theory (2.4). However, the latter possesses another kind of the Galilean symmetry, namely,

\[
h(t, \mathbf{x}) \rightarrow h(t, \mathbf{x} + u \mathbf{t}) , \quad h'(t, \mathbf{x}) \rightarrow h'(t, \mathbf{x} + u \mathbf{t}), \\
v(t, \mathbf{x}) \rightarrow v(t, \mathbf{x} + u \mathbf{t}) - u , \quad v'(t, \mathbf{x}) \rightarrow v'(t, \mathbf{x} + u \mathbf{t})
\]

(it is important here that the random force $\mathbf{F}$ in the equation describing the velocity field (1.6) has a factor $\delta (t - t')$ in its correlation function (1.7)). This symmetry puts restrictions on the form of the counterterms.

Firstly, the monomial $h'(v \cdot \partial) h$ must enter the counterterms only in the form of the invariant combination $h' \nabla^2 h = h' \partial_h h + h'(v \cdot \partial) h$. The first term, however, is forbidden by the real index of divergence (the field $h$ must appear under the spatial derivative). Thus, the second term is also forbidden. Secondly, the monomial $(v' \cdot (v \cdot \partial)v)$ is excluded for the same reason, being a part of the invariant combination $(v' \nabla^2 v)$ with the first term forbidden by the real
index. Finally, the monomial $h'v^2$ is ruled out by the symmetry (3.6) because it is obviously not invariant.

All the remaining counterterms $h'h', h'\partial^2h, h'(\partial h)^2, (v' \cdot \partial^2v), (\partial v')^2$ are present in the action (2.4). Thus, the theory is multiplicatively renormalizable. The renormalized action then can be written in the form:

$$S_R(\Phi) = \frac{1}{2}Z_1D(\partial v')^2 + (v' \cdot \left\{ -\partial v - (v \cdot \partial)v + Z_2 \nu \partial^2v \right\})$$

$$+ \frac{1}{2}Z_3 h'h' + h' \left\{ -\partial h - (v \cdot \partial)h + Z_4 \nu \partial^2h + \frac{1}{2}Z_5 \lambda(\partial h)^2 \right\}. \tag{3.7}$$

Here $Z_i$ are the renormalization constants that depend only on the completely dimensionless parameters $g, w, \lambda$ and absorb the poles in $\varepsilon$. The renormalized action $S_R(\Phi)$ is obtained from the original one (2.4) by the renormalization of the fields $(h \to Z_h h, h' \to Z_h h', v \to Z_v v, v' \to Z_{v'} v')$ and the parameters:

$$\nu_0 = \nu Z_{\nu}, \quad \nu = \nu Z_{\nu}, \quad g_0 = g\mu^2Z_{\mu}, \quad \tilde{\lambda}_0 = \tilde{\lambda}_0 \mu^{\varepsilon/2}Z_{\lambda}, \quad w_0 = w Z_w. \tag{3.8}$$

The amplitude $D$, the coefficients $\lambda$ and $\lambda$ are expressed in renormalized parameters as follows:

$$D = g\mu^4 \mu^2, \quad \lambda = \nu^{3/2} \tilde{\lambda} \mu^{\varepsilon/2}, \quad \lambda = \nu \varepsilon. \tag{3.9}$$

It should be noted that for internal consistency of multiloop calculations we suppose that $g \approx \tilde{\lambda}^2$.

The renormalization constants in the equations (3.7) and (3.8) are subject to the following relations:

$$Z_g = Z_1 Z_2^{-3}, \quad Z_\nu = Z_2, \quad Z_\nu = Z_4 Z_2^{-1}, \quad Z_h = Z_3^{-1/2}, \quad Z_{v'} = Z_3^{1/2},$$

$$Z_{\lambda} = Z_5 Z_3^{-1/2} Z_2^{-3/2}, \quad Z_v = Z_{v'} = 1. \tag{3.10}$$

The renormalization constants $Z_1 Z_5$ are calculated directly from the diagrams, then one finds the constants in equation (3.8) from relations (3.10). We have performed such calculation to the first order in $g$ and $\tilde{\lambda}^2$ (one-loop approximation) using the minimal subtraction (MS) scheme where the renormalization constants have the forms $'Z = 1 +$ only poles in $\varepsilon$". The details of those calculations are given in the appendix. The resulting renormalization constants are as follows:

$$Z_1 = Z_2 = 1 - \frac{1}{16 \varepsilon} \tilde{g}, \quad Z_3 = 1 - \frac{\tilde{\lambda}^2}{w^2} \frac{1}{8 \varepsilon},$$

$$Z_4 = Z_5 = 1 - \frac{1}{4 \varepsilon w(1 + w)} \tilde{g}. \tag{3.11}$$

where $\tilde{g} = g S_d/(2\pi)^d, \tilde{\lambda}^2 = \tilde{\lambda}^2 S_d/(2\pi)^d$, and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere in $d$ dimensions. The results for $Z_1$ and $Z_2$ are in agreement with their analogs (3.10) and (3.11) obtained in [13] within their Wilsonian implementation of the RG (the case of the pure random advection without KPZ nonlinearity).

We calculated the renormalization constants in the one-loop approximation. Nevertheless, some of them are subject to exact relations. Firstly, the equality $Z_1 = Z_2$ is exact due to the fluctuation-dissipation theorem for the original NS equation (1.6) with thermal noise; see [13], appendix B, and [51], section 3.10. These constants also do not depend on the coupling
constant $\tilde{\lambda}$ in all orders of perturbation theory. Indeed, the advection is passive and the fields $h, h'$ cannot affect the behavior of the velocity field $v$. Secondly, for $g = 0$ (the case of the pure KPZ nonlinearity without the advection) the constant $Z_s$ equals unity identically, while the one-loop expression for $Z_s^{-1}$ is exact (see [31, 32] for the proof; the definition of the parameters here is different).

4. RG equations and RG functions

The RG equations are written for the renormalized Green functions $G_R = \langle \Phi \cdots \Phi \rangle_R$:

$$G(e_0, \ldots) = Z^{N_t}_{e_0} Z^{N_w}_{e_0} G_R(e, \mu, \ldots). \quad (4.1)$$

Here $N_t$ and $N_w$ are the numbers of the fields entering into $G$ (we recall that $Z_0 = Z_{\nu'} = 1$); $e_0 = \{g_0, \nu_0, w_0, \tilde{\lambda}_0\}$ is the full set of bare parameters and $e = \{g, \nu, w, \tilde{\lambda}\}$ are their renormalized counterparts; the ellipsis stands for the times, coordinates, momenta, etc.

Let us introduce the differential operation $\tilde{D}_\mu = \mu \partial_\mu\mid_{e_0}$ at fixed bare parameters and apply it to relation (4.1). This gives the basic differential RG equation (see, e.g. [11], section 1.24)

$$\{\mathcal{D}_{RG} + N_t \gamma_h + N_w \gamma_w\} G_R(e, \mu, \ldots) = 0, \quad (4.2)$$

where

$$\mathcal{D}_{RG} \equiv \tilde{D}_\mu + \beta_\mu \partial_\mu + \beta_\nu \partial_\nu + \beta_\lambda \partial_\lambda = \gamma_\nu \mathcal{D}_\nu. \quad (4.3)$$

is the operation $\tilde{D}_\mu$ expressed in renormalized parameters. Here and below, $\mathcal{D}_x \equiv x \partial_x$ for any variable $x$, the anomalous dimensions $\gamma$ are defined as

$$\gamma_F \equiv \tilde{D}_\mu \ln Z_F \quad \text{for any quantity } F. \quad (4.4)$$

The $\beta$-functions for the three coupling constants $g, w,$ and $\tilde{\lambda}$ are found from the definitions and the relations (3.8):

$$\beta_g \equiv \tilde{D}_\mu g = g \left[ -\varepsilon - \gamma_g \right], \quad \beta_\lambda \equiv \tilde{D}_\mu \tilde{\lambda} = \tilde{\lambda} \left[ -\varepsilon/2 - \gamma_\lambda \right], \quad \beta_w \equiv \tilde{D}_\mu w = -w \gamma_w. \quad (4.5)$$

At last, equation (3.10) yield the following relations between the anomalous dimensions (4.4):

$$\gamma_h = -\gamma_3/2, \quad \gamma_h' = \gamma_3/2, \quad \gamma_w = \gamma_4 - \gamma_2, \quad \gamma_0 = \gamma_0' = 0, \quad \gamma_\nu = \gamma_2, \quad \gamma_\nu' = \gamma_1 - 3\gamma_2, \quad \gamma_\lambda = \gamma_5 - 3\gamma_2/2 + \gamma_3/2. \quad (4.6)$$

The anomalous dimension corresponding to a given renormalization constant $Z_F$ can be found from the expression

$$\gamma_F = (\beta_\nu \partial_\nu + \beta_w \partial_w + \beta_\lambda \partial_\lambda) \ln Z_F \simeq \left( \varepsilon \mathcal{D}_g + \varepsilon \mathcal{D}_\lambda / 2 \right) \ln Z_F, \quad (4.7)$$

obtained from the definition (4.4), expression (4.3), and the fact that the renormalization constants depend only on the three completely dimensionless coupling constants $g, w,$ and $\lambda$. Only the leading-order terms in the $\beta$-functions (4.5) were retained in the second part of the relation. The MS scheme in the one-loop approximation yields:

$$\gamma_1 = \gamma_2 = \hat{g}/16, \quad \gamma_3 = \frac{\hat{g}^2}{8w}, \quad \gamma_4 = \gamma_5 = \frac{\hat{g}}{4\varepsilon w(1+w)}. \quad (4.8)$$
where $\hat{g}$ and $\hat{\lambda}$ were defined earlier; the corrections of order $\hat{g}^2$, $\hat{\lambda}^4$ and higher are omitted.

5. Fixed points, scaling regimes, and critical exponents

A long-time large-distance asymptotic behavior of a renormalizable field theory is determined by IR attractive fixed points of the RG equations. The coordinates $g_\ast, \tilde{\lambda}_\ast, w_\ast$ of the fixed points of the theory (2.4) are found from the three equations

$\beta_g(g_\ast, \tilde{\lambda}_\ast, w_\ast) = 0, \quad \beta_{\tilde{\lambda}}(g_\ast, \tilde{\lambda}_\ast, w_\ast) = 0, \quad \beta_w(g_\ast, \tilde{\lambda}_\ast, w_\ast) = 0,$

(5.1)

with the $\beta$ functions from relations (4.5). The type of a fixed point is determined by the matrix

$\Omega(g_\ast) = \{\Omega_\beta = \partial \beta_i / \partial g_\ast\}.$

(5.2)

where $\beta_i$ is the full set of the $\beta$ functions, $g_j = \{g, \tilde{\lambda}, w\}$ is the full set of the coupling constants, and $g_k$ are coordinates of the fixed point. The real parts of all the $\Omega$ matrix eigenvalues are required to be positive for a fixed point to be IR attractive.

Relations (4.5), (4.6) and (4.8) yield the explicit one-loop expressions for the $\beta$ functions:

$$\beta_g = g [-\epsilon - \gamma_g] = -g \left[ \frac{\epsilon}{2} - \frac{\hat{g}}{8} \right],$$

$$\beta_{\tilde{\lambda}} = \tilde{\lambda} [-\epsilon/2 - \gamma_{\tilde{\lambda}}] = -\tilde{\lambda} \left[ \frac{\epsilon}{2} - \frac{3\hat{g}}{16} + \frac{\hat{g}}{4w(w+1)} + \frac{\hat{\lambda}^2}{16w^2} \right],$$

$$\beta_w = -w \gamma_w = -w \hat{g} \left[ \frac{1}{4w(w+1)} - \frac{1}{16} \right].$$

(5.3)

For $\hat{g} = 0$ (pure KPZ model) the expression for $\beta_{\tilde{\lambda}}$ becomes exact (see [31, 32]; the notation used here is different). This fact can hardly remain true for the full model, owing to appearance of new classes of diagrams with propagators (2.5) and new vertices.

The matrix $\Omega$ turns out to be triangular (because $\partial_w \beta_g = \partial_{\tilde{\lambda}} \beta_g = \partial_{\tilde{\lambda}} \beta_w = 0$ for any fixed point) and its eigenvalues are simply given by the diagonal elements $\Omega_\beta = \partial \beta_i / \partial g$, $\Omega_{\tilde{\lambda}} = \partial \beta_{\tilde{\lambda}} / \partial \tilde{\lambda}$, and $\Omega_w = \partial \beta_w / \partial w$.

The fixed points are as follows:

1. The line of Gaussian (free) fixed points: $g_\ast = \tilde{\lambda}_\ast = 0$; $w_\ast$ is an arbitrary number; $\Omega_\beta = -\epsilon, \Omega_{\tilde{\lambda}} = -\epsilon/2, \Omega_w = 0$.
   Here, both the advection and KPZ nonlinearity are irrelevant. Thus, this point corresponds to the Edwards–Wilkinson model.

2. Passive scalar fixed point: $\hat{g}_\ast = 8\epsilon; \tilde{\lambda}_\ast = 0, w_\ast = (-1 + \sqrt{17})/2; \Omega_\beta = \epsilon, \Omega_{\tilde{\lambda}} = -\epsilon/4, \Omega_w = \epsilon/2 + 8\epsilon/(1 + \sqrt{17})^2$.
   This fixed point corresponds to the pure linear passive scalar advection, i.e. the KPZ nonlinearity does not affect the leading-order IR asymptotic behavior (it is irrelevant in the sense of Wilson). These results agree, up to the notation and a misprint in expression for $w_\ast$, with those obtained in [13], equation (3.69). It should be noted that due to the different signs in front of $\epsilon$ in $\Omega_\beta$ and $\Omega_{\tilde{\lambda}}$ the fixed point never becomes IR or UV attractive, it is always unstable.

3. The curve of the fixed points: $g_\ast = 0, \lambda_\ast^2/w_\ast^3 = -8\epsilon, w_\ast$ is an arbitrary number; $\Omega_\beta = -\epsilon, \Omega_{\tilde{\lambda}} = \epsilon, \Omega_w = 0$. 


All of the points on this curve correspond to the pure KPZ model, i.e. the motion of the medium is IR irrelevant. These fixed points never become attractive for the same reasons as the fixed point 2.

4. The new nontrivial fixed point: $\lambda_\varepsilon = 8\varepsilon$, $\lambda_v^2/\varepsilon = -4\varepsilon$, $w_\varepsilon = (-1 +\sqrt{7})/2$; $\Omega_\varepsilon = \varepsilon$, $\Omega_v = \varepsilon/2$, $\Omega_\varepsilon = \varepsilon/2 + 8\varepsilon/(1 + \sqrt{7})^2$.

Strictly speaking, there are two values of $\lambda_\varepsilon$ that satisfy the relation $\lambda^2 = -\varepsilon(-1 +\sqrt{7})^3/2$. However, while in general there is no restrictions on the sign of $\lambda_\varepsilon$, it is uniquely defined by the type of the growth we study (positive growth, e.g. the growth of a bacterial colony, is described by a positive $\lambda_\varepsilon$, while negative growth, e.g. the growth of an acid burn on a surface, is described by a negative $\lambda_\varepsilon$). Thus, there is only one fixed point with such coordinates for any given system and we will refer to it as to a single point.

This fixed point corresponds to a new nontrivial IR scaling regime (universality class), in which the nonlinearity of the model (2.4) and the random motion of the medium are simultaneously important. The point becomes IR attractive when $\varepsilon$ is positive, i.e. when $d < 2$. However, the positive values of $\varepsilon$ lead to an imaginary coupling constant $\lambda_\varepsilon$. This issue will be discussed in depth in section 6.

The critical dimension $\Delta_F$ of a certain IR relevant quantity $F$ in a dynamical model is given by the relation

$$\Delta_F = d_F^\omega + \Delta_\omega d_F^\omega + \gamma_F^\omega, \quad \text{(5.4)}$$

where $\Delta_\omega = 2 - \gamma_\omega^\omega$ is the critical dimension of the frequency, $d_F^\omega$ are the canonical dimensions of $F$ from the table 1, and $\gamma_F^\omega$ is the value of the anomalous dimension from equation (4.4) at the fixed point: $\gamma_F^\omega = \gamma_F(g_\varepsilon, \lambda_\varepsilon, w_\varepsilon); \text{ see, e.g. [11], section 5.16, equations (5.110), (5.111), and [51], equation (2.3).}$

Relations (4.6) and the explicit one-loop expressions (4.8) for the critical dimensions yield:

$$\Delta_h = d/2 - \Delta_\omega/2 + \gamma_h^\omega, \quad \Delta_h' = d/2 + \Delta_\omega/2 - \gamma_h^\omega, \quad \Delta_\omega = 2 - \gamma_\omega^\omega,$$

$$\Delta_\nu = -1 + \Delta_\omega, \quad \Delta_\nu' = d + 1 - \Delta_\omega,$$

where $\gamma_h = -\hat{\lambda}^2/16\omega^3$ and $\gamma_\nu = \hat{g}/16$. These relations give the following expressions for the critical dimensions:

for the line of the fixed points 1:

$$\Delta_h = \frac{-\varepsilon}{2}, \quad \Delta_h' = 2 - \frac{-\varepsilon}{2}, \quad \Delta_\omega = 2, \quad \Delta_\nu = 1, \quad \Delta_\nu' = 1 - \varepsilon, \quad \text{(5.5)}$$

for the fixed point 2:

$$\Delta_h = \frac{-\varepsilon}{4}, \quad \Delta_h' = 2 - \frac{3\varepsilon}{4}, \quad \Delta_\omega = 2 - \frac{-\varepsilon}{2}, \quad \Delta_\nu = \Delta_\nu' = 1 - \frac{-\varepsilon}{2}, \quad \text{(5.6)}$$

for the curve of the fixed points 3:

$$\Delta_h = 0, \quad \Delta_h' = 2 - \varepsilon, \quad \Delta_\omega = 2, \quad \Delta_\nu = 1, \quad \Delta_\nu' = 1 - \varepsilon, \quad \text{(5.7)}$$

for the fixed point 4:

$$\Delta_h = 0, \quad \Delta_h' = 2 - \varepsilon, \quad \Delta_\omega = 2 - \frac{-\varepsilon}{2}, \quad \Delta_\nu = \Delta_\nu' = 1 - \frac{-\varepsilon}{2}, \quad \text{(5.8)}$$
All of the results for critical dimensions are exact except for the values of $\Delta_h$ and $\Delta_{h'}$ in relations (5.8). Indeed, the critical dimensions for the fixed points 1–3 (and $\Delta_\omega, \Delta_v, \Delta_{v'}$ for the fixed point 4) are known exactly due to certain exact relations between renormalization constants for the case of the passive scalar field [13] (see also [51], section 3.10) and for the case of the pure KPZ model [31, 32]. The critical dimensions $\Delta_h$ and $\Delta_{h'}$ for the fixed point 4 are found only in one-loop approximation, thus, calculation of the higher orders of perturbation theory may well affect them.

Although our one-loop results for the RG functions and critical dimensions of the full model are not exact, they reproduce all the known exact results for the special cases. As such, one may hope that in the present case the one-loop approximation may serve as an adequate qualitative estimate. In support of this assumption, one can mention a recent study of the turbulence effects on the critical behaviour, where the use of the functional RG [52] provided a very detailed confirmation of the one-loop perturbative results derived earlier in [45]. It is also worth mentioning that, for the Kraichnan’s model of turbulent advection, the functional renormalization group [53] reproduces the first-order results for the anomalous exponents derived some twenty-five years earlier within various perturbative approaches [54].

To relate the critical dimensions with the critical exponents from (1.1) one has to identify $\Delta_h = -\chi$ and $\Delta_\omega = z$. However, the quantity $\delta_n$ in expression (1.1) is not an ordinary nth order Green function of the primary fields $h(x)$, instead it is a sum of pair correlation functions $\langle h^n(x) h^n(0) \rangle$ of the composite fields $h^n(x)$. In general, renormalization of such ‘composite operators’ (in quantum-field terminology) requires additional, sometimes quite involved, analysis; see, e.g. [10], chapter 12 and [11], sections 3.23–3.28. However, here it is not the case and the analysis is rather simple.

The formal divergence index of the one-irreducible Green function $\Gamma = \langle h^n h \ldots h \rangle_{1-ir}$ with one composite field $h^n$ and arbitrary number of primary fields $h$ is $\delta_\Gamma = 0$. It is a result of the fact that the full canonical dimension of the field $h$ is equal to zero when $\varepsilon = 0$ (see Table 1). Nevertheless, the real divergence index $\delta_\Gamma'$ is, in fact, negative, since any nontrivial diagram of the function $\Gamma$ has at least one external vertex $h'(\partial h)^2$ or $h'(v \cdot \partial) h$ where the field $h$ is under a derivative. This means that at least one external momentum appears in the diagram as an overall factor resulting in a negative index $\delta_\Gamma' = \delta_\Gamma - 1 = -1$. Thus, the operators $h^n$ do not require additional renormalization; their critical dimensions are $\Delta_{h^n} = n\Delta_h$. This means that the relation (1.1) with the dimensions (5.5) and (5.6) correctly describes the scaling behavior.

The above analysis is nearly identical to that given in [20] for Kraichnan’s ensemble, but we reproduce it here because it justifies the expression (1.1) for general $n$ (usually, only the case $n = 2$ is discussed). For more details on composite fields in dynamical models see, e.g. [51], sections 2.1–2.4.

### 6. Discussion and conclusion

The effects of randomly moving medium on the random kinetic growth of an interface were studied. The growth was modelled by the Kardar–Parisi–Zhang stochastic differential equations (1.3) and (1.4). The random motion of the environment was described by the NS stochastic differential equation with thermal noise (1.6) and (1.7).

The full problem is equivalent to the multiplicatively renormalizable theory with the action functional (2.4). The field theoretic RG analysis revealed that there are four possible regimes
of scaling asymptotic behavior related to four possible attractors of the RG equations: the line of Gaussian fixed points (ordinary diffusion with stirring, Edwards–Wilkinson case), purely kinematic fixed point (a passively advected linear scalar field), the curve of the fixed points related to the pure KPZ model (universality class of kinetic growth), and new fully nontrivial non-equilibrium universality class that corresponds to a process in which the motion of the medium and the KPZ nonlinearity are relevant simultaneously (new nontrivial fixed point).

The fixed point coordinates, their regions of stability, and corresponding critical dimensions were calculated to the first order of the expansion in $\varepsilon = 2 - d$ (one-loop approximation). Some of the results for the critical dimensions are exact (valid to all orders in $\varepsilon$).

For $\varepsilon > 0$ the new fixed point is IR attractive and describes a new kind of IR asymptotic behavior of the system. However, its coordinates lie in the unphysical region: the fixed-point value $\hat{\lambda}$ is imaginary. For $\varepsilon < 0$, $\hat{\lambda}$ becomes real, while the point becomes IR repulsive (and UV attractive).

To better illustrate this point, let us recall that the amplitude $C$ in the pair correlator (1.4) was scaled out. Alternatively, we could have scaled out $\lambda_0$ (i.e. put $\lambda_0 = 1$ in the equation (1.3)) and retained a nontrivial amplitude $C$. Then the coupling constant corresponding to the amplitude $C$ (i.e. the coordinate of the new fixed point) would become negative at the fixed point when $\varepsilon > 0$. Thus, the amplitude in the pair correlator (1.4) would have the ‘wrong’ negative sign. It seems that such a fixed point cannot be reached by the RG flow with physical initial conditions. Careful physical interpretation is required for this result.

There are two cases to consider. The new fixed point is either IR attractive and the amplitude $C$ is negative ($\varepsilon > 0$) or the amplitude $C$ is positive but the point is UV attractive ($\varepsilon < 0$).

We recall that exactly the same situation takes place with the nontrivial perturbative fixed point of the usual KPZ model.

For the case $C < 0$, one possible interpretation is provided by the Doi–Peliti formalism, where the original microscopic problem is formulated in terms of the creation-annihilation operators [55, 56]. The terms quadratic in the auxiliary fields can appear in the action functionals with the negative signs; see e.g. [56]. The negative term exactly corresponds to the ‘wrong’ sign of the amplitude $C$ in a stochastic equation.

Moreover, the Doi–Peliti formalism is known to yield stochastic equations (e.g. for the density of particles in the reaction-diffusion processes) with imaginary random noise (see, e.g. [57]). The question of how this noise should be treated and its implications for the interpretation of the fields in the equations is a subject of ongoing discussion [57–63]. For example, in [60] it is stated (with reference to [61]) that purely imaginary multiplicative noise is required for the proper probability interpretation of the path integral. Furthermore, in [60] the effective action for the Cole–Hopf transformed KPZ equation is compared with the effective action for the annihilation reaction-diffusion process $A + A \rightarrow 0$. Both actions are obtained by the Hubbard–Stratonovich auxiliary field loop expansion method; the construction of the latter action also relies on the Doi–Peliti formalism that yields the field theory related to a stochastic equation with an imaginary noise. The effective actions have a strong resemblance; indeed, the only differences are additional term and the coupling constant with a different sign in the KPZ action.

It is also worth mentioning that the original KPZ (1.3) equation can be mapped with the Cole–Hopf transformation onto the one-dimensional Lieb–Liniger model of Bose gas with attraction [14, 64]. Originally, this model describes a system of $n$ identical quantum particles on a ring with repulsive interactions. The change of the ‘sign’ of the interaction turns the model into the transformed KPZ equation. However, a Bose gas with attraction must collapse (see, e.g. [65]). On the contrary, the KPZ equation with the negative amplitude $C$ is related to
the Bose gas with repulsion. This fact seems to imply that it is not a coincidence that the IR attractive fixed point corresponds to the case $C < 0$.

Strictly speaking, the mapping does not guarantee the full equivalence between the KPZ equation and its transformations (the primary fields in one model correspond to composite fields in the other, and vice versa). Thus, it is far from obvious that the RG functions and RG flows are identical for those models. We will not further discuss these fundamental issues here. Nevertheless, we may conclude that these considerations justify the application of our imaginary fixed points to such systems, as they correspond to physical values of their coupling constants. In this connection it should be mentioned that the two-loop result for the Bose gas with repulsion at zero temperature was derived a long time ago in [66]; the results are in agreement with the exact results obtained later for the KPZ model for the perturbative fixed point. Recently, the model of a reaction-diffusion type (directed percolation) coupled to a generalized KPZ equation was studied [24].

It is also worth noting that the proofs of the exactness of the one-loop approximation for the KPZ model are essentially based on the mappings between the KPZ model and the Bose system [31, 32].

From now on, let us consider the case $C > 0$, so that the new fixed point is UV attractive. If so, our perturbative RG analysis fails to predict an IR attractive point that would correspond to observed scaling behavior. Since the RG analysis of the original KPZ equation (1.3) has the same flaw, this question had been widely discussed. The following consensus was achieved.

Existence of the IR repulsive nontrivial fixed point of the KPZ equation is consistent with the assumption that there might be an IR attractive fixed point between the IR repulsive fixed point and the infinity. One can use this assumption and certain, sometimes rather nontrivial, additional dimensionality and symmetry considerations to obtain the exact values of critical exponents for the spatial dimensions $d = 1, 2, 3$ [12, 13, 33]. However, that hypothetical `strong-coupling’ fixed point appeared to be undetectable by any kind of perturbative approach.

To date, only one method seems to be able to reveal that elusive fixed point, namely, the functional (or non-perturbative) RG [34–36]. It does not rely on any kind of perturbation theory, does not feature a small parameter, and is non-perturbative in this sense. Later, more complex modifications of the KPZ problem were successfully studied with the functional RG approach, e.g. the anisotropic equation [27] or the equations with the spatially correlated noise [67–69].

Thus, it would be interesting to study the problem (2.4) with the functional RG. Furthermore, recently, the functional RG was applied to a similar problem, namely, to the model A of equilibrium critical dynamics under influence of the turbulent motion of the environment described by the Kraichnan’s rapid change model [52]. Surprisingly enough, the obtained results are in a full agreement with the one-loop perturbative results [45]. This fact makes it even more tempting to apply the functional RG to the problem (2.4). However, several complications arise in the present case compared with the problem studied in [52].

Firstly, the Kraichnan’s model is a synthetic velocity ensemble with simple Gaussian statistics. The coupling with the NS equation (1.6) considered in the present paper is a more realistic, but more complicated way to take into account the random motion of the environment in the model. Studying the NS equation with the functional RG is a difficult task in itself; see, e.g. the papers [70–73]. Moreover, those papers deal with the problem of turbulence, while the non-perturbative analysis of the NS equation with the thermal noise is still lacking.

Secondly, the nonlinearity in the standard KPZ equation (1.3) is a result of a truncation of the full model that involves all powers of the gradient $(\partial h)^2$, see, e.g. [12, 16]. In the perturbative RG analysis the higher-order terms are omitted, being IR irrelevant. However, this conclusion, based on the analysis of canonical dimensions, is reliable and internally consistent only
within the $\varepsilon$ expansion. Existence of the non-perturbative fixed point means that the simplest nonlinear term $(\partial h)^2$ remains IR relevant for $d > 2$, in contrast to naive dimensional prediction. It is then not impossible that some of the higher order nonlinearities (and probably all of them) are also relevant and should be taken into account. To the best of our knowledge, this problem was never discussed for the pure KPZ model, let alone its generalization with the velocity field.

Since the functional RG approach is highly sensitive to the approximation scheme used, one has to carefully consider the possible Ansatz. The approximation scheme based on the symmetries of the original KPZ equation suggested in [34–36] might be suitable for the present case, but it should be modified properly to include the interaction with the velocity field. In a more general formulation, one could try to use the Ansatz that involves the whole infinite series in $(\partial h)$ for the generating functional of the Green functions. Indeed, the KPZ equation is renormalizable within perturbation theory but once we consider non-perturbative effects previously discarded terms in the KPZ equations may also become important.

The modification of the KPZ model that involves an infinite number of coupling constants was earlier studied by perturbative RG in [18]. The similarly ‘infinite’ model of landscape erosion was also studied by perturbative RG in [20] and by the functional RG in [74]. Inclusion of the velocity field into such models was studied perturbatively in [75, 76].

The formidable task of applying the functional RG to the problem (2.4) is out of the scope of the present study; but the first step would be choosing a suitable Ansatz. Of course, testing such an Ansatz with the infinite series on the original KPZ equation without advection should precede such considerations.
Nevertheless, something can still be said about the fate of possible non-perturbative fixed points of the problem (2.4). The strong-coupling IR attractive fixed point established by the functional RG clearly survives in the full problem (2.4). The corresponding coordinates would have $\hat{g}^* = 0$, $\hat{\lambda}^*$ taken from the pure KPZ model and arbitrary $w^*$ (the point would become a line of points). However, its IR stability is not guaranteed; indeed, the perturbative fixed points corresponding to the pure KPZ model (the fixed points 3) are no longer IR attractive when $\varepsilon > 0$ as they were in the original model.

Moreover, another nontrivial non-perturbative fixed point may emerge in the full problem (2.4). It would be similar to the fixed point 4, i.e. it would be related to the regime where both the advection and the KPZ nonlinearity are relevant. The RG flows depicted on the figures 1 and 2 (the cases of $\varepsilon < 0$ and $\varepsilon > 0$, respectively) are consistent with the existence of two IR attractive fixed points with the negative or zero coordinate $\hat{g}$, and the coordinate $\hat{\lambda}$, bigger than those of the fixed points 4 and 3. One of the two possible non-perturbative fixed points is likely to have three positive eigenvalues of the matrix $\Omega$, i.e. to be IR attractive in all directions.

If one assumes that there is indeed a non-perturbative IR attractive fixed point in the original KPZ model, then the nontrivial perturbative fixed point $\hat{\lambda}^*$ (that becomes IR repulsive when $\varepsilon > 0$) serves as the boundary between two intervals of attraction. Indeed, if the initial conditions for the renormalized coupling constant $\lambda$ is chosen in the interval between the IR attractive Gaussian fixed point (the Edwards–Wilkinson point) and the point $\hat{\lambda}^*$, then the system displays trivial scaling corresponding to the Gaussian point (smooth phase without roughening). In contrast, for the choice $\lambda > \hat{\lambda}$, the RG flow will approach the IR attractive non-perturbative point, i.e. the phase with kinetic roughening will be observed. The latter

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{RG flow diagram of $\beta_g$ and $\beta_{\lambda^2} = \beta_{\hat{\lambda}^2}(\hat{g}, \hat{\lambda}^2, w)$ in the plane $(g, l \ast I)$ where $g = g$ and $I \ast l = \hat{\lambda}^2$. The values of the parameters are $\varepsilon = 1$ and $w^* = (-1 + \sqrt{17})/2$. Fixed points 1, 2, 3 and 4 are marked as empty circles with corresponding numbers in boxes near the circles. It can be seen that the point 4 is now IR attractive.}
\end{figure}
fixed point cannot be sidestepped by the RG flow with the initial condition $\lambda < \lambda_*$ and, thus, the fixed point $\lambda_*$ is the boundary between two basins of attraction, or, in other words, it describes a phase transition between the smooth and rough phases; see, e.g. [77].

In the present case, the pattern of the RG flows in figure 1 suggests that the fixed points 3 and 4 together build a kind of cornerstones of a boundary between the basins of attraction of the IR attractive trivial point and hypothetical strong-coupling unstable attractor, formed by the two non-perturbative fixed points: the original KPZ point and a new point, one of which can also be attractive and the other can only be a saddle point.

However, since now there are two ‘transition fixed points’, it is hardly possible to speak about a sharp phase transition with definite critical dimensions. Furthermore, the full space of coupling constants is now three-dimensional. One can expect a kind of smooth transition with a nearly power-like behaviour, while the exponents are defined by one of the unstable points 3 or 4, depending on the specific initial data of the RG flow.

To conclude with, in this paper we studied the KPZ model coupled to the NS equation with a thermal noise. In the following, it would be interesting to consider ramifications and generalizations of this model with other types of the velocity statistics (NS turbulence with non-local large-scale stirring force) and of the noise in the KPZ equation (the noise quenched, frozen or correlated in space and/or time), and to study the effects of anisotropy and compressibility. It would be highly interesting to investigate the feedback of the advected field on the fluid dynamics (active scalar field) and the effects of the higher-order nonlinear corrections to the plain KPZ model. It is also highly desirable to apply the non-perturbative methods, especially the functional RG, to these problems.

This work remains to the future and is already in progress.

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Appendix. Calculation of the renormalization constants

In this appendix we present all the diagrams contributing to the renormalization constants at one-loop level and briefly outline the technique of their calculation. In order to depict diagrammatic representation of one-irreducible Green functions we use the following graphical notations:

\[ \langle h' h \rangle_0 = \frac{1}{-i \omega + \kappa_0 k^2}, \quad (A.1) \]

\[ \langle h h \rangle_0 = \frac{1}{\omega^2 + \kappa_0^2 k^4}, \quad (A.2) \]

\[ \langle v_i v_j' \rangle_0 = \frac{P_{ij}(k)}{-i \omega + \nu_0 k^2}, \quad (A.3) \]
where $p$ and $k$ are the momenta flowing into the vertex, then corresponding one-loop approximation of one-irreducible Green functions containing UV divergences to be eliminated takes the form:

\begin{align}
\langle u_i'v_j \rangle_0 &= \frac{D_0 k^2 P_{ij}(k)}{\omega^2 + \nu_0^2 k^4}, \\
\langle h'h \rangle_0 &= \frac{p}{k} = ik_j (ip_j) \lambda_0, \\
\langle h'v_i \rangle_0 &= \frac{p}{k} = -ik_j = ip_j, \\
\langle v_i'v_j v_k \rangle_0 &= \frac{k}{k} = i(k_j \delta_{is} + k_s \delta_{ij}),
\end{align}

In present paper we use dimensional regularization and minimal subtraction (MS) scheme, where all renormalization constants have the form:

\begin{align}
Z_i &= 1 + \sum_{n=1}^{\infty} A_{m} \varepsilon^{-n},
\end{align}

Coefficients $A_{m}$ depend only on dimensionless renormalized coupling constants and are determined from the requirement that all one-irreducible Green functions be UV finite, that is, finite in the limit $\varepsilon \to 0$. IR regularization of the diagrams is provided by the cutoff from below in momentum integral at the scale $k = m$ (see (1.7)). Note that UV divergent parts of
diagrams in the MS scheme do not depend on the specific form of IR regularization; the sharp cut-off is the most convenient choice for practical calculation.

Since in our model there are no divergences proportional to the terms with time derivatives, we set all external frequencies to zero in all the diagrams. Then the integration over the loop frequency can always be performed by residues. Remaining integration over the loop momenta can be reduced to the dimensionless scalar integral:

\[ I = \int_{k > m} \frac{d^d k}{(2\pi)^d} \frac{\mu^\varepsilon}{k^{d+\varepsilon}} = S_d \int_{k > m} \frac{d^d k}{(2\pi)^d} \frac{\mu^\varepsilon}{k^{d+\varepsilon}} = S_d \left( \frac{\mu}{m} \right)^\varepsilon = \frac{2\pi}{\varepsilon} + O(\varepsilon^0) \approx \frac{2\pi}{\varepsilon} \]  

(A.14)

by means of the known formulae:

\[ \int d^d k \, k_i f(k) = 0, \quad \int d^d k \, k_i \delta_{ij} f(k) = 0, \]  

(A.15)

\[ \int d^d k \frac{k_i k_j}{k^2} f(k) = \frac{\delta_{ij}}{d} \int d^d k f(k), \]  

(A.16)

\[ \int d^d k \frac{k_i k_j k_m k_n}{k^4} f(k) = \frac{\delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}}{d(d+2)} \int d^d k f(k), \]  

(A.17)

where \( f(k) \) is an arbitrary function depending on \( k \) only and \( S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \) is the area of the unit sphere in \( d \) dimensions. In (A.14) and everywhere below \( \simeq \) denotes equality up to UV finite parts.

The indices contractions in the numerators of the one-loop diagrams integrands can be directly performed ‘by hands’, in more involved cases (like multi-loop calculations) specialized software like, for example, FORM [78], can be applied. In some cases below we will omit \((2\pi)^{d-1}\) and signs of integrals over the loop frequency and momentum for brevity, specifying only their integrands.

Now we have all the tools to proceed directly to calculation of divergent parts of the diagrams.

The integrand of the only diagram contributing to (A.8) reads:

\[ \frac{D^2 k^2 (p + k)^2 P_{cm}(p + k)P_{bs}(k)\delta_{ab}(p_{h\delta_{ac} + p_{c} \delta_{ah}})(-i)(p_{i} \delta_{mn} + p_{m} \delta_{ni})}{(\omega^2 + \nu^2 k^4)(\omega^2 + \nu^2 (p + k)^4)}. \]  

(A.18)

Taking integral over the loop frequency we obtain:

\[ \frac{\pi D^2}{\nu^3} \frac{P_{cm}(p + k)P_{bs}(k)(p_{h\delta_{ac} + p_{c} \delta_{ah}})(p_{i} \delta_{mn} + p_{m} \delta_{ni})}{k^2 + (k + p)^2}. \]  

(A.19)

The divergent part of this diagram is proportional to the second power of external momentum, so we can calculate the contractions in the numerator and drop the contributions with numerator proportional to the higher powers of \( p \).\(^3\)

\(^3\) Note that the explicitly written part of expression (A.20) still includes terms of the order \( O(p^3) \) in the IR limit due to the presence on the external momentum in the denominators. A similar fact will be used later in (A.25) and (A.31) in order to extract the term quadratic in external momenta from the fraction with the numerator linear in \( p \).
\[
\frac{\pi D^2}{v^2(k^2 + (k + p)^2)} \left( 2p_\nu p_n + p_\nu k_n \left( \frac{p \cdot k}{k^2} - \frac{(p \cdot k)}{(p + k)^2} \right) \right) + k_\alpha p_n \left( \frac{p \cdot k}{k^2} - \frac{(p \cdot k)}{(p + k)^2} \right) + k_\alpha k_n \left( \frac{p \cdot k}{k^2} - \frac{(p \cdot k)}{(p + k)^2} + \frac{(p \cdot k)^2}{k^2(p + k)^2} \right) + \delta_{\alpha\mu} \left( 2p^2 \left( \frac{(p \cdot k)^2}{k^2} - \frac{(p \cdot k)^2}{(p + k)^2} \right) \right) + O(p^3).
\] (A.20)

Now we can set \( p = 0 \) in all denominators and calculate resulting integrals making use of the (A.14) and (A.16), (A.17). The final result is:
\[
\frac{\pi D^2 S_\nu}{(2\pi)^d + \nu^2} \frac{\delta_{\alpha\mu} p^2}{4} \rightarrow \frac{\tilde{g}^2}{16\pi^2}p^2 P_{ab}(p) = \frac{\tilde{g}^2}{8\pi}P_{ab}(p). \] (A.21)

In the first step above we have replaced the delta symbol with the transverse projector due to the fact that all momenta flowing through the vertex \( \langle v'v \rangle \) are multiplied by corresponding transverse projectors (the projectors are not shown explicitly in (A.18) in favour of brevity), i.e. the divergent part of this diagram is proportional to \( p^2 P_{ab}(p) \).

The integrand of the only diagram contributing to (A.9) reads:
\[
\frac{Dk^2 P_{cm}(p + k)P_{ab}(k)(p_\nu \delta_{\alpha\nu} + p_\nu \delta_{\alpha\mu})(\rho + (p + k)\delta_{\alpha\mu} + (p + k)\delta_{\alpha\mu})}{(-i\omega + \nu(k + p)^2)(\omega^2 + \nu^2k^4)}, \] (A.22)

which after performing integration over a frequency will take the form:
\[
-\frac{D\pi k^2 P_{cm}(p + k)P_{ab}(k)(p_\nu \delta_{\alpha
u} + p_\nu \delta_{\alpha\mu})(\rho + (p + k)\delta_{\alpha\mu} + (p + k)\delta_{\alpha\mu})}{k^2(k^2 + (k + p)^2)} \] (A.23)

Calculating contractions in numerator and keeping powers of an external momentum no higher than a square we obtain:
\[
-\frac{D\pi}{\nu^2k^2(k^2 + (k + p)^2)} \left( 3p_\nu p_n k^2 + p_\nu k_n \left( 2k^2 - \frac{2(p \cdot k)k^2}{(p + k)^2} \right) - 3k_\alpha p_n (p \cdot k) 
\right. \\
+ k_\alpha k_n \left( \frac{4(p \cdot k)^2}{(p + k)^2} - \frac{2p^2k^2}{(p + k)^2} - \frac{2(p \cdot k)}{k^2(k + p)^2} \right) \\
+ \delta_{\alpha\mu} \left( p^2k^2 - (p \cdot k)^2 \right) + O(p^3). \] (A.24)

The only integral here that can not be calculated directly using (A.14) and (A.16), (A.17) is the second term of (A.24) the numerator of which is linear in \( p \). In order to calculate its divergent part we have to expand the denominator up to a term linear in external momentum:
\[
-\frac{2p_\nu k_n k^2}{k^2(k^2 + (k + p)^2)} = -\frac{2p_\nu k_n k^2}{2k^4} - \frac{2p_\nu k_n k^2(p \cdot k)}{2k^4} + O(p^3). \] (A.25)

The first term here vanishes since it is odd in integration momentum, while the second term can be computed in a standard way. As a result we obtain:
\[
-\frac{D\pi S_\nu}{(2\pi)^d + \nu^2} \left( \frac{1}{2} p_\nu p_n + \frac{1}{8} \delta_{\alpha\mu} p^2 \right) \rightarrow -\frac{\tilde{g}^2 p^2}{4\pi \varepsilon} P_{ab}(p) = -\frac{\tilde{g}^2 p^2}{16\varepsilon} P_{ab}(p). \] (A.26)
In the first step in (A.26) we again dropped the longitudinal part for the reason stated under (A.21).

The diagrams contributing to (A.10) diverge logarithmically, which means that in order to calculate their divergent parts we can set external momentum to zero from the outset. Then the first diagram reads:

$$
\frac{1}{(2\pi)^{d+1}} \int d\omega \int_{k>m} dk \lambda^2 \left( -i \right)_{k} i_{k} i_{m}(-i)_{k_{m}} \frac{k^4}{(\omega^2 + \varepsilon^2 k^4)^2} = \lambda^2 \pi S_d \frac{1}{(2\pi)^{d+1} \varepsilon^2} \approx \frac{\lambda^2}{\varepsilon^3} \frac{1}{4 \varepsilon}. \tag{A.27}
$$

The second diagram does not contribute since in its numerator the transverse projector is contracted with its argument:

$$
\frac{1}{(2\pi)^{d+1}} \int d\omega \int_{k>m} dk \frac{Dk^2P_{sm}(k)i_{k}(-i)_{k_{m}}}{(\omega^2 + \varepsilon^2 k^4)^2} = 0. \tag{A.28}
$$

The first diagram of (A.11) reads:

$$
\frac{1}{(2\pi)^{d+1}} \int d\omega \int_{k>m} dk \frac{i_{k}(-i)(p + k)_i (p + k)_m(-i)_{p_{m}}}{(-i\omega + \varepsilon^2)^2} \left( \omega^2 + \varepsilon^2(k + p)^4 \right) \frac{k^2(k \cdot p)^2}{(k + p)^2(k^4 + (k + p)^4)} = \frac{\pi}{(2\pi)^{d+1} \varepsilon^2} \int_{k>m} dk \frac{k^2(k \cdot p)^2}{(k^4 + (k + p)^4)}. \tag{A.29}
$$

Last term of (A.29) is UV finite because it is proportional to the third power of external momentum. The second and the third terms are calculated in a standard way:

$$
\int_{k>m} dk \frac{k^2(p \cdot k)^2}{2k^4} = (1 + \frac{1}{d})p^2 \int_{k>m} dk \frac{k^2}{2k^4} \approx \frac{\pi}{2 \varepsilon} p^2. \tag{A.30}
$$

In order to extract $p^2$ in the first term we expand denominator in powers of external momenta:

$$
\int_{k>m} dk \left( \frac{1}{2k^4} - \frac{3}{2} \left( \frac{p \cdot k}{k^6} \right) \right) = -\frac{3}{2} \int_{k>m} dk \left( \frac{k^2(p \cdot k)^2}{k^4} \right) \approx \frac{3\pi}{2 \varepsilon} p^2. \tag{A.31}
$$

So the first diagram does not contribute to the corresponding renormalization constant. The second diagram reads:

$$
\frac{1}{(2\pi)^{d+1}} \int d\omega \int_{k>m} dk \frac{Dk^2P_{sm}(k)i_p i_{p_{m}}}{(-i\omega + \varepsilon^2(k + p)^4)(\omega^2 + \varepsilon^2 k^4)} \frac{(k \cdot p)^2 - k^2 p^2}{\nu k^4(\varepsilon(k + p)^2 + \nu k^2)} = \frac{D\pi}{(2\pi)^{d+1}} \int_{k>m} dk \frac{(k \cdot p)^2}{\nu k^4(\varepsilon(k + p)^2 + \nu k^2)}. \tag{A.32}
$$

Setting $p = 0$ in denominator and making use of (A.14) and (A.16) we come to:

$$
\frac{-\pi p^2 DS_d}{(2\pi)^{d+1} \nu(\varepsilon + \nu) \varepsilon} \approx \frac{-\nu p^2 g}{4 w(1 + w)}. \tag{A.33}
$$

The UV divergent parts of the first three diagrams in (A.12) cancel each other, so they give no contribution to the constant $Z_s$. This is a consequence of the Galilean symmetry (3.5) of the pure KPZ model which forbids renormalization of the interaction term [12, 13].

The integrand of the fourth diagram reads:

$$
\lambda Dk^2P_{\omega(k)}i_{\omega}i_{\omega_{l}}(-i)(p + k)_i(-i)(q - k)_i \frac{(\omega^2 + \varepsilon^2 k^4)}{-i\omega + \varepsilon^2(p + k)^4}(\omega^2 + \varepsilon^2(q - k)^4). \tag{A.34}
$$
Calculating contractions keeping only terms proportional to \( pq \) and then setting external momenta in denominator to zero, gives for the corresponding diagram:

\[
\frac{\lambda D}{(2\pi)^{d+1}} \int \frac{d\omega}{k_B} \int_{k_B > m} \frac{d^d k}{(\omega^p + \kappa^2 k^2)(\omega^q + \nu k^2)} = \frac{\lambda D\pi}{(2\pi)^{d+1}} \frac{d\omega}{k_B} \int_{k_B > m} \frac{d^d k}{k^2} \frac{\langle (p \cdot k)(q \cdot k) \rangle - \langle p \cdot q \rangle k^2}{\omega^p + \kappa^2 k^2}(\omega^q + \nu k^2)
\]

\[
= -\pi \lambda D \frac{d\omega}{k_B} \int_{k_B > m} \frac{d^d k}{k^2} \frac{\langle (p \cdot q) \rangle - \langle p \cdot q \rangle k^2}{\omega^p + \kappa^2 k^2}(\omega^q + \nu k^2)
\]

\[
= \lambda \langle p \cdot q \rangle \frac{-\hat{g}}{\varepsilon} \frac{1}{4w(w+1)}. \tag{A.35}
\]

The last two diagrams do not contribute to the constant \( Z_5 \) since they are UV finite. To see this it is sufficient to explicitly calculate the numerators of corresponding integrands and to notice that their expansion starts from the third power of external momenta. The numerator of the integrand of the fifth diagram reads:

\[
\lambda D^2 P_{ab}(k) \langle (p + q)_a(k + q)_b(-i)p_{c}(-i)(k + q)_c \rangle
\]

\[
= \lambda D^2 \frac{k^2 \langle p \cdot q \rangle + k^2 q^2 - \langle p \cdot k \rangle \langle q \cdot k \rangle - \langle q \cdot k \rangle^2}{\langle (p \cdot k) + (q \cdot k) \rangle}. \tag{A.36}
\]

The first square bracket is proportional to the second power of external momenta while the second bracket to at least first power of external momenta. The numerator of the last diagram is given by:

\[
\lambda D^2 P_{ab}(k) \langle (p + q)_a(p - k)_b(-i)q_{c}(-i)(p - k)_c \rangle
\]

\[
= \lambda D^2 \frac{k^2 \langle p \cdot q \rangle + k^2 q^2 - \langle p \cdot k \rangle \langle q \cdot k \rangle - \langle q \cdot k \rangle^2}{\langle (p \cdot q) - (q \cdot k) \rangle}. \tag{A.37}
\]

Again the first square bracket is proportional to the second power of external momenta and the second bracket to at least first power of \( q \). So the total contribution to the constant \( Z_5 \) is given by (A.35).

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