Research Article

Weak Type Estimates of Variable Kernel Fractional Integral and Their Commutators on Variable Exponent Morrey Spaces

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1. Introduction and Main Results

Let \( \Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1}) \) for \( 1 < s \leq \infty \). It satisfies

\[
\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})} = \sup_{x \in \mathbb{R}^n} \left( \int_{S^{n-1}} \|\Omega(x, z')\|^s \, dz' \right)^{1/s} < \infty; \tag{1}
\]

\[
\Omega(x, \mu z) = \Omega(x, z), \tag{2}
\]

\[
\int_{S^{n-1}} \Omega(x, z') \, dz' = 0, \tag{2}
\]

for any \( x, z \in \mathbb{R}^n \), \( \mu > 0 \), where \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) is equipped with the Lebesgue measure \( dz' \).

In 1955, Calderón and Zygmund [1] investigated the \( L^p \) boundedness of the singular integral operator with variable kernels. They found that these operators connect closely with the problem about the second-order linear elliptic equations with variable coefficients. Muckenhoupt and Wheeden [2] subsequently introduced the fractional integral operator with variable kernels, which is defined by

\[
T_{\Omega, \alpha}(f)(x) = \int_{\mathbb{R}^n} \Omega(x, x - y) \, f(y) \, dy. \tag{3}
\]

Muckenhoupt and Wheeden [2] also gave the \( (L^p, L^q) \) boundedness with the power weight of \( T_{\Omega, \alpha} \).

Theorem A (see [2]). Let \( 0 < \alpha < n, 1 < p < n/\alpha \), and \( 1/q = 1/p - \alpha/n \). Suppose that \( \Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1}) \) with \( s > p' \). Then there exists a constant \( C > 0 \) independent of \( f \) such that

\[
\|T_{\Omega, \alpha}f\|_{L^q} \leq C \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})} \|f\|_{L^p}. \tag{4}
\]

It is well known that the fractional integral operators play an important role in harmonic analysis, which greatly promotes the process of the intersection and integration of harmonic analysis and other disciplines.

Given a local integrable function \( b \), the corresponding \( m \)-order commutator is defined by

\[
T_{\Omega, \alpha, b}^m(f)(x) = \int_{\mathbb{R}^n} \Omega(x, x - y) \, [b(x) - b(y)]^m \, f(y) \, dy. \tag{5}
\]

In recent years, the boundedness of singular integral operators with variable kernels has been widely concerned. For example, Ding Lin and Shao [3] obtained the \( L^p \) boundedness of Marcinkiewicz integral operator \( \mu_k \) with variable kernels; Wang [4] proved the boundedness properties of
singular integral operators $T_\alpha$, fractional integral $T_{\alpha,\alpha}$, and parametric Marcinkiewicz integral $\mu^\alpha_\ast$ with variable kernels on the Hardy spaces $H^p(\mathbb{R}^n)$ and weak Hardy spaces $WHP^p(\mathbb{R}^n)$. For the related results of the singular integral operator with variable kernels, the reader is referred to [5–8].

After the paper [9], the variable exponent space theory has been rapidly developed in the past 20 years due to its extensive application in the fields of fluid dynamics and differential equations with nongrowth conditions. For example, in [10], the authors considered the boundedness of fractional integral operators on the Hardy spaces $H_p^\alpha(\mathbb{R}^n)$ and differential equations with nongrowth conditions. For its extensive application in the fields of fluid dynamics, it has been rapidly developed in the past 20 years due to its extensive application in the fields of fluid dynamics

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable function $f$ satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\} \quad (9)$$

$$\eta \in \mathbb{R}_+.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space when equipped with the Luxemburg-Nakano norm above.

The weak Lebesgue space with variable exponent $WL^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable function $f$ satisfying

$$\|f\|_{WL^{p(\cdot)}(\mathbb{R}^n)} = \sup_{\eta > 0} \eta \|\chi_{\{|f| > \eta\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \quad (10)$$

$$\eta \in \mathbb{R}_+.$$

It is easy to see that $\|\cdot\|_{WL^{p(\cdot)}(\mathbb{R}^n)}$ is a quasi-norm; that is, for any $g_1, g_2 \in WL^{p(\cdot)}(\mathbb{R}^n)$, we have

$$\|g_1 + g_2\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \leq 2 \left( \|g_1\|_{WL^{p(\cdot)}(\mathbb{R}^n)} + \|g_2\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \right). \quad (11)$$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximum operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy. \quad (12)$$

Let $\mathcal{B}(\mathbb{R}^n)$ denote the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which satisfies the following conditions:

$$|p(x) - p(y)| \leq \frac{C}{\log(|x|)} \quad (13)$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)} \quad (14)$$

It is proved that the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ as $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ satisfies $p_- > 1$ in [16].

**Remark 1.** For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\lambda > 1$, by Jensen’s inequality, we have $\lambda p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. See Remark 2.13 in [17].

We say an order pair of variable exponents function $(p(\cdot), q(\cdot)) \in \mathcal{B}(\mathbb{R}^n)$, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < \alpha < n/p_+$, and

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n} \quad (15)$$

with $q(\cdot)(n - \alpha)/n \in \mathcal{B}(\mathbb{R}^n)$.

**Remark 2.** (1) The condition $((n - \alpha)/n)q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to saying that there exists $r_0$ with $n/(n - \alpha) < r_0 < \infty$ such that $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$.

(2) $((n - \alpha)/n)q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ implies $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. 
Definition 3 (see [18]). Let \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( u(x, r) : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty) \) be a Lebesgue measurable function; we say \( u \in \mathcal{W}_{q(\cdot)} \) if there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R}^n \) and \( r > 0 \), \( u \) fulfills

\[
\sum_{j=0}^{\infty} \left\| X_{R, j} |f| \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \langle x, z \rangle < C u(x, r). \tag{16}
\]

**Theorem 5.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) be a Lebesgue measurable function; we say \( u \in \mathcal{W}_{p(\cdot), \omega} \) if there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R}^n \) and \( r > 0 \), \( u \) fulfills

\[
\sum_{j=0}^{\infty} \left\| X_{R, j} |f| \right\|_{L^{p(\cdot), \omega}(\mathbb{R}^n)} \langle x, z \rangle < C u(x, r). \tag{17}
\]

**Theorem 6.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfy (13), (14), and (15) with \( 1 = p_- \leq p_+ > n/ \alpha \). If \( \Omega(x, z) \) satisfies (1) and (2) and \( u \) meets with the following condition:

\[
\sum_{j=0}^{\infty} (j + 1)^m \left\| X_{R, j} |f| \right\|_{L^{p(\cdot), \omega}(\mathbb{R}^n)} \langle x, z \rangle < C u(x, r). \tag{22}
\]

Furthermore, \( \Omega(x, z) \) satisfies (1) and (2) and \( u \) meets with the following condition:

\[
\sum_{j=0}^{\infty} (j + 1)^m \left\| X_{R, j} |f| \right\|_{L^{p(\cdot), \omega}(\mathbb{R}^n)} \langle x, z \rangle < C u(x, r). \tag{23}
\]

In particular, \( T_{m, \Omega, \alpha, b} \) is bounded from \( \mathcal{W}_{p(\cdot), \omega} \) to \( \mathcal{W}_{q(\cdot), \omega} \) as \( 1 < p_- \leq p_+ < n/ \alpha \).

### 2. Preliminaries Lemmas

In this section we shall give some lemmas which will be used in the proofs of our main theorems.

**Lemma 8** (see [19]). Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). Define \( p'(\cdot) \) by \( 1/p(\cdot) + 1/p'(\cdot) = 1 \); then there exists a constant \( C > 0 \) such that for any ball \( B \), we have

\[
C^{-1} \leq \frac{1}{|B|} \| X_B |f| \|_{L^{p'(\cdot)}(\mathbb{R}^n)} \| X_B \|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C. \tag{24}
\]

**Lemma 9** (see [20]). Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( s > p_+ \). Defined \( \beta(\cdot) \) by \( 1/p(\cdot) + 1/\beta(\cdot) = 1/s + 1/\beta(\cdot) \), for all measurable functions \( f \) and \( g \), we have

\[
\| fg \|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \leq C \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| g \|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{25}
\]

**Lemma 10** (see [16]). Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). Then \( \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} < C_1 \) if and only if

\[
\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx < C_2. \tag{26}
\]

In particular, if either constant equals 1 we can take the other equal to 1 as well.

By applying the similar method used in the proof of [21, Lemma 4], we can obtain the following result.

**Lemma 11.** Suppose that \( 0 < \theta < \min\{\alpha, n - \alpha\} \), \( x \in \mathbb{R}^n \). Then

\[
\| T_{\Omega, \alpha, \theta} f(x) \| \leq C (n, \alpha, \theta) [M_{\Omega, \alpha, \theta} f(x)]^{1/2} [M_{\Omega, \alpha, \theta} f(x)]^{1/2}, \tag{27}
\]

where

\[
M_{\Omega, \alpha, \theta} f(x) = \sup_{r > 0} \frac{1}{r^{n-\theta}} \int_{|y|\leq r} |\Omega(x, y)| |f(x - y)| \, dy. \tag{28}
\]
Lemma 12. Let $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $1 = p_- \leq p_+ < n/\alpha$. Suppose that $\Omega(x, z)$ satisfies (1) and (2); $q(\cdot)$ is defined by (15). Then there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n)$,
\[ \|T_{\Omega, \alpha, \theta} f\|_{WL^{\frac{\alpha}{\alpha-\theta}}(\mathbb{R}^n)} \leq C \|f\|_{L^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)}. \] (29)

Proof. Let $f \in L^p(\mathbb{R}^n)$. Without loss of generality we may assume that $\|f\|_{L^p(\mathbb{R}^n)} = 1$. Noting that $1 = p_- \leq p_+ < n/\alpha < \infty$, we only need to prove that, for any $\eta > 0$,
\[ \sup_{\eta > 0} \left\| \chi_{\{|f| > \eta\}} \right\|_{L^{\frac{\alpha}{\alpha-\theta}}(\mathbb{R}^n)} \leq C. \] (30)

Since $q_+ < \infty$, by Lemma 10 it will suffice to prove that
\[ \int_{\{|f| > \eta\}} \eta^{\frac{\alpha}{\alpha-\theta}} \, dx \leq C. \] (31)

Fix a $\theta$ with $0 < \theta < \min\{\alpha, n-\alpha\}$ satisfies $1 + (\theta/n)q_+ < 2$. Let $r(x) = 2/(1 + \theta q(x)/n)$. Then $r_+ > 1$. Thus, we have
\[ \frac{1}{p(\cdot)} - \frac{2}{r(\cdot)q(\cdot)} = \frac{\alpha - \theta}{n}, \]
\[ \frac{1}{p(\cdot)} - \frac{2}{r'(\cdot)q(\cdot)} = \frac{\alpha + \theta}{n}. \] (32)

By Lemma II and Young’s inequality, it has
\[ \left| T_{\Omega, \alpha} (f) (x) \right| \leq C \left[ M_{\Omega, \alpha+\theta} f (x) \right]^{1/2} \left[ M_{\Omega, \alpha-\theta} f (x) \right]^{1/2} \]
\[ \leq \frac{\left[ CM_{\Omega, \alpha+\theta} f (x) \right]^{r(x)/2}}{r(x)} \]
\[ + \frac{\left[ CM_{\Omega, \alpha-\theta} f (x) \right]^{r'(x)/2}}{r'(x)} \]
\[ \leq C^{r_+} \left[ M_{\Omega, \alpha+\theta} f (x) \right]^{r(x)/2} \]
\[ + C^{r_-} \left[ M_{\Omega, \alpha-\theta} f (x) \right]^{r'(x)/2}. \] (33)

Noting that $r_+ > 1$, $r'_+ < \infty$, then
\[ \int_{\{|f| > \eta\}} \eta^{\frac{\alpha}{\alpha-\theta}} \, dx \leq \int_{\{|M_{\Omega, \alpha+\theta} f| > 2C\}} \eta^{\frac{\alpha}{\alpha+\theta}} \, dx \]
\[ + \int_{\{|M_{\Omega, \alpha-\theta} f| > 2C\}} \eta^{\frac{\alpha}{\alpha-\theta}} \, dx. \] (34)

Referring to the argument used in the proofs of [16, Theorems 1.8 and 1.9], we can obtain the following inequalities:
\[ \int_{\{|M_{\Omega, \alpha+\theta} f| > 2C\}} \eta^{\frac{\alpha}{\alpha+\theta}} \, dx \leq C, \]
\[ \int_{\{|M_{\Omega, \alpha-\theta} f| > 2C\}} \eta^{\frac{\alpha}{\alpha-\theta}} \, dx \leq C. \] (35)

Then, we have
\[ \int_{\{|T_{\Omega, \alpha, \theta} f| > \eta\}} \eta^{\frac{\alpha}{\alpha-\theta}} \, dx \leq C. \] (36)

Thus
\[ \sup_{\eta > 0} \left\| \chi_{\{|T_{\Omega, \alpha, \theta} f| > \eta\}} \right\|_{L^{\frac{\alpha}{\alpha-\theta}}(\mathbb{R}^n)} \leq C \|f\|_{L^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)}. \] (37)

Lemma 12 is proved. \qed

Lemma 13. Suppose that $b \in L_{\text{loc}}(\mathbb{R}^n), \theta > 0$ with $0 < \alpha - \theta < \alpha < n$. Then, for any $x \in \mathbb{R}^n$,
\[ |T_{\Omega, \alpha, \theta, b} f (x)| \]
\[ \leq C (n, \alpha, \theta) \left[ M_{\Omega, \alpha+\theta} b f (x) \right]^{1/2} \left[ M_{\Omega, \alpha-\theta} f (x) \right]^{1/2}, \] (38)
where
\[ M_{\Omega, \alpha, \theta, b} f (x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|y| < r} |\Omega(x, y)||b(x) - b(y)|^m \, dy. \] (39)

With the similar argument in the proof of [22, Lemma 2], it is easy to draw the above conclusion; the details are omitted here.

Lemma 14. Let $b \in \text{Lip}_p(\mathbb{R}^n), 0 < \beta \leq 1, p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $0 < (\alpha + m \beta)/n \leq 1/p_+$. $q(\cdot)$ is defined as
\[ \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha + m \beta}{n}. \] (40)

If $\Omega(x,z)$ satisfies (1) and (2), then there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n)$,
\[ \|T_{\Omega, \alpha, \theta, b} f\|_{WL^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_p(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \] (41)

Lemma 15. Let $b \in BMO(\mathbb{R}^n), p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $0 < \alpha/n \leq 1/p_+$. If $\Omega(x, z)$ satisfies (1) and (2), $q(\cdot)$ as defined in (15), then there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n)$,
\[ \|T_{\Omega, \alpha, \theta, b} f\|_{WL^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \] (42)

By applying Lemma 13, we can prove Lemmas 14 and 15 with the similar way used in the proof of Lemma 12. Thus, we omit the details here.

Lemma 16 (see [23]). Let $0 < \alpha < n$; if $(p(\cdot), q(\cdot)) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all balls $B$,
\[ \|X_b\|_{L^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)} \|X_a\|_{L^{\frac{\alpha}{\alpha+\theta}}(\mathbb{R}^n)} \leq C. \] (43)
Lemma 17 (see [24]). Suppose \( b \in \text{BMO}(\mathbb{R}^n) \), \( p(\cdot) \in \mathcal{S}(\mathbb{R}^n) \), \( m \) to be a positive integer, \( i, j \in \mathbb{Z} \) with \( i < j \), and then

\[
C^{-1} \| b \|_{\text{BMO}}^m \leq \sup_{\theta} \left\| (b - b_{\theta})^m \chi_{B_{\theta}} \right\|_{L^p(\mathbb{R}^n)} \leq \| b \|_{\text{BMO}}^m,
\]

(44)

\[
\left\| (b - b_{\theta})^m \chi_{B_{\theta}} \right\|_{L^p(\mathbb{R}^n)} \leq C (j - i)^m \| b \|_{\text{BMO}} \| \chi_{B_{\theta}} \|_{L^{p^*}(\mathbb{R}^n)},
\]

where \( B = B(x, r) \), \( B_1 = B(x, 2^{i}r) \).

3. Proofs of Theorems 5–7

Proof of Theorem 5. Let \( f \in \mathcal{M}_{p(\cdot), \omega}(\mathbb{R}^n) \). For any \( z \in \mathbb{R}^n \), \( j \in \mathbb{N} \setminus \{0\} \) and \( r > 0 \), write

\[
f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x),
\]

(45)

where \( f_0 = f_{\mathcal{B}(z,2^j r)} \) and \( f_j = f_{\mathcal{B}(z,2^{j+1} r)} \).

Lemma 12 immediately implies that

\[
\left\| \chi_{\mathcal{B}(z, r)} T_{\Omega, \alpha} f_0 \right\|_{L^{p^*}(\mathbb{R}^n)} \leq C \left\| f_0 \right\|_{L^{p^*}(\mathbb{R}^n)}
\]

(46)

\[
= C \left\| f_{\mathcal{B}(z, 2^j r)} \right\|_{L^{p^*}(\mathbb{R}^n)}.
\]

Note that there exists a constant \( C > 0 \) such that

\[
\chi_{\mathcal{B}(z,2^j r)} \leq CM\chi_{\mathcal{B}(z, r)}.
\]

(47)

In the view of \( 1 \leq p \leq p_* \), \( \omega \in \mathcal{S}(\mathbb{R}^n) \), \( \omega = \chi_{\mathcal{B}(z, 2^j r)} \), so the Hardy-Littlewood maximal operator \( \mathcal{M} \) is bounded on \( L^{q^*}(\mathbb{R}^n) \); it follows that

\[
\left\| \chi_{\mathcal{B}(z, 2^j r)} \right\|_{L^{q^*}(\mathbb{R}^n)} \leq C \left\| M \chi_{\mathcal{B}(z, r)} \right\|_{L^{q^*}(\mathbb{R}^n)}
\]

(48)

\[
\leq C \left\| \chi_{\mathcal{B}(z, r)} \right\|_{L^{q^*}(\mathbb{R}^n)}.
\]

Since

\[
u(z, 2^j r) \leq C \nu(z, r),
\]

(49)

then we have

\[
\frac{1}{\nu(z, r)} \left\| \chi_{\mathcal{B}(z, r)} T_{\Omega, \alpha} f_0 \right\|_{L^{p^*}(\mathbb{R}^n)} \leq C \left\| f_0 \right\|_{\mathcal{M}_{p(\cdot), \omega}}.
\]

(50)

On the other hand, for any \( j \geq 1 \) and \( x \in B(z, r) \), by Hölder’s inequality, we have

\[
\left| T_{\Omega, \alpha} f_j(x) \right| = \left| \int_{B(z,2^{j+1} r)} \frac{\Omega(x, x-y) \chi_{\mathcal{B}(z,2^j r)} f_j(y) dy}{|x-y|^{n-\alpha}} \right|
\]

\[
\leq C (2^j r)^{-(n-\alpha)}
\]

(51)

\[
\cdot \left| \int_{B(z,2^{j+1} r)} \Omega(x, x-y) |f_j(y)| dy \right|
\]

\[
\leq C (2^j r)^{-(n-\alpha)} \| \chi_{\mathcal{B}(z,2^{j+1} r)} \|_{L^{p^*}(\mathbb{R}^n)}
\]

\[
\cdot \left| \left| \chi_{\mathcal{B}(z, x-y)} \chi_{\mathcal{B}(z,2^j r)} f_j \chi_{\mathcal{B}(z,2^j r)} \right|_{L^{p^*}(\mathbb{R}^n)} \right|
\]

(52)

\[
= C 2^{-j(n-\alpha)} r^{-(n-\alpha)} \| \chi_{\mathcal{B}(z,2^{j+1} r)} f_j \chi_{\mathcal{B}(z,2^j r)} \|_{L^{p^*}(\mathbb{R}^n)}.
\]

Thus

\[
I_j = \| \Omega(x, x-y) \chi_{\mathcal{B}(z, 2^j+1 r)} \|_{L^{p^*}(\mathbb{R}^n)}
\]

\[
\leq C \left\| \chi_{\mathcal{B}(z,2^{j+1} r)} \|_{L^{p^*}(\mathbb{R}^n)}
\]

(53)

\[
\cdot \left| \int_{B(z, 2^j r)} \left| \Omega(x, x-y) \right|^{p'} dy \right|^{1/p'}
\]

\[
\leq C \left( \int_{B(z, 2^j r)} \left| \Omega(x, x-y) \right|^{p'} dy \right)^{1/p'}
\]

\[
\cdot \left| \int_{B(z, 2^j r)} \left| \Omega(x, x-y) \right|^{p'} dy \right|^{1/p'}
\]

(54)

\[
\leq C \left( \int_{B(z, 2^j r)} \left| \Omega(x, x-y) \right|^{p'} dy \right)^{1/p'}
\]

\[
\cdot \left| \int_{B(z, 2^j r)} \left| \Omega(x, x-y) \right|^{p'} dy \right|^{1/p'}
\]

(55)

\[
\leq C \left( \int_{B(z, 2^j r)} \left| \chi_{\mathcal{B}(z, 2^j r)} \right|^{p'} dy \right)^{1/p'}
\]

(56)

\[
\cdot \left| \int_{B(z, 2^j r)} \left| \chi_{\mathcal{B}(z, 2^j r)} \right|^{p'} dy \right|^{1/p'}
\]

(57)
Noting that for any \( x \in B(x, z, 2^{j+1}r) \backslash B(x, z, 2^j r) \), \( x \in B(z, r) \), we have
\[
|y| \leq |y - (x - z) + (x + z)| \leq |y - (x - z)| + |(x + z)| \leq 2^{j+2}r
\]
and
\[
|y| \geq |y - (x - z) + (x + z)| \geq |y - (x - z)| - |(x + z)| \geq 2^{j-1}r.
\]
Therefore
\[
B(x - z, 2^{j+1} r) \backslash B(x - z, 2^j r) x \leq B(0, 2^{j+2} r) \backslash B(0, 2^{-1} r).
\]
For any \( j \geq 1 \), we can get
\[
I_j \leq C \left( \int_{B(0,2^{j+2}r) \backslash B(0,2^{j-1}r)} |\Omega(x, x - y)|^s dy \right)^{1/s} \left( \frac{|B(z, 2^{j+1} r)|^{-1/s-a/n}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q}(\mathbb{R}^n)}} \right)
= C \left( \int_{2^j r}^{2^{j+1} r} \int_{|y'|=1} |\Omega(x, y')|^s \lambda^{(n-1)} dy'd\lambda \right)^{1/s} \left( \frac{|B(z, 2^{j+1} r)|^{-1/s-a/n}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q}(\mathbb{R}^n)}} \right)
\leq C \left( \int_{2^j r}^{2^{j+1} r} \lambda^{(n-1)} \int_{|y'|=1} |\Omega(x, y')|^s dy'd\lambda \right)^{1/s} \left( \frac{|B(z, 2^{j+1} r)|^{-1/s-a/n}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q}(\mathbb{R}^n)}} \right).
\]
Since \( \Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1}) \), from (1) and (2), it follows that
\[
I_j \leq C \left( \frac{2^j r}{} \right)^{n/s} \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a} \leq C \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a} \leq C \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a} \left( \frac{2^j r}{} \right)^{-n/s-a}.
\]
Then
\[
T_{\Omega, \alpha} f_j(x) \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
and
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p}(\mathbb{R}^n)} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Thus,
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p}(\mathbb{R}^n)} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Therefore,
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p}(\mathbb{R}^n)} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Applying the quasi-norm \( \| \cdot \|_{W^k} \) on both sides of the above inequality, then
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Note that
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
According to (63), for some \( C > 0 \) independent of \( B(z, r) \), we have that
\[
\frac{1}{u(z, r)} \|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
and
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Therefore,
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Applying the quasi-norm \( \| \cdot \|_{W^k} \) on both sides of the above inequality, then
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Note that
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
According to (63), for some \( C > 0 \) independent of \( B(z, r) \), we have that
\[
\frac{1}{u(z, r)} \|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
and
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Therefore,
\[
\|\chi_{B(z,2^{j+1}r)}f\|_{W^k} \leq C 2^{-j(n-\alpha)} r^{-(n-\alpha)} (2^j r)^{-\alpha}
\]
Remark 1 and (50) yield that
\[
\frac{1}{u(z,r)} \left\| \chi_{B(z,r)} T_{\Omega,\alpha} f \right\|_{WL^{\phi}(\mathbb{R}^n)} \\
\leq C \left( \frac{1}{u(z,r)} \| \chi_{B(z,r)} T_{\Omega,\alpha} f \|_{WL^{\phi}(\mathbb{R}^n)} \right) \\
+ \frac{1}{u(z,r)} \left\| \chi_{B(z,r)} \sum_{j=1}^{\infty} \left| T_{\Omega,\alpha} f_j \right| \right\|_{WL^{\phi}(\mathbb{R}^n)}
\]
(65)

By taking the supremum over \( B(z,r) \), we obtain
\[
\left\| T_{\Omega,\alpha} f \right\|_{WL^{\phi}(\mathbb{R}^n)} \leq C \left\| f \right\|_{M^{\phi}(\mathbb{R}^n)} .
\]

\[
(66)
\]

Proof of Theorem 6. Let \( b \in \text{Lip}_b \) and \( f \in M^{\phi}(\mathbb{R}^n) \). Using the same decomposition of \( f(x) \) as in the proof of Theorem 5, by Lemma 14, we have
\[
\left\| T_{\Omega,\alpha}^{m} f f_{0} \right\|_{WL^{\phi}(\mathbb{R}^n)} \leq C \left\| b \right\|_{\text{Lip}_b} \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} .
\]

(67)

Thus, we get by (49)
\[
\frac{1}{u(z,r)} \left\| \chi_{B(z,r)} T_{\Omega,\alpha}^{m} f f_{0} \right\|_{WL^{\phi}(\mathbb{R}^n)} \\
\leq C \frac{1}{u(z,2r)} \left\| \chi_{B(z,r)} T_{\Omega,\alpha}^{m} f f_{0} \right\|_{WL^{\phi}(\mathbb{R}^n)} \\
\leq C \left\| b \right\|_{\text{Lip}_b} \frac{1}{u(z,2r)} \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\leq C \left\| b \right\|_{\text{Lip}_b} \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} .
\]

(68)

For any \( j \in \mathbb{N} \setminus \{0\} \), \( b \in \text{Lip}_b \), we can obtain by using H"older's inequality,
\[
\left| T_{\Omega,\alpha}^{m} f_j (x) \right| \\
\leq \int_{B(z,2^{j+1}r) \setminus B(z,2^jr)} \frac{\Omega (x, x-y)}{|x-y|^{n-\alpha}} \left| b (x) - b (y) \right|^m \\
\cdot |f (y)| \, dy \leq \left\| b \right\|_{\text{Lip}_b}^m \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \int_{B(z,2^{j+1}r) \setminus B(z,2^jr)} \frac{\Omega (x, x-y)}{|x-y|^{n-\alpha-m\beta}} |f (y)| \, dy \\
\leq C \left\| b \right\|_{\text{Lip}_b}^m \left( 2^j r \right)^{-(n-\alpha-m\beta)} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^jr)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)}
\]
(70)

Note that \( x \in B(z,2^j r) \setminus B(z,2^{j+1}r) \), \( x \in B(z, r) \); we have
\[
B(z,2^j r) \setminus B(z,2^{j+1}r) \\
= B(0,2^{j+1}r) \setminus B(0,2^j r) .
\]

Similar to the estimate of \( I_j \), by the fact of \( 1/p(x) - 1/q(x) = (\alpha + m\beta)/n \), we have that
\[
\Pi_j \leq \left( \int_{B(0,2^j r) \setminus B(0,2^{j+1}r)} \left| \Omega (x, x-y) \right|^q \, dy \right)^{1/q} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
= \left( \int_{2^j r}^{2^{j+1} r} \int_{\mathbb{S}^{n-1}} \left| \Omega (x, y') \right|^q \, dy' \, d\lambda \right)^{1/q} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| B (z, 2^{j+1} r) \right\|_{L^{\phi}(\mathbb{R}^n)}^{1/(q-1)} \\
\cdot \left\| \chi_{B(z,2^j r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^j r)} \right\|_{L^{\phi}(\mathbb{R}^n)} .
\]

(71)

It follows from Lemma 8 that
\[
\Pi_j \leq C \left( 2^j r \right)^{-(\alpha-m\beta)/n} \left\| \Omega \right\|_{L^{\phi}(\mathbb{R}^n)} \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} \left\| \chi_{B(z,2^j r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\leq C \left( 2^j r \right)^{-(\alpha-m\beta)/n} \left\| \Omega \right\|_{L^{\phi}(\mathbb{R}^n)} \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} \left\| \chi_{B(z,2^j r)} \right\|_{L^{\phi}(\mathbb{R}^n)} .
\]

(72)

The above estimates imply that
\[
\left\| T_{\Omega,\alpha}^{m} f_j (x) \right\| \leq C \left\| b \right\|_{\text{Lip}_b}^m \left( 2^j r \right)^{-(n-\alpha-m\beta)} \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^jr)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)} \]
(73)

\[
\left\| \chi_{B(z,2^jr)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| \chi_{B(z,2^{j+1}r)} \right\|_{L^{\phi}(\mathbb{R}^n)} \\
\cdot \left\| f \right\|_{L^{\phi}(\mathbb{R}^n)}
\]
Therefore
\[
X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lip} \ X_{B(z,r)} \sum_{j=1}^{\infty} \| f_{X_{B(z,2^j+1r)}} \|_{L^p(\mathbb{R}^n)} \quad (74)
\]

Applying the quasi-norm \( \| \cdot \|_{W^1(\Omega)} \) on both sides of the above inequality, we get
\[
\left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \leq C \| b \|_{Lip} \ X_{B(z,r)} \sum_{j=1}^{\infty} \| f_{X_{B(z,2^j+1r)}} \|_{W^1(\Omega)} \quad (75)
\]

It implies by (63) that
\[
\frac{1}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \leq C \| b \|_{Lip} \ X_{B(z,r)} \sum_{j=1}^{\infty} \frac{u(z,2^j+1r)}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \quad (76)
\]

From (68), we can arrive at
\[
\frac{1}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \leq C \| b \|_{Lip} \ X_{B(z,r)} \sum_{j=1}^{\infty} \frac{u(z,2^j+1r)}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \quad (77)
\]

By taking the supremum over \( z \in \mathbb{R}^n \) and \( r > 0 \), we conclude that
\[
\left\| T_{\Omega,\alpha,b}^m f \right\|_{W^1(\Omega)} \leq C \| b \|_{Lip} \ \| f \|_{L^p(\Omega)} \quad (78)
\]

This completes the proof of Theorem 6. \( \Box \)

**Proof of Theorem 7.** Let \( b \in \text{BMO}(\mathbb{R}^n), f \in \mathcal{M}_{\lambda_\alpha}(\mathbb{R}^n) \). As in the proof in Theorem 5, write
\[
f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x) . \quad (79)
\]

By Lemma 15, we can obtain
\[
\left\| T_{\Omega,\alpha,b}^m f_0 \right\|_{W^1(\Omega)_{\mathbb{R}^n}} \leq C \| b \|_{\text{BMO}} \ \| f_0 \|_{L^p(\mathbb{R}^n)} . \quad (80)
\]

From (49), we get
\[
\frac{1}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \leq C \| b \|_{\text{BMO}} \ \| f \|_{L^p(\mathbb{R}^n)} . \quad (81)
\]

That is,
\[
\frac{1}{u(z,r)} \left\| X_{B(z,r)} \sum_{j=1}^{\infty} \| T_{\Omega,\alpha,b}^m f_j \|_{W^1(\Omega)} \right\|_{W^1(\Omega)} \leq C \| b \|_{\text{BMO}} \ \| f \|_{L^p(\mathbb{R}^n)} . \quad (82)
\]

Furthermore, for any \( j \in \mathbb{N} \setminus \{0\} \), we have
\[
\left\| T_{\Omega,\alpha,b}^m f_j \right\|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lip} \ \| f \|_{L^p(\mathbb{R}^n)} \quad (83)
\]

For \( f_1 \), Lemmas 9, 16 and Hölder’s inequality assure that
\[
J_1 \leq C \left\| \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} X_{B(z,2^{j+1}r)} \right\|_{L^p(\mathbb{R}^n)} \ \| b \|_{Lip} \ \| f \|_{L^p(\mathbb{R}^n)} \quad (84)
\]

\[
J_2 \leq C \left\| \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} X_{B(z,2^{j+1}r)} \right\|_{L^p(\mathbb{R}^n)} \ \| b \|_{Lip} \ \| f \|_{L^p(\mathbb{R}^n)} \quad (85)
\]
\[ \| \chi_{B(z,2j+1)} \|_{L^p(\mathbb{R}^n)} \times \| \chi_{B(z,2j+1)} f \|_{L^q(\mathbb{R}^n)} \leq C \| b \|_{\text{BMO}}^m \]
\[ \frac{1}{|B(z,r)|} \| \chi_{B(z,r)} \|_{L^q(\mathbb{R}^n)} \times \| \chi_{B(z,r)} f \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{\text{BMO}}^m. \] 

Hence

\[ J_1 \leq C \| b \|_{\text{BMO}}^m \frac{\| \chi_{B(z,2j+1)} f \|_{L^p(\mathbb{R}^n)}}{\| \chi_{B(z,2j+1)} \|_{L^p(\mathbb{R}^n)}}. \]

Now, turn to estimate \( J_2 \). By Lemmas 9, 17 and Hölder's inequality, we have

\[ J_2 \leq C \left\{ \frac{\| \Omega (x, x - y) \|}{|x - y|^{n-\alpha}} \right\} b(y) - b_{B(z,r)} \right\}^m \]
\[ \| \chi_{B(z,2j+1)} \|_{L^p(\mathbb{R}^n)} \times \| \chi_{B(z,2j+1)} f \|_{L^q(\mathbb{R}^n)} \leq C \left\{ \frac{\| \Omega (x, x - y) \|}{|x - y|^{n-\alpha}} \right\} \]
\[ \frac{1}{|B(z,r)|} \| \chi_{B(z,r)} \|_{L^q(\mathbb{R}^n)} \times \| \chi_{B(z,r)} f \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{\text{BMO}}^m. \] 

Thus applying Lemma 16 leads to

\[ J_2 \leq C (j + 1)^m \| b \|_{\text{BMO}}^m \]
\begin{align*}
&\frac{1}{|u(z, r)|} \left\| \chi_{B(z, r)} T_{\alpha, b} f \right\|_{W_{\ell, \theta}} \leq C \|b\|_{\text{BMO}}^m \\
&\left( \frac{1}{|u(z, r)|} \right) \left\| \chi_{B(z, r)} T_{\alpha, b} f_0 \right\|_{W_{\ell, \theta}} \\
&\frac{1}{|u(z, r)|} \left\| \chi_{B(z, r)} \sum_{j=1}^{\infty} |T_{\alpha, b} f_j| \right\|_{W_{\ell, \theta}} \leq C \|b\|_{\text{BMO}}^m \\
&\frac{1}{|u(z, r)|} \left\| \chi_{B(z, r)} \sum_{j=1}^{\infty} |T_{\alpha, b} f| \right\|_{W_{\ell, \theta}} \leq C \|b\|_{\text{BMO}}^m
\end{align*}

Thus, by taking the supremum over $z \in \mathbb{R}^n$ and $r > 0$, we obtain

\begin{equation}
\|T_{\alpha, b} f\|_{W_{\ell, \theta}} \leq C \|b\|_{\text{BMO}}^m \|f\|_{\mathcal{M}_{\ell, \theta}(\alpha, \beta)}.
\end{equation}

The proof of Theorem 7 is completed. \qed

\section*{Data Availability}
No data were used to support this study.

\section*{Conflicts of Interest}
The authors declare that they have no conflicts of interest.

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