The Blackbody Radiation Laws in the AdS$_5 \times S^5$ Spacetime

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In the footsteps of our previous work [1] we generalize the Stefan-Boltzmann and Wien’s displacement laws for the AdS$_5 \times S^5$ spacetime, the background of the AdS/CFT correspondence foremost realization. Our results take into account the AdS$_5 \times S^5$ full dimensionality in the electromagnetic field $A^\mu$ wave equation, which yields the higher-dimensional blackbody characteristic features suggested in literature. In particular, the total radiated power and the spectral radiancy match the original Stefan-Boltzmann and Wien’s displacement laws in the low-energy regime up to available experimental data.

Keywords: Blackbody radiation; Stefan-Boltzmann law; Wien’s law; AdS/CFT correspondence.

I. INTRODUCTION

The AdS/CFT Correspondence proposed by Maldacena in 1997 relates, in its most notable instance, an effective five-dimensional AdS$_5$ gravitational theory (via type IIB string theory) to a four-dimensional Conformal Field Theory ($N = 4$ supersymmetric Yang-Mills theory) on its boundary [2], being the subject of interesting contemporary physical unfoldings [3-14].

The duality expressed in this correspondence is the most successful realization of the Holographic Principle [15-18], which enables to “translate” certain problems in nuclear and condensed matter physics into more tractable ones in the string theory picture [19-23]. Albeit the theories involved are not viable models of the real world, they provide model-building tools for physics beyond the standard model and some of their properties elucidate important questions, such as the quantum behavior of black holes [24-31].

Anti-de Sitter space is the maximally symmetric Einstein’s equations solution with negative (attractive) cosmological constant. In our universe, this constant is certainly not attractive, yet it can be regard as a long-distance regularization of gravity behavior. AdS$_5$ space notable feature relates to its boundary, which looks locally around each of its points like our well-known Minkowski $M^{1,3}$ spacetime, labeled from now on as $\mathcal{M}$ unless otherwise specified. The five additional “compact” dimensions, $S^5$ subspace, come as a dimensional requirement for String Theory mathematical consistency.

Blackbody radiation is described by Planck’s law, which implies its radiative features. In particular, the Stefan-Boltzmann law predicts that the blackbody radiated power is $R(T) = \sigma_b T^4$, with $\sigma_b$ as the Stefan-Boltzmann constant, while the Wien’s displacement law relates the blackbody temperature $T$ to the wavelength $\lambda_m$ corresponding to the wavelength radiancy $R(T, \lambda)$ peak, $\lambda_m T = 2.897 \times 10^{-3}$ m K.

These laws shall depend on the spacetime dimensionality [32, 33], in contrast with observed data. Our results preserve the well-known blackbody radiation laws in the low-temperature regime while obtaining the higher-dimensional theoretical relations in the high-temperature regime, explicitly establishing a connection between the current theoretical unfoldings and our well-acquainted 4-dimensional experience.

The article is organized as follows. In section 2 we lay the blackbody concept and briefly introduce our approach. In Section 3 we single the Minkowski slice out of the AdS$_5$ space and generalize the Stefan-Boltzmann and Wien’s laws for the specific $\mathcal{M} \times S^5$ scenario. Similar steps are taken in section 4 towards the full AdS$_5 \times S^5$ spacetime generalizations. In section 5 we set experimental bounds on the AdS$_5$ conformal coordinate, plotting our main results for these bounds. The final section is devoted to closing remarks and general comments.

II. THE BLACKBODY AND OUR APPROACH

A blackbody is defined as a body with a rich energy spectrum, capable of exciting all frequencies of light by thermalization. As consequence, all blackbodies at the same temperature emit thermal radiation with the same spectrum. For a current review on the matter, see [34].

The standard textbook blackbody approach consists in taking a small bidimensional orifice connecting an isothermal enclosure to its outside as a blackbody surface, a more technical and conceptual discussion on approximating a blackbody for a blackbox is found in [35, 36].

Here, we look for the electromagnetic wave equation solution which fills the $\text{AdS}_5 \times S^5$ spacetime, where Dirichlet boundary conditions are assumed, in order to obtain the radiated spectrum. The general procedure employed...
in this work is analogous to that of \[1\].

The AdS$_5 \times S^5$ spacetime metric can be written in the form

$$ds^2 = R^2 \left[ z^{-2} (dx^\mu dx_\mu + dz^2) + d\Omega_5^2 \right],$$

(1)

and the wave equation to be satisfied by the electromagnetic field $A^\mu$ for a generic metric $g_{ab}$ is

$$(-g)^{-\frac{1}{2}} \partial_a \left[ (-g)^{\frac{1}{2}} g^{ab} \partial_b A^\mu \right] = 0,$$

(2)

where the $a$ and $b$ indices run through all considered dimensions while we keep the $\mu$ indices for the Lorentzian polarizable ones.

Making use of Bose-Einstein statistical prescription and accounting two helicity states related to the propagative aspect of the photons associated to the electromagnetic waves, the blackbody radiation energy density at temperature $T$ is

$$\rho(T) = \frac{2}{V} \sum_i \frac{h\nu_i}{e^{h\nu_i/kT} - 1},$$

(3)

where $V$ is taken as the ordinary 3-dimensional volume, since this is our usual framework and the AdS$_5 \times S^5$ metric is scale invariant.

The spatial boundary conditions determine the photons energy states $h\nu_i$, and the energy density is proportional to the radiancy $R(T)$, the energy rate per unit area, through a geometric speed-of-light factor,

$$R(T) = \frac{c}{4} \rho(T).$$

(4)

### III. $\mathcal{M} \times S^5$ BRANE SCENARIO

The present $\mathcal{M} \times S^5$ setup, in which the conformal coordinate $z = \text{const}$, has effective metric

$$ds^2 = dx^\mu dx_\mu + z^2 d\Omega_5^2,$$

(5)

which means that the electromagnetic field fulfills

$$\left[ \partial^\nu \partial_\nu + \frac{1}{z^2} \Delta_{S^5} \right] A^\mu(x, \Theta) = 0,$$

(6)

where $\Delta_{S^5}$ is the Laplace-Beltrami operator on $S^5$ with the particular property

$$[\Delta_{S^5} + l(l + n - 1)] Y_l(\Theta) = 0,$$

(7)

and $Y_l(\Theta) = Y_{l_1 \cdots l_n}(\theta_1, \cdots, \theta_1)$ are the spherical harmonics in higher dimensions, with $l \equiv l_n$. Since $l_3 \geq l_4 \geq l_5 \geq l_2 \geq |l_1| \geq 0$, for each $l = 0, 1, 2, \ldots$, one has

$$d_l = \frac{1}{4l}(l + 1)(l + 2)(l + 3)(2l + 4)$$

(8)

associated degeneracies.

Solution of (6) is given by variable separation, $A^\mu$ indicates the field polarization, thus

$$A^\mu(x, \Theta) = A^\mu e^{i k^\mu x} Y_l(\Theta),$$

(9)

from which we obtain, by Dirichlet boundary conditions,

$$\left( \frac{2\pi \nu}{c} \right)^2 = \sum_{i=1}^3 n_i^2 \left( \frac{\pi}{L_i} \right)^2 + \frac{l(l + 4)}{z^2}. $$

(10)

where $L_i$ are the usual three-dimensional coordinate lengths and $n_i = 0, 1, 2, \ldots$

Since the $\mathcal{M}$ spatial and $S^5$ contributions may not be on the same foot, particularly for $z \ll L_i$, eq.(11) is better written via (3) as

$$R(T) = \frac{c}{2V} \sum_{i=0}^\infty \sum_{l \neq 0} \frac{h\nu_i}{e^{h\nu_i/kT} - 1}. $$

(11)

The first sum is a single one in $n_i$. Taking it by an integral and considering $\nu_i \equiv \nu(n_i, 0)$ yields

$$R_0(T) \equiv \frac{c}{2V} \sum_{i=0}^\infty \frac{h\nu_i}{e^{h\nu_i/kT} - 1} = \int_0^\infty R_0(T, \nu) d\nu,$$

(12)

where the spectral radiancy $R_0(T, \nu)$ for this range of modes is

$$R_0(T, \nu) = \frac{2\pi h}{c^2} \frac{\nu^3_0}{e^{h\nu_0/kT} - 1}.$$

(13)

Making the variable change $y_0 = h\nu_0/kT$ one integrates \[13\] arriving at

$$R_0(T) = \frac{2\pi c}{(hc)^\alpha} (kT)^4 \int_0^\infty \frac{y_0^3}{e^{y_0} - 1} dy_0,$$

(14)

with the integral expressed by the mathematical identity

$$\int_0^\infty \frac{x^d}{e^x - 1} dx = \Gamma(d + 1) \zeta(d + 1).$$

(15)

Taking $d = 3$ in the above identity, one obtains the ordinary radiancy contribution

$$R_0(T) = \sigma_\beta T^4,$$

(16)

which is the well-known Stephan-Boltzmann law with

$$\sigma_\beta = \frac{2\pi k^5 c}{15} \left( \frac{k}{hc} \right)^3 = 5.67 \cdot 10^{-8} \text{W m}^{-2} \text{K}^{-4}.$$

(17)

Now we consider the case $l \neq 0$ in which both $n_i$ and $l$ contribute to the spectral radiancy. Taking the sum by an integral, with $\nu \equiv \nu(n_i, l)$, one gets

$$R_l(T) \equiv \frac{c}{2V} \sum_{l \neq 0} d_l \frac{h\nu}{e^{h\nu/kT} - 1} = \int_0^\infty R_l(T, \nu) d\nu.$$
and the corresponding spectral radiancy is given by

\[ R_l(T, \nu) = \frac{2}{4\pi} \left( \frac{2\pi z^3}{c} \right)^4 \frac{\Omega_{(d)} \pi z h c}{2e^4} \frac{\nu^4}{e^{h\nu/kT} - 1}, \quad (19) \]

where \( \Omega_{(d)} = 2 \pi^{d/2}/\Gamma(d/2) \) is the solid angle in a \( d \)-dimensional space.

Through the variable change \( y = h\nu/kT \) one gets

\[ R_l(T) = \frac{2\Omega_{(d)} h c (\pi z)^3}{3e^8} \left( \frac{kT}{h} \right)^{9/2} \int_0^\infty \frac{y^8}{e^{3y} - 1} dy, \quad (20) \]

and using (15) the radiancy contribution \( R_l(T) \) due to these modes is

\[ R_l(T) = \sigma_l T^9, \quad (21) \]

\[ \sigma_l = \frac{2\Omega_{(d)} k c}{3} \left( \frac{k}{hc} \right)^8 \Gamma(9) (9)(\pi z)^5. \quad (22) \]

Grouping the computed radiancy contributions one gets for the total blackbody energy rate per unit area

\[ R(T) = \sigma_B T^4 + \sigma_l T^9, \quad (23) \]

which is interpreted as the generalized Stefan-Boltzmann law for the \( M \times S^5 \) subspace.

The nature and validity of the preceding outcome is closely related to the temperature of the blackbody in question. For low temperatures the obtained Stefan-Boltzmann law generalization (23) reduces to its well-known form (16).

Alternatively, one can work with its Taylor series expansion, \( \alpha \in \mathbb{R} \),

\[ J_\alpha(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k! \Gamma(k + \alpha + 1)} \left( \frac{x}{2} \right)^{2k+\alpha}. \quad (34) \]

While the Neumann function is suitably described by

\[ N_\alpha(x) = \frac{J_\alpha(x) \cos(\alpha \pi) - J_{-\alpha}(x)}{\sin(\alpha \pi)}, \quad (35) \]
valid for integer $\alpha$ as a limiting case.

Setting $a$ and $b$ as the boundary conformal coordinates for which (32) matches Dirichlet conditions, $Z(a) = Z(b) = 0$, determines the possible $\kappa$ values that will fit into the spectral relation derived from (30) and (31),

$$\frac{(2\pi \nu)^2}{c^2} = \sum_{i=1}^{3} n_i^2 \left( \frac{\pi}{L_i} \right)^2 + \kappa^2 . \quad (36)$$

The equations concerning the Dirichlet conditions on the conformal coordinates $a$ and $b$

$$c_1 J_{l+2}(\kappa a) + c_2 N_{l+2}(\kappa a) = 0 \quad (37)$$

$$c_1 J_{l+2}(\kappa b) + c_2 N_{l+2}(\kappa b) = 0 \quad (38)$$

have non-trivial solutions only if the determinant

$$J_{l+2}(\kappa a) N_{l+2}(\kappa b) - J_{l+2}(\kappa b) N_{l+2}(\kappa a) = 0 \quad (39)$$

vanishes, thus providing the values for $\kappa$. They are hard to calculate, though. So we abdicate the formal treatment in favor of a more useful approach.

The graphs of these cylindrical $J_m(x)$ and $N_m(x)$ functions look roughly like oscillating sine or cosine decaying functions, with their roots asymptotically (large $x$) periodic. In fact, for sufficiently high argument values

$$J_{l+2}(\kappa z) \approx \sqrt{\frac{2}{\pi \kappa z}} \cos \left( \kappa z - \frac{\pi}{4} (2l + 5) \right) , \quad (40)$$

$$N_{l+2}(\kappa z) \approx \sqrt{\frac{2}{\pi \kappa z}} \sin \left( \kappa z - \frac{\pi}{4} (2l + 5) \right) , \quad (41)$$

so we can reason that

$$\kappa (b - a) \approx \eta \pi , \quad \eta = 0, 1, 2, \ldots \quad (42)$$

Then for each value of $\eta$ there are $\delta \approx 2 \eta \pi/(b - a)$ possible values for $l$, which grants $D_\eta$ degeneracies due to the possible $l_j$ values,

$$D_\eta = \sum_{l=0}^{\delta} d_l = \frac{1}{5!} (\delta + 1)(\delta + 2)(\delta + 3)(\delta + 4)(2\delta + 5) . \quad (43)$$

Analogously, since the $M$ spatial and the $S^5$ conformal contributions are not necessarily of the same order of magnitude, one has

$$R(T) = \frac{c}{2V} \left[ \sum_{\eta=0}^{\infty} + \sum_{\eta \neq 0} \right] \frac{h\nu \eta}{e^{h \nu / kT} - 1} . \quad (44)$$

Working out eq. (41) for $\eta = 0$ leads to the known blackbody relations, while the $\eta \neq 0$ contribution gives

$$R_{\eta}(T) = \frac{c}{2V} \sum_{\eta \neq 0} \frac{D_{\eta} h \nu}{e^{h \nu / kT} - 1} = \int_0^{\infty} R_{\eta}(T, \nu) d\nu \quad (45)$$

with the corresponding spectral radiancy given by

$$R_{\eta}(T, \nu) = \frac{2}{5!} \left( \frac{4 \pi \eta}{c} \right)^5 \frac{\Omega(4) h c}{2 \pi^2 (b - a)} \frac{\mu^4}{e^{h \nu / kT} - 1} . \quad (46)$$

Through the variable change $\nu = h \nu/kT$ one gets

$$R_{\eta}(T) = \frac{\Omega(4) h c}{4 \cdot 5!} \left( \frac{b}{a} - 1 \right) \left( \frac{kT}{\hbar} \right)^{10} \int_0^{\infty} \frac{\eta^9}{e^{\eta} - 1} d\eta , \quad (47)$$

and the radiancy contribution due to these modes is

$$R_{\eta}(T) = \sigma_{\eta} T^{10} , \quad (48)$$

$$\sigma_{\eta} = \frac{\Omega(4) k c^9}{4 \cdot 5!} \left( \frac{k}{\hbar} \right)^9 \Gamma(10) \xi(10) (4a)^6 \left( \frac{b}{a} - 1 \right) . \quad (49)$$

Thus, the total blackbody energy rate per unit area is

$$R(T) = \sigma_a T^4 + \sigma_{\eta} T^{10} , \quad (50)$$

which is understood as the generalized Stefan-Boltzmann law for the $AdS_5 \times S^5$ spacetime.

As before, for low temperatures the above generalization reduces to the well-known Stefan-Boltzmann law, while for sufficiently high temperatures the generalized Stefan-Boltzmann law takes on its higher-dimensional character

$$R(T) = \sigma T^{10} , \quad (51)$$

in agreement with $[32, 33]$. The generalized Wien’s law for the full $AdS_5 \times S^5$ is deduced the same way as before. For a given temperature $T$ there is a $\lambda_m$ for which $R(T, \lambda_m)$ is maximum, namely

$$1 - e^{-\hbar \nu/ kT \lambda_m} = \frac{\hbar c}{kT \lambda_m} \frac{1 + \epsilon_{\eta}(\lambda_m)}{5 + 11 \epsilon_{\eta}(\lambda_m)} , \quad (51)$$

where $\epsilon_{\eta}(\lambda) = R_{\eta}(T, \lambda)/R_4(T, \lambda)$ is the respective wavelength radiancy relative deviation

$$\epsilon_{\eta}(\lambda) = \frac{\pi}{4 \cdot 5!} \left( \frac{b}{a} - 1 \right) \left( \frac{4a}{\lambda} \right)^6 . \quad (52)$$

Then, for large wavelengths one gets out of (51) the usual Wien’s displacement law, while for small wavelengths it reduces to its higher-dimensional behavior.

$$1 - e^{-x_{10}} = \frac{x}{11} ; \quad x_{10} \equiv \frac{\hbar c}{kT \lambda_m} . \quad (53)$$

V. BOUNDS ON THE CONFORMAL COORDINATE VALUES AND GRAPHS

According to precise measurements $[37, 41]$, the wavelength radiancy relative deviation, $\epsilon_{\eta}(\lambda)$, shall be no greater than $1\% \sim 2\%$ for $x^4 = 250 \text{ nm at } T \approx 3000 \text{ K}$. Which sets the upper bound on the conformal coordinate of the $M \times S^5$ scenario at $z_{max} \approx 25 \text{ nm}$, with similar limits for $a$ and $b$ on the $AdS_5 \times S^5$ spacetime.

The obtained Stefan-Boltzmann law generalizations $[25]$ and $[50]$ are plotted in fig. (11) and compared with the standard prediction. For the considered bounds on $z$, $a$ and $b$ the discrepancy between these functions is just noticeable for $T > 10^4 \text{ K}$. 
The generalized Wien’s laws are confronted with the habitual prediction in fig. [2]. The graph depicts a clear higher-dimensional signature for $\lambda_m < \lambda^t$, where [27] and [41] departure from the usual behavior.

The generalized AdS$_5 \times S^5$ Wien’s law discloses a remarkable behavior on its temperature transition range. Its wavelength radiancy displayed in fig. [3] has two distinct maxima separated by a minimum. This feature is accentuated in the temperature transition range for smaller conformal coordinate values.

VI. CONCLUSION AND DISCUSSIONS

In this work we generalize the Blackbody Radiation Laws for the compelling AdS$_5 \times S^5$ spacetime. The temperature for which deviations in the blackbody radiation becomes relevant is inversely proportional to the conformal coordinate values, [23] and [50], while the wavelength for which deviations in the blackbody spectrum becomes important is directly proportional to them, as noted in [28] and [52].

The additive aspect of the generalized Stefan-Boltzmann [23] and Wien’s displacement [27] laws traces back to the temperature induced split structure of [11] and [44], the key conceptual point of our approach [1]. Thus the blackbody temperature sets up how the spacetime takes part on its radiative features.

Our approach yields the suitable higher-dimensional blackbody features in the high-energy regime and is also compatible with our empirically presumed 4-dimensional spacetime, once it reproduces the observed Stefan-Boltzmann and Wien’s displacement law in our ordinary energy scale. In the low energy scale, it is seen that AdS$_5 \times S^5$ (bulk) spacetime as well as its Minkowskian (brane) subspace $M \times S^5$ behave effectively as an $M$ manifold, while in the high energy scale they behave effectively as $M^{1,9}$ and $M^{1,8}$ respectively.

VII. ACKNOWLEDGMENTS

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[1] R. Ramos and H. Boschi-Filho, Physica A, 393 (2014) 261.
[2] J. M. Maldacena, Int. J. Theor. Phys. 38 (1999) 1113-1133, Adv. Theor. Math. Phys. 2 (1998) 231-252.
[3] Itzhaki, N., Maldacena, J. M., Sonnenschein, J. and Yankielowicz, S. Phys. Rev. D 58 (1998) 046004
[4] Maldacena, Juan M., Phys. Rev. Lett. 80 (1998) 4859-4862.
[5] S.S. Gubser and I.R. Klebanov and A.M. Polyakov, Physics Letters B, 428, 12 (1998) 105 - 114.
[6] Edward Witten, Adv. Theor. Math. Phys. 2 (1998) 4859-4862.
[7] Igor R. Klebanov and Edward Witten, Nucl. Phys. B556 (1999) 89-114
[8] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183-386 (2000)
[9] Hawking, S., Maldacena, J. M. and Strominger, A., JHEP 0105 (2001) 001.
[10] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Rev. Lett. 96 (2006) 081602
[11] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96 (2006) 181602.
[12] R. C. Brower, J. Polchinski, M. J. Strassler, C.-I Tan, JHEP 0712 (2007) 005
[13] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, M. Rangamani, JHEP 0802 (2008) 045
[14] Gaiotto, D. and Maldacena, J. M., JHEP 1201 (2012) 189.
[15] Susskind, L., Journal of Mathematical Physics 36, 11 (1995) 63766396.
[16] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) arXiv:hep-th/9802150.
[17] Susskind, Leonard and Witten, Edward, arXiv:hep-th/9805114 (1998).
[18] Bousso, R., Rev. Mod. Phys. 74, 3 (2002) 825874.
[19] Herzog, C. P., J. Phys. A Math. Theor. 42 (2009) 343001.
[20] Alday, L. F. and Maldacena, J. M., JHEP 0706 (2007) 064
[21] Christopher P. Herzog et al, JHEP 0607 (2006) 013
[22] Joshua Erlich, Emanuel Katz, Dam T. Son, Mikhail A. Stephanov, Phys. Rev. Lett. 95 (2005) 261602
[23] G. Policastro, Dan T. Son, Andrei O. Starinets, Phys. Rev. Lett. 87 (2001) 081601
[24] Horowitz, G. T. and Maldacena, J. M. and Strominger, A., Phys. Lett. B 383 (1996) 151-159
[25] Maldacena, J. M. and Strominger, A., Phys. Rev. Lett. 77 (1996) 428-429.
[26] Horowitz, G. T., Lowe, D. A. and Maldacena, J. M., Phys. Rev. Lett. 77 (1996) 430-433
[27] Maldacena, J. M. and Strominger, A., Phys. Rev. D 56 (1997) 4975-4983
[28] Maldacena, J. M. and Strominger, A., JHEP 9812 (1998) 005.
[29] Maldacena, J. M., JHEP 0304 (2003) 021.
[30] Horowitz, G. T. and Maldacena, J. M., JHEP 0402 (2004) 008.
[31] Maldacena, J. M., Science, 6186, 344 (2014) 806-807.
[32] H. Alnes, F. Ravndal and I. K. Wehus, J. Phys. A 40, 14309 (2007).
[33] T. R. Cardoso and A. S. de Castro, Rev. Bras. Ens. Fis., 27, 559 (2005).
[34] R. Lehoucq, Eur. J. Phys. 32 (2011) 14951514.
[35] M. Smerlak, Eur. J. Phys. 32 (2011) 11431153.
[36] A M Garcia-Garcia, Phys. Rev. A 78 023806 (2008).
[37] Mohr, Peter J. et. al., Rev. Mod. Phys. 84, 4, 1527-1605.
[38] V I Sapritsky, Metrologia, 32 (1995) 411-417.
[39] H W Yoon and C E Gibson, Metrologia, 37 (2000) 429-432.
[40] R Friedrich and J Fischer, Metrologia, 37 (2000) 539-542.
[41] H W Yoon, C E Gibson and P Y Barnes, Metrologia, 40 (2003) S172-S176.