Implementation of APOS theory to encourage reflective abstraction on Riemann Sum

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Abstract. The purpose of this study is to explore the implementation of APOS (Action-Process-Object-Schema) in first-year students at higher education as a basis of instructional learning in Riemann Sum to emerge the reflective abstraction. This is qualitative research, and data were taken by tests, interviews, and observation. To construct the Riemann Sum understanding, students came through the Action by estimating the area under the curve with various shapes, looking for the area under the curve through several partitions, and make repetitions. The Process was built by interiorizing the previous action on n partitions. The Object was constructed through the encapsulation mechanism by determining the existence and value of the limit for the Riemann Sum. Schemes occurred through thematization by applying Riemann Sum to contextual situations involving functions of distance and time. The results showed that the success of APOS implementation was determined by genetic decomposition, assistance, reinforcement of preconditions and reinforcement of concepts by lecturers. Some effective scaffoldings to help the construction of reflective abstractions in Riemann Sum are worksheets and guided question from educator.

1. Introduction

Learning mathematics is a transformation process of someone’s knowing and acting [1]. As a transformation process, there has been much research and studies which have shown that there are difficulties in advanced mathematics, especially for the first year students. An in-depth study was conducted by Nardi which is showed that there is cognition tension in the process of transition from school mathematics into advanced mathematics [2]. The study of 79 respondents from 21 countries conducted by Mike OJ et.al showed that 91.1% of respondents said they have problems in this transition. Two-thirds of respondents answered Yes there is a problem, saying that problems arise from differences in the characteristics of school mathematics and advanced mathematics, such as procedural mathematics in school mathematics and conceptual mathematics in advanced mathematics. This research also showed 5 materials that had the lowest understanding, i.e proof, limit, algebraic matrix, Riemann Sum and also sequence and series [3].

At the beginning of advanced mathematics, meetings with mathematical abstractions are an urgent step [2]. This statement is in line with Beth & Piaget who stated that reflective abstraction is very important for the development of advanced mathematical concepts because mathematical constructs are processed through reflective abstraction [4]. To overcome this transition, Nardi suggested that learning on advanced mathematics should cover five characteristics, there are collaborative, mathematically focused, specific, non-prescriptive and non-deficit contexts [2]. Mike said that at least there were four perspectives studies by researchers to analyze the learning in this transition period: ATD
(Anthropological Theory of Didactics), TDS (Theory of Didactical Situations), The Three Worlds of Mathematics and APOS (Action Process Object Schema). The last one, APOS, is a continuation of Piaget's theory of reflective abstraction, applied to advanced mathematical thinking. In the APOS theory, reflective abstraction consists of mental mechanisms of interiorization, encapsulation, coordination, observation, de-encapsulation, and thematization. This mental mechanism will form the mental structure of Action, Process, Object and Schema [5]. APOS theory is an analytical tool that can be used to investigate the understanding of one's mathematical concepts and to describe the development of understanding in the mind of an individual [6].

The APOS theory has been developing for 30 years since it was first reviewed and it is still possible to continue to be refined to get the best identification or diagnosis of how mathematical concepts developed in the minds of students. For this reason, every study on APOS will contribute to the development of concepts through a reflective abstraction process, as a complement to make it perfect. For example, research conducted by Afgani et al about the level of students' understanding in the Limit function based on the APOS theory [7]. The levels referred to in this study are Action, Process, Objects, and Schemes. This is different from the one presented by Arnon et.al which explains that Action Processes, Objects, and Schema do not represent levels, but stages of construction of individual understanding. According to Arnon, the level of abstraction is present in each of these stages [5]. Sopamena et al. identify the characteristics of students' reflective abstraction thinking processes in solving Limit problems [8]. In exploring this characteristic Sopamena refers to Piaget's theory of reflective abstraction and Herskowitz's theory to see cognition process through the cognitive line. It is different from the theoretical reference in this study, where the process of cognition is seen through the mental structure of the APOS.

The purpose of this study was to explore the process of reflective abstraction construction as a mental mechanism in the construction of the Riemann Sum concept through the implementation of the APOS theory. Compared with previous studies, this research will enrich the study of APOS implementation to emerge reflective abstraction, as an important part in the development of advanced mathematical thinking. However, if the previous research was mostly carried out on advanced mathematical material, this study examined the early subject of advanced mathematics, as was done by Maharaj [9] who studied Integral. Exploring the process of conceptualizing the initial concept of students learning in advanced mathematics is important because it will be seen the constraints of students in the transition process of thinking from school mathematics to advanced mathematics.

2. Theoretical Background

2.1. Reflective Abstraction

Reflective abstraction is one of the oldest studies of cognition, derived from Skemp's idea about abstraction [10], continued by Piaget who divided the abstractions into three types, empirical abstraction, pseudo empirical and reflective abstraction [11]. Reflective abstraction is distinguished from empirical or pseudo-empirical abstraction, where reflective abstraction leads to constructive generalization while empirical abstraction leads to inductive generalization. According to Vuyk, the concept of reflective abstraction was increasingly important in Piaget's thought until he replaced the overall structure as a definitive stage of development [12].

Piaget distinguishes various types of construction in reflective abstraction, i.e. interiorization, Coordination, Encapsulation, and Generalization [13]. But even though Piaget has given many examples of reflective abstraction, the process was not detailed in the beginning. Several studies of the reflective abstraction process were carried out through the knowledge level approach as described by Campbell [12]. In this approach, the reflective abstraction process goes through two phases: the abstraction process and the reflective process. The abstraction process is a phase where the infinite implicit traits remain in the lower level which is differentiated and abstracted into their explosive representations. While the reflective process is seen through indicators so that the system can abstract the nature of the processes generated from these indicators.
This study of the reflective abstract process continues to develop until Hershkowitz et al. (2001) states that the abstraction process occurs through the process of recognition, building-with, and construction. These three processes are known as the RBC model [14]. Dreyfus said that for abstraction a process of representation, generalization, and synthesis is needed [15]. Dubinsky through his study for several years managed to convey that the reflective abstraction process occurs through 7 processes: interiorization, encapsulation, de-encapsulation, coordination, reversal, thematization, and generalization [3].

2.2. The learning process that supports the reflective abstraction

It seems that there is enough explanation about the urgency of reflective abstraction to develop children's thinking. Therefore the next study is how to grow reflective abstraction in students. Initially, Piaget argued that reflective abstraction is a personal activity where students can initiate their reflective abstraction. But as the study of reflective abstraction develops, it is evident that there is a relationship between social interaction and the emergence of reflective abstraction, so working in pairs can be used as a strategy to grow this abstraction. Also, other studies show that teachers play a role in initiating reflective abstractions. Further studies show that the curriculum also allows for growing reflective abstraction. Thus Capetta concludes that there are at least four agents who can test the emergence of reflective abstractions, (1) the individual itself, (2) colleagues, (3) the teacher, and (4) Design activities in the curriculum designed to provide challenges and questions students [13]. The need to consider aspects of learning design in generating reflective abstraction is emphasized by Ozmantar & Monaghan which states that the formation of abstraction depends on the context in which the learning activity runs [15].

From the results of the research and theory above, it can be concluded that an educator must design the pedagogical strategy and scaffolding that support reflective abstractions.

2.3. APOS Theory

The APOS theory (Action, Process, Object, and Schema) was developed by Dubinsky in 1983-1984. His first publication was delivered in Helsvy in 1984 when Dubinsky delivered his speech about the different processes of thinking about functions as processes and as objects using computer learning media. The development of the APOS theory was based on Dubinsky in the construction process of Piaget's reflective abstraction [5].

The mathematics idea of an individual moving mentally from Action to Process, and from Process to Object It was discussed by Piaget's regarding the development of cognitive functions. Piaget described functions as maps (applications) which are initially by actions and become processes (that can be measured effectively) and then become objects (allowing variation to apply). Dubinsky interprets this activity as a description of cognitive development that begins with Actions that are inserted into the Process and then encapsulated into Objects where new actions can be applied. This is how Piaget's reflective abstraction becomes the basis for APOS theory development [5].

From the standpoint of APOS, there are mental structures and mental mechanisms. Mental structures built by mental mechanisms. Mental structures consist of Action, Process, Object, and Scheme. Action is interiorized into mental Processes, two mental Processes can be coordinated to form a new Process. A Process can be encapsulated to form the Object mentality, and a scheme can be thematized to become an Object [5]. Every mental mechanism that happened is a reflective abstraction. Harihaan said that there are six kinds of reflective abstraction: interiorization, coordination, reversal, encapsulation, thematization, and generalization [16].

Action is the beginning of how mathematical concepts are understood for the first time, and it consists of step by step instructions that should be done explicitly to transform physical or mental objects. When an individual repeat and reflects on his actions, the object may be internalized into the mental process. A Process is a mental structure that performs the same operation as an action but occurs entirely in the mind of an individual. When individuals become aware of the whole process, realize that transformation can be acted on the process, and/or build such a transformation, the Process is summarized.
(encapsulated) into a mental object. To develop a conceptual understanding, an individual can build many actions, processes, and objects. As these structures are organized and linked into a coherent framework, the individual has formed a scheme for a topic [6].

To explain the thinking process through the mental mechanism, APOS provides pedagogy to apply this theory, which is the stages of ACE: Activity, Classroom Discussion, and Exercise.

3. Methods
This research was a qualitative descriptive study. The subjects were 32 people of the first-year students of Mathematics Education in the academic year 2017/2018. It was conducted at UIN Walisongo Semarang, Central Java, during the first semester of 2018. Data were collected by documentation, observation, test and depth interviews and analyzed by reduction, display, and verification, referred to Miles and Huberman [17]. To implement the ACE approach on teaching learning design, students were classified into seven groups.

4. Results
4.1. Genetic Decomposition
Riemann Sum is one of the basic concepts needed to understand the concept of integral mathematically. The APOS implementation in Riemann Sum starts with constructing the genetic decomposition of the Riemann Sum, by considering Arnon [5], educator experiences, and content characteristics. The genetic decomposition is figured out in Table 1.

| Mental Structure | Mental Mechanism |
|------------------|------------------|
| Action           | 1. Constructing the definite integrals estimation as the area under the curve of a plane  
|                  | 2. Finding the area under the curve that divided into several partitions with the inner approach  
|                  | 3. Repetition of 2nd Action with an outer rectangular approach to eliminate dependence on external cues. |
| Process          | 1. Interiorizing 2nd Action at n-partition  
|                  | 2. Interiorizing Action-3 on the n-partition |
| Object           | 1. Encapsulating the 1st Process and 2nd Process into objects to determine the existence of limit and limit values of Riemann sums, and  
|                  | 2. Encapsulating the 1st, 2nd and 3rd Process into objects |
| Schema           | Thematization by applying the concept of Riemann sum to the function of distance to time and other mathematical situations |

The genetic decomposition as mention in Table 1 guides the educator to develop the learning pedagogy needed.

4.2. Pedagogy design: APOS-based instruction
In this study, the learning design used the ACE approach, as suggested by Dubinsky [5]. Before the main learning is carried out, the teacher identifies students' abilities on the prerequisite material, i.e. the elements and the area of plain. After making sure that all students have mastered the prerequisite material, the activity was carried out in small groups using worksheets as a learning resource. The worksheets were designed referring to the genetic decomposition, as the purpose of the instructions in the worksheet is to encourage the appearance of reflective abstraction, and not to get the correct answer [18]. In this study, there are several worksheets. Worksheet-1 was designed to construct the mental mechanisms of Action-1, Action-2, and Action-3. Worksheet-2 was designed to construct the mental
mechanisms of Process-1 and Object-1, while Worksheet-3 was designed to construct the mental mechanisms of Process-2 and Object-1.

During the Activity, the educator assisted to individuals and groups. This assistance has a big role in identifying whether students go through the expected stages and provide guidance when students had difficulties. From this assistance, it was known that the success of completing the worksheets as a representation of mental mechanisms, was not influenced by student ratings. Students in high groups did not always pass through this mental mechanism well, even in the first action. Conversely, some students come from the lower rating could understand and did the mechanism well.

The step after activity was Classroom Discussion, where students had an opportunity to make a reflection for what they have done and understand those activities. Construction of Object-2 was also carried out in this part, comes through guided questions. In the Exercise, as the last part, students got tasks as their homework.

As a conclusion, there are 3 findings from this research:

a The successful implementation of APOS theory to emerge reflective abstraction is influenced by genetic decomposition, worksheets or learning resources, assistance, and reinforcement by educator as well as strengthening the prerequisite material.

b Students take action by constructing definite integral approximation as an area under the curve with a various region of the plane, finding the area under the curve by divided into several partitions mathematically, with a rectangular approach; and repeating the previous action with a rectangular outer (and middle rectangular approach to eliminate dependency on teacher's side.

c The reflective abstraction process in the material Riemann Sum occurs through the mechanism of interiorization, coordination, encapsulation, and thematization. Interiorization occurs when students did actions that were originally carried out on finite rectangles into \( n \) rectangles. The process is also constructed when students successfully coordinate action with the midpoint approximation. While the object structure is only formed through the encapsulation mechanism and the scheme is formed only through thematization.

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