Thermodynamics of black holes in Einstein-Gauss-Bonnet AdS gravity coupled to Nonlinear Electrodynamics

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Abstract. In an arbitrary dimension $D$, we study quadratic corrections to Einstein-Hilbert action described by the Gauss-Bonnet term. We consider charged black hole solutions with anti-de Sitter (AdS) asymptotics, of interest in the context of gravity/gauge theory dualities (AdS/CFT). The electric charge here is due to the addition of an arbitrary nonlinear electrodynamics (NED) Lagrangian. Due to the existence of a vacuum energy for global AdS spacetime in odd dimensions in the framework of AdS/CFT correspondence, we derive a Quantum Statistical Relation directly from the Euclidean action and not from the First Law of thermodynamics. To this end, we employ a background-independent regularization scheme which consists in supplementing the bulk action with counterterms that depend both on the extrinsic and intrinsic curvatures of the boundary (also known as Counterterms). This procedure results in a consistent inclusion of the vacuum energy in the thermodynamic description for Einstein-Gauss-Bonnet AdS gravity regardless the explicit form of the NED Lagrangian.

1. Introduction
Coupling nonlinear electrodynamics (NED) to gravity is a plausible mechanism to obtain regular black holes. In this respect, static topological black holes in the Born-Infeld (BI) theory [1] coupled to Einstein gravity were derived in Refs.[2, 3]. Other gravitating NED models have been also investigated, e.g., in Ref.[4] for Euler-Heisenberg effective QED, in Ref.[5] for logarithmic ED, and in Ref.[6] for a Lagrangian defined as powers of the Maxwell term.

Within the framework of AdS/CFT correspondence, higher-derivative corrections to either gravitational or electromagnetic action in AdS space modify the dynamics of the strongly coupled dual theory. In particular, in hydrodynamic models, the addition of $R^2$ terms changes the ratio of shear viscosity over entropy density [7], violating the universal bound $1/4\pi$ proposed in Ref.[8]. In turn, it has been proved that higher-derivative terms for Abelian fields in the form of NED do not affect this ratio [9]. Also, in models dual to high $T_c$ superconductivity, higher-curvature terms violate a universal relation between the critical temperature of the superconductor and its...
energy gap [10, 11]. While the Gauss-Bonnet term makes the condensation easier, the inclusion of BI electrodynamics produces the opposite effect [12].

Motivated by the above results in the context of AdS/CFT, we study black hole solutions in Einstein-Gauss-Bonnet gravity with negative cosmological constant coupled to an arbitrary NED theory.

2. Action and equations of motion

We consider gravity minimally coupled to nonlinear electrodynamics in a $D$-dimensional manifold $M$ ($D > 4$), which comes from the action

$$ I_0 = \frac{1}{16\pi G} \int_M d^Dx \sqrt{-g} \left[ R - 2\Lambda + \alpha \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \right) \right] + \int_M d^Dx \sqrt{-g} \mathcal{L}(F^2). \tag{1} $$

The pure gravity part of the action contains the Einstein-Hilbert (EH) action with negative cosmological constant $\Lambda = -(D-1)(D-2)/2\ell^2$ ($\ell$ is the AdS radius) and the Gauss-Bonnet (GB) term. The GB coupling $\alpha$ is a positive constant. NED is described by an arbitrary Lagrangian density $\mathcal{L}(F^2)$ in the quadratic invariant $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}(x)$ is the Abelian field strength associated to the gauge connection $A_{\mu}(x)$ as $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

The equations of motion are then obtained as $\delta I_0/\delta g_{\mu\nu} = 0$ and $\delta I_0/\delta A_{\mu} = 0$, that is,

$$ \mathcal{E}_\mu^\nu \equiv G_\mu^\nu + H_\mu^\nu - 8\pi G T_\mu^\nu = 0, \tag{2} $$

$$ \mathcal{E}_\mu^\nu \equiv \nabla_\nu \left( F^{\mu\nu} \frac{dL}{dF^2} \right) = 0, \tag{3} $$

where $G_\mu^\nu$ is the Einstein tensor with negative cosmological constant and

$$ H_\mu^\nu = -\frac{\alpha}{2} \delta_\mu^\nu \left( R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\lambda\sigma}R_{\alpha\beta\lambda\sigma} \right) + 2\alpha \left( RR_{\mu\nu} - 2R^{\mu\lambda}R_{\lambda\nu} - 2R_{\lambda\sigma}R^{\mu\lambda\sigma} + R^{\mu\alpha\lambda\sigma}R_{\alpha\lambda\sigma} \right) \tag{4} $$

is the Lanczos tensor. The energy-momentum tensor for the matter content is $T_\mu^\nu = \delta_\mu^\nu \mathcal{L} - 4\frac{dL}{dF_\mu} F^{\mu\lambda}F_{\nu\lambda}$.

The GB contribution $H_\mu^\nu$ modifies the asymptotic behavior of the solutions: maximally symmetric spacetimes possess an effective AdS radius $\ell_{\text{eff}}$

$$ R_{\mu\nu}^{\alpha\beta} = -\frac{1}{\ell_{\text{eff}}^2} \delta_{[\mu\nu]}^{[\alpha\beta]}, \tag{5} $$

where $\ell_{\text{eff}}^2$ is given by $\ell_{\text{eff}}^2 = \frac{2\alpha(D-3)(D-4)}{1+\sqrt{1-\frac{2\alpha}{\ell^2}(D-3)(D-4)}}$. For the discussion of the present paper, we shall consider solutions that satisfy Eq. (5) in the asymptotic region, i.e., that tend asymptotically to a constant-curvature spacetime.

3. Generic topological static black hole solution

A static black hole ansatz in the coordinate set $x^m = (t, r, \varphi^n)$ is given by

$$ ds^2 = -f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 \gamma_{mn}(\varphi) d\varphi^m d\varphi^n. \tag{6} $$

The boundary $\partial M$ is located at radial infinity ($r \to \infty$), and it is parameterized by $x^i = (t, \varphi^n)$. The metric $\gamma_{mn}$ with local coordinates $\varphi^n$ describes a $(D-2)$-dimensional Riemann space $\Gamma_{D-2}$.
with constant curvature $k$, where $k = 0, +1$ or $-1$. Solutions possess an event horizon, $r_+$, defined as the largest root of the equation $f(r_+)=0$.

For a static solution, the gauge field is $A_\mu = \phi(r) \delta^\mu_i$, where $\phi(r)$ is an electric potential, what gives the field-strength

$$F_{\mu \nu} = E(r) \left( \delta^\mu_i \delta^\nu_j - \delta^\nu_i \delta^\mu_j \right),$$

(7)

where the electric field is $E(r) = -\phi'(r)$ and the prime denotes radial derivative.

We solve the electric potential in the static ansatz (6) and (7), where $F^2 = -2E^2$, using the only non-vanishing component of the Maxwell-type equation (3). The result is the generalized Gauss’ law

$$E \frac{d\mathcal{L}}{dF^2} \bigg|_{F^2=-2E^2} = -\frac{q}{r^{D-2}}.$$  

(8)

Here, $q$ is an integration constant related to the electric charge.

On the other hand, integrating the electric field, one obtains the electric potential at the distance $r$ measured with respect to radial infinity as $\phi(r) = -\int dv E(v)$. The quantity of physical interest is the electric potential at infinity measured with respect to the event horizon $r_+$ as $\Phi = \phi(\infty) - \phi(r_+)$. The equations of motion $\mathcal{E}_i^\mu = \mathcal{E}_\nu^\nu = 0$ read

$$\frac{16\pi G r^2}{D-2} T_r^r = r \left( f^2 \right)' + (D-3) \left( f^2 - k \right) - (D-1) \frac{r^2}{\ell^2} +$$

$$+ 2\alpha \left( D-3 \right) \left( D-4 \right) \frac{k-f^2}{r} \left[ \left( f^2 \right)' - (D-5) k \frac{f-f^2}{2r} \right].$$  

(9)

Integrating this equation, the metric function is expressed in terms of a function $T(q,r) = \int dv v^{D-2} T_r^r(v)$ and an integration constant $\mu$, and it has two branches,

$$f^2_\pm (r) = k + \frac{2\alpha}{2\alpha(D-3)(D-4)} \left[ 1 \pm \sqrt{1 - 4\alpha \left( D-3 \right) \left( D-4 \right) \left( \frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} + \frac{16\pi G T(q,r)}{(D-2) r^{D-1}} \right)} \right].$$  

(10)

It has been proved in [13] that the branch $f^2_\pm (r)$ is unstable and the corresponding graviton has negative mass. Because of this, henceforth, we consider only the negative branch of the metric, $f(r) \equiv f_-(r)$.

The procedure outlined above can be seen as an algorithm to construct explicit solutions to various NED theories [14]: conformally invariant electrodynamics [6], Born-Infeld electrodynamics [1, 15], logarithmic electrodynamics [4, 5], etc.

4. Surface terms and Counterterm regularization
The space-time manifold possesses a single boundary $\partial\mathcal{M}$ at $r = \infty$ parameterized by the coordinates $x^i$, such that $h_{ij}$ is the induced metric on it,

$$ds^2 = N^2(r) \, dr^2 + h_{ij}(r,x) \, dx^i dx^j.$$  

(11)

In the radial foliation (11), the surface term that comes from the variation of the bulk action, $\delta I_0 = \int_{\partial\mathcal{M}} d^{D-1}x \, \Theta_0$, can be expressed in terms of the extrinsic curvature $K_{ij} = -\frac{1}{2N} h_{ij}$ as

$$\Theta_0 = \frac{1}{8\pi G (D-2)(D-3)} \sqrt{-h} \, \delta_{[ij]}^{jjj} \left[ \frac{1}{2} \left( h^{-1} \delta h \right)^i_k \hat{K}^k_j + \delta K^i_j \right] \times$$

$$\times \left[ \delta^i_{j1} \delta^j_{l2} + 2\alpha \left( D-2 \right) \left( D-3 \right) \left( \frac{1}{2} \hat{R}^i_{j1j2} - \hat{R}^i_{j1j2} \right) \right] + 4\sqrt{-h} \frac{d\mathcal{L}}{dF^2} N F^i \delta_A,$$
where $\sqrt{-g} = N \sqrt{-h}$ and $R^{ij}_{kl}(h)$ is the boundary curvature related to the spacetime Riemann tensor by $R^{ij}_{kl} = R^{ij}_{kl} - K^i_k K^j_l + K^i_l K^j_k$. Here, the tensor $\delta^{[i_1\ldots j_p]}_{[i_1\ldots i_p]}$ denotes the totally antisymmetric product of $p$ Kronecker deltas.

For the present discussion, we consider the counterterms which depend on the extrinsic curvature (it is known as Kounterterm regularization), such that the total action is

$$I = I_0 + c_{D-1} \int_{\partial M} d^{D-1}x B_{D-1}, \tag{12}$$

where $c_{D-1}$ is a given constant. For EH AdS gravity, the Kounterterm series was shown in Refs. [16, 17] as a polynomial of the extrinsic and intrinsic curvatures. In general, Kounterterm series is such that the total variation of the action (12),

$$\delta I = \int_{\partial M} d^{D-1}x \Theta = \int_{\partial M} d^{D-1}x (\Theta_0 + c_{D-1} \delta B_{D-1}), \tag{13}$$

has an extremum on-shell and, at the same time, does not contain divergent terms when $r \to \infty$. In order to cancel the NED part of the surface term, it is a sufficient condition to take $\delta A_i = 0$ at $\partial M$.

For a given dimension, the series $B_{D-1}$ possesses the remarkable property of preserving its form for EGB-AdS gravity [18] and, in general, any theory of the Lovelock type [19].

**Even dimensions ($D = 2n$).** In even dimensions $D = 2n > 4$, the boundary term $B_{2n-1}$ in (12) is given by the $n$-th Chern form [18],

$$B_{2n-1} = 2n \sqrt{-h} \int_0^1 dt \delta^{[j_1\ldots j_{2n-1}]} K_{j_1} (1/2 R_{j_2 j_3} - t^2 K_{j_2} K_{j_3}) \cdots (1/2 R_{j_2n-2 j_{2n-1}} - t^2 K_{j_{2n-2}} K_{j_{2n-1}}), \tag{14}$$

that is the scalar density whose derivative is locally equivalent to the Euler invariant. The integration on the continuous parameter $t$ generates the coefficients when Eq. (14) is expanded as a polynomial. The constant $c_{2n-1}$ in front of the boundary term $B_{2n-1}$ is given in terms of the effective AdS radius as

$$c_{2n-1} = -\frac{1}{16\pi G n (2n-2)!} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (2n-2) (2n-3) \right). \tag{15}$$

**Odd dimensions ($D = 2n + 1$).** The extrinsic regularization developed for EH AdS gravity in $D = 2n + 1$ dimensions [16] can be mimicked for EGB AdS theory, just replacing the AdS radius $\ell$ by the effective one $\ell_{\text{eff}}$ in the boundary terms. Kounterterm regularization provides the explicit form of the boundary term in a compact form thanks to two parametric integrations [18]

$$B_{2n} = 2n \sqrt{-h} \int_0^1 dt \int_0^t ds \delta^{[j_1\ldots j_{2n}]} K_{j_1} (1/2 R_{j_3 j_4} - t^2 K_{j_3} K_{j_4} + s^2 \ell_{\text{eff}}^2 \delta_{j_3 j_4}) \times \cdots$$

$$\times \left( 1/2 R_{j_{2n-1} j_{2n}} - t^2 K_{j_{2n-1}} K_{j_{2n}} + s^2 \ell_{\text{eff}}^2 \delta_{j_{2n-1} j_{2n}} \right). \tag{16}$$
The corresponding constant depends on the GB coupling as

\[ c_{2n} = -\frac{1}{16\pi G} \frac{2(-\ell_{eff}^2)^{n-1}}{n(2n-1)!} \frac{1}{\beta(n, \frac{1}{2})} \left( 1 - \frac{2\alpha}{\ell_{eff}^2} \right)^{2n} \left( 2n - 1 \right) \left( 2n - 2 \right), \]  

where \( \beta(n, \frac{1}{2}) = \frac{2^{2n-1}(n-1)!}{(2n-1)!} \) is the Beta function for those arguments.

5. Conserved quantities

**Electric charge.** For a \( U(1) \) gauge transformation \( \delta \lambda A_\mu = \partial_\mu \lambda, \delta \lambda g_{\mu\nu} = 0 \), the gravitational part of the surface term in Eq.(13) is gauge-invariant, that leads to the conservation law of the Noether current in the form

\[ \delta \lambda I = \int_M d^3 x \partial_\mu J^\mu(\lambda) = 4 \int_M d^3 x \partial_\mu \left( \sqrt{-g} F^{\mu\nu} \frac{dL}{dF^2} \partial_\nu \lambda \right). \]  

For a radial foliation (11) generated by a normal vector \( n_\mu \), the electric charge is \( Q[\lambda] = \int_{\partial M} d^{D-1} x \frac{1}{\sqrt{g}} n_\mu J^\mu(\lambda) \). One can prove that the integrand is a total derivative and use the Stokes’ theorem with a timelike ADM foliation for the line element on \( \partial M \) with the coordinates \( x^i = (t, y^m) \), as

\[ h_{ij} dx^i dx^j = -\tilde{N}^2 dt^2 + \sigma_{mn}(dy^m + \tilde{N}^m dt)(dy^n + \tilde{N}^n dt), \]  

that is generated by the timelike normal vector \( u_i = (u_t, u_m) = (-\tilde{N}, \tilde{0}) \) with \( \sqrt{-k} = \tilde{N} \sqrt{\tilde{\sigma}} \). The metric \( \sigma_{mn} \) describes the geometry of the boundary of spatial section at constant time \( \Sigma_\infty \).

Setting \( \lambda = 1 \), the \( U(1) \) charge for the static black hole metric (6) and the electromagnetic field strength (7) reads

\[ Q = -4\text{Vol}(\Gamma_{D-2}) \lim_{r \to \infty} \left( r^{D-2} E \frac{dL}{dF^2} \right). \]  

Using the generalized Gauss law (8), we get a finite electric charge for an arbitrary NED Lagrangian

\[ Q = 4\text{Vol}(\Gamma_{D-2}) q. \]  

**Black hole mass and vacuum energy.** The Noether current derived from the diffeormorphic invariance, \( \delta \xi I = \int_M d^3 x \partial_\mu J^\mu(\xi) = 0 \) is

\[ J^\mu(\xi) = \Theta^\mu(\xi) + \sqrt{-g} \xi^\mu \mathcal{L}_0 + c_{D-1} \tilde{N} n^\mu \partial_t \left( \tilde{\xi}^i B_{D-1} \right). \]  

The conservation law \( \partial_\mu J^\mu = 0 \) implies the existence of a conserved quantity, which corresponds to the normal component of the current \( J^\mu \). For the action \( I \), the radial component \( J^r = \frac{1}{\tilde{N}} n_\mu J^\mu \) in the foliation (11) is globally a total derivative on \( \partial M \). Then, the conserved charge \( Q[\xi] \) for an asymptotic Killing vector \( \xi \) is expressed as an integral on \( \Sigma_\infty \),

\[ Q[\xi] = \int_{\Sigma_\infty} d^{D-2} y \sqrt{\tilde{\sigma}} u_j \xi^i \left( q^j_i + q^j_{(0)i} \right). \]  

The above splitting to two terms is because \( q^j_i \) produces the mass for black hole solutions and identically vanishes for the vacua of the theory. In turn, the term \( q^j_{(0)i} \) gives rise to a vacuum energy, which is present only in odd dimensions.
**Even dimensions.** In even dimensions, the conserved charge (23) is

\[ q_i^j = \frac{1}{16\pi G (2n-2)!2^{n-2}} \delta^{[j_2\ldots j_{2n-1}]}_{[i_1\ldots i_{2n-1}]} R_i^k \left[ \left( \delta^{[i_1i_4]}_{[j_3j_4]} + 2\alpha (2n-1) (2n-2) R^{i_1i_4}_{j_3j_4} \right) \delta^{[i_5i_6]}_{[j_5j_6]} \ldots \delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} \right] \\
- \left( -r_{\text{eff}}^2 \right)^{n-1} \left( 1 - \frac{2\alpha}{r_{\text{eff}}^2} (2n-2) (2n-3) \right) R_i^{i_1i_3} \ldots R_i^{i_{2n-2}i_{2n-1}} \right], \tag{24} \]

where a NED contribution that vanishes for black holes. For even dimensions \( q_{(0)i}^j = 0 \).

The energy of black hole solution to EGB AdS gravity coupled to NED (6) is computed evaluating the formula (23) for the Killing vector \( q_i \) since \( q \neq 0 \) for maximally symmetric spacetime will have vanishing mass and angular momentum due to the fact that \( q_i = 0 \), such that the vacuum energy will come necessarily from Eq. (28).

We compute the black hole mass from the formula (23),

\[ M = \frac{\text{Vol}(\Gamma_{2n-2})}{16\pi G} \lim_{r \to \infty} r^{2n-2} (f^2)^{r^2 - k} \left[ 1 - 2\alpha (2n-2) (2n-3) \left( \frac{f^2 - k}{r^2} \right)^{n-1} \right]. \tag{25} \]

Taking the asymptotic expansion of the metric function, it is possible to relate \( M \) to the integration constant \( \mu \) as

\[ M = (2n-2) \frac{\text{Vol}(\Gamma_{2n-2}) \mu}{16\pi G}. \tag{26} \]

**Odd dimensions.** In odd dimensions, the Noether charge appears as the sum of two parts, since \( q_{(0)i}^j \) in Eq. (23) is no longer vanishing. The first part takes the form

\[ q_i^j = \frac{1}{16\pi G (2n-1)!2^{n-2}} \delta^{[j_2\ldots j_{2n}]}_{[i_1\ldots i_{2n}]} R_i^k \left[ \left( \delta^{[i_1i_4]}_{[j_3j_4]} + 2\alpha (2n-1) (2n-2) R^{i_1i_4}_{j_3j_4} \right) \delta^{[i_5i_6]}_{[j_5j_6]} \ldots \delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} \right] \\
+ 16\pi G (2n-1)! nc_{2n} \int_0^1 dt \left( R^{i_1i_4}_{j_3j_4} + \frac{t^2}{r_{\text{eff}}^2} \delta^{[i_5i_6]}_{[j_5j_6]} \ldots \delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} \right) \ldots \right], \tag{27} \]

whereas the second one is given by

\[ q_i^{(0)i} = nc_{2n} \int_0^1 dt \delta^{[j_2\ldots j_{2n}]}_{[i_1i_2\ldots i_{2n}]} \left( K^k \delta_j^2 + K^k \delta_j^2 \right) \left( \frac{1}{2} R^{i_1i_4}_{j_3j_4} - t^2 K^j_{i_3} K^{i_4}_{j_4} + \frac{t^2}{r_{\text{eff}}^2} \delta^{i_5}_{j_5} \delta^{i_6}_{j_6} \right) \ldots \right] \times \\
\ldots \times \left( \frac{1}{2} R^{i_1i_4}_{j_3j_4} - t^2 K^j_{i_3} K^{i_4}_{j_4} + \frac{t^2}{r_{\text{eff}}^2} \delta^{i_5}_{j_5} \delta^{i_6}_{j_6} \delta^{i_7}_{j_7} \delta^{i_8}_{j_8} \right). \tag{28} \]

Any maximally symmetric spacetime will have vanishing mass and angular momentum due to the fact that \( q_i^j = 0 \), such that the vacuum energy will come necessarily from Eq. (28).

We compute the black hole mass from the formula (23),

\[ M = \frac{\text{Vol}(\Gamma_{2n-1})}{16\pi G} \lim_{r \to \infty} r^{2n-1} (f^2)^{r^2 - k} \left[ 1 - 2\alpha (2n-1) (2n-2) \left( \frac{f^2 - k}{r^2} \right)^{n-1} \right] + 16\pi G (2n-1)! nc_{2n} \int_0^1 dt \left( \frac{k - f^2}{r^2} + \frac{t^2}{r_{\text{eff}}^2} \right)^{n-1}. \tag{29} \]
The boundary term $c_2 n B_2^2$ plays a double role: it cancels out the divergences in the Noether charge, but also contributes with a finite piece to give the correct result for the mass

$$M = \frac{(2n - 1) \text{Vol}(\Gamma_{2n-1})}{16\pi G} \mu. \quad (30)$$

In turn, the vacuum energy for AAdS black holes, $E_{\text{vac}} = \int_{\Sigma_\infty} d^{D-2} y \sqrt{\sigma} u_t \xi^t q_t^{(0) t}$, takes the form

$$E_{\text{vac}} = 2n(2n-1)!c_2 n \text{Vol}(\Gamma_{2n-1}) \lim \int_0^1 dt \left( f^2 - r \frac{(f^2)'}{2} \right) \left[ k + \left( \frac{r^2}{f_{\text{eff}}'} - f^2 \right) t^2 \right]^{n-1}. \quad (31)$$

When the metric is expanded in the limit $r \to \infty$, we notice that $E_{\text{vac}}$ depends only on the topological parameter $k$, the effective AdS radius and GB coupling, what matches the vacuum energy in EGB gravity obtained in Ref.[18]. That means that, for an arbitrary NED Lagrangian, the fall-off of the electromagnetic field is always such that it does not contribute to the total energy.

It is well-known that a vacuum energy for global AdS spacetime in odd dimensions appears only in background-independent methods. This is important from the semiclassical point of view in order to interpret the Noether charges as thermodynamic variables, and to consistently incorporate the vacuum energy in the definition of internal energy of the system [20], in a similar fashion as in Einstein-BI system [21].

6. Black hole thermodynamics

In Lovelock gravity theory, the entropy $S$ of a charged black hole is not longer a quarter of the horizon’s area, but it still obeys the First Law of thermodynamics

$$\beta dU = dS + \beta \Phi dQ, \quad (32)$$

where $U$ is the internal energy, $\beta$ is the inverse of the Hawking temperature, $\Phi$ is the gauge potential measured at infinity and $Q$ is the electric charge. In general, one identifies the thermodynamic variables in Eq.(32) as the conserved charges of the theory. However, the thermal properties of black holes can only be understood in the semiclassical approximation, where the partition function is given in terms of the classical Euclidean action as $Z = \exp(-I_E)$.

For spacetimes with AdS asymptotics, $I_E$ needs to be regulated to cancel the infrared divergences, such that the thermodynamic information is encoded in the finite part the Euclidean action $I_E$, which satisfies the Quantum Statistical Relation (QSR)

$$I_E = \beta U - \beta \Phi Q - S. \quad (33)$$

Background-subtraction methods for asymptotically AdS spacetimes are useful to extract a finite value from $I_E$, but cannot give rise to a vacuum energy $E_{\text{vac}}$ for global AdS spacetime in odd dimensions. In the context of the AdS/CFT correspondence, the matching of $E_{\text{vac}}$ with the Casimir energy of a boundary CFT is essential to find explicit examples of this gravity/gauge theory duality.

Clearly, the first law (32) is insensitive to the existence of a vacuum energy for asymptotically AdS spacetimes. We prove here for Einstein-Gauss-Bonnet gravity coupled to NED that the QSR is still valid for an internal energy that is shifted as $U = M + E_{\text{vac}}$, with respect to the Hamiltonian mass $M$, what is carried out employing Kounterterm regularization [18].

In the grand canonical ensemble, the variables are the temperature $T$ and the electric potential $\Phi$. The Gibbs free energy $G(T, \Phi) = U - TS - Q\Phi$, that satisfies the differential equation $dG = -SdT - Qd\Phi$, is related to the Euclidean action as $G = I_E/\beta$. 

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In the Euclidean version of black hole metric (6), the Euclidean time \( \tau = it \) appears as identified in a period \( \beta \), which is the inverse of the Hawking temperature \( T \equiv \beta^{-1} = \frac{1}{4\pi} \frac{d^2(r)}{dT} \bigg|_{r=r_+} \). The Wick rotation implies \( I^E = -iI \) for the Euclidean action what, for the black hole ansatz, produces

\[
I_0^E = \frac{\beta \text{Vol}(\Gamma_{D-2})}{16\pi G} \left[ (f^2)' \left( r^{D-2} + 2\alpha (D - 2) (D - 3) (k - f^2) \right) \right]_{r_+}^{\infty} + \\
+ \beta \text{Vol}(\Gamma_{D-2}) \left( 4r^{D-2} \phi E \frac{dL}{dF^2} \right)_{r_+}^{\infty}. \tag{34}
\]

Then, the first line at the horizon \( r_+ \) produces \( -S \), where \( S \) is the standard value of black hole entropy in EGB gravity

\[
S = \frac{\text{Vol}(\Gamma_{D-2})}{4G} r_+^{D-2} \left( 1 + \frac{2k\alpha}{r_+^2} (D - 2) (D - 3) \right), \tag{35}
\]

whereas the second line is \(-\beta Q \Phi\).

The Euclidean action is divergent, and it is clear that any background-subtraction method will not produce a correct Quantum Statistical Relation. Indeed, subtracting the value of \( I_0^E \) evaluated for AdS vacuum (with AdS radius \( \ell_{\text{eff}} \)) gets rid of the divergences at \( r = \infty \) but does not reproduce the mass at the asymptotic region.

**Even dimensions.** The Euclidean boundary term (14) evaluated on the black hole solution plus the bulk term given by Eq. (34), produces the total Euclidean action \( I_{2n}^E = I_0^E + e_{2n-1} \int_{\partial M} d^{2n-1}x B_{2n-1}^E \), or explicitly,

\[
I_{2n}^E = \frac{\beta \text{Vol}(\Gamma_{2n-2})}{16\pi G} \left\{ (f^2)' \left( r^{D-2} + 2\alpha (2n - 2) (2n - 3) (k - f^2) \right) \right\}_{r_+}^{\infty} \\
- \ell_{\text{eff}}^{2n-2} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (2n - 2) (2n - 3) \right) \left[ (f^2)' (f^2 - k)^{n-1} \right]_{r_+}^{\infty} - \beta Q \Phi. \tag{36}
\]

The contribution at infinity from the bulk action combines with the boundary one to produce \( \beta \) times the Noether mass in Eq.(25). In other words, Kounterterm series produces the cancellation of divergences in the asymptotic charges, and the Euclidean action satisfies the QSR

\[
I_{2n}^E = \beta M - \beta Q \Phi - S. \tag{37}
\]

**Odd dimensions.** The Euclidean boundary term is obtained plugging in the static metric ansatz (6) into Eq.(16), what added to the bulk Euclidean action (34) with a suitable coupling constant, \( I_{2n+1}^E = I_0^E + e_{2n} \int_{\partial M} d^{2n}x B_{2n+1}^E \), gives

\[
I_{2n+1}^E = \frac{\beta \text{Vol}(\Gamma_{2n-1})}{16\pi G} \left[ r^{2n-1}(f^2)' \left( 1 - 2\alpha (D - 2) (D - 3) \frac{f^2 - k}{r^2} \right) \right]_{r_+}^{\infty} + \\
+ \beta \text{Vol}(\Gamma_{2n-1}) nc_{2n} (2n - 1)! \left[ r^{2n-1}(f^2)' \int_0^1 \frac{dt}{t^2} \left( \frac{k - f^2}{r^2} + \frac{t^2}{\ell_{\text{eff}}^2} \right)^{n-1} \right]_{r_+}^{\infty} - \beta Q \Phi. \tag{38}
\]
Thus, the contribution coming from radial infinity in first two lines is $\beta M$, where the mass is given by Eq.(29) and the term with parametric integration in the third line is $\beta$ times the vacuum energy (31).

The consistency of the black hole thermodynamics implies the Quantum Statistical Relation

$$I_{2n+1}^E = \beta (M + E_{\text{vac}}) - \beta Q \Phi - S,$$

for a total internal energy $U = M + E_{\text{vac}}$.

7. Conclusions

The interplay between thermodynamic quantities and the conserved charges at infinity can be understood from the evaluation of the Euclidean action, as shown in the present work. In that sense, our approach is more general than simply computing $S$ from local properties of the horizon, e.g., using Wald’s formalism, because it also includes the asymptotic charges.

In this spirit, we have obtained the QSR (33) for black holes in EGB AdS gravity coupled to NED in all dimensions using Kounterterms regularization of the action. The method consistently incorporates the vacuum (Casimir) energy $E_{\text{vac}}$ for AdS spacetime, in the thermodynamic description of the system.

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