ON THE CAUCHY PROBLEM OF 3D INCOMPRESSIBLE NAVIER-STOKES-CAHN-HILLIARD SYSTEM

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Abstract. In this paper, we are concerned with the Cauchy problem for 3D incompressible Navier-Stokes-Cahn-Hilliard equations. First, applying a refined pure energy method, assuming \( \|u_0\|_{H^1} + \|\phi_0\|_{H^1} + \|\nabla \phi_0\|_{H^1} \) is sufficiently small, we obtain the global well-posedness of solutions. Moreover, the optimal decay rates of the higher-order spatial derivatives of the solution are obtained, the \( H^{-s} \) \((0 \leq s \leq \frac{1}{2})\) negative Sobolev norms is shown to be preserved along time evolution and enhance the decay rates.

1. Introduction

In Fluid Mechanics, the incompressible Navier-Stokes equations models the motions of single-phase fluids such as air or water. Sometimes, we also need to understand the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface methods are widely used by many authors to describe the behavior of complex (e.g., binary) fluids [5, 19]. The Model H is a diffuse interface model for incompressible isothermal two-phase flows. This model include the incompressible Navier-Stokes equations for the (averaged) velocity \( u \) nonlinearly coupled with a convective Cahn-Hilliard equation for the (relative) concentration difference \( \phi \).

Suppose that the temperature variations are negligible, taking the density is equal to 1, and let the viscosity \( \nu \) to be constant, the Model H reduces to the incompressible Navier-Stokes-Cahn-Hilliard system [2, 14, 33]

\[
\begin{align*}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla \pi &= K \mu \nabla \phi, \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= \nabla \cdot (M \nabla \mu), \\
\mu &= -\varepsilon \Delta \phi + \zeta F'(\phi),
\end{align*}
\]

in \( \Omega \times (0, \infty) \) with \( \Omega \subseteq \mathbb{R}^3 \) is a bounded domain or the whole space. In equations (1), \( u \) denotes the mean velocity, \( \pi \) represent the pressure and \( \phi \) is an order parameter related to the concentration of the two fluids (e.g. the concentration difference or the concentration of one component), respectively. The quantities \( \mu \geq 0, M \geq 0 \) and \( K \geq 0 \) are positive constants that correspond to the kinematic viscosity of fluid, mobility constant and capillarity (stress) coefficient, respectively. Moreover, \( \mu \) denotes the chemical potential of the mixture which can be given by the variational derivative of the free energy functional

\[
F(\phi) = \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \zeta F'(\phi) \right) dx.
\]
Usually, we take \( F(\phi) = \int_0^\phi f(\xi) d\xi \) as a double-well potential

\[
f(s) = F'(s) = s(s^2 - 1), \quad F(s) = \frac{1}{4}(s^2 - 1)^2,
\]
or a singular free energy

\[
F(\varphi) = \frac{\theta}{2}[(1 + \varphi) \log(1 + \varphi) + (1 - \varphi) \log(1 - \varphi)] - \frac{\theta_c}{2} \varphi^2,
\]
where \( \varphi \in [-1, 1] \) and \( 0 < \theta < \theta_c \). We remark that the double-well structure free energy can be seen as the limiting form of singular free energy. In fact, if \( \frac{\theta}{\theta_c} \to 1 \), (4) reduces to (3).

**Remark 1.1.** Simple calculation shows that

\[
\mathcal{K}_m \nabla \phi = \mathcal{K} \nabla \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \xi F(\phi) \right) - \mathcal{K} \nabla \cdot (\nabla \phi \otimes \nabla \phi).
\]

Hence, (1) can be replaced by

\[
\frac{\partial}{\partial t} u + u \cdot \nabla u - \nu \Delta u + \nabla \tilde{\pi} = -\mathcal{K} \nabla \cdot (\nabla \phi \otimes \nabla \phi),
\]
with \( \tilde{\pi} = \pi - \mathcal{K} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \xi F(\phi) \right) \).

From the analytical viewpoint, incompressible Navier-Stokes-Cahn-Hilliard equations can be seen as the coupling between the incompressible Navier-Stokes equations with the convective Cahn-Hilliard equations. Moreover, the main purpose of this paper is to consider the global well-posedness and time decay rate of solutions for the Cauchy problem incompressible Navier-Stokes-Cahn-Hilliard equations. In the following, we are going to take some time on the researching actualities of Cauchy problems of incompressible Navier-Stokes equations and Cahn-Hilliard equation.

Consider the Cauchy problem of incompressible Navier-Stokes equations

\[
\begin{cases}
\frac{\partial}{\partial t} u - \nu \Delta u + u \cdot \nabla u + \nabla \pi = 0, \\
\text{div} u = 0, \\
u(x, 0) = u_0,
\end{cases}
\]

\( x \in \mathbb{R}^N, \quad N \in \mathbb{Z}^+ \).

There are a lot of classical results on system (5) due to the physical importance, complexity, rich phenomena and mathematical challenges. Especially, the results of the global well-posedness of the Cauchy problem for system (5) in various classical function spaces were obtained for small initial data. Fujita and Kato [27] obtained the global well-posedness for small initial data and the local well-posedness for any initial data in \( H^s(\mathbb{R}^n) \) with \( s \geq \frac{n}{2} - 1 \). Many other interesting and improved results have been established in \( L^p(\mathbb{R}^n) \) by Kato [35], in the Besov space by Cannone [13] and Planchon [46], and in the larger space BMO\(^{-1}\) by Koch and Tataru [37]. Moreover, whether or not the weak solutions to system (5) decay to zero in \( L^2 \) as the time tends to infinity was first posed by Leray in his pioneering paper [39, 40]. The first affirmative answer for strong solutions with small initial data was given by Kato [35]. Schonbek [48] first established the algebraic decay rates for weak solution to system (5). In [48], the author proved that there exists a Leray-Hopf weak solution for system (5) in 3D case with arbitrary data in \( L^1 \cap L^2 \) which satisfies

\[
\|u(x, t)\|_{L^2} \leq C(1 + t)^{-\frac{1}{4}}.
\]

Since then, the method in [48] (Fourier splitting method) was extended by many authors (see e.g., Schonbek [49], Kajikiya and Miyakawa [34], Wiegner [53] and the reference therein). It was
shown that the decay rate of the weak solutions of system (5) in $\mathbb{R}^3$ with large initial data in $L^p \cap L^2$ ($1 \leq p < 2$) is
\[
\|u(x,t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{2}\left(\frac{2}{p} - 1\right)},
\]
which is the same as the decay rate of the heat equation. Recently, together with the introduce of decay indicator $P_r$ and decay character $r^*$ (see [6, 8]), Niche and Schonbek [44] proved that the decay rate of weak solutions for system (5) in $\mathbb{R}^3$ with large initial data in $L^2$ have the following form:
\[
\|u(x,t)\|_{L^2} \leq C(1 + t)^{-\min\{\frac{3}{2}, r^*, \frac{5}{2}\}},
\]
which is also the same as the decay estimate of the heat equation (see [44]).

The Cahn-Hilliard equation
\[
\partial_t u = \Delta(-\Delta u + f(u)),
\]
was originally derived in [11] as a phenomenological equation describing phase transition problems in binary metallic alloys. A large amount of literature has been produced about the Cahn-Hilliard equation in a bounded domain, subject to suitable boundary conditions (see e.g., Elliott and Zheng [22], Schimperna and Pawlow [17], Cherfils, Miranville, Zelik [15], Yin [54], Liu and Wu [42] and the reference therein). However, for the Cauchy problem of Cahn-Hilliard equation, there are only a few papers studied. The early paper on the Cahn-Hilliard equation can be traced back to Caffarelli and Muler [10]. The authors assumed the derivative of potential is Lipschitz continuous and equal to a constant outside a bounded interval in $u$, $u_0(x) \in L^\infty(\mathbb{R}^N, \mathbb{R})$, $\varepsilon\nabla u_0(x) \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, etc., investigated the Cauchy problem of Cahn-Hilliard equation in general $N$ dimensions, obtained an $L^\infty(\mathbb{R}^N, \mathbb{R})$-a priori estimate, which is independent of $\varepsilon > 0$. In 2007, on the basis of Hoff and Smoller’s idea [31, 32] with a slight modification and Fourier splitting method, Liu, Wang and Zhao [43] also considered small data global well-posedness and time decay rate of solutions for the Cauchy problem of Cahn-Hilliard equation in $\mathbb{R}^N$. It is worth pointing out that the authors’ assumptions on the derivative of potential are
\[
f(u) \in C^L(\bar{B}(\bar{u}, 2r), \mathbb{R}), \quad f(u) = O(1)|u - \bar{u}|^l \quad \text{as} \quad u \to \bar{u},
\]
with some positive integer $l$ and $L = \max\{5, N\}$ and $\bar{B}(\bar{u}, 2r) = \{u \in \mathbb{R} : |u - \bar{u}| \leq 2r\}$. Moreover, Cholewa and Rodríguez-Bernal [17] exhibited the dissipative mechanism of the Cauchy problem of Cahn-Hilliard equation in $H^1(\mathbb{R}^N)$. In their paper, the nonlinear term $f = f(x, u)$ satisfies
\[
f(x, u) = g(x) + m(x) + \varphi_0(x, u), \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R},
\]
with
\[
f_0(x, 0) = 0, \quad \partial f_0 \partial u(x, 0) = 0, \quad x \in \mathbb{R}^N,
\]
\[
f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \text{ locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N,
\]
\[
m \in L^r_U(\mathbb{R}^N), \quad \max\left\{\frac{N}{2}, 1\right\} < r \leq \infty,
\]
and
\[
g \in L^p(\mathbb{R}^N) \text{ for some } 1 < p < \infty.
\]

The authors established some results on global existence and global attractor. There are also some papers related to the Cauchy problem of Cahn-Hilliard equation, please refer to [20, 56] and the reference therein. It is worth pointing out that the assumptions imposed on the nonlinear function $f(u) = \int_0^u F(s)ds$ in [10, 43, 17, 20, 56] are too strict. One of the most nature
assumption is \( f(u) = u^3 - u \), which is the derivative of a double-well potential (the other is logarithmic potential). However, we can’t ﬁnd any paper study the Cauchy problem of Cahn-Hilliard equation with double-well potential.

Now, we come back to the incompressible Navier-Stokes-Cahn-Hilliard equation, which is the coupling between the incompressible Navier-Stokes equations with the convective Cahn-Hilliard equations. This coupling of equations was highly studied from the theoretical and mathematical point of view. Till now, there is a large amount of literature on the mathematical analysis of initial-boundary value problem for Navier-Stokes-Cahn-Hilliard system in 2D or 3D case. Abels et. al. [1] [3] [4], Boyer [9], eleuteri, Rocca and Schimperna [21], Gal, Grasselli and Miranville [29], Lam and Wu [38] studied the existence of global weak solution and unique strong solution for Navier-Stokes-Cahn-Hilliard system with non-constant mobility; Gal and Grasselli [28] considered the asymptotic behavior of 2D Navier-Stokes-Navier-Stokes system, proved the existence of global attractor and exponential attractor; Colli, Frigeri and Grasselli [18], Frigeri, Grasselli and Krejci [23], Frigeri and Grasselli [25] and Frigeri, Grasselli and Rocca [26] replaced the chemical potential \((1)\) by the following nonlocal model

\[
\mu = \int_{\Omega} J(x - y)dy\phi - \int_{\Omega} J(x - y)\phi(y)dy + \xi F'(\phi),
\]

obtained the nonlocal Navier-Stokes-Cahn-Hilliard equations with double-well potential or singular free energy, studied the well-posedness and long time behavior of solutions; Bosia and Gatti [7], Cherfils and Madalina [16] and You, Li and Zhang [55] investigated the properties of solutions for Navier-Stokes-Cahn-Hilliard equations with dynamic boundary conditions, studied the well-posedness and long time behavior of solutions respectively; In [57], Zhou and Fan proved the vanishing viscosity limit of solutions for the initial boundary value problem of 2D Navier-Stokes-Cahn-Hilliard equations; Liu and Shen [41], Key and Welfold [36] and Feng [24] studied the numerical approximation of solutions, etc.

There are only few results on the mathematical analysis of the Cauchy problem for incompressible Navier-Stokes-Cahn-Hilliard system. The first related result was obtained by Starovoitov [50]. The author supposed that \( F \) is a suitably smooth double-well potential, studied the qualitative behavior as \( t \to \infty \) of the solutions for 2D Cauchy problem. The author described the \( \omega \)-limit set of the trajectories of the dynamical system generated by the problem, showed that the stationary solution of the problem, which is a homogeneous stationary distribution of one of the components, is asymptotically stable. The other paper on the Cauchy problem was written by Cao and Gal [12]. In their paper, the authors established the global regularity and uniqueness of strong/classical solutions for the Cauchy problem of 2D NS-CH equations with mixed partial viscosity and mobility. We remark that because of the diﬃculty caused by the convective term (the order of \( u \cdot \nabla \phi \) in \( (13) \) is low, one can’t control it by a fourth order linear term directly), the coupling between \( u \) and \( \phi \) (especially the term on the right hand side of \( (11) \)) and the chemical potential (when one study the Cauchy problem of NSCH system with double-well potential, it is so diﬃcult to deal with the linear term of derivative of double-well potential), there’s no paper on the Cauchy problem of three-dimensional incompressible Navier-Stokes-Cahn-Hilliard equations. Thus a natural question is how to study the properties of solutions for the Cauchy problem of 3D Navier-Stokes-Cahn-Hilliard equations. The main purpose of our present paper is devoted to the above problems. That is, we will consider the global existence and the time
Our main results are stated as follows:

\[
\begin{align*}
\frac{\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi}{\parallel \nabla \phi \otimes \nabla \phi\parallel} &= 0, \\
\partial_t \phi + u \cdot \nabla \phi + \Delta^2 \phi &= \Delta (u^3 - u), \\
(u, \phi)|_{x=0} &= (u_0(x), \phi_0(x)).
\end{align*}
\]  

(7)

(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+.

Remark 1.2. We note that the Laplacian operator \((-\Delta)^\delta\) (\(\delta \in \mathbb{R}\)) can be defined through the Fourier transform, namely

\[
(-\Delta)^\delta f(x) = \Lambda^{2\delta} f(x) = \int_{\mathbb{R}^3} |x|^{2\delta} \hat{f}(\xi)e^{2\pi ix \cdot \xi} d\xi,
\]

where \(\hat{f}\) is the Fourier transform of \(f\). Moreover, \(\nabla^l\) with an integral \(l \geq 0\) stands for the usual spatial derivatives of order \(l\). If \(l \leq 0\) or \(l\) is not a positive integer, \(\nabla^l\) stands for \(\Lambda^l\) defined by \([8]\). We also use \(\dot{H}^s(\mathbb{R}^3)\) (\(s \in \mathbb{R}\)) to denote the homogeneous Sobolev spaces on \(\mathbb{R}^3\) with the norm \(\parallel \cdot \parallel_{\dot{H}^s}\) defined by \(\parallel f \parallel_{\dot{H}^s} := \parallel \Lambda^s f \parallel_{L^2}\), and we use \(H^s(\mathbb{R}^3)\) and \(L^p(\mathbb{R}^3)\) \((1 \leq p \leq \infty)\) to describe the usual Sobolev spaces with the norm \(\parallel \cdot \parallel_{H^s}\) and the usual \(L^p\) space with the norm \(\parallel \cdot \parallel_{L^p}\). We will use the notation \(A \lesssim B\) to mean that \(A \leq cB\) for a universal constant \(c > 0\) that only depends on the parameters coming from the problem and the indexes \(N\) and \(s\) coming from the regularity on the data. We also employ \(C\) for positive constant depending additionally on the initial data.

The purpose of this paper is to consider the small data global well-posedness and temporary decay rate of solutions for the Cauchy problem of 3D Navier-Stokes-Cahn-Hilliard equations. Our main results are stated as follows:

Theorem 1.3. Let \(N \geq 1\), assume that \((u_0, \phi_0) \in H^N(\mathbb{R}^3) \times H^{N+1}(\mathbb{R}^3)\), and there exists a constant \(\delta_0 > 0\) such that if

\[
\parallel u_0 \parallel_{H^1} + \parallel \phi_0 \parallel_{H^1} + \parallel \nabla \phi_0 \parallel_{H^1} \leq \delta_0,
\]

then there exists a unique global solution \((u, \phi)\) satisfying that for all \(t \geq 0\),

\[
\begin{align*}
\parallel u(t) \parallel_{H^N}^2 + \parallel \phi \parallel_{H^N}^2 + \parallel \nabla \phi \parallel_{H^N}^2 \\
+ \int_0^t (\parallel \nabla u(s) \parallel_{H^N}^2 + \parallel \Delta \phi \parallel_{H^N}^2 + \parallel \nabla \phi \parallel_{H^N}^2 + \parallel \Delta \nabla \phi \parallel_{H^N}^2 + \parallel \nabla \nabla \phi \parallel_{H^N}^2) ds
\end{align*}
\]

\[
\leq C(\parallel u_0 \parallel_{H^N}^2 + \parallel \phi_0 \parallel_{H^N}^2 + \parallel \nabla \phi_0 \parallel_{H^N}^2).
\]

(10)

If further, \((u_0, \phi_0, \nabla \phi_0) \in \dot{H}^{-s}(\mathbb{R}^3)\) for some \(s \in [0, \frac{1}{2}]\), then for all \(t \geq 0\),

\[
\parallel \Lambda^{-s} u(t) \parallel_{L^2}^2 + \parallel \Lambda^{-s} \phi(t) \parallel_{L^2}^2 + \parallel \Lambda^{-s} \nabla \phi(t) \parallel_{L^2}^2 \leq C,
\]

(11)

and

\[
\parallel \nabla^l u(t) \parallel_{H^{N-l}} + \parallel \nabla^l \phi(t) \parallel_{H^{N-l}} + \parallel \nabla^{l+1} \phi(t) \parallel_{H^{N-l}} \leq C(1 + t)^{-\frac{l+1}{2}}, \quad \text{for} \ l = 0, 1, \ldots, N-1.
\]

(12)

Note that the Hardy-Littlewood-Sobolev theorem implies that for \(p \in [\frac{3}{2}, 2]\), \(L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)\) with \(s = 3\left(\frac{1}{p} - \frac{1}{2}\right) \in [0, \frac{1}{2}]\). Then, on the basis of Theorem 1.3 we easily obtain the following corollary of the optimal decay estimates.
Corollary 1.4. Under the assumptions of Theorem 1.3, if we replace the \( H^{-s}(\mathbb{R}^3) \) assumption by \( (u_0, \nabla \phi) \in L^p(\mathbb{R}^3) \) \((\frac{3}{2} \leq p \leq 2)\), then the following decay estimate holds:

\[
\|\nabla^l u(t)\|_{H^{N-l}} + \|\nabla^l \phi(t)\|_{H^{N-l}} + \|\nabla^l \nabla \phi(t)\|_{H^{N-l}} \leq C(1 + t)^{-\sigma_l}, \quad \text{for } l = 0, 1, \ldots, N - 1,
\]

where

\[
\sigma_l = \frac{3}{2} \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{l}{2}.
\]

Remark 1.5. There’s an amazing phenomenon: although there’s a fourth order equation in system (7), the temporary decay of system (7) satisfies (12) and (13), which is equivalent to the decay rate of second order heat equation. That is because of the introduce of the second order linear term in (19). After introduce this term, the term \( \Delta(\phi^3 - 2\phi) \) can be controlled by \( \Delta^2 \phi \) and \( -\Delta \phi \), the convective term can be controlled by \( -\Delta \phi \). Since both \( \Delta^2 \phi \) and \( -\Delta \phi \) are “good” term, we can use the lower order one to study the decay estimate. Hence, the temporary decay of system (7) is equivalent to the decay rate of second order heat equation.

The main difficulties to consider the Cauchy problem of 3D Navier-Stokes-Cahn-Hilliard equations are how to deal with the convective term \( u \cdot \nabla \phi \), the coupling between \( u \) and \( \phi \) and the linear term of the double-well potential. Since the principle part of (7) is a fourth-order linear term and the convective term is only a first-order nonlinear term, we can’t control a first-order nonlinear term through fourth-order linear term in \( \mathbb{R}^3 \). In order to overcome this difficulty, we borrow a second-order term from the double-well potential, rewrite (7) as

\[
\partial_t \phi + \Delta^2 \phi - \Delta \phi = -u \cdot \nabla \phi + \Delta(\phi^3 - 2\phi).
\]

Hence, one can control the convective term by the last term of the left hand side of equation (14). Moreover, note that (7)_1 is a second order PDE and (7)_3 is a fourth order PDE, it is difficulty to deal with the coupling between \( u \) and \( \phi \). In order to overcome this difficulty, we not only assume the initial data \( \|u_0\|_{H^1} + \|\phi_0\|_{H^1} \) is sufficiently small, but also assume \( \|\nabla \phi_0\|_{H^1} \) is sufficiently small, obtain a priori estimates on \( (u, \phi, \nabla \phi) \). On the other hand, the linear term of the double-well potential also bring trouble to us. When we study the problem in bounded domain, it is easy to control the linear term by using Sobolev’s embedding theorem and the a priori estimates. However, in the whole space \( \mathbb{R}^3 \), since the compactness properties of Sobolev’s embedding theorem is losing, it is difficulty to deal with this linear term. By using the pure energy method \([58, 52]\) of using a family of scaled energy estimates with minimum derivative counts and interpolations among them, we overcome the difficulty caused by the linear term of the double-well potential, obtained the suitable a priori estimates in the negative Sobolev space \( \dot{H}^{-s} \) \((0 \leq s \leq \frac{1}{2})\), prove the well-posedness and optimal decay rate of the Cauchy problem of Navier-Stokes-Cahn-Hilliard equation in the whole spaces.

The structure of this paper is organized as follows. In Section 2, we introduce some preliminary results, which are useful to prove our main results. Section 3 is devoted to establish some refined energy estimates for the solution \( u(x, t) \) of Cauchy problem (7). In Section 4, we prove the local well-posedness of solutions. Section 5 is devoted to derive the evolution of the negative Sobolev norms of the solution. Finally, the proof of Theorem 1.3 is postponed in Section 6.

2. Preliminaries

In this section, we introduce some helpful results in \( \mathbb{R}^3 \). The following Gagliardo-Nirenberg inequality was proved in \([45]\).
δ > (19). Hence, we suppose that for sufficiently small

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1 - \theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.$$  

Here, when \( p = \infty \), we require that \( 0 < \theta < 1 \).

We also introduce the Hardy-Littlewood-Sobolev theorem, which implies the following \( L^p \) type inequality.

**Lemma 2.2** ([51, 30]). Let \( 0 \leq s < \frac{3}{2} \), \( 1 < p \leq 2 \) and \( \frac{1}{2} + \frac{s}{q} = \frac{1}{p} \), then

$$\|f\|_{H^{-s}} \lesssim \|f\|_{L^p}.$$  

The following special Sobolev interpolation lemma will be used in the proof of Theorem 1.3.

**Lemma 2.3** ([52, 51]). Let \( s, k \geq 0 \) and \( l \geq 0 \), then

$$\|\nabla^l f\|_{L^2} \leq \|\nabla^{l+k} f\|_{L^2}^{1-\theta} \|f\|_{H^{-s}}^\theta, \quad \text{with} \quad \theta = \frac{2}{l + k + s}.$$

### 3. Energy estimates

The Cauchy problem of Navier-Stokes-Cahn-Hilliard equations with double-well potential can be rewritten as

$$\begin{aligned}
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \text{div} (\nabla \phi \otimes \nabla \phi), \\
\text{div} u = 0, \\
\partial_t \phi + u \cdot \nabla \phi + \Delta^2 u - \Delta u = \Delta(u^3 - 2u), \\
(u, \phi)|_{t=0} = (u_0(x), \phi_0(x)),
\end{cases}
\end{aligned}$$

where \( u \) and \( \phi \) satisfy (19). Hence, we suppose that for sufficiently small \( \delta > 0 \),

$$\sqrt{\mathcal{E}_0^1(t)} = \|u(t)\|_{H^1} + \|\phi(t)\|_{H^1} + \|\nabla \phi(t)\|_{H^1} \leq \delta.$$  

We begin with the energy estimates including \( u \) and \( \phi \) themselves.

**Lemma 3.1.** Suppose that all assumptions in Theorem 1.3 hold. If \( \sqrt{\mathcal{E}_0^1(t)} \leq \delta \), then for \( k = 0, 1, \cdots, N \), we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla^k u|^2 + |\nabla^k \phi|^2 + |\nabla^k \nabla \phi|^2) dx &+ C(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\phi\|_{L^2}^2 + \|\nabla^{k+2}\nabla \phi\|_{L^2}^2 + \|\nabla^{k+2} \phi\|_{L^2}^2) \\
\lesssim (\delta + \delta^2)(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\phi\|_{L^2}^2 + \|\nabla^{k+2}\phi\|_{L^2}^2 + \|\nabla^{k+2} \phi\|_{L^2}^2).
\end{aligned}$$

**Proof.** Applying \( \nabla^k \) to (19), multiplying the resulting identity by \( \nabla^k u \), and then integrating over \( \mathbb{R}^3 \) by parts, we arrive at

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla^k u\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 &\leq \int_{\mathbb{R}^3} \nabla^k(u \cdot \nabla u) \cdot \nabla^k u dx - \int_{\mathbb{R}^3} \nabla^k(\nabla \cdot (\nabla \phi \otimes \nabla \phi)) \cdot \nabla^k u dx \\
&= I_1 + I_2.
\end{aligned}$$
We shall estimate the two terms in the right hand side of (22). First, for the term $I_1$, by using Hölder’s inequality and Sobolev’s inequality, we deduce that
\[
I_1 = - \int_{\mathbb{R}^3} \nabla^k (u \cdot \nabla u) \cdot \nabla^k u \, dx = - \sum_{0 \leq l \leq k} C_k^l \int_{\mathbb{R}^3} (\nabla^l u \cdot \nabla \nabla^{k-l} u) \cdot \nabla^k u \, dx
\]
(23)
\[
\lesssim \sum_{0 \leq l \leq k} \| \nabla^l u \cdot \nabla \nabla^{k-1-l} u \|_{L^6_{\text{loc}}} \| \nabla^k u \|_{L^6}
\]
\[
\lesssim \sum_{0 \leq l \leq k} \| \nabla^l u \cdot \nabla \nabla^{k-1-l} u \|_{L^6_{\text{loc}}} \| \nabla^{k+1} u \|_{L^2},
\]
If $l \leq \lfloor \frac{k}{2} \rfloor$, through Hölder’s inequality and Lemma 2.1, we derive that
\[
\| \nabla^l u \cdot \nabla \nabla^{k-1-l} u \|_{L^6_{\text{loc}}} \lesssim \| \nabla^l u \|_{L^3} \| \nabla \nabla^{k-1-l} u \|_{L^2}
\]
(24)
\[
\lesssim \| \nabla^{k+1} u \|_{L^2}^{\frac{l}{k+1}} \| \nabla^l u \|_{L^2}^{\frac{1}{k+1}} \| \nabla^{k+1} u \|_{L^2}^{\frac{1}{k+1}} \| \nabla^{k+1} u \|_{L^2}^{\frac{1}{k+1}}
\]
\[
\lesssim \sqrt{\varepsilon_0^l} \| \nabla^{k+1} u \|_{L^2},
\]
where $\alpha$ is defined by
\[
\frac{l}{3} - \frac{1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right) + \left( \frac{k+1}{3} - \frac{1}{2} \right) \frac{l}{k+1},
\]
which implies
\[
\alpha = \frac{k+1}{2(k+1-l)} \in \left( \frac{1}{2}, 1 \right).
\]
If $l \geq \lfloor \frac{k}{2} \rfloor + 1$, by using Hölder’s inequality and Lemma 2.1 again, we find that
\[
\| \nabla^l u \cdot \nabla \nabla^{k-1-l} u \|_{L^6_{\text{loc}}} \lesssim \| \nabla^l u \|_{L^2} \| \nabla \nabla^{k-1-l} u \|_{L^2}
\]
(25)
\[
\lesssim \| \nabla^{k+1} u \|_{L^2}^{\frac{l}{k+1}} \| \nabla^l u \|_{L^2}^{\frac{1}{k+1}} \| \nabla^{k+1} u \|_{L^2}^{\frac{1}{k+1}} \| \nabla^{k+1} u \|_{L^2}^{\frac{1}{k+1}}
\]
\[
\lesssim \sqrt{\varepsilon_0^l} \| \nabla^{k+1} u \|_{L^2},
\]
where $\alpha$ is defined by
\[
\frac{k+1-l}{3} - \frac{1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l}{k+1} + \left( \frac{k+1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right),
\]
we obtain
\[
\alpha = \frac{k+1}{2l} \in \left( \frac{1}{2}, 1 \right].
\]
Combining (23)–(25) together gives
\[
I_1 \lesssim \sqrt{\varepsilon_0^l} \| \nabla^{k+1} u \|_{L^2}^2.
\]
We estimate $I_2$ as
\[
I_2 = - \int_{\mathbb{R}^3} \nabla^k [\nabla \cdot (\nabla \phi \otimes \nabla \phi)] \cdot \nabla^k u \, dx
\]
(27)
\[
\lesssim \sum_{0 \leq l \leq k} \| \nabla^k [\nabla \cdot (\nabla \phi \otimes \nabla \phi)] \|_{L^6_{\text{loc}}} \| \nabla^k u \|_{L^6}
\]
\[
\lesssim \sum_{0 \leq l \leq k} \| \nabla^l \nabla \phi \cdot \nabla^{k-l+1} \phi \|_{L^6_{\text{loc}}} \| \nabla^{k+1} u \|_{L^2}.
\]
If \( l \leq \lceil \frac{k}{2} \rceil \), through Hölder’s inequality and Lemma 2.1, we derive that

\[
\| \nabla^l \nabla \phi \cdot \nabla^{k+1-l} \nabla \phi \|_{L_6} \lesssim \| \nabla^l \nabla \phi \|_{L_6} \| \nabla^{k+1-l} \nabla \phi \|_{L_2} \\
\lesssim \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{1}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{k}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{1}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{k}{k+1}} \\
\lesssim \sqrt{E_0} \| \nabla^{k+1} \nabla \phi \|_{L_2},
\]

(28)

where \( \alpha \) is defined by

\[
\frac{l}{3} - \frac{1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right) + \left( \frac{k+1}{3} - \frac{1}{2} \right) \frac{l}{k+1},
\]

we have

\[
\alpha = \frac{k+1}{2(k+1-l)} \in \left[ \frac{1}{2}, 1 \right).
\]

If \( l \geq \lceil \frac{k}{2} \rceil + 1 \), by using Hölder’s inequality and Lemma 2.1 again, we find that

\[
\| \nabla^l \nabla \phi \cdot \nabla^{k+1-l} \nabla \phi \|_{L_6} \lesssim \| \nabla^l \nabla \phi \|_{L_6} \| \nabla^{k+1-l} \nabla \phi \|_{L_2} \\
\lesssim \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{1}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{k}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{1}{k+1}} \| \nabla^{k+1} \nabla \phi \|_{L_2}^{\frac{k}{k+1}} \\
\lesssim \sqrt{E_0} \| \nabla^{k+1} \nabla \phi \|_{L_2},
\]

(29)

where \( \alpha \) is defined by

\[
\frac{k+1-l}{3} - \frac{1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l}{k+1} + \left( \frac{k+1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right),
\]

which means

\[
\alpha = \frac{k+1}{2l} \in \left( \frac{1}{2}, 1 \right).
\]

Combining (27)-(29) together, we have

\[
I_2 \lesssim \sqrt{E_0} (\| \nabla^{k+1} u \|_{L_2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L_2}^2).
\]

(30)

Applying \( \nabla^k \) to \( \text{19}_3 \), multiplying the resulting identity by \( \nabla^k \phi \), and then integrating over \( \mathbb{R}^3 \) by parts, we arrive at

\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k \phi \|_{L_2}^2 + \| \nabla^{k+2} \phi \|_{L_2}^2 + \| \nabla^{k+1} \phi \|_{L_2}^2 \\
= \int_{\mathbb{R}^3} \nabla^k \Delta (\phi^3 - 2\phi) \cdot \nabla^k \phi \, dx + \int_{\mathbb{R}^3} \nabla^k (u \cdot \nabla \phi) \cdot \nabla^k \phi \, dx \\
=: I_3 + I_4.
\]

(31)
Note that
\[ I_3 = \int_{\mathbb{R}^3} \nabla^k (\phi^3 - 2\phi) \cdot \nabla^{k+2} \phi \, dx \]
\[ = \sum_{0 \leq s \leq k} C_k^s \int_{\mathbb{R}^3} \nabla^s [\phi (\phi - \sqrt{2})] \cdot \nabla^{k-s} (\phi + \sqrt{2}) \cdot \nabla^{k+2} \phi \, dx \]
\[ = \sum_{0 \leq s \leq k} \sum_{0 \leq s \leq k} C_k^s C_s^m \int_{\mathbb{R}^3} \nabla^m \phi \cdot \nabla^{s-m} (\phi - \sqrt{2}) \cdot \nabla^{k-s} (\phi + \sqrt{2}) \cdot \nabla^{k+2} \phi \, dx \]
\[ \lesssim \sum_{0 \leq s \leq k} \sum_{0 \leq s \leq k} C_k^s C_s^m \| \nabla^m \phi \|_{L^6} \| \nabla^{s-m} (\phi - \sqrt{2}) \|_{L^6} \| \nabla^{k-s} (\phi + \sqrt{2}) \|_{L^6} \| \nabla^{k+2} \phi \|_{L^2} \]
\[ \lesssim \sum_{0 \leq s \leq k} \sum_{0 \leq s \leq k} C_k^s C_s^m \| \nabla^m \phi \|_{L^2} \| \nabla^{s-m} \nabla \phi \|_{L^2} \| \nabla^{k-s} \nabla \phi \|_{L^2} \| \nabla^{k+2} \phi \|_{L^2}. \]

If \( 0 \leq s \leq \left[ \frac{k}{2} \right] \), using Lemma 2.1, it yields that
\[ \| \nabla^m \nabla \phi \|_{L^2} \| \nabla^{s-m} \nabla \phi \|_{L^2} \| \nabla^{k-s} \nabla \phi \|_{L^2} \]
\[ \lesssim \| \nabla^\alpha \phi \|_{L^2} \| \nabla^{k+2} \phi \|_{L^2} \frac{m+1}{3} \| \nabla^\alpha \phi \|_{L^2} \| \nabla^{k+2} \phi \|_{L^2} \]
\[ \lesssim \| \phi \|_{H^1}^2 \| \nabla^{k+2} \phi \|_{L^2} \]
\[ \lesssim (\| \phi \|_{H^1}^2 + \| \nabla \phi \|_{H^1}^2) \| \nabla^{k+2} \phi \|_{L^2}, \]
where \( \alpha \) satisfies
\[ \frac{m+1}{3} - \frac{1}{2} = \left( \frac{\alpha}{3} \right) - \frac{1}{2} \times \frac{k+2-m}{k+2} + \left( \frac{k+2}{3} \right) - \frac{1}{2} \times \left( 1 - \frac{k+2-m}{k+2} \right), \]
we easily obtain
\[ \alpha = \frac{k+2}{k+2-m} \in [1, 2). \]

Moreover, if \( \left[ \frac{k}{2} \right] + 1 \leq s \leq k \), using Lemma 2.1, we deduce that
\[ \| \nabla^m \nabla \phi \|_{L^2} \| \nabla^{s-m} \nabla \phi \|_{L^2} \| \nabla^{k-s} \nabla \phi \|_{L^2} \]
\[ \lesssim \| \phi \|_{H^2} \| \nabla^{k+2} \phi \|_{L^2} \]
\[ \lesssim (\| \phi \|_{H^1}^2 + \| \nabla \phi \|_{H^1}^2) \| \nabla^{k+2} \phi \|_{L^2}, \]
where \( \alpha \) is defined by
\[ \frac{k+1-s}{3} - \frac{1}{2} = \left( \frac{\alpha}{3} \right) \times \frac{k+2}{k+2} + \left( \frac{k+2}{3} \right) \times \left( 1 - \frac{s+2}{k+2} \right), \]
which implies
\[ \alpha = \frac{k+2}{s+2} \in [1, 2). \]

Combining (32)-(33) together gives
\[ I_3 \lesssim (\| \phi \|_{H^1}^2 + \| \nabla \phi \|_{H^1}^2) \| \nabla^{k+2} \phi \|_{L^2}^2. \]
Moreover, $I_4$ can be estimated as

$$I_4 = - \int_{\mathbb{R}^3} \nabla^k (u \cdot \nabla \phi) \cdot \nabla^k \phi \, dx$$

$$\lesssim \|\nabla^k \phi\|_{L^6} \left\| \sum_{0 \leq l \leq k} \nabla^l u \cdot \nabla^{k-l} \nabla \phi\right\|_{L^\frac{6}{5}}$$

$$\lesssim \|\nabla^k \nabla \phi\|_{L^2} \left\| \sum_{0 \leq l \leq k} \nabla^l u \cdot \nabla^{k-l} \nabla \phi\right\|_{L^\frac{6}{5}}$$

$$\lesssim \sum_{0 \leq l \leq k} \|\nabla^k \nabla \phi\|_{L^2} \|\nabla^l u\|_{L^2} \|\nabla^{k-l} \nabla \phi\|_{L^3}$$

$$\lesssim \sum_{0 \leq l \leq k} \|\nabla^k \nabla \phi\|_{L^2} \|\nabla^l u\|_{L^2} \|\nabla^{k-l} \nabla \phi\|_{L^2}^{\frac{1}{2}} \|\nabla^{k+2-l} \phi\|_{L^2}^{\frac{1}{2}}.$$

By using Hölder’s inequality and Sobolev’s inequality, it yields that

$$\left\| \nabla^l u\right\|_{L^2} \left\| \nabla^{k-l} \nabla \phi\right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^{k+1-l} \nabla \phi\right\|_{L^2}^{\frac{1}{2}}$$

$$\lesssim \|\nabla^{k+1} u\|_{L^2} \|\nabla \phi\|_{L^2} \left\| \nabla^k \nabla \phi\right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \phi\right\|_{L^2}^{\frac{1}{2} \left(\left\| \nabla^{k+1} \nabla \phi\right\|_{L^2}^{\frac{1}{2}} \|\nabla \phi\|_{L^2}^{\frac{1}{2}}\right)^2}$$

$$\lesssim \|\nabla^k \phi\|_{L^2} + \|\nabla \phi\|_{L^2} \left\| \nabla^{k+1} u\right\|_{L^2} + \|\nabla^k \nabla \phi\|_{L^2} + \|\nabla^{k+1} \nabla \phi\|_{L^2}.$$

Combining (36) and (37) together gives

$$I_4 \lesssim \sqrt{\varepsilon_1 \left(\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} \phi\|_{L^2}^2 + \|\nabla^{k+2} \phi\|_{L^2}^2\right)}.$$

Applying $\nabla^{k+1} \phi$ to (19), multiplying the resulting identity by $\nabla^{k+1} \phi$, and then integrating over $\mathbb{R}^3$ by parts, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\nabla^k \nabla \phi\|_{L^2}^2 + \|\nabla^{k+2} \nabla \phi\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \nabla^{k+1} \Delta (\phi^3 - 2\phi) \cdot \nabla^{k+1} \phi \, dx + \int_{\mathbb{R}^3} \nabla^{k+1} (u \cdot \nabla \phi) \cdot \nabla^{k+1} \phi \, dx$$

$$= I_5 + I_6.$$

We estimate $I_5$ as

$$I_5 = \int_{\mathbb{R}^3} \nabla^{k+1} [\phi (\phi - \sqrt{2}) (\phi + \sqrt{2})] \cdot \nabla^{k+2} \nabla \phi \, dx$$

$$\lesssim \|\nabla^{k+2} \nabla \phi\|_{L^2} \left\| \nabla^{k+1} [\phi (\phi - \sqrt{2}) (\phi + \sqrt{2})]\right\|_{L^2}$$

$$\lesssim \sum_{0 \leq l \leq k+1} \sum_{1 \leq m \leq l} C_{k+1}^l C_l^m \|\nabla^{k+2} \nabla \phi\|_{L^2} \|\nabla^{k+1-l} \phi\|_{L^6} \|\nabla^{l-m} (\phi - \sqrt{2})\|_{L^6} \|\nabla^m (\phi - \sqrt{2})\|_{L^6}$$

$$\lesssim \sum_{0 \leq l \leq k+1} \sum_{1 \leq m \leq l} C_{k+1}^l C_l^m \|\nabla^{k+2} \nabla \phi\|_{L^2} \|\nabla^{k+1-l} \phi\|_{L^2} \|\nabla^{l-m} \nabla \phi\|_{L^2} \|\nabla^m \nabla \phi\|_{L^2}.$$

By using Hölder’s inequality and Sobolev’s inequality, it yields that

$$\left\| \nabla^{k+1-l} \nabla \phi\right\|_{L^2} \left\| \nabla^{l-m} \nabla \phi\right\|_{L^2} \left\| \nabla^m \nabla \phi\right\|_{L^2}$$

$$\lesssim \|\nabla \phi\|_{L^2} \left\| \nabla^{k+1} \nabla \phi\right\|_{L^2} \|\nabla \phi\|_{L^2} \left\| \nabla^{k+1} \nabla \phi\right\|_{L^2} \left\| \nabla \phi\right\|_{L^2}.$$
It then follows from (40) and (41) that

\[ I_5 \lesssim \mathcal{E}^1_0 (\| \nabla^{k+1} \nabla \phi \|_{L^2} + \| \nabla^{k+2} \nabla \phi \|_{L^2}^2). \]

Moreover, \( I_6 \) can be estimated as follows

\[
I_6 = - \int_{\mathbb{R}^3} \nabla^{k+1} (u \cdot \nabla \phi) \cdot \nabla^{k+1} \phi \, dx \\
\lesssim \| \nabla^{k+1} \phi \|_{L^6} \| \sum_{0 \leq l \leq k+1} \nabla^l u \cdot \nabla^{k+1-l} \nabla \phi \|_{L^6}^\frac{3}{2} \\
\lesssim \| \nabla^{k+1} \nabla \phi \|_{L^2} \| \sum_{0 \leq l \leq k+1} \nabla^l u \cdot \nabla^{k+1-l} \nabla \phi \|_{L^6}^\frac{3}{2} \\
\lesssim \sum_{0 \leq l \leq k+1} \| \nabla^{k+1} \nabla \phi \|_{L^2} \| \nabla^l u \|_{L^2} \| \nabla^{k+1-l} \nabla \phi \|_{L^2} \| \nabla^{k+2-l} \nabla \phi \|_{L^2}^\frac{1}{2}.
\]

By using Hölder’s inequality and Sobolev’s inequality, it yields that

\[
\| \nabla^l u \|_{L^2} \| \nabla^{k+1-l} \nabla \phi \|_{L^2}^\frac{3}{2} \| \nabla^{k+2-l} \nabla \phi \|_{L^2}^\frac{3}{2} \lesssim \| \nabla^{k+1} \nabla \phi \|_{L^2} \| \nabla^l u \|_{L^2} \| \nabla^{k+1-l} \nabla \phi \|_{L^2} \| \nabla^{k+2-l} \nabla \phi \|_{L^2}^\frac{1}{2}.
\]

Combining (43) and (44) together gives

\[
I_6 \lesssim \sqrt{\mathcal{E}^1_0} \left( \| \nabla^{k+1} u \|^2_{L^2} + \| \nabla^{k+1} \nabla \phi \|^2_{L^2} + \| \nabla^{k+2} \nabla \phi \|^2_{L^2} \right).
\]

Summing up the estimates for \( I_1-I_6 \), that is, (26), (30), (35), (38), (42) and (45), we deduce

\[
\frac{1}{2} \frac{d}{dt} (\| \nabla^{k+1} u \|^2_{L^2} + \| \nabla^{k+1} \nabla \phi \|^2_{L^2} + \| \nabla^{k+2} \nabla \phi \|^2_{L^2}) \\
+ \| \nabla^{k+1} u \|^2_{L^2} + \| \nabla^{k+1} \phi \|^2_{L^2} + \| \nabla^{k+2} \phi \|^2_{L^2} + \| \nabla^{k+1} \nabla \phi \|^2_{L^2} + \| \nabla^{k+2} \nabla \phi \|^2_{L^2} \\
\lesssim \sqrt{\mathcal{E}^1_0 + \mathcal{E}^1_0} \left( \| \nabla^{k+1} u \|^2_{L^2} + \| \nabla^{k+1} \phi \|^2_{L^2} + \| \nabla^{k+2} \phi \|^2_{L^2} + \| \nabla^{k+1} \nabla \phi \|^2_{L^2} + \| \nabla^{k+2} \nabla \phi \|^2_{L^2} \right)
\]

for \( 0 \leq k \leq N \), this yields the desired result.

\[ \square \]

4. Local well-posedness

The main purpose of this section is to prove the local well-posedness of solution \((u(t), \phi(t), \nabla \phi(t))\) in \( H^1 \)-norm.

We first construct the solution \((u^j, \phi^j)_{j \geq 0}\) by solving iteratively the Cauchy problem:

\[
\begin{align*}
\partial_t u^{j+1} + u^j \cdot \nabla u^{j+1} + \nabla \pi^{j+1} - \Delta u^{j+1} &= -\nabla \cdot (\nabla \phi^j \otimes \nabla \phi^{j+1}), \\
\nabla \cdot u^{j+1} &= 0, \\
\partial_t \phi^{j+1} + \Delta^2 \phi^{j+1} - \Delta \phi^{j+1} + u^j \cdot \nabla \phi^{j+1} &= \Delta [(\phi^j)^2 \phi^{j+1} - 2 \phi^{j+1}], \\
(u^{j+1}, \phi^{j+1})|_{t=0} &= (u_0(x), \phi_0(x)), \quad x \in \mathbb{R}^3,
\end{align*}
\]

(47)
for \( j \geq 0 \), where \((u^0, \phi^0, \nabla \phi^0) \equiv (0, 0, 0)\) holds. One denote \((u^j, \phi^j, \nabla \phi^j)_{j \geq 0}\) in short hand by \((A_j)_{j \geq 0}\) and denote \((u_0, \phi_0, \nabla \phi_0)\) as \(A_0\). In the following, we shall show that \((A_j)_{j \geq 0}\) is a Cauchy sequence in Banach space \(C([0, T_1]; H^1)\) with \(T_1 > 0\) suitable small. Then, by take limit and continuous argument, one propose to prove that \((u, \phi, \nabla \phi)\) is a global solution to Cauchy problem (19).

**Lemma 4.1.** Suppose that all assumptions in Theorem \[1.3\] hold. There are constants \(\varepsilon_1 > 0\), \(T_1 > 0\) and \(M_1 > 0\) such that if \(\|A_0\|_{H^1} \leq \varepsilon_1\), then for each \(j \geq 0\), \(A_j \in C([0, T_1]; H^1)\) is well defined and

\[
\sup_{0 \leq t \leq T_1} \|A^j(t)\|_{H^1} \leq M_1, \quad j \geq 0.
\]

Moreover, \((A^j)_{j \geq 0}\) is a Cauchy sequence in Banach space \(C([0, T_1]; H^1)\), the corresponding limit function denoted by \(A(t)\) belongs to \(C([0, T_1]; H^1)\) with

\[
\sup_{0 \leq t \leq T_1} \|A(t)\|_{H^1} \leq M_1,
\]

and \(A = (u, \phi, \nabla \phi)\) is a solution over \([0, T_1]\) to problem (19). Finally, for the Cauchy problem (19), there exists at most one solution \((u, \phi, \nabla \phi)\) in \(C([0, T_1]; H^1)\) satisfying (19).

**Proof.** We will prove (48) by induction. Through the assumption at initial step, we easily obtain \(A_0 = (0, 0, 0)\), which implies \(j = 0\) holds. Next, let (48) holds for \(j \geq 0\) with \(M_1 > 0\) small enough to be determined later, we are going to prove it is true for \(j + 1\). For this goal, we need some energy estimates on \(A^j+1\). By (17) and (17)3, we can see that for \(k = 1\) and

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \left( \|\nabla u^{j+1}\|_{L^2}^2 + \|\nabla \phi^{j+1}\|_{L^2}^2 + \|\nabla \phi^{j+1}\|_{L^2}^2 + \|\nabla u^{j+1}\|_{L^2}^2 + \|\nabla \phi^{j+1}\|_{L^2}^2 \right) \\
& + \|\nabla \phi^{j+1}\|_{L^2}^2 + \|\nabla \phi^{j+1}\|_{L^2}^2 + \|\nabla \phi^{j+1}\|_{L^2}^2 \\
& = - \int_{\mathbb{R}^3} \nabla (u^j \cdot \nabla u^{j+1}) \cdot \nabla u^{j+1} dx - \int_{\mathbb{R}^3} \nabla (\nabla \phi^j \otimes \nabla \phi^{j+1}) \cdot \nabla u^{j+1} dx \\
& + \int_{\mathbb{R}^3} \nabla \left[(\phi^j)^2 - 2|\phi^j|^2\right] \cdot \nabla \phi^{j+1} \cdot \nabla u^{j+1} dx \\
& + \int_{\mathbb{R}^3} \nabla [((\phi^j)^2 - 2|\phi^j|^2) \cdot \nabla \phi^{j+1} \cdot \nabla \phi^{j+1}] dx \\
& =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{align*}
\]

Since it is trivial for the case \(l = 0\), thus, we may only consider \(l = 1\). For the term \(J_1\), we estimate as follows

\[
J_1 \lesssim \|\nabla u^{j+1}\|_{L^6} \|\nabla (u^j \cdot \nabla u^{j+1})\|_{L^6} \\
\lesssim \sum_{0 \leq s \leq 1} \|\nabla u^j \cdot \nabla^{1-s} u^{j+1}\|_{L^6} \|\nabla^2 u^{j+1}\|_{L^2} \\
\lesssim (\|u^j\|_{L^3} \|\nabla^2 u^{j+1}\|_{L^2} + \|\nabla u^j\|_{L^2} \|\nabla u^{j+1}\|_{L^3}) \|\nabla^2 u^{j+1}\|_{L^2} \\
\lesssim \|u^j\|_{H^1} \|\nabla u^{j+1}\|_{H^1}^2.
\]
The term $J_2$ can be estimated as

\[
J_2 \lesssim \|\nabla u^{j+1}\|_{L^6} \|\nabla \cdot (\nabla \phi^j \otimes \nabla \phi^{j+1})\|_{L^6} \\
\lesssim \sum_{0 \leq s \leq 1} \|\nabla^s \nabla \phi^j \cdot \nabla^{1+s} \nabla \phi^{j+1}\|_{L^6} \|\nabla^2 u^{j+1}\|_{L^2} \\
\lesssim (\|\nabla \phi^j\|_{L^3} \|\nabla^2 \phi^{j+1}\|_{L^2} + \|\nabla \nabla \phi^j\|_{L^2} \|\nabla \nabla \phi^{j+1}\|_{L^2}) \|\nabla^2 u^{j+1}\|_{L^2} \\
\lesssim \|\nabla \phi^j\|_{H^1} (\|\nabla u^{j+1}\|_{H^1}^2 + \|\nabla^2 \phi^{j+1}\|_{H^1}^2).
\]

On the other hand, by using Sobolev's embedding theorem, $J_3$ can be estimated as

\[
J_3 = \int_{\mathbb{R}^3} \nabla [(\phi^j + \sqrt{2})(\phi^j - \sqrt{2})\phi^{j+1}] \cdot \nabla^3 \phi^{j+1} \, dx \\
\lesssim \|\nabla^3 \phi^{j+1}\|_{L^2} \sum_{s=0}^{1} \sum_{m=0}^{s} \|\nabla^m (\phi^j + \sqrt{2}) \cdot \nabla^{s-m} (\phi^j - \sqrt{2}) \cdot \nabla^{1-s} \phi^{j+1}\|_{L^2} \\
\lesssim \sum_{s=0}^{1} \sum_{m=0}^{s} \|\nabla^3 \phi^{j+1}\|_{L^2} \|\nabla^m (\phi^j + \sqrt{2})\|_{L^6} \|\nabla^{s-m} (\phi^j - \sqrt{2})\|_{L^6} \|\nabla^{1-s} \phi^{j+1}\|_{L^6} \\
\lesssim \sum_{s=0}^{1} \sum_{m=0}^{s} \|\nabla^3 \phi^{j+1}\|_{L^2} \|\nabla^{1-s} \phi^{j+1}\|_{L^2} \|\nabla^m \nabla \phi^j\|_{L^2} \|\nabla^{s-m} \nabla \phi^j\|_{L^2} \\
\lesssim \sum_{s=0}^{1} \sum_{m=0}^{s} \|\nabla^3 \phi^{j+1}\|_{L^2} \|\nabla^{1-s} \phi^{j+1}\|_{L^2} \|\nabla^m \phi^j\|_{L^2}^{\theta_2} \|\nabla^2 \phi^j\|_{L^2}^{1-\theta_1} \|\phi^j\|_{L^2}^{\theta_2} \|\nabla^2 \phi^j\|_{L^2}^{1-\theta_2} \\
\lesssim (\|\phi^j\|_{H^1}^2 + \|\nabla \phi^j\|_{H^1}^2) (\|\nabla \phi^{j+1}\|_{H^1}^2 + \|\nabla^2 u^{j+1}\|_{H^1}^2),
\]

where $\theta_1 = \frac{1-m}{2}$ and $\theta_2 = \frac{1+m-s}{2}$. We estimate $J_4$ as

\[
J_4 = \int_{\mathbb{R}^3} \nabla (u^j \cdot \nabla \phi^{j+1}) \cdot \nabla \phi^{j+1} \, dx \\
\lesssim \|\nabla^s u^j \cdot \nabla^{1-s} \nabla \phi^{j+1}\|_{L^6} \|\nabla \phi^{j+1}\|_{L^6} \\
\lesssim (\|u^j\|_{L^4} \|\nabla \phi^{j+1}\|_{L^2} + \|\nabla u^j\|_{L^4} \|\nabla \phi^{j+1}\|_{L^2}) \|\nabla^2 \phi^{j+1}\|_{L^2} \\
\lesssim \|u^j\|_{H^1} (\|\nabla \phi^{j+1}\|_{H^1}^2 + \|\nabla^2 \phi^{j+1}\|_{H^1}^2).
For the term $J_5$, we have

$$J_5 = \int_{\mathbb{R}^3} \nabla^2[(\phi^j + \sqrt{2})(\phi^j - \sqrt{2})\phi^{j+1}] \cdot \nabla^4\phi^{j+1} \, dx$$

$$\lesssim \|\nabla^4\phi^{j+1}\|_{L^2} \|\nabla^2[(\phi^j + \sqrt{2})(\phi^j - \sqrt{2})\phi^{j+1}]\|_{L^2}$$

$$\lesssim \|\nabla^4\phi^{j+1}\|_{L^2} \left( \|\nabla^2(\phi^j + \sqrt{2})(\phi^j - \sqrt{2})\phi^{j+1}\|_{L^\infty} \|\phi^{j+1}\|_{L^\infty} \\
+ \|\phi^j + \sqrt{2}\|_{L^\infty} \|\phi^j - \sqrt{2}\|_{L^\infty} \|\nabla^2\phi^{j+1}\|_{L^2} \\
+ \|\nabla(\phi^j + \sqrt{2})\|_{L^1} \|\nabla(\phi^j - \sqrt{2})\|_{L^6} \|\phi^{j+1}\|_{L^\infty} \\
+ \|\nabla(\phi^j + \sqrt{2})\|_{L^1} \|\phi^j - \sqrt{2}\|_{L^\infty} \|\nabla\phi^{j+1}\|_{L^6} \\
+ \|\phi^j + \sqrt{2}\|_{L^\infty} \|\nabla(\phi^j - \sqrt{2})\|_{L^3} \|\nabla\phi^{j+1}\|_{L^6} \right)$$

$$\lesssim \|\nabla^4\phi^{j+1}\|_{L^2} \left( \|\nabla^2\phi^j\|_{L^2} \|\nabla\phi^j\|_{L^2} \|\nabla\phi^j\|_{L^{3/2}} \|\nabla\phi^{j+1}\|_{L^2} \|\nabla\phi^{j+1}\|_{L^{3/2}} \right)

$$\lesssim \|\nabla\phi^j\|_{H^1}^2 (\|\nabla^3\phi^{j+1}\|_{H^1}^2 + \|\nabla^2\phi^{j+1}\|_{H^1}^2).$$

Moreover, for $J_6$, by Hölder’s inequality, we derive that

$$J_6 = \int_{\mathbb{R}^3} \nabla^2(\phi^j \cdot \nabla\phi^{j+1}) \cdot \nabla^2\phi^{j+1} \, dx$$

$$\leq \|\sum_{s=0}^1 \nabla^s\phi^j \cdot \nabla^{2-s}\nabla\phi^{j+1}\|_{L^6} \|\nabla^3\phi^{j+1}\|_{L^6}$$

$$\lesssim (\|\nabla^2\phi^{j+1}\|_{L^2} \|\nabla^2\phi^{j+1}\|_{L^3}) \|\nabla^3\phi^{j+1}\|_{L^2}$$

$$\lesssim \|\phi^j\|_{H^1} (\|\nabla^3\phi^{j+1}\|_{H^1}^2 + \|\nabla^2\phi^{j+1}\|_{H^1}^2).$$

Combining [50]–[56] together gives

$$\frac{1}{2} \frac{d}{dt} (\|\nabla^2\phi^{j+1}\|_{H^1}^2 + \|\phi^{j+1}\|_{H^1}^2 + \|\nabla\phi^{j+1}\|_{H^1}^2)$$

$$\leq C (\|\nabla^2\phi^{j+1}\|_{H^1}^2 + \|\phi^j\|_{H^1} + \|\nabla\phi^j\|_{H^1} + \|\phi^j\|_{H^1} + \|\nabla\phi^j\|_{H^1})$$

$$\times (\|\nabla\phi^{j+1}\|_{H^1} + \|\nabla\phi^{j+1}\|_{H^1} + \|\nabla\phi^{j+1}\|_{H^1} + \|\nabla^3\phi^{j+1}\|_{H^1} + \|\nabla^2\phi^{j+1}\|_{H^1}).$$
By taking time integration, it yields that
\[
\|u^{j+1}\|^2_{H^1} + \|\phi^{j+1}\|^2_{H^1} + \|\nabla \phi^{j+1}\|^2_{H^1} + \int_0^t \left( \|\nabla u^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \phi^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \nabla \phi^{j+1}(s)\|^2_{H^1} + \|\nabla \nabla \phi^{j+1}(s)\|^2_{H^1} \right) ds \\
\leq C(\|u_0\|^2_{H^1} + \|\phi_0\|^2_{H^1} + \|\nabla \phi_0\|^2_{H^1}) + C \int_0^t \left( \|u^j\|_{H^1} + \|\phi^j\|_{H^1} \right) ds,
\]
which from the inductive assumption implies
\[
\|u^{j+1}\|^2_{H^1} + \|\phi^{j+1}\|^2_{H^1} + \|\nabla \phi^{j+1}\|^2_{H^1} + \int_0^t \left( \|\nabla u^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \phi^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \nabla \phi^{j+1}(s)\|^2_{H^1} + \|\nabla \nabla \phi^{j+1}(s)\|^2_{H^1} \right) ds \\
\leq C \varepsilon^2 + C(M_1 + M_1^2) \int_0^t \left( \|\nabla u^j(s)\|^2_{H^1} + \|\nabla^2 \phi^j(s)\|^2_{H^1} \right) ds \\
+ \|\nabla \phi^j(s)\|^2_{H^1} + \|\nabla^2 \nabla \phi^j(s)\|^2_{H^1} + \|\nabla \nabla \phi^j(s)\|^2_{H^1} ds,
\]
for any $0 \leq t \leq T_1$. Take suitable small $\varepsilon > 0$, $T_1 > 0$ and $M_1 > 0$ such that
\[
\|u^{j+1}\|^2_{H^1} + \|\phi^{j+1}\|^2_{H^1} + \|\nabla \phi^{j+1}\|^2_{H^1} + \int_0^t \left( \|\nabla u^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \phi^{j+1}(s)\|^2_{H^1} \right) ds \\
+ \|\nabla \phi^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \nabla \phi^{j+1}(s)\|^2_{H^1} + \|\nabla \nabla \phi^{j+1}(s)\|^2_{H^1} ds \leq M_1^2,
\]
for any $t \in [0, T_1]$. Hence, (58) is true for $j + 1$ if so for $j$, which means (58) is proved for all $j \geq 0$.

Next, on the basis of (57), we easily obtain
\[
\left| \|A^{j+1}(t)\|^2_{H^1} - \|A^{j+1}(s)\|^2_{H^1} \right| \\
= \left| \int_s^t \frac{d}{d\tau} \|A^{j+1}(\tau)\|^2_{H^1} d\tau \right| \\
\leq C \int_s^t \left( \|A^j(\tau)\|^2_{H^1} + \|A^j(s)\|_{H^1} \right) \left( \|\nabla u^{j+1}(\tau)\|^2_{H^1} + \|\nabla^2 \phi^{j+1}(\tau)\|^2_{H^1} \right) \left( \|\nabla \phi^{j+1}(s)\|^2_{H^1} + \|\nabla^2 \nabla \phi^{j+1}(s)\|^2_{H^1} \right) ds \\
\leq C(M_1 + M_1^2) \int_s^t \left( \|\nabla u^{j+1}(\tau)\|^2_{H^1} + \|\nabla^2 \phi^{j+1}(\tau)\|^2_{H^1} \right) ds \\
+ \|\nabla \phi^{j+1}(\tau)\|^2_{H^1} + \|\nabla^2 \nabla \phi^{j+1}(\tau)\|^2_{H^1} + \|\nabla \nabla \phi^{j+1}(\tau)\|^2_{H^1} dt,
\]
for any $t \in [0, T_1]$. Therefore, due to (55), the time integral in the last inequality is finite, and hence $\|A^{j+1}(t)\|^2_{H^1}$ is continuous in $t$ for each $j \geq 1$. In the following, we also need to consider the convergence of the sequence $(A^{j})_{j \geq 0}$. Taking the difference of (47) for $j$ and $j - 1$, we
obtain
\begin{align}
\partial_t(u^{j+1} - u^j) + u^j \cdot \nabla(u^{j+1} - u^j) + (u^j - u^{j-1}) \cdot \nabla u^j + \nabla(\pi^{j+1} - \pi^j) \\
- \Delta(u^{j+1} - u^j) &= - \nabla \cdot \left( \nabla \phi^j \otimes \nabla (\phi^{j+1} - \phi^j) \right) - \nabla \cdot \left( \nabla (\phi^j - \phi^{j-1}) \otimes \nabla \phi^j \right), \\
\nabla \cdot (u^{j+1} - u^j) &= 0, \\
\partial_t(\phi^{j+1} - \phi^j) + \Delta^2(\phi^{j+1} - \phi^j) - \Delta(\phi^{j+1} - \phi^j) + u^j \cdot \nabla(\phi^{j+1} - \phi^j) \\
+ (u^j - u^{j-1}) \cdot \nabla \phi^j &= \Delta[(\phi^j)^2(\phi^{j+1} - \phi^j)] + \Delta[(\phi^j)^2 - (\phi^{j-1})^2]\phi^j] - 2\Delta(\phi^{j+1} - \phi^j),
\end{align}

Appealing to the same energy estimate as before, we easily obtain
\begin{align}
\frac{1}{2} \frac{d}{dt} \left( ||u^{j+1} - u^j||^2_{H^1} + ||\phi^{j+1} - \phi^j||^2_{H^1} + ||\nabla \phi^{j+1} - \nabla \phi^j||^2_{H^1} \right) \\
+ ||\nabla u^{j+1} - \nabla u^j||^2_{H^1} + ||\nabla^2(\phi^{j+1} - \phi^j)||^2_{H^1} + ||\nabla(\phi^{j+1} - \phi^j)||^2_{H^1} \\
+ ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} \\
\leq C(||u^j - u^{j-1}||_{H^1} + ||\phi^j - \phi^{j-1}||_{H^1} + ||\nabla \phi^j - \nabla \phi^{j-1}||_{H^1} \\
+ ||\nabla \phi^j - \nabla \phi^{j-1}||_{H^1})(||u^j||^2_{H^1} + ||\nabla \phi^j||^2_{H^1} + ||\nabla^2 \phi^j||^2_{H^1} + ||\nabla \phi^j||^2_{H^1} \\
+ C(||u^j||^2_{H^1} + ||\phi^j||^2_{H^1} + ||\phi^{j-1}||_{H^1})(||u^j||^2_{H^1} + ||\phi^j||^2_{H^1} + ||\phi^{j-1}||_{H^1} \\
+ ||\nabla \phi^j - \nabla \phi^{j-1}||^2_{H^1} + ||\nabla(\phi^{j+1} - \phi^j)||^2_{H^1} + ||\nabla^2(\phi^{j+1} - \phi^j)||^2_{H^1} \\
+ ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1},
\end{align}

which is equivalent to
\begin{align}
\frac{1}{2} \frac{d}{dt} (||\mathcal{A}^{j+1} - \mathcal{A}^j||^2_{H^1} + 1) + ||\nabla \mathcal{A}^{j+1} - \nabla \mathcal{A}^j||^2_{H^1} \\
+ ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} \\
\leq C(||\mathcal{A}^j - \mathcal{A}^{j-1}||_{H^1} + ||\mathcal{A}^j - \mathcal{A}^{j-1}||^2_{H^1} + 1) \\
\times (||\nabla \mathcal{A}^{j+1} - \nabla \mathcal{A}^j||^2_{H^1} + ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} \\
+ C(||\mathcal{A}^j||^2_{H^1} + ||\mathcal{A}^j||_{H^1})(||\nabla \mathcal{A}^{j+1} - \nabla \mathcal{A}^j||^2_{H^1} \\
+ ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1}),
\end{align}

On the basis of (58), by taking time integration, it holds that
\begin{align}
|||\mathcal{A}^{j+1} - \mathcal{A}^j||^2_{H^1} + 1 \\
+ \int_0^t (||\nabla \mathcal{A}^{j+1} - \nabla \mathcal{A}^j||^2_{H^1} + ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} ) d\tau \\
\leq C M_1^2 \sup_{0 \leq \tau \leq T_1} (||\mathcal{A}^j - \mathcal{A}^{j-1}||^2_{H^1} + 1) \\
+ C(M_1 + M_1^2) \int_0^t (||\nabla \mathcal{A}^{j+1} - \nabla \mathcal{A}^j||^2_{H^1} + ||\nabla^2(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} + ||\nabla(\nabla \phi^{j+1} - \nabla \phi^j)||^2_{H^1} ) d\tau.
\end{align}

By the smallness of $M_1$, there exists a constant $\lambda \in (0, 1)$ such that
\begin{align}
\sup_{0 \leq \tau \leq T_1} ||\mathcal{A}^{j+1}(\tau) - \mathcal{A}^j(\tau)||^2_{H^1} \leq \lambda \sup_{0 \leq \tau \leq T_1} ||\mathcal{A}^j(\tau) - \mathcal{A}^{j-1}(\tau)||^2_{H^1},
\end{align}
exists in $C([0,T_1];H^1)$, the limit function

$$A(t) = A_0 + \lim_{n \to \infty} \sum_{j=0}^{n} (A^{j+1} - A^j)$$

exists in $C([0,T_1];H^1)$, satisfies

$$\sup_{0 \leq t \leq T_1} \| A(t) \|^2_{H^1} \leq \sup_{0 \leq t \leq T_1} \liminf_{j \to \infty} \| A(t) \|_{H^1} \leq M_1.$$ 

Thus, (49) is proved. Finally, Let $A(t)$ and $\tilde{A}(t)$ are two solutions in $C([0,T_1];H^1)$ satisfying (49). By using the same process as in (62) to prove the convergence of $(A^j)_{j \geq 0}$, we obtain

$$\sup_{0 \leq t \leq T_1} \| A(t) - \tilde{A}(t) \|^2_{H^1} \leq \lambda \sup_{0 \leq t \leq T_1} \| A(t) - \tilde{A}(t) \|^2_{H^1},$$

for $\lambda \in (0,1)$, which implies that $A(t) = \tilde{A}(t)$. Then, we complete the proof of uniqueness and thus the proof of Lemma 4.1 is complete too.



\section{5. Negative Sobolev estimates}

In this section, we derive the evolution of the negative Sobolev norms of the solution to the Cauchy problem (10). In order to estimate the convective term and the double-well potential, we shall restrict ourselves to that $s \in [0,\frac{1}{2}]$.

For the homogeneous Sobolev space, the following lemma holds:

\textbf{Lemma 5.1.} Suppose that all the assumptions in Lemma 3.1 are in force. For $s \in [0,\frac{1}{2}]$, we have

$$\frac{d}{dt} \left( \|u(t)\|^2_{H^{1-s}} + \|\phi(t)\|^2_{H^{1-s}} + \|\nabla \phi(t)\|^2_{H^{1-s}} \right)$$

$$+ \|\nabla u(t)\|^2_{H^{1-s}} + \|\nabla \phi(t)\|^2_{H^{1-s}} + \|\nabla^2 \phi(t)\|^2_{H^{1-s}} + \|\nabla^2 \nabla \phi(t)\|^2_{H^{1-s}}$$

$$\leq E(t)(\|u(t)\|_{H^{1-s}} + \|\phi(t)\|_{H^{1-s}} + \|\nabla \phi(t)\|_{H^{1-s}}),$$

where

$$E(t) = \|\nabla u\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2} + \|\nabla^3 u\|_{L^2} + \|\nabla \phi\|^2_{L^2} + \|\nabla^2 \phi\|^2_{L^2} + \|\nabla^3 \phi\|^2_{L^2}.$$ 

\textbf{Proof.} Applying $\Lambda^{-s}$ to (19), multiplying the resulting identities by $\Lambda^{-s} u$, and then integrating over $\mathbb{R}^3$ by parts, we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\Lambda^{-s} u|^2 dx + \int_{\mathbb{R}^3} |\Lambda^{-s} \nabla u|^2 dx$$

$$= - \int_{\mathbb{R}^3} \Lambda^{-s} (u \cdot \nabla u) \cdot \Lambda^{-s} u dx - \int_{\mathbb{R}^3} \Lambda^{-s} \left[ \nabla \cdot (\nabla \phi \otimes \nabla \phi) \right] : \Lambda^{-s} u dx$$

$$=: K_1 + K_2.$$
If \( s \in [0, \frac{1}{2}] \), then \( \frac{1}{2} + \frac{s}{3} < 1 \) and \( \frac{2}{3} \geq 6 \). By using the estimate (17) of Riesz potential in Lemma 2.2 together with Hölder’s inequality and Young’s inequality, we have

\[
K_1 = \int_{\mathbb{R}^3} \Lambda^{-s}(u \cdot \nabla u) \cdot \Lambda^{-s} u dx
\leq \|\Lambda^{-s}(u \cdot \nabla u)\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|u \cdot \nabla u\|_{L^{2+\frac{s}{3}}} \|\Lambda^{-s} u\|_{L^2}
\leq \|u\|_{L^2} \|\nabla u\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|\nabla u\|_{L^2}^{\frac{s}{3}+s} \|\nabla^2 u\|_{L^2}^{\frac{s}{3}-s} \|\nabla u\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|\Lambda^{-s} u\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2),
\]

and

\[
K_2 = \int_{\mathbb{R}^3} \Lambda^{-s} \nabla \cdot (\nabla \phi \otimes \nabla \phi) \cdot \Lambda^{-s} u dx
\leq \|\Lambda^{-s} \nabla \cdot (\nabla \phi \otimes \nabla \phi)\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|\nabla \cdot (\nabla \phi \otimes \nabla \phi)\|_{L^{2+\frac{s}{3}}} \|\Lambda^{-s} u\|_{L^2}
\leq \|\nabla \phi\|_{L^2} \|\nabla^2 \phi\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|\nabla^2 \phi\|_{L^2}^{\frac{s}{3}+s} \|\nabla^2 \phi\|_{L^2}^{\frac{s}{3}-s} \|\nabla^2 \phi\|_{L^2} \|\Lambda^{-s} u\|_{L^2}
\leq \|\Lambda^{-s} u\|_{L^2} (\|\nabla^2 \phi\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2).
\]

Combining (64)-(66) together gives

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \|\Lambda^{-s} u\|_{L^2}^2 + \int_{\mathbb{R}^3} \|\nabla u\|_{L^2}^2
\leq \|\Lambda^{-s} u\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2).
\]

Applying \( \Lambda^{-s} \) to (19)_3, multiplying the resulting identities by \( \Lambda^{-s} \phi \), integrating over \( \mathbb{R}^3 \) by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^{-s} \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla \phi\|_{L^2}^2
= - \int_{\mathbb{R}^3} \Lambda^{-s} (u \cdot \nabla \phi) \cdot \Lambda^{-s} \phi dx + \int_{\mathbb{R}^3} \Lambda^{-s} \Delta (\phi^3 - 2\phi) \cdot \Lambda^{-s} \phi dx
=: K_3 + K_4.
\]

Using the estimate (17) of Riesz potential in Lemma 2.2 together with Hölder’s inequality and Young’s inequality, we obtain

\[
K_3 = \int_{\mathbb{R}^3} \Lambda^{-s} (u \cdot \nabla \phi) \cdot \Lambda^{-s} \phi dx
\leq \|\Lambda^{-s} (u \cdot \nabla \phi)\|_{L^2} \|\Lambda^{-s} \phi\|_{L^2}
\leq \|u \cdot \nabla \phi\|_{L^{2+\frac{s}{3}}} \|\Lambda^{-s} \phi\|_{L^2}
\leq \|u\|_{L^2} \|\nabla \phi\|_{L^2} \|\Lambda^{-s} \phi\|_{L^2}
\leq \|\nabla u\|_{L^2}^{\frac{s}{3}+s} \|\nabla^2 u\|_{L^2}^{\frac{s}{3}-s} \|\nabla \phi\|_{L^2} \|\Lambda^{-s} \phi\|_{L^2}
\leq \|\Lambda^{-s} \phi\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2),
\]
and

$$K_4 = \int_{\mathbb{R}^3} \Lambda^{-s} \Delta (\phi^3 - 2\phi) \cdot \Lambda^{-s} \phi \, dx$$

$$= \int_{\mathbb{R}^3} \Lambda^{-s} \Delta [\phi (\phi + \sqrt{2})(\phi - \sqrt{2})] \cdot \Lambda^{-s} \phi \, dx$$

$$= - \int_{\mathbb{R}^3} \Lambda^{-s} \nabla [\phi (\phi + \sqrt{2})(\phi - \sqrt{2})] \cdot \Lambda^{-s} \nabla \phi \, dx$$

$$= - \int_{\mathbb{R}^4} \Lambda^{-s} [\phi - \sqrt{2})\phi \nabla \phi + (\phi + \sqrt{2})\phi \nabla \phi + (\phi - \sqrt{2})(\phi + \sqrt{2}) \nabla \phi] \cdot \Lambda^{-s} \nabla \phi \, dx$$

(70)

$$\lesssim \left( \|\Lambda^{-s}[(\phi - \sqrt{2})\phi \nabla \phi]\|_{L^2} + \|\Lambda^{-s}[(\phi + \sqrt{2})\phi \nabla \phi]\|_{L^2} \\
+ \|\Lambda^{-s}[(\phi - \sqrt{2})(\phi + \sqrt{2}) \nabla \phi]\|_{L^2} \right) \|\Lambda^{-s} \nabla \phi\|_{L^2}$$

$$\lesssim \left( \|\phi - \sqrt{2}\|_{L^\infty} \|\phi\|_{L^4} \|\nabla \phi\|_{L^2} \\
+ \|\phi + \sqrt{2}\|_{L^\infty} \|\phi\|_{L^4} \|\nabla \phi\|_{L^2} \right) \|\Lambda^{-s} \nabla \phi\|_{L^2}$$

$$=: (K_{41} + K_{42} + K_{43}) \|\Lambda^{-s} \nabla \phi\|_{L^2}.$$  

Note that

$$K_{41} = \|\phi - \sqrt{2}\|_{L^\infty} \|\phi\|_{L^4} \|\nabla \phi\|_{L^2}$$

(71)

$$\lesssim \|\phi - \sqrt{2}\|_{L^\infty} \|\phi\|_{L^4} \|\nabla \phi\|_{L^2}$$

where we have used (10) in the above inequality. Similarly, we also have

(72)

$$K_{42} + K_{43} \lesssim \|\nabla \phi\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2.$$  

Plugging the estimates (70)-(72) together gives

(73)

$$K_4 \lesssim \left( \|\nabla \phi\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2 \right) \frac{1}{2} \|\Lambda^{-s} \nabla \phi\|_{L^2}.$$  

It then follows from (68), (69) and (73) that

(74)

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{-s} \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla^2 \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla \phi\|_{L^2}^2$$

$$\lesssim \|\Lambda^{-s} \phi\|_{L^2} \left( \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 \right) + \|\Lambda^{-s} \nabla \phi\|_{L^2} \left( \|\nabla \phi\|_{L^2}^2 + \|\nabla^2 \phi\|_{L^2}^2 \right).$$

Applying $\Lambda^{-s} \nabla$ to (19), multiplying the resulting identities by $\Lambda^{-s} \nabla \phi$, integrating over $\mathbb{R}^3$ by parts, we obtain

(75)

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{-s} \nabla \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla^2 \phi\|_{L^2}^2 + \|\Lambda^{-s} \nabla \phi\|_{L^2}^2$$

$$= - \int_{\mathbb{R}^3} \Lambda^{-s} \nabla (u \cdot \nabla \phi) \cdot \Lambda^{-s} \nabla \phi \, dx + \int_{\mathbb{R}^3} \Lambda^{-s} \nabla^3 (\phi^3 - 2\phi) \cdot \Lambda^{-s} \nabla \phi \, dx$$

$$=: K_5 + K_6.$$
Through the estimate (77), Hölder’s inequality and Young’s inequality, we derive that

\[
K_5 = -\int_{\mathbb{R}^3} \Lambda^{-s} \nabla(u \cdot \nabla \phi) \cdot \Lambda^{-s} \nabla \phi \, dx
\leq ||\Lambda^{-s} \nabla(u \cdot \nabla \phi)||_{L^2} ||\Lambda^{-s} \nabla \phi||_{L^2}
\leq ||\nabla(u \cdot \nabla \phi)||_{L^\frac{4}{3+s}} ||\Lambda^{-s} \nabla \phi||_{L^2}
\lesssim (||\nabla u||_{L^\frac{4}{3}} ||\nabla \phi||_{L^2} + ||u||_{L^\frac{4}{3}} ||\nabla^2 \phi||_{L^2}) ||\Lambda^{-s} \nabla \phi||_{L^2}
\lesssim (||\nabla^2 u||_{L^\frac{4}{3+s}} ||\nabla^3 \phi||_{L^2} + ||\nabla u||_{L^\frac{4}{3+s}} ||\nabla^2 u||_{L^\frac{4}{3+s}} ||\nabla^2 \phi||_{L^2}) ||\Lambda^{-s} \nabla \phi||_{L^2}
\lesssim ||\Lambda^{-s} \nabla \phi||_{L^2} (||\nabla u||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2 + ||\nabla^3 \phi||_{L^2}^2 + ||\nabla \phi||_{L^2}^2 + ||\nabla^2 \phi||_{L^2}^2).
\]

Moreover, suppose that 0 \leq l \leq 3 and 0 \leq m \leq l, let a, b, c \in \mathbb{Z}^+ and

\[
a = \min\{m, l - m, 3 - l\}, \quad b = \max\{m, l - m, 3 - l\}, \quad a \leq c \leq b.
\]

Simple calculation shows that a \leq 1. We estimate K_6 as

\[
K_6 = \int_{\mathbb{R}^3} \Lambda^{-s} \nabla^3 (\phi^3 - 2\phi) \cdot \Lambda^{-s} \nabla \phi \, dx
= \int_{\mathbb{R}^3} \Lambda^{-s} \nabla^3 [\phi + \sqrt{2}(\phi - \sqrt{2})] \cdot \Lambda^{-s} \nabla \phi \, dx
\lesssim ||\Lambda^{-s} \nabla \phi||_{L^2} ||\Lambda^{-s} \nabla^3 [\phi + \sqrt{2}(\phi - \sqrt{2})]||_{L^2}
\lesssim ||\Lambda^{-s} \nabla \phi||_{L^2} ||\nabla^3 [\phi + \sqrt{2}(\phi - \sqrt{2})]||_{L^\frac{4}{3+s}}
\lesssim \sum_{0 \leq l \leq 3} \sum_{0 \leq m \leq l} ||\Lambda^{-s} \nabla \phi||_{L^2} ||\nabla^a \phi||_{L^\infty} ||\nabla^b (\phi - \sqrt{2})||_{L^2} ||\nabla^c (\phi + \sqrt{2})||_{L^\frac{4}{3+s}}
\lesssim \sum_{0 \leq l \leq 3} \sum_{0 \leq m \leq l} ||\Lambda^{-s} \nabla \phi||_{L^2} ||\nabla^{a+1} \phi||_{L^2} ||\nabla^{a+2} \phi||_{L^2} ||\nabla^{a+3} \phi||_{L^2} ||\nabla^{c+1} \phi||_{L^\frac{4}{3+s}} ||\nabla^{c+2} \phi||_{L^\frac{4}{3+s}}
\lesssim ||\Lambda^{-s} \nabla \phi||_{L^2} (||\nabla \phi||_{L^2}^2 + ||\nabla^2 \phi||_{L^2}^2 + ||\nabla^3 \phi||_{L^2}^2)^{\frac{1}{2}}.
\]

It follows from (75)-(77) that

\[
\frac{1}{2} \frac{d}{dt} ||\Lambda^{-s} \nabla \phi||_{L^2}^2 + ||\Lambda^{-s} \nabla^2 \phi||_{L^2}^2 + ||\Lambda^{-s} \nabla \phi||_{L^2}^2
\lesssim ||\Lambda^{-s} \nabla \phi||_{L^2} (||\nabla u||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2 + ||\nabla^3 \phi||_{L^2}^2 + ||\nabla \phi||_{L^2}^2 + ||\nabla^2 \phi||_{L^2}^2 + ||\nabla^3 \phi||_{L^2}^2).
\]

Combining (67), (74) and (78) together, we obtain (63) and hence the proof is completed.

\[\square\]

6. Proof of Theorem 1.3

In this section, we shall combine all the energy estimates that we have derived in the previous sections and the Sobolev interpolation to prove Theorem 1.3.

We first close the energy estimates at each l-th level in our weak sense to prove (10). Let \( N \geq 1 \) and 0 \( \leq l \leq m - 1 \) with 1 \( \leq m \leq N \). Summing up the estimates (21) of Lemma 3.1 from
\( k = l \) to \( m \), we easily obtain
\[
\frac{d}{dt} \sum_{l \leq k \leq m} (\| \nabla^k u \|_{L^2}^2 + \| \nabla^k \phi \|_{L^2}^2 + \| \nabla^k \nabla \phi \|_{L^2}^2)
\]
\[+ C_1 \sum_{l \leq k \leq m} (\| \nabla^{k+1} u \|_{L^2}^2 + \| \nabla^{k+1} \phi \|_{L^2}^2 + \| \nabla^{k+2} \phi \|_{L^2}^2 + \| \nabla^{k+3} \phi \|_{L^2}^2)
\]
\[\leq C_2 (\delta + \delta^2) \sum_{l \leq k \leq m} (\| \nabla^{k+1} u \|_{L^2}^2 + \| \nabla^{k+1} \phi \|_{L^2}^2 + \| \nabla^{k+2} \phi \|_{L^2}^2 + \| \nabla^{k+3} \phi \|_{L^2}^2).
\]
Since \( \delta > 0 \) is small, we deduce that there exists a constant \( C_3 > 0 \) such that for \( 0 \leq l \leq m - 1 \),
\[
\frac{d}{dt} \sum_{l \leq k \leq m} (\| \nabla^k u \|_{L^2}^2 + \| \nabla^k \phi \|_{L^2}^2 + \| \nabla^k \nabla \phi \|_{L^2}^2)
\]
\[+ C_3 \sum_{l \leq k \leq m} (\| \nabla^{k+1} u \|_{L^2}^2 + \| \nabla^{k+1} \phi \|_{L^2}^2 + \| \nabla^{k+2} \phi \|_{L^2}^2 + \| \nabla^{k+3} \phi \|_{L^2}^2) \leq 0.
\]
Define \( \mathcal{E}^m_l(t) \) to be \( \frac{1}{C_3} \) times the expression under the time derivative in (80). Hence, we may write (80) as that for \( 0 \leq l \leq m - 1 \),
\[
\frac{d}{dt} \mathcal{E}^m_l(t) + \| \nabla^{l+1} u \|_{H^{m-l}}^2 + \| \nabla^{l+1} \phi \|_{H^{m-l}}^2 + \| \nabla^{l+2} \phi \|_{H^{m-l}}^2 + \| \nabla^{l+3} \phi \|_{H^{m-l}}^2 \leq 0.
\]
Taking \( l = 0 \) and \( m = 1 \) in (81) and integrating directly in time, we deduce that
\[
\| u(t) \|_{H^1}^2 + \| \phi(t) \|_{H^1}^2 + \| \nabla \phi(t) \|_{H^1}^2 \leq \mathcal{E}^1_0(0) \leq \| u_0 \|_{H^1}^2 + \| \phi_0 \|_{H^1}^2 + \| \nabla \phi_0 \|_{H^1}^2.
\]
Then by a standard continuity argument, this closes the a priori estimates (9) if at the initial time \( \| u_0 \|_{H^1}^2 + \| \phi_0 \|_{H^1}^2 + \| \nabla \phi_0 \|_{H^1}^2 \) is sufficiently small. This in turn allows us to take \( l = 0 \) and \( m = N \) in (82), and then integrate it directly in time to obtain (10).

In the following, we prove (11)-(12) for \( s \in [0, \frac{1}{2}] \). Define
\[ \mathcal{E}_{-s}(t) := \| \Lambda^{-s} u(t) \|_{L^2}^2 + \| \Lambda^{-s} \phi(t) \|_{L^2}^2 + \| \Lambda^{-s} \nabla \phi(t) \|_{L^2}^2. \]
For inequality (83), integrating in time, by the bound (10), we have
\[
\mathcal{E}_{-s}(t) \leq \mathcal{E}_{-s}(0) + C \int_0^t E(\tau) \sqrt{\mathcal{E}_{-s}(\tau)} d\tau
\]
\[\leq C_0 \left( 1 + \sup_{0 \leq \tau \leq t} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \right),
\]
which implies (11) for \( s \in [0, \frac{1}{2}] \), that is
\[
\| \Lambda^{-s} u(t) \|_{L^2}^2 + \| \Lambda^{-s} \phi(t) \|_{L^2}^2 + \| \Lambda^{-s} \nabla \phi(t) \|_{L^2}^2 \leq C_0.
\]
Moreover, if \( l = 1, 2, \cdots, N - 1 \), we may use Lemma 2.3 to have
\[ \| \nabla^{l+1} f \|_{L^2} \geq C \| \Lambda^{-s} f \|_{L^2}^2 + \| \nabla f \|_{L^2}^{1+\frac{1}{m-s}}. \]
Then, by this facts and (84), we get
\[
\| \nabla^{l+1} (u, \phi, \nabla \phi) \|_{L^2}^2 \geq C_0 (\| \nabla^l (u, \phi, \nabla \phi) \|_{L^2}^2)^{1+\frac{1}{m-s}}.
\]
Hence, for \( 1, 2, \cdots, N - 1 \),
\[ \| \nabla^{l+1} (u, \phi, \nabla \phi) \|_{H^{N-l-1}}^2 \geq C_0 (\| \nabla^l (u, \phi, \nabla \phi) \|_{H^{N-l}}^2)^{1+\frac{1}{m-s}}. \]
Thus, we deduce from (81) with $m = N$ the following inequality
\begin{equation}
\frac{d}{dt} E_l^N + C_0 (E_l^N)^{1 + \frac{1}{l+1}} \leq 0, \quad \text{for } l = 1, 2, \cdots, N - 1.
\end{equation}
Solving this inequality directly gives
\begin{equation}
E_l^N(t) \leq C_0 (1 + t)^{-l-s}, \quad \text{for } l = 1, 2, \cdots, N - 1,
\end{equation}
which means (12) holds. Hence, we complete the proof of Theorem 1.3.

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