REDUCIBILITY OF THREE DIMENSIONAL SKEW SYMMETRIC SYSTEM WITH
LIOUVILLEAN BASIC FREQUENCIES

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Abstract. In this paper we consider the system $\dot{x} = (A(\epsilon) + \epsilon^m P(t; \epsilon))x$, $x \in \mathbb{R}^3$, where $\epsilon$ is a small parameter, $A, P$ are all $3 \times 3$ skew symmetric matrices, $A$ is a constant matrix with eigenvalues $\pm i\bar{\lambda}(\epsilon)$ and 0, where $\bar{\lambda}(\epsilon) = \lambda + a_{m_0} \epsilon^{m_0} + O(\epsilon^{m_0+1})(m_0 < m)$, $a_{m_0} \neq 0$, $P$ is a quasi-periodic matrix with basic frequencies $\omega = (1, \alpha)$ with $\alpha$ being irrational. First, it is proved that for most of sufficiently small parameters, this system can be reduced to a rotation system. Furthermore, if the basic frequencies satisfy that $0 \leq \beta(\alpha) < r$, where $\beta(\alpha)$ measures how Liouvillean $\alpha$ is, $r$ is the initial analytic radius, it is proved that for most of sufficiently small parameters, this system can be reduced to constant system by means of a quasi-periodic change of variables.

1. Introduction. In this paper we consider the following system:

$$\dot{x} = (A(\epsilon) + \epsilon^m P(t; \epsilon))x, \ x \in \mathbb{R}^3,$$

where $\epsilon$ is a small parameter, $A, P$ are all $3 \times 3$ skew symmetric matrices, $A(\epsilon)$ is a constant matrix with eigenvalues $\pm i\bar{\lambda}(\epsilon)$ and 0, where $\bar{\lambda}(\epsilon) = \lambda + a_{m_0} \epsilon^{m_0} + O(\epsilon^{m_0+1})(m_0 < m)$, $a_{m_0} \neq 0$, $P$ is a quasi-periodic matrix with basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$.

When $m_0 = 0, m = 1, \bar{\lambda}(\epsilon) = \lambda$, Eliasson [10] proved that for most of sufficiently small parameters, analytic quasi-periodic skew symmetric system (1) can be reduced to constant system, if the eigenvalues $\pm i\lambda$ and basic frequencies $\omega$ satisfy the non-resonant condition:

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \ \forall 0 \neq k \in \mathbb{Z}^s,$$

and

$$|\langle k, \omega \rangle - 2\lambda| \geq \frac{\gamma}{|k|^\tau}, \ \forall 0 \neq k \in \mathbb{Z}^s,$$

where $\gamma > 0, \tau > s - 1$ are constants. Moreover, Eliasson [10] showed that such system is generically uniquely ergodic in a topological sense and in particular non-reducible.

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In this paper, we are mainly concerned with the reducibility problems for analytic quasi-periodic skew symmetric system (1) with Liouvillean basic frequencies, i.e., $\omega = (1, \alpha)$, where $\alpha$ is irrational, which means that the Diophantine condition (2) can be eliminated. When $m_0 > 0 (m > m_0)$, our non-degeneracy condition is more weaker. Moreover, the second Melnikov condition is not needed for the reducibility of three dimensional skew symmetric system, we only need the first Melnikov condition.

Let us first recall some well known results and development of reduction theory. Consider the system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n \tag{3}$$

where $A(t)$ is an $n \times n$ matrix which depends on time in a quasi-periodic way with basic frequencies $\omega = (\omega_1, \ldots, \omega_s)$.

**Definition 1.1.** (Reducibility): System (3) is said to be reducible, if there exists a nonsingular quasi-periodic change of variables $x = \Phi(t)y$, such that $\Phi(t)$, $\Phi(t)^{-1}$ and $\dot{\Phi}(t)$ are quasi-periodic and bounded, and such that it transforms the system (3) into a constant system, i.e., a linear system with constant coefficient.

**Definition 1.2.** (Rotations reducibility): System (3) is said to be analytically rotations reducible, if there exists an analytic quasi-periodic transformation $x = \Phi(t)y$, such that system (3) is transformed into a rotation system, i.e., a linear system with so(3, $\mathbb{R}$)-valued coefficients.

**Definition 1.3.** (Non-perburbative reducibility): The non-perburbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constants $(\gamma, \tau)$ of $\omega$ in (2).

For $s = 1$, i.e., periodic case, the classical Floquet theory tells us that there exists a periodic change of variables such that the transformed system is a constant system.

For $s > 1$, i.e., quasi-periodic case, the system is not always reducible. The reducibility of quasi-periodic systems was initiated by Dinaburg and Sinai [7], who proved that the linear Schrödinger equation

$$-y'' + q(\omega t)y = Ey, \quad y \in \mathbb{R} \tag{4}$$

or equivalently the following two-dimensional quasi-periodic system:

$$\dot{x} = y, \quad \dot{y} = (q(\omega t) - E)x \tag{5}$$

is reducible for most of sufficiently large $E$, where the basic frequencies $\omega$ satisfy the Diophantine condition:

$$|\langle k ; \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \quad \forall 0 \neq k \in \mathbb{Z}^s, \tag{6}$$

where $\gamma > 0$, $\tau > s - 1$ are constants.

The reducibility of system (5) implies the existence of absolutely continuous spectrum of the Schrödinger operator $L y = -\frac{d^2}{dt^2} + q(\omega t)y$. Due to its importance in dynamical systems and the greatest relevance in the spectral theory of Schrödinger operator, the reducibility of quasi-periodic systems has been extensively investigated.

Liang-Xu [19] generalized the above results to high dimensional case. Johnson and Sell [15] proved that if the quasi-periodic coefficients matrix $A(t)$ satisfies full
spectrum condition, the system (3) is reducible. Jorba and Simó [16] considered
the following linear differential equations:
\[
\dot{x} = (A + \epsilon P(t, \epsilon)) x, \quad x \in \mathbb{R}^n,
\]
where \( A \) is a constant matrix with different nonzero eigenvalues \( \lambda_1, \ldots, \lambda_n \), \( P(t) \) is a
quasi-periodic matrix with frequencies \( \omega = (\omega_1, \ldots, \omega_s) \), and \( \epsilon \) is a small parameter.
They proved that if
\[
|i(k, \omega) + \lambda_i - \lambda_j| \geq \frac{\gamma}{|k|^\tau}, \quad \forall 0 \neq k \in \mathbb{Z}^s, \quad i, j = 1, \ldots, n,
\]
and
\[
\frac{d}{d\epsilon} (\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon)) \bigg|_{\epsilon=0} \neq 0, \quad i \neq j,
\]
where \( \gamma > 0, \tau > s - 1 \) are constants, \( \bar{\lambda}_i(\epsilon)(i = 1, \ldots, n) \) are eigenvalues of \( A + \epsilon \bar{P}(t, \epsilon) \), \( \bar{P}(t, \epsilon) \) is the average of \( P(t, \epsilon) \) with respect to \( t \), then for most of sufficiently
small parameters \( \epsilon \), the system (7) is reducible. In [17], Jorba and Simó extended
the results of linear system to the nonlinear system, i.e.,
\[
\dot{x} = Ax + h(x, t) + f(x, t; \epsilon), \quad x \in \mathbb{R}^n,
\]
where \( A \) is a constant matrix with different nonzero eigenvalues, \( h = O(x^2) \) as \( x \to 0 \), \( f(x, t; \epsilon) \) is a small perturbation with \( \epsilon \) as a small parameter, \( h \) and \( f \) are
quasi-periodic in \( t \) with frequencies \( \omega = (\omega_1, \ldots, \omega_s) \).

Later, Eliasson [9] proved that all quasi-periodic systems are almost reducible
provided that the system satisfies Diophantine condition and is close to constant.
Eliasson [8] obtained a full measure reducibility result for quasi-periodic Schrödinger
equation. Krikorian [18] generalized the full measure reducibility result to linear
systems with coefficients in Lie algebra of compact semi-simple Lie group. Her-You [13] and Chavaudret [5] established the full measure reducibility with coefficients
in other groups. For the latest reducibility results of resonant cocycles and infi-
nite dimensional quasi-periodic systems, we refer to [2, 3, 6, 11] and the references
therein.

During the development of reducibility of quasi-periodic systems and KAM theory,
a lot of scholars are dedicated to weakening the non-degeneracy condition and non-resonant condition. Xu [28, 29] obtained the reducibility of linear quasi-periodic
system (7) in the case of multiple eigenvalues and more general non-degeneracy conditions, i.e.,
\[
\frac{d^l}{d\epsilon^l} (\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon)) \bigg|_{\epsilon=0} \neq 0, \quad l \geq 1, \quad i \neq j.
\]
Moreover, the Diophantine condition (6) can also be weakened. Rüssmann [23] and
Zhang-Liang [34] obtained the reducibility of Schrödinger equation under Brjuno-Rüssmann non-resonant condition:
\[
|i(k, \omega)| \geq \frac{\gamma}{\Delta(|k|)}, \quad \forall 0 \neq k \in \mathbb{Z}^s,
\]
where \( \gamma > 0, \) and \( \Delta \) is continuous, increasing, unbounded function \( \Delta : [1, \infty) \to [1, \infty) \) such that \( \Delta(1) = 1 \) and
\[
\int_1^\infty \frac{\ln \Delta(t)}{t^2} dt < \infty.
\]
\( \Delta \) is usually called Brjuno-Rüssmann approximation function.
In particular, there have been some interesting aspects for 2-dimensional quasi-
periodic systems. Without imposing any non-degeneracy condition, the reducibility
of two dimensional quasi-periodic system was obtained in [26, 30]. The idea is that if the non-degeneracy condition never occurs, then the non-resonant conditions are always the initial ones; if the non-degeneracy condition appears at some step, it can be kept on in the later steps. These particular phenomena [26, 30] are the nature of 2-dimensional system, which don’t hold in the high-dimensional case.

Moreover, the reducibility of quasi-periodic systems with Liouvillean frequencies has been obtained, i.e., \( \omega = (1, \alpha) \), where \( \alpha \) ia irrational. A. Avila, B. Fayad and R. Krikorian [1] first introduced the method of CD bridge and proved the rotation reducibility of \( SL(2, \mathbb{R}) \) cocycles with one frequency, irrespective of any Diophantine condition on the base dynamics. Zhou and Wang [36] used periodic approximation to study the reducibility of quasi-periodic \( GL(d, \mathbb{R}) \) cocycles with Liouvillean frequencies, which can be viewed as a generalization of [1] to high dimension. Hou and You [14] considered a quasi-periodic linear differential system with two frequencies in \( sl(2, \mathbb{R}) \):

\[
\begin{align*}
\dot{x} &= A(\theta)x, \\
\dot{\theta} &= \omega = (1, \alpha),
\end{align*}
\]

and obtained almost reducibility and rotation reducibility of the above system, provided that the coefficients are analytic and close to constant. Furthermore, if the rotation number of the system and the basic frequencies \( \omega = (1, \alpha) \) satisfy Diophantine condition, the system is reducible. For other interesting results for 2-dimensional systems, we refer to [12, 24] and the references therein. The proof of [14] is based on KAM method, the crucial observation is to analyze the structure of resonant terms, then to eliminate them by Floquet theory. Wang-You-Zhou [25] proved the existence of response solution for quasi-periodically forced harmonic oscillator

\[
\ddot{x} + \lambda^2 x = f(\omega t, x),
\]

with forcing frequencies \( \omega = (1, \alpha) \), where \( \alpha \) is irrational. We [35] obtained the existence of a pair of conjugate quasi-periodic solutions of high dimensional Schrödinger equation with Liouvillean basic frequencies.

These particular phenomena make us realize that the 2-dimensional quasi-periodic system is very special. In this paper we will consider the reducibility for three dimensional skew symmetric system (1) with Liouvillean basic frequencies \( \omega = (1, \alpha) \), where \( \alpha \) is irrational. As the matrix \( A \) has eigenvalues \( \pm i \lambda(\epsilon) \) and 0, the system (1) looks like a two-dimensional system. But we don’t know how to define the rotation number of three dimensional skew symmetric system, the proof method in [14] can’t be adopted, so we consider the system (1) with small parameter.

The quasi-periodic systems in [1, 14] are two dimensional, one can introduce the rotation number naturally and make good use of the property of rotation number. But for three dimensional skew symmetric system (1), it is difficult to define rotation number, which yields that there are essential obstructions in applying the methods in [1, 14] to high dimensional situation. Comparing to [36], the former is discrete case, but this paper is continuous case. The method of periodic approximation for discrete case can’t be adopted to continuous case. Comparing to [25, 35], our proof of this paper is reduced to the reducibility of a class of skew symmetric systems, so that the skew symmetric structure need to be preserved in every KAM steps. Moreover, the system (1) is a more general non-degeneracy quasi-periodic system, so KAM iteration is more complicated. Therefore, our non-degeneracy condition and non-resonant condition for the system are more optimized.
Before stating our results, we first give some notations and definitions. Usually, denote by \( \mathbb{Z} \) and \( \mathbb{Z}_+ \) the sets of integers and positive integers respectively. Denote by \( so(3,\mathbb{R}) \) the set of \( 3 \times 3 \) skew symmetric real matrices.

A function \( f(t) \) is called quasi-periodic with basic frequencies \( \omega_1,\ldots,\omega_s \), if \( f(t) = F(\theta) = F(\theta_1,\ldots,\theta_s) \), where \( F \) is \( 2\pi \)-periodic in all its arguments and \( \theta_j = \omega_j t \) for \( j=1,\ldots,s \). Let \( \omega = (\omega_1,\ldots,\omega_s) \). Thus, \( f(t) = F(\omega t) \). Denote a strip domain of complex space \( \mathbb{C}^s \) by

\[
D(r) = \{ \theta = (\theta_1,\ldots,\theta_s) \in \mathbb{C}^s : |\text{Im}\theta_j| \leq r, j = 1,\ldots,s \}.
\]

Furthermore, if \( F(\theta) \) is analytic with respect to \( \theta \) on \( D(r) \), we say that \( f(t) \) is analytic quasi-periodic on \( D(r) \). Denote by

\[
[f] = \frac{1}{(2\pi)^s} \int_{T_s} F(\theta) d\theta
\]

the average of \( f \). Similarly, a function matrix \( P(t) = (P_{ij})_{n \times n} \) is called analytic quasi-periodic on \( D(r) \), if all \( P_{ij}(t) \) are analytic quasi-periodic on it. Denote by

\[
[P] = ([P_{ij}])_{n \times n}
\]

the average of \( P \).

Let \( \delta > 0 \) and denote by \( \Pi_\delta = (0,\delta] \). If \( f(\epsilon) \) is differentiable with respect to parameters \( \epsilon \in \Pi_\delta \) in the sense of Whitney, define the norm:

\[
|f|_{\Pi_\delta} = \sup_{\epsilon \in \Pi_\delta}(|f(\epsilon)| + |\epsilon f'(\epsilon)|).
\]

If a function

\[
f(t;\epsilon) = \sum_{k \in \mathbb{Z}^s} f_k(\epsilon) e^{i(k,\omega)t}
\]

is analytic quasi-periodic in \( t \) on \( D(r) \), and differentiable with respect to \( \epsilon \in \Pi_\delta \), we define

\[
\|f\|_{r,\Pi_\delta} = \sum_{k \in \mathbb{Z}^s} |f_k(\epsilon)|_{\Pi_\delta} e^{|k|r},
\]

where \( |k| = |k_1| + \cdots + |k_s| \) for \( k = (k_1,\ldots,k_s) \). Denote by \( A_r(\Pi_\delta) = \{ f : \|f\|_{r,\Pi_\delta} < +\infty \} \), which is a Banach algebra under norm \( \| \cdot \|_{r,\Pi_\delta} \).

For any \( K > 0 \), we define the truncating operators \( T_K \) as

\[
T_K f = \sum_{k \in \mathbb{Z}^s, |k| < K} f_k(\epsilon) e^{i(k,\omega)t}
\]

and \( R_K \) as

\[
R_K f = \sum_{k \in \mathbb{Z}^s, |k| \geq K} f_k(\epsilon) e^{i(k,\omega)t}.
\]

For a function matrix \( P(t;\epsilon) = (P_{ij}(t;\epsilon))_{n \times n} \), similarly define a norm by

\[
\|P\|_{r,\Pi_\delta} = \max_{1 \leq i \leq n} \sum_{j=1}^n \|P_{ij}\|_{r,\Pi_\delta}.
\]

We have \( \|P_1 P_2\|_{r,\Pi_\delta} \leq \|P_1\|_{r,\Pi} \|P_2\|_{r,\Pi_\delta} \).

For simplicity, denote

\[
J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Let \( \alpha \in (0,1) \) be irrational and denote by \( \frac{p_n}{q_n} \) the n-th convergence of \( \alpha \). Define

\[
\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},
\]

\[
J_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Let \( \alpha \in (0,1) \) be irrational and denote by \( \frac{p_n}{q_n} \) the n-th convergence of \( \alpha \). Define

\[
\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},
\]
then $\beta(\alpha)$ measures how Liouvillean $\alpha$ is. Notice that $\beta(\alpha)$ has an equivalent definition:

$$\beta(\alpha) = \limsup_{|k| \to \infty} \left| \frac{1}{|k|} \ln \frac{1}{|e^{2\pi ik\alpha} - 1|} \right|. \quad (12)$$

**Theorem 1.4.** Consider the following system:

$$\dot{x} = (A(\epsilon) + \epsilon^m P(t; \epsilon)) x, \quad x \in \mathbb{R}^3,$$

where $\epsilon$ is a small parameter, $A, P \in \mathfrak{so}(3, \mathbb{R})$, $A(\epsilon)$ is a constant matrix with eigenvalues $\pm i \tilde{\lambda}(\epsilon)$ and 0, where $\tilde{\lambda}(\epsilon) = \lambda + a_{m_0} \epsilon^{m_0} + O(\epsilon^{m_0+1})(m_0 < m)$, $a_{m_0} \neq 0$, $P \in \mathcal{A}_r(\Pi_\delta)$ is a quasi-periodic matrix with basic frequencies $\omega = (1, \alpha)$ with $\alpha$ being irrational. Suppose that $(\omega, \lambda)$ satisfy the following non-resonant condition:

$$|\langle k, \omega \rangle - \lambda| \geq \frac{\gamma_0}{(|k| + 1)\tau}, \quad \forall k \in \mathbb{Z}^2, \quad (13)$$

where $\gamma_0 > 0$, $\tau > 2$. Then we have the following two conclusions:

(I) There exists a sufficiently small $\epsilon_0 \in (0, \delta]$, which depends on $r, \gamma_0, \tau$ and $\|P\|_{r, \Pi}$, but not on $\alpha$, and a non-empty Cantor set $\Pi_0 \subset (0, \epsilon_0)$ with positive Lebesgue measure, such that for any $\epsilon \in \Pi_0$, the system (1) is rotations reducible. Moreover, $\text{Meas}((0, \epsilon) \setminus \Pi_*) = O(\epsilon^{m_0+1}), \epsilon \to 0$.

(II) Furthermore, if $\beta(\alpha) = 0$, then for the same $\epsilon_0$ and $\epsilon \in \Pi_*$ in (I), the system (1) is reducible. If $0 < \beta(\alpha) < r$, then there exists a sufficiently small $\epsilon_0 = \epsilon_0(r, \gamma_0, \tau, \|P\|_{r, \Pi}, \beta(\alpha))$ and a non-empty Cantor set $\Pi_* \subset (0, \epsilon_0)$ with positive Lebesgue measure, such that for any $\epsilon \in \Pi_*$ the system (1) is reducible.

**Remark 1.** Another direction is to weaken the Diophantine condition (13) to the Brjuno-Rüssmann’s non-resonant condition:

$$|\langle k, \omega \rangle - \lambda| \geq \frac{\gamma_0}{\Delta(|k| + 1)}, \quad \forall k \in \mathbb{Z}^2,$$

where $\gamma_0 > 0$, and $\Delta$ is the Brjuno-Rüssmann approximation function as in (9). For the KAM theory under Brjuno-Rüssmann’s non-resonant condition, we see [4, 20, 21, 22, 27, 31, 32, 33].

**Remark 2.** When $\beta(\alpha) = 0$, the smallness of $\epsilon_0$ doesn’t depend on $\alpha$, therefore, we not only weakens the Diophantine condition (2) to Liouvillean frequencies, but also improves the results in [10] to be non-perturbative in the case of two dimensional basic frequencies. In this sense, our theorem generalizes partial conclusions of [14] to high dimension.

2. **Proof of the main results.**

2.1. **Outline of the proof.** By Lemma 3.5 in the appendix, any skew symmetric matrix with eigenvalues $\pm i \tilde{\lambda}(\epsilon)$ and 0 is similar to $\tilde{\lambda}(\epsilon) J_1$, where the definition of $J_1$ is given in (10). Without loss of generality, suppose that $A = \tilde{\lambda}(\epsilon) J_1$. We divide the proof of Theorem 1.4 into two big steps. First, we prove that the system (1) can be reduced to a linear system with non-constant coefficients, which has a diagonal form:

$$\frac{dv}{dt} = (A + B^*(\theta)) v, \quad (14)$$

where

$$B^*(\theta) = \begin{pmatrix} 0 & \Xi^*(\theta) & 0 \\ -\Xi^*(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
The first big step can be achieved by infinite steps of KAM iteration. Second, we eliminate the non-resonant terms containing \( \theta \) on the diagonal and transform the system (14) into a linear system with constant coefficients,
\[
\frac{dv^*}{dt} = A^*(\epsilon)v^*,
\]
where
\[
A^*(\epsilon) = \begin{pmatrix}
0 & \Lambda^*(\epsilon) & 0 \\
-\Lambda^*(\epsilon) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
This process can be obtained by only one step if \( 0 \leq \beta(\alpha) < r \). Thus, it follows that the system (1) is reducible.

### 2.2. Homological equation.

Before giving the proof of Theorem 1.4, we first show how to use the method of diagonally dominant to solve the homological equation
\[
\frac{dW}{dt} + W(A + B(\theta) + b(\theta)) - (A + B(\theta) + b(\theta))W = P,
\]
where \( W, P \) are all \( 3 \times 3 \) skew symmetric matrices, \( B(\theta), b(\theta) \) have the same form as \( A \), i.e.,
\[
B(\theta) = \begin{pmatrix}
0 & \Xi(\theta) & 0 \\
-\Xi(\theta) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad b(\theta) = \begin{pmatrix}
0 & \Delta(\theta) & 0 \\
-\Delta(\theta) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
For \( \tau > 2 \), we define
\[
A = \tau + 3, M = \frac{A^4}{2},
\]
and let \((Q_n)\) be the selected subsequence of \( \alpha \) in Lemma 3.2 with this given \( A \). For \( r, \gamma > 0 \), we define
\[
\eta = \frac{\bar{c}}{Q_n^{\frac{1}{2}+\gamma}}, \quad \mathcal{L} = e^{-\gamma\tau\left(\frac{\bar{Q}_n}{Q_n^{\frac{1}{2}}} + Q_n^{\frac{1}{2}+\gamma}\right)},
\]
where \( 0 < \bar{c}, \epsilon_0 < 1 \) are constants, which will be defined later.

**Lemma 2.1.** For every \( \tau > 2, r > 0, \bar{c} < 1 \), there exist \( C_3 = C_3(\tau) \) and \( \varepsilon_1 = \varepsilon_1(\tau, r, \bar{c}) \) with the following properties. For any \( 0 < \sigma < r < \bar{r} \leq r(1 - \eta) \), let \( B(\theta) \in \mathcal{A}_{r}(\Pi) \) with \( R_{Q_n+1}(\theta) = 0 \), \( b(\theta), P(\theta) \in \mathcal{A}_{\bar{r}}(\Pi) \). If
\[
|\langle k, \omega \rangle \pm (\bar{\lambda}(\epsilon) + |\Xi|)| \geq \frac{\gamma}{(|k| + 1)^\tau}, \quad \forall k \in \mathbb{Z}^2, |k| < K,
\]
\[
\|b\|_{r, \Pi} < \frac{\gamma^{A\tau+2}}{2C_3Q_n^{\bar{r}+1}},
\]
and
\[
\|B\|_{r, \Pi} \cdot (\frac{\bar{Q}_n}{Q_n^{\gamma}} + \bar{Q}_n^{\gamma}) \leq \varepsilon_1\gamma(\frac{\bar{Q}_n}{Q_n^{\gamma}} + \bar{Q}_n^{\gamma+1}),
\]
then the homological equation (15) has an approximate solution \( W(\theta; \epsilon) \) with the estimate
\[
\|W\|_{r, \Pi} \leq 2C_3(\tau)\gamma^{-(A\tau+2)}Q_n^{\bar{r}+1}\mathcal{L}^{-\frac{\gamma}{2\bar{c}}}\|P\|_{r, \Pi}.
\]
Moreover, the error term \( P_\epsilon = \langle P_{\epsilon, i,j} \rangle_{3 \times 3} \) with
\[
P_{\epsilon, i,j} = e^{-iB(\theta)}R_\mathcal{K}(e^{iB(\theta)}(P_{i,j}(\theta) + i\Delta(\theta)W_{i,j})),
\]
where \( \partial_r \mathcal{B}(\theta) = -\Xi(\theta) + [\Xi] \), satisfies

\[
\|P_r\|_{\mathcal{F}, \pi, \Pi} \leq 2L^{-\frac{1}{2}} e^{-K\sigma}(\|P\|_{\mathcal{F}, \Pi} + 2\|b\|_{\mathcal{F}, \Pi}\|W_{ij}\|_{\mathcal{F}, \Pi}).
\] (18)

Proof. Let

\[
R = \left( \begin{array}{cc} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{array} \right), \quad S = \left( \begin{array}{cc} R & 0 \\ 0 & 1 \end{array} \right).
\]

Then by

\[
R^{-1} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) R = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right),
\]

we have

\[
S^{-1}AS = \left( \begin{array}{ccc} \bar{\lambda}(\epsilon) i & 0 & 0 \\ 0 & -\bar{\lambda}(\epsilon) i & 0 \\ 0 & 0 & 0 \end{array} \right).
\]

For simplicity, denote

\[
A = S^{-1}AS, \quad B(\theta) = S^{-1}B(\theta)S, \quad b(\theta) = S^{-1}b(\theta)S,
\]

\[
W = S^{-1}WS, \quad P = S^{-1}PS.
\]

Then the equation (15) is equivalent to

\[
\frac{dW}{dt} + W(A + B(\theta) + b(\theta)) - (A + B(\theta) + b(\theta))W = P.
\] (19)

Suppose that \( P \) has the following form:

\[
P(\theta) = \left( \begin{array}{ccc} 0 & a^1(\theta) & a^2(\theta) \\ -a^1(\theta) & 0 & a^3(\theta) \\ -a^2(\theta) & -a^3(\theta) & 0 \end{array} \right),
\]

then

\[
P = S^{-1}PS = \left( \begin{array}{ccc} a^1i & 0 & P_{13} \\ 0 & -a^1i & P_{23} \\ P_{31} & P_{32} & 0 \end{array} \right),
\]

where

\[
P_{13} = \frac{1}{2}(a^2 + a^3) + \frac{i}{2}(a^2 - a^3), \quad P_{23} = \frac{1}{2}(a^2 + a^3) + \frac{i}{2}(a^3 - a^2),
\]

\[
P_{31} = -\frac{1}{2}(a^2 + a^3) + \frac{i}{2}(a^2 - a^3), \quad P_{32} = -\frac{1}{2}(a^2 + a^3) + \frac{i}{2}(a^3 - a^2).
\]

Therefore, the homological equation (19) can be written in the form of components:

\[
\frac{dW_{13}}{dt} - i(\bar{\lambda}(\epsilon) + \Xi(\theta) + \Delta(\theta))W_{13} = P_{13},
\]

\[
\frac{dW_{23}}{dt} + i(\bar{\lambda}(\epsilon) + \Xi(\theta) + \Delta(\theta))W_{23} = P_{23},
\]

\[
\frac{dW_{31}}{dt} + i(\bar{\lambda}(\epsilon) + \Xi(\theta) + \Delta(\theta))W_{31} = P_{31},
\]

\[
\frac{dW_{32}}{dt} - i(\bar{\lambda}(\epsilon) + \Xi(\theta) + \Delta(\theta))W_{32} = P_{32}.
\]

Note that the above homological equations have four cases. Let

\[
\frac{dB(\theta)}{dt} = -\Xi(\theta) + [\Xi],
\]

and

\[
\tilde{P}_{13}(\theta) = e^{i\mathcal{B}(\theta)}P_{13}(\theta), \quad \tilde{P}_{32}(\theta) = e^{i\mathcal{B}(\theta)}P_{32}(\theta),
\]
\[
\tilde{P}_{23}(\theta) = e^{-i\mathcal{B}(\theta)}P_{23}(\theta), \quad \tilde{P}_{31}(\theta) = e^{-i\mathcal{B}(\theta)}P_{31}(\theta).
\]

\[\tilde{W}_{ij} \text{ has a similar definition. Then the homological equation } (19) \text{ is transformed into}
\]
\[
\frac{d\tilde{W}_{ij}}{dt} + i(\tilde{\lambda}(\epsilon) + [\Xi] + \Delta(\theta))\tilde{W}_{ij} = \tilde{P}_{ij}, \quad (20)
\]

Instead of solving the equation (20), we first solve the approximation equation
\[
\mathcal{T}_K\left(\frac{d\tilde{W}_{ij}}{dt} + i(\tilde{\lambda}(\epsilon) + [\Xi] + \Delta(\theta))\tilde{W}_{ij}\right) = \mathcal{T}_K\tilde{P}_{ij}, \quad (21)
\]

If we write
\[
\tilde{W}_{ij} = \sum_{k \in \mathbb{Z}^2, |k| < K} \tilde{W}^k_{ij} e^{i(k, \theta)}, \quad \tilde{P}_{ij} = \sum_{k \in \mathbb{Z}^2, |k| < K} \tilde{P}^k_{ij} e^{i(k, \theta)},
\]

and compare the Fourier coefficients of the equation (21), then for \(|k| < K\), we have
\[
i(|k, \omega| + (\tilde{\lambda}(\epsilon) + [\Xi])\tilde{W}^k_{ij} - i\sum_{|k_1| < K}\Delta^{k-k_1}\tilde{W}^{k_1}_{ij} = \tilde{P}^k_{ij}, \quad (22)
\]

View (22) as a matrix equation
\[
(D + F)W = \mathcal{P},
\]

where
\[
D = \text{diag}(\ldots, i(|k, \omega| + \Omega(\epsilon), \ldots)|k| < K),
\]
\[
\Omega(\epsilon) = (\tilde{\lambda}(\epsilon) + [\Xi]),
\]
\[
F = (-i\Delta^{k_1-k_2})_{|k_1|,|k_2| < K},
\]
\[
\mathcal{W} = (\tilde{W}^k_{ij})_{|k| < K}, \quad \mathcal{P} = (\tilde{P}^k_{ij})_{|k| < K}.
\]

If we denote \(\Gamma_{\tilde{f}} = \text{diag}(\ldots, e^{|k|\tilde{f}}, \ldots)|k| < K\), then
\[
(D + \Gamma_{\tilde{f}}FT_{\tilde{f}}^{-1})\Gamma_{\tilde{f}}W = \Gamma_{\tilde{f}}\mathcal{P}.
\]

Since
\[
|(|k, \omega| - \Omega(\epsilon)| \geq \frac{\gamma}{(|k| + 1)^2}, \forall |k| < K,
\]
then by Lemma 3.4, we have
\[
||D^{-1}||_{\Pi} = \max_{|k| < K} \sup_{\epsilon \in \Pi} \left( \frac{1}{|(|k, \omega| - \Omega(\epsilon)|} + \frac{e^{\frac{d\Omega(\epsilon)}{de}}}{|(|k, \omega| - \Omega(\epsilon)|)^2} \right) \leq C_3(\tau) \gamma^{-A^{r+2}}Q_n^{\theta} 2/2 \leq \frac{1}{4||b||_{\tilde{f}, \Pi}},
\]

where we use the matrix norm
\[
||D||_{\Pi} = \max_{|k| < K} \sup_{\epsilon \in \Pi} \sum_j \left( |D_{ij}(\epsilon)| + \left| \frac{dD_{ij}(\epsilon)}{de} \right| \right),
\]
and \(D_{ij}\) is the \((i, j)\)-th element of the matrix \(D\).

Meanwhile, since the \((k_1, k_2)\)-th element of \(\Gamma_{\tilde{f}}FT_{\tilde{f}}^{-1}\) is \(-ie^{(|k_1| - |k_2|)\tilde{f}}\Delta^{k_1-k_2}\), we obtain that
\[
||\Gamma_{\tilde{f}}FT_{\tilde{f}}^{-1}||_{\Pi} \leq \max_{|k_1| < K} \sum_{|k_2| < K} e^{(|k_1| - |k_2|)\tilde{f}}\Delta^{k_1-k_2} \leq 2||b||_{\tilde{f}, \Pi}.
\]
Thus if \( \|b\|_{n, \Pi} < \frac{\gamma^{\tau+2}}{2\varepsilon_3(\tau)Q_n^{\gamma n+1}} \), we have
\[
\|D^{-1}\Gamma F T^{-1}_{\tau}\|_{\Pi} \leq \|D^{-1}\|_{\Pi} \|\Gamma F T^{-1}_{\tau}\|_{\Pi} < \frac{1}{2},
\]
which implies \( D + \Gamma F T^{-1}_{\tau} \) has a bounded inverse:
\[
\|(D + \Gamma F T^{-1}_{\tau})^{-1}\|_{\Pi} = \|(I + D^{-1}\Gamma F T^{-1}_{\tau})^{-1}D^{-1}\|_{\Pi} \leq \frac{1}{\|D^{-1}\|_{\Pi} - \|D^{-1}\Gamma F T^{-1}_{\tau}\|_{\Pi}} \leq C_3(\tau)\gamma^{-(\tau+2)}Q_n^{6\tau}.
\]
It follows that
\[
\|\tilde{W}_{ij}\|_{\widehat{\Pi}, \Pi} = \sum_{|k|<K} |\tilde{W}_{ij}^k|e^{i|k|j} = \|\Gamma F W\|_{\Pi}
\leq \|(D + \Gamma F T^{-1}_{\tau})^{-1}\|_{\Pi} \|\Gamma F P\|_{\Pi} \leq \|(D + \Gamma F T^{-1}_{\tau})^{-1}\|_{\Pi} \|\tilde{P}_{ij}\|_{\widehat{\Pi}, \Pi}.
\] (23)

Let \( W_{ij}(\theta) = e^{i\theta(\tilde{B}(\theta))}\tilde{W}_{ij}(\theta) \). Then by Lemma 3.3, we have
\[
\|B\|_{\pi(1-\eta), \Pi} \leq C_1(r, \tau, \bar{c})\|\Xi - \langle \Xi \rangle_{r, \Pi} (\tilde{Q}_n/Q_n^{\tau}) + \tilde{Q}_n^{\frac{1}{2}} \).
\]
Therefore, by the assumption (17), if \( \varepsilon_1 < \frac{\varepsilon_0(\tau, \bar{c})}{\varepsilon_2 C_1(r, \tau, \bar{c})} \), we get
\[
\|e^{i\theta B}\|_{\pi(1-\eta), \Pi} \leq e^{\|B\|_{\pi(1-\eta), \Pi}} \leq e^{2C_1\|B\|_{\pi(1-\eta), \Pi}} (\frac{\tilde{Q}_n}{Q_n^{\tau}} + \tilde{Q}_n^{\frac{1}{2}})
\leq e^{2C_1\varepsilon_1\gamma(\frac{\tilde{Q}_n}{Q_n^{\tau}} + \tilde{Q}_n^{\frac{1}{2}})} < \mathcal{L}^{-\frac{\varepsilon_1}{2}}.
\]
By the estimate (23), we have
\[
\|W_{ij}\|_{\pi, \Pi} \leq \mathcal{L}^{-\frac{\varepsilon_1}{2}} \|\tilde{W}_{ij}\|_{\widehat{\Pi}, \Pi} \leq C_3(\tau)\gamma^{-(\tau+2)}Q_n^{6\tau} \mathcal{L}^{-\frac{\varepsilon_1}{2}} \|\tilde{P}_{ij}\|_{\widehat{\Pi}, \Pi},
\]
and
\[
\|W\|_{\pi, \Pi} \leq C_3(\tau)\gamma^{-(\tau+2)}Q_n^{6\tau} \mathcal{L}^{-\frac{\varepsilon_1}{2}} \|P\|_{\pi, \Pi}.
\]
Hence, \( W = SWS^{-1} \) satisfies
\[
\|W\|_{\pi, \Pi} \leq 2C_3(\tau)\gamma^{-(\tau+2)}Q_n^{6\tau} \mathcal{L}^{-\frac{\varepsilon_1}{2}} \|P\|_{\pi, \Pi}.
\]
Moreover, one can verify that the error term \( P_e = (P_{e,ij})_{3x3} \) with
\[
P_{e,ij} = e^{-iB(\theta)}R_K(e^{i\theta(\tilde{B}(\theta))(P_{ij}(\theta) + i\Delta(\theta)W_{ij}))},
\]
satisfies the estimate (18).

2.3. Elimination of non-resonant terms on the off-diagonal block. KAM step. Suppose that we are now in the n-th step, and in what follows the quantities without subscripts refer to those of the n-th step, while the quantities with subscripts “+” denote the corresponding ones of the (n+1)-th step. Thus we consider the following system
\[
\frac{dx}{dt} = (A + B(\theta) + e^\beta P(\theta))x,
\] (24)
where \( A = \tilde{\lambda}(\epsilon)J_1, \ B(\theta) = \Xi(\theta; \epsilon)J_1, \)
\[
\tilde{\lambda}(\epsilon) + \langle \Xi \rangle = \lambda + \epsilon a_m e^{m_0} + O(e^{m_0+1}), m_0 < \beta, a_m \neq 0,
\]
where $\cdot r, \gamma > \beta$

Furthermore, the following estimate holds

$$
\| P \|_{r, \Pi} \leq M, \; \epsilon \in (0, \epsilon_0).
$$

For $\tau > 2$, we define

$$
\mathcal{A} = \tau + 3, \; \mathcal{M} = \frac{\mathcal{A}^4}{2},
$$

and let $(Q_n)$ be the selected subsequence of $\alpha$ in Lemma 3.2 with this given $\mathcal{A}$. For $r, \gamma > 0$, we define

$$
\eta = \frac{\hat{c}}{Q_n^{\frac{1}{2} + \tau}}, \; \mathcal{L} = e^{-\epsilon_0 \gamma r(\hat{Q}_{n+1}^{1/\tau} + \hat{Q}_n^{1/\tau})},
$$

$$
\tau_+ = \tau (1 - \eta)^2, \; \mathcal{L}_+ = e^{-\epsilon_0 \gamma r_+ (\hat{Q}_{n+1}^{1/\tau} + \hat{Q}_n^{1/\tau})},
$$

$$
\mathcal{E}_+ = \mathcal{L}_+ \mathcal{E}, \; K = \left[ \frac{\gamma}{4 \cdot 10^\tau} \max \left\{ \hat{Q}_{n+1}^{1/\tau}, \hat{Q}_n^{1/\tau} \right\} \right],
$$

where $[\cdot]$ denotes its integer part, $0 < \hat{c} < 1$ is a constant which will be defined later, and $\epsilon_0 = \frac{\gamma}{4 \cdot 10^\tau}$ is a constant depending on $\tau, \hat{c}$.

We summarize one KAM step in the following lemma. The key point is to guarantee the non-resonant condition in KAM iteration by adjusting some parameters [16, 17, 19, 28].

**Lemma 2.2. (Step Lemma)** Let us consider the linear quasi-periodic system (24), which is analytic in $t$ and differentiable with respect to $\epsilon$. For any $\tau > 2, 0 < \tau_+ < \tau, 0 < \hat{c} < 1$, there exists $\epsilon_0 = \epsilon_0(\tau, \tau_+, \hat{c}), \epsilon_1 = \epsilon_1(\tau, \tau_+, \hat{c}), J = J(\tau)$ and $T_0 = T_0(\tau, \tau_+, \hat{c})$ such that for $B(\theta) \in \mathcal{A}_r(\Pi)$ with $\mathcal{R}_{Q_{n+1}} B(\theta) = 0$, and $P(\theta) \in \mathcal{A}_r(\Pi), \; \| (k, \omega) \pm (\tilde{\lambda}(\epsilon) + [\Xi]) \| \geq \frac{\gamma}{(\|k\| + 1)^\tau}, \forall k \in \mathbb{Z}^2, |k| < K,$

$$
\tilde{Q}_{n+1} \geq T_0 \cdot \gamma^{-\mathcal{A}/2},
$$

$$
\| B \|_{r, \Pi} (\frac{\tilde{Q}_n^{1/\tau}}{\tilde{Q}_n^{1/\tau} + \hat{Q}_n^{1/\tau}}) \leq \epsilon_1 \gamma (\frac{\tilde{Q}_n^{1/\tau}}{\tilde{Q}_n^{1/\tau} + \hat{Q}_n^{1/\tau}}),
$$

$$
\| \epsilon^2 P \|_{r, \Pi} \leq \epsilon^2 M = \mathcal{E} \leq \epsilon_0 \gamma^2 \mathcal{L},
$$

then there exists an orthogonal transformation $x = \Phi x_+$, which transforms the system (24) into

$$
\frac{dx_+}{dt} = (A + B_+ (\theta) + \epsilon^2 P_+ (\theta)) x_+,
$$

where $B_+ (\theta) = \Xi_+ (\theta) J_1$,

$$
\tilde{\lambda}(\epsilon) + [\Xi_+] = \lambda + a_m \epsilon_m + O(\epsilon_m + 1), \; m_0 < \beta; m_0 \neq 0,
$$

which is defined in $D(\tau_+) \times \Pi$ and satisfies $\mathcal{R}_{Q_{n+2}} B_+ (\theta) = 0$, and

$$
\| \epsilon^{\beta} P_+ \|_{r_+, \Pi} \leq \mathcal{E}_+, \; \| B_+ - B \|_{r_+, \Pi} \leq 2 \mathcal{E}.
$$

Furthermore, the following estimate holds

$$
\| \Phi - \text{id} \|_{r_+, \Pi} \leq \frac{\mathcal{E}_{\Xi_+}^{1/2}}{4}.
$$


Proof. Before giving the proof, we first collect some useful estimates. Let 
\[ \hat{E}_0 = \mathcal{E}, \quad \hat{r}_0 = r(1 - \eta), \]
and define inductively the following sequences
\[ \hat{E}_m = \hat{E}_{m-1}^\frac{\hat{r}_m}{m} = \hat{E}_0^\left(\frac{\hat{r}_m}{m}\right)^m. \]

Let \( m_0 = \min\{ m \in \mathbb{Z}_+ : K\hat{E}_m^1 < 1 \} \), and we define
\[ \sigma_m = \begin{cases} \frac{\eta}{2m+1}, & m < m_0, \\ 2\ln\hat{E}_m - \frac{1}{15K\hat{r}_0}, & m \geq m_0, \end{cases} \]
and \( \hat{r}_m = \hat{r}_{m-1} - 2\hat{r}_0\sigma_{m-1} \).

Once we have these parameters, there exists \( J = J(\tau), \varepsilon_0 = \varepsilon_0(\tau, r_*, \bar{c}) \), and \( T_0 = T_0(\tau, r_*, \bar{c}) \) such that if
\[ \hat{E}_0 \leq \varepsilon_0 \gamma^{J(\tau)} \mathcal{L}, \quad \hat{Q}_{n+1} \geq T_0 \gamma^{-A/2}, \]
then we have the following useful estimates
\[ \hat{E}_0 \leq \min\left\{ \frac{1}{(20r)^{30}}, \left( \frac{\gamma^{A+2}}{16C_2(\tau)} \right)^{30} \right\}, \quad (28) \]
\[ e^{-K\hat{r}_0\sigma_{m}} \leq \hat{E}_m^{-\hat{r}_0 \sigma_{m}}; \quad \hat{E}_m \leq (\hat{r}_0 \sigma_{m})^{15}, \quad (29) \]
where \( C_2(\tau) \) is the constant in Lemma 2.1.

In the following we first check the above estimates. For estimate (28), let \( J(\tau) = [120rA^5] \). By \( \mathcal{L} = e^{-c_0 \gamma r(\frac{Q_{n+1}^2}{2m} + 4\frac{Q_{n+1}^4}{m^2})} \), and the choice of \( (Q_n) : Q_{n+1} \leq \bar{Q}^4 \), there exists \( \varepsilon_2 = \varepsilon_2(\tau, r_*, \bar{c}) \) such that if \( \varepsilon_0 \leq \varepsilon_2 \), we have
\[ \hat{E}_0 \leq \varepsilon_0 \gamma^{J(\tau)} \mathcal{L} \leq \frac{\varepsilon_0 \gamma^{J(\tau)} \mathcal{L}}{(c_0 \gamma r(\frac{Q_{n+1}^2}{2m} + 4\frac{Q_{n+1}^4}{m^2}))^{J(\tau)}} \leq \frac{\varepsilon_2 \gamma^{J(\tau)} \mathcal{L}}{(c_0 \gamma r_*)^{J(\tau)}} \leq \frac{1}{(20r)^{30}}, \]
and
\[ \hat{E}_0 \leq \varepsilon_0 \gamma^{J(\tau)} \mathcal{L} \leq \varepsilon_2^{30}(A^2r^2) \leq \left( \frac{\gamma^{A+2}}{16C_2(\tau)} \right)^{30}. \]

For the first estimate of (29), if \( m \geq m_0 \), by the definition of \( \sigma_m \), the estimate holds apparently. If \( m < m_0 \), by the definition of \( m_0, K \) and \( \sigma_m \), we have
\[ K\sigma_m \hat{r}_0 \geq \left( \frac{1}{\hat{E}_m} \right)^{\mu} \ln\left( \frac{1}{\hat{E}_m} \right) \left( \frac{\gamma}{4 \cdot 10^n} \right) \frac{1}{\hat{Q}_{n+1}^2} \frac{r_\sigma \bar{c}}{Q_n^{3/2}} \geq \ln\left( \frac{1}{\hat{E}_m} \right). \]

For the second estimate of (29), if \( m \geq m_0 \), then
\[ \hat{r}_0 \sigma_m = \frac{2}{15} \ln \frac{1}{\hat{E}_m} \frac{1}{K} \]
\[ = e^{\hat{r}_m \frac{2}{15}} \ln \frac{1}{\hat{E}_m} \frac{1}{\hat{E}_m} \frac{1}{K} \geq \hat{E}_m^{-\hat{r}_0 \sigma_{m}}. \]

If \( m < m_0 \), then
\[ \hat{E}_m^{-\hat{r}_0 \sigma_{m}} \leq \hat{E}_0^{-\hat{r}_0 \sigma_{m}} \leq \hat{E}_0^{-\hat{r}_0 \sigma_{m}} \frac{1}{(Q_{n+1})^{3/2}} \]
\[ \leq \hat{E}_0^{-\hat{r}_0 \sigma_{m}} \frac{1}{(Q_{n+1})^{3/2}} = \hat{E}_0^{-\hat{r}_0 \sigma_{m}}. \]
REDUCIBILITY OF THREE DIMENSIONAL SKEW SYMMETRIC SYSTEM

Now we will prove Step Lemma 2.2 by induction. Let
\[ \frac{du}{dt} = (A + B(\theta) + \epsilon^{\beta_0} \bar{P}_0(\theta))u, \]
where \( u = x, A = \hat{\lambda}(\epsilon)J_1, \beta_0 = \beta, B(\theta) = \Xi(\theta; \epsilon)J_1, \)
\[ \hat{\lambda}(\epsilon) + |\Xi| = \lambda + a_{m_0} \epsilon^{m_0} + O(\epsilon^{m_0+1}), m_0 < \beta_0, a_{m_0} \neq 0, \]
\( B(\theta) \in \mathcal{A}_v(\Pi) \) with \( \mathcal{R}_{Q_{n+1}}B(\theta) = 0, \bar{P}_0(\theta) = P(\theta) \in \mathcal{A}_v(\Pi) \) with
\[ \|\epsilon^{\beta_0} \bar{P}_0\|_{\tau, \Pi} \leq \epsilon^{\beta_0} M_0 = \bar{E}_0, \]
and
\[ \Pi = \left\{ \epsilon \in (0, \epsilon_0) \mid |\langle k, \omega \rangle \pm (\hat{\lambda}(\epsilon) + |\Xi|) \geq \frac{\gamma}{(|k| + 1)^{\tau}}, \forall |k| < K \right\}, \]
where \( \gamma \leq \gamma_0, \tau' > 2\tau + 2. \)

Assume that for \( j = 1, \ldots, \nu \), one can find a quasi-periodic skew symmetric matrix \( W_{\nu-1} \) such that \( \Phi_{\nu-1} = e^{\epsilon^{\beta_0}W_0} \cdots e^{\epsilon^{\beta_{\nu-1}}W_{\nu-1}} \) is an orthogonal transformation and changes the system
\[ \frac{du_{\nu-1}}{dt} = (A + B(\theta) + b_{\nu-1}(\theta) + \epsilon^{\beta_{\nu-1}} \bar{P}_{\nu-1}(\theta))u_{\nu-1}, \]
into
\[ \frac{du_{\nu}}{dt} = (A + B(\theta) + b_{\nu}(\theta) + \epsilon^{\beta_{\nu}} \bar{P}_{\nu}(\theta))u_{\nu}, \]
satisfying \( \mathcal{R}_{Q_{n+1}} \psi_n = 0, \) and
\[ \|\epsilon^{\beta_{\nu}} \bar{P}_{\nu}\|_{\tau_{\nu}, \Pi} \leq \bar{E}_{\nu}, \]
\[ \|b_{\nu} - b_{\nu-1}\|_{\tau_{\nu-1}, \Pi} \leq \bar{E}_{\nu-1}, \]
\[ \|W_{\nu-1}\|_{\tau_{\nu-1}, \Pi} \leq 2C_3(\tau) \gamma^{-(A\tau + 2)}Q_{n+1}^{\beta_{\nu}}L^{-\frac{\pi}{2}} \bar{E}_{\nu-1}. \]

For \( j = \nu + 1 \), one wants to find the matrix function \( W_{\nu} \) such that the orthogonal transformation \( \Phi_{\nu} = e^{\epsilon^{\beta_0}W_0} \cdots e^{\epsilon^{\beta_{\nu}}W_{\nu}} \) reduces the system
\[ \frac{du_{\nu}}{dt} = (A + B(\theta) + b_{\nu}(\theta) + \epsilon^{\beta_{\nu}} \bar{P}_{\nu}(\theta))u_{\nu} \]
into the desired form
\[ \frac{du_{\nu+1}}{dt} = (A + B(\theta) + b_{\nu+1}(\theta) + \epsilon^{\beta_{\nu+1}} \bar{P}_{\nu+1}(\theta))u_{\nu+1}, \]
and satisfies the corresponding estimates.

Suppose that \( \bar{P}_\nu(\theta) = a^1_1(\theta)J_1 + a^2_2(\theta)J_2 + a^3_3(\theta)J_3 \), where the definition of \( J_i (i = 1, 2, 3) \) is given in Lemma 3.5. First, we divide \( \bar{P}_\nu(\theta) \) into three different parts: \( \bar{P}_\nu = T^1_\nu \bar{P}_\nu + T^2_\nu \bar{P}_\nu + \mathcal{R}_K \bar{P}_\nu, \)
where
\[ T^1_\nu \bar{P}_\nu = \begin{pmatrix} 0 & 0 & a^2_2(\theta) \\ 0 & 0 & a^3_3(\theta) \\ -a^1_1(\theta) & -a^2_2(\theta) & 0 \end{pmatrix} \]
and
\[ T^2_\nu \bar{P}_\nu = \begin{pmatrix} 0 & a^1_1(\theta) & 0 \\ -a^1_1(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
satisfying
\[ \| T_K^1 \hat{P}_\nu \|_{\bar{r}_\nu, \Pi} + \| T_K^2 \hat{P}_\nu \|_{\bar{r}_\nu, \Pi} \leq M_\nu, \]  
(33)
and
\[ \| R_K \hat{P}_\nu \|_{\bar{r}_{\nu+1}, \Pi} = \sum_{|k| \geq K} |\hat{P}_\nu(\epsilon)| e^{|k|(|\bar{r}_\nu - 2\bar{r}_0\sigma_\nu|)} \leq e^{-2K\bar{r}_0\sigma_\nu} \| \hat{P}_\nu \|_{\bar{r}_\nu, \Pi} \leq \hat{\xi}_\nu^\pi M_\nu. \]

We rewrite the system (32) as
\[ \frac{d u_{\nu+1}}{dt} = (A + B(\theta) + b_{\nu+1}(\theta) + e^{\beta_\nu} (T_K^1 \hat{P}_\nu + R_K \hat{P}_\nu)) u_\nu, \]  
(34)
where \( b_{\nu+1}(\theta) = b_\nu(\theta) + \hat{b}_\nu(\theta), \hat{b}_\nu(\theta) = e^{\beta_\nu} T_K^2 \hat{P}_\nu. \)

Let \( \hat{\Phi}_\nu : u_\nu = e^{\beta_\nu} W_\nu u_{\nu+1}, \) where \( W_\nu = W_\nu(\omega t; \lambda) \) is a quasi-periodic skew symmetric matrix, which guarantees \( \hat{\Phi}_\nu \) is an orthogonal transformation. Denote \( \hat{A}_\nu(\theta) = A + B(\theta) + b_{\nu+1}(\theta). \) Then, the system (34) is transformed into
\[ \frac{d u_{\nu+1}}{dt} = (\hat{A}_\nu + e^{\beta_\nu} (\hat{A}_\nu W_\nu - W_\nu \hat{A}_\nu + T_K^1 \hat{P}_\nu - \frac{d W_\nu}{dt})) u_{\nu+1} + \hat{P}_{\nu+1} u_{\nu+1}, \]
where
\[ \hat{P}_{\nu+1} = (e^{-\beta_\nu} W_\nu - I + e^{\beta_\nu} W_\nu) \hat{A}_\nu e^{\beta_\nu} W_\nu + \hat{A}_\nu (e^{\beta_\nu} W_\nu - I - e^{\beta_\nu} W_\nu) + e^{\beta_\nu} (T_K^1 \hat{P}_\nu - W_\nu \hat{A}_\nu) (e^{\beta_\nu} W_\nu - I) + e^{\beta_\nu} (e^{-\beta_\nu} W_\nu - I) T_K^1 \hat{P}_\nu e^{\beta_\nu} W_\nu \]
\[ + e^{\beta_\nu} W_\nu e^{\beta_\nu} W_\nu + \frac{d(e^{-\beta_\nu} W_\nu - I + e^{\beta_\nu} W_\nu)}{dt} + \frac{d e^{-\beta_\nu} W_\nu}{dt} (e^{\beta_\nu} W_\nu - I). \]  
(35)
The point is to find \( W_\nu \) such that
\[ \frac{d W_\nu}{dt} - \hat{A}_\nu W_\nu + W_\nu \hat{A}_\nu = T_K^1 \hat{P}_\nu. \]  
(36)

Thanks to the definition of \( b_{\nu+1} \) and (33), one gets \( \| b_{\nu+1} \|_{\bar{r}_\nu, \Pi} \leq e^{\beta_\nu} M_\nu = \hat{\xi}_\nu. \)

From (31), it follows that
\[ \| b_{\nu+1} \|_{\bar{r}_\nu, \Pi} \leq \| b_{\nu+1} - b_\nu \|_{\bar{r}_{\nu+1}, \Pi} + \sum_{j=1}^{\nu} \| b_j - b_{j-1} \|_{\bar{r}_{j-1}, \Pi} \leq 2\hat{\xi}_0. \]  
(37)
Then, by (28), we have
\[ \| b_{\nu+1} \|_{\bar{r}_{\nu+1}, \Pi} \leq 2\hat{\xi}_0 < \frac{\gamma}{2C_3(\tau) Q_{n+2}^0}. \]
which implies that the condition (16) is satisfied. What’s more, since \( K < Q_{n+2} \), it is obvious that \( R_{Q_{n+2}}(b_{\nu+1} - b_\nu) = 0 \), which implies \( R_{Q_{n+2}} b_{\nu+1} = 0 \) by \( R_{Q_{n+2}} b_\nu = 0. \)

On the other hand, by the assumptions of Lemma 2.2, \( B(\theta) \) satisfies (17), \( R_{Q_{n+1}} B(\theta) = 0 \) and
\[ |\langle k, \omega \rangle \pm (\lambda(\epsilon) + [\Xi])| \geq \frac{\gamma}{(|k| + 1)^2}, \quad \forall |k| < K. \]
Then for any $|k| < K$, we can apply Lemma 2.1 to get an approximate solution $W_\nu$ of (36) with the error term $(T^l K \hat{P}_\nu)_e$. That is to say, instead of solving (36), we first solve the approximation equation

$$\frac{dW_\nu}{dt} - A_\nu W_\nu + Y_\nu A_\nu = T^l K \hat{P}_\nu - (T^l K \hat{P}_\nu)_e$$

By Lemma 2.1, we have

$$\|W_\nu\|_{r_\nu, \Pi} \leq 2C_3(\tau)\gamma^{-(A^T + 2)}Q^{6r}_{n+1}L^{-\frac{\beta}{2n}}M_\nu.$$  

Moreover, the error term $(T^l K \hat{P}_\nu)_e = ((T^l K \hat{P}_{\nu,ij})_e)_{3\times 3}$ with

$$(T^l K \hat{P}_{\nu,ij})_e = e^{-i\mathbf{B}(\theta)}R_K(e^{i\mathbf{B}(\theta)}(T^l K \hat{P}_{\nu,ij} + i\Delta_{\nu+1}(\theta)W_{ij})))$$

satisfies

$$\| (T^l K \hat{P}_\nu)_e \|_{r_\nu - r_0\sigma_\nu, \Pi} \leq 2L^{-\frac{\beta}{2n}}e^{-K\hat{r}_0\sigma_\nu}(M_\nu + 2\|b_{\nu+1}\|_{r_\nu - r_0\sigma_\nu}) \cdot \|W\|_{r_\nu - r_0\sigma_\nu} < \frac{\bar{\nu}^2}{8}.$$  

Thus the system (32) becomes

$$\frac{du_{\nu+1}}{dt} = (A + B(\theta) + b_{\nu+1}(\theta) + \epsilon \hat{r}_\nu^{15} \bar{
u} \hat{P}_{\nu+1})u_{\nu+1},$$

where $\hat{P}_{\nu+1} = e^{-i\beta_\nu}(\hat{P}_{\nu+1} + (T^l K \hat{P}_\nu)_e)$.  

Now we estimate the new perturbation term $\hat{P}_{\nu+1}$. Suppose $\|\epsilon^{\beta_\nu} \hat{P}_\nu\|_{r_\nu, \Pi} \leq \epsilon^{\beta_\nu}M_\nu = \bar{E}_\nu$, and $\bar{E}_0$ is sufficiently small such that

$$\|\epsilon^{\beta_\nu} W_\nu\|_{r_\nu, \Pi} \leq 2C_3(\tau)\gamma^{-(A^T + 2)}Q^{6r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu \leq \frac{\bar{\nu}^2}{32} \leq \frac{1}{2},$$

then by $e^W = I + W + \frac{W^2}{2!} + \cdots + \frac{W^n}{n!} + \cdots$, we have

$$\|e^{\pm \epsilon^{\beta_\nu} W_\nu}\|_{r_\nu, \Pi} \leq \frac{1}{1 - \|\epsilon^{\beta_\nu} W_\nu\|_{r_\nu, \Pi}} \leq 2.$$  

Similarly, we have

$$\|e^{\pm \epsilon^{\beta_\nu} W_\nu} - I\|_{r_\nu, \Pi} \leq 4C_3(\tau)\gamma^{-(A^T + 2)}Q^{6r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu \leq \frac{\bar{\nu}^2}{16},$$

$$\|e^{\pm \epsilon^{\beta_\nu} W_\nu} - I - \epsilon^{\beta_\nu} W_\nu\|_{r_\nu, \Pi} \leq 8C_3^2(\tau)\gamma^{-(2A^T + 2)}Q^{12r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu^2 \leq \frac{\bar{\nu}^2}{16},$$

By Cauchy’s estimates and in view of

$$\hat{r}_\nu - \hat{r}_{\nu+1} = 2\hat{r}_0\sigma_\nu,$$

it follows that

$$\left\| \frac{d(\epsilon^{\beta_\nu} W_\nu)}{dt} \right\|_{r_\nu+1, \Pi} \leq \frac{\|\epsilon^{\beta_\nu} W_\nu\|_{r_\nu, \Pi}}{2\hat{r}_0\sigma_\nu} \leq \frac{2C_3(\tau)\gamma^{-(A^T + 2)}Q^{6r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu}{2\hat{r}_0\sigma_\nu} \leq \frac{\bar{\nu}^2}{16},$$

$$\left\| \frac{de^{\epsilon^{\beta_\nu} W_\nu}}{dt} \right\|_{r_\nu+1, \Pi} \leq \frac{4C_3(\tau)\gamma^{-(A^T + 2)}Q^{6r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu}{2\hat{r}_0\sigma_\nu} \leq \frac{\bar{\nu}^2}{16},$$

and

$$\left\| \frac{d(e^{\epsilon^{\beta_\nu} W_\nu} - I + \epsilon^{\beta_\nu} W_\nu)}{dt} \right\|_{r_\nu+1, \Pi} \leq \frac{8C_3^2(\tau)\gamma^{-(2A^T + 2)}Q^{12r}_{n+1}L^{-\frac{\beta}{2n}}\bar{E}_\nu^2}{2\hat{r}_0\sigma_\nu} \leq \frac{\bar{\nu}^2}{8}. $$
Altogether, combining the definition (35) of \( \tilde{\Gamma}_{n+1} \), \( \| \tilde{A}_n \| r_c, \Pi \leq 2 \), and all the above estimates, we obtain

\[
\| \tilde{\Gamma}_{n+1} \| r_{n+1}, \Pi \leq e^{-\frac{15}{16} \beta_n} (\| \tilde{\Gamma}_{n+1} \| r_{n+1}, \Pi + \| (\mathcal{T}_n^1 \tilde{A}_n) e \| r_{n+1}, \Pi )
\]

\[
\leq e^{-\frac{15}{16} \beta_n} \epsilon_n^{16} = M_n^{16} = M_{n+1}.
\]

Notice that there exists \( L \geq m_0 \) such that \( \tilde{S}_L \leq L_+ \tilde{c}_0 = \mathcal{E}_+ \) and \( \tilde{S}_{L-1} > \mathcal{E}_+ \). Once we reach the \( L \)-th step, we stop the above iteration. Let

\[
\Phi = \tilde{\Phi}_L = e^{\beta_n W_0} \cdots e^{\beta_L W_L},
\]

and \( x_+ = u_L, B_+ (\theta) = B (\theta) + b_L (\theta), P_+ (\theta) = \tilde{P}_L (\theta), \beta_+ = (\frac{16}{15}) L \beta, \)

\[
b_L (\theta) = \sum_{j=0}^{L-1} e^{\beta_j} P_j (\theta) = \sum_{j=0}^{L-1} e^{\beta_j} (a_j^1 (\theta))_{|k| < K J_1} = \Delta_L (\theta) J_1.
\]

Then the orthogonal transformation \( x = \Phi x_+ \) reduces the system (24) into

\[
dx_+ dt = (A + B_+ (\theta) + e^{\beta_+} P_+ (\theta)) x_+.
\]

Moreover, since \( K < Q_{n+2} \), it follows that \( R Q_{n+2} B_+ (\theta) = 0 \). By (37) and (38), we get the estimates (26) and (27).

Recall that

\[
r_+ = \tilde{r}_0 (1 - \eta).
\]

By the definition of \( \sigma_m \), we know \( \sigma_{m+1} = \frac{16}{15} \sigma_m \), if \( m \geq m_0 \). Then by the selection of \( L \in \mathbb{N} \), i.e., \( \tilde{S}_L \leq \mathcal{E}_+ \), \( \tilde{S}_{L-1} > \mathcal{E}_+ = L_+ \tilde{c}_0 \), we have

\[
\sum_{j=0}^{L-1} \sigma_j = \sum_{j=0}^{m_0-1} \sigma_j + \sum_{j=m_0}^{L-1} \sigma_j
\]

\[
\leq \frac{\eta}{4} - \frac{2 \ln \tilde{c}_0}{K \tilde{r}_0} \left( \frac{16}{15} \right)^{L-1} - \frac{16}{15} \eta \]

\[
\leq \frac{\eta}{4} - \frac{32 \ln L_+}{15 K \tilde{r}_0} - \frac{32 \ln \tilde{c}_0}{15 K \tilde{r}_0} + 2 \ln \tilde{c}_0 \frac{16}{15} \eta_0.
\]

If \( m_0 \geq 1 \), we can have that

\[
\sum_{j=0}^{L-1} \sigma_j \leq \frac{\eta}{4} - \frac{32 \ln L_+}{15 K \tilde{r}_0} \leq \frac{\eta}{4} + \frac{c}{8} \left( \frac{1}{Q_{n+1}^2} + \frac{1}{Q_{n+1}^2} \right) \leq \frac{\eta}{2}.
\]

If \( m_0 = 0 \), that is to say \( K \tilde{c}_0^{16} < 1 \), we have

\[
\sum_{j=0}^{L-1} \sigma_j \leq - \frac{32 \ln L_+}{15 K \tilde{r}_0} \frac{32 \ln \tilde{c}_0}{15 K \tilde{r}_0} + 2 \ln \tilde{c}_0 \frac{16}{15} \eta_0
\]

\[
= - \frac{32 \ln L_+}{15 K \tilde{r}_0} - \frac{2 \ln \tilde{c}_0}{15 K \tilde{r}_0}.
\]

Suppose that \( \tilde{c}_0 \geq L_+ \), then

\[
\sum_{j=0}^{L-1} \sigma_j \leq - \frac{34 \ln L_+}{15 K \tilde{r}_0} \leq \frac{\eta}{2},
\]

which implies \( \tilde{r}_L \geq r_+ \).
Without loss of generality we suppose that the above iteration to find $\Phi_k$ matrix $L$ series of monotone decreasing numbers, there exists $L_{k+1} = e^{-c_0y_{n+1}} \frac{Q_{n+1}^{2L}}{Q_{n+1}^{2L+1}}$ and $L_{n+1} = e^{-c_0y_{n+1}}(\frac{Q_{n+1}^{2L+1} + Q_{n+1}^{2L+2}}{Q_{n+1}^{2L+1}})$ is a series of monotone decreasing numbers, there exists $L_{k+1}(k > 1)$ such that $\cdots < L_{k+2} < L_{k+1} < \tilde{E}_0 < L_k$. Then we let $\Phi_0 = \Phi_1 = \cdots = \Phi_{k-1} = id$, and apply the above iteration to find $\Phi_k$ from the $k+1$-th step, since $\tilde{E}_0 > L_{k+1}$. Therefore, without loss of generality we suppose that $\tilde{E}_0 \geq L_0$.

Below we prove $B_+(\theta)$ has the same form as $A$. First an arbitrary skew symmetric matrix $P$ can be written in the form

$$P = \mathcal{T}_k^1P + \mathcal{T}_k^2P + R_kP,$$

where

$$\mathcal{T}_k^1P = \begin{pmatrix} 0 & 0 & \alpha^2(\theta) \\ 0 & 0 & \alpha^3(\theta) \\ -\alpha^2(\theta) & -\alpha^3(\theta) & 0 \end{pmatrix}_{|k| < K},$$

and

$$\mathcal{T}_k^2P = \begin{pmatrix} 0 & \alpha^2(\theta) & 0 \\ -\alpha^2(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{|k| < K}.$$

Let $\Delta_L(\theta) = \sum_{j=0}^{L-1} \epsilon_j(\alpha_j^2(\theta))_{|k| < K}$. Due to $b_L(\theta) = \Delta_L(\theta)J_1$, we have

$$B_+(\theta) = B(\theta) + b_L(\theta) = \begin{pmatrix} 0 & \Xi(\theta) + \Delta_L(\theta) & 0 \\ -\Xi(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Xi_+(\theta) & 0 \\ -\Xi_+(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Accordingly, define

$$\Pi_+ = \left\{ \epsilon \in (0, \epsilon_0) \mid |(k, \omega) \pm (\lambda_0(\epsilon) + [\Xi_+])| \geq \frac{\gamma + \gamma_+}{(|k| + 1)^{\frac{\tau_1}{\tau'}}}, \forall |k| < K_+ \right\},$$

where $\gamma_+ = \gamma/2, \tau' > 2\tau + 2$.

**Iteration and convergence.** For any given $0 < r_0 < r, \tau > 2, \gamma > 0, A, M$ are defined as in (25). Since $\frac{\gamma}{\tau'} > 1$, there exists $\tilde{c}_0 = \frac{1}{2}(|\tilde{c}r_0| + 1)$ such that $r > \tilde{c}_0 r_0$. Let

$$\tilde{c} = \frac{1}{12} (1 - \frac{1}{\tilde{c}_0}) < 1, \quad T_1 = \left( \frac{4 \cdot 10'}{\tilde{c}_0 r_0} \right)^A,$$

$$T = \max\{T_0(\frac{\gamma}{2})^{-\frac{\gamma}{2}}, T_1 \gamma^{-A}, 4A^4\}.$$

These constants $\epsilon_0(\tau, r_0, \tilde{c}), \epsilon_1(\tau, r_0, \tilde{c}), J(\tau), T_0(\tau, r_0, \tilde{c})$ are defined as in Lemma 2.2.

Due to the choice of Lemma 3.2, there exists some $n_0 \in \mathbb{Z}_+$ such that $Q_{n_0} \leq T A^4$, whereas $Q_{n_0} \geq T$. Since $Q_{n_0} < T A^4$, we can take $E$ sufficiently small, depending on $r, r_0, \gamma, \tau$, but not on $\alpha$, such that

$$E < \min\{\epsilon_0(\frac{\gamma}{2})^j, \epsilon_1(\frac{\gamma}{2})^j, \frac{1}{T^{120\tau A^4}}\} \leq \min\{\epsilon_0(\frac{\gamma}{2})^j, \epsilon_1(\frac{\gamma}{2})^j, \frac{1}{Q_{n_0}}} \}. \quad (39)$$
For any given $\mathcal{E} > 0$ satisfying (39), we define some sequences inductively:

$$\mathcal{E}_0 = \mathcal{E}, \quad r_0 = r, \quad \gamma_0 = \gamma,$$

$$\gamma_n = \frac{\gamma}{2^n}, \quad \mathcal{L}_{n+1} = e^{-c_0 \gamma_n r_{n+1} \left( \frac{Q_{n_0+n+1}}{Q_{n_0+n+1}^A} + \frac{1}{Q_{n_0+n+1}} \right)}, \quad \mathcal{E}_{n+1} = \mathcal{L}_{n+1} \mathcal{E}_n,$$

$$\eta_n = c Q_{n_0+n}^{-\frac{1}{2}}, \quad r_{n+1} = r_n (1 - \eta_n)^2,$$

$$K_n = \left[ \frac{\gamma_n}{4 \cdot 10^8} \max \left\{ \frac{Q_{n_0+n}}{Q_{n_0+n}^A}, \frac{Q_{n_0+n}^{\frac{3}{2}}}{Q_{n_0+n}^A} \right\} \right].$$

Define

$$\Pi_n = \left\{ \epsilon \in (0, \epsilon_0) \mid \frac{|\langle k, \omega \rangle + (\bar{\lambda}(\epsilon) + \Xi_n)|}{\sqrt{|k| + 1}} \geq \frac{\gamma_n}{2^n}, \forall |k| < K_n \right\},$$

then for any $\epsilon \in \Pi_n$, the corresponding Diophantine condition holds for $|k| < K_n$.

First, we claim that $r_n \geq r_\ast$ for all $n$. In fact, by our selection $Q_{n_0+n} \geq \mathcal{E}_{n_0} \geq T \geq 4^{4^2}$, we have

$$\prod_{k=2}^{\infty} (1 - 2\eta_k) \geq 1 - 4 \sum_{k=2}^{\infty} \eta_k \geq 1 - 8cQ_{n_0+n}^{-\frac{1}{2}} > 1 - 8c,$$

which implies that for any $n \geq 0$,

$$r_n > r(1 - 2\eta_0)(1 - 2\eta_1) \prod_{k=2}^{\infty} (1 - 2\eta_k) > r(1 - 12c) > r_\ast.$$

Second, by the choice of parameters we can verify that $B_n, \bar{Q}_{n_0+n}, P_n$ satisfy that

$$\bar{Q}_{n_0+n} \geq T_0 \gamma^{-A/2}_n,$$

$$\|B_n\|_{r_n, \Pi_n} \left( \frac{\bar{Q}_{n_0+n-1}}{Q_{n_0+n-1}^A} + \frac{\bar{Q}_{n_0+n-1}^{\frac{3}{2}}}{Q_{n_0+n-1}^A} \right) \leq \epsilon_1 \gamma_n \left( \frac{Q_{n_0+n-1}^A}{Q_{n_0+n-1}^A} + \frac{1}{Q_{n_0+n-1}} \right),$$

$$\|\epsilon^\beta \gamma_n P_n\|_{r_n, \Pi_n} \leq \epsilon^\beta \gamma_n M_n = \mathcal{E}_n \leq \epsilon_0 \gamma_n^J \mathcal{E}_n.$$

In the following we first check these above estimates by induction. By $\bar{Q}_{n+1} \geq \bar{Q}_n$ and $\bar{Q}_{n_0} \geq T_0 \gamma_0^{-\frac{A}{2}}$, we have

$$\bar{Q}_{n_0+n} \geq \bar{Q}_{n_0} \geq T_0 \gamma_0^{-\frac{A}{2}} \geq T_0 \gamma_n^{-\frac{A}{2}}.$$

By $Q_{n+1} \geq Q_n, \bar{Q}_{n+1} \geq \bar{Q}_n$ and $\mathcal{E}_0 \leq \epsilon_0 \gamma_n^J$, it follows that

$$\|B_n\|_{r_n, \Pi_n} \left( \frac{Q_{n_0+n-1}^A}{Q_{n_0+n-1}^A} + \frac{\bar{Q}_{n_0+n-1}^{\frac{3}{2}}}{Q_{n_0+n-1}^A} \right) \leq \epsilon_1 \gamma_n \left( \frac{Q_{n_0+n-1}^A}{Q_{n_0+n-1}^A} + \frac{1}{Q_{n_0+n-1}} \right)$$

$$\leq \epsilon_1 \gamma_n \left( \frac{Q_{n_0+n-1}^A}{Q_{n_0+n-1}^A} + \frac{1}{Q_{n_0+n-1}} \right).$$

The definition $\mathcal{E}_n$ and the estimate $\mathcal{E}_0 \leq \epsilon_0 \left( \frac{\gamma}{2} \right)^J$ yields that

$$\mathcal{E}_n = \mathcal{E}_n \mathcal{E}_{n-1} \cdots \mathcal{E}_0 \leq \mathcal{L}_n \frac{1}{2^{(n-1)J}} \mathcal{E}_0$$

$$\leq \mathcal{L}_n \frac{1}{2^{(n-1)J}} \epsilon_0 \left( \frac{\gamma}{2} \right)^J = \epsilon_0 \gamma_n^J \mathcal{E}_n.$$
After setting the parameters, we will iterate KAM iteration infinitely and prove its convergence. For the first step, let $B_0 = 0, P_0 = P, m_0 = m$. By our selection $Q_{n_0} \geq T \geq T_0 \gamma^{-\frac{\tau}{2}}$, and (39), we have

$$\|e^{m_0}P_0\|r_0,\Pi_0 \leq \varepsilon_0^{m_0}M_0 = \mathcal{E}_0 \leq \min\left\{\varepsilon_0\left(\frac{\gamma}{2}\right)^\tau, \frac{1}{Q_{n_0}}\right\}. \tag{40}$$

Meanwhile, we can check (28) and (29) hold. Thus we apply Step Lemma 2.2, and get the transformation $\Phi_0 : D(r_1) \times \Pi_0 \to D(r_0) \times \Pi_0$ such that the system (1) is transformed into

$$\dot{x} = (A(\varepsilon) + B_1(\theta) + e^{m_1}P_1(\theta))x,$$

satisfying

$$\|B_1\|_{r_1,\Pi_0} \leq \mathcal{E}_0 \leq \varepsilon_1 \gamma,$$

and

$$\|e^{m_1}P_1\|_{r_1,\Pi_0} \leq \mathcal{E}_0\mathcal{L} \leq \varepsilon_0 \gamma^\tau \mathcal{L}. \tag{41}$$

Note that the above estimates means that all the conditions of Step Lemma 2.2 for the next step hold.

Inductively, there exists a subset $\Pi_n \subset \Pi$ such that for any $\varepsilon \in \Pi_n$, the following Diophantine condition holds:

$$|\langle k, \omega \rangle \pm (\bar{\lambda}(\varepsilon) + [\Xi^n])| \geq \frac{\gamma_n}{|k| + 1}, \quad \forall |k| < K_n,$$

and for any $\varepsilon \in \Pi_n$ there exists an orthogonal transformation $\Phi_n = e^{i\beta_n}W^n_0 \cdots e^{i\beta_n}W^n_L$ such that

$$\|\Phi_n - \text{id}\|_{r_n,\Pi_n} \leq \frac{\varepsilon_n^{14}}{4}. \tag{42}$$

Let $\Phi^n = \Phi_0 \cdots \Phi_{n-1}$. Then, for any $\varepsilon \in \Pi_n$, the orthogonal transformation $x = \Phi^n x_n$ reduces the system (1) into the following form:

$$\frac{dx_n}{dt} = (A + B^n(\theta) + e^{\beta_n}P_n(\theta))x_n,$$

where

$$B^n(\theta) = \begin{pmatrix} 0 & \Xi^n(\theta) & 0 \\ -\Xi^n(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\|e^{\beta_n}P_n\|_{r_n,\Pi_n} \leq \varepsilon_n^{\beta_n}M_n = \mathcal{E}_n, \quad \|B^{n+1} - B^n\|_{r_{n+1},\Pi_n} \leq 2\mathcal{E}_n.$$ 

By the above estimates we have

$$\|\Phi_n\|_{r_n,\Pi_n} \leq 1 + \frac{\varepsilon_n^{14}}{4}. \tag{43}$$

If $\mathcal{E}_0$ is sufficiently small, then

$$\|\Phi^n\|_{r_n,\Pi_n} \leq \prod_{j=0}^{n-1} \left(1 + \frac{\varepsilon_j^{14}}{4}\right) < \infty,$$

and

$$\|\Phi^n - I\|_{r_n,\Pi_n} \leq \frac{\varepsilon_0^{14}}{2}. \tag{44}$$
Let $\Pi_* = \bigcap_{n \geq 1} \Pi_n$, $\Phi^* = \lim_{n \to \infty} \Phi^n$, $B^* = \lim_{n \to \infty} B^n$. Because $\lim_{n \to \infty} \epsilon^{\beta_n} P_n = 0$, the transformation $x = \Phi^* v$ reduces the system (1) into
\[
\frac{dv}{dt} = \left( A + B^*(\theta) \right) v,
\]
where
\[
B^*(\theta) = \begin{pmatrix}
0 & \Xi^*(\theta) & 0 \\
-\Xi^*(\theta) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Measure estimates.** In the following we give measure estimates of parameters. Let
\[
\hat{A}_n(\epsilon) = [B^n] = \begin{pmatrix}
0 & [\Xi^n] & 0 \\
-[\Xi^n] & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

By the above iteration, we have
\[
\|\hat{A}_n(\epsilon)\| \leq c\epsilon^m, \quad \|\hat{A}'_n(\epsilon)\| \leq c\epsilon^{m-1}.
\]

Recall that
\[
\tilde{\lambda}(\epsilon) = \lambda + a_{m_0}\epsilon^{m_0} + O(\epsilon^{m_0+1}), a_{m_0} \neq 0, m_0 < m,
\]
and
\[
\Pi_n = \left\{ \epsilon \in (0, \epsilon_0) \mid |\langle k, w \rangle \pm (\tilde{\lambda}(\epsilon) + [\Xi^n])| \geq \frac{\gamma_n}{(|k|+1)^{\tau'}}, \forall |k| < K_n \right\},
\]
where $\gamma_n = \gamma_0/2^n$, $\tau' > 2\tau + 2$.

By Lemma 3.10, $\forall \epsilon \in (0, \epsilon_0)$, we have
\[
\text{meas}\left( (0, \epsilon) \setminus \Pi_* \right) \leq c\epsilon^{m_0+1} \frac{\gamma_0}{\gamma_0}, \quad \gamma_n = \frac{\gamma_0}{2^n}.
\]

Note that
\[
(0, \epsilon) \setminus \Pi_* = (0, \epsilon) \setminus \bigcup_{n=0}^{\infty} \Pi_n = \bigcup_{n=0}^{\infty} \left( (0, \epsilon) \setminus \Pi_n \right),
\]
therefore,
\[
\text{meas}\left( (0, \epsilon) \setminus \Pi_* \right) \leq \sum_{n=0}^{\infty} \text{meas}\left( (0, \epsilon) \setminus \Pi_n \right) \leq c\epsilon^{m_0+1}.
\]

### 2.4. Elimination of non-resonant terms on the diagonal block.

By infinite steps of KAM iteration, the system (1) can be reduced to a linear system with non-constant coefficients, which has a diagonal form:
\[
\frac{dv}{dt} = \left( A + B^*(\theta) \right) v,
\]
where
\[
B^*(\theta) = \begin{pmatrix}
0 & \Xi^*(\theta) & 0 \\
-\Xi^*(\theta) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

In this section, we will eliminate all the non-resonant terms containing $\theta$ on the diagonal by only one step. First, notice that $\beta(\alpha)$ has an equivalent definition (12), which implies the equation
\[
\frac{dH(\theta)}{dt} = B^*(\theta) - [B^*]
\]
has an analytic solution if \(0 \leq \beta(\alpha) < r_*\). Let \(v = e^{H(\theta)}v^*\), where
\[
H(\theta) = \begin{pmatrix}
0 & \sum_{0 \neq k \in \mathbb{Z}^2} \Xi^*(k)e^{i(k,\theta)} & 0 \\
-\sum_{0 \neq k \in \mathbb{Z}^2} \Xi^*(k)e^{i(k,\theta)} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then the above system becomes
\[
e^{H(\theta)} \frac{dH(\theta)}{dt} v^* + e^{H(\theta)} \frac{dv^*}{dt} = (A + B^*(\theta)) e^{H(\theta)} v^*.
\]

Because of the commutation \(e^{H(\theta)}\) and \(A, B^*(\theta)\), it follows that
\[
\frac{dv^*}{dt} = (A + [B^*]) v^* = A^* v^*,
\]
where
\[
A^* = \begin{pmatrix}
0 & \Lambda^* & 0 \\
-\Lambda^* & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \Lambda^* = \bar{\lambda}(\epsilon) + [\Xi^*].
\]

Moreover, for any \(\epsilon \in \Pi_4\), we have
\[
\|A^* - A\|_{r_*, \Pi_4} \leq 4 \epsilon_0.
\]

This completes the proof of Theorem 1.4.

3. Appendix.

3.1. Continued fraction expansion. Let \(\alpha \in (0, 1)\) be irrational. Define \(a_0 = 0, \alpha_0 = \alpha\), and inductively for \(k \geq 1,\)
\[
a_k = [\alpha_{k-1}^{-1}], \alpha_k = \alpha_{k-1} - a_k = \left\{ \frac{1}{\alpha_{k-1}} \right\}.
\]

where \([\cdot]\) denotes its integral part, \(\{\cdot\}\) denotes its fractional part.

We define \(p_0 = 0, p_1 = 1, q_0 = 0, q_1 = a_1\), and inductively,
\[
p_k = a_k p_{k-1} + p_{k-2},
\]
\[
q_k = a_k q_{k-1} + q_{k-2}.
\]

The sequence \((q_n)\) is the sequence of denominators of the best rational approximations for \(\alpha\), since it satisfies
\[
\forall 1 \leq k < q_n, \quad \|k\alpha\|_T \geq \|q_{n-1}\alpha\|_T, \quad (40)
\]

and
\[
\|q_n \alpha\|_T \leq \frac{1}{q_{n+1}}, \quad (41)
\]

where we use the norm
\[
\|x\|_T = \inf_{p \in \mathbb{Z}} |x - p|.
\]
3.2. CD bridge. The motivation of this section is to introduce the concept of CD bridge, which first appeared in [1], and give some useful estimates. For detailed proofs we refer to [1, 25].

For any \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), we fix a particular subsequence \( (q_{n_k}) \) in the sequence of the denominators of \( \alpha \), which will be denoted by \((Q_k)\) for simplicity. Denote the sequence \((q_{n_k+1})\) by \((Q_k)\), and denote \((p_{n_k})\) by \((P_k)\).

**Definition 3.1.** (CD bridge [1]) Let \( 0 < A \leq B \leq C \). We say that the pair of denominators \( (q, q_n) \) forms a CD\((A, B, C)\) bridge, if

1. \( q_{i+1} \leq q_i^A, \forall i = l, \ldots, n - 1; \)
2. \( q_i^B \leq q_n \leq q_i^C. \)

**Lemma 3.2.** ([1]) For any \( \lambda \geq 1 \), there exists a subsequence \((Q_k)\) such that \( Q_0 = 1 \) and for each \( k \geq 0 \), \( Q_{k+1} \leq Q_k^A \), and either \( Q_k \geq Q_k^B \), or the pairs \((\bar{Q}_{k-1}, Q_k)\) and the pairs \((Q_k, \bar{Q}_{k+1})\) are both CD\((A, B, C^3)\) bridges.

**Lemma 3.3.** ([1]) For any \( 0 < r_s < r, \tau > 2, \bar{c} < 1 \), there exists \( C_1 = C_1(r_s, \tau, \bar{c}) \), such that if \( f \in \mathcal{A}_r(\Pi) \), then the equation

\[
\partial_2 g(\theta; \epsilon) = -T_{Q_k+1} f(\theta; \epsilon) + [f]
\]

has a solution with

\[
\|g\|_{r(1-\eta), \Pi} \leq C_1(r_s, \tau, \bar{c}) \|f - [f]\|_{r, \Pi} \left(\frac{\bar{Q}_n}{Q_n^A} + \bar{Q}_n^A\right).
\]

**Lemma 3.4.** ([25]) For any \( 0 < \gamma < 1, \tau > 2 \), there exists \( c_2 = c_2(\tau) \), such that if \( \Omega(\epsilon) \in DC_\omega(\gamma, \tau, K, \Pi) \), where

\[
DC_\omega(\gamma, \tau, K, \Pi) = \left\{ \Omega(\epsilon) \mid 0 < \|\Omega(\epsilon)\|_\Pi < 2, |\langle k, \omega \rangle - \Omega(\epsilon)| \geq \frac{\gamma}{(1 + |\epsilon|)}; \forall |k| < \tilde{K} \right\},
\]

and

\[
\tilde{K} = \left\lceil \frac{\sqrt{\gamma}}{4 \cdot 10^r} \max \left\{ \frac{Q_n}{Q_n^A}, \frac{\bar{Q}_n^A}{Q_n^A} \right\} \right\rceil,
\]

then for any \( |k| < \tilde{K} \), we have

\[
|\langle k, \omega \rangle - \Omega(\epsilon)| \geq c_2(\tau) \gamma^{\frac{4r}{\tau} + 1} Q_n^{-3r}.
\]

3.3. The properties of skew symmetric matrices. In this section we give some basic properties of \( so(3, \mathbb{R}) \). First, Denote

\[
J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{42}
\]

Then, \((J_1, J_2, J_3)\) form a basis of \( so(3, \mathbb{R}) \), i.e., \( \forall A \in so(3, \mathbb{R}) \), there exist \( a_1, a_2 \) and \( a_3 \in \mathbb{R} \) such that

\[
A = a_1 J_1 + a_2 J_2 + a_3 J_3.
\]

**Lemma 3.5.** Let \( A = a_1 J_1 + a_2 J_2 + a_3 J_3 \). Then the matrix \( A \) has three eigenvalues \( 0, \pm ia \), where \( a = \sqrt{a_1^2 + a_2^2 + a_3^2} \). If \( x_3 \in \mathbb{R}^3 \) is a unit eigenvector corresponding to eigenvalue zero, then there exist orthogonal unit vectors \( x_1, x_2 \in \mathbb{R}^3 \), such that \( W = (x_1, x_2, x_3) \) is an orthogonal matrix, and

\[
W^{-1}AW = aJ_1.
\]
Proof. First, the characteristic equation of $A$ is 
\[ \det(\lambda I - A) = \lambda[\lambda^2 + (a_1^2 + a_2^2 + a_3^2)], \]
so the eigenvalues are 
\[ \lambda_1 = 0, \lambda_2 = ia, \lambda_3 = -ia, \]
where \( a = \sqrt{a_1^2 + a_2^2 + a_3^2} \).

If \( a = 0 \), let \( W = I \). Then 
\[ W^{-1}AW = I^{-1}AI = 0J_1. \]

If \( a \neq 0 \), note that zero is a simple eigenvalue, the vector 
\[ x_3 = \frac{1}{a}(a_3, -a_2, a_1)^T, \]
satisfies that \( Ax_3 = 0, \|x_3\| = 1 \), so the vector \( x_3 \) is a unit eigenvector corresponding to eigenvalue zero. Next we choose two orthogonal unit vectors \( x_1, x_2 \) on the plane, which are both perpendicular to the vector \( x_3 \), so that \( W = (x_1, x_2, x_3) \) is an orthogonal matrix, and 
\[ B = W^{-1}AW = W^TA(x_1, x_2, x_3) = (W^TAx_1, W^TAx_2, 0). \]
Because orthogonal transformations preserve skew symmetric structure, i.e., they transform skew symmetric matrix into skew symmetric matrix, \( B \) is still a skew symmetric matrix and has the following form:
\[ B = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = mJ_1. \]
Due to similar matrices have the same eigenvalues, we have \( m = a \), i.e., 
\[ W^{-1}AW = aJ_1. \]

Lemma 3.6. If \( Q \) is a skew symmetric matrix, then \( e^Q \) is an orthogonal matrix.

Proof. According to the property of index matrix: if \( A, B \) can be exchanged, i.e., \( AB = BA \), then 
\[ e^A \cdot e^B = e^{A+B}. \]
Therefore by 
\[ e^Q \cdot (e^Q)^T = e^Q \cdot e^{Q^2} = e^{Q+Q^T} = e^{Q-Q} = I, \]
\( e^Q \) is an orthogonal matrix.

Lemma 3.7. Consider the differentiable equation \( \dot{x} = A(t)x \), where \( A(t) \) is a skew symmetric matrix. Let \( x = Q(t)y \), where \( Q(t) \) is an orthogonal matrix, which transforms the above differentiable equation into \( \dot{y} = By \), then \( B \) is still a skew symmetric matrix.

Proof. Let \( x = Q(t)y \). Then 
\[ B = -Q^{-1}\dot{Q} + Q^{-1}AQ. \]
First note that \( Q \) is an orthogonal matrix, so \( Q^{-1}AQ \) is a skew symmetric matrix.
Second, differentiating the equation \( Q^TQ = I \) with respect to \( t \), we have 
\[ \dot{Q}^TQ + Q^T\dot{Q} = 0. \]
And because
\[(−Q^{−1} \dot{Q})^T = −Q^T (Q^{−1})^T = −\dot{Q}^T Q = Q^T \dot{Q} = Q^{−1} \dot{Q},\]
therefore, \(-Q^{−1} \dot{Q}\) is a skew symmetric matrix.

Altogether, \(B\) is still a skew symmetric matrix.

Lemma 3.8. Suppose \(X\) satisfies the homological equation:
\[\dot{X} + XA - AX = Y,\]
where \(A, Y\) are both skew symmetric matrices, then \(X\) is a skew symmetric matrix.

Proof. Let \(X = \sum_k X_k e^{i(k, \theta)}, Y = \sum_k Y_k e^{i(k, \theta)}\). Then comparing the Fourier coefficients of the above homological equation, we have
\[\partial_\omega X_k + X_k A - AX_k = Y_k, \ k \neq 0. \tag{43}\]
Transposing on both sides of the equation gives
\[\partial_\omega X_k^T + A^T X_k^T - X_k^T A^T = Y_k^T, \ k \neq 0, \tag{44}\]
Due to \(A, Y\) are all skew symmetric matrices, the equation (44) is equivalent to
\[\partial_\omega X_k^T - AX_k^T + X_k^T A = -Y_k, \ k \neq 0,\]
i.e.,
\[\partial_\omega (-X_k^T) + (-X_k^T) A - A(-X_k^T) = Y_k, \ k \neq 0. \tag{45}\]
By the equations (43) and (45), \(X_k\) and \(-X_k^T\) are all the solutions of the equation
\[\partial_\omega Z + ZA - AZ = Y_k.\]
According to the uniqueness of the solutions of equations, we have \(X_k = -X_k^T\), i.e., \(X_k\) is a skew symmetric matrix. Therefore, \(X\) is a skew symmetric matrix.

Lemma 3.9. If \(\|Q\| \leq 1/2\), then \(\|e^Q\|^{-1} \leq 2\).

Proof. By \((e^Q)^{-1} = e^{-Q}\), we have
\[(e^Q)^{-1} - I = \sum_{j=1}^{\infty} \frac{(-Q)^j}{j!}.\]
If \(\|Q\| \leq 1/2\), it follows that
\[\|(e^Q)^{-1} - I\| \leq \sum_{j=1}^{\infty} \frac{\|Q\|^j}{j!} \leq \sum_{j=1}^{\infty} \frac{\|Q\|^j}{j!} \leq \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j \leq 1.\]
Therefore,
\[\|(e^Q)^{-1}\| \leq \|(e^Q)^{-1} - I\| + \|I\| \leq 2.\]
3.4. Measure estimates.

Lemma 3.10. Let \( A(\varepsilon) = a_1(\varepsilon)J_1 \) with \( a_1(\varepsilon) = \lambda + a_N \varepsilon^N + O(\varepsilon^{N+1}), a_N \neq 0, N \geq 1 \). Suppose the frequencies \( \omega \in \mathbb{R}^2 \) satisfy the following non-resonant condition:

\[
|\langle k, \omega \rangle \pm \lambda| \geq \frac{\gamma_0}{(|k| + 1)^\tau}, \quad \forall k \in \mathbb{Z}^2,
\]

(46)

where \( \gamma_0 > 0, \tau > 2 \). Suppose \( \hat{A}(\varepsilon) = \hat{a}_1(\varepsilon)J_1 \) is differentiable in \( \varepsilon \) on \( (0, \delta] \), and satisfies that

\[
\|\hat{A}(\varepsilon)\| \leq \mu \varepsilon^{N+1}, \quad \|\hat{A}'(\varepsilon)\| \leq \mu \varepsilon^{N},
\]

where \( \mu \) is a constant and independent of \( \varepsilon \). Let

\[
A_+ = A + \hat{A} = a_+(\varepsilon)J_1, \quad a_+(\varepsilon) = a(\varepsilon) + \hat{a}_1(\varepsilon).
\]

Define

\[
\varphi_k(\varepsilon) = \langle k, \omega \rangle \pm a_+(\varepsilon),
\]

\[
\Pi = \left\{ \varepsilon \in (0, \varepsilon_0) \left| \varphi_k(\varepsilon) \geq \frac{\gamma}{(|k| + 1)^\tau} \quad \forall k \in \mathbb{Z}^2 \right. \right\},
\]

where \( 0 < \gamma \leq \frac{\gamma_0}{2}, \tau' > 2\tau + 2 \), then there exists a sufficiently small \( \varepsilon_0 \in (0, \delta] \), which depends on \( \tau, \tau', A, \mu \), such that \( \forall \varepsilon \in (0, \varepsilon_0] \), we have

\[
\text{meas}\left( (0, \varepsilon) \setminus \Pi \right) \leq c \varepsilon^{N+1} \frac{\gamma}{\gamma_0}.
\]

Proof. For simplicity, we only give the proof of the case: \( \varphi_k(\varepsilon) = \langle k, \omega \rangle - a_+(\varepsilon) \).

Let

\[
g(\varepsilon) = a_+(\varepsilon) - \lambda - a_N \varepsilon^N.
\]

Then there exists a positive constant \( \tilde{\mu} \), depending on \( A, \delta, \mu \), such that

\[
|g(\varepsilon)| \leq \tilde{\mu} \varepsilon^{N+1}, \quad |g'(\varepsilon)| \leq \tilde{\mu} \varepsilon^N, \quad \varepsilon \in (0, \delta].
\]

Moreover,

\[
\varphi_k(\varepsilon) = \langle k, \omega \rangle - \lambda - a_N \varepsilon^N - g(\varepsilon).
\]

Let \( \varepsilon_0 = \min\{\delta, \frac{N[a_N]}{2\tilde{\mu}}\} , 0 < \rho \leq \varepsilon_0 \). Then define

\[
R_k = \left\{ \varepsilon \in (0, \rho) \left| \varphi_k(\varepsilon) < \frac{\gamma}{(|k| + 1)^\tau} \quad k \in \mathbb{Z}^2 \right. \right\}.
\]

Suppose \( \varepsilon \in (0, \rho) \). First, if \( 0 < \varepsilon^N \leq \frac{1}{(N+1)|a_N|} \cdot \frac{\gamma_0}{2(|k| + 1)^\tau} \), then

\[
|a_N \varepsilon^N + g(\varepsilon)| \leq (|a_N| + \tilde{\mu})\varepsilon^N \leq \frac{(|a_N| + \tilde{\mu})\varepsilon^N}{(N+1)|a_N|} \cdot \frac{\gamma_0}{2(|k| + 1)^\tau} \geq \frac{\gamma_0}{2(|k| + 1)^\tau}.
\]

By (46) and \( \gamma \leq \gamma_0/2 \), we have

\[
|\varphi_k(\varepsilon)| \geq |\langle k, \omega \rangle - \lambda| - |a_N \varepsilon^N + g(\varepsilon)| \geq \frac{\gamma_0}{2(|k| + 1)^\tau} \geq \frac{\gamma}{(|k| + 1)^\tau}.
\]

Second, if \( \varepsilon^N \geq \frac{1}{(N+1)|a_N|} \cdot \frac{\gamma_0}{2(|k| + 1)^\tau} \), then

\[
|\varphi_k(\varepsilon)| \geq (N|a_N| - \tilde{\mu})\varepsilon^{N-1} \geq \frac{\gamma_0}{8(|k| + 1)^\tau \varepsilon}.
\]

Therefore,

\[
\text{meas}(R_k) \leq \frac{2\gamma}{(|k| + 1)^\tau} \cdot \frac{8(|k| + 1)^\tau \varepsilon}{\gamma_0} \leq \frac{1}{(N+1)|a_N| \varepsilon^{N+1} \gamma \cdot \frac{\gamma_0}{70} \cdot \frac{1}{(|k| + 1)^\tau - 2\gamma}}.
\]
Since $\tau' > 2\tau + r$, it follows that
\[
\text{meas}\left((0, \rho) \setminus \Pi\right) = \text{meas}\left(\bigcup_{k \in \mathbb{Z}^2} R_k\right) \leq \sum_{k \in \mathbb{Z}^2} \text{meas} (R_k) \leq cN + 1 \frac{2}{\gamma_0}.
\]
where $c = 32(N + 1)|a_N| \sum_{k \in \mathbb{Z}^2} \frac{1}{(|k| + 1)^{\gamma - \gamma_0}}$. \hfill \Box

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