Dynamics of D-branes II. The standard action
— an analogue of the Polyakov action for (fundamental, stacked) D-branes

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Abstract

We introduce a new action \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)} \) for D-branes that is to D-branes as the Polyakov action is to fundamental strings. This ‘standard action’ is abstractly a non-Abelian gauged sigma model — based on maps \( \varphi : (X^A;E;\nabla) \rightarrow Y \) from an Azumaya/matrix manifold \( X^A \) with a fundamental module \( E \) with a connection \( \nabla \) to \( Y \) — enhanced by the dilaton term, the gauge-theory term, and the Chern-Simons/Wess-Zumino term that couples \((\varphi,\nabla)\) to Ramond-Ramond field. In a special situation, this new theory merges the theory of harmonic maps and a gauge theory, with a nilpotent type fuzzy extension. With the analysis developed in D(13.1) (arXiv:1606.08529 [hep-th]) for such maps and an improved understanding of the hierarchy of various admissible conditions on the pairs \((\varphi,\nabla)\) beyond D(13.2.1) (arXiv:1611.09439 [hep-th]) and how they resolve the built-in obstruction to pull-push of covariant tensors under a map from a noncommutative manifold to a commutative manifold, we develop further in this note some covariant differential calculus needed and apply them to work out the first variation — and hence the corresponding equations of motion for D-branes — of the standard action and the second variation of the kinetic term for maps and the dilaton term in this action. Compared with the non-Abelian Dirac-Born-Infeld action constructed in D(13.1) along the same line, the current note brings the Nambu-Goto-string-to-Polyakov-string analogue to D-branes. The current bosonic setting is the first step toward the dynamics of fermionic D-branes (cf. D(11.2): arXiv:1412.0771 [hep-th]) and their quantization as fundamental dynamical objects, in parallel to what happened to the theory of fundamental strings during years 1976–1981.

Key words: D-brane; admissible condition; standard action, enhanced non-Abelian gauged sigma model; Azumaya manifold, \( C^\infty \)-scheme, harmonic map; first and second variation, equations of motion

MSC number 2010: 81T30, 35J20; 16S50, 14A22, 35R01

Acknowledgements. We thank Andrew Strominger, Cumrun Vafa for influence to our understanding of strings, branes, and gravity. C.-H.L. thanks in addition Pei-Ming Ho for a discussion on Ramond-Ramond fields and literature guide and Chenglong Yu for a discussion on admissible conditions; Artan Sheshmani, Brooke Ullery, Ashvin Vishwanath for special/topic/basic courses, spring 2017; Ling-Miao Chou for comments that improve the illustrations and moral support. The project is supported by NSF grants DMS-9803347 and DMS-0074329.
Chien-Hao Liu dedicates this work to
Noel Brady, Hung-Wen Chang, Chongsun Chu, William Grosso, Pei-Ming Ho, Inkang Kim,
who enriched his years at U.C. Berkeley tremendously.*

* (From C.H.L.) It was an amazing time when I landed at Berkeley in the 1990s, following Thurston’s transition to Mathematical Sciences Research Institute (M.S.R.I.). On the mathematical side, representing figures on geometry and topology — 2- and 3-dimensional geometry and topology and related dynamical system (Andrew Casson, Curtis McMullen, William Thurston), 4-dimensional geometry and topology (Robion Kirby), 5-and-above dimensional topology (Morris Hirsch, Stephen Smale), algebraic geometry (Robin Hartshorne), differential and complex geometry (Wu-Yi Hsiang, Shoshichi Kobayashi, Hong-Shi Wu), symplectic geometry and geometric quantization (Alan Weinstein), combinatorial and geometric group theory (relevant to 3-manifold study via the fundamental groups, John Stallings) — seem to converge at Berkeley. On the physics side, several first-or-second generation string-theorists (Orlando Alvarez, Korkut Bardakci, Martin Halpern) and one of the creators of supersymmetry (Bruno Zumino) were there. It was also a time when enumerative geometry and topology motivated by quantum field and string theory started to emerge and for that there were quantum invariants of 3-manifolds (Nicolai Reshetikhin) and mirror symmetry (Alexandre Givental) at Berkeley. Through the topic courses almost all of them gave during these years on their related subject and the timed homeworks one at least attempted, one may acquire a very broad foundation toward a cross field between mathematics and physics if one is ambitious and diligent enough.

However, despite such an amazing time and an intellectually enriching environment, it would be extremely difficult, if not impossible, to learn at least some good part of them without a friend — well, of course, unless one is a genius, which I am painfully not. For that, on the mathematics side, I thank Hung-Wen (Weinstein and Givental’s student) who said hello to me in our first encounter at the elevator at the 9th floor in Evans Hall and influenced my understanding of many topics — particularly symplectic geometry, quantum mechanics and gauge theory — due to his unconventional background from physics, electrical engineering, to mathematics; Bill (Stallings’ student) and Inkang (Casson’s student) for suggesting me a weekly group meeting on Thurston’s lecture notes at Princeton on 3-manifolds.

[Th] W.P. Thurston, The geometry and topology of three-manifolds, typed manuscript, Department of Mathematics, Princeton University, 1979.

which lasted for one and a half year and tremendously influenced the depth of my understanding of Thurston’s work; Noel (Stallings’ student) for a reading seminar on the very thought-provoking French book

[Gr] M. Gromov, Structures métrique pour les variétés riemanniennes, rédigé par J. Lafontaine et P. Pansu, Text Math. 1, Cedic/Fernand-Nathan, Paris, 1980.

for a summer. The numerous other after-class discussions for classes some of them and I happened to sit in shaped a large part of my knowledge pool, even for today. Incidentally, thanks to Prof. Kirby, who served as the Chair for the Graduate Students in the Department at that time. I remember his remark to his staff when I arrived at Berkeley and reported to him after meeting Prof. Thurston: “We have to treat them [referring to all Thurston’s then students] as nice as our own”.

On the physics side, thanks to Chongsun and Pei-Ming (both Zumino’s students) for helping me understand the very challenging topic: quantum field theory, first when we all attended Prof. Bardakci’s course Phys 230A, Quantum Field Theory, based closely on the book

[Ry] L.H. Ryder, Quantum field theory, Cambridge Univ. Press, 1985.

and a second time a year later when we repeated it through the same course given by Prof. Alvarez, followed by a reading group meeting on Quantum Field Theory guided by Prof. Alvarez — another person who I forever have to thank and another event which changed the course of my life permanently. The former with Prof. Bardakci was a semester I had to spend at least three-to-four days a week just on this course: attending lectures, understanding the notes, reading the corresponding chapters or sections of the book, doing the homeworks, occasionally looking into literatures to figure out some of the homeworks, and correcting the mistakes I had made on the returned homework. I even turned in the take-home final. Amazingly, due to the free style of the Department of Mathematics at Princeton University and the visiting student status at U.C. Berkeley, that is the first of the only three courses and only two semesters throughout my graduate student years for which I ever honestly did like a student: do the homework, turn in to get graded, do the final, and in the end get a semester grade back. Special thanks to Prof. Bardakci and the TA for this course, Bogdan Morariu, for grading whatever I turned in, though I was not an officially registered student in that course.

That was a time when I could study something purely for the beauty, mystery, and/or joy of it. That was a time when the future before me seemed unbounded. That was a time when I did not think too much about the less pleasant side of doing research: competitions, publications, credits, ···. That was a time I was surrounded by friends, though only limitedly many, in all the best senses the word ‘friend’ can carry. Alas, that wonderful time, with such a luxurious leisure, is gone forever!
0. Introduction and outline

In this sequel to D(11.1) (arXiv:1406.0929 [math.DG]), D(11.3.1) (arXiv:1508.02347 [math.DG]), D(13.1) (arXiv:1606.08529 [hep-th]) and D(13.2.1) (arXiv:1611.09439 [hep-th]) and along the line of our understanding of the basic structures on D-branes in Polchinski’s TASI 1996 Lecture Notes from the aspect of Grothendieck’s modern Algebraic Geometry initiated in D(1) (arXiv:0709.1515 [math.AG]), we introduce a new action $S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}$ for D-branes that is to D-branes as the (Brink-Di Vecchia-Howe/Deser-Zumino/)Polyakov action is to fundamental strings. This action depends both on the (dilaton field $\rho$, metric $h$) on the underlying topology $X$ of the D-brane world-volume and on the background (dilaton field $\Phi$, metric $g$, $B$-field $B$, Ramond-Ramond field $C$) on the target space-time $Y$; and is naturally a non-Abelian gauged sigma model — based on maps $\varphi: (X^{Az},E;\nabla) \to Y$ from an Azumaya/matrix manifold $X^{Az}$ with a fundamental module $E$ with a connection $\nabla$ to $Y$ — enhanced by the dilaton term that couples $(\varphi,\nabla)$ to $(\rho,\Phi)$, the $B$-coupled gauge-theory term that couples $\nabla$ to $B$, and the Chern-Simons/Wess-Zumino term that couples $(\varphi,\nabla)$ to $(B,C)$ in our standard action $S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}$.

Before one can do so, one needs to resolve the built-in obstruction of pull-push of covariant tensors under a map from a noncommutative manifold to a commutative manifold. Such issue already appeared in the construction of the non-Abelian Dirac-Born-Infeld action (D(13.1) ). In this note, we give a hierarchy of various admissible conditions on the pairs $(\varphi,\nabla)$ that are enough to resolve the issue while being open-string compatible (Sec. 2). This improves our understanding of admissible conditions beyond D(13.2.1). With the noncommutative analysis developed in D(13.1), we develop further in this note some covariant differential calculus for such maps (Sec. 3) and use it to define the standard action for D-branes (Sec. 4). After promoting the setting to a family version (Sec. 5), we work out the first variation — and hence the corresponding equations of motion for D-branes — of the standard action (Sec. 6) and the second variation of the kinetic term for maps and the dilaton term in this action (Sec. 7).

Compared with the non-Abelian Dirac-Born-Infeld action constructed in D(13.1) along the same line, the current standard action is clearly much more manageable. Classically and mathematically and in the special case where the background $(\Phi,B,C)$ on $Y$ is set to vanish, this new theory is a merging of the theory of harmonic maps and a gauge theory (free to choose either a Yang-Mills theory or other kinds of applicable gauge theory) with a nilpotent type fuzzy extension. The current bosonic setting is the first step toward fermionic D-branes (cf. D(11.2): arXiv:1412.0771 [hep-th]) and their quantization as fundamental dynamical objects, in parallel to what happened for fundamental superstrings during 1976–1981; (the road-map at the end: ‘Where we are’).

Convention. References for standard notations, terminology, operations and facts are

(1) Azumaya/matrix algebra: [Ar], [Az], [A-N-T]; (2) sheaves and bundles: [H-L]; with connection: [Bl], [B-B], [D-K], [Ko]; (3) algebraic geometry: [Ha]; $C^\infty$ algebraic geometry: [Jo]; (4) differential geometry: [Eis], [G-H-L], [Hi], [H-E], [K-N]; (5) noncommutative differential geometry: [GB-V-F]; (6) string theory and D-branes: [G-S-W], [Po2], [Po3].

· For clarity, the real line as a real 1-dimensional manifold is denoted by $\mathbb{R}^1$, while the field of real numbers is denoted by $\mathbb{R}$. Similarly, the complex line as a complex 1-dimensional manifold is denoted by $\mathbb{C}^1$, while the field of complex numbers is denoted by $\mathbb{C}$.

· The inclusion $\mathbb{R} \subset \mathbb{C}$ is referred to the field extension of $\mathbb{R}$ to $\mathbb{C}$ by adding $\sqrt{-1}$, unless otherwise noted.
- All manifolds are paracompact, Hausdorff, and admitting a (locally finite) partition of unity. We adopt the *index convention for tensors* from differential geometry. In particular, the tuple coordinate functions on an \( n \)-manifold is denoted by, for example, \((y^1, \cdots, y^n)\). However, no up-low index summation convention is used.

- For this note, ‘*differentiable’*, ‘*smooth’*, and \( C^\infty \) are taken as synonyms.

- \( m \)-matrix vs. manifold of dimension \( m \)

- the Regge slope \( \alpha' \) vs. dummy labelling index \( \alpha \) vs. covariant tensor \( \alpha \)

- \( s \)-section of a sheaf or vector bundle vs. dummy labelling index \( s \)

- \( A_\varphi \)-algebra \( \varphi \) vs. *connection* 1-form \( A_\mu \)

- ring \( R \) vs. \( k \)-th remainder \( R[k] \) vs. Riemann curvature tensor \( R_{ijkl} \)

- boundary \( \partial U \) of an open set \( U \) vs. *partial differentiations* \( \partial, \partial/\partial y^i \)

- \( \text{Spec} \, R \) (:= \{prime ideals of \( R\})\) of a commutative Noetherian ring \( R \) in algebraic geometry vs. \( \text{Spec} \, R \) of a \( C^k \)-ring \( R \) (:= \( \text{Spec} \, R \) := \{\( C^k \)-ring homomorphisms \( R \rightarrow \mathbb{R} \})\)

- morphism between schemes in algebraic geometry vs. \( C^\infty \)-map between \( C^\infty \)-manifolds or \( C^\infty \)-schemes in differential topology and geometry or \( C^\infty \)-algebraic geometry

- group *action* vs. *action* functional for D-branes

- metric tensor \( g \) vs. element \( g' \) in a group \( G \) vs. gauge coupling constant \( g_{\text{gauge}} \)

- sheaves \( F, G \) vs. curvature tensor \( F_\nabla \), gauge-symmetry group \( G_{\text{gauge}} \)

- dilaton field \( \rho \) vs. representation \( \rho_{\text{gauge}} \) of a gauge-symmetry group \( G_{\text{gauge}} \)

- The ‘*support*’ \( \text{Supp} \, (\mathcal{F}) \) of a quasi-coherent sheaf \( \mathcal{F} \) on a scheme \( Y \) in algebraic geometry or on a \( C^k \)-scheme in \( C^k \)-algebraic geometry means the *scheme-theoretical support* of \( \mathcal{F} \) unless otherwise noted; \( \mathcal{I}_Z \) denotes the *ideal sheaf* of a (resp. \( C^k \)-)subscheme \( Z \) of a (resp. \( C^k \)-)scheme \( Y \); \( l(\mathcal{F}) \) denotes the *length* of a coherent sheaf \( \mathcal{F} \) of dimension 0.

- For a sheaf \( \mathcal{F} \) on a topological space \( X \), the notation ‘\( s \in \mathcal{F} \)’ means a local section \( s \in \mathcal{F}(U) \) for some open set \( U \subset X \).

- For an \( \mathcal{O}_X \)-module \( \mathcal{F} \), the *fiber* of \( \mathcal{F} \) at \( x \in X \) is denoted \( \mathcal{F}|_x \) while the *stalk* of \( \mathcal{F} \) at \( x \in X \) is denoted \( \mathcal{F}_x \).

- coordinate-function index, e.g. \((y^1, \cdots, y^n)\) for a real manifold vs. the *exponent of a power*, e.g. \( a_0 y^n + a_1 y^{n-1} + \cdots + a_{r-1} y + a_r \in \mathbb{R}[y] \).

- The current Note D(13.3) continues the study in

  \[ [L-Y8] \quad \text{Dynamics of D-branes I. The non-Abelian Dirac-Born-Infeld action, its first variation, and the equations of motion for D-branes — with remarks on the non-Abelian Chern-Simons/Wess-Zumino term, arXiv:1606.08529 [hep-th].} \)

  \( \text{(D(13.1))} \)

Notations and conventions follow ibidem when applicable.
Outline

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1 Azumaya/matrix manifolds with a fundamental module and differentiable maps therefrom

Basics of maps from an Azumaya/matrix manifold with a fundamental module needed for the current note are collected in this section to fix terminology, notations, and conventions. Readers are referred to [L-Y1] (D(1)), [L-L-S-Y] (D(2)), [L-Y5] (D(11.1)) and [L-Y7] (D(11.3.1)) for details; in particular, why this is a most natural description of D-branes when Polchinski’s TASI 1996 Lecture Note is read from the aspect of Grothendieck’s modern Algebraic Geometry. See also [H-W] and [Wi2].

Azumaya/matrix manifolds with a fundamental module \((X^A, \mathcal{E})\)

From the viewpoint of Algebraic Geometry, a D-brane world-volume is a manifold equipped with a noncommutative structure sheaf of a special type dictated by (oriented) open strings.

Definition 1.1. [Azumaya/matrix manifold with fundamental module] Let \(X\) be a (real, smooth) manifold and \(E\) be a (smooth) complex vector bundle over \(X\). Let

- \(\mathcal{O}_X\) be the structure sheaf of (smooth functions on) \(X\),
- \(\mathcal{O}^C_X := \mathcal{O}_X \otimes_{\mathbb{R}} \mathbb{C}\) be its complexification,
- \(\mathcal{E}\) be the sheaf of (smooth) sections of \(E\), (it’s an \(\mathcal{O}^C_X\)-module), and
- \(\text{End}_{\mathcal{O}^C_X}(\mathcal{E})\) be the endomorphism sheaf of \(\mathcal{E}\) as an \(\mathcal{O}^C_X\)-module (i.e. the sheaf of sections of the endomorphism bundle \(\text{End}_\mathbb{C}(E)\) of \(E\)).

Then, the (noncommutative-)ringed topological space

\[X^A := (X, \mathcal{O}^A_X := \text{End}_{\mathcal{O}^C_X}(\mathcal{E}))\]

is called an Azumaya manifold (or synonymously, a matrix manifold to be more concrete to string-theorists.) It is important to note that non-isomorphic complex vector bundles may give rise to isomorphic endomorphism bundles and from the string-theory origin of the setting, in which \(E\) plays the role of a Chan-Paton bundle on a D-brane world-volume, we always want to record \(E\) as a part of the data in defining \(X^A\). Thus, we call the pair \((X^A, \mathcal{E})\) (or \((X^A, E)\) in bundle notation) an Azumaya/matrix manifold with a fundamental module.

While it may be hard to visualize \(X^A\) geometrically, there in general is an abundant family of commutative \(\mathcal{O}_X\)-subalgebras

\[\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}^A_X\]

that define an abundant family of \(C^\infty\)-schemes

\[X_\mathcal{A} := \text{Spec}^R(\mathcal{A})\]

each finite and germwise algebraic over \(X\). They may help visualize \(X^A\) geometrically.

Definition 1.2. [(commutative) surrogate of \(X^A\)] Such \(X_\mathcal{A}\) is called a (commutative) surrogate of (the noncommutative manifold) \(X^A\). Cf. Figure 1-1.
Without loss of generality, one may assume that $X$ is connected. However even so, a surrogate $X_A$ of $X^{Az}$ in general is disconnected locally over $X$ (and can be disconnected globally as well; cf. Figure 1-1). To keep track of this algebraically, recall the following definition:

**Definition 1.3. [complete set of orthogonal idempotents]** (Cf. e.g. [Ei].) Let $R$ be an (associative, unital) ring, with the identity element $1$. A set of elements $\{e_1, \ldots, e_s\} \subset R$ is called a complete set of orthogonal idempotents if the following three conditions are satisfied

1. idempotent \[ e_i^2 = e_i, \quad i = 1, \ldots, s. \]
2. orthogonal \[ e_i e_j = 0 \quad \text{for} \quad i \neq j. \]
3. complete \[ e_1 + \cdots + e_s = 1. \]

A complete set orthogonal idempotents $\{e_1, \ldots, e_s\}$ is called maximal if no $e_i$ in the set can be further decomposed into a summation $e_i = e' + e''$ of two orthogonal idempotents.

Let $O_X \subset A \subset O_X^{Az}$ be a commutative $O_X$-subalgebra of $O_X^{Az}$ and $X_A$ the associate surrogate of $X^{Az}$. Then, for $U \subset X$ an open set, there is a unique maximal complete set of orthogonal idempotents $\{e_1, \ldots, e_s\}$ of the $C^\infty$-ring $A(U)$ and it corresponds to the set of connected

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1. Unfamiliar physicists may consult [Ar] for basics of Azumaya algebras; see also [Az] and [A-N-T]. Simply put, an Azumaya manifold is topologically a smooth manifold but with a structure sheaf that has fibers Azumaya algebras over $\mathbb{C}$. These fibers are all isomorphic to a matrix ring $M_{r \times r}(\mathbb{C})$ (and hence the synonym matrix manifold) for some fixed $r$ but the isomorphisms involved are not canonical (and hence why the term ‘Azumaya manifold’ is more appropriate mathematically).
components of $X_A | U = \text{Spec}^c (\mathcal{A}(U))$. Up to a relabelling, $e_i$ corresponds the function on $X_A (U)$ that is constant 1 on the $i$-th connected component and 0 on all other connected components.

Finally, we recall also the tangent sheaf and the cotangent sheaf of $X^E$.

**Definition 1.4. [tangent sheaf, cotangent sheaf, inner derivations on $X^E$]** The sheaf of (left) derivations on $\mathcal{O}_X^E$ is denoted by $\mathcal{T}_X^E$ and is called the tangent sheaf of $X^E$. The sheaf of Kähler differentials of $\mathcal{O}_X^E$ is denoted by $\mathcal{T}^E X^E$ and is called the cotangent sheaf of $X^E$. $\mathcal{T}_X^E$ is naturally a (left) $\mathcal{O}_X^E$-module while $\mathcal{T}^E X^E$ is naturally a (left) $\mathcal{O}_X^E$-module. For our purpose, we treat both as $\mathcal{O}_X^C$-modules. There is a natural $\mathcal{O}_X^C$-module homomorphism

$$
\mathcal{O}_X^E \rightarrow \mathcal{T}_X^E
$$

where $[m, \cdot]$ acts on $\mathcal{O}_X^E$ by $m' \mapsto [m, m'] := mm' - m'm$. The image of this homomorphism is called the sheaf/$\mathcal{O}_X^C$-module of inner derivations on $\mathcal{O}_X^E$ and is denoted by $\text{Inn}(\mathcal{O}_X^E)$ or $\text{Inn}(X^E)$. The kernel of the above map is exactly the center $\mathcal{O}_X^C \cdot \text{Id}_E$, canonically identified with $\mathcal{O}_X^C$, of $\mathcal{O}_X^E$. When the choice of a representative of an element of $\text{Inn}(\mathcal{O}_X^E)$ by an element in $\mathcal{O}_X^E$ is irrelevant to an issue, we’ll represent elements of $\text{Inn}(\mathcal{O}_X^E)$ simply by elements in $\mathcal{O}_X^E$.

**When $E$ is equipped with a connection $\nabla$**

From the stringy origin of the setting with $E$ serving as the Chan-Paton bundle on the D-brane world-volume, $E$ is equipped with a gauge field (i.e. a connection) created by massless excitations of open strings. Thus, let $\nabla$ be a connection on $E$. Then $\nabla$ induces a connection $D$ on $\mathcal{O}_X^E := \text{End}_{\mathcal{O}_X^C}(E)$. With respect to a local trivialization of $E$, $\nabla = d + A$, where $A$ is an $\text{End}_{\mathcal{O}_X^C}(E)$-valued 1-form on $X$. Then $D = d + [A, \cdot]$ on $\mathcal{O}_X^E$ under the induced local trivialization. As a consequence, $D$ leaves the center $\mathcal{O}_X^C$ of $\mathcal{O}_X^E$ invariant and restrict the usual differential $d$ on $\mathcal{O}_X^C$.

Once having the induced connection $D$ on $\mathcal{O}_X^E$, one has then $\mathcal{O}_X^C$-module homomorphism

$$
\mathcal{T}_X^C \rightarrow \mathcal{T}_X^E
$$

$$
\xi \mapsto D_\xi
$$

**Lemma 1.5. [D-induced decomposition of $\mathcal{T}_X^E$]** ([DV-M].) One has the short exact sequence

$$
0 \rightarrow \text{Inn}(\mathcal{O}_X^E) \rightarrow \mathcal{T}_X^E \rightarrow \mathcal{T}_X^C \rightarrow 0
$$

split by the above map.

The following two lemmas address the issue of when an idempotent in $\mathcal{O}_X^E$ can be constant under a derivation $\in \mathcal{T}_X^E$.

**Lemma 1.6. [(local) idempotent under $D$]** With the above notations, let $U \subset X$ be an open set, $\xi$ a vector field on $U$, and $\{e_1, \cdots, e_s\}$ be a complete set of orthogonal idempotents of $\mathcal{O}_X^E (U)$. Assume that, say, $D_\xi e_1$ commutes with all $e_i$, $i = 1, \cdots, s$. Then $D_\xi e_1 = 0$. 

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As a dynamical object in space-time, a D-brane moving in a space-time $Y$ from $(\text{differentiable map})$ the associated Azumaya/matrix manifold with a fundamental module. A fold, $E$.

**Definition 1.8.** [map from Azumaya/matrix manifold] Let $X$ be a (real, smooth) manifold, $E$ be a complex vector bundle of rank $r$ over $X$, and $(X^\mathbb{C}, E) := (X, C^\infty(\text{End}_\mathbb{C}(E)), E)$ be the associated Azumaya/matrix manifold with a fundamental module. A map (synonymously, differentiable map, smooth map)

$$\varphi : (X^\mathbb{C}, E) \rightarrow Y$$

from $(X^\mathbb{C}, E)$ to a (real, smooth) manifold $Y$ is defined contravariantly by a ring-homomorphism

$$\varphi^\sharp : C^\infty(Y) \rightarrow C^\infty(\text{End}_\mathbb{C}(E)).$$

Equivalently in terms of sheaf language, let $O_Y$ be the structure sheaf of $Y$. Regard both $O_Y$ and $O_X^\mathbb{C}$ as equivalence classes of gluing system of rings over the topological space $Y$ and $X$ respectively. Then the above $\varphi^\sharp$ specifies an equivalence class of gluing systems of ring-homomorphisms over $\mathbb{R} \subset \mathbb{C}$

$$O_Y \rightarrow O_X^\mathbb{C},$$

which we will still denote by $\varphi^\sharp$.  

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**Lemma 1.7.** [(local) idempotent under inner derivation] With the above notations, let $U \subset X$ be an open set, $m \in O_X^\mathbb{C}(U)$ represent an inner derivation of $O_X^\mathbb{C}(U)$, and $\{e_1, \cdots, e_s\}$ be a complete set of orthogonal idempotents of $O_X^\mathbb{C}(U)$. Assume that, say, $[m, e_1]$ commutes with all $e_i$, $i = 1, \cdots, s$. Then $[m, e_1] = 0$.

**Proof.** Note that the proof of Lemma 1.6 uses only the Leibnitz rule property of $D_\xi$ on $O_X^\mathbb{C}(U)$ and the commutativity property of $D_\xi e_i$ with $e_1, \cdots, e_s$. Since $[m, \cdot]$ satisfies also the Leibniz rule property on $O_X^\mathbb{C}(U)$ and by assumption $[m, e_1]$ commutes with $e_1, \cdots, e_s$, the same proof goes through.

The contraction $\text{End}_{O_X^\mathbb{C}}(E) = E \otimes_{O_X^\mathbb{C}} E^\vee \rightarrow O_X^\mathbb{C}$ defines a trace map

$$\text{Tr} : O_X^\mathbb{C} \rightarrow O_X^\mathbb{C}.$$

One has

$$d\text{Tr} = \text{Tr} D,$$

where $d$ is the ordinary differential on $O_X^\mathbb{C}$.
Through the Generalized Division Lemma à la Malgrange, one can show that $\varphi^\sharp$ extends to a commutative diagram

\[
\begin{array}{c}
O_X^A \xleftarrow{\varphi^\sharp} O_Y \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
O_X \xrightarrow{pr_X^Y} O_{X \times Y},
\end{array}
\]

of equivalence classes of ring-homomorphisms (over $\mathbb{R}$ or $\mathbb{R} \subset \mathbb{C}$, whichever is applicable) between equivalence classes of gluing systems of rings, with

\[
\begin{array}{c}
A_{\varphi} \xleftarrow{f_\varphi^\sharp} O_Y \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
O_X \xrightarrow{pr_X^Y} O_{X \times Y},
\end{array}
\]

a commutative diagram of equivalence classes of ring-homomorphisms between equivalence classes of gluing systems of $C^\infty$-rings. Here, $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ are the projection maps, $O_X \hookrightarrow O_X^A$ follows from the inclusion of the center $O_X^C$ of $O_X^A$, and

\[
A_{\varphi} := O_X(Im.\varphi^\sharp) = Im.\varphi^\sharp.
\]

(Cf. [L-Y7: Theorem 3.1.1] (D(11.3.1)).)

In terms of spaces, one has the following equivalent diagram of maps

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X^A_{\varphi} \xleftarrow{\varphi} Y \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{pr_X} X \times Y \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X \phi \xleftarrow{\varphi} X \times Y,
\end{array}
\]

where $X_{\varphi}$ is the $C^\infty$-scheme $X_{\varphi} := \text{Spec}^R A_{\varphi}$ associated to $A_{\varphi}$.

**Definition 1.9.** [graph of $\varphi$] The push-forward $\tilde{\varphi}_* \mathcal{E} =: \tilde{\mathcal{E}}_{\varphi}$ of $\mathcal{E}$ under $\tilde{\varphi}$ is called the graph of $\varphi$. It is an $O_{X \times Y}^C$-module. Its $C^\infty$-scheme-theoretical support is denoted by $\text{Supp} (\tilde{\mathcal{E}}_{\varphi})$.

**Definition 1.10.** [surrogate of $X^A_{\varphi}$ specified by $\varphi$] The $C^\infty$-scheme $X_{\varphi}$ is called the surrogate of $X^A_{\varphi}$ specified by $\varphi$.

$X_{\varphi}$ is finite and germwise algebraic over $X$ and, by construction, it admits a canonical embedding $\tilde{f}_{\varphi} : X_{\varphi} \to X \times Y$ into $X \times Y$ as a $C^\infty$-subscheme. The image is identical to $\text{Supp} (\tilde{\mathcal{E}}_{\varphi})$. Cf. **Figure 1-2** and **Figure 1-3**.
Figure 1-2. A map $\varphi : (X^{\mathbb{A}_\mathbb{C}}, E) \rightarrow Y$ specifies a surrogate $X_\varphi$ of $X^{\mathbb{A}_\mathbb{C}}$ over $X$. $X_\varphi$ is a $C^\infty$-scheme that may not be reduced (i.e. it may have some nilpotent fuzzy structure thereon). It on one hand is dominated by $X^{\mathbb{A}_\mathbb{C}}$ and on the other dominates and is finite and germwise algebraic over $X$.

Compatibility between the map $\varphi$ and the connection $\nabla$

Up to this point, the map $\varphi : (X^{\mathbb{A}_\mathbb{C}}, E) \rightarrow Y$ and the connection $\nabla$ on $E$ are quite independent objects. A priori, there doesn’t seem to be any reason why they should constrain or influence each other at the current purely differential-topological level. However, when one moves on to address the issue of constructing an action functional for $(\varphi, \nabla)$ as in [L-Y8] (D(13.1)), one immediately realizes that,

- Due to a built-in mathematical obstruction in the problem, one needs some compatibility condition between $\varphi$ and $\nabla$ before one can even begin the attempt to construct an action functional for $(\varphi, \nabla)$.

Furthermore, as a hindsight, that there needs to be a compatibility condition on $(\varphi, \nabla)$ is also implied by string theory:

- We need a condition on $(\varphi, \nabla)$ to encode the stringy fact that the gauge field $\nabla$ on the D-brane world-volume as ‘seen’ by open strings in $Y$ through $\varphi$ should be massless.

We address such compatibility condition on $(\varphi, \nabla)$ systematically in the next section.

2 Pull-push of tensors and admissible conditions on $(\varphi, \nabla)$

When one attempts to construct an action functional for a theory that involves maps from a world-volume to a target space-time, one unavoidably has to come across the notion of ‘pulling back a (covariant) tensor; for example, the metric tensor or a differential form on the target space-time to the world-volume’. In the case where only maps from a commutative world-volume to a (commutative) space-time are involved, this is a well-established standard notion from differential topology. However, in a case, like ours, where maps from a noncommutative world-volume to a (commutative) space-time is involved,

$$\varphi : Space(S) \rightarrow Space(R),$$
Figure 1-3. The equivalence between a map \( \varphi \) from an Azumaya manifold with a fundamental module \( (X, \mathcal{O}_X^{\text{Az}} := \text{End}_{\mathcal{O}_X^Z}(\mathcal{E}), \mathcal{E}) \) to a manifold \( Y \) and a special kind of Fourier-Mukai transform \( \tilde{\mathcal{E}} \in \text{Mod}^C(X \times Y) \) from \( X \) to \( Y \). Here, \( \text{Mod}^C(X \times Y) \) is the category of \( \mathcal{O}_{X \times Y}^C \)-modules.
with the accompanying contravariant ring-homomorphism
\[ S \leftarrow R : \varphi^* \]
there is a built-in mathematical obstruction to such a notion. Here, \( S \) is an (associative, unital) noncommutative ring, \( R \) is a (associative, unital) commutative ring, and \( \text{Space}(S) \) and \( \text{Space}(R) \) are the topological spaces whose function rings are \( S \) and \( R \) respectively.

For the noncommutative ring \( S \), its (standard and functorial-in-the-category-rings) bi-\( S \)-module of Kähler differentials is naturally defined to be
\[ \Omega_{S}^{\text{Kähler}} := \text{Span}_{(S,S)}\{ds \mid s \in S\}/(ds,ds') \text{ for } S \]
while for the commutative ring \( R \) its (standard and functorial-in-the-category-of-commutative-rings) (left) \( R \)-module of Kähler differentials is naturally defined to be
\[ \Omega_{R}^{\text{Kähler}} := \text{Span}_{(R,R)}\{dr \mid r \in R\}/(dr,dr') \text{ for } R \]
with the convention that \( rdr' = (dr')r \) to turn it to a bi-\( R \)-module as well. Treating \( R \) as a ring (that happens to be commutative), it also has the (standard and functorial-in-the-category-rings) bi-\( R \)-module of Kähler differentials
\[ \Omega_{R}^{\text{nc,Kähler}} := \text{Span}_{(R,R)}\{dr \mid r \in R\}/(dr,dr') \text{ for } R \]
exactly like \( \Omega_{S}^{\text{Kähler}} \) for \( S \). There is a built-in tautological quotient homomorphism as bi-\( R \)-modules
\[ \Omega_{R}^{\text{nc,Kähler}} \overset{r_1(dr)r_2}{\longrightarrow} \Omega_{R}^{\text{Kähler}} \]
whose kernel is generated by \( \{rdr' - (dr')r \mid r, r' \in R\} \). Given the map \( \varphi : \text{Space}(S) \to \text{Space}(R) \), one has the following built-in diagram
\[ \Omega_{S}^{\text{Kähler}} \overset{\varphi^*}{\leftarrow} \Omega_{R}^{\text{nc,Kähler}} \]
\[ \Omega_{R}^{\text{Kähler}} \]
where \( \varphi^*(r_1(dr)r_2) = \varphi^*(r_1)d(\varphi^*(r))\varphi^*(r_2) \). The issue is now whether one can extend the above diagram to the following commutative diagram
\[ \Omega_{S}^{\text{Kähler}} \overset{\varphi^*}{\leftarrow} \Omega_{R}^{\text{nc,Kähler}} \]
\[ \Omega_{R}^{\text{Kähler}} \]
\[ \Omega_{R}^{\text{Kähler}} \]
The answer is No, in general. See, e.g., [L-Y5: Example 4.1.20] (D(11.1)) for an explicit counterexample. When \( R \) is a \( C^\infty \)-ring, e.g. the function-ring \( C^\infty(Y) \) of a smooth manifold \( Y \), then the \( R \)-module \( \Omega_{R} \) of differentials of \( R \) is a further quotient of the above module \( \Omega_{R}^{\text{Kähler}} \) of Kähler differentials by additional relations generated by applications of the chain rule on the transcendental smooth operations in the \( C^\infty \)-ring structure of \( R \) ([Jo]; cf. [L-Y5: Sec. 4.1] (D(11.1))). The issue becomes even more involved. In particular, as the counterexample ibidem shows

- [built-in mathematical obstruction of pullback] For a map \( \varphi : (X^A,E) \to Y \), there is no way to define functorially a pull-back map \( \mathcal{T}^*Y \to \mathcal{T}^*X^A \) that takes a (covariant) 1-tensor on \( Y \) to a 1-tensor on \( X^A \). As a consequence, there is no functorial way to pull back a (covariant) tensor on \( Y \) to a tensor on \( X^A \).
Before the attempt to construct an action functional that involves such maps $\varphi$, one has to resolve the above obstruction first. In [L-Y8] (D(13.1)), we learned how to use the connection $\nabla$ to impose a natural admissible condition on $\varphi$ so that the above obstruction is bypassed through the surrogate $X_{\varphi}$ of $X^k$ specified by $\varphi$. With the lesson learned therefrom and further thought beyond [L-Y9] (D(13.2.1)), we propose (Sec. 2.1) in this section a still-natural-but-much-weaker admissible condition on $(\varphi, \nabla)$ that bypasses even the surrogate $X_{\varphi}$ but is still robust enough to construct naturally a pull-push map we need on tensors. It turns out that this much weaker admissible condition remains to be compatible with open strings (Sec. 2.2).

### 2.1 Admissible conditions on $(\varphi, \nabla)$ and the resolution of the pull-push issue

A hierarchy of admissible conditions on $(\varphi, \nabla)$ is introduced. A theorem on how even the weakest admissible condition in the hierarchy can resolve the above obstruction in our case is proved.

#### Three hierarchical admissible conditions

**Definition 2.1.1. [admissible connection on $E$]** Let $\varphi : (X^k, \mathcal{E}) \to Y$ be a map. For a connection $\nabla$ on $\mathcal{E}$, let $D$ be its induced connection on $O_{\mathcal{E}}^{X^k}$. A connection $\nabla$ on $\mathcal{E}$ is called

- $(\ast_1)$-admissible to $\varphi$ if $D_\xi A_{\varphi} \subset \text{Comm}(A_{\varphi})$;
- $(\ast_2)$-admissible to $\varphi$ if $D_\xi \text{Comm}(A_{\varphi}) \subset \text{Comm}(A_{\varphi})$;
- $(\ast_3)$-admissible to $\varphi$ if $D_\xi A_{\varphi} \subset A_{\varphi}$

for all $\xi \in T_{\mathcal{E}} X$. Here, $\text{Comm}(A_{\varphi})$ denotes the commutant of $A_{\varphi}$ in $O_{\mathcal{E}}^{X^k}$.

When $\nabla$ is $(\ast_1)$-admissible to $\varphi$, we will take the following as synonyms:

- $(\varphi, \nabla)$ is an $(\ast_1)$-admissible pair,
- $\varphi$ is $(\ast_1)$-admissible to $\nabla$,
- $\varphi : (X^k, \mathcal{E}; \nabla) \to Y$ is $(\ast_1)$-admissible.

Similarly, for $(\ast_2)$-admissible pair $(\varphi, \nabla)$ and $(\ast_3)$-admissible pair $(\varphi, \nabla)$, ..., etc..

**Lemma 2.1.2. [hierarchy of admissible conditions]**

$\text{Admissible Condition (\ast_3)} \implies \text{Admissible Condition (\ast_2)} \implies \text{Admissible Condition (\ast_1)}$.

**Proof.** Admissible Condition $(\ast_3)$ says that the $O_X$-subalgebra $A_{\varphi} \subset O_{\mathcal{E}}^{X^k}$ is invariant under $D$-parallel transports along paths on $X$. Since $D$-parallel transports on $O_{\mathcal{E}}^{X^k}$ are algebra-isomorphisms, if $A_{\varphi}$ is $D$-invariant, the $O_X$-subalgebra $\text{Comm}(A_{\varphi})$ of $O_{\mathcal{E}}^{X^k}$ must also be $D$-invariant since it is determined by $A_{\varphi}$ fiberwise algebraically. In other words, Admissible Condition $(\ast_3)$ $\implies$ Admissible Condition $(\ast_2)$.

Since $A_{\varphi}$ is commutative, $A_{\varphi} \subset \text{Comm}(A_{\varphi})$. Thus, the inclusion $D \cdot A_{\varphi} \subset D \cdot \text{Comm}(A_{\varphi})$ always holds. This implies that Admissible Condition $(\ast_2)$ $\implies$ Admissible Condition $(\ast_1)$.

**Definition 2.1.3. [strict admissible connection on $E$]** Continuing Definition 2.1.1. Let $F_{\nabla}$ be the curvature tensor of $\nabla$. It is an $O_{\mathcal{E}}^{X^k}$-valued 2-form on $X$. Then, for $\cdot = 1, 2, 3$, $\nabla$ is called strictly $(\ast_\cdot)$-admissible to $\varphi$ if
\[ \nabla \text{ is } (\ast,)-\text{admissible to } \varphi \text{ and } F_\nabla \text{ takes values in } \text{Comm}(A_\varphi) \subset O^A_X. \]

In this case, \((\varphi, \nabla)\) is said to be a \emph{strictly \((\ast,)-\text{admissible pair}\)}.

Clearly, the same hierarchy holds for strict admissible conditions:

\[
\text{strict } (\ast_3) \implies \text{strict } (\ast_2) \implies \text{strict } (\ast_1).
\]

The Strict \((\ast,)-\text{Admissible Condition on } (\varphi, \nabla)\) was introduced in [L-Y8: Definition 2.2.1] (D(13.1)) to define the Dirac-Born-Infeld action for \((\varphi, \nabla)\).

\[\text{Lemma 2.1.4. [commutativity under admissible condition]} \]

Let \(\varphi : (X^A_X, E; \nabla) \rightarrow Y\) be a map.  
(1) If \((\varphi, \nabla)\) is \((\ast_1)-\text{admissible}\), then \([D_{\xi_1}\varphi^\sharp(f_1), \varphi^\sharp(f_2)] = 0\) for all \(f_1, f_2 \in C^\infty(Y)\) and \(\xi \in T_X X\).  
(2) If \((\varphi, \nabla)\) is \((\ast_2)-\text{admissible}\), then \([D_{\xi_1}\varphi^\sharp(f_1), D_{\xi_2}\varphi^\sharp(f_2)] = 0\) for all \(f_1, f_2 \in C^\infty(Y)\) and \(\xi_1, \xi_2 \in T_X X\).

\[\text{Proof.} \text{ Statement (1) is the } (\ast_1)-\text{Admissible Condition itself.} \]

\[\text{For Statement (2), let } f_1, f_2 \in C^\infty(Y) \text{ and } \xi_1, \xi_2 \in T_X X. \text{ Then } [D_{\xi_1}\varphi^\sharp(f_1), \varphi^\sharp(f_2)] = 0 \text{ since } A_\varphi \subset \text{Comm}(A_\varphi). \text{ Thus, applying } D_{\xi_2} \text{ to both sides,} \]

\[ [D_{\xi_2} D_{\xi_1}\varphi^\sharp(f_1), \varphi^\sharp(f_2)] + [D_{\xi_1}\varphi^\sharp(f_1), D_{\xi_2}\varphi^\sharp(f_2)] = 0. \]

The \((\ast_2)-\text{Admissible Condition implies that}\)

\[ D_{\xi_2} D_{\xi_1}\varphi^\sharp(f_1) \in \text{Comm}(A_\varphi). \]

And, hence, \([D_{\xi_2} D_{\xi_1}\varphi^\sharp(f_1), \varphi^\sharp(f_2)] = 0. \text{ Statement (2) follows.} \]

\[\text{Resolution of the pull-push issue under Admissible Condition } (\ast_1)\]

The current theme is devoted to the proof of the following theorem:

\[\text{Theorem 2.1.5. [pull-push under } (\ast_1)-\text{admissible } (\varphi, \nabla)] \]

Let \((\varphi, \nabla)\) be \((\ast_1)-\text{admissible}. \]

\[\text{Then the assignment} \]

\[ \varphi^\circ : \Omega_{C^\infty(Y)} \rightarrow \Omega_{C^\infty(X)} \otimes_{C^\infty(X)} C^\infty(\text{End}_E(E)) \]

\[ f_1 df_2 \mapsto \varphi^\sharp(f_1) D\varphi^\sharp(f_2) \]

\[\text{is well-defined.} \]

The study in [L-Y8: Sec. 4] (D(13.1)) allows one to express \(\varphi^\circ(df)\) locally explicit enough so that one can check that \(\varphi^\circ\) is well-defined when \((\varphi, \nabla)\) is \((\ast_1)-\text{admissible}. \text{ Note that, with Lemma 2.1.2, this implies that if } (\varphi, \nabla) \text{ is either } (\ast_2)- \text{or } (\ast_3)-\text{admissible, then } \varphi^\circ \text{ is also well-defined. We now proceed to prove the theorem.} \]
Lemma 2.1.6. [local expression of $\varphi^\circ(df)$ for $(*_1)$-admissible $(\varphi, \nabla, I)$] Let $(\varphi, \nabla)$ be $(*_1)$-admissible; i.e. $D_\xi A_\varphi \subset \text{Comm}(A_\varphi)$ for all $\xi \in \mathcal{T}_s X$. Let $U \subset X$ be a small enough open set so that $\varphi(U^{k_0})$ is contained in a coordinate chart of $Y$, with coordinate $y = (y^1, \cdots, y^n)$. For $f \in C^\infty(Y)$, recall the germwise-over-$U$ polynomial $R^f[1]$ in $(y^1, \cdots, y^n)$ with coefficients in $\mathcal{O}^*_\xi$ from [LY8: Sec. 4 & Remark/Notation 4.2.3.5] (D.13.1). Then, for $\xi$ a vector field on $U$ and $f \in C^\infty(Y)$, at the level of germs over $U$,

$$(\varphi^\circ(df))(\xi) = R^f[1]|_{y^d \sim D_\xi(\varphi^\circ(y^d))}, \text{ for all multi-degree } d \text{ in } R^f[1].$$

Here, for a multiple degree $d = (d_1, \cdots, d_n)$, $d_i \in \mathbb{Z}_{\geq 0}$, $y^d := (y^{d_1})^{d_1} \cdots (y^{d_n})^{d_n}$ and $y^d \sim D_\xi(\varphi^\circ(y^d))$ means 'replacing $y^d$ by $D_\xi(\varphi^\circ(y^d))$'.

Proof. Denote the coordinate chart of $Y$ in the Statement by $V$. Let $pr_X : X \times Y \to X$, $pr_Y : X \times Y \to Y$ be the projection maps. Recall the induced ring-homomorphism $\hat{\varphi}^\circ : C^\infty(X \times Y) \to C^\infty(\text{End}_C(E))$ over $\mathbb{R} \subset \mathbb{C}$ and the graph $\hat{\varphi}^\circ$ of $\varphi$ and its support $\text{Supp}(\hat{\varphi}^\circ)$ on $X \times Y$. Denote $pr_X^*(f) \in C^\infty(X \times Y)$ still by $f$ when there is no confusion. For clarity, we proceed the proof of the Statement in three steps.

Step (1) How $R^f[1]$ is constructed in [LY8: Sec. 4] (D(13.1)) For any $p \in U$, let $p \in U' \subset U$ be a neighborhood of $p$ in $U$ over which the Generalized Division Lemma à la Malgrange is applied to $f$ on a neighborhood $U' \times V'$ of $\{(p) \times V \cap \text{Supp}(\hat{\varphi}^\circ)\}_{\text{red}} =: \{q_1, \cdots, q_s\}$ in $(X \times Y)/X$ with respect to the characteristic polynomials $\chi^{(i)}_{\varphi} := \det(y^{i_1} \cdot Id_{Y \times Y} - \hat{\varphi}^\circ(y^{i_2})) \in C^\infty(U')[y^1, \cdots, y^n]$ in $C^\infty(U' \times V'$, $i = 1, \cdots, n$. Passing to a smaller open subset if necessary, one may assume that $V'$ is a disjoint union $V'_1 \cup \cdots \cup V'_k$ with $U' \times V'_k$ a neighborhood of $q_k$ and the closure $\overline{V}_1, \cdots, \overline{V}_s$ are all disjoint from each other. Let $1_{\{k\}} \in C^\infty(Y)$ be a smooth functions on $Y$ that takes the value 1 on $V'_k$, and the value zero on $V'_j$, $j \neq k, k = 1, \cdots, s$. (Cf. [LY8: Sec. 4.2.3] (D(13.1)).) Then

$$f|_{U' \times V'} = \sum_{k=1}^{s} 1_{\{k\}} \cdot \left(\sum_d c^{f,k}_{d} y^d + \sum_{i,j=1}^{n} Q^{f,k}_{i,j} \chi^{(i)}_{\varphi} \chi^{(j)}_{\varphi}\right)$$

for some

- $c^{f,k}_{d} \in C^\infty(U')$ for all $k$ and $d$, and $\sum_d c^{f,k}_{d} y^d \in C^\infty(U')[y^1, \cdots, y^n]$ for all $k$;
- $Q^{f,k}_{i,j} \in C^\infty(U' \times V'_k)$ for all $k$ and $i,j$.

In terms of this, over $U'$,

$$R^f[1] = \sum_{k=1}^{s} \varphi^\circ(1_{\{k\}}) \sum_d c^{f,k}_{d} y^d$$

and

$$\varphi^\circ(f)|_{U'} = R^f[1]|_{y^d \sim \varphi^\circ(y^d)} \text{ for all multi-degree } d \text{ in } R^f[1]$$

since $\varphi^\circ(\chi^{(i)}_{\varphi}) = 0$ for $i = 1, \cdots, n$, $\varphi^\circ(f) = \varphi^\circ(pr_X^*(f))$.

Step (2) $\{\varphi^\circ(1_{\{k\}})\}_{k=1}^{s}$ as the maximal complete set of orthogonal $D$-parallel idempotents/$U'$

Since $\varphi^\circ(f)$ depends only on the restriction of $f$, regarded on $X \times Y$, to $\text{Supp}(\hat{\varphi}^\circ)$, one has

$$\varphi^\circ(1_{1} + \cdots + 1_{s}) = \varphi^\circ(1_{X \times Y}) = Id_{\mathcal{E}_x}$$

over $U' \subset X$. Since, in addition, $1_{\{k\}} = 1_{\{1\}}$ for all $k$, $1_{\{k\}}1_{\{k'\}} = 0$ for all $k \neq k'$, and $s = \text{the number of the connected components of } X_{\varphi|\mathcal{U'}} \text{ for } U' \text{ small enough}$, the collection $\{\varphi^\circ(1_{\{1\}}), \cdots, \varphi^\circ(1_{\{s\}})\}$ gives the maximal complete set of orthogonal idempotents in $A_\varphi|\mathcal{U'}$. 

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Furthermore, since $D_ξA_ϕ \subset Comm(A_ϕ)$ for all $ξ \in \mathcal{T}_*X$, $D_ξϕ^\sharp(1_{(k)})$ and $ϕ^\sharp(1_{(k')})$ commute for all $k, k' = 1, \cdots, s$. It follows from Lemma 1.6 that

$$D_ξϕ^\sharp(1_{(1)}) = \cdots = D_ξϕ^\sharp(1_{(s)}) = 0$$

for all $ξ$. In other words, $ϕ^\sharp(1_{(1)}), \cdots, ϕ^\sharp(1_{(s)})$ are $D$-parallel over $U'$.

**Step (3) The evaluation of $(ϕ^\circ(df))(ξ)$ over $U'$** We are now ready to evaluate the $O^f_ξX$-valued derivation $D_ξϕ$ on $f$, locally and germwise over $U'$. Step (1) and Step (2) together imply that, for $ξ \in \mathcal{T}_*X$ and over $U'$,

$$(ϕ^\circ(df))(ξ) := D_ξ(ϕ^\sharp(f))$$

$$= \sum_{k=1}^s ϕ^\sharp(1_{(k)}) \sum_d (ξc_{d}^{f_k}) ϕ^\sharp(y^d) + \sum_{k=1}^s ϕ^\sharp(1_{(k)}) \sum_d cf_{d}^{f_k} D_ξ(ϕ^\sharp(y^d))$$

= Term (I) + Term (II)

since $D_ξϕ^\sharp(1_{(k)}) = 0$ for $k = 1, \cdots, s$.

Note that

$$\text{Term (II)} = R^f[1]|_{y^d \mapsto D_ξ(ϕ^\sharp(y^d))}, \text{ for all multi-degree } d \text{ in } R^f[1].$$

It remains to prove that Term (I) vanishes. But this is the situation studied in [L-Y8: Proposition 4.2.3.1:Proof] (D(13.1)). In essence, since

$$f = \sum_{k=1}^s 1_{(k)} \cdot \left( \sum_d cf_{d}^{f_k} y^d + \sum_{i,j=1}^n Q_{(i,j)}^{f_k} χ_{(i)}^f χ_{(j)}^f \right)$$

on $U' \times V'$ and $f$ on $(X \times Y)/X$ is independent of $X$, one has

$$\text{Term (I)} = ϕ^\sharp \left( \sum_{k} 1_{(k)} \sum_{d} (ξc_{d}^{f_k}) y^d \right) = ϕ^\sharp(ξf) = 0.$$

Here we denote the canonical lifting of $ξ \in \mathcal{T}_*X$ to $\mathcal{T}_*(X \times Y)$, via the product structure of $X \times Y$, by the same notation $ξ$.

This completes the proof.

\[\square\]

**Lemma 2.1.7. [local expression of $ϕ^\circ(df)$ for $(*)_1$-admissible $(ϕ, ∇)$, II]** Let $(ϕ, ∇)$ be $(*)_1$-admissible. Continuing the setting and notations in Lemma 2.1.6.

Then, locally,

$$(ϕ^\circ(df))(ξ) = \sum_{i=1}^n (ϕ^\circ dy^i)(ξ) \otimes \frac{\partial}{\partial y^i} f = \sum_{i=1}^n \left( D_ξϕ^\sharp(y^i) \cdot ϕ^\sharp(\frac{\partial f}{\partial y^i}) \right).$$

Here $\cdot$ is the multiplication in the ring $C^\infty(End_C(E))$ (and will be omitted later when there is no sacrifice to clarity).
Proof. This is a consequence of Lemma 2.1.6. Continuing the setup in the Statement and the proof thereof. Then, since \( \varphi^\sharp(\chi^\sharp_\varphi) = 0 \) for \( i = 1, \cdots, n \), one has

\[
\varphi^\sharp \left( \frac{\partial}{\partial y^j} f \right) = \varphi^\sharp \left( \frac{\partial}{\partial y^j} R^1 \right) = R^f[1] y^d \cdots \varphi^\sharp \left( \frac{\partial}{\partial y^d} y^d \right), \quad \text{for all multi-degree } d \text{ in } R^f[1].
\]

Since \( (\varphi, \nabla) \) is \( (*_1) \)-admissible, \( D\xi \varphi^\sharp(y^j) \) and \( \varphi^\sharp(y^j) \) commute for \( i, j = 1, \cdots, n \). It follows that

\[
\sum_{i=1}^n D\xi \varphi^\sharp(y^j) \cdot \varphi^\sharp \left( \frac{\partial}{\partial y^j} f \right) = R^f[1] y^d \cdots \sum_{i=1}^n D\xi \varphi^\sharp(y^j)(1), \quad \text{for all multi-degree } d \text{ in } R^f[1].
\]

Which is \( \varphi^\circ(df) \) by Lemma 2.1.6. This proves the lemma.

\( \square \)

Proof of Theorem 2.1.5. We now check in two steps that \( \varphi^\circ \) is well-defined. Note that we only need to do so locally over \( X \). Thus, let \( U \subset X \) be the open set in Lemma 2.1.6 such that \( \varphi(U^\Lambda) \) is contained in a coordinate chart \( V \) of \( Y \), with the coordinate \( (y^1, \cdots, y^n) \). Lemma 2.1.7 implies then that the following assignment is the restriction of \( \varphi^\circ \) to \( U \) and hence is independent of the local coordinate \( (y^1, \cdots, y^n) \) on \( V \):

\[
\varphi^\circ : \Omega_{C^\infty(V)} \longrightarrow \Omega_{C^\infty(U)} \otimes_{C^\infty(U)} C^\infty(End_\mathbb{C}(E|U))
\]

\[
f_1 f_2 \longmapsto \varphi^\sharp(f_1) \cdot \sum_{i=1}^n \varphi^\sharp \left( \frac{\partial f_2}{\partial y^j} \right) \cdot D\varphi^\sharp(y^j).
\]

Here, we use again the fact that \( (\varphi, \nabla) \) is \( (*_1) \)-admissible so that the summand \( D\varphi^\sharp(y^j) \cdot \varphi^\sharp \left( \frac{\partial f_2}{\partial y^j} \right) \) in Lemma 2.1.7 is equal to \( \varphi^\sharp \left( \frac{\partial f_2}{\partial y^j} \right) \cdot D\varphi^\sharp(y^j) \) here, with \( f \) replaced by \( f_2 \). It remains to show that \( \varphi^\circ \) is compatible with (a) the commutative Leibniz rule and (b) the chain-rule identities from the \( C^\infty \)-ring structure of \( C^\infty(V) \).

(a) The commutative Leibniz rule For \( f_1, f_2 \in C^\infty \subset C^\infty(V) \), one has

\[
d(f_1 f_2) - f_2 df_1 - f_1 df_2 = 0
\]

in \( \Omega_{C^\infty(V)} \). Under \( \varphi^\circ \), one has

\[
\varphi^\circ \left( d(f_1 f_2) - f_2 df_1 - f_1 df_2 \right) = D\varphi^\sharp(f_1 f_2) - \varphi^\sharp(f_2) D\varphi^\sharp(f_1) - \varphi^\sharp(f_1) D\varphi^\sharp(f_2) = 0
\]

since

\[
D(\varphi^\sharp(f_1 f_2)) = D(\varphi^\sharp(f_1) \varphi^\sharp(f_2)) = (D\varphi^\sharp(f_1)) \varphi^\sharp(f_2) + \varphi^\sharp(f_1) D\varphi^\sharp(f_2),
\]

which is

\[
\varphi^\sharp(f_2) D\varphi^\sharp(f_1) + \varphi^\sharp(f_1) D\varphi^\sharp(f_2)
\]

for \( (\varphi, \nabla) \) \( (*_1) \)-admissible.

(b) The chain-rule identities from the \( C^\infty \)-ring structure Let \( \zeta \in C^\infty(\mathbb{R}^l), \ l \in \mathbb{Z}_{\geq 1} \) and \( f_1, \cdots, f_l \in C^\infty(V) \). Then, one has

\[
d(\zeta(f_1, \cdots, f_l)) - \sum_{k=1}^l (\partial_k \zeta)(f_1, \cdots, f_l) df_k = 0
\]

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in $\Omega_{C^\infty(V)}$. Here, $\partial_k \zeta$ is the partial derivative of $\zeta \in C^\infty(\mathbb{R}^l)$ with respect to its $k$-th argument. Under $\varphi^\circ$, one has

$$
\varphi^\circ \left( d(\zeta(f_1, \cdots, f_l)) - \sum_{k=1}^l (\partial_k \zeta)(f_1, \cdots, f_l) df_k \right)
= \sum_{i=1}^n D\varphi^\sharp(\zeta(f_1, \cdots, f_l)) \otimes \frac{\partial}{\partial y^i} \zeta(f_1, \cdots, f_l)
$$

since, by Lemma 2.1.7 and the ($\ast_1$)-admissibility of $(\varphi, \nabla)$,

$$
D\varphi^\sharp(\zeta(f_1, \cdots, f_l)) = \sum_{i=1}^n D\varphi^\sharp(y^i) \otimes \frac{\partial}{\partial y^i} \zeta(f_1, \cdots, f_l)
$$

This completes the proof of Theorem 2.1.5  

The pull-push $\varphi^\circ$ on tensor product

$\Omega_{C^\infty(Y)} \otimes_{C^\infty(Y)} \cdots \otimes_{C^\infty(Y)} \Omega_{C^\infty(Y)}$

Having the well-defined

$$
\varphi^\circ : \Omega_{C^\infty(Y)} \rightarrow \Omega_{C^\infty(X)} \otimes_{C^\infty(X)} C^\infty(End_C(E)),
$$

it is natural to consider the extension of $\varphi^\circ$ to a correspondence between tensor products

$$
\varphi^\circ : \otimes_{C^\infty(Y)}^k \Omega_{C^\infty(Y)} \rightarrow \left( \otimes_{C^\infty(X)}^k \Omega_{C^\infty(X)} \right) \otimes_{C^\infty(X)} C^\infty(End_C(E))
$$

$$
f_0 df_1 \otimes \cdots \otimes df_k \mapsto \varphi^\sharp(f_0) D\varphi^\sharp(f_1) \otimes \cdots \otimes D\varphi^\sharp(f_k)
$$

Here, the tensor $D\varphi^\sharp(f_1) \otimes \cdots \otimes D\varphi^\sharp(f_k)$ is defined to the tensors of the underlying 1-forms in $\Omega_{C^\infty(X)}$ and multiplication of the coefficients in $C^\infty(End_C(E))$ from each factor. Explicitly, in terms of a local coordinate $(x^1, \cdots, x^m)$ on a chart $U \subset X$,

$$
\varphi^\sharp(f_0) D\varphi^\sharp(f_1) \otimes \cdots \otimes D\varphi^\sharp(f_k)
= \sum_{\mu_1, \cdots, \mu_k=1}^m \varphi^\sharp(f_0) D_{\partial/\partial x^\mu_1} \varphi^\sharp(f_1) \cdots D_{\partial/\partial x^\mu_k} \varphi^\sharp(f_k) \, dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k}.
$$

**Lemma 2.1.8.** [pull-push of (covariant) tensor] For $(\varphi, \nabla)$ ($\ast_1$)-admissible, the above extension of $\varphi^\circ$ to covariant tensors is well-defined.
Proof. In $\bigotimes_{C^{\infty}(Y)} \Omega^{\infty}(Y)$, one has the identities

$$
\begin{align*}
f_0 df_1 \otimes df_2 \otimes \cdots \otimes df_k &= df_1 f_0 \otimes df_2 \otimes \cdots \otimes df_k \\
 &= df_1 \otimes f_0 df_2 \otimes \cdots \otimes df_k = \cdots \\
 &= df_1 \otimes \cdots \otimes df_{k-1} f_0 \otimes df_k = df_1 \otimes \cdots \otimes f_0 df_k = df_1 \otimes \cdots \otimes df_{k-1} f_0 \cdot
\end{align*}
$$

Since the $(\ast_1)$-Admissible Condition implies that $\varphi^\circ(f_0)$ commutes with all of $D\varphi^\circ(f_1)$, \ldots, $D\varphi^\circ(f_k)$, the parallel identities

$$
\begin{align*}
\varphi^\circ(f_0) D\varphi^\circ(f_1) \otimes D\varphi^\circ(f_2) \otimes \cdots \otimes D\varphi^\circ(f_k) &= D\varphi^\circ(f_1) \varphi^\circ(f_0) \otimes D\varphi^\circ(f_2) \otimes \cdots \otimes D\varphi^\circ(f_k) \\
 &= D\varphi^\circ(f_1) \otimes \varphi^\circ(f_0) D\varphi^\circ(f_2) \otimes \cdots \otimes D\varphi^\circ(f_k) = \cdots \\
 &= D\varphi^\circ(f_1) \otimes \cdots \otimes D\varphi^\circ(f_{k-1}) \varphi^\circ(f_0) \otimes D\varphi^\circ(f_k) \\
 &= D\varphi^\circ(f_1) \otimes \cdots \otimes \varphi^\circ(f_0) D\varphi^\circ(f_k) = D\varphi^\circ(f_1) \otimes \cdots \otimes D\varphi^\circ(f_k) \varphi^\circ(f_0)
\end{align*}
$$

hold in $\bigotimes_{C^{\infty}(X)} \bigotimes_{C^{\infty}(X)} C^{\infty}(\text{End}_C(E))$. This proves the lemma. \qed

Note that for a $(\ast_1)$-admissible map $\varphi : (X^k, E; \nabla) \to Y$, since $D_\xi \mathcal{A}_\varphi \subseteq \text{Comm}(\mathcal{A}_\varphi)$ for all $\xi \in \mathcal{T}_X$ and $\text{Comm}(\mathcal{A}_\varphi)$ is itself a (possibly noncommutative) $\mathcal{O}_X^C$-subalgebra of $\mathcal{O}_X^C$, the pull-push $\varphi^\circ \alpha$ of a (covariant) tensor $\alpha$ on $Y$ to $X$ is indeed $\text{Comm}(\mathcal{A}_\varphi)$-valued.

Example 2.1.9. [pull-push of 2-tensor under $(\ast_1)$-admissible $(\varphi, \nabla)$] Let $\varphi : (X^k, E; \nabla) \to Y$ be a $(\ast_1)$-admissible map and $\alpha = \sum_{i,j} a_{ij} dy^i \otimes y^j$ be a 2-tensor on $Y$. Then, with respect to local coordinates $(x^1, \cdots, x^n)$ on $X$ and $(y^1, \cdots, y^n)$ on $Y$, 

$$
\varphi^\circ \alpha = \sum_{\mu, \nu=1}^m \left( \sum_{i,j=1}^n \varphi^\circ(a_{ij}) D_{\frac{\partial}{\partial x^\mu}} \varphi^\circ(y^i) D_{\frac{\partial}{\partial x^\nu}} \varphi^\circ(y^j) \right) dx^\mu \otimes dx^\nu.
$$

Since in general $D_{\partial/\partial x^\mu} \varphi^\circ(y^i) D_{\partial/\partial x^\nu} \varphi^\circ(y^j) \neq D_{\partial/\partial x^\mu} \varphi^\circ(y^j) D_{\partial/\partial x^\nu} \varphi^\circ(y^j)$, $\varphi^\circ$ does not take a symmetric 2-tensor on $Y$ to a $\text{Comm}(\mathcal{A}_\varphi)$-valued symmetric 2-tensor on $X$, nor an antisymmetric 2-tensor on $Y$ to a $\text{Comm}(\mathcal{A}_\varphi)$-valued antisymmetric 2-tensor on $X$. However, after the post-composition with the trace map $\text{Tr} : \mathcal{O}_X^C \to \mathcal{O}_X^C$, $\text{Tr} \varphi^\circ$ does take a symmetric (resp. antisymmetric) 2-tensor on $X$ to an $\mathcal{O}_X^C$-valued symmetric (resp. antisymmetric) 2-tensor on $X$.

Example 2.1.10. [pull-push of higher-rank tensor under $(\ast_1)$-admissible $(\varphi, \nabla)$] Continuing Example 2.1.9. For $\alpha$ a (covariant) tensor on $Y$ of rank $\geq 3$, the trace map no longer help bring symmetric (resp. antisymmetric) tensors to symmetric (resp. antisymmetric) tensors.

The situation gets better for a map $\varphi : (X^k, E; \nabla) \to Y$ that satisfies the stronger $(\ast_2)$-Admissible Condition: $D_\xi \text{Comm}(\mathcal{A}_\varphi) \subseteq \text{Comm}(\mathcal{A}_\varphi)$ for $\xi \in \mathcal{T}_X$.

Lemma 2.1.11. [pull-push of tensor under $(\ast_2)$-admissible $(\varphi, \nabla)$] Let $\varphi : (X^k, E; \nabla) \to Y$ be a $(\ast_2)$-admissible map. Then $\varphi^\circ$ takes a symmetric (resp. antisymmetric) tensor on $Y$ to a $\text{Comm}(\mathcal{A}_\varphi)$-valued symmetric (resp. antisymmetric) tensor on $X$. 

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Proof. In terms of local coordinates \((x^1, \ldots, x^n)\) on \(X\) and \((y^1, \ldots, y^m)\) on \(Y\), a (covariant) \(k\)-tensor

\[
\alpha = \sum_{i_1, \ldots, i_k} \alpha_{i_1 \cdots i_k} dy^{i_1} \otimes \cdots \otimes dy^{i_k}
\]
on \(Y\) is pull-pushed to a \(\text{Comm}(A_\phi)\)-valued \(k\)-tensor

\[
\varphi^\circ \alpha = \sum_{\mu_1, \ldots, \mu_k=1}^m \left( \sum_{i_1, \ldots, i_k=1}^n \varphi^\sharp(\alpha_{i_1 \cdots i_k}) D_{\frac{\partial}{\partial y^{i_1}}} \varphi^\sharp(y^{i_1}) \cdots D_{\frac{\partial}{\partial y^{i_k}}} \varphi^\sharp(y^{i_k}) \right) dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k}.
\]
on \(X\). It follows from Lemma 2.1.4 that, under the \((*)_2\)-Admissible Condition, all the factors

\[
\varphi^\sharp(\alpha_{i_1 \cdots i_k}), D_{\frac{\partial}{\partial x^{\mu_1}}} \varphi^\sharp(y^{i_1}), \cdots, D_{\frac{\partial}{\partial x^{\mu_k}}} \varphi^\sharp(y^{i_k})
\]
in a summand commute among themselves. This implies, in particular, that \(\varphi^\circ\) now takes a symmetric (resp. antisymmetric) tensor on \(Y\) to a \(\text{Comm}(A_\phi)\)-valued symmetric (resp. antisymmetric) tensor on \(X\).

Let \(\wedge^* \mathcal{T}^* Y\) be the sheaf of differential forms on \(Y\). The same proof of Lemma 2.1.11 gives also

**Lemma 2.1.12.** \([\varphi^\circ \text{ and } \wedge]\) Let \(\varphi : (X^A, \mathcal{E}; \nabla) \to Y\) be a \((*)_2\)-admissible map. For \(\alpha, \beta \in \wedge^* \mathcal{T}^* Y\), define the wedge product

\[
\varphi^\circ \alpha \wedge \varphi^\circ \beta
\]
of \(\varphi^\circ \alpha, \varphi^\circ \beta \in (\wedge^* \mathcal{T}^* X)^C \otimes_{\mathcal{O}_X^A} \mathcal{O}_X^A\) by applying the wedge product to the differential forms on \(X\) and multiplication to the \(\mathcal{O}_X^A\)-valued coefficients. Then,

\[
\varphi^\circ(\alpha \wedge \beta) = (\varphi^\circ \alpha) \wedge (\varphi^\circ \beta).
\]

**Remark 2.1.13.** \([\text{admissible condition and Ramond-Ramond field}]\) While the current note will take \((\varphi, \nabla)\) to be \((*)_1\)-admissible most of the time, Example 2.1.9, Example 2.1.10, Lemma 2.1.11 and Lemma 2.1.12 together suggest that when the coupling of D-brane to Ramond-Ramond fields is taken into account, the more natural admissible condition on \((\varphi, \nabla)\) is the stronger \((*)_2\)-Admissible Condition.

### 2.2 Admissible conditions from the aspect of open strings

We address in this subsection the implication of Admissible Condition \((*)_1\) on \((\varphi, \nabla)\) to the mass of the connection \(\nabla\) from the aspect of open strings in the target-space \(Y\).

Let \(\varphi : (X^A, \mathcal{E}; \nabla) \to Y\) be a \((*)_1\)-admissible map. Recall the surrogate \(X_\varphi := \text{Spec}^R A_\varphi\) of \(X^A\) specified by \(\varphi\) and the built-in dominant morphism \(\pi_\varphi : X_\varphi \to X\); cf. Figure 1-2. For \(x \in X\), let \(\{e_1, \cdots, e_s\} \subset A_{\varphi, x}\) the maximal complete set of orthogonal idempotents in the stalk of \(A_{\varphi}\) at \(x\). Then, by \((*)_1\)-admissibility of \((\varphi, \nabla)\) and Lemma 1.6,

\[
D_\xi e_1 = \cdots = D_\xi e_s = 0
\]
for all \(\xi \in (\mathcal{T}_x X)_x\). It follows that
Lemma 2.2.1. [covariantly invariant decomposition of stalks of $\mathcal{E}$] For any $x \in X$, the decomposition

$$\mathcal{E}_x = e_1\mathcal{E}_x + \cdots + e_s\mathcal{E}_x$$

is invariant under $\nabla$. I.e. $\nabla_{\xi}(e_k\mathcal{E}_x) \subset e_k\mathcal{E}_x$, for $k = 1, \ldots, s$ and all $\xi \in (T_xX)_x$.

As a consequence, the connection $\nabla$ on $\mathcal{E}_x$ induces a connection $\nabla^{(k)}$ on each direct summand $e_k\mathcal{E}_x$ of $\mathcal{E}_x$ and one has the direct-sum decomposition

$$(\mathcal{E}_x, \nabla) = (e_1\mathcal{E}_x, \nabla^{(1)}) \oplus \cdots \oplus (e_s\mathcal{E}_x, \nabla^{(s)}).$$

On the other hand, the maximal complete set of orthogonal idempotents $\{e_1, \ldots, e_s\} \subset \mathcal{A}_{\phi,x}$ corresponds canonically and bijectively to the set of connected components of the germ $X_{\phi,x}$ of $X_\phi$ over $x \in X$:

$$X_{\phi,x} = X_{\phi,x}^{(1)} \sqcup \cdots \sqcup X_{\phi,x}^{(s)}.$$ 

Through the built-in inclusion $\mathcal{A}_{\phi,x} \subset \mathcal{O}_{X_{\phi,x}}^x \cdot \mathcal{E}_x$ as the fundamental $\mathcal{O}_{X_{\phi,x}}^x$-module is canonically an $\mathcal{A}_{\phi,x}$-module as well. Since $e_ke_l = 0$ for $k \neq l$, as an $\mathcal{A}_{\phi,x}$-module the direct summand $e_k\mathcal{E}_x$ is supported exactly on $X_{\phi,x}^{(k)}$, for $k = 1, \ldots, s$. The above decomposition of $(\mathcal{E}, \nabla)$ says then geometrically and in terms of physics terminology that the gauge field $\nabla$ on $\mathcal{E}$ has no components that mixes $e_k\mathcal{E}_x$ on $X_{\phi,x}^{(k)}$ and $e_l\mathcal{E}_x$ on $X_{\phi,x}^{(l)}$ for some $k \neq l$; cf. Figure 2-2-1.

Recall now the string-theory origin of D-branes:

- A D-brane is where the end-points of an open string stick to.
- Excitations of open strings create fields on the D-brane.
- As the tension of open strings are constant, the mass of an open string — and hence fields it creates on the D-brane — is proportional to its length. Open strings with arbitrarily small length create massless fields on the brane while open strings with length bounded away from zero create massive fields on the brane.

That the germ $(X_{\phi,x}, \mathcal{E}_x; \nabla)$ over any $x \in X$ is decomposable in accordance with the connected-component decomposition of $X_{\phi,x}$ says that $\nabla$ must be created by open-strings of arbitrarily small length, rather than by those of length bounded away from zero. In other words, $\nabla$ is massless.

In summary:

Corollary 2.2.2. [$(\ast_1)$-Admissible Condition implies massless of $\nabla$] For a $(\ast_1)$-admissible map $\phi : (X^\mathbb{C}, \mathcal{E}; \nabla) \to Y$, the gauge field $\nabla$ on the Chan-Paton sheaf $\mathcal{E}$ on the D-brane (or D-brane world-volume) $X^\mathbb{C}$ is massless from the aspect of open strings in the target-space (or target-space-time) $Y$.

By Lemma 2.1.2, the same holds for $(\ast_2)$-admissible maps and $(\ast_3)$-admissible maps as well.

3 The differential $d\phi$ of $\phi$ and its decomposition, the three basic $\mathcal{O}_X^\mathbb{C}$-modules, induced structures, and some covariant calculus

At the classical level Polyakov string or its generalization, a sigma model, is a theory of harmonic maps on the mathematical side. In this section we construct all the building blocks to generalize the existing theory of harmonic maps to a theory of maps $\phi : (X^\mathbb{C}, \mathcal{E}; \nabla) \to Y$, which describe D-branes. It will turn out that both the connection $\nabla$ and the Admissible Condition $(\ast_1)$ chosen are needed to build up a mathematically sound theory for such maps $\phi$. 

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Figure 2-2-1. When $(\varphi, \nabla)$ is $(*)_1$-admissible, the gauge field $\nabla$ on the Chan-Paton sheaf $E$ on any small neighborhood $U$ of $x \in X$ localizes at each connected branch of $\varphi(U^k)$ from the viewpoint of open strings in $Y$. In other words, $\nabla$ is massless from the open-string aspect. In the illustration, the noncommutative space $X^{Az}$ is expressed as a noncommutative cloud shadowing over its underlying topology $X$, the connection $\nabla$ on $E$ over $X$ is indicated by a gauge field on $X$. Both the gauge field on $X$ and how open strings “see” it in $Y$ are indicated by squiggling arrows $\Rightarrow$. The situation for a general $(\varphi, \nabla)$ (cf. top) and a $(\varphi, \nabla)$ satisfying Admissible Condition $(*)_1$ (cf. bottom) are compared. From the open-string aspect, in the former situation $\nabla$ can have both massless components (which are local fields from the open-string and target-space viewpoint) and massive components (which become nonlocal fields from the open-string and target-space viewpoint), while in the latter situation $\nabla$ has only massless components.
3.1 The differential $d\varphi$ of $\varphi$ and its decomposition induced by $\nabla$

Three kinds of differentials, $d\varphi$, $D\varphi$, and $ad\varphi$, of a map $\varphi$ that naturally appear in the setting are defined and their local expressions are worked out in this subsection.

The differential, the covariant differential, and the inner differential of a map $\varphi$

Let $\varphi : (X, \mathcal{O}_X^{Ae}, \mathcal{E}) \rightarrow Y$ be a map defined contravariantly by an equivalence class of gluing systems of ring-homomorphisms $\varphi^\#: \mathcal{O}_Y \rightarrow \mathcal{O}_X^{Ae}$ over $\mathbb{R} \subset \mathbb{C}$. Then, for any derivation $\eta$ on $\mathcal{O}_X^{Ae}$, the correspondence

\[
\mathcal{O}_Y \rightarrow \mathcal{O}_X^{Ae}, \quad f \mapsto \eta(\varphi^\#(f))
\]

defines an $\mathcal{O}_X^{Ae}$-valued derivation on $\mathcal{O}_Y$. It follows that $\varphi$ induces a correspondence

\[
T_sX^{Ae} \rightarrow \mathcal{O}_X^{Ae} \otimes_{\varphi^\#\mathcal{O}_Y} T_sY, \quad \eta \mapsto d\eta\varphi
\]

that is $\mathcal{O}_X^C$-linear.

**Definition 3.1.1. [differential $d\varphi$ of $\varphi$]** The above $\mathcal{O}_X^C$-linear correspondence is denoted by $d\varphi$ and called the **differential of $\varphi$**.

Recall from Sec. 1 that when $\mathcal{E}$ is equipped with a connection $\nabla$, $\nabla$ induces a connection $D$ on $\mathcal{O}_X^{Ae} := \text{End}_{\mathcal{O}_X^C}(\mathcal{E})$, which in turn induces a splitting

\[
T_sX^C \rightarrow T_sX^{Ae}, \quad \xi \mapsto D\xi
\]

of the exact sequence

\[
0 \rightarrow \text{Inn}(\mathcal{O}_X^{Ae}) \rightarrow T_sX^{Ae} \rightarrow T_sX^C \rightarrow 0.
\]

**Definition 3.1.2. [covariant differential $D\varphi$ of $\varphi$]** Let

$\varphi^*T_sY := \mathcal{O}_X^{Ae} \otimes_{\varphi^\#\mathcal{O}_Y} T_sY$,

regarded as a (left) $\mathcal{O}_X$-module via the built-in inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{Ae}$, be the pull-push of the tangent sheaf $T_sY$ of $X$ to $X$. The **covariant differential**

\[
D\varphi \in C^\infty(T^*X \otimes_{\mathcal{O}_X} \varphi^*T_sY)
\]

of $\varphi$ is the $(\mathcal{O}_X^{Ae}$-valued-derivation-on-$\mathcal{O}_Y$)-valued 1-form on $X$ defined by

\[
(D\xi\varphi)f := D\xi(\varphi^\#(f)) \in C^\infty(\text{End}_C(E))
\]

for $\xi \in C^\infty(T_sX) = \text{Der}(C^\infty(X))$ and $f \in C^\infty(Y)$. In other words, $D\varphi$ takes a tangent vector field on $X$ to a $C^\infty(\text{End}_C(E))$-valued derivation on $C^\infty(Y)$. In the equivalent sheaf format and notations, $D\xi\varphi \in \varphi^*T_sY$ for $\xi \in T_sX$. 

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Definition 3.1.3. [inner differential \(ad \varphi\) of \(\varphi\)] Continuing Definition 3.1.2. Represent elements in \(\text{Inn}(\mathcal{O}^{Ak}_X)\) by elements \(m \in \mathcal{O}^{Ak}_X\). The inner differential \(ad \varphi\) of \(\varphi\) is defined by the \(\mathcal{O}^C_X\)-linear correspondence
\[
\begin{align*}
ad \varphi &: \text{Inn}(\mathcal{O}^{Ak}_X) \rightarrow \varphi^* T_Y \\
m &\mapsto ad_m \varphi := [m, \varphi^*(\cdot)] .
\end{align*}
\]
Since \([\mathcal{O}^C_X, \cdot] = 0\), \(ad_m \varphi\) depends only on the inner derivation \(m\) represents.

By construction,
\[
d_{\eta} \varphi = D_{\zeta_{\eta}} + ad_{m_{\eta}} \varphi ,
\]
for \(\eta = \xi_{\eta} + m_{\eta} \in T_x X^{Ak} \simeq T_x X^C \oplus \text{Inn}(\mathcal{O}^{Ak}_X)\) induced by \(D\).

Local expressions of the covariant differential \(D \varphi\) of \(\varphi\) for \((*_1)\)-admissible \((\varphi, \nabla)\)

Note that if \(\varphi : (X^{Ak}, \xi; \nabla) \rightarrow Y\) is \((*_1)\)-admissible, then \(D_{\xi} \varphi\) is a \(\text{Comm}(A_{\varphi})\)-valued derivation on \(\mathcal{O}_Y\) for \(\xi \in T_x X\). In this case, one has the following lemma and corollary that are simply re-writings of Lemma 2.1.6 and Lemma 2.1.7 respectively.

Lemma 3.1.4. [local expression of \(D \varphi\) for \((*_1)\)-admissible \((\varphi, \nabla), I\)] Let \((\varphi, \nabla)\) be \((*_1)\)-admissible; i.e. \(D_{\xi} A_{\varphi} \subset \text{Comm}(A_{\varphi})\) for all \(\xi \in T_x X\). Let \(U \subset X\) be a small enough open set so that \(\varphi(U^{Ak})\) is contained in a coordinate chart of \(Y\), with coordinates \(y = (y^1, \ldots, y^n)\). For \(f \in C^\infty(Y)\), recall the germwise-over-\(U\) polynomial \(R^f[1]\) in \((y^1, \ldots, y^n)\) with coefficients in \(\mathcal{O}^{Ak}_U\) from [L-Y8: Sec. 4 & Remark/Notation 4.2.3.5] (D.13.1). Then, for \(\xi\) a vector field on \(U\) and \(f \in C^\infty(Y)\), and at the level of germs over \(U\),
\[
(D_{\xi} \varphi) f = R^f[1]_{y^d \mapsto D_{\xi}(\varphi^d(y^d))}, \text{ for all multi-degree } d \text{ in } R^f[1] .
\]
Here, for a multiple degree \(d = (d_1, \ldots, d_n)\), \(d_i \in \mathbb{Z}_{\geq 0}\), \(y^d := (y^1)^{d_1} \cdots (y^n)^{d_n}\) and \(y^d \sim D_{\xi}(\varphi^d(y^d))\) means ‘replacing \(y^d\) by \(D_{\xi}(\varphi^d(y^d))\’.

Corollary 3.1.5. [local expression of \(D \varphi\) for \((*_1)\)-admissible \((\varphi, \nabla), II\)] Assume that \((\varphi, \nabla)\) is \((*_1)\)-admissible. Let \(d_Y\) be the exterior differential on \(Y\). Then
\[
D \varphi = \varphi^o d_Y .
\]
Locally explicitly, let \((e^i)_{i=1, \ldots, n}\) be a local frame on \(Y\) and \((e^i)_{i=1, \ldots, n}\) its dual co-frame. In terms of these dual pair of local frames, \(d_Y = \sum_{i=1}^n e_i \otimes e^i\) under the canonical isomorphism \(T^* Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \simeq T^* Y\). Then
\[
(D_{\xi} \varphi) f = \sum_{i=1}^n (\varphi^o e^i)(\xi) \otimes e_i f = \sum_{i=1}^n (\varphi^o e^i)(\xi) \varphi^j(e_i f) \in \mathcal{O}^{Ak}_X
\]
under the canonical isomorphism \(\mathcal{O}^{Ak}_X \otimes_{\varphi^*, \mathcal{O}_X} \mathcal{O}_Y \simeq \mathcal{O}^{Ak}_X\). In particular, let \((y^1, \ldots, y^n)\) be coordinates of a local chart on \(Y\). Then, locally,
\[
(D_{\xi} \varphi) f = \sum_{i=1}^n (\varphi^o dy^i)(\xi) \otimes \frac{\partial}{\partial y^i} f = \sum_{i=1}^n \left(D_{\xi} \varphi^d(y^i) \cdot \varphi^j(\frac{\partial f}{\partial y^i})\right) .
\]
Here \(\cdot\) is the multiplication in \(\mathcal{O}^{Ak}_X\) (and will be omitted later when there is no sacrifice to clarity).
Local expressions of the inner differential \(ad_\varphi\) of \(\varphi\) for admissible inner derivations

Though for the purpose of defining an action functional for D-branes, \((\varphi, \nabla)\) is the dynamical field of the focus and, hence, the induced covariant derivation \(D\) on \(O_X^A\) may look to play more roles in our discussion, mathematically results related to \(D\) like Lemma 1.6, Lemma 3.1.4, and Corollary 3.1.5 involve only the fact that, for \(\xi \in TX\), \(D_\xi\) satisfies the Leibniz rule on \(O_X^A\) and some additional commutativity assumption. This suggest that similar statements hold if one considers inner derivations on \(O_X^A\) that are compatible to \(\varphi\) in an appropriate sense. Cf. Lemma 1.6 vs. Lemma 1.7. This motivates the setting of the current theme.

**Definition 3.1.6. [admissible inner derivation]** (Cf. Definition 2.1.1.) Let \(m \in Inn(O_X^A)\) be an inner derivation represented by an element of \(O_X^A\). Then \(m\) is called

\[
\begin{align*}
(*_1) \text{-admissible to } \varphi & \quad \text{if } [m, A_\varphi] \subset \text{Comm}(A_\varphi); \\
(*_2) \text{-admissible to } \varphi & \quad \text{if } [m, \text{Comm}(A_\varphi)] \subset \text{Comm}(A_\varphi); \\
(*_3) \text{-admissible to } \varphi & \quad \text{if } [m, A_\varphi] \subset A_\varphi.
\end{align*}
\]

Note that these conditions are independent of the representative chosen in \(O_X^A\) of the inner derivation. The set of all \((*_1)\)-admissible-to-\(\varphi\) inner derivations on \(O_X^A\) form an \(O_X^A\)-module, which will be denoted by \(Inn^\varphi_{(*_1)}(O_X^A)\). Similarly, for \(Inn^\varphi_{(*_2)}(O_X^A)\) and \(Inn^\varphi_{(*_3)}(O_X^A)\).

**Lemma 3.1.7. [hierarchy of admissible conditions on inner derivation]** (Cf. Lemma 2.1.2.)

\[
Inn^\varphi_{(*_3)}(O_X^A) \subset Inn^\varphi_{(*_2)}(O_X^A) \subset Inn^\varphi_{(*_1)}(O_X^A).
\]

**Proof.** Since \(A_\varphi \subset \text{Comm}(A_\varphi)\), it is immediate that \(Inn^\varphi_{(*_2)}(O_X^A) \subset Inn^\varphi_{(*_1)}(O_X^A)\). For the inclusion \(Inn^\varphi_{(*_3)}(O_X^A) \subset Inn^\varphi_{(*_2)}(O_X^A)\), let \(m \in Inn^\varphi_{(*_3)}(O_X^A)\) represented by an element in \(O_X^A\), \(m' \in \text{Comm}(A_\varphi)\), and \(m'' \in A_\varphi\). Then,

\[
[[m, m'], m''] = [[m, m''], m'] + [m, [m', m'']] = 0.
\]

from either the Jacobi identity of Lie bracket or the Leibniz rule for a derivation. The first term vanishes since \([m, m''] \in A_\varphi\) and \([A_\varphi, m'] = 0\). The second term also vanishes since \([m', m''] = 0\). Since \(m' \in \text{Comm}(A_\varphi)\) and \(m'' \in A_\varphi\) are arbitrary, this shows that \([m, \text{Comm}(A_\varphi)] \subset \text{Comm}(A_\varphi)\). This proves the lemma.

**Remark 3.1.8. [Lemma 2.1.2 vs. Lemma 3.1.7]** Note that in Lemma 2.1.2, the implication \((*_3) \Rightarrow (*_2)\) uses \(D\)-parallel transport properties implied by \((*_3)\), which is an analytic technique. Indeed, the proof of Lemma 3.1.7 applies there. Which says that in both situations, the hierarchy is an algebraic consequence.

In terms of this setting and with arguments parallel to the proof of Lemma 3.1.4 and Corollary 3.1.5, one has the following lemma and corollary:
Lemma 3.1.9. [local expression of \( \text{ad} \varphi \) for \( \text{Inn}_{\varphi}(\mathcal{O}_X^A), \ I \)] (Cf. Lemma 3.1.4.)
Let \( \varphi : (X^A, \mathcal{E}) \to Y \) be a map and \( m \in \text{Inn}_{\varphi}(\mathcal{O}_X^A) \) represented by an element of \( \mathcal{O}_X^A \) i.e. \([m, \mathcal{A}_\varphi] \subset \text{Comm}(\mathcal{A}_\varphi)\). Let \( U \subset X \) be a small enough open set so that \( \varphi(U^A) \) is contained in a coordinate chart of \( Y \), with coordinates \( y = (y^1, \ldots, y^n) \). For \( f \in C^\infty(Y) \), recall the germwise-over-\( U \) polynomial \( R^f[1] \) in \( (y^1, \ldots, y^n) \) with coefficients in \( \mathcal{O}_U^A \) from [L-Y8: Sec. 4 & Remark/Notation 4.2.3.5] (D.13.1). Then, at the level of germs over \( U \),

\[
(ad_m \varphi) f = R^f[1]|_{y^d \mapsto \text{ad}_m(\varphi^\flat(y^d))}, \text{ for all multi-degree } d \text{ in } R^f[1].
\]

Here, for a multiple degree \( d = (d_1, \ldots, d_n), d_i \in \mathbb{Z}_{\geq 0} \), \( y^d := (y^1)^{d_1} \cdots (y^n)^{d_n} \) and \( y^d \mapsto \text{ad}_m(\varphi^\flat(y^d)) \) means ‘replacing \( y^d \) by \( \text{ad}_m(\varphi^\flat(y^d)) \)’.

Proof. Recall the proof of Lemma 3.1.4 through the proof of Lemma 2.1.6. With the same setup and notations there, note that for \( m \in \text{Inn}_{\varphi}(\mathcal{O}_X^A), \)

\[
[m, \varphi^\flat(1(k))] = 0, \quad \text{for } k = 1, \ldots, s
\]

by Lemma 1.7. Since \([m, \mathcal{O}_X] = 0\) holds automatically, the lemma follows.

\[\Box\]

Corollary 3.1.10. [local expression of \( \text{ad} \varphi \) for \( \text{Inn}_{\varphi}(\mathcal{O}_X^A), \ II \)] (Cf. Corollary 3.1.5.)
Continuing the setting of Lemma 3.1.9. Recall that \( m \in \text{Inn}_{\varphi}(\mathcal{O}_X^A) \) is represented by an element of \( \mathcal{O}_X^A \). Let \( (y^1, \ldots, y^n) \) be coordinates of a local chart on \( Y \). Then, locally,

\[
(ad_m \varphi) f = \sum_{i=1}^n \left( ad_m \varphi^\flat(y^i) \cdot \varphi^\flat(\frac{\partial}{\partial y^i}) \right) = \sum_{i=1}^n \left( [m, \varphi^\flat(y^i)] \cdot \varphi^\flat(\frac{\partial}{\partial y^i}) \right),
\]

where \( \cdot \) is the multiplication in \( \mathcal{O}_X^A \) (and will be omitted later when there is no sacrifice to clarity). In other words,

\[
ad_m \varphi = \sum_{i=1}^n [m, \varphi^\flat(y^i)] \otimes \frac{\partial}{\partial y^i}.
\]

Proof. The related last part of the proof of Corollary 3.1.5 through the proof of Lemma 2.1.7 works in verbitum with \( D_\xi \) replaced by \( ad_m = [m, \cdot] \). This is simply a re-writing of Lemma 3.1.9 above.

\[\Box\]

Remark 3.1.11. [comparison with differential of ordinary map] As a comparison, let \( u : X \to Y \) be a map between manifolds. Then, \( du \) defines a bundle map \( T_*X \to u^*T_*Y \) that satisfies

\[
du(\xi) f = u_*(\xi) f = \xi(u \circ f) = \xi(u^\sharp(f))
\]

for any \( \xi \in T_*X \) and \( f \in C^\infty(Y) \). In terms of local coordinates \( x = (x^\mu)_{\mu=1,\ldots,m} \) on \( X \) and \( y = (y^i)_{i=1,\ldots,n} \) on \( Y \),

\[
u_*(\frac{\partial}{\partial x^\mu}) = \sum_{i=1}^n \frac{dy^i}{dx^\mu} \frac{\partial f}{\partial y^i}(u(x)).
\]
The above notion of covariant differential $D\varphi$ of $\varphi$ is exactly the generalization of the ordinary differential $du$ of $u$, taking into account the fact that $\varphi$ is now only defined contravariantly through $\varphi^*$ and that the noncommutative structure sheaf $O_X^h$ of $X^h$ is no longer naturally trivial as an $O_X$-module but, rather, is endowed with a natural induced connection $D$ from $\nabla$.

Furthermore, when $(\varphi, \nabla)$ is $(\ast_1)$-admissible or when considering only $\text{Inn}^\ast_1(O_X^h)$, both the covariant differential $D\varphi$ and the inner differential $ad\varphi$ takes the same form as the chain rule in the commutative case.

### 3.2 The three basic $O_X^C$-modules relevant to $D\varphi$, with induced structures

Underlying the notion of covariant differential $D\varphi$ of $\varphi$ are two basic $O_X^C$-modules

- the pull-back tangent sheaf $\varphi^*T_*Y := O_X^h \otimes_{\varphi^* O_Y} T_* Y$, a left $O_Y^h$-module but now regarded as a (left) $O_X^C$-module through the built-in inclusion $O_X^C \hookrightarrow O_X^h$,
- the $O_X^C$-module $T^*X \otimes_{O_X} \varphi^* T_* Y$, where $D\varphi$ lives.

We study them in this subsection after taking a look at another basic but simpler $O_X^C$-module $T^*X \otimes_{O_X} O_X^h$. They play fundamental roles in our variational problem.

**Remark 3.2.0.1.** [structures on the $O_X^C$-algebra $O_X^h$] Recall the connection $D$ on the noncommutative structure sheaf $O_X^h := \mathcal{E}nd_{O_X^C}(\mathcal{E})$ of $X^h$ induced by the connection $\nabla$ on $\mathcal{E}$. The multiplication $\cdot$ in the $O_X^C$-algebra structure of $O_X^h$ defines a nonsymmetric $O_X^h$-valued inner product on $O_X^h$ that is $O_X^C$-bilinear. This inner product is $D$-invariant in the sense that

$$D(m_1 \cdot m_2) = (Dm_1) \cdot m_2 + m_1 \cdot Dm_2,$$

for $m_1, m_2 \in O_X^h$. Together with the built-in trace map

$$\text{Tr} : O_X^h \rightarrow O_X^C,$$

as an $O_X^C$-module-homomorphism, one has a symmetric $O_X^C$-valued inner product on $O_X^h$ defined by the assignment

$$(m_1, m_2) \mapsto \text{Tr}(m_1 \cdot m_2) =: \text{Tr}(m_1 m_2),$$

for $m_1, m_2 \in O_X^h$. This inner product is $O_X^C$-bilinear; and is covariantly constant over $X$ in the sense that

$$d_X (\text{Tr}(m_1 m_2)) = \text{Tr}((Dm_1)m_2) + \text{Tr}(m_1 Dm_2),$$

where $d_X$ is the exterior differential on $X$.

### 3.2.1 The $O_X^h$-valued cotangent sheaf $T^*X \otimes_{O_X} O_X^h$ of $X$, and beyond

Let $X$ be endowed with a (Riemannian or Lorentzian) metric $h$ and $\nabla^h$ be the Levi-Civita connection on $T_*X$ induced by $h$. The corresponding inner product on $T_*X$, its dual $T^*X$, and their tensor products will be denoted $\langle \cdot, \cdot \rangle_h$. The induced connection on the dual $T^*X$ and on the tensor product of copies of $T_*X$ and copies of $T^*X$ will be denoted also by $\nabla^h$. The defining features of $\nabla^h$ are

$$\nabla^h \langle \xi_1, \xi_2 \rangle_h = \langle \nabla^h \xi_1, \xi_2 \rangle_h + \langle \xi_1, \nabla^h \xi_2 \rangle_h$$

(h be $\nabla^h$-covariantly constant),

$$\text{Tor}^h \langle \xi_1, \xi_2 \rangle := \nabla^h_{\xi_1} \xi_2 - \nabla^h_{\xi_2} \xi_1 - [\xi_1, \xi_2] = 0$$

($\nabla^h$ be torsionless),

for all $\xi_1, \xi_2 \in T_*X$. 

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The $\mathcal{O}_X^{Ak}$-valued cotangent sheaf $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ of $X$

The connection $\nabla^h$ on $T^*X$ and the connection $D$ on $\mathcal{O}_X^{Ak}$ together induce a connection

$$\nabla^{(h,D)} := \nabla^h \otimes Id_{\mathcal{O}_X^{Ak}} + Id_{T^*X} \otimes D$$

on $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$. The inner product $\langle \cdot, \cdot \rangle_h$ on $\mathcal{T}_X$ and the inner product $\cdot$ on $\mathcal{O}_X^{Ak}$ together induce an $\mathcal{O}_X^{Ak}$-valued, $\mathcal{O}_X^{Ak}$-bilinear (nonsymmetric) inner product on $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ by extending $\mathcal{O}_X^{Ak}$-bilinearly

$$\langle \omega^1 \otimes m_1, \omega^2 \otimes m_2 \rangle_h := \langle \omega^1, \omega^2 \rangle_h \cdot (m_1^2 - m_2^2),$$

where $\omega^1, \omega^2 \in T^*X$ and $m_1, m_2 \in \mathcal{O}_X^{Ak}$. The trace map $Tr : \mathcal{O}_X^{Ak} \to \mathcal{O}_X^C$ turns this further to an $\mathcal{O}_X^{Ak}$-valued, $\mathcal{O}_X^{Ak}$-bilinear (symmetric) inner product on $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ by the post composition with the above inner product

$$\langle \cdot, \cdot \rangle := Tr(\langle \cdot, \cdot \rangle_h)$$

for $\cdot, \cdot \in T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$. By construction, both inner products are covariantly constant with respect to $\nabla^{(h,D)}$ and they satisfy the Leibniz rules

$$D\langle \cdot, \cdot \rangle_h = \langle \nabla^{(h,D)} \cdot, \cdot \rangle_h + \langle \cdot, \nabla^{(h,D)} \cdot \rangle_h,$$

$$dTr\langle \cdot, \cdot \rangle_h = Tr(D\langle \cdot, \cdot \rangle_h) = Tr(\langle \nabla^{(h,D)} \cdot, \cdot \rangle_h) + Tr(\langle \cdot, \nabla^{(h,D)} \cdot \rangle_h)$$

for $\cdot, \cdot \in T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$.

The sheaf $(\Lambda^\bullet T^*X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ of $\mathcal{O}_X^{Ak}$-valued differential forms on $X$

The setting in the previous theme generalizes to the sheaf $(\Lambda^\bullet T^*X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ of $\mathcal{O}_X^{Ak}$-valued differential forms on $X$, with the 1-forms $\omega^1, \omega^2$ on $X$ there replaced by general differential forms $\alpha^1, \alpha^2$ on $X$. We will use the same notations

$$\nabla^h, \nabla^{(h,D)}, \langle \cdot, \cdot \rangle_h, Tr\langle \cdot, \cdot \rangle_h$$

to denote the connection on $\Lambda^\bullet T^*X$, the connection on $(\Lambda^\bullet T^*X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$, the $\mathcal{O}_X^{Ak}$-valued, $\mathcal{O}_X^{Ak}$-bilinear (nonsymmetric) inner product on $(\Lambda^\bullet T^*X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$, and the $\mathcal{O}_X^{Ak}$-valued, $\mathcal{O}_X^{Ak}$-bilinear (symmetric) inner product on $(\Lambda^\bullet T^*X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ respectively. They satisfy the same Leibniz rule as in the case of $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$.

3.2.2 The pull-back tangent sheaf $\varphi^*\mathcal{T}_Y$

This is the main character among the three basic $\mathcal{O}_X^C$-modules and is slightly subtler than $u^*\mathcal{T}_Y$ in the commutative case (cf. Remark 3.1.11) or $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{Ak}$ in Sec. 3.2.1.

The induced connection and the induced partially-defined inner products

Let $Y$ be endowed with a (Riemannian or Lorentzian) metric $g$ and $\nabla^g$ be the Levi-Civita connection on $\mathcal{T}_Y$ induced by $g$. The corresponding inner product on $\mathcal{T}_Y$ or its dual $\mathcal{T}^*Y$ will be denoted $\langle \cdot, \cdot \rangle_g$. The induced connection on the dual $\mathcal{T}^*Y$ and on the tensor product of copies of $\mathcal{T}_Y$ and copies of $T^*Y$ will be denoted also by $\nabla^g$. The defining features of $\nabla^g$ are

$$\nabla^g(v_1, v_2)_g = \langle \nabla^g v_1, v_2 \rangle_g + \langle v_1, \nabla^g v_2 \rangle_g$$

($g$ be $\nabla^g$-covariantly constant),

$$\text{Tor}_{\nabla^g}(v_1, v_2) := \nabla^g_{v_1} v_2 - \nabla^g_{v_2} v_1 - [v_1, v_2] = 0$$

($\nabla^g$ be torsionless),

for all $v_1, v_2 \in \mathcal{T}_Y$. 

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Lemma 3.2.2.1. [(\(D, \nabla^g\))-induced connection on \(\varphi^*\mathcal{T}_Y\) for \((\ast_1)\)-admissible \((\varphi, \nabla)\)] Assume that \((\varphi, \nabla)\) is \((\ast_1)\)-admissible. Then, the connection \(D\) on \(\mathcal{O}^\mathbb{A}_X\) and the connection \(\nabla^g\) on \(\mathcal{T}_Y\) together induce a connection \(\nabla^{(\varphi, g)}\) on \(\varphi^*\mathcal{T}_Y\), locally of the form

\[
\nabla^{(\varphi, g)} = D \otimes \text{Id}_{\mathcal{T}_Y} + \text{Id}_{\mathcal{O}^\mathbb{A}_X} \cdot \sum_{i=1}^n D\varphi^i(y^i) \otimes \nabla^g \frac{\partial}{\partial y^i}.
\]

\textbf{Proof.} Our construction of an induced connection on \(\varphi^*\mathcal{T}_Y\) is local in nature. As long as a construction is independent of coordinates chosen, the local construction glues to a global construction. Let \(U \subset X\) be a small enough open set so that \(\varphi(U^\mathbb{A}_X)\) is contained in a coordinate chart of \(Y\), with coordinate \(y := (y^1, \cdots, y^n)\). Then the local expression

\[
\nabla^{(\varphi, g); y} := D \otimes \text{Id}_{\mathcal{T}_Y} + \text{Id}_{\mathcal{O}^\mathbb{A}_X} \cdot \sum_{i=1}^n D\varphi^i(y^i) \otimes \nabla^g \frac{\partial}{\partial y^i}.
\]

in the statement defines a connection on \(\varphi^*\mathcal{T}_Y|_U\). We only need to show that it is independent of the coordinate \((y^1, \cdots, y^n)\) chosen.

Let \(z := (z^1, \cdots, z^n)\) be another coordinate on the chart. Then, for \((\varphi, \nabla)\) \((\ast_1)\)-admissible, \(D\varphi^i(y^i) := (D\varphi)y^i\) has a local expression in terms of \(z\)

\[
(D\varphi)y^i := D(\varphi^i(y^i)) = \sum_{j=1}^n D(\varphi^j(z^j)) \otimes \frac{\partial y^i}{\partial z^j}
\]

by Corollary 3.1.5, for \(i = 1, \cdots, n\). It follows that

\[
\nabla^{(\varphi, g); z} := D \otimes \text{Id}_{\mathcal{T}_Y} + \text{Id}_{\mathcal{O}^\mathbb{A}_X} \cdot \sum_{j=1}^n D\varphi^j(z^j) \otimes \nabla^g \frac{\partial}{\partial z^j}
\]

\[
= D \otimes \text{Id}_{\mathcal{T}_Y} + \text{Id}_{\mathcal{O}^\mathbb{A}_X} \cdot \sum_{j=1}^n D\varphi^j(z^j) \otimes \sum_{i=1}^n \frac{\partial y^i}{\partial z^j} \nabla^g \frac{\partial}{\partial y^i}
\]

\[
= D \otimes \text{Id}_{\mathcal{T}_Y} + \text{Id}_{\mathcal{O}^\mathbb{A}_X} \cdot \sum_{i=1}^n D\varphi^i(y^i) \otimes \nabla^g \frac{\partial}{\partial y^i}
\]

\[
= \nabla^{(\varphi, g); y}.
\]

This completes the proof. \(\square\)

Consider next the induced inner products on \(\varphi^*\mathcal{T}_Y\). Completely naturally, one may attempt to combine the multiplication in \(\mathcal{O}^\mathbb{A}_X\) and the inner product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{T}_Y\) to an \(\mathcal{O}^\mathbb{A}_X\)-valued, \(\mathcal{O}^\mathbb{A}_X\)-bilinear (nonsymmetric) inner product on \(\varphi^*\mathcal{T}_Y\) by extending \(\mathcal{O}^\mathbb{A}_X\)-bilinearly

\[
\langle m^1 \otimes v_1, m^2 \otimes v_2 \rangle_g := m^1 m^2 \langle v_1, v_2 \rangle_g = m^1 m^2 \varphi^*(\langle v_1, v_2 \rangle_g),
\]

where \(m^1, m^2 \in \text{Comm}(A_\varphi) \subset \mathcal{O}^\mathbb{A}_X\) and \(v_1, v_2 \in \mathcal{T}_Y\) and the last equality follows from the canonical isomorphism

\[
\mathcal{O}^\mathbb{A}_X \otimes_{\varphi, \mathcal{O}_Y} \mathcal{O}_Y \cong \mathcal{O}^\mathbb{A}_X.
\]

However, for this to be well-defined, it is required that

\[
\langle m^1 \otimes f_1 v_1, m^2 \otimes f_2 v_2 \rangle_g = \langle m^1 \varphi^*(f_1) \otimes v_1, m^2 \varphi^*(f_2) \otimes v_2 \rangle_g,
\]

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i.e.
\[ m^1 m^2 \varphi^2(f_1 f_2 \langle v_1, v_2 \rangle) = m^1 \varphi^2(f_1) m^2 \varphi^2(f_2) \varphi^2(\langle v_1, v_2 \rangle), \]
for all \( m^1, m^2 \in \mathcal{O}_X^\text{tr} \), \( v_1, v_2 \in \mathcal{T}_Y \), and \( f_1, f_2 \in \mathcal{O}_Y \). Which holds if and only if
\[ m^2 \in \text{Comm}(\mathcal{A}_\varphi). \]

What happens if one brings in the trace map \( \text{Tr} : \mathcal{O}_X^\text{tr} \to \mathcal{O}_X^\text{c} \)? In this case,
\[ \text{Tr} (m^1 \otimes f_1 v_1, m^2 \otimes f_2 v_2) = \text{Tr}(m^1 m^2 \varphi^2(f_1 f_2 \langle v_1, v_2 \rangle)) = \text{Tr}(\varphi^2(f_1) m^1 m^2 \varphi^2(f_2 \langle v_1, v_2 \rangle)) \]
by the cyclic-invariance property of \( \text{Tr} \) while
\[ \text{Tr}(m^1 \varphi^2(f_1) \otimes v_1, m^2 \varphi^2(f_2) \otimes v_2) = \text{Tr}(m^1 \varphi^2(f_1) m^2 \varphi^2(f_2 \langle v_1, v_2 \rangle)). \]
The two equal if
\[ \text{either } m_1 \in \text{Comm}(\mathcal{A}_\varphi) \text{ or } m_2 \in \text{Comm}(\mathcal{A}_\varphi). \]

**Definition 3.2.2.2.** [partially-defined inner product \( \langle \cdot, \cdot \rangle_g \) on \( \varphi^* \mathcal{T}_Y \)] The multiplication in \( \mathcal{O}_X^\text{tr} \) and the inner product \( \langle \cdot, \cdot \rangle_g \) on \( \mathcal{T}_Y \) together induce a partially defined, \( \mathcal{O}_X^\text{tr} \)-valued, \( \mathcal{O}_X^\text{c} \)-bilinear (nonsymmetric) inner product on \( \varphi^* \mathcal{T}_Y \) by extending \( \mathcal{O}_X^\text{tr} \)-bilinearly
\[ \langle m^1 \otimes v_1, m^2 \otimes v_2 \rangle_g := m^1 m^2 \otimes \langle v_1, v_2 \rangle_g = m^1 m^2 \varphi^2(\langle v_1, v_2 \rangle), \]
where \( m^1 \in \mathcal{O}_X^\text{tr}, m^2 \in \text{Comm}(\mathcal{A}_\varphi) \in \mathcal{O}_X^\text{tr} \) and \( v_1, v_2 \in \mathcal{T}_Y \) and the last equality follows from the canonical isomorphism \( \mathcal{O}_X^\text{tr} \otimes_{\varphi^*} \mathcal{O}_Y \simeq \mathcal{O}_X^\text{c} \).

**Definition 3.2.2.3.** [partially-defined inner product \( \text{Tr}(\cdot, \cdot)_g \) on \( \varphi^* \mathcal{T}_Y \)] The multiplication in \( \mathcal{O}_X^\text{tr} \), the inner product \( \langle \cdot, \cdot \rangle_g \) on \( \mathcal{T}_Y \), and the trace map \( \text{Tr} : \mathcal{O}_X^\text{tr} \to \mathcal{O}_X^\text{c} \) together induce a partially defined, \( \mathcal{O}_X^\text{c} \)-valued, \( \mathcal{O}_X^\text{c} \)-bilinear (symmetric) inner product on \( \varphi^* \mathcal{T}_Y \) by extending \( \mathcal{O}_X^\text{c} \)-bilinearly
\[ \text{Tr}(m^1 \otimes v_1, m^2 \otimes v_2) := \text{Tr}(m^1 m^2 \otimes \langle v_1, v_2 \rangle_g) = \text{Tr}(m^1 m^2 \varphi^2(\langle v_1, v_2 \rangle)), \]
where either \( m^1 \) or \( m^2 \) is in \( \text{Comm}(\mathcal{A}_\varphi) \), \( v_1, v_2 \in \mathcal{T}_Y \), and the last equality follows from the canonical isomorphism \( \mathcal{O}_X^\text{c} \otimes_{\varphi^*} \mathcal{O}_Y \simeq \mathcal{O}_X^\text{c} \).

By construction, both inner products, when defined, are covariantly constant with respect to \( \nabla(\varphi^* g) \) and one has the Leibniz rules
\[
\begin{align*}
D \langle -, \cdot \rangle_g & = \langle \nabla(\varphi^* g) -, \cdot \rangle_g + \langle -, \nabla(\varphi^* g) \cdot \rangle_g, \\
\text{d}_X \text{Tr} \langle -, \cdot \rangle_g & = \text{Tr}(D \langle -, \cdot \rangle_g) = \text{Tr}(\nabla(\varphi^* g) -, \cdot \rangle_g + \text{Tr}(-, \nabla(\varphi^* g) - \rangle_g),
\end{align*}
\]
whenever the \( \langle -, m \rangle_g \) or \( \text{Tr}(\langle -, m \rangle_g \rangle \) involved are defined.

The following lemma is an immediate consequence of Corollary 3.1.5:
Lemma 3.2.2.4. [sample list of defined inner products for admissible \((\varphi, \nabla)\) \((\ast_1)\)-admissible, the following list of inner products

\[
\langle - , D_\xi \varphi \rangle_g , \quad \text{Tr} \langle - , D_\xi \varphi \rangle_g , \quad \text{Tr} \langle D_\xi \varphi , - \rangle_g
\]

are defined for \(\xi \in T_s X\).

(2) For \((\varphi, \nabla)\) \((\ast_2)\)-admissible, the following additional list of inner products

\[
\langle - , \nabla^{(\varphi,g)}_{\xi_1} \cdots \nabla^{(\varphi,g)}_{\xi_k} D_\xi \varphi \rangle_g , \quad \text{Tr} \langle - , \nabla^{(\varphi,g)}_{\xi_1} \cdots \nabla^{(\varphi,g)}_{\xi_k} D_\xi \varphi \rangle_g , \quad \text{Tr} \langle \nabla^{(\varphi,g)}_{\xi_1} \cdots \nabla^{(\varphi,g)}_{\xi_k} D_\xi \varphi , - \rangle_g
\]

are also defined for \(\xi, \xi_1, \cdots, \xi_k \in T_s X, k \in \mathbb{Z}_{\geq 0}\).

The symmetry properties of the curvature tensor of \(\nabla^{(\varphi,g)}\) on \(\varphi^* T_s Y\)

Let \(\varphi : (X^k, E; \nabla) \to Y\) be a \((\ast_2)\)-admissible map. Let \(F_{\nabla^{(\varphi,g)}}\) be the curvature tensor of the induced connection \(\nabla^{(\varphi,g)}\) on \(\varphi^* T_s Y\) — the \(\mathcal{C}_X^\infty (\varphi^* T_s Y)\)-valued 2-form on \(X\) defined by

\[
F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) := \left( \nabla_{\xi_1}^{(\varphi,g)} \nabla_{\xi_2}^{(\varphi,g)} - \nabla_{\xi_2}^{(\varphi,g)} \nabla_{\xi_1}^{(\varphi,g)} - \nabla_{[\xi_1, \xi_2]}^{(\varphi,g)} \right) s \in \varphi^* T_s Y
\]

for \(\xi_1, \xi_2 \in T_s X\) and \(s \in \varphi^* T_s Y\). (This is \(O_X\)-linear in \(\xi_1, \xi_2\), and \(s\) and, hence, a tensor on \(X\). By construction,

\[
F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) = - F_{\nabla^{(\varphi,g)}}(\xi_2, \xi_1)
\]

for \(\xi_1, \xi_2 \in T_s X\). From Lemma 3.2.2.4, the inner products

\[
\langle \nabla_{\xi_1}^{(\varphi,g)} D_{\xi_2} \varphi , D_{\xi_4} \varphi \rangle_g , \quad \langle \nabla_{\xi_1}^{(\varphi,g)} D_{\xi_2} \varphi , D_{\xi_4} \varphi \rangle_g , \quad \langle \nabla_{\xi_1}^{(\varphi,g)} D_{\xi_2} \varphi , D_{\xi_4} \varphi \rangle_g , \quad \langle F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_3} \varphi , D_{\xi_4} \varphi \rangle_g
\]

are all defined.

Lemma 3.2.2.5. [symmetry property of the curvature tensor of \(\nabla^{(\varphi,g)}\) on \(\varphi^* T_s Y\)] For a \((\ast_2)\)-admissible pair \((\varphi, \nabla)\),

\[
\langle F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_3} \varphi , D_{\xi_4} \varphi \rangle_g
\]

\[
= - \langle D_{\xi_3} \varphi , F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_4} \varphi \rangle_g + [F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) , \langle D_{\xi_3} \varphi , D_{\xi_4} \varphi \rangle_g].
\]

And, hence,

\[
\text{Tr} \langle F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_3} \varphi , D_{\xi_4} \varphi \rangle_g = - \text{Tr} \langle D_{\xi_3} \varphi , F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_4} \varphi \rangle_g
\]

\[
= - \text{Tr} \langle F_{\nabla^{(\varphi,g)}}(\xi_1, \xi_2) D_{\xi_3} \varphi , D_{\xi_4} \varphi \rangle_g = \text{Tr} \langle F_{\nabla^{(\varphi,g)}}(\xi_2, \xi_1) D_{\xi_4} \varphi , D_{\xi_3} \varphi \rangle_g.
\]

Proof. This is a consequence of the Leibniz rule

\[
D \langle - , - \rangle_g = \langle \nabla^{(\varphi,g)} - , - \rangle_g + \langle - , \nabla^{(\varphi,g)} - \rangle_g,
\]
for \(-, \cdot \in \phi^* T_s Y\), when all inner products involved are defined. In detail,

\[
\langle F_{\nabla^{(\phi, g)}}(\xi_1, \xi_2) D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g
\]

\[
= \langle (\nabla^{(\varphi, g)} \nabla^{(\phi, g)} - \nabla^{(\varphi, g)} \nabla^{(\phi, g)} - \nabla^{(\varphi, g)}_{\xi_1} D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g
\]

\[
= D_{\xi_1} (\nabla^{(\varphi, g)} D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g - \langle \nabla^{(\varphi, g)} D_{\xi_1} \varphi, \nabla^{(\phi, g)} D_{\xi_1} \varphi \rangle_g
\]

\[
- D_{\xi_1} (\nabla^{(\varphi, g)}_{\xi_1} D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g + \langle \nabla^{(\phi, g)} D_{\xi_1} \varphi, \nabla^{(\phi, g)} D_{\xi_1} \varphi \rangle_g
\]

\[
- D_{\xi_1} (\nabla^{(\varphi, g)}_{\xi_1} D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g + (D_{\xi_1} D_{\xi_2} - D_{\xi_2} D_{\xi_1} - D_{\xi_1, \xi_2}) \langle D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g
\]

after repeatedly applying the Leibniz rule,

\[
= - \langle D_{\xi_1} \varphi, F_{\nabla^{(\phi, g)}}(\xi_1, \xi_2) D_{\xi_1} \varphi \rangle_g + F_D(\xi_1, \xi_2) \langle D_{\xi_1} \varphi, D_{\xi_1} \varphi \rangle_g .
\]

Note that \( F_D = [F_{\nabla}, \cdot] \) on \( O_X^k \). The lemma follows.

\[\square\]

Covariant differentation and evaluation

**Lemma 3.2.6.** \([D_\xi (m \otimes v f) \text{ vs. } (\nabla^{(\phi, g)}_\xi (m \otimes v)) f ]\) Let \( \xi \in T_s X, f \in O_Y, \) and \( m \otimes v \in \phi^* T_s Y \). Then,

\[
D_\xi (m \otimes v f) = (\nabla^{(\phi, g)}_\xi (m \otimes v)) f + m \sum_{i=1}^n D_\xi \varphi^i (y^i) \otimes \left( \frac{\partial}{\partial y^i} (vf) - \nabla_{\frac{\partial}{\partial y^i}} (vf) \right).
\]

**Proof.**

\[
D_\xi (m \otimes v f) = D_\xi (m \varphi^i (vf)) = D_\xi m \otimes v f + \sum_{i} D_\xi \varphi^i (y^i) \otimes \frac{\partial}{\partial y^i} (vf)
\]

while

\[
(\nabla^{(\phi, g)}_\xi (m \otimes v)) f = (D_\xi m \otimes v + \sum_{i} D_\xi \varphi^i (y^i) \otimes \nabla_{\frac{\partial}{\partial y^i}} v) f.
\]

The lemma follows.

\[\square\]

### 3.2.3 The \( O_X^C \)-module \( T^* X \otimes_{O_X} \phi^* T_s Y \), where \( D \varphi \) lives

Assume that \((\varphi, \nabla)\) is \((*,1)\)-admissible and recall the metric \( h \) on \( X \) and the metric \( g \) on \( Y \). Then the construction in this subsection is a combination of the constructions in Sec. 3.2.1 and Sec. 3.2.2.

The connection \( \nabla^h \) on \( T^* X \), the connection \( D \) on \( O^k_X \), and the connection \( \nabla^g \) on \( T_s Y \) together induce a connection \( \nabla^{(h,\varphi, g)} \) on \( T^* X \otimes_{O_X} \phi^* T_s Y \), locally of the form

\[
\nabla^{(h,\varphi, g)} = \nabla^h \otimes \text{Id}_{O_X^k} \otimes \text{Id}_{T_s Y} + \text{Id}_{T^* X} \otimes D^T \otimes \text{Id}_{T_s Y} + \text{Id}_{T^* X} \otimes \text{Id}_{O_X^k} \cdot \sum_{i=1}^n D \varphi^i (y^i) \otimes \nabla_{\frac{\partial}{\partial y^i}} .
\]
Lemma 3.2.2.1 implies that this is independent of the coordinate \((y^1, \ldots, y^n)\) on coordinate charts chosen and hence well-defined.

The inner product \(\langle \cdot, \cdot \rangle_h\) on \(T^*X\), the multiplication in \(\mathcal{O}_X^\mathbb{C}\), and the inner product \(\langle \cdot, \cdot \rangle_g\) on \(T_Y\) together induce a partially defined, \(\mathcal{O}_X^\mathbb{C}\)-valued, \(\mathcal{O}_X^\mathbb{C}\)-bilinear (nonsymmetric) inner product on \(T^*X \otimes \mathcal{O}_X \varphi^*T_Y\) by extending \(\mathcal{O}_X^\mathbb{C}\)-bilinearily

\[
\langle \omega^1 \otimes m_T^1 \otimes v_1, \omega^2 \otimes m_T^2 \otimes v_2 \rangle_{(h,g)} := \langle \omega^1, \omega^2 \rangle_h \cdot m_T^1 m_T^2 \otimes \langle v_1, v_2 \rangle_g = \langle \omega^1, \omega^2 \rangle_h m_T^1 m_T^2 \varphi^2(\langle v_1, v_2 \rangle_g),
\]

where \(\omega^1, \omega^2 \in T^*X\), \(m_1 \in \mathcal{O}_X^\mathbb{C}\), \(m_2 \in \text{Comm} (\mathcal{A}_\varphi) \subset \mathcal{O}_X^\mathbb{C}\), \(v_1, v_2 \in T_Y\) and the last equality follows from the canonical isomorphism \(\mathcal{O}_X^\mathbb{C} \otimes \varphi^* \mathcal{O}_Y \cong \mathcal{O}_X^\mathbb{C}\). The trace map \(\text{Tr} : \mathcal{O}_X^\mathbb{C} \to \mathcal{O}_X^\mathbb{C}\) gives another partially defined, \(\mathcal{O}_X^\mathbb{C}\)-valued, \(\mathcal{O}_X^\mathbb{C}\)-bilinear (symmetric) inner product on \(T^*X \otimes \mathcal{O}_X \varphi^*T_Y\) by extending \(\mathcal{O}_X^\mathbb{C}\)-bilinearily

\[
\text{Tr} \langle \omega^1 \otimes m_T^1 \otimes v_1, \omega^2 \otimes m_T^2 \otimes v_2 \rangle_{(h,g)} := \text{Tr} \langle \omega^1, \omega^2 \rangle_h \cdot m_T^1 m_T^2 \otimes \langle v_1, v_2 \rangle_g = \text{Tr} \langle \omega^1, \omega^2 \rangle_h m_T^1 m_T^2 \varphi^2(\langle v_1, v_2 \rangle_g),
\]

where \(\omega^1, \omega^2 \in T^*X\), either \(m_1\) or \(m_2\) is in \(\text{Comm} (\mathcal{A}_\varphi)\), \(v_1, v_2 \in T_Y\).

By construction, both inner products, when defined, are covariantly constant with respect to \(\nabla^{(\varphi, \varphi)}\) and one has the Leibniz rules

\[
D \langle \sim, \sim' \rangle_{(h,g)} = \langle \nabla^{(\varphi, \varphi)} \sim, \sim' \rangle_{(h,g)} + \langle \sim, \nabla^{(\varphi, \varphi)} \sim' \rangle_{(h,g)},
\]

\[
d\text{Tr} \langle \sim, \sim' \rangle_{(h,g)} = \text{Tr} \langle D \langle \sim, \sim' \rangle_{(h,g)} \rangle = \text{Tr} \langle \nabla^{(\varphi, \varphi)} \sim, \sim' \rangle_{(h,g)} + \text{Tr} \langle \sim, \nabla^{(\varphi, \varphi)} \sim' \rangle_{(h,g)},
\]

whenever the \(\langle \sim, \sim' \rangle_{(h,g)}\) or \(\text{Tr} \langle \sim, \sim' \rangle_{(h,g)}\) involved are defined.

With all the preparations in Sec. 1–Sec. 3, we are finally ready to construct and study the standard action for D-branes along our line of pursuit.

4 The standard action for D-branes

We introduce in this section the standard action, which is to D-branes as the (Brink-Di Vecchia-Howe/Deser-Zumino)/Polyakov action is to fundamental strings. Abstractly, it is an enhanced non-Abelian gauged sigma model based on maps \(\varphi : (X^\mathbb{C}, \mathcal{E}; \nabla) \to Y\).

The gauge-symmetry group \(C^\infty (\text{Aut}_\mathbb{C}(E))\)

Let \(\text{Aut}_\mathbb{C}(E)\) be the automorphism bundle of the complex vector bundle \(E\) (of rank \(r\)) over \(E\), \(\text{Aut}_\mathbb{C}(E) \subset \text{End}_\mathbb{C}(E)\) canonically as the bundle of invertible endomorphisms; it is a principal \(\text{GL}_r(\mathbb{C})\)-bundle over \(X\). The set

\[
\mathcal{G}_{\text{gauge}} := C^\infty (\text{Aut}_\mathbb{C}(E))
\]

of smooth sections of \(\text{Aut}_\mathbb{C}(E)\) forms an infinite-dimensional Lie group and acts on the space of pairs \((\varphi, \nabla)\) as a gauge-symmetry group:

\[
g' \in \mathcal{G}_{\text{gauge}} : (\varphi, \nabla = d + A) \mapsto (g' \varphi, g' \nabla = d + g' A)
\]

\[
:= (g' \varphi g'^{-1}, d - (dg')g'^{-1} + g' A g'^{-1})
\]

The induced action of \(\mathcal{G}_{\text{gauge}}\) on other basic objects are listed in the lemma below.
Lemma 4.1. [induced action of $G_{gauge}$ on other basic objects]  (All the $G_{gauge}$-actions are denoted by a representation $\rho_{gauge}$ of $G_{gauge}$, if in need.)

(01) on $\mathcal{O}_X^k$:
$$\rho_{gauge}(g')(m) = g'mg'^{-1} \text{ for } m \in \mathcal{O}_X^k.$$ 

(02) on induced connections:
$$D = d + [A, \cdot] \mapsto g'D := d + [g'A, \cdot].$$

(1) on $T^*X \otimes_{\mathcal{O}_X} \mathcal{O}_X^k$:
$$\rho_{gauge}(g') (\omega \otimes m) = \omega \otimes (g'mg'^{-1}) =: g'(\omega \otimes m)g'^{-1}.$$ 

(2) for $\varphi^*T_sY$:
$$\varphi^*T_sY \mapsto g'\varphi^*T_sY$$
$$m \otimes v \mapsto (g'mg'^{-1}) \otimes v =: g'(m \otimes v)g'^{-1}.$$ 

(3) for $T^*X \otimes_{\mathcal{O}_X} \varphi^*T_sY$:
$$T^*X \otimes_{\mathcal{O}_X} \varphi^*T_sY \mapsto T^*X \otimes_{\mathcal{O}_X} g'\varphi^*T_sY$$
$$\omega \otimes m \otimes v \mapsto \omega \otimes (g'mg'^{-1}) \otimes v =: g'(\omega \otimes m \otimes v)g'^{-1}.$$ 

(4) for covariant differential:
$$D\varphi \mapsto g'D\varphi = g'D\varphi g'^{-1}.$$ 

(5) for pull-push:
$$(g'\varphi)^* \alpha = g'\varphi^* \alpha g'^{-1}.$$ 

Proof. The proof is elementary. Let us demonstrate Item (2) as an example.

For $m \otimes v \in \varphi^*T_sY := \mathcal{O}_X^k \otimes_{\varphi^*T_sY} \mathcal{O}_Y$, 
$$\rho_{gauge}(g')(m \otimes v) = \rho_{gauge}(g')(m) \otimes v = (g'mg'^{-1}) \otimes v$$

since $G_{gauge}$ acts on $T_sY$ trivially (i.e. by by the identity map $Id_Y$). The only issue is: Where does $(g'mg'^{-1}) \otimes v$ now live? To answer this, note that, for $f \in C^\infty(Y)$, on one hand
$$\rho_{gauge}(g')(m \otimes f) = \rho_{gauge}(g')(m_{\varphi^*}(f) \otimes v) = (g'm_{\varphi^*}(f)g'^{-1}) \otimes v,$$
while on the other hand
$$\rho_{gauge}(g')(m \otimes f) = (g'mg'^{-1}) \otimes f.$$

It follows that
$$(g'mg'^{-1}) \otimes f$$
$$= (g'm_{\varphi^*}(f)g'^{-1}) \otimes v = (g'mg'^{-1} \cdot g'\varphi^* (f)g'^{-1}) \otimes v = (g'mg'^{-1} \cdot g'\varphi^* (f)) \otimes v.$$ 

Which says that our section $(g'mg'^{-1}) \otimes v$ now lives in $g'\varphi^*T_sY$.

\[\square\]
The standard action for D-branes

Fix a (dilaton field $\rho$, metric $h$) on the underlying smooth manifold $X$ (of dimension $m$) of the Azumaya/matrix manifold with a fundamental module $(X^k, \mathcal{E})$. Fix a background (dilaton field $\Phi$, metric $g$, $B$-field $B$, Ramond-Ramond field $C$) on the target space-(time) $Y$ (of dimension $n$)\footnote{For mathematicians, $\rho$ is a smooth function on $X$, $\Phi$ is a smooth function on $Y$, $B$ is a 2-form and $C$ is a general differential form on $Y$. Such background fields ($\Phi, g, B, C$) on $Y$ are created by massless excitations of closed superstrings on $Y$. The notations for these particular fields are almost already carved into stone in string-theory literature. Which we adopt here.} Here, $h$ and $g$ can be either Riemannian or Lorentzian.

**Definition 4.2. [standard action = enhanced non-Abelian gauged sigma model]**

With the given background fields $(\rho, h)$ on $X$ and $(\Phi, g, B, C)$ on $Y$, the standard action $S_{\text{standard}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)$ for $(*_1)$-admissible pairs $(\varphi, \nabla)$ is defined to be the functional

$$S_{\text{standard}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) := S_{\text{nAGSM}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)$$

$$:= S_{\text{map/kinetic}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) + S_{\text{CS/WZ}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) + S_{\text{gauge/YM}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)$$

with the enhanced kinetic term for maps

$$S_{\text{map/kinetic}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) := \frac{1}{2} T_{m-1} \int_X \Re \left( \Tr (D\varphi, D\varphi) \right) \vol_h + \int_X \Re \left( \Tr \left( d\rho, \varphi^o d\Phi - \varphi^o B \right) \right) \vol_h,$$

the gauge/Yang-Mills term

$$S_{\text{gauge/YM}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) := -\frac{1}{2} \int_X \Re \left( \Tr \left( 2\pi \alpha' F_{\varphi} + \varphi^o B \right) \right) \vol_h$$

and the Chern-Simons/Wess-Zumino term

(if $(\varphi, \nabla)$ is furthermore $(*_2)$-admissible, cf. Remark 2.1.13)

$$S_{\text{CS/WZ}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) \text{ formally } = T_{m-1} \int_X \Re \left( \Tr \left( \varphi^o C \wedge e^{2\pi \alpha' F_{\varphi}} + \varphi^o B \wedge \sqrt{\hat{A}(X^k)/\hat{A}(N_{X^k/Y})} \right) \right) \left( \text{mod} m \right).$$

Here,

(0) On $\Re$ Note that while eigenvalues of $\varphi^f(f)$ are all real ([L-Y5: Sec. 3.1] (D(11.1))) for $f \in \mathcal{O}_Y$, the eigenvalues of $D \xi \varphi^f(f)$, $\xi \in T_a X$, may not be so under the $(*_1)$-Admissible Condition. Thus, $\Tr(\cdots)$ in the integrand of terms in $S_{\text{standard}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)$ are in general $\mathbb{C}$-valued and we take the real part $\Re \Tr(\cdots)$ of it.

(1) The enhanced kinetic term for maps The first summand of $S_{\text{map/kinetic}}^{(\rho, h; \Phi, g, B, C)}$ defines the kinetic energy

$$E_{\nabla}(\varphi) := S_{\text{map/kinetic}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla) := \frac{1}{2} T_{m-1} \int_X \Re \left( \Tr (D\varphi, D\varphi) \right) \vol_h$$

of the map $\varphi$ for a given $\nabla$ and, hence, will be called the kinetic term for maps in the standard action $S_{\text{standard}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)$. When the metric $g$ on $Y$ is Lorentzian, then depending on the convention of its signature $(-, +, \cdots)$ vs. $(+, -, \cdots)$, one needs to add an overall minus $-$ vs. plus $+$ sign. In this note, for simplicity of presentation, we choose $h$ and $g$ to be both Riemannian (i.e. for Euclideanized/Wick-rotated D-branes and space-time).
• The world-volume $X^k$ of D-brane is $m$-dimensional; $T_{m-1}$ is the tension of $(m-1)$-dimensional D-branes. Like the tension of the fundamental string, it is a fixed constant of nature.

• The second summand of $S_{\text{mapkinetic}}^{(\rho,h;\Phi,g)}$

$$S_{\text{dilaton}}(\varphi) := \int_X \text{Re} \left( \text{Tr}(d\rho, \varphi^\circ d\Phi)_h \right) \text{vol}_h,$$

will be called the dilaton term of the standard action $S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla)$.

Note that if let $U$ be small enough and fix a local trivialization of $E|_U$. and assume that $\nabla = d + A$ with respect to this local trivialization. Then $D = d + [A, \cdot]$ and, over $U$ with an orthonormal frame $(e_\mu)_\mu$,

$$\text{Tr}(d\rho, \varphi^\circ d\Phi)_h = \sum_\mu \text{Tr}(d\rho(e_\mu)D_{e_\mu}\varphi^\circ(\Phi)) = \sum_\mu \text{Tr}(d\rho(e_\mu)(e_\mu\varphi^\circ(\Phi) + [A(e_\mu), \varphi^\circ(\Phi)])) = \sum_\mu \text{Tr}(d\rho(e_\mu)(e_\mu\varphi^\circ(\Phi))) .$$

Thus, while $\varphi^\circ d\Phi$ depends on the connection $\nabla$, the integrand $\text{Tr}(d\rho, \varphi^\circ d\Phi)_h \text{vol}_h$ does not. This justifies the dilaton term as a functional of $\varphi$ alone.

In contrast, over $U$ with the above setting, $\text{Tr}(D\varphi, D\varphi)_{(h,g)}$ contains summand

$$\sum_\mu \sum_{i,j} \text{Tr}([A(e_\mu), \varphi^\circ(y^i)] [A(e_\mu), \varphi^\circ(y^j)] \varphi^\circ(g_{ij})) ,$$

which does not vanish in general. Thus, $\text{Tr}(D\varphi, D\varphi)_{(h,g)}$ does depend on the pair $(\varphi, \nabla)$.

(2) The gauge/Yang-Mills term $S_{\text{gaugeYM}}^{(h,B)}(\varphi, \nabla)$ $\alpha'$ is the Regge slope; $2\pi\alpha'$ is the inverse to the tension of a fundamental string.

• $F_\nabla$ is the curvature tensor of the connection $\nabla$ on $E$; $2\pi\alpha'F_\nabla + \varphi^\circ B$ is an $O_X^\mathbb{A}$-valued 2-tensor on $X$; and

$$\|2\pi\alpha'F_\nabla + \varphi^\circ B\|_h^2 := \langle 2\pi\alpha'F_\nabla + \varphi^\circ B, 2\pi\alpha'F_\nabla + \varphi^\circ B \rangle_h$$

from Sec. 3.2.1. Up to the shift by $\varphi^\circ B$, this is a norm-squared of the field strength of the gauge field, and hence the name Yang-Mills term. Note that in $S_{\text{gaugeYM}}^{(h,B)}(\varphi, \nabla)$, $\nabla$ couples with $\varphi$ only through the background $B$-field $B$. When $B = 0$, this is simply a functional $S_{\text{gaugeYM}}^{(h)}(\nabla)$ of $\nabla$ alone.

• In the current bosonic case, the Yang-Mills functional for the gauge term $S_{\text{gaugeYM}}^{(h,B)}(\varphi, \nabla)$ can be replaced any other standard action functional, e.g. Chern-Simons functional, in gauge theories.

(3) The Chern-Simons/Wess-Zumino term $S_{\text{CSWZ}}^{(C,B)}(\varphi, \nabla)$ The coupling constant of Ramond-Ramond fields with D-branes is taken to be equal to the D-brane tension $T_{m-1}$. This choice is adopted from the situation of the Dirac-Born-Infeld action. However, in the current bosonic case, one may take a different constant. As given here, $S_{\text{CSWZ}}^{(C,B)}(\varphi, \nabla)$ is only formal; the anomaly factor $\sqrt{\hat{A}(X^k)/A(N_{X^k/Y})}$ in its integrand remains to be understood in the current situation.
The wedge product of $O^{\mathbb{N}}_X$-valued differential forms was discussed in [L-Y8: Sec.6.1] (D(13.1)). An Ansatz was proposed there in accordance with the notion of ‘symmetrized determinant’ for an $O^{\mathbb{N}}_X$-valued 2-tensor on $X$ in the construction of the non-Abelian Dirac-Born-Infeld action ibidem. Here, we no longer have a direct guide from the construction of the kinetic term $S^{h\psi}_{\text{map},\text{kinetic}}(\varphi, \nabla)$ for maps as to how to define such wedge products. However, just like Polyakov string should be thought of as being equivalent to Nambu-Goto string (at least at the classical level) but technically more robust, here we would think that ‘standard D-branes’ should be equivalent to ‘Dirac-Born-Infeld D-branes’ (at least classically) and, hence, will take the same Ansatz:

**Ansatz [wedge product in the Chern-Simons/Wess-Zumino action]** We interpret the wedge products that appear in the formal expression for the Chern-Simons/Wess-Zumino term $S^{(C, B)}_{\text{CS/WZ}}(\varphi, \nabla)$ through the symmetrized determinant that applies to the above defining identities for wedge product; namely, we require that

$$(\omega^1 \wedge \cdots \wedge \omega^s)(e_1 \wedge \cdots \wedge e_s) = \text{SymDet}(\omega^i(e_j))$$

for $O^{\mathbb{N}}_X$-valued 1-forms $\omega^1, \cdots, \omega^s$ on $X$. Denote this generalized wedge product by $\hat{\wedge}$.

Then, for lower-dimensional D-branes $m = 0, 1, 2, 3$, it is reasonable to assume that the anomaly factor is 1 (i.e. no anomaly) and $S^{(C, B)}_{\text{CS/WZ}}(\varphi, \nabla)$ can be written out precisely.

Locally in terms of a local frame $(e_\mu)_\mu$ on an open set $U \subset X$ and a coordinate $(y^1, \cdots, y^n)$ on a local chart of $Y$, one has: (Assuming that $B = \sum_{i,j} B_{ij} dy^i \otimes dy^j$, $B_{ij} = -B_{ji}$.)

- For $D$-(1)-brane world-point $(m = 0)$:
  $$S^{(C_{(0)})}_{\text{CS/WZ}}(\varphi) = T_{-1} \cdot \text{Tr}(\varphi^0 C_{(0)}) = T_{-1} \cdot \text{Tr}(\varphi^0 (C_{(0)})).$$

- For $D$-particle world-line $(m = 1)$: Assume that $C_{(1)} = \sum_{i=1}^n C_i dy^i$ locally; then
  $$S^{(C_{(1)})}_{\text{CS/WZ}}(\varphi) = T_0 \int_X \text{Re} \left( \text{Tr}(\varphi^0 C_{(1)}) \right) \text{locally} = T_0 \int_U \text{Re} \left( \text{Tr} \left( \sum_{i=1}^n \varphi^0(C_i) \cdot D_{e_i} \varphi^0(y^i) \right) \right) e^1 .$$

Note that as in the case of the dilaton term $S^{(\rho, h, \Phi)}_{\text{dilaton}}(\varphi)$, this is a functional of $\varphi$ alone.

- For $D$-string world-sheet $(m = 2)$: Assume that $C_{(2)} = \sum_{i,j=1}^n C_{ij} dy^i \otimes dy^j$ locally, with $C_{ij} = -C_{ji}$; then
  $$S^{(C_{(0)}, C_{(2)}, B)}_{\text{CS/WZ}}(\varphi, \nabla) = T_1 \int_X \text{Re} \left( \text{Tr}(\varphi^2 C_{(2)} + \varphi^0 C_{(0)} B + 2\pi \alpha' \varphi^0 (C_{(0)}) \otimes F_\varphi) \right)$$

  $$= T_1 \int_X \text{Re} \left( \text{Tr}(\varphi^2 (C_{(2)} + C_{(0)} B) + \pi \alpha' \varphi^0 (C_{(0)}) F_\varphi + \pi \alpha' F_\varphi \varphi^0 (C_{(0)})) \right) \text{locally}$$

  $$= T_1 \int_U \text{Re} \left( \text{Tr} \left( \sum_{i,j=1}^n \varphi^0(C_i + C_{(0)} B_{ij}) D_{e_i} \varphi^0(y^i) D_{e_2} \varphi^0(y^j) + \pi \alpha' \varphi^0 (C_{(0)}) F_\varphi(e_1, e_2) + \pi \alpha' F_\varphi(e_1, e_2) \varphi^0 (C_{(0)}) \right) \right) e^1 \wedge e^2$$

  $$= T_1 \int_U \text{Re} \left( \text{Tr} \left( \sum_{i,j=1}^n \varphi^0(C_i + C_{(0)} B_{ij}) D_{e_i} \varphi^0(y^i) D_{e_2} \varphi^0(y^j) + 2 \pi \alpha' \varphi^0 (C_{(0)}) F_\varphi(e_1, e_2) \right) \right) e^1 \wedge e^2 .$$

Here, the last identity comes from the effect of the trace map $\text{Tr}$. 

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Remark 4.3. [other effects from B-field and Ramond-Ramond field] There are other effects to D-branes beyond just mentioned above from the background B-field and Ramond-Ramond field that have not yet been taken into account in this project so far; e.g. [H-M1], [H-M2], and [H-Y]. They can influence the action for D-branes as well. Such additional effects should be investigated in the future.

Theorem 4.4. [well-defined gauge-symmetry-invariant action] Except the anomaly factor in the Chen-Simons/Wess-Zumino term, which is yet to be understood, the standard action $S_{\text{standard}}^{(\rho,h,\Phi,g,B,C)}(\varphi,\nabla)$ as given in Definition 4.2 for ($*_1$)-admissible pairs $(\varphi,\nabla)$ (and $S_{\text{CS/WZ}}^{(C,B)}(\varphi,\nabla)$ for ($*_2$)-admissible $(\varphi,\nabla)$) is well-defined. Assume that the anomaly factor in the Chen-Simons/Wess-Zumino term transforms also by conjugation as for $\mathcal{O}_X^\mathbb{C}$ under a gauge symmetry, then $S_{\text{standard}}^{(\rho,h,\Phi,g,B,C)}(\varphi,\nabla)$ is invariant under gauge symmetries:

$$S_{\text{standard}}^{(\rho,h,\Phi,g,B,C)}(\varphi,\nabla) = S_{\text{standard}}^{(\rho,h,\Phi,g,B,C)}(g' \varphi, g' \nabla)$$

for $g' \in G_{\text{gauge}} := C^\infty(\text{Aut}_C(E))$. 

Their partial study was done in [L-Y2 : Sec. 6.2] (D(13.1)).

(4) The background B-field The coupling of $(\varphi, \nabla)$ with the background B-field $B$ on $Y$ in the part

$$S_{\text{gauge}}^{(h,B)}(\varphi, \nabla) + S_{\text{CS/WZ}}^{(C,B)}(\varphi, \nabla)$$

of the standard action means that we have to adjust the fundamental module $\mathcal{E}$ on $X$ by a compatible “twisting” governed by $\varphi$ and $B$. With this “twisting”, $\mathcal{E}$ now lives on a gerb over $X$. See [L-Y2] (D(5)) for details and further references. However, since the study of the variational problems in this note is mainly local and focuses on the enhanced kinetic term for maps $S_{\text{map,kinetic}}^{(\rho,h,\Phi,g)}$, we’ll ignore this twisting for the current note to keep the language and expressions simple.
Proof. For the kinetic term for maps

\[ S_{\text{map, kinetic}}^{(h,g)}(\varphi, \nabla) := \frac{1}{2} T_{m-1} \int_X \text{Re} \left( \text{Tr} \langle D\varphi, D\varphi \rangle_{(h,g)} \right) \text{vol}_h, \]

that it is well-defined follows Lemma 3.2.2.4. Under a gauge transformation \( g' \in G_{\text{gauge}} := C^\infty(Aut_C(E)) \) and in terms of local coordinates \((x^1, \ldots, x^m)\) on \(X\) and \((y^1, \ldots, y^n)\) on \(Y\),

\[ g' D g' = \sum_\mu dx^\mu \otimes \sum_i g' D_\mu \varphi^i (\frac{\partial}{\partial y^i}) \otimes \varphi^i \frac{\partial}{\partial y^i} = \sum_\mu dx^\mu \otimes \sum_i \left( g' \left( D_\mu \varphi^i (\frac{\partial}{\partial y^i}) \right) g^{-1} \right) \otimes \varphi^i \frac{\partial}{\partial y^i}. \]

Thus,

\[
\langle g' D g' \varphi, g' D g' \varphi \rangle_{(h,g)} \\
= \sum_{\mu,\nu} \sum_{i,j} h^{\mu\nu} \otimes \left( g' \left( D_\mu \varphi^i (\frac{\partial}{\partial y^i}) \right) g^{-1} \cdot g' \left( D_\nu \varphi^j (\frac{\partial}{\partial y^j}) \right) g^{-1} \right) \otimes \varphi^i \varphi^j g_{ij} \\
= \sum_{\mu,\nu} \sum_{i,j} h^{\mu\nu} \cdot \left( g' \left( D_\mu \varphi^i (\frac{\partial}{\partial y^i}) \right) g^{-1} \cdot g' \left( D_\nu \varphi^j (\frac{\partial}{\partial y^j}) \right) g^{-1} \right) \cdot g' \varphi^i (g_{ij}) g^{-1} \\
= \left( \sum_{\mu,\nu} \sum_{i,j} h^{\mu\nu} \cdot D_\mu \varphi^i (\frac{\partial}{\partial y^i}) \cdot D_\nu \varphi^j (\frac{\partial}{\partial y^j}) \cdot \varphi^i (g_{ij}) \right) g^{-1} \\
= \langle g' D \varphi, D \varphi \rangle_{(h,g)} g^{-1}. \]

It follows that \( \text{Tr} \langle g' D g' \varphi, g' D g' \varphi \rangle_{(h,g)} = \text{Tr} \langle D \varphi, D \varphi \rangle_{(h,g)} \) and, hence,

\[ S_{\text{map, kinetic}}^{(h,g)}(g' \varphi, g' \nabla) = S_{\text{map, kinetic}}^{(h,g)}(\varphi, \nabla). \]

The other terms in \( S_{\text{standard}}^{(h,g;\Phi,B,C)}(\varphi, \nabla) \) do not involve a partially-defined inner product and hence are all defined. That the integrand inside \( \text{Tr} \) all transform by conjugation under a gauge symmetry as for \( O^\infty_X \) follows Lemma 4.1.

This proves the theorem. \( \square \)

Remark 4.5. [gauge-fixing condition] As in any gauge field theory (e.g. [P-S]), understanding how to fix the gauge is an important part of understanding \( S_{\text{standard}}^{(h,g;\Phi,B,C)}(\varphi, \nabla) \).

The standard action as an enhanced non-Abelian gauged sigma model

Recall that, in an updated language and in a form for easy comparison, a sigma model (\(\sigma\)-model, SM) on a (Riemannian or Lorentzian) manifold \((Y,g)\) (of dimension \(n\)) is a field theory on a (Riemannian or Lorentzian) manifold \((X,h)\) (of some dimension \(m\)) with

- **Field:** differentiable maps \( f : X \to Y \),
- **Action functional:**

\[
S_{\text{sigma model}}^{(h,g)}(f) := \pm \frac{1}{2} \int_X \langle df, df \rangle_{(g,h)} \text{vol}_h = \pm \frac{1}{2} \int_X \left\| f^* g \right\|_h^2 \text{vol}_h \\
:= \pm \frac{1}{2} \int_X \sum_{\mu,\nu=1}^m \sum_{i,j=1}^n h^{\mu\nu}(x) g_{ij}(f(x)) \frac{\partial f_i}{\partial x^\mu}(x) \frac{\partial f_j}{\partial x^\nu}(x) \sqrt{|\text{det} h(x)|} \ d^m x,
\]

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in terms of local coordinates \( \mathbf{x} = (x^1, \cdots, x^n) \) on \( X \) and \( \mathbf{y} = (y^1, \cdots, y^n) \) on \( Y \); cf. [GM-L] and see e.g. [C-T] for modern update and further references. (The ± sign depends on the signature of the metric.) At the classical level, this is a theory of harmonic maps; cf. [E-L], [E-S], [Ma], [Sm].

Back to our situation. To begin with, the kinetic term

\[
S_{\text{map:kinetic}}^{(h,g)}(\varphi, \nabla) := \frac{1}{2} T_{m-1} \int_X \text{Re} \left( \text{Tr}(D\varphi, D\varphi)_{(h,g)} \right) \text{vol}_h
\]

qualifies the standard action \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) \) to be regarded as a sigma model, now based on

- **Field**: \((+1)\)-admissible differentiable maps \( \varphi : (X^d, E; \nabla) \to Y \).

The fact that \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) \) is invariant under the gauge symmetry group \( G_{\text{gauge}} := C^\infty(\text{Aut}_C(E)) \) and that the latter is non-Abelian justify that this sigma model is indeed a non-Abelian gauged sigma model (nAGSM). However, compared with, for example, the well-studied \( d = 2, N = (2,2) \) (Abelian) gauged linear sigma model, e.g. [H-V] and [Wi1], the gauge symmetry of \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) \) does not arise from gauging a global group-action on the target space \( Y \). (For this reason, one may call \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) \) a sigma model with non-Abelian gauge symmetry as well.) For D-branes, its additional coupling to the background Ramond-Ramond field \( C \) on \( Y \) is essential ([Po1]) and, hence, the Chern-Simons/Wess-Zumino term \( S_{\text{CS/WZ}}^{(C,B)}(\varphi, \nabla) \). Also, we like our dynamical field \( (\varphi, \nabla) \) to coupled to the background dilaton field \( \Phi \) on \( Y \) as well so that the essence of the other important action — the Dirac-Born-Infeld action — for D-branes can be retained as much as we can. This motivates the dilaton term \( S_{\text{dilaton}}^{(\rho,h;\Phi)}(\varphi) \). In summary,

\[
S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) := S_{\text{nAGSM}}^{(\rho,h;\Phi,g,B)}(\varphi, \nabla) + S_{\text{CS/WZ}}^{(C,B)}(\varphi, \nabla) + S_{\text{dilaton}}^{(\rho,h;\Phi)}(\varphi)
\]

which explains the name enhanced non-Abelian gauged sigma model (nAGSM+).

### 5 Admissible family of admissible pairs \((\varphi_T, \nabla^T)\)

In this section we introduce the notion of one-parameter admissible families of admissible pairs and rephrase the basic settings and results in Sec. 3.2 in a relative format for such a family. Some curvature tensor computations are given for later use. The natural generalization (without work) to two-parameter admissible families of admissible pairs is remarked in the last theme of the section. This prepares us for the study of the variational problem of the enhanced kinetic term for maps \( S_{\text{map:kinetic}}^{(\rho,h;\Phi,g)}(\varphi, \nabla) \) in the standard action \( S_{\text{standard}}^{(\rho,h;\Phi,g,B,C)}(\varphi, \nabla) \) for D-branes.

**Basic setup and the notion of admissible families of admissible pairs \((\varphi_T, \nabla^T)\)**

Let \( T = (-\varepsilon, \varepsilon) \subset \mathbb{R}^1 \), with coordinate \( t \) and \( \varepsilon > 0 \) small, be the one-parameter space and \( \partial_t := \partial/\partial t \) and \( dt \) be respectively the tangent vector field and the 1-form determined by the coordinate \( t \) on \( T \). Let \((X,E)\) be a manifold \( X \) of dimension \( m \) with a complex vector bundle \( E \) of rank \( r \). Recall the structure sheaf \( \mathcal{O}_X \) of \( X \) and the \( \mathcal{O}_X \)-module \( \mathcal{E} \) from \( E \).

Consider the following families of objects over \( T \):

- **Basic setup**
- **Notion of admissible families**
· $X_T := X \times T$, with the structure sheaf $\mathcal{O}_{X_T}$ and regarded as the constant family of manifolds over $T$ determined by $X$. $X_T$ is equipped with the built-in projection maps $pr_X : X_T \to X$ and $pr_T : X_T \times T \to T$. For $U \subset X$ an open set, we will denote by $U_T$ the corresponding open set $U \times T \subset X \times T$ over $T$.

· $T_*X_T :=$ the tangent bundle of $X_T$ and $\mathcal{T}_*X_T :=$ the tangent sheaf of $X_T$; $T^*(X_T/T) :=$ the relative tangent bundle of $X_T$ over $T$ and $\mathcal{T}_*(X_T/T) :=$ the relative tangent sheaf of $X_T$ over $T$; $T^*(X_T/T) :=$ the relative cotangent bundle of $X_T$ over $T$ and $\mathcal{T}^*(X_T/T) :=$ the relative cotangent sheaf of $X_T$ over $T$.

When $X$ is endowed with a (Riemannian or Lorentzian) metric $h$, $h$ induces canonically an inner-product structure on fibers of $T_*(X_T/T)$ and its dual, $T^*(X_T/T)$, over $T$. These induced inner-product structure will be denoted by $(\cdot, \cdot)_h$.

· $E_T := pr_*^*E$ the pull-back vector bundle of $E$ to $X_T$, regarded as the constant $T$-family of vector bundles over $X$ determined by $E$; and $\mathcal{E}_T := pr_*^*\mathcal{E}$ the corresponding $\mathcal{O}_{X_T}$-module, regarded as the constant $T$-family of $\mathcal{O}_X$-modules determined by $\mathcal{E}$.

The projection map $pr_X : X_T \to X$ induces a projection map $pr_E : E_T \to E$ between the total space of bundles in question. $T_*E_T$ (resp. $\mathcal{T}_*E_T$) denotes the tangent space (resp. the tangent sheaf) of the total space of $E_T$.

· $(X^\mathcal{E}_T, \mathcal{E}_T) := (X_T, \mathcal{O}_{X_T}^\mathcal{E} := End_{\mathcal{O}_{X_T}}(\mathcal{E}_T), \mathcal{E}_T)$, regarded as the constant $T$-family of Azumaya/matrix manifolds with a fundamental module determined by $(X^\mathcal{E}_T, \mathcal{E})$. There is a trace map

$Tr : \mathcal{O}_{X_T}^\mathcal{E} \longrightarrow \mathcal{O}_{X_T}^C$

as $\mathcal{O}_{X_T}$-modules, which takes $Id_{\mathcal{E}_T}$ to $r$.

and take the following notational conventions:

· Through the product structure $X_T = X \times T$, a vector field $\xi$ (resp. 1-form $\omega$) on $X$ and the vector field $\partial_t$ on $T$ lift canonically to a vector field (resp. 1-form) on $X_T$, which will still be denoted by $\xi$ (resp. $\omega$) and $\partial_t$ respectively.

· For referral, the restriction of $X_T, X^\mathcal{E}_T, E_T, \cdots_T$ to over $t \in T$ will be denoted $X_t, X^\mathcal{E}_t, E_t, \cdots$, respectively.

**Definition 5.1.** [connection/covariant derivation trivially flat over $T$] A connection $\nabla^T$ on $E_T$ (equivalently, connection/covariant derivative $\nabla^T$ on $\mathcal{E}_T$) is said to be trivially flat over $T$ if the horizontal lifting of $\partial_t$ to $T_*E_T$ lies in the kernel of the map $pr^*_{E_T} : T_*E_T \to T_*E_T$. For such a $\nabla^T$, we will denote the covariant derivative $\nabla^T_{\partial_t}$ simply by $\partial_t$. The curvature tensor of $\nabla^T$ will be denoted by $F_{\nabla^T}$.

Note that any connection on $E_T$ is flat over $T$ and hence, due to the topology of $T$, can be made trivially flat over $T$ after a bundle-isomorphism. Thus the notion of ‘trivially flat’ is only a notational convenience for our variational problem, not a true constraint. However, caution that while $\nabla^T$ is always flat over $T$, its restriction $\nabla^t$ to $X_t$ varies as $t$ varies in $T$. Thus, in general, $F_{\nabla^T}(\partial_t, \cdot) \neq 0$. 

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Definition 5.2. [admissible family of admissible pairs \((\varphi_T, \nabla^T)\)] A \(T\)-family of maps with varying connections from \((X^A, \mathcal{E})\) to \(Y\) is a pair \((\varphi_T, \nabla^T)\), where

\[ \varphi_T : (X^A, \mathcal{E}_T) \rightarrow Y \]

is a map from \((X^A, \mathcal{E}_T)\) to \(Y\) defined contravariantly by a ring-homomorphism

\[ \varphi^\sharp_T : C^\infty(Y) \rightarrow C^\infty(\text{End}_\mathbb{C}(E_T)) \]

over \(\mathbb{R} \subset \mathbb{C}\) and \(\nabla^T\) is a connection on \(\mathcal{E}_T\) that is trivially flat over \(T\). \(\varphi^\sharp_T\) induces a homomorphism

\[ \mathcal{O}_Y \rightarrow \mathcal{O}^A_{X_T} \]

between equivalence classes of gluing systems of rings, which will still be denoted by \(\varphi^\sharp_T\).

Let \(\mathcal{A}_{\varphi_T} \subset \mathcal{O}^A_{X_T} = \mathcal{O}_{X_T} \big( \text{Im} \varphi^\sharp_T \big)\). Then \((\varphi_T, \nabla^T)\) is said to be a \((\ast_i)\)-admissible \(T\)-family of \((\ast_j)\)-admissible pairs if \((\varphi_T, \nabla^T)\) satisfies Admissible Condition \((\ast_i)\) along \(T\) and Admissible Condition \((\ast_j)\) along \(X\), for \(i, j = 1, 2, 3\).

Example 5.3. [(\(\ast_2\))-admissible \(T\)-family of \((\ast_1)\)-admissible pairs] A \((\ast_2)\)-admissible \(T\)-family of \((\ast_1)\)-admissible pairs \((\varphi_T, \nabla^T)\) is a \(T\)-family of maps \(\varphi_T\) with a varying connection \(\nabla^T\) trivially flat over \(T\) such that

\[ (\ast_2) : \partial_t \text{Comm} (\mathcal{A}_{\varphi_T}) \subset \text{Comm} (\mathcal{A}_{\varphi_T}) \quad \text{and} \quad (\ast_1) : \nabla^T_\xi \mathcal{A}_{\varphi_T} \subset \text{Comm} (\mathcal{A}_{\varphi_T}) \]

for all \(\xi \in \mathcal{T}_s(X_T/T)\). Here, \(\text{Comm} (\mathcal{A}_{\varphi_T})\) is the commutant of \(\mathcal{A}_{\varphi_T}\) in \(\mathcal{O}^A_{X_T}\).

Three basic \(\mathcal{O}_{X_T}\)-modules with induced structures

Let \(X\) be endowed with a (Riemannian or Lorentzian) metric \(h\) and \(Y\) be endowed with a (Riemannian or Lorentzian) metric \(g\). Denote the canonically induced inner-product structure from \(h\) and \(g\) on whatever bundle applicable by \(\langle \cdot, \cdot \rangle_h\) and \(\langle \cdot, \cdot \rangle_g\) respectively. Denote the induced connection on \(\mathcal{T}_s(X_T/T)\) and \(\mathcal{T}^*(X_T/T)\) by \(\nabla^h\) and the Levi-Civita connection on \(\mathcal{T}_sY\) by \(\nabla^g\). The associated Riemann curvature tensor is denoted by \(R^h\) and \(R^g\) respectively.

Let \((\varphi_T, \nabla^T)\) be a \((\ast_1)\)-admissible \(T\)-family of \((\ast_1)\)-admissible pairs. The basic \(\mathcal{O}^A_{X_T}\)-modules with induced structures from the setting, as in Sec. 3.2, are listed below to fix notations.

1. \(\mathcal{O}^A_{X_T} : \text{the noncommutative structure sheaf on } X_T\)
   - The induced connection \(D^T\) from \(\nabla^T\), which is also trivially flat over \(T\),
   - An \(\mathcal{O}^A_{X_T}\)-valued, \(\mathcal{O}^C_{X}\)-bilinear (nonsymmetric) inner product from the multiplication in \(\mathcal{O}^A_{X_T}\):
     - an \(\mathcal{O}^C_{X}\)-valued, \(\mathcal{O}^C_{X}\)-bilinear (symmetric) inner product after the post-composition with \(\text{Tr}\).
   - Both inner products are covariantly constant with respect to \(D^T\) and one has the Leiniz rules
     \[ D^T(m^1_T m^2_T) = (D^T m^1_T) m^2_T + m^1_T D^T m^2_T; \]
     \[ d\text{Tr}(m^1_T m^2_T) = \text{Tr}(D^T m^1_T m^2_T) \]
     \[ \text{Tr}((D^T m^1_T) m^2_T) \quad + \quad \text{Tr}(m^1_T D^T m^2_T). \]
(1) \( T^*(X_T/T) \otimes_{O_{X_T}} O_{X_T}^{\mathbb{A}} : O_{X_T}^{\mathbb{A}} \)-valued relative 1-forms on \( X_T/T \)

- The induced connection \( \nabla_{T,(h,D^T)} := \nabla^h \otimes \text{Id} + \text{Id} \otimes D^T \), trivially flat over \( T \).
- An \( O_{X_T}^{\mathbb{A}} \)-valued, \( O_{X_T}^{\mathbb{C}} \)-bilinear (nonsymmetric) inner product \( \langle \cdot, \cdot \rangle_h \); an \( O_{X}^{\mathbb{C}} \)-valued, \( O_{X}^{\mathbb{C}} \)-bilinear (symmetric) inner product \( \text{Tr} \langle \cdot, \cdot \rangle_h \).
- Both inner products are covariantly constant with respect to \( \nabla_{T,(h,D^T)} \) and one has the Leibniz rules
  \[
  D^T \langle \cdot, \cdot' \rangle_h = \langle \nabla_{T,(h,D^T)} \cdot, \cdot' \rangle_h + \langle \cdot, \nabla_{T,(h,D^T)} \cdot' \rangle_h,
  \]
  \[
  d\text{Tr} \langle \cdot, \cdot' \rangle_h = \text{Tr} \langle D^T \cdot, \cdot' \rangle_h = \text{Tr} \langle \nabla_{T,(h,D^T)} \cdot, \cdot' \rangle_h + \text{Tr} \langle \cdot, \nabla_{T,(h,D^T)} \cdot' \rangle_h
  \]
  for \( \cdot, \cdot' \in T^*(X_T/T) \otimes_{O_{X_T}} O_{X_T}^{\mathbb{A}} \).

(2) \( \varphi^*_T \mathcal{T}_Y := O_{X_T}^{\mathbb{A}} \otimes_{O_{Y}} \varphi^*_T \mathcal{T}_Y : O_{X_T}^{\mathbb{A}} \)-valued derivations on \( O_Y \)

- The induced connection \( \nabla_{T,(\varphi^*_{T,Y})} := D^T \otimes \text{Id} + \text{Id} \cdot \sum_{i=1}^n D^T \varphi^*_{T,Y}(y^i) \otimes \nabla^g_{\partial y^i} \) (in local expression), trivially flat over \( T \).
- A partially defined \( O_{X_T}^{\mathbb{A}} \)-valued, \( O_{X}^{\mathbb{C}} \)-bilinear (nonsymmetric) inner product \( \langle \cdot, \cdot \rangle_g \); a partially defined \( O_{X}^{\mathbb{C}} \)-valued, \( O_{X}^{\mathbb{C}} \)-bilinear (symmetric) inner product \( \text{Tr} \langle \cdot, \cdot \rangle_g \).
- Both inner products, when defined, are covariantly constant with respect to \( \nabla_{T,(\varphi^*_{T,Y})} \) and one has the Leibniz rules
  \[
  D^T \langle -, -' \rangle_g = \langle \nabla_{T,(\varphi^*_{T,Y})} -, -' \rangle_g + \langle -, \nabla_{T,(\varphi^*_{T,Y})} -' \rangle_g,
  \]
  \[
  d\text{Tr} \langle -, -' \rangle_g = \text{Tr} \langle D^T (-, -') \rangle_g = \text{Tr} \langle \nabla_{T,(\varphi^*_{T,Y})} (-, -') \rangle_g + \text{Tr} \langle -, \nabla_{T,(\varphi^*_{T,Y})} (-') \rangle_g
  \]
  whenever all \( \langle -'', -''' \rangle_g \) and \( \text{Tr} \langle -'', -''' \rangle_g \) involved are defined.

(3) \( T^*(X_T/T) \otimes_{O_{X_T}} \varphi^*_T \mathcal{T}_Y : (O_{X_T}^{\mathbb{A}} \text{-valued relative 1-form}) \)-valued derivations on \( O_Y \)

This is a combination of the construction in Item (1) and in Item (2).

- The induced connection
  \[
  \nabla_{T,(h,\varphi^*_{T,Y})} = \nabla^h \otimes \text{Id} + \text{Id} \otimes D^T \otimes \text{Id} + \text{Id} \cdot \sum_{i=1}^n D^T \varphi^*_{T,Y}(y^i) \otimes \nabla^g_{\partial y^i}
  \]
  (in local expression), trivially flat over \( T \).
- A partially defined \( O_{X_T}^{\mathbb{A}} \)-valued, \( O_{X}^{\mathbb{C}} \)-bilinear (nonsymmetric) inner product \( \langle \cdot, \cdot \rangle_{(h,g)} \); a partially defined \( O_{X}^{\mathbb{C}} \)-valued, \( O_{X}^{\mathbb{C}} \)-bilinear (symmetric) inner product \( \text{Tr} \langle \cdot, \cdot \rangle_{(h,g)} \).
- Both inner products, when defined, are covariantly constant with respect to \( \nabla_{T,(h,\varphi^*_{T,Y})} \) and one has the Leibniz rules
  \[
  D^T \langle \sim, \sim' \rangle_{(h,g)} = \langle \nabla_{T,(h,\varphi^*_{T,Y})} \sim, \sim' \rangle_{(h,g)} + \langle \sim, \nabla_{T,(h,\varphi^*_{T,Y})} \sim' \rangle_{(h,g)},
  \]
  \[
  d\text{Tr} \langle \sim, \sim' \rangle_{(h,g)} = \text{Tr} \langle D^T \langle \sim, \sim' \rangle_{(h,g)} \rangle
  = \text{Tr} \langle \nabla_{T,(h,\varphi^*_{T,Y})} \sim, \sim' \rangle_{(h,g)} + \text{Tr} \langle \sim, \nabla_{T,(h,\varphi^*_{T,Y})} \sim' \rangle_{(h,g)}
  \]
  whenever the \( \langle \sim'', \sim''' \rangle_{(h,g)} \) and \( \text{Tr} \langle \sim'', \sim''' \rangle_{(h,g)} \) involved are defined.
Curvature tensors with $\partial_t$ and other order-switching formulae

Let $(\varphi_T, \nabla^T)$ be a $(\ast_1)$-admissible $T$-family of $(\ast_1)$-admissible pairs. A very basic step in (particularly the second) variational problem involves passing $\partial_t$ over a differential operator on $X_t$'s. In general, a curvature term appears whenever such passing occurs. In this theme, we collect and prove such formulae we need.

First, passing $\partial_t$ over a differential operator usually means the appearance of a curvature term by the very definition of a curvature tensor:

Lemma 5.4. [curvature tensor with $\partial_t$] Let $(\varphi_T, \nabla^T)$ be a $(\ast_1)$-admissible $T$-family of $(\ast_1)$-admissible pairs. Let $\xi$ be a vector field on an open set $U \subset X$ small enough so that $\varphi_T(U_{\varphi_T})$ is contained in a coordinate chart on $Y$, with coordinates $(y^1, \cdots, y^n)$. The standard lifting of $\xi$ to $U_T$ is denoted also by $\xi$. Note that, by construction, $[\partial_t, \xi] = 0$ and all our connection $\nabla'$ in Theme 3.

The three basic $\mathcal{O}^\ast_X$-modules with induced structures are trivially flat; hence, $F_{\nabla'}(\partial_t, \xi) = \partial_t \nabla^T_{\xi} - \nabla^T_{\partial_t} \partial_t$. One has the following curvature expressions with $\partial_t$ on the basic $\mathcal{O}_{X_T}$-modules: (Below we adopt the convention that the Riemann curvature tensor from a metric is denoted by $R$ while the curvature tensor of a connection in all other bundle situations is denoted by $F$.)

(01) For sections $\omega_T$ of $T^*(X_T/T)$:
$$R_{\nabla'}(\partial_t, \xi) \omega_T = \partial_t \nabla^T_{\xi} \omega_T - \nabla^T_{\partial_t} \partial_t \omega_T = 0.$$

(02) For sections $m_T$ of $\mathcal{O}^\ast_{X_T}$:
$$F_{\nabla'}(\partial_t, \xi) m_T = \partial_t D^T \xi m_T - D^T \xi \partial_t m_T = [(\partial_t, \nabla^T) (\xi), m_T].$$

As a consequence of this, if $(\varphi_T, \nabla^T)$ is furthermore a $(\ast_2)$-admissible $T$-family of $(\ast_2)$-admissible pairs, then
$$(\partial_t \nabla^T) (\xi) \in \text{Inn}^\varphi_{(\ast_1)}(\mathcal{O}^\ast_X) \quad \text{i.e.} \quad [(\partial_t, \nabla^T) (\xi), A_{\varphi_T}] \subset \text{Comm}(A_{\varphi_T}).$$

(1) For sections $\omega_T \otimes m_T$ of $T^*(X_T/T) \otimes_{\mathcal{O}_X} \mathcal{O}^\ast_{X_T}$:
$$F_{\nabla',(h, D^T)}(\partial_t, \xi) (\omega_T \otimes m_T) = \partial_t \nabla^T_{\xi} (h, D^T) (\omega_T \otimes m_T) - \nabla^T_{\partial_t} (h, D^T) \partial_t (\omega_T \otimes m_T) = \omega_T \otimes [(\partial_t, \nabla^T) (\xi), m_T].$$

(2) For sections $m_T \otimes v$ of $\varphi_T^* T_Y := \mathcal{O}^\ast_{X_T} \otimes_{\mathcal{O}_Y} T_Y$:
$$v \text{ on the coordinate chart of } Y \text{ above, with coordinates } (y^1, \cdots, y^n)$$
$$F_{\nabla',(\varphi_T, g)}(\partial_t, \xi) (m_T \otimes v) = \partial_t \nabla^T_{\xi} (\varphi_T, g) (m_T \otimes v) - \nabla^T_{\partial_t} (\varphi_T, g) \partial_t (m_T \otimes v)
\quad = [(\partial_t, \nabla^T) (\xi), m_T] \otimes v + m_T \sum_{i=1}^n [(\partial_t, \nabla^T) (\xi, \varphi_T^*(y^i)), \nabla^g_{\partial y_i} v]
\quad + m_T \sum_{i,j=1}^n \left(D^T_{\xi} \varphi_T^*(y^j) \partial_t \varphi_T^*(y^j) \otimes \nabla^g_{\partial y_i} \nabla^g_{\partial y_i} v - \partial_t \varphi_T^*(y^j) D^T_{\xi} \varphi_T^*(y^j) \otimes \nabla^g_{\partial y_i} \nabla^g_{\partial y_i} v \right).$$

If $(\varphi_T, \nabla^T)$ is furthermore a $(\ast_2)$-admissible $T$-family of $(\ast_1)$-admissible pairs, then the last term has a $Y$-coordinate-free form:
$$The \text{ last term } = m_T \sum_{i,j} \partial_t \varphi_T^*(y^j) D^T_{\xi} \varphi_T^*(y^j) \otimes R^g \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) v = m_T (\varphi_T R^g (\partial_t, \xi)) v.$$
(3) For sections $\omega_T \otimes m_T \otimes v$ of $T^* (X_T/T) \otimes \varphi_T^* \mathcal{T}_Y := T^* (X_T/T) \otimes \mathcal{O}_{X_T}^\mathcal{A}_T \otimes \varphi_T^* \mathcal{O}_Y \mathcal{T}_Y$:

$v$ on the coordinate chart of $Y$ above, with coordinates $(y^1, \cdots, y^n)$

{\begin{align*}
F_{T^*(\varphi_T, g)} (\partial_t, \xi)(\omega_T \otimes m_T \otimes v) \\
= \partial_t \nabla_{\xi}^{T, (\varphi_T, g)} (\omega_T \otimes m_T \otimes v) - \nabla^{T, (\varphi_T, g)} \partial_t (\omega_T \otimes m_T \otimes v) \\
= \omega_T \otimes \left( F_{T^*(\varphi_T, g)} (\partial_t, \xi)(m_T \otimes v) \right).
\end{align*}}

Proof. Statement (01) follows from the fact that $X_T$ is a constant family over $T$. Statement (02), First Part, follows from a computation with respect to an induced local trivialization of $\mathcal{E}_T$ from a local trivialization of $\mathcal{E}$

{\begin{align*}
\partial_t D^T_\xi m_T &= \partial_t (\xi m_T + [A_{\mathcal{E}_T} (\xi), m_T]) \\
&= \xi \partial_t m_T + [\partial_t A_{\mathcal{E}_T} (\xi), m_T] + [A_{\mathcal{E}_T} (\xi), \partial_t m_T] = D^T_\xi \partial_t m_T + [(\partial_t \nabla^T) (\xi), m_T].
\end{align*}}

For Second Part, if $(\varphi_T, \nabla^T)$ is furthermore a ($*2$)-admissible T-family of ($*2$)-admissible pairs, then for $f_1, f_2 \in \mathcal{O}_Y$, by First Part and the ($*2$)-Admissible Condition,

{\begin{align*}
[(\partial_t \nabla^T) (\xi), \varphi_T (f_1), \varphi_T (f_2)] &= [\partial_t D^T_\xi \varphi_T (f_1), \varphi_T (f_2)] - [D^T_\xi \partial_t \varphi_T (f_1), \varphi_T (f_2)] = 0.
\end{align*}}

Which says that $(\partial_t \nabla^T) (\xi) \in \text{Inn}^{\varphi_T} (\mathcal{O}^\mathcal{A}_T)_{X_T}$.

Statement (1) is a consequence of Statement (01) and Statement (02). Statement (3) is a consequence of Statement (01) and a property of the induced connection on a tensor product of $\mathcal{O}_{X_T}^\mathcal{A}_T$-modules with a connection. Let us carry out Statement (2) as a demonstration of the covariant differential calculus involved.

Let $m_T \otimes v \in \varphi_T^* \mathcal{T}_Y$. Then, by Statement (02),

{\begin{align*}
\partial_t \nabla^T_{\xi, (\varphi_T, g)} (m_T \otimes v) &= \partial_t \left( D^T_\xi m_T \otimes v + m_T \sum_i D^T_\xi \varphi_T (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v \right) \\
&= (D^T_\xi \partial_t m_T + [(\partial_t \nabla^T) (\xi), m_T]) \otimes v + (D^T_\xi m_T) \sum_i \partial_t \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v \\
&+ (\partial_t m_T) \sum_i D^T_\xi \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v + m_T \sum_i (D^T_\xi \partial_t \varphi_T^g (y^i) + [(\partial_t \nabla^T) (\xi), \varphi_T^g (y^i)]) \otimes \nabla^g \frac{\partial}{\partial y^i} v \\
&+ m_T \sum_{i,j} D^T_\xi \partial_t \varphi_T^g (y^i) \partial_t \varphi_T^g (y^j) \otimes \nabla^g \frac{\partial}{\partial y^i} \nabla^g \frac{\partial}{\partial y^j} v
\end{align*}}

while

{\begin{align*}
\nabla^T_{\xi, (\varphi_T, g)} \partial_t (m_T \otimes v) &= \nabla^T_{\xi, (\varphi_T, g)} \left( \partial_t m_T \otimes v + m_T \sum_i \partial_t \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v \right) \\
&= D^T_\xi \partial_t m_T \otimes v + (\partial_t m_T) \sum_i D^T_\xi \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v + (D^T_\xi m_T) \sum_i \partial_t \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v \\
&+ m_T \sum_i D^T_\xi \partial_t \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} v + m_T \sum_i \partial_t \varphi_T^g (y^i) D^T_\xi \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} \nabla^g \frac{\partial}{\partial y^i} v
\end{align*}}

Thus,

{\begin{align*}
F_{T^*(\varphi_T, g)} (\partial_t, \xi)(m_T \otimes v) &= (\partial_t \nabla^T_{\xi, (\varphi_T, g)} - \nabla^T_{\xi, (\varphi_T, g)} \partial_t)(m_T \otimes v) \\
&= [(\partial_t \nabla^T) (\xi), m_T] \otimes v + m_T \sum_i [(\partial_t \nabla^T) (\xi), \varphi_T^g (y^i)] \otimes \nabla^g \frac{\partial}{\partial y^i} v \\
&+ m_T \sum_{i,j} \left( D^T_\xi \varphi_T^g (y^i) \partial_t \varphi_T^g (y^j) \otimes \nabla^g \frac{\partial}{\partial y^i} \nabla^g \frac{\partial}{\partial y^j} v - \partial_t \varphi_T^g (y^j) D^T_\xi \varphi_T^g (y^i) \otimes \nabla^g \frac{\partial}{\partial y^i} \nabla^g \frac{\partial}{\partial y^j} v \right).
\end{align*}}
as claimed, after a relabeling of \( i, j \).

If \( (\varphi_T, \nabla^T) \) is furthermore a \((*2)\)-admissible \( T \)-family of \((*1)\)-admissible pairs, then \( D^T_T \varphi_T(y^i) \) and \( \partial_t \varphi_T(y^i) \) commute since \( [D^T_T \varphi_T(y^i), \varphi_T(y^i)] = 0 \) by the \((*1)\)-Admissible Condition along \( X \) and, hence,

\[
0 = \partial_i [D^T_T \varphi_T(y^i), \varphi_T(y^i)] = \left[ \partial_i D^T_T \varphi_T(y^i), \varphi_T(y^i) \right] + \left[ D^T_T \varphi_T(y^i), \partial_i \varphi_T(y^i) \right] = \left[ D^T_T \varphi_T(y^i), \partial_i \varphi_T(y^i) \right]
\]

by the \((*1)\)-Admissible Condition along \( X \) and the \((*2)\)-Admissible Condition along \( T \). The last summand of \( F_{\nabla^T, (\varphi_T, \xi)}(\partial_t, \xi) (m_T \otimes v) \) is then equal to

\[
m_T \sum_{i,j} \partial_i \varphi_T^i \varphi_T^j (y^i) D^T_T \varphi_T (y^i) \odot \left( \nabla^g_{\frac{\partial}{\partial y^i}} \nabla^g_{\frac{\partial}{\partial y^j}} - \nabla^g_{\frac{\partial}{\partial y^j}} \nabla^g_{\frac{\partial}{\partial y^i}} \right) \times v = m_T \left( (\varphi^R_T)(\partial_t, \xi) \right) v.
\]

This proves the lemma.

The following lemma addresses the issue of passing \( \partial_t \) over the covariant differential \( D \varphi_T \) of \( \varphi_T \). Though such passing is not a curvature issue in the conventional sense, it does carry a taste of curvature calculations.

**Lemma 5.5.** \( [\partial_t D^T \varphi_T \text{ versus } \nabla^T, (\varphi_T, \xi)] \partial_t \varphi_T \) \ Let \( (\varphi_T, \nabla^T) \) be a \((*1)\)-admissible \( T \)-family of \((*1)\)-admissible pairs. With the above notation and convention, let \( \xi \) be a vector field on \( X \). Then, for a chart of \( Y \) with coordinates \((y^1, \cdots, y^n)\), one has

\[
\partial_t D^T_T \varphi_T = \nabla^T_{\xi}(\varphi_T, \xi) \partial_t \varphi_T - \left( \text{ad} \otimes \nabla^g \right) \partial_t \varphi_T D^T_T \varphi_T + \sum_{i=1}^n \left[ (\partial_t \nabla^T)(\xi), \varphi_T(y^i) \right] \odot \frac{\partial}{\partial y^i}.
\]

Here, only as a compact notation,

\[
\left( \text{ad} \otimes \nabla^g \right) \partial_t \varphi_T D^T_T \varphi_T := \sum_{i,j=1}^n \left[ \partial_t \varphi_T(y^i), D^T_T \varphi_T(y^j) \right] \odot \nabla^g_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = - \sum_{i,j=1}^n \left[ D^T_T \varphi_T(y^i), \partial_t \varphi_T(y^j) \right] \odot \nabla^g_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i}.
\]

If \( (\varphi_T, \nabla^T) \) is furthermore a \((*2)\)-admissible \( T \)-family of \((*2)\)-admissible pairs, then the last term has a \( Y \)-coordinate-free expression

\[
\text{ad} \left( \partial_t \nabla^T(\xi) \right) \varphi_T.
\]

**Proof.** Under the given setting and by Lemma 5.4 \((0_2)\),

\[
\partial_t D^T_T \varphi_T = \partial_t \left( \sum_i D^T_T \varphi_T(y^i) \odot \frac{\partial}{\partial y^i} \right) = \sum_i \left( D^T_T \partial_t \varphi_T(y^i) + \left[ \partial_t \nabla^T(\xi), \varphi_T(y^i) \right] \odot \frac{\partial}{\partial y^i} \right) + \sum_{i,j} D^T_T \varphi_T(y^i) \partial_t \varphi_T(y^j) \odot \nabla^g_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i}
\]

while

\[
\nabla^T_{\varphi_T, (\varphi_T, \xi)} \partial_t \varphi_T = \nabla^T_{\varphi_T, (\varphi_T, \xi)} \left( \sum_i \partial_t \varphi_T(y^i) \odot \frac{\partial}{\partial y^i} \right) = \sum_i D^T_T \partial_t \varphi_T(y^i) \odot \frac{\partial}{\partial y^i} + \sum_{i,j} \partial_t \varphi_T(y^i) D^T_T \varphi_T(y^j) \odot \nabla^g_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i}.
\]
Thus,
\[ \partial_t D^T_T \varphi - \nabla^T_{\xi}(\varphi_T, g) \partial_t \varphi_T = \sum_i (\partial_i \nabla^T(y)) \varphi_i^T(y') \otimes \frac{\partial}{\partial y'} - \sum_i \partial_i \varphi^T_i(y') \varphi^T_i(y') \otimes \nabla^g \frac{\partial}{\partial y'} \]
Either apply the identity \( \nabla^g \frac{\partial}{\partial y'} = \nabla^g \frac{\partial}{\partial y'} \) to the second term and relabeling \( i, j \) of the third, or apply the identity \( \nabla^g \frac{\partial}{\partial y'} = \nabla^g \frac{\partial}{\partial y'} \) to the third term and relabeling \( i, j \) of the second,
\[ = \sum_i \partial_i \varphi^T_i(y') \varphi^T_i(y') \otimes \nabla^g \frac{\partial}{\partial y'} - \sum_i \partial_i \varphi^T_i(y') \varphi^T_i(y') \otimes \nabla^g \frac{\partial}{\partial y'} \]
This proves the First Statement in Lemma.

The Second Statement in Lemma is a consequence of Corollary 3.1.10 and Lemma 5.4 (0). This proves the lemma.

Before continuing the discussion, we introduce a notion that is needed in the next lemma.

**Definition 5.6. [half-torsion tensor Tor \( \frac{1}{4} \phi \).]** Recall the torsion tensor \( Tor \nabla \) of a connection \( \nabla \) on \( Y \)
\[ Tor \nabla(v_1, v_2) := \nabla_{v_1} v_2 - \nabla_{v_2} v_1 - [v_1, v_2] \]
for \( v_1, v_2 \in \mathfrak{T}_aY \). For the Levi-Civita connection \( \nabla^g \) associated to a metric \( g \) on \( Y \), \( Tor \nabla g = 0 \) by construction. Thus, in this case, for a \( \Phi \in C^\infty(Y) \),
\[ (\nabla^g_{v_1} v_2 - v_1 v_2) \Phi = (\nabla^g_{v_2} v_1 - v_2 v_1) \Phi \]
for \( v_1, v_2 \in \mathfrak{T}_aY \). This defines a symmetric 2-tensor on \( Y \)
\[ Tor \frac{1}{4} \Phi : \mathfrak{T}_aY \times \mathfrak{T}_aY \rightarrow \mathcal{O}_Y \]
\[ (v_1, v_2) \rightarrow (\nabla^g_{v_1} v_2 - v_1 v_2) \Phi \]
called the half-torsion tensor of (the torsion-free connection) \( \nabla^g \) associated to \( \Phi \in C^\infty(Y) \).

The following lemma addresses the issue of passing \( \partial_t \) over ‘evaluation of an \( \mathcal{O}^{\frac{1}{4} \phi}_X \)-valued derivation on \( C^\infty(Y) \)’, and another similar situation:

**Lemma 5.7. [\partial_t((D^T_T \varphi_T) \Phi) versus (\partial_t D^T_T \varphi_T) \Phi; D^T_T((\partial_t \varphi_T) \Phi) versus (\nabla^T_{\xi}(\varphi_T, g) \partial_t \varphi_T) \Phi] \**
Let \( (\varphi_T, \nabla^T) \) be a \( (\ast_1) \)-admissible \( T \)-family of \( (\ast_1) \)-admissible pairs. Continue the notation and
convention in Lemma 5.4. Under the canonical isomorphism \(O_{X_T}^A \otimes \varphi_T^*, O_Y \simeq O_{X_T}^A\),

\[
\partial_t((D_T^T \varphi_T)\Phi) = (\partial_t D_T^T \varphi_T)\Phi + \sum_{i,j} D_T^T \varphi_T(y') \partial_t \varphi_T^e(y') \otimes \left(\frac{\partial}{\partial y} \frac{\partial}{\partial y'} \Phi - \left(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \right) \Phi \right)
\]

\[
= (\partial_t D_T^T \varphi_T)\Phi - (\varphi_T^e \text{Tor} \frac{1}{\varphi_T^e})(\xi, \partial_t); \\
\]

and

\[
D_T^T((\partial_t \varphi_T)\Phi) = (\nabla_T(\varphi_T, T) \partial_t \varphi_T)\Phi + \sum_{i,j} \partial_t \varphi_T^e(y') D_T^T \varphi_T(y') \otimes \frac{\partial}{\partial y} \Phi - \left(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \right) \Phi
\]

\[
= (\nabla_T(\varphi_T, T) \partial_t \varphi_T)\Phi - (\varphi_T^e \text{Tor} \frac{1}{\varphi_T^e})(\partial_t, \xi).
\]

**Proof.** For the first identity,

\[
\partial_t((D_T^T \varphi_T)\Phi) = \partial_t \left(\sum_i D_T^T \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi \right)
\]

\[
= \sum_i \partial_t D_T^T \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi + \sum_{i,j} D_T^T \varphi_T(y') \partial_t \varphi_T^e(y') \otimes \frac{\partial}{\partial y} \Phi.
\]

while

\[
(\partial_t D_T^T \varphi_T)\Phi = \left(\partial_t \sum_i D_T^T \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi \right)
\]

\[
= \left(\sum_i \partial_t D_T^T \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi + \sum_{i,j} D_T^T \varphi_T(y') \partial_t \varphi_T^e(y') \otimes \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \Phi\right) = - (\varphi_T^e \text{Tor} \frac{1}{\varphi_T^e})(\partial_t, \xi)
\]

Thus,

\[
\partial_t((D_T^T \varphi_T)\Phi) - (\partial_t D_T^T \varphi_T)\Phi
\]

\[
= \sum_{i,j} D_T^T \varphi_T^e(y') \partial_t \varphi_T^e(y') \otimes \left(\frac{\partial}{\partial y'} \frac{\partial}{\partial y'} \Phi - \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \Phi\right)
\]

\[
= \sum_{i,j} D_T^T \varphi_T(y') \partial_t \varphi_T^e(y') \otimes \left(\frac{\partial}{\partial y'} \frac{\partial}{\partial y'} \Phi - \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \Phi\right) = - (\varphi_T^e \text{Tor} \frac{1}{\varphi_T^e})(\xi, \partial_t)
\]

and the first identity follows.

For the second identity,

\[
D_T^T((\partial_t \varphi_T)\Phi) = D_T^T \left(\sum_i \partial_t \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi \right)
\]

\[
= \sum_i D_T^T \partial_t \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi + \sum_{i,j} \partial_t \varphi_T^e(y') D_T^T \varphi_T(y') \otimes \frac{\partial}{\partial y} \Phi
\]

while

\[
(\nabla_T(\varphi_T, g) \partial_t \varphi_T)\Phi = \left(\nabla_T(\varphi_T, g) \sum_i \partial_t \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} \Phi\right)
\]

\[
= \left(\sum_i D_T^T \partial_t \varphi_T^e(y') \otimes \frac{\partial}{\partial y'} + \sum_{i,j} \partial_t \varphi_T^e(y') D_T^T \varphi_T^e(y') \otimes \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y'} \Phi\right).
\]
Thus,
\[
D^T_\xi ((\partial_t \varphi_T) \Phi) - (\nabla^T_\xi (\varphi_T, g) \partial_t \varphi_T) \Phi \\
= \sum_{i,j} \partial_i \varphi_T^i (y') D^T_\xi \varphi_T^j (y') \otimes \left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla^g_{\partial_y^i} \frac{\partial}{\partial y^j} \right) \Phi \\
= \sum_{i,j} \partial_i \varphi_T^i (y') D^T_\xi \varphi_T^j (y') \otimes \left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla^g_{\partial_y^i} \frac{\partial}{\partial y^j} \right) \Phi = - (\varphi_T \text{Tor}_{\xi_y}^1 \Phi) (\partial_t, \xi)
\]
and the second identity follows.

This proves the lemma.

\[\square\]

**Remark 5.8. [for \((\ast_2)\)-admissible family of \((\ast_1)\)-admissible pairs]** If \((\varphi_T, \nabla^T)\) is furthermore a \((\ast_2)\)-admissible \(T\)-family of \((\ast_1)\)-admissible pairs, then, as in the proof of Lemma 5.4 (2), \(D^T_\xi \varphi_T^j (y')\) and \(\partial_i \varphi_T^j (y')\) commute for all \(i, j\). In this case,
\[
(\varphi^\circ \text{Tor}_{\xi_y}^1 \Phi) (\xi, \partial_t) = (\varphi^\circ \text{Tor}_{\xi_y}^1 \Phi) (\partial_t, \xi).
\]

**Two-parameter admissible families of admissible pairs**

Let \(T = (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2\), \(\varepsilon > 0\) small, be a two-parameter space with coordinates \((s, t)\). The setting and results above for one-parameter admissible families of admissible pairs generalizes without work to two-parameter admissible of admissible pairs. In particular,

**Definition 5.9. [two-parameter admissible family of admissible pairs]** A \((\ast_2)\)-admissible \(T\)-family of \((\ast_1)\)-admissible maps is a \((\ast_1)\)-admissible map \(\varphi_T : (X^T_{\mathcal{E}}, \mathcal{E}_T; \nabla^T) \rightarrow Y\), where \(\mathcal{E}_T\) is trivially flat over \(T\), such that \(\partial_s \text{Comm} A_{\varphi_T} \subset \text{Comm} (A_{\varphi_T})\) and \(\partial_t \text{Comm} A_{\varphi_T} \subset \text{Comm} (A_{\varphi_T})\).

The following is a consequence of the proof of Lemma 3.2.2.5:

**Lemma 5.10. [symmetry property of \(\text{Tr} \langle F_{\nabla^T_{T, (\varphi_T, g)}} (\partial_s, \xi_2) \partial_t \varphi_T, D^T_\xi \varphi_T \rangle\)** Let \(\varphi_T : (X^T_{\mathcal{E}}, \mathcal{E}_T; \nabla^T) \rightarrow Y\) be a \((\ast_2)\)-admissible \(T\)-family of \((\ast_1)\)-admissible maps. Let \(\xi_2, \xi_4 \in T_s X\) and denote the same for their respective lifting to \(T_s(X_T/T)\). Then,
\[
\text{Tr} \langle F_{\nabla^T_{T, (\varphi_T, g)}} (\partial_s, \xi_2) \partial_t \varphi_T, D^T_\xi \varphi_T \rangle_g = - \text{Tr} \langle \partial_t \varphi_T, F_{\nabla^T_{T, (\varphi_T, g)}} (\partial_s, \xi_2) D^T_\xi \varphi_T \rangle_g \\
= - \text{Tr} \langle F_{\nabla^T_{T, (\varphi_T, g)}} (\partial_s, \xi_2) D^T_\xi \varphi_T, \partial_t \varphi_T \rangle_g = \text{Tr} \langle F_{\nabla^T_{T, (\varphi_T, g)}} (\xi_2, \partial_s) D^T_\xi \varphi_T, \partial_t \varphi_T \rangle_g.
\]

**Proof.** Let \(\xi\) be \(\xi_2\) or \(\xi_4\). Since \(\partial_s \text{Comm} (A_{\varphi_T}) \subset \text{Comm} (A_{\varphi_T})\) and \(\partial_t \text{Comm} (A_{\varphi_T}) \subset \text{Comm} (A_{\varphi_T})\), both \(\partial_s D^T_\xi \varphi_T\) and \(\partial_t \partial_t \varphi_T\) lie in \(\text{Comm} (A_{\varphi_T}) \otimes \varphi_t, \partial_s, T_s Y\). Locally explicitly,
\[
\partial_s D^T_\xi \varphi_T = \sum_i \partial_s D^T_\xi \varphi_T^i (y') \otimes \frac{\partial}{\partial y^i} + \sum_{i,j} D^T_\xi \varphi_T^j (y') \partial_s \varphi_T^j (y') \otimes \nabla^g_{\partial_y^i} \frac{\partial}{\partial y^j} ;
\]
\[
\partial_s \partial_t \varphi_T = \sum_i \partial_s \partial_t \varphi_T^i (y') \otimes \frac{\partial}{\partial y^i} + \sum_{i,j} \partial_t \varphi_T^j (y') \partial_s \varphi_T^j (y') \otimes \nabla^g_{\partial_y^i} \frac{\partial}{\partial y^j} .
\]
Now follow the proof of Lemma 3.2.2.5, but under only the \((*)_1\)-Admissible Condition on \((\varphi_T, \nabla^T)\), to convert \(\text{Tr} (F_{\nabla^T,\varphi_T}) (\partial_s, \xi_2) \partial_t \varphi_T, D^T_{\xi_2} \varphi_T)\) to \(\text{Tr} (\partial_t \varphi_T, F_{\nabla^T,\varphi_T}) (\partial_s, \xi_2) D^T_{\xi_2} \varphi_T)\). Since \(\text{Tr} (-, -')\) is defined as long as one of \(-, -'\) is in \(\text{Comm} (A_T) \otimes \varphi, \nabla, T_* Y\), one realizes that all the terms that appear in the process via the Leibniz rule are defined except

\[- \text{Tr} (\nabla^T_{\xi_2} (\varphi_T), \partial_s D^T_{\xi_2} \varphi_T) + \text{Tr} (\partial_s \partial_t \varphi_T, \nabla^T_{\xi_2} (\varphi_T)) D^T_{\xi_2} \varphi_T).\]

Under the additional \((*)_2\)-Admissible Condition on \((\varphi_T, \nabla^T)\) along \(T\), both \(\partial_s D^T_{\xi_2} \varphi_T\) and \(\partial_s \partial_t \varphi_T\) now lie in \(\text{Comm} (A_T) \otimes \varphi, \nabla, T_* Y\); and the above two exceptional terms become defined.

The lemma follows.

\[\square\]

6 The first variation of the enhanced kinetic term for maps and

\[\cdots\]

Let \((\varphi, \nabla)\) be a \((*)_1\)-admissible pair. Recall the setup in Sec. 5. Let \(T = (-\varepsilon, \varepsilon) \subset \mathbb{R}^1\), for some \(\varepsilon > 0\) small, and \((\varphi_T, \nabla^T)\) be a \((*)_1\)-admissible \(T\)-family of \((*)_1\)-admissible pairs that deforms \((\varphi, \nabla) = (\varphi_T, \nabla^T)|_{t=0}\). We derive in Sec. 6.1 and Sec. 6.2 the first variation formula of the newly introduced enhanced kinetic term for maps

\[S_{\text{map:kinetic}}^{(\rho,h)} (\varphi, \nabla) := \frac{1}{2} \int_X \text{Re} \text{Tr} (D\varphi, D\varphi) \text{vol}_h + \int_X \text{Re} \text{Tr} (d\rho, \varphi^o \Phi) \text{vol}_h\]

in the standard action for D-branes. As the ‘taking the real part’ operation \(\text{Re}(\cdots)\) is a \(\mathcal{O}_X\)-linear operation and can always be added back in the end, we will consider

\[S_{\text{map:kinetic}}^{(\rho,h)} (\varphi, \nabla) \text{C} := \frac{1}{2} \int_X \text{Tr} (D\varphi, D\varphi) \text{vol}_h + \int_X \text{Tr} (d\rho, \varphi^o \Phi) \text{vol}_h\]

so that we don’t have to carry \(\text{Re}\) around.

The first variation of the gauge/Yang-Mills term is analogous to that in the ordinary Yang-Mills theory and the first variation of the Chern-Simons/Wess-Zumino term is an update from [L-Y8: Sec. 6] (D(13.1)). Both are given in Sec. 6.3 under the stronger \((*)_2\)-Admissible Condition.

6.1 The first variation of the kinetic term for maps

Recall the (complexified) kinetic energy \(E^{V^t} (\varphi_t)\) of \(\varphi_t\) for a given \(V^t, t \in T := (-\varepsilon, \varepsilon)\),

\[E^{V^t} (\varphi_t) := S_{\text{map:kinetic}}^{(h,g)} (\varphi_t, V^t) := \frac{1}{2} \int_X \text{Tr} (D^t \varphi_t, D^t \varphi_t) \text{vol}_h.\]

As \(t\) varies, with a slight abuse of notation, denote the resulting function of \(t\) by

\[E^{V^T} (\varphi_T) := S_{\text{map:kinetic}}^{(h,g)} (\varphi_T, V^T) := \frac{1}{2} \int_X \text{Tr} (D^T \varphi_T, D^T \varphi_T) \text{vol}_h,\]

with the understanding that all expressions are taken on \(X_t\) with \(t\) varying in \(T\).
Let \( U \subset X \) be an open set with an orthonormal frame \((e_\mu)_{\mu=1,\ldots,m}\). Let \((e^\mu)_{\mu=1,\ldots,m}\) be the dual co-frame. Assume that \( U \) is small enough so that \( \varphi_T(U^A_T) \) is contained in a coordinate chart of \( Y \), with coordinates \((y^1, \ldots, y^n)\). Then, over \( U \),

\[
\frac{d}{dt} E^{\varphi_T}(C) = \frac{1}{2} T_{m-1} \int_U \partial_t \langle D^T \varphi_T, D^T T \varphi_T \rangle \, \text{vol}_h
\]

\[
= \frac{1}{2} T_{m-1} \int_U \partial_t \langle D^T \varphi_T, D^T \varphi_T \rangle \, \text{vol}_h
\]

\[
= \frac{1}{2} T_{m-1} \int_U \partial_t \sum_{\mu=1}^m \langle D^T e_\mu \varphi_T, D^T e_\mu \varphi_T \rangle \, \text{vol}_h
\]

\[
= T_{m-1} \int_U \partial_t \sum_{\mu=1}^m \langle D^T e_\mu \varphi_T, D^T e_\mu \varphi_T \rangle \, \text{vol}_h
\]

\[
= T_{m-1} \int_U \sum_{\mu=1}^m \langle \nabla^T e_\mu, \partial_t \varphi_T \rangle \, \text{vol}_h
\]

\[
+ T_{m-1} \int_U \sum_{\mu=1}^m \left( \langle \nabla^T \varphi_T, \partial_t \varphi_T \rangle \, \text{vol}_h \right)
\]

\[
= (I.1) + (I.2) + (I.3).
\]

\[(I.1) = T_{m-1} \int_U \sum_{\mu=1}^m \langle \partial_t \varphi_T, D^T e_\mu \varphi_T \rangle \, \text{vol}_h
\]

\[
= T_{m-1} \int_U \sum_{\mu=1}^m \langle \partial_t \varphi_T, D^T e_\mu \varphi_T \rangle \, \text{vol}_h + T_{m-1} \int_U \langle \partial_t \varphi_T, -\sum_{\mu=1}^m \nabla^T e_\mu \varphi_T \rangle \, \text{vol}_h
\]

\[
= (I.1.1) + (I.1.2).
\]

Summand (I.1.1) suggests a boundary term. To really extract the boundary term from it, consider the \( T \)-family of \( \mathbb{C} \)-valued 1-forms on \( U \)

\[
\alpha_T^{(1, \partial_t \varphi_T)} := \partial_t \langle \partial_t \varphi_T, D^T \varphi_T \rangle g
\]

which depends \( C^\infty(U)^C \)-linearly on \( \partial_t \varphi_T \). Let

\[
\xi_T^{(1, \partial_t \varphi_T)} := \sum_{\mu=1}^m \langle \partial_t \varphi_T, D^T e_\mu \varphi_T \rangle g e_\mu
\]

be the \( T \)-family of dual \( \mathbb{C} \)-valued vector fields of \( \alpha_T^{(1, \partial_t \varphi_T)} \) on \( U \) with respect to the metric \( h \). Note that \( \xi_T^{(1, \partial_t \varphi_T)} \) depends \( C^\infty(U)^C \)-linearly on \( \partial_t \varphi_T \) as well. Then

\[(I.1.1) = T_{m-1} \int_U \sum_{\mu=1}^m \langle \xi_T^{(1, \partial_t \varphi_T)}, e_\mu \rangle \, \text{vol}_h
\]

\[
= T_{m-1} \int_U \langle \nabla^h e_\mu \xi_T^{(1, \partial_t \varphi_T)}, e_\mu \rangle \, \text{vol}_h + T_{m-1} \int_U \langle \xi_T^{(1, \partial_t \varphi_T)}, \sum_{\mu=1}^m \nabla^h e_\mu e_\nu \rangle \, \text{vol}_h
\]

The first term is equal to

\[
T_{m-1} \int_U \langle -\text{div} \xi_T^{(1, \partial_t \varphi_T)}, \text{vol}_h \rangle = T_{m-1} \int_U d i \xi_T^{(1, \partial_t \varphi_T)} \, \text{vol}_h = T_{m-1} \int_{\partial U} i \xi_T^{(1, \partial_t \varphi_T)} \, \text{vol}_h
\]

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which is the sought-for boundary term, whose integrand satisfies the requirement that it be $C^\infty(U)^C$-linear on $\partial_1\varphi_T$. The second term is equal to

$$T_{m-1} \int_U \text{Tr} \langle \partial_1 \varphi_T , D_{e_\mu}^T \varphi_T \rangle_g \ vol_h$$

by construction, which is $C^\infty(U)^C$-linear in $\partial_1 \varphi_T$ and hence in a final form.

The integrand of Summand (I.1.2) is already $C^\infty(U)^C$-linear in $\partial_1 \varphi_T$ and hence in a final form.

Summand (I.2) can be re-written as

$$\text{(I.2)} = -T_{m-1} \int_U \text{Tr} \sum_\mu \langle (ad \otimes \nabla^g) \partial_1 \varphi_T D_{e_\mu}^T \varphi_T , D_{e_\mu}^T \varphi_T \rangle_g \ vol_h.$$ 

Thus, its integrand is already $C^\infty(U)^C$-linear in $\partial_1 \varphi_T$ and hence in a final form.

Finally, since the built-in inclusion $O_U^C \subset O_U^D$ identifies $O_U^C$ with the center of $O_U^D$, Summand (I.3) is $C^\infty(U)^C$-linear and hence in its final form.

Altogether, we almost complete the calculation except the issue of whether all the inner products $\text{Tr}(\cdot, \cdot)_g$ that appear in the procedure are truly defined. For this, one notices that wherever such an inner product appears above, at least one of its arguments is either $\partial_1 \varphi_T$ or $D_{e_\mu}^T \varphi_T$, for some $\mu$. It follows from Lemma 3.2.2.4 that they are indeed defined.

In summary,

**Proposition 6.1.1. [first variation of kinetic term for maps]** Let $(\varphi_T, \nabla^T)$ be a $(*_1)$-admissible $T$-family of $(*_1)$-admissible pairs. Then,

$$\frac{d}{dt} E^{\nabla^T}\ (\varphi_T)^C = \frac{d}{dt}\left( \frac{1}{2} T_{m-1} \int_U \text{Tr} \langle D^T \varphi_T , D^T \varphi_T \rangle_{(h,g)} \ vol_h \right)$$

$$= T_{m-1} \int_{\partial U} i_{\xi_{(1, \partial_1 \varphi_T)}} \ vol_h$$

$$+ T_{m-1} \int_U \text{Tr} \langle \partial_1 \varphi_T , (D_{e_\mu}^T \varphi_T \nabla^h_{e_\mu} - \sum_{\mu=1}^m \nabla^T_{(\varphi_T \cdot g)} D_{e_\mu}^T \varphi_T \rangle_g \ vol_h$$

$$- T_{m-1} \int_U \text{Tr} \sum_{\mu=1}^m \langle (ad \otimes \nabla^g) \partial_1 \varphi_T D_{e_\mu}^T \varphi_T \rangle_g \ vol_h$$

$$+ T_{m-1} \int_U \sum_{\mu=1}^m \langle \sum_{i=1}^n \left[ (\partial_1 \nabla^T(\xi_\mu), \varphi_T^T(y_i')) \otimes \frac{\partial}{\partial y_i} \right] D_{e_\mu}^T \varphi_T \rangle_g \ vol_h.$$

Here, the first summand is the boundary term with $\xi_{(1, \partial_1 \varphi_T)} := \sum_{\mu=1}^m \langle \text{Tr} \langle \partial_1 \varphi_T , D_{e_\mu}^T \varphi_T \rangle_g \rangle e_\mu$ $C^\infty(U)^C$-linear in $\partial_1 \varphi_T$; the integrand of the second and the third terms are $C^\infty(U)^C$-linear in $\partial_1 \varphi_T$ and their real part contribute first-order and second-order terms to the equations of motion for $(\varphi, \nabla)$; the integrand of the last term is $C^\infty(U)^C$-linear in $\partial_1 \nabla^T$ and its real part contributes terms, first order in $\varphi$ but zeroth order in the connection 1-from of $\nabla$, to the equations of motion for $(\varphi, \nabla)$ in addition to those from the first variation of the rest part of $S^{(\rho,h,\Phi,g,B,C)}_{\text{standard}}(\varphi, \nabla)$. These lower-order terms contribute to the equations of motion for $(\varphi, \nabla)$ but do not change the signature of the system.
Remark 6.1.2. [for \((*_{2})\)-admissible \(T\)-family of \((*_{2})\)-admissible pairs] If furthermore \((\varphi, \nabla)\) is \((*_{2})\)-admissible and \((\varphi_{T}, \nabla^{T})\) is a \((*_{2})\)-admissible \(T\)-family of \((*_{2})\)-admissible pairs that deforms \((\varphi, \nabla)\), then the third summand of the first variation formula in Proposition 6.1.1 vanishes and the fourth/last summand has a \(Y\)-coordinate-free form

\[
T_{m-1} \int_{U} \sum_{\mu=1}^{m} \langle \text{ad}_{(\partial_{t} \nabla^{T})(e_{\mu})} \varphi_{T}, D_{e_{\mu}}^{T} \varphi_{T} \rangle_{g} \text{vol}_{h}.
\]

In this case, the first variation with respect to \(\varphi\) alone (i.e. setting \(\partial_{t} \nabla^{T} = 0\)), cf. the first two summands, takes the form of a direct formal generalization of the first variation formula in the study of harmonic maps; e.g. [E-L], [E-S], [Ma], [Sm].

### 6.2 The first variation of the dilaton term

We now turn to the (complexified) dilaton term in \(S_{\text{standard}}^{(\rho,h; \Psi, B, C)}(\varphi, \nabla)\).

Let \(\varphi_{T} : (X^{k}, \mathcal{E}_{T}; \nabla^{T}) \rightarrow Y\) be a \((*_{1})\)-admissible \(T\)-family of \((*_{1})\)-admissible pairs. Then, over an open set \(U \subset X\),

\[
S_{\text{dilaton}}^{(\rho,h; \Psi)}(\varphi_{T})^{C} = \int_{U} \text{Tr} \langle d\rho, \varphi_{T}^{2} d\Psi \rangle_{h} \text{vol}_{h} = \int_{U} \text{Tr} \sum_{\mu=1}^{m} \left( d\rho(e_{\mu}) \sum_{i=1}^{n} D_{e_{\mu}}^{T} \varphi_{T}^{2}(y^{i}) \varphi_{T}^{2} \left( \frac{\partial \Psi}{\partial y^{i}} \right) \right) \text{vol}_{h} = \int_{U} \text{Tr} \left( \sum_{\mu} d\rho(e_{\mu}) \left( (D_{e_{\mu}}^{T}(\varphi_{T})) \Psi \right) \right) \text{vol}_{h}.
\]

\[
\frac{d}{dt} S_{\text{dilaton}}^{(\rho,h; \Psi)}(\varphi_{T})^{C} = \int_{U} \text{Tr} \sum_{\mu=1}^{m} d\rho(e_{\mu}) \partial_{t} \left( (D_{e_{\mu}}^{T}(\varphi_{T})) \Psi \right) \text{vol}_{h} = \int_{U} \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \left( (\partial_{t} D_{e_{\mu}}^{T}(\varphi_{T})) \Psi \right) \text{vol}_{h} + \int_{U} \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \sum_{i,j=1}^{n} D_{e_{\mu}}^{T} \varphi_{T}^{2}(y^{i}) \partial_{j} \varphi_{T}^{2}(y^{j}) \otimes \left( \frac{\partial}{\partial y^{i}} \otimes \frac{\partial}{\partial y^{j}} \right) \varphi_{T} \Psi \text{vol}_{h} = (\text{II.1}) + (\text{II.2}).
\]

The integrand of Summand (II.2) is \(C^\infty(U)^{C}\)-linear in \(\partial_{t} \varphi_{T}\) and hence in a final form.

\[(\text{II.1}) = \int_{U} \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \left( (D_{e_{\mu}}^{T}(\varphi_{T}) \otimes \partial_{t} \varphi_{T}) \Psi \right) \text{vol}_{h} - \int_{U} \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \left( (\text{ad} \otimes \nabla^{y})_{\partial_{t} \varphi_{T}} D_{e_{\mu}}^{T}(\varphi_{T}) \Psi \right) \text{vol}_{h} + \int_{U} \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \left( (\sum_{i} (\partial_{t} \varphi_{T})(e_{\mu}), \varphi_{T}^{2}(y^{i}) \otimes \frac{\partial}{\partial y^{i}} \right) \Psi \text{vol}_{h} = (\text{II.1.1}) + (\text{II.1.2}) + (\text{II.1.3}).\]

Both Summand (II.1.2) and Summand (II.1.3) vanish since

\[
\text{Tr}([a,b]c) = 0 \quad \text{if} \quad [b,c] = 0
\]
for $r \times r$ matrices $a, b, c$. 

\[ \text{(II.1.1)} = \int_U \text{Tr} \sum_{\mu} d\rho(e_{\mu}) D_{e_{\mu}}^T((\partial_t \varphi_T)\Phi) \, \text{vol}_h \]

\[ - \int_U \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \sum_{i,j=1}^n \partial_t \varphi_T^i(y^j) D_{e_{\mu}}^T \varphi_T^j(y^i) \otimes \left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \Phi - \left( \nabla^g_{\partial y^i} \frac{\partial}{\partial y^j} \Phi \right) \right) \, \text{vol}_h \]

\[ = \text{(II.1.1.1)} + \text{(II.1.1.2)}. \]

The integrand of Summand (II.1.1.2) is $C^\infty(U)^C$-linear in $\partial_t \varphi_T$ and hence in a final form. It can be combined with Summand (II.2) to give

\[ \text{(II.1.1.2) + (II.2)} \]

\[ = - \int_U \text{Tr} \sum_{\mu} d\rho(e_{\mu}) \sum_{i,j=1}^n [\partial_t \varphi_T^i(y^j), D_{e_{\mu}}^T \varphi_T^j(y^i)] \otimes \left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \Phi - \left( \nabla^g_{\partial y^i} \frac{\partial}{\partial y^j} \Phi \right) \right) \, \text{vol}_h, \]

which again vanishes due to $\text{Tr}$. 

\[ \text{(II.1.1.1) =} \int_U \sum_{\mu} d\rho(e_{\mu}) \text{Tr} D_{e_{\mu}}^T((\partial_t \varphi_T)\Phi) \, \text{vol}_h \]

\[ = \int_U \sum_{\mu} d\rho(e_{\mu}) \, e_{\mu} \text{Tr}((\partial_t \varphi_T)\Phi) \, \text{vol}_h \]

\[ = \int_U \sum_{\mu} e_{\mu} \left( d\rho(e_{\mu}) \text{Tr}((\partial_t \varphi_T)\Phi) \right) \, \text{vol}_h - \int_U \left( \sum_{\mu} e_{\mu} d\rho(e_{\mu}) \right) \text{Tr}((\partial_t \varphi_T)\Phi) \, \text{vol}_h \]

\[ = \text{(II.1.1.1.1) + (II.1.1.1.2)}. \]

The integrand of Summand (II.1.1.2) is $C^\infty(U)^C$-linear in $\partial_t \varphi_T$ and hence in a final form. To extract the boundary term from Summand (II.1.1.1), consider the $T$-family of $\mathbb{C}$-valued 1-forms on $U$ 

\[ \alpha_{(\Pi, \partial_t \varphi_T)}^T := d\rho \text{Tr}((\partial_t \varphi_T)\Phi), \]

which depends $C^\infty(U)^C$-linearly on $\partial_t \varphi_T$. Let 

\[ \xi_{(\Pi, \partial_t \varphi_T)}^T = \sum_{\mu=1}^m \left( d\rho(e_{\mu}) \text{Tr}((\partial_t \varphi_T)\Phi) \right) e_{\mu} \]

be the $T$-family of dual $\mathbb{C}$-valued vector fields of $\alpha_{(\Pi, \partial_t \varphi_T)}^T$ on $U$ with respect to the metric $h$. Note that $\xi_{(\Pi, \partial_t \varphi_T)}^T$ depends $C^\infty(U)^C$-linearly on $\partial_t \varphi_T$ as well. Then

\[ \text{(II.1.1.1)} = \int_U \sum_{\mu} e_{\mu} \langle \xi_{(\Pi, \partial_t \varphi_T)}^T, e_{\mu} \rangle_h \, \text{vol}_h \]

\[ = \int_U \sum_{\mu} \langle \nabla^h e_{\mu} \xi_{(\Pi, \partial_t \varphi_T)}^T, e_{\mu} \rangle_h \, \text{vol}_h + \int_U \langle \xi_{(\Pi, \partial_t \varphi_T)}^T, \sum_{\mu} \nabla^h e_{\mu} e_{\mu} \rangle_h \, \text{vol}_h \]

The first term is equal to

\[ \int_U (- \text{div} \xi_{(\Pi, \partial_t \varphi_T)}^T) \, \text{vol}_h = \int_U d\iota_{\xi_{(\Pi, \partial_t \varphi_T)}^T} \, \text{vol}_h = \int_{\partial U} i_{\xi_{(\Pi, \partial_t \varphi_T)}^T} \, \text{vol}_h. \]
which is the sought-for boundary term, whose integrand satisfies the requirement that it be
$C^\infty(U)^C$-linear in $\partial_t\varphi_T$. The second term is equal to
\[ \int_U d\rho(\sum_\mu \nabla^h_{e_\mu} e_\mu) \: Tr((\partial_t\varphi_T)\Phi) \: vol_h \]
by construction, which is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

In summary,

\textbf{Proposition 6.2.1. [first variation of dilaton term]} Let $(\varphi_T, \nabla^T)$ be a $(*_1)$-admissible
$T$-family of $(*_1)$-admissible pairs. Then,
\[ \frac{d}{dt} S_{\varphi_T}(\varphi_T)^C = \frac{d}{dt} \int_U Tr(d\rho, \varphi_T^2 d\Phi) \: vol_h \]
\[ = \int_{\partial U} \xi_{(1, \partial_t\varphi_T)} vol_h \]
\[ + \int_U \left( d\rho(\sum_{\mu=1}^n \nabla_{e_{\mu}} e_{\mu}) - \sum_{\mu=1}^n e_{\mu} d\rho(e_{\mu}) \right) Tr((\partial_t\varphi_T)\Phi) \: vol_h. \]

Here, the first summand is the boundary term with $\xi_T := \sum_{\mu=1}^n (d\rho(e_{\mu}) Tr((\partial_t\varphi_T)\Phi)) e_{\mu}$
$C^\infty(U)^C$-linear in $\partial_t\varphi_T$: the integrand of the second summand $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and they
contribute additional zeroth-order terms to the equations of motion for $(\varphi, \nabla)$. In particular,
while the dilaton term of the standard action modifies the equations of motion for $(\varphi, \nabla)$, it does not
change the signature of the system.

6.3 The first variation of the gauge/Yang-Mills term and the Chern-Simons/
Wess-Zumino term

To make sure that differential forms on $Y$ of rank $\geq 2$ are pull-pushed to $(\mathcal{O}_X^2$-valued-)differential
forms on $X$ (cf. Lemma 2.1.11), we assume in this subsection that $\varphi_T : (X, \mathcal{E}; \nabla^T) \to Y$ is a
$(*_2)$-family of $(*_2)$-admissible maps. (Note that as the gauge/Yang-Mills term is defined through
a norm-squared, $(*_1)$-admissible family of $(*_1)$-admissible $(\varphi_T, \nabla^T)$ is enough for the derivation
of the first variation formula of the gauge/Yang-Mills term but the result will be slightly messier.)

6.3.1 The first variation of the gauge/Yang-Mills term

Let $(e_1, \cdots, e_m)$ be an orthonormal frame on $U$. Then, over $U,$
\[ S_{gauge/YM}(\varphi_T, \nabla^T)^C := -\frac{1}{2} \int_U Tr||2\pi\alpha^' F_{\nabla^T} + \varphi^\circ B||^2 \: vol_h \]
\[ = -\frac{1}{2} \int_U Tr \sum_{\mu,\nu} \left( (2\pi\alpha^' F_{\nabla^T} + \varphi^\circ B)(e_{\mu}, e_{\nu}) \right)^2 \: vol_h. \]

Applying the following basic identities:
\[ \partial_t F_{\nabla^T}(e_{\mu}, e_{\nu}) = D_{e_{\mu}}((\partial_t \nabla^T)(e_{\nu})) - D_{e_{\nu}}((\partial_t \nabla^T)(e_{\mu})) - (\partial_t \nabla^T)([e_{\mu}, e_{\nu}]), \]
\[ \partial_t((\varphi^\circ B)(e_{\mu}, e_{\nu})) = \sum_{i,j} \partial_i(\varphi^\circ_{i,j})(B_{ij}) D_{e_{\mu}} \varphi^\circ_{i,j}(y') D_{e_{\nu}} \varphi^\circ_{i,j}(y') \]
\[ + \sum_{i,j} \varphi^\circ_{i,j}(B_{ij}) D_{e_{\mu}} \partial_t \varphi^\circ_{i,j}(y') + \sum_{i,j} \varphi^\circ_{i,j}(B_{ij}) D_{e_{\nu}} \partial_t \varphi^\circ_{i,j}(y') \]
\[ + \sum_{i,j} \varphi^\circ_{i,j}(B_{ij}) D_{e_{\mu}} \partial_t \varphi^\circ_{i,j}(y') + \sum_{i,j} \varphi^\circ_{i,j}(B_{ij}) D_{e_{\nu}} \partial_t \varphi^\circ_{i,j}(y') \]

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and proceeding similarly to Sec. 6.1, one has the following results.

\[
\frac{d}{dt} S_{\text{gauge YM}}^{(h; B)}(\varphi_T, \nabla T)^C = -\frac{1}{2} \int_U \partial_t \sum_{\mu, \nu} \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right)^2 \text{vol}_h \\
= - \int_U \sum_{\mu, \nu} \partial_t \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \\
= - \int_U \sum_{\mu, \nu} 2\pi \alpha' \partial_t \left( F_{\varphi_T}(e_\mu, e_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \\
\quad - \int_U \sum_{\mu, \nu} \partial_t \left( \varphi_T^\phi B(e_\mu, e_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \\
= (\text{III.1}) + (\text{III.2}).
\]

(III.1) \(:= - \int_U \sum_{\mu, \nu} 2\pi \alpha' \partial_t \left( F_{\varphi_T}(e_\mu, e_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \)

\[
= - 2\pi \alpha' \int_U \sum_{\mu, \nu} \left( D^T_{e_\mu} ((\partial_t \nabla T)(e_\nu)) - D^T_{e_\nu} ((\partial_t \nabla T)(e_\mu)) - (\partial_t \nabla T)(\nu_\mu, \nu_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \\
\quad - \frac{1}{2} \sum_{\mu, \nu} e^{\nu}([\nu_\nu, \lambda_\lambda])(2\pi \alpha' \varphi_T^\phi B)(e_\mu, e_\lambda) \text{vol}_h.
\]

Here,

\[
\xi^T_{(\text{III}, \partial_t \nabla T)} := \sum_{\mu, \nu} \text{Tr} \left( ((\partial_t \nabla T)(e_\nu) \cdot (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \in T_s(U_T/T)^C
\]

is \(O_U^C\)-linear in \(\partial_t \nabla T\); and the second summand contributes to the equations of motion for \((\varphi, \nabla)\). The latter are standard terms from non-Abelian Yang-Mills theory with additional terms from \(\varphi_T^\phi B\).

(III.2) \(:= - \int_U \sum_{\mu, \nu} \partial_t \left( \varphi_T^\phi B(e_\mu, e_\nu) \right) \cdot \left( (2\pi \alpha' F_{\varphi_T} + \varphi_T^\phi B)(e_\mu, e_\nu) \right) \text{vol}_h \)

\[
= - \int_U \sum_{\mu, \nu} \left( \sum_{i,j} \partial_t \left( \varphi_T^\phi B_{i,j} \right) DT_{e_\mu} \varphi_T^\phi(y') + DT_{e_\nu} \varphi_T^\phi(y') \right) + \sum_{i,j} \varphi_T^\phi \left( DT_{e_\mu} \partial_t \varphi_T^\phi(y') + (\partial_t \nabla T)(e_\mu, \varphi_T^\phi(y')) \right) DT_{e_\nu} \varphi_T^\phi(y') \\
\quad + \sum_{i,j} \varphi_T^\phi \left( DT_{e_\nu} \partial_t \varphi_T^\phi(y') + (\partial_t \nabla T)(e_\nu, \varphi_T^\phi(y')) \right) \right) \text{vol}_h \\
= (\text{III.2.1}) + (\text{III.2.2.1} + (\text{III.2.2.2}) + (\text{III.2.3.1}) + (\text{III.2.3.2})
\]

in the order of the appearance of the five summands after the expansion.
\[(\text{III.2.1}) \quad := - \int_U \text{Tr} \sum_{\mu, \nu} \sum_{i,j} \partial_i (\varphi_T^\beta(B_{ij})) D_{e^\mu}^T \varphi_T^\gamma(y) D_{e^\nu}^T \varphi_T^\delta(y) \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h \]

\[
\quad := - \int_U \text{Tr} \sum_{\mu, \nu} \sum_{i,j} \left((\partial_i \varphi_T) B_{ij} \right) D_{e^\mu}^T \varphi_T^\gamma(y) D_{e^\nu}^T \varphi_T^\delta(y) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h
\]

has an integrand $O_U^C$-linear in $\partial_i \varphi_T$ and hence in a final form.

\[(\text{III.2.2.1}) + (\text{III.2.3.1}) \quad := - \int_U\text{Tr} \sum_{\mu, \nu} \left( \sum_{i,j} \varphi_T^\alpha(B_{ij}) D_{e^\mu}^T \partial_i \varphi_T^\beta(y) D_{e^\nu}^T \varphi_T^\gamma(y) \right) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h \\
\quad = - \int_U \text{Tr} \sum_{\mu, \nu} \sum_{i,j} D_{e^\mu}^T \partial_i \varphi_T^\beta(y) \varphi_T^\alpha(B_{ij}) D_{e^\nu}^T \varphi_T^\gamma(y) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h \\
\quad = - \int_U \sum_{\mu} \left( \sum_{\nu} \sum_{i,j} \partial_i \varphi_T^\beta(y) \varphi_T^\alpha(B_{ij}) D_{e^\nu}^T \varphi_T^\gamma(y) \right) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h \\
\quad = \sum_{\mu} \left( \sum_{\nu} \sum_{i,j} \partial_i \varphi_T^\beta(y) \varphi_T^\alpha(B_{ij}) D_{e^\nu}^T \varphi_T^\gamma(y) \right) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h.
\]

Here,

\[\xi_{(\text{III}, \partial_i \varphi_T)} := \sum_{\mu} \left( \sum_{\nu} \sum_{i,j} \partial_i \varphi_T^\beta(y) \varphi_T^\alpha(B_{ij}) D_{e^\nu}^T \varphi_T^\gamma(y) \right) \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) e_\mu\]

in $\mathcal{T}_s(U_T/T)^C$ is $O_U^C$-linear in $\partial_i \varphi_T$; and the second summand contributes to $\delta S_{\text{standard}}^{\phi, g, B, C}(\varphi, \nabla)/\delta \varphi$-part of the equations of motion for $(\varphi, \nabla)$.

\[(\text{III.2.2.2}) + (\text{III.2.3.2}) \quad := - \int_U \sum_{\mu, \nu} \sum_{i,j} \partial_i \varphi_T^\beta(B_{ij}) \left[(\partial_i \varphi_T^\beta(y)) (e_\mu), \varphi_T^\delta(y) \right] D_{e^\nu}^T \varphi_T^\gamma(y) \\
\quad \cdot \left((2 \pi \alpha' F_{\varphi T} + \varphi_T^\phi B)(e_\mu, e_\nu)\right) \text{vol}_h.
\]

has an integrand $O_U^C$-linear in $\partial_i \nabla T$ and hence in a final form.

In summary,
Proposition 6.3.1.1. [first variation of gauge/Yang-Mills term] Let \((\varphi_T, \nabla^T)\) be a \((*_2)\)-admissible family of \((*_2)\)-admissible pairs. Then

\[
\frac{d}{dt} S_{\text{gauge/YM}}^{(h;B)}(\varphi_T, \nabla^T) = -\frac{1}{2} \frac{d}{dt} \int_U Tr\|2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B\|^2_{h} \text{vol}_h
\]

\[
= -4\pi\alpha' \int_{\partial U} \xi_{(\text{III} T, \varphi_T)}^T \text{vol}_h - 2 \int_{\partial U} \xi_{(\text{III} T, \varphi_T)}^T \text{vol}_h
\]

\[
-4\pi\alpha' \int_U Tr \sum_{\nu} (\partial_t \nabla^T) (e_\nu) \cdot \left( (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(\sum_\mu \nabla^h_{\partial_t \nu} e_\mu, e_\nu) - \frac{1}{2} \sum_{\mu, \lambda} e^e([e_\mu, e_\lambda])(2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(e_\mu, e_\nu) \right) \text{vol}_h.
\]

\[
\left( \int_U Tr \sum_{\mu, \nu} \left( \varphi_T^0(B_{\mu}) \left[ (\partial_t \nabla^T)(e_\mu) \cdot (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(\sum_\nu \nabla^h_{\partial_t \nu} e_\mu, e_\nu) \right] + \varphi_T^0(B_{\mu}) \nabla^h_{\partial_t \mu} \nabla^0_{\partial_t \nu} \right) \right) (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(e_\mu, e_\nu) \] \text{vol}_h
\]

\[
-2 \int_U Tr \sum_\nu \sum_{i,j} \partial_t \varphi_T^0(y) \left( \varphi_T^0(B_{ij}) \left[ (\partial_t \nabla^T)(e_\nu) \cdot (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(\sum_\mu \nabla^h_{\partial_t \mu} e_\mu, e_\nu) \right] - \frac{1}{2} \sum_{\mu, \lambda} e^e([e_\mu, e_\lambda])(2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(e_\mu, e_\nu) \right) \right) \text{vol}_h.
\]

Here,

\[
\xi_{(\text{III} T, \partial_t \nabla^T)}^T := \sum_{\mu, \nu} Tr \left( (\partial_t \nabla^T)(e_\nu) \cdot (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(e_\mu, e_\nu) \right) e_\mu,
\]

\[
\xi_{(\text{III} T, \partial_t \varphi_T)}^T := \sum_{\mu} \left( \sum_\nu \sum_{i,j} \partial_t \varphi_T^0(y) \varphi_T^0(B_{ij}) \left[ (\partial_t \nabla^T)(e_\nu) \cdot (2\pi\alpha' F_{\varphi_T} + \varphi_T^0 B)(e_\mu, e_\nu) \right] \right) e_\mu.
\]

in \(T(U_T/T)^C\), with the first \(O_{\bar{U}}^C\)-linear in \(\partial_t \nabla^T\) and the second \(O_{\bar{U}}^C\)-linear in \(\partial_t \varphi_T\).

6.3.2 The first variation of the Chern-Simons/Wess-Zumino term for lower dimensional D-branes

This is an update of [L-Y8: Sec.6.2] (D(13.1)) in the current setting. Let \(\varphi_T : (X^A, \mathcal{E}_T; \nabla^T) \to Y\) be an \((*_2)\)-family of \((*_2)\)-admissible maps. We work out the first variation of the Chern-Simons/Wess-Zumino term \(S_{\text{CS/WZ}}^{(C;B)}(\varphi, \nabla)\) for the cases where \(m := dim X = 0, 1, 2, 3\). As the details involve no identities or techniques that have not yet been used in Sec. 6.1, Sec. 6.2, and/or Sec. 6.3.1, we only summarize the final results below.
6.3.2.1 D(−1)-brane world-point \((m = 0)\)

For a D(−1)-brane world-point, \(\dim X = 0, \nabla = 0\), and \(S_{\text{CS/WZ}}^{C(0)}(\varphi_T) = T_{-1} \cdot Tr(\varphi_T^2(C(0)))\). It follows that
\[
\frac{d}{dt} S_{\text{CS/WZ}}^{C(0)}(\varphi_T) = T_{-1} Tr (\partial_i(\varphi_T^2(C(0)))) = T_{-1} Tr ((\partial_i\varphi_T)C(0)).
\]

6.3.2.2 D-particle world-line \((m = 1)\)

For a D-particle world-line, \(\dim X = 1\). Let \(e_1\) be the orthonormal frame on an open set \(U \subset X\); \(e^1\) its dual co-frame. Then, over \(U\),
\[
S_{\text{CS/WZ}}^{C(1)}(\varphi_T^0C(1)) = T_0 \int_U Tr \varphi_T^0 C(1) = T_0 \int_U Tr \left( \sum_{i=1}^n \varphi_T^2(C_i) \cdot D^T e_i \varphi_T^2(y^i) \right) e^1.
\]

It follows that
\[
\frac{d}{dt} S_{\text{CS/WZ}}^{C(1)}(\varphi_T) = T_0 \left( \left. Tr \sum_i \partial_i \varphi_T^2(y^i) \varphi_T^2(C_i) \right|_{\partial U} \right) + T_0 \int_U Tr \left( \sum_i \partial_i \varphi_T^2(y^i) D^T e_i \varphi_T^2(C_i) \right) e^1 + T_0 \int_U Tr \left( \sum_i D^T e_i \varphi_T^2(y^i) \cdot (\partial_i \varphi_T) C_i \right) e^1.
\]

6.3.2.3 D-string world-sheet \((m = 2)\)

Denote
\[
\tilde{C}_{(2)} := C_{(2)} + C_{(0)}B = \sum_{ij} (C_{ij} + C_{(0)}B_{ij}) dy^i \otimes dy^j = \sum_{i,j} \tilde{C}_{ij} dy^i \otimes dy^j
\]
in a local coordinate \((y^1, \cdots, y^n)\) of \(Y\). For a D-string world-sheet, \(\dim X = 2\). Let \((e_1, e_2)\) be an orthonormal frame on an open set \(U \subset X\); \((e^1, e^2)\) its dual co-frame. Then, over \(U\),
\[
S_{\text{CS/WZ}}^{C(0),C(2),B}(\varphi_T, \nabla_T)^C = T_1 \int_U Tr \left( \sum_{i,j=1}^n \varphi_T^2(\tilde{C}_{ij}) D^T e_i \varphi_T^2(y^i) D^T e_j \varphi_T^2(y^j) \\
+ \pi \alpha' \varphi_T^2(C_{(0)}) F_{\nabla_T} (e_1, e_2) + \pi \alpha' F_{\nabla_T} (e_1, e_2) \varphi_T^2(C_{(0)}) \right) e^1 \wedge e^2
\]
\[
= T_1 \int_U Tr \left( \sum_{i,j=1}^n \varphi_T^2(\tilde{C}_{ij}) D^T e_i \varphi_T^2(y^i) D^T e_j \varphi_T^2(y^j) + 2 \pi \alpha' \varphi_T^2(C_{(0)}) F_{\nabla_T} (e_1, e_2) \right) e^1 \wedge e^2.
\]
It follows that
\[
\frac{d}{dt}s_{CS/WZ}^{(C_{11}), (C_{21})} (\varphi_T, \nabla^T)^C
\]
\[= T_1 \int_U \text{Tr} \left( \sum_{i,j=1}^{n} \varphi_T^2 (\tilde{C}_{i,j}) D_{e_i}^T \varphi_T^2(y) D_{e_j}^T \varphi_T^2(y') + 2 \pi \alpha' \varphi_T^2 (C_0) F_{\nabla^T} (e_1, e_2) \right) e^1 \wedge e^2
\]
\[+ T_1 \int_{\partial U} i_{IV, \partial_e \varphi_T} (e^1 \wedge e^2) + 2 \pi \alpha' T_1 \int_{\partial U} i_{IV, \partial_e \varphi_T} (e^1 \wedge e^2)
\]
\[+ T_1 \int_U \text{Tr} \left( \sum_{i,j=1}^{n} \partial \varphi_T^2 (y') \left( D_{e_i}^T \varphi_T^2(y') D_{e_j}^T \varphi_T^2(y) \varphi_T^2 (\tilde{C}_{i,j}) \right) - D_{e_j}^T \varphi_T^2(y') \varphi_T^2 (\tilde{C}_{i,j}) \right) (\nabla_{e_1} h + \nabla_{e_2} h) e^1 \wedge e^2
\]
\[+ 2 \pi \alpha' T_1 \int_U \text{Tr} \left( \partial \varphi_T^2 (C_0) \cdot F_{\nabla^T} (e_1, e_2) \right) e^1 \wedge e^2
\]
\[+ 2 \pi \alpha' T_1 \int_U \text{Tr} \left( \varphi_T^2 (C_0) \cdot (\partial \varphi_T - (\varphi_T (e_2)^2 - (\varphi_T (e_1)^2) (\nabla_{e_1} h + \nabla_{e_2} h) - (\partial \varphi_T) (e_1, e_2) \right) e^1 \wedge e^2
\]
\[+ 2 \pi \alpha' T_1 \int_U \text{Tr} \left( D_{e_i}^T \varphi_T^2 (C_0) \cdot (\partial \varphi_T) (e_2) - D_{e_j}^T \varphi_T^2 (C_0) \cdot (\partial \varphi_T) (e_1) \right) e^1 \wedge e^2
\]

Here,
\[
\xi_{IV, \partial_e \varphi_T}^T := \text{Tr} \left( \sum_{i,j=1}^{n} \partial \varphi_T^2 (y') D_{e_i}^T \varphi_T^2 (y') \varphi_T^2 (\tilde{C}_{i,j}) \right) e_1 - \text{Tr} \left( \sum_{i,j=1}^{n} \partial \varphi_T^2 (y') D_{e_j}^T \varphi_T^2 (y') \varphi_T^2 (\tilde{C}_{i,j}) \right) e_2,
\]
\[
\xi_{IV, \partial_e \nabla^T}^T := \text{Tr} \left( \varphi_T^2 (C_0) \cdot (\partial \varphi_T) (e_2) \right) e_1 - \text{Tr} \left( \varphi_T^2 (C_0) \cdot (\partial \varphi_T) (e_1) \right) e_2
\]
in $T_s(U_T/T)^C$, with the first $O_U^C$-linear in $\partial \varphi_T$ and the second $O_U^C$-linear in $\partial \varphi_T$.

6.3.2.4 D-membrane world-volume ($m = 3$)

Denote
\[
\tilde{C}_{(3)} := C_{(3)} + C_{(1)} \wedge B
\]
\[= \sum_{i,j,k} (C_{ijk} + C_i B_{jk} + C_j B_{ki} + C_k B_{ij}) \ dy^i \otimes dy^j \otimes dy^k = \sum_{i,j,k} \tilde{C}_{ijk} dy^i \otimes dy^j \otimes dy^k
\]
in a local coordinate $(y^1, \cdots, y^n)$ of $Y$. For D-membrane world-volume, $\dim X = 3$. Let $(e_1, e_2, e_3)$ be an orthonormal frame on an open set $U \subset X$; $(e^1, e^2, e^3)$ its dual co-frame. Then, over $U$,
\[
S_{CS/WZ}^{(C_{11}), (C_{21})} (\varphi_T, \nabla^T)^C
\]
\[= T_2 \int_U \text{Tr} \left( \sum_{i,j,k=1}^{n} \varphi_T^2 (\tilde{C}_{ijk}) D_{e_i}^T \varphi_T^2 (y^i) D_{e_j}^T \varphi_T^2 (y^j) D_{e_k}^T \varphi_T^2 (y^k) + 2 \pi \alpha' \sum_{(\lambda \mu \nu) \in \mathcal{S}_{pym_3}} \sum_{i=1}^{n} (-1)^{i} \lambda_{\mu \nu} \varphi_T^2 (C_i) D_{\lambda}^T \varphi_T^2 (y^i) F_{\nabla^T} (e_{\mu}, e_{\nu}) \right) e^1 \wedge e^2 \wedge e^3
\]
It follows that
\[
\frac{d}{dt} S_{CS/WZ}^{(C_{(1)}, C_{(2)}, B)}(\varphi_T, \nabla^T)^C
\]

\[
= T_2 \int_U \text{Tr} \left( \sum_{i,j,k=1}^n \varphi^i (\bar{C}_{ijk}) D^T_{C_{ijk}} \varphi^j(y') D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y) \right) + 2\pi \alpha' \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \frac{1}{1!} (-1)^{(\lambda\mu\nu)} \left( \varphi^i_{y'}(C_i) D^T_{C_{ijk}} \varphi^j(y') F_{\varphi_T}(e_\mu, e_\nu) \right) e^1 \wedge e^2 \wedge e^3
\]

\[
= T_2 \int_{\partial U} i_{\xi^T} (\varphi_T) \left( e^1 \wedge e^2 \wedge e^3 \right) + 4\pi \alpha' T_2 \int_{\partial U} i_{\xi^T} (\varphi_T) \left( e^1 \wedge e^2 \wedge e^3 \right)
\]

\[
+ T_2 \int_U \left( \left( \text{Tr} \sum_{i,j,k} \varphi^i_{y'}(\bar{C}_{ijk}) \partial_t \varphi^j_{y'}(y') D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y) \right) e^1
\right.

\[
- \left( \text{Tr} \sum_{i,j,k} \varphi^i_{y'}(\bar{C}_{ijk}) \partial_t \varphi^j_{y'}(y') D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y) \right) e^2
\]

\[
- \left( \text{Tr} \sum_{i,j,k} \varphi^i_{y'}(\bar{C}_{ijk}) \partial_t \varphi^j_{y'}(y') D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y) \right) e^3
\]

\[
\left. \right) \right( \sum_{\mu=1}^3 \nabla^h_{e_\mu} e_\mu \right) e^1 \wedge e^2 \wedge e^3
\]

\[
- T_2 \int_U \text{Tr} \sum_{i,j,k} \partial_t \varphi^i_{y'}(y') \left( D^T_{C_{ijk}} (\varphi^j_{y'}(\bar{C}_{ijk}) D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y)) - D^T_{C_{ijk}} (\varphi^j_{y'}(\bar{C}_{ijk}) D^T_{C_{ijk}} \varphi^k(y') D^T_{C_{ijk}} \varphi^1(y)) \right) e^1 \wedge e^2 \wedge e^3
\]

\[
+ 4\pi \alpha' T_2 \int_U \left( \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \partial_t \varphi^i_{y'}(y') F_{\varphi_T}(e_2, e_3) \right) e^1
\right.

\[
- \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \partial_t \varphi^i_{y'}(y') F_{\varphi_T}(e_1, e_3) \right) e^2
\]

\[
+ \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \partial_t \varphi^i_{y'}(y') F_{\varphi_T}(e_1, e_2) \right) e^3
\]

\[
\left. \right) \right( \sum_{\mu=1}^3 \nabla^h_{e_\mu} e_\mu \right) e^1 \wedge e^2 \wedge e^3
\]

\[
- 4\pi \alpha' T_2 \int_U \text{Tr} \sum_{i,j,k} \partial_t \varphi^i_{y'}(y') \left( D^T_{C_{ijk}} (\varphi^j_{y'}(C_i) F_{\varphi_T}(e_2, e_3)) - D^T_{C_{ijk}} (\varphi^j_{y'}(C_i) F_{\varphi_T}(e_1, e_3)) \right.
\]

\[
+ D^T_{C_{ijk}} (\varphi^j_{y'}(C_i) F_{\varphi_T}(e_1, e_2)) \right) e^1 \wedge e^2 \wedge e^3
\]

\[
+ 2\pi \alpha' T_2 \int_U \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \sum_{i,j,k} (-1)^{(\lambda\mu\nu)} \partial_t \varphi^i_{y'}(C_i) D^T_{C_{ijk}} \varphi^j(y') F_{\varphi_T}(e_\mu, e_\nu) e^1 \wedge e^2 \wedge e^3
\]

\[
+ 2\pi \alpha' T_2 \int_U \left( \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \left( D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_2) - D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_3) \right) \right) e^1
\right.

\[
+ \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \left( D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_1) + D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_1) \right) \right) e^2
\]

\[
+ \left( \text{Tr} \sum_i \varphi^i_{y'}(C_i) \left( D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_2) - D^T_{C_{ijk}} \varphi^j(y') \partial_t \nabla^T(e_1) \right) \right) e^3
\]

\[
\left. \right) \right( \sum_{\mu=1}^3 \nabla^h_{e_\mu} e_\mu \right) e^1 \wedge e^2 \wedge e^3
\]

\[
+ 2\pi \alpha' T_2 \int_U \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \sum_{i,j,k} (-1)^{(\lambda\mu\nu)} \left( \varphi^i_{y'}(C_i) \left( \partial_t \nabla^T(e_\lambda), \varphi^j_{y'} \right) F_{\varphi_T}(e_\mu, e_\nu) \right.
\]

\[
- \varphi^i_{y'}(C_i) D^T_{C_{ijk}} \varphi^j(y') \left( \partial_t \varphi_T(\left[ e_\mu, e_\nu \right] ) \right) e^1 \wedge e^2 \wedge e^3.
\]
Recall

7.1 The second variation of the kinetic term for maps

We work out in this section the second variation formula of the enhanced kinetic term $T$.

Here,

$$\xi^T_{(\nu, \partial \eta; \xi)} := \left( \sum_{i,j,k} \phi_i^p(C_{ijk}) \partial_t \phi_i^p(y') D^T_{ij} \phi^p(y') D^T_{jk} \phi^p(y') \right) e_1$$

$$- \left( \sum_{i,j,k} \phi_i^p(C_{ijk}) \partial_t \phi_i^p(y') D^T_{ij} \phi^p(y') D^T_{jk} \phi^p(y') \right) e_2$$

$$- \left( \sum_{i,j,k} \phi_i^p(C_{ijk}) \partial_t \phi_i^p(y') D^T_{ij} \phi^p(y') D^T_{jk} \phi^p(y') \right) e_3,$$

$$\xi^T_{(\nu, \partial \eta; \xi)} := \left( \sum_{i} \phi_i^p(C_i) \partial_t \phi_i^p(y') F^{\nu T}(e_2, e_3) \right) e_1$$

$$- \left( \sum_{i} \phi_i^p(C_i) \partial_t \phi_i^p(y') F^{\nu T}(e_1, e_3) \right) e_2 + \left( \sum_{i} \phi_i^p(C_i) \partial_t \phi_i^p(y') F^{\nu T}(e_1, e_2) \right) e_3,$$

$$\xi^T_{(\nu, \partial \nu; \nu)} := \left( \sum_{i} \phi_i^p(C_i) \left( D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_2) - D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_3) \right) \right) e_1$$

$$+ \left( \sum_{i} \phi_i^p(C_i) \left( D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_1) + D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_3) \right) \right) e_2$$

$$+ \left( \sum_{i} \phi_i^p(C_i) \left( D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_2) - D^T_{ij} \phi^p(y') (\partial_t \nu^T)(e_1) \right) \right) e_3$$

in $T_s(U_T/T)^C$, with the first two $O_U^C$-linear in $\partial_t \phi$ and the third $O_U^C$-linear in $\partial_t \nabla^T$.

7 The second variation of the enhanced kinetic term for maps

Let $T = (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2$, with coordinate $(s, t)$, and $(\phi_T, \nabla^T)$ be an $(*)$-admissible family of $(*)$-admissible pairs, with $(\phi(0,0), \nabla(0,0)) = (\phi, \nabla)$. Assume further that

$$D_\xi \partial_s A_{\phi_T} \subset \text{Comm}(A_{\phi_T}) \quad \text{for all } \xi \in T_s(X_T/T).$$

We work out in this section the second variation formula of the enhanced kinetic term $S_{\text{map,kinetic}^+}^{(h,g)}(\phi, \nabla)$ in the standard action $S_{\text{standard}}^{(h,g)}(\phi, \nabla)$.

7.1 The second variation of the kinetic term for maps

Recall

$$E^T(\phi_T) := S_{\text{map,kinetic}}^{(h,g)}(\phi_T, \nabla^T) := \frac{1}{2} T_{m-1} \int_X Tr \langle D^T \phi_T, D^T \phi_T \rangle_{(h,g)} vol h,$$

with the understanding that all expressions are taken on $X_{(s,t)}$ with $(s, t)$ varying in $T$.

Let $U \subset X$ be an open set with an orthonormal frame $(e_\mu)_{\mu=1,\ldots,m}$. Let $(e^\mu)_{\mu=1,\ldots,m}$ be the dual co-frame. Assume that $U$ is small enough so that $\phi_T(U T^E)$ is contained in a coordinate chart of $Y$, with coordinates $(y^1, \ldots, y^n)$. Then, as in Sec. 6.1, over $U$,

$$\frac{\partial}{\partial t} E^T(\phi_T) = T_{m-1} \int_U Tr \sum_{\mu=1}^m \left\langle \nabla^T_{e_\mu}(\phi_T), \frac{\partial_t}{\partial s} \phi_T, D^T_{e_\mu} \phi_T \right\rangle_g \ vol \ h$$

$$+ T_{m-1} \int_U Tr \sum_{\mu=1}^m \left\langle (ad \otimes \nabla^g) D^T_{e_\mu} \phi_T, \frac{\partial_t}{\partial s} \phi_T, D^T_{e_\mu} \phi_T \right\rangle_g \ vol \ h$$

$$+ T_{m-1} \int_U \sum_{\mu=1}^m \left\langle \frac{n}{2} \nabla^T(\phi_T)(e_\mu), \frac{\partial_t}{\partial s} \phi_T, D^T_{e_\mu} \phi_T \right\rangle_g \ vol \ h$$

$$= (I^{2.1}) + (I^{2.2}) + (I^{2.3});$$
and

\[ \frac{\partial^2}{\partial s \partial t} E_{\nabla_T}(\varphi_T) = \frac{\partial}{\partial s} (I^{2.1}) + \frac{\partial}{\partial s} (I^{2.2}) + \frac{\partial}{\partial s} (I^{2.3}). \]

Which we now compute term by term.

The term \( \frac{\partial}{\partial s} (I^{2.1}) \)

\[
\frac{\partial}{\partial s} (I^{2.1}) = T_{m-1} \frac{\partial}{\partial s} \int_U \text{Tr} \left( \sum_{\mu=1}^m \left< \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> \right)_g \text{vol}_h
\]

\[
= T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< \partial_s \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
= T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< \partial_s \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< \partial_s \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, \partial_s D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
= (I^{2.1.1}) + (I^{2.1.2}).
\]

(a) Term \( (I^{2.1.1}) \)

\[
(I^{2.1.1}) := T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< \partial_s \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
= T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< \partial_s \nabla_{e_\mu}^{T,(\varphi_T,g)} \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left< F_{\nabla_T,(\varphi_T,g)} (\partial_s, e_\mu) \partial_t \varphi_T, D_{e_\mu}^{T} \varphi_T \right> _g \text{vol}_h
\]

\[
= (I^{2.1.1.1}) + (I^{2.1.1.2}).
\]

For Term \( (I^{2.1.1.1}) \), as in Sec. 6.1 for Summand \( (I.1.1) \), consider the 1-form on \( U_T/T \)

\[
\alpha_{(1^2, \partial_s \partial_t \varphi_T)}^T := \text{Tr} \left( \partial_s \partial_t \varphi_T, D^T \varphi_T \right)_g
\]

and let

\[
\xi_{(1^2, \partial_s \partial_t \varphi_T)}^T := \sum_{\mu=1}^n \text{Tr} \left( \partial_s \partial_t \varphi_T, D_{e_\mu}^T \varphi_T \right)_g e_\mu
\]

be its dual on \( U_T/T \) with respect to \( h \). Then,

\[
(I^{2.1.1.1}) = T_{m-1} \int_{\partial U} \xi_{(1^2, \partial_s \partial_t \varphi_T)}^T \text{vol}_h
\]

\[
+ T_{m-1} \int_U \text{Tr} \left( \partial_s \partial_t \varphi_T, \left( D_{\sum_{\mu} \nabla_{e_\mu}^{T,(\varphi_T,g)} e_\mu} - \sum_{\mu} \nabla_{e_\mu}^{T,(\varphi_T,g)} D_{e_\mu}^T \varphi_T \right) \right)_g \text{vol}_h.
\]

For Term \( (I^{2.1.1.2}) \), recall Lemma 3.2.2.5. Then,

\[
(I^{2.1.1.2}) = - T_{m-1} \int_U \text{Tr} \left( \partial_t \varphi_T, \sum_{\mu} F_{\nabla_T,(\varphi_T,g)} (\partial_s, e_\mu) D_{e_\mu}^T \varphi_T \right)_g \text{vol}_h
\]

\[
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left[ F_{\nabla} (\partial_s, e_\mu), \left( \partial_t \varphi_T, D_{e_\mu}^T \varphi_T \right)_g \right] \text{vol}_h
\]

\[
= - T_{m-1} \int_U \text{Tr} \left( \partial_t \varphi_T, \sum_{\mu} F_{\nabla_T,(\varphi_T,g)} (\partial_s, e_\mu) D_{e_\mu}^T \varphi_T \right)_g \text{vol}_h.
\]

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Here, \[ F_{\theta^T,(\varphi_T,g)}(\partial_s, e_\mu) D^T_{\varphi_T} \varphi_T = (\partial_s \nabla^T_{\varphi_T}(\varphi_T,g) - \nabla^T_{\varphi_T}(\varphi_T,g) \partial_s) \sum_{i=1}^{n} D_{\varphi_T}^i \varphi_T(y^i) \otimes \frac{\partial}{\partial y^i} \]

\[
= \sum_{i=1}^{n} [\langle \nabla^T_{\varphi_T}(\varphi_T,g), D_{\varphi_T}^i \varphi_T(y^i) \rangle \otimes \frac{\partial}{\partial y^i} + \sum_{i=1}^{n} D_{\varphi_T}^i \varphi_T(y^i) \sum_{j=1}^{\text{dim}} [\langle \partial_s \nabla^T_{\varphi_T}(\varphi_T,g), \varphi_T^j(y^j) \rangle \otimes \frac{\partial}{\partial y^j} - \partial_s^2 \varphi_T^j(y^j) D_{\varphi_T}^i \varphi_T(y^i) \otimes \frac{\partial}{\partial y^i} \] 

\] explicitly.

(b) Term (I^2,1.2)

\[
(I^2,1.2) := T_{m-1} \int_U \sum_{\mu} \langle \nabla^T_{\varphi_T}(\varphi_T,g) \partial_t \varphi_T, \partial_s D^T_{\varphi_T} \varphi_T \rangle_g \text{ vol}_h
\]

\[
= T_{m-1} \int_U \sum_{\mu} \langle \nabla^T_{\varphi_T}(\varphi_T,g) \partial_t \varphi_T, \nabla^T_{\varphi_T}(\varphi_T,g) \partial_s \varphi_T \rangle_g \text{ vol}_h
\]

\[
+ T_{m-1} \int_U \sum_{\mu} \langle \nabla^T_{\varphi_T}(\varphi_T,g) \partial_t \varphi_T, (ad \otimes \nabla^g) D^T_{\varphi_T} \varphi_T \partial_s \varphi_T \rangle_g \text{ vol}_h
\]

\[
+ T_{m-1} \int_U \sum_{\mu} \langle \nabla^T_{\varphi_T}(\varphi_T,g) \partial_t \varphi_T, \sum_{i=1}^{n} [\langle \partial_s \nabla^T_{\varphi_T}(\varphi_T,g), \varphi_T^i(y^i) \rangle \otimes \frac{\partial}{\partial y^i} \] \]

As in Sec.6.1, consider the 1-forms on \( U_T/T \),

\[
\alpha^{T}_{(I^2, \partial_t \varphi_T, \nabla^T,(\varphi_T,g))} = \text{ Tr} \langle \partial_t \varphi_T, \nabla^T_{\varphi_T}(\varphi_T,g) \partial_s \varphi_T \rangle_g
\]

\[
\alpha^{T}_{(I^2, \partial_t \varphi_T, D^T_{\varphi_T})} = \text{ Tr} \langle \partial_t \varphi_T, (ad \otimes \nabla^g) D^T_{\varphi_T} \varphi_T \partial_s \varphi_T \rangle_g
\]

\[
\alpha^{T}_{(I^2, \partial_t \varphi_T, \partial_s \nabla^T)} = \text{ Tr} \langle \partial_t \varphi_T, \sum_{i=1}^{n} [\langle \partial_s \nabla^T_{\varphi_T}(\varphi_T,g), \varphi_T^i(y^i) \rangle \otimes \frac{\partial}{\partial y^i} \] \]

and let

\[
\xi^{T}_{(I^2, \partial_t \varphi_T, \nabla^T,(\varphi_T,g))} = \sum_{\mu} \text{ Tr} \langle \partial_t \varphi_T, \nabla^T_{\varphi_T}(\varphi_T,g) \partial_s \varphi_T \rangle_g e_\mu
\]

\[
\xi^{T}_{(I^2, \partial_t \varphi_T, D^T_{\varphi_T})} = \sum_{\mu} \text{ Tr} \langle \partial_t \varphi_T, (ad \otimes \nabla^g) D^T_{\varphi_T} \varphi_T \partial_s \varphi_T \rangle_g e_\mu
\]

\[
\xi^{T}_{(I^2, \partial_t \varphi_T, \partial_s \nabla^T)} = \sum_{\mu} \text{ Tr} \langle \partial_t \varphi_T, \sum_{i=1}^{n} [\langle \partial_s \nabla^T_{\varphi_T}(\varphi_T,g), \varphi_T^i(y^i) \rangle \otimes \frac{\partial}{\partial y^i} \] \]

be their respective dual on \( U_T/T \) with respect to \( h \). Then,

\[
(I^2,1.2) = T_{m-1} \int_{\partial T} i \xi^{T}_{(I^2, \partial_t \varphi_T, \nabla^T,(\varphi_T,g))} + \xi^{T}_{(I^2, \partial_t \varphi_T, D^T_{\varphi_T})} + \xi^{T}_{(I^2, \partial_t \varphi_T, \partial_s \nabla^T)} \text{ vol}_h
\]

\[
+ T_{m-1} \int_U \text{ Tr} \langle \partial_t \varphi_T, (\nabla^T_{\varphi_T}(\varphi_T,g) \nabla^T_{\varphi_T}(\varphi_T,g) \partial_s \varphi_T \rangle_g \text{ vol}_h
\]

\[
+ T_{m-1} \int_U \text{ Tr} \langle \partial_t \varphi_T, (ad \otimes \nabla^g) D^T_{\varphi_T} \varphi_T \partial_s \varphi_T \rangle_g \text{ vol}_h
\]

\[
+ T_{m-1} \int_U \text{ Tr} \langle \partial_t \varphi_T, \sum_{i=1}^{n} [\langle \partial_s \nabla^T_{\varphi_T}(\varphi_T,g), \varphi_T^i(y^i) \rangle \otimes \frac{\partial}{\partial y^i} \] \]

be their respective dual on \( U_T/T \) with respect to \( h \).
The term $\frac{\partial}{\partial s} (I^2.2)$

\[
\frac{\partial}{\partial s} (I^2.2) = T_{m-1} \frac{\partial}{\partial s} \int_U \text{Tr} \sum_{\mu=1}^\infty \left( (ad \otimes \nabla^g) D^T_{g_{\mu}} \partial_\mu \varphi_T, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
= T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( \partial_\mu \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, \partial_\mu D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
= (I^2.2.1) + (I^2.2.2).
\]

(a) Term $(I^2.2.1)$

\[
(I^2.2.1) = T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D^T_{e_\mu} \left( \partial_\mu \varphi_T, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( \sum_{i,j,k} \left[ D^T_{e_\mu} \varphi_T^i (y^j), \partial_\mu \varphi_T^j (y^k) \right] \otimes \nabla^g_{\nabla^g \mu} \frac{\partial}{\partial y^j}, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( \sum_{i,j,k} \left[ D^T_{e_\mu} \varphi_T^i (y^j), \partial_\mu \varphi_T^j (y^k) \right] \otimes \nabla^g_{\nabla^g \mu} \frac{\partial}{\partial y^j}, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( \sum_{i,j,k} \left[ (ad (\partial_\mu \nabla^g))_{(e_\mu)} \varphi_T^i (y^j), \partial_\mu \varphi_T^j (y^k) \right] \otimes \nabla^g_{\nabla^g \mu} \frac{\partial}{\partial y^j}, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h.
\]

The integrand of the first summand captures a related part in the system of equations of motion for $(\varphi, \nabla)$. The integrand of the second summand is tensorial in $\partial_\mu \varphi_T$ and first-order differential operatorial in $\partial_s \varphi_T$. The integrand of the third summand is tensorial in both $\partial_\mu \varphi_T$ and $\partial_s \varphi_T$. The integrand of the fourth summand is tensorial in $\partial_\mu \varphi_T$ and $\partial_s \nabla^T$.

(b) Term $(I^2.2.2)$

\[
(I^2.2.2) = T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, \partial_\mu D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
= T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, \nabla^T_{e_\mu} (\varphi_T, g) \partial_\mu \varphi_T \right)_g \text{vol}_h \\
- T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, (ad \otimes \nabla^g) \partial_\mu \varphi_T, D^T_{e_\mu} \varphi_T \right)_g \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \sum_{\mu} \left( (ad \otimes \nabla^g) D_{e_\mu} \varphi_T \partial_\mu \varphi_T, \sum_i \left[ (\partial_i \nabla^T) (e_\mu), \varphi_T^i (y^j) \right] \otimes \frac{\partial}{\partial y^j} \right)_g \text{vol}_h.
\]

The integrand of the first summand is tensorial in $\partial_\mu \varphi_T$ and first-order differential operatorial in $\partial_s \varphi_T$. The integrand of the second summand is tensorial in both $\partial_\mu \varphi_T$ and $\partial_s \varphi_T$. The integrand of the third summand is tensorial in $\partial_\mu \varphi_T$ and $\partial_s \nabla^T$.

The term $\frac{\partial}{\partial s} (I^2.3)$
\[
\frac{\partial}{\partial s} (I^2.3) = T_{m-1} \frac{\partial}{\partial s} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
= T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \partial_s \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
+ T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
= (I^2.3.1) + (I^2.3.2).
\]

(a) Term (I^2.3.1)

\[
(I^2.3.1) = T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \partial_s \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
= T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
+ T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h.
\]

The integrand of the first summand captures a related part in the system of equations of motion for \((\varphi, \nabla)\). The integrand of the second summand is tensorial in \(\partial_s \varphi_T\) and \(\partial_i \nabla^T\).

(b) Term (I^2.3.2)

\[
(I^2.3.2) = T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
= T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
- T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h \\
+ T_{m-1} \int_U \sum_{\mu=1}^m \left( \sum_{i,j=1}^n \left[ (\partial_i \nabla^T)(e_\mu), \varphi_T^i(y^i) \right] \otimes \frac{\partial}{\partial y^i} \right) \otimes D_{e_\mu} \varphi_T \right)_{g} \text{ vol}_h.
\]

The integrand of the first summand is tensorial in \(\partial_i \nabla^T\) and first-order differential operatorial in \(\partial_s \varphi_T\). The integrand of the second summand is tensorial in \(\partial_i \varphi_T\) and \(\partial_i \nabla^T\). The integrand of the third summand is tensorial in both \(\partial_i \nabla^T\) and \(\partial_s \varphi_T\).

Finally, recall Lemma 3.2.2.4 and note that with the additional assumption at the beginning of this section, all the inner products \(Tr(\cdot, \cdot)\) that appear in the calculation above are defined.

In summary,

**Proposition 7.1.1. [second variation of kinetic term for maps]** Let \((\varphi_T, \nabla^T)\) be a \((\ast_2)\)-admissible \(T\)-family of \((\ast_1)\)-admissible pairs with the additional assumption that
\[ D_{\varphi_T} \subset \text{Comm}(\mathcal{A}_p) \text{ for all } \xi \in T_\ast(X_T/T). \text{ Then,} \]
\[ \frac{\partial}{\partial s} \frac{\partial}{\partial t} E^{\varphi_T}(\varphi_T) \bigg|_{t=0} = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \left( \frac{1}{2} T_{m-1} \sum_U \text{Tr}(D^T \varphi_T, D^T \varphi_T)_g \right) \]
\[ = T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T_T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T_T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T_T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ = T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ = T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ = T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
\[ + T_{m-1} \int_U \left( \frac{\partial}{\partial t} \left( e \varphi_T \right) \right) D^T \varphi_T \]
Here,
\[
\xi^T_{(1^2, \partial_t \varphi_T, \partial_y \varphi_T)} := \sum_{\mu=1}^n Tr \left( \partial_s \partial_t \varphi_T, D^T_{e_{\mu} \varphi_T} \right)_g e_{\mu},
\]
\[
\xi^T_{(1^2, \partial_t \varphi_T, \nabla T(\varphi, g))} = \sum_{\mu} Tr \left( \partial_t \varphi_T, \nabla_{e_{\mu}} T(\varphi, g) \partial_s \varphi_T \right)_g e_{\mu},
\]
\[
\xi^T_{(1^2, \partial_t \varphi_T, D^T_{\varphi_T})} = \sum_{\mu} Tr \left( \partial_t \varphi_T, (ad \otimes \nabla g) D^T_{e_{\mu} \varphi_T} \partial_s \varphi_T \right)_g e_{\mu},
\]
\[
\xi^T_{(1^2, \partial_t \varphi_T, \partial_y \nabla T)} = \sum_{\mu} Tr \left( \partial_t \varphi_T, \sum_{i=1}^n [(\partial_s \nabla T)(e_{\mu}), \varphi^T_i(y^j)] \otimes \frac{\partial}{\partial y^j} \right)_g e_{\mu}
\]
and
\[
F_{\nabla T(\varphi, g)} (\partial_t, e_{\mu}) D^T_{e_{\mu} \varphi_T} = (\partial_s \nabla^T_{e_{\mu}} (\varphi, g) - \nabla^T_{e_{\mu}} (\varphi, g) \partial_s) \sum_{i=1}^n D_{e_{\mu} \varphi_T} (y^j) \otimes \frac{\partial}{\partial y^j}
\]
\[
= \sum_{i=1}^n [(\partial_s \nabla T)(e_{\mu}), D_{e_{\mu} \varphi_T} (y^j)] \otimes \frac{\partial}{\partial y^j} + \sum_{j=1}^n D_{e_{\mu} \varphi_T} (y^j) \sum_{i=1}^n [(\partial_s \nabla T)(e_{\mu}), \varphi^T_i(y^j)] \otimes \nabla^T_{e_{\mu} \varphi_T} \frac{\partial}{\partial y^j}
\]
\[
+ \sum_{j=1}^n D_{e_{\mu} \varphi_T} (y^j) \sum_{j=1}^n \left( D^T_{e_{\mu} \varphi_T} (y^j) \partial_s \varphi_T (y^j) \otimes \nabla^T_{e_{\mu} \varphi_T} \frac{\partial}{\partial y^j} - \partial_s \varphi_T (y^j) D^T_{e_{\mu} \varphi_T} (y^j) \otimes \nabla^T_{e_{\mu} \varphi_T} \frac{\partial}{\partial y^j} \right)
\]

The summands
\[
+ T_{m-1} \int_U Tr \left( \partial_s \partial_t \varphi_T, (D^T_{\sum_{\mu} \nabla^T_{e_{\mu} \varphi_T}} - \sum_{\mu} \nabla^T_{e_{\mu} \varphi_T} D^T_{e_{\mu} \varphi_T} \varphi_T \right)_g \text{vol}_h
\]
\[
+ T_{m-1} \int_U Tr \left( \sum_{\mu} (ad \otimes \nabla g) D^T_{e_{\mu} \varphi_T} (\partial_s \partial_t \varphi_T), D^T_{e_{\mu} \varphi_T} \right)_g \text{vol}_h
\]
\[
+ T_{m-1} \int_U \sum_{\mu=1}^n \left( \sum_{i=1}^n [(\partial_s \partial_t \nabla T)(e_{\mu}), \varphi^T_i(y^j)] \otimes \frac{\partial}{\partial y^j}, D^T_{e_{\mu} \varphi_T} \right)_g \text{vol}_h
\]
will vanish when imposing the equations of motion for \((\varphi, \nabla)\).
If \((\varphi_T, \nabla^T)\) is furthermore a \(\ast_2\)-admissible \(T\)-family of \(\ast_2\)-admissible pairs, Then, the above expression reduces to

\[
\frac{\partial}{\partial s} \frac{\partial}{\partial t} E^{\nabla^T}(\varphi_T)^C = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \left( \frac{1}{2} T_{m-1} \int_U \text{Tr} \left( D^T \varphi_T, (D^T \varphi_T)_{(h,g)} \right) \right)
\]

\[
= T_{m-1} \int_{\partial U} i_{\varphi_T} \langle (\varphi_T, \nabla^T) \rangle_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} i_{\varphi_T} \langle (\varphi_T, \nabla^T) + \varepsilon_{(\varphi_T, \nabla^T)} \rangle_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \text{Tr} \left( \partial_s \partial_t \varphi_T \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \text{Tr} \left( \partial_s \partial_t \varphi_T \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \text{Tr} \left( \partial_s \partial_t \varphi_T \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right)_{g} \text{vol}_h
\]

\[
- T_{m-1} \int_{\partial U} \text{Tr} \left( \partial_s \partial_t \varphi_T \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \left[ \left( \partial_s \partial_t \varphi_T \right) \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right] \otimes \frac{\partial}{\partial y} \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \left[ \left( \partial_s \partial_t \varphi_T \right) \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right] \otimes \frac{\partial}{\partial y} \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \left[ \left( \partial_s \partial_t \varphi_T \right) \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right] \otimes \frac{\partial}{\partial y} \right) \right)_{g} \text{vol}_h
\]

\[
- T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \left[ \left( \partial_s \partial_t \varphi_T \right) \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right] \otimes \frac{\partial}{\partial y} \right) \right)_{g} \text{vol}_h
\]

\[
+ T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \sum_{i,j=1}^{n} \left( \left[ \left( \partial_s \partial_t \varphi_T \right) \left( (D^T \varphi_T, \nabla^T \varphi_T, \partial_t \varphi_T) \right) \right] \otimes \frac{\partial}{\partial y} \right) \right)_{g} \text{vol}_h
\]
If one further imposes the equations of motion on $(\varphi, \nabla)$, then the expression reduces further to

$$
\frac{\partial}{\partial s} \frac{\partial}{\partial t} E^{\nabla} (\varphi T)^C = \frac{\partial}{\partial s} \left( \frac{1}{2} T_{m-1} \int_U \text{Tr} \left( D^T \varphi T, D^T \varphi T \right) (h, g) \, \text{vol}_h \right) \\
= T_{m-1} \int_{\partial U} \left( \frac{\partial}{\partial s} \xi^{(e, \rho, \varphi, T)} \right) \text{vol}_h \\
+ T_{m-1} \int_{\partial U} \left( \frac{\partial}{\partial t} \xi^{(e, \rho, \varphi, T)}, \xi^{(e, \rho, \varphi, T)}, \xi^{(e, \rho, \varphi, T)} \right) \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \left( \partial_s \varphi T, \left( \nabla^{\nabla}_{\sum_{\rho} \nabla_{e_{\rho}}, \varphi T} - \sum_{\rho} \nabla_{\rho} \nabla_{e_{\rho}} \varphi T \right) \partial_s \varphi T \right) \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \left( \partial_t \varphi T, \left( (ad \otimes \nabla^g) D^T_{e_{\rho} \varphi T} \right) \partial_s \varphi T \right) \text{vol}_h \\
- T_{m-1} \int_U \text{Tr} \left( \partial_t \varphi T, \sum_{\mu} F_{\nabla^{\nabla}_{\sum_{\rho} \nabla_{e_{\rho}}}, \varphi T} (\partial_s, e_{\mu}, D^T_{e_{\rho} \varphi T}) \right) \text{vol}_h \\
+ T_{m-1} \int_U \text{Tr} \left( \partial_t \varphi T, \sum_{i=1}^{n} \left( \left( \partial_s \nabla^T (\nabla_{e_{\rho}} \varphi T) \right) \partial_s \varphi T (y^i) \right) \otimes \frac{\partial}{\partial y^i} \right) \text{vol}_h \\
+ T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \sum_{i=1}^{n} \left( \left( \partial_s \nabla^T (\nabla_{e_{\rho}} \varphi T) \right) \partial_s \varphi T (y^i) \right) \otimes \frac{\partial}{\partial y^i} \right) \text{vol}_h \\
- T_{m-1} \int_{\partial U} \sum_{\mu=1}^{m} \left( \sum_{i=1}^{n} \left( \left( \partial_s \nabla^T (\nabla_{e_{\rho}} \varphi T) \right) \partial_s \varphi T (y^i) \right) \otimes \frac{\partial}{\partial y^i} \right) \text{vol}_h \\
+ T_{m-1} \int_{\partial U} \left( \sum_{\mu=1}^{m} \sum_{i,j=1}^{n} \left( \left( \partial_s \nabla^T (\nabla_{e_{\rho}} \varphi T) \right) \partial_s \varphi T (y^i) \right) \otimes \frac{\partial}{\partial y^i} \right) \text{vol}_h .
$$

### 7.2 The second variation of the dilaton term

We now work out the second variation of the (complexified) dilaton term

$$
S^{(\rho, h, \Phi)}_{\text{dilaton}} (\varphi T)^C = \int_U \text{Tr} (d \rho, \varphi T, d \Phi) h \, \text{vol}_h \\
= \int_U \text{Tr} \left( \sum_{\mu} d \rho (e_{\mu}) \left( D^T_{e_{\rho} \varphi T} \Phi \right) \right) \text{vol}_h .
$$

for an $(*_1)$-admissible family of $(*_1)$-admissible pairs $(\varphi T, \nabla^T)$, $T := (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2$ with coordinate $(s, t)$.

It follows from Sec. 6.2 that, due to the effect of the trace map $\text{Tr}$,

$$
\frac{\partial}{\partial t} S^{(\rho, h, \Phi)}_{\text{dilaton}} (\varphi T)^C = \int_U \text{Tr} \sum_{\mu=1}^{m} d \rho (e_{\mu}) \partial_t \left( D^T_{e_{\rho} \varphi T} \Phi \right) \text{vol}_h \\
= \int_U \text{Tr} \sum_{\mu} d \rho (e_{\mu}) D^T_{e_{\rho} \varphi T} \left( \partial_t \varphi T \right) \text{vol}_h .
$$
Thus, due to the effect of the trace map Tr again,

\[
\frac{\partial}{\partial s} \frac{\partial}{\partial t} S^{(\rho,h;\Phi)}_{\text{dilaton}}(\varphi_T)^C = \int_U Tr \sum \frac{dp(\mu)}{\partial_s} D_{\varphi_T}^{(\varphi_T)}((\partial_t \varphi_T)\Phi) \text{vol}_h
\]

\[
= \int_U Tr \sum \frac{dp(\mu)}{\partial_s} D_{\varphi_T}^{(\varphi_T)}((\partial_t \varphi_T)\Phi) \text{vol}_h
\]

\[
= \int_U Tr \sum \frac{dp(\mu)}{\partial_s} D_{\varphi_T}^{(\varphi_T)}((\partial_t \varphi_T)\Phi) \text{vol}_h
\]

\[
+ \int_U Tr \sum \frac{dp(\mu)}{\partial_s} D_{\varphi_T}^{(\varphi_T)}\left(\sum_{i,j} \partial_t \varphi_T^i(y^j) \partial_s \varphi_T^j(y^j) \otimes \left(\frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla g \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right)\Phi\right) \text{vol}_h
\]

\[
= (\Pi^2.1) + (\Pi^2.2).
\]

For Summand \((\Pi^2.1)\), repeating the same argument in Sec. 6.1 for Summand \((I.1.1)\), one concludes that

\[(\Pi^2.1) = \int_{\partial U} i_{\xi^T_{(\Pi^2,\partial_t \varphi_T)\partial_s \varphi_T}} \text{vol}_h
\]

\[+ \int_U \left( \sum \nabla_{\varphi_T} e_{\mu} - \sum \mu dp(\mu) \right) Tr((\partial_s \partial_t \varphi_T)\Phi) \text{vol}_h,
\]

where

\[
\xi^T_{(\Pi^2,\partial_s \varphi_T)} := \sum \mu \left( dp(\mu) Tr((\partial_s \partial_t \varphi_T)\Phi) \right) e_{\mu} \in T_s(U_T/T).
\]

The second summand of Summand \((\Pi^2.1)\) above is the term that captures the \(S^{(\rho,h;\Phi)}_{\text{dilaton}}(\varphi)\)-contribution to the system of equations of motion for \((\varphi, \nabla)\).

With \(\partial_s \partial_t \varphi_T\) replaced by \(\sum_{i,j} \partial_t \varphi_T^i(y^j) \partial_s \varphi_T^j(y^j) \otimes \left(\frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla g \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right)\Phi\), one has similarly

\[(\Pi^2.2) = \int_{\partial U} i_{\xi^T_{(\Pi^2,\partial_t \varphi_T)\partial_s \varphi_T}} \text{vol}_h
\]

\[+ \int_U \left( \sum \nabla_{\varphi_T} e_{\mu} - \sum \mu dp(\mu) \right) Tr\left( \sum_{i,j} \partial_t \varphi_T^i(y^j) \partial_s \varphi_T^j(y^j) \otimes \left(\frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla g \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right)\Phi\right) \text{vol}_h,
\]

where

\[
\xi^T_{(\Pi^2,\partial_t \varphi_T)\partial_s \varphi_T} := \sum \mu \left( dp(\mu) Tr\left( \sum_{i,j} \partial_t \varphi_T^i(y^j) \partial_s \varphi_T^j(y^j) \otimes \left(\frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} - \nabla g \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right)\Phi\right) \right) e_{\mu}
\]

in \(T_s(U_T/T)\). The second summand of Summand \((\Pi^2.2)\) above contributes to the zeroth order terms of the differential operator on \((\partial_s \varphi_T, \partial_t \varphi_T)\) from the second variation of \(S^{(\rho,h;\Phi)}_{\text{standard}}(\varphi, \nabla)\).

In summary,

**Proposition 7.2.1. [second variation of \(S^{(\rho,h;\Phi)}_{\text{dilaton}}(\varphi)^C\)]** For the (complexified) dilaton term

\[
S^{(\rho,h;\Phi)}_{\text{dilaton}}(\varphi)^C := \int_U Tr (dp, \varphi^s d\Phi)_h \text{vol}_h,
\]

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its second variation for a \((\ast_1)\)-admissible family of \((\ast_1)\)-admissible pairs \((\varphi_T, \nabla^T)\), \(T := (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2\) with coordinate \((s, t)\), is given by

\[
\frac{\partial}{\partial s} \frac{\partial}{\partial t} S_{\text{dilaton}}^{(\rho, h; \Phi)}(\varphi_T)^C
= \int_{\partial U} \xi^T_{\Omega^2, \partial_s \varphi_T} + \xi^T_{\Omega^2, \partial_t \varphi_T} \text{vol}_h
+ \int_{U} \left( \sum_{\mu} \nabla_{e_\mu} h e_\mu - \sum_{\mu} e_\mu d\rho(e_\mu) \right) \text{Tr} \left( \left( \frac{\partial}{\partial y^i} - \frac{\partial g}{\partial y^i} \right) \Phi \right) \text{vol}_h 
+ \int_{U} \left( \sum_{\mu} \nabla_{e_\mu} h e_\mu - \sum_{\mu} e_\mu d\rho(e_\mu) \right) \text{Tr} \left( \sum_{i,j} \partial_i \varphi_T^\sharp (y^i) \partial_s \varphi_T^\sharp (y^j) \otimes \left( \frac{\partial}{\partial y^i} - \frac{\partial g}{\partial y^i} \right) \Phi \right) \text{vol}_h ,
\]

where

\[
\xi^T_{\Omega^2, \partial_s \varphi_T} := \sum_{\mu} \left( d\rho(e_\mu) T r \left( \left( \partial_s \varphi_T \right) e_\mu \right) \right) e_\mu
\]

\[
\xi^T_{\Omega^2, \partial_t \varphi_T} := \sum_{\mu} \left( d\rho(e_\mu) T r \left( \sum_{i,j} \partial_i \varphi_T^\sharp (y^i) \partial_s \varphi_T^\sharp (y^j) \otimes \left( \frac{\partial}{\partial y^i} - \frac{\partial g}{\partial y^i} \right) \Phi \right) \right) e_\mu
\]

in \(T_s(U_T/T)\). The integral

\[
\int_{U} \left( \sum_{\mu} \nabla_{e_\mu} h e_\mu - \sum_{\mu} e_\mu d\rho(e_\mu) \right) \text{Tr} \left( \left( \partial_s \varphi_T \right) e_\mu \right) \text{vol}_h
\]

would vanish when imposing the equations of motion of \((\varphi, \nabla)\) after the combination with other Equations-of-Motion capturing parts from the second variation of other terms in \(S_{\text{standard}}^{(\rho, h; \Phi, g, B, C)}(\varphi, \nabla)^C\).
Where we are

The following table summarizes where we are, following the similar steps of fundamental strings

| string theory | D-brane theory |
|---------------|----------------|
| fundamental objects: open or closed string | fundamental objects: Azumaya/matrix $(m-1)$-manifold with a fundamental module with a connection |
| string world-sheet: 2-manifold $\Sigma$ | D-brane world-volume: Azumaya/matrix $m$-manifold with a fundamental module with a connection $(X^{X_A}, E; \nabla)$ |
| string moving in space-time $Y$: differentiable map $f: \Sigma \to Y$ | D-brane moving in space-time $Y$: (admissible) differentiable map $\varphi: (X^{X_A}, E; \nabla) \to Y$ |
| Nambu-Goto action $S_{NG}$ for $f$’s | Dirac-Born-Infeld action $S_{DBI}$ for $(\varphi, \nabla)$’s |
| Polyakov action $S_{Polyakov}$ for $f$’s | standard action $S_{Standard}$ for $(\varphi, \nabla)$’s |
| action for Ramond-Neveu-Schwarz superstrings | ???, cf. [L-Y6: Sec. 5.1] (D(11.2)) |
| action for Green-Schwarz superstrings | ???, cf. [L-Y6: Sec. 5.1] (D(11.2)) |
| quantization | ??? |

(Cf. [L-Y8: Remark 3.2.4: second table]) (D(13.1)). It’s by now a history that as the built-in structure of a string is far richer than that for a point, a physical theory that takes strings as fundamental objects has brought us to where a physical theory that takes only point-particles as fundamental objects cannot reach. Now that a D-brane carries even more built-in structures, are these even-richer-than-string structures all just in vain? Or is a physical theory that takes D-branes as fundamental objects going to lead us to somewhere beyond that from string theories?

Besides a theory in its own right, a theory that takes D-branes as fundamental objects has deep connection with other themes outside. In particular, at low dimensions, that there should be the following connections are “obvious”

1. $(m = 0) \implies$ a new class of matrix models; cf. [L-Y8: Figure 2-1-2] (D(13.1))
2. $(m = 1) \implies$ nature of non-Abelian Ramond-Ramond fields; cf. $e^-$ vs. EM field, [Ja]
3. $(m = 2) \implies$ a new Gromov-Witten type theory; cf. [L-Y3] (D(10.1)), [L-Y4] (D(10.2))

but most details to realize these connections remain far from reach at the moment.

• A reflection at the end of the first decade of the D-project since spring 2007:

    我到為種植,
    我行花未開,
    豈無佳色在,
    留待後人來。

    ~~~ 弘一法師 (李叔同, 1880-1942): *將離淨峰詠菊誌別*

    (English translation by Ling-Miao Chou)
References

[Ar] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Alg. **11** (1969), 532–563.

[Az] G. Azumaya, *On maximally central algebras*, Nagoya Math. J. **2** (1951), 119–150.

[A-N-T] E. Artin, C.J. Nesbitt, and R.M. Thrall, *Rings with minimum condition*, Univ. of Michigan Press, 1944.

[Bl] D. Bleecker, *Gauge theory and variational principles*, Addison-Wesley, 1981.

[B-B] B. Booss and D.D. Bleecker, *Topology and analysis – The Atiyah-Singer Index Formula and gauge-theoretic physics*, English ed. translated by D.D. Bleecker and A. Mader, Springer, 1985.

[B-DV-H] L. Brink, P. Di Vecchia, and P. Howe, *A locally supersymmetric and reparameterization invariant action for the spinning string*, Phys. Lett. **65B** (1976), 471-474.

[B-T] R. Bott and L.W. Tu, *Differential forms in algebraic geometry*, GTM 82, Springer, 1982.

[C-H-L] M. Carmeli, Kh. Hulerhil, and E. Leibowitz, *Gauge fields: classification and equations of motion*, World Scientific, 1989.

[C-T] C.G. Callan, Jr. and L. Thorlacius, *Sigma models and string theory*, in *Particles, strings and supernovae (TASI 88)*, A. Jefficki and C.-I. Tan eds., 795–878, World Scientific, 1989.

[D-E-F-J-K-M-M-W] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, and E. Witten eds., *Quantum fields and strings: a course for mathematicians*, vol. 1 and vol. 2, Amer. Math. Soc. and Inst. Advanced Study, 1999.

[D-K] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Univ. Press, 1990.

[DV-M] M. Dubois-Violette and T. Masson, *SU(n)-connections and noncommutative differential geometry*, J. Geom. Phys. **25** (1998), 104–118. (arXiv:dg-ga/9612017)

[D-Z] S. Deser and B. Zumino, *A complete action for the spinning string*, Phys. Lett. **65B** (1976), 369-373.

[Ei] D. Eisenbud, *Commutative algebra – with a view toward algebraic geometry*, GTM 150, Springer, 1995.

[Eis] L.P. Eisenhart, *Riemannian geometry*, 5th printing, Princeton Univ. Press, 1964.

[E-L] J. Eells, Jr. and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.

[E-S] J. Eells, Jr. and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[G-H-L] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 2nd ed., Springer, 1990.

[G-S-W] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory*, vol. 1: *Introduction*, vol. 2: *Loop amplitudes, anomalies, and phenomenology*, Cambridge Univ. Press, 1987.

[GB-V-F] J.M. Gracia-Bondía, J.C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser, 2001.

[GM-L] M. Gell-Mann and M. Lévy, *The axial vector current in beta decay*, Il Nuovo Cimento **16** (1960), 705–726.

[Ha] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, 1977.

[Hi] N.J. Hicks, *Notes on differential geometry*, Van Nostrand Math. Studies 3, D. Van Nostrand Co., Inc., 1965.

[‘tHo] G. ’t Hooft ed., 50 years of Yang-Mills theory, World Scientific, 2005.

[H-E] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, 1973.

[H-L] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Univ. Press, 2010.

[H-M1] P.-M. Ho and C.-T. Ma, *Effective action for Dp-brane in large RR (p − 1)-form background*, arXiv:1302.6919 [hep-th].

[H-M2] ———, *S-duality for D3-brane in NS-NS and R-R backgrounds*, arXiv:1311.3393 [hep-th].

[H-V] K. Hori and C. Vafa, *Mirror symmetry*, arXiv:hep-th/0002222.

[H-W] P.-M. Ho and Y.-S. Wu, *Noncommutative geometry and D-branes*, Phys. Lett. **B398** (1997), 52 – 60. (arXiv:hep-th/9611233)

[H-Y] P.-M. Ho and C.-H. Yeh, *D-brane in R-R field background*, arXiv:1101.4054 [hep-th].
[Ja] J.D. Jackson, Classical electrodynamics, 2nd ed., John Wiley & Sons, 1990.

[Jo] D. Joyce, Algebraic geometry over $C^\infty$-rings, arXiv:1001.0023v4 [math.AG].

[Ko] S. Kobayashi, Differential geometry of complex vector bundles, Math. Soc. Japan and Princeton Univ. Press, 1987.

[K-N] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. I and vol. II, John Wiley & Sons, 1963 and 1969.

[La] H.B. Lawson, Jr., Lectures on minimal submanifolds, vol. 1, Mono. Mat. 14, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1970.

[Le] R.G. Leigh, Dirac-Born-Infeld action from Dirichlet $\sigma$-model, Mod. Phys. Lett. A4 (1989), 2767–2772.

[L-Y1] C.-H. Liu and S.-T. Yau, Azumaya-type noncommutative spaces and morphism therefrom: Polchinski’s D-branes in string theory from Grothendieck’s viewpoint, arXiv:0709.1515 [math.AG]. (D(1))

[L-S-Y] S. Li, C.-H. Liu, R. Song, S.-T. Yau, Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves, arXiv:0809.2121 [math.AG]. (D(2))

[L-Y2] C.-H. Liu and S.-T. Yau, Nontrivial Azumaya noncommutative schemes, morphisms therefrom, and their extension by the sheaf of algebras of differential operators: D-branes in a B-field background à la Polchinski-Grothendieck Ansatz, arXiv:0909.2291 [math.AG]. (D(5))

[L-Y3] ——–, A mathematical theory of D-string world-sheet instantons, I: Compactness of the stack of Z-semistable Fourier-Mukai transforms from a compact family of nodal curves to a projective Calabi-Yau 3-fold, arXiv:1302.2054 [math.AG]. (D(10.1))

[L-Y4] ——–, A mathematical theory of D-string world-sheet instantons, II: Moduli stack of Z-(semi)stable morphisms from Azumaya nodal curves with a fundamental module to a projective Calabi-Yau 3-fold, arXiv:1310.5195 [math.AG]. (D(10.2))

[L-Y5] ——–, D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds, arXiv:1406.0929 [math.DG]. (D(11.1))

[L-Y6] ——–, D-branes and Azumaya/matrix noncommutative differential geometry, II: Azumaya/matrix supermanifolds and differentiable maps therefrom - with a view toward dynamical fermionic D-branes in string theory, arXiv:1412.0771 [hep-th]. (D(11.2))

[L-Y7] ——–, Further studies on the notion of differentiable maps from Azumaya/matrix manifolds, I. The smooth case, arXiv:1508.02347 [math.DG]. (D(11.3.1))

[L-Y8] ——–, Dynamics of D-branes I. The non-Abelian Dirac-Born-Infeld action, its first variation, and the equations of motion for D-branes — with remarks on the non-Abelian Chern-Simons/Weiss-Zumino term, arXiv:1606.08529 [hep-th]. (D(13.1))

[L-Y9] ——–, More on the admissible condition on differentiable maps $\varphi : (X^A, E; \nabla) \rightarrow Y$ in the construction of the non-Abelian Dirac-Born-Infeld action $S_{DBI}(\varphi, \nabla)$, arXiv:1611.09439 [hep-th]. (D(13.2.1))

[L-Y10] ——–, manuscript in preparation.

[Ma] E. Mazet, La formule de la variation seconde de l’énergie au voisinage d’une application harmonique, J. Diff. Geom. 8 (1973), 279–296.

[M-P] R.S. Millman and G.D. Parker, Elements of differential geometry, Prentice-Hall, 1977.

[Po1] J. Polchinski, Dirichlet-branes and Ramond-Ramond charges, Phys. Rev. Lett. 75 (1995), pp. 4724 - 4727 (hep-th/9510017)

[Po2] ——–, Lectures on D-branes, in “Fields, strings, and duality”, TASI 1996 Summer School, Boulder, Colorado, C. Eathimiou and B. Greene eds., World Scientific, 1997. (arXiv:hep-th/9611050)

[Po3] ——–, String theory, vol. I: An introduction to the bosonic string; vol. II: Superstring theory and beyond, Cambridge Univ. Press, 1998.

[Po4] A.M. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. 103B (1981), 207-210.

[Po5] ——–, Quantum geometry of fermionic strings, Phys. Lett. 103B (1981), 211-213.

[P-S] M.E. Peskin and D.V. Schroeder, An introduction to quantum field theory, Addison-Wesley, 1995.

[Sm] R.T. Smith, The second variation formula for harmonic maps, Proc. Amer. Math. Soc. 47 (1975), 229–236.

[Wi1] E. Witten, Phases of $N = 2$ theories in two dimensions, Nucl. Phys. B403 (1993), pp. 159 - 222. (hep-th/9301042)
[Wi2] ———, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996), pp. 335 - 350. (arXiv:hep-th/9510135)

[Wa] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott Foresmann & Company, 1971.

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