Conditions for up-down asymmetry in the core of tokamak equilibria

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Received 10 January 2014, revised 9 June 2014
Accepted for publication 24 June 2014
Published 30 July 2014

Abstract
A local magnetic equilibrium solution is sought around the magnetic axis in order to identify the key parameters defining the magnetic-surface’s up-down asymmetry in the core of tokamak plasmas. The asymmetry is found to be determined essentially by the ratio of the toroidal current density flowing on axis to the fraction of the external field’s odd perturbation that manages to propagate from the plasma boundary into the core. The predictions are tested and illustrated with experimentally relevant numerical equilibria. Hollow current-density distributions, and hence reverse magnetic shear, are seen to be crucial to bring into the core asymmetry values that are usually found only near the plasma edge.

Keywords: up-down asymmetric equilibria, intrinsic rotation, reversed magnetic-shear configurations

1. Introduction

Plasma turbulence is known to degrade particle and energy confinement in fusion devices, with serious consequences on their performance [1]. However, turbulent transport can be reduced, or even suppressed, in toroidally rotating plasmas by velocity gradients [1–6]. In the absence of external momentum sources, spontaneous (or intrinsic) plasma rotation may arise due to the momentum flux induced by symmetry breaking along magnetic field lines [7–12]. One such symmetry breaking mechanism is yielded by up-down asymmetric equilibria. Unfortunately, early assessments have found that the asymmetry due to the externally shaped plasma boundary largely fails to propagate in to the core [9–11]. The benefits of the induced momentum flux thus appear to be restricted to the outer part of the plasma, with limited success in reducing turbulent transport levels.

In this paper, a local analytic equilibrium is developed near the magnetic axis in order to understand the factors on which the shape of the magnetic surfaces depends and how the asymmetry may be enhanced. We find that reverse magnetic shear configurations can significantly increase the asymmetry on axis and are therefore expected to extend the asymmetry-induced momentum flux deep into the core of tokamak plasmas, thereby improving its confinement properties.

2. Local on-axis equilibria

The distribution of the poloidal-field flux ψ, normalized to an arbitrary constant ψN, in the poloidal section of axisymmetric devices is given by the Grad–Shafranov (GS) equation [13, 14]

\[ -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} - \frac{1}{r^2 \theta} \frac{\partial}{\partial \theta} r R \frac{\partial \psi}{\partial \theta} = R \dot{p}(\psi) + \frac{\dot{Y}(\psi)}{R}, \]

if centrifugal effects are neglected. The static pressure \( p(\psi) \) and the squared poloidal current \( Y(\psi) \) are normalized to \( \mu_0^{-1} \psi_N^2/(a R_0)^2 \) and \( 8 \pi^2 \psi_N^2/(\mu_0 a)^2 \) respectively, the dots denote flux derivatives \( d/d\psi \), and \( r, \theta, \phi \) are right-handed coordinates with the origin displaced by \( R_0 \) from the tokamak’s symmetry axis. The distance \( r \) to the origin is normalized to \( a \), \( \theta \) is a poloidal angle measured from the midplane’s high-field side, \( \phi \) is the toroidal angle, and \( \varepsilon = a/R_0 \) is the inverse aspect ratio, with \( a \) and \( R_0 \) the tokamak’s minor and major radii. The distance \( R \) to the symmetry axis and the height \( Z \) above the midplane, both normalized to \( R_0 \), are

\[ R = 1 - \varepsilon r \cos \theta \quad \text{and} \quad Z = \varepsilon r \sin \theta. \]

We seek a solution to equation (1) in the form [14–16]

\[ \psi(r, \theta) = \psi_0(r) + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \hat{\psi}_n(r, \theta), \]

\[ \psi_n(r, \theta) = \sum_{k=0}^{n} \hat{\psi}_n^{(k)}(r) \cos k\theta + \hat{\psi}_n^{(k)}(r) \sin k\theta. \]
This ansatz is valid for any $r, \theta$, and $\varepsilon$ as long as both series converge, so $\psi_s(r, \theta) \sim 1$ is not formally required. Replacing the ansatz (3) in the GS equation (1) and collecting terms with equal powers of $\varepsilon$, one finds an equation for $\psi_s$, \[
abla^2 \psi_s + r \nabla \psi_s = -\nabla \cdot \left( p_0(r) \hat{Y}_0(r) \right), \tag{4}
\]
in which $p_0(r) = p \left[ \psi_s(r) \right]$, $Y_0(r) = Y \left[ \psi_s(r) \right]$, and the primes denote radial derivatives. Likewise, a sequence of linear inhomogeneous equations, \[
abla^2 \psi_{nk} + r \nabla \psi_{nk} + \left( \nabla^2 \sigma - k^2 \right) \psi_{nk} = b_{nk}, \tag{5}
\]
is found for all other harmonics $\psi_{nk}$ (any of $\psi^{(c)}_{nk}$ or $\psi^{(s)}_{nk}$), with $\sigma(r) = p_0(r) + Y_0(r)$. Each $b_{nk}(r)$ is the $k$th harmonic (either $b^{(c)}_{nk}$ or $b^{(s)}_{nk}$) of the $n^{th}$ source term \[
b_{nk}(r, \theta) = \Delta^m \psi_0 - \hat{p}_n R_n - \hat{Y}_0 \tilde{R}_n - \sum_{m=1}^{n-1} \left( \hat{p}_m R_{n-m} + \hat{Y}_m \tilde{R}_{n-m} - \Delta^m \psi_{n-m} \right), \tag{6}
\]
where the $m$-th order differential operator is defined as \[
\Delta^m = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{R}_m \frac{\partial}{\partial r} \right) - \frac{1}{\varepsilon r} \frac{\partial}{\partial \theta} \left( \tilde{R}_m \frac{\partial}{\partial \theta} \right), \tag{7}
\]
while $\hat{p}_m = (d/ds)^m p(\psi)$, $\hat{Y}_m = (d/ds)^m Y(\psi)$, $R_m = (\partial/\partial s)^m R$, and $\tilde{R}_m = (\partial/\partial s)^m R^{-1}$, evaluated at $\varepsilon = 0$, are radial functions [15, 16]. Because $\psi(r, \theta)$ is intended near the magnetic axis, the latter is made to coincide with the origin demanding that $\nabla \psi$ vanishes at $r = 0$, whence the conditions $\psi_0(0) = \psi_s(0) = \psi^{(e)}_s(0) = 0$. Setting $\psi_0(0) = 0$ makes the poloidal flux vanish at the origin also, being thus defined up to an additive constant.

To obtain $\psi_s(r)$, the source term in equation (4) is taken from the poloidal-plane projection of the force balance, which yields the toroidal current density \[
- J_{\phi}(R, \psi) = R \hat{p}(\psi) + R^{-1} \hat{Y}(\psi) \tag{8}
\]
normalized to $\psi_s / \mu_0 u^2 R_0$. Its zeroth-order term near the origin can be represented by the series \[
- \left[ \hat{p}_0(r) + \hat{Y}_0(r) \right] = J_0 + \frac{J_2}{2} r^2 + \cdots \tag{9}
\]
where $J_0$ is the toroidal current density flowing on axis, $J_1 = - (d/dr) \left[ \hat{p}_0 + \hat{Y}_0 \right]_{|0}$ vanishes since $d/dr = \psi^{(e)}_s’(r)$, and $J_2 = - (d^2/dr^2) \left[ \hat{p}_0 + \hat{Y}_0 \right]_{|0}$, whence $\psi_0(r) = J_0 + \frac{J_2}{2} r^2 + \cdots \tag{10}$

Looking for the next solution $\psi_1(r, \theta)$, the source term \[
b_1(r, \theta) = - \nabla \cdot \left[ \hat{Y}_0 - \hat{p}_0 + \frac{1}{r} \frac{d}{dr} \left( r^2 \psi_0’ \right) \right] \tag{11}
\]
taken from (6) shows that $b^{(c)}_1$ and $b^{(e)}_1$ vanish, whereas \[
b^{(c)}_{11} = \frac{1}{2} \left[ J_0 - 4 \hat{p}_0(0) \right] r^3 + \cdots \tag{12}
\]
Therefore, from equation (5), one finds $\psi^{(c)}_{10} = \psi^{(e)}_{10} = 0$ and \[
\psi^{(c)}_{11} = \frac{1}{16} \left[ J_0 - 4 \hat{p}_0(0) \right] r^3 + \cdots \tag{13}
\]
Similarly, $b^{(c)}_{21}$, $b^{(e)}_{21}$, and $b^{(e)}_{22}$ are seen to vanish also, while \[
b^{(c)}_{20} = B_{20} r^4 + \cdots \quad \text{and} \quad b^{(s)}_{20} = B_{22} r^4 + \cdots \tag{14}
\]
As we shall see, the particular values of $B_{20}$ and $B_{22}$ are not important in what follows. Again, equation (5) yields \[
\psi^{(c)}_{20} = \frac{B_{20} r^4 + \cdots}{16}, \tag{15}
\]
\[
\psi^{(c)}_{22} = \frac{C_0}{2} r^4 + \frac{2B_{22} - \sigma(0)C_0}{24} r^4 + \cdots, \tag{16}
\]
\[
\psi^{(s)}_{22} = \frac{S_0}{2} r^4 - \frac{\sigma(0)S_0 r^4}{24} + \cdots, \tag{17}
\]
and $\psi^{(e)}_{21}(r) = \psi^{(s)}_{21}(r) = 0$, where $C_0$ and $S_0$ are integration constants whose meaning is discussed below.

Thus far, the solutions in equations (10), (13), and (15), plus their combination in the ansatz (3), are valid for any $r$ and $\varepsilon$, if all series converge and sufficient terms are kept. The procedure outlined can therefore proceed to arbitrary powers in $\varepsilon$. However, to get analytically tractable expressions, one must truncate the power series (3) somewhere. This is done dropping all terms $\varepsilon^{m+1}$ (where $m \geq 0$ are integers) smaller than those leading in the combination $\varepsilon^2 \tilde{Y}(r, \theta)$: First, let $\varepsilon \ll 1$ and $r \ll 1$; Then we find $\varepsilon^{m+1} r^{2m+1} \ll \varepsilon^2 r^2$ and all $O(\varepsilon^3)$ terms in the series (3), which have the form $\varepsilon^{m+3} r^{2m+2}$, may be dropped when compared to $\varepsilon^2 r^2 (C_0 \cos 2\theta + S_0 \sin 2\theta) / 4$; Further, the terms $O(\varepsilon^4)$ in equation (15) are also discarded since $\varepsilon^4 r^4 \ll \varepsilon^2 r^2$; If, in addition, one takes $r/\varepsilon \ll 1$, then $\varepsilon^3 r^3 \ll \varepsilon^2 r^2$ and the combination $\varepsilon^3 Y(r, \theta)$ may be dropped altogether because its nonvanishing terms [listed in equation (13)] are all of the form $\varepsilon^3 r^3$; Likewise, $\varepsilon^4 r^4 \ll \varepsilon^2 r^2$ enables all terms $O(\varepsilon^4)$ in equation (10) to be discarded. Finally, the GS solution to lowest order in aspect ratio ($\varepsilon \ll 1$) near the axis ($r \ll \varepsilon$), can be written as \[
\psi(r, \theta) \simeq J_0 \frac{r^2}{4} + \frac{\varepsilon^2}{4} \left( C_0 r^2 \cos 2\theta + S_0 r^2 \sin 2\theta \right). \tag{16}
\]

Notice that $\varepsilon^2 C_0$ and $\varepsilon^2 S_0$ can be of the same size as $J_0$.

### 3. Boundary conditions and plasma profiles

To better understand the physical meaning of the constants \[
C_0 \equiv \psi^{(e)}_{22}(0) \tag{17}
\]
in equation (15), let us write each $\psi_{nk}(r)$ as [15, 17] \[
\psi_{nk}(r) = \frac{\psi_{nk}(r)}{g_n(r)} = \frac{\psi_{nk}(r)}{g_n(r)} - \int_0^r \frac{du}{g_n(u)} \int_0^u \frac{dv}{g_n(v)} b_{nk}(v), \tag{18}
\]
where $g_n(r)$ are homogeneous solutions of equation (5) and $\psi_{nk}(r)$ are boundary conditions at $r_n$. Define also \[
\zeta = \frac{g^{(e)}_n(0)}{g^{(s)}_n(0)} \quad \text{and} \quad \xi = \int_0^r \frac{dv}{u} \left[ \frac{g^{(e)}_n(0)}{u} \right] \int_0^u \frac{dv}{g^{(s)}_n(v)} b^{(s)}_{nk}(v), \tag{19}
\]
and recall that \( b_{22}^{(c)} \) vanishes whereas \( b_{22}^{(e)} \) does not. Hence,
\[
\begin{align*}
C_0 &= \zeta \psi_{22}^{(e)}(r_b) - \xi \quad \text{and} \quad S_0 = \zeta \psi_{22}^{(e)}(r_b) \end{align*}
\] (20)

Next, let \( \psi(r_b, \theta) \) be the angular distribution of the vacuum poloidal-field flux, inferred from external magnetic measurements [18, 19] along some radius \( r_b \) beyond the plasma edge. Then, \( \psi_{22}^{(e)}(r_b) \) and \( \psi_{22}^{(e)}(r_b) \) are the leading terms of the complex 2nd-order Fourier coefficient
\[
C^* + iS^* = \frac{1}{\pi} \int_0^{2\pi} \psi(r_b, \theta) e^{i2\theta} d\theta
\]
\[
= \frac{\epsilon^2}{2} \left[ \psi_{22}^{(e)}(r_b) + i\psi_{22}^{(e)}(r_b) \right] + O(\epsilon^3)
\] (21)

once the ansatz (3) has been recollected. Thus, \( C_0 \propto C^* \) and \( S_0 \propto S^* \), while \( \zeta \) and \( \xi \) in equation (20) depend only on the plasma profiles \( \rho(\psi) \) and \( Y(\psi) \). Interested readers may gain additional insight about equations (16), (19), (20), and (21) by checking the analytic example provided in the appendix.

4. On-axis up-down asymmetry

Setting \( x = r \cos \theta \) and \( y = r \sin \theta \), the equilibrium in equation (16) becomes
\[
\psi(x, y) = \frac{J_0 + \epsilon^2 C_0 x^2 + \frac{\epsilon^2 S_0}{2} xy + \frac{J_0 - \epsilon^2 C_0 y^2}{4}}{4} \quad \text{yields} \quad C_0 \propto C^* \quad \text{and} \quad S_0 \propto S^* \quad \text{for} \quad \epsilon \ll 1.
\] (22)

Its magnetic surfaces are ellipses, all with elongation \( \kappa \) (the major to minor-axis ratio) and tilted by some angle \( \varphi \) such that
\[
\kappa^2 - 1 = \frac{\epsilon^2}{J_0} \sqrt{C_0^2 + S_0^2} \quad \text{and} \quad \tan 2\varphi = \frac{S_0}{C_0}.
\] (23)

Let a closed curve be defined as \( F(x, y) = 0 \), for some function \( F(x, y) \), along which \( \nabla F \) does not vanish and \( \partial F/\partial y = 0 \) at two of its points only (figure 1). These points set the limits \( x_1 \) and \( x_2 \) of the curve’s projection on the line \( y = 0 \) and such curve can thus be split into a top and bottom branch, respectively \( y_1(x) \) and \( y_2(x) \), with \( y_1(x) < y_2(x) \) for \( x_1 < x < x_2 \). So, we define the curve’s asymmetry as the ratio
\[
\eta = \int_{x_1}^{x_2} \left[ y_1(x) - y_2(x) \right] dx \quad \text{of the area shaded in figure 1 to the curve’s total area.}
\] (24)

of the area shaded in figure 1 to the curve’s total area. Once the top and bottom branches are sorted out from the quadratic form (22), equation (24) yields the single value
\[
\eta_0 = \frac{\epsilon^2 |S_0|}{\pi \sqrt{J_0^2 - \epsilon^4 (C_0^2 + S_0^2)}}
\] (25)

for all ellipses sufficiently near the axis. Additionally, if \( \epsilon^4 (C_0^2 + S_0^2) \ll J_0^2 \), the exact equation (25) simplifies to
\[
\eta_0 \approx \frac{\epsilon^2 |S_0|}{\pi |J_0|} \approx \frac{\kappa S^*}{\pi J_0}
\] (26)

Moreover, the relations (23) suffice to write \( \eta_0 \) as
\[
\eta_0 = \frac{\rho_{\text{max}}}{\sqrt{1 + \cot^2 2\varphi}} \quad \text{with} \quad \rho_{\text{max}} = \frac{\kappa^2 - 1}{4\pi \kappa}
\] (27)

the maximum up-down asymmetry at constant \( \kappa \), which is attained when \( \varphi = \pi/4 \) and therefore \( C_0 \ll S_0 \).

Since \( J_0 \) is a property of the core, whereas \( C_0 \) and \( S_0 \) depend on the external field, they may be regarded as independent parameters defining the equilibrium shape: Raising \( J_0 \) in equation (23) produces increasingly circular magnetic surfaces (\( \kappa \sim 1 \)) and thus suppresses locally the shaping imposed externally by \( C^* \) and \( S^* \) via \( C_0 \) and \( S_0 \) in equations (20) and (21); Conversely, the angle \( \varphi \) is set by the external shaping only and does not change with \( J_0 \).

The on-axis safety factor in the cylindrical approximation evaluates to \( q_0 = 2a^2 \psi_{\text{max}} B_0/J_0 \). Therefore,
\[
\eta_0 = \frac{\Sigma^*}{\pi a^2 B_0}
\] (28)

relates \( \eta_0 \) with the ratio of the poloidal-flux’s odd harmonic \( \Sigma^* = \psi_1 S^* \) (in Wb) to the flux of the on-axis toroidal field \( B_0 \) through the poloidal section. Of course, equations (25), (26), and (28) are all estimates of the particular definition (24): higher \( \eta \) does not necessarily imply more momentum flux.

5. Numerical tokamak equilibria

GS solutions are next computed for parameters typical of the TCV device [20], where evidence of momentum flux induced by up-down asymmetry has been reported [21, 22]. Convergent equilibria for finite \( \epsilon \) and \( r \) are obtained by dropping \( O(\epsilon^5) \) terms in the ansatz (3), with equations (5) and (4) yielding a set of equations for \( n, k, \leq 8 \). These are solved using the plasma models
\[
\rho(\psi) = \frac{1}{2} P_0 \left( 1 - \tanh(P_1 \psi - P_2) \right)
\] (29)

\[
\dot{\rho}(\psi) + Y(\psi) = -\left( J_0 + \alpha \psi \right) \exp\left[ -\left( \gamma \psi \right)^2 \right]
\] (30)

where \( P_0, P_1 = 4.2, P_2 = 2.65, J_0, \alpha, \) and \( \gamma \) are constants.
For a given plasma current \( I_p \), the normalized on-axis pressure \( P_0 \) is chosen to keep the normalized \( \beta \) [14] fixed, with \( a = 0.24 \text{ m}, R_0 = 0.91 \text{ m}, \) and \( B_0 = 1.4 \text{ T} \). The remaining parameters are chosen to fit the plasma inside the vessel \((r, \gamma, \psi_N)\) and to adjust its shape \([\psi_{22}^{(2)}]_B, \psi_{22}^{(2)}|_B,\) and \(\psi_{44}^{(2)}|_B = 1, \) all other \(\psi_{nk}|_B = 0, \) with \(\psi_{nk} = \psi_{nk}(R_0)\). These are listed in table 1 for two scenarios (small and large \( I_p \)), each with two configurations (low and high \( J_0 \)). The corresponding numerical equilibria are plotted in figures 2 and 3.

The profiles in figures 2(d) and 3(d) agree with equation (25); similar values of \( \eta \) at the edge result in core asymmetry which is lower for higher values of \( J_0 \). This asymmetry suppression may be thought of as a competition between the imposed external field, which affects \( C_0 \) and \( S_0 \), and the symmetric field induced locally by \( J_0 \). Therefore, to increase the asymmetry on axis, one must decrease \( J_0 \) without making \( I_p \) dwindle to undesirably low values. Indeed, for any current-density profile with a global maximum on axis, the constraint \(|I_0| \lesssim |I_p|/ (\pi a^2)\) places a limit on how much \( J_0 \) can be reduced for a given \( I_p \). This problem is avoided using hollow current-density profiles, as in figure 3(c), where most of the current flows off axis. Besides affording lower \( J_0 \) values (among other advantages for confinement and stability [23, 24]), hollow current profiles induce an asymmetric build-up towards the core: the value of \( \eta \) on axis grows to a local maximum and may become larger than the one at the edge.

The estimate in equation (26) is tested in table 2, where the values of \( J_0, \zeta, \) and \( S^\ast \) with \( \zeta \) and \( S^\ast \) computed from equations (19) and (21) for each configuration in table 1 are used to evaluate \( \eta_0 \). The latter is found to agree rather well with the numerical value \( \eta_0(0) \), which is computed directly from equation (24) for magnetic surfaces near the axis.

Lastly, a large number \((\sim 270)\) of equilibria are computed for a broader set of parameters, with the on-axis current density ranging between 1 and 3 MA m\(^{-2}\), while \( I_p = 75, 200, \) and 350 kA. Other parameters are varied within reasonable bounds to keep the plasma inside the vessel. For each equilibrium, \( S_0 \) is evaluated as \( \psi_{22}^{(0)}(0) \) and the ratio \( \eta(0)/S_0 \) is plotted against \( J_0 \) in figure 4. The agreement with equation (26) is manifest. Also, the dispersion of computed results around the curve, due to nonzero \( \varepsilon^A(C_0^2 + S_0^2) \) in equation (25), is higher for smaller \( J_0 \).

### Table 1. Plasma parameters for numerical TCV-like equilibria: \( I_p \) (kA), \( \psi_N \) (Wb), \( P_0 \) (kPa), and \( J_0 \) (MA m\(^{-2}\)); other parameters are dimensionless.

| \( I_p \) | \( \psi_N \) | \( P_0 \) | \( J_0 \) | \( \alpha \) | \( \gamma \) | \( \psi_{22}^{(2)}|_B \) | \( \psi_{44}^{(2)}|_B \) |
|---|---|---|---|---|---|---|---|
| 76 | 0.01 | 3.74 | 1 | 5.93 | 1.80 | 4 | 5 | 10 |
| 76 | 0.01 | 3.74 | 3 | -7.25 | 1.40 | 3 | 5 | 10 |
| 355 | 0.03 | 17.51 | 1 | 32.54 | 2.08 | 6 | 6 | 200 |
| 355 | 0.03 | 17.51 | 3 | 10.80 | 1.44 | 6 | 6 | 200 |

### 6. Up-down asymmetry, external shaping, and the role of plasma profiles

The idea that up-down asymmetry, driven at the edge by external shaping coils, decreases towards the core as the magnitude of the shaping field drops with increasing distance seems rather intuitive. However, it is disproved by the results plotted in figure 3, where \( \eta_0 \) is seen to grow up to the asymmetry value found at the edge. To understand this, one must keep in mind that up-down asymmetry is a geometric measure of the balance between the shaping produced by symmetric and asymmetric fields, being thus independent of their absolute magnitudes. Indeed, an arbitrarily small asymmetric poloidal-field flux in the core (reflected by \( S_0 \)) can yield arbitrarily high \( \eta_0 \) if the symmetric poloidal-field flux at that location (represented by \( J_0 \)) is suitably decreased, as follows from the relations (26) and as discussed before. Of course, \( \eta_0 \) cannot grow unbounded, since \( J_0^\ast \) cannot be made smaller than \( \varepsilon^A(C_0^2 + S_0^2) \) in the quadratic form (22). Otherwise, the magnetic axis would become hyperbolic, providing in such case a clear illustration of previous theoretical predictions [25].

But the ratio \( S_0/J_0 \) between variables evaluated on-axis is not the only quantity that sets the value \( \eta_0 \). Actually, the on-axis asymmetry is also proportional to the number \( \zeta \) which, according to its definition (20), depends solely on the shape of the radial profile \( \sigma(r) = -dJ_{(0)}^{(0)}/dr, \) where \( J_{(0)}^{(0)} \) is the zeroth-order term of equation (8) in a series of powers of \( \varepsilon \). Recalling again equation (28), the two equilibria in figure 3, and their corresponding rows in table 2, one finds that a three-fold increase in \( q_0 \) raises \( \eta_0 \) only by a factor of roughly \( \varepsilon^2 \), while keeping the odd (or asymmetric) harmonic of the poloidal-field flux at the boundary fixed with the same value \( \mathcal{F} \). Why the growth of \( \eta_0 \) is less than expected stems precisely from the value of \( \zeta \), which decreases by a factor of about \( \varepsilon^2 \) after changing from a less hollow current-density profile to a hollower one.

To understand how the value of \( \zeta \) depends on a particular plasma profile \( \sigma(r) \), one must look closely to the odd harmonic \( \psi_{22}^{(2)}(r) \), as approximated by the third power series in (15). From this expansion near the axis \((\varepsilon \ll 1)\), one readily finds that plasma toroidal current density [related with a non-vanishing \( \sigma(r) \)] contributes to the asymmetric component of the poloidal-field flux only with \( O(\varepsilon^2 r^4) \) and smaller order terms, which can be neglected with respect to the \( O(\varepsilon^2 r^2) \) term arising from the external shaping. Therefore, the value \( \psi_{22}^{(2)}(r) \approx S_0 \) is essentially established by the external shaping and is not significantly changed by toroidal-current-density structures inside the plasma, like the asymmetric current rings enclosing the axis and displayed in figure 5. Conversely, the boundary evaluated value \( \mathcal{F} = \varepsilon^2 \psi_{22}^{(2)}(r_B) \) depends on the toroidal-current-density distribution because the terms of \( \psi_{22}^{(2)}(r) \) involving the profile \( \sigma(r) \) cannot be neglected any more near \( r_B \approx 1 \), allowing one to write it as the sum \( \mathcal{F} = \mathcal{F} + S_0 \) of a plasma and an external contribution. On the other hand, current-density distributions peaked-off-axis are distinguished by positive values of \( dJ_{(0)}^{(0)}/dr = -\sigma(r) \) in the core, which increase as \( J_0 \) gets smaller and the current hollow becomes deeper. Albeit not rigorous near the boundary, the series approximation in (15) indicates that the plasma contribution to \( \psi_{22}^{(2)}(r_B) \) grows with the derivative of the toroidal-current-density profile in the core. Hence, making the current-density...
Figure 2. Magnetic equilibria for small plasma current (rows 1 and 2 in table 1): Equidistant contours start at the same value and are drawn in dark [(a)] and light tones [(b)] for low and high $J_0$ respectively; Midplane profiles of $J(\phi)$ and $\eta$ [(c) and (d)] follow the same tone rule.

Figure 3. Magnetic equilibria for large plasma current (rows 3 and 4 in table 1). All plots drawn as in figure 2.

Table 2. On-axis asymmetry: estimated values ($\eta_0$) against numerical ones [$\eta(0)$]. All variables are dimensionless.

| $q_0$ | $J_0$ | $\varsigma$ | $S^r$ | $\eta_0$ | $\eta(0)$ |
|-------|-------|-------------|------|--------|--------|
| 1     | 2.45  | 6.58        | 3.41 | 0.174  | 0.0574 | 0.0593 |
| 2     | 0.82  | 19.7        | 4.44 | 0.174  | 0.0249 | 0.0252 |
| 3     | 2.45  | 2.07        | 1.50 | 0.209  | 0.0962 | 0.101  |
| 4     | 0.82  | 6.21        | 2.87 | 0.209  | 0.0613 | 0.0636 |

Profile hollower, in order to achieve lower $J_0$ values, results in raising the plasma contribution $S^p$. But, because $S^p$ is being held fixed, this means that the external contribution $S^e$ must decrease, along with $S_0$. This has two consequences: first, the current in the shaping coils drops since less asymmetric external flux is required to keep the same value $S^p$; Secondly, following the second relation in equation (20), the value $\varsigma = \frac{1}{2} \epsilon^2 \frac{\psi S_0}{S^p}$ also diminishes, thus limiting the expected growth in $\eta_0$ caused by the increase in $q_0$ (or, equivalently, the drop in $J_0$).

In summary: Up-down asymmetry in the plasma core is a geometric notion that depends on the relative magnitudes of the symmetric and asymmetric poloidal-field fluxes evaluated on axis, as expressed by the ratio $S_0/J_0$, and not on their absolute values; Plasma currents cannot drive significant contributions to the asymmetric poloidal-field flux near the magnetic axis, which is essentially induced by the external shaping coils and measured by $\frac{1}{2} \epsilon^2 S_0$; Still, asymmetric plasma-current distributions are able to drive an asymmetric poloidal-field flux contribution $S^p$ at the boundary; If $S^p$ is kept constant
S\star measures the ability of the external-field's odd perturbation 'Fundação para a Ciência e Tecnologia' through project Pest-

This work was supported by EURATOM and carried out within the framework of the European Fusion Development Agreement. IST activities were also supported by 'Fundação para a Ciência e Tecnologia' through project Pest-OE/SADG/LA0010/2011. The views and opinions expressed herein do not necessarily reflect those of the European Commission. J. Ball and F. I. Parra were supported by US DOE Grant No. DE-SC008435.

Appendix. Analytic Solovev-type equilibrium.

The validity of the ansatz (3), of the procedure employed to solve the GS equation, and of the approximations leading to its solution displayed in equation (16) can be verified if simple analytic expressions are assigned to the plasma profiles \( \dot{\psi}(\psi) \) and \( \dot{Y}(\psi) \) for which analytic equilibria are known. In addition, this exercise will also illustrate and provide further insight on the concepts embodied in equations (19), (20), and (21).

Let us consider the analytic Solovev equilibrium [29, 30]

\[
\psi(R, Z) = -\frac{1}{8}U R^2 \left( 4Z^2 + R^2 \right) - \frac{1}{2}V \left( R^2 \log R + Z^2 \right) + a_0 R^2 + a_1 Z + a_2 ZR^2 + a_3 R^2 \left( 2Z^2 - R^2 \log R \right) + a_5 \left( 2Z^2 - 9Z^2 R^2 + 3R^2 \left( R^2 - 4Z^2 \right) \log R \right)
\]

(A.1)

which solves the GS equation in equation (1) for constant plasma profiles

\[
\dot{\psi}(\psi) = \epsilon^2 U \quad \text{and} \quad \dot{Y}(\psi) = \epsilon^2 V,
\]

(A.2)

with the coordinate transformation (2) allowing the GS differential operator to be replaced as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) = \frac{\partial^2}{\partial R^2} + \frac{\epsilon^2}{R^2} \frac{\partial^2}{\partial Z^2}
\]

(A.3)

In equation (A.1), each arbitrary coefficient \( a_n \) multiplies an homogeneous solution of the linear GS equation. Of these, \( a_0 \), \( a_1 \), and \( a_2 \) are taken from the on-axis conditions \( \psi(1, 0) = \psi(1, 0) = 0 \). On the other hand,

\[
a_3 = -\frac{1}{2}S_0,
\]

\[
a_4 = 2^{-10} \left[ 23(V - C_0) - 9U - 27c_1 - 15c_2 \right],
\]

(A.4)

\[
a_5 = 2^{-7} \left[ 7V - 9(U + c_1) - 39C_0 - 3c_2 \right], \quad \text{and}
\]

\[
a_6 = 2^{-8} \left( C_0 - U - V + c_1 + c_2 \right)
\]
are written in terms of four new arbitrary constants: $S_0$, $C_0$, $c_1$, and $c_2$. Following equations (8) and (A.2), one defines

$$
\varepsilon^2 U = p_0 \quad \text{and} \quad \varepsilon^2 (U + V) = - J_0, \quad (A.5)
$$

and, invoking again the coordinate transformation (2), the analytic equilibrium in equation (A.1) turns into

$$
\psi(r, \theta) = \frac{J_0}{4} r^2 - \varepsilon J_0 \frac{4}{3} p_0 r^3 \cos \theta + \frac{\varepsilon^2}{2} \left[ B_{20} r^4 + \frac{C_0}{2} r^2 + \frac{B_{22}}{12} r^4 \right] \cos 2\theta + \frac{S_0}{2} r^2 \sin 2\theta + O(\varepsilon^3), \quad (A.6)
$$

where the constants $B_{20}$ and $B_{22}$ are given by

$$
B_{20} = -\frac{1}{8} \left( J_0 + 8 p_0 \right) \quad \text{and} \quad B_{22} = - \frac{1}{8} \left( J_0 + 4 p_0 \right). \quad (A.7)
$$

As expected, the harmonics in equation (A.6) match the leading terms in the general equations (10), (13), and (15), since $p_0$ and $Y_0$ are here assumed to be constants and thus $\sigma(r) = 0$. The homogeneous solutions of the linear equation (5) for $k = 2$ are found to be $g_{2k}(r) \propto r^2$, whence

$$
\xi = \frac{2}{r_B^2} \quad \text{and} \quad \xi = \frac{B_{22}}{6} \frac{1}{r_B^2}, \quad (A.8)
$$

from equations (19). The leading terms of the complex 2nd-order Fourier coefficient, as defined in equation (21), are identified from the solution (A.6) evaluated at $r_B$, giving

$$
\psi^{(c)}(r_B) = \frac{C_0}{2} r_B^2 + \frac{B_{22}}{12} r_B^4 \quad \text{and} \quad \psi^{(s)}(r_B) = \frac{S_0}{2} r_B^2. \quad (A.9)
$$

Equations (A.8) and (A.9) may now combine to arrive at the relations in equation (20), which, recalling equations (21) and (A.7), can also be written in terms of $C*$ and $S*$ as

$$
C_0 = \frac{C^*}{\varepsilon^2 r_B^2} + \frac{1}{16} \left( J_0 + 4 p_0 \right) r_B^2 \quad \text{and} \quad S_0 = \frac{S^*}{\varepsilon^2 r_B^2}. \quad (A.10)
$$

For the particular Solovev example under analysis, the relations (A.10) show how the on-axis coefficients $C_0$ and $S_0$ relate with the poloidal-field flux distribution at the boundary $r_B$, which is described by $C*$ and $S*$. They also clearly show that the even component has an additional contribution arising from the plasma itself, via the term $\xi = \frac{1}{16} \left( J_0 + 4 p_0 \right) r_B^2$, whereas the odd one has none. This fact explains why it is usually much easier to drive ellipticity from the plasma edge towards the magnetic axis than the surface’s tilt needed to achieve the up-down asymmetry.

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