Max-Plus Algebra for Complex Variables and Its Application to Discrete Fourier Transformation

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A generalization of the max-plus transformation, which is known as a method to derive cellular automata from integrable equations, is proposed for complex numbers. Operation rules for this transformation is also studied for general number of complex variables. As an application, the max-plus transformation is applied to the discrete Fourier transformation. Stretched coordinates are introduced to obtain the max-plus transformation whose imaginary part coincides with a phase of the discrete Fourier transformation.

KEYWORDS: Ultradiscretization, complex variables, max-plus algebra, discrete Fourier transformation, cellular automata, stretched coordinates

1. Introduction

The importance of studies on discrete systems generated from integrable equations has recently attracted considerable attention. Because these discrete systems conserve good properties of integrable equations, such systems, known as integrable difference equations and soliton cellular automata, are important as models to describe engineering problems.

Ultradiscretization is introduced as one of the systematic methods to derive discrete systems from differential equations. It is based on an exponential transformation of field variables $u$ into $U$,

$$u = e^{U/\varepsilon}, \quad \varepsilon > 0,$$

(1.1a)

followed by the limiting process $\varepsilon \to +0$. By virtue of a relation

$$\lim_{\varepsilon \to +0} \varepsilon \log \left( e^{A/\varepsilon} + e^{B/\varepsilon} \right) = \max(A, B),$$

(1.1b)

where

$$\max(A, B) = \begin{cases} 
A & (A \geq B), \\
B & (A < B), 
\end{cases}$$

(1.1c)

we can transfer summation and product of positive quantities $a$ and $b$. Let us set $a = e^{A/\varepsilon}$ and $b = e^{B/\varepsilon}$ respectively, and then consider the logarithms of given equations, multiply $\varepsilon$
and take limit of $\varepsilon \to +0$. Variables $A$ and $B$ may depend on parameter $\varepsilon$, and if they remain finite at $\varepsilon \to +0$, we have

$$a + b = e^{A/\varepsilon} + e^{B/\varepsilon} \to \max(A, B), \quad (1.2a)$$

$$ab = e^{(A+B)/\varepsilon} \to A + B. \quad (1.2b)$$

This procedure is usually called “max-plus transformation”. Many nonlinear equations have been transferred to full-discrete systems by this method, and the relations between them have been studied intensively.\(^4\)\(^5\)

However, it is not sufficient to consider the max-plus algebra only for real variables, because all the variables should be positive in order to ensure the relevance of the transformation and limit defined in eqs. (1.1). Since the physical variables can be negative, this restriction is fatal from the viewpoint to discretize solutions of difference equations directly.

In this paper, we aim to introduce a well-defined limiting process for complex variables, which corresponds to a generalization of the max-plus transformation for real numbers. By considering eqs. (1.1) for complex variables, we can obtain a new max-plus type algebra. We can discretize the transformation $U$ of the solutions of difference equations automatically, if the initial values are properly chosen discrete. This method eliminates the difficulty of the max-plus transformation for real variables. Moreover, we can apply the max-plus transformation to discrete Fourier transformation (hereafter we shall abbreviate as DFT), by which we can solve various differential equations.

This paper is organized as follows. In the next section, we shall introduce a max-plus algebra for two complex variables. A generalization of this algebra for more than two variables is given in §3. In §4, we apply this algebra to DFT. The final section is devoted to summary and discussions.

2. Max-Plus Type Transformation for Two Complex Variables

Let us consider the generalization of max-plus transformation for the sum of two complex variables $u_1$ and $u_2$:

$$u \equiv u_1 + u_2. \quad (2.1)$$

Parallel to the method for real variables, we introduce new complex quantities $U_1$ and $U_2$ corresponding respectively

$$u_1 = e^{U_1/\varepsilon} \text{ and } u_2 = e^{U_2/\varepsilon}, \quad (2.2)$$

where $\varepsilon$ is a positive number. Hereafter, we shall write $\varepsilon \log u$ as $W(\varepsilon)$ for simplicity, and are going to derive its explicit expression. Substituting the relations (2.2) into (2.1), we have

$$W(\varepsilon) = \varepsilon \log \left( e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right). \quad (2.3)$$
By the definition of the logarithms on the complex plane, we can write down the real and the imaginary parts of the right-hand side of eq. (2.3) to see

\[ W(\varepsilon) = \varepsilon \ln \left| e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right| + i \varepsilon \arg \left( e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right). \]  

(2.4)

We express \( U_j \) and \( U_2 \) as

\[ U_j = x_j + iy_j \quad (j = 1, 2), \]

where \( x_j, y_j \) are real numbers, and we assume \( x_1 > x_2 \) for the nonce. First, we consider the real part of \( W(\varepsilon) \). From direct calculation, we have

\[ \left| e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right| = \left[ \left( e^{2x_1/\varepsilon} + e^{2x_2/\varepsilon} \right) (1 + \Delta) \right]^{1/2}, \]

\[ \Delta \equiv \sech \frac{x_1 - x_2}{\varepsilon} \cos \frac{y_1 - y_2}{\varepsilon}. \]

(2.5)

From eqs. (2.4) and (2.5), we find the real part of \( W(\varepsilon) \) as

\[ \text{Re} W(\varepsilon) = \frac{\varepsilon}{2} \ln \left( e^{2x_1/\varepsilon} + e^{2x_2/\varepsilon} \right) + \frac{\varepsilon}{2} \ln(1 + \Delta). \]

(2.6)

Due to the fact \( x_1 > x_2 \) and the similar calculation to the max-plus transformation of real variables, the first term of eq. (2.6) yields

\[ \frac{\varepsilon}{2} \ln \left( e^{2x_1/\varepsilon} + e^{2x_2/\varepsilon} \right) \to x_1, \quad \text{as} \ \varepsilon \to +0. \]

We note that \( \Delta \) tends to zero as \( \varepsilon \to 0 \), as long as \( x_1 \neq x_2 \) is satisfied, and \( \ln(1 + \Delta) \) is kept to be bounded. This means that the second term of the right-hand side of eq. (2.6) vanishes as \( \varepsilon \to +0 \). Hence we have

\[ \lim_{\varepsilon \to +0} \text{Re} W(\varepsilon) = x_1 = \text{Re} U_1. \]

(2.7)

Next, we shall consider the imaginary part of \( W(\varepsilon) \), \( \text{Im} W(\varepsilon) \). Because of the relation

\[ \arg \left( e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right) \]

\[ = \arctan \left[ \frac{e^{x_1/\varepsilon} \sin(y_1/\varepsilon) + e^{x_2/\varepsilon} \sin(y_2/\varepsilon)}{e^{x_1/\varepsilon} \cos(y_1/\varepsilon) + e^{x_2/\varepsilon} \cos(y_2/\varepsilon)} \right], \]

we have for an infinitesimal value of \( \varepsilon \) and \( x_1 > x_2 \),

\[ \arg \left( e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right) \simeq \arctan \frac{y_1}{\varepsilon} = \frac{y_1}{\varepsilon}. \]

(2.8)

(2.9)

Multiplying \( \varepsilon \) to the both sides of (2.9) and taking the limit \( \varepsilon \to +0 \), we find

\[ \varepsilon \arg \left( e^{U_1/\varepsilon} + e^{U_2/\varepsilon} \right) \to y_1 = \text{Im} U_1, \quad \text{as} \ \varepsilon \to +0. \]

(2.10)

Hereby from eqs. (2.7) and (2.10), we obtain

\[ \lim_{\varepsilon \to +0} W(\varepsilon) = U_1, \quad \text{for} \ \text{Re} U_1 > \text{Re} U_2. \]

(2.11a)

By similar procedures, we can see the following holds:

\[ \lim_{\varepsilon \to +0} W(\varepsilon) = U_2, \quad \text{for} \ \text{Re} U_1 < \text{Re} U_2. \]

(2.11b)
Lastly, let us consider the formula for the case $x_1 = x_2$. For this case, $\Delta$ defined in (2.5) is given by

$$\Delta = \cos \frac{y_1 - y_2}{\varepsilon}.$$  

As long as $\Delta \neq -1$, that is, a condition

$$\varepsilon \neq \frac{y_1 - y_2}{(2n + 1)\pi}, \quad n: \text{integer},$$  

holds, we can see that $(1 + \Delta)^{\varepsilon/2}$ goes to unity under $\varepsilon \to 0$, and the second term of eq. (2.6) vanishes in the same limit. Then, assuming (2.12) in calculating $\varepsilon \to +0$, $\text{Re} W(\varepsilon)$ yields the same result as in the case $x_1 \neq x_2$. As for $\text{Im} W(\varepsilon)$, eq. (2.8) reduces to

$$\arg \left(e^{U_1/\varepsilon} + e^{U_2/\varepsilon}\right) = \arctan \left[\frac{\sin(y_1/\varepsilon) + \sin(y_2/\varepsilon)}{\cos(y_1/\varepsilon) + \cos(y_2/\varepsilon)}\right]$$

$$= \arctan \tan \frac{y_1 + y_2}{2\varepsilon} = \frac{y_1 + y_2}{2\varepsilon},$$  

(2.13)

and we arrive at a result

$$\lim_{\varepsilon \to +0} W(\varepsilon) = \frac{U_1 + U_2}{2}, \quad \text{for } \text{Re} U_1 = \text{Re} U_2.$$  

(2.14)

Collecting the results (2.11) and (2.14), we obtain

$$\lim_{\varepsilon \to +0} \varepsilon \log u = \begin{cases} 
U_1 & (\text{Re} U_1 > \text{Re} U_2), \\
U_2 & (\text{Re} U_1 < \text{Re} U_2), \\
\frac{U_1 + U_2}{2} & (\text{Re} U_1 = \text{Re} U_2). 
\end{cases}$$  

(2.15)

This relation is considered to be a generalization of max relation defined for real variables. Hereafter, we shall write this operation as $\mathcal{M}(U_1, U_2)$ to distinguish from max$(a, b)$ for real numbers $a$ and $b$.

3. Operation Rules of $\mathcal{M}$ and Its Generalization to More Than Two Variables

We shall compare the operation rules of $\mathcal{M}$ with the max operations for real variables. Hereafter in this section, we assume that $A$, $B$ and $C$ are complex numbers. First, we can immediately see from the definition, the commutative law,

$$\mathcal{M}(A, B) = \mathcal{M}(B, A),$$  

is satisfied as is the case for max operation for real numbers. Secondly, by taking logarithm of a relation

$$a(b + c) = ab + ac,$$

and setting $\varepsilon \to +0$, we can see that the distributive law

$$\mathcal{M}(A + B, A + C) = A + \mathcal{M}(B, C),$$  

(3.2)

is valid for arbitrary variables.
However, the associative law,

$$\mathcal{M}(A, \mathcal{M}(B, C)) = \mathcal{M}(\mathcal{M}(A, B), C),$$

(3.3a)

is true at almost all values except at the case

$$\text{Re } A = \text{Re } B = \text{Re } C.$$

(3.3b)

This is because the result of taking averages usually depends on the order of operations. This situation leads to the difficulty to construct the max-plus algebra for three or more complex variables. Hereafter in this section, we are going to define this operation on general number of variables.

Let us consider a summation of $N$ variables $u_1, \ldots, u_N$:

$$u \equiv u_1 + \cdots + u_N.$$

As in the previous section, we set the variables as

$$u_j = e^{U_j/\varepsilon}, \quad U_j = x_j + iy_j,$$

$$j = 1, \ldots, N.$$  

(3.4)

If the number of the terms which have the largest real parts is two or less, things are the same as the previous section, and we have

$$\mathcal{M}(U_1, \ldots, U_N) = \begin{cases} U_k, & U_k \text{ has the largest real part}, \\ \frac{U_k + U_l}{2}, & U_k \text{ and } U_l \text{ have the largest real parts}. \end{cases}$$

(3.5)

We assume that $U_1, \ldots, U_K$ ($K \geq 3$) have the same largest real parts, denoted as $x$. Calculating $\varepsilon \log u$ under an assumption that $\varepsilon$ is infinitesimal, we have

$$\varepsilon \log u = \varepsilon \log (u_1 + \cdots + u_N)$$

$$\simeq \varepsilon \ln \left| e^{U_1/\varepsilon} + \cdots + e^{U_K/\varepsilon} \right|$$

$$+ i\varepsilon \arg \left( e^{U_1/\varepsilon} + \cdots + e^{U_K/\varepsilon} \right),$$

(3.6)

because all the terms $e^{U_j/\varepsilon}$ ($K + 1 \leq j \leq N$) can be neglected due to the fact

$$e^{(x_j-x)/\varepsilon} \to 0, \quad \text{as } \varepsilon \to +0.$$

Similarly in the previous section, we find an approximation for infinitesimal $\varepsilon$ as

$$\text{Re } (\varepsilon \log u) \simeq \frac{\varepsilon}{2} \left[ \ln e^{2x/\varepsilon} + \ln \sum_{j=1}^{K} \left( \cos \frac{y_j}{\varepsilon} + i \sin \frac{y_j}{\varepsilon} \right) \right]$$

$$= x + \frac{\varepsilon}{2} \ln K + \frac{\varepsilon}{2} \ln(1 + \Delta),$$

(3.7a)
where
\[
\Delta \equiv \frac{1}{K} \sum_{j,l=1 \atop j \neq l}^{K} \cos \frac{y_j - y_l}{\varepsilon}.
\] (3.7b)

We can see that the second term of the right-hand side of (3.7a) goes to zero under \(\varepsilon \to 0\), and it is also the case for the third term, if \(\varepsilon\) satisfies
\[
\frac{1}{K} \sum_{j,l=1 \atop j \neq l}^{K} \cos \frac{y_j - y_l}{\varepsilon} \neq -1.
\] (3.8)

As for the imaginary part of (3.6), because \(U_1, \ldots, U_K\) have the same real part, we have
\[
\text{Im}(\varepsilon \log u) \simeq \varepsilon \arg(e^{U_1/\varepsilon} + \cdots + e^{U_K/\varepsilon}) = \varepsilon \arg(e^{iy_1/\varepsilon} + \cdots + e^{iy_K/\varepsilon}).
\]

Therefore, we obtain a relation
\[
\varepsilon \log u \to x + i \lim_{\varepsilon \to +0} \varepsilon \arg \left( e^{iy_1/\varepsilon} + \cdots + e^{iy_K/\varepsilon} \right).
\]

From the discussions above, we have derived the definition of \(M\) for complex variables \(U_1, \ldots, U_N\). We assume that \(K\) of them, \(U_{j_1}, \ldots, U_{j_K}\), have the same largest real part whose value is \(x\),
\[
U_{j_k} = x + iy_{j_k}, \quad k = 1, \ldots, K.
\] (3.9a)

Then we have
\[
M(U_1, \ldots, U_N)
= \begin{cases} 
U_{j_1}, & \text{for } K = 1, \\
x + i \lim_{\varepsilon \to +0} \varepsilon \arg \left( \sum_{k=1}^{K} e^{iy_{j_k}/\varepsilon} \right), & \text{for } K \geq 2.
\end{cases}
\] (3.9b)

This is the operation of \(M\) for the general number of complex variables. This definition is consistent with eq. (3.5). Unfortunately, although we have defined the \(M\) operation, the associative law (3.3a) is not satisfied. But if the imaginary parts of \(U_j\) \((1 \leq j \leq K)\) is distributed in equal distance, the latter case of (3.9b) can be reduced to
\[
M(U_1, \ldots, U_K) = \frac{1}{K} (U_1 + \cdots + U_K).
\] (3.9c)

The proof of (3.9c) is given in the Appendix.

4. Application of the \(M\) operation to the Formula of DFT

In this section, as an application, we shall apply \(M\) operation for DFT. We shall consider a variable \(u_{k,l}\), which is a discretization of a function \(u(x, t)\), on a region.
\[
-N \leq k \leq N, \quad -M \leq l \leq M.
\] (4.1)
If we apply DFT on \( u_{k,l} \), we have
\[
  u_{k,l} = \sum_{n=-N}^{N} \sum_{m=-M}^{M} c_{n,m} \exp \left( \frac{2\pi i kn}{2N+1} + \frac{2\pi i lm}{2M+1} \right),
\]
(4.2)
where \( c_{n,m} \) is the Fourier coefficient.

Now we define
\[
  \theta = \frac{2\pi kn}{2N+1} + \frac{2\pi lm}{2M+1},
\]
(4.3a)
and \( \epsilon \) as
\[
  \epsilon = \frac{2\pi}{2N+1}.
\]
(4.3b)
In order to transform the phase of DFT in the complex max-plus transformation, we introduce new coordinates, defined by
\[
  \xi = \epsilon n, \quad \nu = \epsilon k,
\]
\[
  \eta = \epsilon m \sqrt{\frac{2N+1}{2M+1}}, \quad \tau = \epsilon l \sqrt{\frac{2N+1}{2M+1}}.
\]
(4.4)
We call these variables stretched coordinates. Substituting (4.4) into (4.3a), we have
\[
  \theta = \frac{\xi \nu + \eta \tau}{\epsilon}.
\]
(4.5)
When we apply the max-plus transformation to \( u_{k,l} \) and \( c_{n,m} \) as
\[
  u_{k,l} = e^{U_{\nu,\tau}/\epsilon}, \quad c_{n,m} = e^{C_{\xi,\eta}/\epsilon},
\]
(4.6)
and setting \( \epsilon \to +0 \), we can get a max-plus equation from eq. (4.2). The limit \( \epsilon \to +0 \) is equivalent to \( N \to \infty \) under a condition where \( M/N \) is finite. We shall assume that the stretched coordinates remain to be constant as \( N \to \infty \). As we can see from eqs. (4.4), the stretched coordinates have discrete values numbered with integers. Then applying suitable scale transformations for these variables, we can assume that the values of \( \xi \) and \( \eta \) are restricted to be integers. If we express \( C_{\xi,\eta} = x_{\xi,\eta} + i y_{\xi,\eta} \), we have
\[
  e^{U_{\nu,\tau}/\epsilon} = \sum_{\xi,\eta \in \mathbb{Z}} e^{x_{\xi,\eta} + i(y_{\xi,\eta} + \xi \nu + \eta \tau)}.
\]
(4.7)
Applying the max-plus transformation for complex variables defined in the previous sections, we obtain
\[
  U_{\nu,\tau} = \mathcal{M} \left( \{ x_{\xi,\eta} + i \phi_{\xi,\eta,\nu,\tau} \} \right),
\]
(4.8)
\[
  \phi_{\xi,\eta,\nu,\tau} = y_{\xi,\eta} + \xi \nu + \eta \tau.
\]
Equation (4.8) means that the application of max-plus transformation to DFT yields the complex number which have the largest amplitude of discrete Fourier coefficient. We note that the trajectory on \( \nu \tau \)-plane, \( U_{\nu,\tau} = \text{const.} \), is presented by the line \( \phi_{\xi,\eta,\nu,\tau} = \text{const.} \), for prescribed \( \xi \) and \( \eta \).
5. Concluding Remarks and Discussions

In this paper, we have proposed a method to construct a max-plus equation for complex variables. By calculating logarithm of variables explicitly for complex variables, a novel operation $\mathcal{M}$ of max-plus type has been introduced. The conventional max-plus transformation has a difficulty that we cannot apply max-plus transformation (1.1) to non-positive definite variables and subtraction terms. For example, an ultradiscretization of the sine-Gordon equation has tried by applying max-plus transformation to a discrete analogue of sine-Gordon equation to avoid this problem.\(^6\) But it seems that this result owes much to the good structure of the original equation. We are convinced that the operation given in this paper dissolves such difficulty and enables us to generate discrete systems from difference equations automatically.

Recently, a way of generalization of max algebra which allows subtraction operation on general field was introduced.\(^7\) The relation between the results listed in this reference and the one presented in this paper is not clear, but the comparison of these methods is one of the worthwhile problems.

Furthermore, we have applied $\mathcal{M}$ to DFT. Introducing stretched coordinates clarified the relation between DFT and complex max-plus transformation. Since discrete Fourier transformation is a common method to solve differential equations, we can expect that solutions of various difference equations are easily discretized by virtue of this method. Moreover, we can expect that $\mathcal{M}$ operation enables us to discretize other types of integral transformation, such as Laplace transformation, and this will yield a systematic way to solve full discrete systems.

At the end of this paper, we shall point out several problems concerning the results. The definition of the operation $\mathcal{M}$ for general number of variables \(^{34}\) does not allow associative law, except for the special case shown in eq. \(^{39c}\). This relation still includes limiting process, and it seems to be difficult to derive the result of transformation for variables given arbitrary. The construction of algebras which satisfy associative law is a future problem.

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Appendix:

We shall prove the formula

$$\mathcal{M}(U_1, \ldots, U_K) = \frac{U_1 + \cdots + U_K}{K},$$

(A-1)
under the condition where all the $U_j$’s have the same real part and their imaginary parts are distributed in equal distance. Let us set

$$U_j = x + iy_j, \quad j = 1, \ldots, K,$$

where

$$y_j = \alpha + \beta(j - 1).$$

The summation of (3.9b) is that of a geometric series with common ratio $e^{i\beta}$. Then we have

$$\sum_{j=1}^{K} e^{iy_j/\varepsilon} = e^{i\alpha/\varepsilon} \sum_{j=1}^{K} e^{i(j-1)\beta/\varepsilon} = \frac{1 - e^{i\beta K/\varepsilon}}{1 - e^{i\beta/\varepsilon}} = \sin(\beta K/2\varepsilon) \exp \left\{ \frac{i}{\varepsilon} \left[ \alpha + \beta(1 - K) \right] \right\}.$$

The argument of this quantity is derived as

$$\arg \left( \sum_{j=1}^{K} e^{iy_j/\varepsilon} \right) = \frac{1}{\varepsilon} \left[ \alpha + \beta(K - 1) \right].$$

Under the condition (A·2), we find from eq. (A·4) that

$$\mathcal{M}(U_1, \ldots, U_K) = x + i \left[ \alpha + \beta(K - 1) \right].$$

This is no other than the arithmetic mean of $U_1, \ldots, U_K$ defined in (A·2). Hereby we have proved eq. (3.9c).
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