SUPERCENTRALIZERS FOR DEFORMATIONS OF THE PIN OSP DUAL PAIR

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Abstract. In recent work, we examined the algebraic structure underlying a class of elements supercommuting with a realization of the Lie superalgebra \(\mathfrak{osp}(1|2)\) inside a generalization of the Weyl Clifford algebra. This generalization contained in particular the deformation by means of Dunkl operators associated with a real reflection group, yielding a rational Cherednik algebra instead of the Weyl algebra. The aim of this work is to show that this is the full supercentralizer, give a (minimal) set of generators, and to describe the relation with the \((\text{Pin}(d), \mathfrak{osp}(2m + 1|2n))\) Howe dual pair.

1. Introduction

In recent work [8], we started the investigation of the algebraic structure underlying a class of elements supercommuting with a realization of the Lie superalgebra \(\mathfrak{osp}(1|2)\) inside a generalization of the Weyl Clifford algebra. This generalization contained in particular the deformation by means of Dunkl operators associated with a real reflection group, yielding a rational Cherednik algebra instead of the Weyl algebra. The aim of this work is to show that this is the full supercentralizer inside the tensor product of rational Cherednik \(H_\kappa\) and a Clifford algebra, give a (minimal) set of generators, and to describe the relation with the \((\text{Pin}(d), \mathfrak{osp}(2m + 1|2n))\) Howe dual pair.

An explicit realisation of \(H_\kappa\) is given by means of Dunkl operators (for the elements of \(V\)) and coordinate variables (the elements of \(V^*\)), which gives a natural (faithful) action on the polynomial space \(S(V^*)\).

We will work over \(\mathbb{C}\), the field of complex numbers with \(i^2 = -1\).

Throughout, \([\cdot, \cdot]\) will denote the skew-supersymmetric operation on a Lie superalgebra or the supercommutator (2.3). The notation \(\{\cdot, \cdot\}\) will denote the antisupercrmmutator (2.3). Moreover, a sign above the comma will sometimes, mostly in Section 4.3, be used to indicate the actual sign used in a(n anti)supercommutator. For instance, if \(a\) and \(b\) are odd then \([a, b] = ab + ba\), so we will write \([a; b]\), while if \(a\) or \(b\) is even, we have \([a; b] = ab - ba\).

Tensor products are assumed to be \(\mathbb{Z}_2\)-graded, unless stated otherwise. The notation \(\circ\) will be used for the supersymmetric tensor product (3.34).

Notations are not final. \(O_u = \tilde{\sigma}_u\) for \(u \in V^*\).

2. Lie superalgebras

2.1. Preliminaries. Denote \(\bar{0}\) and \(\bar{1}\) the elements of \(\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}\), the residue class ring mod 2. The term superspace is used to refer to a \(\mathbb{Z}_2\)-graded vector space \(V = V_0 \oplus V_1\), and superalgebra for a \(\mathbb{Z}_2\)-graded algebra. The parity or \((\mathbb{Z}_2\)-degree of a homogeneous element \(a \in V_j\) is denoted by \(|a| = j \in \mathbb{Z}_2\). The parity reversing

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functor \( \Pi \) sends a superspace \( V = V_0 \oplus V_1 \) to the superspace \( \Pi(V) \) with the opposite \( \mathbb{Z}_2 \)-grading: \( \Pi(V_j) = V_{j+1} \) for \( j \in \mathbb{Z}_2 \).

A Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a superalgebra whose product \([\cdot, \cdot]\) satisfies, for homogeneous elements \( x, y \in \mathfrak{g}_0 \cup \mathfrak{g}_1 \) and \( z \in \mathfrak{g} \),

\[
[x, y] = -(-1)^{|x||y|}[y, x] \quad \text{(super skew-symmetry)}
\]

\[
[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] \quad \text{(super Jacobi identity)}.
\]

A bilinear form \( b \) on a Lie superalgebra \( \mathfrak{g} \) is called invariant if \( b([x, y], z) = b(x, [y, z]) \) for all elements \( x, y, z \in \mathfrak{g} \).

For a superspace \( V \), the general linear Lie superalgebra \( \mathfrak{gl}(V) \) is formed by equipping the associative superalgebra \( A = \text{End}(V) \) with the supercommutator

\[
[a, b] = ab - (-1)^{|a||b|}ba,
\]

for \( a, b \in A \) homogeneous elements for the \( \mathbb{Z}_2 \)-grading on \( A \). Denote \( \mathbb{C}^{m|n} \) for the superspace \( V = V_0 \oplus V_1 \) with \( V_0 = \mathbb{C}^m \) and \( V_1 = \mathbb{C}^n \). In this case, the notation \( \mathfrak{gl}(m|n) \) is used instead of \( \mathfrak{gl}(V) \).

A bilinear form \( b \) on a superspace \( V = V_0 \oplus V_1 \) is called consistent or even, if for \( i, j \in \mathbb{Z}_2 \), one has \( b(V_i, V_j) = 0 \) unless \( i + j = 0 \). A consistent bilinear form \( b \) is said to be supersymmetric (resp. skew-supersymmetric), if \( b|_{V_0 \times V_0} \) is symmetric (resp. skew-symmetric) and \( b|_{V_1 \times V_1} \) is skew-symmetric (resp. symmetric). The subspace where \( b \) restricts to a skew-symmetric form necessarily has even dimension.

If \( b \) is a non-degenerate, consistent, supersymmetric or skew-supersymmetric, bilinear form on \( V \), the orthosymplectic Lie superalgebra \( \mathfrak{osp}(b) = \mathfrak{osp}(V, b) \) is the subalgebra of \( \mathfrak{gl}(V) \) that preserves \( b \).

As Lie superalgebras \( \mathfrak{osp}(V, b) \cong \mathfrak{osp}(\Pi(V), \Pi(b)) \), where \( \Pi(b) \) is the form on \( \Pi(V) \) induced by means of the parity reversing functor \( \Pi \). If \( b \) is supersymmetric, then \( \Pi(b) \) is skew-supersymmetric and vice versa.

For \( V = \mathbb{C}^M^{2n} \) with the standard supersymmetric form, the associated orthosymplectic Lie superalgebra is denoted by \( \mathfrak{osp}(M|2n) \) or \( \mathfrak{osp}^{sk}(M|2n) \).

For \( V = \mathbb{C}^{2n|M} \) with a skew-supersymmetric form, the associated orthosymplectic (or symplectico-orthogonal) Lie superalgebra is \( \mathfrak{sp}(2n|M) \) or \( \mathfrak{osp}^{sk}(2n|M) \).

Given the isomorphism between them via \( \Pi \), the notation \( \mathfrak{osp}(M|2n) \) is sometimes used to refer to either of them.

### 2.2. \( \mathfrak{osp}(1|2) \)

The Lie superalgebra \( \mathfrak{osp}(1|2) \) has a one-dimensional Cartan subalgebra \( h = \{h\} \) and root system \( \Phi = \{\pm 2\delta\} \cup \{\pm \delta\} \), where \( \delta \in h^* \) is the dual of \( h \).

The even subalgebra is \( \mathfrak{osp}(1|2)_0 \cong \mathfrak{sp}(2) \cong \mathfrak{sl}(2) \) with root system \( \{\pm \delta\} \).

The odd root vectors \( e_{\delta}, e_{-\delta} \) satisfy the relations

\[
[e_{\delta}, e_{-\delta}] = (e_{\delta}, e_{-\delta})h_{\delta}, \quad [h, e_{\pm \delta}] = \mp e_{\pm \delta},
\]

where \( h_{\delta} = (\delta, \delta)h \) is the coroot of \( \delta \). Here, \( \langle \cdot, \cdot \rangle \) denotes the unique (up to a constant factor) non-degenerate consistent invariant supersymmetric bilinear form on \( \mathfrak{g} \), and also the (symmetric) non-degenerate restriction of the form to \( h \), and the induced form on \( h^* \).

With the following normalization for the root vectors

\[
F^\pm := e_{\pm \delta}/\sqrt{(e_{\delta}, e_{-\delta})(\delta, \delta)}, \quad E^\pm := \pm [e_{\pm \delta}, e_{\mp \delta}]/(2(e_{\delta}, e_{-\delta})(\delta, \delta)),
\]

and denoting also \( H = h \), we have \( \mathfrak{osp}(1|2) = \text{span}\{H, F^\pm, E^\pm\} \) with the relations that have nonzero right-hand side given by

\[
[F^+, F^-] = H, \quad [H, F^\pm] = \pm F^\pm, \quad [F^\pm, F^\mp] = \pm 2E^\pm,
\]

\[
[E^+, E^-] = H, \quad [H, E^\pm] = \pm 2E^\pm, \quad [F^\pm, E^\mp] = F^\mp.
\]

The even subalgebra is \( \text{span}\{H, E^\pm\} \cong \mathfrak{sl}(2) \).
2.3. Realization and centralizer. Let $A = A_0 \oplus A_1$ be an associative unital superalgebra. Using the supercommutator (2.3), if there are elements $e_\delta, e_{-\delta} \in A_1$ satisfying
\[(2.7)\]
\[[[e_\delta, e_{-\delta}], e_{\pm \delta}] = \pm C e_{\pm \delta},\]
for a non-zero constant $C$, then there is a realization of $\mathfrak{osp}(1|2)$ in $A$. The constant $C$ is related to the bilinear forms on $\mathfrak{osp}(1|2)$ and on $\mathfrak{h}^*$ by $C = (e_\delta, e_{-\delta})(\delta, \delta)$. The elements $e_\delta, e_{-\delta} \in A_1$ can be rescaled as in (2.5) to have the commutation relations (2.6).

Now, assume that we have a realization $\pi : \mathfrak{osp}(1|2) \to A$ of $\mathfrak{osp}(1|2)$ in $A$, with the product given by the supercommutator (2.3) in $A$.

For a subspace $B \subset A$, the supercentralizer in $A$ is
\[(2.8)\]
\[\text{Cent}_A(B) = \{ a \in A \mid [a, b] = 0 \text{ for all } b \in B \} ,\]
The idea is to describe the supercentralizer of $\mathfrak{osp}(1|2)$ in $A$. Hereto, we consider an adjoint action of $\mathfrak{osp}(1|2)$ on $A$:
\[(2.9)\]
\[\mathfrak{osp}(1|2) \times A \to A: g \mapsto (g, a) \mapsto [\pi(g), a].\]
For the action (2.9) of $\mathfrak{osp}(1|2)$, in the representation space $A$, every element of the centralizer $\text{Cent}_A(\mathfrak{osp}(1|2))$ is a copy of the one-dimensional, trivial module. Conversely, the isotypic component of the trivial module is precisely $\text{Cent}_A(\mathfrak{osp}(1|2))$.

The following proposition gives a way to determine $\text{Cent}_A(\mathfrak{osp}(1|2))$ from the centralizer of the even subalgebra $\text{Cent}_A(\mathfrak{osp}(1|2)_0)$. To this end, we consider the following (even) elements in the universal enveloping algebra $U(\mathfrak{osp}(1|2))$:
\[(2.10)\]
\[P_\pm := 1 - F^- F^+ \quad \text{and} \quad P_- := 1 + F^+ F^-,\]
where we use the normalization (2.5).

**Proposition 2.1.** For $\mathfrak{osp}(1|2)$, realized in an associative unital superalgebra $A$, with even subalgebra $\mathfrak{osp}(1|2)_0 \cong \mathfrak{sl}(2)$, one has
\[\text{Cent}_A(\mathfrak{osp}(1|2)) = P_+ \text{Cent}_A(\mathfrak{osp}(1|2)_0) = P_- \text{Cent}_A(\mathfrak{osp}(1|2)_0),\]
with $P_\pm$ acting on $A$ as in (2.9), that is
\[P_\pm : A \to A: a \mapsto P_\pm(a) = a \mp [F^-, [F^+, a]].\]

**Proof.** First, note that, by means of the relations (2.6),
\[(2.11)\]
\[P_- = 1 + F^+ F^- = 1 + (-F^- F^+ + H) = P_+ + H ,\]
so when $[H, \cdot]$ acts by zero, as is the case on $\text{Cent}_A(\mathfrak{osp}(1|2)_0)$, the actions of $P_-$ and $P_+$ coincide.

By definition, we have $\text{Cent}_A(\mathfrak{osp}(1|2)) \subset \text{Cent}_A(\mathfrak{osp}(1|2)_0)$ and $P_+ a = a$ for $a \in \text{Cent}_A(\mathfrak{osp}(1|2))$, hence $\text{Cent}_A(\mathfrak{osp}(1|2)) \subset P_+ \text{Cent}_A(\mathfrak{osp}(1|2)_0)$.

To prove the other inclusion, let $a \in \text{Cent}_A(\mathfrak{osp}(1|2)_0)$, we show that $P_+(a) \in \text{Cent}_A(\mathfrak{osp}(1|2))$. We have, using the super Jacobi identity (2.2),
\[
[F^+, P_+(a)] = [F^+, a] - [F^+, [F^-, [F^+, a]]] \\
= [F^+, a] - ([F^+, F^-], [F^+, a] + [F^-, [F^+, a]]) \\
= [F^+, a] - [H, [F^+, a]] + [F^-, [F^+, a]] \\
= [F^+, a] - [H, [F^+, a]] - [F^+, [H, a]] \\
= [F^+, a] - [F^+, a].
\]
Meanwhile, for $a \in \text{Cent}_A(\mathfrak{osp}(1|2)_0)$, we also have
\[
[F^-, P_+(a)] = [F^-, a] - [F^-, [F^-, [F^+, a]]] \\
= [F^-, a] + [F^-, [F^+, a]]
\]
$= [F^-, a] + [[E^-, F^+], a] + [F^+, [E^-, a]]$

$= [F^-, a] - [F^-, a].$

As $F^\pm$ generate $\mathfrak{osp}(1|2)$, this proves the other inclusion $P_+ \, \text{Cent}_A(\mathfrak{osp}(1|2)_0) \subset \text{Cent}_A(\mathfrak{osp}(1|2))$. \hfill $\Box$

The next result gives a way to obtain, under certain conditions, the centralizer $\text{Cent}_A(\mathfrak{osp}(1|2)_0)$, which can then be used to describe the supercentralizer $\text{Cent}_A(\mathfrak{osp}(1|2))$ by the previous result.

**Proposition 2.2.** Let $\mathfrak{sl}(2)$ be realized in a unital associative algebra $A$ by the elements $e_\alpha, e_{-\alpha}, h_\alpha$ satisfying the commutation relations

$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_{\pm\alpha}] = \pm (\alpha, \alpha)e_{\pm\alpha}.$

If $\text{Cent}_A(\mathfrak{h})$ decomposes into only finite-dimensional irreducible $\mathfrak{sl}(2)$-modules for the adjoint action of $\mathfrak{sl}(2)$ on $A$, then

$$\text{Cent}_A(\mathfrak{sl}(2)) = P_\alpha \, \text{Cent}_A(\mathfrak{h}) = P_{-\alpha} \, \text{Cent}_A(\mathfrak{h}),$$

where

$$P_\alpha = \sum_{k \geq 0} \frac{(-2)^k e^k_{-\alpha}}{(\alpha, \alpha)^k k!(k+1)!},$$

with the elements of $\mathfrak{osp}(1|2)$ acting on $A$ via the action (2.9).

**Proof.** The elements of $\text{Cent}_A(\mathfrak{h})$ form the space of weight zero for the action of $\mathfrak{sl}(2)$ given by (2.9). Assume $\text{Cent}_A(\mathfrak{h})$ decomposes into only finite-dimensional irreducible $\mathfrak{sl}(2)$-modules. For $v \in \text{Cent}_A(\mathfrak{h})$, the sum in (2.12) reduces to a finite one when acting on $v$, and as $e_\alpha P_\alpha v = 0$, which follows via

$$e_\alpha e_{-\alpha}^k e^{k}_\alpha = e^k_{-\alpha} e^{k+1}_\alpha + k e^{k-1}_{-\alpha} (h_\alpha - (\alpha, \alpha)(k-1)/2) e^k_\alpha$$

$$= e^{k+1}_\alpha e_{-\alpha}^k + k e^{k}_{-\alpha} e^k_\alpha (h_\alpha + (\alpha, \alpha)(k+1)/2),$$

so $P_\alpha v$ is a highest weight vector of a finite-dimensional $\mathfrak{sl}(2)$-module with weight zero, hence a trivial module. \hfill $\Box$

**Remark 2.3.** The formulas (2.10) and (2.12) correspond to a so-called “extremal projector”, see for instance [2, 23, 24], when acting on a space of weight zero.

For $\mathfrak{g}$ a basic classical Lie (super)algebra (or generalization thereof), the extremal projector $P$ is an element of an extension of the universal enveloping algebra $U(\mathfrak{g})$ containing formal power series with coefficients in the field of fractions of $U(\mathfrak{h})$, also called a transvector algebra or Mickelsson-Zhelobenko algebra (localization with respect to $\mathfrak{h}$) [24, 21]. The element $P$ is the unique nonzero solution to the equations

$$P^2 = P, \quad e_\alpha P = 0 = P e_{-\alpha}, \quad \text{for all positive roots } \alpha \in \mathfrak{g}. $$

Hence, when acting on a representation space it would project onto highest weight vectors.

However, for Lie superalgebras with isotropic roots, when acting on a space of weight zero the formula for the extremal projector $P$ can result in denominators becoming zero.

We now consider a lemma with some properties of $P_\pm$ that will be used in Section 4.2. Similar properties hold for $P_\alpha$.

**Lemma 2.4.** For a realization of $\mathfrak{osp}(1|2)$ in $A$, and $P_\pm$ given by (2.10), let $a, b, c \in A$, then

$$P_\pm (a + b) = P_\pm (a) + P_\pm (b).$$
If \( a \in \text{Cent}_A(\mathfrak{osp}(1|2)) \), then
\[
P_\pm(a) = a.
\]
If \( a \in \text{Cent}_A(\mathfrak{osp}(1|2)) \) or \( b \in \text{Cent}_A(\mathfrak{osp}(1|2)) \), then
\[
P_\pm(ab) = P_\pm(a)P_\pm(b).
\]
If \( a, c \in \text{Cent}_A(\mathfrak{osp}(1|2)) \), then
\[
P_\pm(abc) = aP_\pm(b)c.
\]

**Proof.** The first relation follows immediately from the bilinearity of the supercommutator and the second relation from the definition of \( \text{Cent}_A(\mathfrak{osp}(1|2)) \).

For the third result, assume \( a \) is a homogeneous element for the \( \mathbb{Z}_2 \)-grading on \( A \). With the action of \( \mathfrak{osp}(1|2) \) on \( A \) given by (2.9), we have
\[
P_+(ab) = ab - F^- F^+(ab) = ab - F^-(F^+(a)b + (-1)^{|a|}aF^+(b)).
\]
On the one hand, if \( a \in \text{Cent}_A(\mathfrak{osp}(1|2)) \), then \( F^\pm(a) = 0 \), so this becomes
\[
P_+(ab) = ab - (-1)^{|a|}(F^-(a)F^+(b) + (-1)^{|a|}a(F^- F^+ b)) = aP_+(b).
\]
On the other hand, if \( b \in \text{Cent}_A(\mathfrak{osp}(1|2)) \), so \( F^\pm(b) = 0 \), this becomes
\[
P_+(ab) = ab - (F^- F^+ a)b = P_+(a)b.
\]
The final relation follows in the same manner. \( \square \)

**Example 2.5.** In the universal enveloping algebra \( U(\mathfrak{osp}(1|2)) \), the centralizer of the even subalgebra is generated by the element \( F^+ F^- - F^- F^+ \) and the constants. This follows by classical invariant theory. Indeed, using (2.6), we have
\[
[E^\pm, F^+ F^- - F^- F^+] = \mp(F^\pm)^2 \pm (F^\pm)^2 = 0.
\]

Now, applying \( P_+ \) and using (2.6), we find
\[
P_+(F^+ F^-) = F^+ F^- - [F^-, [F^+, F^+ F^-]]
= F^+ F^- - [F^-, 2E^+ F^+ - F^+ H]
= F^+ F^- - 2F^+ F^+ + 4E^+ E^- + H^2 - F^+ F^-
= H^2 + 4E^+ E^- - 2F^+ F^-,
\]
with a similar expression for
\[
P_+(F^- F^+) = F^- F^+ - [F^-, [F^-, F^+ F^+]]
= F^- F^+ - [F^-, HF^+ - 2F^- E^+]
= F^- F^+ - F^- F^+ - H^2 - 4E^+ E^- - 2F^- F^+
= -H^2 - 4E^+ E^- - 2F^- F^+.
\]

Combining the two yields that \( P_+(F^+ F^- - F^- F^+) \) is proportional to the quadratic Casimir element of \( U(\mathfrak{osp}(1|2)) \):
\[
(2.14) \quad \Omega_{\mathfrak{osp}} = H^2 + 2(E^+ E^- + E^- E^+) - (F^+ F^- - F^- F^+),
\]
where \( \Omega_{\mathfrak{sl}(2)} = H^2 + 2(E^+ E^- + E^- E^+) \) is the quadratic Casimir element of \( U(\mathfrak{sl}(2)) \). Note that \( F^+ F^- - F^- F^+ \) is related to the \( \mathfrak{osp}(1|2) \) Scasimir element [1]:
\[
(2.15) \quad S = F^+ F^- - F^- F^+ + 1/2,
\]
which squares to \( S^2 = \Omega_{\mathfrak{osp}} + 1/4 \). The Scasimir element \( S \) commutes with the even elements and anticommutes with the odd elements, so it antischur commutes with \( U(\mathfrak{osp}(1|2)) \) since its parity is even. By the above we also have
\[
(2.16) \quad P_\pm(S) = 2\Omega_{\mathfrak{osp}} + 1/2 = 2S^2.
\]
2.4. Generalized symmetries. Let $A$ be an associative unital (super)algebra. We say that an element $a \in A$ is a generalized symmetry of $F \in A$ if there exists $b \in A$ such that $Fa = bF$. Note that $a$ preserves the kernel of $F$.

Extremal projectors and transvector algebras can be used to construct generalized symmetries of either the positive or negative root vectors of a Lie (super)algebra realized in $A$, see also [24, 21]. Note that the extremal projectors can contain fractions of $U(\mathfrak{h})$, so depending on the algebra $A$ one works in, a multiplication by elements of $U(\mathfrak{h})$ can be required to cancel denominators.

We now consider the case of $\mathfrak{osp}(1|2)$, which we will use in Section 4.4 for an explicit realization. Define

$$Q^\pm : A \to A : a \mapsto Q^\pm(a) = (H \pm 1)a \mp F^+[F^\pm, a].$$

Proposition 2.6. Let $a \in A$ be such that $[E^-, a] = bF^-$ for some $b \in A$, then

$$Q^-(a) = (H - 1)a - F^+[F^-, a]$$

is a generalized symmetry of $F^-$. 

Proof. Assume $a$ is homogeneous for the $\mathbb{Z}_2$-grading

$$F^-Q^-(a) = F^-(H - 1)a - F^-F^+[F^-, a]$$

$$= HF^-a - (H - F^+F^-)[F^-, a]$$

$$= (-1)^{|a|}HaF^- + F^+F^-(F^-a - (-1)^{|a|}aF^-)$$

$$= (-1)^{|a|}HaF^- - F^+F^-a - F^+(-1)^{|a|}aF^-$$

$$= (-1)^{|a|}HaF^- - F^+(aE^- + bF^-) - (-1)^{|a|}F^+F^-aF^-$$

$$= (-1)^{|a|}(Ha + (-1)^{|a|}F^+F^- - (-1)^{|a|}F^+b - F^-F^+a)F^-$$

$$= (-1)^{|a|}(Q^-(a) + a - (-1)^{|a|}F^+b)F^-$$

which shows that $Q^-(a)$ is a generalized symmetry of $F^-$. 

A similar result holds for $F^+$ using $Q^+$.  

More generally, we can take $a \in A$ such that for some $b \in A$

$$F^-[F^-, a] = F^-(F^-a - (-1)^{|a|}aF^-) = bF^-,$$

which is the case if for some $c \in A$

$$F^-F^-a = cF^-.$$

3. Dunkl realization

We consider a complex vector space $V \cong \mathbb{C}^d$, for a positive integer $d$. The ring of polynomial functions on $V$ is the symmetric algebra $S(V^*)$ of the dual space $V^*$. In the next sections our focus will be on the dual space $V^*$. The notation $\langle \cdot, \cdot \rangle$ will denote the natural bilinear pairing between a space and its dual.

3.1. Bilinear form. Let $B$ denote a non-degenerate symmetric bilinear form on $V$. The orthogonal group $O := \text{O}(V, B) \cong \text{O}(d, \mathbb{C})$ is the group of invertible linear transformations of $V$ that preserve the form $B$. The action of $O$ on $V$ is naturally extended to $V^*$ as the contragradient action, and in turn also to tensor products of those spaces. There is an isomorphism $\beta : V \to V^* : v \mapsto \beta(v)$ given by

$$\langle \beta(u_1), u_2 \rangle = B(u_1, u_2), \text{ for } u_1, u_2 \in V,$$

which commutes with the action of $O$. Hence, the spaces $V$ and $V^*$ are isomorphic as $O$-modules. We will use $\beta$ also to denote its inverse so $\beta$ becomes an involution of $V \oplus V^*$. Moreover, we will denote by $B$ also the induced bilinear form on $V^*$ given by $B(u, v) = B(\beta(u), \beta(v))$ for $u, v \in V^*$. 
3.2. Bases. If no specific properties are needed, \( v_1^* \ldots v_d^* \) will denote a basis of \( V^* \), dual to a basis \( v_1, \ldots, v_d \) of \( V \), so \((v_p^*, v_k) = \delta_{p,q} \) for \( p,q \in \{1, \ldots, d\} \). In terms of these bases, for \( u \in V^* \) and \( v \in V \) we have

\[
\beta(v) = \sum_{p=1}^{d} B(v, v_p) v_p^*, \quad \beta(u) = \sum_{p=1}^{d} B(u, v_p^*) v_p,
\]

and

\[
v = \sum_{p=1}^{d} B(v, v_p) \beta(v_p^*), \quad u = \sum_{p=1}^{d} B(u, v_p^*) \beta(v_p).
\]

so

\[
v = \sum_{p,q=1}^{d} B(v, v_p) B(v_q^*, v_p^*) v_q, \quad u = \sum_{p,q=1}^{d} B(u, v_p^*) B(v, v_q) v_q^*.
\]

For brevity, we will sometimes denote \( B_{pq} = B(v_p, v_q) \).

When needed, \( y_1, \ldots, y_d \) will denote a basis of \( V \) and \( x_1, \ldots, x_d \in V^* \) its dual basis, such that \( \delta_{j,k} = \langle x_j, y_k \rangle = B(x_j, x_k) = B(y_j, y_k) \). Note that the involution \( \beta \) sends \( y_j \) to \( x_j \) and vice versa.

Let \( V^* = V^+ \oplus V^0 \oplus V^- \) be a Witt decomposition of \( V^* \) where \( V^+ \) and \( V^- \) are complementary maximal \( B \)-isotropic subspaces of \( V^* \) of dimension \( \ell = \lfloor d/2 \rfloor \), and \( V^0 = \emptyset \) for \( d \) even while \( V^0 \) is anisotropic and one-dimensional for \( d \) odd. Let \( z_1^+, \ldots, z_{d/2}^+ \) denote a basis of \( V^+ \) and \( \tilde{z}_1, \ldots, \tilde{z}_{d/2}^- \) a basis of \( V^- \) such that \( B(z_j^+, \tilde{z}_k^-) = \delta_j k/2 \). For odd \( d \) odd, denote \( z_0 \) an element of \( V^0 \) satisfying \( B(z_0, z_0) = 1 \). The dual basis of \( V \) is given by \( w_j^+ := 2\beta(z_j^+) \), satisfying \( \langle z_j^+, w_k^- \rangle = \delta_{j,k} \).

3.3. Clifford algebra. As our focus will be on \( V^* \), we will construct the Clifford algebra associated with \( V^* \) and \( B \). Note that this results in the same Clifford algebra as when using \( V \) and \( B \).

The Clifford algebra \( \mathcal{C} := \mathcal{C}(V^*, B) \) is the quotient of the tensor algebra \( T(V^*) \) by the ideal generated by all elements of the form \( v \otimes v - B(v, v)1 \) for \( v \in V^* \). The quotient map from the embedding \( V^* \to T(V^*) \) gives a canonical map \( \gamma : V^* \to \mathcal{C} \).

We have that \( \mathcal{C} \) is the associative algebra with 1 generated by \( \gamma(V^*) \), subject to the anticommutation relations

\[
\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2B(u, v)1 \quad \text{for} \, u, v \in V^*.
\]

The Clifford algebra inherits the structure of a filtered super algebra from \( T(V^*) \), with the generators \( \gamma(V^*) \) being odd and having filtration degree 1.

For \( u \in V^* \), we will denote \( \gamma_u := \gamma(u) \in \mathcal{C} \), and for \( u_1, \ldots, u_k \in V^* \) we let \( \gamma_{u_1 \ldots u_k} := \gamma_{u_1} \cdots \gamma_{u_k} \in \mathcal{C} \). Moreover, we denote \( \gamma_{u_1 \ldots u_{\ell} \cdots u_n} := \gamma_{u_1} \cdots \gamma_{u_\ell} \cdots \gamma_{u_n} \), where the notation \( \gamma_{u_j} \) indicates that the factor \( \gamma_{v_j} \) is omitted in the product.

For the chosen bases of \( V^* \) we denote \( e_j = \gamma(x_j) \) for \( j \in \{1, \ldots, d\} \), and \( \theta_k^\pm = \gamma(z_k^\pm) \) for \( k \in \{1, \ldots, \ell\} \) with \( \theta_0 = \gamma(z_0) \) for odd \( d \). In \( \mathcal{C} \), they satisfy the relations

\[
e_j e_k + e_k e_j = 2d_{j,k}, \quad \theta_j^\pm \theta_j^\pm + \theta_j^\pm \theta_j^\mp = 0, \quad \theta_j^\pm \theta_k^\pm + \theta_k^\pm \theta_j^\pm = \delta_{j,k},
\]

and when \( d \) is odd, also \( \theta_0^2 = 1 \) and \( \theta_0 \theta_k^\pm + \theta_k^\pm \theta_0 = 0 \).

For a subset \( A \subset \{1, \ldots, d\} \), with elements \( A = \{a_1, a_2, \ldots, a_n\} \) such that \( 1 \leq a_1 < a_2 < \cdots < a_n \leq d \), we denote \( e_A = e_{a_1} e_{a_2} \cdots e_{a_n} \). Let \( e_0 = 1 \), then a basis for \( \mathcal{C} \) as a vector space is given by \( \{e_A \mid A \subset \{1, \ldots, d\}\} \).

We denote the chirality element of the Clifford algebra as

\[
\Gamma := \varepsilon^{d(d-1)/2} e_1 \cdots e_d \in \mathcal{C};
\]

it satisfies \( \Gamma^2 = 1 \) and \( \Gamma \gamma_u = (-1)^{d-1} \gamma_u \Gamma \) for \( u \in V^* \).
Let $A$ denote the anti-symmetrization operator, which has the following action on a multilinear expression in $n$ indices
\begin{equation}
A(f_{u_1 u_2 \cdots u_n}) = \frac{1}{n!} \sum_{s \in S_n} \text{sign}(s)f_{u_{s(1)} \cdots u_{s(n)}} ,
\end{equation}
where $S_n$ is the symmetric group of degree $n$. We have $AA = A$ and
\begin{equation}
A(f_{u_1 u_2 \cdots u_n}) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} f_{u_j A(u_1 \cdots \hat{u_j} \cdots u_n)} .
\end{equation}

The symbol map and its inverse, the quantization map
\begin{equation}
q: \bigwedge(V^*) \to C: u_1 \wedge u_2 \wedge \cdots \wedge u_k \mapsto A(\gamma_{u_1 \cdots u_k}) ,
\end{equation}
are isomorphisms of $O$-modules and filtered super vector spaces [18, Section 2.2.5].

For $u, v, w, x \in V^*$, we have
\begin{align}
A(\gamma_{uv}) &= \gamma_{uv} - B(u, v) \\
A(\gamma_{uvw}) &= \gamma_{uvw} - B(u, v)\gamma_w + B(u, w)\gamma_v - B(v, w)\gamma_u \\
A(\gamma_{uvw}) &= \gamma_{uvw} - B(u, v)\gamma_w\gamma_x + B(u, w)\gamma_v\gamma_x - B(v, w)\gamma_u\gamma_x \\
&\quad - B(u, x)\gamma_v\gamma_w + B(v, x)\gamma_u\gamma_w - B(w, x)\gamma_u\gamma_v + B(u, v)B(w, x) - B(v, w)B(w, x) .
\end{align}

Note that if $u_1, \ldots, u_n \in V^*$ are $B$-orthogonal, then
\begin{equation}
A(\gamma_{u_1 u_2 \cdots u_n}) = \gamma_{u_1 u_2 \cdots u_n} .
\end{equation}

With the commutator, the space $q(\wedge^2(V^*))$ in $C$ forms a realization of the Lie algebra $\mathfrak{so}(V^*, B) \cong \mathfrak{so}(V, B) \cong \mathfrak{so}(d, C)$. We have the following adjoint action on $\gamma(V^*)$:
\begin{equation}
[\gamma_{uv}/2, \gamma_w] = \gamma_u[\gamma_v, \gamma_w]/2 - [\gamma_u, \gamma_w]\gamma_v/2 = B(v, w)\gamma_u - B(u, v)\gamma_v .
\end{equation}

3.4. Spinor space. When $d$ is odd, there is a unique isomorphism class of irreducible $Z_2$-graded $C$-modules, and there are two isomorphism classes of irreducible ungraded $C$-modules, see [18, Theorem 3.10]. When $d$ is even, there are two isomorphism classes of irreducible $Z_2$-graded $C$-modules, and there is a unique isomorphism class of irreducible ungraded $C$-modules.

A model for an irreducible $Z_2$-graded $C$-module $S$ is given by the exterior algebra $\bigwedge(V^+ \oplus V^0)$, where $V^* = V^+ \oplus V^0 \oplus V^-$ is a Witt decomposition of $V^*$. On the space $S$ the elements of $\gamma(V^*)$ act by exterior multiplication and those of $\gamma(V^-)$ by interior multiplication or contraction. In terms of a $B$-isotropic basis of $V^*$ as defined in Section 3.2, $\theta_j^+ \in \gamma(V^-)$ acts as the odd differential operator corresponding to the odd variable $\theta_j^+ \in \gamma(V^+)$. When $d$ is even, $V^0 = \emptyset$ and this gives the complete action of $C$ on $S$. The parity-reversed space $\Pi(S)$ is another irreducible $Z_2$-graded $C$-module, they are isomorphic as ungraded $C$-modules. The spinor space $S$ can be realized explicitly inside the Clifford algebra as
\begin{equation}
S = \bigwedge(V^+) \prod_j (\theta_j^- \theta_j^+) ,
\end{equation}
with the action of $C$ given by Clifford algebra multiplication, since the product $\prod \theta_j^- \theta_j^+$ is annihilated by all $\theta_k^-$.

When $d$ is odd, recall $\theta_0 \in \gamma(V^0)$ with $B(\theta_0, \theta_0) = 1$, we can write $S = S_+ \oplus S_-$ where $S_\pm = \bigwedge(V^+)(1 \pm \theta_0)/2$ are non-isomorphic irreducible ungraded $C$-modules.
The action of $\theta_0$ on $S_\pm$ is given by
\begin{equation}
\theta_0 \cdot \theta = (-1)^k \theta, \quad \text{for } \theta \in \Lambda^k(V^+)/(1 \pm \theta_0)/2 \subset S_\pm,
\end{equation}
and extending by linearity. The sign in the action (3.16) of $\theta_0$ distinguishes between the spaces $S_\pm$. The spinor space $\mathcal{S}$ can be realized explicitly inside the Clifford algebra by letting
\begin{equation}
\mathcal{S}_\pm = \Lambda^k(V^+)/(1 \pm \theta_0)/2 \prod_j (\theta_j^+ \theta_j^-),
\end{equation}
with the action of $\mathcal{C}$ given by Clifford algebra multiplication, since $\theta_0(1 \pm \theta_0)/2 = \pm(1 \pm \theta_0)/2$.

3.5. Reflection group. We fix a finite real reflection group $G \subset \text{O}$. Denote by $\mathcal{S}$ the set of reflections of $G$. For each $s \in \mathcal{S}$, fix $\alpha_s \in V^*$ to be a $-1$ eigenvector for the action of $s$.

By definition, $\beta(\alpha_s) \in V$ is a $-1$ eigenvector for the action of $s$ on $V$. Denote $\alpha_s^\vee := 2\beta(\alpha_s)/B(\alpha_s, \alpha_s)$, then for $v \in V$ and $u \in V^*$, the reflection $s \in \mathcal{S}$ acts as
\begin{equation}
\begin{split}
s(v) &= v - (\alpha_s, v)\alpha_s^\vee, \\
s(u) &= u - \alpha_s(\alpha_s^\vee, u).
\end{split}
\end{equation}

Define $T(V \oplus V^*) \rtimes G$ to be the quotient of $T(V \oplus V^*) \otimes \mathbb{C}[G]$ by the relations
\begin{equation}
(u, g)(v, h) = (u g(v), gh) \quad \text{for } u, v \in T(V \oplus V^*) \text{ and } g, h \in G,
\end{equation}
so in $T(V \oplus V^*) \rtimes G$, we have $g u g^{-1} = g(u)$ for $g \in G$ and $u \in T(V \oplus V^*)$.

We fix a map $\kappa : \mathcal{S} \rightarrow \mathbb{C}$ that is $G$-invariant (for the conjugation action), so that the elements of an orbit all have the same image.

**Definition 3.1.** Define $H_\kappa = H_\kappa(G, V)$ to be the quotient of $T(V \oplus V^*) \rtimes G$ by the relations
\begin{equation}
\begin{split}
[x, u] &= 0 = [y, v], \quad \text{for } y, v \in V \text{ and } x, u \in V^*, \\
[y, x] &= \langle y, x \rangle + \sum_{s \in \mathcal{S}} \langle y, \alpha_s \rangle (\alpha_s^\vee, x) \kappa(s)s, \quad \text{for } y \in V, x \in V^*.
\end{split}
\end{equation}

When $\kappa$ is the zero map, the relations (3.20) reduce to the canonical commutation relations of the Weyl algebra $\mathcal{W} = \mathcal{W}(V)$, so $H_0(G, V) = \mathcal{W}(V) \rtimes G$.

**Remark 3.2.** The algebra $H_\kappa(G, V)$ is called a rational Cherednik algebra, and is a rational degeneration of a double affine Hecke algebra [10, 12]. More generally, a complex reflection group $G \subset \text{GL}(V)$ can be used. Also, there can be an extra parameter $t$ accompanying $\langle y, x \rangle$ in (3.20), which we have taken $t = 1$ here. See [17] for the case where $t \in \mathbb{C}^\times$.

A realization of $H_\kappa$ is given by means of Dunkl operators [9]
\begin{equation}
\mathcal{D}_y = \frac{\partial}{\partial y} + \sum_{s \in \mathcal{S}} \kappa(s) \langle y, \alpha_s \rangle \frac{\alpha_s}{\alpha_s}(1 - s), \quad \text{for } y \in V,
\end{equation}
and coordinate variables (for the elements of $V^*$), which gives a natural, faithful action on the polynomial space $\mathcal{S}(V^*)$.

In the context of a rational Cherednik algebra, the parameter function is usually chosen to be either the opposite sign compared to the one used for Dunkl operators, corresponding to the substitution $\kappa = -c$.

**Lemma 3.3.** For $u, v \in V^*$ or $u, v \in V$, in $H_\kappa$ we have $[\beta(u), v] = [\beta(v), u]$.

**Proof.** Let $u, v \in V^*$, then we have
\begin{equation}
[\beta(u), v] = B(u, v) + 2 \sum_{s \in \mathcal{S}} \frac{B(\alpha_s, u)B(v, \alpha_s)}{B(\alpha_s, \alpha_s)} \kappa(s)s = [\beta(v), u],
\end{equation}
where the last equality follows from $B$ being symmetric. \qed
3.6. Superalgebra. We consider the superspace \( V = \mathbb{C}^{2|1} \) equipped with a non-degenerate, skew-supersymmetric, consistent, bilinear form \( b \). Denote by \( \omega \) the skew-symmetric bilinear form on \( V \), and also its restriction to \( V_0 = \mathbb{C}^2 \), that equals \( b \) on \( V_0 \) and is zero on \( (V \times V) \setminus (V_0 \times V_0) \).

The tensor product \( U = V^* \otimes V \) is again a superspace, inheriting the \( \mathbb{Z}_2 \)-grading from \( V \), so \( U_0 = V^* \otimes V_0 \) and \( U_1 = V^* \otimes V_1 \). There is a natural action of \( O \) on \( U = V^* \otimes V \) as \( O \otimes \text{Id}_V \) where \( \text{Id}_V \) denotes the identity on \( V \). For \( G \subset O \) we consider also another action on \( U \):

\[
(3.23) \quad a_0 : G \times U \rightarrow U : \begin{cases} a_0(g, u \otimes v) = (g \cdot u) \otimes v & \text{for } u \in V^*, v \in V_0 \\ a_0(g, u \otimes v) = u \otimes v & \text{for } u \in V^*, v \in V_1, \end{cases}
\]

where \( g \cdot u \) denotes the action of \( g \in G \) on \( u \in V^* \). The “missing” interaction of \( G \) and \( U_1 \) will be provided by means of the Pin-group inside the Clifford algebra, see (3.27).

The action (3.23) is extended naturally to the tensor superalgebra \( T(U) = \bigoplus_n U^* \otimes \mathbb{C}^n \), which uses \( \mathbb{Z}_2 \)-graded tensor products. Using (3.23), define \( T(U) \rtimes G \) to be the quotient of \( T(U) \otimes \mathbb{C}[G] \) by the relations

\[
(3.24) \quad (u, g)(v, h) = (u a_0(g, v), gh) \quad \text{for } u, v \in T(U), g, h \in G.
\]

Now, we consider the \( G \)-invariant symmetric bilinear map \( \psi^B_{\kappa}(\cdot, \cdot) : V^* \times V^* \rightarrow \mathbb{C}[G] \) defined as

\[
(3.25) \quad \psi^B_{\kappa}(u, v) = 2 \sum_{s \in S} B(\alpha_s, u) B(v, \alpha_s) \kappa(s) s, \quad \text{for } u, v \in V^*.
\]

**Definition 3.4.** Define the superalgebra \( A_{\kappa} \) to be the quotient of \( (T(U) \rtimes G) \) by the relations

\[
(3.26) \quad uv - (-1)^{|u||v|} vu = b_U(u, v) 1 + \psi_{\kappa}(u, v) 1 \quad \text{for } u, v \in U_0 \cup U_1,
\]

where the right-hand side is defined for \( u \otimes w, v \otimes z \in V^* \otimes V = U \) as

\[
b_U(u \otimes w, v \otimes z) = B(u, v)b(w, z), \quad \psi_{\kappa}(u \otimes w, v \otimes z) = \psi^B_{\kappa}(u, v) \omega(w, z).
\]

The superalgebra \( A_{\kappa} \) is generated, as an algebra, by \( U \) and \( G \). We now show that \( A_{\kappa} \) is the tensor product of the rational Cherednik algebra \( H_{\kappa} \) and the Clifford algebra \( \mathcal{C} \).

Fix \( x^+, x^- \in V_0 \) and \( \gamma \in V_1 \) to be a basis of \( V = \mathbb{C}^{2|1} \) satisfying \( b(x^-, x^+) = 1 = -b(x^+, x^-) \) and \( b(\gamma, \gamma) = 2 \), with \( b \) zero for all other combinations.

For the elements of the form \( w \otimes \gamma \in U \), we see that the relations (3.26) correspond precisely to the Clifford algebra relations (3.5). For \( u \in V^* \), we will identify \( \gamma_u = u \otimes \gamma \).

Identifying \( u \in V^* \) with \( u \otimes x^+ \in U \), and \( u \otimes x^- \in U \) with \( \beta(u) \in V \), the relations (3.26) then correspond precisely to (3.20). An element \( v \in V \) then corresponds to \( \beta(v) \otimes x^- \subset U \).

3.7. Double cover. The group \( \text{Pin} := \text{Pin}(V, B) \) is a double cover \( p : \text{Pin} \rightarrow O \) and is realized in the Clifford algebra \( \mathcal{C} \) as the set of products \( \gamma_{u_1} \cdots u_k \) where \( u_j \in V^* \) with \( B(u_j, u_j) = 1 \) [18]. The subgroup \( \text{Spin}(V, B) \) consists of similar products with \( k \) even.

For a reflection \( s \in S \subset O \), denote \( \tilde{s} := \gamma(\alpha_s)/\sqrt{B(\alpha_s, \alpha_s)} \in \text{Pin} \subset \mathcal{C} \), then \( p(\tilde{s}) = s \) and \( p^{-1}(s) = \{ \pm \tilde{s} \} \subset \mathcal{C} \). The preimage of the identity \( \text{Id} \in O \) is \( p^{-1}(\text{Id}) = \{ \pm 1 \} \subset \mathcal{C} \).

Define the pin double cover of \( G \subset O \) as \( \tilde{G} := p^{-1}(G) \subset \text{Pin} \). The conditions for \( \tilde{G} \) to be a non-trivial central extension of \( G \) are in [19]. See [19] also for a presentation in terms of generators and relations for \( G \) and \( \tilde{G} \).
Remark 3.5. Note that this is the version of the Pin-group, and thus of $\tilde{G}$, where the preimages (for the covering map $p$) of a reflection in $O$ have order two (and not four). The (non-isomorphic) other version can be obtained by using elements $u, j \in V^*$ with $B(u, j) = -1$, or by adding a minus sign to the defining relations of the Clifford algebra. We refer to [18, Section 3.7.2] for the definition and the distinction with the group $Pin_c$.

In the superalgebra $A_{C} \cong H_{C} \otimes C$, there is a copy of the group $G \subset H_{C}$ and also of the group $\tilde{G} \subset C$. We use these to define a group homomorphism
\[
\rho: \tilde{G} \rightarrow A: \tilde{s} \mapsto p(\tilde{s}) \tilde{s},
\]
which is extended linearly to a map on the group algebra $\mathbb{C}[\tilde{G}]$. We note that $\rho(\mathbb{C}[\tilde{G}])$ is a quotient of the group algebra $\mathbb{C}[\tilde{G}]$, since the central element (the non-trivial preimage of the identity) is given by the scalar $-1 \in \mathbb{C}$ for the realization of $\tilde{G}$ in $Pin \subset C$, see [17].

Recall that, as an algebra, the superalgebra $A_{C}$ is generated by $U = V^* \otimes V$ and $G$.

Definition 3.6. For $g \in G$ and $u \in A_{C}$, we denote by $g \cdot u$ the action $G \times A_{C} \rightarrow A_{C}$ that is the extension of the natural action of $G \subset O$ on $V^*$, acting as $G \otimes Id_{V}$ on $V^* \otimes V$, and of the action by conjugation on the copy $G \subset H_{C}$.

This action of $G$ is related to the action of $\rho(\tilde{G})$ inside $A_{C}$ as follows.

Lemma 3.7. For $\tilde{g} \in \tilde{G}$ and $u \in A_{C}$ a homogeneous element for the $\mathbb{Z}_2$-grading, in $A_{C}$ we have
\[
\rho(\tilde{g})u\rho(\tilde{g}^{-1}) = (-1)^{|u||\tilde{g}|}p(\tilde{g}) \cdot u.
\]

Proof. Use (3.23), (3.27) and the properties of the Pin-group. \hfill \Box

Finally, for $u \in V^*$, we define the following elements in $\rho(\tilde{G})$:
\[
\tilde{\sigma}_u := \frac{1}{2} \sum_{s \in S} \alpha_s^\vee(u) \kappa(s) s \gamma_{\alpha_s} = \sum_{s \in S} \frac{B(\alpha_s, u)}{\sqrt{B(\alpha_s, \alpha_s)}} \kappa(s) \rho(\tilde{s}).
\]

By (3.26) and (3.22), we have that for $u, v \in V^*$
\[
[\gamma_{\tilde{u}} + \tilde{\sigma}_u, \gamma_{\tilde{v}} + \tilde{\sigma}_v] = [-u, v] - B(u, v) = [\gamma_{\tilde{u}} \tilde{\sigma}_u, \gamma_{\tilde{v}} \tilde{\sigma}_v],
\]
where the last equality follows by Lemma 3.3. Expanding the anticommutators in (3.30) gives rise to the following result (cfr. [8, Lemma 3.10]).

Lemma 3.8. Let $n \in \{1, 2, \ldots, d\}$, and $u_1, \ldots, u_n \in V^*$, then
\[
\mathcal{A}(\tilde{\sigma}_{u_1} \gamma_{u_2} \cdots u_n) = \mathcal{A}(\gamma_{u_1} \tilde{\sigma}_{u_2} \gamma_{u_3} \cdots u_n) = \cdots = \mathcal{A}(\gamma_{u_1} \cdots u_{n-1} \tilde{\sigma}_{u_n}).
\]

Proof. The first non-trivial case, for $n = 2$, follows immediately from (3.30):
\[
\tilde{\sigma}_u \gamma_v - \tilde{\sigma}_v \gamma_u = \gamma_u \tilde{\sigma}_v - \gamma_v \tilde{\sigma}_u.
\]

We can then use this to find for general $n \in \{3, \ldots, d\}$
\[
\mathcal{A}(\tilde{\sigma}_{u_1} \gamma_{u_2} \cdots u_n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \tilde{\sigma}_{u_j} \mathcal{A}(\gamma_{u_1} \cdots \tilde{\sigma}_{u_1} \cdots u_n)
\]
\[
= \frac{1}{n(n-1)} \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} \mathcal{A}(\gamma_{u_1} \cdots \tilde{\sigma}_{u_k} \cdots u_n)
\]
\[
= \frac{1}{n(n-1)} \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} \mathcal{A}(\gamma_{u_1} \cdots \tilde{\sigma}_{u_k} \cdots u_n)
\]
\[
\mathcal{A}(\gamma_{u_1} \tilde{\sigma}_{u_2} \gamma_{u_3} \cdots u_n).
\]
and the other equalities follow by repeated application of the same steps. □

3.8. Lie (super)algebra realizations. The bilinear form \( B \) on \( V \) naturally corresponds to an element of \( (V \otimes V)^* \), a linear map on \( V \otimes V \). Since \( V \) is finite-dimensional and \( B \) is symmetric, we have \( B \in S^2(V^*) \subset V^* \otimes V^* \). As above, let \( v_1^*, \ldots, v_d^* \) denote a basis of \( V^* \), dual to a basis \( v_1, \ldots, v_d \) of \( V \), then

\[
B = \sum_{p,q=1}^d B(v_p, v_q) v_p^* \otimes v_q^* = \sum_{p=1}^d v_p^* \otimes B(v_p) = \sum_{p=1}^d B(v_p) \otimes v_p^*,
\]

which is, by definition, invariant for the action of the group \( O(V, B) \).

Every element of \( V \) corresponds to a copy of \( V^* \) in \( V^* \otimes V \). We can use this to map \( B \in S^2(V^*) \in S^2(V^* \otimes V) \), by viewing \( \nu \in V \) as the map \( \nu : V^* \to V^* \otimes V^* : v \mapsto v \otimes \nu \).

For \( w, z \in V \) homogeneous elements for the \( Z_2 \)-grading, the supersymmetric tensor product is given by

\[
(w \otimes z) = (w \otimes z + (-1)^{|w||z|} z \otimes w)/2,
\]

and we then consider the following elements of \( S^2(V^* \otimes V) \):

\[
(w \otimes z)(B) = \frac{1}{2} \sum_{p,q=1}^d B(v_p, v_q)((v_p^* \otimes w)(v_q^* \otimes z) + (-1)^{|w||z|} (v_q^* \otimes z)(v_p^* \otimes w)).
\]

Under the quotient by the relations (3.26), in the superalgebra \( A_k \) we have:

\[
(w \otimes z)(B) = \sum_{p,q=1}^d B(v_p, v_q)((v_p^* \otimes w)(v_q^* \otimes z) - [v_p^* \otimes w, v_q^* \otimes z]/2)
= \sum_{p,q=1}^d v_p^* \otimes w B(v_p, v_q)v_q^* \otimes z - b(w, z) d/2 - \omega(w, z) \Omega_k,
\]

where we used (3.3) and denote

\[
\Omega_k = \sum_{s \in S} \kappa(s) s,
\]

which is a central element in the group algebra \( \mathbb{C}[G] \).

In the tensor product of a Weyl and Clifford algebra \( W \otimes \mathbb{C} \), the space of invariants for the action (as in Definition 3.6) of \( O \) is generated by the elements of the form \( (w \otimes z)(B) \) [6, Proposition 5.11].

Lemma 3.9. For \( w, z \in V \), in \( A_k \) we have \( [(w \otimes z)(B), \rho(G)] = 0 \).

Proof. This follows from Lemma 3.7 and that \( G \subset O \) preserves \( B \). □

Next, we consider the adjoint action of elements of the form \( (w \otimes z)(B) \) on the space \( U = V^* \otimes V \) in \( A_k \). Recall that \( \gamma \in V \) satisfies \( b(\gamma, \gamma) = 2 \).

Lemma 3.10. Let \( u \in V^* \), while \( \xi_1, \xi_2 \in V_0 \) and \( \eta \in V \). In \( A_k \), we have

\[
[(\xi_1 \otimes \xi_2)(B), u \otimes \eta] = b(\xi_2, \eta) u \otimes \xi_1 + b(\xi_1, \eta) u \otimes \xi_2, \tag{3.38}
\]

and

\[
[(\xi_1 \otimes \gamma)(B), u \otimes \eta] = b(\gamma, \eta) u \otimes \xi_1 + b(\xi_1, \eta)(u \otimes \gamma + 2\tilde{\sigma}_u), \tag{3.39}
\]

where \( \tilde{\sigma}_u \) is given by (3.29).

Proof. Let \( \xi, \eta \in V \) be homogeneous elements for the \( Z_2 \)-grading and \( \xi_1 \in V_0 \), then by (3.36)

\[
[(\xi_1 \otimes \xi)(B), u \otimes \eta] = \sum_{p,q=1}^d B_{pq} [(v_p^* \otimes \xi_1)(v_q^* \otimes \xi), u \otimes \eta] - \omega(\xi_1, \xi) \sum_{s \in S} \kappa(s)[s, u \otimes \eta].
\]
First, note that $\omega(\xi_1, \xi) = 0$ if $\xi \in \mathcal{V}_1$, and $[s, u \otimes \eta] = 0$ for $\eta \in \mathcal{V}_1$. For $\eta \in \mathcal{V}_0$, via (3.23) and (3.18) we have

$$[s, u \otimes \eta] = ((s \cdot u - u) \otimes \eta) s = -\alpha_s^\gamma(u) (\alpha_s \otimes \eta) s.$$

By means of (3.26), we find

$$[(v_p^* \otimes \xi_1)(v_q^* \otimes \xi), u \otimes \eta] = v_p^* \xi_1[v_q^* \otimes \xi, u \otimes \eta] + (-1)^{|\xi||\eta|} v_p^* \xi_1, u \otimes \eta] v_q^* \otimes \xi$$

$$= v_p^* \xi_1(B(v_q^* u)B(\xi, \eta)) + \psi_B^u(v_q^* u)\omega(\xi, \eta)) + \psi_B^u(v_q^* u)\omega(\xi, \eta) v_q^* \otimes \xi.$$

For $\eta = \gamma$, we have $\omega(-, \gamma) = 0$ and the cases where $\eta = \gamma$ now follow by (3.4).

Next, we consider $\eta \in \mathcal{V}_0$. Let $\xi = \xi_2 \in \mathcal{V}_0$. We find using (3.3) and (3.25)

$$\sum_{p,q=1}^d B_{pq} v_p^* \otimes \xi_1 \psi^{\alpha}_u(v_q^*, u) = \sum_{s \in S} \alpha_s^\gamma(u) \kappa(s) \alpha_s \otimes \xi_1 s,$$

$$\sum_{p,q=1}^d \psi^{\alpha}_u(v_p^*, u)B_{pq} v_q^* \otimes \xi_2 = \sum_{s \in S} \alpha_s^\gamma(u) \kappa(s) s \alpha_s \otimes \xi_2.$$

Now, collecting the appropriate terms, and using that $\alpha_s$ is a $-1$ eigenvector of $s \in S$, we have

$$\sum_{s \in S} \alpha_s^\gamma(u) \kappa(s) \alpha_s \otimes (\omega(\xi_1, \xi_2) \eta + \omega(\xi_2, \eta) \xi_1 - \omega(\xi_1, \xi_2) s$$

where $\omega(\xi_1, \xi_2) \eta + \omega(\xi_2, \eta) \xi_1 - \omega(\xi_1, \xi_2) \eta = 0$ for all $\xi_1, \xi_2, \eta \in \mathcal{V}_0 = \mathbb{C}^2$.

Finally, we consider the case $\xi = \gamma \in \mathcal{V}_1$ and $\eta = \xi_2 \in \mathcal{V}_0$. Here, the remaining terms are

$$[(\xi_1 \otimes \gamma)(B), u \otimes \xi_2] = \sum_{p,q=1}^d B_{pq} B(v_p^* u)B(\xi_1, \xi_2) + \psi_B^u(v_q^* u)\omega(\xi_1, \xi_2) v_q^* \otimes \gamma$$

$$= b(\xi_1, \xi_2) u \otimes \gamma + \omega(\xi_1, \xi_2) \sum_{s \in S} \alpha_s^\gamma(u) \kappa(s) s \alpha_s \otimes \gamma.$$

The desired result now follows by (3.29).

When restricted to $\mathcal{V}_0$, in terms of the basis $x^+, x^-$, the relations of Lemma 3.10 can be written as follows. For $v^- \in \mathcal{V}$ and $v^+ \in \mathcal{V}^*$, in $H_\kappa \subset A_\kappa$, we have

(3.40) \quad $$[(x^- \otimes x^+)(B), v^+] = 2\beta(v^+), \quad [(x^+ \otimes x^-)(B), v^-] = -2\beta(v^-),$$

(3.41) \quad $$[(x^+ \otimes x^-)(B), v^+] = v^+, \quad [(x^+ \otimes x^-)(B), v^-] = -v^-.$$  

Rewriting (3.39) for $\xi_1 = x^-$ and $\xi_2 = x^+$, we get for $u \in \mathcal{V}^*$

(3.42) \quad $$\tilde{\sigma}_u = \frac{1}{2}((x^- \otimes \gamma)(B), u) - \gamma_u = \frac{1}{2} \left( \sum_{p=1}^d [v_p, u]_{\gamma_{v_p}} - \gamma_u \right),$$

where $v_1^*, \ldots, v_d^*$ denotes a basis of $V^*$ dual to a basis $v_1, \ldots, v_d$ of $V$. Using (3.26), the expression (3.42) reduces to (3.29).

We can also consider the map $\tilde{\sigma}: V^* \to \rho(\tilde{G})$: $u \mapsto \tilde{\sigma}_u$, where $\tilde{\sigma}_u$ is given by (3.29). We then have the following result for

(3.43) \quad $$\tilde{(\sigma \otimes \gamma)}(B) = \sum_{p,q=1}^d \tilde{\sigma}_{v_p^*} B(v_p, v_q)_{\gamma_{v_q}} = \sum_{p=1}^d \tilde{\sigma}_{v_p} v_p.$$

**Lemma 3.11.** In $A_\kappa$, one has

$$(\tilde{\sigma} \otimes \gamma)(B) = \Omega_\kappa = (\gamma \otimes \tilde{\sigma})(B).$$
Proof. Using (3.29) and (3.4), we have

\[(\bar{\sigma} \otimes \gamma)(B) = \sum_{p,q=1}^{d} B(v_p, v_q)\frac{1}{2} \sum_{s \in S} \alpha^s_p(v_p) \kappa(s) s \gamma_{\alpha_s} \gamma_{v_q} = \sum_{s \in S} \kappa(s) s \gamma_{\alpha_s} \gamma_{\alpha_s}. \]

\[\square\]

Proposition 3.12. For \(z_1, z_2, z_3, z_4 \in V\) homogeneous for the \(\mathbb{Z}_2\)-grading, denoting \(w_1 = b(z_2, z_3)z_1 + (-1)^{\|z_1\| + \|z_2\|} b(z_1, z_3)z_2\) and \(w_2 = b(z_2, z_4)z_1 + (-1)^{\|z_1\| + \|z_2\|} b(z_1, z_4)z_2\), in \(A_{\kappa}\) one has

\[(z_1 \circ z_2)(B), (z_3 \circ z_4)(B)] = (w_1 \circ z_4)(B) + (-1)^{\|z_1\| + \|z_2\|}(z_3 \circ w_2)(B) \]

Proof. Follows by direct computation using Lemma 3.10 and the fact that \(\dim_{\mathbb{C}}(\mathcal{Y}_1) = 1\). For instance, using Lemma 3.10, (3.26), (3.3) and Lemma 3.11, we have

\[[(\xi_1 \circ \gamma)(B), (\xi_2 \circ \gamma)(B)] \]

\[= \sum_{p,q=1}^{d} B(v_p, v_q) \left( [(\xi_1 \circ \gamma)(B), v_p \otimes \xi_2] v_q \otimes \gamma \right) \]

\[= \sum_{p,q=1}^{d} B(v_p, v_q) \left( b(\xi_1, \xi_2)(v_p \otimes \gamma + 2\bar{\sigma}_p v_q \otimes \gamma + (v_p \otimes \xi_2)2(v_q \otimes \xi_1) \right) \]

\[= 2(\xi_2 \circ \xi_1)(B). \]

\[\square\]

Proposition 3.12 shows that the elements \((w \circ z)(B)\) for \(w, z \in V\) form a realization of the Lie superalgebra \(\mathfrak{osp}(V, b) \cong \mathfrak{osp}(1|2)\) in \(A_{\kappa}\). The elements \((w \circ z)(B)\) for \(w, z \in \mathcal{Y}_1 = \mathbb{C}^2\) form a realization of the even subalgebra \(\mathfrak{sp}(\mathcal{Y}_0, \omega) \cong \mathfrak{sl}(2)\) in \(H_{\kappa}\). In particular, using the basis \(x^-, x^+, \gamma\) of \(V\), we have that the elements

\[F^+ := \frac{1}{\sqrt{2}}(x^+ \circ \gamma)(B) = \frac{1}{\sqrt{2}} \sum_{p,q=1}^{d} v_p^* B_{pq} \gamma v_q \]

\[F^- := \frac{1}{\sqrt{2}}(x^- \circ \gamma)(B) = \frac{1}{\sqrt{2}} \sum_{p,q=1}^{d} \beta(v_p^*) B_{pq} \gamma v_q = \frac{1}{\sqrt{2}} \sum_{p=1}^{d} v_p \gamma v_p^* \]

\[(3.45) \quad H := (x^+ \circ x^-)(B) = \sum_{p,q=1}^{d} v_p^* B_{pq} \beta(v_q^*) + \frac{d}{2} + \Omega_{\kappa} = \sum_{p=1}^{d} v_p^* v_p + \frac{d}{2} + \Omega_{\kappa}, \]

\[E^+ := \frac{1}{2}(x^+ \circ x^+)(B) = \frac{1}{2} \sum_{p,q=1}^{d} v_p^* B_{pq} v_q^* \]

\[E^- := \frac{1}{2}(x^- \circ x^-)(B) = \frac{1}{2} \sum_{p,q=1}^{d} \beta(v_p^*) B_{pq} \beta(v_q^*) = \frac{1}{2} \sum_{p,q=1}^{d} v_p B_{pq} v_q \]

satisfy the commutation relations (2.6).

4. Supercentralizers

To describe the supercentralizer of the realization of \(\mathfrak{osp}(1|2)\) in \(A_{\kappa}\) given by (3.45), we first look at the centralizer of its even subalgebra \(\mathfrak{sl}(2)\).

4.1. Centralizer of \(\mathfrak{sl}(2)\). In \(H_{\kappa} \subset A_{\kappa}\), we define \(M_{uv} = M(u, v) := u \beta(v) - v \beta(u)\) for \(u, v \in V^*\). Similar to \((w \circ z)(B)\) in (3.35), every element of \(V^*\) corresponds to a copy of \(V\) in \(V^* \otimes V\), or a copy of \(\mathcal{Y}_0\) in \(V^* \otimes \mathcal{Y}_0\). Using now the skew-symmetric
form $\omega$, we have for $u,v \in V^*$,
\[(u \wedge v)(\omega) = \frac{1}{2}(u\beta(v) - \beta(u)v - v\beta(u) + \beta(v)u)\]
\[= u\beta(v) - v\beta(u) + \frac{1}{2}([\beta(v),u] - [\beta(u),v])\]
\[= u\beta(v) - v\beta(u)\]
\[= \beta(v)u - \beta(u)v,\]

where we used Lemma 3.3. In the Dunkl operator realization, $M(u,v)$ becomes an angular momentum operator where the partial derivative is replaced by a Dunkl operator, see also [11, 4].

By means of the relations (3.40–3.41), it is easily verified that the elements of the form $M(u,v)$, for $u,v \in V^*$, commute with $(w \circ z)(B)$, for $w,z \in Y_0$.

In [7, Theorem 6.5], the authors proved that the centralizer of the $\mathfrak{sl}(2)$ realization inside $H_\kappa$ is the associative subalgebra of $H_\kappa$ generated by the group $G$ and the Dunkl angular momentum operators, that is
\[(4.2)\]
\[\text{Cent}_{H_\kappa}(\mathfrak{sl}(2)) = \langle M_{uv} \mid u,v \in V^* \rangle \rtimes G,\]
where the action of $g \in G$ on $M_{uv}$ is given by $M_{gugv}$ for $u,v \in V^*$ and $g \in G$.

The elements $M(u,v)$ for $u,v \in V^*$ generate a deformation of (the associative algebra generated by) the orthogonal Lie algebra $\mathfrak{so}(V,B) \cong \mathfrak{so}(d)$. The proof proceeds in the same way as the one for [8, Theorem 2.5] or [7, Proposition 6.7].

**Proposition 4.1.** For $u,v,x,y \in V^*$, in $H_\kappa$ one has
\[(4.3)\]
\[\left[ M(u,v), M(x,y) \right] = M(v,x)B_\kappa(u,y) - M(u,x)B_\kappa(v,y) - M(v,y)B_\kappa(u,x) + M(u,y)B_\kappa(v,x),\]

where $B_\kappa = B + \psi_B$, with the latter given in (3.25).

**Proof.** Use $M(u,v) = u\beta(v) - v\beta(u)$, $M(x,y) = x\beta(y) - y\beta(x)$
\[\left[ M(u,v), M(x,y) \right] = u[\beta(v),x]\beta(y) - v[\beta(u),x]\beta(y) - u[\beta(v),y]\beta(x) + v[\beta(u),y]\beta(x) + x[\beta(y),u]\beta(v) + y[\beta(x),u]\beta(v) - y[\beta(x),v]\beta(u)\]
which, using (3.22), equals
\[= M(u,y)[\beta(x),v] - M(v,y)[\beta(x),u] - M(u,x)[\beta(y),v] + M(v,x)[\beta(u),y] + u([\beta(x),v],\beta(y)) - v([\beta(x),u],\beta(y)) - [\beta(y),u],\beta(x))\]
\[+ x([\beta(u),y],\beta(v)) - [\beta(y),\beta(v),\beta(u)]) + y([\beta(u),x],\beta(v)) - [\beta(v),x],\beta(u))).\]

The last two lines vanish by Lemma 4.2, which we prove next. Hence, the desired result follows by (3.26).

**Lemma 4.2.** For $u,v \in V$ and $x^*,y^* \in V^*$, in $H_\kappa$
\[\left[ [x^*,u],v \right] = [[x^*,v],u], \quad [x^*,v],y^*] = [[y^*,v],x^*].\]

**Proof.** Writing out $[[x^*,u],v] - [[x^*,v],u]$ we have
\[(x^*u - ux^*)v - v(x^*u - ux^*) - (x^*v - vx^*)u + u(x^*v - vx^*),\]
where all terms cancel using (3.20). The other identity follows in the same way. \qed
4.2. **Supercentralizer of \(\mathfrak{osp}(1|2)\).** Recall that \(P_\pm\) is given by (2.10), the anti-symmetrization operator by (3.8), and the quantization map by (3.10). We now define the following elements, which, by Proposition 2.1, are in the supercentralizer of \(\mathfrak{osp}(1|2)\) in \(A_\kappa = H_\kappa \otimes C\). An explicit expression is given in Lemma 4.6.

**Definition 4.3.** For a positive integer \(n\) and \(u_1, \ldots, u_n \in V^*\), we define
\[
O_{u_1 \cdots u_n} := -P_\pm(q(u_1 \wedge \cdots \wedge u_n))/2 = -P_\pm(A(\gamma_{u_1 \cdots u_n}))/2 \in A_\kappa,
\]
which is skew-symmetric multilinear in its indices.

Note that the factor \(-1/2\) is chosen to coincide with the definition in [8], and to have a coefficient of 1 for \(M_{uv}\) in (4.7).

**Lemma 4.4.** The group \(\rho(\tilde{G})\) interacts with the elements (4.4) as follows
\[
\rho(\tilde{g})O_{u_1 \cdots u_n} = (-1)^{\tilde{g}^n}O_{\rho(\tilde{g})u_1 \cdots \rho(\tilde{g})u_n}\rho(\tilde{g}),
\]
for \(\tilde{g} \in \tilde{G}\) and \(u_1, \ldots, u_n \in V^*\).

**Proof.** This follows from \(P_\pm\) being an even element of \(U(\mathfrak{osp}(1|2))\) and thus commuting with \(\rho(\tilde{g})\) by Lemma 3.9, that the quantization map (3.10) is a \(G\)-module isomorphism and Lemma 3.7. \(\Box\)

Next, we want to give an explicit expression for the elements (4.4). Recall that \(\tilde{\sigma}_u\) for \(u \in V^*\) is given by (3.29). The case \(n = 1\) of the next result, shows that \(O_u = \tilde{\sigma}_u\) for \(u \in V^*\), see also (4.6).

**Lemma 4.5.** Let \(n \in \{1, 2, \ldots, d\}\), and \(u_1, \ldots, u_n \in V^*\), then
\[
P_\pm(\gamma_{u_1} \cdots \gamma_{u_n}) = (1-n)\gamma_{u_1} \cdots \gamma_{u_n} - 2\sum_{j=1}^{n} \gamma_{u_1} \cdots \tilde{\sigma}_{u_j} \cdots \gamma_{u_n}
\]
\[
- 2\sum_{1 \leq j < k \leq n} (-1)^{j+k-1}(u_j \beta(u_k) - \beta(u_j)u_k)\gamma_{u_1 \cdots \hat{u}_j \cdots \hat{u}_k \cdots u_n}.
\]

**Proof.** For \(v \in V^*\), using (2.10), (3.45), and (3.39), we have
\[
P_\pm(\gamma_v) = \gamma_v - [F^-, [F^+, \gamma_v]] = \gamma_v - [(x^- \circ \gamma)(B), v] = -2\tilde{\sigma}_v.
\]
Now, for \(n \in \{2, 3, \ldots, d\}\) and \(u_1, \ldots, u_n \in V\), we have
\[
P_\pm(\gamma_{u_1} \cdots \gamma_{u_n}) = \gamma_{u_1} \cdots \gamma_{u_n} - [F^-, [F^+, \gamma_{u_1}]\gamma_{u_2} \cdots \gamma_{u_n} - \gamma_{u_1}[F^+, \gamma_{u_2} \cdots \gamma_{u_n}]]
\]
\[
\quad = \gamma_{u_1} \cdots \gamma_{u_n} - [F^-, [F^+, \gamma_{u_1}]\gamma_{u_2} \cdots \gamma_{u_n} - [F^+, \gamma_{u_1}][F^-, \gamma_{u_2} \cdots \gamma_{u_n}]
\]
\[
\quad + [F^-, \gamma_{u_1}][F^+, \gamma_{u_2} \cdots \gamma_{u_n}] - \gamma_{u_1}[F^-, [F^+, \gamma_{u_2} \cdots \gamma_{u_n}]].
\]
By definition (2.10) and relation (3.39), this becomes
\[
P_\pm(\gamma_{u_1} \cdots \gamma_{u_n}) = P_\pm(\gamma_{u_1})\gamma_{u_2} \cdots \gamma_{u_n} + \gamma_{u_1}P_\pm(\gamma_{u_2} \cdots \gamma_{u_n}) - \gamma_{u_1} \cdots \gamma_{u_n}
\]
\[
- 2\sum_{j=2}^{n} (-1)^{j-2}(u_1 \beta(u_j) - \beta(u_1)u_j)\gamma_{u_1 \cdots \hat{u}_j \cdots u_n}.
\]
Applying this formula recursively yields the desired result. \(\Box\)

**Lemma 4.6.** Let \(n \in \{1, 2, \ldots, d\}\), and \(u_1, \ldots, u_n \in V^*\), then
\[
O_{u_1 \cdots u_n} = \frac{n-1}{2}A(\gamma_{u_1 \cdots u_n}) + nA(\tilde{\sigma}_{u_1} \gamma_{u_2 \cdots u_n}) + \frac{n(n-1)}{2}A(M_{u_1 u_2} \gamma_{u_3 \cdots u_n})
\]
\[
= -\frac{(n-1)(n-2)}{4}A(\gamma_{u_1 \cdots u_n}) - n(n-2)A(\tilde{\sigma}_{u_1} \gamma_{u_2 \cdots u_n}) + \frac{n(n-1)}{2}A(O_{u_1 u_2} \gamma_{u_3 \cdots u_n}).
\]
Note that the antisymmetrization in these expressions expands to
\[
O_{u_1\cdots u_n} = \frac{n-1}{2} \mathcal{A}(\gamma_{u_1\cdots u_n}) + \sum_{j=1}^{n} (-1)^{j-1} O_{u_j} \mathcal{A}(\gamma_{u_1\cdots \hat{u}_j \cdots u_n}) \\
+ \sum_{1 \leq j \leq k \leq n} (-1)^{j+k-1} M(u_j, u_k) \mathcal{A}(\gamma_{u_1\cdots \hat{u}_j \cdots \hat{u}_k \cdots u_n}) \\
= -\frac{(n-1)(n-2)}{4} \mathcal{A}(\gamma_{u_1\cdots u_n}) - (n-2) \sum_{j=1}^{n} (-1)^{j-1} O_{u_j} \mathcal{A}(\gamma_{u_1\cdots \hat{u}_j \cdots u_n}) \\
+ \sum_{1 \leq j \leq k \leq n} (-1)^{j+k-1} O_{u_j u_k} \mathcal{A}(\gamma_{u_1\cdots \hat{u}_j \cdots \hat{u}_k \cdots u_n}).
\]

**Proof.** The first expression follows by antisymmetrizing the result of Lemma 4.5 (multiplied by \(-1/2\), using Lemma 3.8 and noting that, by (3.22),
\[
\mathcal{A}(u_j \beta(u_k) - \beta(u_j) u_k) = (u_j \beta(u_k) - \beta(u_j) u_k - u_k \beta(u_j) + \beta(u_k) u_j)/2 = M(u_j, u_k).
\]
For \(u, v \in V^*\), the first expression gives
\[
(4.7) \quad O_{uv} = \mathcal{A}(\gamma_{uv})/2 + 2\mathcal{A}(\mathcal{O}_{u\gamma_v}) + M_{uv}.
\]
For general \(u_1, \ldots, u_n \in V^*\), we can use (4.7) to replace \(M(u_j, u_k)\) in the lemma’s first expression for \(O_{u_1\cdots u_n}\) to find the second. \(\square\)

**Lemma 4.7.** Let \(n \in \{2, \ldots, d\}\), and \(u_1, \ldots, u_n \in V^*\), then
\[
(4.8) \quad \mathcal{A}(\mathcal{O}_{u_1 u_2 \gamma_{u_3\cdots u_n}}) = \mathcal{A}(\gamma_{u_1} \mathcal{O}_{u_2 u_3 \gamma_{u_4\cdots u_n}}) = \cdots = \mathcal{A}(\gamma_{u_1\cdots u_{n-2}} \mathcal{O}_{u_{n-1} u_n}).
\]

**Proof.** This follows by means of (4.7) and then using Lemma 4.5 and the fact that \(M(u_j, u_k)\) commutes with all factors of \(\mathcal{A}(\gamma_{u_1\cdots \hat{u}_j \cdots \hat{u}_k \cdots u_n})\). \(\square\)

**Lemma 4.8.** Let \(n \in \{3, \ldots, d\}\), and \(u_1, \ldots, u_n \in V^*\), then
\[
(n-3)O_{u_1\cdots u_n} = -4(n-2)\mathcal{A}(\mathcal{O}_{u_1 u_2 \gamma_{u_3\cdots u_n}}) + 2(n-1)\mathcal{A}(\mathcal{O}_{u_1 u_2 u_3 \gamma_{u_4\cdots u_n}}).
\]

Note that expanding the antisymmetrization gives the following expressions
\[
\frac{n(n-3)}{4} O_{u_1\cdots u_n} = -(n-2) \sum_{j=1}^{n} (-1)^{j-1} O_{u_j} O_{u_1\cdots \hat{u}_j \cdots u_n} \\
+ \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} O_{u_j u_k} O_{u_1\cdots \hat{u}_j \cdots \hat{u}_k \cdots u_n}.
\]

**Proof.** The result follows by applying \(-P_{k}/2\) to the second expression of Lemma 4.6 and using Lemma 2.4. \(\square\)

The previous result shows that all elements of the form (4.4) can be written in terms of those having one, two or three indices.

In particular, for \(u, v \in V^*\), by (3.11), (4.7) becomes
\[
O_{uv} = u\beta(v) - \beta(u)v + (\gamma_u \gamma_v + B(u, v))/2 + \hat{\sigma}_u \gamma_v + \gamma_u \hat{\sigma}_v \\
= u\beta(v) - v\beta(u) + (\gamma_u \gamma_v - B(u, v))/2 + \hat{\sigma}_u \gamma_v - \hat{\sigma}_v \gamma_u.
\]

Note also that
\[
- P_3(\gamma_u \gamma_v)/2 = O_{uv} - B(u, v)/2.
\]

For \(u, v, w \in V^*\), by (3.12) we have
\[
O_{uvw} = \mathcal{A}(\gamma_{uvw}) + M(v, w)\gamma_u - M(u, w)\gamma_v + M(u, v)\gamma_w \\
+ \hat{\sigma}_u \mathcal{A}(\gamma_{uw}) - \hat{\sigma}_v \mathcal{A}(\gamma_{uw}) + \hat{\sigma}_w \mathcal{A}(\gamma_{uw}).
\]
Proposition 4.9. For \( \mathfrak{osp}(1|2) \) realized in \( A_\kappa = H_\kappa \otimes C \) by the elements (3.45), its supercentralizer \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)) \) is generated by \( \rho(\tilde{G}) \) and the elements \( O_{uv} \) and \( O_{uvw} \) for \( u, v, w \in V^* \).

Proof. By Proposition 2.1, we obtain the centralizer of \( \mathfrak{osp}(1|2) \) by applying the operator \( P_\pm \) to \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)_b) \). In \( H_\kappa \), the centralizer \( \text{Cent}_{H_\kappa}(\mathfrak{osp}(1|2)_b) \) of the even subalgebra \( \mathfrak{osp}(1|2)_b \cong \mathfrak{sl}(2) \) is generated by \( \{ M_{uv} \mid u, v \in V^* \} \) and the group \( G \), see 4.2. As \( H_\kappa \) does not interact with the Clifford algebra part of \( A_\kappa \), we have

\[
\text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)_b) = \text{Cent}_{H_\kappa}(\mathfrak{osp}(1|2)_b) \otimes C.
\]

First, we note that this means the elements \( O_{u_1 \cdots u_n} \) for \( u_1, \ldots, u_n \in V^* \) are in \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)) \) by Definition 4.3. Also, \( \rho(C(\tilde{G})) \subset \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)) \) by Lemma 3.9.

Now, a general element of \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)_b) \) can be written as a sum of terms of the form \( m \in g \) where \( m \) is a product of elements of \( \{ M_{uv} \mid u, v \in V^* \} \), \( c \in C \) and \( g \in G \). Assume \( m \) is a non-zero product of at least one element of \( \{ M_{uv} \mid u, v \in V^* \} \). Then, there are \( u, v \in V^* \) such that \( m = M_{uv}m' \) with \( m' \) a product of elements of \( \{ M_{uv} \mid u, v \in V^* \} \), one fewer than \( m \).

By (4.9) and using Lemma 2.4, we can write

\[
P_\pm(m \in g) = P_\pm(M_{uv}m' \in g)
= P_\pm((O_{uv} - (\gamma_{uv} - B(u, v))/2 - \tilde{\sigma}_u \gamma_v + \tilde{\sigma}_v \gamma_u)m' \in g)
= (O_{uv} + B(u, v)/2)P_\pm(m' \in g) - P_\pm(m' \gamma_{uv}c \in g)/2
- \tilde{\sigma}_u P_\pm(m' \gamma_v c \in g) + \tilde{\sigma}_v P_\pm(m' \gamma_u c \in g),
\]

since \( O_{uv}, \tilde{\sigma}_u, \tilde{\sigma}_v \) are in \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)) \). Hence, by proving the property for the case where \( m \) is a constant, the result follows by induction.

Note that for the case \( m \in g = M(u, v) \) we find in this way

\[
P_\pm(M_{uv}) = 2O_{uv} + 2\tilde{\sigma}_u \tilde{\sigma}_v - 2\tilde{\sigma}_v \tilde{\sigma}_u.
\]

An element \( g \in G \) can be written as a product of reflections in \( S \). For each \( s \in S \), since \( \gamma_{\alpha_s}^2 = B(\alpha_s, \alpha_s) \neq 0 \) in \( C \), we can write

\[s = \gamma_{\alpha_s}^2 s/B(\alpha_s, \alpha_s) = \gamma_{\alpha_s} \rho(\tilde{s})/\sqrt{B(\alpha_s, \alpha_s)}.
\]

In this way, we can write \( g = c_\tilde{g} \tilde{g} \) where \( c_\tilde{g} \in C \) and \( \tilde{g} \in \tilde{G} \). By Lemma 2.4, we have

\[
P_\pm(c \in g) = P_\pm(c c_\tilde{g} \tilde{g}) = P_\pm(c c_g) \tilde{g}.
\]

In particular, for \( s \in S \) this becomes

\[
P_\pm(s) = -2\tilde{\sigma}_s \rho(\tilde{s})/\sqrt{B(\alpha_s, \alpha_s)}.
\]

The elements of \( P_\pm(\text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)_b)) \) are thus given by products of elements in \( \rho(\tilde{G}) \) and elements of the form \( O_{u_1 \cdots u_n} \) for \( u_1, \ldots, u_n \in V^* \) Lemma 4.8 then shows that the latter can be generated by means of those having one, two or three indices, which completes the proof.

Given the interaction with the group \( \rho(\tilde{G}) \) in Lemma 4.4, a minimal set of generators for \( \text{Cent}_{A_\kappa}(\mathfrak{osp}(1|2)) \) requires only a subset of the elements \( O_{uv} \) and \( O_{uvw} \) for \( u, v, w \in V^* \).

4.3. Supercentralizer algebra relations. We can apply \( P_\pm \) to both sides of an equality to determine relations for elements of the centralizer \( O_\kappa \) in this realization. We recall that throughout the \( \mathbb{Z}_2 \)-graded commutator is used \( [a, b] = ab - (-1)^{|a||b|}ba \), where \( a, b \) are homogeneous elements for the \( \mathbb{Z}_2 \)-grading, and we also denote

\[
\{a, b\} = ab + (-1)^{|a||b|}ba.
\]
The following result includes a generalization of [8, Theorem 3.13].

**Proposition 4.10.** Let \( n \in \{2, \ldots, d \} \), and \( u_1, \ldots, u_n \in V^* \), then
\[
A(O_{u_1} O_{u_2} \cdots O_{u_n}) = A(O_{u_1} \cdots O_{u_{n-1}} O_{u_n}) \quad \text{or} \quad A([O_{u_1}, O_{u_2} \cdots O_{u_n}]) = 0,
\]
and
\[
A(O_{u_1} O_{u_2} O_{u_3} \cdots O_{u_n}) = A(O_{u_1} \cdots O_{u_{n-2}} O_{u_{n-1}} O_{u_n}) \quad \text{or} \quad A([O_{u_1}, O_{u_2} \cdots O_{u_n}]) = 0.
\]

**Proof.** By Lemma 3.8, we have \( A(O_{u_1} O_{u_2} \cdots O_{u_n}) = A(\gamma_{u_1} \cdots O_{u_{n-1}} O_{u_n}) \). The first result follows by applying \(-P_{\pm}/2\) and using Lemma 2.4. To see that this implies \( A([O_{u_1}, O_{u_2} \cdots O_{u_n}]) = 0 \), we expand the antisymmetrization \( A(O_{u_1} O_{u_2} \cdots O_{u_n}) = A(O_{u_1} \cdots O_{u_{n-1}} O_{u_n}) \) as
\[
- \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} O_{u_j} O_{u_1} \cdots \hat{u}_j \cdots O_{u_n} = \frac{1}{n} \sum_{j=1}^{n} (-1)^{n-j} O_{u_1} \cdots \hat{u}_j \cdots O_{u_n} O_{u_j}.
\]

Similarly, by Lemma 4.7 we have \( A(O_{u_1} O_{u_2} O_{u_3} \cdots O_{u_n}) = A(\gamma_{u_1} \cdots O_{u_{n-2}} O_{u_{n-1}} O_{u_n}) \) and again the result follows by applying \(-P_{\pm}/2\). \( \square \)

Specific cases of the previous proposition are, for \( u, v, w, z \in V^* \),
\[
(4.16) \quad [O_{uw} \circ O_{w}] - [O_{wu} \circ O_{v}] + [O_{vw} \circ O_{u}] = 0,
\]
\[
(4.17) \quad [O_{uvw} \circ O_{w}] - [O_{uwz} \circ O_{v}] + [O_{uwz} \circ O_{v}] - [O_{uwz} \circ O_{u}] = 0.
\]
Note also that by Proposition 4.10, one can rewrite antisymmetrized products using (4.15), for instance \( A(O_{u_1} O_{u_2} O_{u_3} O_{u_4}) = A([O_{u_1} \hat{u}_2 O_{u_3} O_{u_4}]) / 2 \).

**Lemma 4.11.** For \( u_1, \ldots, u_5 \in V^* \),
\[
O_{u_1} \cdots O_{u_4} = 6 A(O_{u_1} O_{u_2} O_{u_3} O_{u_4}) - 8 A(O_{u_1} O_{u_2} O_{u_3} O_{u_4})
\]
and
\[
O_{u_1} \cdots O_{u_5} = 4 A(O_{u_1} O_{u_2} O_{u_3} O_{u_4} O_{u_5}) + 48 A(O_{u_1} O_{u_2} O_{u_3} O_{u_4} O_{u_5}) - 36 A(O_{u_1} O_{u_2} O_{u_3} O_{u_4} O_{u_5}).
\]

**Proof.** By Lemma 4.8 and Proposition 4.10, we have
\[
(n-3)O_{u_1} \cdots u_n = 2(n-1)A(O_{u_1} O_{u_2} O_{u_3} \cdots O_{u_n}) - 4(n-2)A(O_{u_1} \cdots O_{u_{n-1}} O_{u_n}).
\]
For \( n = 4 \), this gives the desired result. For \( n = 5 \), we have
\[
2O_{u_1} \cdots O_{u_5} = 8A(O_{u_1} O_{u_2} O_{u_3} O_{u_4} O_{u_5}) - 12A(O_{u_1} \cdots O_{u_5}),
\]
where we can use the \( n = 4 \) result to obtain the desired result. \( \square \)

**Proposition 4.12.** For \( a, b, u, v \in V^* \), we have
\[
[O_{ab} \circ O_{uv}] = B(b, u)(O_{ab} + [O_{a} \circ O_{b}]) - B(a, u)(O_{be} + [O_{b} \circ O_{e}])
\]
\[
- [B(b, v)(O_{bu} + [O_{a} \circ O_{b}]) + B(a, v)(O_{bu} + [O_{b} \circ O_{b}])]
\]
\[
+ ([O_{ab} \circ O_{bu}] - [O_{b} \circ O_{ab}] + [O_{ab} \circ O_{ab}] - [O_{a} \circ O_{ab}]) / 2;
\]
or, denoting \( \hat{u} = B(b, a) - B(a) b \) and \( \hat{v} = B(b, v) a - B(a, v) b \),
\[
[O_{ab} \circ O_{uv}] = O_{uv} + [O_{ab} \circ O_{uv}] + [O_{a} \circ O_{b}] + [O_{b} \circ O_{a}]
\]
\[
- [O_{a} \circ O_{b}] - [O_{b} \circ O_{a}].
\]

**Proof.** Using the definition (4.9) together with (3.32) and (3.14)
\[
[O_{ab}, A(\gamma_{uv})] = [\gamma_{ab} / 2, \gamma_{uv}] + [O_{a} \gamma_{b} - O_{b} \gamma_{a}, \gamma_{uv}]
\]
\[
= [\gamma_{a} \gamma_{b}, \gamma_{a} \gamma_{b}] / 2 + \gamma_{u}[\gamma_{a} \gamma_{b}, \gamma_{a} \gamma_{b}] / 2
\]
\[
+ (O_{ab} \gamma_{b} - O_{b} \gamma_{a}) \gamma_{a} \gamma_{b} - \gamma_{a} \gamma_{b} \gamma_{a} O_{b} - \gamma_{a} \gamma_{b} O_{a}
\]
\[
= (B(b, u) \gamma_{a} - B(a, u) \gamma_{b}) \gamma_{v} + \gamma_{u}(B(b, v) \gamma_{a} - B(a, v) \gamma_{b})
\]
\[
+ O_{a} \gamma_{b} \gamma_{u} \gamma_{v} - O_{b} \gamma_{a} \gamma_{u} \gamma_{v} - \gamma_{u} \gamma_{v} \gamma_{a} O_{b} + \gamma_{u} \gamma_{v} \gamma_{b} O_{a}.
\]
By (4.4) and (4.10), applying \(-P_{\pm}/2\) to the previous computation gives

\[
[O_{ab}, O_{uv}] = B(b, u)(O_{av} - B(a, v)/2) - B(a, u)(O_{bu} - B(b, u)/2)
+ B(b, v)(O_{au} - B(a, u)/2) - B(a, v)(O_{bu} - B(b, u)/2)
+ O_{b}(O_{bu} + B(u, v)O_{b} - B(b, v)O_{a} - B(u, v)O_{a})
- O_{b}(O_{bu} + B(u, v)O_{b} - B(a, v)O_{a} + B(u, v)O_{a})
+ (O_{uw} + B(u, a)O_{u} - B(u, a)O_{u} + B(u, v)O_{a})O_{b}
+ (O_{uw} + B(u, b)O_{u} - B(u, b)O_{u} + B(u, v)O_{b})O_{a}.
\]

Collecting the appropriate terms and using (4.17) gives the desired results. □

The next two propositions contain expressions with four or five indices, which are given in Lemma 4.11.

**Proposition 4.13.** Let \(a, b, c, u, v, w \in V^*\), and denote \(\hat{x} = B(b, x)a - B(a, x)b\) for \(x \in \{c, u, v, w\}\). We have

\[
[O_{ab}, O_{uvw}] = O_{avw} + O_{uwv} + O_{uwv}
+ \{O_{a^\dagger} O_{uvw}\} - \{O_{b^\dagger} O_{uwv}\} + \{O_{b^\dagger} O_{uwv}\}
+ [O_{a}, O_{uvw}] - [O_{b}, O_{uwv}],
\]

and

\[
[O_{ab}, O_{cvuw}] = O_{cuvw} + O_{cuvw} + O_{cuuw} + O_{cuuw}
+ \{O_{b^\dagger} O_{cuvw}\} + \{O_{a^\dagger} O_{cuuw}\} + \{O_{b^\dagger} O_{cuvw}\} + \{O_{a^\dagger} O_{cuuw}\}
+ [O_{a^\dagger} O_{cuvw}] - [O_{b^\dagger} O_{cuuw}].
\]

**Proof.** For the first relation, using (4.9) and (3.32)

\[
[O_{ab}, A(\gamma_{xyz})] = [\gamma_{uv}/2, A(\gamma_{xyz})] + [O_{a}, \gamma_{v} - O_{b}, \gamma_{u}, A(\gamma_{xyz})].
\]

On the one hand, by (3.14) we have

\[
[\gamma_{uv}/2, A(\gamma_{xyz})] = \gamma_{xyz} + \gamma_{xzy} + \gamma_{yzx} - B(y, z)\gamma_{\hat{z}} - B(x, z)\gamma_{\hat{y}} - B(x, y)\gamma_{\hat{z}}
- A(\gamma_{xyz}) + A(\gamma_{xzy}) + A(\gamma_{yzx})
\]

where we used that

\[
B(\hat{x}, z) = B(v, x)B(u, z) - B(u, x)B(v, z) = -B(x, \hat{z}).
\]

On the other hand,

\[
[O_{a}, \gamma_{v} - O_{b}, \gamma_{u}, A(\gamma_{xyz})]
= (O_{a}, \gamma_{v} - O_{b}, \gamma_{u})A(\gamma_{xyz}) - A(\gamma_{xyz})(\gamma_{v}O_{b} - \gamma_{u}O_{a})
= [O_{a}, A(\gamma_{xyz})] - [O_{b}, A(\gamma_{xyz})]
+ \{O_{a}, B(v, x)A(\gamma_{yz}) - B(v, y)A(\gamma_{xz}) + B(v, z)A(\gamma_{xy})\}
- \{O_{b}, B(u, x)A(\gamma_{yz}) - B(u, y)A(\gamma_{xz}) + B(u, z)A(\gamma_{xy})\}
= [O_{a}, A(\gamma_{xyz})] - [O_{b}, A(\gamma_{xyz})]
+ \{B(v, x)O_{a} - B(u, x)O_{b}, A(\gamma_{yz})\}
- \{B(v, y)O_{a} - B(u, y)O_{b}, A(\gamma_{xz})\}
+ \{B(v, z)O_{a} - B(u, z)O_{b}, A(\gamma_{xy})\}.
\]

The result follows after applying \(-P_{\pm}/2\).

Similarly, for the second relation, we have

\[
[O_{ab}, A(\gamma_{cvuw})] = [A(\gamma_{ab})]/2 + O_{a}\gamma_{b} - O_{b}\gamma_{a}, A(\gamma_{cvuw})]
\]
where
\[ [\mathcal{A}(\gamma_{ab})/2, \mathcal{A}(\gamma_{cuvw})] = \mathcal{A}(\gamma_{cuvw}) + \mathcal{A}(\gamma_{c\bar{a}uv}) + \mathcal{A}(\gamma_{c\bar{a}uw}) + \mathcal{A}(\gamma_{cu\bar{w}}), \]

and
\[
(O_b^\gamma c - O_c^\gamma b)\mathcal{A}(\gamma_{auvw}) - \mathcal{A}(\gamma_{auvw})(\gamma_b O_c - \gamma_c O_b)
\]
\[ = [O_b^\gamma c \mathcal{A}(\gamma_{cuvw})] - B(c, w)\{O_b^\gamma c \mathcal{A}(\gamma_{auw})\} - [O_c^\gamma b \mathcal{A}(\gamma_{buvw})] - B(b, v)\{O_c^\gamma b \mathcal{A}(\gamma_{auw})\}, \]

while
\[ [O_b^\gamma c \mathcal{A}(\gamma_{cuvw})] = [\mathcal{A}(\gamma_{ab})/2 + O_a^\gamma b - O_b^\gamma a], \]

where
\[ [\mathcal{A}(\gamma_{ab})/2, \mathcal{A}(\gamma_{cuvw})] = B(a, u)\mathcal{A}(\gamma_{bcuvw}) + B(b, c)\mathcal{A}(\gamma_{auvw}) + B(b, v)\mathcal{A}(\gamma_{acuw}), \]

and
\[
(O_a^\gamma b - O_b^\gamma a)\mathcal{A}(\gamma_{cuvw}) - \mathcal{A}(\gamma_{cuvw})(\gamma_a O_b - \gamma_b O_a)
\]
\[ = [O_a^\gamma b \mathcal{A}(\gamma_{bcuvw})] - B(b, c)\{O_a^\gamma b \mathcal{A}(\gamma_{auw})\} - B(b, v)\{O_a^\gamma b \mathcal{A}(\gamma_{auw})\} - [O_b^\gamma c \mathcal{A}(\gamma_{acuw})] - B(a, u)\{O_b^\gamma c \mathcal{A}(\gamma_{cuw})\} . \]

The results follow after applying \(-P_\Delta/2.\)

Moreover, for brevity assuming also \(B(a, c) = 0 = B(b, c),\) we have

**Proposition 4.14.** Let \(a, b, c, u, v, w \in V^*\) be such that the only \(B\)-pairings between \(\{a, b, c\}\) and \(\{u, v, w\}\) that can be non-zero are \(B(a, u), B(b, v), B(c, w).\) We have
\[
[O_{abc} \mathcal{A}(\gamma_{cuvw})] = [−\mathcal{A}(\gamma_{abc})/2 - 3\mathcal{A}(O_{abc} \gamma_{c}) + 3\mathcal{A}(O_{ab} \gamma_{c}) \mathcal{A}(\gamma_{cuvw})].
\]

Proof. We have
\[
[O_{abc} \mathcal{A}(\gamma_{cuvw})] = [−\mathcal{A}(\gamma_{abc})/2 - 3\mathcal{A}(O_{abc} \gamma_{c}) + 3\mathcal{A}(O_{ab} \gamma_{c}) \mathcal{A}(\gamma_{cuvw})].
\]

Now, using (3.12) and (3.13)
\[
[−\mathcal{A}(\gamma_{abc})/2 \mathcal{A}(\gamma_{cuvw})] = B(a, u)\mathcal{A}(\gamma_{bcuvw}) - B(b, v)\mathcal{A}(\gamma_{acuw}) - B(c, w)\mathcal{A}(\gamma_{abuw}) + B(a, u)B(b, v)B(c, w),
\]

while \([-3\mathcal{A}(O_{abc} \gamma_{c}) \mathcal{A}(\gamma_{cuvw})] \) becomes
\[
− [O_a \mathcal{A}(\gamma_{bcuvw})] - \{O_a \mathcal{A}(\gamma_{bcuvw})\} + \{B(b, v)\mathcal{A}(\gamma_{acuw})\} + \{B(c, w)\mathcal{A}(\gamma_{abuw})\} + \{B(a, u)\mathcal{A}(\gamma_{abuw})\} + \{B(b, v)\mathcal{A}(\gamma_{acuw})\} + \{B(c, w)\mathcal{A}(\gamma_{abuw})\}
\]
\[ + [O_b \mathcal{A}(\gamma_{acuw})] - \{O_b \mathcal{A}(\gamma_{acuw})\} + \{B(a, u)\mathcal{A}(\gamma_{bcuvw})\} + \{B(b, v)\mathcal{A}(\gamma_{acuw})\} + \{B(c, w)\mathcal{A}(\gamma_{acuw})\} + \{B(a, u)\mathcal{A}(\gamma_{bcuvw})\} + \{B(b, v)\mathcal{A}(\gamma_{acuw})\} + \{B(c, w)\mathcal{A}(\gamma_{acuw})\}
\]
and
\[ [3\mathcal{A}(O_{abc} \gamma_{c}) \mathcal{A}(\gamma_{cuvw})] = [O_{abc} \mathcal{A}(\gamma_{cuvw})] + B(b, v)\{O_{abc} \mathcal{A}(\gamma_{uw})\} - [O_{abc} \mathcal{A}(\gamma_{cuvw})] + B(b, v)\{O_{abc} \mathcal{A}(\gamma_{uw})\} + [O_{abc} \mathcal{A}(\gamma_{cuvw})] + B(a, u)\{O_{abc} \mathcal{A}(\gamma_{uw})\} . \]

The result follows after applying \(-P_\Delta/2\) and using Proposition 4.13. \qed
4.3.1. Special bases. We will use the following notational conventions when a vector of the chosen bases of Section 3.2 appears as a subscript index in an element of the form (4.4):

An element of the set \( \{1, \ldots, d\} \) will be used to refer to the corresponding element of \( \{x_p\}_{p=1}^d \). For instance, \( O_{12} = O_{x_1x_2} \).

For \( \ell = [d/2] \), an element of the set \( \{1, \ldots, \ell\} \) with \( a + \) or \( - \) above it, will be used to refer to the corresponding element of \( \{z_p\}_{p=1}^{\ell} \), which is (part of) the \( B \)-isotropic basis of \( V^* \). If \( d \) is odd, an index 0 will be used to refer to \( z_0 \). For instance, \( O_{110} = O_{z_1z_2z_0} \).

The relations determined above, reduce to the following for elements of the basis \( \{x_p\}_{p=1}^d \). The proofs are also easier in this case, so they are included for completeness.

**Corollary 4.15.** For \( i, j, k, l, m, n \) distinct elements of the set \( \{1, \ldots, d\} \),

\[
(4.18) \quad [O_{ij}, O_{kl}] = [O_{jk}, O_{il}] + [O_{ij}, O_{kl}]
\]

\[
(4.19) \quad [O_{ij}, O_{kl}] = \frac{1}{2}([O_{ij}, O_{kl}] - [O_{ij}, O_{kl}] + [O_{ij}, O_{kl}])
\]

where the last equality follows by (4.17).

\[
(4.20) \quad [O_{ij}, O_{kl}] = -[O_{ij}, O_{kl}]
\]

Proof. For a 3-element set \( \{a, b, c\} \subset \{1, \ldots, d\} \), Proposition 4.16 below gives

\[
(4.21) \quad [O_{ijkl}, O_{mn}] = [O_{ij}, O_{km} + O_{km}, O_{ij}] - [O_{ij}, O_{km}]
\]

\[
(4.22) \quad [O_{ijk}, O_{l}] = -[O_{ijkl}, O_{m}] + [O_{ijkl}, O_{m}]
\]

\[
(4.23) \quad [O_{ijkl}, O_{m}] = -[O_{ijkl}, O_{m}]
\]

\[
(4.24) \quad [O_{ijkl}, O_{m}] = 2 \left( (O_{ij})^2 + (O_{kl})^2 + (O_{ij})^2 + (O_{kl})^2 \right) - \frac{1}{2}
\]

\[
(4.25) \quad [O_{ijkl}, O_{jkl}] = [O_{ij}, O_{kl}] + [O_{ij}, O_{kl}]
\]

\[
(4.26) \quad [O_{ijkl}, O_{m}] = [O_{ijkl}, O_{m}] + [O_{ijkl}, O_{m}]
\]

\[
(4.27) \quad [O_{ijkl}, O_{m}] = [O_{ijkl}, O_{m}] + [O_{ijkl}, O_{m}]
\]

\[
(4.28) \quad [O_{ijkl}, O_{m}] = -[O_{ijkl}, O_{m}] - [O_{ijkl}, O_{m}]
\]

\[
(4.29) \quad [O_{ijkl}, O_{m}] = -[O_{ijkl}, O_{m}] - [O_{ijkl}, O_{m}]
\]

\[
(4.30) \quad [O_{ijkl}, O_{m}] = [O_{ijkl}, O_{m}] - [O_{ijkl}, O_{m}]
\]

\[
(4.31) \quad (O_{abc})^2 = \frac{1}{4} + (O_{ab})^2 + (O_{ac})^2 + (O_{bc})^2 + (O_{ac})^2 + (O_{bc})^2
\]

For the second relation, we compute

\[
[O_{ijkl}, e_{jmn}] = [-e_{jkl}/2 + 3A(O_{ejkl}) + 3A(O_{ejkl})_e_{jmn}]
\]

\[
= -(O_{ijkl} - O_{kje})e_{jkl} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]

\[
+ (O_{ijkl} - O_{kje})e_{jkl} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]

\[
= O_{ijkl} - e_{jmn}O_{j} - O_{kje}e_{jkl} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]

\[
+ O_{ijkl}e_{jmn} - e_{jkl}O_{j} - O_{kje}e_{jkl} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]

The result follows after applying \( -P_{\pm}/2 \) to both sides and using Proposition 4.13.

Similarly, the final two relations follow from

\[
[O_{ijkl}, e_{j}m_{n}] = [-e_{jkl}/2 + 3A(O_{ejkl}) + 3A(O_{ejkl})e_{jmn}]
\]

\[
= e_{jkl}m_{n} - (O_{ijkl} - O_{kje})e_{jmk} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]

\[
+ (O_{ijkl} - O_{kje})e_{jkl} + O_{ijkl}e_{jmn} - e_{jkl}O_{j} - e_{jkl}O_{k} + e_{jkl}O_{j}
\]
yields the desired result.

Proof. Using again the properties of $u, v, w$ for $\Omega = (4.33)$

$$\{O_{ijkl}, O_{jkmn}\} = -O_{lmn} - \{O_l, O_m\} + \{O_{jk}, O_{jkmn}\}$$

Together with the relations (4.24-4.27) this shows that in $A_k$ a product of elements of the form $O_{uvw}$ for $u, v, w \in V^*$ can be reduced to terms containing at most a single 3-index element.

**Proposition 4.16.** For $n \in \{1, \ldots, d\}$ and $A = \{a_1, a_2, \ldots, a_n\} \subset \{1, \ldots, d\}$

$$(O_A)^2 = (-1)^{n(n-1)/2} \left( \frac{(n-1)(n-2)}{8} - (n-2) \sum_{a \in A} (O_a)^2 - \sum_{\{a,b\} \subset A} (O_{ab})^2 \right).$$

**Proof.** Note that $(e_A)^2 = (-1)^{n(n-1)/2}$ if $|A| = n$. By the last expression of Lemma 4.6

$$O_A e_A = \frac{(n-1)(n-2)}{4} (e_A)^2 - (n-2) \sum_{a \in A} O_a e_a (e_A)^2 - \sum_{\{a,b\} \subset A} O_{ab} e_a e_b (e_A)^2.$$ 

Applying $-P_{\pm}/2$ to both sides and using Lemma 2.4 yields the desired result. 

**Proposition 4.17.** For $u, v, w \in V^*$, we have

$$\{O_{1\ldots d}, O_a\} = 0, \quad [O_{1\ldots d}, O_{uv}] = 0, \quad \{O_{1\ldots d}, O_{uvw}\} = 0.$$ 

Moreover, the expression

$$\Omega = (d-2) \sum_{j=1}^d (O_j)^2 + \sum_{1 \leq j < k \leq d} (O_{jk})^2$$

is central in $\text{Cent}(\mathfrak{osp}(1|2))$.

Note that for $d$ odd, the relations (4.32) imply that $O_{1\ldots d}$ commutes (but not supercommutes as it has odd $Z_2$-degree) with everything in $\text{Cent}(\mathfrak{osp}(1|2))$.

**Proof.** The element $e_{1\ldots d}$ anticommutates with the generators of $\mathcal{C}$. In this way, for $u, v, w \in V^*$, using the expressions (3.29), (4.9), (4.11), we find

$$\{e_{1\ldots d}, O_u\} = 0, \quad [e_{1\ldots d}, O_{uv}] = 0, \quad \{e_{1\ldots d}, O_{uvw}\} = 0,$$

so the relations (4.32) follow after applying $-P_{\pm}/2$.

By Proposition 4.16, we have

$$(O_{1\ldots d})^2 = (-1)^{d(d-1)/2} \left( \frac{(d-1)(d-2)}{8} - (d-2) \sum_{j=1}^d (O_j)^2 - \sum_{1 \leq j < k \leq d} (O_{jk})^2 \right).$$

Using again the properties of $e_{1\ldots d}$, the element $O_{1\ldots d}$ equals

$$O_{1\ldots d} = \frac{1}{2} P_{\pm} (e_{1\ldots d}) = -\frac{1}{2} [e_{1\ldots d}, [F^-, [F^+, e_{1\ldots d}]]] = (F^- F^+ - F^+ F^- - 1/2) e_{1\ldots d}.$$
so, up to the sign $(-1)^{d(d-1)/2}$, the expression $O_{1,\ldots,d}c_{1,\ldots,d}$ equals the $\mathfrak{osp}(1|2)$ Casimir element (2.15). Since $-P_{\pm}(O_{1,\ldots,d}c_{1,\ldots,d})/2 = O_{1,\ldots,d}^2$, we find by (2.16) that, up to constants, $\Omega$ equals the $\mathfrak{osp}(1|2)$ Casimir element (2.14) in $U(\mathfrak{osp}(1|2))$ and thus is central in $\text{Cent}(\mathfrak{osp}(1|2))$. □

Using (4.31), we have

$$\sum_{i<j<k} d(O_{ijk})^2 = \frac{d(d-1)(d-2)}{24} + \frac{(d-1)(d-2)}{2} \sum_{j=1}^d (O_j)^2 + (d-2) \sum_{j<k}^d (O_{jk})^2,$$

so the following combination is also central

$$\frac{(d-1)(d-2)}{2} \sum_{j=1}^d (O_j)^2 + \sum_{1 \leq j < k \leq d} (O_{jk})^2 + \sum_{i<j<k}^d (O_{ijk})^2.$$

**Lemma 4.18.** In $A_{\kappa}$, one has

$$(\bar{\sigma} \otimes \gamma)(B) = \Omega_{\kappa} = (\gamma \otimes \bar{\sigma})(B).$$

*Proof.* Using (3.29) and (3.4), we have

$$(\bar{\sigma} \otimes \gamma)(B) = \sum_{p,q=1}^d \bar{\sigma}_{p\bar{q}} B(v_p, v_q) \gamma_{\bar{q}p}$$

$$= \sum_{p,q=1}^d B(v_p, v_q) \frac{1}{2} \sum_{s \in S} \alpha_s^p (v^*_p) \kappa(s) s \gamma_{\bar{q}s}$$

$$= \sum_{s \in S} \kappa(s) s \gamma_{\alpha_s} \bar{\gamma}_{\alpha_s}/B(\alpha_s, \alpha_s).$$ □

**Corollary 4.19.** For $i, j, k, r, s, t$ distinct elements of the set $\{1, \ldots, d\}$,

$$[O_{ij}, O_{kl}] = O_{jk} + [O_i, O_j] + [O_{ij}, O_i]$$

$$[O_{ij}, O_{kl}] = \frac{1}{2}([O_i, O_{jk}] - [O_j, O_{ik}] - [O_{ij}, O_k] + [O_{ijk}, O_l])$$

$$= [O_i, O_{jk}] - [O_j, O_{ik}],$$

where the last equality follows by (4.17).

$$[O_{ijk}, O_{lmn}] = [O_{ij}, O_{klm}] + [O_{ik}, O_{jlm}] - [O_{ik}, O_{jlm}]$$

$$[O_{ijk}, O_{lmn}] = -[O_{klm}, O_{ijn}] - [O_{jkn}, O_{ijn}]$$

where the last equality follows by (4.17).

$$[O_{ijk}, O_{ij} = 2 ((O_i)^2 + (O_j)^2 + (O_k)^2 + (O_{ij})^2 + (O_{ik})^2 + (O_{jk})^2) - \frac{1}{2}$$

$$[O_{ijk}, O_{ij}] = [O_i, O_i] + [O_{ijk}, O_{ij}] + [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ij}] = O_{jk} + [O_{ij}, O_{jk}] + [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ij}] = [O_i, O_{jk}] + [O_{ijk}, O_{ij}] + [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ij}] = [O_i, O_{ijk}] - [O_j, O_{ijk}] + [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ijk}] = -[O_{ij}, O_{ijk}] - [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ijk}] = -[O_{ijk}, O_{ij}] - [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ijk}] = [O_i, O_{ijk}] - [O_{ijk}, O_{ij}]$$

$$[O_{ijk}, O_{ijk}] = [O_i, O_{ijk}] - [O_{ijk}, O_{ij}].$$
For the $B$-isotropic basis we have the following. Fix $j, k \in \{1, \ldots, \ell\}$ with $j \neq k$ and let $u \in X$ such that $B(z^u_j, u) = 0 = B(z^u_k, u)$, then
\[ [O_{\pm z^u_j}, O_{\pm z^u_k}] = \pm 2(O_{\pm z^u_j} + [O_{\pm z^u_k}, O_{\pm z^u_j}]) - [O_{\pm z^u_j}, O_{\pm z^u_k}], \]
\[ [O_{\pm z^u_j}, O_{\pm z^u_k}] = \pm 2(O_{\pm z^u_j} + [O_{\pm z^u_k}, O_{\pm z^u_j}]) - [O_{\pm z^u_j}, O_{\pm z^u_k}], \]
\[ [O_{ij}, O_{kl}] = \frac{1}{2}(O_i, O_{jk}) - [O_i, O_{kl}] - [O_{ij}, O_{kl}] + [O_{ij}, O_{kl}] = [O_i, O_{jk}] - [O_j, O_{kl}], \]

4.4. Generalized symmetries. Recall (2.17), given in Section 2.4, and the $\mathfrak{osp}(1|2)$-elements (3.45). Denote $X := (x^+ \circ \gamma)(B) = \sum_{j=1}^d x_j e_j$ and $D := (x^- \circ \gamma)(B) = \sum_{j=1}^d y_j e_j$.

**Proposition 4.20.** For $u \in V^*$, the element
\[ R_u := Q^-(\gamma_u) = (H - 1)\gamma_u + X \beta(u) \]
is a generalized symmetry of $D$.

**Proof.** Follows by Proposition 2.6, using (3.39) of Lemma 3.10 to work out
\[ Q^-(\gamma_u) = (H - 1)\gamma_u + F^+[F^-, \gamma_u]. \]

\[ \Box \quad \text{TODO: put in TeX} \]

4.5. Representations and Howe correspondence.

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**APPENDIX**

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