Trilinear Higgs couplings in the two Higgs doublet model with CP violation

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We carry out a detailed analysis of the general two Higgs doublet model with CP violation. We describe two different parametrizations of this model, and then study the Higgs boson masses and the trilinear Higgs couplings for these two parametrizations. Within a rather general model, we find that the trilinear Higgs couplings have a significant dependence on the details of the model, even when the lightest Higgs boson mass is taken to be a fixed parameter. We include radiative corrections in the one-loop effective potential approximation in our analysis of the Higgs boson masses and the Higgs trilinear couplings. The one-loop corrections to the trilinear couplings of the two Higgs doublet model also depend significantly on the details of the model, and can be rather large. We study quantitatively the trilinear Higgs couplings, and show that these couplings are typically several times larger than the corresponding Standard Model trilinear Higgs coupling in some regions of the parameter space. We also briefly discuss the decoupling limit of the two Higgs doublet model.

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I. INTRODUCTION

The Higgs mechanism [1] of spontaneous electroweak symmetry breaking is a necessary ingredient of the Standard Model (SM), which is crucial for its internal consistency. The search for the Higgs boson is, thus, one of the major tasks for the experiments at the upcoming Large Hadron Collider (LHC). In order to confirm the Higgs mechanism as the origin of spontaneous breaking of $SU(2)_L \times U(1)_Y$ gauge symmetry in the SM, not only must the Higgs boson be discovered, but also its trilinear ($\lambda_{HHH}^{SM}$) and quartic ($\lambda_{HHHH}^{SM}$) self couplings must be measured in order to completely reconstruct the Higgs potential. Furthermore, one must also be able to measure the couplings of the Higgs boson to gauge bosons and fermions.

Here we shall be concerned mainly with the trilinear self couplings of the Higgs boson. In the Standard Model, there is only one trilinear self coupling of the Higgs boson which can be written simply in terms of the Higgs boson mass $M_H$ as

$$\lambda_{HHH}^{SM} = \frac{3M_H^2}{v},$$

where $v = 2M_W/g = 246 \text{ GeV}$ is the vacuum expectation value of the neutral component of the Higgs doublet, $M_W$ is the mass of the $W^\pm$, and $g$ is the $SU(2)_L$ gauge coupling, respectively. Several extensions of the SM, such as the minimal supersymmetric standard model (MSSM) [2], and the general two Higgs doublet model (2HDM) [3], have a more complicated Higgs structure. In these models there are several trilinear Higgs couplings, having more complicated dependence on the underlying masses. It is a challenging task to measure these trilinear couplings at the LHC. On the other hand a linear collider could possibly offer much better prospects of measuring these trilinear Higgs couplings [4, 5, 6, 7, 8, 9].

At the proposed International Linear Collider (ILC), if the Higgs boson is not too heavy, the trilinear Higgs coupling can be measured via the Higgs boson pair production $e^+e^- \rightarrow W^+W^-\nu\bar{\nu} \rightarrow H\nu\nu\tau\bar{\nu}$. A precise measurement of the trilinear Higgs self coupling will also make it possible to test extended Higgs models, which have a different structure of the Higgs potential, and hence different trilinear Higgs couplings, as compared to the SM. The 2HDM is the simplest, yet a very general, model with an extended Higgs structure, which leads to various distinct physical effects. In particular, the model can easily accommodate additional CP violation [10, 11, 12, 13], beyond what is generated by...
the Kobayashi–Maskawa (KM) mechanism \[14\]. Furthermore, since there is a distinct possibility of measuring precisely the Higgs boson self couplings at the ILC, there is a motivation to study the radiative corrections to the trilinear self couplings of the Higgs boson.

On the other hand, there are a number of parameters in the potential of the two Higgs doublet model, including those associated with CP violation, which determine the masses and CP properties of the model. In spite of the complicated nature of the Higgs potential, it has been shown in the 2HDM with CP conservation that one-loop corrections to the lightest CP-even Higgs boson self coupling are in general of a non decoupling nature, and can give rise to \(\mathcal{O}(100\%)\) deviations from the SM prediction \[15\]. This happens even when all other couplings of the lightest Higgs boson to gauge bosons and fermions are in good agreement with the SM prediction.

In this paper we shall study in detail the trilinear Higgs couplings in the two Higgs Doublet Model (2HDM) with CP violation. We shall assume that the underlying gauge group is the SM gauge group. After spontaneous breaking of the SM gauge symmetry, the Higgs spectrum of the model consists of three neutral Higgs bosons and two charged Higgs bosons. In the CP-conserving version of the model, two neutral Higgs bosons \((h^0, H^0)\) are CP even, whereas one neutral Higgs boson \((A^0)\) is CP odd. In the CP-conserving case, there are, thus, six allowed trilinear Higgs couplings which can be labelled as \(\lambda_{h\phi h}, \lambda_{hAH}, \lambda_{aHH}, \lambda_{HHH}\) involving the CP-even Higgs bosons, and \(\lambda_{NAA}, \lambda_{HAa}, \lambda_{AaA}\), all even in the number of \(A^0\), involving the CP-odd Higgs boson. When CP is not conserved, the neutral Higgs bosons do not have a definite CP, and there is no such constraint on the couplings involving the \(A^0\) Higgs boson. There, are, thus, a total of ten trilinear Higgs couplings in the case of the 2HDM with CP violation.

Since this is one of the simplest models which goes beyond the KM mechanism of CP violation, it is important to study the impact of CP violation on the Higgs self couplings in this model. We shall, therefore, consider the 2HDM with explicit CP violation in the Higgs sector in this paper. This explicit CP violation is introduced through appropriate complex parameters in the potential of the 2HDM. Furthermore, we shall assume that this explicit CP violation cannot be transformed away by a redefinition of the Higgs fields. Thus, one of the objectives of the present study is to determine the effects of this explicit CP violation in the Higgs sector on various Higgs self couplings in the two Higgs doublet model. In our study of the model, we shall find it convenient to keep the masses of the two lightest neutral Higgs bosons fixed, and then study the dependence on other parameters of the model, which will determine the amount of explicit CP violation in various Higgs self couplings, and also determine the mass of the heaviest Higgs boson. In this way, we do not lay emphasis on the heavy Higgs sector of the theory, whereas at the same time we can exhibit the wide range of values that the trilinear Higgs couplings can assume. In particular, we wish to emphasize that, in contrast to the MSSM, the heavy mass effects in a general 2HDM do not decouple. In order to demonstrate this we shall calculate the one-loop corrected Higgs boson self-couplings using the method of effective potential in a general 2HDM with explicit CP violation as described above. This is in contrast to Ref. \[12\], where all couplings and mass parameters were assumed to be real, thereby precluding the phenomena of explicit CP violation in the Higgs sector as considered in this paper.

The plan of this paper is as follows. In Section \[\text{II}\] we describe the most general two Higgs doublet model with CP violation. We then discuss the spectrum of the Higgs bosons of the model, and the constraints on the parameters when there is CP violation in the model. Here we also discuss how only certain sets of parameters of the Higgs potential can lead to physically consistent models, and discuss different ways in which these parameters can be specified. We delineate the regions of the parameter space where there is no CP violation, corresponding to which one of the neutral Higgs bosons is odd under P. In Section \[\text{III}\] we derive the tree level trilinear couplings between the neutral as well as the charged Higgs bosons of the model, and discuss the correspondence of these couplings with the trilinear Higgs couplings of the minimal supersymmetric standard model. Here we also study numerically the trilinear Higgs couplings, and discuss the domain of the parameter space that is compatible with various theoretical and experimental constraints.

In Section \[\text{IV}\] we use the method of one-loop effective potential to calculate, as a first step, one-loop corrections to the Higgs boson masses. In Section \[\text{V}\] we then calculate the one-loop corrections to the trilinear self couplings of the Higgs bosons. In Section \[\text{VI}\] we carry out a detailed numerical study of the one-loop corrected trilinear Higgs couplings, discuss the magnitudes of different contributions, the dependence on the scale, as well as the decoupling limit, and show that these couplings can be several times larger than the corresponding Standard Model trilinear Higgs coupling in some regions of the parameter space. We summarize our results and conclusions in Section \[\text{VII}\]. Some of the analytical calculations used in our analysis are described in appendices.

II. THE TWO-HIGGS-DOUBLET MODEL

The general 2HDM with the underlying gauge group \(SU(2)_L \times U(1)_Y\) is obtained by extending the Higgs sector of the SM with a second \(SU(2)_L\) Higgs doublet with weak hypercharge \(Y = 1\). Thus, the model contains 4 complex scalar fields which are arranged as \(SU(2)_L\) doublets as follows:

\[
\Phi_i = \begin{pmatrix}
\varphi_i^+ \\
\varphi_i
\end{pmatrix} \quad (Y = +1), \quad i = 1, 2. \quad (2.1)
\]

Using these two Higgs doublets, the most general renormalizable potential for the 2HDM which is invariant un-
under the $SU(2)_L \times U(1)_Y$ gauge group can be written as

$$V_{\text{tree}} = \frac{\lambda_1}{2}(\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger \Phi_2)^2 + \lambda_3(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \frac{1}{2} \left[ \lambda_5(\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} \right]$$

$$+ \left\{ \left[ \lambda_6(\Phi_1^\dagger \Phi_1) + \lambda_7(\Phi_2^\dagger \Phi_2) \right] (\Phi_1^\dagger \Phi_2) + \text{h.c.} \right\}$$

$$- \frac{1}{2} \left\{ m_{11}^2(\Phi_1^\dagger \Phi_1) + m_{12}^2(\Phi_1^\dagger \Phi_2) + \text{h.c.} \right\}$$

$$+ m_{22}^2(\Phi_2^\dagger \Phi_2)), \quad (2.2)$$

where $\lambda_i$ ($i = 1, \ldots, 7$) are dimensionless parameters and the subscript “tree” denotes that (2.2) is a tree-level potential. We note that $\lambda_i$ ($i = 1, \ldots, 4$) are real, whereas $\lambda_i$ ($i = 5, \ldots, 7$) are in general complex parameters. Similarly, $m_{11}^2$ and $m_{12}^2$ are real, whereas $m_{22}^2$ is in general complex. We note that the terms proportional to $\lambda_5$ and $\lambda_7$ have to be constrained, since this potential does not satisfy natural flavor conservation.\[10\], even when each doublet is coupled only to up-type or only to down-type quarks.

When the neutral components of the two Higgs doublets $\Phi_{1,2}$ acquire vacuum expectation values (VEVs), $\phi_{1,2}^0 = v_{1,2}/\sqrt{2}$, the gauge group $SU(2)_L \times U(1)_Y$ breaks down to $U(1)_{\text{em}}$, whereby three of the eight real fields in (2.1) are absorbed by three of the four gauge bosons of $SU(2)_L \times U(1)_Y$, which become massive in the process, leaving behind a massless photon. We can then parametrize the two Higgs doublet fields in (2.1) as

$$\Phi_i = \begin{pmatrix} \varphi_i^+ \\ \chi_i \end{pmatrix} \left( \frac{1}{\sqrt{2}}(v_i + \eta_i + i\lambda_i) \right), \quad i = 1, 2, \quad (2.3)$$

where we have chosen the VEVs of the neutral Higgs fields to be real, and absorbed the relative phase between the two VEVs in the parameters $m_{12}^2, \lambda_5, \lambda_6$ and $\lambda_7$ of $V_{\text{tree}}$. When we substitute the parametrisation (2.3) of the Higgs fields in the potential (2.2), there will be cubic terms in the Higgs fields arising from the quartic couplings $\lambda_i$, which will give rise to trilinear couplings among the Higgs fields. It is these trilinear couplings that we shall study in detail in this paper.

As discussed in the Introduction, in the case of 2HDM with CP conservation, after spontaneous breakdown of the gauge symmetry, we are left with two CP-even Higgs bosons $h^0, H^0$, a CP-odd Higgs boson $A^0$, and a pair of charged Higgs bosons $H^\pm$. However, with CP violation the neutral Higgs bosons $h^0, H^0, A^0$ mix, and it is then more appropriate to define the weak states

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \quad (2.4)$$

to describe the neutral Higgs sector of the CP violating two Higgs doublet model. In (2.4) we have defined

$$\eta_3 = -\sin \beta \chi_1 + \cos \beta \chi_2, \quad (2.5)$$

$$G^0 = \cos \beta \chi_1 + \sin \beta \chi_2, \quad (2.6)$$

where $G^0$ is the would-be neutral Goldstone boson, and $\tan \beta = v_2/v_1$ is the ratio of the vacuum expectation values of the two Higgs fields. We shall take $\tan \beta$ as an independent parameter. The charged Higgs fields and the corresponding charged Goldstone boson are likewise defined as

$$H^\pm = -\sin \beta \varphi_1^\pm + \cos \beta \varphi_2^\pm, \quad (2.7)$$

$$G^\pm = \cos \beta \varphi_1^\pm + \sin \beta \varphi_2^\pm, \quad (2.8)$$

respectively.

A. Basis rotation

The tree-level mass squared matrix of the neutral Higgs bosons can now be defined as

$$M_{ij}^2 = \frac{\partial^2 V}{\partial \eta_i \partial \eta_j}, \quad (2.9)$$

where, after differentiation, all fields are set equal to zero:

$$\eta_1 = \eta_2 = \eta_3 = H^\pm = G^0 = G^\pm = 0.$$  

The physical neutral Higgs states $H_i$, which are the eigenstates of the mass squared matrix (2.9), are then obtained by a rotation $R$

$$H = R \eta, \quad \eta = R^T H, \quad (2.10)$$

or, more explicitly

$$H_i = R_{ij} \eta_j, \quad \eta_j = R_{ij} H_i. \quad (2.11)$$

The rotation matrix $R$ which diagonalizes (2.9),

$$R M^2 R^T = M_{\text{diag}}^2 = \text{diag}(M_1^2, M_2^2, M_3^2), \quad (2.12)$$

with $M_1 < M_2 < M_3$, can be parametrized as

$$R = R_3 R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_3 & \sin \alpha_3 \\ 0 & -\sin \alpha_3 & \cos \alpha_3 \end{pmatrix} \begin{pmatrix} \cos \alpha_2 & 0 & \sin \alpha_2 \\ 0 & 1 & 0 \\ -\sin \alpha_2 & 0 & \cos \alpha_2 \end{pmatrix} \begin{pmatrix} \cos \alpha_1 & \sin \alpha_1 & 0 \\ -\sin \alpha_1 & \cos \alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
where $c_i = \cos \alpha_i$, $s_i = \sin \alpha_i$. Note that Eq. (2.12) can be inverted as

$$\left( M^2 \right)_{ij} = \sum_k R_{ki} M_k^2 R_{kj}. \quad (2.14)$$

By symmetry, $\left( M^2 \right)_{ii} = (M^2)_{jj}$, and, in the general case, this matrix is seen to contain 6 independent parameters. These may be taken as the three masses, and the angles of the rotation matrix.

As discussed in Ref. [17], there are three limits in which there is no CP violation in the 2HDM, corresponding to which one of the neutral Higgs bosons is odd under $\mathcal{P}$. These limits can be characterised in terms of the angles $\alpha_2$ and $\alpha_3$ of the rotation matrix (2.13) as follows:

- $H_1$ is odd: $\quad \alpha_2 = \pm \pi/2, \quad (2.15)$
- $H_2$ is odd: $\quad \alpha_3 = \pm \pi/2, \quad (2.16)$
- $H_3$ is odd: $\quad \alpha_2 = 0, \quad \alpha_3 = 0. \quad (2.17)$

We note that in the latter limit, we have $\alpha_1 = \alpha + \pi/2$, where $\alpha$ is the conventional mixing angle in the CP-even sector of the MSSM [13]. Any CP violation in the 2HDM will also depend on the Yukawa couplings. In the 2HDM II (Model II), where the down-type quarks couple to $\Phi_1$ and the up-type quarks couple to $\Phi_2$, various measures of “maximal” CP violation both in the Higgs-vector boson and Higgs-quark sectors are discussed in [18]. These maxima occur in the “bulk” of the $\alpha_2$–$\alpha_3$ space, typically for $|\alpha_2| = O(\pi/4)$ and $|\alpha_3| = O(\pi/4)$.

### B. Neutral Higgs boson masses

Minimizing the tree-level potential according to

$$\frac{\partial V}{\partial \Phi_i} = 0, \quad i = 1, 2, \quad (2.18)$$

and using the resulting minimization conditions to eliminate $m_{11}^2$ and $m_{22}^2$, one obtains the elements (2.9) of the tree-level mass squared matrix

$$M_{11}^2 = v_1^2 \lambda_1 + v_2^2 \nu + \frac{v_2}{2v_1} \text{Re} \left( 3v_1^2 \lambda_6 - v_2^2 \lambda_7 \right),$$

$$M_{22}^2 = v_2^2 \lambda_2 + v_1^2 \nu + \frac{v_1}{2v_2} \text{Re} \left( -v_1^2 \lambda_6 + 3v_2^2 \lambda_7 \right),$$

$$M_{33}^2 = v^2 \text{Re} \left[ -\lambda_5 + \nu - \frac{1}{2v_1v_2} (v_1^2 \lambda_6 + v_2^2 \lambda_7) \right],$$

$$M_{12}^2 = v_1 v_2 \text{Re} \left( \lambda_3 + \lambda_4 + \lambda_5 - \nu \right) + \frac{1}{2} \text{Re} \left( v_1^2 \lambda_6 + v_2^2 \lambda_7 \right),$$

$$M_{13}^2 = -\frac{1}{2} v \text{Im} \left[ v_2 \lambda_5 + 2v_1 \lambda_6 \right],$$

where we have defined

$$v_1 = v \cos \beta, \quad v_2 = v \sin \beta, \quad \beta \in \left( 0, \frac{\pi}{2} \right), \quad (2.20)$$

and

$$\nu = \frac{1}{2v_1v_2} \text{Re} \left( m_{12}^2 \right), \quad (2.21)$$

with $\nu^2 = v_1^2 + v_2^2 = (246 \text{ GeV})^2$. The squared masses $M_i^2$ of the physical neutral Higgs bosons are obtained as the eigenvalues of the mass squared matrix (2.9). These eigenvalues are solutions of a cubic equation which involves the parameters $\lambda_i$. However, only some set of values of the parameters $\lambda_i$ lead to consistent solutions. In order to identify physically consistent models, we shall, following the approach of Refs. [13, 20, 21], specify physical (tree-level) masses, instead of the $\lambda_i$, as parameters. Depending on how much constrained a model we want to consider, we shall do this in two ways, which we shall denote “approach (A)” and “approach (B)”: 

(A) In this case, which corresponds to $\lambda_6 = \lambda_7 = 0$, two elements of $M^2$ are related via $\tan \beta$ and we cannot therefore take all three masses as independent. Instead, we take the two lightest neutral Higgs boson masses, together with $\tan \beta$ and the three angles $(\alpha_1, \alpha_2, \alpha_3)$ defining the mixing matrix $R$ of Eq. (2.10), as independent parameters. The third (heaviest) Higgs boson mass cannot be determined. In general, the elements $R_{13}$ and $R_{23}$ of the rotation matrix $R$ must be non-zero in order to have CP violation. For consistency, the derived quantities $\text{Im} \lambda_5$ and $\text{Im} m_{12}^2$ must be non-zero.

(B) Here we take $\lambda_6, \lambda_7 \neq 0$. In this case, we take all three neutral Higgs boson masses, $\tan \beta$, together with the mixing matrix $R$ of Eq. (2.10), $\text{Im} \lambda_5$, $\text{Re} \lambda_6$ and $\text{Re} \lambda_7$ as the input parameters.

In either case, specifying the charged Higgs boson mass, $M_{H^\pm}$, as well as the bilinear parameter $\mu^2 = v^2 \nu$, we can determine the remaining $\lambda_i$ through a set of linear relations. They are, thus, unique. Details are given in Appendix A.

We note that in general, there exist multiple minima in the 2HDM potential [11, 22]. However, with our choice of input parameters, including Higgs squared masses, and these being positive, the minimum we are working in is a global one and hence stable [23].
III. TRILINEAR SELF COUPLINGS OF HIGGS BOSONS AT THE TREE LEVEL

In this Section we shall define the trilinear Higgs self couplings, and obtain explicit expressions for them in terms of the parameters of the tree-level Higgs potential of the 2HDM. We shall then study the behavior of these couplings as functions of the various parameters of the model.

A. Trilinear Couplings of Neutral Higgs Bosons

The trilinear self couplings of the neutral Higgs bosons are defined as

\[ \lambda_{ijk} = \frac{-i \partial^3 V}{\partial H_i \partial H_j \partial H_k}, \]  

(3.1)

which are most easily obtained from the corresponding derivatives of \( V \) in \((2.2)\) with respect to the weak fields \( \eta \). The derivatives with respect to \( H_i \) in \((3.1)\) are obtained by noting the useful relation

\[ \frac{\partial}{\partial H_i} = \frac{\partial \eta_i}{\partial H_i} \frac{\partial}{\partial \eta_i} = R_{ij} \frac{\partial}{\partial \eta_j}, \]  

(3.2)

which can be used to go from the \( \eta \) basis to the physical \( H \) basis.

When there is CP violation, the trilinear couplings among the physical Higgs bosons will involve elements of the rotation matrix \( R \). We can then write the trilinear couplings in terms of the derivatives of the potential \((2.2)\) with respect to \( \eta \) and the elements of the rotation matrix \( R \) as

\[ \lambda_{ijk} = \sum_{m \leq n \leq o=1,2,3} R_{i'm} R_{j'n} R_{k'o} \eta_{m} \frac{-i \partial^3 V}{\partial \eta_{m} \partial \eta_{n} \partial \eta_{o}}, \]  

(3.3)

where the indices \( m, n, o \) refer to the weak field basis, and the * denotes a sum over permutations \( P \), \( \{i', j', k'\} = P\{i, j, k\} \), which gives rise to a factor of \( n! \) for \( n \) identical fields. We now proceed to obtain these couplings in an explicit form.

At the tree level, when expressed in terms of \( \lambda_i \), the derivatives in Eq. \((3.3)\) are rather simple. The trilinear couplings \( a_{mno} \) among the weak fields \( \eta \) can be written as (in units of \( -iv \)) \((2.4, 2.5)\):

\[ a_{111} = \frac{1}{2} (\cos \beta \lambda_1 + \sin \beta \Re \lambda_6), \]
\[ a_{112} = \frac{1}{2} (\sin \beta \Re \lambda_{345} + 3 \cos \beta \Re \lambda_6), \]
\[ a_{113} = -\frac{1}{2} (\cos \beta \sin \beta \Im \lambda_5 + (1 + 2 \cos^2 \beta) \Im \lambda_6), \]
\[ a_{122} = \frac{1}{2} (\cos \beta \Re \lambda_{345} + 3 \sin \beta \Re \lambda_7), \]
\[ a_{123} = -\Im \lambda_5 - \cos \beta \sin \beta \Im \lambda_6 + \Im \lambda_7), \]
\[ a_{133} = \frac{1}{2} (\cos \beta (\sin^2 \beta \lambda_1 + \cos^2 \beta \Re \lambda_{345} - 2 \Re \lambda_6 + \cos^2 \beta \Re \lambda_7)), \]
\[ a_{222} = \frac{1}{2} (\sin \beta \lambda_2 + \cos \beta \Re \lambda_7), \]
\[ a_{223} = -\frac{1}{2} (\cos \beta \sin \beta \Im \lambda_5 + (\cos^2 \beta + 3 \sin^2 \beta) \Im \lambda_7), \]
\[ a_{233} = \frac{1}{2} (\sin \beta (\cos^2 \beta \lambda_2 + \sin^2 \beta \Re \lambda_{345} - 2 \Re \lambda_6) + \cos \beta (\sin^2 \beta \Re \lambda_6 + (\cos^2 \beta - 2 \sin^2 \beta) \Re \lambda_7)), \]
\[ a_{333} = \frac{1}{2} (\cos \beta \sin \beta \Im \lambda_5 - \sin^2 \beta \Im \lambda_6 - \cos^2 \beta \Im \lambda_7), \]

(3.4)

where

\[ \lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5. \]  

(3.5)

In order to elucidate the compact notation in Eq. \((3.3)\), we explicitly write the trilinear couplings \( \lambda_{111} \) and \( \lambda_{112} \) in Appendix B.

Rather than studying all the trilinear couplings of the Higgs bosons, we shall here focus on the two couplings \( \lambda_{111} \) and \( \lambda_{112} \) involving the lightest Higgs boson \( H_1 \). In the special case when \( \beta = 0 \) and in addition \( \alpha_2 = \alpha_3 = 0 \), these couplings take the simple form

\[ \lambda_{111} = -3ivc_1c_3s_1 + s_2s_3 \lambda_2 \]
\[ + c_1s_1(c_1s_3 + s_1c_3)\Re \lambda_{345}, \]  

(3.6a)
\[ \lambda_{112} = iv(3c_1s_1(c_1c_3s_1 - s_1s_3) \lambda_2 \]
\[ - [c_1s_2 - s_1c_2 + 2c_1s_1(c_1c_3 - s_1s_3)\Re \lambda_{345}], \]  

(3.6b)

In this limit of \( \lambda_6 = \lambda_7 = 0 \) and \( \alpha_2 = \alpha_3 = 0 \), we note the following:

- For small values of \( \tan \beta \), both \( \lambda_{111} \) and \( \lambda_{112} \) are determined by \( \lambda_1 \) and \( \Re \lambda_{345} \). This is because for small \( \tan \beta \) terms containing \( s_\beta \equiv \sin \beta \) vanish. In actual practice, due to the constraints imposed by \( B \)-physics, the small \( \tan \beta \) limit is not reached.

- For large values of \( \tan \beta \), both \( \lambda_{111} \) and \( \lambda_{112} \) are determined by \( \lambda_2 \) and \( \Re \lambda_{345} \). In this case terms containing \( c_\beta \equiv \cos \beta \) vanish.

Here, the combination \( \Re \lambda_{345} \) is related to \( M_H^2 \) via

\[ M_H^2 = \mu^2 + \frac{1}{2}v^2[\lambda_3 - \Re \lambda_{345} - \Re (\lambda_6 + \lambda_7)]. \]  

(3.7)

It is instructive to compare the couplings \((3.6)\) with the Standard Model trilinear coupling \((1.1)\). While the SM trilinear coupling is given in terms of only one parameter, the mass \( M_H \) of the Higgs boson, those of the 2HDM depend on several parameters, and may actually pass through zero, even for the special case studied in Eq. \((3.6)\). In the general case, we find it convenient to study the dimensionless ratios of the couplings

\[ \xi_1 = \frac{\lambda_{111}}{\lambda_{HHH}}, \quad \xi_2 = \frac{\lambda_{112}}{\lambda_{HHH}}, \]  

(3.8)

where for the reference SM coupling we use the mass \( M_H \) of the lightest Higgs boson of the 2HDM. These dimensionless ratios of the couplings will be calculated and discussed in the following.
1. 2HDM–MSSM correspondence

It is also useful to compare the trilinear Higgs couplings in the 2HDM with the corresponding ones in the minimal supersymmetric standard model. In the MSSM, where there is no CP violation in the Higgs sector at the tree level, there are six trilinear Higgs couplings. The couplings corresponding to \( \lambda_{111} \) and \( \lambda_{112} \) are [6, 26]

\[
\lambda_{hhh} = \frac{3igM_Z}{2\cos\theta_W}\cos(2\alpha)\sin(\alpha + \beta),
\]

\[
\lambda_{hhh} = -\frac{igM_Z}{2\cos\theta_W}[2\sin(2\alpha)\sin(\alpha + \beta)
- \cos(2\alpha)\cos(\alpha + \beta)],
\]

where \( M_Z \) is the mass of the \( Z \), \( \theta_W \) is the Weinberg angle, and \( g \) the \( SU(2)_L \) gauge coupling. The angle \( \alpha \) is the mixing angle in the CP even Higgs sector of the MSSM. In the limit \( \alpha_2 \to 0 \) and \( \alpha_3 \to 0 \), it corresponds to \( \alpha_1 \to \pi/2 \) of the 2HDM.

In the MSSM the trilinear Higgs couplings are controlled by the \( SU(2)_L \) gauge coupling \( g \). In contrast, the trilinear Higgs couplings in the 2HDM arise from the quartic terms of the Higgs potential when the SM gauge symmetry is broken. It is easy to see that in the above limit the couplings in the two models are simply related by the following correspondence

\[
\lambda_1 = \lambda_2 \leftrightarrow \frac{1}{4}(g^2 + g'^2),
\]

\[
\text{Re} \lambda_{345} \leftrightarrow -\frac{1}{4}(g^2 - g'^2),
\]

(together with

\[
\lambda_6 = \lambda_7 = 0,
\]

where \( g' \) is the \( U(1) \) hypercharge coupling. We note that the constraint \( \beta \approx \alpha \approx \pi/2 \) is accessible in approach (A), but not in (B). Using this correspondence, together with \( \alpha_1 \to \alpha + \pi/2 \) and

\[
\frac{1}{4}(g^2 + g'^2)v = \frac{gM_Z}{2\cos\theta_W},
\]

we find

\[
\lambda_{111} \leftrightarrow \lambda_{hhh}, \quad \lambda_{112} \leftrightarrow \lambda_{hhh}. \tag{3.13}
\]

For the relations (3.10) to be satisfied, one needs the non-trivial relations

\[
\lambda_1 = \lambda_2 = -\text{Re} \lambda_{345}. \tag{3.14}
\]

When (3.11) holds, and

\[
\alpha_2 = \alpha_3 = 0,
\]

then (more general relations can be found in Appendix A)

\[
\lambda_1 = \frac{1}{c_{\beta}^2v^2}[s_{\beta}^2M_1^2 + c_{\beta}^2M_2^2 - s_{\beta}^2M_3^2],
\]

\[
\lambda_2 = \frac{1}{s_{\beta}^2v^2}[s_{\beta}^2M_1^2 + c_{\beta}^2M_2^2 - c_{\beta}^2\mu^2],
\]

\[
\text{Re} \lambda_{345} = \frac{1}{c_{\beta}^2s_{\beta}v^2}c_1s_1(M_1^2 - M_2^2) + \frac{\mu^2}{v^2}. \tag{3.16}
\]

We note that Eq. (3.14) can be satisfied when \( \alpha_1 \approx \beta \approx \pi/4 \) and \( M_2 \) and \( \mu \) are both large compared to \( M_1 \):

\[
\frac{\sin 2\alpha_1}{\sin 2\beta} = F \equiv \frac{M_2^2 + M_3^2}{M_2^2 - M_1^2}. \tag{3.17}
\]

In this limit, \( \lambda_1 \) is still positive, \( \lambda_2 \) is negative, and \( \lambda_{345} \) is small. This property can be described as follows. If \( M_{\tilde{g}} \gg M_{Z} \), then \( \alpha \to \beta - \pi/2 \) and the coupling \( \lambda_{hh} \) approaches \( \lambda_{HH}^{SM} \). It follows that in this limit, which is called decoupling limit, \( \lambda_1 \to 0 \). In the present notation, this requires \( \alpha_1 \approx \beta \), with \( M_2 \approx \mu \). (However, the correspondence between the 2HDM and the MSSM described above, fails for \( \tan \beta \) away from unity.) We will return to the issue of decoupling in Sec. \( \alpha \alpha \alpha \)

2. Numerical study

For fixed values of the Higgs boson masses, only a small domain in the parameter space of \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is compatible with positivity [20, 50], and the perturbative unitarity in the Higgs sector [31, 32, 33]. Furthermore, there are experimental constraints from \( B \) physics: \( B \to X_s\gamma \) [34, 35, 36], \( B \to \tau\nu_\tau \) [37, 38], and \( B \to \tau \) oscillations [39, 40, 41]. These do not depend on details of the neutral Higgs sector, but since these processes can get contributions from \( H^\pm \) exchange, they constrain the allowed values of \( \tan \beta \) and \( M_{H^\pm} \). Furthermore, there are experimental constraints that do depend on the neutral Higgs sector. These are \( R_b \), [40, 41], non-observation of a light neutral Higgs boson at LEP [40], and most importantly the constraint arising [20, 42, 43] from \( \Delta \rho \). In contrast to the case of the MSSM, \( g - 2 \) does not play any

\footnote{With \( c_{\beta}^2 = \frac{1}{2} + \epsilon \), one finds to lowest order in \( \epsilon \): \( c_{\beta}^2 = \frac{1}{2} + \epsilon(2\mu^2/M_2^2 - 1) \) and \( M_1^2 = 2\epsilon^2(M_2^2 + \mu^2 - 2\mu^2/M_2^2) \).}
important role here. The experimental constraints on the Higgs sector of the 2HDM are discussed in [17]. Here we shall follow the same approach, defining a $\chi^2$ function

$$\chi^2(\alpha) = \chi^2_{\text{general}} + \sum_i \chi^2_i(\alpha),$$  \hspace{1cm} (3.18)

where the first term, which is independent of $\alpha$, is due to various $B$-physics constraints, $B \rightarrow Xs\gamma, B^{-} \rightarrow \tau\bar{\nu}$, and $B$–$B$ oscillations

$$\chi^2_{\text{general}} = \chi^2_{b \rightarrow s\gamma} + \chi^2_{b \rightarrow \tau\bar{\nu}} + \chi^2_{B \rightarrow B},$$  \hspace{1cm} (3.19)

and the second term is a sum over contributions due to observables $O_i$ that depend on $\alpha$. These are the non-observation of a light neutral Higgs boson at LEP2, the $Z^0 \rightarrow b\bar{b}$ decay rate, and $\Delta\rho$:

$$\chi_i^2(\alpha) = \frac{(O_i^{\text{2HDM}}(\alpha) - O_i^{\text{ref}})^2}{[\sigma(O_i)]^2}.$$  \hspace{1cm} (3.20)

We adopt the same definitions and experimental data as were used in [17], allowing parameters for which $\chi^2 \leq 5.99$, corresponding to a 95% C. L.

Some of the main features of these constraints are worth stressing. First of all, for small values of $\mu^2$, the allowed range of $\tan\beta$ is restricted by unitarity to values below 5–8 [17]. Secondly, for large values of $M_{H^\pm}$, the $\Delta\rho$ constraint requires $M_2 \sim M_3 \sim M_{H^\pm}$, especially for large values $\tan\beta$ [21].

![Diagram](image)

**FIG. 1:** The trilinear coupling ratio $\xi_1$ defined in (3.8) at the tree level plotted as a function of $\alpha_1$ for both the approaches (A) and (B), for two different values of $\alpha_2$: $-\pi/4$, dashed, and $+\pi/4$, solid, both with $\alpha_3 = \pi/12$. The masses of the Higgs bosons are $M_1 = 120$ GeV, $M_2 = M_{H^\pm} = 300$ GeV (red, thin lines), and $M_2 = M_{H^\pm} = 400$ GeV (green, heavy lines). Furthermore, $\mu = 0$ and $\tan\beta = 1$. The numbers in insets give values of $(M_1, M_2, M_{H^\pm}, \mu)$ for approach (A) and for those of $(M_1, M_2, M_{H^\pm}, \mu)$ for approach (B).

Within an allowed domain in the parameter space, the trilinear couplings can have a very strong dependence on the neutral Higgs boson mixing angles, as is illustrated in Fig. 1 where we have plotted $\xi_1$ as a function of $\alpha_1$, keeping $\alpha_2 = \pm\pi/4$ and $\alpha_3 = \pi/12$ fixed.

In this figure, the values of the trilinear coupling are calculated at tree level with $M_1 = 120$ GeV, $\mu = 0$, $\tan\beta = 1$, $\alpha_2 = \pm\pi/4$, and $\alpha_3 = \pi/12$. The upper set of lines in this figure (larger values of the trilinear couplings) refer to the approach (A) of Sec. [11B] where we take $\lambda_6 = \lambda_7 = 0$, and fix $M_2 = M_{H^\pm} = 300$ GeV, or $M_2 = M_{H^\pm} = 400$ GeV, as indicated in the caption. The lower set of lines refer to the approach (B), where we fix $(M_2, M_3, M_{H^\pm})$ equal to $(300, 400, 300)$ GeV, or $(400, 450, 400)$ GeV, with the additional constraints $\text{Im} \lambda_5 = 0$, $\text{Re} \lambda_6 = \text{Re} \lambda_7 = 0$.

From this figure we note a marked qualitative difference between the values of the trilinear coupling calculated via the two approaches of specifying the input parameters. In general, they both lead to solutions for only limited ranges of $\alpha_1$, keeping $M_3$ fixed [approach (B)], the variation of the coupling with $\alpha_1$ is modest, whereas when $\lambda_6 = \lambda_7 = 0$ and $M_3$ is calculated [approach (A)], the variation with $\alpha_1$ is quite strong (upper part of the figure).

The different behavior of the trilinear coupling in the two approaches can be understood as follows. The trilinear couplings are linear functions of the quartic couplings $\lambda_1$ in the Higgs potential (2.2). In both the approaches, (A) and (B), the couplings $\lambda_1$ are rather smooth functions of $\alpha_1$. They are also comparable in magnitude, with the exception that in approach (B), where we fix $M_3$, we also take as input $\text{Im} \lambda_5 = 0$. Thus, the role played by $\text{Im} \lambda_5$ in approach (A) is played by $\text{Im} \lambda_6$ and $\text{Im} \lambda_7$ in approach (B). This switch of role from $\text{Im} \lambda_5$ to $\text{Im} \lambda_6$ and $\text{Im} \lambda_7$ has as a consequence, for the case studied in Fig. 1 that $a_{333}$ (which is quite large) switches sign. Since the coefficient in (3.3) that relates $a_{333}$ to $\lambda_{111}$ is essentially $R_{13}$, with $R_{13} = \sin \alpha_2 = \pm 1/\sqrt{2}$, this sign change of the large coupling $a_{333}$ is the dominant effect which causes the difference between the parameter choices in the approaches (A) and (B) (for the set of parameters considered). In fact, truncating Eq. (3.3) with $a_{333} = 0$, one finds the ratio of the couplings $\xi_1 = 3.8 - 5$ for approach (A), and $3.6 - 3.8$ for approach (B), where we have taken $M_1 = 120$ GeV, $M_2 = 400$ GeV, $\alpha_2 = -\pi/4$, and $\alpha_4 = \pi/12$ for both the approaches. Thus, one may conclude that the two different ways of introducing CP violation in the two Higgs doublet model lead to very different values for the trilinear coupling $\lambda_{111}$.

Furthermore, we note that $\text{Re} \lambda_5$ depends on $\alpha_1$ through $M_3$ only [17]. Thus, in approach (A), with $M_3$ depending on $\alpha_1$, the resulting $\text{Re} \lambda_5$ will also vary, whereas in approach (B), $\text{Re} \lambda_5$ is constant. This contributes to the stronger variation of $\xi_1$ with $\alpha_1$ in approach (A).

In view of the strong dependence of the trilinear couplings on the mixing angles, we shall henceforth average
over these angles within the allowed ranges. In Figs. 2 and 3 we show the tree-level ratios

$$\langle \xi_1 \rangle = \frac{\langle \lambda_{111} \rangle}{\lambda_{HHH}^{SM}} \quad \text{and} \quad \langle \xi_2 \rangle = \frac{\langle \lambda_{112} \rangle}{\lambda_{HHH}^{SM}}, \quad (3.21)$$

for a representative choice of parameter sets. These are averages, obtained by scanning over the $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ space, subjecting all model points to positivity, unitarity, and the experimental constraints discussed above, working within the approach (A) defined in Sec. 11B and normalizing them with respect to the SM coupling, as defined in Eq. 4.8. The scans are performed over $200 \times 200 \times 100$ points.

Before considering the general case of arbitrary CP violation, we first consider, in Fig. 2 a region of parameter space near the limit (2.17) of no CP violation:

$$|\alpha_2| \leq \alpha_0, \quad |\alpha_3| \leq \alpha_0, \quad \alpha_0 = 0.05 \times \pi/2, \quad (3.22)$$

referred to as “minimal CP violation”. From the above discussion (see Eqs. (3.10)–(3.13)), it follows that this corresponds to a domain of parameters close to the CP-conserving Higgs sector of the MSSM. We note that the approach (A), where we determine $M_3$ from $M_1$, $M_2$ and the rotation matrix, does not permit us to go all the way to the limit $\alpha_2 = \alpha_3 = 0$.

In this figure, which is valid for $\mu = 0$, the plots do not extend much beyond $\tan \beta = 5$. This is caused by the constraint of unitarity in the Higgs–Higgs scattering sector [21]. At the higher values of $\tan \beta$, there is also a narrowing of the allowed region, largely due to the constraints following from $\Delta \rho$ and the $B$ physics [20, 21].

The surfaces representing $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ in Fig. 2 are in fact quite reminiscent of the corresponding averages of $\lambda_1$ and $\text{Re} \lambda_{345}$, respectively. The details of these correspondences depend on which domains of the $\alpha$ space (in this case, $\alpha_1$) is populated. Typically, only scattered regions are allowed [20].

On the other hand, in Fig. 3 we have scanned over the full range of values of $\alpha_2$ and $\alpha_3$, compatible with all the constraints. From these figures, we see that in much of the $\tan \beta - M_{H^\pm}$ parameter space, there is a considerable enhancement of the trilinear couplings as compared with the corresponding trilinear coupling in the SM. This is consistent with the special case studied in Fig. 1.

Incorporating CP violation in the 2HDM, we see a considerable change in the behavior of the couplings as we go from Fig. 2 to Fig. 3. This reflects the fact seen in Fig. 1 that the couplings have a strong dependence on the mixing angles, and, thus, on how CP violation is incorporated in the model.

Furthermore, there is a rapid variation of the Higgs coupling ratios around $\tan \beta = O(2)$. As we shall see in the following, this is accompanied by a strong variation as one scans across the different values of $\alpha$.

The trilinear coupling has a rather complicated behaviour across the $\tan \beta - M_{H^\pm}$ plane, despite the smoothing that is implicit in the averaging over $\alpha$. As is evident from Fig. 1 for $\xi_1$, the ratios $\xi_1$ and $\xi_2$ of trilinear couplings also vary considerably over the parameter space of $\alpha$. One measure of this variation is the variance, defined as

$$\sigma_1 = \sqrt{\langle (\xi_1 - \langle \xi_1 \rangle)^2 \rangle}.$$  

In Fig. 4 we display the variances $\sigma_1$ and $\sigma_2$ corresponding to the upper panels of Fig. 3. This quantity (in units of the SM trilinear coupling) is seen to be quite considerable, in particular for $\xi_2$ and for moderate values of $\tan \beta$. 

---

FIG. 2: Trilinear Higgs coupling ratios $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ as defined in Eq. (3.21) at the tree-level, in approach (A), for the values of the Higgs masses $M_1 = 120$ GeV, $M_2 = 300$ GeV and $\mu = 0$. The ratios are plotted for the case of the CP-conserving MSSM-like limit as defined in Eq. (3.22).
FIG. 3: Tree-level trilinear coupling ratios $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ as defined in Eq. (3.21), in approach (A), for the values of the Higgs masses $M_1 = 120$ GeV and $M_2 = 300$ GeV. For the top panel $\mu = 0$, whereas for the bottom panel $\mu = 200$ GeV. This plot is for the general case including CP violation, and is compatible with general constraints on the model.

FIG. 4: Variance of trilinear coupling ratios $\xi_1$ and $\xi_2$ at the tree level. The parameters here correspond to the upper panels ($\mu = 0$) of Fig. 3.

For approach (B), when we keep all three neutral Higgs boson masses fixed, the behaviour is smoother, as seen in Fig. 5. For the parameters considered (note that $\mu = 0$), the values of $\tan \beta$ extend up to about 5. We note that the quantity $\langle \xi_1 \rangle$ is rather similar to that shown in Figs. 2 and 3, increasing more or less monotonically with $\tan \beta$. The ratio $\langle \xi_2 \rangle$ differs more, exhibiting a bulge of rather strong $H_1 H_1 H_2$ coupling for $\tan \beta = 2 - 3$ and $M_{H^\pm} = 450 - 500$ GeV.
We note that in approach (B), with \( \tan \beta \) and cancel against odd parts coming from the third term in (3.24), which are proportional to \( \text{Im} \alpha \). The turn is proportional to \( M \beta \) tan \( \alpha \). These contribute terms which are odd in \( \alpha \). This can be understood in the following manner. For \( i = 1 \), the first two terms in Eq. (3.24) can be written as (in units of \( M_i \))

\[
\begin{align*}
\lambda_{i+} &= \sum_{m=1,2,3} R_m b_m. \\
&= \cos \beta \{ \sin^2 \beta \left( \lambda_1 - \lambda_4 - \text{Re} \lambda_5 \right) + \cos^2 \beta \lambda_3 \\
&+ \cos \beta \sin \beta \left[ (\tan^2 \beta - 2) \text{Re} \lambda_6 + \text{Re} \lambda_7 \right] \}, \\
&= \sin \beta \{ \cos^2 \beta \left( \lambda_2 - \lambda_4 - \text{Re} \lambda_5 \right) + \sin^2 \beta \lambda_3 \\
&+ \cos \beta \sin \beta \left[ \text{Re} \lambda_6 + (\cot^2 \beta - 2) \text{Re} \lambda_7 \right] \}, \\
&= \cos \beta \sin \beta \text{Im} \lambda_5 - \sin^2 \beta \text{Im} \lambda_6 - \cos^2 \beta \text{Im} \lambda_7.
\end{align*}
\]

We note that the index \( m \) (like the indices \( n \) and \( o \) in Eq. (3.23) refers to the weak interaction eigenstates (\( \eta_1, \eta_2 \) and \( \eta_3 \)) of Eq. (2.4)). The coefficients \( b_m \) in (3.24) can be written as (in units of \( -iv \))

\[
\begin{align*}
&= \cos \beta \{ \sin^2 \beta \left( \lambda_1 - \lambda_4 - \text{Re} \lambda_5 \right) + \cos^2 \beta \lambda_3 \\
&+ \cos \beta \sin \beta \left[ (\tan^2 \beta - 2) \text{Re} \lambda_6 + \text{Re} \lambda_7 \right] \}, \\
&= \sin \beta \{ \cos^2 \beta \left( \lambda_2 - \lambda_4 - \text{Re} \lambda_5 \right) + \sin^2 \beta \lambda_3 \\
&+ \cos \beta \sin \beta \left[ \text{Re} \lambda_6 + (\cot^2 \beta - 2) \text{Re} \lambda_7 \right] \}, \\
&= \cos \beta \sin \beta \text{Im} \lambda_5 - \sin^2 \beta \text{Im} \lambda_6 - \cos^2 \beta \text{Im} \lambda_7.
\end{align*}
\]

In Fig. we display the coupling of \( H_1 \) to the charged \( H^+ H^- \) pair, for the same choice of parameters as in Fig. 1. We note that in approach (B), with \( \tan \beta = 1 \) and \( \text{Im} \lambda_5 = 0 \), this coupling has two interesting features: (1) It does not depend on the sign of \( \alpha_2 \); and (2) it passes through zero for \( \alpha_1 = -\pi/4 \). The independence from the sign of \( \alpha_2 \) can be understood in the following manner. For \( i = 1 \), the first two terms in Eq. (3.24) (proportional to \( R_{11} \) and \( R_{12} \)) contribute terms which are odd in \( \alpha_2 \), arising from \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). These odd parts are proportional to \( (M_1^2 - M_2^2) \), and cancel against odd parts coming from the third term in (3.24), which are proportional to \( \text{Im} \lambda_6 + \text{Im} \lambda_7 \), which in turn is proportional to \( M_{13} + M_{23} \).

Furthermore, the vanishing of the coupling at \( \alpha_1 = -\pi/4 \) can be understood as follows. In approach (B), with \( \tan \beta = 1 \) and \( \alpha_1 = -\pi/4 \), we have (see Eqs. (3.1) and (3.2) of Ref.

\[
\begin{align*}
\lambda_1(\alpha_2, \alpha_3) &= \frac{1}{\sqrt{2}} \left[ c_5^2 M_1^2 + (c_4 - s_2 s_3)^2 M_2^2 \right. \\
&+ \left. (s_3 + s_2 c_3)^2 M_2^2 - \mu^2 \right], \\
\lambda_2(\alpha_2, \alpha_3) &= \frac{1}{\sqrt{2}} \left[ c_5^2 M_1^2 + (c_4 + s_2 s_3)^2 M_2^2 \right. \\
&+ \left. (s_3 - s_2 c_3)^2 M_2^2 - \mu^2 \right].
\end{align*}
\]

Thus, for these particular values of \( \tan \beta \) and \( \alpha_1 \), \( \lambda_2(-\alpha_2, \alpha_3) = \lambda_1(\alpha_2, \alpha_3) \), the first two terms in Eq. (3.24) cancel, and we are left with

\[
\lambda_{1+} = R_{13} b_3 = \frac{1}{2} s_2 (\text{Im} \lambda_5 - \text{Im} \lambda_6 - \text{Im} \lambda_7) = 0,
\]

B. Charged–Charged–Neutral Higgs Couplings

Although in this paper we are mainly concerned with the trilinear couplings of the neutral Higgs bosons, here we briefly discuss the trilinear couplings of the charged Higgs boson of the 2HDM [25]. These are given by

\[
\begin{align*}
\end{align*}
\]
where in the last step we have used \( \text{Im} \lambda_5 = 0 \), and the relations (see Eq. (2.19))

\[
\text{Im} \lambda_5 + \text{Im} \lambda_6 + \text{Im} \lambda_7 = -\frac{\sqrt{2}}{v^2} [\mathcal{M}_{13}^2 + \mathcal{M}_{23}^2],
\]

valid for \( \tan \beta = 1 \), and

\[
\mathcal{M}_{13}^2 + \mathcal{M}_{23}^2 = 0,
\]

valid for \( \alpha_1 = -\pi/4 \).

In Fig. 6, we show the coupling of the lightest neutral Higgs bosons to the charged Higgs bosons, averaged over \( \alpha \), in approach (A). Corresponding to what we observed in Fig. 6, these averages over \( \alpha \) also vanish along certain lines in the parameter space.

![Diagram](image)

**FIG. 6:** The trilinear coupling \( \lambda_{1+} \) at the tree level as a function of \( \alpha_1 \), for two different values of \( \alpha_2 \) \((-\pi/4, \text{dashed, and} +\pi/4, \text{solid})\), both with \( \alpha_3 = \pi/12 \). The masses and parameters are the same as in Fig. 1.

In Fig. 7, we show the coupling of the lightest neutral Higgs bosons to the charged Higgs bosons, averaged over \( \alpha \), in approach (A). Corresponding to what we observed in Fig. 6, these averages over \( \alpha \) also vanish along certain lines in the parameter space.

![Diagram](image)

**FIG. 7:** Tree-level trilinear couplings \( \lambda_{1+} \) and \( \lambda_{2+} \) (in units of \(-iv\)) of the two lightest neutral Higgs bosons, \( H_1 \) and \( H_2 \), to a charged pair for \( M_1 = 120 \text{ GeV}, M_2 = 300 \text{ GeV} \) and \( \mu = 0 \). The couplings are averaged over \( \alpha \), and have been plotted for approach (A).

### IV. ONE-LOOP CORRECTIONS TO THE HIGGS BOSON MASSES

#### A. One-loop effective potential

For a realistic computation of the Higgs boson properties in the 2HDM, we must take into account the one-loop radiative corrections to the tree-level potential (2.2). When this is done, the relations among masses and the
couplings of the tree-level potential undergo a change. In the MSSM, this effect, which is dominated by the top (and stop) quark contributions, is known to be very important (see also Ref. [4]). In the 2HDM, there being no one-loop contribution from squarks, the main contributions at one loop are generated by the Higgs bosons. We shall here study the one-loop radiative corrections to the neutral Higgs sector using the method of one-loop effective potential [17]. The one-loop corrections to the tree-level potential (2.2) are given by

$$\Delta V = \frac{1}{64\pi^2} \sum_{\text{bosons}} M^4 \left( \log \frac{M^2}{Q^2} - \frac{3}{2} \right) - \sum_{\text{fermions}} M^4 \left( \log \frac{M^2}{Q^2} - \frac{3}{2} \right),$$

(4.1)

where the sums run over all bosonic and fermionic degrees of freedom, respectively, and $Q^2$ represents the scale at which the couplings are evaluated. As indicated above, the dominant contributions to (4.1) come from the Higgs bosons (neutral and charged), and the top quarks, all of which we will calculate in this section. Taking into account the spin, charge and color degrees of freedom, we obtain

$$\Delta V = \Delta V_{\text{neutral Higgs}} + \Delta V_{\text{charged Higgs}} + \Delta V_{\text{top}}$$

$$= \frac{1}{64\pi^2} \sum_{i=1}^{3} \left[ M_i^4 \left( \log \frac{M_i^2}{Q^2} - \frac{3}{2} \right) + \frac{1}{32\pi^2} M_{H^\pm}^4 \left( \log \frac{M_{H^\pm}^2}{Q^2} - \frac{3}{2} \right) - \frac{3}{16\pi^2} m_i^4 \left( \log \frac{m_i^2}{Q^2} - \frac{3}{2} \right) \right],$$

(4.2)

where all masses are understood to be field-dependent masses. Thus, after differentiation analogous to the one in Eq. (2.9) for the tree-level potential, for example, we do not set the fields $\eta_1$, $\eta_2$ and $\eta_3$ to zero, but instead evaluate

$$\mathcal{M}_{ij}^2(\eta_1, \eta_2, \eta_3) = \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} V(\eta_1, \eta_2, \eta_3, H^\pm, G^\pm),$$

(4.3)

and similarly

$$\mathcal{M}_{H^\pm}^2(\eta_1, \eta_2, \eta_3) = \frac{\partial}{\partial H^\pm} \frac{\partial}{\partial H^\pm} V(\eta_1, \eta_2, \eta_3, H^\pm, G^\pm),$$

(4.4)

and after having performed the differentiation, we set $H^\pm = G^\pm = 0$.

At tree level, the three minimization conditions (2.18) (two real, and one imaginary) can be used to eliminate the bilinear parameters $m_1^2$ and $m_2^2$, as well as relate $\text{Im} \, \lambda_3$ to $\text{Im} \, m_{12}^2$. At the one-loop level, the expressions for $m_1^2$ and $m_2^2$ are modified due to the contributions coming from $\partial \Delta V/\partial \eta_1$ and $\partial \Delta V/\partial \eta_2$. This amounts to adjusting the parameters of the potential such that the minimum is stationary in terms of the same vacuum expectation values $v_1$ and $v_2$ as for the tree-level potential (i.e., $\tan \beta$ remains unchanged). In the following we shall calculate the different contributions to the neutral Higgs boson masses arising from (4.2) using the above procedure.

B. One-loop corrections to Higgs masses

The mass squared matrix for the neutral Higgs bosons will receive contributions from one-loop effects due to neutral Higgs bosons, which can be written as

$$\frac{\partial^2 \Delta V_{\text{neutral Higgs}}}{\partial \eta_i \partial \eta_j} = \frac{1}{64\pi^2} \sum_{i=1}^{3} \left[ \frac{\log M_i^2}{Q^2} \frac{\partial M_i^2}{\partial \eta_i} \frac{\partial M_i^2}{\partial \eta_j} + M_i^2 \left( \log \frac{M_i^2}{Q^2} - 1 \right) \frac{\partial^2 M_i^2}{\partial \eta_i \partial \eta_j} \right],$$

(4.5)

$$\frac{\partial^2 \Delta V_{\text{charged Higgs}}}{\partial \eta_i \partial \eta_j} = \frac{1}{64\pi^2} \sum_{i=1}^{3} \left[ \frac{\log M_{H^\pm}^2}{Q^2} \frac{\partial M_{H^\pm}^2}{\partial \eta_i} \frac{\partial M_{H^\pm}^2}{\partial \eta_j} + M_{H^\pm}^2 \left( \log \frac{M_{H^\pm}^2}{Q^2} - 1 \right) \frac{\partial^2 M_{H^\pm}^2}{\partial \eta_i \partial \eta_j} \right],$$

(4.6)

Finally, there are one-loop corrections due to the top quark, which can be written as

$$\frac{\partial^2 \Delta V_{\text{top}}}{\partial \eta_i \partial \eta_j} = - \frac{3}{2\pi^2} \frac{m_t^2}{v_2^2} \left(2 \log \frac{m_t^2}{Q^2} - 1\right) \delta_{i2} \delta_{j2},$$

(4.7)

where in the last equation we have used $\partial m_t/\partial \eta_i = (m_t/v_2) \delta_{i2}$. We neglect the contributions from lighter fermions.

The full one-loop mass-squared matrix for the neutral Higgs bosons can, thus, be written as a sum of four terms:

$$\mathcal{M}_{ij}^2(\eta_1, \eta_2, \eta_3) = \mathcal{M}_{ij}^2(\eta_1, \eta_2, \eta_3) + \frac{\partial^2 \Delta V_{\text{neutral Higgs}}}{\partial \eta_i \partial \eta_j} + \frac{\partial^2 \Delta V_{\text{charged Higgs}}}{\partial \eta_i \partial \eta_j} + \frac{\partial^2 \Delta V_{\text{top}}}{\partial \eta_i \partial \eta_j}.$$
In Fig. 8 we show, for approach (A), the one-loop corrected Higgs boson masses, corresponding to the trilinear couplings studied in Fig. 1 (i.e., with $\alpha_2$ and $\alpha_3$ held fixed), due to the one-loop radiative corrections discussed above. The superscript ‘LO’ here denotes lowest-order values. The Higgs boson masses are determined as eigenvalues of the matrix $[4\mathbf{8}]$, evaluated for $\eta_1 = \eta_2 = \eta_3 = 0$. The full $\eta$-dependence, to be discussed in the next section, will however be required for evaluating the one-loop trilinear Higgs couplings.

![Approach (A)](image)

**FIG. 8:** One-loop Higgs masses $M_1$, $M_2$, and $M_3$ (in units of GeV) plotted as functions of $\alpha_1$, with $\alpha_2 = \pm \pi/4$, both with $\alpha_3 = \pi/12$. The lowest-order Higgs masses are $M_1^{\text{LO}} = 120$ GeV, with $M_2^{\text{LO}} = M_3^{\text{LO}} = 300$ GeV (thin lines), and $M_1^{\text{LO}} = M_2^{\text{LO}} = 400$ GeV (heavy lines). Here $\mu = 0$ and $\tan \beta = 1$. Dashed lines refer to tree-level values, whereas green and blue lines refer to one-loop-corrected results, with the scale parameter $Q = 2M_1^{\text{LO}}$ and $Q = 500$ GeV, respectively.

The following points are worth noting here: (i) The tree-level value of $M_3$, namely $M_3^{\text{LO}}$, has a strong dependence on $\alpha_1$. This is a consequence of keeping $M_1^{\text{LO}}$, $M_2^{\text{LO}}$, $\alpha_2$ and $\alpha_3$ fixed. (ii) The one-loop corrections have a significant dependence on the scale at which they are evaluated, as is illustrated by comparing the values of the Higgs boson masses at two different values of $Q^2$. (iii) The one-loop correction to the Higgs boson mass can be either positive or negative. (iv) At some of the boundaries of the allowed parameter space, these one-loop corrections become large.

The reason for the strong variation of $M_3$ at the edges of the allowed parameter space can be found from an inspection of the different origin of such edges. These can be described as follows:

1. In approach (A), at tree level, $(M_3^{\text{LO}})^2$ is determined as a rational fraction, where the numerator is linear in $(M_1^{\text{LO}})^2$ and $(M_2^{\text{LO}})^2$ (multiplying rotation matrix elements and $\tan \beta$), whereas the denominator is a linear function in rotation matrix elements and $\tan \beta$ (For an explicit formula, see Eq. (4.16) of Ref. [19]). For critical values of the rotation matrix elements, e.g., when $R_{31} = R_{22} \tan \beta$, the denominator will pass through zero. At such points, $M_3^{\text{LO}}$ diverges, and the allowed parameter space terminates.

2. In approach (A), $M_3^2$ may at some critical values of the rotation matrix elements become equal to $M_2^2$. This happens when either $R_{13} = 0$ or $R_{12} \tan \beta = R_{11}$ [10]. In the former case we have $\alpha_2 = 0$ ($\alpha_1$ and $\alpha_3$ being arbitrary), whereas in the latter we have $\tan \beta = \cot \alpha_1$ ($\alpha_2$ and $\alpha_3$ being arbitrary).

3. Positivity may break down.

4. Perturbative unitarity may break down.

5. Parameter space may be truncated by some experimental constraint.

The first two of these cases are illustrated in Fig. 8. We actually exclude, as ‘unphysical’, regions where $M_1 < 100$ GeV (for $M_1^{\text{LO}} = 120$ GeV) and where either $M_3^{\text{LO}} > 2M_1^{\text{LO}}$ or $M_3 > 2M_2$.

For approach (A), positivity is expressed in terms of simple inequalities, whereas for approach (B), with $\alpha_6$ and $\alpha_7$ non-zero, an involved numerical test is required [20]. Perturbative unitarity in the Higgs–Higgs scattering sector tends to truncate high values of $\tan \beta$, unless $\mu$ is large (comparable to $M_2$) [11].

Having described the radiative corrections to the Higgs masses, we next show in Fig. 9, typical one-loop multiplicative mass corrections, i.e., $M_1/M_1^{\text{LO}}$ and $M_2/M_2^{\text{LO}}$ for the case $M_1^{\text{LO}} = 120$ GeV, $M_2^{\text{LO}} = 300$ GeV, and $\mu = 0$. (This will
be discussed in more detail in Sec. [VII]. The renormalization scale has been chosen as \( Q^2 = (2M_1^{LO})^2 \). As in Section [III], an average over allowed points in \( \alpha \) has been performed.

For \( M_1 \), the loop correction is (for \( Q^2 = (2M_1^{LO})^2 \)) negative at low values of \( \tan \beta \), whereas it is positive for higher values. For \( M_2 \), on the other hand, the correction is smaller, and has no strong dependence on \( \tan \beta \). The negative correction for \( M_1 \) is different from the MSSM, where the one-loop corrections, dominated by the top-quark (squark) contribution, is positive. Here, it is the Higgs sector which dominates, and this contribution has the opposite sign, because of the bosonic nature of the contributing particles in the loop. In actual practice the situation is more complicated, since different Higgs fields may contribute with different signs, depending on how the mass of the respective Higgs boson compares with the scale \( Q \).

In view of the fact that the one-loop correction to \( M_1 \) can be significant, and negative, we impose the constraint of LEP2 non-discovery of a Higgs boson on the one-loop corrected lightest Higgs boson mass. Higgs boson masses below the “magic” 114.4 GeV value are of course allowed, provided the coupling to the \( Z \) and \( b\bar{b} \) (or \( \tau \) pairs) are suppressed.

Treating the loop corrections as a perturbation, we have here imposed the theoretical and experimental constraints on the tree-level parameters. We note that at low values of \( \tan \beta \), the mass of the lightest Higgs boson, \( M_1 \), may be considerably reduced. When we proceed to study the trilinear couplings, we shall therefore impose the LEP2 non-discovery constraint on the loop-corrected mass, \( M_1 \). For a higher scale parameter, though, the mass may increase also at low values of \( \tan \beta \).

V. ONE-LOOP CORRECTIONS TO THE TRILINEAR HIGGS COUPLINGS

The one-loop corrected trilinear Higgs couplings are given by the third order derivatives of the one-loop effective potential with respect to different fields. We shall separately consider the contributions to the potential from the neutral Higgs fields, the charged Higgs fields and the top quark fields. We start the calculation by evaluating the derivatives with respect to the weak fields, which can then be converted to derivatives with respect to the fields corresponding to the mass eigenstates with the help of Eq. (3.2).

The starting point of the calculation of the one-loop corrections to the trilinear Higgs couplings is the expression (3.2). From this we obtain the one-loop radiative corrections to the trilinear Higgs couplings due to neutral Higgs bosons as

\[
\frac{\partial^3 \Delta V_{\text{neutral Higgs}}}{\partial \eta_i \partial \eta_j \partial \eta_k} = \frac{1}{32\pi^2} \sum_{\ell=1}^3 \left[ \frac{1}{M_{\ell}^2} \frac{\partial M_{\ell}^2}{\partial \eta_i} \frac{\partial M_{\ell}^2}{\partial \eta_j} \frac{\partial M_{\ell}^2}{\partial \eta_k} 
\right. 
+ \log \frac{M_{\ell}^2}{Q^2} \left( \frac{\partial^2 M_{\ell}^2}{\partial \eta_i \partial \eta_j} + \frac{\partial^2 M_{\ell}^2}{\partial \eta_j \partial \eta_k} + \frac{\partial^2 M_{\ell}^2}{\partial \eta_i \partial \eta_k} \right) 
\left. + \left( \frac{\partial^2 M_{\ell}^2}{\partial \eta_k \partial \eta_i} \frac{\partial M_{\ell}^2}{\partial \eta_j} \right) + M_{\ell}^2 \left( \log \frac{M_{\ell}^2}{Q^2} - 1 \right) \frac{\partial^3 M_{\ell}^2}{\partial \eta_i \partial \eta_j \partial \eta_k} \right].
\]

The derivatives of squared masses with respect to the weak fields \( \eta_i \) in (5.1) are evaluated and discussed in Appendix [C].
To find one-loop charged Higgs boson correction to the trilinear couplings, we start with field-dependent squared masses, $M_{H^\pm}^2(\eta_1, \eta_2, \eta_3)$, for the charged Higgs boson. Noting that the potential is a function of masses, i.e., $\Delta V_{\text{charged Higgs}} = \Delta V(\eta_1, \eta_2, \eta_3)$, we obtain

$$
\frac{\partial^3 \Delta V_{\text{charged Higgs}}}{\partial \eta_i \partial \eta_j \partial \eta_k} = \frac{1}{16\pi^2} \left[ \frac{1}{M_{H^\pm}^2} \frac{\partial M_{H^\pm}^2}{\partial \eta_i} \frac{\partial M_{H^\pm}^2}{\partial \eta_j} \frac{\partial M_{H^\pm}^2}{\partial \eta_k} 
+ \log \frac{M_{H^\pm}^2}{Q^2} \left( \frac{\partial^2 M_{H^\pm}^2}{\partial \eta_i \partial \eta_j} \frac{\partial M_{H^\pm}^2}{\partial \eta_k} + \frac{\partial^2 M_{H^\pm}^2}{\partial \eta_j \partial \eta_k} \frac{\partial M_{H^\pm}^2}{\partial \eta_i} 
+ \frac{\partial^2 M_{H^\pm}^2}{\partial \eta_k \partial \eta_i} \frac{\partial M_{H^\pm}^2}{\partial \eta_j} \right) \right].
$$

(5.2)

For the one-loop corrections arising from the top-quark, we note that in Model II (which we consider) the up-type quarks couple only to $\Phi_2$ (and not to $\Phi_1$). Thus, the one-loop contribution due to top quark affects only the fields $\eta_2$. We can then write

$$
\frac{\partial^3 \Delta V_{\text{top}}}{\partial \eta_i \partial \eta_j \partial \eta_k} = -\frac{12}{\pi^2} \frac{m_t^4}{v^2} \log \frac{m_t^2}{Q^2} \delta_{i2} \delta_{j2} \delta_{k2}.
$$

(5.3)

VI. QUANTITATIVE RESULTS FOR TRILINEAR HIGGS COUPLINGS

We have seen that the lowest order trilinear couplings are highly dependent on the details of the two Higgs doublet model. This sensitivity on the details of the model is actually enhanced when we take into account the one-loop corrections to the Higgs potential. The loop corrections to the trilinear couplings can vary strongly with the parameters of the model, and also have a considerable dependence on the scale $Q^2$ used in Eqs. (5.1)–(5.3).

A. Different contributions

For fixed values of $\alpha_2$ and $\alpha_3$, as in Fig. 1, we show in Fig. 10 the ratio of the one-loop corrected trilinear Higgs coupling $\lambda_{111}$ and the corresponding tree-level coupling. This ratio is highly dependent on $\alpha_1$ (as is the coupling itself).

![Figure 10](image-url)

FIG. 10: Contributions to the trilinear coupling ratio $\xi_1$ plotted as functions of $\alpha_1$, for $\alpha_2 = \pm \pi/4$, both with $\alpha_3 = \pi/12$. The Higgs mass and parameters are as in Fig. 1. The scale parameter is $Q = 2M_{H^\pm}^{\text{LO}}$. The full correction is shown by the upper, heavy, black lines, whereas partial contributions from neutral and charged Higgs bosons, as well as from top quarks, are shown by the thin, red, green and blue lines.
As compared with the one-loop corrections to the masses, the corrections to the trilinear couplings are relatively much larger. The following two points are worth noting: (i) Most of the one-loop contribution comes from the neutral-Higgs term, Eq. (5.1) (red lines labeled “$H_1, H_2, H_3$”). (ii) The correction can be very large at the boundary of the physically allowed region (in this case, in $\alpha_1$). These regions of very large corrections, which are excluded from averages over $\alpha$, will be discussed further in the following.

In approach (A), when we scan over the values of $\alpha$, there are regions where $M_3$ gets very close to $M_2$. From Ref. [19], we see that (at tree level) this happens when (i) $\alpha_2 \to 0$ or (ii) $\tan \beta \to \cot \alpha_1$, or (iii) $\alpha_2 \to \pm \pi/2$. Case (ii) occurs for $\tan \beta = 1$ and $\alpha_1 = \pi/4$. In this region, the trilinear coupling tends to get very large. This is clearly a region of parameter space where the present approach breaks down. In the notation of Appendix C [see Eq. (C12)], we see that two roots merge ($M_3 \to M_2$) when the complex phases of $s_1$ and $s_2$ differ by $2\pi/3$. Near such points, the derivatives of $s_1$ and $s_2$ with respect to $a$ and $b$, particularly the second and third-order ones, tend to get very large. Actually, such points have been removed from the following figures, and we impose a cut when scanning over $\alpha$ in Sec. VI D.

\[
\min\{(M_3^{LO} - M_2^{LO}), (M_3 - M_2)\} > 5 \text{ GeV.} \quad (6.1)
\]

B. Scale dependence

Just as in the case of one-loop corrections to the masses, we note a strong dependence of the trilinear couplings on the scale parameter that enters the effective potential. This scale plays a role somewhat analogous to the mass scale of superpartners in the effective-potential approach to the MSSM.

In Fig. 11 we study the one-loop corrected trilinear couplings, for both the approaches (A) and (B), and for two sets of Higgs masses within each approach (the same as in Fig. 1). We show how the one-loop corrections depend on the scale $Q$. For this purpose we have considered two timelike values of $Q$, namely, $Q = 2M_1^{LO}$ and $Q = 500$ GeV. As in the case of Fig. 10, we see that in approach (A), the couplings tend to get large near the boundary of the allowed parameter space. (Near $\alpha_1 = \pi/4$, we notice the irregular behavior discussed in Sec. VI A when $M_3 \to M_2$.)

The dependence on the scale is rather involved, but there is a tendency that a lower scale gives a larger correction. This is rather clear in the upper part of the figure, which is a plot for approach (A). For approach (B), however, we have not labeled the individual curves, the main point being to show that there is a considerable uncertainty related to the choice of scale in both approaches.

C. Decoupling

In the context of the trilinear Higgs coupling, decoupling would imply that the self-coupling of the light Higgs boson could be expressed in terms of its mass, as in the Standard Model. This was indeed found to hold for the MSSM, if
the mass is taken to be the loop-corrected one [28, 29].

\[ \beta \tan 5 10 15 20 25 30 35 40 45 50 \] 

\[ \pm H \]

\[ 400 \]

\[ 500 \]

\[ 600 \]

\[ M_1 = 120 \text{ GeV}, M_1^\text{LO} = \mu = 500 \text{ GeV}, \text{ and } Q = 2M_1^\text{LO}. \] 

FIG. 12: Trilinear coupling ratio \( \langle \bar{\xi}_1 \rangle \) as defined by Eq. (6.2). Different quantities in approach (A) with \( M_1^\text{LO} = 120 \text{ GeV}, M_1^\text{LO} = \mu = 500 \text{ GeV}, \) and \( Q = 2M_1^\text{LO}. \) The ratio \( \langle \bar{\xi}_1 \rangle \simeq 1 \) demonstrates approximate decoupling for \( M_2 = \mu \sim M_{H^\pm}. \)

In order to discuss decoupling in the 2HDM, we first need to define, as a reference, a Standard-Model-like limit by imposing the tree-level constraint (3.11), together with Eq. (3.22). The SM-like nature of this limit may be extended to the loop level, if in the loop corrections (4.2) we include only the contributions of the lightest neutral Higgs boson \( H_1 \) and the top quark. Within this framework, we may then study decoupling, by defining the ratio, analogous to (3.21):

\[ \langle \bar{\xi}_1 \rangle = \frac{\langle \lambda_{\text{full}} \rangle}{\langle \lambda_{\text{SM}} \rangle}, \] (6.2)

where

- \( \lambda_{\text{SM}}^{111} \) includes only loop corrections due to \( t\bar{t} \) and \( H_1 \), as would be the case in the SM.
- \( \lambda_{\text{full}}^{111} \) includes all one-loop corrections to the Higgs coupling in the two Higgs doublet model, i.e., also those due to \( H_2, H_3 \) and \( H^\pm \).
- The averaging over \( \alpha_2 \) and \( \alpha_3 \) is constrained by Eq. (3.22) with \( \alpha_0 = 0.025 \times \pi/2 \), whereas \( \alpha_1 \) is left unconstrained.

It should be noted that here we do not insist on a correspondence between the 2HDM and the MSSM, since this would require also \( \alpha_1 \simeq \beta \).

If the ratio (6.2) were to come out as unity, it would be a signal for decoupling: the heavy Higgs bosons do not affect the coupling of the lightest one with itself. In Fig. 12 we display this ratio, for rather high values of \( M_1^\text{LO} \) and \( \mu \): \( M_2^\text{LO} = \mu = 500 \text{ GeV}, \) subjecting all parameters to the general constraints discussed in Sec. III A. For \( M_{H^\pm} \sim M_2 \), the ratio (6.2) is indeed rather close to unity, but for larger values of \( M_{H^\pm} \), it becomes significantly larger. In the region where \( \langle \bar{\xi}_1 \rangle \lesssim 1.20 \), we have also checked that \( M_3 - M_2 < \mathcal{O}(40) \text{ GeV}. \) Thus, in this region of the parameter space, decoupling holds at the loop level to about 10–20%. For larger values of \( M_{H^\pm} \), however, where the model is still consistent, decoupling is strongly violated.

The result displayed in Fig. 12 is quite stable under a change of \( \alpha_0 \) from 0.025 \( \times \pi/2 \) to 0.05 \( \times \pi/2 \). However, the result depends on the choice of scale adopted for evaluating the one-loop potential. For example, for \( Q = 500 \text{ GeV}, \langle \bar{\xi}_1 \rangle \) differs significantly from unity. This is perhaps contrary to expectations, since \( \log(M_{H^\pm}^\text{LO}/Q) \) then vanishes. However, we recall that (i) the one-loop potential also contains non-logarithmic terms, and (ii) the one-loop potential modifies the minimization conditions, i.e., its contribution shifts the values of the soft parameters \( m_{\tilde{t}_1}^2 \) and \( m_{\tilde{t}_2}^2 \) of the tree level potential.

In the 2HDM, in contrast to the MSSM, the decoupling cannot be exact. The reason is obvious: the heavy fields that we would like to neglect, are all bosonic, and thus
all contribute with the same sign to the potential. Furthermore, even if we choose the scale $Q$ such that the logarithms vanish, there would still be the non-logarithmic remainders.

The question of decoupling can be illustrated by the behavior of the $\rho$ parameter, which gets contributions from all pairs of Higgs bosons. These contributions vanish for equal masses, but are large when the masses are very different.

In the limit of no CP violation, custodial symmetry [48] would imply $M_{H^\pm} = M_j$, where $M_j$ is either the mass of the CP odd $\langle A \rangle$ or a CP even $\langle H \rangle$ Higgs particle [49]. In the case that $M_{H^\pm} = A$, orthogonality of the even-sector mixing matrix would protect the $\Delta \rho$ parameter from large corrections. In the present case of CP violation, a higher degeneracy is required, $M_2 = M_3 = M_{H^\pm}$, in order to avoid large contributions to $\Delta \rho$. Since this is in general not satisfied, the experimental constraints on $\Delta \rho$ severely constrain the allowed parameter space [21], and there is no decoupling.

FIG. 13: Approach (A) with $M_{1\text{LO}}^1 = 120$ GeV, $M_{2\text{LO}}^1 = 300$ GeV, $\mu = 0$ and scale $Q = 2M_{1\text{LO}}^1$. Top panel: Trilinear coupling ratios $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ [see Eq. (3.21)]. Middle: one-loop multiplicative mass correction. Bottom: Variance of trilinear coupling ratios $\xi_1$ and $\xi_2$. 
D. The general case

After this exploratory discussion, we are now ready to scan and average over the full range of $\alpha$, compatible with theoretical and experimental constraints on the model, as was done at tree level in Sec. III A. Figs. 13 and 14 are plotted for approach (A) with $M_{1}^{LO} = 120$ GeV, $M_{2}^{LO} = 300$ GeV, $\mu = 0$ (Fig. 13), $\mu = 200$ GeV (Fig. 14) and with scale $Q = 2M_{1}^{LO}$ in both cases.

In these figures, we show the ratios $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ of trilinear couplings in the top panels. The middle panels are devoted to the loop-induced corrections to the masses $M_1$ and $M_2$, whereas the bottom panels show the variance associated with the averaging of $\xi_1$ and $\xi_2$ over the $\alpha$ parameter space.

An important difference between the two cases presented in these figures is the range in $\tan \beta$. As already discussed, for a low value of $\mu$, the high values of $\tan \beta$ would lead to a model that would violate unitarity, and are, therefore, excluded [20].

Comparing the top panels of Figs. 13 and 14 with the corresponding tree-level results shown in Fig. 3, we note the following: (i) The overall shape in the $\tan \beta-M_{H^\pm}$ plane is similar. (ii) The loop-corrected couplings are
somewhat larger, after averaging over $\alpha$, in analogy with the results shown in Fig. 11 for $\alpha_2$ and $\alpha_3$ fixed.

The center panel of Fig. 13 is rather similar to Fig. 9, the only difference being that here the LEP2 constraint is imposed at the loop level. Focusing first on the loop corrections to $M_1$, and comparing the mass correction at the two different values of $\mu$, we note that the case $\mu = 0$ gives a larger reduction at low values of $\tan \beta$ and a larger increase at high values of $\tan \beta$. For $M_2$, on the other hand, the loop correction has only a rather weak dependence on $\tan \beta$ (and $M_{H^\pm}$). For a higher value of the scale $Q$, however, the loop correction to $M_1$ may be positive also at low values of $\tan \beta$.

The trilinear couplings represented in these figures, are of course not physical, they are instead “typical values”, obtained by an averaging over the angles of the mixing matrix $R$ that diagonalizes the mass-squared matrix. As discussed above, the variation over the $\alpha$ parameter space can be considerable, as illustrated by the variance shown in the bottom panels.

A more detailed comparison reveals that the loop corrections are most important for low values of $\tan \beta$. This is particularly so in the case of $\langle \xi_2 \rangle$, where the variance is also quite large. This is in part caused by the ill-defined limit $M_3 \rightarrow M_2$ ($\alpha_2 \rightarrow 0$) discussed above. [Actually, at high values of $M_{H^\pm}$, the $\langle \xi_2 \rangle$ plots are truncated at the top.]

The correlation between these trilinear couplings and the quartic couplings $\lambda_i$ of the tree-level potential or their averages, $\langle \lambda_i \rangle$, is not very simple, but a couple of observations can be made. In Fig. 13, for $\mu = 0$ the enhancement at low values of $\tan \beta$ is correlated with a corresponding enhancement of $\langle \lambda_2 \rangle$, whereas $\langle \lambda_1 \rangle$ is large at high values of $\tan \beta$.

For the $H_1 H_1 H_1$ coupling, represented by $\langle \xi_1 \rangle$, we note that for $\mu = 0$ it is rather large for all values of $\tan \beta$ and $M_{H^\pm}$ where the model is consistent, whereas for $\mu = 200$ GeV it falls to zero for large values of $\tan \beta$.

Finally, in Fig. 15 we show the behavior for somewhat higher value of the soft mass parameter, $\mu = 500$ GeV. We display the averages $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ for two values of $M_2^{LO}$, 300 GeV (as was adopted in the earlier plots) and 500 GeV. Compared to the corresponding panels of
VII. SUMMARY AND CONCLUSIONS

The two Higgs doublet model is a viable extension of the Standard Model Higgs sector. Apart from having a richer structure of Higgs bosons and Higgs couplings, it can accommodate CP violation beyond that of the Kobayashi–Maskawa mechanism in the SM. In this paper we have studied in detail the trilinear couplings of the lightest Higgs boson of this model, including the CP violating effects. Our main objective has been to study the implications of CP violation for the trilinear couplings of the general two Higgs doublet model. We have discussed two different parametrizations of the general two Higgs doublet model, and imposed the constraints on the parameters of the model following from theoretical considerations and from different experiments. Tree-level unitarity, $B$ physics and $\Delta\rho$ are the most constraining ones for the parameters of the two Higgs doublet model with CP violation.

Within the allowed domain of the parameter space of the model, the trilinear couplings can have a very strong dependence on the neutral Higgs boson mixing angles. Because of this strong dependence of the trilinear couplings on the mixing angles, we have presented averages over the allowed ranges for these angles. Together with the variances, these show the range of trilinear couplings that can be expected in the two Higgs doublet model with CP violation.

There is a region of parameter space of the 2HDM, which we call “minimal CP violation,” which corresponds to the domain of parameters close to the Higgs sector of the CP-conserving minimal supersymmetric standard model. In much of this parameter space there is a considerable enhancement of the trilinear couplings of the two Higgs doublet model as compared with the corresponding trilinear coupling of the SM.

We have computed the radiative corrections to the masses of the Higgs bosons and the trilinear couplings in the one-loop effective potential approximation. The one-loop corrected mass of the lightest neutral Higgs boson can be modified by as much as 20%, whereas the heavier ones are less affected by the radiative corrections. The one-loop corrected trilinear couplings are typically enhanced by the loop effects, but so is the sensitivity to the neutral-sector mixing angles. A similar enhancement due to CP violating effects has also been found for the MSSM [50, 51].

To a good approximation, decoupling holds in a limited range of parameters, where the model is close to the CP-conserving MSSM. In general, however, the heavier states also have a considerable impact on the properties of the lightest Higgs boson. In this sense there is no decoupling in the two Higgs doublet model with CP violation.

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APPENDIX A: QUARTIC COUPLINGS EXPRESSED IN TERMS OF MASSES

In the 2HDM, with $\text{Im} \lambda_5 \neq 0$, the two masses $M_1$ and $M_2$, together with $\alpha$ and $\tan \beta$, determine $M_1$. The masses will also be related to the quartic couplings of the Higgs potential, via linear relations that can be unambiguously inverted. It is convenient to distinguish two cases, whether or not $\lambda_6$ and $\lambda_7$ are non-zero.

1. Approach (A), $\lambda_6 = \lambda_7 = 0$

This case is referred to as approach (A) in the text. Providing also $M_{H^\pm}$ and $\mu^2$, in addition to $M_1$, $M_2$ and $\alpha$, all the $\lambda$’s are determined as follows [17, 20]:

\[
\lambda_1 = \frac{1}{c_\beta^2 v^2} \left[ c_1^2 c_2^2 M_1^2 + (c_1 s_2 s_3 + s_1 c_3)^2 M_2^2 \right] + (c_1 s_2 c_3 - s_1 s_3)^2 M_3^2 - s_3^2 \mu^2, \tag{A1}
\]

\[
\lambda_2 = \frac{1}{s_\beta^2 v^2} \left[ s_1^2 c_3^2 M_1^2 + (c_1 c_3 - s_1 s_2 s_3)^2 M_2^2 \right] + (c_1 s_3 + s_1 s_2 c_3)^2 M_3^2 - c_3^2 \mu^2, \tag{A2}
\]

\[
\lambda_3 = \frac{1}{c_\beta s_\beta v^2} \left[ c_1 s_1 (c_2^2 M_1^2 + (s_2^2 s_3^2 - c_3^2) M_2^2) \right] + s_2 c_3 s_3 (c_1^2 - s_1^2)(M_3^2 - M_2^2) \right] + \frac{1}{v^2} [2 M_{H^\pm}^2 - \mu^2], \tag{A3}
\]

\[
\lambda_4 = \frac{1}{v^2} \left[ c_2^2 M_1^2 + c_2^2 s_3^2 M_2^2 + c_2^2 c_3^2 M_3^2 + \mu^2 - 2 M_{H^\pm}^2 \right], \tag{A4}
\]

\[
\text{Re} \lambda_5 = \frac{1}{v^2} \left[ -s_2^2 M_1^2 - c_2^2 s_3^2 M_2^2 - c_2^2 c_3^2 M_3^2 + \mu^2 \right], \tag{A5}
\]

\[
\text{Im} \lambda_5 = -\frac{1}{c_\beta s_\beta v^2} \left[ c_\beta (c_1 c_2 s_2 M_1^2 - c_2 s_3 (c_1 s_2 s_3 + s_1 c_3) M_2^2 + c_2 c_3 (s_1 s_3 - c_1 s_2 c_3) M_3^2) \right] + s_\beta (s_1 c_2 s_2 M_1^2 \tag{A6}
\]

\[
\]
+ c_2 s_3 (c_1 c_3 - s_1 s_2 s_3) M_x^2 - c_2 c_3 (c_1 s_3 + s_1 s_2 c_3) M_y^2},
where c_β = \cos \beta, s_β = \sin \beta.

While M_x^2 is given in terms of M_1^2, M_2^2, R and tan β, it is more transparent not to substitute for M_x^2 in the expressions (A3), (A6). These equations are the analogues of those of [22] for the CP-conserving 2HDM.

2. Approach (B), λ_6 \neq 0, λ_7 \neq 0

In this case, referred to as approach (B) in the text, we take M_1, M_2, M_3, \alpha, M_{H^\pm} and \mu^2, together with Im λ_5, Re λ_5 and Re λ_7, as the input. In order to keep the notation compact, it is convenient to introduce the following abbreviations:

\begin{align*}
\text{Re } \lambda_{345} &= \lambda_3 + \alpha + \text{Re } \lambda_5, \\
\text{Re } \lambda_{567} &= \text{Re } \lambda_5 + \cot \beta \text{ Re } \lambda_6 + \tan \beta \text{ Re } \lambda_7.
\end{align*}

Then, the λ’s can be determined from the following relations:

\begin{align*}
\lambda_1 &= \frac{1}{c^2_β} \left[ \frac{M^2_{11} - s^2_β \mu^2}{v^2} - \frac{s_β}{2 c_β} (3 c^2_β \text{ Re } \lambda_6 - 3 s^2_β \text{ Re } \lambda_7) \right], \\
\lambda_2 &= \frac{1}{s^2_β} \left[ \frac{M^2_{22} - c^2_β \mu^2}{v^2} + \frac{c_β}{2 s_β} (c^2_β \text{ Re } \lambda_6 - 3 s^2_β \text{ Re } \lambda_7) \right], \\
\text{Re } \lambda_{345} &= \frac{1}{c_β s_β} \left[ \frac{c_β s_β \mu^2 + M^2_{12}}{v^2} - \frac{3}{2} (c^2_β \text{ Re } \lambda_6 + s^2_β \text{ Re } \lambda_7) \right], \\
\lambda_4 &= \frac{2}{v^2} [\mu^2 - M^2_{H^\pm}] - \text{Re } \lambda_{567}, \\
\text{Re } \lambda_5 &= \frac{\mu^2 - M^2_{33}}{v^2} - \frac{1}{2 c_β s_β} (c^2_β \text{ Re } \lambda_6 + s^2_β \text{ Re } \lambda_7), \\
\text{Im } \lambda_6 &= -\frac{1}{2 c_β} \left[ \frac{2 M^2_{11}}{v^2} + s_β \text{ Im } \lambda_5 \right], \\
\text{Im } \lambda_7 &= -\frac{1}{2 s_β} \left[ \frac{2 M^2_{22}}{v^2} + c_β \text{ Im } \lambda_5 \right].
\end{align*}

Invoking Eq. (2.14), the M^2_{ij} can be expressed in terms of R and the masses M_1, M_2 and M_3.

**APPENDIX B: TRILINEAR COUPLING FOR SOME SPECIAL CASES**

In order to get a feeling for the trilinear Higgs couplings of the 2HDM, we shall here explicitly write down the two trilinear couplings λ_{111} and λ_{112} for some special cases.

1. The trilinear coupling λ_{111}

We start by writing down Eq. (3.3) explicitly for \{i, j, k\} = \{1, 1, 1\}:

\begin{align*}
λ_{111} &= \sum_{m \leq n \leq 2} R_{1m} R_{1n} R_{1o} a_{mno} \\
&= 3 \{ R^2_{11} [ R_{11} a_{111} + R_{12} a_{112} + R_{13} a_{113} ] + R_{13} [ R^2_{12} a_{122} + R_{13} R_{12} a_{123} + R^2_{13} a_{133} ] \\
&+ R_{12} [ R^2_{12} a_{222} + R_{13} R_{12} a_{223} + R^2_{13} a_{233} ] \\
&+ R^3_{13} a_{333} \}.
\end{align*}

where the factor 3! is due to the fact that we here have couplings to three identical H_1 fields. Substituting for the elements of the rotation matrix, Eq. (2.13), one obtains

\begin{align*}
λ_{111} &= 3! \{ (c_1 c_2^*)^2 [ c_1 c_2 a_{111} + s_1 c_2 a_{112} + s_2 a_{113} ] + c_1 c_2 ([s_1 c_2^*)^2 a_{122} + s_1 s_2 a_{123} + s_2^* a_{133}] \\
&+ s_1 c_2 ([s_1 c_2^*)^2 a_{222} + s_1 s_2 a_{223} + s_2^* a_{233}] + s_2^* a_{333} \}.
\end{align*}

Let us now consider the case of λ_6 = λ_7 = 0, and substitute for the coefficients a_{mno} from Eq. (3.3). In this case, we obtain

\begin{align*}
λ_{111} &= 3! \{ (c_1 c_2) (s_1 c_2^*)^2 [ c_1 c_2 a_{111} + s_1 c_2 a_{112} + s_2 a_{113} ] \\
&+ c_1 c_2 (s_1 c_2^*)^2 a_{122} + s_1 s_2 a_{123} + s_2^* a_{133}] \\
&+ s_1 c_2 ([s_1 c_2^*)^2 a_{222} + s_1 s_2 a_{223} + s_2^* a_{233}] + s_2^* a_{333} \}.
\end{align*}

In the limit of no CP violation, with H_1 being odd, we have c_2 = 0 [see Eq. (2.15)], and the trilinear coupling simplifies further to

\begin{align*}
λ_{111} &= 3 s^3_2 c_β s_β \text{ Im } λ_5, \\
\text{with } s_2 &= \pm 1. Similarly, in the limit of no CP violation, with H_2 being odd, we have s_3 = s_4 = 0, and obtain the simple expression given in Eq. (3.6a), whereas the limit of no CP violation with H_2 being odd, leads to no further simplification of the expression (B3).

2. The trilinear coupling λ_{112}

We shall again first write Eq. (3.3) explicitly for \{i, j, k\} = \{1, 1, 2\}:

\begin{align*}
λ_{112} &= \sum_{m \leq n \leq 2} R_{1m} R_{1n} R_{2o} a_{mno} \\
&= 2 \sum_{m \leq n \leq 2} \{ R_{1m} R_{1n} R_{2o} + R_{1m} R_{2n} R_{1o} \\
&+ R_{2m} R_{1n} R_{1o} \} a_{mno} \\
&= 2 \{ 3 R^2_{11} R_{21} a_{111} + R_{11} R_{12} + R_{12} R_{21} \} a_{112}
\end{align*}
If we now substitute for the elements of the rotation matrix, for the coefficients $\alpha_{mnr}$, and set $\lambda_6 = \lambda_7 = 0$, we obtain the result quoted in Eq. (3.6b).

**APPENDIX C: DERIVATIVES OF $M^2_\ell$**

The loop corrections to the trilinear couplings depend on derivatives of the squared masses $M^2_\ell$ with respect to the weak fields $\eta_i$. In order to obtain these derivatives, a considerable amount of book-keeping is required. The complication arises from the fact that we allow for CP non-conservation and $M^2_\ell$ will thus be determined by the roots of a cubic equation rather than a quadratic one. We start by considering the $M^2_\ell$ as functions of the coefficients of the corresponding cubic eigenvalue equation:

$$\frac{\partial M^2_\ell}{\partial \eta_i} = \frac{\partial}{\partial \eta_i} M^2_\ell(a, b, c)$$

(C1)

where $M^2_\ell = \lambda_\ell$ is a solution of

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0,$$

(C2)

with field-dependent coefficients:

$$a = a(\eta_1, \eta_2, \eta_3), \quad b = b(\eta_1, \eta_2, \eta_3), \quad c = c(\eta_1, \eta_2, \eta_3).$$

(C3)

These coefficients of the cubic eigenvalue equation are obtained from the derivatives (2.7), where, in contrast to Eq. (2.19), we do not set the fields to zero. In terms of the $3 \times 3$ mass squared matrix $M^2$, these coefficients of the cubic equation are given by

$$a = -\text{Tr} M^2, \quad b = \frac{1}{2} \{ (\text{Tr} M^2)^2 - \text{Tr} [(M^2)^2] \}, \quad c = -\det M^2.$$

(C4)

The derivatives (C1) can thus be split up into simpler entities:

$$\frac{\partial M^2_\ell(a, b, c)}{\partial \eta_i} = \frac{\partial M^2_\ell}{\partial a} \frac{\partial a}{\partial \eta_i} + \frac{\partial M^2_\ell}{\partial b} \frac{\partial b}{\partial \eta_i} + \frac{\partial M^2_\ell}{\partial c} \frac{\partial c}{\partial \eta_i}$$

$$= \sum_{\alpha} \frac{\partial M^2_\ell}{\partial a_\alpha} \frac{\partial a_\alpha}{\partial \eta_i},$$

(C5)

where we have introduced the collective notation

$$a_\alpha = \{a, b, c\}.$$

(C6)

The higher derivatives can likewise be written as

$$\frac{\partial^2 M^2_\ell}{\partial \eta_i \partial \eta_j} = \sum_{\alpha} \left[ \sum_{\beta} \frac{\partial^2 M^2_\ell}{\partial a_\alpha \partial a_\beta} \frac{\partial a_\beta}{\partial \eta_i} + \frac{\partial M^2_\ell}{\partial a_\alpha} \frac{\partial^2 a_\alpha}{\partial \eta_i \partial \eta_j} \right],$$

(C7)

$$\frac{\partial^3 M^2_\ell}{\partial \eta_i \partial \eta_j \partial \eta_k} = \sum_{\alpha} \left\{ \sum_{\beta} \left[ \sum_{\gamma} \frac{\partial^3 M^2_\ell}{\partial a_\alpha \partial a_\beta \partial a_\gamma} \frac{\partial a_\gamma}{\partial \eta_i} \frac{\partial a_\beta}{\partial \eta_j} \frac{\partial a_\alpha}{\partial \eta_k} \right] + \frac{\partial^2 M^2_\ell}{\partial a_\alpha \partial a_\beta} \left( \frac{\partial^2 a_\alpha}{\partial \eta_i \partial \eta_j} + \frac{\partial^2 a_\beta}{\partial \eta_j \partial \eta_k} + \frac{\partial^2 a_\alpha}{\partial \eta_i \partial \eta_k} \right) \right\}.$$

(C8)

1. **Cubic equation**

In order to obtain the derivatives of $M^2_\ell$ with respect to the $a_\alpha$ that enter Eqs. (C5), (C7) and (C8), we start by solving the cubic equation (C2) in terms of the following notation. Let

$$q = \frac{1}{3} b - \frac{1}{9} a^2, \quad r = \frac{1}{6} (ab - 3c) - \frac{1}{27} a^3$$

(C9)

and

$$s_1 = [r + \Delta]^{1/3}, \quad s_2 = [r - \Delta]^{1/3},$$

(C10)

with the discriminant

$$\Delta^2 \equiv q^3 + r^2.$$

(C11)

Then the solutions can be written as

$$m^2_\ell = (s_1 + s_2) - \frac{a}{3},$$

$$m^2_2 = -\frac{1}{2} (s_1 + s_2) - \frac{a}{3} + \frac{i\sqrt{3}}{2} (s_1 - s_2),$$

$$m^2_3 = -\frac{1}{2} (s_1 + s_2) - \frac{a}{3} - \frac{i\sqrt{3}}{2} (s_1 - s_2),$$

(C12)

where the $\{m_1, m_2, m_3\}$ refer to the set of masses $\{M_1, M_2, M_3\}$, but not necessarily ordered. In the one-loop contribution to the potential (4.2), only a sum over $\ell$ enters, the order plays no role.

We recall that the coefficients $a_\alpha = \{a, b, c\}$ that enter in the cubic eigenvalue equation (C2) depend on the weak fields $\eta_i$. In the solutions (C12), this dependence can be accessed via $a, s_1$ and $s_2$. It is thus convenient to write (C12) more compactly as

$$m^2_\ell = -\frac{a}{3} + \sum_{r=1}^{2} A_r s_r,$$

(C13)

from which it follows that

$$\frac{\partial m^2_\ell}{\partial a_\alpha} = -\frac{1}{3} \delta_{\alpha 1} + A_r \frac{\partial s_r}{\partial a_\alpha}.$$

(C14)
Likewise, the higher derivatives are given by

$$\frac{\partial^2 m^2}{\partial a_\alpha \partial a_\beta} = A_{\alpha \beta} \frac{\partial^2 s_r}{\partial a_\alpha \partial a_\beta},$$

$$\frac{\partial^4 m^2}{\partial a_\alpha \partial a_\beta \partial a_\gamma} = A_{\alpha \beta \gamma} \frac{\partial^3 s_r}{\partial a_\alpha \partial a_\beta \partial a_\gamma}.$$  \hspace{1cm} (C15)

It remains to obtain the derivatives of $s_r$ with respect to $a_\alpha$. For this purpose, it is useful to think of $s_r$ as a function of the $q$ and $r$ of Eqs. (C9)–(C11). The first derivatives are given by

$$\frac{\partial s_r}{\partial a_\alpha} = \frac{\partial s_r}{\partial q} \frac{\partial q}{\partial a_\alpha} + \frac{\partial s_r}{\partial r} \frac{\partial r}{\partial a_\alpha} = \frac{\partial s_r}{\partial q} \frac{\partial Q^s}{\partial Q^s}.$$  \hspace{1cm} (C16)

where we collectively refer to $q$ and $r$ as

$$Q^s = \{q, r\}.$$  \hspace{1cm} (C17)

In this notation, the higher derivatives can be written as

$$\frac{\partial^2 s_r}{\partial a_\alpha \partial a_\beta} = \frac{\partial^2 s_r}{\partial Q^s \partial Q^t} \frac{\partial Q^s}{\partial a_\alpha} \frac{\partial Q^t}{\partial a_\beta} + \frac{\partial s_r}{\partial Q^s} \frac{\partial^2 Q^s}{\partial a_\alpha \partial a_\beta}.$$  \hspace{1cm} (C18)

and

$$\frac{\partial^3 s_r}{\partial a_\alpha \partial a_\beta \partial a_\gamma} = \frac{\partial^3 s_r}{\partial Q^s \partial Q^t \partial Q^u} \frac{\partial Q^s}{\partial a_\alpha} \frac{\partial Q^t}{\partial a_\beta} \frac{\partial Q^u}{\partial a_\gamma} + \frac{\partial^2 s_r}{\partial Q^s \partial Q^t} \left[ \frac{\partial^2 Q^s}{\partial a_\alpha \partial a_\beta} \frac{\partial Q^t}{\partial a_\gamma} + \frac{\partial^2 Q^s}{\partial a_\beta \partial a_\gamma} \frac{\partial Q^t}{\partial a_\alpha} \right]$$

$$+ \frac{\partial^2 s_r}{\partial Q^s} \left[ \frac{\partial^2 Q^t}{\partial a_\alpha \partial a_\beta} \frac{\partial Q^s}{\partial a_\gamma} + \frac{\partial^2 Q^s}{\partial a_\beta \partial a_\gamma} \frac{\partial Q^t}{\partial a_\alpha} \right]$$

$$+ \frac{\partial^3 s_r}{\partial a_\alpha \partial a_\beta \partial a_\gamma} = \frac{\partial^3 s_r}{\partial a_\alpha \partial a_\beta \partial a_\gamma}.$$  \hspace{1cm} (C19)

Finally, the various derivatives $\partial Q^s / \partial a_\alpha$ are given by

$$\frac{\partial q}{\partial a_\alpha} = \frac{2}{9} a, \quad \frac{\partial r}{\partial a_\alpha} = \frac{1}{6} b - \frac{1}{9} a^2,$$

$$\frac{\partial q}{\partial b} = \frac{1}{3}, \quad \frac{\partial r}{\partial b} = \frac{1}{6} a,$$

$$\frac{\partial q}{\partial c} = 0, \quad \frac{\partial r}{\partial c} = \frac{1}{2}.$$  \hspace{1cm} (C20)

$$\frac{\partial s_1}{\partial a} = \frac{1}{9} \Delta^{2/\sqrt{r + \Delta}}, \quad \frac{\partial s_1}{\partial r} = \frac{1 + \frac{r}{\Delta}}{3^{2/\sqrt{r + \Delta}}},$$

$$\frac{\partial s_2}{\partial q} = -\frac{1}{2} \Delta^{2/\sqrt{r - \Delta}}, \quad \frac{\partial s_2}{\partial r} = \frac{1 - \frac{r}{\Delta}}{3^{2/\sqrt{r - \Delta}}}. \hspace{1cm} (C21)$$

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