EFFECTS OF MAGNETIC FLUX AND OF ELECTRON MOMENTUM ON THE TRANSMISSION AMPLITUDE IN THE AHARONOV-BOHM INTERFEROMETER

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Abstract

A characterization of the two-terminal open-ring Aharonov-Bohm interferometer is made by analyzing the phase space plots in the complex transmission amplitude plane. Two types of plots are considered: type I plot which uses the magnetic flux as the variable parameter and type II plot which uses the electron momentum as the variable parameter. In type I plot, the trajectory closes upon itself only when the ratio $R$ of the arm lengths (of the interferometer) is a rational fraction, the shape and the type of the generated flower-like pattern is sensitive to the electron momentum. For momenta corresponding to discrete eigenstates of the perfect ring (i.e., the ring without the leads), the trajectory passes through the origin a certain fixed number of times before closing upon itself, whereas for arbitrary momenta it never passes through the origin. Although the transmission coefficient is periodic in the flux with the elementary flux quantum as the basic period, the phenomenon of electron transmission is shown not to be so when analyzed via the present technique. The periodicity is seen to spread over several flux units whenever $R$ is a rational fraction whereas there is absolutely no periodicity present when $R$ is an irrational number. In type II plot, closed trajectories passing through the origin a number of times are seen for $R$ being a rational fraction. The case $R = 1$ (i.e., a symmetric ring) with zero flux is rather pathological—it presents a closed loop surrounding the origin. For irrational $R$ values, the trajectories never close.

Keywords: Aharonov-Bohm effect; transmission amplitude; transmittance

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I. INTRODUCTION

The Aharonov-Bohm interferometer (ABI) in a two-terminal configuration has come to stay as a reliable tool in the study of mesoscopic/ nanoscopic systems. Figure 1 presents a schematic diagram of the ABI. The phase of the electron wave function has a topological contribution arising due to the magnetic flux confined within the ring, threading perpendicular to the ring plane. It also has a dynamic contribution arising due to the electron momentum. In a simplistic view it is often assumed to be of the form

\[ t = t_1 + t_2 e^{i\phi} \]

where \( \phi \) is the Aharonov-Bohm (AB) phase and \( t_1, t_2 \) are the TA’s through the two arms of the ring. Whereas this view is appropriate to the original free-space double-slit setup, it is not so to the two-leaded ABI. More complicated dependence on \( \phi \) as well as the dynamical phases \( \chi_1 \) and \( \chi_2 \) contributed by the two arms dictate the behavior of the TA\[1, 2\]. It is the purpose of the present paper to present our analysis of the detailed effects of these phases on the TA. Chen et.al.\[3\] report the effect of \( \phi \) on anti-resonances of the transmittance \( T(=|t|^2) \) in the ABI. They correlate the nature of constant-\( T \) contour plots in the complex electron-energy plane to the variation of \( T \) with the electron energy. Kim et.al.\[4\] study the effects of broken time-reversal symmetry on the transmission zeros of the ABI. They propose a classification scheme of the phase trajectories of \( t \) in the complex \( t \)-plane and study the effect of the variation of the \( \chi \)’s. Taniguchi and Buttiker\[5\], in a pioneering work, analyze
the Friedel phases and the TA phases for a one-dimensional resonant tunnel double barrier, a wire with a side branch and also for the combination of the above two structures.

In this work we consider the transmission properties of the two-leaded AB ring (see Fig. 1). We have repeated the calculation of Xia [6] for the ABI for obtaining the TA. The model and the assumptions that we make are same as the ones made in his work [6]. The wave functions on the various segments are assumed to be of the following form:

\[ \psi_1(x_1) = e^{ikx_1} + ae^{-ikx_1}, \]
\[ \psi_2(x_2) = c_1e^{ik_1x_2} + d_1e^{-ik_2x_2}, \]
\[ \psi_3(x_3) = c_2e^{ik_2x_3} + d_2e^{-ik_1x_3}, \]
\[ \psi_4(x_4) = te^{ikx_4}. \] (1)

The subscripts on the \( \psi \)'s correspond to the labels used on the segments (see Fig. 1). The lengths of the segments marked 2 and 3 are \( l_2 \) and \( l_3 \) and the total circumference of the ring is \( l = l_2 + l_3 \). Here \( x_1, x_2 \) and \( x_3 \) have common origin at A and \( x_4 \) has its origin at B. The boundary conditions used at the junctions are

\[ \psi_1(0) = \psi_2(0) = \psi_3(0), \]
\[ \psi_2(l_2) = \psi_3(l_3) = \psi_4(0), \]
\[ \psi'_1(0) = \psi'_2(0) + \psi'_3(0), \]
\[ \psi'_2(l_2) + \psi'_3(l_3) = \psi'_4(0). \] (2)

Here \( k_1 = k + \eta \) and \( k_2 = k - \eta \) and \( \eta = (2\pi/l)(\Phi/\Phi_0) = (2\pi/l)\theta \), \( \Phi \) denotes the magnetic flux threading the ring and \( \Phi_0 = hc/e \) is the flux quantum. It is convenient to use a dimensionless wave number \( q \) defined by \( k = (2\pi/l)q \) and \( \theta \), the dimensionless flux. We have used the software MAPLE to obtain analytic expression for the TA.

We report on the characteristic features of the complex \( t \)-plane plots when the magnetic flux is made to vary over one or several flux periods. We refer to such plots as type I \( t \)-plot. Next we examine similar plots when the electron momentum is varied. We refer to such plots as type II plots.
II. TYPE I t-PLOTS

We have analyzed the complex $t$-plane plots for a symmetric (i.e., $l_2 = l_3$) as well as for an asymmetric (i.e., $l_2 \neq l_3$) ring. We report our findings under separate headings for these two cases. Note that our usage of the phrases “symmetric” and “asymmetric” ABI is different from the conventional usage.

A. The symmetric ring interferometer

For a symmetric ring ABI, for arbitrary values of $q$, the typical behavior of $T$ as $\theta$ is varied is shown in the curves marked 1 and 2 in Fig. 2. Note the presence of minima of $T$ at integer $q$ values. In contrast, sharp transmission ones with characteristic appearance of resonances arise for values of $q$ in close proximity of integer $q$’s corresponding to discrete energy eigenstates. These resonances were missed in the calculation of Xia [6], but their existence has been anticipated by Cahay et.al. [2]. Note that the perfect ring (i.e., a ring with no attached leads) supports eigenstates of $q$ for integer $q$’s corresponding to energy eigenvalues $E_n = n^2\hbar^2/2\mu l^2$ (corresponding to $q = n$), where $\mu$ is the effective mass of the electron.

FIG. 2: Variation of $T$ with $\theta$. Thick line: $q = 1$, thionine: $q = 1/4$ and dotted line: $q = 1/2$. 

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The typical behavior of $T$ versus $\theta$ (for arbitrary $q$) exhibiting transmission zeros for half odd integer values of $\theta$ and local minima for integer values of $\theta$ is shown in the thin line curve of Fig.2. However, for special values of $q$ in the proximity of integer $q$ values, sharp transmission ones’ appear (thick line in Fig.2).

The flux periodicity of 1 reflected in the pattern of Fig.2 may give the impression that the phenomenon of electron transmission in the ABI is also flux periodic with the same period 1. However, this expectation is not borne out by the following results relating to the TA. For $q = n + 0.01$ ($n$ being an integer), i.e., for $q$-values very close to integer values $n$ corresponding to the transmission ones, the trajectory in the complex $t$-plane is shown in Fig.3. It starts from a point $A$ for $\theta = 0$, covers the path $ABO$ as $\theta$ is made to vary from 0 to $1/2$ and thereafter follows the path $OCD$ as $\theta$ varies from $1/2$ to 1, reaching the point $D$. If $\theta$ is varied further from 1 to 2, the path $DCOBA$ is followed, thus retracing the path to generate a periodic trajectory. Thus two flux periods are necessary to complete one periodic cycle in the $t$-plane. The above description holds also for $q = n - 0.01$; only the shape of the trajectory appears reflected about the real axis. We note that the above two values of $q$ corresponds to energy values—one slightly above and the other slightly below the discrete eigenvalues of the electron energy (in the perfect ring).
FIG. 4: Complex $t$-plane plots as the flux is varied. Curves marked 1, 2, 3 and 4 correspond to $q$ values between 0 and 1/2. The ones marked 5, 6, 7 and 8 correspond to $q$ values between 1/2 and 1.

For $q$ values covering the range $n \leq q \leq n + \frac{1}{2}$, the shape of the trajectory remains unaltered; only the curve gets rotated progressively in the anticlockwise (or clockwise, depending on the direction of the flux) direction to attain the maximum rotation by an angle $\pi/2$. As $q$ is increased further to cover the range $n + \frac{1}{2} \leq q \leq n + 1$, the trajectory has a “flipping point” when it’s shape gets completely reflected about the $\text{Im}(t)$ axis as $q$ crosses the value $n + \frac{1}{2}$, but keeps rotating in the same direction by a further amount of $\pi/2$. The final shape is not back to its initial shape. Further variation of $q$ from $n + 1$ to $n + 2$ is required to generate one complete periodic cycle. This behavior is depicted in Fig. 4.

B. The asymmetric ring interferometer

In order to fix the ideas, we first consider the specific example of a ring with $l_1 : l_2 = 2 : 3$. The cases corresponding to other rational fraction ratios are similar. The first interesting observation is that the trajectories form closed curves only upon varying $\theta$ over five ($= 2 + 3$, the sum of the numbers in the ratio $l_1 : l_2$) flux periods. A typical plot is presented in Fig. 5 for $q = 1/4$ (i.e., $k = \pi/2l$) when the flux is varied in the range $0 \leq \theta \leq 1$. Note that the shape of this curve is similar to the curve in Fig. 3 but the curve does not pass through the origin. In the case of the symmetric
FIG. 5: Complex $t$-plane plot at $q = 1/4$ for $l_2 : l_3 = 2 : 3$ as the flux is varied over (a) one flux period, (b) five flux periods.

ABI, further variation beyond the above range only results in the repetition of this trajectory. However, in the present case, further variation of flux leads to evolution along different directions and when the variation covers five flux-periods (i.e., $0 \leq \theta \leq 5$), a closed flower-like pattern results (see Fig. 6). Further variation of flux leads to reparation of this pattern. This behavior is indicative of an overall periodicity of five in the underlying phenomenon. Note that for this value of $q$, the $T$ versus $\theta$ plot does not reveal this 5-fold periodicity (see Fig. 6). Some plots for a few more typical

FIG. 6: Variation of $T$ with $\theta$ for $q = 1/4$.

$q$ values are shown in Figs. 7. Note that the passage of the trajectory through the origin signifies the presence of a transmission zero.

When the value of the ratio $l_2 : l_3$ is not a rational fraction, the pattern generated
FIG. 7: Typical complex $t$-plane plots. The parameters chosen are: (a) $l_2 = 2/5$, $l_3 = 3/5$, $q = 1/6$, $0 \leq \theta \leq 5$; (b) $l_2 = 2/5$, $l_3 = 3/5$, $q = 1$, $0 \leq \theta \leq 5$; (c) $l_2 = 3/7$, $l_3 = 4/7$, $q = 1/6$, $0 \leq \theta \leq 7$; (d) $l_2 = 3/7$, $l_3 = 4/7$, $q = 2$, $0 \leq \theta \leq 7$.

is not a closed curve. Figure 8 presents an example of such a pattern for $l_2 = 1/\sqrt{5}$ and $l_3 = 1 - 1/\sqrt{5}$. In this case the trajectory never closes and as $\theta$ is varied over longer and longer intervals, the pattern generated turns out progressively denser—the behavior reflected in the figure. Clearly, the phenomenon is not flux-periodic. We now report on the behavior of the complex $t$-plots when, in stead of the flux, the momentum $q$ is made to vary.

III. TYPE II $t$-PLOTS

The complex plane plot of $t$ as the electron momentum is varied have been discussed in the literature earlier\cite{4} for the case of an ABI with an embedded quantum dot and also for an embedded double-barrier. We are not aware of such a calculation for a bare ABI.

For the symmetric ring ABI, we obtain the typical behaviors for different cases
FIG. 8: Complex $t$-plane plot for $l_2 = 1/\sqrt{5}$, $l_3 = 1 - 1/\sqrt{5}$. The plot on the left is drawn for $0 \leq \theta \leq 15$ and the one on the right, for $0 \leq \theta \leq 45$.

FIG. 9: Type II $t$-plots for the symmetric ring ABI. (a) $\theta = 0$, (b) $\theta = 5$.

corresponding to fixed typical values of the flux. Figure 9 displays these behaviors. Note that when the flux is zero (case (a)), the closed curve is an ellipse and it does not pass through the origin indicating the absence of transmission zeros. However, the presence of even a tiny flux causes transmission zeros to appear; thus the plots (cases b, c and d) exhibit curves passing through the origin. This qualitative difference between the two cases is remarkable. We note in passing that increase of flux to higher magnitudes does not alter the appearance shown in Fig. 9b.

The case of an asymmetric ABI with the ratio $l_2 : l_3$ being a rational fraction is shown in Figs. 10. The variation of $q$ over larger and larger intervals does not
FIG. 10: Type II $t$-plots for an asymmetric ring ABI. (a) $\theta = 0$, (b) $\theta = 5$, and $0 \leq q \leq 10$. In both cases, $l_2 = 2/5$ and $l_3 = 3/5$.

FIG. 11: Type II $t$-plots for an ABI with $l_2 = 1/\sqrt{5}$ and $l_2 = 1 - 1/\sqrt{5}$. (a) $\theta = 5$ and $0 \leq q \leq 10$, (b) $\theta = 5$ and $0 \leq q \leq 50$.

generate additional features. In fact, the plot for $0 \leq q \leq 50$ is no different from that presented in Fig. 10. This implies a kind of (rather complicated) $q$-periodicity in the phenomenon.

When the ratio $l_2 : l_3$ is irrational, the type II $t$-plots, just as the type I $t$-plots, do not exhibit closed curves. Figure 11 illustrates this point. Note that as the variation of $q$ spreads over larger intervals, the pattern generated becomes denser. With these results we end with our conclusions.

IV. CONCLUSION

Our results establish the utility of the type I $t$-plots in the characterization of the ABI. The information obtained via both types of plots is very rich and perhaps need
to be pursued further. The fact that the flux periodicity nature of the transmission coefficient $T$ does not imply corresponding periodicity in the underlying phenomenon is clearly brought out in the present calculations. It may be of interest to study the changes arising in the plots when an impurity is embedded in one of the arms of the interferometer. We have obtained a host of new results by way of constant transmission contour plots and also investigated the effect of electric field acting on one of the arms of the interferometer. These results will be reported later.

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