Local and Global Expansion in Random Geometric Graphs

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ABSTRACT

Consider a random geometric 2-dimensional simplicial complex $X$ sampled as follows: first, sample $n$ vectors $u_1, \ldots, u_n$ uniformly at random on $\mathbb{S}^{d-1}$; then, for each triple $i, j, k \in [n]$, add $(i, j, k)$ and all of its subsets to $X$ if and only if $(u_i, u_j) \geq r$, $(u_i, u_k) \geq r$, and $(u_j, u_k) \geq r$. We prove that for every $\varepsilon > 0$, there exists a choice of $d = \Theta(\log n)$ and $r = r(\varepsilon, d)$ so that with high probability, $X$ is a high-dimensional expander of average degree $n^c$ in which each 1-link has spectral gap bounded away from $\frac{1}{2}$.

To our knowledge, this is the first demonstration of a natural distribution over 2-dimensional expanders of arbitrarily small polynomial average degree and spectral link expansion better than $\frac{1}{2}$. All previously known constructions are algebraic. This distribution also furnishes an example of simplicial complexes for which the trickle-down theorem is nearly tight.

En route, we prove general bounds on the spectral expansion of random induced subgraphs of arbitrary vertex transitive graphs, which may be of independent interest. For example, one consequence is an almost-sharp bound on the second eigenvalue of random $n$-vertex geometric graphs on $\mathbb{S}^{d-1}$, which was previously unknown for most $n, d$ pairs.

The full version of this paper can be found here.

CCS CONCEPTS

- Theory of computation → Generating random combinatorial structures;
- Mathematics of computing → Random graphs: Spectra of graphs.

KEYWORDS

random graphs, random geometric graphs, high-dimensional expansion, trace method

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1 INTRODUCTION

A graph $G$ is called a spectral $\lambda$-expander if the second eigenvalue of its normalized adjacency matrix, $\lambda_2(G)$, is at most $\lambda$. More generally, a sequence of graphs of increasing size $(G_n)_{n\in\mathbb{N}}$ is said to be a family of (1-dimensional) spectral $\lambda$-expanders if $\lambda_2(G_n) < \lambda < 1$ as $n \to \infty$, and importantly, this implies that no vertex cut of $G_n$ has sub-constant sparsity. Expanders are an indispensable tool in theoretical computer science and mathematics, underlying advances in pseudorandomness, coding theory, routing algorithms, and more (e.g. [37, 67, 68], see also the survey [35]); similarly, the phenomenon of expansion has enabled the analysis of approximation algorithms, probabilistically checkable proofs, embeddability of metric spaces (e.g. [5, 18, 47]), as well as numerous results in number theory, group theory, and other areas of pure mathematics (see e.g. the survey [53]).

Sparse expander graphs were first shown to exist via the probabilistic method [40, 67]. In fact, even the most extreme version of expansion is ubiquitous: The best possible spectral expansion for a $d$-regular graph is the Ramanujan bound $\lambda = \frac{2\sqrt{d-1}}{d}$, and this bound is achieved (up to an additive $o_d(1)$) by a random $d$-regular graph with high probability [2, 29, 30, 62]. Even an Erdős–Rényi graph $G(n, p)$ forms a “decorated expander” for any $p > \frac{\chi}{n}$, which is to say the graph is an expander when one omits small isolated connected components and dangling trees [8, 26]. The sampling of these graphs may be derandomized [9, 61], so that in addition to several known explicit constructions (see [31, 55, 58, 59], and others) we have a wealth of algorithmic constructions of expander graphs to use in applications.

Higher-dimensional spectral expansion is a generalization of expansion to simplicial complexes. For simplicity, we will for the moment limit ourselves to 2-dimensional spectral expansion, which can be stated easily in terms of simple graphs. A graph $G = (V, E)$ is said to be a 2-dimensional spectral expander if $G$ itself is an expander, and further for every vertex $v \in V$, the induced graph on $v$’s neighbors $G[N(v)]$ (called the “link” of $v$) is a $\lambda$-expander for $\lambda < \frac{1}{2}$. The significance of this is that when the local expansion is $\lambda < \frac{1}{2}$, this is enough to trigger a “trickling down” phenomenon that ensures that $G$ in its entirety is an expander! Hence, higher-dimensional

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1Though even in [2] it is conjectured that random graphs achieve the Ramanujan bound, the first graphs proven achieve this bound were explicit algebraic constructions [55, 58].
expander have the remarkable property that global expansion is witnessed by local expansion. This local-to-global phenomenon has led to a number of recent breakthroughs in theoretical computer science: objects inspired by high-dimensional expanders are crucial in explicit constructions of locally testable codes [19, 45] and quantum LDPC codes [20, 43, 65], and the local-to-global phenomenon has been essential in analyzing Markov chains for a wide variety of sampling problems [4].

The simplest example of a 2-dimensional expander is the complete complex, based on the complete graph $K_n$. Sparse examples are known as well (e.g. [12, 38, 44, 56, 57]), though at first their existence may seem remarkable: such graphs must be globally sparse, and yet the $O(1)$-sized local neighborhood of every vertex must be densely connected to ensure sufficient expansion. This is a delicate balance, and indeed given the state of our knowledge today the phenomenon of sparse high-dimensional expansion seems "rare," in sharp contrast with the ubiquity of 1-dimensional expansion. Only a few sparse constructions are currently known, and many of these constructions are algebraic, inheriting their expansion properties from the groups used to define them (as discussed further in Section 1.3).

A prominent open problem in the area is to identify natural distributions over sparse higher-dimensional expanders [46, 52, 54]; this would be highly beneficial, both for a deeper mathematical understanding and for applications in algorithms and complexity. The simplest distributions immediately fall: random $d$-regular graphs are locally tree-like, and so with high probability $G[N(o)]$ will be an independent set (with $\lambda = 1$) for most $o \in V$. The same is true for an Erdős–Rényi graph $G(n, p)$ when $p \ll \frac{1}{\sqrt{n}}$. Though a number of distributions have been shown to have some higher-dimensional expansion properties [15, 16, 27, 33, 48, 50], they fall short in some sense: either they are quite dense (degree $\Omega(\sqrt{n})$) or fail to satisfy the spectral condition $\lambda < \frac{1}{2}$. In this work, our primary question is the following:

Are there natural, high-entropy distributions over $2$-dimensional expanders of average degree $\ll \sqrt{n}$?

We answer this question in the affirmative. We prove that for any $\varepsilon > 0$ and any large enough $n \in \mathbb{N}$, there exists a choice of $d \in \mathbb{N}$ such that a random $n$-vertex geometric graph on $\mathbb{S}^{d-1}$ with average degree $n'$ is a 2-dimensional expander with high probability.

1.1 Our Results

In order to state our results, we first give some formal definitions.

**Definition 1.1** (Simplicial complex). A $k$-dimensional simplicial complex $X$ is a downward-closed collection of subsets of size at most $k + 1$ over some ground set $X_0$, with a downward-closed weight function $w$. Any $S \in X$ is called a $(|S| - 1)$-face, and the restriction of $X$ to sets of size at most $t + 1 \leq k$ is called the $t$-skeleton of $X$. The degree of $v \in X_0$ is the number of top-level faces that contain it.

For example, the set of all cliques of size at most $k + 1$ in a graph $G$, where the weight of each clique is proportional to the number of $(k + 1)$-cliques it occurs in, defines a $k$-dimensional simplicial complex.

**Definition 1.2** (Link). Let $X$ be a simplicial complex. For any face $S$, the link of $S$ in $X$ is the simplicial complex $X_S$ with weight function $w_S$, consisting of all sets in $X$ which contain $S$, minus $S$: $X_S = \{ T \setminus S \mid T \supseteq S, T \in X \}, \quad w_S(T \setminus S) = w(T) \quad \forall T \in X_S$

For example, in the simplicial complex whose highest order faces are the triangles in a graph $G$, the link of a vertex $v$ is the induced graph on the neighbors of $v$ with its isolated vertices removed.

We are interested in simplicial complexes where the links expand enough to trigger a "local-to-global phenomenon" via the trickling-down theorem, stated below in the 2-dimensional case. 

**Theorem 1.3** (Trickling-down theorem [64]). Let $X$ be a 2-dimensional simplicial complex. If its 1-skeleton is connected, and the second eigenvalue of every link's random walk matrix is at most $\lambda$, then the second absolute eigenvalue of the random walk matrix of the 1-skeleton of $X$ is at most $\frac{\lambda}{1 - \lambda}$.

This theorem explains the significance of $\lambda = \frac{1}{2}$, since when $\lambda < \frac{1}{2}$, local expansion "trickles down" to imply global expansion. We will show that random geometric graphs, in a carefully-chosen parameter regime, have sufficient link expansion.

**Definition 1.4** (Random geometric graph). A random geometric graph $G \sim \text{Geo}_d(n, p)$ is sampled as follows: for each $i \in [n]$, a vector $u_i$ is drawn independently from the uniform distribution over $\mathbb{S}^{d-1}$ and identified with vertex $i$. Then, each edge $(i, j)$ is included if and only if $\langle u_i, u_j \rangle \geq \tau$ where $\tau = \tau(p, d)$ is chosen so that $\Pr[\text{Geo}_d(n, p)(i, j) \in G] = p$.

**Definition 1.5** (Random geometric complex). The random geometric $k$-complex $\text{Geo}_d(k, n, p)$ is the distribution defined by sampling $G \sim \text{Geo}_d(n, p)$ and taking the downward-closure of the complex whose $k$-faces are the cliques of size $(k + 1)$ in $G$.

Our main result proves that there are conditions under which random geometric 2-complexes of degree $n'$ are high-dimensional expanders enjoying the trickling-down phenomenon:

**Theorem 1.6.** For every $\varepsilon \in (0, 1)$, there exist constants $C_{\varepsilon}$ and $\delta = \exp(-O(1/\varepsilon))$ such that when $H \sim \text{Geo}_d(2)(n, n^{-1+\varepsilon})$ for $d = C_{\varepsilon} \log n$, with high probability every vertex link of $H$ is a $(\frac{1}{2} - \delta)$-expander, and hence its 1-skeleton is a $(1 - \frac{4\delta}{1 + 25\delta})$-expander.

**Remark 1.7.** The complexes arising from Theorem 1.6 with high probability have degree bounded by $O(n^{2\varepsilon})$, as the number of triangles a vertex participates in is the square of its degree in the 1-skeleton.

Along the way, we also analyze the spectrum of $G \sim \text{Geo}_d(n, p)$ directly and obtain sharper control of its second eigenvalue in a more general setting, giving bounds on the spectral norm of random geometric graphs in the full high-dimensional ($d \to \infty$) regime. To our knowledge, previous results in this vein are only for $d \sim n^{1/k}$ for fixed integers $k$ [10, 14, 22, 23, 25, 51].

2The trickling-down theorem also generalizes to higher dimensions: sufficiently strong local spectral expansion of only the highest-order links implies global spectral expansion.
Theorem 1.8. Let \( G \sim \text{Geo}_d(n, p) \) and \( \tau := \tau(p, d) \). Then with high probability \( G \) is a \( \mu \)-expander, where
\[
\mu := (1 + o(1)) \cdot \max \left( 1 + o_d(1), \frac{\log^4 n}{\sqrt{np}} \right),
\]
where \( o_d(1) \) denotes a function that goes to 0 as \( d \to \infty \).

In Section 8 of the full version we show that an eigenvalue close to \( \tau \) is achieved (for some \( p, d \)), so Theorem 1.8 is close to sharp. Since in Theorem 1.6 we show that the vertex links of \( G \) have eigenvalue \( \lambda \leq \frac{\tau}{1-d} \), this implies that the trickling-down theorem is tight.

Proposition 1.9 (Trickling-down theorem is tight). For each \( \lambda \in (0, \frac{1}{2}] \) and \( \eta > 0 \) there exists a 2-dimensional expander in which all vertex link eigenvalues are at most \( \lambda \) for which the 1-skeleton is connected with eigenvalue at least \( \frac{\tau}{1-d} - \eta \).

### 1.2 Spectra of Random Restrictions

Theorem 1.8 (and morally Theorem 1.6) is a consequence of a more general theorem that we prove concerning the spectral properties of random restrictions of graphs. We describe this result here, both because it may be of independent interest, and because it may help demystify Theorem 1.6.

Random restriction is a procedure for approximating a large graph \( X \) by a smaller graph \( G \): one selects a random subset of vertices \( S \), and then takes \( G \) to be the induced graph \( X[S] \). The random restriction \( G \) is now a smaller (and often sparser) approximation to \( X \); this idea has been useful in a number of contexts in theoretical computer science (e.g., [3, 7, 32, 36, 42]).

The core question is: to what extent do random restrictions actually inherit properties of the original graph? We will show that if random walks on \( X \) mix rapidly enough, then random restrictions inherit the spectral properties of the original graph.

To see the relevance of this result in our context, notice that a random geometric graph on the sphere is a random restriction of the (infinite) graph with vertex set \( S^d \) and edge set \( \{ (u, v) \mid (u, v) \geq \tau \} \). Theorem 1.6 is then a consequence of the fact that the sphere is itself a 2-dimensional expander.

We state the theorem precisely below.

Definition 1.10 (Random restriction). Suppose \( X \) is a (possibly infinite) graph, and that the simple random walk on \( X \) has unique stationarity distribution \( \rho \). We define an \( n \)-vertex random restriction of \( X \) to be a graph \( G \sim \text{RR}_n(X) \) sampled by sampling \( n \) vertices independently according to \( \rho \), \( S \sim \rho^\otimes n \), then taking \( G = X[S] \) to be the graph induced on those vertices.

We show that if the average degree in \( G \) is not too small, \( \lambda_2(G) \) reflects the rapid mixing of the random walk on \( X \).

Theorem 1.11. Let \( X \) be a (possibly infinite) vertex-transitive graph on which the associated simple random walk has a unique stationarity distribution \( \rho \), and let \( p = \text{Pr}_{G \sim \text{RR}_n(X)}[(i, j) \in E(G)] \) be the marginal edge probability of a \( n \)-vertex random restriction of \( X \). Suppose there exist \( C \geq 1 \) and \( \lambda \in (\frac{1}{\sqrt{\pi}d}, 1] \) such that for any \( k \in \mathbb{N} \), \( k \)-step walks on \( X \) satisfy the following mixing property: for any distribution \( \alpha \) over \( V(X) \),
\[
d_{TV}(\alpha^k, \rho) \leq C \cdot \lambda^k,
\]
where \( X^k \) denotes the \( k \)-step random walk operator on \( X \), and furthermore suppose \( pn \gg C^k \log^4 n \). Then for any constant \( \gamma > 0 \),
\[
\text{Pr}_{G \sim \text{RR}_n(X)} \left[ \left| \lambda_2(\hat{A}_G) \right|, \left| \lambda_\infty(\hat{A}_G) \right| \leq (1 + o(1)) \cdot \max \left( \lambda, \frac{\log^4 n}{\sqrt{pn}} \right) \right] \geq 1 - n^{-\gamma},
\]
where \( \hat{A}_G \) is the (normalized) adjacency matrix of \( G \).

Remark 1.12. It is likely that some of the conditions of Theorem 1.11 could be weakened. The decay of total variation could plausibly be replaced with a (much weaker) assumption about the spectral gap of \( X \); this would not impact our results for \( \frac{d-1}{2} \), but may be useful in other applications. Transitivity is assumed mostly to make the proof of Theorem 1.11 go through at this level of generality; to prove Theorem 1.6 we re-prove a version of Theorem 1.11 for the specific non-transitive case where \( X \) is a link of a vector in the sphere (a spherical cap).

### 1.3 Related Work

We give a brief overview of related work. While so far we have focused on a spectral notion of high-dimensional expanders (HDX), there are two additional notions: coboundary and cosystolic expansion. These are meant to generalize the Cheeger constant, a cut-based measure of graph expansion.

Distributions over high-dimensional expanders. The existence of natural distributions over sparse HDXs has been a question of interest since sparse HDX were first shown to exist (and this was highlighted as an important open problem in e.g., [52, 54]).

The early work of Linial and Meshulam [48] considered the distribution over 2-dimensional complexes in which all edges \( \binom{[n]}{2} \) are included, and each triangle is included independently with probability \( p \); they identified the phase transition at \( p \) for coboundary connectivity for this distribution (see also the follow-ups [6, 49, 60]). This distribution has the drawback that the 1-skeleton of these complexes is \( K_n \), and so the resulting complex is far from sparse.

In [27], the authors show that a union of \( d \) random partitions of \([n]\) into sets of size \( k + 1 \) with high probability produces a geometric expander [34], which is a notion of expansion which measures how much the faces must intersect when the complex is embedded into \( \mathbb{R}^k \). The resulting complexes have disconnected links when \( d \ll \sqrt{n} \), and so they fail to be spectral HDXs.

The work of [50] introduces a distribution over spectral expanders with expansion exactly \( \frac{1}{2} \) by taking a tensor product of a random graph and a HDX; the authors show that down-up walks on these expanders mix rapidly, and [33] introduces a reweighting of these complexes which yields improved mixing time bounds. However, the links in these complexes fail to satisfy \( \lambda < \frac{1}{2} \), and so fall outside of the range of the trickling-down theorem. The same drawback applies to [15, 16]: they show that up-down walks mix on random polylogarithmic-degree graphs given by subsampling a random set of generators of a Cayley graph. However, these graphs do not satisfy the conditions of the trickling-down theorem.

Explicit constructions. One of the first constructions of sparse high-dimensional spectral expanders was the Ramanujan complex of [12, 44, 56, 57], which generalize the Ramanujan expander graphs of [55]. Not only are these spectral expanders, but [24, 38] also show...
that they are co-systolic expanders. These Ramanujan complexes are algebraic by nature, constructed from the Cayley graphs of PSLd(Fq). Other algebraic constructions include that of [39]; the authors analyze the expansion properties of coset complexes for various matrix groups. They achieve sparse spectral expanders, with local expansion arbitrarily close to 0. More recently, [63] extend the coset complex construction to the more general family of Chevalley groups.

A few combinatorial constructions for HDX are also known. [13] prove that objects called (a, b)-expanders are 2-dimensional spectral expanders; they give a graph-product-inspired construction of a family of such expanders, and show that other known complexes [12, 39, 44, 56, 57] are also (a, b)-expanders. Their work is extended by [28] to higher dimensions.

**Applications of HDX.** The local-to-global phenomenon in HDX has already been useful in many settings. [21] use spectral HDXes to construct “agreement expanders,” whose links give rise to local agreement tests: given “shards” of a function that pass a large fraction of the local agreement tests, the authors can conclude the presence of a “global” function g that stitches the shards together. In coding theory, the locally testable codes of [19] and quantum LDPCs of [20, 43, 65] utilize a common simplicial-complex-like structure called the square Cayley complex, whose local-to-global properties are essential in the analysis of these codes.

The local-to-global phenomenon also implies that “down-up” walks on the associated simplicial complex mix (as made formal in [1]). A k-down-up walk is supported k-faces of the simplicial complex, and transitions occur by dropping down into a random (k − 1)-face, then transitioning up to a random k-face (one can also define the “up-down” walk analogously). This local-to-global analysis has recently been influential in the study of mixing times of Markov chains. Several well-studied Markov chains can be recast as the k-down-up random walk of a carefully designed simplicial complex. One notable example is the matroid basis exchange walk, which is an algorithm for sampling independent sets of a matroid (e.g., spanning trees in the graphical matroid). [3] were able to obtain an improved mixing time bound for the basis exchange walk—a significant breakthrough that, due to the local-to-global property, was achieved through the analysis of simple, “local” view of the matroids.

**Random geometric graphs and random kernel matrices.** Random restrictions of metric spaces such as Sd−1 and [−1, 1]d are well-studied in the fixed-dimensional regime, where d = O(1) and n → ∞ (see the survey of Penrose [66]). In our work we are interested in the high-dimensional setting, where d → ∞ with n. The high-dimensional setting was first studied only recently, initiated by [11, 17], and many mysteries remain in this young area of study.

Our Theorem 1.8 is related to the study of kernel random matrices; random n×n matrices whose (i, j)-th entry is given by fδ((ui, uj)), for fδ : R → R and ui, . . . , un sampled independently from some distribution over Sd. The special case of ui ∼ Unif(Sd−1) and fδ(x) = 1[x ≥ τ(p, d)] yields the adjacency matrix of Geoq(n, p). A line of work initiated by [41] studies the spectrum of kernel random matrices [10, 14, 22, 23, 25], and the most recent work [51] characterizes the limiting empirical spectral distribution when d = Θ(n1/k) for k a fixed constant and f can be “reasonably” approximated by polynomials (in a sense that is flexible enough to capture the indicator fδ(x) = 1[x ≥ τ(p, d)]). In comparison with our results, they characterize the entire empirical spectral distribution, but we do not need to restrict d ~ n1/k for integer k, which is crucial for our applications.

## 1.4 Discussion and Open Questions

**Sparser high-dimensional expanders from random restrictions?** As hinted in Section 1.2 the random geometric complex fits in the broader framework of random restrictions of simplicial complexes: starting with a dense high-dimensional expander X, we sample a subset of vertices S of X to produce the sparser induced complex X[S].

We have shown in Theorem 1.11 that XI(S) (to some extent) inherits the spectral properties of X itself, and we’ve leveraged this to show that for any polynomial average degree, one can produce a 2-dimensional expander by taking a random restriction of X the sphere in a particular dimension and with a particular connectivity distance. We hope that Theorem 1.11 (or a strengthening thereof, see Remark 1.12) might help us identify additional natural distributions over sparser and/or higher-dimensional complexes. More specifically,

> Is there a simplicial complex X whose random restrictions yield high-dimensional expanders whose links have eigenvalue < 1/2, of sub-polynomial or polylogarithmic degree?

As a starting point, we remark that geometric graphs on the unit sphere work because the corresponding X itself has link expansion better than 1/2, witnessing that Sd−1 itself is an expander. Some simpler-to-analyze metric spaces do not have this property; for example:

**Shortest path metric in a graph.** Starting with a connected, locally treelike d-regular graph G, consider the geometric graph given by connecting pairs of vertices at distance ≤ 2 in G. The triangle complex on the resulting graph has links which are connected, and further the 1-skeleton is an expander if G is an expander. However, the links cannot identify whether G is an expander, and so link expansion cannot be better than 1/2. To see why, consider a first case where G is a random d-regular graph (expanding), and a second case where G consists of two random d-regular graphs connected by a bridge (non-expanding); because G is locally treelike in both cases, the links in these two cases will be identical.

**The d-dimensional torus with ℓ∞ metric.** Consider the geometric graph X on the d-dimensional torus [−R, R]d with “wrap around” (so that −R is identified with R), in which we connect u, v if ||u − v||∞ ≤ τ. This space is simple to analyze

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4We note that constant average degree would likely require additional work; this is not just because of the polylogarithmic factors appearing in the statement of Theorem 1.11, but because in a random restriction the degree distribution of each vertex is Binom(τ, p) and so when p = Θ(1/n) one will have isolated vertices; this is the same as the phenomenon wherein Erdős-Rényi graphs of degree O(1) are not expanders until one restricts to the giant component.

5Technically, we require this of a reweighting of G where each edge is weighted according to the number of triangles it participates in; concentration phenomena ensure that the expansion of G and this weighted graph are similar.
because of its product structure; the expansion is dictated by the ratio of \( r \) to the side length \( R \), worsening as \( R \) grows relative to \( r \). Each link is simply the box \([-\tau, \tau]^2\) with the same \( \varepsilon_0\)-edge condition, regardless of the value of \( R \). Since \( R \) dictates the global expansion, the link expansion cannot be better than \( \frac{1}{2} \).

By way of contrast, it is not possible to plant the links of the geometric graph on \( \mathbb{S}^{d-1} \) in a nonexpanding graph; for instance, it is possible to determine the radius \( R \) of a sphere of unknown scale given only a link in its geometric graph.

We also remark that Theorem 1.11 could be used to obtain expanders of dimension \( k > 2 \); indeed, it seems that this is within reach even using \( \text{Geo}^2_d(n, p) \). A direct approach, in the case of the sphere, is to perform the conditioning from Section 5 of the full version not only for spherical caps, but for intersections of \( k - 1 \) spherical caps as well; perhaps there is a more elegant alternative approach?

**How faithfully do random geometric graphs discretize continuous manifolds?** One interpretation of Theorem 1.8 is that the random geometric graph \( \text{Geo}_d^k(n, p) \) offers a good approximation (in spectral norm) for the corresponding metric on \( \mathbb{S}^{d-1} \) when \( np \) is large enough relative to \( d \). A natural question is to extend this to other properties of \( \mathbb{S}^{d-1} \); for example, do random geometric graphs offer a good approximation on the rest of the spectrum? Numerical experiments suggest the following (informal) conjecture.

**Conjecture 1.13.** For \( G \sim \text{Geo}_d^k(n, p) \), the spectrum of the normalized adjacency matrix \( A_G \) breaks into a “bulk” portion and an “outlier” portion where every bulk eigenvalue is at most \( O(1/\sqrt{np}) \) in magnitude, and every outlier eigenvalue is “close” to an eigenvalue of the graph on the sphere with an edge between every \( u, v \) with \( \langle u, v \rangle \geq \tau(p, d) \).

A proof of the above conjecture, and an investigation of whether an analogous phenomenon holds on general manifolds, would be very interesting.

**Spectral algorithms for random geometric graphs.** Here, we have given some of the first analyses of the spectral radius of random geometric graphs on the sphere. One appeal of random geometric graphs on the sphere, or in Gaussian space, is that they offer a more natural model for networks arising from data than, e.g., Erdős-Rényi graphs. The idea is that in modern networks, we often think of each node as being representable by a latent feature vector, with nearby nodes having similar features. Hence, geometric graphs are promising as an alternative testbed for rigorous analysis of algorithms. Yet currently, they have not been studied much in such a context, in part because of the absence of tools for their analysis.

A natural question is whether one could build on our work to analyze spectral clustering algorithms in “random geometric block model” graphs.

**Question 1.14.** Suppose \( G \sim \text{RGeo}^k \left( \frac{1}{2} \mathcal{N}(0, \Sigma_1) + \frac{1}{2} \mathcal{N}(\mu, \Sigma_2) \right) \); that is, \( n \) points \( u_1, \ldots, u_n \) are sampled from the uniform mixture over the \( d \)-dimensional Gaussian distributions \( \mathcal{N}(0, \Sigma_1) \) and \( \mathcal{N}(\mu, \Sigma_2) \), then \((i, j) \in E(G)\) if and only if \( ||u_i - u_j|| \leq \varepsilon \). Does spectral clustering recover the component membership of the datapoints?

This question is a more accurate representation of clustering problems arising from real data than, say, the question of applying spectral clustering to recover cluster memberships in the stochastic block model; it would be interesting to understand the conditions (on \( n, d, \delta, \mu, \Sigma_1, \Sigma_2 \)) which guarantee that spectral clustering succeeds.

### 1.5 Overview of the Proof

We now explain how we prove our main theorem, Theorem 1.6, which states that for a complex sampled from \( H \sim \text{Geo}_d^k(n, p) \) for \( p = n^{-1+c} \) with \( 0 < c < 1 \) and \( d = C_d \log n \), with high probability every link of \( H \) is a \( \left( \frac{1}{2} - \delta \right) \)-expander for some \( \delta = \exp(-O(1)) \), and its 1-skeleton is a \( \left( 1 - \frac{4\delta}{1 + 2\delta} \right) \)-expander. By the trickling-down theorem, it suffices for us to prove:

1. All vertices’ corresponding links in \( H \) are \( \left( \frac{1}{2} - \delta \right) \)-expanders with high probability.
2. The 1-skeleton of \( H \) is connected with high probability.

To show Item 2, it is enough to show that some reweighting of the 1-skeleton expands; Item 1 implies that every edge \((i, j)\) must participate in at least one triangle (otherwise the link would be short). Thus, we weight each edge \((i, j)\) by \( 1/\delta \) and then sample \( r \) points \( v_1, \ldots, v_r \) independently and uniformly from a measure-\( p \) cap in \( \mathbb{S}^{d-1} \) centered at some point \( w \) (corresponding to the vector of the link vertex \( i_w \)), placing an edge between every \( i, j \) such that \( \langle v_i, v_j \rangle \geq \tau(p, d) \). Finally, we remove any isolated vertices; here, we’ll show that the graph expands with high probability before removing these isolated vertices, which implies that no isolated vertices have to be removed. For the remainder of the overview, let \( r = r(p, d) \). We’ll show that:

**Theorem 1.15 (Informal Version of Theorem 5.1 of the Full Version).** Let \( G \) be the link of some point \( w \sim \mathbb{S}^{d-1} \) induced by \( v_1, \ldots, v_m \sim \text{cap}_p(w) \). Then with high probability \( G \) is a \( \mu \)-expander where

\[
\mu \sim (1 + o(1)) \cdot \max \left( \frac{\tau}{1 + 1}, \frac{\log^d n}{\sqrt{\varepsilon m}} \right) + o_d(1).
\]

Here \( q = \text{Pr}_{u, v \sim \mathbb{S}^{d-2}} \left[ (u, v) \geq \tau(p, d) \right] \).

Links are essentially random geometric graphs in one lower dimension. Since most of the measure of the cap lies close to its boundary, intuitively the link is distributed almost like a random geometric graph with points drawn independently from the cap boundary, i.e. the shell set \( \text{shell}_p(w) \) of \( \tau(p, d) \). Our proof of Theorem 1.15 must pay attention to the fluctuations in \( \langle v, w \rangle = \tau \), but to simplify
our current discussion we assume each link is in fact a random geometric graph on \( s_{\mu}(w) \), and address the fluctuations later in the overview.

Observe that a uniformly random \( v \) from \( s_{\mu}(w) \) is distributed as \( \tau \cdot v + \sqrt{1 - \tau^2} \cdot u \) where \( u \) is a uniformly random unit vector orthogonal to \( w \). Using this decomposition, we see that \( \langle v, u' \rangle \geq \tau \) if and only if \( \langle u, u' \rangle \geq \tau \). Thus, under our simplifying assumption, the link is distributed exactly like a random geometric graph on \( S^{d-2} \) with inner product threshold \( \tau \). Hence, (up to the difference between \( c_{\mu}(w) \) and \( s_{\mu}(w) \)) to understand link expansion we can study the second eigenvalue of a random geometric graph on the sphere.

**Remark 1.16** (Requiring \( d = \Theta(\log n) \)). In light of Theorem 1.15 (and even the heuristic discussion above), it turns out that \( d = \Theta(\log n) \) is the only regime for which the links can be connected while the 1-skeleton has average degree \( \approx \sqrt{n} \). To see this, we consider the relationship between \( p, \tau, d \), and \( \tau \) have that

\[
p = \Pr_{v, v' \sim S^{d-2}} \left[ \langle v, v' \rangle \geq \tau \right] = \Theta \left( \frac{1}{\tau^2} \right) \cdot \left( 1 - \tau^2 \right)^{\frac{d-1}{2}} \approx \exp(-d\tau^2/2).
\]

(1)

See Lemma 2.8 for a formal argument.\(^7\) Note that the arguments above in conjunction with (1) imply that the probability that two vertices within the link are connected is also roughly

\[
q = \Pr_{u, u' \sim S^{d-2}} \left[ \langle u, u' \rangle \geq \tau \right] = \Theta \left( \frac{1}{\tau^2} \right) \cdot \left( 1 - \frac{\tau^2}{1 + \tau^2} \right)^{\frac{d-1}{2}},
\]

since the link is like a random geometric graph on \( s_{\mu}(w) \).

Connectivity within the links in conjunction with sparsity now requires us to have \( d = \Theta(\log n) \): The number of vertices inside each link concentrates around \( m = np \), so the average degree inside the link is \( qm = qpn \); we must have the average link degree \( qpn \gg 1 \), otherwise the link is likely disconnected. Now, if \( \tau = O(1) \), then \( \tau \approx \frac{1}{\tau^2} \) and \( p \approx q \), so \( qpn \gg 1 \implies p^2n \gtrsim 1 \implies p \gg n^{-1/2} \), ruling out a 1-skeleton with average degree \( \approx \sqrt{n} \). Hence we need \( \tau = \Omega(1) \). Given that \( \tau = \Omega(1) \), (1) implies that to have the average 1-skeleton degree \( \sqrt{n} \gg pn \gg 1 \) we need \( d = \Theta(\log n) \).

**Spectral expansion in random geometric graphs.** We now explain how to prove near-sharp second eigenvalue bounds for random geometric graphs.

**Theorem (Restatement of Theorem 1.8).** Let \( G \sim Geo_d(n, p) \) and \( \tau = \tau(p, d) \). Then with high probability \( G \) is a \( \mu \)-expander, where

\[
\mu := (1 + o(1)) \cdot \max \left( 1 + o_{d+1}(1) \cdot \tau, \frac{\log n}{\sqrt{pn}} \right),
\]

where \( o_{d+1}(1) \) denotes a function that goes to 0 as \( d \cdot \tau(p, d)^2 \to \infty \).

As mentioned above, Theorem 1.8 is a consequence of the more general Theorem 1.11 about the second eigenvalue of random geometric graphs and restrictions of vertex-transitive graphs, and the inner product threshold \( \tau = \tau(p, d) \) appears as the mixing rate of the random walk on \( S^{d-2} \) where a step originating at \( v \) walks to a random vector in \( c_{\mu}(v) \). Via standard concentration arguments applied to the vertex degrees, to prove the above it suffices to bound \( \|A_G - E A_G\| \leq \mu \cdot pn \), where \( A_G \) is the (unnormalized) adjacency matrix of \( G \). We’ll focus on the regime where \( pn \gg \log \log n \), so that \( \mu \approx \tau \).

**Trace method for random geometric graphs.** To bound \( \|A_G - E A_G\| \), we employ the trace method, bounding the expected trace of a power of \( A_G - E A_G \). This is sufficient for the following reason: for convenience, let \( \delta = A_G - E A_G \), and let \( \ell \) be any non-negative, even integer. Since \( \ell \) is even,

\[
\delta = \delta \leq \ell \text{tr}(\delta^\ell).
\]

And so applying Markov’s inequality,

\[
\Pr \left( \|\delta\| \geq e^{\ell} \left( \text{tr}(\delta) \right)^{\frac{1}{\ell}} \right) = \Pr \left( \|\delta\| \geq e^{\ell} \text{tr}(\delta) \right) \leq \exp(-e^{-\ell}).
\]

Thus, our goal reduces to bounding the expectation of \( \text{tr}(\delta^\ell) \) for a sufficiently large even \( \ell \); in particular, if we choose \( \ell > \log n \), then since \( \delta \) has \( n \) eigenvalues, \( \text{tr}(\delta^\ell) \) is a good “soft-max” proxy for \( \|\delta\| \), and we will obtain high-probability bounds.

We now explain why properties of random walks on \( S^{d-1} \) naturally arise when applying the trace method. Concretely, \( \text{tr}(\delta) \) is a sum over products of entries of \( A_G \) corresponding to closed walks of length \( \ell \) in the complete graph \( K_n \) on \( n \) vertices:

\[
\text{tr}(\delta) = \sum_{i_0, \ldots, i_\ell, i_0 \in [n]} (A_G)_{i_0 i_\ell+1}.
\]

The walk \( i_0, i_1, \ldots, i_{\ell-1}, i_0 \) can be represented as a directed graph. When we take the expectation, the symmetry of the distribution means that all sequences \( i_0, \ldots, i_{\ell-1}, i_0 \) which result in the same graph (up to relabeling) give the same value. That is, letting \( W_\ell \) be the set of all such graphs, and for each \( W \in W_\ell \), and all \( W \in W_\ell \) be the number of ways it can arise in the sum above,

\[
\text{tr}(\delta) = \sum_{W \subseteq W_\ell} N_W \cdot \prod_{(i_j, i_{j+1}) \in W} (A_G)_{i_j i_{j+1}}.
\]

To bound this sum, we must bound the expectation contributed by each \( W \in W_\ell \). For the sake of this overview we will consider only the case when \( W = C_\ell \), the cycle on \( \ell \) vertices, as it requires less accounting than the other cases; however it is reasonable to restrict our attention to this case for now, as bounding it already demonstrates our main ideas, and because this term roughly dominates the sum with \( N_C \gg N_W \), for all other \( W \subseteq W_\ell \) at \( \ell = \log \log n \) and \( pn \gg \log \log n \).

We now bound the expectation for the case \( W = C_\ell \); readers uninterested in the finer details may skip to the conclusion in (4). We expand the product using that \( (A_G)_{i_j i_{j+1}} = A_{ij} - p \) (since \( E[A_{ij}] = p \)):

\[
\text{tr}(\delta) = \sum_{\ell} (-p)^{\ell-|T|} \prod_{t \in T} (A_{t,t+1} - p).
\]

\(^7\)Heuristically, it makes sense that \( p = \Pr[\langle v, v' \rangle \geq \tau] = \exp(-\Theta(\tau^2 d^2)) \), because \( \langle v, v' \rangle \) is approximately \( N(0, 1/2) \).

\(^8\)Briefly, this is because whenever \( i_0, \ldots, i_{\ell-1} \) are all distinct elements of \([n]\) the resulting walk’s graph is a cycle, and when \( \ell \gg \log \log n \) indices sampled at random from \([n]\) are all distinct with high probability.
and thus our focus is to understand subgraph probabilities in a random geometric graph. It is not too hard to see that when the edges specified by $T$ form a forest, its subgraph probability is $p^{|T|}$, identical to its counterpart in an Erdős–Rényi graph; the nontrivial correlations introduced by the geometry only play a role when $T$ has cycles. Hence, the sum (3) simplifies,

$$E \left[ \sum_{T \subseteq [T]} (-p)^{|T|} \Pr[(i, i+1) \in T] \right] = \Pr[C_T \text{ is subgraph of } G]$$

where we used that the binomial sum is equal to $(p - p)^0 = 1$.

Hence it remains to estimate the subgraph probability of a length-$\ell$ cycle. We will now see how subgraph probabilities are related to the mixing rate of a random walk on $\mathbb{S}^{d-1}$.

Subgraph probability of a cycle in a random geometric graph. For the cycle $C_T = 0, 1, \ldots, \ell - 1, 0$, by Bayes’ rule:

$$\Pr[C_T \in G] = \prod_{i=0}^{\ell-1} \Pr[(i, i+1) \in G | \forall j < i, (j, j+1) \in G] = p^{\ell-1} \cdot \Pr[(\ell - 1, 0) \in G | \{0, 1, \ldots, \ell - 1 \} \subseteq [T]],$$

since in all but the step $i + 1 = \ell$, the graph in question is a forest. Identifying each $i$ with a point $x_i$ on $\mathbb{S}^{d-1}$, for any choice of $x_0$ the above probability can equivalently be written as

$$p^{\ell-1} \cdot \Pr[x_{\ell-1} | \{x_0, x_1, x_{\ell-1}\} \in U \setminus \mathbb{S}^{d-1}, \{0 \leq i \leq \ell - 2 \}].$$

Denoting with $P$ the transition kernel of the random walk we alluded to earlier, where in one step we walk from a point $x$ to a uniformly random point $\mathbb{S}^{d-1}$, we can write the distribution of $x_{\ell-1} | \{x_0, x_1, x_{\ell-1}\} \in U \setminus \mathbb{S}^{d-1}$ as $p^{\ell-1} \delta_{x_0}$ where $\delta_{x_0}$ refers to the point mass probability distribution supported at $x_0$. In particular, we can write this subgraph probability as:

$$p^{\ell-1} \cdot \Pr[x_{\ell-1} \in \operatorname{cap}_P(x_0)].$$

If $x_{\ell-1}$ were sampled from the uniform distribution $\rho$ on $\mathbb{S}^{d-1}$ then the probability of landing in $\operatorname{cap}_P(x_0)$ would be $p$, which lets us upper bound the subgraph probability by:

$$p^{\ell-1} \cdot \left( p + d_{TV} \left( p^{\ell-1} \delta_{x_0}, \rho \right) \right)$$

The terms for more complicated subgraphs $W^* \in \mathcal{W}_T$ also similarly depend on the mixing properties of $P$ via subgraph probabilities. Our next goal then is to understand the mixing properties of $P$.

**Remark 1.17.** To prove Theorem 1.11 about random restrictions, the same strategy is used to relate subgraph probabilities with mixing rate of the random walk on the original graph we start with.

**Mixing properties of $P$.** We show that the walk over $\mathbb{S}^{d-1}$ with transition kernel $P$ contracts the TV distance by coupling this discrete walk with the continuous Brownian motion $U_t$ over $\mathbb{S}^{d-1}$. Then via a known log-Sobolev inequality for Brownian motion on spheres, we can prove the following contraction property for $P$.

**Theorem 1.18 (Informal version of Theorem 4.6 of the full version).** For any probability measure $\alpha$ over $\mathbb{S}^{d-1}$ and integer $k \geq 0$,

$$d_{TV} \left( P_k \alpha, \rho \right) \leq (1 + o_{d, \alpha}((1) \cdot \tau))^{k-1} \cdot \frac{\log 1}{\sqrt{2} \log p},$$

where $P_k$ denotes the transition kernel in which every $x \in \mathbb{S}^{d-1}$ walks to a uniformly random point in the measure-$P$ cap around it and $o_{d, \alpha}(1)$ denotes a function that goes to 0 as $d, \alpha \to \infty$.

We leave the details to Section 4 of the full version, but in brief, the reason we are able to execute this coupling is that the probability mass in $P \delta_{x_0}$ concentrates around $\text{shell}_{1/2}(x_0)$, and most of the $(1/\log 1/d - 1/2)$-step Brownian motion starting from $x_0$ concentrates at $\text{shell}_{1/8}(x_0)$, so when $t = 1/\log 1/d$ the operators $P$ and $U_t$ have similar action.

We can now apply Theorem 1.18 to bound $d_{TV} \left( P^{\ell-1} \delta_{x_0}, \rho \right)$ with $\sigma = \delta_{x_0}$ and $k = \ell - 1$:

$$d_{TV} \left( P^{\ell-1} \delta_{x_0}, \rho \right) \leq (1 + o(1))^{\ell-2} \cdot \frac{1}{\sqrt{2} \log 1/p}.$$
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