An Approximation Method and Its Application to the Ablowitz-Kaup-Newell-Segur Type Linear Scattering Problem by Discretizing the Initial Wave Packet

Hironobu Fujishima¹ *, and Tetsu Yajima² †

¹Optics R&D Center, CANON INC., 23-10 Kiyohara Kogyoudanchi, Utsunomiya, Tochigi 323-3298 Japan
²Department of Information Systems Science, Graduate School of Engineering, Utsunomiya University, Yoto 7-1-2, Utsunomiya, Tochigi 321-8585, Japan

A method to approximately analyze the Zakharov-Shabat eigenvalue problem associated with soliton equations which belong to the Ablowitz-Kaup-Newell-Segur system is presented. Approximated scattering data can be derived under desired accuracy by splitting the initial wave packet into many small bins and considering transfer matrices which connect the Jost functions in each interval. As an application, box-type initial conditions are considered for the nonlinear Schrödinger equation. The proposed method can extract information such as the number of solitons which appear in the final state. Furthermore, we derived the parameter conditions which generates double pole solitons.

KEYWORDS: nonlinear integrable equations, initial value problem, Zakharov-Shabat eigenvalue problem, double-pole soliton, approximate method

1. Introduction

The theory of solitons has played a prominent role in development of mathematical physics. Soliton phenomena can be observed in various systems, where nonlinearity plays a significant role, such as fluid mechanics, plasma physics, or nonlinear optics.¹ In the context of low temperature physics, the macroscopic wave functions of the Bose-Einstein condensed systems are known to obey a kind of the nonlinear Schrödinger (NLS) equation whose nonlinear term represents the bi-particle collision of constituent atoms.²

Time evolution of above mentioned nonlinear systems is investigated by solving the initial value problems of corresponding soliton equations. The inverse scattering transform (IST) is a useful method which can deal with such problems.³ This method is based on a scattering problem of a set of auxiliary linear equations, which are associated with the original soliton equation:

\[ \Psi_x = M\Psi, \]  

\[ \Psi_t = N\Psi, \]       

(1.1a)  

(1.1b)

where the quantities \( M \) and \( N \) are matrices or operators including the unknown functions of the soliton
equation, the spectral parameter and the wave function $\Psi$ representing an auxiliary field obeying appropriate boundary conditions. An important step of the IST method is to analyze the spatial equation (1.1a) as a scattering problem whose potential term is given by the initial condition of the unknown function. The wave function $\Psi$ and the spectral parameter correspond to eigenfunction and eigenvalue, respectively. This is called the Zakharov-Shabat (ZS) problem. Once the ZS problem is solved, the time-evolved wave function is easily obtained through eq. (1.1b) and the solution of the Cauchy problem is provided by virtue of the Gel’fand-Levitan-Malchenko (GLM) equation. The GLM equation clearly shows that the number of discrete eigenvalues determines that of solitons to be generated in the asymptotic future.

Usually, the ZS problems accompanied with general initial conditions are evidently difficult ones and it is rarely possible for us to predict even how many solitons remain in the final state, except for some specific initial conditions. In this paper, we aim to construct a method to derive approximate eigensets of the ZS problem for the soliton equations which can be formulated as the Ablowitz-Kaup-Newell-Segur (AKNS) system with arbitrary precisions. This can be realized by discretizing the initial wave packet into small bins and solving the scattering problem in each interval successively. Our method can straightforwardly be applied to other AKNS equations due to the common structure of the ZS problem.

This paper is organized as follows. In the next section, we shall briefly summarize the IST method and the ZS problem using the NLS equation as an example. In §3, we shall present our method and explain how to extract approximated scattering data. In §4, we will derive distributions of eigenvalues for the NLS equation with double box-type initial conditions. We will also show this simple application leads to some non-trivial results including conditions for generating double-pole solitons and crucial roles played by interfering radiations from each box-like pulse. Results of numerical simulation are shown in §5. The final section is devoted to discussions and concluding remarks.

2. Summary of the IST method and the ZS problem

We shall give a minimum explanation on the IST method and the ZS problem for later need. Throughout this paper, we take the NLS equation as an illustration. The example is mainly based on the NLS equation:

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi. \quad (2.1)$$

For the NLS equation, the matrices $M$ and $N$ in (1.1) are given as

$$M = \begin{pmatrix} -i\xi & i\psi^* \\ i\psi & i\xi \end{pmatrix}, \quad (2.2a)$$
\[ N = \begin{pmatrix} 2i\xi^2 - i|\psi|^2 & \psi^* - 2i\xi \psi^* \\ -\psi^* - 2i\xi \psi & -2i\xi^2 + i|\psi|^2 \end{pmatrix}, \] (2.2b)

where $\xi$ is the spectral parameter. Equations (1.1a) and (2.2a) completely define the ZS problem for the NLS equation. Other soliton equations belonging to the AKNS system have similar ZS problems.

We shall introduce a usual boundary condition for $\psi$:

\[ \psi \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \] (2.3)

By this boundary condition, each element of the wave function $\Psi$ must become a plane wave. As the fundamental solutions, we can select two sets of functions $\{\phi, \bar{\phi}\}$ and $\{\chi, \bar{\chi}\}$ called the Jost functions, which satisfy boundary conditions

\[ \begin{align*}
\phi(x; \xi) & \rightarrow \begin{pmatrix} e^{-i\xi x} \\ 0 \end{pmatrix}, & \phi(x; \xi) & \rightarrow \begin{pmatrix} 0 \\ e^{i\xi x} \end{pmatrix}, & & \text{as } x \rightarrow -\infty, \tag{2.4a} \\
\chi(x; \xi) & \rightarrow \begin{pmatrix} 0 \\ e^{i\xi x} \end{pmatrix}, & \chi(x; \xi) & \rightarrow \begin{pmatrix} e^{-i\xi x} \\ 0 \end{pmatrix}, & & \text{as } x \rightarrow +\infty. \tag{2.4b}
\end{align*} \]

The Jost functions are related by each other as

\[ \begin{align*}
\phi(x; \xi) &= a(\xi)\bar{\chi}(x; \xi) + b(\xi)\chi(x; \xi), \\
\bar{\phi}(x; \xi) &= \bar{a}(\xi)\chi(x; \xi) - \bar{b}(\xi)\bar{\chi}(x; \xi). \tag{2.5}
\end{align*} \]

The coefficient functions are called the scattering data and $a(\xi)$ can be analytically continued to the upper half-plane $\text{Im} \xi > 0$.

From eqs. (2.4) and (2.5), we can see that the Jost function $\phi(x; \xi)$ satisfies an asymptotic form

\[ \phi(x; \xi) = \begin{pmatrix} a(\xi)e^{-i\xi x} \\ b(\xi)e^{i\xi x} \end{pmatrix}, \quad \text{as } x \rightarrow \infty. \tag{2.6} \]

When the function $a(\xi)$ has $N$ simple zeros $\xi = \xi_1, \xi_2, \ldots, \xi_N$ on the upper half-plane, there appear $N$ solitons in the asymptotic future and each $\xi$ determines the characteristics of each soliton. We need to know $a(\xi)$ to extract information on solitons in the asymptotic future. By eq. (2.6), we can find this is equivalent to calculate $\phi(x; \xi)$ at $x \rightarrow \infty$ under the boundary condition eq. (2.4a).

### 3. Discretization of the Initial Wave Packet and Approximated Scattering Data

In this section, we shall consider the ZS problem of the NLS equation:

\[ \Psi_x = M \Psi, \quad M = \begin{pmatrix} -i\xi & i\psi^* \\ i\psi & i\xi \end{pmatrix}. \tag{3.1} \]

Since the spectral parameter $\xi$ is a time-independent quantity, we can take $\psi$ in eq. (3.1) to be the initial value of the unknown wave packet $\psi(x, 0)$.
The major difficulty in analyzing (3.1) for general initial conditions comes from the fact that \( \psi(x, 0) \) depends on the coordinate \( x \). In order to overcome this difficulty, we shall split the support of \( \psi(x, 0) \) into many small intervals:

\[
I_j : x_j \leq x < x_{j+1} \quad (j = 1, \ldots, N),
\]

and approximate \( \psi(x, 0) \) to take a constant value in each interval. We shall introduce a set of functions \( \psi_j \):

\[
\psi_j(x) = \begin{cases} 
V_j & x \in I_j, \\
0 & x \notin I_j.
\end{cases}
\]

The initial value \( \psi(x, 0) \) is now approximated as

\[
\psi(x, 0) \approx \sum_{j=1}^{N} \psi_j(x),
\]

\[
= \begin{cases} 
V_j & (x \in I_j, \ j = 1, 2, \ldots, N), \\
0 & \text{(otherwise)}.
\end{cases}
\]

Assuming \( \psi(x, 0) \) belongs to the class of rapidly decreasing functions, we can approximately consider that \( \psi(x, 0) \) has a compact support. Within each interval, eq. (3.1) reads

\[
\Psi_x = M_j \Psi, \quad M_j = \begin{pmatrix} -i\xi & iV_j^* \\
iV_j & i\xi \end{pmatrix}.
\]

We can solve eq. (3.6) for \( x \) satisfying \( x \in I_j \) as

\[
\Psi(x) = T(X)\Psi(x_j), \quad T(X) = \exp(XM_j),
\]

where \( X \equiv x - x_j \) and the matrix \( T(X) \) is explicitly written as

\[
T(X) = \begin{pmatrix} \cos KX - i(\xi/K) \sin KX & i(V_j^*/K) \sin KX \\
i(V_j/K) \sin KX & \cos KX + i(\xi/K) \sin KX \end{pmatrix},
\]

\[
K = \sqrt{\xi^2 + |V_j|^2}.
\]

We shall denote the width of the \( j \)-th bin as

\[
x_{j+1} - x_j = L_j,
\]

and we can see that the Jost function satisfies the relation

\[
\Psi(x_{N+1}) = T\Psi(x_1),
\]

\[
T = T(L_N)T(L_{N-1}) \cdots T(L_2)T(L_1).
\]
The matrix $T$ is interpreted as a transfer matrix which connects two asymptotic forms in $x \to \pm \infty$. Recalling eq. (2.4a) and the fact we truncate $\psi(x; \xi)$ to be supported only in the region $x_1 \leq x \leq x_{N+1}$, we derive a relation

$$\phi(x_{N+1}; \xi) = T \phi(x_1; \xi) = e^{-i \xi x_1} T \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(3.11)

$$= \begin{pmatrix} a(\xi)e^{-i \xi x_{N+1}} \\ b(\xi)e^{i \xi x_{N+1}} \end{pmatrix}.$$  

(3.12)

Thus we have the approximated expressions of scattering data in terms of transfer matrix as

$$a(\xi) = e^{i L \xi} T_{11},$$

(3.13a)

$$b(\xi) = e^{-i (x_1 + x_{N+1}) \xi} T_{21},$$

(3.13b)

where the parameter $L$ is defined as $L = L_1 + L_2 + \cdots + L_N$. By considering the initial packet as a set of constant functions, we can obtain explicit expression of the scattering data $a(\xi)$ and $b(\xi)$ for any initial values provided they belong the class of rapidly decreasing functions. Thus, desired information which characterizes solitons in the asymptotic future can be extracted from $a(\xi)$ with arbitrary precision by properly adjusting the width of each bin $L_j$.

4. Applications

In this section, we shall apply the method proposed in the previous section to simple initial wave packets. We approximately solve the ZS problems for these initial conditions and investigate the corresponding final states.

Throughout this section, we shall assume that initial conditions are real-valued, which means initial wave packets are static. Thus the spectral parameter $\xi$ is expected to be pure imaginary. Since the zeros of $a(\xi)$ should be located in upper half plane of $\xi$, we shall find the discrete eigenvalues under a condition

$$\xi = i \eta, \quad (\eta > 0).$$

(4.1)

4.1 Single box-type initial condition

We consider a box-type initial condition whose width is $L$:

$$\psi(x, 0) = \begin{cases} V_0 & (0 \leq x \leq L), \\ 0 & \text{(otherwise)}, \end{cases}$$

(4.2)
where \( V_0 \) is a real number. From eqs. (3.7b) and (3.13a), the scattering data \( a(\xi) \) is derived as
\[
a(\xi) = e^{iKL}(\cos KL - i\frac{\xi}{K} \sin KL),
\]
\( K = \sqrt{V_0^2 + \xi^2}. \) (4.3)

Setting \( a(\xi) \) to be zero, we find that the zeros of \( a(\xi) \) can be derived from a set of relations:
\[
\sqrt{V_0^2 - \eta^2} = -\eta \tan(L \sqrt{V_0^2 - \eta^2}), \quad (|V_0| > \eta). \quad (4.4a)
\]
\[
\sqrt{\eta^2 - V_0^2} = -\eta \tanh(L \sqrt{\eta^2 - V_0^2}), \quad (|V_0| < \eta). \quad (4.4b)
\]

Since the solution of eq. (4.4b) does not satisfy the condition \( \eta > 0 \), we shall eliminate it and consider only eq. (4.4a). Introducing \( A \) and \( u \) as
\[
A = V_0L, \quad u = \eta L, \quad (4.5)
\]
we can omit the parameter \( L \). Thus the equation we should consider becomes
\[
\sqrt{A^2 - u^2} = -u \tan(\sqrt{A^2 - u^2}), \quad (4.6a)
\]
\[
0 < u < A. \quad (4.6b)
\]

This means that we should find the intersection of curves \( y = u \tan \sqrt{A^2 - u^2} \) and \( y = \sqrt{A^2 - u^2} \) on the first quadrant.

For sufficiently minute value of \( V_0 \), the value of \( \sqrt{A^2 - u^2} \) stays in the interval \( (0, \pi/2) \) and the right-hand side of (4.6a) is kept negative. In such a case, there exists no solution and no soliton remains at \( t \to \infty \). As the value of \( V_0 \) increases, \( \sqrt{A^2 - u^2} \) can exceed \( \pi/2 \), and solutions of eq. (4.6a) appear. It is clear that the condition where eq. (4.6) has a solution or more is \( A > \pi/2 \) (\( V_0 > \pi/(2L) \)). A typical situations for these cases are shown in Fig. 1 After a brief consideration, we can see that if the potential height \( V_0 \) satisfies
\[
(n - \frac{1}{2})\frac{\pi}{L} < V_0 \leq (n + \frac{1}{2})\frac{\pi}{L}, \quad (n: \text{a positive integer),} \quad (4.7)
\]
the number of solitons which will be generated asymptotically should be \( n \).

### 4.2 Double box-type initial condition

Next, we shall consider an initial condition
\[
\psi(x, 0) = \begin{cases} 
V_0, & (0 < x < L, \ L + w < x < 2L + w), \\
0, & \text{(otherwise),}
\end{cases} \quad (4.8)
\]
which means two identical pulses, each of which has a common amplitude \( V_0 \) and width \( L \), are located with a separation \( w \) (Fig. 2). In this case, by using eq. (4.1), we can find that the scattering data \( a(\xi) \) is
Fig. 1. Typical cases concerning of eq. (4.6). (a) The case of $A = 1.2$. (b) The case of $A = 1.8$. The thick curves are the graphs of $y = \sqrt{A^2 - u^2} - 2u$ in $u > 0$, and the thin curves are the graphs of $y = -u \tan \sqrt{A^2 - u^2}$. The dashed line in (b) denotes the value of $u$ where $\sqrt{A^2 - u^2} = \pi/2$ and the right-hand side of eq. (4.6a) diverges.

Fig. 2. The double box-type initial condition (4.8)

given by

$$a(i\eta) = e^{-2\eta L} \left\{ \left[ \cos K L + \frac{\eta}{K} \sin K L \right]^2 - \frac{V_0^2}{K^2} e^{-2\eta w} \sin^2 K L \right\}. \quad (4.9a)$$

$$K = \sqrt{V_0^2 - \eta^2}. \quad (4.9b)$$

If we introduce $A$ and $u$ as in eqs. (4.5), the zeros of $a(i\eta)$ can be derived from

$$\left[ \cos K + \frac{u}{K} \sin K \right]^2 = \frac{A^2}{K^2} e^{-2\eta w/L} \sin^2 K, \quad (4.10a)$$

$$K = \sqrt{A^2 - u^2}. \quad (4.10b)$$
Fig. 3. The distribution of the zeros of scattering datum $a(i\eta)$ for various values of $V_0$ under the given $w$. (a) The case $w = 0.1$. (b) The case $w = 1.5$. The dashed line expresses the two boundaries of the allowed region for the solutions, given by eq. (4.10c).

The values of $u$ are restricted to

$$0 < u < A,$$  \hspace{1cm} \text{(4.10c)}

because there are no positive $\eta$ satisfying this relation if $\eta > V_0$, following the discussion in deriving eq. (4.6b).

Let us consider two limiting cases. When two initial pulses are sufficiently separated, it is natural to expect the number of solitons which remain in the course of time is twice as many as that of a single initial pulse case, because the amplitude of diffusing radiation is generally so small that the interaction between two pulses hardly affect asymptotically. This observation is confirmed by taking a limit $w \to \infty$ in eq. (4.9a). This operation makes the final term in eq. (4.9a) vanished. In this limit, the function $a$ given by eq. (4.9a) coincides with the square of the scattering datum of eq. (4.3) under the condition (4.1). In the opposite limit, $w \to 0$, the two initial pulses are fused together into a single pulse whose width is $2L$. In fact, eq. (4.9a) coincides with eq. (4.3) if $L$ is replaced by $2L$.

For appropriately chosen value of $w$, the analysis of the eigenvalue problem provides non-trivial solutions where the final term of eq. (4.9a) plays an essential role. We show in Fig. 3 the curves which satisfy eq. (4.10a) on $A$-$u$ plane. We have chosen the values of separation as $w = 0.1L$ in Fig. 3 (a), and $w = 1.5L$ in (b). In Fig. 3 (a), we can see that there is no solution for $A \leq 0.8$. This means that an initial wave packet with too small amplitude all transforms into diffusing waves known as radiation. As the quantity $A(\sim V_0)$ increases, a solution of eq. (4.10a) appears. This can be realized for $\frac{\pi}{4} \leq A \leq \frac{3\pi}{4}$. This solution gives one asymptotically remaining soliton. The soliton is expected to be located at the center of the two initial pulses, because there is space reflection symmetry. For larger values of $A$, the
number of remaining solitons increases monotonically.

In Fig. 3 (b), we can see that there is a qualitatively different result which cannot be observed in the previous case. In this case, the quantity \( \eta \) is not always given as a single-valued function of \( A \) on every branch. After having one solution for \( \frac{\pi}{4} \leq A \leq 2.2 \), eq. (4.10a) is observed to have two roots around \( A \sim 2.2 \). The smaller solution of \( u \) for this value of \( A \) gives a double-pole solution. We have presented a graph of \( a(i\eta) \) in Fig. 4 for the value of \( A \approx 2.2 \), the smallest value of \( A \) where the tangent of the curves shown in Fig. 3 (b) is parallel to the \( u \)-axis. Let us denote this value of \( A \) as \( A_0 \). When \( A = A_0 \), the number of remaining soliton is two. As soon as \( A \) exceeds this value, the number of solitons becomes three, however small the excess is. As the value of \( A \) becomes larger, the number of remaining solitons decrease to be two for \( A \approx \frac{3\pi}{4} \). Thus, the number of solitons which appears in the course of time is not a simple monotonic function of the amplitude of the initial pulse for moderate value of \( w \).

5. Numerical Simulation

In this section, we shall show results of numerical simulation on the initial value problem described in the previous subsection. By numerically integrating the NLS equation (2.1), we solve the initial value problem under the double box-type initial condition eq. (4.8) horizontally shifted so that the center of the valley coincides the origin \( x = 0 \). We set the width of the valley \( w \) to be 1.5\( L \) and vary the common potential height \( V_0 \) for each time.

First, we examine the case where \( V_0 = 1.5/L \). In this case, we have only one solution so that we expect only one soliton in the final state. Figure 5 is the absolute square of the wave amplitude \( |\psi(x, t)|^2 \) at \( t = 100 \). Though we can observe three pulses, the height of the two smaller peaks across the center peak keeps falling to fade away. It seems to remain only one soliton at the center at the limit \( t \to \infty \).

Secondly, we raise the potential height to be \( V_0 = 2.3/L \), which is slightly larger value than the critical value for a double pole soliton but smaller than the upper threshold \( V_0 = \frac{3\pi}{4} \). Therefore, we
expect three remaining solitons in the far distant future. Figure 6 is the absolute square of the wave amplitude $|\psi(x, t)|^2$ at $t = 65$. We can observe three sharp pulses. In this case, the two smaller peaks at both sides never diffuse with time.

Thirdly, we set the potential height to be $V_0 = 2.5/L$, which exceeds the boundary value of $V_0 = \frac{3\pi}{4}$, and two solitons are predicted to survive. Figure 7 is the absolute square of the wave amplitude $|\psi(x, t)|^2$ at $t = 50$. We can admit two large peaks around the origin as expected. They keep alternately splitting and fusing together, like a breather.

Our numerical simulation could not catch the features of the double pole soliton probably because the condition for generating the double pole soliton has a zero measure. Other results of the numerical simulation, however, show good agreement with theoretical predictions and strongly support the validity of the presented approximation method.
6. Discussions and Concluding Remarks

In this paper, we have addressed an approximation method to solve the Zakharov-Shabat eigenvalue problem. As we have seen in the previous sections, the presented method is effective in solving equations under non-soliton initial conditions. For illustration, we have considered the initial value problem of the NLS equation of self-focusing type under box-type initial conditions. We found the interplay between the decaying tails from the initial pulses can affect the asymptotic behaviors, and succeeded in making qualitative predictions including the number of remaining solitons and conditions under which the initial wave becomes double-pole solitons.

The method presented in this paper can be considered to be especially useful in solving the initial value problem of a soliton equation. By taking sufficiently small intervals which split the initial wave packet, we can solve the Zakharov-Shabat eigenvalue problem in arbitrary precision. This fact is significant in the viewpoint of application. Generally, it is not easy to generate exact solitons as an initial condition and one of the most feasible initial conditions is the Gaussian-type wave packet in plasma physics or the BEC system. The proposed method can extract information such as the number of solitons which appears asymptotically in real systems.

At the end of this article, we shall refer to the possibilities of extensions of the proposed method. Though the examples are given for the NLS equation cases, this method can be applied to various soliton equations which belong to the AKNS system. In addition, we can expect more extensions to the integrable equations which belong to other systems, such as the Kaup-Newell\(^9\) or the Wadati-Konno-Ichikawa\(^10\) systems. These extensions should be considered as future problems.

Acknowledgment

The authors express their sincere gratitude to Professor Ralph Willox of the University of Tokyo and Professor Ken-ichi Maruno of Waseda University for their interest to this work and stimulated
discussions. One of the authors H. F. thanks to the hospitality of Utsunomiya University for offering opportunities of fruitful discussions.
References

1) E. Infeld and G. Rowlands: *Nonlinear Waves, Solitons and Chaos* (Cambridge Univ. Press, New York, 1990).
2) L.P. Pitaevskii and S. Stringari: *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).
3) M. J. Ablowitz and H. Segur: *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
4) V. E. Zakharov and A. B. Shabat: Sov. Phys. JETP 34 (1972) 62.
5) I. M. Gel’fand and B. M. Levitan: Amer. Math. Soc. Trans. 1 (1955) 253.
6) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: Phys. Rev. Lett. 31 (1973) 125.
7) E. Olmedilla: Physica D 25 (1987) 330.
8) J. Satsuma and N. Yajima: Prog. Theor. Phys. Suppl. 55 (1974) 284.
9) D. J. Kaup and A.C. Newell: J. Math. Phys. 19 (1978) 798.
10) M Wadati, K. Konno and Y. H. Ichikawa: J. Phys. Soc. Jpn. 46 (1979) 1698.