ON THE SCALAR CURVATURE OF CONSTANT MEAN CURVATURE HYPERSURFACES IN SPACE FORMS

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Abstract. In this paper we study the behavior of the scalar curvature $S$ of a complete hypersurface immersed with constant mean curvature into a Riemannian space form of constant curvature, deriving a sharp estimate for the infimum of $S$. Our results will be an application of a weak Omori-Yau maximum principle due to Pigola, Rigoli and Setti [17].

1. Introduction

In a classical paper, Klotz and Osserman [10] characterized totally umbilical spheres and circular cylinders as the only complete surfaces immersed into the Euclidean 3-space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and whose Gaussian curvature does not change sign. Later on, Hoffman [8] and Tribuzy [19] gave an extension of that result to the case of surfaces with constant mean curvature in the Euclidean 3-sphere $\mathbb{S}^3$ and in the hyperbolic space $\mathbb{H}^3$, respectively. Specifically, putting together the results of those authors in a single statement, one gets the following result (see also [5, Proposition 3.3]).

**Theorem 1.** Let $\Sigma$ be a complete surface immersed into a 3-dimensional space form $\mathbb{M}^3_c$ ($c = 0, 1, -1$) with constant mean curvature $H$. If its Gaussian curvature $K$ does not change sign, then $\Sigma$ is either a totally umbilical surface or $K = 0$ and

(a) $c = 0$ and $\Sigma$ is a circular cylinder $\mathbb{R} \times \mathbb{S}^1(r) \subset \mathbb{R}^3$, with $r > 0$,

(b) $c = 1$ and $\Sigma$ is a flat torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r) \subset \mathbb{S}^3$, with $0 < r < 1$,
(c) \( c = -1 \) and \( \Sigma \) is a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1+r^2}) \times S^1(r) \subset \mathbb{H}^3 \), with \( r > 0 \).

As a nice application of Theorem 1, one gets the following consequence for the infimum of the Gaussian curvature of \( \Sigma \).

Theorem 2. Let \( \Sigma \) be a complete surface immersed into a 3-dimensional space form \( \mathbb{M}_c^3 \) (\( c = 0, 1, -1 \)) with constant mean curvature \( H \) such that \( H^2 + c > 0 \), and let \( K \) stand for its Gaussian curvature. Then

(i) either \( \inf \Sigma K = H^2 + c \), and \( \Sigma \) is a totally umbilical surface,

(ii) or \( \inf \Sigma K \leq 0 \), with equality if and only if

(a) \( c = 0 \) and \( \Sigma \) is a circular cylinder \( \mathbb{R} \times S^1(r) \subset \mathbb{R}^3 \), with \( r > 0 \),

(b) \( c = 1 \) and \( \Sigma \) is a flat torus \( S^1(\sqrt{1-r^2}) \times S^1(r) \subset \mathbb{S}^3 \), with \( 0 < r < 1 \),

(c) \( c = -1 \) and \( \Sigma \) is a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1+r^2}) \times S^1(r) \subset \mathbb{H}^3 \), with \( r > 0 \).

Actually, it follows from the Gauss equation of the surface that \( K \leq H^2 + c \) on \( \Sigma \), with equality at the umbilical points of \( \Sigma \). Therefore, \( \inf \Sigma K \leq H^2 + c \) with equality if and only if \( \Sigma \) is totally umbilical. This proves part (i). Moreover, if \( \inf \Sigma K < H^2 + c \) then it must be \( \inf \Sigma K \leq 0 \) necessarily. Otherwise, one would have \( K \geq \inf \Sigma K > 0 \) which is not possible by Theorem 1 since the non-totally umbilical surfaces in (a), (b) and (c) are all flat. This shows that \( \inf \Sigma K \leq 0 \). Finally, if equality holds, \( \inf \Sigma K = 0 \), then \( K \geq 0 \) and the result follows from Theorem 1.

Rotational surfaces show that the estimate in Theorem 2 is sharp. For instance, let us consider the Delaunay rotational surfaces in the Euclidean space. For a given constant \( H \neq 0 \), we may consider the family of unduloids in \( \mathbb{R}^3 \) with constant mean curvature \( H \), which are given by the following parametrization

\[
(s, \theta) \mapsto (x_B(s), y_B(s) \cos \theta, y_B(s) \sin \theta)
\]

where \( 0 < B < 1 \) and

\[
x_B(s) = \int_0^s \frac{1 + B \sin(2Ht)}{\sqrt{1 + B^2 + 2B \sin(2Ht)}}\,dt
\]

\[
y_B(s) = \frac{\sqrt{1 + B^2 + 2B \sin(2Hs)}}{2|H|}.
\]

The first fundamental form of these surfaces is \( ds^2 + y_B(s)^2 d\theta^2 \) and the Gaussian curvature is then

\[
K_B(s, \theta) = K_B(s) = \frac{y_B''(s)}{y_B(s)} = \frac{4H^2B(B + \sin(2Hs))(1 + B \sin(2Hs))}{(1 + B^2 + 2B \sin(2Hs))^2}.
\]

Therefore, for these examples we have \( \inf K_B = -4H^2B/(1 - B)^2 < 0 \), and for a given \( \varepsilon > 0 \) there exists \( 0 < B < 1 \) such that \( \inf K_B = -\varepsilon < 0 \).
It is worth pointing out that the proof of Theorem 1 (and hence Theorem 2) strongly depends on the conformal structure of the two-dimensional surface \( \Sigma \), and cannot be extended to higher dimensions. Our objective in this paper is, using an alternative approach, to extend Theorem 2 to the case of \( n \)-dimensional hypersurfaces, with \( n \geq 3 \) (see Theorem 3 and Corollary 4 below). As a consequence of Theorem 3, in Corollary 6 we give a generalization of Theorem 1.5 in [1] to the case of complete parabolic hypersurfaces in space forms.

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**2. Preliminaries**

Let us denote by \( \mathbb{M}_c^{n+1} \) the standard model of an \((n+1)\)-dimensional Riemannian space form with constant curvature \( c \), \( c = 0,1,-1 \). That is, \( \mathbb{M}_c^{n+1} \) denotes the Euclidean space \( \mathbb{R}^{n+1} \) when \( c = 0 \), the Euclidean sphere
\[
\mathbb{S}^{n+1} = \{ x \in \mathbb{R}^{n+2} : \|x\|^2 = 1 \} \subset \mathbb{R}^{n+2},
\]
when \( c = 1 \), and the hyperbolic space \( \mathbb{H}^{n+1} \) when \( c = -1 \). In this last case, it will be appropriate for us to use the Minkowskian model of the hyperbolic space. Write \( \mathbb{R}_1^{n+2} \) for \( \mathbb{R}^{n+2} \), with canonical coordinates \((x_0, x_1, \ldots, x_{n+1})\), endowed with the Lorentzian metric
\[
\langle \cdot, \cdot \rangle_1 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.
\]
The \((n+1)\)-dimensional hyperbolic space \( \mathbb{H}^{n+1} \) is the complete simply connected Riemannian manifold with sectional curvature \(-1\), which is realized as the hyperboloid
\[
\mathbb{H}^{n+1} = \{ x \in \mathbb{R}_1^{n+2} : \langle x, x \rangle_1 = -1, x_0 > 0 \} \subset \mathbb{R}_1^{n+2}
\]
endowed with the Riemannian metric induced from \( \mathbb{R}_1^{n+2} \). In order to simplify our notation, when \( c = \pm 1 \) we agree to denote by \( \langle \cdot, \cdot \rangle \), without distinction, both the Euclidean metric on \( \mathbb{R}^{n+2} \) and the Lorentzian metric (1) on \( \mathbb{R}_1^{n+2} \). We also agree to denote by \( \langle \cdot, \cdot \rangle \) the corresponding Riemannian metric induced on \( \mathbb{M}_c^{n+1} \hookrightarrow \mathbb{R}^{n+2} \).

Let us consider \( \psi : \Sigma^n \hookrightarrow \mathbb{M}_c^{n+1} \) an isometric immersion of an \( n \)-dimensional orientable Riemannian manifold \( \Sigma \), and denote by \( A \) its second fundamental form (with respect to a globally defined normal unit vector field \( N \)) and by \( H \) its mean curvature, \( H = (1/n)\text{tr}(A) \). In the general \( n \)-dimensional case, instead of the curvature, it will be more appropriate to deal with the so called traceless second fundamental form of the hypersurface, which is given by \( \Phi = A - HI \), where \( I \) denotes the identity operator on \( \mathcal{X}(\Sigma) \). Observe that \( \text{tr}(\Phi) = 0 \) and \( |\Phi|^2 = \text{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0 \),
with equality if and only if $\Sigma$ is totally umbilical. For that reason, $\Phi$ is also called the total umbilicity tensor of $\Sigma$.

As is well known, the curvature tensor $R$ of the hypersurface is given by the Gauss equation, which can be written in terms of $\Phi$ as

$$R(X,Y)Z = (c + H^2)(\langle X, Z \rangle Y - \langle Y, Z \rangle X) + \langle \Phi X, Z \rangle \Phi Y - \langle \Phi Y, Z \rangle \Phi X + H(\langle \Phi X, Z \rangle Y - \langle Y, Z \rangle \Phi X + \langle X, Z \rangle \Phi Y - \langle \Phi Y, Z \rangle X)$$

for $X, Y, Z \in \mathcal{X}(\Sigma)$. In particular, the Ricci and the scalar curvatures of $\Sigma$ are given, respectively, by

$$\text{Ric}(X,Y) = (n-1)(c + H^2)\langle X, Y \rangle + (n-2)H\langle \Phi X, Y \rangle - \langle \Phi X, \Phi Y \rangle,$$

and

$$S = n(n-1)(c + H^2) - |\Phi|^2.$$

### 2.1. Stochastic completeness and the Omori-Yau maximum principle.

For the proof of our results in higher dimension, we will make use of a weaker version of the Omori-Yau maximum principle. Following the terminology introduced by Pigola, Rigoli and Setti in [17], the Omori-Yau maximum principle is said to hold on an $n$-dimensional Riemannian manifold $\Sigma^n$ if, for any smooth function $u \in C^2(\Sigma)$ with $u^* = \sup_{\Sigma} u < +\infty$ there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}}$ in $\Sigma$ with the properties

$$\begin{align*}
(i) & \quad u(p_k) > u^* - \frac{1}{k}, \\
(ii) & \quad |\nabla u(p_k)| < \frac{1}{k}, \quad \text{and} \quad (iii) \quad \Delta u(p_k) < \frac{1}{k}.
\end{align*}$$

In this sense, the classical result given by Omori and Yau in [15, 20] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [17, Example 1.13], a sufficiently controlled decay of the radial Ricci curvature of the form

$$\text{Ric}\Sigma(\nabla \varrho, \nabla \varrho) \geq -C^2 G(\varrho)$$

where $\varrho$ is the distance function on $\Sigma$ to a fixed point, $C$ is a positive constant, and $G : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying

$$\begin{align*}
(i) & \quad G(0) > 0, \quad (ii) \quad G'(t) \geq 0, \quad (iii) \quad \int_0^{+\infty} 1/\sqrt{G(t)} = +\infty \quad \text{and} \quad (iv) \quad \limsup_{t \to +\infty} tG(\sqrt{t})/G(t) < +\infty,
\end{align*}$$

suffices to imply the validity of the Omori-Yau maximum principle. In particular, and following the terminology introduced by Bessa and Costa in [3], the Omori-Yau
maximum principle holds on a complete Riemannian manifold whose Ricci curvature has *strong quadratic decay* [6], that is, with

\[ \text{Ric}_\Sigma \geq -C^2(1 + \rho^2 \log(\rho + 2)). \]

On the other hand, as observed also by Pigola, Rigoli and Setti in [17], the validity of Omori-Yau maximum principle on \( \Sigma^\ast \) does not depend on curvature bounds as much as one would expect. For instance, the Omori-Yau maximum principle holds on every Riemannian manifold which is properly immersed into a Riemannian space form with controlled mean curvature (see [17, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, and following again the terminology introduced in [17], the *weak* Omori-Yau maximum principle is said to hold on a (not necessarily complete) \( n \)-dimensional Riemannian manifold \( \Sigma \) if, for any smooth function \( u \in C^2(\Sigma) \) with \( u^\ast = \sup_{\Sigma} u < +\infty \) there exists a sequence of points \( \{p_k\}_{k \in \mathbb{N}} \) in \( \Sigma \) with the properties

\[ u(p_k) > u^\ast - \frac{1}{k}, \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}. \]

As proved by Pigola, Rigoli and Setti [16], the fact that the weak Omori-Yau maximum principle holds on \( \Sigma \) is equivalent to the stochastic completeness of the manifold (see also [17, Theorem 3.1]). In particular, the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold (see also [7, Corollary 6.4]).

### 3. Statement of the main results

Now we are ready to state the following extension of Theorem 2 to the case of \( n \)-dimensional hypersurfaces, with \( n \geq 3 \).

**Theorem 3.** Let \( \Sigma \) be a stochastically complete hypersurface immersed into an \( (n + 1) \)-dimensional space form \( M^{n+1}_c \) (\( c = 0, 1, -1 \) and \( n \geq 3 \)) with constant mean curvature \( H \) such that \( H^2 + c > 0 \), and let \( S \) stand for its scalar curvature. Then

(i) either \( \inf_{\Sigma} S = n(n - 1)(c + H^2) \) and \( \Sigma \) is a totally umbilical hypersurface,

(ii) or

\[ \inf_{\Sigma} S \leq \frac{n(n - 2)}{2(n - 1)} \left( 2(n - 1)c + nH^2 + |H|\sqrt{n^2H^2 + 4(n - 1)c} \right). \]

Moreover, the equality holds and this infimum is attained at some point of \( \Sigma \) if and only if

(a) \( c = 0 \) and \( \Sigma \) is an open piece of a circular cylinder \( \mathbb{R} \times S^{n-1}(r) \subset \mathbb{R}^{n+1} \), with \( r > 0 \),

(b) \( c = 1 \) and \( \Sigma \) is an open piece of either a minimal Clifford torus \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \subset S^{n+1} \), with \( k = 1, \ldots, n - 1 \), or a constant mean curvature torus \( S^1(\sqrt{n-r^2}) \times S^{n-1}(r) \subset S^{n+1} \), with \( 0 < r < \sqrt{(n-1)/n} \),

(c) \( c = -1 \) and \( \Sigma \) is an open piece of a minimal sphere \( S^{n-1}(r) \subset S^n \), with \( r > 0 \),

(d) \( c = -2 \) and \( \Sigma \) is an open piece of a minimal sphere \( S^{n-1}(r) \subset S^n \), with \( 0 < r < \sqrt{(n-1)/n} \),

(e) \( c = -3 \) and \( \Sigma \) is an open piece of a minimal sphere \( S^{n-1}(r) \subset S^n \), with \( 0 < r < \sqrt{(n-1)/n} \),

(f) \( c = -4 \) and \( \Sigma \) is an open piece of a minimal sphere \( S^{n-1}(r) \subset S^n \), with \( 0 < r < \sqrt{(n-1)/n} \).
(c) \( c = -1 \) and \( \Sigma \) is an open piece of a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r) \subset \mathbb{H}^{n+1} \), with \( r > 0 \).

In the particular case where \( \Sigma^n \) is complete (which happens, for instance, when \( \Sigma^n \) is properly immersed), we obtain the following consequence.

**Corollary 4.** Let \( \Sigma^n \) be a complete hypersurface immersed into an \((n+1)\)-dimensional space form \( \mathbb{M}^{n+1}_c \) \((c = 0, 1, -1 \text{ and } n \geq 3)\) with constant mean curvature \( H \) such that \( H^2 + c > 0 \). Then

(i) either \( \inf_{\Sigma} S = n(n-1)(c + H^2) \) and \( \Sigma \) is a totally umbilical hypersurface, 

(ii) or

\[
\inf_{\Sigma} S \leq \frac{n(n - 2)}{2(n - 1)} \left( 2(n - 1)c + nH^2 + |H|\sqrt{n^2H^2 + 4(n - 1)c} \right).
\]

Moreover, the equality holds and this infimum is attained at some point of \( \Sigma \) if and only if

(a) \( c = 0 \) and \( \Sigma \) is a circular cylinder \( \mathbb{R} \times S^{n-1}(r) \subset \mathbb{R}^{n+1} \), with \( r > 0 \),

(b) \( c = 1 \) and \( \Sigma \) is either a minimal Clifford torus \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n - k)/n}) \subset S^{n+1} \), with \( k = 1, \ldots, n-1 \), or a constant mean curvature torus \( S^1(\sqrt{1 - r^2}) \times S^{n-1}(r) \subset S^{n+1} \), with \( 0 < r < \sqrt{(n - 1)/n} \),

(c) \( c = -1 \) and \( \Sigma \) is a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r) \subset \mathbb{H}^{n+1} \), with \( r > 0 \).

On the other hand, it follows from (4) that \( \inf_{\Sigma} S = n(n-1)(c + H^2) - \sup_{\Sigma} |\Phi|^2 \). Therefore, Theorem 3 (as well as Corollary 4) can be re-written equivalently in terms of the total umbilicity tensor as follows.

**Theorem 5.** Let \( \Sigma^n \) be a stochastically complete hypersurface immersed into an \((n+1)\)-dimensional space form \( \mathbb{M}^{n+1}_c \) \((c = 0, 1, -1 \text{ and } n \geq 3)\) with constant mean curvature \( H \) such that \( H^2 + c > 0 \), and let \( \Phi \) stand for its total umbilicity tensor. Then

(i) either \( \sup_{\Sigma} |\Phi| = 0 \) and \( \Sigma \) is a totally umbilical hypersurface,

(ii) or

\[
\sup_{\Sigma} |\Phi| \geq \alpha_H = \frac{\sqrt{n}}{2\sqrt{n - 1}} \left( \sqrt{n^2H^2 + 4(n - 1)c} - (n - 2)|H| \right) > 0.
\]

Moreover, the equality holds and this supremum is attained at some point of \( \Sigma \) if and only if

(a) \( c = 0 \) and \( \Sigma \) is an open piece of a circular cylinder \( \mathbb{R} \times S^{n-1}(r) \subset \mathbb{R}^{n+1} \), with \( r > 0 \),

(b) \( c = 1 \) and \( \Sigma \) is an open piece of either a minimal Clifford torus \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n - k)/n}) \subset S^{n+1} \), with \( k = 1, \ldots, n-1 \), or a constant mean curvature torus \( S^1(\sqrt{1 - r^2}) \times S^{n-1}(r) \subset S^{n+1} \), with \( 0 < r < \sqrt{(n - 1)/n} \),
(c) \( c = -1 \) and \( \Sigma \) is an open piece of a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r) \subset \mathbb{H}^{n+1}, \) with \( r > 0. \)

In particular, we get the following consequence, which gives a generalization of Theorem 1.5 in [1] to complete parabolic hypersurfaces in space forms.

**Corollary 6.** Let \( \Sigma^n \) be a complete parabolic hypersurface immersed into an \( (n+1) \)-dimensional space form \( \mathbb{M}^{n+1}_c \) (\( c = 0, 1, -1 \) and \( n \geq 3 \)) with constant mean curvature \( H \) such that \( H^2 + c > 0, \) and let \( \Phi \) stand for its total umbilicity tensor. Then

(i) either \( \sup_{\Sigma} |\Phi| = 0 \) and \( \Sigma \) is a totally umbilical hypersurface,

(ii) or

\[
\sup_{\Sigma} |\Phi| \geq \alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H| \right) > 0
\]

with equality if and only if

(a) \( c = 0 \) and \( \Sigma \) is a circular cylinder \( \mathbb{R} \times S^{n-1}(r), \) with \( r > 0, \)

(b) \( c = 1 \) and \( \Sigma \) is either a minimal Clifford torus \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}), \)

with \( k = 1, \ldots, n-1, \) or a constant mean curvature torus \( S^1(\sqrt{1-r^2}) \times S^{n-1}(r), \) with \( 0 < r < \sqrt{(n-1)/n}, \)

(c) \( c = -1 \) and \( \Sigma \) is a hyperbolic cylinder \( \mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r), \) with \( r > 0. \)

4. **Proof of the main results**

The proof of our results is based on a Simons type formula for the Laplacian of the function \( |\Phi|^2, \) which has already been used by several authors. For the sake of completeness, we include here its derivation, following Nomizu and Smyth [13]. A standard tensor computation implies that

\[
\frac{1}{2} \Delta |\Phi|^2 = \frac{1}{2} \Delta \langle \Phi, \Phi \rangle = |\nabla \Phi|^2 + \langle \Phi, \Delta \Phi \rangle.
\]

Here \( \nabla : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma) \) denotes the covariant differential of \( \Phi, \)

\[
\nabla \Phi(X,Y) = (\nabla_Y \Phi)X = \nabla_Y (\Phi X) = \Phi(\nabla_Y X), \quad X, Y \in \mathcal{X}(\Sigma),
\]

and \( \Delta \Phi : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma) \) is the rough Laplacian,

\[
\Delta \Phi(X) = \text{tr}(\nabla^2 \Phi(X, \cdot, \cdot)) = \sum_{i=1}^{n} \nabla^2 \Phi(X, E_i, E_i),
\]

where \( \{E_1, \ldots, E_n\} \) is a local orthonormal frame on \( \Sigma. \) Observe that, in our notation, \( \nabla^2 \Phi(X, Y, Z) = (\nabla_Z \nabla \Phi)(X, Y). \) Let us assume that the mean curvature \( H \) is constant. In that case, \( \nabla \Phi = \nabla A, \) which is symmetric by the Codazzi equation of the hypersurface and, hence, \( \nabla^2 \Phi \) is also symmetric in its two first variables,

\[
\nabla^2 \Phi(X, Y, Z) = \nabla^2 \Phi(Y, X, Z), \quad X, Y, Z \in \mathcal{X}(\Sigma).
\]
With respect to the symmetries of $\nabla^2 \Phi$ in the other variables, it is not difficult to see that

$$\nabla^2 \Phi(X, Y, Z) = \nabla^2 \Phi(X, Z, Y) - R(Z, Y) \Phi X + \Phi(R(Z, Y)X).$$

Thus, using the Gauss equation (2) it follows from here that

$$\Delta \Phi(X) = \sum_{i=1}^{n} (\nabla^2 \Phi(E_i, E_i, X) - R(E_i, X) \Phi E_i + \Phi(R(E_i, X)E_i))$$

$$= \operatorname{tr}(\nabla_X(\nabla \Phi)) - H|\Phi|^2 X + (n(c + H^2) - |\Phi|^2) \Phi X + nH \Phi^2 X,$$

where we have used the facts that trace commutes with $\nabla_X$ and that $\operatorname{tr}(\nabla \Phi) = 0$. Therefore, by (8) we conclude that

$$\frac{1}{2} \Delta |\Phi|^2 = |\nabla \Phi|^2 + nH \operatorname{tr}(\Phi^3) - |\Phi|^2(|\Phi|^2 - n(c + H^2)).$$

We will also need the following auxiliary result, known as Okumura lemma, which can be found in [14] and [1, Lemma 2.6].

**Lemma 7.** Let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^{n} a_i = 0$. Then

$$- \frac{n-2}{\sqrt{n(n-1)}} \left( \sum_{i=1}^{n} a_i^2 \right)^{3/2} \leq \sum_{i=1}^{n} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}.$$

Moreover, equality holds in the right-hand (respectively, left-hand) side if and only if $(n-1)$ of the $a_i$’s are nonpositive (respectively, nonnegative) and equal.

4.1. **Proof of Theorem 5 (or, equivalently, Theorem 3).** Since $\operatorname{tr}(\Phi) = 0$, we may use Lemma 7 to estimate $\operatorname{tr}(\Phi^3)$ as follows

$$|\operatorname{tr}(\Phi^3)| \leq \frac{n-2}{\sqrt{n(n-1)}} |\Phi|^3,$$

and then

$$nH \operatorname{tr}(\Phi^3) \geq -n|H||\Phi|^3 \geq -\frac{n(n-2)}{\sqrt{n(n-1)}} |H||\Phi|^3.$$

Using this in (11), we find

$$\frac{1}{2} \Delta |\Phi|^2 \geq |\nabla \Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\Phi|^3 - |\Phi|^2(|\Phi|^2 - n(c + H^2))$$

$$\geq -|\Phi|^2 P_H(|\Phi|),$$

where

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} |H|x - n(c + H^2).$$
Observe that, since $H^2 + c > 0$, the polynomial $P_H(x)$ has a unique positive root given by
\[
\alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H| \right).
\]

If $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then by applying (11) to the function $|\Phi|^2$ we know that there exists $\{p_k\}_{k \in \mathbb{N}}$ in $\Sigma$ such that
\[
\lim_{k \to \infty} |\Phi|(p_k) = \sup_\Sigma |\Phi|, \quad \text{and} \quad \Delta |\Phi|^2(p_k) < 1/k,
\]
which jointly with (11) implies
\[
1/k > \Delta |\Phi|^2(p_k) \geq -2|\Phi|^2(p_k)P_H(|\Phi|(p_k)).
\]
Taking limits here, we get $0 \geq -2(\sup_\Sigma |\Phi|)^2P_H(\sup_\Sigma |\Phi|)$, that is
\[
(\sup_\Sigma |\Phi|^2)^2P_H(\sup_\Sigma |\Phi|) \geq 0.
\]

It follows from here that either $\sup_\Sigma |\Phi| = 0$, which means that $|\Phi| = \text{constant} = 0$ and the hypersurface is totally umbilical, or $\sup_\Sigma |\Phi| > 0$ and then $P_H(\sup_\Sigma |\Phi|) \geq 0$. In the latter, it must be $\sup_\Sigma |\Phi| \geq \alpha_H$, which gives the inequality in (ii). Moreover, assume that equality holds, $\sup_\Sigma |\Phi| = \alpha_H$. In that case, $P_H(|\Phi|) \leq 0$ on $\Sigma$, which jointly with (11) implies that $|\Phi|^2$ is a subharmonic function on $\Sigma$. Therefore, if there exists a point $p_0 \in \Sigma$ at which this supremum is attained, then $|\Phi|^2$ is a subharmonic function on $\Sigma$ which attains its supremum at some point of $\Sigma$ and, by the maximum principle, it must be constant, $|\Phi| = \text{constant} = \alpha_H$. Thus, (11) becomes trivially an equality,
\[
\frac{1}{2}\Delta |\Phi|^2 = 0 = -|\Phi|^2P_H(|\Phi|).
\]

From here we obtain that $\nabla \Phi = \nabla A = 0$, that is, the second fundamental form of the hypersurface is parallel. If $H = 0$ (which can occur only when $c = 1$) then by a classical local rigidity result by Lawson \cite{Lawson} Proposition 1] we know that $\Sigma^n$ is an open piece of a minimal Clifford torus of the form $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \subset S^{n+1}$, with $k = 1, \ldots, n-1$, which trivially satisfies $|\Phi| = \text{constant} = \alpha_0 = \sqrt{n}$. If $H \neq 0$ then from the equality in (11) we also obtain the equality in Okumura lemma (Lemma\cite{Okumura}), which implies that the hypersurface has exactly two constant principal curvatures, with multiplicities $(n-1)$ and 1. Then, by the classical results on isoparametric hypersurfaces of Riemannian space forms \cite{Ono, Ono1, Ono2, Ono3} we conclude that $\Sigma$ must be an open piece of one of the three following standard product embeddings:

(a) $\mathbb{R}^{n-1} \times S^1(r) \subset \mathbb{R}^{n+1}$ or $\mathbb{R} \times S^{n-1}(r) \subset \mathbb{R}^{n+1}$ with $r > 0$, if $c = 0$;
(b) $S^1(\sqrt{1-r^2}) \times S^{n-1}(r) \subset S^{n+1}$, with $0 < r < 1$, if $c = 1$; and
(c) $\mathbb{H}^{n-1}(-\sqrt{1 + r^2}) \times S^1(r) \subset \mathbb{H}^{n+1}$, with $0 < r < 1/\sqrt{n(n-2)}$ (recall that $H^2 > -c = 1$), or $\mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r) \subset \mathbb{H}^{n+1}$, with $r > 0$, if $c = -1$. Obviously, in all the examples above $|\Phi| = \text{constant} = \sup_\Sigma |\Phi|$. A detailed analysis of the value of the constant $|\Phi|$ for these examples shows that when $c = 0$ $|\Phi| = \sqrt{n(n-1)}|H| > \alpha_H$ for the standard products $\mathbb{R}^{n-1} \times S^1(r)$, whereas $|\Phi| = \sqrt{n}|H|/\sqrt{n-1} = \alpha_H$ for the standard products $\mathbb{R} \times S^{n-1}(r)$, with $r > 0$. On the other hand, when $c = 1$ we can see that

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 + 4(n-1)} + (n-2)|H| \right) > \alpha_H$$

for the standard products $S^1(-\sqrt{1 - r^2}) \times S^{n-1}(r)$ if $r > \sqrt{(n-1)/n}$, whereas

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 + 4(n-1)} - (n-2)|H| \right) = \alpha_H$$

if $0 < r < \sqrt{(n-1)/n}$. Finally, when $c = -1$ we have that

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 - 4(n-1)} + (n-2)|H| \right) > \alpha_H$$

for the standard products $\mathbb{H}^{n-1}(-\sqrt{1 + r^2}) \times S^1(r)$, with $0 < r < 1/\sqrt{n(n-2)}$, whereas

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 - 4(n-1)} - (n-2)|H| \right) = \alpha_H$$

for the standard products $\mathbb{H}^1(-\sqrt{1 + r^2}) \times S^{n-1}(r)$, with $r > 0$. For the details, see Appendix. This finishes the proof of Theorem 5.

4.2. **Proof of Corollary 4** As in the previous proof, instead of proving Corollary 4 we will prove its equivalent statement in terms of the total umbilicity tensor. That is, we will show that, under the assumptions of Corollary 4, it holds that

(i) either $\sup_\Sigma |\Phi| = 0$ and $\Sigma$ is a totally umbilical hypersurface,

(ii) or

$$\sup_\Sigma |\Phi| \geq \alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( \sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H| \right) > 0,$$

and the equality holds and this supremum is attained at some point of $\Sigma$ if and only if one has (a), (b) or (c).

Obviously, if $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then we can estimate

$$H\langle \Phi X, X \rangle \geq -|H||\langle \Phi X, X \rangle| \geq -|H||\Phi||X|^2 \geq -|H|\sup_\Sigma |\Phi||X|^2,$$
and 
\[ \langle \Phi X, \Phi X \rangle \leq |\Phi|^2 |X|^2 \leq (\sup_{\Sigma} |\Phi|)^2 |X|^2, \]
for \( X \in \mathcal{X}(\Sigma) \). Then, by (3) we obtain for every \( X \in \mathcal{X}(\Sigma) \),
\[
\text{Ric}(X, X) = (n - 1)(c + H^2)|X|^2 + (n - 2)H \langle \Phi X, X \rangle - \langle \Phi X, \Phi X \rangle \\
\geq \left( (n - 1)(c + H^2) - (n - 2)|H| \sup_{\Sigma} |\Phi| - (\sup_{\Sigma} |\Phi|)^2 \right) |X|^2.
\]
Therefore, if \( \sup_{\Sigma} |\Phi| < +\infty \) then the Ricci curvature of \( \Sigma \) is bounded from below by the constant
\[
C = (n - 1)(c + H^2) - (n - 2)|H| \sup_{\Sigma} |\Phi| - (\sup_{\Sigma} |\Phi|)^2.
\]
Since \( \Sigma \) is complete, the classical Omori-Yau maximum principle holds on \( \Sigma \) and the result follows directly from Theorem 5 (or, equivalently, Theorem 3).

**Remark 8.** The proof of Corollary 4 has been inspired by the estimates of the Ricci curvature for submanifolds into a Riemannian space given by Asperti and Costa in [2]. We refer the reader to that paper for other general Ricci estimates.

4.3. **Proof of Corollary 6.** For the proof of Corollary 6, first recall that the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold. Then, by the first part of Theorem 5 we obtain that either \( \sup_{\Sigma} |\Phi| = 0 \) and \( \Sigma \) is a totally umbilical hypersurface, or \( \sup_{\Sigma} |\Phi| \geq \alpha_H \). Moreover, if equality holds, \( \sup_{\Sigma} |\Phi| = \alpha_H \), then as in the proof above we have \( P_H(|\Phi|) \leq 0 \) and \( |\Phi|^2 \) is a subharmonic function on \( \Sigma \) which is bounded from above. Since \( \Sigma \) is parabolic, it must be constant, \( |\Phi| = \text{constant} = \alpha_H \). The proof then finishes as in Theorem 5 by observing that the standard Riemannian products \( \mathbb{R} \times S^{n-1}(r) \), \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \), \( S^{l}(\sqrt{1-r^2}) \times S^{n-1}(r) \) and \( H^1(-\sqrt{1+r^2}) \times S^{n-1}(r) \) are all parabolic. For \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \) and \( S^l(\sqrt{1-r^2}) \times S^{n-1}(r) \) this is clear because they are compact. For the other cases, it follows from the fact that any standard Riemannian product \( \mathbb{R} \times M \) with \( M \) compact is parabolic (see [9, Subsection 2.1]).

**Appendix**

In this section we will briefly compute the value of \( |\Phi| \) for the standard examples which appears in Theorem 5 and Corollary 6. In the Euclidean space \( (c = 0) \), apart from the totally umbilical hypersurfaces, the easiest constant mean curvature hypersurfaces are the standard product embeddings of the form \( \mathbb{R}^{n-k} \times S^k(r) \hookrightarrow \mathbb{R}^{n+1} \), for a given radius \( r > 0 \) and integer \( k \in \{1, \ldots, n-1\} \). Its principal curvatures are given by
\[
\kappa_1 = \cdots = \kappa_{n-k} = 0, \quad \kappa_{n-k+1} = \cdots = \kappa_n = \frac{1}{r},
\]
and its constant mean curvature $H$ is given by $nH = k/r$. For these examples

$$|\Phi| = \frac{\sqrt{k(n-k)}}{\sqrt{nr}} = \frac{\sqrt{n(n-k)}}{\sqrt{k}}|H| \quad \text{and} \quad \alpha_H = \frac{\sqrt{n}}{\sqrt{n-1}}|H|.$$ 

In particular, $|\Phi| = \alpha_H$ if and only $k = n-1$, and $|\Phi| > \alpha_H$ otherwise.

When $c = 1$, let us consider the standard immersions $S^1(\sqrt{1-r^2}) \hookrightarrow \mathbb{R}^2$ and $S^{n-1}(r) \hookrightarrow \mathbb{R}^n$, for a given radius $0 < r < 1$, and take the product immersion $S^1(\sqrt{1-r^2}) \times S^{n-1}(r) \hookrightarrow S^{n+1} \subset \mathbb{R}^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \frac{r}{\sqrt{1-r^2}}, \quad \kappa_2 = \cdots = \kappa_n = -\frac{\sqrt{1-r^2}}{r},$$

and its constant mean curvature is given by

$$H = H(r) = \frac{nr^2 - (n-1)}{nr\sqrt{1-r^2}}. \tag{13}$$

In this case,

$$|\Phi| = \frac{\sqrt{n-1}}{r\sqrt{n(1-r^2)}}$$

where, by (13),

$$r^2 = \frac{2(n-1) + nH^2 \pm |H|\sqrt{n^2H^2 + 4(n-1)}}{2n(1+H^2)}$$

where we choose the sign $-$ or $+$ according to $r^2 \leq (n-1)/n$ or $r^2 > (n-1)/n$. Therefore,

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left((n-2)|H| \pm \sqrt{n^2H^2 + 4(n-1)}\right),$$

where we use the same criterion for the sign. In particular, $|\Phi| = \alpha_H$ when $r^2 \leq (n-1)/n$, and $|\Phi| > \alpha_H$ when $r^2 > (n-1)/n$.

Finally, when $c = -1$ let us consider the standard immersions $\mathbb{H}^{n-k}(-\sqrt{1+r^2}) \hookrightarrow \mathbb{H}^{n-k+1}_1$ and $S^k(r) \hookrightarrow \mathbb{R}^{k+1}$, for a given radius $r > 0$ and integer $k \in \{1, \ldots, n-1\}$, and take the product immersion $\mathbb{H}^{n-k}(-\sqrt{1+r^2}) \times S^k(r) \hookrightarrow \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_{n-k} = \frac{r}{\sqrt{1+r^2}}, \quad \kappa_{n-k+1} = \cdots = \kappa_n = \frac{\sqrt{1+r^2}}{r},$$

and its constant mean curvature is given by

$$H = \frac{nr^2 + k}{nr\sqrt{1+r^2}}. \tag{14}$$
We are interested in the cases where $k = 1$ and $k = n - 1$. Observe that when $k = 1$, $H^2 > 1$ if and only if $r < 1/\sqrt{n(n-2)}$. In that case

(15) \[ |\Phi| = \frac{\sqrt{n-1}}{r \sqrt{n(1+r^2)}} \]

where, by (14),

\[ r^2 = \frac{2 - nH^2 + |H| \sqrt{n^2 H^2 - 4(n-1)}}{2n(H^2 - 1)}. \]

Thus,

\[ |\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( (n-2)|H| + \sqrt{n^2 H^2 - 4(n-1)} \right) > \alpha_H. \]

On the other hand, when $k = n - 1$ we have that $H^2 > 1$ for every $r > 0$ and $|\Phi|$ is also given by (15), where now, by (14), $r^2$ is given by

\[ r^2 = \frac{2(n-1) - nH^2 + |H| \sqrt{n^2 H^2 - 4(n-1)}}{2n(H^2 - 1)}. \]

Therefore, in this case we have for every $r > 0$

\[ |\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}} \left( (n-2)|H| - \sqrt{n^2 H^2 - 4(n-1)} \right) = \alpha_H. \]

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