Theory of Higgs Modes in \(d\)-Wave Superconductors

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Introduction.—In the past few decades, collective excitation in the field of superconductivity has attracted much attention. Due to the angular and radial excitations in the Mexican-hat potential of free energy [1], two types of the collective excitations emerge: Nambu-Goldstone [2–7] and Higgs [8–11] modes, which describe the phase and amplitude fluctuations of the superconducting order parameter respectively. These modes are decoupled in equilibrium state as they represent mutually orthogonal excitations [1]. The Nambu-Goldstone mode corresponds to gapless Goldstone boson due to the spontaneous breaking of continuous \(U(1)\) symmetry by order parameter \(\phi\). For the Higgs mode, theoretical studies in conventional \(s\)-wave superconductors reveal an excitation gap at long wavelength, twice of the superconducting gap [11, 8]. Whereas being charge neutral and spinless, this mode has long been experimentally elusive. Until recently, thanks to advanced ultrafast terahertz pump-probe technique, the Higgs mode in conventional superconductors has been observed from the optically excited oscillation of the superfluid density in second-order regime [12–13]. The most convincing evidence comes from the triggered resonance at the optical frequency which equals to superconducting gap [13, 14]. The Higgs mode has since stimulated a lot of experimental interest in a variety of contexts, extending to unconventional superconductors. Particularly, recent experiments in cuprate superconductor reported similar oscillation of superfluid density in the second-order optical response [15, 16], indicating the observation of the Higgs mode in high-\(T_c\) superconductors.

Despite experimental activity, the theoretical description of the Higgs mode in high-\(T_c\) superconductors remains wide open [13, 20]. In high-\(T_c\) superconductors, a finite amplitude of the order parameter persists up to \(T^*\), which is well above \(T_c\) [21, 22], indicating the existence of the Higgs mode not only in superconducting phase but also in pseudogap one. Moreover, compared with the \(s\)-wave case, the high-\(T_c\) superconductivity with the \(d\)-wave order parameter and hence the lower rotating symmetry supports the additional Higgs modes. Early symmetry analysis revealed the existence of the breathing and rotating Higgs modes [19], which for the case of \(d_{x^2-y^2}\)-wave order parameter correspond to \(d_{x^2-y^2}\)-wave and \(d_{xy}\)-wave amplitude fluctuations, respectively. The breathing Higgs mode is expected to be similar to conventional \(s\)-wave Higgs mode, whereas the rotating one is unique. Recent experiment in cuprate superconductor has reported that the direction of the order parameter detaches from that of lattice [20]. A detectable response of rotating Higgs mode to the external probe is therefore expected. Nevertheless, the energy spectra and dynamic properties for both Higgs modes in \(d\)-wave superconductors are still unclear in the literature. Considering the growing experimental findings, a full theoretical investigation on these modes becomes imperative.

In this Letter, we use the microscopic gauge-invariant kinetic equation (GIKE) approach [27, 28] to investigate the Higgs modes for \(d\)-wave order parameter. For the first time, we provide the analytic expressions for energy spectra of both breathing and rotating Higgs modes. Then, investigation on their dynamic properties is presented. The breathing Higgs mode is optically visible in the second-order regime, irrelevant of the optical polarization direction. Whereas the rotating Higgs mode is optically inactive, we show that this mode responds to magnetic field in the linear regime, suggesting a possible detection by magnetic resonance experiment in the pseudogap phase. It is interesting to find that the charge-neutral rotating Higgs mode, which does not manifest itself in the electric measurement, generates a thermal Hall current by magnetic field in the presence of temperature gradient in pseudogap phase, providing a unique scheme for its detection.
Hamiltonian.—We begin our analysis with a free two-dimensional BC-lik Hamiltonian

$$H = \sum_{k, s=\uparrow, \downarrow} \xi_k c_{ks}^\dagger c_{ks} - \sum_{kk'} g_{kk'} c_{k\uparrow}^\dagger c_{k\uparrow} c_{-k'\downarrow} c_{-k'\downarrow},$$

where $\xi_k = \varepsilon_k - \mu$ and $\varepsilon_k = k^2/(2m)$ with $m$ and $\mu$ being the effective mass and chemical potential, respectively; $g_{kk'}$ denotes the pairing interaction. The order parameter is determined by $\Delta_{kk} = -\sum_k g_{kk'} (c_{-k\downarrow} c_{k\uparrow} + c_{k\uparrow} c_{-k\downarrow})$. This leads to the Bogoliubov quasiparticle energy $E_k = \sqrt{\xi_k^2 + \Delta_{kk}^2}$. In the present analysis, we approximately take the pairing interaction $g_{kk'}$ around the Fermi surface (i.e., $k = k' = k_F$) so that the order parameter only has angular dependence of the momentum. In the presence of the translational and time-reversal symmetries \cite{29}, the pairing parameter in high-$T_c$ superconductors persists up to $T^*$, which is well above $T_c$. In this finite or- derness, our calculation of its scattering terms due to spatial inhomogeneity read

$$\frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} = \frac{i}{2} \left\{ \varepsilon E \tau_3 - (\nabla R - 2ieA\tau_3)\hat{\Delta}_k (R), \partial_t \rho_{k\uparrow \downarrow} \right\}$$

with the electric field $E = -\nabla R (e\phi + \mu H)$ and magnetic field $B = \nabla \times A$. By the Meissner effect, the magnetic field $B$ is expelled from superconductor in superconducting phase. Whereas in pseudogap phase where the superfluid density vanishes, $B$ can be applied. Here, we consider a perpendicular $B$ to the conducting layer.

The diffusion terms due to spatial inhomogeneity read

$$\frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} = -\frac{1}{2} \left\{ \varepsilon E \tau_3 - (\nabla R - 2ieA\tau_3)\hat{\Delta}_k (R), \partial_t \rho_{k\uparrow \downarrow} \right\}$$

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To determine the nonequilibrium property/ fluctuations, one needs to solve $\delta\rho_k$ from the GIKE

$$\frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} + \frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} \bigg|_{\text{coh}} + \frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} \bigg|_{\text{diff}} = \frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} \bigg|_{\text{scat}}.$$  

The coherent terms are given by

$$\frac{\partial}{\partial t} \rho_{k\uparrow \downarrow} \bigg|_{\text{coh}} = \frac{1}{2} \left\{ \varepsilon E \tau_3 - (\nabla R - 2ieA\tau_3)\hat{\Delta}_k (R), \partial_t \rho_{k\uparrow \downarrow} \right\}$$

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consists of breathing $\delta \Delta_B(R)$ and rotating $\delta \Delta_R(R)$ parts.

**Breathing Higgs mode.**—In CM frequency and momentum space $[R = (t, \mathbf{R}) \rightarrow q = (\omega, \mathbf{q})]$, we find the optical response of the breathing Higgs mode at weak scattering (for details, see Supplemental Material [30]).

\[
(\omega_B^2 - \omega^2) \delta \Delta_B(q) = \left( \frac{e E_q \cdot i \mathbf{q}}{m} \right) 8u_\omega \Delta_0 + \nu_F^2 \left[ \frac{2}{i \omega \Omega} \left( \frac{e E_q}{i \omega \Omega} \right) \cdot e E_q + 2 \left( \frac{e E_q}{i \omega \Omega} \right) \cdot e A_q + e^2 \mathbf{A}_q^2 \right] \Delta_0 d_\omega,
\]

where $\mathbf{A}_q = \int d q' A_q B_{q'-q'}$; $i \omega \Omega = i \omega + \Gamma_p$ and $i \omega \Omega = i \omega + \Gamma_H$, with $\Gamma_p (\Gamma_H)$ being the relaxation rate of nonequilibrium $\hbar \rho_{k0} (\hbar \rho_{kz})$ from the scattering; the energy spectrum

\[
\omega_B = 2 \Delta_0 \sqrt{\frac{\sum_k \cos^4(2\theta_k) a_k z_{\omega,B,k}}{\sum_k \cos^2(2\theta_k) a_k z_{\omega,B,k}}},
\]

with $a_k = \frac{1 - \frac{f(\theta_k)}{2 E_k}}{2 E_k}$ and $z_{\omega,B,k} = \frac{1}{i \omega \Omega_{\omega,B,k}}$. $u_\omega$ and $d_\omega$ are the dimensionless coefficients for the linear and second-order responses [see Supplemental Material, Eqs. (S31) and (S32) [30], respectively.

In this response function, it is first noted from the left-hand side that due to the lower rotational symmetry for the $d$-wave order parameter [i.e., $\cos^4(2\theta_k) \leq \cos^2(2\theta_k)$ in Eq. (11)], the energy spectrum of its breathing Higgs mode $\omega_B < 2 \Delta_0$, in contrast to the $s$-wave case where the Higgs-mode energy appears at $2 \Delta_0$. Particularly, through a numerical calculation of Eq. (11), we find $\omega_B \sim 1.7 \Delta_0$ for a wide range of parameter choice ($T$ and $\Delta_0$). Furthermore, the first term on the right-hand side of Eq. (11) denotes the linear response, which vanishes in the long-wavelength limit ($q = 0$) as it should be for the charge-neutral mode. The second term, i.e., the second-order response, which originates from the drive terms [Eq. (5)], is finite at $q = 0$, similar to the $s$-wave case [28]. Particularly, this response is totally irrelevant of optical polarization direction, consistent with the experimental finding [18]. This actually is very natural considering the fact that the optical field in the long-wavelength limit is unaware of the relative momentum of the pairing electrons (i.e., pairing symmetry).

In the optical detection, for the case with the multi-cycle terahertz pulse which possesses a stable phase as well as a narrow frequency bandwidth, one approximately has $E_q, A_q \sim \delta(\omega - \Omega) \delta(q)$, and then, from Eq. (10), $\delta \Delta_B$ in the time domain becomes

\[
\delta \Delta_B(t) = \frac{e^2 \nu_F^2 \left[ A_0^2 + \frac{2F \phi^2 A_0^2}{m^2 \rho_{p0} (1 + \Gamma_p)} \right] d_{2\Omega} \Delta_0 e^{2i\Omega t}}{\omega_B^2 - (2\Omega)^2},
\]

which oscillates at twice optical frequency and exhibits a resonance at $2\Omega = \omega_B$, similar to the $s$-wave case but with $\omega_B < 2 \Delta_0$. As for the case with a short terahertz pulse which possesses the broad frequency bandwidth, no pole emerges from the electromagnetic fields in Eq. (10). In this circumstance, by neglecting the damping poles from $1/\omega \Omega$, and $1/\omega \Omega$, the residual poles in $\delta \Delta_B$ ($\omega$) come from $\pm \omega_B$, leading to an oscillating behavior of $\delta \Delta_B(t)$ at frequency $\omega_B$ in the time domain.

**Rotating Higgs mode.**—We also find the optical response of the rotating Higgs mode at weak scattering (for details, see Supplemental Material [30]).

\[
[\omega_R^2(q) - \omega^2] \delta \Delta_R = \left[ \frac{[i vec(q \times E_q)] \cdot z}{m} \right] (2 \Delta_0) s_\omega,
\]

where $s_\omega$ is the dimensionless response coefficient [see Supplemental Material, Eq. (S33) [30]]; the energy spectrum

\[
\omega_R(q) = \sqrt{m_R^2 + q^2 v_F^2 z^2/4},
\]

and

\[
m_R^2 = \frac{\Delta_0^2 \sum_k \sin^2(4\theta_k) [\partial^2 \phi_0(f(E_k))] / E_k^2}{\sum_k \sin^2(2\theta_k) \xi_k^2 a_k / E_k^2},
\]

and $z = [\sum_k \sin^2(2\theta_k) \xi_k^2 a_k / E_k^2] / [\sum_k \sin^2(2\theta_k) \xi_k^2 a_k / E_k^2]$. Then, it is found that the energy spectrum of the rotating Higgs mode $\omega_R$ exhibits an excitation gap $m_R [m_R \propto |\partial \phi_0(f(E_k))|]$ in the long-wavelength limit, which vanishes at $T = 0$ K but becomes finite at $T \neq 0$. This can be understood as follows. The rotating Higgs mode here is in fact a Goldstone boson due to the spontaneous breaking of the rotational symmetry by the formation of the $d$-wave order parameter [31], and hence, is gapless at $T = 0$ K according to the Goldstone theorem [3]. Whereas at finite temperature, due to the interaction with the thermally excited Bogoliubov quasiparticle, the rotating Higgs mode acquires a finite excitation gap, and thus, does not violate the Mermin-Wagner theorem which rules out the long-range order (gapless excitation) at $T \neq 0$ in two-dimensional systems. In addition, the energy dispersion of the rotating Higgs mode exhibits the $s$-wave structure, which is because the collective excitation in the long-wavelength limit is unaware of the pairing symmetry. The linear optical response, i.e., the right-hand side of Eq. (13), vanishes in the long-wavelength limit ($q = 0$) as this mode is charge neutral. Moreover, no second-order optical response is found in our calculation. Consequently, the rotating Higgs mode is in fact optically invisible, in contrast to the breathing one.
Actually, the finite response of the rotating Higgs mode, in principle, needs the chiral-symmetry breaking, which can be achieved by the magnetic field. Specifically, in the pseudogap phase, considering the case with magnetic field, we find the response of the rotating Higgs mode (for details, see Supplemental Material [30])

\[
[\omega_R^2(q) - \omega^2] \delta \Delta_R = -\frac{e B_{z,q}}{m} \frac{\Delta_0^3}{i \omega \mu} c_1, \tag{16}
\]

with \(c_1\) being the response coefficient [see Supplemental Material, Eq. (S34) [30]]. Then, it is observed that the rotating Higgs mode responds to magnetic field in the linear regime as expected, directly suggesting its possible detection by magnetic resonance measurement.

Finally, we show an interesting property of the rotating Higgs mode. Inspired by recent experiment where a negative thermal Hall signal contributed by unidentified charge-neutral excitation is discovered in the pseudogap phase of cuprate superconductors [32], we calculate the thermal response of the rotating Higgs mode. Specifically, in the presence of both temperature gradient and magnetic field in the pseudogap phase, we find (for details, see Supplemental Material [30])

\[
(\omega_R^2 - \omega^2) \delta \Delta_R = \frac{\varepsilon_g}{2m} \left( \nabla_R T \times i \mathbf{q} \right) \cdot \nabla \Delta_0 \partial_T \Delta_0 - \frac{e B_z \Delta_0^3}{i \omega H_\mu} c_1, \tag{17}
\]

with \(c_2\) being the corresponding response coefficient [see Supplemental Material, Eq. (S35) [30]]. The first term on the right-hand side of above equation, by temperature gradient of the order parameter \(\partial_T \Delta_0\), provides a transverse drive effect on the rotating-Higgs-mode excitation. Nevertheless, this term alone does not provide any finite thermal current due to the chiral symmetry [after chiral transformation \(\delta \Delta_R \rightarrow -\delta \Delta_R\), momentum \(\mathbf{q}\) in Eq. (17) changes sign]. The magnetic field, i.e., second term on the right-hand side, breaks this symmetry, and a finite thermal Hall current can therefore be achieved. To calculate this current, we define the bosonic field \(\phi_q(t) = \delta \Delta_R(t, \mathbf{q})/D\) with \(D\) being the scale parameter, and construct its Lagrangian from the equation of motion [Eq. (17)]:

\[
L = |\partial_t \phi_q(t)|^2 - \omega_R^2 |\phi_q(t)|^2 + \left[ M_q \phi_q^*(t) + h.c. \right]. \tag{18}
\]

Here, \(M_q = F_{q, \omega_R}/D\) with \(F_{q, \omega_R}\) labeling the terms on the right-hand side of Eq. (17), in which we have neglected the trivial damping pole by taking \(\omega \approx \omega_R\). It is noted that the Lagrangian in Eq. (18) is a typical Klein-Gordon one in the field theory [33]. Therefore, within the standard path integral method (for details, see Supplemental Material [30]), a finite thermal Hall current density, induced by the rotating Higgs mode, is directly derived:

\[
\mathbf{j}_E = \frac{\left( \nabla_R T \times e \mathbf{B} \right)}{2mT} (\lambda_R \Delta_0 \partial_T \Delta_0), \tag{19}
\]

where \(\lambda_R = \sum_q \frac{e B_{z,q}^2}{m D_T} \frac{2}{\omega_R^2} \left\{ \frac{2 \omega_R + 1}{2 \omega_R} - \frac{\partial \Delta_0}{\partial_T \Delta_0} \right\} \) with \(n_R = \frac{1}{e R(T)}\) being the Bose-distribution function. We emphasize this thermal Hall current, induced by the charge-neutral rotating Higgs mode, is totally different from the conventional thermal Hall current for electronic carriers which is generated by Lorentz force. The current here is generated due to temperature gradient of the order parameter and chiral-symmetry breaking by magnetic field as mentioned above. Particularly, due to the negative sign in \(\partial_T \Delta_0\), the thermal Hall current here possesses the opposite sign to the conventional one.

![FIG. 1: (Color online) Temperature dependence of \(k_{xy}/T\) from numerical calculation of Eq. (19) with \(\Delta_0 = \gamma (T^* - T)^\alpha\). In the calculation, we choose a case in the overdoped regime which is far from the antiferromagnetic phase. \(T^* = 57\) K. Other used parameters can be found in the Supplemental Material (see Table SI [30]).](image.png)

The temperature dependence of the thermal Hall conductivity \(k_{xy} = \mathbf{j}_E/(\mathbf{T} \cdot \mathbf{E})\) with \(T\) being the thickness of the single layer in cuprate superconductors] requires the specific behavior of the ground state (i.e., pairing interaction) to obtain \(\Delta_0(T)\), and this goes beyond the scope of present analysis. Nevertheless, in Eq. (19), with the increase of \(T\) from 0 to \(T^*\), \(n_R\) increases, whereas the order parameter part \(\Delta_0 \partial_T \Delta_0\), in general, drops slowly at the beginning and then rapidly near \(T_c\). Thus, a peak behavior in the temperature dependence of \(|k_{xy}|\) can be expected. To justify this analysis, we approximately take \(\Delta_0 \sim (T^* - T)^\alpha\) and perform a numerical calculation of Eq. (19). The numerical results for different \(\alpha\) are plotted in [Fig. 1]. As expected, with the increase of temperature from \(T = 0\) K, \(k_{xy}/T\) first increases and then decreases, leading to a peak behavior observed around \(T = 12\) K. The experimental finding in the pseudogap phase so far lies in the regime with \(T > 14\) K and only the decrease of \(|k_{xy}/T|\) is observed [32]. In this regime, the experimentally observed linear dependence on magnetic field and temperature dependence as well as negative sign of \(k_{xy}/T\) in the pseudogap phase show good agreement with our results. Therefore, we conjecture that the experimentally observed unidentified charge-neutral excitation
in the pseudogap phase \[32\] is the rotating Higgs mode. It is noted that notwithstanding the fact that our computation in Fig. 1 extends to \(T = 0\) K, our result is valid only in the pseudogap regime with \(T_c < T < T^*\).

In conclusion, within the GIKE approach, we have analytically derived the energy spectra of both breathing and rotating Higgs modes of the \(d\)-wave order parameter for the first time. Then, investigations on their rich dynamic properties have been carried out. Particularly, for the unique rotating Higgs mode in \(d\)-wave superconductors, it is interesting to find that with longitudinal temperature gradient in pseudogap phase, by magnetic field, this charge-neutral mode generates a thermal Hall current, which is likely to capture recent experimental finding in pseudogap phase of cuprate superconductors \[32\].

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[29] The translational symmetry leads to \(g_{k k'} = g_{k - k'}\), whereas \(g_{k - k'} = g_{k' - k}\) by time-reversal symmetry.
[30] See Supplemental Material for additional details of the GIKE and its solution as well as the derivation of the thermal Hall current within path integral method.
[31] As the practical formation of the \(d\)-wave order parameter with a certain direction spontaneously breaks the rotational symmetry, according to Goldstone theorem \[R\] there exists a gapless Goldstone boson, which describes the rotational fluctuation of the direction of the \(d\)-wave order parameter. For the case with \(d_{x^2 - y^2}\) order parameter, one finds \(\Delta_k = \Delta_0 \cos 2[\theta_k R - (\alpha R)]\), and hence, for small fluctuation \(\alpha (R)\), \(\delta \Delta_k \approx 2 \Delta_0 \alpha (R) \sin (2\theta_k)\), equivalent to the rotational Higgs mode \(\delta \Delta_R\).
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I. GAUGE-INVAR IANT KINETIC EQUATION

We first emphasize that by performing the gauge transformation \( \rho_k(R) \to e^{-i\tau_3 x(R)} \rho_k(R)e^{i\tau_3 x(R)} \), the GIKE [Eq. (4) in the main text] is gauge-invariant under the gauge transformation first revealed by Nambu:

\[
\begin{align*}
 eA_\mu &\to eA_\mu - \partial_\mu x(R), \\
 \delta \theta(R) &\to \delta \theta(R) + 2\chi(R),
\end{align*}
\]

where the four vectors \( A_\mu = (\phi, A) \) and \( \partial_\mu = (\partial_t, -\nabla R) \).

Therefore, we can make the unitary transformation \( \rho_k(R) \to e^{i\tau_3 \delta \theta(R)/2} \rho_k(R)e^{-i\tau_3 \delta \theta(R)/2} \) mentioned in the main text, and the GIKE becomes

\[
\begin{align*}
 \partial_t \rho_k + i \left[ (\xi_k + \mu_{\text{eff}}) \tau_3 + \Delta_k \tau_1 + \frac{e\nabla R}{4} \tau_3 \rho_k + \frac{k}{2m} \tau_3 \nabla R \rho_k \right] - \left[ \frac{p_s e \nabla R}{8m} \tau_3 \rho_k \right] &+ \frac{k}{2m} \left[ \frac{eB}{4m} \tau_3 \partial_t \rho_k \right] = \frac{k}{2m} \left[ \frac{eB}{4m} \tau_3 \partial_t \rho_k \right] = \partial_t \rho_{k'}^{\text{scat}}, \\
- \frac{i}{8} \left[ (\nabla R + ip_s \tau_3)(\nabla R + ip_s \tau_3) \Delta_k \tau_1, \partial_t \partial_k \rho_k \right] + \frac{k}{2m} \left[ \frac{eB}{4m} \tau_3 \partial_t \rho_k \right] = \partial_t \rho_{k'}^{\text{scat}},
\end{align*}
\]

where \( \Delta_k = \Delta_k + \delta \Delta_k \); the gauge-invariant \( p_s = \nabla R \delta \theta - 2eA \) and \( \mu_{\text{eff}} = \partial_t \delta \theta/2 + e\phi + \mu_H + p_s^2/(8m) \). It is noted that in Eq. (S3), the phase fluctuation is effectively removed from the order parameter and manifests itself by \( p_s \) and \( \mu_{\text{eff}} \). Moreover, after the transformation, the equation of the order parameter [Eq.(3) in the main text] becomes

\[
\begin{align*}
 \sum_{k'} g_{kk'} \rho_{k'} &= -\Delta_k, \\
 \sum_{k'} g_{kk'} \rho_{k'}^2 &= 0.
\end{align*}
\]

Equation (S4) gives the gap equation and hence the Higgs mode. Whereas Eq. (S5) determines the phase fluctuation as revealed in our previous work. We emphasize that the calculation of the amplitude fluctuations is irrelevant of the phase fluctuation, since they represent mutually orthogonal excitations and hence are decoupled.

It can be demonstrated that the charge conservation is satisfied in the GIKE approach for \( d \)-wave superconductivity in clean limit (\( \partial_t \rho_{k'}^{\text{scat}} = 0 \)). The gauge-invariant density and current read

\[
\begin{align*}
 \hat{n} &= \sum_k (1 + 2\rho_{k3}) \text{ and } \hat{j} = \sum_k \left( \frac{k}{m} \rho_{k3} \right) \text{, respectively. By taking the summation of the } \tau_3 \text{ component of Eq. (S3) over } k, \text{ one has}
\end{align*}
\]

\[
\partial_t \left( \sum_k \rho_{k3} \right) + \nabla R \cdot \left( \sum_k \left( \frac{k}{m} \rho_{k3} \right) \right) = \sum_k 2\Delta_k \rho_{k2}.
\]

By using Eqs. (S1) and (S5), the right-hand side of above equation \( \sum_k 2\Delta_k \rho_{k2} = -\sum_{kk'} g_{kk'}^2 (\rho_{k3})^2 \rho_{k2} = -\sum_{kk'} \rho_{k3}(2g_{kk'}^2) \rho_{k2} = 0 \) in which we have substituted \( g_{kk'} = \delta_{kk'} \). Then, we immediately obtain the charge conservation \( \partial_t \delta \hat{n} + \nabla R \cdot \hat{j} = 0 \). The charge conservation of the gauge-invariant kinetic theory is natural, since it has been proved long time ago by Nambu via the generalized Ward identity that the gauge invariance in the superconducting states is equivalent to the charge conservation.

II. SCATTERING

We next present the scattering terms \( \partial_t \rho_{k}^{\text{scat}} \) in Eq. (S5). Since the electron-phonon scattering is weak at low temperature, we mainly consider the electron-impurity scattering. The specific impurity scattering terms, extending to \( d \)-wave superconductivity from the \( s \)-wave case, are written as

\[
\begin{align*}
 \partial_t \rho_{k}^{\text{scat}} = -[S_k(\cdot, \cdot) - S_k(\cdot, \cdot) + h.c.],
\end{align*}
\]
with
\[ S_k(>,<) = n_i \sum_{k'} \int_{-\infty}^{t} dt' V_{k-k'} \tau_3 e^{i(t'-t)\hat{H}_k} [1 - \rho_{k'}(t')] V_{k-k'} \tau_3 \rho_k(t' e^{-(t'-t)\hat{H}_k}. \] (S8)

Here, \( n_i \) is the impurity density and \( V_{k-k'} \) stands for the electron-impurity interaction; \( \hat{H}_k = \xi_{k+\tau_3} p_x, \tau_3 + \Delta_{k0} \tau_1 \) denotes the BCS-like Hamiltonian in the presence of the intrinsic phase fluctuation \( \delta \theta_i \). \( p_{x,i} = \nabla R \delta \theta_i \).

It is noted that this scattering term is non-Markovian. In order to turn this non-Markovian scattering into the Markovian one for further calculation, one needs to carefully handle the Markovian approximation in a variety of situations of the external probe, as the previous work for \( s \)-wave superconductivity revealed. However, in high-\( T_c \) superconductors, there exists large intrinsic phase fluctuation \( \delta \theta_i \) in the pseudogap phase. The specific behaviors of this fluctuation is still unclear in the literature, making it hard to rigorously calculate the scattering effect here. In the present analysis, we take the relaxation-time approximation:
\[ \partial_t \rho_k \big|_{\text{scat}} = -\Gamma_p \delta \rho_0 \tau_0 - \Gamma_H \delta \rho_k - \tau_0 - \Gamma_H \delta \rho_0 + \tau_+. \] (S9)

in which the \( \tau_3 \) component of the scattering terms is ruled out by the charge conservation in the impurity scattering. Here, \( \Gamma_p (\Gamma_H) \) denotes the relaxation rate of the nonequilibrium \( \delta \rho_{k0} (\delta \rho_{kz}) \). We emphasize that in the main text, the scattering term/effect is added only for completeness. It plays an insignificant role in our results, since we focus on the excitation and response to external probe.

### III. SOLUTION OF GIKE

Considering a weak probe, we take the expansion \( \delta \rho_k = \delta \rho_k^{(1)} + \delta \rho_k^{(2)} \) with \( \delta \rho_k^{(1)} \) and \( \delta \rho_k^{(2)} \) being the first and second order responses to the external probe.

#### A. Linear response

The first-order GIKE reads
\[
\begin{align*}
\partial_t \delta \rho_k^{(1)} + i [\Xi \tau_3 + \Delta_{k0} \tau_1, \delta \rho_k^{(1)}] + i [\mu_{\text{eff}} \tau_3 + \delta \Delta_k \tau_1, \rho_k^{(0)}] &+ \frac{1}{2} \{ \nabla R \Gamma_{R, T} \delta \rho_k^{(0)} + \nabla R \delta \rho_k^{(0)} \} + \frac{1}{\delta m} \{ \nabla R \cdot p \tau_3, \tau_3 \rho_k^{(0)} \} \\
+ \frac{1}{2} \{ e \xi \tau_3, \partial_k \rho_k^{(0)} \} &- \frac{1}{2} \{ \nabla R \Gamma_{R, T} \Delta_{k0} \tau_1 \} + \nabla R \delta \Delta_k \tau_1 - p \tau_2 \Delta_{k0} \partial_k \rho_k^{(0)} \} - \frac{i}{8} \{ \nabla R \nabla R \delta \Delta_k \tau_1 - \nabla R P_s \Delta_{k0} \tau_2, \partial_k \partial_k \rho_k^{(0)} \} \\
+ \frac{1}{4} \{ v_k \times B \} [\tau_3, \tau_3 \partial_k \rho_k^{(0)}] &+ \frac{1}{2} \{ v_k \times B \} [\tau_3, \tau_3 \partial_k \rho_k^{(0)}] = -\Gamma_p \delta \rho_0 \tau_0 - \Gamma_H \delta \rho_k - \tau_0 - \Gamma_H \delta \rho_k + \tau_+, \quad (S10)
\end{align*}
\]

where \( \Xi = \xi - e \nabla n / 4 \) and \( v_k = k / m \); \( \mu_{\text{eff}}^{(1)} \) and \( \delta \Delta_k \) are corresponding first-order parts in \( \mu_{\text{eff}} \) and \( \Delta_k \), respectively. In CM frequency and momentum space \( [R = (t, R) \rightarrow q = (\omega, q)] \), the components of Eq. \( (S10) \) are given by

\[
\begin{align*}
(i \omega + \Gamma_p) \delta \rho_k^{(1)} &= -v_k \cdot \nabla R \tau_0 \delta \rho_k^{(0)} + i v_k \cdot q \delta \rho_k^{(0)} - i q \cdot \partial_k \delta \rho_k^{(1)} - \partial \Delta_{k0} \nabla R T \cdot \partial k \rho_k^{(0)} - e q \cdot \partial k \rho_k^{(0)} \tau_0, \\
(i \omega + \Gamma_H) \delta \rho_k^{(1)} &= -2 \Xi_k \delta \rho_k^{(1)} - i q \cdot \partial_k \rho_k^{(0)} - 2 \Xi_k \delta \rho_k^{(0)} - (v_k \times B_q) \cdot \partial_k \rho_k^{(0)} \tau_0, \\
(q \cdot \partial_k \rho_k^{(0)} \tau_0) + i q \cdot \partial_k \rho_k^{(0)} \tau_0 &= -2 \Xi_k \delta \rho_k^{(0)} - 2 \Xi_k \delta \rho_k^{(0)} - 2 \Xi_k \delta \rho_k^{(0)} - 2 \Xi_k \delta \rho_k^{(0)} - (v_k \times B_q) \cdot \partial_k \rho_k^{(0)} \tau_0, \\
(i \omega + \Gamma_H) \delta \rho_k^{(1)} &= 2 \Delta_{k0} \delta \rho_k^{(1)} + 2 \Xi_k \delta \rho_k^{(1)} - 2 \Xi_k \delta \rho_k^{(0)} - 2 \Xi_k \delta \rho_k^{(0)} - q q \delta \Delta_k^{(1)} / 4 - \partial_k \partial_k \rho_k^{(0)} \tau_0. \quad (S12)
\end{align*}
\]

Since the longitudinal vector potential does not exist in either optical response or static magnetic response, one finds \( q \cdot \partial_k \rho_k^{(0)} = -q^2 \delta \theta_q + iq \cdot 2 e A_q = -q^2 \delta \theta_q \), which in the long-wavelength limit can be neglected. Then, substituting Eqs. \( (S11)-(S13) \) into Eq. \( (S14) \), one has

\[
\begin{align*}
[4 \Xi_k + 4 (\Delta_{k0})^2 + (i \omega H)^2] \delta \rho_k^{(1)} + 2 \Delta_{k0} \delta \rho_k^{(1)} \left[ \Gamma_H + \frac{(v_k \cdot q)^2}{\omega_p} \right] &= -2 \Delta_{k0} \frac{v_k \cdot q}{\omega_p} [\partial \Delta_{k0} \nabla R T \cdot \partial k \rho_k^{(0)} - v_k \cdot \nabla R T \partial k \rho_k^{(0)}] \\
- i q \cdot \partial_k \rho_k^{(0)} + \frac{i q q}{2} \delta \Delta_k^{(1)} - e q \cdot \partial_k \rho_k^{(0)} - \frac{\Delta_{k0}}{2} i q q \cdot \partial_k \rho_k^{(0)} + v_k \cdot B_q \cdot \partial_k \rho_k^{(0)} + (v_k \cdot B_q) \cdot (2 \Delta_{k0} \partial_k \rho_k^{(0)} - \Xi_k \partial_k \rho_k^{(0)}) \\
- 2 \rho_k^{(0)} \omega_H \delta \Delta_k^{(1)} + 2 \rho_k^{(0)} \omega_H \mu_{\text{eff}}^{(1)} - \frac{1}{4} i q q \omega_H \delta \Delta_k^{(1)} + \partial_k \partial_k \rho_k^{(0)} \tau_0. \quad (S15)
\end{align*}
\]
Considering the weak scattering and long-wavelength case ($\Delta_0 > \Gamma_H, v_\omega q$), we neglect the scattering and CM momentum part on the left-hand side of the above equation. Then, one immediately obtains the solution of $\delta \rho_{k_2}^{(1)}$, from which $\delta \rho_{k_1}^{(1)}$ by Eq. \[S13\] is derived:

$$i \omega_H \delta \rho_{k_1}^{(1)} = -i \omega_H \delta \Delta_{k_1}^{(1)} H_k^{(0)} + E_k^{(1)} - P_k^{(1)}.$$  \hfill (S16)

Here, $P_k^{(1)} = 4 \Xi_k z_{\omega, k} i \omega H \rho_{k_1}^{(0)} \mu_{\text{eff}}^{(1)} ; H_k^{(0)}$ and $E_k^{(1)}$ read

$$H_k^{(0)} = -4 \Xi_k z_{\omega, k} \rho_{k_3}^{(0)} - \Xi_k z_{\omega, k} \|q\| \boldsymbol{\alpha} \cdot \partial_h \rho_{k_1}^{(0)}/2 - 4 \Delta_{k_0} \Xi_k z_{\omega, k} (v_k \cdot q)(q \cdot \partial_h) \rho_{k_1}^{(0)}/(i \omega p i \omega H),$$

$$E_k^{(1)} = \Xi_k z_{\omega, k} \left\{4 \Delta_{k_0} (v_k \cdot q)/\omega_H [\nabla_R T \cdot (\partial_T \Delta_{k_0} \partial_h \rho_{k_1}^{(0)}/\omega_H) - eE_q \cdot \partial_h \rho_{k_1}^{(0)}] + i q\rho_{k_1}^{(0)} \Delta_{k_0} : (\Delta_{k_0} \partial_h \partial_h \rho_{k_1}^{(0)} + \Xi_k \partial_h \partial_h \rho_{k_1}^{(0)} - 2 (v_k \times \mathbf{B}) \cdot (\Delta_{k_0} \partial_h \rho_{k_1}^{(0)} - \Xi_k \partial_h \rho_{k_1}^{(0)}) - i q\rho_{k_1}^{(0)} \Delta_{k_0} / 4 : \partial_h \partial_h \rho_{k_1}^{(0)}/(v_k \times \mathbf{B}_q) \cdot \partial_h \rho_{k_1}^{(0)}/2.\right\}$$ \hfill (S17)

Finally, from Eqs. (8) and (9) in the main text, the breathing and rotating Higgs modes are derived:

$$i \omega_H \delta \Delta_{k_1}^{(1)}/D_2 = -i \omega_H \sum_k \cos(2 \theta_k) \delta \rho_{k_1}^{(1)} = i \omega_H \delta \Delta_{k_1}^{(1)} \sum_k \cos^2(2 \theta_k) H_k^{(0)} - \sum_k \cos(2 \theta_k) E_k^{(1)},$$

$$i \omega_H \delta \Delta_{k_1}^{(1)}/D_2 = -i \omega_H \sum_k \sin(2 \theta_k) \delta \rho_{k_1}^{(1)} = i \omega_H \delta \Delta_{k_1}^{(1)} \sum_k \sin^2(2 \theta_k) H_k^{(0)} - \sum_k \sin(2 \theta_k) E_k^{(1)},$$ \hfill (S19-20)

where we have taken care of the particle-hole symmetry to remove terms with the odd order of $\xi_k$ in the summation of $k$. From Eq. \[S41\] in equilibrium state, one has

$$\frac{1}{D_2} = \sum_k \cos^2(2 \theta_k) \frac{1 - 2 f(E_k)}{2 E_k},$$ \hfill (S21)

and its equivalent variation [see Eq. \[S37\] in the following section]

$$\frac{1}{D_2} = \sum_k \sin^2(2 \theta_k) \left[ \frac{\sum_k 1 - 2 f(E_k)}{2 E_k} - \frac{\Delta_{k_0}^2}{E_k^2} \nabla_{E_k} f(E_k) \right].$$ \hfill (S22)

Substituting Eqs. \[S21\] and \[S22\] into Eqs. \[S19\] and \[S20\] respectively, we obtain the linear responses of the breathing and rotating Higgs modes to the external probe in the main text. Particularly, for optical response, $E \neq 0$, $A \neq 0$, $B \approx 0$ and $\nabla_R T = 0$; for magnetic response, $E = 0$, $A \neq 0$, $B \neq 0$ and $\nabla_R T = 0$; for thermal-Hall investigation, $E = 0$, $A \neq 0$, $B \neq 0$ and $\nabla_R T \neq 0$.

### B. Second-order response

In second-order regime, we only focus on the optical response in long-wavelength limit ($q = 0$) where $\delta \Delta_{k_1}^{(1)}$ vanishes. Then, the second-order GIKE reads

$$\partial_h \delta \rho_{k_2}^{(2)} + i [\xi_{k_3} + \Delta_{k_0} \tau_1, \delta \rho_{k_2}^{(2)}] + i [\mu_{\text{eff}}^{(2)} \tau_3 + \delta \Delta_{k_2}^{(2)} \tau_1, \delta \rho_{k_2}^{(0)}] + \frac{1}{2} \{ eE_{\tau_3} + \sum p_{\delta} \Delta_{k_0} \tau_2, \partial_h \rho_{k_2}^{(1)} \} + i \frac{1}{2} \left[ \sum p_{\delta} \Delta_{k_0} \tau_1, \partial_h \rho_{k_2}^{(0)} \right] = \partial_h \rho_{k_2}^{(2)} \text{ scat},$$ \hfill (S23)

where $\partial_h \rho_{k_2}^{(2)} = -\Gamma_{\rho} \delta \rho_{k_0}^{(2)} \tau_0 - \Gamma_H \delta \rho_{k_2}^{(2)} \tau_1 - \Gamma_H \delta \rho_{k_2}^{(2)} \tau_2 - \Gamma_H \delta \rho_{k_2}^{(2)} \tau_3$, $\mu_{\text{eff}}^{(2)}$ and $\delta \Delta_{k_2}^{(2)}$ are corresponding second-order parts in $\mu_{\text{eff}}$ and $\Delta_{k_2}$ respectively. In CM frequency and momentum space $[\tau = (t, \mathbf{R}) \rightarrow q = (\omega, \mathbf{q})]$, with the linear solution $\delta \rho_{k_2}^{(1)} = -[eE_{\tau} \cdot \partial_h \rho_{k_2}^{(0)}/(i \omega p)] \tau_3$ in the long-wavelength limit for optical response [see Eqs. \[S11\], \[S14\]], the components of Eq. \[S23\] are given by

$$i \omega_H \delta \rho_{k_1}^{(2)} = 0,$$ \hfill (S24)

$$i \omega_H \rho_{k_2}^{(2)} = 2 \Delta_{k_0} \delta \rho_{k_2}^{(2)} - eE_q \cdot \partial_h \rho_{k_0}^{(1)},$$ \hfill (S25)

$$i \omega_H \delta \rho_{k_1}^{(2)} = -2 \xi_k \delta \rho_{k_1}^{(2)},$$ \hfill (S26)

$$i \omega_H \delta \rho_{k_2}^{(2)} = 2 (\xi_k \delta \rho_{k_1}^{(2)} - \Delta_{k_0} \delta \rho_{k_3}^{(2)} - \rho_{k_1}^{(0)} \Delta_{k_2}^{(2)} + \rho_{k_1}^{(0)} \rho_{k_2}^{(0)} - \sum_{pq} \Delta_{k_0} \sum_{pq} \partial_h \rho_{k_1}^{(0)} - \Delta_{k_0} \rho_{k_1}^{(0)} \cdot \partial_h \rho_{k_1}^{(0)}/2.\right\}$$ \hfill (S27)
In analogy to above derivation of linear response, one can straightforwardly obtain the solution of $\delta \rho_{k1}^{(2)}$:

$$
\delta \rho_{k1}^{(2)} = -2\xi_k z_{\omega,k} (2\rho_{k1}^{(0)} \mu_{\text{eff}}^{(2)} - 2\rho_{k3}^{(0)} \Delta_{k1}^{(2)}) - 2\xi_k Y_k^{(2)} ,
$$

(S28)

with $Y_k^{(2)} = z_{\omega,k} \Delta_{k0} (\frac{2}{\omega q} c E_q \partial_k \delta \rho_{k0}^{(1)} - A_q \partial_k \delta \theta_{k0}^{(0)} + 2 A_q \partial_k \delta \rho_{k0}^{(1)}).

From Eq. (S26) as well as Eqs. (8) and (9) in the main text, the breathing and rotating Higgs modes are derived:

$$
\delta \Delta_{B}^{(2)} / D_2 = - \sum_k \cos(2\theta_k) \delta \rho_{k1}^{(2)} = - \delta \Delta_{B}^{(2)} \sum_k \cos^2(2\theta_k) 4\xi_k z_{\omega,k} \rho_{k3}^{(0)} + \sum_k \cos(2\theta_k) 2\xi_k Y_k^{(2)} ,
$$

(S29)

$$
\delta \Delta_{R}^{(2)} / D_2 = - \sum_k \sin(2\theta_k) \delta \rho_{k1}^{(2)} = - \delta \Delta_{R}^{(2)} \sum_k \sin^2(2\theta_k) 4\xi_k z_{\omega,k} \rho_{k3}^{(0)} + \sum_k \sin(2\theta_k) 2\xi_k Y_k^{(2)} ,
$$

(S30)

where we have taken care of the particle-hole symmetry to remove terms with the odd order of $\xi_k$ in the summation of $\textbf{k}$. Substituting Eqs. (S21) and (S22) into Eqs. (S29) and (S30) respectively, we obtain the second-order optical responses of the breathing and rotating Higgs modes. Particularly, the last term in Eq. (S30) is zero after the summation of $\theta_k$, indicating the vanishing second-order optical response of the rotating Higgs mode.

The corresponding response coefficients in the main text are given by

$$
u_{\omega} = \frac{\sum_k \xi_k^2 \cos^2(2\theta_k) z_{\omega,k} \theta_{\epsilon,k} (\xi_k a_k)}{\omega \omega H \sum_k \cos^2(2\theta_k) z_{\omega,k} a_k},
$$

(S31)

$$
\rho_{\omega} = \frac{\sum_k \xi_k^2 \cos^2(2\theta_k) \theta_{\epsilon,k}^2 (\xi_k a_k)}{\sum_k \cos^2(2\theta_k) a_k z_{\omega,k}},
$$

(S32)

$$
\omega_{\omega} = \frac{\Delta_{k0}^2 \sum_k \xi_k^2 / E_k \sin^2(4\theta_k) z_{\omega,k} \theta_{\epsilon,k} a_k}{i \omega \omega H \sum_k \xi_k^2 / E_k^2 \sin^2(2\theta_k) z_{\omega,k} a_k},
$$

(S33)

$$
c_1 = \frac{\sum_k \sin^2(4\theta_k) (\xi_k / E_k^2)^2 \left[ -2 \partial_{E_{k'}} f(E_{k'}) \right]}{\sum_k \sin^2(2\theta_k) (\xi_k / E_k^2)^2 a_k},
$$

(S34)

$$
c_2 = \frac{\sum_k \sin^2(4\theta_k) (1 / E_k^2)^2 \Delta_{k0} \partial_{\epsilon,k} (\Delta_{k0} a_k)}{\sum_k 2 \sin^2(2\theta_k) (\xi_k / E_k^2)^2 a_k}.
$$

(S35)

IV. ROTATIONAL SYMMETRY

Equation (S36) in equilibrium state reads

$$
\Delta_{k0} = \sum_k g_{kk'} \Delta_{k0} \frac{1 - 2 f(E_{k'})}{2 E_{k'}},
$$

(S36)

which exhibits the rotational symmetry of the polar axis of \textbf{k}. Then, in the above equation, by first replacing $\theta_k$ and $\theta_{k'}$ with $\theta_k - \alpha$ and $\theta_{k'} - \alpha$ and then taking the derivative of $\alpha$ on both sides of the equation, one obtains an equivalent variation of Eq. (S36) after setting $\alpha = 0$:

$$
\Delta_0 \sin(2\theta_k) = \sum_k g_{kk'} \Delta_0 \sin(2\theta_{k'}) \left[ \frac{\xi_k^2}{E_{k'}^2} \frac{1 - 2 f(E_{k'})}{2 E_{k'}} - \frac{\Delta_{k0}^2}{E_{k'}^2} \partial_{E_{k'}} f(E_{k'}) \right].
$$

(S37)

V. THERMAL HALL CURRENT

From the Lagrangian of the bosonic field $\phi_q(t)$ [Eq. (18) in the main text], one has the action

$$
S = \int dt \int d\textbf{q} L(\phi_q, \phi_q^*, \partial_t \phi_q, \partial_t \phi_q^*).
$$

(S38)

Considering the variation $\delta \phi_q^* S = 0$, one obtains

$$
0 = \int dt \int d\textbf{q} \left[ \frac{\partial L}{\partial \phi_q^*} \delta \phi_q^* + \frac{\partial L}{\partial (\partial_t \phi_q^*)} \delta (\partial_t \phi_q^*) \right] = \int dt \int d\textbf{q} \left\{ \frac{\partial L}{\partial \phi_q^*} \delta \phi_q^* - \partial_t \left[ \frac{\partial L}{\partial (\partial_t \phi_q^*)} \right] \delta \phi_q^* + \partial_t \left[ \frac{\partial L}{\partial (\partial_t \phi_q^*)} \delta \phi_q^* \right] \right\}.
$$

(S39)
The last term vanishes since $\delta \phi_q$ is zero at temporal beginning and end. Then, we arrive the Euler-Lagrange equation of motion $\partial_t \left[ \frac{\partial}{\partial (\phi_q)} \right] - \frac{\partial}{\partial \phi_q} = 0$. Substituting the specific Lagrangian [Eq. (18) in the main text] into this equation, one recovers Eq. (17) in the main text.

To calculate the thermal property of the bosonic field, we map the action in Eq. (S38) into the imaginary-time one and then obtain

$$S = \int_0^\beta d\tau \int dq [\phi_q(\tau)D^{-1}(\tau, q)\phi_q(\tau) + M_q^*\phi_q(\tau) + M_q\phi^*_q(\tau)],$$  

where $\beta = 1/T$ and the Green function $D(\tau, q) = [\partial^2 - \omega_R^2(q)]^{-1}$. Then, following the standard generating functional method within the path integral approach, the thermal current density is written as

$$j_E = \frac{T}{Z_0} \int_0^\beta d\tau' \sum_{q'} \omega_R(q')v_q \langle \langle \phi^{*}_q(\tau')\phi_q(\tau')e^{-S} \rangle \rangle$$

$$= \frac{T}{Z_0} \int_0^\beta d\tau' \sum_{q'} \omega_R(q')v_q \langle \langle \delta_{\tau, \tau'}\delta_{\tau, \tau'} e^{-\int_0^\beta d\tau [\phi^*_q(\tau)D^{-1}(\tau, q)\phi_q(\tau) - \tilde{J}_q\phi_q(\tau) - \tilde{J}_q^*\phi^*_q(\tau)]} \rangle \rangle$$

$$= \frac{T}{Z_0} \int_0^\beta d\tau' \sum_{q'} \omega_R(q')v_q \delta_{\tau, \tau'} \delta_{\tau, \tau'} e^{\text{Tr}[J_qD(\tau, q)J_q^*]}|_{J = J^* = 0}$$

$$= \frac{T}{Z_0} \int_0^\beta d\tau' \sum_{q'} \omega_R(q')v_q \delta_{\tau, \tau'} \delta_{\tau, \tau'} e^{\text{Tr}[J_qD(\tau, q)J_q^*]}|_{J = J^* = 0},$$

where $\tilde{J}_q = J_q - M_q^*$ with $J_q$ being the generating functional; $\delta_{\tau, \tau'}$ denotes the functional derivative; $Z_0$ stands for the partition function and $v_q = \partial_q \omega_R(q)$ represents the group velocity. In Matsubara frequency space, considering the weak external probe, the above equation becomes

$$j_E = \sum_{n, q} \omega_R v_q M_q M_q^* \exp \left[ \frac{1}{\beta} \sum_{n', q'} \frac{M_q^* M_q^*}{(i\omega_n')^2 - \omega_R^2} \right] \simeq \sum_{n, q} \omega_R v_q M_q M_q^* \exp \left[ \frac{1}{\beta} \sum_{n'} \frac{M_q^* M_q^*}{(i\omega_n')^2 - \omega_R^2} \right]$$

$$= \frac{(v_B T \times c \epsilon B)}{mT} \left( \frac{\Delta_0}{D} \right)^2 \sum_{q} \frac{c_1 c_2 \epsilon_B^2 \epsilon q}{4 \Delta_0} \frac{\Delta_0^2}{\Gamma_p^2 + \omega_R^2} \frac{\Delta_0^2}{\Gamma_p^2 + \omega_R^2} \frac{2 \omega_R}{T}$$

(S42)

Then, after the summation of the Matsubara frequency, Eq. (19) in the main text is obtained.

| $m/m_0$ | $a^*$ | $E_F$ (meV) | $E_0$ (meV) | $2\Gamma$ | $T^*$ (K) | $5t^{1/2}$ |
|--------|-------|-------------|-------------|-----------|---------|-----------|
| $\Gamma_p/E_0$ | 0.2 | $\Gamma_R/E_0$ | 0.1 | $D/E_0$ | 0.02 | $t$ (nm) | 1.3$^a$ |

$^a$ Ref. [5]; $^b$ Ref. [4]; $^c$ Ref. [3]; $^d$ Ref. [10]; $^e$ Ref. [11]

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