THE $\beta$-TRANSFORMATION WITH A HOLE

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Abstract. This paper extends those of Glendinning and Sidorov [3] and of Hare and Sidorov [6] from the case of the doubling map to the more general $\beta$-transformation. Let $\beta \in (1, 2)$ and consider the $\beta$-transformation $T_\beta(x) = \beta x \pmod{1}$. Let $J_\beta(a, b) := \{x \in (0, 1) : T^n_\beta(x) / \notin (a, b) \text{ for all } n \geq 0\}$. An integer $n$ is bad for $(a, b)$ if every $n$-cycle for $T_\beta$ intersects $(a, b)$. Denote the set of all bad $n$ for $(a, b)$ by $B_\beta(a, b)$. In this paper we completely describe the following sets:

$D_0(\beta) = \{(a, b) \in [0, 1)^2 : J_\beta(a, b) \neq \emptyset\}$,
$D_1(\beta) = \{(a, b) \in [0, 1)^2 : J_\beta(a, b) \text{ is uncountable}\}$,
$D_2(\beta) = \{(a, b) \in [0, 1)^2 : B_\beta(a, b) \text{ is finite}\}$.

1. Introduction

Let $\beta \in (1, 2)$ and let $T_\beta : [0, 1) \to [0, 1)$ denote the $\beta$-transformation, that is $T_\beta(x) = \beta x \pmod{1}$.

Our main object of study is the avoidance set of a hole:

$J_\beta(a, b) = \{x \in (0, 1) : T^n_\beta(x) / \notin (a, b) \text{ for all } n \geq 0\}$,

where $0 < a < b < 1$. This is the set of points whose orbits are disjoint from the “hole” $(a, b)$. The map $T_\beta$ restricted to $J_\beta(a, b)$ is what is referred to as an open map, or a map with a hole. Intuitively, if $(a, b)$ is small then $J_\beta(a, b)$ should be large, and vice versa. This paper aims to generalise results about this set from the case of the doubling map to the more general $\beta$-transformation for $\beta \in (1, 2)$.

Define as follows:

$D_0(\beta) = \{(a, b) \in [0, 1)^2 : J_\beta(a, b) \neq \emptyset\}$,
$D_1(\beta) = \{(a, b) \in [0, 1)^2 : J_\beta(a, b) \text{ is uncountable}\}$,
$D_2(\beta) = \{(a, b) : B_\beta(a, b) \text{ is finite}\}$.

Here an integer $n$ is bad for $(a, b)$ if every $n$-cycle for $T_\beta$ intersects $(a, b)$, and so $B_\beta(a, b)$ is the set of all bad $n$ for $(a, b)$.

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These sets were fully described for the doubling map by Glendinning and Sidorov [3] ($D_0(2)$ and $D_1(2)$) and by Hare and Sidorov [6] ($D_2(2)$). This work showed that specific 0-1 words known as balanced words are very important in these descriptions. In this paper we discuss in Section 3 precisely why these words are important, together with how and to what extent the results transfer from the doubling map to the general case. However, for general $\beta$ balanced words alone do not suffice, and so Section 4 describes this difference and is completely new.

2. $\beta$-EXPANSIONS AND COMBINATORICS ON WORDS

Much of the study of these avoidance sets involves combinatorics on words. We therefore include the basic definitions from combinatorics on words here — see [8, Chapter 2] for a more thorough discussion. We will be considering words on the alphabet $\{0, 1\}$. Given two finite words $u = u_1 \ldots u_n$ and $v = v_1 \ldots v_m$ we denote by $uv$ their concatenation $u_1 \ldots u_n v_1 \ldots v_m$. In particular $u^k = u \ldots u$ ($k$ times) and $u^\infty = \lim_{k \to \infty} u^k$. We denote the length of $u$ by $|u|$ and the number of 1s in $u$ by $|u|_1$. To compare words we use the lexicographic order: a finite or infinite word $u$ is lexicographically smaller than a word $v$ (that is, $u \prec v$) if either $u_1 < v_1$ or there exists $k > 1$ with $u_i = v_i$ for $1 \leq i < k$ and $u_k < v_k$.

A finite or infinite word $w$ is said to be balanced if for any two factors $u$ and $v$ of $w$ of equal length, we have that $||u|_1 - |v|_1| \leq 1$. A finite word $w$ is called cyclically balanced if $w^2$ is balanced.

We also introduce the following notation. Given a finite word $w$ and a subword $u$ of $w$, we denote by $u$-max the lexicographically maximal shift of $w$ that begins with the word $u$. Similarly we denote by $u$-min the lexicographically minimal shift of $w$ that begins with the word $u$. For example, given $w = 10100$, we have $0$-max = 01010 and 1-min = 10010.

In order to use combinatorics on words in the context of the $\beta$-transformation, we recall that $T_\beta$ is conjugate to the shift map on a subset of $\Sigma = \{0, 1\}^\mathbb{N}$. This arises by writing a number $x$ as

$$x = \sum_{i \geq 1} x_i \beta^{-i},$$

with $x_i \in \{0, 1\}$, as first studied in [10]. In particular, we consider the greedy $\beta$-expansion of $x$, namely the expansion with $x_i = \lfloor \beta T^{-1} x \rfloor$. Informally the greedy expansion is given by taking a 1 whenever possible. We denote the set of possible (“admissible”) sequences $(x_i)_{i=1}^\infty$ by $X_\beta$.

Consider the expansion of 1 given by $d_i = \lfloor \beta T^{-1} x \rfloor$. If this sequence is infinite (i.e. does not end in 0$^\infty$) then set $d_i = \tilde{d}_i$. If $\tilde{d}_i$ is
finite then let $k = \max \{ j : \tilde{d}_j \neq 0 \}$ and set $d_1d_2\cdots = (\tilde{d}_1 \cdots \tilde{d}_{k-1}0)^\infty$.

This is the periodic quasi-greedy expansion of 1. Then as shown by Parry in \cite{9}, we have

$$\mathfrak{X}_\beta = \{(x_i)_{i=1}^\infty : x_jx_{j+1}x_{j+2}\cdots \preceq d_1d_2d_3\cdots \text{ for all } j \in \mathbb{N}\}.$$

We will denote the quasi-greedy expansion of 1 by $1\cdot$, akin to a decimal point. This is to avoid confusion between the real number 1, the sequence $10^\infty$, and this quasi-greedy expansion $1\cdot$.

Throughout this paper we will refer to a point $x \in (0,1)$ and its expansion $(x_i)_{i=1}^\infty \in \mathfrak{X}_\beta$ interchangeably. The only possible ambiguity here is where a point has a finite expansion; that is to say $(x_i) = u10^\infty$ for some finite admissible word $u$. Here we naturally have $u10^\infty = u01\cdot$. Generally in such cases we will use the finite expansion by default and will specify if this is not the case.

We also make use of the idea of extremal pairs, linked to the study of Lorenz maps through kneading invariants – see \cite{7} and \cite{4}.

**Definition 2.1** (Extremal pairs). A pair $(s, t)$ of finite \{0, 1\} words is said to be an extremal pair if the following inequalities do not hold for any $k, \ell > 0$:

$$s^\infty \preceq \sigma^k s^\infty \prec t^\infty,$$

$$s^\infty \prec \sigma^\ell t^\infty \preceq t^\infty.$$

In our context we will always have that $t$ is a cyclic permutation of $s$, and so these two inequalities combine into one. Notice that we do not require that $s_1 = 0$ and $t_1 = 1$ as in \cite{3}. However as an immediate consequence of the definition and the cyclic permutation requirement we have that $s = u0$-max and $t = u1$-min for some word $u$. The intuitive description is that given an orbit, we take two neighbouring points of that orbit: the rest of the orbit cannot fall in between these two points.

Given an extremal pair $(S, T)$, we naturally have $S, T \in \{0, 1\}^n$ and so denote more fully the pair as $(S(0,1), T(0,1))$. However, it is entirely possible to take another extremal pair $(s, t)$ and use this pair as an alphabet to gain the pair $(S(s, t), T(s, t))$. These words then belong to \{s, t\}^n. We call such a pair $(S(s, t), T(s, t))$ a descendant of $(s, t)$. It is shown in \cite{3, Proposition 2.1} that all such descendants are themselves extremal pairs.

We further define an extremal pair $(s, t)$ to be maximal if firstly there does not exist any point $x$ such that the orbit of $x$ is contained in one of either $[0, s^\infty)$ or $(t^\infty, 1)$, and secondly there does not exist a distinct extremal pair $(\tilde{s}, \tilde{t})$ such that $(s^\infty, t^\infty) \subset (\tilde{s}^\infty, \tilde{t}^\infty)$. Equivalently, an extremal pair $(s, t)$ is maximal if $\mathcal{J}_\beta(s^\infty, t^\infty) = \{\sigma^n s^\infty : n \in \mathbb{N}\}$. 

2.1. Admissible Sturmian sequences for the $\beta$-transformation.

For the case $\beta = 2$ the maximal extremal pairs are formed from balanced (Sturmian) words, as shown in [3]. Hence, the first issue is to establish which Sturmian sequences are admissible for which $\beta$. A detailed exposition on Sturmian sequences may be found in the book by Lothaire [8, §2] and the survey paper by Vuillon [12]. There are several ways of defining Sturmian sequences. We will do so using the Farey tree as the tree structure allows for easier proofs later.

**Definition 2.2** (Farey tree). We construct the Farey tree inductively. Take 0 and 1 as initial sequences, with associated fractions $0/1$ and $1/1$ respectively. These initial sequences are said to be neighbouring, and more generally a sequence in the tree is considered to be a neighbour to its parents (and children).

Given two neighbouring words $w_{\gamma_1}$ and $w_{\gamma_2}$ such that $w^\infty_{\gamma_1} < w^\infty_{\gamma_2}$, concatenate them to make a child word $w_{\gamma_2}w_{\gamma_1}$, and combine the associated fractions $\gamma_1 = a_1/b_1$ and $\gamma_2 = a_2/b_2$ to make $\gamma_1 \oplus \gamma_2 = (a_1+a_2)/(b_1+b_2)$. The limit points of the resultant tree correspond to irrational $\gamma$ and are called Sturmian sequences.

For rational $\gamma = p/q$, we obtain a finite cyclically balanced word $w_{\gamma}$ of length $q$, and we may define $X_{p/q}$ to be the finite set

$$X_{p/q} = \{\sigma^n(w^\infty_{\gamma}) : n \in \mathbb{N}\}.$$ 

These sets $X_{\gamma}$ has been well studied in [2]. For rational $\gamma$ the word $w_{\gamma}$ given by the Farey tree turns out to be the maximal cyclic shift; that is to say $w^\infty_{\gamma}$ is the maximal element of $X_{\gamma}$. The analogous definition of $X_{\gamma}$ for irrational $\gamma$ begets an infinite set of Hausdorff dimension 0.

Each rational $\gamma$ gives rise to two distinct infinite balanced words. Given $w_{\gamma} = w_{\gamma_1}w_{\gamma_1}$, these words are $w^\infty_{\gamma_1}$ and $w_{\gamma_2}w^\infty_{\gamma}$. Naturally if $\gamma_1 < \gamma_2$ then $w_{\gamma_1} < w_{\gamma_2}$. Therefore we may define a function $\gamma(\beta)$ to be the maximal admissible Sturmian word for a given $\beta$. This is non-decreasing, with the effect that for a given $\beta$, $w^\infty_{\gamma}$ is admissible if and only if $\gamma \leq \gamma(\beta)$.

The exact description of this function is as follows:

**Lemma 2.3** (Admissible Sturmian words). We have $\gamma(\beta) = \gamma \in \mathbb{Q}$ for every $\beta$ such that $1^\cdot \in [w^\infty_{\gamma}, w_{\gamma_2}w^\infty_{\gamma}]$.

**Proof.** This follows immediately from the fact that $w_{\gamma}$ is the maximal element of $X_{\gamma}$: any $\beta$ with $1^\cdot \prec w^\infty_{\gamma}$ must therefore have $w^\infty_{\gamma}$ inadmissible and $\gamma(\beta) < \gamma$. Similarly, if $1^\cdot \succ w_{\gamma_2}w^\infty_{\gamma}$ then in order to be a valid greedy expansion of 1 we must have that $1^\cdot$ begins $w^\infty_{\gamma_2}$, which implies that $\gamma(\beta) > \gamma$. Therefore $\gamma(\beta) = \gamma \in \mathbb{Q}$ if and only if $1^\cdot \in [w^\infty_{\gamma}, w_{\gamma_2}w^\infty_{\gamma}]$. \(\square\)
Example 2.4. For example, $\gamma(\beta) = 1/2$ for every $\beta$ such that $1 \cdot \in [\langle 10 \rangle \infty, 1\langle 10 \rangle \infty]$. This corresponds to $\beta \in [\varphi, 1.8019 \ldots]$ where $\varphi$ denotes the golden ratio. This is the largest plateau visible in Figure 1.

As can be seen in Figure 1, $\gamma(\beta)$ is a devil’s staircase: it is continuous, non-decreasing, and has zero derivative almost everywhere. The same function arises when considering digit frequencies for $\beta$-expansions, as described by Boyland et al. in [1].

We can associate an extremal pair with each rational $\gamma$. Given the word $w_\gamma$, take $s = 0$-max and $t = 1$-min. These pairs may themselves be constructed using a tree structure: given two neighbouring pairs $(s_\gamma_1, t_\gamma_1)$ and $(s_\gamma_2, t_\gamma_2)$ with associated rationals $0 < \gamma_1 < \gamma_2 < 1$, their child is $(s_\gamma_2 s_\gamma_1, t_\gamma_1 t_\gamma_2)^\infty$.

We remark that these pairs satisfy the following:

$$s_\gamma_1 t_\gamma_1^\infty < (s_\gamma_2 s_\gamma_1^\infty, t_\gamma_2 t_\gamma_1^\infty) < s_\gamma_2^\infty.$$

This is shown for balanced words in [6, Lemma 3.2]. We remark that the result is actually a property of the tree construction method, not

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1Note the reason for excluding $\gamma = 0$ or 1 is that here $s$ and $t$ are not well defined.
specifically of the words themselves: we simply take the tree for balanced words and map 0 and 1 to the left and right roots of our alternative tree, and provided the left root is less than the right root, the result will hold.

2.2. \textbf{Level \( n \) “balanced” words.} For describing \( D_0(\beta) \), the normal balanced words are sufficient. However, to describe \( D_1(\beta) \) one must consider higher level “balanced” words. These are defined in [3] and for completeness’ sake we repeat the discussion here. These words are not themselves balanced but are derived from balanced words.

Consider \( p/q \in (0, 1) \). Define a function \( \rho_{p/q} : \{0, 1\} \rightarrow \{0, 1\} \) by

\[
\rho_{p/q}(0) = s_{p/q}, \\
\rho_{p/q}(1) = t_{p/q},
\]

where \( s = 0\)-max and \( t = 1\)-min are the Sturmian extremal pairs associated to \( p/q \) as defined in the previous section. These are thus the descendants of Sturmian pairs.

Then for \( r \in (\mathbb{Q} \cap (0, 1))^n \) we may define as follows:

\[
s_r = \rho_{r_1}\rho_{r_2}\ldots\rho_{r_n}(0), \\
t_r = \rho_{r_1}\rho_{r_2}\ldots\rho_{r_n}(1).
\]

By taking limits this definition may be extended to \( r \in (\mathbb{Q} \cap (0, 1))^\mathbb{N} \) and to \( r \in (\mathbb{Q}^{n-1} \times \mathbb{R}) \cap (0, 1)^n \). We define a function \( \gamma(\beta) \) to be the maximal admissible higher level “balanced” word for a given \( \beta \in (1, 2) \).

This will be a vector:

\[
\gamma(\beta) \in \left( \bigcup_{n=1}^{\infty} (\mathbb{Q} \cap (0, 1))^n \right) \cup \left( \bigcup_{n=1}^{\infty} (\mathbb{Q}^{n-1} \times \mathbb{R}) \cap (0, 1)^n \right).
\]

The function \( \gamma(\beta) \) defined above corresponds to \( \gamma(\beta)_1 \). Essentially on each plateau of \( \gamma(\beta) \), we define a new devil’s staircase giving \( \gamma(\beta)_2 \).

Each plateau of this will then give rise to a further devil’s staircase for \( \gamma(\beta)_3 \), and so the process continues. Throughout this text we shall refer to \( \gamma(\beta)_1 = \gamma(\beta) \): we will need the vector when discussing \( D_1(\beta) \), but only the scalar is needed for \( D_0(\beta) \) and \( D_2(\beta) \).

We will describe descendants of balanced and level \( n \) “balanced” extremal pairs as being \emph{Farey descendants}. The vector \( r \) then functions almost as a coordinate system, telling you which pair you are descended from.

2.3. \textbf{Bad \( n \).} As in [6], we say that a natural number \( n \) is \emph{bad} for \( (a, b) \) if every \( n \)-cycle for \( T_\beta \) intersects the hole \( (a, b) \).

In the \( \beta = 2 \) case, it is natural to discard \( n = 2 \) as there is only one 2-cycle, making for an uninteresting definition. As \( \beta \) decreases, gradually each \( n \) will have fewer cycles and thus once we have only one \( n \)-cycle remaining we wish to discard this \( n \). For even \( n = 2k \) this
occurs when \(1^* = (10^{k-2}10^k)\infty\) and for odd \(n = 2k + 1\) this occurs at \(1^* = (10^{k-1}10^k)\infty\). For each \(\beta \in (1,2)\) let \(N_\beta\) denote the least \(n\) such that there exists at least two \(n\)-cycles for \(\beta\).

Then let \(B_\beta(a,b)\) denote the set of \(n > N_\beta\) such that \(n\) is bad for \(T_\beta\). Then we define

\[
D_2(\beta) = \{(a,b) : B_\beta(a,b) \text{ is finite}\}.
\]

3. Transfer of results from the doubling map

In this section we transfer what results we can from the case of the doubling map as studied by Glendinning and Sidorov in [3] and Hare and Sidorov in [6].

3.1. Large and small \(a\) and \(b\). For the doubling map one may restrict to \((a,b) \in (1/4,1/2) \times (1/2,3/4)\) without losing any interesting behaviour. This section covers the analogue of this restriction for \(\beta \in (1,2)\).

**Lemma 3.1 (Large \(a\)).** If \(a > 1/\beta\) then \((a,b) \in D_2(\beta)\).

**Proof.** As for \(\beta = 2\), write \(a = 10^k1\ldots\) for some \(k \geq 0\). The consider the following subshift:

\[
A = \{w \in X_\beta : w_i = 1 \implies w_{i+j} = 0 \text{ for } j = 1, \ldots, k+1\}.
\]

Then \(A \subset J_\beta(a,b)\). A contains periodic orbits \((10^m)\) for any \(m > k+1\), thus \((a,b)\) has only finitely many bad \(n\). \(\square\)

By definition we have that \(D_2(\beta) \subset D_1(\beta) \subset D_0(\beta)\), so \((a,b) \in D_1(\beta) \cup D_0(\beta)\) whenever \(a > 1/\beta\) also.

The restriction for small \(b\) is more different to the \(\beta = 2\) case as it involves \(\gamma(\beta)\).

**Lemma 3.2 (Small \(b\)).** Suppose \(b < \inf X_\gamma(\beta)\). Then \((a,b) \in D_2(\beta)\).

**Proof.** We show this result for \(\gamma(\beta) \in \mathbb{Q}\); the case where \(\gamma(\beta) \in \mathbb{R}\) then follows by taking limits. Write \(\inf X_\gamma(\beta) = u^\infty\) and let \(u_\gamma\) denote the smallest shift of the left Farey parent of \(\gamma(\beta)\). Then \(b = u^Kv\) for some \(K\) and some \(v < u\). Consider the following shift:

\[
B_K = \{w \in X_\beta : w \text{ is made of blocks } u^k u_\gamma \text{ for any } k > K\}.
\]

Clearly \(B_K \subset J_\beta(a,b)\). As \(|u|\) and \(|u_\gamma|\) are coprime and we are allowed any \(k > K\), this shift \(B\) will contain periodic orbits of any suitably long length. Thus \((a,b)\) has finitely many bad \(n\). \(\square\)

We have the following corollaries:

**Corollary 3.3.** \(J_\beta(0,\inf X_\gamma(\beta),\inf X_\gamma(\beta)) = X_\gamma(\beta)\).
Proof. Because $\inf X_{\gamma(\beta)} = \sigma(0 \inf X_{\gamma(\beta)})$, it is clear that any point strictly below $0 \inf X_{\gamma(\beta)}$ must fall into the hole $(0 \inf X_{\gamma(\beta)}, \inf X_{\gamma(\beta)})$. Then the result follows from the above lemma by considering the limit of $B_K$ as $K \to \infty$.

\[ \square \]

Corollary 3.4. $J_{\beta}(1/\beta, 1 \inf X_{\gamma(\beta)}) = X_{\gamma(\beta)}$.

Proof. This hole is a preimage of $(0, \inf X_{\gamma(\beta)})$ which by the previous corollary has avoidance set $X_{\gamma(\beta)}$.

These results describe what are essentially the easy cases, where $a \geq 1/\beta$ or $b \leq \inf X_{\gamma(\beta)}$. The interesting behaviour that is more difficult to describe thus occurs within this region for $(a, b)$:

$$I_{\beta} = (0 \inf X_{\gamma(\beta)}, 1/\beta) \times (\inf X_{\gamma(\beta)}, 1 \inf X_{\gamma(\beta)})$$

Notice that as $\beta \to 2$, $I_{\beta}$ approaches as expected the $(1/4, 1/2) \times (1/2, 3/4)$ region seen for the doubling map. However as $\beta \to 1$, $I_{\beta} \to (0, 1)^2$.

3.2. Extremal pairs. We now commence to transfer results from the doubling map. Essentially, if an extremal pair is admissible for a given $\beta$, then all results involving that extremal pair will still hold for that $\beta$. We formalise this as follows.

Theorem 3.5. Let $\beta \in (1, 2)$. Suppose $(s, t)$ is an extremal pair such that $\{s, t\}^\infty$ is admissible for $\beta$ and $(s^\infty, t^\infty) \in I_{\beta}$. Let $u$ and $v$ be words such that $s = uv$ and $t = vu$. Then for any $\epsilon > 0$, we have

1. $J_{\beta}(s^\infty, t^\infty) \supseteq \{\sigma^n s^\infty : n \geq 0\}$,
2. $J_{\beta}(s^\infty, ts^\infty - \epsilon)$ and $J_{\beta}(st^\infty + \epsilon, t^\infty)$ are uncountable.

If additionally $j = |u|$ and $q = |s|$ are coprime, then

3. $(s^\infty, ts^\infty - \epsilon)$, $(s^\infty + \epsilon, t^\infty)$ and $(st^\infty + \epsilon, ts^\infty - \epsilon)$ have finitely many bad $n$.

If $(s, t)$ is maximal extremal, then

4. $J_{\beta}(s^\infty, t^\infty) = \{\sigma^n s^\infty : n \geq 0\}$.

If $(s, t)$ is the Farey descendant of a maximal pair, then

5. $J_{\beta}(s^\infty, t^\infty)$ is countable,
6. $J_{\beta}(s^\infty, ts^\infty), J_{\beta}(st^\infty, t^\infty), J_{\beta}(st^\infty, ts^\infty)$ and $J_{\beta}(st^\infty, t^\infty)$ are countable.

Proof. These results are shown for the case of balanced pairs (or $n$-th level balanced in the place of Farey descendants) in [3] and [6]. We collect them here in a bid to make clearer precisely what combinatorial property of words each result is relying upon, and so for the sake of clarity we repeat the arguments here and alter them as necessary to encompass the general case.

Let $(s, t)$ be an extremal pair. Item (1) – that $\{\sigma^n s^\infty : n \geq 0\} \subseteq J_{\beta}(s^\infty, t^\infty)$ – follows immediately from the definition.
For item (2), to show that $J_\beta(s^\infty, ts^\infty - \epsilon)$ is uncountable, we follow [3] Lemma 2.2. Let $N \in \mathbb{N}$ and define

$$W_N = \{\sigma^i w : w \text{ is composed of blocks } ut^m \text{ with } m > N\}.$$  

Because $\{s, t\}$ is admissible, we know that $W_N$ is admissible for all $N$. Furthermore $W_N$ is shift invariant and has positive entropy (and therefore positive Hausdorff dimension). For any $\epsilon > 0$, there exists an $N$ such that $ts^\infty - \epsilon < ts^N$. Then we claim $W_N \subset J_\beta(s^\infty, ts^\infty - \epsilon)$. To see this, notice that $ut^m = s^m u$. By extremality, the only shifts we need be concerned about are those beginning with $s$ or $t$. Any shift beginning $s$ will be of the form $s^i u < s^\infty$, so avoids the hole. Any shift beginning $t$ either has multiple $ts$ and so avoid the hole, or begins $ts^m u$ for some $m > N$. This therefore also avoids the hole for large enough $N$.

The case $J_\beta(st^\infty + \epsilon, t^\infty)$ is similar, using shifts with $t^m v = vs^m$. Either of these shifts will then avoid $(st^\infty + \epsilon, ts^\infty - \epsilon)$. This leads immediately to item (3), following [4] Theorem 3.6. Notice that the orbit $(ut^m)$ has period $mq + j$. If $j$ and $q$ are coprime, then for every $\ell$ there exists $k$ such that $\ell \equiv kj \mod q$. So by considering points of the form

$$w = (ut^{m_1} ut^{m_2} \ldots ut^{m_k})^\infty,$$

for sufficiently large $m_i > N$, we can create orbits of any sufficiently large length which avoid the hole. Thus whenever $|u|$ and $|s|$ are coprime, we have that $(s^\infty, ts^\infty - \epsilon), (st^\infty + \epsilon, t^\infty)$ and $(st^\infty + \epsilon, ts^\infty - \epsilon)$ have finitely many bad $n$.

Item (4) is immediately clear from the definition of maximal extremality combined with (1), and therefore in fact functions as an equivalent definition of maximal extremal.

For (5), we follow [3] Lemma 2.12 and use induction to show that $J_\beta(s^\infty, t^\infty)$ is countable for Farey descendants of maximal pairs. The result clearly holds for 0th level descendants; that is to say for the maximal pair itself. Assume the claim holds for all $k$th level descendants $(s_k, t_k) = (s_{(r_1, \ldots, r_k)}, t_{(r_1, \ldots, r_k)})$. We show it must then hold for the $(k + 1)$st level. Write $r_{k+1} = p_{k+1}/q_{k+1}$. Note firstly that as $J_\beta(s_k^\infty, t_k^\infty)$ is countable, we wish to show that all but countably many points of $(s_k^\infty, t_k^\infty)$ must fall into $(s_{k+1}^\infty, t_{k+1}^\infty)$.

Any word $(s_{k+1}, t_{k+1})$ is by definition a balanced word on the alphabet $\{s_k, t_k\}$, with length in this alphabet $q_{k+1}$. The only shifts of $s_{k+1}$ that fall into $(s_{k+1}^\infty, t_{k+1}^\infty)$ are those beginning with $s_k$ or $t_k$. Label these (in order) as $x_1, \ldots, x_{q_{k+1}}$. Balanced words correspond to ordered orbits as discussed in [2] and [5]. This means that any interval $[x_i, x_{i+1}]$ will be mapped by $\sigma^{q_{k+1}}$ to some other interval $[x_j, x_{j+1}]$, and by repeatedly applying $\sigma^{q_{k+1}}$ we will cycle through all possible $j \in \{1, \ldots, q_{k+1} - 1\}$. One of these intervals is $[s_{k+1}^\infty, t_{k+1}^\infty]$. Therefore all but countably many points in $(x_1, x_{q_{k+1}})$ will fall into $(s_{k+1}^\infty, t_{k+1}^\infty)$. 
The only remaining possibilities are points in \((s^\infty_k, x_1)\) and points in \((x_{qk+1}, t^\infty_k)\). Applying \(\sigma^{qk+1}\) to these intervals maps them to \((s^\infty_{k+1}, x_1)\) and \((x_j, t^\infty_{k+1})\) respectively for some \(i > 1\) and \(j < qk+1\). Thus again by applying \(\sigma^{qk+1}\) repeatedly we see that all but countably many points must fall into \((s^\infty_{k+1}, t^\infty_{k+1})\).

Therefore item (5) holds and \(J_\beta(s^\infty, t^\infty)\) is countable for Farey descendents of maximal pairs.

The final item (6) is more complex, with a degree of subtlety as to why the result holds only for Farey descendants, not for either all extremal pairs or only maximal extremal pairs. We follow [3, Theorem 2.13]. Suppose \((s, t)\) is a Farey descendant of a maximal extremal pair \((u, v)\). Consider \(J_\beta(s^\infty, t^\infty)\). This is a countable subset of \(\{u, v\}^N\).

Then take any point in \([ts^\infty, t^\infty]\). By applying \(\sigma^q\) repeatedly we can see that all but countably many points in this interval must fall into \((s^\infty, ts^\infty)\). Hence \(J_\beta(s^\infty, ts^\infty) \setminus J_\beta(s^\infty, t^\infty)\) is countable. Therefore \(J_\beta(s^\infty, ts^\infty)\) is countable.

Similarly, consider any point in \([s^\infty, sts^\infty]\). Apply \(\sigma^q\) repeatedly, and we see that all but countably many points must fall into \((sts^\infty, ts^\infty)\). Therefore we have that \(J_\beta(sts^\infty, ts^\infty) \setminus J_\beta(s^\infty, ts^\infty)\) must be countable. Hence \(J_\beta(sts^\infty, ts^\infty)\) is countable.

The cases with \(s\) and \(t\) reversed are similar.

Remark 3.6. It is key to note that this result relies very strongly on \(J_\beta(s^\infty, t^\infty)\) being countable, which holds only by item (5). This is where the proof fails if the pair \((s, t)\) is not a Farey descendant of a maximal extremal pair.

Remark 3.7 (Continuity of boundaries). Notice that the boundaries of \(D_0(\beta), D_1(\beta)\) and \(D_2(\beta)\) are continuous in \((a, b)\), in the sense that if we have a sequence \((a_i, b_i)\) in \(\partial D_j(\beta)\) converging to a limit \((a, b)\), then this limit point belongs to \(\partial D_j(\beta)\) also. This allows us to extend results to include the points arising as limits of maximal extremal pairs.

We therefore wish to establish which extremal pairs are maximal for a given \(\beta\). The above results will then combine to delimit the boundaries of \(D_0(\beta), D_1(\beta)\), and \(D_2(\beta)\), with minor modifications for cases where for example \(s^\infty\) is admissible but \(st^\infty\) is not. Each set has a continuous boundary consisting of a countable set of plateaus given by \([s^\infty, st^\infty]\) in the case of \(D_0(\beta)\) and \(D_2(\beta)\) and by \([s^\infty, sts^\infty]\) in the case of \(D_1(\beta)\), as shown in Figure 2. Notice that given a maximal pair \((s, t)\), we have – up to a set of measure zero given by the limit points – that

\[
[s^\infty, st^\infty] = \bigcup_{(s, t)} [s^\infty_r, s^\infty_t s^\infty_r],
\]

2Recall we consider maximal extremal pairs to be Farey descendents of themselves, so this result does hold for maximal extremal pairs.
Figure 2. Lemma 3.5 for a maximal extremal pair \((s, t)\). The dark grey shows points in \(D_2(\beta)\), the light grey shows points in \(D_0(\beta)\), and the white region shows where the Farey descendants of \((s, t)\) will lie, so these points are in \(D_0(\beta)\) and may or may not be in \(D_1(\beta)\).

where \((s_r, t_r)\) are the Farey descendants of \((s, t)\). This follows easily from Lemma 2.10 of [3] with minor modifications for general \(\beta\).

This ensures that once we have the correct maximal pairs, \(D_1(\beta)\) is well defined.

If it is the case that a pair \((s, t)\) is such that \(s^\infty\) is admissible but \(st^\infty\) is inadmissible, the above results are not significantly disrupted. Any inadmissible sequence must be replaced by the largest admissible sequence that is less than the intended inadmissible sequence. The results showing that a point is not in \(D_i(\beta)\) for some \(i\) will clearly still apply as we have less admissible sequences meaning \(J_\beta\) will if anything be smaller than previously shown. The difficulty is when we want to show that \(J_\beta\) is large, as we must ensure the inadmissibility has not removed too much of \(J_\beta\).

Considered the balanced pairs \((s, t) = (0\text{-max}, 1\text{-min})\) discussed in the previous section. As shown by Glendinning and Sidorov, these pairs (when admissible) are maximal extremal. Each pair is admissible when the associated \(\gamma\) is less than \(\gamma(\beta)\). Notice that in the context of this problem, we consider these particular pairs because they are maximal extremal: that they are balanced is a side effect, not the reason for interest.

For \(\beta = 2\), the holes formed from these pairs will customarily have two distinct preimages, formed by appending either a 0 or a 1 to both endpoints of the hole. As \(\beta\) decreases, the preimage formed by appending a 1 becomes inadmissible and so a particular hole may have a unique preimage. Because \(J_\beta(a, b)\) is invariant under \(T_\beta\), this means
that all results pertaining to the original hole will also apply to its unique preimage. This leads us to the following conclusion:

**Lemma 3.8.** Suppose $\gamma(\beta) \in [\frac{1}{n+1}, \frac{1}{n}]$ and $0 < k \leq n$. Let $(s, t) = (0\text{-max}, 1\text{-min})$ be the maximal extremal balanced pair corresponding to $\gamma < \gamma(\beta)$. Then the pairs $(0^k\text{-max}, 0^{k-1}\text{-min})$ are also maximal extremal.

Notice that when $\gamma(\beta) \in [\frac{1}{n+1}, \frac{1}{n}]$, we have that 1. begins with $10^{n-1}$. This gives the correct range of $0 < k \leq n$ to ensure a unique preimage. Also note that $\inf X_{\gamma(\beta)}$ begins with $0^n$, so as one would expect these pairs will fall into the region $I_{\beta} = (0 \inf X_{\gamma(\beta)}, 1/\beta) \times (\inf X_{\gamma(\beta)}, 1 \inf X_{\gamma(\beta)})$.

This means that by considering all suitable $\gamma$ and $k$, balanced pairs will cover the range $(0^{n+1}, 01\text{-}) \times (0^{n-1}, 1 \inf X_{\gamma(\beta)})$. However these are all available balanced pairs, and so we cannot expect the remaining region $R_{\beta} = (0 \inf X_{\gamma(\beta)}, 0^n1) \times (\inf X_{\gamma(\beta)}, 0^{n-1})$ to involve balanced pairs.

Figure 3 shows the balanced pairs giving $D_i(\beta)$ for $I_{\beta} \setminus R_{\beta}$, with $\beta \approx 1.427$. Note that $D_1(\beta)$ is shown by the dark grey and the white areas “between” the light and dark grey. These white areas do exist but are so small as to be barely visible, therefore the inset image shows a magnification as indicated. Notice how the overall image has the same section repeated three times at different scales. This corresponds to the shifting of the balanced words as in Lemma 3.8 above. Furthermore there are vertical intervals that appear to be jumps, at $a = 1/\beta^k$, such that $\partial D_2(\beta) = \partial D_1(\beta) = \partial D_0(\beta)$. This corresponds to where $s^\infty$ is admissible but $st^\infty$ is inadmissible.

4. **The region $R_{\beta} = (0 \inf X_{\gamma(\beta)}, 0^n1) \times (\inf X_{\gamma(\beta)}, 0^{n-1})$**

The previous sections have described $D_0(\beta), D_1(\beta)$ and $D_2(\beta)$ for $a \leq 0 \inf X_{\gamma(\beta)}$ and for $a \geq 0^n1$. In countably many cases the remaining region $R_{\beta}$ is empty. This occurs precisely when $1 = w_\gamma^\infty$ for

$$\gamma = \frac{k}{(n+1)k-1} \in (1/(n+1), 1/n],$$

with $k, n \in \mathbb{N}$. For these values of $\beta$, the description of the $D_i(\beta)$ is already complete and needs only balanced pairs. The doubling map is one of these exceptional cases ($n = k = 1$), as is the golden ratio $\beta = (1 + \sqrt{5})/2$ ($n = 2, k = 1$).

For the remaining $\beta$ we hence need to find the maximal extremal pairs that fall into the region $R_{\beta}$.

As it happens, the required extremal pairs for $R_{\beta}$ are defined using balanced words as per the following algorithm. For some rational $\gamma \in (1/(n+1), 1/n)$, denote by $u_\gamma$ the minimal shift of the balanced word formed from $\gamma$. Then create a Farey-like tree of words beginning with
0 and \( u_\gamma \) as the roots of this tree. Just as in a standard Farey tree, neighbours \( u_1 \) and \( u_2 \) such that \( u_1^\infty < u_2^\infty \) are combined to form \( u_2u_1 \).

As \( \gamma \in (1/(n + 1), 1/n) \), the resultant words (excepting roots 0 and \( u \)) must contain \( 0^n+1 \). Therefore, define \( s = 0^n+1\)-max and \( t = 0^n1\)-min. These by definition form an extremal pair. We claim that for the right combinations of \( \gamma \) and \( \beta \), these are the maximal pairs.

We begin by showing that if the pairs defined above fall into \( R_\beta \), then they must be maximal. We do this by induction, exploiting the tree structure of the definition.

**Lemma 4.1.** For any minimal cyclically balanced \( u \) associated to \( p/q \), consider the pair \( (s,t) = (0u^k, u0u^{k-1}) \). If \( \beta \) satisfies \( (s^\infty, t^\infty) \in R_\beta \), then \( J_\beta[s^\infty, t^\infty] = \emptyset \): that is to say, \( (s,t) \) is a maximal extremal pair.
Proof. Consider a point \( x \in (0, 1) \). We know by Corollary 4.3 that \( \mathcal{J}_{\beta}[0, \inf X_{\gamma}(\beta), \inf X_{\gamma}(\beta)] \) is empty, so we may restrict to \( x \) in this region. This region overlaps the hole under consideration, so restrict \( x \) to \( [\inf X_{\gamma}(\beta), s^\infty] = [\inf X_{\gamma}(\beta), (0u^k)^\infty] \). By applying the shift map we may restrict to \( \inf X_{\gamma}(\beta), (u^k0)^\infty \). Again this overlaps the hole, so restrict to \( [t^\infty, \sigma s^\infty] = [(u0u^{k-1})^\infty, (u^k0)^\infty] \). Then apply \( \sigma^i \) repeatedly and restricting each time, we may conclude that any point still remaining must itself be equal to \( t^\infty \). \( \square \)

Remark 4.2. Following the above proof it is also easy to see that pairs \((s, t) = (0u0^{k-1}, u0^k)\) must also be maximal extremal: notice that in this case it is simpler as \( t^\infty = \sigma s^\infty \).

Lemma 4.3. Suppose two maximal extremal pairs \((s_1, t_1)\) and \((s_2, t_2)\) are Farey neighbours in a tree generated by \( 0 \) and \( u \) for a minimal cyclically balanced word \( u \). Then the pair \((s, t) = (s_2s_1, t_1t_2)\) is also maximal.

Proof. This has been shown for the tree of balanced words in [11, Lemma 2.5] and again is more a property of the tree construction than of the specific words. We repeat the proof here for completeness’ sake. To show maximality of \((s, t)\), we aim to show that \( \mathcal{J}_\beta[s^\infty, t^\infty] = \emptyset \). Consider \( x \in (0, 1) \). We know by maximal extremality that \( \mathcal{J}_\beta[s^\infty_1, t^\infty_1] = \emptyset \), so we may restrict to \( x \in [s^\infty_1, t^\infty_1] \). We know by the tree construction that \( s^\infty_1 < s^\infty \) and \( t^\infty_1 < t^\infty \), so we may restrict to \( x \in [s^\infty_1, s^\infty] \). Then \( \sigma^{|s_1|}(x) \in [s^\infty_1, t^\infty_1] \), so restrict again. Continuing this process, we see that the only possible point avoiding \( [s^\infty, t^\infty] \) must be \( s^\infty_1 \). But then this shifts to \( t^\infty_1 \in [s^\infty, t^\infty] \). \( \square \)

The above two lemmas combine to imply that if \((s, t)\) is a suitable admissible extremal pair as described, then \((s, t)\) is maximal extremal.

We now discuss which \( \gamma \) are associated with which \( \beta \). We describe this in two ways: firstly, by giving the set of correct \( \beta \) for a particular \( \gamma \) and secondly by giving the correct \( \gamma \) in terms of a particular \( \beta \).

Lemma 4.4. Let \( w \) and \( u \) denote the maximal and minimal cyclic shifts of the balanced word associated to \( \gamma \in \mathbb{Q} \) with \( \gamma \neq 1/n \) and Farey parents given by \( \gamma_1 < \gamma_2 \). Then for

\[
1 \cdot [\in (w_{\gamma_1})^\infty, w_{\gamma_1}^\infty] \iff \gamma(\beta) \in [\gamma_1, \gamma),
\]

we have that the admissible pairs \((s, t)\) from the Farey tree formed by \( 0 \) and \( w_\gamma \) are maximal extremal pairs.

Proof. Firstly, notice that the given interval is precisely the region where we have \( \gamma(\beta) \in [\gamma_1, \gamma) \). Therefore at least part of the Farey tree generated by \( 0 \) and \( w_\gamma \) will be admissible and give pairs \((s, t)\) satisfying \([s^\infty, t^\infty] \subset R_\beta \). Outside of these values of \( \gamma(\beta) \) we have that either the entirety of the tree will be inadmissible or the sequences will fall below \( R_\beta \).
As 1 increases towards \( w_\gamma^\infty \), more and more of the tree from 0 and \( u_\gamma \) becomes admissible. Therefore for every \( k \), there exists \( \beta \) such that the pair \([0u^k, u0u^{k-1}]\) is an admissible extremal pair with \((s^\infty, t^\infty) \in R_\beta\), and so in the limit the entire tree gives maximal extremal pairs.

**Lemma 4.5.** Let \( \gamma(\beta) \) be rational with continued fraction expansion \([0; a_1, a_2, \ldots, a_n]\), with \( a_n > 1 \). Then \( \gamma(\beta) \in [\gamma_1, \gamma] \) if and only if \( \gamma \) has continued fraction expansion given by \([0; a_1, \ldots, a_{2k+1}]\) with \( 2k + 1 < n \) or by \([0; a_1, \ldots, a_n - 1, 1, k]\) for \( k \geq 1 \) if \( n \) is odd and \([0; a_1, \ldots, a_n, k]\) for \( k \geq 1 \) if \( n \) is even.

**Proof.** As is well known, the odd convergents in the continued fraction expansion of a number give a decreasing sequence of overapproximations of that number. It is also well known that given a rational \( \gamma \) with continued fraction expansion \([0; b_1, \ldots, b_n]\) with \( b_n > 1 \), its Farey parents are given by \([0; b_1, \ldots, b_{n-1}]\) and \([0; b_1, \ldots, b_n - 1]\). If \( n \) is odd, then \([0; b_1, \ldots, b_{n-1}]\) must therefore be the left Farey parent. Therefore it follows immediately that the odd convergents of \( \gamma(\beta) \) will satisfy \( \gamma(\beta) \in [\gamma_1, \gamma] \). Those given by \([0; a_1, \ldots, a_n - 1, 1, k]\) for \( n \) odd or \([0; a_1, \ldots, a_n, k]\) for \( n \) even are then further overapproximations.

The case \( \gamma = 1/n \) is excluded simply because it becomes subsumed in other cases: for example, the word 000101 may be consider as 00(01)² with \( \gamma = 1/2 \) or as 0(00101) with \( \gamma = 2/5 \).

It remains to explain why these intervals cover almost all of \( R_\beta \). To see this, suppose that

\[
1 \in [(w_{\gamma_1})^\infty, w_{\gamma}^\infty].
\]

Then for almost every \( \beta \) in this range there exists a maximal pair \((s,t)\) in the tree from 0 and \( u_\gamma \) such that \( s^\infty \) is admissible and \( st^\infty \) is inadmissible. Then consider the greatest admissible sequence in \([s^\infty, st^\infty]\). This will end in 1· so may be rewritten as a finite sequence.

**Lemma 4.6.** The greatest admissible finite sequence for \( \gamma \) described above is equal to 0u_{\gamma_2}.

**Proof.** \( s^\infty \) is admissible so clearly the sequence \( st^\infty \) becomes inadmissible with the very first \( t \). Therefore, to be admissible we should truncate from the maximal shift of \( s \) and replace the preceding 0 with a 1. We know that \( s \) begins \( 0u_\gamma \) and must be the maximal shift beginning this way, so consider \( u_\gamma = u_1 \ldots u_q = u_{\gamma_1}u_{\gamma_2} \). There exists \( k \) such that \( \sigma^k u_\gamma \) begins \( w_{\gamma_2} \), which will be the point at which to truncate the sequence. Then because \( u_\gamma \) is balanced, we have that \( u_1 \ldots u_{k-2}1 = u_{\gamma_2}^3 \). This proves the lemma.

The cases where \( s^\infty \), \( st^\infty \) or a limit point of the tree end in 1· and so may themselves be rewritten as finite sequences are similar.

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\(^3\)This follows from Lothaire [3, Proposition 2.2.2].
The descendants of the above maximal pairs will be given by taking an $n$-th level Sturmian pair $(s_r, t_r)$ and applying the map $m : 0 \rightarrow 0$, $1 \rightarrow u_\gamma$. This completes the description of $D_1(\beta)$.

Figure 4 shows an approximation to $D_1(\beta)$ in the region $R_\beta$ for $\beta \approx 1.427$. For this value of $\beta$, the region $R_\beta$ is very small, and $D_1(\beta)$ is once again too small to see distinctly.

5. Summary

We summarise the results in the following theorem.
Theorem 5.1. Let $\beta \in (1,2)$ satisfy $1/2 \in [1/(n + 1), 1/n)$. Then as depicted in Figure 2, we have that

- For any $a > 1/\beta$, $(a, b) \in D_2(\beta)$;
- For any $b < \inf X_{\gamma(\beta)}$, $(a, b) \in D_2(\beta)$;
- The boundary of $D_0(\beta)$ is given by joining points $(s^\infty, ts^\infty)$, $(s^\infty, t^\infty)$ and $(st^\infty, t^\infty)$ where $(s, t)$ are given by maximal extremal pairs for $\beta$;
- The boundary of $D_1(\beta)$ is given by joining the points $(s^\infty, ts^\infty)$ to $(sts^\infty, ts^\infty)$ and the points $(st^\infty, tst^\infty)$ to $(st^\infty, t^\infty)$, where $(s, t)$ are given by Farey descendants of maximal extremal pairs for $\beta$;
- The boundary of $D_2(\beta)$ is given by joining points $(s^\infty, ts^\infty)$, $(st^\infty, ts^\infty)$, and $(st^\infty, t^\infty)$ where $(s, t)$ are given by maximal extremal pairs for $\beta$.

The maximal extremal pairs for $\beta$ are given by

- Shifts of the balanced word $w_\gamma$ whenever $\gamma \leq \gamma(\beta)$, given by $(0^k\text{-max}, 0^{k-1}\text{-min})$ for $0 < k \leq n$;
- Pairs $(0^n\text{-max}, 0^{n-1}\text{-min})$ formed from admissible words taken from a Farey tree with roots $0$ and $u_\gamma$, where $u_\gamma$ is the minimal shift of the balanced word corresponding to $\gamma \neq 1/k$ and $\gamma$ satisfies $\gamma(\beta) \in [\gamma_1, \gamma_2]$.

If a maximal extremal pair $(s, t)$ has $s^\infty$ admissible and $st^\infty$ inadmissible, then the above results hold with any inadmissible sequences replaced by the greatest admissible sequence in $[s^\infty, st^\infty]$.

In summary, the boundaries of $D_i(\beta)$ consist of a countable set of plateaus which are closely linked to the set of maximal extremal pairs for $\beta$, as explained in Section 3. The maximal extremal pairs are mainly balanced words, but for almost every $\beta \in (1,2)$ there is a small region where there are no admissible balanced words. In this region the maximal extremal pairs are formed by taking certain inadmissible balanced words and adding a 0 to make them admissible, as described in Section 4. We include some pictures of $D_i(\beta)$ for different values of $\beta$; see Figures 5 and 6.

We note the following result.

Lemma 5.2. Let $\gamma(\beta) \in [1/(n + 1), 1/n)$. If $(a, b) \in D_i(\beta)$ then $b - a < C_i(\beta)$ where

$$C_0(\beta) = \frac{\beta^{n-1}(\beta - 1)}{\beta^{n+1} - 1},$$

$$C_1(\beta) = C_2(\beta) = \frac{\beta - 1}{\beta^2}.$$
Figure 5. $D_2(\beta)$ (dark grey), $D_1(\beta)$ (white + dark grey) and $D_0(\beta)$ (light grey + white + dark grey) for the region $I_\beta$ with $1 \cdot = (10010000)\infty$, $\beta \approx 1.427$, $\gamma(\beta) = (1/4, 1/2)$.

Proof. Each $C_i(\beta)$ is given by
\[
\sup\{b - a : (a, b) \in D_i(\beta)\} = \max\{b - a : (a, b) \text{ is a corner of } D_i(\beta)\}.
\]
For $D_1(\beta)$ and $D_2(\beta)$ this is equal to
\[
\max\{ts^\infty - s^\infty : (s, t) \text{ is maximal extremal for } \beta\}.
\]
Then $ts^\infty - s^\infty = t - s$. Recall that $t = u1\text{-min}$ and $s = u0\text{-max}$ for some word $u$. Therefore $t - s$ will be maximised when $u$ is the empty word. Then the only maximal extremal pairs with $s$ beginning 0 and $t$ beginning 1 are balanced pairs. For any balanced pair, we have $t = 10w$.
and $s = 01w$ for some word $w$. Thus $t - s = 10 - 01 = 1/\beta - 1/\beta^2$, giving the required result for $C_1(\beta)$ and $C_2(\beta)$.

For $C_0(\beta)$, we need to maximise $t^\infty - s^\infty$ and so similarly may conclude that $t = 10w$ and $s = 01w$. This quantity will clearly be maximised when $w$ is as short as possible. Therefore given $\gamma(\beta) \in [1/(n + 1), 1/n)$, we see that this will be maximised when $(s, t) = (10^n, 010^{n-1})$ as this is the shortest possible admissible $w$. Then $t^\infty - s^\infty$ gives $C_0(\beta)$ as stated.

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REFERENCES

[1] P. Boyland, A. de Carvalho, and T. Hall. On digit frequencies in $\beta$-expansions, 2013. Preprint, see http://arxiv.org/abs/1308.4437.
[2] S. Bullett and P. Sentenac. Ordered orbits of the shift, square roots, and the devil’s staircase. Math. Proc. Camb. Phil. Soc., 115(03):451–481, 1994.
[3] P. Glendinning and N. Sidorov. The doubling map with asymmetrical holes. Ergodic Theory and Dynamical Systems, FirstView:1–21, 2014. Available on CJO2013. doi:10.1017/etds.2013.98.
[4] P. Glendinning and C. T. Sparrow. Prime and renormalisable kneading invariants and the dynamics of expanding Lorenz maps. Phs. D, 62:22–50, 1993.
[5] L. Goldberg and C. Tresser. Rotation orbits and the Farey tree. Ergodic Theory and Dynamical Systems, 16:1011–1029, 1996.
[6] K. G. Hare and N. Sidorov. On cycles for the doubling map which are disjoint from an interval. Monatsh. Math., 175:347–365, 2014.
[7] J. H. Hubbard and C. T. Sparrow. The classification of topologically expansive Lorenz maps. Comm. Pure Appl. Math., 43:431–443, 1990.
[8] M. Lothaire. Algebraic combinatorics on words, volume 90 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2002.
[9] W. Parry. On the $\beta$-expansions of real numbers. Acta Math. Acad. Sci. Hung., 11:401–416, 1960.
[10] A. Rényi. Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hung., 8:477–493, 1957.
[11] N. Sidorov. Supercritical holes for the doubling map. Acta Mathematica Hungarica, 143:298–312, 2014.
[12] L. Vuillon. Balanced words. Bull. Belg. Math. Soc. Simon Stevin, 10(5):787–805, 2003.

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