Joint universality of periodic zeta-functions with multiplicative coefficients. II

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Received: July 19, 2020 / Revised: February 19, 2021 / Published online: May 1, 2021

Abstract. In the paper, a joint discrete universality theorem for periodic zeta-functions with multiplicative coefficients on the approximation of analytic functions by shifts involving the sequence \(\{\gamma_k\}\) of imaginary parts of nontrivial zeros of the Riemann zeta-function is obtained. For its proof, a weak form of the Montgomery pair correlation conjecture is used. The paper is a continuation of [A. Laurinčikas, M. Teke, Joint universality of periodic zeta-functions with multiplicative coefficients, Nonlinear Anal. Model. Control, 25(5):860–883, 2020] using nonlinear shifts for approximation of analytic functions.

Keywords: joint universality, nontrivial zeros of the Riemann zeta-function, periodic zeta-function, space of analytic functions, weak convergence.

1 Introduction

It is well known that some zeta- and \(L\)-functions, and even some classes of Dirichlet series, for example, the Selberg-Steuding class, see [29, 32], are universal in the Voronin sense, i.e., a wide class of analytic functions can be approximated by one and the same zeta-function. For example, in the case of the Riemann zeta-function \(\zeta(s), s = \sigma + it\), analytic nonvanishing functions on the strip \(D = \{s \in \mathbb{C}: 1/2 < \sigma < 1\}\) are approximated by shifts \(\zeta(s + i\tau), \tau \in \mathbb{R}\) (continuous case), or shifts \(\zeta(s + ikh), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, h > 0\) (discrete case); see [1, 6, 13, 24, 32].

The above shifts are very simple, \(\tau\) and \(kh\) occur in them linearly. It turned out that the approximation remains valid also with more general shifts. A significant progress in this direction was made by Pańkowski [31] using the shifts \(\zeta(s + i\varphi(\tau))\) and \(\zeta(s + i\varphi(k))\)

\textsuperscript{1}The research of the author is funded by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).

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with \( \varphi(\tau) = \tau^\alpha \log^\beta \tau \) and a wide class of reals \( \alpha \) and \( \beta \). The papers [22] and [35] are also devoted to approximation of analytic functions by generalized shifts of zeta-functions. In [5], the shifts \( \zeta(s + i\gamma_k) \) were applied, where \( \{\gamma_k: k \in \mathbb{N}\} = \{\gamma_k: 0 < \gamma_1 < \cdots < \gamma_k < \gamma_{k+1} < \cdots\} \) is the sequence of imaginary parts of nontrivial zeros of the Riemann zeta-function.

Universality in the Voronin sense also has its joint version. In the joint case, a collection of analytic functions is approximated simultaneously by a collection of shifts of zeta- or \( L \)-functions. The first joint universality theorem belongs to Voronin who proved [36] the joint universality of Dirichlet \( L \)-functions \( L(s, \chi_j), j = 1, \ldots, r \). Obviously, in joint universality theorems, the approximating shifts must be in some sense independent. Voronin required [36] for this the pairwise nonequivalence of Dirichlet characters, i.e., in fact, he considered joint universality of different Dirichlet \( L \)-functions. On the other hand, as it was observed by Pańkowski [31], the independence of approximating shifts of Dirichlet \( L \)-functions can be ensured by different functions \( \varphi_j(\tau) \) in shifts \( L(s + i\varphi_j(\tau), \chi_j) \) or \( L(s + i\varphi_j(k), \chi_j) \) even with the same characters \( \chi_j \). This observation extends significantly classes of jointly universal functions. For example, the joint universality with generalized shifts was obtained in [16] and [20].

In general, joint universality of zeta-functions was widely studied, and many results are known; see, for example, general results obtained in [7–11,14,26,30] and other papers by authors of the mentioned works. In this note, we focus on joint universality of so-called periodic zeta-functions with generalized shifts involving the sequence \( \{\gamma_k: k \in \mathbb{N}\} \) of imaginary parts of nontrivial zeros of the function \( \zeta(s) \). We will mention some joint universality results involving the latter sequence. Note that the behaviour of the sequence \( \{\gamma_k\} \), as of nontrivial zeros of \( \zeta(s) \), is very complicated, and at the moment, its known properties are not sufficient for the proof of universality. Therefore, in [5], the conjecture that, for \( c > 0 \),

\[
\sum_{\gamma_k, \gamma_l \leq T} 1 \ll T \log T \quad \text{if} \quad |\gamma_k - \gamma_l| < c/\log T
\]

was introduced. This conjecture is inspired by the Montgomery pair correlation conjecture [28] that

\[
\sum_{2\pi \alpha_1/\log T \leq \gamma_k - \gamma_l \leq 2\pi \alpha_2/\log T} 1 \sim \left( \int_{\alpha_1}^{\alpha_2} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right) du + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T,
\]

where \( \alpha_1 < \alpha_2 \) are arbitrary real numbers, and

\[
\delta(\alpha_1, \alpha_2) = \begin{cases} 1 & \text{if } 0 \in [\alpha_1, \alpha_2], \\ 0 & \text{otherwise}. \end{cases}
\]

Now we will state a joint universality theorem for Dirichlet \( L \)-functions involving the sequence \( \{\gamma_k\} \) obtained in [18]. Denote by \( K \) the class of compact subsets of the strip \( D \) with connected complements, and by \( H_0(K) \) with \( K \in K \) the class of continuous nonvanishing functions on \( K \) that are analytic in the interior of \( K \).
Theorem 1. Suppose that \( \chi_1, \ldots, \chi_r \) are pairwise nonequivalent Dirichlet characters, and estimate (1) is true. For \( j = 1, \ldots, r \), let \( K_j \in \mathcal{K} \) and \( f_j(s) \in H_0(K_j) \). Then, for every \( \varepsilon > 0 \) and \( h > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| L(s + ih\gamma_k, \chi_j) - f_j(s) \right| < \varepsilon \right\} > 0.
\]

Moreover “\( \liminf \)” can be replaced by “\( \lim \)” for all but at most countably many \( \varepsilon > 0 \).

Here \( \# A \) denotes the cardinality of the set \( A \), and \( N \) runs over the set \( \mathbb{N} \).

Now we recall the definition of the periodic zeta-function, which is an object of investigation of the present note. Let \( a = \{a_m : m \in \mathbb{N}\} \) be a periodic sequence of complex numbers with minimal period \( q \in \mathbb{N} \). Then the periodic zeta-function \( \zeta(s; a) \) is defined, for \( \sigma > 1 \), by the Dirichlet series

\[
\zeta(s; a) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}
\]

and has an analytic continuation to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue

\[
\frac{1}{q} \sum_{l=1}^{q} a_l.
\]

The sequence \( a \) is called multiplicative if \( a_1 = 1 \) and \( a_{mn} = a_m a_n \) for all coprimes \( m, n \in \mathbb{N} \). If \( 0 < \alpha \leq 1 \) is a fixed number, then the function

\[
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}, \quad \sigma > 1,
\]

and its meromorphic continuation are called the periodic Hurwitz zeta-function. In [15] and [3], under hypothesis (1), joint universality theorems involving sequence \( \{\gamma_k\} \) for the pair consisting from the Riemann and Hurwitz zeta-functions and their periodic analogues, respectively, were obtained, while in [23], such theorems were proved for Hurwitz zeta-functions.

For \( j = 1, \ldots, r \), let \( a_j = \{a_{jm} : m \in \mathbb{N}\} \) be a periodic sequences of complex numbers with minimal period \( q_j \in \mathbb{N} \), and let \( \zeta(s; a_j) \) be the corresponding zeta-function. The main result of the paper is the following theorem.

Theorem 2. Suppose that the sequences \( a_1, \ldots, a_r \) are multiplicative, \( h_1, \ldots, h_r \) are positive algebraic numbers linearly independent over the field of rational numbers, and estimate (1) is true. For \( j = 1, \ldots, r \), let \( K_j \in \mathcal{K} \) and \( f_j(s) \in H_0(K_j) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| \zeta(s + ih_j\gamma_k; a_j) - f_j(s) \right| < \varepsilon \right\} > 0.
\]

Moreover “\( \liminf \)” can be replaced by “\( \lim \)” for all but at most countably many \( \varepsilon > 0 \).

In [21], joint continuous universality theorems for periodic zeta-functions with shifts defined by means of certain differentiable functions were obtained.
2 The sequence \( \{ \gamma_k \} \)

From the functional equation for the Riemann zeta-function
\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s)
\]
it follows that \( \zeta(-2m) = 0 \) for all \( m \in \mathbb{N} \), and the zeros \( s = -2m \) of \( \zeta(s) \) are called trivial. Moreover, it is known that \( \zeta(s) \) has infinitely many of so-called complex nontrivial zeros \( \rho_k = \beta_k + i \gamma_k \) lying in the strip \( \{ s \in \mathbb{C}: 0 < \sigma < 1 \} \). The famous Riemann hypothesis, one of seven Millennium problems, asserts that \( \beta_k = 1/2 \), i.e., all nontrivial zeros lie on the critical line \( \sigma = 1/2 \). There exists a conjecture that all nontrivial zeros of \( \zeta(s) \) are simple.

We recall some properties of the sequence \( \{ \gamma_k \} \) with \( k \in \mathbb{N} \).

By the definition, a sequence \( \{ x_k : k \in \mathbb{N} \} \subset \mathbb{R} \) is called uniformly distributed modulo 1, if, for every subinterval \( (a, b] \subset (0, 1] \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} I_{(a,b]}(\{ x_k \}) = b - a,
\]
where \( I_{(a,b]} \) is the indicator function of \( (a, b] \), and \( \{ u \} \) denotes the fractional part of \( u \in \mathbb{R} \). Though the sequence \( \{ \gamma_k \} \) is distributed irregularly, the following statement is true for it.

**Lemma 1.** The sequence \( \{ \gamma_k : k \in \mathbb{N} \} \) with every \( a \in \mathbb{R} \setminus \{ 0 \} \) is uniformly distributed modulo 1.

**Proof.** Proof of the lemma is given in [33], and in the above form, was applied in [5].

For convenience, we recall the Weyl criterion on the uniform distribution modulo 1; see, for example, [12].

**Lemma 2.** A sequence \( \{ x_k : k \in \mathbb{N} \} \subset \mathbb{R} \) is uniformly distributed modulo 1 if and only if, for every \( m \in \mathbb{Z} \setminus \{ 0 \} \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.
\]

Obviously, the uniform distribution modulo 1 of the sequence shows its nonlinear character.

The following statement is well known; see, for example, [34].

**Lemma 3.** For \( k \to \infty \),
\[
\gamma_k = \frac{2\pi k}{\log k} \left( 1 + o(1) \right).
\]
3 Limit theorems

Denote by $H(D)$ the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta. We will derive Theorem 2 from a limit theorem on the weak convergence of probability measures in the space

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r}.$$  

Therefore, we start with a certain probability model.

Let $\mathcal{B}(\mathbb{X})$ be the Borel $\sigma$-field of the space $\mathbb{X}$, and $\mathbb{P}$ denote the set of all prime numbers. Define

$$\Omega = \prod_{p \in \mathbb{P}} \mathbb{X}_p,$$

where $\mathbb{X}_p = \{ s \in \mathbb{C} : |s| = 1 \}$ for all $p \in \mathbb{P}$. Then $\Omega$ is a compact topological Abelian group. Moreover, let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then again $\Omega^r$ is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure $m^r_H$ can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m^r_H)$. Denote by $\omega(p)$ the $p$th component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \ldots, r$. For brevity, let $\omega = (\omega_1, \ldots, \omega_r) \in \Omega^r$, $\omega_1 \in \Omega_1, \ldots, \omega_r \in \Omega_r$, $\underline{a} = (a_1, \ldots, a_r)$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m^r_H)$, define the $H^r(D)$-valued random element

$$\zeta(s, \omega; \underline{a}) = \left( \zeta(s, \omega_1; a_1), \ldots, \zeta(s, \omega_r; a_r) \right),$$

where

$$\zeta(s, \omega_j; a_j) = \prod_{p \in \mathbb{P}} \left( 1 + \sum_{l=1}^{\infty} \frac{a_{jp} \omega_j^l(p)}{p^l s} \right), \quad j = 1, \ldots, r.$$  

Note that the latter products, for almost all $\omega_j$, are uniformly convergent on compact subsets of the strip $D$. Since the periodic sequences $a_j$, $j = 1, \ldots, r$, are bounded, the proofs of the above assertions completely coinside with those of Lemma 5.1.6 and Theorem 5.1.7 from [13]. More general results are given in [1]. Denote by $P_\zeta$ the distribution of the random element $\zeta(s, \omega; \underline{a})$, i.e.,

$$P_\zeta(A) = m^r_H \{ \omega \in \Omega^r : \zeta(s, \omega; \underline{a}) \in A \}, \quad A \in \mathcal{B}(H^r(D)).$$

Put $\underline{h} = (h_1, \ldots, h_r)$, and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_N(A) = \frac{1}{N} \# \{ 1 \leq k \leq N : \zeta(s + i\gamma_k \underline{h}; \underline{a}) \in A \},$$

where

$$\zeta(s; \underline{a}) = \left( \zeta(s; a_1), \ldots, \zeta(s; a_r) \right).$$

In this section, we will prove the following limit theorem.
Theorem 3. Suppose that the sequences $a_1, \ldots, a_r$ are multiplicative, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate (1) is valid. Then $P_N$ converges weakly to $P_\infty$ as $N \to \infty$.

We start the proof of Theorem 3, as usual, with a limit lemma in the space $\Omega^r$. In this lemma, the uniform distribution modulo 1 of the sequence $\{\gamma_k a\}, a \in \mathbb{R} \setminus \{0\}$, and the property of the numbers $h_1, \ldots, h_r$ essentially are applied.

For $A \in B(\Omega^r)$, define

$$Q_N(A) = \frac{1}{N} \# \{1 \leq k \leq N: ((p^{-ih_1 \gamma_k}: p \in \mathbb{P}), \ldots, (p^{-ih_r \gamma_k}: p \in \mathbb{P})) \in A \}.$$

Before the statement of a limit theorem for $Q_N$, we recall one result of Diophantine type.

Lemma 4. Suppose that $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ are algebraic numbers such that the logarithms $\log \lambda_1, \ldots, \log \lambda_r$ are linearly independent over $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \ldots, \beta_r$, not all zero, we have

$$|\beta_0 + \beta_1 \log \lambda_1 + \cdots + \beta_r \log \lambda_r| > H^{-C},$$

where $H$ is the maximum of the heights of $\beta_0, \beta_1, \ldots, \beta_r$, and $C$ is an effectively computable number depending on $r$ and the maximum of the degrees of $\beta_0, \beta_1, \ldots, \beta_r$.

The lemma is the well-known Baker theorem on logarithm forms; see, for example [2].

Lemma 5. Suppose that $h_1, \ldots, h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then $Q_N$ converges weakly to the Haar measure $m_H$ as $N \to \infty$.

Proof. As usual, we apply the Fourier transform method. The characters of the group $\Omega^r$ are of the form

$$\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_j p}(p),$$

where the star “*” shows that only a finite number of integers $k_{jp}$ are distinct from zero. Therefore, the Fourier transform of $Q_N$ is

$$g_N(k_1, \ldots, k_r) = \int_{\Omega^r} \left( \prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_j p}(p) \right) \, dQ_N,$$

where $k_j = (k_{jp}: k_{jp} \in \mathbb{Z}, p \in \mathbb{P}), j = 1, \ldots, r$. Thus, by the definition of $Q_N$,

$$g_N(k_1, \ldots, k_r) = \frac{1}{N} \sum_{k=1}^N \prod_{j=1}^r \prod_{p \in \mathbb{P}}^* p^{-ih_j k_{jp} \gamma_k}$$

$$= \frac{1}{N} \sum_{k=1}^N \exp \left\{ -i \gamma_k \sum_{j=1}^r h_j \sum_{p \in \mathbb{P}}^* k_{jp} \log p \right\}. \quad (2)$$

Nonlinear Anal. Model. Control, 26(3):550–564
Obviously,
\[ g_N(0, \ldots, 0) = 1. \] (3)

Now, suppose that \( k \neq (0, \ldots, 0) \). Then there exists \( j \in \{1, \ldots, r\} \) such that \( k_j \neq 0 \). Thus, there exists a prime number \( p \) such that \( k_j p \neq 0 \). Define
\[ a_p = \sum_{j=1}^{r} h_j k_j p. \]

Then, in view of a property of the numbers \( h_1, \ldots, h_r \), we have \( a_p \neq 0 \). The numbers \( a_p \) are algebraic, and the set \( \{\log p: p \in \mathbb{P}\} \) is linearly independent over \( \mathbb{Q} \). Therefore, by Lemma 4,
\[ \sum_{j=1}^{r} h_j \sum_{p \in \mathbb{P}}^{*} k_j p \log p = \sum_{p \in \mathbb{P}}^{*} a_p \log p \neq 0. \]

Hence, in virtue of Lemma 1, the sequence
\[ \left\{ \frac{1}{2\pi} \gamma_k a_{k_1, \ldots, k_r}; k \in \mathbb{N} \right\} \]
is uniformly distributed modulo 1. This, together with (2) and Lemma 2, shows that, in the case \((k_1, \ldots, k_r) \neq (0, \ldots, 0)\),
\[ \lim_{N \to \infty} g_N(k_1, \ldots, k_r) = 0. \]

Thus, in view of (3),
\[ \lim_{N \to \infty} g_N(k_1, \ldots, k_r) = \begin{cases} 1 & \text{if } (k_1, \ldots, k_r) = (0, \ldots, 0), \\ 0 & \text{if } (k_1, \ldots, k_r) \neq (0, \ldots, 0), \end{cases} \]
and the lemma is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure \( m^r_H \).

Lemma 5 implies a limit lemma in the space \( H^r(D) \) for absolutely convergent Dirichlet series. Let, for a fixed \( \theta > 1/2 \),
\[ v_n(m) = \exp \left\{ - \left( \frac{m}{n} \right)^{\theta} \right\}, \quad m, n \in \mathbb{N}, \]
and
\[ \zeta_n(s; a_j) = \sum_{m=1}^{\infty} \frac{a_j m v_n(m)}{m^s}, \quad j = 1, \ldots, r. \]

Then the latter series are absolutely convergent for \( \sigma > 1/2 \). Actually, since \( v_n(m) \ll m^{-L/n^{\theta}} \) with every \( L > 0 \), the latter series are absolutely convergent even in the whole
complex plane. For $B(H^r(D))$, define

$$V_{N,n}(A) = \frac{1}{N} \# \{ 1 \leq k \leq N : \zeta_n(s + \imath h_k; \mathfrak{a}) \in A \},$$

where

$$\zeta_n(s; \mathfrak{a}) = (\zeta_n(s; \mathfrak{a}_1), \ldots, \zeta_n(s; \mathfrak{a}_r)).$$

Moreover, let

$$\zeta_n(s, \omega; \mathfrak{a}_j) = \sum_{m=1}^{\infty} a_{jm} \omega_j(m)v_n(m) m^{-s}, \quad j = 1, \ldots, r,$$

$$\zeta_n(s, \omega; \mathfrak{a}) = (\zeta_n(s, \omega_1; \mathfrak{a}_1), \ldots, \zeta_n(s, \omega_r; \mathfrak{a}_r)),$$

and let $u_n : \Omega^r \to H^r(D)$ be given by the formula

$$u_n(\omega) = \zeta_n(s, \omega; \mathfrak{a}).$$

**Lemma 6.** Suppose that $h_1, \ldots, h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then $V_{N,n}$, as $N \to \infty$, converges weakly to a measure $V_n = \lim_{n \to \infty} m_H u_n^{-1}$, where

$$m_H u_n^{-1}(A) = m_H (u_n^{-1} A), \quad A \in B(H^r(D)).$$

**Proof.** Since the series for $\zeta_n(s, \omega_j; \mathfrak{a}_j)$ are absolutely convergent for $\sigma > 1/2$, the function $u_n$ is continuous, hence $(B(\Omega^r), B(H^r(D)))$-measurable. Therefore, the measure $V_n$ is defined correctly. The definitions of $Q_N$, $V_{N,n}$, and $u_n$ imply the equality $V_{N,n} = Q_N u_n^{-1}$. Therefore, the lemma follows from Lemma 5 and a preservation of weak convergence under continuous mappings; see [4, Thm. 5.1].

The limit measure $V_n$ in Lemma 6 is independent on $h$ and $\{\gamma_k\}$ and has a good convergence property, which is the next lemma.

**Lemma 7.** Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative. Then $V_n$ converges weakly to $P_\varLambda$ as $n \to \infty$.

**Proof.** In [17], the weak convergence for

$$\hat{P}_T(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta(s + \imath \tau; \mathfrak{a}) \in A \}, \quad A \in B(H^r(D)),$$

was considered, and it was obtained its weak convergence to $P_\varLambda$ as $T \to \infty$, and that $V_n$ also converges weakly to $P_\varLambda$ as $n \to \infty$. In other words, $V_n$ and $\hat{P}_T$ have the same limit measure $P_\varLambda$. \(\square\)

In view of Lemma 7, to prove Theorem 3, it suffices to show that $P_N$, as $N \to \infty$, and $V_n$, as $n \to \infty$, have a common limit measure. For this, a certain closeness of $\zeta(s; \mathfrak{a})$ and $\zeta_n(s; \mathfrak{a})$ is needed.

There exists a sequence $\{ K_l : l \in \mathbb{N} \} \subset D$ of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$
A. Laurinčikas et al.

\[ K_l \subset K_{l+1}, \text{ for all } l \in \mathbb{N}, \text{ and if } K \subset D \text{ is a compact set, then } K \subset K_l \text{ for some } l. \]

Then, putting, for \( g_1, g_2 \in H(D) \),

\[
\rho(g_1; g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},
\]

we have a metric in \( H(D) \) inducing its topology of uniform convergence on compacta. Hence,

\[
\rho(g_1; g_2) = \max_{1 \leq j \leq r} \rho(g_{1j}; g_{2j}),
\]

\( g_1 = (g_{11}, \ldots, g_{1r}), g_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D), \)

is a metric in \( H^r(D) \) inducing its product topology. Note that, in the proof of the next lemma, the multiplicativity of the sequences \( a_j, j = 1, \ldots, r \), is not used.

**Lemma 8.** Suppose that estimate (1) is true. Then, for every positive \( h_1, \ldots, h_r \) and \( a_1, \ldots, a_r \),

\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \rho(\zeta(s + ih_\gamma k; a), \zeta_n(s + ih_\gamma k; a)) = 0.
\]

**Proof.** By the definitions of the metrics \( \rho \) and \( \rho \), it is sufficient to show that, for every compact set \( K \subset D \),

\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} \left| \zeta(s + ih_j \gamma k; a_j) - \zeta_n(s + ih_j \gamma k; a_j) \right| = 0,
\]

\( j = 1, \ldots, r. \) The equality of type (5) was already used in [3], therefore, only for fullness, we give remarks on its proof.

Thus, let \( h > 0 \) and \( a \) be arbitrary. We consider \( \zeta(s + ih \gamma k; a) \) and \( \zeta_n(s + ih \gamma k; a) \). Let \( \theta \) be as in the definition of \( v_n(m) \). Then the representation

\[
\zeta_n(s; a) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z; a) l_n(z) \frac{dz}{z},
\]

where

\[
l_n(z) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) n^z,
\]

is valid. Hence, for \( \theta_1 < 0, \)

\[
\zeta_n(s; a) - \zeta(s; a) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s + z; a) l_n(z) \frac{dz}{z} + R_n(s; a),
\]

http://www.journals.vu.lt/nonlinear-analysis
where
\[ R_n(s; a) = \frac{a l_n(1 - s)}{1 - s}, \]
and \( a \) is the residue of \( \zeta(s; a) \) at the point \( s = 1 \). Let \( K \subset D \) be an arbitrary compact set, and \( \varepsilon > 0 \) be such that \( 1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon \) for \( s \in K \). Then, in view of (6), for \( s = \sigma + iv \in K \),
\[ |\zeta_n(s; a) - \zeta(s; a)| \ll \int_{-\infty}^{\infty} \left| \zeta(s - \theta_1 + it; a) \right| \left| \frac{l_n(-\theta_1 + it)}{|-\theta_1 + it|} \right| dt + |R_n(s; a)|. \]

Hence, taking \( t \) in place of \( t + v \) and \( \theta_1 = \sigma - \varepsilon - 1/2 \), we have
\[ \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s + ih\gamma_k; a) - \zeta_n(s + ih\gamma_k; a)| \ll I + Z, \quad (7) \]

where
\[ I = \int_{-\infty}^{\infty} \left( \frac{1}{N} \sum_{k=1}^{N} \left| \zeta\left(\frac{1}{2} + \varepsilon + ih\gamma_k + it; a\right) \right| \right) \sup_{s \in K} \left| \frac{l_n(1/2 + \varepsilon - s + it)}{1/2 + \varepsilon - s + it} \right| dt \]
and
\[ Z = \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |R_n(s + ih\gamma_k; a)|. \]

Estimate (1) is applied for estimation of the first factor of the integrated function in the integral \( I \). It is well known that, for \( \tau \in \mathbb{R} \),
\[ \int_{0}^{T} \left| \zeta\left(\frac{1}{2} + \varepsilon + i\tau + it; a\right) \right|^2 dt \ll_{\varepsilon} T(1 + |\tau|). \quad (8) \]
The same estimate is also true for the derivative of \( \zeta(s; a) \). Let \( \delta = c h(\log \gamma_N)^{-1} \) and
\[ N_\delta(h\gamma_k) = \sum_{\gamma_k, \gamma_l \leq \gamma_N, |\gamma_l - \gamma_k| < \delta} 1. \]

Then, in view of (1) and Lemma 3,
\[ \sum_{k=1}^{N} N_\delta(h\gamma_k) = \sum_{\gamma_k, \gamma_l \leq \gamma_N} \sum_{|\gamma_k - \gamma_l| < c(\log \gamma_N)^{-1}} 1 \ll \gamma_N \log \gamma_N \ll N. \]
This, (6) and an application of the Gallagher lemma connecting discrete and continuous mean squares for some function, see Lemma 1.4 of [27], give

\[
\sum_{k=1}^{N} \left| \zeta \left( \frac{1}{2} + \varepsilon + i h\gamma_k + it; a \right) \right| \leq \left( \sum_{k=1}^{N} N\delta(h\gamma_k) \sum_{k=1}^{N} N^{-1}\delta(h\gamma_k) \left| \zeta \left( \frac{1}{2} + \varepsilon + i h\gamma_k + it; a \right) \right|^{2} \right)^{1/2} \leq N^{1/2} \left( \frac{1}{\delta} \int_{h\gamma_1}^{h\gamma_N} \left| \zeta \left( \frac{1}{2} + \varepsilon + i \tau + it; a \right) \right|^{2} d\tau \right)^{1/2} \leq \varepsilon, h, K N \left( 1 + |t| \right).
\]

Therefore, the classical estimate for the gamma-function and the definition of \( l_n(s) \) show that

\[
I \ll \varepsilon, h, K n^{-\varepsilon} \quad \text{and} \quad Z \ll h, K n^{1/2 - 2\varepsilon} \log N \frac{N}{N}.
\]

This, together with (7), proves (5), thus (4).

\[ \square \]

**Proof of Theorem 3.** We will use the random element language. Denote by \( X_n = X_n(s) \) the \( H^r(D) \)-valued random element having the distribution \( V_n \), where \( V_n \) is the limit measure in Lemma 6. Then, by Lemma 7,

\[
X_n \xrightarrow{D_{n \to \infty}} P_{\zeta}, \quad \text{(9)}
\]

where \( D \) means the convergence in distribution. Now, let the random variable \( \eta_N \) be defined on a certain probability space with a measure \( \mu \), and

\[
\mu \{ \eta_N = \gamma_k \} = \frac{1}{N}, \quad k = 1, \ldots, N.
\]

Define the \( H^r(D) \)-valued random element

\[
X_{N,n} = X_{N,n}(s) = \zeta_n(s + i h\eta_N; a).
\]

Then, in virtue of Lemma 7,

\[
X_{N,n} \xrightarrow{D_{N \to \infty}} X_n, \quad \text{(10)}
\]

Let

\[
Y_N = Y_N(s) = \zeta(s + i h\eta_N; a).
\]
Then Lemma 8 implies that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \mu \{ \rho(Y_n(s), X_{N,n}(s)) \geq \varepsilon \} \leq \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \rho(\zeta(s + i\hbar\gamma_k; a), \zeta_n(s + i\hbar\gamma_k; a)) = 0.$$ 

Therefore, this, (9), (10) and Theorem 4.2 of [4] show that $Y_N \overset{D}{\to} P_\zeta$, and the theorem is proved.

4 Proof of Theorem 2

We start with the explicit form of the support of the measure $P_\zeta$. Recall that the support of a probability measure $P$ is a minimal closed set $S_P$ such that $P(S_P) = 1$.

Let $S = \{ g \in H(D): g(s) \neq 0 \text{ or } g(s) \equiv 0 \}$. 

**Lemma 9.** The support of the measure $P_\zeta$ is the set $S^r$.

**Proof.** The space $H^r(D)$ is separable. Therefore [4],

$$\mathcal{B}(H^r(D)) = \mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D)).$$

From this it follows that it suffices to consider the measure $P_\zeta$ on the rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \ldots, A_r \in \mathcal{B}(H(D)).$$

Denote by $m_{jH}$ the Haar measure on $\Omega_j$, $j = 1, \ldots, r$. Then the Haar measure $m_{rH}$ is the product of the measures $m_{1H}, \ldots, m_{rH}$. These remarks imply the equality

$$P_\zeta(A) = m_{rH} \{ \omega \in \Omega^r: \zeta(s, \omega; a) \in A \} = m_{1H} \{ \omega_1 \in \Omega_1: \zeta(s, \omega_1; a_1) \in A_1 \} \cdots m_{rH} \{ \omega_r \in \Omega_r: \zeta(s, \omega_r; a_r) \in A_r \}. \quad (11)$$

It is known [19] that the support of

$$P_\zeta(A_j) = m_{jH} \{ \omega_j \in \Omega_j: \zeta(s, \omega_j; a_j) \in A_j \}, \quad j = 1, \ldots, r,$$

is the set $S$. Therefore, (11) and the minimality of the support prove the lemma.

**Proof of Theorem 2.** The theorem is corollary of Theorem 3, the Mergelyan theorem on the approximation of analytic functions by polynomials [25], and Lemma 9, and it is standard. By the Mergelyan theorem, there exist polynomials $p_1(s), \ldots, p_r(s)$ such that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| f_j(s) - a^{p_j(s)} \right| < \frac{\varepsilon}{2}. \quad (12)$$
In view of Lemma 9, the set
\[ G_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \right\} \]
is an open neighbourhood of an element of the support of the measure \( P_\Sigma \). Hence,
\[ P_\Sigma(G_\varepsilon) > 0. \] (13)
Therefore, by Theorem 3 and the equivalent of weak convergence of probability measures in terms of open sets,
\[ \liminf_{N \to \infty} P_N(G_\varepsilon) \geq P_\Sigma(G_\varepsilon) > 0. \]
This, the definitions of \( P_N \) and \( G_\varepsilon \), together with inequality (12), prove the first part of the theorem.

For the proof of the second part of the theorem, we define one more set
\[ \hat{G}_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}. \]
Then \( \hat{G}_\varepsilon \) is a continuity set of the measure \( P_\Sigma \) for all but at most countably many \( \varepsilon > 0 \), moreover, in view of (12), the inclusion \( G_\varepsilon \subset \hat{G}_\varepsilon \) is valid. Therefore, Theorem 3, the equivalent of weak convergence of probability measures in terms of continuity sets and (13) lead the inequality
\[ \lim_{N \to \infty} P_N(\hat{G}_\varepsilon) = P_\Sigma(\hat{G}_\varepsilon) > 0 \]
for all but at most countably many \( \varepsilon > 0 \). This, the definitions of \( P_N \) and \( \hat{G}_\varepsilon \) prove the second part of the theorem.

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