On an Old Question of Erdős and Rényi Arising in the Delay Analysis of Broadcast Channels

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Abstract

Consider a broadcast channel with \( n \) users, where different users receive different messages, and suppose that each user has to receive \( m \) packets. A quantity of interest here, introduced by Sharif and Hassibi (2006-7) \cite{SharifHassibi2006}, \cite{SharifHassibi2007}, is the (packet) delay \( D_{m,n} \), namely the number of channel uses required to guarantee that all users will receive \( m \) packets.

For the case of a homogeneous network, where in each channel use the transmitter chooses a user at random, i.e. with probability \( 1/n \), and sends him/her a packet, the same quantity \( D_{m,n} \) had already appeared in the coupon collector context, in the works of Newman and Shepp (1960) \cite{NewmanShepp1960} and of Erdős and Rényi (1961) \cite{ErdosRenyi1961}.

A problem of particular interest in wireless communications, related to the delay \( D_{m,n} \), is to determine its behavior as \( n \) and \( m \) grow large. Regarding this problem, Sharif and Hassibi \cite{SharifHassibi2006}, \cite{SharifHassibi2007} managed to calculated the asymptotics of the mean value \( \mathbb{E}[D_{m,n}] \), as \( n \to \infty \), for the cases (a) \( m = \ln n \) and (b) \( m = (\ln n)^\rho \), \( \rho > 1 \). It is somehow surprising that Erdős and Rényi \cite{ErdosRenyi1961} had, also, raised the question of the determination of the asymptotic profile of \( D_{m,n} \) for large \( m \) and \( n \).

In this article we determine the asymptotics of the moments of \( D_{m,n} \), as well as its limiting distribution, in the “supercritical case” where \( m \) grows faster than \( \ln n \) and in the “critical case” \( m \sim \beta \ln n \).

Keywords. (Packet) delay in a broadcasting fading channel; opportunistic scheduling; homogeneous network; urn problems; coupon collector’s problem (CCP); limiting distribution; Gumbel distribution; incomplete Gamma function.

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1 Introduction

In their works [8], [10] M. Sharif and B. Hassibi consider a single-antenna broadcast fading channel with \( n \) users, where the transmission is packet-based. They define the (packet) delay \( D_{m,n} \) as the minimum number of channel uses that guarantees all \( n \) users to successfully receive \( m \) packets (clearly, \( D_{m,n} \geq mn \)). Sharif and Hassibi consider an “opportunistic” scheduling, where in each channel use the transmitter sends the packet to the user with the best channel conditions, i.e. the highest signal-to-noise ratio (SNR). As it turns out [10], the opportunistic scheduling maximizes the sum rate (or throughput), namely the rate of successful message delivery of the broadcast channel.

The case where all users have the same SNR is referred as the homogeneous network case. Here, in each channel use the transmitter chooses the \( j \)-th user, \( j = 1, 2, \ldots, n \), with probability \( \frac{1}{n} \). It follows that \( D_{m,n} \) becomes a classical quantity related to urn problems and, in particular, to the coupon collector’s problem (CCP): Suppose \( n \) equally likely coupons are sampled independently with replacement. Then, \( D_{m,n} \) is the number of trials needed until all \( n \) coupons are detected at least \( m \) times. If the coupon probabilities are not equal, then the CCP-quantity \( D_{m,n} \) corresponds to the delay of a heterogeneous network. In the present paper, however, we will only consider the homogeneous network case.

In the CCP context, the random variable \( D_{m,n} \) first appeared in the works [6] of D.J. Newman and L. Shepp (1960), and [4] of P. Erdős and A. Rényi (1961). Both works focused on the asymptotic behavior of \( D_{m,n} \) as \( n \to \infty \), while \( m \) stays fixed. Newman and Shepp [6] obtained that

\[
\mathbb{E}[D_{m,n}] = n \ln n + (m - 1) n \ln \ln n + nC_m + o(n) \tag{1.1}
\]

as \( n \to \infty \), where \( C_m \) is a constant depending on \( m \). Roughly speaking, formula (1.1) tells us that, on the average, the detection of all \( n \) coupons at least once “costs” \( n \ln n + O(n) \), while each additional detection (of all coupons) raises the cost by \( n \ln \ln n + O(n) \).

Soon after, Erdős and Rényi [4] went a step further and determined the limit distribution of \( D_{m,n} \), as well as the exact value of the constant \( C_m \). They proved that for every real \( y \) one has

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \frac{D_{m,n} - n \ln n - (m - 1)n \ln \ln n}{n} \leq y \right\} = \exp \left( - \frac{e^{-y}}{(m - 1)!} \right) \tag{1.2}
\]

or, equivalently,

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \frac{D_{m,n}}{n} - \ln n - (m - 1) \ln \ln n + \ln(m - 1)! \leq y \right\} = e^{-e^{-y}}. \tag{1.3}
\]

Noticing that in the right-hand side of (1.3) we have the standard Gumbel distribution function, whose expectation is \( \gamma = 0.5772 \cdots \) (the Euler-
Mascheroni constant), it is not hard to justify that the constant in (1.1) must be
\[ C_m = \gamma - \ln (m - 1)! \quad (1.4) \]
At the end of their paper, Erdős and Rényi [4] have included the following comment:
"It is an interesting problem to investigate the limiting distribution of \( \nu_m(n) \) when \( m \) increases with \( n \), but we can not go into this question here."
The quantity denoted by \( \nu_m(n) \) in [4] is nothing but \( D_{m,n} \).
Formulas (1.1), (1.4), and (1.3) hint that the case where \( m \) grows like \( \ln n \) is expected to be "critical" (as opposed to the "subcritical" case where \( m \) grows slower than \( \ln n \)) in the sense that it is the smallest growth of \( m \) (with respect to \( n \)) which seems to affect the leading asymptotic behaviors of \( \mathbb{E}[D_{m,n}] \) and \( D_{m,n} \).
Sharif and Hassibi [8], [10] were also interested in the behavior of \( \mathbb{E}[D_{m,n}] \) as both \( m \) and \( n \) grow to infinity (although they did not seem to be aware of the paper of Erdős and Rényi). In fact, they managed to show [8], [10] that:
(i) If \( m = \ln n \) and \( n \to \infty \), then
\[ \mathbb{E}[D_{m,n}] = \alpha n \ln n + O(n \ln \ln n), \quad (1.5) \]
where \( \alpha \) is the (unique) solution of the equation \( \alpha - \ln \alpha = 2 \) in the interval \((1, \infty)\) (\( \alpha \approx 3.146 \)).
(ii) If \( m = (\ln n)^\rho \), where \( \rho > 1 \) is fixed and \( n \to \infty \), then
\[ \mathbb{E}[D_{m,n}] = nm + o(mn), \quad (1.6) \]
They also showed that if \( m \to \infty \), while \( n \) stays fixed, then \( \mathbb{E}[D_{m,n}] = nm + o(m) \), but this fact had beed already known to Newman and Shepp [6] (Sharif and Hassibi, though, seem to be aware of the fact that (1.6) remains valid whenever \( m \) grows faster than \( \ln n \)).
The reader may have noticed that, since \( m \) and \( n \) are integers, the above mentioned equalities \( m = \ln n \) and \( m = (\ln n)^\rho \), strictly speaking, cannot be satisfied; they only make sense asymptotically (e.g., \( m = \ln n \) can be interpreted as \( m = \ln n + O(1) \)).
In Section 2 of the present paper we calculate the (leading) asymptotics of the moments of \( D_{m,n} \) in the cases (i) \( m \gg \ln n \) and (ii) \( m \sim \beta \ln n \), where \( \beta > 0 \) (as usual, the notation \( A(n) \gg B(n) \) means that \( B(n)/A(n) \to 0 \) as \( n \to \infty \)). Then, in Section 3, under some mild restrictions on the growth of \( m \) in the aforementioned cases (i) and (ii), namely (i’) \( m \gg \ln^3 n \) and (ii’) \( m = \beta \ln n + o(\sqrt{\ln n}) \), we determine the limiting distribution of \( D_{m,n} \), answering the question of Erdős-Rényi [4] for these cases. As we will see, the quantity \( D_{m,n} \), appropriately normalized, converges in distribution to a Gumbel random variable. This may not sound surprising, but the challenging part is to obtain the right normalization of \( D_{m,n} \).
1.1 Analyzing $D_{m,n}$ via Poissonization

Suppose that, for $j = 1, \ldots, n$, we denote by $X_j$ the number of trials needed in order to detect the $j$-th coupon $m$ times. Then, it is clear that $X_j$ is a negative binomial random variable, with parameters $m$ and $1/n$, and

$$D_{m,n} = \max_{1 \leq j \leq n} X_j.$$  

However, the above formula for $D_{m,n}$ is not very convenient, since the $X_j$’s are not independent. Fortunately, there is a clever “Poissonization technique” (see, e.g., [7]) from which we can get more insight about $D_{m,n}$.

Let $N(t), t \geq 0$, be a Poisson process with rate 1. We imagine that each Poisson event associated to this process is a sampled coupon, so that $N(t)$ is the number of sampled coupons at time $t$. Next, for $j = 1, \ldots, n$, let $N_j(t)$ be the number of detections of the $j$-th coupon at time $t$. Then, the processes $N_j(t), j = 1, \ldots, n$, are independent Poisson processes with rates $1/n$ [7] and it is clear that $N(t) = N_1(t) + \cdots + N_n(t)$. If $T_j, j = 1, \ldots, n$, denotes the time of the $m$-th event of the process $N_j$, then $T_1, \ldots, T_n$ are independent (being associated to independent processes) and

$$\Delta_{m,n} := \max_{1 \leq j \leq n} T_j \quad (1.7)$$

is the time when all different coupons have been detected at least $m$ times.

Now, for each $j = 1, \ldots, n$, the random variable $T_j$ (being a sum of $m$ independent exponential variables with parameter $1/n$) is Erlang with parameters $m$ and $1/n$, hence

$$P\{T_j > t\} = S_m(t/n)e^{-t/n}, \quad t \geq 0, \quad (1.8)$$

where $S_m(y)$ denotes the $m$-th partial sum of the Taylor-Maclaurin series of $e^y$, namely

$$S_m(y) := 1 + y + \frac{y^2}{2!} + \cdots + \frac{y^{m-1}}{(m-1)!} + \sum_{l=0}^{m-1} \frac{y^l}{l!} \quad (1.9)$$

It follows from (1.7), (1.8), and the independence of the $T_j$’s that the distribution function of $\Delta_{m,n}$ is

$$F_\Delta(t) = P\{\Delta_{m,n} \leq t\} = P\{T_1 \leq t\}^n = \left[1 - S_m(t/n)e^{-t/n}\right]^n, \quad t \geq 0. \quad (1.10)$$

It remains to relate $\Delta_{m,n}$ to $D_{m,n}$. Clearly,

$$\Delta_{m,n} = \sum_{k=1}^{D_{m,n}} U_k, \quad (1.11)$$
where \( U_1, U_2, \ldots \) are the interarrival times of \( N(t) \). Since \( N(t) \) is a Poisson process of rate 1, the \( U_j \)'s are independent exponential random variables with parameter 1. Furthermore, it is clear that \( D_{m,n} \) is independent of the \( U_j \)'s [7].

One consequence of formula (1.11) is that, given \( D_{m,n} \), the variable \( \Delta_{m,n} \) is Erlang with parameters \( D_{m,n} \) and 1, i.e.

\[
\mathbb{P} \{ \Delta_{m,n} > t \mid D_{m,n} \} = S_{D_{m,n}}(t)e^{-t}, \quad t \geq 0. \tag{1.12}
\]

In other words, the conditional probability density of \( \Delta_{m,n} \) given \( D_{m,n} \) is

\[
f_{\Delta \mid D}(t) = \mathbb{P} \{ \Delta_{m,n} \in dt \mid D_{m,n} \} = \frac{t^{D_{m,n}-1}}{(D_{m,n} - 1)!} e^{-t}, \quad t \geq 0. \tag{1.13}
\]

We can take expectations in (1.12) and obtain

\[
\mathbb{E} \left[ S_{D_{m,n}}(t) \right] e^{-t} = \mathbb{P} \{ \Delta_{m,n} > t \} = 1 - F_\Delta(t), \quad t \geq 0, \tag{1.14}
\]

where \( F_\Delta(t) \) is given in (1.10). If we, then, differentiate (1.14) with respect to \( t \), we obtain

\[
\mathbb{E} \left[ \frac{t^{D_{m,n}-1}}{(D_{m,n} - 1)!} \right] e^{-t} = F_\Delta'(t) =: f_\Delta(t), \quad t \geq 0. \tag{1.15}
\]

From (1.13) we also have

\[
\mathbb{E} \left[ g(\Delta_{m,n}) \mid D_{m,n} \right] = \frac{1}{(D_{m,n} - 1)!} \int_0^\infty g(t) t^{D_{m,n}-1} e^{-t} dt, \tag{1.16}
\]

where \( g(t) \) is any function for which the integral makes sense. Taking expectations in (1.16) yields

\[
\mathbb{E} \left[ g(\Delta_{m,n}) \right] = \mathbb{E} \left[ \frac{1}{(D_{m,n} - 1)!} \int_0^\infty g(t) t^{D_{m,n}-1} e^{-t} dt \right], \tag{1.17}
\]

and if

\[
\mathbb{E} \left[ \frac{1}{(D_{m,n} - 1)!} \int_0^\infty |g(t)| t^{D_{m,n}-1} e^{-t} dt \right] < \infty, \tag{1.18}
\]

then Fubini’s theorem allows us to interchange expectation and integral in (1.17) and obtain

\[
\mathbb{E} \left[ g(\Delta_{m,n}) \right] = \int_0^\infty g(t) \mathbb{E} \left[ \frac{t^{D_{m,n}-1}}{(D_{m,n} - 1)!} \right] e^{-t} dt. \tag{1.19}
\]

Of course, (1.19) is, also, an immediate consequence of (1.15).
Let us look at some examples. If \( g(t) = t^z \) for some complex \( z \), then (1.16) becomes

\[
E \left[ \Delta_{m,n}^z \mid D_{m,n} \right] = \frac{1}{(D_{m,n} - 1)!} \int_0^\infty t^{D_{m,n}+z-1} e^{-t} dt = \frac{\Gamma(D_{m,n} + z)}{(D_{m,n} - 1)!},
\]

(1.20)

where \( \Gamma(\cdot) \) is the Gamma function. And since \( D_{m,n} \geq mn \), the above integral converges for \( \Re(z) > -mn \).

In the case where \( z = r \), a positive integer, formula (1.20) becomes

\[
E \left[ \Delta_{m,n}^r \mid D_{m,n} \right] = \frac{(D_{m,n} + r - 1)!}{(D_{m,n} - 1)!} = D_{m,n} (D_{m,n} + 1) \cdots (D_{m,n} + r - 1),
\]

or

\[
E \left[ \Delta_{m,n}^r \mid D_{m,n} \right] = D_{m,n}^r,
\]

(1.21)

where we have used the notation

\[
M^{(r)} := M(M + 1) \cdots (M + r - 1), \quad r \geq 1.
\]

(1.22)

If \( z = -r \), where \( r = 1, 2, \ldots, (mn - 1) \), formula (1.20) yields

\[
E \left[ \Delta_{m,n}^{-r} \mid D_{m,n} \right] = \frac{(D_{m,n} - r - 1)!}{(D_{m,n} - 1)!} = \frac{1}{(D_{m,n} - r)(D_{m,n} - r + 1) \cdots (D_{m,n} - 1)}.
\]

(1.23)

Now, taking expectations in (1.20) and invoking (1.10) yields (after integrating by parts)

\[
E \left[ \Gamma \left( \frac{D_{m,n} + z}{D_{m,n} - 1} \right) \right] = E \left[ \Delta_{m,n}^z \right]
\]

\[= z \int_0^\infty \left\{ 1 - \left[ 1 - S_m(t/n)e^{-t/n} \right]^n \right\} t^{z-1} dt, \quad \Re(z) > 0,
\]

\[= zn^z \int_0^\infty \left\{ 1 - \left[ 1 - S_m(\tau)e^{-\tau} \right]^n \right\} \tau^{z-1} d\tau, \quad \Re(z) > 0.
\]

(1.24)

In the case where \( z = r \) is a positive integer formula (1.21) becomes

\[
E \left[ D_{m,n}^{(r)} \right] = E \left[ \Delta_{m,n}^r \right] = rn^r \int_0^{\infty} \left\{ 1 - \left[ 1 - S_m(\tau)e^{-\tau} \right]^n \right\} \tau^{r-1} d\tau.
\]

(1.25)

The quantity \( E \left[ D_{m,n}^{(r)} \right] \) is called the \( r \)-th rising moment of \( D_{m,n} \). For \( r = 1 \) and \( r = 2 \) formula (1.20) gives

\[
E[D_{m,n}] = E[\Delta_{m,n}] = n \int_0^{\infty} \left\{ 1 - \left[ 1 - S_m(\tau)e^{-\tau} \right]^n \right\} d\tau,
\]

(1.26)
\[ E[D^{(2)}_{m,n}] = E[D_{m,n}(D_{m,n} + 1)] = E[\Delta_{m,n}^2] = 2n^2 \int_0^\infty \{1 - [1 - S_m(\tau)e^{-\tau}]^n\} \tau d\tau \] (1.27)

(Thus \( V[D_{m,n}] = V[\Delta_{m,n}] - E[\Delta_{m,n}] \)). Formulas (1.26) and (1.27) were first derived in [1] by a different approach.

Let us also notice that by taking expectations in (1.23) we obtain

\[ E\left[ \frac{1}{D_{m,n} - r}(D_{m,n} - r + 1) \cdots (D_{m,n} - 1) \right] = E[D_{m,n}^{-r}] \] (1.28)

for \( r = 1, 2, \ldots, (mn - 1) \). In particular,

\[ E\left[ \frac{1}{\Delta_{m,n}} \right] = E\left[ \frac{1}{D_{m,n} - 1} \right] \] (1.29)

and

\[ E\left[ \frac{1}{\Delta_{m,n}^2} \right] = E\left[ \frac{1}{(D_{m,n} - 2)(D_{m,n} - 1)} \right] = E\left[ \frac{1}{D_{m,n} - 2} \right] - E\left[ \frac{1}{D_{m,n} - 1} \right]. \] (1.30)

Suppose now that \( g(t) = e^{zt} \). Then (1.16) becomes

\[ E[e^{z\Delta_{m,n}} | D_{m,n}] = \frac{1}{(D_{m,n} - 1)!} \int_0^\infty t^{D_{m,n} - 1} e^{-(1-z)t} dt = \frac{1}{(1-z)^{D_{m,n}}}, \quad \Re(z) < 1 \] (1.31)

(first we show that the second equation in (1.31) holds for real \( z < 1 \) and then we use analytic continuation).

Hence, by taking expectations in (1.31) and invoking (1.10) we obtain (after integrating by parts once)

\[ E\left[ \frac{1}{(1-z)^{D_{m,n}}} \right] = E[e^{z\Delta_{m,n}}] = 1 + z \int_0^\infty \{1 - [1 - S_m(t/n)e^{-t/n}]^n\} e^{zt} dt, \quad \Re(z) < \frac{1}{n} \]
\[ = 1 + zn \int_0^\infty \{1 - [1 - S_m(\tau)e^{-\tau}]^n\} e^{nz\tau} d\tau, \quad \Re(z) < \frac{1}{n}. \] (1.32)

2 **Asymptotics of the moments of \( D_{m,n} \) as \( m, n \to \infty \)**

For typographical convenience we set

\[ F_m(x) := 1 - S_m(x)e^{-x}, \quad x \geq 0, \] (2.1)
where $S_m(\cdot)$ is given by (1.9). Observe that
\[ F_m(x) = \mathbb{P}\{\Theta_m \leq x\}, \] (2.2)
where $\Theta_m$ is an Erlang random variable with parameters $m$ and 1.

In view of (2.1), formula (1.24) can be written as
\[
E\left[ \Gamma \left( D_{m,n} + z \right) \right] = E\left[ \Delta_{m,n}^z \right] = zn^z \int_0^\infty \left[ 1 - F_m(\tau) \right]^n \tau^{z-1} d\tau, \quad \Re(z) > 0.
\] (2.3)

In order to determine the asymptotic behavior of the integral in (2.3) one has to locate a relatively narrow interval, say $[x_1, x_2]$ of values of $x$ in which the values of $S_m(x)e^{-x} = 1 - F_m(x)$ change from $\gg 1/n$ to $\ll 1/n$, so that $F_m(x_1)^n \to 0$ and $F_m(x_2)^n \to 1$ as $n \to \infty$.

One observation, somehow relevant to the above task, is that (as long as $m \geq 2$) the unique point of inflection of $1 - F_m(x)$ (and of $F_m(x)$) is located at $x = m - 1$.

Now, since $\Theta_m$ of (2.2) can be expressed as a sum of $m$ independent exponential random variables with parameter 1, we can apply the Berry-Esseen Theorem and obtain the estimate
\[
\left| 1 - F_m(x) - \Phi \left( \frac{m - x}{\sqrt{m}} \right) \right| \leq \frac{C}{\sqrt{m}}, \quad x \in \mathbb{R}, \ m \in \mathbb{N} := \{1, 2, \ldots\}, \tag{2.4}
\]
where $\Phi(\cdot)$ is the standard normal distribution function and $C > 0$ is independent of both $x$ and $m$. For example, one implication of (2.4) is
\[
1 - F_m(m + O(\sqrt{m})) \gg \frac{1}{n}, \quad n \to \infty. \tag{2.5}
\]

In the case where $|m-x|/\sqrt{m} \to \infty$ the Berry-Esseen estimate (2.4) becomes too crude. For instance, if $(m-x)/\sqrt{m} \to -\infty$, then $\Phi((m-x)/\sqrt{m})$ may become much smaller than the error bound $C/\sqrt{m}$. Fortunately, in this case one can use a nice asymptotic formula due to F.G. Tricomi (1950) \[\] regarding the incomplete Gamma function.

First, let us notice that formula (1.9) implies that $S_m'(y) = S_{m-1}(y)$, hence (as we have already noticed in (1.13))
\[
\frac{d}{dy} \left[ S_m(y)e^{-y} \right] = S_{m-1}(y)e^{-y} - S_m(y)e^{-y} = -y^{m-1} (m-1)! e^{-y}. \tag{2.6}
\]

Thus, by integrating (2.6) from $x$ to $\infty$ we obtain
\[
1 - F_m(x) = S_m(x)e^{-x} = \frac{1}{(m-1)!} \int_x^\infty y^{m-1} e^{-y} dy = \frac{\Gamma(m,x)}{(m-1)!}, \tag{2.7}
\]
where $\Gamma(\cdot, \cdot)$ is the upper incomplete Gamma function.
Tricomi [11] (see, also, [5]) has shown that if
\[ x - \mu \to \infty \quad \text{and} \quad \frac{\sqrt{\mu}}{x - \mu} \to 0, \quad (2.8) \]
then
\[ \Gamma(\mu + 1, x) = \frac{x^{\mu+1}e^{-x}}{x - \mu} \left[ 1 - \frac{\mu}{(x - \mu)^2} + \frac{2\mu}{(x - \mu)^3} + O\left( \frac{\mu^2}{(x - \mu)^4} \right) \right]. \quad (2.9) \]
Formula (2.9) holds for complex \( \mu \) and \( x \) as long as the argument of the quantity \( \sqrt{\mu}/(x - \mu) \) ultimately remains between \( -3\pi/4 \) and \( 3\pi/4 \). In fact, Tricomi [11] has given the complete asymptotic expansion of \( \Gamma(\mu + 1, x) \), but, for our purposes formula (2.9) is more than sufficient. Indeed, by using (2.9) in (2.7) with \( \mu = m - 1 \) and invoking Stirling’s formula we can conclude that
\[ 1 - F_m(x) = S_m(x)e^{-x} = \frac{1}{\sqrt{2\pi}} e^{-(x-m)} \left( \frac{x}{m} \right)^m \frac{\sqrt{m}}{x-m+1} \left[ 1 + O\left( \frac{m}{(x-m)^2} \sqrt{\frac{1}{m}} \right) \right], \quad (2.10) \]
provided that (2.8) is satisfied, namely
\[ x - m \to \infty \quad \text{and} \quad \frac{\sqrt{m}}{x-m} \to 0 \quad (2.11) \]
(in (2.10) we have used the notation \( a \lor b \) for the maximum of \( a \) and \( b \)).

### 2.1 The “supercritical” case \( m \gg \ln n \)

In the case where \( m \) grows faster than \( \ln n \), it turns out that the desired jump of \( F_m(x)^n \) from 0 to 1 happens in the interval \( m \leq x \leq m[1 + \varepsilon(n)] \), where \( \varepsilon(n) \) is given by (2.35) below. This is the main ingredient in the proof of the theorem that follows.

**Theorem 1.** Let \( z \) be a fixed complex number with \( \Re(z) > 0 \) and assume that \( m = m(n) \) is such that \( m(n) \gg \ln n \), i.e.
\[ \frac{\ln n}{m(n)} \to 0 \quad \text{as} \quad n \to \infty. \quad (2.12) \]
Then
\[ \mathbb{E} \left[ \frac{\Gamma(D_{m,n} + z)}{(D_{m,n} - 1)!} \right] = \mathbb{E} \left[ \Delta_{m,n}^z \right] \sim n^z m^z, \quad n \to \infty \quad (2.13) \]
(recall that the variable \( \Delta_{m,n} \) has been introduced in Subsection 1.1).
Proof. The substitution $\tau = m\xi$ in the integral of (2.3) yields

$$E \left[ \frac{\Gamma(D_{m,n} + z)}{\Gamma(D_{m,n} - 1)!} \right] = E \left[ \Delta_{m,n}^z \right] = zn^z I(z),$$  \hspace{1cm} (2.14)

where

$$I(z) := \int_0^\infty [1 - F_m(m\xi)^n] \xi^{z-1} d\xi.$$  \hspace{1cm} (2.15)

We split $I(z)$ as

$$I(z) = I_0(z) + I_1(z),$$  \hspace{1cm} (2.16)

where

$$I_0(z) := \int_0^{1+\varepsilon} [1 - F_m(m\xi)^n] \xi^{z-1} d\xi, \quad I_1(z) := \int_{1+\varepsilon}^{\infty} [1 - F_m(m\xi)^n] \xi^{z-1} d\xi,$$  \hspace{1cm} (2.17)

where $\varepsilon = \varepsilon(n)$ is positive and approaches 0 as $n \to \infty$. The specific form of $\varepsilon(n)$ will be decided later in the proof.

Using the Berry-Esseen estimate (2.4) and applying bounded convergence to the first integral in (2.17) we obtain immediately that

$$\lim_{n} I_0(z) = \int_0^1 \xi^{z-1} d\xi = \frac{1}{z}.$$  \hspace{1cm} (2.18)

Therefore, in view of (2.14), (2.15), (2.16), (2.17), and (2.18), the proof of (2.13) will be completed if we show that we can choose $\varepsilon = \varepsilon(n)$ so that

$$\lim_{n} I_1(z) = 0.$$  \hspace{1cm} (2.19)

From (2.17) we have

$$|I_1(z)| \leq \int_{1+\varepsilon}^{\infty} [1 - F_m(m\xi)^n] \xi^{\sigma-1} d\xi, \quad \text{where } \sigma := \Re(z) > 0.$$  \hspace{1cm} (2.20)

Now, in view of (2.21),

$$F_m(m\xi)^n = \left[ 1 - S_m(m\xi)e^{-m\xi} \right]^n \geq 1 - nS_m(m\xi)e^{-m\xi}.$$  \hspace{1cm} (2.21)

Hence, (2.20) implies

$$|I_1(z)| \leq n \int_{1+\varepsilon}^{\infty} S_m(m\xi)e^{-m\xi} \xi^{\sigma-1} d\xi.$$  \hspace{1cm} (2.22)

Next we notice that for $\xi \geq 1$ formula (1.9) implies easily that

$$S_m(m\xi) \leq m \frac{(m\xi)^{m-1}}{(m-1)!} = m^{m+1} m! \xi^{m-1}.$$  \hspace{1cm} (2.23)
Thus, by using (2.23) in (2.22) we get
\[ |I_1(\varepsilon)| \leq \frac{m^{m+1}}{m!} \int_{1+\varepsilon}^{\infty} \xi^{\sigma-2}e^{-m\xi}d\xi = \frac{m^{m+1}}{m!} e^{-m\Lambda}, \]  
where we have set
\[ \Lambda := \int_{\varepsilon}^{\infty} (1+t)^{\sigma-2}e^{-m[t-\ln(1+t)]}dt. \]  
(2.24)

For the function \( \rho(t) := t - \ln(1+t) \) we have
\[ \rho(0) = \rho'(0) = 0 \quad \text{and} \quad \rho'(t) = 1 - \frac{1}{t+1} > 0 \quad \text{for} \ t \in (0, \infty), \]  
(2.25)
so that \( \rho(t) \) is strictly increasing on \([0, \infty)\). Therefore, for the integral in (2.25) we have the estimate
\[ \Lambda < 2 \int_{\varepsilon}^{1} (1+t)^{\sigma-2}e^{-m[t-\ln(1+t)]}dt < 2^{\sigma-1} \int_{\varepsilon}^{1} e^{-m[t-\ln(1+t)]}dt, \]  
(2.26)

as long as \( m \) is sufficiently large.

Now, from (2.27) (and the fact that \( \rho''(0) = 1 \)) it also follows that for all sufficiently small \( \delta > 0 \) we must have
\[ \rho(t) \geq \delta t^2 \quad \text{for every} \ t \in [0, 1]. \]  
(2.27)

If we choose such a \( \delta \), then (2.28) implies
\[ \Lambda < 2^{\sigma-1} \int_{\varepsilon}^{1} e^{-m\delta^2}dt < 2^{\sigma-1} \int_{\varepsilon}^{\infty} e^{-m\delta^2}dt = \frac{2^{\sigma-1}}{\sqrt{\delta m}} \int_{\varepsilon\sqrt{\delta m}}^{\infty} e^{-x^2}dx. \]  
(2.28)

Thus, in view of (2.24) and (2.30) (and Stirling’s formula) the limit (2.19) will hold if we can find an \( \varepsilon = \varepsilon(n) \) (with \( \varepsilon(n) \to 0 \) as \( n \to \infty \)) such that
\[ \int_{\varepsilon\sqrt{\delta m}}^{\infty} e^{-x^2}dx = o \left( \frac{1}{n} \right), \quad n \to \infty. \]  
(2.29)

In order to satisfy (2.31) it is clearly necessary to have
\[ \varepsilon \sqrt{m} \to \infty \quad \text{as} \ n \to \infty. \]  
(2.30)

Now, if (2.32) is satisfied, then it is not hard to see (e.g., by applying L’Hôpital’s rule) that
\[ \int_{\varepsilon\sqrt{\delta m}}^{\infty} e^{-x^2}dx \sim \frac{1}{2\varepsilon \sqrt{\delta m}} e^{-m\varepsilon^2}, \]  
(2.33)
Hence, under (2.32), formula (2.31) is equivalent to
\[
\frac{1}{\varepsilon \sqrt{m}} e^{-m \delta \varepsilon^2} = o \left( \frac{1}{n} \right), \quad n \to \infty. \tag{2.34}
\]

Notice that if \( m = m(n) \) does not grow faster than \( \ln n \), then it is not possible to simultaneously satisfy \( \varepsilon(n) \to 0 \), (2.32), and (2.34). This indicates that without the assumption (2.12) formula (2.13) may not hold.

To satisfy (2.34) we can pick a \( \kappa > 1 \) and then take
\[
\varepsilon = \varepsilon(n) = \sqrt{\frac{\kappa \ln n}{\delta m}}. \tag{2.35}
\]

Since (2.31) is equivalent to (2.34), we can conclude that (2.31) is satisfied if \( \varepsilon(n) \) is chosen as above (also it is clear that, under assumption (2.12), the above choice of \( \varepsilon(n) \) satisfies \( \lim_n \varepsilon(n) = 0 \) as well as (2.32)). \( \Box \)

In the case where \( z = r \in \mathbb{N} \), formula (2.13) becomes
\[
\mathbb{E} \left[ D_{m,n}^{(r)} \right] = \mathbb{E} \left[ \Delta_{m,n}^{r} \right] \sim n^r m^r, \quad n \to \infty, \tag{2.36}
\]
from which it follows that
\[
\mathbb{E} \left[ D_{m,n}^{(r)} \right] \sim n^r m^r, \quad n \to \infty. \tag{2.37}
\]

In particular,
\[
\mathbb{E} [D_{m,n}] \sim nm, \quad n \to \infty, \tag{2.38}
\]
which is in agreement with the corresponding result which appeared in [8] and [10].

Notice that, roughly speaking, formula (2.38) tells us that, on the average, if all \( n \) coupons have already been detected \( m \) times, where \( m \gg \ln n \), then each additional detection (of all coupons) “costs” \( n \). Apart from being interesting by itself, formula (2.35) is used in the proof of Theorem 4 below.

**Remark 1.** Suppose \( n \) is fixed. Then formula (2.13) is still valid, where now \( m \to \infty \), i.e. if \( \Re(z) > 0 \), then
\[
\mathbb{E} \left[ \frac{\Gamma(D_{m,n} + z)}{(D_{m,n} - 1)!} \right] = \mathbb{E} \left[ \Delta_{m,n}^{z} \right] \sim n^z m^z, \quad m \to \infty; \tag{2.39}
\]
in particular, for \( r \in \mathbb{N} \) we get
\[
\mathbb{E} \left[ D_{m,n}^{(r)} \right] = \mathbb{E} \left[ \Delta_{m,n}^{r} \right] \sim n^r m^r, \quad m \to \infty. \tag{2.40}
\]

We can prove (2.39) by slightly modifying the proof of Theorem 1. Here is the key element in the modified proof: Instead of (2.35) we now pick
a (constant, but otherwise arbitrary) $\varepsilon > 0$. The Berry-Esseen theorem implies (as $m \to \infty$)

$$F_m(m\xi) = \Phi \left( \frac{m\xi - m}{\sqrt{m}} \right) + O\left( \frac{1}{\sqrt{m}} \right) = \Phi \left( (\xi - 1)\sqrt{m} \right) + O\left( \frac{1}{\sqrt{m}} \right),$$

(2.41)

uniformly in $\xi$. Thus,

$$F_m(m\xi)^n = O\left( \frac{1}{\sqrt{m}} \right), \quad \text{if } 0 \leq \xi \leq 1 - \varepsilon,$$

(2.42)

while

$$F_m(m\xi)^n = 1 + O\left( \frac{1}{\sqrt{m}} \right), \quad \text{if } \xi \geq 1 + \varepsilon.$$

(2.43)

### 2.2 The critical case $m \sim \beta \ln n$

In the case where $m$ grows like $\ln n$, the formula (2.13) is no longer true. As we will see in the following lemma, the reason is that, when $m \sim \beta \ln n$ for some $\beta > 0$, in contrast with the supercritical case, the values of $x$ at which $S_m(x)e^{-x} = 1 - F_m(x)$ changes from $\gg 1/n$ to $\ll 1/n$ are quite away from $m$.

**Lemma 1.** Suppose $m = m(n), n = 1, 2, \ldots,$ is a given sequence of positive integers such that

$$m(n) \sim \beta \ln n, \quad n \to \infty,$$

(2.44)

where $\beta > 0$ is a fixed constant, and let $\alpha$ be the unique solution of the equation

$$\alpha - \beta \ln \alpha = \beta - \beta \ln \beta + 1$$

(2.45)

in the interval $(\beta, \infty)$ (thus $\alpha > \beta$). Then, there are two sequences $a_1 = a_1(n) \to 0$ and $a_2 = a_2(n) \to 0$, with $a_1(n) < a_2(n)$ such that (as $n \to \infty$):

If $\chi_1 = \chi_1(n) := \alpha \ln n + a_1(n) \ln n$, then $S_m(\chi_1)e^{-\chi_1} \gg \frac{1}{n}$,

(2.46)

while

if $\chi_2 = \chi_2(n) := \alpha \ln n + a_2(n) \ln n$, then $S_m(\chi_2)e^{-\chi_2} \ll \frac{1}{n}$.

(2.47)

**Proof.** Notice that, in view of (2.44), (2.45), (2.46), and (2.47), if $x = \chi_1$ or $x = \chi_2$, then the conditions (2.11) are satisfied. Thus, (2.10) implies

$$S_m(\chi_j)e^{-\chi_j} = \frac{1}{\sqrt{2\pi}} \frac{e^{-(\chi_j-m)}}{\sqrt{m}} \left( \frac{\chi_j}{m} \right)^m \frac{m}{\chi_j - m} \left[ 1 + O\left( \frac{1}{\ln n} \right) \right], \quad j = 1, 2.$$ 

(2.48)
We will now analyze each of the three factors appearing in the right-hand side of (2.48). But, before we start, let us express assumption (2.44) in the equivalent form

\[ m(n) := \beta \ln n + b(n) \ln n, \quad \text{where} \ b(n) \to 0 \]  

(2.49)

(the sequence \( b = b(n) \) is assumed given).

For the factor \( m/(\chi_j - m) \) it follows immediately that

\[ \frac{m}{\chi_j - m} \sim \frac{\beta}{\alpha - \beta}. \]  

(2.50)

For the factor \( e^{-(\chi_j - m)}/\sqrt{m} \) we have, in view of (2.46), (2.47), and (2.49),

\[ e^{-(\chi_j - m)}\sqrt{m} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha - \beta} e^{-(a_j - b) \ln n}. \]  

(2.51)

Finally, we analyze the factor \( (\chi_j/m)^m \):

\[ \left( \frac{\chi_j}{m} \right)^m = \left( \frac{\alpha + a_j}{\beta + b} \right)^m = \left( \frac{\alpha}{\beta} \right)^m \left( 1 + \frac{a_j}{\alpha} \right)^m \left( 1 + \frac{b}{\beta} \right)^m \]

\[ = \left( \frac{\alpha}{\beta} \right)^{\beta \ln n + b \ln n} \left( 1 + \frac{a_j}{\alpha} \right)^{\beta \ln n + b \ln n} \]

\[ = e^{(\ln \alpha - \ln \beta) b \ln n} \left( 1 + \frac{a_j}{\alpha} \right)^{\beta \ln n + b \ln n} \]

\[ = e^{(\ln \alpha - \ln \beta) b \ln n} \left( 1 + \frac{a_j}{\alpha} \right)^{\beta \ln n + b \ln n}, \]  

(2.52)

If we now use (2.50), (2.51), and (2.52) in (2.48) and invoke (2.45), we obtain

\[ S_m(\chi_j) e^{-\chi_j} \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta}}{\alpha - \beta} \frac{e^{A_j(n) \ln n}}{n}, \quad j = 1, 2, \]  

(2.53)

where for typographical convenience we have set

\[ A_j(n) := (\ln \beta - \ln \alpha + 1) b - (\beta + b) \ln \left( 1 + \frac{b}{\beta} \right) \]

\[ - a_j + (\beta + b) \ln \left( 1 + \frac{a_j}{\alpha} \right) - \frac{1}{2} \ln \ln n, \quad j = 1, 2. \]  

(2.54)

Our assumption \( b = b(n) \to 0, \) as \( n \to \infty, \) implies immediately that

\[ B = B(n) := (\ln \alpha - \ln \beta + 1) b - (\beta + b) \ln \left( 1 + \frac{b}{\beta} \right) \to 0. \]  

(2.55)

For the sequences \( a_1 \) and \( a_2 \) we need \( a_j \to 0. \) Thus,

\[ - a_j + (\beta + b) \ln \left( 1 + \frac{a_j}{\alpha} \right) = - \left( 1 - \frac{\beta}{\alpha} \right) a_j + o(a_j), \quad n \to \infty \]  

(2.56)
(recall that $0 < \beta/\alpha < 1$).

Substituting (2.55) and (2.56) in (2.54) yields

$$A_j(n) = B(n) - \frac{1}{2} \ln \ln n \left( 1 - \frac{\beta}{\alpha} \right) a_j + o(a_j) \quad (2.57)$$

and from formula (2.57) it is clear that, since $B(n)$ is a given sequence with $B(n) \to 0$, we can choose a sequence $a_1 = a_1(n)$, with $a_1(n) \to 0$ so that $A_1(n) \ln n \to \infty$. For example, just take

$$a_1(n) = -\frac{2\alpha}{\alpha - \beta} \left( |B(n)| + \frac{\ln \ln n}{2 \ln n} \right).$$

Likewise, we can choose a sequence $a_2 = a_2(n)$, with $a_2(n) \to 0$ so that $A_2(n) \ln n \to -\infty$. For example,

$$a_2(n) = \frac{2\alpha}{\alpha - \beta} \left( |B(n)| + \frac{\ln \ln n}{2 \ln n} \right).$$

Therefore, in view of (2.53), we have demonstrated that there are sequences $a_1(n) \to 0$ and $a_2(n) \to 0$ which satisfy (2.46) and (2.47) respectively. ■

**Remark 2.** Formula (2.45) can be written as $\alpha = \beta + 1 + \beta \ln(\alpha/\beta)$ and from this, together with the fact that $\alpha > \beta > 0$, it is obvious that $\alpha > \beta + 1$. Also, $\alpha$ increases with $\beta$ (as it is easy to check that $d\alpha/d\beta > 0$) and $\alpha \to 1^+$ as $\beta \to 0^+$. Thus, $\alpha$ can take any value in $(1, \infty)$.

It is remarkable that the equation (2.45) also appears, in a different context, in Sharif and Hassibi [9].

**Remark 3.** It is rather obvious that if $\tilde{a}_1(n)$ is a sequence such that $\tilde{a}_1(n) \to 0$ and $\tilde{a}_1(n) \leq a_1(n)$, where $\chi_1(n) := \alpha \ln n + a_1(n) \ln n$ satisfies (2.46), then $\tilde{\chi}_1(n) := \alpha \ln n + \tilde{a}_1(n) \ln n$ also satisfies (2.46). Likewise, if $\tilde{a}_2(n)$ is a sequence such that $\tilde{a}_2(n) \to 0$ and $\tilde{a}_2(n) \geq a_2(n)$, where $\chi_2(n) := \alpha \ln n + a_2(n) \ln n$ satisfies (2.47), then $\tilde{\chi}_2(n) := \alpha \ln n + \tilde{a}_2(n) \ln n$ also satisfies (2.47).

We are now ready to give the (leading) asymptotic behavior of the moments of $D_{m,n}$ as $n \to \infty$ in the critical case.

**Theorem 2.** Let $z$ be a fixed complex number with $\Re(z) > 0$ and assume that

$$m = m(n) \sim \beta \ln n \quad \text{for some constant } \beta > 0. \quad (2.58)$$

Then,

$$\mathbb{E} \left[ \frac{\Gamma(D_{m,n} + z)}{(D_{m,n} - 1)!} \right] = \mathbb{E} \left[ \Delta_{m,n}^z \right] \sim \alpha^z n^z (\ln n)^{\tilde{z}} \sim \left( \frac{\alpha}{\beta} \right) \frac{\alpha^z}{\beta} m^z, \quad n \to \infty, \quad (2.59)$$
where $\alpha$ is the unique solution of the equation (2.45) in the interval $(\beta, \infty)$.

**Proof.** We will prove the theorem by adapting the proof of Theorem 1. From formula (2.3) we have

$$E \left[ \frac{\Gamma(D_{m,n} + z)}{(D_{m,n} - 1)!} \right] = E \left[ \Delta_{m,n}^z \right] = n^z m^z J(z),$$

(2.60)

where

$$J(z) := \int_0^\infty [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi.$$  

(2.61)

This time we split $J(z)$ as

$$J(z) = J_1(z) + J_2(z) + J_3(z),$$

(2.62)

where

$$J_1(z) := \int_0^{\chi_1/m} [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi,$$

(2.63)

$$J_2(z) := \int_{\chi_1/m}^{\chi_2/m} [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi,$$

(2.64)

and

$$J_3(z) := \int_{\chi_2/m}^\infty [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi,$$

(2.65)

where $\chi_1$ and $\chi_2$ are as in (2.46) and (2.47) respectively. Notice that (2.46), (2.47), and (2.58) imply that

$$\frac{\chi_j}{m} = \frac{\alpha}{\beta} + o(1) \quad \text{as} \quad n \to \infty.$$  

(2.66)

From the display (2.46) and the fact that $S_m(x)e^{-x}$ is decreasing on $[0, \infty)$ we get (in view of (2.66)) that

$$nS_m(m\xi)e^{-m\xi} \to \infty, \quad \text{if} \quad \xi \in \left[0, \frac{\alpha}{\beta} - \varepsilon\right],$$

(2.67)

for any given $\varepsilon > 0$. Consequently,

$$\lim_n F_m(m\xi)^n = \lim_n \left[ 1 - S_m(m\xi)e^{-m\xi} \right] = 0 \quad \text{for all} \quad \xi \in \left[0, \frac{\alpha}{\beta} - \varepsilon\right].$$

(2.68)

Let us break the integral $J_1(z)$ of (2.63) as

$$J_1(z) = \int_0^{\beta - \varepsilon} [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi + \int_{\frac{\alpha}{\beta} - \varepsilon}^{\chi_1/m} [1 - F_m(m\xi)^n] z\xi^{z-1} d\xi.$$  

(2.69)

By (2.66) and the fact that $F_m(x)$ is a distribution function, the second integral in the right-hand side of (2.69) is bounded by $|z|((\alpha/\beta)^{\sigma-1}[\varepsilon + o(1)],$
where, as in (2.20), \( \sigma = \Re(z) \). Therefore, in view of (2.68) and the fact that \( \varepsilon \) is arbitrary, we can conclude from (2.69) that 

\[
J_1(z) \to \int_0^{\beta} z^{\xi - 1} d\xi = \left( \frac{\alpha}{\beta} \right)^z \quad \text{as} \quad n \to \infty.
\]

(2.70)

Now, the integral \( J_2(z) \) of (2.64) is very easy to treat. We have

\[
|J_2(z)| \leq |z| \int_{\frac{\alpha}{\beta} + o(1)}^{\alpha + o(1)} \xi^{\sigma - 1} e^{-m \xi} d\xi,
\]

(2.71)

which implies immediately that

\[
J_2(z) \to 0 \quad \text{as} \quad n \to \infty.
\]

(2.72)

Thus, in view of (2.60), (2.61), (2.62), (2.63), (2.64), (2.65), (2.70), and (2.71), in order to complete the proof of (2.59) it remains to show that we can choose \( a_2 = a_2(n) \) (recall (2.47) and Remark 3) so that

\[
\lim_{n \to \infty} J_3(z) = 0.
\]

(2.73)

In the same way we derived formula (2.24) in the proof of Theorem 1, we can now get

\[
|J_3(z)| \leq |z| \int_{\frac{\alpha}{\beta} + a_2}^{\alpha + a_2} \xi^{\sigma - 2} e^{-m \xi} \xi^{m e^{-m \phi(\xi)}} d\xi,
\]

(2.74)

Since \( z \) is fixed, application of Stirling’s formula in (2.74) yields

\[
|J_3(z)| \leq C n e^{m(\xi - \ln(\xi))} \int_{\frac{\alpha}{\beta} + a_2}^{\alpha + a_2} \xi^{\sigma - 2} \xi^{m e^{-m \phi(\xi)}} d\xi \quad \text{for some constant} \quad C > 0.
\]

(2.75)

Let us set

\[
\phi(\xi) := \xi - \ln(\xi).
\]

(2.76)

Recall that \( \alpha/\beta > 1 \), while \( a_2 = a_2(n) \) (see Lemma 1) is a sequence approaching 0, which, without loss of generality (in view of Remark 3), can be assumed positive. And since \( \phi'(\xi) = 1 - \xi^{-1} \) and \( \phi''(\xi) = \xi^{-2} \), it follows that \( \phi(\xi) \) is convex and increasing on \([\alpha/\beta + a_2, \infty)\). Therefore,

\[
\phi(\xi) \geq \phi \left( \frac{\alpha}{\beta} + a_2 \right) + \phi' \left( \frac{\alpha}{\beta} + a_2 \right) \left( \xi - \frac{\alpha}{\beta} - a_2 \right) \quad \text{for} \quad \xi \geq \frac{\alpha}{\beta} + a_2,
\]

(2.77)

and, hence, by using (2.76) and (2.77) in (2.75), we obtain

\[
|J_3(z)| \leq C n e^{m(\xi - \ln(\xi))} \int_{\frac{\alpha}{\beta} + a_2}^{\alpha + a_2} \xi^{\sigma - 2} \xi^{m e^{-m \phi(\xi)}} d\xi,
\]

(2.78)
or

\[ |J_3(z)| \leq C n e^{-m \left[ 1 - \phi \left( \frac{\alpha}{\beta} + a_2 \right) \right]} \sqrt{m} \int_0^\infty \left( t + \frac{\alpha}{\beta} + a_2 \right)^{\sigma-2} e^{-m \phi' \left( \frac{\alpha}{\beta} + a_2 \right) t} dt \]

\[ \leq K n e^{-m \left[ 1 - \phi \left( \frac{\alpha}{\beta} + a_2 \right) \right]} \sqrt{m} \int_0^\infty e^{-m \phi' \left( \frac{\alpha}{\beta} + a_2 \right) t} dt = K n e^{-m \left[ 1 - \phi \left( \frac{\alpha}{\beta} + a_2 \right) \right]} \sqrt{m} \phi' \left( \frac{\alpha}{\beta} + a_2 \right) \]

(2.78)

for some constant \( K > 0. \)

Now, in view of (2.76) and the fact that \( a_2 \to 0 \) we have

\[ \phi \left( \frac{\alpha}{\beta} + a_2 \right) = \frac{\alpha}{\beta} + a_2 - \ln \left( \frac{\alpha}{\beta} + a_2 \right) = \frac{\alpha}{\beta} + a_2 - \ln \left( \frac{\alpha}{\beta} \left( 1 + \frac{\beta}{\alpha} a_2 \right) \right) \]

\[ = \frac{\alpha}{\beta} - \ln \left( \frac{\alpha}{\beta} \right) + a_2 - \frac{\beta}{\alpha} a_2 + O \left( a_2^2 \right) \]

\[ = \frac{\alpha - \beta \ln \alpha + \beta \ln \beta}{\beta} + \left( 1 - \frac{\beta}{\alpha} \right) a_2 + O \left( a_2^2 \right) \]

\[ = 1 + 1 + \left( 1 - \frac{\beta}{\alpha} \right) a_2 + O \left( a_2^2 \right), \]

(2.79)

where the last equality follows from (2.45). Also,

\[ \phi' \left( \frac{\alpha}{\beta} + a_2 \right) = 1 - \left( \frac{\alpha}{\beta} + a_2 \right)^{-1} = 1 - \frac{\beta}{\alpha} \frac{1}{1 + \frac{\beta}{\alpha} a_2} = 1 - \frac{\beta}{\alpha} + O \left( a_2 \right). \]

(2.80)

Substituting (2.79) and (2.80) in (2.78) yields

\[ |J_3(z)| \leq K n e^{-m \left[ \frac{1}{\alpha} + (1 - \frac{\beta}{\alpha}) a_2 + O \left( a_2^2 \right) \right]} \sqrt{m} \left[ 1 - \frac{\beta}{\alpha} + O \left( a_2 \right) \right], \]

(2.81)

Let us recall that \( 0 < \beta/\alpha < 1 \) and \( m = m(n) = \beta \ln n + b(n) \ln n, \) with \( b(n) \to 0. \) Thus, the denominator of the fraction in the right-hand side of (2.81) approaches \( \infty \) as \( n \to \infty. \) As for the numerator of that fraction, if we choose a sequence \( a_2 \) so that

\[ a_2(n) \gg \left| b(n) \right| + \frac{\ln \ln n}{\ln n} \]

(2.82)

(this choice is legitimate in view of Remark 3), then

\[ n e^{-m \left[ \frac{1}{\alpha} + (1 - \frac{\beta}{\alpha}) a_2 + O \left( a_2^2 \right) \right]} = n e^{-\left[ 1 + \beta \left( 1 - \frac{\beta}{\alpha} \right) a_2 + o(a_2) \right]} \ln n = e^{-\left[ \beta \left( 1 - \frac{\beta}{\alpha} \right) a_2 + o(a_2) \right]} \ln n \]

(2.83)
and, hence,
\[ ne^{-m\left[\frac{a}{n} + (1-\frac{a}{n})a_2 + O\left(a_2^2\right)\right]} \to 0 \quad \text{as} \quad n \to \infty. \]  \hfill (2.84)

Finally, by using (2.84) in (2.81) we deduce that \( J_3(z) \to 0 \) as \( n \to \infty \). Therefore, there is a sequence \( a_2(n) \) for which (2.73) is satisfied, and the proof is finished. \hfill ■

In the case where \( z = r \in \mathbb{N} \), formula (2.59) becomes
\[ \mathbb{E}\left[D_{m,n}^r\right] = \mathbb{E}\left[\Delta_{m,n}^r\right] \sim \alpha^r n^r (\ln n)^r \sim \left(\frac{\alpha}{\beta}\right)^r n^r m^r, \quad n \to \infty, \]  \hfill (2.85)

from which it follows that
\[ \mathbb{E}\left[D_{m,n}^r\right] \sim \alpha^r n^r (\ln n)^r \sim \left(\frac{\alpha}{\beta}\right)^r n^r m^r, \quad n \to \infty. \]  \hfill (2.86)

In particular,
\[ \mathbb{E}\left[D_{m,n}\right] \sim \alpha n \ln n \sim \frac{\alpha}{\beta} nm, \quad n \to \infty, \]  \hfill (2.87)

which is in agreement with the corresponding result appeared in [8] and [10]. Roughly speaking, formula (2.87) tells us that, on the average, if all \( n \) coupons have already been detected \( m \) times, where \( m \sim \beta \ln n \), then each additional detection (of all coupons) “costs” \( (\alpha/\beta)n \). Apart from being interesting by itself, formula (2.87) is used in the proof of Theorem 6 below.

3 The limiting distribution of \( D_{m,n} \)

Let us sketch our strategy for determining the limiting distribution of \( D_{m,n} \).

We start with the observation that formulas (1.11) and (1.25) hint that, under a suitable normalization the limiting distributions of \( \Delta_{m,n} \) and \( D_{m,n} \) should coincide. Hence, we can first try to find the limiting distribution of \( \Delta_{m,n} \), which, thanks to (1.10), seems an easier problem, and from that obtain the limiting distribution of \( D_{m,n} \) (with the help of the “Converging Together Lemma” — see below). In order, though, to determine the limiting distribution of \( \Delta_{m,n} \), we first need to come up with its correct normalization, and this, in view of (1.10), can be accomplished, if we manage to find an expression \( t \) for which \( S_m(t/n)e^{-t/n} \sim Q/n \), where \( Q \) is some quantity which is independent of \( n \). This task is more delicate than the one of the previous section, where we had to determine an interval of values of \( x \) in which \( S_m(x)e^{-x} \) changes from \( \gg 1/n \) to \( \ll 1/n \). For this reason one expects that, in order to achieve the desired asymptotics, i.e. \( S_m(t/n)e^{-t/n} \sim Q/n \), we may need to impose some mild restrictions on \( m(n) \).
Another thing worth repeating here is that, in order to obtain the limiting distribution of $D_{m,n}$ from the limiting distribution of $\Delta_{m,n}$ via the Converging Together Lemma, it is necessary to have an estimate for the growth of $E[D_{m,n}]$. Hence, formulas (2.38) and (2.87) play a key role in the proofs of Theorems 4 and 6 below.

Finally, let us mention that by using the approach of this section, one can, also, give an alternative proof of the formula (1.2) of Erdős and Rényi [4].

### 3.1 The limiting distribution of $D_{m,n}$ in the supercritical case

**Theorem 3.** Let $\Delta_{m,n}$ be a random variable whose distribution function is given by (1.10), where $m = m(n)$, $n = 1, 2, \ldots$, is a sequence of positive integers such that

$$\frac{(\ln n)^3}{m(n)} \to 0 \quad (3.1)$$

(in other words, $m \gg \ln^3 n$). Then, for any fixed $y \in \mathbb{R}$ we have

$$\lim_{n \to \infty} \mathbb{P} \left\{ \frac{\Delta_{m,n} - nm - n\sqrt{m}\sqrt{2\ln n - \ln \ln n}}{n \sqrt{m/2 \ln n}} \leq y \right\} = \exp \left( -\frac{e^{-y}}{2\sqrt{\pi}} \right) \quad (3.2)$$

or, equivalently,

$$\sqrt{2m \ln n} \left( \frac{\Delta_{m,n}}{nm} - 1 \right) - 2 \ln n + \frac{\ln \ln n}{2} + \ln \left( 2\sqrt{\pi} \right) \xrightarrow{d} G, \quad (3.3)$$

where $G$ is the standard Gumbel random variable and the symbol $\xrightarrow{d}$ denotes convergence in distribution.

**Proof.** For typographical convenience we set

$$\lambda = \lambda(n) := m(n) + \sqrt{m(n)\sqrt{2\ln n - \ln \ln n}} + \frac{m(n)}{\sqrt{2 \ln n}} y, \quad (3.4)$$

where $y$ is a fixed real number. Then, $\lambda > 0$ for all sufficiently large $n$ and (1.10) implies

$$\mathbb{P} \left\{ \frac{\Delta_{m,n} - nm - n\sqrt{m}\sqrt{2\ln n - \ln \ln n}}{n \sqrt{m/2 \ln n}} \leq y \right\} = \mathbb{P} \{ \Delta_{m,n} \leq n\lambda \}
\quad = \left[ 1 - S_m(\lambda)e^{-\lambda} \right]^n, \quad (3.5)$$

where we have suppressed the dependence of $m$ and $\lambda$ on $n$ for typographical convenience.

We can, now, use (2.10) in order to compute the asymptotics of $S_m(\lambda)e^{-\lambda}$. First we notice that from (3.4) we have $(\lambda - m) \to \infty$ and, furthermore,

$$\frac{\sqrt{m}}{\lambda - m} = \frac{1}{\sqrt{2 \ln n - \ln \ln n} + \frac{y}{\sqrt{2 \ln n}}} \sim \frac{1}{\sqrt{2 \ln n}}. \quad (3.6)$$
hence (2.11) is satisfied (for $x = \lambda$) and, consequently, formula (2.10) yields

$$S_m(\lambda)e^{-\lambda} \sim \frac{1}{\sqrt{2\pi}} \left( \frac{\lambda}{m} \right)^m \frac{\sqrt{m}}{\lambda - m} e^{-(\lambda - m)}. \quad (3.7)$$

Next, by invoking (3.4), formula (3.7) becomes

$$S_m(\lambda)e^{-\lambda} \sim \left( 1 + \frac{\sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}}}{\sqrt{m}} \right)^m e^{-\sqrt{m}\left( \sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}} \right)} \quad (3.8)$$

Now, in view of (3.1),

$$\left( 1 + \frac{\sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}}}{\sqrt{m}} \right)^m = e^m \ln \left( 1 + \frac{\sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}}}{\sqrt{m}} \right) \sim e^{\sqrt{m}\left( \sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}} \right)} - \frac{1}{2} \left( \sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}} \right)^2 \sim e^{\sqrt{m}\left( \sqrt{2\ln n - \ln \ln n + \frac{y}{\sqrt{2\ln n}}} \right)} - \ln n + \frac{\ln n}{2} - y. \quad (3.9)$$

Substituting (3.9) in (3.8) we obtain

$$S_m(\lambda)e^{-\lambda} \sim e^{-\ln n + \frac{\ln n}{2} - y} \sim e^{-\frac{\ln n + \ln n - y}{2\sqrt{\pi} \sqrt{\ln n}}} = \frac{1}{n} e^{-y} \quad (3.10)$$

and, therefore, formula (3.2) follows by using (3.10) in (3.5) and letting $n \to \infty$. 

By a straightforward adaptation of the above proof we can cover the case where, instead of the assumption (3.1), we allow the slightly more general condition $m \gg (\ln n)^p$, with $1 < p < 3$. However, the formulas get considerably messier.

Theorem 3 is by itself interesting. However, our ultimate goal is to prove a similar statement for the variable $D_{m,n}$. In order to relate $D_{m,n}$ to $\Delta_{m,n}$, we will use the following well-known lemma (see, e.g., [2], [3]).

Converging Together Lemma. Let $X_n$ and $Y_n$, $n = 1, 2, \ldots$, be two sequences of random variables such that

$$X_n \xrightarrow{d} X \quad \text{and} \quad (Y_n - X_n) \xrightarrow{d} 0, \quad (3.11)$$

where $X$ is some random variable. Then

$$Y_n \xrightarrow{d} X. \quad (3.12)$$

We are now ready to give the limiting distribution of $D_{m,n}$.
**Theorem 4.** Let \( m = m(n), n = 1, 2, \ldots \), be a sequence of positive integers such that \( m \gg \ln^3 n \). Then, for any fixed \( y \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{D_{m,n} - mn - n\sqrt{m/2} \ln n}{n \sqrt{2 \ln n}} \leq y \right) = \exp \left( -\frac{e^{-y}}{2\sqrt{\pi}} \right)
\]

or, equivalently,

\[
\sqrt{2m \ln n} \left( \frac{D_{m,n}}{nm} - 1 \right) - 2\ln n + \frac{\ln \ln n}{2} + \ln \left( 2\sqrt{\pi} \right) \xrightarrow{d} G,
\]

where \( G \) is the standard Gumbel random variable.

**Proof.** Let us set

\[
Z_{m,n} := \frac{\Delta_{m,n} - D_{m,n}}{\sqrt{D_{m,n}}}. \tag{3.15}
\]

Then

\[
\mathbb{E} \left[ Z_{m,n}^2 \right] = \mathbb{E} \left[ \frac{\Delta_{m,n}^2}{D_{m,n}} \right] - 2 \mathbb{E} \left[ \Delta_{m,n} \right] + \mathbb{E} \left[ D_{m,n} \right] = \mathbb{E} \left[ \frac{\Delta_{m,n}^2}{D_{m,n}} \right] - \mathbb{E} \left[ D_{m,n} \right],
\]

where the second equality follows from (1.26). Now, in view of (1.21) and (1.22),

\[
\mathbb{E} \left[ \frac{\Delta_{m,n}^2}{D_{m,n}} \right] D_{m,n} = \frac{1}{D_{m,n}} \mathbb{E} \left[ \Delta_{m,n}^2 \mid D_{m,n} \right] = \frac{D_{m,n}^{(2)}}{D_{m,n}} = D_{m,n} + 1 \tag{3.17}
\]

and hence

\[
\mathbb{E} \left[ \frac{\Delta_{m,n}^2}{D_{m,n}} \right] = \mathbb{E} \left[ D_{m,n} \right] + 1. \tag{3.18}
\]

Therefore, by substituting (3.18) in (3.16) we obtain that

\[
\mathbb{E} \left[ Z_{m,n}^2 \right] = 1 \tag{3.19}
\]

(actually, in view of (1.11) and the fact that \( D_{m,n} \geq mn \), one can show that \( Z_{m,n} \) converges in distribution to a standard normal random variable, as \( m \to \infty \) or \( n \to \infty \)).

Now, from (3.15) we get

\[
|\Delta_{m,n} - D_{m,n}| = |Z_{m,n}| \sqrt{D_{m,n}}
\]

and, hence, in view of (3.19),

\[
\mathbb{E} \left[ |\Delta_{m,n} - D_{m,n}| \right] = \mathbb{E} \left[ |Z_{m,n}| \sqrt{D_{m,n}} \right] \leq \sqrt{\mathbb{E} \left[ Z_{m,n}^2 \right] \mathbb{E} \left[ D_{m,n} \right]} = \sqrt{\mathbb{E} \left[ D_{m,n} \right]}.
\]

(3.20)
Thus, in view of (2.38),

\[
E \left[ \left| \Delta_{m,n} - D_{m,n} \right| \right] \leq \frac{\sqrt{E[D_{m,n}^2]}}{n \sqrt{m/2 \ln n}} \to 0. \tag{3.21}
\]

It follows that we can apply the Converging Together Lemma to the sequences

\[
X_n := \frac{\Delta_{m,n} - nm - n \sqrt{m/2 \ln n} \ln \ln n}{n \sqrt{m/2 \ln n}}
\]

and

\[
Y_n := \frac{D_{m,n} - nm - n \sqrt{m/2 \ln n} \ln \ln n}{n \sqrt{m/2 \ln n}}
\]

and conclude that their limiting distributions coincide.

\[\blacksquare\]

**Remark 4.** In the somehow extreme case where \( m \to \infty \), while \( n \) stays fixed, the limiting distribution of \( \Delta_{m,n} \) follows directly from (1.7): Since each \( T_j, j = 1, 2, \ldots, n \) is Erlang with parameters \( m \) and \( 1/n \), the Central Limit Theorem yields

\[
\frac{T_j - mn}{n \sqrt{m}} \xrightarrow{d} Z, \quad m \to \infty, \tag{3.22}
\]

where \( Z \) is the standard normal random variable. Then, formula (1.7) and the independence of the \( T_j \)'s imply immediately that

\[
\frac{\Delta_{m,n} - nm}{n \sqrt{m}} \xrightarrow{d} \max\{Z_1, Z_2, \ldots, Z_n\}, \quad m \to \infty, \tag{3.23}
\]

where \( Z_1, Z_2, \ldots, Z_n \) are independent standard normal variables (also, in view of (1.26), we can obtain that \( E[D_{m,n}] = E[\Delta_{m,n}] \sim nm \) as \( m \to \infty \)). However, in the case of a fixed \( n \), formula (3.21) in the proof of Theorem 4 fails and, hence, we cannot conclude that \( D_{m,n} \) and \( \Delta_{m,n} \) have the same limiting distributions. Actually, in the trivial case \( n = 1 \) we obviously have \( D_{m,1} = m \), while from (3.23) we see that \( (\Delta_{m,1} - m)/\sqrt{m} \xrightarrow{d} Z \).

### 3.2 The limiting distribution of \( D_{m,n} \) in the critical case

We will now consider the case

\[
m = m(n) = \beta \ln n + b(n) \ln n, \quad \text{where} \quad b(n) = o\left(\frac{1}{\sqrt{\ln n}}\right), \tag{3.24}
\]

i.e. \( m = \beta \ln n + o(\sqrt{\ln n}) \) (as usual, \( \beta > 0 \) is a fixed constant). The restriction on the order of \( b(n) \) is imposed in order to keep the formulas relatively simple (we believe that by a straightforward adaptation of the
proof of Theorem 5 below we can cover the more general case \( b(n) \ll (\ln n)^{-p} \) for a fixed \( p \in (0, 1/2) \); however, the formulas will get considerably messier.

**Theorem 5.** Let \( \Delta_{m,n} \) be a random variable whose distribution function is given by (1.10), where \( m = m(n) \), \( n = 1, 2, \ldots \), is a sequence of positive integers satisfying (3.24). Then, for any fixed \( y \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \frac{\Delta_{m,n} - \alpha n \ln n - \frac{\alpha(\alpha - \beta - 1)}{\beta(\alpha - \beta)} b(n) n \ln n + \frac{\alpha}{\alpha - \beta} n \ln n}{\frac{\alpha}{\alpha - \beta} n} \leq y \right\} = \exp \left( -\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta}}{\alpha - \beta} e^{-y} \right),
\]

(3.25)

where \( \alpha \) is given by (2.45) (recall that \( \alpha > \beta \)).

**Proof.** We will follow the approach of the proof of Theorem 3. For typographical convenience we set

\[
\chi = \chi(n) := \alpha \ln n + \frac{\alpha(\alpha - \beta - 1)}{\beta(\alpha - \beta)} b(n) n \ln n - \frac{\alpha}{2(\alpha - \beta)} n \ln n + \frac{\alpha}{\alpha - \beta} y,
\]

(3.26)

where \( y \) is a fixed real number. Since \( b(n) = o(1) \), the quantity \( \chi \) is positive for all sufficiently large \( n \) and, hence, (1.10) implies

\[
\mathbb{P} \left\{ \frac{\Delta_{m,n} - \alpha n \ln n - \frac{\alpha(\alpha - \beta - 1)}{\beta(\alpha - \beta)} b(n) n \ln n + \frac{\alpha}{\alpha - \beta} n \ln n}{\frac{\alpha}{\alpha - \beta} n} \leq y \right\} = \mathbb{P} \{ \Delta_{m,n} \leq n\chi \} = \left[ 1 - S_m(\chi)e^{-\chi} \right]^n.
\]

(3.27)

Next, we write (3.26) as

\[
\chi(n) = \alpha \ln n + a(n) \ln n,
\]

(3.28)

where

\[
a = a(n) := \frac{\alpha(\alpha - \beta - 1)}{\beta(\alpha - \beta)} b(n) - \frac{\alpha}{2(\alpha - \beta)} \ln n + \frac{\alpha}{\alpha - \beta} y,
\]

(3.29)

so that, our assumption (3.24) implies

\[
a(n) = o \left( \frac{1}{\ln n} \right)
\]

(3.30)

(in particular, \( a(n) \to 0 \) as \( n \to \infty \)). We can, therefore, invoke formulas (2.53) and (2.54) appearing in the proof of Lemma 1 and obtain

\[
S_m(\chi)e^{-\chi} \sim \frac{1}{n} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta}}{\alpha - \beta} e^{A(n)\ln n},
\]

(3.31)
where

\[ A(n) := (\ln \alpha - \ln \beta + 1)b - (\beta + b) \ln \left(1 + \frac{b}{\beta}\right) \]

\[ - a + (\beta + b) \ln \left(1 + \frac{a}{\alpha}\right) - \frac{1}{2} \ln \ln n \]  

\[ - \alpha - \beta - \frac{1}{2} \beta \ln n + \ln \ln n + C \xrightarrow{d} G, \]  

(3.32)

(here, \( a = a(n) \) is, of course, given by (3.29)).

Now, in view of (3.24) and (3.30), formula (3.32) implies

\[ A(n) = (\ln \alpha - \ln \beta)b - \frac{1}{2} \ln \ln n - \alpha - \beta a + o \left(\frac{1}{\ln n}\right). \]  

(3.33)

We can, then, substitute (3.29) in (3.33) and get

\[ A(n) = (\ln \alpha - \ln \beta)b - \frac{\alpha - \beta - 1}{\beta} b - \frac{y}{\ln n} + o \left(\frac{1}{\ln n}\right), \]  

(3.34)

or, in view of (2.45),

\[ A(n) = - \frac{y}{\ln n} + o \left(\frac{1}{\ln n}\right), \]  

(3.35)

Thus, by substituting (3.35) in (3.31) we obtain

\[ S_m(\chi)e^{-\chi} \sim \frac{1}{n} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta}}{\alpha - \beta} e^{-y}. \]  

(3.36)

Therefore, formula (3.25) follows by using (3.36) in (3.27) and then letting \( n \to \infty \). ■

Let us notice that formula (3.25) can be, also, expressed equivalently as

\[ \frac{\alpha - \beta}{\alpha} \frac{\Delta_{m,n}}{n} - (\alpha - \beta) \ln n - \frac{\alpha - \beta - 1}{\beta} b(n) \ln n + \frac{\ln \ln n}{2} + C \xrightarrow{d} G. \]  

(3.37)

with

\[ C := \frac{1}{2} \ln \left(\frac{2\pi (\alpha - \beta)^2}{\beta}\right) \]  

(3.38)

(as usual, \( G \) is the standard Gumbel random variable). Furthermore, in view of (3.24), formula (3.37) can be also written as

\[ \frac{\alpha - \beta}{\alpha} m \left(\frac{\Delta_{m,n}}{nm} - 1\right) + \frac{b(n)}{\beta} \ln n + \frac{\ln \ln n}{2} + C \xrightarrow{d} G. \]  

(3.39)

As in the supercritical case, in the critical case too, under the above normalization \( \Delta_{m,n} \) and \( D_{m,n} \) have the same limiting distributions:
Theorem 6. Let \( m = m(n), n = 1, 2, \ldots \), be a sequence of positive integers satisfying (3.24). Then, for any fixed \( y \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} \mathbb{P} \left\{ D_{m,n} - \alpha n \ln n - \frac{\alpha(\alpha-1)}{\beta(\alpha-\beta)} b(n) \ln n + \frac{\alpha}{\alpha-\beta} \frac{\ln n}{\ln \ln n} \leq y \right\} = \exp \left( -\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta}}{\alpha - \beta} e^{-y} \right),
\]

where \( \alpha \) is given by (2.45). Equivalently,

\[
\frac{\alpha - \beta}{\alpha} m \left( D_{m,n} - 1 \right) + \frac{b(n)}{\beta} \ln n + \frac{\ln n}{2} + C \xrightarrow{d} G,
\]

where \( C \) is given by (3.38) and \( G \) is the standard Gumbel random variable.

The proof of Theorem 6 is exactly the same as the proof of Theorem 4. The only difference here is that, in order to justify the analog of estimate (3.21), instead of (2.38), we now use (2.87).

### 3.3 Comparison of the formulas (3.14) and (3.41)

As we saw in the previous subsections, in the critical case \( m = \beta \ln n + o(\sqrt{\ln n}) \) the limiting behavior of \( D_{m,n} \) is given by (3.41), while in the supercritical case \( m \gg \ln^3 n \) the limiting behavior of \( D_{m,n} \) is given by (3.14). In this short subsection we will enquire whether these two behaviors can, in some sense, be “bridged.”

Heuristically, one expects that the transition from the critical to the supercritical case can be observed by letting \( \beta \to \infty \).

Let us first notice that the left-hand side of (3.14) as well as the left-hand side of (3.41) consist of four terms. Of these four terms, the third, namely \( \ln \ln n/2 \), is common in both formulas, while the second terms do not agree, although they both contain the factor \( \ln n \).

From the fact that \( \alpha > \beta \) it follows that if \( \beta \to \infty \), then \( \alpha \to \infty \). Now formula (2.45) implies

\[
\frac{\alpha}{\beta} - 1 = \ln \left( \frac{\alpha}{\beta} \right) + \frac{1}{\beta},
\]

hence

\[
\frac{\alpha}{\beta} - 1 = \ln \left( \frac{\alpha}{\beta} \right) + o(1) \quad \text{as} \quad \beta \to \infty
\]

from which we get that

\[
\frac{\alpha}{\beta} \to 1 \quad \text{as} \quad \beta \to \infty.
\]
Let us set
\[ u := \frac{\alpha}{\beta} - 1. \quad (3.45) \]

Then, (3.42) can be written as
\[ u = \ln (1 + u) + \frac{1}{\beta}, \quad (3.46) \]

Since (in view of (3.44)) \( u = o(1) \) as \( \beta \to \infty \), formula (3.46) implies
\[ u = u - \frac{u^2}{2} + \frac{1}{\beta} + O \left( u^3 \right), \quad \beta \to \infty \]
i.e.
\[ u = \sqrt{\frac{2}{\beta}} + O \left( u^{3/2} \right), \quad \beta \to \infty. \quad (3.47) \]

Since \( u \gg u^{3/2} \) as \( \beta \to \infty \), formula (3.47) can be written as
\[ u = \sqrt{\frac{2}{\beta}} + O \left( \frac{1}{\beta^{3/4}} \right), \quad \beta \to \infty, \quad (3.48) \]
or, in view of (3.46),
\[ \frac{\alpha}{\beta} - 1 = \sqrt{\frac{2}{\beta}} + O \left( \frac{1}{\beta^{3/4}} \right), \quad \beta \to \infty, \quad (3.49) \]

which implies immediately that
\[ \frac{\alpha - \beta}{\sqrt{\beta}} = \sqrt{2} + O \left( \frac{1}{\beta^{3/4}} \right), \quad \beta \to \infty. \quad (3.50) \]

Using (3.50) in (3.38) we obtain that
\[ C = \frac{1}{2} \ln \left( \frac{2\pi(\alpha - \beta)^2}{\beta} \right) \to \frac{1}{2} \ln (4\pi) = \ln (2\sqrt{\pi}) \quad \text{as} \quad \beta \to \infty. \quad (3.51) \]

Therefore, the fourth term of the left-hand side of (3.41) (namely \( C \)) approaches the corresponding (fourth) term of the left-hand side of (3.14), namely \( \ln (2\sqrt{\pi}) \), as \( \beta \to \infty \).

Finally, regarding the first term of the left-hand side of (3.41) we have (in view of (3.44) and (3.50))
\[ \frac{\alpha - \beta}{\alpha} \sim \frac{\alpha - \beta}{\alpha} \beta \ln n \sim (\alpha - \beta) \ln n = \frac{\alpha - \beta}{\sqrt{\beta}} \sqrt{\beta} \ln n \sim \sqrt{2\beta} \ln n \quad (3.52) \]

and this is in “asymptotical agreement” with the corresponding factor of the first term of the left-hand side of (3.14) (in the case where \( m \sim \beta \ln n \)) since
\[ \sqrt{2m \ln n} \sim \sqrt{2\beta \ln^2 n} \sim \sqrt{2\beta} \ln n. \quad (3.53) \]
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