Einstein–Cartan Algebroids and Black Holes in Solitonic Backgrounds

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October 23, 2018

Abstract

We construct a new class of exact solutions describing spacetimes possessing Lie algebroid symmetry. They are described by generic off-diagonal 5D metrics embedded in bosonic string gravity and possess nontrivial limits to the Einstein gravity. While we focus on nonholonomic vielbein transforms of the Schwarzschild metrics to 5D ansatz with solitonic backgrounds, much of the analysis continues to hold for more general configurations with nontrivial Lie algebroid structure and nonlinear connections. We carefully investigate some examples when the anchor structure is related to 3D solitonic interactions. The approach defines a general geometric method of constructing exact solutions with various type of symmetries and new developments and applications of the Lie algebroid theory.

Keywords: Lie algebroids, exact solutions, gravitational solitons, nonholonomic vielbeins, black holes.

PACS Classification:
02.40.-k, 04.20.Jb, 04.50.+h, 05.45.Yu, 04.70.-s, 04.90.+e

2000 AMS Subject Classification:
17B99, 53A40, 83C20, 83E99

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1 Introduction

It has been widely investigated the black hole geometry and physics for 2D, 3D, 4D and extra dimensional gravity theories \(^1\); see, for instance, Refs. \(^2\). The bulk of such exact solutions define asymptotically flat spacetimes, or spaces with nontivial cosmological constant, possessing Killing symmetry and constructed for the Minkowski, or (anti) De Sitter, backgrounds. It is less known how the black hole objects can be self–consistently defined for non–constant curvature backgrounds and what kind of symmetries and physical properties may characterize such solutions. In a series of recent papers, we studied certain classes of metrics for black ellipsoids \(^3\) \(^4\) \(^5\), black holes and wormholes in solitonic, spinorial, noncommutative and different \(^6\)

\(^1\) we shall write 2D for two dimensional, 3D for three dimensional and so on...
types of nonholonomic backgrounds \cite{6,4} and locally anisotropic deformations of the Taub-NUT metrics \cite{7}. The new class of solutions possess an explicit nonlinear polarization of constants in the metric and linear connection coefficients, induced from extra dimension and/or by specific off–diagonal gravitational interactions, and related to a special class of nonholonomic frame transforms (vierbeins) with associated nonlinear connection (in brief, N–connection) structure. Such solutions can be constructed in string/brane gravity, and in general relativity, and constrained to define asymptotically flat spacetimes. At least in a finite region, they are characterized by generalized Lie group structure equations emphasizing certain type of noncommutative symmetries (investigated in Ref. \cite{8}) or, for a different class of constructions, by Lie algebroid symmetries.

In this paper, inspired by the Lie algebroid theory with applications to mechanics and classical field theory \cite{9,10,11,12,13} and recent approaches to the theory of gauge fields and gravity \cite{14,15,16}, we want to address essentially the following purposes:

Given an Einstein–Cartan manifold with the metric and affine connections satisfying the field equations of string gravity, to construct new classes of exact solutions describing spacetimes with Lie algebroid structure (gravitational algebroids). Such solutions will be parametrized by generic off–diagonal metric ansatz, anholonomic moving frames and generalized affine connections related to certain three/two dimensional solitonic gravitational configurations\textsuperscript{2} with nontrivial limits to the Schwarzschild black hole metric. We shall analyze the conditions when the solutions can be constrained to describe gravitational and matter field interactions in the framework of the Einstein gravity. There will be presented new motivations for the algebroid theory and developed a new type of Lie algebroid geometry on nonholonomic manifolds provided with nonlinear connection structure derived from the gravity theory.\textsuperscript{3}

The paper is organized as follows: In section 2, we outline the basic Lie algebroid constructions behind the Einstein–Cartan spacetimes provided with nontrivial nonholonomic vielbeins with associated N–connection structure and consider the effective field equations with string corrections. Section 3 is devoted to a geometric method of constructing 5D and 4D off–diagonal solutions for gravitational algebroids. In section 4, there are elaborated two classes of exact solutions with trivial conformal factors generated by nonholonomic Lie algebroid deformations of the Schwarzschild solution to certain

\textsuperscript{2}In brief, we shall write 3D; for four dimensions, 4D, and so on...

\textsuperscript{3}A manifold is nonholonomic if it is provided with a nonintegrable distribution of submanifolds. In this paper, we shall emphasize the constructions for a special class when the distribution defines the nonlinear connection structure.
type of gravitational solitonic spacetimes. The conclusions are presented in section 5. The Appendix outlines the main steps of constructing solutions on nonholonomic manifolds. Additionally to the solutions investigated in section 4, we briefly analyze two examples of metrics with nontrivial conformal factors in Appendix B.

2 Riemann–Cartan algebroids

In this paper, for simplicity, we shall work with real, paracompact and necessary smooth class manifolds and maps and with locally trivial bundle spaces.

2.1 Lie algebroids and N–connections

The standard definition of a Lie algebroid \( \mathcal{A} \equiv (E, [,], \rho) \) is related to a vector bundle \( \mathcal{E} = (E, \pi, M) \), with a surjective map \( \pi : E \to M \) of the total spaces \( E \) to the base manifold \( M \), of respective dimensions \( \dim E = n + m \) and \( \dim M = n \). The algebroid structure is stated by the anchor map \( \rho : E \to TM \) (\( TM \) is the tangent bundle to \( M \)) and a Lie bracket on the \( C^\infty(M) \)–module of sections of \( E \), denoted \( \text{Sec}(E) \), such that

\[
[X, fY] = f[X,Y] + \rho(X)(f)Y
\]

for any \( X, Y \in \text{Sec}(E) \) and \( f \in C^\infty(M) \). The anchor also induces a homomorphism of \( C^\infty(M) \)–modules \( \rho : \text{Sec}(\mathcal{A}) \to \mathcal{X}^1(M) \) where \( \wedge^r(M) \) and \( \mathcal{X}^r(M) \) will denote, respectively, the spaces of differential \( r \)–forms and \( r \)–multivector fields on \( M \).

In order to investigate geometric models of gravity and string theories with nonholonomic frame (vielbein) structure, one does not work on a vector bundle \( E \), or a tangent bundle \( TM \), but on a general manifold \( V \), \( \dim V = n + m \), which is a (pseudo) Riemannian spacetime, or a more general one with possible torsion and nonmetricity fields (for the purposes of this paper, we shall consider it to be a Riemann–Cartan, or Einstein, manifold; see explicit constructions and references in [3]). A Lie algebroid structure can be modelled locally on a spacetime \( V \) by considering a Whitney type sum

\[
TV = hV \oplus vV
\]

(1)

defining a splitting into certain conventional horizontal (h) and vertical (v) subspaces. In this case, the anchor is defined as a map \( \hat{\rho} : V \to hV \) and the Lie bracket structure is considered on the spaces of sections \( \text{Sec}(vV) \).

\[\text{on Lie algebroids geometry and applications, see Refs.} \quad [9, 10, 11, 12, 13]\]
Roughly speaking, we consider Riemann–Cartan manifolds admitting a locally fibered structure induced by the splitting (1) when the Lie algebroid constructions are usual ones but with formal substitutions $E \rightarrow V$ and $M \rightarrow hV$. In general, a such structure is not integrable which mean that we work on a nonholonomic manifold, see details in Refs. [17, 16].

The global decomposition (1) is equivalent to an exact sequence

$$0 \rightarrow vV \rightarrow TV \rightarrow TV/vV \rightarrow 0,$$

giving a morphism $N : TV \rightarrow vV$ such that $N \circ i$ is the unity in the vertical subbundle $vV$ (the kernel $\ker \pi^\top \cong vV$, for $\pi^\top : TV \rightarrow hV$) where $i : vV \rightarrow TV$ is the inclusion mapping. The morphism $N$ defines a nonlinear connection (in brief, $N$–connection) on the spacetime $V$. The manifolds provided with $N$–connection structure are called $N$–anholonomic [3].

Let us state the typical notations for abstract (coordinate) indices of geometrical objects defined with respect to an arbitrary (or coordinate) local basis, i. e. with respect to a system of reference. For a local basis on $V$, we write $e_\alpha = (e_i, v_a)$. The small Greek indices $\alpha, \beta, \gamma, ...$ are considered to be general ones, running values $1, 2, \ldots, n + m$ and $i, j, k, ...$ and $a, b, c, ...$ respectively label the geometrical objects on the base and typical "fiber" and run, correspondingly, the values $1, 2, \ldots, n$ and $1, 2, \ldots, m$. The dual base is denoted by $e^\alpha = (e^i, v^a)$. The local coordinates of a point $u \in V$ are written $u = (x, u)$, or $u^a = (x^i, u^a)$, where $u^a(u)$ is the $a$-th coordinate with respect to the basis $(v_a)$ and $(x^i)$ are local coordinates on $hV$ with respect to $e_i$. By $u^a(x)$, one denotes sections of $vV$ over $M$. We shall use "boldface" symbols in order to emphasize that the objects are defined on spaces provided with $N$–connection structure.

A $N$–connection $N$ is described by its coefficients,

$$N = N^a_\dot{\alpha}(u)dx^\dot{\alpha} \otimes \frac{\partial}{\partial u^a} = N^b_i(u)e^i \otimes v_b,$$

where we underlined the indices defined with respect to the local coordinate basis

$$e_\underline{\alpha} = \partial_\underline{\alpha} = \partial/\partial u^\underline{\alpha} = (e_\underline{\dot{\alpha}} = \partial/\partial x^\dot{\alpha}, v_\underline{\alpha} = \partial_\underline{\alpha} = \partial/\partial u^\underline{\alpha})$$

and its dual

$$e^\underline{\alpha} = du^\underline{\alpha} = (e^\dot{\alpha} = dx^\dot{\alpha}, e^\underline{a} = du^\underline{a}).$$

We can also consider a 'vielbein’ transform

$$e_\alpha = e^\underline{\alpha}(u)e_\underline{\alpha} \quad \text{and} \quad e^\alpha = e^\underline{\alpha}(u)e^\underline{\alpha} \quad \text{(2)}$$
given respectively by a nondegenerate matrix $e_\beta^\alpha(u)$ and its inverse $e^{\alpha}_{\beta}(u)$. Such matrices parametrize maps from a local coordinate frame and co-frame, respectively, to any general frame $e_\alpha = (e_i, v_a)$ and co-frame $e^\alpha = (e^i, v^a)$. The class of linear connections consists, for instance, a case of linear dependence on $\mathbf{u}$, i.e. $N^a_i(x, u) = \Gamma^a_i(x)u^i$.

In local form, the Lie algebroid structure on the manifold $V$ is stated by its structure functions $\rho^i_a(x)$ and $C^c_{ab}(x)$ defining the relations

$$\rho(v_a) = \rho^i_a(x)e_i = \rho^a_i(x)\partial_i,$$

$$[v_a, v_b] = C^c_{ab}(x)v_c$$

and subjected to the structure equations

$$\rho^i_a \frac{\partial \rho^j_b}{\partial x^j} - \rho^j_b \frac{\partial \rho^i_a}{\partial x^j} = \rho^j_c C^c_{ab} \text{ and } \sum_{cyclic(a,b,c)} \left( \rho^j_a \frac{\partial C^d_{bc}}{\partial x^j} + C^d_{af}C^f_{bc} \right) = 0.$$ (5)

For simplicity, we shall omit underlying of coordinate indices if it will not result in ambiguities. Such equations are standard ones for the Lie algebroids but defined on N–anholonomic manifolds. In brief, we call such spaces to be Lie N–algebroids.

By straightforward computations, we can prove that the Lie algebroid and N–connection structures prescribe a subclass of preferred local frames related by subclass of matrix transforms (2) linearly depending on $N^a_i(x, u)$, with the coefficients

$$e_\alpha^a(u) = \begin{bmatrix} e_i^a(u) & N^b_i(u)e_b^a(u) \\ 0 & e_a^a(u) \end{bmatrix}$$ (6)

and

$$e^{\beta}_{\alpha}(u) = \begin{bmatrix} e^i_{\alpha}(u) & -N^b_k(u)e^k_{\alpha}(u) \\ 0 & e^a_{\alpha}(u) \end{bmatrix}.$$ (7)

Such transforms generate N–adapted frames

$$e_\alpha = (e_i = \frac{\partial}{\partial x^i} - N^b_i v_b, v_b)$$ (8)

and dual coframes

$$e^\alpha = (e^i, v^b = v^b + N^b_i dx^i),$$ (9)

for any $v_b = e^b_i \partial_i$ satisfying the condition $v_c | v^b = \delta^b_c$. In a particular case, we can take $v_b = \partial_b$. 

6
We note that the operators $e_\alpha$ (8) and $e^a$ (9) are the so-called "N-elongated" partial derivatives and differentials which define a N-adapted differential calculus on N-anholonomic manifolds. In the structure equations (5), we have $e_i \rho^b_a \rightarrow \partial_i \rho^b_a$ because the structure functions $\rho^i_a(x)$ and $C^f_{ab}(x)$ do not depend on v-variables $u^a$. For trivial N-connections, we can put $N^a_i = 0$ and obtain the usual Lie algebroid constructions.

The curvature of a N-connection $\Omega \equiv -N_v$ is defined as the Nijenhuis tensor

$$N_v(X, Y) \equiv [vX, vY] + vv[X, Y] - v[X, vY] - v[Y, vX]$$

for any $X, Y \in \mathfrak{X}(\mathcal{V})$ associated to the vertical projection "v" defined by this N-connection, i.e.

$$\Omega = \frac{1}{2} \Omega^b_{ij} e_i \wedge e_j \otimes v^b$$

with the coefficients

$$\Omega^b_{ij} = \rho^b_{ij} = \rho_{ij}^a N^a_i = e_j N^a_i - e_i N^a_j + N^b_i v^b_i (N^a_j) - N^a_j v^b_i (N^a_i).$$

The vielbeins (8) satisfy certain nonholonomy (equivalently, anholonomy) relations

$$[e_\alpha, e_\beta] = W^\gamma_{\alpha\beta} e_\gamma$$

with nontrivial anholonomy coefficients $W^a_{jk} = \Omega^a_{jk}(x, u)$, $W^b_{ie} = v_e N^b_i (x, u)$ and $W^b_{ae} = C^b_{ae}(x)$ reflecting the fact that the Lie algebroid is N-anholonomic.

We can write down the Lie algebroid and N-connection structures in a compatible form by introducing the "N-adapted" anchor

$$\hat{\rho}^j_i (x, u) \equiv e^j_i (x, u) e_a \rho^b_{a\beta}(x, u) \rho^a_{a\beta}(x)$$

and "N-adapted" (boldfaced) structure functions

$$C^f_{ag}(x, u) = e^f_i (x, u) e_a \rho^b_{a\beta}(x, u) e_{g\beta}(x, u) C^f_{gb}(x),$$

respectively, into formulas (8), (10) and (11). We conclude that the Lie algebroids on N-anholonomic manifolds are defined by the corresponding sets of functions $\hat{\rho}^j_i (x, u)$ and $C^f_{ag}(x, u)$ with additional dependencies on v-variables $u^b$ for the N-adapted structure functions. For such generalized Lie N-algebroids, the structure relations became

$$\hat{\rho}(v^b) = \hat{\rho}^j_i (x, u) e_i,$$

$$[v_d, v_b] = C^f_{db}(x, u) v_f$$
and the structure equations of the Lie N–algebroid are written

\[
\sum_{\text{cyclic}(a,b,e)} \left( \hat{\rho}_{\alpha}^{i} e_{j}^a (C_{be}^{f}) + C_{ag}^{f} C_{be}^{a} - C_{b'e'} \hat{\rho}_{a}^{j} Q_{f'}^{e' e} \right) = 0,
\]

for \( Q_{f'}^{e' e} = e_{e'}^{f'} e_{e'}^{f} e_{j}^{f} e_{j}^{f} e_{j}^{f} e_{j}^{f} \) with the values \( e_{e'}^{f'} \) and \( e_{e'}^{f} \) defined by the N–connection. The Lie N–algebroid structure is characterized by the coefficients \( \hat{\rho}_{\alpha}^{i} \) \( (x,u) \) and \( C_{ag}^{f} \) \( (x,u) \) stated with respect to the N–adapted frames (8) and (9).

A Riemann–Cartan algebroid (in brief, RC–algebroid) is a Lie algebroid \( A = (V, [\cdot, \cdot], \rho) \) associated to a N–anholonomic manifold \( V \) provided with a N–connection \( N \), symmetric metric \( g(u) \) and linear connection \( \Gamma(u) \) structures resulting in a metric compatible covariant derivative \( D \), when \( Dg = 0 \), but, in general, with non–vanishing torsion. \(^5\) In this work, we shall investigate some classes of metrics \( g(u) \) and linear connections \( \Gamma(u) \) modelling RC–algebroids as exact solutions of the field equations in string or Einstein gravity.

**2.2 Metrics and linear connections on RC–algebroids**

Let us consider a metric tensor \( g \) on the manifold \( V \) with the coefficients defined with respect to a local coordinate co–basis \( du^\alpha = (dx^i, du^a) \),

\[
g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta
\]

where

\[
g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_{e}^{i} N_{j}^{h} h_{ab} & N_{j}^{e} h_{ae} \\ N_{e}^{i} h_{be} & h_{ab} \end{bmatrix}.
\]

Performing the vielbein transform \( e_{\alpha} = e_{\alpha}^{\mu} \partial_{\mu} \) and \( e^{\beta} = e^{\beta} du^{\beta} \) with the matrix coefficients defined respectively by (6) and (7), we write equivalently the metric \( g \) in the form

\[
g = g_{\alpha\beta}(u) e^{\alpha} \otimes e^{\beta} = g_{ij}(u) e^{i} \otimes e^{j} + h_{cb}(u) v^{c} \otimes v^{b},
\]

where \( g_{ij} = g(e_{i}, e_{j}) \) and \( h_{cb} = g(v_{c}, v_{b}) \) and \( e_{\nu} = (e_{\nu}, v_{b}) \) and \( e^{\mu} = (e^{i}, v^{b}) \) are, respectively, just the vielbeins (8) and (9).

\(^5\)We consider that reader is familiar with the main concepts from differential geometry; in general form, for N–anholonomic spaces, the torsion is defined below by the formula (S).
If the manifold $V$ is (pseudo) Riemannian, there is a unique linear connection (the Levi–Civita connection) $\nabla$, which is metric, $\nabla g = 0$, and torsionless, $\nabla T = 0$, but this connection is not adapted to the nonintegrable distribution induced by $N^b_i(u)$. In order to construct exact solutions parametrized by generic off–diagonal metrics, or to investigate nonholonomic frame structures in gravity models with nontrivial torsion, it is more convenient to work with more general classes of linear connections which are N–adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients $W^\gamma_{\alpha\beta}$. For a corresponding subset of constraints, the solutions can be related to N–anholonomic configurations in general relativity, see discussions and references from [3, 4].

A distinguished connection (d–connection) $D = \{\Gamma^\alpha_{\beta\gamma}\}$ on $V$ is a linear connection conserving under parallelism the Whitney sum \(1\). This means that a d–connection $D$ may be represented by h- and v–components in the form $\Gamma^i_{jk} = (L^i_{jk}, B^i_{jc}, B^i_{bc})$, stated with respect to N–elongated frames \(2\) and \(3\), defining a N–adapted splitting into h– and v–covariant derivatives, $D = hD + vD$, where $hD = (L, L)$ and $vD = (B, B)$.

A distinguished tensor (in brief, d–tensor; for instance, a d–metric \(17\)) formalism and d–covariant differential and integral calculus can be elaborated \(4\) for spaces provided with general N–connection, d–connection and d–metric structure by using the mentioned type of N–elongated operators. The simplest way to perform a d–tensor covariant calculus is to use N–adapted differential forms with the coefficients defined with respect to \(3\) and \(8\), for instance, $\Gamma^\alpha_{\beta} = \Gamma^\alpha_{\beta\gamma} e^\gamma$.

The torsion

$$T^i = D e^i = d e^i + \Gamma^i_{\beta\gamma} \wedge e^\beta$$

of a d–connection $D$ has the irreducible h– v– components (d–torsions),

$$T^i_{jk} = L^i_{[jk]}, \quad T^i_{ja} = -T^i_{aj} = B^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji},$$

$$T^a_{bi} = T^a_{ib} = v_b(N^a_i) - L^a_{bi}, \quad T^a_{bc} = B^a_{[bc]}.$$  \hspace{1cm} \text{(19)}$$

This is the result of a straightforward calculation.

On RC–algebroids, the Levi–Civita linear connection\(^6\) $\nabla = \{\nabla \Gamma^a_{\beta\gamma}\}$ is also not adapted to the global splitting \(1\). Such nonholonomic manifolds can be characterized by a different type of linear connections: For instance, there is a preferred canonical d–connection structure $\hat{\Gamma}$ constructed only from the metric and N–connection coefficients $[g_{ij}, h_{ab}, N^a_i]$ and satisfying the metricity conditions $\hat{D}g = 0$ and $\hat{T}^i_{jk} = 0$ and $\hat{T}^a_{bc} = 0$. This can be checked by

\(^6\)by definition, satisfying the metricity and zero torsion conditions
straightforward calculations with respect to the \( N \)-adapted bases \((9)\) and \((8)\) if we take

\[
\hat{\Gamma}^\alpha_{\beta\gamma} = \nabla^\alpha_{\beta\gamma} + \hat{P}^\alpha_{\beta\gamma}
\]  

(20)

with the deformation \( d \)-tensor

\[
\hat{P}^\alpha_{\beta\gamma} = (P^i_{jk} = 0, P^a_{bk} = v_b(N^a_k), P^i_{jc} = -\frac{1}{2}g^{ik}\Omega^a_{kj} h_{ca}, P^a_{bc} = 0),
\]

where \( \nabla^\alpha_{\beta\gamma} \) are the coefficients of the Levi–Civita connection. The torsion of the connection (20) is denoted \( \hat{T}^\alpha_{\beta\gamma} \).

It should be noted that, in general, the torsion components \( \hat{T}^i_{ja}, \hat{T}^a_{ji}, \) and \( \hat{T}^a_{bi} \) are not zero. This is an anholonomic frame (or, equivalently, off–diagonal metric) frame effect.

In explicit form, the \( h–v \)-components of the canonical \( d \)-connection \( \hat{\Gamma}^\gamma_{\alpha\beta} \) = (\( \hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{B}^i_{jc}, \hat{B}^a_{bc} \)), are given by formulas

\[
\hat{L}^i_{jk} = \frac{1}{2}g^{ir}[e_k(g_{jr}) + e_j(g_{kr}) - e_r(g_{jk})],
\]

(21)

\[
\hat{L}^a_{bk} = v_b(N^a_k) + \frac{1}{2}h^{ac}[e_k(h_{bc}) - h_{dc}v_b(N^d_k) - h_{db}v_c(N^d_k)],
\]

\[
\hat{B}^i_{jc} = \frac{1}{2}g^{ik}v_c(g_{jk}),
\]

\[
\hat{B}^a_{bc} = \frac{1}{2}h^{ad}[v_c(h_{bd}) + v_b(h_{cd}) - v_d(h_{bc})].
\]

They present a ‘minimal’ generalization of the Levi–Civita connection for the nonholonomic (pseudo) Riemannian manifolds, which in the case of nontrivial nonholonomy coefficients \( W^{\gamma}_{\alpha\beta} \) (10), resulting in nontrivial \( d \)-torsion components (19), consist in a subclass of Riemann–Cartan manifolds provided with \( N \)-connection structure.

The formulas (10), (19) and (20) and (21) are defined on the nonholonomic spacetime \( V \) and contain the partial derivative operator \( v_c = \partial/\partial u^c \).

We can emphasize the Lie \( N \)-algebroid structure on the space of sections \( Sec(vV) \) by substituting \( v_c = \hat{\rho}^i_c(x)\partial/\partial x^i \) (using the anchor map (3)), or, in \( N \)-adapted form, by working with ”boldface” operators \( v_c \rightarrow v_c = \hat{\rho}^i_c(x,u)e_i \) (see formulas (13) and (5)). A such ”anchoring” of formulas defines canonical maps for \( d \)-metrics, anholonomic frames, \( d \)-connections and \( d \)-torsions from \( V \) to \( Sec(vV) \). For instance, we can define the anchored map for the ”contravariant” \( h \)-part of the \( d \)-metric (17),

\[
h^{cb}(u) \ v_c \otimes \ v_c \rightarrow h^{cb}(u) \ \hat{\rho}^i_c \ \hat{\rho}^j_b \ e_i \otimes \ e_j
\]
modelling a \( h \)-metric \( N^{hij} = h^{cb}(u) \ \hat{\rho}^i_c \ \hat{\rho}^j_b \). We can consider certain canonical anchors \( \hat{\rho}^i_b \) when \( N^{hij} = g^{ij} \).
By anchoring the N–elongated differential operators, we can define and compute (substituting $v_c$ by $\hat{\rho}_c e_i$ into (21)) the canonical d–connection $\rho^\gamma_{\alpha\beta}$ on $\text{Sec}(vV)$ stating a canonical map $\hat{\Gamma}^\gamma_{\alpha\beta} \rightarrow \rho^\gamma_{\alpha\beta}$.

### 2.3 Curvature on nonholonomic RC–algebroids

In a similar form, with respect to N–adapted bases, we can compute the h–v–coefficients of the curvature

\[ R^\alpha_{\beta} \div \text{d} \Gamma^\alpha_{\beta} = \text{d} \Gamma^\alpha_{\beta} - \Gamma^\gamma_{\beta} \wedge \Gamma^\alpha_{\gamma} \]

(i. e. d–curvatures) of a d–connection $\Gamma^\alpha_{\gamma}$:

\[
\begin{align*}
R^i_{hjk} &= e_k (L^i_{hj}) - e_j (L^i_{hk}) + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - B^i_{ha} \Omega_k^a, \\
R^a_{bik} &= e_k (L^a_{bik}) - e_j (L^a_{bik}) + L^c_{bj} L^a_{ce} - L^c_{bk} L^a_{cj} - B^a_{bc} \Omega_k^c, \\
R^i_{jka} &= v_a (L^i_{jka}) - D_k (B^i_{ja}) + B^i_{ja} T^k_{ca}, \\
R^b_{ka} &= v_a (L^b_{ka}) - D_k (B^b_{ka}) + B^b_{ka} T^c_{ka}, \\
R^i_{jbc} &= v_c (B^i_{jb}) - v_b (B^i_{jc}) + B^h_{jb} B^i_{hc} - B^h_{jc} B^i_{hb}, \\
R^a_{bcd} &= v_d (B^a_{bc}) - v_c (B^a_{bd}) + B^c_{bc} B^a_{ed} - B^b_{ed} B^a_{ec}. 
\end{align*}
\]

The "anchored" curvature is computed by the same formulas with $v_c \rightarrow \hat{\rho}_c e_i$ for any given d–connection $\rho^\gamma_{\alpha\beta}$. For the curvature of the canonical d–connection, we have to use the anchored components of (21), for instance,

\[ \mathcal{R}^c_{bka} = \hat{\rho}_a e_i (\rho L^c_{bk}) - \rho D_k (\rho B^c_{ba}) + \rho B^c_{bd} \rho T^c_{ka}, \]

where we denote by the "calligraphic" symbol $\mathcal{R}$ the RC–algebroid anchored curvature. In a similar form, we can map all components of (22) (we omit such details in this work). This mean that on a N–anholonomic RC–algebroid $V$ we can work with the curvature $R^\alpha_{\beta}$ or we can transfer the constructions on $\text{Sec}(vV)$ and work with $\mathcal{R}^\alpha_{\beta}$.

Contracting the components of (22), we define the Ricci d–tensor

\[ \hat{\mathcal{R}}_{\alpha\beta} \div \mathcal{R}_{\alpha\beta} \]

with h–v–components

\[ R_{ij} \div \hat{R}_{ij}, \quad R_{ia} \div -\hat{R}_{ika}, \quad R_{ai} \div \hat{R}_{aib}, \quad R_{ab} \div \hat{R}_{abc}. \]

and the scalar curvature

\[ \hat{\mathcal{R}} \div g^{\alpha\beta} \hat{\mathcal{R}}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}. \]
The Einstein d–tensor is computed in standard form,

$$E_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R}.$$  

We shall denote the anchored versions of the Ricci and Einstein d–tensors, respectively, by $\hat{R}_{\alpha\beta}$ and $E_{\alpha\beta}$.

### 2.4 String gravity and N–anholonomic manifolds

Let us consider the strength $\hat{H}_{\nu\lambda\rho} = e_{\nu} B_{\lambda\rho} + e_{\rho} B_{\nu\lambda} + e_{\lambda} B_{\rho\nu}$ (antisymmetric torsion of the antisymmetric tensor $B_{\rho\nu} = -B_{\nu\rho}$ from the bosonic model of string theory with dilaton field $\Phi$, see details, for instance, in [18, 19]) and introduce the torsion

$$H_{\nu\lambda\rho} = \hat{H}_{\nu\lambda\rho} + \hat{Z}_{\nu\lambda\rho}$$

where the deformation

$$\hat{Z}_{\nu\lambda} = \hat{Z}_{\nu\lambda\rho} e^{\rho} = e_{\lambda} \hat{T}_{\nu} - e_{\nu} \hat{T}_{\lambda} + \frac{1}{2} (e_{\nu} e_{\lambda} \hat{T}_{\rho}) e^{\gamma}$$  \hspace{1cm} (24)

may be computed by using N–adapted differential forms and the interior product $\iota$”. We denote the energy–momentums of fields by

$$\hat{\Upsilon}_{\alpha\beta} = \Sigma^{[\text{mat}]}_{\alpha\beta} + \Sigma^{[T]}_{\alpha\beta}$$

where $\Sigma^{[\text{mat}]}_{\alpha\beta}$ is the source from any possible matter fields and $\Sigma^{[T]}_{\alpha\beta}(\hat{T}_{\nu}, \Phi)$ contains contributions of torsion and dilatonic fields.

The dynamics of sigma model of bosonic string gravity with generic off–diagonal metrics, effective matter and torsion is defined by the system of field equations

$$\hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} = k \hat{\Upsilon}_{\alpha\beta},$$  \hspace{1cm} (25)

$$\hat{D}^{\nu}(H_{\nu\lambda\rho}) = 0,$$  \hspace{1cm} (26)

where $k = \text{const}$, and the Euler–Lagrange equations for the matter fields are considered on the background $V$.

Let us consider a well known ansatz in string theory for the $H$–field when

$$H_{\nu\lambda\rho} = \hat{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho} = \lambda_{[H]} \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho}$$  \hspace{1cm} (27)

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_{[H]} = \text{const}$, which satisfies the field equations (26) for $H_{\nu\lambda\rho}$. In this work, the ansatz (27) is chosen for a
5D N–anholonomic background with $\hat{Z}_{\nu\lambda\rho}$ defined by the d–torsions for the canonical d–connection. So, the values $\hat{H}_{\nu\lambda\rho}$ are constrained to solve the equations (27) for a fixed value of the cosmological constant $\lambda$. Additionally deformed by a class of N–anholonomy constrains. As a result, a diagonal (with respect to (8) and (9)) source $\hat{\Upsilon}_{\alpha\beta} = \text{diag}\{\hat{\Upsilon}_\alpha\}$ is parametrized in the form

$$\hat{\Upsilon}_{\alpha\beta} = \{\hat{\Upsilon}_1 + \frac{\lambda^2[H]}{4}, \hat{\Upsilon}_2 + \frac{\lambda^2[H]}{4}, \hat{\Upsilon}_3 + \frac{\lambda^2[H]}{4}, \hat{\Upsilon}_4 + \frac{\lambda^2[H]}{4}, \hat{\Upsilon}_5 + \frac{\lambda^2[H]}{4}\}$$

where $\hat{\Upsilon}_\alpha$ are defined by certain matter fields contributions and $\lambda^2[H]/4$ states string contributions.

In terms of differential forms, the equations (28) are written

$$\eta_{\alpha\beta\gamma} \wedge \hat{\mathcal{R}}^{\beta\gamma} = \hat{\Upsilon}_\alpha,$$

where, for the volume 4–form $\eta \doteq \ast 1$ with the Hodje operator $\ast$, $\eta_{\alpha} \doteq e_{\alpha} \rfloor \eta$, $\eta_{\alpha\beta} \doteq e_{\beta} \rfloor \eta_{\alpha}$, $\eta_{\alpha\beta\gamma} \doteq e_{\gamma} \rfloor \eta_{\alpha\beta}$, ..., $\hat{\mathcal{R}}^{\beta\gamma}$ is the curvature 2–form and $\Upsilon_\alpha$ denote all possible matter sources. The deformation of connection (20) defines a deformation of the curvature tensor with respect to the curvature of the Levi–Civita connection, $\nabla \mathcal{R}^{\beta\gamma}$. The gravitational field equations (28) transforms into

$$\eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{R}^{\beta\gamma} + \eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{Z}^{\beta\gamma} = \hat{\Upsilon}_\alpha,$$

where $\nabla \mathcal{Z}^{\beta\gamma} = \nabla \mathcal{P}^{\beta\gamma} + \mathcal{P}^{\beta\gamma} \wedge \mathcal{P}^{\alpha\gamma}$.

A subclass of solutions of the gravitational field equations for the canonical d–connection defines also solutions of the Einstein equations for the Levi–Civita connection if and only if

$$\eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{Z}^{\beta\gamma} = 0$$

and $\hat{\Upsilon}_\alpha = \nabla \Upsilon_\alpha$, (i.e. the effective source is the same one for both type of connections). Such constraints have to be imposed in order to select some classes of solutions in general relativity from certain ones constructed by applying the canonical d–connection (one has to solve some systems of first order partial differential equations, or, for certain classes of solutions, the constraints (30) reduce to a system of algebraic equations).

### 3 Gravitational Algebroid Configurations

We introduce a class of generic off–diagonal metrics (depending on 3, or 4, variables) for which the vacuum Einstein equations with generalizations to a
certain type of string and matter field corrections are completely integrable. There stated certain parametrizations when such exact solutions define Lie algebroid configurations.

### 3.1 The 5D and 4D ansatz

We consider a five dimensional (5D) Einstein–Cartan spacetime $\mathbf{V}$ provided with a $N$–connection structure $\mathbf{N} = [N^i_4(u^\alpha), N^5_i(u^\alpha)]$ where the local coordinates are labeled $u^\alpha = (x^i, u^4 = v, u^5)$, for $i = 1, 2, 3$. Let us formulate some general conditions when a class of exact solutions of the field equations of the bosonic string gravity (25) and (26) depending on holonomic variables and on one anholonomic (equivalently, anisotropic) variable $u^4 = v$ can be constructed in explicit form. Every coordinate from a set $u^\alpha$ may be time like, a 3D space one, or extra dimensional. The partial derivatives will be denoted in the form $\alpha^\ast = \partial \alpha / \partial x^i$, $\alpha^\ast = \partial \alpha / \partial x^2$, $\alpha^\ast = \partial \alpha / \partial x^3$, $\alpha^\ast = \partial \alpha / \partial v$.

We consider a class of metrics

$$g^{[\omega]} = \omega^2(x^i, v) \tilde{g}_{\alpha\beta}(x^i, v) \, du^\alpha \otimes dv^\beta,$$

with the coefficients $\tilde{g}_{\alpha\beta}$ parametrized by the ansatz

$$
\begin{bmatrix}
g_1 + w_{11}h_4 + n_1h_5 & w_{12}h_4 + n_1n_2h_5 & w_{13}h_4 + n_1n_3h_5 & (w_1 + \zeta_1)h_4 & n_1h_5 \\
w_{21}h_4 + n_2n_1h_5 & g_2 + w_{22}h_4 + n_2h_5 & w_{23}h_4 + n_2n_3h_5 & (w_2 + \zeta_2)h_4 & n_2h_5 \\
w_{31}h_4 + n_3n_1h_5 & w_{32}h_4 + n_3n_2h_5 & g_3 + w_{33}h_4 + n_3h_5 & (w_3 + \zeta_3)h_4 & n_3h_5 \\
(w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & h_4 & 0 \\
n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5
\end{bmatrix}
$$

for $w_{ij} = w_j w_j + \zeta_i \zeta_j$, a conformal factor

$$\omega^2(x^i, v) = \omega_0^2(x^2, x^3) \eta_0(x^i, v), \, \eta_0 > 0,$$

and functions

$$g_i = g_i(x^k)\eta_i(x^k),$$

for $g_1 = \epsilon = \pm 1, \eta_i = 1, g_j = q_j(x^k)\eta_j(x^k)$;

$$h_a = q_a(x^k)\eta_a(x^i, v).$$

The $N$–coefficients $\mathbf{[S]}$ and $\mathbf{[I]}$ are parametrized in the form $N^4_i = w_i(x^i, v)$ and $N^5_i = n_i(x^k, v)$ for the indices running the values $k, j, \ldots = 1, 2, 3; \hat{k}, \hat{j}, \ldots = 2, 3$ and $a, b, \ldots = 4, 5$. Such 5D metrics possess also a second order anisotropy $\mathbf{[II]}$ when the $N$–coefficients on the second ‘shell’ (with four holonomic, $(x^i, u^5)$, and one anholonomic, $u^4$, coordinates) are stated by nontrivial $N^5_i = \zeta_i(x^i, v)$ and, for simplicity, $\zeta_5 = 0$ (the indices with ‘hat’
take values like $\hat{i} = 1, 2, 3, 5)$. One shall be considered metrics with smooth limits $\eta_0, \eta_\alpha \to 1$ which do not change the signature. We suppose, that such limits result into certain well known exact solutions of the Einstein equations (for instance, into the Schwarzschild metric and/or its imbedding into higher dimensional spacetimes).

The metric (31) defined by the ansatz (32) can be represented equivalently in the form (17),

$$g^{[\omega]} = \omega^2 \left[ \epsilon (dx^1)^2 + g_2 (dx^2)^2 + g_3 (dx^3)^2 + h_4 (e^4)^2 + h_5 (e^5)^2 \right],$$ (35)

where 

$$e^4 = du^4 + w_i dx^i$$

and 

$$e^5 = du^5 + n_i dx^i.$$

The nontrivial components of the 5D Einstein equations defined by the Ricci $d$–tensor (23), $\hat{R}_{\alpha\beta} = (\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{S}_{ab})$, for the d–metric (17) and corresponding canonical d–connection $\hat{\Gamma}_{\alpha\beta}^\gamma$ (21) are stated by the formulas, see details on a similar calculus in the Appendix to Ref. [4],

$$R_2^2 = R_3^3 = \frac{1}{2} R_1^1 =$$

$$= -\frac{1}{2g_2g_3} \left[ g_3^{**} - \frac{g_2^* g_3}{2g_2} - \frac{(g_3^*)^2}{2g_3} + g_2^* \frac{g_3'}{2g_3} - \frac{(g_2')^2}{2g_2} \right] = -\Upsilon_4 (x^2, x^3),$$

$$S_4^4 = S_5^5 = -\frac{1}{2h_4h_5} \left[ h_5^{**} - h_5^* \left( \ln \sqrt{|h_4 h_5|} \right)^* \right] = -\Upsilon_2 (x^2, x^3, v),$$ (37)

$$R_{4i} = -w_5 \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0,$$ (38)

$$R_{5i} = -\frac{h_5}{2h_4} \left[ n_i^{**} + \gamma n_i^* \right] = 0,$$ (39)

where

$$\alpha_i = \partial_i h_5^* - h_5^* \frac{\partial_i \ln \sqrt{|h_4 h_5|}}{h_4 h_5}, \beta = h_5^{**} - h_5^* \left( \ln \sqrt{|h_4 h_5|} \right)^*,$$

$$\gamma = 3h_5^*/2h_5 - h_4^*/h_4,$$

for $h_4^* \neq 0, h_5^* \neq 0$ (the cases with vanishing $h_4^*$ or $h_5^*$ should be analyzed additionally) if there are satisfied the conditions

$$\hat{\delta}_i h_4 = 0 \text{ and } \hat{\delta}_i \omega = 0$$ (41)

for $\hat{\delta}_i = \partial_i - (w_i + \zeta_i) \partial_4 + n_i \partial_5$ when the values $\zeta_i = (\zeta_i, \zeta_5 = 0)$ are to be defined for the solutions of (41).

Both conditions (41) are satisfied, for instance, if

$$\omega^{p_1/p_2} = h_4 (p_1 \text{ and } p_2 \text{ are integers}),$$ (42)
and \( \zeta_i \) is the solution of the equations

\[
\partial_i \omega - (w_i + \zeta_i) \omega^* = 0. \tag{43}
\]

Here we note that there are different possibilities to solve the conditions (41). For instance, if \( \omega = \omega_1 \omega_2 \), we can consider that \( h_4 = \omega_1^{p_1/p_2} \omega_2^{p_3/p_4} \) for some integers \( p_1, p_2, p_3 \) and \( p_4 \).

The Einstein equations (23) for the ansatz (32) are compatible for non-vanishing sources and if and only if the nontrivial components of the Ricci d–tensor (23) can be solved in general gravitational field equations (25) (equivalently, (28)) for the ansatz (32) and (33), with respect to the frames (36) and (37), are any functions of type

\[
\hat{\Upsilon}_2 = \hat{\Upsilon}_3 = \Upsilon_2(x^2, x^3, v), \quad \hat{\Upsilon}_4 = \hat{\Upsilon}_5 = \Upsilon_4(x^2, x^3) \text{ and } \hat{\Upsilon}_1 = \Upsilon_2 + \Upsilon_4. \tag{44}
\]

This follows from the fact that the nontrivial components of the Einstein d-tensor \( \hat{G}^a_{\hat{b}} \) satisfy the conditions

\[
\hat{G}^1 = -(\hat{R}^2_2 + \hat{R}^4_4), \quad \hat{G}^2 = \hat{G}^3 = -\hat{R}^1_4(x^2, x^3, v), \quad \hat{G}^4 = \hat{G}^5 = -\hat{R}^2_2(x^2, x^3).
\]

Parametrizations of sources in the form (44) can be satisfied for quite general distributions of matter, torsion and dilatonic fields (see Refs. 4, 3, 7 for an analyze of such configurations; in this paper, we shall consider that there are given certain values \( \Upsilon_2 \) and \( \Upsilon_4 \) which vanish in the vacuum cases).

There are proofs 3, 4, 8, see also the Appendix, that the system of gravitational field equations (23) (equivalently, (23)) for the ansatz (32) and nontrivial components of the Ricci d–tensor (23) can be solved in general form if there are given certain values of functions \( g_2(x^2, x^3) \) (or, inversely, \( g_3(x^2, x^3) \), \( h_4(x^i, v) \) (or, inversely, \( h_5(x^i, v) \)) and of sources \( \Upsilon_2(x^2, x^3, v) \) and \( \Upsilon_4(x^2, x^3) \).

Let us denote the local coordinates \( u^\alpha = (x^i, u^a) \) with \( x^i = (x^1, x^2) \), \( x^3 = (x^2, x^3) \), \( u^a = (u^4 = v, u^5) \) and consider arbitrary signatures \( \epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \) (where \( \epsilon_\alpha = \pm 1 \)). Summarizing the results outlined in the Appendix, for the nondegenerated cases (when \( h_4 \neq 0 \) and \( h_5 \neq 0 \) and, for simplicity, for a trivial conformal factor \( \omega \)), we formulate an explicit result for 5D exact solutions of the system (30)–(33) and (36):

Any off–diagonal metric

\[
\delta s^2 = \epsilon_1(dx^1)^2 + \epsilon_3 g_k^k dx^k (dx^k)^2 + \\
\epsilon_4 h_2^2(x^i) \left[ f^*(x^i, v) \right]^2 |\gamma_T(x^i, v)| (\delta v)^2 + \epsilon_5 f^2(x^i, v) (\delta u^5)^2, \\
\delta v = dv + w_k(x^i, v) dx^k, \quad \delta u^5 = du^5 + n_k(x^i, v) dx^k, \tag{45}
\]

16
with the coefficients being of necessary smooth class, where \( g_\hat{k}(x^i) \) is a solution of the 2D equation (36) for a given source \( \Upsilon_4(x^3) \),

\[
\varsigma_\Upsilon(x^i, v) = \varsigma_4(x^i, v) = 1 - \frac{\epsilon_4}{16} h_0^2(x^i) \int \Upsilon_2(x^\hat{k}, v)[f^2(x^i, v)]^2 dv
\]

(46)

and the N–connection coefficients \( N^4_i = w_i(x^k, v) \) and \( N^5_i = n_i(x^k, v) \),

\[
w_i = -\frac{\partial_i \varsigma_\Upsilon(x^k, v)}{\varsigma_\Upsilon^*(x^k, v)}
\]

(47)

and

\[
n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{[f^* (x^i, v)]^2}{[f (x^i, v)]^2} \varsigma_\Upsilon (x^i, v) dv,
\]

(48)

define an exact solution of the system of Einstein equations (36)–(39) for arbitrary nontrivial functions \( f(x^i, v) \) (with \( f^* \neq 0 \), \( h_0^2(x^i) \), \( \varsigma_4(x^i) \), \( n_{k[1]}(x^i) \) and \( n_{k[2]}(x^i) \), and sources \( \Upsilon_2(x^\hat{k}, v), \Upsilon_4(x^3) \)) and any integration constants and signatures \( \epsilon_\alpha = \pm 1 \) which have to be defined by certain boundary conditions and physical considerations.

Any metric (45) with \( h_4^* \neq 0 \) and \( h_5^* \neq 0 \) has the property to be generated by a function of four variables \( f(x^i, v) \) with emphasized dependence on the anisotropic coordinate \( v \), because \( f^* \equiv \partial_v f \neq 0 \) and by arbitrary sources \( \Upsilon_2(x^\hat{k}, v), \Upsilon_4(x^3) \). The rest of arbitrary functions not depending on \( v \) have been obtained in result of integration of partial differential equations. This fix a specific class of metrics generated by the relation (68) and the first formula in (72). We can generate also a different class of solutions with \( h_4^* = 0 \) by considering the second formula in (67) and respective formulas in (72). The ”degenerated” cases with \( h_4^* = 0 \) but \( h_5^* \neq 0 \) and inversely, \( h_4^* \neq 0 \) but \( h_5^* = 0 \) are more special and request a proper explicit construction of solutions. Nevertheless, such type of solutions are also generic off–diagonal and they could be of substantial interest.

The sourceless case with vanishing \( \Upsilon_2 \) and \( \Upsilon_4 \) is defined by the statement: Any off–diagonal metric (45) with \( \varsigma_\Upsilon = 1 \), \( h_0^2(x^i) = h_0^2 = \text{const} \), \( w_i = 0 \) and \( n_k \) computed as in (48) but for \( \varsigma_\Upsilon = 1 \), defines a vacuum solution of 5D Einstein equations for the canonical d–connection (21). By imposing additional constraints on arbitrary functions from \( N^5_i = n_i \) and \( N^5_i = w_i \), in order to satisfy the conditions (30), we can select just those off–diagonal gravitational configurations when the Levi–Civita connection and the canonical d–connections are related to the same class of solutions of the vacuum Einstein equations, see details in Ref. [4].
Finally, one should be noted that we can reduce the constructions to a 4D manifold provided with local coordinates \((x^2, x^3, u^4, u^5)\) if we exclude dependencies on \(x^1\) and do not consider terms with indices taking values \(i = 1\). This way, one can be generated exact 4D exact solutions: for certain parametrizations one get metrics with Lie N–algebroid symmetries.

### 3.2 Gravitational Lie algebroid configurations

Let us analyze the conditions when a subclass of d–metrics of type (17) (for 5D, a subclass of metrics of type (35)) models a Lie algebroid provided with N–connection structure. We write

\[
g = g^{\alpha\beta}(u) e_\alpha \otimes e_\beta = g^{ij}(u) e_i \otimes e_j + h^{ab}(u) v_b \otimes v_b,
\]

where \(v_a = \partial/\partial u^a\) satisfy the Lie N–algebroid conditions \(v_av_b - v_vv_a = C^d_{ab}(x,u)v_d\) of type (14) with \(e_i\) being of type (8). Using the anchor map (13) with \(\hat{\rho}_a^i(x) = e_a^\alpha(x)\rho^i_\alpha(x)\rho^j_\beta(x)\)

\[
(49)
\]

defined in the form (11) by some matrices of type (6) and (7), we can write the canonical relation

\[
g^{ij}(u) = h^{ab'}(u) \hat{\rho}^{i}_{ab'}(u) \hat{\rho}^{j}_{ab'}(u).
\]

(50)

As a result, the h–component of the d–metric can be represented

\[
g^{ij}(u) e_i \otimes e_j = h^{ab'}(u) e^i_\alpha(u) e^a_\beta(x)\rho^i_\beta(x) e^b_\gamma(x)\rho^j_\gamma(x) e_i \otimes e_j = h^{ab'}(u) \rho^{i}_{ab'}(x) \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b},
\]

where \(\rho^{i}_{ab'}(x) = e^a_\beta(x)\rho^i_\beta(x)\). We conclude that a metric (17) admits a Lie algebroid type structure with the structure functions \(\rho^{i}_{ab'}(x)\) and \(C^d_{ab}(x,u)\) if and only if the contravariant h–component of the corresponding d–metric, with respect to the local coordinate basis, can be parametrized in the form

\[
g^{ij}(u) = h^{ab'}(u) \rho^{i}_{ab'}(x) \rho^{j}_{ab'}(x).
\]

(51)

The anchor \(\rho^{i}_{ab'}(x)\) may be treated as a vielbein transform depending on \(x\)–coordinates lifting the horizontal components of the contravariant metric on the \(v\)–subspace. The Lie type structure functions \(C^d_{ab}(x,u)\) define certain anholonomy relations for the basis \(v_a\). On a general Lie N–algebroids we shall consider any set of coefficients \(\hat{\rho}^{i}_{ab'}(u)\) and \(C^d_{ab}(x,u)\) not obligatory subjected to the data (49).
Let us analyze an example of more general conditions when a metric (31) (equivalently, a d–metric (35)) defines a class of anchor maps (50). On N–anholonomic manifolds, it is more convenient to work with the N–adapted relations (50) than with (51). For effectively diagonal d–metrics, such anchor map conditions must be satisfied both for $\eta_\alpha = 1$ and nontrivial values of $\eta_\alpha$, i.e.

$$
\begin{align*}
    g^i &= h^4 \left( \tilde{\rho}_4^i \right)^2 + h^5 \left( \tilde{\rho}_5^i \right)^2, \\
    q^i &= q^4 \left( \tilde{\rho}_4^i \right)^2 + q^5 \left( \tilde{\rho}_5^i \right)^2, \\ & \quad \text{for} \eta_\alpha \to 1,
\end{align*}
\tag{52}
$$

where $g^i = 1/g_i$, $h^a = 1/h_a$, $\eta^\alpha = 1/\eta_\alpha$ and $q^\alpha = 1/q_\alpha$. The real solutions of (52) are

$$
\begin{align*}
    ( \tilde{\rho}_4^i )^2 &= q^i q_4 H_4^i, \\
    ( \tilde{\rho}_5^i )^2 &= -q^i q_5 H_4^i, \\
\end{align*}
\tag{53}
$$

where

$$
H_a^i = \eta_a \frac{1 - \eta_4/\eta_i}{\eta_5 - \eta_4},
$$

$\eta_1 = 1$, for any parametrization (34) with a set of values of $q^\alpha$ and $\eta^\alpha$ for which $( \tilde{\rho}_a^i )^2 > 0$. We note that the conformal factor $\omega^2$ is not related to such equations and their solutions which mean that the gravitational “boldfaced” anchor structures are conformally invariant, for such classes of metrics. The nontrivial anchor coefficients can be related to a general solution of type (45), (47) and (48) by formulas

$$
g_1 = \epsilon_1, \quad \epsilon_\hat{\rho} g_\hat{\rho}(x^i) = q_\hat{\rho} \eta_\hat{\rho}, \quad \epsilon_4 h_0^2 [ f^* ]^2 | \zeta_T | = q_4 \eta_4, \quad \epsilon_5 f^2 = q_5 \eta_5
$$

with the functions stated in explicit form by considering nonholonomic deformations of some already known solutions.

The final step in constructing such classes of metrics is to define $C^d_{ab}(x,u)$ from the algebraic relations defined by the first equation in (15) with given values for $\tilde{\rho}_a^i$, see (53), and defined N–elongated operators $e_i$. In result, the second equation in (15) will be satisfied as a consequence of the first one. This restrict the classes of possible v–frames, $v_b = e_b^\#(x,u)\partial/\partial u_b^\#$, where $e_b^\#(x,u)$ have to satisfy the algebraic relations (14). We conclude, that the Lie N–algebroid structure impose certain algebraic constraints on the coefficients of vielbein transforms. Such spacetimes are with preferred frame structure which can be taken into account by explicit constructions with respect to N–adapted frames and distinguishing the anchor $\tilde{\rho}_a^i$ and Lie type $C^d_{ab}(x,u)$ structures functions.
4 Gravitational Solitonic Schwarzschild Algebroids

We construct in explicit form two classes of solutions of the gravitational field equations (36)–(39) and (41) with nontrivial Lie algebroid and N–connection structure describing generalizations of the 4D Schwarzschild metric to generic off–diagonal 5D and 4D ansatz with nontrivial 3D gravitational solitonic backgrounds. For simplicity, we study here d–metrics with trivial conformal factors $\omega^2 = 1$ (see Appendix B for similar two solutions with $\omega^2 \neq 1$).

As the starting point for our considerations, we consider the 5D metric

$$ds^2 = \epsilon d\chi^2 - a^2(p) \left( dp^2 + d\theta^2 + \sin^2 \theta d\varphi^2 \right) + b^2(p) dt^2$$

with extra dimension coordinate $\chi$ defining a trivial extension of the Schwarzschild spacetime for $a^2(p) = a^2(\xi) = \frac{\zeta^2}{\xi^2}(\xi + 1)^2, b^2(p) = b^2(\xi) = \left( \frac{\xi - 1}{\xi + 1} \right)^2$ (55)

with $dp = d\xi/\xi$, where $\xi = \zeta/\zeta_g$ is related with the usual radial coordinate by formula $r = \zeta \left( 1 + r_g/4\zeta \right)^2$, for $\zeta_g = r_g/4$ with $r_g = 2G[4]m_0/c^2$ being the 4D Schwarzschild radius of a point particle of mass $m_0$; $G[4] = 1/M_{P[4]}^2$ is the 4D Newton constant expressed via the Planck mass $M_{P[4]}$ (in general, we may consider that $M_{P[4]}$ may be an effective 4D mass scale which arises from a more fundamental scale of the full, higher dimensional spacetime); we set $c = 1$. The 4D part of (54) expressed in terms of functions on $\xi$ is just the Schwarzschild solution in isotropic spherical coordinates [20].

The diagonal metric (54) defines a vacuum solution of the Einstein equations for the Levi–Civita connection. A such solution is with Killing symmetry and asymptotically flat. By N–anholonomic frame transforms (6) and (7) of this metric, we shall generate new classes of exact solutions describing Lie N–algebroid configurations with a nontrivial solitonic background.

4.1 Stationary off–diagonal solutions

Let us firstly analyze in details how we can generate a stationary solution (with the coefficients not depending on the time like variable $u^5 = t$) by deforming nonholonomically the metric (54) to a generic off–diagonal metric (52) (equivalently, to a d–metric, (35)) with trivial conformal factor $\omega^2 = 1$.
for a set of local coordinates \((x^\alpha = (\chi, p, \theta), u^\alpha = (v = \varphi, t))\). We write

\[
q_1 = \epsilon \rightarrow g_1 = \epsilon,
\]

\[
q_2 = -a^2(p) \rightarrow g_2 = q_2(p) \eta_2(p, \theta),
\]

\[
q_3 = -a^2(p) \rightarrow g_3 = q_3(p) \eta_3(p, \theta),
\]

\[
q_4 = -a^2(p) \sin^2 \theta \rightarrow h_4 = q_4(p, \theta) \eta_4(p, \theta, \varphi),
\]

\[
q_5 = b^2(p) \rightarrow h_5 = q_5(p) \eta_5(p, \theta, \varphi),
\]

where the non–deformed values are stated by the coefficients (55) and the "polarization" functions \(\eta_{2,3}(p, \theta)\) have to be found as a solution of type (56), (57), or (58), depending explicitly of the type of source \(\Upsilon_2(p, \theta, \varphi)\) and vacuum boundary conditions, and the "polarization" functions \(\eta_{4,5}(p, \theta, \varphi)\) are solutions of the equations (64), (65), or (66), depending on the type of source \(\Upsilon_2(p, \theta, \varphi)\) and vacuum boundary conditions. This class of solutions can be represented in the form (45)

\[
\delta s^2 = \epsilon(d\chi)^2 - a^2(p)\eta_2(p, \theta)(dp)^2 - a^2(p)\eta_3(p, \theta)(d\theta)^2
\]

\[
-\eta_4^2(p, \theta) |f^*(p, \theta, \varphi)|^2 |\zeta_T(p, \theta, \varphi)| (\delta \varphi)^2 + f^2(p, \theta, \varphi)(\delta t)^2,
\]

for

\[
\delta \varphi = d\varphi + w_1(\chi, p, \theta, \varphi) d\chi + w_2(\chi, p, \theta, \varphi) dp + w_3(\chi, p, \theta, \varphi) d\theta,
\]

\[
\delta t = dt + n_1(\chi, p, \theta, \varphi) d\chi + n_2(\chi, p, \theta, \varphi) dp + n_3(\chi, p, \theta, \varphi) d\theta,
\]

where we parametrize

\[
f^2(p, \theta, \varphi) = b^2(p)\eta_5(p, \theta, \varphi), h_0^2(p, \theta) = a^2(p)/b^2(p)
\]

and

\[
h_0^2(p, \theta) |f^*(p, \theta, \varphi)|^2 |\zeta_T(p, \theta, \varphi)| = a^2(p) \sin^2 \theta \eta_4(p, \theta, \varphi)
\]

with the \(N\)–connection coefficients \(w_k\) and \(n_k\) computed respectively by the integrals (59) and (60).

If we choose the integration functions to be of sooth class related to certain distributions of matter, the \(d\)–metric (57) has the diagonal coefficients very similar to those for the Schwarzschild metric (54) with the coefficients \(a^2(p)\) and \(b^2(p)\) multiplied respectively on certain polarization \(\eta\)–functions but (roughly speaking) embedded into a 5D background of string gravity with nontrivial torsion and nonholonomic deformation to a preferred frame structure with associated \(N\)–connection. We analyze here an interesting physical case of non–perturbative gravitational background defined by \(f^2(p, \theta, \varphi)\), resulting in a static locally anisotropic polarization \(\eta_5 = f^2/b^2(p)\), related to a soliton solution of the Kadomtsev–Petviashvili (KdP) equation or (2+1)
sine-Gordon (SG) equation (Refs. 21, 22, 3 contain original results, basic references and methods for handling such non-linear equations with solitonic solutions). In the KdP case, the function \( \eta_5(p, \theta, \varphi) \) satisfies the equation

\[
\eta_5^{**} + \epsilon (\eta_5 - 6\eta_5' \eta_5'' + \eta_5''')' = 0, \quad \epsilon = \pm 1,
\]

while in the most general SG case \( \eta_5(p, \theta, \varphi) \) satisfies

\[
\pm \eta_5^{**} \mp \eta_5'' \mp \eta_5''' = \sin(\eta_5).
\]

For simplicity, we can also consider less general versions of the SG equation where \( \eta_5 \) depends on only one (e.g. \( \varphi \) and \( x_1 \)) variable. We use the notation \( \eta_5 = \eta_5^{stn} \) or \( \eta_5 = \eta_5^{sg} \) with ”stn = KP”, or = SG, depending if \( \eta_5 \) satisfies equation (58), or (59) respectively.

For a stated solitonic form for \( h_5 = h_5^{stn} = b^2(p)\eta_5^{stn} \), with \( b^2(p) \) taken as for the Schwarzschild metric, \( h_4 \) can be computed

\[
h_4 = h_4^{stn} = h_4^{[0]} \left[ \left( \sqrt{|h_5^{stn}(p, \theta, \varphi)|} \right)^{stn} \right]^2 \tag{60}
\]

where \( h_4^{[0]} \) is a constant (see formula (68) in the Appendix). This allows to define \( \eta_4^{stn}(p, \theta, \varphi) \) and \( f^{stn}(p, \theta, \varphi) \), which (by using the \( f \)-function) result in off–diagonal terms \( w_5^{stn}(\chi, p, \theta, \varphi) \) and \( n_5^{stn}(\chi, p, \theta, \varphi) \). The 3D solitonic character of such \( N \)-connection coefficients can be substantially modified by presence of the source \( \Upsilon_2(p, \theta, \varphi) \). If \( \Upsilon_2 \to 0 \), one has \( \varsigma_T \to 1 \) and we can put \( w_5^{stn} \to 0 \) but \( n_5^{stn} \) would preserve solitonic contributions.

The mentioned 3D solitonic background posses a specific nontrivial torsion related both to the \( B \)-field in string gravity with cosmological constant approximations (27) and d–torsions (19) of the canonical d–connection. We can compute such values by associated to the coefficients of (57) for any type of solitonic (KdP or SG) background. At the first step one defines the coefficients of the canonical d–connection \( \hat{\Gamma}_\alpha^{\beta \gamma} \) then of \( \hat{\Gamma}_\alpha^{\beta \gamma} \) which allows to compute the deformations \( \hat{\Gamma}_\nu^{\lambda \rho} \) and, at the second step, we find the \( N \)-anholonomically deformed string torsion

\[
\hat{\mathbf{H}}^{stn}_{\nu \lambda \rho} = \lambda_{[H]} \sum_{\gamma} \left( g^{stn}_{\alpha \beta} \hat{\mathbf{Z}}_{\nu \lambda \rho} - \hat{\mathbf{Z}}_{\nu \lambda \rho} \right)
\]

for a stated value of \( \lambda_{[H]} \). This way, by using the solitonic coefficients of the d–metric (57) and mentioned procedure of computation of \( \hat{\mathbf{H}}^{stn}_{\nu \lambda \rho} \) we generate a class of exact solutions of the field equations of the bosonic string gravity (25) and (26) defined by \( N \)-anholonomic maps to 3D soliton backgrounds. As a matter of principle, we can restrict the polarizations \( \eta_\alpha \) and
the integration functions of type \( h_0^i(x^i) \), \( \zeta_{4[0]}(x^i) \), \( n_{k[1]}(x^i) \) and \( n_{k[2]}(x^i) \) to the form satisfying the conditions which in 4D selects just the Einstein type metrics (vacuum ones or with certain effective matter modifications). Some new examples and discussion of the former obtained solutions satisfying such constraints are given in Ref. [4]. Here we emphasize that the considered method of N–anholonomic transforms is a powerful one which states a geometric procedure of constructing general classes of solutions with generic off–diagonal metrics, torsions and nonholonomic frames.

We developed our procedure of constructing new solutions in gravity by starting from the Schwarzschild metric which is asymptotically flat and posses Killing symmetry. The resulting N–anholonomically deformed metrics do not have, in general, such properties. As a matter of principle, we can chose the mentioned set of polarization functions to tend asymptotically (for a certain effective radial coordinate \( r \to \infty \)) to the Minkowski 4D metric trivially embedded into 5D, very similarly to the Schwarzschild case. Nevertheless, even in a such case, our solutions will have a very different symmetry properties at least in a finite region of the 5D (or 4D) spacetime. In this paper, we investigate the conditions when such N–anholonomic spacetimes may be characterized by Lie algebroid configurations. The nontrivial anchor coefficients are easy to be computed for such configurations by using the data and any solitonic solution:

\[
(\hat{\rho}_1^a)^2 = -\epsilon a^2(p) \sin^2 \theta \ H_1^2(p), \quad (\hat{\rho}_2^a)^2 = \sin^2 \theta \ H_2^2(p), \quad (\hat{\rho}_3^a)^2 = \sin^2 \theta \ H_3^2(p),
\]

\[
(\hat{\rho}_4^a)^2 = \frac{b^2(p)}{a^2(p)} H_4^2(p), \quad (\hat{\rho}_5^a)^2 = -\frac{b^2(p)}{a^2(p)} H_5^2(p),
\]

where

\[
H_i^a(p, \theta, \varphi) = \eta_i^{stn}(p, \theta, \varphi) \frac{1 - \eta_i^{stn}(p, \theta, \varphi) / \eta_i(p, \theta)}{\eta_i^{stn}(p, \theta, \varphi) - \eta_i^{stn}(p, \theta, \varphi)},
\]

for \( \eta_i(p, \theta) = 1; a = 3, 4 \) and \( i = 1, 2, 3 \). Such anchor coefficients can be zero, when \( b^2(p) = 0 \) and for certain polarizations one could be \( (\hat{\rho}_a^2)^2 < 0 \) in some spacetime regions. We have to exclude such regions for the real valued solutions, or to redefine the type of anchor maps in order to obtain to generate only real metrics and connections. A Lie algebroid configuration is completely established after fixing a frame of reference (in general, nonholonomic) in the \( v \)–subspace, defining the values \( C^a_{bc}(p, \theta) \), see (14).

Finally, in this section, we address this important physical question: The Schwarzschild solution defines a 4D black hole. Should the N–anholonomic deformations define similar objects? In general, the black hole character of the solutions is not preserved under such transforms (not preserving the spherical Killing symmetry). The singularities of any exact solution can be
investigated by an explicit computation of the components of the Riemann d–tensors with a corresponding anchoring of formulas. We omit such cumbersome formulas and their analysis in this paper. Nevertheless, it is almost obvious that there is a subclass of N–anholonomic and/or solitonic transforms preserving the type of certain black hole configurations (even the solutions are deformed to generic off–diagonal configurations). This follows from the fact that in the vicinity of the black hole singularity we can define some infinitesimal nonholonomic maps with smooth coefficients which preserve all singular properties of the curvature but induce certain additional smooth off–diagonal corrections and the same structure of the metric coefficients but defined with respect to N–adapted frames. This way we defined the so–called black ellipsoid solutions. For some special cases (see details in Refs.), one can select certain black hole configurations on nontrivial backgrounds, in our case, of stationary solitonic character.

4.2 Time–depending solitonic backgrounds

We also deform nonholonomically the metric to a generic off–diagonal metric (equivalently, to a d–metric, with trivial conformal factor $\omega^2 = 1$) but for a set of local coordinates $x^\alpha = (\chi, p, \theta)$ and $u^a = (v = t, \varphi)$ when

\begin{align*}
q_1 &= \epsilon \to g_1 = \epsilon, \\
q_2 &= -a^2(p) \to g_2 = g_2(p) \eta_2(p, \theta), \\
q_3 &= -a^2(p) \to g_3 = g_3(p) \eta_3(p, \theta), \\
q_4 &= b^2(p) \to h_4 = q_5(p) \eta_4(p, \theta, t), \\
q_5 &= -a^2(p) \sin^2 \theta \to h_5 = q_5(p, \theta) \eta_5(p, \theta, t).
\end{align*}

These data are different from by emphasizing the "anisotropic" dependence on the time coordinate $t$ instead of the angular one, as it was on $\varphi$ for the previous solution. In this case, the "polarization" functions $\eta_4,5(p, \theta, t)$ are solutions of the equations, or (67), and (69), in their turn depending on the type of a variable in time source $\Upsilon_2(p, \theta, t)$ and vacuum boundary conditions. This is another class of d–metrics

\begin{equation}
\delta s^2 = \epsilon(d\chi)^2 - a^2(p)\eta_2(p, \theta)(dp)^2 - a^2(p)\eta_3(p, \theta)(d\theta)^2 + h_0^2(p, \theta)\left[f^*(p, \theta, t)\right]^2 \left|\delta r (p, \theta, t)\right|^2 - f^2(p, \theta, t) (\delta \varphi)^2,
\end{equation}

for

\begin{align*}
\delta t &= dt + w_1(\chi, p, \theta, t) d\chi + w_2(\chi, p, \theta, t) dp + w_3(\chi, p, \theta, t) d\theta, \\
\delta \varphi &= d\varphi + n_1(\chi, p, \theta, t) d\chi + n_2(\chi, p, \theta, t) dp + n_3(\chi, p, \theta, t) d\theta,
\end{align*}

for
where we parametrize

\[ f^2(p, \theta, t) = b^2(p)\eta_5(p, \theta, t), \]
\[ h_0^2(p, \theta) = b^2(p)/a^2(p) \sin^2 \theta \]

and

\[ h_0^2(p, \theta) [f^*(p, \theta, t)]^2 |\varsigma_\tau(p, \theta, t)| = b^2(p) \eta_4(p, \theta, t) \]

with the N–connection coefficients \( w_k \) and \( n_k \) computed respectively by the integrals (47) and (48).

The solitonic background of the d–metric (61) is given by

\[ h_5 = h_5^{stn} = -a^2(p) \sin^2 \theta \eta_5^{stn} \]

and

\[ h_4 = h_4^{stn} = h_{[0]}^2 \left( \sqrt{|h_5^{stn}(p, \theta, t)|} \right)^2 \]

defined by the 3D solitonic equation (58), or (59), for the new set of coordinates, when \( h_{[0]} = \text{const} \), see formula (68) in the Appendix. This allows to define \( \eta_1^{stn}(p, \theta, t) \) and \( f^{stn}(p, \theta, t) \), which (by using the \( f \)–function) result in off–diagonal terms (i. e. in N–connection coefficients) \( w_k^{stn}(\chi, p, \theta, t) \) (47) and \( n_k^{stn}(\chi, p, \theta, t) \) (48).

This class of metrics is characterized by dynamical anchor maps,

\[ \left( \hat{\rho}_1^1 \right)^2 = c b^2(p) H_1^1, \]
\[ \left( \hat{\rho}_2^2 \right)^2 = -\frac{b^2(p)}{a_2(p)} H_2^2, \]
\[ \left( \hat{\rho}_3^3 \right)^2 = -\epsilon a^2(p) \sin^2 \theta H_3^3, \]
\[ \left( \hat{\rho}_4^4 \right)^2 = -\frac{b^2(p)}{a_2(p)} H_4^2, \]
\[ \left( \hat{\rho}_5^5 \right)^2 = \sin^2 \theta H_5^5, \]

where

\[ H_a^i(p, \theta, t) = \eta_1^{stn}(p, \theta, t) \frac{1 - \eta_1^{stn}(p, \theta, t)/\eta_1(p, \theta)}{\eta_5^{stn}(p, \theta, t) - \eta_5^{stn}(p, \theta, t)}, \]

for \( \eta_1(p, \theta) = 1; a = 3, 4 \) and \( i = 1, 2, 3 \). We have to exclude some spacetime regions from consideration (where the solutions became complex valued) or to redefine the type of anchor maps in order to generate only real valued metrics. A dynamical Lie algebroid configuration is completely established after fixing a frame of reference (in general, nonholonomic) in the \( v \)–subspace, defining the values \( C_{bc}^a(p, \theta) \), see (14), in a form compatible with (15).

The constructed in this section exact solutions, in general form, depend on certain type functions on variables \( x^i \) obtained from the procedure of integrating systems of partial equations. In the particular case of Schwarschild solution, the result of such integration were certain constants which are defined from the boundary conditions like the asymptotic limit to the Newton potential for the gravitational field, with Killing spherical symmetry, and
asyptotically Minkowski spacetime. A such result with integration constants is possible for the corresponding diagonal ansatz for metric reducing the vacuum Einstein equations to an effective nonlinear second order partial differential equation.

The generic off–diagonal ansatz considered in paper results in systems of nonlinear partial differential equations. We proved that by corresponding geometric methods such equations can be solved in a quite general form which depends not only on arbitrary constants but also on certain classes of functions depending on one, two, three and four variables (for 5D configurations). Roughly speaking, this means that we can extend the Schwarzschild solution to a very general background which can be constrained in various forms in order to describe different type of nonlinear interactions, for instance, 3D solitons, or certain nonholonomic Lie algebroid configurations. Nevertheless, even in such cases, the solutions depends on some functions on $x^i$. This is characteristic for various classes of solutions of the systems of nonlinear equations. For instance, in Refs. [14, 15], there are investigated the conditions when a "resonable" theory of gravity is defined from a 2D Poisson setting and related Lie algebroids. In this work, we considered the problem of selecting "reasonable" off–diagonal spacetime possessing Lie N–algebroid symmetries. Such metrics still depend on some classes of functions (we call them gravitational polarizations). One can fix an explicit system of reference and choose the Lie algebroid structure functions to have a limit to certain Lie type structure constants, following some physical prescriptions on symmetry and boundary conditions, i. e. a particular exact solution distinguished from a set of general ones. The priority of the method developed in this work is that the gravitational algebroids can be defined as certain general nonlinear configurations but not only as particular ones with integration constants.

For some very special classes of functions $\eta$, $w$, and $n$, the d–metric may define certain black hole like configurations with polarized and variable in time constants if we constrain the configurations to mimic the Schwarzschild metric embedded into solitonically perturbed 5D spacetime. But, in general, such solutions do not possess a black hole character and describe a nonlinear gravitational solitonic dynamics related to a Lie N–algebroid configuration. In the Appendix B there are analyzed two classes of d–metrics with nontrivial conformal factor $\omega$.

5 Conclusions and Discussion

In this paper, we have examined a new class of 5D metric ansatz defining exact solutions in string gravity and possessing nontrivial 4D vacuum and
nonvacuum limits to the Einstein gravity. Such models define spacetimes characterized by Lie algebroid symmetries and prescribed vielbein structures with associated nonlinear connections. While our analysis has mainly focused on the properties of nonholonomic deformations of the Schwarzschild metric to generic off–diagonal solutions with 3D solitonic configurations, much of the constructions hold true for more general background metrics and symmetries. We parametrized such solutions in general form possessing an explicit dependence on arbitrary integration functions (on 1, 2, or 3 variables) and constants. We computed the nontrivial coefficients of the nonlinear connections and anchor maps defining nonholonomic Lie algebroid structures. Stating the systems of reference and the boundary conditions, we can define in explicit form the polarization of constants and metric coefficients induced by extra dimensional and/or generic off–diagonal solitonic gravitational interactions and nonholonomic constraints.

The bulk of astrophysical and cosmological applications of exact solutions are related to spherically symmetric and asymptotically Minkowski spacetimes. Such constructions are technically more easy to be handled and generalized to extra dimensions and/or quantum gravity. In another turn, nonlinear gravitational and matter field configurations and interactions, with string/brane corrections depending on 2–4 variables and with generalized (non–Killing) symmetries are very important for elaboration new types of models with non–perturbative vacuum in modern high energy physics and gravity. In this work we emphasized possible 3D solitonic deformations of the Schwarzschild solutions in the presence of extra dimensions, off–diagonal/ nonholonomic interactions and bosonic string corrections.

One should worry that arbitrary nonholonomic deformations to certain Lie algebroid spacetimes will not preserve the former black hole character of the solution. Nevertheless, it is always possible to define a smooth subclass of such solitonic deformations which preserve the singular structure of the curvature tensor but 'slightly' modify the horizons, polarize the constants and move the black hole solutions on the extra dimension and/or time like coordinates.

The mentioned approach requests a more sophisticate geometric techniques and methods. The geometry of nonlinear connections and moving frames have in this case a new realization in terms of Lie algebroid structures related to the symmetries of gravitational field equations. This suggests new directions of investigation both in Lie algebroid mathematics and the geometry of classical and quantum fields.

We briefly comment and compare our results with the previous applications of algebroid methods in mechanics and classical field theory [10] [11] [12] [13], and in string, gravity and gauge theories [14] [15] [16]. It should
be emphasized that our gravitational algebroid constructions are derived as exact solutions from string and Einstein gravity being elaborated for nonholonomic manifolds \cite{3, 4, 17}. They are quite different from the usual Lie algebroids defined for vector or (co) tangent bundles with trivial (vanishing) nonlinear connection structure \cite{9}. The first applications of Lie algebroids to mechanics and jet extensions of Lie algebroids for classical fields were performed by using the fiber bundle formalism and geometrization of the Euler-Lagrange equations (for instance, in terms of Poincare-Cartan forms, or Ehressman connections). In our investigations, we emphasized that any Lagrangian/Hamiltonian configuration results in canonical nonlinear connection and adapted metric and linear connection structures which transform the Lie algebroid constructions to be nonholonomic ones. In this paper, and in Refs. \cite{17, 16}, we gave certain explicit examples when the commutative and noncommutative geometric configurations and almost sympletic and algebroid structures can be derived as exact solutions in gravity. This presents additional arguments and a new understanding of algebroid geometry and certain applications for constructing new kinds of gauge theories by replacing Lie algebras by Lie or Courant algebroids and searching for a reasonable theory of gravity.

The final conclusion of this paper is that one could be constructed certain Lie algebroid models of spacetimes defined as exact solutions in gravity but such generic off-diagonal gravitational configurations are generated as nonholonomic manifolds, i.e. as spacetimes provided with nonintegrable distributions possessing new types of symmetries and nonlinear connection structure. To investigate the geometric and physical properties of such vacuum and nonvacuum gravitational algebroid spacetimes is one of the aims of our further researches.

Acknowledgement: The work is supported by a sabbatical fellowship of the Ministry of Education and Research of Spain.

A N–anholonomic Deformations and Exact Solutions

In a series of papers, see \cite{3, 4} and presented there references, we elaborated the anholonomic frame method of constructing exact solutions with generic off–diagonal metrics (depending on 2-4 variables) in general relativity, gauge gravity and certain extra dimension generalizations. In this Appendix, we outline the results which are necessary for constructing exact solutions characterized by Lie algebroid symmetries and nontrivial nonlinear connection
We sketch five steps of generating solutions of the system of second order nonlinear partial differential equations (36)–(39) and (43) for given certain values of functions $g_2(x^2, x^3)$ (or, inversely, $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or, inversely, $h_5(x^i, v)$), $\omega(x^i, v)$ and of sources $\Upsilon_2(x^2, x^3, v)$ and $\Upsilon_4(x^2, x^3)$:

1. The general solution of equation (36) may be represented in the form

\[ \varpi = g_{[0]} \exp[a_2 \tilde{x}^2(x^2, x^3) + a_3 \tilde{x}^3(x^2, x^3)], \]

were $g_{[0]}, a_2$ and $a_3$ are some constants and the functions $\tilde{x}^{2,3}(x^2, x^3)$ define any coordinate transforms $x^{2,3} \rightarrow \tilde{x}^{2,3}$ for which the 2D line element becomes conformally flat, i.e.

\[ g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \varpi(x^2, x^3) [(dx^2)^2 + \epsilon(dx^3)^2], \]

where $\epsilon = \pm 1$ for a corresponding signature. For the coordinates $\tilde{x}^{2,3}$, the equation (36) transform into

\[ \varpi (\varpi'' + \varpi') - \varpi - \varpi'' = 2 \varpi^2 \Upsilon_4(\tilde{x}^2, \tilde{x}^3) \]

or

\[ \ddot{\psi} + \psi'' = 2 \Upsilon_4(\tilde{x}^2, \tilde{x}^3), \]

for $\psi = \ln|\varpi|$. The form of solutions of (64) depends on the source $\Upsilon_4$. As a particular case we can consider that $\Upsilon_4 = 0$. We also can prescribe that $g_2 = g_3$ and get the equation (64) for $\psi = \ln|g_2| = \ln|g_3|$. If we select the case when $g_2' = 0$, for a given $g_2(x^2)$, we obtain from (36)

\[ g_3^{\cdot \cdot} - \frac{g_2 g_3^{\cdot}}{2g_2} - \frac{(g_3')^2}{2g_3} = 2g_2 g_3 \Upsilon_4(x^2, x^3) \]

which can be integrated explicitly for given values of $\Upsilon_4$. Similarly, we can generate solutions for a prescribed $g_3(x^3)$ in the equation

\[ g_2'' - \frac{g_2' g_3'}{2g_3} - \frac{(g_3')^2}{2g_2} = 2g_2 g_3 \Upsilon_4(x^2, x^3). \]

We note that a transform (63) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case with $\Upsilon_4 = 0$ the equation (36) is solved by arbitrary two functions $g_2(x^2)$ and $g_3(x^2)$. 

29
2. For $\Upsilon_2 = 0$, the equation (37) relates two functions $h_4(x^i, v)$ and $h_5(x^i, v)$ following two possibilities:

a) to compute

$$
\sqrt{|h_5|} = h_{5[1]}(x^i) + h_{5[2]}(x^i) \int \sqrt{|h_4(x^i, v)|} dv, \quad h_4^*(x^i, v) \neq 0;
$$

$$
h_{5[1]}(x^i) + h_{5[2]}(x^i) v, \quad h_4^*(x^i, v) = 0, \quad (67)
$$

for some functions $h_{5[1,2]}(x^i)$ stated by boundary conditions;

b) or, inversely, to compute $h_4$ for a given $h_5(x^i, v), h_5^* \neq 0$,

$$
\sqrt{|h_4|} = h_{[0]}(x^i) (\sqrt{|h_5(x^i, v)|})^*, \quad (68)
$$

with $h_{[0]}(x^i)$ given by boundary conditions. We note that the sourceless equation (37) is satisfied by arbitrary pairs of coefficients $h_4(x^i, v)$ and $h_{5[0]}(x^i)$. Solutions with $\Upsilon_2 \neq 0$ can be generated by an ansatz of type

$$
h_5[\Upsilon_2] = h_5, \quad h_4[\Upsilon_2] = \varsigma_4(x^i, v) h_4, \quad (69)
$$

where $h_4$ and $h_5$ are related by formula (67), or (68). Substituting (69), we obtain

$$
\varsigma_4(x^i, v) = \varsigma_{4[0]}(x^i) - \int \Upsilon_2(x^2, x^3, v) \frac{h_4 h_5}{4 h_5^*} dv, \quad (70)
$$

where $\varsigma_{4[0]}(x^i)$ are arbitrary functions. We have to put $\varsigma_{4[0]}(x^i) = 1$ in order to have compatibility with the sourceless case $\Upsilon_2 \rightarrow 0$.

3. The exact solutions of (38) for $\beta \neq 0$ are defined from an algebraic equation, $w_i \beta + \alpha_i = 0$, where the coefficients $\beta$ and $\alpha_i$ are computed as in formulas (40) by using the solutions for (36) and (37). The general solution is

$$
w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \quad (71)
$$

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients $w_k$ could be arbitrary functions on $(x^i, v)$. For the vacuum Einstein equations this is a degenerated case imposing the the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the $h_4$ and $h_5$ are related as in the formula (68) but with $h_{[0]}(x^i) = \text{const.}$
4. Having defined $h_4$ and $h_5$ and computed $\gamma$ from (40) we can solve the equation (39) by integrating on variable "v" the equation $n_i^* + \gamma n_i = 0$. The exact solution is

$$n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [h_4/\sqrt{|h_5|}]^3 dv, \ h_5^* \neq 0;$$

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \ h_5^* = 0;$$

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/\sqrt{|h_5|}]^3 dv, \ h_4^* = 0,$$

for some functions $n_{k[1,2]}(x^i)$ stated by boundary conditions.

5. The exact solution of (43) is given by some arbitrary functions $\zeta_i = \zeta_i(x^i, v)$ if both $\partial_i \omega = 0$ and $\omega^* = 0$, we chose $\zeta_i = 0$ for $\omega = \text{const}$, and

$$\zeta_i = -w_i + (\omega^*)^{-1} \partial_i \omega, \ \omega^* \neq 0,$$

$$(\omega^*)^{-1} \partial_i \omega, \ \omega^* \neq 0, \text{ for vacuum solutions.}$$

B Solitonic deformations with nontrivial conformal factors

We illustrate here that similar methods, applied in Section 4 can be used for generating d–metrics of type (35) with nontrivial conformal factors $\omega$.

B.1 Almost stationary metrics

Let we consider how the Schwarzschild metric (54) can deformed nonholonomically to a d–metric, (35)) with nontrivial conformal factor $\omega^2$ for a set of local coordinates $(x^a = (t, p, \theta), u^a = (v = \varphi, \chi)$. We consider deformations of type

$$q_1 = 1 \rightarrow g_1 = 1, \ q_2 = -a^2(p)/b^2(p) \rightarrow g_2 = q_2(p) \eta_2(p, \theta),$$

$$q_3 = -a^2(p)/b^2(p) \rightarrow g_3 = q_3(p) \eta_3(p, \theta),$$

$$q_4 = a^2(p) \sin^2 \theta \rightarrow h_4 = q_4(p, \theta) \eta_4(p, \theta, \varphi),$$

$$q_5 = e b^{-2}(p) \rightarrow h_5 = q_5(p) \eta_5(p, \theta, \varphi),$$

(74)
with nontrivial deformations of the conformal factor
\[ \omega_0^2 = b^2(p) \rightarrow \omega^2 = \eta_0(p, \theta, \varphi)\omega_0^2(p) \]
where the non–deformed values are stated by the coefficients [55]. The “polarization” functions \( \eta_{2,3}(p, \theta) \) have to be found as a solution of type (64), (65), or (66), depending explicitly of the type of source \( \Upsilon_4(p, \theta) \) and vacuum boundary conditions. The “polarization” functions \( \eta_{4,5}(p, \theta, \varphi) \) are solutions of the equations (68), or (67), and (69), in their turn depending on the type of source \( \Upsilon_2(p, \theta, \varphi) \) and vacuum boundary conditions. This class of solutions can be represented in the form (45)
\[
\begin{align*}
\delta s^2 &= \eta_0(p, \theta, \varphi)\omega_0^2(p)[(dt)^2 - \frac{a^2(p)}{b^2(p)}\eta_2(p, \theta)(dp)^2 - \frac{a^2(p)}{b^2(p)}\eta_3(p, \theta)(d\theta)^2 - h_0^2(p, \theta)[f^*(p, \theta, \varphi)]^2 |\chi_T(p, \theta, \varphi)| (\delta \varphi)^2 + \epsilon f^2(p, \theta, \varphi)(\delta \chi)^2], \quad (75)
\end{align*}
\]
for
\[
\begin{align*}
\delta \varphi &= d\varphi + w_1(t, p, \theta, \varphi) \, dt + w_2(t, p, \theta, \varphi) \, dp + w_3(t, p, \theta, \varphi) \, d\theta, \\
\delta \chi &= d\chi + n_1(t, p, \theta, \varphi) \, dt + n_2(t, p, \theta, \varphi) \, dp + n_3(t, p, \theta, \varphi) \, d\theta,
\end{align*}
\]
where we parametrize
\[
\begin{align*}
f^2(p, \theta, \varphi) &= b^{-2}(p)\eta_5(p, \theta, \varphi), h_0^2(p, \theta) = a^2(p))
\end{align*}
\]
and
\[
\begin{align*}
h_0^2(p, \theta)[f^*(p, \theta, \varphi)]^2 |\chi_T(p, \theta, \varphi)| &= \frac{a^2(p)}{b^2(p)} \sin^2 \theta \eta_4(p, \theta, \varphi)
\end{align*}
\]
with the N–connection coefficients \( w_k \) and \( n_k \) computed, respectively, as certain integrals (47) and (48). We call such spacetimes to be almost stationary because the d–metric coefficients do not depend on time coordinate \( t \) but the N–connection coefficients and related nonholonomic frames of reference may possess a such dependence. For a stated solitonic form for \( h_5 = h_5^{stn} = b^{-2}(p)\eta_5^{stn} \), with \( b^2(p) \) taken as for the Schwarzschild metric, \( h_4 \) can be computed
\[
\begin{align*}
h_4 = h_4^{stn} = h_{[0]}^2 \left[ \left( \sqrt{|h_5^{stn}(p, \theta, \varphi)|} \right)^{\star} \right]^2
\end{align*}
\]
where \( h_{[0]} \) is a constant (see formula (68) in the Appendix).
We satisfy the conditions (11) if we choose any conformal factor
\[
\omega = \eta_0(p, \theta, \varphi)\omega_0^2(p) = (h_4^{stn})^{p_2/p_1}
\]
for some integers \( p_1 \) and \( p_2 \) and defining \( \zeta_i \) as solutions of the equations

\[
\partial_i \omega - (w_i + \zeta_i) \omega^* = 0
\]

for given solitonic values \( w_i^{stn} \) and \( \omega \).

The nontrivial anchor coefficients are defined by the values \( q_\alpha = 1/q_\alpha \) and \( \eta_\beta = 1/\eta_\beta \), stated by (74), introduced in formulas (53) for \( (\hat{\rho}_a)^2 \) corresponding to \( H^1_a(p, \theta, \varphi) \).

### B.2 Metrics with explicit extra dimension polarization

This type of deformations transforms the metric (54) into a d–metric (35) via re–parametrizations

\[
q_1 = 1 \rightarrow g_1 = 1, \quad q_2 = -\frac{a^2(p)}{b^2(p)} \rightarrow g_2 = q_2(p) \eta_2(p, \theta),
\]

\[
q_3 = -\frac{a^2(p)}{b^2(p)} \rightarrow g_3 = q_3(p) \eta_3(p, \theta),
\]

\[
q_4 = \epsilon b^{-2}(p) \rightarrow h_4 = q_4(p) \eta_4(p, \theta, \chi),
\]

\[
q_5 = -\frac{a^2(p)}{b^2(p)} \sin^2 \theta \rightarrow h_5 = q_5(p, \theta) \eta_5(p, \theta, \chi),
\]

for the local coordinates \( x^\alpha = (t, p, \theta) \), \( u^a = (v = \chi, \phi) \). Such data are stated for the "anisotropic" dependence on the extra dimension coordinate \( \chi \) with the "polarization" functions \( \eta_{4,5}(p, \theta, t) \) being solutions of the equations (68), or (67), and (69). This class of d–metrics (45) is written

\[
\delta s^2 = \eta_0(p, \theta, \chi) \omega_0^2(p)((dt)^2 - \frac{a^2(p)}{b^2(p)} \eta_2(p, \theta) (dp)^2 - \frac{a^2(p)}{b^2(p)} \eta_3(p, \theta) (d\theta)^2 + \epsilon h_0^2(p, \theta) [f^* (p, \theta, \chi)]^2 |\xi_T (p, \theta, \chi) | (\delta \chi)^2 - f^2 (p, \theta, \chi) (\delta \varphi)^2],
\]

for

\[
\delta \chi = d\chi + w_1(t, p, \theta, \chi) dt + w_2(t, p, \theta, \chi) dp + w_3(t, p, \theta, \chi) d\theta,
\]

\[
\delta \varphi = d\varphi + n_1(t, p, \theta, \chi) dt + n_2(t, p, \theta, \chi) dp + n_3(t, p, \theta, \chi) d\theta,
\]

where we parametrize

\[
f^2(p, \theta, \chi) = \frac{b^2(p)}{a^2(p)} \sin^2 \theta \eta_5(p, \theta, \chi), h_0^2(p, \theta) = b^{-2}(p)
\]

and

\[
h_0^2(p, \theta) [f^* (p, \theta, \chi)]^2 |\xi_T (p, \theta, \chi) | = b^{-2}(p) \eta_4(p, \theta, \chi)
\]
with the N–connection coefficients $w_k$ and $n_k$ computed respectively by the integrals (47) and (48).

The solitonic background of the d–metric (61) is given by $h_5 = h_5^{stn} = - \left[ b^2(p) \sin^2 \theta / a^2(p) \right] \eta_5^{stn}$ and

$$h_4 = h_4^{stn} = h_0^2 \left( \sqrt{\eta_5^{stn}(p, \theta, \chi)} \right)^2$$

defined by the 3D solitonic equation (58), or (59), for the new set of coordinates, when $h_{[0]} = const$. This allows to define $\eta_4^{stn}(p, \theta, \chi)$ and $f^{stn}(p, \theta, \chi)$, which (by using the $f$–function) result in off–diagonal terms (i. e. in N–connection coefficients) $w_k^{stn}(t, p, \theta, \chi)$ (47) and $n_k^{stn}(t, p, \theta, \chi)$ (48).

We satisfy the conditions (41) if we choose a conformal factor

$$\omega^2 = \eta_0(p, \theta, \chi) \omega_0^2(p) = (h_4^{stn})^{p_2/p_1}$$

for some integers $p_1$ and $p_2$ and defining $\zeta_i$ as solutions of the equations

$$\partial_i \omega - (w_i + \zeta_i) \omega^* = 0$$

for given solitonic values $w_i = w_i^{stn}$ and $\omega = \omega^{stn}$.

The nontrivial anchor coefficients are defined by the values $q^\alpha = 1/q_\alpha$ and $\eta_\beta = 1/\eta_\beta$, stated by (76), introduced in formulas (53) for $(\bar{\rho}_a^i)^2$ with $H_a^i(p, \theta, \chi)$.

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