Possibility of the structured mixed phases at first order phase transitions in neutron stars is reexamined by taking into account the charge screening effect. The Maxwell construction is shown to be not conflicted with the Gibbs conditions once the Coulomb potential is properly taken into account. Taking the hadron-quark deconfinement transition as an example, we explicitly demonstrate a mechanical instability of the geometrical structure of the structured mixed phase by the charge screening effect. In this case we have effectively the picture given by the Maxwell construction.

1. Introduction

It is now commonly accepted that various phase transitions may occur in compact star interiors or during the gravitational collapse from progenitor stars. Possibilities of the meson (pion and kaon) condensations and the hadron-quark deconfinement transition at high-density matter or the liquid-gas transition at subnuclear density have been studied by many authors. These phase transitions may have some implications to compact star phenomena, and it has been expected that recent progress in observations might reveal such new forms of matter.

Such phase transitions are of the first order in most cases and the Maxwell construction has been applied to get the equation of state (EOS) in phase equilibrium; there appears a separation of spatially bulk phases in the mixed-phase with the equal pressure. Glendenning demonstrated a possibility of the structured mixed phases (SMP) in such systems by invoking the proper treatment based on the Gibbs conditions. Where the charge
density as well as the baryon-number density are inhomogeneous. Subsequently, many authors have demonstrated energetic preference of SMP and its existence in a wide density region, disregarding effects of inhomogeneity of the particle configurations and/or the electric field \(^2\). The geometrical structure of SMP looks like droplets, rods or slabs as in the nuclear pasta phase \(^3,4\).

The Gibbs conditions require the pressure balance and the equality of the chemical potentials between two phases, denoted by I and II, for phase equilibrium \(^5, a\). For a multi-component system with more than one chemical potential, as is common in neutron-star matter, we must impose the equality condition for each chemical potential in order to fulfill the condition of the physico-chemical equilibrium. More definitely, we, hereafter, consider the charge chemical potential (\(\mu_Q\)) and the baryon-number chemical potential (\(\mu_B\)) respecting two conservation laws in neutron-star matter: 
\[
\mu_Q^I = \mu_Q^{II} \quad \text{and} \quad \mu_B^I = \mu_B^{II}.
\]

On the other hand, the first condition is not fulfilled in the Maxwell construction, since the local charge neutrality is implicitly imposed, while only the global charge neutrality must be satisfied. When we naively apply the Gibbs conditions instead of the Maxwell construction, we can see that there appears SMP in a wide density region and there is no constant-pressure region in EOS.

SMP, if exists, may have phenomenological implications on compact stars through e.g., glitches, neutrino opacity, gamma-ray burst or mass of hybrid stars.

In this talk we address a controversial issue about the relevance of SMP, by taking the hadron-quark deconfinement transition as an example \(^6\). We shall see that the Debye screening effects greatly modify the mechanical stability of SMP. In the absence of SMP we effectively recover the picture of phase equilibrium given by the Maxwell construction where two bulk phases are separated without spoiling the Gibbs conditions.

2. Bulk calculations and finite-size effects

Consider SMP consisting of two phases I and II, where we assume spherical droplets of phase I with the radius \(R\) to be embedded in the matter of phase II and two phases are clearly separated by sharp boundaries. We divide the whole space into the equivalent Wigner-Seitz cells with the radius \(R_W\) (see Fig.1). The volume of the cell is \(V_W = 4\pi R^3_W/3\) and that of the droplet is \(V = 4\pi R^3/3\).

\(^a\)We consider here matter at zero temperature.
A bulk calculation proceeds as follows. For a given volume fraction $f = (R/R_W)^3$, the total energy $E$ may be written as the sum of the volume energy $E_V$, the Coulomb energy $E_C$ and the surface energy $E_S$,

$$E = E_V + E_C + E_S. \quad (1)$$

We further assume, for simplicity, that baryon number ($\rho_B^{\alpha}$) and charge ($\rho_Q^{\alpha}$) densities are uniform in each phase $\alpha$, $\alpha = I, II$. Then, $E_V$ can be written as $E_V/V_W = f e^I(\rho_B^{I}) + (1 - f)e^{II}(\rho_B^{II})$ in terms of the energy densities $e^\alpha$, $\alpha = I, II$. The surface energy $E_S$ may be represented as $E_S/V_W = f \times 4\pi\sigma/R$ in terms of the surface tension $\sigma$. The Coulomb energy $E_C$ is given by

$$E_C/V_W = f \times \frac{16\pi^2}{15} (\rho_Q^I - \rho_Q^{II})^2 R^2. \quad (2)$$

The optimal value of $R_D$ is determined by the minimum condition,

$$\left. \frac{\partial (E/V_W)}{\partial R} \right|_f = 0, \quad (3)$$

for a given $f$ (see Fig. 2). Since $E_V$ does not depend on $R$, we can always find a minimum as a result of the competition between the Coulomb and the surface energies, satisfying the well-known relation, $E_S = 2E_C$.

However, such bulk calculations have been proved to be too crude for the discussions of SMP. Instead, a careful consideration of the interface of two phases is required. As a defect of the bulk calculations they ignore the finite size effects. In particular, they have the inconsistent treatment of the Coulomb potential; they do not use the Poisson equation, so that the charge density profiles are assumed ab initio to be constants and the Coulomb potential is assumed to be $1/r$. If one properly solves the Poisson equation, one should have the screening effect as a result of the rearrangement of the

Figure 1. Equal droplets of the phase I embedded in the phase II (right panel), and the geometrical structure of the Wigner-Seitz cell (left panel)
Figure 2. Schematic view of the total energy and each contribution in the bulk calculations (solid curves). Screening effect reduces the Coulomb energy, shown by the thick arrow.

charge-density distribution. Hence, the radius $R_D$ should be not too large, compared with the Debye screening length $\lambda_D^{-2} = \sum_i (\lambda_D^i)^{-2}$,

$$\frac{1}{\lambda_D^2} = 4\pi Q_i \frac{\partial \rho_{ch}}{\partial \mu_i},$$

in order the above treatment to be justified, the suffix $i$ runs over the particle species. Otherwise, the Coulomb energy is reduced by the screening effect, which should lead to a mechanical instability of SMP in some cases (Fig. 2). In the case of the hadron-quark deconfinement transition, $\lambda_D^q \simeq 5\text{fm}$ and $\lambda_D^p, \lambda_D^e$ are of the same order as $\lambda_D^q$, for a typical density with $\mu_B \simeq 1\text{GeV}$. We shall see in the following that $R_D$ is typically of the same order as $\lambda_D \sim \lambda_D^q$, and the mechanical stability of the droplet is much affected by the screening effect.

3. Mechanical instability of the geometrical structure of SMP

3.1. Thermodynamic potential for hadron - quark deconfinement transition

In the following we consider thermodynamics for non-uniform systems. The situation is the same as described in Fig. 1: the phase I in the domain $D_I$ consists of $u, d, s$ quarks and electrons and the phase II in the domain $D^{II}$ neutrons, protons and electrons. These phases should be clearly separated
by the narrow boundary layer $D_S$ with the width $\sim d_s \leq 1 \text{ fm}$ due to the non-perturbative effect of QCD. We treat such narrow boundary as the sharp one ($\partial D$) with the surface tension parameter $\sigma_{\text{QCD}}$ by using the bag model picture, while the value of $\sigma_{\text{QCD}}$ is poorly known. We shall see that the Debye screening length $\lambda_D$ is much longer than $d_s$ and thereby the introduction of the sharp boundary should be reasonable \(^b\).

Then, the thermodynamic potential per cell is given by a density functional \(^7\),

$$\Omega = E[\rho] - \mu^I_i \int_{D^I} d\vec{r} \rho^I_i - \mu^{II}_i \int_{D^{II}} d\vec{r} \rho^{II}_i,$$

where $E[\rho]$ is the energy of the cell and consists of four contributions:

$$E[\rho] = \int_{D^I} d\vec{r} \epsilon^I_{\text{kin+str}}[\rho^I_i] + \int_{D^{II}} d\vec{r} \epsilon^{II}_{\text{kin+str}}[\rho^{II}_i] + 4\pi R^2 \sigma_{\text{QCD}} + E_V. \quad (6)$$

The first two terms are given by the kinetic and strong interaction energies, and the Coulomb interaction energy $E_V$ is expressed in terms of particle densities,

$$E_V = \frac{1}{2} \int d\vec{r} d\vec{r}' e Q_i \rho_i(\vec{r}) Q_j \rho_j(\vec{r}') |\vec{r} - \vec{r}'|,$$

with $Q_i$ being the particle charge ($Q = -e < 0$ for the electron).

The equations of motion are given by $\delta \Omega/\delta \rho^\alpha_i = 0$ and written as

$$\mu^\alpha_i = \frac{\partial \epsilon^\alpha_{\text{kin+str}}}{\partial \rho^\alpha_i} - N^{\text{ch,} \alpha}_i V^\alpha(\vec{r}), \quad N^{\text{ch,} \alpha}_i = Q_i^\alpha / e, \quad (8)$$

with the electric potential $V^\alpha(\vec{r})$:

$$V(\vec{r}) = - \int d\vec{r}' e Q_i \rho_i(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \equiv \begin{cases} V^I(\vec{r}), & \vec{r} \in D^I \\ V^{II}(\vec{r}), & \vec{r} \in D^{II}. \end{cases} \quad (9)$$

Thus chemical potentials $\mu^\alpha_i$ for charged particles have values depending on the electric state of the phase as well as on its chemical composition. Actually it is sometimes called the electro-chemical potential to stress this fact. \(^5\)

\(^b\)This treatment is also similar to the Gibbs geometrical surface. \(^5\)
3.2. Gauge invariance

The thermodynamic potential enjoys the invariance under a gauge transformation, \( V(\vec{r}) \rightarrow V(\vec{r}) - V^0 \) and \( \mu_i^\alpha \rightarrow \mu_i^\alpha + N_i^{\text{ch},\alpha} V^0 \), with an arbitrary constant \( V^0 \). Hence the chemical potential \( \mu_i^\alpha \) acquires physical meaning only after gauge fixing \(^c\).

Here we reconsider the Gibbs conditions and the Maxwell construction. As has been mentioned, on the first glance the Maxwell construction looks as contradicting the Gibbs conditions, especially the equilibrium condition for the charge chemical potential \( \mu_Q(= \mu_e) \) in our context. However, correctly speaking, when we say \( \mu_e^I \neq \mu_e^{II} \) within the Maxwell construction, it means nothing but the difference in the electron number density \( n_e \) in two phases, \( n_e^I \neq n_e^{II} \); this is because \( n_e = \mu_e^3/(3\pi^2) \), if the Coulomb potential is absent. Once the Coulomb potential is taken into account, using eq. (8), \( n_e \) can be written as

\[
  n_e^\alpha = \frac{(\mu_e^\alpha - V^\alpha)^3}{3\pi^2}.
\]

Thus we may have \( \mu_e^I = \mu_e^{II} \) and \( n_e^I \neq n_e^{II} \) simultaneously, with the different values of \( V, V^I \neq V^{II} \) (see Fig. 3).

![Figure 3](image)

Figure 3. Relation between the charge chemical potential \( \mu_Q(= \mu_e) \) and the electron number density \( n_e \) in the presence of the Coulomb potential \( V \). Fulfilling the Gibbs conditions, \( \mu_{eI} = \mu_{eII} \), \( \mu_e^I = \mu_e^{II} \), we can change \( n_e \) in two phases as in the Maxwell construction, if \( V \) changes from one phase to another.

Applying Laplacian (\( \Delta \)) to the l.h.s. of eq. (9) we recover the Poisson

\(^c\)Note that \( V = 0 \) is a conventional choice in the usual treatment of uniform matter, while any constant is possible there.
equation ($\vec{r} \in D^{\alpha}$),
\[
\Delta V^{\alpha}(\vec{r}) = 4\pi e^2 \rho^{\text{ch},\alpha}(\vec{r}) \equiv \frac{4\pi e Q_i^\alpha \rho_i^\alpha(\vec{r})}{V_{\alpha}^\text{ref}}.
\]
(11)
The charge density $\rho^{\text{ch},\alpha}(\vec{r})$ as a function of $V^{\alpha}(\vec{r})$ is determined by the equations of motion (8). Thus eq. (11) is a nonlinear differential equation for $V^{\alpha}(\vec{r})$. The boundary conditions are
\[
V^I = V^{II}, \quad \nabla V^I = \nabla V^{II}, \quad \vec{r} \in \partial D,
\]
(12)
where we have neglected a small contribution of the surface charge accumulated at the interface of the phases. We also impose the condition,
\[
\nabla V^{II} = 0,
\]
that implies that each cell must be charge neutral.

3.3. Results

The Debye screening parameter is introduced by the Poisson equation, if one expands the charge density in $\delta V^{\alpha}(\vec{r}) = V^{\alpha}(\vec{r}) - V_{\alpha}^\text{ref}$ around a reference value $V_{\alpha}^\text{ref}$. Then eq. (11) renders
\[
\Delta \delta V^{\alpha}(\vec{r}) = 4\pi e^2 \rho^{\text{ch},\alpha}(V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}) + (\kappa^{\alpha}(V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}))^2 \delta V^{\alpha}(\vec{r}) + \ldots,
\]
(13)
with the Debye screening parameter,
\[
(\kappa^{\alpha}(V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}))^2 = \frac{4\pi e^2}{\partial \rho^{\text{ch},\alpha} / \partial V|_{V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}}} = \frac{4\pi e Q_i^\alpha Q_j^\alpha \partial \rho_i^\alpha / \partial \mu_j^\alpha|_{V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}}}{V_{\alpha}^\text{ref}}.
\]
(14)
Then we calculate contribution to the thermodynamic potential of the cell up to $O(\delta V^{\alpha}(\vec{r})^2)$. The “electric field energy” of the cell (7) can be written by way of the Poisson equation (13) as
\[
E_V = \int_{D^I} d\vec{r} \delta V^I + \int_{D^{II}} d\vec{r} \delta V^{II} = \int_{D^I} \frac{(\nabla V^I(\vec{r}))^2}{8\pi e^2} d\vec{r} + \int_{D^{II}} \frac{(\nabla V^{II}(\vec{r}))^2}{8\pi e^2} d\vec{r},
\]
(15)
that is, in the case of unscreened approximations, usually called the Coulomb energy. Besides the terms given by (15), there are another contributions arising from effects associated with the inhomogeneity of the electric potential profile, through implicit dependence of the particle densities on $V^{I,II}(\vec{r})$. We will call them “correlation terms”, $\omega^\alpha_{\text{cor}} = c^{\alpha}_{\text{kin,str}} - \mu_i^\alpha \rho_i^\alpha$.

We obtain the corresponding correlation contribution to the thermodynamic potential $\Omega_{\text{cor}} = \int_{D^I} d\vec{r} \omega^I_{\text{cor}} + \int_{D^{II}} d\vec{r} \omega^{II}_{\text{cor}}$:
\[
\omega^\alpha_{\text{cor}} = c^{\alpha}_{\text{kin,str}}(\rho_i^\alpha(V_{\alpha}^\text{ref}) - \mu_i^\alpha \rho_i^\alpha(V_{\alpha}^\text{ref}) - \rho^{\text{ch},\alpha}(V_{\alpha}^\text{ref})V_{\alpha}^\text{ref}) + V_{\alpha}^\text{ref} \Delta V^{\alpha}(\vec{r}) \frac{4\pi e^2}{\partial \rho^{\text{ch},\alpha} / \partial V|_{V^{\alpha}(\vec{r}) = V_{\alpha}^\text{ref}}} + \frac{(\kappa^{\alpha}(V_{\alpha}^\text{ref}))^2 (\delta V^{\alpha}(\vec{r}))^2}{8\pi e^2} + \ldots.
\]
(16)
where we also used eqs. (13) and (14). In general $V_{\text{ref}}^I \neq V_{\text{ref}}^{II}$ and they may depend on the droplet size. Their proper choice should provide appropriate convergence of the above expansion in $\delta V(\vec{r})$. Taking $V_{\text{ref}}^I = V_{\text{ref}}^{II} = V_{\text{ref}} = \text{const}$ we find

$$\omega_{\text{cor}}^\alpha = \frac{(\kappa^\alpha(V_{\text{ref}}))^2(V^\alpha(\vec{r}) - V_{\text{ref}})^2}{8\pi e^2},$$

(17)

except an irrelevant constant.

For given baryon-number chemical potential $\mu_B$ and charge chemical potential $\mu_Q$, all the particle chemical potentials $\mu_i$ can be represented in terms of $\mu_B$ and $\mu_Q(=\mu_e)$ with the help of the chemical equilibrium conditions:

$$\mu_u - \mu_s + \mu_e = 0, \ \mu_d = \mu_s,$$

$$\mu_n = \mu_p + \mu_e,$$

(18)

in each phase and

$$\mu_B \equiv \mu_n = 2\mu_d + \mu_u,$$

(19)

at the boundary.

Then particle number densities $\rho_i$ are represented as functions of $\mu_B$, $\mu_Q$ and the Coulomb potential $V$, due to the equations of motion. Substituting them in the Poisson equation (11), we can solve it with the proper boundary conditions (12); note that $\nabla V = 0$ at the boundary of the Wigner-Seitz cell provides us with another relation between $\mu_B$ and $\mu_Q$.

Thus we eventually have the density profiles of all the particles for

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![Figure 4. Dimensionless value of the thermodynamic potential per droplet volume. Solid lines are given for $f = 0.5$ and dashed lines for $f = 1/100$. The ratio of the screening lengths of two phases, $\alpha_0 = \lambda_0^I/\lambda_0^J$, is fixed as one. $\xi$ is a dimensionless radius of the droplet, $\xi \equiv R/\lambda_0^D$, with $\lambda_0^D \approx 5$ fm in this calculation. See text for further details.](image-url)
given density or the baryon-number chemical potential $\mu_B$. In Fig. 4 we demonstrate the radius ($R$) dependence of the total thermodynamic potential per droplet volume for the case of spherical droplets, \( \delta \tilde{\omega}_{\text{tot}}/\beta_0 = (\tilde{\epsilon}_V + \tilde{\epsilon}_I + \tilde{\omega}_{\text{cor}}^I + \tilde{\omega}_{\text{cor}}^\Lambda + \tilde{\epsilon}_S)/\beta_0 \), given by the sum of partial contributions, where tilde denotes each quantity scaled by the droplet volume $V = 4\pi R^3/3$ and $\beta_0$ is a typical quantity with the dimension of the energy density. Preparing some wide range for the value of the surface tension parameter $\sigma_{QCD}$, $\beta_1 \propto \sigma_{QCD}^6$, we present two cases of $f$, $f = 0, 0.5$.

The label “C” is given for reference to show the previous non-selfconsistent case, where the Coulomb potential is not screened see Fig. 2. We can see that only in the limit of $f \ll 1$ and $R \ll \lambda_D$, we are able to recover this case. The “e.m.” curve shows the partial contribution to the thermodynamic potential, $\tilde{\epsilon}_{\text{e.m.}}/\beta_0 \equiv (\tilde{\epsilon}_V + \tilde{\epsilon}_S)/\beta_0$, ignoring correlation terms. Comparing these curves we can see how the screening effect changes the thermodynamic potential: we can see that the minima at the “e.m.” curves disappear already at $\beta_1 > 0.03$, corresponding to unphysically small $\sigma_{QCD} \sim$ several MeV·fm$^{-2}$. However, the correlation energy gives a sizable contribution to allow the minimum for larger value of $\sigma_{QCD}$. Consequently, the minimum totally disappears between $\beta_1 = 0.1$ and $\beta_1 = 0.5$, which may be interpreted as $10 < \sigma_{QCD} < 50$(MeV·fm$^{-2}$) in this calculation. Thus we have seen a mechanical instability of the droplet for the medium values of $\sigma_{QCD}$, which might be in the physically meaningful range.

4. Summary and Concluding remarks

In this talk we addressed an issue about SMP at the first order phase transitions in multicomponent systems, like in neutron-star matter. We have studied a so called “contradiction” between the Gibbs conditions and the Maxwell construction extensively discussed in previous works. We have demonstrated that this contradiction is resolved if one correctly takes into account the difference in the “meaning” of the chemical potentials used in the two approaches: the different values of the electron chemical potentials in the Maxwell construction and the ones used in the Gibbs conditions do not contradict each other if one properly takes into account the electric field.

We have presented a framework based on the density functional theory to describe thermodynamics in the non-uniform systems. The Coulomb potential is properly included and particle density profiles are consistently determined with the Poisson equation.
Taking the hadron-quark deconfinement transition in high-density matter as an example, we have demonstrated the importance of the Debye screening effect, which is a consequence of the above treatment. With a numerical example, we have seen that the screening effect gives rise to a mechanical instability for realistic values of the surface tension parameter of $\sigma_{QCD}$. In this case we may effectively recover the picture given by the Maxwell construction, where the phase separation of two bulk phases arises.

Our framework is rather general and it may be applicable to any first order phase transition, e.g. the liquid-gas phase transition at subnuclear density $^8$.

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