Entanglement measures and non-equilibrium dynamics of quantum many-body systems: a path integral approach

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We present a path integral formalism for expressing matrix elements of the density matrix of a quantum many-body system between any two coherent states in terms of standard Matsubara action with periodic(anti-periodic) boundary conditions on bosonic(fermionic) fields. We show that this enables us to express several entanglement measures for bosonic/fermionic many-body systems described by a Gaussian action in terms of the Matsubara Green function. We apply this formalism to compute various entanglement measures for the two-dimensional Bose-Hubbard model in the strong-coupling regime, both in the presence and absence of Abelian and non-Abelian synthetic gauge fields, within a strong coupling mean-field theory. In addition, our method provides an alternative formalism for addressing time evolution of quantum-many body systems, with Gaussian actions, driven out of equilibrium without the use of Keldysh technique. We demonstrate this by deriving analytical expressions of the return probability and the counting statistics of several operators for a class of integrable models represented by free Dirac fermions subjected to a periodic drive in terms of the elements of their Floquet Hamiltonians. We provide a detailed comparison of our method with the earlier, related, techniques used for similar computations, discuss the significance of our results, and chart out other systems where our formalism can be used.

I. INTRODUCTION

The density matrix \( \hat{\rho} \) is of central importance for describing properties of any quantum statistical mechanical system. For a many-body quantum system with a Hamiltonian \( \hat{H} \), it is given by

\[
\hat{\rho} = \exp[-\beta \hat{H}]/Z, \quad Z = \text{Tr} \exp[-\beta \hat{H}],
\]

where \( \text{Tr} \) denotes the sum over all possible field configurations of the system, \( \beta = 1/(k_B T_0) \), \( k_B \) is the Boltzmann constant, and \( T_0 \) is the temperature. Its importance stems from the fact that any physical observable pertaining to such a system, described by an operator \( \hat{O} \), satisfies \( \langle \hat{O} \rangle = \text{Tr} [\hat{\rho} \hat{O}] \). It is well known that such statistical mechanical many-body systems may be described using path integrals. Indeed, the partition function \( Z \) of a quantum many-body system or expectation of any of its operators, \( \langle \hat{O} \rangle \), is conveniently described as a coherent state path integral over of bosonic or fermionic fields describing its constituent particles. It is therefore useful to have an expression for the matrix element of the density matrix between two arbitrary coherent states: \( \rho_{f,i} = \langle \phi_f | \hat{\rho} | \phi_i \rangle \), where \( | \phi_f \rangle \) and \( | \phi_i \rangle \) are bosonic or fermionic coherent states. However, computation of \( \rho_{f,i} \) cannot be trivially carried out even for the simplest Gaussian systems. The key problem that one faces in trying to evaluate such a quantity is that it cannot be directly expressed in terms of a path integral with standard periodic or anti-periodic boundary conditions on bosonic or fermionic fields. This difficulty stems from the fact that unlike expression of \( Z \), the presence of fixed initial and final coherent states \( | \phi_f \rangle \) and \( | \phi_i \rangle \) does not allow for a straightforward implementation of these boundary conditions. Consequently, it is not clear if one can use Matsubara formalism for such computations.

There are at least two key quantities where the knowledge of \( \rho_{f,i} \) proves to be useful. The first is the entanglement entropy of a quantum system which can be directly expressed in terms of the reduced density matrix \( \rho_\text{red} \) which can be obtained from \( \hat{\rho} \) by tracing out appropriate degrees of freedom. There are several measures for a system’s entanglement; the most frequently used ones are the Von-Neumann entropy \( S \) and the Renyi entropies \( S_n \). These are most easily computed by separating the closed quantum system into a subsystem and a bath and integrating out the bath degrees of freedom. In terms of the reduced density matrix \( \rho_\text{red} \), obtained from \( \hat{\rho} \) by integrating out the bath degrees of freedom, \( S \) and \( S_n \) are expressed as

\[
S = -\text{Tr}[\rho_\text{red} \ln \rho_\text{red}],
\]

\[
S_n = \frac{1}{1-n} \ln[\text{Tr}(\rho_\text{red}^n)],
\]

where one has the relation \( S = \lim_{n \to 1} S_n \). A scheme for computation of \( S_n \) has been pointed out in the context of one-dimensional (1D) conformal quantum field theories in Refs. 12,13 by introducing replica fields. Here we note that the knowledge of the matrix element \( \rho_{f,i} \) enables us to compute \( \rho_{f,i}^{\text{red}} \) and thus allows for a direct computation of \( S_n \). The key difficulty in carrying this out in the context of general bosonic or fermionic theories stems, once again, from one’s inability to express \( \rho_{f,i} \) in terms of correlation functions that can be computed using Matsubara formalism. In contrast, the Hamiltonian formulation for computing matrix elements of \( \rho_\text{red} \) for systems with Gaussian action indicates that such matrix elements can be expressed in terms of Gaussian correlators computed using standard Green function...
methods. However, such a formulation works only if the system Hamiltonian is quadratic. To the best of our knowledge, the results of Refs. 14,15 have not been derived using path integral methods. Such a derivation would be extremely useful since there are several systems, such as the Bose-Hubbard model treated with strong coupling mean-field approximation or the non-linear sigma model in the large-$N$ limit, where the effective action of the system can be brought to a Gaussian form. In contrast, due to complicated frequency dependence of the effective action or presence of constraint conditions, there are no simple Gaussian Hamiltonians that describe such systems. The knowledge of $\rho_{fi}$ (and hence $\rho_{f1}^{red}$) for such systems may throw light on their entanglement measures.

The second instance where the knowledge of $\rho_{fi}$ would prove useful is in the field of non-equilibrium quantum dynamics of many-body systems. The study of out-of-equilibrium dynamics of quantum systems has been a subject of active theoretical and experimental research in recent years. The key quantity which controls such dynamics is the evolution operator $\hat{U}$ for a quantum system given by

$$\hat{U}(t, 0) = T e^{-i \int_0^t dt' \hat{H}(t')/\hbar}$$

where $t$ is the final time up to which we track the evolution, $\hbar$ is the Planck’s constant, $T$ is the time ordering operator, and the initial time is set to zero without loss of generality. From Eqs. (4) and (1), one finds that $U_{fi}(t) = \langle \phi_f | \hat{U}(t, 0) | \phi_i \rangle$ can be obtained from $\rho_{fi}$ (with $\hat{H} \rightarrow \tilde{\hat{H}}(\tau)$ in Eq. (1) in which case $\tilde{\hat{\rho}} = T \exp[-i \int_0^\beta \tilde{\hat{H}}(\tau')d\tau']/Z$) via a Wick rotation $t \rightarrow -i \beta \hbar$. We note any correlation function of a non-equilibrium system (and hence any physical observable) can be computed from the knowledge of $\hat{U}$. This relation is usually not utilized due to the practical difficulty associated with such Wick rotation; however for cases where $\rho_{fi}$ is analytically known, this method can provide a different technique for computing the evolution operator without having to resort to Keldysh formulation. This is particularly useful for Gaussian actions subjected to periodic drives where they are expected to provide analytic expression for the system’s Floquet Hamiltonian. However, the relation between $\rho_{fi}$ and $U_{fi}$, to the best of our knowledge, has not been utilized so far for computing physical quantities in driven many-body systems.

In this work, we present a formalism for computing matrix element of the density matrix of a many-body quantum system between two arbitrary coherent states. The main results that we obtain from our analysis are as follows. First, we provide a general framework for computing $\rho_{fi}$ for arbitrary fermionic and bosonic systems in terms of their correlation functions which can be computed using standard Matsubara formalism. Our method shows that this is possible in principle for arbitrary quantum systems; however, practical computation are most easily done for systems with Gaussian actions. We note that such actions need not correspond to non-interacting systems with quadratic Hamiltonian but can arise out of mean-field or large-$N$ approximations of strongly interacting quantum systems. Second, we use this formalism to compute several measures of entanglement such as Von-Neumann and Renyi entropies of strongly interacting bosons in an optical lattice, both without and in the presence of Abelian and non-Abelian artificial gauge fields, within a strong coupling mean-field theory. For bosons without gauge fields, our results regarding $S_n$ and $S$ qualitatively agree with those of Refs. 29,30. Moreover, we also compute the entanglement negativity of such bosons both in the presence and absence of artificial gauge fields using our formalism. To the best of our knowledge, entanglement measures for boson systems with artificial gauge fields and entanglement negativity for strongly interacting bosons with or without gauge fields have not been computed so far. Since $S_2$ has been experimentally measured for superfluid bosons, we expect these computations to have direct relevance to future experiments. Third, we use our formalism to compute the Floquet Hamiltonian for a class of integrable many-body models represented by free Dirac fermions subjected to periodic drive according to a square pulse protocol and show that it agrees with the known result in the literature. We then use it to obtain exact analytical expressions for the return probability and the counting statistics for the fermionic density and order parameter operators for such driven Dirac fermions in terms of the elements of the Floquet Hamiltonian. We note that such expressions have been obtained earlier for quench protocols; our results constitute a generalization of these results to periodically driven systems. We use them to discuss the behavior of these quantities for Ising model in a transverse field, represented by 1D Dirac fermions, near its critical point. Finally, we summarize our main results, point out other physical systems where our formalism can be applied, and discuss the relation of our results with those in the existing literature.

The plan of the rest of the paper is as follows. In Sec. II we detail the general formalism for expressing the matrix element of the density matrix of quantum many-body systems between two arbitrary coherent states and chart out how the results obtained can be used to compute several measures of entanglement entropies and the entanglement negativity for these systems. We shall also discuss the application of this formalism to address non-equilibrium dynamics of driven quantum systems. This is followed by Sec. III, where we use this formalism to compute the Renyi and Von-Neumann entropies and the entanglement negativity of the Bose-Hubbard model both in the presence and absence of Abelian and non-Abelian gauge fields. Next, in Sec. IV, we obtain analytical expressions of the return probability and the counting statistics of fermionic number and order parameter operators for a class of integrable quantum many-body systems driven out of equilibrium by a periodic drive. Fi-
nally, we summarize our results, discuss their applica-
tibility to other systems, and conclude in Sec. V. Further
details of some aspects of our calculations and the rela-
tion of our results to those in the existing literature are
discussed in the appendices.

II. GENERAL FORMALISM

In this section, we provide a path-integral based for-
malism for computing the matrix elements of a many-
body density matrix between two arbitrary coherent
states. This is done in Sec. II A and is followed by its
application for computing Von-Neumann and Renyi en-
semble density matrix bodies. This is charted out in Sec. II C.

A. Matrix elements of a density matrix

In this subsection, we shall compute \( \rho_{fi} \) where the
coherent states \(|\phi_f\rangle\) and \(|\phi_i\rangle\) can either be bosonic or
fermionic (represented by Grassmann functions). To
compute this matrix element, we express the density ma-
xibers, is charted out in Sec. II C.

\[ \rho_{fi} = \frac{1}{Z} \int \prod_{k=1}^{N} d\phi_k^* d\phi_k \exp \left\{ -\sum_{k=1}^{N} \sum_{\alpha} \phi_{ka} \right\} + \sum_{k=2}^{N} \left[ \sum_{\alpha} \phi_{ka}^* \phi_{k-1a} - \epsilon H(\phi_k^*, \phi_{k-1}) \right] + \sum_{\alpha} (\phi_{fa}^* \phi_{N-1a} - \epsilon H(\phi_f^*, \phi_{N-1}) - H(\phi^*_1, \phi_1)) \right\}, \]  

(5)

where \( \epsilon = \beta/N \) and \( N \to \infty \), \( k \) denotes index for
time slices, \( \alpha \) stands for spatial and spin indices of
the fields, and we use the short-hand notation \( d\phi_k^* d\phi_k = \prod_{\alpha} d\phi_{ka}^* d\phi_{ka} \). The \( \phi_k \)'s are c-numbers for bosons
and Grassmann numbers for fermions. We now introduce
additional variables, \( \phi_{N\alpha} = \eta \phi_{\alpha} \) and \( \phi_{N\alpha}^* = \phi_{fa} \) and use

\[ 1 = \int d\phi_N^* d\phi_N \prod_{\alpha} [\delta(\phi_{N\alpha} - \phi_{fa}) \delta(\phi_{N\alpha} - \eta \phi_{\alpha})] \]

\[ = e^{\eta \phi_f} \int d\phi_N^* d\phi_N \int d\lambda^* d\lambda \exp \left\{ -\sum_{\alpha} |\phi_{N\alpha}|^2 \right\}

+ \sum_{\alpha} (\phi_{fa}^* \phi_{N-1a} - \epsilon H(\phi_f^*, \phi_{N-1}) - \epsilon H(\phi^*_1, \phi_1)) \right\}, \]  

(6)
where

\[
\hat{W}[\phi^\dagger, \phi] = \ln \int D[\lambda^*, \lambda] e^{W[\lambda^*, \lambda]} + \sum_{n=1}^{\infty} \frac{1}{(n!)^2}
\times \sum_{\{\alpha, \alpha_i\}} \phi^*_f \phi_{\alpha n} \cdots \phi^*_f \phi_{\alpha n} \hat{W}^{(2n)}_{\alpha_1 \cdots \alpha_n, \alpha'_1 \cdots \alpha'_n} \phi_{\alpha i_1} \cdots \phi_{\alpha i_n},
\tag{13}
\]

with

\[
\hat{W}^{(2n)}_{\alpha_1 \cdots \alpha_n, \alpha'_1 \cdots \alpha'_n} = (-\eta)^n \langle \lambda_{\alpha_1} \cdots \lambda_{\alpha_n} 
\times \lambda^*_{\alpha'_1} \cdots \lambda^*_{\alpha'_n} \rangle S_{\alpha \alpha'}[\lambda^*, \lambda], c
\tag{14}
\]

and \(S_{\alpha \alpha'}[\lambda^*, \lambda] = -W[\lambda^*, \lambda]\). We note that Eqs. (12-14) constitute an expression of \(\rho_f\), for any generic fermionic and bosonic systems in terms of their correlation functions which can in principle be computed using Matsubara formulation. In practice, however, these correlators get quite complicated with increasing order.

To make further analytical progress, we now concentrate on systems that can be represented by Gaussian actions,

\[
S[\phi^*, \phi] = \int_0^\beta d\tau d\tau' \sum_{\alpha, \alpha'} \phi_\alpha^*(\tau) G_{\alpha \alpha'}(\tau - \tau') \phi_{\alpha'}(\tau'),
\tag{15}
\]

where \(G_{\alpha \alpha'}(\tau - \tau') = \langle \phi_\alpha(\tau) \phi^*_{\alpha'}(\tau') \rangle\). We note that this does not necessarily restrict our analysis to non-interacting systems since \(G\) may include self-energy terms. Hence the Gaussian action \(S\) may represent several interacting and constrained systems for which two-particle and higher-order Green functions can be neglected compared to the single-particle one. This is typically possible when the system concerned can be treated using a large-\(N\) approximation or within mean-field theory. Concrete examples include the Bose-Hubbard model in its superfluid and Mott insulating phases near the critical point treated within a strong coupling mean-field approximation, the \(O(N)\) non-linear sigma model in the large-\(N\) limit, and spin models such as the \(d = 1\) Ising model or the \(d = 2\) Kitaev model which have exact free fermionic representations.

For the Gaussian action (15) one easily finds

\[
e^{W[\lambda^*, \lambda]} = e^{-\sum_{\alpha, \alpha'} \lambda^*_\alpha G_{\alpha \alpha'}(0^+) \lambda_{\alpha'},
\]

\[
e^{W[\phi^\dagger, \phi]} = [\det G(0^+)]^{-\eta} e^{-\eta \sum_{\alpha, \alpha'} \phi^*_f \phi_{\alpha n} G_{\alpha \alpha'}^{-1}(0^+) \phi_{\alpha n},
\tag{16}
\]

and

\[
\rho_f = [\det G(0^+)]^{-\eta} e^{\eta \sum_{\alpha \alpha'} \phi^*_f \phi_{\alpha n} L_{\alpha \alpha'} \phi_{\alpha n},
\]

\[
L_{\alpha \alpha'} = \delta_{\alpha \alpha'} - G_{\alpha \alpha'}^{-1}(0^+).
\tag{17}
\]

Here we have used \(G(\beta) = \eta G(0)\) and interpreted \(G(0)\) as \(G(0^+)\) as can be inferred from a careful analysis of the discrete-time path integral (see, e.g., Appendix A1). Note that \(G^{-1}(0^+)\) should be understood as the inverse matrix of \(G(0^+)\), i.e. \(\sum_{\alpha \alpha'} G_{\alpha \alpha'}(0^+) G_{\alpha' \alpha}^{-1}(0^+) = \delta_{\alpha \alpha'}\).

One can verify that the density operator has the correct normalization, since its trace

\[
\frac{1}{Z} \int d\phi^* d\phi e^{-\sum_{\alpha} \phi_\alpha^2} \langle \eta \phi | e^{-\beta \hat{H}} | \phi \rangle
\]

\[= [\det G(0^+)]^{-\eta} \int d\phi^* d\phi e^{-\sum_{\alpha, \alpha'} \phi_{\alpha n}^* G_{\alpha \alpha'}^{-1}(0^+) \phi_{\alpha n}},
\tag{18}
\]

is equal to unity.

The matrix \(L\) in (17) can be more explicitly written using

\[
G_{\alpha \alpha'}(0^+) = \delta_{\alpha \alpha'} + G_{\alpha \alpha'}(0^-),
\]

\[
G_{\alpha \alpha'}(0^-) = \frac{1}{\beta} \sum_{\omega_n} G_{\alpha \alpha'}(i\omega_n) e^{i\omega_n 0^+},
\tag{19}
\]

where the first equality follows from boson/fermion (anti)commutation relations and \(\omega_n\) (a integer) is a Matsubara frequency. Thus Eqs. (17) and (19) provide us with an analytic expression for the matrix elements of \(\hat{\rho}\) for bosonic/fermionic systems with Gaussian action in terms of their Matsubara one-particle Green functions.

The above arguments may be easily generalized to cases where the action of the system breaks \(U(1)\) symmetry,

\[
S[\phi^*, \phi] = \frac{1}{2} \int_0^\beta d\tau d\tau' \sum_{\alpha, \alpha'} (\phi_\alpha(\tau), \phi_{\alpha'}(\tau'))
\times G^{-1}(\tau - \tau') \left( F_{\alpha \alpha'} \phi_{\alpha}(\tau) \right),
\tag{20}
\]

where

\[
G_{\alpha \alpha'}(\tau - \tau') = \begin{pmatrix} (\phi_\alpha(\tau) \phi^*_{\alpha'}(\tau')) & (\phi_\alpha(\tau) \phi_{\alpha'}(\tau')) \\ (\phi^*_{\alpha}(\tau) \phi_{\alpha}(\tau')) & (\phi^*_{\alpha}(\tau) \phi^*_{\alpha'}(\tau')) \end{pmatrix}
\]

\[
= \frac{G_{\alpha \alpha'}(\tau - \tau')} \left( F_{\alpha \alpha'}(\tau - \tau') \eta G_{\alpha \alpha'}(\tau' - \tau') \right).
\tag{21}
\]

A straightforward calculation in the same lines as charted out above leads to

\[
\rho_f = [\det G(0^+)]^{-\eta/2} e^{\frac{\eta}{2} \sum_{\alpha, \alpha'} \Phi^*_f \eta L_{\alpha \alpha'} \Phi_{\alpha'}},
\tag{22}
\]

where

\[
L_{\alpha \alpha'} = I_0 \delta_{\alpha \alpha'} - I_0 G_{\alpha \alpha'}^{-1}(0^+), \quad I_0 = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}
\tag{23}
\]

and \(\Phi_\alpha = (\phi_{\alpha a}, \phi^*_f)^T\), \(\Phi^*_f = (\phi^*_f, \phi_{\alpha a})\). The \(2 \times 2\) matrix \(G_{\alpha \alpha'}(0^+)\) can be written as

\[
G_{\alpha \alpha'}(0^+) = \begin{pmatrix} G_{\alpha \alpha'}(0^+) & F_{\alpha \alpha'}(0^+) \\ \eta F^*_{\alpha \alpha'}(0^+) & \eta G_{\alpha \alpha'}(0^+) \end{pmatrix}.
\tag{24}
\]

In the next section, we shall use Eqs. (17) and (22) to compute several entanglement measurements for these systems. Moreover, these equations shall be used to compute correlation functions in driven bosonic/fermionic systems in Sec. II C.
B. Entanglement measures

In this section, we are going to relate expressions for several entanglement measures for ground states of many-body quantum systems whose action is Gaussian and is given by Eq. (15), to its Matsubara Green function \( G(0^+) \) at zero temperature. This can be done directly from Eqs. (19) and (24) via the introduction of replica fields\(^{12,14} \); this procedure is charted out in App. B. Here we connect Eqs. (19) and (24) to standard methods used in the literature for such computation\(^{4,7} \).

In what follows, we consider a lattice model with Gaussian action and broken \( U(1) \) symmetry for which \( F \neq 0 \); results for systems with no \( U(1) \) symmetry breaking can be obtained from our analysis by setting \( F = 0 \). One can easily check that the density matrix obtained in the preceding section yields the correct value of the equal-time correlator

\[
\langle \hat{\phi}_i \hat{\phi}_j \rangle = \text{Tr}(\hat{\rho} \hat{\phi}_i \hat{\phi}_j) = \int d\phi \hat{\phi}_i \hat{\phi}_j \rho(\phi) \phi_i \phi_j, \tag{25}
\]

where \( i \) and \( j \) denote sites of the lattice (we ignore the spin index). Using (22) for the matrix element of the density matrix between the coherent states \( |\eta \phi \rangle \) and \( |\phi \rangle \), one obtains

\[
\langle \hat{\phi}_i \hat{\phi}_j \rangle = K \int d\phi \hat{\phi}_i \hat{\phi}_j e^{-\sum_{\ell} |\phi_{\ell}|^2} \Phi_i \Phi_j \Phi^\dagger_{i \ell} \Phi^\dagger_{j \ell}, \tag{26}
\]

where \( \Phi_i = (\phi_i, \phi_{\ell}) \) and \( K = |\det G(0^+)|^{-1/2} \). Performing the integral gives \( \langle \hat{\phi}_i \hat{\phi}_j \rangle = G_{ij}(0^+) \) as expected. A similar analysis gives

\[
\langle \hat{\phi}_i \hat{\phi}_j \hat{\phi}_k \rangle = K \int d\phi \hat{\phi}_i \hat{\phi}_j \hat{\phi}_k e^{-\sum_{\ell} |\phi_{\ell}|^2} \Phi_i \Phi_j \Phi_k \Phi^\dagger_{i \ell} \Phi^\dagger_{j \ell} \Phi^\dagger_{k \ell} = F_{ij}(0^+). \tag{27}
\]

Given the structure (24) of the matrix \( G_{ij}(0^+) \), the quadratic form appearing in Eqs. (26) and (27) can be diagonalized by a Bogoliubov transformation.\(^{5,6} \) In Fourier space,

\[
\frac{1}{2} \sum_k \Phi_k^\dagger \hat{G}^{-1}_{kk}(0^+) \Phi_k = \frac{1}{2} \sum_k \Psi_k^\dagger \left( \begin{array}{cc} \lambda_k^{-1} & 0 \\ 0 & \eta \lambda_k^{-1} \end{array} \right) \Psi_k = \sum_k \psi_k^\dagger \lambda_k^{-1} \psi_k, \tag{28}
\]

where \( \Psi_k = (\hat{\psi}_k, \hat{\psi}_{-k}) \), \( \lambda_k \geq 0 \) and the sum runs over all momenta \( k \) of the first Brillouin zone.

We now first consider the case where \( \hat{\psi}_k \) is a bosonic field. We note that in this case the correlations of \( \hat{\psi}_k \) may be thought as those of a free charged scalar fields whose Hamiltonian is given by

\[
\hat{H}_{\text{scalar}} = \sum_k \left( \hat{\Pi}^\dagger_k \hat{\Pi}_k + \frac{1}{4 \lambda_k} \hat{\psi}_k^\dagger \hat{\psi}_k \right), \tag{29}
\]

where \( \left[ \hat{\psi}_k, \hat{\Pi}_k \right] = \left[ \hat{\psi}_k^\dagger, \hat{\Pi}_k^\dagger \right] = i \delta_{k,-k'} \) (and all other commutators vanishing). From (29) one easily obtains

\[
\langle \hat{\psi}_k \hat{\psi}_k^\dagger \rangle = \delta_{k,k} \lambda_k \text{ and } \langle \hat{\Pi}_k \hat{\Pi}^\dagger_k \rangle = \delta_{k,k} / 4 \lambda_k.
\]

We note that since the action of the system is Gaussian, the correlations are completely specified by the \( \lambda_k \)'s; thus all the quantities including entanglement entropy \( S \) corresponding to such an action are identical to that obtained from \( H_{\text{scalar}} \).

It is convenient to rewrite \( \hat{\psi}^{(t)} \) and \( \hat{\Pi}^{(t)} \) in terms of four real operators \( \hat{\psi}_i, \hat{\Pi}_i (i = 1, 2) \) defined by

\[
\hat{\psi}_k = \frac{1}{\sqrt{2}} (\hat{\psi}_{1k} + i \hat{\psi}_{2k}),
\]

\[
\hat{\Pi}_k = \frac{1}{\sqrt{2}} (\hat{\Pi}_{1k} - i \hat{\Pi}_{2k}),
\]

and \( \hat{\psi}_k \hat{\psi}_{-k} = \hat{\psi}_{k-} \hat{\psi}_{-k} = \hat{\Pi}_{k-} \hat{\Pi}_{-k} = i \delta_{ij} \delta_{k,-k'} \).

The Hamiltonian can then be written as the sum of two uncoupled harmonic oscillators,

\[
\hat{H}_{\text{scalar}} = \frac{1}{2} \sum_{i=1}^2 \sum_k \left( \hat{\Pi}_{1k} \hat{\Pi}_{-1k} + \frac{1}{4 \lambda_k} \hat{\psi}_{-k} \hat{\psi}_k \right). \tag{31}
\]

The computation of the Von-Neumann entropy \( S \) for each oscillator is straightforward.\(^7 \) One chooses a subsystem \( A \) which is a part of the full system and construct the covariance matrix \( A_{jj'} \) with \( j, j' \) ∈ \( A \) given by

\[
A_{jj'} = \begin{pmatrix} \langle \hat{\psi}_j \hat{\psi}_{j'} \rangle & 0 \\ 0 & \langle \hat{\Pi}_j \hat{\Pi}_{j'} \rangle \end{pmatrix} \begin{pmatrix} A_{jj'} & 0 \\ 0 & B_{jj'} \end{pmatrix} = \begin{pmatrix} \lambda_{jj'} & 0 \\ 0 & \lambda_{jj'} \end{pmatrix},
\]

\[
B_{jj'} = \int \frac{dk}{2 \pi} e^{i k (r_j - r_{j'})},
\]

where the integral \( \int = \int \frac{dk}{2 \pi} e^{i k \xi} \) is over the first Brillouin zone. The symplectic eigenvalues \( \nu \) of the covariance matrix \( \Lambda \) then yields the boson entanglement entropy\(^4,7 \) (and hence \( B \) is identical to \( A \)). Thus the only non-trivial correlation function is \( \langle \hat{\psi}_k \hat{\psi}_{-k}^\dagger \rangle \) and there is no need to consider the harmonic oscillator (29). The entanglement Hamiltonian is determined by requiring that it reproduces the correlation matrix \( A_{jj'} \) of the subsystem, which gives

\[
S_f = -\sum_\ell \left[ \nu_\ell \ln \nu_\ell + (1 - \nu_\ell) \ln (1 - \nu_\ell) \right]. \tag{34}
\]
where \( \nu \varepsilon = 1 - n \varepsilon \) denotes the eigenvalues of \( A \) and \( n \varepsilon = 1/(e^{\varepsilon} + 1) \), i.e., \( \varepsilon = 2 \arctanh(2 \nu - 1) \).

We note that this procedure also allows us to compute the Renyi entropies for Gaussian systems. In the basis where the entanglement Hamiltonian \( \hat{H}_e = \sum \varepsilon \alpha_\ell \hat{a}_\ell \hat{a}_\ell^\dagger \) is diagonal, the reduced density matrix reads

\[
\bar{\rho}_{\text{red}} = e^{-\bar{H}_e} = \prod_\ell (1 - \varepsilon \alpha_\ell) \varepsilon^{-\nu \varepsilon} e^{-\nu \varepsilon \alpha_\ell \hat{a}_\ell \hat{a}_\ell^\dagger}.
\]

The occupation number operator \( \hat{a}_\ell \hat{a}_\ell^\dagger \) has all positive integer eigenvalues for bosons and 0, 1 for fermions. The \( n \)th Renyi entropy (3) is given by \( 4,7 \)

\[
S_n = -\frac{1}{n-1} \sum_\ell \ln \frac{1 - \varepsilon \alpha_\ell}{1 - \varepsilon \nu \varepsilon}. \tag{36}
\]

One can also compute the entanglement negativity \( N_b \) for bosonic systems from the covariance matrix constructed in Eq. (32). To this end, we note that for computing \( N_b \) for any bosonic many-body systems one needs to identity two subsystems \( L_1 \) and \( L_2 \). One then creates a partial transposed density matrix, \( \rho^T \), which is defined as \( 8 \)

\[
\rho^T_{fi} = \langle \phi_i^{L_1}, \phi_i^{L_2\dagger} | \hat{\rho}^T | \phi_i^{L_1}, \phi_i^{L_2\dagger} \rangle. \tag{37}
\]

Such a partial transposition has been studied for a harmonic chain and it is known that it amounts to the transposition of the canonical momenta in subsystem \( L_2 \): \( \Pi^{L_1} \rightarrow -\Pi^{L_2} \). Thus one needs to change the sign of the momentum correlators between the two subsystems. The covariance matrix can now be written as

\[
\Lambda^T = \begin{pmatrix}
0 & 0 & A_{L_1L_1}^{L_2L_2} & A_{L_1L_2}^{L_1L_2} \\
0 & 0 & 0 & A_{L_2L_2}^{L_1L_2} \\
B_{L_1L_1}^{L_2L_1} & -B_{L_1L_2}^{L_1L_2} & 0 & 0 \\
-B_{L_2L_1}^{L_2L_2} & B_{L_2L_2}^{L_2L_2} & 0 & 0
\end{pmatrix}, \tag{38}
\]

where \( A_{ij}^{jk} = [A_{ij}^{jk}] \) involve field (canonical momentum) correlators defined in Eq. (32) with \( j \in L_1 \) and \( j' \in L_2 \). The eigenvalues of this covariance matrix are denoted by \( \nu^T_\ell \). From Eq. (33), we find \( \nu^T_\ell < 1 \) occurs if \( \rho^T \) has negative eigenvalues. Thus, in terms of these eigenvalues, one can define the entanglement negativity for bosons as

\[
N_b = -\sum_\ell \ln \text{Min}[1, \nu^T_\ell]). \tag{39}
\]

For fermions, partial transposition is more complicated than for bosons. Indeed it has been shown in Ref. 9 that a transposition of fermionic fields with a generic Gaussian action does not keep the action Gaussian. Moreover such a transformation, performed twice, does not lead to the starting Gaussian Hamiltonian or action. To remedy this, a different transformation which amounts to replacement of the partial transformation by partial time reversal operation has been defined in Ref. 10. However, to the best of our knowledge, it is not fully understood if this new measure accurately reflects the presence of entanglement for generic fermionic systems. We shall not address this issue further in this work.

C. Non-equilibrium dynamics

In this subsection, we shall outline the application of our approach to quantum systems taken out of equilibrium \( \text{via} \) a drive with a given protocol. To this end, we shall compute \( U_{fi}(t,0) = \langle \phi_f | U(t,0) | \phi_i \rangle \) where \( U \) is given by Eq. (4). To compute \( U_{fi} \) using the formalism developed in Sec. II A, we analytically continue to imaginary time using \( t \rightarrow -i\tau \). Denoting \( t = -i\beta \hbar \), we find

\[
U_{fi}(\beta,0) = \langle \phi_f | T_{\tau} e^{-\int_0^{\beta} d\tau \hat{H}(\tau)} | \phi_i \rangle \tag{40}
\]

where \( \hat{H}(\tau) = \hat{H}(t' = -i\tau) \) is the analytically continued Hamiltonian. We now retrace the steps outlined in Sec. II A. The difference that arises in such a procedure is that the Hamiltonian can in principle be \( \tau \) dependent. However, this issue does not lead to any major complication. One finds that the evolution operator can be written as

\[
U_{fi}(\beta,0) = e^{\eta \phi_i^*}, \int D[\lambda^*, \lambda] \int D[\phi^*, \phi] \exp \left\{ -S[\phi^*, \phi] + \sum_\alpha \left[ \eta \lambda^*_\alpha (\phi_\alpha(0) - \phi_\alpha(0)) - (\phi^*_\alpha(\beta) - \phi^*_\alpha(\beta)) \lambda_\alpha \right] \right\} \tag{41}
\]

with \( \phi(\tau) = \eta \phi^*(0) \) and where the Euclidean action \( S[\phi^*, \phi] \) can now be explicitly time dependent.

The time-evolution operator \( U_{fi}(\beta,0) \) can be explicitly computed when the dynamics of the system is governed by a Gaussian action,

\[
S[\phi^*, \phi] = \frac{1}{2} \int dt dt' \sum_{\alpha, \alpha'} (\phi^*_{\alpha}(t), \phi_{\alpha}(t)) \times G_{\alpha\alpha'}^{-1}(t, t') \left( \phi_{\alpha'}(t'), \phi_{\alpha'}(t') \right). \tag{42}
\]

Denoting by \( G(\tau, \tau') \) the analytically-continued Green function, one then finds

\[
U_{fi}(\beta,0) = \mathcal{N} e^{\frac{1}{2} \sum_{\alpha, \alpha'} \Phi^*_{\alpha} L_{\tau\alpha'\beta}(\tau) \Phi_{\alpha'}}, \tag{43}
\]

\[ L(\beta) = I_0 \delta_{\alpha\alpha'} - I_0 [G_{\alpha\alpha'}(\tau = 0^+, \tau' = 0)]^{-1} I_0, \]

where \( \mathcal{N} = (\det G^{-1})^{-\eta/2}(\det G(0^+, 0))^{-\eta/2} \). The two component field \( \Phi_\alpha \) and the matrix \( I_0 \) are defined in Eq. (23). The temperature dependence of \( L(\beta) \) comes via the dependence of \( G \) on \( \beta \). The analytic continuation to real time finally gives

\[
U_{fi}(t,0) = \mathcal{N} e^{\frac{1}{2} \sum_{\alpha, \alpha'} \Phi^*_{\alpha} L_{\tau\alpha'\beta}(t) \Phi_{\alpha'}} \tag{44}
\]

where \( L(t) = L(\beta = it) \). We note that the field \( \Phi_\alpha \) mixes \( \phi_i \) and \( \phi_f \). In Sec. IV we shall see how one can express
the evolution operator in a more natural way in terms of \( \Phi_{f} = (\phi_{f}, \phi_{f}^{*})^{T} \) and \( \Phi^{f}_{i} = (\phi^{*}_{f}, \phi_{f}) \).

Next, we note a couple of features of our computation. First, we point out that the computation of \( U \) does not require the use of a Keldysh contour even though we are addressing the dynamics of a non-equilibrium system with time-dependent Hamiltonian. This can be understood as follows. A computation of \( U \) involves matrix elements of \( \hat{U} \) between two different coherent states. It does not require these two states to be identical; consequently, we do not need to construct a contour for implementing this restriction. Second, our method requires an analytic continuation to real time at the end of the calculation. This can be trivially done if \( G(0^{+}, 0) \) is analytically known; however, it is a significant challenge to carry this out numerically. This constitutes one of the main difficulties concerning the application of our formalism to non-equilibrium systems.

To demonstrate the application of the above-mentioned steps, in Appendix A2 we consider a single degree of freedom with a time-dependent Hamiltonian. This example demonstrates, albeit for a very simple model Hamiltonian, that the analytic continuation between real and imaginary times along with the formalism developed in Sec. II A allows us to compute matrix elements of the evolution operator between any two arbitrary coherent states in a driven system without using Keldysh formalism. Indeed for time evolution following sudden quenches an analogous formulation for 1D quantum systems has been presented in Ref. 32. Our formulation generalizes the work of Ref. 32 to arbitrary quench protocols and for quantum systems in arbitrary dimensions. However, we note that the formalism is only effective when the time-dependent Green function of the system is analytically known. Thus it can be most easily used in practice for piecewise constant drive protocols such as periodic square pulses or kicks, where the time ordering involved in the definition of \( \hat{U} \) does not preclude analytical treatment. We shall use this formalism to compute experimentally relevant quantities of periodically driven integrable many-body systems in Sec. IV.

III. BOSE-HUBBARD MODEL

In this section, we shall apply the results obtained in Sec. II B to compute several entanglement measures of the two-dimensional Bose-Hubbard model which can be described by a quadratic action within a strong coupling mean-field theory. In what follows, we shall obtain entanglement measures for this model both in its pristine form and in the presence of artificial gauge fields. The former topic shall be discussed in Sec. III A while the latter shall be addressed in Sec. III B.

### A. Entanglement for the pristine Bose-Hubbard model

The Hamiltonian of the Bose-Hubbard model is given by \( \hat{H} = \hat{H}_{0} + \hat{H}_{1} \),

\[
\hat{H}_{1} = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left( \alpha_{\mathbf{r}}^{\dagger} \alpha_{\mathbf{r}'} + h.c. \right),
\]

\[
\hat{H}_{0} = \sum_{\mathbf{r}} \left[ -\mu \alpha_{\mathbf{r}}^{\dagger} \alpha_{\mathbf{r}} + \frac{U}{2} \left( \alpha_{\mathbf{r}}^{\dagger} \alpha_{\mathbf{r}}^{\dagger} \alpha_{\mathbf{r}} \alpha_{\mathbf{r}} - 1 \right) \right],
\]

where \( \alpha_{\mathbf{r}}^{\dagger} \) denotes the boson annihilation operator at site \( \mathbf{r} \), \( t \) is the nearest-neighbor hopping amplitude for the bosons, \( U \) is their on-site interaction potential, and \( \mu \) is the chemical potential. For \( t/U \ll 1 \), the ground state of the model is a Mott insulating state of bosons. At \( t = t_{c} \), the bosons undergo a superfluid-insulator transition and the ground state for \( t > t_{c} \) is a correlated superfluid. The precise value of \( t_{c}/U \) is well-known from quantum Monte-Carlo (QMC) studies; it turns out that \( t_{c}/U \ll 1 \) in any dimension. Thus the study of the Mott and the superfluid phases near the transition point requires addressing the properties of the model in the strongly correlated regime where \( U \gg t \).

An analytic treatment of the Bose-Hubbard model usually involves standard mean-field theory where the kinetic energy term is treated within mean-field approximation, projection operator approach, and slave boson technique. All of these methods use the local nature of the interaction and involves treatment of the kinetic energy term using different approximations. However, none of them allows for a direct access to the momentum-space properties near the superfluid-insulator critical point, either in the Mott or the superfluid phases. Such information is particularly relevant for computation of the momentum-distribution function which can be directly measured experimentally. In contrast, the strong coupling expansion technique, which was developed in Ref. 16 and applied to Bose-Hubbard model in the presence of artificial gauge fields in Refs. 27,28, provides a direct access to the momentum-space Green function in the strong coupling regime. The momentum distribution function computed using this technique provides a near-exact match with experimental measurements carried out in the superfluid and Mott phases near the critical point. In what follows, we shall use this technique to obtain the action of the bosons and use it to obtain several entanglement measures both in the Mott and the SF phases near the critical point.
where \( \epsilon_k = -2t(\cos k_x + \cos k_y) \) is the boson kinetic energy, \( g_0 \) is the coefficient of the quartic interaction term, and \( G_0 \) is the local single-particle zero-temperature Green function in the Mott limit (\( t=0 \)) given by

\[
G_0(\omega_n) = \frac{-n_0}{i\omega_n + E_h(n_0)} + \frac{n_0+1}{i\omega_n - E_p(n_0)}.
\]

(47)

Here \( n_0 = n_0(\mu/U) \) is the ground state boson occupation number in the Mott limit and \( E_p(n_0) = -\mu + Un_0 \) and \( E_h = -\mu - Un_0 - 1 \) denotes the energy cost of adding (removing) a boson to (from) the ground state at \( t=0 \). In what follows, to comply with the notations of Ref. 16, we define the Green function as \( G_0(\omega_n) = -\langle \psi(\omega_n)\psi\rangle \) for the local propagator. Following Ref. 16 we approximate \( S_1 \) within a mean-field theory as

\[
S_{1MF} = \frac{g_0\Delta_0^2}{2} \int_0^\beta d\tau \int d^d r [\phi(r,\tau)^2 + \phi^*(r,\tau)^2 + 4|\phi(r,\tau)|^2]
\]

(48)

where \( \Delta_0^2 = \langle \phi(r,\tau)^2 \rangle = |G_0^{-1}(0) + z\ell|/g_0 \) in the superfluid phase and vanishes in the Mott phase, and \( z = 2d = 4 \) is the coordination number of the square lattice. We note that the mean-field theory used here differs from its standard weak-coupling counterpart since \( S_0 \) captures the effect of strong correlation through \( G_0 \).

In the Mott phase, \( S_{1MF} = 0 \) and the action of the bosons is given by \( S_0 [Eq. (46)] \). The Green function of the bosons can be read off from (46) and is given by

\[
G(\omega_n, k) = \frac{z_k}{i\omega_n - E^+_k} + \frac{1-z_k}{i\omega_n - E^-_k},
\]

\[
E^{\pm}_k = -\mu_0 + \frac{1}{2} \left( \epsilon_k \pm \sqrt{\epsilon_k^2 + U^2 + 4U\epsilon_k(n_0 + 1/2)} \right),
\]

\[
z_k = [E^+_k + \mu_0 + U(n_0 + 1/2)]/[E^+_k - E^-_k],
\]

(49)

where \( \mu_0 = \mu - U(n_0 - 1/2) \). A straightforward calculation thus yields \( G(\tau = 0^+, k) = z_k \). Using the results of Sec. II A we therefore find

\[
\rho_{j_1j_2} = K\sum_k \phi_{j_k}(1-z_k)\phi_{j^*k},
\]

(50)

where \( K = [\det G(0^+)]^{-1} = \prod_k z_k^{-1}, \) and

\[
\langle \phi_k \phi^*_k \rangle = z_k = \lambda^{Mott}_k.
\]

(51)

From Eq. (32), one can now construct the covariance matrix \( \chi_{jj'} \). A numerical diagonalization of this matrix yields the eigenvalues \( \nu_j^{Mott} \). The Von-Neumann entropy \( S_0 \) and the \( n^{th} \) Rényi entropy can then be computed using Eqs. (33) and (36) respectively. One can also compute the entanglement negativity by numerically diagonalizing the partial transposed density matrix (38) and then using (39). We emphasize that all the measures of entanglement, within the strong-coupling mean-field theory, can be directly linked to \( z_k \) and thus in turn to the Matsubara Green functions of the bosons. In the superfluid phase, the boson action is approximated by the quadratic action \( S = S_0 + S_{1MF} \). The Green function reads\textsuperscript{16}

\[
G^s(\omega_n, k) = \frac{[i\omega_n + \mu_0 + U(n_0 + 1/2)](i\omega_n - z_k)(i\omega_n - z_k^*)}{(\omega_n^2 + E_k^+(\omega_n^2 + E_k^-))},
\]

\[
F^s(\omega_n, k) = g_0\Delta_0^2 \omega_n^2 + [\mu_0 + U(n_0 + 1/2)]^2
\]

\[
eq \frac{(\omega_n^2 + E_k^+(\omega_n^2 + E_k^-)^2)}{(\omega_n^2 + E_k^+(\omega_n^2 + E_k^-)^2)},
\]

(52)

where \( G^s \) and \( F^s \) denotes the diagonal and off-diagonal parts of \( C_{jj'}^s = \langle \phi_j \phi^*_j \rangle = \langle G(0^-) \rangle_{jj'} \). the boson Green function \( G_{1SF}^s \) in the superfluid phase and \( E_k^s \) is chosen to be a positive quantity for all \( k \). The Green function \( G_{1SF}^s \), computed within the strong coupling mean-field analysis, has four poles at \( \pm E_k^\pm \) corresponding to two different types of mode: a gapless sound mode \( \omega = \pm E_k^\pm \) with a linear dispersion at small \( k \) and a gapped Higgs mode.
In the absence of superfluidity, \( \Delta_0 = 0 \) and \( F_k = 0 \), we recover the expression obtained in the Mott phase:

\[
\lambda_k = C_k = \langle \phi_k \phi_k^* \rangle.
\]

We are now in a position to construct the covariance matrix (32), find its eigenvalues \( \nu_c^{SF} \) and deduce the Von-Neuman and Renyi entropies as well as the entanglement negativity in the SF phase (see Sec. II B).

A plot of the Von-Neuman entropy of the pristine bosons, \( S_b \), obtained from the method detailed above is shown in Figs. 1 and 2. We have chosen a subsystem in the form of a cylindrical region with circumference \( R = L_y/a = 30 \); this ensures one has periodic boundary condition along \( y \). The length of the cylinder \( L_x = L a \) is chosen to be \( L = 30 \) (\( a \) is the lattice spacing), where the system size corresponds to \( L_x = L_y = 60 a \). The chemical potential \( \mu / U \) for all the plots is chosen so that we follow a line of constant boson density as we change \( t / U \). We find that our results reproduce the expected behavior of \( S_b \) as a function of \( t / U \). \( S_b \) peaks at the critical point and obeys an area law,

\[
S_b = \left\{ \begin{array}{ll}
A \left( \frac{L_x}{a} \right) & \text{(Mott phase)} \\
A' \left( \frac{L_x}{a} \right) + B' \ln \left( \frac{L_x}{a} \right) & \text{(superfluid phase)}
\end{array} \right.,
\]

in accordance with standard expectation.

Our work here qualitatively agrees with that of Refs. 29,30 with one important difference. This difference can be understood from Fig. 3 where the subleading part of \( S_b \) in the superfluid phase is plotted as function of \( t / U \). It is well-known that in a superfluid whose excitations are given by Goldstone modes, \( S_b = A' (L_x/a) + B' \ln (L_x/a) + \ldots \) where the ellipsis denotes \( O(1/L) \) terms as shown in the right panel of Fig. 2. Moreover, within the gapless Bogoliubov approximation, \( b' \) is expected to be a universal constant independent of \( t / U \). However, in our case, for correlated superfluid near the critical point which has both gapped and gapless modes, \( B' \) turns out to be a monotonic function of \( t / U \); its dependence on \( t / U \) is shown in Fig. 3. We note that the deviation of \( B' \) from its universal value is a signature of the presence of gapped modes in the system which arise due to strong correlations.
bosonic systems in the presence of artificial Abelian gauge fields. These fields are typically generated using additional Raman lasers coupled to the bosons and can be either Abelian or non-Abelian. In this subsection, we shall discuss the Bose-Hubbard Hamiltonian for the bosons in a 2D optical lattice the presence of an Abelian gauge field; the case of non-Abelian gauge fields will be discussed in Sec. III C. It is well-known that such a Hamiltonian is given by

$$\hat{H} = \hat{H}_0 - \sum_{\langle rr' \rangle} \left( b_r^\dagger b_{r'} t e^{i e^*} J_t \, d^r \hat{A} / \hbar c + \text{h.c.} \right). \quad (56) $$

Here $\hat{A}_r = B_0(0, x_i)$ (where $r_i = (x_i, y_i)$) is the synthetic vector potential and $e^* B_0$ is the effective magnetic field whose strength can be tuned by varying the intensity of the Raman lasers. In what follows, we shall treat the cases for which the effective magnetic field corresponds to flux of $\Phi_0/q$ through the lattice: $e^* B_0 a^2 / (\hbar c) = 2\pi p/q$, where $\Phi_0 = \hbar c / e^*$ is the flux quanta and $c$ denotes speed of light. We shall also choose $p/q$ to be a rational fraction. We note that the creation of synthetic gauge fields amounts to creation of the vector potential $\hat{A}_r$ using light-atom coupling; this is in contrast to standard electromagnetic fields where the fields are the physical entities and choosing vector potentials to describe them involves freedom of gauge choice.

The strong coupling expansion developed in Ref. 16 can be used to obtain the boson Green function in their Mott phase in the presence of such Abelian gauge fields. As shown in Ref. 27, the action of such a system can be written in terms of a $q$ component bosonic field $\Psi(\omega_n, k) = (\psi_0(\omega_n, k), \psi_1(\omega_n, k), \ldots, \psi_{q-1}(\omega_n, k))^T \equiv \psi(k)$ where $\psi_0(\omega_n, k) = \psi(\omega_n, k + 2\pi a / q x)$. In the Mott phase, this action is given by

$$S_0 = \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \Psi^\dagger(k) \left[ -G_0^{-1}(\omega_n) I + J_q(k) \right] \Psi(k). \quad (57) $$

In this notation $k = (k_x, k_y)$ lies within the magnetic Brillouin zone defined by $-\pi / a \leq k_y \leq \pi / a$ and $-\pi / (qa) \leq k_x \leq \pi / (qa)$, $J_q(k)$ is a $q \times q$ tridiagonal matrix whose upper off-diagonal (diagonal) components are $-te^{-ik_y a}(-t\cos(k_x a + 2\pi a / q))$ for $\alpha = 0, 1, \ldots, q - 1$, and $I$ is the $q \times q$ unit matrix. The diagonalization of $J_q(k)$ leads to $q$ bands with energy dispersion $\epsilon^{(2)}_{\alpha}(k)$. For example, in the simplest case where $q = 2$ (corresponding to half flux quanta through each lattice plaquette) there are two bands with energy dispersion $\epsilon^{(2)}_{0}(k) = +(-)t \sqrt{\cos^2 k_x + \cos^2 k_y}$. These bands lead to $q$ energy minima within the magnetic Brillouin zone of the bosons. The Green function of the bosons in the Mott phase can be written as $G(\omega_n, k) = [G_0^{-1}(\omega_n) I - J_q(k)]^{-1}$. Noting...
has bosons in the presence of the Abelian gauge fields, one analysis of pristine bosons in Sec. III A, we find that for restricted to the magnetic Brillouin zone. Following earlier where $\Psi$ and (17), we find
diagonal elements are given by $Z$ where

$$
G(\omega_n, k) = U_q(k)^\dagger G_D(\omega_n, k) U_q(k),
$$

(58)

$$
G_D(\omega_n, k) = \sum_{\alpha=0}^{q-1} \left( \frac{1 - z_{\alpha}(k)}{i\omega_n - E_{\alpha}^-(k)} + \frac{z_{\alpha}(k)}{i\omega_n - E_{\alpha}^+(k)} \right),
$$

where $U_q(k)$ is the matrix which diagonalizes $J_q(k)$ and $z_{\alpha}(k)$ and $E_{\alpha}^\pm(k)$ are obtained from Eq. (49) by substituting $\epsilon_k \rightarrow \epsilon_{\alpha}^q(k)$. Using the fact that $U_q(k)$ is independent of $\omega_n$, one finds

$$
G(\tau = 0^+, k) = U_q(k)^\dagger Z_q(k) U_q(k),
$$

(59)

where $Z_q$ is a $q \times q$ dimensional diagonal matrix whose diagonal elements are given by $z_{\alpha}^q(k)$. Using Eqs. (59) and (17), we find

$$
\rho_f(1)_{fi} = e \sum_{\alpha=0,1,..q-1} \psi_{\alpha}(k) |1 - 1/2\psi_{\alpha}(k)| \psi_{\alpha}(k),
$$

(60)

where $\Psi(k) = U_q(k) \Psi(k)$ and the sum over $k$ is restricted to the magnetic Brillouin zone. Following earlier analysis of pristine bosons in Sec. III A, we find that for bosons in the presence of the Abelian gauge fields, one has

$$
\langle \psi_{\alpha}(k) \psi_{\alpha}^*(k) \rangle = z_{\alpha}(k) = \lambda_{\alpha}^{\text{Mott}}(k).
$$

(61)

The covariance matrix can then be formed by using Eq. (32) and all measures of entanglement may be computed from its eigenvalues using Eqs. (33), (35), and (39).

The plot of the Von-Neumann and second Renyi entropies in the presence of Abelian gauge field, $S^A_1$ and $S^A_2$, is shown in Fig. 6 as a function of $t/U$ for several values of $q$ and $L_z/a = 30$. We find that these entropies show qualitatively similar behavior as a function of $t/U$; in particular they display peaks around the critical point. The entanglement negativity, shown in Fig. 7, exhibits a similar behavior. It is to be noted, however, that $N^A_q$ shows a slight stronger dependence on $t/U$ in the Mott phase. In all of the plots, we have chosen $\mu$ to correspond to the tip of the Mott lobe for all $q$.

The increase of all measures of entanglement with increasing $q$ for a given $t/U$ in the Mott phase can be qualitatively understood as follows. The presence of the magnetic field with a flux $2\pi/q$ per plaquette leads to $q$ equal-amplitude peaks of the momentum distribution $n_k$ at $k = Q_{q,n} = (0, 2\pi n/q)$ with $n = 0, 1, ..q-1$. This indicates that the real space correlation of bosons $n_{\tau} = \langle \tilde{b}_\tau^\dagger b_\tau \rangle$, for any given $\tau = 0$, receives its main contribution from $q$ wave-vectors ($Q_{q,n}$). Thus with increasing $q$, $n_{\tau}$ receives contribution from smaller non-zero momenta. This leads to longer-ranged boson correlations in real space with increasing $q$. Therefore one expects to have a strongly entangled boson system with increasing $q$ leading to larger value of $S_b$, $S_2$ or $N_b$ for larger $q$. This expectation is corroborated by the results shown in Figs. 6 and 7.

C. Non-Abelian gauge fields

In this section, we consider the Mott phase of bosons in the presence of a non-Abelian gauge field. There are concrete proposals of realization for $^{87}$Rb atomic gases by inducing an effective spin-orbit coupling between two hyperfine states of the atoms. In the presence of such Raman lasers can be written

![Fig. 6: Plot of the Von-Neumann entropy $S^A_1$ (top panel) and the second Renyi entropy $S^A_2$ (bottom panel) in the presence of an Abelian gauge field as a function of $t/\tau_c(q)$ for $L \equiv L_z/a = 30$. The case $q = 1$ corresponds to zero effective magnetic field and reproduces the corresponding results for pristine bosons. All other parameters are chosen as in Fig. 10. See text for details.](image)

![Fig. 7: Plot of the entanglement negativity $N^A_q$ as a function of $t/\tau_c(q)$. All other parameters are same as in Fig. 6. See text for details.](image)
\[
\hat{H}_0' = \sum_{\mathbf{r}a} \left[ -\mu \hat{n}_{\mathbf{r}a} + U \hat{n}_{\mathbf{r}a} (\hat{n}_{\mathbf{r}a} - 1) / 2 \right] + \lambda U \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}1} \hat{n}_{\mathbf{r}2} - \sum_{\langle \mathbf{r} \mathbf{r}' \rangle a} t_{\mathbf{a}} \hat{b}_{\mathbf{r}a} \hat{b}_{\mathbf{r}'a},
\]

where \( \hat{b}_{\mathbf{r}a} \) denotes the bosons annihilation operator on the site with coordinates \( \mathbf{r} = (x, y) \) on a 2D square lattice, \( a = 1, 2 \) correspond to the index of the hyperfine states, \( \hat{n}_{\mathbf{r}a} = \hat{b}_{\mathbf{r}a}^\dagger \hat{b}_{\mathbf{r}a} \) is the boson number operator, \( U(\lambda U) \) is the interaction strength between the bosons in same (different) hyperfine states, and \( t_{\mathbf{a}} \) (with \( t_1 = t \) and \( t_2 = \eta t \)) denotes the nearest neighbor hopping amplitudes. In the presence of the Raman lasers inducing a Rashba spin-orbit coupling, the additional terms in the boson Hamiltonian are given, in terms of a two component boson field \( \hat{\Psi}_{\mathbf{r}} = (\hat{b}_{\mathbf{r}1}, \hat{b}_{\mathbf{r}2})^T \), by

\[
\hat{\mathbf{H}}_1' = i \gamma \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} \hat{\Psi}_{\mathbf{r}}^\dagger \hat{\sigma}_z \cdot (\hat{\sigma} \times \mathbf{d}_{\mathbf{r} \mathbf{r}'}) \hat{\Psi}_{\mathbf{r}'} - \sum_{\mathbf{r}} \Omega \hat{\Psi}_{\mathbf{r}}^\dagger \hat{\sigma}_y \hat{\Psi}_{\mathbf{r}},
\]

where \( \hat{\sigma}_z \), \( \hat{\sigma}_y \), and \( \hat{\sigma}_x \) are Pauli matrices.

The first term represents the lattice analogue of the Rashba spin-orbit coupling generated by the Raman lasers, \( \mathbf{d}_{\mathbf{r} \mathbf{r}'} \) is a unit vector along the \( x-y \) plane between the neighboring sites \( \mathbf{r} \) and \( \mathbf{r}' \), \( \Omega \) is the hyperfine state dependent shift in the chemical potential of the bosons. Here we have neglected additional on-site terms \( \sim \sigma_y \) arising due to boson-laser interaction; these terms can always be made small by adjusting the detuning of the lasers.

Following exactly similar analysis as that for the pris-tine bosons, it is possible to develop a strong-coupling expansion for the bosons in the presence of non-Abelian gauge fields. Such an analysis was carried out in Ref. 28. The on-site single-particle Green function, computed from the on-site terms in \( \hat{H}_0 \) for \( n_0 = 1 \), is given by

\[
G_1(\omega_n) = \frac{-1}{i\omega_n + E_1} + \frac{2}{i\omega_n + E_1 - U},
\]

\[
G_2(\omega_n) = \frac{1}{i\omega_n - E_2},
\]

\[
E_1 = \mu + \Omega, \quad E_2 = \mu - \Omega - \lambda U.
\]

Using this one can write the action of the bosons in the Mott phase as

\[
S_{\text{Mott}}^{\text{NA}} = -\sum_{a,b=1,2} \frac{1}{\omega_n} \sum_{\mathbf{r}} \sum_{\omega_n} \int d^2k (2\pi)^2 \Phi^\dagger(\omega_n, \mathbf{k})
\]

\[
\times \Phi^{-1}(\omega_n, \mathbf{k}) \hat{g}(\omega_n, \mathbf{k}) G_{\text{NA}}^{-1}(\omega_n, \mathbf{k}),
\]

where \( \hat{g}(\omega_n, \mathbf{k}) \) is a unit vector along the \( z \) plane between \( \mathbf{k} \) and \( \mathbf{k}' \). The corresponding boson Green function can be written as

\[
G_{\text{NA}}(\omega_n, \mathbf{k}) = \frac{i\omega_n + U + E_1}{\prod_{i=0,2} (\omega_n - \Lambda_{ik})} \left( \begin{array}{c}
-G_{1}^{-1}(\omega_n) + \epsilon_k \\
\Delta_k
\end{array} \right)
\]

\[
= \sum_{j=1,2} \frac{\Lambda_{jk} + U + E_1}{(\Lambda_{jk} - \Lambda_{j+1k})(\Lambda_{jk} - \Lambda_{j+2k})} \left( \begin{array}{c}
-G_{2}^{-1}(\Lambda_{jk}) + \epsilon_k \\
\Delta_k
\end{array} \right),
\]

where \( \Lambda_{ik} \), for \( i = 0, 1, 2 \), are solutions of the cubic equation \((G_1^{-1} - \epsilon_k)(G_2^{-1} - \epsilon_k) - |\Delta_k|^2 = 0 \). We have checked numerically that in the Mott phase, this equation has three real roots out of which two are positive. In the Mott limit, these two positive roots correspond to energies \( U_0 - E_1 \) and \( E_2 \) as can be read off from Eq. (64). Denoting these positive roots by \( \Lambda_1 \) and \( \Lambda_2 \), one obtains, after performing a Matsubara sum over \( \omega_n \),

\[
G_{\text{NA}}(0^+, \mathbf{k}) = \sum_{j=1,2} \frac{\Lambda_{jk} + U + E_1}{(\Lambda_{jk} - \Lambda_{j+1k})(\Lambda_{jk} - \Lambda_{j+2k})} \left( \begin{array}{c}
-G_{2}^{-1}(\Lambda_{jk}) + \epsilon_k \\
\Delta_k
\end{array} \right),
\]

where \( j \) is a cyclic variable with \( j \in Z \text{Mod} 3 \) and \( S_{\text{Mott}}^{\text{T}} = (S_{\text{Mott}}^{\text{T}})^* \). Using this, and following the method outlined in Sec. II A, we find that for any two arbitrary boson coherent states \( |\Phi_f \rangle \) and \( |\Phi_i \rangle \),

\[
\rho_{fi}^{\text{NA}} = e^{\sum_k \Phi_f^\dagger(k)(1 - |G(0^+, \mathbf{k})|^{-1})} \Phi_i(k),
\]

where \( \phi_f^\dagger(k) = [\phi_{f1}^\dagger(k), \phi_{f2}^\dagger(k)] \) and \( \Phi_i(k) \) are two com-
component Bose fields. Using this density matrix one obtains the Boson correlation matrix in the presence of a non-Abelian gauge field as
\[ \langle \phi^*_\sigma (k) \phi_{\sigma'}(k) \rangle = S^{\sigma\sigma'}_k. \] (69)

Using this correlation matrix, one can now use the procedure of Sec. II B to obtain the covariance matrices of the bosons and hence different measures of entanglement entropies. The eigenvalues of the correlation matrix \( G_{\sigma\sigma}(0^+, k) \), as discussed in Sec. II B, yield the frequency \( \lambda_k \) of the effective oscillator Hamiltonian. A straightforward but cumbersome calculation shows that
\[ \lambda_k^{(1)} = -1 + \sum_\sigma S^{\sigma\sigma}_k, \quad \lambda_k^{(2)} = 1. \] (70)

Note that only one of these eigenvalues depends on system parameters such as \( t/U, \gamma/U \). This property can be easily verified in the Mott limit and holds for all values of \( t/U \) and \( \gamma/U \) in the Mott phase. Using these eigenvalues, one can compute the covariance matrix using Eq. (32) and numerically compute the different entanglement measures using Eqs. (33), (36), and (39).

The result of such computation is shown in Figs. 8 and 9. In Fig. 8, we plot the Von-Neumann \( S_{s}^{\text{NA}} \) and the Renyi \( S_{2}^{\text{NA}} \) entropies as a function of both \( \gamma/U \) and \( t/U \). We find that the entanglement peaks as one approaches the critical point either by increasing \( t \) or \( \gamma \). A similar feature is seen for entanglement negativity \( N_{b}^{\text{NA}} \). We note that our computation indicates that for correlated boson systems the entanglement negativity has a qualitatively similar characteristic to Renyi and Von-Neumann entropies.

IV. NON-EQUILIBRIUM DYNAMICS FOR INTEGRABLE MODELS

In this section, we demonstrate the application of our method to a class of integrable fermionic quantum models driven out of equilibrium by a periodic square pulse array. In Sec. IV A, we define the model and obtain its Floquet Hamiltonian under periodic drive. This is followed by the computation of the return probability in Sec. IV B and counting statistics of number operator in Sec. IV C.

A. Floquet Hamiltonian

The Hamiltonian of free fermion models obeying Dirac-like equation in \( d \)-dimensions can be written as
\[ \hat{H} = \sum_k \psi_k^\dagger \hat{H}_k \psi_k, \]
\[ \hat{H}_k = (g(t) - b_k)\sigma_3 + \Delta_k \sigma_1 \] (71)
where the sum over \( k \) extends over half the Brillouin zone (defined, e.g., by \( k_x \geq 0 \)). \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) denotes Pauli-matrices in particle-hole space, and \( g(t), b_k \) and \( \Delta_k \) are parameters of the Hamiltonian whose precise forms depend on the system that \( \hat{H} \) represents. We note that Eq. (71) provides a representation of the low-energy theory for Dirac quasiparticles in graphene and on surface of topological insulators; moreover, it gives an exact fermionic representation of spin models such as Ising model in \( d = 1 \) and Kitaev model in \( d = 2 \). The two-component field \( \psi_k \) is either of the form \( \psi_k = (\tilde{c}_k, \tilde{\tilde{c}}_{-k})^T \) (for the case of graphene or 11 quasiparticles and 2D Kitaev model) or of the form \( \psi_k = (\tilde{c}_k, \tilde{\tilde{c}}_{-k})^T \) for superconductors and Ising model in a transverse field. In the second case, we perform a particle-hole transformation \( \tilde{c}_{-k} \leftrightarrow \tilde{\tilde{c}}_{-k} \) thus allowing us to return to the first case. In what follows we study the properties of this model under a periodic drive protocol,
\[ g(t) = \begin{cases} g_1 & \text{if } 0 \leq t \leq T/2, \\ g_2 & \text{if } T/2 < t \leq T, \end{cases} \] (72)
where \( T = 2\pi/\omega_D \) is the drive period and \( \omega_D \) is the drive frequency. Our aim is to show that our method enables one to semi-analytically compute counting statistics and return probability of such a system in terms of Floquet eigenvalues and eigenvectors.

We begin by computing the Floquet Hamiltonian of the model by using the formalism developed in Sec. II C.
To this end, we first note that the evolution operator of the system for the protocol given by Eq. (72), at \( t = T \), can be written as

\[
\hat{U}(T, 0) = \hat{U}(T, T/2)\hat{U}(T/2, 0) = e^{-i\hat{H}[g_2]T/2b}e^{-i\hat{H}[g_1]T/2b},
\]

where \( \hat{H}[g_{1(2)}] \) is given by Eq. (71) with \( g(t) = g_{1(2)} \). To compute the Floquet Hamiltonian we first obtain the matrix elements of \( \hat{G}_k \) corresponding to \( \hat{H}_k \). By Eq. (78) with \( \varphi_k \), where \( \varphi_k = (\varphi_{1k}, \varphi_{2k})^T \) is a two-component Grassmann field.

To obtain this matrix element, we must compute the Green function \( g(0^+) \) corresponding to \( \hat{H} \). To this end, we first note that \( \hat{H}_k \) can be brought into a diagonal form via the transformation \( \hat{H}_k^D = \Lambda_k \hat{H}_k \Lambda_k^+ \) where

\[
\Lambda_k = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix},
\]

\[
E_k(g_1) = \sqrt{(g_1 - b_k)^2 + \Delta_k^2},
\]

\[
u_k[v_k] = \frac{1}{\sqrt{2}} \sqrt{1 + [-g_1 - b_k] / E_k}.
\]

In the diagonal basis, it is easy to see that the Green function \( G_D(k; \omega_\nu) = -(i\omega_\nu - \tau_3 E_k)^{-1} \) which leads to

\[
G_D(k; \tau = 0^+) = \frac{1}{\beta} \sum_{\omega_\nu} G_D(k; \omega_\nu) e^{-i\omega_\nu \tau} = (I + \exp[-\sigma_3 E_k(g_1)])^{-1}.
\]

Rotating back to the original basis, we find

\[
[U_{12}(\beta, 0) = \langle \phi_1 | e^{-\hat{H}[g_1]0^+} | \phi_2 \rangle = e^{-\sum_k \Phi_{1k}^\dagger \Phi_{1k}(g_1)T/2b - \sum_k \Phi_{2k}^\dagger \Phi_{2k}(g_1)T/2b}, \]

\[
\Phi_{1k}(g_1, T) = \left( \begin{array}{c} 1 + u_k^2 e^{-\beta E_k(g_1)} + v_k^2 e^{\beta E_k(g_1)} \\ 2u_k v_k e^{-\beta E_k(g_1)} + u_k^2 e^{\beta E_k(g_1)} \end{array} \right) = \left( \begin{array}{c} 1 + u_k^2 e^{-\beta E_k(g_1)} + v_k^2 e^{\beta E_k(g_1)} \\ 2u_k v_k e^{-\beta E_k(g_1)} + u_k^2 e^{\beta E_k(g_1)} \end{array} \right).
\]

Substituting Eq. (77) in Eq. (74) and rotating back to the last line, the product matrix \( \mathbb{L}_k(g_2, T) \) provides an analytic expression for the evolution operator in real time we find

\[
\mathbb{L}_k(g_2, T) \mathbb{L}_k(g_1, T) = \left( \begin{array}{cc} -\cos \varphi_k^{(1)} + i n_{1k}^{(1)} \sin \varphi_k^{(1)} & i n_{1k}^{(1)} \sin \varphi_k^{(1)} \\ i n_{3k}^{(1)} \sin \varphi_k^{(1)} & -\cos \varphi_k^{(1)} - i n_{1k}^{(1)} \sin \varphi_k^{(1)} \end{array} \right),
\]

and plays a central role in our analysis. It is given by

\[
\mathbb{L}_k(g_2, T) \mathbb{L}_k(g_1, T) = \left( \begin{array}{cc} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{array} \right),
\]

where

\[
\beta_k = -i\frac{n_{3k}^{(1)} \sin \varphi_k^{(1)} (\cos \varphi_k^{(2)} - i n_{1k}^{(1)} \sin \varphi_k^{(2)})}{\sin \varphi_k^{(2)} + n_{3k}^{(1)} \sin \varphi_k^{(2)} (\cos \varphi_k^{(1)} + i n_{1k}^{(1)} \sin \varphi_k^{(1)})},
\]

\[
\alpha_k = \cos \varphi_k^{(1)} (\cos \varphi_k^{(2)} - n_{1k}^{(1)} (n_{3k}^{(1)} \sin \varphi_k^{(2)}(\cos \varphi_k^{(1)} + i n_{1k}^{(1)} \sin \varphi_k^{(1)})) - i(n_{1k}^{(1)} \sin \varphi_k^{(1)} \cos \varphi_k^{(2)} + n_{1k}^{(1)} \sin \varphi_k^{(1)} \cos \varphi_k^{(2)}),
\]

(81)
Its eigenvalue and eigenvectors can be easily found to be
\[ \nu_k^a = e^{ia\epsilon_k^F T}, \]
\[ \frac{u_k^a}{v_k^a} = i \frac{\beta_k}{\Im(\alpha_k) - a \sin[\epsilon_k^F T]}, \]
where \( a = \pm \) and the Floquet eigenvalues \( \epsilon_k^F \) are given by
\[ \epsilon_k^F T = \arccos[\cos \varphi_k^{(1)} \cos \varphi_k^{(2)}]
- \delta_{n_k^1, n_{-k}^2} \sin \varphi_k^{(1)} \sin \varphi_k^{(2)}]. \]

These results agree with those derived in Ref. 26 from standard Hamiltonian methods. We shall use these results in Secs. IV B and IV C to compute the return probability and counting statistics of the number operator.

## B. Return Probability

In this section, we consider the computation of the return probability using our method. This quantity, for a quench protocol, is identical to the Loschmidt echo \( \mathcal{G}(t) = \langle \hat{U}(t, 0) \rangle \) computed for transverse field Ising model in Ref. 42 and is related to the work statistics of the driven quantum system. Here, we consider the fermion system introduced in the previous section and whose dynamics is governed by the Hamiltonian (71). We assume that the initial state is \( |n\rangle \equiv \{|n_k, n_{-k}\rangle\} \) where \( n_k \) and \( n_{-k} \) are the occupation numbers of the single-particle states with momentum \( k \) and \(-k\). Note that in the case of superconductor, where the two-component field \( \tilde{\psi}_k = (\tilde{\epsilon}_k, \tilde{c}_k^\dagger)^T \) becomes \( \psi_k = (\tilde{\epsilon}_k, \tilde{c}_k^\dagger)^T \) after the particle-hole transformation \( \tilde{c}_{-k} \rightarrow \tilde{c}_k^\dagger \), the occupation number \( n_{-k} \) actually refers to the number of holes: \( n_{-k} = 0 \) (1) if the electron state with momentum \( -k \) is occupied (empty).

The probability amplitude that the system returns to its initial state after a time \( t \) is given by
\[
\mathcal{P}_n(t) = \langle n | \hat{U}(t, 0) | n \rangle = \sum_{k_1, k_2} \phi_{k_1}^n \phi_{k_2}^{n_{-k}} \phi_{k_1}^{\dagger} \phi_{k_2}^{\dagger},
\]
where we use the notation \( \phi_k^1 = \phi_k \), \( \phi_k^2 = \phi_{-k} \), \( n_k^1 = n_k \) and \( n_k^2 = n_{-k} \). The functional integral in (86) is Gaussian and can be performed using Wick’s theorem. Let us write the “action” as
\[
\mathcal{I} = \sum_k \left( \Phi_k^1 \Phi_k^2 \right) D_k^{-1}(t) \left( \Phi_k^1 \Phi_k^2 \right)^\dagger,
\]
where the matrix
\[
D_k^{-1}(t) = \begin{pmatrix}
1 & \mathcal{M}_{k,11}(t) & \mathcal{M}_{k,12}(t) \\
0 & 1 & \mathcal{M}_{k,21}(t) \\
0 & 0 & 1
\end{pmatrix}
\]
satisfies \( \det D_k^{-1}(t) = 1 \) and \( D_k(t) \) is simply deduced from \( D_k^{-1}(t) \) by changing \( M_{k,i,j}(t) \) into \( -M_{k,i,j}(t) \). We then obtain
\[
\mathcal{P}_n(t) = \prod_k \left( \Phi_k^1 \Phi_k^2 \right) \mathcal{M}_{k,11}(t) \mathcal{M}_{k,12}(t) = \prod_k \mathcal{M}_{k,12}(t),
\]
where \( \langle \cdot \rangle \) denotes an average with the action (87).

Thus the return probability after a period \( T \) of the drive protocol (72) is determined by the matrix \( \mathcal{M}_k(T) = -\mathcal{L}_k(g_2, T) \mathcal{L}_k(g_1, T) \) defined by (80). For the state defined by \( n_k = a \) and \( n_{-k} = b \) \((a, b = 0, 1)\), we find that the probability \( \mathcal{P}_{ab}(T) \) takes the simple expression
\[
\mathcal{P}_{00}(T) = 1, \quad \mathcal{P}_{11}(T) = \prod_k \mathcal{M}_{k,11}(T) = 1,
\]
\[
\mathcal{P}_{10}(T) = \prod_k \alpha_k, \quad \mathcal{P}_{01}(T) = \prod_k \alpha_k^*.
\]

We note that the states \( \{|0_k, 0_{-k}\rangle\} \) and \( \{|1_k, 1_{-k}\rangle\} \), which correspond to the empty and maximum density states, respectively, do not evolve in time. In the case of a superconductor, the state \( |0_k, 0_{-k}\rangle \) has all electron states with momentum \( k \) empty while all states with momentum \( -k \) are occupied (since \( 0_{-k} \) denotes the number of holes with momentum \(-k\)). Being the state with minimal total momentum, and the total momentum being conserved, this state does not evolve in time (a similar analysis applies to the state of maximum total momentum, \(|\{1_k, 1_{-k}\}\rangle\)).

Thus we find that the return probability of a fermionic system with Gaussian action driven by an arbitrary protocol can be obtained from its thermal Green function \( G(0^+) \) via its analytic continuation to real time. The probability amplitude depends on \( \alpha_k \) and \( \alpha_k^* \); thus our analysis ties the return probability to the matrix elements of the unitary evolution operator.

In Fig. 10 we show
\[
\ln |\mathcal{P}_{01}(T)| = L \int \frac{d^d k}{(2\pi)^d} \ln |\alpha_k|
\]
Ising model as a function of the drive period

FIG. 10: Plot of $\ln|P_{01}(T)| / L$ for the $d = 1$ transverse field Ising model as a function of the drive period $T$ for several transverse fields $g_1$ and $g_2$. The plot clearly shows the difference in behavior at large $T$ between protocols which do and do not cross the critical point. See text for details.

for the specific case of the one-dimensional transverse field Ising model for which $b_k = \cos k$ and $\Delta_k = \sin k$. We assume the starting state to correspond to $g = g_1 = 10$ which is the ground state deep inside the ferromagnetic phase. The dynamics of the model is studied using the square pulse protocol defined in Eq. (72) with several $g_2$. We find that the return probability computed using this protocol falls into two distinct categories. For protocol where the dynamics never crosses the critical point ($g_2 \geq g_c = 1$), the return probability has a higher value for slow enough protocols. This is shown in Fig. 10 where $\ln|P_{01}(T)| / L$ is plotted as a function of the drive time $T = 2\pi / \omega_D$. In contrast, the return probability reaches a much lower value for dynamics which cross the critical point ($g_2 \leq g_c = 1$). Moreover for $g_2 < g_c$, the return probability shows a stronger non-monotonic behavior as a function of $T$.

These features can be understood by noting that for both near-adiabatic and sudden protocols, the system is expected to remain close to its original ground state when it does not pass through a critical point. Thus the return probability should be close to unity for such drives. However, for finite $\omega_D$, where the drive frequency matches the energy gap at some $k$, one expects significant excitation production leading to weak non-monotonicity of $\ln|P_{01}|$.

In contrast, for drives which take the system through the critical point, there is no adiabaticity and $|P_{01}(T) > 1| \leq |P_{01}(T) \ll 1|$. However, even in this case, excitation production is maximal at intermediate frequency, where one can produce excitations at maximal number of $k$ modes. Thus the return probability shows a dip close to $\omega_D \simeq 1$; the precise position of this dip depends on $g_2$.

We note that our results indicate that the position of a critical point in a quantum system can be inferred from its return probability as a function of the drive amplitude for slow enough drive frequencies. To elucidate this feature, we consider

$$\langle \ln|P_{01}(T)| \rangle = \frac{1}{T_f - T_i} \int_{T_i}^{T_f} dT \ln|P_{01}(T)|, \quad (92)$$

where $T_i$ and $T_f$ are the lowest and highest time periods between which the return probability is summed over. For our plots $T_f \simeq 200$ and $T_i = 20$ so that $\langle \ln|P_{01}(T)| \rangle$ reflects the drive time averaged value of the return probability for large $T$. A plot of $\langle \ln|P_{01}(T)| \rangle / L$ as a function of $1/g_2$, shown in Fig. 11, clearly indicates the difference between drives with $g_2 > 1$ and $g_2 < 1$. We note that for large $g_2 > 1$, the system is gapped; thus the large $T$ limit of the return probability remains constant as $g_2$ is increased. The gapped nature of the system ensures that all excitations are produced near the critical point at large $T$. Therefore increasing $g_2$ does not change the value of $\langle \ln|P_{01}(T)| \rangle / L$.

C. Probability distribution of operators: Counting Statistics

In this section, we compute the probability distribution of a quadratic operator $\hat{O}$ for the fermionic model considered in the preceding sections. The probability distribution is given by $P(O, t) = \text{Tr}[\hat{\rho}(t) \delta(O - \hat{O})]$, where $\hat{\rho}(t)$ denotes the density matrix of the system. The corresponding characteristic function is defined by\textsuperscript{13–15}

$$P(O, t) = \int_{-\infty}^{\infty} df \frac{d}{\sqrt{2\pi}} e^{-ifO} C(f, t),$$

$$C(f, t) = \int_{-\infty}^{\infty} dO e^{ifO} P(O, t). \quad (93)$$

For a system in a pure state, $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$, the characteristic function can be written as

$$C(f, t) = |\langle \Psi(t)| e^{i\hat{O}t} |\Psi(t)\rangle| = \langle \Psi(0)| \hat{U}^\dagger(t, 0) e^{if\hat{O}} \hat{U}(t, 0)|\Psi(0)\rangle, \quad (94)$$

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig10}
\caption{Plot of $\ln|P_{01}(T)| / L$ for the $d = 1$ transverse field Ising model as a function of the drive period $T$ for several transverse fields $g_1$ and $g_2$. The plot clearly shows the difference in behavior at large $T$ between protocols which do and do not cross the critical point. See text for details.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig11}
\caption{Plot of $\langle \ln|P_{01}(T)| \rangle / L$ as a function of $1/g_2$ for a fixed $g_1 = 10$ indicating the presence of critical point at $g_2 = 1$. See text for details.}
\end{figure}
where $\hat{U}(t, 0)$ is the time evolution operator and $|\Psi(0)\rangle$ the initial state at time $t = 0$.

Let us first compute the characteristic function associated with the density operator $\hat{n}_k = c_k^\dagger \hat{c}_k - c_k \hat{c}_k^\dagger + 1$ (which in the case of a superconductor corresponds to the electron density $c_k^c \hat{c}_k + \hat{c}_k^c c_k$ before the particle-hole transformation). The matrix element of the operator $\hat{O}_{\pm k} = \pm \hat{c}_k \hat{c}_k \hat{c}_k \hat{c}_k$ (with $k$ restricted to half the Brillouin zone, e.g. $k_x \geq 0$) is given by

$$\langle \phi_1 | e^{i\hat{O}_k} | \phi_2 \rangle = e^{i\sum_{k' \neq k} \phi_{1k'} \phi_{2k'}} + e^{i\phi_{1k} \phi_{2k}},$$

$$\langle \phi_1 | e^{i\hat{O}_{-k}} | \phi_2 \rangle = e^{i\sum_{k' \neq k} \phi_{1k'} \phi_{2k'}} + e^{i\phi_{1k} \phi_{2k}}, \quad (95)$$

where

$$O_k = \begin{pmatrix} e^{if/2} & 0 \\ 0 & e^{-if} \end{pmatrix}, \quad O_{-k} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-if} \end{pmatrix}. \quad (96)$$

are $2 \times 2$ matrices. This implies

$$\langle \phi_1 | e^{i\hat{n}_k} | \phi_2 \rangle = e^{i\Phi_{1k} \phi_{2k'}} + e^{i\Phi_{1k} \phi_{2k}},$$

$$\hat{O}_{\pm k} = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{-if} \end{pmatrix}. \quad (97)$$

The characteristic function (94) for the operator $\hat{O} = \hat{n}_k$ can be written as

$$C_{\hat{n}_k}(f, t) = \int \prod_{\ell=1}^4 D\Phi_{1\ell} D\Phi_{f\ell} e^{-\sum_{i, k} \phi_{ik\ell}^2} \langle \Psi(0) | \phi_1 \rangle$$

$$\times \langle \phi_1 | \hat{U}(t, 0) | \phi_2 \rangle \langle \phi_2 | e^{if\hat{n}_k} | \phi_3 \rangle$$

$$\times \langle \phi_3 | \hat{U}(t, 0) | \phi_4 \rangle \langle \phi_4 | \Psi(0) \rangle. \quad (98)$$

Assuming that the system is initially in the number state $|n\rangle \equiv \{|0_0, 0_{-k}\}$, we find

$$C_{\hat{n}_k}(f, t) = \int \prod_{\ell=1}^4 D\Phi_{1\ell} D\Phi_{f\ell} \prod_{k_1, k_2} \Phi_{1k_1} \Phi_{2k_1} \Phi_{3k_2} \Phi_{4k_2}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell}^2} \phi_{ik\ell} \phi_{ik\ell}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

$$\times \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

$$\times \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

where the matrix $O_{\hat{n}_k}$ is defined by (97) and we use the same notations as in Sec. IV.B. Integrating out $\Phi_2$ and $\Phi_3$, we obtain

$$C_{\hat{n}_k}(f, t) = \int D\Phi_{1\ell} D\Phi_{f\ell} D\Phi_{g\ell} D\Phi_{h\ell} \prod_{k_1, k_2} \Phi_{1k_1} \Phi_{2k_1} \Phi_{3k_2} \Phi_{4k_2}$$

$$\times \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

$$\times \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell} \phi_{ik\ell}$$

$$\times e^{-\sum_{k_1, k_2} \phi_{ik\ell} \phi_{ik\ell}}$$

Thus the probability distribution after a period $T$ of the drive protocol (72) is determined by the matrix $M_k(T) = -\mathcal{L}(q_2, T)\mathcal{L}(g_1, T)$ defined by (80). The functional integral in (100) is Gaussian and can be easily performed (see Sec. IV.B). One finds

$$C_{\hat{n}_k}^{00}(f, T) = C_{\hat{n}_k}^{01}(f, T) = e^{if},$$

$$C_{\hat{n}_k}^{10}(f, T) = |\alpha_k|^2 e^{i2f} + |\beta_k|^2,$$

$$C_{\hat{n}_k}^{01}(f, T) = |\alpha_k|^2 + |\beta_k|^2 e^{i2f}, \quad (101)$$

where $C_{\hat{n}_k}$ denotes the characteristic function for the initial state $|\Psi(0)\rangle = \{|0_0, 0_{-k}\}$, etc. From (93) we finally obtain

$$P_{\hat{n}_k}^{10}(n_k, T) = P_{\hat{n}_k}^{11}(n_k, T) = \delta(n_k - 1),$$

$$P_{\hat{n}_k}^{01}(n_k, T) = |\alpha_k|^2 \delta(n_k - 2) + |\beta_k|^2 \delta(n_k - 2). \quad (102)$$

The trivial expression of $P_{\hat{n}_k}^{00}(n_k, T)$ and $P_{\hat{n}_k}^{11}(n_k, T)$ comes from the fact that the states $|0_0, 0_{-k}\rangle$ and $|1_{1k}, 1_{-k}\rangle$ have no dynamics (see Sec. IV.B). By contrast the states $|1_{1k}, 0_{-k}\rangle$ and $|0_0, 1_{-k}\rangle$ have non-trivial dynamics since $|1_{1k}, 0_{-k}\rangle = c_{1k}^\dagger |0_{k'}, 0_{-k'}\rangle$ and $|0_0, 1_{-k}\rangle = c_{\tilde{k}}^\dagger |0_{k'}, 0_{-k'}\rangle$ are not eigenstates of the Hamiltonian and mix in the time evolution. Noting that these two states are eigenstates of $\hat{n}_k$ with eigenvalues 2 and 0, respectively, the expression of $P_{\hat{n}_k}^{10}(n_k, T)$ and $P_{\hat{n}_k}^{01}(n_k, T)$ can then directly be deduced from the return probability computed in Sec. IV.B (e.g., $|\alpha_k|^2$ is the probability that the system initially prepared in the state $|1_{1k}, 0_{-k}\rangle$ returns to the same state after time $T$).

In the more general case where the initial state is defined by

$$|\Psi(0)\rangle = \prod_k (a_k c_k^\dagger + b_k c_k^\dagger) |0_{k'}, 0_{-k'}\rangle \quad (103)$$

(with $|a_k|^2 + |b_k|^2 = 1$), the characteristic function is given by

$$C_{\hat{n}_k}(f, T) = |\alpha_k|^2 \beta_k^* - b_k \alpha_k^* |^2 + e^{i2f} |\alpha_k|^2 \beta_k^* + b_k \alpha_k^* |^2. \quad (104)$$

A similar analysis can be done to order parameter $\Delta_k = \hat{c}_k \hat{c}_k$ (which in the case of a superconductor corresponds to $\Delta_k = c_{\tilde{k}}^\dagger c_{\tilde{k}}$ before the particle-hole transformation). It is convenient to consider the real and imaginary parts of $\Delta_k$,

$$\Delta_k' = \frac{1}{2} (\Delta_k + \Delta_k^*),$$

$$\Delta_k'' = \frac{1}{2} (\Delta_k - \Delta_k^*), \quad (105)$$

which have real expectation values. For the computation of the characteristic functions, the relevant matrices are

$$O_{\Delta_k'} = \begin{pmatrix} 1 & if/2 \\ if/2 & 1 \end{pmatrix}, \quad O_{\Delta_k''} = \begin{pmatrix} 1 & f/2 \\ -f/2 & 1 \end{pmatrix}. \quad (106)$$
Assuming that the system is initially in the state (103), one finds
\[
C_{\hat{\Delta}^k}(f, T) = 1 + i f \Re[(|b_k|^2) - |a_k|^2] \alpha_k \beta_k
+ a_k^* b_k (\alpha_k^2 - \beta_k^2)],
\]
and in turn
\[
P_{\hat{\Delta}^k}(\Delta_k, T) = \delta(\Delta_k - \delta(\Delta_k)) \Re(|b_k|^2) - |a_k|^2 \alpha_k \beta_k
+ a_k^* b_k (\alpha_k^2 - \beta_k^2)],
\]
\[
P_{\hat{\Delta}^k}(\Delta_k', T) = \delta(\Delta_k') - \delta(\Delta_k') \Re(|b_k|^2) - |a_k|^2 \alpha_k \beta_k
+ a_k^* b_k (\alpha_k^2 - \beta_k^2)].
\]

The expectation value of the order parameter at time \(T\) is therefore given by
\[
\langle \Psi(T) | \hat{\Delta}_k' | \Psi(T) \rangle = \int d\Delta_k' P_{\hat{\Delta}^k}(\Delta_k', T) \Delta_k'^* \Psi(T)
\]
\[
= \Re[(|b_k|^2) - |a_k|^2] \alpha_k \beta_k
+ a_k^* b_k (\alpha_k^2 - \beta_k^2)]
\]
and
\[
\langle \Psi(T) | \hat{\Delta}_k'' | \Psi(T) \rangle = \int d\Delta_k'' P_{\hat{\Delta}^k}(\Delta_k'', T) \Delta_k''^* \Psi(T)
\]
\[
= \Im[(|a_k|^2 - |b_k|^2) \alpha_k \beta_k
+ a_k^* b_k (\alpha_k^2 + \beta_k^2)],
\]
or, equivalently,
\[
\langle \Psi(T) | \hat{\Delta}_k | \Psi(T) \rangle = \langle \Psi(T) | \hat{\Delta}_k | \Psi(T) \rangle
\]
\[
= \Re[(|b_k|^2) - |a_k|^2] \alpha_k^* \beta_k^* + a_k^* b_k \alpha_k^2 - \alpha_k b_k \beta_k^2.
\]

Equations (101,102,104) and (107,108) represent the main result of this section. They yield the characteristic functions and probability distributions of the quadratic operators \(\hat{\Delta}_k\) and \(\hat{\Delta}_k\) in terms of the elements of the Floquet Hamiltonian of the system for any arbitrary state \(|\psi(0)\rangle\) that can be constructed out of superposition of \(|1_k, 0_{-k}\rangle\) and \(|0_k, 1_{-k}\rangle\).

Finally we show that the characteristic functions exhibit a clear signature of the dynamics of the system. In Figs. 12 and 13 we plot \(C_{\hat{\Delta}_k}(f = i, T)\) and \(C_{\hat{\Delta}_k}(f = i, T)\) for the one-dimensional transverse field Ising model as a function of \(k\) (measured in units of the lattice spacing \(a\)) and the drive period \(T = 2\pi/\omega_D\) (in units of \(\hbar/\gamma\)). The square pulse protocol has \(g_1 = 0.01\) while \(g_2\) is varied between 0.01 and 10 as shown in Figs. 12 and 13. The initial state is assumed to be given by (103) with
\[
a_k = \frac{E_k(g_1) - g_1 + \cos k}{N E_k(g_1)},
\]
\[
b_k = \frac{\sin k}{N E_k(g_1)}
\]
(\(N\) is a normalization constant), which corresponds to the ground state deep in the paramagnetic phase. For small \(T\), i.e. for very fast drive (\(\hbar \omega_D \gg J\)) and momentum \(k\) (in units of \(a^{-1}\)). Each of the plots corresponds to \(g_1 = 0.01\). The top left (right) panel corresponds to \(g_2 = 0.01(0.4)\) and the bottom left (right) panel to \(g_2 = 0.99(10)\).

These irregular oscillations are however not present in \(C_{\hat{\Delta}_k}\) when \(\hbar \omega_D \ll J\) and \(g_2 \gg 1\). This can be understood as follows. For \(g_2 \gg g_1\) (ferromagnetic phase; \(g_1 = 1\) being the critical point) and \(g_1 \ll g_2\) (paramagnetic phase), both cycles of the square pulses correspond to a gapped Hamiltonian with a large gap. As a result, the system does not absorb significant energy during the drive. Therefore, the drive-induced excitations of the system are mostly fluctuations of the phase of \(\langle \Delta_k \rangle\). Thus \(\langle \hat{\Delta}_k \rangle\), which does not depend on this phase, does not show oscillations at low frequencies. In contrast, the order parameter receives contribution from these phase fluctuations, which explains the irregular oscillations observed in the bottom right plot of Fig. 13.
as a function of the drive period $T$. Systems with Gaussian Hamiltonians in this respect reproduce the ones derived earlier for systems of their Matsubara Green functions. Our results allow one to express the matrix elements of the density matrix of a many-body quantum system in terms of elements of the Floquet Hamiltonian.

For systems with Gaussian actions, we are thus able to express several measures of entanglement entropy in terms of their Matsubara Green functions. Results in this respect reproduce the ones derived earlier for systems with Gaussian Hamiltonians. We stress however that our method allows us to compute these measures for systems with Gaussian action which may not have a quadratic Hamiltonian. As an example, we have considered strongly interacting bosons in the framework of the Bose-Hubbard model, both in its pristine form and in the presence of Abelian and non-Abelian gauge fields.

Our analysis also allows us to compute the matrix elements of the evolution operator $U(t,0)$ for a driven Gaussian system using analytic continuation $t \to -i\beta\hbar$. In systems where one can compute the Green function $G(0^+)$ at any finite temperature $T_0$ analytically so that the Wick rotation can be carried out in a straightforward manner, we are therefore able to compute the dynamical correlation functions without resorting to Keldysh formalism. In contrast, if the Green function of the system is only known numerically, it might be difficult to carry out the Wick rotation and this is one of the drawbacks of this approach. To illustrate our approach we have computed the return probability and the counting statistics of density and order parameter operators for Dirac fermions subjected to a periodic drive with square pulse protocol. Our analysis allows us to obtain analytic expressions in terms of elements of the Floquet Hamiltonian.

We have presented a formalism for computing the matrix element of the density matrix of a many-body quantum system between two arbitrary coherent states. This formalism shows that it is possible, at least in principle, to express such matrix elements in terms of the Matsubara correlation functions of the system. In practice, such computation can be straightforwardly carried out for systems with a Gaussian action. This does not necessarily require a quadratic Hamiltonian; for example, the Bose-Hubbard model treated within strong coupling mean-field theory can be represented by a Gaussian action but not a quadratic Hamiltonian. One of our main results is that for systems with Gaussian actions these matrix elements are completely determined by the Matsubara Green function $G(0^+)$. For systems with Gaussian actions, we are thus able to express several measures of entanglement entropy in terms of their Matsubara Green functions. Our results in this respect reproduce the ones derived earlier for systems with Gaussian Hamiltonians. We stress however that our method allows us to compute these measures for systems with Gaussian action which may not have a quadratic Hamiltonian. As an example, we have considered strongly interacting bosons in the framework of the Bose-Hubbard model, both in its pristine form and in the presence of Abelian and non-Abelian gauge fields. Our results not only demonstrate the usefulness of our methods but also provide, to the best of our knowledge, the first calculation of entanglement entropy of bosons in their Mott phase in the presence of synthetic gauge fields. We note that the second Renyi entropy of pristine bosons has recently been measured, we therefore expect our computations to be useful for similar experiments carried out on bosons in the strong coupling regime in the presence of synthetic gauge fields.

Our analysis also allows us to compute the matrix elements of the evolution operator $U(t,0)$ for a driven Gaussian system using analytic continuation $t \to -i\beta\hbar$. In systems where one can compute the Green function $G(0^+)$ at any finite temperature $T_0$ analytically so that the Wick rotation can be carried out in a straightforward manner, we are therefore able to compute the dynamical correlation functions without resorting to Keldysh formalism. In contrast, if the Green function of the system is only known numerically, it might be difficult to carry out the Wick rotation and this is one of the drawbacks of this approach. To illustrate our approach we have computed the return probability and the counting statistics of density and order parameter operators for Dirac fermions subjected to a periodic drive with square pulse protocol. Our analysis allows us to obtain analytic expressions in terms of elements of the Floquet Hamiltonian. We reproduce earlier results as special cases and also find that these measures, in the presence of a suitable drive protocol, can point out the location of the critical point.

The analysis carried out in this work is expected to find applications in several other theoretical models. Examples include quantum rotor models treated with the large-$N$ approximation. We note that while there is some progress in understanding the dynamics of these models in the paramagnetic and ferromagnetic phases, the behavior of the driven system near its critical point is still not resolved. Second, the Renyi entropy near the critical point has not been understood analytically. Moreover, much of the formalism developed here for Dirac fermions may be applied to a class of models used to describe spin-liquids within RVB mean-field theories; in particular we expect our method to yield analytical results concerning the dynamics. Third, we hope to address the out-of-equilibrium dynamics and entanglement measures in Luttinger liquids whose action can be written in a quadratic form in terms of bosonic degrees of freedom.

In conclusion, we have developed a formalism which allows one to express the matrix elements of the density matrix of a many-body quantum system in terms of its Matsubara correlation functions. This allows us to compute several entanglement measures and address out-of-equilibrium dynamics of several fermionic and bosonic systems with Gaussian actions. In particular, we have used our formalism to compute entanglement measures in strongly interacting bosons described by the Bose-Hubbard model in the presence of synthetic gauge fields within a strong-coupling mean-field approach. We have also obtained semi-analytic expressions for return probability and counting statistics of Gaussian operators for Dirac fermions subjected to a periodic drive in terms of elements of the Floquet Hamiltonian. We note that our analysis may be useful for entanglement related exper-

![FIG. 13: $C_\Delta'(f = i,T)$ for the one-dimensional Ising model as a function of the drive period $T = 2\pi/\omega_D$ (in units of $\hbar/J$) and momentum $k$ (in units of $a^{-1}$). All other parameters are same as in Fig. 12.](image)
iments on bosons and may also be used for analysis of non-equilibrium dynamics and entanglement entropy of several other systems as discussed above.

Acknowledgments

RG acknowledges CSIR SPM fellowship for support and ND thanks I. Frérot for correspondence.

Appendix A: Density matrix for a single degree of freedom

1. Equilibrium case

We consider a single, bosonic or fermionic, degree of freedom described by the Hamiltonian $\hat{H} = \omega \hat{c}^{\dagger}\hat{c}$ with $\hat{c}^{\dagger} - \hat{c} = 1$. Using $|\phi\rangle = e^{\eta \hat{c}}|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the vacuum state ($\langle\hat{c}|\text{vac}\rangle = 0$), a straightforward calculation gives

$$\rho_{fi} = \frac{1}{Z} \langle \phi| e^{-\beta \hat{H}} |\phi\rangle = \frac{1}{Z} \exp\{ -\beta \omega \phi^{*}_i \phi_i \} \quad (A1)$$

with the partition function $Z = (1 - \eta e^{-\beta \omega})^{-1}$. Let us now reproduce this result from the path integral formalism discussed in Sec. II A. For the problem at hand, the discrete-time path integral (7) reads

$$Z = \int \rho_{fi} \prod_{k=1}^{N} d\phi_k^* d\phi_k \int d\lambda^* d\lambda \times e^{-\sum_{k,k'=1}^{N} \phi_k^* S_{kk'} \phi_{k'} - (\phi_N^* - \phi_1^*) \lambda + \lambda^* (\phi_N - \phi_1)} \quad (A2)$$

where

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\eta a \\ -a & 1 & 0 & \cdots & 0 \\ 0 & -a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} \quad (A3)$$

is a $N \times N$ matrix and $a = 1 - (\beta/N)\omega$. Integrating out $\phi_k$ and $\phi_k^*$, we find

$$U_{fi} = \frac{(\det S)^{-\eta}}{Z} e^{\eta \phi_i^* \phi_i} \int d\lambda^* d\lambda e^{-\lambda^* S_{NN}^{1} \lambda + \phi_N^* \lambda - \eta \lambda^* \phi_i}$$

Using $\det S = 1 - \eta a^N$ and

$$S^{-1} = \frac{1}{1 - \eta a^N} \begin{pmatrix} 1 & \eta a^{N-1} & \eta a^{N-2} & \cdots & \eta a \\ a & 1 & \cdots & \cdots & \cdots \\ a^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a^{N-1} \\ a^{N-1} & \cdots & \cdots & 1 & \eta a^{N-1} \end{pmatrix}$$

we finally obtain

$$U_{fi} = \frac{1}{Z} e^{\eta \phi_i^* \phi_i}, \quad (A6)$$

since $\lim_{N \to \infty} a^N = e^{-\beta \omega}$, in agreement with (A1). The present derivation makes it clear that the path integral involves the Matsubara Green function.

2. Out-of-equilibrium case

We now consider the case where the energy $\omega(t)$ is time-dependent: $\hat{H}(t) = \omega(t) \hat{c}^{\dagger}\hat{c}$. The evolution operator satisfies $\hat{H}(t)\hat{U}(t,0) = i\partial_t \hat{U}(t,0)$ and is given by

$$\hat{U}(t,0) = e^{-f(t)\hat{c}^{\dagger}\hat{c}}, \quad f(t) = i \int_0^t dt' \omega(t'). \quad (A7)$$

A straightforward computation then gives

$$U_{fi}(t,0) = \langle \phi_f|\hat{U}(t,0)|\phi_i\rangle = \exp\{e^{-f(t)}\phi_i^* \phi_i\} \quad (A8)$$

In the path-integral formalism, the dynamics of the system is governed by the real-time action

$$S = \int dt \phi^* (t)[i\partial_t - \omega(t)]\phi(t) \quad (A9)$$

The imaginary-time Green function satisfies

$$[\partial_{\tau} + \bar{\omega}(\tau)]G(\tau,\tau') = \delta(\tau - \tau'), \quad (A10)$$

the solution of which is

$$G(\tau,\tau') = e^{-\int_{\tau}^{\tau'} dt' \omega(\tau')}[\theta(\tau - \tau') (1 + \eta \tau) + \theta(-\tau + \tau') \eta \tau], \quad (A11)$$

where $\bar{\omega}(\tau)$ is the analytic continuation of $\omega(t)$ to imaginary time $\tau = it$. The value of $n$ is determined from the boundary conditions $G(\tau,\beta) = n G(\tau,0)$ and $G(\beta,\tau') = \eta G(0,\tau')$ (which follow from $\phi^{\lambda}(\beta) = \eta \phi^{\lambda}(0)$ in the Euclidean path integral), i.e.

$$n = \frac{1}{e^{f(\beta)} - \eta}, \quad f(\beta) = \int_0^\beta d\tau \bar{\omega}(\tau) \quad (A12)$$
Equation (43) then gives
\[ U_{J_1}(\beta, 0) = \mathcal{N} \exp \{ -f(\beta) \phi^* \phi \} \]  
(13)
and therefore Eq. (A8) after analytic continuation to real time. We do not compute the normalization factor \( \mathcal{N} \) which does play an essential role in the dynamics of the system.

**Appendix B: Relation to previous results**

In this appendix, we show that our formulation reproduces well-known earlier results on entanglement measures of many-body systems with a Gaussian Hamiltonian.\(^{12,14,15,50}\)

\[
S_n = \frac{1}{1-n} \ln \text{Tr}(\bar{\rho}^{n}) = \frac{1}{1-n} \ln \left[ \frac{1}{Z^n} \int d\phi^n_1 d\phi_1 \cdots d\phi^n_n d\phi_n e^{-\Phi^\dagger M \Phi} \right],
\]
(B2)

where \( \Phi^\dagger = (\phi^*_1, \ldots, \phi^*_n) \) and the matrix \( M \) (in block matrix form) is given by

\[
M_{ij} = i \delta_{ij} + \eta \frac{C}{I + \eta C} \delta_{i,1} \delta_{j,2} + \frac{C}{I + \eta C} \times \delta_{i,j-1} (1 - \delta_{i,1} \delta_{j,2}) + \frac{C}{I + \eta C} \delta_{n,i} \delta_{j,1}.\]
(B3)

Also it can be seen that \( Z = \det(1 + \eta C)^n \). So Eq. (B2) can finally be written as

\[
S_n = \frac{1}{1-n} \ln \left\{ \frac{\det[I - \eta(C/(I + \eta C))^{n-1}] - \eta^n}{\det(I + \eta C)^n} \right\}.\]
(B4)

This expression can be simplified to yield

\[
S_n = \frac{1}{1-n} \text{Tr} \ln[(I + \eta C)^n - \eta^n C^n],
\]
(B5)

which is the Renyi entropy expression obtained in Ref. 7. The results for the \( U(1) \) symmetry broken case may also be obtained by carrying out exactly the same analysis after switching to a basis which diagonalizes \( G \). The result is given by Eq. (B5) with \( C \to C_d \) where \( C_d \) denotes the correlation matrix in the diagonal basis and has contribution from both \( C \) and \( F \).

Next, we derive the results concerning the correlation functions \( C \) and \( F \) for fermions obtained earlier in Refs. 14,15. To this end, we first note that in the diagonal basis the correlation matrix, as obtained from Eq. (22), is given by

\[
C_d = \begin{pmatrix}
1 - \eta u & 0 \\
0 & n
\end{pmatrix},
\]
(B6)

First, we show that Eqs. (17) and (22) lead to the well-known expressions of the \( n^{th} \) Renyi entropy \( S_n \). For this we use the replica field technique introduced in Ref. 12. We start with our expression of the density matrix for a \( U(1) \) unbroken symmetry Hamiltonian,

\[
\rho_A(\phi_f, \phi_i) = \langle \phi_f | \hat{\rho} | \phi_i \rangle = \frac{1}{Z^n} \eta \sum_{ij} \phi^*_i (1 - G^{-1}(0^+))_{ij} \phi^*_j,
\]
(B1)

where \( \eta = 1[-1] \) for bosons[fermions] and \( C_{jj'} = \eta(G_{jj'}(0^+)) \).

To compute \( S_n \) we now introduce \( n \) replica fields \( \phi_1 \ldots \phi_n \). Using these one can write

\[
\rho_A(\phi_f, \phi_i) = \langle \phi_f | \hat{\rho} | \phi_i \rangle = \frac{1}{Z^n} \eta \sum_{ij} \phi^*_i \phi^*_j (C[1 + \eta C]^{-1})_{ij} \phi^*_j \phi^*_i.
\]
(B1)

where \( n \) is a diagonal matrix whose eigenvalues are obtained by diagonalizing the correlation matrix of bosons or fermions. In contrast, in the off-diagonal basis, the correlation matrix \( C \), as obtained in Refs. 14,15, is given by

\[
C = \begin{pmatrix}
I - \eta C & F \\
F^* & C
\end{pmatrix}.
\]
(B7)

We note that \( C_d \) and \( C \) must be related by a diagonalizing matrix \( U \): \( C_d = U C U^\dagger \). Moreover, the elements of \( U \) can be constructed out of \( 2L \) component eigenvector \( \psi_0 = (u, v) \) of \( C \), one can write \( C \psi_0 = C_d \psi_0 \). This leads to

\[
(1 - \eta C)u + Fv = (1 - \eta n)u, \quad \eta F^*u + Cv = nv.
\]
(B8)

Assuming \( F \) to be real, this gives, for fermions,

\[
\left( C - \frac{1}{2} \right) u + Fv = - \left( n - \frac{1}{2} \right) u, \quad \left( C - \frac{1}{2} \right) v = \left( n - \frac{1}{2} \right) v.
\]
(B10)

Introducing \( \phi = u + v, \psi = u - v \), we finally obtain

\[
\left( C - \frac{1}{2} - F \right) \phi = - \left( n - \frac{1}{2} \right) \psi, \quad \left( C - \frac{1}{2} + F \right) \psi = - \left( n - \frac{1}{2} \right) \phi.
\]
(B12)
This set of equations can be combined into one, as,

\[
(C - \frac{1}{2} - F) (C - \frac{1}{2} + F) \phi = (n - \frac{1}{2})^2 \phi. \quad \text{(B14)}
\]

This is exactly Peschel’s equation with the identification

\[ n = \text{Diag}(\epsilon_{\ell}) \text{ where } (n_{\ell} - \frac{1}{2})^2 = \frac{1}{4} \tanh^2(\epsilon_{\ell}/2) \text{ and } \epsilon_{\ell} \text{ are }
\]

the eigenvalues of the entanglement spectrum. This can be easily seen from the fact that \( \nu_{\ell} \) are the eigenvalues of the covariance matrix and they are related to \( \epsilon_{\ell} \) by \( n_{\ell} = 1/(\epsilon_{\ell}^2 + 1) \) for fermions. The choice of the negative root of the above equation leads to the results of Sec. II.B.
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