Minuscule embeddings

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Abstract

We study embeddings $J \to G$ of simple linear algebraic groups with the following property: the simple components of the $J$ module $\text{Lie}(G)/\text{Lie}(J)$ are all minuscule representations of $J$. One family of examples occurs when the group $G$ has roots of two different lengths and $J$ is the subgroup generated by the long roots. We classify all such embeddings when $J = \text{SL}_2$ and $J = \text{SL}_3$, show how each embedding implies the existence of exceptional algebraic structures on the graded components of $\text{Lie}(G)$, and relate properties of those structures to the existence of various twisted forms of $G$ with certain relative root systems.

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*In memory of T.A. Springer
11. Introduction

In this paper we study embeddings $J \rightarrow G$ of simple linear algebraic groups over a field such that the simple factors of the composition series of the $J$-module $\text{Lie}(G)/\text{Lie}(J)$ are all minuscule representations of $J$. We call such embeddings minuscule.

Recall that minuscule representations of a split, simple group $J$ over a field of characteristic zero are the irreducible representations whose weights for a maximal split torus lie in a single orbit for the Weyl group. (Unlike Bourbaki [1, Ch VI, §1, Ex 24] or [2, Ch VIII, §7.3], we consider the trivial representation to be minuscule.) Over a general field, they are the irreducible representations whose highest weight is minimal for the partial ordering on the set of dominant weights (given by $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of positive roots). Alternatively, they are the irreducible representations whose weights $\lambda$ satisfy $\langle \lambda, \alpha^\vee \rangle \in \{0, 1, -1\}$ for all roots $\alpha$ [2, Ch VIII, §7.3, Prop 6]. Each minuscule representation is determined by its central character, and the number of minuscule representations is equal to the order of the finite center $Z(J)$. For the group $J = \text{SL}_2$ only the trivial and the standard two dimensional representation are minuscule.

Minuscule embeddings arise naturally in several different contexts:

- When $G$ is a split group which has two root lengths and $J$ is the subgroup generated by the long roots. Indeed, for a short root $\alpha$ and a long root $\beta$, the pairing $\langle \alpha, \beta^\vee \rangle$ is in $\{0, \pm 1\}$. See section 7.
- When $J$ is the $A_1$ subgroup generated by the highest root of $G$ [1, Ch VI, §1] and its negative, as in [3, Prop. 3.3]. This is up to conjugacy the unique $A_1$ subgroup of $G$ with Dynkin index 1 [4, Th. 2.4]. See section 3.
- Several rows of the Magic Triangle in [5] can be viewed in terms of minuscule embeddings, where $J = \text{SL}_2$ or $\text{SL}_3$ and $G$ is exceptional of type $E$, $F$, or $G$. See sections 3 and 8.
- When $G$ has a relative root system with two root lengths such that the long roots have multiplicity 1 in $\text{Lie}(G)$ and form the root system of $J$, see sections 6, 7, 10, and 11.

This paper includes a classification of the minuscule embeddings $J \rightarrow G$ over $k$, for $J = \text{SL}_2$, $\text{SL}_3$, and $\text{Spin}_{4,4}$ (the split simply-connected group of type $D_4$). We will assume, throughout this paper, that the characteristic of $k$ is not equal to 2 or 3, so that in particular Proposition 2.2 applies. Much of our work involves the study of the centralizer $Z_G(J)$ and its representations $W_F$, which are defined in the next section. These representations have exceptional invariant tensors, which were studied in detail by T.A. Springer [6], [7], [8, §38], and it is a pleasure to dedicate this paper to his memory. We leverage knowledge of those tensor structures to give criteria for the existence of algebraic groups with relative root systems of type $BC_1$, $G_2$, and $F_4$. 
Regarding related work: After we had written this paper, we learned from Alberto Elduque of Vинberg’s paper [9], where what we call a minuscule embedding $\text{SL}_3 \to G$ is studied as a “short $\text{SL}_3$-structure on $\mathfrak{g}$”. Sections 8 and 9 have substantial overlap with [9]; one could view this material as a perspective on Springer’s monograph [6].

In another direction, the recent paper [10] begins with an isotropic semisimple group $G$ and also deduces algebraic structures on some subspaces of $\text{Lie}(G)$. In yet another direction, the papers [11] and [12] study embeddings $J \to G$ such that the nonzero weights of the representation $\mathfrak{g}/J$ of $J$ are all roots of $J$.

2. Generalities

A minuscule embedding $J \to G$ gives a grading of the Lie algebra $\mathfrak{g}$ of $G$ over $k$, where the summands are indexed by the characters of the center $Z(J)$ of $J$. Since the center is a finite group scheme of multiplicative type over $k$, its Cartier dual $C = \text{Hom}(Z(J), \mathbb{G}_m)$ is a finite étale group scheme. For each character $\chi$ in $C$ we let $V_{\chi}$ be the minuscule representation of $J$ whose weights restrict to $\chi$ on $Z(J)$. If $\chi \neq \chi'$, then the difference of the highest weights of $V_{\chi}$ and $V_{\chi'}$ is not in the root lattice, so $\text{Ext}^1_J(V_{\chi}, V_{\chi'}) = 0$ [13, II.2.14] and we have a direct sum decomposition

$$\mathfrak{g}/J = \bigoplus_{\chi \in C} V_{\chi} \otimes W_{\chi} \quad (2.1)$$

as representations of $J$. We note that each vector space $W_{\chi}$ is a linear representation of the centralizer $Z_G(J)$ of $J$ in $G$. For the minuscule embeddings which correspond to the long root subgroups of a split adjoint group with two root lengths, the centralizer $Z_G(J)$ is equal to the finite center $Z(J)$, $W_0 = 0$ and for each nonzero $\chi$, $W_{\chi} = \chi$.

We can make this decomposition more uniform by considering the Vinberg grading of $\mathfrak{g}$ given by the action of the finite group scheme $Z(J)$. For nonzero $\chi$, the component $\mathfrak{g}(\chi)$ is the representation $V_{\chi} \otimes W_{\chi}$ of the centralizer $G(0) = JZ_G(J)$. For $\chi = 0$ the component $\mathfrak{g}(0)$ is the Lie algebra of $G(0)$.

Let $H$ be the connected component of the centralizer $Z_G(J)$ and let $\mathfrak{h} = \text{Lie}(H)$. Since the intersection of $J$ and $Z_G(J)$ in $G(0)$ is the finite center $Z(J)$, if we assume that the characteristic of $k$ does not divide the order of $Z(J)$, the Lie algebra $\mathfrak{g}(0) = J + \mathfrak{h}$ decomposes as a direct sum. In summary:

**Proposition 2.2.** Assume that $J \to G$ is a minuscule embedding and that the finite group scheme $Z(J)$ has order prime to the characteristic of $k$. Then we have the decomposition

$$\mathfrak{g} = (J \otimes 1) \oplus (1 \otimes \mathfrak{h}) \oplus \bigoplus_{0 \neq \chi \in C} (V_{\chi} \otimes W_{\chi}) \quad (2.3)$$

as representations of $J \times Z_G(J)$.

3. $A_1$ case: Minuscule embeddings of $\text{SL}_2$

Let $G$ be a split, simple group of adjoint type over $k$, of rank at least two. In this section, we will construct a minuscule embedding $\text{SL}_2 \to G$ (generalizing the one studied over $\mathbb{C}$ in [3]), and will show that all such embeddings are conjugate.
The construction of a minuscule embedding of $SL_2$ is given as follows. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup of $G$, and let $\beta$ be the highest root, which is the highest weight of $T$ on the adjoint representation $\mathfrak{g}$. The 1-dimensional weight spaces $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$ generate a 3-dimensional Lie subalgebra of $\mathfrak{g}$, which is isomorphic to $\mathfrak{sl}_2$. A fixed embedding of $SL_2$ sends the standard generators $E$ and $F$ of $\mathfrak{sl}_2$ to compatible basis elements of $\mathfrak{g}_\beta$ and of $\mathfrak{g}_{-\beta}$ respectively. This embedding is minuscule. Indeed, a maximal torus $S$ in $SL_2$ is the image of the co-root $\beta^\vee$, and for any positive root $\alpha$ which is not equal to $\beta$ we have $\langle \beta^\vee, \alpha \rangle = 0$ or $\langle \beta^\vee, \alpha \rangle = 1$. Hence the only representations of $SL_2$ which occur in the quotient $\mathfrak{g}/\mathfrak{sl}_2$ are the standard and the trivial representation.

**Theorem 3.1.** Every minuscule embedding $SL_2 \rightarrow G$ is conjugate to the embedding given above.

**Proof.** If we have an embedding of $SL_2$, then we may conjugate it by an element of $G$ so that the restriction to a maximal torus $S$ of $SL_2$ lies in $T$, and is a dominant co-character $\nu$ with respect to $B$. Since the embedding is minuscule, for all positive roots $\alpha$, we have $\langle \nu, \alpha \rangle = 0, 1, 2$, and there is a unique positive root such that $\langle \nu, \alpha \rangle = 2$. Since the multiplicity of each simple root in $\alpha$ is less than or equal to its multiplicity in the highest root $\beta$, we must have $\langle \nu, \beta \rangle = 2$. Then the sub Lie algebra $\mathfrak{sl}_2$ is given by $\mathfrak{g}_{-\beta} + \text{Lie}(S) + \mathfrak{g}_\beta$ and $\nu = \beta^\vee$ is the associated co-root. We have therefore conjugated any embedding to have the same image as our standard embedding with equality on the maximal torus $S$. To finish the proof, we observe that the centralizer of $S$ acts transitively on the basis elements in the one dimensional $k$-vector space $\mathfrak{g}_\beta$. Indeed, the centralizer of $S$ contains the maximal torus $T$. Since $G$ is adjoint and the root $\beta$ can be extended to give a root basis of the character group of $T$, there is a co-character $\mu : \mathbb{G}_m \rightarrow T$ which satisfies $\langle \mu, \beta \rangle = 1$.

That is, every minuscule embedding $SL_2 \rightarrow G$ is up to conjugacy the unique $A_1$ subgroup of $G$ with Dynkin index 1 [4, Th. 2.4].

For a fixed minuscule embedding $SL_2 \rightarrow G$, we wish to determine the centralizer $H$ in $G$ and the full stabilizer $M$ in $\text{Aut}(G)$. The calculation of the centralizer $H$ follows the argument in [3, §2], but to determine the structure of the full stabilizer (which has connected component $H$) we need to consider the action of outer automorphisms of $G$. Fix a pinning of the simple root space with respect to $B$ and let $\Sigma$ be the group of all pinned automorphisms of $G$. This is a finite group, which is trivial unless $G$ is of type $A_n$ with $n \geq 2$, $D_n$ with $n \geq 4$, or $E_6$. In all but one of these cases, the group $\Sigma$ has order 2. When $G$ has type $D_4$, the group $\Sigma$ has order 6 and is isomorphic to the permutation group on 3 letters. The group $\Sigma$ permutes the simple roots, via the automorphisms of the Dynkin diagram. Since the multiplicity of a simple root in the highest root $\beta$ depends only on its orbit under $\Sigma$, the group $\Sigma$ fixes the highest root. Hence $\Sigma$ acts on the highest root space $\mathfrak{g}_\beta$. In all cases but type $A_{2n}$, the group $\Sigma$ acts trivially on $\mathfrak{g}_\beta$, whereas in the case of $A_{2n}$, it acts by the non-trivial character. This follows from the following more general result.

**Lemma 3.2.** Let $\Sigma$ be the group of all pinned automorphisms of $G$, and let $\alpha$ be a root fixed by $\Sigma$. Then $\Sigma$ acts trivially on the root space $\mathfrak{g}_\alpha$, except in the case when $G$ has type $A_{2n}$, where $\Sigma$ acts on $\mathfrak{g}_\alpha$ by the sign character.
**Proof.** We compute the trace of each non-trivial element \( \sigma \) in \( \Sigma \) in two ways. The first uses the grading of \( \mathfrak{g} \) into eigenspaces for \( \sigma \). When \( \sigma \) has order two, it suffices to determine the dimension of the fixed algebra. For \( g = \mathfrak{sl}_{2n} \), the fixed algebra is \( \mathfrak{sp}_{2n} \) and the trace of \( \sigma \) is \( 2n + 1 \). For \( \mathfrak{g} = \mathfrak{sl}_{2n+1} \) the fixed algebra is \( \mathfrak{so}_{2n+1} \) and the trace of \( \sigma \) is \(-2n\). For \( \mathfrak{g} = \mathfrak{so}_{2n} \) the fixed algebra is \( \mathfrak{so}_{2n-1} \) and the trace of \( \sigma \) is \( 2n^2 - 5n + 2 \). Finally, for \( \mathfrak{g} = \mathfrak{e}_6 \) the fixed algebra is \( \mathfrak{f}_4 \) and the trace of \( \sigma \) is 26. For \( \sigma \) of order 3 acting on \( \mathfrak{so}_8 \), the trace of \( \sigma \) is 7. (The fixed algebras are determined in \([2, Exercise VIII.5.13]\), for example.)

We can also compute the trace of \( \sigma \) using the Cartan decomposition \( \mathfrak{g} = \mathfrak{t} + \sum \alpha \mathfrak{g}_\alpha \).

The full automorphism group of \( G \) is isomorphic to the semi-direct product \( G \Sigma \). This acts transitively on the set of minuscule embeddings \( \mathfrak{sl}_2 \to G \), and the stabilizer \( M \) of our fixed embedding is an extension

\[
1 \to H \to M \to \Sigma \to 1. \quad (3.3)
\]

When \( G \) is not of type \( A_{2n} \) this extension is split. Indeed, the group \( \Sigma \) fixes the minuscule embedding described above. We shall see that it is not split for type \( A_{2n} \).

| \( G \) | \( M \) | \( W \) |
|-------|-------|-------|
| \( \text{PGL}_{n+2} \) | \( \text{GL}_n \times 2 \) | \( V_n + V_n^\vee \) |
| \( \text{SO}_{2n+5} \) | \( \text{SL}_2 \times \text{SO}_{2n+1} \) | \( V_2 \otimes V_{2n+1} \) |
| \( \text{Sp}_{2n+2} / \mu_2 \) | \( \text{Sp}_{2n} \) | \( V_{2n} \) |
| \( \text{SO}_{2n+4} / \mu_2 \) | \( (\text{SL}_2 \times \text{SO}_{2n})/\Lambda \mu_2 \) | \( V_2 \otimes V_{2n} \) |
| \( \text{SO}_8 / \mu_2 \) | \( (\text{SL}_2^2 / \prod \mu_2 = 1), S_3 \) | \( V_2 \otimes V_2 \otimes V_2 \) |
| \( G_2 \) | \( \text{SL}_2 \) | \( \text{Sym}^3(V_2) = V_4 \) |
| \( F_4 \) | \( \text{Sp}_6 \) | \( \Lambda^3(V_6)_0 = V_{14} \) |
| \( E_6 / \mu_3 \) | \( (\text{SL}_6 / \mu_3).2 \) | \( \Lambda^3(V_6) = V_{20} \) |
| \( E_7 / \mu_2 \) | \( \text{Spin}_{12} / \mu_2 \) | \( V_{32} \) (half-spin) |
| \( E_8 \) | \( E_7 \) | \( V_{56} \) (minuscule) |

Table 1: For a minuscule \( \text{SL}_2 \) in \( G \), the group \( M \) and its representation \( W \)

The action of \( \text{SL}_2 \times M \) on \( \mathfrak{g} \) decomposes as in \((2.3)\) as a direct sum of representations

\[
\mathfrak{g} = \mathfrak{sl}_2 \otimes 1 + 1 \otimes \mathfrak{m} + V_2 \otimes W \quad (3.4)
\]
where \( \mathfrak{m} \) is the adjoint representation of (the disconnected reductive group) \( M \). The center of \( M \) is isomorphic to \( \mu_2 \) and the map \((\text{SL}_2 \times M) \to \text{Aut}(G)\) has kernel the diagonally embedded \( \mu_2 \).

Table 1 lists the groups \( M \) of automorphisms of \( G \) which fix the minuscule embedding and their irreducible symplectic representations \( W \). The connected component of \( M \) is the centralizer \( H \) of the embedding in \( G \). For \( G = \text{PGL}_{n+2}, \) \( H = \text{GL}_n \) is the Levi subgroup of a Siegel parabolic in \( \text{Sp}(W) = \text{Sp}_{2n} \) and \( M \) is its normalizer. This is a semi-direct product when \( n \) is even, by Lemma 3.2. When \( n \) is odd, the exact sequence \( 1 \to H \to M \to \mathbb{Z}/2\mathbb{Z} \to 1 \) is not split – the smallest order of an element in the normalizer which does not lie in \( H \) is 4.

4. \( A_1 \) case: \( M \)-Invariant tensors on \( W \)

Fix a minuscule embedding \( \text{SL}_2 \to G \) associated to the highest root \( \beta \). The co-character \( \beta^\vee \) gives a 5-term grading on \( \mathfrak{g} \): 
\[
\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.
\]  
(4.1)

Each summand is a representation of \( M = H.S \), which fixes the minuscule embedding. The subalgebra \( \mathfrak{g}_0 \) is the Lie algebra of the reductive subgroup \( H.S \) of \( G \), and the eigencomponents \( \mathfrak{g}_2 \) and \( \mathfrak{g}_{-2} \) are the highest and lowest weight spaces for the torus. Both have dimension 1 with a chosen basis element (the images of the elements \( E \) and \( F \) in \( \mathfrak{sl}_2 \)) and give the trivial representation of \( M \). The Lie bracket \( \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_2 \) gives a non-degenerate alternating bilinear form \( \langle \cdot, \cdot \rangle \) on \( W \) which is \( M \)-invariant via 
\[
[w, w'] = \langle w, w' \rangle E \quad \text{for } w, w' \in W,
\]  
(4.2)
so \( W \) is a symplectic representation of \( M \).

We have already defined an \( M \)-invariant alternating bilinear form \( \langle \cdot, \cdot \rangle \) on \( W \) in (4.2). Using the chosen basis element (which is the image of \( F \)) of \( \mathfrak{g}_{-2} \) we can define an \( M \)-invariant quartic form \( q \) on \( W \) by the formula 
\[
(\text{ad } w)^4 F = q(w) E \quad \text{for } w \in W.
\]

For \( G \) not of type \( A_n \), there is a unique simple root \( \gamma \) that is not orthogonal to \( \beta \) and \( W \) is, as a subspace of \( \mathfrak{g} \), a sum of the root subalgebras \( \mathfrak{g}_a \) for \( a \) such that, when written as a sum of simple roots, the coefficient of \( \gamma \) is 1. By [14, Th. 2f], there is an open orbit in \( W \) under \( H.T \), equivalently, under the group generated by \( H \) and the image of the coroot \( \beta^\vee \). As \( \beta^\vee \) acts by scalars on \( W \), we find that there is an open \( H \)-orbit in \( \mathbb{P}(W) \), whence \( k[W]^H = k[f] \) for a (possibly constant) homogeneous \( f \).

When \( G \) has type \( C_n, M = \text{Sp}(W) \). Because the nonzero vectors in \( W \) are a single \( \text{Sp}(W) \)-orbit, this representation has no invariant symmetric tensors of degree greater than zero, and in particular \( q = 0 \). In all other cases, \( q \) is a non-zero quartic that generates the ring of \( M \)-invariant polynomials on \( W \). Note that in the case when \( G \) has
type $A_n$ the subgroup $H$ fixes a quadratic form $q_2$ on $W$ [3, Prop 6.1]. However, the form $q_2$ is not $M$-invariant: the quotient $M/H$ acts non-trivially and the first non-trivial invariant is the quartic $q = q^2_2$.

For types $B$, $D$, and $E$, it is a theorem [15, Th. 27] that $q$ and $\langle \; , \; \rangle$ satisfy the algebraic identities defining a Freudenthal triple system as in [16], [17], or [18]. In the simplest case, when $G$ is split of type $D_4$, $M = (\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 / \prod \mu_2 = 1).S_3$, and $W = V_2 \otimes V_2 \otimes V_2$ is the tensor product of the natural two dimensional representations. The quartic form $q$ is Cayley’s hyperdeterminant from [19]. In another simple case, when $G$ is split of type $E_6$, $H$ is $(\text{SL}_6 / \mu_3).2$ and $W = \wedge^2 k^6$. The quartic form is described in [20, p. 83] or [21, p. 4773]. When $G$ is split of type $E_8$, $q$ is the famous $E_7$ quartic in 56 variables described in, for example, [22] and [23].

**Theorem 4.3.** For $G$, $M$, and $W$ as in Table 1, $M$ is the subgroup of $\text{GL}(W)$ that stabilizes the two tensors $\langle \; , \; \rangle$ and $q$.

**Proof.** This is clear for type $C_n$, where $q = 0$ and $M = \text{Sp}(W)$ is the subgroup of $\text{GL}(W)$ stabilizing the non-degenerate symplectic form. For type $A_n$, the stabilizer of $\langle \; , \; \rangle$ and the quadratic form $q_2$ is the Levi subgroup $H$ of a Siegel parabolic in $\text{Sp}(W)$, and the stabilizer of the quartic form $q = q^2_2$ is its normalizer $M = H.2$. In the remaining cases, the stabilizer of $q$ in $\text{GL}(W)$ has been determined, for example, in [21, §9] and for $G$ of type $E_8$ in [18]. This is the subgroup $\mu_4.M$; the subgroup $\mu_2.M = M$ also stabilizes the bilinear form $\langle \; , \; \rangle$. □

5. $A_1$ case: Twisting and tensor structures

We now change our notation and let $G$ be a simple group of adjoint type over $k$ with a minuscule embedding $\text{SL}_2 \rightarrow G$. (We use $G_0$ to denote its split form, which is the group studied in the previous sections.) For example, suppose that the group $G$ has a relative root system of type $BC_1$ over $k$. Such a $G$ has a maximal split torus $S \cong \mathbb{G}_m$ of dimension one, whose non-trivial characters on $\mathfrak{g}$ are $\{ \pm 1, \pm 2 \}$. If we assume further that the long root spaces $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ have dimension one, then $\mathfrak{g}_{-2} + \text{Lie}(S) + \mathfrak{g}_2$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. If we fix this isomorphism, the corresponding embedding $\text{SL}_2 \rightarrow G$ is minuscule. We will want to identify these groups of rank one with certain tensor structures over $k$.

Choose an isomorphism 

$$\phi : G_0 \rightarrow G.$$ 

of algebraic groups over the separable closure $\overline{k}$. Then for every element $\sigma$ in the Galois group of $\overline{k}$ over $k$, the composition

$$a(\sigma) = \phi^{-1} \circ \sigma(\phi)$$

is an automorphism of $G_0$ over $\overline{k}$. This gives a 1-cocycle on the Galois group of $\overline{k}$ over $k$ with values in $\text{Aut}(G_0)(\overline{k})$. Since the minuscule embeddings of $\text{SL}_2$ into $G_0$ form a single orbit for the automorphism group, we may modify our chosen isomorphism $\phi$ by
an automorphism of \( G_0 \) so that it induces the identity map on the embedded subgroup \( \text{SL}_2 \) in \( G_0 \) and \( G \). Then \( a(\sigma) \) lies in the stabilizer \( M_0 \) of the minuscule embedding and defines a 1-cocycle on the Galois group with values in \( M_0(\overline{k}) \). The image of this cocycle under the map \( M_0 \to \text{Aut}(W_0, (\cdot)_0, q_0) \) determines a pure form \( M \) of the stabilizer \( M_0 \) over \( k \), or equivalently, a form \( W \) of the tensor structure we have studied on \( W_0 \). The image under the map \( H^1(k, M_0) \to H^1(k, \text{Aut}(G_0)) \) determines the isomorphism class of the twisted group \( G \), and the twisted representation \( W \) of \( M \) occurs in the decomposition of its Lie algebra as in (3.4).

The map \( M_0 \to \Sigma \) in (3.3) sends the 1-cocycle \( a \) to a 1-cocycle \( \iota(a) \) with values in \( \Sigma \). Recalling that \( \Sigma \) is isomorphic to the symmetric group on \( d \) letters for some \( d \), \( \iota(a) \) determines a degree \( d \) étale \( k \)-algebra \( K \) up to \( k \)-algebra isomorphism. The groups \( G \) and \( H \) are of inner type if and only if \( a \) is in the image of the map \( H^1(k, H_0) \to H^1(k, M_0) \), equivalently, if and only \( K \) is “split”, i.e., is isomorphic to a product of copies of \( k \).

In case \( K \) is not split, we twist sequence (3.3) by the 1-cocycle \( \iota(a) \) to obtain an exact sequence of group schemes

\[
1 \to H_q \to M_q \to \Sigma_q \to 1.
\]

Here, \( H_q \) is a quasi-split form of \( H_0 \) (the unique quasi-split group that is an inner form of \( H \)) and \( \Sigma_q \) is a not-necessarily-constant étale group scheme. Put \( a_q \) for the image of \( a \) under the twisting isomorphism \( H^1(k, M_0) \to H^1(k, M_q) \). By construction, \( \iota(a_q) = 0 \), so \( a_q \) is the image of a 1-cocycle \( b_q \) with values in \( H_q(\overline{k}) \).

6. \( A_1 \) case: \( k \)-forms and groups with a relative root system of type \( BC_1 \)

We follow the notation of the preceding section, i.e., we consider an adjoint simple group \( G \) with a minuscule \( \text{SL}_2 \). Such a group is obtained by twisting \( G_0 \) by a 1-cocycle \( z \) with values in \( M_0(\overline{k}) \). We now describe concrete interpretations of the resulting form \( H \) of the identity component of \( M \) in terms of other algebraic structures, and indicate the correspondence between isotropy of \( H \) (i.e., possible Tits indexes) and properties of that structure.

We keep a specific focus on conditions for \( H \) to be anisotropic, equivalently, for \( G \) to have a relative root system of type \( BC_1 \). Note that, if \( H \) contains a split torus of rank one, then the quartic form \( q \) must vanish on each of its non-trivial eigenspaces; that is, \( q \) does not represent zero, then \( H \) is anisotropic. We prove the converse when \( G \) has type \( D_4 \).

Consider first the case \( G_0 = \text{PGL}_{n+2} \). If \( G \) is inner, then \( z \) is the image of some \( z_0 \in H^1(k, \text{GL}_n) \), so is trivial by Hilbert’s theorem 90. Hence we cannot find an inner twisting with \( H \) anisotropic, and indeed a \( G \) of inner type \( A \) with a minuscule \( \text{SL}_2 \) is split.

Suppose now that \( G \) is not inner, so it is an inner form of the quasi-split group \( PU_{n+2} \) corresponding to a quadratic extension \( K \) of \( k \) as at the end of the preceding section, and \( H_q \) is isomorphic to the unitary group \( U_n \). The set \( H^1(k, U_n) \) classifies non-degenerate Hermitian spaces \( W \) of rank \( n \) over the quadratic field extension \( K \), i.e.,
\[ H = U(h) \] for some such form. The group \( G \) will have a relative root system of type \( BC_1 \) over \( k \) if and only if \( H \) is anisotropic if and only if \( h \) is anisotropic. The group \( G \) is the projective unitary group of the Hermitian space \( W + N \), where \( N \) is a split Hermitian space of dimension 2. The decomposition of the Lie algebra over \( k \) as in (2.3) is

\[ \text{su}(W + N) = \text{sl}_2 \otimes 1 + 1 \otimes \text{u}(W) + V_2 \otimes W. \]

Note that \( \text{sl}_2 \cong \text{su}(N) \).

Looking at Table 1 in the previous section, and using the fact that \( H^1(k, \text{SL}_n) = H^1(k, \text{Sp}_{2n}) = 1 \), we see that there are no groups \( G \) with a relative root system of type \( BC_1 \) for inner forms of the split groups \( B_n, C_n, G_2 \), and \( F_4 \), as well as for inner forms of the split group \( E_6 \). However, the quasi-split groups \( D_n, 2D_n, 3D_4, 4D_4, 5E_6, E_7, \) and \( E_8 \) have inner forms over certain fields \( k \) with a relative root system of type \( BC_1 \). We can make this more explicit by studying certain algebraic structures on the representation \( W \) of \( H \).

We now consider in some detail the case where \( H \) has type \( D_4 \), i.e., where \( q_0 \) is Cayley’s hyperdeterminant. The group \( \Sigma \) is a copy of the symmetric group on three letters. As above, there is a natural map of \( \text{Res} \) \( \text{Aut}(W_0, \langle \cdot, \cdot \rangle_0, q_0), \) and by [15, Cor. 49] or [21, Cor. 9.10] the latter group is generated by the image of \( M_0 \) and \( \mu_4 \) acting as scalars. By Galois descent, all twisted forms of the hyperdeterminant triple system are obtained by this construction.

For this case, we can prove the following.

**Proposition 6.1.** For a twisted form \( q \) of the hyperdeterminant, the automorphism group of \( q \) is isotropic if and only if \( q \) represents zero.

**Proof.** Over \( \overline{k} \), \( q \) is isomorphic to the hyperdeterminant \( q_0 \). We leverage the study of the \( H \)-orbits in the projective variety \( q_0 = 0 \) as described in [24], or see [25] for a more geometric viewpoint. Specifically, there is a unique minimal closed \( H_0 \)-invariant subvariety \( X \), the \( H_0 \)-orbit of \( x_0 \). A smooth point of the variety \( q_0 = 0 \) is in the \( H(\overline{k}) \)-orbit of \( v := x_{a_1+a_2} + x_{a_2+a_3} + x_{a_3+a_4} \), where \( x_a \) denotes a generator for \( \mu_a \) and we have numbered the simple roots \( a_1, \ldots, a_4 \) of \( D_4 \) as in [1] so that \( a_2 \) corresponds to the central vertex of the Dynkin diagram. Combining \( q_0 \) and \( \langle \cdot, \cdot \rangle_0 \), we find an \( H_0 \)-invariant symmetric trilinear map \( t_0 : W_0 \times W_0 \times W_0 \to W_0 \) such that \( \langle t_0(w, w, w), w \rangle = q_0(w) \) for all \( w \in W_0 \). Because \( v \) is a smooth point, \( t_0(v, v, v) \neq 0 \), and it follows from the \( H_0 \)-invariance of \( t_0 \) that \( t_0(v, v, v) \) is in the \( k \)-span of \( x_{a_2} \), i.e., belongs to \( X \). Looking now at \( H \) and \( q \) over \( k \), if the variety \( q = 0 \) is nonempty, then we take \( v \) to be a smooth point and observe that \( X(k) \) contains \( t(v, v, v) \) so is nonempty. Then the stabilizer of \( t(v, v, v) \) in \( H \) is a parabolic subgroup and \( H \) is isotropic. \( \Box \)

We can exhibit inner forms of quasi-split groups of type \( D_4 \) with a relative root system of type \( BC_1 \). Let \( K \) be the cubic étale algebra determined by \( \alpha \), so \( H_q \) is the group \( \text{Res}_{K/k} \text{SL}_2 \). The inner forms of \( H_q \) are isomorphic to \( \text{Res}_{K/k}(S_3) \). The inner forms of \( H_q \) are isomorphic to \( \text{Res}_{K/k}(S_3) \). For \( Q \) a quaternion algebra with center \( K \) such that the corestriction of \( Q \) to \( k \) (which is a central simple \( k \)-algebra of dimension 8) is a
matrix algebra. (When $G_0$ is split, $K = k \times k \times k$ and $Q$ corresponds to three quaternion algebras $(Q_1, Q_2, Q_3)$ over $k$ such that the tensor product $Q_1 \otimes Q_2 \otimes Q_3$ is a matrix algebra.) Explicitly, by [8, 43.9], the quaternion algebra $Q$ has Hilbert symbol $(a, b)_K$ with $b \in k^\times$ and $a \in K^\times$ such that $N_{k/K}(a) = 1$. When $K$ is a field, $H$ will be anisotropic if and only if $Q$ is a division algebra. When $K = k \times k \times k$, $H$ will be anisotropic if each of the quaternion algebras $Q_1$, $Q_2$, and $Q_3$ is a division algebra over $k$; in particular, the Brauer group of $k$ must contain a Klein 4-group. It follows from Tits’s Witt-type theorem that every isotropic group of type $^3D_4$ or $^4D_4$ arises in this way, see [26].

Now suppose that $G_0$ is quasi-split of type $D_n$ for some $n \geq 5$; the Galois action on the Dynkin diagram determines the quadratic étale $k$-algebra $K$. The group $H_q$ is isomorphic to $(\text{SL}_2 \times \text{SO}(q))/\mu_2$ for $q$ a sum of $n - 2$ hyperbolic planes and the 2-dimensional orthogonal space $K$ with norm $N_{K/k}$. Every inner twist $H$ of $H_q$ is of the form $(\text{SL}_2(Q) \times \text{SO}(h))/\mu_2$ for a (possibly split) quaternion $k$-algebra $Q$ and a skew-hermitian form $h$ on a $Q$-module $V$ of rank $n - 1$ such that $h$ has discriminant $K$ in the sense of [8, §10]. Such a group $H$ is isotropic if and only if $h$ represents 0, i.e., if and only if there is some nonzero $v \in V$ such that $h(v, v) = 0$, see [27, §17.3].

Next suppose that $G_0$ is quasi-split of type $^2E_6$, which determines a quadratic field extension $K$ of $k$ and the quasi-split group $H_q$ is $\text{SU}_6/\mu_3$. Every inner form $H$ of $H_q$ is $\text{SU}(B, \tau)/\mu_3$ for $B$ a central simple $K$-algebra of dimension 6, and $\tau$ an involution on $B$ that restricts to the nontrivial $k$-automorphism of $K$ and such that the discriminant algebra $D(B, \tau)$ defined in [8, §10.E] is split. Indeed, the Brauer class of the discriminant algebra is the Tits algebra for the representation $W$. Because $B^{\text{res}}$ is Brauer-equivalent to $D(B, \tau) \otimes K$ by [28] or [8, 10.30], it follows that $B = M_2(B_0)$ for some central simple $K$-algebra $B_0$ of dimension 3 whose corestriction to $k$ is a matrix algebra. Such a group $H$ is isotropic if and only if $\tau(b)b = 0$ for some nonzero $b$, i.e., if and only if $\tau$ is isotropic in the sense of [8, 6.3]. Alternatively, one can view $\tau$ as the involution adjoint to a hermitian form $h$ on a rank 2 $B_0$-module $V$ as in [8, §4.A], in which case we have: $H$ is isotropic if and only if $h(v, v) = 0$ for some nonzero $v \in V$.

When $G_0$ is split of type $E_7$, $H_0$ is a half-spin group and $W_0$ is the half-spin representation, i.e., $H_0$ is the image of $\text{Spin}_{12} \rightarrow \text{GL}(W_0)$. Every inner form $H$ of $H_0$ is isogenous to $\text{SO}(A, \sigma)$ where $A$ is a central simple $k$-algebra of dimension 12 and $\sigma$ is an orthogonal involution with trivial discriminant such that the even Clifford algebra $C(A, \sigma)$ as defined in [8, §8] has one split component (namely the action on $W$). Such pairs $(A, \sigma)$ have recently been described more explicitly, see [29]. Such an $H$ is isotropic if and only if the involution $\sigma$ is isotropic, i.e., if and only if $\sigma(a)a = 0$ for some nonzero $a \in A$.

Remarks 6.2 (for $G$ of type $E_7$). See [30, Prop. 3] for a description of the $H_0$-orbits on $W_0$.

To provide an anisotropic form $H$ of $H_0$, it is sufficient to produce an anisotropic 12-dimensional quadratic form in $\bar{q}k$ over some $k$. This is easily done using Pfister’s explicit description of such forms from [31].

Finally when $G_0$ is split of type $E_8$, $W_0$ is the 56-dimensional minuscule repre-
sentation of $H_{0}$, the split simply-connected group of type $E_7$. Each inner form $H$ of $H_{0}$ has a corresponding 56-dimensional representation $W$ over $k$ and we obtain then twisted forms of the Freudenthal triple system arising in the split case. As above, if $H$ is isotropic, then $q(w) = 0$ for some nonzero $w \in W$. See [32, §7] for the structures in $W$ corresponding to parabolic subgroups of $H$ and, for example, [33] for a discussion of the variety $q = 0$ defined by the vanishing of the quartic form.

Remarks 6.3 (for $G$ of type $E_8$). (i): See [24], [32, Th. 7.6], or [33] for a description of the $H$-orbits in $W$. The description in [32] describes $k$-points on the projective homogeneous spaces for $H$ in terms of inner ideals in $W$, i.e., subspaces $I$ such that $t(I, I, W) \subseteq I$.

(ii): Diverse constructions of anisotropic pure inner forms $H$ exist, see for example [34, Prop. 2(B)], [35, Example 7.2], [36, Appendix A], or [37, Cor. 10.17].

(iii): Groups with relative root systems of type $BC_1$, viewed from the angle of Lie algebras with a 5-term grading as in (4.1), have been studied in the context of structurable algebras as in [38] and [39].

(iv): When $G$ has type $D_4$, we proved (Prop. 6.1) that if $q$ represents zero, then $H$ is isotropic. No proof on the same outline is possible in the $E_8$ case, as we illustrate with examples. Specifically, first note that there are groups of type $E_8$ with semisimple anisotropic kernel of type $D_6$ or $E_6$. For such groups, $H$ is isotropic with semisimple anisotropic kernel of the same type, and the corresponding form $q$ represents zero, i.e., there is a smooth $k$-point $v$ on the hypersurface $q = 0$. The groups $H$ with anisotropic kernel of type $E_6$ correspond to a $W$ containing a 1-dimensional inner ideal but no 12-dimensional inner ideal; those with anisotropic kernel of type $D_6$ correspond to a $W$ containing a 12-dimensional inner ideal but no 1-dimensional inner ideal. Therefore, there cannot be a deterministic mechanical procedure to construct from $v$ an inner ideal of $W$, in contrast to the $D_4$ case where the $k$-span of $t(v, v, v)$ provides a 1-dimensional inner ideal.

(v): Our methods do fail to capture four possibilities with relative root system of type $BC_1$, corresponding to $G$ having one of the following Tits indexes:

In these cases, the long roots have multiplicity 7, 8, 10, and 14 respectively.

Remark. The paper [40] gives results related to the case where $G$ has relative root system of type $BC_2$ and $J = SL_2 \times SL_2$.

7. Minuscule embeddings and relative root systems

Let $G$ be a split, simple adjoint algebraic group with roots of different lengths. As mentioned in the introduction, the embedding $J \rightarrow G$ is minuscule when $J$ is the subgroup generated by the long root subgroups. There are four cases to consider.

For type $B_n$, $G$ is the split adjoint group $SO_{2n+1}$ and $J$ is the subgroup which fixes a non-isotropic line in the standard representation $V_{2n+1}$, with orthogonal complement
$V_{2n}$. This gives an isomorphism of $J$ with the split even orthogonal group $SO_{2n}$. The action of $J$ on $\mathfrak{g}/\mathfrak{j}$ is given by the standard representation $V_{2n}$.

For type $C_n$, $G$ is the adjoint group $Sp_{2n}/\mu_2$ and $J$ is the subgroup stabilizing a decomposition of the symplectic space into non-degenerate planes. The group $J$ is isomorphic to the split group $SL_3^\mathbb{Q}/\Delta\mu_2$, and the action of $J$ on $\mathfrak{g}/\mathfrak{j}$ is of the four dimensional representations $V_2^\mathbb{Q} \otimes V_2^\mathbb{Q}$, with $1 \leq i < j \leq n$.

For type $G_2$, the subgroup $J$ is isomorphic to $SL_3$ and its action on $\mathfrak{g}/\mathfrak{s}_k$ is by the direct sum of the three dimensional representations $V_3$ and $V_3^\vee$.

For type $F_4$, the subgroup $J$ is isomorphic to the split group $Spin_{10,4}$ and its action on $\mathfrak{f}_4/\mathfrak{spin}_{10,4}$ is by the direct sum of the three eight dimensional representations $V_8$, $V_8^\vee$, and $V_8''$.

In all four cases, the centralizer of $J$ in $G$ is the center $Z(J)$, which is isomorphic to $\mu_2$, $(\mu_2)^{n-1}$, $\mu_3$, and $(\mu_2)^2$ respectively. The pinned outer automorphism group of $J$ is isomorphic to the symmetric group $S_k$ with $k = 2, n, 2, 3$ respectively, and the normalizer of $J$ in $G$ is equal to $JS_k$. We should emphasize that in all these cases, we are only establishing the existence of a minuscule embedding, not the uniqueness up to conjugation in $G$ as we did for $SL_2$. For example, the adjoint group $PGL_3(k)$ acts on the set of minuscule embeddings $SL_3 \rightarrow G_2$ over $k$ by its action by conjugation on $SL_3$. Only the conjugates by the subgroup $SL_3(k)/\mu_3(k)$ yield conjugate embeddings. Hence the conjugacy classes of embeddings form a principal homogeneous space for the quotient group $k^\times/k^\times_3$. In all the four cases, $J$ is isomorphic to the split group mentioned, but the isomorphism is not unique and $J$ has inner automorphisms which do not come from conjugation in $G$.

We now consider the case where $G$ need not be split over $k$, but has a relative root system of type $B_n$, $C_n$, $G_2$, or $F_4$. Let $S$ be a maximal split torus in $G$ and assume that the long root spaces for $S$ acting on the Lie algebra $\mathfrak{g}$ all have dimension one. Note that this hypothesis is automatic in case $G$ is split, because root spaces for a maximal torus all have dimension one. It also holds when the relative root system is of type $G_2$ or $F_4$, as one can see by comparing the table of relative root systems from [41, pp. 129–135] with Tables 2 and 3.

Let $J \rightarrow G$ be the subgroup generated by $S$ and the long root groups. Then the subgroup $J$ is given above, and its action on $\mathfrak{g}/\mathfrak{j}$ decomposes as a direct sum of minuscule representations. Indeed, the remaining weights for $S$ are the short roots, and they are the weights which occur in the minuscule representations of $J$. These minuscule representations of $J$ will now occur with higher multiplicity in $\mathfrak{g}/\mathfrak{j}$, as the short root spaces will have multiplicity greater than one when $G$ is not split.

Let $H$ be the centralizer of $J$ in $G$. Since the ranks of $J$ and $G$ are the same (they both have maximal split torus $S$) the subgroup $H$ is anisotropic. It contains the anisotropic kernel of $G$ as its connected component, as the anisotropic kernel must act trivially on each long root space. Since $H$ centralizes the torus $S$, it acts linearly on
each short root space $W_\alpha \subset \mathfrak{g}$, and the isomorphism class of the representation $W_\alpha$ depends only on the orbit of the short root $\alpha$ under the action of the Weyl group of $S$ in $J$.

For $G$ with relative root system of type $B_n$ there is a single orbit of the Weyl group of $J = \text{SO}_{2n}$ on the set of $2n$ short roots. The action of $J \times H$ on the quotient $\mathfrak{g}/(\mathfrak{j} + \mathfrak{h})$ is given by the tensor product $V_{2n} \otimes W$, where $W$ is the orthogonal representation of $H$ on the short root space $W_\alpha$ with $\alpha = e_1$.

For $G$ with relative root system of type $C_n$ there are $\binom{n}{2}$ orbits of the Weyl group of $J = (\text{SL}_2)^n/\Delta\mu_2$ on the set of $4\binom{n}{2}$ short roots. The action of $J \times H$ on the quotient $\mathfrak{g}/(\mathfrak{j} + \mathfrak{h})$ is given by the direct sum of representations $\sum_{1 \le i < j \le n} (V_i \otimes V_j) \otimes W_{ij}$, where $W_{ij}$ is the orthogonal representation of $H$ on the short root space $W_\alpha$ with $\alpha = e_i + e_j$. Although these representations are not isomorphic, they are exchanged by the outer automorphism group of $J$, so all have the same dimension.

For $G$ with relative root system of type $G_2$ there are two orbits of the Weyl group of $J = \text{SL}_3$ on the set of 6 short roots. The action of $J \times H$ on the quotient $\mathfrak{g}/(\mathfrak{j} + \mathfrak{h})$ is given by the direct sum of representations $V_1 \otimes W + V'_3 \otimes W'^\vee$, where $W$ is the representation on one of the short root spaces. We will see in §10 that $W$ and its dual $W'^\vee$ have dimensions either 1, 3, 9, or 27, and that both have an $H$-invariant cubic form.

For $G$ with relative root system of type $F_4$ there are three orbits of the Weyl group of $J = \text{Spin}_{3,4}$ on the set of 24 short roots. The action of $J \times H$ on the quotient $\mathfrak{g}/(\mathfrak{j} + \mathfrak{h})$ is given by the direct sum of representations $V_8 \otimes W + V'_8 \otimes W' + V''_8 \otimes W''$, where $W, W', W''$ are three orthogonal representations of the same dimension. This dimension is either 1, 2, 4, or 8, see §11.

Here is an example where the relative root system has type $B_n$ and the long root spaces have dimension one. Let $V$ be a non-degenerate orthogonal space over $k$ of odd dimension $d$ and rank $n$, so $d \geq 2n + 1$. Let $X$ and $X'$ be a pair of dual maximal isotropic subspaces of dimension $n$, and let $W = X + X'$ be the corresponding non-degenerate subspace of dimension $2n$. Then $V = W + W^\perp$ and the adjoint group $G = \text{SO}(V)$ has a relative root system of type $B_n$. The long roots have multiplicity one and give the subgroup $J = \text{SO}(W) = \text{SO}_{2n}$. The short roots have multiplicity equal to the dimension of $W^\perp$ and the centralizer $H = \text{O}(W^\perp)$ of $J$ acts on the short root spaces by the standard representation. The decomposition of the Lie algebra as in (2.3) is

$$\mathfrak{so}(V) = \mathfrak{so}(W) + \mathfrak{so}(W^\perp) + W \otimes W^\perp.$$  

A similar decomposition occurs for orthogonal spaces of even dimension $d \geq 2n + 2$, where $n$ is the rank.

An example where the relative root system has type $C_n$ and the long root spaces have dimension one comes from the real groups $G$ that act on tube domains. Here $n$ is the rank of the domain. We will assume $n \geq 3$, as the cases where $n = 2$ are already
covered by the $B_n$ case above. There are then three groups $G = \text{Sp}_{2n}/\mu_2$, $G = \text{PU}_{n,n}$, and $G = \text{SO}_{n,n}/\mu_2$, together with the exceptional group $E_{7,3}/\mu_2$ which only occurs when $n = 3$. In the first case $G$ is split, $H$ is the center of $J$, and the orthogonal representations $W_{ij}$ all have dimension one. In the second case, $G$ is quasi-split, $H = U_1^n/\mu_2$, and the orthogonal representations $W_{ij}$ all have dimension 2. In the third case, $H = (SU_2)^n/\Delta\mu_2$ and the orthogonal representations $W_{ij}$ all have dimension four. In the exceptional case, $H$ is the compact form $\text{Spin}_8$ of $\text{Spin}_{4,4}$ and the orthogonal representations $W, W'$ and $W''$ all have dimension 8.

8. $A_2$ case: Minuscule embeddings of $\text{SL}_3$

In this section, our objective is to describe the minuscule embeddings of $\text{SL}_3$ into split, simple groups $G$ of adjoint type over $k$. (We will use this description to give a classification of groups with a relative root system of type $G_2$.) If we have such an embedding, with centralizer $H$, we obtain a $\mu_3$-decomposition of the Lie algebra of $G$ as in (2.3):

$$\mathfrak{g} = \mathfrak{sl}_3 + \mathfrak{h} + V_3 \otimes V + V'_3 \otimes V''.$$

Restricting the minuscule embedding $\text{SL}_3 \rightarrow G$ to an embedded $\text{SL}_2 \hookrightarrow \text{SL}_3$ that is itself minuscule provides a minuscule embedding $\text{SL}_2 \hookrightarrow G$. Indeed, the restriction of the standard representation $V_3$ (and its dual) is the direct sum of the standard representation of $\text{SL}_2$ and the trivial representation, and the restriction of the adjoint representation $\mathfrak{sl}_3$ is the direct sum of the adjoint representation $\mathfrak{sl}_2$, two copies of the standard representation and one copy of the trivial representation. Hence the decomposition of $\mathfrak{g}$ under $\text{SL}_2$ is as in (3.4):

$$\mathfrak{g} = \mathfrak{sl}_2 + \mathfrak{m} + V_2 \otimes W.$$

Let $S$ be the centralizer of $\text{SL}_2$ in $\text{SL}_3$, which is a split torus of dimension one and has character group isomorphic to $\mathbb{Z}$. We can fix an isomorphism with $\mathbb{G}_m$, so that the characters of $S$ on $V_3$ are 1, 1, −2 and the characters of $S$ on $V_3'$ are −1, −1, 2. It follows that the characters of $S$ on $\mathfrak{sl}_3 \subset V_3 \otimes V_3'$ are 3, 3, 0, 0, 0, 0, −3, −3. The centralizer of $S$ in $\text{SL}_3$ is isomorphic to $\text{GL}_2$. Since $S$ centralizes $\text{SL}_2$, it is contained in the stabilizer $M$ of the minuscule embedding of $\text{SL}_2$ and acts on the two representations $\mathfrak{m}$ and $W$.

From the decomposition of $\mathfrak{g}$ into representations of $\text{SL}_3 \times H$ we see that the only characters of $S$ that appear (with multiplicities) in $\mathfrak{g}$ are $\{-3, -2, -1, 0, 1, 2, 3\}$. Since the intersection of $S$ and $\text{SL}_3$ is the center $\mu_2$, the torus $S$ acts by even characters on $\mathfrak{m}$ and by odd characters on $W$. Therefore $S$ acts by the three characters −2, 0, 2 on $\mathfrak{m}$, and by the four characters −3, −1, 1, 3 on $W$. The characters 3 and −3 only appear in the summand $\mathfrak{sl}_3$, so each appears with multiplicity 2 in $\mathfrak{g}$. Hence the characters 3 and −3 each appear in $W$ with multiplicity one, and the characters 1 and −1 each appear with multiplicity equal to dim $V$. The multiplicities of the characters 2 and −2 in the representation of $S$ on $\mathfrak{m}$ are also equal to dim $V$. By counting dimensions, this gives the multiplicity of the trivial character of $S$ in $\mathfrak{m}$, and we see that the centralizer of $S$ in $M$ is isomorphic to $S.H$. 

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Since $S$ has only three weights on $m$, the inclusion $S \to M$ is a minuscule co-character. The centralizer $S.H$ of $S$ is a Levi subgroup of a parabolic in $M$, whose abelian unipotent radical is isomorphic to the representation $V$ of $H$. Moreover, the action of this centralizer of $S$ on the symplectic representation $W$ of $M$ stabilizes two lines, where $S$ acts by the characters 3 and $-3$ and $H$ acts trivially. Since we know $M$ and the representations $m$ and $W$ from our previous classification of minuscule embeddings of $\text{SL}_2$, these conditions determine the co-character $\mu : S \to G$ up to conjugacy (when it exists). If we fix a maximal split torus $T$ in $G$ and a set of positive roots $\alpha_i$, there is no loss of generality in assuming that $\mu$ is a dominant co-character of $T$, and we can describe $\mu$ by giving the inner products $\langle \mu, \alpha_i \rangle$ for all $i$.

When $G$ is exceptional, the co-character $\mu$ has inner product 1 with a unique simple root $\alpha$, which has multiplicity 3 in the highest root, and inner product 0 with all other simple roots. When $G$ has type $D_4$, $\mu$ has inner product 1 with the three simple roots $\alpha_i$ which have multiplicity 1 in the highest root, and inner product 0 with the remaining simple root. When $G$ has type $B_n$ or $D_n$ for $n \geq 5$, $\mu$ has inner product 1 with the two simple roots $\alpha_1$ and $\alpha_3$ and inner product 0 with the remaining simple roots. See Figure 1 for an illustration. (When $G$ has type $A_{n-1}$, the minuscule $\text{SL}_3$ is the subgroup of $\text{PGL}_n$ stabilizing a subspace of dimension 3, and $\text{sl}_3 + \mathfrak{h} + V_3 \otimes V$ is a parabolic subalgebra of $\mathfrak{g}$. We ignore this degenerate case, cf. [9, Th. 4.8].)

![Figure 1: Dynkin diagrams with circles around those simple roots $\alpha$ such that $\langle \mu, \alpha \rangle \neq 0$.](image)

Having determined the co-character $\mu$, we obtain a seven term grading of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}(-3) + \mathfrak{g}(-2) + \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1) + \mathfrak{g}(2) + \mathfrak{g}(3)$$

(8.1)

where the summands are representations of the centralizer of $S$, which is isomorphic to $\mathfrak{(GL}_2 \times H)/\mu_3$. The summand $\mathfrak{g}(1)$ is isomorphic to the representation $V_2 \otimes V$, the summand $\mathfrak{g}(2)$ is isomorphic to $\text{det} \otimes V^\vee$, and the summand $\mathfrak{g}(3)$ is isomorphic to $V_2 \otimes \text{det}$ as a representation of $\mathfrak{gl}_2$ (tensor the trivial representation of $H$). The centralizer of $\mu_3 \hookrightarrow S$ is then isomorphic to $(\text{SL}_3 \times H)/\mu_3$, and this gives a minuscule embedding of $\text{SL}_3$. We describe the centralizer $H$ and the representations $V$ and $V^\vee$ of $H$ in Table 2, cf. [5] and [9, Table 2].
Since the relevant node or nodes on the Dynkin diagram are stable under graph automorphisms, we find that the full stabilizer \( M \) of the minuscule embedding \( \text{SL}_3 \to G \) in \( \text{Aut}(G) \) is \( O_{2n-2} \times \text{GL}_1 \) for type \( D_{n+2} \), \( (\text{GL}_1)^{N=1} \) for type \( D_4 \), and \( (\text{SL}_3 \times \text{SL}_3)/\mu_3 \) for type \( E_6 \).

9. \( A_2 \) case: invariant tensors

We now approximately follow the path of §4, except with a minuscule embedding \( \text{SL}_3 \to G \) as in the previous section. We assume here that \( G \) is of type \( F_4 \) or \( E_6 \). (The tiny cases where \( G \) has type \( G_2 \) or \( D_4 \) have similar outcomes but involve ad hoc arguments that we omit here.)

Let \( G' \) be the subgroup of \( G \) generated by \( H \) and the root subgroups \( G_{\pm \alpha} \) where \( \alpha \) is the unique simple root such that \( \langle \mu, \alpha \rangle \neq 0 \) as in Figure 1. It is semisimple. The coefficient of \( \alpha \) in the highest root of \( G' \) is 1, so \( \mu \) gives a 3-grading \( g' = g'(-1) \oplus g'(0) \oplus g'(1) \) such that \( g'(1) = V \) and \( g'(-1) \) is the dual of \( V \) as a representation of \( H \). (This can be seen by exactly the same deduction as the observation that \( g(1) = V_2 \otimes V \) in (8.1), appealing to [14, Th. 2].) The subalgebra \( g' \) is called the stock in [9].

By the same argument as in §4, \( k[V]^H = k[f] \) for some homogeneous \( f \). In case \( k = \mathbb{C} \), a routine calculation with weights shows that \( f \) has degree 3. As in [21, pp. 4767, 4768], one deduces that \( \text{deg } f = 3 \) in all cases. (The argument in [21] is uniform and relies on [42]. Alternatively, one can calculate by hand in each case.)

Looking from a different angle, the 3-grading shows that \( g'(1) \oplus g'(-1) \) is a Jordan pair, meaning that the quadratic maps \( Q_\epsilon : g'(\epsilon) \to \text{Hom}_k(g'(\epsilon), g'(\epsilon)) \) defined by

\[
Q_\epsilon(x)(y) := (\text{ad } x)^2 y \quad \text{for } x \in g'(\epsilon) \text{ and } y \in g'(\epsilon)
\]

for \( \epsilon = \pm 1 \) satisfy certain identities; see [43] for an extensive theory. This is the point of view of [6, esp. §2], [44], [45], and [46, Ch. 11]; it can be viewed in the context of the Tits-Kantor-Koecher construction of Lie algebras. (Yet another angle is pursued in [47], where the authors allow the representation \( g/sl_3 \) of \( \text{SL}_3 \) to include also copies of
\(\mathfrak{sl}_3\) in addition to copies of \(V_3\) and \(V'_3\), and use this to construct a structurable algebra from \(\mathfrak{g}\).

Given a Jordan algebra \(J\), one can construct from it a Jordan pair \((J, J)\), and the Jordan pair \(\mathfrak{g}'(1) \oplus \mathfrak{g}'(-1)\) is of this form, see [6, §14, esp. 14.31] or [9, Prop. 4.2]. In each case \(J\) is a cubic Jordan algebra. Specifically:

- For \(G\) of type \(E_8\), \(G'\) is of type \(E_7\) and \(J\) is a 27-dimensional exceptional Jordan algebra, sometimes called an *Albert algebra*.
- For \(G\) of type \(E_7\), \(G'\) is of type \(D_6\) and \(J\) is the Jordan algebra of 6-by-6 alternating matrices with norm the Pfaffian, as in [6, 14.19].
- For \(G\) of type \(E_6\), \(G'\) is of type \(A_5\) and \(J\) is the Jordan algebra of 3-by-3 matrices with norm the determinant, as in [6, 14.16].
- For \(G\) of type \(F_4\), \(G'\) is of type \(C_3\) and \(J\) is the Jordan algebra of 3-by-3 symmetric matrices with norm the determinant, as in [6, 14.17].

Alternatively, \(J\) is the Jordan algebra of 3-by-3 hermitian matrices with entries in a composition algebra \(C\) of dimension 8, 4, 2, or 1 respectively.

10. \(A_2\) case: \(k\)-forms and groups with relative root system of type \(G_2\)

We now describe \(k\)-forms of the groups appearing in the previous section. As in §6, we put a subscript 0 on the groups involved to indicate the split group.

The automorphism group \(H'_0\) of the Jordan algebra structure on \(V_0\) is the subgroup of \(H_0\) fixing the identity element \(e \in V\), see [6, 14.11] or [48, Th. 4]. Moreover, \(H_0\) has a central \(\mu_3\) that acts as scalars on \(V\). It follows that the stabilizer of the line \(ke\) in \(\mathbb{P}(V_0)\) is \(\mu_3 \times H'_0\). On the other hand, the \(H_0\)-orbit of \(ke\) is dense; it is the collection of lines \(kv\) such that \(f(v) \neq 0\). Therefore, the natural map \(H^1(k, \mu_3 \times H'_0) \to H^1(k, H_0)\) is surjective as in [36, 9.11] (cf. [49]), and twisting \(G_0\) by a cocycle with values in \(H_0\) amounts to twisting separately by a cocycle with values in \(H'_0\) and by a cocycle with values in \(\mu_3\). The latter twist does not affect the isomorphism class of the resulting \(H\) and therefore by Tits’s Witt-type theorem does not affect the isomorphism class of the resulting twist \(G\) of \(G_0\). In summary, the twists of \(G_0\) by a cocycle with values in \(H_0\) can be obtained by twists by cocycles with values in \(H'_0\). In particular, the twist \(V\) of \(V_0\) so obtained will be a Jordan algebra, and the generic norm on \(V\) will be a cubic form \(f\) invariant under \(H\).

Thus we can determine the groups \(G\) with a relative root system of type \(G_2\) such that the long roots have multiplicity one. We use a method similar to our determination of the groups with a relative root system of type \(BC_1\). Namely, the split torus in \(G\) together with the long root groups generate a minuscule \(\text{SL}_3 \to G\). Let \(G_0\) be the split inner form of \(G\) over \(k\), and let \(\text{SL}_3 \to G\) be a minuscule embedding as described in the previous section, associated to the co-character \(\mu\). Let \(M_0\) be the stabilizer of this embedding in \(\text{Aut}(G_0)\), so \(M_0\) has connected component the group \(H_0\) tabulated in Table 2. Since all minuscule embeddings of \(\text{SL}_3\) into \(G_0\) are conjugate over \(\overline{k}\) we may choose an isomorphism \(\phi : G_0 \to G\) over \(\overline{k}\) which is the identity on the embedded
This gives a cohomology class in $H^1(k, \text{M}_0(k))$, which determines the isomorphism class of $H$ and $G$. The question is whether we can find such a class so that the corresponding form $H$ of $H_0$ is anisotropic.

For these Jordan algebras, the following are equivalent by [8, 37.12, 38.3]:

1. The cubic form $f$ on $V$ represents zero.
2. The algebra $V$ has zero divisors.
3. $H$ is isotropic.

That is, $G$ will have relative root system of type $G_2$ if and only if $f$ does not represent zero, if and only if $V$ is a division algebra.

In the cases where $\dim V = 6$ or 15 (i.e., $G_0 = F_4$ or $E_7$), the equivalent conditions hold. This can be seen by Jordan algebra methods, as is done in [8, 37.12]. It can be seen also by Galois descent, because inner twists of $\text{SL}_3$ and $\text{SL}_6/\mu_2$ are isotropic.

In the remaining cases, anisotropic forms of $H$ exist. Specifically, for the case $\dim V = 9$, find a field $k$ and a central associative division $k$-algebra $A$ of dimension $3^2$. The algebra $V$ with underlying vector space $A$ and product $a \cdot b = \frac{4}{3}(ab + ba)$ where juxtaposition denotes the associative multiplication in $A$ is a Jordan algebra without zero divisors. Moreover, adjoining an indeterminate $t$ to $k$, there is an Albert algebra over $k(t)$ with no zero divisors, namely the “first Tits construction” denoted $J(A, t)$, compare [34, Prop. 3(B)].

Remark. In case $G_0 = E_6$, the group $M_0$ with identity component $H_0$ has two components, and the same reasoning applies for twisting by a cocycle with values in $M_0$. Twisting $V_0$ by a 1-cocycle with values in $M_0$ whose image in $H^1(k, M_0/H_0)$ is a quadratic field extension $K$ of $k$ gives a Jordan algebra with underlying vector space the $\tau$-symmetric elements of $(B, \tau)$ where $B$ is a central simple $K$-algebra and $\tau$ is a unitary involution on $B$ whose restriction to $K$ is the nontrivial $k$-algebra automorphism.

11. $D_4$ case: minuscule embeddings of $\text{Spin}_{4,4}$ and groups with relative root system of type $F_4$

Like the case of $G_2$, the groups with a relative root system of type $F_4$ are all exceptional. We obtain a minuscule embeddings of the long root subgroup $\text{Spin}_{4,4} \rightarrow G$, which in the split cases gives the following decomposition of $\mathfrak{g}$ as in (2.3):

$$\mathfrak{g} = \mathfrak{spin}_{4,4} + \mathfrak{h} + V_8 \otimes W + V'_8 \otimes W' + V''_8 \otimes W''.$$ (11.1)

Table 3 lists the centralizers $H$ and the dimensions of the three orthogonal representations $W$, $W'$, and $W''$ of $H$ which occur, cf. [5]. Note that there is a copy of the symmetric group $\Sigma$ on 3 letters in $G$ normalizing $J$ and acting as outer automorphisms on $J$. (This is true in the case where $G = F_4$ as in [8, 23.13, 26.5, 38.7], and the other embeddings $J \rightarrow G$ factor through an $F_4$ subgroup.) So $\Sigma$ acts on $JH$ and permutes the three $V_8 \otimes W$ summands; in this sense the three summands are interchangeable.

We omit a “top down” analysis reconstructing the algebraic structure on $W$, although it is natural to think of it as a symmetric composition algebra as defined in [8, §34].
| $G$   | $H$                 | $\dim W$ |
|-------|---------------------|----------|
| $F_4$ | $\mu_2 \times \mu_2$ | 1        |
| $E_6/\mu_3$ | $(\mathbb{G}_m)^2$ | 2        |
| $E_7/\mu_2$ | $\left(\text{SL}_2\right)^3/\Delta \mu_2$ | 4        |
| $E_8$ | $\text{Spin}_{4,4}$ | 8        |

Table 3: For minuscule Spin$_{4,4}$ in $G$, the centralizer $H$ and its representation $W$

Alternatively, the additive decomposition (11.1) is familiar from the theory of structurable algebras as in [39, p. 1869, (c)], which takes a tensor product $C_1 \otimes C_2$ with $C_1$ an octonion algebra and $C_2$ any composition algebra and constructs from it a Lie algebra $\mathfrak{g}$ with the same decomposition (11.1). For a different view, see [50], [51]. In such ways, one can reconstruct Table 3 “from the ground up”.

In the non-split case, the group $H$ will be anisotropic if and only if the quadratic norm form on $W$ does not represent zero over $k$, or equivalently, when the composition algebra is a division algebra. This will occur for the split group $F_4$, the quasi-split group $^{2}E_6$, as well as certain inner forms of $E_7$ and $E_8$. In these cases, the short root spaces have dimension 1, 2, 4, and 8 respectively as in Table 3.

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