Uniform asymptotics for discrete orthogonal polynomials on infinite nodes with an accumulation point

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Abstract

In this paper, we develop the Riemann-Hilbert method to study the asymptotics of discrete orthogonal polynomials on infinite nodes with an accumulation point. To illustrate our method, we consider the Tricomi-Carlitz polynomials $f_{\alpha}(z)$ where $\alpha$ is a positive parameter. Uniform Plancherel-Rotach type asymptotic formulas are obtained in the entire complex plane including a neighborhood of the origin, and our results agree with the ones obtained earlier in [SIAM J. Math. Anal 25 (1994)] and [Proc. Amer. Math. Soc. 138 (2010)].

Keywords: Uniform asymptotics; Tricomi-Carlitz polynomials; Riemann-Hilbert method; Airy function.

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1 Introduction

Discrete orthogonal polynomials arise in many fields of mathematical physics, such as random matrix theory and quantum mechanics. There has been a considerable amount of interest in the asymptotic analysis of these orthogonal polynomials, and various methods have been developed for this purpose; see [24] and [28].

In 2007, Baik et al. [1] studied the asymptotics of discrete orthogonal polynomials with respect to a general weight function by using the Riemann-Hilbert approach. The starting point of their investigation is the interpolation problem (IP) for discrete orthogonal polynomials, introduced in Borodin and Boyarchenko [5]. The IP is then turned into a Riemann-Hilbert problem (RHP), and the Deift-Zhou method for oscillating RHP applies; see, e.g., [8, 9, 10]. The work of Baik et al. furnishes an important step forward of the method of Deift and Zhou.

Much attention has been attracted lately. For example, Wong and coworkers considered cases with finite nodes [7, 19, 20], and infinite nodes [23, 27] regularly distributed. A common feature in these work is termed global asymptotics, with global referring to the domains of uniformity.

Quite recently, Bleher and Liechty [3, 4] made a major modification to the method in the treatment of the so-called band-saturated region endpoints, when they were considering the large-$N$ asymptotics of a system of discrete orthogonal polynomials with respect to the varying exponential weight $e^{-NV(x)}$ on the regular infinite lattice of mesh $1/N$, where $V(x)$ is a real analytic function with sufficient growth at infinity. Here regular infinite lattice means that the infinite nodes are equally spaced.

In this paper, we study the asymptotics of discrete orthogonal polynomials on infinite nodes with an accumulation point. We illustrate our method by concentrating on the Tricomi-Carlitz polynomials. It is worth noting that there are other polynomials share such a structure. For example, there is a class of sieved Pollaczek polynomials defined by a second-order difference equation; see [26]. A significant fact is that the corresponding orthogonal measure consists of an absolutely continuous part, and a discrete part having infinite many mass points with an accumulation point.

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The Tricomi-Carlitz polynomials are also of interest. Initially, Tricomi \[25\] introduced a class of non-orthogonal polynomials \(t_n^{(\alpha)}(x)\), related to the Laguerre polynomials via \(t_n^{(\alpha)}(x) = (-1)^n L_n^{(x-\alpha-n)}(x)\). They are explicitly given by
\[
t_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!}, \quad n = 0, 1, 2, \ldots ;
\]
(1.1)
cf. \[17\] \[21\]. We note that each \(t_n^{(\alpha)}(x)\) is of degree \([\alpha]\), and the polynomials satisfy the following recurrence relation
\[(n + 1)t_{n+1}^{(\alpha)}(x) - (n + \alpha)t_n^{(\alpha)}(x) + x t_n^{(\alpha)}(x) = 0, \quad n \geq 1,
\]
(1.2)
with initial values \(t_0^{(\alpha)}(x) = 1\) and \(t_1^{(\alpha)}(x) = \alpha\). Carlitz \[6\] revisited these polynomials, and found that if one set
\[f_n^{(\alpha)}(x) = x^n t_n^{(\alpha)}(x^2),
\]
(1.3)
then \(f_n^{(\alpha)}(x)\) possess the following orthogonality
\[
\int_{-\infty}^{\infty} f_m^{(\alpha)}(x) f_n^{(\alpha)}(x) d\psi^{(\alpha)}(x) = h_n \delta_{mn}, \quad h_n = \frac{2e^{\alpha}}{(n + \alpha)n!},
\]
(1.4)
where \(\alpha\) is a positive number, \(\psi^{(\alpha)}(x)\) is the step function with jumps
\[
d\psi^{(\alpha)}(x) = \frac{(k + \alpha)^{k-1} e^{-k}}{k!} \quad \text{at} \quad x = \pm x_k,
\]
(1.5)
and the nodes \(x_k = (k + \alpha)^{-1/2}\) for \(k = 0, 1, 2, \ldots\). It is readily verified that the Tricomi-Carlitz polynomials satisfy the recurrence relation
\[(n + 1)f_{n+1}^{(\alpha)}(x) - (n + \alpha)x f_n^{(\alpha)}(x) + f_n^{(\alpha)}(x) = 0, \quad n \geq 1,
\]
(1.6)
with initial values \(f_0^{(\alpha)}(x) = 1\), and \(f_1^{(\alpha)}(x) = \alpha x\). From (1.6) we see that a symmetry relation holds, namely,
\[f_n^{(\alpha)}(z) = (-1)^n f_n^{(\alpha)}(-z).
\]
(1.7)
Also, if we denote the monic polynomials by
\[\pi_n(z) := f_n^{(\alpha)}(x)/\gamma_n,
\]
(1.8)
then the leading coefficient of \(f_n^{(\alpha)}(x)\) is
\[\gamma_n = \prod_{k=0}^{n-1} \frac{k + \alpha}{k + 1} = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha) \Gamma(n + 1)}.
\]
(1.9)
Moreover, the Tricomi-Carlitz polynomials are also related to the random walk polynomials \(r_n(x, \alpha)\), which were discovered by Karlin and McGregor in \[14\] to the study of a birth and death process. For more information on orthogonal polynomials, we refer to \[24\] and \[2\].

Asymptotic behavior of these polynomials was first investigated by Goh and Wimp in \[12\] for \(f_n^{(\alpha)}(y/\sqrt{\alpha})\), and in \[13\] for \(f_n^{(\alpha)}(y/\sqrt{n})\). Later, López and Temme \[21\] took \(f_n^{(\alpha)}(x)\) as an example to approximate polynomials in terms of the Hermite polynomials. The present paper is also inspired by the work of Lee and Wong. In \[17\], Lee and Wong derived an asymptotic expansion for \(f_n^{(\alpha)}(t/\sqrt{\nu})\) by using the difference equation method, which holds for \(t\) in \([0, \infty)\), where \(\nu = n + 2\alpha - 1/2\). In another paper \[18\], an alternative integral method is used to derive the expansion, along with asymptotic formulas for the extreme zeros. In both treatments they obtained the uniform asymptotics, in terms of the Airy function, at and around the turning point \(t = 2\).

It is worth noting that the previous uniform results are obtained on the real line, while an advantage of the Riemann-Hilbert approach lies in that the uniform asymptotic approximations can be obtained in overlapping domains covering the whole complex plane. In the present case, the nodes are not regularly distributed, with
the origin being the accumulation point. Even worse, the mass in (1.5) also shows a singularity at the origin. Hence, it is of interest to see the influence of these singularities on the asymptotic behavior at the origin.

The objective of this paper is to derive the uniform asymptotics of the Tricomi-Carlitz polynomials, based only on the weight function, and using several of the techniques developed by Baik et al. and Bleher and Liechty. The rest of this paper is arranged as follows: In Section 2, we state our main results, and introduce the basic interpolation problem. In Section 3, we will consider the transformation, which converts the interpolation problem into an equivalent Riemann-Hilbert problem (RHP). In Section 4, we give the matrix transformation $V$ that normalizes the RHP for $R(z)$ presented in Section 3 by using a function $g(z)$, which is related to the logarithmic potential of the equilibrium measure. Several auxiliary functions, such as the $\phi$-functions and $D$-functions, are also studied in Section 4. Then, we factorize the jump matrix in the RHP for $V(z)$ and construct the global parametrix $N(z)$ in Section 5. In Section 6, we study local parametrices and the last transformation $V \to S$. The proofs of Theorem 2.1 and Theorem 2.2 are presented in Section 7 and 8, and we compare our formulas with previous results in Section 9.

## 2 Statement of Results

It is well-known that the zero distribution plays an important role in the asymptotic analysis of the polynomials; see [10] and [11]. Note that the asymptotic zero distribution of the Tricomi-Carlitz polynomials is the Dirac point mass at zero; see Goh and Wimp [12]. In the present paper, we study the large-$n$ asymptotics of the rescaled Tricomi-Carlitz polynomials $f_n^{(a)}(n^{-1/2}z)$, of which the density function has already been given as

$$
\psi(x) = \begin{cases} 
\frac{1}{\pi} \left( \frac{4 \arctan(|x|/\sqrt{4 - x^2}) - \sqrt{4 - x^2}}{|x|^3} \right), & |x| \leq 2, \\
\frac{2}{|x|^3}, & |x| > 2;
\end{cases}
$$

(2.1)

see [13] [16].

To state the asymptotic behavior of the rescaled polynomials, we need to introduce some notations. Let the $g$-function be the logarithmic potential defined by

$$
g(z) := \int_{-\infty}^{\infty} \log(z - s)\psi(s)ds \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R},
$$

(2.2)

with the branch chosen such that $\arg(z - s) \in (-\pi, \pi)$, and the so-called $\phi$-function be given by

$$
\phi(z) := l/2 - g(z) \quad \text{for } z \in \mathbb{C}_\pm,
$$

(2.3)

where $l := 2 \int_{-\infty}^{\infty} \log|2 - s|\psi(s)ds$ is the Lagrange multiplier; cf. [13,10] below. Also, we introduce the auxiliary function

$$
\tilde{\phi}(z) := \int_{\gamma} \left( -g'(s) + \frac{2\pi i}{s^3} \right) ds \quad \text{for } z \in C_\pm,
$$

(2.4)

where the path of integration lies entirely in the regions $z \in \mathbb{C} \setminus (-\infty, 2]$ except for the initial point. In Section 6, we will show that the function

$$
\tilde{f}_n(z) := \left( -\frac{3}{2} \tilde{\phi}(z) \right)^{2/3}
$$

(2.5)

is analytic in a neighborhood of $z = 2$. The last function we need is

$$
D(z) := \frac{\Gamma(\alpha - n/z^2)}{\sqrt{2\pi e^{n/z^2}}} \left( -\frac{n}{2\pi} \right)^{n/z^2 - \alpha + 1/2} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R},
$$

(2.6)

where the branch is chosen such that $-1/z^2 = e^{\pm \pi i}/z^2$ for $z \in \mathbb{C}_\pm$, with $\arg z \in (-\pi, \pi)$.

Now we are ready to present our main results. In view of the symmetries [14,7] and $f_n^{(a)}(z) = f_n^{(a)}(\bar{z})$, we only need to present the asymptotic formulas for $\pi_n(n^{-1/2}z)$ in the first quadrant of the complex plane. The asymptotic formulas are stated in the five closed regions $A_\delta$, $B_\delta$, $C_{\delta,1}$, $C_{\delta,2}$ and $D_\delta$ depicted in Figure 1.
Figure 1: Asymptotic regions for $\pi_n(n^{-1/2}z)$ in the first quadrant.

**Theorem 2.1.** Let $\alpha > 0$ and $k_n = \sqrt{n/\alpha} + \delta$. Then there exists $\delta_0 > 0$ such that for all $0 < \varepsilon < \delta \leq \delta_0$, the following holds (see Figure 1): 

(a) For $z$ in the outside region $A_\delta \setminus U(0, \varepsilon)$, where $U(0, \varepsilon)$ is the disk of radius $\varepsilon$, centered at the origin:

$$
\pi_n(n^{-1/2}z) = \frac{\Gamma(\alpha)e^{n/2}}{\sqrt{2\pi n^{\alpha/2+1/2}}} D^{-1}(z)(z^2 - 4)^{-1/4} \times \left\{ \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} e^{-n\phi(z) - \alpha \pi i + \pi i/2} + \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} e^{n\phi(z) + \alpha \pi i} \right\} (1 + O(1/n)).
$$

(b) For $z$ in the region $B_\delta \setminus U(0, \varepsilon)$:

$$
\pi_n(n^{-1/2}z) = \frac{\Gamma(\alpha)e^{n/2}}{\sqrt{2\pi n^{\alpha/2+1/2}}} (z^2 - 4)^{-1/4} \times \left\{ \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} e^{-n\phi(z) - \alpha \pi i + \pi i/2} + \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} e^{n\phi(z) + \alpha \pi i} \right\} (1 + O(1/n)) + O(e^{-nRe \phi/n}).
$$

(c) For $z$ in the Airy region $C_{\delta,1} \cup C_{\delta,2}$:

$$
\pi_n(n^{-1/2}z) = \frac{\Gamma(\alpha)e^{n/2}}{\sqrt{2\pi n^{\alpha/2+1/2}}} [A(z, n) (1 + O(1/n)) + B(z, n) (1 + O(1/n))],
$$

where

$$
A(z, n) = \left[ \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} - \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} \right] (z^2 - 4)^{-1/4} (\tilde{f}_n(z))^{-1/4}
$$

and

$$
B(z, n) = \left[ \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} + \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^{2n-1/2} \right] (z^2 - 4)^{-1/4} (\tilde{f}_n(z))^{1/4}
$$

Theorem 2.1.
Figure 2: The contour near $z = 0$.

(d) For $z$ in the region $D_{\delta}$:

$$
\pi_n(n^{-1/2}) = \frac{\Gamma(\alpha)e^{n/2}}{\sqrt{2\pi n^{\alpha/2 - 1/2}}} \times \left[ D^{-1}(z)(z^2 - 4)^{-1/4} e^{-\alpha\phi(z) - \frac{\alpha}{2} + \frac{1}{2}} \right].
$$

(2.12)

In the disk $U(0, \varepsilon)$ of radius $\varepsilon > 0$, centered at the origin, we have the following uniform asymptotic approximation:

**Theorem 2.2.** For $z \in U(0, \varepsilon) \cap \mathbb{C}_+$; see Figure 2, it holds

$$
\pi_n(n^{-1/2}) = \frac{\Gamma(\alpha)}{\sqrt{2\pi n^{\alpha/2 - 1/2}}} \times \left[ D^{-1}(z)(z^2 - 4)^{-1/4} e^{-\alpha\phi(z) - \frac{\alpha}{2} + \frac{1}{2}} \right].
$$

(2.13)

where branches are chosen such that \( \arg(z + 2), \arg(2 - z) \) and $\phi(z) = z + \sqrt{z^2 - 1}$ is a analytic function in $\mathbb{C} \\setminus \{-1, 1\}$ and behaves like $2z$ at infinity; see (5.12). The formula for $z \in U(0, \varepsilon) \cap \mathbb{C}_-$ is obtained from (2.13) by taking complex conjugate.

To derive the main results, we begin with the basic IP for the Tricomi-Carlitz polynomials. Following [1], one can formulate the IP for a $2 \times 2$ matrix-value function $Y(z)$ with the properties:

(Y1) $Y(z)$ is analytic in $\mathbb{C} \setminus \{0, \pm x_0, \pm x_1, \ldots, \pm x_k, \ldots\}$;

(Y2) at each $\pm x_k$, $k = 0, 1, \ldots$, $Y(z)$ has a simple pole, and satisfies

$$
\text{Res}_{z = \pm x_k} Y(z) = \lim_{z \to \pm x_k} Y(z) \begin{pmatrix} 0 & w_d(z) \\ 0 & 0 \end{pmatrix},
$$

(2.14)

where

$$
w_d(z) = \frac{(1/z^2)^{1/2} - \alpha}{\Gamma(1/z^2 + 1 - \alpha)} e^{-1/z^2 + \alpha}
$$

(2.15)

and the branch is chosen such that $w_d(z)$ is analytic in $\mathbb{C} \setminus i\mathbb{R}$ and takes positive values for $\mathbb{R} \setminus \{0\}$;

(Y3) as $z \to \infty$,

$$
Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};
$$

(2.16)
(Yₙ) \( Y(z) \) has the following behavior as \( z \to 0 \),
\[
Y(z) = O \left( \frac{1}{\log |z|} \right), \quad z \not\in \left[ -1/\sqrt{\alpha}, 1/\sqrt{\alpha} \right]. \tag{2.17}
\]

By the well-known theorem of Fokas, Its and Kitaev [11], we have

**Theorem 2.3.** The unique solution of the IP for \( Y \) is given by
\[
Y(z) = \begin{pmatrix}
\pi_n(z) \\
\gamma_n^{-1} h_n^{-1} \pi_n^{-1}(z) \\
\gamma_n^{-2} h_n^{-1} \pi_n^{-2}(z)
\end{pmatrix}
\begin{pmatrix}
\sum_{k=0}^{\infty} \frac{\pi_n(x_k) w_d(x_k)}{z - x_k} \\
\sum_{k=0}^{\infty} \frac{\pi_n(-x_k) w_d(-x_k)}{z + x_k} \\
\sum_{k=0}^{\infty} \frac{\pi_n(-x_k) w_d(-x_k)}{z + x_k}
\end{pmatrix}, \tag{2.18}
\]

where \( \pi_n(z) \) is the monic Tricomi-Carlitz polynomials of degree \( n \), \( \gamma_n \) and \( h_n \) are defined in (1.9) and (1.4), respectively.

## 3 Riemann-Hilbert problem

In this section, we will make a sequence of transformations \( Y \to U \to R \), to convert the basic IP into a RHP. The first transformation is the following rescaling of variable
\[
U(z) := n^{\sigma_3/2} Y(n^{-1/2} z), \tag{3.1}
\]
where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the Pauli matrix. Let \( X \) denote the set defined by
\[
X := \{ \pm X_k \}_{k=0}^{\infty}, \quad \text{where } X_k = n^{1/2} x_k. \tag{3.2}
\]
The \( \pm X_k \)'s are called nodes, and they all lie in the interval \( (-\sqrt{n/\alpha}, \sqrt{n/\alpha}) \). It is readily seen that at each node \( \pm X_k \)
\[
\text{Res}_{z = \pm X_k} U(z) = \lim_{z \to \pm X_k} U(z) \begin{pmatrix} 0 & w(z) \\ 0 & 0 \end{pmatrix}, \tag{3.3}
\]
where
\[
w(z) := n^{1/2} w_d(n^{-1/2} z), \tag{3.4}
\]
and as \( z \to \infty \),
\[
U(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{3.5}
\]

Define
\[
\Pi(z) := \sin \theta(z)/\gamma(z), \tag{3.6}
\]
where
\[
\theta(z) := n\pi/z^2 - \pi\alpha \quad \text{and} \quad \gamma(z) := -2n\pi/z^3. \tag{3.7}
\]
Note that for each \( X_k \in X \)
\[
\Pi(\pm X_k) = 0 \quad \text{and} \quad \frac{[\sin \theta(\pm X_k)]'}{\gamma(\pm X_k)} = (-1)^k. \tag{3.8}
\]
Moreover, we introduce the upper triangular matrices
\[
U_k(z) := \begin{pmatrix} 1 & -\frac{w(z)}{\Pi(z)} e^{\pm i\theta(z)} \\ 0 & 1 \end{pmatrix}, \tag{3.9}
\]
For Proof.

Proposition 3.1. Define \( (3.10) \). To reduce the IP to a RHP, we shall employ an idea of Bleher and Liechty [3, 4] with some modifications.

\( X \) has no pole at \( l \), \( \Omega = \Omega \). Note that here we choose the contour like \( \Sigma \) and \( \Omega \). Similarly, we also have \( \text{Res} \). Consider any \( X \) and the lower triangular matrices \( \delta \).

\[ \begin{pmatrix} 1 & 0 \\ \frac{1}{(z_m(z))^2}e^{\pm i\theta(z)} & 1 \end{pmatrix} \]. (3.10)

Let \( \delta \) be a small positive number and \( k_n = \sqrt{n/\alpha} + \delta \), we divide the complex plane into nine parts: \( \Omega = \Omega_{t,+} \cup \Omega_{t,-} \cup \Omega_{r,+} \cup \Omega_{r,-} \cup \Omega_{\infty} \) by the contour \( \Sigma = (-k_n, k_n) \cup \Sigma_{t,+} \cup \Sigma_{t,-} \cup \Sigma_{r,+} \cup \Sigma_{r,-} \); see Figure 3. Note that here we choose the contour like \( \Sigma_{t,\pm} \) and \( \Sigma_{r,\pm} \) so as to avoid the zeros of \( \Pi(z) \) and \( w(z) \); see (3.9) and (3.10). To reduce the IP to a RHP, we shall employ an idea of Bleher and Liechty [3, 4] with some modifications. Define

\[ R(z) := U(z) \times \begin{cases} U_+(z), & z \in \Omega_{t,+} \cup \Omega_{t,-}, \\ U_-(z), & z \in \Omega_{t,-} \cup \Omega_{r,+}, \\ U_+^\lambda(z), & z \in \Omega_{t,+} \cup \Omega_{r,-}, \\ U_-^\lambda(z), & z \in \Omega_{t,-} \cup \Omega_{r,+}, \\ 1, & z \in \Omega_{\infty}. \end{cases} \] (3.11)

**Proposition 3.1.** For each \( \pm X_k \in X \), the singularity of \( R(z) \) at \( \pm X_k \) is removable; that is, \( \text{Res}_{z=\pm X_k} R(z) = 0 \).

**Proof.** For \( z \in \Omega_{r,\pm} \), it follows from (3.11) that

\[ R_{11}(z) = U_{11}(z), \quad R_{12}(z) = U_{12}(z) - U_{11}(z) \frac{w(z)}{\Pi(z)} e^{\pm i\theta(z)}. \] (3.12)

Consider any \( X_k \in (0, 2) \). By (3.8) and (3.12), the residue of \( R_{12}(z) \) at \( X_k \) is given by

\[ \text{Res}_{z=X_k} R_{12}(z) = w(X_k) U_{11}(X_k) - U_{11}(X_k) \frac{w(X_k)}{(-1)^k} (-1)^k = 0. \] (3.13)

Similarly, we also have \( \text{Res}_{z=X_k} R_{22}(z) = 0 \), thus

\[ \text{Res}_{z=X_k} R(z) = 0 \quad \text{for} \quad X_k \in (0, 2). \] (3.14)

On the other hand, it is readily seen that for \( z \in \Omega_{r,\pm} \),

\[ R_{11}(z) = U_{11}(z)\Pi(z)^{-1} - w(z)^{-1}U_{12}(z)e^{\pm i\theta(z)}, \quad R_{12}(z) = U_{12}(z)\Pi(z). \] (3.15)

For \( X_k \in (2, k_n) \), since the pole of the entry \( R_{12}(z) \) at \( X_k \) is canceled by the zero of the function \( \Pi(z) \), \( R_{12}(z) \) has no pole at \( X_k \). Moreover, from (3.8) and (3.15) we have

\[ \text{Res}_{z=X_k} R_{11}(z) = (-1)^k U_{11}(X_k) - \frac{1}{w(X_k)} U_{11}(X_k) w(X_k) (-1)^k = 0. \] (3.16)
Similarly, it can be shown that $\text{Res}_{z=X_k} R_{22}(z) = 0$, thus,

$$
\text{Res}_{z=X_k} R(z) = 0, \quad X_k \in \{2, k_n\}.
$$

(3.17)

In the same way, we obtain that for each $-X_k \in X$

$$
\text{Res}_{z=-X_k} R(z) = 0,
$$

(3.18)

hence, $R(z)$ has no pole at $\pm X_k \in X$.

Note that this transformation makes $R_+ (z)$ and $R_- (z)$ continuous on the interval $(-k_n, k_n)$. As a consequence, we have created several jump discontinuities on the contour $\Sigma$ in the complex plane. It is easily verified that $R(z)$ is a solution of the following RHP:

(R1) $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$;

(R2) for $z \in \Sigma$, $R(z)$ satisfies

$$
R_+(z) = R_-(z) J_R(z),
$$

where for $z$ on the real line

$$
J_R(z) = \begin{cases}
1 & , z \in (-k_n, -2), \\
-2i\gamma(z)/w(z) & , z \in (-2, 0), \\
1 & , z \in (0, 2), \\
2i\gamma(z)/w(z) & , z \in (2, k_n),
\end{cases}
$$

(3.19)

for $z \in \Sigma_{\ell, \pm} \cup \Sigma_{\ell, \pm}^\Delta$

$$
J_R(z) = \begin{cases}
U_+(z)^{-1}, & z \in \Sigma_{\ell, +}, \\
U_-(z), & z \in \Sigma_{\ell, -}, \\
U_+^\Delta(z)^{-1}, & z \in \Sigma_{\ell, +}^\Delta, \\
U_-^\Delta(z), & z \in \Sigma_{\ell, -}^\Delta,
\end{cases}
$$

(3.20)

for $z \in \Sigma_{r, \pm} \cup \Sigma_{r, \pm}^\Delta$

$$
J_R(z) = \begin{cases}
U_-(z)^{-1}, & z \in \Sigma_{r, +}, \\
U_+(z), & z \in \Sigma_{r, -}, \\
U_-^\Delta(z)^{-1}, & z \in \Sigma_{r, +}^\Delta, \\
U_+^\Delta(z), & z \in \Sigma_{r, -}^\Delta,
\end{cases}
$$

(3.21)

and

$$
J_R(z) = \begin{cases}
\Pi(z) & , z \in -2 + (0, \delta)i \cup 2 - (0, \delta)i, \\
- w(z)e^{i\theta(z)} & , z \in -2 + (0, \delta)i \cup 2 - (0, \delta)i, \\
2i\gamma(z)e^{-i\theta(z)} & , z \in -2 - (0, \delta)i \cup 2 + (0, \delta)i;
\end{cases}
$$

(3.22)
\((R_3)\) as \(z \to \infty\),

\[
R(z) = (I + O (1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};
\]

\((R_4)\) \(R(z)\) has the following behavior as \(z \to 0\),

\[
R(z) = O \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\log |z|}{\log |z|}.
\]

4 The transformations \(R \to T \to Q\)

To normalize the behavior at infinity, we introduce the first transformation

\[
T(z) := e^{-nl/2} \sigma_3 R(z) e^{-ng(z) + nl/2} \sigma_3.
\]

Now, we need some properties of the function \(g(z)\). In a similar manner as in [19], the derivative of \(g(z)\) can be calculated explicitly to give

\[
g'(z) = \frac{4}{z^3} \log(z + \sqrt{z^2 - 4}) + \frac{\sqrt{z^2 - 4}}{z^2} - \frac{4}{z^3} \log 2 \mp \frac{2\pi}{z^3}, \quad \text{for } z \in \mathbb{C}_\pm.
\]

Proposition 4.1. The function \(g'(z)\) satisfies

\[
g_+'(x) = \mp \pi i \psi(x), \quad x \in (-2, 2);
\]

\(g'_\pm(x)\) refer to the limiting values from the upper and lower half planes, respectively. Also, it follows from (4.12) and (4.13) that

\[
g_+ + g_-(x) \begin{cases} = l, & x \in (-2, 2), \\ > l, & x \in (-\infty, -2) \cup (2, +\infty); \end{cases}
\]

where \(l = 2 \int_{-\infty}^{\infty} \log |2 - s| \psi(s) ds\) is the Lagrange multiplier; cf. (4.40). Moreover, \(g(z)\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\), and

\[
g(z) = \log z + O(1/z) \quad \text{as } z \to \infty.
\]

It is readily seen that \(T(z)\) solves the following RHP:

\((T_1)\) \(Y(z)\) is analytic in \(\mathbb{C} \setminus \Sigma\);

\((Y_2)\) for \(z \in \Sigma\)

\[
T_+(z) = T_-(z) J_T(z),
\]

where

\[
J_T(z) = \begin{cases} e^{n(g_-(z) - l/2) \sigma_3} J_R(z) e^{-n(g_+(z) - l/2) \sigma_3} & \text{for } z \in \mathbb{R}, \\ e^{n(g(z) - l/2) \sigma_3} J_R(z) e^{-n(g(z) - l/2) \sigma_3} & \text{for } z \in \Sigma \setminus \mathbb{R}; \end{cases}
\]

\((T_3)\) as \(z \to \infty\),

\[
T(z) = I + O(1/z);
\]

\((T_4)\) \(T(z)\) has the behavior as \(z \to 0\),

\[
T(z) = O \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\log |z|}{\log |z|}.
\]
In particular, for $z \in \mathbb{R}$ we have

$$J_T(z) = \begin{cases}
(\begin{array}{cc}
0 & e^{-2n\pi i/z^2} \\
e^{-2n\pi i/z^2} & 0
\end{array})
, & z \in (-\infty, -k_n), \\
(\begin{array}{cc}
e^{2i(\theta(z)+\pi\alpha)}/2 & 0 \\
2i\gamma(z) & 0
\end{array})
, & z \in (-k_n, -2),
\end{cases}$$

$$= \begin{cases}
(\begin{array}{cc}
e^{-2n\phi_-(z)} & 2i\gamma(z)w(z) \\
0 & e^{-2n\phi_+(z)}
\end{array})
, & z \in (-2, 0),
\end{cases}$$

$$= \begin{cases}
(\begin{array}{cc}
e^{-2i(\theta(z)+\pi\alpha)}/2 & 0 \\
2i\gamma(z) & 0
\end{array})
, & z \in (0, 2),
\end{cases}$$

$$= \begin{cases}
(\begin{array}{cc}
e^{-2i(\theta(z)+\pi\alpha)}/2 & 0 \\
2i\gamma(z) & 0
\end{array})
, & z \in (2, k_n),
\end{cases}$$

$$= \begin{cases}
(\begin{array}{cc}
e^{-2n\pi i/z^2} & 0 \\
0 & e^{2n\pi i/z^2}
\end{array})
, & z \in (k_n, +\infty).
\end{cases}$$

Note that for the jump matrix on $(0, 2)$, we have the following factorization

$$\begin{pmatrix}
e^{-2n\phi_-(z)} & 2i\gamma(z)w(z) \\
0 & e^{-2n\phi_+(z)}
\end{pmatrix} = \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix},$$

$$= \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix},$$

which allows us to reduce the jump matrix $J_T(z)$ to a simpler one on the line segment $(0, 2)$. Also, note that for $z \in \Sigma_{r,+}$, the $(1,1)$ entry of $J_T(z)$ is $1 - e^{-2n\theta(z)}(1 - e^{-2i\theta(z)})$, and it behaves as $1 / 2i\gamma(z)$ for large $n$. Now we are in a position to introduce the second transformation $T \rightarrow Q$:

$$Q(z) := T(z) \times \begin{pmatrix}
-\mathcal{V}_\Delta(z), & z \in \Omega_{t,+}^\Delta \cup \Omega_{r,-}^\Delta \\
\mathcal{V}_\Delta(z), & z \in \Omega_{t,-}^\Delta \cup \Omega_{r,+}^\Delta \\
\mathcal{V}_-(z), & z \in \Omega_{t,+} \cup \Omega_{r,-} \\
I, & z \in \Omega_{\infty},
\end{pmatrix}$$

where

$$\mathcal{V}_\pm(z) := \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix},$$

$$\mathcal{V}_\Delta(z) := \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \\
0 & 1
\end{pmatrix}.$$

Note that for $z \in \Sigma \setminus (-2, 2)$, we expect that

$$J_Q(z) \sim I.$$

However, for $z \in \Sigma_{r}^\Delta$, since $1 - e^{-2i\theta(z)} = O(1)$ when $z = O(k_n)$, the jump matrix $J_Q(z) \neq I$ near the critical point $z = k_n$. The reason is that the interval $(2, k_n)$ is a saturated region, so special attention must be paid to the edges of the saturated regions (c.f. the contour $\Sigma_{r,\pm}^\Delta$ or $\Sigma_{r,\pm}^\Delta$; see [1] and [27].

Next, we introduce some auxiliary functions, known as the $D$-functions, which are analogous to the function used in [1] to remove the jumps of RHP near the edges of the saturated regions. Recall

$$D(z) := \frac{\Gamma(\alpha - n/z^2)}{e^{n/z^2} \sqrt{2\pi}} \left( -\frac{n}{z^2} \right)^{n/z^2 - \alpha + 1/2}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\tilde{D}(z) := \frac{\sqrt{2\pi}}{\Gamma(1 + n/z^2 - \alpha)e^{n/z^2}} \left( \frac{n}{z^2} \right)^{n/z^2 - \alpha + 1/2},$$

introduced in [1], and we let
where the branches are chosen as \(\arg z \in (-\pi, \pi)\) in (4.15), while in (4.14), \(-1/z^2 = e^{\pm \pi i}/z^2\) for \(z \in \mathbb{C}_\pm\). It is readily seen that the function \(\tilde{D}(z)\) is a non-zero analytic function on \(\mathbb{C} \setminus (-\infty, 0]\) such that
\[
\tilde{D}(z) = D(z)(1 - e^{\pm 2i\theta(z)}), \quad z \in \mathbb{C}_\pm.
\] (4.16)
Moreover, we have as \(n \to \infty\),
\[
\tilde{D}(z) = 1 + O(1/n),
\] (4.17)
holding uniformly for \(z\) in any compact set \(K \subset \mathbb{C} \setminus (-\infty, 0]\), and as \(z \to \infty\),
\[
D(z) = \frac{\Gamma(n)}{\sqrt{2\pi}} n^{1/2 - \alpha} (-z^2)^{n/2} + O(1/z).
\] (4.18)
The main difference between the function \(D(z)\) and that in [19], lies in the behavior as \(z\) goes to infinity, therefore another function is needed in order to normalize the behavior at infinity.

Define
\[
E(z) := \sqrt{\frac{2\pi}{\Gamma(n)}} (4 - z^2)^{1/2 - \alpha}, \quad z \in \mathbb{C} \setminus (-\infty, -2) \cup (2, +\infty),
\] (4.19)
where we always take the principal branches of \((2 - z)^{1/2 - \alpha}\) and \((z + 2)^{1/2 - \alpha}\). It is based on two observations. The first is that,
\[
D_+(z)E_+(z) = D_-(z)E_-(z) \times \begin{cases} e^{2n\pi i/z^2}, & z \in (2, +\infty), \\ e^{-2n\pi i/z^2}, & z \in (-\infty, -2), \end{cases}
\] (4.20)
which is exactly the jumps of \(J_T(z)\) for \(z \in (-\infty, -k_n) \cup (k_n, +\infty)\); see also (4.10), and \(D(z)E(z)\) tends to \(n^{1/2 - \alpha}\) as \(z \to \infty\). The second observation is the fact that
\[
[E_-(z)\tilde{D}_-(z)]^{-\sigma_3}J_T(z)[E_+(z)\tilde{D}_+(z)]^{\sigma_3} \sim I,
\] (4.21)
uniformly for \(z \in (-k_n, -2) \cup (2, k_n)\).

Similarly, we set
\[
\tilde{D}(z) := \sqrt{\frac{2\pi}{\Gamma(n)}} \left(\frac{n}{(-z)^2}\right)^{n/2 - \alpha + 1/2},
\] (4.22)
which is analytic in \(z \in \mathbb{C} \setminus [0, +\infty)\), with the branch taken as \(\arg(-z) \in (-\pi, \pi)\), and it satisfies
\[
\tilde{D}(z) = D(z)(1 - e^{\pm 2i\theta(z)}), \quad z \in \mathbb{C}_\pm.
\] (4.23)
Moreover,
\[
\tilde{E}(z) := \sqrt{\frac{2\pi}{\Gamma(n)}} (z^2 - 4)^{1/2 - \alpha}, \quad z \in \mathbb{C} \setminus (-\infty, 2),
\] (4.24)
and
\[
\tilde{E}(z) := \sqrt{\frac{2\pi}{\Gamma(n)}} (-z - 2)^{1/2 - \alpha}(2 - z)^{1/2 - \alpha}, \quad z \in \mathbb{C} \setminus (-2, +\infty).
\] (4.25)

For simplicity, we also bring in the notations
\[
w_0(z) := 2i\gamma(z)w(z)/E^2(z),
\] (4.26)
and
\[
\tilde{w}(z) := -2i\gamma(z)w(z)/\tilde{E}^2(z), \quad \tilde{w}(z) := -2i\gamma(z)w(z)/\tilde{E}^2(z).
\] (4.27)
As we have mentioned before, the density function \(\psi(x)\) attains its upper constraint at \(x = \pm 2\), the so-called band-saturated region endpoints; see (2.7). Furthermore, since \(\psi(x)\) is not differentiable at the point \(x = \pm 2\), the function \(\phi(z)\) is not analytic there, nor can we construct our local parametrix there (such as the Airy parametrix) by using \(\phi(z)\). Therefore, for our future analysis, a few more auxiliary functions are needed. To this aim, we resume the function \(\phi(z)\) in (2.7), and define
\[
\phi(z) := \int_{-2}^{z} \left(-g'(s) \pm \frac{2\pi i}{s^3}\right) ds \quad \text{for} \quad z \in \mathbb{C}_\pm,
\] (4.28)
which is analytic in \(\mathbb{C} \setminus [-2, +\infty)\). The functions \(\tilde{\phi}(z)\) and \(\hat{\phi}(z)\) will play an important role in our argument, and the following are some of their properties.
Proposition 4.2. With \( \phi(z) \) defined in (2.20), the following connection formulas between the \( \phi \)-function and the \( \tilde{\phi} \)-function (\( \hat{\phi} \)-function) hold:

\[
\tilde{\phi}(z) = \phi(z) \pm \frac{\pi i}{z^2}
\]

and

\[
\tilde{\phi}(z) = \phi(z) \mp \left( \frac{1}{z^2} - 1 \right) \pi i
\]

for \( z \in C_{\pm} \).

If \( x \in (2, +\infty) \), we have

\[
\text{Im} \tilde{\phi}(x) = 0, \quad \text{and} \quad \text{Re} \tilde{\phi}(x) < 0.
\]

Similarly, for \( x \in (-\infty, -2) \),

\[
\text{Im} \tilde{\phi}(x) = 0, \quad \text{and} \quad \text{Re} \tilde{\phi}(x) < 0.
\]

Moreover, for \( x \in (-2, 2) \), we have

\[
\text{Re} \phi(x \pm i\varepsilon) < 0.
\]

Proof. For \( z \in C_+ \), we have from (2.20) and (2.4)

\[
\tilde{\phi}(z) = \int_2^z \left( -g'(s) - \frac{2\pi i}{s^3} \right) ds = -g(z) + g_+(2) + \frac{\pi i}{z^2} - \frac{\pi i}{4} = \phi(z) + \frac{\pi i}{z^2},
\]

where in the last step we have made use of the result \( g_+(2) = l/2 \pm \pi i \int_2^{+\infty} \psi(s) ds = l/2 \pm \pi i/4 \).

Also, it follows that for \( z \in C_- \)

\[
\tilde{\phi}(z) = -g(z) + g_-(2) - \frac{\pi i}{z^2} + \frac{\pi i}{4} = \phi(z) - \frac{\pi i}{z^2},
\]

thus proving (4.29). For \( z \in C_+ \), (4.30) can be proven in a similar way.

Since by (4.29) \( \text{Im} \tilde{\phi}(z) = \text{Im} \{ \phi(z) \pm \pi i/z^2 \} \), from (2.3) we get \( \text{Im} \tilde{\phi}(x) = 0 \). Also, note that for \( x \in (2, +\infty) \) the upper constraint on the density \( \psi(x) \) implies

\[
\text{Re} \tilde{\phi}(x) = l/2 - \int_{-\infty}^{+\infty} \log|x-s|\psi(s) ds < 0.
\]

Similarly, we can prove (4.32).

On account of (4.3), we obtain for \( x \in (-2, 2) \)

\[
\text{Re} \phi(x \pm i\varepsilon) = -\text{Re} \int_2^x g_+(s) ds - \text{Re} \int_x^{x \pm i\varepsilon} g'(s) ds = -\text{Re} \int_x^{x \pm i\varepsilon} g'(s) ds = -\pi \varepsilon \psi(x) + O(\varepsilon^2) < 0.
\]

We conclude this section with a calculation of \( l \) given in (2.3). First we note that \( \tilde{\phi}(z) \) is analytic in \( C \setminus (-\infty, 2] \). Coupling (4.2) with (2.4) gives

\[
\tilde{\phi}(z) = \frac{2}{z^2} \log(z + \sqrt{z^2 - 4}) - \log(z + \sqrt{z^2 - 4}) + (1 - \frac{2}{z^2}) \log 2 + \frac{\sqrt{z^2 - 4}}{2z}
\]

for \( z \in C \setminus (-\infty, 2] \). In view of (2.4), we have

\[
l/2 - g(z) - \tilde{\phi}(z) \pm \frac{\pi i}{z^2} = 0 \quad \text{for} \quad z \in C_{\pm}.
\]

Now let \( z \to \infty \); on account of (2.2), (4.38) and (4.39), we obtain

\[
l = \lim_{z \to \infty} 2(\log(z) + \tilde{\phi}(z)) = 1.
\]
5 The transformation $Q \to V$ and the parametrix for the outside region

Using the functions introduced in Section 4, we take the third transformation $Q \to V$

$$V(z) := n^{(\alpha-1/2)\beta} Q(z) E(z)^{\alpha} \times \begin{cases} \tilde{D}(z) \sigma_3, & z \in \Omega_{t,\pm} \cup \Omega_{t,\pm}^\Delta, \\ \hat{D}(z) \sigma_3, & z \in \Omega_{r,\pm} \cup \Omega_{r,\pm}^\Delta, \\ D(z) \sigma_3, & z \in \Omega_\infty; \end{cases} \quad (5.1)$$

see [19] and [27] for similar transformations, and see Figure 3 for the regions.

**Proposition 5.1.** The matrix-valued function $V(z)$ has the following jumps on the contour $\Sigma$:

$$V_+(z) = V_-(z) J_V(z), \quad (5.2)$$

where

$$J_V(z) = \begin{cases} \begin{pmatrix} 0 & -w_0(z)/\tilde{D}(z) \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in (-2, 0), \\ \begin{pmatrix} 1 & 0 \\ \frac{D(z)}{\tilde{D}(z)} e^{2n\tilde{\phi}(z)} & 1 \end{pmatrix}, & z \in (-k_n, -2), \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{t,+}, \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{t,-}, \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{t,\pm}, \end{cases} \quad (5.3)$$

and

$$J_V(z) = \begin{cases} \begin{pmatrix} 0 & w_0(z)/\tilde{D}(z) \\ -\tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in (0, 2), \\ \begin{pmatrix} 1 & 0 \\ \tilde{D}^2(z)/w_0(z) & 1 \end{pmatrix}, & x \in (2, k_n), \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{r,+}, \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{r,-}, \\ \begin{pmatrix} 1 \frac{\tilde{D}(z)}{D(z)} e^{2n\tilde{\phi}(z)} & 1 \\ \tilde{D}^2(z)/w_0(z) & 0 \end{pmatrix}, & z \in \Sigma_{r,\pm}, \end{cases} \quad (5.4)$$
It is readily seen that $V(z)$ satisfies the following RHP:

$(V_1)$ $V(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$;

$(V_2)$ for $z \in \Sigma$, $V(z)$ satisfies

$$V_+(z) = V_-(z)J_V(z);$$

$(V_3)$ as $z \to \infty$,

$$V(z) = I + O(1/z);$$

$(V_4)$ as $z \to 0$, $V(z)$ has the behavior

$$V(z) = O(\log|z|);$$

$(V_5)$ as $z \to \pm 2$, $V(z)$ has the behavior

$$V(z) = O(1)(z^2 - 4)^{(1/2 - \alpha)\sigma_3}.$$

From Proposition 4.2, we can choose $\delta$ sufficiently small so that

$$\text{Re} \phi(z) < 0, \quad \text{Re} \tilde{\phi}(z) > 0 \quad \text{and} \quad \text{Re} \tilde{\phi}(z) > 0$$

in both the upper and lower lens regions. These together with (4.32)–(4.33), (4.16) and (4.23), imply that all jumps on the contour $\Sigma$ are exponentially close to the identity matrix, provided that they are bounded away from the segment $(-2, 2)$. Moreover, for $x \in (-2, 2)$, the functions $-w_0(x)/\hat{D}^2(x)$ and $w_0(x)/\hat{D}^2(x)$ are approximated by $-(4 - x^2)^{2\alpha - 1}e^L$ as $n \to \infty$, where $L = \log(\sqrt{2\pi} \Gamma^2(\alpha)e^{\alpha}/\sqrt{\pi})$. It is therefore natural to expect that for large $n$, the solution of the RHP for $V(z)$ may behave asymptotically like the solution of the following RHP for $N(z)$:

$(N_1)$ $N(z)$ is analytic in $\mathbb{C}\setminus[-2, 2];$

$(N_2)$ for $x \in (-2, 2),$

$$N_+(x) = N_-(x)\begin{pmatrix} 0 & -(4 - x^2)^{2\alpha - 1}e^L \\ (4 - x^2)^{1 - 2\alpha}e^{-L} & 0 \end{pmatrix};$$

$(N_3)$ as $z \to \infty,$

$$N(z) = I + O(1/z).$$

This problem can be solved explicitly, and its solution is given by

$$N(z) = \begin{pmatrix} (z^2 - 4)^{1/4 - \alpha}\varphi(z/2)^{2\alpha - 1/2} & -ie^L(z^2 - 4)^{\alpha - 3/4}\varphi(z/2)^{1/2 - 2\alpha} \\ ie^{-L}(z^2 - 4)^{1/4 - \alpha}\varphi(z/2)^{3\alpha - 3/2} & (z^2 - 4)^{\alpha - 3/4}\varphi(z/2)^{3/2 - 2\alpha} \end{pmatrix},$$

where $\text{arg}(z \pm 2) \in (-\pi, \pi)$, and

$$\varphi(z) := z + \sqrt{z^2 - 1}$$

with a branch cut along $[-1, 1]$ and $\varphi(z) \to 2z$ as $z \to \infty$; see also [13].

6 Local parametrices and the final transformation $V \to S$

In this section, we will consider local parametrices in small disks $U(0, \varepsilon)$, $U(2, \varepsilon)$ and $U(-2, \varepsilon)$, respectively, centered at the origin and at the end points of bands which are adjacent to a saturated region.
6.1 Parametrix at endpoints of the saturated-band region

We seek a local parametrix \( V_{loc}(z) \) defined on \( U(2, \varepsilon) \cup U(-2, \varepsilon) \) such that

(V\(_{loc,1}\)) \( V_{loc}(z) \) is analytic in \( U(2, \varepsilon) \cup U(-2, \varepsilon) \setminus \Sigma \);

(V\(_{loc,2}\)) for \( z \in (U(2, \varepsilon) \cup U(-2, \varepsilon)) \cap \Sigma \),

\[
(V_{loc})_+(z) = (V_{loc})_-(z)J_V(z);
\]

(6.1)

(V\(_{loc,3}\)) for \( z \in \partial U(2, \varepsilon) \cup \partial U(-2, \varepsilon) \),

\[
V_{loc}(z)N^{-1}(z) = I + O(1/n) \quad \text{as} \ n \to \infty;
\]

(6.2)

(V\(_{loc,4}\)) as \( z \to \pm 2 \), \( V_{loc}(z) \) has the behavior

\[
V(z) = O(1) (z^2 - 4)^{(1/2-\alpha)\sigma_3}.
\]

(6.3)

At first, we construct the parametrix near the point \( z = 2 \). The jumps \( J_V(z) \) are given by

\[
J_V(z) = \begin{cases} 
0 & \text{if} \ z \in (2 - \varepsilon, 2), \\
-\tilde{D}^2/w_0 & \text{if} \ z \in (2 - \varepsilon, 2), \\
1 & \text{if} \ z \in (2, 2 + i\varepsilon), \\
\tilde{w}/(\tilde{D}^2e^{2n\tilde{g}}) & \text{if} \ z \in (2, 2 - i\varepsilon), \\
1 & \text{if} \ z \in (2, 2 + \varepsilon).
\end{cases}
\]

(6.4)

If we set

\[
P(z) = V_{loc}(z) \times \begin{cases} 
(\tilde{w}/\tilde{D})^{\sigma_3}e^{-n\tilde{g}(z)\sigma_3} & \text{for} \ z \in U(2, \varepsilon), \\
((-\tilde{w}/\tilde{D})^{\sigma_3}e^{-n\tilde{g}(z)\sigma_3} & \text{for} \ z \in U(-2, \varepsilon),
\end{cases}
\]

(6.5)

then the jump conditions on \( P(z) \) become

\[
P_+(z) = P_-(z)J_P(z),
\]

(6.6)

where for \( z \in U(2, \varepsilon) \)

\[
J_P(z) = \begin{cases} 
0 & \text{if} \ z \in (2 - \varepsilon, 2), \\
1 & \text{if} \ z \in (2, 2 + i\varepsilon), \\
1 & \text{if} \ z \in (2, 2 - i\varepsilon), \\
1 & \text{if} \ z \in (2, 2 + \varepsilon);
\end{cases}
\]

(6.7)

see Figure 4

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comparing the RHP for $\Psi$ with the RHP for $P$, it is readily seen that $\tilde{\phi}(z)$. Since by (4.38) $\omega z = \omega e^{2\pi i/3}$, and it satisfies the following jumps

\[ \Psi(z) := \begin{cases} 
  \left( \begin{array}{cc} 
  \text{Ai}(z) & \omega^2 \text{Ai}(\omega^2 z) \\
  i\text{Ai}'(z) & i\omega \text{Ai}'(\omega^2 z) 
  \end{array} \right) & \text{for arg } z \in (0, \pi/2), \\
  \left( \begin{array}{cc} 
  -\omega \text{Ai}(\omega z) & \omega^2 \text{Ai}(\omega^2 z) \\
  -i\omega^2 \text{Ai}'(\omega z) & i\omega \text{Ai}'(\omega^2 z) 
  \end{array} \right) & \text{for arg } z \in (\pi/2, \pi), \\
  \left( \begin{array}{cc} 
  -\omega^2 \text{Ai}(\omega^2 z) & -\omega \text{Ai}(\omega z) \\
  -i\omega \text{Ai}'(\omega z) & -i\omega^2 \text{Ai}'(\omega z) 
  \end{array} \right) & \text{for arg } z \in (-\pi, -\pi/2), \\
  \left( \begin{array}{cc} 
  \text{Ai}(z) & -\omega \text{Ai}(\omega z) \\
  i\text{Ai}'(z) & -i\omega^2 \text{Ai}'(\omega z) 
  \end{array} \right) & \text{for arg } z \in (-\pi/2, 0), 
\end{cases} \tag{6.8} \]

where $\omega = e^{2\pi i/3}$, and it satisfies the following jumps

\[ \Psi_+(z) = \Psi_-(z) \begin{cases} 
  \left( \begin{array}{cc} 
  0 & -1 \\
  1 & 0 
  \end{array} \right) & \text{for } z \in (-\infty, 0), \\
  \left( \begin{array}{cc} 
  1 & 0 \\
  -1 & 1 
  \end{array} \right) & \text{for } z \in (\infty e^{-\pi i/2}, 0) \cup (\infty e^{\pi i/2}, 0), \\
  \left( \begin{array}{cc} 
  1 & -1 \\
  0 & 1 
  \end{array} \right) & \text{for } z \in (0, \infty). 
\end{cases} \tag{6.9} \]

To construct our parametrix, we recall the function

\[ \tilde{f}_n(z) = \left( -\frac{3}{2} n\tilde{\phi}(z) \right)^{2/3}; \tag{6.10} \]

see [Z]. Since by (18) \[ \tilde{\phi}(z) = -\frac{2}{3} (z-2)^{3/2} + O((z-2)^2), \tag{6.11} \]
it is readily seen that $\tilde{f}_n(z)$ is a one-to-one mapping of $U(2, \varepsilon)$ onto a neighborhood of the origin. For $z \in U(2, \varepsilon)$, comparing the RHP for $\Psi$ with the RHP for $P$ invokes us to seek a solution to the RHP for $V_{loc}$ in the form of

\[ V_r(z) = E_r(z)\sigma_2\Psi(\tilde{f}_n(z))\sigma_2 e^{n\sigma_3 (\tilde{D}/\tilde{\sigma}^{1/2})^{\sigma_3}}, \tag{6.12} \]
where the Pauli matrix \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and the matrix-valued function \( E_r(z) \) is analytic in \( U(2, \varepsilon) \), to be determined later. Note that

\[
\sigma_2 \Psi(\tilde{f}_n(z)) \sigma_2 e^{i \tilde{\sigma}_3} = \frac{1}{\sqrt{n}} (\tilde{f}_n(z))^{\sigma_3/4} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right)^{-1} (I + O(1/n))
\]

(6.13)

and

\[
(\tilde{D}/\tilde{w}^{1/2})^{\sigma_3} N(z)(\tilde{D}/\tilde{w}^{1/2})^{-\sigma_3} = \frac{(z^2 - 4)^{(1/2-\alpha)} \sigma_3}{(z^2 - 4)^{1/4}} \left( \begin{array}{cc} \varphi(z/2)^{2\alpha-1/2} & -i \varphi(z/2)^{1/2-2\alpha} \\ i \varphi(z/2)^{2\alpha-3/2} & \varphi(z/2)^{3/2-2\alpha} \end{array} \right) (I + O(1/n)),
\]

(6.14)

which holds uniformly for \( z \in \partial U(2, \varepsilon) \) as \( n \to \infty \). Thus, the condition \( V_{\text{loc}, \alpha} \) with (6.13) and (6.14) leads us to set

\[
E_r(z) = \sqrt{n} (\tilde{D}/\tilde{w}^{1/2})^{-\sigma_3} m_r(z)(\tilde{f}_n(z))^{-\sigma_3/4},
\]

(6.15)

where

\[
m_r(z) = \frac{(z^2 - 4)^{(1/2-\alpha)} \sigma_3}{(z^2 - 4)^{1/4}} \left( \begin{array}{cc} \varphi(z/2)^{2\alpha-1/2} & -i \varphi(z/2)^{1/2-2\alpha} \\ i \varphi(z/2)^{2\alpha-3/2} & \varphi(z/2)^{3/2-2\alpha} \end{array} \right) \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right).
\]

(6.16)

It can be shown that \( E_r(z) \) has no jumps across the interval \((0, 2)\), and in addition that it has a removable singularity at \( z = 2 \).

A similar construction gives the parametrix at the point \( z = -2 \). Namely, if we set

\[
\tilde{f}_n(z) = \left( -\frac{3}{2} n \tilde{\phi}(z) \right)^{2/3},
\]

(6.17)

which is analytic in \( U(-2, \varepsilon) \), and is a one-to-one mapping of \( U(-2, \varepsilon) \) onto the neighborhood of the origin. For \( z \in U(-2, \varepsilon) \), the jumps \( J_P(z) \) are given by

\[
J_P(z) = \begin{cases} 
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in (-2, -2 + \varepsilon), \\
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & z \in (-2, -2 + i\varepsilon), \\
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & z \in (-2, -2 - i\varepsilon), \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in (-2 - \varepsilon, -2); 
\end{cases}
\]

(6.18)
see Figure 5. These jump conditions are satisfied by the function \( \sigma_1 \Psi(-z)\sigma_1 \), where the pauli matrix \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Hence, we can take for \( z \in U(-2, \epsilon) \)

\[
V_i(z) = E_i(z)\sigma_1 \Psi(f_n(z))\sigma_1 e^{\alpha \phi(z)}(\hat{D}/(-\hat{w}))^{1/2} \sigma_3,
\]

where

\[
E_i(z) = \sqrt{\pi}(\hat{D}/(-\hat{w}))^{-1/2} m_i(z)(f_n(z))^{-1/4}
\]

and

\[
m_i(z) = \frac{\left[(2 - z)(2 - z)\right]^{(1/4 - \alpha)\sigma_3}}{(2 - z)^{1/4}(2 - z)^{1/4}} \begin{pmatrix} \varphi(-z/2)^{2\alpha - \frac{1}{2}} & i\varphi(-z/2)^{1/2 - 2\alpha} \\ -i\varphi(-z/2)^{1/2 - 2\alpha} & \varphi(-z/2)\varphi(-z/2)^{3/2 - 2\alpha} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\]

### 6.2 Parametrix at the point \( z = 0 \)

As \( z \to 0 \) for \( z \in \Sigma \), \( J_V(z) - I \) is no longer small. Hence we are led to the following parametrix for \( V_0(z) \) at the origin:

1. **(V_{0,1})** \( V_0(z) \) is analytic in \( U(0, \epsilon) \setminus \Sigma \);
2. **(V_{0,2})** for \( z \in U(0, \epsilon) \cap \Sigma \), \( V_0(z) \) shares the same jumps as \( V(z) \); cf. 5.3 and 5.21; see Figure 8 for an illustration of the contours;
3. **(V_{0,3})** for \( z \in \partial U(0, \epsilon) \), the matching condition holds:

\[
V_0(z)N^{-1}(z) = I + O(1/n) \quad \text{as } n \to \infty.
\]

Next, we proceed to find an approximating RHP, of which the solution \( V_{0,loc}(z) \) is the leading order approximation of \( V_0(z) \). Indeed, on account of 4.38 and 120, we observe that \( e^{-\alpha \phi} \) is exponentially small and \( D/\hat{D} \) tends to 1 exponentially and uniformly for \( z \in \Sigma_{r,\pm} \), so long as \( \arg z \) keeps a distance to \( \pm \pi/2 \). By simplifying the jump conditions for the RHP for \( V \), we wish to find a matrix-valued function \( V_{0,loc}(z) \) such that

1. **(V_{loc,1})** \( V_{loc}(z) \) is analytic in \( U(0, \epsilon) \setminus \Sigma \);
2. **(V_{loc,2})** for \( z \in U(0, \epsilon) \cap \Sigma \),

\[
(V_{loc})_+^0(z) = (V_{loc})_0^0(z)J_{V_{loc}}^0(z),
\]

where

\[
J_{V_{loc}}^0(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -(4 - z^2)^{1-2\alpha} e^{-L + 2\phi} & 1 \end{pmatrix} & \text{for } z \in U(0, \epsilon) \cap (\Sigma_{r,\pm} \cup \Sigma_{r,\pm}), \\
\begin{pmatrix} 0 & 1 \\ -(4 - z^2)^{1-2\alpha} e^{-L} & 0 \end{pmatrix} & \text{for } z \in (-\epsilon, \epsilon),
\end{cases}
\]

with \( \arg(2 - z) \in (-\pi, \pi) \) and \( \arg(z + 2) \in (-\pi, \pi) \);
3. **(V_{loc,3})** for \( z \in \partial U(0, \epsilon) \),

\[
V_{loc}^0(z)N^{-1}(z) = I + O(1/n) \quad \text{as } n \to \infty.
\]

A solution is readily found as

\[
V_{loc}^0(z) := \begin{cases} 
N(z) \begin{pmatrix} 1 & 0 \\ -(4 - z^2)^{1-2\alpha} e^{-L + 2\phi} & 1 \end{pmatrix} & \text{for } z \in I, \\
N(z) \begin{pmatrix} 1 & 0 \\ (4 - z^2)^{1-2\alpha} e^{-L + 2\phi} & 1 \end{pmatrix} & \text{for } z \in II, \\
N(z) \begin{pmatrix} 1 & 0 \\ (4 - z^2)^{1-2\alpha} e^{-L + 2\phi} & 1 \end{pmatrix} & \text{for } z \in U(0, \epsilon) \setminus (I \cup II),
\end{cases}
\]

where I and II are depicted in Figure 8.
It is worth noting that

\[ J_{V_{0 \text{loc}}}(z)J_{V}^{-1}(z) = I + O(z^2/n), \]  

uniformly for \( z \in \{U(0, \varepsilon) \cap \Sigma \} \cup \partial U(0, \varepsilon). \) Following the discussion in [10, Section 7], from (6.27) we conclude that

\[ V_0(z) = V_{0 \text{loc}}(z)(I + O(1/n)), \]  

uniformly for \( z \in U(0, \varepsilon). \)

**Remark 6.1.** An alternative derivation of (6.28) is given as follows. Let \( H(z) := V_0(z) \left( V_{0 \text{loc}} \right)^{-1}(z) \) for \( z \in U(0, \varepsilon) \setminus \Sigma. \) It is readily seen from (6.27) that the jump for \( H \) fulfills \( J_H(z) = I + O(1/n) \) for \( z \in U(0, \varepsilon) \cap \Sigma, \) and that \( H(z) = I + O(1/n) \) on \( \partial U(0, \varepsilon). \) Using the single-layer potential techniques, we seek \( H(z) \) of the form

\[ H(z) = I + \frac{1}{2\pi i} \int_{\Sigma_\varepsilon} \frac{\mu_H(t)dt}{t-z}, \quad z \in U(0, \varepsilon) \setminus \Sigma, \]  

where \( \Sigma_\varepsilon = U(0, \varepsilon) \cap \Sigma. \) The RHP for \( H(z) \) is then reduced to the singular integral equation for the new unknown matrix function \( \mu_H(\tau): \)

\[ \mu_H(\tau) = \left[ -\frac{1}{2} \mu_H(\tau) + \frac{1}{2\pi i} \text{p.v.} \int_{\Sigma_\varepsilon} \frac{\mu_H(t)dt}{t-\tau} \right] (J_H(\tau) - I) + (J_H(\tau) - I), \quad \tau \in \Sigma_\varepsilon. \]

It is seen that the operator in the square brackets are bounded operator in \( L_2(\Sigma_\varepsilon). \) Hence, for large \( n, \) bearing in mind that \( J_H(\tau) - I = O(1/n), \) we see that the nonhomogeneous equation is contractive. A unique solution is then determined such that \( \mu_H(\tau) = O(1/n) \) on \( \Sigma_\varepsilon. \) Accordingly, it is readily verified that \( H(z) = I + O(1/n) \) in \( U(0, \varepsilon), \) and (6.28) follows.

It is worth mentioning that refinements to (6.28) can be obtained by picking up the later terms in (6.27).

### 6.3 The transformation \( V \to S \)

Let

\[ \tilde{V}(z) := \begin{cases} 
V_l(z), & z \in U(-2, \varepsilon), \\
V_0(z), & z \in U(0, \varepsilon), \\
V_r(z), & z \in U(2, \varepsilon), \\
N(z), & \text{otherwise}.
\]  

(6.29)

It is readily verified that the matrix-valued function

\[ S(z) := V(z)\tilde{V}^{-1}(z) \]  

(6.30)

is a solution of the RHP:

\( (S_1) \) \( S(z) \) is analytic in \( \mathbb{C} \setminus \Sigma_S \) (see Figure 4);
\[ S_+(z) = S_-(z) J_S(z), \] (6.31)

where

\[ J_S(z) = \tilde{V}_-(z) J_V(z) \tilde{V}_+^{-1}(z) = \begin{cases} V_1(z) N^{-1}(z), & z \in \partial U(-2, \epsilon), \\ V_0(z) N^{-1}(z), & z \in \partial U(0, \epsilon), \\ V_r(z) N^{-1}(z), & z \in \partial U(2, \epsilon), \\ N(z) J_V(z) N^{-1}(z), & \text{otherwise}; \end{cases} \] (6.32)

\((S_4)\) as \( z \to \infty \),

\[ S(z) = I + O(1/z). \] (6.33)

It follows from the matching condition of the local parametrices and the definition of \( \phi \) that

\[ J_S(z) = \begin{cases} I + O(1/n), & z \in \partial(U(0, \epsilon) \cup U(-2, \epsilon) \cup U(2, \epsilon)) \cup (-2 + \epsilon, \epsilon) \cup (\epsilon, 2 - \epsilon), \\ I + O(e^{-cn}), & \text{otherwise}, \end{cases} \] (6.34)

where \( c \) is a positive constant, and the error term is uniform for \( z \in \Sigma_S \). Hence, we have

\[ \| J_S(z) - I \|_{L^2; L^\infty(\Sigma_S)} = O(1/n). \] (6.35)

Then, applying the standard procedure of norm estimation of Cauchy operators \( C \) and \( C_- \) (cf. [9], [10]), it follows that

\[ S(z) = I + C(\Delta_S + u_S \Delta_S), \] (6.36)

where \( \Delta_S := J_S(z) - I \) and \( u_S := (I - C \Delta_S)^{-1}(C_- \Delta_S) \). Furthermore, by using the technique of deformation of contours and (6.35), we have

\[ S(z) = I + O(1/n) \] (6.37)

uniformly for \( z \in \mathbb{C} \).

7. Proof of Theorem 2.1

Since \( V(z) = S(z) \tilde{V}(z) \) by (6.30), it follows that

\[ V_{11}(z) = S_{11}(z) \tilde{V}_{11}(z) + S_{12}(z) \tilde{V}_{21}(z) \] (7.1)

and

\[ V_{12}(z) = S_{11}(z) \tilde{V}_{12}(z) + S_{12}(z) \tilde{V}_{22}(z). \] (7.2)

Moreover, from (6.37) it follows that

\[ S_{11}(z) = 1 + O(1/n) \quad \text{and} \quad S_{12}(z) = O(1/n). \] (7.3)
Region $A_δ \setminus U(0,ε)$. At first, let us consider $z ∈ A_δ$ while $z ∉ U(0,ε)$. Note by Theorem 2.3 and 5.1 that
\[
π_n(n^{-1/2}z) = n^{-n/2}U_{11}(z).
\] (7.4)
Recalling the definition of $\tilde{V}(z)$ in 6.29, we have from 4.11, 4.12, 5.1 and 7.3
\[
π_n(n^{-1/2}z) = e^{n/2n^{1/2-2-α}}V_{11}(z)(D(z)E(z))^{-α}e^{-nφ} \\
= e^{n/2n^{1/2-2-α}}(D(z)E(z))^{-1}e^{-nφ}N_{11}(1 + O(1/n)).
\] (7.5)
A combination of this with $E(z) = \tilde{E}(z)e^{-πi(1/2-α)}$, 4.24 and 5.11 gives 2.7.

Region $B_δ \setminus U(0,ε)$. From 5.1, 4.12, 4.1 and 5.11, it follows that
\[
U(z) = e^{n/2σ_μn^{1/2-α}}V(z)[E(z)\tilde{D}(z)]^{-σ_1}[V_{-}(z)]^{-1}e^{-nφ}[U_{-}(z)]^{-1} \quad \text{for } z ∈ Ω_{r+}.
\] (7.6)
Then, on account of 6.29 we have
\[
e^{-n/2n^{1/2-α-1/2}π_n(n^{-1/2}z)} = [e^{-nφ}N_{11}/(E\tilde{D}) + e^{nφ}E\tilde{D}/(2iγw)N_{12}]S_{11}(z) \\
+ [e^{-nφ}N_{21}/(E\tilde{D}) + e^{nφ}E\tilde{D}/(2iγw)N_{22}]S_{12}(z).
\] (7.7)
From 4.17, we note that $\tilde{D}(z) = 1 + O(1/n)$ for $z ∈ B_δ$, which may be neglected. Hence, we obtain 2.8 by 4.19, 5.11 and 7.3.

Region $C_{δ_1} ∪ C_{δ_2}$. Next, let us now consider $z$ in the Airy region. For simplicity, we only consider the case $z ∈ C_{δ_1}$; that is, $z ∈ Σ_{r+}$ and arg $\tilde{f}_n(z) ∈ (π/2, π)$. Again by 7.6 and 6.29, we note that
\[
e^{-n/2n^{1/2-α+1/2}π_n(n^{-1/2}z)} = [[V_r(z)]_{11}S_{11}(z) + (V_r(z)21S_{12}(z))e^{-nφ}/(E\tilde{D}) \\
+ [(V_r(z)]_{12}S_{11}(z) + (V_r(z)22S_{12}(z))e^{nφ}/E\tilde{D}/(2iγw)].
\] (7.8)
Recalling the well-known formula of the Airy functions [22] (9.2.11], one can see that
\[
σ_2Ψ(\tilde{f}_n(z))σ_2 = \begin{pmatrix}
A(\tilde{f}_n(z)) & B(\tilde{f}_n(z)) \\
 πA(\tilde{f}_n(z)) & πB(\tilde{f}_n(z))
\end{pmatrix}
\begin{pmatrix}
-\frac{i}{2} & -\frac{i}{2} \\
 \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\quad \text{for arg } \tilde{f}_n(z) ∈ (π/2, π).
\] (7.9)
Inserting 7.9 into 6.12, and combining 4.29, 4.24, 4.27 and 7.3 yields 2.9 for $z ∈ C_{δ_1}$. Following the same argument as given above, one can establish 2.10 for $z ∈ C_{δ_2}$.

Region $D_δ$. Similarly, it can be shown that
\[
U(z) = e^{n/2σ_μn^{1/2-α}}V(z)[E(z)\tilde{D}(z)]^{-σ_1}[V_{+}(z)]^{-1}e^{-nφ}σ_1[U_{+}(z)]^{-1} \quad \text{for } z ∈ Σ_{r+}.
\] (7.10)
Again by 6.29, we have
\[
e^{-n/2n^{1/2+α-1/2}π_n(n^{-1/2}z)} = [S_{11}(z)N_{11}(z) + S_{12}(z)N_{21}(z)]2γ(\Pi(z))e^{-iθ(z)}e^{-nφ(z)} \\
+ [S_{11}(z)N_{12}(z) + S_{12}(z)N_{22}(z)]2iγ(\tilde{D}(z))e^{nφ(z)}.
\] (7.11)
Note that $2γ(\Pi(z))e^{-iθ(z)}/D(z)$ and $e^{nφ(z)}$ is exponentially small as $n → ∞$. From 4.19 and 7.3, we obtain 2.12. This completes the proof of Theorem 2.1.

8 Proof of Theorem 2.2
The proof is similar to that of Theorem 2.1. For instance, as $z ∈ A_δ ∩ U(0,ε)$, 7.3 and the first equality of 7.25 are still valid. We need only to replace $V_0(z)$ with $V_{0φ}(z)$ up to an error of order $O(1/n)$. Then 2.13 follows accordingly by using the Stirling’s formula.

The other cases, such as $z ∈ B_δ ∩ U(0,ε)$ can be justified similarly by tracing back all the transformations. Again 2.13 follows from a simplification of 7.7. Here use has been made of the relations between the auxiliary functions, established in Section 4.
9 Discussion and comparison with known results

The Riemann-Hilbert approach has proven an powerful tool in dealing with uniform asymptotics. An example of its powerfulness is the uniform approximation in a neighborhood of the origin. We given in Theorem 2.2 the leading behavior. Refinements are achievable if we pick more terms for the matrix functions of its.

To conclude this paper, we will do a insistence check by comparing our formulas in Theorem 2.1 with the results previously obtained in [12] and [17]. Moreover, note by (4.38) that

\[ \tilde{\phi}(n^{1/2}x) \sim 1/2 - \log(n^{1/2}x) + 2 \log(n^{1/2}x)/(nx^2) \] as \( n \to \infty \). Thus, we have

\[ \pi_n(x) \sim \frac{\Gamma(\alpha) \nu^{-1/2} e^{1/2} x^{\nu}}{\Gamma(\alpha - 1/2)} \]

In view of (3.4), it is readily seen that (2.12) agrees with (9.1).

Next, we consider the case when \( z = 2 + O(n^{-2/3}) \). First, we introduce the notation \( t = (\nu/n)^{1/2}z \), where \( \nu = n + 2\alpha - 1/2 \). From (2.3), it can be shown that

\[ \left[ \varphi(z/2)^{2\alpha-1/2} - \varphi(z/2)^{-2\alpha+1/2} \right] (z^2 - 4)^{-1/4} (\tilde{f}_n(z))^{-1/4} = O(\nu^{-1/6}) \]

and

\[ \left[ \varphi(z/2)^{2\alpha-1/2} + \varphi(z/2)^{-2\alpha+1/2} \right] (z^2 - 4)^{-1/4} (\tilde{f}_n(z))^{1/4} = \sqrt{2} \nu^{1/6}(1 + O(\nu^{-2/3})) \]

Furthermore, we have \( \tilde{f}_n(z) = \nu^{2/3}\zeta(t) + O(1/\nu) \), where \( \zeta(t) \) is the function introduced by Lee and Wong [17 (2.15)]. From (2.9) and (17), it follows that

\[ f_n^{(\alpha)}(\nu^{-1/2}t) = n^{-1/3-n/2} e^{\nu t/2} \left\{ \cos(\pi\alpha - \nu t/2) \left[ \text{Ai}(\nu^{2/3}\zeta(t)) + O(\nu^{-1/3}) \right] + \sin(\pi\alpha - \nu t/2) \left[ \text{Bi}(\nu^{2/3}\zeta(t)) + O(\nu^{-1/3}) \right] \right\} \]

which exactly agrees with (3.7) in Lee and Wong [17] if we note that \((4\zeta(t)/(t^2 - 4))^{1/4} = 1 + O(\nu^{-2/3})\).

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