Transient perturbation growth of vortex rings

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Abstract. Perturbations to a vortex ring with largest transient energy growth are calculated and applied to drive the vortex ring to evolve. Optimal initial perturbations appear at azimuthal wave numbers \( \beta = 0 \) and \( \beta = 9 \). The former was described as ‘axial flow’ and is not related to the breakdown of the ring, while the latter is critical to the ring breakdown process as observed in previous direct numerical simulation (DNS) with random initial noise input. In the present work, DNS is conducted to study the nonlinear development of the optimal perturbations, and good agreement with past studies is achieved.

1. Introduction
In the wake of a wind turbine, helical vortices are the dominant vortical structures. These vortices and their instability and breakdown to turbulence have been ubiquitously observed, in both numerical simulations and field measurements of large offshore wind farms [1, 2]. The previous study of perturbation developments in helical vortices has been focused on analytical analyses of short-wave perturbations and very few efforts have been devoted to global numerical analyses. If the perturbation in a helical vortex is short enough in the axial direction, it can be approximated by modes with high azimuthal wavenumbers in a vortex ring, which is axisymmetric and therefore enables a Bi-Global numerical analyses. Further more, most of the pervious works on helical vortices are about its instabilities and breakdown, while the transient dynamics have not been investigated. As shown in the present work, the transient growth of perturbations can be dominant over short time intervals.

Vortex rings are ubiquitous in nature. Classical examples can be found both in the propulsion of fish and salps, through the undulatory motion of the body and tail or the ejection of fluid through an orifice, and in the generation of lift during insect flight [3] [4]. Aquatic mammals such as dolphins and whales have also been observed to produce vortex rings, resulting from the exhalation of air through their blowholes [5]. The stability and development of vortex rings have been the subject of numerous theoretical [6, 7, 8, 9, 10, 11] and experimental [12, 13, 14, 15] studies.

The instability of a vortex ring was first documented as early as 1939 and there are now many known mechanisms by which a ring may become unstable [12]. An inviscid instability associated with a strained field, induced by either an external source of noise, or other portions of the curved ring itself, has been commonly observed. This instability can manifest itself in both long and short wave forms. The long wave instability was first found in a counter rotating pair of vortices [6], and the wavelength is typically in the order of the distance between the two vortices, or of
the vortex ring’s radius [16, 17]. As this instability develops, the vortices are seen to connect and changes to the flow’s topology are observed [16]. The short wave instability features the amplification of azimuthal bending waves of a wavelength comparable to that of the vortex core size [7]. This instability has been described to be the result of resonant interaction between the strain and Kelvin waves whose azimuthal wave numbers are separated by one [8]. A pair of vortex rings travelling in the same direction are also subject to a ‘leapfrog’ instability, which induces the leapfrog of the pursuing ring towards its leading counterpart, eventually resulting in the merge of the pair [18].

In recent years, direct numerical simulation (DNS) has been utilised to study the formation, evolution and breakdown of vortex rings [9, 10, 11]. James et al. (1996) investigated the formation and post formation characteristics of a vortex ring and revealed the process of mixing and entrainment [9]. Bergdorf et al. (2007) performed DNS of the turbulent decay of vortex rings and Archer et al. (2008) studied the temporal development of single vortex rings at various Reynolds numbers and core thicknesses [10, 11]. Both provided a detailed description of the development of the azimuthal instability, highlighting the importance of the formation of secondary vortical structures around the vortex core. These structures were noted to be responsible for depositing hairpin vortices into the ring’s wake, a leading factor in vortex breakdown, where the interaction between the ring and its recycled wakes increases the growth rate of the instability. Further to this, certain azimuthal wave numbers were found to hold a greater significance in the instability’s development than others. The onset of the instability was observed in the excitation of wave numbers 6 and 8 for thick and thin core rings respectively, and, on the evolution of the secondary vortical structures, energy was then seen to be transferred to both their 2nd and 3rd harmonics [10]. The independent growth of a narrow band of wave numbers was also noted to determine the number of standing waves about the vortex core. Values of 6, 8, 9 and 10 were recorded, dependent on the ring’s Reynolds number and core thickness [11]. Although such previous studies have given insight into the turbulent decay of vortex rings, the use of DNS has been limited to studying effects of a random initial noise input.

To the author’s knowledge, very few studies have considered the global optimal transient or non-normal dynamics in vortex ring flow. The establishment and description of such ‘optimal’ perturbation modes will reveal the development of a vortex ring perturbed by coherent and energetic noises. In the present work, the optimal perturbations are calculated and used to perturb the vortex ring in DNS. Further to the rolling of hairpin vortices at high azimuthal wave numbers observed in previous DNS, the regeneration of vortex rings is found at a relatively low azimuthal wave number.

2. Transient Growth Methodology
Assuming that the fluid is both Newtonian and incompressible, the dynamics of the ring are governed by the incompressible Navier Stokes (NS) equations,

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Re^{-1} \nabla^2 \mathbf{u},
\]

\[
\nabla \cdot \mathbf{u} = 0
\]

where \( \mathbf{u} \) and \( p \) denote the velocity vector field and the modified pressure respectively, and \( Re \) represents the Reynolds number based on the maximum velocity and radius of the ring.

The flow field is then decomposed into the summation of an axisymmetric base flow and a perturbation, \(( \mathbf{u}, p ) = ( \mathbf{U}, P ) + ( \mathbf{u}', p' )\). When the magnitude of perturbation is significantly smaller than the base flow, the perturbation variables can then be governed by the linearised NS
equations,

\[ \partial_t \mathbf{u}' - (\mathbf{U} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{U} = -\nabla p' + Re^{-1}\nabla^2 \mathbf{u}', \]
\[ \nabla \cdot \mathbf{u}' = 0. \]  \hspace{1cm} (3)

Since the base flow is homogeneous in the azimuthal direction, the perturbation variables can thus be further decomposed as

\[ \mathbf{u}'(x, r, \theta, t) = \sum_{\beta=0}^{\infty} \mathbf{u}_\beta(x, r, t) e^{i\beta \theta}, \]  \hspace{1cm} (4)

where \( \mathbf{u}'_\beta \) denotes the Fourier decomposed mode with an integer azimuthal wave number \( \beta \), and \( x, r \) and \( \theta \) representing the axial, radial and azimuthal coordinates in a cylindrical system, respectively. Owing to the linearisation of the governing equations, the dynamics of modes with different wave numbers are decoupled and can be calculated individually. The Fourier decomposed modes will henceforth be referred to as ‘perturbations’ and the subscript \( \beta \) is omitted for clarity.

The transient energy growth over a time horizon, \( \tau \), is given by the ratio of the final and initial kinetic energies of the perturbation,

\[ G = \frac{E(\tau)}{E(0)}, \text{ where } E = \int_\Omega \mathbf{u}' \cdot \mathbf{u}' dV \]  \hspace{1cm} (5)

denotes the kinetic energy of the perturbation integrated over the computational domain \( \Omega \). The most energetic initial perturbation over the given time period, and thus the perturbation of predominate interest, is defined as the perturbation that maximises this ratio. This optimal perturbation and the corresponding value of \( G \) are calculated by applying an Arnoldi method to a Krylov sequence built by iteratively integrating the linearised NS equations and the adjoint [19, 20, 30]. This is a well established method, which is not elaborated on here.

3. Modelling and discretisation

A Lamb-Oseen vortex is adopted in the problem’s initialisation as it has been widely considered in the stability studies of vortex flow [21]. The non-dimensionalised streamwise, vortical and azimuthal velocity components of the Lamb-Oseen vortex can be written as

\[ U = U_0 + \frac{(1 - r)/2\pi}{x^2 + (r - 1)^2} \left[ 1 - \exp\left(-\frac{(x^2 + (r - 1)^2)}{R_0^2}\right) \right], \]  \hspace{1cm} (6)
\[ V = \frac{x/2\pi}{x^2 + (r - 1)^2} \left[ 1 - \exp\left(-\frac{(x^2 + (r - 1)^2)}{R_0^2}\right) \right], \]  \hspace{1cm} (7)
\[ W = 0, \]  \hspace{1cm} (8)

where \( U_0 = -0.225 \) is applied to limit the self-induced translation of the ring, and \( R_0 \) denotes the initial radius of the vortex. A value of \( R_0 = 0.2 \) is applied throughout this work and, in each \( x - r \) plane, the vortex is initially centred at \((0,1)\). The governing NS and linearised NS equations are discretised over the domain via Semtex. Semtex is a quadrilateral spectral element code that uses the standard nodal Gauss-Lobato-Legendre basis functions and Fourier expansions in a homogeneous direction to provide three-dimensional solutions [22]. We used 4272 elements for DNS. The second order split step method developed by Karniadakis et al. [23] is adopted for time integration. The mesh is significantly refined uniformly about the region encompassing the vortex’s trajectory in a rectangular region to ensure accuracy. Views of the entire domain and
of the mesh’s refinement used in the computation of the base flow and transient growth studies are presented in Figures 1(a) and 1(b) respectively. In each element, the variables are further expanded using a polynomial function $P$ with order 6, at which the transient growth converges to four significant figures in a convergence check as displayed in Table 1. The base flow and transient growth calculations are 2D and serial. The DNS of perturbation developments are 3D and each simulation takes around 10 hours on a 64-core cluster.

![Figure 1.](image)

**Figure 1.** (a) Entire domain and (b) detailed view of mesh surrounding vortex core region used in base flow and transient growth studies.

| $P$ | $G$     | Relative Difference (%) |
|-----|---------|-------------------------|
| 3   | $7.041 \times 10^3$ | 1.35                    |
| 4   | $6.904 \times 10^3$ | 0.62                    |
| 5   | $6.909 \times 10^3$ | 0.54                    |
| 6   | $6.947 \times 10^3$ | 0.00                    |
| 7   | $6.947 \times 10^3$ | 0.00                    |
| 8   | $6.947 \times 10^3$ | -                       |

**Table 1.** Convergence of the perturbation ($\beta = 0$) growth rate $G$ with respect to polynomial order $P$.

4. Base Flow

The unperturbed development of the vortex ring flow is obtained by two-dimensional (2D) DNS over a time period of $\tau = 25$ and saved every $dT = 0.002$ for optimal transient growth calculations. A quick relaxation process is undertaken beforehand, considering that the Lamb-Oseen vortex profile along a curved axis is not an exact solution of the NS equations [10, 11, 20]. Therefore a vortex ring that satisfies the governing equations can be obtained after relaxation. The simplicity of the flow allows the application of symmetry in the simulation set up. The velocity boundary conditions of $U = U_0 = 0.225$, $V = 0$ and $W = 0$ are prescribed on the far-field boundaries to account for the induced axial motion generated by the vortex ring’s velocity field and the Reynolds number is set to 7500. The High-order pressure boundary condition as described in [23] is used on the far-field boundaries. These values are applied in all subsequent simulations unless otherwise stated.
Initially, the vortex is torus-shaped. As the vortex is evolved through to $t = 25$, it is seen to expand and decay due to the gradual diffusion of vorticity. This process has been described in detail by Archer et al. [11] for the evolution of a range of laminar vortex rings at various core thicknesses and Reynolds numbers.

5. Linear Transient Growth Study
In transient growth calculations, at each step when integrating the governing equations, the base flow is obtained by a Lagrangian interpolation of the saved slices [24]. In transient growth, each combination of the time horizon $\tau$ and azimuthal wavenumber $\beta$ produces a different trajectory of flow kinetic energy and hence must be varied to find the global optimum. Simulations were run for $0 \leq \beta \leq 20$ and $0 < \tau \leq 25$. $\tau$ was capped at $\tau = 25$ as, at too large a value, even the main vortex decays significantly. The logarithmic scaled growth rate $G$ is illustrated in Figure 2.

The largest value of transient growth is observed at $\beta = 0$ and $\tau = 25$, with another noticeable local maxima presenting at $\beta = 9$ for the same value of $\tau$. $\tau = 25$ is adopted as the time horizon for further investigation into the form and development of the optimal perturbations, due to its relative dominance over the largest values of growth recorded. The two predominant optimum perturbations of wave numbers $\beta = 0$ and $\beta = 9$ at $\tau = 25$ are discussed below:

At $\tau = 25$ and $\beta = 0$, the optimal growth reaches $6.9 \times 10^3$, almost 3 times greater than that of the next closest simulation ($2.4 \times 10^3$ at $\tau = 25$ and $\beta = 9$). The $\beta = 0$ mode has previously been observed experimentally by Maxworthy (1977), who described it as ‘axial flow’ along the circumferential axis of the vortex core and is the focus of a series of experimental studies conducted by Naitoh et al. (2002) [25, 26]. Archer et al. (2008) documented its prominence in the decay of a thin core ring at an intermediate Reynolds number of 7500. This revealed that the mode laid relatively dormant throughout the laminar phase of the ring’s evolution, before growing significantly during the ring’s transition to a turbulent regime. Such axial flow is not related to the formation of secondary vortical structures associated with the process of ring breakdown.

The large value of transient growth recorded for the $\beta = 9$ mode is also well matched to Archer et al. (2008). During the ring’s transition phase, Archer observed that the azimuthal instability (as a sum of all its individual modes) takes the form of 9 standing waves around
the vortex core. It should be noted, however, that the \( \beta = 9 \) mode has only been shown to hold significance to this specific vortex ring, and that the number of standing waves present in the azimuthal direction varies according to the ring’s slenderness ratio and Reynolds number. Alongside the \( \beta = 9 \) perturbation, modes \( \beta = 8 \), \( \beta = 10 \) and \( \beta = 1 \) (and \( \beta = 0 \)) also presented energy growth of a similarly large order (\( \sim 10^3 \)) and azimuthal wave forms incorporating 8 and 10 standing waves have both been shown to develop in previous DNS [11].

Distributions of the optimal perturbations relative to the positioning of the base flow are presented in the left hand column of Figure 3. It is observed that each optimal perturbation takes an almost identical position inside and in front of the ring along its trajectory. These perturbations are then re-evolved over the same time horizon to depict their final (linearly amplified) form at \( t = 25 \), as displayed in the right hand column of Figure 3. It is seen that the disturbances are drawn in through the ring and wrapped around its core. The final form of each perturbation is that of a roughly elliptic pair of oppositely signed regions of vorticity, positioned either side of the vortex ring’s core. The \( \beta = 9 \) mode takes the most compact form out of the three disturbances, perhaps a reflection of its larger wave number. Such 2D linear analysis, however, is inherently limited in studying the development of infinitesimally small perturbations. To reveal the nonlinear development of perturbations, 3D DNS are carried out in the following section.

6. 3D DNS

As discussed in Section 1, previous numerical investigations reported that the development of secondary vortical structures played a key role in the observed ring breakdown process [10, 11]. These secondary structures have also been visualised experimentally [13]. In order to reveal the activation of these vortical structures by initial perturbations, 3D DNS of the base flow initially seeded by the optimal initial perturbations with a small magnitude (at ratios of 0.01 \( \rightarrow \) 0.1) are carried out. The \( \beta = 0 \) mode is not presented here as this work focuses on the secondary vortical structures and induced breakdown of the vortex ring, rather than that of axial flow. The

Figure 3. Contours of azimuthal vorticity for the optimal initial perturbations at \( \tau = 25 \) and (a) \( \beta = 0 \) and (c) \( \beta = 9 \); optimal outcomes at \( t = \tau = 25 \) and (b) \( \beta = 0 \) and (d) \( \beta = 9 \). Contour levels are selected to highlight the structures relative to the development of the base flow.
Figure 4. Iso-surface of $\lambda_2 = -1$ for the optimally perturbed flow with $\beta = 9$, coloured by streamwise velocity (red for most positive and blue most negative), at (a) $t = 10$, (b) $t = 12$, (c) $t = 14$ and (d) $t = 16$.

The development of the flow initially perturbed by the optimal perturbation with $\beta = 9$ is illustrated in Figure 4. After 10 time units, the vortex ring’s core has deformed from the circular form, into a standing wave pattern incorporating 9 waves (see Figure 4(a)). As the flow develops, these deformations are seen to be severely pronounced and it becomes important to distinguish between inner and outer core vorticity. In this work ‘inner core’ vorticity is used to describe the original (deformed) core as pictured in Figure 4(a). The term ‘outer core’ vorticity is then used to refer to any secondary vortical structures that form around the original vortex core. This has previously been termed as ‘halo’ vorticity [11]. In Figure 4(b) it is shown that 9 distinct saddles of outer core vorticity are formed on the peaks and troughs of the standing wave’s outer and inner rims respectively. These saddles are isolated from each other at first and act on the vortex’s inner core so as to increase its deformation. Contrasting views have previously been given as to how the saddles develop, but in this case they are observed to result from the stretching of the outer rim regions of the inner core [10, 11]. As time reaches $t = 14$ (Figure 4(c)), the continual growth of the saddles and the progressive formation of a pair of secondary vortical loops at each saddle point are depicted. At $t = 16$ (Figure 4(d)), the neighbouring loops attach to form one interwoven mesh of outer core vorticity, the same secondary vortical structure as reported in previous DNS works [10, 11]. Here the inner core deformation is now appreciable as the inner core is forced into weaving its way through the centre of each loop [10]. It should also be noted that, throughout the time period pictured, a steady increase of vortical filaments ejected into the ring’s wake is observed, which is the first process of ring decay.

Despite the application of a unique initial perturbation, the similarity of the described mechanism to previous studies is clear. This is because this process is derived from the mode that undergoes the largest transient growth, and it is hence also likely to have been the dominant mode observed in previous DNS visualisations of related vortex rings.

7. Conclusion
An investigation aimed to identify the perturbation modes of largest transient growth and to also describe the nonlinear development of perturbations to a vortex ring were undertaken. A simplified model, obtained by rotating the Lamb-Oseen vortex profile in the azimuthal direction, at a Reynolds number of 7500 was established. The non-linear DNS of an unperturbed vortex ring was then completed and applied as a base flow in an investigation into the linear transient growth of a perturbed vortex ring. A well established time stepper approach, using the Arnoldi’s
method, was applied in this study to simulate the growth of perturbations at various wave numbers $\beta$ at several time horizons $\tau$. Optimal perturbations (those of largest transient growth) were identified at $\beta = 0$ and $\beta = 9$ at $\tau = 25$.

The optimal perturbations for each wave number were then combined with the base flow and evolved again over the same time horizon in a non-linear 3D DNS. From this, the resulting iso-surface plots enabled the visualisation and description of the azimuthal instability’s development, and this process was found to be in good agreement with the descriptions provided by Bergdorf et al. (2007) and Archer et al. (2008). The present work indicates that the previously observed breakup of vortex rings in DNS can be attributed to the transient dynamics of perturbations stemming from the centre region of the ring. The $\beta = 0$ mode was not presented as this work chose to focus on the formation of secondary vortical structures and induced breakdown of the vortex ring, which are not associated with the $\beta = 0$ mode.

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