Incidence bounds for complex algebraic curves on Cartesian products

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Abstract

We prove bounds on the number of incidences between a set of algebraic curves in $\mathbb{C}^2$ and a Cartesian product $A \times B$ with finite sets $A, B \subset \mathbb{C}$. Similar bounds are known under various conditions, but we show that the Cartesian product assumption leads to a simpler proof. This assumption holds in a number of interesting applications, and with our bound these applications can be extended from $\mathbb{R}$ to $\mathbb{C}$. We also obtain more precise information in the bound, which is used in several recent papers [13, 15, 21].

The proof is a new application of the polynomial partitioning technique introduced by Guth and Katz [10].

1 Introduction

Not many incidence bounds have been proved over the complex numbers. The quintessential incidence bound of Szemerédi and Trotter for points and lines in $\mathbb{R}^2$ was generalized to $\mathbb{C}^2$ by Tóth [20] and Zahl [22]. It states that for a finite set $P$ of points in $\mathbb{C}^2$ and a finite set $L$ of lines in $\mathbb{C}^2$, the number of incidences, denoted by $I(P, L) := |\{(p, \ell) \in P \times L : p \in \ell\}|$, satisfies

$$I(P, L) = O \left( |P|^{2/3} |L|^{2/3} + |P| + |L| \right).$$

The Szemerédi-Trotter bound was generalized to algebraic (and even continuous) curves in $\mathbb{R}^2$ by Pach and Sharir [12], but their result has not yet been fully extended to $\mathbb{C}^2$. Solymosi and Tao [18] and Zahl [22] did prove complex versions, but only for algebraic curves satisfying certain restrictions. These restrictions include the requirement that the intersections of the curves are transversal (i.e., the curves have distinct tangent lines at their intersection points), which does not hold in many potential applications.

We prove a Pach-Sharir-like incidence bound for algebraic curves in $\mathbb{C}^2$, under the assumption that the point set is a Cartesian product $A \times B$ with $A, B \subset \mathbb{C}$, but without the restrictions of [18, 22] on the curves. Like in [12, 18, 22], the curves must satisfy a degrees-of-freedom condition, which can come in different forms. Theorem 1 uses what is probably the most convenient condition. We have worked out in detail the dependence on
The parameters of this condition, which is of interest in certain applications, in particular [21, 15]. In Section 5 we also state a version of this bound for well-behaved surfaces in \( \mathbb{R}^4 \), which is used in [13].

**Theorem 1.** Let \( A \) and \( B \) be finite subsets of \( \mathbb{C} \) with \( |A| = |B| \), and set \( \mathcal{P} = A \times B \). Let \( \mathcal{C} \) be a finite set of algebraic curves in \( \mathbb{C}^2 \) of degree at most \( d \), such that any two points of \( \mathcal{P} \) are contained in at most \( M \) curves of \( \mathcal{C} \), with \( M \geq d \). Then

\[
I(\mathcal{P}, \mathcal{C}) = O(d^{4/3}M^{1/3} |\mathcal{P}|^{2/3} |\mathcal{C}|^{2/3} + M \log M |\mathcal{P}| + d^4 |\mathcal{C}|).
\]

Although the assumption that the point set is a Cartesian product is very restrictive, it is satisfied in a number of interesting problems. We give several examples of such applications in Section 6. These include an answer to a question of Elekes [6] related to sum-product estimates, and a generalization to \( \mathbb{C} \) of a recent result of Sharir, Sheffer, and Solymosi [16] on distinct distances between lines. More sophisticated applications can be found in the recent works [13, 15, 21], which were in fact the original motivation for this paper.

We begin in Section 2 with the elementary proof of the real analogue of our main theorem, which is not a new result, but serves as an introduction to our main proof. In Section 3 we collect the technical tools that we will use. Then, in Section 4 we give our main proof, and in Section 5 we prove several corollaries, including Theorem 1 above. Finally, in Section 6 we give three applications.

## 2 Warmup: Points and curves in \( \mathbb{R}^2 \).

As a warmup for the complex case, we first give a proof of the corresponding statement for incidences between real algebraic curves and a Cartesian product in \( \mathbb{R}^2 \). This is not a new result, as it follows from the work of Pach and Sharir [12]. The proof given here is, however, much simpler, because the product structure allows for a trivial partitioning of the plane. A similar approach was taken by Solymosi and Vu [19].

**Theorem 2.** Let \( A \) and \( B \) be finite subsets of \( \mathbb{R} \) and \( \mathcal{P} = A \times B \). Let \( \mathcal{C} \) be a finite set of algebraic curves in \( \mathbb{R}^2 \) of degree \( d \) such that any two points of \( \mathcal{P} \) are contained in at most \( M \) curves of \( \mathcal{C} \). We assume that no curve in \( \mathcal{C} \) contains a horizontal or vertical line, that \( d^4 |\mathcal{C}| \leq M |\mathcal{P}|^2 \), and that \( |A| \leq |B| \) and \( d |\mathcal{C}| \geq M |B|^2 / |A| \). Then

\[
I(\mathcal{P}, \mathcal{C}) = O(d^{2/3}M^{1/3} |\mathcal{P}|^{2/3} |\mathcal{C}|^{2/3}).
\]

**Proof.** Let \( r \) be a real number, to be chosen at the end of the proof, satisfying \( d \leq r \leq |A| \). We cut \( \mathbb{R} \) in \( O(r) \) points that are not in \( A \), splitting \( \mathbb{R} \) into \( O(r) \) intervals so that each interval contains \( O(|A|/r) \) elements of \( A \) (this is possible because \( r \leq |A| \)). Similarly, we obtain \( O(r) \) cutting points not in \( B \) that split \( \mathbb{R} \) into at most \( O(r) \) intervals, each containing \( O(|B|/r) \) elements of \( B \) (using \( r \leq |A| \leq |B| \)). This gives a partition of \( \mathbb{R}^2 \) into \( O(r^2) \) open cells (which are rectangles) and a closed boundary (consisting of \( O(r) \) lines). Each cell contains \( O(|A||B|/r^2) = O(|\mathcal{P}|/r^2) \) points of \( \mathcal{P} \), while the boundary is disjoint from \( \mathcal{P} \).

We need to bound the number of cells that a curve \( C \in \mathcal{C} \) can intersect. The curve has \( O(d^2) \) connected components (see Lemma 4 below), and it has at most \( d \) intersection points with each of the \( O(r) \) cell walls by Bézout’s Inequality (see Lemma 3 below), using the fact
that it contains no horizontal or vertical line. Thus the $O(d^2)$ connected components are “cut” in $O(dr)$ points. By wiggling the cutting lines slightly, we can ensure that they do not hit a curve in a singularity, since algebraic curves have finitely many singularities (see Section 3). Therefore, each cut increases the number of connected components by at most one (this wouldn’t be true if a cutting point were a singularity). Thus any $C \in \mathcal{C}$ intersects $O(d^2 + dr) \leq O(dr)$ (using $d \leq r$) of the $O(r^2)$ cells.

Let $I_1$ be the subset of incidences $(p, C)$ such that $p$ is the only incidence of $C$ in the cell containing $p$, and let $I_2$ be the subset of incidences $(p, C)$ such that $C$ has at least one other incidence in the cell that contains $p$. Then, since a curve intersects $O(dr)$ cells, we have

$$I_1 = O(dr|\mathcal{C}|).$$

On the other hand, given two points in one cell, there are by assumption at most $M$ curves in $\mathcal{C}$ that contain both points. Thus, in a cell with $k$ points there are at most $2M \left( \binom{k}{2} \right) = O(Mk^2)$ incidences from $I_2$. Therefore, summing over all cells, we have

$$I_2 = O \left( r^2 \cdot M \left( \frac{|\mathcal{P}|}{r^2} \right)^2 \right) = O \left( \frac{M|\mathcal{P}|^2}{r^2} \right).$$

Choosing $r^3 = \frac{M|\mathcal{P}|^2}{d|\mathcal{C}|}$ gives

$$I(\mathcal{P}, \mathcal{C}) = I_1 + I_2 = O \left( d^{2/3}M^{1/3}|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} \right).$$

We have to verify that $d \leq r$ and $r \leq |A|$. The first follows from the assumption $d^4|\mathcal{C}| \leq M|\mathcal{P}|^2$ and

$$r = d^{-1/3} \left( \frac{M|\mathcal{P}|^2}{|\mathcal{C}|} \right)^{1/3} \geq d^{-1/3} \left( \frac{d^4|\mathcal{C}|}{|\mathcal{C}|} \right)^{1/3} = d.$$

The second follows from the assumption $d|\mathcal{C}| \geq M|B|^2/|A|$ and

$$r^3 = \frac{M|\mathcal{P}|^2}{d|\mathcal{C}|} \leq \frac{M|\mathcal{P}|^2}{M|B|^2/|A|} = |A|^3.$$

This completes the proof.

Theorem 2 has several conditions which can be simplified in various ways to make the statement more suitable for application. We state one version here as an example, but we refer to Section 3 for the proof (which is identical to that of the complex version presented there).

Corollary 3. Let $A$ and $B$ be finite subsets of $\mathbb{R}$ with $|A| = |B|$, and $\mathcal{P} = A \times B$. Let $\mathcal{C}$ be a finite set of algebraic curves in $\mathbb{R}^2$ of degree at most $d$, such that any two points of $\mathcal{P}$ are contained in at most $M$ curves of $\mathcal{C}$. Then

$$I(\mathcal{P}, \mathcal{C}) = O \left( d^{2/3}M^{1/3}|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + M \log M|\mathcal{P}| + d^2|\mathcal{C}| \right).$$

1This fact can be obtained more directly using Lemma 6 below, by noting that the lines are defined by a polynomial $f$ of degree $O(r)$, so $\mathcal{C}\setminus Z(f)$ has $O(dr)$ connected components. However, we have used the argument above because it will play a crucial role in the proof of our main theorem.
3 Definitions and tools

3.1 Definitions

We introduce a few definitions and basic facts from algebraic geometry in some detail, because the subtleties of real and complex varieties play a role in our proof. A variety in \( \mathbb{C}^D \) is a set of the form

\[
Z_{\mathbb{C}^D}(f_1, \ldots, f_m) := \{(z_1, \ldots, z_D) \in \mathbb{C}^D : f_i(z_1, \ldots, z_D) = 0 \text{ for } i = 1, \ldots, m\},
\]

for polynomials \( f_i \in \mathbb{C}[z_1, \ldots, z_D] \). Such sets are normally called affine varieties (or just zero sets), but since this is the only type of variety that we consider, we refer to them simply as varieties. Similarly, we define a real variety to be a zero set of the form

\[
Z_{\mathbb{R}^D}(f_1, \ldots, f_m) := \{(x_1, \ldots, x_D) \in \mathbb{R}^D : f_i(x_1, \ldots, x_D) = 0 \text{ for } i = 1, \ldots, m\}
\]

with polynomials \( f_i \in \mathbb{R}[x_1, \ldots, x_D] \). We refer to [11] for definitions of the complex dimension of a complex variety \( V \), denoted by \( \text{dim}_{\mathbb{C}}(V) \), and we refer to [3, Section 5.3] for a careful definition of the real dimension of a real variety \( W \), denoted by \( \text{dim}_{\mathbb{R}}(W) \). Alternatively, one can locally view a real variety as a real manifold (around any nonsingular point, see below), and the real dimension equals the dimension in the manifold sense (more precisely, it is the maximum dimension at any nonsingular point).

A complex algebraic curve in \( \mathbb{C}^2 \) is a variety \( V \) with \( \text{dim}_{\mathbb{C}}(V) = 1 \), or equivalently, a set of the form \( Z_{\mathbb{C}^2}(f) \) for \( f \in \mathbb{C}[z_1, z_2] \setminus \{0\} \). A variety \( W \subset \mathbb{R}^D \) is a real algebraic curve in \( \mathbb{R}^D \) if \( \text{dim}_{\mathbb{R}}(W) = 1 \), and it is a real algebraic surface if \( \text{dim}_{\mathbb{R}}(W) = 2 \). A real curve in \( \mathbb{R}^2 \) can be equivalently defined as any infinite set of the form \( Z_{\mathbb{R}^2}(f) \) with \( f \in \mathbb{R}[x, y] \setminus \{0\} \). For a curve \( C \) in \( \mathbb{C}^2 \) or \( \mathbb{R}^2 \), we define the degree \( \deg(C) \) to be the minimum degree of a polynomial defining the curve. For convenience, we will occasionally use the notion of a semialgebraic curve in \( \mathbb{R}^2 \), which is a subset of a real curve defined by polynomial inequalities; in particular, if we remove a finite point set from a real curve, the connected components of the remainder are semialgebraic curves.

A curve \( C \subset \mathbb{C}^2 \) is irreducible if there is an irreducible \( f \) such that \( C = Z_{\mathbb{C}^2}(f) \). An irreducible component of an algebraic curve \( C \subset \mathbb{C}^2 \) is an irreducible algebraic curve \( C' \) such that \( C' \subset C \). A curve in \( \mathbb{C}^2 \) of degree \( d \) has a decomposition as a union of at most \( d \) irreducible components.

We also need to consider singularities of curves, but only for real curves in \( \mathbb{R}^2 \) or \( \mathbb{R}^4 \). For a curve in \( \mathbb{R}^D \), we define a point on the curve to be a singularity if it does not have a neighborhood in which the curve is a real manifold of dimension one (a more algebraic definition can be found in [11, Lecture 14]). For a curve in \( \mathbb{R}^2 \), a singularity of \( Z_{\mathbb{R}^2}(f) \) can be equivalently defined as a point \( p \in \mathbb{R}^2 \) such that \( f(p) = (\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0 \). A key fact that we need is that the number of singularities of a curve in \( \mathbb{R}^2 \) of degree \( d \) is less than \( d^2 \). More precisely, define the branches of a singularity in some small neighborhood to be the connected components of the curve in that neighborhood after removing the singularity. Then the total number of branches over all singularities, for any choice of neighborhoods, is at most \( d^2 \) (see [9, Chapter 3]). For a curve in \( \mathbb{R}^4 \), we only need the fact that the number of singularities is finite (see [18, Proposition 4.4]).
3.2 Intersection bounds

In this paper we will frequently have to bound the size of the intersection of two varieties, both over $\mathbb{C}$ and over $\mathbb{R}$. The prototype for such intersections bounds is the following lemma. Here we consider the degree of a finite point set to be its size, so the lemma says that the intersection of two curves is either a finite set of bounded size, or a curve of bounded degree.

**Lemma 4** (Bézout’s Inequality). If $C_1$ and $C_2$ are algebraic curves in $\mathbb{C}^2$ or $\mathbb{R}^2$, then
\[ \deg(C_1 \cap C_2) \leq \deg(C_1) \cdot \deg(C_2). \]

With the right definition of degree, this inequality can be extended to varieties in $\mathbb{C}^D$, but in $\mathbb{R}^D$, the inequality may fail in this form. Nevertheless, various cautious bounds on the number of connected components of intersections of real varieties have been proved, which can often serve the same purpose; see for instance [3, Chapter 7]. We will use the following recent result of Barone and Basu [1]. It gives a refined bound when the defining polynomials of the variety have different degrees, which is crucial in our proofs. We state it in a similar way to Basu and Sombra [4, Theorem 2.5], with some modifications based on the more general form in [1]. We simplify the statement of the bound somewhat using the following (non-standard) definition.

**Definition 5.** Let $V = Z_{\mathbb{R}^D}(g_1, \ldots, g_m)$ have dimension $k_m$, with $\deg(g_1) \leq \cdots \leq \deg(g_m)$. Write $\dim_{\mathbb{R}}(Z_{\mathbb{R}^D}(g_1, \ldots, g_i)) = k_i$ and $k_0 = D$. We define the Barone-Basu degree of $V$ by
\[ \deg_{BB}(V) = \prod_{i=1}^m \deg(g_i)^{k_{i-1} - k_i}. \]

**Lemma 6** (Barone-Basu). Let $V = Z_{\mathbb{R}^D}(g_1, \ldots, g_m)$ with $\deg(g_1) \leq \cdots \leq \deg(g_m)$. Let $h \in \mathbb{R}[x_1, \ldots, x_D]$ with $\deg(h) \geq \deg(g_m)$. Then the number of connected components of both $V \cap Z_{\mathbb{R}^D}(h)$ and $V \setminus Z_{\mathbb{R}^D}(h)$ is
\[ O \left( \deg_{BB}(V) \cdot \deg(h)^{\dim_{\mathbb{R}}(V)} \right). \]

In the ideal case where each $k_i = D - i$, this would be a natural generalization of Lemma [4]. On the other hand, if $\deg(g_i) \leq d$ for each $i$, we get the bound $O(d^{D-k_m} \deg(h)^{k_m})$, without any individual conditions on the $g_i$. The fact that $h$ can be arbitrary allows for the following trick to deal with more polynomials in the role of $h$: To bound the number of connected components of, say, $Z_{\mathbb{R}^D}(g_1, \ldots, g_m) \setminus Z_{\mathbb{R}^D}(h_1, h_2)$, one can simply set $h := h_1^2 + h_2^2$ and use the lemma.

We need a simple fact about the surface in $\mathbb{R}^4$ associated to a curve in $\mathbb{C}^2$.

**Lemma 7.** Let $C \subset \mathbb{C}^2$ be an algebraic curve of degree $d$. Then the associated real surface $S$ in $\mathbb{R}^4$ has $\deg_{BB}(S) \leq d^2$.

**Proof.** Let $f(x, y)$ be a minimum-degree polynomial such that $C = Z_{\mathbb{C}^2}(f)$. Let
\[ g_1(x_1, x_2, x_3, x_4) := \text{Re}(f(x_1 + ix_2, x_3 + ix_4)), \quad g_2(x_1, x_2, x_3, x_4) := \text{Im}(f(x_1 + ix_2, x_3 + ix_4)) \]
be the associated real polynomials defining $S$. Both have degree at most $d$. Write $k_0 := 4$, $k_1 := \dim_{\mathbb{R}}(Z_{\mathbb{R}^4}(g_1))$, and $k_2 := \dim_{\mathbb{R}}(Z_{\mathbb{R}^4}(g_1, g_2))$ as in Definition [5]. We clearly have $k_2 = 2$. Then $k_1 \in \{2, 3, 4\}$, and, whichever it is, we get
\[ d_1^{k_0 - k_1} \cdot d_2^{k_1 - k_2} \leq d^2. \]

This completes the proof. \(\square\)
3.3 Polynomial partitioning

Our proof relies on the following technique introduced by Guth and Katz \[10\].

**Lemma 8** (Polynomial partitioning). *Let \(A\) be a finite subset of \(\mathbb{R}^2\). For any \(r \in \mathbb{R}\) with \(1 \leq r \leq |A|^{1/2}\) there exists a polynomial \(f \in \mathbb{R}[x,y]\) of degree \(O(r)\) such that \(\mathbb{R}^2 \setminus Z_{\mathbb{R}^2}(f)\) has \(O(r^2)\) connected components, each containing \(O(|A|/r^2)\) points of \(A\).*

In the proof of Theorem \[2\] we in fact used a trivial partitioning on \(\mathbb{R}\): For \(A \subset \mathbb{R}\) and any \(1 \leq r \leq |A|\), there is a subset \(X \subset \mathbb{R} \setminus A\) of size \(O(r)\) such that \(\mathbb{R} \setminus X\) has \(O(r)\) connected components, each containing \(O(|A|/r^2)\) points of \(A\). Moreover, the points of \(X\) have some “wiggle room”, in the sense that they can be varied in some small neighborhood without affecting the partitioning property. We now show that a real algebraic curve can be partitioned in a similar way.

Such a partitioning would not be possible for arbitrary continuous curves with self-intersections, or for algebraic curves of arbitrary degree. If we take an arbitrary point set in general position and connect any two points by a line, the union of the lines is an algebraic curve of high degree that cannot be partitioned with a small number of cutting points on the curve. However, on an algebraic curve of bounded degree \(D\), one can control the number of self-intersections (singularities) of the curve in terms of \(D\), and this allows us to partition it into \(O(D^2)\) pieces.

**Lemma 9** (Partitioning a real algebraic curve). *Let \(C \subset \mathbb{R}^2\) be an algebraic curve of degree \(D\), containing a finite set \(A\). Then there exists \(X \subset C \setminus A\) of \(O(D^2)\) points, such that \(C \setminus X\) consists of \(O(D^2)\) connected semialgebraic curves, each containing \(O(|A|/D^2)\) points of \(A\). Moreover, each point \(p \in X\) has a small open neighborhood on \(C\) such that any point of that neighborhood could replace \(p\) without affecting the partitioning property.*

**Proof.** Around every singularity \(p\) of \(C\), choose a sufficiently small closed ball \(B_p\) with boundary circle \(R_p\), so that \(B_p\) contains no other singularities of \(C\), and no point of \(A\) other than possibly \(p\) itself. We put the points of \(C \cap R_p\) into \(X\) for each \(p\). For each singularity \(p\), \(|C \cap R_p|\) is at most the number of branches of \(C\) at \(p\) in the neighborhood \(B_p\) (as defined above), and the total sum of these numbers is at most \(D^2\). Hence we have put at most \(D^2\) points into \(X\). The points of \(X\) are themselves not singularities, so removing a point of \(X\) increases the number of connected components by at most one, since around such a point \(C\) is a one-dimensional manifold.

By Lemma \[6\] \(C\) has at most \(O(D^2)\) connected components, so removing the points of \(X\) cuts \(C\) into \(O(D^2)\) connected semialgebraic curves. Each of these semialgebraic curves either contains at most one point of \(A\) (a singularity), or it is simple (i.e., it has no self-intersections). We can cut these simple curves at a total of \(O(D^2)\) points, so that every resulting curve contains \(O(|A|/D^2)\) points of \(A\), and no cutting point is in \(A\). Adding these cutting points to \(X\) completes the proof. It should be clear that shifting the cutting points within a small open neighborhood will not affect the proof. \(\square\)
4 Points and curves in $\mathbb{C}^2$

In this section we prove our main incidence bound for points and curves in $\mathbb{C}^2$; in Section 5, we will deduce Theorem 1. Throughout most of the proof, we identify $\mathbb{C}$ with $\mathbb{R}^2$, and we work with the surfaces in $\mathbb{R}^4$ associated to the curves in $\mathbb{C}^2$, but we do keep in mind the fact that these surfaces come from complex curves.

We use the following terminology for the cell decomposition of $\mathbb{R}^4$ obtained by partitioning: a $k$-cell is a connected set of dimension $k$ that will be used in the final cell decomposition; a $k$-wall is a $k$-dimensional variety that cuts out the $(k+1)$-cells, but that is itself to be decomposed into $k$-cells; a $k$-gap is a $k$-dimensional variety that also helps to cut out the $(k+1)$-cells, but does not contain any incidences, so does not need to be partitioned further. To summarize: $\mathbb{R}^4$ is partitioned into 4-cells by 3-walls and 3-gaps; each 3-wall is then partitioned into 3-cells by 2-walls and 2-gaps; the 2-walls are then partitioned into 2-cells using only 1-gaps.

**Theorem 10.** Let $A_1$ and $A_2$ be finite subsets of $\mathbb{C}$ and $P = A_1 \times A_2$. Let $\mathcal{C}$ be a finite set of algebraic curves of degree $d$ in $\mathbb{C}^2$, such that any two points of $P$ are contained in at most $M$ curves of $\mathcal{C}$. Assume that no curve in $\mathcal{C}$ contains a horizontal or vertical line. Also assume that $d^8 |C| \leq M |P|^2$, and that $|A_1| \leq |A_2|$ and $d^2 |C| \geq M |A_2|^2 / |A_1|$. Then

$$I(P, C) = O(d^{1/3} M^{1/3} |P|^{2/3} |C|^{2/3}).$$

*Proof.* Let $\mathcal{S}$ be the set of surfaces $S$ in $\mathbb{R}^4$ associated to the curves $C$ of $\mathcal{C}$ in $\mathbb{C}^2$. By Lemma 7, the surfaces $S \in \mathcal{S}$ have $\deg_{BB}(S) \leq d^2$. As in the proof of Theorem 2, we partition the space, see how the varieties intersect the cells, and then use a simple counting argument. We note that the counting is exactly as in Theorem 2 with the parameters $d$ and $r$ from that proof replaced by the parameters $d^2$ and $r^2$ in this proof.

**Partitioning.** We partition $\mathbb{R}^4$ into $O(r^4)$ cells and some gaps, so that each cell contains $O(|P|/r^4)$ points of $P$, and the gaps contain no points of $P$. We assume that $d^2 \leq r^2 \leq |A_1|$.

We use Lemma 8 to get polynomials $f_1, f_2$ of degree $r \leq |A_1|^{1/2}$ so that $C_i := Z_{\mathbb{R}^2}(f_i)$ partitions $\mathbb{R}^2$ into $r^2$ cells, each containing $O(|A_i|/r^2)$ points of $A_i$. Then we use Lemma 9 to partition $C_1$ and $C_2$, obtaining sets $X_i \subset C_i \setminus A_i$ with $|X_i| = O(r^2)$, so that $C_i \setminus X_i$ consists of $O(r^2)$ connected components, each containing $O(|A_i|/r^2)$ points of $A_i$.

The 3-walls $C_1 \times \mathbb{R}^2$ and $\mathbb{R}^2 \times C_2$ partition $\mathbb{R}^4$ into $O(r^4)$ 4-cells, each containing $O(|P|/r^4)$ points of $P$. The 3-wall $C_1 \times \mathbb{R}^2$ is partitioned by the 2-wall $C_1 \times C_2$, combined with the 2-gap $X_1 \times \mathbb{R}^2$; similarly, $\mathbb{R}^2 \times C_2$ is partitioned by the 2-wall $C_1 \times C_2$ and the 2-gap $\mathbb{R}^2 \times X_2$. Thus the 3-walls are partitioned into $O(r^4)$ 3-cells, each containing $O(|P|/r^4)$ points of $P$. The gaps are not partitioned further. The 2-wall $C_1 \times C_2$ is partitioned by the 1-gaps $X_1 \times C_2$ and $C_1 \times X_2$, again resulting in $O(r^4)$ cells, each containing $O(|P|/r^4)$ points of $P$. This completes the partitioning. Altogether there are $O(r^4)$ cells, each containing $O(|P|/r^4)$ points of $P$.

**Intersections.** We now show that any surface $S \in \mathcal{S}$ intersects $O(d^2 r^2)$ of the $O(r^4)$ cells, and we do this separately for the 4-cells, 3-cells, and 2-cells.

*4-cells:* Note that $(C_1 \times \mathbb{R}^2) \cup (\mathbb{R}^2 \times C_2) = Z_{\mathbb{R}^4}(f_1 f_2)$. We apply Lemma 8 to deduce that $S \setminus Z_{\mathbb{R}^4}(f_1 f_2)$ has

$$O(\deg_{BB}(S) \cdot \deg(f_1 f_2)^{\dim_k(S)}) = O(d^2 \cdot (2r)^2)$$

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connected components; here \( \deg_{BB}(S) \leq d^2 \) by Lemma 7 and the condition of Lemma 6 (that \( \deg(g_m) \leq \deg(h) \), in this case \( \deg(g_2) \leq \deg(f_1 f_2) \), where \( g_2 \) is one of the two real polynomials defining \( S \)) follows from the assumption \( d^2 \leq r^2 \). This means that \( S \) intersects \( O(d^2 r^2) \) of the 4-cells.

3-cells: Set \( S_1 := S \cap (C_1 \times \mathbb{R}^2) \). Note that \( \dim_{\mathbb{R}}(S_1) \leq 1 \), because for any point \( p \in \mathbb{R}^2 \) the fiber \((p \times \mathbb{R}^2) \cap S\) is finite, since it corresponds to the intersection of the vertical line \( p \times \mathbb{C} \) in \( \mathbb{C}^2 \) with a complex curve that does not contain any vertical line. If \( \dim_{\mathbb{R}}(S_1) = 0 \), then by Lemma 6 \( S_1 = S \cap Z_{\mathbb{R}^4}(f_1) \) consists of \( O(d^2 \cdot r^2) \) points, so it intersects at most that many cells. Hence we can assume \( \dim_{\mathbb{R}}(S_1) = 1 \). To see how many 3-cells inside \( C_1 \times \mathbb{R}^2 \) are intersected by \( S_1 \), we separately consider its intersection with the 2-wall \( C_1 \times C_2 \), and with the 2-gap \( X_1 \times \mathbb{R}^2 \).

The fact that \( \dim_{\mathbb{R}}(S_1) = 1 \) implies that \( \deg_{BB}(S_1) = O(d^2 r) \). Therefore, by Lemma 6 \( S_1 \setminus (C_1 \times C_2) = S_1 \setminus Z_{\mathbb{R}^4}(f_2) \) has

\[
O(\deg_{BB}(S_1) \cdot \deg(f_2)^{\dim_{\mathbb{R}}(S_1)}) = O(d^2 r \cdot r)
\]

connected components. Hence the wall \( C_1 \times C_2 \) cuts \( S_1 \) into \( O(d^2 r^2) \) connected semialgebraic curves.

Now consider the gap \( X_1 \times \mathbb{R}^2 \). For \( p \in X_1 \), we have \( S_1 \cap (p \times \mathbb{R}^2) \subset S \cap (p \times \mathbb{R}^2) \), and \( S \cap (p \times \mathbb{R}^2) \) is finite, again because \( S \) comes from a complex curve that does not contain a vertical line \( p \times \mathbb{C} \). Since we can write \( p \times \mathbb{R}^2 = Z_{\mathbb{R}^4}((x_1 - p_x)^2 + (x_2 - p_y)^2) \), it follows from Lemma 6 that \( \dim_{\mathbb{R}}(S \cap (p \times \mathbb{R}^2)) = O(d^2 \cdot 2^2) \). Thus the curve \( S_1 \) has

\[
|S_1 \cap (X_1 \times \mathbb{R}^2)| = O(d^2 \cdot |X_1|) = O(d^2 r^2)
\]

points of intersection with this gap. Moreover, using the “wiggle room” for the points in \( X_1 \) mentioned in Lemma 9 we can assume that none of the points in \( S_1 \cap (X_1 \times \mathbb{R}^2) \) is a singularity of \( S_1 \). Hence, removing such a point increases the number of connected components by at most one (which would not quite be true at a singularity). With that it finally follows that \( S_1 \) intersects \( O(d^2 r^2) \) of the 3-cells inside \( C_1 \times \mathbb{R}^2 \).

A symmetric argument gives the same bounds for \( S_2 := S \cap (\mathbb{R}^2 \times C_2) \), so altogether we get that \( S \) intersects \( O(d^2 r^2) \) of the 3-cells inside \( \mathbb{R}^2 \times C_2 \). Note that \( X_1 \) should be chosen so that \( X_1 \times \mathbb{R}^2 \) avoids the singularities of \( S_1 \) for all \( S \in \mathcal{S} \) simultaneously, but this is possible since there are finitely many points to avoid, while there is infinite wiggle room.

2-cells: Set \( S_3 := S \cap (C_1 \times C_2) \). The 2-wall \( C_1 \times C_2 \) is partitioned by the 1-gaps \( X_1 \times C_2 \) and \( C_1 \times X_2 \). As above we have \( |S_3 \cap (p \times C_2)| = O(d^2) \) for \( p \in X_1 \), so we get \( |S_3 \cap (X_1 \times C_2)| = O(d^2 r^2) \) and similarly \( |S_3 \cap (C_1 \times X_2)| = O(d^2 r^2) \). Finally, we can write \( S_3 = S \cap Z_{\mathbb{R}^4}(f_1^2 + f_2^2) \), so \( S_3 \) has \( O(d^2 \cdot (2r)^2) \) connected components. Again each cut increases the number of connected components by at most one, so altogether \( S_3 \) intersects \( O(d^2 r^2) \) of the 2-cells.

\(^2\)Note that \( X_1 \times \mathbb{R}^2 \) is defined by a polynomial \( g \) of degree \( 2 |X_1| = O(r^2) \). Thus, applying Lemma 6 to \( S_1 \setminus Z_{\mathbb{R}^4}(g) \) gives \( O(d^2 r \cdot r^2) \), which is too large. This is why we need a more refined argument, using the specific nature of \( X_1 \times \mathbb{R}^2 \).

\(^3\)We could do slightly better by observing that the intersection of the complex curve and the line is at most \( d \), but it would not bring any benefit, and it would cause a problem in Corollary 14 below.
Counting. Let $I_1$ be the subset of incidences $(p, S)$ such that $p$ is the only incidence of $S$ in the cell containing $p$, and let $I_2$ be the subset of incidences $(p, S)$ such that $S$ has at least one other incidence in the cell that contains $p$. Then (much like in the proof of Theorem 2)

$$I_1 = O(d²r²|S|)$$

and

$$I_2 = O(r^4 \cdot M \cdot \left(\frac{|P|}{r^4}\right)^2) = O(M \cdot \frac{|P|^2}{r^4}).$$

Setting $r^6 = \frac{M|P|^2}{|S|}$ gives

$$I(P, S) = O(d^{4/3}M^{1/3}|P|^{2/3}|S|^{2/3}).$$

We need to assure that $d^2 \leq r^2 \leq |A_1|$; this follows from the two assumptions of the theorem, by the same calculation as in the proof of Theorem 2 (with $d$ and $r$ replaced by $d^2$ and $r^2$).

5 Corollaries of Theorem 10

In this section we show how to remove the somewhat impractical conditions of Theorem 10. The techniques used here are mostly standard. We use the Kővári-Sós-Turán theorem, which bounds the number of edges in a graph not containing a complete bipartite graph \(K_{2,L}\). See Bollobás [2, Theorem IV.9] for the version stated here, with dependence on \(L\). We will apply this to the incidence graph of the points and curves, which is the bipartite graph whose vertex sets are the points and the curves, with an edge between any point and curve that are incident.

**Theorem 11.** Let \(G\) be a bipartite graph with vertex set \(X \cup Y\). Suppose that \(G\) contains no \(K_{2,L}\), i.e., for any two vertices in \(X\), there are at most \(L\) vertices in \(Y\) connected to both. Then

$$|E(G)| = O(L^{1/2}|X||Y|^{1/2} + |Y|).$$

Note that the counting step in the proofs of Theorems 2 and 10 could also be done using Theorem 11. Similarly, we could use the general Kővári-Sós-Turán theorem (or generalize our counting step) to obtain an incidence bound for curves whose incidence graph does not contain a \(K_{k,L}\), i.e., curves with “\(k\) degrees of freedom” as in [12]. Because the applications we have in mind are all about curves with two degrees of freedom, and because it would considerably complicate the calculations, we have not included this generalization.

We now deduce our first practical version of Theorem 10.

**Corollary 12.** Let \(A\) and \(B\) be finite subsets of \(\mathbb{C}\) with \(|A| \leq |B|\), and let \(P = A \times B\). Let \(C\) be a finite set of algebraic curves in \(\mathbb{C}^2\) of degree \(d\) without common components, such that any two points of \(P\) are contained in at most \(M\) curves of \(C\), and \(M \geq d\). Then

$$I(P, C) = O(d^{4/3}M^{1/3}|P|^{2/3}|C|^{2/3} + d^{-1}M|A|^{-1/2}|B|^{5/2} + d^4|C|).$$
Corollary 13.

Proof. If a curve in \( C \) contains horizontal or vertical lines, we split them off from the curve and account for their incidences separately. Let \( C_1 \) be the set of horizontal and vertical lines contained in curves of \( C \), and let \( C_2 \) be the curves that remain after these lines have been removed. Since the curves have no common components, each horizontal or vertical line occurs at most once, and any point is contained in at most two such lines, so

\[
I(\mathcal{P}, \mathcal{C}_1) \leq 2|\mathcal{P}|.
\]

We can apply Theorem 10 to obtain a bound on \( I(\mathcal{P}, \mathcal{C}_2) \), unless we have \( d^8|\mathcal{C}| > M|\mathcal{P}|^2 \) or \( d^2|\mathcal{C}| < M|B|^2/|A| \).

Suppose that \( d^8|\mathcal{C}| > M|\mathcal{P}|^2 \). Since the incidence graph with vertex set \( \mathcal{P} \cup \mathcal{C} \) contains no \( K_{2,M} \), we can use Theorem 11 to get

\[
I(\mathcal{P}, \mathcal{C}) = O(M^{1/2}|\mathcal{P}||\mathcal{C}|^{1/2} + |\mathcal{C}|) = O(d^4|\mathcal{C}|).
\]

Suppose that \( d^2|\mathcal{C}| < M|B|^2/|A| \). Because the curves do not have common components, any two curves intersect in at most \( d \) points by Bézout’s Inequality. Thus the incidence graph contains no \( K_{d^2+1,2} \), so by Theorem 11 we have

\[
I(\mathcal{P}, \mathcal{C}) = O((d^2)^{1/2}|\mathcal{C}||\mathcal{P}|^{1/2} + |\mathcal{P}|) = O(d^{-1}M|A|^{-1/2}|B|^{5/2}).
\]

Combining these bounds finishes the proof. \(\square\)

With a little more work, we can remove the condition that no two curves have a common component, with almost no effect on the bound. This is Theorem 11.

Corollary 13. Let \( A \) and \( B \) be finite subsets of \( \mathbb{C} \) with \(|A| = |B|\), and \( \mathcal{P} = A \times B \). Let \( \mathcal{C} \) be a finite set of algebraic curves in \( \mathbb{C}^2 \) of degree at most \( d \), such that any two points of \( \mathcal{P} \) are contained in at most \( M \) curves of \( \mathcal{C} \), with \( M \geq d \). Then

\[
I(\mathcal{P}, \mathcal{C}) = O(d^{4/3}M^{1/3}|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + M \log M|\mathcal{P}| + d^4|\mathcal{C}|).
\]

Proof. We have to deal with horizontal or vertical lines in the curves, and with the case \( d^2|\mathcal{C}| < M|\mathcal{P}|^{1/2} \); the case \( d^8|\mathcal{C}| > M|\mathcal{P}|^2 \) can be treated as in Corollary 12.

Let \( \mathcal{C}_1 \) be the \emph{multiset} of horizontal and vertical lines contained in curves of \( \mathcal{C} \), and let \( \mathcal{C}_2 \) be the curves that remain after these lines have been removed. In total there are at most \( d|\mathcal{C}| \) lines in \( \mathcal{C}_1 \) (counted with multiplicity). The lines that contain at most one point of \( \mathcal{P} \) together give at most \( d|\mathcal{C}| \) incidences. The lines that contain at least two points of \( \mathcal{P} \) have multiplicity at most \( M \) by assumption, so a point on such a line is contained in at most \( 2M \) such lines (with multiplicity), resulting in at most \( 2M|\mathcal{P}| \) incidences. Hence

\[
I(\mathcal{P}, \mathcal{C}_1) = O(M|\mathcal{P}| + d|\mathcal{C}|).
\]

Now suppose \( d^2|\mathcal{C}| < M|\mathcal{P}|^{1/2} \). We split each curve in \( \mathcal{C}_2 \) into its at most \( d \) irreducible components. The components that contain at most one point of \( \mathcal{P} \) give altogether at most \( d|\mathcal{C}| \) incidences. Let \( \mathcal{C}^* \) be the \emph{multiset} of components that contain at least two points of \( \mathcal{P} \). A curve in \( \mathcal{C}^* \) has multiplicity at most \( M \) by assumption.

Let \( \mathcal{C}_{ij} \) be the \emph{set} of curves in \( \mathcal{C}^* \) that have multiplicity between \( 2^i \) and \( 2^{i+1} \) and degree between \( 2^j \) and \( 2^{j+1} \). The sum of all the degrees of all the components of the curves in \( \mathcal{C} \) is
at most \(d|\mathcal{C}|\), so the number of curves of degree at least \(2^j\) that occur with multiplicity at least \(2^i\) is bounded by \(d|\mathcal{C}|/2^{i+j}\). Thus
\[
|C_{ij}| \leq d|\mathcal{C}|/2^{i+j} \leq d^{-1}M|\mathcal{P}|^{1/2}/2^{i+j},
\]
and two distinct curves in \(C_{ij}\) intersect in at most \(2^{j+1}\cdot 2^{i+1} = 4 \cdot 2^{2j}\) points by Lemma 4. Hence the incidence graph contains no \(K_{(4\cdot 2^j+1),2}\), so Theorem 11 gives
\[
I(\mathcal{P}, C_{ij}) = O \left( (2^{2j})^{1/2} (d^{-1}M|\mathcal{P}|^{1/2}/2^{i+j})|\mathcal{P}|^{1/2} + |\mathcal{P}| \right) = O \left( 2^{-i}d^{-1}M|\mathcal{P}| + |\mathcal{P}| \right).
\]
Therefore,
\[
I(\mathcal{P}, C^*) \leq \sum_{i=1}^{\log M} \sum_{j=1}^{\log d} 2^{i+1}I(\mathcal{P}, C_{ij}) = O \left( \sum_{i=1}^{\log M} \sum_{j=1}^{\log d} d^{-1}M|\mathcal{P}| + 2^i|\mathcal{P}| \right)
\]
\[
= O \left( d^{-1}M \log M \log d|\mathcal{P}| + M \log d|\mathcal{P}| \right).
\]
Together with \(I(\mathcal{P}, C_2) = I(\mathcal{P}, C^*) + O(d|\mathcal{C}|)\), this completes the proof. \(\square\)

Theorem 10 can also be applied to surfaces in \(\mathbb{R}^4\), under the following condition. We say that a surface \(S\) in \(\mathbb{R}^4\) has good projections if for every \(p \in \mathbb{R}^2\), the fibers \(\pi_1^{-1}(p) \cap S\) and \(\pi_2^{-1}(p) \cap S\) are finite, where \(\pi_1, \pi_2\) are the standard projections of \(\mathbb{R}^4\) onto its first two and last two coordinates, respectively. See [13] for an application of this theorem.

**Corollary 14.** Let \(A\) and \(B\) be finite subsets of \(\mathbb{R}^2\) with \(|A| = |B|\), and \(\mathcal{P} = A \times B \subset \mathbb{R}^4\). Let \(\mathcal{S}\) be a finite set of surfaces in \(\mathbb{R}^4\) that are defined by polynomials of constant degree and that have good projections. Assume that any two points of \(\mathcal{P}\) are contained in at most \(M\) surfaces of \(\mathcal{S}\), and any two surfaces of \(\mathcal{S}\) intersect in at most \(M\) points of \(\mathcal{P}\), for some constant \(M\). Then
\[
I(\mathcal{P}, \mathcal{S}) = O \left( |\mathcal{P}|^{2/3} |\mathcal{S}|^{2/3} + |\mathcal{P}| + |\mathcal{S}| \right).
\]

**Proof.** It is easy to check that in the proof of Theorem 10 the good-projections property is the only property of surfaces associated to complex curves that we used. In the proofs of Corollaries 12 and 13, we used the fact that two curves either have finite intersection, or they have a common component, which is not true for surfaces. But the assumption that two surfaces of \(\mathcal{S}\) intersect in at most \(M\) points of \(\mathcal{P}\) lets us apply Theorem 11 exactly as in those proofs. Given that we assume \(M\) to be constant, the stated bound follows. \(\square\)

**Remark 15.** Finally, we note that, like in most partitioning proofs, our results also hold for subgraphs of the incidence graph. Specifically, let \(I\) be a subset of the incidences such that the corresponding subgraph of the incidence graph contains no \(K_{2,M}\), and consider the proof of Theorem 10. The partitioning and intersection steps can proceed exactly as before. In the counting step, one chooses \(I_1, I_2\) as subsets of \(I\), and finishes just as before. Therefore, the incidence bounds in the previous corollaries hold also for the incidences in \(I\). This observation is used in the applications in [13, 15, 21]. (We have left it out of the main statements for the sake of clarity.)
6 Applications

We now show several examples of applications in which the assumption that the point set is a Cartesian product is satisfied. We do not work out these examples in the greatest generality here, but merely give some samples that should illustrate the usefulness of our bounds.

Rich transformations. Elekes [6, 5] introduced various questions of the following form: Given a group $G$ of transformations on some set $X$ and an integer $k$, what is the maximum size of

$$R_k(S) := \{ \varphi \in G : |\varphi(S) \cap S| \geq k \}$$

for a finite subset $S \subset X$? The work of Guth and Katz [10] involved this question for $X = \mathbb{R}^2$ and $G$ the group of Euclidean isometries of $\mathbb{R}^2$. Solymosi and Tardos [18] gave the bound $|R_k(A)| = O(|A|^4/k^3)$ when $X = \mathbb{C}$ and $G$ is the group of linear transformations from $\mathbb{C}$ to $\mathbb{C}$, and the bound $|R_k(A)| = O(|A|^6/k^5)$ when $X = \mathbb{C}$ and $G$ is the group of Möbius transformations from $\mathbb{C}$ to $\mathbb{C}$.

We give one example to illustrate how our incidence bound can be used for this type of problem. We consider the group of Möbius transformations $(cz + a)/(dz + b)$ for which $c = 0, d = 1$; i.e., the inversion transformations.

**Theorem 16.** Let $A \subset \mathbb{C}$ be finite and let $R_k(A)$ be the set of inversion transformations $\varphi_{ab}(z) = a/(z + b)$, with $a, b \in \mathbb{C}$ and $a \neq 0$, for which $|\varphi_{ab}(A) \cap A| \geq k$. Then

$$|R_k(A)| = O\left(\frac{|A|^4}{k^3}\right).$$

**Proof.** Let $\mathcal{P} := A \times A$. Define $C_{ab} := Z_{\mathbb{C}^2}(y(x + b) - a)$ and set $\mathcal{C} := \{ C_{ab} : \varphi_{ab} \in R_k(A) \}$. Then for every $C_{ab} \in \mathcal{C}$ we have $|C_{ab} \cap \mathcal{P}| \geq k$.

The curves $C_{ab}$ are clearly distinct. Suppose that two points $(x, y), (x', y') \in \mathbb{C}^2$ lie on the curve $C_{ab}$. Then we have $y(x + b) = a = y'(x' + b)$, so

$$(y - y')b = y'x' - yx.$$ 

If $y \neq y'$, then $b$ is determined by this equation, and $a$ is determined by $a = y(x + b)$. If $y' = y \neq 0$, then we have $x' = x$, a contradiction. If $y = 0$, we would have $a = 0$, also a contradiction. Thus at most two curves $C_{ab}$ pass through any two points $(x, y), (x', y')$.

Corollary 13 (or Theorem 1) then gives

$$k \cdot |\mathcal{C}| \leq I(\mathcal{P}, \mathcal{C}) = O\left(|A|^2 + |\mathcal{C}|^{2/3} + |A|^2 + |\mathcal{C}|\right).$$

This implies $|R_k(A)| = |\mathcal{C}| = O\left(|A|^4/k^3\right)$.

Elekes-Nathanson-Ruzsa-type problems. In [7], Elekes, Nathanson, and Ruzsa considered generalizations of sum-product inequalities over $\mathbb{R}$. Their proofs converted these problems into incidence problems for points and curves over $\mathbb{R}$, with the point set being a Cartesian product. So these problems are well-suited to our incidence bounds over $\mathbb{C}$.

For instance, [7] proved that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function and $A \subset \mathbb{R}$ a finite set, then

$$\max\{|A + A|, |f(A) + f(A)|\} = \Omega(n^{5/4}) \quad \text{and} \quad |A + f(A)| = \Omega(n^{5/4}),$$
where \( f(A) := \{ f(a) : a \in A \} \). For a rational function \( f \in \mathbb{R}(x) \), the same bounds can be deduced by splitting up the graph of \( f \) into convex and concave pieces (the number of which is bounded in terms of the degree of \( f \)). Over \( \mathbb{C} \), it is not clear what the analogue of a convex function would be, but for polynomials or rational functions, these bounds could be generalized to \( \mathbb{C} \). We do this for the specific function \( f(x) = 1/x \), thereby solving Problem 2.10 in Elekes’s survey \(^{[6]}\).

**Theorem 17.** Let \( A \subset \mathbb{C} \) be finite and write \( 1/A := \{ 1/a : a \in A \} \). Then
\[
\max\{|A + A|, |1/A + 1/A|\} = \Omega(n^{5/4}) \quad \text{and} \quad |A + 1/A| = \Omega(n^{5/4}).
\]

**Proof.** Set \( \mathcal{P} := (A + A) \times (1/A + 1/A) \),
\[
C_{ab} := Z_{\mathbb{C}^2}((x-a)(y-1/b) - 1),
\]
and \( \mathcal{C} := \{ C_{ab} : a, b \in A \} \). The curve \( C_{ab} \) equals the graph \( y = 1/(x - a) + 1/b \). Then each of the \( |A|^2 \) curves \( C_{ab} \) has \( |C_{ab} \cap \mathcal{P}| \geq |A| \), since for every \( a' \in A \) we have \( x = a + a' \in A + A \) and
\[
\frac{1}{x-a} + \frac{1}{b} = \frac{1}{a'} + \frac{1}{b} = y \in 1/A + 1/A,
\]
so \( (x, y) \in C_{ab} \).

We now check that the curves in \( \mathcal{C} \) meet the conditions of Corollary \(^{[12]}\). The curves \( C_{ab} \) are clearly distinct and irreducible, so do not have common components. Suppose that two points \( (x, y), (x', y') \in \mathbb{C}^2 \) lie on the curve \( C_{ab} \). Then \( x \neq a, x' \neq a \). We have
\[
y - y' = \frac{1}{x-a} - \frac{1}{x'-a'},
\]
so \( (y - y')(x - a)(x' - a) = x' - x \). This implies \( y \neq y' \), since otherwise we would also get \( x' = x \). Then we get
\[
a^2 - (x + x')a + \left( xx' - \frac{x - x'}{y - y'} \right) = 0.
\]
At most two \( a \) satisfy this equation, and \( a \) determines \( b \) by \( y = 1/(x - a) - 1/b \). Thus at most two curves \( C_{ab} \) pass through the points \( (x, y), (x', y') \).

By Corollary \(^{[12]}\) we get (the second and third term have no effect)
\[
|A| \cdot |A|^2 \leq I(\mathcal{P}, \mathcal{C}) = O \left( (|A|^2)^{2/3} (|A + A| \cdot |1/A + 1/A|^2)^{2/3} \right).
\]
This gives \( |A + A| \cdot |1/A + 1/A| = \Omega(|A|^{5/2}) \).

For the second statement, we define \( C_{ab}^* := Z((x - 1/a)(y - b) - 1) \), which is \( |A| \)-rich on \( (A + 1/A) \times (A + 1/A) \). The conditions of Corollary \(^{[13]}\) can be checked in a similar way, so
\[
|A|^3 = O \left( (|A|^2)^{2/3} (|A + 1/A|^2)^{2/3} \right),
\]
which gives \( |A + 1/A| = \Omega(|A|^{5/4}) \). \( \square \)
**Elekes-Rónyai-type problems.** Elekes and Rónyai [8] introduced another class of questions that lead to incidence problems on Cartesian products in a natural way. The strongest result in this direction was recently obtained by Raz, Sharir, and Solymosi in [14], and it states the following. Let \( f(x, y) \in \mathbb{R}[x, y] \) be a polynomial of constant degree, let \( A, B \subset \mathbb{R} \) with \(|A| = |B| = n\), and write \( f(A, B) := \{ f(a, b) : a \in A, b \in B \} \). Then we have \(|f(A, B)| = \Omega(n^{4/3})\), unless \( f \) is of the form \( g(h(x)+k(y)) \) or \( g(h(x)\cdot k(y)) \), with \( g, h, k \in \mathbb{R}[z] \). In other words, \( f \) is an “expander” unless it has a special form. A generalization of this statement is proved in [15], using our Theorem 1.

Again, the typical approach to these problems is by converting them into incidence problems between points and curves, with the points forming a Cartesian product. The general analysis is considerably more difficult; in particular, the curves can actually have many common components, and one needs to show that they do not have too many common components, unless \( f \) has a special form.

To illustrate how our incidence bounds can extend such results to \( \mathbb{C} \), we establish a simple case, where the polynomial does not have the special form. Moreover, this case is a nice geometric question. It was first considered in [8], and the real equivalent of the bound below was obtained by Sharir, Sheffer, and Solymosi [16], whose proof we follow here.

Consider the “Euclidean distance” defined by \( D(p, q) := (p_x - q_x)^2 + (p_y - q_y)^2 \) for \( p = (p_x, p_y), q = (q_x, q_y) \in \mathbb{C}^2 \), and write \( D(A, B) := \{ D(a, b) : a \in A, b \in B \} \).

**Theorem 18.** Let \( L_1, L_2 \) be two lines in \( \mathbb{C}^2 \), and \( A \subset L_1, B \subset L_2 \) with \(|A| = |B| = n\). Then
\[
|D(A, B)| = \Omega(n^{4/3}),
\]
unless \( L_1 \) and \( L_2 \) are parallel or orthogonal.

**Proof.** If the lines are not parallel or orthogonal, we can assume that \( L_1 \) is the \( x \)-axis, and that \( L_2 \) contains the origin and is not vertical. Then the lines can be parametrized by \( p(x) = (x, 0) \) and \( q(y) = (y, my) \), for some \( m \in \mathbb{C} \setminus \{0\} \). The distance is then given by
\[
f(x, y) := D(p(x), q(y)) = (x - y)^2 + m^2y^2.
\]
We will show that the polynomial \( f \) is an expander in the sense of Elekes-Rónyai.

Set \( \mathcal{P} := A \times A \),
\[
C_{bb'} := Z_{\mathbb{C}^2}(f(x, b) - f(y, b')),
\]
and \( \mathcal{C} := \{ C_{bb'} : b, b' \in B \} \). The equation of \( C_{bb'} \) is
\[
(x - b)^2 - (y - b')^2 = m^2(b^2 - b'^2),
\]
which defines a hyperbola, unless \( b = b' \). The curves of the form \( C_{bb} \) have altogether at most \( n^2 \) incidences, so we can safely ignore them. A quick calculation shows that any two points are contained in at most two hyperbolas \( C_{bb'} \) with \( b \neq b' \). Thus, by Corollary [13] we have
\[
I(\mathcal{P}, \mathcal{C}) = O\left( (|A|^2)^{2/3}(|B|^2)^{2/3} + |A|^2 + |B|^2 \right) = O(n^{8/3}).
\]

Writing \( f^{-1}(c) := \{(a, b) \in A \times B : f(a, b) = c\} \) and using Cauchy-Schwarz gives
\[
I(\mathcal{P}, \mathcal{C}) + n^2 \geq \left| \{(a, b, a', b') \in (A \times B)^2 : f(a, b) = f(a', b') \} \right|
\[
= \sum_{c \in f(A, B)} |f^{-1}(c)|^2 \geq \frac{1}{|f(A, B)|} \left( \sum_{c \in f(A, B)} |E_c| \right)^2 = \frac{n^4}{|f(A, B)|}.
\]
Therefore \(|f(A, B)| = \Omega(n^4/I(\mathcal{P}, \mathcal{C})) = \Omega(n^{4/3})\).
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