CHEVALLEY’S AMBIGUOUS CLASS NUMBER FORMULA FOR AN ARBITRARY TORUS

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Abstract. In this paper we obtain Chevalley’s ambiguous class number formula for an arbitrary torus $T$ over a global field. The classical formula of C.Chevalley may be recovered by setting $T = \mathbb{G}_m$ in our formula. A key ingredient of the proof is the work of X.Xarles on groups of components of Néron-Raynaud models of tori.

1. Introduction

Let $F$ be a global field and let $S$ be a nonempty finite set of primes of $F$ containing the archimedean primes in the number field case. Further, let $C_{F,S}$ denote the $S$-ideal class group of $F$ and let $K/F$ be a finite Galois extension with Galois group $G$. We will write $C_{K,S}$ for the $S_K$-ideal class group of $K$, where $S_K$ is the set of primes of $K$ lying above the primes in $S$. Now let $\mathcal{O}_{K,S}$ denote the ring of $S_K$-integers of $K$ and, for each prime $v$ of $F$, let $e_v$ denote the ramification index of $v$ in $K/F$. The order of a finite set $M$ will be denoted by $[M]$.

In his 1933 thesis [5], C.Chevalley obtained the following ambiguous class number formula, which has become well-known.

Theorem 1.1. We have

$$\frac{[C^G_{K,S}]}{[C_{F,S}]} = \frac{[H^1(G, K^*/\mathcal{O}_{K,S}^*)]}{[H^1(G, \mathcal{O}_{K,S}^*)]} \prod_{v \in S} e_v.$$ 

In this paper we extend the above formula to an arbitrary torus $T$ over $F$ (the preceding formula may then be recovered from our Main Theorem below by setting $T = \mathbb{G}_{m,F}$ there).

Let $\tilde{T} = \text{Spec} \mathcal{O}_{K,S}$, let $T$ be an arbitrary torus over $F$ and let $\tilde{T}^o$ denote the identity component of the Néron-Raynaud model of $T_K$ over

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Then $G$ acts on the class group $C_{T,K,S} = C(\widetilde{T}^\circ)$. Now, for each prime $v \notin S$, we fix once and for all a prime $w = w_v$ of $K$ lying above $v$ and write $\kappa(v)$ (resp. $\kappa(w)$) for the residue field of $F$ (resp. $K$) at $v$ (resp. $w$). Let $G_w$ be the decomposition group of $w$ in $K/F$. For each $v \notin S$, we will write $d_v$ for the dimension of the largest split subtorus of $T_{F_v}$, where $F_v$ is the completion of $F$ at $v$. Now let $\Phi_w$ denote the scheme of connected components of the reduction of $\widetilde{T}$ at $w$. There exist maps
\[ H^1(G, T(K)/\widetilde{T}^\circ(\widetilde{U})) \to \bigoplus_{v \notin S} H^1(G_{w_v}, \Phi_{w_v}(\kappa(w_v))) \]
and
\[ H^1(G, \widetilde{T}^\circ(\widetilde{U})) \to H^1(G, T(K)). \]
We denote their kernels by $H^1(G, T(K)/\widetilde{T}^\circ(\widetilde{U}))'$ and $H^1(G, \widetilde{T}^\circ(\widetilde{U}))'$, respectively. Finally, let $U = \text{Spec } O_{F,S}$ and let $T^\circ$ denote the identity component of the Néron-Raynaud model of $T$ over $U$.

**Main Theorem.** We have
\[ \left[ C_{T,K,S}^G \right] = \left[ H^1(G, T(K)/\widetilde{T}^\circ(\widetilde{U}))' \right] \prod_{v \notin S} e_v^d_v \ell_v \]
\[ \left[ H^1(G, \widetilde{T}^\circ(\widetilde{U}))' \right] \left[ \widetilde{T}^\circ(\widetilde{U})^G : T^\circ(U) \right]^{-1}, \]
where the $\ell_v$ are certain local factors described in Lemma 3.5.

We should remark that some authors have discussed the norm map for the trivial torus $G_F$ (see Section 5). However, the capitulation map for arbitrary tori has not been discussed previously. In Section 4 we specialize our main result to the case of norm tori and obtain some interesting new formulas.

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**2. Preliminaries**

Let $F$ be a global field, i.e. $F$ is a finite extension of $\mathbb{Q}$ (the “number field case”) or is finitely generated and of transcendence degree 1 over a finite field of constants (the “function field case”). Let $S$ be a nonempty
finite set of primes of $F$ containing the archimedean primes in the number field case and let $O_{F,S}$ denote the ring of $S$-integers of $F$. We will write $U = \text{Spec } O_{F,S}$. Further, for any non-archimedean prime $v$ of $F$, $F_v$ will denote the completion of $F$ at $v$ and $O_v$ will denote the ring of integers of $F_v$. The residue field of $O_v$ will be denoted by $\kappa(v)$. Further, we will write $j : \text{Spec } F \to U$ for the inclusion of the generic point of $U$ and, for each closed point $v \in U$, $i_v : \text{Spec } \kappa(v) \to U$ will denote the similar map at $v$.

Let $K/F$ be a finite Galois extension of global fields with Galois group $G$. We fix a separable algebraic closure $\overline{F}$ of $F$ containing $K$ and write $G_K$ for $\text{Gal}(\overline{F}/K)$. The set of primes of $K$ lying above the primes in $S$ will be denoted by $S_K$. For each prime $v$ of $F$, we fix once and for all a prime $w$ of $K$ lying above $v$ and write $G_w$ for the decomposition group of $w$ in $K/F$. When necessary, we will write $w_v$ for $w$. The inertia subgroup of $G_w$ will be denoted by $I_w$. Now, for each $v$ as above, we fix a prime $w = w_v$ of $F$ lying above $w = w_v$. Then $F_w$ is a separable algebraic closure of $F_v$ containing $K_w$. We $I_w = \text{Gal}(F_w/F_v^{nr})$ and $I_v = \text{Gal}(F_w/F_v^{nr})$, where $K_v^{nr}$ (resp. $F_v^{nr}$) is the maximal unramified extension of $K_w$ (resp. $F_v$) inside $K/F$. Clearly, $I_v$ is a subgroup of $I_w$ and there exist natural isomorphisms $I_v/I_w = \text{Gal}(K_w^{nr}/F_v^{nr}) = I_w$. We will write $G(w)$ (or $G(w_v)$, if necessary) for $\text{Gal}(\kappa(w)/\kappa(v))$, which we will identify with the quotient $G_w/I_w$. Further, we will write $e_v$ for the ramification index of $v$ in $K$.

Now, for any finite set of primes $S'$ of $F$ containing $S$, set

$$A_{F,S'} = \prod_{v \in S'} F_v \times \prod_{v \notin S'} O_v.$$ 

Then the ring of adeles of $F$ is by definition

$$A_F = \varprojlim_{S' \supset S} A_{F,S'}.$$ 

Let $H$ be a $U$-group scheme of finite type with smooth generic fiber. The class set of $H$ is by definition the double coset space

$$C(H) = H(F) \backslash H(A_F) / H(A_{F,S}).$$

The cardinality of this set, when finite (which is the case if $H_F$ is a torus over $F$ [6, §3]), is called the class number of $H$. If $H = \mathbb{G}_{m,U}$, $C(H)$ may be identified with the $S$-ideal class group of $F$.

Let $O_{K,S}$ be the ring of $S_K$-integers of $K$, let $\tilde{U} = \text{Spec } O_{K,S}$ and let $\tilde{j} : \text{Spec } K \to \tilde{U}$ be the inclusion of the generic point. Now let $T$ be an $F$-torus and let $\tilde{T}$ denote the Néron-Raynaud model of $T_K$ over $\tilde{U}$. Thus $\tilde{T}$ is a smooth and separated $\tilde{U}$-group scheme which
is locally of finite type and represents the sheaf \( \tilde{j}_*T_K \) on \( \widetilde{\mathcal{U}}_{\text{sm}} \). See [2, §10] for more details. The Néron-Raynaud model of \( T \) over \( U \) will be denoted by \( \mathcal{T} \). Let \( \mathcal{T}^\circ \) (resp. \( T^\circ \)) denote the (fiberwise) identity component of \( \mathcal{T} \) (resp. \( T \)). Then \( \mathcal{T}^\circ \) is an affine smooth \( U \)-group scheme of finite type. For each prime \( w \notin S_K \), let \( \Phi_w = i_w^*(\mathcal{T}/\mathcal{T}^\circ) \) denote the sheaf of connected components of \( \mathcal{T} \) at \( w \). Then \( \Phi_w \) is represented by an étale \( \kappa(w) \)-scheme of finite type and hence completely determined by the \( G_{\kappa(w)} \)-module \( \Phi_w(\kappa(w)) \). For each prime \( v \notin S \), \( \Phi_v \) will denote the sheaf of connected components of \( T \) at \( v \). If \( v \notin S \) and \( w = w_v \) is the prime of \( K \) lying above \( v \) fixed previously, there exists a canonical capitulation map

\[
\delta_v : \Phi_v(\kappa(v)) \to \Phi_w(\kappa(w))^{G(w)}
\]

We will need the Nisnevich topology. It is a Grothendieck topology on \( U \) stronger than the Zariski topology but weaker than the étale topology. It was introduced in [16] in order to generalize the well-known cohomological interpretation \( C_{F,S} = H^1_{\text{ét}}(U, \mathbb{G}_m) \) of the \( S \)-ideal class group of \( F \) to arbitrary, generically smooth, \( U \)-group schemes \( \mathcal{H} \) of finite type. That is, the following holds:

\[
C(\mathcal{H}) = H^1_{\text{Nis}}(U, \mathcal{H}).
\]

See [17, esp. pp.281-289] for a partial account of the above theory\(^1\). In this paper, the \( S \)-class group \( C_{T,F,S} \) of \( T \) over \( F \) is defined to be \( C(T^\circ) \) (as noted above, \( T^\circ \) is of finite type and not only generically smooth but in fact smooth). Its cardinality will be denoted by \( h_{T,F,S} \).

Remark 2.1. One can also define a class group \( C_{T,F,S} = C(\mathcal{H}) \) for some other integral model of finite type \( \mathcal{H} \) of \( T \). We choose the model \( T^\circ \) because, with this choice, the well-known exact sequence

\[
1 \to \mathcal{O}^*_K \to K^* \to \mathcal{I}_{K,S} \to C_{K,S} \to 0,
\]

where \( \mathcal{I}_{K,S} \) denotes the group of fractional \( S \)-ideals of \( K \), admits the natural generalization (5) below (this will play a fundamental role in [8]). Assume now that \( F \) is a number field. V.Voskresenskii [21, §20] and B.Kunyavskii et al. [10, p.47] have defined (for \( F \) a number field and \( S \) equal to the set of archimedean primes of \( F \)) a class group of \( T \)

\(^1\) The proof of this proposition is valid in the function field case as well, as pointed out by D.Lorenzini.

\(^2\) Readers wishing to learn more about the non \( K \)-theoretic applications of the Nisnevich topology, and who are not prepared to wait for the publication of a proper survey, are advised to obtain a copy of [16].
using a certain “standard model” \( \mathcal{H} = \mathcal{T}_0 \) of \( T \) which (unlike the Néron-Raynaud model \( \mathcal{T} \) of \( T \)) is of finite type (but can be non-smooth). The corresponding class number \( h(\mathcal{T}_0) \) is minimal among all class numbers \( h(\mathcal{T}_1) \) as \( \mathcal{T}_1 \) runs over the family of all integral models of \( T \) of finite type (see [21, p.197]). Since \( T^\circ \) is such a model (see above), one has \( h(\mathcal{T}_0) \big| h_{T,F,\infty} \). When \( T \) splits over a tamely ramified extension of \( F \), the identity components of \( T \) and \( \mathcal{T}_0 \) are canonically isomorphic [10, Theorem 3, p.49]. In this case, therefore, if \( \mathcal{T}_0 \) has connected fibers (which is the case, for example, if \( T \) is a quasi-split \( F \)-torus of the form \( T = R_{K/F}(\mathbb{G}_{m,K}^d) \), where \( K/F \) is a tamely ramified Galois extension of number fields [10, p.48]), then \( h(\mathcal{T}_0) \) and \( h_{T,F,\infty} \) coincide.

We will write \( X = \text{Hom}(T(F), F^*) \) for the \( GF \)-module of characters of \( T \). For any \( GF \)-lattice \( Y \), we will write \( Y^\vee \) for the \( \mathbb{Z} \)-linear dual of \( Y \), i.e., \( Y^\vee = \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) \).

If \( f : A \to B \) is a homomorphism of abelian groups with finite kernel and cokernel, we define

\[
q(f) = \frac{[\text{Coker}f]}{[\text{Ker}f]}. 
\]

This function is multiplicative on short exact sequences, in the sense that if \( f^\bullet : X^\bullet \to Y^\bullet \) is a map of short exact sequences and \( q(f^r) \) is defined for \( r = 1, 2, 3 \), then \( q(f^2) = q(f^1)q(f^3) \).

If \( M \) is any abelian group and \( m \) is any positive integer, \( M_m \) will denote its \( m \)-torsion subgroup and we will write \( M/mM \). Applying \( \text{Tor}^\mathbb{Z}(\mathbb{Z}/m, -) \) to a given exact sequence \( 0 \to A \to B \to C \to 0 \) of abelian groups, we obtain an exact sequence

\[
0 \to A_m \to B_m \to C_m \xrightarrow{c} A/m \to B/m \to C/m \to 0,
\]

where \( c \) is a certain “connecting homomorphism”. See, e.g., [22, §3.1.1, p.66]. The (Pontryagin) dual of \( M \) is by definition \( M^D = \text{Hom}(M, \mathbb{Q}/\mathbb{Z}) \). If \( G \) is a group and \( M \) is a (left) \( G \)-module, \( M^D \) will be endowed with its natural \( G \)-action (see, e.g., [1, p.94]). Further, \( M_G \) (resp. \( M^G \)) will denote the largest quotient (resp. subgroup) of \( M \) on which \( G \) acts trivially. It is not difficult to check that \( (M^D)^G = (M_G)^D \). Further, if \( M \) is finite and \( G \) is pro-cyclic, then \( [M_G] = [M^G] \) (see, e.g., [1, proof of Proposition 11, p.109]). It follows that \( [(M^D)^G] = [M^G] \) (we will use the latter fact often when referring to [23, Corollary 2.18, p.175]). We also note that, if \( H \) is a normal subgroup of \( G \), then there exists a canonical isomorphism \( \mathbb{Z}[G]^H = \mathbb{Z}[G/H] \). In addition to the above, we will need the following well-known facts: if \( M = \mathbb{Z}[G] \otimes Y \) for some
abelian group $Y$, then the Tate groups $\hat{H}_0(G, M) = \hat{H}_0^0(G, M) = 0$ (see, e.g., [1, Proposition 6, p.102]). If $Y$ is a torsion-free abelian group, then $\text{Tor}_1^G(Y, -) = 0$. Further, if $G$ is a torsion group which acts trivially on $Y$, then $H^1(G, Y) = \text{Hom}(G, Y) = 0$.

3. Proof of the main theorem

There exists a natural exact sequence of sheaves on $\tilde{U}_{\text{sm}}$:

$$0 \to \tilde{T}^o \to \tilde{T} \to \bigoplus_{w \notin S_K} (i_w)_* \Phi_w \to 0.$$ 

The preceding exact sequence induces an exact sequence

(4)
$$1 \to \tilde{T}^o(\tilde{U}) \to T(K) \to \bigoplus_{w \notin S_K} \Phi_w(\kappa(w)) \to H^1_{\text{ét}}(\tilde{U}, \tilde{T}^o) \to H^1_{\text{ét}}(\tilde{U}, \tilde{T}).$$

Lemma 3.1. There exists a canonical isomorphism

$$\text{Ker} \left[ H^1_{\text{ét}}(\tilde{U}, \tilde{T}^o) \to H^1_{\text{ét}}(\tilde{U}, \tilde{T}) \right] = H^1_{\text{Nis}}(\tilde{U}, \tilde{T}^o) = C_{T,K,S}.$$

Proof. Since $\tilde{T} = j_* T_K$ as étale sheaves, the Cartan-Leray spectral sequence

$$H^p_{\text{ét}}(\tilde{U}, R^q j_* T_K) \implies H^{p+q}_{\text{ét}}(K, T)$$

yields an injection $H^1_{\text{ét}}(\tilde{U}, \tilde{T}) = H^1_{\text{ét}}(\tilde{U}, j_* T_K) \hookrightarrow H^1_{\text{ét}}(K, T)$. Thus $H^1_{\text{ét}}(\tilde{U}, \tilde{T})$ may be replaced with $H^1(K, T)$ in the statement of the lemma. Now [17, 1.44.2, p.286] completes the proof. \hfill \Box

By the lemma and (2), (4) induces a canonical exact sequence of $G$-modules

(5)
$$1 \to \tilde{T}^o(\tilde{U}) \to T(K) \to \bigoplus_{w \notin S_K} \bigoplus_{w|v} \Phi_w(\kappa(w)) \to C_{T,K,S} \to 0.$$ 

We will identify the $G$-module $\bigoplus_{w|v} \Phi_w(\kappa(w))$ with the $G$-module coinduced by the $G_{w_v}$-module $\Phi_{w_v}(\kappa(w_v))$. Thus, by Shapiro’s lemma,

$$H^i(G, \bigoplus_{w|v} \Phi_w(\kappa(w))) = H^i(G_{w_v}, \Phi_{w_v}(\kappa(w_v)))$$

for every $i \geq 0$. Note also that, since $I_{w_v}$ acts trivially on $\Phi_{w_v}(\kappa(w_v))$, $\Phi_{w_v}(\kappa(w_v))^{G_{w_v}} = \Phi_{w_v}(\kappa(w_v))^{G(w_v)}$.

Clearly, (5) induces an exact sequence

$$C_{T,K,S}^G \xrightarrow{\alpha} H^1(G, T(K)/\tilde{T}^o(\tilde{U})) \xrightarrow{\beta} \bigoplus_{w \notin S} H^1(G_{w_v}, \Phi_{w_v}(\kappa(w_v))).$$

3 All tensor products in this paper are taken over $\mathbb{Z}$. 

Define
\[(C^G_{T,K,S})_{\text{trans}} = \text{Ker}(\alpha)\]
and
\[(6) \quad H^1(G, T(K)/\widetilde{T}(\widetilde{U}))' = \text{Ker}(\beta).\]
Then
\[(7) \quad [C^G_{T,K,S}] = [(C^G_{T,K,S})_{\text{trans}}] \left[ H^1(G, T(K)/\widetilde{T}(\widetilde{U}))' \right].\]

Remark 3.2. If \(T\) splits over \(K\), i.e., \(T_K = \mathbb{G}^d_{m,K}\) for some \(d\), then \(H^1(G_w, \Phi_w(\kappa(w_v))) = \text{Hom}(G_w, \mathbb{Z}^d) = 0\). In this case, therefore,
\[H^1(G, T(K)/\widetilde{T}(\widetilde{U}))' = H^1(G, T(K)/\widetilde{T}(\widetilde{U})).\]

Now there exists a canonical \textit{capitulation map}
\[j_{T,K/F}: C_{T,F,S} \rightarrow C^G_{T,K,S}\]
which is the composite
\[H^1_{\text{Nis}}(U, T^o) \xrightarrow{\text{ad}} H^1_{\text{Nis}}(U, \pi_*\pi^*T^o) \hookrightarrow H^1_{\text{Nis}}(\widetilde{U}, \pi^*T^o)^G \xrightarrow{\text{bc}} H^1_{\text{Nis}}(\widetilde{U}, T^o)^G,\]
where \(\pi: \widetilde{U} \rightarrow U\) is the canonical map, “ad” is induced by the adjoint morphism \(T^o \rightarrow \pi_*\pi^*T^o\), the middle injection is the first nontrivial map in the exact sequence of terms of low degree belonging to the Cartan-Leray spectral sequence \(H^p_{\text{Nis}}(U, R^q\pi_*\pi^*T^o) \Rightarrow H^{p+q}_{\text{Nis}}(\widetilde{U}, \pi^*T^o)\) [17, 1.22.1, p.270] and the map “bc” is induced by the base change morphism \(\pi^*T^o = T^o \times_U \widetilde{U} \rightarrow T^o\) [2, §7.2, Theorem 1(i), p.176]. It is not difficult to check that the image of \(j_{T,K/F}\) is contained in \((C^G_{T,K,S})_{\text{trans}}\).

We will write \(j'_{T,K/F}: C_{T,F,S} \rightarrow (C^G_{T,K,S})_{\text{trans}}\) for the induced map. Then (7) may be rewritten as
\[(8) \quad \frac{[C^G_{T,K,S}]}{[C_{T,F,S}]} = q(j'_{T,K/F}) \left[ H^1(G, T(K)/\widetilde{T}(\widetilde{U}))' \right].\]

Now there exists a natural exact commutative diagram
\[(9) \quad \begin{array}{ccccccccc}
0 & \longrightarrow & T(F)/T^o(U) & \longrightarrow & \bigoplus_{v \in S} \Phi_v(\kappa(v)) & \longrightarrow & C_{T,F,S} \\
& | \gamma \downarrow & & | \oplus_{v \in S} \delta_v & & | j'_{T,K/F} \\
0 & \longrightarrow & (T(K)/\widetilde{T}(\widetilde{U}))^G & \longrightarrow & \bigoplus_{v \in S} \Phi_{w_v}(\kappa(w_v))^{G(w_v)} & \longrightarrow & (C^G_{T,K,S})_{\text{trans}}
\end{array}\]
where, for each \( v \notin S \), \( \delta_v \) is the capitulation map (1) and \( \gamma \) is induced by \( \bigoplus_{v \notin S} \delta_v \).

**Lemma 3.3.** For each \( v \notin S \), there exists an isomorphism of finite groups

\[
\Ker \left[ \Phi_v(\kappa(v)) \xrightarrow{\delta_v} \Phi_{w_v}(\kappa(w_v))^G(w_v) \right] = H^1(I_{w_v}, T(K_{w_v}^{nr}))^{G_v(v)}.
\]

Here \( I_{w_v} \) is identified with \( \text{Gal}(K_{w_v}^{nr}/F_v^{nr}) \). In particular, if \( v \) is unramified in \( K/F \), then \( \delta_v \) is injective.

**Proof.** Set \( w = w_v \). Since \( \Ker \delta_v \) is a torsion group (cf. [7, Theorem 1]), we have \( \Ker \delta_v = \Ker \delta' \), where \( \delta' : \Phi_v(k(v))_{\text{tors}} \to \Phi_{w_v}(\kappa(w))_{\text{tors}} \) is the induced map. Since \( \Phi_v(k(v))_{\text{tors}} \) is finite by [23, Corollary 2.18], we conclude that \( \Ker \delta_v \) is finite as well. Now, by [23, Corollary 2.18] and [13, Corollary I.2.4, p.35], there exists a natural exact commutative diagram

\[
\begin{array}{ccc}
\Phi_v(k(v))_{\text{tors}} & \xrightarrow{\sim} & H^1(I_{v}, T)^{G_k(v)} \\
\downarrow \delta' & & \downarrow \text{Res} \\
\Phi_{w_v}(\kappa(w))_{\text{tors}}^G & \xrightarrow{\sim} & H^1(I_{w_v}, T)^{G_k(v)}
\end{array}
\]

where the right-hand vertical map is induced by the restriction map \( H^1(I_v, T) \to H^1(I_{w_v}, T) \). The result is now immediate from the inflation-restriction exact sequence. \( \square \)

An \( F \)-torus \( T \) is said to have good reduction at a nonarchimedean prime \( v \) of \( F \) if \( T_{\kappa(v)} \) is a torus over \( \mathcal{O}_v \). The following are equivalent conditions (see [15, (1.1), p.462]): (a) \( T \) has good reduction at \( v \) as defined above; (b) \( T_{\kappa(v)} \) is a torus over \( \kappa(v) \); (c) \( I_v \) acts trivially on \( X \); (d) \( T \) splits over an unramified extension of \( F_v \).

Since only finitely many primes of \( F \) can ramify in a splitting field of \( T \), we see that \( T \) has good reduction at all but finitely many primes of \( F \).

**Lemma 3.4.** Let \( v \) be a nonarchimedean prime of \( F \) which is unramified in \( K/F \) and where \( T \) has good reduction. Then the capitulation map

\[
\delta_v : \Phi_v(\kappa(v)) \to \Phi_{w_v}(\kappa(w_v))^G(w_v)
\]

The commutativity of this diagram follows by examining the proofs of Lemma 2.13, Proposition 2.14, Lemma 2.17 and Corollary 2.18 in [23]. The key point is that the map \( H^1(I_v, X) \to H^1(I_{w_v}, X) \) (notation as in [op.cit.]) which corresponds to the map \( \text{Ext}^1_Z(\Phi_w, Z) \to \text{Ext}^1_Z(\Phi_v, Z) \) under the isomorphism of [op.cit., Proposition 2.14] is induced by the norm map \( N_{I_v} : X^{I_{w_v}} \to X^{I_{v}} \) (see, especially, the diagram at the bottom of p.174 of [op.cit.]).
is an isomorphism.

Proof. Proposition 3.2 of [3] and functoriality immediately reduce the proof to the case $T = G_{m,F}$, where $\delta_v : \mathbb{Z} \to \mathbb{Z}$ is multiplication by the ramification index $e_v$ of $v$ in $K/F$. The lemma is clear in this case. □

Lemmas 3.3 and 3.4 show that the map $\bigoplus_{v \notin S} \delta_v$ in diagram (9) has a finite kernel and cokernel. Therefore

$$q(\bigoplus_{v \notin S} \delta_v) = \prod_{v \notin S} q(\delta_v)$$

is defined and diagram (9) yields

$$\prod_{v \notin S} q(\delta_v) = q(\gamma)q(j'_{T,K/F}).$$

Consequently, (8) yields the identity

$$(10) \quad \frac{[C_{T,K,S}^{G}]}{[C_{T,F,S}]} = \frac{[H^1(G,T(K)/\tilde{T}^o(\tilde{U}))]}{q(\gamma)} \cdot \prod_{v \notin S} q(\delta_v).$$

It remains to compute the factors $q(\delta_v)$ (for $v \notin S$) and $q(\gamma)$. We compute first the local factors $q(\delta_v)$.

Let $v \notin S$. The inertia group $I_v$ acts on $X$ through a finite quotient $J_v$ (say), and we have a natural map

$$N_v : X \to X^{I_v}, \chi \mapsto \sum_{g \in J_v} \chi^g.$$

Let $\tilde{T}_v$ be the $F_v$-torus corresponding to the subgroup $\text{Ker} N_v$ of $X$ and let $T_v^*$ be the largest split subtorus of $T_{F_v}$. Let $d_v$ be the dimension of $T_v^*$. Set $w = w_v$. Since $H^1(G(w), \mathbb{Z}) = \text{Hom}(G(w), \mathbb{Z}) = 0$, Corollary 3.3 of [3] yields an exact commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \Phi_v^*(\kappa(v)) & \rightarrow & \Phi_v(\kappa(v)) & \rightarrow & \tilde{\Phi}_v(\kappa(v)) & \rightarrow & 0 \\
& \downarrow & \delta_v^* & & \downarrow & \delta_v & & \downarrow & \\
0 & \rightarrow & \Phi_w^*(\kappa(w))^{G(w)} & \rightarrow & \Phi_w(\kappa(w))^{G(w)} & \rightarrow & \tilde{\Phi}_w(\kappa(w))^{G(w)} & \rightarrow & 0.
\end{array}
$$

The left-hand vertical map $\delta_v^*$ may be identified with the map $\mathbb{Z}^{d_v} \rightarrow \mathbb{Z}^{d_v}, (n_i)_{1 \leq i \leq d_v} \mapsto (e_i n_i)_{1 \leq i \leq d_v}$. Thus

$$q(\delta_v^*) = e_v^{d_v}.$$ 

On the other hand, by [23, Corollary 2.19(b), p.175] and the fact that $[M_{G_{\kappa(v)}}] = [M^{G_{\kappa(v)}}]$ for any finite $G_{\kappa(v)}$-module $M$, $\tilde{\Phi}_v(\kappa(v))$ and
\(\hat{\Phi}_w(\kappa(w))\) are finite \(G_{k(v)}\)-modules and
\[
q(\hat{\delta}_v) = \left[ \hat{\Phi}_w(\kappa(w))^{G_{k(v)}} \right] / \left[ \hat{\Phi}_v(\kappa(w))^{G_{k(v)}} \right] = \left[ H^1(I_{wv}, \text{Ker} \ N_v)^{G_{k(v)}} \right] / \left[ H^1(I_{v}, \text{Ker} \ N_v)^{G_{k(v)}} \right].
\]

Thus the following holds.

**Lemma 3.5.** Let \(v \notin S\). Then
\[
q(\delta_v) = e_v^{d_v} \left[ \frac{H^1(I_{wv}, \text{Ker} \ N_v)^{G_{k(v)}}}{H^1(I_{v}, \text{Ker} \ N_v)^{G_{k(v)}}} \right]. \square
\]

**Remark 3.6.** For \(v \notin S\), set \(w = w_v\). If \(T\) splits over \(K\), then
\(H^1(I_{wv}, \text{Ker} \ N_v) = \text{Hom}(I_{wv}, \text{Ker} \ N_v) = 0\)
since \(I_{wv}\) is torsion and \(\text{Ker} \ N_v\) is a torsion-free abelian group with trivial \(I_{wv}\)-action. Thus, the inflation-restriction exact sequence shows that \(H^1(I_{wv}, \text{Ker} \ N_v) = H^1(I_{w}, \text{Ker} \ N_v)\) and \(G_{k(v)}\) acts on this group through the quotient \(G(w) = G_{k(v)} / G_{k(w)}\). Thus, in this case, the formula of the lemma has the simpler form
\[
q(\delta_v) = e_v^{d_v} \left[ \frac{H^1(I_{wv}, \text{Ker} \ N_v)^{G(w)}}{H^1(I_{wv}, \text{Ker} \ N_v)^{G(w)}} \right].
\]

Now set
\[
H^1(G, \tilde{T}^\circ(\tilde{U}))' = \text{Ker} \left[ H^1(G, \tilde{T}^\circ(\tilde{U})) \to H^1(G, T(K)) \right],
\]
where the map involved is induced by the natural map \(\tilde{T}^\circ(\tilde{U}) \to T(K)\). Further, recall the map
\[
\gamma : T(F)/T^\circ(U) \to (T(K)/\tilde{T}^\circ(\tilde{U}))^G
\]
from diagram (9).

**Lemma 3.7.** We have
\[
q(\gamma) = \left[ \frac{H^1(G, \tilde{T}^\circ(\tilde{U}))'}{\tilde{T}^\circ(\tilde{U})^G : T^\circ(U)} \right].
\]

**Proof.** The commutativity of the exact diagram
\[
0 \longrightarrow T^\circ(U) \longrightarrow T(F) \longrightarrow T(F)/T^\circ(U) \longrightarrow 0
\]
\[
0 \longrightarrow \tilde{T}^\circ(\tilde{U})^G \longrightarrow T(F) \longrightarrow (T(K)/\tilde{T}^\circ(\tilde{U}))^G \longrightarrow H^1(G, \tilde{T}^\circ(\tilde{U})){'}
\]
yields canonical isomorphisms
\[
\text{Ker}(\gamma) = \tilde{T}^\circ(\tilde{U})^G / T^\circ(U)
\]
and

\[ \text{Coker}(\gamma) = H^1(G, \tilde{T}^o(\tilde{U}))'. \]

The lemma is now clear. \qed

We can now state our generalization of Chevalley's ambiguous class number formula.

**Theorem 3.8.** We have

\[ \left[ \frac{C^G_{T,K,S}}{C_{T,F,S}} \right] = \frac{\left[ H^1(G, T(K)/\tilde{T}^o(\tilde{U})))' \right] \prod_{v \notin S} q(\delta_v)}{\left[ H^1(G, \tilde{T}^o(\tilde{U})))' \right] \left[ \tilde{T}^o(\tilde{U})^G : T^o(U) \right]^{-1}}, \]

where \( H^1(G, T(K)/\tilde{T}^o(\tilde{U}))' \) and \( H^1(G, \tilde{T}^o(\tilde{U})))' \) are given by (6) and (11), respectively, and, for each \( v \notin S \), \( q(\delta_v) \) is given by Lemma 3.5.

**Proof.** This is immediate from (10) and Lemma 3.6. \qed

The next formula is closer to Chevalley’s original formula than that of the theorem.

**Corollary 3.9.** Assume that \( T_K \) has good reduction over \( \tilde{U} \), i.e., that \( S_K \) contains all primes of bad reduction for \( T_K \). Then

\[ \left[ \frac{C^G_{K,T,S}}{C_{T,F,S}} \right] = \frac{\left[ H^1(G, (X^{G_K})^\vee \otimes (K^*/\mathcal{O}_{K,S}^*))') \right] \prod_{v \notin S} q(\delta_v)}{\left[ H^1(G, (X^{G_K})^\vee \otimes \mathcal{O}_{K,S}^*)') \right] \left[ ((X^{G_K})^\vee \otimes \mathcal{O}_{K,S}^*)^G : T^o(U) \right]^{-1}}. \]

**Proof.** By hypothesis \( \tilde{T}^o \) is a torus over \( \tilde{U} \), whence

\[ \tilde{T}^o = \text{Hom}_{\tilde{U}}(X(\tilde{T}^o), \mathbb{G}_m) \]

as sheaves on \( \tilde{U}_{\text{et}} \), where \( X(\tilde{T}^o) \) is the sheaf of characters of \( \tilde{T}^o \). Now, since \( X(\tilde{T}^o) \) is an étale locally constant \( \mathbb{Z} \)-constructible sheaf on \( \tilde{U} \),

\[ X(\tilde{T}^o)(\tilde{U}) = X^{G_K} \] (see, e.g., [12, p.156]). Consequently,

\[ \tilde{T}^o(\tilde{U}) = \text{Hom}(X^{G_K}, \mathcal{O}_{K,S}^*) = (X^{G_K})^\vee \otimes \mathcal{O}_{K,S}^*. \]

The corollary is now immediate. \qed

**Remark 3.10.** The corollary is valid for any set \( S \) verifying our general assumptions if \( T \) splits over \( K \). In this case \( X^{G_K} = X \) and, using
Remark 3.2, the formula of the corollary has the following simpler form:

\[
\frac{[C^G_{K,T,S}]}{[C_{T,F,S}]} = \frac{[H^1(G, X^\vee \otimes (K^*/\mathcal{O}^*_{K,S}))]}{[H^1(G, X^\vee \otimes \mathcal{O}^*_{K,S})]} \prod_{v \notin S} q(\delta_v) \left[\left((X^\vee \otimes \mathcal{O}^*_{K,S})^G : T^o(U)\right)\right]^{-1}.
\]

4. Norm tori

Define a $G_F$-module $X$ by the exactness of the sequence

(12) \[0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}G \to X \to 0,\]

where $\varepsilon$ is given by $\varepsilon(1) = \sum_{\sigma \in G} \sigma$. This exact sequence induces an exact sequence of $F$-tori

(13) \[0 \to T \to R_{K/F}(\mathbb{G}_{m,K}) \xrightarrow{N} \mathbb{G}_{m,F} \to 0,\]

where $R_{K/F}(\mathbb{G}_{m,K})$ is the Weil restriction of $\mathbb{G}_{m,K}$, $N$ is induced by the norm map $K \to F$ and $T = R_{K/F}^{(1)}(\mathbb{G}_{m,K})$ is the corresponding norm torus. Note that $T_K = \mathbb{G}_{m,K}^{n-1}$, where $n = [K : F]$. For any $v \notin S$, let $w = w_v$ be the prime of $K$ lying above $v$ fixed previously. We will write $f_v$ for the residue class degree $[\kappa(w) : \kappa(v)]$ and $G^\text{ab}_w$ for the largest abelian quotient of $G_w$, i.e., $G^\text{ab}_w = G_w/G'_w$, where $G'_w$ is the commutator subgroup of $G_w$. Note that $G(w)$ is a cyclic group of order $f_v$ and $[G_w] = e_v f_v$. Further, since $G_w/I_w = G(w)$ is abelian, $I_w$ contains $G'_w$.

The dimension $d_v$ of the largest split subtorus of $T_{F_v}$ is the rank of $X^{G_w}$. Since $H^1(G_w, \mathbb{Z}) = 0$ and $\mathbb{Z}G$ is a free (right) $\mathbb{Z}[G_w]$-module of rank $[G : G_w]$, (12) shows that $d_v = [G : G_w] - 1$. We will now compute $q(\delta_v) = e_v^{d_v}/[H^1(I_w, \text{Ker} N_v)^{G(w)}]$ (see Remark 3.6). Recall the norm element $N_{I_w} = \sum_{\tau \in I_w} \tau \in \mathbb{Z}[G]$. Since $\mathbb{Z}[G] = \mathbb{Z}[I_w]^{[G : I_w]}$, the multiplication-by-$N_{I_w}$ map $\mathbb{Z}[G] \to \mathbb{Z}[G]^{I_w}$ is surjective. Further, since $H^1(I_w, \mathbb{Z}) = 0$, the canonical map $\mathbb{Z}[G]^{I_w} \to X^{I_w}$ is surjective as well. It follows that $N_v : X \to X^{I_w}$ is surjective. Consequently, we have a natural exact sequence of $I_w$-modules

\[0 \to \text{Ker} N_v \to X \xrightarrow{N_v} X^{I_w} \to 0.\]

Taking $I_w$-cohomology of the above exact sequence, we obtain a natural exact sequence of $G(w)$-modules

(14) \[0 \to X^{I_w}/e_v \to H^1(I_w, \text{Ker} N_v) \to H^1(I_w, X) \to 0\]
and therefore an exact sequence of abelian groups

\[(15)\]

\[H^1(G(w), X^{I_w}/e_v)^D \xrightarrow{\partial^D} (H^1(I_w, X)^{G(w)})^D \rightarrow (H^1(I_w, \text{Ker} N_v)^{G(w)})^D \]

\[\rightarrow ((X^{I_w}/e_v)^{G(w)})^D \rightarrow 0,\]

where \(\partial : H^1(I_w, X)^{G(w)} \rightarrow H^1(G(w), X^{I_w}/e_v)\) is the “connecting homomorphism” appearing in the long \(G(w)\)-cohomology sequence arising from (14) (see, e.g., [1, end of §2, p.97] for a general description of \(\partial\)).

**Lemma 4.1.** There exists an isomorphism

\[(H^1(I_w, X)^{G(w)})^D = I_w/G'_w.\]

**Proof.** We will need the dual of the well-known inflation-restriction-transgression exact sequence [19, p.51], namely

\[(16)\]

\[H^2(G(w), X^{I_w})^D \rightarrow (H^1(I_w, X)^{G(w)})^D \rightarrow H^1(G_w, X)^D \rightarrow H^1(G(w), X^{I_w})^D \rightarrow 0.\]

Since \(H^1(I_w, \mathbb{Z}) = 0\), the exact sequence (12) induces an exact sequence of \(G(w)\)-modules

\[0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G(w)]^{[G:G_w]} \rightarrow X^{I_w} \rightarrow 0.\]

Taking \(G(w)\)-cohomology of the preceding exact sequence and using Shapiro’s lemma together with [4, p.250] (along with the periodicity of the cohomology of cyclic groups), we obtain isomorphisms

\[H^1(G(w), X^{I_w})^D = H^2(G(w), \mathbb{Z})^D = G(w)\]

and

\[H^2(G(w), X^{I_w})^D = H^3(G(w), \mathbb{Z})^D = H^1(G(w), \mathbb{Z})^D = 0.\]

On the other hand, taking \(G_w\)-cohomology of the exact sequence of \(G_w\)-modules

\[0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G_w]^{[G:G_w]} \rightarrow X \rightarrow 0\]

we obtain, using Shapiro’s lemma and [4, p.250] again, isomorphisms

\[H^1(G_w, X)^D = H^2(G_w, \mathbb{Z})^D = G_w^\text{ab}.\]

Thus (16) is isomorphic to an exact sequence

\[0 \rightarrow (H^1(I_w, X)^{G(w)})^D \rightarrow G_w^\text{ab} \rightarrow G(w) \rightarrow 0.\]
It is not difficult to check that the map $G_w^{ab} \to G(w)$ appearing above is the canonical projection map $G_w/G'_w \to G_w/I_w$. The lemma is now immediate.

**Lemma 4.2.** $[(X^I_w/e_v)^{G(w)}] = e_v^D(f_v, e_v)$. Further, there exists an isomorphism

$$H^1(G(w), X^I_w/e_v)^D = G(w)_{e_v}.$$  

**Proof.** Since: $\mathbb{Z}[G] = \mathbb{Z}[G_w]^{[G:G_w]}$ as $G_w$-modules, $\mathbb{Z}[G_w]I_w = \mathbb{Z}[G(w)]$ canonically and $\text{Tor}_1^\mathbb{Z}(X^I_w, \mathbb{Z}/e_v) = 0$, we see that (12) yields an exact sequence of $G(w)$-modules

$$0 \to \mathbb{Z}/e_v \to (\mathbb{Z}[G(w)] \otimes \mathbb{Z}/e_v)^{[G:G_w]} \to X^I_w/e_v \to 0.$$  

We conclude, using Shapiro’s lemma, that there exists an exact sequence

$$0 \to \mathbb{Z}/e_v \to (\mathbb{Z}[G(w)] \otimes \mathbb{Z}/e_v)^{G(w)} \to \text{Hom}(G(w), \mathbb{Z}/e_v) \to 0.$$  

The first assertion of the lemma is now clear. On the other hand, there exist isomorphisms

$$H^1(G(w), X^I_w/e_v)^D = H^2(G(w), \mathbb{Z}/e_v)^D = (H^2(G(w), \mathbb{Z})/e_v)^D = (H^2(G(w), \mathbb{Z})^D)_{e_v} = G(w)_{e_v}$$

(to obtain the second equality, take $G(w)$-cohomology of the sequence $0 \to \mathbb{Z} \xrightarrow{e_v} \mathbb{Z} \to \mathbb{Z}/e_v \to 0$). This completes the proof. □

We now return to the exact sequence (15). The map labeled $\partial^D$ there corresponds, under the isomorphisms of Lemmas 4.1 and 4.2, to the connecting homomorphism $c: G(w)_{e_v} \to I_w/G'_w = (I_w/G'_w)/e_v$ appearing in the exact sequence

$$0 \to I_w/G'_w \to (G_w^{ab})_{e_v} \to G(w)_{e_v} \xrightarrow{c} I_w/G'_w,$$

which is the beginning of the exact sequence (3) (with $m = e_v$) associated to the short exact sequence $0 \to I_w/G'_w \to G_w^{ab} \to G(w) \to 0$. Therefore

$$[\text{Coker}(\partial^D)] = \left[\frac{(G_w^{ab})_{e_v}}{f_v, e_v}\right].$$

Now the exactness of (15) together with the first assertion of Lemma 4.2 yield the identity

$$[H^1(I_w, \text{Ker} N_v)^{G(w)}] = e_v^D\left[(G_w^{ab})_{e_v}\right].$$

\footnote{This is a bit tedious since it involves checking the commutativity of a certain diagram whose horizontal arrows are duals of inflation maps and whose vertical maps are duals of composites of connecting homomorphisms of the form $H^1 \to H^2$ with integral duality isomorphisms.}
Thus, by Remark 3.6, the following (simple) formula holds.

**Lemma 4.3.** \( q(\delta_v) = [(G_{w_v}^{ab})_{e_v}]^{-1}. \) □

We now compute the global factors entering in the formula of Remark 3.10.

Since \( \text{Tor}_1(\mathbb{Z},-) = 0 \), there exists an exact sequence of \( G \)-modules

\[
0 \to X^\vee \otimes (K^*/\mathcal{O}_{K,S}^*) \to \mathbb{Z}[G] \otimes (K^*/\mathcal{O}_{K,S}^*) \to K^*/\mathcal{O}_{K,S}^* \to 0
\]

which is induced by the \( \mathbb{Z} \)-linear dual of (12). Now, taking Tate cohomology \( \widehat{H}^r \) of the above exact sequence and using Shapiro’s lemma, we obtain an isomorphism

\[
(17) \quad H^1(G, X^\vee \otimes (K^*/\mathcal{O}_{K,S}^*)) = \widehat{H}^0(G, K^*/\mathcal{O}_{K,S}^*).
\]

Arguing similarly, we obtain an isomorphism

\[
H^1(G, X^\vee \otimes \mathcal{O}_{K,S}^*) = \widehat{H}^0(G, \mathcal{O}_{K,S}^*) = \mathcal{O}_{F,S}^*/N_{K/F}(\mathcal{O}_{K,S}^*).
\]

On the other hand, (13) shows that \( H^1(G, T(K)) = F^*/N_{K/F}(K^*) \). Consequently, the map \( H^1(G, X^\vee \otimes \mathcal{O}_{K,S}^*) \to H^1(G, T(K)) \) corresponds to the canonical map

\[
\mathcal{O}_{F,S}^*/N_{K/F}(\mathcal{O}_{K,S}^*) \to F^*/N_{K/F}(K^*).
\]

Thus, we have a canonical isomorphism

\[
H^1(G, X^\vee \otimes \mathcal{O}_{K,S}^*) = \mathcal{O}_{F,S}^*/N_{K/F}(\mathcal{O}_{K,S}^*).
\]

Setting

\[
(18) \quad W_{F,S} = \mathcal{O}_{F,S}^* \cap N_{K/F}(K^*)
\]

we conclude that

\[
(19) \quad \left[ H^1(G, X^\vee \otimes \mathcal{O}_{K,S}^*) \right] = \left[ \widehat{H}^0(G, \mathcal{O}_{K,S}^*) \right] / \left[ \mathcal{O}_{F,S}^* : W_{F,S} \right].
\]

Next, (12) induces an exact sequence of \( G \)-modules

\[
0 \to X^\vee \otimes \mathcal{O}_{K,S}^* \to \mathbb{Z}[G] \otimes \mathcal{O}_{K,S}^* \to \mathcal{O}_{K,S}^* \to 0
\]

where the right-hand map is induced by the \( \mathbb{Z} \)-linear map \( \mathbb{Z}[G] \to \mathbb{Z}, \sigma \mapsto 1 \). The preceding exact sequence induces an exact sequence

\[
0 \to (X^\vee \otimes \mathcal{O}_{K,S}^*)^G \to \mathcal{O}_{K,S}^* \xrightarrow{N_{\mathcal{O}}} \mathcal{O}_{F,S}^*,
\]

where \( N_{\mathcal{O}} \) is the restriction of \( N_{K/F} \) to \( \mathcal{O}_{K,S}^* \). We conclude that

\[
(20) \quad \left[ (X^\vee \otimes \mathcal{O}_{K,S}^*)^G : T^3(U) \right] = \left[ \text{Ker}(N_{\mathcal{O}}) : T^3(U) \right]
\]
Remarks 4.4. (a) If (13) extends to an exact sequence
\[ 0 \rightarrow T^o \rightarrow R\sigma_{K,S/\mathcal{O}_{F,S}}(\mathbb{G}_{m,\mathcal{O}_{K,S}}) \xrightarrow{N_{\mathcal{O}}} \mathbb{G}_{m,\mathcal{O}_{F,S}} \]
of identity components of Néron-Raynaud models (cf. \[15, \text{Lemma 3.1}\]), then the index (20) is clearly 1.

(b) The proofs of Lemma 3.7 and Corollary 3.9, together with diagram (9) (see also Remark 3.10), show that the index
\[ \left[ (X^\vee \otimes \mathcal{O}_{K,S}^*)^G : T^o(U) \right] = \left[ \tilde{T}^o(\tilde{U})^G : T^o(U) \right] = \left[ \text{Ker}(\gamma) \right] \]
divides \( \prod_{v \in S} \left[ \Phi_v(\kappa(v))_{\text{tors}} \right] \). But \[23, \text{Corollary 2.18, p.175}\] and Lemma 4.1 above show that \( \left[ \Phi_v(\kappa(v))_{\text{tors}} \right] = \left[ I_{w_v} : G'_{w_v} \right] \) for any \( v \). Consequently, \( \left[ \text{Ker}(N_{\mathcal{O}}) : T^o(U) \right] \) divides \( \prod_{v \in S} \left[ I_{w_v} : G'_{w_v} \right] \), i.e., the latter integer is an upper bound for (20).

Combining Remark 3.10, Lemma 4.3 and formulas (17)-(20), we obtain Chevalley’s ambiguous class number formula for a norm torus:

**Theorem 4.5.** Let \( K/F \) be a Galois extension of global fields of degree \( n \) and Galois group \( G \). Let \( T = R_{K/F}^{(1)}(\mathbb{G}_{m,K}) \) be the corresponding norm torus. Then
\[
\frac{\left[ (C_{K,S}^{n-1})^G \right]}{h_{T,F,S}} = \frac{\left[ \hat{H}^0(G, K^*/\mathcal{O}_{K,S}^*) \right] \left[ \mathcal{O}_{F,S}^*: W_{F,S} \right] \left[ \text{Ker}(N_{\mathcal{O}}) : T^o(U) \right]}{\left[ \hat{H}^0(G, \mathcal{O}_{K,S}^*) \right] \prod_{v \in S} \left[ (G_{w_v}^{ab})_{e_v} \right]},
\]
where \( W_{F,S} \) is the group (18). \( \square \)

Next we need a definition. Let \( G \) be any finite group and let \( M \) be a left \( G \)-module such that both \( \hat{H}^0(G, M) \) and \( H^1(G, M) \) are finite. We define
\[
h(G, M) = \frac{\hat{H}^0(G, M)}{H^1(G, M)}.
\]
When \( G \) is cyclic, \( h(G, M) \) is known as the Herbrand quotient of \( M \).

We now divide Chevalley’s original formula (Theorem 1.1 of the Introduction) by the formula in the preceding theorem and obtain\(^6\)

**Corollary 4.6.** With the notations of the theorem, we have
\[
\frac{h_{T,F,S}}{h_{F,S}} = \frac{\left[ (C_{K,S}^{n-1})^G : C_{K,S}^G \right] h(G, \mathcal{O}_{K,S}^*) \prod_{v \in S} \left\{ \left[ (G_{w_v}^{ab})_{e_v} \right] \right\}}{\left[ \mathcal{O}_{F,S}^*: W_{F,S} \right] h(G, K^*/\mathcal{O}_{K,S}^*) \left[ \text{Ker}(N_{\mathcal{O}}) : T^o(U) \right]},
\]

\(^6\) We embed \( C_{K,S} \) in \( C_{K,S}^{n-1} \) diagonally.
where \( h(G, \mathcal{O}_{K,S}^*) \) and \( h(G, K^*/\mathcal{O}_{K,S}^*) \) are given by (21).

Assume now that \( K/F \) is a cyclic extension of degree \( n \). The exact sequence \( 1 \to \mathcal{O}_{K,S}^* \to K^* \to K^*/\mathcal{O}_{K,S}^* \to 1 \) and Hilbert’s Theorem 90 show that
\[
h(G, K^*/\mathcal{O}_{K,S}^*) = h(G, K^*)/h(G, \mathcal{O}_{K,S}^*)
\]
(see, e.g., [1, Proposition 10, p.109]). On the other hand, we have the well-known formula
\[
h(G, \mathcal{O}_{K,S}^*) = \frac{1}{n} \prod_{v \in S} [K_w : F_v]
\]
(see [20, proof of Theorem 8.3, p.179]). Finally, \( (G_{ab})_{e_v} = e_v \) for any \( v \not\in S \). Using the above, we deduce from Corollary 4.6 the following formula.

**Corollary 4.7.** Let \( K/F \) be a cyclic extension of global fields of degree \( n \) and Galois group \( G \). Let \( T = R_{K/F}^{(1)}(\mathbb{G}_m, K) \) be the corresponding norm torus. Then
\[
\frac{h_{T,F,S}}{h_{F,S}} = \frac{\left( (C_{K,S}^{n-1})^G : C_{K,S}^G \right) \prod_{v \in S} [K_w : F_v]^2 \prod_{v \notin S} e_v^2}{n^2 \left[ \mathcal{O}_{F,S}^* : W_{F,S} \right] [F^* : N_{K/F}(K^*)] [\ker(N_{\mathcal{O}}) : \mathcal{O}^c(U)]}.
\]

Assume now that \( K/F \) is a quadratic extension, i.e., \( n = 2 \). Clearly,
\[
\left( (C_{K,S}^{n-1})^G : C_{K,S}^G \right) = 1.
\]
Let \( \mu \) be the number of primes in \( S \) which do not split in \( K \), i.e., those \( v \in S \) such that \( [K_w : F_v] = 2 \), and let \( \nu \) be the number of primes of \( F \) not in \( S \) which ramify in \( K \). Then
\[
\prod_{v \in S} [K_w : F_v]^2 \prod_{v \notin S} e_v^2 = 4^{\mu+\nu}.
\]
Thus Corollary 4.7 yields

**Corollary 4.8.** Let \( K/F \) be a quadratic extension of global fields and let \( T = R_{K/F}^{(1)}(\mathbb{G}_m, K) \) be the corresponding norm torus. Then
\[
[\mathcal{O}_{F,S}^* : W_{F,S}] [F^* : N_{K/F}(K^*)] [\ker(N_{\mathcal{O}}) : \mathcal{O}^c(U)] \cdot h_{T,F,S} = 4^{\mu+\nu-1} h_{F,S},
\]
where \( \mu \) is the number of primes in \( S \) which do not split in \( K \) and \( \nu \) is the number of primes of \( F \) which ramify in \( K \) but are not in \( S \). In particular, \( h_{T,F,S} \) divides \( 4^{\mu+\nu-1} h_{F,S} \) if \( \mu + \nu \geq 1 \). □
Remark 4.9. Another interesting example, which we hope to discuss at length in [8], involves the quotient torus $S = R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_{m,F}$. This torus is dual to $R_{K/F}^{(1)}(\mathbb{G}_{m,K})$, and is isomorphic to it if $K/F$ is cyclic (see [11, Lemma 4.1, p.201]). In particular, the formula of Corollary 4.7 holds true with $S$ in place of $T$ (if $K/F$ is cyclic).

5. Concluding remarks

In [14, Theorem, p.135] M.Morishita, generalizing work of T.Ono, obtained a formula for the class number of the norm “torus” $T' = R_{O_{K,S}/O_{F,S}}^{(1)}$, associated to $O_{K,S}/O_{F,S}$, i.e., $T'$ is the kernel of the norm map $N_O: R_{O_{K,S}/O_{F,S}}(\mathbb{G}_{m,O_{K,S}}) \to \mathbb{G}_{m,O_{F,S}}$. Since in general the exact sequence (13) above does not extend in the sense of Remark 4.4(a) (not even in the tamely ramified case. Cf. [18, Lemma 6.7, p.28]), $T'$ and $T'$ need not coincide (indeed, $T'$ need not have connected fibers). It follows that Morishita’s formula and our own (see, e.g., Corollary 4.6) are not immediately comparable. It is an interesting problem, however, to explore the deeper connections that certainly exist between the approach of [14] and that of this paper. We hope to carry out this project in a not-too-distant future.

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We use quotation marks because, in general, $T'$ is not a torus (it is a torus if $O_{K,S}/O_{F,S}$ is étale).
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