LOWER BOUNDS FOR ESTIMATES OF THE SCHRÖDINGER MAXIMAL FUNCTION

XIUMIN DU, JONGCHON KIM, HONG WANG, AND RUIXIANG ZHANG

Abstract. We give new lower bounds for $L^p$ estimates of the Schrödinger maximal function by generalizing an example of Bourgain.

1. Introduction

Let

$$e^{it\Delta} f(x) = (2\pi)^{-n/2} \int e^{i(x \cdot \xi + t|\xi|^2)} \hat{f}(\xi) \, d\xi$$

denote the solution to the free Schrödinger equation

$$\begin{cases}
    iu_t - \Delta u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R} \\
    u(x, 0) = f(x), & x \in \mathbb{R}^n.
\end{cases}$$

We are interested in the value of $\bar{\gamma}_{n,p}$, the infimum of the numbers $\gamma_{n,p}$ such that the following Schrödinger maximal estimate holds:

$$\left(1.1\right) \sup_{0 < t \leq R} \left\| e^{it\Delta} f \right\|_{L^p(B^n(0, R))} \lesssim R^{\gamma_{n,p}} \|f\|_{L^2}, \quad \forall f : \text{supp} \hat{f} \subset B^n(0, 1).$$

Here $A \lesssim B$ denotes $A \leq C \varepsilon R^C$ for some constant $C > 0$ and positive results by Kenig–Ponce–Vega [11, $n = 1$], D.–Guth–Li [5, $n = 2$] and D.–Z. [8, $n \geq 3$], it is known that

$$\left(1.2\right) \bar{\gamma}_{n,p} = \max \left\{ n \left( \frac{1}{p} - \frac{n}{2(n+1)} \right), 0 \right\}$$

for any $p \geq 1$ when $n = 1, 2$, and $1 \leq p \leq 2$ when $n \geq 3$. Also, from the Stein-Tomas Fourier restriction theorem it follows that $\bar{\gamma}_{n,p} = 0$ for $p \geq \frac{2(n+2)}{n}$. However, it remains as an interesting problem to determine $\bar{\gamma}_{n,p}$ for $2 < p < \frac{2(n+2)}{n}$ when $n \geq 3$.

It may seem plausible that $\left(1.2\right)$ should hold for any $p \geq 1$ and $n \geq 1$. However, we disprove this for a certain range of $p$ when $n \geq 3$. Our main result is the following lower bound for $\bar{\gamma}_{n,p}$.

Theorem 1.1. Let $n \geq 3$ and $p \geq 2$. For every integer $1 \leq m \leq n$,

$$\bar{\gamma}_{n,p} \geq \frac{n+m}{2} \left( \frac{1}{p} - \frac{n}{2(n+1)} \right) + \frac{m}{2(m+1)}.$$
The example that proves Theorem 1.1 is built upon Bourgain’s example [2] that provides the lower bound for the case \( m = n \). For the case \( 1 \leq m < n \), we take Bourgain’s example in the intermediate dimension \( m \) and then “fatten” it to a function on \( \mathbb{R}^n \).

We state two special cases of Theorem 1.1 as a corollary.

**Corollary 1.2.** If \( \bar{\gamma}_{n,p} = n\left( \frac{1}{p} - \frac{m}{2(n+1)} \right) \), then

\[
p \leq p_0(n) := 2 + \frac{4}{(n-1)(n+2)}.\]

If \( \bar{\gamma}_{n,p} = 0 \), then

\[
p \geq p_1(n) := \max_{m \in \mathbb{Z}, 1 \leq m \leq n} 2 + \frac{4}{n - 1 + m + n/m}.
\]

**Remark 1.3.** Note that \( p_0(n) < \frac{2(n+1)}{n} < p_1(n) \) when \( n \geq 3 \). Therefore, (1.2) fails for \( p_0(n) < p < p_1(n) \) when \( n \geq 3 \).

Finally, we remark that some upper bounds for \( \bar{\gamma}_{n,p} \) can be obtained from weighted Fourier restriction estimates, c.f. [3]. In particular, we refer the reader to [7] for such estimates with \( p = 2(n+1)/n \), which was obtained via the polynomial partitioning method [8, 9] and refined Strichartz estimates [3, 6]. For \( p > 2(n+1)/n \), one can get new upper bounds by using an additional ingredient, the fractal \( L^2 \) restriction estimate [3]. However, it seems that new ingredients are still needed to get sharp results. We do not explore along this direction in the current paper.

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## 2. An example that proves Theorem 1.1

Theorem 1.1 is a consequence of the following.

**Proposition 2.1.** Let \( m, n \) be integers with \( 1 \leq m \leq n \). For any \( R > 1 \), there exists \( f \in L^2(\mathbb{R}^n) \) with \( \hat{f} \) supported in the annulus \( \{ \xi \in \mathbb{R}^n : |\xi| \sim R \} \) satisfying the following property: There is a set \( E \subset B^n(0,1) \) of measure comparable to \( R^{-\frac{m}{2-n}} \) such that for every \( x \in E \),

\[
\left| e^{it\Delta} f(x) \right| \geq R^{\frac{m}{2-n}} R^{\frac{m}{2-n}} \quad \text{for some } t = \frac{x_1}{2R} + O(R^{-3/2}).
\]

**Proof.** We write \( \bar{x} = (x, x') \in \mathbb{R}^m \times \mathbb{R}^{n-m} \) and \( \bar{\xi} = (\xi, \xi') \in \mathbb{R}^m \times \mathbb{R}^{n-m} \).

We briefly recall an estimate for the example \( f_0 \in L^2(\mathbb{R}^n) \) from [2], where \( \hat{f}_0 \) is supported in the annulus \( \{ \xi \in \mathbb{R}^n : |\xi| \sim R \} \); There is a set \( E_0 \subset B^n(0,1) \) of measure comparable to 1 such that for every \( x \in E_0 \),

\[
(2.1) \quad \left| e^{it\Delta} f_0(x) \right| \geq R^{\frac{m}{2-n}} R^{\frac{m}{2-n}} \quad \text{for some } t = \frac{x_1}{2R} + \tau \text{ with } |\tau| \leq \frac{1}{10} R^{-3/2}.
\]
Let $E$ be given by so that

$$This finishes the proof of Theorem 1.1.

(2.3)

See also [12] for a different example based on [1], which provides an estimate essentially the same as (2.1).

Let $\chi = \chi[-\frac{1}{2}, \frac{1}{2}]$ be the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let $f_1(x')$ be given by

$$\hat{f}_1(\xi') = \prod_{j=m+1}^{n} R^{-\frac{1}{2}} \chi \left(R^{-\frac{1}{2}}(\xi_j - R)\right),$$

so that $\|f_1\|_{L^2(\mathbb{R}^n)} = 1$. The choice of the function $f_1$ is motivated by the example from [2]. Note that

$$|e^{it\Delta} f_1(x')| = (2\pi)^{-(n-m)/2} \prod_{j=m+1}^{n} R^{\frac{1}{2}} \left| \int_{[-\frac{1}{2}, \frac{1}{2}]} e^{i(R^{1/2}\xi_j(x_j + 2Rt) + tR^2)} d\xi_j \right|.

When $|t + \frac{x_j}{2R}| \leq \frac{1}{2} R^{-3/2}$ and $|x_j - x_1| \leq \frac{1}{2} R^{-1/2}$ for each $m < j \leq n$, there is little cancellation in the above integral and therefore

(2.2)

$$|e^{it\Delta} f_1(x')| \gtrsim R^{\frac{n-m}{4}}.

We take $f$ to be the tensor product of $f_0$ and $f_1$, i.e.,

$$f(\bar{x}) := f_0(x)f_1(x').

Let $E$ be the set given by

$$E = \{(x, x') \in B^n(0, 1) : x \in E_0 \text{ and } \max_{m<j\leq n} |x_j - x_1| \leq \frac{1}{2} R^{-1/2}\}.$

It follows that the measure of the set $E$ is comparable to $R^{-\frac{n-m}{2}}$. Moreover, for any $\bar{x} = (x, x') \in E$, we have by (2.1) and (2.2),

$$\frac{|e^{it\Delta} f(\bar{x})|}{\|f\|_{L^2}} \geq \frac{|e^{it\Delta} f_0(x)|}{\|f_0\|_{L^2}} \frac{|e^{it\Delta} f_1(x')|}{\|f_1\|_{L^2}} \gtrsim R^{\frac{m-n}{2} + R^{\frac{n-m}{4}}}

for some $t$ satisfying $|t + \frac{x_j}{2R}| \leq \frac{1}{10} R^{-3/2}$.

We proceed to the proof of Theorem 1.1. It follows from Proposition 2.1 that,

(2.3)

$$\left\| \sup_{0< t \leq R} |e^{it\Delta} f| \right\|_{L^p(B^n(0, 1))} \gtrsim R^{-\frac{m-n}{2} + \frac{n-m}{2} + \frac{1}{p} - \frac{1}{p} - 1} \|f\|_2.

Theorem 1.1 follows from (2.3) by scaling. Define the function $g \in L^2(\mathbb{R}^n)$ by

$$\hat{g}(\xi) = R^{\frac{n-m}{2} + \frac{n-m}{2} + \frac{1}{p} - \frac{1}{p} - 1} \|g\|_2.

so that $\hat{g}$ is supported in the annulus $|\xi| \sim 1$ and $\|g\|_{L^2} = \|f\|_{L^2}$. By parabolic rescaling, we have

$$|e^{it\Delta} f(x)| = R^{\frac{n-m}{2} + \frac{n-m}{2} + \frac{1}{p} - \frac{1}{p} - 1} |e^{itR^2\Delta} g(Rx)|.

Hence, by (2.3),

$$\left\| \sup_{0< t \leq R} |e^{it\Delta} g| \right\|_{L^p(B^n(0, R))} = R^{\frac{n-m}{2} + \frac{n-m}{2} + \frac{1}{p} - \frac{1}{p} - 1} \sup_{0< t \leq R} |e^{it\Delta} f| \left\|_{L^p(B^n(0, 1))} \gtrsim R^{-\frac{m-n}{2} + \frac{n-m}{2} + \frac{1}{p} - \frac{1}{p} - 1} \|g\|_2.

This finishes the proof of Theorem 1.1.
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UNIVERSITY OF MARYLAND, COLLEGE PARK, MD
E-mail address: xdu@math.umd.edu

UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BC
E-mail address: jkim@math.ubc.ca

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
E-mail address: hongwang@mit.edu

UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI
E-mail address: ruixiang@math.wisc.edu