2D String Theory as Normal Matrix Model

Sergei Yu. Alexandrov, Vladimir A. Kazakov, and Ivan K. Kostov

We show that the $c = 1$ bosonic string theory at finite temperature has two matrix-model realizations related by a kind of duality transformation. The first realization is the standard one given by the compactified matrix quantum mechanics in the inverted oscillator potential. The second realization, which we derive here, is given by the normal matrix model. Both matrix models exhibit the Toda integrable structure and are associated with two dual cycles (a compact and a non-compact one) of a complex curve with the topology of a sphere with two punctures. The equivalence of the two matrix models holds for an arbitrary tachyon perturbation and in all orders in the string coupling constant.
1. Introduction

The $c = 1$ string theory\(^1\) has been originally constructed in the early 90’s as the theory of random surfaces embedded into a one-dimensional spacetime [2]. Since then it became clear that this is only one of the realizations of a more universal structure, which reappeared in various mathematical and physical problems. Most recently, it was used for the description of the 2D black hole [3,4,5,6] in the context of the topological string theories on Calabi-Yau manifolds with vanishing cycles [7] and $\mathcal{N} = 1$ SYM theories [8].

The $c = 1$ string theory can be constructed as the collective field theory for a one-dimensional $N \times N$ hermitian matrix field theory known also as Matrix Quantum Mechanics (MQM). This construction represents the simplest example of the strings/matrix correspondence. The collective excitations in the singlet sector of MQM are massless “tachyons” with various momenta, while the non-singlet sectors contain also winding modes.

The singlet sector of MQM can be reduced to a system of $N$ nonrelativistic fermions in the upside-down gaussian potential. Thus the elementary excitations of the $c = 1$ string can be represented as collective excitations of free fermions near the Fermi level. The tree-level string-theory $S$-matrix can be extracted by considering the propagation of infinitesimal “pulses” along the Fermi sea and their reflection off the “Liouville wall” [9,10]. The (time-dependent) string backgrounds are associated with the possible profiles of the Fermi sea [11].

An important property of the $c = 1$ string theory is its integrability. The latter has been discovered by Dijkgraaf, Moore and Plesser [12] in studying the properties of the tachyon scattering amplitudes. It was demonstrated in [12] that the partition function of the $c = 1$ string theory in the case when the allowed momenta form a lattice, as in the case of the compactified Euclidean theory, is a tau function of Toda hierarchy [13]. The operators associated with the momentum modes in the string theory have been interpreted in [12] as Toda flows. A special case represents the theory compactified at the self-dual radius $R = 1$. It is equivalent to a topological theory that computes the Euler characteristic of the moduli space of Riemann surfaces [14]. When $R = 1$ and only in this case, the partition function of the string theory has alternative realization as a Kontsevich-type model [12,15].

In this paper we will show that there is another realization of the $c = 1$ string theory by the so-called Normal Matrix Model (NMM). This is a complex $N \times N$ matrix model in which the integration measure is restricted to matrices that commute with their hermitian conjugates. The normal matrix model has been studied in the recent years in the context of the laplacian growth problem, the integrable structure of conformal maps, and the quantum Hall droplets [16,17,18,19,20,21]. It has been noticed that both matrix models, the singlet sector of MQM and NMM, have similar properties. Both models can be reduced to systems

\(^1\) As a good review we recommend [1].
of nonrelativistic fermions and possess Toda integrable structure. There is however an essential difference: the normal matrix model describes a compact droplet of Fermi liquid while the Fermi sea of the MQM is non-compact. Furthermore, the perturbations of NMM are introduced by a matrix potential while these of MQM are introduced by means of time-dependent asymptotic states. In this paper we will show that nevertheless these two models are equivalent.

The exact statement is that the matrix quantum mechanics compactified at radius \( R \) is equivalent to the normal matrix model defined by the probability distribution function 

\[
\exp\left[ \frac{1}{\hbar} W_R(Z, Z^\dagger) \right],
\]

where

\[
W_R(Z, Z^\dagger) = \text{tr}(ZZ^\dagger)^R - \frac{R}{2 \hbar} \text{tr} \log(ZZ^\dagger) - \sum_{k \geq 1} t_k \text{tr}Z^k - \sum_{k \geq 1} t_{-k} \text{tr}Z^{\dagger k}. \tag{1.1}
\]

More precisely, the grand canonical partition function of MQM is identical to the canonical partition function of NMM at the perturbative level, by which we mean that the genus expansions of the two free energies coincide. This will be proved in two steps. First, we will show that the non-perturbed partition functions are equal. Then we will use the fact that both partition functions are \( \tau \)-functions of Toda lattice hierarchy, which implies that they also coincide in presence of an arbitrary perturbation.

We also give a unified geometrical description of the two models in terms of a complex curve with the topology of a sphere with two punctures. This complex curve is analog of the Riemann surfaces arising in the case of hermitian matrix models. However, for generic \( R \) it is not an algebraic curve. The curve has two dual non-contractible cycles, which determine the boundaries of the supports of the eigenvalue distributions for the two models. The normal matrix model is associated with the compact cycle while the MQM is associated with the non-compact cycle connecting the two punctures. We will construct a globally defined one-form whose integrals along the two cycles give the number of eigenvalues \( N \) and the derivative of the free energy with respect to \( N \).

The paper is organized as follows. In the next section we will remind the realization of the \( c = 1 \) string theory as a fermionic system with chiral perturbations worked out in [11]. We will stress on the calculation of the free energy, which will be needed to establish the equivalence with the NMM. In Sect. 3 we construct the NMM having the same partition function. In Sect. 4 we consider the quasiclassical limit and give a unified geometrical description of the two models.
2. Matrix quantum mechanics, free fermions and integrability

2.1. Eigenfunctions and fermionic scattering

In absence of winding modes, the 2D string theory is described by the singlet sector of Matrix Quantum Mechanics in the double scaling limit with Hamiltonian

$$\hat{H}_0 = \frac{1}{2} \text{tr}(-\hbar^2 \frac{\partial^2}{\partial X^2} - X^2).$$

(2.1)

The radial part of this Hamiltonian is expressed in terms of the eigenvalues $x_1, \ldots, x_N$ of the matrix $X$. The wave functions in the $SU(N)$-singlet sector are completely antisymmetric and thus describe a system of $N$ nonrelativistic fermions in the inverse gaussian potential. The dynamics of the fermions is governed by the Hamiltonian

$$\hat{H}_0 = \frac{1}{2} \sum_{i=1}^N (\hat{p}_i^2 - \hat{x}_i^2),$$

(2.2)

where $p_i$ are the momenta conjugated to the fermionic coordinates $x_i$.

To describe the incoming and outgoing tachyonic states, it is convenient to introduce the "light-cone" coordinates in the phase space

$$\hat{x}_\pm = \frac{\hat{x} \pm \hat{p}}{\sqrt{2}}$$

(2.3)

satisfying the canonical commutation relations

$$[\hat{x}_+, \hat{x}_-] = -i\hbar.$$

(2.4)

In these variables the one-particle Hamiltonian takes the form

$$\hat{H}_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+).$$

(2.5)

We can work either in $x_+$ or in $x_-$ representation, where the theory is defined in terms of fermionic fields $\psi_\pm(x_\pm)$ respectively. The solutions with a given energy are $\psi_\pm^E(x_\pm, t) = e^{-iEt} \psi_\pm^E(x_\pm)$ with

$$\psi_\pm^E(x_\pm) = \frac{1}{\sqrt{2\pi\hbar}} e^{\mp \frac{i}{\hbar} \phi_0 x_\pm + \frac{i}{2} \frac{\left| E - \frac{1}{2} \right|}{\hbar}}$$

(2.6)

where the phase factor $\phi_0(E)$ will be determined below. These solutions form two complete systems of $\delta$-function normalized orthonormal states under the condition that the domain
of the definition of the wave functions is a semi-axis. The left and right representations are related to each other by a unitary operator which is the Fourier transform on the half-line

\[ [\hat{S}\psi_+]_-(x_-) = \int_0^\infty dx_+ S(x_-, x_+)\psi_+(x_+). \]  

(2.7)

The integration kernel \( S(x_-, x_+) \) is either a sine or a cosine depending on the boundary conditions at the origin. For definiteness, let us choose the cosine kernel

\[ S(x_-, x_+) = \sqrt{2\pi\bar{\hbar}} \cos(\frac{1}{\bar{\hbar}}x_+x_-). \]  

(2.8)

Then the operator \( \hat{S} \) is diagonal on the eigenfunctions (2.6) with given energy

\[ [\hat{S}\psi^E_+]_-(x_-) = e^{-\frac{i}{\bar{\hbar}}\phi_0} \mathcal{R}(E)\psi^E_-(x_-), \]  

(2.9)

where

\[ \mathcal{R}(E) = \hbar^i E \sqrt{\frac{2}{\pi}} \cosh \left( \frac{\pi}{2} \left( \frac{1}{\bar{\hbar}}E - i/2 \right) \right) \Gamma \left( \frac{i}{\bar{\hbar}}E + 1/2 \right). \]  

(2.10)

Since the operator \( \hat{S} \) is unitary, i.e.

\[ \overline{\mathcal{R}(E)}\mathcal{R}(E) = \mathcal{R}(-E)\mathcal{R}(E) = 1, \]  

(2.11)

it can be absorbed into the function \( \phi_0(E) \). This fixes the phase of the eigenfunctions as

\[ \phi_0(E) = -i\hbar \log \mathcal{R}(E). \]  

(2.12)

In fact, the operator \( \hat{S} \) appears in our formalism as the fermionic \( S \)-matrix describing the scattering off the inverse oscillator potential and the factor \( \mathcal{R}(E) \) gives the fermionic reflection coefficient.

Let us introduce the scalar product between left and right states as follows

\[ \langle \psi_- | \hat{S} | \psi_+ \rangle = \int_0^\infty dx_+ dx_- \overline{\psi_-}(x_-)S(x_-, x_+)\psi_+(x_+). \]  

(2.13)

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2 In order to completely define the theory, we should define the interval where the phase-space coordinates \( x_\pm \) are allowed to take their values. One possibility is to allow the eigenvalues to take any real value. The whole real axis corresponds to a theory whose Fermi sea consists of two disconnected components on both sides of the maximum of the potential. To avoid the technicalities related to the tunneling phenomena between the two Fermi seas, we will define the theory by restricting the eigenvalues to the positive real axis. The difference between the two choices is seen only at the non-perturbative level.
Then the in- and out-eigenfunctions are orthonormal with respect to this scalar product

\[ \langle \psi^E_- | \hat{S} | \psi^E_+ \rangle = \delta(E - E'). \] (2.14)

It is actually this relation that defines the phase factor containing all information about the scattering.

The ground state of the Hamiltonian (2.5) can be constructed from the wave functions (2.6) and describes the linear dilaton background of string theory \[10\]. The tachyon perturbations can be introduced by changing the asymptotics of the wave functions at \( x_\pm \to \infty \) to

\[ \Psi^E_\pm(x_\pm) \sim e^{\mp \frac{i}{\hbar} R \sum t_k x_\pm^{k/R}} x_\pm^{\pm \frac{1}{2} E - \frac{i}{\hbar}}. \] (2.15)

The exact phase contains also a constant mode \( \mp \frac{1}{2\pi} \phi(E) \) as in (2.6) and negative powers of \( x_\pm^{1/R} \) which are determined by the orthonormality condition

\[ \langle \Psi^E_- | \hat{S} | \Psi^E_+ \rangle = \delta(E - E'). \] (2.16)

The constant mode \( \phi(E) \) of the phase of the fermion wave function contains all essential information about the perturbed system.

2.2. Cut-off prescription and density of states

To find the density of states, we introduce a cut-off \( \Lambda \) by confining the phase space to a periodic box

\[ x_\pm + 2\sqrt{\Lambda} \equiv x_\pm. \] (2.17)

which can be interpreted as putting a reflecting wall at distance \( \sqrt{\Lambda} \). This means that at the points \( x_+ = \sqrt{\Lambda} \) and \( x_- = \sqrt{\Lambda} \) the reflected wave function coincides with the incoming one

\[ [\hat{S} \Psi](\sqrt{\Lambda}) = \Psi(\sqrt{\Lambda}). \] (2.18)

Applied to the wave functions (2.15), this condition gives an equation for the admissible energies

\[ e^{\mp \phi(E)} = e^{-\mp \frac{1}{2\pi} V(\Lambda)} A^{\pm E}, \quad V(\Lambda) = R \sum (t_k + t_{-k}) \Lambda^{k/2R} \] (2.19)

where we neglected the negative powers of \( \Lambda \). It is satisfied by a discrete set of energies \( E_n \) defined by the relation

\[ E_n \log \Lambda - \phi(E_n) = 2\pi \hbar n + V(\Lambda), \quad n \in \mathbb{Z}. \] (2.20)

Taking the limit \( \Lambda \to \infty \) we find the density of states (in units of \( \hbar \))

\[ \rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi} \frac{d\phi(E)}{dE}. \] (2.21)
2.3. The partition function of the compactified 2D string theory

Knowing the density of states, we can calculate the grand canonical partition function $Z$ of the quantum-mechanical system compactified at Euclidean time $\beta = 2\pi R$. If $\mu$ is the chemical potential, then the free energy $F = \hbar^2 \log Z$ of the ensemble of free fermions is given by

$$F(\mu, t) = \hbar \int_{-\infty}^{\infty} dE \rho(E) \log \left[ 1 + e^{-\frac{i\hbar}{\beta}(\mu+E)} \right]. \tag{2.22}$$

Integrating by parts and dropping out the $\Lambda$-dependent non-universal piece, we obtain

$$F(\mu, t) = -\frac{\beta}{2\pi} \int_{-\infty}^{\infty} dE \frac{\phi(E)}{1 + e^{\frac{i\hbar}{\beta}(\mu+E)}}. \tag{2.23}$$

We close the contour of integration in the upper half plane and take the integral as a sum of residues of the thermal factor

$$F(\mu, t) = i\hbar \sum_{n \geq 0} \phi \left( i\hbar \frac{n + \frac{1}{2}}{R} - \mu \right). \tag{2.24}$$

We will be interested only in the perturbative expansion in $1/\mu$ and neglect the non-perturbative terms $\sim e^{-\frac{i\hbar}{\pi} \mu}$, as well as non-universal terms in the free energy (regular in $\mu$). Therefore, for the case of zero tachyon couplings we can retain only the $\Gamma$-function in the reflection factor $\tau_0 \left( \mu, \frac{s}{2}\right)$ and choose the phase $\phi_0$ as

$$\phi_0(E) = -i\hbar \log \Gamma \left( \frac{i}{\hbar} E + 1/2 \right). \tag{2.25}$$

It is known [12] that the tachyon scattering data for MQM are generated by a $\tau$-function of Toda lattice hierarchy where the coupling constants $t_k$ play the role of the Toda times. The $\tau$-function is related to the constant mode $\phi(\mu + i\hbar/2R)$ by [11]

$$e^{\frac{i\hbar}{\beta} \phi(-\mu)} = \frac{\tau_0 \left( \mu + i\hbar \frac{s}{2R} \right)}{\tau_0 \left( \mu - i\hbar \frac{s}{2R} \right)}. \tag{2.26}$$

Taking into account (2.24) we conclude that the partition function of the perturbed MQM is equal to the $\tau$-function:

$$Z(\mu, t) = \tau_0(\mu, t). \tag{2.27}$$

The discrete space parameter $s \in \mathbb{Z}$ along the Toda chain is related to the chemical potential $\mu$. More precisely, $s$ corresponds to an imaginary shift of $\mu$ [3,11]:

$$\tau_s(\mu, t) = \tau_0 \left( \mu + i\hbar \frac{s}{R}, t \right). \tag{2.28}$$

This fact was used to rewrite the Toda equations as difference equations in $\mu$ rather than in the discrete parameter $s$. It will be also used below to prove the equivalence of 2D string theory to the normal matrix model.

3 We neglect the nonperturbative terms associated with the cuts of the function $\phi(E)$.
3. 2D string theory as normal matrix model

In this section we will show that the partition function of 2D string theory with
tachyonic perturbations can be rewritten as a normal matrix model. It is related to the
original matrix quantum mechanics by a kind of duality transformation. In fact, we will
give two slightly different normal matrix models fulfilling this goal.

Consider the following matrix integral

\[ Z_{NMM}(N, t, \alpha) = \int_{[Z, Z^\dagger] = 0} dZ dZ^\dagger \left[ \det(ZZ^\dagger) \right]^{(R-1)/2+\frac{R}{2\hbar}} e^{-\frac{1}{\hbar} \text{tr}[(ZZ^\dagger)^R - V_+(Z) - V_-(Z^\dagger)]}, \tag{3.1} \]

where the integral goes over all complex matrices satisfying \([Z, Z^\dagger] = 0\) and the potentials
are given by

\[ V_\pm(z) = \sum_{k \geq 1} t_{\pm k} z^k. \tag{3.2} \]

We made the dependence on the Planck constant explicit because later we will need to
analytically continue in \(\hbar\). The integral (3.1) defines a \(\tau\)-function of Toda hierarchy [20].
We remind the proof of this statement in Appendix A, where we also derive the string
equation specifying the unique solution of Toda equations. This string equation coincides
(up to change \(\hbar \to i\hbar\)) with that of the hierarchy describing the 2D string theory [6,11].
Therefore, if we identify correctly the parameters of the two models, the \(\tau\)-functions should
also be the same. Namely, we should relate the chemical potential \(\mu\) of 2D string theory
to the size of matrices \(N\) and the parameter \(\alpha\) of the normal matrix model. There are two
possibilities to make such identification.

3.1. Model I

The first possibility is realized taking a large \(N\) limit of the matrix integral (3.1). Namely, we will prove that the full perturbed partition function of MQM is given by the
large \(N\) limit of the partition function (3.1) with \(\alpha = R\mu - \hbar N\) and a subsequent analytical
continuation \(\hbar \to i\hbar\)

\[ Z_{\hbar}(\mu, t) = \lim_{N \to \infty} Z_{ih}^{NMM}(N, t, R\mu - i\hbar N). \tag{3.3} \]

The necessity to change the Planck constant by the imaginary one follows from the com-
parison of the string equations of two models as was discussed above. More generally,
the \(\tau\)-function (2.28) for arbitrary \(s\) is obtained from the partition function (3.1) in the
following way

\[ \tau_{s, \hbar}(\mu, t) = \lim_{N \to \infty} Z_{ih}^{NMM}(N + s, t, R\mu - i\hbar N). \tag{3.4} \]

Let us stress that in spite of the large \(N\) limit taken here, we obtain as a result the full
(and not only dispersionless) partition function of the \(c = 1\) string theory.
To prove this statement we will use the fact that both partition functions are $\tau$-functions of the Toda lattice hierarchy with times $t_{\pm k}$, $k = 1, 2, \ldots$. Since the unperturbed $\tau$-function provides the necessary boundary conditions for the unique solution of Toda equations, it is sufficient to show that the unperturbed partition functions of the two models coincide and the $s$-parameters of two $\tau$-functions are identical.

The integral (3.1) reduces to the product of the normalization coefficients of the orthogonal polynomials, eq. (A.5). From the definition of the scalar product (A.4) it follows that when all $t_{\pm k} = 0$ the orthogonal polynomials are simple monomials and the normalization factors $h_n$ are given by

$$h_n(0, \alpha) = \frac{1}{2\pi i} \int_C d^2ze^{-\frac{1}{\hbar}R(z\bar{z})}(z\bar{z})^{(R-1)/2 + \frac{\alpha}{R} + n}. \quad (3.5)$$

Up to constant factors and powers of $\hbar$, we find

$$h_n(0, \alpha) \sim \Gamma \left( \frac{\alpha}{\hbar R} + \frac{n + R - \frac{\alpha}{R}}{2} \right). \quad (3.6)$$

We see that the result coincides with the reflection coefficient given in (2.25). Therefore, taking the analytical continuation of parameters as in (3.3), we have

$$\lim_{N \to \infty} \hbar^2 \log Z_{ihh}^{NMM}(N, 0, R\mu - i\hbar N) = \lim_{N \to \infty} i\hbar \sum_{n=0}^{N-1} \phi_0 \left( i\hbar \frac{N - n - \frac{\alpha}{R}}{2} - \mu \right) \quad (3.7)$$

$$= i\hbar \sum_{n=0}^{\infty} \phi_0 \left( i\hbar \frac{n + \frac{\alpha}{R}}{2} - \mu \right).$$

This coincides with the unperturbed free energy $F(\mu, 0)$ from (2.24), what proves (3.3) for all $t_{\pm k} = 0$.

Thus, it remains to show that the $s$-parameter of the $\tau$-function describing the normal matrix model is associated with $\mu$. From (A.17) we see that it coincides with $N$. Hence it trivially follows from (3.4) that

$$\tau_{s, \hbar}(\mu, t) = \lim_{N \to \infty} Z_{ihh}^{NMM} \left( N, t, R \left( \mu + \frac{i\hbar s}{R} \right) - i\hbar N \right) = \tau_{0, \hbar} \left( \mu + \frac{i\hbar s}{R}, t \right), \quad (3.8)$$

what means that the $\tau$-function defined in (3.4) possesses the characteristic property (2.28).

3.2. Model II

In fact, one can even simplify the representation (3.3) what will provide us with the second realization of 2D string theory in terms of NMM. Let us consider the matrix model...
for $\alpha = 0$. We claim that the partition function $Z_{ih}(N, t, 0)$ in the canonical ensemble, analytically continued to imaginary Planck constant and

$$N = -\frac{i}{\hbar} R\mu,$$

(3.9)

coincides with the partition function of 2D string theory in the grand canonical ensemble:

$$Z_{\bar{\hbar}}(\mu, t) = Z_{i\bar{\hbar}}^{N\text{MM}} (\mu, t).$$

(3.10)

Indeed, as for the first model, they are both given by $\tau$-functions of Toda hierarchy. After the identification (3.9), the $s$-parameters of these $\tau$-functions are identical due to (A.17) and (2.28). Therefore, it only remains to show that the integral $Z_{ih}(N, 0)$ without potential is equal to the unperturbed partition function (2.24). In this case the method of orthogonal polynomials gives

$$Z_{i\bar{\hbar}}^{N\text{MM}} (N, 0) = \prod_{n=0}^{N-1} R \left( -i\hbar \frac{n + \frac{1}{2}}{R} \right),$$

(3.11)

where the $R$ factors are the same as in MQM, as was shown in (3.6). Then we write

$$Z_{i\bar{\hbar}}^{N\text{MM}} (N, 0) = \Xi(0)/\Xi(N), \quad \text{where} \quad \Xi(N) = \prod_{n=N}^{\infty} R \left( -i\hbar (n + \frac{1}{2})/R \right).$$

(3.12)

$\Xi(0)$ is a constant and can be neglected, whereas $\Xi(N)$ can be rewritten as

$$\Xi(N) = \prod_{n=0}^{\infty} R \left( -i\hbar N/R - i\hbar (n + \frac{1}{2})/R \right).$$

(3.13)

Taking into account the unitarity relation (2.11) and substituting $N$ from (3.9), we obtain

$$Z_{i\bar{\hbar}}^{N\text{MM}} (-\frac{i}{\hbar} R\mu, 0) \sim \Xi^{-1} (-\frac{i}{\hbar} R\mu) = \prod_{n=0}^{\infty} R \left( \mu + i\hbar (n + \frac{1}{2})/R \right).$$

(3.14)

Thus $\hbar^2 \log Z_{i\bar{\hbar}}^{N\text{MM}} (-\frac{i}{\hbar} R\mu, 0)$ coincides with the free energy (2.24) with all $t_k = 0$. Note that the difference in the sign of $\mu$ does not matter since the free energy is an even function of $\mu$ (up non-universal terms). Since the two partition functions are both solutions of the

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4 From now on we will omit the last argument of $Z_{i\bar{\hbar}}^{N\text{MM}}$ corresponding to the vanishing parameter $\alpha$.

5 Moreover, this sign can be correctly reproduced from (2.23) if closing the integration contour in the lower half plane.
Toda hierarchy, the fact that they coincide at $t_k = 0$ implies that they coincide for arbitrary perturbation.

We conclude that the grand canonical partition function of the $c = 1$ string theory equals the canonical partition function of the normal matrix model (3.14) for $\alpha = 0$ and $N = -\frac{i}{\hbar}R\mu$. It is also clear that we can identify the operators of tachyons in two models

$$\text{Tr } X^+_n/R \leftrightarrow \text{Tr } Z^n, \quad \text{Tr } X^-_n/R \leftrightarrow \text{Tr } (Z^\dagger)^n.$$

(3.15)

4. Geometrical meaning of the duality between the two models

The quasiclassical limit of most of the solvable matrix models has a nice geometrical interpretation. Namely, the free energy in this limit can be parameterized in terms of the periods of a holomorphic 1-form around the cycles of an analytical curve [22,23,24,25]. Recently this geometrical picture appeared also in the context of the topological strings on singular Calabi-Yau manifolds and supersymmetric gauge theories [26,27,28].

In this section we will show that both MQM and NMM have in the quasiclassical limit a similar geometrical interpretation in terms of a one-dimensional complex curve, which is topologically a sphere with two punctures. Here by complex curve we understand a complex manifold with punctures and given behavior of the functions on this manifold at the punctures. Each of the two matrix models corresponds to a particular real section of this curve coinciding with one of its two non-contractible cycles, and the duality between them is realized by the exchange of the cycles of the curve.

4.1. The dispersionless limit

The quasiclassical limit $\hbar \to 0$ corresponds to the dispersionless limit of the Toda hierarchy where it has a description in terms of a classical dynamical system. The Lax operators are considered as $c$-functions of the two canonically conjugated coordinates $\mu$ and $\omega$, where $\omega$ is the quasiclassical limit of the shift operator $\hat{\omega} = e^{\partial/\partial s}$. Moreover, the two Lax operators can be expressed as series in $\omega$. The solutions of the dispersionless hierarchy correspond to canonical transformations in the phase space of the dynamical system. In the case of MQM this is the transformation from the coordinates $\mu$ and $\log \omega$ to $x_+$ and $x_-$. The particular solution of the Toda hierarchy that appears in our problem also satisfies the dispersionless string equation. In the case of MQM this is nothing but the equation of the profile of the Fermi sea in the phase space [11]

$$x_+ x_- = \sum_{k \geq 1} kt_{\pm k} x_{\pm k}/R + \mu + \sum_{k \geq 1} v_{\pm k} x_{\pm k}/R.$$

(4.1)

To get the string equation for NMM it is enough to make the substitution following (3.13)

$$x_+ \leftrightarrow z^R, \quad x_- \leftrightarrow \bar{z}^R$$

(4.2)

and $\mu = \hbar N/R$ as explained in the previous section. (We took into account that $\hbar$ from [3.3] should be replaced here by $-i\hbar$ so that the factor $i$ is canceled.) Then the string equation describes the contour $\gamma$ bounding the region $D$ in the complex $z$-plane filled by the eigenvalues of the normal matrix.

6 Here we consider the case when the eigenvalues are distributed in a simply connected domain.
4.2. NMM in terms of electrostatic potential

In this section we will introduce a function of the spectral variables $z$ and $\bar{z}$, which plays a central role in the NMM integrable structure. This function has several interpretations in the quasiclassical limit. First, it can be viewed as the generating function for the canonical transformation mentioned above, which maps variables $\mu$ and $\log \omega$ to the variables $z$ and $\bar{z}$. Second, it gives the phase of the fermion wave function at the Fermi level after the identification (4.2).

There is also a third, electrostatic interpretation, which is geometrically the most explicit and which we will follow in this section. According to this interpretation, the eigenvalues distributed in the domain $\mathcal{D}$ can be considered to form a charged liquid with the density

$$\rho(z, \bar{z}) = \frac{1}{\pi} \partial_z \bar{z} W_R(z, \bar{z}) = \frac{R^2}{\pi} (z \bar{z})^{R-1}.$$ (4.3)

Let $\varphi(z, \bar{z})$ be the potential of the charged eigenvalue liquid, which is a harmonic function outside the domain $\mathcal{D}$ and it is a solution of the Laplace equation inside the domain

$$\varphi(z, \bar{z}) = \begin{cases} \varphi(z) + \tilde{\varphi}(\bar{z}), & z \notin \mathcal{D}, \\ (z \bar{z})^R, & z \in \mathcal{D}. \end{cases}$$ (4.4)

To fix completely the potential, we should also impose the asymptotics of the potential at infinity. This asymptotics is determined by the coupling constants $t_{\pm n}, n = 1, 2, ...$ and can be considered as the result of placing a dipole, quadrupole etc. charges at infinity.

The solution of this electrostatic problem is obtained as follows. The continuity of the potential $\varphi(z, \bar{z})$ and its first derivatives leads to the following conditions to be satisfied on the boundary $\gamma = \partial \mathcal{D}$

$$\varphi(z) + \tilde{\varphi}(\bar{z}) = (z \bar{z})^R,$$ (4.5)

$$z \partial_z \varphi(z) = \bar{z} \partial_{\bar{z}} \tilde{\varphi}(\bar{z}) = Rz^R \bar{z}^R.$$ (4.6)

Each of two equations (4.6) can be interpreted as an equation for the contour $\gamma$. Since we obtain two equations for one curve, they should be compatible. This imposes a restriction on the holomorphic functions $\varphi(z)$ and $\tilde{\varphi}(\bar{z})$. The solutions for these chiral parts of the potential can be found comparing (4.6) with the string equation (4.1). In this way we have

$$\varphi(z) = \hbar N \log z + \frac{1}{2} \phi + R \sum_{k \geq 1} t_k z^k - R \sum_{k \geq 1} \frac{1}{k} v_k z^{-k},$$

$$\tilde{\varphi}(\bar{z}) = \hbar N \log \bar{z} + \frac{1}{2} \phi + R \sum_{k \geq 1} t_{-k} z^k - R \sum_{k \geq 1} \frac{1}{k} v_{-k} z^{-k}.$$ (4.7)

See also [29] for another interesting interpretation which appears useful in string field theory.
The zero mode $\phi$ is fixed by the condition Eq. (4.5). However, it is easier to use another interpretation. As we mentioned above, the holomorphic functions $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ coincide with the phases of the fermionic wave functions and in particular their constant modes are the same. Then, taking the limit $\hbar \to 0$ of (2.26), in the variables of NMM we find

$$\phi = -\hbar \frac{\partial}{\partial N} \log Z_{\text{NMM}}. \quad (4.8)$$

4.3. The complex curve

Instead of considering the function $\varphi(z, \bar{z})$ harmonic in the exterior domain $\bar{D} = \mathbb{C}/D$, we can introduce one holomorphic function $\Phi$ defined in the double cover of $\bar{D}$. The doubly covered domain $\bar{D}$ is topologically equivalent to a two-sphere with two punctures (at $z = \infty$ and $\bar{z} = \infty$), which we identify with the north and the south poles. It can be covered by two coordinate patches associated with the north and south hemispheres and parameterized by $z$ and $\bar{z}$. They induce a natural complex structure so that the sphere can be viewed as a complex manifold. The patches overlap in a ring containing the contour $\gamma$ and the transition function between them $\bar{z}(z)$ (or $z(\bar{z})$) is defined through the string equation Eq. (4.6).

The string equation can also be considered as a defining equation for the manifold. We start with the complex plane $\mathbb{C}^2$ with flat coordinates $z$ and $\bar{z}$. Then the manifold is defined as solution of equation (4.6), thus providing its natural embedding in $\mathbb{C}^2$.

To completely define the complex curve, we should also specify the singular behavior at the punctures of the analytic functions on this curve. We fix it to be the same as the one of the holomorphic parts of the potential Eq. (4.7).

We define on the complex curve the following holomorphic field

$$\Phi \overset{\text{def}}{=} \begin{cases} 
\Phi_+(z) = \varphi(z) - \frac{1}{2}(z\bar{z}(z))^R & \text{in the north hemisphere}, \\
\Phi_-(\bar{z}) = -\varphi(\bar{z}) + \frac{1}{2}(z(\bar{z})\bar{z})^R & \text{in the south hemisphere}.
\end{cases} \quad (4.9)$$

Its analyticity follows from equation (4.3), which now holds on the entire curve, since $z$ and $\bar{z}$ are no more considered as conjugated to each other.

The field $\Phi$ gives rise to a closed (but not exact) holomorphic 1-form $d\Phi$, globally defined on the complex curve. Actually, this is the unique globally defined 1-form with given singular behavior at the two punctures. Therefore, we can characterize the curve by periods of this 1-form around two conjugated cycles $A$ and $B$ defined as follows. The cycle $A$ goes along the ring where both parameterizations overlap and is homotopic to the contour $\gamma$. The cycle $B$ is a path going from the puncture at $z = \infty$ to the puncture at $\bar{z} = \infty$.

The integral around the cycle $A$ is easy to calculate using (4.7) and (4.6) by picking up the pole

$$\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{2\pi i} \oint_\gamma d\Phi_+(z) = \frac{1}{2\pi i} \oint_{\gamma^{-1}} d\Phi_-(\bar{z}) = \hbar N, \quad (4.10)$$
where we indicated that in \( \bar{z} \)-coordinates the contour should be reversed.

To find the integral along the non-compact cycle \( B \), one should introduce a regularization by cutting it at \( z = \sqrt{\Lambda} \) and \( \bar{z} = \sqrt{\Lambda} \). Then we obtain

\[
\int_B d\Phi = \int_{\sqrt{\Lambda}}^{z_0} d\Phi_+(z) + \int_{\bar{z}(z_0)}^{\sqrt{\Lambda}} d\Phi_-(\bar{z}) = \Phi_-(\sqrt{\Lambda}) - \Phi_+(\sqrt{\Lambda}),
\]

where \( z_0 \) is any point on the cycle and we used (4.5). Taking into account the definition (4.3) and the explicit solution (4.7), we see that the result contains the part vanishing at \( \Lambda \to \infty \), the diverging part which does not depend on \( N \) and, as non-universal, can be neglected, and the contribution of the zero mode \( \phi \). Since \( \phi \) enters \( \Phi_+ \) and \( \Phi_- \) with different signs, strictly speaking, it is not a zero mode for \( \Phi \). As a result, it does not disappear from the integral, but is doubled. Finally, using (4.8) we get

\[
\int_B d\Phi = \hbar \frac{\partial}{\partial N} \log Z_{\text{NMM}}.
\]

In Appendix B we derive this formula using the eigenvalue transfer procedure similar to one used in [28], or in [25] in a similar case of the two matrix model equivalent to our \( R = 1 \) NMM. Thus, the free energy of NMM can be reproduced from the monodromy of the holomorphic 1-form around the non-compact cycle on the punctured sphere.

Since the string equations coincide, it is clear that the solution of MQM is described by the same analytical curve and holomorphic 1-form. To write the period integrals in terms of MQM variables, it is enough to change the coordinates according to (4.2) and \( \hbar N = R\mu \) in all formulas. In this way we find

\[
\frac{1}{2\pi i} \oint_A d\Phi = R\mu, \quad \int_B d\Phi = -\frac{1}{R} \frac{\partial \mathcal{F}}{\partial \mu}.
\]

The last integral can be also obtained from the integral over the Fermi sea. Indeed, we can choose a point on the contour of the Fermi sea and split the integral, which actually calculates the area of the sea, into three parts

\[
\frac{1}{R} \frac{\partial \mathcal{F}}{\partial \mu} = -\int_{\text{F.s.}} dx_+ dx_- = \int_{x_0}^{\sqrt{\Lambda}} x_-(x_+) dx_+ + \int_{x_-(x_0)}^{\sqrt{\Lambda}} x_+(x_-) dx_- + x_0 x_-(x_0).
\]

Since \( \partial_+ \varphi = x_-(x_+) \) and \( \partial_- \bar{\varphi} = x_+(x_-) \) (see (4.9)), the last expression is equal to the integral (4.11) of \( d\Phi \) around the (reversed) cycle \( B \). This derivation gives an independent check that the free energies of NMM and MQM do coincide.

\[\text{8} \quad \text{To get the second integral from (4.12), one should take into account that also } \hbar \to i\hbar.\]
Fig. 1: Symbolic representation of the complex curve and the two real sections along the cycles $A$ and $B$. The filled regions symbolize the Fermi seas of the two matrix models.

4.4. Duality

We found that the solutions of both models are described by the same complex curve. The curve is characterized by a pair of conjugated cycles: the compact cycle $A$ encircling one of the punctures and the non-compact cycle $B$ connecting the two punctures. The parameter of the free energy $\mu$ or $N$ and the derivative of the free energy itself are given by the integrals of the unique globally defined holomorphic 1-form along the $A$ and $B$ cycles, correspondingly.

Now one can ask: what is the difference between the two models? Is it seen at the level of the curve? In fact, both models, NMM and MQM can be associated with two different real sections of the complex curve. Indeed, in NMM the variables are conjugated to each other, whereas in MQM they are real. Therefore, let us take the interpretation where the curve is embedded into $\mathbb{C}^2$ and consider its intersection with two planes. The first plane is defined by the condition $z^* = \bar{z}$ and the second one is given by $z^* = z$, $\bar{z}^* = \bar{z}$. In the former case we get the cycle $A$, whereas in the latter case the intersection coincides with the cycle $B$ (see fig. 1). One can think about the planes as the place where the eigenvalues of NMM and MQM, correspondingly, live (with the density given by (4.3)). Then the intersections describe the contours of the regions filled by the eigenvalues. We see that for NMM it is given by the compact cycle $A$, and for MQM the Fermi sea is bounded by the cycle $B$ of the curve.

Therefore, the duality between the two models can be interpreted as the duality exchanging the cycles of the complex curve describing the solution. Under this exchange
the real variables of one model go to the complex conjugated variables of another model, and the grand canonical free energy is replaced by the canonical one.

This duality can be seen even more explicitly if one rewrites the relations (4.13) in terms of the canonical free energy of MQM defined as \( F = \mathcal{F} + \hbar R \mu M \), where \( M = -\frac{1}{\hbar R} \frac{\partial \mathcal{F}}{\partial \mu} \) is the number of eigenvalues. Then they take the following form

\[
\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{\hbar} \frac{\partial F}{\partial M}, \quad \int_B d\Phi = \hbar M. \tag{4.15}
\]

They have exactly the same form as (4.10) and (4.12) provided we change the cycles. In particular, the numbers of eigenvalues are given in the two models as integrals around the dual cycles.

5. Conclusions and problems

We have shown that the tachyonic sector of the compactified \( c = 1 \) string theory is described in terms of a normal matrix model. This matrix model is related by duality to the traditional representation in terms of Matrix Quantum Mechanics. More precisely, the canonical partition function of NMM is equal to the grand canonical partition function of MQM, provided we identify the number of eigenvalues \( N \) of NMM with the chemical potential \( \mu \) in MQM. This holds for any compactification radius. Moreover, the duality between the two matrix models has a nice geometrical interpretation. In the quasiclassical limit the solutions of the two models can be described in terms of a complex curve with the topology of a sphere with two punctures. The Matrix Quantum Mechanics and the Normal Matrix Model are associated with two real sections of this curve which are given by the two non-contractible cycles on the punctured sphere. The duality acts by exchanging the two cycles.

The duality we have observed relates a fermionic problem with non-compact Fermi sea and continuous spectrum to a problem with compact Fermi sea and discrete spectrum. The solutions of the two problems are obtained from each other by analytic continuation. Thus, the reason for the duality can be seen in the common analytic structure of both problems. We also emphasize that the duality relates the partition functions in the canonical and the grand canonical ensembles, which are related by Legendre transform. This is natural regarding that it exchanges the cycles of the complex curve.

Let us mention two among the many unsolved problems related to the duality between these two models. First, it would be interesting to generalize the above analysis to the case when the eigenvalues of NMM form two or more disconnected droplets. For \( R = 1 \) this is the analog of the multicut solutions of the two-matrix model, considered recently in [25]. In terms of MQM this situation would mean the appearance of new, compact components.
of the Fermi sea. The corresponding complex curve has the topology of a sphere with two punctures and a number of handles.

Second, we would like to generalize the correspondence between the two matrix models in such a way that it incorporates also the winding modes. For this we should understand how to introduce the winding modes in the Normal Matrix Model. Whereas in MQM they appear when we relax the projection to the singlet sector, in NMM this could happen when we relax the normality condition \([Z, \bar{Z}] = 0\).

Up to now there were two suggestions how to include both the tachyon and winding modes within a single matrix model. In [11], a 3-matrix model was proposed with interacting two hermitian matrices and one unitary matrix. This model can be seen as a particular reduction of Euclidean compactified MQM. The model correctly describes the cases of only tachyon or only winding perturbations, though its validity in the general case is still to be proven. For the particular case of the self-dual radius and the multiples of it, a 4-matrix model was recently proposed in [8]. It is based on the old observation of [30] about the geometry of the ground ring of \(c=1\) string theory. It would be interesting to find the relation of this model to our approach, at least in the above-mentioned particular cases. The general understanding of this important problem is still missing.

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Appendix A. Toda description of the perturbed system

Here we show that the partition function (3.1) is a \(\tau\)-function of the Toda lattice hierarchy (for details see [31]). We will do it following the standard method of orthogonal polynomials. First, we write the integral (3.1) as integral over the eigenvalues \(z_1, ..., z_N\)

\[
Z_{\text{h}}^{\text{NMM}}(N, t, \alpha) = \frac{1}{N!} \int \prod_{k=1}^{N} w_{\alpha}(z_k, \bar{z}_k) \Delta(z) \Delta(\bar{z}) \quad (A.1)
\]

with measure

\[
w_{\alpha}(z, \bar{z}) = \frac{d^2 z}{2\pi i} e^{-\frac{1}{i} [((z \bar{z})^R - V_+(z) - V_-(\bar{z})]} (z \bar{z})^{(R-1)/2+\frac{1}{4}}. \quad (A.2)
\]
Then we introduce a set of bi-orthogonal polynomials

\begin{equation}
P_n^+(z) = \frac{1}{n!\sqrt{h_n}} \int_C \prod_{k=1}^{n} \frac{w_\alpha(z_k, \bar{z}_k)}{h_{k-1}(t, \alpha)} \Delta(z)\Delta(\bar{z}) \prod_{k=1}^{n} (z - z_k), \tag{A.3}
\end{equation}

and similarly for \( P_n^-(\bar{z}) \). Here \( \Delta(z) = \prod_{k<l}(z_k - z_l) \) is the Vandermonde determinant and normalization factors \( h_k \) are determined by the orthonormality condition

\begin{equation}
\langle P_n^-|P_m^+\rangle_w \equiv \int_C w_\alpha(z, \bar{z}) P_n^-(\bar{z})P_m^+(z) = \delta_{n-m}. \tag{A.4}
\end{equation}

Then the partition function (A.1) reduces to the product of the normalization factors

\begin{equation}
\mathcal{Z}_h^{NMM}(N, t, \alpha) = \prod_{n=0}^{N-1} h_n(t, \alpha). \tag{A.5}
\end{equation}

The operators of multiplication by \( z \) and \( \bar{z} \) are represented in the basis of orthogonal polynomials by the infinite matrices

\begin{equation}
z P_m^+(z) = \sum_m L_{nm} P_m^+(z), \quad \bar{z} P_n^-(\bar{z}) = \sum_m P_n^-(\bar{z})L_{mn}. \tag{A.6}
\end{equation}

with

\begin{align}
L_{n,n+1} &= \bar{L}_{n+1,n} = \sqrt{h_n/h_{n+1}}, \\
L_{nm} &= \bar{L}_{mn} = 0, \quad m > n + 1. \tag{A.7}
\end{align}

Differentiation of the orthogonality relation (A.4) with respect to coupling constants \( t_k \) gives

\begin{align}
\hbar \frac{\partial P_n^+(z)}{\partial t_k} &= -\sum_{m=0}^{n-1} (L^k)_{nm} P_m^+(z) - \frac{1}{2}(L^k)_{nn} P_n^+(z), \\
\hbar \frac{\partial P_n^+(z)}{\partial t_{-k}} &= -\sum_{m=0}^{n-1} (\bar{L}^k)_{nm} P_m^+(z) - \frac{1}{2}(\bar{L}^k)_{nn} P_n^+(z) \tag{A.8}
\end{align}

and similarly for \( P_n^-(\bar{z}) \). Let us define now a wave function which is a vector \( \Psi = \{\Psi_n\} \) with elements

\begin{equation}
\Psi_n(t; z) = P_n^+(z)e^{\frac{i}{\hbar}V_+(z)}. \tag{A.9}
\end{equation}

Then the equations (A.8) lead to the following eigenvalue problem

\begin{equation}
z\Psi = L\Psi, \quad \hbar \frac{\partial \Psi}{\partial t_k} = H_k\Psi, \quad \hbar \frac{\partial \Psi}{\partial t_{-k}} = H_{-k}\Psi, \tag{A.10}
\end{equation}

where we introduced the Hamiltonians

\begin{equation}
H_k = (L^k)_+ + \frac{1}{2}(L^k)_0, \quad H_{-k} = -(\bar{L}^k)_- - \frac{1}{2}(\bar{L}^k)_0. \tag{A.11}
\end{equation}
Here the subscripts $0/−/+\) denote diagonal/lower/upper triangular parts of the matrix. From the commutativity of the second derivatives, it is easy to find the Zakharov-Shabat zero-curvature condition

$$\hat{h} \frac{\partial H_k}{\partial t_l} - \hat{h} \frac{\partial H_l}{\partial t_k} + [H_k, H_l] = 0. \quad (A.12)$$

It means that the Hamiltonians generate commuting flows and the perturbed system is described by the Toda Lattice hierarchy.

In addition, one can obtain the string equation for the hierarchy. It follows from the Ward identity

$$z \frac{\partial \Psi}{\partial z} = \frac{R}{\hat{h}} \bar{L} RL^R \Psi - \left[ (R + 1)/2 + \frac{\alpha}{\hat{h}} \right] \Psi \quad (A.13)$$

and can be written as

$$[L^R, \bar{L}^R] = \hat{h}. \quad (A.14)$$

Here we should understand the operators $L^R$ as analytical continuation of the operators in integer powers.

The Toda structure leads to an infinite set of PDE’s for the coefficients of the operators $L$ and $\bar{L}$. The first of these equations can be written for the normalization factors and is known as Toda equation

$$\hat{h}^2 \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{-1}} \log h_n = \frac{h_n}{h_{n-1}} - \frac{h_{n+1}}{h_n}. \quad (A.15)$$

The quantity $h_n$ is known to be related to the $\tau$-function of the Toda hierarchy by

$$h_n = \frac{\tau_{NMM}^{\text{NMM}}}{\tau_n^{\text{NMM}}}. \quad (A.16)$$

Taking into account (A.3), this gives for the partition function

$$Z_h^{\text{NMM}} (N, t, \alpha) = \tau_{N, \hat{h}}^{\text{NMM}} (t, \alpha). \quad (A.17)$$

**Appendix B. Derivation of the formula for the cycle $B$**

The formula (4.12) can be obtained by means of the procedure of transfer of an eigenvalue from the point $(z_1, \bar{z}_1 = \bar{z}(z_1))$ belonging to the boundary $\gamma$ of the spot to $\infty$ (or rather to the cut-off point $\sqrt{\Lambda}$), worked out for the case $R = 1$ in [25]. On the double cover described in subsection 4.3, $z_1$ and $\bar{z}_1$ are considered as complex coordinates of the point in two patches. We find from (A.1) that the free energy in the large $N$ limit changes during this transfer as follows:

$$\hat{h} \frac{\partial}{\partial N} \log Z_h^{\text{NMM}} = -(z_1 \bar{z}_1)^R + V_+(z_1) + V_-(\bar{z}_1) + \hat{h} \sum_{m=2}^{N} \log [(z_1 - z_m)(\bar{z}_1 - \bar{z}_m)], \quad (B.1)$$
where all $z_m$’s are taken at their saddle point values. Here we took into account that the determinant in the matrix integral (B.1) does not contribute in the quasiclassical limit. In this limit the integral (A.1) leads to the following saddle point equations [25]

\[
R_z R_{-1}^R - V_+^R(\bar{z}) = \bar{G}(\bar{z}), \quad R_{-1}^R \bar{z} - V_+^R(z) = G(z),
\]

(B.2)

where $G(z)$ and $G(\bar{z})$ are the resolvents of the one dimensional distributions of $z_k$’s and $\bar{z}_k$’s, respectively. They have the following asymptotics at large $z$ and $\bar{z}$:

\[
G(z) \to \frac{\hbar N}{z}, \quad \bar{G}(\bar{z}) \to \frac{\hbar N}{\bar{z}}.
\]

(B.3)

Rewriting the sum in (B.1) as an integral with the measure given by the eigenvalue density and then expressing it through the resolvents, we find

\[
\hbar \frac{\partial}{\partial N} \log Z^{NMM} = -(z_1 \bar{z}_1)^R + V_+(z_1) + V_-(\bar{z}_1) + \int \frac{dz}{2\pi i} G(z) \log(z_1 - z) + \int \frac{d\bar{z}}{2\pi i} \bar{G}(\bar{z}) \log(\bar{z}_1 - \bar{z}),
\]

(B.4)

where the contours of integration encircle the whole support of the distribution of eigenvalues on the physical sheets of the functions $\bar{z}(z)$ and $z(\bar{z})$, respectively. [10]

Due to (B.2) we do not have any singularities at $z \to \infty$ ($\bar{z} \to \infty$) except poles. Blowing up the contour we pick up the logarithmic cut and obtain

\[
\hbar \frac{\partial}{\partial N} \log Z^{NMM} = -(z_1 \bar{z}_1)^R - R \int_{z_1}^{\sqrt{\Lambda}} \frac{dz}{z} (z \bar{z}(z))^R - R \int_{\bar{z}_1}^{\sqrt{\Lambda}} \frac{d\bar{z}}{\bar{z}} (z(\bar{z}) \bar{z})^R.
\]

(B.5)

The first term in the r.h.s. can be regrouped with the other two terms, giving (up to a nonuniversal term $\sim \Lambda^R$)

\[
\hbar \frac{\partial}{\partial N} \log Z^{NMM} = \int_{z_1}^{\sqrt{\Lambda}} \left[ R(z \bar{z}(z))^R \frac{dz}{z^2} - \frac{1}{2} d(z \bar{z})^R \right] - \int_{\bar{z}_1}^{\sqrt{\Lambda}} \left[ R(z(\bar{z}) \bar{z})^R \frac{d\bar{z}}{\bar{z}} - \frac{1}{2} d(z \bar{z})^R \right],
\]

(B.6)

which immediately yields (4.11) if we take into account (4.6) and the definition (4.9). Note that (B.2) is essentially the same as (4.6) if one uses the explicit expansion (4.7) for the holomorphic parts of the potential. Therefore, they define the same functions $\bar{z}(z)$ and $z(\bar{z})$. On the complex curve described in the subsection 4.3, the formula (B.6) reduces to the period integral (4.12) of the holomorphic differential $d\Phi$.

---

9 We view $\bar{z}(z)$ and $z(\bar{z})$ as analytical functions similarly to the interpretation of subsection 4.3.

10 Note that the point of infinite branching at the origin related to the singularities of $z^R$ and $\bar{z}^R$ should not appear on the physical sheets.
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