Decompositions of Complete Multigraphs into Cyclic Designs

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Abstract—Let $v$ and $2$ be positive integer, $\lambda K_v$ denote a complete multigraph. A decomposition of a graph $G$ is a set of subgraphs of $G$ whose edge sets partition the edge set of $G$. The aim of this paper, is to decompose a complete multigraph $4K_v$ into cyclic $(v-1)$-cycle system according to specified conditions. As the main consequence, construction of decomposition of $8K_v$ into cyclic Hamiltonian wheel system, where $v \equiv 2 \pmod 4$, is also given. The difference set method is used to construct the desired designs.

Keywords—Cyclic design; Hamiltonian cycle, Near four factor, Wheel graph.

I. INTRODUCTION

Throughout this paper, all graphs consider finite and undirected. A decomposition of a complete graph of order $v$ by its subgraphs belonging to an assigned multiset $Y$. An $(G,Y)$-design is a decomposition of the graph $G$ into subgraphs subject to specified conditions.

A cycle of a graph $G$ is called Hamiltonian when its vertices passes through all the vertex set of $G$. An m-cycle, written $C_{m}=(c_0, c_1, \ldots, c_{m-1})$, consists of $m$ distinct vertices $\{c_0, c_1, \ldots, c_{m-1}\}$ and $m$ edges $\{c_i c_{(i+1)} \pmod m\}$, for $0 \leq i \leq m-2$ and $c_0 c_{(m-1)}$. An m-cycle of a graph $G$ is called Hamiltonian when its vertices passes through all the vertex set of $G$. An $m$-path, written $[c_0, c_1, \ldots, c_{(m-1)}]$, consists of $m$ distinct vertices $\{c_0, c_1, \ldots, c_{(m-1)}\}$ and $m-1$ edges $\{c_i c_{(i+1)} \pmod m\}$, for $0 \leq i \leq m-2$. An m-cycle system of a graph $G$ is a collection of m-cycles. If $G=K_v$ then such m-cycle system is called m-cycle system of order $v$ and is also said a simple when its cycles are all distinct.

An automorphism group on $(G,Y)$-design is a bijections on $V(G)$ fixed $Y$. An $(G,Y)$-design is a cyclic if it admit automorphism group acting regularly on $V(G)$ [1]. For a cyclic $(G,Y)$-design, we can assume that $V(G)=Z_v$. So, the automorphism can be represented by

$\alpha : i \mapsto i+1 \pmod v$ or $\alpha : (0,1,\ldots, v-1)$

A starter set of a cyclic $(G,Y)$-design is a set of subgraphs of $G$ that generates all subgraphs of $Y$ by repeated addition of $1$ modular $v$.

A complete multigraph of order $v$, denoted by $\lambda K_v$, is obtained by replacing each edge of $K_v$ with $\lambda$ edges. The problem which concerned in the decomposition of the complete multigraph into subgraphs has received much attention in recent years. The necessary and sufficient conditions for decomposing $\lambda K_v$ into cycles of order $\lambda$ and cycles of prime order have been established by [2]. While, the existence theorem of m-cycle system of $\lambda K_v$ has been proved for all values of $\lambda$ in [3]. For the important case of $\lambda=1$, the existence question for m-cycle system of order $v$ has been completely settled by [4] in the case $m$ odd and by [5] in the case $m$ even. Moreover, the cyclic m-cycle system of order $v$ for $m=3$, denoted by $CTS(v,\lambda)$, has been constructed by [6] and for a cyclic Hamiltonian cycle system of order $v$ was proved when $v$ is an odd integer but $v \neq 15$ and $v \neq p^a a$ with $p$ a prime and $a > 1$ [7].

On the other hand, the necessary and sufficient conditions for decomposing $\lambda K_v$ into cycle and star graphs have been investigated by [8].

A four-factor of a graph $G$ is a spanning subgraph whose vertices have a degree 4. While a near-four-factor is a spanning subgraph in which all vertices have a degree four with exception of one vertex (isolated vertex) which has a degree zero [9].

In this paper, we propose new type of cyclic cycle system that is called cyclic near Hamiltonian cycle system of $4K_v$, denoted $CNHC(4K_v,C_{(v-1)})$. This is obtained by combination a near-four-factors and cyclic $(v-1)$-cycle system of $4K_v$ when $v \equiv 2 \pmod 4$. Furthermore, the construction of $CNHC(4K_v,C_{(v-1)})$ will be employed to decompose $8K_v$ into Hamiltonian wheels.

II. PRELIMINARIES

In our paper, all graphs considered have vertices in $Z_v$. We will use the difference set method to construct the desired designs. The difference between any two distinct vertices $a$ and $b$ in $\lambda K_v$ is $|a-b|$, arithmetic (mod $v$). Given $C_m = \{c_0, \ldots, c_{m-1}\}$ an $m$-cycle, the differences from
\( C_m \) are the multiset \( \Delta(C_m) = \{ \pm |c_i - c_{i-1}| \mid i = 1, 2, \ldots, m \} \) where \( c_0 = c_m \). Let \( F = \{ B_1, B_2, \ldots, B_s \} \) be an \( m \)-cycles of \( \lambda K_v \), the list of differences from \( F \) is \( \Delta(F) = \bigcup_{i=1}^{s} \Delta(B_i) \).

The orbit of cycle \( C_m \), denoted by \( orb(C_m) \), is the set of all distinct \( m \)-cycles in the collection \( \{ C_m + k \mid k \in Z \} \). The length of \( orb(C_m) \) is its cardinality, i.e., \( orb(C_m) = k \) where \( k \) is the minimum positive integer such that \( C_m + k = C_m \). A cycle orbit of length \( v \) on \( \lambda K_v \) is said full or otherwise short. [10]

The stabilizer of a subgraph \( H \) of a graph \( G \) of order \( v \) by \( \text{stab}(H) = \{ z \in Z_v \mid z + H = H \} \) and \( H \) has trivial stabilizer when \( \text{stab}(H) = \{0\} \). One may easily deduce the following result.

For presenting a cyclic \( m \)-cycle system of \( \lambda K_v \), it sufficient to construct a starter set, i.e., \( m \)-cycle system of representations for its cycle orbits. As particular consequences of the theory developed in [11] we have:

Lemma 1. Let \( H \) be a subgraph of \( G \) and \( |\text{stab}(H)| > 1 \). Then each nonzero integer in \( \Delta H \) appears a multiple of \( |\text{stab}(H)| \) times.

Lemma 2. Let \( \delta \) be a multiset of subgraphs of \( \lambda K_v \) and every subgraph of \( \delta \) has trivial stabilizer. Then \( \delta \) is a starter of cyclic \( \left( \lambda K_v, \delta \right) \)-design if and only if \( |\Delta \delta| \) covers each nonzero integer of \( Z_v \), exactly \( t \) times.

III. CYCLIC NEAR HAMILTONIAN CYCLE SYSTEM

Definition 1. A full cyclic near Hamiltonian cycle system of the \( 4K_v \), denoted by \( CNHC(4K_v, C_{n-1}) \), is a cyclic \( (v - 1) \)-cycle system of \( 4K_v \) graph, that satisfies the following conditions:

1. The cycle in row \( r \) form a near-4-factor with focus \( r \).
2. The cycles associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the \( 4K_v \), \( CNHC(4K_v, C_{n-1}) \), it is sufficient to provide a starter set that satisfies a near-4-factor. We give here example to explain the above definition.

Example 1. Let \( G = 4K_{14} \) and \( F = \{ C_{13}, C_{13} \} \) is a set of 13-cycles of \( G \) such that:

\[
C_{13} = (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8),
\]

\[
C_{13} = (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7).
\]

Firstly, it is easy to observe that each non zero element in \( Z_{14} \) occurs exactly twice in the 13-cycles of \( F \). Since, the cycle graph is 2-regular graph, then every vertex has a degree 4 except a zero element (isolated vertex) has a degree zero. Thus, it is satisfies the near-4-factor with focus zero element. Secondly, the list of differences set of the set \( F \) is listed in Table I.

| 13-cycles          | Difference set          |
|--------------------|-------------------------|
| (1,13,2,12,3,11,4,5,10,6,9,7,8) | \{2,123,11,4,10,5,6,8,7,1,13\} |
| (13,8,12,9,11,10,4,3,5,2,6,1,7) | \{5,9,4,10,3,11,2,1,13,6,8,1\} |

It can be seen from the Table I. \( \Delta(F) = \Delta(C_{13}) \cup \Delta(C_{13}) \) covers each nonzero element in \( Z_{14} \) exactly four times. Since the cycles set \( F \) has trivial stabilizer based on Lemma 1, then the set \( F = \{ C_{13}, C_{13} \} \) is the starter set of \( CNHC(4K_{14}, C_{13}) \) by Lemma 2.

Therefore, \( CNHC(4K_{14}, C_{13}) \) is an \((14 \times 2)\) array design and cycles set \( F = \{ C_{13}, C_{13} \} \) in the first row generates all cycles in \((14 \times 2)\) array by repeated addition of 1 modular 14 as shown in the Table II.

| Focus | \( CNHC(4K_{14}, C_{13}) \)          |
|-------|-------------------------------------|
| \( r = 0 \) | (1,13,2,12,3,11,4,5,10,6,9,7,8) (13,8,12,9,11,10,4,3,5,2,6,1,7) |
| \( r = 1 \) | (20,3,13,4,12,5,6,11,7,10,8,9) (0,9,13,10,12,11,5,4,6,3,7,2,8) |
| \( r = 2 \) | (3,14,05,13,6,7,12,8,11,9,10) (1,10,0,11,13,12,6,5,7,4,3,9) |
| \( i \) | \( i \) | \( i \) |
| \( r = 13 \) | (0,12,1,2,10,3,4,5,5,6,7,8,6,7) (12,7,11,8,10,9,3,2,4,15,0,6) |

Throughout the paper, a near Hamiltonian cycle of order \((v - 1)\) will be represented as connected paths, we mean that \( C_{r-1} = \{ c_{(1,1)}, c_{(1,2)}, \ldots, c_{(1,n)} \} \), where \( c_{(1,i)} \) and \( c_{(1,n)} \) are \((2n)\)-paths such that:

\[
P_{(n,1,2)} = \{ c_{(1,1)}, c_{(1,2)}, \ldots, c_{(n,1)}, c_{(2,1)} \},
\]

\[
P_{(n,2,1)} = \{ c_{(1,2)}, c_{(2,1)}, \ldots, c_{(2,n)}, c_{(4,1)} \}.
\]

Let the vertex sets of \( P_{(n,1,2)} \) and \( P_{(n,2,1)} \) are \( \{ U_{1=1}^{n-1}c_{(1,i)}, U_{1=1}^{n-1}c_{(2,i)} \}, \{ U_{1=1}^{n-1}c_{(1,i)}, U_{1=1}^{n-1}c_{(4,i)} \} \), respectively. And the list of difference sets of \( P_{(n,1,2)} \) and \( P_{(n,2,1)} \) will be calculated as follows:

\[
\Delta(P_{(n,1,2)}) = \Delta_{1}(P_{(n,1,2)}) \cup \Delta_{2}(P_{(n,1,2)}) \cup \Delta_{3}(P_{(n,1,2)}) \text{ such that:}
\]

\[
\Delta_{1}(P_{(n,1,2)}) = \{ \pm c_{(1,i)} - c_{(1,j)} \mid 1 \leq i < j \leq n \},
\]

\[
\Delta_{2}(P_{(n,1,2)}) = \{ \pm c_{(1,i)} - c_{(1,j)} \mid 1 \leq i < j \leq n - 1 \},
\]

\[
\Delta_{3}(P_{(n,1,2)}) = \{ \pm c_{(4,i)} - c_{(4,j)} \mid 1 \leq i < j \leq n \}.
\]
And we define \( \Delta(c_{1}, p_{1}, p_{2}, c_{1}) \) and \( \Delta(p_{1}, p_{2}, p_{3}, c_{1}) \) as follows:

\[
\Delta(c_{1}, p_{1}, p_{2}, c_{1}) = \pm |c_{1} - c_{1}|, \\
\Delta(p_{1}, p_{2}, p_{3}, c_{1}) = \pm |p_{1} - c_{1}|
\]

So, the list of difference of \( C_{n+1} \) shall be represented as follows:

\[
\Delta(C_{n+1}) = \Delta(p_{1}, p_{2}, p_{3}, c_{1}) \cup \Delta(p_{1}, p_{2}, p_{3}, c_{1}) \cup \\
\Delta(c_{1}, p_{1}, p_{2}, c_{1}) \cup \Delta(c_{1}, p_{1}, p_{2}, c_{1})
\]

Now we are able to provide our main result.

Theorem 1. There exists a full cyclic near Hamiltonian cycle system of \( 4K_{n} \), \( CNHC(4K_{n}, C_{n+1}) \), when \( n = 2 + 2n > 2 \).

Proof. Suppose \( F = \{ C_{n+1}, C_{n+1}^{*} \} \) is a set of near Hamiltonian cycles of \( 4K_{n} \) where \( C_{n+1} = \{ P_{1}, P_{2}, P_{3}, C_{1} \} \) and \( C_{n+1}^{*} = \{ 2n + 1, P_{1}, P_{2}^{*}, P_{3}^{*} \} \).

Such that:

- \( P_{1}^{*} = [4n + 1, 2n + 1, 3, \ldots, 3n + 2 + i] \) = \( \{ U_{i}^{n+1} n + i, 1 \leq i \leq 2n \} \)
- \( P_{2}^{*} = [n + 2, 3n + 1, n + 1, 2n + 1, 2n + 2] \)
- \( P_{3}^{*} = [4n + 1, 2n + 2, 4n + 2, 3n + 3, 3n + 4, 3n + 5, \ldots, 3n + 2n + 1 + i] \)
- \( P_{4}^{*} = [n + 2, n + 3, \ldots, 2n + 1, n + 1, 2n + 1] \)

We will divide the proof into two parts as follows:

Part 1. In this part will be proved that \( F \) satisfies a near-four-factor. We shall calculate the vertex set of \( C_{n+1} \) and \( C_{n+1}^{*} \) such that:

\[
V(C_{n+1}) = V(p_{1}^{*}) \cup V(p_{2}^{*}) \cup \{ 1 \}, \\
V(C_{n+1}^{*}) = V(p_{1}^{*}) \cup V(p_{2}^{*}) \cup \{ 2n + 1 \}
\]

\[
U_{i}^{p_{1}} = \{ 4n + 2 - i, 1 \leq i \leq n \} = \{ 4 + 1, 4n, \ldots, 3n + 2 \} \\
U_{i}^{p_{2}} = \{ i + 1, 1 \leq i \leq n \} = \{ 2, 3, \ldots, n + 1 \} \\
U_{i}^{p_{3}} = \{ n + 1, i + 1, 1 \leq i \leq n \} = \{ n + 2, n + 3, \ldots, 2n + 1 \} \\
U_{i}^{p_{4}} = \{ 3n + 2 - i, 1 \leq i \leq n \} = \{ 3n + 1, 3n + 2, \ldots, 2n + 2 \}
\]

From above equations, it is easy to notice that \( V(C_{n+1}) \) covers each nonzero element of \( Z_{4n+2} \) exactly once.
\[ \Delta_0 \left( P_{n \times 2}^{m+1} \right) = U_{n+1}^{m+1} \cup \{ 2n, 2n+2 \} \]

- \[ \Delta_0 \left( P_{n \times 2}^{m+1} \right) = U_{n+1}^{m+1} \cup \{ 2n, 2n+2 \} \]

\[ \Delta_0 \left( P_{n \times 2}^{m+1} \right) = U_{n+1}^{m+1} \cup \{ 2n, 2n+2 \} \]

As clearly shown, in the equations (10), every nonzero element in \( Z_{4n+2} \) appears twice except \( \{ 2n, 2n+2 \} \) appear three times in \( \Delta(P_{n \times 2}^{m+1}) \). Based on Lemma 1, the cycles \( \{ C_{4n+1}, C_{4n+1} \} \) have trivial stabilizer. One can easily note that \( \Delta(F) = \Delta(C_{4n+1}) \cup \Delta(C_{4n+1}) \) covers each nonzero integer in \( Z_{4n+4} \) four times. Thus, \( F = \{ C_{4n+1}, C_{4n+1} \} \) is the starter cycles of cyclic (\( v \)-1)-cycle system of \( K_{n}^{1} \) by Lemma 2. Hence, the cycles set \( F = \{ C_{4n+1}, C_{4n+1} \} \) generates a full near Hamilton cycle system of \( K_{n}^{1} \) by adding one modular \( v \) when \( v = 4n + 2, n \geq 2 \).

### IV. CYCLIC HAMILTONIAN WHEEL SYSTEM

A wheel graph of order \( m \), denoted by \( W_{m} \), consists of a singleton graph \( K_{1} \) and a cycle graph of order \( m-1 \), in which the \( K_{1} \) is connected to all the vertices of \( C_{m-1} \)-written \( K_{1} + C_{m-1} \) or \( c_{0} + \{ c_{1}, c_{2}, \ldots, c_{m-1} \} \). An \( m \)-wheel contains \( 2(m-1) \) edges such that the edge set of \( W_{m} \) is \( E(W_{m}) = E(K_{1}) \cup E(C_{m-1}) \).

An \( m \)-wheel system of graph \( G \) is a decomposition of edge set of \( G \) into collection \( \mathcal{W} \) of \( W_{m} \) of edge-disjoint \( m \)-wheels. Similar to the cyclic system, an \( m \)-wheel system of \( \mathcal{K}_{m} \) is a cyclic if \( \mathcal{W}(\mathcal{K}_{m}) \) is \( Z_{e} \) and if \( W_{m} = c_{0} + \{ c_{1}, c_{2}, \ldots, c_{m-1} \} \) implies that \( W_{m} = \mathcal{W}(\mathcal{K}_{m}) \) covers each nonzero element of \( Z_{m} \) exactly four times. Now we want to find the list of differences from \( \{ K_{1}, K_{1} \} \) as a follows

\[ \Delta(K_{1}, K_{1}) = \{ \pm c_{i} - c_{j} | 1 \leq i \leq m \} \]

such that the \( c_{j} \) appears twice except \( \{ c_{j} \} \) appear three times in \( \Delta(C_{4n+1}) \). \( \{ c_{j} \} \) \( Z_{14} \) four times. Thus, \( \mathcal{W} = \{ W_{4}, W_{4} \} \) is the starter set of \( \mathcal{K}_{14} \).

Then \( \mathcal{W}(\mathcal{K}_{14}, W_{4}) \) is an \( (14 \times 2) \) array design where all its wheels can be generated by repeated addition 1 (modular 14) on the starter set \( \mathcal{W} \) as shown in the Table III.

| \( \mathcal{W}(\mathcal{K}_{14}, W_{4}) \) |
|---|
| 0 + (1, 1, 2, 3, 12, 13, 14, 5, 6, 7, 8) |
| 1 + (2, 3, 13, 14, 12, 5, 6, 11, 10, 8, 9) |
| 2 + (3, 4, 14, 15, 13, 6, 7, 12, 8, 11, 10) |
| 3 + (0, 12, 11, 12, 6, 5, 4, 9, 8, 7, 6, 5) |

The following theorem proves the existence of \( \mathcal{W}(\mathcal{K}_{4n+2}, \mathcal{W}) \).

Theorem 2. There exists a full cyclic Hamiltonian wheel system of \( \mathcal{K}_{m} \), \( \mathcal{W}(\mathcal{K}_{m}, \mathcal{W}) \), for \( v = 4n + 2, n > 2 \).

Proof. We have to present a starter set \( \mathcal{W} = \{ K_{1} + C_{4n+1}, K_{1} + C_{4n+1} \} \) of \( \mathcal{W}(\mathcal{K}_{4n+2}, \mathcal{W}) \) such
that the cycles associated with the wheels in $W$ satisfy a near-four-factor with focus a singleton graph.

Suppose $W = \{0 + C_{2n+1}, 0 + C_{4n+1} \}$ is a set of Hamiltonian wheels of $3K_{2n+2}$ where

$$C_{2n+1} = (1, P_{2n+1}^{2n+1}),$$
$$C_{4n+1} = (2n + 1, P_{2n+1}^{2n+1}, P_{4n+1}^{2n+1}).$$

Such that:

- $P_{2n+1}^{2n+1} = [4n + 1, 2, 4n, 3, \ldots, 3n + 2, n + 1].$
- $P_{4n+1}^{2n+1} = [2n + 1, 2n + 1, 3n + 2, n + 1].$

Now, we want to prove $W = \{K_n + C_{2n+1}, K_n + C_{4n+1} \}$ is a $\lambda K_n, W$-difference system. To do this, it is enough to show that the list of differences

$$\Delta W = \{\Delta(C_{2n+1}) \cup \Delta(C_{4n+1}) \cup \Delta(K_{1, (4n+1)}^{n}) \cup \Delta(K_{1, (4n+1)}^{n+1})\}$$

covers each element of $\{Z_{4n+2} - \{0\}\}$ eight times. Firstly, as indicated in Theorem 1, the list of differences of $\{C_{4n+1}, C_{4n+1}\}$ cover each nonzero element in $Z_{4n+2}$ exactly four times.

Secondly, the list of differences of $\{K_{1, (4n+1)}^{n}, K_{1, (4n+1)}^{n+1}\}$ is $\{\pm c_i - 0 | i \in C_{4n+1}\}.$ Since $V(C_{4n+1}) = Z_{4n+2} - \{0\}$ then $\{c_i - 0 | i \in C_{4n+1}\} = Z_{4n+2} - \{0\}.$ Because of $Z_{4n+2} = \{Z_{4n+2} - \{0\}\},$ then $\Delta(K_{1, (4n+1)}^{n}) = \{\pm c_i - 0 | i \in C_{4n+1}\}$ covers each nonzero element of $Z_{4n+2}$ twice. Likewise, we repeat the same strategy on cycle $K_{1, (4n+1)}^{n+1}$ to find $\Delta(K_{1, (4n+1)}^{n+1})$. Also, it is an easy matter to check that $\Delta(K_{1, (4n+1)}^{n+1}) = \Delta(K_{1, (4n+1)}^{n}).$

Linking together the above list of differences, we see that $\Delta W$ covers each nonzero element of $Z_{4n+2}$ eight times. On the other hand, each wheel graph in $W$ has trivial stabilizer based on Lemma 1. Therefore, $W$ is the starter set of $\text{CHWS}(3K_{2n+2}, W)$, by Lemma 2. One can be generated $\text{CHWS}(3K_{2n+2}, W)$ by repeated addition $1$ modular $\nu$ on $W$. □

V. CONCLUSION

In this paper, we have provided new designs $\text{CNHC}(4K_{2n}, C_{2n+1})$ and $\text{CHWS}(3K_{2n+2}, W)$ where $\nu \equiv 2(\text{mod} 4)$. These designs are interested in a decomposition of complete multigraph into cyclic $(\nu - 1)$-cycle and cyclic $(\nu)$-wheel graphs, respectively. We have also proved the existence of these designs by constructed the starter set for each of them. Moreover, one can ask if $\text{CNHC}(2\lambda K_{2n}, C_{2n+1})$ and $\text{CHWS}(2\lambda K_{2n+2}, W)$ can be constructed for the case $\nu \equiv 2, 4 (\text{mod} 4)$ and $\lambda > 2$.

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