A new Bernstein’s Inequality and the 2D Dissipative Quasi-Geostrophic Equation

Qionglei Chen $^1$, Changxing Miao $^2$, Zhifei Zhang $^3$

$^1, ^2$Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, P. R. China.
(chenqionglei@iapcm.ac.cn and miao changxing@iapcm.ac.cn)
$^3$ School of Mathematical Science, Peking University,
Beijing 100871, P. R. China.
(zfzhang@math.pku.edu.cn)

Abstract

We show a new Bernstein’s inequality which generalizes the results of Canonne-Planchon, Danchin and Lemarié-Rieusset. As an application of this inequality, we prove the global well-posedness of the 2D quasi-geostrophic equation with the critical and super-critical dissipation for the small initial data in the critical Besov space, and local well-posedness for the large initial data.

Mathematics Subject Classification (2000): 76U05, 76B03, 35Q35

Keywords: Bernstein’s inequality, Besov space, Littlewood-Paley decomposition, Quasi-Geostrophic equation.

1 Introduction

We are concerned with the 2D dissipative quasi-geostrophic equation

\[(QG)_\alpha \left\{ \begin{array}{l}
\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\theta(0, x) = \theta_0(x). 
\end{array} \right. \tag{1.1}\]

Here $\alpha \in [0, \frac{1}{2})$, $\kappa > 0$ is the dissipative coefficient, $\theta(t, x)$ is a real-valued function of $t$ and $x$. The function $\theta$ represents the potential temperature, the fluid velocity $u$ is determined from $\theta$ by a stream function $\psi$

\[(u_1, u_2) = \left( -\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right), \quad (-\Delta)^{\frac{1}{2}} \psi = -\theta. \tag{1.2}\]

A fractional power of the Laplacian $(-\Delta)^\beta$ is defined by

\[(-\Delta)^{\frac{\beta}{2}} f(\xi) = |\xi|^{2\beta} \hat{f}(\xi),\]

where $\hat{f}$ denotes the Fourier transform of $f$. We rewrite (1.2) as

\[u = (\partial_{x_2}(-\Delta)^{-\frac{1}{2}}\theta, -\partial_{x_1}(-\Delta)^{-\frac{1}{2}}\theta) = \mathcal{R}^\perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta),\]
where $R_k$, $k = 1, 2$, is the Riesz transform defined by

$$\tilde{R}_k f(\xi) = -i\xi_k/|\xi| \hat{f}(\xi).$$

$(QG)_\alpha$ is an important model in geophysical fluid dynamics, they are special cases of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. There exists deep analogy between the equation (1.1) with $\alpha = \frac{1}{2}$ and the the 3D Navier-Stokes equations. For more details about its background in geophysics, see [8, 21]. The case $\alpha > \frac{1}{2}$ is called the subcritical case, the case $\alpha = \frac{1}{2}$ is critical, and the case $0 \leq \alpha < \frac{1}{2}$ is supercritical. In the subcritical case, Constantin and Wu[9] proved the existence of global in time smooth solutions. In the critical case, Constantin, Cordoba, and Wu[10] proved the existence and uniqueness of global smooth solution on the spatial periodic domain under the assumption of small $L^\infty$ norm. Recently, Chae and Lee[5] studied the super-critical case and proved the global well-posedness for small data in the Besov spaces $\dot{B}^{2-2\alpha}_{2,1}$. Very recently, Cordoba-Cordoba[13], Ning[17, 18] studied the existence and uniqueness in the Sobolev spaces $H^s, s \geq 2 - 2\alpha, \alpha \in [0, \frac{1}{2}]$. Wu [24, 25] studied the well-posedness in general Besov space $B^{s}_{p,q}, s > 2(1 - \alpha), p = 2^N$. Many other relevant results can also be found in [4, 11, 12].

One purpose of this paper is to study the well-posedness of the 2D dissipative quasi-geostrophic equation in the critical Besov space $B^{\frac{2}{p}+1-2\alpha}_{p,q}, p \geq 2, q \in [1, \infty)$. If we use the standard energy method as in [5, 26], we need to establish the lower bound for the term generated from the dissipative part

$$\int_{\mathbb{R}^2} \Lambda^{2\alpha} \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta dx \geq 2^{2\alpha j} \|\Delta_j \theta\|_p^p, \quad p \geq 2,$$

where $\Delta_j$ is the frequency localization operator at $|\xi| \approx 2^j$ (see Section 2). For $p = 2$, this is a direct consequence of Plancherel formula. In the case $\alpha = 1$, it is proved by Cannone and Planchon[3]. To generalize (1.3) to general index $\alpha, p$, it is sufficient to show the following Bernstein’s inequality

$$c_p 2^{2\alpha j} \|\Delta_j f\|_p \leq \|\Lambda^{\alpha}(|\Delta_j f|^\frac{p}{2})\|_2^\frac{p}{2} \leq C_p 2^{2\alpha j} \|\Delta_j f\|_p, \quad p > 2,$$

which together with an improved positivity Lemma 3.3 in [18] (see also Section 3, Lemma 3.3) will imply (1.3). We should point out that (1.4) is proved by Lemarié-Rieusset[19] in the case $\alpha = 1$, and by Danchin [14] when $p$ is any even integer. On the other hand, Wu [26] gives a formal proof for general index. The first purpose of this paper is to present a rigorous proof of Theorem 3.4 in [26] which plays a key role in Wu’s paper.

**Theorem 1.1 (Bernstein’s inequality)** Let $p \in [2, \infty)$ and $\alpha \in [0, 1]$. Then there exist two positive constants $c_p$ and $C_p$ such that for any $f \in S'$ and $j \in \mathbb{Z}$, we have

$$c_p 2^{2\alpha j} \|\Delta_j f\|_p \leq \|\Lambda^{\alpha}(|\Delta_j f|^\frac{p}{2})\|_2^\frac{p}{2} \leq C_p 2^{2\alpha j} \|\Delta_j f\|_p.$$

The second purpose is to study the well-posedness of the 2D dissipative quasi-geostrophic equation in the critical Besov space $B^{\frac{2}{p}+1-2\alpha}_{p,q}$ by using Theorem 1.1 and Fourier localization technique.
Theorem 1.2 Assume that \((\alpha, p, q) \in (0, \frac{1}{2}] \times [2, \infty) \times [1, \infty)\). If \(\theta_0\) belongs to \(B^\sigma_{p,q}\) with 
\(\sigma = \frac{2}{p} + 1 - 2\alpha\), then there exists a positive real number \(T\) such that a unique solution to the
2D dissipative quasi-geostrophic equation \(\theta(t, x)\) exists on \([0, T) \times \mathbb{R}^2\) satisfying
\[
\theta(t, x) \in C([0, T]; B^\sigma_{p,q}) \cap \dot{L}^1(0, T; \dot{B}^{\frac{2}{p}+1}_{p,q}),
\]
with the time \(T\) bounded from below by
\[
\sup \{ T' > 0 : \|(1 - e^{-\kappa c^2(2\alpha^2 + \tau T)}\frac{1}{2} 2^{j\sigma} \|\Delta_j \theta_0\|_{l^p(\mathbb{Z})} \leq c\kappa \}.
\]
Furthermore, if \(\|\theta_0\|_{B^\sigma_{p,q}} \leq \epsilon\kappa\) for some positive number \(\epsilon\), then we can choose \(T = +\infty\).

Remark 1.1 It is pointed out that the homogeneous Besov space \(\dot{B}^\sigma_{p,q}\) is important as it gives
the important scaling invariant function space. In fact, if \(\theta(t, x)\) and \(u(t, x)\) are solutions of
(1.1), then \(\theta_0(t, x) = \lambda^{2\alpha-1} \theta(\lambda^{2\alpha-1} t, \lambda x)\) and \(u_0(t, x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha-1} t, \lambda x)\) are also solutions of (1.1). The \(\dot{B}^\sigma_{p,q}\) norm of \(\theta(t, x)\) is invariant under this scaling. Moreover, for the global existence result, the smallness assumption is imposed only on the homogenous norm of the initial data.

Remark 1.2 The result of Theorem 1.2 for the case \((p, q) = (2, 1)\) corresponds to the result of Chae and Lee[5] in the critical Besov space \(B^{2-2\alpha}_{2,1}\). In the case \((p, q) = (2, 2)\), it corresponds to the result of Ning[17] in the Sobolev space \(H^{2-2\alpha}\). On the other hand, thanks to the embedding relationship:
\[
B^s_{2,1} \subseteq H^s \subseteq B^s_{p,q}, \quad \text{for } q > 2,
\]
our result improves the results of [5] and [17].

Remark 1.3 Wu[24, 25] proved the well-posedness of (1.1) for the initial data in the subcritical Besov space \(B^s_{p,q}\) with \(s > 2 - 2\alpha, p = 2^N\). We obtain the well-posedness in the critical Besov space \(B^{\frac{2}{p}+1-2\alpha}_{p,q}\), and get rid of the restriction on \(p = 2^N\).

Remark 1.4 Very recently, Miura[20] proved the local well-posedness of (1.1) for the large initial data in the critical Sobolev space \(H^{2-2\alpha}\). His result is a particular case of Theorem 1.2, and our proof is simpler(see section 4.2).

Notation: Throughout the paper, \(C\) denotes various “harmless” large finite constants, and \(c\) denotes various “harmless” small constants. We shall sometimes use \(X \lesssim Y\) to denote the estimate \(X \leq CY\) for some \(C\). \(\{c_j\}_{j \in \mathbb{Z}}\) denotes any positive series with \(\ell^1(\mathbb{Z})\) norm less than or equals to 1. We shall sometimes use the \(\|\cdot\|_{L^p}\) to denote \(L^p(\mathbb{R}^d)\) norm of a function.

2 Littlewood-Paley decomposition

Let us recall the Littlewood-Paley decomposition. Let \(\mathcal{S}(\mathbb{R}^d)\) be the Schwartz class of rapidly decreasing functions. Given \(f \in \mathcal{S}(\mathbb{R}^d)\), its Fourier transform \(\mathcal{F}f = \hat{f}\) is defined by
\[
\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.
\]
Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^d)$, supported respectively in $\mathcal{B} = \{ \xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3} \}$ and $\mathcal{C} = \{ \xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}$ such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Setting $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, we define the frequency localization operator as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^jy)f(x-y)dy,$$

$$S_j f = \sum_{k \leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^jy)f(x-y)dy.$$

Informally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \approx 2^j\}$, while $S_j$ is a frequency projection to the ball $\{|\xi| \leq 2^j\}$. One easily verifies that with our choice of $\varphi$

$$\Delta_j \Delta_k f \equiv 0 \text{ if } |j-k| \geq 2 \text{ and } \Delta_j(S_{k-1}f \Delta_k f) \equiv 0 \text{ if } |j-k| \geq 5. \quad (2.1)$$

Now we give the definitions of the Besov spaces.

**Definition 2.1** Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{ f \in \mathcal{Z}'(\mathbb{R}^d); \|f\|_{\dot{B}^s_{p,q}} < \infty \}.$$

Here

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}}, & \text{for } q \in \mathbb{R}, \\
\|\Delta_j f\|_p, & \text{for } q = \infty, \end{cases}$$

and $\mathcal{Z}'(\mathbb{R}^d)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^d) = \{ f \in \mathcal{S}(\mathbb{R}^d); \partial^\gamma \hat{f}(0) = 0; \forall \gamma \in \mathbb{N}^d \text{ multi-index} \}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

**Definition 2.2** Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the inhomogenous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{ f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{B^s_{p,q}} < \infty \}.$$

Here

$$\|f\|_{B^s_{p,q}} = \begin{cases} \left( \sum_{j \geq 0} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}} + \|S_0(f)\|_p, & \text{for } q \in \mathbb{R}, \\
\sup_{j \geq 0} \|\Delta_j f\|_p + \|S_0(f)\|_p, & \text{for } q = \infty. \end{cases}$$

If $s > 0$, then $B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}$ and $\|f\|_{B^s_{p,q}} \approx \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}$. We refer to [1, 23] for more details.

Next let’s recall Chemin-Lerner’s space-time space which will play an important role in the proof of Theorem 1.2.
Definition 2.3 Let $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$, $I \subset \mathbb{R}$ is an interval. The homogeneous mixed time-space Besov space $\tilde{L}^r(I; \tilde{B}^s_{p,q})$ is the space of the distribution such that

$$\tilde{L}^r(I; \tilde{B}^s_{p,q}) = \{ f \in \mathcal{D}(I; \mathcal{Z}'(\mathbb{R}^d)); \|f\|_{\tilde{L}^r(I; \tilde{B}^s_{p,q})} < +\infty \}.$$ 

Here

$$\|f(t)\|_{\tilde{L}^r(I; \tilde{B}^s_{p,q})} = \left\| 2^{sj} \left( \int_I \|\Delta_j f(\tau)\|_p^r d\tau \right) \right\|_{\ell^q(\mathbb{Z})},$$

(usual modification if $r, q = \infty$). We also need the inhomogeneous mixed time-space Besov space $\tilde{L}^r(I; B^s_{p,q})$, $s > 0$, whose norm is defined by

$$\|f(t)\|_{\tilde{L}^r(I; B^s_{p,q})} = \|f(t)\|_{L^r(I; L^p(x))} + \|f(t)\|_{\tilde{L}^r(I; \tilde{B}^s_{p,q})}.$$

For the convenience, we sometimes use $\tilde{L}^r I (\tilde{B}^s_{p,q})$ and $\tilde{L}^r I (\tilde{B}^s_{p,q})$ to denote $\tilde{L}^r I (0, T; \tilde{B}^s_{p,q})$ and $\tilde{L}^r I (0, \infty; \tilde{B}^s_{p,q})$, respectively. The direct consequence of Minkowski’s inequality is that

$$\tilde{L}^r I (\tilde{B}^s_{p,q}) \subseteq \tilde{L}^r I (\tilde{B}^s_{p,q}) \text{ if } r \leq q \text{ and } \tilde{L}^r I (\tilde{B}^s_{p,q}) \subseteq \tilde{L}^r I (\tilde{B}^s_{p,q}) \text{ if } r \geq q.$$

We refer to [7] for more details.

Let us state some basic properties about the Besov spaces.

Proposition 2.1 (i) We have the equivalence of norms

$$\|D^k f\|_{B^s_{p,q}} \sim \|f\|_{B^s_{p,q+k}}, \text{ for } k \in \mathbb{Z}^+.$$

(ii) Interpolation: for $s_1, s_2 \in \mathbb{R}$ and $\theta \in [0, 1]$, one has

$$\|f\|_{B^\theta s_{1}+(1-\theta)s_{2}_{p,q}} \leq \|f\|_{B^{s_{1}}_{p,q}} \|f\|_{B^{s_{2}}_{p,q}}^{(1-\theta)},$$

and the similar interpolation inequality holds for inhomogeneous Besov space.

(iii) Embedding: If $s > \frac{d}{p}$, then

$$B^s_{p,q} \hookrightarrow L^\infty;$$

If $p_1 \leq p_2$ and $s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}$, then

$$B^{s_1}_{p_1,q_1} \hookrightarrow B^{s_2}_{p_2,q_2}; \quad B^s_{p,\min(p,2)} \hookrightarrow H^s_{p} \hookrightarrow B^s_{p,\max(p,2)}.$$

Here $H^s_{p}$ is the inhomogeneous Sobolev space.

Proof: The proof of (i) – (iii) is rather standard and one can refer to [23].

Finally we introduce the well-known Bernstein’s inequalities which will be used repeatedly in this paper.

Lemma 2.2 Let $C$ be a ring, and $B$ a ball, $1 \leq p \leq q \leq +\infty$. Assume that $f \in \mathcal{S}'(\mathbb{R}^d)$, then for any $|\gamma| \in \mathbb{Z}^+ \cup \{0\}$ there exist constants $C$, independent of $f$, $j$ such that

$$\|\partial^\gamma f\|_q \leq C\lambda^{|\gamma|+\frac{d}{p}-\frac{d}{q}} \|f\|_p \quad \text{if supp} \hat{f} \subset \lambda B, \quad (2.2)$$

$$\|f\|_p \leq C \sup_{|\beta|=|\gamma|} \lambda^{-|\gamma|} \|\partial^\beta f\|_p \leq C \|f\|_p \quad \text{if supp} \hat{f} \subset \lambda C. \quad (2.3)$$

Proof: The proof can be found in [6].
3 A new Bernstein’s inequality

Firstly, we will give certain kind of Bernstein’s inequality which can be found in [19, Chapter 29].

**Proposition 3.1** Let $2 < p < \infty$. Then there exist two positive constants $c_p$ and $C_p$ such that for every $f \in \mathcal{S}'$ and every $j \in \mathbb{Z}$, we have

$$c_p2^{2j} \| \Delta_j f \|_p \leq \| \nabla (|\Delta_j f|^\frac{p}{2}) \|_2^{\frac{2}{p}} \leq C_p2^{2j} \| \Delta_j f \|_p. \quad (3.1)$$

Naturally, we want to establish a generalization of (3.1) for the fractional differential operator $\Lambda^\alpha(0 < \alpha < 1)$ which is defined by $\Lambda^\alpha f = \mathcal{F}^{-1}(|\xi|\alpha \hat{f})$. However it seems nontrivial, since for $p > 2$, the spectrum of $|\Delta_j f|^\frac{p}{2}$ can’t be included in a ring although $\text{sup} \Delta_j f$ is localized in $|\xi| \approx 2^j$. This section is devoted to prove Theorem 1.1. For this purpose, we first need the following priori lemma.

**Lemma 3.2** Let $p \in [1, \infty), s \in [0, p) \cap [0, 2)$. Suppose that $\ell, r, m$ satisfy

$$1 < \ell \leq r < \infty, \quad 1 < m < \infty, \quad \frac{1}{\ell} = \frac{1}{r} + \frac{p-1}{m}.$$ 

Then for $f(u) = |u|^p$, the following estimate holds:

$$\| f(z) \|_{\dot{B}_{\ell,2}^s} \leq C_p \| z \|_{\dot{B}_{\ell,2}^{p-1}} \| z \|_{\dot{B}_{r,2}^s}. \quad (3.2)$$

**Proof:** Let us first recall the equivalence norm of Besov spaces: for $0 \leq s < 2$, $1 \leq \ell, q \leq \infty$

$$\| v \|_{\dot{B}_{\ell,2}^s} \triangleq \left( \int_0^\infty t^{-sq} \sup_{|y| \leq t} \| \tau_y v + \tau_{-y} v - 2v \|_q \frac{dt}{t} \right)^{\frac{1}{q}},$$

where $\tau_y v(x) = v(x+y)$. In the special case when $0 \leq s < 1$, we also have

$$\| v \|_{\dot{B}_{r,2}^s} \triangleq \left( \int_0^\infty t^{-sq} \sup_{|y| \leq t} \| \tau_y v - v \|_q \frac{dt}{t} \right)^{\frac{1}{q}}.$$ 

It is not difficult to check that

$$|f^{[s]}(z_1) - f^{[s]}(z_2)| \leq C \left\{ \begin{array}{l}
|z_1|^{p-[s]-1} + |z_2|^{p-[s]-1} |z_1 - z_2|, \quad p \geq [s] + 1, \\
|z_1 - z_2|^{p-[s]}, \quad p < [s] + 1.
\end{array} \right. \quad (3.3)$$

where $f^{[s]}(z) = D_{1,s} f(z)$. For simplicity we set $u_+ \triangleq \tau_{+y} u$. We divide the proof of Lemma 3.2 into two cases.

**Case 1** $p \geq 2$. We write

$$\tau_y f(u) + \tau_{-y} f(u) - 2f(u) = f(u_+) + f(u_-) - 2f(u)$$

$$= f'(u)(u_+ + u_- - 2u) + \sum_{\pm} (u_+ - u) \int_0^1 [f'(\lambda u_+ + (1-\lambda)u) - f'(u)]d\lambda, \quad (3.4)$$
where $\theta = \frac{m \tau}{m + r}$. Then by the previous equivalence norm of Besov spaces, we have
\[
\|f(u)\|_{\dot{B}^s_{2,2}} \leq C\|u\|_{\dot{B}^s_{r,2}}^p \|u\|_{\dot{B}^s_{m,\infty}}^{p-1}.
\]

Thanks to the interpolation inequality
\[
\|u\|_{\dot{B}^2_{2,4}} \leq \|u\|_{\dot{B}^{s}_{r,2}} \|u\|_{\dot{B}^{s}_{m,\infty}},
\]
and the inclusion map $L^m \hookrightarrow \dot{B}^{0}_{m,\infty}$, we obtain
\[
\|f(u)\|_{\dot{B}^s_{r,2}} \leq C\|u\|_{\dot{B}^s_{r,2}} \|u\|_{\dot{B}^s_{m,\infty}}^{p-1}.
\]

Case 2 $p \leq 2$. (3.3) and (3.4) imply that
\[
|f(u_+) + f(u_-) - 2f(u)| \leq f'(u)|u_+ + u_- - 2u| + C\sum_{\pm} |u_{\pm} - u|^p.
\]

In the same way as leading to (3.5), we can deduce that
\[
\|f(u)\|_{\dot{B}^s_{r,2}} \leq C(\|u\|_{\dot{B}^s_{m,\infty}}^{p-1} \|u\|_{\dot{B}^s_{r,2}} + \|u\|_{\dot{B}^s_{r,2}}^p)
\leq C(\|u\|_{\dot{B}^s_{m,\infty}}^{p-1} \|u\|_{\dot{B}^s_{r,2}}^p + \|u\|_{\dot{B}^s_{r,2}} \|\dot{u}\|_{\dot{B}^s_{r,2}}) \leq C\|u\|_{\dot{B}^s_{r,2}}^{p-1} \|u\|_{\dot{B}^s_{r,2}}.
\]

Collecting (3.5) and (3.6), the lemma is proved. \qed

Remark 3.1 In fact, the inequality holds for all $p \in [1, \infty)$, $s \in [0, p)$. But in order to make the presentation lighter, we only give the proof of the case $s \in [0, p) \cap [0, 2)$, and the other cases can be treated in the same way.

Now let’s come back to the proof of Theorem 1.1. By homogeneity and scaling, it is enough to prove the inequality for $j = 0$. According to the definition of Besov spaces, we have
\[
\|\Delta^{\alpha}(\Delta_0 f)^{\frac{p}{2}}\|_2 \geq \|\Delta_0 f\|_{\dot{B}^s_{2,2}}^{\frac{p}{2}}.
\]

Applying Lemma 3.2 to the right hand side of (3.7) yields that for $2 \leq p < \infty$, $\alpha \in [0, 1]$
\[
\|\Delta_0 f\|_{\dot{B}^s_{2,2}} \leq C_p \|\Delta_0 f\|_{\dot{B}^s_{p,2}} \|\Delta_0 f\|_{\dot{B}^s_{p,2}}^{\alpha}.
\]
If we choose $M$ such that

$$C_p 2^{-M\varepsilon} \leq \frac{1}{2} C_p,$$

we conclude that

$$c_p \| f_0 \|_{p}^{\frac{p}{p-2}} \leq \| \Lambda^\alpha(|f_0|^{\frac{p}{2}}) \|_2.$$  \hfill (3.14)

This completes the proof of Theorem 1.1. \hfill \Box
Finally let us recall the following improved positivity Lemma.

**Lemma 3.3** Suppose that $s \in [0, 2]$, and $f, \Lambda^s f \in L^p(\mathbb{R}^2)$, $p \geq 2$. Then

$$
\int_{\mathbb{R}^2} |f|^{p-2} f \Lambda^s f \, dx \geq \frac{2}{p} \int_{\mathbb{R}^2} (\Lambda^s |f|^{p/2})^2 \, dx.
$$

(3.15)

**Proof:** The proof can be found in [18].

**4 The proof of Theorem 1.2**

In this section, we will prove Theorem 1.2. We divided it into two parts.

**4.1 Global well-posedness for small initial data**

**Step 1. A priori estimates**

Taking the operator $\Delta_j$ on both sides of (1.1), we have

$$
\partial_t \Delta_j \theta + \kappa \Lambda^{2\alpha} \Delta_j \theta + u \cdot \nabla \Delta_j \theta = [u, \Delta_j] \cdot \nabla \theta.
$$

Multiplying by $p|\Delta_j \theta|^{p-2} \Delta_j \theta$ and integrating with respect to $x$ yield that

$$
\frac{d}{dt} \|\Delta_j \theta\|_p^p + \kappa p \int_{\mathbb{R}^2} \Lambda^{2\alpha} \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx + p \int_{\mathbb{R}^2} u \cdot \nabla \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx = p \int_{\mathbb{R}^2} [u, \Delta_j] \cdot \nabla \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx.
$$

(4.1)

Since $\text{div} u = 0$, by integration by parts we infer that

$$
\int_{\mathbb{R}^2} u \cdot \nabla \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx = 0.
$$

(4.2)

Thanks to Lemma 3.3 and Theorem 1.1, we deduce that

$$
p \int_{\mathbb{R}^2} \Lambda^{2\alpha} \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx \geq 2 \int_{\mathbb{R}^2} (\Lambda^{\alpha} |\Delta_j \theta|^{p/2})^2 \, dx \geq c_p 2^{2\alpha_j} \|\Delta_j \theta\|^p_p.
$$

(4.3)

Summing up (4.1)–(4.3) and Hölder inequality yield that

$$
\frac{d}{dt} \|\Delta_j \theta\|_p + 2 \kappa c_p 2^{2\alpha_j} \|\Delta_j \theta\|_p \leq C \|[u, \Delta_j] \cdot \nabla \theta\|_p.
$$

which together with Gronwall’s inequality implies that

$$
\|\Delta_j \theta\|_p \leq e^{-\kappa c_p t 2^{2\alpha_j}} \|\Delta_j \theta_0\|_p + C e^{-\kappa c_p t 2^{2\alpha_j}} \|[u, \Delta_j] \cdot \nabla \theta\|_p.
$$

(4.4)

where the sign $*$ denotes the convolution of functions defined in $\mathbb{R}^+$, in details

$$
e^{-\kappa c_p t 2^{2\alpha_j}} \ast f \triangleq \int_0^t e^{-\kappa c_p (t-\tau) 2^{2\alpha_j}} f(\tau) \, d\tau.
$$
Taking the $L^r(0,T)$ norm, $1 \leq r \leq \infty$, $T \in (0,\infty]$, and using Young’s inequality to obtain
\[
\| \Delta_j \theta \|_{L^r_T(L^p)} \leq \| e^{-KCp_{2\alpha_\sigma}} \|_{L^r_T(\| \Delta_j \theta_0 \|_p + C\| u, \Delta_j \| \cdot \nabla \theta \|_{L^1_T(L^p)})}.
\] (4.5)
Multiplying $2^{i}\theta$ on both sides of (4.5), then taking $\ell^q(Z)$ norm, we obtain
\[
\| \theta \|_{\ell^q(\mathcal{B}_{p,q}^j, \mathcal{B}_{p,q}^{j+1})} \leq \kappa^{-1/r} \left( \| \theta_0 \|_{\mathcal{B}_{p,q}^j} + \| 2^{i}\theta \|_{L^1(\mathcal{B}_{p,q}^{j+1})} \right).
\] (4.6)
where we used the fact that
\[
\| e^{-KCp_{2\alpha_\sigma}} \|_{L^r_T} \leq \left( \frac{1 - e^{-RCp_{2\alpha_\sigma}T}}{RCp_{2\alpha_\sigma}} \right)^{1/r}, \quad \text{for} \quad 1 \leq r \leq \infty,
\] (4.7)
and $\sigma = \frac{2}{p} + 1 - 2\alpha$. On the other hand, it follows from Proposition 5.3 that
\begin{align*}
\| 2^{i}\theta \|_{L^1(\mathcal{B}_{p,q}^j, \mathcal{B}_{p,q}^{j+1})} & \leq C\| u \|_{L^2(\mathcal{B}_{p,q}^j, \mathcal{B}_{p,q}^{j+1})} \| \theta \|_{L^2(\mathcal{B}_{p,q}^j, \mathcal{B}_{p,q}^{j+1})} \\
& \leq C\| \theta \|_{L^\infty(\mathcal{B}_{p,q}^j)} \| \theta \|_{L^1(\mathcal{B}_{p,q}^{j+1})},
\end{align*}
(4.8)
where in the last inequality we have used the interpolation and the fact that
\[
\| u \|_{L^r(\mathcal{B}_{p,q}^j)} = \| \mathcal{R}_k \theta \|_{L^r(\mathcal{B}_{p,q}^j)} \leq C\| \theta \|_{L^r(\mathcal{B}_{p,q}^j)}, \quad \text{for} \quad s \in \mathbb{R}, \quad (r,p,q) \in [1,\infty]^3,
\] (4.9)
since $\| \Delta_j \mathcal{R}_k \theta \|_p \approx \| \overline{\Delta_j} \mathcal{R}_k \Delta_j \theta \|_p \leq C\| \Delta_j \theta \|_p$ for all $1 \leq p \leq \infty$, here $\overline{\Delta_j} = (\Delta_{j-1} + \Delta_j + \Delta_{j+1})$. Combining (4.6) and (4.8), we get
\[
\| \theta \|_{L^r(\mathcal{B}_{p,q}^j, \mathcal{B}_{p,q}^{j+1})} \leq \kappa^{-1/r} \left( \| \theta_0 \|_{\mathcal{B}_{p,q}^j} + C\| \theta \|_{L^\infty(\mathcal{B}_{p,q}^j)} \| \theta \|_{L^1(\mathcal{B}_{p,q}^{j+1})} \right).
\] (4.10)
On the other hand, it follows from ([13], Corollary 2.6) that
\[
\| \theta(t,x) \|_p \leq \| \theta_0(x) \|_p, \quad t \geq 0,
\] (4.11)
which together with (4.10) implies that
\[
\| \theta(t) \|_{L^\infty(\mathcal{B}_{p,q}^j)} + c_1 \kappa \| \theta(t) \|_{L^1(\mathcal{B}_{p,q}^{j+1})} \leq 2\| \theta_0 \|_{\mathcal{B}_{p,q}^j} + C\| \theta \|_{L^\infty(\mathcal{B}_{p,q}^j)} \| \theta \|_{L^1(\mathcal{B}_{p,q}^{j+1})}.
\] (4.12)

**Step 2. Approximation solutions and uniform estimates**

Let us define the sequence $\{ \theta^{(n)}, u^{(n)} \}_{n \in \mathbb{N}_0}$ by the following systems:
\[
\begin{cases}
\partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \kappa(-\Delta)^{\alpha} \theta^{(n+1)} = 0, \quad x \in \mathbb{R}^2, t > 0, \\
u^{(n)} = \mathcal{R}^\perp \theta^{(n)}, \\
\theta^{(n+1)}(0,x) = \theta_0^{(n+1)}(x) = \sum_{j \leq n+1} \Delta_j \theta_0(x).
\end{cases}
\] (4.13)
Setting \((\theta(0), u(0)) = (0, 0)\) and solving the linear system, we can find \(\{\theta^n, u^n\}_{n \in \mathbb{N}_0}\) for all \(n \in \mathbb{N}_0\). As in Step 1, we can deduce that
\[
\left\|\theta^{(n+1)}(t)\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n+1)}(t)\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} 
\leq 2\left\|\theta_0^{(n+1)}\right\|_{B^p_{p,q}} + C_2(c_1 \kappa)^{-1} \left(\left\|\theta^{(n)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})}\right) 
\times \left(\left\|\theta^{(n+1)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n+1)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})}\right),
\]
(4.14)

If we take \(\epsilon > 0\) such that \(\|\theta_0\|_{B^p_{p,q}} \leq \epsilon \kappa, \epsilon \leq \frac{c_1}{\kappa c_2}\), then for all \(n\), we will show
\[
\left\|\theta^{(n)}(t)\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n)}(t)\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} \leq 4\|\theta_0\|_{B^p_{p,q}}.
\]
(4.15)

In fact, assume that \(\left\|\theta^{(k)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(k)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} \leq 4\|\theta_0\|_{B^p_{p,q}}\) for \(k = 0, \cdots, n\). It follows from (4.14) that
\[
\left\|\theta^{(n+1)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n+1)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} 
\leq 2\|\theta_0\|_{B^p_{p,q}} + C_2(c_1 \kappa)^{-1} \left|\theta_0\right|_{B^p_{p,q}} \left(\left\|\theta^{(n+1)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n+1)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})}\right) 
\leq 2\|\theta_0\|_{B^p_{p,q}} + \frac{1}{2} \left(\left\|\theta^{(n+1)}\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n+1)}\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})}\right),
\]
(4.16)

which implies (4.15). Summing up (4.11) and (4.15), we finally get for all \(n\),
\[
\left\|\theta^{(n)}(t)\right\|_{L^\infty(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} + c_1 \kappa \left\|\theta^{(n)}(t)\right\|_{L^1(B^p_{p,q} \cap \dot{B}^{2+1}_{p,q})} \leq 4\|\theta_0\|_{B^p_{p,q}}.
\]
(4.17)

**Step 3. Compactness arguments and Existence**

We will show that, up to a subsequence, the sequence \(\{\theta^n\}\) converges in \(\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^2)\) to a solution \(\theta\) of (1.1). The proof is based on compactness arguments. First we show that \(\partial_t \theta^n\) is uniformly bounded in the space \(L^\infty(B^{-2\alpha}_{p,q})\). By (4.13), \(\partial_t \theta^{(n+1)}\) satisfies the equation
\[
\partial_t \theta^{(n+1)} = -\nabla \cdot (u^n \theta^{(n+1)}) - \kappa \left((-\Delta)^\alpha \theta^{(n+1)}\right).
\]

Then thanks to Proposition 5.1 with \(p \neq \infty\), we get
\[
\left\|\partial_t \theta^{(n+1)}\right\|_{L^\infty(B^{-2\alpha}_{p,q})} \lesssim \left\|\theta^{(n+1)}\right\|_{L^\infty(B^0_{p,q})} + \left\|u^n\right\|_{L^\infty(L^p)} \left\|\theta^{(n+1)}\right\|_{L^\infty(B^\sigma_{p,q})} + \left\|\theta^{(n+1)}\right\|_{L^\infty(L^p)} \left\|u^n\right\|_{L^\infty(B^\sigma_{p,q})} \lesssim \left\|\theta^{(n+1)}\right\|_{L^\infty(B^\sigma_{p,q})} + \left\|\theta^n\right\|_{L^\infty(B^0_{p,q})} \left\|\theta^{(n+1)}\right\|_{L^\infty(B^\sigma_{p,q})} < \infty,
\]

where we have used the fact: for \(s > 0\), \(B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}\), and the inclusion map \(B^\sigma_{p,q} \subset B^0_{p,q}\). We remark that the above inequality can be obtained also by Proposition 5.2 with \(s = -2\sigma\), \(s_1\) be an any number such that \(0 < s_1 < \frac{2}{p}\). Now let us turn to the proof of the existence. Observe that for any \(\chi \in C_c^\infty(\mathbb{R}^2)\), the map: \(u \mapsto \chi u\) is compact from \(B^\sigma_{p,q}(\mathbb{R}^2)\) into \(L^p(\mathbb{R}^2)\).
This can be proved by noting that the map $u \mapsto \chi u$ is compact from $H^s_p$ into $H^{s'}_p$ for $s' > s$, $p < \infty$, and the embedding relation $B^\sigma_{p,q} \hookrightarrow B^{\sigma - \epsilon}_{p,2} \hookrightarrow H^{\sigma - \epsilon}_p$ (by Proposition 2.1(iii)). Thus by the Lions-Aubin compactness theorem (see [22]), we can conclude that there exists a subsequence $\{\theta^{(n_k)}\}$ and a function $\theta$ so that
\[
\lim_{n_k \to +\infty} \theta^{(n_k)} = \theta \quad \text{in} \quad L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}^2).
\]
Moreover, the uniform estimate (4.17) allows us to conclude that
\[
\theta(t, x) \in \tilde{L}^\infty(0, \infty; B^\sigma_{p,q}) \cap \tilde{L}^1(0, \infty; B^{2+1}_{p,q}),
\]
and
\[
\|\theta(t)\|_{L^\infty(B^\sigma_{p,q})} + \|\theta(t)\|_{\tilde{L}^1(B^{2+1}_{p,q})} \leq 4\|\theta_0\|_{B^\sigma_{p,q}}.
\]
Then by a standard limit argument, we can prove that the limit function $\theta(t, x)$ satisfies the equation (1.1) in the sense of distribution.

We still have to prove $\theta(t, x)$ belongs to $C(\mathbb{R}^+; B^\sigma_{p,q})$. Our idea comes from [15]. We observe that
\[
\partial_t \Delta_j \theta = -\kappa \Lambda^{2\alpha} \Delta_j \theta - \Delta_j \nabla \cdot (u \theta) \tag{4.18}
\]
For fixed $j$, the right hand side of (4.18) belongs to $L^\infty(0, \infty; B^\sigma_{p,q})$, which can be easily proved by using Lemma 2.2. Therefore, we infer that $\partial_t \Delta_j \theta \in L^\infty(0, \infty; B^\sigma_{p,q})$ for fixed $j$, which implies that each $\Delta_j \theta$ is continuous in time in $B^\sigma_{p,q}$. On the other hand, note that
\[
\|\theta\|_{L^\infty(B^\sigma_{p,q})} = \left( \sum_{j \in \mathbb{Z}} \sup_{t \geq 0} (2^{j/2} \|\Delta_j \theta\|_{L^p})^q \right)^{1/q} < +\infty,
\]
which implies that $\sum_{|j| \leq n} \Delta_j \theta$ converges uniformly in $L^\infty(\mathbb{R}^+; B^\sigma_{p,q})$ to $\theta(t, x)$. Hence, $\theta(t, x) \in C(\mathbb{R}^+; B^\sigma_{p,q})$.

**Step 4. Uniqueness**

Assume that $\theta' \in \tilde{L}^\infty(B^\sigma_{p,q}) \cap \tilde{L}^1(B^{2+1}_{p,q})$ is another solution of (1.1) with the same initial data $\theta_0(x)$. Let $\delta \theta = \theta - \theta'$ and $\delta u = u - u'$. Then $(\delta \theta, \delta u)$ satisfy the following equations
\[
\begin{cases}
\partial_t \delta \theta + u \cdot \nabla \delta \theta + \delta u \cdot \nabla \theta' + \kappa (\Delta \delta \theta) = 0, & x \in \mathbb{R}^2, t > 0, \\
\delta u = R \perp \delta \theta, \\
\delta \theta(0, x) = 0.
\end{cases} \tag{4.19}
\]
Following the same way as a priori estimates, we can deduce that
\[
\frac{d}{dt} \|\Delta_j \delta \theta\|_p + 2\kappa c p 2^{2\alpha j} \|\Delta_j \delta \theta\|_p \leq C \left( \|[u, \Delta_j] \cdot \nabla \delta \theta\|_p + \|\Delta_j(\delta u \cdot \nabla \theta')\|_p \right),
\]
which together with Gronwall’s inequality leads to
\[
\|\Delta_j \delta \theta\|_p \leq C e^{-\kappa c p 2^{2\alpha j} t} * (\|[u, \Delta_j] \cdot \nabla \delta \theta\|_p + \|\Delta_j(\delta u \cdot \nabla \theta')\|_p). \tag{4.20}
\]
Choose a positive number $\eta$ such that $\frac{2\alpha}{p} < \eta < \frac{2}{p}$. Thanks to Proposition 5.2, (4.9), and interpolation, we get

$$
\| \delta u \cdot \nabla \theta' \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta})} \lesssim \| \delta u \|_{L_T^{q/2}(B_{p,q}^{\frac{2}{q}+\eta+\frac{2\alpha}{p}})} \| \theta' \|_{L_T^{q/2}(B_{p,q}^{\frac{2}{q}+1-\frac{2\alpha}{p}})} 
$$

$$
\lesssim \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta+2\alpha})} \| \theta' \|_{L_T^p(B_{p,q}^{\frac{2}{p}+1-\frac{2\alpha}{p}})} 
$$

$$
\lesssim \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta})} \| \frac{1}{p} \|_{L_T^p(B_{p,q}^{\frac{2}{q}+\eta+2\alpha})} \| \theta' \|_{L_T^p(B_{p,q}^{\frac{2}{q}+1-\frac{2\alpha}{p}})}. 
$$

(4.21)

On the other hand, thanks to Proposition 5.3, (4.9) we have

$$
\| [u, \Delta_j] \cdot \nabla \delta \theta \|_{L_T^p(L^p)} \lesssim c_j 2^{-j(\frac{2}{p}+\eta)} \| \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+1-\alpha})} \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta+\alpha})} 
$$

$$
\lesssim c_j 2^{-j(\frac{2}{p}+\eta)} \| u \|_{L_T^p(B_{p,q}^{\frac{2}{p}+1-\alpha})} \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta})} \| \frac{1}{p} \|_{L_T^p(B_{p,q}^{\frac{2}{q}+\eta+2\alpha})} \| \theta \|_{L_T^p(B_{p,q}^{\frac{2}{q}+1-\frac{2\alpha}{p}})}. 
$$

(4.22)

where $\|c_j\|_{L^q(Z)} \leq 1$. Taking $L^\infty(L^1)$, respectively norm on time, and using Young’s inequality, then multiplying $2^j(\frac{2}{p}+\eta)(2^j(\frac{2}{p}+\eta+2\alpha)$, respectively) on both sides of (4.20), then taking $\ell^q(Z)$ norm, we have

$$
Z(T) \triangleq \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta})} + \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta+2\alpha})} 
$$

$$
\lesssim \| 2^j(\frac{2}{p}+\eta) \|[u, \Delta_j] \cdot \nabla \delta \theta \|_{L_T^p(L^p)} \| \theta \|_{L_T^p(B_{p,q}^{\frac{2}{q}+\eta})} \| \delta \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+\eta})} 
$$

$$
\lesssim \left( \| \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+1-\frac{2\alpha}{p}})} + \| \theta \|_{L_T^p(B_{p,q}^{\frac{2}{p}+1-\alpha})} \right) Z(T), 
$$

(4.23)

where we have used (4.21) and (4.22) in the last inequality. Now it is clear that two terms in the bracket of the right hand side of (4.23) tend to 0 as $T$ goes to 0. Therefore, if $T$ has been chosen small enough, then it follows from (4.23) that $Z \equiv 0$ on $[0,T]$ which implies that $\delta \theta \equiv 0$. Then by a standard argument, we can show that $\delta \theta(t,x) = 0$ in $[0,\infty) \times \mathbb{R}^2$, i.e. $\theta(t,x) = \theta'(t,x)$.

This completes the proof of global well-posedness. 

☐

4.2 Local well-posedness for large initial data

Now we prove the local well-posedness for the large initial data. As the existence result will be essentially followed from the a priori estimate. For simplicity, we only present a priori estimate of the solution $\theta(t,x)$.

Let us return to (4.5). Taking $r = 2$ in (4.5), multiplying $2^j(\frac{2}{p}+1-\alpha)$ on both sides of (4.5), then taking $\ell^q(Z)$ norm and applying Proposition 5.3 and (4.9), we get

$$
\| \theta \|_{L_T^2(B_{p,q}^{\frac{2}{p}+1-\alpha})} \lesssim C_3 \kappa^{-\frac{1}{2}} \left( \| E_j(T) \|_{L_T^2(L^p)} \| \Delta_j \theta_0 \|_{\ell^q(Z)} + \| u \|_{L_T^2(B_{p,q}^{\frac{2}{p}+1-\alpha})} \| \theta \|_{L_T^2(B_{p,q}^{\frac{2}{p}+1-\alpha})} \right) 
$$

$$
\lesssim C_3 \kappa^{-\frac{1}{2}} \left( \| E_j(T) \|_{L_T^2(L^p)} \| \Delta_j \theta_0 \|_{\ell^q(Z)} + \| \theta \|_{L_T^2(B_{p,q}^{\frac{2}{p}+1-\alpha})} \right), 
$$

(4.24)
where
\[ E_j(T) \triangleq 1 - e^{-\kappa c p^{2\alpha_j} T}. \]

Set
\[ T_0 \triangleq \sup \left\{ T' > 0; \left( \sum_{j \in \mathbb{Z}} E_j(T') \frac{1}{2} 2^{j\sigma q} \| \Delta_j \theta_0 \|_p^q \right)^\frac{1}{q} \leq \frac{\kappa}{2C^2} \right\}. \]

Then the inequality (4.24) implies that there holds for \( T \in [0, T_0] \)
\[ \| \theta(t) \|_{L^\infty_T(B^s_{p,q})} + c_1 \kappa \| \theta(t) \|_{L_T^\infty (B^{s+1}_{p,q})} \leq 2 \| \theta_0 \|_{B^s_{p,q}} + C \| \theta \|_{L_T^\infty (B^{s+1}_{p,q})} \leq C \| \theta_0 \|_{B^s_{p,q}}. \]

Combining with (4.11), we obtain for \( T \in [0, T_0] \)
\[ \| \theta(t) \|_{L^\infty_T(B^s_{p,q})} + c_1 \kappa \| \theta(t) \|_{L_T^\infty (B^{s+1}_{p,q})} \leq 2 \| \theta_0 \|_{B^s_{p,q}} + c \| \theta \|_{L_T^\infty (B^{s+1}_{p,q})} \leq C \| \theta_0 \|_{B^s_{p,q}}. \]

This completes the proof of local well-posedness. \( \square \)

5 Appendix

Firstly, we recall the paradifferential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see [2]). The paraproduct between \( u \) and \( v \) is defined by
\[ T_u v \triangleq \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v. \]

We then have the following formal decomposition:
\[ uv = T_u v + T_v u + R(u, v), \quad (5.1) \]
with
\[ R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \bar{\Delta}_j v \quad \text{and} \quad \bar{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}. \]

The decomposition (5.1) is called the Bony’s paraproduct decomposition.

Now we state some results about the product estimates in Besov spaces.

**Proposition 5.1** Let \( s > -\frac{d}{p}, \ 2 \leq p \leq \infty, \ 1 \leq q \leq \infty \). Then
\[ \| uv \|_{B^s_{p,q}} \lesssim \| u \|_{B^s_{p,q}} \| v \|_{B^{s+1}_{p,q}} + \| v \|_{B^s_{p,q}} \| u \|_{B^{s+1}_{p,q}}. \quad (5.2) \]
Proof: Using lemma 2.2, we have
\[ \|S_0(uv)\|_p \leq \|S_0(\mu v)\|_p \leq C\|u\|_p\|v\|_p \lesssim \|u\|_p\|v\|_{B_{p,q}^{\frac{d}{p}+s}}. \] (5.3)

Then using the Bony’s paraproduct decomposition and the property of quasi-orthogonality (2.1), for fixed \( j \geq 0 \), we have
\begin{align*}
\Delta j(uv) &= \sum_{|k-j|\leq 4} \Delta j(S_{k-1}u\Delta k v) + \sum_{|k-j|\leq 4} \Delta j(S_{k-1}u\Delta k u) + \sum_{k\geq j-2} \Delta j(\Delta k u\bar{\Delta} k v) \\
&\triangleq I + II + III.
\end{align*} (5.4)

We shall estimate the above three terms separately. Using Young’s inequality and Lemma 2.2, we get
\[ \|\Delta j(S_{k-1}u\Delta k v)\|_p \lesssim 2^{dj}\|S_{k-1}u\|_p\|\Delta k v\|_p, \]

Thus we have
\[ 2^{dj}\|I\|_p \lesssim \|u\|_p \sum_{|k-j|\leq 4} 2^{(j-k)(\frac{d}{p}+s)}2^{k(\frac{d}{p}+s)}\|\Delta k v\|_p \lesssim c_j\|u\|_p\|v\|_{B_{p,q}^{\frac{d}{p}+s}}, \] (5.5)

where the \( \ell^q(Z) \) norm of \( c_j \) is less than or equals to 1. Similarly to \( II \), we have
\[ 2^{dj}\|II\|_p \lesssim c_j\|v\|_p\|u\|_{B_{p,q}^{\frac{d}{p}+s}}, \] (5.6)

Now we turn to estimate \( III \). From Lemma 2.2, Young’s inequality, and Hölder inequality we have
\[ \|\Delta j(\Delta k u\bar{\Delta} k v)\|_p \lesssim 2^{2 dj}\|\Delta j(\Delta k u\bar{\Delta} k v)\|_p \lesssim 2^{2 dj}\|\Delta k u\|_p\|\bar{\Delta} k v\|_p. \]

So, we get by \( \ell^1(Z) - \ell^q(Z) \) convolution,
\[ 2^{dj}\|III\|_p \lesssim \|u\|_p \sum_{k\geq j-2} 2^{(j-k)(\frac{d}{p}+s)}2^{k(\frac{d}{p}+s)}\|\bar{\Delta} k v\|_p \lesssim c_j\|u\|_p\|v\|_{B_{p,q}^{\frac{d}{p}+s}}, \] (5.7)

where we have used the fact \( s + \frac{d}{p} > 0 \). Summing up (5.3), (5.5)-(5.7), we obtain the desired inequality (5.2). \( \square \)

**Proposition 5.2** Let \( s > \frac{d}{p} - 1, s < s_1 < \frac{d}{p}, 2 \leq p \leq \infty, 1 \leq q \leq \infty, \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1, \) and \( u \) be a solenoidal vector field. Then
\[ \|u \cdot \nabla v\|_{L_t^q(B_{p,q}^{s_1})} \lesssim \|u\|_{L_t^{r_1}(B_{p,q}^{s_1})}\|\nabla v\|_{L_t^{r_2}(B_{p,q}^{\frac{d-s_1}{2}})}. \] (5.8)

If \( s_1 = \frac{d}{p} \) or \( s_1 = s \), \( q \) has to be equal to 1.

**Proof:** Throughout the proof, the summation convention over repeated indices \( i \in [1, d] \) is used. Similarly to the proof of Proposition 5.1, we will estimate separately each part of the Bony’s paraproduct decomposition of \( u_i \partial_i v \).
By Lemma 2.2, we have
\[
\| \Delta_j (S_{k-1} u^j \Delta_k \partial_t v) \|_{L^r_t (L^p)} \lesssim \| S_{k-1} u \|_{L^r_t (L^\infty)} \| \Delta_k \nabla v \|_{L^2_t (L^p)}
\]
\[
\lesssim \sum_{k' \leq k-2} 2^{j (\frac{d}{p} - s_1) k'} \gamma s_1 k' \| \Delta_k' u \|_{L^r_t (L^p)} \| \Delta_k \nabla v \|_{L^2_t (L^p)}
\]
\[
\lesssim 2^{j (\frac{d}{p} - s_1) k} \| u \|_{L^r_t (\tilde{B}^s_{p,q})} \| \Delta_k \nabla v \|_{L^2_t (L^p)},
\]
where the fact $s_1 < \frac{d}{p}$ has been used in the last inequality. Hence, we get
\[
2^{s_j} \| \sum_{|j-k| \leq 4} \Delta_j (S_{k-1} u^j \partial_t \Delta_k v) \|_{L^r_t (L^p)} \lesssim \| u \|_{L^r_t (\tilde{B}^s_{p,q})} \sum_{|j-k| \leq 4} 2^{(j-k) s_2} 2^{(s + \frac{d}{p} - s_1) k} \| \Delta_k \nabla v \|_{L^2_t (L^p)}
\]
\[
\lesssim c_j \| u \|_{L^r_t (\tilde{B}^s_{p,q})} \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)},
\]
where $\| c_j \|_{\mathcal{E}(Z)} \leq 1$. Since $\text{div} = 0$ and $p \geq 2$, Lemma 2.2 applied yields that
\[
\| \Delta_j (\Delta_k u \tilde{\Delta}_k \partial_t v) \|_{L^1_t (L^p)} \lesssim 2^{j (\frac{d}{p} + 1)} \| \Delta_j (\Delta_k u \tilde{\Delta}_k v) \|_{L^1_t (L^\infty)}.
\]
Thus by Hölder inequality and $\frac{d}{p} + 1 + s > 0$, we have
\[
2^{s_j} \| \sum_{k \geq j-2} \Delta_j (\Delta_k u^j \tilde{\Delta}_k \partial_t v) \|_{L^r_t (L^p)} \lesssim \sum_{k \geq j-2} 2^{j (\frac{d}{p} + 1 + s)} \| \Delta_k u \|_{L^r_t (L^p)} \| \tilde{\Delta}_k v \|_{L^2_t (L^p)}
\]
\[
\lesssim \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)} \sum_{k \geq j-2} 2^{(j-k) (\frac{d}{p} + 1 + s)} 2^{k s_1} \| \Delta_k u \|_{L^2_t (L^p)}
\]
\[
\lesssim c_j \| u \|_{L^r_t (\tilde{B}^s_{p,q})} \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)},
\]
On the other hand, due to $s_1 > s$, we get
\[
\| \Delta_j (\Delta_k u^j S_{k-1} \partial_t \Delta_k v) \|_{L^r_t (L^p)} \lesssim \sum_{k' \leq k-2} 2^{(\frac{d}{p} + 1 + s - s_1) k'} 2^{(s - s_1) k'} \| \Delta_k' v \|_{L^2_t (L^p)} \| \Delta_k u \|_{L^r_t (L^p)}
\]
\[
\lesssim 2^{(s - s_1) k} \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)} \| \Delta_k u \|_{L^r_t (L^p)}.
\]
Then we have
\[
2^{s_j} \| \sum_{|j-k| \leq 4} \Delta_j (\Delta_k u^j S_{k-1} \partial_t \Delta_k v) \|_{L^r_t (L^p)}
\]
\[
\lesssim \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)} \sum_{|j-k| \leq 4} 2^{s(j-k)} 2^{s_1 k} \| \Delta_k u \|_{L^r_t (L^p)}
\]
\[
\lesssim c_j \| v \|_{L^2_t (\tilde{B}^s_{p,q} + \frac{d}{p} + 1 - s_1)} \| u \|_{L^r_t (\tilde{B}^s_{p,q})}.
\]
Summing up (5.9)-(5.11), the desired inequality (5.8) is proved. □
Finally we give the commutator estimate.

**Proposition 5.3** Let $1 \leq p, q \leq \infty$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$, $\rho < 1$, $\gamma > -1$ and $u$ be a solenoidal vector field. Assume in addition that

$$\rho - \gamma + d \min \left(1, \frac{2}{p}\right) > 0 \quad \text{and} \quad \rho + \frac{d}{p} > 0.$$ 

Then the following inequality holds:

$$\| [u, \Delta_j] \cdot \nabla v \|_{L^r_t(L^p)} \lesssim c_j 2^{-j \left(\frac{d}{p} + \rho - 1 - \gamma\right)} \| \nabla u \|_{L^1_t(B^d_{p,q} - \rho - 1)} \| \nabla v \|_{L^2_t(B^d_{p,q} - \gamma - 1)}.$$  

(5.12)

where $c_j$ denotes a positive series with $\| c_j \|_{\ell^d(Z)} \leq 1$. In the above, we denote

$$[u, \Delta_j] \cdot \nabla v = \sum_{1 \leq i \leq d} u^i \Delta_j \partial_i v - \Delta_j (u^i \partial_i v).$$

If $\rho = 1$, $\| \nabla u \|_{L^1_t(B^d_{p,q} - \rho - 1)}$ has to be replaced by $\| \nabla u \|_{L^1_t(B^d_{p,q} - \rho - 1) \cap L^1_t(L^\infty)}$. If $\gamma = -1$, $\| \nabla v \|_{L^2_t(B^d_{p,q} - \gamma - 1)}$ has to be replaced by $\| \nabla v \|_{L^2_t(B^d_{p,q} - \gamma - 1) \cap L^1_t(L^\infty)}$.

**Proof:** The proof is a straightforward adaptation of Lemma A.1 in [16] which is a version of the commutator estimate in Besov space.

**Acknowledgements** We would like to thank Professors H. Smith and T. Tao so much for their helpful discussion and suggestions. The authors are also deeply grateful to the referee for their valuable advices. Q. Chen and C. Miao were partly supported by the National Natural Science Foundation of China.

**References**

[1] Bergh J., L"ofstrom J.: Interpolation spaces, An Introduction, New York: Springer-Verlag, 1976.

[2] Bony J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. **14**, 209-246(1981).

[3] Cannone M., Planchon F.: More Lyapunov functions for the Navier-Stokes equations, in Navier-Stokes equations: Theory and Numerical Methods, R. Salvi Ed., Lectures notes in pure and applied Mathematics, Marcel-Dekker **223**, 19-26(2001).

[4] Chae D.: The quasi-geostrophic equation in the Triebel-Lizorkin spaces. Nonlinearity **16**, 479-495(2003).

[5] Chae D., Lee J.: Global well-posedness in the super-critical dissipative quasi-geostrophic equations. Commun. Math. Phys. **233**, 297-311(2003).

[6] Chemin J.-Y.: Perfect incompressible fluids. Oxford University Press, New York, 1998.

[7] Chemin J.-Y., Lerner N.: Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. J. Differential Equations **121**, 314-328(1995).
[8] Constantin P., Majda A. J., Tabak E.: Formation of strong fronts in the 2-D quasi-
geostrophic thermal active scalar. Nonlinearity 7, 1495-1533(1994).

[9] Constantin P., Wu J.: Behavior of solutions of 2D quasi-geostrophic equations. SIAM
J. Math. Anal. 30, 937-948 (1999).

[10] Constantin P., Cordoba D., Wu J.: On the critical dissipative quasi-geostrophic equa-
tion. Indiana Univ. Math. J. 50, Special Issue, 97-107 (2001).

[11] Córdoba D.: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equa-
tion. Ann. of Math. 148, 1135-1152(1998).

[12] Córdoba D., Fefferman C.: Growth of solutions for QG and 2D Euler equations. J. Amer. Math. Soc. 15, 665-670(2002).

[13] Córdoba A., Córdoba D.: A maximum principle applied to quasi-geostrophic equations.
Commum. Math. Phys. 249, 511-528(2004).

[14] Danchin R.: Poches de tourbillon visqueuses. J. Math. Pures Appl. 76(9), 609-
647(1997).

[15] Danchin R.: Global existence in critical spaces for compressible Navier-Stokes equations.
Invent. Math. 141, 579-614(2000).

[16] Danchin R.: Density-dependent incompressible viscous fluids in critical spaces. Proc.
Roy. Soc. Edinburgh Sect. A 133, 1311-1334(2003).

[17] Ju N.: Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic
equations in the Sobolev space. Commun. Math. Phys. 251, 365-376(2004).

[18] Ju N.: The maximum principle and the global attractor for the dissipative 2D quasi-
geostrophic equations. Commun. Math. Phys. 255,161-181(2005).

[19] Lemarié-Rieusset P. G.: Recent developments in the Navier-stokes problem. Chapman
& Hall/CRC, London, 2002.

[20] Miura H.: Dissipative quasi-geostrophic equation for large initial data in the critical
Sobolev space, Commun. Math. Phys., in press.

[21] Pedlosky J.: Geophysical Fluid Dynamics. New York: Springer-Verlag, 1987.

[22] Teman R.: Navier-Stokes equations, Theory and Numerical analysis. New York: North-
Holland, 1979.

[23] Triebel H.: Theory of Function Spaces. Monograph in mathematics, Vol.78, Basel:
Birkhauser Verlag, 1983.

[24] Wu J.: Global solutions of the 2D dissipative quasi-geostrophic equation in Besov
spaces. SIAM J. Math. Anal. 36, 1014-1030(2004).

[25] Wu J.: The two-dimensional quasi-geostrophic equation with critical or supercritical
dissipation. Nonlinearity 18, 139-154(2005).
[26] Wu J.: Lower bounds for an integral involving fractional laplacians and the generalized Navier-Stokes equations in Besov spaces. Commun. Math. Phys. 263, 803-831 (2006).