Non-modal linear stability analysis of miscible viscous fingering in a Hele-Shaw cell

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Abstract

For miscible viscous fingering (VF) in a Hele-Shaw cell or in two dimensional homogeneous porous media, the transient growth of disturbances is investigated by non-modal linear stability analysis (NMA). Due to the non-autonomous nature of the linearized perturbed equations, the linear stability theory prohibits using the normal mode analysis. The linearized perturbed equations for Darcy’s law coupled with a convection-diffusion equation is discretized using finite difference method. The resultant matrix valued initial value problem is then solved by fourth order Runge-Kutta method, followed by a singular value decomposition (SVD) of the propagator matrix. We demonstrate the dominant perturbation that experiences the maximum amplification within the linear regime which lead to the transient growth. This feature was previously unattained in the existing linear stability methods for miscible VF. To explore the relevance of the optimal perturbation obtained from non-modal analysis of the physical system, we performed direct numerical simulations (DNS) using a highly accurate pseudo-spectral method. From DNS results it is observed that at early time the amplification of the perturbation decays before it starts growing, and this completely agrees with the NMA results. It is also shown that the onset of instability in the linear regime obtained by the non-modal theory is meticulously agreed with that of the DNS results, as compared to other existing linear stability analyses.
I. INTRODUCTION

The Saffman-Taylor or VF instability, which occurs when a less viscous fluid is injected into a more viscous fluid, is a classical hydrodynamic instability phenomena\[1, 2\]. Such instability occurs in many physical processes such as, enhanced oil recovery\[2\], spreading of surfactant coated thin liquid films\[3\], liquid chromatography\[4\], dispersion in aquifers\[5\] etc. VF also has been used as an effective mechanism to enhance the mixing of two fluids in a micro channel \[6\]. The VF instability for miscible fluids were studied experimentally and theoretically by many researchers, such as Hill \[7\], Slobod and Thomas \[8\], Perkins et al. \[9\], Tan and Homsy \[10\] are few to name. In recent times many substantial contributions \[11–15\] have been made in terms of developing new mathematical tools or numerical recipes for better understanding of this phenomenon. The main mathematical challenge arises in the linear stability analysis (LSA) of miscible VF due to the unsteady base state. Thus to analyze its LSA one needs to solve a non-autonomous system of the linearized differential operators. For the time-periodic problems one can invoke the Floquet theory \[16\]. But in an arbitrary time-dependent problem, Farrell and Ioannou \[17\] observed that the disturbances can no longer be set proportional to exp(\(\sigma t\)), where \(\sigma\) is the growth rate. As a result, the linear stability theory of the non-autonomous system prohibits to use normal mode analyses.

The following two approaches are common in practice for analyzing the linear stability of miscible VF to find the onset of fingering: (a) quasi steady state approximation (QSSA) and (b) initial value problem (IVP). In QSSA, one reduces to an autonomous system by freezing the base state at some specific time, preceded by the assumption that the base state evolves much more slowly than the perturbations and apply modal analysis, while in IVP approach one can find the full solution of the non-autonomous linearized problem for some representative initial conditions. Tan and Homsy\[10\] compared IVP calculations and non-linear simulations with the predictions from QSSA and reported that QSSA does not perform well at early times. Although the IVP-approach predicts the early time behavior
better than QSSA, there are two major challenges in this method. The first one being how large the disturbances must grow before they become observable, and the other one is the choice of representative initial condition. The chosen random initial condition may not necessarily correspond to the one that gives optimum growth of the perturbation. Also, there is much disagreement on how the growth of the perturbations to be measured from the IVP analysis. Moreover, the random initial conditions, which are supposed to be localized within the diffusive layer, perturb the system over the entire computational domain.

Ben et al. [11] performed a linear stability analysis for viscous fingering instability using a spectral analysis method without invoking QSSA assumption in a self-similar coordinate, so that the diffusing base state solution to the problem is frozen in time, and the resultant system is solved numerically. Recently, Kim [12] analyzed the viscous fingering of a miscible slice and compared the predicted growth rates of QSSA with those obtained using the spectral analysis method. He found that the spectral analysis method predicts the system to be initially unconditionally stable and becomes unstable at later time, while in contrary, QSSA [10] prediction reveals an initially unstable situation. As time progresses the predicted growth rates of the two methods get closer. A similar approach was also adopted by Pramanik and Mishra [14] to study the effect of the Korteweg stresses on miscible VF using a self-similar QSSA (SS-QSSA).

However, the eigenmodes obtained from both SS-QSSA and QSSA are non-orthogonal, which indicates the possibility for transient growth of the disturbances [18, 19]. It is a well-known fact that modal analysis does not address the phenomenon of transient growth in time [19]. In particular, it depends on the spectral properties of the underlying linear operator. For a non-normal operator (i.e., the operator that does not commute with its adjoint) an initial perturbation can grow in time by large factors before decaying, even if all the eigenmodes of the operator are damping. Earlier observations of transient growth due to the non-normality of the linearized operators were reported in the studies of instability in parallel shear flows [20, 21], many atmospheric and laboratory flows [17, 22] and in a review
article of Govindarajan and Sahu\cite{15}(also see the references therein). Most importantly, in all these hydrodynamic instability problems, the transient growth of infinitesimal small perturbations has been studied about a steady base state.

In recent times, few studies \cite{3, 23, 26} have been done for the unsteady base flow using the transient growth analysis. However, the literatures pertinent to the present study of viscous fingering phenomenon are discussed briefly in the following paragraph. Rapaka \textit{et al.} \cite{27, 28} used NMA to determine the optimal perturbations with maximum amplification for density driven fingerings by singular value decomposition (SVD) method. Later, Doumenc \textit{et al.} \cite{29} and Daniel \textit{et al.} \cite{30} used the ‘direct-adjoint looping’ (DAL) method \cite{18, 22} for studying the transient growth of perturbations in Rayleigh-B´ernard-Marogani convection and density driven fingerings, respectively. Additionally, Daniel \textit{et al.} \cite{30} observed that the onset of convection in physical systems is due to suboptimal perturbations localized within the diffusive layer and hence proposed a modified optimization procedure (MOP).

Despite the fact that transient growth analysis is known in the various hydrodynamic instability problems mentioned above, to the best of authors’ knowledge, the transient growth of the perturbations is yet to be investigated for miscible VF. The non-autonomous linear equations and the non-orthogonal SS-QSSA eigenmodes, which may lead to the transient growth of perturbations, motivate us to pursue the non-modal analysis in such a system. To quantify the maximum amplification of the perturbations we solve a matrix valued IVP, obtained from Darcy’s equation coupled with a convection-diffusion equation, by fourth order explicit Runge-Kutta method to obtain the propagator matrix. Using SVD of the propagator matrix, the singular values and the right singular vectors are obtained. The obtained singular value and right singular vector provide the optimal amplification and the optimal initial conditions, respectively. At the early time, a substantial transient growth of the perturbations is observed to exist in miscible VF, which was not shown in the existing literature \cite{11, 12, 14} of miscible VF. We also perform the direct numerical simulations using a highly accurate Fourier spectral method \cite{34}. The results of the amplifications obtained
FIG. 1: Schematic of the flow configuration with coordinate system. Initially the interface is flat (dotted line) and then a wave like infinitesimal small perturbation is applied.

by non-modal theory are in very good agreement with those observed from the DNS.

The paper is organized as follows: The governing equations along with appropriate boundary and initial conditions are presented in Sec. II. The rudiments of classical non-modal linear stability analysis is described, along with numerical scheme in Sec. III. This part also describes the non-modal approach via singular value decomposition method. Sec. IV contains the details about the DNS computations and non-linear growth rate. Sec. V discusses our numerical findings from NMA with previously studied linear stability analyses of VF phenomenon as well as the DNS.

II. MODEL AND FORMULATION

The classical rectilinear displacement of miscible fluids in a Hele-Shaw cell is considered [10]. The schematic of the flow is given in Fig. 1. Fluids are assumed to be neutrally buoyant, incompressible and the porous medium or Hele-Shaw cell is homogeneous with constant permeability and the dispersion is isotropic.
A. Governing Equations

The non-dimensional governing equations for the prescribed two dimensional flow in a reference frame moving with velocity $U$ are given by

\begin{align}
\nabla \cdot \mathbf{u} &= 0, \quad (1) \\
\nabla p &= -\mu(c)(\mathbf{u} + e_x), \quad (2) \\
\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c &= \nabla^2 c, \quad (3)
\end{align}

where $\mathbf{u} = (u, v)$ is the Darcy’s velocity, $c$ is the concentration of the solvent, $\mu(c)$ is the viscosity, and $e_x$ is the unit vector along the $x-$ direction. The characteristic scale used for velocity, length and time are $U$, $D/U$, and $D/U^2$, respectively, where $D$ is the dispersion tensor (which is assumed to be constant in all spatial directions) and $U$ is the uniform velocity at which the fluid with viscosity $\mu_1$ displaces the other fluid of viscosity $\mu_2$ to the right. The only parameter that entering the dimensionless viscosity-concentration relation is the log-mobility ratio $R$, defined as, $R = \ln(\mu_2/\mu_1)$, and the viscosity and concentration are related by an Arrhenius type relationship, i.e., $\mu(c) = \exp(Rc)$. The coupled Eqs. (1) - (3) are associated with following initial and boundary conditions

Initial conditions:

\begin{align}
\mathbf{u} &= (0, 0), \quad c(x, y, t = 0) = \begin{cases}
0, & x \leq 0 \\
1, & x > 0
\end{cases}, \forall y, \text{ the Heaviside unit step function.} \quad (4)
\end{align}

Boundary conditions:

\begin{align}
\mathbf{u} &= (0, 0), \quad \frac{\partial c}{\partial x} = 0, \quad |x| \to \infty, \text{ streamwise direction,} \quad (5) \\
\frac{\partial c}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 0, \forall x, \text{ spanwise direction.} \quad (6)
\end{align}

B. Base State

The base state of the flow is assumed to be pure diffusion of the concentration along the axial direction (i.e., no advection $\mathbf{u} = 0$). Hence, the time dependent base state solution of
Eqs. (1)-(3) on infinite domain with the no-flux boundary conditions for the concentration is as follows,

\begin{align*}
    u_0 &= (u_0, v_0) = 0, \quad (7) \\
    \mu_0 &= \mu_0(c_0) = \mu(x, t), \quad (8) \\
    p_0(x, t) &= -\int_{-\infty}^{x} \mu_0(s, t) ds, \quad (9) \\
    c_0(x, t) &= \frac{1}{2} [1 + \text{erf}(x/2\sqrt{t})], \quad (10)
\end{align*}

where \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt \) is the error function. On transforming the current coordinates \((x, y, t)\) to a new coordinate system \((\xi, y, t)\), the base state concentration (Eq. (10)) becomes

\[ c_0(\xi) = \frac{1}{2} [1 + \text{erf}(\xi/2)], \quad (11) \]

where \( \xi := x/\sqrt{t} \) is the similarity variable transformation. Advantage of working with \((\xi, y, t)\) coordinate systems is that, one can pretend that as if the base state is time independent.

III. LINEAR STABILITY ANALYSIS

In this section different linear stability techniques, non-modal theory, IVP, and the numerical methods used to solve the resultant linear problems are presented. For understanding the non-normality of the disturbance matrices obtained from linear stability theory, the numerical abscissa and the spectral abscissa is discussed. The response of the system to small disturbances is obtained by linearizing Eqs. (1) - (3) according to

\[ u(\xi, y, t) = u_0(\xi) + u'(\xi, y, t), \quad c(\xi, y, t) = c_0(\xi) + c'(\xi, y, t), \quad (12) \]

where \( u'(\xi, y, t), c'(\xi, y, t) \) represent infinitesimal disturbances in the velocity and concentration, respectively. We seek disturbances of the form \( (u', c')(\xi, y, t) = (u', c')(\xi, t)e^{i\kappa y} \), describing a function which propagates and evolves in time in the streamwise direction while exhibiting sinusoidal character in the transverse direction. Using viscosity-concentration
relationship $\mu(c) = e^{Rc}$, the disturbances $u'$ and $c'$ are determined from the following set of equations representing a disturbance of transverse wave number $k$,

$$\left[ D + \frac{R}{2\sqrt{\pi}} \exp \left( -\frac{\xi^2}{4} \right) \frac{\partial}{\partial \xi} \right] u'(\xi, t) = k^2 Rtc'(\xi, t), \tag{13}$$

$$\left[ t \frac{\partial}{\partial t} - \xi \frac{\partial}{\partial \xi} - D \right] c'(\xi, t) = -\frac{\sqrt{t}}{2\sqrt{\pi}} \exp \left( -\frac{\xi^2}{4} \right) u'(\xi, t), \tag{14}$$

where $D := \frac{\partial^2}{\partial \xi^2} - tk^2$. The associated boundary conditions are $c', u' \to 0, |\xi| \to \infty$. Note that $u', c'$ are not periodic in the ‘$\xi$’ variable due to the far field conditions i.e., $c', u' \to 0, |\xi| \to \infty$. So the functions $u', c'$ cannot be represented as a superposition of sinusoidal perturbations in the ‘$\xi$’ variable. The analytic solution for the coupled system of partial differential Eqs. (13) - (14), for a diffusive base state is unattainable. Hence, a numerical solution method need to be used for solving these coupled equations. In this paper finite difference method is adopted.

A. Numerical Scheme

For a given $k$ and $R$, we discretized Eqs. (13) - (14) over a finite computational domain $[-100, 100]$, with a uniform mesh size $h$ and $n + 2$ number of grid points. All the derivatives were approximated by central difference formula. The discretized version of Eqs. (13) - (14) with the boundary conditions $u'(t) \big|_{i=0} = 0 = c'(t) \big|_{i=0}$ and $u'(t) \big|_{i=n+1} = 0, c'(t) \big|_{i=n+1} = 1$ can be cast into the form

$$\frac{dq(t)}{dt} = A(t)q(t), \tag{15}$$
where \( q_i(t) = c'(\xi_i, t) \) and \( A(t) = M_3 + M_4 M_1^{-1} M_2 \), an \( n \times n \) matrix. Here each matrix \( M_j, j = 1, 2, 3, 4 \) is either a diagonal matrix or a tridiagonal matrix and are given by,

\[
M_1(i, j) = \begin{cases} 
\frac{1}{h^2} - \frac{R}{4h^2} (c_0(j + 1) - c_0(j - 1)), & i = j + 1 \\
-\frac{2}{h^2} - k^2 t, & i = j \\
\frac{1}{h^2} + \frac{R}{4h^2} (c_0(j + 1) - c_0(j - 1)), & i = j - 1, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
M_2(i, j) = \begin{cases} 
k^2 R t, & i = j \\
0, & \text{otherwise,}
\end{cases}
\]

\[
M_3(i, j) = \begin{cases} 
\frac{1}{h^2} - \frac{\xi(j)}{4h t}, & i = j + 1 \\
-\frac{2}{h^2} - k^2, & i = j \\
\frac{1}{h^2} + \frac{\xi(j)}{4h t}, & i = j - 1, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
M_4(i, j) = \begin{cases} 
-\left( \frac{c_0(j+1) - c_0(j-1)}{2\sqrt{t}} \right), & i = j \\
0, & \text{otherwise.}
\end{cases}
\]

The linearized operator \( A(t) \) is non autonomous since it explicitly contains the time \( t \), besides being dependent on the temporal behavior of the base states. For the computational purpose the unstable interface is set at \( \xi = 0 \). The initial value problem, Eq. (15) is solved using Runge-Kutta fourth order explicit method for time integration with a relative error of order \( O(10^{-10}) \). The matlab built-in routine \texttt{ode45} is used to implement the Runge-Kutta method for solving the IVPs, and sub-routine \texttt{eigs} is used to finding the eigenvalues in modal analysis.

In order to avoid the possible appearance of non-conjugate pair of complex eigenvalues all the numerical computations are carried out with a quadruple precision, instead of double precision floating points. Computational domain is taken sufficiently large so that the numerical results remain unaffected by the truncation of infinite physical domain into finite domain in numerical algorithms. To ensure that our numerical quantities, \textit{viz.} singular values, eigenvalues, etc., that characterize the instability, are independent of spatial domain a series of numerical calculations performed for different \( L \) ranging from 50 to 120. The
FIG. 2: Dependence on the size of the computational domain on growth rate obtained from NMA, for $R = 3$, $k = 0.2$. Beyond the domain length $L = 90$, there are almost no change.

Parameter values are chosen to be $R = 3, k = 0.2$, because this corresponds to the most dangerous mode determined by QSSA [10]. The obtained numerical results are compared in Fig. 2 and it is observed that for $L \geq 90$ there is no change in the growth rates calculated from NMA which can be found from the explanations in Sec. III D. Thus, for optimal result the computational domain is chosen $[-100, 100]$ in all our calculations. Numerical convergence of finite difference method has been tested with different refinement of grid points, and the optimal computational time and accuracy were obtained with the spatial step size 0.2. As a verification of our numerical code we reproduce the results of Rapaka et al. [27]. Due to the possible singularity at $t = 0$, all the numerical integrations are performed starting from the initial time $t = 0.001$ to various final time. Although we restrict our study of the optimal perturbations and maximum amplification for $R = 3$, the analogous transient growth dynamics of perturbations can be observed for other positive values of $R$. The next
section (Sec. III B) describes the importance of NMA for the present problem by using the discretized Eq. (15) and explain about the results of IVP calculations in \((x,t)\) and \((\xi,t)\) co-ordinate systems.

**B. Initial Value Calculation**

For a given \(k\) and \(R\), the spatial discretization of Eqs. (13)-(14) using central difference scheme yields the non-autonomous system (similar equation can be obtained in \((x,t)\) coordinate system)

\[
\frac{dc'(t)}{dt} = A(t)c',
\]

were \(A(t)\) is the linearized disturbance matrix. Eq. (16) is solved with following random initial condition for both \((x,t)\) and \((\xi,t)\) coordinates :

\[
c'(\omega,t_0) = \delta \ast \text{rand}(\omega), \text{ where } \omega = \xi \text{ or } x,
\]

where \(\text{rand}(\omega)\) represents a random number generator between \(-1\) and \(1\), and an infinitesimal parameter \(\delta\) determines the amplitude of the perturbation, which is chosen to be \(O(10^{-3})\).

The objective of performing IVP in both the coordinate systems is to describe the structure of perturbed concentration in each coordinate system. The initial time \(t_0 = 10^{-5}\) has been chosen for comparing the structure of perturbed concentration obtained from \((x,t)\) and \((\xi,t)\) coordinates. Fig. 3(a) depicts that the concentration eigenfunction determined by solving the IVP in \((\xi,t)\) domain converges rapidly to the dominant eigenmode \(\exp(-\xi^2/4)\) [11].

On the other hand the concentration eigenfunction obtained from the IVP calculation in \((x,t)\) coordinate system took longer time to converge to the dominant mode, in comparison to calculations in \((\xi,t)\) coordinate system (See Fig. 3(b)). Both the results shown in Fig. 3 are obtained by averaging over 10 random initial conditions. Thus, the localized eigenfunctions in \((\xi,t)\) coordinates converge rapidly to the exact solution, thereby giving an accurate disturbance growth rate at small times. With this observation, rest of the numerical
FIG. 3: Disturbance concentration profile for $R = 3, k = 0$ at different times, obtained from the IVP in (a) $(\xi, t)$ coordinate system, (b) $(x, t)$ coordinate system.

calculations are performed in the self-similar $(\xi, t)$ domain. We perform IVP calculations in $(\xi, t)$ domain with dominant eigenmode $\exp(-\xi^2/4)$ as the initial condition and the obtained results are discussed in section V B 1.

C. Spectral Abscissa and Numerical Abscissa

The behavior of the eigenvalues of the linearized disturbance matrix $A(t)$ determine the instability of a given perturbation. But, for the highly non-normal operators, e.g., in bounded shear flows [19, 20], the classical normal mode analysis, which assumes an exponential time dependence of $A(t)$, fails to predict the instability appropriately. The temporal eigenmodes of the time dependent operator $A(t)$ are not defined [17] and to determine the asymptotic behavior (by the Lyapunov exponents) of Eq. (16) is beyond the scope of this paper. To quantify the non-normality of $A(t)$, we freeze $A(t)$ at different times and consider following two quantities namely the spectral abscissa and the numerical abscissa, denoted by $\alpha(A(t))$.
FIG. 4: (Color online) Spectral abscissa (red-circles) and numerical abscissa (blue-squares) for \( k = 0.2, \ R = 3, \) at different time (a) \( t = 10^{-8}, \) (b) \( t = 10^{-5}, \) (c) \( t = 1, \) (d) \( t = 10 \) . Also shown the eigenvalue with largest real part (black dot) and imaginary axis as bold continuous vertical line.

and \( \eta(A(t)) \), respectively, and are defined as,

\[
\alpha(A(t)) := \max\{\Re(\lambda(A(t)))\},
\]

\[
\eta(A(t)) := \max\{\Re(\lambda(A(t) + A'(t))/2)\}
\]

Here \( \lambda(\cdot) \) represents the spectrum of respective matrices and \( \Re(\cdot) \) denotes the real part.

It can be verified that for a normal matrix \( \alpha(A(t)) = \eta(A(t)) \) and the main objective of modal analysis is to study the spectral abscissa and the corresponding eigenmodes. But in a non-normal system, spectral abscissa only give asymptotic stability. Using the matlab GUI EigTool \[33\], the spectral and numerical abscissa for the evolution difference matrix \( A(t) \) are plotted in Fig. 4 at different time with fixed wave number \( k = 0.2 \) and \( R = 3 \). It is observed from Figs. 4(a) and 4(b) that at early times, the spectral abscissa and the numerical abscissa differ from each other in the order of \( 10^7 \) and \( 10^4 \). This reveals that at early time, these disturbance operators are highly non-normal. At later time \( A(t) \) is very close to a normal matrix (see Figs. 4(c) and 4(d)). This early time structure of non-normality of \( A(t) \) inspire us to investigate the transient growth of perturbations and the onset time of fingering, which
shall be presented in Sec. V B.

D. Non-modal Stability Analysis

In this section we give a brief description of non-modal linear stability theory and explain the necessary mathematical background for calculating the growth rate and onset time of perturbations. Let us assume a solution of Eq. (16) as

$$c'(t) := \Phi(t_0; t)c'_0,$$  \hspace{1cm} (20)

where $c'(t_0) = c'_0$ is an arbitrary initial condition. Here $\Phi(t_0; t)$ is called the propagator matrix, because it propagates the information forward from the initial time $t_0$ to time $t$. On substituting Eq. (20) into Eq. (16) we obtain a matrix differential equation,

$$\frac{d}{dt}\Phi(t_0; t) = A(t)\Phi(t_0; t),$$  \hspace{1cm} (21)

with the initial condition $\Phi(t_0; t_0) = I$, where $I$ is the identity matrix of the same order of the matrix $A(t)$. Thus instead of solving a vector differential equation with random initial condition, now it needs to solve a matrix differential equation, Eq. (21), with deterministic initial condition, i.e., $\Phi(t_0; t_0) = I$.

The stability analysis is performed based on the amplification magnitude of the perturbations over a prescribed finite time interval. In the stability analysis of a physical system with time dependent base state, the growth or decay of disturbances are only meaningful in reference to the evolved base state. In 1961 Shen observed in his study of unsteady parallel shear flow that if both the disturbance and the base state are decaying, but the latter one with faster rate, the disturbance relative to the basic state would appear to be amplified at a later instant. Conversely, if a disturbance grows in time, but the base state grows faster than that, then the disturbance will appear to decay in time. So it is necessary that the growth or decay of a disturbance should be measured only by a comparison with the basic flow. With these observations Shen introduced “momentary stability” and defined
the appropriate measure for stability analysis. Similar measures have also been adopted by Matar and Troian [3, 24] in their studies of spreading of surfactant on a thin film and Bestehorn and Firoozabadi [25] in their study of the dissolution of CO$_2$ in saline aquifers.

Following the work of Shen [23], we would like to introduce the quantities such as amplification magnitude and transient growth rate of the disturbances that will be used to measure the time dependent growth rate of the perturbations. The perturbation magnitude for the flow variable $f$ can be defined as

$$M_f(t) := \frac{E_f(t)}{E_{f_0}(t)},$$  \hspace{1cm} (22)

where $f_0 = c_0$, or $u_0$, and $f = c'$, or $u'$ and $E_f(t) := \|f(t)\|^2 = \int_{-\infty}^{\infty} f^2(x,t)dx$, $E_{f_0}(t) := \|f_0(t)\|^2 = \int_{-\infty}^{\infty} f_0^2(x,t)dx$. The sensitivity of the infinitesimal disturbances introduced at time $t_0$ is measured from the ratio of the perturbation magnitude $M_f(t)$ at time $t$ to its initial value $M_f(t_0)$. Consider the normalized amplification $\Psi_f(t)$ defined by

$$\Psi_f(t) := \frac{M_f(t)}{M_f(t_0)} = \frac{\left[ \frac{E_f(t)}{E_f(t_0)} \right]}{\left[ \frac{E_{f_0}(t)}{E_{f_0}(t_0)} \right]} = \frac{G_f(t)}{G_{f_0}(t)}.$$ \hspace{1cm} (23)

The time-dependent growth rate is defined as [3, 24, 25] \hspace{1cm} 

$$\sigma(t) = \frac{1}{\Psi_f(t)} \frac{d\Psi_f(t)}{dt} = \frac{1}{G_f} \frac{dG_f}{dt} - \frac{1}{G_{f_0}} \frac{dG_{f_0}}{dt} = \sigma_f(t) - \sigma_{f_0}(t).$$ \hspace{1cm} (24)

As $c_0$ in $(\xi,t)$ coordinate is independent of time and $u_0 = 0$, this implies $\sigma_{f_0}(t) = 0$, further Eq. (24) reduces to

$$\sigma(t) = \frac{1}{G_f} \frac{dG_f}{dt}.$$ \hspace{1cm} (25)

In order to implement the non-modal linear stability analysis, the propagator matrix approach of Rapaka et al. [27, 28] is used. With the limitation of such propagator operator approach, only the optimal amplification associated with the concentration disturbances is used to find the growth rate. To measure the transient growth, we consider the perturbation magnitude at time $t$ normalized to the perturbation magnitude at $t_0$. As the perturbation
equations are linear, without loss of generality it can be assumed that $\| c'(t_0) \| = 1$ and using Eq. (20), Eq. (23) reduces to

$$
\Psi'(t) = E'(t)/E'(t_0) = \| c'(x, t) \|^2 = \langle \Phi(t_0; t)^* \Phi(t_0; t)c', c' \rangle, \tag{26}
$$

where $\langle \cdot, \cdot \rangle$ is the standard $L^2$ inner product. In non-modal analysis, we would like to find the perturbations that lead to the maximal amplification. Thus the objective of study in NMA is to find the amplification that optimized over all possible initial conditions, i.e.,

$$
G(t) = G(t, k, R) := \max_{c'_0} \Psi'(t) = \max_{c'_0} \| \Phi(t_0; t)c'_0 \|_2 = \| \Phi(t_0; t) \| = \sup_j s_j(t), \tag{27}
$$

where $s_j$’s are the singular values of $\Phi(t_0; t)$ that is the eigenvalues of $\Phi(t_0; t)^* \Phi(t_0; t)$ and can be found by singular value decomposition (SVD) of $\Phi(t_0; t)$. Thus, the quantities that determine the optimal amplification are eigenvalues of $\Phi(t_0; t)^* \Phi(t_0; t)$. It must be noted that we do not calculate the $E_c'(t)$ explicitly, instead, we are using the classical definition of amplification magnitude of disturbances to finding the optimum amplification and the optimum perturbations by SVD. It may be noted that there exist other possible ways to describe such transient growth instability [30, 37]. For example, using DAL approach Daniel et al. [30] modified the definition of $E_f(t)$ (given in Eq. (22)) to accommodate both concentration and velocity disturbances i.e., $E_c(t) := \int_{-\infty}^{\infty} [c^2(x, t) + u^2(x, t)] \, dx$ for density-stratified flow. It was concluded that for such flow problem, the amplification measure $E_c'(t)$ was sufficient for their stability analysis. In an another study of Rayleigh-Bénard-Maragoni convection, Doumenc et al. [29] observed that the velocity perturbations are slave to the temperature perturbation when the time derivative of velocity was dropped out from their model equations. Hence they discussed the stability criteria based on the optimal amplification associated to the temperature disturbances only. Thus following the above observations and since the linearized equations (Eqs. (13)-(14)) do not contain the explicit time derivative of velocity, the concentration disturbances are pertinent in the present problem. Hence the optimal amplification associated to the concentration disturbances is used for the non-modal
stability analysis. Below we define the growth rate and the critical time for the quantitative investigation of the transient growth of perturbations.

**Definition III.1** (Stability criterion and Growth rate). The instantaneous growth rate $\sigma$ at time $t$ of a mode with wave number $k$ can be defined as:
\[
\sigma = \sigma(R, k, t) := \frac{1}{G(t)} \frac{dG(t)}{dt}.
\]  
(28)
Thus from Eq. 28, a given perturbation is stable if $\sigma(t) < 0$ and unstable for $\sigma(t) > 0$. With this definition of growth rate the critical time or the onset of instability for the given perturbation can be found.

**Definition III.2** (Critical time). For a given perturbation the critical time is denoted by $t_c(k, R)$ and can be defined as the first instance of time when $\frac{dG(t)}{dt} \bigg|_{t=t_c} = 0$.

As noted by Doumenc et al. [29] that the optimum amplification (see Eq. (27)) has to be determined with subject to the following constraints, (a) the disturbance energy at initial time $t_0$ is equal to unity, (b) the disturbance must satisfy the linear governing equations along with the boundary conditions in the time interval $[t_0, t]$. Because of a possible singularity at $t = 0$ of the difference matrix $A(t)$ in Eq. (16), the starting time $t_0$ cannot be zero. With this observation i.e., $G(t_0) = \| \Phi(t_0; t_0) \| = 1$, from Eq. (28), a relationship between the optimum amplification $G(t)$ and the growth rate $\sigma(t)$ can be obtained as [26]
\[
G(t) = \exp \left[ \int_{t_0}^{t} \sigma(s) ds \right].
\]  
(29)

**IV. DIRECT NUMERICAL SIMULATIONS**

In this section we look to validate the transient growth obtained from NMA by solving the coupled nonlinear equations. We solve the nonlinear problem using a highly accurate pseudospectral method based on vorticity-stream function formulation of Eqs. [1-3], proposed by Tan and Homsy [34]. This method has been employed successfully to obtain highly
accurate results for various viscous fingering problems [5]. Writing the unknown variables as, $\psi(x,y,t) - \psi_0(x,t) = \psi'(x,y,t)$, $\omega(x,y,t) - \omega_0(x,t) = \omega'(x,y,t)$, $c(x,y,t) - c_0(x,t) = c'(x,y,t)$, the vorticity-stream function form of Eqs. (1)-(3) in terms of the perturbation quantities can be represented by,

\[
\nabla^2 \psi' = -\omega', \quad (30)
\]

\[
\omega' = -R \left( \nabla \psi' \cdot \nabla (c_0 + c') + \frac{\partial c'}{\partial y} \right), \quad (31)
\]

\[
\frac{\partial c'}{\partial t} + \frac{\partial \psi'}{\partial y} \left( \frac{\partial c_0}{\partial x} + \frac{\partial c'}{\partial x} \right) - \frac{\partial \psi'}{\partial x} \frac{\partial c'}{\partial y} = \nabla^2 c', \quad (32)
\]

where the vorticity $\omega = (\nabla \times \mathbf{u}) \cdot e_z$, $e_z$ being the unit vector normal to the $(x,y)$ plane. Here the base-state flow is the same as discussed in Sec. II B, i.e. $\psi_0 = \omega_0 = 0$, $c_0$ is given by Eq. (7) and the primes correspond to their perturbation quantities (not necessarily infinitesimal).

The non-dimensional width of the computational domain becomes $\text{Pe} = UH/D$, the Péclet number and the corresponding length of the domain is $A \cdot \text{Pe}$, where $A = L/H$, and $L,H$ as depicted in Fig. 1. Periodic boundary conditions are applied in the transverse direction. In the longitudinal direction, we work with the periodic extension of a displacement front. In this paper the computational domain is chosen in such a way that $\text{Pe} = 512$ and $A = 8$ with $1024 \times 128$ grid points discretizing the domain. The time integration is performed by taking time stepping $10^{-3}$.

The simulations are performed with $t_0 = 0.1$ and initial concentration perturbation of the form,

\[
c'(x,y,0) = \epsilon \text{rand}(\cdot) \sin(ky). \quad (33)
\]

Here, $k$ is the wave number, $\text{rand}(\cdot)$ represents a random number between 0 and 1 at the diffusive interface, $\epsilon$ is the amplitude of the perturbation, which is taken to be $10^{-2}$. Assuming that the perturbations grow exponentially, the growth rate of perturbations, $\sigma_{DNS}$, can be computed at every time from,

\[
\sigma_{DNS} = \frac{1}{2} \frac{d[\ln(E_{DNS}(t))]}{dt}. \quad (34)
\]
Here, $E_{DNS}(t) \equiv \int_{0}^{A-Pe} \int_{0}^{Pe} [c'(x, y, t)]^2 dx dy$ represents the magnitude of the perturbed concentration. The DNS results are averaged over ten different random realizations of the initial concentration perturbations, Eq. (33). For our analysis DNS are performed for seven different wave numbers and the results obtained are discussed in section V B.

V. NUMERICAL RESULTS AND DISCUSSION

A. Findings from non-normal analysis

First we explain the optimal amplification of a given perturbation measured from $(x, t)$ and $(\xi, t)$ coordinate systems by non-modal analysis. For the unsteady base state $c_0$ (see Eq. (10)), given a value of log-mobility ratio $R$, time $t$ and wave number $k$, the non-modal analysis determines the optimum amplification over all possible perturbations. Then the spatial structure of the obtained optimal perturbations is described. We illustrate that the optimal perturbations are localized about the base state, which is in contrary to the random initial perturbations that perturbs the system across the entire domain.

1. Amplification

Fig. 5(a) illustrates the optimal amplification obtained from both $(x, t)$ and $(\xi, t)$ coordinate systems. It is observed that in $(x, t)$ coordinate system any infinitesimal perturbation amplifies immediately. But for a physical system under present study, the effect of diffusion is prominent at early times, hence the amplitude of the perturbations must decay before it starts growing. This physical characterization of the system is well captured by NMA in $(\xi, t)$ coordinate system. This is a possible reason for linear stability analysis in the self-similar coordinate system $(\xi, t)$ to give a reasonable explanation to physical situation. Since definition of amplification $G(t)$ (see Eq. (27)) is obtained by taking all admissible perturbations, it includes the mode that provide maximum growth in SS-QSSA. Therefore, we need
FIG. 5: (a) Optimum amplification profile in \((x, t)\) (dotted line) and \((\xi, t)\) (continuous line) coordinate systems with \(k = 0.2\), and \(R = 3\). Also shown the onset of fingering (black dot) determined by non-modal analysis in \((\xi, t)\) domain. (b) Optimal amplification in \((\xi, t)\) coordinate system, for \(R = 3\) and \(k = 0.01, 0.1, 0.3\).

not to calculate the magnitude of the amplification of disturbances explicitly. Thus, the curve \(G(t)\) vs \(t\) can be thought of as an envelope over optimal initial conditions.

In Fig. 3(b) we plot the optimal amplification that the system can observe for fixed choice of \(R = 3\) and various wave numbers. The obtained amplifications can give an alternate definition for onset of the instability. The onset \(t_e(k)\), for a given perturbation with wave number \(k\), is the moment when amplification starts to grow. We observe that the range of wave number between 0.1 and 0.3 dominate most, and the most unstable wave number predicted from the modal analysis \([10]\) lies within this interval. Thus, it can be claimed that the non-modal theory not only determine the transient growth, but also is a generalization of modal and IVP analyses. It is also observed when \(k\) approaches 0 (zero), Eqs. (13) - (14) with boundary conditions (5) - (6) become

\[
\left[ t \frac{\partial}{\partial t} - \frac{\xi}{2} \frac{\partial}{\partial \xi} - \frac{\partial^2}{\partial \xi^2} \right] c'(\xi, t) = 0 \tag{35}
\]

From Fig. 5(b) it can be noted that, unlike density fingering\([30]\), in the present study, the amplification is a non-constant function of time for any non-negative wave number.
This is in good agreement with the previously studied linear stability analysis of VF in the self-similar coordinate systems \[11, 12, 14\].

![Contour plot of optimum initial perturbation at time $t = 0.125$, $k = 0.2$, and $R = 3$ (dotted line correspond to negative contour lines and continuous line corresponds to positive contour lines). Inset shows the base state concentration $c_0$.](image)

**FIG. 6:** Contour plot of optimum initial perturbation at time $t = 0.125$, $k = 0.2$, and $R = 3$ (dotted line correspond to negative contour lines and continuous line corresponds to positive contour lines). Inset shows the base state concentration $c_0$.

2. **Optimal Perturbations**

One of the most important aspects of the non-modal analysis is to investigate the disturbances that produce maximum response during the fingering process. It is not clear from the existing linear stability analyses of miscibleVF what will be the dominant perturbations at early time on the order of $t \sim t_c$. This section also explains the transient period physically, by examining the optimal perturbations and eigenmodes obtained from SS-QSSA.

The maximal amplification $G(t) = \|\Phi(t_0, t)\|$ can be found by computing its singular value decomposition or Schmidt decomposition \[36\],

$$\Phi(t_0, t) = U_{[t_0,t]} \sum_{[t_0,t]} V^*_{[t_0,t]}.$$

(36)
where $V^*$ represents the adjoint of the matrix $V$. The largest singular value is the optimal amplification for the given perturbation i.e., the first entry of the diagonal matrix $\sum_{[t_0,t]}$, and the corresponding column of $V$ (first right singular vector) is the optimal perturbation (initial condition), denoted by $V_{opt}$. This optimal initial condition evolves into the first column of $U$ (the first left singular vector), denoted by $U_{opt}$. This is also evident from the singular value decomposition of $\Phi(t_0,t)$, i.e., $\Phi(t_0,t)v_1 = s_{1,1}u_1$, where $s_{1,1}$ is the first entry of the diagonal matrix $\sum_{[t_0,t]}$. Thus the leading vectors of $V$ are the useful inputs for selecting the correct representative for optimal initial conditions. Fig. 6 shows the contour of the initial perturbation of concentration that is most amplified at $t = 0.125$. As expected the perturbation is localized within the diffusive layer and overcome the random initial condition which perturbs the system everywhere. In Fig. 7(a) we plot the normalized optimal perturbation $V_{opt}$ for fixed $k = 0.2$ at times $t = 0.125, 0.25, 1, 2, 4.125$. It is observed that these right singular vectors are situated around the base state and gradually resolves to the dominant eigenmode calculated from modal analysis that discussed by Ben et al. [11]. In Fig. 7(b) the normalized $V_{opt}$, for various wave numbers is shown at critical times obtained as defined in section III.2. Two important observations are noted from the structure of the optimal perturbations determined from non-modal theory: (i) the optimal perturbation at their respective critical time are the identical to the eigenmode predicted by SS-QSSA, irrespective of any wave number chosen, and, (ii) at early times the SS-QSSA eigenmode and the optimal perturbations differ substantially, which is not shown for brevity. This reflects that although SS-QSSA captures the early times behavior better than QSSA and predicts the onset of instability, it fails to capture the transient growth of perturbations. It is noted that at later time the optimal initial conditions are independent of wave numbers $k$ at their respective critical time $t_c$ and are composed of the dominant SS-QSSA modes. Hence, a physically relevant transient period for VF can be defined as $[t_0, t_c]$, where $t_0$ is the time when the time integration starts and $t_c$ is the critical time obtained from NMA.
FIG. 7: Optimal normalized perturbation $V_{opt}$ (a) for $k = 0.2$ and $R = 3$ at different times, with the eigenfunction calculated from SS-QSSA at $t = 3.5$, (b) for $R = 3$ and $k = 0.15, 0.175, 0.2, 0.225$ at their respective critical times calculated from definition III.2.

B. Comparison of non-modal theory with SS-QSSA, IVP and DNS

In the previous section (Sec. V A), it is shown that the structure of the optimal perturbation describes about the regime of transient growth. Also from the discussion of numerical and spectral-abscissa (see Sec. III.C), we noted that at early times there exists significant difference between $\alpha(A(t))$ and $\eta(A(t))$, which is of order $O(10^7)$ (see Fig. 4). To explore the relevance of this optimal perturbation to the physical system, we compare the results of non-modal analysis with those obtained from SS-QSSA, IVP and DNS.

1. Dispersion Curves and Growth rates

Fig. 8 illustrates the dispersion curves obtained from non-modal analysis, SS-QSSA, IVP in $(\xi, t)$ domain, and DNS at $t = 3.5$. The relevance of plotting the dispersion curves at $t = 3.5$ is that it is close to the onset of fingering obtain from non-modal analysis (see Fig. 10). It is to be noted that in Fig. 8 (also in Fig. 10) the time for SS-QSSA calculation must be interpreted as diffusion time, where the base is frozen. It is evident from Fig. 23.
FIG. 8: Comparison of dispersion curves obtained from SS-QSSA (line with asterisk), IVP (line with square), NMA (continuous line), and DNS (circles) for $R = 3$ at time $t = 3.5$.

that non-modal analysis captures the onset of the instability more accurately than any other linear stability methods. Although both IVP and SS-QSSA show the flow is stable, the non-modal analysis meticulously agreeing to DNS calculation. Moreover it is shown that the threshold wave number (the least wave number at which $\sigma > 0$) and the cut-off wave number $k_c$ (wave number for which $\sigma$ changes from positive to negative) obtained from non-modal analysis are more close to DNS than those obtained from SS-QSSA and IVP. Thus, there exists significant transient effect of linearized non-normal operators in VF and the present nonlinear simulations agree with the non-modal linear stability results at early times. With these agreement of growth rate in NMA and DNS, we plot the amplification obtained from DNS and non-modal theory in Fig. 9 for the wave number $k = 0.25$, which amplifies most at early time. The amplification for DNS calculations are obtained by Eq. (29). It is shown that non-modal theory has an excellent agreement with DNS calculations
up to $t = 15$. This confirms the physically relevance of the transient growth calculation from non-modal theory and the instance, when the two curves in Fig. 9 deviate from each other can be marked as the threshold of nonlinear convection.

2. Optimal growth and dominant wave numbers

In an experiment or in any physical process a perturbation consists of combination of different wave numbers, so it is important to calculate the onset of instability which incorporate all possible wave numbers. This will characterize what will be the optimum growth at a given time. Fig. 10(a) depicts that the onset time obtained from modal, non-modal, and DNS are 4.6, 3.2, and 2.8, respectively. One of the important observation from optimal growth calculation is that the effect of the diffusion at early time is well described by non-modal and DNS, whereas SS-QSSA fails to observe this physical phenomenon effectively. The reason that NMA explains the physical phenomenon appropriately is attributed to the fact that at early times the linearized matrix $A(t)$ is highly non-normal, which is not re-
FIG. 10: (a) Optimum growth $\sigma_{\text{max}} \cdot t$ over all possible wave numbers $k$, for $R = 3$, (b) Neutral stability curve comparison for $R = 3$: Continuous lines for SS-QSSA and dashed lines for NMA.

ported in earlier linear stability analyses. Since at very early stage of the evolution of the perturbations, the growth rate in NMA and SS-QSSA are all negative, so it does not affect in instability.

Thus at early stage $t \ll 1$ in the linear regime optimal growth is damped before it starts to grow. The instance when both convection and diffusion terms are balanced, i.e., when $\sigma = 0$ is termed as the neutral stability condition. It is often required and useful for various purpose to identify the unstable and stable regime precisely. Fig. 10(b) represents the neutral stability curve determined by modal and non-modal theory. Again, the qualitative features remain the same, but there is a quantitative difference at early times and hence the stable regime identified by modal analysis becomes unstable for NMA. The lowest point on the curves corresponds to the critical time and critical wave number. This clearly depicts that although the critical wave numbers obtained from both modal and non-modal analyses are the same, but the critical time is smaller in non-modal theory than modal analysis.
VI. CONCLUSIONS

A novel non-modal linear stability analysis is presented to study the transient growth of perturbation in miscible viscous fingering. Recent works [3, 27, 30] suggested that in the unsteady base flow problem, the transient response of the perturbations can be significant. The existing literature did not study the transient effect of time dependent base state in the case of viscosity unstable problems. Our approach of non-modal analysis of VF is based on finding the propagator matrix and singular value decomposition method [27, 28]. This method predicts the dominant perturbations that experience the maximum amplification within the linear regime. The present linear stability analysis based on sounder mathematical basis have two most important features: (i) it determines the optimal amplification of the perturbation that is imperceptible in both SS-QSSA and IVP analyses and (ii) identifies the physical mechanism of the VF instability which is in very good agreement with DNS results. Additionally, the structure of the initial perturbations which lead to the optimal amplification is also illustrated, along with all the qualitative information of flow instability that SS-QSSA and IVP together provides. Unlike the random perturbations which disturb the system everywhere, the initial perturbations obtained from NMA are localized within the diffusive zone, which is in agreement with the spectral analysis performed by Ben et al. [11]. Although matrix differential method is used in the present study but the DAL method [22, 29, 30] can be an useful alternate way to study the transient growth. Using this classical non-modal analysis, it will be interesting to investigate the viscous fingering effects in CO₂ sequestration problem to find the suitable time and length scale for onset of convection, which will be the future aim of our study.
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[1] P. G. Saffman and G. Taylor, “The penetration of a fluid into a medium or Hele-Shaw cell containing a more viscous liquid,” Proc. Soc. London, Ser A, 245:312-329, (1958).

[2] G. M. Homsy, “Viscous fingering in porous media,” Annu. Rev. Fluid Mech. 19, 271-311 (1987).

[3] O.K. Matar, S.M. Troian, “Spreading of a surfactant monolayer on a thin liquid film: Onset and evolution of digitated structures,” Chaos. 9(1):141-153 (1999).

[4] M. Mishra, M. Martin, and A. De Wit, “Differences in miscible viscous fingering of finite width slices with positive or negative log mobility ratio,” Phys. Rev. E 78, 066306 (2008).

[5] A. De Wit, Y. Bertho, M. Martin, “Viscous fingering of miscible slices,” Phys. Fluids 17, 054114 (2005).

[6] B. Jha, L. Cueto-Felgueroso, and R. Juanes, “Quantifying mixing in viscously unstable porous media flows,” Phys. Rev. E 84, 066312 (2011).

[7] S. Hill, “Chanelling in packed columns,” Chem. Eng. Sci. 1, 247-53 (1952).

[8] R.L. Slobod and R.A. Thomas, “Effect of transverse diffusion on fingering in miscible displacement,” Soc. Pet. Eng. J. 3, 9-13 (1963).

[9] T. K. Perkins, O.C. Jhonston, and C.H. Hoffman, “Mechanics of viscous fingering in miscible system,” Soc. Pet. Eng. J. 5, 301 (1965).

[10] C. T. Tan, G. M. Homsy, “Stability of miscible displacements in porous media: Rectilinear flow,” Phys. Fluids 29, 3549 (1986).
[11] Y. Ben, E. A. Demekhin, and H. C. Chang, “A spectral theory for small amplitude miscible fingering,” Phys. Fluids 14, 999 (2002).

[12] M. C. Kim, “Linear stability analysis on the onset of the viscous fingering of a miscible slice in a porous media,” Advances in Water Resources, 35, 1-9 (2012).

[13] M. Mishra, A. De Wit, K. C. Sahu, “Double diffusive effects on pressure-driven miscible displacement flows in a channel,” J. Fluid Mech., 712, 579-597 (2012).

[14] S. Pramanik and M. Mishra, “Linear stability analysis of Korteweg stresses effect on the miscible viscous fingering in porous media,” Phys. Fluids 25, 074104 (2013).

[15] R. Govindarajan and K. C. Sahu, “Instabilities in Viscosity-Stratified Flow,” Annu. Rev. Fluid Mech. 46, 331-353 (2014).

[16] F. Bauer and J.A. Nohel, “The Qualitative Theory of Ordinary Differential Equations: An Introduction,” Dover (1969).

[17] B.F. Farrell and P.J. Ioannou, “Generalized Stability Theory. Part II: Nonautonomous Operators,” J. Atmos. Sci. 53, 2041-2053 (1996).

[18] P. J. Schmid, “Non-modal stability theory,” Annu. Rev. Fluid Mech. 39, 128 -162 (2007)

[19] P.J. Schmid and D.S. Henningson, “Stability and Transition in Shear Flows,” Springer publication (2001).

[20] S. C. Reddy, P. J. Schmid, and D. S. Henningson, “Pseudospectra of the Orr-Sommerfeld Operator,” SIAM J. APPL. MATH., 53, 1, 15-47 (1993).

[21] S. C. Reddy, P.J. Schmid, J.S. Baggett, and D.S. Henningson, “On stability of streamwise streaks and transition thresholds in plane channel flows,” J. Fluid Mech. 365, 269-303 (1998).

[22] P. Corbett, A. Bottaro, “Optimal perturbations for boundary layers subject to stream-wise pressure gradient,” Phys. Fluids 12, 120 (2000).

[23] S. F. Shen, Some considerations of the laminar stability of incompressible time-dependent basic flows,” J. Aerosp. Sci. 28, 397 (1961).
[24] O. K. Matar and S. M. Troian, “The development of transient fingering patterns during the spreading of surfactant coated films,” Phys. Fluids 11, 3232-3246 (1999).

[25] M. Bestehorn and A. Firoozabadi, “Effect of fluctuations on the onset of density-driven convection in porous media,” Phys. Fluids 24, 114102 (2012).

[26] N. Tilton, D. Daniel, and A. Riaz, “The initial transient period of gravitationally unstable diffusive boundary layers developing in porous media,” Phys. Fluids, 25, 092107 (2013).

[27] S. Rapaka, S. Chen, R. J. Pawar, Philip H. Stauffer, Dongxiao Zhang, “Non-modal growth of perturbations in density-driven convection in porous media,” J. Fluid Mech. 609, 285-303 (2008).

[28] S. Rapaka, R. J. Pawar, P. H. Stauffer, D. Zhang, and S. Chen, “Onset of convection over a transient base-state in anisotropic and layered porous media,” J. Fluid Mech. 641, 227-244 (2009).

[29] F. Doumenc, T. Boeck, B. Guerrier, and M. Rossi, “Transient RayleighBnardMarangoni convection due to evaporation: a linear non-normal stability analysis,” J. Fluid Mech. 648, 521-539 (2010).

[30] D. Daniel, N. Tilton, and A. Riaz, “Optimal perturbations of gravitationally unstable, transient boundary layers in porous media,” J. Fluid Mech. 727, 456-487 (2013).

[31] O. Manickam and G. M. Homsy “Fingering instabilities in vertical miscible displacement flows in porous media ,” J. Fluid Mech. 288, 75-102 (1995).

[32] D.A. Neild and A. Bejan, “Convection in Porous Media,” Springer-verlag Newyork Inc. (1992,page 14).

[33] T.G. Wright, “EigTool. http://www.comlab.ox.ac.uk/pseudospectra/eigtool/,” (2002).

[34] C. T. Tan and G. M. Homsy, “Simulation of non-linear viscous fingering in miscible displacement,” Phys. Fluids 31, 1330 (1988).

[35] H. Vitoshkin and A. Yu. Gelfgat, “Non-modal disturbances growth in a viscous mixing layer flow,” Fluid Dyn. Res. 46, 041414 (2014).
[36] G.H. Golub and C.F. Van Loan, “Matrix Computation,” Hindustan Book Agency, 3rd edition (2007).

[37] A.C. Slim, and T.S. Ramakrishnan, “Onset and cessation of time-dependent, dissolution-driven convection in porous media,” Phys. Fluids 22, 124103 (2010).