The regularity of the boundary of vortex patches revisited

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Abstract

We provide a new argument to show the persistence of boundary smoothness of vortex patches for the vorticity form of Euler’s equation, quite in the spirit of the well-known Bertozzi-Constantin approach. Our argument avoids the use of defining functions and, surprisingly, yields persistence of boundary smoothness of vortex patches for transport equations in the plane given by velocity fields which are convolution of the density with an odd kernel, homogeneous of degree $-1$ and of class $C^2$ off the origin. This allows the velocity field to have non-trivial divergence.

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1 Introduction

The vorticity form of the Euler equation is

$$\begin{align*}
\partial_t \omega + v \cdot \nabla \omega &= 0, \\
v(\cdot, t) &= \omega(\cdot, t) \ast \nabla^\perp N, \\
\omega(z, 0) &= \omega_0(z),
\end{align*}$$

where $\omega(z, t)$ is vorticity at the point $z$ at time $t$, $z = x + iy \in \mathbb{C} = \mathbb{R}^2$, $t \in \mathbb{R}$, and $N(z) = (1/2\pi) \log |z|$ is the fundamental solution of the laplacian in the plane. Then $\nabla^\perp N(z) = (i/2\pi)z/|z|^2$. It is a deep result of Yudovich [1] that [2] is well posed in $L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, in the sense that there exists a unique weak solution of [1] for each given initial condition in $L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$.

When one takes as initial condition the characteristic function $\rho_0 = \chi_{D_0}$ of a bounded domain $D_0$, then $\omega(\cdot, t) = \chi_{D_t}(\cdot)$, where $D_t$ is a bounded domain. This follows from the
fact that (1) is a transport equation and one refers to such a solution as a vortex patch. In the eighties the problem of deciding if boundary smoothness persisted for all times was an open challenging question. Chemin proved the following [Ch].

**Theorem** (Chemin, 1993). If $D_0$ is a bounded simply connected domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, then the weak solution of (1) with initial condition $\chi_{D_0}$ is of the form $\omega(z,t) = X_{D_0}(z)$ with $D_t$ a simply connected domain of class $C^{1+\gamma}$ for all times $t \in \mathbb{R}$.

Indeed in [Ch] one proves a more general statement and the proof depends on paradifferential calculus. In [BC] a shorter proof, based on classical analysis methods, was devised. See also [Se].

The proof in [BC] takes for granted that a local in time solution exists (see Chapter 8 of [MB]) and then continues by proving an a priori inequality for the relevant quantities giving the smoothness of $D_t$. These quantities are defined in terms of the defining function for $D_t$ obtained by transporting a given defining function of $D_0$. The a priori bounds are obtained by resorting to an appropriate commutator, which allows to perform a control in terms of $\|\nabla v(\cdot, t)\|_\infty$. The second element in the proof, as in [Ch], is a logarithmic estimate for $\|\nabla v(\cdot, t)\|_\infty$, which follows from a simple property of convolution even homogeneous smooth singular integrals of Calderón-Zygmund type.

Inspired by conversations with J.Garnett we avoid the use of defining functions and prove directly an a priori inequality for the Hölder seminorm of order $\gamma$ of the gradient of the local in time solution $X(\cdot, t)$ of the contour dynamics equation (see (10)). This follows by bringing in a commutator, which appears after an application of Whitney’s extension theorem. The proof follows by proving a logarithmic inequality which estimates $\|\nabla v(\cdot, t)\|_\infty$ in terms of $\|\nabla X(\cdot, t)\|_{\gamma, \partial D_0}$ and the Lipschitz constant of the inverse of $X(\cdot, t)$.

Our proof is not shorter, nor essentially different in its basic elements from that in [BC], but it is more direct in that avoids defining functions. It has a surprising consequence: it works not only for the vorticity equation, but for all transport equations of the form

$$\begin{align*}
\partial_t \omega + v \cdot \nabla \omega &= 0, \\
v(\cdot, t) &= \omega(\cdot, t) * k, \\
\omega(z, 0) &= \chi_{D_0}(z),
\end{align*}$$

(2)

where $k : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is an odd kernel, homogeneous of degree $-1$, of class $C^2$ off the origin, and $D_0$ is a simply connected domain with boundary of class $C^{1+\gamma}$. These kernels produce velocity fields $v$ with non-zero divergence and thus the incompressibility assumption in Chemin’s theorem plays no role. For these more general equations a well-posedness Yudovich theory is not yet available and so uniqueness holds at present only in the class of patches with smooth boundary.

A formal statement of our main result is as follows.
**Theorem.** If $D_0$ is a bounded simply connected domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, then there exists a weak solution of (2) with initial condition $\chi_{D_0}$ of the form $\omega(z,t) = X_{D_t}(z)$ with $D_t$ a simply connected domain of class $C^{1+\gamma}$ for all times $t \in \mathbb{R}$. This solution is unique in the class of characteristic functions of domains of class $C^{1+\gamma}$.

This has also been proved in any dimension in [CMOV] via an intricate construction of appropriate defining functions.

2 The a priori estimate

Parametrize the boundary $\partial D_0$ by a mapping $h: \mathbb{T} \to \partial D_0$, of class $C^{1+\gamma}(\mathbb{T},\mathbb{R}^2)$ satisfying the bilipschitz condition

$$c_0^{-1}|u-v| \leq |h(u)-h(v)| \leq c_0|u-v|, \quad u, v \in \mathbb{T}$$

for some positive constant $c_0$. The mapping $h$ induces a norm in the vector space of $C^{1+\gamma}$ functions on the boundary of $D_0$ with values in the plane defined on $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$ by

$$\|X\|_{1+\gamma} = \|X\|_{\infty,\partial D_0} + \|\nabla X\|_{\gamma,\partial D_0},$$

where

$$\|X\|_{\infty,\partial D_0} = \sup\{|X(\alpha)| : \alpha \in \partial D_0\}$$

and

$$\|\nabla X\|_{\gamma,\partial D_0} = \sup\{\left|\frac{d}{du}X(h(u)) - \frac{d}{du}X(h(v))\right| : u, v \in \mathbb{T}, u \neq v\}.$$
the trajectory of the particle that at time 0 is at the point \( \alpha \in \partial D_0 \) and \( X(\partial D_0, t) \) is a Jordan curve of class \( C^{1+\gamma} \) which encloses a simply connected domain \( D_t \). The function \( \omega(z, t) = \chi_{D_t}(z) \) is a weak solution of equation (2).

Thus
\[
\frac{d}{dt} X(\alpha, t) = v(X(\alpha, t), t),
\]
\[
X(\alpha, t) = \alpha \in \mathbb{R}^2,
\]
is the flow associated with the Lipschitz velocity field \( v(\cdot, t) = \chi_{D_t}(\cdot) \ast k \) and coincides with the solution of the CDE for \( \alpha \in \partial D_0 \). What remains to be proven is that the maximal time of existence of the solution of the CDE is \( T = \infty \) and for that we need an a priori inequality. By (5)
\[
X(h(u), t) = h(u) + \int_0^t v(X(h(u), s), s) \, ds, \quad u \in \mathbb{T}.
\]
Hence
\[
\frac{d}{du} X(h(u), t) = \frac{dh}{du}(u) + \int_0^t \nabla v(X(h(u), s), s) \left( \frac{d}{du} X(h(u), s) \right) \, ds.
\]

Clearly \( \frac{d}{du} X(h(u), s) \) is a tangent vector to \( \partial D \) at the point \( X(h(u), s) \). We would like to show that
\[
\nabla v(X(h(u), s), s) \left( \frac{d}{du} X(h(u), s) \right)
\]
is a commutator. We need three lemmas. The first works in \( \mathbb{R}^n \). The notion of jet is in \[S, p. 176\], although the term jet is not used there. Given a subset \( S \subset \mathbb{R}^n \) and a vector field \( \vec{F} : S \to \mathbb{R}^n \) set
\[
\| \vec{F} \|_{\gamma,S} = \sup \{ \| \vec{F}(x) - \vec{F}(y) \| : x, y \in S, x \neq y \}.
\]

**Lemma 1.** Let \( \Gamma \) be a hypersurface of dimension \( n-1 \) and of class \( C^{1+\gamma} \) in \( \mathbb{R}^n \). Let \( N(x) = (N_1(x), \ldots, N_n(x)) \) a normal field on \( \Gamma \) of class \( C^\gamma(\Gamma) \). Then \( (0, N(x)) \) is a jet of class \( C^{1+\gamma}(\Gamma) \) with constant \( A\|N\|_{\gamma,\Gamma}, A \) a constant depending only on \( \gamma \). In other words,
\[
\sup_{x, y \in \Gamma, x \neq y} |(y - x) \cdot N(x)| \leq A\|N\|_{\gamma,\Gamma} |y - x|^{1+\gamma}.
\]

**Proof.** Assume, without loss of generality, that \( x = 0 \) and \( N(0) = (0, \ldots, 0, N_n(0)) \) with \( N_n(0) > 0 \). Set \( \delta^{-\gamma} = 2\frac{\|N\|_{\gamma,\Gamma}}{|N_n(0)|} \).

If \( |y| \geq \delta \), then
\[
|y \cdot N(0)| \leq |y||N(0)| = |y|2\|N\|_{\gamma,\Gamma} \delta^\gamma \leq 2\|N\|_{\gamma,\Gamma} |y|^{1+\gamma},
\]
as required.

If \( y \in B(0, \delta) \cap \Gamma \), then

\[
|N(y) - N(0)| \leq \|N\|_{\gamma, \Gamma} \delta^\gamma = \frac{|N(0)|}{2}.
\]

Hence the tangent hyperplane to \( \Gamma \) at \( y \) forms an angle of less than 30° with the horizontal hyperplane. Consequently \( B(0, \delta) \cap \Gamma \) is the graph of a function \( y_n = \varphi(y') \), \( y' = (y_1, \ldots, y_{n-1}) \), of class \( C^{1+\gamma} \), defined on the projection \( U \) of \( B(0, \delta) \cap \Gamma \) on \( \{ y \in \mathbb{R}^n : y_n = 0 \} \). Note that \( U \) is open in \( \mathbb{R}^{n-1} \) and that, for \( y \in B(0, \delta) \cap \Gamma \), the segment with extremes 0 and \( y' \) is contained in \( U \) (by the implicit function theorem). It is also clear that \( |\nabla \varphi(y')| \leq 1 \), \( y' \in U \), because of the inclination of tangent hyperplanes with respect to the horizontal hyperplane.

If \( y \in B(0, \delta) \cap \Gamma \) we have

\[
|y \cdot N(0)| \leq |\varphi(y')||N(0)| \leq \left( \sup_{|z'| \leq |y'|, z' \in U} |\nabla \varphi(z')| \right) |y'| |N(0)| \leq \|\nabla \varphi\|_{\gamma, \Gamma} |y'|^{1+\gamma} |N(0)|.
\]

Set

\[
M(y') = (-\partial_1 \varphi(y'), \ldots, -\partial_{n-1} \varphi(y'), 1), \quad y' \in U,
\]

which is a normal vector to \( \Gamma \) at \( y = (y', \varphi(y')) \). Hence

\[
M(y') = \frac{N(y)}{N_n(y)}, \quad y \in B(0, \delta) \cap \Gamma,
\]

and

\[
\|\nabla \varphi\|_{\gamma, U} = \|M\|_{\gamma, U} = \left\| \frac{N(y)}{N_n(y)} \right\|_{\gamma, U}.
\]

Take \( y, z \in B(0, \delta) \cap \Gamma \). Then

\[
|y - z| = (|y' - z'|^2 + |\varphi(y') - \varphi(z')|^2)^{1/2} \leq \sqrt{2}|y' - z'|.
\]

One has

\[
|N(y) - N(z)| \leq \frac{1}{|N_n(y)|} |N(y) - N(z)| + |N(z)| |N_n(z) - N_n(y)| |N(y)||N_n(z)|.
\]

By the definition of \( \delta \)

\[
|N_n(y)| \geq |N_n(0)| - |N_n(y) - N_n(0)| \geq |N(0)| - \|N\|_{\gamma, \Gamma} \delta^\gamma = \frac{|N(0)|}{2}.
\]

Thus the right hand side of (7) is not greater than

\[
2^{3+\gamma/2} \frac{\|N\|_{\gamma, \Gamma}}{|N(0)|} |y' - z'|^\gamma.
\]
Therefore
\[ \frac{|y \cdot N(0)|}{|y|^{1+\gamma}} \leq \|\nabla \varphi\|_{\gamma,\Gamma} |N(0)| \leq 2^{3+\gamma/2} \|N\|_{\gamma,\Gamma} \]
and the lemma follows with \( A = 2^{3+\gamma/2} \).

\[ \square \]

**Lemma 2.** Let \( \Gamma \) a Jordan curve of class \( C^{1+\gamma} \) in the plane and \( \tau(x) = (\tau_1(x), \tau_2(x)) \), \( x \in \Gamma \), a tangent vector to \( \Gamma \) at \( x \) with \( \tau \in C^{\gamma}(\Gamma, \mathbb{R}^2) \). Then there exists \( g(x) = (g_1(x), g_2(x)) \in C_\gamma(\mathbb{R}^2, \mathbb{R}^2) \) such that \( g = \tau \) on \( \Gamma \), \( \|g\|_{\gamma,\mathbb{R}^2} \leq C \|\tau\|_{\gamma,\Gamma} \) for a positive constant \( C \) depending only on \( \gamma \), and \( \text{div} \ g = 0 \) in \( \mathbb{R}^2 \).

**Proof.** The field \( (-\tau_2(x), \tau_1(x)) \) is normal to \( \Gamma \) at each \( x \in \Gamma \). By the previous lemma \( (0, -\tau_2(x), \tau_1(x)) \) is a \( C^{1+\gamma} \) jet on \( \Gamma \) with constant \( C_\gamma \|\tau\|_{\gamma,\Gamma} \) and by Whitney’s extension theorem there is a function \( \varphi \in C^{1+\gamma}(\mathbb{R}^2, \mathbb{R}) \) such that
\[ \varphi = 0, \quad \partial_1 \varphi = -\tau_2, \quad \partial_2 \varphi = \tau_1, \quad \text{on} \ \Gamma \]
and \( \|\nabla \varphi\|_{\gamma,\mathbb{R}^2} \leq C_0 \|\tau\|_{\gamma,\Gamma} \), with \( C_0 \) an absolute constant [S, Theorem 4, p.177]. Therefore
\[ \tau(x) = (\partial_2 \varphi(x), -\partial_1 \varphi(x)), \quad x \in \Gamma. \]
Set \( g = (\partial_2 \varphi, -\partial_1 \varphi) \), so that \( g \) has no divergence in the plane and \( \|g\|_{\gamma,\mathbb{R}^2} \leq C \|\tau\|_{\gamma,\Gamma} \), where \( C = C_0 C_\gamma \) depends only on \( \gamma \).

**Lemma 3.** Let \( \tau(x) = (\tau_1(x), \tau_2(x)) \in C^{\gamma}(\Gamma) \) be a tangent field on \( \Gamma = \partial D_s \). Then
\[ (8) \quad \nabla v(x, s)(\tau(x)), \quad x \in \Gamma, \]
is the restriction to \( \Gamma \) of the commutator
\[ \int_{D_s} \nabla v(x-y, s)(g(x) - g(y)) \, dy, \quad x \in \mathbb{R}^2, \]
where \( g \) is the field given by Lemma [S].

**Proof.** The first component of \( \tau \) is
\[ (9) \quad \partial_1 v_1(x, s)\tau_1(x) + \partial_2 v_1(x, s)\tau_2(x) = (\partial_1 k_1 * \chi)(x)\tau_1(x) + (\partial_2 k_1 * \chi)(x)\tau_2(x) \]
with \( \chi = \chi_{D_s} \). In the sense of distributions
\[ \partial_1 k_1 = \text{p.v.} \partial_1 k_1 + c_1 \delta_0, \quad c_1 = \int_{|x|=1} k_1(x) x_1 \, d\sigma(x) \]
and
\[ \partial_2 k_1 = \text{p.v.} \partial_2 k_1 + c_2 \delta_0, \quad c_2 = \int_{|x|=1} k_1(x) x_2 \, d\sigma(x). \]
Let $g$ be the field given by Lemma 2. Since $(\partial_1 k_1 \ast \chi)g_1 = (\text{p.v. } \partial_1 k_1 \ast \chi)g_1 + c_1 \chi g_1$ and $\partial_1 k_1 \ast (\chi g_1) = \text{p.v. } \partial_1 k_1 \ast (\chi g_1) + c_1 \chi g_1$, we get

$$(\partial_1 k_1 \ast \chi)g_1 = (\text{p.v. } \partial_1 k_1 \ast \chi)g_1 - \text{p.v. } \partial_1 k_1 \ast (\chi g_1) + \partial_1 k_1 \ast (\chi g_1).$$

Similarly

$$(\partial_2 k_1 \ast \chi)g_2 = (\text{p.v. } \partial_2 k_1 \ast \chi)g_2 - \text{p.v. } \partial_2 k_1 \ast (\chi g_2) + \partial_2 k_1 \ast (\chi g_2).$$

Note that $\partial_1 k_1 \ast (\chi g_1) = k_1 \ast \partial_1 (\chi g_1) = \partial_1 (g \partial_1 \chi + \chi \partial_1 g_1)$ and so, recalling that $g$ has vanishing divergence,

$$\partial_1 k_1 \ast (\chi g_1) + \partial_2 k_1 \ast (\chi g_2) = k_1 \ast (g \cdot \nabla \chi + \chi \text{ div } g) = k_1 \ast (g \cdot \nabla \chi) \equiv 0$$

since $g|\Gamma = \tau$ is tangent to $\Gamma$ and $\nabla \chi$ normal to $\Gamma$ at each point of $\Gamma$. Thus if $x \in \Gamma = \partial D_s$

$$(\partial_1 k_1 \ast \chi)(x)\tau_1(x) + (\partial_2 k_1 \ast \chi)(x)\tau_2(x) = \int_{D_s} \partial_1 k_1(x - y)(g_1(x) - g_1(y)) dy + \int_{D_s} \partial_2 k_1(x - y)(g_2(x) - g_2(y)) dy$$

is indeed a sum of two scalar commutators. One has a similar formula for the second component of $\nabla v(x, s)(\tau(x))$ and the lemma follows.

Therefore, by the usual commutator estimate in the Hölder seminorm (Lemma, p.26) or [BGLV, Lemma 3.2, p.3799],

$$\|\nabla v(x, s)(\tau(x))\|_{\gamma, \Gamma} \leq C \left(1 + \|\nabla v(\cdot, s)\|_\infty\right) \|g\|_{\gamma, \mathbb{R}^2} \leq C \left(1 + \|\nabla v(\cdot, s)\|_\infty\right) \|	au\|_{\gamma, \Gamma}.$$

From (6) we see that

$$\left\|\frac{d}{du}X(h(u), t)\right\|_{\gamma, T} \leq \left\|\frac{d}{du}h(u)\right\|_{\gamma, T} + C \int_0^t (1 + \|\nabla v(\cdot, s)\|_\infty) \left\|\frac{d}{du}X(h(u), s)\right\|_{\gamma, T} ds,$$

and then, by Gronwall’s lemma, we get the a priori inequality

$$\left\|\frac{d}{du}X(h(u), t)\right\|_{\gamma, T} \leq \left\|\frac{d}{du}h(u)\right\|_{\gamma, T} \exp \left(C \int_0^t (1 + \|\nabla v(\cdot, s)\|_\infty) ds\right).$$

We also have an a priori inequality for

$$b(t) \equiv \inf_{\alpha \neq \beta} \frac{|X(\alpha, t) - X(\beta, t)|}{|\alpha - \beta|} = \frac{1}{\sup_{\alpha \neq \beta} \frac{|\alpha - \beta|}{|X(\alpha, t) - X(\beta, t)|}} \geq \exp \left(-\int_0^t \|\nabla v(\cdot, s)\|_\infty ds\right).$$

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Setting
\[ q(t) = \frac{\| \frac{d}{du}X(h(u), t) \|_{\gamma,T}}{b(t)^{1+\gamma}}, \quad -T < t < T, \]
we finally obtain
\[ q(t) \leq C \exp \left( C \int_0^t \left( 1 + \| \nabla v(\cdot, s) \|_{\infty} \right) ds \right), \quad 0 < t < T. \]

3 The logarithmic inequality

Let \( K \in C^1(\mathbb{R}^2 \setminus \{0\}) \) be a real function, homogeneous of degree \(-2\) and even. Set
\[ Tf(x) = \text{p.v.} \int_{\mathbb{R}^2} K(x - y)f(y) \, dy, \]
so that \( T \) is a convolution, smooth, homogeneous, even Calderón–Zygmund operator.

Let \( X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^2), D_0 \) a domain of class \( C^{1+\gamma} \). Assume that \( X \) is bilipschitz onto the image and let \( b \) stand for the inverse of Lipschitz norm of the inverse mapping, as in [H]. Since \( X(\partial D_0) \) is a Jordan curve, the simply connected domain \( D \) enclosed by the curve satisfies \( \partial D = X(\partial D_0) \).

The maximal singular integral of a function \( f \) is
\[ T^*f(x) = \sup_{\varepsilon > 0} \int_{|y-x| > \varepsilon} K(x - y)f(y) \, dy, \quad x \in \mathbb{R}^2. \]

**Lemma 4.** We have
\[ T^*(\chi_D)(x) \leq A \left( 1 + \log^+ \left( |D|^{1/2} \left\| \frac{d}{du}[X(h(u))]|_{\gamma,T} \frac{1}{b^{1+\gamma}} \right\| \right) \]
\[ = A \left( 1 + \log^+ \left( |D|^{1/2} q(t) \right) \right), \quad x \in \mathbb{R}^2, \]
where \( A \) is a constant depending only on \( \gamma \) and the kernel \( K \).

**Proof.** By an elementary argument [MOV, p.409-410] it is enough to prove the inequality for points \( x \in \partial D \). Without loss of generality one can assume that \( x = 0 = X(0) \) and that the unitary normal vectors to \( \partial D_0 \) and \( \partial D \) at 0 are \((0, 1)\). Define \( \delta \) by
\[ \delta^{-\gamma} = \frac{1}{b^{1+\gamma}} \left\| \frac{d}{du}[X(h(u))]|_{\gamma,T} \right\|. \]

Then for \( \varepsilon > 0 \)
\[ \int_{|y| > \varepsilon} K(y)\chi_D(y) \, dy = \int_{|y| > \varepsilon} K(y)\chi_D(y) \, dy + \int_{|D|^{1/2} > |y| > \delta} K(y)\chi_D(y) \, dy \]
\[ + \int_{|y| > |D|^{1/2}} K(y)\chi_D(y) \, dy. \]

(11)
The third term is estimated by
\[ \int_{|y|>|D|^{1/2}} \frac{1}{|y|^2} \chi_D(y) \, dy \leq 1 \]
and the second by
\[ 2\pi \sup_{|y|=1} |K(y)| \log \left( \frac{|D|^{1/2}}{\delta} \right) \leq A \log^+ \left( |D|^{1/2} \| \frac{d}{du} [X(h(u))] \|_{\gamma,T} \frac{1}{\rho^{1+\gamma}} \right). \]

It remains to estimate the first term in the right hand side of (11). Set
\[ D_{\varepsilon,\delta} = \{ y \in D : \varepsilon < |y| < \delta \}, H^- = \{ y = (y_1,y_2) \in \mathbb{R}^2 : y_2 < 0 \} \text{ and } S_\rho = \{ y \in \mathbb{R}^2, |y| = \rho \}, \rho > 0. \]
Since an even kernel with vanishing integral on the unit circumference has also vanishing integral on half a circumference, we get
\[
\int_{D_{\varepsilon,\delta}} K(y) \, dy = \int_\varepsilon^\delta \left( \int_{\{\theta \in S^1 : \rho \theta \in D\}} K(\theta) \, d\sigma(\theta) \right) \frac{d\rho}{\rho} \\
= \int_\varepsilon^\delta \left( \int_{\{\theta \in S^1 : \rho \theta \in D\}} K(\theta) \, d\sigma(\theta) - \int_{S^1 \cap H^-} K(\theta) \, d\sigma(\theta) \right) \frac{d\rho}{\rho} \\
= \int_\varepsilon^\delta \left( \int_{\{\theta \in S^1 : \rho \theta \in (D \setminus H^-) \cup (H^- \setminus D)\}} K(\theta) \, d\sigma(\theta) \right) \frac{d\rho}{\rho}
\]
and
\[
\left| \int_{D_{\varepsilon,\delta}} K(y) \, dy \right| \leq \sup_{|\theta|=1} |K(\theta)| \int_0^\delta \sigma \{ \theta \in S^1 : \rho \theta \in (D \setminus H^-) \cup (H^- \setminus D) \} \frac{d\rho}{\rho}.
\]
Since the domains $D \setminus H^-$ and $H^- \setminus D$ are “tangential”
\[ \sigma \{ \theta \in S^1 : \rho \theta \in (D \setminus H^-) \cup (H^- \setminus D) \} \leq A \frac{1}{\rho} \sup \{|X_2(\alpha)| : \alpha \in \partial D_0 \text{ and } |X(\alpha)| = \rho\}. \]
Take $|X(\alpha)| = \rho < \delta$. If $\alpha = h(u)$ and $0 = h(u_0)$, then
\[
|X_2(\alpha)| = |X_2(h(u)) - X_2(h(u_0))| \leq \| \frac{d}{du} [X_2(h(u))] \|_{\gamma,T} |u - u_0|^{1+\gamma}
\leq \| \frac{d}{du} [X(h(u))] \|_{\gamma,T} \frac{1}{b^{1+\gamma}} |X(\alpha)|^{1+\gamma}
= \| \frac{d}{du} [X(h(u))] \|_{\gamma,T} \frac{1}{b^{1+\gamma}} \rho^{1+\gamma}.
\]
Thus
\[
\left| \int_{D_{\varepsilon, \delta}} K(y) \, dy \right| \leq A \left\| \frac{d}{du} [X(h(u))] \right\|_{\gamma, T} \frac{1}{\delta^{1+\gamma}} \int_0^\delta \rho^{\gamma-1} \, d\rho
= A \left\| \frac{d}{du} [X(h(u))] \right\|_{\gamma, T} \frac{\delta^{\gamma}}{\delta^{1+\gamma}} = A. \quad \square
\]

Having at our disposition the a priori estimate of section 2 and the logarithmic estimate it is a standard matter to complete the proof. We remark that \(\nabla v(\cdot, t)\) is a \(2 \times 2\) matrix with entries which are either constant multiples of \(\chi_{D_t}\) or even homogeneous smooth Calderón-Zygmund operators applied to \(\chi_{D_t}\). Inserting the a priori estimate (10) into the logarithmic inequality we obtain
\[
\left\| \nabla v(\cdot, t) \right\|_\infty \leq C + C \int_0^t (1 + \left\| \nabla v(\cdot, s) \right\|_\infty) \, ds, \quad t \in (0, T).
\]

The factor \(|D|^{1/2}\) in the inequality of Lemma 3 causes no trouble because of the standard estimate
\[
\left\| \nabla X(\cdot, t) \right\|_\infty \leq \exp \int_0^t \left\| \nabla v(\cdot, s) \right\|_\infty \, ds, \quad t \in (0, T).
\]

Gronwall’s lemma then yields
\[
\left\| \nabla v(\cdot, t) \right\|_\infty \leq C \exp( Ct ), \quad t \in (0, T).
\]

Hence, for \(t \in (0, T)\),
\[
\frac{d}{du} [X(h(u), t)]_{\gamma, T} \leq C \exp(C \exp( Ct )) \leq C \exp(C \exp(CT)),
\]
and
\[
b(t) \geq C^{-1} \exp(- C \exp(Ct)) \geq C^{-1} \exp(- C \exp(CT)).
\]

Consequently \(T = \infty\).

4 Appendix: local in time existence for the CDE

Our first remark is that if \(k : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2\) is an odd kernel, homogeneous of degree \(-1\), differentiable off the origin, then
\[
k = \partial_1(x_1 k) + \partial_2(x_2 k), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}.
\]

This follows straightforwardly from Euler’s theorem on homogeneous functions.
Assume that $\omega(z, t) = \chi_D(t)$ is a weak solution of the general equation (2). The velocity field is

$$v(\cdot, t) = \chi_D(t) \ast k = \chi_D(t) \ast \left( \frac{\partial}{\partial x}(xk) + \frac{\partial}{\partial y}(yk) \right)$$

$$= \frac{\partial}{\partial x}\chi_D(t) \ast (xk) + \frac{\partial}{\partial y}\chi_D(t) \ast (yk).$$

Since $\nabla\chi_D(t) = -\vec{n}d\sigma = idz$, where $d\sigma$ is the arc-length measure on the curve $\partial D_t$ and $dz = dz_{\partial D_t}$, we have

$$\frac{\partial}{\partial x}\chi_D(t) = -n_1d\sigma = -dy \quad \text{and} \quad \frac{\partial}{\partial y}\chi_D(t) = -n_2d\sigma = dx.$$

Setting $z = x + iy$ and $w = u + iv$ we get

$$v(z, t) = -\int_{\partial D_t} (x - u)k(z - w) \, dv + \int_{\partial D_t} (y - v)k(z - w) \, dw$$

$$= \int_{\partial D_t} k(z - w)\langle -i(z - w), dw \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product in the plane.

The flow mapping is the solution of the ODE

$$\frac{d}{dt}X(\alpha, t) = v(X(\alpha, t), t),$$

$$X(\alpha, t) = \alpha \in \mathbb{R}^2.$$

Substituting the expression (13) for the velocity field in (14) and making the change of variables $w = X(\beta, t)$ we get

$$\frac{d}{dt}X(\alpha, t) = v(X(\alpha, t), t) = \int_{\partial D_0} k(X(\alpha, t) - w)\langle -i(X(\alpha, t) - w), dw \rangle,$$

$$= \int_{\partial D_0} k(X(\alpha, t) - X(\beta, t))\langle -i(X(\alpha, t) - X(\beta, t)), \nabla X(\beta, t)(d\beta) \rangle.$$
In fact $\nabla X(\beta, t)$ can be understood more intrinsically as the differential $DX(\beta, t)$ of the mapping $X(\cdot, t)$ as a differentiable mapping from the differentiable curve $\partial D_0$ onto the differentiable curve $\partial D_t$. This mapping takes a tangent vector to $\partial D_0$ at the point $\beta$ into a tangent vector to $\partial D_t$ at the point $X(\beta, t)$. The point is that this operation involves only first derivatives of $X(\cdot, t)$ on $\partial D_0$.

Consider the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$ endowed with the norm (3) and the open set $\Omega$ consisting of those mappings in $C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$ satisfying the bilipschitz condition (4). For each $X \in \Omega$ define

$$F(X)(\alpha) = \int_{\partial D_0} k(X(\alpha) - X(\beta)) \langle -i(X(\alpha) - X(\beta)), \nabla X(\beta)(d\beta) \rangle, \quad \alpha \in \partial D_0. \tag{15}$$

The CDE is the autonomous ODE in $\Omega$

$$\frac{dX}{dt} = F(X), \quad X(\cdot, 0) = I, \tag{16}$$

where $I$ is the identity mapping on $\partial D_0$.

Of course one has to check that $F(X)$ belongs to $C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$ for each $X \in \Omega$. After that one wants to apply the existence and uniqueness theorem of Picard, and for that one needs to check that $F$ is locally Lipschitz in $\Omega$.

All these facts are verified routinely and depend essentially on the fact that the kernel appearing in (15) is of class $C^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2)$ and homogeneous of degree 0. The reader may consult [MB, Chapter 8] for the case of the vorticity equation. For instance, when you take a derivative in $\alpha$ in (15) to prove that $F(X)$ is in $C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$, then you get an expression of the form

$$\int_{\partial D_0} H(X(\alpha) - X(\beta)) L(\nabla X(\beta), d\beta), \tag{17}$$

where $H(\cdot)$ is a kernel of homogeneity $-1$ of class $C^1(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2)$ and $L(\nabla X(\beta), d\beta)$ a linear expression in $\partial X_j/\partial \beta_k$ and $d\beta_j$. Hence (17) may be understood as a (non-convolution) singular integral on the curve $\partial D_0$ applied to a linear combination of the $\partial X_j/\partial \beta_k$, which are functions satisfying a Hölder condition of order $\gamma$. Then the result is a function of $\alpha$ in the same space.

An analogous situation appears in checking that $F$ is locally Lipschitz in $\Omega$.

The reader may also consult [BGLV], where full details are provided in a similar context.

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