A note on open-chain transfer matrices from $q$-deformed $su(2|2)$ $S$-matrices

Rajan Murgan

Physics Department,
Gustavus Adolphus College,
800 West College Avenue, St. Peter, MN 56082 USA

Abstract

In this note, we perform Sklyanin’s construction of commuting open-chain/boundary transfer matrices to the $q$-deformed $SU(2|2)$ bulk $S$-matrix of Beisert and Koroteev and a corresponding boundary $S$-matrix. This also includes a corresponding commuting transfer matrix using the graded version of the $q$-deformed bulk $S$-matrix. Utilizing the crossing property for the bulk $S$-matrix, we argue that the transfer matrix for both graded and non-graded versions contains a crucial factor which is essential for commutativity.
1 Introduction

Centrally extended $su(2|2)$ algebra (two copies) \cite{1,2} is a key symmetry in investigations of integrability in AdS/CFT. (Readers are urged to refer to \cite{3} for reviews.) It leads to bulk $S$-matrix \cite{1} that obeys (twisted) Yang-Baxter equation (YBE). (Also refer to \cite{4} for the corresponding bulk $S$-matrix that obeys standard YBE.) Such a $S$-matrix can be used to prove \cite{1,5,6} a conjectured set of asymptotic Bethe equations \cite{7} for the spectrum of gauge/string theory. In connection to the open string/spin chain sector of AdS/CFT \cite{8}-\cite{23}, these results have also been generalized to cases with boundaries, where the corresponding boundary $S$-matrices have been derived. Hofman and Maldacena \cite{12} proposed boundary $S$-matrices corresponding to open strings attached to maximal giant graviton \cite{24} in $AdS_5 \times S^5$ that describe the reflection of world-sheet excitations for two cases: $Y = 0$ and $Z = 0$ giant graviton branes, hence generalizing the scattering theory of magnons in the planar limit of the AdS/CFT correspondence by including boundaries. In \cite{15}, related boundary $S$-matrices (that indeed obey standard boundary Yang Baxter equation (BYBE) \cite{25,26}) were derived. Recently, transfer matrices for open chain for AdS/CFT were derived \cite{19} and subsequently the corresponding all loop Bethe ansatz equations have been presented by Galleas in \cite{21}.

A $q$-deformation of the above mentioned centrally-extended $su(2|2)$ algebra of AdS/CFT have been proposed by Beisert and Koroteev \cite{27}. They derived the corresponding $q$-deformed bulk $S$-matrix, which they related to a deformation \cite{28} of the one-dimensional Hubbard model \cite{29}. The related factorizable boundary $S$-matrices that obey the standard BYBE have been derived recently by using the Zamolodchikov-Faddeev (ZF) algebra \cite{30}. While derivation of all loop Bethe ansatz equations for this case is an interesting problem (analogous to that given by Galleas for the $su(2|2)$ case in \cite{21}), construction of commuting open-chain/boundary transfer matrices (as observed by Sklyanin \cite{31}) is crucial for such a derivation. The fact that the $q$-deformed case of $SU(2|2)$ matrices is not as widely explored in literature relative to the $q = 1$ case, has motivated us to consider this problem and present the material in a relatively more unified way. The $q$-deformed bulk $S$-matrix is not of the difference form and has a peculiar crossing property \cite{27}, thus motivating one to consider the generalization of Sklyanin’s construction \cite{31}. Although such a construction for the undeformed $su(2|2)$ algebra was given \cite{19}, in this note, we consider an analogous construction for the $q$-deformed case, given that there might be details that deserve generalizations which could not have emerged by considering the undeformed case alone. Indeed, we rely on the crossing property obeyed by the $q$-deformed bulk $S$-matrix to show that the transfer matrix contains a crucial additional factor which is essential for commutativity. Such a factor emerges for both the graded and non-graded versions of the $q$-deformed bulk $S$-matrices. These factors reduce to that obtained for the undeformed case as $q \rightarrow 1$ \cite{19}.
The outline of this note is as follows. In Section 2, we review the $q$-deformed bulk $S$-matrix that obeys standard YBE. Next, closely following the method outlined in [15], we reformulate another derivation of the crossing symmetry for the $q$-deformed bulk $S$-matrix [27] in Section 3. In Section 4, we construct two different commuting open-chain transfer matrices, first with non-graded $q$-deformed bulk $S$-matrix and the second, with graded version of the corresponding bulk $S$-matrix. Next, we argue an extra factor is necessary for both versions in order for the transfer matrices to commute. We conclude in Section 5 with a brief discussion of our results.

2 The $q$-deformed $SU(2|2)$-invariant bulk $S$-matrix

We first briefly review the action of the symmetry generators (three Cartan generators $h_j$, three simple positive roots $E_j$ and three simple negative roots $F_j$, $j = 1, 2, 3$) on the ZF operators, which we will denote by $A_i^\dagger(p)$, $i = 1, 2, 3, 4$ following [4, 15]. The generators $E_2, F_2$ are fermionic, while the remaining ones are bosonic. (Readers are urged to refer to [27, 30] for more detailed discussions.)

2.1 Bulk ZF algebra

The action of the symmetry generators on the ZF operators can be obtained from the following two requirements: The one-particle states $A_i^\dagger(p)|0\rangle$, $|0\rangle$ being the vacuum state, must form a fundamental representation of the symmetry algebra (see Eq. (2.55) in [27]); and multi-particle states must form higher (reducible) representations. Finally, together with the fact that the symmetry generators annihilate the vacuum state, the action of these generators on ZF operators is obtained. We review the results below which are reproduced from [30]:

A. The nontrivial commutators of the Cartan generators with the ZF operators are given by

$$
\begin{align*}
h_1 A_1^\dagger(p) &= -A_1^\dagger(p) + A_1^\dagger(p) h_1, \\
h_1 A_2^\dagger(p) &= A_3^\dagger(p) + A_2^\dagger(p) h_1, \\
h_3 A_3^\dagger(p) &= -A_3^\dagger(p) + A_3^\dagger(p) h_3, \\
h_3 A_4^\dagger(p) &= A_4^\dagger(p) + A_4^\dagger(p) h_3, \\
h_3 A_2^\dagger(p) &= -A_4^\dagger(p) + A_2^\dagger(p) h_2, \\
h_3 A_1^\dagger(p) &= A_2^\dagger(p) + A_1^\dagger(p) h_2, \\
h_2 A_3^\dagger(p) &= -(C - \frac{1}{2}) A_3^\dagger(p) + A_3^\dagger(p) h_2, \\
h_2 A_4^\dagger(p) &= -(C + \frac{1}{2}) A_4^\dagger(p) + A_4^\dagger(p) h_2,
\end{align*}
$$

(2.1)

$$
C = C(p) \text{ denotes the value of the corresponding central charge}
$$

$$
C = -\frac{1}{2} h_1 - h_2 - \frac{1}{2} h_3
$$

(2.2)
The remaining such commutators are trivial, \( h_j A^\dagger_j(p) = A^\dagger_j(p) h_j \).

B. The nontrivial commutators of the bosonic simple roots with the ZF operators are given by

\[
E_1 A^\dagger_1(p) = \frac{q^{1/2}}{h} A^\dagger_2(p) q^{-h_{11}/2} + q^{-1/2} A^\dagger_1(p) E_1, \quad E_1 A^\dagger_2(p) = \frac{q^{1/2}}{h} A^\dagger_2(p) E_1, \\
E_3 A^\dagger_3(p) = q^{-1/2} A^\dagger_3(p) q^{-h_{31}/2} + q^{1/2} A^\dagger_3(p) E_3, \quad E_3 A^\dagger_3(p) = q^{-1/2} A^\dagger_3(p) E_3, \\
F_1 A^\dagger_1(p) = q^{-1/2} A^\dagger_1(p) q^{-h_{11}/2} + q^{1/2} A^\dagger_1(p) F_1, \quad F_1 A^\dagger_1(p) = q^{-1/2} A^\dagger_1(p) F_1, \\
F_3 A^\dagger_3(p) = q^{1/2} A^\dagger_3(p) q^{-h_{31}/2} + q^{-1/2} A^\dagger_3(p) F_3, \quad F_3 A^\dagger_3(p) = q^{1/2} A^\dagger_3(p) F_3. \quad (2.3)
\]

The remaining such commutators are trivial,

\[
E_1 A^\dagger_\alpha(p) = A^\dagger_\alpha(p) E_1, \quad F_1 A^\dagger_\alpha(p) = A^\dagger_\alpha(p) F_1, \quad \alpha = 3, 4, \\
E_3 A^\dagger_\alpha(p) = A^\dagger_\alpha(p) E_3, \quad F_3 A^\dagger_\alpha(p) = A^\dagger_\alpha(p) F_3, \quad \alpha = 1, 2. \quad (2.4)
\]

C. Finally, the commutators of the fermionic generators with the ZF operators are given by

\[
E_2 A^\dagger_2(p) = e^{-ip/2} \left[ a(p) A^\dagger_4(p) q^{-h_{23}/2} + q^{-(C+\frac{1}{2})/2} A^\dagger_2(p) E_2 \right], \\
E_2 A^\dagger_3(p) = e^{-ip/2} \left[ b(p) A^\dagger_1(p) q^{-h_{32}/2} - q^{-(C-\frac{1}{2})/2} A^\dagger_3(p) E_2 \right], \\
F_2 A^\dagger_1(p) = e^{ip/2} \left[ c(p) A^\dagger_3(p) q^{-h_{13}/2} + q^{-(C-\frac{1}{2})/2} A^\dagger_1(p) F_2 \right], \\
F_2 A^\dagger_4(p) = e^{ip/2} \left[ d(p) A^\dagger_2(p) q^{-h_{42}/2} - q^{-(C+\frac{1}{2})/2} A^\dagger_4(p) F_2 \right], \quad (2.5)
\]

and

\[
E_2 A^\dagger_4(p) = e^{-ip/2} q^{-(C-\frac{1}{2})/2} A^\dagger_1(p) E_2, \quad E_2 A^\dagger_4(p) = -e^{-ip/2} q^{-(C+\frac{1}{2})/2} A^\dagger_4(p) E_2, \\
F_2 A^\dagger_2(p) = e^{ip/2} q^{-(C+\frac{1}{2})/2} A^\dagger_1(p) F_2, \quad F_2 A^\dagger_2(p) = -e^{ip/2} q^{-(C-\frac{1}{2})/2} A^\dagger_2(p) F_2. \quad (2.6)
\]

The functions \( a(p), b(p), c(p), d(p) \) are given below. (Refer to [27] and [30] for details used to set these functions.)

\[
a = \sqrt{g} \gamma q^{-C}, \\
b = \sqrt{g} \frac{1}{x^\gamma} \left( x^- - q^{2C-1} x^+ \right), \\
c = i\sqrt{g} \frac{q^{-C+\frac{1}{2}}}{x^+}, \\
d = i\sqrt{g} \frac{q^{-\frac{1}{2}}}{\gamma} \left( q^{2C+1} x^- - x^+ \right). \quad (2.7)
\]
with
\[ e^{ip} = \frac{x^+}{q x^-}. \] (2.8)
Together with (2.8), one also need the following constraint to determine \( x^\pm(p) \),
\[ \frac{x^+}{q} + \frac{q}{x^+} - q x^- - \frac{1}{q x^-} + ig(q - q^{-1}) \left( \frac{x^+}{q x^-} - \frac{q x^-}{x^+} \right) = \frac{i}{q}. \] (2.9)
As in [27], we leave \( \gamma \) unspecified at this point. We also recall that expressions for \( a \) and \( d \) in (2.7) differ from those in [27] by factors of \( q^{\mp C} \).

### 2.2 Nonzero matrix elements

The two-particle S-matrix (up to a phase) is determined by demanding that the symmetry generators commute with two-particle scattering: from \( J A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle \) where \( J \) is a symmetry generator, and assuming that \( J \) annihilates the vacuum state, one arrives at linear combinations of \( A_i(p_2) A_j^\dagger(p_1)|0\rangle \) in two different ways, by applying the ZF relation,
\[ A_i^\dagger(p_1) A_j^\dagger(p_2) = S^{i'j'}_{ij}(p_1, p_2) A_{j'}(p_2) A_{i'}(p_1), \] (2.10)
and the symmetry relations (2.1) - (2.6) in different orders. The consistency condition yields a system of linear equations for the \( S \)-matrix elements. The bulk \( S \)-matrix can then be defined by
\[ S = S^{i'j'}_{ij} e_{i'j'} \otimes e_{ij}, \] (2.11)
where \( S^{i'j'}_{ij} \) represents the bulk \( S \)-matrix elements and summation over repeated indices is implied. Noting that \( e_{ij} \) is the usual elementary \( 4 \times 4 \) matrix whose \((i, j)\) matrix element is 1, and all others are zero, (2.11) represents the arrangement of the matrix elements into a \( 16 \times 16 \) matrix. Below, we list the nonzero matrix elements for the \( q \)-deformed bulk \( S \)-matrix, reproduced from [30],
\[
\begin{align*}
S^{aa}_{aa} &= A, & S^{\alpha\alpha}_{\alpha\alpha} &= D, \\
S^{ab}_{ab} &= \frac{A - B}{q + q^{-1}}, & S^{ba}_{ab} &= \frac{q^{-\epsilon_{ab}} A + q^{\epsilon_{ab}} B}{q + q^{-1}}, \\
S^{\alpha\beta}_{\alpha\beta} &= \frac{D - E}{q + q^{-1}}, & S^{\beta\alpha}_{\alpha\beta} &= \frac{q^{-\epsilon_{\alpha\beta}} D + q^{\epsilon_{\alpha\beta}} E}{q + q^{-1}}, \\
S^{\alpha\beta}_{ab} &= \frac{q^{(\epsilon_{ab} - \epsilon_{\alpha\beta})/2} \epsilon_{ab} \epsilon^{\alpha\beta}}{q + q^{-1}}, & S^{ab}_{\alpha\beta} &= \frac{q^{(\epsilon_{\alpha\beta} - \epsilon_{ab})/2} \epsilon^{ab} \epsilon_{\alpha\beta}}{q + q^{-1}}, \\
S^{aa}_{a\alpha} &= \mathcal{L}, & S^{aa}_{a\alpha} &= \mathcal{K}, & S^{\alpha\alpha}_{a\alpha} &= \mathcal{H}, & S^{aa}_{a\alpha} &= \mathcal{G},
\end{align*}
\] (2.12)
where \(a, b \in \{1, 2\}\) with \(a \neq b\); \(\alpha, \beta \in \{3, 4\}\) with \(\alpha \neq \beta\); and

\[
\mathcal{A} = A_{21}^{BK} = S_0 q^{C_2 - C_1} e^{i(p_2 - p_1)/2} \frac{x_1^+ - x_2^-}{x_1^+ - x_2^-},
\]

\[
\mathcal{B} = B_{21}^{BK} = S_0 q^{C_2 - C_1} e^{i(p_2 - p_1)/2} \frac{x_1^+ - x_2^-}{x_1^+ - x_2^-} \left(1 - (q + q^{-1})q^{-1}x_1^+ - x_2^- \right) x_1^+ - x_2^- x_1^- - s(x_2^-)\right),
\]

\[
\mathcal{C} = q^{-(C_1 + C_2 - 1)/2} C_{21}^{BK} = S_0 (q + q^{-1}) i q^{(C_2 - 5C_1 - 2)/2} e^{i(p_2 - 2p_1)/2} \frac{q^{-1}x_1^+ - (q - q^{-1})}{x_1^- - s(x_2^-)} x_1^- - s(x_2^-),
\]

\[
\mathcal{D} = -S_0,
\]

\[
\mathcal{E} = E_{21}^{BK} = -S_0 \left(1 - (q + q^{-1})q^{-2C_1 - 1} e^{-ip_1} \frac{x_1^+ - x_2^-}{x_1^- - x_2^-} x_1^+ - s(x_2^-)\right),
\]

\[
\mathcal{F} = q^{(C_1 + C_2 - 1)/2} F_{21}^{BK} = -S_0 (q + q^{-1}) i q^{(5C_2 - C_1 - 2)/2} e^{i(2p_2 - p_1)/2} x_1^- - \frac{s(x_1^+)}{x_1^+ - s(x_2^-)} x_1^+ - s(x_2^-),
\]

\[
\mathcal{G} = G_{21}^{BK} = S_0 q^{-C_1 - 1/2} e^{-ip_1/2} x_1^+ - x_2^+ \frac{x_1^+ - x_2^+}{x_1^- - x_2^-},
\]

\[
\mathcal{H} = q^{(C_1 - C_2)/2} H_{21}^{BK} = S_0 q^{(C_1 - C_2)/2} \frac{\gamma_2 x_1^+ - x_1^-}{\gamma_1 x_1^- - x_2^+},
\]

\[
\mathcal{K} = q^{-(C_1 - C_2)/2} K_{21}^{BK} = S_0 q^{(C_2 - C_1)/2} e^{i(p_2 - p_1)/2} \frac{\gamma_1 x_2^+ - x_2^-}{\gamma_2 x_1^- - x_2^+},
\]

\[
\mathcal{L} = L_{21}^{BK} = S_0 q^{C_2 + 1/2} e^{ip_2/2} x_1^- - x_2^- \frac{x_1^- - x_2^-}{x_1^- - x_2^-},
\]

\[(2.13)\]

where \(A_{21}^{BK}, B_{21}^{BK}, \ldots \) denote the amplitudes \(A_{12}, B_{12}, \ldots \) in Table 2 of [27], respectively, with labels 1 and 2 interchanged. \(S_0\) is the overall scalar factor (denoted by \(R^0\) in [27]). The function \(s(x)\) is the “antipode map” defined by [27]

\[
s(x) = \frac{1 - ig(q - q^{-1})x}{x + ig(q - q^{-1})},
\]

\[(2.14)\]

which has the limit \(s(x) \to 1/x\) for \(q \to 1\). Furthermore, \(C_1 \equiv C(p_1), C_2 \equiv C(p_2)\) are determined from [27]

\[
q^{2C} = \frac{1}{q} \left(1 - ig(q - q^{-1})x^+ \right) = q \left(1 + ig(q - q^{-1})/x^+ \right).
\]

\[(2.15)\]
As pointed out in [30], the amplitudes \( \mathcal{C}, \mathcal{F}, \mathcal{H} \) and \( \mathcal{K} \) have extra factors involving powers of \( q \) with respect to the amplitudes given in [27]. Nevertheless, it has been verified that the above given \( S \)-matrix still satisfies the (standard) Yang-Baxter equation

\[
S_{12}(p_1, p_2) S_{13}(p_1, p_3) S_{23}(p_2, p_3) = S_{23}(p_2, p_3) S_{13}(p_1, p_3) S_{12}(p_1, p_2).
\] (2.16)

even without those extra factors. We use the standard convention \( S_{12} = S \otimes I, S_{23} = I \otimes S \), and \( S_{13} = P_{12} S_{23} P_{12} \), where \( P_{12} = P \otimes I \), \( P \) is the permutation matrix, and \( I \) is the four-dimensional identity matrix. In addition, as for the undeformed \( (q = 1) \) matrix, the \( q \)-deformed matrix has the following unitarity property

\[
S_{12}(p_1, p_2) S_{21}(p_2, p_1) = I,
\] (2.17)

provided that the bulk scalar factor obeys

\[
S_0(p_1, p_2) S_0(p_2, p_1) = 1
\] (2.18)

where \( S_{21} = P_{12} S_{12} P_{12} \), as well as the crossing property [27] which we shall describe in more detail in the following Section.

3 Crossing symmetry

In this Section, we reformulate the derivation of crossing equation for the \( q \)-deformed bulk \( S \)-matrix given in [27], following closely the method outlined in [15] in terms of ZF operators. This property is needed to construct the commuting open-chain transfer matrices. Following [15], we begin by defining the “singlet” operator

\[
I(p) = C^{ij}(p) A_i^\dagger(p) A_j^\dagger(\bar{p}) \equiv \alpha(p) A_1^\dagger(p) A_2^\dagger(\bar{p}) + \beta(p) A_2^\dagger(p) A_1^\dagger(\bar{p}) + \sqrt{q} A_3^\dagger(p) A_4^\dagger(\bar{p})
- \frac{1}{\sqrt{q}} A_4^\dagger(p) A_3^\dagger(\bar{p}),
\] (3.1)

where the functions \( \alpha(p), \beta(p) \) are yet to be determined. Hence, \( C(p) \) is the \( 4 \times 4 \) matrix

\[
C(p) = \begin{pmatrix}
0 & \alpha(p) & 0 & 0 \\
\beta(p) & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{q} \\
0 & 0 & -\frac{1}{\sqrt{q}} & 0
\end{pmatrix}.
\] (3.2)

\( \bar{p} = -p \) denotes the antiparticle momentum, with [27]

\[
x^\pm(\bar{p}) = s(x^\pm(p)),
\] (3.3)
Functions $\alpha(p), \beta(p)$ are determined by the conditions that the singlet operator commutes with the fermionic generators. Indeed, using (2.5) and (2.6), the condition $E_2 I(p)|0\rangle = I(p) E_2 |0\rangle = 0$ leads to

$$\alpha(p) = -e^{-ip/2} \sqrt{q} b(p) a(p) = e^{ip/2} \sqrt{q} b(\bar{p}) a(p) = i \sqrt{q} \text{sign}(p),$$

$$\beta(p) = -e^{ip/2} \frac{1}{\sqrt{q}} b(\bar{p}) = e^{-ip/2} \frac{1}{\sqrt{q}} b(p) = -i \frac{1}{\sqrt{q}} \text{sign}(p).$$ (3.4)

We remark that (3.2) reduces to the matrix $C(p)$ in [15] as $q \to 1$. We also note the following property of the matrix $C(p)$,

$$C(-p) = -C(p)^{-1}$$ (3.5)

which is needed to construct the desired transfer matrix. A particular choice for $\gamma(p)$ that evidently appears in $a(p), b(p), c(p), d(p)$ as given in (2.7) is

$$\gamma(p) = \sqrt{-q^C e^{ip/2} (x^+(p) - x^-(p))}$$ (3.6)

which can be used to verify the crossing property that follows, namely (3.9) and (3.10) below. $C$ in (3.6) again refers to the value of central charge. The expression for $\gamma$ used here is essentially the same as (2.65) in [27] up to a certain constant. This difference is presumably due to the fact that our expressions for $a$ and $d$ differ from those in [27] by factors of $q^\pm C^2$.

With regard to crossing property (3.9) and (3.10), (3.6) is also consistent with the bulk $S$-matrix presented in the last Section where the amplitudes $C, F, H, K$ have extra factors involving powers of $q$. As noticed in [27], (3.6) possesses nice properties analogous to that of the undeformed case [32]. Having found the matrix $C(p)$ (as given by (3.2) and (3.4)), we now begin the reformulation of crossing property for the $q$-deformed bulk $S$-matrix following closely the method outlined in [15] for the $q = 1$ case. As pointed out in [15] for $q = 1$ case, the requirement that the singlet operator scatter trivially with a particle, along with (2.10) and (3.1) lead to the following,

$$A_i^\dagger(p_1) I(p_2) = C^{jk}(p_2) A_j^\dagger(p_1) A_k^\dagger(p_2) A_k^\dagger(\bar{p}_2)$$

$$= C^{jk}(p_2) S^{ij'}_{ij}(p_1, p_2) A_{j'}^\dagger(p_2) A_k^\dagger(p_1) A_k^\dagger(\bar{p}_2)$$

$$= C^{jk}(p_2) S^{ij'}_{ij}(p_1, p_2) S^{nk'}_{nk}(p_1, \bar{p}_2) A_{j'}^\dagger(p_2) A_{k'}^\dagger(p_2) A_{k'}^\dagger(\bar{p}_2) A_{n'}^\dagger(p_1)$$

$$= I(p_2) A_i^\dagger(p_1)$$ (3.7)

1 Similar matrix is given in [27]

2 One could also use the relation [27] $\gamma(p) \gamma(\bar{p}) = -i(q^C e^{ip/2} - q^{-C} e^{-ip/2})$ to verify crossing property (3.9) and (3.10). Again, this expression is essentially the same as in [27] up to a certain constant presumably due to the difference in our expressions for $a$ and $d$ compared to those in [27].
that implies
\[ C^{jk}(p_2) S^{i'j'}(p_1, p_2) S^{i''k'}(p_1, \bar{p}_2) = C^{j'k'}(p_2) \delta_i^{i''}, \quad (3.8) \]

One can re-write the above equation in matrix notation as
\[ S_{21}^{t_2}(p_1, p_2) C_2(p_2) S_{12}(p_1, \bar{p}_2) C_2(p_2)^{-1} = I. \quad (3.9) \]

which is the desired crossing property for the \( q \)-deformed bulk \( S \)-matrix. The following equivalent form of (3.9) can be obtained by applying the permutation and exchanging \( p_1 \) and \( p_2 \),
\[ S_{21}^{t_1}(p_2, p_1) C_1(p_1) S_{21}(p_2, \bar{p}_1) C_1(p_1)^{-1} = I. \quad (3.10) \]

In (3.9) and (3.10) above, \( C_1 = C \otimes I \), \( C_2 = I \otimes C \); \( t_1 \) and \( t_2 \) are the transposition in the first and the second space respectively. Using (2.12), (2.13), (3.2) and (3.4) in (3.9), one obtains,
\[ S_0(p_1, p_2) S_0(p_1, \bar{p}_2) = \frac{1}{f(p_1, p_2)} \quad (3.11) \]

for the bulk scalar factor where \(^{3}\)
\[ f(p_1, p_2) = \frac{1}{q} \frac{(s(x_1^+ - x_2^-)(x_1^+ - x_2^+))}{(s(x_1^- - x_2^-)(x_1^- - x_2^+))}. \quad (3.12) \]

It would be interesting to find a solution of (3.11).

4 Transfer matrix

In this section, we present the Sklyanin’s construction (31) of the open chain transfer matrix. Bulk and boundary \( S \)-matrices are the two main building blocks of the transfer matrix. While the \( q \)-deformed bulk \( S \)-matrix is as given in (2.12) and (2.13) which obeys the standard YBE (2.16), the \( q \)-deformed right boundary \( S \)-matrix \( R^-(p) \) is a diagonal matrix found in (30) for the \( Y = 0 \) giant graviton brane,
\[ R^-(p) = \text{diag}(-e^{ip} \gamma(p), e^{-ip} \gamma(-p), 1, 1). \quad (4.1) \]

in a basis where the standard (right) boundary Yang-Baxter equation (BYBE) (23, 24)
\[ S_{12}(p_1, p_2) R_1^-(p_1) S_{21}(p_2, -p_1) R_2^-(p_2) = R_2^-(p_2) S_{12}(p_1, -p_2) R_1^-(p_1) S_{21}(-p_2, -p_1) \quad (4.2) \]

\(^{3}\)Refer to (27) for a number of other equivalent forms for \( f(p_1, p_2) \)
is satisfied. Note that in the $q \to 1$ limit, (4.1) reduces to the corresponding undeformed boundary $S$-matrix in [15]. We also recall from [30],

$$x^\pm(-p) = -\frac{1}{s(x^+(p))}$$

which is crucial to study boundary scattering. The following monodromy matrices can then be constructed from the bulk $S$-matrix,

$$T_a(p; \{p_i\}) = S_{aN}(p, p_N) \cdots S_{a1}(p, p_1),$$

$$\hat{T}_a(p; \{p_i\}) = S_{1a}(p_1, -p) \cdots S_{Na}(p_N, -p),$$

where $\{p_1, \ldots, p_N\}$ are arbitrary “inhomogeneities” associated with each of the $N$ quantum spaces, and the auxiliary space is denoted by $a$. The quantum-space “indices” are suppressed from the monodromy matrices. Further, the “decorated” right boundary $S$-matrix given by

$$T_a^-(p; \{p_i\}) = T_a(p; \{p_i\}) R_a^-(p) \hat{T}_a(p; \{p_i\})$$

also satisfies the BYBE, i.e.,

$$S_{ab}(p_a, p_b) T_a^-(p_a; \{p_i\}) S_{ba}(p_b, -p_a) T_a^-(p_b; \{p_i\}) = T_b^-(p_b; \{p_i\}) S_{ab}(p_a, -p_b) T_a^-(p_a; \{p_i\}) S_{ba}(-p_b, -p_a),$$

because of (4.2) and the following relations obeyed by the monodromy matrices,

$$S_{ab}(p_a, p_b) T_a(p_a; \{p_i\}) T_b(p_b; \{p_i\}) = T_b(p_b; \{p_i\}) T_a(p_a; \{p_i\}) S_{ab}(p_a, p_b),$$

$$S_{ba}(-p_b, -p_a) \hat{T}_a(p_a; \{p_i\}) \hat{T}_b(p_b; \{p_i\}) = \hat{T}_b(p_b; \{p_i\}) \hat{T}_a(p_a; \{p_i\}) S_{ba}(-p_b, -p_a),$$

$$\hat{T}_a(p_a; \{p_i\}) S_{ba}(p_b, -p_a) T_b(p_b; \{p_i\}) = T_b(p_b; \{p_i\}) S_{ba}(p_b, -p_a) \hat{T}_a(p_a; \{p_i\}).$$

Following Sklyanin [31], we assume that the open-chain transfer matrix is of the double-row form

$$t(p; \{p_i\}) = \text{tr}_a R_a^+(p) T_a^-(p; \{p_i\})$$

$$= \text{tr}_a R_a^+(p) T_a(p; \{p_i\}) R_a^-(p) \hat{T}_a(p; \{p_i\}),$$

where the trace is taken over the auxiliary space, and the left boundary $S$-matrix $R^+(p)$ is chosen to ensure the essential commutativity property

$$[t(p; \{p_i\}), t(p'; \{p_i\})] = 0$$

(4.9)
for arbitrary values of \( p \) and \( p' \). Making use of the unitarity and crossing properties (2.17), (3.9) and (3.10), we find that the commutativity property is indeed obeyed, provided that \( R^+(p) \) obeys

\[
S_{21}(p_2, p_1)^{t_{12}} R^+_1(p_1)^{t_1} C_1(p_1)^{-1} S_{21}(p_2, -p_1)^{t_2} C_1(p_1) R^+_2(p_2)^{t_2} = R^+_2(p_2)^{t_2} C_2(p_2)^{-1} S_{12}(p_1, -p_2)^{t_1} C_2(p_2) R^+_1(p_1)^{t_1} S_{12}(-p_2, -p_1)^{t_{12}}. \tag{4.10}
\]

where as defined in (3.3), \( \bar{p} \) denotes the antiparticle momentum. In obtaining this result, we also make use of (3.5) and the following identity

\[
f(p_1, p_2) = f(-p_2, -p_1) \tag{4.11}
\]

for the function defined in (3.12). Using (3.9) and (3.10), (4.10) can be simplified to yield

\[
S_{12}(p_1, p_2) M_1^{-1} R_1^+(-p_1) S_{21}(p_2, -p_1) M_2^{-1} R_2^+(-p_2) = M_2^{-1} R_2^+(-p_2) S_{12}(p_1, -p_2) M_1^{-1} R_1^+(-p_1) S_{21}(-p_2, -p_1), \tag{4.12}
\]

where the matrix \( M \) is given by

\[
M = C(p)^t C(p) = \text{diag} \left( -1/q, -q, 1/q, q \right). \tag{4.13}
\]

where \( C(p)^t \) is the transpose of \( C(p) \). In obtaining (4.12), we make use of the identities

\[
f(p_1, p_2) = f(-p_2, -p_1) \tag{4.14}
\]

and

\[
M_1 S_{12}(p_1, p_2) M_2^{-1} = M_2^{-1} S_{12}(p_1, p_2) M_1. \tag{4.15}
\]

or equivalently

\[
M_1^{-1} S_{12}(p_1, p_2) M_2 = M_2 S_{12}(p_1, p_2) M_1^{-1}. \tag{4.16}
\]

Comparing the \( R^+(p) \) relation (4.12) with the \( R^-(p) \) relation (4.7), we conclude that the left boundary \( S \)-matrix is given by

\[
R^+(p) = M R^-(-p), \tag{4.17}
\]

where \( M \) is given by (4.13). We emphasize that this matrix \( M \) is essential in order for the transfer matrix (4.8) to have the commutativity property (4.9). We have verified (4.9) numerically for small numbers of sites.

\[4\]Equivalent relation appears in [19].

\[5\]Similar matrices also appear in the construction of open-chain transfer matrices in [34] and [35].
We also find that if we work instead with corresponding graded quantities \(^6\) with the following parity assignments

\[ p(1) = p(2) = 0, \quad p(3) = p(4) = 1, \] (4.18)

and define the graded bulk \(S\)-matrix by (see, e.g., \([5]\))

\[ S^g(p_1, p_2) = \mathcal{P}^g \mathcal{P} S(p_1, p_2), \] (4.19)

where \(\mathcal{P}^g\) is the graded permutation matrix

\[ \mathcal{P}^g = \sum_{i,j=1}^4 (-1)^{p(i)p(j)} e_{i\,j} \otimes e_{j\,i}, \] (4.20)

the transfer matrix

\[ t(p; \{p_i\}) = \text{str}_a R^+_a(p) T_a(p; \{p_i\}) R^-_a(p) \hat{T}_a(p; \{p_i\}). \] (4.21)

satisfies the commutativity property (4.9) provided \(R^+(p)\) is given by (4.17) with \(M = \text{diag}(1/q, q, 1/q, q)\), namely

\[ t(p; \{p_i\}) = \text{str}_a MR^-_a(-p) T_a(p; \{p_i\}) R^-_a(p) \hat{T}_a(p; \{p_i\}). \] (4.22)

In (4.22), \(\text{str}\) denotes the supertrace, the monodromy matrices are formed as in (4.4) except with the graded \(S\)-matrix (4.19) using the graded tensor product (instead of the ordinary tensor product), and \(R^-(p)\) is again given by (4.1), which also satisfies the graded BYBE. We have again numerically verified (4.9) for (4.22) with Mathematica for small numbers of sites.

5 Discussion

We have presented a commuting open-chain transfer matrix given by (4.8) constructed from the \(q\)-deformed \(SU(2|2)\) bulk and boundary \(S\)-matrices, where \(T_a(p; \{p_i\})\) and \(\hat{T}_a(p; \{p_i\})\) are given by (4.4), and \(R^+(p)\) is given by (4.17), which contains the factor \(M\) (4.13). Alternatively, using graded version of the bulk \(S\)-matrix, we also constructed a transfer matrix (4.22) which still seems to include similar extra factor (unlike the undeformed case where such a factor can be avoided using graded versions of the \(S\)-matrices). These transfer matrices reduce to that obtained for the undeformed case when \(q \to 1\) \([15]\).  

\(^6\)See for example \([33]\) and \([34]\).
An interesting problem is to solve the open versions of the deformed Hubbard models based on the deformed boundary $S$-matrix and the corresponding transfer matrix. It will also be interesting to construct corresponding open chain transfer matrices that exclude the extra factor $M$. Perhaps such a construction will be more convenient for the formulation of the Bethe-Yang equation on an interval with boundaries. We hope to be able to address these issues in the future.

Acknowledgments

I would like to thank R. I. Nepomechie for his collaboration on earlier related projects.

References

[1] N. Beisert, “The $su(2|2)$ dynamic $S$-matrix,” *Adv.Theor.Math.Phys.* **12**, 945 (2008) [arXiv:hep-th/0511082].
N. Beisert, “The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry,” *J. Stat. Mech.* **0701**, P017 (2007) [arXiv:nlin/0610017].

[2] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, “The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring,” *J. Phys.* **A40**, 3583 (2007) [arXiv:hep-th/0609157].

[3] A.A. Tseytlin, “Spinning strings and AdS/CFT duality,” in Ian Kogan Memorial Volume, *From Fields to Strings: Circumnavigating Theoretical Physics*, M. Shifman, A. Vainshtein, and J. Wheater, eds. (World Scientific, 2004) [arXiv:hep-th/0311139].
N. Beisert, “The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory and integrability,” *Phys. Rept.* **405**, 1 (2005) [arXiv:hep-th/0407277];
K. Zarembo, “Semiclassical Bethe ansatz and AdS/CFT,” *Comptes Rendus Physique* **5**, 1081 (2004) [*Fortsch. Phys.* **53**, 647 (2005)] [arXiv:hep-th/0411191];
J. Plefka, “Spinning strings and integrable spin chains in the AdS/CFT correspondence,” *Living Rev. Rel.* **8**, 9 (2005) [arXiv:hep-th/0507136];
J.A. Minahan, “A brief introduction to the Bethe ansatz in $\mathcal{N} = 4$ super-Yang-Mills,” *J. Phys.* **A39**, 12657 (2006);
K. Okamura, “Aspects of Integrability in AdS/CFT Duality,” [arXiv:0803.3999].

[4] G. Arutyunov, S. Frolov and M. Zamaklar, “The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring,” *JHEP* **0704**, 002 (2007) [arXiv:hep-th/0612229].
[5] M.J. Martins and C.S. Melo, “The Bethe ansatz approach for factorizable centrally extended $S$-matrices,” *Nucl. Phys.* **B785**, 246 (2007) [arXiv:hep-th/0703086].

[6] M. de Leeuw, “Coordinate Bethe Ansatz for the String $S$-Matrix,” *J. Phys.* **A40**, 14413 (2007) [arXiv:0705.2369].

[7] N. Beisert and M. Staudacher, “Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings,” *Nucl. Phys.* **B727**, 1 (2005) [arXiv:hep-th/0504190].

[8] D. Berenstein and S.E. Vazquez, “Integrable open spin chains from giant gravitons,” *JHEP* **0506**, 059 (2005) [arXiv:hep-th/0501078].

[9] T. McLoughlin and I. Swanson, “Open string integrability and AdS/CFT,” *Nucl. Phys.* **B723**, 132 (2005) [arXiv:hep-th/0504203].

[10] A. Agarwal, “Open spin chains in super Yang-Mills at higher loops: Some potential problems with integrability,” *JHEP* **0608**, 027 (2006) [arXiv:hep-th/0603067]; K. Okamura and K. Yoshida, “Higher loop Bethe ansatz for open spin-chains in AdS/CFT,” *JHEP* **0609**, 081 (2006) [arXiv:hep-th/0604100].

[11] N. Mann and S.E. Vazquez, “Classical open string integrability,” *JHEP* **0704**, 065 (2007) [arXiv:hep-th/0612038].

[12] D.M. Hofman and J.M. Maldacena, “Reflecting magnons,” *JHEP* **0711**, 063 (2007) [arXiv:0708.2272].

[13] H.Y. Chen and D.H. Correa, “Comments on the Boundary Scattering Phase,” *JHEP* **0802**, 028 (2008) [arXiv:0712.1361].

[14] C. Ahn, D. Bak and S.J. Rey, “Reflecting Magnon Bound States,” *JHEP* **0804**, 050 (2008) [arXiv:0712.4144].

[15] C. Ahn and R.I. Nepomechie, “The Zamolodchikov-Faddeev algebra for open strings attached to giant gravitons,” *JHEP* **0805**, 059 (2008) [arXiv:0804.4036].

[16] N. Beisert and F. Loebbert, “Open Perturbatively Long-Range Integrable $gl(N)$ Spin Chains,” *Adv. Sci. Lett.* **2**, 261 (2009) [arXiv:0805.3260].

[17] L. Palla, “Issues on magnon reflection,” *Nucl.Phys.* **B808**, 205 (2009) [arXiv:0807.3646].

[18] D.H. Correa and C.A.S. Young, “Reflecting magnons from D7 and D5 branes,” *J.Phys.* **A41**, 455401 (2008) [arXiv:0808.0452].
[19] R. Murgan and R. I. Nepomechie, “Open-chain transfer matrices for AdS/CFT,” *JHEP* **0809**, 085 (2008) [arXiv:0808.2629].

[20] R. I. Nepomechie and E. Ragoucy, “Analytical Bethe ansatz for the open AdS/CFT $SU(1|1)$ spin chain,” *JHEP* **0812**, 025 (2008) [arXiv:0810.5015].

[21] W. Galleas, “The Bethe ansatz equations for reflecting magnons,” [arXiv:0902.1681].

[22] R. I. Nepomechie, “Bethe ansatz equations for open spin chains from giant gravitons,” *JHEP* **0905**, 100 (2009) [arXiv:0903.1646].

[23] D.H. Correa and C.A.S. Young, “Finite size corrections for open strings/open chains in planar AdS/CFT,” [arXiv:0905.1700].

[24] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” *JHEP* **0006**, 008 (2000) [arXiv:hep-th/0003075]; M.T. Grisaru, R.C. Myers and O. Tafjord, “SUSY and Goliath,” *JHEP* **0008**, 040 (2000) [arXiv:hep-th/0008015]; A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” *JHEP* **0008**, 051 (2000) [arXiv:hep-th/0008016].

[25] I.V. Cherednik, “Factorizing particles on a half line and root systems,” *Theor. Math. Phys.* **61**, 977 (1984).

[26] S. Ghoshal and A.B. Zamolodchikov, “Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory,” *Int. J. Mod. Phys.* **A9**, 3841 (1994) [arXiv:hep-th/9306002].

[27] N. Beisert and P. Koroteev, “Quantum Deformations of the One-Dimensional Hubbard Model,” *J. Phys.* **A41**, 255204 (2008). [arXiv:0802.0777].

[28] F.C. Alcaraz and R.Z. Bariev, “Interpolation between Hubbard and supersymmetric $t−J$ models. Two-parameter integrable models of correlated electrons,” *J. Phys.* **A32**, L483 (1999) [arXiv:cond-mat/9908265].

[29] F.H.L. Essler, H. Frahm, F. Göhmann, A. Klümper and V.E. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, 2005).

[30] R. Murgan and R.I. Nepomechie, “$q$-deformed $su(2|2)$ boundary $S$-matrices via the ZF algebra,” *JHEP* **0806**, 096 (2008) [arXiv:0805.3142].

[31] E.K. Sklyanin, “Boundary conditions for integrable quantum systems,” *J. Phys.* **A21**, 2375 (1988).
[32] G. Arutyunov and S. Frolov, “On string S-matrix, bound states and TBA,” *JHEP* **0712**, 024 (2007) [arXiv:0710.1568].

[33] A. Foerster and M. Karowski, “The supersymmetric t-J model with quantum group invariance,” *Nucl. Phys.* **B408**, 512 (1993);
A. González-Ruiz, “Integrable open-boundary conditions for the supersymmetric t-J model. The quantum group invariant case,” *Nucl. Phys.* **B424**, 468 (1994) [arXiv:hep-th/9401118];
R.H. Yue, H. Fan and B.Y. Hou, “Exact diagonalization of the quantum supersymmetric $SU_q(n|m)$ model,” *Nucl. Phys.* **B462**, 167 (1996) [cond-mat/9603022];
M. Shiroishi and M. Wadati, “Integrable Boundary Conditions for the One-Dimensional Hubbard Model,” *J. Phys. Soc. Jpn.* **66**, 2288 (1997) [arXiv:cond-mat/9708011];
X.-W. Guan, “Algebraic Bethe ansatz for the one-dimensional Hubbard model with open boundaries,” *J. Phys.* **A33**, 5391 (2000) [arXiv:cond-mat/9908054];
D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, “General boundary conditions for the $sl(N)$ and $sl(M|N)$ open spin chains,” *J. Stat. Mech.* **P08005**, 1 (2004) [math-ph/0406021].

[34] A.J. Bracken, X.-Y. Ge, Y.-Z. Zhang and H.-Q. Zhou, “Integrable open-boundary conditions for the $q$-deformed supersymmetric $U$ model of strongly correlated electrons,” *Nucl. Phys.* **B516**, 588 (1998) [arXiv:cond-mat/9710141].

[35] L. Mezincescu and R.I. Nepomechie, “Integrable open spin chains with nonsymmetric $R$ matrices,” *J. Phys.* **A24**, L17 (1991).