WONG-ZAKAI APPROXIMATION AND SUPPORT THEOREM FOR SEMILINEAR SPDES WITH FINITE DIMENSIONAL NOISE IN THE WHOLE SPACE

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Abstract. In this paper we consider the following stochastic partial differential equation (SPDE) in the whole space:
\[ du(t, x) = [a^{ij}(t, x) \partial_{ij}u(t, x) + f(u, t, x)] dt + \sum_{k=1}^{m} g_k(u(t, x)) dw_k(t). \]
We prove the convergence of a Wong-Zakai type approximation scheme of the above equation in the space \( C^\theta([0, T], H^\gamma_p(\mathbb{R}^d)) \) in probability, for some \( \theta \in (0, 1/2), \gamma \in (1, 2), \) and \( p > 2. \) We also prove a Stroock-Varadhan’s type support theorem.

To prove the results we combine V. Mackeviˇ cius ideas from his papers on Wong-Zakai theorem and the support theorem for diffusion processes with N.V. Krylov’s \( L_p \)-theory of SPDEs.

1. Introduction

It was first noted and proved by E. Wong and M. Zakai in [32] (see also [31]) that if \( w_n \) is a sequence of approximations of a standard Wiener process \( w, \) then, under certain conditions on \( w_n, \sigma, x_0, \) the solution of the equation
\[ dx_n(t) = \sigma(x_n(t)) dw_n(t), \quad x_n(0) = x_0 \]
converges locally uniformly a.s. to the solution of the equation
\[ dx(t) = \sigma(x(t)) dw(t) + 1/2 (\sigma D\sigma)(x(t)) dt, \quad x(0) = x_0. \]
Here, \( D\sigma \) is the derivative of \( \sigma. \) After this result was discovered, there has been an extensive research around this phenomenon in stochastic differential equations (SDEs). Let us mention two articles that were the starting point of this work. In [22] V. Mackeviˇ cius proved a Wong-Zakai type theorem for an SDE driven by a multidimensional semimartingale essentially by integrating by parts in a stochastic integral, and later I. Gyöngy in [8] generalized his result using the same technique. By the way, one can modify the above approximation scheme, so that the limiting equation does not have the correction term \( 1/2 (\sigma D\sigma)(x(t)) dt. \) In particular, by the same method of [22] we have \( y_n \to y \) as \( n \to \infty \) locally uniformly in probability, where
\[ dy_n(t) = \sigma(y_n(t)) dw_n(t) - 1/2 (\sigma D\sigma)(y_n(t)) dt, \quad y_n(0) = x_0, \]
\[ dy(t) = \sigma(y(t)) dw(t), \quad y(0) = x_0. \]
This fact can be used in practice to approximate certain SDEs.

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It turns out that a similar phenomenon occurs in stochastic evolution equations that are, roughly speaking, described as follows:

\[ \frac{du(\omega, t)}{dt} = [A(\omega, t)u(\omega, t) + B(\omega, t, u(\omega, t))] dt + \sum_{k=1}^{m} G^k(\omega, t, u(\omega, t)) dw^k(t). \]  

(1.1)

Here, \(A(\omega, t), B(\omega, t, \cdot), G^k(\omega, t, \cdot)\) are mappings between some Banach spaces, \(A(\omega, t)\) is a linear map, and \(w^k\) is a sequence of independent standard Wiener processes, where \(k = 1, \ldots, m\), and \(m\) is finite. In this case, one expects that the sequence of solutions of equations with regularized noise \(w^k_n\) converges to the solution of the equation with some correction term. There is a number of papers with Wong-Zakai type theorems established for various instances of operators \(A, B, G^k\). Here, we discuss only those articles that cover the case when \(A(\omega, t)\) is a second order elliptic operator. The case when \(G^k(\omega, t, \cdot)\) is a linear operator was covered in many articles such as [1], [2], [3], [7], [9] - [11]. The rate of convergence was obtained in [12] (see also [27]) and [13]. Results for the nonlinear case can be found in [6], [25], [28], [29]. We also mention papers [4], [5], [14], [15] on one-dimensional parabolic SPDEs with infinite-dimensional noise (i.e. \(m = \infty\)). Of these works only in [15] is the noise term linear in \(u\), and in the others it is semilinear.

Let \(d \geq 1\) be an integer, and let \(\mathbb{R}^d\) be a Euclidean space of points \(x = (x_1, \ldots, x_d)\). In this article we consider an SPDE in \(\mathbb{R}^d\) of the following form:

\[ du(t, x) = \left[a^{ij}(t, x)D_{ij}u(t, x) dt + f(u(t, x))\right] dt + \sum_{k=1}^{m} g^k(u(t, x)) dw^k(t), \quad u(0, x) = u_0(x). \]  

(1.2)

Here and throughout this article we assume the summation with respect to indexes \(i, j\). The assumptions are stated in the Section 2. Let us just mention that \(a\) is a uniformly nondegenerate bounded matrix-valued function, which is Lipschitz in \(x\), and \(f(u, t, x)\) is a 'first-order' term. We construct Wong-Zakai approximations by regularizing \(w^k\) and subtracting the Stratonovich correction term:

\[ du_n(t, x) = \left[a^{ij}(t, x)D_{ij}u_n(t, x) + f(u_n(t, x))\right] dt - \frac{1}{2} \sum_{k=1}^{m} (g^k D g^k)(u_n(t, x))] dt + \sum_{k=1}^{m} g^k(u_n(t, x)) dw^k_n(t), \quad u_n(0, x) = u_0(x). \]  

(1.3)

We prove the convergence of the approximation sequence to the solution of (1.2) in the space \(C^{\gamma/2 - 1/p}([0, T], X)\) in probability, where \(X = H^{2-\mu}_p(\mathbb{R}^d)\) is the space of Bessel potentials, and \(p > 2 + d\), and \(\gamma\) and \(\mu\) are numbers such that \(1 - d/p > \mu > \gamma > 2/p\). In addition, we also prove a Stroock-Varadhan's type support theorem for the solution of the equation (1.2). Here, we will use V. Mackevičius approach to the characterization of the topological support of a diffusion process (see also [24]). In [23] he showed
that for SDEs the support theorem can be proved using a Wong-Zakai type
approximation result combined with Girsanov’s theorem. Later, in [10] (see
also [11]) I. Gyöngy, adopting methods from [23], proved a support theorem
for a linear SPDE on the whole space with unbounded coefficients and a finite
dimensional noise term. In [25] and [30] support theorems were proved for
SPDEs in a Hilbert space \( H \) driven by an \( H \)-valued Wiener process. Let
us also mention papers [4] and [5] where support theorems were established
for solutions of a nonlinear heat equation and Burgers equation driven by a
space-time white noise.

We briefly explain how the aforementioned articles on Wong-Zakai prob-
lems differ from this work. The results of [25], [28], [29] imply the con-
vergence of Wong-Zakai approximations of equations of the type (1.2) in
the space \( C([0,T],H) \), where \( H \) is a Hilbert space, and the convergence
holds either in probability or in distribution. Perhaps, the closest to ours
result was proved in [6]. For any positive integer \( p \), and \( \kappa \in (0,1-d/p) \),
pathwise convergence in the space \( C^{\kappa/2}([0,T],H^p_\kappa(R^d)) \) of Wong-Zakai
approximations was proved for the equation (1.2) with \( a^{ij} = \delta_{ij} \), \( f \equiv 0 \), \( h \equiv 0 \)
and a space-dependent nonlinearity in the stochastic term (i.e. \( g^k(u(t,x)) \) is
replaced by \( g^k(x,u(t,x)) \)). In addition, the authors explained why a similar
result should be true for a nonlinear SPDE of divergence form (i.e \( \Delta u \)
is replaced by \( D_ia^{ij}(x)D_ju(t,x) \)). However, in [6] it is assumed that for
each \( y \in \mathbb{R} \) the function \( x \to g^k(x,y) \) has a compact support. This rules out
the case that we are interested in. Finally, in the papers [4], [5], [14] SPDEs
are not considered on the whole space.

Let us delineate the key steps of the proof of the Wong-Zakai type theo-
rem of this article. First, following V. Mackevičius (see [22]), we split \( w^k_n \) by a
‘regular part’ \( w^k_n - w^k \) and the noise term \( w^k \). Since \( w^k_n - w^k \) converges
to 0 as \( n \to \infty \) (see Appendix A), it makes sense to integrate by parts in
the integral \( g^k(u_n(t,x))d[w^k_n(t) - w^k(t)] \). It turns out that to get rid of
‘divergent’ terms from the equation (1.2), one needs to integrate by parts
one more time. As a result, we obtain that a function related to the error
of the approximation scheme satisfies a certain SPDE. We show that this
error is small by using an a priori estimate from N.V. Krylov’s \( L_p \)-theory
of SPDEs, and this allows us to prove the desired convergence.

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SPDEs in honour of G. Da Prato”, where he had an opportunity to present
the results of this paper and discuss it with other participants.

2. Statement of the main results

Basic notations and definitions. Let \( (\Omega, \mathcal{F}, P) \) be a complete probability
space, and let \( (\mathcal{F}_t, t \geq 0) \) be an increasing filtration of \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \)
containing all \( P \)-null sets of \( \Omega \). By \( \mathcal{P} \) we denote the predictable \( \sigma \)-field
generated by \((\mathcal{F}_t, t \geq 0)\). Let \(m\) be a positive integer, and \(\{w^i(t), t \in \mathbb{R}, i = 1, \ldots, m\}\) be a sequence of independent standard Wiener processes such that \(w^i(t) = 0, t \leq 0 \forall i\).

Denote when it makes sense

\[ D_t = \frac{\partial}{\partial x_t}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \]

and let \(u_{xx}\) be the matrix of second order derivatives of \(u\). For a function \(f : \mathbb{R} \to \mathbb{R}\), and any integer \(k \geq 2\), we denote

\[ Df = \frac{df}{dx}, \quad D^k f = \frac{d^k f}{dx^k}. \]

Let \(B(\mathbb{R}^d)\) be the space of bounded Borel functions, \(C^k(\mathbb{R}^d)\) be the space of bounded \(k\) times differentiable functions with bounded derivatives up to order \(k\), \(C_0^\infty(\mathbb{R}^d)\) be the space of infinitely differentiable functions with compact support, \(C^{k+\alpha}(\mathbb{R}^d)\) be the usual Hölder space, where \(k\) is a nonnegative integer, and \(\alpha \in (0, 1)\). For a Banach space \(X\), and finite \(T > 0\), we denote by \(C^k(\mathbb{R}^d)\) the space of \(X\)-valued functions that are Hölder continuous in the time variable. For \(p \in (1, \infty)\), we denote by \(L_p(\mathbb{R}^d)\) the space of \(L_p\)-integrable functions, and by \(W_p^k(\mathbb{R}^d)\) and \(W_p^{r,k}([0, T] \times \mathbb{R}^d)\) we mean the usual Sobolev space and parabolic Sobolev space. We introduce the spaces of Bessel potentials as follows:

\[ H_p^\gamma(\mathbb{R}^d) := (1 - \Delta)^{-\gamma/2} L_p(\mathbb{R}^d), \quad H^\gamma_p(\mathbb{R}^d, t_2) := (1 - \Delta)^{-\gamma/2} L_p(\mathbb{R}^d, t_2). \]

Here, \(\gamma \in \mathbb{R}\), and \(t_2\) is the set of all sequences of real numbers \(h = (h^k, k = 1, \ldots)\) such that \(|h|_{t_2}^2 = \sum_{k=1}^\infty |h^k|^2 < \infty\). For a distribution \(f\) and a sequence of distributions \(h = (h^k, k = 1, \ldots)\), we denote when it makes sense

\[ ||f||_{\gamma, p} = ||(1 - \Delta)^{\gamma/2} f||_p, \quad ||h||_{\gamma, p} = |||(1 - \Delta)^{\gamma/2} h||_{t_2}||_p. \]

where \(|| \cdot ||_p\) stands for the \(L_p\) norm. For a distribution \(f\) and a test function \(g \in C_0^\infty(\mathbb{R}^d)\), we denote the action of \(f\) on \(g\) by \((f, g)\).

By \(N(\ldots)\) we denote a constant depending only on the parameters inside the parenthesis. A constant \(N\) might change from inequality to inequality. In some cases where it is clear what parameters \(N\) depends on, we will omit listing them.

The following facts about \(H^\gamma_p(\mathbb{R}^d)\) spaces will be used in the sequel sometimes without mentioning them. First, for a nonnegative integer \(\gamma\), the spaces \(W_p^\gamma(\mathbb{R}^d)\) and \(H^\gamma_p(\mathbb{R}^d)\) coincide as sets and have equivalent norms, i.e there exists \(N(d, p, \gamma) > 0\) such that, for all \(u \in H^\gamma_p(\mathbb{R}^d)\),

\[ N||u||_{\gamma, p} \leq ||u||_{W_p^\gamma(\mathbb{R}^d)} \leq N^{-1}||u||_{\gamma, p}. \]

Second,

\[ ||f||_{\gamma_1, p} \leq ||f||_{\gamma_2, p}. \]
if $\gamma_1 \leq \gamma_2, p > 1$. The proof of these facts and the detailed discussion of $H_p^\gamma(\mathbb{R}^d)$ spaces can be found in Chapter 13 of [19].

For any stopping time $\tau$, we denote $(0, \tau] = \{(\omega, t) : 0 < t \leq \tau(\omega)\}$, and
\[
\mathbb{H}_p^\gamma(\tau) := L_p((0, \tau], \mathcal{P}, H_p^\gamma(\mathbb{R}^d)), \quad \mathbb{I}_p^\gamma(\tau, l_2) := L_p((0, \tau], \mathcal{P}, H_p^\gamma(\mathbb{R}^d, l_2)),
\]
\[
\mathbb{I}_p(\tau) := L_p((0, \tau], \mathcal{P}, L_p(\mathbb{R}^d)).
\]

We define stochastic Banach spaces $\mathcal{H}_p^\gamma(\tau)$.

**Definition 2.1.** For any $p \geq 2, \gamma \in \mathbb{R}$, and any stopping time $\tau$, we write that $u \in \mathcal{H}_p^\gamma(\tau)$ if

1. $u$ is a distribution-valued process, and $u \in \cap_{T > 0} \mathbb{H}_p^\gamma(\tau \wedge T)$,
2. $u_{xx} \in \mathbb{H}_p^{\gamma - 2}(\tau)$, $u(0, \cdot) \in L_p(\Omega, \mathcal{F}_0, H_p^{\gamma - 2/p}(\mathbb{R}^d))$,
3. there exist $f \in \mathbb{H}_p^{\gamma - 2}(\tau)$ and a sequence of distributions $h = (h^k, k = 1, \ldots) \in \mathbb{H}_p^{\gamma - 1}(\tau, l_2)$ such that, for any $\phi \in C_0^\infty(\mathbb{R}^d)$, and any $t \geq 0, \omega \in \Omega$,
\[
(u(t \wedge \cdot, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^{t \wedge \tau} (f(s, \cdot), \phi) \, ds + \sum_{k=1}^{\infty} \int_0^{t \wedge \tau} (h^k(s, \cdot), \phi) \, dw^k(s),
\]
(2.1)

The norm is defined in the following way:
\[
||u||_{\mathcal{H}_p^\gamma(\tau)} := ||u_{xx}||_{\mathbb{H}_p^{\gamma - 2}(\tau)} + ||f||_{\mathbb{H}_p^{\gamma - 2}(\tau)} + ||h||_{\mathbb{H}_p^{\gamma - 1}(\tau, l_2)} + (E||u(0, \cdot)||_{H_p^{\gamma - 2/p, p}}^p)^{1/p}.
\]

For $u \in \mathcal{H}_p^\gamma(\tau)$, we denote $\mathbb{D}u := f$, $\mathbb{S}u := h$.

**Remark 2.2.** By Remark 3.2 of [17] if $h \in \mathbb{H}_p^\gamma(\tau, l_2)$, for some $\gamma \in \mathbb{R}, p \geq 2$, then, for any $\phi \in C_0^\infty(\mathbb{R}^d)$, and any number $T > 0$, the series of stochastic integrals $\sum_{k=1}^{\infty} \int_0^T (h^k(s, \cdot), \phi) \, dw^k(s)$ converges uniformly in $t$ on $[0, T]$ in probability.

**Remark 2.3.** It was showed in Theorem 3.7 of [17] that, for any $\gamma \in \mathbb{R}, p \geq 2$, $\mathcal{H}_p^\gamma(\tau)$ is a Banach space. Also by the same theorem if $T > 0$ is finite, and $\tau \leq T$ is a stopping time, then, for any $v \in \mathcal{H}_p^\gamma(\tau)$,
\[
||v||_{\mathcal{H}_p^\gamma(\tau)} \leq N(d, T)||v||_{\mathcal{H}_p^\gamma(\tau)}.
\]
It follows that, for any bounded stopping time $\tau$, we may replace $||u_{xx}||_{\mathbb{H}_p^{\gamma - 2}(\tau)}$ by $||u||_{\mathcal{H}_p^\gamma(\tau)}$ in the definition of the norm of $\mathcal{H}_p^\gamma(\tau)$ and obtain an equivalent norm.

**Remark 2.4.** Let $p > 2, T > 0$ be finite, and let $\theta$ and $\mu$ be numbers such that $1 > \mu > \theta > 2/p$. Also let $\tau \leq T$ be a stopping time. Then, by
Theorem 7.2 of [17], for any $u \in H_p^\beta(\tau)$, there exists a modification of $u$ such that we have $u \in C^{\theta/2-1/p}(0,T, H_p^{-\mu}(\mathbb{R}^d))$, for any $\omega$. In addition, 

$$E||u||_{C^{\theta/2-1/p}(0,T, H_p^{-\mu}(\mathbb{R}^d))} \leq N(d,p,\theta,\mu,T,E||u||_{H_p^\beta(\tau)}).$$

Further, by the embedding theorem for $H_p^\beta$ spaces, for any non-integer $\nu$ such that $\nu \in (0, \gamma - \mu - d/p)$, we have $u \in C^{\theta/2-1/p}(0,T, C^\nu(\mathbb{R}^d))$, for any $\omega$.

**Definition 2.5.** We say that $u$ is a solution of (1.2) of class $H_p^\beta(\tau)$ if $u \in H_p^\beta(\tau)$ with 

$$D u(t,x) = a^{ij}(t,x)D_{ij}u(t,x) + f(u,t,x),$$

$$S u(t,x) = (g^k(u(t,x))), k = 1,\ldots,m,\text{ } u_0 \in H_p^{-2/p}(\mathbb{R}^d).$$

Note that this implies that $f(u,t,x) \in \mathbb{H}_p^{\gamma-2}(\tau)$, and $g^k(u(t,x)) \in \mathbb{H}_p^{\gamma-1}(\tau), k = 1,\ldots,m$.

**Assumptions.** Fix some finite $p \geq 2, T > 0$.

(A1) $a^{ij}(t,x) = a^{ij}(\omega,t,x), i,j = 1,\ldots,d$ are real-valued $\mathcal{F} \times B(\mathbb{R}^d)$ measurable functions. In addition, there exists $\lambda > 0$ such that, for all $t \geq 0, x, \xi \in \mathbb{R}^d, i,j,\omega,$

$$\lambda|\xi|^2 \leq a^{ij}(t,x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

(A2a) For any $\varepsilon > 0$, there exists a constant $\kappa_\varepsilon > 0$ such that, for all $i,j,t,\omega$,

$$|a^{ij}(t,x) - a^{ij}(t,y)| \leq \varepsilon$$

if $|x - y| \leq \kappa_\varepsilon$.

(A2b) There exists a constant $L > 0$ such that, for all $i,j,t,x,y,\omega$,

$$|a^{ij}(t,x) - a^{ij}(t,y)| \leq L|x - y|.$$

(A3) $f(u,t,x)$ is a real-valued function defined for any $\omega \in \Omega, u \in H_p^1(\mathbb{R}^d), t \geq 0, x \in \mathbb{R}^d$ such that the following assumptions hold:

(i) for any $u \in H_p^1(\mathbb{R}^d)$, $f(u,t,x)$ is a predictable $L_p(\mathbb{R}^d)$-valued function;

(ii) $f(0,\cdot,\cdot) \in L_p(T)$;

(iii) there exists a constant $K > 0$ such that, for any $u,v \in H_p^1(\mathbb{R}^d), t,x,\omega$, we have

$$||f(u,t,x) - f(v,t,x)||_p \leq K||u - v||_{1,p}.$$

(A4a) For each $k \in \{1,\ldots,m\}, g^k(x) = c_kx, x \in \mathbb{R}$, where $c_k \in \mathbb{R}$.

(A4b) For each $k \in \{1,\ldots,m\}, g^k \in C^2(\mathbb{R})$, and $g^k(0) = 0$.

(A5) $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{2-2/p}(\mathbb{R}^d))$.

(A6) For any $k$, $w_n^k$ is an $\mathcal{F}_t$-adapted piecewise $C^1_{loc}$ approximation of $w^k$, and, for any $k,l$, we denote

$$\delta w_n^k(t) = w^k(t) - w^k_n(t),$$

$$s_n^k(t) = \int_0^t \delta w_n^k(r) dw_n^l(r) - \delta_k t/2.$$
We assume that the following holds:
(i) for any ε ∈ (0, 1/2), k, l,
\[ ||\delta w^k_n||_{C^\epsilon[0,T]} + ||s^{kl}_n||_{C^\epsilon[0,T]} \rightarrow 0 \]
as \( n \rightarrow \infty \) in probability;
(ii) for any \( q > 0, k, l \),
\[ \lim_{R \rightarrow \infty} \sup_n P( \int_0^T |Ds^{kl}_n(t)|^q \, dt > R ) = 0. \]

**Remark 2.6.** Let \( h(x) \) be a Lipschitz function such that \( h(0) = 0 \), and \( c(t, x), b^i(t, x) \) be \( \mathcal{P} \times B(\mathbb{R}^d) \)-measurable functions, and \( f(t, x) \in L_p(T), p \geq 2 \). For any \( u \in H^{1,p}_\gamma(\mathbb{R}^d) \), \( t, x, \omega \), we put
\[
f(u, t, x) = b^i(t, x) D_i u(x) + c(t, x) h(u(x)) + f(t, x).
\]
It is easy to see that \( f(u, t, x) \) satisfies (A3)(p).

**Remark 2.7.** It is proved in Appendix A that the polygonal approximation defined by (5.2) and the smoothing approximation given by (5.1) satisfy the assumption (A6).

**Remarks on the existence and uniqueness of solutions of (1.2)**

**Remark 2.8.** We fix any numbers \( \gamma \in \mathbb{R}, p \geq 2, T > 0 \) and take any stopping time \( \tau \leq T \). In this remark we state the existence and uniqueness theorem for the following SPDE:
\[
dv(t, x) = [a^{ij}(t, x) D_{ij} v(t, x) + b(v, t, x)] \, dt
+ \sum_{k=1}^\infty h^k(v, t, x) \, dw^k(t), \quad v(0, x) = v_0(x).
\]
(2.2)

Here, \( a^{ij}(t, x), b(v, t, x), h^k(v, t, x) \) are real-valued functions defined for any \( \omega \in \Omega, t \geq 0, x \in \mathbb{R}^d, u \in H^{\gamma+2}_p(\mathbb{R}^d) \). We repeat almost word for word the statement of Theorem 5.1 of [17] with \( \sigma^{ik} \equiv 0 \).

Fix a number \( \nu \in [0, 1) \) such that \( \nu = 0 \) if \( \gamma = \pm 1, \pm 2, \ldots \); otherwise \( \nu > 0 \) and is such that \( |\gamma| + \nu \) is not an integer. We define
\[
B^{\gamma|+\nu} = \begin{cases} 
B(\mathbb{R}^d) & \text{if } n = 0, \\
C^{\gamma|-1,1}(\mathbb{R}^d) & \text{if } \gamma = \pm 1, \pm 2, \ldots, \\
C^{\gamma|+\nu}(\mathbb{R}^d) & \text{otherwise}.
\end{cases}
\]

Here, \( C^{\gamma|-1,1}(\mathbb{R}^d) \) is the space of \( |\gamma| - 1 \) continuously differentiable functions whose derivatives of order \( |\gamma| - 1 \) are Lipschitz continuous on \( \mathbb{R}^d \).

**Assumption 1.** For any \( \omega \in \Omega, t \geq 0, x, \xi \in \mathbb{R}^d \),
\[
\lambda |\xi|^2 \leq a^{ij}(t, x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2,
\]
where \( \lambda > 0 \).
Assumption 2. For any \( \varepsilon > 0, i, j \), there exists a number \( \kappa_\varepsilon > 0 \) such that, for all \( \omega, t \),
\[
|a^{ij}(t, x) - a^{ij}(t, y)| \leq \varepsilon
\]
if \( |x - y| \leq \kappa_\varepsilon \).

Assumption 3. For any \( i, j \), the functions \( a^{ij}(t, x) \) are real-valued \( \mathcal{P} \times \mathbb{R}^d \)-measurable functions, and, for any \( \omega, t \), we have \( a^{ij}(t, \cdot) \in B^{(\gamma)} \). In addition, for any \( \omega, t, i, j \),
\[
|||a^{ij}(t, \cdot)|||_{B^{(\gamma)}} \leq K.
\]

Assumption 4. For any \( u \in H_p^{\gamma+2}(\mathbb{R}^d) \), the functions \( b(u, t, x) \) and \( h(u, t, x) \) are predictable as functions taking values in \( H_p^{\gamma}(\mathbb{R}^d) \) and \( H_p^{\gamma+1}(\mathbb{R}^d, l_2) \) respectively.

Assumption 5. \( b(0, \cdot, \cdot) \in \mathbb{H}_p^{\gamma}(\tau) \), and \( h(0, \cdot, \cdot) \in \mathbb{H}_p^{\gamma+1}(\tau, l_2) \).

Assumption 6. The functions \( b \) and \( h \) are continuous in \( u \in H_p^{\gamma+2}(\mathbb{R}^d) \). Moreover, for any \( \varepsilon > 0 \), there exists a constant \( K_\varepsilon > 0 \) such that, for any \( u, v \in H_p^{\gamma+2}(\mathbb{R}^d) \), \( \omega, t \in [0, T] \), we have
\[
||b(u, t, \cdot) - b(v, t, \cdot)||_{\gamma,p} + ||h(u, t, \cdot) - h(v, t, \cdot)||_{\gamma+1,p} \
\leq \varepsilon ||u - v||_{\gamma+2,p} + K_\varepsilon ||u - v||_{\gamma,p}.
\]

Theorem 1.5 of [17]. Let Assumption 1 - 6 be satisfied, and let \( v_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{\gamma+2-2/p}(\mathbb{R}^d)) \). Then, the equation (2.2) has a unique solution \( v \) of class \( \mathcal{H}_p^{\gamma+2}(\tau) \). Moreover, the following estimate holds:
\[
||v||_{H_p^{\gamma+2}(\tau)} \leq N(||b(0, \cdot, \cdot)||_{\mathbb{H}_p^{\gamma}(\tau)}^{p} + ||h(0, \cdot, \cdot)||_{\mathbb{H}_p^{\gamma+1}(\tau, l_2)}^{p} + E||v_0||_{\gamma+2-2/p,p}^{p}),
\]
where \( N = N(d, \gamma, \nu, p, \lambda, K, T, \kappa_\varepsilon, K_\varepsilon) > 0 \).

We fix any finite \( p \geq 2, T > 0 \) and assume \( (A1), (A2a), (A3)(p), (A4a), (A5)(p) \). Then, by Remark 2.8 with \( \gamma = 0 \),
\[
b(u, t, x) = f(u, t, x), \quad h(u, t, x) = (c_k u(t, x), k = 1, \ldots, m),
\]
there exists a unique solution \( u \) of class \( \mathcal{H}_p^{2}(T) \) of the equation (1.2).

Next, assume \( (A2b) \) and \( (A4b) \) instead of \( (A2a) \) and \( (A4a) \). Again, by Remark 2.8 with \( \gamma = -1 \),
\[
b(u, t, x) = f(u, t, x), \quad h(u, t, x) = (g^k(u(t, x)), k = 1, \ldots, m),
\]
the equation (1.2) has a unique solution \( u \) of class \( \mathcal{H}_p^{1}(T) \), and there exists a constant \( N \) independent of \( u \) such that
\[
||u||_{\mathcal{H}_p^{1}(T)} \leq N(||f(0, \cdot, \cdot)||_{L_p(T)}^{p} + E||u_0||_{1-2/p}^{p}).
\]

We will show that the solution \( u \) of (1.2) actually belongs to \( \mathcal{H}_p^{2}(T) \) by using the so-called bootstrap method. Let
\[
b(t, x) = f(u, t, x), \quad h(t, x) = (g^k(u(t, x)), k = 1, \ldots, m),
\]
and consider the equation (2.2) with \( b(t, x) \) and \( h(t, x) \) instead of \( b(u, t, x) \) and \( h(v, t, x) \) respectively. We claim that \( b(t, x) \in \mathbb{L}_p(T) \), and \( h(t, x) \in \mathbb{H}_p^1(T) \). By Theorem 3.7 of [17], \( u(t, \cdot) \in H^1_d(\mathbb{R}^d) \), for any \( t \in [0, T], \omega \). Then, by \((A3)(p)\) and the estimate of Remark 2.3, we have

\[
E \int_0^T ||b(t, \cdot)||_p^p \, dt \leq NE \int_0^T ||f(0, t, \cdot)||_p^p \, dt + NE \int_0^T ||u(t, \cdot)||_p^p \tag{2.5}
\]

\[
\leq N||f(0, \cdot, \cdot)||_{L_p(T)}^p + N||u||_{\mathbb{H}_p^1(T)}^p.
\]

By the same argument and Lemma 6.1, and \((A4b)\) we get that \( g^k(u(t, x)) \in \mathbb{H}_p^1(T) \) in the following way:

\[
E \int_0^T ||g^k(u(t, \cdot))||_{1,p}^p \, dt \leq NE \int_0^T ||u(t, \cdot)||_{1,p}^p \, dt \leq N||u||_{\mathbb{H}_p^1(T)}^p. \tag{2.6}
\]

Thus, \( h(t, x) \in \mathbb{H}_p^1(T, l_2) \). Then, by the theorem stated in Remark 2.8 with \( \gamma = 0 \),

\[
b(t, x) = f(u, t, x), \quad h(t, x) = (g^k(u(t, x)), k = 1, \ldots, m)
\]

the equation (2.2) has a unique solution \( v \) of class \( \mathcal{H}^2_p(\tau) \). Since \( \mathcal{H}^2_p(\tau) \subset \mathcal{H}^1_p(T) \), \( v \) is also a solution of (1.2) of class \( \mathcal{H}^1_p(T) \), and, hence, \( v \equiv u \) as elements of \( \mathcal{H}^2_p(T) \). Moreover, by (2.3) - (2.6) we have

\[
||u||_{\mathcal{H}^2_p(T)}^p \leq N(||b||_{L_p(T)}^p + ||h||_{\mathbb{H}_p^1(T, l_2)}^p + E||u_0||_{L_2(-2/p, p)}^p \tag{2.7}
\]

\[
\leq N||f(0, \cdot, \cdot)||_{L_p(T)}^p + NE||u_0||_{L_2(-2/p, p)}^p.
\]

**Statement of the main results.** Here is the statement of a Wong-Zakai type theorem.

**Theorem 2.9.** Let \( m = \{1, 2, \ldots\} \), \( T > 0 \), \( p > d + 2 \), \( \theta \in (0, 1) \) be numbers. Assume the following:

(i) \((A1), (A3) (p), (A5) (p), (A6)\) hold;

(ii) either \((A2a), (A4a)\) or \((A2b), (A4b)\) hold;

(iii) \( D^2 g \in C^\theta(\mathbb{R}) \).

Let \( u \) and \( u_n \) be the unique solutions of class \( \mathcal{H}^2_p(T) \) of the equations (1.2) and (1.3) respectively (see Remark 2.12 (i)). Then, for any numbers \( \mu \) and \( \gamma \) such that \( 1 - d/p > \mu > \gamma > 2/p \), we have

\[
||u - u_n||_{C^{\gamma/2 - 1/p}([0, T], H^2_p(-\nu(\mathbb{R}^d)))} \to 0
\]
as \( n \to \infty \) in probability.

Here is the statement of the support theorem.

**Theorem 2.10.** Assume the conditions of Theorem 2.9 and assume that \( a^{ij}(t, x), f(u, t, x), u_0(x) \) are nonrandom functions. Take any numbers \( \gamma \) and \( \mu \) such that \( 1 - d/p > \mu > \gamma > 2/p \). Let \( \mathcal{H}(T) \) be the set of all \( \mathbb{R}^m \)-valued functions \( h = (h^k, k = 1, \ldots, m) \) such that each \( h^k \) is a Lipschitz function on \([0, T], \) and \( h^k(0) = 0 \). For any \( h \in \mathcal{H}(T) \), we set \( \mathcal{R}(h) \) to be the unique
solution of class $W_p^{1,2}([0, T] \times \mathbb{R}^d)$ (see Remark 2.12 (ii)) of the following PDE:

\[
\partial_t z(t, x) = a^{ij}(t, x) D_{ij} z(t, x) + f(z, t, x) - 1/2 \sum_{k=1}^{m} (g^k D g^k)(z(t, x)) \\
+ \sum_{k=1}^{m} g^k(z(t, x)) D h^k(t), \quad z(0, x) = u_0(x).
\]

We denote $\mathcal{R} = \{\mathcal{R} h : h \in \mathcal{H}(T)\}$ and let $\mathcal{R}_c$ be the closure of $\mathcal{R}$ in the space $\mathcal{V}(T) := C^{\gamma/2-1/p}([0, T], H_p^{2-\mu}(\mathbb{R}^d))$. Let $P \circ u^{-1}_{\mathcal{V}(T)}$ be the distribution of the solution of (1.2) in the space $\mathcal{V}(T)$, and let supp $P \circ u^{-1}_{\mathcal{V}(T)}$ be its support. Then, supp $P \circ u^{-1}_{\mathcal{V}(T)} = \mathcal{R}_c$.

To prove the main results we will use the following approximation theorem.

**Theorem 2.11.** Assume the conditions of Theorem 2.9. Fix any $\alpha, \beta \in \mathbb{R}$, and let $v$ and $v_n$ be the solutions of class $\mathcal{H}_p^2(T)$ (see Remark 2.12 (i) ) of the following SPDEs:

\[
dv(t, x) = [a^{ij}(t, x) D_{ij} v(t, x) + f(v, t, x)] dt \\
+ (\alpha + \beta) \sum_{k=1}^{m} g^k(v(t, x)) d w^k(t), \quad v(0, x) = u_0(x),
\]

\[
dv_n(t, x) = [a^{ij}(t, x) D_{ij} v_n(t, x) + f(v_n, t, x) + \alpha \sum_{k=1}^{m} g^k(v_n(t, x)) D w^k_n(t) \\
- (\alpha^2/2 + \alpha \beta) \sum_{k=1}^{m} (g^k D g^k)(v_n(t, x))] dt + \beta \sum_{k=1}^{m} g^k(v_n(t, x)) d w^k(t),
\]

\[
v_n(0, x) = u_0(x).
\]

Then, we have

\[
||v - v_n||_{C^{\gamma/2-1/p}([0, T], H_p^{2-\mu}(\mathbb{R}^d))} \to 0
\]
as $n \to \infty$ in probability.

**Remark 2.12.** Let $p \geq 2, T > 0$ be finite. We assume (A1), (A3)(p), (A5)(p), (A6), and assume either (A2a), (A4a) or (A2b), (A4b).

(i) Here, we show that the equation (2.10) has a unique solution of class $\mathcal{H}_p^2(T)$. We use the same reasoning that we used to show that (1.2) has a unique solution of class $\mathcal{H}_p^2(T)$. This time, we set

\[
b(u, t, x) = f(u, t, x) - (\alpha^2/2 + \alpha \beta) \sum_{k=1}^{m} (g^k D g^k)(u(x)) + \alpha \sum_{k=1}^{m} g^k(u(x)) D w^k_n(t),
\]

\[
h(u, t, x) = (\beta g^k(u(x)), k = 1, \ldots, m), \quad u \in H_p^1(\mathbb{R}^d).
It is easily seen that we only need to check Assumption 6 (with \( \gamma = 0 \)) of Remark 2.8. For any \( z, v \in H^1_p(\mathbb{R}^d) \), and any \( t \in [0, T], \omega \), we have
\[
||b(z, t, \cdot) - b(v, t, \cdot)||_p \leq ||f(z, t, \cdot) - f(v, t, \cdot)||_p + \alpha \sum_{k=1}^{m} ||g^k(z(\cdot)) - g^k(v(\cdot))||_p |Dw^k_n(t)|
\]
\[
+ N(\alpha, \beta) \sum_{k=1}^{m} ||(g^k Dg^k)(z(\cdot)) - (g^k Dg^k)(v(\cdot))||_p \leq \tilde{K} ||z - v||_{1,p},
\]
where
\[
\tilde{K} = K + N(\alpha, \beta) \sum_{k=1}^{m} (||Dg^k||_\infty ||Dw^k_n||_{L_\infty[0,T]} + ||Dg^k||^2_\infty + ||g^k D^2 g^k||_\infty),
\]
and \( K \) is the constant from (A3)(p). Hence, Assumption 6 with \( \gamma = 0 \) holds.

(ii) We apply Remark 2.8 with \( \gamma = 0 \) and
\[
b(u, t, x) = f(u, t, x) - 1/2 \sum_{k=1}^{m} (g^k Dg^k)(u(x)) + \sum_{k=1}^{m} g^k(u(x)) Dh^k(t),
\]
\[
h(u, t, x) \equiv 0, \ u \in H^1_p(\mathbb{R}^d).
\]
Then, (2.8) has a unique solution \( \mathcal{R}(h) \) of class \( \mathcal{H}_p^2(T) \). Since \( \mathcal{R}(h) \) is a nonrandom function, we have \( \mathcal{R}(h) \in W^{1,2}_p([0, T] \times \mathbb{R}^d) \).

3. Auxiliary Results.

**Lemma 3.1.** Let \( \theta \in (0, 1) \), \( h \in C^{1+\theta}(\mathbb{R}) \), and \( h(0) = 0 \). Let \( \rho \) be a \( C_0^\infty(\mathbb{R}) \) function such that \( \int_\mathbb{R} \rho(y) dy = 1 \). Denote \( \rho_\varepsilon(x) = 1/\varepsilon \rho(x/\varepsilon) \),
\[
h_\varepsilon(x) = (h * \rho_\varepsilon)(x) - (h * \rho_\varepsilon)(0),
\]
where \( * \) stands for the convolution. Then, the following assertions hold:

(i) \( h_\varepsilon(0) = 0 \), and, for any \( x \in \mathbb{R} \),
\[
|h(x) - h_\varepsilon(x)| \leq N(\rho)||Dh||_\infty \varepsilon;
\]

(ii) for any \( k = \{0, 1, \ldots\} \),
\[
||D^{k+1}h_\varepsilon||_{C^0} \leq N(\rho, \theta, k)||Dh||_{C^0} 1/\varepsilon^k.
\]

**Proof.** (i) The proof is standard.

(ii) Clearly, for any \( k \),
\[
D^{k+1}h_\varepsilon(x) = 1/\varepsilon^k \int D^k \rho(y) \ Dh(x - \varepsilon y) dy,
\]
and from this the claim easily follows. \( \Box \)

Denote when it makes sense
\[
Lu(t, x) = a^{ij}(t, x) D_{ij} u(t, x),
\]
\[
Mu(t, x) = a^{ij}(t, x) D_i u(t, x) D_j u(t, x).
\]
Lemma 3.2. Assume the conditions and notations of Theorem 2.11. Let
$h^{kl} : \mathbb{R} \rightarrow \mathbb{R}$ be a function of class $C^2_{loc}(\mathbb{R})$.

Denote
$$\bar{v}_n(t, x) := v_n(t, x) - v(t, x)$$
$$+ \alpha \sum_{k=1}^{m} g^k(v_n(t, x)) \delta u^k_n(t) - \alpha^2 \sum_{k,l=1}^{m} h^{kl}(v(t, x)) s_n^{kl}(t).$$

Then, there exist constants $N_k(\alpha, \beta), k = 1, \ldots, 13$ such that, for any $\omega \in \Omega$, $t \in [0, T]$, $\psi \in C^0_b(\mathbb{R}^d)$, the function $\bar{v}_n$ satisfies the following equation:

$$(\bar{v}_n(t, \cdot), \psi) = \int_0^t \left( L\bar{v}_n(s, \cdot) + \sum_{q=1}^{10} N_q F_{q,n}(s, \cdot), \psi \right) ds + \sum_{r=1}^{m} \int_0^t (N_{11} H_{1,n}^r(s, \cdot) + N_{12} H_{2,n}^r(s, \cdot) + N_{13} H_{3,n}^r(s, \cdot), \psi) dw^r(s),$$

where

- $F_{1,n}(s, x) = \sum_{k=1}^{m} (D^2 g^k)(v_n(s, x)) M v_n(s, x) \delta u^k_n(s)$,
- $F_{2,n}(s, x) = \sum_{k,l=1}^{m} (D^2 h^{kl})(v(s, x)) M v(s, x) s_n^{kl}(s)$, $F_{3,n}(s, x) = f(v_n(s, x)) - f(v, s, x)$,
- $F_{4,n}(s, x) = \sum_{k=1}^{m} f(v_n(s, x))(Dg^k)(v_n(s, x)) \delta u^k_n(s)$,
- $F_{5,n}(s, x) = \sum_{k,l=1}^{m} f(v, s, x)( Dh^{kl})(v(s, x)) s_n^{kl}(s)$,
- $F_{6,n}(s, x) = \sum_{k,l=1}^{m} (g^l Dg^k)(v_n(s, x)) \delta u^k_n(s)$,
- $F_{7,n}(s, x) = \sum_{k,l=1}^{m} ((g^l)^2 D^2 g^k)(v_n(s, x)) \delta u^k_n(s)$,
- $F_{8,n}(s, x) = \sum_{k,l=1}^{m} [(g^l Dg^k)(v_n(s, x)) - (g^l Dg^k)(v(s, x))] Ds_n^{kl}(s)$,
- $F_{9,n}(s, x) = \sum_{k,l=1}^{m} [(g^l Dg^k)(v(s, x)) - h^{kl}(v(s, x))] Ds_n^{kl}(s)$,
- $F_{10,n}(s, x) = \sum_{k,l,r=1}^{m} ((g^l)^2 D^2 h^{kl})(v(s, x)) s_n^{kl}(s)$,
- $H_{1,n}^r(s, x) = [g^r(v_n(s, x)) - g^r(v(s, x))]$. 

\( H_{1,n}^r(s, x) = \frac{1}{2} (v_n(s, x) - v(s, x))^2 \) for all \( s \in [0, T] \).
the third term in the equation (2.10) by
we will get a ‘boundary’ term, an integral and a mutual quadratic variation
\((< A, B >)\) of convenience, in this proof we omit the dependence of functions on the
argument \(x\).

**Proof.** In the proof we assume the summation with respect to indexes \(k, l, r \in \{1, \ldots, m\}\).

For any two real-valued continuous semimartingales \(A(t), B(t), t \geq 0\) by
\(< A, B > (t)\) we denote their mutual quadratic variation. For the sake
of convenience, in this proof we omit the dependence of functions on the
argument \(x\).

**Step 1.** Following V. Mackevičius in [22] and I. Gyöngy in [8] we will split
the third term in the equation (2.10) by \(\alpha g^k(v_n(t)) \, dw^k(t)\) and \(\alpha g^k(v_n(t)) \, d[w_n^k(t) - w^k(t)]\). Then we will integrate by parts in the second integral. From this
we will get a ‘boundary’ term, an integral and a mutual quadratic variation
term, which is also an integral. The ‘boundary’ term will be subtracted from
\(v_n - v\), and the integrals, if necessary, will be further decomposed via Itô’s
formula and integration by parts.

First, we subtract the equation (2.9) from (2.10), and we formally write
the ‘stochastic part’ of the difference in the following way:

\[
\alpha g^k(v_n(t)) \, dw^k_n(t) + \beta g^k(v_n(t)) \, dw^k(t) - (\alpha + \beta)g^k(v(t)) \, dw^k(t) = \alpha g^k(v_n(t)) \, d[w_n^k(t) - w^k(t)] + (\alpha + \beta)[g^k(v_n(t)) - g^k(v(t))] \, dw^k(t).
\]

Fix any \(\psi \in C^\infty_0(\mathbb{R}^d)\). Then, by the above,

\[
(v_n(t) - v(t), \psi) = \sum_{q=1}^5 I_n^{(q)}(t), \quad (3.4)
\]

where

\[
I_n^{(1)}(t) = \int_0^t (L[v_n(s) - v(s)], \psi) \, ds,
\]

\[
I_n^{(2)}(t) = \int_0^t ([f(v_n, s) - f(v, s)], \psi) \, ds,
\]

\[
I_n^{(3)}(t) = \alpha \int_0^t (g^k(v_n(s)), \psi) \, d[w_n^k(s) - w^k(s)],
\]

\[
I_n^{(4)}(t) = (\alpha + \beta) \int_0^t ([g^k(v_n(s)) - g^k(v(s))], \psi) \, dw^k(s),
\]

\[
I_n^{(5)}(t) = -\frac{\alpha^2}{2} + \alpha \beta \int_0^t ((g^k Dg^k)(v_n(s)), \psi) \, ds.
\]

Next, we assume that the support of \(\psi\) is contained in some ball \(B_R(0) := \{x \in \mathbb{R}^d : |x| \leq R\}\), and denote \(H := L_2(B_R(0))\). We introduce a process

\[
v^*(s) := Lv_n(s) + f(v_n, s) + \alpha g^k(v_n(s)) \, Dw^k_n(s) - (\alpha^2/2 + \alpha \beta)(g^k Dg^k)(v_n(s))
\]
and a linear functional

\[ \phi(h) := \int h(x)\psi(x) \, dx, \; h \in H. \]

By Remark 2.3 \( v_n \) is a predictable \( H \)-valued process. Then, the same holds for \( v^* \) by (A3)(p) and (A4)(a) or (A4)(b). Observe that the formula (3.1) of [21] holds with \( V = H, u \equiv v_n, v^* = v^* \), and \( \sigma^k = g^k(v_n(\cdot)) \). It is easy to check that all the conditions of Theorem 3.1 of [21] are satisfied, and, hence, we may apply Itô’s formula to \( \phi(g^k(v_n(t)), t \geq 0 \). We obtain that this process is a semimartingale, and, moreover, for any \( \omega, t \), the following holds:

\begin{align*}
(g^k(v_n(t)), \psi) &= (g^k(v_n(0)), \psi) + V(t) \\
& \quad + \beta \int_0^t ((g^j D g^k)(v_n(s)), \psi) \, dw^j(s),
\end{align*}

where \( V(t), t \geq 0 \) is a process of locally bounded variation.

Next, by the integration by parts formula for semimartingales we have

\[ I_n^{(3)}(t) = -\alpha(g^k(v_n(t)), \psi) \delta w^k_n(t) + I_n^{(3,1)}(t) + I_n^{(3,2)}(t), \]

where

\[ I_n^{(3,1)}(t) = \int_0^t \delta w^k_n(s) \, d(g^k(v_n(s)), \psi), \]

\[ I_n^{(3,2)}(t) = \alpha < (g^k(v_n(\cdot)), \psi), w^k(\cdot) > (t). \]

By (3.5) we get

\[ I_n^{(3,2)}(t) = \alpha \beta \int_0^t ((g^k D g^k)(v_n(s), \psi) \, ds. \]

In the sequel we omit testing the equations with \( \psi \in C^\infty_0(\mathbb{R}^d) \).

Again, by Itô’s formula we get

\[ I_n^{(3,1)}(t) = \sum_{q=1}^6 I_n^{(3,1,q)}(t), \]

where

\[ I_n^{(3,1,1)}(t) = \alpha \int_0^t L v_n(s) D g^k(v_n(s)) \delta w^k_n(s) \, ds, \]

\[ I_n^{(3,1,2)}(t) = \alpha \int_0^t f(v_n, s) D g^k(v_n(s)) \delta w^k_n(s) \, ds, \]

\[ I_n^{(3,1,3)}(t) = \alpha^2 \int_0^t (g^j D g^k)(v_n(s)) \delta w^k_n(s) \, Dw^j_n(s) \, ds, \]

\[ I_n^{(3,1,4)}(t) = -\alpha (\alpha^2 / 2 + \alpha \beta) \int_0^t (g^j D g^j D g^k)(v_n(s)) \delta w^k_n(s) \, ds, \]

\[ I_n^{(3,1,5)}(t) = \alpha \beta^2 / 2 \int_0^t ((g^j)^2 D^2 g^k)(v_n(s)) \delta w^k_n(s) \, ds, \]
\[ I_n^{(3,1,0)}(t) = \alpha \beta \int_0^t (g^l D g^k)(v_n(s)) \delta w_n^k(s) \, dw^l(s). \]

Next, observe that
\[ \delta w_n^k(s) Dw_n^l(s) = D s_n^{kl}(s) + \delta_{kl}/2, \]
and, hence,
\[ I_n^{(5)}(t) + I_n^{(3,2)}(t) + I_n^{(3,1,3)}(t) = R_n(t) := \alpha^2 \int_0^t (g^l D g^k)(v_n(s)) D s_n^{kl}(s) \, ds. \]

**Step 2.** In turns out that all integrals from Step 1 that we obtained after integration by parts are 'under control' except \( R_n(t) \). To handle this term we rewrite it as follows:
\[ R_n(t) = \sum_{q=1}^3 R_n^{(q)}(t), \]
where
\[ R_n^{(1)}(t) = \alpha^2 \int_0^t [(g^l D g^k)(v_n(s)) - (g^l D g^k)(v(s))] D s_n^{kl}(s) \, ds, \]
\[ R_n^{(2)}(t) = \alpha^2 \int_0^t [(g^l D g^k)(v(s)) - h^{kl}(v(s))] D s_n^{kl}(s) \, ds, \]
\[ R_n^{(3)}(t) = \alpha^2 \int_0^t h^{kl}(v(s)) D s_n^{kl}(s) \, ds. \]

By the way, in the proof of Theorem 2.11 the function \( h^{kl} \) will be a suitable approximation of \( g^l D g^k \).

Next, we integrate by parts in \( R_n^{(3)}(t) \), and, since \( s_n^{kl} \) has a locally bounded variation, there is no mutual quadratic variation term. Then, we get
\[ R_n^{(3)}(t) = \alpha^2 h^{kl}(v(t)) s_n^{kl}(t) + R_n^{(3,1)}(t), \]
(3.8)
\[ R_n^{(3,1)}(t) := -\alpha^2 \int_0^t s_n^{kl}(s) dh^{kl}(v(s)). \]

Applying Itô’s formula, we obtain
\[ R_n^{(3,1)}(t) = \sum_{q=1}^4 R_n^{(3,1,q)}(t), \]
where
\[ R_n^{(3,1,1)}(t) = -\alpha^2 \int_0^t L v(s)(D h^{kl}(v(s))) s_n^{kl}(s) \, ds, \]
\[ R_n^{(3,1,2)}(t) = -\alpha^2 \int_0^t f(v,s)(D h^{kl}(v(s))) s_n^{kl}(s) \, ds, \]
\[ R_n^{(3,1,3)}(t) = -\alpha^2 (\alpha + \beta) \int_0^t (g^r D h^{kl}(v(s))) s_n^{kl}(s) \, dw^r(s), \]
Then, for any $f$ into the term $R$ be a Banach space. For $\theta$ note that the terms $I$ speaking, as integral of $L\bar{v}_n$ plus some error terms:

\[
I_n^{(1)}(t) + I_n^{(3,1,1)}(t) + R_n^{(3,1,1)}(t)
\]

\[
= \int_0^t (L[\bar{v}_n(s) - v(s)] + \alpha(Dg^k(v_n(s))L\bar{v}_n(s)\delta w^k_n(s) - \alpha^2(Dh^{kl}(v_n(s))L\bar{v}(s))s^{kl}_n(s)) \, ds
\]

\[
= \int_0^t (L\bar{v}_n(s) - \alpha F_{1,n}(s) + \alpha^2 F_{2,n}(s)) \, ds.
\]

We show how the rest of the terms on the right hand side of (3.3) relate to the ones that appeared in the proof. We have

\[
I_n^{(2)}(t) = N_3 \int_0^t (F_{3,n}(s), \psi) \, ds,
\]

\[
I_n^{(3,1,2)}(t) = N_4 \int_0^t (F_{4,n}(s), \psi) \, ds,
\]

\[
R_n^{(3,1,2)}(t) = N_5 \int_0^t (F_{5,n}(s), \psi) \, ds,
\]

\[
I_n^{(3,1,4)}(t) = N_6 \int_0^t (F_{6,n}(s), \psi) \, ds,
\]

\[
I_n^{(3,1,5)}(t) = N_7 \int_0^t (F_{7,n}(s), \psi) \, ds,
\]

\[
R_n^{(1)}(t) = N_8 \int_0^t (F_{8,n}(s), \psi) \, ds,
\]

\[
R_n^{(2)}(t) = N_9 \int_0^t (F_{9,n}(s), \psi) \, ds,
\]

\[
R_n^{(3,1,4)}(t) = N_{10} \int_0^t (F_{10,n}(s), \psi) \, ds,
\]

\[
I_n^{(4)}(t) = N_{11} \int_0^t (H_{1,n}^r(s), \psi) \, dw^r(s),
\]

\[
I_n^{(3,1,6)}(t) = N_{12} \int_0^t (H_{2,n}^r(s), \psi) \, dw^r(s),
\]

\[
R_n^{(3,1,3)}(t) = N_{13} \int_0^t (H_{3,n}^r(s), \psi) \, dw^r(s).
\]

Note that the terms $I_n^{(3,1,3)}(t), I_n^{(3,2)}(t), I_n^{(5)}(t)$ were not lost, but absorbed into the term $R_n(t)$. \hfill \Box

**Lemma 3.3.** Let $\alpha$ and $\tilde{\alpha}$ be numbers such that $0 < \alpha < \tilde{\alpha} < 1$, and let $X$ be a Banach space. For $\theta \in (0, 1)$, and $t > 0$, we denote $V_\theta^t = C^\theta([0, t], X)$. Then, for any $f \in V_\theta^T$, the function $t \to ||f||_{V_\theta^t}$ is continuous on $[0, T]$. 
Proof. Denote

\[ h(\xi, \nu) := \frac{||f(\xi) - f(\nu)||_X}{|\xi - \nu|^\alpha}, \xi \neq \nu, \quad h(\xi, \xi) := 0, \]

\[ x(t) := \sup_{\xi \in [0, t]} ||f(\xi)||_X, \]

\[ y(t) := \sup_{\xi, \nu \in [0, t]} h(\xi, \nu), t \in [0, T]. \]

Take any numbers 0 < s < t < T and write

\[ ||f||_{V^\beta t} - ||f||_{V^\beta s} \leq |x(t) - x(s)| + |y(t) - y(s)|. \] (3.9)

Hence, it suffices to show that both \( x(\cdot) \) and \( y(\cdot) \) are continuous functions on \([0, T]\).

Since \( f \in C^\beta([0, T], X) \), for any \( \xi, \nu \in [0, T] \) such that \( \xi \neq \nu \), we have

\[ |h(\xi, \nu)| \leq ||f||_{V^\beta T} |\xi - \nu|^{\beta - \alpha}. \] (3.10)

Hence, \( h(\xi, \nu) \) is continuous on \([0, T] \times [0, T]\). Observe that both functions \( x(\cdot) \) and \( y(\cdot) \) are nondecreasing on \([0, T]\) and, hence, at worst they have countably many jump discontinuities. Then, since, both \( h(\cdot, \cdot) \) and \( ||f(\cdot)||_X \) are continuous on their domains, it follows that \( x \) and \( y \) are continuous functions on \([0, T]\). \( \square \)

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 2.11. It is assumed that \( i, j \in \{1, \ldots, d\} \), and \( k, l, r \in \{1, \ldots, m\} \), and the summation with respect to these indexes is omitted.

We fix arbitrary \( \varepsilon \in (0, 1) \), and denote by \( N \) a constant independent of \( \varepsilon \) and \( n \). As before, \( N \) might change from inequality to inequality.

Take any \( R > 0 \) and denote

\[ \mathcal{V}(t) := C^{\gamma/2 - 1/p}([0, t], H^{2-\mu}_p(\mathbb{R}^d)), \]

\[ \eta_n(\varepsilon) := \inf\{t \geq 0 : \sum_{k=1}^m ||\delta w_n^k||_{C^{\gamma/2 - 1/p}[0, t]} + \sum_{k,l=1}^m ||s^k_l||_{C^{\gamma/2 - 1/p}[0, t]} \geq \varepsilon\}, \]

\[ \gamma_n := \inf\{t \geq 0 : ||v_n - v||_{\mathcal{V}(t)} \geq 1\}, \]

\[ \sigma_n(R) := \inf\{t \geq 0 : ||v||_{\mathcal{V}(t)} + \sum_{k,l=1}^m \int_0^t |D s^k_l(s)|^p ds \geq R\}, \]

\[ \tau_n(\varepsilon, R) := \eta_n(\varepsilon) \wedge \gamma_n \wedge \sigma_n(R) \wedge T. \]

In Steps 1 - 5, we omit the dependence of stopping times on \( \varepsilon \) and \( R \).

By Remark 2.4, for any \( \tilde{\gamma} \in (\gamma, \mu) \), we have \( v, v_n \in C^{\tilde{\gamma}/2 - 1/p}([0, T], H^{2-\mu}_p(\mathbb{R}^d)) \), for any \( \omega \). Hence, by Lemma 3.3 the process \( ||z||_{\mathcal{V}(t)}, t \in [0, T] \) has continuous paths, for \( z \in \{v, v_n - v\} \). Then, \( \sigma_n \) and \( \gamma_n \) are well-defined stopping times, and, for any \( \omega \),

\[ ||v_n - v||_{\mathcal{V}(\gamma_n)} = 1, \quad ||v_n - v||_{\mathcal{V}(\tau_n)} \leq 1, \quad ||v_n||_{\mathcal{V}(\tau_n)} \leq N. \] (4.1)
3.2. Observe that all the conditions of this lemma hold. Then, by the a priori estimate of Remark 2.8 with $\gamma$ terms that appear in the lemma. Let $\bar{v}$ where we have

$$ \sup_{s \leq \tau_n} ||z(s, \cdot)||_X \leq N, \quad (4.2) $$

where $X = C^1(\mathbb{R}^d)$ or $X = H^1_\theta(\mathbb{R}^d)$, and $z \in \{v, v - v_n, v_n\}$.

For any $k, l$, we set $h^{kl}_\varepsilon$ to be an approximation of $g^l Dg^k$ given by (3.1) with $\varepsilon^{1/2}$ in place of $\varepsilon$. Observe that if (A4a) holds, we have

$$ h^{kl}_\varepsilon(x) = c_k c_l x. $$

We state some properties of $h^{kl}_\varepsilon$ that will be used below. In case when (A4)(a) holds, all of these inequalities follow directly from the previous paragraph. Otherwise, one obtains them by using Lemma 3.1. By what was just said, for any $x \in \mathbb{R}, k, l$, we have

$$ |(g^l Dg^k - h^{kl}_\varepsilon)(x)| \leq N \varepsilon^{1/2} |x|; \quad (4.3) $$

$$ |h^{kl}_\varepsilon(x)| \leq N|x|; \quad (4.4) $$

$$ ||D^{q+1} h^{kl}_\varepsilon||_{C^q} \leq N \varepsilon^{-q/2}, \quad q = \{0, 1, 2, \ldots\}. \quad (4.5) $$

Next, to prove this theorem we apply Lemma 3.2 and then estimate the terms that appear in the lemma. Let $\bar{v}_n$ be the function defined in Lemma 3.2. Observe that all the conditions of this lemma hold. Then, by the a priori estimate of Remark 2.8 with $\gamma = 0$,

$$ b(t, x) = \sum_{q=1}^{10} F_{q,n}(t, x), \quad h^r(t, x) = \sum_{q=1}^{3} H^r_{q,n}(t, x), $$

we have

$$ ||\bar{v}_n||_{H^2_p(\tau_n)}^p \leq N \sum_{q=1}^{13} I_{q,n}, \quad (4.6) $$

where

$$ I_{1,n} = E \int_0^{\tau_n} ||(D^2 g^k)(v_n(s, \cdot)) M v_n(s, \cdot)||_p^p |\delta w^k_n(s)|^p ds, $$

$$ I_{2,n} = E \int_0^{\tau_n} ||(D^2 h^{kl}_\varepsilon)(v(s, \cdot)) M v(s, \cdot)||_p^p |s^{kl}_n(s)|^p ds, $$

$$ I_{3,n} = E \int_0^{\tau_n} ||f(v_n, s, \cdot) - f(v, s, \cdot)||_p^p ds, $$

$$ I_{4,n} = E \int_0^{\tau_n} ||f(v_n, s, \cdot)(Dg^k)(v_n(s, \cdot))||_p^p |\delta w^k_n(s)|^p ds, $$

$$ I_{5,n} = E \int_0^{\tau_n} ||f(v, s, \cdot)(D h^{kl}_\varepsilon)(v(s, \cdot))||_p^p |s^{kl}_n(s)|^p ds, $$

$$ I_{6,n} = E \int_0^{\tau_n} ||(g^l Dg^k)(v_n(s, \cdot))||_p^p |\delta w^k_n(s)|^p ds, $$

and

$$ I_{7,n} = E \int_0^{\tau_n} ||((g^l)^2 D^2 g^k)(v_n(s, \cdot))||_p^p |\delta w^k_n(s)|^p ds, $$
\[
I_{8,n} = E \int_0^{\tau_n} \left| (g^I Dg^k)(v_n(s, \cdot)) - (g^I Dg^k)(v(s, \cdot)) \right|^p |Ds^k_n(s)|^p ds,
\]

\[
I_{9,n} = E \int_0^{\tau_n} \left| (g^I Dg^k)(v(s, \cdot)) - h^k_\varepsilon(v(s, \cdot)) \right|^p |Ds^k_n(s)|^p ds,
\]

\[
I_{10,n} = E \int_0^{\tau_n} \left| (g^I D^2h^k_\varepsilon)(v(s, \cdot)) \right|^p |s^k_n(s)|^p ds,
\]

\[
I_{11,n} = E \int_0^{\tau_n} \left| g^I(s, \cdot) - g^I(0, \cdot) \right|^p |\delta w_n^k(s)|^p ds,
\]

\[
I_{12,n} = E \int_0^{\tau_n} \left| (g^I Dg^k)(v_n(s, \cdot)) \right|^p |\delta w_n^k(s)|^p ds,
\]

\[
I_{13,n} = E \int_0^{\tau_n} \left| (g^I D^2h^k_\varepsilon)(v(s, \cdot)) \right|^p |s^k_n(s)|^p ds.
\]

**Step 1.** We estimate the terms \( I_{6,n} - I_{10,n} \). First, by \((A4a)\) or \((A4b)\) we have

\[
I_{6,n} + I_{7,n} \leq NE \int_0^{\tau_n} \left( ||v_n(s, \cdot)||^p_p + ||v_n(s, \cdot)||^2_p \right) |\delta w_n^k(s)|^p ds.
\]

Using \((4.2)\), we obtain

\[
I_{6,n} + I_{7,n} \leq N\varepsilon^p.
\] (4.7)

Similarly,

\[
I_{8,n} \leq NE \int_0^{\tau_n} \left| v_n(s, \cdot) - v(s, \cdot) \right|^p |Ds^k_n(s)|^p ds,
\] (4.8)

\[
I_{10,n} \leq N\varepsilon^{p/2},
\] (4.9)

where in the last inequality we also used \((4.5)\).

Next, by \((4.3)\) and \((4.2)\), and the fact that \(\tau_n \leq \sigma_n\), we get

\[
I_{9,n} \leq N\varepsilon^{p/2} E \int_0^{\tau_n} \left| v(s, \cdot) \right|^p |Ds^k_n(s)|^p ds
\]

\[
\leq N\varepsilon^{p/2} E \int_0^{\tau_n} |Ds^k_n(s)|^p ds \leq N\varepsilon^{p/2}.
\] (4.10)

**Step 2.** We move on to the terms \( I_{3,n} - I_{5,n} \). By \((A3)(p)\) we obtain

\[
I_{3,n} \leq NE \int_0^{\tau_n} \left| v_n(s, \cdot) - v(s, \cdot) \right|^p \|Ds^k_n(s)\|^p ds.
\] (4.11)

Similarly, by splitting \( f(u, t, x) \) by \( f(u, t, x) - f(0, t, x) \) and \( f(0, t, x) \) and using \((A3)(p)\), we get

\[
I_{4,n} + I_{5,n} \leq N\varepsilon^p E \int_0^{\tau_n} \left( ||v(s, \cdot)||^p_{L^p} + ||v_n(s, \cdot)||^p_{L^p} + ||f(0, s, \cdot)||^p_{L^p} \right) ds \leq N\varepsilon^p.
\] (4.12)

Note that in the last inequality we also used \((4.2)\).

**Step 3.** We estimate \( I_{1,n} \) and \( I_{2,n} \). First, by \((4.5)\) we have

\[
I_{1,n} + I_{2,n} \leq N\varepsilon^{p/2} E \int_0^{\tau_n} \left( ||Mv_n(s, \cdot)||^p_p + ||Mv(s, \cdot)||^p_p \right) ds.
\] (4.13)
Next, by Cauchy-Schwartz inequality, for any $\omega, s$,

$$||M v_n(s, \cdot)||_p^p \leq N ||v_n(s, \cdot)||_{1,2p}^2.$$  

Using the embedding theorem for $H^r_p$ spaces (see, for instance, Theorem 13.8.7 in [19]), we get, for $u \in \{v, v_n\}$,

$$||u||_{1,2p} \leq N ||u||_{1+d/(2p),p} \leq N ||u||_{2-\mu,p},$$

where the last inequality holds, since $d/(2p) + \mu < 1$. By this and the definition of $\tau_n$ we get

$$I_{1,n} + I_{2,n} \leq \varepsilon^{p/2}. \quad (4.14)$$

**Step 4.** We deal with $I_{11,n} - I_{13,n}$. The term $I_{11,n}$ can be estimated by Lemma 6.2. Let us check its assumptions. In the notations of this lemma, for $\omega \in \Omega$, $s \in [0, \tau_n]$, $x \in \mathbb{R}^d$, we put

$$u(x) = v(s, x), \quad v(x) = v_n(s, x), \quad g(x) = g^k(x).$$

Recall that by Remark 2.4 we have $v(s, \cdot), v_n(s, \cdot) \in H^1_p(\mathbb{R}^d)$, for all $\omega$, $s \in [0, \tau_n]$. Also the condition (6.1) holds because $\tau_n \leq \sigma_n$. Then, by Lemma 6.2

$$I_{11,n} \leq NE \int_0^{\tau_n} ||v(s, \cdot) - v_n(s, \cdot)||_1^p ds. \quad (4.15)$$

Next, to handle the terms $I_{12,n}$ and $I_{13,n}$ we use Lemma 6.1. Note that $g^k(0) = 0$, for each $k$, so that this lemma is applicable. Then, by (4.5) and (4.2), we have

$$I_{12,n} + I_{13,n} \leq N \varepsilon^{p/2} E \int_0^{\tau_n} (||v_n(s, \cdot)||_1^p + ||v(s, \cdot)||_1^p) ds \leq N \varepsilon^{p/2}. \quad (4.16)$$

**Step 5.** We combine all the estimates (4.6) - (4.16) and obtain that

$$||\bar{v}_n||_{H^p(\tau_n)}^p \leq N \varepsilon^{p/2} + NE \int_0^{\tau_n} ||v_n(s, \cdot) - v(s, \cdot)||_1^p (1 + |Ds_n^{kl}(s)|^p) ds. \quad (4.17)$$

Next, we fix any stopping time $\tau$. Clearly, in (4.17) we may replace $\tau_n$ by $\tau \land \tau_n$ and, moreover, this replacement will not change the constant $N$. Using this combined with Remark 2.4, we get

$$E||v_n - v||_{V(\tau \land \tau_n)}^p \leq N (J_{1,n} + J_{2,n} + \varepsilon^{p/2})$$

$$+ NE \int_0^{\tau \land \tau_n} ||v_n - v||^{p}_{V(s)} (1 + |Ds_n^{kl}(s)|^p) ds, \quad (4.18)$$

where

$$J_{1,n} = E||g^k(v_n)\delta w_n^k||_{V(\tau_n)}^p,$$

$$J_{2,n} = E||h_x^{kl}(v) s_n^{kl}||_{V(\tau_n)}^p.$$  

First, note that we were able to replace $||v_n(s, \cdot) - v(s, \cdot)||_1^p$ by $||v_n - v||_{V(s)}$ because $\mu < 1$. Second, recall that by Lemma 3.3 $||v_n - v||_{V(s)}, s \geq 0$ is a process with continuous paths, and, hence, the integral on the right hand side of (4.18) is well-defined.
Let us estimate $J_{1,n}$ and $J_{2,n}$. By the product rule inequality in Hölder spaces we get

$$J_{1,n} \leq N\varepsilon^p E||g^k(v_n)||_{V(\tau_n)}^p.$$ 

Since $2 - \mu > 1 + d/p$, we may apply Lemma 6.3, and we obtain that $J_{1,n}$ is less than

$$N\varepsilon^p (E||v_n||_{V(\tau_n)}^p + E||v_n||_{V(\tau_n)}^{2p} + E||v_n||_{V(\tau_n)}^{(2+\theta)p}).$$

By (4.1) we may replace the terms involving $v_n$ by $N$. Using the same argument and (4.5), we obtain

$$J_{2,n} \leq N\varepsilon^p ||Dh^{kl}_\varepsilon||_{C^{1+\theta}} \leq N\varepsilon^p/2.$$ 

We point out that Lemma 6.3 is applicable because $h^{kl}_\varepsilon(0) = 0 = g^k(0)$. It follows from the last paragraph and (4.18) that

$$E x(\tau) \leq N\varepsilon^{p/2} + NE \int_0^\tau x(s) (1 + |Ds_n^{kl}(s)|^p)I_{s \leq \tau} ds, \quad (4.19)$$

where

$$x(s) := ||v_n - v||_{V(s \wedge \tau_n)}^p,$$

and $N$ depends on $T$. Then, by a stochastic variant of Gronwall’s inequality (see, for instance, Lemma 4 in [22]), we obtain

$$E||v_n - v||_{V(\tau_n)}^p \leq N\varepsilon^{p/2} \quad (4.20)$$

because the integral $\int_0^{\tau_n} |Ds_n^{kl}(s)|^p ds$ is bounded by $R$, for any $\omega$.

Next, let $A_n = \{\gamma_n < \eta_n \wedge \sigma_n \wedge T\}$. Then, by (4.1) we have

$$P(\gamma_n < \eta_n \wedge \sigma_n \wedge T) = E||v_n - v||_{V(\tau_n)}^p I_{A_n} \leq E||v_n - v||_{V(\tau_n)}^p I_{A_n}.$$ 

Therefore, by (4.20)

$$P(\gamma_n < \eta_n \wedge \sigma_n \wedge T) \leq N\varepsilon^{p/2}. \quad (4.21)$$

Step 6. Finally, we prove the convergence in probability. First, recall that $v \in V(T)$, for any $\omega$. Then, for any $\varepsilon > 0$, one can choose $R > 0$ such that

$$P(||v||_{V(T)} \geq R) < \varepsilon.$$ 

Then, by what was just said and (A6)(ii), for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_n P(\sigma_n(R) < T) < \varepsilon. \quad (4.22)$$

Next, for any $\delta > 0$, we have

$$P(||v_n - v||_{V(T)} \geq \delta) \leq P(||v_n - v||_{V(\tau_n)} \geq \delta) \quad (4.23)$$

$$+ P(\gamma_n < \eta_n(\varepsilon) \wedge \sigma_n(R) \wedge T) + P(\eta_n(\varepsilon) < T) + P(\sigma_n(R) < T).$$

Using Chebyshev’s inequality and (4.20), we get that the first term on the right hand side of (4.23) tends to 0 as $n \to \infty$. The same holds for the second and the last term by (4.21) and (4.22) respectively. By (A6) the
third term tends to 0 as $n \to \infty$. Thus, by what was just said, for any $\delta, \varepsilon > 0$, we obtain

$$
\lim_{n \to \infty} P(||v_n - v||_{V(T)} \geq \delta) \leq N(\varepsilon + (1 - \delta^p)\varepsilon^{p/2}),
$$

and this implies the claim.

**Proof of Theorem 2.9.** We set $\alpha = 1, \beta = 0$, and let $v$ and $v_n$ be the unique solutions of class $H^2_p(T)$ of equations (2.9) and (2.10) respectively. Clearly, we have $v \equiv u$ and $v_n \equiv u_n$ as elements of $H^2_p(T)$. Then, the result follows directly from Theorem 2.11.

**Proof of Theorem 2.10. Step 1.** First, we prove the inclusion $R_{cl} \subset P \circ u^{-1}|_{V(T)}$. We use the argument from [24]. Let $w^k_n$ be the polygonal approximation defined by (5.2) with $\gamma = 1$. Fix some $h \in H(p, T)$. In the equation (2.9) we set $\alpha = -1$, $\beta = 1$, and we replace $f(z,t,x)$ by $\bar{f}(z,t,x) := f(z,t,x) + \sum_{k=1}^{m} g^k(z(x)) Dh^k(t) - 1/2 \sum_{k=1}^{m} (g^k Dg^k)(z(x)), z \in H^1_p(\mathbb{R}^d)$.

Note that $\bar{f}(z,t,x)$ satisfies the assumption (A3)(p). Then by Remark 2.12 (i) there exists a unique solution $v_n$ of class $H^2_p(T)$ of the following SPDE:

$$
dz(t,x) = [a_{ij}(t,x) D_{ij} z(t,x) + f(z,t,x) + \sum_{k=1}^{m} g^k(z(t,x)) Dh^k(t) - \sum_{k=1}^{m} g^k(z(t,x)) Dw^k_n(t) + \sum_{k=1}^{m} g^k(z(t,x)) dw^k(t), z(0,x) = u_0(x).$$

Then, by Theorem 2.11, for any $\delta > 0$, and $n$ large, we have

$$
P(||v_n - R(h)||_{V(T)} < \delta) > 0. \quad (4.24)
$$

Next, we denote

$$
w^k(t,n) := w^k(t) - w^k_n(t) + h^k(t),
$$

$$\rho_n := \prod_{k=1}^{m} \exp(\int_0^T (Du^k_n(t) - Dh^k(t)) dw^k(t) - 1/2 \int_0^T |Du^k_n(t) - Dh^k(t)|^2 dt).$$

Let $P_n$ be a measure on $(\Omega, \mathcal{F})$ given by

$$
dP_n(\omega) := \rho_n(\omega) dP(\omega).
$$

To apply Girsanov’s theorem, we need to verify that $E\rho_n = 1$. By [20] we only need to check that

$$
\lim_{\varepsilon \downarrow 0} \varepsilon \ln E \exp(1 - \varepsilon \int_0^T |Du^1_n(t) - Dh^1(t)|^2 dt) < \infty. \quad (4.25)
$$
Using a simple inequality $ab \leq \delta a^2 + \delta^{-1}b^2$ combined with the fact that $Dh^1$ is a bounded function on $[0, T]$, for any $\varepsilon \in (0, 1)$, we get

$$
\frac{1 - \varepsilon}{2} \int_0^T |Dw_n^1(t) - Dh(t)|^2 dt
\leq Y := (1/2 - \varepsilon/4 - \varepsilon^2/4) \int_0^T |Dw_n^1(t)|^2 dt + N/\varepsilon + N,
$$

where $N$ is independent of $\varepsilon$. Next, observe that by the scaling property of Wiener process, for $t \in [l/n, (l+1)/n)$, the random variable

$$
Dw_n^1(t) = n[w^1(l/n) - w^1((l-1)/n)]
$$

is distributed as $\sqrt{n}\zeta$, where $\zeta$ is a standard normal variable. Hence, by the independence of increments of Wiener process, the random variable

$$
\int_0^T |Dw_n^1(t)|^2 dt = \sum_{l=0}^{[Tn]-1} \int_{(l+1)/n}^{l/n} |Dw_n^1(t)|^2 dt
$$

has the chi-square distribution with $\kappa := [Tn]$ degrees of freedom. Then, by the explicit representation of the moment generation function of this distribution we obtain

$$
\lim_{\varepsilon \downarrow 0} \varepsilon \ln E \exp(Y) \leq N \lim_{\varepsilon \downarrow 0} (\varepsilon \ln(1/\varepsilon) + 1) < \infty,
$$

and this proves (4.25).

Thus, Girsanov’s theorem is applicable, and $P_n$ is a probability measure on $(\Omega, \mathcal{F})$, and, for any $n$, $\{\bar{w}^k(t, n), t \geq 0, k = 1, \ldots, m\}$ is a sequence of independent standard Wiener processes with respect to $\{\mathcal{F}_t, t \leq T\}$ on $(\Omega, \mathcal{F}, P_n)$.

Next, let $\mathcal{H}_p^2(T, n)$ be the stochastic Banach space defined on the probability space $(\Omega, \mathcal{F}, P_n)$ with $w^k$ replaced by $\bar{w}^k_n, k = 1, \ldots, m$. Observe that $v_n$ is the unique solution of class $\mathcal{H}_p^2(T, n)$ of the following SPDE:

$$
dz(t, x) = [a^{ij}(t, x)D_{ij} z(t, x) + f(z(t, x))] dt
+ \sum_{k=1}^m g^k(z(t, x)) dw^k(t, n), \quad z(0, x) = u_0(x). \quad (4.26)
$$

Claim that

$$
P_n \circ v_n^{-1}|_{\mathcal{V}(T)} = P \circ u^{-1}|_{\mathcal{V}(T)}. \quad (4.27)
$$

To prove this one can invoke an argument similar to the one used for SDEs. Here, we use our Wong-Zakai theorem (see Theorem 2.9). First, may assume that $u_0 \equiv 0$ (see the proof of Theorem 5.1 of [17]). Second, by Theorem 2.9 and Portmanteau theorem one may replace the equations (1.2) and (4.26) by their Wong-Zakai approximation schemes such that the approximation of $w^k$ is given by (5.2) with $\gamma = 1$. It is well-known, that each Wong-Zakai approximation is a fixed point of a contraction operator on $\mathcal{H}_p^2(T)$ (see, for instance Theorem 5.1 and Theorem 6.3 of [17]). Now the claim follows from
Picard iteration method in $H^2_p(T)$ and the embedding theorem for $H^s_p(T)$ spaces (see Remark 2.4).

Finally, by (4.24) combined with the fact that $P_n$ is absolutely continuous with respect to $P$, and (4.27) we obtain that, for any $\delta > 0$,

$$P \circ u^{-1}|_{V(T)}(\{z \in V(T) : ||z - R(h)||_{V(T)} \leq \delta\}) > 0.$$  

This proves the announced inclusion.

**Step 2.** We prove the reverse inclusion. Let $u_n$ be the unique solution of class $H^2_p(T)$ of (1.3). Then, by Theorem 2.9 and Portmanteau theorem

$$P \circ u^{-1}|_{V(T)}(R_{cl}) \geq \lim_{n \to \infty} P(u_n \in R_{cl}) = 1.$$  

This implies the assertion.

5. **Appendix A.**

**Definition 5.1.** The process $w^k_n(t), t \geq 0$ is called the smoothing of $w^k$ if

$$w^k_n(t) = \int_0^1 w^k(t - r/n^\gamma) \, dr,$$  

for some $\gamma > 0$.

**Definition 5.2.** Let $\gamma \in \{1, 2, \ldots\}$ and set $h = n^{-\gamma}$. Then, the process $w^k_n(t), t \in [0, T]$ is called the polygonal approximation of $w^k$ if

$$w^k_n(t) = w^k((l - 1)h) + 1/h(t - (l - 1)h)(w^k(lh) - w^k((l - 1)h)),$$  

for $t \in [lh, (l + 1)h)$, and $l \in \{0, 1, 2, \ldots\}$.

The following lemma is similar to Proposition 6.3.1 and Proposition 6.4.1 of [27].

**Lemma 5.3.** Let $p > 0$, $\theta \in (0, 1/2)$ be numbers. Assume that $w^k_n$ is either given by (5.1) or (5.2). Then, for any $k, l$, the following assertions hold.

(i) For any $\varepsilon > 0$,

$$E||\delta w^k_n||_{C[0,T]}^p \leq N(p, T, \varepsilon)n^{-\gamma p/2+\varepsilon}.$$  

(ii) For any $\theta' \in (0, \theta)$,

$$E||\delta w^k_n||_{C^{1/2-\theta}[0,T]}^p \leq N(p, T, \theta, \theta')n^{-\gamma \theta' p}.$$  

(iii) For any $\varepsilon > 0$,

$$E||s^k_{nl}||_{C[0,T]}^p \leq N(p, T, \varepsilon)n^{-\gamma p/2+\varepsilon}.$$  

(iv)

$$E\int_0^T |Ds^k_{nl}(t)|^p \, dt \leq N(p, T).$$  

(vi) For any $\theta' \in (0, \theta)$,

$$E||s^k_{nl}||_{C^{1/2-\theta}[0,T]}^p \leq N(p, T, \theta, \theta')n^{-\gamma \theta' p}.$$
Proof. Clearly, it suffices to prove all the assertions with \( \gamma = 1 \). For any \( f : \mathbb{R} \to \mathbb{R} \), we denote its modulus of continuity as follows:

\[
\rho_f(h, T) = \sup_{t, s \in [0, T]; |t-s| \leq h} |f(t) - f(s)|.
\]

For the sake of convenience, in the proofs (i), (ii) we denote \( w := w^k, w_n := w^k_n \).

(i) Polygonal approximation. For \( t \in [l/n, (l + 1)/n) \), we have

\[
|\delta w_n(t)| \leq |w(t) - w((l - 1)/n)| + |w(l/n) - w((l - 1)/n)|. \tag{5.3}
\]

Then,

\[
|\delta w_n(t)| \leq 2\rho_w(1/n, T). \tag{5.4}
\]

Smoothing. Clearly,

\[
\delta w_n(t) = \int_0^1 (w(t - r/n) - w(t))\, dr. \tag{5.5}
\]

Hence, (5.4) holds. By Theorem 2.3.2 of [16], for any \( \alpha > 0 \), there exists a positive random variable \( N_{\alpha, T} \) such that, for any \( r > 0 \), \( EN_{\alpha, T}^r < \infty \), and

\[
\rho_w(h, T) \leq N_{\alpha, T} h^{1/2 - \alpha}, \forall \omega \in \Omega, h \in [0, T]. \tag{5.6}
\]

Thus, in both cases the assertion follows from (5.6).

(ii) Polygonal approximation. Fix any \( \alpha \in (0, \theta) \). First, we consider the case when \( |t - s| \geq 1/n, t, s \in [0, T] \). The argument in this case works for both approximations. By (5.4) and (5.6) we have

\[
1/(t - s)^{1/2 - \theta} |\delta w_n(t) - \delta w_n(s)| \leq 2n^{1/2 - \theta} \sup_{t \in [0, T]} |\delta w_n(t)| \leq 2N_{\alpha, T} n^{\alpha - \theta}.
\]

Next, we take any \( t, s \in [0, T] \) such that \( |t - s| < 1/n \). There are two subcases: either

\[
t, s \in I, I \in \{[l/n, (l + 1)/n), (l/n, (l + 1)/n]\} \tag{5.7}
\]

or

\[
l/n < s \leq (l + 1)/n \leq t < (l + 2)/n, \tag{5.8}
\]

for some \( l \).

To handle (5.7) we write

\[
|\delta w_n(t) - \delta w_n(s)| \leq |w_n(t) - w_n(s)| + |w(t) - w(s)|. \tag{5.9}
\]

Using (5.6) and the fact that \( |t - s| \leq 1/n \), we get

\[
|w(t) - w(s)| \leq N_{\alpha, T} n^{\alpha - \theta} |t - s|^{1/2 - \theta}. \tag{5.10}
\]

Next, by (5.2) and (5.6) we obtain

\[
|w_n(t) - w_n(s)| \leq |t - s| n \rho_w(1/n, T) \leq N_{\alpha, T} n^{\alpha - \theta} |t - s|^{1/2 - \theta}. \tag{5.11}
\]

Then, the claim in this subcase follows from (5.9) - (5.11).
We move to (5.8). By what was proved in the first subcase we get
\[ |w_n(t) - w_n(s)| \]
\[ \leq |w_n(t) - w_n((l + 1)/n)| + |w_n(s) - w_n((l + 1)/n)| \]
\[ \leq N_{\alpha,T} n^{\alpha - \theta} |t - (l + 1)/n|^{1/2 - \theta} + |s - (l + 1)/n|^{1/2 - \theta} \]
\[ \leq 2N_{\alpha,T} n^{\alpha - \theta} (t - s)^{1/2 - \theta}. \]
This combined with (5.9) and (5.10) proves the claim.

Smoothing. By what was said above we only need to consider the case when \( t - s \in (0,1/n) \). Then, we have
\[ |w_n(t) - w_n(s)| = |(\int_{t-1/n}^{t} - \int_{s-1/n}^{s})w(r) dr| \]
\[ \leq (\int_{s-1/n}^{t-1/n} + \int_{s}^{t})|w(r)| dr \leq 2|t - s| \max_{r \in [0,T]} |w(r)|. \]
As above, we finish the argument by combining this with (5.9) and (5.10) as follows:
\[ 1/(t - s)^{1/2 - \theta} |\delta w_n(t) - \delta w_n(s)| \]
\[ \leq \max_{r \in [0,T]} |w(r)| n^{-(1/2 + \theta)} + N_{\alpha,T} n^{\alpha - \theta} \]
\[ \leq N_{\alpha,T} (T^{1/2 - \alpha} + 1)n^{\alpha - \theta}. \]

(iii) This estimate was proved in Proposition 6.3.1 and Proposition 6.4.1 of [27].

(iv) Smoothing. First, by Hölder’s inequality we have
\[ E \int_{0}^{T} |D_s w_n^l(t)|^p dt \leq n^p \int_{0}^{T} M_{1,n}^{l/2}(t) M_{2,n}^{l/2}(t) dt, \tag{5.12} \]
where
\[ M_{1,n}(t) = \int_{0}^{1} E|w^k(t - r/n) - w^k(t)|^{2p} dr, \]
\[ M_{2,n}(t) = E|w^l(t) - w^l(t - 1/n)|^{2p}. \]
By the scaling property of Wiener process
\[ M_{i,n}(t) \leq N(p)n^{-p}, i = 1, 2, \]
and this combined with (5.12) implies the claim.

Polygonal approximation. This time, for \( t \in [(l/n), (l + 1)/n) \), we have
\[ |\delta w_n^l(t)| \leq M_{i,n}(t) := |w^l(t) - w^l((l - 1)/n)| + |w^l(l/n) - w^l((l - 1)/n)|, \]
\[ D_s w_n^l(t) = n[w^l(l/n) - w^l((l - 1)/n)]. \]
Then, by Cauchy-Schwartz inequality and the scaling property of Wiener process we get
\[ E \int_{0}^{T} |s_n^{ij}|^p dt \]
\[ \leq \sum_{i=0}^{[T_n]-1} \int_{[i/n]}^{(i+1)/n} (EM_{i,n}(t))^{1/2} (E|Dw_n^i(t)|^{2p(t)})^{1/2} dt \leq N(p, T). \]

(v) By Hölder’s inequality, for any \( r > 1 \), and \( r' = r/(r - 1) \), we have
\[ E||Ds_n^{kl}||_{L^\infty[0,T]}^p \leq M_{1,n}M_{2,n}, \]
where
\[ M_{1,n} = (E||\delta u_n^k||_{C([0,T])}^p)^{1/r}, \quad M_{2,n} = (E||Du_n^l||_{L^\infty[0,T]}^{pr})^{1/r}. \]
By (i) and (5.6)
\[ M_{1,n} \leq N(p, r, \varepsilon, T)n^{-p/2+\varepsilon}, \]
\[ M_{2,n} \leq n^p(E||\rho u_n(1/n, T)||_{pr})^{1/r'} \leq N(p, r, \varepsilon, T)n^{p/2+\varepsilon}. \]

Then, by the above, for any \( \varepsilon > 0 \),
\[ E||Ds_n^{kl}||_{L^\infty[0,T]}^p \leq N(p, \varepsilon, T)n^\varepsilon. \] (5.13)

By the interpolation inequality (see, for example, Theorem 3.2.1 in [18]), for any \( \lambda > 0 \),
\[ ||s_n^{kl}||_{C^{1/2-\theta}} \leq N(\theta, T)(\lambda^{1/2+\theta}||Ds_n^{kl}||_{L^\infty[0,T]} + \lambda^{\theta-1/2}||s_n^{kl}||_{C([0,T])}). \]

We finish the proof by setting \( \lambda = 1/n \) and using (iii) combined with (5.13).

\[ \square \]

6. APPENDIX B.

Lemma 6.1. Let \( g \in C^1_{1\infty}(\mathbb{R}) \), \( Dg \in L^\infty(\mathbb{R}) \), \( g(0) = 0 \), and \( u \in H^1_p(\mathbb{R}^d) \). Then,
\[ ||g(u(\cdot))||_{1,p} \leq N(d, p)||Dg||_{\infty}||u||_{1,p}. \]

Proof. Recall that the spaces \( W^1_p(\mathbb{R}^d) \) and \( H^1_p(\mathbb{R}^d) \) coincide as sets and have equivalent norms. By this and the chain rule in \( W^1_p(\mathbb{R}^d) \) we may write
\[ ||g(u(\cdot))||_{1,p} \leq N(d, p)(||g(u(\cdot))||_{p} + ||D_1u(\cdot)Dg(u(\cdot))||_{p}) \]
\[ \leq N(||Dg||_{\infty}||u||_{p} + ||D_1u||_{p}) \]
\[ = N||Dg||_{\infty}||u||_{W^1_p(\mathbb{R}^d)} \leq N||Dg||_{\infty}||u||_{1,p}. \]

\[ \square \]

Lemma 6.2. Let \( u, v \in H^1_p(\mathbb{R}^d) \) and assume that
\[ ||Dg||_{\infty} + ||D^2g||_{\infty} \leq K, \quad ||D_1u||_{\infty} \leq R. \] (6.1)

Then,
\[ ||g(u(\cdot)) - g(v(\cdot))||_{1,p} \leq N(d, p)K(1 + R)||u - v||_{1,p}. \]
Proof. By the argument of the proof of Lemma 6.1 we have
\[\|g(u(\cdot)) - g(v(\cdot))\|_{1,p} \leq N(d,p)(\|g(u(\cdot)) - g(v(\cdot))\|_p + \|D_i u(\cdot) D g(u(\cdot)) - D_i v(\cdot) D g(v(\cdot))\|_p)\]
\[\leq N K \|u - v\|_p + N \|D_i u(\cdot) D g(u(\cdot)) - D g(v(\cdot))\|_p + N \|D g(v(\cdot)) (D_i u(\cdot) - D_i v(\cdot))\|_p.\]

By this and (6.1) we obtain the assertion of the lemma. 

\[\square\]

Lemma 6.3. Let \( p > d, \delta \in (0,1), \gamma \in (d/p,1), \tau > 0 \) be numbers. Let \( g : \mathbb{R} \to \mathbb{R} \) be a function such that \( D g \in C^{1+\theta}([0,\tau], H^{1+\gamma}[\mathbb{R}^d]) \) and \( g(0) = 0 \). Denote
\[\mathcal{B} := C^{\delta}([0,\tau], H^{1+\gamma}_p(\mathbb{R}^d))\]
and take any \( u \in \mathcal{B} \). Then, there exists a constant \( N \) independent of \( g \) and \( u \) such that
\[\|g(u)\|_B \leq N \|D g\|_{C^{1+\theta}}(\|u\|_B + \|u\|_B^2 + \|u\|_B^{2+\theta}).\]

Proof. We only show how to prove the estimate of the Hölder’s seminorm. The sup norm estimate can be obtained in the same way.

For the sake of convenience, we denote \( u_t = u(t,\cdot) \), and we omit the dependence of \( u \) on the spatial variable \( x \). We also denote \( N_g = \|D g\|_{C^{1+\theta}} \).

Take any \( s, t \in [0,\tau] \) such that \( s \neq t \). First, by the fact that \((1 - D_i)\) is a strongly elliptic differential operator of order 1 we obtain (see Theorem 13.3.10 of [19])
\[\|g(u_t) - g(u_s)\|_{1+\gamma,p} \leq N(J^{(1)} + J^{(2)}), \tag{6.2}\]
where
\[J^{(1)} = \|g(u_t) - g(u_s)\|_{\gamma,p},\]
\[J^{(2)} = \|D g(u_t) D_i u_t - D g(u_s) D_i u_s\|_{\gamma,p}.\]

Recall that, by the elementary embedding (see Section 2) we may replace \( \gamma \) by 1 in the expression for \( J^{(1)} \). Since \( \gamma > d/p \), by the embedding theorem for \( H^p_\theta(\mathbb{R}^d) \) spaces (see, for instance, Theorem 13.8.1 of [19]) we have
\[\sup_{t \leq T} \|D_i u_t\|_\infty \leq N \|u\|_B.\]

Then, by Lemma 6.2 and what was just said we obtain
\[J^{(1)} \leq N N_g(1 + \|u\|_B) \|u_t - u_s\|_{1,p}. \tag{6.3}\]

Next, we split \( J^{(2)} \) in three pieces and use triangle inequality as follows:
\[J^{(2)} \leq J^{(2,1)} + J^{(2,2)} + J^{(2,3)}, \tag{6.4}\]
where
\[J^{(2,1)} = \|(D g(u_t) - D g(0))(D_i u_t - D_i u_s)\|_{\gamma,p},\]
\[J^{(2,2)} = \|D_i u_s (D g(u_t) - D g(u_s))\|_{\gamma,p},\]
\[J^{(2,3)} = \|D g(0)\| \|u_t - u_s\|_{\gamma,p}.\]
It is well-known that $H^\gamma_p(\mathbb{R}^d)$ is a multiplication algebra because $\gamma > d/p$ (see, for example, Theorem 1 in Section 4.6.1 of [26]). Then, we get

$$J^{(2,1)} \leq N ||Dg(u_t) - Dg(0)||_{1,1+p} \leq N ||u_t - u_s||_{1,1+p},$$

$$J^{(2,2)} \leq N ||u_s||_{1,1+p} ||Dg(u_t) - Dg(u_s)||_{1,1+p}.$$ 

To handle $J^{(2,1)}$ we estimate $||Dg(u_t) - Dg(0)||_{1,1+p}$ via Lemma 6.1. By this we have

$$J^{(2,1)} \leq NN_g ||u_t - u_s||_{1,1+p}.$$ (6.5)

Next, using the fact that $Dg^2 \in C^0$, we obtain

$$||Dg(u_t) - Dg(u_s)||_{1,1+p} \leq ||Dg(u_t) - Dg(u_s)||_{1,1+p} + N ||D^2g(u_s) (D_t u_t - D_t u_s)||_{1,1+p} + N ||D_t u_t (D^2g(u_t) - D^2g(u_s))||_{1,1+p} \leq NN_g ||u_t - u_s||_{1,1+p} + N N_g ||u_t - u_s||_{1,1+p}.$$ 

Again, by the embedding theorem we may replace $||u_t - u_s||_{1,1+p}$ by $||u_t - u_s||_{1,1+p}$. Then, by the above we have

$$J^{(2,2)} \leq NN_g ||u_s||_{1,1+p} ||u_t - u_s||_{1,1+p} + ||u_t||_{1,1+p} ||u_t - u_s||_{1,1+p}.$$ (6.6)

The assertion follows from (6.2) - (6.6). 

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