Anticommutativity and \( n \)-schemes

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Abstract

The purpose of this paper is two-fold. A first and more concrete aim is to give new characterizations of equivalence distributive Goursat categories (which extend 3-permutable varieties) through variations of the little Pappian Theorem involving reflexive and positive relations. A second and more abstract aim is to show that every finitely complete category \( \mathcal{E} \) satisfying the \( n \)-scheme is locally anticommutative.

1 Introduction and Preliminaries

In this section we recall some basic definitions and results from the literature, needed throughout the article.

1.1 \( n \)-schemes

For a sublattice \( L \) of an equivalence lattice \( \text{Eq}A \), Gumm’s Shifting Lemma is stated as follows. Given congruences \( R, S \) and \( T \) on the same algebra \( X \) in \( V \) such that \( R \land S \leq T \), whenever \( x, y, z, t \) are elements in \( X \) with \( (x, y) \in R \land T, (x, t) \in S, (y, z) \in S \) and \( (t, z) \in R \), it then follows that \( (t, z) \in T \). We display this condition as

\[
\begin{array}{c}
  x & S & t \\
  \hline
  R & R & T \\
  y & S & z
\end{array}
\]

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A variety $\mathcal{V}$ of universal algebras satisfies the Shifting Lemma precisely when it is congruence modular, this meaning that the lattice of congruences on any algebra in $\mathcal{V}$ is modular. In particular, since any 3-permutable variety is congruence modular, it always satisfies the Shifting Lemma.

Recall from [11] that a sublattice $L$ of an equivalence lattice $\text{Eq}A$ satisfies the Triangular scheme if for each $R, S, T \in L$ with $R \wedge S \leq T$ and for $x, y, z \in A$ such that $\langle x, y \rangle \in T$, $\langle x, z \rangle \in S$, $\langle z, y \rangle \in R$ we have $\langle z, y \rangle \in T$. This can be visualized as follows:

A sublattice $L$ of $\text{Eq}A$ satisfies the $n$-scheme if for each $R, S, T \in L$ with $R \wedge S \leq T$ and for $x, y, z_1, \cdots, z_n \in A$ such that

$$\langle x, y \rangle \in R, \langle x, z_1 \rangle \in S, \langle z_1, z_2 \rangle \in T, \langle z_2, z_3 \rangle \in S, \cdots, \langle z_{n-1}, y \rangle \in S$$

for $n$ odd and $\langle z_{n-1}, y \rangle \in T$ for $n$ even we have $\langle x, y \rangle \in T$. These schemes can be also visualized but, contrary to the previous cases, classes of the same congruence fail to be parallel:

A sublattice $L$ of $\text{Eq}A$ satisfies the little Pappian Theorem, [21] if given congruences $R, S_i$, and $T$ on the same algebra $X$ in $L$ such that $R \wedge S_i \leq T$, whenever $x, y, u, z, x', y', z'$ are element in $X$ with $(u, y'), (x, z) \in S_1$, $(x', x), (u, z') \in S_1$, $\langle x, y \rangle \in T$. This can be visualized as follows:

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\( S_2, (x', u), (u, z), (y', x), (x, z') \in R \) and \( (z, z') \in T \), then \( (x', y') \in T \):

\[
\begin{align*}
S_2 & \quad R \\
\downarrow & \quad T \\
\downarrow & \quad S_1 \\
x & \quad R \\
\downarrow & \quad z' \\
(x') & \quad R \\
\downarrow & \quad T \\
y' & \quad S \\
\downarrow & \quad u \\
x & \quad R \\
\downarrow & \quad z' \\
y & \quad R \\
\downarrow & \quad S \\
x & \quad \downarrow \\
(y', x) & \quad (x, z') \quad \in \quad R \quad \wedge \quad S \quad \leq \quad T
\end{align*}
\]

Similarly, on identifying \( S_2 \) with \( T \) and \( u \) with \( z' \) we obtain a sublattice \( L \) of \( EqA \) satisfies the scheme-1 if given congruences \( R, S \) and \( T \) on the same algebra \( X \) in \( L \) such that \( R \wedge S \leq T \), one has

\[
\begin{align*}
S_2 & \quad R \\
\downarrow & \quad T \\
\downarrow & \quad S_1 \\
x & \quad R \\
\downarrow & \quad z' \\
(x') & \quad R \\
\downarrow & \quad T \\
y' & \quad S \\
\downarrow & \quad u \\
x & \quad R \\
\downarrow & \quad z' \\
y & \quad R \\
\downarrow & \quad S \\
x & \quad \downarrow \\
(y', x) & \quad (x, z') \quad \in \quad R \quad \wedge \quad S \quad \leq \quad T
\end{align*}
\]

### 1.2 Anticommutative categories

Our categories will always be regular, in the sense of Barr [2]; we recall that a category is regular if it has finite limits, each arrow factors as a regular epi followed by a mono, and regular epis are pull-back stable. (It turns out that in a regular category the kernel pair of an arrow always has a coequalizer, given by the regular epi part of the factorization of the arrow) In a regular category, it is possible to compose relations. If \((R, r_1, r_2)\) is a relation from \( X \) to \( Y \) and \((S, s_1, s_2)\) a relation from \( Y \) to \( Z \), their composite \( SR \) is a relation from \( X \) to \( Z \) obtained as the regular image of the arrow

\[
(r_1 \pi_1, s_2 \pi_2) : \ R \times_YS \longrightarrow X \times Z,
\]

where \((R \times_YS, \pi_1, \pi_2)\) is the pullback of \( r_2 \) along \( s_1 \). The composition of relations is then associative, thanks to the fact that regular epimorphisms are assumed to be pullback
stable. A relation $E$ on $X$ is called positive when it is of the form $E = R^2 R$ for some relation $R \to X \times Y$. Recall that a category is said to be pointed if it admits a zero object $0$, i.e., an object which is both initial and terminal. A point in a category $E$ is a split epimorphism $p : A \to X$ together with a fixed splitting $s : X \to A$, usually depicted as

$$\begin{array}{ccc}
A & \xrightarrow{s} & X.
\end{array}$$

Let $E$ be an arbitrary category. The category $Pt_E(X)$ of points of $E$ over $X$ is the category of pointed objects of the comma category $E \downarrow X$, that is, $Pt_E(X) = (X, 1_X) \downarrow (E \downarrow X)$.

Explicitly, objects of this category are triples $(A, p, s)$ where $A$ is an object of $E$ and $p : A \to X$ and $s : X \to A$ are morphisms in $E$ with $p \circ s = 1_X$. A morphism $f : (A, p, s) \to (B, q, t)$ in $Pt_E(X)$ is a morphism $f : A \to B$ in $E$ such that $q \circ f = p$ and $f \circ s = t$. The category $Pt_E(X)$ is always pointed, where the zero-object is $(X, 1_X, 1_X)$, and if $E$ is finitely complete, then so is $Pt_E(X)$. Recall that two morphisms $f : A \to C$ and $g : B \to C$ in a pointed category $E$ with binary products are said to commute if there exists a morphism $\rho : A \times B \to C$ such that $\rho \circ \iota_1 = f$ and $\rho \circ \iota_2 = g$, where $\iota_1 : A \to A \times B$ and $\iota_2 : B \to A \times B$ are the canonical product inclusions.

Two morphisms $f : X \to Z$ and $g : Y \to Z$ in a pointed category $E$ are said to be disjoint if for any commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{a} & Y \\
\downarrow{b} \quad & \quad \downarrow{g} \\
X & \xrightarrow{f} & Z
\end{array}$$

we have $g \circ a = 0 = f \circ b$. This brings us to the main definition of this paper: A pointed category $E$ with binary products is a called anticommutative if every pair of commuting morphisms are disjoint.

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2 Majority Categories and Goursat Categories

For a regular category $\mathcal{E}$ the property of being a majority category can be equivalently defined as follows (see [16]): for any reflexive relations $R, S$ and $T$ on the same object $X$ in $\mathcal{E}$, the inequality

$$R \land (ST) \leq (R \land S)(R \land T)$$

holds. We then observe that any regular majority category satisfies the 3-scheme and, consequently, also the 2-scheme and Shifting Lemma):

**Lemma 2.1.** The $n$-scheme holds true in any regular majority category $\mathcal{E}$.

**Proof.** Given equivalence relations $R, S$ and $T$ on the same object such that $R \land S \leq T$, then

$$R \land (S,T)_n \leq (R \land S)(R \land T) \cdots (R \land S) \leq T$$

for $n$ odd and

$$R \land (S,T)_n \leq (R \land S)(R \land T) \cdots (R \land T) \leq T$$

for $n$ even. Here $(S,T)_n$ denotes the composite $STST \cdots$ of $S$ and $T$, $n$ times. □

**Corollary 2.2.** Let $\mathcal{E}$ be a regular majority category.

1. The little Pappian Theorem holds true in $\mathcal{E}$.
2. The scheme-1 holds true in $\mathcal{E}$.

A variety $\mathcal{V}$ of universal algebras is called 3-permutable when the strictly weaker equality $RSR = SRS$ holds. Such varieties are characterized by the existence of two quaternary operations $p$ and $q$ satisfying the identities $p(x, y, y, z) = x$, $p(u, u, v, v) = q(u, u, v, v)$, $q(x, y, y, z) = z$ (see [10]). The notions of 3-permutability can be extended from varieties to regular categories by replacing congruences with (internal) equivalence relations, allowing one to explore some interesting new (non-varietal) examples. Regular categories that are 3-permutable are usually called Goursat categories. As examples of Goursat categories we have: compact groups, topological groups, torsion-free abelian groups, reduced commutative rings. It is well-known that any 3-permutable variety is congruence modular, thus the Shifting Lemma and 3-scheme hold. This result also extends to the regular categorical context.
Theorem 2.3. [10] Let \( \mathcal{E} \) be a regular category. The following statements are equivalent:

(i) \( \mathcal{E} \) is a Goursat category;

(ii) \( \forall R, S \in \text{Equiv}(X), RSR = SRS \in \text{Equiv}(X), \) for any \( X; \)

(iii) every relation \( P \to X \times Y \) in \( \mathcal{E}, PP^oPP^o = PP^o; \)

(iv) every reflexive relation \( F \) in \( \mathcal{E}, F^oF \in \text{Equiv}(X); \)

(v) every reflexive and positive relation in \( \mathcal{E} \) is an equivalence relation.

Let us begin with the following observation:

Proposition 2.4. Let \( \mathcal{E} \) be an equivalence distributive Goursat categories.

(1) The Little Pappian Theorem holds true in \( \mathcal{E} \) when \( S_i \) is a reflexive relation and \( R \) and \( T \) are equivalence relations.

(2) The scheme-1 holds true in \( \mathcal{E} \) when \( S \) is a reflexive relation and \( R \) and \( T \) are equivalence relations.

Proof. The proof of this result is based on that of Proposition 5.3 in [12] which claims that a Goursat category satisfies the Shifting Lemma, 2-scheme and 3-scheme when \( S \) is a reflexive relation and \( R \) and \( T \) are equivalence relations.

We prove (1). Let \( R \) and \( T \) be equivalence relations and let \( S_i \) be a reflexive relation on an object \( X \) such that \( R \cap S_i \subseteq T. \) Suppose that \( x, y, z, u, x', y', z' \) are elements in \( X \) related as in (1.1). We are going to show that \( (x', y') \in T. \)

We apply 2-scheme to

\[
\begin{array}{c}
T \\
\downarrow R \\
\downarrow S_i \\
\downarrow u \\
\end{array}
\]

We now apply the Shifting Lemma to

\[
\begin{array}{c}
T \\
\downarrow R \\
\downarrow S_i \\
\downarrow x \\
\end{array}
\]

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Next we apply 2-scheme to

We now apply the Shifting Lemma to

It follows that, \((x', u), (u, z) \in T\) and \((z, z'), (z', x) \in T\), \((x, y') \in T\). We conclude that \(x'Ty'\) \((T \text{ is transitive})\), as desired. \(\Box\)

We are now ready to prove the main result in this section:

**Theorem 2.5.** Let \(E\) be a regular category. The following conditions are equivalent:

1. \(E\) is an equivalence distributive Goursat category;
2. the Little Pappian Theorem holds true in \(E\) when \(S_i\) is a reflexive relation and \(R\) and \(T\) are reflexive and positive relations;
3. the scheme-1 holds true in \(E\) when \(S\) is a reflexive relation and \(R\) and \(T\) are reflexive and positive relations.

**Proof.** (1) \(\Rightarrow\) (2) This implication follows from the fact that reflexive and positive relations are necessarily equivalence relations in the Goursat context (Theorem 2.3) and from Proposition 2.4.

(2) \(\Rightarrow\) (3) Obvious. (3) \(\Rightarrow\) (1) We shall prove that for any reflexive relation \(E\) on \(X\) in \(E\), \(EE^o = E^o E\) (see Theorem 2.3(iv)). Suppose that \((x, y) \in EE^o\). Then, for some \(z\) in \(X\), one has that \((z, x) \in E\) and \((z, y) \in E\). Consider the reflexive and positive relations.
\[ R = EE^o \text{ and } T = E^o E, \text{ and the reflexive relation } E \text{ on } X. \text{ Then we have:} \]

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
T \quad \text{ } R \\
\text{x} \quad \text{y} \\
\text{y} \quad \text{E} \\
\text{E} \quad \text{E} \\
\text{y} \quad \text{R} \\
\text{R} \quad \text{E} \\
\text{z} \\
\text{y} \quad \text{z} \\
\end{array}
\]

\[ \text{to conclude that } (x, y) \in E^o E. \text{ Having proved that } EE^o \leq E^o E \text{ for every reflexive relation } E, \text{ the equality } EE^o = E^o E \text{ follows immediately.} \]

\[ \square \]

3 Locally Anticommutative Categories

The fibration of points \( \pi : Pt(\mathcal{E}) \rightarrow \mathcal{E} \) classifies many central notions in categorical algebra, such as, Mal’tsev categories: a finitely complete category \( \mathcal{E} \) is Mal’tsev if and only if every fibre \( Pt_{\mathcal{E}}(X) \) of the fibration of points is unital, strongly unital or subtractive \[3\].

**Definition 3.1.** \[15\] A category \( \mathcal{E} \) is locally anticommutative if for any object \( X \) in \( \mathcal{E} \), the category \( Pt_{\mathcal{E}}(X) \) is anticommutative.

**Proposition 3.2.** If \( \mathcal{D} \) is any finitely complete category which satisfies the \( n \)-scheme and \( U : \mathcal{E} \rightarrow \mathcal{D} \) is any conservative functor (i.e., reflects isomorphisms) which preserves pullbacks and equalizers, then \( \mathcal{E} \) satisfies the \( n \)-scheme.

Note that the assumptions on the functor \( U \) imply that it preserves monomorphisms, and that if \( R \) is an equivalence relation in \( \mathcal{E} \), then \( U(R) \) the relation obtained by applying \( U \) to the representative of \( R \) is an equivalence relation in \( \mathcal{D} \).

**Proof.** Let \( R, S, T \) are equivalence relations on an object \( X \) in \( \mathcal{E} \) such that \( R \wedge S \leq T \) and for \( x, y, z_1, \cdots, z_n \) are related such that

\[
\langle x, y \rangle \in R, \langle x, z_1 \rangle \in S, \langle z_1, z_2 \rangle \in T, \langle z_2, z_3 \rangle \in S, \cdots, \langle z_{n-1}, y \rangle \in S
\]
for $n$ odd and $(z_{n-1}, y) \in T$ for $n$ even. Then we are required to show that $xTy$, which is equivalent to showing that in the pullback diagram

$$
\begin{array}{ccc}
P & \rightarrow & T \\
p_2 \downarrow & & \downarrow t \\
R & \rightarrow & X \times X
\end{array}
$$

$p_2$ is an isomorphism. Applying $U$ to the diagram above, we obtain a pullback diagram in $\mathbb{D}$. The assumptions on $U$ easily imply that the canonical morphism $U(X \times X) \rightarrow U(X) \times U(X)$ is a monomorphism, which implies that $(U(P), U(p_1), U(p_2))$ form a pullback of $U(t)$ along $(U(x), U(y))$. Since

$$(U(x), U(y)) \in U(R), \langle U(x), U(z_1) \rangle \in U(S), \langle U(z_1), U(z_2) \rangle \in U(T),$$

$$\langle U(z_2), U(z_3) \rangle \in U(S), \cdots, \langle U(z_{n-1}), U(y) \rangle \in U(S)$$

for $n$ odd and $\langle U(z_{n-1}), U(y) \rangle \in U(T)$ for $n$ even. Since $\mathbb{D}$ satisfies the $n$-scheme $(U(x), U(y))$ factors through $T$, which implies that $U(p_2)$ is an isomorphism, so that $p_2$ is an isomorphism since $U$ reflects isomorphisms.

**Corollary 3.3.**

(i) If $\mathcal{E}$ is a finitely complete category which satisfies the $n$-scheme, then so does $\mathcal{E} \downarrow X$ and $X \downarrow \mathcal{E}$ for any object $X$. In particular, it follows that $Pt_{\mathcal{E}}(X)$ satisfies the $n$-scheme if $\mathcal{E}$ does.

(ii) Every finitely complete category $\mathcal{E}$ satisfying the $n$-scheme is locally anticommutative.

**Proof.** The proof follows from the fact that the codomain-assigning functor $X \downarrow \mathcal{E} \rightarrow \mathcal{E}$ and the domain-assigning functors $\mathcal{E} \downarrow X \rightarrow \mathcal{E}$ and $Pt_{\mathcal{E}}(X) \rightarrow \mathcal{E}$ satisfy the conditions of Proposition 3.2.

**Proposition 3.4.** If $\mathbb{D}$ is any finitely complete category which satisfies the scheme-1 and $U : \mathcal{E} \rightarrow \mathbb{D}$ is any conservative functor (i.e., reflects isomorphisms) which preserves pullbacks and equalizers, then $\mathcal{E}$ satisfies the scheme-1.
Proof. Let $R, S, T$ are equivalence relations on an object $X$ in $E$ such that $R \wedge S \leq T$ and for $x, y, z, x', u, y', z'$ are related as follows

we show that $x'Ty'$.

We apply Proposition 3.2 (3-scheme) to

Next, We apply Proposition 3.2 (2-scheme) to

It follows that, $(x', z') \in T$, $(z', y) \in T$ and $(y, y') \in T$, we conclude that $x'Ty'$ ($T$ is transitive), as desired. \hfill \Box

Corollary 3.5.

(i) If $E$ is a finitely complete category which satisfies the scheme-1, then so does $E \downarrow X$ and $X \downarrow E$ for any object $X$. In particular, it follows that $Pt_{E}(X)$ satisfies the the scheme-1 if $E$ does.

(ii) Every finitely complete category $E$ satisfying the the scheme-1 is locally anticommutative.
References

[1] B. R. Amougou Mbarga, Triangular scheme revisited in the light of $n$-permutable categories, *Earthline Journal of Mathematical Sciences* 6(1) (2021), 105-116. https://doi.org/10.34198/ejms.6121.105116

[2] M. Barr, Exact categories, in: *Exact Categories and Categories of Sheaves*, 1-120, Lecture Notes in Math., vol. 236, Springer, Berlin, 1971. https://doi.org/10.1007/BFb0058580

[3] F. Borceux and D. Bourn, Meta-theorems, in: *Mal’cev, Protomodular, Homological and Semi-Abelian Categories*, Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, Dordrecht, 2004. https://doi.org/10.1007/978-1-4020-1962-3_1

[4] S. Burris and H.P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, 78, Springer-Verlag, New York-Berlin, 1981.

[5] A. Carboni, J. Lambek and M. C. Pedicchio, Diagram chasing in Mal’cev categories, *Journal of Pure and Applied Algebra* 69 (1991), 271-284. https://doi.org/10.1016/0022-4049(91)90022-T

[6] A. Carboni, G. M. Kelly and M. C. Pedicchio, Some remarks on Mal’tsev and Goursat categories, *Appl. Categor. Struct.* 1 (1993), 385-421. https://doi.org/10.1007/BF00872942

[7] I. Chajda and E.K. Horváth, A scheme for congruence semidistributivity, *Discuss. Math. Gen. Algebra Appl.* 23 (2003), 13-18. https://doi.org/10.7151/dmgaa.1060

[8] I. Chajda, E.K. Horváth and G. Czédli, Trapezoid lemma and congruence distributivity, *Math. Slovaca* 53 (2003), 247-253.

[9] M. Gran, *Notes on regular, exact and additive categories*, Summer School on Category Theory and Algebraic Topology, Ecole Polytechnique Fédérale de Lausanne, 11-13 September 2014.

[10] M. Gran and D. Rodelo, *Beck-chavalley condition and Goursat categories*, 2013. arxiv:1512.04066v1
[11] M. Gran, D. Rodelo and I. T. Nguefeu, Variations of the shifting lemma and Goursat categories, *Algebra Univers.* 80 (2019), Paper No. 2. 
https://doi.org/10.1007/s00012-018-0575-z

[12] D. Rodelo and I. Tchoffo Nguefeu, Facets of congruence distributivity in Goursat categories, 2020. arXiv:1909.10211v2

[13] B. Jonnson, Algebras whose congruence lattices are distributive, *Math. Scand.* 21 (1967), 110-121. https://doi.org/10.7146/math.scand.a-10850

[14] J. Hagemann and A. Mitschke, On $n$-permutable congruences, *Algebra Univers.* 3 (1973), Article number: 8. https://doi.org/10.1007/BF02945100

[15] M. Hoefnagel, Anticommutativity and the triangular lemma, 2020. arXiv:2008.00486v2

[16] M. Hoefnagel, Majority categories, *Theory Appl. Categ.* 34 (2019), 249-268.

[17] M. Hoefnagel, Characterizations of majority categories, *Appl. Categor. Struct.* 28 (2020), 113-134. https://doi.org/10.1007/s10485-019-09571-z

[18] M. Hoefnagel, A categorical approach to lattice-like structures, Ph.D. thesis, 2018

[19] P.-A. Jacqmin and D. Rodelo, Stability properties characterising $n$-permutable categories, *Theory Appl. Categ.* 32 (2017), Paper No. 45, 1563-1587.

[20] K. A. Kearnes and E. W. Kiss, The triangular principle is equivalent to the triangular scheme, *Algebra Univers.* 54 (2005), 373-383. https://doi.org/10.1007/s00012-005-1954-9

[21] H. Peter Gumm, Geometrical methods in congruence modular algebras, *Mem. Amer. Math. Soc.* 45 (1983). http://dx.doi.org/10.1090/memo/0286

[22] H. Peter Gumm, The little Desarguesian theorem for algebras in modular varieties, *Proc. Amer. Math. Soc.* 80 (1980), 393-397. 
https://doi.org/10.1090/S0002-9939-1980-0580991-6

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