Abstract

Using the recent solution of the triple Pomeron coupling in the QCD dipole picture as a closed string amplitude with six legs, its analytical form in terms of hypergeometric functions and numerical value are derived.

1. The triple Pomeron coupling has since a long time catalyzed a series of phenomenological and theoretical problems. In practice, it is associated to high-mass diffractive scattering at high-energy which has been observed experimentally and studied in various hadron-hadron reactions. More recently, the observation of high-mass diffraction at the virtual photon vertex of lepton-proton interactions at HERA has led to a new domain of investigation where the non-negligible virtuality $Q^2$ of the photon implies a (at least partial) connection with perturbative QCD. On the theoretical point of view, the coupling between three Pomerons in an S-Matrix framework has its counterpart in the six-gluon amplitude in QCD which, in the framework of the BFKL dynamics has been derived in Refs. and obtained in Ref. as a $1 \rightarrow 2$ dipole vertex in the related ($1/N_c$ limit of the) QCD dipole framework. Recently we reported a derivation of the high-mass diffractive dissociation which shows explicitly the implication of the QCD...
triple Pomeron coupling in the evaluation of the physical diffractive cross-section. The result could be interpreted as follows:

\[
G_{3P}^{eff}(k) \approx \frac{1}{k} g_{3P} \left( \frac{\alpha}{\pi} \right)^3 N_c \left( \frac{2a(y)}{\pi} \right)^3,
\]

where \( k \) is the momentum transfer at the proton vertex, \( G_{3P}^{eff} \) represents the effective triple-Pomeron coupling in an S-matrix phenomenological framework. The factor \( \left( \frac{2a(y)}{\pi} \right)^3 = \left\{ \frac{7}{2} \alpha N_c \zeta(3) y \right\}^{-3} \) is a logarithmic correction to the Pomeron behaviour \( e^{\alpha P y} \) as a function of the rapidity \( y \) and the QCD triple Pomeron coupling \( g_{3P} \) is given by \( I_{3P}(1,1,1) \), where

\[
I_{3P}(h_1, h_2, h_3) = \int d^2 \rho_1 d^2 \rho_2 d^2 \rho_3 \left| \rho_1 \right|^{h_1} \left| 1 - \rho_1 \right|^{-h_2 + h_3} \left| \rho_2 \right|^{h_2} \left| 1 - \rho_2 \right|^{-h_3} \times \\
\left| \rho_3 \right|^{h_3} \left| 1 - \rho_3 \right|^{-h_1} \left| 1 - \rho_1 \rho_2 \rho_3 \right|^{h_1 - h_2 + h_3 - 2},
\]

where \( h_i = 1 + 2i \nu_i, i = 1, 2, 3 \) are the complex conformal dimensions of the three \( SL(2, \mathbb{C}) \) eigenvectors associated to the three coupled Pomerons. In the same paper, another integral \( g_{2P} \) multiplies the triple Pomeron coupling for describing the coupling of two of the Pomerons to the target in the particular case of a dipole target. One finds

\[
g_{2P} = \int d^2 \rho_1 d^2 \rho_2 \left| \rho_1 \right|^{-1} \left| 1 - \rho_1 \right|^{-1} \left| \rho_2 \right|^{-1} \left| 1 - \rho_2 \right|^{-1} \left| 1 - \rho_1 \rho_2 \right|^{-1}.
\]

The purpose of our paper is to give both an analytical formula and a numerical evaluation of the integrals \( g_{2P} \) and \( g_{3P} \). As we shall see this requires an extensive use of the recently shown connection between QCD dipole vertices and Shapiro-Virasoro amplitudes which describe the tree-level amplitudes of a closed string theory.

The plan of the paper is the following: in section 2, we recall the connection of the QCD triple Pomeron coupling function with the adequate string amplitudes and give the string interpretation of \( g_{2P}(\gamma) \) and \( g_{3P} \). In section 3, we use the known relation between closed and open strings to express the quadruple (resp. sextuple) integrals corresponding to \( g_{2P} \) (resp. \( g_{3P} \)) in terms of products of double (resp. triple) integrals in \([0,1]\) real intervals. This gives an immediate answer for \( g_{2P} \), while the more involved solution for \( g_{3P} \) is treated in section 4 using a topological characterization of the open string amplitudes. Conclusions and some mathematical by-products of the obtained
relations involving generalized hypergeometric functions are proposed in the final section 5.

2. In the paper [4], a general relation was established between $1 \to p$ dipole vertices and dual Shapiro-Virasoro amplitudes $B_{2p+2}$ describing tree level amplitudes of a closed string theory. In particular, the $1 \to 2$ vertex, which is equivalent to the triple Pomeron coupling in the $1/N_c$ limit of BFKL dynamics, can be identified with the $B_{6}$ Shapiro-Virasoro [9] amplitude with appropriate variables. Indeed, one writes in the Bardacki-Ruegg formalism [10, 11]:

$$B_{N} = \int \prod_{i=2}^{N-2} d^{2} \rho_{i} \prod_{i=2}^{N-2} | \rho_{i} |^{2K_{ii}} \prod_{2 \leq i < j \leq N-1} | 1 - \rho_{i} \cdots \rho_{j-1} |^{2K_{ij}}, \quad (4)$$

where the exponents $2K_{ij}$ play the rôle of scalar products of external momenta in the string framework. Note that the definition of $K_{ij}$ can be extended [11] to the full set of indices $\{i = 1, \cdots, N\}$ by the conditions

$$\sum_{i=1}^{N} K_{ij} = 0; \quad K_{ii} = 2, \quad (5)$$

which allow one to put the expression for $B_{N}$ in a completely cyclic and $SL(2, \mathbb{C})$-symmetrical form known as the Koba Nielsen conformal invariant parametrization [12, 11]

$$B_{N} = \int \prod_{i=1}^{N} d^{2} z_{i} \left[ \frac{dz_{\alpha}dz_{\beta}dz_{\delta}}{|z_{\alpha\beta}z_{\beta\delta}z_{\delta\alpha}|^{2}} \right]^{-1} \prod_{i<j}^{N} | z_{ij} |^{2K_{ij}}, \quad (6)$$

where $z_{kl} = z_{k} - z_{l}$, and the factor $[\cdots]^{-1}$ corresponds to the formal division by the infinite “volume” of the invariance group $SL(2, \mathbb{C})$ leaving only $N - 3$ variables for integration, the other ones $\{z_{\alpha}, z_{\beta}, z_{\delta}\}$ chosen arbitrarily. The choice $\{0, 1, \infty\}$ and a suitable reparametrization of variables [11] ensures the equality of formulae (4) and (6).

Using definition (4) and the expressions (2) and (3), it is easy to identify the exponents $K_{ij}$ corresponding to the Pomeron couplings. In particular one finds:

$$g_{2p} = B_{5} \left\{ K_{ij} = -\frac{1}{2} \forall i \neq j; \quad K_{ii} = 2 \right\}$$

$$g_{3p} = B_{6} \left\{ K_{12} = K_{34} = K_{45} = 0; \quad K_{ij} = -\frac{1}{2} \forall (ij) \neq (12), (34), (56); \quad K_{ii} = 2 \right\} \quad (7)$$
A comment is in order about the choice of the set $K_{ij}$ in formula (3). It is well-known \[11\] that $B_N$ reflects the sphere topology of the surface spanned by the closed string described by the tree-level amplitude. As such, the couples $(ij), K_{ij} = 0$ in $B_6$ are arbitrary partitions in pairs of the six indices. The choice in formula (2) is thus for convenience. As we shall see, this arbitrariness is not true for the disk topology of the open string amplitudes $A_6$ which we will now consider.

3. There exists a remarkable relation between closed and open string tree-level amplitudes \[13\],

$$B_N = \sum_{O,O'} A_N(O) A_N(O') e^{i\pi f(O,O')}, \quad (8)$$

where $O, O'$ denote the possible independent orderings of the external legs of an open string tree-level amplitude on the boundary of a disk which represents the world-surface spanned by the open string. Indeed, only non-equivalent orderings are those which are not related by cyclic and/or reversal symmetry \[11\]. The phase factors $e^{i\pi f(O,O')}$ are defined \[13\] by the appropriate analytic continuation in the complex plane of the integration variables. One writes

$$A_N = \int_0^1 \prod_{i=2}^{N-2} d\rho_i \prod_{i=2}^{N-2} (\rho_i)^{K_{ii}} \prod_{2 \leq i < j \leq N-1} (1 - \rho_i \cdots \rho_j - 1)^{K_{ij}}, \quad (9)$$

and, for a given ordering $O$, $A_N(O)$ is obtained from (3) by the permutation $\rho_i \rightarrow \rho_{O(i)}$. It is crucial for our calculation that the amplitudes $B_N$ are thus reduced to products of $N - 3$ dimensional integrals on the segment $[0,1]$.

Let us now write the results for the amplitudes $B_5$ and $B_6$. After proper definition of the integral contours and phase factors, the results for generic amplitudes \[13\] are as follows:

$$B_5 = \sin \pi K_{12} \sin \pi K_{34} A_5(12345) A_5(21435)$$
$$+ \sin \pi K_{13} \sin \pi K_{24} A_5(13245) A_5(31425), \quad (10)$$

and

$$B_6 = \sin \pi K_{12} \sin \pi K_{45} A_6(123456)$$
$$\times \{ A_6(215346) \sin \pi K_{35} + A_6(215436) \sin \pi (K_{13} + K_{35}) \}$$
$$+ \text{permutations of (234)}. \quad (11)$$
From (10) and the symmetry of the exponents corresponding to $g_{2P}$, see (7), it is clear that the amplitudes $A_5(O)$ are all equal. The solution of the integral (3) can thus be expressed as follows:

\[
g_{2P} = 2 \left\{ A_5(12345) \right\}^2
\]

\[
= 2 \left\{ \int_0^1 d\rho_1 d\rho_2 \rho_1^{-\frac{1}{2}} (1 - \rho_1)^{-\frac{1}{2}} \rho_2^{-\frac{1}{2}} (1 - \rho_2)^{-\frac{1}{2}} (1 - \rho_1 \rho_2)^{-\frac{1}{2}} \right\}^2
\]

\[
= 2\pi \left\{ \sum_m \left[ \frac{\Gamma(\frac{1}{2} + m)}{\Gamma(1 + m)} \right]^3 \right\}^2
= 2\pi^4 \left\{ \sum \frac{\Gamma(\frac{1}{2} + m)}{\Gamma(1 + m)} \right\}^2
\]

\[
= \frac{1}{8\pi^2} \Gamma^8(1/4) \approx 378.145..., \quad (12)
\]

where $3F_2(z)$ is the well-known generalized hypergeometric function.

The case of $g_{3P}$ is less straightforward, since the appropriate conditions (7) introduce a partial dissymmetry between the exponents. While the topology of the sphere inherent to the dual amplitude $B_6$ leaves the zero exponents (e.g. $K_{12}=K_{34}=K_{45}=0$ in formula (7)) arbitrary, the various amplitudes $A_6$ appearing in formula (11) and characterized by the disc topology depend on this choice. By a simple investigation of the topologically different cases, see Fig.1, one is led to five a-priori independent configurations, namely

\[
\begin{align*}
A_{VI}^I &= A_6(123456) & A_{VI}^{II} &= A_6(134562) \\
A_{VI}^{III} &= A_6(124653) & A_{VI}^{IV} &= A_6(135246) \\
A_{VI}^V &= A_6(146253).
\end{align*}
\]

Indeed, as shown in Fig.1, these are the only unequivalent topological orderings when a particular partition of the set 123456 into pairs is singled out with zero exponents, the other being equal and non-zero.

However, the remaining symmetry of the exponents expressed by the relations (7) is useful to restrict further the number of independent amplitudes $A_6$ to be computed. By considering the “closed-open” string relations (11) starting with the different orderings (13) in the first place (remember that this redefinition is allowed by the sphere topology of the initial $B_6$) one gets the remarkable relations

\[
A_{VI}^{IV} = 2A_{VI}^{II}; \quad A_{VI}^V = 2A_{VI}^{III} = 4A_{VI}^I. \quad (14)
\]
Finally using these relations to simplify the expression (11) for $g_{3P}$ one can write
\[ g_{3P} \equiv 4 \, A_{6}^{II} \times A_{6}^{V}, \]  
(15)
or any other equivalent expression involving the different amplitudes related by (14).

4. Using the resulting relation (15) and the appropriate definitions (9), one finds the following expressions
\[ A_{6}^{II} = \int_{0}^{1} d\rho_{3} \prod_{i=2}^{4} (\rho_{i})^{-\frac{1}{2}} \prod_{i=2}^{4} (1 - \rho_{i})^{-\frac{1}{2}} (1 - \rho_{2}\rho_{3}\rho_{4})^{-\frac{1}{2}}, \]  
(16)
for which both analytic expressions and numerical evaluations can be performed.

First, by mere expansion of $(1 - \rho_{2}\rho_{3}\rho_{4})^{-\frac{1}{2}}$ one gets:
\[ A_{6}^{II} \equiv \pi^{3} \, {}_{4}F_{3}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; 1, 1, 1; z = 1), \]  
(17)
where ${}_{4}F_{3}(z)$ is the well-known generalized hypergeometric function. The evaluation of the amplitude $A_{6}^{V}$ requires more care due to the coupling factors between all integration variables. Keeping aside the integration over $\rho_{3}$ in expression (16), one writes
\[ A_{6}^{V} = \pi^{2} \int_{0}^{1} d\rho_{3} (1 - \rho_{3})^{-\frac{1}{2}} \left\{ {}_{2}F_{1}(\frac{1}{2}, 1; 1; \rho_{3}) \right\}^{2} \equiv \pi^{2} \int_{0}^{1} d\rho_{3} (1 - \rho_{3})^{-1} {}_{3}F_{2} \left( \frac{1}{2}, \frac{1}{2}, 1, 1; 1, \rho_{3}^{-1}; -\frac{\rho_{3}^{2}}{4(1 - \rho_{3})} \right), \]  
(18)
using a known identity between hypergeometric functions [14]. The Mellin-Barnes representation yields
\[ A_{6}^{V} = \int \frac{ds}{2i\pi} \frac{\Gamma^{2}(-s) \Gamma^{4}(\frac{1}{2} + s)}{\Gamma^{2}(1 + s)} \equiv G_{4,4}^{2,4} \left( -1 \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \]  
(19)
where $G_{4,4}^{2,4}$ is the Meijer function [15] and the $\psi$ functions come from the residues of the doubles poles in the integration over $s$. Note that $A_6^V$ can also be expressed as a derivative of a $4F_3$ with respect to the parameters

$$A_6^V = 4\pi^2 \frac{\partial}{\partial \epsilon} \left\{ \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)}{\Gamma(1 - \epsilon)} 4F_3\left(\frac{1}{2} \right. \left. - \epsilon; 1/2, 1/2; 1 - \epsilon, 1, 1; 1\right) \right\}_{\epsilon=0}. \quad (20)$$

All in all the final result reads

$$g_{3P} \equiv 4A_6^{II} A_6^V = 16\pi^5 \left\{ \sum_m \frac{\Gamma^4\left(\frac{1}{2} + m\right)}{\Gamma^4(1 + m)\Gamma^4\left(\frac{1}{2}\right)} \right\} \times \left\{ \sum_n \frac{\Gamma^4\left(\frac{1}{2} + n\right)}{\Gamma^4(1 + n)\Gamma^4\left(\frac{1}{2}\right)} \left( \psi(1+n) - \psi\left(\frac{1}{2} + n\right) \right) \right\} \approx 7766, \quad (21)$$

where we have explicitely written the expression of the $4F_3$ in (17). Equation (15) was quoted in our previous paper [6]. It coincides with expressions found independently using a different method in [16] and with the numerical evaluation of [17].

5. Some comments of physical and mathematical nature are in order. On the physical ground, it has been shown in [6] that the couplings $g_{2P}$ and principally $g_{3P}$, the triple Pomeron coupling in the QCD dipole picture appear in the calculation of the high-mass diffraction cross-section off the virtual photon, for which interesting data taken at HERA exist. It is quite remarkable that the high value obtained for $g_{3P}$ may compensate the smallness of the perturbative six-gluon factor $(\alpha_s)^3$ and give rise to a large effective coupling $G_{3P}^{\text{eff}}$ of (1). As noticed in [6], this may help in predicting a quite sizeable cross-section, opening the way towards an unified description of diffractive and non-diffractive proton structure functions [18].

On a more mathematical point of view, the relation between string amplitudes and specific hypergeometric functions show some interesting features. It has been noticed [19] that integrals of the type

$$I_{p+1}(z) = \int \prod_i d^2 u_i \left| u_i \right|^{\alpha_i} \left| 1 - u_i \right|^{\beta_i} \left| 1 - \prod_j u_j z \right|^{\alpha_0} \quad (22)$$

can be expressed explicitly in terms of known combinations of hypergeometric functions $p+1F_p(z)$. When some coefficients $\alpha$ and $\beta$ are equal degeneracies
may occur which leads to the appearance of derivatives. In this respect, we note that, using (20), our result (21) may also be expressed as
\[ g_3P(1) \equiv 16\pi^5 4F_3\left(1\over2, 1\over2, 1\over2; 1, 1, 1; 1\right) \times \]
\[ \times \frac{\partial}{\partial \epsilon} \left\{ \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)}{\Gamma\left(1 - \epsilon\right)} 4F_3\left(1\over2 - \epsilon, 1\over2, 1\over2, 1\over2; 1, 1, 1; 1\right) \right\} \bigg|_{\epsilon=0}. \] (23)

In order to illustrate how powerful is the method based on string amplitudes, we comment on the mathematical identities implied by the (14) which lead to highly non trivial relations between special functions. Among them, the relation \( A_{IV}^6 = 2A_{II}^6 \) leads to the identity
\[ \int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} 2F_1\left(1\over2, 1\over2, 1; x\right) 2F_1\left(1, 1\over2; 3\over2, x\right) = 2 \times 4F_3\left(1\over2, 1\over2, 1\over2, 1\over2; 1, 1, 1; 1\right). \] (24)

The relation \( A_{V}^6 = 2A_{III}^6 \) gives an integral identity between squares of elliptic functions, namely
\[ \int_0^1 (1-x)^{-\frac{1}{2}} \left\{ 2F_1\left(1\over2, 1\over2, 1; x\right) \right\}^2 = 2 \int_0^1 x^{-\frac{1}{2}} \left\{ 2F_1\left(1\over2, 1\over2, 1; x\right) \right\}^2, \] (25)

where the functions \( 2F_1 \) appear after partial integration over \( \rho_3 \) and \( \rho_4 \) in formula (13), and the elliptic function of first kind \( K(k) = \frac{\pi}{2} 2F_1^2\left(1\over2, 1\over2; 1; k^2\right) \).

Noting that
\[ A_{I}^6 = \int_0^1 \prod_{i=2}^4 d\rho_i \ (\rho_3)^{-\frac{1}{2}} (\rho_4)^{-\frac{1}{2}} (1-\rho_2)^{-\frac{1}{2}} (1-\rho_4)^{-\frac{1}{2}} \]
\[ \times (1-\rho_2\rho_3)^{-\frac{1}{2}} (1-\rho_3\rho_4)^{-\frac{1}{2}} (1-\rho_2\rho_3\rho_4)^{-\frac{1}{2}}, \] (26)

and using Mellin Barnes representations, the relation \( A_{V}^6 = 4A_{I}^6 \) implies
\[ A_{V}^6 = G_{4,4}^{2,4}\left(-1 | \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0, 0 \right) \equiv 4 \sqrt{\pi} G_{4,4}^{2,4}\left(-1 | \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0, 0 \right) = 4A_{I}^6. \] (27)
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Figure 1
The 5 topologically unequivalent open-string amplitudes $A_6(O)$
The dashed lines indicate the missing exponents $K_{ij} \equiv 0$, see text.
