Explicit Expressions for a Family of Bell Polynomials and Derivatives of Some Functions

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Abstract. In the paper, the authors first inductively establish explicit formulas for derivatives of the arc sine function, then derive from these explicit formulas explicit expressions for a family of Bell polynomials related to the square function, and finally apply these explicit expressions to find explicit formulas for derivatives of some elementary functions.

1. Introduction

Throughout this paper, we denote the set of all positive integers by \( \mathbb{N} \).

It is general knowledge that the \( n \)-th derivatives of the sine and cosine functions for \( n \in \mathbb{N} \) are

\[
\sin^{(n)} x = \sin \left( x + \frac{\pi}{2} n \right) \quad \text{and} \quad \cos^{(n)} x = \cos \left( x + \frac{\pi}{2} n \right). \tag{1.1}
\]

In [18, 19], among other things, the following explicit formulas for the \( n \)-th derivatives of the tangent and cotangent functions were inductively established:

\[
\tan^{(n)} x = \frac{1}{\cos^{n+1} x} \left\{ \frac{1}{2} \alpha_{n,\frac{1+(-1)^n}{2}} \sin \left[ \frac{1+(-1)^n}{2} x + \frac{1+(-1)^n \pi}{2} \right] \right. \\
\left. + \sum_{i=1}^{\frac{n-1-1+(-1)^n}{2}} \alpha_{n,2i+\frac{1+(-1)^n}{2}} \sin \left( 2i + \frac{1+(-1)^n}{2} x + \frac{1+(-1)^n \pi}{2} \right) \right\} \tag{1.2}
\]

and

\[
\cot^{(n)} x = \frac{1}{\sin^{n+1} x} \left\{ \frac{1}{2} \beta_{n,\frac{1+(-1)^n}{2}} \cos \left[ \frac{1+(-1)^n}{2} x \right] \right. \\
\left. + \sum_{i=1}^{\frac{n-1-1+(-1)^n}{2}} \beta_{n,2i+\frac{1+(-1)^n}{2}} \cos \left( 2i + \frac{1+(-1)^n}{2} x \right) \right\}, \tag{1.3}
\]

where

\[
\alpha_{p,q} = (-1)^{\frac{p-1}{2}} \frac{p-1}{2} \left[ 1-(-1)^{p-q} \right] \sum_{\ell=0}^{\frac{p-q-1}{2}} (-1)^{\ell} \binom{p+1}{\ell} \left( \frac{p-q-1}{2} + \ell+1 \right)^{p} \tag{1.4}
\]

2010 Mathematics Subject Classification. 11B83, 26A06, 26A09, 26A24, 26C05, 26C99, 33B90.

Key words and phrases. explicit formula; derivative; Bell polynomial; Faà di Bruno formula; elementary function.

This paper was typeset using AMS-LaTeX.
\[\beta_{p,q} = (-1)^{\frac{p+1}{2}} [1 - (-1)^pq] \sum_{\ell=0}^{p-q-1} (-1)^\ell \binom{p+1}{\ell} \left( \frac{p-q-1}{2} - \ell + 1 \right)^p \] (1.5)

for \( p > q \geq 0 \). These formulas have been applied in [18, 20, 21].

In [23, Theorem 2] and its formally published paper [27, Theorem 2.2], the following explicit formula for the \( n \)-th derivative of the exponential function \( e^{\pm 1/t} \) was inductively obtained:

\[
\left( e^{\pm 1/t} \right)^{(n)} = (-1)^n e^{\pm 1/t} \sum_{k=0}^{n-1} (\pm 1)^{n-k} L(n, n-k) t^k, \tag{1.6}
\]

where

\[
L(n, k) = \frac{(n-1) n!}{(k-1) k!} \]

are called Lah numbers in combinatorics. By the way, Lah number \( L(n, k) \) were discovered by Ivo Lah in 1955 and it counts the number of ways a set of \( n \) elements can be partitioned into \( k \) nonempty linearly ordered subsets. The formula (1.6) was also recovered in [3] and have been applied in [8, 10, 14, 15, 16, 22, 24, 25] respectively.

In combinatorics, Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) for \( n \geq k \geq 1 \), are defined by

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n} \left( x_i^\ell_i \right). \tag{1.8}
\]

See [2, p. 134, Theorem A]. The famous Faà di Bruno formula may be described in terms of Bell polynomials of the second kind \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) by

\[
\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)). \tag{1.9}
\]

See [2, p. 139, Theorem C]. This is an effective tool to compute the \( n \)-th derivatives of some composite functions. However, generally it is not an easy matter to explicitly find Bell polynomials \( B_{n,k} \).

In this paper, motivated by inductive deductions and extensive applications of the formulas (1.2), (1.3), and (1.6), we first inductively establish explicit formulas for the \( n \)-th derivatives of the functions \( \arcsin x \) and \( \arccos x \), then derive from these explicit formulas explicit expressions of Bell polynomials \( B_{n,k}(x, 1, 0, \ldots, 0) \), and finally apply these explicit expressions to compute the \( n \)-th derivatives of some elementary functions involving the square function \( x^2 \).

2. Explicit formulas for derivatives of \( \arcsin x \) and \( \arccos x \)

In this section, we will inductively establish explicit formulas for the \( n \)-th derivatives of the functions \( \arcsin x \) and \( \arccos x \). Essentially, we will find explicit formulas for the \( n \)-th derivatives of the function \( \frac{1}{\sqrt{1-x^2}} \).
Theorem 2.1. For \( k \in \mathbb{N} \) and \( x \in (-1, 1) \), the \( n \)-th derivatives of the functions \( \arcsin x \) and \( \arccos x \) may be computed by

\[
\arcsin^{(2k-1)} x = - \arccos^{(2k-1)} x = \sum_{i=0}^{k-1} a_{2k-1,2i} \frac{x^{2i}}{(1-x^2)^{k+i-1/2}} \tag{2.1}
\]
and

\[
\arcsin^{(2k)} x = - \arccos^{(2k)} x = \sum_{i=0}^{k-1} a_{2k,2i+1} \frac{x^{2i+1}}{(1-x^2)^{k+i+1/2}}, \tag{2.2}
\]

where

\[
a_{2k-1,0} = [(2k-3)!!]^2, \tag{2.3}
\]
\[
a_{2k,1} = [(2k-1)!!]^2, \tag{2.4}
\]
\[
a_{k+1,k} = (2k-1)!!, \tag{2.5}
\]
and

\[
a_{m,k} = \frac{(m+k-2)!(m-1)!}{2^{m-k-2}k!} \tag{2.6}
\]

for \( m \geq k + 2 \geq 3 \).

Proof. It is easy to obtain that

\[
(\arcsin x)' = \frac{1}{(1-x^2)^{1/2}} \quad \text{and} \quad (\arcsin x)'' = \frac{x}{(1-x^2)^{3/2}}.
\]

This means the special case \( k = 1 \) in (2.3) and (2.5). Therefore, the formulas (2.1) and (2.2) are valid for \( k = 1 \).

Assume that the formulas (2.1) and (2.2) are valid for \( k > 1 \). By this inductive hypothesis and a direct differentiation, we have

\[
\arcsin^{(2k)} x = \left[ \arcsin^{(2k-1)} x \right]' = \left[ \sum_{i=0}^{k-1} a_{2k-1,2i} \frac{x^{2i}}{(1-x^2)^{k+i-1/2}} \right]'
\]

\[
= \sum_{i=0}^{k-1} a_{2k-1,2i} \left( \frac{x^{2i}}{(1-x^2)^{k+i-1/2}} \right)'(1-x^2)^{k+i-1/2} - x^{2i} \left( (1-x^2)^{k+i-1/2} \right)'(1-x^2)^{2(k+i-1/2)}
\]

\[
= \sum_{i=0}^{k-1} a_{2k-1,2i} \left[ \frac{2ix^{2i-1}(1-x^2)^{k+i-1/2} + 2(k+i-1/2)x^{2i+1}(1-x^2)^{k+i-3/2}}{(1-x^2)^{2(k+i-1/2)}} \right]
\]

\[
= \sum_{i=0}^{k-1} \left[ a_{2k-1,2i} \frac{2ix^{2i-1}}{(1-x^2)^{k+i-1/2}} + a_{2k-1,2i} \frac{2(k+i-1/2)x^{2i+1}}{(1-x^2)^{k+i+1/2}} \right]
\]

\[
= \sum_{i=0}^{k-2} a_{2k-1,2(i+1)} \frac{2(i+1)x^{2i+1}}{(1-x^2)^{k+i+1/2}} + \sum_{i=0}^{k-1} a_{2k-1,2i} \frac{2(k+i-1/2)x^{2i+1}}{(1-x^2)^{k+i+1/2}}
\]

\[
= \sum_{i=0}^{k-2} \left[ 2(i+1)a_{2k-1,2(i+1)} + \frac{k+i-1}{2} a_{2k-1,2i} \right] \frac{x^{2i+1}}{(1-x^2)^{k+i+1/2}}
\]

\[
+ 2 \left( \frac{3k}{2} \right) a_{2k-1,2k-2} \frac{x^{2k-1}}{(1-x^2)^{2k-1/2}}
\]

and
and it is not difficult to inductively conclude (2.4). 

\[
\arcsin^{(2k+1)} x = \left[\arcsin^{(2k)} x\right]' = \sum_{i=0}^{k-1} a_{2k,2i+1} \frac{x^{2i+1}}{(1-x^2)^{k+i+1/2}}
\]

\[
= \sum_{i=0}^{k-1} a_{2k,2i+1} \left(\frac{(x^{2i+1})' (1-x^2)^{k+i+1/2} - x^{2i+1} [(1-x^2)^{k+i+1/2}]}{(1-x^2)^{2(k+i+1/2)}}\right)
\]

\[
= \sum_{i=0}^{k-1} a_{2k,2i+1} \left[\frac{(2i+1)x^{2i} (1-x^2)^{k+i+1/2} + 2(k+i+1/2)x^{2i+2} (1-x^2)^{k+i-1/2}}{(1-x^2)^{2(k+i+1/2)}}\right]
\]

\[
= \sum_{i=0}^{k-1} a_{2k,2i+1} \frac{(2i+1)x^{2i}}{(1-x^2)^{k+i+1/2}} + \sum_{i=1}^{k} a_{2k,2i-1} \frac{2(k+i-1/2)x^{2i}}{(1-x^2)^{k+i+1/2}}
\]

\[
= \sum_{i=0}^{k-1} \left[(2i+1)a_{2k,2i+1} + 2(k+i-1/2)a_{2k,2i-1}\right] \frac{x^{2i}}{(1-x^2)^{k+i+1/2}}
\]

\[
+ a_{2k,1} \frac{1}{(1-x^2)^{k+1/2}} + 2 \left(2k - \frac{1}{2}\right) a_{2k,2k-1} \frac{x^{2k}}{(1-x^2)^{2k+1/2}}.
\]

Comparing the above two formulas with

\[
\sum_{i=0}^{k-1} a_{2k,2i+1} \frac{x^{2i+1}}{(1-x^2)^{k+i+1/2}} \quad \text{and} \quad \sum_{i=0}^{k} a_{2k+1,2i} \frac{x^{2i}}{(1-x^2)^{k+i+1/2}}
\]

respectively yields the recursion formulas

\[
a_{2k,2k-1} = (4k-3)a_{2k-1,2k-2}, \quad \text{(2.7)}
\]

\[
a_{2k,2i+1} = 2(i+1)a_{2k-1,2i+1} + (2k+2i-1)a_{2k-1,2i}, \quad \text{(2.8)}
\]

for \(0 \leq i < k - 1\), and

\[
a_{2k+1,0} = a_{2k,1}, \quad \text{(2.9)}
\]

\[
a_{2k+1,2k} = (4k-1)a_{2k,2k-1}, \quad \text{(2.10)}
\]

\[
a_{2k+1,2i} = (2i+1)a_{2k,2i+1} + (2k+2i-1)a_{2k,2i-1} \quad \text{(2.11)}
\]

for \(1 \leq i \leq k - 1\).

From (2.3), (2.7), and (2.10), it is easy to derive that

\[
a_{2k+1,2k} = (4k-1)!! \quad \text{and} \quad a_{2k,2k-1} = (4k-3)!!,
\]

which may be unified into (2.5) for \(k \geq 2\).

From

\[
a_{3,0} = a_{2,1} = 1 = (1!!)^2, \quad a_{5,0} = a_{4,1} = 9 = (3!!)^2,
\]

\[
a_{7,0} = a_{6,1} = 225 = (5!!)^2, \quad a_{9,0} = a_{8,1} = 11025 = (7!!)^2,
\]

it is not difficult to inductively conclude (2.4).

Letting \(i = k - 2\) and \(i = k - 1\) for \(k \geq 2\) in (2.8) and (2.11) respectively yields

\[
a_{2k,2k-3} = 2(k-1)a_{2k-1,2k-2} + (4k-5)a_{2k-1,2k-4}
\]

and

\[
a_{2k+1,2k-2} = (2k-1)a_{2k,2k-1} + (4k-3)a_{2k,2k-3}.
\]
Combining these two recurrence formulas with (2.4) and (2.5) and recurring give
\[ a_{k+3,k} = (2k + 1)!! \sum_{\ell=1}^{k+1} \ell = (2k + 1)!! \frac{(k + 1)(k + 2)}{2}, \quad k \geq 0. \]  
(2.12)

Taking \( i = k - 3 \) and \( i = k - 2 \) for \( k \geq 3 \) in (2.8) and (2.11) respectively yields
\[ a_{2k,2k-5} = 2(k-2)a_{2k-1,2k-4} + (4k-7)a_{2k-1,2k-6} \]
and
\[ a_{2k+1,2k-4} = (2k - 3)a_{2k,2k-3} + (4k - 5)a_{2k,2k-5}. \]
Combining these two recurrence formulas with (2.4) and (2.5) and recurring give
\[ a_{k+5,k} = (2k + 3)!! \sum_{\ell=1}^{k+1} \ell(\ell + 1)(\ell + 2) \frac{2}{2} = (2k + 3)!! \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{8} \]
(2.13)
for \( k \geq 0 \).

Similarly as above, by induction, we obtain
\[ a_{k+\ell,k} = \frac{(2k + \ell - 2)!!}{2^{\ell-2}} \prod_{i=1}^{\ell-1} (k+i) = \frac{(2k + \ell - 2)!!}{2^{\ell-2}} \frac{(k + \ell - 1)!}{k!}, \quad \ell \geq 2. \]  
(2.14)

Letting \( \ell = m - k \) in (2.14) leads to (2.6). The proof of Theorem 2.1 is complete. \( \square \)

The formulas (2.1) and (2.2) may be straightforwardly unified as the following corollary.

**Corollary 2.1.** For \( n \in \mathbb{N} \) and \( x \in (-1,1) \), the \( n \)-th derivatives of the functions \( \arcsin x \) and \( \text{arccos} x \) may be computed by
\[ \arcsin^{(n)}(x) = -\text{arccos}^{(n)}(x) \]
\[ = \frac{1}{2^{n+1}} \frac{(-1)^n}{n!} \sum_{i=0}^{n-1} a_{n,2i+\frac{1}{2}(n-1)i} \frac{x^{2i+\frac{1}{2}(n-1)i}}{(1-x^2)^{i+\frac{1}{2}n+\frac{1}{2}}}, \]  
(2.15)
where \( a_{n,2i+\frac{1}{2}(n-1)i} \) are defined by (2.3), (2.4), (2.5), and (2.6).

From the formula (2.15), we may derive the \( n \)-th derivative of the elementary function \( \frac{1}{\sqrt{1-x^2}} \) as follows.

**Corollary 2.2.** For \( n \in \mathbb{N} \) and \( x \in (-1,1) \), the \( n \)-th derivatives of the function \( \frac{1}{\sqrt{1-x^2}} \) may be computed by
\[ \left( \frac{1}{\sqrt{1-x^2}} \right)^{(n)} = \frac{1}{2^{n+1}} \frac{(-1)^n}{n!} \sum_{i=0}^{n-1} a_{n,2i+\frac{1}{2}(n-1)i} \frac{x^{2i+\frac{1}{2}(n-1)i}}{(1-x^2)^{i+\frac{1}{2}n+\frac{1}{2}}}, \]  
(2.16)
where \( a_{n,2i+\frac{1}{2}(n-1)i} \) are defined by (2.3), (2.4), (2.5), and (2.6).
3. Explicit expressions of Bell polynomials

In this section, by virtue of Corollary 2.2, we will derive explicit expressions of Bell polynomials $B_{n,k}(x, 1, 0, \ldots, 0)$.

**Theorem 3.1.** For $n \in \mathbb{N}$, Bell polynomials $B_{n,k}$ satisfy

\[
B_{2n-1,n-1}(x, 1, 0, \ldots, 0) = 0, \quad (3.1)
\]

\[
B_{2n,n}(x, 1, 0, \ldots, 0) = (2n - 1)!!, \quad (3.2)
\]

\[
B_{n,k}(x, 1, 0, \ldots, 0) = 0, \quad 1 \leq k < \frac{1}{2}\left[n - \frac{1 - (-1)^n}{2}\right], \quad (3.3)
\]

\[
B_{n,k}(x, 1, 0, \ldots, 0) = \frac{a_{n+1,2k-n}}{(2k-1)!!} x^{2k-n}, \quad n \geq k > \frac{1}{2}\left[n - \frac{1 - (-1)^n}{2}\right], \quad (3.4)
\]

where $a_{n,k}$ are defined by (2.3), (2.4), (2.5), and (2.6).

**Proof.** Let $v = v(x) = 1 - x^2$. Then, by Faà di Bruno formula (1.9), we have

\[
\frac{d^n}{dx^n}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{v}}\right)^{(k)} \sum_{\ell=0}^{k-1} \left(\frac{1}{2 + \ell}\right) \frac{1}{v^{k+1/2}} B_{n,k}(-2x, 2, 0, \ldots, 0).
\]

By the formula

\[
B_{n,k}(a x_1, a b x_2, \ldots, a b^{n-k+1} x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})
\]

in [2, p. 135], we have

\[
B_{n,k}(-2x, -2, 0, \ldots, 0) = (-2)^k B_{n,k}(x, 1, 0, \ldots, 0).
\]

Therefore, we have

\[
\frac{d^n}{dx^n}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sum_{k=1}^{n} \prod_{\ell=0}^{k-1} (2\ell + 1) \frac{1}{v^{k+1/2}} B_{n,k}(x, 1, 0, \ldots, 0)
\]

\[
= \sum_{k=1}^{n} (2k - 1)!! \frac{(2k-1)!}{(1-x^2)^{k+1/2}} B_{n,k}(x, 1, 0, \ldots, 0).
\]

Comparing this with the formula (2.16) reveals that

\[
\sum_{k=1}^{2n} \frac{(2k-1)!!}{(1-x^2)^{k+1/2}} B_{2n,k}(x, 1, 0, \ldots, 0) = \frac{1}{(1-x^2)^n} \sum_{k=0}^{n} a_{2n+1,2k} \frac{x^{2k}}{(1-x^2)^k+1/2}
\]

\[
(3.6)
\]
and

\[ \sum_{k=1}^{2n-1} \frac{(2k-1)!!}{(1-x^2)^{k+1/2}} B_{2n-1,k}(x, 1, 0, \ldots, 0) = \frac{1}{(1-x^2)^n} \sum_{k=0}^{n-1} a_{2n,2k+1} x^{2k+1} \]  

for \( n \in \mathbb{N} \). Multiplying on both sides of (3.6) and (3.7) by \((1-x^2)^{2n+1/2}\) gives

\[ \sum_{k=1}^{2n} (2k-1)!!(1-x^2)^{2n-k} B_{2n,k}(x, 1, 0, \ldots, 0) = \sum_{k=0}^{n} a_{2n+1,2k} x^{2k}(1-x^2)^{n-k} \]  

and

\[ \sum_{k=1}^{2n-1} (2k-1)!!(1-x^2)^{2n-k} B_{2n-1,k}(x, 1, 0, \ldots, 0) = \sum_{k=0}^{n-1} a_{2n,2k+1} x^{2k+1}(1-x^2)^{n-k} \]  

for \( n \in \mathbb{N} \). Equating these two equations finds that

(1) when \( n > k \), Bell polynomials \( B_{2n,k}(x, 1, 0, \ldots, 0) = 0 \);

(2) when \( n > k + 1 \), Bell polynomials \( B_{2n-1,k}(x, 1, 0, \ldots, 0) = 0 \).

These two results may be unified as the formula (3.3).

Making use of the formula (3.3), the formulas (3.8) and (3.9) are reduced to

\[ \sum_{k=n}^{2n} (2k-1)!!(1-x^2)^{2n-k} B_{2n,k}(x, 1, 0, \ldots, 0) \]

\[ = \sum_{\ell=0}^{n} (2n + 2\ell - 1)!!(1-x^2)^{n-\ell} B_{2n,n+\ell}(x, 1, 0, \ldots, 0) \]

\[ = \sum_{k=0}^{n} a_{2n+1,2k} x^{2k}(1-x^2)^{n-k} \]

and

\[ \sum_{k=n-1}^{2n-1} (2k-1)!!(1-x^2)^{2n-k} B_{2n-1,k}(x, 1, 0, \ldots, 0) \]

\[ = (2n-3)!!(1-x^2)^{n+1} B_{2n-1,n-1}(x, 1, 0, \ldots, 0) \]

\[ + \sum_{\ell=0}^{n-1} (2n + 2\ell - 1)!!(1-x^2)^{n-\ell} B_{2n-1,n+\ell}(x, 1, 0, \ldots, 0) \]

\[ = \sum_{k=0}^{n-1} a_{2n,2k+1} x^{2k+1}(1-x^2)^{n-k} \]

for \( n \in \mathbb{N} \). Equating the above equations figures out the formula (3.1),

\[ B_{2n,n+k}(x, 1, 0, \ldots, 0) = \frac{a_{2n+1,2k}}{(2n+2k-1)!!} x^{2k}, \quad 0 \leq k \leq n, \]  

(3.10)
and

\[ B_{2n-1,n+k}(x,1,0,\ldots,0) = \frac{a_{2n,2k+1}}{(2n+2k-1)!!} x^{2k+1}, \quad 0 \leq k \leq n-1. \quad (3.11) \]

The formulas (3.10) and (3.11) may be reformulated as (3.2) and (3.4). The proof of Theorem 3.1 is complete. \(\square\)

4. Explicit formulas for the \(n\)-th derivatives of some functions

In this section, with the help of Theorem 3.1, we will discover explicit formulas for the \(n\)-th derivatives of some elementary functions.

**Theorem 4.1.** For \(\ell \in \mathbb{N}\), we have

\[
(\arctan x)^{(2\ell)} = \frac{(-1)^\ell x}{(1+x^2)^{\ell+1}} \sum_{k=0}^{\ell-1} (-1)^k \frac{[2(k+\ell)]!!}{[2(k+\ell) - 1]!!} a_{2\ell,2k+1} \left( \frac{x^2}{1+x^2} \right)^k \]

and

\[
(\arctan x)^{(2\ell-1)} = \frac{(-1)^{\ell-1}}{(1+x^2)^{\ell}} \sum_{k=0}^{\ell-1} (-1)^k \frac{[2(k+\ell-1)]!!}{[2(k+\ell-1) - 1]!!} a_{2\ell-1,2k} \left( \frac{x^2}{1+x^2} \right)^k,
\]

where \(a_{n,k}\) are defined by (2.3), (2.4), (2.5), and (2.6).

**Proof.** Let \(v = v(x) = 1 + x^2\). Then, by Faá di Bruno formula (1.9) and the formula (3.5), we obtain

\[
(\arctan x)^{(n)} = \left( \frac{1}{1+x^2} \right)^{(n-1)} \sum_{k=1}^{n-1} \left( \frac{1}{v} \right)^{(k)} B_{n-1,k}(2x,2,0,\ldots,0) \]

\[
= \sum_{k=1}^{n-1} (-1)^k \frac{k!}{v^{k+1}} 2^k B_{n-1,k}(x,1,0,\ldots,0) \]

\[
= \sum_{k=1}^{n-1} (-1)^k \frac{(2k)!!}{(1+x^2)^{k+1}} B_{n-1,k}(x,1,0,\ldots,0). \]

Hence, by Theorem 3.1, it follows that (1) when \(n = 2\ell\), we have

\[
(\arctan x)^{(2\ell)} = \left( \frac{1}{1+x^2} \right)^{(2\ell-1)} \sum_{k=1}^{2\ell-1} (-1)^k \frac{(2k)!!}{(1+x^2)^{k+1}} B_{2\ell-1,k}(x,1,0,\ldots,0) \]

\[
= \sum_{k=\ell}^{2\ell-1} (-1)^k \frac{(2k)!! a_{2\ell,2(k-\ell)+1}}{(2k-1)!!} \frac{x^{2k-2\ell+1}}{(1+x^2)^{k+1}};
\]
(2) when $n = 2\ell - 1$, we have

$$\frac{d^{2\ell - 1} e^{\pm x^2}}{dx^{2\ell - 1}} = e^{\pm x^2} \sum_{k=\ell}^{2\ell - 1} \frac{(\pm 2)^k}{(2k-1)!!} a_{2\ell - 1, 2k} x^{2k}.$$  

The proof of Theorem 4.2 is complete. □

Remark 4.1. After this paper was completed on 20 March 2014, the authors searched out on 27 March 2014 the papers [1, 9] in which several formulas for the $n$-th derivatives of the inverse tangent function were established and discussed.

Theorem 4.2. For $\ell \in \mathbb{N}$, we have

$$\frac{d^{2\ell} e^{\pm x^2}}{dx^{2\ell}} = (\pm 2)^{\ell} e^{\pm x^2} \sum_{k=0}^{\ell} \frac{(\pm 2)^k}{[2(k+\ell) - 1]!!} a_{2\ell + 1, 2k} x^{2k},$$  

(4.3)

and

$$\frac{d^{2\ell - 1} e^{\pm x^2}}{dx^{2\ell - 1}} = (\pm 2)^{\ell} x e^{\pm x^2} \sum_{k=\ell}^{n-1} \frac{(\pm 2)^k}{[2(k+\ell) - 1]!!} a_{2\ell + 1, k + 1} x^{2k},$$  

(4.4)

where $a_{n,k}$ are defined by (2.3), (2.4), (2.5), and (2.6).

Proof. Let $u = u(x) = x^2$. Then, by Faà di Bruno formula (1.9) and the formula (3.5), we acquire

$$\frac{d^n e^{\pm x^2}}{dx^n} = \sum_{k=1}^{n} \frac{d^k e^{\pm u}}{du^k} B_{n,k}(2x, 2, 0, \ldots, 0) = e^{\pm x^2} \sum_{k=1}^{n} (\pm 2)^k B_{n,k}(x, 1, 0, \ldots, 0).$$

Hence, by Theorem 3.1, it follows that

(1) when $n = 2\ell$, we have

$$\frac{d^{2\ell} e^{\pm x^2}}{dx^{2\ell}} = e^{\pm x^2} \sum_{k=1}^{2\ell} \frac{(-1)^{2\ell-k}}{(2k-1)!!} a_{2\ell, k} x^{2k}.$$  

(2) when $n = 2\ell - 1$, we have

$$\frac{d^{2\ell - 1} e^{\pm x^2}}{dx^{2\ell - 1}} = e^{\pm x^2} \sum_{k=\ell}^{2\ell - 1} \frac{(-1)^{2\ell-k}}{(2k-1)!!} a_{2\ell - 1, k} x^{2k}.$$  

The proof of Theorem 4.2 is complete. □
Theorem 4.3. For $\ell \in \mathbb{N}$, we have

\[
\frac{d^{2\ell} \sin(x^2)}{dx^{2\ell}} = 2^\ell \sum_{k=0}^{\ell} \frac{2^k a_{2\ell+1,2k}}{[2(k+\ell) - 1]!!} x^{2k} \sin\left(x^2 + \frac{\pi}{2}(k+\ell)\right),
\]

(4.5)

\[
\frac{d^{2\ell-1} \sin(x^2)}{dx^{2\ell-1}} = 2^\ell \sum_{k=0}^{\ell-1} \frac{2^k a_{2\ell,2k+1}}{[2(k+\ell) - 1]!!} x^{2k+1} \sin\left(x^2 + \frac{\pi}{2}(k+\ell)\right),
\]

(4.6)

\[
\frac{d^{2\ell} \cos(x^2)}{dx^{2\ell}} = 2^\ell \sum_{k=0}^{\ell} \frac{2^k a_{2\ell+1,2k}}{[2(k+\ell) - 1]!!} x^{2k} \cos\left(x^2 + \frac{\pi}{2}(k+\ell)\right),
\]

(4.7)

\[
\frac{d^{2\ell-1} \cos(x^2)}{dx^{2\ell-1}} = 2^\ell \sum_{k=0}^{\ell-1} \frac{2^k a_{2\ell,2k+1}}{[2(k+\ell) - 1]!!} x^{2k+1} \cos\left(x^2 + \frac{\pi}{2}(k+\ell)\right),
\]

(4.8)

where $a_{n,k}$ are defined by (2.3), (2.4), (2.5), and (2.6).

Proof. Let $u = u(x) = x^2$. Then, by Faà di Bruno formula (1.9) and the formulas (3.5) and (1.1), we gain

\[
\frac{d^n \sin(x^2)}{dx^n} = \sum_{k=1}^{n} \frac{d^k \sin u}{du^k} \underbrace{B_{n,k}(2x, 2, 0, \ldots, 0)}_{n-k-1} = \sum_{k=1}^{n-k-1} \sin\left(x^2 + \frac{\pi}{2}k\right) 2^k B_{n,k}(x, 1, 0, \ldots, 0).
\]

Accordingly, by Theorem 3.1, it follows that

(1) when $n = 2\ell$, we have

\[
\frac{d^{2\ell} \sin(x^2)}{dx^{2\ell}} = \sum_{k=1}^{2\ell} \sin\left(x^2 + \frac{\pi}{2}k\right) 2^k B_{2\ell,k}(x, 1, 0, \ldots, 0) = \sum_{k=1}^{2\ell} 2^k \sin\left(x^2 + \frac{\pi}{2}k\right) a_{2\ell+1,2k} (2k-1)!! 2^{2(k-1)};
\]

(2) when $n = 2\ell - 1$, we have

\[
\frac{d^{2\ell-1} \sin(x^2)}{dx^{2\ell-1}} = \sum_{k=1}^{2\ell-1} \sin\left(x^2 + \frac{\pi}{2}k\right) 2^k B_{2\ell-1,k}(x, 1, 0, \ldots, 0) = \sum_{k=1}^{2\ell-1} 2^k \sin\left(x^2 + \frac{\pi}{2}k\right) a_{2\ell,2k+1} (2k-1)!! 2^{2(k-1)}+1.
\]

By the formulas in (1.1), if replacing the sine by the cosine in the above arguments, all results are also valid. The proof of Theorem 4.3 is complete. \qed

Theorem 4.4. For $\ell \in \mathbb{N}$, we have

\[
\left(\ln \frac{1 + x}{1 - x}\right)^{(2\ell)} = \frac{2x}{(1 + x^2)^{\ell+1}} \sum_{k=0}^{\ell-1} \frac{[2(k+\ell)]!! a_{2\ell,2k+1}}{[2(k+\ell) - 1]!!} \left(\frac{x^2}{1 - x^2}\right)^k
\]

(4.9)
and
\[
\left( \ln \frac{1+x}{1-x} \right)^{(2\ell-1)} = \frac{2}{(1+x^2)^\ell} \sum_{k=0}^{2\ell-2} \frac{[2(k+\ell-1)]!!}{[2(k+\ell-1)-1]!!} a_{2\ell-1,2k} \left( \frac{x^2}{1-x^2} \right)^k ,
\]
where \( a_{n,k} \) are defined by (2.3), (2.4), (2.5), and (2.6).

Proof. Let \( u = u(x) = x^2 \). Then, by Faà di Bruno formula (1.9) and the formula (3.5), we obtain
\[
\left( \ln \frac{1+x}{1-x} \right)^{(n)} (\ln 1 + x^{1-x} - x^{1-x})^{(2\ell-1)} = 2 \left( \frac{1}{1-x^2} \right)^{(n-1)}
\]

Hence, by Theorem 3.1, it follows that
(1) when \( n = 2\ell \), we have
\[
\left( \ln \frac{1+x}{1-x} \right)^{(2\ell)} = 2 \left( \frac{1}{1-x^2} \right)^{(2\ell-1)}
\]

(2) when \( n = 2\ell - 1 \), we have
\[
\left( \ln \frac{1+x}{1-x} \right)^{(2\ell-1)} = 2 \left( \frac{1}{1-x^2} \right)^{(2\ell-2)}
\]

The proof of Theorem 4.4 is complete.

Remark 4.2. Since
\[
\left( \ln \frac{1+x}{1-x} \right)^' = \frac{2}{1-x^2} = \frac{1}{x+1} - \frac{1}{x-1},
\]
the $n$-th derivative of $\ln \frac{1 + x}{1 - x}$ may also be computed by
\[
\left( \ln \frac{1 + x}{1 - x} \right)^{(n)} = (-1)^{n-1}(n-1)! \left[ \frac{1}{(x+1)^n} - \frac{1}{(x-1)^n} \right], \quad n \in \mathbb{N}. \quad (4.11)
\]

Similarly,
\[
\frac{d^n \ln(1 - x^2)}{dx^n} = (-1)^{n-1}(n-1)! \left[ \frac{1}{(x+1)^n} + \frac{1}{(x-1)^n} \right], \quad n \in \mathbb{N}. \quad (4.12)
\]

**Theorem 4.5.** For $\ell \in \mathbb{N}$, we have
\[
\frac{d^{2\ell} \ln(1 + x^2)}{dx^{2\ell}} = \frac{(-1)^{\ell-1}}{(1 + x^2)^{\ell}} \sum_{k=0}^{\ell} (-1)^k \frac{2(k + \ell - 1)!!}{2(k + \ell) - 1}! a_{2\ell+1,2k}(n,1,0,\ldots,0),
\]
and
\[
\frac{d^{2\ell-1} \ln(1 + x^2)}{dx^{2\ell-1}} = \frac{(-1)^{\ell-2}}{(1 + x^2)^{\ell}} \sum_{k=0}^{\ell-1} (-1)^k \frac{2(k + \ell - 1)!!}{2(k + \ell) - 1}! a_{2\ell,2k+1}(n,1,0,\ldots,0),
\]
where $a_{n,k}$ are defined by (2.3), (2.4), (2.5), and (2.6).

**Proof.** Let $u = u(x) = x^2$. Using Faa di Bruno formula (1.9) and the formula (3.5) yields
\[
\frac{d^n \ln(1 + x^2)}{dx^n} = \sum_{k=1}^{n} \ln(1 + u)^{(k)} \frac{d^k}{dx^k} B_{n,k}(2x,2,0,\ldots,0) = \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{(1+u)^k} 2^k B_{n,k}(x,1,0,\ldots,0) = 2^{n-k-1} \sum_{k=1}^{n} (-1)^{k-1} \frac{(2k-2)!!}{(1+x^2)^k} B_{n,k}(x,1,0,\ldots,0).
\]

Consequently, by Theorem 3.1, it follows that

1. when $n = 2\ell$, we have
\[
\frac{d^{2\ell} \ln(1 + x^2)}{dx^{2\ell}} = 2^{2\ell} \sum_{k=1}^{\ell} (-1)^{k-1} \frac{(2k-2)!!}{(1+x^2)^k} B_{2\ell,k}(x,1,0,\ldots,0) = 2^{2\ell} \sum_{k=1}^{\ell} (-1)^{k-1} \frac{(2k-2)!!}{(1+x^2)^k} a_{2\ell+1,2(k-\ell)} x^{2(k-\ell)};
\]

2. when $n = 2\ell - 1$, we have
\[
\frac{d^{2\ell-1} \ln(1 + x^2)}{dx^{2\ell-1}} = 2^{2\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} \frac{(2k-2)!!}{(1+x^2)^k} B_{2\ell-1,k}(x,1,0,\ldots,0) = 2^{2\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} \frac{(2k-2)!!}{(1+x^2)^k} a_{2\ell,2(k-\ell)+1} x^{2(k-\ell)+1}.
\]

The proof of Theorem 4.5 is complete. \qed
Theorem 4.6. Let \( \alpha \notin \{0\} \cup \mathbb{N} \). For \( \ell \in \mathbb{N} \), we have
\[
\frac{d^{2\ell}}{dx^{2\ell}}[(1 \pm x^2)^\alpha] = \frac{(\pm 2)^\ell}{(1 \pm x^2)^{\ell - \alpha}} \sum_{k=0}^{\ell} \frac{\sum_{m=1}^{k+\ell} (\alpha - m + 1)}{2(k+\ell) - 1!!} a_{2\ell+1,2k} \left( \frac{x^2}{1 \pm x^2} \right)^k
\]  
and
\[
\frac{d^{2\ell-1}}{dx^{2\ell-1}}[(1 \pm x^2)^\alpha] = \frac{(\pm 2)^\ell x}{(1 \pm x^2)^{\ell - \alpha}} \sum_{k=0}^{\ell-1} \frac{\sum_{m=1}^{k+\ell} (\alpha - m + 1)}{2(k+\ell) - 1!!} a_{2\ell,2k+1} \left( \frac{x^2}{1 \pm x^2} \right)^k,
\]
where \( a_{n,k} \) are defined by (2.3), (2.4), (2.5), and (2.6).

Proof. Let \( u = u(x) = \pm x^2 \). Using Faá di Bruno formula (1.9) and the formula (3.5) brings out
\[
\frac{d^n}{dx^n}[(1 \pm x^2)^\alpha] = \sum_{k=1}^{n} (1 + u)^{\alpha} \cdot B_{n,k}(\pm 2x, \pm 2, 0, \ldots, 0)
\]
\[
= \sum_{k=1}^{n} \prod_{m=1}^{k} (\alpha - m + 1) (1 + u)^{\alpha - k} (-2)^k B_{n,k}(x, 1, 0, \ldots, 0)
\]
\[
= \sum_{k=1}^{n} \left[ (\pm 2)^k \prod_{m=1}^{k} (\alpha - m + 1) \right] (1 \pm x^2)^{\alpha - k} B_{n,k}(x, 1, 0, \ldots, 0).
\]
As a result, by Theorem 3.1, it follows that
1. when \( n = 2\ell \), we have
\[
\frac{d^{2\ell}}{dx^{2\ell}}[(1 \pm x^2)^\alpha] = \sum_{k=1}^{2\ell} \left[ (\pm 2)^k \prod_{m=1}^{k} (\alpha - m + 1) \right] (1 \pm x^2)^{\alpha - k} B_{2\ell,k}(x, 1, 0, \ldots, 0)
\]
\[
= \sum_{k=\ell}^{2\ell} \left[ (\pm 2)^k \prod_{m=1}^{k} (\alpha - m + 1) \right] (1 \pm x^2)^{\alpha - k} a_{2\ell+1,2k} x^{2(k-\ell)};
\]
2. when \( n = 2\ell - 1 \), we have
\[
\frac{d^{2\ell-1}}{dx^{2\ell-1}}[(1 \pm x^2)^\alpha] = \sum_{k=1}^{2\ell-1} \left[ (\pm 2)^k \prod_{m=1}^{k} (\alpha - m + 1) \right] (1 \pm x^2)^{\alpha - k} B_{2\ell-1,k}(x, 1, 0, \ldots, 0)
\]
\[
= \sum_{k=\ell}^{2\ell-1} \left[ (\pm 2)^k \prod_{m=1}^{k} (\alpha - m + 1) \right] (1 \pm x^2)^{\alpha - k} a_{2\ell,2k+1} x^{2(k-\ell)+1}.
\]
The proof of Theorem 4.6 is complete. \(\Box\)

Remark 4.3. In general, the \( n \)-th derivatives of the function \( h(x) = f(x^2) \) may be expressed as
\[
h^{(2\ell)}(x) = \sum_{k=0}^{\ell} \frac{1}{2(k+\ell) - 1!!} a_{2\ell+1,2k} x^{2k} f^{(k+\ell)}(x^2)
\]  
(4.17)
and
\[ h^{(2\ell-1)}(x) = \sum_{k=0}^{\ell-1} \frac{1}{[2(k+\ell)-1]!!} a_{2\ell,2k+1} x^{2k+1} f^{(k+\ell)}(x^2), \]
where \( \ell \in \mathbb{N} \) and \( a_{n,k} \) are defined by (2.3), (2.4), (2.5), and (2.6).

5. Miscellanea

By Faà di Bruno formula (1.9), we may establish

\[ -(\tan x)^{(n-1)} = (\ln \cos x)^{(n)} \]
\[ = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\cos^k x} B_{n,k} \left( \cos \left( x + \frac{\pi}{2} \right), \ldots, \cos \left( x + (n-k+1)\frac{\pi}{2} \right) \right) \]

and

\[ (\cot x)^{(n-1)} = (\ln \sin x)^{(n)} \]
\[ = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\sin^k x} B_{n,k} \left( \sin \left( x + \frac{\pi}{2} \right), \ldots, \sin \left( x + (n-k+1)\frac{\pi}{2} \right) \right). \]

It is possible that, by comparing and equating these derivatives with the formulas (1.2) and (1.3), we may discover explicit expressions for Bell polynomials

\[ B_{n,k} \left( \cos \left( x + \frac{\pi}{2} \right), \cos \left( x + 2\frac{\pi}{2} \right), \ldots, \cos \left( x + (n-k+1)\frac{\pi}{2} \right) \right) \]
and
\[ B_{n,k} \left( \sin \left( x + \frac{\pi}{2} \right), \sin \left( x + 2\frac{\pi}{2} \right), \ldots, \sin \left( x + (n-k+1)\frac{\pi}{2} \right) \right). \]

These results may be applied to procure explicit formulas for the \( n \)-th derivatives of the functions \( e^{\pm \sin x} \) and \( e^{\pm \cos x} \).

Utilizing Faà di Bruno formula (1.9) and the formulas (1.1) and (3.5), we obtain

\[ [\sin(e^{\pm x})]^{(n)} = \sum_{k=1}^{n} \sin^{(k)}(e^{\pm x}) B_{n,k}( (\pm 1)e^{\pm x}, (\pm 1)^2 e^{\pm x}, \ldots, (\pm 1)^{n-k+1} e^{\pm x} ) \]
\[ = (\pm 1)^n \sum_{k=1}^{n} \sin \left( e^{\pm x} + \frac{\pi}{2} k \right) e^{\pm k x} B_{n,k}(1,\ldots,1) \]
\[ = (\pm 1)^n \sum_{k=1}^{n} S(n,k) \sin \left( e^{\pm x} + \frac{\pi}{2} k \right) e^{\pm k x} \]

and

\[ [\cos(e^{\pm x})]^{(n)} = (\pm 1)^n \sum_{k=1}^{n} S(n,k) \cos \left( e^{\pm x} + \frac{\pi}{2} k \right) e^{\pm k x}, \]

where

\[ B_{n,k}(1,\ldots,1) = S(n,k) \]
may be found in [2, p. 135] and

\[ S(n,k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n \]
is called Stirling number of the second kind which may be combinatorially interpreted as the number of partitions of the set \{1, 2, \ldots, n\} into \(k\) non-empty disjoint sets. For more information on Stirling numbers of the second kind \(S(n, k)\), please refer to [2, 4, 5, 6, 7, 11, 12, 13, 14, 17] and closely related references therein.

**Remark 5.1.** This paper is a slightly revised version of the preprint [26].

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