ALGEBRAIC COBORDISM OF CLASSIFYING SPACES

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Abstract. We define algebraic cobordism of classifying spaces, \( \Omega^*(BG) \) and \( G \)-equivariant algebraic cobordism \( \Omega^*_G(-) \) for a linear algebraic group \( G \). We prove some properties of the coniveau filtration on algebraic cobordism, denoted \( F^j(\Omega^*(-)) \), which are required for the definition to work. We show that \( G \)-equivariant cobordism satisfies the localization exact sequence. We calculate \( \Omega^*(BG) \) for algebraic groups over the complex numbers corresponding to classical Lie groups \( GL(n) \), \( SL(n) \), \( Sp(n) \), \( O(n) \) and \( SO(2n+1) \). We also calculate \( \Omega^*(BG) \) when \( G \) is a finite abelian group. A finite non-abelian group for which we calculate \( \Omega^*(BG) \) is the quaternion group of order 8. In all the above cases we check that \( \Omega^*(BG) \) is isomorphic to \( MU^*(BG) \).

1. Introduction

Complex cobordism \( MU^*(-) \) is a cohomology theory represented by the Thom spectrum \( MU \) in the classical stable homotopy category. Voevodsky has defined a version of algebraic cobordism, \( MGL^*(-) \) for arbitrary schemes which can be found in [17]. We call \( MGL^*(-) \) as motivic cobordism. \( MGL^*(-) \) is a bigraded cohomology theory represented by the motivic Thom spectrum \( MGL \) in the Morel-Voevodsky motivic homotopy category. Levine and Morel have constructed algebraic cobordism \( \Omega^*(-) \) [8], based on the arguments of Quillen in [13]. For a quasi-projective schemes \( X \), the ring \( \Omega^*(X) \) is constructed as

\[
\Omega^*(X) = \{[f : Y \to X]/(relations)\}
\]

where \( f \) is a projective morphism of pure codimension from a smooth variety \( Y \) to \( X \). Some details of this construction are given in section 2. Levine has recently showed that for smooth quasi-projective schemes, the definition of algebraic cobordism \( \Omega^*(-) \) agrees with the definition of motivic cobordism \( MGL^{2*}(\cdot) \) [7].

It turns out that the algebraic cobordism of a field is isomorphic to the Lazard ring, which is isomorphic to the complex cobordism of a point ([14], 2.8). This naturally leads to the question of finding varieties whose algebraic cobordism is isomorphic to the complex cobordism of the realization of the variety over the complex field. Classifying spaces of groups is a good class of varieties to consider. As a topological space, classifying spaces are infinite dimensional for most groups. So one needs to carefully define an analogue of classifying spaces in algebraic geometry. Burt Totaro has defined and studied Chow rings of classifying spaces of linear algebraic groups in [15]. Let \( G \) be a linear algebraic group. Let \( G \) act on \( \mathbb{A}^N \), i.e. consider an \( N \)-dimensional representation of \( G \). Let \( S \) be a (Zariski) closed subset of \( \mathbb{A}^N \) of codimension \( j \) such that \( G \) acts freely on \( \mathbb{A}^N - S \). Totaro defined Chow rings of the classifying space of \( G \) by showing that the cycle class map

\[
CH^*(X) \to MU^*(X) \otimes MU^* \mathbb{Z}
\]

in [14]. In all the examples in [15], it turns out that the cycle class map \( CH^*(BG) \to MU^*(BG) \otimes MU^* \mathbb{Z} \) is an isomorphism. Algebraic cobordism relates to Chow rings in exactly similar fashion [8], i.e.

\[
CH^*(X) \cong \Omega^*(X) \otimes \mathbb{Q} \mathbb{Z}.
\]

Hence it is very natural to expect that the canonical map from algebraic cobordism to complex cobordism is an isomorphism whenever the cycle class map is an isomorphism. This paper is an attempt to check

\[
\Omega^*(BG) \cong MU^*(BG)
\]
for a few groups $G$ such as classical Lie groups, products of finite cyclic groups and the quaternion group. We define $\Omega^*(BG)$ for a linear algebraic groups $G$. Unlike Chow groups, $\Omega^i(A^n_S - S)$ are not independent of the choice of $S$. A reason being, $CH^i(X)$ is zero when $i < 0$ whereas $\Omega^i(X)$ is non-zero even when $i < 0$. Instead, we show that quotients of $\Omega^i(A^n_S - S)$ by the coniveau filtration are independent of the choices of $S$ and $A^n_S$ [Theorem 3.7]. Similar to Edidin-Graham [3], we define $G$-equivariant algebraic cobordism for a $G$-scheme $X$. We show that $\Omega^*_G(-)$ satisfies the localization exact sequence.

Let $BP^*(-)$ be the Brown-Peterson cohomology for a prime $p$. Yagita has defined $\Omega^*_BP(-)$, the algebraic Brown-Peterson theory in the section 8 of [19] and has checked $\Omega^*_BP(G) \cong BP^*(BG)$ for a few groups. We check that our result implies Yagita’s result. $MGL^*(BG(n))$ has been calculated in [5].

The structure of this chapter is as follows. Section 2 gives a brief introduction to the theory $\Omega^*(-)$. In section 3 we define algebraic cobordism of classifying spaces similar to as done in [15]. This definition uses the coniveau filtration on cobordism ring. We will also prove some properties of the coniveau filtration in this section. Computations of $\Omega^*(BG)$ for some reductive algebraic groups $G$ over complex numbers corresponding to the classical Lie groups are shown in section 4. Computations for products of the finite cyclic groups $G$ are done in section 5. Section 6 gives a way of calculating $\Omega^*(X)$ when $X$ can be partitioned into affine spaces. We define equivariant algebraic cobordism in section 7 and calculate $\Omega^*(BQ)$ for the group of quaternions $Q$ in section 8.

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2. BRIEF INTRODUCTION TO ALGEBRAIC COBORDISM

In this section, we will give two equivalent definitions of $\Omega^*(-)$ extracted from the chapter 2 of [8] and from [9] and state some of the properties of $\Omega^*(-)$.

Throughout this chapter, assume that $k$ is a field of characteristic zero and $k$ admits resolution of singularities. Let $Sch/k$ denote the category of equidimensional quasi-projective schemes of finite type over $k$. By a scheme, we always mean an element of $Sch/k$. Let $Sm/k$ denote a full subcategory of $Sch/k$ whose objects are also smooth. Let $X \in Sch/k$. As said in the introduction, 

$$\Omega_*(X) = \{ \text{bordism cycles} \}/(\text{relations}).$$

Define a bordism cycle on $X$ to be a family $[f : Y \to X, L_1, L_2, ..., L_r]$ where $Y$ is in $Sm/k$, $f$ is projective morphism and $L_i$’s are line bundles over $Y$. Define the degree of such a bordism cycle to be $\text{dim}_k(Y) - r$. Let $Z_*(X)$ be the free abelian group generated by the isomorphism classes of the bordism cycles.

We will now start imposing relations on $Z_*(X)$ so that we will have a working definition of Chern classes of vector bundles and the formal group law to be the universal formal group law. Firstly we will impose the dimensional constraint. Let $R_*^{\text{dim}}(X) \subset Z^*(X)$ be the subgroup generated by cycles of the form $[Y \to X, L_1, ..., L_r]$ where $\text{dim}_k(Y) < r$. Define,

$$\overline{Z}_*(X) = Z_*(X)/R_*^{\text{dim}}(X).$$

On $\overline{Z}_*(X)$ we will now impose the relation which will help us in defining the Chern classes. Let $R_*^{\text{sect}}(X) \subset \overline{Z}_*(X)$ be the subgroup generated by all the elements of the form $[(Y \to X, L)] - [Z \to Y \to X])$, where $L$ is a line bundle over $Y$, $s : Y \to L$ is a section transversal to the zero section and $Z \to Y$ is a closed subvariety of the zeros of $s$. Note that $Z$ is smooth as $s$ is transversal to the zero section. Define,

$$\mathcal{M}_*(X) = \overline{Z}_*(X)/R_*^{\text{sect}}(X).$$

Now we are ready to define the algebraic pre-bordism $\overline{\Omega}_*(X)$. We say the two cycles $[f : Y \to X]$ and $[g : Z \to X]$ are elementary bordant if there exist $W \in Sm/k$ and a projective morphism $h : W \to X \times P^1$ transversal to $X \times \{0\}$ and $X \times \{1\}$ such that $h : h^{-1}(X \times \{0\}) \to X$ is isomorphic to $[f : Y \to X]$ and $[h : h^{-1}(X \times \{1\}) \to X]$ is isomorphic to $[g : Z \to X]$. Let
\( \mathcal{R}_*^{cob}(X) \subset \mathcal{M}_*(X) \) be the subgroup generated by elements of the form \( ([f : Y \to X] - [g : Z \to X]) \) where \([f]\) and \([g]\) are elementary bordant to each other. Define Algebraic Pre-bordism by,

\[
\overline{\Omega}_*(X) = \mathcal{M}_*(X)/\mathcal{R}_*^{cob}(X).
\]

Given a line bundle \( L \) over \( X \), we define the Chern class homomorphism \( c_1(L) : \overline{\Omega}_*(X) \to \overline{\Omega}_{* - 1}(X) \) by \( c_1(L)([f : Y \to X]) = [f : Y \to X, f^*(L)] \). Note that \( \mathcal{R}_*^{cob}(X) \) forces \( c_1(L) \) to be locally nilpotent for all \( L \). i.e. given any cycle \([f : Y \to X]\), there exist \( n > 0 \) such that \((c_1(L))^n([f : Y \to X]) = 0\).

Now we will enforce the formal group law on \( \overline{\Omega}_*(X) \), which will give us the theory of algebraic bordism. Let

\[
F(x, y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j
\]

be the universal formal group law with coefficients in the Lazard ring \( L_\ast \). \( L_\ast \) is the quotient of the formal power series ring \( \mathbb{Z}[[a_{11}, a_{12}, \ldots]] \) by the relations enforced via properties of \( F(-,-) \) \([1], 2.6\). \( L_\ast \) is a graded ring and the degree of \( a_{ij} \) is \((i+j-1)\). Let \( \mathcal{R}^{FGL}_* \subset L_\ast \otimes \overline{\Omega}_*(X) \) be the subgroup generated by elements of the form \([F(c_1(L), c_1(M))(x)] - [c_1(L \otimes M)(x)]\); where \( L, M \) are line bundles over \( X \) and \( x \in \overline{\Omega}_*(X) \). Local nilpotency of \( c_1(-) \) ensures that \( \mathcal{R}^{FGL}_* \) is well defined. We define the algebraic bordism of \( X \) by

\[
\Omega_*(X) = L_\ast \otimes \overline{\Omega}_*(X)/< L_\ast, \mathcal{R}^{FGL}_* >
\]

The theory of algebraic bordism can also be constructed by allowing the ‘double point degeneration’ in the elementary bordism relations \([2] \). A formulation is as follows.

Let \( Y \) be in \( \text{Sch}/k \). A morphism \( \pi : Y \to \mathbb{P}^1 \) is a double point degeneration over \( 0 \in \mathbb{P}^1 \) if

\[
\pi^{-1}(0) = A \cup B
\]

where \( A \) and \( B \) are smooth Cartier divisors intersecting transversely in \( Y \). The intersection \( D = A \cap B \) is called the double point locus of \( \pi \) over \( 0 \in \mathbb{P}^1 \). Let \( N_{A/D} \) and \( N_{B/D} \) denote the normal bundles of \( D \) in \( A \) and \( B \) respectively. Since \( \mathcal{O}_D(A + B) \) is trivial, we have

\[
N_{A/D} \otimes N_{B/D} \cong \mathcal{O}_D.
\]

Hence the two projective bundles \( \mathcal{P}(\mathcal{O}_D \oplus N_{A/D}) \to D \) and \( \mathcal{P}(\mathcal{O}_D \oplus N_{B/D}) \to D \) are isomorphic. Let \( \mathcal{P}(\pi) \to D \) denote the either of the two.

Let \( p_1 \) and \( p_2 \) denote the projections to the first and the second factor of \( X \times \mathbb{P}^1 \) respectively. Let \( W \) be a smooth variety and let \( h : W \to X \times \mathbb{P}^1 \) be a projective morphism. Suppose the composition

\[
h_2 := p_2 \circ h : W \to \mathbb{P}^1
\]

is a double point degeneration over \( 0 \). Let \([A \to X], [B \to X] \) and \([\mathcal{P}(h_2) \to X] \) be the elements in \( Z_*(X) \) obtained from the fiber \( h_2^{-1}(0) \). Without loss of generality, say \( 1 \) is a regular value of \( p_2 \circ h \) and say \([p_1 \circ h : h^{-1}(X \times \{1\}) \to X] \) is isomorphic to \([f : Y \to X]\). Define an associated double point relation over \( X \) by

\[
[f : Y \to X] - [A \to X] - [B \to X] + [\mathcal{P}(h_2) \to X].
\]

Let \( \mathcal{R}_*^{dpr}(X) \) be the subgroup generated by all associated double point relations over \( X \). Define the double point bordism theory \( \omega_*(-) \) by

\[
\omega_*(X) := Z_*(X)/\mathcal{R}_*^{dpr}(X).
\]

Then there is a canonical isomorphism

\[
\omega_*(X) \cong \Omega_*(X).
\]

\( \Omega_*(-) \) satisfies all the standard properties of an algebraic homology theory such as the existence of push forwards along proper morphisms and the existence of pull backs under smooth morphism. To be precise, \( \Omega_*(X) \) is an oriented Borel-Moore functor of geometric type on the category \( \text{Sch}/k \). Axioms defining such a functor are given in \([3] \), 2.11, 2.2.9.
If $X$ is in $Sm/k$ and of pure dimension $n$, then define the algebraic cobordism of $X$ by,

$$\Omega^*(X) := \Omega_{n-*}(X).$$

Having constructed $\Omega^*(-)$, we will now state a few important properties of $\Omega^*(-)$.

2.1. Localization Sequence. Let $X$ be in $Sch/k$. For $Z$, a closed subscheme of codimension $r$ in $X$, let $i : Z \to X$ and $j : X - Z \to X$ be inclusions. Then we have an exact sequence, ([8], 3.2.7)

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j_*} \Omega_*(X - Z) \to 0$$

if both $X$ and $Z$ are smooth, we can write this sequence in the cohomological notation as

$$\Omega^i(Z) \xrightarrow{i^*} \Omega^{i+r}(X) \xrightarrow{j^*} \Omega^{i+r}(X - Z) \to 0$$

2.2. Homotopy Invariance. Let $X \in Sch/k$. Let the dimension of $X$ be $d$ and let $p : E \to X$ be a vector bundle over $X$. Then $p_* : \Omega_*(X) \to \Omega_{*+d}(E)$ is an isomorphism, ([8], 3.6.3).

2.3. Projective Bundle Formula. Let $X \in Sch/k$. If $E$ is a rank $n + 1$ bundle over $X$, then there is an isomorphism $\Phi_{X,E} : \oplus_{i=0}^{n+1} \Omega_{-*+j_i}(X) \to \Omega_{*}(P(E))$. Here $\Phi_{X,E}$ is described as $\oplus \xi^j \circ q^*$, where $q : P(E) \to X$ is the projection and $\xi$ denotes the $c_1(O(1))$ operator for a canonical line bundle $O(1)$ over $P(E)$, ([8], 3.5.2).

This implies that $\Omega^*(CP^n) = \Omega^*[x]/(x^{n+1})$ where $x = c_1(O(1))(id)$ as $\Omega^{n+1}(CP^n) = 0$.

2.4. Universality. ([8], 1.1.2) gives a definition of an oriented cohomology theory over the category $Sm/k$. Algebraic cobordism satisfies these axioms and in fact is the universal oriented cohomology theory (8, 7.1.3). Hence given any other theory $h^*(-)$ satisfying these axioms, there is a unique map between functors $\Omega^*(-) \to h^*(-)$, which induces homomorphism of the graded rings $\Omega^*(X) \to h^*(X)$ for all $X$ in $Sm/k$. In particular there is a canonical graded ring homomorphism $\Omega^*(X) \to CH^*(X)$, and in fact $\Omega^*(k) \to CH^*(k)$ induces an isomorphism ([8], 4.5.1)

$$CH^*(X) \cong \Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}. $$

$\Omega^*$ is isomorphic to the Lazard ring $L^*$ (with cohomological grading) which is generated over $\mathbb{Z}$ by the monomials in $a_{11}, a_{12}, a_{21}, ...$ where the degree of $a_{ij}$ is $-(i + j - 1)$. The map

$$\Omega^* \to CH^* \cong \mathbb{Z}$$

is the one sending each $a_{ij}$ to zero.

Given a smooth variety $X$, then $MU^{2*}(X(\mathcal{C}))$ satisfies oriented cohomology theory axioms, where $X(\mathcal{C})$ is the topological space of complex points over $X$. Hence by the universality there is a natural map $\Omega^*(X) \to MU^{2*}(X(\mathcal{C}))$. This map factors via another natural map, $\Omega^*(X) \to MGL^{2*,-}(X)$, i.e.

$$\Omega^*(X) \to MGL^{2*,-}(X) \to MU^{2*}(X(\mathcal{C})).$$

The composition of the isomorphism $CH^*(X) \cong \Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}$ with the natural map $\Omega^*(X) \to MU^{2*}(X(\mathcal{C}))$ gives the cycle class map

$$CH^*(X) \to MU^*(X) \otimes_{MU} \mathbb{Z}$$

constructed in [13].

In a similar way $\Omega_*(-)$ is the universal functor among oriented Borel-Moore functors over the category of varieties and there exists maps such as

$$\Omega_*(X) \to CH_*(X) ; \Omega_*(X) \to MU_{2*}^{BM}(X)$$

between the corresponding homology theories.
3. Coniveau Filtration

In this section we will prove some properties of the coniveau filtration. In particular, we will show that the $i$th Chern class of a vector bundle lies in the $i$th level of this filtration. We will define algebraic cobordism ring for classifying spaces of linear algebraic groups. The idea involved is to extend Totaro’s definition for Chow rings given in [15] to the case of algebraic cobordism. A difference in the algebraic cobordism case is that we need to pass through the coniveau filtration on algebraic cobordism. Vishik [16] has showed that the coniveau filtration is multiplicative.

For $X \in Sm/k$, define the coniveau filtration on $\Omega^i X$ by

$$F^j(\Omega^i X) := \{ x \in \Omega^i X | x \text{ restricts to zero in } \Omega^i(X - S) \text{ for some closed subspace } S \subset X \text{ of the codimension at least } j \}.$$

That is, $x \in F^j(\Omega^i X)$ if and only if, there is a closed $S \subset X$ of codimension at least $j$ such that $i^*(x) = 0$ for the inclusion $i: X - S \hookrightarrow X$. Hence we get the filtration

$$\Omega^i X = F^0(\Omega^i X) \supset F^1(\Omega^i X) \supset F^2(\Omega^i X) \supset \cdots$$

**Lemma 3.1.** Let $f: X \rightarrow Y$ be a proper map and let $r = \text{dim}(Y) - \text{dim}(X)$, then $f_*(F^j(\Omega^i(X))) \subset F^{j+r}(\Omega^{i+r}(Y))$ and if $g : X \rightarrow Y$ is a smooth morphism, then $g^*(F^j(\Omega^i(Y))) \subset F^j(\Omega^i(X))$.

**Proof:** Let $g : X \rightarrow Y$ be a smooth morphism, and $\alpha \in F^j(\Omega^i(Y))$. Let $\alpha|_{Y - S_Y}$ be zero for some closed $S_Y \subset Y$ of codimension $d$ which is greater equal $j$. Let $S_X \subset X$ be the pull-back of $S_Y \subset Y$ along $g$. Hence $\text{codim}(S_X \subset X) \geq \text{codim}(S_Y \subset Y) \geq j$. Now consider the following commutative diagram,

\[
\begin{array}{ccc}
\Omega^{i-d}(S_Y) & \xrightarrow{i^Y} & \Omega^i(Y) \\
\downarrow{g^*} & & \downarrow{g^*} \\
\Omega^{i-d}(S_X) & \xrightarrow{i^X} & \Omega^i(X - S_Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
\Omega^{i-d}(S_X) & \xrightarrow{i^X} & \Omega^i(X - S_X) \\
\downarrow{f_*} & & \downarrow{f_*} \\
\Omega^{i-d+r}(S_Y) & \xrightarrow{i^Y} & \Omega^{i+r}(Y - S_Y) \\
\end{array}
\]

Now consider the following commutative diagram

\[
\begin{array}{ccc}
\Omega^{i-d}(S_Y) & \xrightarrow{i^Y} & \Omega^i(Y) \\
\downarrow{g^*} & & \downarrow{g^*} \\
\Omega^{i-d}(S_X) & \xrightarrow{i^X} & \Omega^i(X - S_Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
\Omega^{i-d}(S_X) & \xrightarrow{i^X} & \Omega^i(X - S_X) \\
\downarrow{f_*} & & \downarrow{f_*} \\
\Omega^{i-d+r}(S_Y) & \xrightarrow{i^Y} & \Omega^{i+r}(Y - S_Y) \\
\end{array}
\]

Chasing the diagram in a similar way as in the pullback case gives the result. □

**Remark 3.2.** For $X \in Sm/k$, $\Omega^*(X)$ is generated as an $\Omega^*$-module by the unit element $1 \in \Omega^0(X)$ and by elements of degree greater than zero in $\Omega^*(X)$ ([15, 1.2.14]). This implies that if $x \in F^j(\Omega^i(X))$, then there exist $z_i : Z_i \rightarrow X$ and $a_i \in \Omega^*$ such that $[z_i] \in \Omega^{\geq 3}(X)$ and $\alpha = \Sigma_i a_i [z_i]$.

In particular, this implies that the coniveau filtration is multiplicative ([16, 5.2]).

$$F^i(\Omega^*(X)) \cdot F^j(\Omega^*(X)) \subset F^{i+j}(\Omega^*(X))$$

**Lemma 3.3.** Let $X \in Sm/k$. The natural map from algebraic cobordism to Chow rings factors through the coniveau filtration, i.e.

$$\Omega^i(X) \rightarrow \Omega^i(X)/F^{i+1}(\Omega^i(X)) \rightarrow CH^i(X)$$
Lemma 3.5. Let \( X \in \text{Sm}/k \). Then
\[
\frac{\Omega^i(X)}{F^j(\Omega^i(X))} \cong \frac{\Omega^i(X - S_0)}{F^j(\Omega^i(X - S_0))}
\]
for a closed \( S_0 \subset X \) and codimension of \( S_0 \) is at least \( j \).

Proof: Consider the inclusion map \( i : X - S_0 \hookrightarrow X \). The localization sequence implies that the map \( i^* : \Omega^i(X) \to \Omega^i(X - S_0) \) is surjective. Let \( \alpha \in \Omega^i(X) \) and \( i^*(\alpha) \in \Omega^i(X - S_0) \). Let \( \overline{\alpha} \) and \( \overline{i^*(\alpha)} \) be their images in the respective quotients.

Suppose \( \overline{\alpha} = 0 \).
\[ \Rightarrow \alpha \text{ restricted to } \Omega^i(X - S) \text{ is zero for some closed } S \subset X \text{ of codimension at least } j. \]
\[ \Rightarrow \alpha \text{ is zero in } \Omega^i(X - S_0 - S) \text{ i.e. } \overline{i^*(\alpha)} \text{ is zero.} \]

Suppose that \( \overline{i^*(\alpha)} \text{ is zero.} \)
\[ \Rightarrow \text{There is some closed } S \subset X \text{ of codimension at least } j \text{ such that } i^*(\alpha) \text{ is zero in } \Omega^i(X - S_0 - S). \]
\[ \Rightarrow \alpha \text{ is zero in } \Omega^i(X - S_0 - S), \text{ i.e. } \overline{\alpha} \text{ is zero.} \]

Hence \( i^* : \frac{\Omega^i(X)}{F^j(\Omega^i(X))} \to \frac{\Omega^i(X - S_0)}{F^j(\Omega^i(X - S_0))} \) is an isomorphism.

Lemma 3.6. Let \( X \in \text{Sm}/k \) and \( p : E \to X \) be a vector bundle over \( X \) then,
\[
\frac{\Omega^i(E)}{F^j(\Omega^i(E))} \cong \frac{\Omega^i(X)}{F^j(\Omega^i(X))}
\]

Proof: We have that the map \( p^* : \Omega^*(X) \to \Omega^*(E) \) is an isomorphism from the homotopy invariance. First we see that \( p^* : F^j(\Omega^i(X)) \subset F^j(\Omega^i(E)) \). For, let \( x \in F^j(\Omega^i(X)) \). Let \( x \) be represented by a map \( [f : Y \to X] \) and \( x|_{X - S_X} = 0 \) for closed \( S_X \subset X \) of codimension at least \( j \). Then \( e = p^*(x) = [f^E : f^*(E) \to E] \). Let \( S_E \) denote the space \( E|_S \). Then \( S_E \) is a closed subset of \( E \) of codimension at least \( j \) in \( E \). Then \( x|_{X - S_X} = 0 \) implies \( e|_{E - S_E} = 0 \) (from lemma 3.4). Hence \( p^* : F^j(\Omega^i(X)) \subset F^j(\Omega^i(E)) \). This implies \( p^* : \frac{\Omega^i(X)}{F^j(\Omega^i(X))} \to \frac{\Omega^i(E)}{F^j(\Omega^i(E))} \) is a surjection.

Now we will show that \( p^* \) is an injection. Note that if \( s_0 : X \to E \) is the zero section, then \( s_0^* \) is the inverse map of \( p^* \). This is because as in the above set up, \( f^E : f^*(E) \to E \) is transversal to \( s_0 : X \to E \), and pulls back to \( f : Y \to X \).

We have \( e = p^*(x) \). To show an injection, we need to show if \( e \in F^j(\Omega^i(E)) \) then \( x \in F^j(\Omega^i(X)) \). By remark 3.2 we can write \( e = \sum a_i \cdot e_i \), where \( a_i \in \Omega^* \) and \( e_i \in \Omega^{2j}(E) \). As \( p^* \) is isomorphism, let \( e_i = p^*(x_i) \). Then \( x = \sum a_i \cdot x_i \). As each \( x_i \in \Omega^{2j}(X), x \in F^j(\Omega^i(X)) \).

□
Definition 3.6. Let $G$ be a linear algebraic group over a field $k$ admitting resolution of singularities. Let $V_j$ be any representation of $G$ over $k$ such that $G$ acts freely outside a $G$-invariant closed subset $S_j \subset V_j$ of codimension at least $j$. Suppose that the geometric quotient $\tilde{V}_j \rightarrow S_j$ exists as an element of $\text{Sm}/k$. Then we define

$$\Omega^j(BG) = \lim_{j \rightarrow \infty} \frac{\Omega^j(\tilde{V}_j/S_j)}{\text{F}^j(\Omega^j(\tilde{V}_j/S_j))}$$

For a given $j$, we call a pair $(V_j, S_j)$ in the above definition as a $j$-admissible pair over $X$. The following theorem will show that $\Omega^j(BG)$ is well-defined. We follow the technique used in [15].

Theorem 3.7. For $G$, $V$, and $S$ as above $Q^j(\Omega^j(BG)) := \frac{\Omega^j(\tilde{V}_j/S_j)}{\text{F}^j(\Omega^j(\tilde{V}_j/S_j))}$ is independent of the choice of the representation $V_j$ and a closed subset $S_j$ of codimension at least $j$.

Proof:
Having fixed a representation, independence of the choice of $S$ is established in Lemma 3.4. To obtain independence over a choice of representation, consider any 2 representations $V$ and $W$. Assume that $G$ acts freely outside $S_V$ of codimension at least $j$ in $V$ and outside $S_W$ of codimension at least $j$ in $W$. Also assume quotients $(V - S_V)/G$ and $(W - S_W)/G$ exists as elements of $\text{Sm}/k$. Then consider the direct sum $V \oplus W$. The quotient $((V - S_V) \times W)/G$ exists as an element of $\text{Sm}/k$ being a vector bundle over $(V - S_V)/G$ and so also $((W - S_W) \times V)/G$. Independence over the choice of $S$ for the representation $V \oplus W$ implies that

$$Q^j(((V - S_V) \times W)/G) \cong Q^j(((W - S_W) \times V)/G).$$

And Lemma 3.4 above shows that $Q^j$ of a vector bundle is same as that of the base scheme. Hence both $V$ and $W$ have isomorphic quotients $Q^j$ in indices less than equal to $j$, proving independence of the choice of representation.

Lemma 3.8. Let $X \in \text{Sm}/k$. Let $p : L \rightarrow X$ be a line bundle over $X$. Then $p^*([f : Y \rightarrow X, f^*(L)]) = [s_0 \circ f : Y \rightarrow L]$ where $s_0 : X \rightarrow L$ is the zero section.

Proof: Consider the following commutative diagram

$$\begin{array}{ccc}
  f^*(L) & \longrightarrow & L \\
  \downarrow q & & \downarrow p \\
  Y & \rightarrow & X
\end{array}$$

We get that

$$p^*([f : Y \rightarrow X, f^*(L)]) = [f^* : f^*(L) \rightarrow L, q^*(f^*(L))].$$

Let $y \in Y$ and let $q^{-1}(y)$ denote the fiber over $y$ in $f^*(L)$. Then the fiber over $y$ in $q^*(f^*(L))$ is precisely $q^{-1}(y) \times q^{-1}(y)$. Hence the diagonal map $\Delta : q^{-1}(y) \rightarrow q^{-1}(y) \times q^{-1}(y)$ induces the diagonal section $s_\Delta : f^*(L) \rightarrow q^*(f^*(L))$, transversal to the zero section. The zeros of $s_\Delta$ is precisely the closed subscheme given by the image of the zero section $s_0 : Y \rightarrow f^*(L)$. This implies

$$p^*([f : Y \rightarrow X, f^*(L)]) = [s_0 \circ f : Y \rightarrow L].$$

Lemma 3.9. For $X \in \text{Sm}/k$ and a vector bundle $p : E \rightarrow X$ of rank $n$, $p^*(c_n(E)) = [s_0 : X \rightarrow E]$ where $s_0$ is the zero section.

Proof: Proof is by induction on $n$. Lemma 3.8 implies the case $n = 1$. Let $P(E)$ be the projectivization of $E$ and let $t : P(E) \rightarrow X$ be the associated projective bundle map. Then we have an exact sequence of bundles over $P(E)$

$$0 \longrightarrow E_1 \stackrel{h_1}{\longrightarrow} t^*(E) \stackrel{h_2}{\longrightarrow} E_{n-1} \longrightarrow 0$$
where $E_1$ is the $O(-1)$ bundle over $P(E)$ and rank of $E_{n-1}$ is $n-1$. Let $q_{n-1} : E_{n-1} \to P(E)$ and $q_1 : E_1 \to P(E)$ denote the bundle maps. Note that if $q : t^*(E) \to P(E)$ is the bundle map associated with $t^*(E)$, then $q = q_{n-1} \circ h_2$. We have a cartesian diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{h_1} & t^*(E) \\
q_1 \downarrow & & \downarrow \quad h_1 \\
P(E) & \xrightarrow{s_0^{n-1}} & E_{n-1}
\end{array}
\]

where $s_0^{n-1}$ is the zero section of the bundle $E_{n-1}$. Then

\[
q^*(c_n(t^*(E))) = q^*(c_1(E_1)c_{n-1}(E_{n-1})) = h_2^*(h_2^{n-1}(c_1(E_1))) = h^*(s_0^{n-1}(E_1))) \text{ by induction}
\]

\[
= h_1^*(q_1^*(c_1(E_1))) \text{ by } \Box
\]

\[
= [s_0^* : P(E) \to t^*(E)]
\]

where the final map $s_0^*$ is the inclusion of the zero section of $t^*(E)$. Finally note that the pullback of $[s_0 : X \to E]$ under the map $t^*(E) \to E$ is $[s_0^* : P(E) \to t^*(E)]$. Hence we have shown that $p^*(c_n(E)) = [s_0 : X \to E]$.

Lemma 3.10. For $X \in S^m/k$ and a vector bundle $p : E \to X$, $c_i(E) \in F^i(\Omega^i(X))$.

Proof: We prove this by an induction on the rank of $E$. Let $E$ be a rank $n$ vector bundle over $X$. Let $s_0 : X \to E$ be the zero section. Then $p^*(c_n(E)) = [s_0 : X \to E]$. Hence $p^*(c_n(E)) \in F^i(\Omega^i(E))$. By lemma 3.5, $c_n(E) \in F^n(\Omega^n(X))$.

For smaller $c_i$'s, we use an inductive definition of the smaller $c_i$'s. Let $E_0$ be $E - s_0(X)$.

There is a canonical rank $n-1$ bundle $E_{n-1}^0$ over $E_0$, (where the fiber over a point $(x,v) \in E_0$ is the space orthogonal to $v$ in $E_x$). We have a bundle map $p : E_0 \to X$ and the localization sequence implies that $p^* : \Omega^*(X) \to \Omega^*(E_0)$ is an isomorphism in degrees less than or equal to $n-1$. And then $c_i(E_0) = (p^*)^{-1}(c_i(E_{n-1}))$ for $i < n$. Lemma 3.4 implies that for $j \leq n$, $\Omega^*(E_0)/F^j(\Omega^*(E_0))$ equals $\Omega^*(E)/F^j(\Omega^*(E))$ and lemma 3.5 implies that $p^*$ induces an isomorphism between $\Omega^*(X)/F^j(\Omega^*(X))$ and $\Omega^*(E)/F^j(\Omega^*(E))$. Hence by induction $p : E \to X$, $c_i(E) \in F^i(\Omega^i(X))$.

The Chern classes in algebraic cobordism are defined using the projective bundle formula, but that definition is same as the one used above. Consider $t : P(E) \to X$, where $P(E)$ is the projectivisation of $E$. There is an exact sequence $0 \to O(-1)(P(E)) \to t^*(E) \to E_{n-1} \to 0$. For the line bundle $E_1 = O(-1)(P(E))$, the space $E_0$ used in the above proof is the total space of $L$ minus the zero section. If we look at $t^*(E)$ as a bundle over $E_1$, i.e. pull-back of $E_{n-1}$ over $L$, then its restriction to $E_0$ is precisely $E_{n-1}^0$ used in the above proof. $c_1(t^*(E)) = c(E_1)c(E_{n-1}^0)$ implies $c_i(E) = (p^*)^{-1}(c_i(E_{n-1}^0))$ for $i < n$ as the multiples of $c_1(E_1)$ are mapped to zero in $\Omega^*(E_0)$.

4. Classical Lie Groups

In this section, we assume the base field to be the complex numbers. In this section, we will first prove a lemma which helps in finding generators of $\Omega^*(X)$ from that of $CH^*(X)$. We will use this to show that the algebraic cobordism of $BGL(n), BSL(n), BSp(n), BO(n)$ and $BSO(2n+1)$ respectively maps isomorphically to the complex cobordism of these spaces.

In (19), section 9, Yagita has considered the groups that satisfy $\Omega^*_{BP}(BG) \cong BP^*(BG)$. Here $BP^*(-)$ is Brown-Peterson cohomology for a prime $p$ and $\Omega^*_{BP}(-)$ is algebraic Brown-Peterson theory defined in section 8 of (19). There is a relation $\Omega^*_{BP}(X) \cong BP^* \otimes_{\Omega^*_{BP}} \Omega^*(X)_{(p)}$ where $\Omega^*(X)_{(p)} := \Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}$ is the localisation of the ring $\mathbb{Z}$ at the prime $p$. Hence our calculations also implies results in (19), section 9).

Hu and Kriz have calculated $MGL^*(-)(BGL(n))$ in (5), section 2).
 Firstly, we will look at the situation in $MU^*(-)$ which tells us what answers to expect and why.

$$MU^*(BGL(1)) = MU^*(\mathbb{C}P^\infty) = MU^*[x];$$
where $x \in MU^2(\mathbb{C}P^\infty)$ is the Euler class of the canonical bundle $\gamma_1$ over $\mathbb{C}P^\infty$. For $X$ denoting the product of $n$ copies of $\mathbb{C}P^\infty$,

$$MU^*(X) \cong MU^*[x_1, \ldots, x_n]$$

where each $x_i \in MU^2(X)$ is the pullback along the projection on the $i$th co-ordinate of the Euler class of $\gamma_1$. Then

$$MU^*(BGL(n)) \cong MU^*[c_1, \ldots, c_n]$$

where each $c_i$ corresponds to the $i$th Chern class of the canonical bundle $\gamma_n$ over $BGL(n)$ which is homotopy equivalent to $Gr(n, \infty)$. For each $i$, the pullback of $c_i$ along the embedding of torus $GL(1)^n \to GL(n)$ is the $i$th symmetric polynomial in $x_j$’s. $BSL(n)$ is homotopy equivalent to a $GL(1)$-bundle over $BGL(n)$. This $GL(1)$-bundle is the complement of the zero section in the total space of $det(\gamma_n)$ over $BGL(n)$. This helps in concluding using the topological version of lemma [8,1] that

$$MU^*(BSL(n)) \cong MU^*[c_2, \ldots, c_n].$$

Similarly, we have $MU^*(BSp(1)) = MU^*(BGL(2)) = MU^*[c_2]$ where $c_2$ is contained in $MU^4(BSp(1))$. And

$$MU^*(BSp(n)) \cong MU^*[c_2, \ldots, c_{2n}]$$

where $c_{2i}$ is the $2i$th Chern class of the standard representation of $Sp(n)$. If $Y$ is the product of $n$ copies of $BSp(2)$, then

$$MU^*(Y) \cong MU^*[y_1, \ldots, y_n]$$

where each $y_j \in MU^4(Y)$. The pullback of $c_{2i}$ along the diagonal inclusion $SU(2)^n \to Sp(n)$ is the $i$th symmetric polynomial in $y_j$’s.

The $BO(n)$ case is done by Wilson in [18]. He has shown that $MU^*(BO(n))$ is isomorphic to the quotient of $MU^*(BGL(n))$ by the relations $c_i = c_i^*$ where $c_i^*$ is the $i$th Chern class of the dual of the standard representation of $GL(n)$. These relations are forced because the standard representation of $O(n)$ is self dual.

$$MU^*(BO(n)) \cong MU^*[c_1, \ldots, c_n]/(c_1 - c_1^*, \ldots, c_n - c_n^*).$$

The expression for $c_i^*$ can be obtained as follows. Let $z_j = [-1](x_j)$ be the inverse of $x_j$ under the formal group law. Then pullback of $c_i^*$ to $MU^*(X)$ is the $i$th symmetric polynomial in $z_j$’s i.e. in $[-1](x_j)$’s which can be written in terms of the symmetric polynomials in $x_j$’s. The $i$th symmetric polynomial in $x_j$’s is the pullback of $c_i$. This gives the expression for $c_i^*$ as a formal power series in $c_i$’s with coefficients in $MU^*$. The topological version of lemma [5,1] implies that

$$MU^*(BO(2n + 1)) \cong MU^*[c_2, \ldots, c_n]/(c_2 - c_2^*, \ldots, c_n - c_n^*).$$

Now we will start our calculations.

**Lemma 4.1.** Let $X \in Sm/k$. If $CH^*X$ is generated as an abelian group by some elements $x_1, x_2, \cdots$ and if $y_1, y_2, \cdots \in \Omega^*X$ map to $x_i \in CH^*X$ then $\Omega^*X$ is generated as an $\Omega^*$-module by $y_1, y_2, \cdots$.

**Proof:** We have a natural map $\Omega^*X \to CH^*X$ and there is an isomorphism $CH^*X = \Omega^*X \otimes_{\Omega^*} \mathbb{Z}$. This implies that the kernel of the map $\Omega^*X \to CH^*X$ is $\Omega_{<0} \cdot \Omega^*(X)$. We prove the lemma by induction on decreasing degree of $y \in \Omega^*X$. Let the image of $y$ in $CH^*X$ be $x = \Sigma n_i x_i$. Then $y - \Sigma n_i y_i$ is mapped to zero in the Chow ring. Hence $y - \Sigma n_i y_i = \Sigma u_j z_j$ where $u_j \in \Omega_{<0}$. Hence each of the $z_j$ have the degree higher than the degree of $y$. As cobordism is zero in the degrees higher than dimension of $X$, we are through by induction. □
We use this lemma to calculate the algebraic cobordism of $BGL(n), BSL(n), Sp(n), BO(n)$ and $BSO(2n + 1)$. The Chow rings of these spaces have been calculated in [13] and then also in [11] using an unified approach termed as the ‘stratification’.

$CH^*(BGL(n)) \cong \mathbb{Z}[c_1, c_2, ..., c_n]$, where each $c_i$ is the $i$th Chern class of the canonical $n$-bundle $\gamma_n$ over $BGL(n)$ in the Chow ring. Since the canonical map $\Omega^* X \to CH^* X$ is functorial, for any vector bundle $E$ over $X$, it maps the Chern classes of $E$ in $\Omega^* (X)$ to the Chern classes of $E$ in $CH^* X$. Note that Levine-Morel defined the Chern classes as maps $c_i : \Omega^* X \to \Omega^{n+i} X$. Hence the Chern classes of $\gamma_n$ over $G\ell(n, \infty)$. Now consider the natural map $\Omega^* (BGL(n)) \to MU^* (BGL(n)) \cong MU^*[c_1, c_2, ..., c_n]$.

We have that the natural map $\Omega^* (pt) \to MU^* (pt)$ is an isomorphism. Hence any non-zero polynomial or power series relation between $c_i$'s with coefficients in $\Omega^*$ will be mapped to non-zero relation between $c_i$'s with coefficients in $MU^*$. But since there is no such relation in the complex cobordism ring of $BGL(n)$, there is no such relation in algebraic cobordism too. Hence

(13) $\Omega^* (BGL(n)) \cong \Omega^*[c_1, c_2, ..., c_n]$.

The exact similar arguments, i.e. obtaining the Chern classes as the generators using the Chow ring and using the complex cobordism to show there is no relation between Chern classes implies that

(14) $\Omega^* (BSL(n)) \cong \Omega^*[c_2, ..., c_n]$

(15) $\Omega^* (BSp(2n)) \cong \Omega^*[c_2, c_4, ..., c_{2n}]$.

We can use this method for $BO(n)$ and $BSO(2n + 1)$ too using the calculations of $MU^* BO(n)$ done in [13].

We have $CH^* BO(n) \cong \mathbb{Z}[c_1, ..., c_n]/(2c_i = 0; i \text{ odd})$. This implies that $\Omega^* (BO(n))$ as a module over $\Omega^*$ is generated by the monomials $c_1, ..., c_n$; Chern classes in algebraic cobordism of standard representation of $BO(n)$. Now we obtain the relations among these Chern classes. Let $c_i^*$ be the Chern classes for the dual of standard representation of $GL(n)$. As the standard representation $O(n) \to GL(n)$ is self-dual, pullbacks of $c_i$ and $c_i^*$ are equal. So, we get $c_i = c_i^*$ in algebraic cobordism too.

Now consider the natural map $\Omega^* (BO(n)) \to MU^* (BO(n))$ and the isomorphism $\Omega^* \to MU^*$. Absence of any polynomial or power series relation besides $c_i = c_i^*$ in $MU^* (BO(n))$ implies that there are no more relations between $c_i$'s other than $c_i = c_i^*$ in algebraic cobordism too. Hence

(16) $\Omega^* (BO(n)) \cong \frac{\Omega^*[c_1, ..., c_n]}{< c_i = c_i^* >}$

Similarly we get that

(17) $\Omega^* (BSO(2n + 1)) \cong \frac{\Omega^*[c_2, c_3, ..., c_{2n+1}]}{< c_i = c_i^* >}$.

5. Products of finite cyclic groups

In this section we continue to assume that the base field is the complex numbers. We will calculate $\Omega^* (BG)$ when $G$ is a product of finite cyclic groups. $CH^* (BG)$ for such $G$ is calculated in [15] and $MU^* (BG)$ is calculated in [6]. As said before, Yagita has calculated $\Omega^*_{BP} (BG)$ for $G$ a product of finite abelian $p$-groups and our calculations imply results in [19], section 9).

In this section, we will follow the approach used in [6] and in the process show that a version of Kunneth formula for Chow rings proved by Totaro in [18] also holds in algebraic cobordism.

**Lemma 5.1.** Let $X \in Sm/k$ and $p : E \to X$ be a vector bundle of rank $r$. Let $E_0$ be a complement of the zero section of $E$. Then the pullback homomorphism $\Omega^* (X) \to \Omega^* (E_0)$ is always surjective and its kernel is generated by the top Chern class $c_r(E)$. 
PROOF: This follows from the localization sequence, Let \( s_0 : X \to E \) be the zero section and let \( j : E_0 \to E \) be the inclusion map. Let \( p_0 : E_0 \to X \) be the restriction of \( p \). The localization sequence corresponding to \( j \) is

\[
\Omega^*(X) \xrightarrow{s_0^*} \Omega^{*+r}(E) \xrightarrow{j^*} \Omega^{*+r}(E_0) \to 0.
\]

By the homotopy invariance, we have an isomorphism \( p^* : \Omega^*(X) \to \Omega^*(E) \). Hence by the localization sequence, the pullback along \( p_0 \), denoted by \( p_0^* : \Omega^*(X) \to \Omega^*(E_0) \) is always surjective. To show that the kernel of this surjection is generated by \( c_r(E) \), we need to show \( s_0^*(x) = p^*(c_r(E)) \) for \( x \in \Omega^*(X) \). Lemmas 5.2 and 5.3 show exactly this. Hence proved.

\[ \Omega^*(\mathbb{C}P^\infty) = \Omega^*[x] \], where \( x \) is the first Chern class of \( \mathcal{O}(1) \) bundle over \( \mathbb{C}P^\infty \), i.e. \( x = c_1(\mathcal{O}(1))(id) \). By \( [n](x) \), we mean the addition of \( x \) with itself \( n \) times, under the formal group law in \( \Omega^*(\mathbb{C}P^\infty) \), \( F(a,b) = a + b + \sum_{i,j>0} a_i a_j b_j \). Then

\[
[2](x) = F(x,x) = 2x + \sum_{i,j>0} a_{ij} x^{i+j} = 2x + a_{11} x^2 + 2a_{12} x^3 + \cdots .
\]

\[
[3](x) = F(x,[2](x)) = 3x + 3a_{11} x^2 + (a_{11}^2 + 8a_{12}) x^3 + \cdots
\]

and \( [n](x) = F(x,[n-1](x)) = nx + \) higher order terms.

**Lemma 5.2.** \( \Omega^*(B\mathbb{Z}_n) = \Omega^*[x]/([n](x)) \).

**Proof:** If \( E \) is the total space of \( \mathcal{O}(n) \) over \( \mathbb{C}P^\infty \) and \( s_0 : \mathbb{C}P^\infty \to E \) is the zero section then, \( B\mathbb{Z}_n \) is \( E - \text{Im}(s_0) \).

The localization sequence implies that

\[
\Omega_*(\mathbb{C}P^\infty) \xrightarrow{j} \Omega_*(E) \to \Omega_*(B\mathbb{Z}_n) \to 0
\]

Considering the isomorphism \( \Omega_*(E) \cong \Omega_*(\mathbb{C}P^\infty) \), we get that, the map \( j \) is the multiplication by \( c_1(\mathcal{O}(n)) \).

From the formal group law, \( c_1(\mathcal{O}(n)) = [n](x) \). Hence

\[
\Omega^*(B\mathbb{Z}_n) = \Omega^*[x]/([n](x)).
\]

**Lemma 5.3.** Let \( X \) and \( Y \) be in \( \text{Sch}/k \), and that \( Y \) can be partitioned into open subsets of an affine space, then the cup product map \( \Omega^*(X) \otimes \Omega^*(Y) \to \Omega^*(X \times Y) \) is surjective.

**Proof:** Using the localization sequence for an open subset \( U \) of \( \mathbb{A}^n \), if \( Z \) is the complement of \( U \), \( \Omega^*(Z) \to \Omega^*(A) \to \Omega^*(U) \to 0 \) implies that \( \Omega^*(A) \to \Omega^*(U) \) is a surjection. The homotopy invariance property says that \( \Omega^*(X) \otimes \Omega^*(\mathbb{A}^n) \cong \Omega^*(X \times \mathbb{A}^n) \). As \( X \times U \) is an open subset of \( X \times \mathbb{A}^n \), the localization sequence implies that \( \Omega^*(X \times \mathbb{A}^n) \to \Omega^*(X \times U) \) is also surjection. Hence the commutative diagram

\[
\begin{array}{ccc}
\Omega^*(X) \otimes \Omega^*(\mathbb{A}^n) & \xrightarrow{\cong} & \Omega^*(X \times \mathbb{A}^n) \\
\downarrow & & \downarrow \\
\Omega^*(X) \otimes \Omega^*(U) & \to & \Omega^*(X \times U)
\end{array}
\]

implies that \( \Omega^*(X) \otimes \Omega^*(U) \to \Omega^*(X \times U) \) is surjective.

Suppose that \( Y \) can be partitioned into open subsets of affine space. Let \( U \) be an open subset among that partition and \( Z \) be its complement in \( Y \).
by induction on the number of open subsets in the partition.

Theorem 5.5. \( B \mathbb{Z}/n \) can be taken to be the complement of zero section in \( O(n) \) bundle on \( \mathbb{C}P^\infty \). It can be successively approximated by the complement of the zero section in the \( O(n) \) bundle on the finite dimensional 1-Grassmannians, i.e. complex projective spaces [15], 1.4. Since \( \mathbb{C}P^n \)'s can be partitioned into affine spaces, each of the approximations of \( B \mathbb{Z}/n \) can be partitioned into open subsets of affine spaces.

If \( X \) is an infinite CW complex with finite skeletons \( X_n \), then these skeletons induce a filtration on \( MU^*(X) \), i.e. we say \( x \in MU^*(X) \) is in the \( n^{th} \) level of filtration if if its zero when restricted to \( X_n \). From [3], if \( MU^*(X) \) has no elements of infinite filtrations then

\[
\tag{20} MU^*(B \mathbb{Z}/n \times X) \cong MU^*(B \mathbb{Z}/n) \otimes_{MU^*} MU^*(X);
\]

\[
\tag{21} MU^*(B \mathbb{Z}/n \times B \mathbb{Z}/m) \cong MU^*(B \mathbb{Z}/n) \otimes_{MU^*} MU^*(B \mathbb{Z}/m).
\]

Lemma 5.4. \( \Omega^*(B \mathbb{Z}/m) \otimes_{\Omega^*} \Omega^*(B \mathbb{Z}/n) \cong \Omega^*(B(\mathbb{Z}/m \times \mathbb{Z}/n)). \)

Proof: For a complex smooth variety \( X \), we have a natural map \( \Omega^*(X) \to MU^*(X(\mathbb{C})). \) This map is an isomorphism for \( X = B\mathbb{Z}/n \) and \( Y = B\mathbb{Z}/m \). Also note that

\[
MU^*(X) \otimes_{MU^*} MU^*(Y) \to MU^*(X \times Y)
\]

is an isomorphism. Now consider the diagram

\[
\begin{CD}
\Omega^*(X) \otimes_{\Omega^*} \Omega^*(Y) @>>> \Omega^*(X \times Y) \\
@VVV @VVV \\
MU^*(X) \otimes_{MU^*} MU^*(Y) @>>> MU^*(X \times Y)
\end{CD}
\]

The left vertical map is an injection and the lower horizontal map is an isomorphism in case \( X \) and \( Y \) are classifying spaces of cyclic groups. This implies that the upper horizontal map is an injection. Surjection is proved by the lemma [5,3]. Hence \( \Omega^*(B \mathbb{Z}/m) \otimes_{\Omega^*} \Omega^*(B \mathbb{Z}/n) \cong \Omega^*(B(\mathbb{Z}/m \times \mathbb{Z}/n)) \)

Hence now we have,

**Theorem 5.5.** If \( G = \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \ldots \times \mathbb{Z}/n_r \) then \( \Omega^*(BG) = \frac{\Omega^*[x_1, x_2, \ldots, x_r]}{[n_1](x_1), \ldots, [n_r](x_r)} \) where \( x_i \in \Omega^1(BG) \) for all \( i \).

### 6. Cellular spaces

Let \( X \in Sch/k \) and suppose there exists a filtration \( \phi = Z_0 \subset Z_1 \subset \ldots \subset Z_n = X \) where \( U_i = Z_{i+1} - Z_i \) is an affine space \( \mathbb{A}^N_k \) (a cell). We call such \( X \) a cellular space. ([3], 5.2.11) shows that \( \Omega_*(X) \to \bigoplus U_i \Omega_*(U_i) \) induced by restriction maps is a surjection over any base field allowing resolution of singularities. We will show its an isomorphism over the complex numbers using the natural map to the complex cobordism. As expected, we get

\[
\Omega_*(X) \cong MU_*(X).
\]
The statement is known for a long time now, but for the sake of completeness we provide the proof. For the next theorem, we assume that the base field is the complex numbers.

**Theorem 6.1.** For above $X$, $\Omega_r(X)$ is isomorphic to $\bigoplus_{U \in I} \Omega_r(U)$, where $I$ is set of cells of $X$. In fact, the map $\Omega_*(X) \to MU_2(X)$ is an isomorphism.

**Proof:** When there is just one cell, statement follows trivially. We now proceed by induction on the number of cells. Suppose the statement holds for $Z$ and we want to add a cell $U$ of dimension $n$ to get $X$, i.e. $Z \hookrightarrow X \leftarrow U$. Let $\bar{U}$ be closure of $U$ in $X$. And $\bar{U}'$ be a resolution of singularities of $\bar{U}$. Consider the diagram

\[
\begin{array}{c}
Z \\ \downarrow i \\
\bar{U}' \\
\downarrow f \\
\bar{U} \\
\downarrow j \\
(Pt)
\end{array}
\]

This gives the following commutative diagram on cobordism groups,

\[
\begin{array}{c}
\Omega_r(Z) \\
\downarrow i_* \\
\Omega_r(U') \\
\downarrow f_* \\
\Omega_r(U) \\
\downarrow j_* \\
\Omega_r(X) \\
0
\end{array}
\]

\[
\begin{array}{c}
\Omega_r(Z) \\
\downarrow i_* \\
\Omega_r(U') \\
\downarrow f_* \\
\Omega_r(U) \\
\downarrow j_* \\
\Omega_r(X) \\
0
\end{array}
\]

Hence $i_* \circ \tilde{f}^*$ gives a section of $\Omega_r(U)$ in $\Omega_r(X)$. Which implies $\Omega_r(X) \cong \Omega_r(U) \oplus Im(\Omega_r(Z))$.

To prove that $\Omega_r(Z)$ injects into $\Omega_r(X)$ we use the natural isomorphism from $\Omega_*(X)$ to $MU_*(X)$.

As the natural map $\Omega_*(\mathbb{A}^n) \to MU_*(\mathbb{A}^n)$ is an isomorphism, by induction, we get that the map $\Omega_*(Z) \to MU_*(Z)$ is an isomorphism. We have a commutative diagram,

\[
\begin{array}{c}
\Omega_r(Z) \\
\downarrow \\
\Omega_r(U) \\
\downarrow \\
\Omega_r(U) \\
\downarrow \\
\Omega_r(Z) \\
0
\end{array}
\]

\[
\begin{array}{c}
\Omega_r(Z) \\
\downarrow \\
\Omega_r(U) \\
\downarrow \\
\Omega_r(U) \\
\downarrow \\
\Omega_r(Z) \\
0
\end{array}
\]

Since $U$ is a complex affine space $\mathbb{A}^N$, $MU_{2i+1}(U)$ is zero for all $i$. Hence we have that the map $MU_{2r}(Z) \to MU_{2r}(X)$ is injective. Hence we get that the map $\Omega_r(Z) \to \Omega_r(X)$ is also injective. Hence

$$\Omega_*(X) \cong \bigoplus_{U \in I} \Omega_*(U).$$

In particular $\Omega_*(X) \to MU_*(X)$ is an isomorphism. □

Note that the isomorphism in the above theorem is not unique and depends on the choice of resolution of singularities $\bar{U}'$.

We can apply this to projective homogeneous varieties which have the Schubert cell decomposition. In particular we can apply this to Grassmanians.
7. Equivariant cobordism

Let $G$ be a linear algebraic group. By a $G$-scheme, we mean an element of $\text{Sch}/k$ equipped with an action of $G$. In this section we define $G$-equivariant algebraic cobordism similar to $G$-equivariant Chow ring defined by Edidin and Graham in [3]. We show that $G$-equivariant algebraic cobordism satisfies the localization exact sequence.

Let $G$ be any linear algebraic group $G$ and a let $X$ be a smooth $G$-scheme. We say that an action of $G$ on $X$, denoted by say $\sigma$, is linearized if there exist a line bundle

$$\pi : L \to X$$

over $X$ such that, the action $\sigma$ of $G$ on $X$ lifts to an action on $L$ ([12], Definition 1.6). In particular, we have a commutative diagram

$$
\begin{array}{ccc}
G \times L & \xrightarrow{\Sigma} & L \\
\downarrow{\pi} & & \downarrow{\pi} \\
G \times X & \xrightarrow{\sigma} & X.
\end{array}
$$

Let $(V_j, S_j)$ be a $j$-admissible pair as in definition 3.6. Let $U$ be $V_j - S_j$. Whenever the action of $G$ is linearized for a line bundle which is relatively ample for the projection $X \times U \to U$ and $U/G$ is known to be quasi-projective, then the quotient $(X \times U)/G$ is quasi-projective ([12], proposition 7.1). From now on, we will take this as a definition for $G$-linearized action. Furthermore, if both $X$ and $U$ are smooth, then the quotient $(X \times U)/G$ is also smooth.

**Definition 7.1.** For a linear algebraic group $G$ and a smooth scheme $X$ equipped with a $G$-linearized action and any $j$-admissible pair $(V_j, S_j)$ as in definition 3.6, we define

$$\Omega^i_G(X) = \lim_j \frac{\Omega^i_j(X \times (V_j - S_j))}{F^j(\Omega^i_j(X \times (V_j - S_j))}.$$ (25)

**Lemma 7.2.** Let $G$ be a linear algebraic group with normal subgroup $H$ and let $X$ be a smooth $G$-scheme with a $G$-linearized action. Suppose the action of $H$ on $X$ is free and the quotient scheme $X/H$ exists. Then

$$\Omega^i_G(X) \cong \Omega^i_{G/H}(X/H).$$

**Proof:** Let $(V_j, S_j)$ be a $j$-admissible pair over $G$ for a given $j$.

$$\Omega^i_G(X) = \lim_j \frac{\Omega^i_j(X \times (V_j - S_j))}{F^j(\Omega^i_j(X \times (V_j - S_j)))}$$

$$= \lim_j \frac{\Omega^i_j((X \times (V_j - S_j))/H)}{F^j(\Omega^i_j((X \times (V_j - S_j))/H))}$$

$$= \lim_j \frac{\Omega^i_j((X/H) \times (V_j - S_j))}{F^j(\Omega^i_j((X/H) \times (V_j - S_j)))}$$

where the last equality follows because $(X \times (V_j - S_j))/H$ is isomorphic to $(X/H) \times (V_j - S_j)$ as the action of $H$ on $X$ is free.

The final expression is by definition isomorphic to $\Omega^i_{G/H}(X/H)$, which is required.

Now we have to check that the $G$-equivariant localization sequence also holds.

**Theorem 7.3.** Let $X$ be a smooth scheme with a $G$-linearized action. Let $Z$ be its closed $G$-subscheme of codimension $r$ and let $U$ be the complement $Z$ in $X$. Then

$$\Omega^i_G(Z) \to \Omega^{i+r}_G(X) \to \Omega^{i+r}_G(U) \to 0.$$
PROOF: Let \((V, S)\) be a \(j\)-admissible pair over \(G\) for a given \(j\). Let \(X'\) denote the scheme \(\frac{X \times (V - S)}{G}\) and similarly denote by \(Z'\) and \(U'\) the subschemes of \(X'\) corresponding to \(Z\) and \(U\). To prove the lemma, it is enough to show that for each \(j\)

\[
\begin{array}{c}
\Omega'(Z')/F_j(\Omega'(Z)) & \to & \Omega^{i+r}(X')/F_j(\Omega^{i+r}(X')) & \to & \Omega^{i+r}(U')/F_j(\Omega^{i+r}(U')) & \to & 0.
\end{array}
\]

This follows from analysing the following commutative diagram

\[
\begin{array}{ccc}
\Omega'(Z') & \xrightarrow{i_*} & \Omega^{i+r}(X') & \xrightarrow{j^*} & \Omega^{i+r}(U') & \to & 0 \\
F_j(\Omega'(Z)) & \xrightarrow{i'_*} & F_j(\Omega^{i+r}(X')) & \xrightarrow{j'^*} & F_j(\Omega^{i+r}(U')) & \to & 0 \\
\end{array}
\]

\(\text{Im}(i_*) = \text{Ker}(j^*)\) implies \(\text{Im}(i'_*) \subset \text{Ker}(j'^*)\).

Now we will show the containment the other way. Let \(x \in \Omega^{i+r}(X')\). Let its image in the quotient \(\frac{\Omega^{i+r}(X')}{F_j(\Omega^{i+r}(X'))}\) be \(\overline{x}\) and \(j^* (x) = u\). Suppose \(j'^*\) maps \(\overline{x}\) to zero, then we need to show \(\overline{x}\) is the image of some \(\overline{z} \in \frac{\Omega'(Z')}{F_j(\Omega'(Z))}\), where \(z \in \Omega'(Z')\).

The idea is to find another cobordism cycle \(x' \in F_j(\Omega^{i+r}(X'))\) s.t. \(j^*(x') = u\). So that we can write \(x - x' = i_*(z)\), and then taking the images in the lower row gives \(\overline{x} = i'_*(\overline{z})\). We find \(x'\) as follows.

\(j'^*\) maps \(\overline{x}\) to zero implies that \(u \in F_j(\Omega^{i+r}(U'))\). Hence there is a subset \(S\) of \(U'\) s.t., it is closed in \(U'\), its codimension in \(U'\) denoted by \(r_S\) is greater equal \(j\) and \(u_{U - S} = 0\). This means \(u = i^*_S(s)\) for some \(s \in \Omega^*(S)\). Let \(\overline{S}\) be the closure of \(S\) in \(X'\). We have a cartesian square

\[
\begin{array}{ccc}
\overline{S} & \xrightarrow{\overline{f}} & X' \\
\downarrow & & \downarrow \quad j \\
S & \xrightarrow{f} & U'
\end{array}
\]

where all the arrows are inclusions and it gives a commutative diagram of the respective cobordism groups

\[
\begin{array}{ccc}
\Omega^{i+r - r_S}(\overline{S}) & \xrightarrow{i^*_{\overline{S}}} & \Omega^{i+r}(X') \\
\downarrow f^* & & \downarrow j^* \\
\Omega^{i+r - r_S}(S) & \xrightarrow{i^*_S} & \Omega^{i+r}(U').
\end{array}
\]

There exists \(\overline{S}\) such that, \(s = f^*(\overline{S})\) as \(f^*\) is a surjection by the localization sequence. And we set \(x' = i^*_{\overline{S}}(\overline{S})\). The above commutative diagram implies that

\(j^*(x') = i^*_S(f^*(\overline{S})) = u\).

Hence we have found a correct \(x'\) we needed and hence proved.

\[\square\]

8. THE QUATERNION GROUP

In this section, we again assume that the base field is the complex numbers. Let \(Q = \{\pm 1, \pm i, \pm j, \pm k\}\) be the group of quaternions. In this section we will calculate \(\Omega^*(BQ)\).

The group \(Q\) has 4 one-dimensional representations \(I, g_i, g_j, g_k\). \(g_i\) denotes the homomorphism \(Q \to \{\pm 1\} \subset \mathbb{C}\), which maps \(i\) to \(-1\) and \(j, k\) to \(1\). It also has a unique 2-dimensional irreducible
complex representation $\rho$. Let $L_i, L_j,$ and $L_k$ respectively denote the line bundles over $BQ$ corresponding to $\varrho_i, \varrho_j, \varrho_k$. And let $V$ be the rank-2 vector bundle corresponding to $\rho$. Relations between these representations are given in [2]. Written in terms of corresponding vector bundles, these are

\begin{align*}
(30) & \quad L_i \otimes L_j \cong L_k, \\
(31) & \quad L_i \otimes L_i \cong \mathbb{I} \text{ and similar for } j \text{ and } k. \\
(32) & \quad L_i \otimes V \cong V \text{ and similar for } j \text{ and } k. \\
(33) & \quad L_i \oplus L_j \oplus L_k \oplus \mathbb{I} \cong V \otimes V. \\
(34) & \quad \det(V) \cong \mathbb{I}.
\end{align*}

As shown in [2], $H^*(BQ, \mathbb{Z})$ is generated by $c_1(L_i) = x', c_1(L_j) = y'$ and $c_2(V) = z'$. The above mentioned relations in representations force the relations on these generators. We get for $x', y' \in H^2(BQ, \mathbb{Z})$ and $z' \in H^4(BQ, \mathbb{Z})$,

\begin{equation}
H^*(BQ, \mathbb{Z}) = \mathbb{Z}(x', y', z')/\langle 2x', 2y', 8z', x'^2, y'^2, x'y' - 4z' \rangle. 
\end{equation}

We will first show

$$CH^*(BQ) \cong H^{2*}(BQ, \mathbb{Z}).$$

The argument was outlined in ([4], 1.1.3).

**Lemma 8.1**. $CH^*(BQ) \cong H^{2*}(BQ, \mathbb{Z})$.

**Proof**: The natural map $CH^*(BQ) \to H^{2*}(BQ, \mathbb{Z})$ is an isomorphism for $*=1,2$ by ([15], 3.2). Consider the following commutative diagram of the localisation sequences for the representation $V$.

\[
\begin{array}{c}
CH^1_Q(\{0\}) \xrightarrow{-c_2(V)} CH^i+2_Q(\mathbb{A}^2) \xrightarrow{-e_2(V)} CH^{i+2}_Q(\mathbb{A}^2 - \{0\}) \to 0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
H^2_Q(\{0\}) \xrightarrow{-c_2^H(V)} H^{2i+4}_Q(\mathbb{A}^2) \xrightarrow{-e_2^H(V)} H^{2i+4}_Q(\mathbb{A}^2 - \{0\}) \to 0.
\end{array}
\]

Note that $Q$ acts freely on $\mathbb{A}^2 - \{0\}$, and $(\mathbb{A}^2 - \{0\})/Q$ has the dimension 2. Hence we get that for $i \geq 1$, the map given by multiplication by $c_2(V)$ is a surjective map from $CH^i(BQ) \cong CH^i_Q(\{0\})$ to $CH^i_Q(\{0\})$. We know that the multiplication by $c_2^H(V)$ is always an isomorphism from $H^i_Q(\{0\})$ to $H^{2i+4}_Q(\{0\})$. Hence the commutative diagram reduces to

\[
\begin{array}{c}
CH^1_Q(\{0\}) \xrightarrow{-c_2(V)} CH^{i+2}_Q(\{0\}) \to 0 \\
\downarrow \quad \quad \quad \downarrow \\
H^2_Q(\{0\}) \xrightarrow{-c_2^H(V)} H^{2i+4}_Q(\{0\}) \to 0.
\end{array}
\]

As we already know that the natural map $CH^*(BQ) \to H^{2*}(BQ, \mathbb{Z})$ is an isomorphism for $*=1,2$, we have the result by induction.

\[
\boxed{\text{by \cite{15}}}.
\]

Hence the lemma implies that $\Omega^*(BQ)$ is generated as an $\Omega^*$-module by $c_1(L_i) = x$, $c_1(L_j) = y$ and $c_2(V) = z$. What now remains is to find all the relations on these generators. As before, let $F(a, b) = a + b + \Sigma_{i > 0, j > 0} a_i b_j$ be the formal group law of the complex cobordism. Whenever convenient, we will denote $F(a, b)$ by $a + Fb$. The idea is to understand $MU^*(BQ)$ first, and then show that $\Omega^*(BQ) \to MU^*(BQ)$ is an isomorphism. Mesnaoui in [10] has calculated
$MU^*(BQ)$ explicitly. He has shown that there is an ideal $I$ generated by certain six homogeneous power series in $MU^*[[c_1(L), c_1(L_1), c_2(V)]]$ such that

$$MU^*(BQ) = MU^*[[c_1(L_i), c_1(L_j), c_2(V)]]/I.$$ 

We will make a standard abuse of the splitting principle to explain what these six power series are in a different way. The idea is following. Given a base space $X$, and a rank $n$ vector bundle $p : E \to X$, the projective bundle formula implies that $MU^*(X)$ injects into $MU^*(F(E))$, for the flag bundle of $E$, $q_1 : F(E) \to X$. If $q : P(E) \to X$ is the projectivisation, then there is an exact sequence $0 \to L_1 \to q_1^*(E) \to E_{n-1} \to 0$ for line bundle $L_1$. So $c(q_1^*(E)) = c(L_1)c(E_{n-1})$. Repeating the argument for $E_{n-1}$ and so on, we can claim that there are line bundles $L_i$ over $F(E)$ such that $q^*(E)$ splits as a sum $L_i$’s.

Using this for $V$ over $BQ$, there is a space $p : X \to BQ$ such that $MU^*(BQ)$ injects in $MU^*(X)$ and $c(p^*(V)) = c(M)c(N)$, for some line bundles $M$ and $N$ over $X$, where $c(\cdot)$ is the total Chern class operator. Let $a$ and $b$ respectively denote the pullbacks of $c_1(L_i)$ and $c_1(L_j)$ in $X$ and $m$ and $n$ respectively denote the first Chern classes of $M$ and $N$. We will now use the formal group law to find the relations in cobordism from the relations in representations.

We will start with relation 34. Here, it is interesting to note that for a vector bundle $E$, the equation $c_1(det(E)) = c_1(E)$, which is true in $H^*(-, \mathbb{Z})$ and $CH^*(-)$ does not hold in the cobordism theory in general. If $E$ splits as a sum of line bundles $\oplus_i L_i$, then $det(E) = \oplus_i L_i$. Because $H^*(-, \mathbb{Z})$ and $CH^*(-)$ have the additive formal group law, in those theories we get

$$c_1(\oplus_i L_i) = c_1(\oplus L_i).$$

But the formal group law in the cobordism theory, $F(-, -)$, is not additive and $det(V) = 1$ implies that

$$F(m, n) = 0 \text{ i.e. } m + F n = 0.$$ 

We can transform the equation $F(m, n) = 0$ completely algebraically by using the relations among $a_i$’s to make it look like $m + n = P(mn)$ where $P(-)$ is a formal power series with coefficients in $MU^*$. This is done as follows.

**Lemma 8.2.** Let $MU^*(BGL(1)) = MU^*[[c_1]]$. If $[-1][c_1]$ denotes the inverse of $c_1$ under the formal group law, then there exist a formal power series $P(-) \in MU^*[[c_1]]$ such that $c_1 + [-1][c_1] = P(c_1 \cdot [-1][c_1])$

**Proof:** Let $f : GL(1) \to SL(2)$ be a map given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. Let $e_1$ and $e_2$ be the Chern classes of the standard representation of $SL(2)$. Then $MU^*(BSL(2)) \cong MU^*[[e_2]]$, and let $e_1 = P(e_2)$. As the pullbacks of $e_1$ and $e_2$ are $(c_1 + [-1][c_1])$ and $(c_1 \cdot [-1][c_1])$ respectively, we have the result.

Again note here that $e_1 \in MU^*(BSL(2))$ is not zero. But a fact that all the coefficients of $P(-)$ are actually in $MU^{\leq 0} \cong \Omega^{\leq 0}$ implies that the image of $e_1$ in $H^*(BSL(2))$ is zero and the image of $e_1$ in $CH^*(BSL(2))$ is zero. Hence we have

$$m + n = P(mn).$$

Looking at its image in $MU^*(BQ)$, it implies

$$c_1(V) = P(c_2(V)).$$

The relation 32 implies that $c_1(L_i \otimes V) = c_1(V)$ and $c_2(L_i \otimes V) = c_2(V)$. Which translates as

$$F(a, m) + F(a, n) = m + n \text{ and } F(a, m)F(a, n) = mn.$$ 

$$F(b, m) + F(b, n) = m + n \text{ and } F(b, m)F(b, n) = mn.$$ 

The existence of $P(-)$ already implies that $P(F(a, m)F(a, n)) = P(mn)$, so relations $F(a, m) + F(a, n) = m + n$ and $F(b, m) + F(b, n) = m + n$ are already implied by $F(a, m)F(a, n) = mn$. 


and \( F(b,m)F(b,n) = mn \) respectively. Hence the third relation in representations gives only 2 independent relations in cobordism.

\begin{align}
(43) & \quad F(a,m)F(a,n) = mn. \\
(44) & \quad F(b,m)F(b,n) = mn.
\end{align}

The relation \([31]\) implies that

\begin{align}
(45) & \quad F(a,a) = 0. \\
(46) & \quad F(b,b) = 0.
\end{align}

Relation \([33]\) gives four relations in the cobordism by comparing all 4 Chern classes of the two bundles. Comparing the first Chern class gives

\begin{equation}
(47) \quad a + b + c = [2](m) + [2](n).
\end{equation}

Here by \( c \), we mean pullback of \( c_1(L_k) = F(a,b) \). Comparing the second Chern class gives,

\begin{equation}
(48) \quad ab + bc + ca = [2](m) \cdot [2](n).
\end{equation}

Comparing the third Chern class gives

\begin{equation}
(49) \quad abc = 0.
\end{equation}

As the fourth Chern class of the two vector bundles is zero, it does not give any relation.

We will now prove a higher dimensional version of lemma \([8.2]\) to show that relations \([47]\) and \([49]\) are implied by relation \([48]\).

**Lemma 8.3.** Let \( MU^*(BGL(1)^3) \cong MU^*[r,s,t] \). Let \( u = [-1](r+s+t) \). Let \( d_1 = r+s+t+u, \ d_2 = rs + rt + ru + st + su + tu, \ d_3 = rst + stu + rtu + rsu, \ d_4 = rstu. \) Then there exist formal power series \( P_1(-,-) \) and \( P_3(-,-) \) such that

\[ d_1 = P_1(d_2, d_4) \text{ and } d_3 = P_3(d_2, d_4). \]

**Proof:**

Consider the map \( r : GL(1)^3 \to Sp(2) \) given by

\begin{equation}
(50) \quad (u,v,w) \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} (uvw)^{-1}
\end{equation}

This induces the map \( r^* : MU^*[c_2,c_4] \to MU^*[y_1,y_2,y_3] \). Let the standard representation of \( Sp(2) \) be \( \theta \). We know that \( MU^*(BSp(2)) \cong MU^*[c_2(\theta), c_4(\theta)] \). Let \( c_3(\theta) = P_1(c_2(\theta), c_4(\theta)) \) and \( c_3(\theta) = P_3(c_2(\theta), c_4(\theta)) \). The pullback of \( \theta \) under \( r \) is \( r^*\theta = \phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \varphi \) where \( \phi_i \) is the projection \( GL(1)^3 \to GL(1) \) and \( \varphi \) is the dual of \( \phi_1 \otimes \phi_2 \otimes \phi_3 \).

We have \( c_1(\phi_1) = r, \ c_1(\phi_2) = s, \ c_1(\phi_3) = t \) and \( c_1(\varphi) = u \). Then for each \( i \), the expression \( d_i \) is the \( i \)th Chern of \( r^*\theta \). Hence we have,

\[ d_1 = P_1(d_2, d_4) \text{ and } d_3 = P_3(d_2, d_4). \]

Hence we get

\[ a + b + c = P_1(ab + bc + ca, 0) \text{ and } abc = P_3(ab + bc + ca, 0) \]

\[ [2](m) + [2](n) = P_1([2](m) \cdot [2](n), 0) \text{ and } 0 = P_3([2](m) \cdot [2](n), 0). \]

This implies that relation \([18]\) implies relations \([17]\) and \([19]\).

The sixth and the final relation in the cobordism of \( BQ \) is obtained as follows. Tensoring the two sides of \( L_i + L_j + L_k + 1 = V^2 \) by \( V \) gives that

\[ 4V = V^3. \]

The relation we need is \( c_2(4V) = c_2(V^3). \)
Let \( \eta \) denote the standard representation \( SL(2) \). We have \( MU^*(BSL(2)) \cong MU^*[c_2] \). So let \( c_2(4\eta) = G(c_2) \) and \( c_2(\eta^3) = H(c_2) \) where \( G(-) \) and \( H(-) \) are formal power series. Pulling back \( G(-) \) and \( H(-) \) to \( MU^*(BQ) \) via \( Q \subset SL(2) \), we have
\[
(51) \quad G(c_2(V)) = H(c_2(V))
\]
This gives us the six relations, namely, \[43\], \[44\], \[45\], \[46\], \[47\] and \[51\]. As all these relations are symmetric in \( m \) and \( n \), we can write them in terms of \( a, b, mn \) and \( m + n \). Then using \( m + n = P(mn) \) we can write these relations in terms of \( a, b \) and \( mn \). Taking their images in \( MU^*(BQ) \), i.e. replacing \( a \) by \( x \), \( b \) by \( y \) and \( mn \) by \( z \), we get the six equations in \( MU^*(BQ) \). These are precisely the six power series which generate the ideal \( I \) in \( MU^*[[x, y, z]] \) so that \( MU^*(BQ) \cong MU^*[[x, y, z]]/I \).

**Theorem 8.4.** \( \Omega^*(BQ) \cong MU^*(BQ) \).

**Proof:** We have already established \( MU^*(BQ) \cong MU^*[[x, y, z]]/I \) where \( I \) is generated by the six relations mentioned above. Observe that we can obtain these six relations in \( \Omega^*[[x, y, z]] \) by using exactly the same arguments as above because we already have \( MU^*(BG) \cong \Omega^*(BG) \) for \( G = GL(n), SL(n) \) and \( Sp(n) \). And now we go back to the argument in section \[11\]. Under the map \( \Omega^*(BQ) \to MU^*(BQ) \) a relation \( f(c_1(L_i), c_1(L_j), c_2(V)) = 0 \) in \( \Omega^*(BQ) \) is mapped to a relation \( f(c_1(L_i), c_1(L_j), c_2(V)) = 0 \) in \( MU^*(BQ) \). As there are no relations in \( MU^*(BQ) \) other than the six mentioned above, the same is true in \( \Omega^*(BQ) \), hence
\[
(52) \quad \Omega^*(BQ) \cong MU^*(BQ)
\]

**Remark:** It is complicated to obtain the coefficients of these six power series. But we can at least see what are the images of these relations in \( CH^*(BQ) \cong H^*(BQ) \). The images of \( x, \ y \) and \( z \) in \( H^*(BQ) \) are \( x', \ y' \) and \( z' \) respectively. The images of relations \[43\] and \[44\] are \( x'^2 = y'^2 = 0 \). The images of relations \[45\] and \[46\] are \( 2x' = 2y' = 0 \). The image of relation \[47\] is \( x'y' = 4z' \). And finally the image of relation \[51\] is \( 8z' = 0 \).

The Chow ring of the classifying space of an iterated wreath product \( \mathbb{Z}/p\mathbb{Z}/p\cdots\mathbb{Z}/p \) has been calculated in \[15\]. Let \( D = D_8 \) denote the dihedral group of order 8. Observe that \( D \cong \mathbb{Z}/2\mathbb{Z}/2 \). Hence we have
\[
CH^*(BD) \cong \mathbb{Z}[x_D, y_D, z_D]/(2x_D = 2y_D = 4z_D = 0; x_Dy_D = 2z_D)
\]
where \( x_D, y_D \in CH^1(BD) \) and \( z_D \in CH^2(BD) \). Hence by lemma \[11\] we have the structure of \( \Omega^*(BD) \) as an \( \Omega^* \)-module.

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