Proof Verification Can Be Hard! *

Naveen Sundar Govindarajulu² & Selmer Bringsjord¹,²

Department of Computer Science¹, Department of Cognitive Science²
Rensselaer Polytechnic Institute (RPI)
Troy NY 12180 USA
govinn2@rpi.edu  •  selmer@rpi.edu

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The generally accepted wisdom in computational circles is that pure proof verification is a solved problem and that the computationally hard elements and fertile areas of study lie in proof discovery.¹ This wisdom presumably does hold for conventional proof systems such as first-order logic with a standard proof calculus such as natural deduction or resolution. But this folk belief breaks down when we consider more user-friendly/powerful inference rules. One such rule is the restricted ω-rule, which is not even semi-decidable when added to a standard proof calculus of a nice theory.² While presumably not a novel result, we feel that the hardness of proof verification is under-appreciated in most communities that deal with proofs. A proof-sketch follows.

We set some context before we delve into the sketch. The formal machinery and conventions follow (Ebbinghaus et al. 1984). We assume the standard apparatus of first-order logic and that we are concerned only with theories of arithmetic, and machine checking and discovery of proofs of theorems of arithmetic. A theory Γ is said to be negation-incomplete (incomplete) iff there is at least one ϕ such that Γ ⊢ ϕ and Γ ⊬ ¬ϕ. As readers will recall, Gödel’s first incompleteness theorem states that any sufficiently strong theory of arithmetic that has certain desirable attributes is incomplete. Peano Arithmetic (PA) is one of the smallest incomplete theories that covers all of standard arithmetic. One way to surmount incompleteness is to add more user-friendly (or mathematician-friendly) rules of inference.

The ω-rule is one such rule of inference. The ω-rule can be added in order to complete proof calculi; specifically, the ω-rule renders PA complete. This infinitary rule is of the following form:

\[
\frac{\phi(0), \phi(1), \ldots}{\forall x \, \phi(x)} \text{ ω-rule}
\]

The above rule has an infinite number of premises and is clearly not suitable for implementation. A restricted ω-rule is a finite form of the rule which still keeps PA complete.

Assume that we have machines operating over representations of numerals and proofs. Then if we have a machine m_ϕ, which for all n ∈ ℤ and the formula ϕ with one free variable, it produces a proof of ϕ(n) from some set of axioms Γ. That is, m_ϕ : n → ρ(Γ, φ(n)).³

Given this, one form of the restricted ω-rule is as follows:

\[
\frac{\forall x \, \phi(x)}{m_\phi} \text{ restricted ω-rule}
\]

Though the restricted ω-rule can (as just seen) be written down in full, complete checking of the rule is beyond any machine implementation, since in the general case, a proof verification system that handles the rule would be able to check in all possible cases whether the program supplied halts with the correct proof. A simple proof of this limit is given in the appendix. We feel that this limiting result demonstrates that proof representation and proof verification in mathematics can be a fertile area of study involving a rich interplay between expressibility and computational costs.

¹ This abstract was accepted for a presentation at Computability in Europe 2014: http://cie2014.inf.elte.hu/?Accepted_Papers.
² Conjecture generation in our experience is also commonly regarded to be genuinely difficult.
³ Nice theories are consistent, decidable, and allow representations (Ebbinghaus, Flum & Thomas 1984). Roughly put, if a theory allows representations, it can prove facts about the primitive-recursive relations and functions. (See Smith 2007.) A formal system (a theory Γ and a proof calculus ρ) is decidable/semi-decidable/not-semi-decidable if the decision problem Γ ⊢ ρ γ is decidable/semi-decidable/not-semi-decidable.
³ An accessible reference for the ω-rule is (Baker, Ireland & Smaill 1992). All these results, except the main argument in the present abstract, are available in (Ebbinghaus et al. 1984, Franzén 2004).
Appendix: Proof

**Theorem:** \( (\text{PA}, \rho^{\omega}) \) is not-semi-decidable

**Proof.** Let \( (\text{PA}, \rho^{\omega}) \) denote the formal system comprised of PA with a standard proof calculus \( \rho \) augmented with the restricted \( \omega \)-rule. Assume that we are only talking about Turing machines which output exactly one of \{yes, no\} on all inputs, or else go on forever without halting, e.g., loops. The inputs are numerals which encode natural numbers.

**Given:** \( (\text{PA}, \rho^{\omega}) \) is negation-complete and all its theorems are true on the standard model \( \langle \mathbb{N}; 0, S, +, 1 \rangle \).

The following three statements can be coded up as arithmetic statements in the language of PA.

1. Machine \( m \) on input \( n \) halts with yes
2. Machine \( m \) on input \( n \) halts with no
3. Machine \( m \) on input \( n \) loops

For any machine \( m \) and any input \( n \), exactly one of the above is true in the standard model and therefore a theorem in \( (\text{PA}, \rho^{\omega}) \).

**Assumption 1:** \( (\text{PA}, \rho^{\omega}) \) is semi-decidable. That is, we have a machine \( G \) which on input \( \langle p, q \rangle \) outputs yes if \( p \) represents a proof in \( (\text{PA}, \rho^{\omega}) \) of the statement encoded by \( q \); otherwise outputs no or loops.

If **Assumption 1** holds, then we can have a machine \( H \) which on input \( \langle m, n \rangle \) decides if machine \( m \) halts on input \( n \); i.e., \( H \) solves the halting problem. The machine \( H \) is specified below as an algorithm.

**Algorithm for Machine \( H \)**

**Input:** \( \langle m, n \rangle \)

**Output:** Does \( m \) halt on \( n \)?

**Initialization:**

**init**

\( q_1 = \) “Arithmetic Statement encoding that \( m \) on input \( n \) halts with yes”;

\( q_2 = \) “Arithmetic Statement encoding that \( m \) on input \( n \) halts with no”;

\( q_3 = \) “Arithmetic Statement encoding that \( m \) on input \( n \) does not halt or loops.”;

\( H \) is composed of three parallel threads, exactly one of which halts. If any of the threads halts, \( H \) halts.

**Thread 1**

Do a breadth-first search for a proof \( p \) such that \( G \) on \( \langle p, q_1 \rangle \) halts with yes

**Thread 2**

Do a breadth-first search for a proof \( p \) such that \( G \) on \( \langle p, q_2 \rangle \) halts with yes

**Thread 3**

Do a breadth-first search for a proof \( p \) such that \( G \) on \( \langle p, q_3 \rangle \) halts with yes

**Algorithm 1:** Program \( H \)

**Algorithm for Breadth-First Search for Finding a Proof for \( q \)**

Assume we have an lexicographic ordering of strings \( \langle p_0, p_1, \ldots \rangle \). In the first iteration we run \( G \) on \( \langle p_0, q \rangle \) for one step. In the next iteration, we run \( G \) \( \langle p_0, q \rangle \) for one more step and we also run \( G \) on \( \langle p_1, q \rangle \) for one step. We continue in this fashion until we hit a \( p \) such that \( G \) on \( \langle p, q \rangle \) stops with yes.

**Algorithm 2:** Breadth-First Search for a Proof

One of the three threads in \( H \) will halt. Therefore \( H \) decides the halting problem. We have arrived at a contradiction by supposing **Assumption 1**, which can be now be discarded, and our main thesis is established. ■

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