Infinite dimensional moment map geometry and closed Fedosov’s star products.*

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Abstract

We study the Cahen-Gutt moment map on the space of symplectic connections of a symplectic manifold. Given a Kähler manifold \((M, \omega, J)\), we define a Calabi-type functional \(\mathcal{F}\) on the space \(\mathcal{M}_\Theta\) of Kähler metrics in the class \(\Theta := [\omega]\). We study the space of zeroes of \(\mathcal{F}\). When \((M, \omega, J)\) has non-negative Ricci tensor and \(\omega\) is a zero of \(\mathcal{F}\), we show the space of zeroes of \(\mathcal{F}\) near \(\omega\) has the structure of a smooth finite dimensional submanifold. We give a new motivation, coming from deformation quantization, for the study of moment maps on infinite dimensional spaces. More precisely, we establish a strong link between trace densities for star products (obtained from Fedosov’s type methods) and moment map geometry on infinite dimensional spaces. As a byproduct, we provide, on certain Kähler manifolds, a geometric characterization of a space of Fedosov’s star products that are closed up to order 3 in \(\nu\).

Keywords: Symplectic connections, Moment map, Deformation quantization, closed star products, Kähler manifolds.

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The idea that curvature can be seen as moment map on various infinite dimensional spaces has been a great source of inspiration in mathematics, see Atiyah-Bott [1], Donaldson [11, 10].

We give a new motivation, coming from deformation quantization [4], for the study of moment maps on infinite dimensional spaces. More precisely, we establish a strong link between trace densities for star products (obtained from Fedosov’s type methods) and moment map geometry on infinite dimensional spaces. We study the Cahen-Gutt moment map on the space of symplectic connections [6]. As a byproduct, we provide, on certain Kähler manifolds, a geometric characterization of a space of Fedosov’s star products that are closed up to order 3 in $\nu$.

Consider a symplectic manifold $(M, \omega)$. The space $\mathcal{E}(M, \omega)$ of symplectic connections (torsion-free connection $\nabla$ with $\nabla \omega = 0$) on $(M, \omega)$ is in a natural way, an infinite dimensional symplectic manifold. The group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of $(M, \omega)$ acts symplectically on $\mathcal{E}(M, \omega)$. Cahen and Gutt [6] computed a moment map

$$\mu : \nabla \in \mathcal{E}(M, \omega) \mapsto \mu(\nabla) \in \mathcal{C}^\infty(M)$$

for this action (see Theorem 2.2 below). Fox [16] defines the notion of extremal symplectic connections. Those are connections that are critical points of the squared $L^2$-norm of $\mu$. 

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Let \((M, \omega, J)\) be a closed Kähler manifold and set \(\Theta\) the Kähler class of \(\omega\). Denote by \(\mathcal{M}_\Theta\) the space of Kähler forms in the class \(\Theta\), it is a Fréchet (infinite dimensional) manifold. We define a Calabi type functional \(\mathcal{F}\) on \(\mathcal{M}_\Theta\) that is the square of the \(L^2\)-norm of \(\mu\) (where both the \(L^2\)-norm and \(\mu\) are taken with respect to the point where we are in \(\mathcal{M}_\Theta\)). Our Theorem 1 gives the structure of the space of zeroes of \(\mathcal{F}\) in a neighbourhood of a Kähler metric which is a zero of \(\mathcal{F}\) with Ricci tensor non-negative everywhere.

**Theorem 1.** Assume the closed Kähler manifold \((M, \omega, J)\) has a Ricci tensor which is non-negative everywhere. If \(\omega\) is a zero of \(\mathcal{F}\), then there exists an open neighbourhood \(U\) of \(\omega\) in \(\mathcal{M}_\Theta\) such that the only zeroes of \(\mathcal{F}\) inside \(U\) is the \(H_0(M, J)\)-orbit of \(\omega\), where \(H_0(M, J)\) is the reduced automorphism group of \((M, \omega, J)\).

Deformation quantization as defined in [4] is a formal associative deformation of the Poisson algebra \((C^\infty(M), \{\cdot, \cdot\})\) of a Poisson manifold \((M, \pi)\) in the direction of the Poisson bracket. The deformed algebra is the space \(C^\infty(M)[[\nu]]\) of formal power series of smooth functions with composition law \(*\) called star product. The existence of star products was obtained first in the symplectic case by Dewilde-Lecomte [9], Fedosov [12] and Omori-Maeda-Yoshioka [23] and finally in the Poisson case by Kontsevitch [21].

Star products on symplectic manifolds admit a trace [14, 13, 22, 19] that is a \(\nu\)-linear functional defined on formal series of smooth functions with compact support with values in \(\mathbb{R}[[\nu]]\) that vanishes on the star commutators. A trace for a star product \(*\) is determined by its trace density \(\rho^{(\nu)} \in C^\infty(M)[\nu^{-1}, \nu]] : \)

\[
\text{tr}^\nu(F) = \int_M F \rho^{(\nu)}(\omega^n) \nu^n.
\]

Closed star products in the sense of Connes, Flato, Sternheimer [8] are star products for which \(\rho^{(\nu)} \equiv 1\) is a trace density up to order \(\frac{\dim M}{2}\) in \(\nu\).

We consider star products on symplectic manifolds obtained by the geometric Fedosov’s construction. On a symplectic manifold \((M, \omega)\), for any choice of symplectic connection \(\nabla\) and formal power series \(\Omega\) of closed 2-forms the Fedosov’s method produces a star product, denoted in the sequel by \(*_{\nabla, \Omega}\). Our main observation is that \(\mu(\nabla)\) is the first non-trivial term of a trace density for \(*_{\nabla, 0}\).

The above Theorem [4] gives in fact the local structure of a space of “natural” Fedosov’s star products on a Kähler manifold that are closed up to order 3 in \(\nu\). Let \((M, \omega, J)\) be a closed Kähler manifold with \(\Theta := [\omega]\). To any \(\tilde{\omega} \in \mathcal{M}_\Theta\), one associates the Fedosov’s star product \(*_{\nabla, 0, \tilde{\omega}}\) with \(\nabla\) the corresponding Levi-Civita connection.

**Theorem 2.** Let \((M, \omega, J)\) be a closed Kähler manifold with Levi-Civita connection \(\nabla\) and Ricci tensor everywhere non negative. Assume the Fedosov’s star product \(*_{\nabla, 0}\) is closed up to order 3 in \(\nu\).

Then, there exists an open neighbourhood \(U\) of \(\omega\) in \(\mathcal{M}_\Theta\) such that if \(\tilde{\omega} \in U\) gives rise to the star product \(*_{\nabla, 0, \tilde{\omega}}\) closed up to order 3 in \(\nu\) then there is an \(f \in H_0(M, J)\) inducing an isomorphism

\[
f^* : (C^\infty(M)[[\nu]], *_{\nabla, 0}) \xrightarrow{\cong} (C^\infty(M)[[\nu]], *_{\nabla, 0}).
\]
Finally, we consider more general Fedosov’s star products and the Wick star product obtained by Bordemann-Waldmann [5]. In both case, we observe that the first non-trivial term of a trace density can be interpreted as a moment map on an infinite dimensional manifold.

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2 Symplectic connections

Consider a symplectic manifold \((M,\omega)\) of dimension \(2n\). A symplectic connection \(\nabla\) on \((M,\omega)\) is a torsion-free connection such that \(\nabla \omega = 0\). There always exists a symplectic connection on a symplectic manifold but this connection is not unique. Indeed, given a symplectic connection \(\nabla\) on \((M,\omega)\), any other symplectic connection on \((M,\omega)\) is of the form

\[
\nabla' X Y = \nabla X Y + A(X) Y
\]

where \(A\) is a section of \(\Lambda^1(M) \otimes \text{End}(TM,\omega)\) satisfying \(A(X) Y - A(Y) X = 0\). Equivalently, \(\omega(A(X) Y, Z)\) is a completely symmetric 3-tensor, i.e. a section of \(S^3 T^* M\). Then, the space \(\mathcal{E}(M,\omega)\) of symplectic connections is the affine space

\[
\mathcal{E}(M,\omega) = \nabla + \Gamma(S^3 T^* M)
\]

for some \(\nabla \in \mathcal{E}(M,\omega)\).

We assume from now that \((M,\omega)\) is closed. Then, there is a natural symplectic form on \(\mathcal{E}(M,\omega)\). For \(A, B \in T\mathcal{E}(M,\omega)\), seen as elements of \(\Gamma(S^3 T^* M)\), one defines

\[
\Omega^\mathcal{E}_\nabla (A, B) := 2 \int_M \text{tr}(\hat{\wedge} B) \wedge \frac{\omega^{n-1}}{(n-1)!} = - \int_M \Lambda^{kl} \text{tr}(A(e_k) B(e_l)) \frac{\omega^n}{n!},
\]

where \(\hat{\wedge}\) is the product on \(\Lambda^1(M) \otimes \text{End}(TM,\omega)\) induced by the usual \(\wedge\)-product on forms and the composition on the endomorphism part, \(\Lambda^{kl}\) is defined by \(\Lambda^{kl} \omega_{li} = \delta^k_l\) for \(\omega_{li} := \omega(e_l, e_i)\) for a frame \(\{e_k\}\) of \(T_x M\). The 2-form \(\Omega^\mathcal{E}\) is a symplectic form on \(\mathcal{E}(M,\omega)\).

Remark 2.1. The constant 2 appearing in the definition of \(\Omega^\mathcal{E}\) is introduced to fit with the form defined in [6]. Indeed Cahen and Gutt defined \(\Omega^\mathcal{E}\) for \(A, B \in T\mathcal{E}(M,\omega)\), seen as element of \(\Gamma(S^3 T^* M)\), by

\[
\Omega^\mathcal{E}_\nabla (A, B) := \int_M \Lambda^{i_1j_1} \Lambda^{i_2j_2} \Lambda^{i_3j_3} A_{i_1j_2j_3} B_{j_1j_2j_3} \frac{\omega^n}{n!}.
\]
A positive almost complex structure $J$ compatible with $\omega$ induces a positive almost complex structure $J^E$ compatible with $\Omega^E$ defined by

$$ (J^E(A))(X) := -JA(JX)JY, \quad \forall A \in T_{\nabla}\mathcal{E}(M, \omega). \tag{1} $$

And, as usual, one defines the associated Riemannian metric $G^J$ on $\mathcal{E}(M, \omega)$ by

$$ G^J(A, B) := \Omega(A, J^E B) = \int_M g^{KL} \text{tr}(JA(JB(e_k)) \frac{\omega^n}{n!}). \tag{2} $$

There is a natural symplectic action of the group of symplectic diffeomorphisms on $\mathcal{E}(M, \omega)$. For $\varphi$, a symplectic diffeomorphism, we define an action

$$ (\varphi, \nabla)_X Y := \varphi_* (\nabla_{\varphi^{-1}X} \varphi^{-1} Y), \tag{3} $$

for all $X, Y \in TM$ and $\nabla \in \mathcal{E}(M, \omega)$.

Recall that a Hamiltonian vector field is a vector field $X_F$ for $F \in C^\infty(M)$ such that

$$ i(X_F)\omega = dF. $$

We denote by $\text{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of the symplectic manifold $(M, \omega)$. The group $\text{Ham}(M, \omega)$ is a Lie group with Lie algebra the space $C^\infty_0(M)$ of normalised smooth functions $[3]$, i.e., smooth functions $F$ such that $\int_M F \omega^n = 0$.

The action defined in Equation (3) restricts to an action of the group $\text{Ham}(M, \omega)$. Let $X_F$ be a Hamiltonian vector field with $F \in C^\infty_0(M)$, the fundamental vector field on $\mathcal{E}(M, \omega)$ associated to this action is

$$\begin{align*}
(X_F)^E(Y) Z &:= \frac{d}{dt} \big|_{t=0} \varphi^F_t \cdot \nabla = (\mathcal{L}_{X_F} \nabla)(Y) Z = \nabla^2_{(Y,Z)} X_F + R^\nabla(X_F, Y) Z,
\end{align*}$$

where $R^\nabla(U, V) W := [\nabla_U, \nabla_V] W - \nabla_{[U, V]} W$ is the curvature tensor of $\nabla$. A moment map for the action of $\text{Ham}(M, \omega)$ on $\mathcal{E}(M, \omega)$ is a map $\tilde{\mu} : \mathcal{E}(M, \omega) \to C^\infty(M)$, where $C^\infty(M)$ is viewed, using the $L^2$-product, as a subspace of the dual of $C^\infty_0(M)$, satisfying

$$\frac{d}{dt} \big|_{t=0} \int_M \tilde{\mu}(\nabla + tA) F \frac{\omega^n}{n!} = \Omega^E((X_F)^E, A) = \Omega^E(\mathcal{L}_{X_F} \nabla, A), \tag{4}$$

where $\nabla \in \mathcal{E}(M, \omega)$ and $A \in T_{\nabla}\mathcal{E}(M, \omega)$.

Denote by $\text{Ric}^\nabla$ the Ricci tensor of $\nabla$ defined by

$$\text{Ric}^\nabla(X, Y) := \text{tr}[V \mapsto R^\nabla(V, X) Y]$$

for all $X, Y \in TM$.

**Theorem 2.2** (Cahen-Gutt [13]). The map $\tilde{\mu} : \mathcal{E}(M, \omega) \to C^\infty(M)$ defined by

$$\tilde{\mu}(\nabla) := (\nabla^2_{(e_p, e_q)} \text{Ric}^\nabla)(e^p, e^q) + P(\nabla)$$

where $\{e_k\}$ is a frame of $T_xM$, $\{e^1\}$ is the symplectic dual frame of $\{e_k\}$ (that is $\omega(e_k, e^l) = \delta^l_k$) and $P(\nabla)$ is the function defined by $P(\nabla) \frac{\omega^n}{n!} := \frac{1}{2} \text{tr}(\text{Ric}^\nabla(\ldots) \circ \text{Ric}^\nabla(\ldots)) \circ \frac{\omega^{n-2}}{(n-2)!}$, is a moment map for the action of $\text{Ham}(M, \omega)$ on $\mathcal{E}(M, \omega)$. 

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Remark 2.3. It is possible to extend the above theorem to non compact symplectic manifolds. Of course, the symplectic form $\Omega^E$ is only defined on $L^2$-section of $S^3T^*M$ for the $L^2$-norm induced by $G^J_E$.

Remark 2.4. On a closed symplectic manifold, since $(\nabla^2_{(e_p,e_q)}Ric^\nabla)(e^p,e^q)$ has vanishing integral the constant $\mu_0 := \int_M \tilde{\mu}(\nabla)\frac{\omega^n}{n!}$ is a topological constant. Hence, $\mu_0$ does not depend on the symplectic connection.

Remark 2.5. One can develop the expression of $P(\nabla)$ and obtain

$$\tilde{\mu}(\nabla) := (\nabla^2_{(e_p,e_q)}Ric^\nabla)(e^p,e^q) - \frac{1}{2}Ric^\nabla_{pq}(Ric^\nabla)^{pq} + \frac{1}{4}R^\nabla_{pqrs}(R^\nabla)^{pqrs}$$

where $R^\nabla_{pqrs} := \omega(R^\nabla(e_p,e_q)e_r,e_s)$ for a frame $\{e_k\}$ of $T_x M$ and $(R^\nabla)^{pqrs} := \omega(R^\nabla(e^p,e^q)e^r,e^s)$ for the symplectic dual frame $\{e^l\}$ of $\{e_k\}$.

Definition 2.6. On a closed symplectic manifold $(M,\omega)$, we will call the (normalized) moment map the map $\mu$ defined by

$$\mu : \mathcal{E}(M,\omega) \to C^\infty_0(M) : \nabla \mapsto \tilde{\mu}(\nabla) - \mu_0.$$  

We are interested in zeroes of the normalised moment map $\mu$ when the symplectic manifold is Kähler. On a Kähler manifold $(M,\omega,J)$, one has a particular choice of symplectic connection : the Levi-Civita connection $\nabla$. Using the second Bianchi identity, one has $\Lambda^k_{\ell l}Ric^\nabla(e_k,\cdot) = -\frac{1}{2}dScal^\nabla(J\cdot)$ where $Scal^\nabla$ is the scalar curvature. The moment map reduces to

$$\mu(\nabla) = -\frac{1}{2}\Delta Scal^\nabla - \frac{1}{2}Ric^\nabla_{pq}(Ric^\nabla)^{pq} + \frac{1}{4}R^\nabla_{pqrs}(R^\nabla)^{pqrs} - \mu_0.$$  

Of course the connection is fixed in $\mathcal{E}(M,\omega)$. But one can vary other datas of the Kähler manifold such as the complex structure or the Kähler form. The advantage is that the space of complex structures compatible with $\omega$ and the space of Kähler forms in the same Kähler class as $\omega$ are “smaller” (still infinite dimensional) than $\mathcal{E}(M,\omega)$.

Let us give some examples of zeroes of $\mu$ on Kähler manifolds.

Exemple 2.7. Assume $(M,\omega,J)$ is a Kähler manifold of dimension 2. As pointed out by D.J. Fox [16], the moment map evaluated at the Levi-Civita connection $\nabla$ reduces to

$$\mu(\nabla) = -\frac{1}{2}\Delta Scal^\nabla.$$  

So that $\mu(\nabla) = 0$ if and only if the scalar curvature is constant.

Exemple 2.8. On closed Kähler manifolds with constant holomorphic sectionnal curvature, the Levi-Civita connection provides a zero of the moment map.

In [16], Fox studies the functional on $\mathcal{E}(M,\omega)$ defined by the $L^2$-norm of $\mu$. He describes its critical points when the symplectic manifold has dimension 2.

In Section [1] we consider a Kähler manifold and define a similar functional on the space of Kähler metrics in the same Kähler class and study its zeroes.
3 The space $\mathcal{J}(M, \omega)$

In this section, we consider a closed Kähler manifold $(M, \omega, J)$. We study the Levi-Civita map $\text{lc} : \mathcal{J}_{\text{int}}(M, \omega) \to \mathcal{E}(M, \omega)$ where $\mathcal{J}_{\text{int}}(M, \omega)$ is the space of integrable complex structures on $M$ compatible with $\omega$ and $\text{lc}(\tilde{J}) := \nabla\tilde{J}$ is the Levi-Civita connection of the metric $g_{\tilde{J}}(\cdot, \cdot) := \omega(\cdot, \tilde{J}\cdot)$ with $\tilde{J}$ integrable.

The goal of this section is to obtain conditions implying a certain non-degeneracy of $\text{lc}^*\Omega^E$. This is a technical condition that will show up in our study of zeroes of $\mu$ in Section 4 (see Proposition 4.6).

3.1 The Kähler structure on $\mathcal{J}(M, \omega)$

First, let us describe the space $\mathcal{J}(M, \omega)$ and its tangent space.

**Definition 3.1.** Let $(M, \omega)$ be a symplectic manifold. A compatible almost complex structure $J$ on $(M, \omega)$ is a section of $\text{End}(TM)$ such that $J^2 = -\text{Id}$, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot)$ is a Riemannian metric. We set $\mathcal{J}(M, \omega)$ the space of all almost complex structures compatible with $\omega$.

The tangent space to $\mathcal{J}(M, \omega)$ at a point $J$ is

$$T_J\mathcal{J}(M, \omega) = \{A \in \text{End}(TM, \omega) \text{ s.t. } AJ + JA = 0\}.$$  

The space $\mathcal{J}(M, \omega)$ admits a Kähler structure. For $A, B \in T_J\mathcal{J}(M, \omega)$, we define a metric $G_J$ and a complex structure $I_J$ by:

$$G_J^2(A, B) := \int_M \text{tr}(AB)\frac{\omega^n}{n!} \text{ and } I_J(A) := JA.$$  

The associated Kähler form is

$$\Omega^2_J(A, B) := G_J^2(JA, B) = \int_M \text{tr}(JAB)\frac{\omega^n}{n!}.$$  

The Hamiltonian diffeomorphisms group acts symplectically on $\mathcal{J}(M, \omega)$ by

$$\varphi \cdot J := \varphi_* \circ J \circ \varphi_*^{-1} \text{ for all } \varphi \in \text{Ham}(M, \omega).$$  

Donaldson [11] and Fujiki [17] showed that this action admits a moment map given by the Hermitian scalar curvature. We use this moment map picture in the final section so we briefly recall its construction.

The fundamental vector fields for this action are given by

$$(X_F)^{\mathcal{J}} := \mathcal{L}_{X_F}J, \text{ for } F \in C_0^\infty(M).$$  

Given a compatible $J \in \mathcal{J}(M, \omega)$, consider the Chern connection $\nabla$ on $(TM, J)$. It induces a complex connection $\nabla^{\Lambda^n}$ on $\Lambda^n(TM, J)$ (recall that $n = \frac{\dim M}{2}$). Denote by $R^{(n)}$ the
curvature of $\nabla^{\Lambda_n}$. The Hermitian scalar curvature is the function $\text{Scal}(J)$ (depending on $J$) defined by

$$\text{Scal}(J) := 2.R^{(n)} \wedge \frac{\omega^{n-1}}{(n-1)!}. $$

The Hermitian scalar curvature is a moment map for the action of $\text{Ham}(M, \omega)$ on $\mathcal{J}(M, \omega)$, [11] and [17]. That is, if $A := \frac{d}{dt}|_0 J_t \in T_J \mathcal{J}(M, \omega)$, then for all $F \in C^\infty_0(M)$ we have:

$$\Omega^\mathcal{J}_J((X_F)^*J, A) = \frac{d}{dt}|_0 \int_M F \text{Scal}(J_t) \frac{\omega^n}{n!}. \quad (5)$$

**Definition 3.2.** We denote by $\mathcal{J}_{\text{int}}(M, \omega)$ the subspace of integrable complex structures in $\mathcal{J}(M, \omega)$.

We describe the tangent space of $\mathcal{J}_{\text{int}}(M, \omega)$.

**Proposition 3.3.** Let $J \in \mathcal{J}_{\text{int}}(M, \omega)$, if $A \in T_J \mathcal{J}(M, \omega)$ is tangent to $\mathcal{J}_{\text{int}}(M, \omega)$ then the 2-tensor

$$J(\nabla A(X)Y) - (\nabla A)(JX)Y$$

is symmetric in $X, Y$.

**Proof.** Consider $J_t \in \mathcal{J}_{\text{int}}(M, \omega)$ such that $\frac{d}{dt}|_0 J_t = A$. Because each $J_t$ is integrable, the Nijenhuis tensor $N_{J_t} = 0$. Differentiating this equation at $t = 0$ gives:

$$J(\nabla A(X)Y) - (\nabla A)(JX)Y = J(\nabla A(Y)X) - (\nabla A)(JY)X.$$

Then, the vector fields valued 2-tensor $J\nabla A(X)Y - (\nabla A)(JX)Y$ is symmetric. \qed

### 3.2 The map $\text{lc} : \mathcal{J}_{\text{int}}(M, \omega) \to \mathcal{E}(M, \omega)$

We define the **Levi-Civita map** to be the map

$$\text{lc} : \mathcal{J}_{\text{int}}(M, \omega) \to \mathcal{E}(M, \omega) : J \mapsto \nabla^J$$

which associates to an integrable complex structure $J$ compatible with $\omega$, the Levi-Civita connection $\nabla^J$ of the Kähler metric $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$.

**Lemma 3.4.** The map $\text{lc}$ is equivariant with respect to the group of symplectic diffeomorphisms of $(M, \omega)$. That is:

$$\text{lc}(\varphi \cdot J) = \varphi \cdot \text{lc}(J)$$

for all $\varphi \in \text{Symp}(M, \omega)$ and $J \in \mathcal{J}_{\text{int}}(M, \omega)$.

**Proof.** It is a straightforward computation that $\nabla$ is the Levi-Civita connection of the metric $g_J(\cdot, \cdot) := \omega(\cdot, J \cdot)$ if and only if $\varphi \nabla$ is the Levi-Civita connection of the metric $g_{\varphi \cdot J}(\cdot, \cdot) := \omega(\cdot, \varphi \cdot J \cdot)$. \qed
In the sequel of the paper, we will be concerned by Kähler manifolds satisfying a certain non-degeneracy condition.

**Definition 3.5.** A Kähler manifold \((M, \omega, J)\) is said to satisfy **Condition C** if for \(F \in C^\infty_0(M)\):

\[(lc^*\Omega_E)_J(\mathcal{L}_{X^F}J, J\mathcal{L}_{X^F}J) = 0, \forall H \in C^\infty_0(M) \Rightarrow \mathcal{L}_{X^F}J = 0.\]

**Remark 3.6.** The condition \(\mathcal{L}_{X^F}J = 0\) is equivalent to \(\mathcal{L}_{X^F}\nabla = 0\) (i.e. \(lc_{s,J}(\mathcal{L}_{X^F}J) = 0\)).

We now exhibit a geometric condition in term of the Ricci tensor of the Kähler manifold which implies the above Condition C. For this, we study carefully the differential of the Levi-Civita map: \(lc_{s,J} : T_J\mathcal{J}_{int}(M, \omega) \rightarrow T_{lc(J)}E(M, \omega)\).

**Proposition 3.7.** Let \(A \in T_J\mathcal{J}_{int}(M, \omega)\) and write \(B \in T_\nabla E(M, \omega)\) such that \(B = lc_{s,J}(A)\).

Then:

1. \(B\) is the unique solution to the equation

\[B(X)Y + JB(X)JY = -\nabla JAX(Y).\]

2. if \(JA \in T_J\mathcal{J}_{int}(M, \omega)\), then:

\[lc_{s,J}(JA)(X)Y = -J^E B(X)Y + \frac{1}{2}(J(\nabla A)(JX)Y) + (\nabla A)(X)Y.\]

**Proof.** Consider \(J_t\) in \(\mathcal{J}_{int}(M, \omega)\) with \(\frac{d}{dt}|_0 J_t = A\). Set \(\nabla_t := lc(J_t)\), we want to compute \(B := \frac{d}{dt}|_0 lc(J_t)\). By definition, \(\nabla_t J_t = 0\). Differentiating this relation at \(t = 0\) gives:

\[0 = \frac{d}{dt}|_0 (\nabla_t J_t)(X)Y = B(X)Y - J(B(X)Y) + (\nabla A)(X)Y. \tag{6}\]

Suppose that there are two solutions \(B\) and \(B'\) of \((\ref{6})\). The difference \(C := B - B'\) satisfies \(C(X)Y - J(C(X)Y) = 0\). It means that the connection \(\nabla + C\) preserves the complex structure \(J\). Moreover, \(\nabla + C\) is symplectic. So that, \(\nabla\) and \(\nabla + C\) are torsion free connections preserving \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\). It implies \(C = 0\).

For the second statement, set

\[\tilde{B}(X)Y := -J^E B(X)Y + \frac{1}{2}(J(\nabla A)(JX)Y) + (\nabla A)(X)Y.\]

By Proposition 3.5, \(\tilde{B}(X)Y\) is symmetric in \(X\) and \(Y\). One checks by direct computations that \(\tilde{B}(X)\) sits in \(End(TM, \omega)\). Moreover,

\[J\tilde{B}(X)Y - \tilde{B}(X)JY = \nabla(JA)(X)Y.\]

It implies \(\tilde{B} = lc_{s,J}(JA)\). \qed
Remark 3.8. The formula of point 2 in the above Proposition 3.7 gives the defect for lc\(_s,J\) to be complex anti-linear for the complex structures \(I_J\) on \(T_J\mathcal{J}_{int}(M,\omega)\) and \(J^\mathcal{E}\) on \(T_{\text{lc}(J)}\mathcal{E}(M,\omega)\).

Proposition 3.9. Let \((M,\omega,J)\) be a Kähler manifold. If the Ricci tensor \(\text{Ric}^\mathcal{E}\) is everywhere non-negative (i.e. \(\forall x \in M : \text{Ric}^\mathcal{E}(X_x,X_x) \geq 0 \forall X_x \in T_xM\)), then \((M,\omega,J)\) satisfies the Condition C.

Proof. Consider \(A\) and \(JA \in T_J\mathcal{J}_{int}(M,\omega)\). We show that \(\Omega_{\text{lc}(J)}^E(lc_sJ(A),lc_sJ(JA)) \leq 0\) with equality if and only if \(lc_sJ(A) = 0\).

Write \(B = lc_sJ(A)\) and \(\tilde{B} := lc_sJ(JA)\). By Proposition 3.7, we know the \(J\)-anti-linear part of \(B\) and \(\tilde{B}\). We write

\[
B(X)Y = \frac{1}{2}(-\nabla(JA)(X)Y + B(X)^JY), \tag{7}
\]

\[
\tilde{B}(X)Y = \frac{1}{2}((\nabla A)(X)Y) + \tilde{B}(X)^JY. \tag{8}
\]

where \(B(X)^J\) and \(\tilde{B}(X)^J\) denotes the \(J\)-linear part of the fields of endomorphisms \(B(X)\) and \(\tilde{B}(X)\). Again by Proposition 3.7 one checks that \(\tilde{B}(X)^J = -B(JX)^J\). We can now compute

\[
(lc^E\Omega_{\text{lc}(J)})_J(A,JA) = \Omega_{\text{lc}(J)}^E(B,\tilde{B}),
\]

\[
= \frac{1}{4} \int_M \Lambda^{kl} \text{tr}((\nabla JA)(e_k)(\nabla A(e_l)))\frac{\omega^n}{n!} + \int_M \Lambda^{kl} \text{tr}(B(e_k)^J B(Je_l)^J)\frac{\omega^n}{n!},
\]

\[
= \frac{1}{4} \int_M \Lambda^{kl} \text{tr}((\nabla JA)(e_k)(\nabla A(e_l)))\frac{\omega^n}{n!} + \int_M g^{kl} \text{tr}(B(e_k)^J B(e_l)^J)\frac{\omega^n}{n!}. \tag{9}
\]

The second term of the last equality is negative definite because the trace induces a negative definite bilinear form on \(J\)-linear endomorphisms of \((TM,\omega)\). It remains to deal with the first term.

\[
\int_M \Lambda^{kl} \text{tr}((\nabla JA)(e_k)(\nabla A(e_l)))\frac{\omega^n}{n!} = \int_M \text{tr}(JA\Lambda^{lk} R(e_k,A)(e_l))\frac{\omega^n}{n!},
\]

\[
= \frac{1}{2} \int_M \text{tr}(JA\Lambda^{lk} R(e_k,A))\frac{\omega^n}{n!},
\]

\[
= \int_M \text{tr}(JA\rho A)\frac{\omega^n}{n!} - \int_M \text{tr}(JAA\rho)\frac{\omega^n}{n!}
\]

\[
= 2 \int_M \text{tr}(JA\rho A)\frac{\omega^n}{n!},
\]

where \(\rho \in \text{End}(TM,\omega)\) denotes the Ricci endomorphism defined by \(\omega(\rho X,Y) := \text{Ric}(X,Y)\). One has

\[
\text{tr}(JAA\rho A) = \Lambda^{lk} \omega(JA\rho Ae_k,e_l) = \Lambda^{lk} \omega(JAAe_l,\rho Ae_k) = -2g^{kl} \text{Ric}(Ae_k,Ae_l) \leq 0.
\]
So that \( \Omega^E (lc_\ast J(A), lc_\ast J(A)) \leq 0 \). Moreover, if \( \Omega^E (lc_\ast J(A), lc_\ast J(A)) = 0 \), then \( B(X)^J = 0 \) and also \( \tilde{B}(X)^J = 0 \) for all \( X \). But, this implies the tensor \( (\nabla A)(X,Y) \) to be symmetric in \( X,Y \). Hence, we have

\[
0 = (lc^\ast \Omega^E)_J(A, JA) = \frac{1}{4} \int_M g^{kl} \text{tr}((\nabla A)(e_k)(\nabla A(e_l))).
\]

Because the trace is positive definite on \( J \)-anti-linear endomorphisms, it implies \( \nabla A = 0 \) and thus \( lc_\ast J(A) = 0 \).

Now, we have to conclude that \((M, \omega, J)\) satisfies the Condition C. We consider \( A = \mathcal{L}_{X_f} J \) for \( F \in C^\infty_0(M) \) such that \( \Omega^E (lc_\ast J, \mathcal{L}_{X_f} J, lc_\ast J(\mathcal{L}_{X_f} J)) = 0 \) for all \( H \in C^\infty_0(M) \). Taking \( H = F \), we showed above that necessarily \( lc_\ast J(\mathcal{L}_{X_f} J) = 0 \). By the equivariance of the map \( lc \), we have \( \mathcal{L}_{X_f} \nabla = 0 \). Then, by remark 3.6 it means \( \mathcal{L}_{X_f} J = 0 \). \( \square \)

**Lemma 3.10.** The 2-form \( lc^\ast \Omega^E \) is \( I \)-invariant, \( I \) being the complex structure on \( \mathcal{J}_{\text{int}}(M, \omega) \). That is if \( A, A' \) and \( JA, JA' \in T_J \mathcal{J}_{\text{int}}(M, \omega) \) then

\[
(lc^\ast \Omega^E)_J(JA, JA') = (lc^\ast \Omega^E)_J(A, A').
\]

**Proof.** From the equations 7 and 8, one checks directly that \( lc^\ast \Omega^E \) is \( I \)-invariant. \( \square \)

**Remark 3.11.** By the Lemma 3.10 the Condition C is equivalent to : for \( F \in C^\infty_0(M) \):

\[
(lc^\ast \Omega^E)_J(\mathcal{L}_{X_f} J, J \mathcal{L}_{X_f} J) = 0, \forall H \in C^\infty_0(M) \Rightarrow \mathcal{L}_{X_f} J = 0.
\]

**Remark 3.12.** The Proposition 3.9 (and its proof) and the Lemma 3.10 means that : if \((M, \omega, J)\) has Ricci tensor positive definite at all points of \( M \), then the 2-form \( (lc^\ast \Omega^E)_J \) restricted on the subspace

\[
\mathcal{T} := \{ \mathcal{L}_{X_f} J + J \mathcal{L}_{X_f} J \mid F, H \in C^\infty_0(M) \} \leq T_J \mathcal{J}_{\text{int}}(M, \omega)
\]

is a symplectic form and \( I_J \) is a compatible negative complex structure with \( (lc^\ast \Omega^E)_J \).

The Condition C is stable by complex transformation.

**Proposition 3.13.** Assume the closed Kähler manifold \((M, \omega, J)\) satisfies the Condition C. Let \( f \) be a diffeomorphism of \( M \) preserving \( J \) (i.e. \( f_\ast \circ J = J \circ f_\ast \)), then the Kähler manifold \((M, f^\ast \omega, J)\) also satisfies the Condition C.

**Proof.** Let \( \tilde{J} \in \mathcal{J}_{\text{int}}(M, \omega) \), define \( f^{-1} \circ \tilde{J} := f^{-1}_\ast \circ \tilde{J} \). Then, \( f^{-1} \circ \tilde{J} \in \mathcal{J}_{\text{int}}(M, f^\ast \omega) \). Because it is clearly invertible, \( f^{-1} \) induces a bijection \( f^{-1} : \mathcal{J}_{\text{int}}(M, f^\ast \omega) \to \mathcal{J}_{\text{int}}(M, \omega) \). Its differential acts on \( A \in T_J \mathcal{J}_{\text{int}}(M, \omega) \) by \((f^{-1})_\ast_J(A) = f^{-1}_\ast A \circ f_\ast \). So that, using \( f^{-1} \circ J = J \) we have for \( F \in C^\infty(M) \):

\[
(f^{-1})_\ast_J(\mathcal{L}_{X_f} J) = \mathcal{L}_{f^{-1} X_{f^\ast}} J = \mathcal{L}_{X_{f^\ast f^{-1}}} J,
\]

where we keep track of the symplectic form with respect to the Hamiltonian vector fields are taken from.
In the same vein, \( f^{-1} \) induces a symplectomorphism
\[
f^{-1} : (\mathcal{E}(M, \omega), \Omega^{E(M,\omega)}) \to (\mathcal{E}(M, f^*\omega), \Omega^{E(M,f^*\omega)}),
\]
defined by \((f^{-1}.\nabla)_X Y := f_*^{-1}(\nabla_{f_*X} f_*Y)\). Moreover, for \( \tilde{J} \) in \( \mathcal{J}_{int}(M, \omega) \), one checks that \( f^*\omega(\cdot, f^{-1}.\tilde{J}) = f^*g^\tilde{J}(\cdot, \cdot) \). So that, \( f^{-1} \) commutes with the Levi-Civita maps:
\[
f^{-1} \circ \text{lc} = \text{lc}' \circ f^{-1}.
\]
where \( \text{lc}' : \mathcal{J}_{int}(M, f^*\omega) \to \mathcal{E}(M, f^*\omega) \) is the Levi-Civita map.

All this proves that
\[
(\text{lc}'^*\Omega^{E(M,f^*\omega)})_J \left( \mathcal{L}_{X^\perp} f^*\omega, J \mathcal{L}_{X^\perp} f^*\omega, J \right) = (\text{lc}^*\Omega^{E(M,\omega)})_J \left( \mathcal{L}_{X^\perp} f^*\omega, J \mathcal{L}_{X^\perp} f^*\omega, J \right).
\]
Then \((M, f^*\omega, J)\) also satisfies the Condition C, as desired.

4 A Calabi type functional on the space of Kähler metrics

We study the zeroes of the moment map \( \mu \) on a closed Kähler manifold \((M, \omega, J)\). Instead of looking for zeroes that are symplectic connections with respect to the fixed symplectic form \( \omega \), we will consider Levi-Civita connections associated to the Kähler forms \( \tilde{\omega} \) in the same Kähler class than \( \omega \).

4.1 The norm squared moment map functional

We consider a closed Kähler manifold \((M, \omega, J)\). Let \( \Theta \) be the Kähler class of \( \omega \) and denote by \( \mathcal{M}_\Theta \) the set of Kähler forms in the class \( \Theta \). By the classical \( dd^c \)-lemma,
\[
\mathcal{M}_\Theta := \{ \omega_\phi = \omega + dd^c \phi \text{ s.t. } \phi \in C_0^\infty(M), \omega_\phi(\cdot, J\cdot) \text{ is positive definite } \},
\]
where \( dd^c F := -dF \circ J \). The condition \( \omega_\phi(\cdot, J\cdot) \) being positive definite is open. Then, \( \mathcal{M}_\Theta \) is a Fréchet manifold modeled on \( C_0^\infty(M) \).

Definition 4.1. We define the \( \mathcal{F} \)-functional to be the map
\[
\mathcal{F} : \mathcal{M}_\Theta \to \mathbb{R} : \omega_\phi \mapsto \mathcal{F}(\omega_\phi) := \int_M (\mu^\phi(\nabla^\phi))^2 \omega^\phi_{n!},
\]
where \( \nabla^\phi \) is the Levi-Civita connection of the Kähler metric \( g_\phi(\cdot, \cdot) := \omega_\phi(\cdot, J\cdot) \) and \( \mu^\phi \) is the normalised moment map on the space \( \mathcal{E}(M, \omega_\phi) \) described in Section 2.
This new point of view of varying the Kähler form in a fixed Kähler class is intimately related to the one of varying the symplectic connection of the fixed symplectic form ω. This relation is the key of the remainder of the paper.

Consider a smooth one-parameter family φ : ] − ε, ε[ → C_0^∞(M) : t ↦ φ(t) for some ε ∈ ℝ⁺ such that the 2-form ω_φ(t) := ω + dd^cφ(t) is a smooth path inside M_∅. All the forms ω_φ(t) are symplectomorphic to each other. Indeed, set X_t := −grad_φ(t)(φ) the gradient vector field of φ(t) with respect to g_φ(t) (that is g_φ(t)(grad_φ(t)(φ), ·) = dφ). Then the one parameter family of diffeomorphisms f_t integrating the time-dependent vector field X_t satisfies

\[ f_t^*ω_φ(t) = ω. \] (9)

Consider f_t as in the above equation (9). Then, the natural action of f_t⁻¹ on J produces a path

\[ J_t := f_t⁻¹ J := f_t⁻¹ J f_t \in J_{int}(M, ω). \]

Define the associated Kähler metric \( g_{J_t}(·, ·) := ω(·, J_t·) \) and denote by \( \nabla^{J_t} \) its Levi-Civita connection. Then, \( \nabla^{J_t} \) and \( \nabla^{φ(t)} \) are related by the following formula:

**Proposition 4.2.** With the above notations, we have that \( \nabla^{J_t} \in \mathcal{E}(M, ω) \) and

\[ \nabla^{J_t} = f_t⁻¹ \nabla^{φ(t)}, \]

where \( (f_t⁻¹ \nabla^{φ(t)})YZ = f_t⁻¹ \nabla^{φ(t)}_{f_t Y} f_t Z \).

**Proof.** Because \( (M, ω, J_t) \) is Kähler, then \( \nabla^{J_t} \) preserves \( ω \). The equation \( \nabla^{J_t} = f_t⁻¹ \nabla^{φ(t)} \), follows from the observation that \( g_{J_t} = f_t^* g_{φ(t)} \).

**Lemme 4.3.** Let \( φ : ] − ε, ε[ → C_0^∞(M) : t ↦ φ(t) \) for some \( ε ∈ ℝ^+_0 \) be a smooth map such that the 2-form \( ω_φ(t) := ω + dd^cφ(t) \) belongs to \( M_∅ \). If \( f_t \) is the family of diffeomorphisms defined by equation (7). Denote by \( \nabla^{φ(t)} \) (resp. \( \nabla^{J_t} \)) the Levi-Civita connection of the Kähler metric \( g_{φ(t)} \) (resp. \( g_{J_t} \)). Then,

\[ μ(\nabla^{J_t}) = f_t^* μ(φ(t))(\nabla^{φ(t)}). \]

**Proof.** By Proposition 4.2, we have \( \nabla^{J_t} = f_t⁻¹ \nabla^{φ(t)} \). In terms of the curvature tensors, it means that

\[ R^{\nabla^{J_t}} = f_t^* R^{\nabla^{φ(t)}}. \]

Since \( f_t^* ω_φ(t) = ω \), a direct computation leads to

\[ μ(\nabla^{J_t}) = f_t^* μ(φ(t))(\nabla^{φ(t)}). \]

The above equation implies that the integral

\[ \int_M μ(φ(t))(\nabla^{φ(t)}) \frac{ω^n_φ(t)}{n!} \]

does not depend on \( φ(t) \). So that the normalised moment maps satisfy

\[ μ(\nabla^{J_t}) = f_t^* μ(φ(t))(\nabla^{φ(t)}). \]

\[ \square \]
4.2 An associated elliptic operator

We show the problem of finding zeroes of $\mu^\phi(\nabla^\phi)$ on $\mathcal{M}_\Theta$ is an elliptic partial differential problem. We mean that its linearization is given by an elliptic differential operator with smooth coefficients and with leading term the operator $\Delta^3$, the cube of the Laplacian.

**Proposition 4.4.** *The moment map $\mu^\phi(\nabla^\phi)$ depends analytically on the Kähler potential $\phi$ and its derivatives up to order 6.*

**Proof.** Recall that

$$\mu^\phi(\nabla^\phi) = -\frac{1}{2}\Delta^\phi \text{Scal}^\nabla^\phi - \frac{1}{2}R_{pq} (\text{Ric}^\nabla^\phi)^pq + \frac{1}{4}R_{pqrs} (R^\nabla^\phi)^{pqrs} - \mu_0.$$  

The curvature tensor $R^{\nabla^\phi}$ depends analytically on the Kähler potential $\phi$ and its derivatives up to order four. Indices are raised using the inverse of the Kähler form which depends analytically on $\phi$ and its derivatives up to order 2. The leading term $-\frac{1}{2}\Delta^\phi \text{Scal}^\nabla^\phi$ involves derivatives of the Kähler potential up to order 6.

**Definition 4.5.** Let $\omega + dd^c \phi \in \mathcal{M}_\Theta$. We define the operator $D^\phi : C^\infty(M) \to C^\infty(M)$ by:

$$D^\phi(\psi) := \frac{d}{dt}|_0 \mu^\phi(t\psi)(\nabla^\phi+t\psi)$$  (10)

for any $\psi \in C^\infty(M)$

**Proposition 4.6.**

1. The operator $D^\phi$ is an elliptic partial differential operator of order 6 null on constants with smooth coefficients depending analytically on the derivatives of $\phi$ up to order 6.

2. If $(M,\omega_\phi,J)$ satisfies the condition (C) and $\mu^\phi(\nabla^\phi) = 0$, then $\psi \in C^\infty(M)$ is in the kernel of $D^\phi$ if and only if $\mathcal{L}_{X^\phi_\psi} J = 0$, where $X^\phi_\psi$ denotes the Hamiltonian vector field of $\psi$ with respect to $\omega_\phi$.

**Proof.**

1. From Proposition 4.4 we deduce the analytic dependence of the coefficient of $D^\phi$ on $\phi$ and its derivatives up to order 6. The leading term of $D^\phi$ occurs in $\frac{d}{dt}|_0 \Delta^\phi+t\psi \text{Scal}^\nabla^\phi+t\psi$. Recall that

$$\frac{d}{dt}|_0 \text{Scal}^\nabla^\phi+t\psi = -(\Delta^\phi)^2 \psi - (dd^c \psi)pq (\rho^\nabla^\phi)^{pq},$$

$$\frac{d}{dt}|_0 \Delta^\phi+t\psi F = (dd^c F)pq (dd^c \psi)^{pq},$$

where $\rho^\nabla^\phi(\cdot,\cdot) := \text{Ric}^\nabla^\phi(J\cdot,\cdot)$ is the Ricci form and indices are rised using the symplectic form $\omega_\phi$. Then, we have

$$D^\phi \psi = \frac{1}{2}(\Delta^\phi)^3 \psi + \text{lower order terms}.$$  

It means $D^\phi$ is an elliptic operator.
2. We assume $\phi = 0$ for simplicity. By hypothesis we know $(M, \omega, J)$ satisfies the condition (C) and $\mu(\nabla) = 0$. We compute

$$\int_M D_\psi \psi_n \omega^n \frac{1}{n!} = \frac{d}{dt} |_0 \int_M \mu (\nabla^2 \psi) \psi_2 \omega^n \frac{1}{n!}$$

for $\psi_1, \psi_2 \in C^\infty(M)$ (where $D$ stands for $D^\phi$ with $\phi = 0$). Set $f_t$ the family of diffeomorphisms of $M$ generated by $X_t := -\text{grad}^\psi(\psi_1)$ so that $f_t^* \omega_{\psi_1} = \omega$ for small $t$. Then, by Lemma 4.3,

$$\int_M D_\psi \psi_2 \omega^n \frac{1}{n!} = \frac{d}{dt} |_0 \int_M (f_t^{-1})^* \mu (\nabla^J) \psi_2 \omega^n \frac{1}{n!},$$

for the family of complex structures $J_t := (f_t^{-1})^*J$. Using the moment map property of $\mu$ and the fact that $\mu(\nabla) = 0$, we obtain :

$$\frac{d}{dt} |_0 \int_M (f_t^{-1})^* \mu (\nabla^J) \psi_2 \omega^n \frac{1}{n!} = \Omega(\mathcal{L}_{X_{\psi_2}}, \psi, \frac{d}{dt} |_0 \nabla^J).$$

Now, $\nabla^J_t = \text{lc}(J_t)$ and $J_t = f_t^{-1}J_t f_t$. So that, $\frac{d}{dt} J_t = \mathcal{L}_{f_t^{-1}X_t} J_t$ and, by direct computation, one checks that $f_t^{-1}X_t = -J_tX_{\psi_1}$. So that $\frac{d}{dt} J_t = -\mathcal{L}_{J_tX_{\psi_1}} J_t = -J_t \mathcal{L}_{X_{\psi_1}} J_t$. Then, we have

$$\Omega(\mathcal{L}_{X_{\psi_2}}, \psi, \frac{d}{dt} |_0 \nabla^J) = -\Omega(\mathcal{L}_{X_{\psi_2}}, \psi, \frac{d}{dt} |_0 \nabla^J, \mathcal{L}_{X_{\psi_1}} J_t).$$

Because $(M, \omega, J)$ satisfies the condition C, we see that $\int_M D_\psi \psi_2 \omega^n \frac{1}{n!} = 0$ for all $\psi_2$ if and only if $\mathcal{L}_{X_{\psi_1}} J_t = 0$.

\[\square\]

**Exemple 4.7.** Consider the flat torus $(\mathbb{C}^n/\mathbb{Z}^{2n}, \omega_{\text{std}}, i)$ with its flat connection $\nabla$. From the computations of point 1 in the above proof, at $\omega = \omega_{\text{std}}$, we have

$$D^\phi \psi = \frac{1}{2}(\Delta^\phi)^2 \psi.$$

**Definition 4.8.** The operator $(D^\phi)^*$ is the formal adjoint of the operator $D^\phi$ with respect to the Kähler form $\omega_{\phi}$. That is $(D^\phi)^*$ is defined by the equation

$$\int_M F D^\phi G \omega_{\phi} \frac{n}{n!} = \int_M (D^\phi)^* F G \omega_{\phi} \frac{n}{n!}.$$

Because $(D^\phi)^*$ is the formal adjoint of the operator $D^\phi$, the following Proposition is obvious.

**Proposition 4.9.** The operator $(D^\phi)^*$ is an elliptic partial differential operators of order 6 with smooth coefficients depending analytically on the derivatives of $\phi$ up to order 12.
4.3 Manifold structure for $\mathcal{F}^{-1}(0)$

We analyse the structure of the space of zeroes of $\mathcal{F}$ or equivalently the space of $\omega_\phi \in \mathcal{M}_\Theta$ such that $\mu^\phi(\nabla^\phi) = 0$, when $(M, \omega_\phi, J)$ satisfies the condition (C). The key property is that the Hessian of $\mathcal{F}$ at a zero is given by a non-negative elliptic operator.

**Proposition 4.10.** The map $\mathcal{F} : \mathcal{M}_\Theta \to \mathbb{R}$ is smooth and its differential at $\omega_\phi$ evaluated at $\psi \in T_{\omega_\phi} \mathcal{M}_\Theta \simeq C_0^\infty(M)$ is

$$d\mathcal{F}_\phi(\psi) := \int_M (2(D^\phi)^* \mu^\phi(\nabla^\phi) - \Delta^\phi(\mu^\phi(\nabla^\phi))^2) \psi \frac{\omega^\phi_n}{n!}.$$ 

Moreover, if $\omega_\phi$ is a zero of $\mathcal{F}$, then it is a critical point of $\mathcal{F}$ and the Hessian $d^2\mathcal{F}_\phi$ of $\mathcal{F}$ at $\omega_\phi$ is given by

$$d^2\mathcal{F}_\phi(\psi, \psi) := \int_M (2(D^\phi)^* D^\phi \psi_1) \psi_2 \frac{\omega^\phi_n}{n!},$$

for $\psi_1, \psi_2 \in C_0^\infty(M)$

**Proof.** The fact that $\mathcal{F}$ is smooth directly follows from Proposition 4.6.

Now, we compute the differential of $\mathcal{F}$:

$$\frac{d}{dt} \frac{d}{dt} \mathcal{F}(\omega_{\phi+tv}) = 2 \int_M \left( \frac{d}{dt} \mu^\phi_{t} \mu^\phi_{t}(\nabla^\phi_{t} \psi) \right) \mu^\phi(\nabla^\phi) \frac{\omega^\phi_n}{n!} + \int_M \mu^\phi(\nabla^\phi)^2 \frac{d}{dt} \mu^\phi_{t \psi} \frac{\omega^\phi_n}{n!}$$

$$= 2 \int_M (D^\phi \psi) \mu^\phi(\nabla^\phi) \frac{\omega^\phi_n}{n!} - \int_M \mu^\phi(\nabla^\phi)^2 \Delta^\phi \psi \frac{\omega^\phi_n}{n!},$$

$$= \int_M (2(D^\phi)^* \mu^\phi(\nabla^\phi) - \Delta^\phi(\mu^\phi(\nabla^\phi))^2) \psi \frac{\omega^\phi_n}{n!}.$$

Because it is symmetric, the Hessian of $\mathcal{F}$ is determined by $d^2\mathcal{F}_\phi(\psi, \psi)$ for $\psi \in C_0^\infty(M)$. We compute

$$d^2\mathcal{F}_\phi(\psi, \psi) = \frac{d^2}{dt^2} \left. \mathcal{F}(\omega_{\phi+tv}) \right|_{t=0}$$

$$= \frac{d}{dt} \int_M (2(D^\phi_{t+\psi})^* \mu^\phi_{t+\psi}(\nabla^\phi_{t+\psi}) - \Delta^\phi_{t+\psi}(\mu^\phi_{t+\psi}(\nabla^\phi_{t+\psi}))^2) \psi \frac{\omega^\phi_n}{n!}.$$ 

Since $\mu^\phi(\nabla^\phi) = 0$, we have

$$d^2\mathcal{F}_\phi(\psi, \psi) = \int_M (2(D^\phi)^* \frac{d}{dt} \mu^\phi_{t+\psi}(\nabla^\phi_{t+\psi}) \psi \frac{\omega^\phi_n}{n!} = \int_M (2(D^\phi)^* D^\phi \psi) \psi \frac{\omega^\phi_n}{n!}.$$ 

$\square$
From now on, we will work in a neighbourhood of a given $\omega \in \mathcal{M}_\Theta$. Let $U \subset C^\infty_0(M)$ be a convex neighbourhood of the origin such that if $\phi \in U$ then $\omega_\phi \in \mathcal{M}_\Theta$. Then $d\mathcal{F}$ induces a smooth map of Fréchet spaces

$$\tilde{d\mathcal{F}} : U \to C^\infty_0(M)$$

defined for $\phi \in U$ by

$$\tilde{d\mathcal{F}}(\phi) := 2(D^\phi)^* \mu^\phi(\nabla^\phi) - \Delta^\phi(\mu^\phi(\nabla^\phi))^2 - \int_M 2(D^\phi)^* \mu^\phi(\nabla^\phi) - \Delta^\phi(\mu^\phi(\nabla^\phi))^2 \omega^n.$$

Moreover, from Proposition 4.10, $\omega_\phi$ with $\phi \in U$ is a critical point of $\mathcal{F}$ if and only if $d\mathcal{F}(\phi) \equiv 0$.

We will now extend the map $\tilde{d\mathcal{F}}$ to a smooth map defined on suitable Sobolev spaces. Denote by $\|\cdot\|_l$ the $L^2$-norm induced by the Kähler metric $g$ on $C^\infty(M)$. Let $l > 0$ be an integer, for $K \in C^\infty(M)$, $\nabla^l K$ is a section of $T^*M^\otimes l$. The Kähler metric $g$ induces a scalar product and then a $L^2$-norm $\|\cdot\|_{g,l}$ on $\Gamma(T^*M^\otimes l)$ we define the $l$-th Sobolev norm $\|\cdot\|_l$ on $C^\infty(M)$ by:

$$\|H\|_l := \left(\|H\|_2^2 + \sum_{j=1}^l \|\nabla^j H\|_{g,j}^2\right)^{\frac{1}{2}}.$$ 

The $l$-th Sobolev space $H^l(M)$ is defined to be the completion of $C^\infty(M)$ for the $l$-th Sobolev norm. The $H^l(M)$ are Hilbert spaces. The subspaces $H^l_0(M)$ denote the closure of $C^\infty_0(M)$ in $H^l(M)$, they are also Hilbert spaces.

**Proposition 4.11.** For $l \geq \frac{\dim(M)}{2} + 12$, the map $\tilde{d\mathcal{F}}$ extends to a smooth map

$$\tilde{d\mathcal{F}} : \tilde{U} \subset H^l_0(M) \to H^{l-12}_0(M),$$

for $\tilde{U}$ a neighbourhood of the origin in $H^l_0(M)$.

**Proof.** When $l \geq \frac{\dim(M)}{2} + 12$, the Sobolev embedding’s Theorem states the inclusions $H^l(M) \hookrightarrow C^{12}(M)$ and $H^{l-12}_0(M) \hookrightarrow C^0(M)$ are continuous, see for example [2]. Since $	ilde{d\mathcal{F}}$ is continuous for the Fréchet topology on $U$, it extends to a continuous map defined on a neighbourhood $V$ of the origin in $C^{12}(M)$ with value in $C^0(M)$. By Propositions 4.6 and 4.9 $\tilde{d\mathcal{F}}(\phi)$ depends analytically on $\phi$ and its derivatives of order at most 12 so that if $\phi \in V \cap H^l_0(M)$ then $\tilde{d\mathcal{F}}(\phi)$ sits in $H^{l-12}_0(M)$. Then, the restriction $\tilde{d\mathcal{F}} : V \cap H^l_0(M) \to H^{l-12}_0(M)$ is a smooth map of Hilbert manifolds. \qed

To finish this subsection, we will assume $\omega$ is a zero of $\mathcal{F}$ satisfying the condition C and we show that the zeroes of $\mathcal{F}$ around $\omega$ form a manifold. We will write $D$ for $D^\phi$ when $\phi = 0$. 

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Let \((M, \omega, J)\) be a Kähler manifold and consider its reduced automorphism group \(H_0(M, J)\) that is the connected (finite dimensional) Lie group whose Lie algebra \(\mathfrak{h}_0\) consists of real holomorphic vector fields \(Z\) (i.e. \(\mathcal{L}_Z J = 0\)) of the form \(X_H + JX_K\) for \(H\) and \(K \in C_0^\infty(M)\).

If \(\omega\) is a zero of \(\mathcal{F}\), then Lemma 4.3 with a family \(f_t \in H_0(M, J)\) shows that all the points in the \(H_0(M, J)\)-orbit \(O\) of \(\omega\) are zeroes of \(\mathcal{F}\). The tangent space to \(O\) is the space of \(K \in C_0^\infty(M)\) such that there exists \(H \in C_0^\infty(M)\) with \(X_H + JX_K \in \mathfrak{h}_0\). Moreover, if \(\omega\) satisfies the Condition C, then, by Proposition 3.13, all points in \(O\) satisfy this condition. Our main Theorem is:

**Theorem 1.** Assume the Kähler manifold \((M, \omega, J)\) has a Ricci tensor which is non-negative everywhere.

If \(\omega\) is a zero of \(\mathcal{F}\), then there exists an open neighbourhood \(U\) of \(\omega\) in \(\mathcal{M}_\Theta\) such that the only zeroes of \(\mathcal{F}\) inside \(U\) is the \(H_0(M, J)\)-orbit of \(\omega\), where \(H_0(M, J)\) is the reduced automorphism group of \((M, \omega, J)\).

The proof of Theorem 1 is a direct corollary of the following local statement.

**Theorem 4.12.** Let \((M, \omega, J)\) be a Kähler manifold satisfying the Condition C and such that \(\omega\) is a zero of \(\mathcal{F}\). Then, \(d\overline{\mathcal{F}}(0) = 0\) and \((d\overline{\mathcal{F}})^{-1}(0)\) is a submanifold of \(\overline{U}\) (shrinking \(U\) if necessary) which is the intersection of \(U\) with the \(H_0(M, J)\)-orbit of \(\omega\).

**Proof.** We use the implicit function theorem for Hilbert manifolds. When \(\omega\) is a zero of \(\mathcal{F}\), it is also a zero of \(\mu\), so that \(\overline{\mathcal{F}}(0) = 0\).

Let \(l \geq \frac{\dim(M)}{2} + 12\) be an integer. From Proposition 4.10, the differential at 0 of \(\overline{\mathcal{F}}\) evaluated at \(\psi \in H_0^\ast(M)\) is given by

\[
d_0 \overline{\mathcal{F}}(\psi) = 2D^* D\psi - \int_M 2D^* D\psi \frac{\omega^n}{n!} = 2D^* D\psi,
\]

where \(D^* D\) is extended by continuity to an operator \(H_0^\ast(M) \rightarrow H_0^{\ast-12}(M)\) still denoted by \(D^* D\).

By Proposition 4.6, the self-adjoint operator \(D^* D\) is elliptic with smooth coefficients. So that, \(\ker(D^* D) \subset C_0^\infty(M)\) is finite dimensional and for \(s \in \mathbb{Z}\) we have the isomorphism of Hilbert space:

\[
H_0^\ast(M) \cong D^* D(H_0^{\ast+12}(M)) \oplus \ker(D^* D),
\]

the summands are orthogonal for the \(L^2\)-product and the projections are continuous. So that the differential of \(d\overline{\mathcal{F}}\) has finite dimensional kernel \(\ker(D^* D)\) and image equal to \(D^* D(H_0^\ast(M))\). This is enough to use the implicit function theorem on Hilbert spaces and concludes that \(\overline{\mathcal{F}}^{-1}(\ker(D^* D))\) is a submanifold of \(\overline{U}\) whose tangent space is isomorphic to \(\ker(D^* D)\).

To conclude the proof, just consider \(O\) the \(H_0(M, J)\)-orbit of \(\omega\) and the intersection \(O \cap U\) seen as a subset of \(\overline{U}\). Any point in \(O \cap U\) satisfy the condition (C). By Proposition
the tangent space of $\mathcal{O} \cap U$ at some point $\phi$ contains $\text{ker}((D^\phi)^*D^\phi) \cong \text{ker}(D^*D)$. Moreover, $d\mathcal{F}$ is constant on $\mathcal{O} \cap U$. So that $d\mathcal{F}^{-1}(\text{ker}(D^*D)) = d\mathcal{F}^{-1}(0) = \mathcal{O} \cap U$. The proof is over.

**Proof of Theorem 4.1.** Since a Kähler manifold with non-negative Ricci tensor everywhere satisfies the Condition C, we can use the above Theorem 4.12. So that, there exists a neighbourhood $U$ of $\omega$ in $M_{\Theta}$ such that $(d\mathcal{F})^{-1}(0) = U \cap \mathcal{O}$. Then, the zeroes of $\mathcal{F}$ restricted to $U$ are elements of $\mathcal{O}$.

### 4.4 Critical points

We compute $d\mathcal{F}$ from a different point of view as in Proposition 4.10 to obtain an equation for critical points of $\mathcal{F}$ similar to the one of extremal Kähler metric. This equation was already pointed out in Fox’s paper [16] but in different settings.

**Theorem 4.13.** Let $(M, \omega, J)$ be a Kähler manifold satisfying the Condition C. Then, the form $\omega$ is a critical point of $\mathcal{F}$ if and only if $L_{X_{\mu(\mathcal{V})}} \nabla = 0$, that is, if and only if the Hamiltonian vector field $X_{\mu(\mathcal{V})}$ is a Killing vector field.

Let us compute the first variation of $\mathcal{F}(\omega_\phi)$.

**Proposition 4.14.** Let $\omega_{\phi(t)}$ be a smooth path in $M_{\Theta}$ with $\phi(0) = 0$. Then,

$$
\frac{d}{dt} \mathcal{F}(\omega_{\phi(t)}) = -2\Omega^\mathcal{E}\left(lc_{\ast}J_t(\mathcal{L}_{X_{\mu(\mathcal{V})}} J_t), lc_{\ast}J_t(\mathcal{L}_{X_{\mu(\mathcal{V})}^{\omega_{\phi(t)}}} J_t)\right).
$$

**Proof.** Let $f_t \in \text{Diff}_0(M)$ defined as in equation (9). With the help of the Lemma 4.3 we compute :

$$
\frac{d}{dt} \mathcal{F}(\omega_{\phi(t)}) = \frac{d}{dt} \int_M (\mu(\nabla_{\phi(t)}))^2 \frac{\omega^n}{n!},
$$

where the last equality follows from the fact that $\mu$ is a moment map on $\mathcal{E}(M, \omega)$.

Now, $\nabla_{J_t} = lc(J_t)$ and $J_t = f_t^{-1}J := f_{t^{-1}}J f_t$. So that, $\frac{d}{dt}J_t = \mathcal{L}_{f_{t^{-1}}X_t}J_t$ and $f_{t^{-1}}X_t = -J_tX_t^{\omega_{f_t^{-1}\phi}}$. Then,

$$
\frac{d}{dt} \mathcal{F}(\omega_{\phi(t)}) = -2\Omega^\mathcal{E}\left(lc_{\ast}J_t(\mathcal{L}_{X_{\mu(\mathcal{V})}^{\omega_{\phi(t)}}} J_t), lc_{\ast}J_t(\mathcal{L}_{X_{f_t^{-1}\phi}^{\omega_{\phi(t)}}} J_t)\right).
$$
Proof of Theorem 4.13. We assume \((M, \omega, J)\) satisfies Condition C. The form \(\omega\) is a critical point if and only if for all \(H \in C_0^\infty(M)\) we have \(\frac{d}{dt}|_{t=0} F(\omega tH) = 0\). By proposition 4.14, this means

\[
\Omega^f \left( \text{lc}_J(J \mathcal{L}_{X_\mu}) J, \text{lc}_J(J \mathcal{L}_{X_\mu}) J \right) = 0
\]

for all \(H \in C^\infty(M)\). Now, Condition C implies that \(\mathcal{L}_{X_\mu} J = 0\).

\[\square\]

Corollary 4.15. Let \((M, \omega, J)\) be a closed Kähler manifold which is Ricci flat. Then \(\omega\) is a critical point of \(F\) if and only if it is a zero of \(\mathcal{F}\).

Proof. A Ricci flat Kähler manifold satisfies the Condition C. If \(\omega\) is a critical point of \(\mathcal{F}\), then \(\mathcal{L}_{X_\mu} J = 0\), by Theorem 4.13. On a Ricci flat manifold \(\mathcal{L}_{X_\mu} J = 0\) implies \(\mu(\nabla) = 0\) (the function \(\mu\) being normalized), so that \(\mathcal{F}(\omega) = 0\).

\[\square\]

5 Deformation quantization

We will now provide a new motivation for the study of various moment maps on infinite dimensional symplectic manifolds. We exhibit a link between the moment map \(\mu\) on the space of symplectic connections and the trace density of the Fedosov’s star products.

Moreover, we also mention other examples of star products, for which the trace density is linked to moment maps on infinite dimensional symplectic manifolds.

5.1 Closed Fedosov’s star products

The moment map \(\mu\) and its zeroes do have a nice interpretation in terms of Fedosov’s star products.

Consider the space \(C^\infty(M)[[\nu]]\) of formal power series of smooth functions. A star product \([1]\) on a symplectic manifold is a \(\mathbb{R}[[\nu]]\)-bilinear map

\[
* : C^\infty(M)[[\nu]] \times C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]] : (F, H) \mapsto F * H = \sum_{r=0}^{\infty} \nu^r C_r(F, H),
\]

such that :

- * is associative,
- the \(C_r\) are \(\mathbb{R}[[\nu]]\)-linear bidifferential operators,
- \(C_0(F, H) = FH\) and \(C_1^{-1}(F, H) := C_1(F, H) - C_1(H, F) = \{F, H\}\) where \(\{F, H\} := -\omega(X_F, X_H)\), for \(F, H \in C^\infty(M)\),
- \(F * 1 = F = 1 * F\), for all \(F \in C^\infty(M)\).
In [12], Fedosov gave a geometric construction of star products $\ast_{\nabla, \Omega}$ on symplectic manifold using a symplectic connection $\nabla$ and a formal series of closed 2-forms $\Omega \in \nu \Omega^2(M)[[\nu]]$. In this subsection, we only consider Fedosov’s star products obtained with $\Omega = 0$. Concretely, the first terms up to order 3 in $\nu$ are described by the formula

$$F \ast_{\nabla,0} H = FH + \frac{\nu^2}{2} \{F, H\} + \frac{\nu^2}{4} \Lambda^{i_1j_1} \Lambda^{i_2j_2} (\nabla^2 F)_{i_1i_2} (\nabla^2 H)_{j_1j_2} + \frac{\nu^3}{48} S^3_{\nabla}(F, H) + O(\nu^4),$$

(12)

where $F, H \in C^\infty(M)$ and, denoting by $\mathcal{L}_X \nabla$ the symmetric 3-tensor $\omega(\mathcal{L}_X \nabla (\cdot), \cdot)$,

$$S^3_{\nabla}(F, H) := \Lambda^{i_1j_1} \Lambda^{i_2j_2} \Lambda^{i_3j_3} (\mathcal{L}_X \nabla)_{i_1i_2i_3} (\mathcal{L}_H \nabla)_{j_1j_2j_3},$$

One checks that the $\ast_{\nabla,0}$-commutator is given by

$$[F, H]_{\ast_{\nabla,0}} = F \ast_{\nabla,0} H - H \ast_{\nabla,0} F = \nu \{F, H\} + \frac{\nu^2}{2} S^3_{\nabla}(F, H) + O(\nu^4),$$

Remark 5.1. In [13] the notion of “natural” star products was introduced. Such natural star products determine the symplectic connection. Star products obtained via the Fedosov’s method are natural in the sense of [13].

Let $\ast$ be a star product on a symplectic manifold. A trace for $\ast$ is a $\mathbb{R}[[\nu]]$-linear map

$$\text{tr} : C^\infty_c(M)[[\nu]] \to \mathbb{R}[[\nu]],$$

satisfying $\text{tr}(F \ast H) = \text{tr}(H \ast F)$ for all $F, H \in C^\infty_c(M)[[\nu]]$, where $C^\infty_c(M)$ denotes the space of smooth functions with compact support.

**Proposition 5.2** (Fedosov [14, 13], Nest-Tsigan [22], Gutt-Rawnsley [19]). Any star product $\ast$ on a symplectic manifold $(M, \omega)$ admit a trace. More precisely, there exists $\rho \in C^\infty(M)[[\nu]]$ such that

$$\text{tr}(F) := \int_M F \rho \frac{\omega^n}{n!},$$

(13)

for all $F \in C^\infty_c(M)[[\nu]]$. The function $\rho$ is called a trace density.

Moreover, any two traces for $\ast$ differ from each other by multiplication with a formal constant $C \in \mathbb{R}[[\nu^{-1}, \nu]]$.

A star product is called closed up to order $l$ in $\nu$ if the map $F \mapsto \int_M F \frac{\omega^n}{n!}$ satisfies the trace property up to order $l$ in $\nu$, i.e.

$$\int_M F \ast H \frac{\omega^n}{n!} = \int_M H \ast F \frac{\omega^n}{n!} + O(\nu^{l+1}), \quad \text{for all } F, H \in C^\infty_c(M)[[\nu]].$$

(14)

Fedosov [15] gives an algorithm to compute the trace density of the star product $\ast_{\nabla,0}$. He computes explicitly the first non trivial term. Here we prove that the equivariant moment map property of $\mu$ implies that $\mu(\nabla)$ is the first non trivial term of a trace density for $\ast_{\nabla,0}$.
Proposition 5.3 (Fedosov [15]). Let \((M, \omega)\) be a closed symplectic manifold. If \(F, H \in C^\infty(M)\), then \(\rho := 1 + \frac{\nu^2}{2!} \mu(\nabla)\) satisfies

\[
\int_M (F \ast_{\nabla,0} H - H \ast_{\nabla,0} F) \rho \frac{\omega^n}{n!} \equiv 0 \mod \nu^4.
\]

Consequently, \(\ast_{\nabla,0}\) is closed up to order 3 in \(\nu\) if and only if \(\nabla\) is a zero of the normalised moment map \(\mu\).

Proof. We compute

\[
\int_M [F, H] \ast_{\nabla,0} \rho \frac{\omega^n}{n!} = \frac{\nu^3}{24} \left( \Omega^E(\mathcal{L}_{\mathcal{X}_F} \nabla, \mathcal{L}_{\mathcal{X}_H} \nabla) + \int_M \{F, H\} \mu(\nabla) \frac{\omega^n}{n!} \right) + O(\nu^4).
\]

Because \(\mu\) is an equivariant moment map on \(\mathcal{E}(M, \omega)\), equation (4) says:

\[
\Omega^E(\mathcal{L}_{\mathcal{X}_H} \nabla, \mathcal{L}_{\mathcal{X}_F} \nabla) = - \int_M \{F, H\} \mu(\nabla) \frac{\omega^n}{n!}.
\]

It concludes the proof. \(\square\)

Let \((M, \omega, J)\) be a closed Kähler manifold. One associate naturally the Fedosov’s star product \(\ast_{\nabla,0}\) for \(\nabla\) the Levi-Civita connection. Assume \(\tilde{\omega} \in \mathcal{M}_\Theta\), with \(\Theta := [\omega]\), is such that \(f^* \tilde{\omega} = \omega\) for an \(f \in H_0(M, J)\). Set \(\tilde{\nabla}\) the Levi-Civita connection of \(\tilde{\omega}\). One checks \(\nabla = f^{-1} \tilde{\nabla}\). Because of that, see [12], the pull-back by \(f^*\) gives an isomorphism of star product

\[
f^* : (C^\infty(M)[[\nu]], \ast_{\tilde{\nabla},0}) \xrightarrow{\cong} (C^\infty(M)[[\nu]], \ast_{\nabla,0}).
\]

Our Theorem 1 translates in terms of Fedosov’s star product in the following way.

Theorem 2. Let \((M, \omega, J)\) be a closed Kähler manifold with Levi-Civita connection \(\nabla\) and Ricci tensor everywhere non negative. Assume the Fedosov’s star product \(\ast_{\nabla,0}\) is closed up to order 3 in \(\nu\).

Then, there exists an open neighbourhood \(U\) of \(\omega\) in \(\mathcal{M}_\Theta\) such that if \(\tilde{\omega} \in U\) gives rise to the star product \(\ast_{\tilde{\nabla},0}\) closed up to order 3 in \(\nu\) then there is an \(f \in H_0(M, J)\) inducing an isomorphism

\[
f^* : (C^\infty(M)[[\nu]], \ast_{\tilde{\nabla},0}) \xrightarrow{\cong} (C^\infty(M)[[\nu]], \ast_{\nabla,0}).
\]

Proof. Because \(\ast_{\nabla,0}\) is closed up to order 3, \(\mathcal{F}(\omega) = 0\). Now, we can use Theorem 1 which states that in a neighbourhood of \(\omega\) in \(\mathcal{M}_\Theta\) the zeroes of \(\mathcal{F}\) are the \(H_0(M, J)\)-orbit of \(\omega\). As pointed out above, an element \(f \in H_0(M, J)\) produces the desired isomorphism of star product algebra. \(\square\)

Exemple 5.4. Consider the flat torus \((\mathbb{C}^n/\mathbb{Z}^{2n}, \omega_{std}, i)\) with its flat connection \(\nabla\). The corresponding Fedosov’s star product \(\ast_{\nabla,0}\) is closed (i.e. the integral is a trace functional). Since \(\text{Ric}^\nabla \equiv 0\), the group \(H_0(M, J) = 0\). By the above Theorem 2 \(\omega_{std}\) is isolated from any other \(\omega_\phi \in \mathcal{M}_{\{\omega_{std}\}}\) such that \(\ast_{\nabla,0}\) is closed.

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Exemple 5.5. Consider $\mathbb{CP}^n, \omega_{FS}, J$ the complex projective space with its Fubini-Study metric $\omega_{FS}$ and standard $J$. Denote by $\nabla$ the Kähler connection, it is known that $*_\nabla,0$ is closed. Because $\text{Ric}^\nabla$ is positive definite at any point, Theorem 2 states that in a neighbourhood $U$ of $\omega_{FS}$ in $M[\omega_{FS}]$ the only other closed Fedosov’s star products $*_\tilde{\nabla},0$ associated to $\tilde{\omega} \in U$ are obtained by the pull-back by a transformation of $SL(n+1, \mathbb{C})$.

5.2 Other closed star products and moment maps

The link between trace densities for star products and moment maps on infinite dimensional symplectic manifolds in Proposition 5.3 involves more star products than just the Fedosov’s star products $*_\nabla,0$. We give here two additional examples.

5.2.1 Wick star products of Bordemann-Waldmann

On a Kähler manifold, Bordemann and Waldmann [5] adapted the Fedosov’s construction to produce star products of Wick type. That is star products defined by series of bidifferential operators $C_j$ for which the first argument is differentiated in holomorphic directions while the other is differentiated in anti-holomorphic directions (for their construction, the star product is defined on $C_\infty(M, \mathbb{C})[[\nu]]$ and the $C_j$’s are $\mathbb{C}[[\nu]]$-bilinear).

Theorem 5.6 (Bordemann-Waldmann [5]). On the Kähler manifold $(M, \omega, J)$, there exists a star product of Wick type $*_\omega,J$. Their proof is constructive in the sense that the $C_j$’s defining $*_\omega,J$ can be obtained recursively. The product $F *_\omega,J H$ for $F, H \in C_\infty(M, \mathbb{C})$ is described by the formula

$$ F *_\omega,J H := FH + \nu \Lambda^\alpha_\beta \partial_\alpha F \partial_\beta H + \nu^2 \Lambda^{\alpha_1 \beta_1} \Lambda^{\alpha_2 \beta_2} (\nabla^2 F)_{\alpha_1 \alpha_2} (\nabla^2 H)_{\beta_1 \beta_2} + O(\nu^3), $$

where the above expression is written in local holomorphic coordinate $\{z_\alpha\}$ and their conjugates $\{\bar{z}_\beta\}$.

On a closed Kähler manifold, one observe that for $F, H \in C_\infty(M, \mathbb{R})$

$$ \int_M [F, H]_{*_\omega,J} \omega^n = \frac{\nu^2}{4} i \Omega^J_{\omega_J} (\mathcal{L}_F J, \mathcal{L}_H J) + O(\nu^3), $$

where $\Omega^J_\omega$ is the symplectic form introduced in Subsection 3.1 on the space $\mathcal{J}(M, \omega)$ of almost complex structure compatible with $\omega$. Using the equivariant moment map property of the Hermitian scalar curvature of Equation (5), we give the first non-trivial term of a trace density for $*_\omega,J$. It was already computed by Karabegov [20].

Proposition 5.7 (Karabegov [20]). Let Scal($J$) be the scalar curvature of the Kähler manifold $(M, \omega, J)$. If $F, H \in C_\infty(M)$, then $\rho := 1 + \frac{\nu}{4} \text{Scal}(J)$ satisfies

$$ \int_M (F *_\omega,J H - H *_\omega,J F) \rho \omega^n n! \equiv 0 \mod O(\nu^3). $$

Consequently, $*_\omega,J$ is closed up to order 2 in $\nu$ if and only if the Kähler manifold $(M, \omega, J)$ is of constant scalar curvature.
Proof. For $F, H \in C^\infty(M)$:

$$
\int_M (F \ast_{\omega,J} H - H \ast_{\omega,J} F) \rho^n = i \frac{\nu^2}{4} \left( \Omega_J^2(\mathcal{L}_{X_F} J, \mathcal{L}_{X_H} J) + \int_M \{F, H\} \text{Scal}(J) \frac{\omega^n}{n!} \right) + O(\nu^3).
$$

Using the moment map equation (5), we see the right hand side is in $O(\nu^3)$. \hfill \Box

Now, we fix the complex structure of $(M, \omega, J)$ and let vary the Kähler form inside $\mathcal{M}_\Theta$. One defines the Calabi functional

$$
\text{Cal} : \mathcal{M}_\Theta \to \mathbb{R} : \omega_{\phi} \mapsto \int_M (\text{Scal} \nabla^\phi)_{\phi} \frac{\omega^n}{n!},
$$

where as before $\nabla^\phi$ is the Levi-Civita connection of $g_{\phi}$. Critical points of $\text{Cal}$ are called extremal Kähler metrics.

**Proposition 5.8** (Calabi [7]). Extremal Kähler metrics in $\mathcal{M}_\Theta$ form a submanifold of $\mathcal{M}_\Theta$ whose connected component are $H_0(M, J)$-orbit.

When a Kähler metric $\omega$ is of constant scalar curvature, then it is extremal and any metrics in its $H_0(M, J)$-orbit is of constant scalar curvature. In terms of star product, it translates into the following:

**Corollary 5.9.** Let $(M, \omega, J)$ be a closed Kähler manifold. Consider the star products $\ast_{\tilde{\omega}, J}$ for $\tilde{\omega} \in \mathcal{M}_\Theta$. Then, the $\tilde{\omega} \in \mathcal{M}_\Theta$ such that $\ast_{\tilde{\omega}, J}$ is closed up to order 2 form a submanifold of $\mathcal{M}_\Theta$ whose connected components are $H_0(M, J)$-orbits.

### 5.2.2 Fedosov’s star products $\ast_{\nabla, \nu \chi}$

Here, we consider Fedosov’s star products $\ast_{\nabla, \nu \chi}$ on a symplectic manifold $(M, \omega)$ built using the data of a symplectic connection and a closed 2-form $\nu \chi \in \nu \Omega^2(M) \subset \nu \Omega^2(M)[[\nu]]$.

For $F, H \in C^\infty(M)$, one has

$$
F \ast_{\nabla, \nu \chi} H = FH + \nu \{F, H\} + \nu^2 \left( \frac{1}{2} \chi(X_F, X_H) + k' \Lambda^i_{j1} \Lambda^j_{l2} (\nabla^2 F)_{i1j1} (\nabla^2 H)_{j1l2} \right) + O(\nu^3).
$$

The $\ast_{\nabla, \nu \chi}$-commutator writes

$$
[F, H]_{\ast_{\nabla, \nu \chi}} = \nu \{F, H\} + \nu^2 \chi(X_F, X_H) + O(\nu^3).
$$

And one can check that the trace density for $\ast_{\nabla, \nu \chi}$ writes $\rho = 1 + \nu \Lambda^i_{ij} \chi_{ij} + O(\nu^2)$.

We recall a moment map construction from Donaldson [10]. Assume now that $\chi$ is non-degenerate (i.e. a symplectic form). Consider $\mathcal{M} := \text{Diff}_0(M)$ the identity component of the group of diffeomorphisms of $M$. Then the tangent space $T_f \mathcal{M} = \Gamma(f^*TM)$. The symplectic form $\Omega^\mathcal{M}$ on $\mathcal{M}$ is defined by

$$
\Omega^\mathcal{M}_f(U, V) := \int_M \chi(f(x))(U_x, V_x) \frac{\omega^n}{n!},
$$

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for \( f \in \mathcal{M} \) and \( U, V \in \Gamma(f^*TM) \). The group \( \text{Ham}(M, \omega) \) acts on the right on \( \text{Diff}_0(M) \) preserving \( \Omega^M \). The fundamental vector fields of this action are \((X_F)^*M := f_*X_F \) for \( F \in C_0^\infty(M) \). Define the map \( \mu : \mathcal{M} \rightarrow C^\infty(M) \) by
\[
\mu(f)\frac{\omega^n}{n!} = -(f^*\chi) \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]

Donaldson \cite{10} shows this map is a moment map, that is
\[
\frac{d}{dt} |_{t=0} \int_M F \mu(f_t)\frac{\omega^n}{n!} = \Omega^M_0((X_F)^*M, V),
\]
for \( F \in C_0^\infty(M) \), \( f_t \) a smooth path in \( \text{Diff}_0(M) \) and \( V := \frac{d}{dt} |_{t=0} f_t \in \Gamma(f_0^*TM) \).

In conclusion, when \( \chi \) is non-degenerate, the trace density for \( \ast_{\nabla, \nu f^*\chi} \), where \( f \in \text{Diff}_0(M) \) writes \( \rho^{\ast_{\nabla, \nu f^*\chi}} := 1 + \nu^2 \mu(f) + O(\nu^2) \).

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