Fermions in spherical field theory

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We derive the spherical field formalism for fermions. We find that the spherical field method is free from certain difficulties which complicate lattice calculations, such as fermion doubling, missing axial anomalies, and computational problems regarding internal fermion loops.

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1 Overview

Spherical field theory is a new non-perturbative method for studying quantum field theory. It was introduced in [1] and was used to describe the interactions of scalar boson fields. In this paper we show how to extend the spherical field method to fermionic systems.

The central idea of spherical field theory is to treat a $d$-dimensional system as a set of coupled one-dimensional systems. This is done by expanding field configurations of the functional integral in terms of spherical partial waves. Regarding each partial wave as a distinct field in a new one-dimensional theory, we interpret the functional integral as a time-evolution equation, where the radial distance in the original theory serves as the time parameter. For a purely bosonic system the time-evolution equation corresponds with a multidimensional partial differential equation. In the case of a purely fermionic system, we find that the time evolution is described by a system of first-order ordinary differential equations. In future work we will study mixed systems with both bosons and fermions which are described by coupled partial differential equations.

Unlike lattice methods, spherical field theory yields an expansion which, at any order, corresponds with a continuous system. It is therefore able to avoid problems associated with discrete approximation methods. There is no doubling of fermion states, and we find the correct axial anomaly. Further...
thermore internal fermion loops present no special computational difficulties and is included in the dynamics of the time-evolution equation. Detailed examples of such calculations will be presented in a forthcoming paper. We anticipate that spherical field methods will be useful in the study of non-perturbative fermionic systems, especially chiral fermions and phenomena related to fermion loop processes.

The organization of this paper is as follows. We begin with a brief description of Grassmanian path integrals and the fermionic analog of the Feynman-Kac formula. We then generate the spherical expansion for free fermion theory in two Euclidean dimensions with sources and derive the spherical field time-evolution operator and generating functional. By functional differentiation with respect to the sources, we obtain the spherical field formalism for general interacting theories. Next we check that spherical field theory produces the correct axial anomaly. We then show how to write the time-evolution equation as a matrix system and comment on the utility of spherical field methods in studying fermionic systems. Although our analysis is done in two dimensions, the extension to higher dimensions is straightforward.

2 Grassmanian path integrals

Let $\bar{\psi}_i(t)$ and $\psi_j(t)$ be Grassman-valued functions where $i, j = 1, \ldots, N$. Let $V(\bar{\psi}_i, \psi_j, t)$ be a polynomial in $\bar{\psi}_i$ and $\psi_j$, ordered such that all $\bar{\psi}$'s are placed on the right and all $\psi$'s are placed on the left. In [2] it is shown that

$$
\text{Tr} \left[ T \exp \left\{ - \int_{t_1}^{t_2} dt V(a_i^+, a_i^-, t) \right\} \right] \propto \int d\Psi d\bar{\Psi} \int_{\bar{\psi}(t_1) = -\psi(t_1) = \Psi} d\bar{\psi} D\bar{\psi} \exp \left\{ - \int_{t_1}^{t_2} dt \left[ \sum_{k=1}^{N} \frac{\bar{\psi}_k}{\psi_k} \frac{d\psi_k}{dt} + V(\bar{\psi}_i, \psi_j, t) \right] \right\},
$$

where the trace is performed over the space spanned by vectors of the form

$$
| s_1 \cdots s_N \rangle \quad s_1 = 0, 1; \cdots s_N = 0, 1;
$$

and

$$
\langle s_1' \cdots s_N' | s_1 \cdots s_N \rangle = \delta_{s_1' s_1} \cdots \delta_{s_N' s_N}
$$

$$
a_i^+ | s_1 \cdots s_i \cdots s_N \rangle = (-1)^{s_1 + \cdots + s_{i-1} - 1} \delta_{s_i, 0} | s_1 \cdots 1 \cdots s_N \rangle
$$

$$
a_i^- | s_1 \cdots s_i \cdots s_N \rangle = (-1)^{s_1 + \cdots + s_{i-1} - 1} \delta_{s_i, 1} | s_1 \cdots 0 \cdots s_N \rangle.
$$
This is the fermionic version of the Feynman-Kac formula. A more recent derivation using fermionic coherent states can be found in [3] and [4]. The antiperiodic boundary conditions imposed at \( t_I \) and \( t_F \) follow as a consequence of computing the trace. We note that these are in fact special conditions. More general boundary constraints produce ambiguities which depend on the specific discrete approximation used to obtain the continuum limit.

It is not clear that such antiperiodic boundary conditions can be generalized in a coordinate-independent manner for functional integrals over higher dimensional regions. The rigorous theory of Grassmanian functional integration has not developed to the point where we can answer such questions. Nevertheless functional integration is a convenient method for deriving useful field-theoretic results, although in a somewhat heuristic fashion. In this analysis we use the functional integral to deduce the spherical field formalism for fermions. Although we will be careless with regard to boundary conditions, in the end we explicitly check that the spherical field method produces the correct generating functional for free field theory. By functional differentiation with respect to the external sources, we conclude that the spherical field formalism is valid for general interacting theories.

3 Spherical fermions

Let us consider Euclidean field theory in two dimensions. We will use both cartesian and polar coordinates,

\[
\vec{t} = (t \cos \theta, t \sin \theta) = (x, y).
\]

In Euclidean space the gamma matrices satisfy

\[
\{ \gamma^i, \gamma^j \} = -2\delta^{ij},
\]

and we choose the representation

\[
\vec{\gamma} = i\vec{\sigma}.
\]

Let us start by constructing the spherical field Hamiltonian. We first decompose the fermion fields,

\[
\psi = \begin{bmatrix} \psi^\uparrow(\vec{t}) \\ \psi^\downarrow(\vec{t}) \end{bmatrix} = \sum_{n=0,\pm 1,\ldots} \sqrt{2n+1} \frac{\psi_n^\uparrow(t)e^{in\theta}}{\sqrt{2\pi}} = \sum_{n=0,\pm 1,\ldots} \sqrt{2n+1} \frac{\psi_n^\downarrow(t)e^{in\theta}}{\sqrt{2\pi}}.
\]
\[ \bar{\psi} = \begin{bmatrix} \bar{\psi}^+ (\vec{t}) \\ \bar{\psi}^\perp (\vec{t}) \end{bmatrix} = \sum_{n=0, \pm 1, \ldots} \left[ \frac{1}{\sqrt{2\pi}} \bar{\psi}_{n}^+ (t) e^{in\theta} - \frac{1}{\sqrt{2\pi}} \bar{\psi}_{n}^\perp (t) e^{in\theta} \right]. \tag{8} \]

Using
\[ \bar{\sigma} \cdot \vec{\nabla} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{i\theta} \left( \frac{\partial}{\partial t} + i \frac{\partial}{\partial \vec{r}} \right) \tag{9} \end{bmatrix}, \]

we have
\[ \bar{\sigma} \cdot \vec{\nabla} \left[ \frac{1}{\sqrt{2\pi}} \psi_{n}^+ (t) e^{in\theta} \right] = \frac{1}{\sqrt{2\pi}} \left[ \left( \frac{\partial \psi_{n}^+}{\partial t} + \frac{n}{t} \chi_{n} \right) e^{i(n+1)\theta} \right]. \tag{10} \]

The Euclidean action for free field theory with external sources, \( \eta \) and \( \bar{\eta} \), is
\[ S = -i \int d\theta dt t \left( \bar{\psi} (i \vec{\gamma} \cdot \vec{\nabla} - m) \psi + \bar{\psi} \eta + \bar{\eta} \psi \right). \tag{11} \]

In terms of partial waves,\(^4\)
\[ S = -i \int dt \sum_{n=0, \pm 1, \ldots} \left[ -\bar{\psi}_{n-1}^+ \left( \frac{\partial \psi_{n+1}^+}{\partial t} + \frac{n+1}{t} \chi_{n} \right) - \bar{\psi}_{n-1}^\perp \left( \frac{\partial \psi_{n+1}^\perp}{\partial t} - \frac{n}{t} \chi_{n}^\perp \right) - m \left( \bar{\psi}_{n-1} \psi_{n}^+ + \bar{\psi}_{n-1} \chi_{n} \right) \psi_{n}^\perp + \bar{\psi}_{n-1}^\perp \eta_{n}^\perp + \bar{\eta}_{n}^\perp \psi_{n}^+ + \bar{\eta}_{n}^\perp \psi_{n}^\perp \right]. \tag{12} \]

The generating functional is therefore
\[ \int \left( \prod_{i,n} \mathcal{D} \psi_{n}^i \mathcal{D} \bar{\psi}_{n}^i \right) \exp \left\{ \int_0^\infty dt \sum_{n=0, \pm 1, \ldots} G_{n} \right\}, \tag{13} \]

where \( G_{n} \) is defined as
\[ -\bar{\psi}_{n}^+ \left( \frac{\partial \psi_{n+1}^+}{\partial t} + \frac{n+1}{t} \chi_{n+1} \right) - \bar{\psi}_{n-1}^\perp \left( \frac{\partial \psi_{n+1}^\perp}{\partial t} - \frac{n}{t} \chi_{n}^\perp \right) \tag{14} \]
\[ -m \left( \bar{\psi}_{n} \psi_{n}^+ + \bar{\psi}_{n} \chi_{n} \right) + \bar{\psi}_{n-1} \eta_{n}^\perp + \bar{\psi}_{n} \eta_{n}^\perp + \bar{\eta}_{n} \psi_{n}^+ + \bar{\eta}_{n-1} \psi_{n}^\perp. \]

If we now define \( \bar{\psi}_{n}^i = t \psi_{n}^i \), the generating functional is, up to an overall constant,
\[ \int \left( \prod_{i,n} \mathcal{D} \psi_{n}^i \mathcal{D} \bar{\psi}_{n}^i \right) \exp \left\{ \int_0^\infty dt \sum_{n=0, \pm 1, \ldots} G_{n}^i \right\}, \tag{15} \]

\(^4\)We expand \( \eta \) and \( \bar{\eta} \) into partial waves in the same manner as \( \psi \) and \( \bar{\psi} \).
where $G'_n$ is

$$
-\psi_{-n}^{-\uparrow} \left( \frac{\partial \psi_{n+1}^{-\uparrow}}{\partial t} + \frac{n+1}{t} \psi_{n+1}^{-\uparrow} \right) - \psi_{-n}^{\downarrow} \left( \frac{\partial \psi_{n+1}^{\downarrow}}{\partial t} - \frac{n}{t} \psi_{n}^{\downarrow} \right) - m \left( \tilde{\psi}_{-n}^{\uparrow} \psi_{n}^{\uparrow} + \tilde{\psi}_{-n}^{\downarrow} \psi_{n}^{\downarrow} \right) + \tilde{\psi}_{-n}^{\uparrow} \eta_{n}^{\uparrow} + \tilde{\psi}_{-n-1}^{\downarrow} \eta_{n+1}^{\downarrow} + t \eta_{-n}^{\uparrow} \psi_{n}^{\uparrow} + t \eta_{-n-1}^{\downarrow} \psi_{n+1}^{\downarrow}. $$

(16)

Our goal is to find an equivalent expression for (15), in analogy with the Feynman-Kac formula (1). We start by defining a linear vector space. For each finite subset $S \subset \{ \sigma_{i} \mid \sigma_{i} = 0, \pm 1, \pm 2, \cdots, i = \downarrow, \uparrow \}$

we assign a vector $|S\rangle$ satisfying the following orthogonality and normalization conditions,

$$
\langle S'|S \rangle = \begin{cases} 0 & \text{if } S \neq S' \\ 1 & \text{if } S = S'. \end{cases}
$$

(18)

For later convenience we define a lexicographic order, namely,

$$
s_{i}^{n} < s_{i'}^{n'}\iff
\begin{cases} n < n' & \text{or } n = n', i = \downarrow, \text{ and } i' = \uparrow. \end{cases}
$$

(19)

Let $\Sigma$ be the linear space spanned by all such vectors $|S\rangle$. Let us define operators $a_{n}^{\downarrow+}$ and $a_{n}^{\downarrow-}$ by the following relations,

$$
a_{n}^{\downarrow-} |S \rangle = \begin{cases} 0 & \text{if } s_{i}^{n} \notin S \\ (-1)^{\#} |S - s_{i}^{n} \rangle & \text{if } s_{i}^{n} \in S, \end{cases}
$$

$$
a_{n}^{\downarrow+} |S \rangle = \begin{cases} 0 & \text{if } s_{i}^{n} \in S \\ (-1)^{\#} |S \cup s_{i}^{n} \rangle & \text{if } s_{i}^{n} \notin S, \end{cases}
$$

(21)

where $\#$ is the number of elements in $S$ which are less than $s_{i}^{n}$. Comparing (15) with (1), we make the correspondences,

$$
\psi_{n}^{\downarrow}, \psi_{-n-1}^{\downarrow} \leftrightarrow a_{n}^{\downarrow-}, a_{n}^{\downarrow+} \quad \psi_{n+1}^{\downarrow}, \psi_{-n}^{\downarrow} \leftrightarrow a_{n+1}^{\downarrow-}, a_{n+1}^{\downarrow+}
$$

(22)

5We denote the conjugate of $a_{n}^{\downarrow-}$ as $a_{n}^{\downarrow+}$. Although $a_{n}^{\downarrow-}$ corresponds with a partial wave with orbital angular momentum $n$, we note that $a_{n}^{\downarrow+}$ corresponds with a partial wave with orbital angular momentum $-n - 1$ or $-n + 1$, depending on $i$. 


and define

\[ Z[\bar{n}, \eta] = \lim_{\varepsilon \to 0} \frac{\text{Tr} \left[ T \exp \left\{ - \int_0^\infty dt \sum_{n=0, \pm 1, \ldots} H_n^\varepsilon(\bar{n}, \eta, t) \right\} \right]}{\text{Tr} \left[ T \exp \left\{ - \int_0^\infty dt \sum_{n=0, \pm 1, \ldots} H_n^\varepsilon(0,0,t) \right\} \right]} , \quad (23) \]

where

\[ H_n^\varepsilon = \frac{n+1}{T} a_{n+1}^+ a_n^− - \frac{n}{T} a_n^+ a_n^− + (m + \varepsilon) a_{n+1}^+ a_n^− + (m + \varepsilon) a_n^+ a_{n+1}^− \quad (24) \]

The traces in (23) are performed over the space \( \Sigma \). In practise it is not necessary to explicitly compute these traces, since only the \( \varepsilon \to 0 \) prescription picks out the correct ground state. We will refer to (23) as the fermionic spherical field ansatz. For notational ease we will suppress the \( \varepsilon \) terms. We now show that \( Z[\bar{n}, \eta] \) is the correct generating functional for free field theory.

We note that \( \Sigma \) can be decomposed as a tensor product space with the identification

\[ |S\rangle \leftrightarrow \bigotimes_{n=0, \pm 1, \ldots} |S \cap \{s_n^+, s_n^+\} \rangle . \quad (25) \]

Since \( H_n(\bar{n}, \eta, t) \) acts upon only the \( n \)-th component in the tensor product, we can write \( Z[\bar{n}, \eta] \) as a product

\[ Z[\bar{n}, \eta] = \prod_{n=0, \pm 1, \ldots} Z_n[\bar{n}_n^+, \bar{n}_{n+1}^\pm, \eta_n^\pm, \eta_{n+1}^\pm] , \quad (26) \]

where

\[ Z_n[\bar{n}_n^+, \bar{n}_{n+1}^\pm, \eta_n^\pm, \eta_{n+1}^\pm] = \frac{\text{Tr} \left[ T \exp \left\{ - \int_0^\infty dt H_n(\bar{n}, \eta, t) \right\} \right]}{\text{Tr} \left[ T \exp \left\{ - \int_0^\infty dt H_n(0,0,t) \right\} \right]} . \quad (27) \]

In (27) the trace is performed over the four-dimensional space spanned by the vectors

\[ |\emptyset\rangle, \{s_n^\pm\}, \{s_n^\pm\}, \{s_n^\pm, s_{n+1}^\pm\} . \quad (28) \]

In Appendix 1 we derive the result,

\[ Z = \exp \left\{ \sum_{n=0, \pm 1, \ldots} \int dt_1 dt_2 \left[ t_1 \bar{n}_n^+ (t_1) t_1 \bar{n}_{n-1}^+ (t_1) \right] M_n(t_1, t_2) \left[ \begin{array}{c} t_2 \eta_n^+ (t_2) \\ t_2 \eta_{n+1}^+ (t_2) \end{array} \right] \right\} , \quad (29) \]
where the matrix $\mathbf{M}_n(t_1, t_2)$ is defined as

$$
\theta(t_1 - t_2) \begin{bmatrix}
    mK_n(|m| t_1) I_n(|m| t_2) & |m| K_n(|m| t_1) I_{n+1}(|m| t_2) \\
    |m| K_{n+1}(|m| t_1) I_n(|m| t_2) & mK_{n+1}(|m| t_1) I_{n+1}(|m| t_2)
\end{bmatrix}
$$

(30)

$$
+ \theta(t_2 - t_1) \begin{bmatrix}
    mI_n(|m| t_1) K_n(|m| t_2) & -|m| I_n(|m| t_1) K_{n+1}(|m| t_2) \\
    -|m| I_{n+1}(|m| t_1) K_n(|m| t_2) & mI_{n+1}(|m| t_1) K_{n+1}(|m| t_2)
\end{bmatrix}
$$

for $m \neq 0$, and

$$
\theta(t_1 - t_2) \begin{bmatrix}
    0 & \frac{\theta(-n-\frac{1}{2}) t^n_1}{t^n_{2}+1} \\
    \frac{\theta(n+\frac{1}{2}) t^n_2}{t_{2}+1} & 0
\end{bmatrix} - \theta(t_2 - t_1) \begin{bmatrix}
    0 & \frac{\theta(-n-\frac{1}{2}) t^n_1}{t^n_{2}+1} \\
    \frac{\theta(n+\frac{1}{2}) t^n_2}{t_{2}+1} & 0
\end{bmatrix}
$$

(31)

for $m = 0$.

Let us now compare these results with the known results for free field theory. The two-point free field correlator is

$$
\Delta^{ij}(\vec{t}) = \int \frac{d^2 \vec{k}}{(2\pi)^2} \frac{e^{i\vec{k} \cdot \vec{t}}}{\vec{k}^2 + m^2} e^{i\vec{k} \cdot \vec{t}}
$$

(32)

Integrating over $\vec{k}$, we find

$$
\Delta^{ij}(\vec{t}) = \frac{1}{2\pi} \begin{bmatrix}
    {mK_0(|m| t) + |m| K_1(|m| t) \cdot (\sigma^x \cos \theta + \sigma^y \sin \theta)} \\
    \frac{mK_0(|m| t)}{m} |m| e^{-i\theta} K_1(|m| t)
\end{bmatrix}
$$

(33)

for $m \neq 0$. When $m \to 0$ we find

$$
\Delta^{ij}(\vec{t}) = \frac{1}{2\pi} \begin{bmatrix}
    0 & e^{-i\theta} \frac{1}{t} \\
    e^{i\theta} \frac{1}{t} & 0
\end{bmatrix}
$$

(34)

From (33) and (34) it is straightforward to recover the spherical correlation functions in (30) and (31). We conclude that the fermionic spherical field ansatz produces the correct generating functional for free fermions. These correlation functions are part of the spherical Feynman rules for fermions. For future reference we have written these in a more convenient format in Appendix 2. By functional differentiation with respect to the sources $\eta$ and $\bar{\eta}$, we conclude that the spherical field formalism is valid for general interacting theories.

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\(^6\) $I_i$ and $K_i$ are the $i^{th}$ order modified Bessel functions of the first and second kinds respectively.
4 Axial anomaly

We now show that the spherical field formalism yields the correct form for the axial anomaly. We consider free massless fermions, again in two Euclidean dimensions. Let us define

\[ S_{\mu\nu}(\vec{t}) = \left\langle 0 \left| V^\mu(\vec{t}) A^\nu(0) \right| 0 \right\rangle_E, \tag{35} \]

where \( V^\mu \) and \( A^\mu \) are the vector and axial vector currents, and the subscript \( E \) is intended as a reminder that our Euclidean correlation function is defined as the analytic continuation of the corresponding time-ordered function in Minkowski space.\(^7\) We note that

\[ \left\langle 0 \left| V^\mu(\vec{t}) \partial_\nu A^\nu(0) \right| 0 \right\rangle_E = -\partial_\nu \left\langle 0 \left| V^\mu(\vec{t}) A^\nu(0) \right| 0 \right\rangle_E = -\partial_\nu S_{\mu\nu}. \tag{36} \]

The one-loop process corresponding with \( S_{\mu\nu} \) carries a logarithmic divergence proportional to \( \varepsilon^{\mu\nu} \), and the regulated value of \( S_{\mu\nu} \) will depend on our definitions of \( V^\mu \) and \( A^\mu \) as operator products. In our discussion here we will remove this ambiguity by considering the symmetric combination \( S_{\mu\nu} + S_{\nu\mu} \).

Let us define vector and axial vector currents,

\[ V^\mu(\vec{t}) = \bar{\psi}(\vec{t}) \gamma^\mu \psi(\vec{t}) = i \bar{\psi}(\vec{t}) \sigma^\mu \psi(\vec{t}) \tag{37} \]
\[ A^\mu(\vec{t}) = \bar{\psi}(\vec{t}) \gamma^\mu \gamma^5 \psi(\vec{t}) = i \bar{\psi}(\vec{t}) \sigma^\mu \sigma^z \psi(\vec{t}) \tag{38} \]

Expanding the currents in terms of partial waves, we have

\[ V^\mu(\vec{t}) = \frac{1}{2\pi} \sum_{k,n=0,\pm1,\cdots} e^{i(k-n)\theta} \left[ v^\mu_{\uparrow\downarrow} \bar{\psi}_{-n}(t) \psi^\dagger_{k}(t) + v^\mu_{\downarrow\uparrow} \bar{\psi}_{-n}(t) \psi^\dagger_{k}(t) \right] \tag{39} \]
\[ A^\mu(\vec{t}) = \frac{1}{2\pi} \sum_{k,n=0,\pm1,\cdots} e^{i(k-n)\theta} \left[ a^\mu_{\uparrow\downarrow} \bar{\psi}_{-n}(t) \psi^\dagger_{k}(t) + a^\mu_{\downarrow\uparrow} \bar{\psi}_{-n}(t) \psi^\dagger_{k}(t) \right], \tag{40} \]

where

\[ v^1_{\uparrow \downarrow} = i, \ v^1_{\downarrow \uparrow} = i, \ v^2_{\uparrow \downarrow} = 1, \ v^2_{\downarrow \uparrow} = -1 \tag{41} \]
\[ a^1_{\uparrow \downarrow} = -i, \ a^1_{\downarrow \uparrow} = i, \ a^2_{\uparrow \downarrow} = -1, \ a^2_{\downarrow \uparrow} = -1. \tag{42} \]

\(^7\)Euclidean correlation functions can also be defined without analytic continuation. However this involves new complications which are described in [5].
From (39) and (40), we have

\[ S^{\mu\nu}(\vec{t}) = \frac{1}{(2\pi)^2} \left( e^{2i\theta} v_\mu^\mu \psi^{\dagger}_1(\vec{t})\psi^{\dagger}_1(t) + e^{-2i\theta} v_\mu^\mu \psi^{\dagger}_{-1}(\vec{t})\psi^{\dagger}_{-1}(t) \right) \begin{bmatrix} a_\nu^\nu \psi^{\dagger}_0(0)\psi_0^\dagger(0) \\ + a_{\nu\nu} \psi^{\dagger}_0(0)\psi_0^\dagger(0) \end{bmatrix} \right|_E. \] (43)

Recalling the correlator results from the previous section, we have

\[ S^{\mu\nu}(\vec{t}) = \frac{1}{(2\pi)^2} \left( e^{2i\theta} v_\mu^\mu a_\nu^\nu + e^{-2i\theta} v_\mu^\mu a_{\nu\nu} \right), \] (44)

and so

\[- \partial_\nu S^{\mu\nu} - \partial_\nu S^{\nu\mu} = -\frac{4i}{(2\pi)^2} \delta^{\mu_1} \left\{ \frac{\partial}{\partial x} \left( \frac{2xy}{(x^2+y^2)^2} \right) - \frac{\partial}{\partial y} \left( \frac{x^2-y^2}{(x^2+y^2)^2} \right) \right\} \] (45)

\[+ \frac{4i}{(2\pi)^2} \delta^{\mu_2} \left\{ \frac{\partial}{\partial x} \left( \frac{x^2-y^2}{(x^2+y^2)^2} \right) + \frac{\partial}{\partial y} \left( \frac{2xy}{(x^2+y^2)^2} \right) \right\}. \]

Integrating with any smooth test function we recognize these terms as derivatives of the Dirac delta distribution,

\[- \partial_\nu S^{\mu\nu} - \partial_\nu S^{\nu\mu} = -\frac{4i}{(2\pi)^2} \delta^{\mu_1} \left[ -\pi \partial_2 \delta(\vec{t}) \right] + \frac{4i}{(2\pi)^2} \delta^{\mu_2} \left[ -\pi \partial_1 \delta(\vec{t}) \right] \] (46)

\[= \frac{i}{\pi} \varepsilon^{\mu\nu} \partial_\nu \delta(\vec{t}). \]

We conclude that

\[ \int d^2 t \ e^{i\vec{p} \cdot \vec{t}} \left( \left\langle 0 \left| V^{\nu}(\vec{t}) \partial_\nu A^\mu(0) \right| 0 \right\rangle_E - \left\langle 0 \left| \partial_\nu V^{\nu}(\vec{t}) A^\mu(0) \right| 0 \right\rangle_E \right) = \frac{1}{\pi} \varepsilon^{\mu\nu} p_\nu. \] (47)

If we now choose to maintain conservation of the vector current, we have

\[ \int d^2 t \ e^{i\vec{p} \cdot \vec{t}} \left\langle 0 \left| V^{\mu}(\vec{t}) \partial_\nu A^\nu(0) \right| 0 \right\rangle_E = \frac{1}{\pi} \varepsilon^{\mu\nu} p_\nu, \] (48)

which is the desired result for the axial anomaly (see [6]). This should not be surprising. With \( \partial_\nu A^\nu \) placed at the origin, the calculation we have done is the same as that of standard field theory in position space. For the case when \( \partial_\nu A^\nu \) is not at the origin, our spherical field calculation generates an expansion of

\[ S^{\mu\nu}(\vec{t} - \vec{t}') = \left\langle 0 \left| V^{\mu}(\vec{t}) A^\nu(\vec{t}') \right| 0 \right\rangle_E \] (49)

in terms of sums of spherical waves. The partial sums of this expansion converge pointwise (except at \( \vec{t} = \vec{t}' \)) to the result (44), and we again get the correct axial anomaly.
5 Matrix representation

Spherical fermion fields can be studied using ordinary matrices. Unlike the spherical bosonic system which corresponds with a multidimensional partial differential equation, the spherical fermionic system corresponds with a set of coupled first-order ordinary differential equations. We illustrate some basic methods here using the free fermion system.

Let us define the following column vectors

\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = |\emptyset\rangle, \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \{s^\dagger_{n+1}\}, \quad \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = \{s^\dagger_n\}, \quad \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \{s^\dagger_n, s^\dagger_{n+1}\}.
\]

We now write the Hamiltonian and creation and annihilation operators as matrices,

\[
H_n(0,0,t) = \begin{bmatrix}
\frac{1}{t} & 0 & 0 & 0 \\
0 & -\frac{n}{t} & m & 0 \\
0 & m & \frac{n+1}{t} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
a^\dagger_{n+1} = \begin{bmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \quad a^\dagger_n = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
a^{\uparrow\uparrow}_n = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \quad a^\uparrow_n = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

As an illustrative example we calculate the correlation function

\[
\langle 0 | \psi^\dagger_1(t) \psi^\dagger_0(0) | 0 \rangle_E
\]

for the case \( m \neq 0 \). Let

\[
U(t_2,t_1) = T \exp \left\{ - \int_{t_1}^{t_2} dt \, H_0(0,0,t) \right\}.
\]
Using the fact that
\[
\frac{\partial}{\partial t_2} U(t_2, t_1) = -H_0(0, 0, t_2) \cdot U(t_2, t_1),
\]  
(55)
we find
\[
U(t_2, t_1) |\{s\uparrow_1\} = |m| t_1 \cdot \left[ \begin{array}{c} K_0(|m| t_1)I_1(|m| t_2) \\ + I_1(|m| t_1)K_0(|m| t_2) \end{array} \right] |\{s\uparrow_1\} \right. 
+ m t_1 \cdot \left. \left[ \begin{array}{c} -K_0(|m| t_1)I_0(|m| t_2) \\ + I_0(|m| t_1)K_0(|m| t_2) \end{array} \right] |\{s\uparrow_0\} \right),
\]  
(57)
\[
U(t_2, t_1) |\{s\uparrow_0\} = |m| t_1 \cdot \left[ \begin{array}{c} K_1(|m| t_1)I_0(|m| t_2) \\ + I_1(|m| t_1)K_0(|m| t_2) \end{array} \right] |\{s\uparrow_0\} \right. 
+ m t_1 \cdot \left. \left[ \begin{array}{c} -K_1(|m| t_1)I_1(|m| t_2) \\ + I_1(|m| t_1)K_1(|m| t_2) \end{array} \right] |\{s\uparrow_1\} \right),
\]  
(58)
\[
U(t_2, t_1) \left| \{s\uparrow_n, s\uparrow_{n+1}\} = \frac{\Omega}{t_2} \left( \{s\uparrow_n, s\uparrow_{n+1}\} \right). \right.
\]  
(59)
Therefore
\[
\lim_{t_i \to 0^+} \text{Tr} U(t_f, t_i) = 1 + \lim_{t_i \to 0^+} |m| t_i K_1(|m| t_i)I_0(|m| t_f) 
= \lim_{t_f \to \infty} I_0(|m| t_f)
\]  
(60)
and
\[
\lim_{t_i \to 0^+} \text{Tr} U(t_f, t) a_i^{\dagger} U(t_f, t_i) a_i^{\dagger} = \lim_{t_f \to \infty} |m| \cdot \left[ K_1(|m| t)I_0(|m| t_f) + I_1(|m| t)K_0(|m| t_f) \right].
\]  
(61)
We conclude that
\[
\langle 0 | \psi_i(t) \bar{\psi}_i(0) | 0 \rangle_E = \lim_{t_i \to 0^+} \frac{\text{Tr} U(t_f, t_i) a_i^{\dagger} U(t, t_i) a_i^{\dagger}}{\text{Tr} U(t_f, t_i)}
= |m| K_1(|m| t),
\]  
(62)
which agrees with (30).

Calculations in interacting theories are done in a similar manner. There we encounter new terms in the Hamiltonian which can be found by functional differentiation with respect to the sources $\eta$ and $\bar{\eta}$. Several detailed examples of interacting systems will be presented in later work.

6 Summary

In this paper we have extended the spherical field formalism to include fermionic systems. Since spherical field theory deals with continuous systems, it avoids problems associated with discrete approximation methods, such as fermion doubling and cancelled axial anomalies. The spherical field method should therefore be useful in studying chiral fermion systems.

We have shown that the time evolution of spherical fermion fields can be modelled using matrices and is described by a set of coupled first-order ordinary differential equations. We recall that in lattice field theory Grassmanian variables are associated with each fermionic degree of freedom at each lattice site. In spherical field theory, however, Grassmanian creation and annihilation operators are associated with each spherical partial wave. Although we have not analyzed interacting systems in this work, we anticipate that for comparable levels of accuracy spherical field theory will require manipulating much smaller anticommuting algebras. Since large sets of anticommuting variables present serious computational problems, spherical field theory may yield significant improvements in the numerical calculation of fermionic interactions.

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Appendix 1

In this appendix we derive the results (29), (30), and (31). Let us define

$$W_n = \ln Z_n.$$  

(63)
Using the notation,
\[ \langle A \rangle_t = \frac{\text{Tr} \left[ T \exp \left\{ - \int _t ^{\infty} dt' H_n(\eta, \eta, t') \right\} \right] \text{Tr} \left[ T \exp \left\{ - \int _0 ^t dt' H_n(\eta, \eta, t') \right\} \right]}{\text{Tr} \left[ T \exp \left\{ - \int _0 ^{\infty} dt' H_n(\eta, \eta, t') \right\} \right]}, \]
we note that
\[ W_n \frac{\delta}{\delta \eta _{n+1} (t)} = \ln Z_n \frac{\delta}{\delta \eta _{n+1} (t)} = \langle a _n ^{\dagger +} \rangle_t. \tag{65} \]
Differentiating with respect to \( t \), we find
\[ \frac{d}{dt} \left[ W_n \frac{\delta}{\delta \eta _{n+1} (t)} \right] = \langle [H_n(\eta, \eta, t), a _n ^{\dagger +}] \rangle_t = \langle -\frac{n}{t} a _n ^{\dagger +} + ma _{n+1} ^{\dagger +} + t\eta _{-n} ^{\dagger} \rangle_t. \tag{66} \]
Differentiating again, we get
\[ \frac{d^2}{dt^2} \left[ W_n \frac{\delta}{\delta \eta _{n+1} (t)} \right] = \langle \left( \frac{n}{t^2} a _n ^{\dagger +} + \frac{n}{t^2} \eta _{-n} ^{\dagger} - \frac{n}{t^2} \eta _{-n} ^{\dagger} + (n - 1)\eta _{-n} ^{\dagger} - \frac{dt}{dt} \eta _{-n} ^{\dagger} - mt\eta _{-n-1} ^{\dagger} \right) \rangle_t. \tag{67} \]

We now combine (65), (66), and (67),
\[ \left[ t^2 \frac{d^2}{dt^2} - t \frac{d}{dt} - (n^2 + 2n + m^2 t^2) \right] W_n \frac{\delta}{\delta \eta _{n+1} (t)} = \langle n t^2 \eta _{-n} ^{\dagger} - t^3 \frac{d}{dt} \eta _{-n} ^{\dagger} - mt^3 \eta _{-n-1} ^{\dagger} \rangle_t = nt^2 \eta _{-n} ^{\dagger} - t^3 \frac{d}{dt} \eta _{-n} ^{\dagger} - mt^3 \eta _{-n-1} ^{\dagger}. \tag{68} \]
By similar steps, we find
\[ \left[ t^2 \frac{d^2}{dt^2} - t \frac{d}{dt} - (n^2 - 1 + m^2 t^2) \right] W_n \frac{\delta}{\delta \eta _n (t)} = -(n + 1)t^2 \eta _{-n-1} ^{\dagger} - t^3 \frac{d}{dt} \eta _{-n-1} ^{\dagger} - mt^3 \eta _{-n} ^{\dagger}. \tag{69} \]
\[ \left[ t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - (n^2 + m^2 t^2) \right] \frac{\delta}{\delta \eta _{-n} (t)} W_n = (n + 1)t \eta _{n+1} ^{\dagger} + t^2 \frac{d}{dt} \eta _{n+1} ^{\dagger} - mt^2 \eta _n ^{\dagger}. \tag{70} \]
\[ \left[ t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - ((n + 1)^2 + m^2 t^2) \right] \frac{\delta}{\delta \eta _{-n-1} (t)} W_n = -nt \eta _n ^{\dagger} + t^2 \frac{d}{dt} \eta _n ^{\dagger} - mt^2 \eta _{n+1} ^{\dagger}. \tag{71} \]
For $m \neq 0$ the general solution for $W_n$ is
\[
\int dt_1 dt_2 \left[ t_1 \bar{t}_{-n}(t_1) \quad t_1 \bar{t}_{-n-1}(t_1) \right] \mathbf{M}_n(t_1, t_2) \left[ \begin{array}{c} t_2 \eta^i(t_2) \\ t_2 \bar{t}_{n+1}(t_2) \end{array} \right] + F \left[ \int dt \int K_n(|m| t) \bar{t}_{-n}(t), \ldots, \int dt I_{n+1}(|m| t) \bar{t}_{n+1}(t) \right],
\]
where $F$ is the general homogeneous solution and $\mathbf{M}_n(t_1, t_2)$ is given by
\[
\theta(t_1 - t_2) \left[ \begin{array}{cc} m K_n(|m| t_1) I_n(|m| t_2) & m K_n(|m| t_1) I_{n+1}(|m| t_2) \\ m K_{n+1}(|m| t_1) I_n(|m| t_2) & m K_{n+1}(|m| t_1) I_{n+1}(|m| t_2) \end{array} \right] + \theta(t_2 - t_1) \left[ \begin{array}{cc} m I_n(|m| t_1) K_n(|m| t_2) & -m I_n(|m| t_1) K_{n+1}(|m| t_2) \\ -m I_{n+1}(|m| t_1) K_n(|m| t_2) & m I_{n+1}(|m| t_1) K_{n+1}(|m| t_2) \end{array} \right]
\]
for $m \neq 0$, and
\[
\theta(t_1 - t_2) \left[ \begin{array}{cc} \theta(-\frac{n-1}{2}) t_1^n & 0 \\ 0 & \frac{\theta(n+1) t_1^n}{t_1^n} \end{array} \right] - \theta(t_2 - t_1) \left[ \begin{array}{cc} 0 & \frac{\theta(n+1) t_1^n}{t_1^n} \\ \theta(-\frac{n-1}{2}) t_1^n & 0 \end{array} \right]
\]
for $m = 0$. The function $F$ depends on eight anticommuting variables and is therefore a polynomial of degree at most eight. It is straightforward to check that for $t_1 \neq 0$, the limits
\[
\lim_{t \to 0^+} \frac{\delta}{t_1 \delta \bar{t}_{-n}(t_1)} Z_n \left. \frac{\bar{\delta}}{t \delta \bar{t}_{-n}(t)} \right|_{\eta = \bar{\eta} = 0}
\]
and
\[
\lim_{t \to \infty} \frac{\delta}{t_1 \delta \bar{t}_{-n}(t_1)} Z_n \left. \frac{\bar{\delta}}{t \delta \bar{t}_{-n}(t)} \right|_{\eta = \bar{\eta} = 0}
\]
are well-defined. Corresponding limits for the other two-point correlators are also well-defined, and the same holds true for other correlators such as
\[
\lim_{t \to 0^+} \frac{\delta}{t_1 \delta \bar{t}_{-n-1}(t_1) t_2 \delta \bar{t}_{-n}(t_2)} Z_n t_2 \left. \frac{\bar{\delta}}{t_2 \delta \bar{t}_{n+1}(t_2) t \delta \bar{t}_{-n}(t)} \right|_{\eta = \bar{\eta} = 0}
\]
and
\[
\lim_{t \to \infty} \frac{\delta}{t_1 \delta \bar{t}_{-n-1}(t_1) t_2 \delta \bar{t}_{-n}(t_2)} Z_n t_2 \left. \frac{\bar{\delta}}{t_2 \delta \bar{t}_{n+1}(t_2) t \delta \bar{t}_{-n}(t)} \right|_{\eta = \bar{\eta} = 0}
\]
provided that $t_1, t_2, t_3 \neq 0$. From this and the fact that $W_n$ vanishes when $\eta = \bar{\eta} = 0$, we conclude that $F = 0$ and we obtain (29).
Appendix 2

The following is a list of the spherical correlation functions. For $m \neq 0$ we have

$$
\left< 0 \left| \psi_{n}^{\dagger}(t_{1}) \bar{\psi}_{-n}(t_{2}) \right| 0 \right>_{E} = \theta(t_{1} - t_{2}) m K_{n}(|m| t_{1}) I_{n}(|m| t_{2}) + \theta(t_{2} - t_{1}) m I_{n}(|m| t_{1}) K_{n}(|m| t_{2}),
\tag{79}
$$

$$
\left< 0 \left| \psi_{n+1}^{\dagger}(t_{1}) \bar{\psi}_{-n-1}(t_{2}) \right| 0 \right>_{E} = \theta(t_{1} - t_{2}) m K_{n+1}(|m| t_{1}) I_{n+1}(|m| t_{2}) + \theta(t_{2} - t_{1}) m I_{n+1}(|m| t_{1}) K_{n+1}(|m| t_{2}),
\tag{80}
$$

$$
\left< 0 \left| \psi_{n}^{\dagger}(t_{1}) \bar{\psi}_{-n-1}(t_{2}) \right| 0 \right>_{E} = \theta(t_{1} - t_{2}) |m| K_{n}(|m| t_{1}) I_{n+1}(|m| t_{2}) - \theta(t_{2} - t_{1}) |m| I_{n}(|m| t_{1}) K_{n+1}(|m| t_{2}),
\tag{81}
$$

$$
\left< 0 \left| \psi_{n+1}^{\dagger}(t_{1}) \bar{\psi}_{-n}(t_{2}) \right| 0 \right>_{E} = \theta(t_{1} - t_{2}) |m| K_{n+1}(|m| t_{1}) I_{n}(|m| t_{2}) - \theta(t_{2} - t_{1}) |m| I_{n+1}(|m| t_{1}) K_{n}(|m| t_{2}),
\tag{82}
$$

For $m = 0$,

$$
\left< 0 \left| \psi_{n}^{\dagger}(t_{1}) \bar{\psi}_{-n-1}(t_{2}) \right| 0 \right>_{E} = \left[ \frac{\theta(t_{1} - t_{2}) \theta(-n - \frac{1}{2})}{\theta(t_{2} - t_{1}) \theta(n + \frac{1}{2})} \right] \frac{\nu_{n}^{2}}{\nu_{n+1}^{2}},
\tag{83}
$$

$$
\left< 0 \left| \psi_{n+1}^{\dagger}(t_{1}) \bar{\psi}_{-n}(t_{2}) \right| 0 \right>_{E} = \left[ \frac{\theta(t_{1} - t_{2}) \theta(n + \frac{1}{2})}{\theta(t_{2} - t_{1}) \theta(-n - \frac{1}{2})} \right] \frac{\nu_{n+1}^{2}}{\nu_{n}^{2}},
\tag{84}
$$

$$
\left< 0 \left| \psi_{n}^{\dagger}(t_{1}) \bar{\psi}_{-n}(t_{2}) \right| 0 \right>_{E} = \left< 0 \left| \psi_{n+1}^{\dagger}(t_{1}) \bar{\psi}_{-n-1}(t_{2}) \right| 0 \right>_{E} = 0.
\tag{85}
$$

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\footnote{See comments on Euclidean correlation functions immediately following (85).}
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