Nonautonomous ultradiscrete hungry Toda lattice and a generalized box-ball system

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Abstract
A nonautonomous version of the ultradiscrete hungry Toda lattice with a finite lattice boundary condition is derived by applying reduction and ultradiscretization to a nonautonomous two-dimensional discrete Toda lattice. It is shown that the derived ultradiscrete system has a direct connection to the box-ball system with many kinds of balls and finite carrier capacity. Particular solutions to the ultradiscrete system are constructed by using the theory of some sort of discrete biorthogonal polynomials.

Keywords: integrable systems, biorthogonal polynomials, discrete two-dimensional Toda lattice

1. Introduction

The box-ball system (BBS) is a soliton cellular automaton composed of an infinite array of boxes and a finite number of balls [24], known as one of the most important ultradiscrete integrable systems. The time evolution equation of the original BBS

\[ U_n^{t+1} = \min \left( 1 - U_n^{t}, \sum_{j=-\infty}^{n-1} (U_j^t - U_j^{t+1}) \right), \tag{1} \]

where \( U_n^t \in \{0, 1\} \) denotes the number of balls in the \( n \)th box at time \( t \), is derived from the discrete KdV lattice through ultradiscretization [27, 29]. The left side of figure 1 shows an example of the time evolution of the original BBS (1), in which ‘1’ and ‘.’ denote a ball and an empty box, respectively. We can observe that three blocks of balls move from left to right and interact with each other like solitons.

There is another time evolution equation for the original BBS:
$E(t+1) = E(t) - Q(t) + Q(t+1), \quad (2a)$

$E(t) = E(t) + Q(t) + Q(t+1), \quad (2b)$

for $n = 0, 1, \ldots, N-1$ with the finite lattice boundary condition

$E_{-1} = E_{N-1} = +\infty \quad (2c)$

for all $t \in \mathbb{Z}$, where

- $Q_n(t)$: the number of balls in the $n$th block at time $t$;
- $E_n(t)$: the number of empty boxes between the $n$th and $(n+1)$st blocks of balls at time $t$;
- $N$: the number of the blocks of balls.

It is known that equations (2) are derived from the discrete Toda lattice

$q_n(t+1) + e_{n+1}(t+1) = q_n(t) + e_n(t), \quad (2a)$

$q_n(t+1) - e_{n-1}(t+1) = q_{n+1}(t) - e_n(t), \quad (2b)$

with the finite lattice condition

$e_{-1}(t) = e_{N-1}(t) = 0 \quad (2c)$

through ultradiscretization [19]. The right side of figure 1 shows an example of the time evolution of the ultradiscrete Toda lattice (2), in which the initial values are chosen to correspond to the initial state of the original BBS on the left side.

We can introduce some extended rules to the original BBS. The time evolution equations of the extended BBSs are derived from the nonautonoumous discrete KP lattice through reduction and ultradiscretization, or from the geometric crystal for $\hat{sl}_M$ through crystallization [6, 8]. We have already known the following correspondences between the BBS with an extended rule and an ultradiscrete Toda type system:

(i) The BBS with many kinds of balls and the ultradiscrete hungry Toda lattice [26];
(ii) The BBS with a carrier of balls whose capacity is finite and the nonautonomous ultradiscrete Toda lattice [15];
(iii) The BBS with boxes whose capacity is greater than one and a generalized ultradiscrete Toda lattice [14].

Figure 1. An example of the time evolution of (left) the original BBS (1) and (right) the ultradiscrete Toda lattice with the finite lattice boundary condition (2).
In addition, there are studies on the relation between the BBS and the ultradiscrete Toda lattice for the case of a periodic boundary condition [7], and for the case in which the number of balls in each box can take any real value [5].

In this paper, we wish to derive and study an ultradiscrete Toda type system corresponding to the BBS with both the rules (i) and (ii). To this end, in section 2, we first consider the theory of biorthogonal polynomials and derive a nonautonomous version of the discrete two-dimensional Toda lattice (nd-2D-Toda lattice). It is known that the theory of (bi)orthogonal functions is a very useful tool for deriving and analyzing many Toda type systems and their solutions [2, 10, 16, 18, 20–23]. In addition to these known results, we will show, based on the previous studies [1, 28, 30], that the nd-2D-Toda lattice is derived as compatibility conditions for spectral transformations of biorthogonal polynomials. Since the discrete two-dimensional Toda lattice hierarchy yields many discrete Toda type systems by imposing reduction conditions, we will also be able to derive many nonautonomous discrete Toda type systems by using the techniques developed in the present paper. In section 3, we will impose \((M, 1)\)-reduction condition for biorthogonal polynomials and the nd-2D-Toda lattice, and derive a nonautonomous version of the discrete hungry Toda lattice (ndh-Toda lattice). Further, we will impose a finite lattice boundary condition to the ndh-Toda lattice and give a particular solution coming from a determinant structure of the biorthogonal polynomials. We will also give a condition for the positivity of the solution. In section 4, we will ultradiscretize the ndh-Toda lattice and its solution, and prove that the derived ultradiscrete system is another time evolution equation of the generalized BBS. Section 5 is devoted to concluding remarks.

2. Biorthogonal polynomials and semi-infinite lattice equations

In this section, we consider the theory of biorthogonal polynomials, and derive the nd-2D-Toda lattice with a semi-infinite lattice boundary condition as compatibility conditions for spectral transformations of the biorthogonal polynomials.

2.1. Definitions and determinant representations

Let \( B : \mathbb{C}[z] \times \mathbb{C}[z] \rightarrow \mathbb{C} \) be a bilinear form. Let us consider polynomial sequences \( \{ \phi_n(z) \}_{n=0}^{\infty} \) and \( \{ \psi_n(z) \}_{n=0}^{\infty} \) satisfying the following properties:

(i) \( \deg \phi_n(z) = \deg \psi_n(z) = n \);

(ii) The polynomials \( \phi_n(z) \) and \( \psi_n(z) \) are monic; i.e. the leading coefficients of \( \phi_n(z) \) and \( \psi_n(z) \) are one;

(iii) The biorthogonal relation with respect to \( B \)

\[
B[\phi_m(z), \psi_n(z)] = h_n \delta_{m,n}, \quad h_n \neq 0, \quad m, n = 0, 1, 2, \ldots
\]

holds, where \( \delta_{m,n} \) is the Kronecker delta.

We call the polynomial sequences \( \{ \phi_n(z) \}_{n=0}^{\infty} \) and \( \{ \psi_n(z) \}_{n=0}^{\infty} \) the pair of monic biorthogonal polynomial sequences with respect to \( B \).

Note that, since both \( \{ \phi_n(z) \}_{n=0}^{\infty} \) and \( \{ \psi_n(z) \}_{n=0}^{\infty} \) span \( \mathbb{C}[z] \), the biorthogonal relation (3) is equivalent to

\[
B[\phi_n(z), z^m] = h_n \delta_{m,n}, \quad B[z^n, \psi_n(z)] = h_n \delta_{m,n},
\]

\[
n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n.
\]
Theorem 2.1. The pair of monic biorthogonal polynomial sequences \( \{\phi_n(z)\}_{n=0}^{\infty} \) and \( \{\psi_n(z)\}_{n=0}^{\infty} \) has the determinant representations

\[
\phi_0(z) = 1, \quad \phi_n(z) = \frac{1}{\tau_n} \begin{vmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n-1} & 1 \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n-1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} & z^{n-1} \\ \tau_0 & \tau_1 & \cdots & \tau_{n-1} & z^n \end{vmatrix}, \quad n = 1, 2, 3, \ldots.
\]

\[\tau_n = |\mu_{ij}|_{i,j=0}^{n-1}, \quad n = 1, 2, 3, \ldots.\]

Here, we assume that \( \tau_n \neq 0 \) for all \( n = 1, 2, 3, \ldots \). The constant \( h_n \) in the biorthogonal relation (4) is given by

\[
h_n = \frac{\tau_{n+1}}{\tau_n}.
\]

**Proof.** Let \( c_{n,j} \) be the coefficients of the polynomial \( \phi_n(z) \):

\[
\phi_n(z) = z^n + \sum_{i=0}^{n-1} c_{n,i} z^i.
\]

Then, the biorthogonal relation (4) gives the linear equation

\[
\begin{pmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} & \mu_{n,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} & \mu_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1} & \mu_{n,n-1} \\ \mu_{0,n} & \mu_{1,n} & \cdots & \mu_{n-1,n} & \mu_{n,n} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ c_{n,n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_n \end{pmatrix}.
\]

Applying Cramer’s rule yields the relation \( 1 = \tau_n h_n/\tau_{n+1} \), which implies \( h_n = \tau_{n+1}/\tau_n \) and the determinant representation of \( \phi_n(z) \). The determinant representation of \( \psi_n(z) \) is also given in the same manner. \( \square \)

The proof of theorem 2.1 also claims that, for a bilinear form \( \mathcal{B} \) on \( \mathbb{C}[z] \), there is a unique pair of monic biorthogonal polynomial sequences \( \{\phi_n(z)\}_{n=0}^{\infty} \) and \( \{\psi_n(z)\}_{n=0}^{\infty} \) if exists.
Example 2.1. We give a simple example. Consider

$$
\mathcal{B}[\phi(z), \psi(z)] := \int_0^1 \phi(x^\alpha)\psi(x^\beta) \, dx,
$$

where $\alpha$ and $\beta$ are nonzero real numbers satisfying $\alpha i + \beta j \neq -1$ for all $i, j = 0, 1, 2, \ldots$ Then, we have

$$
\mu_{ij} = \int_0^1 x^{\alpha i + \beta j} \, dx = \frac{1}{\alpha i + \beta j + 1}.
$$

Since the determinant $\tau_n$ becomes a Cauchy determinant, we obtain

$$
\tau_n = \frac{\prod_{0 \leq i < j \leq n-1} (\alpha j - \alpha i)(\beta j - \beta i)}{\prod_{i=0}^{n-1} (\alpha i + \beta j + 1)}.
$$

Hence $\tau_n \neq 0$ for all $n = 1, 2, 3, \ldots$. The corresponding monic biorthogonal polynomials are calculated as

$$
\phi_n(z) = (-1)^n \sum_{k=0}^n \binom{n}{k} (\frac{\alpha k + \beta j + 1}{an + \beta j + 1})(-z)^k,
$$

$$
\psi_n(z) = (-1)^n \sum_{k=0}^n \binom{n}{k} (\frac{\alpha i + \beta k + 1}{ai + \beta n + 1})(-z)^k.
$$

These polynomials satisfy

$$
\mathcal{B}[\phi_m(z), \psi_n(z)] = \int_0^1 \phi_m(x^\alpha)\psi_n(x^\beta) \, dx = h_n \delta_{m,n},
$$

where

$$
h_n = \frac{\tau_{n+1}}{\tau_n} = \frac{\alpha^n \beta^n (n!)^2}{(an + \beta n + 1) \prod_{i=0}^{n-1} (\alpha j + \beta n + 1)(an + \beta j + 1)}.
$$

As a special case, let $\alpha = \beta = 1$. Then, we obtain

$$
\phi_n(z) = \psi_n(z) = (-1)^n \sum_{k=0}^n \binom{2n}{k} \binom{n+k}{n} (\frac{2n}{n}) (\frac{n+k}{n}) (-z)^k,
$$

$$
\frac{1}{2n+1} \frac{(2n)}{n} \frac{(n+k)}{n} (-z)^k
$$

$$
h_n = \frac{1}{2n+1} \frac{(2n)}{n} \frac{(n+k)}{n} (-z)^k
$$

satisfy the orthogonality relation

$$
\mathcal{B}[\phi_m(z), \psi_n(z)] = \int_0^1 \phi_m(x^\alpha)\psi_n(x^\beta) \, dx = h_n \delta_{m,n}.
$$
\[ \int_{0}^{1} \hat{\phi}_n(x) \hat{\phi}_n(x) \, dx = \frac{1}{2n + 1} \delta_{m,n}. \]

The polynomials \( \hat{\phi}_n(z) \) are known as the shifted Legendre polynomials; \( \hat{\phi}_n((z + 1)/2) \) become the ordinary Legendre polynomials. The example thus gives a biorthogonal generalization of the Legendre polynomials.

**Remark 2.1.** Theorem 2.1 is an analogue of the classical result of orthogonal polynomials. There should be many previous works including this result in several fields. For example, in the field of integrable systems, Adler and Moerbeke [1] studied the biorthogonal polynomials, which they called string-orthogonal polynomials, and the determinant representation is written in section 3 of their paper. Konhauser [11] studied the biorthogonal polynomials like example 2.1 in detail.

Iserles and Nørsett [9] also studied biorthogonal polynomials. Although the definition of the biorthogonal polynomials in their paper is slightly different from the one in this paper, their result (theorem 1) is very similar to theorem 2.1.

### 2.2. Spectral transformations

From the pair of the monic biorthogonal polynomial sequences \( \{\phi_n(z)\}_{n=0}^{\infty} \) and \( \{\psi_n(z)\}_{n=0}^{\infty} \) with respect to \( \mathcal{B} \), we can construct new monic biorthogonal polynomials

\[ \phi_n(z) := \frac{\phi_{n+1}(z) + q_n^* \phi_n(z)}{z - s^*}, \quad q_n^* := -\frac{\phi_{n+1}(s^*)}{\phi_n(s^*)}, \quad (6a) \]

\[ \psi_n(z) := \frac{\psi_{n+1}(z) + q_n \psi_n(z)}{z - s^*}, \quad q_n := -\frac{\psi_{n+1}(s^*)}{\psi_n(s^*)}, \quad (6b) \]

for \( n = 0, 1, 2, \ldots \), where \( s^* \) and \( s^* \) are any complex parameters satisfying \( \phi_n(s^*) \neq 0 \) and \( \psi_n(\tilde{s}^*) \neq 0 \) for all \( n \). Since \( \phi_{n+1}(s^*) + q_n^* \phi_n(s^*) = 0 \) and \( \psi_{n+1}(s^*) + q_n \psi_n(s^*) = 0 \), both \( \phi_n(z) \) and \( \psi_n(z) \) are monic \( n \)th degree polynomials. The new polynomials are biorthogonal with respect to the bilinear forms \( \mathcal{B}^* \) and \( \mathcal{B}^! \) defined by

\[ \mathcal{B}^*[\varphi', \varphi'] := \mathcal{B}[(z - s^*)\varphi', \varphi'], \quad \mathcal{B}^![\varphi', \varphi'] := \mathcal{B}[\varphi', (z - s^*)\varphi'], \quad i, j = 0, 1, 2, \ldots . \quad (7) \]

We can readily verify that the biorthogonal relations

\[ \mathcal{B}^*[\phi_n^*(z), \varphi^m] = \mathcal{B}[\phi_{n+1}(z) + q_n^* \phi_n(z), \varphi^m] = \frac{q_n^* \tau_{n+1}}{\tau_n} \delta_{m,n}, \]

\[ \mathcal{B}^![\varphi^m, \psi_n^*(z)] = \mathcal{B}[\varphi^m, \psi_{n+1}(z) + q_n \psi_n(z)] = \frac{q_n \tau_{n+1}}{\tau_n} \delta_{m,n}, \]

indeed hold for \( n = 0, 1, 2, \ldots \) and \( m = 0, 1, \ldots, n \). The relations (6) are analogues of the Christoffel transformation for monic orthogonal polynomials.

Next, let us derive a relation between the polynomial sequences \( \{\phi_n(z)\}_{n=0}^{\infty} \) and \( \{\phi_n^!(z)\}_{n=0}^{\infty} \) satisfying the biorthogonal relation

\[ \mathcal{B}^![\phi_n^!(z), \varphi^m] = \frac{q_n \tau_{n+1}}{\tau_n} \delta_{m,n}, \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n. \]
For each \( n = 0, 1, 2, \ldots \), write \( \phi_n(z) \) as a linear combination of \( \phi_0(z), \phi_1(z), \ldots, \phi_n(z) \):
\[
\phi_{n+1}(z) = \phi_{n+1}^\dagger(z) + \sum_{i=0}^n c_{n+1,i} \phi_i^\dagger(z).
\]
Then we have, if \( n \geq 1 \),
\[
B[\phi_{n+1}(z), z - s^\dagger] = B^\dagger \left[ \phi_{n+1}^\dagger(z) + \sum_{i=0}^n c_{n+1,i} \phi_i^\dagger(z), 1 \right] = c_{n+1,0} \frac{q_0^\dagger \tau_1}{\tau_0} = 0.
\]
Since \( q_0^\dagger \tau_1 / \tau_0 \neq 0 \), that implies \( c_{n+1,0} = 0 \). In the same manner, the equation \( B[\phi_{n+1}(z), (z-s^\dagger)z^\circ] = 0 \) implies \( c_{n+1,m} = 0, m = 0, 1, \ldots, n-1 \), by induction on \( m \). Finally,
\[
B[\phi_{n+1}(z), (z-s^\dagger)z^\circ] = B^\dagger \left[ \phi_{n+1}^\dagger(z) + c_{n+1,n} \phi_n^\dagger(z), z^\circ \right] = c_{n+1,n} \frac{q_n^\dagger \tau_{n+2}}{\tau_{n+1}} = \frac{\tau_{n+2}}{\tau_{n+1}}.
\]
Thus \( c_{n+1,n} = \tau_n \tau_{n+2} q_n^\dagger \tau_{n+1})^2 \neq 0 \). We can show by similar discussions that the relation
\[
\psi_{n+1}(z) = \psi_{n+1}^\dagger(z) + \frac{\tau_n \tau_{n+2}}{q_n^\dagger \tau_{n+1})^2} \psi_n^\dagger(z)
\]
also holds.

We summarize the results above.

**Theorem 2.2.** Let \( B, B^\dagger \) be bilinear forms on \( \mathbb{C}[z] \) connected by the relations (7). Suppose that \( \{\phi_n(z)\}_{n=0}^\infty \) and \( \{\phi_n(z)\}_{n=0}^\infty \) are the pairs of the monic biorthogonal polynomial sequences with respect to \( B, B^\dagger \), respectively. Then, these polynomial sequences satisfy the following relations:

\[
(z-s^\dagger)\phi_n^\dagger(z) = \phi_{n+1}^\dagger(z) + q_n^\dagger \phi_n(z), \quad q_n^\dagger = -\frac{\phi_{n+1}(s^\dagger)}{\phi_n(s^\dagger)}, \quad (8a)
\]
\[
\phi_{n+1}(z) = \phi_{n+1}^\dagger(z) + e_n^\dagger \phi_n(z), \quad e_n^\dagger = \frac{\tau_n \tau_{n+2}}{q_n^\dagger (\tau_{n+1})^2}, \quad (8b)
\]
\[
(z-s^\dagger)\psi_n^\dagger(z) = \psi_{n+1}^\dagger(z) + q_n^\dagger \psi_n(z), \quad q_n^\dagger = -\frac{\phi_{n+1}(s^\dagger)}{\psi_n(s^\dagger)}, \quad (8c)
\]
\[
\psi_{n+1}(z) = \psi_{n+1}^\dagger(z) + e_n^\dagger \psi_n(z), \quad e_n^\dagger = \frac{\tau_n \tau_{n+2}}{q_n^\dagger (\tau_{n+1})^2}, \quad (8d)
\]

for \( n = 0, 1, 2, \ldots \).

We should remark that the relations \((8b)\) and \((8d)\) are analogues of the Geronimus transformation for monic orthogonal polynomials.

Let us define the moments of \( B^\dagger \) and \( B^\dagger \)
\[
\mu_i^\dagger := B^\dagger \left[ z^i, z^\dagger \right], \quad \mu_i^\dagger := B^\dagger \left[ z^i, z^\dagger \right], \quad i, j = 0, 1, 2, \ldots ,
\]
and the determinants
\[
\tau_0^\dagger := 1, \quad \tau_0^\dagger := 1, \quad \tau_n^\dagger := |\mu_{i,j}|_{i,j=0}^{n-1}, \quad n = 1, 2, 3, \ldots ,
\]
Then, from (7), we have
\[ \mu_{ij} = \mu_{i+1,j} - s^i \mu_{ij}, \quad \mu_{ij}^\dagger = \mu_{ij+1} - s^j \mu_{ij}. \] (9)

Applying the elementary column-additions (or row-additions) to the determinant representations of the monic biorthogonal polynomials (5) and using the relations (9), we have
\[ \phi_n(s^\ast) = (-1)^n \frac{\tau_n^\ast}{\tau_n}, \quad \psi_n(s^\dagger) = (-1)^n \frac{\tau_n^\dagger}{\tau_n}. \]

Hence, the variables appear in (8) are rewritten as
\[ q_n^\ast = \frac{\tau_n \tau_{n+1}}{\tau_{n+1} \tau_n}, \quad e_n^\dagger = \frac{\tau_{n+2} \tau_n}{\tau_{n+1} \tau_{n+1}}, \quad q_n^\dagger = \frac{\tau_n \tau_{n+1}}{\tau_{n+1} \tau_{n+1}}, \quad e_n^\ast = \frac{\tau_{n+2} \tau_n}{\tau_{n+1} \tau_{n+1}}. \]

**Remark 2.2.** The results in this section were firstly derived by Tsujimoto and Kondo [30] based on the work by Adler and Moerbeke [1]. However, they studied only the autonomous discrete two-dimensional Toda lattice; the role of nonautonomous parameters is not discussed in the paper.

### 2.3. Nonautonomous discrete semi-infinite two-dimensional Toda lattice

In this section, we consider the chain of spectral transformations (8) and derive the nd-2D-Toda lattice as their compatibility conditions. This derivation is a generalization of the one for the nd-Toda lattice from monic orthogonal polynomials [16].

Let us introduce discrete time variables \( k_1, k_2, t_1, t_2 \) into bilinear forms as follows:
\[ \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[\xi', \zeta'] := \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[\zeta', \zeta'], \] (10a)
\[ \mathcal{B}^{(k_1,k_2+1),(t_1,t_2)}[\xi', \zeta'] := \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[\zeta', \zeta^{t_1+1}], \] (10b)
\[ \mathcal{B}^{(k_1,k_2+1),(t_1,t_2)}[\zeta', \zeta'] := \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[(\zeta - \tau^{t_1}_1)\zeta', \zeta'], \] (10c)
\[ \mathcal{B}^{(k_1,k_2+1),(t_1+1,t_2)}[\zeta', \zeta'] := \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[\zeta', (\zeta - \tau^{t_2}_2)\zeta'], \] (10d)
for all \( i,j = 0, 1, 2, \ldots \), where \( s^{(t_2)}_1 \) and \( s^{(t_1)}_2 \) are parameters chosen at each \( t_1 \) and \( t_2 \), respectively.

Then, the moment
\[ \mu^{(t_1,t_2)}_{ij} := \mathcal{B}^{(0,0),(t_1,t_2)}[\zeta', \zeta'] \]
has the relations
\[ \mu^{(t_1+1,t_2)}_{ij} = \mu^{(t_1,t_2)}_{i+1,j}, \quad \mu^{(t_1,t_2+1)}_{ij} = \mu^{(t_1,t_2)}_{ij} - s^{(t_2)}_1 \mu^{(t_1,t_2)}_{ij} + s^{(t_1)}_2 \mu^{(t_1,t_2)}_{i+1,j}. \]

Let \( \{ \phi^{(k_1,k_2),(t_1,t_2)}_n(z) \}_{n=0}^{\infty} \) be one of the pair of the monic biorthogonal polynomial sequences with respect to \( \mathcal{B}^{(k_1,k_2),(t_1,t_2)} \):
\[ \mathcal{B}^{(k_1,k_2),(t_1,t_2)}[\phi^{(k_1,k_2),(t_1,t_2)}_n(z), \zeta^m] \equiv \tau^{(k_1,k_2),(t_1,t_2)}_{n+1} \delta_{m,n}, \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n, \]
where
\[ \tau_0^{(k_1,k_2,t_1,t_2)} := 1, \quad \tau_n^{(k_1,k_2,t_1,t_2)} := \|e_n^{(t_1,t_2)}\|_{l_j=0}^{n-1}, \quad n = 1, 2, 3, \ldots \]

The polynomials \( \phi_n^{(k_1,k_2,t_1,t_2)}(z) \) also have the relations
\[ \phi_n^{(k_1,k_2,t_1,t_2)}(0) = \frac{\tau_n^{(k_1+1,k_2,t_1,t_2)}}{\tau_n^{(k_1,k_2,t_1,t_2)}}, \quad \phi_n^{(k_1,k_2,t_1,t_2)}(1) = \frac{\tau_{n+1}^{(k_1,k_2,t_1+1,t_2)}}{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}. \]

From theorem 2.2, the polynomials \( \phi_n^{(k_1,k_2,t_1,t_2)}(z) \) satisfy
\[ z\phi_{n+1}^{(k_1,k_2,t_1,t_2)}(z) = \phi_n^{(k_1,k_2,t_1,t_2)}(z) + \psi_n^{(k_1,k_2,t_1,t_2)}(z), \quad (11a) \]
\[ \phi_{n+1}^{(k_1,k_2-1,t_1,t_2)}(z) = \phi_n^{(k_1,k_2,t_1,t_2)}(z) + \psi_n^{(k_1,k_2,t_1,t_2)}(z), \quad (11b) \]
\[ (z−x_1)\phi_{n+1}^{(k_1,k_2,t_1,t_2)}(z) = \phi_n^{(k_1,k_2,t_1,t_2)}(z) + \psi_n^{(k_1,k_2,t_1,t_2)}(z), \quad (11c) \]
\[ \phi_{n+1}^{(k_1,k_2,t_1,t_2-1)}(z) = \phi_n^{(k_1,k_2,t_1,t_2)}(z) + \psi_n^{(k_1,k_2,t_1,t_2-1)}(z), \quad (11d) \]

for \( n = 0, 1, 2, \ldots \), where
\[ q_n^{(k_1,k_2,t_1,t_2)} = \frac{\tau_n^{(k_1,k_2,t_1,t_2)} - \tau_{n+1}^{(k_1,k_2,t_1+1,t_2)}}{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}, \quad \hat{q}_n^{(k_1,k_2,t_1,t_2)} = \frac{\tau_n^{(k_1,k_2,t_1,t_2)} + \tau_{n+1}^{(k_1,k_2,t_1+1,t_2)}}{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}, \quad (12a) \]
\[ \tilde{q}_n^{(k_1,k_2,t_1,t_2)} = \frac{\tau_n^{(k_1,k_2,t_1,t_2)} - \tau_{n+1}^{(k_1,k_2,t_1+1,t_2)}}{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}, \quad \tilde{q}_n^{(k_1,k_2,t_1,t_2)} = \frac{\tau_n^{(k_1,k_2,t_1,t_2)} + \tau_{n+1}^{(k_1,k_2,t_1+1,t_2)}}{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}, \quad (12b) \]

Example 2.2. Let \( \omega(x) \) be a positive weight function on the interval (0, 1), \( \alpha \) and \( \beta \) be positive integers. Define the bilinear forms as
\[ \mathcal{B}^{(k_1,k_2,t_1,t_2)}[\phi(z), \psi(z)] := \int_0^1 \phi(x^\alpha)\psi(x^\beta)\omega^{(k_1,k_2,t_1,t_2)}(x) dx, \]
where
\[ \omega^{(k_1,k_2,t_1,t_2)}(x) := \omega(x)x^{\alpha k_1 + \beta k_2} \prod_{t=0}^{t_1-1}(x^\alpha - s_1^{(t)}) \prod_{t=0}^{t_2-1}(x^\beta - s_2^{(t)}). \]

Then, the bilinear forms \( \mathcal{B}^{(k_1,k_2,t_1,t_2)} \) satisfy relations (10). Suppose that \( s_1^{(t_1)} \) and \( s_2^{(t_2)} \) are non-negative for all \( t_1, t_2 \in \mathbb{Z} \). Then, since \( \omega^{(k_1,k_2,t_1,t_2)}(x) \) become also positive weight functions on the interval (0, 1), it is shown that, by applying Konhauser’s result [11, theorem 2.3], the pair of the monic biorthogonal polynomials \( \{\phi_n^{(k_1,k_2,t_1,t_2)}(z)\}_{n=0}^\infty \) and \( \{\psi_n^{(k_1,k_2,t_1,t_2)}(z)\}_{n=0}^\infty \) with respect to \( \mathcal{B}^{(k_1,k_2,t_1,t_2)} \) exist and \( \tau_n^{(k_1,k_2,t_1,t_2)} \neq 0 \).

Let us consider a particular case: \( \omega(x) = 1 \) and \( s_1^{(t_1)} = s_2^{(t_2)} = -1 \) for all \( t_1, t_2 \in \mathbb{Z} \). Then, the moments are calculated as
\[ \mu_{ij}^{(t_1,t_2)} = \int_0^1 x^{\alpha i + \beta j} (x^\alpha + 1)^i (x^\beta + 1)^j \frac{1}{\alpha(i + 1) + \beta(j + 1) + 1}. \]

Substitute this into the determinant \( \tau_{n}^{(t_1,t_2,t_3)} \), then (12) give a particular solution to the nd-2D-Toda lattice that will be derived in what follows. Especially, if \( t_1 = t_2 = 0 \), then we readily obtain

\[ \tau_{n}^{(k_1,k_2,0,0)} = \frac{\prod_{0 \leq j < n-1} (\alpha j + \alpha)(\beta j + \beta)}{\prod_{0 \leq j < n} (\alpha(k_1 + j) + \beta(k_2 + j) + 1)}. \]

Therefore, one can explicitly write a nontrivial solution to the autonomous case. On the other hand, it may be hard to obtain a closed form of \( \tau_{n}^{(k_1,k_2,1,0)} \) in the nonautonomous case.

Using the semi-infinite bidirectional matrices

\[
R^{(k_1,k_2,n)} := \left( q_{ij}^{(k_1,k_2,n)} \delta_{ij} + \delta_{i+1,j} \right)_{i,j=0}^{\infty}, \quad L^{(k_1,k_2,n)} := \left( q_{ij}^{(k_1,k_2,n)} \delta_{ij} + \delta_{i+1,j} \right)_{i,j=0}^{\infty},
\]

and the semi-infinite vector

\[
\phi^{(k_1,k_2,n)} := \left( \begin{array}{c} \phi_0^{(k_1,k_2,n)}(z) \\ \phi_1^{(k_1,k_2,n)}(z) \\ \vdots \end{array} \right),
\]

the relations (11) are rewritten as

\[
\begin{align*}
z \phi^{(k_1+1,k_2,n)}(z) &= R^{(k_1,k_2,n)} \phi^{(k_1,k_2,n)}(z), \quad (13a) \\
\phi^{(k_1,k_2-1,n)}(z) &= L^{(k_1,k_2-1,n)} \phi^{(k_1,k_2,n)}(z), \quad (13b) \\
(z - \delta_{k_1,1})(\phi^{(k_1,k_2,n+1)}(z) &= R^{(k_1,k_2,n)} \phi^{(k_1,k_2,n+1)}(z), \quad (13c) \\
\phi^{(k_1,k_2,n-1)}(z) &= L^{(k_1,k_2,n-1)} \phi^{(k_1,k_2,n)}(z). \quad (13d)
\end{align*}
\]

From (13a) and (13b), we have

\[
\begin{align*}
z \phi^{(k_1+1,k_2,n)}(z) &= R^{(k_1+1,k_2,n)} R^{(k_1,k_2,n)} \phi^{(k_1+1,k_2,n)}(z) \\
&= R^{(k_1+1,k_2,n)} L^{(k_1,k_2,n)} \phi^{(k_1+1,k_2,n)}(z),
\end{align*}
\]

whose each element gives

\[
\begin{align*}
z \phi_n^{(k_1+1,k_2,n)}(z) &= \phi_{n+1}^{(k_1+1,k_2,n)}(z) \\
&= (q_n^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z) + q_{n+1}^{(k_1,k_2,n)} \phi_{n+1}^{(k_1,k_2,n+1)}(z) + q_n^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z)) \\
&= (q_n^{(k_1,k_2,n)} + q_{n+1}^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z) + q_n^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z)) \\
&= q_n^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z) + q_n^{(k_1,k_2,n)} \phi_n^{(k_1,k_2,n+1)}(z)
\end{align*}
\]

for \( n = 0, 1, 2, \ldots \). Hence, we obtain the matrix equation
which is equivalent to the scalar equations

\begin{align}
q_n^{(k_1,k_2+1,j_1,j_2)} + e_{n-1}^{(k_1,k_2,j_1,j_2)} &= q_n^{(k_1,k_2,j_1,j_2)} + e_n^{(k_1,k_2,j_1,j_2)}, \\
q_n^{(k_1,k_2+1,j_1,j_2)} e_{n}^{(k_1,k_2,j_1,j_2)} &= q_n^{(k_1,k_2,j_1,j_2)} e_{n+1}^{(k_1,k_2,j_1,j_2)},
\end{align}

with the boundary condition

\[ e_{-1}^{(k_1,k_2,j_1,j_2)} = 0 \]  

for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \). The discrete equations (14) are the compatibility conditions for the spectral transformations (11a) and (11b), and called the discrete two-dimensional Toda lattice.

Similar calculations for each pair of the spectral transformations (13) also yield

\[ z(z - s_1^{(1)}) \phi^{(k_1,k_2,t_1+1,t_2)}(z) = R^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} \phi^{(k_1,k_2,t_1,t_2)}(z), \]

\[ z^{(k_1,k_2,t_1+1)} \phi^{(k_1,k_2,t_1+1,t_2)}(z) = R^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} \phi^{(k_1,k_2,t_1+1,t_2)}(z), \]

\[ R^{(k_1,k_2,t_1,t_2)} R^{(k_1,k_2,t_1+1,t_2)} = R^{(k_1+1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} \]

\[ L^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} = R^{(k_1,k_2,t_1,t_2)} L^{(k_1,k_2,t_1,t_2)} \]

\[ L^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1+1,t_2)} = R^{(k_1,k_2,t_1+1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} \]

\[ L^{(k_1,k_2,t_1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} = L^{(k_1,k_2,t_1+1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} \]

In the same manner, we obtain the matrix equations

\[ R^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} = R^{(k_1+1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} \]

\[ L^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1,t_2)} = R^{(k_1,k_2,t_1,t_2)} L^{(k_1,k_2,t_1,t_2)} \]

\[ L^{(k_1,k_2,t_1+1,t_2)} R^{(k_1,k_2,t_1+1,t_2)} = R^{(k_1,k_2,t_1+1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} \]

\[ L^{(k_1,k_2,t_1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} = L^{(k_1,k_2,t_1+1,t_2)} L^{(k_1,k_2,t_1+1,t_2)} \]

which are equivalent to the scalar equations

\[ q_n^{(k_1,k_2,t_1+1,j_1,j_2)} + q_{n+1}^{(k_1,k_2,t_1,j_1,j_2)} = q_n^{(k_1,k_2,t_1,j_1,j_2)} + q_{n+1}^{(k_1+1,k_2,t_1,j_2)}, \]

\[ q_n^{(k_1,k_2,t_1+1,j_1,j_2)} q_{n}^{(k_1,k_2,t_1,j_1,j_2)} = q_n^{(k_1,k_2,t_1,j_1,j_2)} q_{n}^{(k_1+1,k_2,t_1,j_2)}, \]

\[ q_n^{(k_1,k_2+1,j_1,j_2)} + e_{n-1}^{(k_1,k_2,t_1+1,j_1,j_2)} = q_n^{(k_1,k_2,t_1+1,j_1,j_2)} + e_{n}^{(k_1,k_2,t_1,j_2)}, \]
\begin{align}
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2+1,t_1,t_2)} = \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2+1)} + \epsilon_{-n-1}^{(k_1,k_2,t_1,t_2)} = \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2+1,t_1,t_2)} + \epsilon_{n}^{(k_1,k_2,t_1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2+1)} + \epsilon_{n-1}^{(k_1,k_2,t_1,t_2)} = \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2+1,t_1,t_2)} + \epsilon_{n}^{(k_1,k_2,t_1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2+1)} + \epsilon_{n-1}^{(k_1,k_2,t_1,t_2)} = \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2+1,t_1,t_2)} + \epsilon_{n}^{(k_1,k_2,t_1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2+1)} + \epsilon_{n-1}^{(k_1,k_2,t_1,t_2)} = \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2+1,t_1,t_2)} + \epsilon_{n}^{(k_1,k_2,t_1,t_2)},
\end{align}

for \( n = 0, 1, 2, \ldots \) with the boundary condition (14c) and

\[ \frac{\partial}{\partial t_{-1}} \phi_{n}^{(k_1,k_2,t_1,t_2)} = 0 \]

for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \).

In these discrete equations, the parameters \( s_1^{(n)} \) and \( s_2^{(n)} \) do not appear explicitly. The parameters are, in fact, embedded into boundary conditions as follows.

2.3.1. **For equation (16).** Subtraction of (11a) from (11c) yields the relation

\[ (z - s_1^{(n)}) \phi_{n}^{(k_1,k_2,t_1+1,t_2)}(z) = z \phi_{n}^{(k_1,k_2+1,t_1,t_2)}(z) + \alpha_{n}^{(k_1,k_2,t_1,t_2)} \phi_{n}^{(k_1,k_2+1,t_1,t_2)}(z), \]

where

\[ \alpha_{n}^{(k_1,k_2,t_1,t_2)} := \frac{\phi_{n}^{(k_1,k_2,t_1+1,t_2)}(0)}{\phi_{n}^{(k_1,k_2+1,t_1,t_2)}(0)} = s_1^{(n)} \frac{\phi_{n}^{(k_1+1,k_2,t_1+1,t_2)}(z^{(n)})}{\phi_{n}^{(k_1,k_2,t_1,t_2)}}. \]

The relation (21) induces

\[ \frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1+1,t_2)} = -s_1^{(n)} \frac{\phi_{n}^{(k_1,k_2,t_1+1,t_2)}(z^{(n)})}{\phi_{n}^{(k_1,k_2+1,t_1,t_2)}} \]

By using the variable \( \alpha_{n}^{(k_1,k_2,t_1,t_2)} \), equations (16) are rewritten as

\begin{align}
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2)} = \phi_{n}^{(k_1,k_2,t_1+1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1+1,t_2)} = \phi_{n}^{(k_1,k_2+1,t_1,t_2)}, \\
\frac{\partial}{\partial t_n} \phi_{n}^{(k_1,k_2,t_1,t_2+1)} = \phi_{n}^{(k_1,k_2+1,t_1,t_2)},
\end{align}

for \( n = 0, 1, 2, \ldots \) with the boundary condition

\[ \frac{\partial}{\partial t_{-1}} \phi_{n}^{(k_1,k_2,t_1,t_2)} = -s_1^{(n)} \]

for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \). Note that equation (23a) is readily transformed into the bilinear equation
and equations (23b) and (23c) are obvious identical equations of \( r_n^{(k_1,k_2,t_1,t_2)} \).

2.3.2. For equation (17). Subtraction of (11b) from (11c) yields the relation
\[
(z - s_1^{(t_1)}) \phi_n^{(k_1,k_2,t+1,t_2)}(z) = \phi_n^{(k_1,k_2,-1,t_1,t_2)}(z) + b_n^{(k_1,k_2,-1,t_1,t_2)} \phi_n^{(k_1,k_2,t_1,t_2)}(z),
\]
where
\[
b_n^{(k_1,k_2,t_1,t_2)} := q_n^{(k_1,k_2+1,t_1,t_2)} - e_n^{(k_1,k_2,t_1,t_2)}.
\]
The relation (25) induces
\[
b_n^{(k_1,k_2,t_1,t_2)} = -\frac{\phi_n^{(k_1,k_2,t_1+1,t_2)}}{\phi_n^{(k_1,k_2+1,t_1,t_2)}}(t_1) \left( \frac{\phi_n^{(k_1,k_2,t_1+1,t_2)}}{\phi_n^{(k_1,k_2+1,t_1,t_2)}} \right) = \frac{r_n^{(k_1,k_2+1,t_1,t_2)}}{r_n^{(k_1,k_2,t_1,t_2)}}(t_1) \frac{r_n^{(k_1,k_2,t+1,t_1,t_2)}}{r_n^{(k_1,k_2+1,t_1,t_2)}}.
\]
By using the variable \( b_n^{(k_1,k_2,t_1,t_2)} \), equations (17) are rewritten as
\[
q_n^{(k_1,k_2+1,t_1,t_2)} = b_n^{(k_1,k_2,t_1,t_2)} + e_n^{(k_1,k_2,t_1,t_2)}, \quad q_n^{(k_1,k_2+1,t_1,t_2)} b_n^{(k_1,k_2,t_1,t_2)} = q_n^{(k_1,k_2,t_1,t_2)} b_n^{(k_1,k_2+1,t_1,t_2)}, \quad q_n^{(k_1,k_2+1,t_1,t_2)} e_n^{(k_1,k_2,t_1,t_2)} = q_n^{(k_1,k_2,t_1,t_2)} e_n^{(k_1,k_2+1,t_1,t_2)},
\]
for \( n = 0, 1, 2, \ldots \) with the boundary condition
\[
b_0^{(k_1,k_2,t_1,t_2)} = q_0^{(k_1,k_2+1,t_1,t_2)} - s_1^{(t_1)} = q_0^{(k_1,k_2,t_1,t_2)}
\]
for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \). Equation (26a) is transformed into the bilinear equation
\[
\frac{r_n^{(k_1,k_2+1,t_1,t_2)}}{r_n^{(k_1,k_2,t_1,t_2)}} \phi_n^{(k_1,k_2+1,t_1+1,t_2)} = \frac{r_n^{(k_1,k_2+1,t_1+1,t_2)}}{r_n^{(k_1,k_2,t_1,t_2)}} \phi_n^{(k_1,k_2,t_1+1,t_2)} + \frac{r_n^{(k_1,k_2,t_1+1,t_2)}}{r_n^{(k_1,k_2+1,t_1+1,t_2)}} \phi_n^{(k_1,k_2+1,t_1+1,t_2)}.
\]

2.3.3. For equation (18). Subtraction of (11d) from (11a) yields the relation
\[
(z - s_1^{(t_1)}) \psi_n^{(k_1,k_2,t_1+1,t_2)}(z) = \psi_n^{(k_1,k_2,t_1-1,t_2)}(z) + d_n^{(k_1,k_2,t_1-1,t_2)} \phi_n^{(k_1,k_2,t_1,t_2)}(z),
\]
where
\[
d_n^{(k_1,k_2,t_1,t_2)} := q_n^{(k_1,k_2+1,t_1+1,t_2)} - e_n^{(k_1,k_2+1,t_1+1,t_2)}.
\]
The relation (27) induces
\[
d_n^{(k_1,k_2,t_1,t_2)} = -\frac{\phi_n^{(k_1,k_2+1,t_1+1,t_2)}}{\phi_n^{(k_1,k_2,t_1+1,t_2)}}(0) = \frac{\phi_n^{(k_1,k_2+1,t_1+1,t_2)}}{\phi_n^{(k_1,k_2,t_1+1,t_2)}}(0) = \frac{r_n^{(k_1,k_2+1,t_1+1,t_2)}}{r_n^{(k_1,k_2,t_1+1,t_2)}} \frac{r_n^{(k_1,k_2,t_1+1,t_2+1)}}{r_n^{(k_1,k_2+1,t_1+1,t_2+1)}}.
\]
By using the variable \( d_n^{(k_1,k_2,t_1,t_2)} \), equations (18) are rewritten as
\[
q_n^{(k_1,k_2,t_1+1,t_2+1)} = d_n^{(k_1,k_2,t_1,t_2)} + e_n^{(k_1,k_2,t_1,t_2)},
\]
(28a)
\[ q_n^{(k_1,k_2,t_1,t_2+1)}(z) = q_n^{(k_1,k_2,t_1,t_2)}(z), \quad (28b) \]

\[ d_n^{(k_1,k_2,t_1,t_2)} e_n^{(k_1,k_2,t_1,t_2)} = d_n^{(k_1,k_2,t_1,t_2)} e_n^{(k_1,k_2,t_1,t_2)}, \quad (28c) \]

for \( n = 0, 1, 2, \ldots \) with the boundary condition

\[ d_0^{(k_1,k_2,t_1,t_2)} = d_0^{(k_1,k_2,t_1,t_2)} = 0 \quad (28d) \]

for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \). Equation (28a) is transformed into the bilinear equation

\[ i_n^{(k_1,k_2,t_1,t_2)}(z_1^{(k_1+1,k_2,t_1,t_2)} + 1) = i_n^{(k_1,k_2,t_1,t_2)}(z_1^{(k_1+1,k_2,t_1,t_2)} + 1) + i_n^{(k_1,k_2,t_1,t_2)}(z_1^{(k_1+1,k_2,t_1,t_2)} + 1). \]

2.3.4. For equation (19). Subtraction of (11d) from (11b) yields the relation

\[ \phi_n^{(k_1,k_2,t_1,t_2)}(z) = \phi_n^{(k_1,k_2,t_1,t_2)}(z) + f_n^{(k_1,k_2,t_1,t_2)}(z), \quad (29) \]

where

\[ f_n^{(k_1,k_2,t_1,t_2)} := e_n^{(k_1,k_2,t_1,t_2)} - e_n^{(k_1,k_2+1,t_1,t_2)}. \]

Since the same discussion for another one of the pair of the monic biorthogonal polynomial sequences \( \{\phi_n^{(k_1,k_2,t_1,t_2)}\}_{n=0}^\infty \) leads us to the ‘dual’ version of the bilinear equation (24)

\[ i_n^{(k_1,k_2,t_1,t_2+1)}(z_1^{(k_1+1,k_2,t_1,t_2)}) = i_n^{(k_1,k_2,t_1,t_2)}(z_1^{(k_1+1,k_2,t_1,t_2)}) - \frac{1}{t_n+1} \sum_{n=1}^{t_n+2} i_n^{(k_1,k_2,t_1,t_2)}(z_1^{(k_1+1,k_2,t_1,t_2)}), \quad (30) \]

the relation (29) induces

\[ f_n^{(k_1,k_2,t_1,t_2)} = \frac{z_n^{(k_1,k_2,t_1,t_2+1)}}{t_n+1} \phi_n^{(k_1,k_2,t_1,t_2+1)}(z), \quad (31a) \]

By using the variable \( f_n^{(k_1,k_2,t_1,t_2)} \), equations (19) are rewritten as

\[ e_n^{(k_1,k_2,t_1,t_2+1)} = f_n^{(k_1,k_2,t_1,t_2)} e_n^{(k_1,k_2+1,t_1,t_2)}, \quad (31b) \]

\[ e_n^{(k_1,k_2,t_1,t_2+1)} = f_n^{(k_1,k_2,t_1,t_2)} e_n^{(k_1,k_2+1,t_1,t_2)}, \quad (31c) \]

for \( n = 0, 1, 2, \ldots \) with the boundary condition
\[ f_0^{(k_1,k_2,t_1,t_2)} = -s_0^{(t_2)} \left( k_1,k_2,t_1,t_2 \right) \frac{s_0^{(t_2)}}{r_1^{(k_1,k_2,t_1,t_2+1)}} - s_1^{(t_2)} \left( k_1,k_2,t_1,t_2 \right) \frac{s_1^{(t_2)}}{r_1^{(k_1,k_2,t_1,t_2)}} = -s_0^{(t_2)} \left( k_1,k_2,t_1,t_2 \right) \frac{s_0^{(t_2)}}{r_1^{(k_1,k_2,t_1,t_2+1)}} \]
\[ = \frac{g_0}{r_1^{(k_1,k_2,t_1,t_2)}} \frac{s_0^{(t_2)}}{r_1^{(k_1,k_2,t_1,t_2)}} - s_0^{(t_2)} \left( k_1,k_2,t_1,t_2 \right) \]
\[ \text{(31d)} \]

for all \( k_1, k_2, t_1 \) and \( t_2 \). Equation (31a) is transformed into the bilinear equation (30).

2.3.5. For equation (20). Subtraction of (11d) from (11c) yields the relations
\[ (z - s_0^{(t_2)}) \phi_n^{(k_1,k_2,t_1+1,t_2)}(z) = \phi_n^{(k_1,k_2,t_1,t_2+1)}(z) + g_n^{(k_1,k_2,t_1,t_2+1)} \phi_n^{(k_1,k_2,t_1,t_2)}(z), \]
\[ \text{where} \]
\[ g_n^{(k_1,k_2,t_1,t_2+1)} := \frac{g_n^{(k_1,k_2,t_1,t_2+1)}}{r_n^{(k_1,k_2,t_1,t_2)}} - \phi_n^{(k_1,k_2,t_1,t_2)}. \]

The relation (32) induces
\[ g_n^{(k_1,k_2,t_1,t_2+1)} = -\frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{\phi_n^{(k_1,k_2,t_1,t_2+1)}} - \frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{\phi_n^{(k_1,k_2,t_1,t_2+1)}} = -\frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{r_n^{(k_1,k_2,t_1,t_2)}} - \frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{r_n^{(k_1,k_2,t_1,t_2)}}. \]

By using the variable \( g_n^{(k_1,k_2,t_1,t_2+1)} \), equations (20) are rewritten as
\[ \tilde{q}_n^{(k_1,k_2,t_1,t_2+1)} = g_n^{(k_1,k_2,t_1,t_2+1)} + \phi_n^{(k_1,k_2,t_1,t_2)}, \]
\[ \tilde{g}_n^{(k_1,k_2,t_1,t_2+1)} = g_n^{(k_1,k_2,t_1,t_2+1)} + \frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{r_n^{(k_1,k_2,t_1,t_2+1)}} - \frac{g_n^{(k_1,k_2,t_1,t_2+1)}(t_1)}{r_n^{(k_1,k_2,t_1,t_2+1)}}. \]

for \( n = 0, 1, 2, \ldots \) with the boundary condition
\[ g_0^{(k_1,k_2,t_1,t_2)} = \tilde{q}_0^{(k_1,k_2,t_1,t_2)} - s_0^{(t_1)} = \tilde{q}_0^{(k_1,k_2,t_1,t_2)}. \]

for all \( k_1, k_2, t_1, t_2 \in \mathbb{Z} \). Equation (33a) is transformed into the bilinear equation
\[ \sum_{n=1}^{\infty} g_n^{(k_1,k_2,t_1,t_2+1)} (t_1) = \sum_{n=1}^{\infty} g_n^{(k_1,k_2,t_1,t_2+1)} (t_1) + \sum_{n=1}^{\infty} g_n^{(k_1,k_2,t_1,t_2+1)} (t_1) + \sum_{n=1}^{\infty} g_n^{(k_1,k_2,t_1,t_2+1)} (t_1) \]

In this paper, we call the system of (14), (23), (26), (28), (31) and (33) the nd-2D-Toda lattice. A particular solution to the system has already been given in example 2.2.

Remark 2.3. The matrix equations of the nd-2D-Toda lattice containing the new variables, e.g. \( a_n^{(k_1,k_2,t_1,t_2)} \), are derived as follows. Let us define semi-infinite matrices
\[ I := (\delta_{ij}^{(0)}), \quad \tilde{I} := (\delta_{i+1,j}^{(0)}), \quad L := (\delta_{ij}^{(0)}), \quad \tilde{L} := (\delta_{i+1,j}^{(0)}), \quad Q^{(k_1,k_2,t_1,t_2)} := (q_i^{(k_1,k_2,t_1,t_2)}), \quad \tilde{Q}^{(k_1,k_2,t_1,t_2)} := (\tilde{q}_i^{(k_1,k_2,t_1,t_2)}). \]

Then, the bidiagonal matrices are written as
\[ R^{(k_1,k_2,t_1,t_2)} = \tilde{Q}^{(k_1,k_2,t_1,t_2)} + \tilde{I} \text{ and } \tilde{R}^{(k_1,k_2,t_1,t_2)} = \tilde{Q}^{(k_1,k_2,t_1,t_2)} + I. \]

By using the matrix notation above, the definition of the variable \( a_n^{(k_1,k_2,t_1,t_2)} \) (22) is rewritten as
\[ \mathcal{A}^{(k_1,k_2,t_1,t_2)} = (a_j^{(k_1,k_2,t_1,t_2)}), \quad \mathcal{B}^{(k_1,k_2,t_1,t_2)} = (b_i^{(k_1,k_2,t_1,t_2)}). \]
Here, the subdiagonal elements of equation (15a) gives
\[
Q^{(k_1,k_2,t_1+1,t_2)}_j + \tilde{t}Q^{(k_1,k_2,t_1,t_2)}_j = \tilde{Q}^{(k_1+1,k_2,t_1,t_2)}_j + \tilde{t}Q^{(k_1,k_2+1,t_1,t_2)}_j.
\]

Since $\mathcal{I}_2 = 1$, we have
\[
Q^{(k_1,k_2,t_1+1,t_2)}_j + \tilde{t}Q^{(k_1,k_2,t_1,t_2)}_j = \tilde{Q}^{(k_1+1,k_2,t_1,t_2)}_j + \tilde{t}Q^{(k_1,k_2+1,t_1,t_2)}_j.
\]

Hence, we obtain the matrix equation corresponding to equation (23c):
\[
\mathcal{R}^{(k_1,k_2,t_1,t_2)} = \mathcal{R}^{(k_1+1,k_2,t_1,t_2)} - \mathcal{R}^{(k_1+1,k_2+1,t_1,t_2)}
= \mathcal{R}^{(k_1,k_2+1,t_1,t_2)} - \mathcal{R}^{(k_1,k_2,t_1+1,t_2)}
= (Q^{(k_1+1,k_2,t_1,t_2)} - Q^{(k_1,k_2,t_1,t_2)}) \mathcal{R}^{(k_1,k_2,t_1,t_2)}
= \tilde{t}(\tilde{Q}^{(k_1,k_2,t_1,t_2)} - Q^{(k_1,k_2,t_1,t_2)}) \mathcal{R}^{(k_1,k_2,t_1,t_2)}
= \tilde{t} \mathcal{R}^{(k_1,k_2,t_1,t_2)} \mathcal{I} \mathcal{R}^{(k_1,k_2,t_1,t_2)}.
\]

Note that $\mathcal{I} \mathcal{R}^{(k_1,k_2,t_1,t_2)} \mathcal{I} = (d_{j+1} \delta_{j-1})_{j=0}^{\infty}$. We should remark that the boundary conditions containing the parameters $s^{(n)}_1$ and $s^{(n)}_2$ cannot appear in the matrix equations. Although the matrix equations are important from the point of view of the matrix analysis, the scalar equations derived in this section play more important roles in the present paper, especially for the ultradiscretization in section 4.

3. $(M, 1)$-reduction

In this section, we consider a special case of the chain of the monic biorthogonal polynomials: the bilinear forms satisfy the condition $\mathcal{B}^{(k_1+M,k_2,t_1,t_2)} = \mathcal{B}^{(k_1,k_2,t_1+1,t_2)}$ for all $k_1$, $k_2$, $t_1$ and $t_2$, where $M$ is a positive integer. The condition is equivalent to
\[
\mathcal{B}^{(k_1,k_2,t_1,t_2)}[z^{i+M}, z] = \mathcal{B}^{(k_1,k_2,t_1,t_2)}[z^i, z^{j+1}], \quad i, j = 0, 1, 2, \ldots \tag{34}
\]

3.1. Nonautonomous discrete semi-infinite hungry Toda lattice

Introduce a new linear functional $\mathcal{L}^{(k_1,k_2,t_1,t_2)}: \mathbb{C}[z] \rightarrow \mathbb{C}$ by
\[
\mathcal{L}^{(k_1,k_2,t_1,t_2)}[z^i] := \mathcal{B}^{(k_1,k_2,t_1,t_2)}[z^i, 1], \quad i = 0, 1, 2, \ldots
\]

Since the relation
\[
\mathcal{L}^{(k_1,k_2,t_1,t_2)}[z^{i+M}] = \mathcal{B}^{(k_1,k_2,t_1,t_2)}[z^{i+M}, 1] = \mathcal{B}^{(k_1,k_2,t_1,t_2)}[z^i, z]
\]
holds, the pair of the monic biorthogonal polynomial sequences $(\phi_n^{(k_1,k_2,t_1,t_2)}(z))_{n=0}^{\infty}$ and $(\psi_m^{(k_1,k_2,t_1,t_2)}(z))_{m=0}^{\infty}$ with respect to $\mathcal{B}^{(k_1,k_2,t_1,t_2)}$ satisfy $(M, 1)$-biorthogonal relation with respect to $\mathcal{L}^{(k_1,k_2,t_1,t_2)}$,
\[
\mathcal{L}^{(k_1,k_2,t_1,t_2)}[\phi_n^{(k_1,k_2,t_1,t_2)}(z)] \psi_m^{(k_1,k_2,t_1,t_2)}(z^M)] = \frac{\tau_{n+1}^{(k_1,k_2,t_1,t_2)}}{\tau_n^{(k_1,k_2,t_1,t_2)}} \delta_{m,n},
\quad m, n = 0, 1, 2, \ldots
\]
Example 3.1. For example 2.2, consider the case of \( \alpha = 1 \) and \( \beta = M \); i.e.

\[
\mathcal{L}^{(k_1, k_2, t_1, t_2)}[\phi(z), \psi(z)] := \int_0^1 \phi(x)\psi(x^M)\omega^{(k_1, k_2, t_1, t_2)}(x) \, dx,
\]

\[
\omega^{(k_1, k_2, t_1, t_2)}(x) := \omega(x)k_1^{(x + M t_1)} \prod_{\tau=0}^{n-1} (x - s_1^{(\tau)}) \prod_{\tau=0}^{n-1} (x^M - s_2^{(\tau)}).
\]

Then, it is readily verified that the bilinear forms \( \mathcal{B}^{(k_1, k_2, t_1, t_2)} \) satisfy the condition (34).

Remark 3.1. The \((M, 1)\)-biorthogonal polynomials for the classical Hermite, Laguerre, and Jacobi weights were studied by Konhauser [12], Madhekar and Thakare [13, 25].

Hereafter, we will fix \( k_2 \) and \( t_1 \) to zero and consider only the time variables \( k_1 \) and \( t_2 \); we will simply write \( \mathcal{L}^{(k, t)} \) and \( \phi_n^{(k, t)}(z) \) instead of \( \mathcal{L}^{(k, 0, 0, t)} \) and \( \phi_n^{(k, 0, 0, t)}(z) \), respectively. We will also omit \( k_2 \) and \( t_1 \) for all the other variables in the same manner. Then, the determinant representation of \( \phi_n^{(k, t)}(z) \) is given by

\[
\phi_0^{(k, t)}(z) = 1, \quad \phi_n^{(k, t)}(z) = \frac{1}{\tau_n^{(k, t)}} \begin{vmatrix} \mu_0^{(k)} & \mu_1^{(k)} & \cdots & \mu_{k-M(n-1)}^{(k)} & 1 \\ \mu_1^{(k)} & \mu_1^{(k+M)} & \cdots & \mu_{k+M(n-1)}^{(k)} & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{k-n-1}^{(k)} & \mu_{k-n-1}^{(k+M)} & \cdots & \mu_{k+n-1}^{(k+M(n-1))} & z^{n-1} \\ \mu_k^{(k)} & \mu_k^{(k+nM)} & \cdots & \mu_k^{(k+nM(n-1))} & z^n \end{vmatrix},
\]

where

\[
\mu_m^{(k)} := \mathcal{L}^{(0, 0)}[z^m], \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, 2, \ldots.
\]

Note that the moment \( \mu_m^{(k)} \) satisfies the relations

\[
\mathcal{L}^{(k, t)}[z^m] = \mu_m^{(k)}, \quad \mu_m^{(k+1)} = \mu_m^{(k+M)} - s^{(k)} \mu_m^{(k)},
\]

where we simply write \( s^{(k)} \) instead of \( s^{(k)}_2 \).

From the discussion in section 2, the monic \((M, 1)\)-biorthogonal polynomials \( \{\phi_n^{(k, t)}(z)\}_{n=0}^{\infty} \) satisfy the following relations

\[
\phi_n^{(k, t)}(z) = \phi_{n+1}^{(k, t)}(z) + d_n^{(k, t)} \psi_n^{(k, t)}(z),
\]

\[
\phi_{n+1}^{(k, t)}(z) = \phi_{n+1}^{(k, t)}(z) + e_n^{(k, t)} \phi_n^{(k, t)}(z),
\]

\[
\phi_{n+1}^{(k, t)}(z) = \phi_{n+1}^{(k, t)}(z) + f_n^{(k, t)} \phi_{n+1}^{(k, t)}(z),
\]

\[
\phi_{n+1}^{(k, t)}(z) = \phi_{n+1}^{(k, t)}(z) + g_n^{(k, t)} \phi_{n+1}^{(k, t)}(z).
\]
where

\begin{align}
q_{n}^{(k,j)} &= \frac{\tau_{n}^{(k,j)}}{\tau_{n+1}^{(k+1,j)}} , \quad \epsilon_{n}^{(k,j)} = \frac{\tau_{n+2}^{(k+1,j)}}{\tau_{n+1}^{(k+1,j)}}, \\
\tilde{q}_{n}^{(k,j)} &= \frac{\tau_{n+1}^{(k,j)}}{\tau_{n+1}^{(k+1,j+1)}} , \quad \tilde{\epsilon}_{n}^{(k,j)} = \frac{\tau_{n+2}^{(k+1,j+1)}}{\tau_{n+1}^{(k+1,j+1)}}. 
\end{align}

(37a)

\begin{align}
d_{n}^{(k,j)} &= \frac{\tau_{n}^{(k,j+1)}}{\tau_{n+1}^{(k,j+1)}} , \quad f_{n}^{(k,j)} = -\frac{\tau_{n+2}^{(k,j+1)}}{\tau_{n+1}^{(k,j+1)}}. 
\end{align}

(37b)

We omitted the variable \( d_{n}^{(k,j-1,j)} \) and its related relations and variables, because we will not use them in the subsequent discussion. The compatibility conditions for (36) give the recurrence relations

\begin{align}
q_{n}^{(k+1,j)} &= d_{n}^{(k,j)} + \tilde{q}_{n}^{(k,j)}, \quad \epsilon_{n}^{(k+1,j)} = f_{n}^{(k,j)} + \tilde{\epsilon}_{n}^{(k,j)}, \\
d_{n+1}^{(k,j)} &= d_{n}^{(k,j)} q_{n+1}^{(k+1,j)} , \quad f_{n+1}^{(k,j)} = f_{n}^{(k,j)} \frac{e_{n+1}^{(k,j)}}{e_{n+1}^{(k+1,j)}}, \\
\tilde{q}_{n}^{(k+1,j)} &= \tilde{q}_{n}^{(k,j)} \frac{q_{n+1}^{(k+1,j)}}{q_{n}^{(k+1,j)}}, \quad \tilde{\epsilon}_{n}^{(k+1,j)} = \tilde{\epsilon}_{n}^{(k,j)} \frac{e_{n+1}^{(k+1,j)}}{e_{n+1}^{(k+1,j)}}. 
\end{align}

(38c)

for \( n = 0, 1, 2, \ldots \) with the boundary condition

\begin{align}
d_{0}^{(k,j)} &= q_{0}^{(k,j)}, \quad f_{0}^{(k,j)} = -\frac{e_{0}^{(k,j)} \xi(t)}{\prod_{j=0}^{M-1} q_{0}^{(k+j,j)} - s(t)}, \quad \tilde{q}_{0}^{(k,j)} = \frac{e_{0}^{(k,j)} \prod_{j=0}^{M-1} q_{0}^{(k+j,j)}}{\prod_{j=0}^{M-1} q_{0}^{(k+j,j)} - s(t)}. 
\end{align}

(38d)

for all \( k, \ t \in \mathbb{Z} \). A particular solution to the system (38) can be constructed in the same manner as example 2.2 with \( \alpha = 1 \) and \( \beta = M \) (see also example 3.1).

We should remark that, if \( s(t) = 0 \) for all \( t \), then \( f_{0}^{(k,j)} = 0, e_{n}^{(k,j)} = \tilde{e}_{n}^{(k,j)} \) and \( L^{(k+1,j)} = L^{(k+1,j)} \) hold for all \( k, \ t \) and \( n \). Therefore, equations (38) are reduced to

\begin{align}
q_{n}^{(k,M)} &= d_{n}^{(k,j)} + \tilde{q}_{n}^{(k,j)}, \quad \epsilon_{n}^{(k,M)} = f_{n}^{(k,j)} \frac{q_{n+1}^{(k+1,M)}}{q_{n}^{(k+1,M)}}, \\
\tilde{q}_{n}^{(k,M)} &= \tilde{q}_{n}^{(k,j)} \frac{q_{n+1}^{(k+1,M)}}{q_{n}^{(k+1,M)}}, \quad \tilde{\epsilon}_{n}^{(k,M)} = \tilde{\epsilon}_{n}^{(k,j)} \frac{e_{n+1}^{(k+1,M)}}{e_{n+1}^{(k+1,M)}}. 
\end{align}

where we omitted the time variable \( t \). Elimination of \( d_{n}^{(k,j)} \) yields

\begin{align}
q_{n}^{(k+1,M)} + \epsilon_{n}^{(k+1,j)} &= q_{n}^{(k,M)} + \epsilon_{n}^{(k,M)}, \quad \tilde{q}_{n}^{(k+1,j)} = q_{n+1}^{(k+1,j)} + \epsilon_{n+1}^{(k+1,j)}. 
\end{align}

(39)

The system (39) is called the discrete hungry Toda lattice, which is the reason why we call the system (38) the ndh-Toda lattice.

3.2. Nonautonomous discrete finite hungry Toda lattice

In this section, we consider the ndh-Toda lattice (38) with the finite lattice boundary condition

\begin{align}
\tau_{n}^{(k,j)} &= 0 \quad \text{if} \ n > N \quad \text{for all} \ k, \ t \in \mathbb{Z}, \ \text{where the lattice size} \ N \ \text{is a positive integer. By imposing the boundary condition, the pair of the semi-infinite biorthogonal polynomial sequences} \ \{ \phi_{n}^{(k,j)}(z) \}_{n=0}^{\infty} \ \text{and} \ \{ \psi_{n}^{(k,j)}(z) \}_{n=0}^{\infty} \ \text{are reduced to the pair of finite polynomial sequences} \ \{ \phi_{n}^{(k,j)}(z) \}_{n=0}^{N} \ \text{and} \ \{ \psi_{n}^{(k,j)}(z) \}_{n=0}^{N}. \ \text{Since}
\end{align}

(40)
\[ e_{N-1}^{(k,j)} = e_N^{(k,j)} = 0 \] holds from (37), the spectral transformations (36b) and (36c) for \( n = N - 1 \) read
\[ \phi_N^{(k,M,j)}(z) = \phi_N^{(k,+1)}(z) = \phi_N^{(k,j)}(z). \] The ‘dual’ relation
\[ \psi_N^{(k+,+1)}(z) = \psi_N^{(k,j)}(z) \]
also holds.

By using the \( N \times N \) bidiagonal matrices
\[
L^{(k,j)} := \begin{pmatrix}
1 & e_0^{(k,j)} & 0 & \cdots & 0 \\
0 & 1 & e_1^{(k,j)} & \cdots & 0 \\
& 0 & 1 & \ddots & \vdots \\
& & \ddots & \ddots & e_{N-2}^{(k,j)} \\
& & & 0 & 1
\end{pmatrix},
\]
\[
R^{(k,j)} := \begin{pmatrix}
1 & q_0^{(k,j)} & 0 & \cdots & 0 \\
0 & 1 & q_1^{(k,j)} & \cdots & 0 \\
& 0 & 1 & \ddots & \vdots \\
& \ddots & \ddots & \ddots & q_{N-2}^{(k,j)} \\
& & & 0 & 1
\end{pmatrix},
\]
and the \( N \)-dimensional vectors
\[
\phi^{(k,j)}(z) := \begin{pmatrix}
\phi_0^{(k,j)}(z) \\
\phi_1^{(k,j)}(z) \\
\vdots \\
\phi_{N-1}^{(k,j)}(z)
\end{pmatrix},
\]
\[
\phi_N^{(k,j)}(z) := \begin{pmatrix}
0 \\
\vdots \\
0 \\
\phi_{N-1}^{(k,j)}(z)
\end{pmatrix},
\]
the spectral transformations (36a)–(36c) with the finite lattice boundary condition (40) are written as
\[
z \phi^{(k,+1)}(z) = R^{(k,j)} \phi^{(k,j)}(z) + \phi_N^{(k,j)}(z),
\]
\[
\phi^{(k,-M,j)}(z) = L^{(k,-M,j)} \phi^{(k,j)}(z),
\]
\[
\phi^{(k,j-1)}(z) = L^{(k,j-1)} \phi^{(k,j)}(z),
\]
respectively. Hence, we have
\[
z \phi^{(k,+1)}(z) = \tilde{L}^{(k,+1,j)} R^{(k,j+1)} \phi^{(k,j+1)}(z) + \tilde{L}^{(k,+1,j)} \phi_N^{(k,j+1)}(z)
= R^{(k,j)} \tilde{L}^{(k,j)} \phi^{(k,j+1)}(z) + \phi_N^{(k,j)}(z),
\]
\[
\phi^{(k,j)}(z) = \tilde{L}^{(k,j)} \phi^{(k,M,j+1)}(z)
= L^{(k,j)} \phi^{(k,M,j+1)}(z).\]
Note that, from (42), \( L^{(k+1,t)} \phi_n^{(k+1)}(z) = \phi_n^{(k+1)}(z) = \phi_n^{(k)}(z) \) holds. Therefore, the compatibility conditions are written in the same matrix form also for the finite lattice case:

\[
\tilde{L}^{(k+1,t)} R^{(k+1,t)} = R^{(k,t)} \tilde{L}^{(k,t)} R^{(k+1,t)} = L^{(k,t)} \tilde{L}^{(k+1,t)}, \tag{43}
\]

Now consider upper Hessenberg matrices of the form \( H^{(k,t)} := L^{(k,t)} R^{k+M-1,t} R^{k+M-2,t} \ldots R^{k,t} \).

By using the matrix relations (43), we find

\[
\tilde{L}^{(k,t)} H^{(k,t+1)} = \tilde{L}^{(k,t)} L^{(k+1,t)} R^{k+M-1,t+1} R^{k+M-2,t+1} \ldots R^{k,t+1)
\]

By using the matrix relations (43), we find

\[
\tilde{L}^{(k,t)} H^{(k,t+1)} = \tilde{L}^{(k,t)} L^{(k+1,t)} R^{k+M-1,t+1} R^{k+M-2,t+1} \ldots R^{k,t+1)
\]

Since \( \tilde{L}^{(k,t)} \) is regular, this implies that the two upper Hessenberg matrices \( H^{(k,t+1)} \) and \( H^{(k,t)} \) are similar. Therefore, we can say that the ndh-Toda lattice (38) with the finite lattice boundary condition (40) gives recurrence relations for computing iterations of similarity transformations of the upper Hessenberg matrices.

To construct solutions for the system above, let us consider the theory of ‘discrete’ \((M, 1)\)-biorthogonal polynomials \( \{ \phi_n^{(k)}(z) \}_{n=0}^N \) and \( \{ \psi_n^{(k)}(z) \}_{n=0}^N \) with respect to \( L^{(k,t)} \); i.e. the polynomials satisfy

\[
L^{(k,t)} \left[ z^n \phi_n^{(k)}(z) \right] = L^{(k,t)} \left[ z^m \psi_n^{(k)}(z) \right] = \frac{\delta_{n,m}}{\phi_n^{(k)}(z)} \delta_{n,m},
\]

\[
n = 0, 1, \ldots, N - 1, \quad m = 0, 1, \ldots, n,
\]

for all \( k, t \in \mathbb{Z} \). We can prove the following theorem, which is an analogue of Gauss quadrature for orthogonal polynomials.

**Theorem 3.1.** Suppose that the monic polynomial \( \psi_N^{(k)}(z) \) has simple zeros \( z_0^{(k)}, z_1^{(k)}, \ldots, z_{N-1}^{(k)} \). Then, there exist constants \( \psi_{n,r}^{(k)} \in \mathbb{C} \), \( r = 0, 1, \ldots, N - 1 \) and \( n = 0, 1, \ldots, M - 1 \), satisfying

\[
L^{(k,t)}[\pi(z)] = \sum_{r=0}^{N-1} \sum_{n=0}^{M-1} \psi_{n,r}^{(k,t)} \left( z_r^{(k)} \right)^{1/M} \left( z_r^{(k)} \right)^{-2\pi i/n/M}
\]

for all \( \pi(z) \in \mathbb{C}[z] \) where \( (z_r^{(k)})^{1/M} \) denotes one of the Mth roots of \( z_r^{(k)} \).

**Proof.** Let us consider the Lagrange interpolating polynomial

\[
L(z) := \sum_{r=0}^{N-1} \sum_{n=0}^{M-1} \pi \left( z_r^{(k)} \right)^{1/M} \left( z_r^{(k)} \right)^{-2\pi i/n/M} \left( \psi_N^{(k)}(z) \right)^{1/M} \left( z - z_r^{(k)} \right)^{1/M} \left( z - z_r^{(k)} \right)^{-2\pi i/n/M},
\]
where \( \cdot \) indicates the differentiation with respect to \( z \). Note that we can factorize the polynomial \( \psi_N^{(k,l)}(z^M) \) into

\[
\psi_N^{(k,l)}(z^M) = \prod_{r=0}^{N-1} \prod_{v=0}^{M-1} \left( z - (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right).
\]

Hence \( L(z) \) is a polynomial at most degree \( MN - 1 \) satisfying

\[
L \left( (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right) = \pi \left( (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right),
\]

\( r = 0, 1, \ldots, N - 1 \), \( v = 0, 1, \ldots, M - 1 \).

This implies the existence of a polynomial \( P(z) \) (may be zero, if \( \deg \pi(z) < MN \)) satisfying

\[
\pi(z) - L(z) = P(z)\psi_N^{(k,l)}(z^M).
\]

By using the relation above and the biorthogonal relation (45), we find

\[
\mathcal{L}^{(k,l)}[\pi(z)] = \mathcal{L}^{(k,l)}[L(z) + P(z)\psi_N^{(k,l)}(z^M)]
\]

\[
= \mathcal{L}^{(k,l)}[L(z)]
\]

\[
= \sum_{r=0}^{N-1} \sum_{v=0}^{M-1} \psi_N^{(k,l)}(z_r^{(l)}) \left( (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right),
\]

\[
\mathcal{L}^{(k,l)} \left[ \frac{\psi_N^{(k,l)}(z^M)}{\psi_N^{(k,l)}} \right] \left[ \frac{(z_r^{(l)})^{1/M} e^{-2\pi iv/M}}{\pi(z_r^{(l)})^{1/M} e^{-2\pi iv/M}} \right],
\]

which completes the proof.

Theorem 3.1 leads us to a representation of the moment:

\[
\mu_m^{(l)}(s) = \mathcal{L}^{(0,0)} \left[ z^m \prod_{r=0}^{N-1} \prod_{v=0}^{M-1} (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right]
\]

\[
= \sum_{r=0}^{N-1} \sum_{v=0}^{M-1} W_{r,v}^{(0,0)} \left( (z_r^{(l)})^{1/M} e^{-2\pi iv/M} \right)^m \prod_{r=0}^{N-1} \prod_{v=0}^{M-1} (z_r^{(l)})^{1/M} e^{-2\pi iv/M}
\]

\[
= \sum_{r=0}^{N-1} \sum_{v=0}^{M-1} \left( \sum_{v=0}^{M-1} W_{r,v}^{(0,0)} e^{-2\pi iv/M} \right)^m \prod_{r=0}^{N-1} \prod_{v=0}^{M-1} (z_r^{(l)})^{1/M} e^{-2\pi iv/M},
\]

(46)

where \( z_r := z_r^{(0)} \). Let us introduce new constants

\[
W_r^{(m)} := \sum_{v=0}^{M-1} W_{r,v}^{(0,0)} e^{-2\pi iv/M}, \quad r = 0, 1, \ldots, N - 1, \quad m = 0, 1, 2, \ldots.
\]

(47)

Then, the representation of the moment (46) is rewritten as

\[
\mu_m^{(l)} = \sum_{r=0}^{N-1} W_r^{(m)} \prod_{r=0}^{N-1} \prod_{v=0}^{M-1} (z_r^{(l)})^{1/M} e^{-2\pi iv/M},
\]

(48)

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We should remark that \( w^{(m)}_r = w^{(m \mod M)}_r \) holds for all \( m = 0, 1, 2, \ldots \) and there is a one-to-one correspondence between the constants \( \{w^{(0,0)}_r\}_{r=0,1,\ldots,M-1} \) and \( \{w^{(m)}_r\}_{r=0,1,\ldots,N-1} \) via the definition (47), that is the discrete Fourier transform.

Substituting the moment representation (48) into the determinant \( r_n^{(k,t)} \) (35), we find

\[
\tau_n^{(k,t)} = \det(\tilde{V}_n^{(k)} \mathcal{D}^{(k,t)} V_n),
\]

where

\[
\tilde{V}_n^{(k)} := \begin{pmatrix}
 w^{(k)}_0 & w^{(k)}_1 & \cdots & w^{(k)}_{N-1} \\
 w^{(k+1)}_0 & w^{(k+1)}_1 & \cdots & w^{(k+1)}_{N-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 w^{(k+n-1)}_0 & w^{(k+n-1)}_1 & \cdots & w^{(k+n-1)}_{N-1}
\end{pmatrix},
\]

\[
\mathcal{D}^{(k,t)} := \text{diag}
\begin{pmatrix}
 \frac{k}{M} & \frac{k}{M} & \cdots & \frac{k}{M} \\
 \frac{1}{M} & \frac{1}{M} & \cdots & \frac{1}{M} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{1}{M} & \frac{1}{M} & \cdots & \frac{1}{M}
\end{pmatrix},
\]

\[
V_n := \begin{pmatrix}
 1 & z_0 & \cdots & z_{n-1} \\
 1 & z_1 & \cdots & z_{n-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 1 & z_{N-1} & \cdots & z_{N-1}
\end{pmatrix}.
\]

Applying the Binet–Cauchy formula and the expansion formula for the Vandermonde determinant to (49), we obtain

\[
\tau_n^{(k,t)} = \sum_{0 \leq \tau_1 < \tau_2 < \cdots < \tau_{n-1} \leq N-1} \mathcal{V}^{(k)}_{\tau_1, \tau_2, \ldots, \tau_{n-1}} \prod_{j=0}^{n-1} \left( \frac{k/M}{\tau_j} \prod_{\tau = 0}^{t-1} (\zeta_{\tau} - s(\tau)) \right) \prod_{0 \leq \tau < j \leq N-1} (\zeta_{\tau} - z_j),
\]

where

\[
\mathcal{V}^{(k)}_{\tau_1, \tau_2, \ldots, \tau_{n-1}} := \begin{pmatrix}
 w^{(k)}_0 & w^{(k)}_1 & \cdots & w^{(k)}_{N-1} \\
 w^{(k+1)}_0 & w^{(k+1)}_1 & \cdots & w^{(k+1)}_{N-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 w^{(k+n-1)}_0 & w^{(k+n-1)}_1 & \cdots & w^{(k+n-1)}_{N-1}
\end{pmatrix}.
\]

Let us derive a sufficient condition for the positivity \( \tau_n^{(k,t)} > 0 \) for all \( k, t \in \mathbb{Z} \) and \( n = 1, 2, \ldots, N \). Hereafter, for simplicity, we suppose that

- \( z_0, z_1, \ldots, z_{N-1} \) are all real numbers satisfying \( 0 < z_0 < z_1 < \cdots < z_{N-1}; \)
- all the \( M \)th roots \( \zeta_0^{1/M}, \zeta_1^{1/M}, \ldots, \zeta_{N-1}^{1/M} \) are chosen as real numbers;
- the parameter \( s(t) \) is chosen as \( s(t) < z_0 \) for all \( t \in \mathbb{Z} \).

These assumptions are natural for an application to matrix eigenvalue algorithms; the parameter \( s(t) \) should be chosen as above for the dqds algorithm [3], which is
equivalent to the nd-Toda lattice, in order to guarantee the positivity of the variables. In addition, if \( V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} > 0 \) for all \( k \in \mathbb{Z} \) and all \( n \)-tuples \((r_0, r_1, \ldots, r_{n-1})\) satisfying \( 0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N - 1 \), then it is obvious that the conditions are sufficient for the positivity of \( \tau_n(k) \). Since \( w_r^{(m)} = w_r^{(m \mod M)} \) implies \( V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} = \cdot \), the number of the conditions \( V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} > 0 \) is finite: there are \( M^N \sum_{n=1}^{N} \binom{N}{n} = M(2^N - 1) \) conditions.

We will rewrite the condition \( V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} > 0 \) in a simpler form. First, if \( n = 1 \), then \( V_{t_0}^{(k)} = w_{r_0}^{(k)} > 0 \), i.e., all \( w_r^{(m)} \) must be positive. Next, if \( n = 2, 3, \ldots, N \), then the elementary row-additions yield

\[
V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} = w_{t_0}^{(k)} W_{r_0}^{(k)} W_{r_1}^{(k)} \ldots W_{r_{n-1}}^{(k)} = w_{t_0}^{(k)} W_{r_0}^{(k)} W_{r_1}^{(k)} \ldots W_{r_{n-1}}^{(k)}
\]

where

\[
w_{t_0}^{(m)} := \frac{w_{t_0}^{(m+1)}}{w_{r_1}^{(m)}} \frac{w_{r_1}^{(m+1)}}{w_{t_1}^{(m)}} \ldots \frac{w_{r_{n-1}}^{(m+1)}}{w_{r_1}^{(m)}} \frac{w_{r_{n-1}}^{(m+1)}}{w_{r_1}^{(m)}} = \frac{1}{w_{t_0}^{(m)}} \frac{w_{r_0}^{(m)}}{w_{t_1}^{(m)}} \ldots \frac{w_{r_{n-1}}^{(m)}}{w_{r_1}^{(m)}} \frac{w_{r_{n-1}}^{(m)}}{w_{r_1}^{(m)}} = \cdot
\]

In the same manner, we can show by induction on \( n \) that

\[
V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} = w_{t_0}^{(k)} W_{r_0}^{(k+1)} \ldots W_{r_{n-1}}^{(k+1)}
\]

where \( w_{t_0, r_1, \ldots, r_{n-1}}^{(m)} \) is defined recursively by

\[
w_{t_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)} := \frac{1}{w_{t_0}^{(m)}} \frac{w_{t_0}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} \frac{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} \frac{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} = \cdot
\]

From equation (51) with the condition \( V_{t_0, r_1, \ldots, r_{n-1}}^{(k)} > 0 \), it is readily induced by induction on \( n \) that all \( w_{t_0, r_1, \ldots, r_{n-1}}^{(m)} \) must be positive. Thus we obtain the following theorem.

**Theorem 3.2.** Suppose that all \( w_r^{(m)} \) are positive and the relation

\[
\frac{w_{t_0}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} \frac{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} > \frac{w_{t_0}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}} \frac{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}}{w_{r_0, r_1, \ldots, r_{n-2}, r_{n-1}}^{(m+1)}}
\]

holds for all \( m = 0, 1, \ldots, M - 1 \) and all \( n \)-tuples \((r_0, r_1, \ldots, r_{n-1})\) satisfying \( 0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N - 1 \), \( n = 2, 3, \ldots, N \). Then,
\[ t^{(k,l)}_{n} = \sum_{0 \leq r_{0} < r_{1} < \cdots < r_{n-1} \leq N-1} \left( W^{(k)}_{r_{0},r_{1}} W^{(k+1)}_{r_{1},r_{2}} \cdots W^{(k+n-1)}_{r_{n-1},r_{n}} \right) \times \prod_{j=0}^{n-1} \left( z_{r_{j}}^{k/M} \prod_{\tau=0}^{t-1} \left( z_{r_{j}} - s^{(\tau)} \right) \right) \prod_{0 \leq i < j \leq n-1} \left( z_{r_{j}} - z_{r_{i}} \right) > 0 \]  
(54)

for all \( k, t \in \mathbb{Z} \) and \( n = 1, 2, \ldots, N \).

**Corollary 3.1.** A solution to the ndh-Toda lattice (38) with the finite lattice boundary condition (41) is given by (37) and (54). If the positivity condition (53) is satisfied, then the variables \( q^{(k,l)}_{n}, \epsilon^{(k,l)}_{n}, \zeta^{(k,l)}_{n} \) and \( d^{(k,l)}_{n} \) are always positive. Furthermore, if the parameter \( s^{(l)} \) is chosen as \( s^{(l)} \leq 0 \), then \( f^{(k,l)}_{n} \) is always nonnegative.

**Remark 3.2.** Since we have assumed that \( z_{0}, z_{1}, \ldots, z_{N-1} \) are all simple in theorem 3.1, this is not the general solution to the ndh-Toda lattice under the finite lattice condition. In addition, the solution to the initial value problem should be studied in future works.

Finally, we discuss the asymptotic behaviour of the solution as \( t \to \infty \). Since, from the assumption, \( 0 < z_{0} - s^{(l)} < z_{1} - s^{(l)} < \cdots < z_{N-1} - s^{(l)} \) holds for all \( t \in \mathbb{Z} \), we find

\[ t^{(k,l)}_{n} \sim \mathcal{V}^{(k)}_{N-n,N-n+1,\ldots,N-1} \prod_{j=0}^{n-1} \left( z_{N-n-j}^{k/M} \prod_{\tau=0}^{t-1} \left( z_{N-n+j} - s^{(\tau)} \right) \right) \prod_{0 \leq i < j \leq n-1} \left( z_{N-n+j} - z_{N-n-i} \right) \]

as \( t \to \infty \). Hence, we have the following asymptotic behaviour of the solution given by (37) and (50):

\[ q^{(k,l)}_{n} \sim \frac{\mathcal{V}^{(k)}_{N-n,N-n+1,\ldots,N-1} \mathcal{V}^{(k+1)}_{N-n-1,N-n+2,\ldots,N-1}}{\mathcal{V}^{(k)}_{N-n-1,N-n+1,\ldots,N-1} \mathcal{V}^{(k+1)}_{N-n,N-n+2,\ldots,N-1}} \frac{1}{z_{N-n-1}^{k/M}}, \]  
(55a)

\[ e^{(k,l)}_{n} \sim \mathcal{V}^{(k)}_{N-n-2,N-n,\ldots,N-1} \prod_{\tau=0}^{t} \frac{z_{N-n-\tau} - s^{(\tau)}}{z_{N-n-1} - s^{(\tau)}} \to 0 \]  
(55b)

as \( t \to \infty \), where

\[ \mathcal{V}^{(k)}_{N-n-2,N-n,\ldots,N-1} \frac{\mathcal{V}^{(k)}_{N-n-1,N-n+1,\ldots,N-1}}{(\mathcal{V}^{(k)}_{N-n-1,N-n,\ldots,N-1})^{2}} \frac{z_{N-n-2}^{k/M}}{z_{N-n-2}^{(k+1)/M}} \prod_{j=0}^{n} \frac{z_{N-n-1+j} - z_{N-n-2}}{z_{N-n-1+j} - z_{N-n-1}}. \]

Especially, we also have

\[ \prod_{j=0}^{M-1} q^{(k+1,l)}_{n} \to z_{N-n-1} \]

as \( t \to \infty \).

The results indicate that the upper Hessenberg matrix \( H^{(k,l)} \) (44) goes to an upper triangular matrix whose \((n,n)\)-entry is \( z_{N-n} \) as \( t \to \infty \), \( n = 0, 1, \ldots, N-1 \). Since \( H^{(k,l)} \) and \( H^{(k,l+1)} \) are similar, it is revealed that \( z_{0}, z_{1}, \ldots, z_{N-1} \) are the eigenvalues of \( H^{(k,l)} \). That is, the recurrence relations of the ndh-Toda lattice (39) with the finite lattice boundary condition (41) give an eigenvalue algorithm for upper Hessenberg matrices that can be factorized.
into a product of bidiagonal matrices as (44). Its convergence speed depends on the value 
\((z_{N-n-2} - e^{(l)})/z_{N-n-1} - e^{(l)}), n = 0, 1, \ldots, N - 2, \) by (55b). This means that an appropriate
choice of the parameter \(s^{(l)}\) may improve the convergence speed.

**Remark 3.3.** The recurrence relations (38) with (41) are essentially same as the eigen-
value algorithm for totally nonnegative Hessenberg matrices proposed by Fukuda et al [4, algorithm 1]. Therefore, we can say that we have investigated another theoretical aspect of
the eigenvalue algorithm from the viewpoint of discrete integrable systems and biortho-
gonal polynomials.

4. **Ultradiscretization**

In this section, we ultradiscretize the ndh-Toda lattice and its solution in section 3, and give
a proof of a connection between the derived ultradiscrete system and the generalized BBS.

4.1. **Nonautonomous ultradiscrete finite hungry Toda lattice**

For the variables and parameter of the ndh-Toda lattice (38), we consider the transformations
from the variables to new variables denoted by capital letters as follows: 
\(q^{(k,t)}_n = e^{-Q^{(k,t)}_n}/\epsilon, e^{(k,t)}_n = e^{-E^{(k,t)}_n}/\epsilon, f^{(k,t)}_n = e^{-F^{(k,t)}_n}/\epsilon, g^{(k,t)}_n = e^{-G^{(k,t)}_n}/\epsilon,\) and
\(h^{(k,t)}_n = -e^{-H^{(k,t)}_n}/\epsilon,\) where \(\epsilon\) is a positive parameter. Since there is an ultradiscretization formula
\[
\lim_{\epsilon \to 0} -\epsilon \log(p_1 e^{-A}/\epsilon + p_2 e^{-B}/\epsilon) = \min(A, B),
\]
where \(p_1\) and \(p_2\) are positive numbers, applying these transformations and taking a limit 
\(\epsilon \to +0\) yield piecewise linear recurrence relations

\[
Q^{(k,t+1)}_n = \min(D^{(k,t)}_n, E^{(k,t)}_n), \quad (56a)
\]

\[
E^{(k,t+1)}_n = \min(F^{(k,t)}_n, E^{(k+M,t)}_n), \quad (56b)
\]

\[
D^{(k,t)}_{n+1} = D^{(k,t)}_n - Q^{(k,t+1)}_n + Q^{(k,t)}_{n+1}, \quad (56c)
\]

\[
F^{(k,t)}_{n+1} = F^{(k,t)}_n - E^{(k,t+1)}_n + E^{(k,t)}_{n+1}, \quad (56d)
\]

\[
\tilde{E}^{(k+1,t)}_n = \tilde{E}^{(k,t)}_n - Q^{(k+1,t+1)}_n + Q^{(k+1,t)}_{n+1}, \quad (56e)
\]

\[
\tilde{E}^{(k,t+1)}_n = \tilde{E}^{(k+M,t)}_n - Q^{(k+1,t+1)}_n + E^{(k,t)}_{n+1}, \quad (56f)
\]

for \(n = 0, 1, 2, \ldots\) with the boundary condition

\[
D^{(k,t)}_0 = Q^{(k,t)}_0, \quad (56g)
\]

\[
F^{(k,t)}_0 = E^{(k,t)}_0 + \max \left(0, S^{(t)} - \sum_{j=0}^{M-1} Q^{(k+j,t)}_0 \right), \quad (56h)
\]

\[
\tilde{E}^{(k,t)}_0 = E^{(k,t)}_0 + \max \left(0, \sum_{j=0}^{M-1} Q^{(k+j,t)}_0 - S^{(t)} \right) \quad (56i)
\]
for all \( k, t \in \mathbb{Z} \). We call the system (56) the nonautonomous ultradiscrete hungry Toda lattice (nuh-Toda lattice). In addition, we also impose the finite lattice condition corresponding to (41):

\[
E_{N-1}^{(k,t)} = \tilde{E}_{N-1}^{(k,t)} = +\infty.
\]  

(57)

A solution to the nuh-Toda lattice (56) with the finite lattice condition (57) is constructed from the solution (37) and (54) to the ndh-Toda lattice (38) with (41). Consider the transformations of variables \( t_n^{(k,t)} = e^{-t_n^{(k,t)}} \) and \( z_n = p_n e^{-Z_n} \) and \( w_n^{(m)} = e^{-w_n^{(m)}} \) and the limit procedure \( \epsilon \to +0 \), where \( p_n \) is a positive constant satisfying \( p_n < p_{n+1} \) if \( Z_n = Z_{n+1} \). Note that, since we assume the inequality \( 0 < z_0 < z_1 < \cdots < z_{N-1} \) in section 3, the new variable \( Z_n \) satisfies \( Z_0 > Z_1 > \cdots > Z_{N-1} \). To apply the transformations of variables, \( t_n^{(k,t)} \) must be positive; i.e. the condition (53) must be satisfied. This means that the new variables satisfy the relation

\[
w_n^{(m)} + w_{n+1}^{(m+1)} + \frac{1}{M} Z_{n+1} = W_n^{(m+1)} + \frac{1}{M} Z_{n+1} = \cdots = W_{n+1}^{(m+1)} + \frac{n-1}{M} Z_{n+1}.
\]  

(58)

for all \( m = 0, 1, \ldots, M-1 \) and all \( n \)-tuples \( (r_0, r_1, \ldots, r_{n-1}) \) satisfying \( 0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N - 1, n = 1, 2, 3, \ldots, N \). We should remark that the formula

\[
\lim_{\epsilon \to +0} -\epsilon \log(p_1 e^{-A_1/\epsilon} - p_2 e^{-B_1/\epsilon}) = \begin{cases} 
A & \text{if } A < B \text{ or } A = B \text{ and } p_1 > p_2, \\
\text{indefinite} & \text{if } A > B \text{ or } A = B \text{ and } p_1 < p_2,
\end{cases}
\]

holds, where \( p_1 \) and \( p_2 \) are positive numbers. The latter indefinite result is the cause of the negative problem of ultradiscretization. However, if the condition (58) is satisfied, then it is assured that we can always use the former result for (52). Hence we obtain, by induction on \( n \),

\[
w_n^{(m)} + w_{n+1}^{(m+1)} + \frac{1}{M} Z_{n+1} = W_n^{(m+1)} + \frac{1}{M} Z_{n+1} = \cdots = W_{n+1}^{(m+1)} + \frac{n-1}{M} Z_{n+1}.
\]  

(59)

We should remark that, by using (59), the condition (58) is simply rewritten as

\[
w_n^{(m)} + w_{n+1}^{(m+1)} + \frac{1}{M} Z_{n+1} = W_n^{(m+1)} + \frac{1}{M} Z_{n+1} = \cdots = W_{n+1}^{(m+1)} + \frac{n-1}{M} Z_{n+1}.
\]

Hence, we obtain the following theorem.

**Theorem 4.1.** If the conditions \( Z_0 > Z_1 > \cdots > Z_{N-1} \) and

\[
w_n^{(m)} - w_{n+1}^{(m+1)} + W_n^{(m+1)} - W_{n+1}^{(m+1)} + \frac{Z_{n+1} - Z_n}{M} \geq 0
\]

are satisfied for all \( m = 0, 1, \ldots, M-1 \) and all pairs \( (r_0, r_1) \) satisfying \( 0 \leq r_0 < r_1 \leq N - 1 \), then we have, from (54) and (59),

\[
T_{0}^{(k,t)} = 0,
\]

\[
T_{n}^{(k,t)} = \min_{0 \leq r_0 < r_1 < \cdots < r_{n-1} < N-1} \left( \sum_{j=0}^{n-1} \left( W_{r_j}^{(k)} + k + (M+1)j \right) Z_{r_j} + \sum_{\tau=0}^{j-1} \min(Z_{r_\tau}, S_{r_\tau}) \right),
\]

\( n = 1, 2, \ldots, N \).

By using the function \( T_n^{(k,t)} \), a solution to the nuh-Toda lattice (56) with the finite lattice condition (57) is given by, from (37),
\[ Q_n^{(k,i)} = T_n^{(k,i)} - T_{n+1}^{(k,i)} + T_{n+1}^{(k+1,i)} - T_n^{(k+1,i)}, \]
\[ F_n^{(k,i)} = T_{n+2}^{(k,i)} - T_{n+1}^{(k,i)} + T_{n+1}^{(k+1,i)} - T_n^{(k+1,i+1)}, \]
\[ E_n^{(k,i)} = T_{n+2}^{(k,i)} - T_{n+1}^{(k,i)} + T_{n+1}^{(k+1,i)} - T_n^{(k+1,i)}, \]
\[ D_n^{(k,i)} = T_{n+2}^{(k,i)} - T_{n+1}^{(k,i)} + T_{n+1}^{(k+1,i+1)} - T_n^{(k+1,i+1)}, \]
\[ F_n^{(k,i)} = T_{n+2}^{(k,i+1)} - T_{n+1}^{(k,i+1)} + T_{n+1}^{(k+1,i+1)} - T_n^{(k+1,i+1)} + S^{(i)}. \]

### 4.2. Connection to the generalized BBS

Finally, we prove a correspondence between the generalized BBS and the nuh-Toda lattice (56).

The time evolution equation of the BBS with \(M\) kinds of balls and the carrier of capacity \(S^{(i)} > 0\) at time \(t\) is given by an \((M + 1)\)-reduced nonautonomous ultradiscrete KP lattice:

\[ U_n^{(k,i+1)} = U_n^{(k,i)} - X_n^{(k,i)} + X_{n+1}^{(k,i+1)}, \quad (60a) \]
\[ V_{n+1}^{(k,i)} = V_n^{(k,i)} + X_n^{(k,i)} - X_{n+1}^{(k,i+1)}, \quad (60b) \]
\[ X_n^{(k,i)} = \min_{j=0,1,\ldots,M} \left( \sum_{j=0}^{M-1} U_n^{(k,j,i)} + \sum_{j=M-1}^{M} V_n^{(k,j+1,i)} \right), \quad (60c) \]

where \(U_n^{(k,M+1,i)} = U_n^{(k,i)}\) and \(V_n^{(k,M+1,i)} = V_n^{(k,i)}\) for all \(k, t, n \in \mathbb{Z}\), with the boundary condition

\[ U_n^{(0,i)} = 1, \quad V_n^{(0,i)} = S^{(i)}, \quad (60d) \]
\[ U_n^{(k,i)} = 0, \quad V_n^{(k,i)} = 0, \quad k = 1, 2, \ldots, M, \quad (60e) \]

for \(|n| \gg 1\). We should choose the initial values of the system (60) to satisfy

\[ \sum_{j=0}^{M} U_n^{(j,i)} = 1 \]

for all \(n \in \mathbb{Z}\). Then, it is readily shown that the relations

\[ \sum_{j=0}^{M} U_n^{(j,i)} = 1, \quad \sum_{j=0}^{M} V_n^{(j,i)} = S^{(i)} \]

hold for all \(n, t \in \mathbb{Z}\). The variables denote

- \(U_n^{(0,i)} \in \{0, 1\}\): the number of empty spaces in the \(n\)th box at time \(t\);
- \(U_n^{(k,i)} \in \{0, 1\}\): the number of balls with index \(k\) in the \(n\)th box at time \(t\), \(k = 1, 2, \ldots, M\);
- \(V_n^{(0,i)} \in \{0, 1, \ldots, S^{(i)}\}\): the number of empty spaces in the carrier at the \(n\)th box from time \(t\) to \(t+1\);
- \(V_n^{(k,i)} \in \{0, 1, \ldots, S^{(i)}\}\): the number of balls with index \(k\) in the carrier at the \(n\)th box from time \(t\) to \(t+1\), \(k = 1, 2, \ldots, M\).

The left side of figure 2 shows an example of the time evolution of the BBS (60) with \(M = 3\) kinds of balls and carrier capacity \(S^{(i)} = 6\) for all \(t\), in which ‘1’, ‘2’, ‘3’, and ‘ ’, denote a ball with index 1, 2, 3, and an empty box, respectively. Each box can contain only
one ball. In the followings, we regard ‘an empty space’ as a ball with index 0. Then, we can explain the evolution rule of the BBS (60) as follows: From time \( t \) to \( t + 1 \), the carrier of capacity \( S^{(i)} \) moves from left to right. When the carrier passes each box, if the box contains a ball with index \( k \), then the carrier exchanges the ball with a ball in the carrier whose index is the smallest in the carrier’s balls, where the index order is defined on \( \mathbb{Z}/(M + 1)\mathbb{Z} \) as the smallest index is \( k + 1 \); e.g. if \( k = 2 \), then the index order is \( 3 < 4 < 5 < \cdots < M < 0 < 1 < 2 \). Figure 3 also illustrates the exchange rule by examples.

Now, we can prove the following theorem.

**Theorem 4.2.** For the BBS with \( M \) kinds of balls and carrier capacity \( S^{(i)} \) at each time \( t \), let

- \( Q^{(K, j)}_{n, t} \) be the number of balls with index \( k \) in the \( n \)th block of balls at time \( t \), \( k = 1, 2, \ldots, M \);
- \( E^{(1, j)}_{n, t} \) be the number of empty boxes between the \( n \)th and \((n + 1)\)st blocks of balls at time \( t \).

Then, the variables \( Q^{(K, j)}_{n, t} \) and \( E^{(1, j)}_{n, t} \) satisfy the nuh-Toda lattice (56) with the finite lattice boundary condition \( E^{(1, j)}_{M, t} = +\infty \).

The right side of figure 2 shows an example of the time evolution of the nuh-Toda lattice with \( M = 3 \), \( N = 3 \) and \( S^{(i)} = 6 \) for all \( t \). The initial values of the nuh-Toda lattice are chosen to correspond to the initial state of the BBS. The solution given by theorem 4.1 with the parameters \( Z_0 = 7, Z_1 = 5, Z_2 = 3 \), \( W^{(0)}_0 = 1, W^{(1)}_0 = 5/3, W^{(2)}_0 = 1/3, W^{(1)}_1 = 6, W^{(1)}_2 = 19/3, W^{(2)}_1 = 17/3, W^{(2)}_2 = 13, W^{(3)}_2 = 13 \), and \( W^{(2)}_2 = 12 \) corresponds to the time evolution in figure 2. Notice that the balls in each block must be arranged in ascending order of indices from left to right. For example, ‘1222333112’ is composed of two blocks ‘1222333’ and ‘1112’.

**Proof of theorem 4.2.** We will also show the roles of the other variables.

- \( D^{(K, j)}_{n, t} \): the maximum number of balls with index \( k \) that the carrier can put into boxes as the part of the \( n \)th block of balls at time \( t + 1 \), \( k = 1, 2, \ldots, M \);
- \( D^{(K, j)}_{n, t} \): the number of boxes between the leftmost position of the balls with index \( k \) corresponding to the variables \( Q^{(K, j)}_{n, t+1} \) and \( O^{(K, j)}_{n+1} \), \( k = 1, 2, \ldots, M \);
- \( E^{(1, j)}_{n, t} \): the number of boxes between the leftmost position of the empty boxes corresponding to the variables \( E^{(1, j)}_{n+1, t} \) and \( E^{(1, j)}_{n+1} \);
- \( E^{(1, j)}_{n, t} \): the sum of the value \( E^{(1, j)}_{n, t} \) and the number of empty spaces in the carrier after passing the \( n \)th block of balls from time \( t \) to \( t + 1 \).
See also figure 4. In this proof, we will use the following simple formulae:

\[ - \min(-A, -B) = \max(A, B), \]
\[ A + \min(B, C) = \min(A + B, A + C). \]

First, since the carrier gets \( Q(k, t) \) balls with index \( k \) from the 0th block of balls, the boundary condition (56g) gives the number of balls with index \( k \) that the carrier can put into boxes as the part of the 0th block of balls at time \( t + 1 \). The carrier puts \( Q(k, t + 1) \) balls with index \( k \) and gets \( Q_{n+1}^{k,j} \) balls with index \( k \) between the leftmost position of the \( n \)th and \((n + 1)\)st blocks of empty boxes. Hence, the recurrence relation (56e) indeed calculates the value of \( D_n^{k,j} \) for its role correctly.

Next, since the carrier exchanges \( \min(\sum_{k=1}^{M} Q_{0}^{k,j}, S^{(i)}) \) balls in the 0th block of balls with empty spaces in the carrier, the value of \( E_0^{(1,j)} \) is given by

\[
E_0^{(1,j)} = E_0^{(1,j)} + \sum_{k=1}^{M} Q_{0}^{k,j} - \min\left(\sum_{k=1}^{M} Q_{0}^{k,j}, S^{(i)}\right)
\]
\[
= E_0^{(1,j)} + \max\left(0, \sum_{k=1}^{M} Q_{0}^{k,j} - S^{(i)}\right).
\]

This is the boundary condition (56i). Then, it is obvious that the recurrence relations (56e) and (56f) indeed calculate the value of \( E_n^{k,j} \) for its role. Similarly, the value of \( F_0^{(1,j)} \) should be

\[
F_0^{(1,j)} = E_0^{(1,j)} + S^{(i)} - \min\left(\sum_{k=1}^{M} Q_{0}^{k,j}, S^{(i)}\right)
\]
\[
= E_0^{(1,j)} + \max\left(0, S^{(i)} - \sum_{k=1}^{M} Q_{0}^{k,j}\right).
\]

which coincides with the boundary condition (56h). Here, let \( Y_n^{(1,j)} \) be the number of empty spaces in the carrier just after passing the \( n \)th block of balls at time \( t \); i.e.

\[
F_n^{(1,j)} = E_n^{(1,j)} + Y_n^{(1,j)}.
\]

Then, since the carrier puts \( \min(E_n^{(1,j)}, S^{(i)} - Y_n^{(1,j)}) \) balls into the \( n \)th block of empty boxes at time \( t \), the number of empty spaces in the carrier just before the carrier passes the \((n + 1)\)st
block of balls is given by
\[ Y_{n+1}^{(1)} = \min(F_n^{(1)} - S(i)) - \left(F_n^{(1, n+1)} - \max(0, F_n^{(1, n+1)} - S(i))\right) \]

and
\[ F_{n+1}^{(1)} = E_{n+1}^{(1)} + Y_{n+1}^{(1)} = F_n^{(1)} - E_n^{(1, n+1)} + E_{n+1}^{(1)} , \]

which coincides with the recurrence relation (56d).

Notice that the value of \( E_n^{(k, j)} \) gives the maximum number of boxes in which the carrier can put the balls with index \( k \) as the part of the \( n \)th block of balls at time \( t+1 \). Hence, the value of \( Q_n^{(k, r+1)} \) is given by the recurrence relation (56a). We also notice that \( E_n^{(1, n+1)} \) gives the maximum number of boxes which can be consistent with empty boxes at time \( t+1 \); i.e. \( E_n^{(1, n+1)} = E_n^{(1, n+1) + M, i} \) if the carrier has a sufficient number of empty spaces. Further, by the discussion in the previous paragraph, we find that the sum of ‘the number of empty boxes that the carrier passes with no balls between the \( n \)th and \( (n+1) \)th blocks of balls’ and ‘the number of empty spaces in the carrier just before passing the \( (n+1) \)th block of balls’ is equal to
\[ \max(0, F_n^{(1, j)} - S(i)) + \min(F_n^{(1, j)} - S(i)) = F_n^{(1, j)} - \min(F_n^{(1, j)} - S(i)) + \min(F_n^{(1, j)} - S(i)) = F_n^{(1, j)} . \]

Therefore, the value of \( F_n^{(1, j)} \) also gives the maximum number of boxes which can be consistent with \( n \)th block of empty boxes at time \( t+1 \), and the recurrence relation (56d) indeed calculates the value of \( E_n^{(1, n+1)} \). \[ \square \]
5. Concluding remarks

In this paper, we have derived the nuh-Toda lattice and constructed its particular solution under the finite lattice boundary condition by using the theory of biorthogonal polynomials. Further, we have proven that the nuh-Toda lattice is another time evolution equation of the BBS with many kinds of balls and finite carrier capacity.

Several problems are left for future works. In subsequent papers, we are going to discuss the following topics.

Firstly, after imposing \((M,1)\)-reduction condition to biorthogonal polynomials, we have only discussed the time evolution for \(t_2\)-direction. However, there is another time variable \(t_1\) and exists another nonautonomous version of the discrete hungry Toda lattice. We will be able to ultradiscretize the system and consider a corresponding BBS-like cellular automaton. Investigating and analyzing this novel cellular automaton, its solutions, and relations to the BBS discussed in this paper are interesting problems.

Secondly, it is known that the nonautonomous discrete Toda type systems give good numerical algorithms [17]. As mentioned in section 3, the ndh-Toda lattice (38) is same as the eigenvalue algorithm for totally nonnegative Hessenberg matrices proposed by Fukuda et al [4, algorithm 1], and we have discussed its asymptotic behaviour for a special case by analyzing the solution (50). Then, we have a natural question: Does the recurrence relation of the ndh-Toda lattice for \(t_1\)-direction also give a good numerical algorithm? By using techniques similar to that used in this paper, we will be able to construct a particular solution, perform asymptotic analysis, and give an answer to this question. Investigating relations between the algorithms of \(t_1\)-direction and \(t_2\)-direction is also an interesting problem.

Thirdly, we will be able to impose other reduction conditions to the nd-2D-Toda lattice. For example, as a generalization of the qd algorithm, which is same as the (autonomous) discrete Toda lattice, the multiple dqd algorithm is proposed by Yamamoto and Fukaya [31]. The multiple dqd algorithm is an eigenvalue algorithm for matrices decomposed to the product of \(M_1\) upper bidiagonal matrices and \(M_2\) lower bidiagonal matrices. We expect that nonautonomous versions of the multiple dqd algorithm are derived from the nd-2D-Toda lattice by imposing \((M_1, M_2)\)-reduction, i.e. \(g^{(k_1,k_2,t_1,t_2)} = g^{(k_1,k_2+M_2,t_1,t_2)}\). There must be many examples and applications not limited to it.

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