Optimal Contact Points for an Octopus Arm

Simone Cacace¹ · Anna Chiara Lai² · Paola Loreti²

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Abstract
We investigate the optimality of the configurations of a tentacle-like soft manipulator ensuring a planar force-closure condition. In particular, the optimization is performed with respect to both the control strategies and to the set of contact points ensuring a planar, frictionless, first-order force-closure of the target. The case of an elliptic target object is discussed in detail, and numerical simulation are presented.

Keywords Grasping · Soft manipulators · Optimal control strategies · Force-closure

Introduction
In this paper, we investigate the optimality of the configurations of a tentacle-like soft manipulator ensuring a planar force closure condition on an elliptic target. The novelty consists in considering the set of contact points as an unknown of the optimization problem. Indeed, the contact points on the target object are selected to yield force-closure configurations, while minimizing an integral cost on the curvature of the manipulator. We adopt an optimal control theoretic approach: optimal contact points and optimal configuration satisfy stationarity conditions for a constrained minimization problem. In particular, the objective cost functional accounts for an integral cost on the curvature, an non-interpenetration (penalized) constraint, and the grasping task. The minimization is performed subject to an equilibrium condition for the manipulator, and to force-closure conditions for the target contact points.

In [1], we considered prescribed contact subregions on general objects. Here, we consider planar, stationary, frictionless grasps of elliptic targets— with positive eccentricity. This choice is motivated by the fact that we see the case of an ellipse as a projection of the three-dimensional problem of grasping cylindrical and ellipsoidal objects.

We borrow from [2] the study of a multi-target optimal control problem: the main difference is that in [2], the target points were fixed a priori, whereas in the present work, the target points are provided as part of the optimization problem.

The control model underlying our study was introduced in [3]. It is a control model for a planar string, modeling the symmetry axis of an octopus arm. Its dynamics encompasses an exact inextensibility constraint, bending moment, curvature constraints and distributed curvature controls. In this framework, several optimization problems were addressed. They include optimal reachability [3, 4], obstacle avoidance [1, 5], and grasping problems [2, 6]. In particular, [1] addresses the problem of touching a prescribed portion of the boundary of a target object while avoiding interpenetration and minimizing a quadratic cost on the controls. In [2], the contact sub-region of the manipulator is treated as an unknown of the problem and, at the same time, the target is allowed to be disconnected.

Here, we look for optimal configurations ensuring force-closure grasps. The force-closure property is a classical
geometric condition on the contact points ensuring the grasp of an object to be stable with respect external disturbances: in particular, a grasp achieves force-closure if any external wrench can be balanced by wrenches at the contact points [7–10]. For planar objects, four contact points are necessary to guarantee a first-order force-closure grasp. Of course, the contact points ensuring a force-closure grasp are in general non-unique. Then, one may wonder what are the best contact points to target with the soft-manipulator, to optimize some cost associated to the resulting grasping configuration. Based on this observation, we aim to select the contact points, the related force-closure configurations, and, ultimately, the optimal controls, minimizing a given integral cost.

We proceed as follows. In the next section, we recall our model for a tentacle-like soft manipulator, the associated equilibria and a class of optimal stationary grasping problems. Following section is devoted to multi-target reachability problems. In the next section, we address the geometry of the problem. We provide some explicit force-closure conditions for elliptic targets and we present a class of ellipses as cross-sections of three-dimensional objects. In the next section, we present our optimization problem for the force-closure of an ellipse, we discuss the related optimality system, and we provide a projected gradient method for its numerical solution. Finally in the following section, we report the results of some experiments, confirming the effectiveness of the proposed method. In the last section, we draw our conclusions.

We conclude this introduction with the literature that mostly inspired our work. The papers [11–14] provide a general introduction on soft robotics and related motion planning problems. From a modeling point of view, we refer to [15–19]. A survey on grasping problems for soft manipulators can be found in [20]. Our work is based on an optimal control theoretic approach, constrained reachability problems in a similar fashion are addressed in [21, 22], where a finite number of degrees of freedom is taken into account. Here, we carry on a formal analysis in an infinite-dimensional setting, encompassing at once continuous manipulators as well as hyper-redundant, discrete manipulators [1]. The power of a rigorous derivation of the optimality system consists in its suitability for extensions to a purely dynamic setting, from which a time-varying, optimal control can be synthesized as in [5], where an optimal dynamic reachability problem is addressed. To the best of our knowledge, the present variational approach to optimal grasping problems for soft manipulators with non-uniform curvature constraints is new. We point out that the optimality system is then numerically solved via a descent gradient method. Some information useful for the parameter tuning can be recovered by the numerical analysis of the optimization process. The drawback of this first-order method consists in a possibly slow convergence to the stationary points that, in case of lack of convexity of the Lagrangian, can result in local minima. In this regard, also in view of the smoothness of the constraints, the search of optimal grasping controls could benefit from metaheuristic methods [23–25]. In particular, the application of a method like the particle swarm optimization (PSO) to the residual of the optimality system (rather than to the Lagrangian itself) could produce a speedup in the computation. We are not aware of data available in the literature for the specific problem under exam, and we postpone this investigation in a future work.

An Optimal Control Problem for Multi-touch Configurations

We introduce our model and we set the problem of optimizing the contact of the arm with a general, possibly disconnected target set, while avoiding an obstacle. The obstacle can represent either an external element in the workspace or an object to be grasped without interpenetration, in the latter case, it suffices to assume the target point set to belong to the boundary of the obstacle.

A Control Model for an Octopus Arm

Our model was introduced in [3]. It consists in a control model for an octopus arm, characterized by an axial symmetry, a non-uniform thickness and a fixed endpoint. We assumed an exact inextensibility constraint and a bending moment, representing the natural resistance of the body to deformations. Also, a constraint prevents the body to bend above a fixed threshold. Finally, the bending can be pointwise forced by the controller. By virtue of the axial symmetry, we specialized the problem to the evolution of the symmetry axis on the plane. In particular, the axis is modelled as an inextensible string. The mass and the bending constraints/controls of the whole manipulator are projected on the symmetry axis. They, respectively, result in a non-uniform mass and suitably weighted curvature constraints and controls, see [4] for details on this projection.

More precisely, the model includes: (i) a curvature constraint, forcing the curvature of the axis under a fixed (non-uniform) threshold \(\omega\); (ii) a curvature control, forcing the signed curvature to the quantity \(\omega u\), where \(u \in [-1, 1]\) is the control map; (iii) a bending moment, forcing the
uncontrolled manipulator to arrange on a straight line. On the other hand, the inextensibility constraint is exact.

We consider the curve \( q(s,t) : [0,1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^2 \) parametrizing the symmetry axis of the manipulator in arclength coordinates, and the associated inextensibility multiplier \( \sigma(s,t) \in \mathbb{R} \), playing the role of the tension of the string. We denote by \( q_s, q_{ss}, q_t \) partial derivatives in space and time, respectively. The scalar quantity \( \|q_{ss}\| \) is the curvature of \( q \). The product \( q_s \times q_{ss} := q_s \cdot q_{ss} \) represents the signed curvature, where the symbol \( q_{ss} \perp \) denotes the counterclockwise orthogonal vector to \( q_{ss} \). We describe in detail the constraints and the control.

1. **Inextensibility constraint**: \( \|q_s\| = 1 \).
2. **Curvature constraint**: \( \|q_{ss}\| \leq \omega \). The penalization elastic potential is \( \nu(\|q_{ss}\|^2 - \omega^2)\| \), where the function \( \nu \) is a non-uniform elastic constant.
3. **Curvature control**: \( q_s \times q_{ss} = \omega u \). The penalization elastic potential is \( \mu(\omega u - q_s \times q_{ss})^2 \), where the function \( \mu \) is a non-uniform elastic constant.

Then, we consider the following Lagrangian:

\[
\mathcal{L}(q, \sigma) := \int_0^1 \left( \frac{1}{2} \rho \|q_t\|^2 - \frac{1}{2} \sigma(\|q_s\|^2 - 1) \right) ds + \frac{1}{4} \nu(\|q_{ss}\|^2 - \omega^2)^2 - \frac{1}{2} \epsilon \|q_{ss}\|^2 + \frac{1}{2} \mu(\omega u - q_s \times q_{ss})^2 ds.
\]

We refer to [3] for a rigorous justification of \( \mathcal{L} \) and for the complete derivation, via the least action principle, of the equations of motion of \( q \) (and \( \sigma \)) from \( \mathcal{L} \).

In this paper, we are interested in the optimization, with respect to the control \( u \), of the equilibrium configuration of \( \mathcal{L} \), namely the shape of the manipulator provided by the solution \( q \) of the stationary control system (refer to [3] for the proof)

\[
\begin{aligned}
q_{ss} &= \omega u \frac{q_t}{\|q_t\|} & \text{in } (0,1) \\
\|q_s\|^2 &= 1 & \text{in } (0,1) \\
q(0) &= (0,0) & \text{subject to(1), } |u(s)| \leq 1 & \text{for } s \in (0,1), s_i \in I, i = 1, \ldots, N.
\end{aligned}
\]

(1)

where \( \omega := \mu \omega / (\mu + \epsilon) \leq \omega \) represents the effective curvature threshold resulting from the balancing between the bending moment and the curvature control in \( \mathcal{L} \). Note that the curvature constraint is embedded in the first equation of (1), since \( |u| \leq 1 \) and the inextensibility constraint \( \|q_s\| = 1 \) imply \( \|q_{ss}\| = \omega u \leq \omega \).

**Optimality Conditions for a Multi-target, Obstacle Avoidance Reachability Problem**

We recall from [2], an optimal control problem involving the simultaneous contact with a point set target, an obstacle avoidance task and the minimization of a quadratic cost in the control. Let \( \Omega_0 \) be an open subset of \( \mathbb{R}^2 \) representing the obstacle, and let \( \Omega_1 := \{p_1, \ldots, p_N \} \) be a set of \( N \) fixed points, possibly located on \( \partial \Omega_0 \). Note that the condition \( \Omega_1 \subset \partial \Omega_0 \) is assumed when \( \Omega_0 \) is a target object that needs to be grasped without interpenetration.

We also consider \( N \) unknowns \( s = (s_1, \ldots, s_N) \subset I := [\gamma, 1 - \gamma] \) for a.e. \( \gamma \) such that \( \|s_i - s_j\| \leq \epsilon \) for any \( i, j \). The set \( S \) belongs to the interior of the parametrization interval of the manipulator; in other words, \( q(s_i) \) is a point of the manipulator, for all \( i = 1, \ldots, N \). Since \( \gamma > 0 \), then \( q(s_i) \) is not any of the two endpoints of the manipulator. The set \( q(S) \subset \mathbb{R}^2 \) represents the set of points of the manipulator touching the \( N \) target points. Then \( S \), and consequently \( q(S) \), is an unknown, since the contact subregion of the manipulator with the target is not prescribed a priori (as, instead, for [1, 6]). We let the manipulator adapt to choose the best contact points as a result of the following optimization process:

\[
\text{min} \quad \frac{1}{2} \int_0^1 u^2(s) ds + \frac{1}{2\tau_1} \int_0^1 \mathcal{O}(q(s)) ds + \frac{1}{2\tau_1} \sum_{i=1}^N \|q(s_i) - p_i\|^2,
\]

subject to (1), \( |u(s)| \leq 1 \) for \( s \in (0,1), s_i \in I, i = 1, \ldots, N \). (2)

The first integral term represents a quadratic cost on the controls. The second integral term encompasses the obstacle avoidance task: the function \( \mathcal{O} : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a non-negative function with support \( \Omega_0 \), acting as repulsive potential. The effect of this integral term is then precisely to push the points of the manipulator outside \( \Omega_0 \), according to a penalization parameter \( \tau_0 > 0 \). Finally, the third term penalizes, according to \( \tau_1 > 0 \), the distance of the (unknown) point of the manipulator \( q(s_i) \) from the target point \( p_i \), for all \( i = 1, \ldots, N \).

To address (2), we recall that (1) implies the identity \( u^2(s) = \|q_{ss}(s)\|^2/\omega^2(s) \) and the equivalence \( |u(s)| \leq 1 \) if and only if \( \|q_{ss}(s)\| \leq \omega(s) \) for a.e. \( s \in (0,1) \). Therefore, the optimization problem is equivalent to the following:
\[
\min \frac{1}{2} \int_0^1 \frac{\|q\|^2}{\bar{a}^2(s)} ds + \frac{1}{2\tau_0} \int_0^1 \mathcal{O}(q(s)) ds \\
\text{subject to} \\
q(0) = (0,0), q_1(0) = (0,-1), \|q_2(s)\|^2 = 1, \|q_3(s)\| \leq \bar{a}(s) \text{ for } s \in (0,1)
\]
in which the explicit dependence on the controls is dropped by virtue of the constraint (1).

**Force-Closure Conditions for an Ellipse**

We consider first-order force-closure conditions for frictionless contact points, in the case of elliptic boundary of the target object—see the end of the section for some motivating examples. Force-closure conditions are geometric conditions on the disposition of contact points on the boundary of the object, to ensure the immobility of the object despite external disturbances. In other words, the contact points are placed so that the associated (normal) contact forces are able to counteract a disturbance wrench. We consider frictionless contact forces.

More precisely, we denote by \( p = (p_1, p_2) \in \partial \Omega_0 \) a generic contact point. We set the reference frame as the center of \( \Omega_0 \), which is assumed to be an ellipse with semiaxes \( 0 < b \leq a \), described by the equation

\[
E(s) := E(x_1, x_2) = \left( \frac{x_1}{a} \right)^2 + \left( \frac{x_2}{b} \right)^2 - 1 = 0.
\]

We represent \( p \) in polar coordinates, so that for some \( \theta \in [0, 2\pi) \):

\[
p = p(\theta) = (a \cos(\theta), b \sin(\theta)).
\]

Denoting by \( \mathbf{n}(p) \in \mathbb{R}^2 \) the inward normal vector to the ellipse at \( p \),

\[
\mathbf{n}(p) := -\nabla E(p) = -2\left( \frac{\cos(\theta)}{a}, \frac{\sin(\theta)}{b} \right)^T,
\]

\( (T \text{ denotes the transpose}) \) we consider frictionless contact forces of the form \( \mathbf{f}(p) = \mathbf{f}(p_1, p_2) \) with \( f \geq 0 \). Then, the wrench associated to \( p \) is the vector \( \mathbf{w}(p) \in \mathbb{R}^3 \) defined by

\[
\mathbf{w}(p) := \left( \begin{array}{c} \mathbf{f}(p) \\
p \times \mathbf{f}(p) \end{array} \right) = \left( \begin{array}{c} b \cos(\theta) \\
a \sin(\theta) \left( a^2 - b^2 \right) \cos(\theta) \sin(\theta) \end{array} \right).
\]

Following [10], given \( N \) contact points \( \{p_1, \ldots, p_N\} \in \partial \Omega_0 \), we say that \( \Omega_0 \) is in (first-order) force-closure if the set of wrenches \( \{\mathbf{w}(p_1), \ldots, \mathbf{w}(p_N)\} \) positively spans \( \mathbb{R}^3 \), i.e., for all \( \mathbf{x} \in \mathbb{R}^3 \), there exist \( a_1, \ldots, a_N \geq 0 \) such that

\[
\mathbf{x} = \sum_{i=1}^N a_i \mathbf{w}(p_i).
\]

Denoting by \( \mathcal{W}_{FC} \) the \( 3 \times N \) matrix whose columns are the wrenches, the force-closure condition can be equivalently stated as follows:

\[
\text{rank } \mathcal{W}_{FC} = 3,
\]

\[
\mathcal{W}_{FC} \mathbf{y} = 0 \text{ for some } \mathbf{y} \in \mathbb{R}^N, y_i > 0, i = 1, \ldots, N.
\]

Note that the full-rank condition (4) cannot be satisfied if \( a = b \), namely when \( \Omega_0 \) is a circle. Moreover, the number \( N \) of contact points must be equal to or greater than 4, which is the minimal number of generators of a three dimensional conic hull.

**Ellipses as Cross-sections of Three-Dimensional Objects**

In this section, we embed the reference system of the manipulator in \( \mathbb{R}^3 \), so that the workspace of the manipulator belongs the plane \( z = 0 \) and the \( x \) and \( y \) axis are oriented according to (1). We consider two classes of three-dimensional objects: a family of cylinders and a family of ellipsoids. For each member of each class, we derive an elliptic target, as the intersection between the three-dimensional object and the workspace of the manipulator. Such elliptic intersection is the region of the object that can be actually reached by the manipulator.

We begin by modelling a collection of cylindrical objects. When a cylinder is intersected with the workspace, we obtain an elliptic target object. More precisely, we parametrize a collection of cylinders of radius \( r > 0 \) whose symmetry axes have \( (c, 0) = (c_1, c_2, 0) \), as common point. We assume the symmetry axis to be parallel to the \( yz \)-plane and we describe its orientation via the parameter \( \varphi \in (-\pi/2, \pi/2) \)—the case \( \varphi = 0 \) corresponds to a vertical cylinder, see Fig. 1.

More precisely, let

\[
C_{r,\varphi} := \{(c_1 + r \cos(\theta), c_2 + r \sin(\theta), z \cos(\theta)), \theta \in [0, 2\pi], z \in \mathbb{R}\}.
\]

For \( \varphi \in (-\pi/2, \pi/2) \) define

\[
C_{r,\varphi} := (c, 0)^T + R_x(\varphi)(C_{r,\varphi} - (c, 0)^T) \\
= \{(c_1 + r \cos(\theta), c_2 + r \sin(\theta)) - z \cos(\theta), r \sin(\theta) \sin(\theta) \\
+ z \cos(\theta), \theta \in [0, 2\pi], z \in \mathbb{R}\},
\]

where \( R_x(\varphi) \) is the rotation matrix about the \( x \)-axis with angle of rotation \( \varphi \).
The associated target ellipse is hence given by the intersection $\Omega_{r,\phi} := C_{r,\phi} \cap \{(x_1,x_2,0), x_1,x_2 \in \mathbb{R}\}$. By a direct computation,

$$\Omega_{r,\phi} := \{(x_1,x_2) \in \mathbb{R}^2 \mid \left(\frac{x_1 - c_1}{r}\right)^2 + \left(\frac{x_2 - c_2}{r(\cos \phi + \tan \phi)}\right)^2 = 1\}$$

that is, $\Omega_{r,\phi}$ is the ellipse centered in $c$ with semiaxes $a = a_r := r$ and $b = b_{r,\phi} := r(\cos \phi + \tan \phi)$. Note that if $\phi = 0$, then $\Omega_{r,\phi}$ is a circle.

Consider now a family of ellipsoids, with center $(c,0)$ and with semiaxes $a_1, a_2, a_3 > 0$. For brevity, let us collect $a_1, a_2$ and $a_3$ in the three-dimensional vector $a := (a_1, a_2, a_3)$. Given the ellipsoid $E_{a,0}$ implicitly defined by

$$E_{a,0} := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \left(\frac{x_1 - c_1}{a_1}\right)^2 + \left(\frac{x_2 - c_2}{a_2}\right)^2 + \left(\frac{x_3 - c_3}{a_3}\right)^2 = 1\}$$
\[
\left(\frac{x_1 - c_1}{a_1}\right)^2 + \left(\frac{x_2 - c_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1
\]

we consider the family of rotations of \( E_{a,0} \) about the symmetry axes that is parallel to the x-axis, see Fig. 2. We have
\[ E_{a,\varphi} := (c, 0^T) + R_\varphi(\varphi)(E_{a,\varphi} - (c, 0)^T). \]

We recall that we defined \( R_\varphi(\varphi) \) as the rotation matrix about the x-axis. More precisely, for \( \varphi \in [-\pi, \pi] \), we have the parametrization in polar coordinates
\[
E_{a,\varphi} := \{(c_1 + a_1 \sin \varphi \cos \theta, c_2 + a_2 \sin \varphi \cos \theta, c_3 + a_3 \cos \varphi) \mid \theta \in [0, 2\pi), \varphi \in [0, \pi) \}.
\]
The intersection with the \( z = 0 \) plane yields the ellipse
\[
\Omega_{a,\varphi} := \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \left(\frac{x_1 - c_1}{a_1}\right)^2 + \left(\frac{x_2 - c_2}{a_2}\right)^2 + \left(\frac{1}{a_3^2 \sin^2 \varphi + a_2^2 \cos^2 \varphi}\right)x_3 = 1 \right\}
\]
that is, \( \Omega_{a,\varphi} \) is the ellipse centered in \( c \) with semiaxes \( a = a_1 \) and \( b = \|a_2, a_3, \varphi\| := a_2 a_3 \sqrt{a_2^2 \sin^2 \varphi + a_2^2 \cos^2 \varphi}. \]

**Optimal Grasping Strategies**

In this section, we show how to suitably embed the force-closure condition (4)–(5) in the multi-target optimization problem with obstacle avoidance (3) presented in “Optimality conditions for a multi-target, obstacle avoidance reachability problem”. Moreover, we discuss the related optimization system and we provide a projected gradient method for its numerical solution.

We consider again the case of an elliptic obstacle \( \Omega_0 \) with semi-axes \( 0 < b < a \), but possibly rotated by a given angle \( \theta_0 \in [0, 2\pi) \) and translated to the point \( c_0 \in \mathbb{R}^2 \). Then, we settle on the minimal requirement of \( N = 4 \) target points, looking for a point set \( \Omega := \{p_1, \ldots, p_4\} \subset \partial \Omega_0 \) of the form
\[
p_i = p(\theta_i) = c_0 + R_0(a \cos(\theta_i), b \sin(\theta_i))^T, \quad \theta_i \in [0, 2\pi), \quad i = 1, \ldots, 4,
\]
with \( R_0 = \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ -\sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \), satisfying the force-closure condition.

To this end, we consider the wrenches of \( \Omega_0 \), expressed in local coordinates as in “Force-closure conditions for an ellipse” and scaled by the constant factor \( \sqrt{\frac{2}{ab}} \), i.e.,
\[
w_i := w(p_i) = -(b \cos(\theta_i), a \sin(\theta_i), (a^2 - b^2) \cos(\theta_i) \sin(\theta_i)),
\]
and we denote by \( W \) the \( 3 \times 3 \) matrix whose columns are the components of the first three wrenches \( w_1, w_2, w_3 \). Moreover, we consider its formal inverse \( W^{-1} \), obtaining, by direct computation
\[
W^{-1} = \frac{1}{\det W} \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \quad \text{with } \det W = w_1 \cdot w_2 \times w_3,
\]
whose rows are given by the components of the vectors
\[
\tilde{w}_i := \begin{pmatrix} -a(a^2 - b^2) \sin(\theta_{i+1}) \sin(\theta_{i+2}) (\cos(\theta_{i+1}) - \cos(\theta_{i+2})) \\ b(a^2 - b^2) \cos(\theta_{i+1}) \cos(\theta_{i+2}) (\sin(\theta_{i+1}) - \sin(\theta_{i+2})) \\ -ab \sin(\theta_{i+1} - \theta_{i+2}) \end{pmatrix},
\]
i = 1, 2, 3,
where the indices are meant to cycle when out of bounds (e.g., for \( i = 3 \) we set \( i + 1 = 1 \) and \( i + 2 = 2 \)). Note that, for \( i = 1, 2, 3 \), the wrench \( w_i \) depends on \( \theta_i \) only, whereas \( \tilde{w}_i \) depends on the whole triplet \( (\theta_i, \theta_{i+1}, \theta_{i+2}) \).

Then, it is easy to see that the force-closure condition (4)–(5) is equivalent to the following:
\[
\det W \neq 0, \quad \text{sign} (\det W) \tilde{w}_i \cdot w_4 \leq 0, \quad i = 1, 2, 3.
\]

We now collect the unknown angles of the target points in the set \( \Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\} \subset [0, 2\pi) \), and we define the function
\[
F(\Theta) := \frac{1}{2} \sum_{i=0}^3 F_i(\Theta),
\]
where
\[
F_0(\Theta) = \max_{0, \epsilon_0} \left\{ 0, \epsilon_0 - |\det W| \right\},
\]
\[
F_i(\Theta) = \begin{cases} \max \left\{ 0, \tilde{w}_i \cdot w_4 \right\} \quad \text{if } \det W > 0, \\ \min \left\{ 0, \tilde{w}_i \cdot w_4 \right\} \quad \text{otherwise}, \end{cases}
\]
i = 1, 2, 3,
and
\[
\epsilon_0 := \epsilon_0(a, b) := \epsilon \max \left\{ |\det W(\theta_1, \theta_2, \theta_3)| \mid \text{s.t. } (\theta_1, \theta_2, \theta_3) \in [0, 2\pi]^3 \right\}
\]
is a fixed parameter, for some \( \epsilon \in (0, 1) \).

We remark that \( F \) is a non-negative function and, by construction, it follows that \( \Theta \) is an absolute minimizer (achieving \( F(\Theta) = 0 \)) if and only if the corresponding target points satisfy the force-closure condition (6) with \( |\det W| \geq \epsilon_0 \).

We also remark that \( \epsilon_0 \) only depends on \( (a, b) \), and it is well defined by virtue of compactness arguments, in
We observe that the third term in the functional (8) provides a quite involved cross coupling of the unknowns \((q, S, \Theta)\), in particular its minimization forces the tentacle \(q\) to grasp the obstacle, and also the simultaneous attraction between the contact set \(S\) and the target set \(\Theta\).

The next step is to obtain necessary optimality conditions for problem (9) in a form suitable for numerical computations, i.e. avoiding for simplicity the KKT conditions associated to the inequality constraint \(\|q_{ss}\| \leq \bar{\omega}\). This is done via relaxation, by introducing an auxiliary variable \(z\) satisfying the equivalent equality constraint

\[
\|q_{ss}\|^2 - \bar{\omega}^2 + z = 0 \quad \text{with} \; z \geq 0,
\]

and by defining the following augmented Lagrangian

\[
\mathcal{L}(q, \sigma, S, \Theta, z, \lambda) := G(q, S, \Theta) + \frac{1}{2} \int_0^1 \sigma(\|q_s\|^2 - 1)ds \\
+ \frac{1}{2} \int_0^1 \lambda(\|q_{ss}\|^2 - \bar{\omega}^2 + z)ds \\
+ \frac{1}{4\rho_\lambda} - 2pt \int_0^1 -5pt(\|q_{ss}\|^2 - \bar{\omega}^2 + z)^2ds,
\]

where \(\sigma : [0, 1] \rightarrow \mathbb{R}\) is the Lagrange multiplier for the exact inextensibility constraint \(\|q_s\| = 1\). The function \(\lambda : [0, 1] \rightarrow \mathbb{R}\) and the constant \(\rho_\lambda > 0\) are respectively the multiplier and penalty parameter related to the relaxed constraint on the curvature (10). Then, the optimization problem (9) is equivalent to the minimization of \(\mathcal{L}\), subject to \(S \subset I_r\) and \(z \geq 0\) (note that, due to the \(2\pi\)-periodicity of \(\{w_i\}_{i=1,...,4}\), \(w_{i+4} = 1, \ldots, 3\) and det \(\mathcal{V}\) in the definition of \(F(\Theta)\), no constraint is actually required on \(\Theta\)). Such optimization process can be performed employing the method of multipliers ([26, 27], see also [28] and the references therein for the infinite-dimensional case), starting from \(\lambda(0) \equiv 0\) and iterating on \(k \geq 0\) up to convergence

\[
\begin{aligned}
&\left(\hat{q}(k+1), \hat{\sigma}(k+1), \hat{s}(k+1), \hat{\Theta}(k+1), \hat{z}(k+1)\right) = \\
&\arg\min_{q, \sigma, \Theta} \mathcal{L}(q, \sigma, S, \Theta, z, \lambda(k)) \\
&\text{subject to} \quad S \subset I_r, \; z \geq 0
\end{aligned}
\]

\[
\lambda(k+1) = \lambda(k) + \frac{1}{\rho_\lambda}(\|q_{ss}(k)\|^2 - \bar{\omega}^2 + z(k+1)).
\]

The optimization in (12) with respect to \(z \geq 0\) yields the following variational inequality

\[
\int_0^1 \left(\lambda(k) \rho_\lambda + \|q_{ss}\|^2 - \bar{\omega}^2 + z(\nu - z)\right)ds \geq 0, \quad \forall \nu \geq 0,
\]

whose solution is given almost everywhere by

\[
z(q, \lambda(k)) := \max \left\{-\lambda(k) \rho_\lambda - \|q_{ss}\|^2 + \bar{\omega}^2, 0\right\}.
\]
Plugging the above expression in (12), we get
\[
\begin{align*}
\left\{ \begin{array}{l}
(q^{(k+1)}, \bar{\sigma}^{(k+1)}, \bar{S}^{(k+1)}, \bar{\Theta}^{(k+1)}) = \\
\arg \min_{q, \sigma, \Theta} \mathcal{L}(q, \sigma, S, \Theta, z(q, \lambda^{(k)}), \lambda^{(k)}) \\
S \subset I_r \\
\lambda^{(k+1)} = \max \left\{ \lambda^{(k)} + \frac{1}{\rho} \left( \|q^{(k+1)}\|^2 - \omega^2 \right), 0 \right\}
\end{array} \right. \\
\mathcal{L}'^{(k)}(q, \sigma, S, \Theta) = \left( \Lambda^{(k)}(q^{(k)})q^{(k)} - \sigma q^{(k)} \right)
\end{align*}
\] (13)

On the other hand, the solution \((q^{(k+1)}, \bar{\sigma}^{(k+1)}, \bar{S}^{(k+1)}, \bar{\Theta}^{(k+1)})\) of the optimization sub-problem for the reduced Lagrangian
\[
\mathcal{L}^{(k)}(q, \sigma, S, \Theta) := \mathcal{L}(q, \sigma, S, \Theta, z(q, \lambda^{(k)}), \lambda^{(k)})
\]
(14)
satisfies the following optimality system
\[
\begin{align*}
\begin{cases}
\left( \Lambda^{(k)}(q^{(k)})q^{(k)} \right)_{s_i} - \sigma q^{(k)} \\
+ \frac{1}{\tau_0} \nabla \mathcal{O}(s) + \frac{1}{\tau_1} \sum_{i=1}^4 (q(s) - p(\theta_i)) \delta_i(s) = 0 \\
q(0) = 0, \quad q_s(0) = (0, -1), \quad q_{ss}(1) = 0, \quad q_{sas}^* = 0, \\
\sigma(1) = 0,
\end{cases}
\end{align*}
\] (0, 1)
\]
\[\|q_s\|^2 = 1 \quad \text{in } (0, 1)
\]
\[\frac{1}{\tau_1} (q(s_i) - p(\theta_i)) \cdot q_s(s_i) (w_i - s_i) \geq 0, \quad \forall w_i \in I_r, i = 1, \ldots, 4
\]
\[\frac{1}{\tau_1} (q(s_i) - p(\theta_i)) \cdot p(\theta_i) + \frac{1}{\tau_2} \frac{\partial}{\partial \theta_i} F(\Theta) = 0 \quad i = 1, \ldots, 4
\]
(15)

Then, given an initial guess \((q^{(0)}, \sigma^{(0)}, S^{(0)}, \Theta^{(0)})\) satisfying the boundary conditions and the constraints, we iterate on
\[n \geq 0 \] up to convergence
\[
\begin{cases}
\begin{bmatrix} q^{(n+1)} \\ \sigma^{(n+1)} \\ S^{(n+1)} \end{bmatrix} = \begin{bmatrix} q^{(n)} \\ \sigma^{(n)} \\ S^{(n)} \end{bmatrix} - \alpha \mathcal{L}'^{(k)}(q^{(n)}, \sigma^{(n)}, S^{(n)}, \Theta^{(n)}),
\end{cases}
\] (17)

with \[
\mathcal{L}^{(k)}(q^{(k)}):= \frac{1}{\omega^2} + \max \left\{ \lambda^{(k)} + \frac{1}{\rho} \left( \|q^{(k+1)}\|^2 - \omega^2 \right), 0 \right\}
\]

As remarked in [2] (see also [3]), the first equation and the boundary conditions in (15) emerge from the optimization of \(\mathcal{L}^{(k)}\) with respect to \(q\). Similarly, optimizing \(\mathcal{L}^{(k)}\) in \(S\) and \(\Theta\), respectively, we recover (in order) the inextensibility constraint, four variational inequalities for the contact set (due to the constraint \(S \subset I_r\)) and four equations for the target set.

The solution of (15) is approximated as in [2], using a projected gradient descent method. More precisely, we first compute the partial Fréchet derivatives of \(\mathcal{L}^{(k)}\):
Table 1 Elliptic obstacle settings

| Test | $a$  | $b$  | $\theta_0$ |
|------|------|------|-------------|
| 1    | 0.2  | 0.06 | $-15^\circ$ |
| 2    | 0.23 | 0.16 | $90^\circ$  |
| 3    | 0.16 | 0.06 | $75^\circ$  |
| 4    | 0.16 | 0.06 | $45^\circ$  |

is the component-wise projection on $I_x$ ensuring a feasible contact set at each iteration. We refer to [2] for further details on the discretization of the optimality system (15), and on the actual implementation of the full algorithm, including both (17) and the method of multipliers (13).

**Numerical Experiments**

In this section, we present several numerical experiments, showing the ability of the proposed method to obtain optimal grasping configurations for the manipulator, satisfying the force-closure condition.

Let us first define the setting for the tests. We always choose the curvature threshold function $\bar{\omega}(s) = \frac{1-0.9s}{(1-0.9s)+(0.1-0.09s)} \cdot 2\pi(3+s^2)$, corresponding to (1) with $\omega(s) = 2\pi(3+s^2)$, $\mu(s) = 1-0.9s$ and $\epsilon(s) = 0.1 - 0.09s$. We also assume that the manipulator has unit length and it is discretized with 201 nodes. Then, we set $\gamma = \frac{1}{200}$, namely equal to the mesh size, so that the interval $I_x$ for the contact set $S$ contains all the grid nodes except the end points. The elliptic obstacle $\Omega_0$ is centered at the point $c_0 = (0.1,-0.3)$, while its semi-axes and rotation are chosen as summarized in Table 1. Moreover, for each pair $(a, b)$, we numerically compute $\varepsilon_0(a, b)$ in (7) choosing $\varepsilon = \frac{1}{2}$.

As initial guess for (17), we always choose a profile $q^{(0)}$ surrounding the obstacle, whereas $q^{(0)} \equiv 0$ and $S^{(0)}$ corresponds to a small random perturbation of equally spaced points along the manipulator. Accordingly, the target set $\Theta^{(0)}$ is obtained by a small random perturbation of the projection of $q^{(0)}(S^{(0)})$ on the ellipse, with the exception of Test 1 where $\Theta^{(0)}$ is chosen randomly.

Finally, the step size for the gradient descent method is set to $\alpha = 5 \cdot 10^{-3}$, while the penalty parameters $\tau_0$, $\tau_1$, $\tau_2$ and $\rho_2$ are slowly decreased using the continuation method discussed in [2]. In particular, at each iteration of the optimization process, we keep the relationship $\tau_0 < \tau_1 < \tau_2$ to prioritize, in order, the obstacle avoidance, the contact, the force-closure, and finally the curvature minimization. This results in a greater mobility of $S$ and $\Theta$ in the grasping task.

In Test 1, we consider, for comparison purposes, the case of frozen target points as in [2], removing from (9) the optimization with respect to $\Theta$. In particular, the chosen target set does not satisfy the force-closure condition.

In Fig. 4, we report the results of the optimization. More precisely, in Fig. 4a, we show the obstacle (grey), the target points (black circles), the optimal contact points (yellow circles) and the optimal configuration $q$ of the manipulator, while Fig. 4b represents the corresponding signed curvature (thicker line) as a function of $s \in [0, 1]$, and the thresholds $\pm \bar{\omega}(s)$ (thin lines).

Moreover, in Fig. 4c, we show the evolution of the contact set $S$ versus the total number of iterations to reach convergence.

We clearly observe a temporary saturation of $S$, which corresponds to the higher priority given to the minimization of the distance between the contact set and the target set. In the remaining part of the evolution, once the manipulator has grasped the obstacle, the optimization proceeds with a sliding of $s_T, s_R, s_L$ toward the free end. This reflects an attempt of the manipulator to unroll itself to reduce also the curvature contribution in (8), which has already reached the threshold $\bar{\omega}$ is some regions (see again Fig. 4b), due to the large eccentricity of the obstacle. Nevertheless, the resulting configuration of the manipulator looks quite “unnatural” (see again Fig. 4a), we can somehow perceive a great effort of the controls in the grasping task, and this is ultimately related to the lack of mobility of the target set.

In Test 2, we consider the case of an obstacle with a lower eccentricity, and we perform the optimization of the complete functional (8) with respect to $(q, S, \Theta)$. The results are reported in Fig. 5, where we also show the convergence history of the target set (Fig. 5d) and of the four terms related to the force-closure condition (6) (Fig. 5e).

Here, it is interesting to observe that the force-closure condition is satisfied and preserved during the first part of the optimization, which is mostly devoted to the adjustment of $\Theta$ (note indeed in Fig. 5c that the contact set $S$ remains almost unchanged).

From now on, the optimization proceeds focusing on the contact and curvature terms in (8), as confirmed by the sliding of $S$ in Fig. 5c. This slightly violates the condition $\text{sign} \left( \det W \right) \mathbf{w}_1 \cdot \mathbf{w}_1 \leq 0$ and also produces a decrease in the term $| \det W |$. In the remaining iterations, the missing condition is recovered, while the determinant of $W$ is locked above the bound $\varepsilon_0$, both with small adjustments of $\Theta$.

Finally, comparing Fig. 5a, b, we readily observe the correspondence between the contact points and the faster transitions in the curvature of the manipulator. Since the grasping is imposed on $q(S)$ only, the manipulator need not to retrace the profile (hence the curvature) of the obstacle. This is more apparent in the part of $q$ between the last two contact points.
Test 3 and Test 4 are more challenging than the previous one. The obstacle has now a larger eccentricity and it is differently rotated, which forces higher values in the curvature of the manipulator, as observed in Test 1. Moreover, in these two tests, the initial guess for the target set is such that the force-closure condition is more and more violated at the beginning of the optimization. The results are similar, as reported in Figs. 6 and 7 respectively.

In both cases, we observe that one contact occurs at the point of the obstacle furthest from the anchor point of the manipulator, precisely on the major axis of the ellipse, where the curvature achieves its largest value.

Moreover, in both tests, we recognize a relevant sliding of the contact set, especially for the contact point near the free end of the manipulator, aiming at avoiding adherence to the obstacle, so to reduce the total curvature as much as possible. This behavior is more apparent in Test 4, due to the different orientation of the obstacle. In particular, we also observe some adjustment in the first contact point, which

Fig. 4 Test 1, optimal grasping configuration (a), optimal curvature (b) and convergence history of the optimal contact set (c)
Fig. 5 Test 2, optimal grasping configuration (a), optimal curvature (b) and convergence history of (c) the optimal contact set, (d) optimal target set, (e) force-closure condition.
Fig. 6 Test 3, optimal grasping configuration (a), optimal curvature (b) and convergence history of (c) the optimal contact set, (d) optimal target set, (e) force-closure condition
Fig. 7 Test 4, optimal grasping configuration (a), optimal curvature (b) and convergence history of c the optimal contact set, d optimal target set, e force-closure condition
prevents the curvature from changing sign in the first part of the manipulator, as in Test 3.

Conclusions

In this paper, we addressed optimal force-closure configurations for a soft manipulator. The underlying model corresponds to the equilibria of a dynamical system, which in turn models the evolution of a tentacle-like manipulator subject to inextensibility and curvature constraint, a bending moment and a distributed curvature control, introduced in [5]. The stationary setting taken into exam here constraints the configurations of the manipulator to a system of controlled ordinary differential equations (1). The goal is to select an admissible force-closure configuration (namely a solution of (1)) which avoids interpenetration with an elliptic target while minimizing a quadratic cost on the controls. Our approach is based on optimal control theory and it consists in the constrained minimization of an appropriate cost functional, see the problem (9). Then, a discretization algorithm is proposed for the numerical solution. Finally, some test cases are taken into exam. The observation of the resulting optimal solutions and the analysis of the optimization process confirmed the consistency of the proposed approach. Also they shed some light on how the parameters impact the selection of the optimal configuration.

The present paper is part of an ongoing investigation on optimal control strategies for soft-manipulators. This is a relatively new line of research, exploiting a variational approach to embed (internal and external) reaction forces in the system and, at the same time, to minimize of the costs associated to the execution of some reachability and grasping tasks by a tentacle-like soft manipulator. The main novelty, compared with previous works by the authors, is the optimization of configurations that not only optimally wrap the target while avoiding interpenetration, but also satisfy a force-closure condition. A far from trivial aspect of the problem is the definition an appropriate cost functional, the functional \( G \) introduced in (8), which also required a geometrical analysis of the force-closure condition. Further directions of research include: the study of more general shapes of the target, higher order force-closure conditions, the introduction of friction in the contact forces, and an extension of the above techniques to a fully dynamical setting.

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