0. Introduction

0.1. This paper consists of two parts. In the first part, assuming the log Minimal Model Program (which is currently only known to be true in \( \dim \leq 3 \)), we construct the complete moduli of “stable pairs” \((X, B)\) of projective schemes with divisors that generalize the moduli space of \(n\)-pointed stable curves \(M_{g,n}\) to arbitrary dimension. The construction itself is a direct generalization of that of [Ale96b] where it was given in the case of surfaces, and is based in part on ideas from [KSB88, Kol90, Vie95].

0.2. In the second part of the paper we study the singularities of stable quasiabelian varieties and stable quasiabelian pairs \((X, B)\) that appear in [AN96] as limits of abelian varieties. We show that the singularities are semi log canonical. This implies, via Kollár’s Ampleness Lemma, that over \(\mathbb{C}\) if there exists a compactification of the moduli space \(A_g\) of principally polarized abelian varieties by stable quasiabelian pairs, then it is in fact projective.

We give more examples of situations where log canonical singularities appear naturally in connection with complete moduli problems. One of them is the minimal and toroidal compactifications of quotients \(D/\Gamma\) of bounded symmetric domains by arithmetic groups. We point out the fact, which could be obvious to specialists had they known the definitions, that they all have log canonical singularities and that the minimal (=Baily-Borel) compactification is the log canonical model of any toroidal compactification when the “boundary” divisor \(B\) is correctly defined.

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1. Definitions for singularities

Definition 1.1. Let $X$ be a normal variety (not necessarily irreducible) defined over an algebraically closed field $k$ of any characteristic, and let $B_1 \ldots B_m$ be distinct reduced divisors on $X$. Denote $\sum B_j$ by $B$. Let $i : U \hookrightarrow X$ be the inclusion of the nonsingular part and denote

$$O(N(K + B)) = i_*O(N(K_U + B|_U)),$$

where $O(K_U)$ is the canonical sheaf, the top exterior power of $\Omega^1_U$, and $N$ is an integer.

One says that the pair $(X, B)$ has log canonical singularities if

(i). $O(N(K + B))$ is invertible for some $N > 0$ (one then says that $K + B$ is $\mathbb{Q}$-Cartier).

(ii). for any birational morphism from a normal scheme $f : Y \to X$ one has

$$f_*O_Y(N(K_Y + f^{-1}B + \sum E_i)) = O_X(N(K_X + B)),$$

where $E_i$ are exceptional divisors of $f$.

Remark 1.2. The above definition can be formulated also for the case of a divisor $B = \sum b_j B_j$ with rational coefficients $b_j$ by requiring $N$ to be divisible enough.

1.3. An equivalent way would be to use log codiscrepancies – the coefficients $a_i$ appearing in the following natural formula:

$$f^*(K_X + B) = K_Y + f^{-1}B + \sum a_i E_i$$

The log codiscrepancies depend only on the divisors $E_i$ themselves, i.e. the corresponding discrete valuations of the function field, and not on the model $Y$ chosen. Indeed, every two models $Y_1$ and $Y_2$ are comparable since they are both dominated by a third normal variety $Y_3$ – take for example the component of the normalization of $Y_1 \times_Y Y_2$ which dominates $Y$.

Definition 1.4. The singularities are log canonical if all log codiscrepancies are $\leq 1$. They are log terminal if $a_i < 1$, klt if $a_i < 1$ and $b_j < 1$. And they are canonical (resp. terminal) if $B$ is empty and $a_i \leq 0$ (resp. $a_i < 0$).

Remark 1.5. In the above definition one usually assumes $Y$ to be non-singular, and then one needs the embedded resolution of singularities and hence characteristic 0. This does not appear to be necessary. Still, without the resolution of singularities the situation becomes somewhat cumbersome. For example, it is not absolutely obvious that the next definition is equivalent
to, or even implies, the previous one (this is obvious with resolution of singularities).

**Definition 1.6.** Let \((X, B)\) be as above. We say that this pair has **pre log canonical singularities** if there exists a proper birational morphism from a nonsingular variety \(f : Y \to X\) such that

(i). \(\mathcal{O}(N(K + B))\) is invertible for some \(N > 0\).
(ii). the exceptional set of \(f\) is a union of divisors \(E_i\).
(iii). \(\bigcup f^{-1}B_j \cup E_i\) has normal crossings.
(iv). \(f_*\mathcal{O}_Y(N(K_Y + f^{-1}B + \sum E_i)) = \mathcal{O}_X(N(K_X + B))\).

Another important class of singularities is the following.

**Definition 1.7.** Let \(X\) be a reduced variety (not necessarily irreducible) defined over an algebraically closed field \(k\) of any characteristic, and let \(B_1 \ldots B_m\) be distinct reduced divisors on \(X\), denote \(\sum B_j\) by \(B\). In addition, assume that \(X\) is quasi-projective over \(k\).

Let \(i : U \hookrightarrow X\) be the union of the open locus of Gorenstein points of \(X\) not contained in \(B\) and the nonsingular locus of \(X\). Denote \(\mathcal{O}(N(K + B)) = i_*\mathcal{O}(N(K_U + B|_U))\), where \(N\) is an integer, and \(\mathcal{O}(K_U)\) is the restriction of the dualizing sheaf \(\omega_U\) of a projective closure of \(U\) ([Har77, III.7]).

One says that the pair \((X, B)\) has **semi log canonical singularities** if

(i). \(X\) satisfies the Serre’s condition \(S_2\).
(ii). \(X\) is Gorenstein in codimension 1.
(iii). none of the irreducible components of \(B_j\) is contained in the singular locus of \(X\).
(iv). the closed subscheme \(\text{cond}(\nu)\) of \(X^\nu\) corresponding to the conductor of normalization \(\nu : X^\nu \to X\) is a union of reduced divisors.
(v). \(\mathcal{O}(N(K + B))\) is invertible for some \(N > 0\).
(vi). the pair \((X^\nu, \nu^{-1}B + \text{cond}(\nu))\) has log canonical singularities.

In the same way as above, one can define pre semi log canonical singularities.

**Remark 1.8.** We note a certain lack of symmetry in the definitions of \(\mathcal{O}(N(K + B))\) for log and semi log canonical cases. However, they coincide if \(X\) is both normal and quasi-projective.

**Definition 1.9.** Under the assumptions above, we will say that \(K + B\) is ample if the sheaf \(\mathcal{O}_X(N(K + B))\), equivalently \(\mathcal{O}_{X^\nu}(N(K + B + \text{cond}(\nu)))\), is an ample invertible sheaf for some integer \(N > 0\). In this case the pair \((X, B)\) is called the **log canonical model**.

Let us now try to see what is the most general situation where the previous definitions still work. The main thing to understand is the canonical sheaf. The rest transfers over in a pretty straightforward way.
Let us fix a regular Noetherian scheme $S$ (for example spectrum of $\mathbb{Z}$ or a DVR) and consider a reduced scheme $X$ flat and of finite type over $S$. Let us assume that $\pi : X \to S$ is smooth in codimension 1. Denoting by $i : U \to X$ the embedding of this smooth locus, we can set

$$O_X(K_{X/S}) = i_*O_X(K_{U/S}),$$

where $O_X(K_{U/S})$ is the top exterior power of $\Omega^1_{U/S}$. We can now define (pre) log canonical singularities of a pair $(X, B = \sum B_j)$, where $B_j \subset X$ are closed codimension 1 subschemes of $X$, by copying the definitions from section 2. In particular, for log canonical singularities we require $X$ to be normal.

For (pre) semi log canonical we need to assume that $X/S$ is quasi-projective and that the normalization $X^\nu/S$ is smooth in codimension 1.

As one can see, in these definitions we use the regular scheme $S$ only as “the beginning of coordinates”, something to start measuring from.

Let us push the limits even little further. Clearly, the definition of (pre) log canonical singularities is stable under $\acute{e}$tale maps. Therefore, they transfer directly to algebraic spaces and algebraic stacks. If $R_X \twoheadrightarrow U_X$ is an equivalence relation or a groupoid defining $X$, and $R_{B_j}, U_{B_j}$ are the closed subschemes corresponding to $B_j$, then we say that the pair $(X, B)$ has (pre) log canonical singularities if the same holds for $(U_X, U_B)$.

2. Moduli of stable pairs in general

The purpose of this section is to describe a construction of complete and projective moduli spaces for stable $n$-dimensional pairs which generalize the usual moduli of stable $n$-pointed curves. This will be done assuming a series of conjectures the main of which is the log Minimal Model Program in dimension $n + 1$. These conjectures are theorems only when $n + 1 = 3$, so only in the case of surfaces the results are not hypothetical, and this case was considered in detail in [Ale96b].

Where possible, we work in general context, over a fixed base scheme. The bulk of this material, however, applies only to the case of an algebraically closed field of characteristic 0 because of the Minimal Model Program.

We would like to point out that the general framework of what is described here has already been essentially understood in [KSB88], [Kol90] and [Ale96b]. Many important ideas also come from [Vie95].

We first remind what a stable $n$-pointed curve is.

**Definition 2.1.** A stable $n$-pointed curve over an algebraically closed field is a collection $(C; P_1 \ldots P_m)$, where

1. $C$ is a connected projective curve and $P_1 \ldots P_m$ are points on $C$.
2. (condition on singularities) $C$ is reduced and has nodes only, and $P_1 \ldots P_m$ all lie in the nonsingular part.
(2). (numerical condition) for every smooth rational curve \( E \subset C \), \( E \) has at least 3 special points: one of \( P_1 \) or the nodes; and for every smooth elliptic curve or a rational curve with one node \( E \subset C \), \( E \) has at least 1 special point.

A stable \( n \)-pointed curve over a scheme \( S \) is a flat projective morphism \( \pi : (C; P_1 \ldots P_m) \to S \), with \( P_i \subset C \) closed subschemes and each \( P_i \to S \) also flat, whose every geometric fiber is a stable \( n \)-pointed curve over a field \( k = \bar{k} \).

The moduli stack of stable \( n \)-pointed curves is proper, and it is coarsely represented by a projective scheme \( M_{g,n} \), see [Knu83].

**Question.** What is the analog of this in higher dimensions?

One definitely has to consider a collection consisting of a connected projective scheme \( X \) plus \( m \) closed subschemes. We have two basic choices: they could be points or divisors. Here, we choose divisors: \( B_1 \ldots B_m \).

The numerical condition (2) above can be reformulated by saying “\( K_X + \sum P_i \) is ample”. We can now directly transfer this to dimension \( n \) if we understand what \( K + B = K_X + \sum B_j \) is.

Finally, the condition on the singularities. This is the trickiest of the three. The answer comes from the log Minimal Model Program theory: the singularities of \((X, B)\) have to be semi log canonical.

We are now ready to introduce our main object.

**Definition 2.2.** A stable pair over an algebraically closed field is a collection \((X; B_1 \ldots B_m)\), where

1. \( X \) is a connected projective not necessarily irreducible variety and \( B_1 \ldots B_m \) are reduced divisors on \( X \).
2. (condition on singularities) the pair \((X, B)\) has semi log canonical singularities.
3. (numerical condition) \( K + B \) is ample.

A stable pair over a scheme \( S \) of level \( N \) is a flat projective morphism \( \pi : (X; B_1 \ldots B_m; \mathcal{L}) \to S \), with \( B_i \subset X \) closed subschemes, each \( B_i \to S \) also flat and \( \mathcal{L} \) an invertible sheaf on \( X \), whose every geometric fiber is a stable pair over a field \( k = \bar{k} \) and such that the restriction of \( \mathcal{L} \) on each geometric fiber coincides with \( \mathcal{O}(N(K + B)) \). We say that two pairs \((X_1, B_1; \mathcal{L}_1)\) and \((X_2, B_2; \mathcal{L}_2)\) are isomorphic if there exists an isomorphism of \((X_1, B_1)\) and \((X_2, B_2)\) over \( S \) that induces a fiber-wise isomorphism of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

**Conjecture 2.3** (Boundedness Conjecture). For every positive rational number \( C \) there exist

1. a positive integer \( N > 0 \) with the property that for every stable \( n \)-dimensional stable pair \((X, B)\) with \((K + B)^n = C \) the sheaf \( \mathcal{O}(N(K + B)) \) is invertible.
(ii). a scheme $S$ of finite type over the base scheme and a flat projective family $(X; B_1 \ldots B_m)$ whose geometric fibers include all stable $n$-dimensional pairs of level $N$ with $(K + B)^n = C$.

This has been shown to be true only in dimension 2 ([Ale94]) and trivially in dimension 1.

**Definition 2.4.** We now fix a rational number $C$ and an integer $N$ as above and define the functor

$$M^N_C(S) = \left\{ \text{stable } n\text{-dimensional pairs over } S \text{ of level } N \text{ with } (K + B)^n = C \right\} / \simeq$$

and the moduli stack by the same formula but without dividing by isomorphisms, and by giving $M^N_C(S)$ the groupoid structure in a natural way.

There are other possible definitions for the moduli functor, see f.e [Ale96b].

At this point we can choose a certain scheme in a product of Hilbert schemes with the universal family that contains all interesting for us stable pairs. The next step is to separate the stable pairs from wrong fibers, and for this we need to know that our functor is locally closed in the following sense.

For every flat projective family $(X, B) \to S$ there exist locally closed subschemes $S_i \subset S$ with the following universal property:

- A morphism of schemes $T \to S$ factors through $\bigsqcup_{i} S_i$ iff $(X, L) \times_{S} T \to T$ belongs to $M^N_C(T)$.

For our functor this property follows from the following conjecture of Shokurov ([Sho92]).

**Conjecture 2.5** (Inversion of log Adjunction). Let $(X, B) \to S$ be a flat 1-dimensional family. Assume that there exists an invertible sheaf $L$ on $X$ whose restriction on each fiber coincides with $O_X(N(K_X + B))$ as in definition 1.7. Then the $S_2$-fication of the pair $(X_0, B_0)$ has semi log canonical singularities iff the pair $(X, B + X_0)$ has semi log canonical singularities in a neighborhood of $X_0$.

**Remark 2.6.** $X_0$ is $S_2$ iff $X$ is $S_3$. In many cases the varieties are Cohen-Macaulay, so taking the $S_2$-fication is unnecessary.

2.7. One direction of this conjecture (going from the family to the central fiber) is easy and the proof for the case when $X$ is irreducible can be found in [Kol92, ch.17]. The general case can be easily deduced from that by taking the normalization.

The same reference contains the proof of the opposite direction (the inversion) assuming the log Minimal Model Program in dimension $n + 1$. It also contains several special cases where it can be proved without log MMP, using the Kawamata-Viehweg vanishing theorem only.
The inversion of adjunction conjecture implies that if the sheaves $\mathcal{O}_X(N(K_X + B))$ are locally free and can be put together in a flat family then the semi log canonical property is stable under generizations.

Indeed, it follows from the definition that for a general fiber $X_t$ the pair $(X, B + X_0 + X_t)$ is still semi log canonical. Then $(X_t, B_t)$ is semi log canonical by the easy direction of log adjunction.

The question when exactly the sheaves $\mathcal{O}_X(N(K_X + B))$ can be put together in a flat family is rather delicate. It follows from a technical result of Kollár, see e.g. [Ale96b].

At this point we can pick a sub-family in our universal family that contains exactly our stable pairs. What remains is to take a quotient by the pre equivalence relation (or a groupoid) which is given by the action of the projective linear group $PGL$. This groupoid is easily seen to be flat. It also has a quasifinite stabilizer because stable pairs have finite automorphism groups by [Iit82]. The next separateness property implies that the stabilizer is in fact finite. In this situation the quotient exists as a separated algebraic space. Nowadays, there are several convenient references for this statement, for example [Kol95] and [KM95]. As a result, one obtains a coarse moduli space $M^N_{\mathbb{C}}$ as a separated algebraic space of finite type, and we are already working over an algebraically closed field $k$ of char 0 since we used the log MMP.

**Theorem 2.9.** Let us assume the inversion of log adjunction conjecture. Let $(X', B') \to S \setminus 0$ be a 1-dimensional family without the central fiber which is a stable pair over $S \setminus 0$. Then it can be completed to a stable pair over $S$ in no more than one way up to an isomorphism.

**Proof.** Let $(X, B) \to S$ be one such completion. By the inversion of log adjunction we know that $(X, B + X_0)$ is semi log canonical, possibly after shrinking $S$. Assume first that the scheme $X$ is irreducible, so that the singularities are in fact canonical. Then for any proper birational morphism from a normal variety $f : Y \to X$ and for every positive integer $d$ we have by definition

$$f_* \mathcal{O}_Y(dN(K_Y + f^{-1}B + f^{-1}X_0 + \sum E_i)) = \mathcal{O}_X(dN(K_X + B + X_0))$$

Here the following three circumstances are important:

1. $f^{-1}X_0 + \sum E_i$ is in fact the central fiber of $Y$ with the reduced structure.
2. the divisor $X_0$ is relatively trivial.
3. the divisor $K + B$ is relatively ample, so that the family $(X, B) \to S$ can be computed as a Proj of a big graded ring of relative sections of $\mathcal{O}(dN(K + B))$.

As a result of this, we obtain

$$X = \text{Proj}_{d \geq 0} \oplus \mathcal{O}_Y(dN(K_Y + f^{-1}B + Y_{0,\text{red}})),$$
where $\pi$ denotes the morphism $Y \to S$.

But this means that the family $(X, B)$ can be uniquely reconstructed from $(Y, f^{-1}B)$. Now, given two families $(X_1, B_1)$ and $(X_2, B_2)$, we can find a normal variety $Y$ which dominates both of them. By uniqueness, we have a canonical isomorphism $(X_1, B_1) \to (X_2, B_2)$.

This completes the case when $X$ is irreducible. In general, the above argument shows the uniqueness of $(X^\nu, B + \text{cond}(\nu))$, and $(X, B)$ is uniquely recoverable from that.

Next, we would like to prove that this algebraic space is in fact proper. For this, we have to check the corresponding property for our functor $\mathcal{M}^N_C$.

2.10. The pair $(X, B)$ above is the log canonical model of $(Y, f^{-1}B + Y_{0,\text{red}})$. So, the argument actually followed from the uniqueness of the log canonical model. Vice versa, assume that we have the log Minimal Model available. Start with arbitrary compactification $(X, B) \to S$ of a stable pair over $S \setminus 0$. Take the normalization. For each irreducible component apply the Semistable Reduction Theorem (of course, char 0 is necessary for that) to obtain, after a finite ramified base change and resolution of singularities, a family with the reduced central fiber such that the irreducible components of the central fiber, $\text{cond}(\nu)$, exceptional divisors of resolution and $B_j$ intersect transversally. Note that it is possible to choose the same base change that works for every irreducible component. And then just find the log canonical model applying log MMP.

In fact, we don’t need all the results of log MMP but only the following conjecture and only in the 1-dimensional semistable case. After that, glue the irreducible components back together. That will be the desired family over a finite ramified cover of $S$. This proves that our functor and the moduli space are proper.

**Conjecture 2.11 (Existence of log Canonical Model).** Let $\pi : (Y, B) \to S$ be a projective morphism and assume that

(i). the singularities of $(Y, B)$ are log canonical.

(ii). restriction of $O_Y(N(K + B)$ on each generic fiber is big (contains an ample divisor).

Then the ring of $O_S$-modules

$$\oplus_{d \geq 0} \pi_* O_Y(dN(K_Y + B))$$

is finitely generated.

2.12. The last step is to show that the moduli space $M^N_C$ is projective. This follows by the Kollár’s Ampleness Lemma, see [Kol90]. The input data for this statement is

(i). $M$ has to be a proper algebraic space of finite type over an algebraically closed field field $k$ of characteristic 0.
(ii). On a finite cover of $M$ there has to exist a projective polarized family $(X, B)$ whose every fiber has semi log canonical singularities (Kollár considered the case $B = \emptyset$ but the generalization to the case of reduced $B$ is immediate). For example, this happens when $M$ is a coarse moduli space for some functor of polarized varieties, as in our case.

(iii). The polarization has to be functorial, i.e. compatible with base changes. In our case, the polarization $\mathcal{O}(N(K_{X/S} + B))$ has this property.

3. Examples of log canonical singularities

The following examples should be in any introductory article on log canonical singularities but surprisingly they aren’t.

**Lemma 3.1.** Let $X = T_{\text{Nemb}}(\Delta)$ be a torus embedding over a field $k$ defined by a rational partial polyhedral cone decomposition and $B = \sum B_j$ be the sum of divisors corresponding to the 1-dimensional faces of the fan $\Delta$. Then the pair $(X, B)$ has pre log canonical singularities, and $B$ (i.e. the pair $(B, 0)$) has pre semi log canonical singularities.

**Proof.** The basic formula of the theory of torus embeddings for the canonical sheaf is

$$\omega_X(B) \simeq \mathcal{O}_X$$

Every torus embedding has a toric resolution of singularities $f : Y \rightarrow X$ such that $f^{-1}B \cup E_i$ has normal crossings, where $E_i$ are the exceptional divisors of $f$. Here $f^{-1}B \cup E_i$ is the union of divisors corresponding to 1-dimensional faces of the fan of $Y$. Therefore,

$$f^*\mathcal{O}(K_X + B) \simeq f^*\mathcal{O}_X \simeq \mathcal{O}_Y \simeq \mathcal{O}(K_Y + f^{-1}B + \sum E_i)$$

and the singularities of the pair $(X, B)$ are pre log canonical.

The normalization $B^\nu$ of $B$ is a disjoint union of torus embeddings, and $\text{cond}(\nu)$ is again the union of divisors corresponding to the 1-dimensional faces. This shows that $B$ has pre semi log canonical singularities.

Therefore, every time when toric geometry is used, log canonical singularities show up. One of such situations is the following theorem of Mumford [Mum77, 3.4.4.2].

**Theorem 3.2.** Let $\Gamma$ be a neat arithmetic group acting on a bounded symmetric complex domain $D$. Let $(D/\Gamma)^*$ be the Baily-Borel compactification of $D/\Gamma$ and $\overline{D/\Gamma}$ be any of the toroidal compactifications. Denote the boundaries of these compactifications by $\Delta^*, \overline{\Delta}$ respectively. Then

$$(D/\Gamma)^* = \text{Proj}_{d \geq 0} H^0(d(K_{(D/\Gamma)^*} + \Delta^*))$$

$$= \text{Proj}_{d \geq 0} H^0(d(K_{\overline{D/\Gamma}} + \overline{\Delta}))$$

**Corollary 3.3.** $((D/\Gamma)^*, \Delta^*)$ is the log canonical model of $(D/\Gamma, \overline{\Delta})$, and they both have log canonical singularities.
The above formula in fact is one of the definitions of a log canonical model. We remind that a group $\Gamma$ is called neat if eigenvalues of each element of $\Gamma$ generate a torsion-free subgroup of $\mathbb{C}^*$. The quotient space $D/\Gamma$ by a neat group is nonsingular.

What about the general case? It is easy, all one has to do is use the Hurwitz formula (cf. [Kol, 3.16]).

Every arithmetic group contains a neat subgroup $\Gamma_0 \subset \Gamma$ of finite index. Let $D_j$ be the irreducible ramification divisors of $D/\Gamma_0 \to D/\Gamma$ on $D/\Gamma$ with ramification indices $n_j$. Then we immediately obtain the following

**Theorem 3.4.**

$$(D/\Gamma)^* = \text{Proj}_{d \geq 0} H^0(d(K_{(D/\Gamma)}^* + \Delta^* + \sum (1 - 1/n_j)D_j))$$

$$= \text{Proj}_{d \geq 0} H^0(d(K_{D/\Gamma} + \Delta + \sum (1 - 1/n_j)D_j^*))$$

**Corollary 3.5.** $((D/\Gamma)^*, \Delta^* + \sum (1 - 1/n_j)D_j^*)$ is the log canonical model of $(D/\Gamma, \Delta + \sum (1 - 1/n_j)D_j)$, and they both have log canonical singularities.

**Example 3.6.** The compactification $\overline{A}_1 = \mathbb{P}^1$ of the moduli space $A_1$ of elliptic curves does not have log general type:

$$\deg(K_{\mathbb{P}^1} + P_\infty) = -2 + 1 < 0,$$

so it is not a log canonical model of anything. However, the sum becomes positive when one adds the terms $\frac{1}{n_i}P_i$ corresponding to the elliptic curves with automorphisms. This answers the footnote of Mumford appearing on the same page as theorem [Mum77, 4.2].

Another situation is the stable quasiabelian varieties and pairs appearing as the limits of abelian varieties. We refer the reader to [AN96, Ale96a] for their definition. The very construction for them is toric, so not surprisingly we have

**Lemma 3.7.** Let $P_0$ is a SQAV. Then $P_0$ has pre semi log canonical singularities.

**Proof.** By construction ([AN96]) there exists an étale map $\tilde{P}_0 \to P_0$, and $\tilde{P}_0$ is a union of divisors in a torus embedding $\tilde{P}$ corresponding to the 1-dimensional faces of the fan. The statement now follows from 3.1. \qed

$P_0$ in [AN96] appears as a central fiber of a one-dimensional degenerating normal family $P/S$ of abelian varieties. Over $\mathbb{C}$, $P$ is a quotient of a torus embedding (which is locally of finite type) by a group $\mathbb{Z}^g$ acting freely in the classic topology.

**Lemma 3.8.** The family $P$ itself has log canonical singularities.
Proof. Indeed, the general fiber of $P/S$ is smooth, so all the “bad” discrepancies lie over the central fiber. By 3.1 the pair $(P, P_0)$ has log canonical singularities, i.e. the corresponding discrepancies are $a_i \leq 1$. But the discrepancies of $(P, 0)$ have to be less than $a_i$ by at least the multiplicities of $f^*P_0$ along the exceptional divisors. Since $P_0$ is Cartier, these multiplicities are at $\geq 1$ and the discrepancies of $P$ are $\leq 0$. 

In the principally polarized case a SQAV by [AN96] comes with a natural theta divisor $\Theta$.

Remark 3.9. An easy generalization of the last lemma is that a pair $(P_0, \varepsilon \Theta_0)$ has semi log canonical singularities for $\varepsilon \ll 1$ in char 0. For this one simply has to notice that $\Theta$ does not entirely contain any of the strata $P_0$: [AN96, 3.28]. A more interesting is the following.

Theorem 3.10. A principally polarized stable quasiabelian pair $(P_0, \Theta_0)$ over $\mathbb{C}$ has semi log canonical singularities.

Proof. For the abelian varieties this result is a theorem of Kollár [Kol93]. The present proof is the adaptation of the proof of that theorem to our situation.

By [AN96] every stable quasiabelian pair appears as the central fiber in a 1-dimensional family $\pi : (P, \Theta) \to S = D_\varepsilon$ with abelian general fiber over a small disk. We denote by $I$ the ideal defining $0 \in D_\varepsilon$.

If we prove that the pair $(P, \Theta + P_0)$ has log canonical singularities then we would be done by the easy direction of the “inversion of log adjunction theorem” (see [Kol92, ch.17] or 2.5).

The locus $Z$ of non-log canonical singularities of $(P, \Theta + P_0)$ coincides with the locus of non-log terminal singularities of the pair $(P, (1 - \varepsilon)(\Theta + P_0))$ for $0 < \varepsilon \ll 1$. We will apply the Kawamata-Viehweg vanishing theorem in the following Nadel’s form (see f.e. [Kol, 2.16]):

Theorem 3.11. Let $X$ be a normal and proper variety and $N$ a line bundle on $X$. Assume that $N \equiv K_X + \Delta + M$, where $M$ is nef and big $\mathbb{Q}$-Cartier divisor and $\Delta$ effective $\mathbb{Q}$-Cartier divisor with coefficients $< 1$. Then there is an ideal sheaf $J \subset O_X$ such that

$$\text{Supp}(O_X/J) = \{x \in X \mid (X, \Delta) \text{ is not log terminal at } x\}$$

We will apply this theorem in the relative situation to the proper morphism $\pi : P \to S$. We have

$$K_P + \Theta + P_0 = K_P + (1 - \varepsilon)(\Theta + P_0) + \varepsilon(\Theta + P_0),$$

$K_P, P_0$ are relatively trivial and $\Theta$ is relatively ample. Therefore, by the above there exists an ideal $J \subset O_P$ supporting the locus $Z$ where the pair $(P, \Theta + P_0)$ is not log canonical, and $R^1\pi_*J(\Theta + P_0) = 0$. Therefore, the following map is surjective

$$\pi_*O_P(K_P + \Theta + P_0) \to \pi_*O_Z(K_P + \Theta + P_0)$$
Since in the nonsingular case the statement holds by Kollár’s theorem, after shrinking $S$ the support of $Z_{\text{red}}$ will be contained in the central fiber. Therefore $Z$ is a closed complex-analytic subspace of $P_0 = P \times R/I^n$ for some $n \geq 0$, where $R$ is the ring of germs of analytic functions at 0. Moreover, $Z$ is a closed subspace of the theta divisor $\Theta$. Indeed, as in the proof of lemma 3.7, theorem 3.1 the pair $(P, (1-\varepsilon)P_0)$ is log canonical, therefore the pair $(P, (1-\varepsilon)P_0)$ is log terminal.

By theorem 4.6 of [AN96] we have $H^0(\mathcal{O}_{P_0}(\Theta_0)) = 1$ and $H^i(\mathcal{O}_{P_0}(\Theta_0)) = 0$ for $i > 0$. This implies $R^i\pi_*\mathcal{O}_P(\Theta) = 0$ for $i > 0$ and $\pi_*\mathcal{O}_P(\Theta) = \mathcal{O}_S$.

We have $\mathcal{O}(K_P) \simeq \mathcal{O}(P_0) \simeq \mathcal{O}$, so $\mathcal{O}(K_P + \Theta + P_0) \simeq \mathcal{O}(\Theta)$. Since $Z$ is a closed subspace of $\Theta$, $\phi$ has to be the zero map. On the other hand, $\pi_*\mathcal{O}_Z(\Theta) \not\simeq 0$ for any proper subspace $Z \subset P_n$. In the nonsingular case this is concluded by the semi-continuity argument and the fact that the abelian variety acts transitively by translations. In our situation, there is the action of a semiabelian group $G/S$, and although it is not transitive, still the intersection of translations $g(\Theta)$ by sections $g \in G$ is empty: [AN96, 3.28], and this implies $\pi_*\mathcal{O}_Z(\Theta) \not\simeq 0$. 

**Corollary 3.12.** Over $\mathbb{C}$, if the compactification of the moduli space $A_g$ by the pairs $(P_0, \Theta_0)$ exists, it is projective.

**Proof.** This follows by applying the Ampleness Lemma of Kollár, cf. 2.12. 

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