ATOMIC DECOMPOSITIONS OF MIXED NORM BERGMAN SPACES ON TUBE TYPE DOMAINS

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ABSTRACT. We use the author’s previous work on atomic decompositions of Besov spaces with spectrum on symmetric cones, to derive new atomic decompositions for Bergman spaces on tube type domains. It is related to work by Ricci and Taibleson who derived decompositions for classical Besov spaces from atomic decompositions of Bergman spaces on the upper half plane. Moreover, for this class of domains our method is an alternative to classical results by Coifman and Rochberg, and it works for a larger range of Bergman spaces.

1. INTRODUCTION

In this paper we suggest a new approach to atomic decompositions for the Bergman spaces on tube type domains. Atomic decompositions have previously been obtained for the upper half plane (unit disc) in [20, 19], and for the unit ball [21, 7] as well as other bounded symmetric domains [10, 17]. These decompositions have typically been attained by investigating oscillations of the Bergman kernel and in most cases the atoms are samples of the Bergman kernel. In the case of the unit ball such oscillations can be estimated for all parameters, but for higher rank spaces it is more complicated. The issue is connected to the question of boundedness of the Bergman projection on bounded symmetric domains, which is still an open problem. Advances to answer this problem have recently been made in the case of tube type domains in [11] and for general domains in [18]. In the case of tube type domains over forward light cones the problem is now solved [5].

In this paper we will concentrate on tube type domains, and we use Fourier-Laplace extensions to transfer known atomic decompositions for Besov spaces [6] to Bergman spaces. This allows us to narrow the gap in the atomic decompositions from [10] in the case of tube type domains. Moreover, these decompositions are for mixed norm Bergman spaces which have not previously been dealt with. We would like to mention that this work seems to be in the reverse direction of the paper [20] which uses atomic decompositions for the mixed norm Bergman spaces on the upper half plane to get atomic decompositions for the Besov spaces. See also [14] for related work for Bergman spaces on the unit ball in relation to Besov spaces on the Heisenberg group.

2. SYMMETRIC CONES

For an introduction to symmetric cones we refer to the book [13]. Let $V$ be a Euclidean vector space over the real numbers of finite dimension $n$. A subset $\Omega$ of
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V is a cone if \( \lambda \Omega \subseteq \Omega \) for all \( \lambda > 0 \). Assume \( \Omega \) is open and convex, and define the open dual cone \( \Omega^* \) by

\[
\Omega^* = \{ y \in V \mid (x,y) > 0 \text{ for all non-zero } x \in \Omega \}.
\]

The cone \( \Omega \) is called symmetric if \( \Omega = \Omega^* \), and the automorphism group

\[
\mathcal{G}(\Omega) = \{ g \in \text{GL}(V) \mid g\Omega = \Omega \}
\]

acts transitively on \( \Omega \). Notice that the group \( \mathcal{G}(\Omega) \) is semisimple. Define the characteristic function of \( \Omega \) by

\[
\varphi(x) = \int_{\Omega^*} e^{-(x,y)} \, dy,
\]

then

\[
\varphi(gx) = |\det(g)|^{-1} \varphi(x).
\]

Also,

\[
f \mapsto \int_{\Omega} f(x)\varphi(x) \, dx
\]

defines a \( \mathcal{G}(\Omega) \)-invariant measure on \( \Omega \). The connected component \( \mathcal{G}_0(\Omega) \) of \( \mathcal{G}(\Omega) \) has Iwasawa decomposition

\[
\mathcal{G}_0(\Omega) = KAN
\]

where \( K = \mathcal{G}_0(\Omega) \cap \text{O}(V) \) is compact, \( A \) is abelian and \( N \) is nilpotent. The unique fixed point in \( \Omega \) for the mapping \( x \mapsto \nabla \log \varphi(x) \) will be denoted \( e \), and we note that \( K \) fixes \( e \). The connected solvable subgroup \( H = AN \) of \( \mathcal{G}_0(\Omega) \) acts simply transitively on \( \Omega \) and the integral (1) thus also defines the left-Haar measure on \( H \).

Denote by \( S(V) \) the space of rapidly decreasing smooth functions with topology induced by the semi-norms

\[
\| f \|_k = \sup_{|\alpha| \leq k} \sup_{x \in V} |\partial^\alpha f(x)| (1 + |x|)^k.
\]

Here \( \alpha \) is a multi-index, \( \partial^\alpha \) denotes usual partial derivatives of functions, and \( k \geq 0 \) is an integer. For \( f \in S(V) \) the Fourier transform is defined by

\[
\hat{f}(w) = \frac{1}{(2\pi)^{n/2}} \int_V f(x) e^{-i(x,w)} \, dx \text{ for } w \in V.
\]

The convolution

\[
f \ast g(x) = \int_V f(y)g(x-y) \, dy
\]

of functions \( f, g \in S(V) \) satisfies

\[
\hat{f} \ast \hat{g}(w) = \hat{f}(w)\hat{g}(w).
\]

The space \( S'(V) \) of tempered distributions is the linear dual of \( S(V) \). For functions on \( V \) define \( \tau_x f(y) = f(y-x), f^x(y) = f(-y) \) and \( f^*(y) = \overline{f(-y)} \). Convolution of \( f \in S'(V) \) and \( \phi \in S(V) \) is defined by

\[
f \ast \phi(x) = f(\tau_x \phi^x).
\]

As usual, the Fourier transform extends to tempered distributions by duality. The space of rapidly decreasing smooth functions with Fourier transform vanishing on \( \Omega \) is denoted \( S_\Omega \). It is a closed subspace of \( S(V) \) and will be equipped with the subspace topology.

The space \( V \) can be equipped with a Jordan algebra structure such that \( \overline{\Omega} \) is identified with the set of all squares. This gives rise to the notion of a determinant.
Theorems 3.1 and 3.2.

Let \( \psi \) be such that \( \hat{\psi} \) is compactly supported. Then there is an index set \( I \), a set \( \{ (h_i, x_i) \}_{i \in I} \subseteq H \times V \), a Banach sequence space \( b^p,q(I) \), a set of continuous functionals \( \{ c_i : B^p,q \to C \}_{i \in I} \), and a constant \( C > 0 \) such that

\[
\begin{align*}
(1) & \quad f(x) = \sum_{i \in I} c_i(f) \frac{1}{\sqrt{\det h_i}} \hat{\psi}(h_i^{-1}(x-x_i)) \text{ with convergence in norm in } B^p,q \ \\
(2) & \quad \| c_i(f) \|_{b^p,q} \leq C \| f \|_{B^p,q}
\end{align*}
\]

3. Besov spaces related to symmetric cones

The cone \( \Omega \) can be identified as a Riemannian manifold \( \Omega = G_0(\Omega)/K \) where \( K \) is the compact group fixing \( e \). The Riemannian metric in this case is defined by

\[
(u, v)_y = (g^{-1}u, g^{-1}v)
\]

for \( u, v \) tangent vectors to \( \Omega \) at \( y = ge \). Denote the balls of radius \( \delta \) centered at \( x \) by \( B_{\delta}(x) \). For \( \delta > 0 \) and \( \lambda \geq 2 \) the points \( \{ x_j \} \) are called a \( (\delta, \lambda) \)-lattice if

\[
\begin{align*}
(1) & \quad \{ B_{\delta}(x_j) \} \text{ are disjoint, and} \\
(2) & \quad \{ B_{\lambda\delta}(x_j) \} \text{ cover } \Omega.
\end{align*}
\]

We now fix a \( (\delta, \lambda) \)-lattice \( \{ x_j \} \) with \( \delta = 1/2 \) and \( \lambda = 2 \). Then there are functions \( \psi_j \in S_{\Omega} \), such that \( 0 \leq \hat{\psi}_j \leq 1 \), \( \text{supp}(\hat{\psi}_j) \subseteq B_2(x_j) \), \( \hat{\psi}_j \) is one on \( B_{1/2}(x_j) \) and \( \sum_j \hat{\psi}_j = 1 \) on \( \Omega \). Using this decomposition of the cone, the Besov space norm for \( 1 \leq p, q < \infty \) and \( \nu \in \mathbb{R} \) is defined in [1] by

\[
\| f \|_{B^p,q} = \left( \sum_j \Delta(x_j)^{-\nu} \| f * \psi_j \|_p^q \right)^{1/q}.
\]

The Besov space \( B^p,q \) consists of the equivalence classes of tempered distributions \( f \) in \( (S_{\Omega})' \simeq \{ f \in S'(V) \mid \text{supp}(\hat{f}) \subseteq \overline{\Omega} \}/S'_{\partial \Omega} \) for which \( \| f \|_{B^p,q} < \infty \).

Define the index

\[
\tilde{q}_{\nu,p} = \frac{\nu + n/R - 1}{(n/Rp') - 1}
\]

if \( n/R > p' \) and set \( \tilde{q}_{\nu,p} = \infty \) if \( n/R \leq p' \). The following results from [1] states when the Besov spaces are included in the space of tempered distributions \( S'(V) \).

Lemma 3.1. Let \( \nu > 0 \), \( 1 \leq p < \infty \) and \( 1 \leq q < \tilde{q}_{\nu,p} \). Then for every \( f \in B^p,q \) the series \( \sum_j f * \psi_j \) converges in the space \( S'(V) \), and the correspondence

\[
f + S'_{\partial \Omega} \mapsto f^\delta = \sum_j f * \psi_j
\]

is continuous, injective and does not depend on the particular choice of \( \{ \psi_j \} \).

The main result from [6] is that the quasiregular representation of the group \( H \times V \) can be used to obtain atomic decompositions for these Besov spaces. We summarize the result in

Theorem 3.2. Let \( \psi \in S_{\Omega} \) be such that \( \hat{\psi} \) is compactly supported. Then there is an index set \( I \), a set \( \{ (h_i, x_i) \}_{i \in I} \subseteq H \times V \), a Banach sequence space \( b^p,q(I) \), a set of continuous functionals \( \{ c_i : B^p,q \to C \}_{i \in I} \), and a constant \( C > 0 \) such that

\[
\begin{align*}
(1) & \quad f(x) = \sum_{i \in I} c_i(f) \frac{1}{\sqrt{\det h_i}} \hat{\psi}(h_i^{-1}(x-x_i)) \text{ with convergence in norm in } B^p,q \\
(2) & \quad \| c_i(f) \|_{b^p,q} \leq C \| f \|_{B^p,q}
\end{align*}
\]
(3) if \( \{ \lambda_i \} \in b^{p,q}_\nu \) then

\[
    f(x) = \sum \lambda_i \frac{1}{\sqrt{\det(h_i)}} \psi(h_i^{-1}(x - x_i))
\]

is in \( B^{p,q}_\nu \) and \( \|f\|_{B^{p,q}_\nu} \leq C\|\{ \lambda_i \}\|_{b^{p,q}_\nu} \).

**Remark 3.3.** At this stage it is appropriate to describe the sequence of points \( \{(h_i, x_i)\}_{i \in I} \) and the sequence space \( b^{p,q}_\nu \) in some detail. For this one chooses a covering \( \{U_i\}_{i \in I} \) of the space \( \Omega \times V \). This covering is chosen such that each \( U_i \) is a translate of a fixed relatively compact neighbourhood \( U \) of \( \{e\} \times \{0\} \) by some element \( (h_i, x_i) \) of the semidirect product \( H \rtimes V \). Moreover, the sets \( U_i \) have the finite overlapping property, that is, there is an \( N \) such that each set \( U_i \) overlap at most \( N \) others. A sequence \( \{\lambda_i\} \) is in \( b^{p,q}_\nu \) if

\[
    \|\{\lambda_i\}\|_{b^{p,q}_\nu} := \left( \int_{\Omega} \left( \int_{V} \sum_{i \in I} |\lambda_i| |1_{U_i}(x,t)|^p \, dt \right)^{q/p} \Delta(x)^{\nu-qn/(2R)-n/R} \, dx \right)^{1/q}
\]

is finite. If \( p = q \) this is a \( \Delta^{\nu-pn/(2R)-n/R} \)-weighted \( \ell^p \)-space.

### 4. Bergman spaces on tube type domains.

In this section we introduce the Bergman spaces on the tube type domains, and describe the isomorphism between a range of Besov spaces and Bergman spaces.

Let \( T = \{ z = x + iy \mid x \in V, y \in \Omega \} \) be the tube type domain related to the symmetric cone \( \Omega \). For \( 1 \leq p, q < \infty \) and \( \nu > 0 \) define the weighted Lebesgue space \( L^{p,q}_\nu \) on the tube type domain to consist of the equivalence classes of measurable functions on \( \Omega \) for which the norm

\[
    \|F\|_{L^{p,q}_\nu} := \left( \int_{\Omega} \left( \int_{V} |F(x + iy)|^p \, dx \right)^{q/p} \Delta(x)^{\nu-qn/(2R)-n/R} \, dy \right)^{1/q}
\]

is finite. Here \( dx \) and \( dy \) denote the usual Lebesgue measures on \( V \) and \( \Omega \). The mixed norm Bergman space \( A^{p,q}_\nu \) on \( \Omega \) consists of the holomorphic functions in \( L^{p,q}_\nu \). It is well-known that this is a reproducing kernel Banach space, that is, for every \( z \in T \) the mapping \( F \mapsto F(z) \) is continuous from \( A^{p,q}_\nu \) to \( \mathbb{C} \).

The special case \( p = q = 2 \) and \( \nu = n/R \) is the usual Bergman space and the reproducing kernel in this case is

\[
    B(z, w) = B_{n/r}(z, w) = c(\nu)\Delta \left( \frac{z - w}{\overline{w}} \right)^{-2n/r},
\]

which will be called the Bergman kernel.

Following [1] we now define the Fourier-Laplace extensions of elements in the Besov spaces. This extension only works for Besov spaces which can be naturally imbedded in the usual space of tempered distributions. This introduces a restriction in the range of indices that can be used. We summarize the results from [1] that we need.

Define the Fourier-Laplace extension of a tempered distribution \( f \) whose Fourier transform \( \hat{f} \) is supported on \( \overline{\Omega} \) by

\[
    \mathcal{E}f = \int \hat{f}(w)e^{iz \cdot w} \, dw
\]
for \( z \in T \). For \( 1 \leq q < \bar{q}_{\nu,p} \), the Besov space can be identified with a space of such distributions, and therefore we can define

\[
\tilde{E}f = \mathcal{E} \sum_j f \ast \psi_j
\]

for \( f \in B_{p,q}^{\nu} \). Define the index

\[
q_{\nu,p} = \min(p,p')^{\nu + n/R - 1} \frac{n/R - 1}{n/R - 1}
\]

when \( n > R \) and set \( q_{\nu,p} = \infty \) when \( n = 1 \). Notice that \( 2 < q_{\nu,p} \leq \bar{q}_{\nu,p} \) so when \( 1 \leq q < q_{\nu,p} \) all elements of the Besov space \( B_{p,q}^{\nu} \) can be identified with tempered distributions whose Fourier-Laplace extensions are in the Bergman space \( A_{p,q}^{\nu} \). We have

**Theorem 4.1.** If \( \nu > n/R - 1 \), \( 1 \leq p < \infty \) and \( 1 \leq q < q_{\nu,p} \), then the mapping \( \tilde{E} : B_{p,q}^{\nu} \to A_{p,q}^{\nu} \) is an isomorphism. Moreover,

\[
\lim_{y \to 0} F(x + iy) = f(x)
\]

in both \( S'(V) \) and \( B_{p,q}^{\nu} \).

5. **Atomic decomposition of Bergman spaces**

In this section we merge the results from [6] and [1] to obtain atomic decompositions for Bergman spaces on the tube type domains. This will give an alternate approach to the atomic decompositions found in [10]. This new approach allows for a large class of atoms, and moreover, through the Paley-Wiener theorem the decay properties of these atoms are quite well known. Note, that the decomposition from [10] uses samples of the Bergman kernel, but the Bergman kernel is not among the possible atoms with the new approach. The paper [20] goes in the opposite direction and uses [10] to obtain atomic decompositions the Besov spaces. It would of course be interesting to investigate how to completely align the two methods and to determine exactly which atoms can be moved from the Besov setting and to the Bergman setting. In this paper we are clearly only dealing with a subset of possible atoms.

Let \( F \in A_{p,q}^{\nu} \) then \( f = \tilde{E}^{-1} F \) is in \( B_{p,q}^{\nu} \) with equivalent norms and can be decomposed as

\[
f(x) = \sum_{i \in I} c_i(f) \frac{1}{\sqrt{\det(h_i)}} \psi(h_i^{-1}(x - x_i)).
\]

Since \( \psi \in S_0 \) with compactly supported Fourier transform is in every Besov space, we get

\[
\psi_i(z) := \tilde{E}(\psi(h_i^{-1}(\cdot - x_i)))(z) = \tilde{E}\psi(h_i^{-1}(z - x_i)).
\]

This results in the following atomic decompositions.

**Theorem 5.1.** Let \( \nu > n/R - 1 \), \( 1 \leq p < \infty \) and \( 1 \leq q < q_{\nu,p} \). There is a sequence \( \{d_i\} \) of functionals and atoms \( \{\psi_i\} \) parameterized by appropriate \( \{(h_i,x_i)\} \subseteq \mathcal{H} \times V \) such that

1. if \( F \in A_{p,q}^{\nu} \) then

\[
F(z) = \sum_{i \in I} d_i(F) \frac{1}{\sqrt{\det(h_i)}} \psi_i(z).
\]
where \( d_i(F) = c_i(E_i^{-1} F) \) satisfy that \( \| \{ d_i(F) \} \|_{\mathcal{B}^{p,q}} \leq C \| F \|_{\mathcal{A}^{p,q}} \), \( \quad (2) \) and if \( \{ d_i \} \in \mathcal{B}^{p,q} \), then
\[
F(z) = \sum_{i \in I} d_i \frac{1}{\sqrt{\det(h_i)}} \psi_i(z).
\]
is in \( \mathcal{A}^{p,q} \) and \( \| F \|_{\mathcal{A}^{p,q}} \leq C \| \{ d_i \} \|_{\mathcal{B}^{p,q}} \).

**Remark 5.2.** Notice that for the case of cones of rank 2 (for example the forward light cones) this theorem can be extended to the larger range \( 1 \leq q < \tilde{q}_{p,p} \) which is the entire range of \( q \) for which Laplace extensions can be defined. See [5].

6. Comparison with previous results and some open problems

To demonstrate how this work extends the range of Bergman spaces for which atomic decompositions can be found, we now compare our atomic decompositions to classical results due to Coifman and Rochberg \[10\] in the special case of tube type domains. Their results only work for rank one spaces, but with minor modifications this issue can be addressed via the Forelli-Rudin estimates from Theorem 4.1 in [12] or Corollary II.4 in [4]. The latter result was used in [3] to correct and generalize the atomic decompositions of Coifman and Rochberg to also include two non-symmetric domains.

We first summarize the atomic decompositions from [10, 3]. Let \( V \) be an open convex cone in \( \mathbb{R}^m \) and let \( F \) be a \( V \)-valued Hermitian form on \( \mathbb{C}^n \). The open subset of of elements \((z, w)\) in \( \mathbb{C}^n \times \mathbb{C}^n \) for which \( \text{Im}(z) - F(w, w) \in V \) is called a Siegel domain of type II. The domain is called symmetric if it is also a symmetric space. Let \( D \) be a symmetric Siegel domain of type II and let \( B \) denote the associated Bergman kernel. Coifman and Rochberg use a parametrization of Bergman spaces that differs from the one used earlier in this paper. In their notation the Bergman space \( \mathcal{A}^p \) consists of holomorphic functions for which the following norm is bounded
\[
\| F \|_{\mathcal{A}^p} = \int_D |F(z)|^p B(z, z)^{-r} dz.
\]

**Theorem 6.1.** Let \( p \geq 1 \) and assume that \( -\epsilon_D + \gamma_D (p - 1) < r < \infty \). Given \( \theta > p(1 - \epsilon_D) + \epsilon_D + \gamma_D - 2 - r \) there is a lattice \( \{ \xi_i \} \) in \( D \) and a constant \( C > 0 \) such that for \( F \in \mathcal{A}^p \) we have
\[
F(z) = \sum_i \lambda_i(F) \left( \frac{B(z, \xi_i)^2}{B(\xi_i, \xi_i)} \right)^{\frac{r-p}{p}} \left( \frac{B(z, \xi_i)}{B(\xi_i, \xi_i)} \right)^{\frac{\theta}{p}},
\]
and \( \sum_i |\lambda_i(F)|^p \leq C \| F \|_{\mathcal{A}^p}^p \). Moreover, if \( \{ \lambda_i \} \in \ell^p \) then the series
\[
F(z) = \sum_i \lambda_i \left( \frac{B(z, \xi_i)^2}{B(\xi_i, \xi_i)} \right)^{\frac{r-p}{p}} \left( \frac{B(z, \xi_i)}{B(\xi_i, \xi_i)} \right)^{\frac{\theta}{p}}.
\]
defines a function in \( \mathcal{A}^p \) and \( \| F \|_{\mathcal{A}^p} \leq C \| \{ \lambda_i \} \|_{\ell^p} \).

**Remark 6.2.** The constants are given by \( \epsilon_D = 1/G \) and \( \gamma_D = (R - 1)a/(2G) \), where \( G \) is the genus and \( a \) is another structural constant. See, for example, [12] for a full explanation of these constants.
We will now establish the range of $r$ which work for tube type domains. The connection between the number $\nu$ from Theorem 5.1 in the special case of $p = q$ and the number $r$ from Theorem 6.1 is

$$\nu = \frac{2nr}{R} + \frac{n}{R}.$$ 

Moreover, the structural constants $\epsilon_D$ and $\gamma_D$ for tube type domains are

$$\epsilon_D = \frac{R}{2n} \quad \text{and} \quad \gamma_D = \frac{1}{2} - \epsilon_D.$$ 

With this in mind the conditions $\nu > \frac{p}{n}$ and $p < \frac{\nu + n/R - 1}{n/R - 1} + 1$ from Theorem 5.1 rewrite into

$$r > \max\{-\epsilon_D, -\frac{3}{2} + p(1 - \epsilon_D) + \epsilon_D - \frac{p}{2}\}.$$ 

The result from Theorem 6.1 works for

$$r > \max\{-\epsilon_D + \gamma_D, -\frac{3}{2} + p(1 - \epsilon_D)\},$$

when restricted to the case $\theta = 0$. Since for tube type domains over cones $\epsilon_D = R/(2n) \leq 1/2$ and $p \geq 1$ we see that the atomic decompositions in Theorem 5.1 work for a larger range than those of Theorem 6.1.

**Remark 6.3.** The reason we restrict to the $\theta = 0$ when comparing the two methods is, that the the atomic decompositions provided by Laplace extensions is connected to the discrete series representation of the automorphism group on the tube type domain, and we might as well transfer this result to the bounded realization of the domain. Therefore Theorem 5.1 can be transfered to Bergman spaces on the bounded symmetric domain. It thus makes sense to compare with the version of Theorem 6.1 which also can be transfered to the bounded realization, that is, $\theta = 0$.

We finish this section with a list of open problems connected to the results of this paper.

**Problem 1.** The atoms from Theorem 5.1 do not include samples of the Bergman kernel as in [10, 3]. The reason is that the atoms we use are extensions of compactly supported smooth functions, and by the Paley-Wiener Theorem these cannot include the Bergman kernel. This means that Theorem 5.1 does not include Theorem 6.1 as a special case. We believe that it is possible to overcome this issue. The atomic decompositions in [6] build on irreducible, unitary, and integrable group representations, and therefore a much larger class of atoms for the Besov spaces can be used via [14]. It would be interesting to see if the Laplace extensions of this larger class of atoms for the Besov spaces would include the Bergman kernel in order to obtain Theorem 6.1 as a special case of Theorem 5.1. This question would be of interest even on the upper half plane.

**Remark 6.4.** Upon completion of this work the author was made aware of a related paper by D. Békollé, J. Gonessa and C. Nana [2]. They obtain atomic decompositions for the exact same range of Bergman spaces. In their work the atoms are indeed samples of the Bergman kernel. Problem 1 thus formulates one possible approach to uncovering the connection between their result and the present paper.
Problem 2. Another way to get a larger set of atoms including samples of the Bergman kernel is to apply the coorbit theory \cite{9} as has been done for the unit ball in \cite{7,16}. The integral operator with positive kernel from Theorem II.7 in \cite{4} which was used to derive Theorem 6.1 would play a crucial role in this approach (as it did on the unit ball \cite{7,15}), so we predict this approach would work for the same range of parameters as Theorem 6.1. It would be interesting to see if the use of Theorem II.7 in \cite{4} could be avoided or refined in the context of coorbits in order to get to the same range of Bergman spaces as in Theorem 5.1.

Problem 3. The approach highlighted in this paper could be used in the setting of the unit ball by using Laplace extensions of Besov spaces on the Heisenberg group mentioned in \cite{15}. It also seems possible to extend our approach to all bounded symmetric domains via work in \cite{18}.

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