One-loop surface tensions of (supersymmetric) kink domain walls from dimensional regularization

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Abstract. We consider domain walls obtained by embedding the (1 + 1)-dimensional $\phi^4$-kink in higher dimensions. We show that a suitably adapted dimensional regularization method avoids the intricacies found in other regularization schemes in both supersymmetric and non-supersymmetric theories. This method allows us to calculate the one-loop quantum mass of kinks and surface tensions of kink domain walls in a very simple manner, yielding a compact $d$-dimensional formula which reproduces many of the previous results in the literature. Among the new results is the non-vanishing one-loop correction to the surface tension of a (2 + 1)-dimensional $\mathcal{N} = 1$ supersymmetric kink domain wall with chiral domain-wall fermions.

1. Introduction

One of the simplest situations where one can study quantum corrections to non-trivial background fields is the calculation of the quantum mass of (1 + 1)-dimensional solitons with exactly known fluctuation spectra [1]–[6]. One-loop corrections can be obtained by computing the difference of the sums (and integrals) of zero-point energies in the soliton background and in the topologically trivial vacuum. The regularization of these sums is a surprisingly delicate matter whose subtleties have been investigated only rather recently, starting with the observation [7] that, for example, a simple energy–momentum cut-off leads to incorrect results if the same cut-off is used in
the topologically distinct sectors. This has been an actual problem in the calculation of the quantum mass of supersymmetric (SUSY) solitons \[8\]–[12]. On the other hand, the extension of the mode-number cut-off regularization method introduced by Dashen et al \[2\], which begins by discretizing the problem by means of a finite volume, to fermions, turns out to lead to new subtleties concerning the choice of boundary conditions which may or may not entail a contamination through energies localized at the boundaries \[13\]–[16].

However, there do exist methods which give correct results that can be formulated \textit{a priori} in the continuum. In \[13\] it has been shown that the derivative of the quantum kink mass with respect to the mass of elementary scalar bosons is less sensitive and can be calculated by energy cut-off regularization, leading to a result for the quantum mass of SUSY kinks that agrees with \textit{S}-matrix factorizations \[17, 18\], also validating previous results obtained by Schonfeld who considered mode-number regularization of the kink–antikink system \[19\], and by \[20\]–[22] using a finite mass formula in terms of only the discrete modes. In \[23\]–[26] another viable continuum approach was developed that is based on subtracting successive Born approximations for scattering phase shifts. In \[14\] the authors introduced SUSY-preserving higher (space) derivative terms in the action and obtained the correct one-loop results for the energy and the central charge from simple Feynman graphs. In addition, heat-kernel and zeta-function regularization methods have been applied successfully to this problem \[27, 28\].

In \[29\] it has been shown that dimensional regularization through embedding kinks as domain walls in extra dimensions reproduces the known result for the bosonic kink mass, but it was concluded that this method may be difficult to generalize.

In this work, we extend the analysis of \[29\] and demonstrate that dimensional regularization also allows one to calculate the surface tensions of kink domain walls in a way that is far simpler than the methods used previously. Moreover, the consideration of domain walls gives insight into where precisely naive cut-off regularization fails, and resolves its ambiguities by observing that finite ambiguities become divergences in higher dimensions. Requiring finiteness in \((d + 1)\) dimensions thus fixes the finite ambiguities in \((1 + 1)\) dimensions. In this way we confirm the recent observation in \[30\] that the defective energy cut-off method can be repaired by using smooth cut-offs, or sharp cut-offs as limits of smooth ones.

Through dimensional regularization we derive a remarkably compact formula for surface tensions that unifies the diverse results on kink domain walls in \((2 + 1)\) and \((3 + 1)\) dimensions, and yields a finite result even in \((4 + 1)\) dimensions. We discuss the effects of using different renormalization schemes and confirm (most of the) previous one-loop results in the literature on kink domain walls in \((2 + 1)\) and \((3 + 1)\) dimensions.

We also show that this method of dimensional regularization works for the SUSY case by rederiving the quantum mass of the \((1 + 1)\) SUSY kink, and find a new result for a \((2 + 1)\)-dimensional SUSY kink domain wall with chiral domain-wall fermions, which unlike its \((3 + 1)\)-dimensional analogue has non-zero quantum corrections.

2. Bosonic kink and domain walls

2.1. Bosonic kink and dimensional regularization

In \((1 + 1)\) dimensions, a real \(\varphi^4\) theory with spontaneously broken \(Z_2\) symmetry \((\varphi \to -\varphi)\)

\[
\mathcal{L} = -\frac{1}{2} (\partial_\mu \varphi_0)^2 - \frac{\lambda_0}{4} (\varphi_0^2 - \mu_0^2/\lambda_0)^2
\] (1)
has topologically non-trivial solutions to the field equations with finite energy: solitons called ‘kinks’, which interpolate between the two degenerate vacuum states $\varphi = \pm \mu_0 / \sqrt{\lambda_0} \equiv \pm v_0$. A kink/anti-kink at rest at $x = x_0$ is given classically by [1]

$$\varphi_{K,R} = \pm v_0 \tanh(\mu_0(x - x_0)/\sqrt{2}).$$

Embedding the kink solution in $(d + 1)$ dimensions instead of $(1 + 1)$ dimensions gives a domain wall separating the two distinct vacua. This is no longer a finite-energy solution—its energy is proportional to the transverse volume $L^{d-1}$, with classical energy density (surface tension)

$$M_0 / L^{d-1} = 2\sqrt{2}\mu_0^3 / 3\lambda_0.$$  (3)

In $(d + 1) \leq 4$ dimensions, (1) is renormalizable or super-renormalizable, and upon specifying one’s renormalization conditions, quantum corrections to the energy density should be calculable in perturbation theory without ambiguity. Some authors are somewhat cavalier with regard to fixing the meaning of the parameters of the theory through the renormalization conditions, making their results basically meaningless: since the lowest order involves two parameters, any one- or two-loop result is correct in some renormalization scheme.

In $(1 + 1)$ dimensions, where kinks correspond to particles with a calculable quantum mass determined by the parameters of the Lagrangian, the most frequently used renormalization scheme consists of demanding that the tadpole diagrams cancel in their entirety, while $\lambda = \lambda_0$ and $\varphi = \varphi_0$.

Such a renormalization scheme can still be used in $(2 + 1)$ dimensions, whereas in $(3 + 1)$ dimensions there is finally the need to renormalize the coupling constant non-trivially in order to absorb all one-loop divergences. In the following we shall concentrate on the particularly natural scheme which fixes the coupling constant renormalization such that in addition to the absence of tadpole diagrams the renormalized mass of the elementary scalar is equal to the pole of its propagator.

Wavefunction renormalization $\varphi_0 = \sqrt{Z} \varphi$ is finite to one-loop order in $(3 + 1)$ dimensions and to all orders in lower dimensions and it is therefore not mandatory for the one-loop corrections to the energies of kinks and kink domain walls. For simplicity we choose $Z = 1$ for now, postponing the discussion of schemes with non-trivial $Z$ to section 2.2.2.

With $\lambda_0 = Z\lambda = \lambda + \delta \lambda, v_0^2 \equiv \mu_0^2 / \lambda_0 = Z v^2 = v^2 + \delta v^2$, the renormalized Lagrangian for elementary excitations $\eta$ around $\varphi = v$ then reads

$$\mathcal{L} = -\frac{1}{2}(\partial_{\mu} \eta)^2 - \mu^2 \eta^2 - \mu \lambda \eta^4 - \frac{1}{4} \lambda \eta^4 + \mu \lambda^2 \delta v^2 \eta + \frac{1}{2}(\lambda \delta v^2 - 2v^2 \delta \lambda) \eta^2$$

$$- \frac{1}{4} \delta \lambda (\eta^4 + 4v \eta^3) - \frac{1}{2}(\lambda + \delta \lambda)(\delta \nu)^2 + \frac{1}{2} \delta \lambda \delta v^2 (\eta^2 + 2\eta v),$$  (4)

which shows that the renormalized mass of the elementary boson at tree-graph level is $m^2 = 2\mu^2$.

The requirement that tadpole graphs are completely cancelled by the counter-term proportional to $\eta$ fixes $\delta v^2$ at the one-loop level

$$\delta v^2 = -3i\hbar \int \frac{dk_0}{(2\pi)^{d+1}} \frac{d^d k}{k^2 + m^2 - i\epsilon} = 3\hbar \int \frac{d^d k}{(2\pi)^d} \frac{1}{2[k^2 + m^2]^{1/2}}.$$  (5)

Using dimensional regularization, where for Euclidean momenta $k_E$

$$\int d^d k_E (k_E^2 + M^2)^{-\alpha} = \pi^{\nu}(M^2)^{\nu-\alpha} \Gamma(\alpha - \nu)/\Gamma(\alpha),$$  (6)
and writing \( d = 1 + s \) so that \( s \) denotes the number of spatial dimensions orthogonal to the kink axis, we have (setting \( \hbar = 1 \) henceforth)

\[
\delta v^2 = \frac{3}{2\pi} (1 + s) \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(4\pi)^{\frac{s}{2}}} \int_0^\infty \! dk \,(k^2 + m^2)^{\frac{s}{2}-\frac{3}{2}},
\]

which is written in a form that will shortly turn out to be convenient.

Calculating the one-loop correction to the pole mass of the elementary bosons involves local seagull diagrams that are exactly cancelled by \( \delta v^2 \) and a non-local diagram with two three-vertices. According to (4) the renormalized mass \( m \) will be equal to the pole mass, if the latter diagram evaluated on-shell (OS) is cancelled by the counter-term \( \propto \delta \lambda \eta^2 \). This determines \( \delta \lambda \) as

\[
\delta \lambda = \frac{9\lambda^2}{(4\pi)^{\frac{s}{2}}} \int_0^1 \! dx \,[1 - x(1 - x)]^{\frac{s-2}{2}} \int_0^\infty \! \! dk \,(k^2 + m^2)^{\frac{s}{2}-\frac{3}{2}},
\]

which is written in a form that will shortly turn out to be convenient.

In (1+1) dimensions, the one-loop quantum corrections to the mass of a kink are determined by the functional determinant of the differential operator describing fluctuations around the classical solution (2) compared to that of the trivial vacuum, formally leading to a sum over zero-point energies which contribute according to

\[
M^{(1)} = M_0 + \frac{\hbar}{2} \left( \sum \omega - \sum \omega' \right) + O(\lambda)
\]

where \( \omega \) and \( \omega' \) are the eigenfrequencies of fluctuations around a kink and the vacuum, respectively. The individual sums as well as their difference are ultraviolet divergent. The latter divergence is removed by the counter-terms obtained by rewriting the bare kink mass \( M_0 \) in terms of renormalized parameters

\[
M_0 = \frac{2\sqrt{2}}{3} (v_0^3)^{3/2} \lambda_0^{1/2} = \frac{m_0^3}{3\lambda_0} = \frac{m^3}{3\lambda} + \delta M
\]

with

\[
\delta M = m \delta v^2 + \frac{m^3}{6\lambda^2} \delta \lambda \equiv \delta_v M + \delta_\lambda M
\]

or, equivalently, by evaluating the counter-terms to the potential as given by (4) in the kink background

\[
\delta M = \frac{\lambda}{2} \delta v^2 \int_{-\infty}^\infty \! \! dx \left[ \varphi_k^2(x) - \frac{m^2}{2\lambda} \right] + \frac{\delta \lambda}{4} \int_{-\infty}^\infty \! \! dx \left[ \varphi_k^2(x) - \frac{m^2}{2\lambda} \right]^2.
\]
regularization procedures by a finite amount. In fact, it has been shown that cut-off regularization can be repaired by using smooth cut-offs [30] which are, in fact, also required in the calculation of Casimir energies in order that sums over zero-point energies there can be evaluated by means of the Euler–McLaurin formula [31]. The limit of a sharp cut-off differs from a straightforward sharp cut-off by a delta-function peak in the spectral density at the integration boundary which must not be omitted. A completely different procedure using sharp cut-offs which depend on the coordinate $x$ has recently been proposed in [32], and independently in [16]. This ‘local mode regularization’ has been used in [32] to calculate the local distribution of the quantum energies of $(1 + 1)$-dimensional solitons.

In the following, we shall, however, employ dimensional regularization, which has been shown in [29] to correctly reproduce the quantum mass of the bosonic $(1 + 1)$-dimensional kink, and also consider the higher-dimensional kink domain walls†. By analytic continuation of the number $s$ of extra transverse dimensions of a kink domain wall, no further regularization is needed. In the vacuum this is indeed consistent with standard (isotropic) dimensional regularization over $s + 1$ spatial dimensions, as its formulae continue to apply if one first integrates over a subset of dimensions.

Denoting the momenta pertaining to the $s$ transverse dimensions by $\ell$ and reserving $k$ for the momentum along the kink, i.e. perpendicular to the kink domain wall, the energy of the latter per transverse volume $L^s$ follows from (9)

$$
\frac{M^{(1)}}{L^s} = \frac{m^3}{3\lambda} + \frac{1}{2} \sum_B \int_{-\infty}^{\infty} \frac{d^s \ell}{(2\pi)^s} \sqrt{\omega_B^2 + \ell^2} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{s+1}} \sqrt{k^2 + \ell^2 + m^2 \delta'_K(k)} + \delta M
$$

(13)

where the discrete sum is over the normalizable states $B$ of the $(1 + 1)$-dimensional kink with energy $\omega_B$, and the integral is over the continuum part of the spectrum.

The spectrum of fluctuations for the $(1 + 1)$-dimensional kink is known exactly [1]. It consists of a zero mode, a bound state with energy $\omega_B^2/m^2 = 3/4$ and scattering states in a reflectionless potential for which the phase shift $\delta_K(k) = -2 \arctan(3mk/(m^2 - k^2))$ in the kink background provides the difference in the spectral density between the kink and trivial vacuum

$$
\int_{-\infty}^{\infty} dx (|\phi_k(x)|^2 - 1) = \delta'_K(k) = -\frac{3m^2}{2} \frac{2k^2 + m^2}{(k^2 + m^2)(k^2 + m^2/4)}.
$$

(14)

The zero mode ($\omega_B = 0$), which trivially does not contribute to the mass of a kink because of its vanishing energy, corresponds to a massless mode with energy $\sqrt{k^2}$ for $s \neq 0$, but does also not contribute to the energy densities of kink domain walls in dimensional regularization, because in the latter integrals without a mass scale vanish. However, it does contribute in cut-off regularization, as we shall discuss further in what follows.

The leading divergence in the last integral of (13) matches the divergence in $\delta \lambda M$ and can be combined with it using (7) to give (with $x \equiv k/m$)

$$
\frac{M^{(1)}}{L^s} = \frac{m^3}{3\lambda} + \frac{\Gamma(-1-s)}{\Gamma(-\frac{1}{2}) (4\pi)^{\frac{s}{2}}} \left\{ \frac{1}{2} \left( \frac{3}{4} \right)^{\frac{s+1}{2}} + \frac{3}{4\pi} \int_{-\infty}^{\infty} dx (x^2 + 1) \left( \frac{s+1}{2} \right) \left[ \frac{-1}{4x^2 + 1} + s \right] \right\} + \delta \lambda M.
$$

(15)

† Dimensional regularization adapted to domain-wall configurations was, in fact, discussed a long time ago in [33], however, without giving concrete results for the surface tension, and in [34, 35] in combination with zeta-function techniques.
Here the first term inside the braces is the contribution from the bound state with non-zero energy.

In the limit $s \to 0$, which corresponds to the $(1 + 1)$-dimensional kink, where one may renormalize ‘minimally’ by putting $\delta \lambda M = 0$, one obtains

$$\Delta M^{(1)}_{s=0} \equiv M^{(1)}_{s=0} - \frac{m^3}{3\lambda} = \frac{m}{2\sqrt{3}} - \frac{3m}{2\pi},$$

reproducing the well known DHN result [2]. It is interesting to note that it is the last term in (16) that would be missed in a sharp-cut-off calculation (see [7]) and that now comes from the last term in the square brackets of (15). The latter arises because the counter-term due to $\delta v^2$ no longer matches all of the divergences of the integrals involving $\delta K$ for $s > 0$, but dimensional regularization gives a finite result as $s \to 0$.

In energy cut-off regularization this term can be recovered by implementing the cut-off as $\delta K(k) \to \delta K(k)\theta(\Lambda - k)$ which gives a Dirac-delta in the spectral density by differentiating $\theta$ [30] and a finite contribution because the scattering phase $\delta K(k)$ decays only as $1/k$ at large momenta. The need for such subtle corrections is nicely avoided by dimensional regularization: for sufficiently negative transverse dimensionality $s$ the ultraviolet behaviour of the scattering phases in the longitudinal direction is made harmless.

For $s = 1, 2, 3$, the integral in (15) is divergent and gives poles in dimensional regularization, but as the final results will show, these divergences are cancelled by the other terms in (15): for $s = 1, 3$, they come from the bound state contribution, whereas for $s = 2$, they are provided by $\delta \lambda M$.

However, naive cut-off regularization would give rise to problems which in fact point to the necessity of its modification as in [30]. In contrast to dimensional regularization, cut-off regularization leads to singularities for linear and quadratic divergences. Let us consider as an example the $(2 + 1)$ case, i.e. $s = 1$. Using a sharp cut-off in the $k$-integral of (13) and $\delta M = \delta v M$, one can combine these integrals yielding

$$\frac{M^{(1)}_{s=1}}{L} = \frac{m^3}{3\lambda} + \int_{-\infty}^{\infty} \frac{d\ell}{2\pi} \left\{ \frac{1}{2} \sqrt{\ell^2} + \frac{1}{2} \sqrt{\ell^2 + 3m^2/4} \right\} - \frac{1}{\pi} \left[ \sqrt{\ell^2 / m^2 + \sqrt{\ell^2 + 3m^2/4}} \arctan \left( \sqrt{3 + 4\ell^2/m^2} \right) \right].$$

In this expression, the quadratic divergences cancel (for which it is necessary that the kink zero mode is not omitted!) but because $\arctan(x) = \pi/2 - 1/x + O(1/x^2)$ for large $x$ the terms in the square bracket also contain linear divergences that do not cancel. However, if the $k$-integral in (13) is evaluated with a cut-off that is obtained from a smooth cut-off through a limiting procedure, the Dirac-delta peak in the spectral density [30] contributes the additional term

$$\lim_{\Lambda_k \to \infty} \int_{-\infty}^{\infty} \frac{d\ell}{2\pi} \frac{-3m\sqrt{A_k^2 + \ell^2 + m^2}}{2\pi \Lambda_k}$$

where we have used $\delta K(\Lambda_k) \sim 3m/\Lambda_k$. This renders the complete result finite, and equal to that obtained in dimensional regularization.

Our study of domain walls thus resolves the ambiguities previously found in the calculation of the kink mass. Finite ambiguities in $(1 + 1)$ dimensions become divergences in $(d + 1)$ dimensions with $d > 1$. Requiring finiteness in $(d + 1)$ dimensions fixes the finite ambiguities in $(1 + 1)$ dimensions.

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Table 1. Special cases of $2F_1$ in (19) for the values $s$ of physical interest.

| $s$ | $2F_1\left(\frac{2-s}{2}, \frac{1}{2}; \frac{3}{2}; -\kappa\right)$ |
|-----|------------------------------------------------------------------|
| 0   | $\arctan(\sqrt{\kappa})/\sqrt{\kappa}$                         |
| 1   | $\text{Arsinh}(\sqrt{\kappa})/\sqrt{\kappa}$                   |
| 2   | 1                                                                 |
| 3   | $\frac{1}{2}[\sqrt{1+\kappa} + \text{Arsinh}(\sqrt{\kappa})/\sqrt{\kappa}]$ |

2.2. The surface tension of bosonic kink domain walls

For $d > 1$, it is straightforward to extract the finite answers for the one-loop surface tensions of the bosonic kink domain walls by expanding $s$ around integer values, which leads to elementary integrals. But instead of giving these individual results, some of which have been obtained previously, we shall aim at covering them all together.

2.2.1. Renormalization schemes with $Z = 1$. First we shall consider renormalization schemes where the wavefunction renormalization constant is kept at $Z = 1$ so that $\varphi = \varphi_0$, which is a valid and convenient choice at all loop orders for $s < 2$ and to one-loop order for $s = 2$.

For general non-integer $s$, the integral in (15) can be expressed in terms of the same hypergeometric function that appeared in the counter-term $\delta\lambda$, equation (8), which was chosen so as to let $m$ coincide with the physical pole mass of the elementary scalar bosons. This leads to the following remarkably compact formula for the energy densities of $s$-dimensional bosonic kink domain walls:

$$\frac{\Delta M^{(4)}_{OS}}{L^s} = \frac{m^{s+1}}{(4\pi)^{\frac{s+2}{2}}} \left\{(s + 2) \left(\frac{3}{4}\right)^{\frac{s}{2}} 2F_1\left(\frac{2-s}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{3}\right) - 3\right\},$$

where $m$ is the physical (pole) mass of the elementary scalar, and the term proportional to $-3$ is produced by the term proportional to $s$ in (15). This is a finite expression for $-1 < s < 4$. (The minimal renormalization (MR) scheme where $Z_\lambda = 1$, which is only possible for $s < 2$, is obtained by replacing $(2 + s)$ in the first term by 1.)

For the integer values of $s$ of physical interest, the hypergeometric function in (19) can be reduced to the elementary functions given in Table 1.

In the $(3 + 1)$-dimensional case, one has $2F_1(0, \ldots) \equiv 1$, giving a zero for the content of the braces in equation (19), but multiplying a pole of the gamma function. Here one has to expand around $s = 2$, for which one needs the following, easily derivable relation:

$$\lim_{\epsilon \to 0} \Gamma(\epsilon) \left[2F_1\left(\epsilon, \frac{1}{2}; \frac{3}{2}; -\kappa\right) - 1\right] = \sum_{n=1}^{\infty} \frac{(-\kappa)^n}{n(2n+1)} = -\ln(1 + \kappa) - \frac{2}{\sqrt{\kappa}} \arctan(\sqrt{\kappa}) + 2. \quad (20)$$

The numerical results for $s = 0, 1, 2, 3$ following from (19) are given in Table 2 for both the physical OS renormalization scheme and, where applicable, the minimal one with $\delta\lambda = 0$ (MR).

† Using, for example, formula (3.259.3) of [36] together with the linear transformation formulae (9.131), (9.132).
Table 2. One-loop contributions to the quantum mass of the bosonic kink \((s = 0)\) and to the surface tension of \(s\)-dimensional domain walls for the OS, and MR schemes, both with wavefunction renormalization \(Z = 1\).

\[
\Delta M^{(1)}
\]

\[
\frac{1}{L^s} \frac{m^{s+1}}{

| s | (OS) | (MR) |
|---|---|
| 0 | \(\frac{1}{2\sqrt{3}} - \frac{3}{2\pi} \approx -0.189\) | \(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \approx -0.333\) |
| 1 | \(\frac{3}{32\pi} (3\ln 3 - 4) \approx -0.0210\) | \(\frac{3}{32\pi} (\ln 3 - 4) \approx -0.0866\) |
| 2 | \(\frac{3}{16\pi^2} - \frac{1}{8\pi\sqrt{3}} \approx -0.00397\) | — |
| 3 | \(\frac{9(4 - 5\ln 3)}{(32\pi)^2} \approx -0.00133\) | — |

2.2.2. Renormalization schemes with non-trivial \(Z\). Because the kink represents a stationary point of the action, a non-trivial wavefunction renormalization \(\varphi_0 = \sqrt{Z} \varphi\) does not introduce additional terms in (12), and correspondingly leaves the form of the counter-terms in (10) and (11) unchanged (see the remarks at the end of section 5.4 in [1]). It does, however, modify the values of \(\delta v^2\) and \(\delta \lambda\) in these expressions, and thus changes the numerical result for \(\Delta M^{(1)}\).

In the OS scheme with a non-trivial \(Z = 1 + \delta Z\), the equation defining \(\delta v^2\) is obtained by replacing \(\delta v^2 \rightarrow \delta v^2 - v^2 \delta Z\) in the left-hand side of (5) and the equation defining \(\delta \lambda\) by the substitution \(\delta \lambda \rightarrow \delta \lambda + \lambda \delta Z\) in the left-hand side of (8)†. For any \(\delta Z\) these replacements in the OS scheme preserve the relation \(\lambda = m^2/(2v^2)\), but with the definition of \(m\) fixed, that changes the coupling appearing in the classical expression \(M_{cl} = m^3/(3\lambda)\) according to \(\lambda = \lambda|_{Z=1}(1 - \delta Z)\). The extra contribution to \(\Delta M^{(1)}\) is therefore simply \(+M_{cl}\delta Z\).

A natural refinement of the OS scheme, where \(m\) is given by the physical (pole) mass, is to require that the residue of this pole be unity. This leads to

\[
\delta Z = -9\lambda \frac{m^{s-2}}{(4\pi)^{s+2}} \Gamma\left(\frac{4 - s}{2}\right) \int_0^1 dx x(1 - x)[1 - x(1 - x)]^{s-1} = 9\lambda \frac{m^{s-2}}{(4\pi)^{s+2}} \Gamma\left(\frac{4 - s}{2}\right) \left[\frac{3}{4}\right]^{s-2} 2F_1\left(\frac{2 - s}{2}; \frac{1}{2}; \frac{3}{4}; \frac{1}{3}\right) - \frac{4}{3} 2F_1\left(\frac{4 - s}{2}; \frac{1}{2}; \frac{3}{2}; -\frac{1}{3}\right)
\]

(21)

Curiously enough, with the help of the Gauss recursion relations [36] the particular combination of hypergeometric functions in this expression can be recast in a form proportional to (19),

\[
\delta Z = 2\lambda \frac{m^{s-2}}{(4\pi)^{s+2}} \Gamma\left(\frac{2 - s}{2}\right) \left\{(s + 2)\left(\frac{3}{4}\right)\right\}^\frac{s-2}{2} 2F_1\left(\frac{2 - s}{2}; \frac{1}{2}; \frac{3}{2}; -\frac{1}{3}\right) - 3\}
\]

(22)

† In the latter case there is a contribution proportional to \(\delta Z\) from the kinetic term, while the seagull graph now cancels against a counter-term with \(\delta v^2 - v^2 \delta Z\) instead of a counter-term with only \(\delta v^2\).
Table 3. One-loop contributions to the quantum mass of the bosonic kink \((s = 0)\) and to the surface tension of \(s\)-dimensional domain walls for the OS scheme with normalized residue (OSR) and the ZM scheme.

| \(s\) | \(\Delta M^{(1)}(\text{OSR})/L^s/m^{s+1}\) | \(\Delta M^{(1)}(\text{ZM})/L^s/m^{s+1}\) |
|---|---|---|
| 0 | \(\frac{2}{3\sqrt{3}} - \frac{2}{\pi} \approx -0.252\) | \(\frac{1}{3\sqrt{3}} - \frac{19}{16\pi} \approx -0.234\) |
| 1 | \(\frac{5}{32\pi}(3\ln 3 - 4) \approx -0.0350\) | \(-\frac{1}{64\pi^2} - \frac{1}{32\pi^3} \approx -0.00733\) |
| 2 | \(\frac{3}{8\pi^2} - \frac{1}{4\pi\sqrt{3}} \approx -0.00795\) | \(-\frac{3}{32\pi^2} \approx -0.00296\) |
| 3 | \(\frac{21(4 - 5\ln 3)}{(32\pi)^2} \approx -0.00310\) | \(-\frac{20 - 9\ln 3}{(32\pi)^2} \approx -0.00296\) |

The energy densities of kink domain walls in an OS renormalization scheme with physical pole mass and unit residue (OSR) are thus given by the simple conversion formula

\[
\frac{\Delta M^{(1)}_{\text{OSR}}}{L^s} = \frac{\Delta M^{(1)}_{\text{OS}}}{L^s} + \frac{m^3}{3\lambda} \delta_Z = \frac{s + 4 \Delta M^{(1)}_{\text{OS}}}{3L^s}
\]

and the particular results for the values \(s\) of interest are listed in table 3.

For the sake of comparison with previous results in the literature, table 3 also includes another widely used renormalization scheme [37], where the mass is renormalized at zero momentum (ZM) according to \(m^2_{\text{ZM}} = \Gamma^{(1)}(0)\) with \(\Gamma^{(1)}(k^2)\) the inverse propagator to one-loop order and \(\delta_Z\) is chosen such that \([\partial \Gamma^{(1)}/\partial k^2](0) = 1\). In this scheme, formula (19) is replaced by

\[
\frac{\Delta M^{(1)}_{\text{ZM}}}{L^s} = \frac{m^{s+1}}{(4\pi)^\frac{s+2}{2}} \frac{2\Gamma(\frac{s+1}{2})}{s + 1} \left\{ \left( \frac{3}{4} \right)^\frac{s}{2} \, _2F_1\left( \frac{2 - s}{2}, \frac{1}{2}; \frac{3}{2}, \frac{1}{3} \right) + \frac{3}{4}(s - 3) - \frac{1}{16}(s + 1)(2 - s) \right\},
\]

where the very last term within the braces is the contribution of \(\delta_Z\).

The surface tension of \(\varphi^4\) domain walls has been calculated in the ZM scheme to one-loop order in \((3 + 1)\) dimensions in [34] by considering the energy splitting of the two lowest states in a finite volume using zeta-function techniques, and our result completely agrees with that. Our result is also consistent with the older [38] result using \(\epsilon\)-expansion (in the limit \(\epsilon \to 0\)) which employed yet another renormalization scheme that is closer (but not identical) to an \(\overline{\text{MS}}\) scheme. We do not, however, agree with the ZM-scheme result reported in [39] nor with its correction in [40].†

In \((2 + 1)\) dimensions, the surface tension of the kink domain wall has been calculated to one-loop order in [35], and in [41] to two-loop order in the ZM scheme. Our one-loop ZM result reproduces that given in [41], while the one-loop result of [35] cannot be directly

† The latter reports the same result as that contained in [38] (for \(\epsilon \to 0\)), while formulating different renormalization conditions amounting to the ZM scheme at one-loop order.
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compared with ours as it re-expresses the ZM result in terms of the physical pole mass without using the coupling of either our OS or OSR schemes. We also agree with the most recent work [42], where the \((2+1)\)-dimensional kink domain-wall energy density was calculated using the Born approximation methodology of [23, 43] in the MR scheme. Compared to [42], the present calculation in dimensional regularization turns out to be considerably simpler and more straightforward, as the former has to exert some care in identifying ‘half-bound’ states and to employ certain non-trivial sum rules for phase shifts. On the other hand, the methods presented in [23, 43] will also be useful in cases where one can only determine phase shifts numerically.

Finally, comparing the size of the one-loop corrections in the four different renormalization schemes considered in tables 2 and 3, one notices that the corrections are largest in the MR scheme and significantly smaller in the other schemes, with the ZM and OSR results being rather close, but with noticeable differences.

These issues are of relevance in practical applications, and, indeed, the surface tension of the \(\phi^4\) kink domain wall can be related to universal quantities that can be investigated by lattice simulations of the Ising model and experimentally in binary mixtures [41]. In a comparison of the field-theoretic results with lattice studies, the different definitions of mass in the OS and in the ZM schemes correspond to the true (exponential) correlation length and to the second moment of the correlation function, respectively, both of which can be found in the literature (see, e.g. [44] and references therein).

Of perhaps mere academic interest is the case of kink domain walls in five dimensions \((s = 3)\) where our formulae still give finite results. In five dimensions, \(\phi^4\) theory is of course no longer renormalizable, though it may still be of interest as an effective theory.

3. Supersymmetric kink and domain walls

3.1. The SUSY kink and domain string

In \((1+1)\) and \((2+1)\) dimensions \((s = 0\) and \(1)\), the model (1) has the SUSY extension [45, 46]

\[
\mathcal{L} = -\frac{1}{2}[(\partial_\mu \phi)^2 + U(\phi)^2] + \bar{\psi} \gamma^\mu \partial_\mu \psi + U'(\phi) \bar{\psi} \psi
\]

(25)

where \(\psi\) is a Majorana spinor, \(\bar{\psi} = \psi^T \gamma^0\) and

\[
U(\phi) = \sqrt{\lambda_0 / 2} (\phi^2 - v_0^2), \quad v_0^2 \equiv \mu_0^2 / \lambda_0.
\]

(26)

(In \((1+1)\) dimensions, \(U \propto \sin(\sqrt{\gamma} \phi / 2)\) gives the sine-Gordon model, which is, however, not renormalizable in \((2+1)\) dimensions.)

Imbedding the SUSY kink in \((2+1)\) dimensions gives a domain wall centred about a one-dimensional string on which the fermion mass vanishes (since \(U'(\phi_K) \propto \phi_K\) vanishes at the centre of the kink). In the following we shall succinctly refer to this particular domain wall as the ‘domain string’, postponing a brief discussion of higher-dimensional kink domain walls to the next section.

Going from \((1+1)\) to \((2+1)\) dimensions, the discrete symmetry content of (25) in fact changes. In \((1+1)\) dimensions, (25) has the \(Z_2\) symmetry \(\phi \rightarrow -\phi, \psi \rightarrow \gamma^5 \psi\) with \(\gamma^5 = \gamma^0 \gamma^1\).

In \((2+1)\) dimensions, on the other hand, \(\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \propto \pm 1\), and the sign of the fermion mass term can no longer be reversed by \(\psi \rightarrow \gamma^5 \psi\). By the same token, (25) breaks parity, because a sign change of one of the spatial \(\gamma\) matrices cannot be effected by an equivalence transformation,
but leads to the other of the two inequivalent irreducible representations of a Clifford algebra in odd space-time dimensions.

In what follows we shall consider the quantum corrections to both the mass of the SUSY kink and the tension of the domain string, together. In both cases we shall continue to use a renormalization scheme where we put $Z_q = 1 = Z_v$ at one-loop order. For this reason we have already dropped a subscript 0 for the unrenormalized fields in (25). We shall, however, consider the possibility of (finite) coupling constant renormalization, again by requiring that the renormalized mass of elementary scalars and fermions be given by the physical pole mass, together with the requirement of vanishing tadpoles, which fixes $\delta \nu^2$.

Inclusion of the fermionic tadpole loop replaces 3 by $(3 - 2)$ in (5) so that compared to the bosonic result we have

$$\delta \nu^2|_{SUSY} \equiv \delta \tilde{\nu}^2 = \frac{1}{3} \delta \nu^2|_{\text{bos.}}.$$  

(When useful we distinguish quantities in the SUSY case by tildes.)

In the OS scheme, the SUSY version of (8) is obtained by the replacement $\delta \nu^2|_{SUSY} \equiv \delta \tilde{\nu}^2 = \frac{1}{3} \delta \nu^2|_{\text{bos.}}$.

$$9m^2 \to 9m^2 - 2(2m^2 + \frac{1}{2}g^2)|_{q^2 = -m^2} = 6m^2,$$

and thus

$$\delta \tilde{\lambda} = \frac{2}{3} \delta \lambda|_{\text{bos.}}.$$  

In a Majorana representation of the Dirac matrices in terms of the usual Pauli matrices $\tau^k$ with $\gamma^0 = -i\tau^2, \gamma^1 = \tau^3, \gamma^2 = \tau^1$ (added for $s = 1$) and $C = \tau^2$ so that $\psi = (\psi^+ \psi^-)$ with real $\psi^+(x, t)$ and $\psi^-(x, t)$, the equations for the bosonic and fermionic normal modes with frequency $\omega$ and longitudinal momentum $\ell$ (non-zero only when $s = 1$) in the kink background $\varphi = \varphi_K$ read

$$[-\partial_x^2 + U'' + UU''']\eta = (\omega^2 - \ell^2)\eta,$$

$$\left(\partial_x + U\right)\psi^+ + i(\omega + \ell)\psi^- = 0, \quad (28)$$

$$\left(\partial_x - U\right)\psi^- + i(\omega - \ell)\psi^+ = 0. \quad (29)$$

Acting with $(\partial_x - U')$ on (28) and eliminating $\psi^-$ as well as $\varphi' = -U$ shows that $\psi^+$ satisfies the same equation as the bosonic fluctuation $\eta$. Compared to $\psi^+$, the component $\psi^-$ has a continuous spectrum whose modes differ by an additional phase shift $\theta = -2\arctan(m/k)$ when traversing the kink from $x_1 = -\infty$ to $+\infty$, which is determined only by $U'(\varphi_K(x_1 = \pm \infty)) = \pm m$. Correspondingly, the difference of the spectral densities of the $\psi^+$-fluctuations in the kink and in the trivial vacuum equals that of the $\eta$-fluctuations, given in (14), whereas that of $\psi^-$-fluctuations is obtained by replacing $\delta \kappa \to \delta \kappa + \theta'$.

In the sum over zero-point energies for the one-loop quantum mass of the kink (when $s = 0$),

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_d + \frac{1}{2} \left(\sum \omega_B - \sum \omega_B'\right) - \frac{1}{2} \left(\sum \omega'_F - \sum \omega'_F\right) + \delta \tilde{\mathcal{M}}, \quad (30)$$

one thus finds that the bosonic contributions from the continuous spectrum are cancelled by the fermionic contributions except for the additional contribution involving $\theta'(k)$ in the spectral density of the $\psi^-$ modes.

† The counter-term $\frac{1}{2} \lambda \delta \nu^2 \nu^2$ induced by the tadpole with a fermionic loop cancels only those contributions to the bosonic self-energy due to a fermionic loop which contain one propagator. The remaining contributions have two propagators and are proportional to the bosonic contribution to the self-energy.
The discrete bound states cancel exactly, apart from the subtlety that the fermionic zero mode should be counted as half a fermionic mode [15]. In strictly \((1 + 1)\) dimensions, the zero modes do not contribute simply because they carry zero energy, and for \(s > 0\), where they become massless modes, they do not contribute in dimensional regularization.

In a cut-off regularization in \(s = 1\), as we have already discussed and shall further discuss in the following, they in fact do play a role. Remarkably, the half-counting of the fermionic zero mode for \(s = 0\) has an analogue for \(s = 1\) where the bosonic and fermionic zero modes of the kink correspond to massless modes with energy \(|\omega| = |\ell|\). From (28) and (29) one finds that the fermionic kink-zero mode \(\psi^+ \propto \varphi_K', \psi^- = 0\) is a solution only for \(\omega = +\ell\). It therefore cancels only half of the contributions from the bosonic kink-zero mode which for \(s = 1\) have \(\omega = \pm \ell\). For \(s = 1\) one thus finds that the fermionic zero mode of the kink corresponds to a chiral (Majorana–Weyl) fermion on the \((s = 1)\)-dimensional domain string [47, 48, 49]†.

In dimensional regularization, however, the kink zero modes and their massless counterparts for \(s > 0\) can be dropped, and the energy density of the SUSY domain wall reads

\[
\frac{\tilde{M}^{(1)}}{L^s} = \frac{m^3}{3\lambda} - \frac{1}{4} \int \frac{dk \, d^s \ell}{(2\pi)^{s+1}} \sqrt{k^2 + \ell^2 + m^2 \theta'(k)} + \delta \tilde{M},
\]

where

\[
\theta'(k) = \frac{2m}{k^2 + m^2}.
\]

With \(\delta_v \tilde{M} = \frac{1}{2} \delta_v M\) the logarithmic divergence in the integral in (31) as \(s \to 0\) gets cancelled. A naive cut-off regularization at \(s = 0\) would actually lead to a total cancellation of the \(k\)-integral with the counter-term \(\delta_v \tilde{M}\), giving a vanishing quantum correction in renormalization schemes with \(\lambda = \lambda_0\). In dimensional regularization there is now, however, a mismatch for \(s \neq 0\) and a finite remainder in the limit \(s \to 0\) proportional to \(s \Gamma(-s/2)\). Including the optional \(\lambda\)-renormalization the final result reads

\[
\frac{\tilde{M}^{(1)}}{L^s} = \frac{m^3}{3\lambda} - \frac{m^{s+1}}{(4\pi)^{s+1/2}} \frac{2\Gamma(\frac{2-s}{2})}{s + 1} + \delta_\lambda \tilde{M}.
\]

In the MR scheme one has \(\delta_\lambda \tilde{M} = 0\), whereas in the more physical OS scheme, where \(m\) is the pole mass of the elementary bosons as well as fermions, one has \(\delta_\lambda \tilde{M} = \frac{2}{3} \delta_\lambda M\), yielding

\[
\frac{\Delta \tilde{M}^{(1)}}{L^s} = \frac{m^{s+1} \Gamma(\frac{2-s}{2})}{(4\pi)^{s+1/2}} \left\{ \left( \frac{3}{4} \right)^{\frac{s^2}{2}} 2F_1\left( \frac{2-s}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{3}{4} \right) - \frac{2}{s+1} \right\}.
\]

The respective results for the \((1 + 1)\)-dimensional SUSY kink \((s = 0)\) and for the \((s = 1)\)-dimensional SUSY kink domain ‘wall’ (domain string) are given in table 4(a); for completeness, the corresponding results for the OSR and ZM schemes are given in table 4(b). Again we find that there is much faster apparent convergence in the OS scheme compared to the MR one where only the tadpoles are subtracted, with the OSR and ZM results lying in between.

† Choosing a different sign for \(\gamma_1\) reverses the allowed sign of \(\ell\) for these fermionic modes and thus their chirality (with respect to the domain-string world sheet). This corresponds to the other, inequivalent representation of the Clifford algebra in \((2 + 1)\) dimensions.
Table 4. (a) One-loop contributions to the quantum mass of the SUSY kink \((s = 0)\) and to the surface tension of the \((s = 1)\)-dimensional SUSY kink domain ‘wall’ for the OS and MR schemes. (b) The same for the OS mass scheme with normalized residue (OSR) and the ZM scheme.

| \(s\) | \(\frac{\Delta M^{(1)}}{L^s} / m^{s+1}\) |
|------|----------------------------------|
| (a)  | \(\text{(OS)}\) | \(\text{(MR)}\) |
| 0    | \(\frac{1}{6\sqrt{3}} - \frac{1}{2\pi} \approx -0.063\) | \(-\frac{1}{2\pi} \approx -0.159\) |
| 1    | \(\frac{1}{8\pi} (\ln 3 - 1) \approx +0.004\) | \(-\frac{1}{8\pi} \approx -0.040\) |
| (b)  | \(\text{(OSR)}\) | \(\text{(ZM)}\) |
| 0    | \(\frac{2}{9\sqrt{3}} - \frac{5}{6\pi} \approx -0.137\) | \(-\frac{53}{2\pi 72} \approx -0.117\) |
| 1    | \(\frac{1}{24\pi} (5\ln 3 - 7) \approx -0.020\) | \(-\frac{129}{8\pi 72} \approx -0.016\) |

In the literature, to the best of our knowledge, only the case of a SUSY kink \((s = 0)\) in the MR scheme has been considered and dimensional regularization reproduces the result obtained previously by \([13, 14, 19, 21, 25]\). However, a (larger) number of papers have missed the \(-m/(2\pi)\) contribution because of the (mostly implicit) use of the inconsistent energy cut-off scheme \([8]–[12]\) or have obtained different answers because of the use of boundary conditions that accumulate a finite amount of energy at the boundaries \([7, 50]\). The former result is, however, now generally accepted and, in the case of the super-sine-Gordon model (where the same issues arise with the same results) is in agreement with \(S\)-matrix factorization \([18]\).

In \([30]\) the correct SUSY kink mass has also been obtained by employing a smooth energy (momentum) cut-off, the necessity of which becomes apparent, as in the purely bosonic case, by considering the \((2 + 1)\)-dimensional domain wall. Using a naive cut-off for \(s = 1\) one finds quadratic divergences which cancel only upon inclusion of the zero modes (which become massless modes in \((2 + 1)\) dimensions). As we have discussed above, unlike the other bound states, these modes do not cancel because the fermionic zero mode becomes a chiral fermion on the domain-string world sheet and thus cancels only half of the bosonic zero (massless) mode contribution, yielding

\[
\int_{\Lambda_k} \int_{0}^{\infty} \frac{d\ell}{2\pi} \left\{ \frac{1}{2} \sqrt{\ell^2 - \int_{-\Lambda_k}^{\Lambda_k} \frac{dk}{2\pi} \left[ \sqrt{k^2 + \ell^2 + m^2} - \frac{m}{\sqrt{k^2 + \ell^2 + m^2}} \right] - \frac{1}{\sqrt{k^2 + \ell^2 + m^2}}} \right\} \left\{ \frac{1}{2} \sqrt{\ell^2 - \int_{0}^{\infty} \frac{d\ell}{2\pi} \left[ \frac{\ell}{2 \pi \arctan \frac{\ell}{m}} \right]} \right\} \sim \int_{0}^{\infty} \frac{d\ell}{2\pi} \frac{m}{\ell^2 + m^2} \tag{35}
\]

\[\]
which is, however, still linearly divergent. Smoothing out the cut-off in the \( k \)-integral does pick an additional (and for \( s = 0 \) the only) contribution \(-m/(2\pi)\), which is now necessary to have a finite result for \( s = 1 \). This finite result then reads

\[
\frac{M_{s=1}^{(1)}}{L} = -\frac{1}{\pi} \int_0^\infty \frac{d\ell}{2\pi} \left( m - \ell \arctan \frac{m}{\ell} \right) = -\frac{m^2}{8\pi} \tag{36}
\]

in agreement with the result obtained above in dimensional regularization.

### 3.2. SUSY kink domain walls in \((3+1)\) dimensions

For completeness we shall also briefly discuss kink domain walls in the \((3+1)\)-dimensional Wess–Zumino-model [51]. In accordance with [52, 53] we shall demonstrate that in this model there is no non-trivial quantum correction to the surface tension.

A Wess–Zumino model with a spontaneously broken \( Z_2 \) symmetry now requires two real scalar fields to pair up with the now four-component Majorana spinor. For the classical Lagrangian we choose

\[
\mathcal{L} = -\frac{1}{2} (\partial A)^2 - \frac{1}{2} (\partial B)^2 - V(A, B) - \frac{1}{2} \bar{\psi} \left[ \partial + \sqrt{2} \lambda (A + i\gamma_5 B) \right] \psi
\]

\[
V(A, B) = \frac{\lambda}{4} (A^2 - B^2 - \nu^2)^2 + \lambda A^2 B^2, \tag{37}
\]

where \( A \) is a real scalar with non-vanishing vacuum expectation value, while \( B \) is a real pseudoscalar without one. For \( B = 0 \) the potential coincides with that of the kink model (1), and correspondingly a classical domain-wall solution is given by \( A_K(x) = \varphi(x_1) \) and all other fields zero.

As is well known [54, 55], in the \((3+1)\)-dimensional Wess–Zumino model there is only one non-trivial renormalization constant \( Z \) for the kinetic term, which implies \( \mu^2 = Z \mu_0^2 \) and \( \lambda = Z^{3/2} \lambda_0 \) and thus a vanishing counter-term \( \delta M \) for the kink wall energy density.

The fluctuation equations for \( \eta = A - A_K \), \( B \), and \( \psi \) read

\[
\partial^2 \eta - (U'^2 + UU'') \eta = 0
\]

\[
\partial^2 B - (\lambda A_K^2 + \mu^2) B = 0
\]

\[
[\partial + U'] \psi = 0, \tag{38}
\]

with \( U \) as in (26). \( A_K \) satisfies the Bogomolnyi equation \( A_K' = -U(A_K) \), and the \( x \)-dependent parts of the \( \eta \) and \( B \) field equations factorize as \(-\partial_x - U'(\partial_x + U')\) and \(-\partial_x + U'(\partial_x - U')\), respectively.

Both the \( \eta \) and \( B \) fluctuation equations involve reflectionless potentials of the form

\[
- \partial_z^2 - \frac{n(n+1)}{\cosh^2 z} + n^2, \tag{39}
\]

where \( z := \frac{m\sqrt{2}}{2} \).

The kink fluctuation modes \( \phi_k(x) \) correspond to \( n = 2 \), and the \( \eta \) fluctuations are given by the former multiplied by plane waves with momentum \( \ell = (\ell_2, \ell_3) \) in the trivial directions. Their spectrum thus consists of one massless mode and one massive mode localized on the domain wall with \( \omega_0^2(\ell) = \ell^2 \) and \( \omega_B^2(\ell) = \frac{3}{2} m^2 + \ell^2 \) and delocalized ones with \( \omega_k^2(\ell) = k^2 + \ell^2 + m^2 \).
The \(x\)-dependence of the \(B\)-fluctuations, on the other hand, involves the potential (39) with \(n = 1\), like the fluctuation equations for the sine-Gordon soliton, but with different energies according to

\[
\left(-\partial_z^2 - \frac{2}{\cosh^2 z} + 1\right)s(z) = \left[\frac{4}{m^2}(\omega^2 - \ell^2) - 3\right]s(z).
\] (40)

The spectrum of the sine-Gordon system is now shifted by \(\ell^2 + \frac{3}{2}m^2\) so that the sine-Gordon zero mode matches the bound state of the kink, and the continuous parts of the spectrum also coincide. The spectrum of the \(B\)-fluctuations thus equals that of the \(\eta\)-fluctuations apart from the absence of the massless (zero) mode. The spectral densities for the delocalized modes are, however, different and the bosonic contribution to the one-loop surface tension reads

\[
\frac{\Delta^b M^{(1)}}{L^s} = \frac{1}{2} \int \frac{d^s \ell}{(2\pi)^s} \left(\omega_0(\ell) + 2\omega_B(\ell) + \int \frac{dk}{2\pi} \delta_k(\omega)[\delta_k(k) + \delta_{SG}(k)]\right)
\] (41)

where \(s = 2 - \epsilon\).

Choosing the Majorana representation for the Dirac matrices

\[
\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix},
\] (42)

and writing \(\psi\) in terms of two two-component spinors \(e^{-i\omega t + \ell \cdot \mathbf{x}}(\psi_A, \psi_B)\), the fermionic fluctuation equation of (38) becomes

\[
(\partial_x + U')\psi_A + i[\omega + \ell] \psi_B = 0 \quad \text{and} \quad i[\omega - \ell] \psi_A + (\partial_x - U') \psi_B = 0,
\] (43) (44)

where \(\ell = \tau_1 \ell_2 + \tau_3 \ell_3\). Through (43), \(\psi_B\) can be expressed algebraically in terms of \(\psi_A\), except when \(\omega^2 = \ell^2\), and inserting into (44) shows that the latter satisfies the same fluctuation equation as the bosonic fluctuation \(\eta\). Using that \((\partial_x + U')\phi_k = \omega_{Kink}s_k\), one finds that \(\psi_B\) has the same spectrum as the \(B\) fluctuations.

For the massless (zero) mode \((\omega_{Kink} = 0)\) only \((\partial_x + U')\psi_A = 0\) in (43) has a normalizable solution, which is located at the domain wall. The other equation, \((\partial_x - U')\psi_B = 0\), has normalizable solutions only if boundaries for the \(x\)-direction were introduced, and would be localized there.

As a result, the fermionic contribution to the one-loop correction of the domain-wall tension becomes identical to the bosonic one, but with a negative sign,

\[
\frac{\Delta^f \hat{M}^{(1)}}{L^s} = -\frac{\Delta^b \hat{M}^{(1)}}{L^s}.
\] (45)

In perfect agreement with the non-renormalization theorem of the superpotential (which does not apply at the lower dimensions considered above), there is no quantum correction to the classical value of the surface tension of the SUSY kink domain wall in \((3 + 1)\) dimensions.

This cancellation of the quantum corrections can also be linked to the cancellation of quantum corrections to the \(N = 2\) SUSY kink mass \([13, 14]\).

Such a cancellation is also to be expected for \((4 + 1)\)-dimensional SUSY theories with domain walls. In contrast to \((2 + 1)\) dimensions, in \((4 + 1)\) dimensions there are no Majorana fermions, so one needs to extend the SUSY algebra to involve a Dirac fermion. From the point of view of the \((1 + 1)\)-dimensional kink, this will imply \(N = 4\) supersymmetry. On the then four-dimensional domain wall one may have chiral fermions, but as pointed out in \([48]\), these domain-wall fermions necessarily come in pairs containing both chiralities.
4. Conclusion

In this paper we have shown that dimensional regularization allows one to compute the one-loop contributions to the quantum energies of bosonic and SUSY kinks and kink domain walls in a very simple manner. The ambiguities associated with ultraviolet regularization observed in the $(1+1)$-dimensional kinks have been shown to be eliminated by considering their extension to kink domain walls in higher dimensions.

For the bosonic kink domain walls, which are of interest also in the context of condensed matter physics, we have derived a compact $d$-dimensional formula, which reproduces and (mostly) confirms existing results in the literature, and we have also discussed in detail the dependence on particular renormalization schemes.

In the SUSY case, we confirmed previous results in $(1+1)$ and $(3+1)$ dimensions. While in the latter case quantum corrections to the surface tension vanish, we have obtained a non-trivial one-loop correction for a $(2+1)$-dimensional $N=1$ SUSY kink domain wall with chiral domain-wall fermions. The non-trivial quantum corrections to the supersymmetry algebra in the $(1+1)$- and $(2+1)$-dimensional models will be discussed in a forthcoming publication.

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Note added in proof. After this article had been proofread, the author of [40] informed us that he now agrees with our results on the bosonic domain walls. An erratum to [40] will appear in Physical Review D.

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