A New Formulation for HQET

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We proposed a new formulation for heavy quark effective theory (HQET), whose Lagrangian is hermitian and has a manifest reparametrization invariance. As an application, we calculated the semileptonic and nonleptonic inclusive heavy hadron decay rates up to second order mass corrections and found that there are no kinetic energy correction terms.

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I. INTRODUCTION

The heavy quark effective theory (HQET) is an useful tool in studies of heavy quark systems. In infinite heavy quark mass limit, there exists a heavy flavour-spin symmetry of QCD. To derive a HQET theory from QCD, one needs to integrate out the high frequency modes of the heavy quark field to retain the low frequency modes. The high frequency modes with scales larger than two times heavy quark mass can be integrated out by employing the equation of motion (EOM) method or the functional integration (FI) method. The Lagrangians of these two HQET theories contain non-hermitian terms which would lead to imaginary mass corrections. Nevertheless, this problem did not receive too much attentions in literature. We will show in Section II that the non-hermitian terms can be regularized by further integrating out the high frequency modes with scales larger than heavy quark mass.

The heavy quark momentum can be factorized into the heavy quark velocity part and the residual momentum part. Reparameterizing the heavy quark velocity and the residual momentum would not change the effective theory. It implies that the coefficients of mass correction terms in HQET Lagrangian can be fixed by reparameterization. It was noted that the EOM and FI theories have no manifest invariance under reparameterization. Another HQET theory was derived by employing the Foldy-Wouthuysen (FW) transformation. It is unclear that whether the theory is invariant under reparameterization.

The non-hermitian terms and lacking a manifest reparameterization invariance should be avoided for a self-consistent HQET theory. We will show that a new formulation for HQET could simultaneously resolve these flaws. The remainder of this paper is organized as follows. Section II will explain the cause that leads to the non-hermitian terms, and Section III will present the derivation of a hermitian Lagrangian from QCD. In Section IV, we will show that the hermitian Lagrangian is manifestly invariant under reparameterization. Section V devotes to evaluation of mass corrections for inclusive $B$ decays. An Appendix enumerates the mass correction terms up to $O(1/M_Q^3)$.

II. THE NON-HERMITICITIAN TERMS

The happens of non-hermitian terms in EOM and FI theories could be better understood by investigating the non-relativistic reduction of an electron interacting with electromagnetic fields. For simplicity, we consider the case that the EM fields are static. The equation of motion for an electron under static Coulomb potentials takes the form

\[ i\frac{\partial \psi}{\partial t} = \left[ \vec{\alpha} \cdot \vec{\pi} + \beta m + eV \right] \psi, \tag{1} \]

where $V$ represents the Coulomb potential, $m$ denotes the electron mass, $\psi$ is the electron wave function, $(\vec{\alpha})_i = \gamma^0 \gamma^i$ with $i = 1, \cdots, 3$, $\beta = \gamma^0$ and $\vec{\pi} = -i\vec{\nabla}$. In the nonrelativistic limit $E \sim m + p^2/2m$, it is convenient to use the large $\phi$ and small $\chi$ components of $\psi$ defined as

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]

(2)

to recast (1) into two coupled equations

\[ i\frac{\partial \phi}{\partial t} = \vec{\sigma} \cdot \vec{\pi} \chi + eV \phi + m\phi, \]
\[ i\frac{\partial \chi}{\partial t} = \vec{\sigma} \cdot \vec{\pi} \phi + eV \chi - m\chi, \tag{3} \]
where we have employed the representation
\[
\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}.
\]
(4)

As times evolve, the potential contributions are smeared out by the large value of electron mass, \(m\). To avoid this, one may introduce slowly varying functions of times \(\Phi\) and \(X\) defined as
\[
\Phi = e^{imt} \phi, \quad X = e^{imt} \chi,
\]
(5)
whose equations are easily derived
\[
i \frac{\partial \Phi}{\partial t} = \vec{\sigma} \cdot \vec{\pi} X + eV \Phi,
\]
\[
i \frac{\partial X}{\partial t} = \vec{\sigma} \cdot \vec{\pi} \Phi + eVX - 2mX.
\]
(6)

Because that \(eV \ll 2m\), we are legal to approximate \(X\) into
\[
X = \frac{1}{2m + \pi^0} \vec{\sigma} \cdot \vec{\pi} \Phi
\approx \left[ \frac{\vec{\sigma} \cdot \vec{\pi}}{2m} - \frac{\pi^0 \vec{\sigma} \cdot \vec{\pi}}{4m^2} + \cdots \right] \Phi,
\]
(7)
and substitute the expanded \(X\) into the first equation of (6) to obtain
\[
i \frac{\partial \Phi}{\partial t} = eV \Phi + \frac{[\vec{\sigma} \cdot \vec{\pi} V] \vec{\sigma} \cdot \vec{\pi} + V(\vec{\sigma} \cdot \vec{\pi})^2} {4m^2} \Phi,
\]
(8)
where
\[
[\vec{\sigma} \cdot \vec{\pi} V] \vec{\sigma} \cdot \vec{\pi} = \vec{\pi} V \cdot \vec{\pi} + i \vec{\sigma} \cdot (\vec{\pi} V \times \vec{\pi})
\]
(9)

After rewriting (8) into Hamiltonian
\[
H = \Phi^\dagger eV \Phi + \Phi^\dagger \frac{[\vec{\sigma} \cdot \vec{\pi} V] \vec{\sigma} \cdot \vec{\pi} + V(\vec{\sigma} \cdot \vec{\pi})^2} {4m^2} \Phi,
\]
(10)
we note that the Darwin term in the above Hamiltonian
\[
O_D = \frac{e^2}{4m^2} \Phi^\dagger \vec{\pi} V \cdot \vec{\pi} \Phi
\]
(11)
is nonhermitian
\[
O_D^\dagger = \frac{e^2}{4m^2} (\vec{\pi} \Phi^\dagger \cdot \vec{\pi} V) \Phi \neq O_D.
\]
(12)
One can add up \(O_D^\dagger\) and \(O_D\), and divide the sum by 2 and perform integration by parts by ignoring the surface terms to derive a hermitian Darwin term
\[
O_D^R = \frac{e^2}{8m^2} \Phi^\dagger [(\vec{\pi})^2 V] \Phi.
\]
(13)
Or equivalently, one can make use of the renormalized \(\Phi\) with expression
\[
\Phi_{NR} = (1 + \frac{(\vec{\pi})^2}{8m^2} + \cdots) \Phi
\]
(14)
to obtain the regularized Darwin term \(O_D^R\). The regularized Darwin term \(O_D^R\) is in fact the second spatial variations of \(V\) due to the jittery motions of electron with Compton wavelength \(\delta \vec{r} \sim 1/m\)
\[
< V(\vec{r} + \delta \vec{r}) > \approx < V(\vec{r}) > + \frac{1}{6m^2} < (\vec{\pi})^2 V >,
\]
(15)
where the bracket means integration with electron wavefunctions and the first order variational term vanishes due to the assumption that the wave functions are spherically symmetric.

The above example exhibits the cause of the nonhermitian terms. Similarly, we can show the non-hermitian terms in the HQET theories derived by EOM or FI. The equation of motion for the heavy quark field $\psi$ is

$$(iD - MQ)\psi = 0, \quad (22)$$

where $MQ$ denotes the heavy quark mass and $iD$ is the covariant derivative $iD = i\partial - gA^aT^a$. At energies much below than $MQ$ scale, $\psi$ is not an appropriate variable. One invokes field redefinition $Q(x) = \exp(iMQv \cdot x)\psi(x)$ to remove the large phase factor $MQv$ from the wave function. The $v$ variable represents the heavy quark velocity. Rewriting $(18)$ in terms of $Q(x)$ yields

$$(iD - 2MQ\frac{(1 - \gamma)}{2})Q = 0. \quad (17)$$

By imposing condition $v^2 = 1$, one can separate $Q$ into large $h$ and small $H$ components

$$Q = 1 + \frac{\gamma}{2}Q + \frac{1 - \gamma}{2}Q \equiv h + H. \quad (18)$$

Substituting $(18)$ into $(17)$ and multiplying $(1 - \gamma)/2$ from the left of $(17)$ yields

$$H = \frac{1}{2MQ + iD||} (iD_\perp)h \quad (19)$$

with $D|| = v \cdot D$ and $D_\perp = D - \gamma D||$. Using $(18)$ and $(19)$ leads to

$$Q = [1 + \frac{1}{2MQ + iD||}(iD_\perp)]h. \quad (20)$$

Substituting $(20)$ into $(17)$ and expanding it up to $O(1/Q^2)$, one then arrives at

$$iD||h = \left[-\frac{1}{2MQ}{[-D^2_\perp + \frac{1}{2}\sigma \cdot G]} - \frac{1}{4MQ^2} [i\sigma_{\alpha\beta}v_\lambda G^{\alpha\lambda}D^\beta_\perp + iD_\parallel \sigma_{\alpha\beta}G^{\alpha\beta} - iD_\parallel D_\perp^2 + v_\alpha G^{\alpha\beta}D_\perp^\beta] + O(\frac{1}{MQ}) \right]h, \quad (21)$$

where $\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu} + i\sigma^{\mu\nu}$ and $[iD^{\mu}, D^{\nu}] = -iG^{\mu\nu}$ have been used. Note that the Darwin term (the last term in the second line of $(21)$) is nonhermitian. Following the route in previous QED example, we may regularize the Darwin term by renormalized large components $h'$

$$h' = (1 + \frac{1}{8MQ}iD^2_\perp + \ldots)h. \quad (22)$$

The equation of motion for $h'$ thereby takes the form

$$iD||h' = \left[-\frac{1}{2MQ}{[-D^2_\perp + \frac{1}{2}\sigma \cdot G]} + \frac{1}{8MQ^2} [i\sigma_{\alpha\beta}v_\lambda \{D^\alpha_\perp, G^{\beta\lambda}\} + v_\alpha \{D^\beta_\perp, G^{\alpha\beta}\}] + O(\frac{1}{MQ}) \right]h'. \quad (23)$$

The Darwin term in the second line of $(23)$ correspond to the relativistic effects of Zitterbewegungen from the jittery motions of the heavy quark with Compton wavelength $\lambda_Q \approx 1/MQ$. This implies that the large components $h$ still contains high frequency modes whose scales are larger than $MQ$. These high frequency modes should be integrated out to have a low energy effective theory. It concludes that the mass corrections receive two kinds of contributions: the first kind of contributions comes from the integration out of heavy antiquark degrees of freedom, and the second kind of contributions arises from the integration out of the high frequency modes of the heavy quark degrees of freedom. Only both kinds of contributions having been carefully taking into account can result in a hermitian theory. The integration out of the high frequency modes is equivalent to the renormalization for the heavy quark field $(22)$. 


This means that the renormalized field $h'$ contains only low frequency modes with scales not larger than $M_Q$ and is shown responsible for low energy physics. A systematic method, which can derive a hermitian Lagrangian as well as an appropriate effective field, is very intriguing in theory and phenomenology. To develop this method is the main purpose of this paper.

To reveal the eligibility of the unrenormalized large components $h$, we support two examples. The first example is the spin sum of $h$ in free theory. By definition (20), the spin sum takes expression

$$
\sum_{\lambda} h(\lambda) h(\lambda) = \frac{1 + \hat{p}}{2} \sum_{\lambda} Q(\lambda) Q(\lambda) \frac{1 + \hat{p}}{2} ,
$$

where $\lambda$ denotes the spin of the summed spinors. The spin sum over $Q$ is equal to

$$
\sum_{\lambda} Q(\lambda) Q(\lambda) = \frac{1 + \hat{p}}{2} + \frac{\hat{k}}{2M_Q} ,
$$

where $k$ means the residual momentum whose magnitude is much smaller than $M_Q$. Substituting (25) into (24) yields

$$
\sum_{\lambda} h(\lambda) h(\lambda) = \frac{1 + \hat{p}}{2} (1 - \frac{\hat{k}^2}{4M_Q^2}) \tag{26}
$$

It is noted that the spin sum of the HQET effective spinor $h_v$ is equal to

$$
\sum_{\lambda} h_v(\lambda) h_v(\lambda) = \frac{1 + \hat{p}}{2} . \tag{27}
$$

This example shows that the $h$ propagator differs from the $h_v$ propagator

$$
S_{h_v} = \frac{i}{v \cdot \hat{k} + i \epsilon} \frac{1 + \hat{p}}{2} \tag{28}
$$

by a factor $(1 - \frac{\hat{k}^2}{4M_Q^2})$, which is just twice the reversal renormalization factor in (22). We take the free heavy quark spinor as the second example. Let $u_Q$ denote a free full heavy quark spinor whose energy is $E_Q$ and mass $M_Q$ and spatial momentum $\vec{k}$. $u_Q$ can be expressed in terms of its rest frame spinor as

$$
u_Q = \left( \begin{array}{c} \sqrt{E_Q + M_Q} \\
\frac{\vec{\sigma} \cdot \vec{k}}{\sqrt{2M_Q(M_Q + E_Q)}} \end{array} \right) \phi^{(\alpha)} \phi^{(\alpha)} \right) \tag{29}
$$

where

$$
\phi^{(1)} = \left( \begin{array}{c} 1 \\
0 \end{array} \right) , \quad \phi^{(2)} = \left( \begin{array}{c} 0 \\
1 \end{array} \right) \tag{30}
$$

denote the rest frame spinors. In the static approximation, we can expand $E_Q$

$$
E_Q = \sqrt{M_Q^2 + \vec{k}^2} \approx M_Q - \frac{\vec{k}^2}{2M_Q} , \tag{31}
$$

where $\vec{k}^2 = -k_\perp^2$ and $k_\perp = (0, \vec{k})$ have been used. Under this approximation, the full spinor $u_Q$ becomes

$$
u_Q = \sqrt{1 - \frac{\vec{k}^2}{4M_Q^2}} \left( \begin{array}{c} \phi^{(\alpha)} \\
\frac{\vec{k} \cdot \vec{\sigma}}{2M_Q - \vec{k} \cdot \vec{\sigma} \cdot \vec{k}} \phi^{(\alpha)} \end{array} \right) . \tag{32}
$$

From (32), we can identify the large components $h$ and small components $H$ of $u_Q$

$$
h = \sqrt{1 - \frac{\vec{k}^2}{4M_Q^2}} \phi , \quad H = \frac{\vec{k} \cdot \vec{\sigma}}{2M_Q - \vec{k} \cdot \vec{\sigma} \cdot \vec{k}} h . \tag{33}
$$
Spinors $\phi$ and $u_Q$ are well normalized $\bar{\phi}\phi = \bar{u}_Q u_Q = 1$, while the large components $h$ has an incorrect normalization $\bar{h}h = 1 - (\frac{k}{2M_Q})^2$ as pointed out in [8]. Finally, we emphasize that from the equations of motion of $Q$ (19) we can directly derive the relation between $h$ and $H$ as

$$H = \left[ \frac{1}{2M_Q} - i\delta \right] h$$

and on-shell condition for $Q$

$$[iD + \frac{(iD)^2}{2M_Q}] Q = 0 \ .$$

### III. CONSTRUCTION OF THE EFFECTIVE THEORY

#### A. Derivation of The Effective Field

One may match QCD to HQET at the heavy quark mass scale by requiring that the 1PI Green’s functions of the two theories describe the same physics. The simplest way to achieve this is by setting the external quark momenta on shell [7]. The LSZ reduction formula for a heavy quark fermion in momentum space is

$$S(P_Q, \ldots) = \frac{-i}{\sqrt{Z_Q}} u_Q(P_Q/M_Q) \frac{(P_Q - M_Q)}{2M_Q} \ldots \int dx e^{iP_Q \cdot x} <0|T[\psi(x) \ldots]|0 > |P_Q = M_Q^2$$

$$= \frac{-i}{\sqrt{Z_Q}} Q(v + k/M_Q) \frac{\bar{k}}{2M_Q} \ldots \int dx e^{i\bar{k} \cdot x} <0|T[Q(x) \ldots]|0 > |v \cdot k = -k^2/2M_Q$$

Eq. (36) is equivalent to the equation of motion $[iD - M_Q(1 - \gamma^\parallel)] Q(x) = 0$. Being a projection operator, $\Lambda^+$ obeys the identity

$$(\Lambda^+)^2 = \Lambda^+$$

which implies on-shell condition

$$[k^\parallel + \frac{k^2}{2M_Q}] Q = 0$$

with $k^\parallel = v \cdot k$. The inverse operator of $\Lambda^+$ is the negative energy projection operator

$$\Lambda^- = \frac{(1 - \gamma^\parallel)}{2} - \frac{\bar{k}}{2M_Q}$$

defined by identity

$$\Lambda^+ + \Lambda^- = 1 \ .$$
In order to derive \( h_v \) which respects the physics in the limit \( M_Q \to \infty \), we define projectors

\[
\Lambda_v^\pm = \frac{1 \pm \frac{\not{k}}{2}}{2} \equiv \lim_{M_Q \to \infty} \Lambda^\pm .
\] (43)

Note that operators \( \Lambda_v^+ \) (\( \Lambda_v^- \)) are the infinite mass limit of the energy projection operators \( \Lambda^+ (\Lambda^-) \). \( \Lambda_v^\pm \) satisfy identities

\[
(\Lambda_v^\pm)^2 = \Lambda_v^\pm .
\] (44)

By \( 1 = \Lambda_v^+ + \Lambda_v^- \), we recast \( Q \) to be

\[
Q = \Lambda_v^+ Q + \Lambda_v^- Q = h + H .
\] (45)

From (38) and (45) we arrive at

\[
H = \left[ \frac{1}{2 M_Q - \frac{k}{2 M_Q}} \right] h ,
\] (46)

and

\[
Q = \left[ \frac{1}{1 - \frac{k}{2 M_Q}} \right] h .
\] (47)

In literature, people always stop at this point and identify \( h \) as \( h_v \). However, as point out in last Section, \( h \) is not identical to \( h_v \). It is natural to assume that \( h_v \) is the limit \( M_Q \to \infty \) of \( h \) and two spinors are proportional. The first assumption comes from \( h_v \equiv \lim_{M_Q \to \infty} Q \), while the second is based on the fact that both \( h \) and \( h_v \) are projected out by \( \Lambda_v^+ \). In this way, we argue that \( h = [1 + \omega] h_v \) with ansatz \( \bar{\omega} = \omega \) and \( \not{k} \omega = \omega \). To derive \( \omega \), we note identities

\[
\Lambda^+ = \sum Q \bar{Q} ,
\] (48)

\[
\Lambda_v^+ = \sum h_v \bar{h}_v ,
\] (49)

where summations over spin indices of the spinors are implied. Eqs. (48) and (49) hold if and only if \( \bar{Q}Q = \bar{h}_v h_v = 1 \). Eq. (49) comes directly from definition, the limit \( M_Q \to \infty \) of (48). Substituting (47) and \( h = [1 + \omega] h_v \) into (48) and using (49) for \( h_v \), we get the equation for \( \omega \)

\[
(\omega)^2 + 2 \omega + \frac{1 + \frac{\not{k}}{2}}{2} = (1 - \frac{k}{2 M_Q}) (1 + \frac{\not{k}}{2} + \frac{k}{2 M_Q}) (1 - \frac{k}{2 M_Q}) .
\] (50)

With the help of on-shell condition (40), (50) is recast as

\[
(\omega)^2 + 2 \omega + \left( \frac{k}{2 M_Q} \right)^2 (1 + \frac{k}{2}) = 0 .
\] (51)

The above equation is easily solved

\[
\omega = -1 + \sqrt{1 + T}
\] (52)

with \( T = -\left( \frac{k}{2 M_Q} \right)^2 (1 + \frac{k}{2}) \). We then obtain

\[
h = \sqrt{1 - \left( \frac{k}{2 M_Q} \right)^2 (1 + \frac{k}{2})} h_v .
\] (53)

The relation between \( h \) and \( h_v \) is consistent with that between \( h \) and \( h' \) found in last Section.

Combining (47) and (53), we get

\[
Q = \sqrt{\frac{1 + \frac{k}{2 M_Q}}{1 - \frac{k}{2 M_Q}}} \Lambda_v h_v \equiv \Lambda (w = v + k/M_Q, v) h_v .
\] (54)
Note that (54) is just the Lorentz transformation between two spinors with relative velocity \( k/M_Q \). The transformation operator \( \Lambda(w = v + k/M_Q, v) \) is identical to the Lorentz boost in the spinor representation \( \tilde{\Lambda}(w, v) = 1 + \frac{\vec{k}/(2M_Q)}{1 + \vec{k}/(2M_Q)} \Lambda^+ Q \).  

In presence of interactions, the Lorentz boost interpretation of (54) is invalid. The reverse transformation from \( h_v \) into \( Q \) can be derived in a similar way. The result is

\[
h_v = \sqrt{\frac{1 - \vec{k}/(2M_Q)}{1 + \vec{k}/(2M_Q)}} \Lambda^+ Q .
\]

(56)

Transforming (54) into coordinate space by replacements \( \vec{k} \rightarrow iD \), one obtains

\[
Q(x) = \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} \Lambda^+_v h_v(x).
\]

(57)

Field \( Q(x) \) is consistent with the field derived by Luke and Manohar \[5\]

\[
\Psi_v(x) = \tilde{\Lambda}(v + iD/M_Q, v) h_v(x).
\]

(58)

By employing (54) and (57), we arrive at the matching between \( Q \) and \( h_v \) at scale equal to \( M_Q \)

\[
S(k, \ldots) = \frac{-i}{\sqrt{Z_Q}} \bar{h}_v(v)(\frac{\vec{k}}{2M_Q} - \frac{1 - \vec{k}}{2} \ldots) \int dx e^{ik \cdot x} \langle 0 | T [h_v(x) \ldots] | 0 \rangle_{v \cdot k = -k^2/2M_Q} ,
\]

(59)

which is different from (41) in [10] by a factor \( \sqrt{Z(k)} = \sqrt{(1 - \vec{k}^2/(4M_Q^2))} \). This is because the effective field \( h_v^{KO} \) employed in [10] is the unrenormalized large components \( h_v^{KO} = h = \sqrt{Z(k)}h_v \).

By matching QCD and HQET at 2PI and quark-gluon-quark interaction Green functions, we can derive the HQET Lagrangian

\[
L = \bar{\psi} (iD - M_Q) \psi = \bar{Q} (iD - 2M_Q \Lambda^-_v) Q = \bar{h}_v \Lambda^+_v \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} (iD - 2M_Q \Lambda^-_v) \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} \Lambda^+_v h_v^+ .
\]

(60)

Note that the HQET Lagrangian is hermitian. This can be inspected by explicitly performing hermitian conjugation and integration by parts.

### IV. THE VELOCITY REPARAMETERIZATION TRANSFORMATION

#### A. Field Transformation

The heavy quark momentum \( P_Q \) makes no differences for parameterizations \( P_Q = M_Q v + k \) or \( P_Q = M_Q v' + k' \). It was found that the HQET Lagrangian should be invariant under reparameterizations for the velocity \( v \rightarrow v' \) and for the residual momentum \( k \rightarrow k' \). We show the following theorem: If \( v \) and \( v' \) relate each other as \( v' = v + \delta v \) with \((v')^2 = v^2 = 1 \) and \( v \cdot \delta v + (\delta v)^2 / 2 = 0 \), then \( h_{v'} \) relates to \( h_v \) as

\[
h_{v'} = \sqrt{\frac{1 + \delta \vec{k}/2}{1 - \delta \vec{k}/2}} \Lambda^+_v h_v .
\]

(61)

and
We show this explicitly below. The successive transformations denote the transformation from \( h \) to \( v \). The proof is completed. The importance of this transformation is the association of successive transformations. If we replace \( \delta v \) with \( k/M_Q \) in the transformation (54), we thus derive the transformation from \( h \) to \( v \). The proof of \( Q(P_Q = M_Qv' + k') \) identical to \( Q(P_Q = M_Qv + k) \) is also trivial by noting that

\[
h_v = \frac{1 + \delta \phi /2}{1 - \delta \phi /2} \Lambda_v^+ h_v,
\]

(64)

\[
Q(P_Q = M_Qv + k) = \frac{1 + \delta \phi /2}{1 - \delta \phi /2} \Lambda_v^+ h_v,
\]

(65)

\[
Q(P_Q = M_Qv' + k') = \frac{1 + \delta \phi /2}{1 - \delta \phi /2} \Lambda_v^+ h_v.'
\]

(66)

and \( M_Q \delta v = k - k' \). It results in the identity

\[
Q(P_Q = M_Qv' + k') = \sqrt{\frac{1 + \delta \phi /2}{1 - \delta \phi /2}} \Lambda_v^+ h_v
\]

(67)

The proof is completed. The importance of this transformation is the association of successive transformations. If we denote the transformation from \( h_v \) into \( h_{v'=v+\delta v} \) by \( h_{v'} = L(v, v')h_v \), then we can have \( L(v, v''') = L(v', v'')L(v, v') \). We show this explicitly below. The successive transformations \( v \to v' = v + \delta v_1 \) followed by \( v' \to v'' = v' + \delta v_2 = v + \delta v_1 + \delta v_2 \) would have the transformations in the effective field as

\[
h_v \to h_v' = \sqrt{\frac{1 + \delta \phi /2}{1 - \delta \phi /2}} \frac{1 + \delta \phi /2}{1 - \delta \phi /2} h_v^+
\]

(68)

and

\[
h_v' \to h_v'' = \sqrt{\frac{1 + \delta \phi /2}{1 - \delta \phi /2}} \frac{1 + \delta \phi /2}{1 - \delta \phi /2} h_v'^+
\]

(69)

\[
= \sqrt{\frac{1 + (\delta \phi_1 + \delta \phi_2)/2}{1 - (\delta \phi_1 + \delta \phi_2)/2}} \frac{1 + (\delta \phi_1 + \delta \phi_2)/2}{1 - (\delta \phi_1 + \delta \phi_2)/2} h_v^+.
\]

(70)

**B. Reparameterization Invariance**

We now show that the reparameterization invariance is trivial and manifest for the Lagrangian \( Q \). By previous theorem, it is also straightforward to prove that the effective Lagrangian in terms of \( Q \) is invariant under transformations \( v \to v' = v + \delta v, M_Q \delta v = k - k' \) and \( Q^+(M_Qv' + k')(x) = \exp (iM_Q \delta v \cdot x)Q^+(M_Qv + k)(x) \)
\[ L^+ = \bar{Q}^+(P_Q = M_Q v + k) (\not{\partial} - 2M_Q \Lambda_v^-) Q^+(P_Q = M_Q v + k) = \bar{Q}^+(P_Q = M_Q v' + k') (\not{\partial} - 2M_Q \Lambda_{v'}^-) Q^+(P_Q = M_Q v' + k') . \] (71)

It is also trivial to prove the above invariance for effective Lagrangian in terms of \( h_v \)
\[ L^+ = \bar{Q}^+(P_Q = M_Q v' + k') (\not{\partial} - 2M_Q \Lambda_{v'}^-) Q^+(P_Q = M_Q v' + k') \]
\[ = h_v^+ \Lambda_v^+ \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} (\not{\partial} - 2M_Q \Lambda_v^-) \Lambda_v^- h_v^+ \]
\[ = h_v^+ \Lambda_v^+ \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} e^{-iM_Q \delta v - x} \Lambda_v^- h_v^+ \]
\[ = h_v^+ \Lambda_v^+ \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} \Lambda_v^- h_v^+ \]
\[ = \bar{Q}^+(P_Q = M_Q v + k) (\not{\partial} - 2M_Q \Lambda_v^-) Q^+(P_Q = M_Q v + k) \] (72)

\[ e^{-iM_Q \delta v - x} \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} e^{iM_Q \delta v - x} \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} = \sqrt{\frac{1 + iD/(2M_Q)}{1 - iD/(2M_Q)}} . \] (73)

C. Zero recoil limit

The matrix element of \( \langle H_v | \bar{Q} Q | H_v \rangle \) appears much simpler in the zero recoil limit, \( v' = v \). With (54) and (55), we can derive the following identities
\[ \langle H_v | \bar{Q} Q | H_v \rangle = \langle H_v | \bar{v}_v h_v | H_v \rangle , \] (74)
\[ \langle H_v | \bar{Q} \gamma^\mu Q | H_v \rangle = \langle H_v | \bar{v}_v (\gamma^\mu + i \frac{D}{M_Q} v) | H_v \rangle , \] (75)
\[ \langle H_v | \bar{Q} \gamma^5 Q | H_v \rangle = \langle H_v | \bar{v}_v (\gamma^5 - \gamma^5 \frac{iD}{M_Q} v) | H_v \rangle , \] (76)
\[ \langle H_v | \bar{Q} \sigma^\mu \nu Q | H_v \rangle = \epsilon^{\mu \nu \alpha \beta} \langle H_v | \bar{v}_v \gamma_\alpha \gamma_\beta (v_\beta + \frac{iD}{M_Q} v) | H_v \rangle . \] (77)

Notes that the pseudoscalar currents vanish.

V. APPLICATIONS

We apply the HQET Lagrangian (60) to calculate the rates for inclusive semileptonic and nonleptonic heavy hadron decays up to second order mass corrections. According to operator product expansion (OPE) (12–15), the decay rate of a heavy hadron \( H_b \) containing a \( b \) quark is expanded in terms of local operators with increasing dimensions
\[ \Gamma_{H_b} = \langle H_b | c_1 \bar{b} b + c_G \bar{b} \gamma^5 G b + \cdots | H_b \rangle , \] (78)

where \( c_1 \) and \( c_G \) are short distance coefficients. The momentum carried by \( H_b \) is chosen as \( P_{H_b} = M_{H_b} v \) with \( M_{H_b} \) the \( H_b \) mass and \( v \) its velocity. \( | H_b \rangle \) means the eigenstate of the full HQET Lagrangian (60) with normalization \( \langle H_b(P') | H_b(P) \rangle = v^0 (2\pi)^3 \delta^3 (v' - v) \). The next step is to expand the matrix elements of the local operators into inverse powers of the \( b \)-quark mass. Because of (74) the matrix element \( \langle H_b | \bar{b} b | H_b \rangle \) is transformed into \( \langle H_b | \bar{v}_v h_v | H_b \rangle \). There is no mass corrections in \( \langle H_b | \bar{b} b | H_b \rangle \). This differs from the conventional result
\[ \langle H_b | \bar{b} b | H_b \rangle = 1 - \frac{\mu^2_b (H_b) - \mu^2_G (H_b)}{2m_b^2} + O\left(\frac{1}{m_b^4}\right) , \] (79)
where $\mu_2^2(H_b)$ and $\mu_3^2(H_b)$ parameterize the matrix elements of the kinetic-energy and the chromo-magnetic operators, respectively. The matrix element $\langle H_b|\bar{b}b|H_b\rangle$ has coefficient of $O(1/m_b^2)$ and its leading mass corrections are of $O(1/m_b^4)$ will be ignored. With these considerations, we derive the semileptonic decay rate

$$\Gamma_{sl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left\{ z_0 \left( \frac{m_c}{m_b} \right) - 2 z_1 \left( \frac{m_c}{m_b} \right) \frac{\mu_2^2}{m_b^2} + \cdots \right\}, \quad (80)$$

which is free from $\mu_\tau$ uncertainty in contrast with the usual formula

$$\Gamma_{sl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left\{ z_0 \left( \frac{m_c}{m_b} \right) \left( 1 - \frac{\mu_2^2 - \mu_3^2}{2m_b^2} \right) - 2 z_1 \left( \frac{m_c}{m_b} \right) \frac{\mu_2^2}{m_b^2} + \cdots \right\}. \quad (81)$$

The difference is due to different accounts for the time dilation in the Fermi motion of the heavy quark in the heavy hadron rest frame. Recall that the weak decay widths of muon

$$\Gamma(\mu \rightarrow e\nu \bar{\nu}) = \frac{G_F^2 m_\mu^5}{192\pi^3} z_0 \left( \frac{m_c}{m_\mu} \right). \quad (82)$$

One can see that our result is more reasonable because the leading term $\langle H_b|\bar{b}b|H_b\rangle$ is normalized to unity in our HQET theory. We can derive the nonleptonic decay rate

$$\Gamma_{nl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{qb}|^2 |V_{cb}|^2 N_c \left\{ w_1 z_0 \left( \frac{m_q}{m_b} \right) - 2 w_2 (z_1 \left( \frac{m_q}{m_b} \right) + z_2 \left( \frac{m_q}{m_b} \right) \frac{\mu_2^2}{m_b^2} + \cdots \right\}, \quad (83)$$

which is different from the conventional result

$$\Gamma_{nl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{qb}|^2 |V_{cb}|^2 N_c \left\{ w_1 z_0 \left( \frac{m_q}{m_b} \right) \left( 1 - \frac{\mu_2^2}{2m_b^2} \right) - 2 w_2 (z_1 \left( \frac{m_q}{m_b} \right) + z_2 \left( \frac{m_q}{m_b} \right) \frac{\mu_2^2}{m_b^2} + \cdots \right\}. \quad (84)$$

The Wilson coefficients are defined as

$$w_1 = c_1^2 + c_2^2 + \frac{c_1 c_2}{2N_c}, \quad w_2 = \frac{c_1 c_2}{2N_c} \quad (85)$$

with $c_1 = (c_+ + c_2)/2$ and $c_2 = (c_+ - c_-)/2$ and the phase space factors are

$$z_0(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \log x, \quad z_1(x) = (1 - x)^4, \quad z_2(x) = (1 - x)^3. \quad (86)$$

VI. CONCLUSION

We have regularized the non-hermitian terms in the EOM type HQET theories \[\] up to $O(1/M_Q^2)$. We found that the large components of the heavy quark field are not appropriate variables beyond leading order mass corrections. Only the renormalized large components, whose high frequency modes have been integrated out, can be a relevant effective field for low energies. In terms of the renormalized large components, the HQET Lagrangian \[\] is hermitian to all orders in $1/M_Q$ and contains manifest reparameterization invariance.

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APPENDIX A: MASS EXPANSION LAGRANGIAN

We discuss the mass expansion of the HQET Lagrangian derived in \[\]. The HQET Lagrangian $L$ is expanded into mass correction terms as

$$L = \sum_{n=0}^{\infty} \frac{L^{(n)}}{(2M_Q)^n} \quad (A1)$$

where the first leading terms $L^{(n)}, n = 0, 1, 2, 3$ are enumerated as follows
\[ L^{(0)} = \overline{h}_{\nu} i D_{\parallel} h_{\nu} , \] (A2)

\[ L^{(1)} = \overline{h}_{\nu} \left[ -D_{\parallel}^{2} - D^{2} + \frac{1}{2} \sigma_{\alpha\beta} G^{\alpha\beta} \right] h_{\nu} , \] (A3)

\[ L^{(2)} = \overline{h}_{\nu} \left[ -2iD^{3} + \frac{1}{2} \left( v_{\alpha} \left[ D_{\beta}, G^{\alpha\beta} \right] + i \sigma_{\alpha\beta} v_{\lambda} \left[ D^{\beta}, G^{\lambda\alpha} \right] \right) \right] h_{\nu} , \] (A4)

\[ L^{(3)} = \overline{h}_{\nu} \left[ D^{2}(D^{2} + D_{\parallel}^{2}) + \frac{1}{2} G^{2} + \frac{1}{2} \sigma \cdot G D_{\parallel}^{2} - \{ D^{2}, \sigma \cdot G \} ight. \\
\left. + \sigma_{\alpha\beta} \left( D_{\lambda} \{ D^{\beta}, G^{\lambda\alpha} \} + [ D^{\beta}, G^{\lambda\alpha} ] D_{\lambda} - i G^{\lambda\alpha} G_{\lambda}^{\beta} \right) \\
\frac{i}{4} \gamma^{5} \epsilon_{\alpha\beta\lambda\rho} G^{\alpha\beta} G^{\lambda\rho} \right] h_{\nu} . \] (A5)

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