NNLO unquenched calculation of the $b$ quark mass.

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By combining the first unquenched lattice computation of the $B$-meson binding energy with the recently calculated two–loop contribution to the lattice HQET mass, we determine the $\overline{\text{MS}}$ $b$–quark mass, $m_b$ at the NNLO. We find $m_b = (4.26 \pm 0.03 \pm 0.04 \pm 0.05) \, \text{GeV}$. The inclusion of the two–loop effects is one of the steps necessary to extract $m_b$ with a precision better than $\mathcal{O}(\Lambda_{\text{QCD}})$, which is the uncertainty due to the presence of an IR renormalon singularity in the perturbative series of the residual mass. Our results have been obtained on a sample of 60 lattices of size $24^3 \times 40$ at $\beta = 5.6$, using the unquenched Wilson action with two degenerate sea quarks. The quark propagators have been computed using the unquenched links generated by the T$\chi$L Collaboration.

1. Introduction and Motivation.

Quark masses are fundamental parameters of QCD which cannot be determined by theoretical considerations only. Moreover, they are very important for phenomenology because enter many theoretical predictions of physical quantities like the CKM matrix elements, the CP violation parameter $\epsilon'/\epsilon$, the $\Delta I = 1/2 \, K \to \pi \pi$ amplitude, the $B$ meson semileptonic decays, the $B - \bar{B}$ mixing . . . etc. Quark masses cannot be measured directly since quarks are confined in the hadrons. Therefore, a short distance definition of the quark mass, which is scale and scheme dependent, must be adopted.

In this paper, we report on the first unquenched HQET lattice calculation of the $b$ quark mass. The main idea is to combine unquenched HQET lattice computations of the $B$-meson propagator with recent NNLO analytical formulae of the matching of the continuum $\overline{\text{MS}}$ quark mass to the lattice one. We stress that both the unquenched lattice simulation and the NNLO matching are necessary ingredients to improve the accuracy of the results: the former is necessary to control potentially large vacuum polarization contributions to the $B$ meson propagator and the latter is crucial to cancel the renormalon ambiguities in the matching.

2. How to determine the $b$-quark mass on the lattice.

The key idea is to match the $b$-quark propagator in QCD to its lattice HQET counterpart. Since the lattice HQET is an effective theory, the relation between these propagators is, to lowest order in $1/m_b$,

$$S^{-1}(p, m_b; \mu) = C(m_b/\mu, \alpha_s) \, S_L^{-1}((v \cdot k)a; \mu) \quad (1)$$

where $p = m_b \, v + k$ is the external momentum of the $b$-quark, $v$ is its velocity, $k$ is the residual momentum with $|k| \ll m_b$, $\mu$ is the renormalization point, $a$ is the lattice spacing and $C(m_b/\mu, \alpha_s)$ is the Wilson coefficient. Note that, as expansion parameter of the HQET, we have chosen the quark mass $m_b$. In this way, all the mass dependence is factorized in the coefficient function $C$. The method to fix $C$ is well known: it consists in calculating the $b$-quark propagator in QCD and in the lattice HQET to a given order in $\alpha_s$, comparing both expressions at a fixed $\mu = \mu_0$ and extracting $C(m_b/\mu_0)$. Renormalization group can
then be used to evolve this function to lower scales. The important point is that by rewriting eq. (1) in terms of the pole mass, $m_{pole}$, it is easy to find the relation

$$m_{pole} = m_b + \sum_{n=0}^{\infty} (\alpha_s(a))^{n+1} \frac{X_n}{a}$$

(2)

where the last term is the perturbative expansion of the residual mass generated in the lattice HQET, $\delta m$. The coefficients $X_n$ are functions of $\ln(m_b a)$.

On the other hand, the HQET mass formula, to lowest in $1/m_b$,

$$M_B - \mathcal{E} = m_b + \mathcal{O}(1/m_b)$$

allows us to write the unknown mass $m_b$ in eq. (3) in terms of the physical B-meson mass, $M_B$. The so-called binding energy, $\mathcal{E}$, is independent of $m_b$

$$m_{pole} = M_B - \mathcal{E} + \sum_{n=0}^{\infty} (\alpha_s(a))^{n+1} \frac{X_n}{a}.$$  

(4)

$\mathcal{E}$ is not a physical quantity because it diverges linearly as $a \to 0$. Since the pole mass is finite in this limit, this divergence is cancelled by the last term of eq. (1). In practice this cancellation is not perfect and hence one cannot take $a$ too small. As a large $n_f$ calculation demonstrates (see [3]), the series in eq. (1) has IR renormalon singularities, the same as the pole mass. In other words, the coefficients $X_n$ grow as $n!$ as $n \to \infty$. This behaviour gives rise to ambiguities of $\mathcal{O}(\Lambda_{QCD}^2 / m_b)$.

In order to avoid this problem, it is convenient to use a short distance definition of the b-quark mass, $m_b$, rather than to eq. (7).

Putting all together, we find

$$m_b(m_b) = \left( M_B - \mathcal{E} + \frac{2.1173}{a} \alpha_s(m_b) + \frac{1}{a} \left[ 3.197 \ln(m_b a) - 1.514 \alpha_s(m_b)^2 \right] \times \left[ 1 - \frac{4}{3} \frac{\alpha_s(m_b)}{\pi} - 9.58 \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 \right] \right)$$

(6)

where we have taken $n_f = 2$, the number of sea quarks in our simulation. Note that the remaining ambiguity in our result is only $\mathcal{O}(\Lambda_{QCD}^2 / m_b)$ which is beyond our precision. By computing $\mathcal{E}$ through lattice simulations of the HQET and by using the experimental values of the B-meson masses, we can extract $m_b$.

3. Lattice computation of $\mathcal{E}$.

As well known, $\mathcal{E}$ can be extracted by studying the large time behaviour of the two–point function of the B meson, $C_2$ in the HQET,

$$C_2(t) \to Z \ e^{-\mathcal{E} t}$$

(7)

The light quarks are described by the Wilson action. In order to improve the isolation of the ground state, we use cube and double cube smeared axial currents as interpolating operators of the B meson [19]. The actual value of $\mathcal{E}$ has some dependence on the cube size and the smearing type due to contamination from excited states.

To obtain our best estimate, we compare different methods and account this systematic effect in the final error. We use the Standard Method, in which we base our results on the best cube, defined as the one which yields the largest and flattest effective mass plateau. In practice, we have few cube sizes (only two in our simulation, 7 and 9), and hence the best cube is difficult to find. To improve our results, we also use the Multifit Method which consists in performing a global fit of the data for all smearing types and cubes sizes imposing that the binding energy be the same for all of them. In order to reduce the effect of excited states, it is convenient to fit the data to a two-state form of $C_2$ rather than to eq. (6).
4. Analysis and results.

We performed an unquenched Wilson simulation with two degenerate sea quarks at two values of their mass, \(k_{\text{sea}} = 0.1575\) and \(k_{\text{sea}} = 0.1580\), at \(\beta = 5.6\) on a \(24^3 \times 40\) lattice. The sample is of 60 configurations at four values of the valence light quark masses. We compute the quark propagators and correlation functions from the gluon configurations generated by the \(T\chi L\) Collaboration. Since our simulation is unquenched, the procedure to measure the lattice quantities, including the lattice spacing, for each \(k_{\text{sea}}\). Only at this point, the quantities expressed in physical units can be extrapolated as a function of the sea quark masses. The reason is that a change of the coupling constant, and hence, may induce a rapid variation of the value of the coupling. We compute the quark masses. The reason is that a change of the value of the lattice spacing. Therefore, lattice results from different values of \(k_{\text{sea}}\) are not directly comparable until they have been converted to physical units. We find, from \(m_{K^*}\),

\[
\left. a^{-1}\right|_{m_{K^*}} = \{2.51(6), 2.54(6)\} \text{ GeV}
\]

and for the binding energy

\[
\begin{align*}
    aE_{B_d} &= \{0.588(11)(5), 0.606(15)(2)\} \\
    aE_{B_s} &= \{0.620(8)(4), 0.632(12)(2)\}
\end{align*}
\]

for the two values of \(k_{\text{sea}}\). More details will be given in (I).

We have carefully studied different sources of systematic errors in eq. (I). We find that the dependence of \(\overline{m}_b(\overline{m}_b)\) on the unknown \(\Lambda_{QCD}^{n_f=2}\) is small. Since the quenched value is \(\Lambda_{QCD}^{n_f=0} \sim 250 \text{ MeV}\) (II), and the physical one is expected to be larger, we varied the value of \(\Lambda_{QCD}^{n_f=2}\) in the range \([250, 350]\) MeV. We also tried other options to evaluate \(\alpha_s^{n_f=2}\) (III). In the numerical evaluation, we also used eq. (II) expanded to \(\mathcal{O}(\alpha_s^2)\). In this way, we obtain a rough estimate of higher-order terms which is found less than 3%. Chiral extrapolations are also under control because

the b–quark masses obtained from the \(B_d\) and \(B_s\) mesons are nicely compatible. Results from different smearing methods are also compatible, showing that the ground state has been well isolated. Finally, the dependence on \(k_{\text{sea}}\) is very small. This allows us to take for our best estimate of \(\overline{m}_b(\overline{m}_b)\) the value obtained at the lightest one, \(k_{\text{sea}} = 0.1580\). Taking into account all errors, we find

\[
\overline{m}_b(\overline{m}_b) = (4.26 \pm 0.03 \pm 0.04 \pm 0.05) \text{ GeV}
\]

where the first error is statistical, the second is systematic and the third is an estimate of the effects of higher-order corrections in the matching. It is interesting to compare our result with our old quenched numbers (I) at three different lattice spacings \(a^{-1} = 2.0(2), 2.9(3)\) and 3.8(3) GeV, which have been reanalyzed in (III) to include the two–loop correction to the matching (II).

\[
\overline{m}_b(\overline{m}_b) = \{4.34(5), 4.29(7), 4.25(7)\} \text{ GeV}
\]

where the error comes from the lattice uncertainties. In this case, an educated estimate of higher-order corrections is \(0.10\) GeV. As can be seen, they are in good agreement with our unquenched result. Given the present uncertainties, in particular the small physical volume used to obtain the result for \(a^{-1} = 3.8\) GeV, we are not in the position to attempt the extrapolation of the results in (I).

REFERENCES

1. M. Crisafulli et al., Nucl. Phys. B457 (1995) 594; V. Giménez et al., Phys. Lett. B393 (1997) 124.
2. G. Martinelli and C.T. Sachrajda, hep-lat/9812001.
3. M. Beneke and V. M. Braun, Nucl. Phys. B426 (1994) 301; M. Beneke, hep-ph/9807443 and references therein.
4. N. Gray et al, Z. Phys. C48 (1990) 673; K.G. Chetyrkin and M. Steinhauser, hep-ph/9907509.
5. V. Giménez et al., Rome preprint 99/1269, in preparation.
6. M. Lüscher et al, Nucl. Phys. B389 (1993) 247; ibid. B413 (1994) 481.