CLT FOR NON-HERMITIAN RANDOM BAND MATRICES WITH VARIANCE PROFILES

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Abstract. We show that the fluctuations of the linear eigenvalue statistics of a non-Hermitian random band matrix of increasing bandwidth \( b_n \) with a continuous variance profile \( w_\nu(x) \) converges to a \( N(0, \sigma^2_f(\nu)) \), where \( \nu = \lim_{n \to \infty} (2b_n/n) \in [0, 1] \) and \( f \) is the test function. When \( \nu \in (0, 1] \), we obtain an explicit formula for \( \sigma^2_f(\nu) \), which depends on \( f \) and variance profile \( w_\nu \). When \( \nu = 1 \), the formula is consistent with Rider and Silverstein (2006) [33]. We also independently compute an explicit formula for \( \sigma^2_f(0) \) i.e., when the bandwidth \( b_n \) grows slower compared to \( n \). In addition, we show that \( \sigma^2_f(\nu) \to \sigma^2_f(0) \) as \( \nu \downarrow 0 \).

Keywords: Random band matrices, random matrices with a variance profile, central limit theorem, linear eigenvalue statistics.

1. Introduction

In this article, we consider the linear eigenvalue statistics of random non-Hermitian band matrices with a variance profile. Let \( M \) be an \( n \times n \) random non-Hermitian matrix and \( \lambda_i(M); 1 \leq i \leq n \) be its eigenvalues. Define the empirical spectral measure (ESM) of \( M \) as

\[
\mu_M = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(M)},
\]

where \( \delta_x \) is unit point mass at \( x \). It was shown, in a series of papers, that if the entries of \( M \) are i.i.d. random variables with zero mean and unit variance, then asymptotically \( \mu_M/\sqrt{n} \) converges to the uniform density on the unit disc in \( \mathbb{C} \) [19, 5, 16, 38, 39]. However, if the entries are not identically distributed, the limiting law may be different. In particular, when the entries of the matrix are multiplied by some predetermined weights, the matrix is called a random matrix with a variance profile. Limiting ESM of such matrices were found in [13].

In an analogous way to classical probability, limiting ESM is the law of large numbers for random matrices. One might be interested in finding fluctuations of such convergence after proper scaling, which is the central limit theorem (CLT) in classical probability. In the case of random matrices, we would be studying CLT of the sequence of random measures (the ESMs). One way to study such object is by studying \( \int f \, d\mu_M \) for some test function \( f \). This brings the question from the space of random measures to the space of real/complex valued random variables. More precisely, we define the linear eigenvalue statistics of \( M \) with respect to a test function \( f \) as

\[
\mathcal{L}_f(M) := n \int f \, d\mu_M = \sum_{i=1}^{n} f(\lambda_i(M)),
\]

\[
\mathcal{L}_f^\Delta(M) := \mathcal{L}_f(M) - nf(0).
\]

We consider the limiting distribution of \( \mathcal{L}_f^\Delta(M) \). Limiting behavior of such quantities were studied in [22, 37, 27, 11 section 5] for Hermitian matrices; and in [4, 26, 21, 35] for Hermitian band matrices.

In this article, we consider non-Hermitian matrices \( M \) whose entries are complex valued random variables. Distributional limit of such objects was found in [30, 7, 32, 33, 34], which was later extended in [2, 31, 8, 25]. CLT for polynomial \( f \) and real valued \( M \) were studied in [30]. More recently, CLT for products of random matrices were established in [14, 20]; and words of random matrices were studied in [15].

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In both the cases [33, 30], the matrix $M$ was a full matrix without any variance profile. Recently, for polynomial test functions, it was shown that $L_f^n(M)$ for random symmetric/non-symmetric matrices with a variance profile converges to $\mathcal{N}(0, \sigma_f^2)$ in total variation norm [1]. However, since the results in [1] were stated in a very broad context, the exact expression of $\sigma_f^2$ was difficult to find.

The main contribution of this article is calculating $\sigma_f^2$ for random band matrices with a variance profile. In [33], the variance was calculated in the process of proving the CLT. The same procedure does not yield the variance in our case. So, the proof of CLT and calculation of variance is done using two separate methods. An $n \times n$ non-periodic (and periodic) band matrix of bandwidth $b_n$ is obtained by keeping $2b_n$ many off-diagonal vectors around the main diagonal (and around the corners), and setting rest of the off-diagonal vectors to zero.

![Figure 1. Blue lines represent the non-zero diagonal vectors.](image)

A precise definition of random band matrix is given in the Definition 2.1. In particular, we show that if we have a periodic band matrix with $2b_n + 1 = n$, then our results are consistent with that of [33]. In this context, we would also like to mention that while in full matrix case the unscaled $L_f^n(M)$ converges to a Gaussian distribution, in band case we need to scale it by $\sqrt{(2b_n + 1)/n}$. This shows a significant difference in between full and band matrices. In the first case $\text{Var}(L_f^n(M))$ remains constant, while in the latter case it grows as $O(n/b_n)$.

The article is organized as follows. In Section 2, we enlist the notations and definitions. The main theorem is formulated in Section 3, and the proof is given in Section 4. In the process of the proof, we need the norm of the random matrix to be bounded almost surely, which is discussed in appendix A. Variance of the limiting distribution is calculated in Section 5.

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2. **Preliminaries and notations**

For convenience, we do not indicate the size of a matrix in its name. For example, to denote an $n \times n$ matrix $A$, we simply write $A$ instead of $A_{n \times n}$. In addition, throughout this paper we use the following notations:

- $\{e_1, e_2, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$
- $I_j$ is a band index set as defined in the Definition 2.1
- $I_j$ is a band diagonal matrix as defined in the Definition 2.1
- $a_{ij}$ := $i$th entry of the matrix $A$
- $a_k$ := $k$th column of the matrix $A$
**Definition 2.1** (Band matrix with a variance profile). Let \( \nu \in (0, 1] \), and \( w : \mathbb{R} \to [0, \infty) \) be a piece-wise continuous function supported on \([-1/2, 1/2]\) such that it is continuous at 0 and \( \int_{-1/2}^{1/2} w(x) \, dx = 1 \). Define a \( 1/\nu \) periodic function \( w_{\nu} \) and a non-periodic function \( w_0 \) as follows

\[
\begin{align*}
  w_{\nu}(x) &= w(x), \quad \forall \, x \in \left[-\frac{1}{2\nu}, \frac{1}{2\nu}\right], \quad \text{and} \quad w_{\nu}\left(x + \frac{1}{\nu}\right) = w_{\nu}(x) \\
  w_0(x) &= w(x), \quad \forall \, x \in \mathbb{R}.
\end{align*}
\]

In particular, \( w_{\nu}(x) \xrightarrow{\nu \to 0} w_0(x) \) on any compact subset of \( \mathbb{R} \). Let \( \{x_{ij} : 1 \leq i, j \leq n\} \) be a set of i.i.d. random variables for each \( n \), and \( b_n = (c_n - 1)/2 \).

1. When \( \lim_{n \to \infty} (c_n/n) = \nu \in (0, 1] \), define a periodic band matrix \( M_{\nu}^{c_n} \) of bandwidth \( b_n \) as

\[
m_{ij} = \frac{1}{\sqrt{c_n}} E_{ij} \sqrt{w_{\nu} \left( \frac{1 - j}{c_n} \right)}.
\]
(2) When $c_n = o(n)$, define non-periodic band matrix $M_0^\otimes$ of bandwidth $b_n$ as

$$m_{ij} = \frac{1}{\sqrt{c_n}} x_{ij} \sqrt{w_0 \left( \frac{i-j}{c_n} \right)}.$$  

and a periodic band matrix $M_0^\otimes$ of bandwidth $b_n$ as

$$m_{ij} = \frac{1}{\sqrt{c_n}} x_{ij} \sqrt{w_0 \left( \frac{i-j}{c_n} \right)} + \frac{1}{\sqrt{c_n}} x_{ij} \sqrt{w_0 \left( \frac{i-j+n}{c_n} \right)} + \frac{1}{\sqrt{c_n}} x_{ij} \sqrt{w_0 \left( \frac{i-j-n}{c_n} \right)}.$$  

In this context, let us also define the band index set $I_j$, and the band diagonal matrix $I_j$ as follows

$$I_j := \begin{cases} 
\{1 \leq i \leq n : \min\{|i-j|, n-|i-j|\} \leq b_n\} & \text{if } M \text{ is periodic} \\
\{1 \leq i \leq n : |i-j| \leq b_n\} & \text{if } M \text{ is non-periodic}
\end{cases}$$

$$I_j := \begin{cases} 
\sum_{i \in I_j} e_i e_i^t w_\nu \left( \frac{i-j}{c_n} \right) & \text{if } \nu \in (0, 1] \\
\sum_{i \in I_j} e_i e_i^t w_0 \left( \frac{i-j}{c_n} \right) & \text{if } \nu = 0
\end{cases}.$$  

In particular, we observe that if $\mathbb{E}[x_{ij}] = 0$ and $\mathbb{E}[|x_{ij}|^2] = 1$ for all $1 \leq i, j \leq n$, then

$$\mathbb{E}[m_{ij}m_j^*] = \frac{1}{c_n} I_j,$$

where $m_j$ is the $j$th column of $M$, which is one of the $M_0^\otimes, M_0^\circ, M_0^\circ$ in the above.

In the above definition, we notice that if we take $w(x) \equiv 1$, then it yields band matrices without any variance profile i.e., identical variances. We also observe that if the periodic band matrix is a full matrix then $\nu = 1$.

We would also like to mention that when $\nu \in (0, 1]$, we are considering only periodic band matrices; and when $\nu = 0$, we are considering both periodic and non-periodic band matrices. In short, we are not considering non-periodic band matrices when $\nu \in (0, 1]$. The CLT may still be true for non-periodic band matrices with $\nu \in (0, 1]$. However, our method of variance calculation does not work in this case; as outlined in Remarks 5.1, 5.2.

**Definition 2.2** (Poincaré inequality). A complex random variable $\xi$ is said to satisfy Poincaré inequality with constant $\alpha \in (0, \infty)$ if for any differentiable function $h : \mathbb{C} \to \mathbb{C}$, we have $\text{Var}(h(\xi)) \leq \frac{1}{\alpha} \mathbb{E}[|h'(\xi)|^2]$. Here $\mathbb{C}$ is identified with $\mathbb{R}^2$.

Here are some properties of Poincaré inequality

(1) If $\xi$ satisfies Poincaré inequality with constant $\alpha$, then $c \xi$ also satisfies Poincaré inequality with constant $\alpha/c^2$ for any $c \in \mathbb{R}\setminus\{0\}$.

(2) If two independent random variables $\xi_1, \xi_2$ satisfy the Poincaré inequality with the same constant $\alpha$, then for any differentiable function $h : \mathbb{C}^2 \to \mathbb{C}$, $\text{Var}(h(\xi)) \leq \frac{1}{\alpha} \mathbb{E}[|\nabla h|^2]$ i.e., $\xi := (\xi_1, \xi_2)$ also satisfies Poincaré inequality with the same constant $\alpha$. 

\[\]
(3) [Lemma 4.4.3] If \( \xi \in \mathbb{C}^N \) satisfies Poincaré inequality with constant \( \alpha \), then for any differentiable function \( h : \mathbb{C}^N \rightarrow \mathbb{C} \)

\[
\mathbb{P}( |h(\xi) - \mathbb{E}[h(\xi)]| > t ) \leq K \exp \left\{ - \frac{\sqrt{\alpha t}}{\sqrt{2 \| \nabla h \|_2}} \right\},
\]

where \( K = -2 \sum_{i=1}^{\infty} 2^i \log(1 - 2^{-2i-1}) \). Here \( \mathbb{C}^N \) is identified with \( \mathbb{R}^{2N} \). In the above, \( \| \nabla h(x) \|_2 \) denotes the \( \ell^2 \) norm of the \( N \)-dimensional vector \( \nabla h(x) \) at \( x \in \mathbb{C}^N \); and \( \| \nabla h \|_2 = \sup_{x \in \mathbb{C}^N} \| \nabla h(x) \|_2 \). In particular if \( h : \mathbb{C}^N \rightarrow \mathbb{C} \) is a Lipschitz function with lipschitz constant \( \| h \|_{Lip} \), then \( \| \nabla h \|_2 \leq \| h \|_{Lip} \).

For example, Gaussian random variables and compactly supported continuous random variables satisfy Poincaré inequality.

3. Main result

**Condition 3.1.** Let \( \{x_{ij} : 1 \leq i, j \leq n\} \) be an i.i.d. set of complex-valued continuous random variables, and \( M \) be one of the random band matrices as defined in the Definition 2.1. Assume that

(i) \( \mathbb{E}[x_{ij}] = 0 \) and \( \mathbb{E}[|x_{ij}|^2] = 1 \) for all \( 1 \leq i, j \leq n \),

(ii) \( x_{ij} \)'s are continuous random variables with bounded density and satisfy the Poincaré inequality with some universal constant \( C \). In particular, \( \mathbb{E}[|x_{ij}|^k] \leq C k^k \) for all \( k \geq 2 \) and for some universal constant \( C \).

(iii) Either of the following is true.

(a) \( n \geq c_n \geq \log^3 n \), and \( \mathbb{E}[x_{ij}^k] = 0 \) for all \( 1 \leq i, j \leq n \); \( k \in \mathbb{N} \),

(b) \( \lim_{n \to \infty} (c_n/n) = \nu \in (0, 1] \) and \( \mathbb{E}[x_{ij}^2] = 0 \) for all \( 1 \leq i, j \leq n \).

Here the Poincaré inequality is assumed for both \( c_n = o(n) \) as well as \( c_n = \Omega(n) \) to unify the proof. However in the latter case i.e., \( c_n = \Omega(n) \), the proof may go through using the techniques of [33] without Poincaré inequality. Also in (iii)(a), it suffices to take \( c_n > \log^{2+\epsilon} n \) (see (4.14)) for some \( \epsilon > 0 \). But we assume \( c_n > \log^3 n \) for simplicity.

The above conditions implies that \( \| M \| \leq \rho \) for some fixed \( \rho \) almost surely as \( n \to \infty \). We shall discuss this in more details in appendix A.

**Theorem 3.2.** Let \( M \) be one of the \( n \times n \) random band matrices of bandwidth \( b_n \) as defined in the Definition 2.1 such that condition 3.1 holds. Let \( f_1, f_2, \ldots, f_k : \mathbb{C} \rightarrow \mathbb{C} \) be analytic on \( \mathbb{D}_{\rho+\gamma} \) for some \( \tau > 0 \) and bounded elsewhere.

Then as \( n \to \infty \),

\[
\sqrt{\frac{c_n}{n}} \left( \mathcal{L}_{f_1}^\Delta(M), \mathcal{L}_{f_2}^\Delta(M), \ldots, \mathcal{L}_{f_k}^\Delta(M) \right) \overset{d}{\to} \mathcal{N}_k(0, \Sigma, \Upsilon),
\]

where \( \Upsilon = 0 \),

\[
\Sigma_{ij} = -\frac{\nu}{4\pi^2} \sum_{k \in \mathbb{Z}} \int_{\partial D_1} \int_{\partial D_1} f_i(z) f_j(\eta) \hat{w}_\nu(k) \frac{dz d\eta}{(z \bar{\eta} - \hat{w}_\nu(k))^2}
\]

if \( \nu = \lim_{n \to \infty} (c_n/n) \in (0, 1] \) and \( M \) is periodic,

and

\[
\Sigma_{ij} = -\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\partial D_1} \int_{\partial D_1} f_i(z) f_j(\eta) \hat{w}_0(t) \frac{dz d\eta dt}{(z \bar{\eta} - \hat{w}_0(t))^2}
\]

if \( \lim_{n \to \infty} (c_n/n) = 0 \).

Here

\[
\hat{w}_\nu(k) = \int_{-1/2}^{1/2} w_\nu(x) e^{2\pi ikx} \, dx = \int_{-1/2}^{1/2} w(x) e^{2\pi ikx} \, dx,
\]

and
\[
\hat{w}_0(t) = \int_{-1}^{1} e^{2\pi itx} w_0(x) \, dx = \int_{-1/2}^{1/2} e^{2\pi itx} w(x) \, dx.
\]

We give the proof in Section 3 and variance is calculated in Section 4. Before going into the proof, we would like to make some remarks about the above theorem. First of all, the theorem is stated for \(\mathcal{L}_f^2(M) = \mathcal{L}_f(M) - nf(0)\); not for \(\mathcal{L}_f^2(M) := \mathcal{L}_f(M) - \mathbb{E}[\mathcal{L}_f(M)]\). To the best of our knowledge, the circular law for random band matrices is not known for \(c_n = o(n)\). If the circular law is true for random band matrices, we would asymptotically have

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\mathcal{L}_f(M)] = \frac{1}{\pi} \int_{\mathbb{D}_1} f(z) \, dR(z) \, d\mathfrak{S}(z) = f(0) = \frac{1}{2\pi i} \int_{\partial_{\mathbb{D}_1}} \frac{f(z)}{z} \, dz.
\]

Then asymptotically we would have \(\mathcal{L}_f^2(M) \approx \mathcal{L}_f^2(M)\).

Secondly, if \(M_f^{\hat{w}}\) is a non-Hermitian full matrix with a continuous variance profile \(w(x)\), then \(\nu = 1\). Limiting ESM of such matrices was discovered in [12]. The following corollary provides a CLT for such matrices.

**Corollary 3.3.** Let \(M\) be a non-Hermitian random matrix with a variance profile \(w(x)\) as defined in Definition 2.1 such that condition 3.1 holds. Let \(f_1, f_2, \ldots, f_k : \mathbb{C} \to \mathbb{C}\) be analytic on \(\mathbb{D}_{1+\tau}\), for some \(\tau > 0\) and bounded elsewhere.

Then \((\mathcal{L}_f^2(M), \mathcal{L}_f^2(M), \ldots, \mathcal{L}_f^2(M)) \xrightarrow{d} N_0(0, \Sigma, \Upsilon)\) as \(n \to \infty\), where \(\Upsilon = 0\),

\[
\Sigma_{ij} = -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \int_{\partial_{\mathbb{D}_1}} \int_{\partial_{\mathbb{D}_1}} f_i(z)\overline{f_j(\eta)} \left(\frac{1}{1 - z \overline{\eta}}\right)^2 \, dz \, d\overline{\eta},
\]

\[
\hat{w}(k) = \int_{-1/2}^{1/2} w(x) e^{2\pi ikx} \, dx.
\]

It should be noted that we can also have \(\nu = 1\) without having a full matrix; for example, by replacing \(o(n)\) many off diagonals of a full matrix by zeros. The above corollary along with Theorem 3.2 asserts that the limiting Gaussian distribution will be unchanged by doing so. In addition to the above corollary, we discuss a few more particular cases.

(I) If we have the full matrix with i.i.d. entries, then \(\nu = 1\), and \(w(x) = 1\). In that case, \(\hat{w}(k) = 1\) for \(k = 0\).

As a result,

\[
\Sigma_{ij} = -\frac{1}{4\pi^2} \int_{\partial_{\mathbb{D}_1}} \int_{\partial_{\mathbb{D}_1}} f_i(z)\overline{f_j(\eta)} \left(\frac{1}{1 - z \overline{\eta}}\right)^2 \, dz \, d\overline{\eta}
\]

\[
= -\frac{1}{4\pi^2} \int_{\partial_{\mathbb{D}_1}} \int_{\partial_{\mathbb{D}_1}} f_i(z)\overline{f_j(\eta)} \left[\frac{1}{\pi} \int_{\partial_{\mathbb{D}_1}} \frac{dR(\zeta) \, d\mathfrak{S}(\zeta)}{(\zeta - z)^2(\zeta - \overline{\eta})^2}\right] \, dz \, d\overline{\eta}
\]

\[
= \frac{1}{\pi} \int_{\partial_{\mathbb{D}_1}} f_i(\zeta)\overline{f_j(\zeta)} \left(\frac{1}{\pi} \int_{\partial_{\mathbb{D}_1}} dR(\zeta) \, d\mathfrak{S}(\zeta)\right) \, dz.
\]

The above is the same as the expression obtained in [33]. In particular, if \(f(z) = z^k\), then \(\text{Var}(\lim_{n \to \infty} \mathcal{L}_f^2(M)) = k\) for all \(k \in \mathbb{N}\). A numerical evidence of this fact is outlined in table 1.

(II) Let \(\nu \in (0, 1]\) and \(M_f^{\hat{w}}\) be a periodic band matrix as defined in Definition 2.1 with variance profile \(w(x) \equiv 1\). Consider the monomial test function \(f(z) = z^k\). Then as a consequence of Theorem 3.2 we have \(\sqrt{c_n/n} \mathcal{L}_f^2(M) \xrightarrow{d} N(0, \nu, (f), 0)\), where \(\nu_{\nu}(f) = l\nu \sum_{k \in \mathbb{Z}} \sin^{\nu(k\pi\nu)}\),

and \(\text{sinc}(t) = \sin(t)/t\). The above equality follows from the fact that

\[
\hat{w}_\nu(k) = \int_{-1/2}^{1/2} w_\nu(x) e^{2\pi ikx} \, dx = \int_{-1/2}^{1/2} e^{2\pi ikx} \, dx = \text{sinc} (k\pi \nu).
\]
(III) Let \( f(z) = z^l \) and \( \nu = 0 \), then 
\[ \sqrt{c_n/n\mathcal{L}_f^2(M)} \xrightarrow{d} N(0,\mathcal{V}_0(f),0) \] 

where 
\[ \mathcal{V}_0(f) = \frac{l}{\pi} \int_{\mathbb{R}} \sin^l(t) \, dt = \frac{l}{(l-1)!} \sum_{i=0}^{[l/2]} (-1)^i \binom{l}{i} \left( \frac{l}{2} - i \right)^{l-1}. \] 

(3.1) 

The above follows from the second part of the Theorem 3.2 and the fact that 
\[ \hat{w}_0(t) = \int_{-1/2}^{1/2} e^{2\pi itx} \, dx = \text{sinc}(\pi t). \] 

(3.2) 

The last equality of (3.1) was obtained by establishing a connection to Irwin-Hall distribution which is outlined in Section 5.4. In addition if \( l \) is even, one can also write 
\[ \mathcal{V}_0(f) = \frac{l}{(l-1)!} A(l-1, l/2 - 1), \] 

where \( A(n, m) \) is an Eulerian number. In combinatorics, \( A(n, m) \) counts the number of permutations of the numbers 1, 2, ..., \( n \) in which exactly \( m \) elements are greater than the previous element.

(IV) We have the following table regarding integrals of the \( \text{sinc}(\cdot) \) function [29].

\[
\begin{align*}
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}(t) \, dt &= 1 \\
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^2(t) \, dt &= 1 \\
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^3(t) \, dt &= \frac{3}{4} \\
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^4(t) \, dt &= \frac{2}{3} \\
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^5(t) \, dt &= \frac{115}{192} \\
\frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^6(t) \, dt &= \frac{11}{20}.
\end{align*}
\] 

(3.3)

From the above equations, we obtain

\[
\begin{align*}
\mathcal{V}_0(z) &= 1, \quad \mathcal{V}_0(z^2) = 2, \quad \mathcal{V}_0(z^3) = \frac{9}{4} = 2.25 \\
\mathcal{V}_0(z^4) &= \frac{8}{3} \approx 2.67, \quad \mathcal{V}_0(z^5) = \frac{575}{192} \approx 2.9948 \\
\mathcal{V}_0(z^6) &= \frac{33}{10} = 3.3 \\
\ldots
\end{align*}
\]

Figure 3 shows some simulations done in Python. We have taken \( n \times n \) periodic random band matrix with i.i.d. complex \( N(0,1) \) variables. We run the simulation for test functions \( f(z) = z^l \) and varying bandwidth.

| \( f(z) \)   | \( z \)    | \( z^2 \)   | \( z^3 \)   | \( z^4 \)   | \( z^5 \)   | \( z^6 \)   |
|-------------|------------|-------------|-------------|-------------|-------------|-------------|
| \( b_n = n^{0.3} \) | 1.0018     | 2.0357      | 2.4068      | 2.6585      | 3.0184      | 3.3746      |
| \( b_n = n/2 \)  | 1.0767     | 1.8385      | 3.1995      | 3.8756      | 4.6096      | 5.9000      |

Table 1. Numerical values are the variances of \( \sqrt{c_n/n\mathcal{L}_f^2(M)} \), calculated from 500 iterations in each case.
4. PROOF OF THEOREM 3.2

We adopt the methods based on [33, 35]. Let us define the event

$$\Omega_n := \{ \| M \| \leq \rho \},$$

where $\rho$ is the same as in Lemma A.3. From Lemma A.3 we also have that

$$P(\Omega_n^c) \leq K \exp \left( -\frac{\alpha c_n 3\rho}{2\omega} \right).$$

So, if $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{D}_{\rho+\tau}$, then on the event $\Omega_n$ we may write

$$\mathcal{L}_f(M) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} \sum_{i=1}^{n} \frac{f(z)}{z - \lambda_i(M)} \, dz = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} f(z) \text{tr} R_z(M) \, dz,$$

and

$$\mathcal{L}_f^\Delta(M) = \mathcal{L}_f(M) - nf(0) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} f(z) \text{tr} R_z^\Delta(M) \, dz,$$

where $\text{tr} R_z^\Delta(M) := \text{tr} R_z(M) - \frac{n}{z}$. While the above is true for any such function $f$, the readers may think of $f$ as a linear combination of $f_1, \ldots, f_k$ from theorem 3.2. Let us define

$$\hat{R}_z(M) := R_z(M) 1_{\Omega_n}.$$

Now we decompose $\text{tr} R_z^\Delta(M)$ as

$$\text{tr} R_z^\Delta(M) = \text{tr} R_z(M) 1_{\Omega_n} - E[\text{tr} R_z(M) 1_{\Omega_n}]$$
In the due course, we would like to show that
\[ P \text{ which implies that} \]
\[ \sup_{z \in \partial \mathbb{D}_{\rho + \tau}} |\text{tr} R_z(M)\mathbf{1}_{\Omega_n^c}| \overset{P}{\to} 0 \]
\[ \sup_{z \in \partial \mathbb{D}_{\rho + \tau}} \left| \mathbb{E}[\text{tr} \hat{R}_z(M)] - \frac{n}{\rho} \right| \to 0, \]
and \( \hat{R}_z^c(M) \) converges to an appropriate Gaussian process.

**Proof of 4.2** First of all since the entries of \( M \) are continuous random variables, \( \mathbb{P}(\|M\| = \rho + \tau) = 0 \). Which implies that \( \mathbb{P} \left( \sup_{z \in \partial \mathbb{D}_{\rho + \tau}} |\text{tr} R_z(M)| < \infty \right) = 1 \). Therefore

\[ \mathbb{P} \left( \sup_{z \in \partial \mathbb{D}_{\rho + \tau}} |\text{tr} R_z(M)\mathbf{1}_{\Omega_n^c}| > \epsilon \right) \leq \mathbb{P}(\Omega_n^c) \to 0, \text{ as } n \to \infty. \]

**Proof of 4.3** To prove (4.3) we expand \( \text{tr} R_z(M) \) on \( \Omega_n \) for \( z \in \partial \mathbb{D}_{\rho + \tau} \) as follows

\[ \text{tr} R_z(M) = \frac{n}{\rho} + \sum_{l=1}^{\left\lfloor \log^2 n \right\rfloor} \frac{1}{z^{l+1}} \text{tr} M^l + \frac{1}{z^{\left\lfloor \log^2 n \right\rfloor + 1}} \text{tr}[R_z(M)M^\lfloor \log^2 n \rfloor]. \]

On \( \Omega_n \), the last term is bounded by \( nz^{-1}(\rho/|z|)^{\log^2 n} \) for all \( z \in \partial \mathbb{D}_{\rho + \tau} \).

Now from the condition 3.1(ii, iii) and boundedness of \( \nu(x) \), we shall show that for any \( l \geq 3 \)

\[ |\mathbb{E}[\text{tr} M^l]| \leq C n^{-l/6} l^{3l-1/2} = C \left( (l^{18} n^{-1})^{\nu^{-3}} \right)^{l/6}, \]

where \( C \) is a universal constant.

Under 3.1(iii)(a), it follows that \( \mathbb{E}[\text{tr} M^l] = \sum \mathbb{E}[m_{i_1 i_2} m_{i_2 i_3} \cdots m_{i_l i_1}] = 0 \). Otherwise under condition 3.1(iii)(b), \( \mathbb{E}[m_{ij}] = 0 = \mathbb{E}[m_{ij}^n] \). Therefore non-trivial contributions in \( \mathbb{E}[\text{tr} M^l] \) (in terms of \( n \)) are obtained when each random variable appears at least three times. There are at most \( l^l \) different ways to partition \( l \) in which each partition size at least three. In each case, \( \{|i_1, i_2, \ldots, i_l| \} \leq n^{1/3} \), and the expectation is bounded by \( C l^2 \) (by condition 3.1(ii)). On the other hand, due to normalization by \( c_n^{-1/2} \), the denominator is \( c_n^{-1/2} = \Omega((\nu n)^{1/2}) \). Combining the two, we have the result (4.5).

Using the equation (4.3) along with Hölder’s inequality, we have

\[ \mathbb{E}[|\text{tr} M^l|\mathbf{1}_{\Omega_n}] \leq n \mathbb{E}[\|M\|^{2l}]^{1/2} \mathbb{E}[\Omega_n^{1/2}] \]
\[ \leq n \left[ \rho^l + \sqrt{KT(l + 1)} \left( \frac{\alpha c_n}{2\omega} \right)^{-1/2} \exp \left( -\frac{\alpha c_n}{2\omega} \right) \right] K \exp \left( -\frac{\alpha c_n}{2\omega} \right) \]
\[ = \rho^l \exp \left( -\frac{\alpha c_n}{2\omega} + \frac{2\omega(l+1)}{\alpha c_n} \right)^{1/2} K n \exp \left( -\frac{\alpha c_n}{2\omega} \right), \]
\[ \leq C \left[ \rho^l + \left( \frac{2\omega(l+1)}{\alpha c_n} \right)^{1/2} \right] n^{-2}, \]
(4.6)

for \( l \leq \log^2 n \) and large enough \( n \). Here \( C \) is a universal constant.
In addition, \( \mathbb{E}[\text{tr} M] = 0 = \mathbb{E}[\text{tr} M^2] \). As a result taking expectation in \( (4.4) \) after multiplying by \( 1_{\Omega_n}, \) and using \( (4.6), (4.5) \) we have
\[
\left| \mathbb{E}[\text{tr} R(z)(M)] - \frac{n}{2} \right| \leq C_{\tau,\rho} n^{-1/2},
\]
where for all \( z \in \partial \mathbb{D}_{\rho+\tau}, \) the constant \( C_{\tau,\rho} \) depends uniformly on \( \tau \) and \( \rho. \) This completes the proof of \( (4.3) \).

Therefore, asymptotically we may write
\[
(4.7) \quad \mathcal{L}^{\tau}_{R}(M) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} f(z) \text{tr} R^o_\tau(M) \, dz = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} f(z) \text{tr} \hat{R}^o_\tau(M) \, dz + o(1).
\]

Thus, proving the Theorem \[3.2\] is equivalent to proving the following proposition.

**Proposition 4.1.** The sequence \( \{ \frac{\sqrt{c_n}}{n} \text{tr} \hat{R}^o_\tau(M) \}_n \) is tight in the space of continuous functions on \( \partial \mathbb{D}_{\rho+\tau}, \) and converges in distribution to a Gaussian process with covariance kernel
\[
\sigma(z, \eta) = \left\{ \begin{array}{ll}
\nu \sum_{k \in \mathbb{Z}} \frac{\hat{w}_o(k)}{(\epsilon^2 - \hat{w}_o(k))^2} & \text{if } \nu = \lim_{n \to \infty} (c_n/n) \in (0,1] \text{ and } M \text{ is periodic} \\
\int_{\mathbb{R}} \frac{\hat{w}_o(t)}{(\epsilon^2 - \hat{w}_o(t))^2} \, dt & \text{if } \lim_{n \to \infty} (c_n/n) = 0.
\end{array} \right.
\]

**Remark 4.2.** Note that since \( w \) is continuous and \( \int w(x) \, dx = 1, \) we have \( |\hat{w}_o(k)|, |\hat{w}_o(t)| \leq 1 \) for all \( k \in \mathbb{Z}, \, t \in \mathbb{R}. \) Here \( \hat{w} \) stands for the Fourier transform of \( w. \) Therefore \( \sigma(z, \eta) \) is well defined for \( z, \eta \in \partial \mathbb{D}_1 \) and analytic on \( \mathbb{D}_1 \pm \epsilon \) for any \( \epsilon > 0. \) On the other hand, \( f_s \) are analytic. Therefore the value of \( \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_{\rho+\tau}} f(z) \sigma(z, \eta) \, dz \, d\bar{\eta} \) is same for any \( \epsilon > 0. \) This justifies the integrals in the Theorem \[3.2\] are over \( \partial \mathbb{D}_1 \) instead of \( \partial \mathbb{D}_{\rho+\tau}. \)

Now, we move to the proof of Proposition \[4.1\]. By Cramér-Wold device, it suffices to show that for any \( \{ z_i, 1 \leq i \leq q \} \subset \partial \mathbb{D}_{\rho+\tau} \) and \( \{ \theta_i, \beta_i : 1 \leq i \leq q \} \subset \mathbb{C} \) for which
\[
(4.9) \quad \sqrt{\frac{c_n}{n}} \sum_{i=1}^{q} \left[ \theta_i \text{tr} \hat{R}^o_{z_i}(M) + \beta_i \text{tr} \hat{R}^o_{\bar{z}}(M) \right] 1_{\Omega_n}
\]
is real; converges to a Gaussian random variable with mean zero and variance \( \sum_{i,j} \theta_i \beta_j \sigma(z_i, z_j). \) In this Section, we shall show that it converges to a Gaussian process with mean zero and unknown variance. The exact computation of variance is done in Section \[3.3\].

We use the martingale difference technique from Lemma \[3.1\] to establish the above. Let \( \mathcal{E}_k \) be the averaging with respect to the \( k \)th column of \( M, \) and \( \mathbb{E}_k[] = \mathbb{E}_{k+1} \mathcal{E}_{k+2} \ldots \mathcal{E}_n[.] = \mathbb{E}_{k+1} \mathcal{E}_{k+1}[.] \). Clearly, \( \mathbb{E}_0[.] = \mathbb{E}[.] \) and \( \mathbb{E}_n[.] = [.]. \) We write \( \text{tr} \hat{R}^o_{\tau}(M) \) as
\[
(4.10) \quad \text{tr} \hat{R}^o_{\tau}(M) = \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) \text{tr} \hat{R}^o_{\tau}(M) = \sum_{k=1}^{n} \mathbb{E}_k \left\{ (\text{tr} \hat{R}^o_{\tau}(M))_{\tau} \right\} =: \sum_{k=1}^{n} S_{z,k}(M),
\]
where \( \xi_{\tau} = \xi - \mathcal{E}_k[.]. \) Clearly, \( \{ S_{z,k}(M) \}_{1 \leq k \leq n} \) is a martingale difference sequence with respect to the filtration \( \mathcal{F}_{n,k} = \sigma\{ m_j : 0 \leq j \leq k \}. \) Rewrite \( (4.9) \) as
\[
\sum_{i=1}^{q} \left[ \theta_i \text{tr} \hat{R}^o_{z_i}(M) + \beta_i \text{tr} \hat{R}^o_{\bar{z}}(M) \right] = \sum_{k=1}^{n} \left\{ \sum_{i=1}^{q} \theta_i S_{z_i,k}(M) + \beta_i S_{\bar{z},k}(M) \right\} =: \sum_{k=1}^{n} \xi_{n,k}.
\]
Clearly, \( \{ \xi_{n,k} \}_{1 \leq k \leq n} \) is a martingale difference sequence with respect to the filtration \( \mathcal{F}_{n,k}. \) Notice that \( \mathbb{E}[\mathcal{F}_{n,k-1}] = \mathbb{E}_{k-1}[.]. \) Therefore, condition (i) of Lemma \[3.1\] is equivalent to
show that replacing \( R \) entry of (4.14) thus proving the Proposition 4.1 is equivalent to showing the tightness of \( \{ (4.12), (4.13) \). We do the following general reductions before the proof.

\[ c_n \sum_{k=1}^{n} \mathbb{E}[\xi_{n,k}^2 1_{|\xi_{n,k}| > \delta}] \leq \frac{c_n}{n \delta^2} \sum_{k=1}^{n} \mathbb{E}[\xi_{n,k}^4] \leq \frac{(2\delta)^3 c_n}{n \delta^2} \sum_{k=1}^{n} \sum_{i=1}^{q} (|\theta_i|^4 + |\beta_i|^4)\mathbb{E}[|S_{z,k}(M)|^4] \rightarrow 0. \]

(4.11)

And the condition (ii) of Lemma \([B.1]\) is equivalent to

\[ c_n \sum_{k=1}^{n} \mathbb{E}_{k-1} \left[ S_{z,k}(M)S_{z,k}(M) \right] \xrightarrow{p} 0, \]

(4.12)

and

\[ c_n \sum_{k=1}^{n} \mathbb{E}_{k-1} \left[ S_{z,k}(M)S_{z,k}(M^*) \right] \xrightarrow{p} \sigma(z_i, z_j). \]

(4.13)

Thus proving the Proposition \([4.1]\) is equivalent to showing the tightness of \( \{ \sqrt{n} \text{tr} \hat{R}_z(M) \}_{n} \) and (4.11), (4.12), (4.13). We do the following general reductions before the proof.

Let \( M^{(k)} \) be obtained by setting the \( k \)th column of \( M \) to zero, and \( \hat{R}_z(M^{(k)}) := R_z(M^{(k)})\mathbb{1}_{\Omega_{n,k}}, \) where \( \Omega_{n,k} = \{ \|M^{(k)}\| \leq \rho \}. \) The resolvent identity yields \( \text{tr} R_z(M) = \text{tr} R_z(M^{(k)}) + e_k^t R_z(M^{(k)}) R_z(M) m_k. \) We show that replacing \( R_z(M^{(k)})\mathbb{1}_{\Omega_n} \) by \( R_z(M^{(k)})\mathbb{1}_{\Omega_{n,k}} \) in (4.10) is asymptotically same in probability.

\[ \mathbb{E} \left[ \sum_{k=1}^{n} \mathbb{E}_{k} \left[ \text{tr} R_z(M^{(k)})\mathbb{1}_{\Omega_n} - \text{tr} R_z(M^{(k)})\mathbb{1}_{\Omega_{n,k}} \right]_k \right] \xrightarrow{p} 0, \]

\[ \leq 2 \mathbb{E} \left[ \sum_{k=1}^{n} \mathbb{E}_{k} \left[ \text{tr} R_z(M^{(k)})\mathbb{1}_{\Omega_n^c \cap \Omega_{n,k}} \right] \right] \leq 2 n^2 \tau \mathbb{P}(\Omega_n^c) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \]

Therefore using resolvent identity and Lemma \([B.2]\) we have

\[ \text{tr} \hat{R}_z(M) = \text{tr} \hat{R}_z(M^{(k)}) + e_k^t \hat{R}_z(M^{(k)}) \hat{R}_z(M) m_k \]

\[ = \text{tr} \hat{R}_z(M^{(k)}) + \frac{e_k^t \hat{R}_z(M^{(k)})^2 m_k}{1 + e_k^t \hat{R}_z(M^{(k)}) m_k} \]

\[ = \text{tr} \hat{R}_z(M^{(k)}) - \frac{\partial}{\partial z} \log \{1 + \delta_k(z)\}, \]

where

\[ \delta_k(z) := e_k^t \hat{R}_z(M^{(k)}) m_k. \]

We notice that \( \delta_k(z) \) is a product of two independent random variables. Intuitively, conditioned on \( \{m_j : j \in \{1, 2, \ldots, n\} \setminus k\}, \delta_k(z) \rightarrow 0 \) almost surely. We give the exact estimate below.

Define \( h(m_k) = e_k^t \hat{R}_z(M^{(k)}) m_k = \delta_k(z). \) Then using the property \([3]\) in Definition \([2.2]\) and the fact that \( \sum_{j=1}^{n} |\hat{R}_z(M^{(k)})_{ij}|^2 \leq \|\hat{R}_z(M^{(k)})\|^2 \leq \tau^{-2} \) for \( z \in \partial \mathbb{D}_{\rho + \tau}, \) we have

\[ \mathbb{P} \left( |\delta_k(z)| > t \mid \hat{R}_z(M^{(k)}) \right) \leq K \exp \left( -\tau \sqrt{\frac{\omega c_n}{2\omega}} \right), \]

(4.14)

where \( \omega = \sup_x w(x) \). The factor \( c_n/\omega \) is obtained by using Definition \([2.2](1)\) and the fact that each entry of \( M \) is scaled by the variance profile \( w \) and \( 1/\sqrt{c_n} \). Now, taking \( t_n = c_n^{-1/8} \) and using the fact that \( c_n \geq \log^3 n \) we have \( \sum_n \exp(-\tau \sqrt{\frac{\omega c_n}{2\omega}} t_n) < \infty. \) Hence by Borel–Cantelli lemma, we have \( \delta_k(z) \xrightarrow{a.s.} 0. \) In addition, we also get that

\[ \mathbb{E}[|\delta_k(z)|^p] \leq K p! \left( \frac{2\omega}{\tau^2 \alpha c_n} \right)^{p/2} \forall z \in \partial \mathbb{D}_{\rho + \tau}. \]
We notice that
\begin{equation}
(4.15) \quad \| \text{tr} \hat{R}_z(M^{(k)}) \|_\infty = \text{tr} \hat{R}_z(M^{(k)}) - \mathcal{E}_k[\text{tr} \hat{R}_z(M^{(k)})] = 0.
\end{equation}

Because \( M^{(k)} \) is independent of the \( k \)th column \( m_k \). Consequently,
\begin{equation}
S_{z,k}(M) = \mathbb{E}_k \{ (\text{tr} R_z(M))_k^\circ \} = -\mathbb{E}_k \left[ \frac{\partial}{\partial z} \log \{ 1 + \delta_k(z) \} \right]_k^n.
\end{equation}

Now we show that the above expectation exists. Let \( p_{\delta_k} \) be the pdf of \( 1 + \delta_k(z) | \hat{R}_z(M^{(k)}) \). Since the pdf of each \( x_{ij} \) are bounded (condition [3.1(ii)]), by Young’s convolution inequality, \( \| p_{\delta_k} \|_\infty \leq C \sqrt{n} \). So for any \( 1 \leq s < 2 \) and large enough \( n \),
\begin{align*}
\mathbb{E}[|1 + \delta_k(z)|^{-s}| \hat{R}_z(M^{(k)})] &= \int_{D_{1/2}} r^{-s} p_{\delta_k}(r, \theta) r \, dr \, d\theta + \int_{D_{1/2}^c} r^{-s} p_{\delta_k}(r, \theta) r \, dr \, d\theta \\
&\leq C \frac{2^{s-2}}{2-s} \tau \sqrt{cn} + K 2^{s-1} \\
&\leq C' \frac{2^{s-2}}{2-s} \tau \sqrt{cn},
\end{align*}
where \( C' \) is a universal constant. Taking further expectation,
\begin{equation}
(4.16) \quad \mathbb{E}[|1 + \delta_k(z)|^{-s}] \leq C' \frac{2^{s-2}}{2-s} \tau \sqrt{cn}, \quad \forall 1 \leq s < 2.
\end{equation}

Similarly also using (4.14), \( \mathbb{E}[|\log(1 + \delta_k(z))|] \leq C' \tau \sqrt{cn} \). Therefore, we have
\begin{equation}
(4.17) \quad S_{z,k}(M) = \mathbb{E}_k \left\{ (\text{tr} \hat{R}_z(M))_k^\circ \right\} = -\frac{\partial}{\partial z} \mathbb{E}_k[(\log(1 + \delta_k(z)))_k^n],
\end{equation}
where the switching between \( \mathbb{E}_k \) and \( \partial/\partial z \) can be justified by the dominated convergence theorem and (4.16). The above technique rewrites \( S_{z,k}(M) \) in terms of derivative of an analytic function, which will allow us to estimate \( S_{z,k}(M) \) via the following basic fact from complex analysis;
\begin{equation}
(4.18) \quad |g'(z)| = \left| \frac{1}{2\pi i} \oint_{z+\partial D_\tau/3} \frac{g(\zeta) \, d\zeta}{(\zeta - z)^2} \right| \leq \frac{5}{\pi \tau^2} \oint_{z+\partial D_\tau/3} |g(\zeta)| \, d\zeta,
\end{equation}
where \( g \) is analytic on \( z + D_\tau/2 \).

4.1. **Proof of tightness.** This part is similar to Section 3.2 in [33]. In this subsection, we show that \( \{ \sqrt{cn/\text{tr} \hat{R}_z^2(M)} \}_n \) is tight in \( \mathcal{C}(\partial D_{\rho+\tau}) \), the space of continuous functions on \( \partial D_{\rho+\tau} \). In other words, for any \( \epsilon > 0 \) there exists a compact set \( \mathcal{K}(\epsilon) \subset \mathcal{C}(\partial D_{\rho+\tau}) \) such that
\begin{equation}
(4.19) \quad \mathbb{P} \left( \sqrt{\frac{cn}{n}} \text{tr} \hat{R}_z^2(M) \in \mathcal{K}(\epsilon)^c \right) < \epsilon.
\end{equation}

However, the compact sets in the space of continuous functions are the space of equicontinuous functions. Let us define,
\begin{equation}
\Xi_n := \{ |\delta_k(z)| < c_n^{-1/8}, \quad \forall 1 \leq k \leq n \}.
\end{equation}

By (4.14), and simple union bound
\begin{equation}
\mathbb{P}(\Xi_n^c) \leq K \exp \left( \log n - \tau \sqrt{\frac{\alpha}{2\omega}} c_n^{3/8} \right) = o(1).
\end{equation}

The last equality follows from the assumption that \( c_n \geq \log^3 n \). Consequently,
\begin{equation}
\mathbb{P} \left( \sqrt{\frac{cn}{n}} \text{tr} \hat{R}_z^2(M) \in \mathcal{K}(\epsilon)^c \right) \leq \mathbb{P} \left( \sqrt{\frac{cn}{n}} \left| \frac{\text{tr} \hat{R}_z^2(M) - \text{tr} \hat{R}_z^2(M)}{z - \eta} \right| > K \right).
\end{equation}
In the view of (4.15), we have
\[
\frac{\text{tr} \hat{R}^\circ_z(M) - \text{tr} \hat{R}^\circ_z(M)}{z - \eta} \left| 1_{\epsilon_n > K} \right| + o(1), 
\]
for any \( K > 0 \). Now, (4.19) follows from Markov’s inequality and the following condition;
\[
\text{E} \left\{ c_n \left| \frac{\text{tr} \hat{R}^\circ_z(M) - \text{tr} \hat{R}^\circ_z(M)}{z - \eta} \right|^2 1_{\epsilon_n} \right\} \leq C,
\]
uniformly for all \( z, \eta \in \partial \mathcal{D}_{\rho+\tau} \) and \( n \in \mathbb{N} \). In what follows, the methods are similar to [33]. For the sake of completeness, we outline it here.

Using the resolvent identity, we can write
\[
\text{tr} \hat{R}_z(M) - \text{tr} \hat{R}_\eta(M) = (\eta - z)\text{tr}[\hat{R}_z(M)\hat{R}_\eta(M)].
\]
Thus, in the view of (4.10)
\[
(\eta - z)^{-1} \left[ \text{tr} \hat{R}^\circ_z(M) - \text{tr} \hat{R}^\circ_\eta(M) \right] = \sum_{k=1}^n E_k \{ (\text{tr}[\hat{R}_z(M)\hat{R}_\eta(M)])\}.
\]
Using resolvent identity and Lemma B.2
\[
\text{tr}[\hat{R}_z(M)\hat{R}_\eta(M) - \hat{R}_z(M)\hat{R}_\eta(M)] = \text{tr}[\hat{R}_z(M)\hat{R}_\eta(M)]
\]
when \( 3 \in (1 + \delta_k(z))(1 + \delta_k(\eta)) \). Therefore we can rewrite (4.21) as
\[
(\eta - z)^{-1} \left[ \text{tr} \hat{R}^\circ_z(M) - \text{tr} \hat{R}^\circ_\eta(M) \right] = \sum_{k=1}^n E_k \{ [\Theta_1(k) + \Theta_2(k) + \Theta_3(k)]\}.
\]
Since \( \{[\Theta_i(k)]\} \) is a martingale difference sequence,
\[
\text{E} \sum_{k=1}^n E_k \{ [\Theta_1(k) + \Theta_2(k) + \Theta_3(k)]\}^2 = \sum_{k=1}^n \text{E} \{ E_k \{ [\Theta_1(k) + \Theta_2(k) + \Theta_3(k)]\}^2 \}.
\]
Therefore, proving (4.20) is equivalent to showing that
\[
c_n \text{E} \{ [\Theta_i(k)]^2 \} \leq C', \quad 1 \leq k \leq n, \quad i = 1, 2, 3,
\]
uniformly for all \( n \in \mathbb{N} \) and \( z, \eta \in \partial \mathcal{D}_{\rho+\tau} \). However, applying the same method as described in (4.14), we can get similar tail estimates for \( \epsilon_k \hat{R}_z(M)\hat{R}_\eta(M^k)m_k \) etc. (with \( \tau^2 \) or \( \tau^3 \) in the rhs of (4.14)). Now since \( \delta_k(z) \leq c_n^{-1/8} \) on \( \Xi_n \), using the estimate \( (1 + \delta_k(z))^{-1} \leq 2 \) in \( \text{E} \{ [\Theta_i(k)]^2 \} \), we have (4.22).
4.2. Proof of (4.11). Expanding \( \log \{1 + \delta_k(z)\} \) up to two terms and using (4.14), (4.17), (4.18) we have

\[ E[|S_{z,k}(M)|^4] = O(c_n^{-2}). \]

Substituting the above in (4.11), we obtain

\[ \sum_{k=1}^{n} E[\xi_{n,k}^2 1_{|\xi_{n,k}| > \delta}] = \delta^{-2} O(c_n^{-1}), \]

which proves the result.

4.3. Proof of (4.12). Using condition 3.1iii), (4.14) and expanding \( \log \{1 + \delta_k(z)\} \) up to two terms, we see that

\[ \mathcal{E}_k \{ \mathbb{E}_k[\log \{1 + \delta_k(z)\}] \} \mathbb{E}_k[\log \{1 + \delta_k(\eta)\}] = O(c_n^{-2}). \]

Thus, using (4.17), (4.18) and the above we have

\[ \frac{c_n}{n} \sum_{k=1}^{n} \mathbb{E}_{k-1} [S_{z,k}(M) S_{z,k}(M)] = O(c_n^{-1}), \]

which proves (4.12).

4.4. Proof of (4.13). Expanding \( \log \{1 + \delta_k(z)\} \) up to two terms and using (4.14), (4.17) we have

\[ E_{k-1}[S_{z,k}(M) S_{\bar{\eta},k}(M^*)] = \frac{\partial^2}{\partial z \partial \bar{\eta}} D_k(z, \bar{\eta}), \]

where

\[ D_k(z, \bar{\eta}) = \mathcal{E}_k \left\{ \mathbb{E}_k[\log \{1 + \delta_k(z)\}] \mathbb{E}_k[\log \{1 + \delta_k(\eta)\}] \right\} \]

\[ = \frac{1}{c_n} e_k \mathbb{E}_k [R_z(M^{(k)})] \mathbb{E}_k [R_\eta(M^{(k)})] e_k + O(c_n^{-2}) \]

\[ = \frac{1}{c_n} T_k(z, \bar{\eta}) + O(c_n^{-2}). \]

As a result, (4.13) becomes

\[ \frac{c_n}{n} \sum_{k=1}^{n} \mathbb{E}_{k-1} [S_{z,k}(M) S_{\bar{\eta},k}(M^*)] = \frac{\partial^2}{\partial z \partial \bar{\eta}} \left[ \frac{1}{n} \sum_{k=1}^{n} T_k(z, \bar{\eta}) \right] + O(c_n^{-1}) \]

\[ = \frac{\partial^2}{\partial z \partial \bar{\eta}} U(z, \bar{\eta}) + O(c_n^{-1}). \]

Now since \( \|R_z(M^{(k)})\| \leq 2T^{-1} \) on \( \mathbb{D}_r^{c+\tau/2} \), \( U(z, \bar{\eta}) \) is a sequence of uniformly bounded analytic functions on \( \mathbb{D}_r^{c+\tau/2} \). Therefore by Vitali’s theorem (eg. [40, Theorem 5.21]), proving (4.13) equivalent to show that \( U(z, \bar{\eta}) \) converges in probability.

Since \( \{S_{z,k}(M)\} \) is a martingale difference sequence, we have \( E[S_{z,k}(M) S_{\bar{\eta},l}(M^*)] = 0 \) if \( k \neq l \). As a result,

\[ \mathbb{E} \left\{ \sum_{k=1}^{n} \mathbb{E}_{k-1} [S_{z,k}(M) S_{\bar{\eta},k}(M^*)] \right\} = \mathbb{E} \left\{ \sum_{k=1}^{n} S_{z,k}(M) \sum_{l=1}^{n} S_{\bar{\eta},l}(M^*) \right\} = \mathbb{E}[\text{tr} \hat{R}_z^2(M) \text{tr} \hat{R}_\eta^2(M^*)]. \]

Limit of the above is calculated in Section 5 Here we show that \( \text{Var}(U(z, \bar{\eta})) \rightarrow 0. \)

Recall

\[ T_k(z, \bar{\eta}) = e_k \mathbb{E}_k \left[ \hat{R}_z(M^{(k)}) \right] \mathbb{E}_k \left[ \hat{R}_\eta(M^{(k)})^* \right] e_k. \]

Let \( \epsilon_n = 2/c_n \), and \( \phi_{\epsilon_n, k} : \mathbb{R}^{2n} \rightarrow [0, 1] \) be a smooth function such that

\[ \phi_{\epsilon_n, k} |\|M^{(k)}\| \leq \rho \equiv 1, \]

\[ \phi_{\epsilon_n, k} |\|M^{(k)}\| \geq \rho + \epsilon_n \equiv 0, \]
\[
\frac{\partial \phi_{e_n,k}}{\partial x_i} \leq \frac{2}{\epsilon_n}, \quad \forall 1 \leq i \leq 2n, x \in \mathbb{R}^{2n}.
\]

Let us define \( \hat{R}_z(M^{(k)}) = R_z(M^{(k)})\phi_{e_n,k} \) and
\[
\hat{T}_k(z, \bar{\eta}) = \epsilon_k^i \mathbb{E}_k \left[ \hat{R}_z(M^{(k)}) \right] \mathcal{I}_k \mathbb{E}_k \left[ \hat{R}_\eta(M^{(k)})^* \right] e_k.
\]

We note the following estimate
\[
|T_k(z, \bar{\eta}) - \hat{T}_k(z, \bar{\eta})| \leq \frac{20}{(\tau - \epsilon_n)^2} \phi_{e_n,k} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}}.
\]

Using the Lemma A.3, we have
\[
\mathbb{E} \left[ |T_k(z, \bar{\eta}) - \hat{T}_k(z, \bar{\eta})|^2 \right] \leq \frac{K}{(\tau - \epsilon_n)^4} \exp \left( -\sqrt{\frac{\alpha c_n}{2\omega^2}} \right).
\]

For notational simplicity, let us denote
\[
u := \mathbb{E}_k \left[ R_z(M^{(k)}) \right] \sqrt{\mathcal{I}_k}, \quad v := \mathbb{E}_k \left[ R_\eta(M^{(k)})^* \right],
\]
\[
\hat{\nu} := \mathbb{E}_k \left[ \hat{R}_z(M^{(k)}) \right] \sqrt{\mathcal{I}_k}, \quad \hat{v} := \mathbb{E}_k \left[ \hat{R}_\eta(M^{(k)})^* \right],
\]
where \( \sqrt{\mathcal{I}_k} \) denotes the diagonal matrix by taking square root of each entry of the diagonal matrix \( \mathcal{I}_k \). Then \( \hat{T}_k(z, \bar{\eta}) = \sum_{s \in I_k} \hat{\nu}_s \hat{v}_s \). Since \( x_{ij} \)’s satisfy Poincaré inequality, we have
\[
\text{Var}(\hat{T}_k(z, \bar{\eta})) \leq \frac{1}{\alpha} \sum_{j=1}^{k-1} \sum_{i \in I_j} \mathbb{E} \left[ \frac{\partial \hat{T}_k(z, \bar{\eta})}{\partial x_{ij}} \right]^2 + \mathbb{E} \left[ \frac{\partial \hat{T}_k(z, \bar{\eta})}{\partial x_{ij}} \right]^2.
\]

The first sum stops at \( k - 1 \) because \( \hat{T}_k(z, \bar{\eta}) \) is constant as a function of \( k, k + 1, \ldots, n \) columns of \( M \).

On the other hand, we have
\[
\frac{\partial R_z(M^{(k)})_{ks}}{\partial m_{ij}} = R_z(M^{(k)})_{ki} R_z(M^{(k)})_{js}, \quad \frac{\partial R_z(M^{(k)})_{sk}}{\partial m_{ij}} = 0.
\]

Consequently,
\[
\frac{\partial \hat{\nu}_s}{\partial m_{ij}} = \hat{\nu}_{ki} \hat{v}_{js} + u_{ks} \frac{\partial \phi_{e_n,k}}{\partial m_{ij}} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}},
\]
\[
\frac{\partial \hat{v}_s}{\partial m_{ij}} = v_{sk} \frac{\partial \phi_{e_n,k}}{\partial m_{ij}} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}},
\]
\[
\frac{\partial (\hat{T}_k(z, \bar{\eta}))}{\partial m_{ij}} = \sum_{s \in I_k} \hat{\nu}_s \hat{v}_s + \sum_{s \in I_k} (u_{ks} \hat{v}_s + \hat{\nu}_s v_{sk}) \frac{\partial \phi_{e_n,k}}{\partial m_{ij}} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}}.
\]

Denoting \( \tilde{y}_k = \sum_{s \in I_k} \hat{\nu}_s \hat{v}_s \) and using the facts that \( \|\hat{\nu}\|, \|\hat{v}\| \leq (\tau - \epsilon_n)^{-1} \), we have
\[
\sum_{j=1}^{k-1} \sum_{s \in I_j} \left| \hat{\nu}_{ki} \hat{v}_{js} \right|^2 = \sum_{j=1}^{k-1} |\hat{\nu}_{ki} \tilde{y}_k|^2 \leq \frac{1}{2} \|\hat{\nu}_k\|^2 \|\tilde{y}_k\|^2 \leq (\tau - \epsilon_n)^{-6},
\]
\[
\sum_{j=1}^{k-1} \sum_{s \in I_j} (u_{ks} \hat{v}_s + \hat{\nu}_s v_{sk}) \frac{\partial \phi_{e_n,k}}{\partial m_{ij}} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}}^2 \leq \frac{8}{\epsilon_n} \sum_{j=1}^{k-1} \sum_{s \in I_j} \left| \hat{\nu}_s \right|^2 \left| v_{sk} \right|^2 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}} \leq 8nc_n^3 (\tau - \epsilon_n)^{-4} 1_{\{\rho < \|M^{(k)}\| < \rho + \epsilon_n\}}.
\]

Using the above estimates, [4.1], and the fact that \( \partial m_{ij}/\partial x_{ij} = O(\epsilon_n^{-1/2}) \) we have,
\[
\text{Var} \left( \frac{1}{n} \sum_{k=1}^{n} \hat{T}_k(z, \bar{\eta}) \right) \leq \frac{1}{n} \sum_{k=1}^{n} \text{Var}(\hat{T}_k(z, \bar{\eta})) \leq \frac{2}{\alpha c_n(\tau - \epsilon_n)^6} + \frac{K n c_n^2}{(\tau - \epsilon_n)^4} \exp \left( - \sqrt{\frac{\alpha c_n^3 \rho}{2 \omega}} \right).
\]

No using the estimate \((4.23)\) and the assumption \(c_n > \log^3 n\), we conclude that

\[
\text{Var}(U(z, \bar{\eta})) = \text{Var} \left( \frac{1}{n} \sum_{k=1}^{n} T_k(z, \bar{\eta}) \right) \leq \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ (T_k(z, \bar{\eta}) - \bar{T}_k(z, \bar{\eta}))^2 \right] + \text{Var} \left( \frac{1}{n} \sum_{k=1}^{n} \bar{T}_k(z, \bar{\eta}) \right) \leq \frac{2}{\alpha c_n(\tau - \epsilon_n)^6} + \frac{K n c_n^2}{(\tau - \epsilon_n)^4} \exp \left( - \sqrt{\frac{\alpha c_n^3 \rho}{2 \omega}} \right) \to 0, \text{ as } n \to \infty.
\]

5. Calculation of the variance

Let us first find the variance for monomial test functions. Let us define \(f(z) = z^l; \ l \geq 2\).

5.1. Case I: \(\nu = \lim_{n \to \infty} \frac{c_n}{n} \in (0, 1]\). The matrix in Definition \(2.1\) is periodic.

\[
\lim_{n \to \infty} \frac{c_n}{n} \mathbb{E} \left[ \mathcal{L}_f(M) \mathcal{L}_f(M) \right] = \lim_{n \to \infty} \frac{c_n}{n} \mathbb{E} \left[ \text{tr} M^l \text{tr} M^l \right] = \lim_{n \to \infty} \frac{c_n}{n} \mathbb{E} \left\{ \sum_{i_1, \ldots, i_l=1}^{n} m_{i_1i_2} m_{i_2i_3} \cdots m_{i_li_1} \right\} \left\{ \sum_{i_1, \ldots, i_l=1}^{n} \tilde{m}_{i_1j_2} \tilde{m}_{j_2j_3} \cdots \tilde{m}_{j_1j_l} \right\}.
\]

In the above expression, the maximum contribution (in terms of \(n\)) occurs when all the indices in the loop \(i_1 \to i_2 \to i_3 \to \cdots \to i_l \to i_1\) are distinct and the loop overlaps with the loop \(j_1 \to j_2 \to j_3 \to \cdots \to j_l \to j_1\). The reasoning is similar to \((4.5)\). Once the indices \(i_1 \to i_2 \to i_3 \to \cdots \to i_l \to i_1\) are fixed, the loop \(j_1 \to j_2 \to j_3 \to \cdots \to j_l \to j_1\) must be same as the loop \(i_1 \to i_2 \to i_3 \to \cdots \to i_l \to i_1\). However, they can overlap in \(l\) different ways by rotating \(i_1 \to i_2 \to i_3 \to \cdots \to i_l \to i_1\).

Now the first index \(i_1\) can be chosen in \(n\) different ways. After that, while choosing the remaining \((l-1)\) many indices, due to the band matrix structure, each index has to be within \(\pm b_n\) neighborhood of the previous index such that the final index \(i_l\) is also within \(\pm b_n\) neighborhood of the first index \(i_1\) as well.

This last condition imposes an additional constraint which is not present in the full matrix cases. However, these band constraints are completely handled by the weight profile \(w_\nu\). Therefore using the structures of \(m_{i,j}\)s from Definition \(2.1\) we have

\[
\lim_{n \to \infty} \frac{c_n}{n} \mathbb{E} \left[ \mathcal{L}_f(M) \mathcal{L}_f(M) \right] = \lim_{n \to \infty} \frac{c_n}{n} \frac{\ln}{c_n} \sum_{1 \leq i_2, i_3, \ldots, i_l \leq n} w_\nu \left( \frac{i_1 - i_2}{c_n} \right) w_\nu \left( \frac{i_2 - i_3}{c_n} \right) \cdots w_\nu \left( \frac{i_l - i_1}{c_n} \right) = l \lim_{n \to \infty} \frac{1}{c_n} \sum_{1 \leq i_2, i_3, \ldots, i_l \leq n} w_\nu \left( \frac{i_1 - i_2}{c_n} \right) w_\nu \left( \frac{i_2 - i_3}{c_n} \right) \cdots w_\nu \left( \frac{i_l - i_1}{c_n} \right)
\]

\[
(5.1) = l \int_{[0,1/c_n]^l} w_\nu(t_1 - t_2) w_\nu(t_2 - t_3) \cdots w_\nu(t_l - t_1) \ dt_2 dt_3 \cdots dt_l
\]

\[
= l w^{(l)}_\nu(0)
\]

\[
(5.2) = l \nu \sum_{k \in \mathbb{Z}} \tilde{w}_\nu(k)^l,
\]

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where \((l)\) denotes the \(l\) fold convolution, and

\[
\hat{w}_\nu(k) = \int_{-1/2}^{1/2} w_\nu(x) e^{2\pi ikx} \, dx = \int_{-1/2}^{1/2} w(x) e^{2\pi ikx} \, dx
\]

is the \(k\)th Fourier coefficient of \(w_\nu\). Note that in any \(l\) fold convolution, the integral is taken over \(l-1\) many variables only (as in \ref{5.1}). In our context, this can also be explained by the fact that for each chosen index \(i_1\), we have freedom to choose the indices \(i_2, \ldots, i_l\) only.

**Remark 5.1.** The integral \ref{5.1} is taken over \([0, 1/\nu]^{l-1}\), because \(t_i \in [0, n/c_n] \rightarrow [0, 1/\nu]\) for all \(2 \leq i \leq l\). However, the values of the variables \(t_{i+1} - t_i\) may fall outside \([0, 1/\nu]\) in principle. But using the 1/\(\nu\) periodicity of \(w_\nu\), we can bring it back to \([0, 1/\nu]\). The above calculation does not work for non-periodic band matrices while \(c_n = \Omega(n)\).

5.2. **Case II:** \(\lim_{n \to \infty} \frac{c_n}{n} = 0\). The band matrix in Definition \ref{2.1} is periodic. In that case, we compute

\[
\begin{align*}
\lim_{n \to \infty} \frac{c_n}{n} E \left[ L_f(M) L_f(M) \right] &= \lim_{n \to \infty} \frac{c_n}{n} E \left[ \sum_{i_1, \ldots, i_l=1}^{n} \left[ \frac{m_{i_1} m_{i_2} \cdots m_{i_l}}{c_n} \right] \sum_{i_1, \ldots, i_l=1}^{n} \left[ \frac{m_{i_1} j_2 m_{i_2} j_3 \cdots m_{i_l} j_l}{c_n} \right] \right] \\
&= \lim_{n \to \infty} \frac{1}{c_n} \int_{\mathbb{R}^{l-1}} w_0(t_1 - t_2) w_0(t_2 - t_3) \cdots w_0(t_{l-1} - t_1) \, dt_1 \, dt_2 \, \ldots \, dt_l \\
&= \lim_{n \to \infty} \frac{1}{c_n} \int_{\mathbb{R}^{l-1}} w_0(-t_2) w_0(t_2 - t_3) \cdots w_0(t_{l-1} - t_1) \, dt_1 \, dt_2 \, \ldots \, dt_l \\
&= \lim_{n \to \infty} \frac{1}{c_n} \int_{\mathbb{R}^{l-1}} \hat{w}_0(t) \, dt,
\end{align*}
\]

where

\[
\hat{w}_0(t) = \int_{\mathbb{R}} e^{2\pi i tx} w_0(x) \, dx = \int_{-1/2}^{1/2} e^{2\pi i tx} w(x) \, dx.
\]

**Remark 5.2.** In the above, if the band matrix was not periodic, we can make the above integration over \(\mathbb{R}^{l-1}\) i.e., \(l\) fold convolution, by taking \(t_1\) far off from the origin. We can do that because of \(t_1 \in (0, n/c_n) \rightarrow (0, \infty)\). Thus, the above calculation will also go through for non-periodic band matrices. However, the same approach cannot be implemented in \ref{5.1}, as in that case \(t_1 \in (0, n/c_n) \rightarrow (0, 1/\nu)\). So, we need the matrix to be periodic when \(c_n = \Omega(n)\).

5.3. **Covariance kernel of \(\text{tr} \tilde{R}_\nu(M)\).** In the view of \ref{4.5}, we notice that if \(k \neq l\) then

\[
\frac{c_n}{n} E \left[ \text{tr} \tilde{M}^k \text{tr} \tilde{M}^* \right] = O \left( \frac{1}{n^2} \int \frac{1}{1 - 1/\nu - 1/\nu(n-3)k^{l-l/6}} \right).
\]

If \(k = l = 1\), then

\[
\lim_{n \to \infty} \frac{c_n}{n} E \left[ \text{tr} \tilde{M} \text{tr} \tilde{M}^* \right] = w_\nu(0).
\]

Therefore using \ref{5.2}, Lemma \ref{A.2} for \(z, \eta > \rho\), and proceeding as \ref{4.4}, \ref{4.6} on page 9 we have,

\[
\begin{align*}
&\lim_{n \to \infty} \frac{c_n}{n} E \left[ \text{tr} \tilde{R}_\nu^z(M) \text{tr} \tilde{R}_\nu^\eta(M^*) \right] \\
&\quad = \lim_{n \to \infty} \frac{c_n}{n} \left( E \left[ \text{tr} \tilde{R}_\nu^z(M) \text{tr} \tilde{R}_\nu^\eta(M^*) \right] - \frac{n^2}{z\eta} \right) \\
&\quad = \lim_{n \to \infty} \frac{c_n}{n} \sum_{l=1}^{\infty} (z\eta)^{-l-1} E \left[ \text{tr} \tilde{M}^l \text{tr} \tilde{M}^{*l} \right]
\end{align*}
\]
Using the inversion formula,

\[ \frac{w_\nu(0)}{(z\eta)^2} + \nu \sum_{l=2}^{\infty} \sum_{k \in \mathbb{Z}} l \frac{\hat{w}_\nu(k)^l}{(z\eta)^{l+1}} \]

\[ = \frac{w_\nu(0)}{(z\eta)^2} + \nu \sum_{k \in \mathbb{Z}} \hat{w}_\nu(k)(z\eta)^{-2} \left[ \left(1 - \frac{\hat{w}_\nu(k)}{z\eta}\right)^{-2} - 1 \right] \]

\[ = \nu \sum_{k \in \mathbb{Z}} (z\eta - \hat{w}_\nu(k))^2, \]

where the last expression follows from the fact that \( w_\nu(0) = \nu \sum_{k \in \mathbb{Z}} \hat{w}_\nu(k) \). If we take \( \nu \downarrow 0 \) in the above expression, we obtain

\[ \int_{\mathbb{R}} \frac{\hat{w}_0(t)}{(z\eta - \hat{w}_0(t))^2} \, dt. \]

Alternatively when \( z, \eta > \rho \), using (5.5),

\[ \lim_{n \to \infty} \frac{c_n}{n} \left\{ \mathbb{E}[\text{tr}\, \hat{R}_z(M) \text{tr}\, \hat{R}_\eta(M)] - \frac{n^2}{z\eta} \right\} \]

\[ = \frac{n}{\sum_{l=1}^{\infty} (z\eta)^{-l-1} l \int_{\mathbb{R}} \hat{w}_0(t)^l \, dt} \]

\[ = \int_{\mathbb{R}} \frac{\hat{w}_0(t)}{(z\eta)^2} \left(1 - \frac{\hat{w}_0(t)}{z\eta}\right)^{-2} \, dt \]

\[ = \int_{\mathbb{R}} \frac{\hat{w}_0(t)}{(z\eta - \hat{w}_0(t))^2} \, dt. \]

### 5.4. Connection to Irwin-Hall distribution & Eulerian numbers.

Suppose the weight profile \( w(x) \equiv 1 \). Then (5.4) can be written as \( l_{\gamma_1} \), where

\[ \gamma_1 := \mathbb{P}(|U_2 + U_3 + \cdots + U_l| \leq 1/2), \]

and \( U_i \overset{i.i.d.}{\sim} \text{Unif}[-1/2, 1/2] \). Let \( S_{l-1} = \sum_{i=2}^{l} U_i \) and \( p_{l-1}(x) \) be the pdf of \( S_{l-1} \). Since \( S_{l-1} + \frac{l-1}{2} \) follows the Irwin-Hall distribution, the density of \( S_{l-1} \) is given by

\[ p_{l-1}(x) = \frac{1}{(l-2)!} \sum_{i=0}^{\lfloor x+(l-1)/2 \rfloor} (-1)^i \binom{l-1}{i} \left( x + \frac{l-1}{2} - i \right)^{l-1}, \quad x \in \left[ -\frac{l-1}{2}, \frac{l-1}{2} \right]. \]

For a geometric derivation of the above formula, see [28]. On the other hand, the characteristic function of \( S_{l-1} \) is given by

\[ \mathbb{E}[e^{itS_{l-1}}] = \left\{ \mathbb{E}[e^{itU_1}] \right\}^{l-1} = (\text{sinc}(t/2))^{l-1}. \]

Using the inversion formula,

\[ p_{l-1}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \text{sinc}^{l-1}(t/2) \, dt = \frac{1}{\pi} \int_{\mathbb{R}} e^{-itx} \text{sinc}^{l-1}(t) \, dt. \]

Therefore,

\[ \gamma_l = \mathbb{P}(|S_{l-1}| \leq 1/2) = \int_{-1/2}^{1/2} p_{l-1}(x) \, dx = \frac{1}{\pi} \int_{\mathbb{R}} \text{sinc}^{l-1}(t) \, dt = p_l(0) \]

\[ = \frac{1}{(l-1)!} \sum_{i=0}^{\lfloor l/2 \rfloor} (-1)^i \binom{l}{i} \left( \frac{l}{2} - i \right)^{l-1} = A(l-1, l/2 - 1) \frac{(l-1)!}{(l-1)!}, \]

where \( A(l-1, l/2 - 1) \) is an Eulerian number for even \( l \). \( A(n, m) \) counts the number of permutations of \( 1, 2, \ldots, n \) in which exactly \( m \) elements are bigger than the previous element. The above establishes (3.1).
In this section we discuss about norm of random non-Hermitian matrices. A sharp almost sure bound on the spectral radius of non-Hermitian random matrices can be found in [17, 19].

**Theorem A.1.** [18] Let \( M = (m_{ij})_{n \times n} \) be a sequence of \( n \times n \) random matrices with \( m_{ij} ; 1 \leq i, j \leq n \) real valued i.i.d. for each \( n \). Assume that for each \( n \),

(i) \( \mathbb{E}[m_{11}] = 0 \)

(ii) \( \mathbb{E}[m_{11}^2] = \sigma^2 \)

(iii) \( \mathbb{E}||m_{11}|^p|| \leq p^p \sigma^p \) for all \( p \geq 2 \) and for some \( c > 0 \).

Let

\[
\rho_n = \max_{1 \leq i \leq n} \{ |\lambda_i(M/\sqrt{n})| \}.
\]

Then \( \limsup_{n \to \infty} \rho_n \leq \sigma \) almost surely.

We see that in our context if a full random matrix satisfies the condition 3.1 then it also satisfies the above condition and as a result, \( ||M|| \leq 1 \) almost surely as \( n \to \infty \). However, the above theorem does not take a variance profile into account. The following theorem from [6] estimates the norm of a symmetric random matrix with a variance profile.

**Theorem A.2.** [6 Corollary 3.5] Let \( X \) be a \( n \times n \) real symmetric matrix with \( X_{ij} = \xi_{ij}w_{ij} \), where \( \{\xi_{ij} : i \geq j\} \) are independent centered random variables and \( \{w_{ij} : i \geq j\} \) are give scalars. If \( \mathbb{E}[|\xi_{ij}|^{2p}] \leq C p^{\beta/2} \) for some \( C, \beta > 0 \) and all \( p, i, j \), then

\[
\mathbb{E}\|X\| \leq C' \max_i \left( \sum_{j=1}^n w_{ij}^2 + C' \max_i \max_j |w_{ij}| \log^{(\beta/1)} n \right),
\]

where \( C \) depends on \( C, \beta \) only.

Using the above theorem along with Poincaré inequality, we have the following lemma.

**Lemma A.3.** Let \( M \) be an \( n \times n \) random matrix as in the Theorem 3.2. Then there exists \( \rho \geq 1 \) such that

\[
\mathbb{P}(\|M\| > \rho/4 + t) \leq K \exp\left(-\sqrt{\frac{\alpha cn}{2\omega}} t\right), \quad \forall t > 0,
\]

where \( K > 0 \) is a universal constant. In particular,

\[
\mathbb{P}(\|M\| > \rho \text{ infinitely often}) = 0,
\]

and

\[
\mathbb{E}[\|M\|^l] \leq \rho^l + K\Gamma(l + 1) \left(\frac{\alpha cn}{2\omega}\right)^{-l/2} \exp\left(\sqrt{\frac{\alpha cn}{2\omega}} \rho\right).
\]

**Proof.** First of all, we may write \( M = M_R + iM_I \), where both \( M_R \) and \( M_I \) are real valued matrices. Then we can estimate \( ||M|| \leq ||M_R|| + ||M_I|| \). Therefore without loss of generality, let us consider \( M \) be a real valued matrix. Consider

\[
\tilde{M} := \begin{bmatrix} O & M \\ M^* & O \end{bmatrix},
\]

and apply theorem A.2 on \( \tilde{M} \) to obtain the same bound for \( \mathbb{E}[\|	ilde{M}\|](= \mathbb{E}[\|M\|]) \). In our case, \( w_{ij}^2 = \frac{1}{c_n} w((i - j)/n) \) or \( w_{ij}^2 = \frac{1}{c_n} w_0((i - j)/n) \) as described in Definition 2.1. Since \( c_n \geq \log^3 n \) and \( w \) is a piece-wise continuous function, \( \lim_{n \to \infty} \sum_{j=1}^n w_{ij}^2 = \int w(x) \, dx = 1 \) and \( \max_{i,j} |w_{ij}| \log^{(3/1)} n \to 0 \). As a result, there exists \( \rho \geq 1 \) such that

\[
\limsup \mathbb{E}[\|M\|] \leq \rho/4.
\]

Here we note that we need \( c_n \) to grow at least as \( \log n \). In fact, this is a sharp condition. Otherwise, the matrix norm may be unbounded [10, 23].
Secondly, \( \|M\| \leq \sqrt{\sum_{i,j} |m_{ij}|^2} \) implies that \( h(M) := \|M\| \) is a Lipschitz function. Therefore applying the properties of Poincaré inequality as described in Definition 2.2 we have

\[
P \left( \|M\| - \mathbb{E}\|M\| > t \right) \leq K \exp \left( -\sqrt{\frac{\alpha c_n t}{2\omega}} \right)
\]

i.e., \( P(\|M\| > \rho/4 + t) \leq K \exp \left( -\sqrt{\frac{\alpha c_n t}{2\omega}} \right) \),

where \( \omega = \sup_x w(x) \).

Equation (A.2) can be seen from the equation (A.1) along with the application of Borel-Cantelli lemma.

We finally would like to remark that although the Theorem A.2 gives a constant bound on the norm of the matrix with a variance profile, the constant is not that sharp unlike Theorem A.1. However, we expect that for matrices with continuous variance profile, the correct norm bound should be \( \limsup \sup \rightarrow \infty \sum_{i=1}^n w_{ij}^2 \). This was remarked in [24, Remark 4.11]. In our case, this limit is equal to 1. As we have mentioned in remark 4.2 that eventually it suffices to take \( z, \eta \in \partial D_1 \) only.

APPENDIX B.

Here we list down the two key ingredients; martingale difference CLT, and Sherman-Morrison formula. Interested readers may find the proofs in the included references.

**Lemma B.1.** [9] Theorem 35.12 Let \( \{\xi_{n,k}\}_{1 \leq k \leq n} \) be a martingale difference array with respect to a filtration \( \{\mathcal{F}_{k,n}\}_{1 \leq k \leq n} \). Suppose for any \( \delta > 0 \),

(i) \( \lim_{n \to \infty} \sum_{k=1}^n \mathbb{E}[\xi_{n,k}^2 1_{\xi_{n,k} > \delta}] = 0 \),

(ii) \( \sum_{k=1}^n \mathbb{E}[\xi_{n,k}^2] 1_{|\mathcal{F}_{k,n-1}|} \overset{p}{\to} \sigma^2 \) as \( n \to \infty \).

Then \( \sum_{k=1}^n \xi_{n,k} \overset{d}{\to} N(0, \sigma^2) \).

**Lemma B.2.** [36], Sherman-Morrison formula. Let \( A \) and \( A + ve_k^t \) be two invertible matrices, where \( v \in \mathbb{C}^n \). Then

\[
(A + ve_k^t)^{-1}v = \frac{A^{-1}v}{1 + e_k^t A^{-1}v}.
\]

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