RINGS OF GLOBAL SECTIONS IN TWO-DIMENSIONAL SCHEMES

Holger Brenner, Bochum

Abstract

In this paper we study the ring of global sections $\Gamma(U, \mathcal{O})$ of an open subset $U = D(I) \subseteq \text{Spec } A$, where $A$ is a two-dimensional noetherian ring. The main concern is to give a geometric criterion when these rings are finitely generated, in order to correct an invalid statement of Schenzel in [7].

1 Introduction

Let $A$ be a noetherian ring with an ideal $I \subseteq A$ and $U = D(I) \subseteq \text{Spec } A$ the corresponding open subset. If $U$ is an affine scheme, then the ring of global sections $B = \Gamma(U, \mathcal{O}_X)$ –which is also called the ideal-transform $T(I)$– is of finite type over $A$. The converse is by no means true, in dimension two however we have the following result due to Eakin et. al. ([4], Theorem 3.2): Suppose $A$ is a local excellent Cohen-Macaulay domain of dimension two, and let $I$ be an ideal of height one. Then (among other characterizations) $D(I)$ is affine if and only if $B$ is noetherian if and only if $B$ is of finite type over $A$.

Schenzel states in [7], Theorem 4.1 and 4.2, that this holds more general for two-dimensional excellent local domains. However, this is not true, as the following example shows.

Example. Let $X = \text{Spec } A$ be an affine excellent irreducible surface which is regular outside one single closed point $P$ and such that in the normalization two points $P_1, P_2$ lie over $P$. Outside these points the normalization mapping $\tilde{X} \rightarrow X$ is isomorphic.

Let $Y = V(I)$ be the image of an irreducible curve $Y'$ passing through $P_1$, but not through $P_2$. Then $U = X - Y$ is not affine, since the preimage of $Y$ consists of the curve $Y'$ and of the isolated point $P_2$. On the other hand, $U = X - Y$ is normal and isomorphic to $\tilde{X} - Y' - P_2$, so the rings of global sections are identical. Since $\tilde{X}$ is normal, this ring equals also the ring of global sections of $\tilde{X} - Y'$. $\tilde{X}$ is a normal excellent affine surface, thus the complement of a curve is affine, and $B$ is finitely generated. For an explicit example see below.

1In fact the result was stated under the somewhat weaker conditions that the normalization is finite and the local rings of the normalization are analytically irreducible, instead of excellent.

2The mistake in [7] is at the end of the proof of theorem 4.1, where the statement $T \subseteq T_N$ is wrong.
In this paper we give a criterion for two-dimensional local rings to decide the finiteness of the ring of global sections of \( U = D(I) \), \( I \) an ideal of height one. The criterion is based on the combinatoric of the components in the completion \( \hat{A} \) of \( A \). It says that in case \( U \) is not affine the ring of global sections of \( U \) is not finitely generated if and only if there exists an irreducible component of \( \text{Spec} \hat{A} \) where \( U \) is affine and a component where \( U \) is not affine such that their intersection is one-dimensional.

The criterion is (due to the connectedness theorem of Hartshorne) seen to be fulfilled in case \( A \) is Cohen-Macaulay, thus we recover the result of Eakin et. al. as a corollary (Cor. 2.4). Another consequence is that if \( D(I) \) is non-affine and connected, then \( T(I) \) is not noetherian (Cor. 2.3).

In the third section we extend the result to the non-complete case and describe the conditions used in the criterion in terms of the normalization.

## 2 The complete case

Let \( X = \text{Spec} A \) be the spectrum of a local complete noetherian ring \( A \) of dimension 2, and let \( P \) denote the closed point. Let \( X_j = V(p_j) = \text{Spec} A/p_j \) be the irreducible components of \( X \) corresponding to the minimal primes \( p_j, j \in J \). Let \( I \) be an ideal in \( A \), \( Y = V(I) \) and \( U = D(I) \). \( U \) is affine if and only if \( U_j = U \cap X_j \) is affine on every component, and this is due to the theorem of Lichtenbaum-Hartshorne (see [3], 8.2.1) the case if and only if \( \text{ht} I(A/p_j) \leq 1 \) for every \( j \in J \). Thus \( U \) is not affine if and only if there exists a two-dimensional component \( X_j \) where \( Y_j = Y \cap X_j \) consists just of the single point \( P \).

We want to know for an ideal \( I \) of height one whether the ring of global sections of \( D(I) \) is finitely generated. If \( D(I) \) is affine, this is the case, so we suppose furtheron that \( D(I) \) is not affine. We divide \( J = J_0 \cup J_1 \) in such a way, that for \( j \in J_1 \) the open subsets \( U_j \subset X_j \) are affine and for \( j \in J_0 \) not. Thus the \( X_j, j \in J_0 \), are the two-dimensional components of \( X \) where \( Y_j \) is just the closed point. The affineness of \( U \) is equivalent with \( J_0 = \emptyset \).

Put \( a_0 = \bigcap_{j \in J_0} p_j \) and \( a_1 = \bigcap_{j \in J_1} p_j \) and \( X_0 = \text{Spec} A/a_0 \), \( X_1 = \text{Spec} A/a_1 \). We denote the structure sheaves on these closed subschemes of \( X \) with \( \mathcal{O}_{X_i}, i = 0, 1 \).

Furthermore we put \( U_i = U \cap X_i, i = 0, 1 \), considered as an open subset in \( X_i \) with the induced scheme structure, put \( B_i = \Gamma(U_i, \mathcal{O}_i) \). \( U_1 = \text{Spec} B_1 \subset X_1 \) is affine, \( U_0 \) is not affine. The closed embedding \( X_1 \hookrightarrow X \) yields a (closed) restriction map \( \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U_1, \mathcal{O}_1) = B_1 \).

Finally, let \( b = a_0 + a_1 \leq m \) and \( R = A/b \). \( R \) is a zero- or one-dimensional local complete noetherian ring, let \( Z = \text{Spec} R \) und \( Z^\times = D(m) \subset Z \). The dimension
of $Z = V(b) = X_0 \cap X_1$ is the crucial point for $\Gamma(U, \mathcal{O}_X)$ to be noetherian or not.

For our proof we have to put on $A$ the condition $S_1$ of Serre, meaning that every associated prime of $A$ is minimal, equivalently that every zero-divisor lies in a minimal prime or that every ideal of height one contains a non-zero-divisor. This is fulfilled for example if $A$ is reduced.

**Theorem 2.1.** Let $A$ be a two-dimensional complete local noetherian ring, fulfilling the condition $S_1$. Let $I$ be an ideal of height one and suppose that $U = D(I)$ is not affine. Then the following are equivalent.

1. $\Gamma(U, \mathcal{O}_X)$ is not of finite type.
2. $\Gamma(U, \mathcal{O}_X)$ is not noetherian.
3. The image of $\Gamma(U, \mathcal{O}_X) \to \Gamma(U_1, \mathcal{O}_1)$ is not noetherian.
4. The intersection $Z$ of $X_0$ and $X_1$ is one-dimensional.

**Proof.** The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are clear. (1) $\Rightarrow$ (4). Suppose $Z = \{P\}$ is only the closed point. Then $U$ is the disjoint union of $U_0$ and $U_1$ (both closed hence open in $U$). Thus we have

$$\Gamma(U, \mathcal{O}_X) = \Gamma(U_0, \mathcal{O}_0) \oplus \Gamma(U_1, \mathcal{O}_1).$$

Since $U_1$ is affine, the second component is of finite type. Since $U_0 = X_0 - \{P\}$, the mapping $A/a_0 \to \Gamma(U_0, \mathcal{O}_X)$ is also of finite type, see Lemma 2.2 (1).

So we have to show (4) $\Rightarrow$ (3). We denote the image of $\Gamma(U, \mathcal{O}_X) \to \Gamma(U_1, \mathcal{O}_1)$ by $C$.

Let $h \in A$ be an element such that in $Z = \text{Spec } R$ we have $V(h) = V(m) = \{P\}$. Thus $1/h$ is a function defined on $Z^\times = Z - \{P\} = D(h)$. Since $Z^\times \hookrightarrow U_1$ is a closed embedding and since $U_1$ is affine, there exists a function $q \in B_1 = \Gamma(U_1, \mathcal{O}_1)$ with $q|_{Z^\times} = 1/h$. 

3
Let $a \in b \subset A$ be a regular element (i.e. a non-zero-divisor) inside the describing ideal of $Z$. The functions $aq^n$ are defined on $U_1$ and the restrictions to $Z^\times$ are zero, thus they are extendible to $Z$. Since $Z \hookrightarrow X_0$ is closed and $X_0$ is affine, these functions are also extendible to $X_0$ and in particular to $U_0$. So we may assume that these functions are defined on $U$ and we see that they lie in $C$.

Consider in $C$ the ideal $(a, aq, aq^2, aq^3, \ldots)$ spanned by this functions, and suppose that it is finitely generated. Then we have an equation

$$aq^{n+1} = a_naq^n + \ldots + a_1aq + a_0$$

with $a_i \in C \subset B_1$. We may assume that $a_i \in \Gamma(U, O_X)$. Since $a$ is regular in $A$, it is also a regular in $A/a_0$. (For if $ax \in a_i = \bigcap_{j \in J_i} p_j$, we have $ax \in p_j$ for all $j \in J_i$ and thus $x \in p_j$ for all $j \in J_i$, so $x = 0 \mod a_i$.) Since the restriction $A/a_i = \Gamma(X_1, O_1) \rightarrow \Gamma(U_1, O_1)$ is injective, $a$ is also a regular element in $B_1$.

This yields in $B_1$ (on $U_1$) the equation $q^{n+1} = a_nq^n + \ldots + a_1q + a_0$. This equation restricted to $Z^\times \subseteq U_1$ yields an integral equation for $q = 1/h$ over $R[a_i'] \subseteq R_h$, where the $a_i'$ denote the restrictions of $a_i$ on $R_h = \Gamma(Z^\times, O_Z)$.

We claim that the $a_i'$ are integral over $R$: Consider the elements $a_i \in \Gamma(U, O_X)$ as functions on $U_0$ as elements of $B_0$. Since $U_0 = X_0 - \{P\}$, the $a_i \in B_0$ are integral over $A/a_0 = \Gamma(X_0, O_0)$, see Lemma 2.2. The closed embeddings $(Z^\times \subset Z) \hookrightarrow (U_0 \subset X_0)$ show that the $a_i'$ are integral over $R = \Gamma(Z, O_Z)$. It follows that $q |_Z = 1/h$ would be integral over $R$, but this is not possible.

**Lemma 2.2.** Let $A$ be a local noetherian ring of dimension two fulfilling $S_1$. Let $m$ be the maximal ideal and $B = \Gamma(D(m), O)$ the ring of global sections. Then the following hold.

1. $A \rightarrow B$ is of finite type.
2. If furthermore all components of $\text{Spec} \, A$ have dimension two, $B$ is even finite over $A$.

**Proof.** We first proof the second part, using [6], 5.11.4 (or [2], 2.5.). A point $x \in \text{Ass} \, O_X$ has height zero, for every ideal of bigger height contains a regular element. The closure $\bar{x}$ is a two-dimensional component and therefore the point $P$ has codimension two on it.

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\[ X_0 \gets Z \leftrightarrow X_1 \]
\[ U \]
\[ U_0 \leftrightarrow Z^\times \leftrightarrow U_1 \]
\[ \nabla \]
The first part follows from the second part. The one-dimensional components of $X$ meet the other components only in the closed point, thus the punctured curves are connected components of $W = D(m)$. These are affine and of finite type.

We deduce from the theorem two corollaries.

**Corollary 2.3.** Let $A$ be a local complete noetherian ring of dimension two fulfilling $S_1$. Let $I$ be an ideal of height one. If $U = D(I)$ is connected and $\Gamma(U, O_X)$ is of finite type, then $U$ is affine.

**Proof.** Suppose $U$ is not affine, then in the partition described above $U_0$ is not empty, and $U_1$ is not empty since $I$ is of height one. Put $Z = X_0 \cap X_1$. Since $U$ is connected, $U_0$ and $U_1$ are not disjoint, thus $Z$ does not consist only of the closed point, it must be a curve. Then due to the theorem the ring of global sections can not be noetherian.

We recover the result of Eakin et. al. in the Cohen-Macaulay case.

**Corollary 2.4.** Let $A$ be a local complete noetherian Cohen-Macaulay ring of dimension two. Let $I$ be an ideal of height one. Then $U = D(I)$ is affine if and only if its ring of global sections is of finite type (or noetherian).

**Proof.** Again, suppose $U$ to be not affine, put $X = X_0 \cup X_1$ as before with the describing ideals $a_0$ and $a_1$. Then $a_0 \cap a_1$ is nilpotent, thus due to the connectedness theorem of Hartshorne (see [5], theorem 18.12) the ideal $a_0 + a_1$ has height one. Since it describes the intersection, $Z = X_0 \cap X_1$ is one-dimensional and $\Gamma(U, O_X)$ is not noetherian.

**Example.** Of course, $U = D(I)$ can be affine without being connected. $A = K[[x, y, z]]/(xy)$ is Cohen-Macaulay ($K$ a field), the complement of the common axis $V(x, y)$ is affine, but not connected.

**Remark.** We may associate to a complete local ring of dimension two a graph $\Gamma$ in such a way, that for each irreducible two-dimensional component we associate a point, and two points are connected by an edge if and only if the intersection of the corresponding components is one-dimensional. Then an open subset as above yields a partition $\Gamma = \Gamma_0 \cup \Gamma_1$, and the ring of global sections is noetherian if and only if there is no edge between points of $\Gamma_0$ and of $\Gamma_1$.

### 3 Interpretation in the Normalization

We want to extend the result from the complete case to the general case. Suppose we are given a curve $V(I) \subseteq \text{Spec } A$ where $A$ is a two-dimensional noetherian domain. Then $\Gamma(D(I), O)$ is of finite type if this is true in every (closed) point $x \in \text{Spec } A$, see [1]. Furthermore, we have the following Lemma.
Lemma 3.1. Let \( A \longrightarrow A' \) be faithfully flat and let \( U \subseteq \text{Spec} \, A \) denote an open subset with preimage \( U' \). Then \( B = \Gamma(U, O) \) is of finite type over \( A \) if and only if \( B' = \Gamma(U', O') \) is of finite type over \( A' \).

Proof. We have \( B' = B \otimes_A A' \) due to flatness. This yields the first implication. If \( B' \) is of finite type, we may assume that it is generated by finitely many elements of \( B \), thus there is a surjection \( A'[T_1, \ldots, T_n] \longrightarrow B' = B \otimes_A A' \) induced by \( A[T_1, \ldots, T_n] \longrightarrow B \). Due to faithfull, this must also be surjective. \( \Box \)

Therefore the condition in the theorem that \( \Gamma(U, O) \) is of finite type is preserved by passing to the completion, and we may skip in Cor. 2.4 the assumption of completeness.

So we take a look at the condition that the intersection of two components in the completion is one-dimensional, and we want to describe it in terms of the normalization of \( A \). For this we recall some correspondences between normalization and completion, see [6], 7.6.1 and 7.6.2. Let \( X \) be the spectrum of a local excellent domain \( A \) with completion \( \hat{X} \) and normalization \( \tilde{X} \). Then the normalization of \( \hat{X} \) equals the completion of \( \tilde{X} \) (semilocal), and this consists of connected components being the normalizations of the irreducible components of \( \hat{X} \) and the completion of the localizations of \( \tilde{X} \) as well. In particular, there is a correspondence between the irreducible components of \( \hat{X} \) and the closed points of \( \tilde{X} \).

For a closed subset \( C \subseteq X \) the completion of \( C \) equals the preimage of \( C \) in \( \hat{X} \) yielding a canonical inclusion \( \hat{C} \subseteq \hat{X} \). The irreducible components of \( \hat{C} \) correspond again to closed points of \( \tilde{C} \), but this is of course not the preimage of \( C \) in the normalization \( \tilde{X} \).

Lemma 3.2. Let \( A \) be an excellent local domain of dimension two, \( P_0 \in \tilde{X} \) the closed point on \( \tilde{X} \) corresponding to the irreducible component \( X_0 \) of the completion \( \hat{X} \). Let \( C \subseteq X \) be an irreducible curve and let \( D \subseteq X \) be the preimage of \( C \) without the isolated points.

1. There exists an irreducible component of \( \hat{C} \) on \( X_0 \) if and only if \( P_0 \) is not an isolated point on \( \varphi^{-1}(C) \) (\( \varphi : \hat{X} \longrightarrow X \) normalization map).

2. The irreducible component \( C_0 \) of \( \hat{C} \) lies on \( X_0 \) if and only if there exists a point \( R \in \hat{D} \) over \( P_0 \) mapping to the point \( Q_0 \in \tilde{C} \) corresponding to \( C_0 \).

3. The component \( C_0 \) of \( \hat{C} \) connects the irreducible components \( X_1 \) and \( X_2 \) of \( \hat{X} \) if and only if the corresponding point \( Q_0 \in \tilde{C} \) is reached by points \( R_1, R_2 \in \hat{D} \) lying over \( P_1 \) and \( P_2 \).

Proof. (1). We consider the mapping (completion) \( \hat{X}_0 \longrightarrow \hat{X}_{P_0} \), where \( \hat{X}_{P_0} \) means the localization at \( P_0 \). The preimage of \( C \subseteq X \) in \( \hat{X}_{P_0} \) is just the closed point if and only if this is true in \( \hat{X}_0 \), and this is the case if and only if \( C \) is zero-dimensional on \( X_0 \).
Definition. Let $X$ denote a reduced irreducible noetherian scheme, $\varphi : \tilde{X} \to X$ its normalization, $P \in X$ a closed point and $P_1, P_2 \in \tilde{X}$, $\varphi(P_1) = \varphi(P_2) = P$. We call an irreducible curve $C \subset X$ a melting curve for the points $P_1$ and $P_2$ if and only if $P_1, P_2$ are not isolated on $\varphi^{-1}(C)$ and there exist points $R_1, R_2 \in D (D$ as in lemma 3.2) over $P_1, P_2$ mapping to one common point $Q \in C$.

Theorem 3.3. Let $X = \text{Spec } A$, where $A$ is an excellent local domain of dimension two. Then the intersection of the components $X_1$ and $X_2$ on $\tilde{X}$ is one-dimensional if and only if there exists a melting curve for $P_1, P_2 \in \tilde{X}$.

Proof. If $C$ is a melting curve for $P_1$ and $P_2$ with common point $Q$ as in the definition, then the previous proposition says that the corresponding component $C_0$ lies on $X_1$ and $X_2$, thus the intersection is one-dimensional.

For the converse, let $C_0$ be an irreducible curve on $X_1 \cap X_2$ with prime ideal $q \subset A$ of height one. Then $p = q \cap A$ is also of height one. For $q$ is not a normal point of $A$, since on the normalization there are at least two points above it. Then also $p$ is not a normal point, because the normal locus commutes with completion under the condition of excellence (see [6], 7.8.3.1.) Thus $\text{ht } p = 1$, $C = V(p)$ is a curve, $C_0$ a component of its completion and we may apply the previous proposition. \qed

Proposition 3.4. Let $P_1, P_2$ be two closed points in the normalization $\tilde{X}$ over $P \in \text{Spec } A$, where $A$ is a two-dimensional noetherian domain. Then the following hold.

1. If there exist two different irreducible curves $C_1, C_2$ with $P_i \in C_i = V(q_i)$ on $\tilde{X}$ such that $q_1 \cap A = q_2 \cap A = p$, then $C = V(p)$ is a melting curve for $P_1, P_2$.
2. If $C$ is normal (or analytically irreducible) and $P_1$ and $P_2$ are not isolated on $\varphi^{-1}(C)$, then $C$ is a melting curve.
3. If $P_1, P_2 \in C'$ is irreducible and $\varphi(C') = C$ is a melting curve, then $\varphi|_{C'} : C' \to C$ is not birational. A melting curve lies in the non-normal locus.

Proof. (1) Both mappings $C_1 \to C$ and $C_2 \to C$ are surjective, and this is then also true for the normalizations. Thus for any closed point $Q \in C$ there are points on $C_i$ over $P_i$ mapping to $Q$. 

7
(2) If $C$ is analytically irreducible, then any closed point of $\tilde{D}$ maps to the only closed point of $\tilde{C}$.

(3) Suppose $C' \to C$ is birational. Then the normalizations of these curves are the same, and different points cannot be identified. If the generic point of a curve $C$ is normal, then $D$ consists just of one irreducible component, and $D \to C$ is birational. □

**Examples.** We give some typical examples of (non-)melting curves to illustrate the cases the previous proposition is talking about. They are given by mappings $\mathbb{A}^2_K \to \mathbb{A}^2_K$ such that the affine plane is the normalization of the image ($K$ is a field).

(1) $(x, y) \mapsto (x, y^3 - y, y^2 - 1)$. This identifies the two different curves $V(y - 1)$ and $V(y + 1)$. The common image curve $C$ is a melting curve.

(2) $(x, y) \mapsto (x, y^2, xy)$. The line $V(x)$ is melted with itself, identifying the points $(0, 1)$ and $(0, -1)$. $V(x) \to V(r, t)$ is not birational, $C$ is a melting curve.

(3) $(x, y) \mapsto (x, y^2, y((y - 1)^2 + x^2)((y + 1)^2 + x^2), xy)$. This identifies only the two points. $V(x)$ is birational with its image $C$, thus $C$ is not a melting curve.
(4) Consider the mapping \((x, y) \mapsto (x, y^2, y(x^2 - y^2(y + 1)))\) followed by the identification of the points \((0, 0, 0)\) and \((-1, 0, 0)\). Then \(D = V(x^2 - y^2(y + 1)) \mapsto C\) is not birational, but \(C (= \text{the image of } D)\) is not a melting curve for their common point. Thus the necessary condition in prop. 3.4 (3) is not sufficient.

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Holger Brenner
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
brenner@cobra.ruhr-uni-bochum.de