Optimal control analysis of a multigroup SEAIHRD model for COVID-19 epidemic

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Abstract
The COVID-19 pandemic has threatened public health and caused substantial economic loss to most countries worldwide. A multigroup susceptible–exposed–asymptomatic–infectious–hospitalized–recovered–dead (SEAIHRD) compartment model is first constructed to model the spread of the disease by dividing the population into three age groups: young (aged 0–19), prime (aged 20–64), and elderly (aged 65 and over). Then, we develop a free terminal time, partially fixed terminal state optimal control problem to minimize deaths and costs associated with hospitalization and the implementation of different control strategies. And the optimal strategies are derived under different assumptions about medical resources and vaccination. Specifically, we explore optimal control strategies for reaching herd immunity in the COVID-19 outbreak in a free terminal time situation to evaluate the effect of nonpharmaceutical interventions (NPIs) and vaccination as control measures. The transmission rate of SARS-CoV-2 is calibrated by using real data in the United States at the early stage of the epidemic. Through numerical simulation, we conclude that the outbreak of COVID-19 can be contained by implementing appropriate control of the prime age population and relatively strict control measures for young and elderly populations. Within a specific period, strict control measures should be implemented before the vaccine is marketed.

Keywords
COVID-19, intervention strategy, optimal control

1 | INTRODUCTION
The spread of acute respiratory syndrome coronavirus 2 (SARS-CoV-2) has caused more than 4.5 million deaths worldwide, with more than 230 million confirmed cases reported globally by September 2021. It has had a huge social influence and caused economic loss to most countries in the world. While the debate continues about how SARS-CoV-2 is transmitted, infection control guidelines have stated that most respiratory virus transmissions, including contact, droplet, airborne, fomite, and so forth (WHO, 2020). Several nonpharmaceutical interventions (NPIs) measures have been implemented to reduce the transmission of this disease in many countries. Including washing hands, wearing masks, maintaining social distance when leaving the house, quarantining at home, closing schools, working from home, and restricting public gatherings. Even though intervention can effectively control the spread of the disease, the cost of applying these control measures is very high. Therefore, it is urgent to predict the dynamics of the epidemic, evaluate various intervention measures, and develop the best management strategies to balance the trade-off between the losses caused by the spread of disease and the cost of intervention during the outbreak.

Since the outbreak, many scholars have used clinical case data and mathematical models to estimate important epidemiological parameters of COVID-19 (Hauser et al., 2020; He et al., 2020b; Li et al., 2020; Liu et al., 2020; Salje et al., 2020). Establishing an epidemic disease model is very important for predicting and controlling the epidemic scenario. The susceptible–infected–recovered (SIR) compartment model, established by Kermack and McKendrick (1927), brought the mathematical modeling of infectious diseases to a new level. Since then, many different epidemic models based on the classical SIR compartment model have been developed to simulate the dynamics of infectious diseases, such as Susceptible-Infectious-Recovered-Susceptible (SIRS) (Vargas-De-León, 2011), Susceptible-Exposed-Infectious-Recovered (SEIR) (Almeida, 2018), Susceptible-Exposed-Asymptomatic-Infectious-Recovered (SEAIR) (Zhang & Jin, 2012), susceptible–exposed–asymptomatic–infectious–hospitalized–recovered–dead (SEAIHRD; Ledder & Homp, 2023;43:62–77.
2022), and so forth. These models can better predict the spread of infectious diseases with incubation periods by adding compartments, such as exposed and asymptomatic, to the SIR model. Many studies have applied the SEIR model and its extensions to predict and control COVID-19. For example, He et al. (2020a) proposed an improved SEIR model to describe the evolution of the epidemic in Hubei province, China; and Hao et al. (2020) built a SEPAIHR model to reconstruct the full-spectrum dynamics of COVID-19 in Wuhan, based on 32,583 laboratory-confirmed cases.

Optimal control theory plays an essential role in the research and control strategy of various infectious diseases. This theory has proven to be a successful tool in understanding ways to curtail the spread of infectious diseases by devising optimal disease intervention strategies. Revelle et al. (1967) first applied optimal control theory to the disease control process to determine the optimal control strategies. Fast et al. (2015) designed optimal strategies to minimize the expected total cost of disease, including the cost of control measures. Recently, optimal control models have been used to study intervention strategies to reduce the spread of COVID-19. Alvarez et al. (2020) developed the optimal control strategy to control the spread of COVID-19 while minimizing the output costs of the lockdown based on the SIR compartment model. Perkins and España (2020) performed an optimal control analysis to determine optimal intervention strategies to control COVID-19, and the results of different control strategies under different parameters were analyzed. Djidjou-Demasse et al. (2020) explored the optimal control problem of COVID-19 to develop the best control strategies before vaccines are developed. Ketcheson (2021) proposed a control strategy to achieve herd immunity by controlling the number of infections to avoid greater deaths resulting from too many hospitalizations.

Many studies on COVID-19 have shown differences in contact rates, genetic factors, and other aspects among people of different ages (Davies et al., 2020; Hauser et al., 2020). According to the study by Lau et al. (2020), age is an important factor in the transmission of the virus by analyzing reported cases between March and early May 2020 in the state of Georgia. Davies et al. (2020) also found that susceptibility to infection in adults over 20 years is roughly twice that of individuals under 20. Because different age groups have different impacts on the social economy, it is necessary for decision makers to consider age when implementing different intervention strategies. However, there have been few studies on strategies for different age groups.

In this article, we developed a multigroup SEAIHRD compartment model and divided the total population into three age groups: young population (aged 0–19), prime population (aged 20–64), and elderly population (aged 65 and over). Then, we formulate and solve the optimal intervention strategies in a free terminal time optimal control framework. The objective is to minimize deaths and costs associated with implementing an age-based control strategy under different medical resources and vaccine assumptions. We also calculate the different control strategies under the three scenarios of limited, unlimited medical resources, and having vaccines developed 1 year later. This will enable decision makers to control the spread of the epidemic while minimizing the cost of containment and the impact on economic activity.

This article makes three contributions. First, we have developed a new multigroup SEAIHRD model and derived the expression of the basic reproduction number for the model. Second, we prove the existence and characterization of the solution to the optimal control problem under the proposed model by considering different groups. Third, the model parameters are calibrated using real epidemic data in the United States, making the model more consistent with reality. To the best of our knowledge, this is the first study to calibrate the parameters and study the optimal control problem under a multigroup SEAIHRD model.

The structure of the article is organized as follows. Section 2 describes a multigroup SEAIHRD compartment model, and Section 3 proposes the intervention optimal control strategy for the multigroup SEAIHRD compartment model. Numerical simulations and empirical analysis are presented in Section 4. Finally, Section 5 summarizes the conclusions.

## 2 | THE MODEL

### 2.1 | General SEAIHRD model

The general SEAIHRD compartment model consists of the following seven compartments:

- Susceptible ($S$): The number of individuals who are not infected by the disease.
- Exposed ($E$): The number of individuals who have been infected by the disease but are not yet infectious.
- Asymptomatic infected ($A$): The number of individuals who are asymptomatic after being exposed and are capable of infecting others.
- Symptomatic infected ($I$): The number of individuals who develop symptoms after being exposed and are capable of infecting others.
- Hospitalized ($H$): The number of individuals with severe symptoms requiring hospitalization.
- Recovered ($R$): The number of recovered individuals, and it is assumed that the individuals have become immune.
- Dead ($D$): The number of deaths due to the disease. As infectious individuals with severe symptoms are hospitalized, this compartment only accounts for those transferred from the hospitalized compartment.

The transition diagram of the SEAIHRD model is illustrated in Figure 1, and a system of ordinary differential equations for the SEAIHRD model is given as follows:

\[
\frac{dS}{dt} = -\gamma S,\\
\frac{dE}{dt} = \gamma S - \theta E,\\
\frac{dA}{dt} = \theta E - \alpha A - \delta A,\\
\frac{dI}{dt} = \alpha A - \beta I,\\
\frac{dH}{dt} = \beta I,\\
\frac{dR}{dt} = \mu H,\\
\frac{dD}{dt} = \mu H.
\]
Transmission diagram of SEAIHRD (Susceptible-Exposed-Asymptomatic-Infectious-Hospitalized-Recovered-Dead) model.

**FIGURE 1** Transmission diagram of SEAIHRD (Susceptible-Exposed-Asymptomatic-Infectious-Hospitalized-Recovered-Dead) model.

**TABLE 1** Description of each parameter in SEAIHRD (Susceptible-Exposed-Asymptomatic-Infectious-Hospitalized-Recovered-Dead) model.

| Parameter | Description |
|-----------|-------------|
| $\alpha$  | The rate of transmission between susceptible and asymptomatic infected individuals. |
| $\beta$   | The rate of transmission between susceptible and symptomatic infected individuals. |
| $\theta$  | Waiting rate to viral shedding. |
| $\rho$    | Proportion of symptomatic infected cases. |
| $\eta$    | The average time from onset to hospitalization of infected individuals. |
| $\delta$  | The average time between hospitalization and death. |
| $r_A$     | Recovery rate of asymptomatic infections. |
| $r_I$     | Recovery rate of symptomatic infections. |
| $r_D$     | The average time to recover after hospitalization. |

The ordinary differential equations for this model are given by

\[
\begin{align*}
\frac{dS_i}{dt} &= -S_i \Gamma_i, \\
\frac{dE_i}{dt} &= S_i \Gamma_i - \theta E_i, \\
\frac{dA_i}{dt} &= (1 - \rho_i) \theta E_i - r_A A_i, \\
\frac{dI_i}{dt} &= \rho_i \theta E_i - (\eta_i + r_I) I_i, \\
\frac{dH_i}{dt} &= \eta_i I_i - (\delta_i + r_H) H_i, \\
\frac{dR_i}{dt} &= r_A A_i + r_I I_i + r_H H_i, \\
\frac{dD_i}{dt} &= \delta_i H_i, \\
\end{align*}
\]

(1)

with initial value $S_i(0) = S_i^0, E_i(0) = E_i^0, A_i(0) = A_i^0, I_i(0) = I_i^0, H_i(0) = H_i^0, R_i(0) = R_i^0$, and $D_i(0) = D_i^0$, for $i = 1, \ldots, n$.

Let $C = (c_{ij})_{n \times n}$ denote the contact matrix. According to Guo et al. (2006), contact matrix $C$ is irreducible since the infected individuals in each group can infect susceptible individuals of any other group through contact. The feasible domain $\Omega = \{(S_i, E_i, A_i, I_i, H_i, R_i, D_i) \in \mathbb{R}^7_+ \mid S_i + E_i + A_i + I_i + H_i + R_i + D_i = 2006, i = 1, \ldots, n\}$.

There exists a disease-free equilibrium $x_0$ on the boundary of $\Omega$:

\[
x_0 = (S_1^0, \ldots, S_n^0, 0, \ldots, 0, 0, \ldots, 0),
\]

where $S_i^0$ is the initial population of group $i$ and $\sum_{i=1}^n S_i^0 = 1$.

2.3 Basic reproduction number $R_0$

Let $\mathcal{F}$ be the rate of appearance of new infections in each compartment. $\mathcal{V}^+$ and $\mathcal{V}^-$ denote the rate of transfer of individuals into and out of each compartment by all other means, respectively. Then, $\mathcal{F}$, $\mathcal{V}^+$, and $\mathcal{V}^-$ are defined as

\[
\mathcal{F} = \begin{pmatrix}
0 \\
S_i \Gamma_i \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}_{7 \times 1}, \quad \mathcal{V}^+ = \begin{pmatrix}
0 \\
-(1 - \rho_i) \theta E_i \\
\rho_i \theta E_i \\
\eta_i I_i \\
r_A A_i + r_I I_i + r_H H_i \\
\delta_i H_i \\
\end{pmatrix}_{7 \times 1}.
\]
\[
\mathcal{V}^{-} = \\
\begin{pmatrix}
S_i \Gamma_i \\
\theta_i E_i \\
r_i^A A_i \\
(\eta_i + r_i^H) H_i \\
0 \\
0
\end{pmatrix}_{n \times 1}.
\]

Let \( \mathcal{V} = \mathcal{V}^{-} - \mathcal{V}^{+} \), then we get

\[
\mathcal{V} = \\
\begin{pmatrix}
S_i \Gamma_i \\
\theta_i E_i \\
r_i^A A_i - (1 - \rho_i) \theta_i E_i \\
(\eta_i + r_i^H) H_i - \rho_i \eta_i H_i \\
- r_i^A A_i - r_i^H H_i \\
- \delta_i H_i
\end{pmatrix}_{n \times 1}.
\]

The infected compartments are \( E, A, \) and \( I, \) then

\[
F_0 = \\
\begin{pmatrix}
0 \\
S_i \Gamma_i \\
0
\end{pmatrix}_{3 \times 1}, \quad V_0 = \\
\begin{pmatrix}
\theta_i E_i \\
(\eta_i + r_i^H) H_i - \rho_i \eta_i H_i \\
- r_i^A A_i - r_i^H H_i
\end{pmatrix}_{3 \times 1}.
\]

Then, the Jacobian matrix of \( F_0 \) and \( V_0 \) at the disease-free equilibrium point \( x_0 \) are

\[
F = \\
\begin{pmatrix}
0 & 0 & S_i' \sigma_1 \sigma_1 & \cdots & S_i' \sigma_n \sigma_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & S_i' \sigma_1 \sigma_1 & \cdots & S_i' \sigma_n \sigma_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n},
\]

\[
V = \\
\begin{pmatrix}
\theta_i & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \theta_a & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & - (1 - \rho_i) \theta_a & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & - (1 - \rho_i) \theta_a & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \eta_i + r_i^H & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & - \rho_i \theta_a & 0 & \cdots & \eta_i + r_i^H
\end{pmatrix}_{n \times n}.
\]

By using the next-generation matrix (van den Driessche & Watmough, 2002), we obtain the basic reproduction number

\[
R_0 = \rho(FV^{-1}),
\]

where \( \rho \) denotes the spectral radius.

The basic reproduction number \( R_0 \) is an important parameter for understanding the severity of the epidemic, and \( R_0 = 1 \) is of great significance as the threshold for the disappearance of infectious diseases. When \( R_0 > 1 \), the disease will cause an epidemic, and the number of infected individuals will increase. In contrast, the number of infected individuals decreases monotonically to 0, and the disease dies out when \( R_0 \leq 1 \).

### 3 | OPTIMAL CONTROL PROBLEM

This section considers a free terminal time, partially fixed terminal state optimal control problem, with NPIs control, vaccination control, and limited medical supplies.

Let \( u_i \) be the intensity of NPIs, and \( v_i \) denote the vaccination distribution rate in group \( i \) for \( i = 1, \ldots, n \). By definition, \( u_i(t) \) and \( v_i(t) \) always satisfy

\[
0 \leq u_i(t) \leq u_{\text{max}}, 0 \leq v_i(t) \leq v_{\text{max}},
\]

where \( u_i(t) = 0 \) means that there are no NPIs implemented and people are free to move about; \( u_i(t) = u_{\text{max}} \) denotes full NPIs and all individuals are isolated at home. \( v_i(t) = 0 \) means no vaccination and \( v_i(t) = v_{\text{max}} \) means full vaccination. Then, the control set is

\[
\Lambda = \{(u_i, v_i) \in L^1(0,T)(u_i(t), v_i(t)) \}
\]

\[
\in [0, u_{\text{max}}] \times [0, v_{\text{max}}], \forall t \in [0, T], i = 1, \ldots, n,
\]

where \( T \) is the terminal time, which is a free decision variable.

Herd immunity is an important public health phenomenon. Herd immunity acts as a barrier against the disease. After the population has reached herd immunity, epidemics can slow down or even end. So we set the terminal states to be the population at which herd immunity was achieved, that is,

\[
S_i(T) = S_i^{\text{herd}}, \ i = 1, \ldots, n,
\]

where \( S_i^{\text{herd}} \) is the number of different populations reaching herd immunity.

Then, we modify model 2 in Section 2.2 by including NPIs control \( u_i(t) \) and vaccination control \( v_i(t) \), and the model
becomes
\[
\begin{align*}
\frac{dS_i}{dt} &= -(1 - u_i)S_i \Gamma_i - v_i S_i(0), \\
\frac{dE_i}{dt} &= (1 - u_i)S_i \Gamma_i - \partial_i E_i, \\
\frac{dA_i}{dt} &= (1 - \rho_i)\partial_i E_i - r^A_i A_i, \\
\frac{dI_i}{dt} &= \rho_i \partial_i E_i - (\eta_i + r^A_i)I_i, \\
\frac{dH_i}{dt} &= \eta_i I_i - (\delta_i + r^H_i)H_i, \\
\frac{dR_i}{dt} &= r^A_i A_i + r^I_i I_i + r^H_i H_i + v_i S^0_i, \\
\frac{dD_i}{dt} &= \delta_i H_i,
\end{align*}
\]
with initial conditions: \( S_i(0) = S^0_i, E_i(0) = E^0_i, A_i(0) = A^0_i, I_i(0) = I^0_i, H_i(0) = H^0_i, R_i(0) = R^0_i, \) and \( D_i(0) = D^0_i, \) for \( i = 1, \ldots, n. \)

The objective is to minimize the total cost of control, hospitalization, death, and vaccination with a medical resource constraint and free terminal time. That is, the optimal control problem is to minimize the objective function
\[
J(u, v, T) = \sum_{i=1}^{n} \Delta_i D_i(T) + \int_{0}^{T} \sum_{j=1}^{n} (\Theta_j H_j(t)) \]
subjected to Model (13) with partially fixed terminal state constraints:
\[
S_i(T) = S^\text{end}_i, \quad i = 1, \ldots, n, \tag{15}
\]

where \( \Phi_{ij} \) is the impact of NPIs control on economic activity, and \( \Phi_{2j} \) is the other loss of economic activity caused by NPIs control since it takes time to restore employment and resume business activities after the imposition of a lockdown. The square term is used because a strict lockdown would take longer to recover and cause greater economic damage. \( \Psi_{1j} \) is the cost of vaccine production, and \( \Psi_{2j} \) is the cost of vaccine inoculation. The square term is used because it is more expensive to vaccinate more people simultaneously.

To avoid the collapse of health care system, a state constraint is added to control the maximum number of hospitalizations:
\[
\sum_{i=1}^{n} H_i(t) \leq B, \tag{16}
\]
where \( B \) is the total number of beds in the hospital.

### 3.1 Existence of optimal control

For simplicity, we consider the optimal control problem and corresponding cost functional of the group \( k, \) for \( k \in [1, n]. \)

**Theorem 1.** For the initial value of the state equation and objective function \( J(u_k, v_k, T), \) there exists an optimal control \( u^*_k(t), v^*_k(t) \) such that \( J(u^*_k, v^*_k, T) = \min_{(u_k, v_k) \in \Phi_k} J(u_k, v_k, T), \) for \( k \in [1, n]. \)

**Proof.** We verify Theorem 1 according to the theorem of Fleming and Rishel (2012). Let \( g(t, x, \mu) \) be the state function, where \( x \in \mathbb{R}^m \) is the \( m \)-dimensional state vector and \( \mu \in \mathbb{R}^n \) is an \( n \)-dimensional control variable. The remaining verification conditions are as follows:

1. The first partial derivative of the state equation \( g(t, x, \mu) \) is continuous, and there is a constant \( C \) such that \( |g(t, 0, 0)| \leq C, |g_x(t, x, \mu)| \leq C(1 + |\mu|), |g_{xx}(t, x, \mu)| \leq C. \)
2. The set of all solutions to the system with corresponding control functions in \( \Omega \) is nonempty.
3. \( g(t, x, \mu) = \alpha(t, x) + \beta(t, x)\mu. \)
4. The control set \( \Omega = [0, u_{\max}] \times [0, v_{\max}] \) is convex, compact, and closed.
5. The integrand of the cost functional is convex on \( \Omega. \)

The above conditions are verified individually as follows: The right-hand side of the state equation is
\[
g(t, u) = \begin{bmatrix}
-(1 - u_k)S_k \Gamma_k - v_k S^0_k \\
(1 - u_k)S_k \Gamma_k - \partial_k E_k \\
(1 - \rho_k)\partial_k E_k - r^A_k A_k \\
\rho_k \partial_k E_k - (\eta_k + r^A_k)I_k \\
\eta_k I_k - (\delta_k + r^H_k)H_k \\
r^A_k A_k + r^I_k I_k + r^H_k H_k + v_k S^0_k \\
\delta_k H_k
\end{bmatrix}, \tag{17}
\]

Clearly, \( g(t, x, \mu) \) has a first-order continuous partial derivative, and there exists a constant \( C \) such that \( |g(t, 0, 0)| \leq C. \) Taken the partial derivative of \( g(t, x, \mu) \) with respect to \( x \) and
\[ |g_\mu(t, x, \mu)| = \begin{bmatrix}
-(1 - u_k)\Gamma_k & 0 & -(1 - u_k)S_kc_{1,k}\alpha_k & -(1 - u_k)S_kc_{2,k}\beta_k & 0 & 0 & 0 \\
(1 - u_k)\Gamma_k & -(1 - \rho_k)\Theta_k & (1 - u_k)S_kc_{1,k}\alpha_k & (1 - u_k)S_kc_{2,k}\beta_k & 0 & 0 & 0 \\
0 & 0 & -r_k^A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\eta_k - r_k^I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & r_k^I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_k 
\end{bmatrix}, \tag{18} \]

Because all variables are bounded, there definitely exists a constant \( C \) such that

\[ |g(t, 0, 0)| \leq C, |g_\mu(t, x, \mu)| \leq C(1 + |\mu|), |g_\mu(t, x, \mu)| \leq C. \tag{20} \]

Thus, condition 1 is true.

According to condition 1, Model (13) has a constant control solution and the control set \( \Omega \) is clearly not empty, so condition 2 is true.

\[ g(t, x, \mu) \] can be rewritten as

\[ g(t, x, \mu) = \begin{bmatrix}
-S_k\Gamma_k & S_0^T \\
-S_k\Gamma_k & 0 \\
(1 - \rho_k)\Theta_k - r_k^A A_k & 0 \\
(1 - \rho_k)\Theta_k - (\eta_k + r_k^I)I_k & 0 \\
\eta_h - (\delta_k + r_k^H)H_k & 0 \\
r_k^A A_k + r_k^I I_k + r_k^H H_k & 0 \\
\delta_k H_k & 0 
\end{bmatrix} \begin{bmatrix}
u_k 
\end{bmatrix}. \tag{21} \]

Thus, condition 3 is true. By definition, condition 4 is true. Let

\[ f(t, x, (1 - a)\tilde{u}_k + a\tilde{u}_k) = \Theta_k H_k + \Phi_{1,k}u_k + \Phi_{2,k}u_k^2 + \Psi_{1,k}v_k + \Psi_{2,k}v_k^2, \tag{22} \]

To prove that the integrand of the cost function is convex, we only need to prove that for any \( a \in (0, 1) \), \( \tilde{u}_k = (\tilde{u}_k, \tilde{v}_k) \), and \( \tilde{u}_k = (u_k, v_k) \):

\[ f(t, x, (1 - a)\tilde{u}_k + a\tilde{u}_k) \leq (1 - a)f(t, x, \tilde{u}_k) + af(t, x, \tilde{u}_k). \tag{23} \]

That is,

\[ f(t, x, (1 - a)\tilde{u}_k + a\tilde{u}_k) = \Theta_k H_k + \Phi_{1,k}(1 - a)\tilde{u}_k + a\tilde{u}_k + \Phi_{2,k}(1 - a)\tilde{u}_k + a\tilde{v}_k)^2 + \Psi_{1,k}(1 - a)\tilde{v}_k + a\tilde{v}_k)^2 + \Psi_{2,k}(1 - a)\tilde{v}_k + a\tilde{v}_k)^2, \tag{24} \]

and

\[ (1 - a)f(t, x, \tilde{u}_k) + af(t, x, \tilde{u}_k) = (1 - a)(\Theta_k H_k + \Phi_{1,k}\tilde{u}_k + \Phi_{2,k}\tilde{u}_k^2 + \Psi_{1,k}\tilde{v}_k + \Psi_{2,k}\tilde{v}_k^2) + a(\Theta_k H_k + \Phi_{1,k}\tilde{u}_k + \Phi_{2,k}\tilde{u}_k^2 + \Psi_{1,k}\tilde{v}_k + \Psi_{2,k}\tilde{v}_k^2). \tag{25} \]

Then, we have

\[ f(t, x, (1 - a)\tilde{u}_k + a\tilde{u}_k) - ((1 - a)f(t, x, \tilde{u}_k) + af(t, x, \tilde{u}_k)) = \Phi_{2,k}a(a - 1)\tilde{u}_k + a\tilde{v}_k)^2 + \Psi_{2,k}a(a - 1)\tilde{v}_k + a\tilde{v}_k)^2 \leq 0. \tag{26} \]

Then condition 5 is true, thus proving Theorem 1.

\[ \Box \]

### 3.2 Characterization of optimal control

The necessary conditions that optimal solutions must satisfy are derived from Pontryagin’s maximum principle (Pontryagin, 2018).

The Hamiltonian of this optimal control problem is defined as

\[ H(x, u, \lambda) = \sum_{k=1}^{n} \Theta_k H_k + \Phi_{1,k}u_k + \Phi_{2,k}u_k^2 + \Psi_{1,k}v_k + \Psi_{2,k}v_k^2 + \lambda_{1,k}(-(1 - u_k)S_k\Gamma_k - v_k S_0^T) + \lambda_{2,k}((1 - u_k)S_k\Gamma_k - \Theta_k E_k) + \lambda_{3,k}((1 - \rho_k)\Theta_k E_k - r_k^A A_k) + \lambda_{4,k}(\rho_k\Theta_k E_k - (\eta_k + r_k^I)I_k), \tag{22} \]

To prove that the integrand of the cost function is convex, we only need to prove that for any \( a \in (0, 1) \), \( \tilde{u}_k = (\tilde{u}_k, \tilde{v}_k) \), and \( \tilde{u}_k = (u_k, v_k) \):

\[ f(t, x, (1 - a)\tilde{u}_k + a\tilde{u}_k) \leq (1 - a)f(t, x, \tilde{u}_k) + af(t, x, \tilde{u}_k). \tag{23} \]
the state equation

\[
\begin{align*}
0 & = \sum_{i=1}^{n} \left( w_{11}^k u_k + w_{12}^k (v_{\text{max}} - u_k) \right) \\
& + w_{21}^k v_k + w_{22}^k (v_{\text{max}} - v_k) - \varepsilon (B - \sum_{i=1}^{n} H_i),
\end{align*}
\]

where \( \lambda = (\lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}, \lambda_{4,k}, \lambda_{5,k}, \lambda_{6,k}, \lambda_{7,k})' \) is called the adjoint variable.

Then, the Lagrangian is

\[
L(x, u, \lambda) = H(x, u, \lambda) - \sum_{k=1}^{n} \left( w_{11}^k u_k + w_{12}^k (u_{\text{max}} - u_k) \\
+ w_{21}^k v_k + w_{22}^k (v_{\text{max}} - v_k) \right) - \varepsilon (B - \sum_{i=1}^{n} H_i),
\]

where \( \lambda \) and \( \varepsilon \) is the Lagrange multiplier, and \( w_{ij}^k \geq 0 \) is the penalty factor, which satisfies

\[
\begin{align*}
w_{11}^k u_k &= w_{12}^k (u_{\text{max}} - u_k) = 0 \quad \text{at } u_k = u_k^*, \\
w_{21}^k v_k &= w_{22}^k (v_{\text{max}} - v_k) = 0 \quad \text{at } v_k = v_k^*.
\end{align*}
\]

The following theorem gives the characterization of optimal control:

**Theorem 2.** Given an optimal control pair \((u_k^*, v_k^*)\), an optimal terminal time \( T^* \), and corresponding solution \( S_k^*, E_k^*, A_k^*, R_k^*, H_k^*, R_k^*, D_k^* \) to the state equations that minimizes \( J(u_k, v_k, T) \), and the adjoint variables \( \lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}, \lambda_{4,k}, \lambda_{5,k}, \lambda_{6,k}, \lambda_{7,k} \) satisfying the adjoint system:

\[
\frac{d\lambda}{dt} = - \frac{dL}{dx},
\]

that is,

\[
\begin{align*}
\lambda_{1,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) \Gamma_k, \\
\lambda_{2,k}' &= (\lambda_{k,2} - (1 - \rho_k \lambda_{k,3} - \rho_k \lambda_{k,4}) \Theta_k, \\
\lambda_{3,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) S_k C_{k,k} \lambda_{k}, \\
\lambda_{4,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) S_k E_{k,k} \lambda_{k}, \\
\lambda_{5,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) S_k H_{k,k} \lambda_{k}, \\
\lambda_{6,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) S_k R_{k,k} \lambda_{k}, \\
\lambda_{7,k}' &= (\lambda_{k,1} - \lambda_{k,2})(1 - u_k) S_k D_{k,k} \lambda_{k}, \\
\lambda_{k,8}' &= (\lambda_{k,4} - \lambda_{k,5}) \eta_k + (\lambda_{k,4} - \lambda_{k,6}) \rho_k - P_k, \\
\lambda_{k,9}' &= (\lambda_{k,5} - \lambda_{k,6}) \theta_k + (\lambda_{k,5} - \lambda_{k,7}) \theta_k - \Theta_k - \varepsilon, \\
\lambda_{k,10}' &= 0, \\
\lambda_{k,11}' &= 0,
\end{align*}
\]
with final conditions

\[ S_i(T) = S_i^{\text{herd}}, \quad (32) \]

The transversality condition for the terminal time is

\[ \lambda_{1,k}(T) = \xi_k, \lambda_{k,2}(T) = \lambda_{k,3}(T) = \lambda_{k,4}(T) = \lambda_{k,5}(T) = \lambda_{k,6}(T) = \lambda_{k,7}(T) = \Delta_k, \quad (33) \]

and

\[ H + \xi_k + \Delta_k = 0 \text{ at } t = T^*, \quad (34) \]

where \( \xi \) is a Lagrange multiplier.

Furthermore, \( u^*_k \) and \( v^*_k \) are represented by

\[ u^*_k = \min \left( \max \left( 0, \frac{(\lambda_{2,k} - \lambda_{1,k}) S_k^{\text{opt}} - \Phi_{1,k}}{2 \Phi_{2,k}} \right), u_{\max} \right), \]

\[ v^*_k = \min \left( \max \left( 0, \frac{(\lambda_{1,k} - \lambda_{6,k}) S_k^{\text{opt}} - \Psi_{1,k}}{2 \Psi_{2,k}} \right), v_{\max} \right) \quad (35) \]

for \( k \in [1, n]. \)

\textbf{Proof.} According to Pontryagin’s maximum principle, there exist adjoint variables \( \lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}, \lambda_{k,4}, \lambda_{k,5}, \lambda_{k,6}, \lambda_{k,7} \) satisfying

\[ \lambda'_{k,1} = \frac{dL \text{, } x \text{, } \lambda}{dx}, \quad \lambda'_{k,2} = \frac{dL \text{, } y \text{, } \lambda}{dy}, \quad \lambda'_{k,3} = \frac{dL \text{, } u \text{, } \lambda}{du}, \quad \lambda'_{k,4} = \frac{dL \text{, } v \text{, } \lambda}{dv}, \]

\[ \lambda'_{k,5} = \frac{dL \text{, } \lambda \text{, } \lambda}{d\lambda}, \quad \lambda'_{k,6} = \frac{dL \text{, } \lambda \text{, } \lambda}{d\lambda}, \quad \lambda'_{k,7} = \frac{dL \text{, } \lambda \text{, } \lambda}{d\lambda}. \]

From \( \frac{dL \text{, } x \text{, } \lambda}{dx} = 0 \) and \( \frac{dL \text{, } y \text{, } \lambda}{dy} = 0 \), we get

\[ u^*_k = \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} + w_{11}^k - w_{12}^k - \Phi_{1,k}}{2 \Phi_{2,k}}, \quad \lambda'_{k,1} = \lambda'_{k,2} = \lambda'_{k,3} = \lambda'_{k,4} = \lambda'_{k,5} = \lambda'_{k,6} = \lambda'_{k,7} = 0. \]

Then \( u^*_k \) is analyzed by considering the following three cases:

(1) On the set \( \{ t \mid u^*_k = 0 \} \), \( w_{11}^k > 0, w_{12}^k = 0 \), then

\[ u^*_k = \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} + w_{11}^k - \Phi_{1,k}}{2 \Phi_{2,k}} = 0 \quad (37) \]

which implies that

\[ \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} - \Phi_{1,k}}{2 \Phi_{2,k}} \leq 0. \quad (38) \]

(2) On the set \( \{ t \mid 0 < u^*_k < u_{\max} \} \), \( w_{11}^k = w_{12}^k = 0 \), then

\[ u^*_k = \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} - \Phi_{1,k}}{2 \Phi_{2,k}}. \quad (39) \]

(3) On the set \( \{ t \mid u^*_k = u_{\max} \} \), \( w_{11}^k = 0, w_{12}^k > 0 \), then

\[ u^*_k = \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} - w_{12}^k - \Phi_{1,k}}{2 \Phi_{2,k}} = u_{\max}, \quad (40) \]

which implies that

\[ u^*_k = \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} - \Phi_{1,k}}{2 \Phi_{2,k}} \geq u_{\max}. \quad (41) \]

Then the optimal control pair \((u^*_k, v^*_k)\) is obtained

\[ u^*_k = \min \left( \max \left( 0, \frac{(\lambda_{k,2} - \lambda_{k,1}) S_k^{\text{opt}} - \Phi_{1,k}}{2 \Phi_{2,k}}, u_{\max} \right), \right) \quad (42) \]

\[ v^*_k = \min \left( \max \left( 0, \frac{(\lambda_{1,k} - \lambda_{6,k}) S_k^{\text{opt}} - \Psi_{1,k}}{2 \Psi_{2,k}}, v_{\max} \right), \right) \quad (43) \]

From the characterization of optimal control, it can be seen that optimal control is continuous with respect to time.

4  |  NUMERICAL ANALYSIS

In this section, the transmission rate is calibrated using real data in the United States after the specification of several parameters, and then different optimal control scenarios are simulated.

According to the study by Davies et al. (2020), susceptibility to infection in adults over 20 years is roughly twice that of individuals under 20 years of age, and the death rate varies by age. Therefore, we divided the total population into three groups according to age: 0–19 years old (young population), 20–64 years old (prime population), and over 65 years old (elderly population), that is, \( n = 3 \).

We adopt the contact matrix data of different age groups provided by Prem et al. (2017). The contact matrix is divided into four groups: home, school, workplace, and other locations. We assume that the strictest NPIs adopted are equivalent to home quarantine, then \((1 - u_t)c_{ij}\) in Model 12 matrix is

\begin{table}[h]
\centering
\caption{The cost under different strategies}
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Cost} & \textbf{Strategy I} & \textbf{Strategy II} & \textbf{Strategy III} \\
\hline
Control cost\,(trillion) & $1.6759$ & $6.5376$ & $6.7035$ \\
Hospitalization cost\,(trillion) & $0.4597$ & $0.4332$ & $0.3836$ \\
Death cost\,(trillion) & $5.4480$ & $5.0515$ & $4.5697$ \\
Vaccination cost\,(billion) & - & - & $2.2905$ \\
Hospitalization cost\,(trillion) & $0.4597$ & $0.4332$ & $0.3836$ \\
Total cost\,(billion) & $7.5836$ & $12.0223$ & $11.6591$ \\
\hline
\end{tabular}
\end{table}
is represented as

\[ c_{ij}^H + (1 - u_i)(c_{ij}^S + c_{ij}^W + c_{ij}^O), \]  \hspace{1cm} (44)

where \( c_{ij}^H, c_{ij}^S, c_{ij}^W, \) and \( c_{ij}^O \) denote the contact matrix at home, school, workplace, and other locations, respectively.

We use the US per capita income as the parameter of economic activity \( \Phi_{1,k} \). Since neither the young nor the elderly population have an income, we set it to 0.
The objective function becomes

\[
J(u_i, v_i, T) = \int_{0}^{T} \left[ \Theta(H_1(t) + H_2(t) + H_3(t)) + \Phi_1 S_0^1 u_1(t) + \Phi_2 S_0^2 u_2(t) + \Phi_3 S_0^3 u_3(t) + \Psi_1 S_0^1 v_1(t) + \Psi_2 S_0^2 v_2(t) + \Psi_3 S_0^3 v_3(t) \right] dt \\
+ \Delta_1 D_1(T) + \Delta_2 D_2(T) + \Delta_3 D_3(T),
\]

where \(S_0^1, S_0^2,\) and \(S_0^3\) represent the total population of these three groups, respectively.

### 4.1 Specification of the parameters

The US population and contact matrix data are used for the numerical simulation. We assume that the number of exposed individuals in all three groups at the initial moment is 100.

The percentage of symptomatic infections for each age group are taken from Davies et al. (2020), which are 21% (aged 0–19), 70% (aged 20–64), and 70% (aged 65 and over), so we set \(\rho_1 = 0.21, \rho_2 = 0.7,\) and \(\rho_3 = 0.7.\) To be consistent with other studies (He et al., 2020b; McAloon et al., 2020), we assume the incubation period to be 6 days and set \(\theta_1 = \theta_2 = \theta_3 = 0.167.\) He et al. (2020b) report that the infectiousness declined rapidly within 7 days of symptom onset, which implies that \(r_{A1} = r_{A2} = r_{A3} = 0.143\) and \(r_{I1} = r_{I2} = r_{I3} = 0.143.\) According to CDC (2020), we calculated the average number of days of hospitalization, which is \(r_{H1} = 0.357, r_{H2} = 0.196, r_{H3} = 0.140.\)

As in CDC (2020), the duration of infection from onset to hospitalization is assumed to be 2, 6, and 4 days for young, prime, and elderly populations, respectively. We adopt the probability of hospitalization after onset from Lin et al.
and set it to 19%. Then $\eta_1 = 0.095$, $\eta_2 = 0.032$, and $\eta_3 = 0.048$ is obtained. According to CDC (2020), the proportion of deaths among those hospitalized in the young, prime, and elderly populations are 0.7%, 4.0%, and 18.8%, and the average number of days from hospitalization to death is 10, 17.7, and 16 days, which implies $\delta_1 = 0.0007$, $\delta_2 = 0.0023$, and $\delta_3 = 0.0118$.

Using the constant Value of a Statistical Life Year (VSLY) in Robinson et al. (2021), we assume that the loss from death for young, prime, and elderly populations are $\Delta_1 = 13.41$.
The young population and elderly population essentially do not affect most of society’s production and economic activities. Therefore, we assume that the economic cost of control for both the young and the elderly population is 0. We use 2019 US per capita income (BEA, 2021) as the economic activity cost of control for the prime population, which is $156 per person per day. We set the other loss of economic activity $\Phi_2 = 30$ per person per day. Even if the government requires people to be completely isolated at home, public
FIGURE 14 Infections curves for people aged 0–19, 20–64, and over 65 (from left to right)

FIGURE 15 Hospitalization curves for people aged 0–19, 20–64, and over 65 (from left to right)

FIGURE 16 Cumulative death curves for people aged 0–19, 20–64, and over 65 (from left to right)

service personnel also have outdoor activities. Simultaneously, ordinary residents need to purchase daily necessities at regular intervals, so a complete blockade is impossible. Therefore, we set the maximum value of control $u_{\text{max}} = 0.95$.

The maximum vaccination rate $v_{\text{max}}$ is set to 0.01. That is, 1% of the population can be vaccinated per day. The cost of vaccine production $\Psi_1$ is assumed to be 50$ per person, and the cost of vaccine inoculation $\Psi_2$ is assumed to be 30$.

Finally, the maximum number of hospitalizations $B$ is set to be 0.003, which implies no more than 3 per 1000 of the population. Table 2 summarizes the parameter values used in numerical analysis.
4.2 Calibration of transmission rate

In order to simulate the spread of the epidemic more consistent with the real-world situation, we calibrate the transmission rate $R_0$ by using the data in the United States from January 22, 2020 to April 18, 2020, provided by CDC’s COVID Date Tracker.

We chose the data at the early stage of the epidemic in the United States because the government did not take any measures to deal with the epidemic at this moment. Therefore, the dynamic of COVID-19 at the early stage is the most natural. Since the COVID-19 testing method was not perfect at the beginning of the epidemic, we believe that only the death data can reflect the spread of the SARS-CoV-2 virus at the beginning of the epidemic. A national emergency concerning COVID-19 was declared on March 13, 2020, so we chose the data before April 12.

According to CDC (2020), the infectiousness of asymptomatic individuals relative to symptomatic individuals is 75%. Davies et al. (2020) report that adults over 20 are about twice as susceptible to infection as those under 20. Calibrated with the above-mentioned real-world death data, we get $\beta_1 = 0.663$, $\beta_2 = \beta_3 = 1.325$, and $\alpha_1 = 0.497$, $\alpha_2 = \alpha_3 = 0.994$. In this case, $R_0 = 5.47$. This result is similar to Sanche et al. (2020), which is 5.7 (95% CI 3.8–8.9), and Ke et al. (2021), which is 5.8. Figure 2 shows the comparison of real death data with our model. The scatter in the figure is the real death data. The dashed line in the figure is the fitting result of the model, and the solid line is the predicted deaths of the model.

4.3 Numerical simulation

The following three different optimal control strategies are investigated in this section:

- Strategy I: minimize the cost of death, hospitalization, and NPI measures without hospitalization constraint.
- Strategy II: minimize the cost of death, hospitalization, and NPI control measures with hospitalization constraint.
- Strategy III: minimize the cost of death, hospitalization, vaccination, and NPI control measures with hospitalization constraint.
In order to observe the real effect of each strategy, we also simulated the development of the epidemic after reaching herd immunity and stopping the control. That is, we simulated the epidemic’s evolution from the outbreak’s beginning to 2 years later.

We first consider the scenario under strategy I, and the result are shown in Figures 3-7. It is a counterfactual scenario in which all patients who need to be hospitalized can be treated. In this situation, additional deaths due to limited medical resources are not taken into account. Under this strategy, herd immunity was reached after 133 days. That is the position of the gray dotted line in Figure 3. It can be seen from the figure that the NPIs control has been basically lifted before the herd immunity is reached, and the epidemic will not break out again after the NPIs control is completely lifted after the herd immunity. After simulating the development of the outbreak for 2 years (730 days), the total loss of the epidemic was 7.5836 trillion dollars, of which the cost of hospitalization was 0.4597 trillion dollars, the economic loss due to NPIs control was 1.6759 trillion dollars, and the loss caused by the death of the population was 5.4480 trillion dollars.

In strategy II, herd immunity was reached after 337 days from the beginning of the epidemic. Figures 8-12 show the result of strategy II. Strategy II is the most realistic scenario. Strategy II assumes that medical resources are limited, and it is necessary to strictly control the number of inpatients and not exceed the total number of beds. After simulating the development of the epidemic for 2 years (730 days), the total loss of the epidemic was 12.0223 trillion dollars, of which the cost of hospitalization was 0.4332 trillion dollars, the economic loss due to NPIs control was 6.5376 trillion dollars, and the economic loss due to population deaths was 5.0515 trillion dollars. As can be seen from the cost of death in strategies I and II, even with unlimited medical resources in strategy I, the final number of deaths is still less in strategy II with stricter NPIs control.

In strategy III, we assume that a vaccine will be available 9 months after the outbreak. The results of strategy III are shown in Figure 13-18. The vaccine development is set to 9 months because the development of the COVID-19 vaccine began in March 2020, and the vaccine was successfully developed 9 months later in December. Under this strategy, herd immunity was reached 337 days after the outbreak began. After simulating the development of the epidemic for 2 years (730 days), the total loss caused by the epidemic was 11.6591 trillion dollars, of which the cost of hospitalization was 0.3836 trillion dollars, the economic loss due to NPIs control was 6.7035 trillion dollars, and the economic loss due to deaths was 4.5697 trillion dollars, and the vaccination cost was 2.2905 billion dollars.

Table 3 reports the cost of different strategies based on numerical simulation. According to Table 3, strategy 1 has the least economic loss. It is also reasonable that unlimited medical resources can minimize the number of deaths, and herd immunity can be achieved without resorting to strict NPIs control. The economic loss of strategy III is less than that of strategy II, and strategy III can reach herd immunity earlier than strategy II, thus ending the epidemic earlier, which provides valuable time for economic recovery after the epidemic.

The control curves under strategies I and II are shown in Figures 7 and 12. It can be seen from these figures that in the case of limited medical resources in strategy II, the duration of the outbreak was significantly longer, which is more than twice that of strategy I. In strategy II, all groups must be locked down continuously during the epidemic, and only the prime population can be lifted in advance to recover the economy more quickly. Then, the lockdown is lifted for the young population, while the elderly population needs to remain in lockdown throughout (strongest control) until herd immunity. In both sets of strategies, the lockdown of the prime population will be lifted as soon as possible by implementing a strict lockdown on the young and elderly populations to restore the economy earlier. As can be seen from the number of infections and hospitalizations for strategies I and II (Figures 8-10, 13-17), both strategies choose to weaken a huge wave of outbreaks into two smaller waves by imposing NPI controls. This indicates that the optimal control strategy can effectively reduce the outbreak of the epidemic. (Figures 11 and 18)

In strategy III, where vaccines are introduced, the situation is different. Strict NPIs control should be implemented for the entire population when a vaccine is known to be available after a certain period. As can be seen from Figures 17 and 18, the strictest NPIs control was applied to all three groups until about 20 days before the successful development of the vaccine. NPIs control was then completely lifted, and herd immunity was reached through vaccination. This will reduce the economic losses and death toll and bring the pandemic to an end earlier, speeding up economic recovery.

5 | CONCLUSION

We construct a multigroup SEAIHRD compartment model by splitting the population into three age groups: young (aged 0–19), prime (aged 20–64), and elderly population (aged 65 and over). By applying optimal control theory to this model, we develop a free terminal time, partially fixed terminal state optimal control problem to minimize deaths and costs associated with hospitalization, and the implementation of the control strategy under different assumptions about medical resources and vaccination. Specifically, we investigate optimal control strategies for the COVID-19 epidemic in order to evaluate the effect of NPIs and vaccination as control measures. The transmission rate of SARS-CoV-2 is calibrated by using real data in the United States at the early stage of the epidemic.

The numerical results show that, even if no vaccine is developed, proper control of the prime population and relatively strict control measures for the young and elderly population could significantly mitigate the spread of the disease and significantly reduce the number of deaths. It is possible to reduce the duration of NPIs control in the prime population by extending the control time of the young and the elderly population, restoring economic activity earlier without collapsing the health system. However, under the
assumption that an effective vaccine would become available in a certain time, the best strategy is to keep the entire population under strict control until vaccination begins.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest that could be perceived as prejudicing the impartiality of the research reported.

REFERENCES

Almeida, R. (2018). Analysis of a fractional SEIR model with treatment. Applied Mathematics Letters, 84, 56–62.

Alvarez, F. E., Argente, D., & Lippi, F. (2020). A simple planning problem for COVID-19 lockdown (Technical report). National Bureau of Economic Research.

Bartsch, S. M., Ferguson, M. C., McKinnell, J. A., O’shea, K. J., Wedlock, P. T., Siegmund, S. S., & Lee, B. Y. (2020). The potential health care costs and resource use associated with COVID-19 in the United States: A simulation estimate of the direct medical costs and health care resource use associated with COVID-19 infections in the United States. Health Affairs, 39(6), 927–935.

BEA. (2021). Personal income by county and metropolitan area, 2019. CDC. (2020). COVID-19 pandemic planning scenarios [EB/OL]. https://www.cdc.gov/coronavirus/2019-ncov/hcp/planning-scenarios.html

Davies, N. G., Klepac, P., Liu, Y., Prem, K., Jit, M., & Eggo, R. M. (2020). Age-dependent effects in the transmission and control of COVID-19 epidemics. Nature Medicine, 26(8), 1205–1211.

Djidjou-Demasse, R., Michalakis, Y., Choisy, M., Sforzana, M. T., & Alizon, S. (2020). Optimal COVID-19 epidemic control until vaccine deployment. medRxiv.

Fast, S. M., Gonzalez, M. C., & Markuzon, N. (2015). Cost-effective control of infectious disease outbreaks accounting for societal reaction. PLoS One, 10(8), e0136059.

Fleming, W. H., & Rishel, R. W. (2012). Deterministic and stochastic optimal control (Vol. 1). Springer Science & Business Media.

Guo, H., Li, M. Y., & Shi, Z. (2006). Global stability of the endemic equilibrium of multigroup SIR epidemic models. Canadian Applied Mathematics Quarterly, 14(3), 259–284.

Hao, X., Cheng, S., Wu, D., Wu, T., Lin, X., & Wang, C. (2020). Reconstruction of the full transmission dynamics of COVID-19 in Wuhan. Nature, 584(7821), 420–424.

Hauser, A., Courotte, M. J., Marogossian, C. C., Konstantinouidis, G., Low, N., Althaus, C. L., & Riou, J. (2020). Estimation of SARS-CoV-2 mortality during the early stages of an epidemic: A modeling study in Hubei, China, and six regions in Europe. PLoS Medicine, 17(7), e1003189.

He, S., Peng, Y., & Sun, K. (2020a). SEIR modeling of the COVID-19 and its dynamics. Nonlinear Dynamics, 101(3), 1667–1680.

He, X., Lau, W., Deng, X., Wang, J., Hao, X., Lau, Y. C., Wong, J. Y., Guan, Y., Tan, X., Mo, X., Chen, Y., Liao, B., Chen, W., Hu, F., Zhang, Q., Zhong, M., Wu, Y., Zhao, L., … Leung, G. M. (2020b). Temporal dynamics in viral shedding and transmissibility of COVID-19. Nature Medicine, 26(9), 1491–1493.

Ke, R., Romero-Severson, E., Sanche, S., & Hengartner, N. (2021). Estimating the reproductive number R0 of SARS-CoV-2 in the United States and eight European countries and implications for vaccination. Journal of Theoretical Biology, 517, 110621.

Kermack, W. O., & McKendrick, A. G. (1927). A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 115(772), 700–721.

Ketcheson, D. I. (2021). Optimal control of an SIR epidemic through finite-time non-pharmaceutical intervention. Journal of Mathematical Biology, 83(1), 1–21.

Lau, M. S. Y., Grenfell, B., Thomas, M., Bryan, M., Nelson, K., & Lopman, B. (2020). Characterizing superspreading events and age-specific infectiousness of SARS-CoV-2 transmission in Georgia, USA. Proceedings of the National Academy of Sciences, 117(36), 22430–22435.

Ledger, G., & Hopm, M. (2022). Using a COVID-19 model in various classroom settings to assess effects of interventions. PRIMUS, 32(2), 278–297.

Li, Q., Guan, X., & Wu, P. (2020). Early transmission dynamics in Wuhan, China, of novel coronavirus—Infected pneumonia. New England Journal of Medicine, 382(13), 1199–1207.

Lin, Q., Zhao, S., & Gao, D. (2020). A conceptual model for the outbreak of coronavirus disease 2019 (COVID-19) in Wuhan, China with individual reaction and governmental action. International Journal of Infectious Diseases, 93, 211–216.

Liu, Y., Yan, L.-M., & Wan, L. (2020). Viral dynamics in mild and severe cases of COVID-19. Lancet Infectious Diseases, 20(6), 656–657.

McAloon, C., Collins, Á., & Hunt, K. (2020). Incubation period of COVID-19: A rapid systematic review and meta-analysis of observational research. BMJ Open, 10(8), e039652.

Perkins, T. A., & España, G. (2020). Optimal control of the COVID-19 pandemic with non-pharmaceutical interventions. Bulletin of Mathematical Biology, 82(9), 1–24.

Pontyragin, L. S. (2018). Mathematical theory of optimal processes. Routledge.

Prem, K., Cook, A. R., & Jit, M. (2017). Projecting social contact matrices in 152 countries using contact surveys and demographic data. PLoS Computational Biology, 13(9), e1005697.

Revelle, C. S., Lynn, W. R., & Feldmann, F. (1967). Mathematical models for the economic allocation of tuberculosis control activities in developing nations. American Review of Respiratory Disease, 96(5), 893–909.

Robinson, L. A., Sullivan, R., & Shogren, J. F. (2021). Do the benefits of COVID-19 policies exceed the costs? Exploring uncertainties in the age-VSL relationship. Risk Analysis, 41(5), 761–770.

Salje, H., Kiem, C. T., & Lefrançq, N. (2020). Estimating the burden of SARS-CoV-2 in France. Science, 369(6500), 208–211.

Sanche, S., Lin, Y. T., Xu, C., Romero-Severson, E., Hengartner, N., & Ke, R. (2020). High contagiousness and rapid spread of severe acute respiratory syndrome coronavirus 2. Emerging Infectious Diseases, 26(7), 1470–1477.

van den Driessche, P., & Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Mathematical Biosciences, 180(1-2), 29–48.

Vargas-De-León, C. (2011). On the global stability of SIS, SIR and SIRS epidemic models. Mathematical Biosciences, 230(1), 110621.

Zong, K., & Luo, C. (2023). Mathematical theory of COVID-19 epidemic. Optimal control analysis of a multigroup SEAIHRD model for COVID-19 epidemic. Risk Analysis, 43, 62–77. https://doi.org/10.1111/risa.14027

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