ON THE SMOOTH RIGIDITY OF ALMOST-EINSTEIN MANIFOLDS WITH NONNEGATIVE ISOTROPIC CURVATURE

HARISH SESHADRI

Abstract. Let $(M^n, g)$, $n \geq 4$, be a compact simply-connected Riemannian manifold with nonnegative isotropic curvature. Given $0 < l \leq L$, we prove that there exists $\varepsilon = \varepsilon(l, L, n)$ satisfying the following: If the scalar curvature $s$ of $g$ satisfies

\[ l \leq s \leq L \]

and the Einstein tensor satisfies

\[ |Ric - \frac{s}{n} g| \leq \varepsilon \]

then $M$ is diffeomorphic to a symmetric space of compact type.

This is a smooth analogue of the result of S. Brendle that a compact Einstein manifold with nonnegative isotropic curvature is isometric to a locally symmetric space.

1. Introduction

A Riemannian manifold $(M, g)$ is said to have nonnegative isotropic curvature if

\[ R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0 \]

for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$.

In the case of strict inequality above we say that the manifold has positive isotropic curvature. Recently S. Brendle proved that a compact Einstein manifold with nonnegative isotropic curvature has to be a locally symmetric space of compact type. In this note we relax the restriction that the metric is Einstein to the condition that the Einstein tensor is small in norm and obtain the following smooth rigidity result:

Theorem 1.1. Let $(M^n, g)$, $n \geq 4$, be a compact simply-connected Riemannian manifold with nonnegative isotropic curvature. Given $0 < l \leq L$, there exists $\varepsilon = \varepsilon(l, L, n)$ satisfying the following: If the scalar curvature $s$ of $g$ satisfies

\[ l \leq s \leq L \]

and the Einstein tensor satisfies

\[ |Ric - \frac{s}{n} g| \leq \varepsilon \]

then $M$ is diffeomorphic to a symmetric space of compact type.

This result was inspired by the paper of P. Petersen and T. Tao where it is proved that “almost” quarter-pinching of sectional curvatures again leads to smooth rigidity as above. The main difference between their conclusion and ours is that

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symmetric spaces of rank $\geq 2$ are allowed in our case, while almost quarter-pinching gives only rank-1 spaces.

We remark that for any $L, \varepsilon$ the conditions $s \leq L$ and $|Ric - \frac{c}{n}g| \leq \varepsilon$ can be achieved just by rescaling the metric by a large constant. In particular, consider the connected sum $S^{n-1} \times S^1 \# S^{n-1} \times S^1$ which admits a metric with positive isotropic curvature by [3]. Rescaling this metric gives the two bounds above. However this manifold does not support a locally symmetric metric (irreducible or reducible) of compact type. This is seen by observing that the fundamental group of the latter space has to contain an abelian subgroup of finite index. Hence the lower bound on scalar curvature is necessary. On the other hand it is not known if just positive Ricci curvature and nonnegative isotropic curvature already imply that the underlying compact manifold is diffeomorphic to a locally symmetric space, even without the assumption of simple-connectivity.

A few remarks about the proof. Let $(M, g)$ be a manifold satisfying the hypotheses of Theorem 1.1. The main parts of the proof are obtaining a two-sided bound on sectional curvature and a lower bound on the injectivity radius of $(M, g)$. An uniform upper bound on diameter is immediate since the Ricci curvature is uniformly positive for $\varepsilon$ small enough. The bound on sectional curvature is the content of Lemma 2.1. The injectivity radius bound is non-trivial and follows from a theorem of Petrunin - Tuschmann. To apply their result one needs finite second homotopy group which is guaranteed by positive isotropic curvature. To deal with nonnegative isotropic curvature we use the results of H. Seshadri [6] and S. Brendle [1] which allow us to reduce the nonnegative case to the positive case.

2. proof of Theorem 1.1

We begin with a simple but useful lemma. Let $c \in \mathbb{R}$. By $K^{iso} \geq c$
we mean that

$$K^{iso}(e_i, e_j, e_k, e_l) := R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl} \geq c$$

for every orthonormal 4-frame $\{e_i, e_j, e_k, e_l\}$.

Lemma 2.1. Given $c, C \in \mathbb{R}$, there exists $b = b(c, C, n)$ such that if $(M^n, g)$ is a Riemannian manifold with

$$K^{iso} \geq c, \quad s \leq C,$$

then the norm of the Weyl tensor $W$ is bounded by $b$:

$$|W| \leq b.$$

Proof. The proof is similar to that of Proposition 2.5 of [3]. Note that

$$K^{iso}(e_i, e_j, e_k, e_l) + K^{iso}(e_i, e_j, e_l, e_k) := 2(R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl}).$$

From this it follows that $4s$ can be expressed as a sum of $n(n - 1)$ isotropic curvatures, Since we have an upper bound on $s$ and a lower bound on $K^{iso}$, we get an upper bound $b_1 = b_1(c, C, n)$ for $K^{iso}$. We have a lower bound on $K^{iso}$ by hypothesis and hence we have two-sided bounds on

$$4W_{ijkl} = 4R_{ijkl} = K^{iso}(e_i, e_j, e_l, e_k) - K^{iso}(e_i, e_j, e_k, e_l).$$
depending only on $c, C$ and $n$. Since this holds for an arbitrary orthonormal 4-frame, we can apply the above bound to the 4-frame
\[
\left\{ e_i, \frac{1}{\sqrt{2}}(e_j - e_l), e_k, \frac{1}{\sqrt{2}}(e_j + e_l) \right\}
\]
to see that $|W_{ijkj} - W_{ijkl}| \leq b_2(c, C, n)$. Since $\sum_p W_{ipkp} = 0$, this implies that $|W_{ipkp}| \leq b_3(c, C, n)$. Hence $|W| \leq b_4(c, C, n)$.

□

An immediate corollary of Lemma 2.1 is that an upper bound on scalar curvature, a lower bound on isotropic curvature and an upper bound on the norm of the Ricci tensor gives a bound on the norm of the Riemann curvature tensor. This applies, in particular, to a metric satisfying the hypotheses of Theorem 1.1.

The first restriction we impose on $\varepsilon$ is that $\varepsilon \leq \frac{1}{2n}$. This implies that the Ricci curvature is uniformly positive:

(2.1) \[ Ric \geq \frac{1}{2n} g. \]

We claim that we can find such an $\varepsilon$ if we assume that $(M, g)$ has positive isotropic curvature. Then, by Micallef-Moore [2], $\pi_2(M) = 0$. Theorem 0.4 of [3] states that the injectivity radius of a compact simply-connected Riemannian $n$-manifold with finite second homotopy group, bounded sectional curvature $|K| \leq a$ and positive Ricci curvature $Ric > bg$ has a positive lower bound on injectivity radius dependent only on $a, b$ and $n$. The comment following Lemma 2.1 and (2.1) give us the required bounds on curvature.

Remark: If $M$ is even-dimensional, then one has the following alternative proof for a lower bound on injectivity radius $inj$. If $inj \to 0$, then all the characteristic numbers of $M$, in particular the Euler characteristic, would have to vanish. On the other hand, a simply-connected Riemannian $n$-manifold with positive isotropic curvature has to be homeomorphic to the $n$-sphere [2]. This contradiction shows that collapse cannot occur in even-dimensions.

Now suppose that there is no $\varepsilon$ for which the conclusion of Theorem 1.1 holds. Then we get a sequence of $(M_i, g_i)$ of Riemannian $n$-manifolds, none of which is diffeomorphic to a symmetric space of compact type, with uniformly bounded sectional curvatures and diameter (by Myers-Bonnet, since (2.1) holds) and injectivity radius bounded below. As in [4] we can assume that a subsequence converges in the $C^\infty$ topology to a smooth complete Riemannian manifold $(M, g)$. This manifold will have to be Einstein, of finite diameter (hence compact) and of nonnegative isotropic curvature. By [1], $(M, g)$ is isometric to a symmetric space of compact type or flat. Since $M_i$ is diffeomorphic to $M$ for large $i$ and $M_i$ is simply-connected, $M$ cannot be flat. Hence $M_i$ is diffeomorphic to a symmetric space of compact type for large $i$, which is a contradiction.

Hence we have established the existence of

(2.2) \[ \varepsilon_p = \varepsilon_p(l, L, n) \]

which yields the conclusion in the presence of positive isotropic curvature.

Next consider the general case of nonnegative isotropic curvature.
Lemma 2.2. Let \((M^n, g)\) be a compact simply-connected Riemannian manifold with nonnegative isotropic curvature. Suppose that

\[ 0 < l \leq s \leq L, \quad |\text{Ric}_g - \frac{s}{n} g|_g \leq \varepsilon \]

for some \(0 < l \leq L\) and \(\varepsilon \leq \frac{2l}{n}\).

Let \((N^k, h)\) be an irreducible factor in the de Rham decomposition of \(N\). If \(k = 2\) or \(3\), \(N\) is diffeomorphic to \(S^2\) or \(S^3\). If \(k \geq 4\), then \((N, h)\) has nonnegative isotropic curvature and

\[ 0 < \frac{2l}{n} < s \leq L, \quad |\text{Ric}_h - \frac{s}{k} h|_h \leq \varepsilon. \]

Proof. The statement about \(k = 2\) or \(3\) follows from the description of reducible manifolds with nonnegative isotropic curvature given by M. Micallef and M. Wang (Theorem 3.1, [3]). If \(k \geq 4\), note that

\[
|\text{Ric}_g - \frac{s}{n} g|_g^2 \geq |\text{Ric}_h - \frac{s}{k} h|_h^2
= |\text{Ric}_h - \frac{s}{k} h + k^2 |\frac{s}{k} h - \frac{s}{n} g|^2.
\]

Hence

\[ |\text{Ric}_h - \frac{s}{k} h|_h \leq \varepsilon. \]

and

\[ s \geq \frac{k}{n} s - \varepsilon \geq \frac{4l}{n} - \frac{2l}{n} = \frac{2l}{n}. \]

Moreover, since \((M, g)\) has positive Ricci curvature, so does each irreducible component and hence \(s \leq s \leq L\). □

We can now complete the proof of the theorem by induction. The proof for the first nontrivial dimension \(n = 4\) is the same as that for the inductive step, so we assume that the result is true in all dimensions less than \(n\). Let \((M^n, g)\) be a manifold as in Theorem 1.1 with the norm of the Einstein tensor being smaller than

\[ \varepsilon_r(l, n) := \min \left\{ \frac{2l}{n}, \varepsilon \left(\frac{2l}{n}, L, 4\right), ..., \varepsilon \left(\frac{2l}{n}, L, n - 1\right) \right\}. \]

If \((M, g)\) is reducible, it is enough to prove that each irreducible component of \((M, g)\) is diffeomorphic to a symmetric space of compact type. Let \((N^k, h)\), \(1 \leq k \leq n - 1\) be such a component. By Lemma 2.2 and the inductive hypothesis we are done.

Suppose \((M, g)\) is irreducible. We claim that if the Einstein tensor of \(g\) is \(\frac{1}{2} \varepsilon_p(\frac{1}{2}, 2L, n)\)-small (where \(\varepsilon_p\) is defined by (2.2)) then we have the desired conclusion. By the results of [6] and [1] the following holds: Either \((M, g)\) is diffeomorphic to a symmetric space with nonconstant sectional curvature or we can find a metric \(\bar{g}\) with positive isotropic curvature as close (in the \(C^\infty\) topology) to \(g\) as we want. Choose \(\bar{g}\) so close to \(g\) that

\[ 0 < \frac{1}{2} \leq s_{\bar{g}} \leq 2L, \quad |\text{Ric}_{\bar{g}} - \frac{s}{n} \bar{g}|_{\bar{g}} \leq \varepsilon_p. \]

Since \((M, \bar{g})\) has positive isotropic curvature and satisfies the above bounds, it is diffeomorphic to a symmetric space by our earlier result.
Finally we choose
\[ \varepsilon(l, L, n) = \min \left\{ \frac{1}{2} \varepsilon_p \left( \frac{l}{2}, 2L, n \right), \varepsilon_p(l, L, n) \right\}. \]

\[ \square \]

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Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

E-mail address: harish@math.iisc.ernet.in