Analytic functional calculus for two operators

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Abstract
This paper is a survey devoted to the transformations

\[ C \mapsto \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} C R_{2,\mu} \, d\mu \, d\lambda, \]

\[ C \mapsto \frac{1}{2\pi i} \int_{\Gamma} g(\lambda) R_{1,\lambda} C R_{2,\lambda} \, d\lambda, \]

where \( R_{1,(\cdot)} \) and \( R_{2,(\cdot)} \) are pseudo-resolvents acting in a Banach space, i.e., the resolvents of bounded, unbounded, or multivalued linear operators, and \( f \) and \( g \) are analytic functions; here \( \Gamma_1, \Gamma_2, \) and \( \Gamma \) surround the singular sets (spectra) of the pseudo-resolvents \( R_{1,(\cdot)}, R_{2,(\cdot)}, \) and the both, respectively. Several applications are considered: a representation of the impulse response of a second-order linear differential equation with operator coefficients, a representation of the solution of the Sylvester equation, and properties of the differential of the ordinary functional calculus.

Keywords  Functional calculus · Pseudo-resolvent · Extended tensor products · Meromorphic functional calculus · Sylvester’s equation · Impulse response · Differential of the functional calculus · Transformator

Mathematics Subject Classification  47A60 · 47A80 · 47B49 · 39B42 · 34A30

1 Introduction

In this article, the constructions and facts concerning the substitution of two linear operators into analytic functions of two or one variables are collected together and presented from a unified point of view and in unified notation. Many of the results

Communicated by M. S. Moslehian.

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and ideas of this article are known in special cases; thus, it is largely a review than an original research paper.

Let $A$ and $B$ be matrices of the sizes $n \times n$ and $m \times m$, respectively, and $p(\lambda, \mu) = \sum_{i,j=0}^{N} c_{ij} \lambda^i \mu^j$ be a polynomial of two variables. The result $p(A, B)$ of the substitution of these matrices into $p$ is usually understood as the transformation $C \mapsto \sum_{i,j=0}^{N} c_{ij} A^i B^j$, which acts on matrices $C$ of the size $n \times m$, or as a special block matrix. This article is devoted to generalizations and applications of this construction.

First, the matrices $A$ and $B$ can be replaced by bounded linear operators acting in (infinite-dimensional) Banach spaces $X$ and $Y$, respectively. In this case, the number of interpretations of the object $p(A, B)$ increases. The most natural abstract interpretation is considering $p(A, B)$ as an operator acting in the completion of the algebraic tensor product $[31, 67, 124] X \otimes Y$ with respect to some cross-norm. This interpretation covers many spaces of functions of two variables. For example [31, 67, 124], $L_1[a, b] \otimes \mathbb{R} L_1[c, d]$ is isometrically isomorphic to $L_1[a, b] \times [c, d]$, and $C[a, b] \otimes \mathbb{C} C[c, d]$ is isometrically isomorphic to $C[a, b] \times [c, d]$. However, unfortunately, $L_\infty[a, b] \otimes \mathbb{C} L_\infty[c, d]$ is isomorphic only to a subspace of the space $L_\infty[a, b] \times [c, d]$. Another example that does not fit directly into the scheme of tensor products is the interpretation of $p(A, B)$ as the transformation $C \mapsto \sum_{i,j=0}^{N} c_{ij} A^i B^j$ of operators $C : Y \rightarrow X$; the reason is that $X \otimes Y^*$ is isomorphic only to the subspace of operators $C : Y \rightarrow X$ having a finite-dimensional image. From the point of view of applications, the last example seems to be the most important. Therefore, in all cases, we call the operator that corresponds to $p(A, B)$ a transformator in accordance with the tradition [53] accepted for the mappings of the type $C \mapsto \sum_{i,j=0}^{N} c_{ij} A^i B^j$ acting on the operators $C$. To cover the last example and some others, our exposition is based on the notion of an extended tensor product (Sect. 5) proposed in [97].

Second, one can replace the polynomial $p$ by an analytic function $f$. In this case, it is convenient to define $f(A, B)$ by means of a contour integral. For our main interpretation of $f(A, B)$ as a transformator acting on operators $C : Y \rightarrow X$, the relevant formula looks as follows:

$$f(A, B) C = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) (\lambda I - A)^{-1} C(\mu I - B)^{-1} \, d\mu \, d\lambda,$$

where $\Gamma_1$ and $\Gamma_2$ surround the spectra of $A$ and $B$, respectively. We call a correspondence of the type $f \mapsto f(A, B)$ that maps functions $f$ to transformators $f(A, B)$ a functional calculus.

From the algebraic point of view, the functional calculus $f \mapsto f(A, B)$ possesses properties of the tensor product $\varphi_1 \otimes \varphi_2$ of two ordinary functional calculi:

$$\varphi_1(f) = \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) (\lambda I - A)^{-1} \, d\lambda$$

and

\begin{align*}
\varphi_2(f) &= \frac{1}{2\pi i} \int_{\Gamma_2} f(\mu) (\mu I - B)^{-1} \, d\mu.
\end{align*}
\[
\varphi_2(f) = \frac{1}{2\pi i} \int_{\Gamma_2} f(\mu)(\mu 1 - B)^{-1} \, d\mu,
\]
where \( \Gamma_1 \) and \( \Gamma_2 \) surround the spectra of \( A \) and \( B \), respectively. To emphasize this fact, we use the notation \( \varphi_1 \otimes \varphi_2 \) and write \( [(\varphi_1 \otimes \varphi_2)f]C \) instead of \( f(A, B)C \) when the transformator \( f(A, B) \) acts in an extended tensor product. An important and nontrivial property of the transformation \( \varphi_1 \otimes \varphi_2 \) is the spectral mapping theorem (Theorem 39).

Third, the basic properties of the functional calculus \( \varphi_1 \otimes \varphi_2 \) are preserved when one replaces the resolvents \( \lambda \mapsto (\lambda 1 - A)^{-1} \) and \( \mu \mapsto (\mu 1 - B)^{-1} \) of operators by pseudo-resolvents [70], i.e. operator-valued functions \( R(\cdot) \) that satisfy the Hilbert identity:

\[
R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu.
\]

This generalization enables one to cover some additional applications. For example, a special case of a pseudo-resolvent is [70] the resolvent of an unbounded operator, and the most general example of a pseudo-resolvent is the resolvent of a linear relation or, in other terminology, a multivalued linear operator [8, 9, 16, 22, 41, 60, 101].

Moreover, in this article, we adhere to the point of view that a pseudo-resolvent is just as fundamental as an operator (bounded, unbounded or multivalued) that generates it. The reason for this is due to the fact that, when speaking about unbounded operators and linear relations, we often actually work with their resolvents. For example, an unbounded operator is a generator of a strongly continuous or analytic semigroup if and only if [39, 70, 140] its resolvent satisfies a special estimate of the decrease rate at infinity; in [79, Theorem 2.25] and [119, VIII.7], the natural convergence of unbounded operators is defined as the convergence of their resolvents in norm; and in [117, 118], a function \( f \) of unbounded operators \( A \) and \( B \) is defined as an (unbounded) operator \( f(A, B) \) that possesses the following property: there exist sequences of bounded operators \( A_n \) and \( B_n \) such that the resolvents of \( A_n \), \( B_n \), and \( f(A_n, B_n) \) converge in norm to the resolvents of \( A \), \( B \), and \( f(A, B) \), respectively. Another argument (not used in this article) is that there is no analogue of unbounded and multivalued operators in Banach algebras, but, nevertheless, there are evident analogues of the resolvents of such operators. The last idea is employed in [20, 21, 26, 139] for the investigation of unbounded operators using tools of \( C^\ast \)-algebras.

This approach enables one to extend the notion of \( f(A, B) \) to meromorphic functions \( f \) (Theorem 42): the result of the action of a meromorphic function on \( A \) and \( B \) is defined as a new pseudo-resolvent; thus, there is no need to discuss which operator it is generated by.

Many important applications are connected with special cases of construction (*) and their modifications. For example, it often occurs that the function \( f \) depends on the difference or the sum of its arguments: the transformator \( C \mapsto AC - CB \) generated by the function \( f(\lambda, \mu) = \lambda - \mu \) is related to the Sylvester equation
(Sect. 10), and the transformator \( C \mapsto e^{A^t}Ce^{Bt} \) generated by the function \( f(\lambda, \mu) = e^{(\lambda+\mu)} \) is connected with the stability theory of differential equations [5, 25].

The version (Sect. 8)

\[
\begin{align*}
f^{[1]}(A, B)C = \frac{1}{2\pi i} \int_G f(\lambda)(\lambda 1 - A)^{-1} C(\lambda 1 - B)^{-1} d\lambda
\end{align*}
\]

of functional calculus (*) often occurs in applications; it involves functions \( f \) of one variable. For example, expression (***) with \( f(\lambda) = e^{\lambda t} \) forms the principal part of the representation of a solution of the second-order differential equation \( \frac{d^2}{dt^2} 1 - A)C^{-1}(\frac{d}{dt} 1 - B)y = 0 \) (Sect. 9). Furthermore (Sect. 11), the differential of the ordinary functional calculus \( A \mapsto f(A) \) at a point \( A \) can be represented in the form: (Theorem 71)

\[
df(\cdot, A) = f^{[1]}(A, A).
\]

It turns out that (Theorem 45) \( f^{[1]}(A, B)C \) coincides with (*) provided \( f^{[1]} \) is understood as the divided difference:

\[
f^{[1]}(\lambda, \mu) = \begin{cases} 
\frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\
f'(\lambda), & \text{if } \lambda = \mu, \\
0, & \text{if } \lambda = \infty \text{ or } \mu = \infty.
\end{cases}
\]

When choosing the level of generality of our exposition, we proceed from the following principles. First, in order that a specialist in the classical operator theory can use the article, we try to minimize the explicit mention of Banach algebras and linear relations (multivalued operators), at least in main statements. At the same time, when using the operator language, we aspire the maximal generality and, in particular, whenever possible, we consider the case of an arbitrary pseudo-resolvent (and, therefore, implicitly, the cases of unbounded operators and linear relations). Second, we try not to fall outside the framework of the theory of analytic functions of operators, and therefore, for example, we do not discuss issues related to generators of semigroups. Third we avoid, as far as possible, the explicit use of operator pencils \([107] \lambda \mapsto \lambda F - G \) with \( F \neq 1 \) instead of the operators \( A \) and \( B \), because this approach leads to very cumbersome formulae. Finally, we confine ourselves to the consideration of functions of two variables, assuming that a generalization to the cases of three and more variables does not cause significant difficulties.

The literature on the subject under discussion is extremely extensive. Therefore, the bibliography can not be made completely comprehensive; undoubtedly, the presented references reflect authors’ tastes and interests. Some additional references can be found in the cited articles and books.

Sections 2–5 outline preliminary information. Here we recall and refine notation and the main facts in a convenient form. In Sect. 2, the terminology connected with Banach algebras and their properties is recalled. In Sect. 3, the basic properties of algebras of analytic functions of one and two variables are described. In Sect. 4, we
discuss the notion of pseudo-resolvent and recall the construction of the functional calculus of analytic functions of one variable (Theorems 25, 26) including the spectral mapping theorem (Theorem 27). In Sect. 5, the definition of the extended tensor product is given, the main examples are described, and the construction of the functional calculus of operator-valued analytic functions of one variable (Theorems 28, 29) is recalled as well as the relevant spectral mapping theorem (Theorem 31).

In Sect. 6, we present the construction of the functional calculus (*) of functions of two variables (Theorems 32, 33, 34), and prove the corresponding spectral mapping theorem (Theorem 39). In Sect. 7, these results are extended to meromorphic functions. A well-known example of a meromorphic function of an operator is a polynomial of an unbounded operator (a polynomial has a pole at infinity, while the point at infinity belongs to the extended spectrum of an unbounded operator). This example shows that the result of applying a meromorphic function cannot be a bounded transformator; as a convenient tool for its description, we use not the resulting object itself, but its resolvent, while we interpret the extended singular set of this resolvent as its spectrum (Theorem 42).

In Sect. 8, we discuss modified variant (**) of the functional calculus of functions of two variables. The connection between functional calculi (*) and (**) is described (Theorem 46) as well as some properties of functional calculus (**).

The subsequent Sections are devoted to applications. In Sect. 9, the pencil \( \lambda \mapsto \lambda^2 E + \lambda F + H \) of the second order is considered; it is induced by the equation \( E\ddot{y}(t) + F\dot{y}(t) + Hy(t) = 0 \); we assume that the pencil admits a factorization, i.e., it can be represented as a product of two linear pencils. In such a case, the solution of the differential equation is expressed by a transformation of the kind (**) (Theorem 54). In Sect. 10, we discuss the properties of the transformator \( W : C \rightarrow Z \) generated by the Sylvester equation \( AZ - ZB = C \) (Theorem 64). Finally, in Sect. 11, it is shown that the differential of the ordinary functional calculus \( A \mapsto f(A) \) is also a kind of transformator (**) (Theorem 71).

## 2 Banach algebras

In this section, we clarify the terminology connected with Banach algebras [18, 70, 122] and recall some of their properties.

In this article, all linear spaces and algebras are assumed to be complex.

Let \( X \) and \( Y \) be Banach spaces. We denote by \( \mathcal{B}(X,Y) \) the set of all bounded linear operators \( A : X \rightarrow Y \). When \( X = Y \), we use the shorthand symbol \( \mathcal{B}(X) \). The symbol \( 1 = 1_X \) stands for the identity operator. We adhere to the following notations: \( X^* \) denotes the conjugate space of \( X \); \( \langle x, x^* \rangle \) denotes the value of the functional \( x^* \in X^* \) on \( x \in X \), and \( \langle x^{**}, x^* \rangle \) is the value of \( x^{**} \in X^{**} \) on \( x^* \in X^* \); \( A^* \) denotes the conjugate operator of \( A \in \mathcal{B}(X,Y) \). The preconjugate of an operator \( A \in \mathcal{B}(Y^*,X^*) \) is an operator \( A^0 \in \mathcal{B}(X,Y) \) such that \( (A^0)^* = A \).

The unit [18, 70, 122] of an algebra \( \mathcal{B} \) is an element \( 1 \in \mathcal{B} \) such that \( 1A = A1 = A \) for all \( A \in \mathcal{B} \). If an algebra \( \mathcal{B} \) has a unit, it is called an algebra with a unit or unital.
A subset $R$ of an algebra $B$ is called a subalgebra if $R$ is stable under the algebraic operations (addition, scalar multiplication, and multiplication), i.e., $A + B$, $\lambda A$, $AB \in R$ for all $A, B \in R$ and $\lambda \in \mathbb{C}$. If the unit $1$ of an algebra $B$ belongs to its subalgebra $R$, then $R$ is called a subalgebra with a unit or a unital subalgebra.

Let $B$ be a non-unital algebra. The set $\tilde{B} = \mathbb{C} \oplus B$ with the componentwise linear operations and the multiplication $(a, A)(b, B) = (ab, aB + bA + AB)$ is obviously an algebra with the unit $(1, 0)$, where $0$ is the zero of the algebra $B$. The algebra $\tilde{B}$ is called the algebra derived from $B$ by adjoining a unit or an algebra with an adjoint unit. Usually, the symbol $1$ stands for the element $(1, 0)$, and the symbol $a1 + A$ denotes the element $(a, A)$. If $B$ is a normed algebra, the formula $\|A / C_0 - B\| = \|a\| + \|A\|$ defines a norm on $\tilde{B}$. It is clear that $\tilde{B}$ is complete provided that $B$ is complete. If $B$ is unital, then $\tilde{B}$ means the algebra $B$ itself.

**Theorem 1** Let $B$ be a unital Banach algebra and $A, B \in B$. If $A$ is invertible and
\[ \|B\| \cdot \|A^{-1}\| < 1, \]
then the element $A - B$ is also invertible and
\[ (A - B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1}BA^{-1} + \ldots \]

In this case
\[
\|(A - B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|B\| \cdot \|A^{-1}\|}, \\
\|(A - B)^{-1} - A^{-1}\| \leq \frac{\|B\| \cdot \|A^{-1}\|^2}{1 - \|B\| \cdot \|A^{-1}\|}, \\
\|(A - B)^{-1} - A^{-1} - A^{-1}BA^{-1}\| \leq \frac{\|B\|^2 \cdot \|A^{-1}\|^3}{1 - \|B\| \cdot \|A^{-1}\|}.
\]

**Proof** We consider the series
\[ A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1}BA^{-1} + \ldots \]
We represent the series in the form
\[ A^{-1}(1 + BA^{-1} + BA^{-1}BA^{-1} + BA^{-1}BA^{-1}BA^{-1} + \ldots). \]
Since $\|BA^{-1}\| \leq \|B\| \cdot \|A^{-1}\| < 1$, the series converges absolutely. We denote its sum by $C$. It is straightforward to verify that $C$ coincides with the inverse of $A - B$.

Estimates (1) follow from the geometric series formula $\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}$, $|q| < 1$. For example, the proof of the second estimate is as follows:
\[
\| (A - B)^{-1} - A^{-1} \| \leq \sum_{k=1}^{\infty} \| B \|^k \cdot \| A^{-1} \|^k + 1 = \frac{\| B \| \cdot \| A^{-1} \|^2}{1 - \| B \| \cdot \| A^{-1} \|}.
\]

Let \( B \) be a (nonzero) unital algebra. We call the spectrum (in the algebra \( B \)) of an element \( A \in B \) the set of all \( \lambda \in \mathbb{C} \) such that the element \( \lambda I - A \) is not invertible. The spectrum of the element \( A \) is denoted by the symbol \( \sigma(A) \). The complement \( \rho(A) = \mathbb{C} \setminus \sigma(A) \) is called the resolvent set of \( A \). The function \( R_\lambda = (\lambda I - A)^{-1} \) is called the resolvent of the element \( A \).

**Proposition 2** ([70, Theorem 4.8.1]) The resolvent \( R_\lambda \) of any element \( A \in B \) satisfies the Hilbert identity

\[
R_\lambda - R_\mu = - (\lambda - \mu) R_\lambda R_\mu, \quad \lambda, \mu \in \rho(A).
\]

**Proposition 3** ([18, ch. 1, § 2.5], [122, Theorem 10.13]) The spectrum of any element \( A \) of a nonzero unital Banach algebra \( B \) is a compact and nonempty subset of the complex plane \( \mathbb{C} \).

Let \( A \) and \( B \) be algebras. A mapping \( \varphi : A \to B \) is called [18] a morphism of algebras if

\[
\varphi(A + B) = \varphi(A) + \varphi(B),
\]

\[
\varphi(zA) = z \varphi(A),
\]

\[
\varphi(AB) = \varphi(A) \varphi(B).
\]

If, in addition, \( A \) and \( B \) are unital and

\[
\varphi(1_A) = 1_B,
\]

\( \varphi \) is called a morphism of unital algebras. If \( A \) and \( B \) are Banach algebras [18, 70, 122] and, in addition, the morphism \( \varphi \) is continuous, then \( \varphi \) is called a morphism of Banach algebras.

A unital subalgebra \( R \) of a unital algebra \( B \) is called [18, ch. 1, § 1.4] full if every \( B \in R \) that is invertible in \( B \) is also invertible in \( R \). Since the inverse is unique, this condition is equivalent to the following one: if for \( B \in R \) there exists \( D \in B \) such that \( BD = DB = 1 \), then \( D \in R \).

**Example 1** Let \( X \) be a Banach space. The set \( B_0(X^*) \) of all operators having a preconjugate is a full subalgebra of the algebra \( B(X^*) \).

**Proposition 4** ([18, ch. 1, § 2.5]) The closure of a full subalgebra of a Banach algebra is also a full subalgebra. The closure of the least full subalgebra of a Banach algebra that contains a set \( M \) is the least full closed subalgebra that contains \( M \).

An algebra \( B \) is called commutative if \( AB = BA \) for all \( A, B \in B \).
A character of a unital commutative algebra $B$ [18, ch. 1, § 1.5] is a morphism $\chi : B \to \mathbb{C}$ of unital algebras.

A character of a commutative non-unital algebra $B$ [18, ch. 1, § 1.5] is a morphism of (non-unital) algebras $\chi : B \to \mathbb{C}$. If an algebra $B$ is non-unital, we denote by $\chi_0 : B \to \mathbb{C}$ that is equal to zero on all elements of $B$. We call $\chi_0$ the zero character. We stress that the zero character $\chi_0$ exists only if the algebra $B$ is non-unital.

**Proposition 5** All characters of a non-unital algebra $B$ are uniquely extendable to characters of the algebra $\widehat{B}$ derived from $B$ by adjoining a unit; the extension is defined by the formula $\chi(x1 + A) = x + \chi(A)$. Conversely, the restriction of any character of the algebra $\widehat{B}$ to $B$ is a character of the algebra $B$. In particular, the zero character $\chi_0$ is the restriction of the character $x1 + A \mapsto x$; we will denote it by the same symbol $\chi_0$.

**Proof** The proof is obvious.

We denote by $X(B)$ the set of all nonzero characters of a commutative algebra $B$ (unital or non-unital), and we denote by $\widetilde{X}(B)$ the set of all characters of a commutative algebra $B$ (including the zero character $\chi_0$ if the algebra is non-unital).

If an algebra $B$ is unital, then $\widetilde{X}(B)$ obviously coincides with $X(B)$. The set $X(B)$ is called [18] the character space of the algebra $B$.

**Theorem 6** ([18, ch. 1, § 3.3, Proposition 3]) Let $B$ be a unital commutative Banach algebra. Then for all $A \in B$,

$$\sigma(A) = \{ \chi(A) : \chi \in \widetilde{X}(B) \}.$$  

**Corollary 7** ([18, ch. 1, § 3, Theorem 1]) Every character of a commutative Banach algebra is continuous; namely, its norm is less than or equal to unity.

**Corollary 8** In a unital commutative Banach algebra $B$, the spectrum continuously depends on an element; more precisely, if $A, B \in B$ and $\|A - B\| < \varepsilon$, then $\sigma(B)$ is contained in the $\varepsilon$-neighborhood of $\sigma(A)$.

**Proof** The proof is obvious.

**3 Algebras of analytic functions**

This section is a preparation for the discussion of analytic functional calculi. Here we collect some preliminaries on algebras of analytic functions defined on subsets of $\mathbb{C}$ and $\mathbb{C}^2$.

We denote by $\overline{\mathbb{C}}$ the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane $\mathbb{C}$, and we denote by $\mathbb{C}^2$ the Cartesian product $\mathbb{C} \times \mathbb{C}$.
Proposition 9 \ Let \( \sigma_1, \sigma_2 \subseteq \overline{C} \) be closed sets, and let an open set \( W \subseteq \overline{C}^2 \) contain \( \sigma_1 \times \sigma_2 \). Then there exist open sets \( U, V \subseteq \overline{C} \) such that \( \sigma_1 \times \sigma_2 \subseteq U \times V \subseteq W \).

**Proof** \ For an arbitrary \( \lambda \in \sigma_1 \), we consider the set \( \{ \lambda \} \times \sigma_2 \). We consider a finite cover of \( \{ \lambda \} \times \sigma_2 \) by the sets of the form \( U_i \times V_i \), where \( U_i, V_i \subseteq \overline{C} \) are open and \( U_i \times V_i \subseteq W \). We put \( \tilde{U} = \bigcap_i U_i \) and \( \tilde{V} = \bigcup_i V_i \). It is clear that the set \( \tilde{U} \times \tilde{V} \subseteq W \) also covers the set \( \{ \lambda \} \times \sigma_2 \), namely, \( \{ \lambda \} \subseteq \tilde{U}, \sigma_2 \subseteq \tilde{V} \).

Furthermore, we cover every subset of the form \( \{ \lambda \} \times \sigma_2 \) of the set \( \sigma_1 \times \sigma_2 \) by a set of the form \( \tilde{U} \times \tilde{V} \subseteq W \). We choose a finite subcover \( \{ \tilde{U}_k \times \tilde{V}_k \} \) and put \( U = \bigcup_k \tilde{U}_k, V = \bigcap_k \tilde{V}_k \). Obviously, \( \sigma_1 \times \sigma_2 \subseteq U \times V \subseteq W \). \( \square \)

Proposition 10 \ Let \( U_1, U_2 \subseteq \overline{C} \) be open sets. Then for every compact set \( N \subseteq U_1 \times U_2 \) there exist compact sets \( N_1 \subseteq U_1 \) and \( N_2 \subseteq U_2 \) such that \( N \subseteq N_1 \times N_2 \).

**Proof** \ It is sufficient to take for \( N_1 \) the image of the set \( N \) under the projection \( (\lambda, \mu) \rightarrow \lambda \) onto the first coordinate, and for \( N_2 \) the image of the set \( N \) under the projection \( (\lambda, \mu) \rightarrow \mu \) onto the second coordinate. \( \square \)

We recall [57, 127] that a function \( f : U \subseteq \overline{C} \rightarrow B, \infty \in U \), is called analytic at infinity if \( f \) can be expanded in a power series \( f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k}{\lambda^k} \) in a neighborhood of infinity; a function \( f : U \subseteq \overline{C}^2 \rightarrow C, (\infty, \infty) \in U \), is called analytic at \( (\infty, \infty) \) if \( f \) can be expanded in a power series \( f(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{c_{km}(\lambda - \lambda_0)^k}{\mu^m} \) in a neighborhood of \( (\infty, \infty) \); a function \( f : U \subseteq \overline{C}^2 \rightarrow C, (\lambda_*, \infty) \in U \), is called analytic at \( (\lambda_*, \infty) \), where \( \lambda_* \in C \), if \( f \) can be expanded in a power series \( f(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{c_{km}(\lambda - \lambda_0)^k}{\mu^m} \) in a neighborhood of \( (\lambda_*, \infty) \).

Let \( K \) be a closed subset of \( \overline{C}^2 \) or \( \overline{C} \) and \( B \) be a unital Banach algebra. We denote by \( O(K, B) \) the set of all analytic functions \( f : U \rightarrow B \), where \( U \) is an open neighborhood of the set \( K \) (it is implied that the neighborhood \( U \) may depend on \( f \)). We accept that the domain of an analytic function may be disconnected. Two functions \( f_1 : U_1 \rightarrow B \) and \( f_2 : U_2 \rightarrow B \) are called equivalent if there exists an open neighborhood \( U \subseteq U_1 \cap U_2 \) of the set \( K \) such that \( f_1 \) and \( f_2 \) coincide on \( U \), i.e. \( f_1(\lambda) = f_2(\lambda) \) for all \( \lambda \in U \). It can be easily shown that it is really an equivalence relation. Thus, strictly speaking, elements of \( O(K, B) \) are classes of equivalent functions. The notation \( O(K, \mathbb{C}) \) is abbreviated to \( O(K) \).

Proposition 11 \ The set \( O(K, B) \) is an algebra with respect to the pointwise operations with the unit \( u(\lambda) = 1, \lambda \in \overline{C} \) or \( \lambda \in \overline{C}^2 \), respectively.

**Proof** \ The proof is obvious. \( \square \)

Proposition 12 \ (a) For \( f \in O(K, B) \), the following conditions are equivalent:

1. the function \( f \) is invertible in the algebra \( O(K, B) \);
2. the element \( f(\lambda) \in B \) is invertible at all points \( \lambda \in K \);
3. the element \( f(\lambda) \) is invertible at all points \( \lambda \in U \), where \( U \supset K \) is an open set.
The spectrum of a function \( f \in \mathcal{O}(K, B) \) in the algebra \( \mathcal{O}(K, B) \) is given by the formula
\[
\bigcup_{\lambda \in K} \sigma(f(\lambda)).
\]

**Proof** The proof is obvious. \( \square \)

We recall the definition of the natural topology on the algebra \( \mathcal{O}(K, B) \), see [18, ch. 1, § 4.1] for more details.

For each open set \( U \supset K \), we denote by \( \mathcal{O}(U, B) \) the linear space of all analytic functions \( f: U \to B \). We endow \( \mathcal{O}(U, B) \) with the topology of compact convergence [17, ch. X, § 3.6], [123, ch. III, § 3]. In this topology, a fundamental system of neighborhoods of zero is formed by the sets:

\[
T(N, \delta) = \{ f \in \mathcal{O}(U, B) : \|f(\lambda)\| < \delta \ \text{for} \ \lambda \in N \},
\]

where \( N \subset U \) is compact and \( \delta > 0 \); clearly, when \( N \) enlarges, the neighborhood \( T(N, \delta) \) shrinks; therefore, it is enough to consider only those sets \( N \), the interiors of which contain \( K \). There are evident canonical mappings \( g_U: \mathcal{O}(U, B) \to \mathcal{O}(K, B) \). The mappings \( g_U \) are not always injective. Nevertheless, by an abuse of language, we will regard \( \mathcal{O}(U, B) \) as subspaces of the space \( \mathcal{O}(K, B) \).

We endow \( \mathcal{O}(K, B) \) with the inductive topology [123, ch. II, § 6] induced by the mappings \( g_U \) (one may restrict himself to a decreasing sequence of open sets \( U \supset K \)). A fundamental system of neighborhoods of zero in \( \mathcal{O}(K, B) \) consists of all balanced, absorbent, and convex sets \( W \subseteq \mathcal{O}(K, B) \) such that the inverse image \( g_U^{-1}(W) \) is a neighborhood of zero in \( \mathcal{O}(U, B) \). Thus, for all \( U \supset N \supset K \) such that the interior of the compact set \( N \) contains \( K \), the inverse image \( g_U^{-1}(W) \) must contain the set \( T(N, \delta) \subset \mathcal{O}(U, B) \).

We recall [123, ch. 2, Theorem 6.1] that a linear mapping \( \varphi: \mathcal{O}(K, B) \to \mathcal{E} \), where \( \mathcal{E} \) is a Banach space, is continuous if and only if all the compositions \( \varphi \circ g_U: \mathcal{O}(U, B) \to \mathcal{E} \), where \( U \supset K \), are continuous, i.e. for any neighborhood \( W \subseteq \mathcal{E} \) of zero, there exist a compact set \( N \subset U \) and a number \( \delta > 0 \) such that the interior of \( N \) contains \( K \) and \( \varphi \circ g_U(T(N, \delta)) \subseteq W \). We note that since \( \mathcal{E} \) is a Banach space, it is sufficient to restrict ourselves to the consideration of \( \varepsilon \)-neighborhoods of zero as \( W \). Below, by an abuse of language, we denote \( \varphi \circ g_U \) by the abbreviated symbol \( \varphi \).

**Proposition 13** ([72, Proposition 1.3]) Let \( U_1 \subseteq \overline{C} \) and \( U_2 \subseteq \overline{C} \) be open sets. Then the natural image of the algebraic tensor product \( \mathcal{O}(U_1) \otimes \mathcal{O}(U_2) \) is everywhere dense in \( \mathcal{O}(U_1 \times U_2) \).
4 Pseudo-resolvents and functional calculi

We call a functional calculus a mapping that converts functions to operators (or transformators). Of course, the most interesting are functional calculi that possess special properties (for example, morphisms of algebras).

A pseudo-resolvent is a function that takes values in a Banach algebra and satisfies the Hilbert identity (2), like a resolvent. In this section, we recall that every pseudo-resolvent generates a functional calculus (Theorems 25, 26) which is a morphism of algebras and possesses the property of preserving the spectrum (Theorem 27).

Let $B$ be a Banach algebra and $U \subseteq \mathbb{C}$ be a non-empty subset. A function (family) $\lambda \mapsto R_\lambda$ defined on $U$ and taking values in $B$ is called [70, ch. 5, § 2] a pseudo-resolvent if it satisfies the Hilbert identity

$$R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu, \quad \lambda, \mu \in U.$$  \hspace{1cm} (3)

A pseudo-resolvent is called [9, p. 103] maximal if it cannot be extended to a larger set with the preservation of the Hilbert identity (3). Below (Theorem 16) we will see that every pseudo-resolvent can be extended to a unique maximal one. The maximal set $\rho(R_\lambda)$, on which it is possible to extend the pseudo-resolvent $R_\lambda$, is called a regular set of the original pseudo-resolvent. The complement $\sigma(R_\lambda) = \mathbb{C} \setminus \rho(R_\lambda)$ of the regular set $\rho(R_\lambda)$ is called [6, 9, p. 103] the singular set.

Example 2 The examples of pseudo-resolvents are: (a) the resolvent of an element of a unital Banach algebra (Proposition 14); (b) a constant function $\lambda \mapsto N$, where $N \in B$ is an arbitrary element whose square equals zero (Proposition 23); (c) in particular, the identically zero function; (d) the resolvent of a closed linear operator [70, Theorem 5.8.1]; (e) the resolvent of a linear relation [8, 9, 16, 22, 41, 60, 101]; the last example is the most general, because every pseudo-resolvent is a resolvent of some linear relation [9, Theorem 5.2.4], [60, Proposition A.2.4]; (f) direct sums of pseudo-resolvents from the previous examples.

The simplest but the most important example of a maximal pseudo-resolvent is given in the following proposition.

Proposition 14 ([97, Proposition 17]) The resolvent of an arbitrary element $A \in B$ is a maximal pseudo-resolvent, i.e., it cannot be extended to a set larger than $\rho(A)$ with the preservation of the Hilbert identity (3).

We note that identity (3) can be equivalently rewritten in the form (here $1$ is an adjoint unit if the original algebra has no one)

$$R_\lambda(1 + (\lambda - \mu)R_\mu) = R_\mu.$$  \hspace{1cm} (4)

We recall that $\tilde{B}$ denotes the algebra derived from $B$ by adjoining a unit. If $B$ is unital, then $\tilde{B}$ means the algebra $B$ itself.
Proposition 15 ([70, Corollary 1 of Theorem 5.8.4]) Let $R_\lambda, R_\mu \in \mathcal{B}$ be two commuting elements that satisfy identity (3). Then the element $1 + (\lambda - \mu)R_\mu \in \widetilde{\mathcal{B}}$ is necessarily invertible.

Theorem 16 ([70, Theorem 5.8.6]) Each pseudo-resolvent whose domain contains at least one point $\mu \in \mathbb{C}$ can be extended to a maximal pseudo-resolvent; this extension is unique. The domain of the maximal extension is the set of all $\lambda \in \mathbb{C}$ such that the element $1 + (\lambda - \mu)R_\mu$ is invertible in $\widetilde{\mathcal{B}}$. This extension can be defined by the formula

$$R_\lambda = R_\mu \left(1 + (\lambda - \mu)R_\mu\right)^{-1} = \left(1 + (\lambda - \mu)R_\mu\right)^{-1} R_\mu.$$ (5)

We denote the original pseudo-resolvent and its continuation to a maximal pseudo-resolvent by the same symbol $R_{(\cdot)}$. Moreover, we will generally assume that all pseudo-resolvents under consideration are already extended to maximal pseudo-resolvents.

Corollary 17 ([70, Theorem 5.8.2], [22, ch. 6, § 1]) The domain of a maximal pseudo-resolvent is an open set and the maximal pseudo-resolvent is an analytic function (with values in $\mathcal{B}$). More precisely, in a neighborhood of any point $\mu \in \rho(R_{(\cdot)})$, the maximal pseudo-resolvent admits the Taylor series expansion:

$$R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1}.$$ (6)

Proposition 18 ([70, Theorem 5.8.3]) A pseudo-resolvent $R_{(\cdot)}$ in a Banach algebra $\mathcal{B}$ is a resolvent of some element $A \in \mathcal{B}$ if and only if $\mathcal{B}$ is unital and the element $R_\mu$ is invertible for at least one (and, consequently, for all) $\mu \in \rho(R_{(\cdot)})$. A pseudo-resolvent $R_{(\cdot)}$ in $\mathcal{B}(X)$, where $X$ is a Banach space, is a resolvent of some unbounded operator $A : D(A) \subset X \to X$ if and only if the operator $R_\mu : X \to \text{Im}R_\mu$, where $\text{Im}R_\mu$ is the image of $R_\mu$, is invertible for at least one (and, consequently, for all) $\mu \in \rho(R_{(\cdot)})$. In this case $A = \lambda 1 - (R_\lambda)^{-1}$ for all $\lambda \in \rho(R_{(\cdot)})$.

In a similar way, a linear relation can also be reconstructed from the value of its resolvent at one point. Thus, the resolvent contains all information about the linear relation or about the operator that generates it. On the other hand, the conditions on unbounded operators and linear relations are often imposed in terms of their resolvents (the nonemptiness of the resolvent set, the estimate of the rate of decrease at infinity etc.). Besides, functions of linear relations and of unbounded operators are often expressed directly via their resolvents. For this reason, the resolvent can be regarded as a more fundamental object than the operator or the relation that generates it. This is the viewpoint that we adhere to in this article.
We fix a pseudo-resolvent \( R(\cdot) \). We denote by \( B_{\mathcal{R}} \) the smallest closed subalgebra of the algebra \( B \) that contains all elements \( R_\lambda, \lambda \in \rho(R(\cdot)) \), of the extension of the pseudo-resolvent \( R(\cdot) \) to a maximal pseudo-resolvent.

**Proposition 19** ([97, Proposition 21]) The algebra \( B_{\mathcal{R}} \) coincides with the closure of the linear span of the family of all elements \( R_\lambda, \lambda \in \rho(R(\cdot)) \), and is commutative.

If the algebra \( B_{\mathcal{R}} \) does not contain the unit of the algebra \( B \) (this is certainly the case if \( B \) is not unital), then we denote by \( \tilde{B}_{\mathcal{R}} \) the algebra \( B_{\mathcal{R}} \) with an adjoint unit from \( B \) (or from the algebra \( \tilde{B} \) with an adjoint unit). Adjoining \( 1 \) to \( B_{\mathcal{R}} \) we obtain a closed subalgebra, because the sum of a closed subspace and a one-dimensional subspace is a closed subspace. If \( B_{\mathcal{R}} \) contains the unit of the algebra \( B \), the symbol \( \tilde{B}_{\mathcal{R}} \) is understood to be \( B_{\mathcal{R}} \).

**Proposition 20** ([97, Theorem 22]) The subalgebra \( \tilde{B}_{\mathcal{R}} \) is commutative and full.

**Proposition 21** If a character \( \chi \) of the algebra \( B_{\mathcal{R}} \) equals zero at least at one element \( R_\mu, \mu \in \rho(R(\cdot)) \), then it is identically equals zero on \( B_{\mathcal{R}} \), i.e., coincides with \( \chi_0 \).

**Proof** The proof follows from formula (5) and the description of \( B_{\mathcal{R}} \) (Proposition 19) as the closure of the linear span of the family \( R_\lambda, \lambda \in \rho(R(\cdot)) \). \( \square \)

We say that a sequence of maximal pseudo-resolvents \( R_{n,\cdot} \) converges to a maximal pseudo-resolvent \( R(\cdot) \) if there exists a point \( \mu \in \mathbb{C} \) such that all the pseudo-resolvents \( R_{n,\cdot} \) are defined at \( \mu \) (for \( n \) sufficiently large) and the sequence \( R_{n,\mu} \) converges to \( R_{\mu} \) in norm, cf. [79, Theorem 2.25]. The following lemma shows that this definition does not depend on the choice of the point \( \mu \in \mathbb{C} \).

**Lemma 22** Let a sequence \( R_{n,\cdot} \) of maximal pseudo-resolvents converge to a maximal pseudo-resolvent \( R(\cdot) \) at a point \( \mu \in \mathbb{C} \) (it is assumed that the pseudo-resolvents \( R_{n,\mu} \) are defined at the point \( \mu \) for all \( n \) large enough). Then for any point \( \lambda \in \rho(R(\cdot)) \), the sequence \( R_{n,\mu} \) is defined for all \( n \) large enough and converges to \( R_\lambda \) in norm.

Moreover, given a compact set \( \Gamma \subset \rho(R(\cdot)) \), the elements \( R_{n,\lambda} \) are defined at all \( \lambda \in \Gamma \) for \( n \) large enough and converge to \( R_\lambda \) uniformly with respect to \( \lambda \in \Gamma \).

**Proof** Let \( R_{n,\mu} \) converge to \( R_{\mu} \). By Theorem 16, the element \( 1 + (\lambda - \mu)R_{\mu} \) is invertible for all \( \lambda \in \Gamma \). Since the function \( \lambda \mapsto 1 + (\lambda - \mu)R_{\mu} \) is continuous, from Theorem 1 it follows that

\[
\min_{\lambda \in \Gamma} \| (1 + (\lambda - \mu)R_{\mu})^{-1} \| > 0.
\]

Since \( R_{n,\mu} \) converges to \( R_{\mu} \), the sequence \( \lambda \mapsto 1 + (\lambda - \mu)R_{n,\mu} \) converges to \( \lambda \mapsto 1 + (\lambda - \mu)R_{\mu} \) uniformly with respect to \( \lambda \in \Gamma \). Therefore, again, by Theorem 1, the elements \( 1 + (\lambda - \mu)R_{n,\mu} \) are invertible for all \( \lambda \in \Gamma \) provided \( n \) is large enough; in
this case, by estimate (1), the inverses also converge uniformly. It remains to apply formula (5).

We note that the limit of a sequence of resolvents of bounded operators can be the resolvent of a unbounded operator, see [118, Lemma 7].

**Proposition 23** ([70, Theorem 5.9.2]) Let a pseudo-resolvent $R_{(\cdot)}$ admit an analytic continuation in a neighborhood of the point $\infty$. Then there exist elements $P, A, N \in B_R$ such that

$$
N^2 = 0, \quad P^2 = P, \quad AP = PA = A, \quad NP = PN = 0
$$

and the expansion of the pseudo-resolvent into the Laurent series with center $\infty$ has the form

$$
R_{\lambda} = -N + \frac{P}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \frac{A^3}{\lambda^4} + \ldots
$$

We call the extended regular set $\bar{\rho}(R_{(\cdot)}) \subseteq \overline{\mathbb{C}}$ of a pseudo-resolvent $R_{(\cdot)}$ (in the algebra $B$) either the regular set $\rho(R_{(\cdot)})$ or the union $\rho(R_{(\cdot)}) \cup \{\infty\}$; more precisely, we add the point $\infty$ to $\bar{\rho}(R_{(\cdot)})$ if the algebra $B$ is unital, the regular set $\rho(R_{(\cdot)})$ contains a (deleted) neighborhood of $\infty$, and $\lim_{\lambda \to \infty} \lambda R_{\lambda} = 1$. We call the extended singular set of the pseudo-resolvent the complement $\sigma(R_{(\cdot)}) = \overline{\mathbb{C}} \setminus \bar{\rho}(R_{(\cdot)})$ of the extended regular set.

**Proposition 24** The following properties of a maximal pseudo-resolvent are equivalent:

(a) $\infty \in \bar{\rho}(R_{(\cdot)})$;

(b) the maximal pseudo-resolvent is the resolvent of some element $A \in B_R$ (see Proposition 18);

(c) the algebra $B$ is unital and the subalgebra $B_R$ contains the unit of the algebra $B$.

**Proof** The equivalence of (a) and (b) is proved in [97, Proposition 23].

Let assumption (b) be fulfilled, i.e. $R_{\lambda} = (\lambda 1 - A)^{-1}$, where $A \in B_R$. Then, by virtue of Theorem 1, the Laurent series expansion

$$
(\lambda 1 - A)^{-1} = \frac{1}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \ldots
$$

holds in a neighborhood of infinity, which shows that $1 \in B_R$, i.e., assumption (c) holds.

Let assumption (c) be fulfilled, i.e. let the subalgebra $B_R$ contain the unit of the algebra $B$. There are no identically zero characters on a unital commutative algebra, because $\chi(1) = 1$. Therefore, by Proposition 21, $\chi(R_{\mu}) \neq 0$ for all $\chi \in \hat{X}(B_R)$ and $\mu \in \rho(R_{(\cdot)})$. Hence, by Theorem 6, all values $R_{\mu}$ of the pseudo-resolvent are
invertible. Then, by virtue of Proposition 18, the pseudo-resolvent is the resolvent of some element $A \in \mathcal{B}_R$, i.e. assumption (b) holds.

Below, in this section, we assume that $X$ is a Banach space and we are given a maximal pseudo-resolvent $R(\cdot)$ in $\mathcal{B}(X)$.

Let $\sigma$ and $\Sigma$ be disjoint closed subsets of $\overline{C}$. A contour $\Gamma$ is called [70, ch. V, § 5.2] an oriented envelope of the set $\sigma$ with respect to the set $\Sigma$ if $\Gamma$ is an oriented boundary of an open set $U$ that contains $\sigma$ and is disjoint from $\Sigma$. Thus, $\Gamma$ surrounds the set $\sigma$ in the counterclockwise direction and surrounds the set $\Sigma$ in the clockwise direction.

**Theorem 25** Let $\infty \notin \bar{\sigma}(R(\cdot))$ and the mapping $\phi : O(\sigma(R(\cdot))) \to \mathcal{B}_R$ be defined by the formula

$$
\phi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda) d\lambda,
$$

(7)

where $\Gamma$ (see left Fig. 1) is an oriented envelope of the singular set $\sigma(R(\cdot))$ with respect to the point $\infty$ and the complement of the domain of the function $f$. Then $\phi$ is a continuous morphism of unital algebras.

The morphism $\phi$ maps the function $u(\lambda) = 1$ to the identity operator $1_X$. The function $v_1(\lambda) = \lambda$ is mapped by $\phi$ to the operator $A \in \mathcal{B}(X)$ that generates the maximal pseudo-resolvent $R(\cdot)$ in accordance with Proposition 24; and the function $r_{\lambda_0}(\lambda) = \frac{1}{\lambda_0 - \lambda}$, where $\lambda_0 \in \rho(R(\cdot))$, is mapped by $\phi$ to $R_{\lambda_0}$.

**Proof** The proof is analogous to that of the theorem on analytic functional calculus for bounded operators [18, ch. 1, § 4, Theorem 3], [70, Theorem 5.2.5], [122, Theorem 10.27].

When it is desirable to stress that the functional calculus $\phi$, considered in Theorem 25, is generated by the resolvent of an operator $A \in \mathcal{B}(X)$, we use the notation $R_{A,\lambda}$ instead of $R_{\lambda}$, the notation $\phi_A$ instead of $\phi$, and the notation $f(A)$ instead of $\phi(f)$.

**Theorem 26** Let $\infty \in \bar{\sigma}(R(\cdot))$ and the mapping $\phi : O(\bar{\sigma}(R(\cdot))) \to \tilde{\mathcal{B}}_R$ be defined by the formula

\[ \phi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda) d\lambda, \]
\[ \varphi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda \, d\lambda + f(\infty)1, \]  

(8)

where \( \Gamma \) is an oriented envelope of the extended singular set \( \sigma(R_{(1)}) \) with respect to the complement of the domain of \( f \) (see right Fig. 1). Then \( \varphi \) is a continuous morphism of unital algebras.

The morphism \( \varphi \) maps the function \( u(\lambda) = 1 \) to the identity operator \( 1 \). The function \( r_{\lambda_0}(\lambda) = \frac{1}{\lambda_0 - \lambda} \), where \( \lambda_0 \in \rho(R_{(1)}) \), is mapped by \( \varphi \) to \( R_{\lambda_0} \).

**Proof** The proof is analogous to that of the theorem on analytic functional calculus for unbounded operators [70, Theorem 5.11.2].

When it is desirable to stress that the functional calculus \( \varphi \) considered in Theorem 26 is generated by a pseudo-resolvent \( R_{(1)} \), we will use the notation \( \varphi_{R_{(1)}} \) instead of \( \varphi \) and the notation \( f(R_{(1)}) \) instead of \( \varphi(f) \).

A unified notation for formulae (7) and (8) is suggested in [70]:

\[ \varphi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda \, d\lambda + \delta f(\infty)1, \]

where \( \delta = 0 \) if \( \Gamma \) does not enclose \( \infty \), and \( \delta = 1 \) if \( \Gamma \) encloses \( \infty \).

The following theorem is a version of the spectral mapping theorem for the case of pseudo-resolvents.

**Theorem 27** For any function \( f \in O(\sigma(R_{(1)})) \) we have the equality

\[ \sigma_{B}(\varphi(f)) = \bigcup_{\lambda \in \sigma(R_{(1)})} f(\lambda). \]

**Proof** The proof is analogous to that of the spectral mapping theorem for unbounded operators [70, Theorem 5.12.1] and for linear relations [9, Theorem 5.2.17].

An analytic functional calculus for bounded operators was created in [37, 38, 135]. It was carried over to unbounded operators in [36, 70] and to linear relations in [8, 9, 22, 41, 60].

### 5 Extended tensor products

The notion of an extended tensor product is a generalization of the notion of a completion of an algebraic tensor product with respect to a uniform cross-norm [31, 34, 58, 67, 124]. It enables one to extend some constructions which are natural for the usual tensor products to supplementary applications. We recall an example which is the most important for our applications. It is known (see, e.g., [56]) that in the case of finite-dimensional Banach spaces \( X \) and \( Y \), the space \( B(Y, X) \) can be naturally identified with the tensor product \( X \otimes Y^* \). If \( X \) and \( Y \) are infinite-
dimensional, then $X \otimes Y^*$ corresponds only to the subspace of $B(Y, X)$ consisting of operators that have a finite-dimensional image. Therefore, the completion of $X \otimes Y^*$ with respect to any reasonable norm cannot coincide with the whole $B(Y, X)$. Nevertheless, $B(Y, X)$ can be represented (see Example 3(e) below) as an extended tensor product $X \bar{\otimes} Y^*$, which enables one to treat it almost as an ordinary tensor product.

We denote by $X \otimes Y$ the usual tensor product of linear spaces $X$ and $Y$. To distinguish $X \otimes Y$ from its extensions, we call $X \otimes Y$ an algebraic tensor product.

Let $X$ and $Y$ be Banach spaces. A norm $\| \cdot \|_\alpha$ on $X \otimes Y$ is called a cross-norm if

$$\|x \otimes y\|_\alpha = \|x\| \cdot \|y\|$$

for all $x \in X$ and $y \in Y$. We denote by $X \bar{\otimes}_\alpha Y$ the completion of the tensor product $X \otimes Y$ by the cross-norm $\alpha$.

Every element $v^* = \sum_{i=1}^{m} x_i^* \otimes y_i^* \in X^* \otimes Y^*$ induces the linear functional

$$v^* : \sum_{k=1}^{n} x_k \otimes y_k \mapsto \sum_{i=1}^{m} \sum_{k=1}^{n} \langle x_k, x_i^* \rangle \cdot \langle y_k, y_i^* \rangle$$

on the space $X \otimes Y$. We define the norm $\| \cdot \|_{\alpha^*}$ on $X^* \otimes Y^*$, conjugate to the cross-norm $\alpha$, by the formula

$$\|v^*\|_{\alpha^*} = \sup \{ |\langle v, v^* \rangle| : v \in X \otimes Y, \|v\|_{\alpha} \leq 1 \}.$$ 

A cross-norm $\alpha$ is called $*$-uniform if $\alpha^*$ is finite and is a cross-norm.

The space $B(X) \otimes B(Y)$ has the natural structure of an algebra. Every element $T = \sum_{i=1}^{m} A_i \otimes B_i \in B(X) \otimes B(Y)$ induces the linear operator

$$T : \sum_{k=1}^{n} x_k \otimes y_k \mapsto \sum_{i=1}^{m} \sum_{k=1}^{n} (A_i x_k) \otimes (B_i y_k)$$

in $X \otimes Y$. A cross-norm $\alpha$ on the space $X \otimes Y$ induces the norm $\bar{\alpha}$ of the operator $T \in B(X) \otimes B(Y)$ by the formula

$$\|T\|_{\bar{\alpha}} = \sup \{ \|Tv\| : v \in X \otimes Y, \|v\|_{\alpha} \leq 1 \}.$$ 

A cross-norm $\alpha$ is called [124] uniform if $\bar{\alpha}$ is finite and is a cross-norm. Every uniform cross-norm is $*$-uniform, see [130].

Let $X$ and $Y$ be Banach spaces. We call an extended tensor product [97] of $X$ and $Y$ a collection of three objects: a Banach space $X \overline{\otimes} Y$ (which we briefly refer to as the extended tensor product) and two (not necessarily closed) full unital subalgebras $B_0(X)$ and $B_0(Y)$ of the algebras $B(X)$ and $B(Y)$, respectively, that satisfy assumptions (A), (B), and (C) listed below.

(A) We are given a linear mapping $j : X \otimes Y \to X \overline{\otimes} Y$. We denote $j(x \otimes y)$ briefly by the symbol $x \overline{\otimes} y$. It is assumed that
\[ \|x \otimes y\|_{X \odot Y} = \|x\|_X \cdot \|y\|_Y \] (10)

for all \( x \in X \) and \( y \in Y \).

(B) We are given a linear mapping \( J : X^* \otimes Y^* \to (X \otimes Y)^* \). We denote briefly \( J(x^* \otimes y^*) \) by the symbol \( x^* \otimes y^* \). It is assumed that
\[
\langle x^* \otimes y^*, x^* \otimes y^* \rangle = \langle x, x^* \rangle \langle y, y^* \rangle
\] (11)

for all \( x^* \in X^*, \ y^* \in Y^*, \ x \in X, \) and \( y \in Y \), and
\[
\|x^* \otimes y^*\|_{(X \odot Y)^*} = \|x^*\|_X \cdot \|y^*\|_Y.
\] (12)

for all \( x^* \in X^* \) and \( y^* \in Y^* \).

(C) We are given a morphism of unital algebras \( \mathfrak{J} : B_0(X) \otimes B_0(Y) \to B(X \otimes Y) \). We denote \( \mathfrak{J}(A \otimes B) \) briefly by the symbol \( A \boxtimes B \). It is assumed that
\[
(\mathfrak{J}(A \otimes B))(x \otimes y) = (Ax) \otimes (By)
\] (13)

for all \( A \in B_0(X), \ B \in B_0(Y), \ x \in X, \) and \( y \in Y \), and
\[
(\mathfrak{J}(A \otimes B))^*(x^* \otimes y^*) = (A^* x^*) \otimes (B^* y^*)
\] (14)

for all \( A \in B_0(X), \ B \in B_0(Y), \ x^* \in X^*, \) and \( y^* \in Y^* \), and
\[
\|A \boxtimes B\|_{B(X \otimes Y)} = \|A\|_{B(X)} \cdot \|B\|_{B(Y)}
\] (15)

for all \( A \in B_0(X) \) and \( B \in B_0(Y) \).

Example 3  (a) Let \( \mathfrak{K} \) be a cross-norm on an algebraic tensor product \( X \otimes Y \). We take for \( X \odot Y \) the completion \( X \odot Y \) of the space \( X \otimes Y \) with respect to the cross-norm \( \mathfrak{K} \), and we take for \( B_0(X) \) and \( B_0(Y) \) the algebras \( B(X) \) and \( B(Y) \), respectively. In such a case, assumption (12) means that the cross-norm \( \mathfrak{K} \) is \(*\)-uniform, and assumption (15) means that the cross-norm \( \mathfrak{K} \) is uniform.

(b) Let \( X \) and \( Y \) be Banach spaces. We denote by \( K(X, Y) \) the Banach space of all bilinear forms \( K : X \times Y \to \mathbb{C} \) that are bounded with respect to the norm \( \|K\| = \sup \{ |K(x, y)| : \|x\| \leq 1, \|y\| \leq 1 \} \). To represent \( K(X, Y) \) as an extended tensor product \( X^* \otimes Y^* \), we take for \( B_0(X^*) \) and \( B_0(Y^*) \) the subalgebras of algebras \( B(X^*) \) and \( B(Y^*) \) consisting of all operators that have a preconjugate, see Example 1. We define the mappings \( j, J, \) and \( \mathfrak{J} \) by the rules (extended by linearity):
\[
\begin{align*}
[x^* \otimes y^*](x, y) &= \langle x, x^* \rangle \langle y, y^* \rangle, \\
(x^* \otimes y^* \otimes K) &= \overline{K}(x^*, y^*), \\
[(A \boxtimes B)K](x, y) &= K(A^0 x, B^0 y),
\end{align*}
\]

where \( \overline{K} \) is the canonical extension [7] of \( K \) to \( X^{**} \times Y^{**} \). This example can be considered as a special case of the following one, because \( K(X, Y) \) is naturally isomorphic to \( (X \odot_\pi Y)^* \), where \( \pi \) is the projective cross-norm.

(c) Let \( X \) and \( Y \) be Banach spaces, and \( X \odot_\pi Y \) be a completion of the space \( X \otimes Y \) with respect to a uniform cross-norm \( \mathfrak{K} \). The conjugate space \( (X \odot_\pi Y)^* \) can be
regarded as an extended tensor product $X^* \otimes Y^*$ if one takes for $B_0(X^*)$ and $B_0(Y^*)$ the subalgebras of the algebras $B(X^*)$ and $B(Y^*)$ consisting of all operators that have a preconjugate. We notice that this example is a generalization of the previous one, because $K(X, Y) \cong (X \otimes \pi Y)^*$, where $\pi$ is the projective cross-norm [31, 67, 124].

We define $j : X^* \otimes Y^* \to X^* \otimes Y^* = (X \otimes \pi Y)^*$ as the canonical embedding (9).

Next, we define $J : X^{**} \otimes Y^{**} \to (X^* \otimes Y^*)^* = (X \otimes \pi Y)^**$. To this end, we assign to each functional $w^* \in X^* \otimes Y^* = (X \otimes \pi Y)^*$ the bilinear form $K_{w^*}(x, y) = \langle x \otimes y, w^* \rangle$ on $X \times Y$. For $\sum_{k=1}^n x_{k}^{**} \otimes y_{k}^{**} \in X^{**} \otimes Y^{**}$, we set

$$\left\langle J\left(\sum_{k=1}^n x_{k}^{**} \otimes y_{k}^{**}\right), w^* \right\rangle = \sum_{k=1}^n K_{w^*}(x_{k}^{**}, y_{k}^{**}),$$

where $\overline{K_{w^*}}$ is the canonical extension of the bilinear form $K_{w^*}$ to $X^{**} \times Y^{**}$.

We define the operator $\mathfrak{J}(\sum_{k=1}^n A_k \otimes B_k) \in B((X \otimes \pi Y)^*)$ as the conjugate of the operator $\sum_{k=1}^n A_k \otimes B_k : X \otimes \pi Y \to X \otimes \pi Y$.

(d) Let $X = L_\infty[a, b]$ and $Y = L_\infty[c, d]$. By example (c), the space $L_\infty[a, b] \times [c, d]$ can be regarded as the extended tensor product $L_\infty[a, b] \otimes L_\infty[c, d]$ (we recall that the space $L_\infty[a, b]$ is the conjugate of the space $L_1[a, b]$). We notice that one should take for $B_0(X)$ and $B_0(Y)$ the subalgebras of the algebras $B(X)$ and $B(Y)$ consisting of all operators that have a preconjugate. We define the mappings $j, J$, and $\mathfrak{J}$ by the rules (extended by linearity):

$$(x \otimes y^*)y = x\langle y, y^* \rangle,$$

$$\langle U, x^* \otimes y^{**} \rangle = \langle y^{**}, U^* x^* \rangle,$$

$$(A \otimes B)U = AUB^0,$$

where $B^0$ is the preconjugate of $B$. Note that in this example the subalgebra $B_0(Y^*)$ can be regarded as $B(Y)$, but the action of $B(Y)$ on $U \in B(Y, X)$ should be understood as contravariant, i. e.,

$$(A_1 \otimes B_1) \left( (A_2 \otimes B_2) U \right) = A_1 A_2 U B_2 B_1.$$

Example (e) is the most important example of extended tensor product from the point of view of applications of this paper.

Below, in this section, we assume that we are given an extended tensor product $X \otimes Y$ of Banach spaces $X$ and $Y$, and a pseudo-resolvent $R(\cdot)$ in the algebra $B_0(Y)$.

**Theorem 28** ([97, Theorem 26]) Let us assume that $\infty \not\in \sigma(R(\cdot))$. Let the mapping $\Phi : O(\sigma(R(\cdot)), B_0(X)) \to B(X \otimes Y)$ be defined by the formula
\[ \Phi(F) = \frac{1}{2\pi i} \int_{\Gamma} F(\lambda) d\lambda, \]

where \( \Gamma \) is an oriented envelope of the singular set \( \sigma(R_{(\cdot)}) \) of the pseudo-resolvent with respect to the point \( \infty \) and the complement of the domain of \( F \). Then \( \Phi \) is a continuous morphism of unital algebras.

For all \( A \in \mathcal{B}_0(X) \) and \( h \in \mathcal{O}(\sigma(R_{(\cdot)})) \), the morphism \( \Phi \) maps the function \( F(\lambda) = Ah(\lambda) \) to the operator \( A \boxtimes \varphi(h) \), where \( \varphi \) is defined as in Theorem 25.

We stress that the function \( F \) in (16) takes its values in \( \mathcal{B}_0(X) \), but not in \( \mathbb{C} \).

**Theorem 29** ([97, Theorem 27]) Let us assume that \( \infty \in \bar{\sigma}(R_{(\cdot)}) \). Let the mapping \( \Phi : \mathcal{O}(\bar{\sigma}(R_{(\cdot)}), \mathcal{B}_0(X)) \to \mathcal{B}(X \boxtimes Y) \) be defined by the formula

\[ \Phi(F) = \frac{1}{2\pi i} \int_{\Gamma} F(\lambda) d\lambda + F(\infty) \boxtimes 1, \]

where \( \Gamma \) is an oriented envelope of the extended singular set \( \bar{\sigma}(R_{(\cdot)}) \) of the pseudo-resolvent with respect to the complement of the domain of \( F \). Then \( \Phi \) is a continuous morphism of unital algebras.

For all \( A \in \mathcal{B}_0(X) \) and \( h \in \mathcal{O}(\bar{\sigma}(R_{(\cdot)})) \), the morphism \( \Phi \) maps the function \( F(\lambda) = Ah(\lambda) \) to the operator \( A \boxtimes \varphi(h) \), where \( \varphi \) is defined as in Theorem 26.

**Theorem 30** ([97, Theorem 41]) Let \( F \in \mathcal{O}(\bar{\sigma}(R_{(\cdot)}), \mathcal{B}_0(X)) \). We define the operator \( \Phi(F) \) by formula (16) if \( \infty \notin \bar{\sigma}(R_{(\cdot)}) \); and we define the operator \( \Phi(F) \) by formula (17) if \( \infty \in \bar{\sigma}(R_{(\cdot)}) \). Then the operator \( \Phi(F) : X \boxtimes Y \to X \boxtimes Y \) is not invertible if and only if for some \( \lambda \in \bar{\sigma}(R_{(\cdot)}) \) the operator \( F(\lambda) \in \mathcal{B}_0(X) \) is not invertible.

**Theorem 31** ([97, Theorem 42]) Let \( F \in \mathcal{O}(\bar{\sigma}(R_{(\cdot)}), \mathcal{B}_0(X)) \). We define the operator \( \Phi(F) \) by formula (16) if \( \infty \notin \bar{\sigma}(R_{(\cdot)}) \); and we define the operator \( \Phi(F) \) by formula (17) if \( \infty \in \bar{\sigma}(R_{(\cdot)}) \). Then the spectrum of the operator \( \Phi(F) : X \boxtimes Y \to X \boxtimes Y \) is given by the formula

\[ \sigma(\Phi(F)) = \bigcup_{\lambda \in \bar{\sigma}(R_{(\cdot)})} \sigma(F(\lambda)). \]

### 6 Functional calculus \( \varphi_1 \boxtimes \varphi_2 \)

In this section, we define and discuss the product \( \varphi_1 \boxtimes \varphi_2 \) of functional calculi \( \varphi_1 \) and \( \varphi_2 \) that were defined in Sect. 4; it acts in the extended tensor product \( X \boxtimes Y \). Keeping in mind the space \( \mathcal{B}(Y, X) \) (see Example 3(e)) as the main example of an extended tensor product, we call operators acting in \( X \boxtimes Y \) transformators.

Below in this section, we assume that we are given an extended tensor product \( X \boxtimes Y \) of Banach spaces \( X \) and \( Y \), and we are given pseudo-resolvents \( R_{1,\cdot} \) and \( R_{2,\cdot} \) in the algebras \( \mathcal{B}_0(X) \) and \( \mathcal{B}_0(Y) \), respectively.
Theorem 32  Let $\infty \not\in \bar{\sigma}(R_{1,(-)})$ and $\infty \not\in \bar{\sigma}(R_{2,(-)})$. Let the mapping $\phi_1 \boxtimes \phi_2 : O(\sigma(R_{1,(-)}) \times \sigma(R_{2,(-)})) \to B(X \boxtimes Y)$ be defined by the formula
\[
(\phi_1 \boxtimes \phi_2)f = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu)R_{1,\lambda} \boxtimes R_{2,\mu} \, d\mu \, d\lambda,
\]
where $\Gamma_1$ and $\Gamma_2$ are oriented envelopes of the sets $\sigma(R_{1,(-)})$ and $\sigma(R_{2,(-)})$ with respect to the point $\infty$ and the complements $\overline{C \setminus U_1}$ and $\overline{C \setminus U_2}$, respectively; here $U_1 \times U_2$ is an open neighborhood of $\sigma(R_{1,(-)}) \times \sigma(R_{2,(-)})$ that is contained in the domain of the function $f$ (see Proposition 9). Then $\phi_1 \boxtimes \phi_2$ is a continuous morphism of unital algebras.

Proof  The proof is analogous to that of Theorem 34, see below. $\square$

Theorem 33  Let $\infty \not\in \bar{\sigma}(R_{1,(-)})$, but $\infty \in \bar{\sigma}(R_{2,(-)})$. Let the mapping $\phi_1 \boxtimes \phi_2 : O(\sigma(R_{1,(-)}) \times \bar{\sigma}(R_{2,(-)})) \to B(X \boxtimes Y)$ be defined by the formula
\[
(\phi_1 \boxtimes \phi_2)f = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu)R_{1,\lambda} \boxtimes R_{2,\mu} \, d\mu \, d\lambda,
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty)R_{1,\lambda} \boxtimes 1_Y \, d\lambda,
\]
where $\Gamma_1$ is an oriented envelope of the set $\sigma(R_{1,(-)})$ with respect to the point $\infty$ and the complement $\overline{C \setminus U_1}$, and $\Gamma_2$ is an oriented envelope of the extended singular set $\bar{\sigma}(R_{2,(-)})$ with respect to the complement $\overline{C \setminus U_2}$; here $U_1 \times U_2$ is an open neighborhood of the set $\sigma(R_{1,(-)}) \times \bar{\sigma}(R_{2,(-)})$ that is contained in the domain of the function $f$ (see Proposition 9). Then $\phi_1 \boxtimes \phi_2$ is a continuous morphism of unital algebras.

Proof  The proof is analogous to that of Theorem 34, see below. $\square$

Theorem 34  Let $\infty \in \bar{\sigma}(R_{1,(-)})$ and $\infty \in \bar{\sigma}(R_{2,(-)})$. Let the mapping $\phi_1 \boxtimes \phi_2 : O(\bar{\sigma}(R_{1,(-)}) \times \bar{\sigma}(R_{2,(-)})) \to B(X \boxtimes Y)$ be defined by the formula

\[\]
\[(\varphi_1 \boxtimes \varphi_2)f = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu)R_{1,\lambda} \otimes R_{2,\mu} \, d\lambda \, d\mu \]
\[+ \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty)R_{1,\lambda} \mathbf{1}_Y \, d\lambda \]
\[+ \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \mu)\mathbf{1}_X \otimes R_{2,\mu} \, d\mu + f(\infty, \infty)\mathbf{1}_{X \otimes Y}, \]

where \(\Gamma_1\) and \(\Gamma_2\) are oriented envelopes of the sets \(\bar{\sigma}(R_{1,\cdot})\) and \(\bar{\sigma}(R_{2,\cdot})\) with respect to the complements \(\mathbb{C} \setminus U_1\) and \(\mathbb{C} \setminus U_2\), respectively; here \(U_1 \times U_2\) is an open neighborhood of the set \(\bar{\sigma}(R_{1,\cdot}) \times \bar{\sigma}(R_{2,\cdot})\) that is contained in the domain of the function \(f\) (see Proposition 9). Then \(\varphi_1 \boxtimes \varphi_2\) is a continuous morphism of unital algebras.

For all \(g \in \mathcal{O}(\bar{\sigma}(R_{1,\cdot}))\) and \(h \in \mathcal{O}(\bar{\sigma}(R_{2,\cdot}))\) the morphism \(\varphi_1 \boxtimes \varphi_2\) maps the function \(f(\lambda, \mu) = g(\lambda)h(\mu)\) to the transformator \(\varphi_1(g) \boxtimes \varphi_2(h)\), where \(\varphi_1\) and \(\varphi_2\) are scalar functional calculi (Theorem 26) generated by the pseudo-resolvents \(R_{1,\cdot}\) and \(R_{2,\cdot}\).

**Proof** For each \(\mu \in U_2\), we consider the operator
\[G(\mu) = \varphi_1(f(\cdot, \mu)) = \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu)R_{1,\lambda} \, d\lambda + f(\infty, \mu)\mathbf{1}_X. \tag{18}\]

By Theorem 26, for any fixed \(\mu \in U_2\), the correspondence \(f \mapsto G(\mu)\) preserves the three operations: addition, scalar multiplication, and multiplication. We change the interpretation: formula (18) defines a mapping \(f \mapsto G\) from \(\mathcal{O}(\bar{\sigma}(R_{1,\cdot}) \times \bar{\sigma}(R_{2,\cdot}))\) to \(\mathcal{O}(\bar{\sigma}(R_{2,\cdot}), \mathcal{B}_0(X))\). Since the three operations in \(\mathcal{O}(\bar{\sigma}(R_{2,\cdot}), \mathcal{B}_0(X))\) are understood in the pointwise sense, it follows that the correspondence \(f \mapsto G\) is a morphism of algebras.

In accordance with Theorem 29 we put
\[ \Phi_1(G) = \frac{1}{2\pi i} \int_{\Gamma_2} G(\mu) \mathcal{R}_{2,\mu} \, d\mu + G(\infty) \mathcal{1}_Y \]

\[ = \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu) R_{1,\lambda} \, d\lambda + f(\infty, \mu) \mathcal{1}_X \right) \mathcal{R}_{2,\mu} \, d\mu \]

\[ + \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \, d\lambda + f(\infty, \infty) \mathcal{1}_X \mathcal{1}_Y \]

\[ = \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu) R_{1,\lambda} \, d\lambda \right) \mathcal{R}_{2,\mu} \, d\mu \]

\[ + \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \, d\lambda + f(\infty, \infty) \mathcal{1}_X \mathcal{1}_Y \]

By Theorem 29, the correspondence \( G \mapsto \Phi_1(G) \) also preserves the three operations. Clearly, the mapping \( \varphi_1 \mathcal{R}_2 \) from the formulation of the theorem is the composition of the correspondences \( f \mapsto G \) and \( G \mapsto \Phi_1(G) \), and, by what has been proved, is a morphism of algebras.

The continuity is evident.

The second assertion is verified by direct calculations.

When it is desirable to stress that the functional calculus \( \varphi_1 \mathcal{R}_2 \) in Theorems 32, 33, and 34 is generated by pseudo-resolvents \( R_{1,()} \) and \( R_{2,()} \), we will use the notation \( \varphi_{R_{1,()}} \mathcal{R}_{R_{2,()}} \) instead of \( \varphi_1 \mathcal{R}_2 \), and we will use the notation \( f(R_{1,()}) \mathcal{R}_{R_{2,()}} \) instead of \( (\varphi_1 \mathcal{R}_2)(f) \). If the pseudo-resolvents \( R_{1,()} \) and \( R_{2,()} \) are generated by operators \( A \) and \( B \) (see Proposition 24), we will use the notations \( R_{A,()} \) and \( R_{B,()} \), \( \varphi_A \mathcal{R}_B \), and \( f(A, B) \).

To present the definitions of \( \varphi_1 \mathcal{R}_2 \) from Theorems 32, 33, and 34 in a unified form, it is convenient to use the notation.
\[
(\varphi_1 \boxplus \varphi_2)(f) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxplus R_{2,\mu} \, d\mu \, d\lambda
+ \delta_i \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \boxplus 1_Y \, d\lambda
+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \mu) 1_X \boxplus R_{2,\mu} \, d\mu
+ \delta_1 \delta_2 f(\infty, \infty) 1_X \boxplus 1_Y,
\]

where \( \delta_i = 1 \) if \( \Gamma_i \) encloses \( \infty \), and \( \delta_i = 0 \) in the opposite case, \( i = 1, 2 \).

We list the results of the action of \( \varphi_1 \boxplus \varphi_2 \) on some frequently encountered functions.

**Corollary 35** Under the assumptions of Theorems 32, 33, and 34, the morphism \( \varphi_1 \boxplus \varphi_2 \) maps the function \( u(\lambda, \mu) = 1 \) to the unit \( 1_X \boxplus 1_Y \) of the algebra \( B(X \boxplus Y) \); the function \( r_{1,\lambda_0}(\lambda, \mu) = \frac{1}{\lambda_0 - \lambda} \), where \( \lambda_0 \in \rho(R_{1,\cdot}) \), is mapped by the morphism \( \varphi_1 \boxplus \varphi_2 \) to the transformator \( R_{1,\lambda_0} \boxplus 1_Y \); the function \( r_{2,\mu_0}(\lambda, \mu) = \frac{1}{\mu_0 - \mu} \), where \( \mu_0 \in \rho(R_{2,\cdot}) \), is mapped by the morphism \( \varphi_1 \boxplus \varphi_2 \) to the transformator \( 1_X \boxplus R_{2,\mu_0} \); the function \( r_{\lambda_0,\mu_0}(\lambda, \mu) = \frac{1}{(\lambda_0 - \lambda)(\mu_0 - \mu)} \), where \( \lambda_0 \in \rho(R_{1,\cdot}) \) and \( \mu_0 \in \rho(R_{2,\cdot}) \), is mapped by the morphism \( \varphi_1 \boxplus \varphi_2 \) to the transformator \( R_{1,\lambda_0} \boxplus R_{2,\mu_0} \).

Under the assumptions of Theorems 32 and 33, the morphism \( \varphi_1 \boxplus \varphi_2 \) maps the function \( c_1(\lambda, \mu) = \lambda \) to the transformator \( A \boxplus 1_Y \), where \( A \) is the operator that generates the maximal pseudo-resolvent \( R_{1,\cdot} \) in accordance with Proposition 24.

Under the assumptions of Theorem 32, the morphism \( \varphi_1 \boxplus \varphi_2 \) maps the function \( c_2(\lambda, \mu) = \mu \) to the transformator \( 1_X \boxplus B \), where \( B \) is the operator that generates the maximal pseudo-resolvent \( R_{2,\cdot} \) in accordance with Proposition 24; if \( v_0 \not\in \sigma(A) \pm \sigma(B) \), then the function \( r_{v_0}(\lambda, \mu) = \frac{1}{v_0 - \lambda + \mu} \) is mapped by the morphism \( \varphi_1 \boxplus \varphi_2 \) to the transformator \( (v_0 1_X \boxplus 1_Y - A \boxplus 1_Y = 1_X \boxplus B)^{-1} \).

**Proof** We restrict ourselves to proving the last assertion. By Theorem 32, we have that the function \( (\lambda, \mu) \mapsto v_0 - \lambda + \mu \) is mapped by the morphism \( \varphi_1 \boxplus \varphi_2 \) to the transformator \( v_0 1_X \boxplus 1_Y - A \boxplus 1_Y = 1_X \boxplus B \). Since \( \varphi_1 \boxplus \varphi_2 \) is a morphism of algebras, the reciprocal function is mapped to the inverse transformator.

**Theorem 36** ([93, Theorem 4.4]) Let \( g \in O\left(\overline{\sigma(R_{1,\cdot})} \times \overline{\sigma(R_{2,\cdot})}\right) \) and \( f \in O\left(g(\overline{\sigma(R_{1,\cdot})} \times \overline{\sigma(R_{2,\cdot})})\right) \). Then the transformator \( (\varphi_1 \boxplus \varphi_2)(f \circ g) \) is the function \( f \) of the transformator \( (\varphi_1 \boxplus \varphi_2)(g) \):

\[
(\varphi_1 \boxplus \varphi_2)(f \circ g) = \frac{1}{2\pi i} \int_{\Gamma_3} f(v) \left(1_X \boxplus 1_Y - (\varphi_1 \boxplus \varphi_2)(g)\right)^{-1} \, dv,
\]

where \( \Gamma_3 \) is an oriented envelope of the spectrum \( \sigma((\varphi_1 \boxplus \varphi_2)(g)) \) with respect to the complement of the domain of \( f \).

**Proof** We have \( (\delta_1, \delta_2 = 0, 1) \)
\((\varphi_1 \boxtimes \varphi_2)(f \circ g) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(g(\lambda, \mu)) R_{1, \lambda} R_{2, \mu} \, d\mu \, d\lambda \)

\(= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{f(v)}{v - g(\lambda, \mu)} \, dv \left[ R_{1, \lambda} R_{2, \mu} \right] \, d\lambda \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{R_{1, \lambda} R_{2, \mu}}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

(we represent the compositions by means of Cauchy’s Integral Formula)

\((\text{here we interchange the orders of integration})\)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{R_{1, \lambda} R_{2, \mu}}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

\(= \frac{1}{2\pi i} \int_{\Gamma_1} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - g(\lambda, \mu)} \, d\mu \, d\lambda \right] \, dv \)

(further, by Theorems 32, 33, 34, it follows that)
\[= \frac{1}{2\pi i} \int_{\Gamma_3} f(v) \left( (v1_X \otimes 1_Y - (\varphi_1 \otimes \varphi_2)(g))^{-1} \right. \]
\[\left. - \delta_1 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_{1,\lambda}}{v - g(\lambda, \infty)} \, d\lambda \right. \]
\[\left. - \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{v - g(\infty, \mu)} \, d\mu \right. \]
\[\left. + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_3} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_{1,\lambda}}{v - g(\lambda, \infty)} \, d\lambda \right] \right] \, dv \]
\[+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_3} f(v) \left[ \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{v - g(\infty, \mu)} \, d\mu \right] \, dv \]
\[+ \delta_1 \delta_2 \left( g(\infty, \infty) \right) 1_{X \otimes Y} \]
\[= \frac{1}{2\pi i} \int_{\Gamma_3} f(v) (v1_X \otimes 1_Y - (\varphi_1 \otimes \varphi_2)(g))^{-1} \, dv. \]
\]

**Corollary 37** Let \( A \in \mathbf{B}(X) \) and \( B \in \mathbf{B}(Y) \). Let \( f \in O(\sigma(A) \pm \sigma(B)) \). Then
\[
\frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda \pm \mu) R_{A, \lambda \otimes B, \mu} \, d\mu \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_3} f(v) (v1_X \otimes 1_Y - A \otimes B)^{-1} \, dv,
\]
where \( \Gamma_3 \) is an oriented envelope of \( \sigma(A) \pm \sigma(B) \) with respect to the complement of the domain of \( f \).

**Proof** This is a special case of Theorem 36 for \( g(\lambda, \mu) = \lambda \pm \mu \). \( \square \)

**Example 4** (a) Let \( A \in \mathbf{B}(X) \), \( B \in \mathbf{B}(Y) \), and \( C \in \mathbf{B}(Y, X) \). We take for \( X \otimes Y \) the space \( \mathbf{B}(Y, X) \). Let \( f_1(\lambda, \mu) = \lambda + \mu \), \( f_2(\lambda, \mu) = \lambda - \mu \), \( f_3(\lambda, \mu) = \lambda^2 \mu^m \). Then, by Theorem 32 and Corollaries 35 and 37, one has
\[
\left( (\varphi_A \otimes \varphi_B) f_1 \right) C = AC + CB,
\]
\[
\left( (\varphi_A \otimes \varphi_B) f_2 \right) C = AC - CB,
\]
\[
\left( (\varphi_A \otimes \varphi_B) f_3 \right) C = A^2 C - CB^2,
\]
\[
\left( (\varphi_A \otimes \varphi_B) f_2 \right) \left( (\varphi_A \otimes \varphi_B) f_3 \right) C = A^2 C - CB^2,
\]
\[
\left( (\varphi_A \otimes \varphi_B) f_3 \right) C = A^2 CB^m.
\]

(b) Let \( A \in \mathbf{B}(X) \) and \( B \in \mathbf{B}(Y) \). By Corollary 37 and the formula \( e^{\lambda t} e^{\mu t} = e^{(\lambda + \mu)t} \), one has (cf. [11, 56, 68, Theorem 10.9])
\[
e^{At \otimes e^{Bt}} = e^{(A \otimes 1_Y + 1_X \otimes B)t}.
\]

Similarly, from
\[
\begin{align*}
\cos(\lambda + \mu)t &= \cos \lambda t \cos \mu t - \sin \lambda t \sin \mu t, \\
\sin(\lambda + \mu)t &= \cos \lambda t \sin \mu t + \cos \mu t \sin \lambda t
\end{align*}
\]

one has

\[
\begin{align*}
\cos((A \otimes 1_Y + 1_X \otimes B)t) &= \cos At \cos Bt - \sin At \sin Bt, \\
\sin((A \otimes 1_Y + 1_X \otimes B)t) &= \cos At \sin Bt + \cos Bt \sin At.
\end{align*}
\]

We proceed to the discussion of spectral mapping theorems.

**Theorem 38** Let \( f \in \mathcal{O}(\bar{\sigma}(R_{1,()}) \times \bar{\sigma}(R_{2,})) \). Then the transformator 
\((\varphi_1 \otimes \varphi_2)(f) : X \otimes Y \rightarrow X \otimes Y\) is not invertible if and only if \( f(\lambda, \mu) = 0 \) for at least one pair of points \( \lambda \in \bar{\sigma}(R_{1,}) \) and \( \mu \in \bar{\sigma}(R_{2,}) \).

**Proof** For each \( \mu \in U_2 \), we consider operator (18). By Theorem 27, the operator 
\( G(\mu) : X \rightarrow X \) is not invertible if and only if \( f(\lambda, \mu) = 0 \) for at least one 
\( \lambda \in \bar{\sigma}(R_{1,}) \). Furthermore, by Theorem 30, operator (19) is not invertible if and 
only if \( G(\mu) \) is not invertible for at least one \( \mu \in \bar{\sigma}(R_{2,}) \). Combining (in the 
opposite order) these results, we arrive at the desired statement. \( \square \)

**Theorem 39** Let \( f \in \mathcal{O}(\bar{\sigma}(R_{1,}) \times \bar{\sigma}(R_{2,})) \). Then the spectrum of the transformator 
\((\varphi_1 \otimes \varphi_2)f : X \otimes Y \rightarrow X \otimes Y\) is given by the formula 
\[
\bar{\sigma}((\varphi_1 \otimes \varphi_2)f) = \{ f(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,}), \mu \in \bar{\sigma}(R_{2,}) \}.
\]

**Proof** We take an arbitrary \( v \in \mathbb{C} \). By the definition of the spectrum, the number \( v \) 
belongs to the set \( \bar{\sigma}((\varphi_1 \otimes \varphi_2)f) \) if and only if the transformator 
\( v1_{X \otimes Y} - (\varphi_1 \otimes \varphi_2)(f) \) is not invertible.

We denote by \( u \) the function belonging to \( \mathcal{O}(\bar{\sigma}(R_{1,}) \times \bar{\sigma}(R_{2,})) \) that equals 1 
identically. By Theorems 32, 33, and 34, we have 
\[
(\varphi_1 \otimes \varphi_2)(u) = 1_{X \otimes Y},
\]
whence, 
\[
v1_{X \otimes Y} - (\varphi_1 \otimes \varphi_2)(f) = (\varphi_1 \otimes \varphi_2)(vu - f).
\]
We apply Theorem 38: the transformator 
\((\varphi_1 \otimes \varphi_2)(vu - f) \) is not invertible if and only if 
\( vu(\lambda, \mu) - f(\lambda, \mu) = 0 \) for some \( \lambda \in \bar{\sigma}(R_{1,}) \) and \( \mu \in \bar{\sigma}(R_{2,}) \) or, in other 
words, \( v \in \{ f(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,}), \mu \in \bar{\sigma}(R_{2,}) \} \). \( \square \)

We denote by \( B_{R_1, R_2} \) the closure in the algebra \( B(X \otimes Y) \) of the set of all 
transformators \((\varphi_1 \otimes \varphi_2)f \), where \( f \in \mathcal{O}(\bar{\sigma}(R_{1,}) \times \bar{\sigma}(R_{2,})) \). We stress that by 
Corollary 35 the set \( B_{R_1, R_2} \) always contains the unit \( 1_{X \otimes Y} \) of the algebra \( B(X \otimes Y) \).
Corollary 40  The set $B_{R_1, R_2}$ is a full commutative subalgebra of the algebra $B(X \overline{\times} Y)$ of all transformators acting in $X \overline{\times} Y$.

Proof  Clearly, the image under $\phi_1 \Box \phi_2$ of the unital commutative algebra (see Proposition 11) $O(\sigma(R_1, \cdot)) \times \sigma(R_2, \cdot)$ is a unital commutative subalgebra.

Let the transformator $(\phi_1 \Box \phi_2)f$ be invertible in the algebra $B(X \overline{\times} Y)$. By Theorem 38, this means that $f(\lambda, \mu) \neq 0$ for all $\lambda \in \sigma(R_1, \cdot)$ and $\mu \in \sigma(R_2, \cdot)$. By Theorem 32, 33, and 34, the inverse of $(\phi_1 \Box \phi_2)f$ is the transformator $(\phi_1 \Box \phi_2)\left(\frac{1}{f}\right)$.

It remains to apply Proposition 4.

Functional calculus (an analogue of Theorem 32) was first constructed in the tensor product of Banach spaces for bounded operators and polynomial functions in [125]. In [72, Theorem 2.4], an analogue of Theorem 32 was proved in the tensor product of Banach spaces for bounded operators and arbitrary analytic functions; in [72, Theorem 3.1], an analogue of Theorem 34 was also proved for unbounded operators and analytic functions. See also the initial version [71] of article [72].

There are several versions of the spectral mapping theorem (Theorem 39) in the tensor products of Banach spaces. In [19], it was shown that the spectrum of the tensor product $A \otimes B$ of two bounded operators acting in a Hilbert space is the set $\sigma(A) \times \sigma(B)$. In [105, Theorem 10], it was established a spectral mapping theorem for the transformators $x \mapsto \sum_{j=1}^{N} f_j(u)xg_j(v)$ of elements $x$ of a unital Banach algebra ($u$ and $v$ are fixed elements of the same Banach algebra). This result was extended to arbitrary polynomial functions $f$ of two variables in [63, Theorem 3.3], see also [62]; another equivalent version was proved in [40, Theorem 3.4]. The results of [19, 40, 62, 63, 105] cover Theorem 39 only in the case of pseudo-resolvents generated by bounded operators.

In [72, Theorem 3.2], it was proved an analogue of Theorem 39 for unbounded operators, see also [71]. In [72] operators act in the tensor product of Banach spaces.

For the case of arbitrary extended tensor products (in particular, in the case of transformators acting in $B(X, Y)$) and arbitrary pseudo-resolvents (namely, the resolvents of linear relations), Theorems 34 and 39 seem to be new.

In [93, Lemma 4.1, Lemma 4.2, Theorem 4.4], one can found a modern exposition of Theorems 32, 36, and 39 for the case of matrices and their applications.

Functions of the transformator $A \otimes 1 \pm 1 \otimes B$ (see Corollary 37) were investigated in [11, 12, 47].

7 Meromorphic functional calculus

A meromorphic function of a bounded operator is an unbounded operator or a linear relation (provided a pole of the function is contained in the spectrum). According to our approach, we identify such an object with its resolvent.

Let $U$ be an open subset of $\mathbb{C}^2$ and $f : U \to \mathbb{C}$. The function $f$ is called [127, ch. IV, § 15.43] meromorphic if: (i) $f$ is analytic on a set $U \setminus M$, where $M$ is a nowhere dense closed subset of $U$, (ii) $f$ cannot be analytically continued to any
point of \( M \), (iii) for any point \( \zeta \in M \), there exist a connected open neighborhood \( V \) of \( \zeta \) and an analytic function \( q_\zeta : V \to \mathbb{C} \) such that the function \( p_\zeta = f \cdot q_\zeta \) is analytic in \( V \cap (\mathbb{C} \setminus M) \) and can be extended analytically into \( V \), and \( q_\zeta \) equals zero only on \( V \cap M \). Clearly, \( q_\zeta(\zeta) = 0 \). The set \( M \) is called the polar set of the function \( f \). It consists of points of two types: if \( q_\zeta \) can be chosen so that \( p_\zeta(\zeta) \neq 0 \) (and therefore, \( \lim_{z \to \zeta} f(z) = \infty \)), then \( \zeta \) is called a pole; if for any choice of \( q_\zeta \) one has \( p_\zeta(\zeta) = 0 \), then \( \zeta \) is called a point of indeterminacy. In an arbitrary neighborhood of a point of indeterminacy, the function \( f \) takes any value from \( \mathbb{C} \) [127].

For example, for the function \( f(\lambda, \mu) = \lambda \mu \), the points of indeterminacy are \((0, \infty)\) and \((\infty, 0)\), for the function \( f(\lambda, \mu) = \frac{\lambda}{\mu} \), the points of indeterminacy are \((0, 0)\) and \((\infty, \infty)\), and for the function \( f(\lambda, \mu) = \lambda - \mu \), the point of indeterminacy is \((\infty, \infty)\).

Assume that we are given an extended tensor product \( X \otimes Y \) of Banach spaces \( X \) and \( Y \), and we are given pseudo-resolvents \( R_{1,(\cdot)} \) and \( R_{2,(\cdot)} \) in the algebras \( B_0(X) \) and \( B_0(Y) \), respectively.

We consider a function \( f \) that is meromorphic in an open neighborhood \( U \subseteq \mathbb{C}^2 \) of the set \( \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \) and has no points of indeterminacy in \( U \). We consider the subset

\[
\{ f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \} = \{ f(\lambda, \mu) : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \}
\]

of the set \( \mathbb{C}^2 \). The set \( f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \) is compact, being the image under the continuous function \( f \) of the compact set \( \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \).

**Lemma 41** For any \( v \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \), the set

\[
(\mathbb{C}^2 \setminus U) \cup \{ (\lambda, \mu) \in U : f(\lambda, \mu) = v \}
\]

is closed in \( \mathbb{C}^2 \) and does not intersect \( \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \). Moreover, for any closed set \( W \subseteq \mathbb{C} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \), the set

\[
(\mathbb{C}^2 \setminus U) \cup \{ (\lambda, \mu) \in U : f(\lambda, \mu) \in W \}
\]

is closed in \( \mathbb{C}^2 \) and does not intersect \( \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \).

**Proof** The set \( \mathbb{C}^2 \setminus U \) is closed, being a complement of an open one. The set \( \{ (\lambda, \mu) \in U : f(\lambda, \mu) \in W \} = f^{-1}(W) \) is closed in \( U \), being the inverse image of the closed set \( W \) under the continuous function \( f \). This implies that limit points of the set \( f^{-1}(W) \) either belong to \( f^{-1}(W) \) or to the complement \( \mathbb{C}^2 \setminus U \). Thus, set \((22)\) is closed.

We show that set \((22)\) is disjoint from \( \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \). Actually, if \( f(\lambda, \mu) = v \in W \) for some \( (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \), then \( v \in f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \), which contradicts the assumption. If \( (\lambda, \mu) \notin U \), then \( (\lambda, \mu) \notin \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \) by the definition of \( U \). \( \square \)
For all \( v \in \mathbb{C} \setminus \mathcal{f}(\sigma(R_{1,1}), \sigma(R_{2,1})) \), we set
\[
S_v = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{v - f(\lambda, \mu)} R_{1,\lambda} \otimes R_{2,\mu} \, d\mu \, d\lambda
\]
\[
+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{v - f(\lambda, \infty)} R_{1,\lambda} \otimes 1 \, d\lambda
\]
\[
+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{v - f(\infty, \mu)} 1 \otimes R_{2,\mu} \, d\mu + \frac{\delta_1 \delta_2}{v - f(\infty, \infty)} 1 \otimes 1,
\]
where \( \Gamma_i \) is an oriented envelope of the spectrum \( \sigma(R_{i,1}) \); \( \delta_i = 1 \) if \( \Gamma_i \) encloses \( \infty \), and \( \delta_i = 0 \) otherwise; \( i = 1, 2 \). By Lemma 41, the function \( h_v(\lambda, \mu) = \frac{1}{v - f(\lambda, \mu)} \) belongs to \( \mathcal{O}(\sigma(R_{1,1}) \times \sigma(R_{2,1})) \). Therefore, \( S_v \) can be regarded as the image under the morphism \( \varphi_1 \otimes \varphi_2 \) of the function \( h_v \):
\[
S_v = (\varphi_1 \otimes \varphi_2) h_v.
\]

We call \( S_v \) the resolvent of the function \( f \) of \( R_{1,\lambda} \) and \( R_{2,\lambda} \). When we need to stress the dependence of \( S_v \) from \( R_{1,1} \) and \( R_{2,1} \), we denote transformator (23) by \( S_v(R_{1,1}, R_{2,1}) \).

**Theorem 42** Let a function \( f \) be meromorphic in an open neighborhood \( U \subseteq \mathbb{C}^2 \) of the set \( \sigma(R_{1,1}) \times \sigma(R_{2,1}) \) and have no points of indeterminacy in \( U \). Then the family
\[
S_v, \quad v \notin \mathcal{f}(\sigma(R_{1,1}), \sigma(R_{2,1})),
\]
defined by formula (23), is a maximal pseudo-resolvent. In particular,
\[
\sigma(S_{1,1}) = f(\sigma(R_{1,1}), \sigma(R_{2,1})).
\]

Equality (24) can be regarded as an analogue of the spectral mapping theorem.

**Proof** We show that, on the set \( \mathbb{C} \setminus \mathcal{f}(\sigma(R_{1,1}), \sigma(R_{2,1})) \), the Hilbert identity holds:
\[
S_{v_1} - S_{v_2} = -(v_1 - v_2) S_{v_1} S_{v_2}, \quad v_1, v_2 \notin \mathcal{f}(\sigma(R_{1,1}), \sigma(R_{2,1})).
\]

We note that
\[
\frac{1}{v_1 - f(\lambda, \mu)} - \frac{1}{v_2 - f(\lambda, \mu)} = -\frac{v_1 - v_2}{(v_1 - f(\lambda, \mu))(v_2 - f(\lambda, \mu))}.
\]

Applying the morphism \( \varphi_1 \otimes \varphi_2 \) to this identity we arrive at the Hilbert identity (25).

We verify that the pseudo-resolvent \( S_{1,1} \) is maximal. The validity of the Hilbert identity implies that
\[ \sigma(S_v) \subseteq f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})). \]

To prove the reverse inclusion, we fix a point \( v \in \mathbb{C} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \). By Theorem 16, the pseudo-resolvent \( S_{\cdot} \) can be extended to points \( \eta \in \mathbb{C} \) in which the transformator \( 1 + (\eta - v)S_v \) is invertible. By Theorems 32, 33, 34, and 39, and formula (23) we have

\[ \sigma(S_v) = \left\{ \frac{1}{v - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}. \]

It follows that

\[
\sigma(1 + (\eta - v)S_v) = \left\{ 1 + \frac{\eta - v}{v - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\} \\
= \left\{ \frac{\eta - f(\lambda, \mu)}{v - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}.
\]

From this formula it is seen that the transformator \( 1 + (\eta - v)S_v \) is invertible if and only if

\[ 0 \not\in \left\{ \frac{\eta - f(\lambda, \mu)}{v - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}, \]

which is equivalent to

\[ \eta \not\in \left\{ f(\lambda, \mu) : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}. \]

Thus, from \( \eta \not\in f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \) it follows that \( \eta \not\in \sigma(S_{\cdot}) \).

It remains to show that \( \infty \) belongs to both sides of (24) simultaneously.

First, we assume that \( \infty \not\in f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \). Then, since the set \( f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \) is closed, one can take a closed neighborhood \( W \subseteq \mathbb{C} \) of infinity that is disjoint from \( f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) \). By definition, \( S_{\cdot} \) is defined for all \( v \in W \setminus \{\infty\} \); besides, by Lemma 41, one can assume that the contours \( \Gamma_1 \) and \( \Gamma_2 \) in (23) do not depend on \( v \in W \setminus \{\infty\} \). According to the definition of the extended singular set, we calculate the limit (see Theorems 32, 33, and 34):

\[
\lim_{v \to \infty} vS_v = \lim_{v \to \infty} \left( \frac{1}{2\pi i} \int_{\Gamma_1} \frac{v}{v - f(\lambda, \mu)} R_{1,\lambda} \otimes R_{2,\mu} \, \mathrm{d}\mu \, \mathrm{d}\lambda \right) \\
+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{v}{v - f(\lambda, \infty)} R_{1,\lambda} \otimes 1 \, \mathrm{d}\lambda \\
+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{v}{v - f(\infty, \mu)} 1 \otimes R_{2,\mu} \, \mathrm{d}\mu + \frac{\delta_1 \delta_2 v}{v - f(\infty, \infty)} \, 1 \otimes 1 \\
= \lim_{v \to \infty} (\phi_1 \otimes \phi_2) \left( \frac{v}{v - f(\cdot, \cdot)} \right) = \lim_{v \to \infty} (\phi_1 \otimes \phi_2) u = 1 \otimes 1,
\]

because the functions \( \frac{v}{v - f(\cdot, \cdot)} \) converge to \( u \) as \( v \to \infty \) uniformly on \( \Gamma_1 \times \Gamma_2 \); where \( u(\lambda, \mu) = 1 \). Consequently, we arrive at \( \infty \not\in \bar{\sigma}(S_{\cdot}) \).
Conversely, assume that $\infty \not\in \bar{\sigma}(S_i)$. By the definition of the extended singular set, this means that $S_i$ is defined in a deleted neighborhood $W$ of infinity and

$$
\lim_{v \to \infty} vS_v = \lim_{v \to \infty} \left( \frac{1}{2\pi i} \int_{\Gamma_1} \int_{\Gamma_2} \frac{v}{v-f(\lambda, \mu)} R_{1,\lambda} R_{2,\mu} \, d\mu \, d\lambda + \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{v}{v-f(\lambda, \infty)} R_{1,\lambda} 1 \, d\lambda + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{v}{v-f(\infty, \mu)} 1 R_{2,\mu} \, d\mu + \frac{\delta_1 \delta_2 v}{v-f(\infty, \infty)} 1 1 \right)
$$

(26)

By the definition of $S_i$, we have that $W \cap f(\bar{\sigma}(R_{1,\cdot}), \bar{\sigma}(R_{2,\cdot})) = \emptyset$. We show that $\infty \not\in f(\bar{\sigma}(R_{1,\cdot}), \bar{\sigma}(R_{2,\cdot}))$.

We assume the contrary: let $f(\lambda_*, \mu_*) = \infty$ for a point $(\lambda_*, \mu_*) \in \bar{\sigma}(R_{1,\cdot}) \times \bar{\sigma}(R_{2,\cdot})$. Then, by Theorem 39, $0 \in \bar{\sigma}(S_v)$ for any $v \in W$. We also have $0 \in \bar{\sigma}(vS_v)$ for $v \in W$. By Corollaries 40 and 8, it follows that

$$0 \in \bar{\sigma}(\lim_{v \to \infty} vS_v),$$

which contradicts (26). □

Corollary 43 below gives an affirmative answer to the question posed in [117, 118] about the independence of the definition of $f(A, B)$ for unbounded operators $A$ and $B$ from the choice of sequences of bounded operators $A_n$ and $B_n$ whose resolvents converge to the resolvents of $A$ and $B$, respectively.

**Corollary 43** Let the sequences of pseudo-resolvents $R_{n,1,\cdot}$ and $R'_{n,1,\cdot}$ converge (see the definition on p. 11) to the same pseudo-resolvent $R_{1,\cdot}$, and let the sequences of pseudo-resolvents $R_{n,2,\cdot}$ and $R'_{n,2,\cdot}$ converge to the same pseudo-resolvent $R_{2,\cdot}$. Then both the sequence $S_v(R_{n,1,\cdot}, R_{n,2,\cdot})$ and the sequence $S_v(R'_{n,1,\cdot}, R'_{n,2,\cdot})$ converge to $S_v(R_{1,\cdot}, R_{2,\cdot})$.

**Proof** We make use of definition (23). By Lemma 22, $R_{n,1,\cdot}$ and $R'_{n,1,\cdot}$ converge to $R_{1,\cdot}$ uniformly on $\Gamma_1$, and $R_{n,2,\cdot}$ and $R'_{n,2,\cdot}$ converge to $R_{2,\cdot}$ uniformly on $\Gamma_2$. From formula (23) it is seen that $S_v(R_{n,1,\cdot}, R_{n,2,\cdot})$ and $S_v(R'_{n,1,\cdot}, R'_{n,2,\cdot})$ converge to $S_v(R_{1,\cdot}, R_{2,\cdot})$. □

The theory of meromorphic functions of one operator originates from the polynomial functional calculus for unbounded operators constructed in [136], see also an exposition in [39, ch. VII, § 9]. Meromorphic functional calculus for one operator was constructed in [60]. A spectral mapping theorem for a polynomial of a linear relation was proved in [22, Theorem VI.5.4].

Polynomial functions of two unbounded operators acting in the tensor product of Banach spaces were defined in [72, Theorem 3.4]; in particular, a spectral mapping
theorem was established, see [72, Theorem 3.13]. Other analogues of the spectral mapping theorem for analytic functions (including polynomials) of unbounded operators acting in the tensor product of Banach spaces were obtained in [117, Theorem 1] and [118, Theorem 4].

Meromorphic functions of two pseudo-resolvents acting in an arbitrary extended tensor product were not considered earlier.

8 Functional calculus $\varphi_1 \boxtimes \varphi_2$

In this section, we assume that we are given an extended tensor product $X \boxtimes Y$ of Banach spaces $X$ and $Y$, and we are given pseudo-resolvents $R_{1,(\cdot)}$ and $R_{2,(\cdot)}$ in the algebras $B_0(X)$ and $B_0(Y)$, respectively.

We define the mapping $\varphi_1 \boxtimes \varphi_2$ acting on functions $f \in O(\overline{\sigma(R_{1,(\cdot)})} \cup \overline{\sigma(R_{2,(\cdot)})})$ of one variable by the formula

$$(\varphi_1 \boxtimes \varphi_2)f = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R_{1,\lambda} \boxtimes R_{2,\lambda} \, d\lambda,$$  \hspace{1cm} (27)

where $\Gamma$ is an oriented envelope of $\overline{\sigma(R_{1,(\cdot)})} \cup \overline{\sigma(R_{2,(\cdot)})}$ with respect to the complement of the domain of $f$.

Let $U \subseteq \overline{\mathbb{C}}$ be an open set and $f : U \to \mathbb{C}$ be an analytic function. We call the divided difference [46, 77] of the function $f$ the function $f^{[1]} : U \times U \to \mathbb{C}$ defined by the formula

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ f'(\lambda), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda = \infty \text{ or } \mu = \infty. \end{cases} \hspace{1cm} (28)$$

Example 5 We give examples of divided differences of some functions:

- $v_1^{[1]}(\lambda, \mu) = 1,$
- $v_2^{[1]}(\lambda, \mu) = \lambda + \mu,$
- $v_n^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \mu^{n-1},$
- $v_{1/2}^{[1]}(\lambda, \mu) = \frac{1}{\sqrt{\lambda} + \sqrt{\mu}},$
- $r_1^{[1]}(\lambda, \mu) = \frac{1}{(\lambda_0 - \lambda)(\lambda_0 - \mu)},$
- $r_n^{[1]}(\lambda, \mu) = \frac{v_n^{[1]}(\lambda_0 - \lambda, \lambda_0 - \mu)}{(\lambda_0 - \lambda)^n(\lambda_0 - \mu)^n}.$

The Taylor series for the divided difference of a function $f$ at a point $(\lambda_0, \mu_0)$ has the form
\[ f^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(\lambda_0)}{(n+1)!} v^{[1]}_{n+1}(\lambda - \lambda_0, \mu - \lambda_0) \]
\[ = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(\lambda_0)}{(n+1)!} \sum_{i=0}^{n} (\lambda - \lambda_0)^{n-i}(\mu - \lambda_0)^i, \]

where \( v_n(\lambda) = \lambda^n \). In particular, for \( \exp(\lambda) = e^{\lambda t} \) and \( \exp^{[1]}(\lambda) = \lambda e^{\lambda t} \), one has
\[ \exp^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} v^{[1]}_{n+1}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \sum_{i=0}^{n} \lambda^{n-i} \mu^i, \]

\[ \exp^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{n!} v^{[1]}_{n+1}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{n} \lambda^{n-i} \mu^i. \]

**Proposition 44** Let \( U \subseteq \mathbb{C} \) be an open set and \( f: U \to \mathbb{C} \) be an analytic function. Then the function \( f^{[1]} \) is analytic in \( U \times U \).

**Proof** The analyticity of \( f^{[1]} \) at a finite point \((\lambda, \mu)\), \( \lambda \neq \mu \), is evident. The analyticity at the points of the form \((\lambda, \infty)\) and \((\infty, \mu)\), where \( \lambda, \mu \in \mathbb{C} \), is also evident. We verify the analyticity of \( f^{[1]} \) at points of the form \((\lambda, \lambda)\).

Next, we consider the case of a finite point \((\lambda, \lambda)\), \( \lambda \neq \infty \). We expand \( f \) in the Taylor series about \( \lambda_0 \):
\[ f(\lambda) = \sum_{n=0}^{\infty} c_n (\lambda - \lambda_0)^n. \]

For \( \lambda \neq \mu \) close to \( \lambda_0 \), we have
\[ f^{[1]}(\lambda, \mu) = \sum_{n=1}^{\infty} c_n v^{[1]}_n(\lambda - \lambda_0, \mu - \lambda_0), \]
where \( v^{[1]}_n(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2} \mu + \cdots + \mu^{n-1} \). This series determines an analytic function in a neighborhood of the point \((\lambda_0, \lambda_0)\). Clearly, \( f^{[1]}(\lambda_0, \lambda_0) = f'(\lambda_0) \).

Finally, we consider the case of the point \((\infty, \infty)\). We expand \( f \) in the Laurent series in a neighborhood of \( \infty \):
\[ f(\lambda) = \sum_{n=0}^{\infty} c_n \frac{1}{\lambda^n}. \]

This formula shows that for \( \lambda \neq \mu \) close to \( \infty \), one has
\[ f^{[1]}(\lambda, \mu) = -\sum_{n=1}^{\infty} c_n \frac{v_n^{[1]}(\lambda, \mu)}{\lambda^n \mu^n}, \]

where \( v_n^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2} \mu + \cdots + \mu^{n-1} \). This series determines an analytic function in a neighborhood of the point \((\infty, \infty)\). Clearly, \( f^{[1]}(\infty, \infty) = 0 \). \( \square \)

**Theorem 45** Let \( f \in O(\bar{\sigma}(R_{1,()}) \cup \bar{\sigma}(R_{2,())) \). Then (strictly speaking, in this formula, \( f^{[1]} \) is understood to be the canonical projection of the function \( f^{[1]} \in O[\bar{\sigma}(R_{1,()} \cup \bar{\sigma}(R_{2,()))] \) into \( O(\bar{\sigma}(R_{1,()} \times \bar{\sigma}(R_{2,()))] \)

\[ (\varphi_1 \boxtimes \varphi_2)f = (\varphi_1 \boxtimes \varphi_2)f^{[1]}. \]

The spectrum of the transformator \( (\varphi_1 \boxtimes \varphi_2)f : X \boxtimes Y \to X \boxtimes Y \) is given by the formula

\[ \sigma((\varphi_1 \boxtimes \varphi_2)f) = \{ f^{[1]}(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,(}), \mu \in \bar{\sigma}(R_{2,()} \} \}. \]

**Proof** We take contours \( \Gamma_1 \) and \( \Gamma_2 \) such that both are oriented envelopes of \( \bar{\sigma}(R_{1,(}) \cup \bar{\sigma}(R_{2,(}) \) with respect to the complement of the domain of the function \( f \), and \( \Gamma_2 \) lies outside of \( \Gamma_1 \) (so that \( \lambda - \mu \) does not vanish for \( \lambda \in \Gamma_1 \) and \( \mu \in \Gamma_2 \)), see Fig. 2. We make use of the definition:

\[ (\varphi_1 \boxtimes \varphi_2)f^{[1]} = \frac{1}{(2\pi i)^3} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu)R_{1,\lambda} \boxtimes R_{2,\mu} \, d\lambda \, d\mu \]

\[ + \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} f^{[1]}(\lambda, \infty)R_{1,\lambda} \boxtimes 1 \, d\lambda \]

\[ + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} f^{[1]}(\infty, \mu)1 \boxtimes R_{2,\mu} \, d\mu \]

\[ + \delta_1 \delta_2 f^{[1]}(\infty, \infty)1 \boxtimes 1 \]

\[ = \frac{1}{(2\pi i)^3} \int_{\Gamma_1} \int_{\Gamma_2} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} R_{1,\lambda} \boxtimes R_{2,\mu} \, d\lambda \, d\mu. \]

We represent the last integral as the sum of two iterated integrals:
By the Cauchy integral formula, for the internal integral in (29), we have
\[
\frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{\lambda - \mu} R_{2,\mu} \, d\mu = R_{2,\lambda},
\]
and by the Cauchy integral theorem, for the internal integral in (30), we have
\[
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\mu - \lambda} R_{1,\lambda} \, d\lambda = 0,
\]
since, by the assumption, the contour \( \Gamma_1 \) does not surround the singularities of the function \( \lambda \to \frac{1}{\mu - \lambda}, \mu \in \Gamma_2, \) and the singularities of the pseudo-resolvent \( \lambda \to R_{1,\lambda}. \)

Thus, the original integral takes the form
\[
\frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) R_{1,\lambda} \otimes R_{2,\lambda} \, d\lambda = [(\varphi_1 \Box \varphi_2) f].
\]

The second formula from the statement of the theorem follows from Theorem 39. □

**Theorem 46** Let \( A \in \mathcal{B}(X), B \in \mathcal{B}(Y), \) and \( f \in \mathcal{O}\left(\sigma(A) \cup \sigma(B)\right). \) Then
\[
\varphi_A(f) \Box 1 - 1 \Box \varphi_B(f) = \left[ (\varphi_A \Box \varphi_B)f \right] (A \Box 1 - 1 \Box B) = (A \Box 1 - 1 \Box B) \left[ (\varphi_A \Box \varphi_B)f \right],
\]
where the functional calculi \( \varphi_A \) and \( \varphi_B \) are constructed by \( A \) and \( B \) respectively.

**Proof** The proof follows from the identity
\[
f(\lambda) - f(\mu) = f^{[1]}(\lambda, \mu)(\lambda - \mu) = (\lambda - \mu)f^{[1]}(\lambda, \mu)
\]
and Theorems 45 and 32. □

In the following corollary, we describe a representation for the increment of an analytic function.

**Corollary 47** Let \( A, B \in \mathcal{B}(X) \) and \( f \in \mathcal{O}\left(\sigma(A) \cup \sigma(B)\right). \) Then
\[
f(A) - f(B) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{A,\lambda}(A - B) R_{B,\lambda} \, d\lambda,
\]
and the operator
\[
X = ((\varphi_A \Box \varphi_B)f^{[1]}) 1
\]
satisfies the Sylvester equation.
We apply the formulae from Theorem 46 to the operator $C = 1$, assuming that $X = Y$. We have (taking into account that $C = 1$)

$$AX - XB = f(A) - f(B).$$

For $f = \exp_{\lambda}$, the first formula is described in [138, p. 978].

**Proof** We apply the formulae from Theorem 46 to the operator $C = 1$, assuming that $X = Y$. We have (taking into account that $C = 1$)

$$(\varphi_A(f) \boxtimes 1 - 1 \boxtimes \varphi_B(f)) C = \varphi_A(f) C - C \varphi_B(f)$$

$$= f(A) C - C f(B) = f(A) - f(B),$$

$$(A \boxtimes 1 - 1 \boxtimes B) AC - CB = A - B,$$

$$[(\varphi_A \Box \varphi_B) f] (A \boxtimes 1 - 1 \boxtimes B) C = [(\varphi_A \Box \varphi_B) f] (A - B)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{A, \lambda}(A - B) R_{B, \lambda} d\lambda,$$

$$(A \boxtimes 1 - 1 \boxtimes B) [(\varphi_A \Box \varphi_B) f] C = (A \boxtimes 1 - 1 \boxtimes B) [(\varphi_A \Box \varphi_B) f] C$$

$$= A \left( [(\varphi_A \Box \varphi_B) f] C \right) 1 - \left( [(\varphi_A \Box \varphi_B) f] 1 \right) B.$$

One of the primary ideas [33, 43, 68, 96, 100, 110, 112] of approximate calculation of an analytic function $f$ of an operator or a pseudo-resolvent consists in an approximation of $f$ by a polynomial or a rational function. To apply this idea for the calculation of $(\varphi_1 \Box \varphi_2)f$, it is necessary at least to be able to compute $(\varphi_1 \Box \varphi_2)f$ when $f$ is a monomial or an elementary rational function. Just this case is considered in Corollary 48 below.

**Corollary 48** (a) If the pseudo-resolvents $R_{1,(\cdot)}$ and $R_{2,(\cdot)}$ are generated by the operators $A$ and $B$, respectively, then

$$(\varphi_A \Box \varphi_B) (v_n) = A^{n-1} \boxtimes 1 + A^{n-2} \boxtimes B + \cdots + 1 \boxtimes B^{n-1},$$

where $v_n(\lambda) = \lambda^n$.

(b) If $\lambda_0 \in \rho(R_{1,(\cdot)}) \cap \rho(R_{2,(\cdot)})$, then

$$(\varphi_1 \Box \varphi_2)(r_1) = R_{1, \lambda_0} \boxtimes R_{2, \lambda_0},$$

where $r_1(\lambda) = \frac{1}{\lambda_0 - \lambda}$.

(c) If the extended singular sets of the pseudo-resolvents $R_{1,(\cdot)}$ and $R_{2,(\cdot)}$ are disjoint, then

$$(\varphi_1 \Box \varphi_2)(r_n) = -\left( R_{1, \lambda_0} \boxtimes 1 - 1 \boxtimes R_{2, \lambda_0} \right) \left( R_{1, \lambda_0} \boxtimes 1 - 1 \boxtimes R_{2, \lambda_0} \right)^{-1},$$

where $r_n(\lambda) = \frac{1}{(\lambda_0 - \lambda)}$. If, in addition, the pseudo-resolvents $R_{1,(\cdot)}$ and $R_{2,(\cdot)}$ are generated by the operators $A$ and $B$, respectively, then

$$(\varphi_A \Box \varphi_B)(r_n) = \left[ (\lambda_0 1 - A)^{n-1} \boxtimes 1 + \cdots + 1 \boxtimes (\lambda_0 1 - B)^{n-1} \right] \left[ R_{A, \lambda_0} \boxtimes R_{B, \lambda_0} \right].$$
Proof It suffices to make use of Example 5 and to apply Theorem 45 and Corollary 35.

Corollary 49 Let \( g, h \in \mathbf{O}(\sigma(R_1, \cdot) \cup \sigma(R_2, \cdot)) \). Then

\[
(\varphi_1 \boxdot \varphi_2)(gh) = \left[ (\varphi_1 \boxdot \varphi_2)(g) \right] (1 \boxdot \varphi_2(h)) + \left[ (\varphi_1 \boxdot \varphi_2)(h) \right] \left[ (\varphi_1 \boxdot \varphi_2)(g) \right],
\]

where \( (gh)(\lambda) = g(\lambda)h(\lambda) \).

Proof By Theorem 45, this formula is equivalent to the identity

\[
(\varphi_1 \boxdot \varphi_2)(gh)[1] = \left[ (\varphi_1 \boxdot \varphi_2)[1](g) \right] (1 \boxdot \varphi_2(h)) + \left[ (\varphi_1 \boxdot \varphi_2)[1](h) \right] \left[ (\varphi_1 \boxdot \varphi_2)[1](g) \right].
\]

For \( \lambda \neq \mu \), one has

\[
g(\lambda)h(\lambda) - g(\mu)h(\mu) \quad \text{or} \quad \frac{g(\lambda)h(\lambda) - g(\mu)h(\lambda)}{\lambda - \mu} = \frac{g(\lambda)h(\lambda) - g(\mu)h(\lambda)}{\lambda - \mu} + \frac{g(\mu)h(\lambda) - g(\mu)h(\mu)}{\lambda - \mu} = \frac{g(\lambda) - g(\mu)}{\lambda - \mu} h(\lambda) + g(\mu) \frac{h(\lambda) - h(\mu)}{\lambda - \mu}.
\]

Taking into account the ability of passages to the limits as \( \lambda - \mu \to 0 \) and \( \lambda - \mu \to \infty \) this formula can be rewritten as

\[
(gh)[1](\lambda, \mu) = g[1](\lambda, \mu)h(\mu) + g(\lambda)h[1](\lambda, \mu).
\]

It remains to apply Theorems 32, 33, and 34.

The function \( \beta_{g, h} : U \times U \to \mathbb{C} \) defined by the formula:

\[
\beta_{g, h}(\lambda, \mu) = \begin{cases} 
\frac{g(\lambda)h(\mu) - h(\lambda)g(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\
g'(\lambda)h(\mu) - h'(\lambda)g(\mu), & \text{if } \lambda = \mu, \\
0, & \text{if } \lambda = \infty \text{ or } \mu = \infty,
\end{cases}
\]
is similar to the divided difference. It is generated by two analytic functions \( g, h : U \to \mathbb{C} \). By analogy with [64, 65, 92], we call the function \( \beta_{g, h} \) the Bezoutian. The Bezoutian is a difference-differential analogue of the Wronskian. For example, the Bezoutian of the functions sin and cos is sinc(\( \lambda - \mu \)) = \( \frac{\sin(\lambda - \mu)}{\lambda - \mu} \). We note that the Bezoutian can be expressed in terms of divided differences:

\[
\beta_{g, h}(\lambda, \mu) = g[1](\lambda, \mu)h(\mu) - h[1](\lambda, \mu)g(\mu).
\]

(In particular, this formula and Proposition 44 imply that \( \beta_{g, h} \) is an analytic function.) The converse is also true:
\[ g_{1}(\lambda, \mu) = \beta_{g, h}(\lambda, \mu), \]

where \( u(\lambda) = 1 \).

**Corollary 50** Let \( g, h \in O(\bar{\sigma}(R_{1}(.)) \cup \bar{\sigma}(R_{2}(.))) \) and let \( h(\lambda) \neq 0 \) for \( \lambda \in \bar{\sigma}(R_{1}(.)) \cup \bar{\sigma}(R_{2}(.)) \). Then

\[
(\varphi_{1} \Box \varphi_{2}) \left( \frac{g}{h} \right) = \left[ (\varphi_{1} \Box \varphi_{2})(\beta_{g, h}) \right] \left[ \varphi_{1}(h) \Box \varphi_{2}(h) \right]^{-1} \left[ (\varphi_{1} \Box \varphi_{2})(\beta_{g, h}) \right],
\]

where \( \left( \frac{g}{h} \right)(\lambda) = \frac{g(\lambda)}{h(\lambda)} \).

**Proof** The proof is analogous to that of Corollary 49 and follows from the formula:

\[
\left[ \frac{g}{h} \right]^{1} (\lambda, \mu) = \frac{\beta_{g, h}(\lambda, \mu)}{h(\lambda)h(\mu)}.
\]

**Proposition 51** Let \( X \) be a Banach space. For \( R_{1}(.) \) and \( R_{2}(.) \) we take the resolvent \( R(\lambda) = R_{A}(\lambda) \) of the same operator \( A \in B(X) \). Let an operator \( C \in B(X) \) commute with at least one value \( R_{\mu} \) of the pseudo-resolvent \( R(\lambda) \). Then

\[
\left[ (\varphi_{A} \Box \varphi_{A})f \right] C = f'(A)C = Cf'(A).
\]

**Proof** We note that, by virtue of Theorem 16, \( C \) commutes with all values \( R_{\lambda} \) of the pseudo-resolvent \( R(\lambda) \).

By the definition and commutativity, we have

\[
\left[ (\varphi \Box \varphi)f \right] C = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R_{\lambda}C R_{\lambda} d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R^{2}_{\lambda} C d\lambda
\]

\[
= \left( \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R^{2}_{\lambda} d\lambda \right) C.
\]

Passing to the limit in the Hilbert identity (3) we obtain the relation:

\[
R^{2}_{\lambda} = -R'_{\lambda}, \quad \lambda \notin \sigma(R(\lambda)).
\]

Substituting this identity into the previous equality and integrating by parts, we obtain
We note that the divided differences \[ f^{[1]}(A,B) \] of the operators \( A \) and \( B \) are also closely related to the calculation of functions of block triangular matrices [27, 28, 55, 68, 113].

9 The impulse response

In subsequent sections, we discuss some applications.

In this section, the previous results are applied to the representation of the impulse response of a second-order differential equation. Here we regard the space \( \mathcal{B}(Y,X) \) as an extended tensor product (Example 3(e)). Therefore, for example, the action of the transformator \( \varphi_1(g) \boxtimes \varphi_2(h) \) on the operator \( C \in \mathcal{B}(Y,X) \) results in the operator \( \varphi_1(g)C\varphi_2(h) \).

Let \( X \) and \( Y \) be Banach spaces and \( E,F,H \in \mathcal{B}(Y,X) \). A function \( \lambda \mapsto \lambda^2 E + \lambda F + H \), where \( \lambda \in \mathbb{C} \), is called [45, 74, 91, 107, 114] a square pencil. The resolvent set of the pencil is the set \( \rho(E,F,H) \) of all \( \lambda \in \mathbb{C} \) such that the operator \( \lambda^2 E + \lambda F + H \) is invertible. The spectrum is the complement \( \sigma(E,F,H) = \mathbb{C} \setminus \rho(E,F,H) \) and the resolvent is the function

\[
R_\lambda = (\lambda^2 E + \lambda F + H)^{-1}, \quad \lambda \in \rho(E,F,H).
\]

The main sources [107, 114, 137] of square pencils are second-order differential equations of the form:

\[
E\ddot{y}(t) + F\dot{y}(t) + Hy(t) = 0,
\]

where \( y : \mathbb{R} \to Y \). In this section, it is always assumed that the operator \( E \) is invertible. We note that even if \( E \) is invertible, the multiplication of the Eq. (32) by \( E^{-1} \) is not always desirable. For example, the operators \( E, F, H \) are often assumed [91, 126] to be self-adjoint, but the multiplication by \( E^{-1} \) may cause to the loss of this property.

We recall the following proposition.

**Proposition 52** (See, for example, [102, Theorem 16]) Let the operator \( E \) be invertible. Then, the solution of the initial value problem

\[
[((\varphi_1 \boxtimes \varphi_2)f)(C) = -\left( \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R_\lambda d\lambda \right) \]

\[
= \left( \frac{1}{2\pi i} \int_{\Gamma} f'(\lambda)R_\lambda d\lambda \right) \]

\[
= \varphi(f')C.
\]
\[ Ey(t) + Fy(t) + Hy(t) = 0, \]
\[ y(0) = y_0, \]
\[ y'(0) = y_1 \]

can be represented in the form
\[ y(t) = \hat{T}(t)E y_0 + T(t)(Ey_1 + F y_0), \]
where
\[ T(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda) (\lambda^2 E + \lambda F + H)^{-1} d\lambda, \]
\[ \hat{T}(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp^{(1)}(\lambda) (\lambda^2 E + \lambda F + H)^{-1} d\lambda, \]
\( \Gamma \) is an oriented envelope of the pencil spectrum \( \sigma(E,F,H) \), and
\[ \exp(\lambda) = e^{i\lambda}, \quad \exp^{(1)}(\lambda) = \lambda e^{i\lambda}. \]

The function \( T \) is called the impulse response. It can be shown that \( \hat{T} \) is its derivative.

A factorization of the pencil is the representation of its resolvent in the form:
\[ R_{\lambda} = R_{1,\lambda} CR_{2,\lambda}, \tag{33} \]
where \( R_{1,(\cdot)} \) and \( R_{2,(\cdot)} \) are pseudo-resolvents acting in \( X \) and \( Y \), respectively, and \( C \in \mathcal{B}(Y,X) \). It is assumed that \( \rho(R_{1,(\cdot)}) \cap \rho(R_{2,(\cdot)}) \supseteq \rho(E,F,H) \).

**Proposition 53** Let the operator \( E \) be invertible. Then we have \( C = E^{-1} \) in formula (33), and the pseudo-resolvents \( R_{1,(\cdot)} \) and \( R_{2,(\cdot)} \) are the resolvents of some operators \( A_1 \) and \( A_2 \).

**Proof** By Proposition 23, we have
\[ R_{1,\lambda} = -N_1 + \frac{P_1}{\lambda} + \frac{A_1}{\lambda^2} + \frac{A_1^2}{\lambda^3} + \ldots, \]
\[ R_{2,\lambda} = -N_2 + \frac{P_2}{\lambda} + \frac{A_2}{\lambda^2} + \frac{A_2^2}{\lambda^3} + \ldots \]
Hence,
\[ R_{1,\lambda} CR_{2,\lambda} = N_1 CN_2 - \frac{P_1 CN_2 + N_1 CP_2}{\lambda} \]
\[ + \frac{-A_1 CN_2 + P_1 CP_2 - N_1 CA_2}{\lambda^2} + \ldots \]
On the other hand, in a neighborhood of \( \infty \), by Theorem 1, we have
\[ R_\lambda = \frac{E^{-1}}{\lambda^2} - \frac{E^{-1}FE^{-1}}{\lambda^3} + \cdots \]

Therefore

\[ N_1 CN_2 = 0, \quad P_1 CN_2 + N_1 CP_2 = 0, \]
\[ -A_1 CN_2 + P_1 CP_2 - N_1 CA_2 = E^{-1}. \]

Multiplying the second equation on the left by \( A_1 P_1 \) (keeping in mind the identities \( P^2 = P \), \( AP = PA = A \) and \( NP = PN = 0 \) from Proposition 23), we arrive at

\[ A_1 CN_2 = 0. \]

Similarly, multiplying the second equation on the right by \( A_2 \), we have

\[ N_1 CA_2 = 0. \]

Substituting these results into the third equation, we obtain

\[ P_1 CP_2 = E^{-1}. \]

Because of the invertibility of \( E \), it follows that the projectors \( P_1 \) and \( P_2 \) coincide with \( 1 \), and \( C = E^{-1} \). Consequently (by the identity \( NP = PN = 0 \), see Proposition 23), we have \( N_1 = 0 \) and \( N_2 = 0 \). It follows that \( \lim_{\lambda \to \infty} \lambda R_{1,\lambda} = 1 \) and \( \lim_{\lambda \to \infty} \lambda R_{2,\lambda} = 1 \). By Proposition 24(c), these mean that the pseudo-resolvents \( R_{1,()} \) and \( R_{2,()} \) are the resolvents of the operators \( A_1 \) and \( A_2 \).

\textbf{Theorem 54} Let the operator \( E \) be invertible, and the square pencil admit factorization (33). Then the impulse response \( T \) and its derivative \( _T \) can be represented in the form

\[ T(t) = (\varphi_1 \boxdot \varphi_2)(\exp t)C = (\varphi_1 \boxdot \varphi_2)(\exp^{(1)} t)C, \]
\[ _T(t) = (\varphi_1 \boxdot \varphi_2)(\exp^{(1)} t)C = (\varphi_1 \boxdot \varphi_2)(\exp^{(1)} t)C, \]

where \( \exp^{(1)}(\lambda, \mu) = \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu} \) for \( \lambda \neq \mu \).

\textbf{Proof} The proof follows from Proposition 52 and Theorem 45.

\textbf{Corollary 55} The spectra of the transformators \( C \rightarrow T(t) \) and \( C \rightarrow _T(t) \) in the algebra \( B(B(Y,X)) \) are equal to

\[ \{ \exp^{(1)}(\lambda, \mu) : \lambda \in \sigma(A), \mu \in \sigma(B) \}, \]
\[ \{ \exp^{(1)}(\lambda, \mu) : \lambda \in \sigma(A), \mu \in \sigma(B) \}, \]

respectively.

\textbf{Proof} The proof follows from Theorems 39 and 54.
Remark 56 (a) In article [82], for the approximate calculation of expressions of the type 
\[ C_0 u_1 C_2 u_2 C_1 \exp \left( \frac{\lambda - \mu}{2} t \right) \], it is suggested to use the representation (written in other notations; the representation can be verified directly)
\[
\exp^{[1]}(\lambda, \mu) = (e^{\lambda t} + e^{\mu t}) \frac{\tanh \left( \frac{\lambda - \mu}{2} t \right)}{\frac{\lambda - \mu}{2} t}, \quad \lambda \neq \mu.
\]
By Theorem 32, \((\varphi_1 \otimes \varphi_2)(e^{\lambda t} + e^{\mu t})\) is \(\varphi_1(\exp) \otimes 1 + 1 \otimes \varphi_2(\exp)\). By Corollary 37, the operator \((\varphi_1 \otimes \varphi_2)\left( \frac{\tanh \left( \frac{\lambda - \mu}{2} t \right)}{\frac{\lambda - \mu}{2} t} \right)\) is the function \(\tau(z) = \frac{\tanh \left( \frac{z}{2} \right)}{z}\) of the transformer \((A \otimes 1 - 1 \otimes B) t\). The function \(\tau\) is analytic in the circle \(|z| < \pi\). In [82], for its computation, it is suggested to use the Taylor polynomials or rational approximations.

(b) Formulae
\[
\exp^{[1]}(\lambda, \mu) = e^{(\lambda + \mu) t} \frac{\sinh \left( \frac{\lambda - \mu}{2} t \right)}{\frac{\lambda - \mu}{2} t}, \quad \exp^{[1]}(\lambda, \mu) = e^{\mu t} e^{(\lambda - \mu) t} - 1 \frac{\lambda - \mu}{\lambda - \mu},
\]
where \(\lambda \neq \mu\), assuming a similar usage, are suggested in [68, formula (10.17)].

(c) We present two formulae that enables one to apply similar ideas for the calculation of \(\exp^{[1]}(\lambda, \mu)\):
\[
\exp^{[1]}(\lambda, \mu) = e^{\lambda t} + e^{\mu t} \frac{\lambda e^{\lambda t} - \mu e^{\mu t}}{\lambda - \mu} = e^{\lambda t} + e^{\mu t} \frac{\lambda - \mu}{\lambda - \mu} = e^{\lambda t} + e^{\mu t} \exp^{[1]}(\lambda, \mu).
\]
Interchanging \(\mu\) and \(\nu\) in this equality, and then calculating the arithmetic mean, we arrive at the symmetric formula
\[
\exp^{[1]}(\lambda, \mu) = \frac{1}{2} (e^{\lambda t} + e^{\mu t}) \left( 1 + (\lambda + \mu) \frac{\tanh \left( \frac{\lambda - \mu}{2} t \right)}{\frac{\lambda - \mu}{2} t} \right), \quad \lambda \neq \mu.
\]

Corollary 57 Let \(E = 1\). Then
\[
\mathcal{T}(t + s) = \mathcal{T}_1(t) \mathcal{T}(s) + \mathcal{T}(t) \mathcal{T}_2(s),
\]
where
\[ T_1(t) = \varphi_1(\exp_t) = \frac{1}{2\pi i} \int_{\Gamma_1} \exp(\lambda) R_{1,\lambda} \, d\lambda, \]
\[ T_2(t) = \varphi_2(\exp_t) = \frac{1}{2\pi i} \int_{\Gamma_2} \exp(\mu) R_{2,\mu} \, d\mu. \]

**Proof** This is a special case of Corollary 49.

Issues related to factorization of square pencils are widely discussed in the literature [30, 80, 84, 85, 88, 91, 99, 107, 129, 137]. The factorization of an operator pencils of an arbitrary order is discussed in [54, 59, 61, 76, 95, 107, 108, 141, 142].

Estimates of the norms of operators \((\varphi_1 \otimes \varphi_2)(f)C\) are obtained in [48, 49]; special attention is paid to \(T(t)\) and \(\dot{T}(t)\). Estimates of the norm of \(e^{(A \otimes 1 + 1 \otimes B)t}\) are given in [11].

### 10 The transformator \(W\) and the Sylvester equation

It often arises the problem of calculating the transformator \(W = (\varphi_1 \otimes \varphi_2)w\), where

\[ w(\lambda, \mu) = \frac{1}{\lambda - \mu}. \]

As a rule, it is equivalent to solving the Sylvester equation. In this section, we discuss some properties of the transformator \(W\).

Let \(X \otimes Y\) be an extended tensor product of Banach spaces \(X\) and \(Y\), and \(R_{1,()}\) and \(R_{2,()}\) be pseudo-resolvents in the algebras \(B_0(X)\) and \(B_0(Y)\), respectively. We assume that the extended singular sets \(\sigma(R_{1,()}\) and \(\sigma(R_{2,()})\) are disjoint. We consider the transformator \(W\) defined by the formula:

\[ W = (\varphi_1 \otimes \varphi_2)w, \]

where (note that the function \(w\) is meromorphic with the point of indeterminacy \((\infty, \infty)):\)

\[ w(\lambda, \mu) = \frac{1}{\lambda - \mu}. \]

If necessary, we use the more detailed notation \(W_{\varphi_1, \varphi_2}\) or \(W_{A,B}\).

---

\( \sigma_1 \Gamma_1 \)
\( \sigma_2 \Gamma_2 \)
\( \sigma_1 \Gamma_2 \)
\( \sigma_2 \Gamma_1 \)

**Fig. 3** Various options of the arrangement of the contours \(\Gamma_1\) and \(\Gamma_2\) and the extended singular sets, see the proof of Proposition 58.
Proposition 58 We assume that the extended singular sets $\sigma(R_{1,\omega})$ and $\sigma(R_{2,\omega})$ of the pseudo-resolvents $R_{1,\omega}$ and $R_{2,\omega}$ are disjoint. Then

$$W = \frac{1}{2} (\varphi_1 \boxtimes \varphi_2)(\text{sgn}_{1/2})$$

(35)

where the function $\text{sgn}_{1/2}$ is equal to 1 in an open neighborhood of the extended singular set $\sigma(R_{1,\omega})$ and is equal to $-1$ in an open neighborhood of the extended singular set $\sigma(R_{2,\omega})$. The transformator $W$ can be represented in the form

$$W = \frac{1}{2\pi i} \int_\Gamma R_{1,\lambda} \boxtimes R_{2,\lambda} \, d\lambda,$$

(36)

where $\Gamma$ is an oriented envelope of $\sigma(R_{1,\omega})$ with respect to $\sigma(R_{2,\omega})$.

**Proof** It is easy to verify that $\text{sgn}^{|\lambda|}_{1/2} = 2w$. Therefore, formula (35) follows from Theorem 45.

We calculate $(\varphi_1 \boxtimes \varphi_2)w$. To be definite, we assume that $\infty \notin \sigma(R_{2,\omega})$. We assume that the oriented envelope $\Gamma_1$ of the set $\sigma(R_{1,\omega})$ and the oriented envelope $\Gamma_2$ of the set $\sigma(R_{2,\omega})$ are located, as shown in Fig. 3. In particular, $\lambda - \mu$ is not equal to zero for $\lambda \in \Gamma_1$ and $\mu \in \Gamma_2$. We have (note that in the representation (20) for the function $w$, we have $d_1 = d_2 = 0$):

$$W = (\varphi_1 \boxtimes \varphi_2)w = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{\lambda - \mu} R_{1,\lambda} \boxtimes R_{2,\mu} \, d\mu \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_1} R_{1,\lambda} \boxtimes \left(\frac{1}{2\pi i} \int_{\Gamma_2} \frac{R_{2,\mu}}{\lambda - \mu} \, d\mu \right) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} R_{1,\lambda} \boxtimes R_{2,\lambda} \, d\lambda.$$

Obviously, $\Gamma_1$ is an oriented envelope of $\sigma(R_{1,\omega})$ with respect to $\sigma(R_{2,\omega})$.

**Remark 59** According to Corollary 37 the following representation also holds:

$$W = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{v} (v1_X \boxtimes 1_Y - A \boxtimes 1_Y + 1_X \boxtimes B)^{-1} \, dv,$$

where $\Gamma$ is an oriented envelope of $\sigma(A) - \sigma(B)$ with respect to zero, provided the spectra of $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are disjoint.

Proposition 60 ([13, 66, 89, 94, Lemma 2.2], [104]) Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, and the embeddings $\sigma(A) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda < \rho \}$ and $\sigma(B) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda > \rho \}$ hold for some $\rho > 0$. Then

$$W = - \int_0^\infty e^{At} \boxtimes e^{-Bt} \, dt.$$
\[ w(\lambda, \mu) = - \int_0^\infty e^{\lambda t} e^{-\mu t} dt. \]

It is valid for \( \lambda \in U \) and \( \mu \in V \), where the neighborhoods \( U \supset \sigma(A) \) and \( V \supset \sigma(B) \) are sufficiently small. Moreover, we may assume that the integral converges uniformly for \( \lambda \in U \) and \( \mu \in V \). We substitute this integral into the formula:

\[ W = (\varphi_1 \boxtimes \varphi_2) w = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} w(\lambda, \mu) R_{A, \lambda} \boxtimes R_{B, \mu} \, d\mu \, d\lambda \]

from Theorem 32 assuming that \( \Gamma_1 \subset U \) and \( \Gamma_2 \subset V \). By the uniform convergence of the last integral, we may interchange the order of integration:

\[ W = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \left( - \int_0^\infty e^{\lambda t} e^{-\mu t} dt \right) R_{A, \lambda} \boxtimes R_{B, \mu} \, d\mu \, d\lambda \]

\[ = - \int_0^\infty \left( \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} e^{\lambda t} e^{-\mu t} R_{A, \lambda} \boxtimes R_{B, \mu} \, d\mu \, d\lambda \right) dt \]

\[ = - \int_0^\infty e^{\lambda t} e^{-\mu t} dt. \]

\[\square\]

**Proposition 61 (\cite{13, Theorem 9.1})** Let the embeddings \( \sigma(R_{1, \lambda}) \subset \{ \lambda \in \mathbb{C} : |\lambda| < \rho \} \) and \( \sigma(R_{2, \lambda}) \subset \{ \lambda \in \mathbb{C} : |\lambda| > \rho \} \) hold for some \( \rho > 0 \). Then

\[ W = - \sum_{n=0}^\infty A^n \boxtimes R_{2,0}^{n+1}, \]

where the operator \( A \in \mathcal{B}(X) \) generates \( R_{1, \lambda} \) according to Proposition 24.

**Proof** We make use of the representation

\[ w(\lambda, \mu) = - \sum_{n=0}^\infty \frac{\lambda^n}{\mu^{n+1}}. \]

It is valid for \( \lambda \in U \) and \( \mu \in V \), where the neighborhoods \( U \supset \sigma(A) \) and \( V \supset \sigma(R_{2, \lambda}) \) are sufficiently small. Moreover, we may assume that the series converges uniformly for \( \lambda \in U \) and \( \mu \in V \). We substitute this series into the formula:

\[ (\varphi_1 \boxtimes \varphi_2)(w) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} w(\lambda, \mu) R_{1, \lambda} \boxtimes R_{2, \mu} \, d\mu \, d\lambda \]

\[ + \frac{\delta}{2\pi i} \int_{\Gamma_1} w(\lambda, \infty) R_{1, \lambda} \boxtimes 1 \, d\lambda \]

from Theorem 33 assuming that \( \Gamma_1 \subset U \) and \( \Gamma_2 \subset V \). By the uniform convergence of the series (and by \( w(\lambda, \infty) = 0 \)), we have
\[
(\varphi_1 \boxtimes \varphi_2)(w) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} -\sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^{n+1}} R_{A,\lambda} \boxtimes R_{2,\mu} \, d\mu \, d\lambda \\
= -\sum_{n=0}^{\infty} \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\lambda^n}{\mu^{n+1}} R_{A,\lambda} \boxtimes R_{2,\mu} \, d\mu \, d\lambda \\
= -\sum_{n=0}^{\infty} A^n \boxtimes R_{2,0}^{n+1}.
\]

\[\square\]

Theorem 62 below reduces the calculation of \([\varphi_1 \boxtimes \varphi_2](f)\) to the calculation of \(\varphi_1(f)\) and \(\varphi_2(f)\) provided \(W(C)\) is known; it is a version of Theorem 46.

**Theorem 62** Let the extended singular sets \(\overline{\sigma}(R_{1,\cdot})\) and \(\overline{\sigma}(R_{2,\cdot})\) of the pseudoresolvents \(R_{1,\cdot}\) and \(R_{2,\cdot}\) be disjoint, and \(f \in \mathcal{O}(\overline{\sigma}(R_{1,\cdot}) \cup \overline{\sigma}(R_{2,\cdot}))\). Then

\[
[(\varphi_1 \boxtimes \varphi_2)f] C = [\varphi_1(f) \boxtimes 1 - 1 \boxtimes \varphi_2(f)] W(C).
\]

In the special case, when \(B(Y, X)\) is taken as the extended tensor product (see Example 3(e)),

\[
[(\varphi_1 \boxtimes \varphi_2)f] C = \varphi_1(f) \cdot W(C) - W(C) \cdot \varphi_2(f).
\]

**Proof** By Theorem 45

\[
[(\varphi_1 \boxtimes \varphi_2)f] C = [(\varphi_1 \boxtimes \varphi_2)[f^{[1]}]] C \\
= (\varphi_1 \boxtimes \varphi_2)((f \otimes u)w)C - (\varphi_1 \boxtimes \varphi_2)((u \otimes f)w)C,
\]

where \((f \otimes u)(\lambda, \mu) = f(\lambda), (u \otimes f)(\lambda, \mu) = f(\mu)\). From Theorems 32, 33, and 34 it follows that

\[
[(\varphi_1 \boxtimes \varphi_2)f] C = (\varphi_1(f) \boxtimes 1)[(\varphi_1 \boxtimes \varphi_2)(w)] C \\
- (1 \boxtimes \varphi_2(f))[(\varphi_1 \boxtimes \varphi_2)(w)] C \\
= (\varphi_1(f) \boxtimes 1) W(C) - (1 \boxtimes \varphi_2(f)) W(C).
\]

\[\square\]

In Corollary 63, we present a version of Theorem 62. It suggests the reverse order of operations, which enables one to apply the transformator \(W\) only once; namely, first, \(\varphi_1(f)\) and \(\varphi_2(f)\) are calculated, and then \(W(\cdot)\) is applied.

**Corollary 63** Let the extended singular sets \(\overline{\sigma}(R_{1,\cdot})\) and \(\overline{\sigma}(R_{2,\cdot})\) be disjoint. Then

\[
[(\varphi_1 \boxtimes \varphi_2)f] C = W(\varphi_1(f) \cdot C - C \cdot \varphi_2(f)).
\]
Proof. It is sufficient to note that the transformators \( u_1 \) and \( u_2 \), and \( W = (\phi_1 \otimes \phi_2)(w) \) commute, and then apply Theorem 62.

Let \( A \in \mathcal{B}(X) \) and \( B \in \mathcal{B}(Y) \). The equation

\[
AZ - ZB = C
\]

for the unknown \( Z \in \mathcal{B}(Y, X) \) with the free term \( C \in \mathcal{B}(Y, X) \) is called the (continuous) Sylvester equation [13, 73, 131]. The Sylvester equation is connected with a number of applications [5, 10, 25, 45, 75, 83, 132] and is widely discussed in the literature.

**Theorem 64** Let \( A \in \mathcal{B}(X) \) and \( B \in \mathcal{B}(Y) \). Equation (37) has a unique solution \( Z \in \mathcal{B}(Y, X) \) for all \( C \in \mathcal{B}(Y, X) \) if and only if the spectra of the operators \( A \) and \( B \) are disjoint. This solution coincides with the operator \( W(C) \).

**Proof.** By Theorem 32 and Corollary 35, the transformator \( Z \to AZ - ZB \) is equal to \( (\phi_1 \otimes \phi_2)f \), where \( f(\lambda, \mu) = \lambda - \mu \). By Theorem 39, its spectrum is equal to \( \sigma(A) - \sigma(B) \). Therefore, the transformator is invertible if and only if \( 0 \not\in \sigma(A) - \sigma(B) \). By Theorem 32, the inverse transformator is \( (\phi_1 \otimes \phi_2)w \), where \( w(\lambda, \mu) = \frac{1}{\lambda - \mu} \).

The equation

\[
Z - AZB = C
\]

is called the (discrete) Sylvester equation [73] or the Stein equation [93]. Its theory is similar to the theory of equation (37).

**Theorem 65** Let \( A \in \mathcal{B}(X) \) and \( B \in \mathcal{B}(Y) \). Equation (38) has a unique solution \( Z \in \mathcal{B}(Y, X) \) for all \( C \in \mathcal{B}(Y, X) \) if and only if the product of the spectra of the operators \( A \) and \( B \) does not contain 1. This solution coincides with the operator \( [(\phi_1 \otimes \phi_2)(s)](C) \), where

\[
s(\lambda, \mu) = \frac{1}{1 - \lambda \mu}.
\]

**Proof.** By Theorem 32 and Corollary 35, the transformator \( Z \to Z - AZB \) is equal to \( (\phi_1 \otimes \phi_2)f \), where \( f(\lambda, \mu) = 1 - \lambda \mu \). By Theorem 39, its spectrum is equal to \( 1 - \sigma(A)\sigma(B) \). Therefore, the transformator is invertible if and only if \( 1 \not\in \sigma(A)\sigma(B) \). By Theorem 32, the inverse transformator is \( (\phi_1 \otimes \phi_2)s \).

**Remark 66** Let us return to Eq. (37) and discuss the case, where \( A \) and \( B \) are unbounded operators or linear relations. The natural hypothesis is as follows: if the extended singular sets of the resolvents of \( A \) and \( B \) are disjoint (and thus \( A \) or \( B \) is a bounded operator), then Eq. (37) has a unique solution, which is determined by the transformator \( W \). The problem is: How one can interpret Eq. (37)? We discuss some variants.
First, we assume that $A$ and $B$ are linear relations with non-empty resolvent sets, and the extended spectra of $A$ and $B$ are disjoint. We assume that $C \in \mathcal{B}(Y, X)$ and a solution $Z \in \mathcal{B}(Y, X)$ of Eq. (37) is of interest.

To begin with, we show that without loss of generality one can assume that the inverse operators of $A$ and $B$ are everywhere defined bounded operators. Since the extended spectra of $A$ and $B$ are closed and disjoint, there exists $m \notin \sigma(A) \cup \sigma(B)$. We rewrite (37) in the form:

$$-vZ + AZ + vZ - ZB = C,$$

and then in the form (with the invertible coefficients $v1 - A$ and $v1 - B$)

$$-(v1 - A)Z + Z(v1 - B) = C.$$

See [60, Proposition A.1.1, p. 281] or [101, Theorem 36] for a justification of the last equality in the case of linear relations.

We consider the case, where $\infty \in \sigma(A)$. Since the relation $A$ is invertible, its range coincides with the whole of $X$, and the image of zero $\text{Im}_0A = \{ x : (0,x) \in A \}$ is zero (otherwise the left side of Eq. (37) is not an operator). Therefore, $A$ is an operator (not a relation). We call an operator $Z \in \mathcal{B}(Y, X)$, whose range is contained in the domain of the operator $A$ (otherwise the domains of the left and the right sides of Eq. (37) are different), a solution of Eq. (37) provided it satisfies the equation.

Since $\infty \notin \sigma(B)$, $B$ is a bounded linear operator, see Proposition 24. Multiplying (37) on the right by $B^{-1}$, we arrive at

$$AZB^{-1} - ZBB^{-1} = CB^{-1}.$$

According to [101, Theorem 16] we rewrite this equation in the form
\[ AZB^{-1} - Z1_{\text{Im}_0B} = CB^{-1}, \]

where \( 1_{\text{Im}_0B} = \{ (y_1, y_2) \in Y \times Y : (0, y_1 - y_2) \in B \} \). Since the kernel of the operator \( Z \) contains \( \text{Im}_0B \), the last equation can be rewritten as

\[ AZB^{-1} - Z = CB^{-1}. \]

By Theorem 65, this equation has a unique solution \( Z \) for an arbitrary \( CB^{-1} \) if

\[ 0 \notin \{ 1 - \lambda \mu : \lambda \in \sigma(A), \mu \in \sigma(B^{-1}) \} = \{ 1 - \frac{\lambda}{\mu} : \lambda \in \sigma(A), \mu \in \sigma(B) \}, \]

which is the case, because the extended spectra of \( A \) and \( B \) are disjoint. Multiplying the last equation on the right by \( B \), we obtain

\[ AZB^{-1}B - ZB = CB^{-1}B, \]

or (according to [101, Theorem 16] and \( \text{Ker}B = 0 \))

\[ AZ - ZB = C. \]

So, \( Z \) is a solution of original Eq. (37) as well.

We consider another case: let \( B \) be an invertible unbounded operator with the dense domain \( \text{Dom}B \) (in particular, \( \infty \in \sigma(B) \)). We call an operator \( Z \in \mathcal{B}(Y, X) \) a solution if

\[ AZy - ZBy = Cy \]

for all \( y \in \text{Dom}B \). Let us seek a solution of Eq. (37) in the form \( Z = VB^{-1} \), where \( V \in \mathcal{B}(Y, X) \) is the new unknown operator. Substituting \( Z = VB^{-1} \) into Eq. (37), we obtain

\[ AVB^{-1} - VB^{-1}B = C, \quad (40) \]

or

\[ AVB^{-1} - V1_{\text{Dom}_B} = C, \]

where \( 1_{\text{Dom}_B} \) is the identity operator with the domain \( \text{Dom}B \). By our definition of a solution, the last equation is equivalent to the equation

\[ AVB^{-1} - V = C. \]

Obviously, it has a unique solution \( V \). Returning to equivalent equation (40), we see that the operator \( Z = VB^{-1} \) is a solution of the original equation.

For matrices, Theorem 64 was first proved in [134]. For the case of bounded operators, an independent proof of its sufficient part was obtained in [23, 90, 121]. For a Hilbert space, a necessary and sufficient condition for the solvability of the Sylvester equation was first obtained in [29], see also [52, p. 54]. For matrices, an analogue of Theorem 65 was proved, for example, in [93].
The representation for the solution of the Sylvester equation in the form of contour integral (36) was first published in [121], see also Example 4. Estimates of the solution of the Sylvester equation are given in [48, 50, 51].

The Sylvester equation (37) with unbounded operator coefficients $A$ and $B$ was considered in [2, 3, 42, 86, 103, 116, 128].

The spectral theory of the transformator $W$ was first investigated in [121]. A generalization of the transformator $W$ corresponds to the function $w(\lambda, \mu) = \frac{1}{\lambda - \mu}$ is the inverse of the transformator $v^{[1]}(A, B)$, where $v^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2} \mu + \cdots + \mu^{n-1}$. It is discussed in [15, 44, 50].

11 The differential of the functional calculus

Let $X$ be a Banach space. The (Fréchet) differential of a nonlinear transformator $f : D(f) \subseteq B(X) \rightarrow B(X)$ at a point $A \in D(f)$ is defined to be a linear transformator $df(\cdot, A) : B(X) \rightarrow B(X)$ depending on the parameter $A$ that possesses the property

$$f(A + \Delta A) = f(A) + df(\Delta A, A) + o(\|\Delta A\|).$$

(41)

It is assumed that the domain $D(f)$ of the transformator $f$ contains a neighborhood of $A$. We recall standard properties of the differential.

**Proposition 67** ([4, § 2.2.2], [35, 8.2.1]) Let a transformator $g : B(X) \rightarrow B(X)$ be differentiable at a point $A \in B(X)$ and a transformator $f : B(X) \rightarrow B(X)$ be differentiable at the point $g(A) \in B(X)$. Then the composition $f \circ g$ is differentiable at the point $A$, and

$$d(f \circ g)(\cdot, A) = df[dg(\cdot, A), g(A)].$$

**Corollary 68** ([4, § 2.3.4], [35, 8.2.3]) Let a transformator $f : B(X) \rightarrow B(X)$ be continuously differentiable (i.e., $df(\cdot, A)$ exists and depends on $A$ continuously in norm) in a neighborhood of a point $A \in B(X)$ and let the transformator $df(\cdot, A)$ be invertible. Then the inverse transformator of $f$ is defined and differentiable in a neighborhood of the point $B = f(A)$, and the differential of the inverse transformator is equal to the inverse of the original differential:

$$df^{-1}(\cdot, B) = [df(\cdot, A)]^{-1}.$$
\[
\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - (A + \Delta A))^{-1} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1} d\lambda + df(\Delta A, A) + o(\|\Delta A\|).
\]

We note that
\[
(\lambda \mathbf{1} - (A + \Delta A))^{-1} = (\lambda \mathbf{1} - A - \Delta A)^{-1} = R_{\lambda}(1 - \Delta A \cdot R_{\lambda})^{-1} = (1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}.
\]

Based on this formula, we adopt the following definition.

Let \( R_{\lambda} \) be a pseudo-resolvent in the algebra \( B(X) \). We call the \textit{perturbation of} \( R_{\lambda} \) \textit{by an operator} \( \Delta A \in B(X) \) the function
\[
T_{\lambda} = R_{\lambda}(1 - \Delta A \cdot R_{\lambda})^{-1} = (1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}.
\]

**Proposition 69** For any perturbation \( \Delta A \in B(X) \) the function
\[
T_{\lambda} = R_{\lambda}(1 - \Delta A \cdot R_{\lambda})^{-1} = (1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}
\]
is a pseudo-resolvent.

**Proof** We verify the Hilbert identity for all \( \lambda \) and \( \mu \) such that \( T_{\lambda} \) and \( T_{\mu} \) are defined. We have
\[
T_{\lambda} - T_{\mu} + (\lambda - \mu)T_{\lambda}T_{\mu} = (1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} - R_{\mu}(1 - \Delta A \cdot R_{\mu})^{-1} \\
+ (\lambda - \mu)(1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} R_{\mu}(1 - \Delta A \cdot R_{\mu})^{-1} \\
= (1 - R_{\lambda} \cdot \Delta A)^{-1} \left[ R_{\lambda}(1 - \Delta A \cdot R_{\mu}) - (1 - R_{\lambda} \cdot \Delta A) R_{\mu} + (\lambda - \mu) R_{\lambda} R_{\mu} \right] \\
\times (1 - \Delta A \cdot R_{\mu})^{-1} \\
= (1 - R_{\lambda} \cdot \Delta A)^{-1} \left[ R_{\lambda} - R_{\mu} + (\lambda - \mu) R_{\lambda} R_{\mu} \right] \\
\times (1 - \Delta A \cdot R_{\mu})^{-1} = 0.
\]

\( \square \)

**Remark 70** We note briefly an additional reasoning in favor of definition (43). Let \( R_{\lambda} \) be the resolvent of a linear relation \( A \), i.e., \( R_{\lambda} = (\lambda \mathbf{1} - A)^{-1} \). We show that
\[
(1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} = (\lambda \mathbf{1} - A - \Delta A)^{-1}.
\]

Obviously (for details, see [60, Proposition A.1.1] or [101, Proposition 12]):
\[
(1 - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} = R_{\lambda}^{-1}(1 - R_{\lambda} \cdot \Delta A)^{-1} = [(\lambda \mathbf{1} - A)(1 - R_{\lambda} \cdot \Delta A)]^{-1}.
\]

Furthermore, since the image of the operator \( R_{\lambda} \cdot \Delta A \) is contained in the image of \( R_{\lambda} \), which is equal to the domain of the relation \( A \), by virtue of [101, Theorem 36(a)], we can develop the internal parentheses:
\[
\left[ \lambda I - A - (\lambda I - A)R_{\lambda} \cdot \Delta A \right]^{-1} = \left[ \lambda I - A - (\lambda I - A)(\lambda I - A)^{-1} \cdot \Delta A \right]^{-1}.
\]

We note that

\[
(\lambda I - A)(\lambda I - A)^{-1}
\]

is equal to the linear relation:

\[
1 \times 1 = 1 \times \text{Im}_{\Omega}A = \{(x_1, x_2) \in X \times X : (0, x_1 - x_2) \in A \}.
\]

Obviously,

\[
(\lambda I - A - 1 \times \text{Im}_{\Omega}A \cdot \Delta A)^{-1} = (\lambda I - A - \Delta A)^{-1}.
\]

Let \( R_{(.)} \) be a pseudo-resolvent in the algebra \( B(X) \). We define the differential \( df(\cdot, R_{(.)}) \) of the mapping (which is a generalization of (42))

\[
R_{(.)} \rightarrow f(R_{(.)}) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} \, d\lambda + \delta f(\infty) 1,
\]

(44)

by means of the formula

\[
\frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda}(1 - \Delta A \cdot R_{\lambda})^{-1} \, d\lambda + \delta f(\infty) 1
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} \, d\lambda + \delta f(\infty) 1 + df(\Delta A, R_{(.)}) + o(\|\Delta A\|).
\]

Theorem 71 Let \( R_{(.)} \) be a pseudo-resolvent in the algebra \( B(X) \), and \( f \in O(\sigma(R_{(.)})) \). Then the differential of mapping (44) possesses the representation

\[
df(\Delta A, R_{(.)}) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} \Delta A R_{\lambda} \, d\lambda.
\]

In other words,

\[
df(\cdot, R_{(.)}) = (\varphi \Box \varphi)(f),
\]

where \( \varphi \) is the functional calculus generated by the pseudo-resolvent \( R_{(.)} \).

Proof We assume that

\[
\|\Delta A\| \cdot \|R_{\lambda}\| < 1, \quad \lambda \in \Gamma.
\]

By Theorem 1, we have

\[
\left\| \left[ R_{\lambda} \cdot (1 - \Delta A \cdot R_{\lambda})^{-1} - R_{\lambda} \cdot \Delta A \cdot R_{\lambda} \right] \right\| \leq \frac{\|R_{\lambda}\|^3 \cdot \|\Delta A\|^2}{1 - \|R_{\lambda}\| \cdot \|\Delta A\|}.
\]

Therefore
Proposition 72 (\cite{14, Theorem 2.1}) Let $A \in \mathcal{B}(X)$ and $f \in \mathcal{O}(\sigma(A))$. Then
\[
df(A \Delta A - \Delta A A, A) = \varphi_A(f) \Delta A - \Delta A \varphi_A(f),
\]
where the functional calculus $\varphi_A$ is generated by the operator $A$.

**Proof** The proof follows from Theorems 46 and 71.

Proposition 73 (\cite{68, Theorem 3.3}) Let $g, h \in \mathcal{O}(\sigma(R(\chi)))$. Then
\[
d(gh)(\Delta A, R(\chi)) = dg(\Delta A, R(\chi)) h(R(\chi)) + g(R(\chi)) dh(\Delta A, R(\chi)),
\]
where $(gh)(\lambda) = g(\lambda) h(\lambda)$.

**Proof** The proof follows from Corollary 49.

Corollary 74 (\cite{122, Theorem 10.36}, see also \cite{120}) Let $f \in \mathcal{O}(\sigma(R(\chi)))$. We assume that $\Delta A$ commutes with at least one value $R_\mu$ of the pseudo-resolvent $R(\chi)$. Then
\[
df(\Delta A, R(\chi)) = \varphi(f') \Delta A = \Delta A \varphi(f').
\]

**Proof** The proof follows from Proposition 51.

Theorem 75 Let $R(\chi)$ be a pseudo-resolvent in the algebra $\mathcal{B}(X)$ and $f \in \mathcal{O}(\sigma(R(\chi)))$. Then the differential of mapping (44) admits the representation
\[
df(\Delta A, R(\chi)) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu) R_\lambda \Delta A R_\mu \, d\mu \, d\lambda,
\]
where the divided difference $f^{[1]}$ is defined by formula (28), the contours $\Gamma_1$ and $\Gamma_2$ are oriented envelopes of the extended singular set $\sigma(R(\chi))$ with respect to the complement $\mathbb{T} \setminus U$, and $U$ is the domain of the function $f$.

The spectrum of the transformator $df(\cdot, R(\chi)) : \mathcal{B}(X) \to \mathcal{B}(X)$ is given by the formula
\[
\sigma[df(\cdot, R(\cdot))] = \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(R(\cdot)) \}. \tag{46}
\]

**Proof** Theorem 71 shows that the differential \( df(\Delta A, R(\cdot)) \) is the operator \( [(\phi \Box \phi)(f)] \Delta A \). By Theorem 45,

\[
[(\phi \Box \phi)(f)] \Delta A = [(\phi \Box \phi)f^{[1]}] \Delta A.
\]

It remains to apply Theorem 32.

Formula (46) follows from Theorem 39. \( \square \)

**Example 6** Let \( A \in \mathcal{B}(X) \). The following corollaries are consequences of Example 5 and Theorem 75.

The differential of the transformator \( v_2(A) = A^2 \) is given by the formula:

\[
dv_2(\Delta A, A) = A \cdot \Delta A + \Delta A \cdot A, \tag{47}
\]

and its spectrum at a point \( A \in \mathcal{B}(X) \) is equal to

\[
\sigma[dv_2(\cdot, A)] = \{ \lambda + \mu : \lambda, \mu \in \sigma(A) \}. \nonumber
\]

The differential of the mapping \( r_1(R(\cdot)) = R_{\lambda_0} \) is given by the formula:

\[
dr_1(\Delta A, R(\cdot)) = R_{\lambda_0} \cdot \Delta A \cdot R_{\lambda_0}, \nonumber
\]

and its spectrum at a point \( R(\cdot) \) is equal to

\[
\sigma[dr_1(\cdot, R(\cdot))] = \left\{ \frac{1}{(\lambda_0 - \lambda)(\lambda_0 - \mu)} : \lambda, \mu \in \sigma(R(\cdot)) \right\}. \nonumber
\]

The differential of the transformator \( \exp(t)(A) = e^{At} \) is given by the formula:

\[
d\exp(t)(\Delta A, A) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{n} A^{n-i} \Delta A A^i, \nonumber
\]

and its spectrum at a point \( A \in \mathcal{B}(X) \) is equal to

\[
\sigma[d\exp(t)(\cdot, A)] = \{ \exp^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}. \nonumber
\]

The differential of the transformator \( \exp^{(1)}(t)(A) = A e^{At} \) is given by the formula:

\[
d\exp^{(1)}(t)(\Delta A, A) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{n} A^{n-i} \Delta A A^i, \nonumber
\]

and its spectrum at a point \( A \in \mathcal{B}(X) \) is equal to

\[
\sigma[d\exp^{(1)}(t)(\cdot, A)] = \{ \exp^{(1)[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}. \nonumber
\]
We note special formulae for the differentials of the transformators $\exp_t(A) = e^{At}$ and $\exp_t^{(1)}(A) = Ae^{At}$.

**Proposition 76** The differentials of the transformators $\exp_t(A) = e^{At}$ and $\exp_t^{(1)}(A) = Ae^{At}$ at a point $A \in \mathbf{B}(X)$ can be calculated by means of the formulae

$$d \exp_t(\Delta A, A) = \int_0^t e^{(t-s)\Delta}Ae^{sA} \, ds = \int_0^t \exp_{t-s}(A)\Delta \exp_s(A) \, ds,$$

(48)

$$d \exp_t^{(1)}(\Delta A, A) = \exp_t(A)\Delta A + \int_0^t \exp_{t-s}(A)\Delta A \exp_s(A) \, ds.$$  

(49)

**Proof** For $\lambda \neq \mu$, we have

$$\frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu} = 1 - \frac{e^{\lambda s}e^{\mu(t-s)}}{\lambda - \mu} \bigg|_{s=0}^t = 1 - \frac{1}{\lambda - \mu} \int_0^t \frac{d}{ds} \left[ e^{\lambda s}e^{\mu(t-s)} \right] \, ds =$$

$$= \int_0^t e^{\lambda s}e^{\mu(t-s)} \, ds.$$

By continuity, the same representation of $\exp_t^{[1]}(\lambda, \mu)$ holds for all finite $\lambda$ and $\mu$. Hence, from Theorems 75, 32, and 39 it follows (48). Formula (49) follows from (34). $\square$

An integral representation of the differential of Green’s function of the bounded solutions problem for the equation $x'(t) = Ax(t) = f(t)$, similar to (48), was established in [98].

The differentials of inverse functions are defined by the inverse transformators, see Corollary 68. To calculate them and their spectra, one can use the following theorem.

**Theorem 77** Let $R_{A,(\cdot)}$ be the resolvent of an operator $A \in \mathbf{B}(X), \ f \in \mathbf{O}(\sigma(A))$, and $0 \notin \bigcup \{f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A)\}$. Then the differential of the transformator $B \mapsto f^{-1}(B)$ at the point $B = f(A)$ is given by the formula

$$df^{-1}(\Delta B, B) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{f^{[1]}(\lambda, \mu)} R_{A,\lambda} \Delta B R_{A,\mu} d\mu d\lambda$$

(50)

(note that the right-hand side is expressed in terms of $A$, not $B$), where the contours $\Gamma_1$ and $\Gamma_2$ are oriented envelopes of the spectrum $\sigma(A)$ with respect to the point $\infty$ and the complement of the domain of the function $f$.

The spectrum of the transformator $df^{-1}(\cdot, B) : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$ is given by the formula

$$\sigma[df^{-1}(\cdot, B)] = \bigcup \left\{ \frac{1}{f^{[1]}(\lambda, \mu)} : \lambda, \mu \in \sigma(A) \right\}.$$
Proof  By Theorem 75,
\[ df(\Delta A, A) = \frac{1}{(2\pi i)^3} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu) R_{A, \lambda} \Delta A R_{A, \mu} d\mu d\lambda, \]
\[ \sigma[\nu(\cdot, A) - \sigma(A)] = \left\{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \right\}. \]
Since \( 0 \not\in \bigcup \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \} \), the transformator \( df(\cdot, A) \) is invertible. By Corollary 68, the inverse transformator is the differential of the mapping \( B \mapsto f^{-1}(B) \), which is the inverse mapping of \( A \mapsto f(A) \). By Theorem 32, the inverse transformator is given by formula (50). By Theorem 39, its spectrum is given by formula (51).

Example 7  Let the spectrum of an operator \( B \in \mathcal{B}(X) \) be contained in \( \mathbb{C}\backslash(-\infty, 0] \). The following corollaries are consequences of Example 6 and Theorem 77.

From (47) it is clear that the differential \( dv_{1/2}(\Delta B, B) \) of the transformator \( v_{1/2}(B) = \sqrt{B} \) satisfies the continuous Sylvester equation:
\[ \sqrt{B} \cdot dv_{1/2}(\Delta B, B) + dv_{1/2}(\Delta B, B) \sqrt{B} = \Delta B. \]
Therefore
\[ dv_{1/2}(\Delta B, B) = W_{\sqrt{B}, -\sqrt{B}}(\Delta B), \]
where the transformator \( W_{\sqrt{B}, -\sqrt{B}} \) is constructed by means of the functional calculus generated by the operators \( \sqrt{B} \) and \( -\sqrt{B} \). The spectrum of the differential of the transformator \( v_{1/2}(B) = \sqrt{B} \) at the point \( B \) is equal to
\[ \sigma[dv_{1/2}(\cdot, B)] = \left\{ \frac{1}{\lambda + \mu} : \lambda, \mu \in \sqrt{\sigma(B)} \right\}. \]
The spectrum of the differential of the transformator \( f : B \mapsto \log B \), where \( \log \) denotes the principal value, at the point \( B \) is equal to
\[ \sigma[\nu(\cdot, B)] = \left\{ \frac{1}{\exp^{[1]}(\lambda, \mu)} : \lambda, \mu \in \log(\sigma(B)) \right\}. \]

Remark 78  The equation
\[ AZ + ZB + ZCZ + D = 0 \quad (52) \]
with \( A, B, C, D \in \mathcal{B}(X) \) and unknown \( Z \in \mathcal{B}(X) \) is called [73, 86, 87] the Riccati equation. It arises in control theory [5, 115]. The differential \( dZ = dZ(\Delta A, \Delta B, \Delta C, \Delta D; Z) \) of the solution \( Z \) of Riccati equation (52) satisfies [73, p. 135] the continuous Sylvester equation.
\[(A + ZC)dZ + dZ(B + CZ) = -\Delta D - \Delta A Z - Z \Delta B - Z \Delta C Z.\]

So,
\[
dZ(\Delta A, \Delta B, \Delta C, \Delta D; Z) = W_{A+ZC, -B-CZ}(-\Delta D - \Delta A Z - Z \Delta B - Z \Delta C Z).
\]

For pseudo-resolvents generated by bounded operators, Theorem 71 was proved in [122, Theorem 10.38], see also [14, formula (2.3)], [24] and [133]. For matrices, representation (45) was given in [93, Theorem 5.1]. For matrices, formula (46) was proved in [68, Theorem 3.9], its weaker version was previously obtained in [81, Lemma 2.1]. Formula (48) was first obtained in [78, formula (1.8)], see also [10, ch. 10, § 14], [68, formula (10.15)], [81, example 2], [109, 111, 138]. Differentials connected with some specific functions \(f\) were investigated in [1, 32, 68, 69, 81, 82, 138, 143]; special attention was paid to estimates of their norms which helps to find the condition number of the transformator \(A \mapsto f(A)\). Properties of differentials of higher orders were investigated in [14, 106, 111].

Acknowledgements
V. G. Kurbatov (corresponding author) was supported by the Russian Scientific Foundation under Grant no. 19-12-00095. I. V. Kurbatova was supported by the Russian Foundation for Basic Research under research project No. 19-01-00732 A. M. N. Oreshina was supported by the Russian Foundation for Basic Research and Lipetsk region under research project no. 19-48-480009-r_a.

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