Dynamical Behavior of SEIR-SVS Epidemic Models with Nonlinear Incidence and Vaccination

Xiao-mei FENG$^{1,2,\dagger}$, Li-li LIU$^{3,4}$, Feng-qin ZHANG$^1$

$^1$School of Mathematics and Informational Technology, Yuncheng University, Yuncheng 044000, China
(E-mail: xiaomei...0529@126.com)

$^2$School of Mathematics and Informational Sciences, Shaanxi Normal University, Xi’an 710062, China

$^3$Complex Systems Research Center, Shanxi University, Taiyuan 030006, China

$^4$Shanxi Key Laboratory of Mathematical Techniques and Big Data Analysis on Disease Control and Prevention, Shanxi University, Taiyuan 030006, China

Abstract For some infectious diseases such as mumps, HBV, there is evidence showing that vaccinated individuals always lose their immunity at different rates depending on the inoculation time. In this paper, we propose an age-structured epidemic model using a step function to describe the rate at which vaccinated individuals lose immunity and reduce the age-structured epidemic model to the delay differential model. For the age-structured model, we consider the positivity, boundedness, and compactness of the semiflow and study global stability of equilibria by constructing appropriate Lyapunov functionals. Moreover, for the reduced delay differential equation model, we study the existence of the endemic equilibrium and prove the global stability of equilibria. Finally, some numerical simulations are provided to support our theoretical results and a brief discussion is given.

Keywords vaccination; age-structured epidemic model; delay differential equation model; stability; Lyapunov functional

2000 MR Subject Classification 34G20; 34K20; 35B40; 35B65; 35B35

1 Introduction

Mathematical models have been used extensively to study the transmission dynamics of infectious diseases and design or evaluate control measures on the spread of diseases. It is well known that vaccination is an effective strategy against infectious diseases. Most epidemic models are compartmental models described by systems of ordinary differential equations (ODEs). It is generally assumed that individuals from a compartment move to another compartment are homogeneous in ODE models. For instance, vaccinated individuals have the same immunity periods regardless of individual heterogeneity. However, it is pointed out that for some infectious diseases such as mumps vaccinated individuals always lose their immunity at different rates depending on the inoculation time$^3$. Therefore, in modeling, it should be more reasonable to introduce vaccination age (i.e., the time individuals spend in the vaccinated class) to describe different immunity waning rates. If we introduce the vaccination age, it will be formulated as an age-dependent epidemic model.

Manuscript received February 1, 2021. Accepted on September 30, 2021.
This paper is supported by The National Natural Science Foundation of China [12026236, 12026222, 12061079, 11601293, 12071418], Science and Technology Activities Priority Program for Overseas Researchers in Shanxi Province [20210049], The Natural Science Foundation of Shanxi Province [201901D211160, 201901D211461, 201901D111295].

$^\dagger$Corresponding author.
The earlier age-structured epidemic models were proposed by Kermack and McKendrick\cite{18, 19} and the mathematical theory was developed in the 80-90’s of last century\cite{16, 34}. Since then, many research studies about age-structured epidemic models (such as infection age\cite{4, 14, 26, 27, 36}, latency age\cite{8, 33}, vaccination age\cite{17, 22, 30}, relapse age\cite{23} et al.) have been published. On the other hand, some age structure models are established to study pest population control if there is no effective vaccine\cite{15, 25}. Here, we just list some recent works. Iannelli, Martcheva and Li\cite{17} proposed a two-strain epidemic model with vaccination age in which the first strain can infect individuals already infected by the second one. Li, Wang and Ghosh\cite{22} considered a two-dimensional SIS model with vaccination and the rate at which vaccine loses its protective properties with time depends upon vaccination age. Shen and Xiao\cite{30} studied a multi-group SVEIR epidemiological model with the vaccination age and infection age. These results suggest that vaccination is helpful for disease control by decreasing the basic reproduction number, either enhancing the vaccination rate or lengthening the duration of vaccination protection.

It is pointed out that the transmission of an infectious disease depends on both the population behavior and the infectivity of the disease\cite{9}. Mathematically, it is described by the incidence rate of the disease (i.e., the average number of new cases of the disease per unit time). Although bilinear and standard incidence rates are frequently used in classical epidemic models, its nonlinearity was used as the form $Sg(I)$ at disease modelling in \cite{2}, where $g(I)$ is a nonlinear function. Some other forms of nonlinear incidence rates were proposed and studied\cite{29, 35}.

In this paper, we focus on a SEIR-SVS epidemic model with vaccination age and nonlinear incidence of the form $Sg(I)$. Moreover, if we describe the rate at which vaccinated individuals lose immunity by a step function, the age-structure model reduces to a delay differential equation (DDE) model. Our goal is to construct appropriate Lyapunov functionals to show that the so-called basic reproduction number provides the threshold to determine the global dynamics of the age-structured and DDE models.

The paper is organized as follows. In section 2, the age structured and DDE models are formulated. In section 3, for the vaccine-age-structured model, the positivity and boundedness of solutions, the asymptotic smoothness and existence of global attractors of the semi-flows generated by the model, and the existence of equilibria are studied. In sections 4 and 5, by constructing suitable Lyapunov functions, the global asymptotic stability of the age-structured and DDE models are established. In section 6, some numerical simulations on the general age-structured model are carried out. The paper ends with a brief discussion in section 7.

2 Formulation of the Models

The population is divided into four classes, the susceptible $S(t)$, the exposed $E(t)$, the infectious $I(t)$, the removed $R(t)$, and the vaccinated $v(a,t)$ at time $t$ with vaccine-age $a$. The vaccine-age-structured epidemic model with nonlinear incidence is as follows

\[
\frac{dS(t)}{dt} = A - (\mu + p)S(t) - S(t)g(I(t)) + \int_0^\infty \delta(a)v(a,t)da, \quad t > 0,
\]

\[
\frac{dE(t)}{dt} = S(t)g(I(t)) - (\mu + \varepsilon)E(t), \quad t > 0,
\]

\[
\frac{dI(t)}{dt} = \varepsilon E(t) - (\mu + \sigma + \omega)I(t), \quad t > 0,
\]

\[
\frac{dR(t)}{dt} = \omega I(t) - \mu R(t), \quad t > 0,
\]

\[
\frac{\partial v(a,t)}{\partial t} + \frac{\partial v(a,t)}{\partial a} = -[\mu + \delta(a)]v(a,t), \quad t > 0, \quad a > 0,
\]

(2.1)
with the boundary condition
\[ v(0, t) = pS(t), \quad (2.2) \]
and initial conditions
\[ S(0) = S_0 > 0, \quad E(0) = E_0 > 0, \quad I(0) = I_0 > 0, \quad R(0) = R_0, \quad v(a, 0) = v_0(a) \in L^1_+(0, \infty). \quad (2.3) \]
Here, all parameters are assumed to be positive and their biological interpretations are listed in Table 1.1.

| Parameters | Interpretations |
|------------|-----------------|
| \( A \)    | the recruitment rate of the susceptible class |
| \( \mu \)  | the per capita natural death rate |
| \( p \)    | the vaccination rate coefficient for the susceptible class |
| \( \varepsilon \) | the conversional rate from the latent class |
| \( \sigma \) | the additional death rate induced by the infectious diseases |
| \( \omega \) | the recovery rate from the infectious class |
| \( \delta(a) \) | the rate of per vaccinated capita with vaccine-age \( a \) losing immunity and returning to the susceptible class |

The function \( g(I) \) satisfies the following assumptions:

(H) \( g(0) = 0, \ g'(I) > 0, \ g''(I) \leq 0 \) for \( I > 0 \).

Since it is impossible that the age of the vaccinated individuals is infinite, the reasonable condition for \( v(a, t) \) is
\[ v(\infty, 0) = v_0(\infty) = 0. \quad (2.4) \]
By the continuity of solutions of (2.1), it is necessary to require that \( pS_0 = v_0(0) \).

Notice that the variable \( R(t) \) does not appear in other equations. Then we may only consider the following reduced system
\[
\begin{align*}
\frac{dS(t)}{dt} &= A - (\mu + p)S(t) - S(t)g(I) + \int_0^\infty \delta(a)v(a, t)da, \quad t > 0, \\
\frac{dE(t)}{dt} &= S(t)g(I(t)) - (\mu + \varepsilon)E(t), \quad t > 0, \\
\frac{dI(t)}{dt} &= \varepsilon E(t) - (\mu + \alpha)I(t), \quad t > 0, \\
\frac{\partial v(a, t)}{\partial t} + \frac{\partial v(a, t)}{\partial a} &= -[\mu + \delta(a)]v(a, t), \quad t > 0, \quad a > 0,
\end{align*}
\]
with the boundary condition
\[ v(0, t) = pS(t), \]
and the initial conditions
\[ S(0) = S_0 > 0, \quad E(0) = E_0 > 0, \quad I(0) = I_0 > 0, \quad v(a, 0) = v_0(a) \in L^1_+(0, \infty). \]
where \( \alpha = \sigma + \omega \).

Along the characteristic line \( t - a = \text{constant} \), integrating the fourth equation of (2.5) with the boundary and initial conditions gives
\[
\begin{align*}
v(a, t) &= \begin{cases} 
pS(t - a)e^{-\int_0^a [\mu + \delta(\xi)]d\xi}, & 0 < a \leq t, \\
v_0(a - t)e^{-\int_{a-t}^0 [\mu + \delta(\xi)]d\xi}, & a \geq t > 0.
\end{cases} \quad (2.6)
\end{align*}
\]
From (2.4) and (2.6) we have

$$v(\infty, t) = 0.$$  \hspace{1cm} (2.7)

Indeed, a vaccinated individual can have the perfect immunity against the infection within a period following vaccination. After the period the vaccine wanes gradually, the vaccinated individual loses the immunity finally, and becomes susceptible. For example, the clinic experiments verified that the neutralizing-antibody response to live SARS-CoV-2 virus was maintained up to 180 days after vaccination and it will be weakened gradually\[1\]. Based on this fact, the vaccinated class can be divided into two subclasses, which are with perfect and weak immunity, respectively. In order to describe this situation, we choose the losing rate of immunity as the following step function:

$$\delta(a) = \begin{cases} 0, & \text{for } a \leq \tau, \\ \gamma, & \text{for } a > \tau, \end{cases}$$

where \(\tau\) is the period of the perfect immunity, \(\gamma\) is the rate at which vaccine wanes.

Denote

$$V(t) = \int_{\tau}^{\infty} v(a, t)da.$$ 

Then it is the number of vaccinated individuals with weak immunity at time \(t\). From the third equation of (2.5) we have

$$\frac{dV}{dt} = \int_{\tau}^{\infty} \frac{\partial v(a, t)}{\partial a} da - \int_{\tau}^{\infty} [\mu + \delta(a)] v(a, t)da $$

$$= v(\tau, t) - (\mu + \gamma)V,$$

where \(v(\infty, t) = 0\) is used. It follows from (2.6) that \(v(\tau, t) = pS(t - \tau)e^{-\mu t}\) for \(t \geq \tau\) Then system (2.5) will become as follows

$$\frac{dS}{dt} = A - (\mu + p)S - Sg(I) + \gamma V,$$

$$\frac{dE}{dt} = Sg(I) - (\mu + \epsilon)E,$$

$$\frac{dI}{dt} = \epsilon E - (\mu + \alpha)I,$$

$$\frac{dV}{dt} = pS(t - \tau)e^{-\mu t} - (\mu + \gamma)V.$$ \hspace{1cm} (2.8)

The initial conditions for model (2.8) take the form

$$S(\theta) = \varphi(\theta), \hspace{0.5cm} E(0) = E_0 > 0, \hspace{0.5cm} I(0) = I_0 > 0, \hspace{0.5cm} V(0) = V_0 > 0,$$

$$\varphi(\theta) \geq 0, \hspace{0.5cm} \theta \in [-\tau, 0], \hspace{0.5cm} \varphi(0) > 0,$$ \hspace{1cm} (2.9)

where \((\varphi(\theta), E_0, I_0, V_0) \in (C[-\tau, 0] \times \mathbb{R}_+^3, \mathbb{R}_+^1),\) and \(\mathbb{R}_+^n = \{(x_1, x_2, \cdots, x_n) : x_i \geq 0, i = 1, 2, \cdots, n\}\).

In the following we will investigate dynamic behaviors of system (2.5) with conditions (2.2) and (2.3) and system (2.8) with condition (2.9).

### 3 Global Analysis of the Age-structured Model (2.5)

In this section, we consider the positivity and boundedness of solutions of the age-structured model (2.5) with conditions (2.2) and (2.3), the existence of the endemic equilibrium, the asymptotic smoothness of the semiflow, and prove the global stability of the disease-free and endemic equilibria of (2.5) by constructing appropriate Lyapunov functionals.
3.1 Positivity and Boundedness

This subsection devotes to the preliminaries, including positivity and boundedness.

The function $\delta(a)$ satisfies the following assumptions:

(F1) $\delta(a) \in L^1_+$ with essential infimum and supremum, i.e., $\delta^{\inf}$ and $\delta^{\sup}$;

(F2) $\delta(a)$ is Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz coefficient $M_3$.

The state space is $X = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times L^1_+(0, \infty)$ with the norm $\| (x_1, x_2, x_3, x_4) \|_X = x_1 + x_2 + x_3 + \int_0^{\infty} x_4(a)da$ for any point $(x_1, x_2, x_3, x_4) \in X$. The standard theory implies that model (2.5) always has a unique nonnegative solution. Furthermore, $X$ is positively invariant and model (2.5) generates a continuous semiflow $\Phi : \mathbb{R}^+ \times X \rightarrow X$.

Suppose that $x(t)$ is a solution with the initial condition $x(0) = x_0 \in X$. Then $x(t) = \Phi_t(x_0) = \Phi(t, x_0) = (S, E, I, v(t, \cdot))$. The total population $N(t)$ is given by $N(t) = \| x(t) \|_X = S + E + I + \int_0^{\infty} v(t, a)da$.

Let $N^* = A/\mu$. Then the semi-flow $\Phi$ has the following property.

**Proposition 3.1.** Let $x_0 \in X$, then $\Phi$ is point dissipative, i.e., there is a bounded set that attracts all points in $X$.

**Proof.** Note that

$$\frac{d}{dt} \| \Phi_t(x_0) \|_X = \frac{d}{dt} N(t) = \frac{d}{dt} S(t) + \frac{d}{dt} E(t) + \frac{d}{dt} I(t) + \frac{\partial}{\partial t} \int_0^{\infty} v(a, t)da.$$ 

From model (2.5), we have

$$\frac{d}{dt} N(t) = A - \mu(S + E + I) - \int_0^{\infty} \left[ \mu + \delta(a)v(a, t) + \frac{\partial}{\partial t} v(a, t) \right]da$$

$$\leq A - \mu \left( S + E + I + \int_0^{\infty} v(a, t)da \right)$$

$$= A - \mu N(t).$$

Using the variation of constant formula, one has

$$\frac{d}{dt} N(t) \leq \frac{A}{\mu} - e^{-\mu t} \left( \frac{A}{\mu} - \| x_0 \|_X \right).$$

Thus, $\limsup_{t \to \infty} N(t) \leq A/\mu$, which implies that $\Phi$ is point dissipative. This completes the proof. \qed

As a direct consequence of Proposition 3.1, we have the following results.

**Proposition 3.2.** There exists a constant $C$ satisfying $C \geq \max\{\| x_0 \|_X, A/\mu \}$ such that $S(t), E(t), I(t), \int_0^{\infty} v(t, a)da \leq C$ for all $t \geq 0$.

The next proposition provides a positive asymptotic lower bound for model (2.5).

**Proposition 3.3.** Let $x_0 \in X$. Then

$$\liminf_{t \to \infty} S(t) \geq m_S, \quad \liminf_{t \to \infty} E(t) \geq m_E, \quad \liminf_{t \to \infty} I(t) \geq m_I.$$

**Proof.** It follows from the first equation of model (2.5) that

$$\frac{d}{dt} S(t) = A - (\mu + p)S(t) - S(t)g(I) + \int_0^{\infty} \delta(a)v(a, t)da.$$
\[ \geq A - (\mu + p)S(t) - S(t)g'(0), \]

which implies that

\[ \liminf_{t \to \infty} S(t) \geq \frac{A}{\mu + p + g'(0)} \equiv m_S. \] (3.1)

Next, using the second equation of model (2.5) and (3.1), one has

\[ \frac{d}{dt} E(t) \geq g'(0)m_S - (\mu + \epsilon)E. \]

Obviously, it has

\[ \liminf_{t \to \infty} E(t) \geq \frac{m_Sg'(0)}{\mu + \epsilon} \equiv m_E. \] (3.2)

Then, combining the third equation of model (2.5) and (3.2) yields that

\[ \frac{d}{dt} I(t) \geq \epsilon m_E - (\mu + \alpha)I. \]

This implies that

\[ \liminf_{t \to \infty} I(t) \geq \frac{m_E\epsilon}{\mu + \alpha} \equiv m_I. \] (3.3)

The proof is finished.

\[ \square \]

**Proposition 3.4.** There exist two constants \( T \) and \( \epsilon \) such that \( v(0, t) > \epsilon \) for \( t \geq T \).

**Proof.** It follows from \( v(0, t) = pS(t) \) and (3.1) that there exists a \( T \) such that

\[ v(0, t) \geq pm_S \equiv \epsilon \]

holds. This completes the proof. \[ \square \]

### 3.2 Asymptotic Smoothness and Existence of Global Attractors

In this subsection, the asymptotic smoothness and existence of global attractors of model (2.5) are investigated. The main result is shown by using the following result, which comes from Lemma 3.2.3 in [13].

**Lemma 3.5.** The semiflow \( \Phi_t(x_0) = \Phi^1_t(x_0) + \Phi^2_t(x_0) : \mathbb{R}^+ \times X \to X \) is asymptotically smooth if \( \Phi^1_t(x_0) \) and \( \Phi^2_t(x_0) \) satisfy the following conditions:

(i) There exists a continuous function \( f(t, h) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( \lim_{t \to \infty} f(t, h) = 0 \) and \( \|\Phi^1_t(x_0)\|_X \leq f(t, h) \) if \( \|x_0\|_X \leq h \);

(ii) There exists \( T_b \geq 0 \) such that \( \Phi^2_t(B) \) has compact closure for each \( t \geq T_b \) and any bounded closed set \( B \subset X \).

By using Lemma 3.5, we have the following theorem.

**Theorem 3.6.** The semiflow \( \Phi \) is asymptotically smooth.
Proof. According to Lemma 3.5, we define two maps $\Phi_1^1(x_0)$ and $\Phi_2^1(x_0)$ on $X$ satisfying $\Phi_t(x_0) = \Phi_1^1(x_0) + \Phi_2^1(x_0)$ by $\Phi_1^1(x_0) = (0, 0, 0, v_1(t, t))$ and $\Phi_2^1(x_0) = (S, E, I, v_2(t, t))$, where

$$v_1(a, t) := \begin{cases} 0, & t > a \geq 0; \\ v(a, t), & a \geq t \geq 0 \end{cases} \quad \text{and} \quad v_2(a, t) := \begin{cases} v(a, t), & t > a \geq 0; \\ 0, & a \geq t \geq 0. \end{cases}$$

**Step 1.** To show that condition (i) of Lemma 3.5 holds. For $h > 0$, let $f(t, h) = e^{-(\mu + \delta \inf)t}$. Obviously, $\lim_{t \to \infty} f(t, h) = 0$. Now, we show that $\|\Phi_1^1(x_0)\|_X \leq f(t, h)$ for $\|x_0\|_X \leq h$.

It follows from the definition of $\Phi_1^1(x_0)$ and $\mu + \delta(a) \geq \mu + \delta \inf$ that

$$\|\Phi_1^1(x_0)\|_X = 0 + 0 + \int_0^\infty v_1(a, t)da$$

$$= \int_t^\infty v_0(a - t)e^{-\int_{x_0}^{0-}((\mu + \delta(\xi))d\xi}da$$

$$= \int_0^\infty v_0(\tau)e^{-\int_{x_0}^{0+}((\mu + \delta(\xi))d\xi}d\tau$$

$$\leq e^{-\int_{x_0}^{0+}((\mu + \delta(\xi))d\xi}da$$

$$= e^{-(\mu + \delta \inf)t}\int_0^{\infty} v_0(\alpha)d\alpha$$

$$\leq he^{-(\mu + \delta \inf)t} = f(t, h).$$

**Step 2.** To prove that condition (ii) of Lemma 3.5 is valid. According to Proposition 3.2, $S, E$ and $I$ remain in the compact set $[0, A/\mu] \subset [0, C]$. Now, we show that $v_2(a, t)$ remains in a precompact set $K$ of $L^1(0, \infty)$, which has the compact closure and is independent of $x_0$. To do so, we only need to verify that $v_2(a, t)$ satisfies the following conditions in [31]:

(i) $\sup_{v_2(a, t) \in K} \int_0^\infty v_2(a, t)da \leq \infty$;

(ii) $\lim_{h \to 0+} \int_h^\infty v_2(a, t)da = 0$ uniformly in $v_2(a, t) \in K$;

(iii) $\lim_{h \to 0+} \int_0^\infty |v_2(a + h, t) - v_2(a, t)|da = 0$ uniformly in $v_2(a, t) \in K$;

(iv) $\lim_{h \to 0+} \int_0^h v_2(a, t)da = 0$ uniformly in $v_2(a, t) \in K$.

Combining assumption (G1) and Proposition 3.2, one has

$$0 \leq v_2(a, t) = \begin{cases} pS(t - a)e^{-\int_0^{t-a}((\mu + \delta(\xi))d\xi}, & t > a, \\ 0, & a \geq t \end{cases}$$

$$\leq pCe^{-(\mu + \delta \inf)a}, \quad (3.4)$$

which implies that conditions (i), (ii) and (iv) are valid for $v_2(a, t)$. Now, we check that $v_2(a, t)$ also satisfies condition (iii). For sufficiently small $h \in (0, t)$, one has

$$\int_0^\infty |v_2(a + h, t) - v_2(a, t)|da$$

$$= \int_0^t |v_2(a + h, t) - v_2(a, t)|da$$

$$= \int_0^{t-h} |v(0, t - h) - v(0, t) - v(0, t - a) + v(0, t - a)|da$$

$$+ \int_0^{t-h} |v(0, t - a) - v(0, t) - v(0, t - a) - v(0, t)|da$$

$$= \int_0^{t-h} |v(0, t - a) - v(0, t) - v(0, t - a) - v(0, t)|da$$

$$+ \int_0^{t-h} |v(0, t - a) - v(0, t) - v(0, t - a) - v(0, t)|da$$

(3.5)
Following Proposition 3.2 and Assumption \((G_1)\), we have \(v(0, t) \leq pC\) and \(e^{-\int_0^s (\mu + \delta(\xi))d\xi} \leq e^{-(\alpha + \delta_{int})a} \leq 1\). Thus, (3.5) becomes

\[
\int_0^\infty |v_2(a + h, t) - v_2(a, t)| da = pC \int_0^{t-h} |e^{-\int_0^{s+h} (\mu + \delta(\xi))d\xi} - e^{-\int_0^{s} (\mu + \delta(\xi))d\xi}| da \\
+ \int_0^{t-h} e^{-(\mu + \delta_{int})a} |v(0, t - a - h) - v(0, t - a)| da + pCh.
\]

(3.6)

It follows from the monotonicity of \(e^{-\int_0^s (\mu + \delta(\xi))d\xi}\) that

\[
\int_0^{t-h} |e^{-\int_0^{s+h} (\mu + \delta(\xi))d\xi} - e^{-\int_0^{s} (\mu + \delta(\xi))d\xi}| da \\
= \int_0^{t-h} e^{-\int_0^{s+h} (\mu + \delta(\xi))d\xi} da - \int_0^{t-h} e^{-\int_0^{s} (\mu + \delta(\xi))d\xi} da \\
= \int_0^{h} e^{-\int_0^{s} (\mu + \delta(\xi))d\xi} da - \int_{t-h}^{t} e^{-\int_0^{s} (\mu + \delta(\xi))d\xi} da \\
\leq \int_0^{h} e^{-\int_0^{s} (\mu + \delta(\xi))d\xi} da \leq h.
\]

(3.7)

Using Proposition 3.2 and Assumption \((G_1)\), we know that \(|S'|\) is bounded by \(M_S = A + (\mu + p)C + g'(0)C + \delta_{sup} C\). Thus,

\[
|v(0, t - a - h) - v(0, t - a)| = p|S(t - a - h) - S(t - a)| \leq pM_S h.
\]

(3.8)

Combining (3.7) and (3.8) into (3.6) yields

\[
\int_0^\infty |v_2(a + h, t) - v_2(a, t)| da \leq 2pCh + pM_S h \int_0^{t-h} e^{-(\mu + \delta_{int})a} da \\
= 2pCh + \frac{pM_S h}{\mu + \delta_{int}}.
\]

Obviously, \(\lim_{h \to 0^+} \int_0^\infty |v_2(a + h, t) - v_2(a, t)| da = 0\). Thus, \(v_2(a, t)\) remains in a precompact set \(K\). Consequently, for any bounded closed set \(B \subset X\), \(\Phi^2(t, B) \subset [0, C] \times [0, C] \times [0, C] \times K\). Hence, \(\Phi^2_t\) has compact closure.

Combining Steps 1-2, one can conclude that the semiflow \(\Phi\) is asymptotically smooth. 

Applying the results on the existence of global attractors in Theorem 2.6, we conclude the following proposition.

**Proposition 3.7.** The semi-flow \(\{\Phi(t)\}_{t \geq 0}\) has a global attractor \(T\) contained in \(X\), which attracts bounded sets of \(X\).
3.3 Existence of Equilibria

In this subsection, we consider the existence of equilibria. It follows from the equation of \( v(a, t) \) of (2.5) that \( e^{-\int_0^\infty [\mu + \delta(\xi)]d\xi} \) is the probability for an individual to stay in the vaccinated class for a time units. Denote

\[
\eta = \int_0^\infty \delta(a)e^{-\int_0^\infty [\mu + \delta(\xi)]d\xi} da, \tag{3.9}
\]

then it is easy to see that \( \eta < 1 \), and it represents the probability of leaving the vaccinated class for an individual losing immunity but being alive. Further, it follows from the boundary condition \( v(0, t) = pS(t) \) that \( p\eta \) is the per capita rate at which vaccinated individuals return to the susceptible class.

Now, we give the main result of this subsection.

**Theorem 3.8.** System (2.5) with conditions (2.2) and (2.3) always has a disease-free equilibrium \( P_{01}(S_{01}, 0, 0, v_0(a)) \). When \( R_{01} > 1 \), besides \( P_{01} \), it also has a unique endemic equilibrium \( P^*_1(S^*_1, E^*_1, I^*_1, v^*(a)) \), where \( R_{01} \) is defined in (3.10).

**Proof.** Obviously, system (2.5) with (2.2) and (2.3) always has a disease-free equilibrium \( P_{01}(S_{01}, 0, 0, v_0(a)) \), where

\[
S_{01} = \frac{A}{\mu + (1 - \eta)p}, \quad v_0(a) = ps_{01}e^{-\int_0^\infty [\mu + \delta(\xi)]d\xi}.
\]

Denote

\[
R_{01} = \frac{\varepsilon g'(0)S_{01}}{(\mu + \varepsilon)(\mu + \alpha)} = \frac{\varepsilon Ag'(0)}{(\mu + \varepsilon)(\mu + \alpha)(\mu + (1 - \eta)p)}. \tag{3.10}
\]

The endemic equilibrium \( P^*_1(S^*_1, E^*_1, I^*_1, v^*(a)) \) with \( I^*_1 > 0 \) of model (2.5) with (2.2) and (2.3) satisfies the following equilibrium equations

\[
\begin{aligned}
A - (\mu + p)S^*_1 - S^*_1g(I^*_1) + \int_0^\infty \delta(a)v^*(a)da &= 0, \\
S^*_1g(I^*_1) - (\mu + \varepsilon)E^*_1 &= 0, \\
\varepsilon E^*_1 - (\mu + \alpha)I^*_1 &= 0. & \tag{3.11}
\end{aligned}
\]

From the last two equations of (3.11) we have

\[
E^*_1 = \frac{\mu + \alpha}{\varepsilon}I^*_1
\]

and

\[
v^*(a) = ps^*_1e^{-\int_0^\infty [\mu + \delta(\xi)]d\xi}.
\]

Substituting them into the first two equations of (3.11) respectively gives

\[
\begin{aligned}
A - [\mu + (1 - \eta)p]S^*_1 - S^*_1g(I^*_1) &= 0, \\
S^*_1g(I^*_1) - \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon}I^*_1 &= 0. & \tag{3.12}
\end{aligned}
\]

which is equivalent to the following equations

\[
\begin{aligned}
A - [\mu + (1 - \eta)p]S^*_1 - \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon}I^*_1 &= 0, \\
S^*_1g(I^*_1) - \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon}I^*_1 &= 0. & \tag{3.13}
\end{aligned}
\]
From the first equation of (3.13) we know that \( I_1^* < \varepsilon A/[(\mu + \varepsilon)(\mu + \alpha)] \), and from the second equation of (3.13) we have

\[
S_1^* = \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon} \frac{I_1^*}{g(I_1^*)}.
\]

Substituting it into the first equation of (3.13) gives

\[
g(I_2) = \frac{[\mu + (1 - \eta)p(\mu + \varepsilon)(\mu + \alpha)]I_2^*}{\varepsilon A - (\mu + \varepsilon)(\mu + \alpha)I_2^*}.
\]

Notice that \( I = \varepsilon A/[(\mu + \varepsilon)(\mu + \alpha)] \) is a vertical asymptote of function \( h_1(I) \). For function \( h_1(I) \), we have

\[
h'_1(I) = \frac{[\mu + (1 - \eta)p(\mu + \varepsilon)(\mu + \alpha)]\varepsilon A}{[\varepsilon A - (\mu + \varepsilon)(\mu + \alpha)I]^2} > 0, \quad h''_1(I) = \frac{2[\mu + (1 - \eta)p(\mu + \varepsilon)^2(\mu + \alpha)^2\varepsilon A}{[\varepsilon A - (\mu + \varepsilon)(\mu + \alpha)I]^3} > 0
\]

for \( 0 < I < \varepsilon A/[(\mu + \varepsilon)(\mu + \alpha)] \). Thus, \( h_1(I) \) passes through the point \((0, 0)\), and is increasing and concave up in \((0, \varepsilon A/[(\mu + \varepsilon)(\mu + \alpha)])\). By assumptions for the function \( g(I) \), when \( g'(0) > h'_1(0) = [\mu + (1 - \eta)p(\mu + \varepsilon)(\mu + \alpha)]/\varepsilon A \), i.e., \( R_{01} > 1 \), equation (3.14) has a unique root \( I_1^* \) in the interval \((0, \varepsilon A/[(\mu + \varepsilon)(\mu + \alpha)])\). It implies that (3.13) (i.e., (3.12)) has a unique positive solution \((S_1^*, I_1)\) when \( R_{01} > 1 \), where \( S_1^* = (\mu + \varepsilon)(\mu + \alpha)I_1^*/[\varepsilon g(I_1^*)] \). Correspondingly, model (2.5) has a unique endemic equilibrium \( P_1^* (S_1^*, E_1^*, I_1^*, v^*(a)) \) when \( R_{01} > 1 \).

### 3.4 Global Stability

In order to simplify the process of proving the global stability of the equilibria, we introduce the following lemmas.

**Lemma 3.9.** Suppose that the function \( w(a, t) \) is a solution of the partial differential equation

\[
\frac{\partial w(a, t)}{\partial t} + \frac{\partial w(a, t)}{\partial a} = -\theta(a)w(a, t), \quad a > 0, \quad t > 0
\]

with conditions

\[
w(0, t) = \Phi(t), \quad t > 0; \quad w(a, 0) = \Psi(a), \quad a > 0,
\]

and that function \( \bar{w}(a) \) satisfies the following ordinary differential equation

\[
\frac{d\bar{w}(a)}{da} = -\theta(a)\bar{w}(a).
\]

Define a functional

\[
\bar{L} = \int_0^\infty q(a) \left[ \int_{\bar{w}(a)}^{w(a, t)} \frac{u - \bar{w}(a)}{u} du \right] da,
\]

where \( q(a) \in C^1[0, \infty) \), then the derivative of function \( \bar{L} \) with respect to \( t \) can be expressed as

\[
\frac{d\bar{L}}{dt} = [q(0)\bar{w}(0)] \int_1^\Phi(t)/\bar{w}(0) \frac{u - 1}{u} du + \int_0^\Phi(t)/\bar{w}(0) \frac{d[q(a)\bar{w}(a)]}{da} \left[ \int_1^{w(a, t)}/\bar{w}(a) \frac{u - 1}{u} du \right] da.
\]

**Proof.** It is easy to see that the solution of (3.15) with condition (3.16) is

\[
w(a, t) = \begin{cases} \Phi(t - a) \exp \left( -\int_0^a \theta(\xi)d\xi \right), & 0 < a \leq t, \\ \Psi(a - t) \exp \left( -\int_0^{a-t} \theta(\xi)d\xi \right), & a \geq t > 0. \end{cases}
\]
Using (3.18), the functional \( \bar{L} \) can be rewritten as

\[
\bar{L} = \int_0^t q(a) \left( \int_{\bar{w}(a)}^{\bar{w}(a,t)} \frac{u - \bar{w}(a)}{u} \, du \right) da + \int_t^\infty q(a) \left( \int_{\bar{w}(a)}^{\Phi(t-a) \exp \left( - \int_0^t \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(a)}{u} \, du \right) da
\]

\[
= \int_0^t q(a) \left( \int_{\bar{w}(a)}^{\Phi(t-a)} \exp \left( - \int_0^t \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(a) \right) \, du \, da
\]

\[
+ \int_t^\infty q(a) \left( \int_{\Phi(t-a)}^{\Phi(a-t)} \exp \left( - \int_0^t \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(a) \right) \, du \, da
\]

\[
= \int_0^t q(t-b) \left( \int_{\bar{w}(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db
\]

\[
+ \int_0^\infty q(t-b) \left( \int_{\Phi(t-b)}^{\Phi(t-b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db \text{(3.19)}
\]

Then

\[
\frac{d\bar{L}}{dt} = q(0) \int_{\bar{w}(0)}^{\Phi(t)} \frac{u - \bar{w}(0)}{u} \, du + \int_0^t dq(t-b) \left( \int_{(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db
\]

\[
+ \int_0^t q(t-b) \left( \frac{d}{dt} \int_{\bar{w}(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db
\]

\[
+ \int_0^\infty dq(t-b) \left( \frac{d}{dt} \int_{\bar{w}(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db
\]

\[
+ \int_0^t q(t-b) \left( \frac{d}{dt} \int_{\bar{w}(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right) \, db \text{(3.20)}
\]

Since the function \( \bar{w}(a) \) satisfies (3.17), we have

\[
\frac{d}{dt} \left( \int_{\bar{w}(t-b)}^{\Phi(b) \exp \left( - \int_0^{t-b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t-b)}{u} \right)
\]

\[
= \frac{d}{dt} \left( \Phi(b) \exp \left( - \int_0^{t-b} \theta(\xi) \, d\xi \right) - \bar{w}(t-b) \right)
\]

\[
+ \Phi(b) \exp \left( - \int_0^{t-b} \theta(\xi) \, d\xi \right) \frac{\theta(t-b) \bar{w}(t-b)}{u}
\]

\[
= - \theta(t-b) \Phi(b) \exp \left( - \int_0^{t-b} \theta(\xi) \, d\xi \right) \frac{u - \bar{w}(t-b)}{u} \, du. \text{(3.20)}
\]

Similarly,

\[
\frac{d}{dt} \left( \int_{\bar{w}(t+b)}^{\Phi(b) \exp \left( - \int_0^{t+b} \frac{\theta(\xi) \, d\xi}{u} \right) u - \bar{w}(t+b)}{u} \right)
\]

\[
= - \theta(t+b) \Phi(b) \exp \left( - \int_0^{t+b} \theta(\xi) \, d\xi \right) \frac{u - \bar{w}(t+b)}{u} \, du. \text{(3.21)}
\]
Using (3.20) and (3.21), we have

\[
\frac{dL}{dt} = q(0) \int \frac{\Phi(t)}{\tilde{w}(0)} \frac{u - \tilde{w}(0)}{u} \, du + \int_0^t \frac{dq(t - b)}{dt} \left[ \int \frac{\Phi(t)}{\tilde{w}(t - b)} \frac{u - \tilde{w}(t - b)}{u} \, du \right] \, db
\]

\[- \int_0^t q(t - b) \theta(t - b) \left[ \int \frac{\Phi(t)}{\tilde{w}(t - b)} \frac{u - \tilde{w}(t - b)}{u} \, du \right] \, db
\]

\[+ \int_0^\infty \frac{dq(t + b)}{dt} \left[ \int \frac{\Phi(t)}{\tilde{w}(t + b)} \frac{u - \tilde{w}(t + b)}{u} \, du \right] \, db
\]

\[- \int_0^t q(t + b) \theta(t + b) \left[ \int \frac{\Phi(t)}{\tilde{w}(t + b)} \frac{u - \tilde{w}(t + b)}{u} \, du \right] \, db.
\]

From (3.17) we have

\[
\frac{dq(a)}{dt} - q(a) \theta(a) = \frac{1}{\tilde{w}(a)} \frac{d[q(a) \tilde{w}(a)]}{da}.
\]

Hence,

\[
\frac{dL}{dt} = q(0) \int \frac{\Phi(t)}{\tilde{w}(0)} \frac{u - \tilde{w}(0)}{u} \, du
\]

\[+ \int_0^t \frac{d[q(t - b) \tilde{w}(t - b)]}{dt} \left[ \int \frac{\Phi(t)}{\tilde{w}(t - b)} \frac{u - 1}{u} \, du \right] \, db
\]

\[+ \int_0^\infty \frac{d[q(t + b) \tilde{w}(t + b)]}{dt} \left[ \int \frac{\Phi(t)}{\tilde{w}(t + b)} \frac{u - 1}{u} \, du \right] \, db
\]

\[= q(0) \int \frac{\Phi(t)}{\tilde{w}(0)} \frac{u - \tilde{w}(0)}{u} \, du
\]

\[+ \int_0^t \frac{d[q(a) \tilde{w}(a)]}{da} \left[ \int \frac{\Phi(t - a)}{\tilde{w}(a)} \frac{u - 1}{u} \, du \right] \, da
\]

\[+ \int_0^\infty \frac{d[q(a) \tilde{w}(a)]}{da} \left[ \int \frac{\Phi(a - t)}{\tilde{w}(a)} \frac{u - 1}{u} \, du \right] \, da.
\]

Furthermore, using (3.18), we obtain

\[
\frac{dL}{dt} = [q(0) \tilde{w}(0)] \int_1^{\Phi(0)/\tilde{w}(0)} \frac{u - 1}{u} \, du + \int_0^\infty \frac{d[q(a) \tilde{w}(a)]}{da} \left[ \int \frac{\tilde{w}(a, t) \tilde{w}(a)}{\tilde{w}(a)} \frac{u - 1}{u} \, du \right] \, da.
\]

This completes the proof of Lemma 3.9.

Lemma 3.10. Assume that the function \( f(x) \) satisfies condition (H), then we have the following statements:
(i) $f(x) < f'(0)x$ for $x > 0$.

(ii) For an arbitrary positive number $x^*$ the following inequality holds:

$$\left[1 - \frac{f(x)}{f(x^*)}\right]\left[\frac{xf(x^*)}{x^*f(x)} - 1\right] < 0 \quad \text{for } x > 0 \text{ and } x \neq x^*.$$ 

Proof. This lemma can easily be understood geometrically. In the following we give an analytical proof.

(i) $f''(x) < 0$ implies that the function $f(x)$ is convex in $(0, \infty)$, then any tangent line of function $f(x)$ is above the graph of $f(x)$. Since both the graph of $f(x)$ and its tangent line at $(0, f(0)) = (0, 0)$ pass through the origin, we know that $f(x) < f'(0)x$ for $x > 0$.

(ii) Define a function

$$h(x) = f(x) - \frac{f(x^*)}{x^*}x,$$

then $h(0) = h(x^*) = 0$. By Rolle’s Theorem, there is a $\xi \in (0, x^*)$ such that $h'(\xi) = 0$.

Since $h''(x) = f''(x) < 0$ for $x > 0$, i.e., $h(x)$ is convex, it follows that $h'(x)$ is decreasing in $(0, \infty)$, so $\xi$ is the only zero of $h'(x)$ in $(0, \infty)$. It implies that $h'(x) < 0$ for $x > x^* > \xi$, then from $h(x^*) = 0$ we know that $h(x) < 0$ for $x > x^*$. Moreover, from $h(0) = h(x^*) = 0$ we have $h(x) > 0$ for $0 < x < x^*$. Thus, it follows that $h(x) > 0$ for $0 < x < x^*$ and that $h(x) < 0$ for $x > x^*$, that is, $f(x) > f'(x^*)x$ for $0 < x < x^*$, and $f(x) < f'(x^*)x$ for $x > x^*$.

Again, $f'(x) > 0$ implies that $f(x) < f(x^*)$ for $0 < x < x^*$ and $f(x) > f(x^*)$ for $x > x^* > 0$. This implies that Lemma 3.10(ii) is true. The proof of Lemma 3.10 is completed.

With respect to the global stability of equilibria of model (2.5), we have the following theorem.

**Theorem 3.11.** For system (2.5), the disease-free equilibrium $P_{01}$ is globally stable if $R_{01} \leq 1$, the endemic equilibrium $P_1^*$ is globally stable if $R_{01} > 1$.

Proof. We first prove the global stability of the disease-free equilibrium $P_{01}$. Define a Lyapunov functional

$$L_{11} = \int_{S_{01}} S \frac{u - S_{01}}{u} du + E + \frac{\mu + \varepsilon}{\varepsilon} I + \int_{0}^{\infty} q(\alpha) \left[ \int_{v_0(\alpha)}^{v(\alpha, t)} \frac{u - v_0(\alpha)}{u} du \right] da,$$

where $q(\alpha) \in C^1[0, \infty)$ is to be determined below. Then, applying the equalities $A = (\mu + p)S_{01} - npS_{01}$ and $v_0(0) = pS_{01}$ and Lemma 3.9, the derivative of $L_{11}$ with respect to time $t$ along the solution of system (2.5) is given by

$$\frac{dL_{11}}{dt} = \left(1 - \frac{S_{01}}{S}\right)\left[ - (\mu + p)(S - S_{01}) - Sg(I) + \int_{0}^{\infty} \delta(a)v(a, t)da - npS_{01} \right]$$

$$+ \left[ Sg(I) - \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon} I \right] + [q(0)v_0(0)] \int_{1}^{pS(\varepsilon)/v_0(0)} \frac{u - 1}{u} du$$

$$+ \int_{0}^{\infty} d[q(\alpha)v_0(\alpha)] \left[ \int_{1}^{v(\alpha, t)/v_0(\alpha)} \frac{u - 1}{u} du \right] da$$

$$= - (\mu + p)\left(\frac{S - S_{01}}{S}\right)^2 + \left[S_{01}g(I) - \frac{(\mu + \varepsilon)(\mu + \alpha)}{\varepsilon} I \right]$$

$$+ \left(1 - \frac{S_{01}}{S}\right)\left[ \int_{0}^{\infty} \delta(a)v(a, t)da - npS_{01} \right]$$

$$+ [q(0)v_0(0)] \left(\frac{S}{S_{01}} - 1 - \ln \frac{S}{S_{01}}\right) + \int_{0}^{\infty} \frac{d[q(\alpha)v_0(\alpha)]}{da} \left[ \frac{v(a, t)}{v_0(\alpha)} - 1 - \ln \frac{v(a, t)}{v_0(\alpha)} \right] da.$$
In order to eliminate the term \( \int_0^\infty \delta(a) v(a, t) da \) in \( dL_{11}/dt \), we choose

\[
q(a) = \frac{\int_a^\infty \delta(\xi) e^{-\int_a^\xi [\mu + \delta(\zeta)] d\zeta} d\xi}{e^{-\int_a^\infty [\mu + \delta(\zeta)] d\zeta}} = \frac{\eta - \int_a^\infty \delta(\xi) e^{-\int_a^\xi [\mu + \delta(\zeta)] d\zeta} d\xi}{e^{-\int_a^\infty [\mu + \delta(\zeta)] d\zeta}}
\]  \quad (3.22)

to satisfy

\[
\frac{d}{da} (q(a)v_0(a)) = -\delta(a)v_0(a) \quad \text{and} \quad q(0) = \eta.
\]

Notice that \( \eta v_0(0) = p\eta S_0 = \int_0^\infty \delta(a)v_0(a) da \), then

\[
\frac{dL_{11}}{dt} = - (\mu + p) \frac{(S - S_0)^2}{S} + \left[ S_0 g(I) - \frac{\varepsilon}{\varepsilon} (\mu + \alpha) \right] I
\]

\[
+ \int_0^\infty \delta(a)v_0(a) \left[ \frac{S_0}{S(t)} + \frac{S(t)}{S_0} - 1 - \frac{S_0 v(a, t)}{S(t)v_0(a)} + \ln \frac{S_0 v(a, t)}{S(t)v_0(a)} \right] da
\]

\[
= - (\mu + (1 - \eta)p) \frac{(S - S_0)^2}{S} + \left[ S_0 g(I) - \frac{\varepsilon}{\varepsilon} (\mu + \alpha) \right] I
\]

\[
+ \int_0^\infty \delta(a)v_0(a) \left[ 1 - \frac{S_0 v(a, t)}{S(t)v_0(a)} + \ln \frac{S_0 v(a, t)}{S(t)v_0(a)} \right] da.
\]

According to assumption (H) for function \( g(I) \), from Lemma 3.10(i) we have that \( g(I) < g'(0)I \) for \( I > 0 \). Hence,

\[
\frac{dL_{11}}{dt} \leq - [\mu + (1 - \eta)p] \frac{(S - S_0)^2}{S} + \frac{\varepsilon}{\varepsilon} (\mu + \alpha) (R_{01} - 1) I
\]

\[
+ \int_0^\infty \delta(a)v_0(a) \left[ 1 - \frac{S_0 v(a, t)}{S(t)v_0(a)} + \ln \frac{S_0 v(a, t)}{S(t)v_0(a)} \right] da.
\]

Notice that \( 1 - x + \ln x \leq 0 \) for \( x > 0 \) and the equality holds if and only if \( x = 1 \), then \( L_{11} \leq 0 \) as \( R_{01} \leq 1 \) and \( L_{11} = 0 \) if and only if \( (S(t), E(t), I(t), v(a, t)) = (S_0, 0, 0, 0, v_0(a)) \). Therefore, by the Lyapunov asymptotic stability theorem\(^{[12]} \) the disease-free equilibrium \( P_{01} \) is globally stable if \( R_{01} \leq 1 \).

Next, we prove the global stability of the endemic equilibrium \( P^*_v \). Define another Lyapunov functional

\[
L_{12} = \int_{S^*_1}^S \frac{u - S^*_1}{u} du + \int_{E^*_1}^E \frac{u - E^*_1}{u} du + \frac{\mu + \varepsilon}{\varepsilon} \int_{I^*_1}^I \frac{g(u) - g(I^*_1)}{g(u)} du
\]

\[
+ \int_0^\infty q(a) \left[ \int_{v^*(a)}^{v(a, t)} \frac{u - v^*(a)}{u} du \right] da,
\]

where \( q(a) \) is a positive and continuous function to be determined later, and the monotonicity of \( g(I) \) ensures that the integral \( \int_{I^*_1}^I \frac{g(u) - g(I^*_1)}{g(u)} du \) in \((0, \infty)\) has only one extremum which is a global minimum at \( I^*_1 \), then \( L_{12} \) is positive definite with respect to \((S^*_1, E^*_1, I^*_1, v^*(a))\). Applying Lemma 3.10 and the following equalities:

\[
A = (\mu + p)S^*_1 + S^*_1 g(I^*_1) - \int_0^\infty \delta(a)v^*(a) da,
\]

\[
\mu + \varepsilon = \frac{S^*_1 g(I^*_1)}{E^*_1}, \quad \mu + \alpha = \frac{\varepsilon E^*_1}{I^*_1}, \quad v^*(0) = pS^*_1, \quad v(0, t) = pS(t),
\]

the derivative of \( L_{12} \) with respect to time \( t \) along solutions of (2.5) is given by

\[
\frac{dL_{12}}{dt} = \left(1 - \frac{S^*_1}{S}\right) \left\{ - (\mu + p)(S - S^*_1) - [Sg(I) - S^*_1 g(I^*_1)] + \int_0^\infty \delta(a) |v(a, t) - v^*(a)| da \right\}
\]
\[ + (1 - \frac{E_1}{E}) \left[ Sg(I) - \frac{S_1 g(I^*_1)}{E_1} E \right] + (\mu + \varepsilon) \left[ 1 - \frac{g(I^*_1)}{g(I)} \right] \left( E - \frac{E^*_1}{E^*_1} I \right) \]

\[ + [g(0)v^*(0)] \int_1^{\infty} \frac{u-1}{u} du + \int_0^\infty \frac{d(q(a)v^*(a))}{da} \left[ \int_1^{v(a,t)/v^*(a)} \frac{u-1}{u} du \right] da \]

\[ = -(\mu + p) \frac{(S - S_1)^2}{S} + S_1 g(I^*_1) \left[ \frac{g(I)}{g(I^*_1)} \right] \left[ \frac{I g(I^*_1)}{I^*_1 g(I)} - 1 \right] \]

\[ + S_1 g(I^*_1) \left[ 3 - \frac{S_1}{S} - \frac{E g(I^*_1)}{E g(I)} - f E_1 S g(I) \frac{E S_1 g(I^*_1)}{S} \right] + \Delta, \]

where

\[ \Delta = \left(1 - \frac{S_1}{S}\right) \int_0^\infty \delta(a)[v(a,t) - v^*(a)]da + [g(0)v^*(0)] \left( \frac{S}{S_1} - 1 - \ln \frac{S}{S_1} \right) \]

\[ + \int_0^\infty \frac{d(q(a)v^*(a))}{da} \left[ \frac{v(a,t)}{v^*(a)} - 1 - \ln \frac{v(a,t)}{v^*(a)} \right] da. \]

If we still choose function \( q(a) \) defined in (3.22), then

\[ \frac{d}{da} [q(a)v^*(a)] = -\delta(a)v^*(a) \quad \text{and} \quad q(0) = \eta. \]

Further, from \( q(0)v^*(0) = \eta p S_1 = \int_0^\infty \delta(a)v^*(a)da \) we get

\[ \Delta = \int_0^\infty \delta(a)v^*(a) \left[ \frac{S_1}{S} + \frac{S}{S_1} - 1 - \frac{S_1 v(a,t)}{S(t) v^*(a)} + \ln \frac{S v(a,t)}{S(t) v^*(a)} \right] da \]

\[ = \eta p \frac{(S - S_1)^2}{S} + \int_0^\infty \delta(a)v^*(a) \left[ 1 - \frac{S_1 v(a,t)}{S(t) v^*(a)} + \ln \frac{S v(a,t)}{S(t) v^*(a)} \right] da. \]

Therefore,

\[ \frac{dL_{12}}{dt} = -[\mu + (1 - \eta)p] \frac{(S - S_1)^2}{S} + S_1 g(I^*_1) \left[ \frac{g(I)}{g(I^*_1)} \right] \left[ \frac{I g(I^*_1)}{I^*_1 g(I)} - 1 \right] \]

\[ + S_1 g(I^*_1) \left[ 3 - \frac{S_1}{S} - \frac{E g(I^*_1)}{E g(I)} - f E_1 S g(I) \frac{E S_1 g(I^*_1)}{S} \right] \]

\[ + \int_0^\infty \delta(a)v^*(a) \left[ 1 - \frac{S_1 v(a,t)}{S(t) v^*(a)} + \ln \frac{S v(a,t)}{S(t) v^*(a)} \right] da. \]

Notice that \( 1 - x + \ln x \leq 0 \) for \( x > 0 \) and the equality holds if and only if \( x = 1 \), then, it follows from Lemma 3.10(ii) that \( L_{12}^\prime \leq 0 \) and \( L_{12} = 0 \) if and only if \( (S(t), E(t), I(t), V(t)) = (S_1^*, E_1^*, I_1^*, v^*(a)) \). Therefore, by the Lyapunov asymptotic stability theorem \([12]\) the endemic equilibrium \( P_1^* \) is globally stable if \( R_{01} > 1 \). The proof of Theorem 3.11 is completed. □

4 Global Stability of the Delay Model

According to the fundamental theory of functional differential equations\([12]\), it is easy to show the following property.

**Proposition 4.1.** Model (2.8) has a unique solution \((S(t), E(t), I(t), V(t))\) satisfying the initial conditions (2.9), and all solutions of model (2.8) with initial conditions (2.9) are defined on \([0, \infty)\) and remain positive for all \( t \geq 0 \).
In the following, we consider the existence of the endemic equilibrium of the delay system (2.8), and prove the global stability of the disease-free and endemic equilibria by constructing Lyapunov functionals.

It is easy to see that model (2.8) always has a disease-free equilibrium $P_{02}(S_{02}, 0, V_0)$, where

$$S_{02} = \frac{A}{\mu + p[1 - \gamma e^{-\mu \tau}/(\mu + \gamma)]}, \quad V_0 = \frac{pe^{-\mu \tau}}{\mu + \alpha}S_{02}. $$

Denote

$$R_{02} = \frac{\varepsilon g'(0)S_{02}}{(\mu + \varepsilon)(\mu + \alpha)} = \frac{\varepsilon A g'(0)}{(\mu + \varepsilon)(\mu + \alpha)[\mu + p[1 - \gamma e^{-\mu \tau}/(\mu + \gamma)]]}. $$

The endemic equilibrium $P^*_2(S^*_2, E^*_2, I^*_2, V^*_2)$ with $I^*_2 > 0$ of model (2.8) satisfies the following equilibrium equations

$$A - (\mu + p)S^*_2 - S^*_2g(I^*_2) + \gamma V^*_2 = 0, \quad S^*_2g(I^*_2) - (\mu + \varepsilon)E^*_2 = 0, \quad \varepsilon E^*_2 - (\mu + \alpha)I^*_2 = 0, \quad pe^{-\mu \tau}S^*_2 - (\mu + \gamma)V^*_2 = 0. \quad (4.1)$$

From the last three equations of (4.1) we have

$$S^*_2 = \frac{(\mu + \varepsilon)(\mu + \alpha)I^*_2}{\varepsilon g(I^*_2)}, \quad E^*_2 = \frac{(\mu + \alpha)I^*_2}{\varepsilon}, \quad V^*_2 = \frac{pe^{-\mu \tau}}{\mu + \gamma} \cdot \frac{(\mu + \varepsilon)(\mu + \alpha)I^*_2}{\varepsilon g(I^*_2)}. \quad (4.2)$$

Substituting them into the first equation of (4.1) gives

$$g(I^*_2) = \left[\mu + p \left(1 - \gamma e^{-\mu \tau}/\mu + \gamma\right)\right] \frac{(\mu + \varepsilon)(\mu + \alpha)I^*_2}{\varepsilon A - (\mu + \varepsilon)(\mu + \alpha)I^*_2} =: h_2(I^*_2). \quad (4.3)$$

Similar to the analysis for equation (3.14) in Section 3, we know that equation (4.3) has a unique root $I^*_2$ in the interval $(0, \varepsilon A/((\mu + \varepsilon)(\mu + \alpha)))$ as $R_{02} > 1$. Further, it follows from (4.2) that model (2.8) has a unique endemic equilibrium $P^*_2(S^*_2, E^*_2, I^*_2, V^*_2)$ as $R_{02} > 1$. Therefore, with respect to the existence of equilibria of system (2.8) we have

**Theorem 4.2.** System (2.8) always has the disease-free equilibrium $P_{02}(S_{02}, 0, 0, V_0)$. When $R_{02} > 1$, besides $P_{02}$, it also has a unique endemic equilibrium $P^*_2(S^*_2, E^*_2, I^*_2, V^*_2)$, where

$$S_{02} = \frac{A}{\mu + p[1 - \gamma e^{-\mu \tau}/(\mu + \gamma)]}, \quad V_0 = \frac{pe^{-\mu \tau}}{\mu + \alpha}S_{02}, $$

$$S^*_2 = \frac{(\mu + \alpha)I^*_2}{g(I^*_2)}, \quad E^*_2 = \frac{(\mu + \alpha)I^*_2}{\varepsilon}, \quad V^*_2 = \frac{pe^{-\mu \tau}}{\mu + \gamma} \cdot \frac{(\mu + \alpha)I^*_2}{g(I^*_2)}, $$

and $I^*_2$ is determined by equation (4.3).

With respect to the global stability of equilibria of model (2.8), we have

**Theorem 4.3.** For system (2.8), the disease-free equilibrium $P_{02}$ is globally stable if $R_{02} \leq 1$, the endemic equilibrium $P^*_2$ is globally stable if $R_{02} > 1$.

**Proof.** We first prove the global stability of the disease-free equilibrium $P_{02}$. Define a Lyapunov functional

$$L_{21} = \int_{S_{02}}^{\frac{S_0}{u}} \frac{u - S_{02}}{u} du + E + \frac{\mu + \varepsilon}{\varepsilon} I$$

$$+ \frac{\gamma}{\mu + \gamma} \int_{V_0}^{V} \frac{u - V_0}{u} du + \gamma V_0 \int_{t-\tau}^{t} \left[\frac{S(\theta)}{S_{02}} - 1 - \ln \frac{S(\theta)}{S_{02}}\right] d\theta.$$
Then, applying the equalities $A = (\mu + p)S_0 + \gamma V_0$ and $pe^{-\mu T} = (\mu + \gamma)V_0/S_0$, the derivative of $L_{21}$ with respect to time $t$ along solutions of system (2.8) is given by
\[
\frac{dL_{21}}{dt} = \left(1 - \frac{S_0}{S}\right) \left[-(\mu + p)(S - S_0) - Sg(I) + \gamma(V - V_0)\right] + Sg(I) - \left(\frac{\mu + \varepsilon(\mu + \alpha)}{\mu}\right)I + \gamma(V_0) \left[\frac{S(t) - \gamma(V - V_0)}{S_0} V_0 - V\right]
\]
\[
+ \gamma V_0 \left[\frac{S(t)}{S_0} - \frac{S(t - \tau)}{S_0} - \ln S(t) + \ln S(t - \tau)\right]
\]
\[
= - (\mu + p) (S - S_0) + \left[\frac{Sg(I) - \left(\frac{\mu + \varepsilon(\mu + \alpha)}{\mu}\right)I}{S}\right]
\]
\[
+ \gamma V_0 \left[\frac{S(t)}{S_0} + \frac{S_0 V(t)}{S(t)V_0} \right] - \frac{V_0 S(t - \tau)}{V(t)S_0} + \ln \frac{S(t - \tau)}{S(t)}\right]
\]
\[
= - (\mu + p) (S - S_0) + \frac{\gamma V_0 (S - S_0)^2}{S} + \gamma V_0 \left[2 - \frac{\gamma V_0 (S - S_0)^2}{S} \right] + (\mu + \alpha)(R_0 - 1)I
\]
\[
+ \frac{\gamma V_0 (S - S_0)^2}{S} + \gamma V_0 \left[2 - \frac{\gamma V_0 (S - S_0)^2}{S} \right] + \ln \frac{S(t - \tau)}{S(t)}\right].
\]

Applying Lemma 3.10(i) and the equality $V_0/S_0 = pe^{-\mu T}/(\mu + \gamma)$ yields
\[
\frac{dL_{21}}{dt} \leq - \left(\mu + p \left(1 - \frac{e^{-\mu T}}{\mu + \gamma}\right)\right) (S - S_0) + \left[\frac{Sg(I) - \left(\frac{\mu + \varepsilon(\mu + \alpha)}{\mu}\right)I}{S}\right]
\]
\[
+ \gamma V_0 \left[\frac{2 - \frac{V_0 S(t - \tau)}{V(t)S_0} + \ln \frac{S(t - \tau)}{S(t)}\right]
\]
\[
= - \left(\mu + p \left(1 - \frac{e^{-\mu T}}{\mu + \gamma}\right)\right) (S - S_0) + (\mu + \alpha)(R_0 - 1)I
\]
\[
+ \gamma V_0 \left[2 - \frac{V_0 S(t - \tau)}{V(t)S_0} + \ln \frac{S(t - \tau)}{S(t)}\right].
\]

According to Lemma 3.1 in [21], $2 - \frac{S_0 V(t)}{S(t)V_0} - \frac{V_0 S(t - \tau)}{V(t)S_0} + \ln \frac{S(t - \tau)}{S(t)} \leq 0$ and the equality holds if and only if $V(t)/V_0 = S(t)/S_0 = S(t - \tau)/S_0$. Then $L_{21}' \leq 0$ as $R_0 \leq 1$ and $L_{21}' = 0$ if and only if $(S(t), E(t), I(t), V(t)) = (S_0, 0, 0, V_0)$. Hence, it follows from the Lyapunov asymptotic stability theorem[12] that the endemic equilibrium $P_0$ is globally stable if $R_0 \leq 1$.

Next, we prove the global stability of the endemic equilibrium $P_2^*$. Define the other Lyapunov functional by
\[
L_{22} = \int_{S_0}^S \frac{u - S_2}{u} du + \int_{E_2}^E \frac{E - E_2}{u} du + \frac{\mu + \varepsilon}{\mu} \int_{I_2}^I \frac{g(u) - g(I_2)}{g(u)} du
\]
\[
+ \frac{\gamma}{\mu + \gamma} \int_{V^*}^V \frac{u - V^*}{u} du + \gamma V^* \int_{I_2}^I \frac{S(\theta)}{S_2} - 1 - \ln \frac{S(\theta)}{S_2} d\theta.
\]

The monotonicity of $g(I)$ ensures that the integral $\int_{I_2}^I \frac{g(u) - g(I_2)}{g(u)} du$ in $(0, \infty)$ has only one extremum which is a global minimum at $I_2$. Then $L_{22}$ is positive definite with respect to $(S_2, I_2, V^*)$. Applying the equalities
\[
A = (\mu + p)S_2 + S_2 g(I_2) - \gamma V^*, \quad \mu + \varepsilon = \frac{S_2 g(I_2)}{E_2}, \quad \mu + \alpha = \frac{\varepsilon E_2}{I_2}, \quad pe^{-\mu T} = \frac{(\mu + \gamma)V^*}{S_2},
\]
the derivative of $L_{22}$ with respect to time $t$ along solutions of (2.8) is given by
\[
\frac{dL_{22}}{dt} = \left(1 - \frac{S_2^*}{S}\right) \left\{ - (\mu + p)(S - S_2^*) - \left[ Sg(I) - S_2^*g(I_2^*) \right] + \gamma(V - V^*) \right\}
\]
\[
+ \left[ 1 - \frac{E_2}{I} \right] \left[ Sg(I) - \frac{S_2^*g(I_2^*)}{I_2^*} \right] + (\mu + \gamma) \left[ 1 - \frac{g(I_2^*)}{g(I)} \right] \left( E - \frac{E_2}{I_2^*} \right)
\]
\[
+ \gamma \left(1 - \frac{V^*}{V}\right) \left[ S(t - \tau)/S_2^* - V^* - V \right] + \gamma V^* \left[ S(t)/S_2^* - S(t - \tau)/S_2^* - \ln \frac{S(t)}{S_2^*} + \ln \frac{S(t - \tau)}{S_2^*} \right]
\]
\[
= - (\mu + p) \frac{(S - S_2^*)^2}{S}
\]
\[
+ S_2^*g(I_2^*) \left\{ \left[ 3 - \frac{S_2^*}{S} \right] - \frac{E_2Sg(I)}{ES_2^*g(I_2^*)} - \frac{E_2g(I)}{E_2^*g(I)} \right\} + \left[ 1 - \frac{g(I_2^*)}{g(I)} \right] \left[ I_2^*g(I) - 1 \right]
\]
\[
+ \gamma V^* \left\{ \left[ (S - S_2^*)^2 \right] + \left[ 2 - \frac{S_2^*V(t)}{S(t)V^*} \right] \frac{V^*S(t - \tau)}{V(t)S_2^*} + \ln \frac{S(t - \tau)}{S(t)} \right\}.
\]

Applying the equality $\gamma V^*/S_2^* = p\gamma e^{-\mu T}/(\mu + \gamma)$ gives
\[
\frac{dL_{22}}{dt} = - \left[ \mu + \left(1 - \frac{\gamma e^{-\mu T}}{\mu + \gamma}\right)p \right] \frac{(S - S_2^*)^2}{S} + S_2^*g(I_2^*) \left[ 1 - \frac{g(I_2^*)}{g(I)} \right] \left[ I_2^*g(I) - 1 \right]
\]
\[
+ \gamma V^* \left[ 2 - \frac{S^*V(t)}{S(t)V^*} \right] \frac{V^*S(t - \tau)}{V(t)S_2^*} + \ln \frac{S(t - \tau)}{S(t)}.
\]

Since $2 - \frac{S_2^*V(t)}{S(t)V^*} - \frac{V^*S(t - \tau)}{V(t)S_2^*} + \ln \frac{S(t - \tau)}{S(t)} \leq 0$ and the equality holds if and only if $V(t)/V^* = S(t)/S_2^* = S(t - \tau)/S_2^*$, it follows from Lemma 3.10(ii) that $L_{22} \leq 0$ and $L_{22}' = 0$ if and only if $(S(t), E(t), I(t), V(t)) = (S_2^*, E_2^*, I_2^*, V^*)$. Therefore, by the Lyapunov asymptotic stability theorem\textsuperscript{[12]} the endemic equilibrium $P_2^*$ is globally stable if $R_{02} > 1$. The proof of Theorem 4.3 is completed. \hfill $\Box$

5 Numerical Simulations

In this section, we provide some numerical simulations of the general model (2.5) to verify the feasibility of theoretical results. The main parameter values are given in Table 2.

| Parameters | Interpretations | Values | References |
|------------|----------------|-------|------------|
| $A$        | the recruitment rate of the susceptible class | 0.01  | Assumed    |
| $\mu$      | the per capita natural death rate | 0.021 | [7],[38]   |
| $\varepsilon$ | the conversion rate from the latent class | 0.025 | [11]       |
| $\sigma$   | the additional death rate induced by the infectious diseases | 0.002 | [5]        |
| $\omega$   | the recovery rate from the infectious class | 0.025 | [37]       |

Model (2.5) is applied to the epidemic of HBV status in China. According to the reference\textsuperscript{[10]}, there is a assumption that the rate of waning vaccine-induced immunity is 0.1 and the immunity
protective period is about 10 years ([10]). Then \( \delta(a) \) is taken as \( \delta(a) = 0.1 \sin\left(\frac{(a-5)\pi}{10}\right) \). Using parameter values in Table 2, this subsection concerns on the influence of the adults’ vaccination coverage rate on the dynamical behaviors of model (2.5). Take \( p = 0.15 \) and \( R_{01} = 1.25 > 1 \). According to Theorem 3.11, there exists a unique globally stable endemic equilibrium, which is simulated in Fig. 1. We have \( S^*_1 = 0.06, \ E^*_1 = 0.94, \ I^*_1 = 0.60 \), and the proportion of vaccination class is the distribution function of \( a \), particularly, \( v^*(0) = 0.035 \) and \( v^*(10) = 0.012 \). Moreover, Fig. 2 shows the distribution of the vaccination population at the endemic equilibrium for all time and a range of age. According to Figs. 1-2, we can see that a lower vaccination rate will lead to the appearance of a stable endemic equilibrium.

If we increase the vaccination rate of adults to \( p = 0.8 \), after some computations, we obtain \( R_{01} = 0.25 < 1 \). It follows from Theorem 3.11 that model (2.5) tends to the disease-free equilibrium which is globally stable (See Fig. 3). Furthermore, one has \( S_{01} = 0.029, \ E_{01} = 0, \ I_{01} = 0 \), and the proportion of the vaccinated class is the distribution function of \( a \), particularly, \( v_0(0) = 0.023 \) and \( v_0^*(10) = 0.008 \). A comparison between Fig. 1 and Fig. 3 implies that increasing adults’ vaccination coverage will be an effective strategy to reduce and control the incidence of some epidemic diseases.

![Figure 5.1](image)

**Figure 5.1.** The global asymptotically stability of the endemic equilibrium of model (2.5) with the initial values \((0.1, 0.04, 0.02, 0.02)\). Here \( p = 0.15 \) and \( R_0 = 1.25 > 1 \).
6 Discussion

In this paper, we considered two classes of epidemic models with vaccination by incorporating the vaccine age into a SIR epidemic model with nonlinear incidence. One is model (2.5) with vaccine age, which is constrained by the boundary condition (2.2) and the initial condition (2.3):
the other one is model (2.8) with stage structure of vaccination. From the derivation of system (2.8) we know that (2.8) is a special case of (2.5), but they include partial differential equations and delay differential equations, respectively. The thresholds (i.e. \( R_{01} \) and \( R_{02} \)) determining their dynamical behaviors were found. The obtained results show that the two models have the same dynamical behaviors, but there exist some differences on the associated analysis since the types of the two models are different.

On the other hand, since (2.8) is a special case of (2.5), we can also obtain the threshold \( R_{02} \) from the expression of \( R_{01} \). In fact, substituting the step function \( \delta(a) \) into (3.9) yields

\[
\eta = \int_0^\tau \delta(a) e^{-\int_0^a [\mu+\delta(\xi)]} \, da + \int_\tau^\infty \delta(a) e^{-\int_a^\infty [\mu+\delta(\xi)]} \, da
\]

\[= \gamma \int_\tau^\infty \delta(a) e^{-[\mu+\gamma]a-\gamma\tau} \, da = \frac{\gamma e^{-\mu\tau}}{\mu+\gamma}.
\]

Then, \( R_{02} \) can be obtained from \( R_{01} \).

For epidemic models, the basic reproduction number is defined as the expected number of secondary cases produced by one infective host in an entirely susceptible population\[6, 32\]. It is easy to see that, in the absence of vaccination, both basic reproduction numbers of models (2.5) and (2.8) are \( \frac{Ag'(0)}{\mu+\alpha} \)[20], where \( A/\mu \) represents the size of the entirely susceptible population. When vaccination is incorporated, from the epidemiological meaning of \( \eta \) we know that a fraction, \( (1-\eta)p/\mu + (1-\eta)p \), of susceptible individuals is transferred to the vaccinated class and cannot be infected. Then the size of the entirely susceptible population becomes

\[
\frac{A}{\mu} \left[ 1 - \frac{(1-\eta)p}{\mu + (1-\eta)p} \right] = \frac{A}{\mu + (1-\eta)p}.
\]

Therefore, the threshold

\[
R_{01} = \frac{Ag'(0)}{(\mu + \alpha)[\mu + (1-\eta)p]}
\]

is the basic reproduction number of model (2.5). Further, according to the relation between the two models, \( R_{02} \) is the basic reproduction number of model (2.8).

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