LEBESGUE MIXED NORM ESTIMATES FOR BERGMAN PROJECTORS: FROM TUBE DOMAINS OVER HOMOGENEOUS CONES TO HOMOGENEOUS SIEGEL DOMAINS OF TYPE II

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Abstract. We present a transference principle of Lebesgue mixed norm estimates for Bergman projectors from tube domains over homogeneous cones to homogeneous Siegel domains of type II associated to the same cones. This principle implies improvements of these estimates for homogeneous Siegel domains of type II associated with Lorentz cones, e.g. the Pyateckii-Shapiro Siegel domain of type II.

1. Introduction

Let $D$ be a domain in $\mathbb{C}^n$ and $dv$ the Lebesgue measure defined in $\mathbb{C}^n$. We denote by $P$ the Bergman projector i.e., the orthogonal projector of the Hilbert space $L^2(D, dv)$ onto its closed subspace $A^2(D, dv)$ consisting of holomorphic functions on $D$. It is well-known that $P$ is an integral operator defined on $L^p(D, dv)$ whose kernel $B(.,.)$, called the Bergman kernel, is the reproducing kernel of $A^2(D, dv)$. In this work, we consider the case where $D$ is a homogeneous Siegel domain of type II and we are interested in the values of $p \geq 1$ for which the Bergman projector $P$ can be extended as a bounded operator on $L^p(D, dv)$. More generally, we investigate the values $1 \leq p, q \leq \infty$ for which the Bergman projector extends to a bounded operator on Lebesgue mixed norm spaces $L^{p,q}(D)$.

In fact, C. Nana [16] determined a range of values $1 \leq p, q \leq \infty$ for which the Bergman projector of a homogeneous Siegel domain of type II extends as a bounded operator on Lebesgue mixed norm spaces $L^{p,q}(D)$. He even considered the case where the Lebesgue measure $dv$ is replaced by standard weighted measures. Earlier in a joint work [17] with B. Trojan, the same author considered the particular case of tube domains over homogeneous cones (homogeneous Siegel domains of type I). The purpose of the present paper is to present a transference principle to deduce mixed norm estimates for Bergman projectors on homogeneous Siegel domains of type II from analogous estimates on tube domains over associated cones. As an application, the results of [16] can be obtained as consequences of the results of [17].

Key words and phrases. Homogeneous cones - Homogeneous Siegel domains of type II - Bergman spaces - Bergman projectors - Box operator.
2. Description of homogeneous cones and homogeneous Siegel domains of type II. Statement of the main results

In this section, we recall the description of a homogeneous cone within the framework of $T$-algebras. Next, we introduce homogeneous Siegel domains of type II and state our main results.

2.1. Homogeneous cones. We use the same notations as in [9] and [17]. We denote by $\mathcal{U}$ a (real) matrix algebra of rank $r$ with canonical decomposition

$$\mathcal{U} = \bigoplus_{1 \leq i,j \leq r} \mathcal{U}_{ij}$$

such that $\mathcal{U}_{ij} \mathcal{U}_{jk} \subseteq \mathcal{U}_{ik}$ and $\mathcal{U}_{ij} \mathcal{U}_{lk} = \{0\}$ if $j \neq l$. We assume that $\mathcal{U}$ has a structure of $T$-algebra (in the sense of [18]) in which an involution is given by $x \mapsto x^\star$. This structure implies that the subspaces $\mathcal{U}_{ij}$ satisfy: $\mathcal{U}_{ii} = \mathbb{R}c_i$ where $c_i^2 = c_i$ and $\dim \mathcal{U}_{ij} = n_{ij} = n_{ji}$. Also, the matrix

$$e = \sum_{j=1}^{r} c_j$$

is a unit element for the algebra $\mathcal{U}$.

Let $\rho$ be the unique isomorphism from $\mathcal{U}_{ii}$ onto $\mathbb{R}$ with $\rho(c_i) = 1$ for all $i = 1, \ldots, r$. We shall consider the subalgebra

$$\mathcal{T} = \bigoplus_{1 \leq i \leq j \leq r} \mathcal{U}_{ij}$$

of $\mathcal{U}$ consisting of upper triangular matrices and let

$$H = \{ t \in \mathcal{T} : \rho(t_{ii}) > 0, \ i = 1, \ldots, r \}$$

be the subgroup of upper triangular matrices whose diagonal elements are positive.

Denote by $V$ the vector space of "Hermitian matrices" in $\mathcal{U}$

$$V = \{ x \in \mathcal{U} : \ x^\star = x \}.$$ 

If we set

$$n_i = \sum_{j=1}^{i-1} n_{ji}, \quad m_i = \sum_{j=i+1}^{r} n_{ij},$$

then

$$\dim V = n = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i.$$ 

The vector space $V$ becomes a Euclidean space with the inner product

$$(x|y) = tr (xy^\star)$$
where

\[ \text{tr} (x) = \sum_{i=1}^{r} \rho(x_{ii}). \]

Next we define

\[ \Omega = \{ ss^* : s \in H \}. \]

By a theorem of Vinberg ([18, p. 384]), \( \Omega \) is an open convex homogeneous cone containing no entire straight lines, in which the group \( H \) acts simply transitively via the transformations

\[ \pi(w) : uu^* \mapsto \pi(w)[uu^*] = (wu)(u^*w^*) \quad (w, u \in H). \]

Thus, to every element \( y \in \Omega \) corresponds a unique \( t \in H \) such that

\[ y = \pi(t)[e]. \]

Like in [17], we shall adopt the notation:

\[ t \cdot e = \pi(t)[e]. \]

We shall assume that \( \Omega \) is irreducible, and hence rank \( \Omega \) = \( r \). All homogeneous convex cones can be constructed in this way ([18, p. 397]).

As in [17], we denote by \( Q_j \) the fundamental rational functions in \( \Omega \) given by

\[ Q_j(y) = \rho(t_{jj})^2, \quad \text{when} \quad y = t \cdot e \in \Omega. \]

We consider the matrix algebra \( \mathcal{U}' \) which differs from \( \mathcal{U} \) only on its grading, in the sense that

\[ \mathcal{U}'_{ij} = \mathcal{U}_{r+1-i,r+1-j} \quad (i, j = 1, ..., r). \]

It is proved in [18] that \( \mathcal{U}' \) is also a \( T \)-algebra and \( V' = V \) where \( V' \) is the subspace of \( \mathcal{U}' \) consisting of Hermitian matrices. We define accordingly its subalgebra

\[ \mathcal{T}' = \bigoplus_{1 \leq i \leq j \leq r} \mathcal{U}'_{ij} \]

of \( \mathcal{U} \) consisting of lower triangular matrices and the subgroup \( H' \) of \( \mathcal{T}' \) whose diagonal elements are positive. We have

\[ \mathcal{T}' = \{ t^* : t \in \mathcal{T} \} \quad \text{and} \quad H' = H^* = \{ t^* : t \in H \}. \]

The corresponding homogeneous cone coincides with the dual cone of \( \Omega \), namely

\[ \Omega^* = \{ \xi \in V' : (x|\xi) > 0, \quad \forall x \in \overline{\Omega} \setminus \{0\} \}. \]

One also has

\[ \Omega^* = \{ t^*t : t \in H \}. \]

(See [18, p. 390]).

For \( \xi = t^*t \in \Omega^* \), we shall define

\[ Q_j^*(\xi) = \rho(t_{jj}^2). \]
The group $H'$ acts simply transitively on the cone $\Omega^*$ via the transformations
\begin{equation}
\pi(w^*) : u^*u \mapsto \pi(w^*)[u^*u] = (w^*u^*)(uw) \quad (w^*, u^* \in H').
\end{equation}

We write
\begin{equation}
t^* \cdot e = \pi(t^*)[e] \quad (t^* \in H').
\end{equation}

We have the following identity.
\begin{equation}
Q_j^∗(t^* \cdot e) = Q_j(t \cdot e).
\end{equation}

In the sequel, we shall use the following notations: for all $x \in \Omega$, $\xi \in \Omega^*$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r$,
\begin{equation}
Q^\alpha(x) = \prod_{j=1}^{r} Q_j^{\alpha_j}(x) \quad \text{and} \quad (Q^*)^\alpha(\xi) = \prod_{j=1}^{r} (Q_j^*)^{\alpha_j}(\xi).
\end{equation}

We identify a real number $\beta$ with the vector $(\beta, \ldots, \beta) \in \mathbb{R}^r$ and we write
\begin{equation}
Q^\beta(x) = \prod_{j=1}^{r} Q_j^{\beta_j}(x) \quad \text{and} \quad (Q^*)^\beta(\xi) = \prod_{j=1}^{r} (Q_j^*)^{\beta_j}(\xi),
\end{equation}
\begin{equation}
Q^{\alpha+\beta}(x) = \prod_{j=1}^{r} Q_j^{\alpha_j+\beta_j}(x) \quad \text{and} \quad (Q^*)^{\alpha+\beta}(\xi) = \prod_{j=1}^{r} (Q_j^*)^{\alpha_j+\beta_j}(\xi).
\end{equation}

We put $\tau = (\tau_1, \tau_2, \ldots, \tau_r) \in \mathbb{R}^r$ with
\begin{equation}
\tau_i = 1 + \frac{1}{2}(m_i + n_i).
\end{equation}

Let $x \in \Omega$, we have for $j = 1, \ldots, r$
\begin{equation}
Q_j(\pi(t)[x]) = Q_j(t \cdot e)Q_j(x).
\end{equation}

Therefore, for any $t \in H$,
\begin{equation}
Q^r(\pi(t)[x]) = \det \pi(t)Q^r(x)
\end{equation}
since
\begin{equation}
\det \pi(t) = Q^r(t \cdot e).
\end{equation}

(See [18, p. 388]). The above properties are also valid if we replace $Q_j$ by $Q_j^*$ and $x \in \Omega$ by $\xi \in \Omega^*$. In particular, for all $\xi \in \Omega^*$ and $t^* \in H'$, we have for $j = 1, \ldots, r$
\begin{equation}
Q_j^∗(\pi(t^*)[\xi]) = Q_j^∗(t^* \cdot e)Q_j^∗(\xi).
\end{equation}
2.2. **Homogeneous Siegel domains of type II.** Let $V^\mathbb{C} = V + iV$ be the complexification of $V$. Then each element of $V^\mathbb{C}$ is identified with a vector in $\mathbb{C}^n$. The coordinates of a point $z \in \mathbb{C}^n$ are arranged in the form

\begin{equation}
z = (z_{11}, z_2, z_{22}, ..., z_r, z_{rr})
\end{equation}

where

\begin{equation}
z_j = (z_{1j}, ..., z_{j-1,j}), \quad j = 2, ..., r
\end{equation}

and

\begin{equation}
z_{jj} \in \mathbb{C}, \quad z_{ij} = (z_{ij}^{(1)}, ..., z_{ij}^{(n_{ij})}) \in \mathbb{C}^{n_{ij}}, \quad 1 \leq i < j \leq r.
\end{equation}

For all $j = 1, ..., r$ we denote $e_{jj} = z$, where $z_{jj} = 1$ and the other coordinates are equal to zero and we denote

\begin{equation}
e = \sum_{j=1}^r e_{jj} = (1, 0, 1, ..., 0, 1).
\end{equation}

Let $m \in \mathbb{N}$. For each row vector $u \in \mathbb{C}^m$, we denote $u'$ the transpose of $u$. Given $m \times m$ Hermitian matrices $\tilde{H}_{11}, \tilde{H}_2, \tilde{H}_{22}, ..., \tilde{H}_r, \tilde{H}_{rr}$ such that for every $j = 1, ..., r$, we have

\begin{equation}
u \tilde{H}_{jj} \bar{v} \in \mathbb{C}, \quad u \tilde{H}_{jj} \bar{v} \in \mathbb{C}^{n_{jj}},
\end{equation}

we define a $\Omega$-Hermitian, homogeneous form $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ as

\begin{equation}
F(u, v) = (u \tilde{H}_{11} \bar{v}', u \tilde{H}_2 \bar{v}', u \tilde{H}_{22} \bar{v}', ..., u \tilde{H}_r \bar{v}', u \tilde{H}_{rr} \bar{v}'), \quad (u, v) \in \mathbb{C}^m \times \mathbb{C}^m
\end{equation}

such that

(i) $F(u, u) \in \overline{\Omega}$;

(ii) $F(u, u) = 0$ if and only if $u = 0$;

(iii) for every $t \in H$, there exists $\tilde{t} \in GL(m, \mathbb{C})$ such that

\begin{equation}
t \cdot F(u, u) = F(\tilde{t}u, \tilde{t}u).
\end{equation}

The point set

\begin{equation}
D(\Omega, F) = \{(z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \Im m z - F(u, u) \in \Omega\}
\end{equation}

in $\mathbb{C}^{n+m}$ is called a Siegel domain of type II associated to the open convex homogeneous cone $\Omega$ and to the $\Omega$–Hermitian, homogeneous form $F$. Recall that if $m = 0$, the domain $D$ is a tube type Siegel domain or a homogeneous Siegel domain of type I, associated with the cone $\Omega$, or the tube domain over the homogeneous cone $\Omega$, considered by the authors of \cite{17}.

Using (7), we write

\begin{equation}
F(u, u) = (F_{11}(u, u), F_{2}(u, u), F_{22}(u, u), ..., F_r(u, u), F_{rr}(u, u))
\end{equation}

where for $i = 1, ..., r$ and $j = 2, ..., r,$

\begin{equation}F_{ii}(u, u) = u \tilde{H}_{ii} \bar{u}', \quad F_j(u, u) = u \tilde{H}_j \bar{u}' = (F_{1j}(u, u), ..., F_{j-1,j}(u, u))
\end{equation}
and for $1 \leq i < j \leq r$ and $\lambda = 1, ..., n_{ij},$

$$F_{ij}(u, u) = (F_{ij}^{(1)}(u, u), ..., F_{ij}^{(n_{ij})}(u, u)), \quad F_{ij}^{(\lambda)}(u, u) = u \tilde{H}_{ij}^{(\lambda)} u'.$$

The space $\mathbb{C}^m$ decomposes into the direct sum of subspaces $\mathbb{C}^{b_1} \oplus ... \oplus \mathbb{C}^{b_r}$ on which are concentrated the Hermitian forms $F_{ij}$, that is, with appropriate coordinates, we have for $i = 1, ..., r,$

$$\tilde{H}_{ii} = \text{diag}(0_{(b_i)}, ..., 0_{(b_i)}, I_{(b_i)}, 0_{(b_i+1)}, ..., 0_{(b_r)})$$

where $0_{(b_i)}$ and $I_{(b_i)}$ denote respectively the null matrix and the identity matrix of the vector space $\mathbb{C}^{b_k}$ for all $k = 1, ..., r$. (See for instance [19, pp. 127-129].)

In the sequel, we denote $b$ the vector

$$b = (b_1, ..., b_r) \in \mathbb{N}^r.$$

and we denote $dv$ the Lebesgue measure in $\mathbb{C}^m$. Let $\nu = (\nu_1, ..., \nu_r) \in \mathbb{R}^r$. For all $(x + iy, u) \in D$, we shall consider the measure

$$dV_{\nu}(x + iy, u) = Q^{\nu - \frac{b}{2} - \tau}(y - F(u, u))dxdydv(u).$$

We denote by $L^p_{\nu}(D)$, $1 \leq p \leq \infty$, the Lebesgue space $L^p(D, dV_{\nu}(z, u))$. The weighted Bergman space $A^p_{\nu}(D)$ is the (closed) subspace of $L^p_{\nu}(D)$ consisting of holomorphic functions. In order to have a non-trivial subspace, we take $\nu = (\nu_1, ..., \nu_r) \in \mathbb{R}^r$ such that $\nu_i > \frac{m_i + b_i}{2}, \quad i = 1, ..., r$. (See [16].)

The orthogonal projector of the Hilbert space $L^2_{\nu}(D)$ on its closed subspace $A^2_{\nu}(D)$ is the weighted Bergman projector $P_{\nu}$. We recall that $P_{\nu}$ is defined by the integral

$$P_{\nu}f(z, u) = \int_D B_{\nu}((z, u), (w, v)) f(w, v)dV_{\nu}(w, v), \quad (z, u) \in D,$$

where for a suitable constant $d_{\nu,b},$

$$B_{\nu}((z, u), (w, v)) = d_{\nu,b}Q^{\nu - \frac{b}{2} - \tau}\left(z / 2^t - w / 2^t - F(u, v)\right)$$

is the weighted Bergman kernel i.e., the reproducing kernel of $A^2_{\nu}(D)$. (See [3, Proposition II.5].) The scalar product $\langle \cdot, \cdot \rangle_{\nu}$ is given by

$$\langle f, g \rangle_{\nu} = \int_D f(z, u) \overline{g(z, u)}dV_{\nu}(z, u).$$

Let us now introduce mixed norm spaces. For $1 \leq p \leq \infty$ and $1 \leq q < \infty$, let $L^{p,q}_{\nu}(D)$ be the space of measurable functions on $D$ such that

$$\|f\|_{L^{p,q}_{\nu}(D)} := \left(\int_{\mathbb{C}^m} \int_{\Omega + F(u, u)} \left(\int_V |f(x + iy, u)|^p dx\right)^{\frac{q}{p}} Q^{\nu - \frac{b}{2} - \tau}(y - F(u, u))dxdydv(u)\right)^{\frac{1}{q}}$$

is finite (with obvious modification if $p = \infty$). As before, we call $A^{p,q}_{\nu}(D)$ the (closed) subspace of $L^{p,q}_{\nu}(D)$ consisting of holomorphic functions. Note that for $p = q$, the
Lebesgue mixed norm space $L_{\nu}^{p,q}(D)$ coincides with the Lebesgue space $L_{\nu}^{p}(D)$ and the mixed norm Bergman space $A_{\nu}^{p,q}(D)$ coincides with the Bergman space $A_{\nu}^{p}(D)$.

The unweighted case corresponds to $\nu = \tau + \frac{b}{2}$.

2.3. Example. The homogeneous (non-symmetric) Siegel domain of type II introduced by Pyateckii-Shapiro is associated to the spherical cone

$$\Gamma := \{(y_{11}, y_{12}, y_{22}) \in \mathbb{R}^3 : Q_1(y) = y_{11} > 0, Q_2(y) = y_{22} - \frac{(y_{12})^2}{y_{11}} > 0\}$$

and to the $\Gamma-$Hermitian, homogeneous form

$$F : \mathbb{C}^2 \rightarrow \mathbb{C}^3, (u, v) \mapsto F(u, v) = (0, 0, u\bar{v}).$$

In this domain $D(\Gamma, F)$, we have

$$n = 3, \ r = 2, \ n_1 = m_2 = 0, \ n_2 = m_1 = 1, \ b_1 = 0, \ b_2 = 1, \ \tau = \left(\frac{3}{2}, \frac{3}{2}\right).$$

The (unweighted) Bergman kernel of $D(\Gamma, F)$ has the following expression:

$$B((z, u), (w, v)) = CQ_1^{-3}\left(\frac{z - \bar{w}}{2i} - F(u, v)\right)Q_2^{-4}\left(\frac{z - \bar{w}}{2i} - F(u, v)\right)$$

which can be written

$$B((z, u), (w, v)) = C\left(\frac{z_{11} - \bar{w}_{11}}{2i}\right)^{-3}\left(\frac{z_{22} - \bar{w}_{22}}{2i} - u\bar{v} - \frac{(z_{12} - \bar{w}_{12})^2}{2i}\right)^{-4}.$$

For $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ such that $\nu_j > \frac{1}{2}, \ j = 1, 2$, the associated (weighted) Bergman kernel is given by

$$B_{\nu}((z, u), (w, v)) = d_{\nu}Q_1^{-\nu_1 - \frac{3}{2}}\left(\frac{z - \bar{w}}{2i} - F(u, v)\right)Q_2^{-\nu_2 - 2}\left(\frac{z - \bar{w}}{2i} - F(u, v)\right) .$$

2.4. Statement of the results. The main result of our paper is the following.

**Theorem 2.1.** Let $\nu = (\nu_1, ..., \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + n_j + b_j}{2}, \ j = 1, ..., r$. Assume that the Bergman projector $P_{\nu - \frac{b}{2}}$ of the tube domain $T_\Omega$ over the homogeneous cone $\Omega$ is bounded on $L_{\nu - \frac{b}{2}}^{p,q}(T_\Omega)$. Then the Bergman projector $P_{\nu}$ of the homogeneous Siegel domain $D$ of type II associated to $\Omega$ and to the $\Omega-$Hermitian homogeneous form $F$ is bounded on $L_{\nu}^{p,q}(D)$.

As a consequence, the following result of [16] for $D$ is a consequence of the corresponding result of [17] for tube domains over homogeneous cones (see Theorem 4.7 below).
Theorem 2.2. Let \( \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j + n_j + b_j}{2} \), \( j = 1, \ldots, r \). We set \( q_\nu := 1 + \min_{1 \leq j \leq r} \frac{\nu_j - \nu_j^\prime}{\frac{n_j}{2}} \). The Bergman projector \( P_\nu \) extends to a bounded operator on \( L^p,q_\nu(D) \) for
\[
\begin{cases}
0 \leq \frac{1}{p} < \frac{1}{q} < 1 - \frac{1}{q_\nu'} \\
\frac{1}{q_\nu'} < \frac{1}{q} < 1 - \frac{1}{q_\nu} 
\end{cases}
\]

Our theorem implies improvements of \( L^p,q_\nu \) estimates for Bergman projectors in homogeneous Siegel domains of type II associated to Lorentz cones for some particular values of \( \nu \in \mathbb{R}^2 \). An interesting case is the Pyateckii-Shapiro Siegel domain of type II defined in Example 2.3. For this domain, the problem under study was investigated in [15, section 6] and a particular case of Theorem 2.1 was used there. We point out that the underlying spherical cone is isomorphic to the Lorentz cone of \( \mathbb{R}^3 \).

Theorem 2.3. Let \( \nu = (\nu_1, \nu_2) \in \mathbb{R}^2 \) such that \( \nu_1 > \frac{n}{2} - 1 \) and \( \nu_2 > 0 \). The weighted Bergman \( P_\nu \) of the tube domain \( T_{\Lambda_n} \) is bounded in \( L^p,q_\nu(T_{\Lambda_n}) \) for the following values of \( p, q \) and \( \nu \).

1. \( \frac{n-2}{\nu_2 + \frac{n}{2} - 1} < p < \frac{n-2}{\nu_2 + \frac{n}{2} - 1 - \nu_2} \), \( q'_\nu(p) < q < q_\nu(p) \) provided \( 0 < \nu_2 < \frac{n}{2} - 1 \);
2. \( 1 \leq p \leq \infty \), \( q'_\nu(p) < q < q_\nu(p) \) provided \( \nu_2 \geq \frac{n}{2} - 1 \);
3. \( 2 \leq p \leq \frac{2n}{n-2} \), \( p < p_\nu \) and \( q'_\nu(p) < q < 2q_\nu \);
4. \( p_\nu > p > \frac{2n}{n-2} \) and \( 2 < q < q_\nu \) provided \( \frac{n-2}{2n} < \nu_2 < \frac{n}{2} - 1 \);
5. \( p > \frac{2n}{n-2} \) and \( 2 < q < q_\nu \) provided \( \nu_2 \geq \frac{n}{2} - 1 \);
6. the couples \( (p, q) \) obtained by complex interpolation from the previous couples.
Figure 1.

Figure 2.
Figures 1, 2, 3 and 4 illustrate the regions of boundedness of the Bergman projector $P_\nu$ of $\mathcal{T}_\Lambda$. An application of Theorem 2.1 (the transference principle) then gives the following result (which improves [15, Theorem 6.2.15] and [16, Theorem 2.3] for $D = D(\Gamma, F)$):
Theorem 2.4. Let $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $\nu_1 > \frac{1}{2}$ and $\nu_2 > 1$. The Bergman projector $P_\nu$ of the Pyateckii-Shapiro domain $D(\Gamma, F)$ extends to a bounded operator on $L^{p,q}_\nu(D(\Gamma, F))$ for the following values of $p$ and $q$:

1. $1 \leq p \leq \infty$ and $\frac{2p\nu_2}{2p\nu_2 - 1} < q < 2p\nu_2$;
2. $2 \leq p \leq 6$ and $\frac{\nu_2}{\nu_2 - \frac{2p}{2p'}} < q < 4\nu_2$;
3. $p > 6$ and $2 < q < \frac{2p'}{2p'} - 1$;
4. the couples $(p, q)$ obtained by symmetry and complex interpolation from the previous couples.

The couples $(p, q)$ described in the previous theorem are represented in the figure depicted below.

Further improvements will follow for homogeneous Siegel domains of type II associated to symmetric cones if one could solve the conjecture stated in [3] for tube domains over symmetric cones.

The plan of the sequel of the present paper is as follows. In section 3, we review the analysis on homogeneous cones and on homogeneous Siegel domains of type II. Most of the results in this section are taken from [16]. In section 4, we restrict to tube domains over homogeneous cones and we introduce the action of the Box operator, generalizing results from [4] for Lorentz cones. For these domains, we exhibit a necessary and sufficient condition for the boundedness of the Bergman projector in terms of a Hardy type inequality for the Box operator. In section 5, we prove the Hardy type inequality for homogeneous Siegel domains of type II and we
apply it to prove the main result, namely Theorem 2.1. The proofs of Theorem 2.3 and Theorem 2.4 are given in section 6.

3. Analysis on homogeneous cones and in homogeneous Siegel domains of type II

Let \( n \geq 3 \) and \( D \) be a homogeneous Siegel domain of type II associated to the homogeneous cone \( \Omega \) and the \( \Omega \)-Hermitian form \( F \).

3.1. Basic results on homogeneous cones and in homogeneous Siegel domains of type II

In this section, we recall the following results whose proofs are essentially in [17].

**Lemma 3.1.** [17, Corollary 4.16] Let \( \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j}{2}, \quad j = 1, \ldots, r \). Then

\[
\int_{\Omega} e^{-\langle \xi | y \rangle} Q^{-\tau}(y) dy = \Gamma_{\Omega}(\nu)(Q^*)^{-\nu}(\xi), \quad \xi \in \Omega^*,
\]

where \( \Gamma_{\Omega}(\nu) \) denotes the gamma integral [17] in the cone \( \Omega \).

**Remark 3.2.** It is well-known that the fundamental rational functions \( Q_j \) (resp. \( Q_j^* \)) can extended as a zero-free analytic function \( Q_j(\frac{z}{i}) \) (resp. \( Q_j^*(\frac{z}{i}) \)) on the tube domain \( V + i\Omega \) (resp. \( V' + i\Omega^* \)). In particular, it follows from Lemma 3.1 that if \( \zeta \in V' + i\Omega^* \) and \( \nu_j > \frac{m_j}{2}, \quad j = 1, \ldots, r \), we set

\[
\int_{\Omega} e^{i\langle \zeta | y \rangle} Q^{-\tau}(y) dy = \Gamma_{\Omega}(\nu)(Q^*)^{-\nu}\left(\frac{\zeta}{i}\right).
\]

**Lemma 3.3.** [17, Lemma 4.19] Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r \). For all \( y \in \Omega \), the integral

\[
J_{\mu\lambda}(y) = \int_{\Omega} Q^\mu(y + v)Q^{\lambda-\tau}(v) dv
\]

is finite if and only if

\[
\lambda_j > \frac{m_j}{2}, \quad \mu_j + \lambda_j < \frac{-n_j}{2}, \quad j = 1, \ldots, r.
\]

In this case, there is a positive constant \( M_{\mu\lambda} \) such that

\[
J_{\mu\lambda}(y) = M_{\mu\lambda}Q^{\mu+\lambda}(y).
\]

**Lemma 3.4.** [17, Lemma 4.20] Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \). The integral

\[
J_{\alpha}(y) = \int_{V} \left| Q^{-\alpha}\left(\frac{x + iy}{i}\right) \right| dx \quad (y \in \Omega)
\]

converges if and only if \( \alpha_j > 1 + n_j + \frac{m_j}{2}, \quad j = 1, \ldots, r \). In this case, there is a positive constant \( c_{\alpha} \) such that

\[
J_{\alpha}(y) = c_{\alpha}Q^{-\alpha+\tau}(y).
\]
The following two results were stated in [16].

**Corollary 3.5.** $A^p,q_v(D)$ is a Banach space.

**Lemma 3.6.** Assume that $\mu = (\mu_1, ..., \mu_r)$ and $\nu = (\nu_1, ..., \nu_r)$ belong to $\mathbb{R}^r$ and satisfy $\mu_j, \nu_j > \frac{m_j + b_j}{2}$, $j = 1, ..., r$. The subspace $(A^p_v \cap A^q_v)(D)$ is dense in $A^p,q_v(D)$.

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$. Following [5], we shall denote $L^2_{(-\nu)}(\Omega^* \times \mathbb{C}^m)$ the Hilbert space of functions $g : \Omega^* \times \mathbb{C}^m \to \mathbb{C}$ such that:

i) for all compact subset $K_1$ of $\mathbb{C}^n$ contained in $\Omega^*$ and for all compact subset $K_2$ of $\mathbb{C}^m$, the mapping $u \mapsto g(\cdot, u)$ is holomorphic on $K_2$ with values in $L^2(K_1, -\nu)$, where

$$L^2(K_1, -\nu) = \{ f : K_1 \to \mathbb{C} : \int_{K_1} |f(\xi)|^2 (Q^*)^{-\nu + \frac{j}{2}}(\xi) d\xi < \infty \};$$

ii) the function $g \in L^2(\Omega^* \times \mathbb{C}^m)$, $(Q^*)^{-\nu + \frac{j}{2}}(\xi)e^{-2(F(\cdot, u)\xi)} d\xi d\nu(u))$.

We then define by

$$\mathcal{L}g(z, u) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} e^{i(z|x|)}g(\xi, u) d\xi$$

the "Laplace transform" of any function $g \in L^2_{(-\nu)}(\Omega \times \mathbb{C}^m)$. Now, we recall the Plancherel-Gindikin result found in [5] Theorem II.2 which is a generalization of the Paley-Wiener Theorem [17] Theorem 5.1.

**Theorem 3.7.** Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r$ with $\nu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$. A function $G$ belongs to $A^p_v(D)$ if and only if $G = \mathcal{L}g$, with $g \in L^2_{(-\nu)}(\Omega^* \times \mathbb{C}^m)$. Moreover there is a positive constant $e_{\nu,b}$ such that

$$\|G\|_{A^p_v(D)} = e_{\nu,b} \|g\|_{L^2_{(-\nu)}(\Omega^* \times \mathbb{C}^m)}.$$

3.2. Integral operators associated to the Bergman projector. Let $\mu, \alpha \in \mathbb{R}^r$ and $f$ be a bounded function with compact support on $D$. We consider the integral operators:

$$T_{\mu,\alpha}f(z, u) = Q^{\alpha}(\Im mz - F(u, u)) \int_D B_{\mu+\alpha}((z, u), (w, t))f(w, t) dV_{\mu}(w, t)$$

and

$$T_{\mu,\alpha}^+f(z, u) = Q^{\alpha}(\Im mz - F(u, u)) \int_D |B_{\mu+\alpha}((z, u), (w, t))| f(w, t) dV_{\mu}(w, t).$$

**Theorem 3.8.** Let $\alpha, \nu, \mu \in \mathbb{R}^r$ and $1 \leq p, q \leq \infty$. Then the operator $T_{\mu,\alpha}^+$ extends boundedly to $L^p_v,q(D)$ whenever for all $j = 1, \ldots, r$ the parameters satisfy the following:

$$\mu_j + \alpha_j > \frac{1}{2}(m_j + n_j + b_j).$$
and 
\[ \mu_j q - \nu_j > (q - 1) \left( \frac{m_j}{2} + \frac{b_j}{2} \right) + \frac{n_j}{2}, \quad \alpha_j q + \nu_j > \frac{m_j}{2} + \frac{b_j}{2} + (q - 1) \frac{n_j}{2}. \]

**Proof.** We follow the scheme of [16, Proof of Theorem 2.1]. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r \) be such that \( \mu_j > \frac{m_j + b_j}{2}, \ j = 1, \ldots, r. \)

We shall denote \( U = \{(t, u) : u \in \mathbb{C}^m, t \in \Omega + F(u, u)\}. \) We define the measure \( \nu_\mu, \mu \in \mathbb{R}^r, \) on \( U \) by 
\[ d\nu_\mu(t, u) = Q^{\mu - \frac{1}{2} - \tau}(t - F(u, u))dtdv(u) \]
and we define \( L^q_\mu(U) \) as the space of all \( g : U \rightarrow \mathbb{C} \) with norm given by
\[ \|g\|^q_{L^q_\mu(U)} = \int_{\mathbb{C}^m} \int_{\Omega + F(u, u)} |g(t, u)|^q d\nu_\mu(t, u) \]
\[ = \int_{\mathbb{C}^m} |g(y + F(u, u), u)|^q Q^{\mu - \frac{1}{2} - \tau}(y)dydv(u). \]

We will need the following result.

**Proposition 3.9.** [16] Proposition 5.2 Let \( u, s \in \mathbb{C}^m; \ y \in \Omega + F(u, u) \) and \( t \in \Omega + F(s, s). \) For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r, \) the integral
\[ I_\lambda(y, u, t) = \int_{\mathbb{C}^m} Q^{-\lambda}(y + t + F(s, s) - 2 \Re F(u, u))dv(s) \]
converges if \( \lambda_j - b_j > \frac{n_j}{2}, \ j = 1, \ldots, r. \) In this case, there is a positive constant \( C_\lambda \) such that
\[ (19) \quad I_\lambda(y, u, t) = C_\lambda Q^{-\lambda + b}(y - F(u, u) + t). \]

Next, we shall use the following notation: for all \( u, s \in \mathbb{C}^m, \)
\[ A = \Re F(u, s). \]

Also we shall denote
\[ f_{y,u}(x) = f(x + iy, u). \]

Thus, for \( f \in L^{p,q}_\mu(D), \) using Minkowski’s inequality for integrals, Young’s inequality and Lemma 3.4 we get
\[ \|T_{\mu,\alpha}^+ f\|_{L^{p,q}_\mu(D)} \leq C \|R_{\mu,\alpha} g\|_{L^q_\mu(U)} \]
where for \( s \in \mathbb{C}^m \) and \( t \in \Omega + F(s, s), \)
\[ g(t, s) = \|f_{t,s}\|_p \]
and \( R_{\mu,\alpha} \) is the integral operator with positive kernel defined on \( L^q_\mu(U) \) by
\[ (20) \quad R_{\mu,\alpha} g(y, u) = Q^\alpha(y - F(u, u)) \int_{\mathbb{C}^m} \int_{\Omega + F(s, s)} Q^{-\mu - \alpha - \frac{1}{2}}(y - 2A + t)g(t, s)d\nu_\mu(t, s). \]

Observe that \( R_{\mu,\alpha} \) is a self-adjoint operator if \( \alpha = 0 \) and \( \mu = \nu. \)
To prove Theorem 3.8, it suffices therefore to prove the boundedness of the operator $R_{\mu,\alpha}$ on $L_q^2(U)$.

**Theorem 3.10.** Let $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ such that $\mu_j + \alpha_j > \frac{m_j+n_j+b_j}{2}$, $j = 1, \ldots, r$. The operator $R_{\mu,\alpha}$ is bounded on $L_q^2(U)$ whenever

$$
\mu_j q - \nu_j > (q-1) \left( \frac{m_j}{2} + \frac{b_j}{2} \right) + \frac{n_j}{2}, \quad \alpha_j q - \nu_j > \frac{m_j}{2} + \frac{b_j}{2} + (q-1) \frac{n_j}{2}.
$$

**Proof.** We will use Schur’s Lemma (See [13]). The kernel of the operator $R_{\mu,\alpha}$ relative to the measure $dV_{\nu}(t,s)$ is given by

$$
N(y,u;t,s) = Q^\alpha(y - F(u,u))Q^{-\mu - \alpha - \frac{b}{2}}(y - 2A + t)Q^{\mu - \nu}(t - F(s,s))
$$

and it is positive. By Schur’s Lemma, it is sufficient to find a positive and measurable function $\varphi$ defined on $U$ such that

$$
\int_{\mathbb{C}^m} \int_{\Omega + F(u,u)} N(y,u;t,s) \varphi(y,u) q dV_{\nu}(y,u) = C \varphi(t,s)^q
$$

and

$$
\int_{\mathbb{C}^m} \int_{\Omega + F(s,s)} N(y,u;t,s) \varphi(t,s) q' dV_{\nu}(t,s) = C \varphi(y,u)^{q'}
$$

We take as test functions $\varphi(t,s) = Q^\gamma(t - F(s,s))$ where $\gamma = (\gamma_1, \ldots, \gamma_r) \in \mathbb{R}^r$ has to be determined. The left-hand side of (21) equals

$$
K(t,s) = Q^{\mu - \nu}(t - F(s,s)) \int_{\Omega} I_{-\mu - \alpha - \frac{b}{2}}(t,s,y)Q^{\gamma q + \nu + \alpha - \frac{b}{2} - \nu}(y) dy.
$$

Using (19), we get

$$
K(t,s) = CQ^{\mu - \nu}(t - F(s,s)) \int_{\Omega} Q^{-\mu - \alpha - \frac{b}{2}}(y + t - F(s,s))Q^{\gamma q + \nu + \alpha - \frac{b}{2} - \nu}(y) dy.
$$

An application of Lemma 3.3 gives that (23) holds whenever

$$
-\nu_j - \alpha_j + \frac{m_j}{2} + \frac{b_j}{2} < \gamma_j < \frac{\mu_j - \nu_j - \frac{n_j}{2}}{q}, \quad j = 1, \ldots, r.
$$

Likewise, (22) holds when

$$
-\frac{\mu_j + \frac{m_j}{2} + \frac{b_j}{2}}{q'} < \gamma_j < \frac{\alpha_j - \frac{n_j}{2}}{q'}, \quad j = 1, \ldots, r.
$$

For these intervals to be non-empty, we need $\mu_j + \alpha_j > \frac{m_j+n_j+b_j}{2}$, $j = 1, \ldots, r$.

The identities (21) and (22) are simultaneously satisfied if each $\gamma_j$, $j = 1, \ldots, r$, satisfy the following condition

$$
\gamma_j \in \left[ -\frac{\nu_j - \alpha_j + \frac{m_j}{2} + \frac{b_j}{2}}{q'}, \frac{\mu_j - \nu_j - \frac{n_j}{2}}{q} \right] \bigcap \left[ -\frac{\mu_j + \frac{m_j}{2} + \frac{b_j}{2}}{q'}, \frac{\alpha_j - \frac{n_j}{2}}{q} \right].
$$
The intersection in (24) is not empty if \( \frac{-\nu_j - \alpha_j + \frac{m_j}{2} + \frac{b_j}{2}}{q} < \frac{\alpha_j - \frac{n_j}{2}}{q} \) and \( \frac{-\mu_j - \frac{m_j}{2} + \frac{b_j}{2}}{q} < \frac{\mu_j - \nu_j - \frac{n_j}{2}}{q} \); that is for any \( j = 1, \ldots, r \),

\[ \alpha_j q + \nu_j > (q - 1) \frac{n_j}{2} + \frac{m_j}{2} + \frac{b_j}{2} \quad \text{and} \quad \mu_j q - \nu_j > \frac{n_j}{2} + (q - 1) \left( \frac{m_j}{2} + \frac{b_j}{2} \right). \]

Corollary 3.11. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r \) such that \( \mu_j > \frac{m_j + n_j + b_j}{2}, \ j = 1, \ldots, r \) and \( \nu_j > \frac{m_j + b_j}{2}, \ j = 1, \ldots, r \). Assume that \( 1 \leq p, q \leq \infty \). Then \( P^+_\mu \) is bounded on \( L_{\nu}^{p,q}(D) \) whenever for all \( j = 1, \ldots, r \), we have

\[ \frac{\nu_j - \frac{m_j}{2} - \frac{b_j}{2} + \frac{n_j}{2}}{\mu_j - \frac{m_j}{2} - \frac{b_j}{2}} < q < 1 + \frac{\nu_j - \frac{m_j}{2} - \frac{b_j}{2}}{\frac{b_j}{2}}. \]

In this case, the Bergman projector \( P^+_\mu \) extends to a bounded operator from \( L_{\nu}^{p,q}(D) \) onto \( A_{\nu}^{p,q}(D) \).

Proof. Just take \( \alpha = 0 \) in Theorem 3.8. The case \( q = 1 \) is left as an exercise: also cf. [5]. □

Remark 3.12. Let \( k \) be a positive integer, let \( \rho \in \mathbb{R}^r \) be such that \( \rho_j > 0 \) for every \( j = 1, \ldots, r \). Let \( 1 \leq p \leq \infty \) and \( 2 < q < \infty \). In view of Corollary 3.11, the operator \( P^+_{\nu+k\rho} \) (and hence the Bergman projector \( P_{\nu+k\rho} \)) is bounded on \( L_{\nu+k\rho}^{p,q}(D) \) for \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j + n_j + b_j}{2}, \ j = 1, \ldots, r \) and \( k \) large.

Proposition 3.13. Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r \) be such that \( \nu_j > \frac{m_j + b_j}{2}, \ j = 1, \ldots, r \). Assume \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r \) and \( p, q \) are such that \( P^+_\mu \) extends as a bounded operator in \( L_{\nu}^{p,q}(D) \). Then

(i) the Bergman space \( A_{\nu}^{p,q}(D) \) is the closed linear span of the set \( \{ B_{\mu}(\cdot, (w, t)) \mid (w, t) \in D \} \). In particular, \( B_{\mu}(\cdot, (w, t)) \in L_{\nu}^{p,q}(D) \).

(ii) \( P^+_\mu \) is the identity on \( A_{\nu}^{p,q}(D) \); in particular, \( P^+_\mu(L_{\nu}^{p,q}(D)) = A_{\nu}^{p,q}(D) \).

Proof. (i) We start by establishing that \( B_{\mu}(\cdot, (w, t)) \in A_{\nu}^{p,q}(D) \) for all \( (w, t) \in D \). Let \( f, g \in C_c(D) \) (continuous functions on \( D \) with compact support). Then

\[ \langle P^+_\mu g, f \rangle_{\nu} = \int_D P^+_\mu g(z, u) \overline{f(z, u)} dV_{\nu}(z, u) = \langle g, P^+_\mu f \rangle_{\nu} \]

where

\[ P^+_\mu f(z, u) = Q^{\mu - \nu}(\Im z - F(u, u)) \int_D B_{\mu}((z, u), (w, t)) f(w, t) dV_{\nu}(w, t) \]

i.e.

\[ P^+_\mu f = T_{\nu, \mu - \nu} f \]
according to (17). By density of $C_c(D)$ in $L^{p',q'}_\nu(D)$ and continuity of $P^*_\mu$ on $L^{p',q'}_\nu(D)$, we conclude that

$$P^*_\mu f = T_{\nu,\mu - \nu} f, \; \forall f \in L^{p',q'}_\nu(D). \quad (27)$$

Now, the boundedness of $P^*_\mu$ on $L^{p',q'}_\nu(D)$ means boundedness of $T_{\nu,\mu - \nu}$ on $L^{p',q'}_\nu(D)$. Therefore, $B_\mu(\cdot, (w, t)) \in L^{p,q}_\nu(D)$ for all $(w, t) \in D$.

To conclude, we show that for $f \in L^{p',q'}_\nu(D)$ such that

$$\langle f, B_\mu(\cdot, (w, t)) \rangle_\nu = 0, \; \forall (w, t) \in D, \quad (28)$$

we also have $\langle f, F \rangle_\nu = 0$ for all $F$ in a dense subspace of $A^{p,q}_\nu(D)$. Now, identities $\langle f, B_\mu(\cdot, (w, t)) \rangle_\nu = 0$ imply that

$$T_{\nu,\mu - \nu} f = 0.$$ 

Thus, for all $F \in A^{p,q}_\nu(D) \cap A^{2,2}_\mu(D)$, we have

$$\langle f, F \rangle_\nu = \langle f, P^*_\mu f, F \rangle_\nu = \langle T_{\nu,\mu - \nu} f, F \rangle_\nu = 0.$$ 

(ii) To prove $(ii)$, we just observe that $P^*_\mu$ is the identity on the subspace $(A^{p,q}_\nu \cap A^{2,2}_\mu(D)$ which is dense on the space $A^{p,q}_\nu(D)$ by Lemma 3.6.

4. The action of the Box operator in tube domains over convex homogeneous cones

In this section, we restrict to the case where the $\Omega$–Hermitian form $F$ is identically zero and $m = 0$. We denote $T_\Omega$ the tube domain over the homogeneous cone $\Omega$ of rank $r$, i.e. $T_\Omega = V + i\Omega$. For $\nu \in \mathbb{R}^r$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$, let $L_{p,q}^\mu(T_\Omega)$ be the space of measurable functions on $T_\Omega$ such that

$$||f||_{L_{p,q}^\mu(T_\Omega)} := \left( \int_\Omega \left( \int_V |f(x + iy)|^p dx \right)^{\frac{q}{p}} Q^{\nu - \tau}(y) dy \right)^{\frac{1}{q}}$$

is finite (with obvious modification if $p = \infty$).

We call $A^{p,q}_\nu(T_\Omega)$ the subspace of $L^{p,q}_\nu(T_\Omega)$ consisting of holomorphic functions. For $A^{p,q}_\nu(T_\Omega) \neq \{O\}$, we assume in the sequel that $\nu = (\nu_1, \ldots, \nu_r)$ is such that $\nu_j > \frac{m_j}{2}$ for each $j = 1, \ldots, r$.

We call $P_\nu$ the weighted Bergman projector on $T_\Omega$ and $B_\nu$ the associated weighted Bergman kernel on $T_\Omega$.

**Definition 4.1.** Let $\rho \in \mathbb{R}^r$ be such that $\rho_j > 0$ for every $j = 1, \ldots, r$. We say that $\rho$ is an $\Omega$–integral vector if the fundamental compound function $(Q^*)^\rho(\xi)$ is a polynomial in $\xi$.

In the sequel, we fix an $\Omega$-integral vector $\rho$. We shall now adapt some proofs of [4, section 6] to tube domains over open convex homogeneous cones.
**Definition 4.2.** The generalized wave operator (the Box) $\Box = \Box_x$ on the cone $\Omega$ is the differential operator defined by the equality

$$\Box_x e^{i(x[|\xi|]} = (Q^*)^p(\xi)e^{i(x[|\xi|]} \text{ where } \xi \in \mathbb{R}^n.$$ 

When applied to a holomorphic function on the tube domain $T_\Omega$ over the cone $\Omega$, we have $\Box = \Box_x = \Box_x$ where $z = x + iy$. In view of Remark 3.2, for every $\mu \in \mathbb{R}^r$ such that $\mu_j > \frac{m_j}{2}$, $j = 1, \ldots, r$, we have

$$\Box \left( Q^{-\mu} \left( \frac{z}{i} \right) \right) = c_{\mu} Q^{-\mu-\rho} \left( \frac{z}{i} \right), \quad z \in T_\Omega.$$ 

**Lemma 4.3.** For every $f \in A^{2,2}_\nu(T_\Omega)$ and every $h \in H$, if we set $f_h(z) = f(hz)$, then

$$\Box(f_h) = Q^\rho(h \cdot e)(\Box f)_h.$$ 

**Proof.** In view of the previous theorem, we have

$$f_h(z) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} e^{i(\pi[h]z[|\xi|]} g(\xi)d\xi = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} e^{i(\pi[h]z[|\xi|]} g(\xi)d\xi$$

for some $g \in L^2(\nu, \Omega^*)$. The definition of $\Box$ and identity (6) give

$$\Box(f_h)(z) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} (Q^*)^\rho(\pi[h^*]z[|\xi|]} e^{i(\pi[h^*]z[|\xi|]} g(\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}} (Q^*)^\rho(h^* \cdot e) \int_{\Omega^*} (Q^*)^\rho(\pi[h]z[|\xi|]} e^{i(\pi[h]z[|\xi|]} g(\xi)d\xi$$

$$= Q^\rho(h \cdot e)(\Box f)_h(z).$$

since by (3), $Q^\rho(h^* \cdot e) = Q(h \cdot e).$ \hfill $\square$

**Proposition 4.4.** The Box operator $\Box^k$ is bounded from $A^{p,q}_\nu(T_\Omega)$ to $A^{p,q}_{\nu+kq}(T_\Omega)$ for all $1 \leq p, q \leq \infty$ and $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j}{2}$, $j = 1, \ldots, r$.

**Proof.** It suffices to prove the proposition for $k = 1$. The other cases are obtained by induction on $k$. We denote by $d$ the invariant distance on the cone $\Omega$. The Cauchy integral formula for derivatives implies that, if $f$ is holomorphic,

$$|\Box f(x + ie)| \leq C \int_{|\xi|<1, \, d(y,e)<1} |f(x - \xi + in)|d\xi d\eta.$$ 

Hence by the Minkowski integral inequality,

$$\|\Box f(\cdot + ie)\|_p \leq C \int_{d(y,e)<1} \|f(\cdot + in)\|_p Q^{-\tau}(\eta)d\eta.$$ 

Here we have introduced the $H$-invariant measure on the cone $\Omega$:

$$dm(\eta) = Q^{-\tau}(\eta)d\eta.$$
We recall that $Q(\eta) \sim Q(y)$ if $d(\eta, y) < 1$. (See for instance [17 section 4]). Let $f$ be in the dense subspace $(A^{p,q}_\nu \cap A^2_\nu)(T_\Omega)$ and let $y = h \cdot e$ with $h \in H$. A change of variables combined with the previous lemma gives that

$$||\Box f(\cdot + iy)||_p \leq C Q(y)^{-\rho} \int_{d(\eta,y)<1} ||f(\cdot + i\eta)||_p dm(\eta).$$

Then

$$||\Box f||_{A^{p,q}_{\nu + q\rho}(T_\Omega)}^q \leq C \int_\Omega \left( \int_{d(\eta,y)<1} ||f(\cdot + i\eta)||_p dm(\eta) \right)^q Q^{\nu - \tau}(y) dy$$

$$\leq C \int_\Omega \left( \int_{d(\eta,y)<1} ||f(\cdot + i\eta)||_p dm(\eta) \right) \left( \int_{d(\eta,y)<1} dm(\eta) \right)^{q-1} Q^{\nu - \tau}(y) dy$$

by the Hölder inequality. Since

$$(30) \int_{d(\eta,y)<1} dm(\eta) = \int_{d(\eta,e)<1} dm(\eta) = \text{const.},$$

we obtain

$$||\Box f||_{A^{p,q}_{\nu + q\rho}(T_\Omega)}^q \leq C \int_\Omega \left( \int_{d(\eta,y)<1} ||f(\cdot + i\eta)||_p Q^{\nu - \tau}(\eta) d\eta \right) Q^{\nu - \tau}(y) dy$$

$$\leq C' \int_\Omega \left( \int_{d(\eta,y)<1} ||f(\cdot + i\eta)||_p Q^{\nu - \tau}(\eta) d\eta \right) Q^{\nu - \tau}(y) dy = C' ||f||_{A^{p,q}_\nu(T_\Omega)}.$$ 

since $Q_j(y) \sim Q_j(\eta)$ when $d(\eta, y) < 1$. An application of the Fubini-Tonelli Theorem and of identity (30) gives that

$$||\Box f||_{A^{p,q}_{\nu + q\rho}(T_\Omega)}^q \leq C \int_\Omega \left( \int_{d(\eta,y)<1} Q^{\nu - \tau}(y) dy \right) ||f(\cdot + i\eta)||_p Q^{\nu - \tau}(\eta) d\eta$$

$$\leq C' \int_\Omega ||f(\cdot + i\eta)||_p Q^{\nu - \tau}(\eta) d\eta = C' ||f||_{A^{p,q}_\nu(T_\Omega)}.$$ 

We show next the relations between properties of $\Box$ and boundedness of $P_\nu$. They will give us a necessary condition for the boundedness of $P_\nu$. Our considerations are based on the identity stated in the next lemma. We set

$$M_k f(x + iy) = Q^{kp}(y) f(x + iy).$$

**Lemma 4.5.** Let $f$ be a continuous function with compact support in $T_\Omega$. Then, for every positive integer $k$,

$$(31) \Box^k (P_\nu f) = \gamma_{\nu,k} P_{\nu+k\rho}(M_k f),$$

where $\gamma_{\nu,k}$ is a non-zero constant and $\Box^k = \Box \circ \ldots \circ \Box$ $k$ times.
Proposition 4.6. Assume that $\mathbb{P}_\nu$ is bounded on $L_{\nu}^{p,q}(T_\Omega)$. Then for every positive integer $k$, $\Box^k$ is an isomorphism of $A_{\nu}^{p,q}(T_\Omega)$ onto $A_{\nu+k,p}^{p,q}(T_\Omega)$. Moreover, if $f \in A_{\nu}^{p,q}(T_\Omega)$,

\begin{equation}
\mathbb{P}_\nu \circ M_k(\Box^k f) = \gamma_{\nu,k} f.
\end{equation}

Proof. Clearly, $M_{-k}$ is an isometric isomorphism of $L_{\nu}^{p,q}(T_\Omega)$ onto $L_{\nu+k,p}^{p,q}(T_\Omega)$ with $(M_k)^{-1} = M_{-k}$. Let $f$ be a continuous function with compact support in $T_\Omega$. By (31),

\[ \mathbb{P}_{\nu+k} f = \gamma_{\nu,k}^{-1} \Box^k \circ \mathbb{P}_\nu \circ M_k f. \]

By density, $P_{\nu+k}$ is bounded on $L_{\nu+k,p}^{p,q}(T_\Omega)$. We then have the following commutative diagram.

\[
\begin{array}{ccc}
L_{\nu}^{p,q}(T_\Omega) & \xrightarrow{\mathbb{P}_\nu} & A_{\nu}^{p,q}(T_\Omega) \\
M_{-k} \downarrow & & \downarrow \gamma_{\nu,k}^{-1} \Box^k \\
L_{\nu+k,p}^{p,q}(T_\Omega) & \xrightarrow{P_{\nu+k}} & A_{\nu+k}^{p,q}(T_\Omega)
\end{array}
\]

where each map is continuous. By Remark 3.12 and Proposition 3.13, $\mathbb{P}_{\nu+k}$ is onto; then also $\Box^k$ is onto.

The rest of the proof goes in the following order. We first prove that $\Box^k$ is one to one for $p = q = 2$ (in which case $P_\nu$ is obviously bounded). Then we prove (32) and finally we show that $\Box^k$ is one to one for general $p, q$.

Let $f = Lg \in A_{\nu}^{2}(T_\Omega)$. Then

\[ \Box^k f = C_k L(Q^{k,p} g) \in A_{\nu+2kp}^{2}(T_\Omega). \]

By Theorem 3.7, if $\Box^k f = 0$, then $g = 0$ a.e. and hence $f = 0$.

In order to prove (32), it suffices to take $f \in (A_{\nu}^{p,q} \cap A_{\nu}^{2,q})(T_\Omega)$, since the left-hand side of (32) involves continuous operators.

Calling $G$ the left-hand side of (32), then $G \in (A_{\nu}^{p,q} \cap A_{\nu}^{2,2})(T_\Omega)$ and, by the commutativity of the diagram above,

\[ \Box^k G = \gamma_{\nu,k} \mathbb{P}_{\nu+k,p}(\Box^k f) = \gamma_{\nu,k} \Box^k f. \]

By the injectivity of $\Box^k$ on $A_{\nu}^{2,2}(T_\Omega)$, we conclude that $G = \gamma_{\nu,k} f$.

Finally assume that $f \in A_{\nu}^{p,q}(T_\Omega)$. Then the assumption $\Box^k f = 0$ implies that $f = 0$ by (32). \hfill \Box

The following theorem was proved in [17, Theorem 6.8].

Theorem 4.7. Let $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + n_j + b_j}{2}$ for each $j = 1, \ldots, r$. We set $q_\nu := 1 + \min_{1 \leq j \leq r} \frac{\nu_j - m_j}{2}$. Let $1 \leq p \leq \infty$ and $2 \leq q < \infty$. Then the weighted Bergman projector $\mathbb{P}_\nu$ is bounded from $L_{\nu}^{p,q}(T_\Omega)$ to $A_{\nu}^{p,q}(T_\Omega)$ for

\[
\begin{cases}
0 \leq \frac{1}{p} \leq \frac{1}{2} & \text{or} & \frac{1}{2} \leq \frac{1}{p} \leq 1 \\
\frac{1}{q_\nu p'} < \frac{1}{q} < 1 - \frac{1}{q_\nu p'}
\end{cases}
\]
The following statement is a direct consequence of Proposition 4.6 and Theorem 4.7.

**Corollary 4.8.** For the values of \( p, q \) given in Theorem 4.7, \( \Box^k \) is an isomorphism of \( A^{p,q}_\nu(T_\Omega) \) onto \( A^{p,q}_{\nu+k\rho}(T_\Omega) \) for every positive integer \( k \).

A partial converse of Corollary 4.8 also holds. (See [1, 3] for tube domains over symmetric cones and [4] for tube domains over Lorentz cones).

**Theorem 4.9.** Assume that \( 1 \leq p < \infty \) and \( 2 \leq q < \infty \) and that for some \( k \geq k_0 \), the inequality

\[
||f||_{A^{p,q}_\nu(T_\Omega)} \leq C||\Box^k f||_{A^{p,q}_{\nu+k\rho}(T_\Omega)}
\]

holds. Then \( P_\nu \) is bounded on \( L^{p,q}_\nu(T_\Omega) \).

5. PROOF OF THEOREM 2.1

In this section, we start by proving the Hardy-type inequality for the homogeneous Siegel domain \( D \) of type II. The notations are those of section 2.

**Theorem 5.1.** Let \( \nu \in \mathbb{R}^r \) be such that \( \nu_i > \frac{m_i+n_i+h_i}{2}, \; i = 1, \ldots, r \). Let \( 1 \leq p < \infty \) and \( 2 \leq q < \infty \). Assume that there exists a positive integer \( k \) and a positive constant \( C = C(k, p, q, \nu) \) such that for all \( f \in A^{p,q}_\nu(D) \), the following Hardy type inequality holds.

\[
\begin{align*}
&\int_{\mathbb{C}^n} \int_{\Omega + F(u,u)} \left( \int_V |f(x + iy, u)|^p dx \right)^{\frac{2}{p}} Q^{\nu-\tau-\frac{\nu}{2}}(y - F(u, u)) dy dv(u) \\
&\leq C \int_{\mathbb{C}^n} \int_{\Omega + F(u,u)} \left( \int_V |\Box^k f(x + iy, u)|^p dx \right)^{\frac{2}{p}} Q^{\nu+k\rho-\tau-\frac{\nu}{2}}(y - F(u, u)) dy dv(u).
\end{align*}
\]

Then the Bergman projector \( P_\nu \) of \( D \) admits a bounded extension to \( L^{p,q}_\nu(D) \).

**Proof.** We adapt the proof of [3, Theorem 1.3], for tube domains over symmetric cones. In this reference, \( \nu \) is a real number. We want to prove the existence of some constant \( C \) such that, for \( f \in (L^{p,q}_\nu \cap L^{2,2}_\nu)(D) \), we have the inequality

\[
||P_\nu f||_{A^{p,q}_\nu(D)} \leq C||f||_{L^{p,q}_\nu(D)}.
\]

Consider such an \( f \) with \( ||f||_{L^{p,q}_\nu(D)} = 1 \). Call \( G = P_\nu f \). By Fatou’s Lemma, it is sufficient to prove that the functions \( G_\epsilon(z, u) := G(z + i\epsilon u, u) \), which belong to \( A^{p,q}_\nu(D) \), have norms uniformly bounded. So using (33), it is sufficient to show that \( \Box^k G_{\epsilon} \) is uniformly in \( L^{p,q}_{\nu+k\rho}(D) \). To prove this, we apply (29) to obtain the identity

\[
\Box^k G_{\epsilon}(z, u) = C \int_D B_{\nu+k\rho}((z + i\epsilon u, u), (w, t)) f(w, t) dV_\nu(w, t).
\]

In view of Proposition 4.4, it suffices to prove that \( P_\nu \) is bounded on \( L^{p,q}_{\nu+k\rho}(D) \) for \( k \) large. We apply Remark 3.12 to conclude. \( \square \)
By Theorem 5.1, it suffices to prove the existence of a positive integer \( k \) such that the Hardy type inequality (33) is valid. If we make the change of variable \( y' = y - F(u, u) \), this inequality takes the following form.

\[
\int_{\mathbb{C}^n} \left( \int_{V} \left( \int_{V} |f(x + iy' + iF(u, u), u)|^p dx \right)^\frac{2}{p} Q^{\nu - \frac{k}{2}}(y')dy' \right) dv(u)
\]

\[
\leq C \int_{\mathbb{C}^n} \left( \int_{V} \left( \int_{V} |\Box_{\mathbb{C}}^k f(x + iy' + iF(u, u), u)|^p dx \right)^\frac{2}{p} Q^{\nu + kq_0 - \frac{k}{2}}(y')dy' \right) dv(u).
\]

For any fixed \( u \in \mathbb{C}^n \), we consider the holomorphic function \( f_u : T_\Omega \rightarrow \mathbb{C} \) defined by

\[
f_u(x + iy') = f(x + iy' + iF(u, u), u).
\]

Also observe that the property \( f \in A_\nu^{p,q}(D) \) can be expressed in the following form.

\[
||f||_{A_\nu^{p,q}(D)}^q = \int_{\mathbb{C}^n} \left( \int_{V} \left( \int_{V} |f(x + iy' + iF(u, u), u)|^p dx \right)^\frac{2}{p} Q^{\nu - \frac{k}{2}}(y')dy' \right) dv(u) < \infty.
\]

An application of the Fubini Theorem gives that, for almost all \( u \in \mathbb{C}^n \), the function \( f_u \) belongs to \( A_\nu^{p,q}(T_\Omega) \).

Assume that the Bergman projector \( P_{\nu - \frac{k}{2}} \) of \( T_\Omega \) is bounded on \( A_\nu^{p,q}(T_\Omega) \). By Proposition 4.6, this implies that for every positive integer \( k \), \( \Box^k \) is an isomorphism of \( A_\nu^{p,q}(T_\Omega) \) onto \( A_{\nu + kq_0 - \frac{k}{2}}^{p,q}(T_\Omega) \). So there exists a positive constant \( C \) such that

\[
\int_{\Omega} \left( \int_{V} |f_u(x + iy)|^p dx \right)^\frac{2}{p} Q^{\nu - \frac{k}{2}}(y)dy
\]

\[
\leq C \int_{\Omega} \left( \int_{V} |\Box_{\mathbb{C}}^k f_u(x + iy)|^p dx \right)^\frac{2}{p} Q^{\nu + kq_0 - \frac{k}{2}}(y)dy
\]

for almost all \( u \in \mathbb{C}^n \). An integration with respect to \( u \) finishes the proof.

6. The case of the Pyateckii-Shapiro Siegel domain of type II

6.1. Bergman projections and Besov-type spaces in tube domains over symmetric cones. Let \( \Omega \) be a symmetric cone of rank \( r \) in a Euclidean Jordan algebra \( V \). We describe a Littlewood-Paley decomposition adapted to to the geometry of \( \Omega \). Referring to [2], we call \( d \) the invariant distance in \( \Omega \). Let \( \{ \xi_j \} \) be a fixed \((\frac{1}{2}, 2)\)-lattice in \( \Omega \) and let \( B_j \) be the d-ball \( B_1(\xi_j) \) with centre \( \xi_j \) and radius 1. These balls \( \{ B_j \} \) form a covering of \( \Omega \). We choose a real function \( \varphi_0 \in C_c^\infty(B_2(e)) \) such that

\[
0 \leq \varphi_0 \leq 1, \quad \text{and} \quad \varphi_0|_{B_1(e)} \equiv 1.
\]

We write \( \xi_j = g_j e \), for some \( g_j \in T \). Then, we can define \( \varphi_j(\xi) = \varphi_0(g_j^{-1}\xi) \), so that

\[
\varphi_j \in C_c^\infty(B_2(\xi_j)), \quad 0 \leq \varphi_j \leq 1, \quad \text{and} \quad \varphi_j|_{B_j} \equiv 1.
\]
We assume that $\xi_0 = e$ to avoid ambiguity of notation. By the finite intersection property of the lattice $\{\xi_j\}$, there exists a constant $c > 0$ such that

$$\frac{1}{c} \leq \Phi(\xi) := \sum_j \varphi_j(\xi) \leq c.$$  

We define the function $\psi_j$ by $\hat{\psi}_j = \frac{\varphi_j}{\Phi}$. The Besov-type spaces, $B^{p,q}_\nu$, $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, adapted to this Littlewood-Paley decomposition are defined as the equivalence classes of tempered distributions which have finite seminorms

$$\|f\|_{B^{p,q}_\nu} = \left[ \sum_j (Q^* - \nu_j)(\xi_j) \|f * \psi_j\|^q_p \right]^{\frac{1}{q}}.$$  

When $n = 1$, and $\xi_j = 2^j$, the norm (34) corresponds to the classical Besov space $B^{-\nu/q}_p$ ($\mathbb{R}$). We denote by $S_\Omega$ the space of Schwartz functions $f : V \to \mathbb{C}$ with $\text{Supp} \hat{f} \subset \overline{\Omega}$. One basic tool is a special decomposition for functions in $S_\Omega$,

$$f = \sum_j f * \psi_j, \text{ for all } f \in S_\Omega.$$  

Moreover, we call $D_\Omega$ the subspace of $S_\Omega$ consisting of those functions whose support is compact in $\Omega$. We point out that the subspace $D_\Omega$ is dense in $B^{p,q}_\nu$.

We further refer to [10], [11] and [2]. We normalize the Fourier transform by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^n} \int_V e^{-i(\xi|\xi)} f(x) dx, \text{ for } \xi \in V$$

and like in section 3, we define the Laplace transform $\mathcal{L}$ by

$$\mathcal{L}(f)(z) = \frac{1}{(2\pi)^\frac{n}{2}} \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi, \text{ for } z \in T_{\Omega}.$$  

We call $\mathcal{C}$ the operator $C(f) = \mathcal{L}(\hat{f})$. We call $p'$ the conjugate index of $p$ and for $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r$, we adopt the following notations.

$$p_\nu = \min(p, p');$$

$$p_\nu = 1 + \min_{j=1, \cdots, r} \frac{\nu_j + \frac{n_j}{2}}{\nu_j - \nu_j};$$

$$q_\nu = 1 + \min_{j=1, \cdots, r} \frac{\nu_j - \frac{m_j}{2}}{\nu_j};$$

$$q_\nu(p) = p_\nu q_\nu;$$

$$\tilde{q}_{\nu,p} = \min_{j=1, \cdots, r} \frac{\nu_j + \frac{n_j}{2}}{\nu_j - \nu_j + \frac{m_j}{2} + (1 + \frac{m_j}{2})}.$$
We record the following theorem due to D. Debertol. (See for instance [10] Theorem 4.6, [11] Theorem 1.2 and Corollary 4.7).

**Theorem 6.1.** Let \( \nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j}{2} \), \( j = 1, \ldots, r \) and let \( 1 < p < \infty \) and \( 1 < q < \tilde{q}_{\nu,p} \). Then, for every function \( f \) in \( V \) such that \( \|f\|_{B^p_q} < \infty \) and \( f = C \). Moreover we have

\[
\begin{align*}
(1) \quad & \lim_{y \to 0, y \in \Omega} F(\cdot + iy) = f \quad \text{both in} \quad \mathcal{S}'(V) \quad \text{and in} \quad B^p_q; \\
(2) \quad & \|f\|_{B^p_q} \lesssim \|F\|_{A^p_q}.
\end{align*}
\]

D. Debertol also proved the following theorem found in [10] Theorem 5.8 and [11] Theorem 1.3.

**Theorem 6.2.** Let \( \nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j}{2} \), \( j = 1, \cdots, r \) and \( 1 < p < p', \quad q'(p) < q < \tilde{q}_{\nu,p} \). The following assertions are equivalent.

1. The Bergman projector \( P_\nu \) of \( T_\Omega \) admits a bounded extension from \( L^p_q(T_\Omega) \) to \( A^p_q(T_\Omega) \).
2. The operator \( C \) is an isomorphism from \( B^p_q \) to \( A^p_q(T_\Omega) \).

The following two results are generalizations of [2] Theorem 4.11 and [2] Lemma 4.14. For the analysis on symmetric cones, we also refer to [12].

**Theorem 6.3.** Let \( \nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r \) such that \( \nu_j > \frac{m_j}{2} \), \( j = 1, \ldots, r \) and \( 1 \leq p, \quad s < \infty \). Assume that there exist a number \( \delta > 0 \), a vector \( \mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r \) with \( \mu_j > 0, = 1, \ldots, r \) and a constant \( C = C(\mu, \delta) > 0 \) such that the estimate

\[
\left\| \sum_j f_j \right\|_p \leq C \left[ \sum_j (Q^\nu)^{-\mu} (\xi_j) e^{\delta(\xi_j|e)} \|f_j\|_p \right]^{\frac{1}{2}}
\]

holds for every finite sequence \( \{f_j\} \subset L^p(V) \) with supp \( \hat{f}_j \subset B_2(\xi_j) \). We assume that the index \( q \) satisfies one of the following conditions.

(i) \( 1 \leq q \leq s \) and \( q < s \min_{j=1,\ldots,r} \frac{\nu_j - m_j}{\mu_j} \);

(ii) \( s < q < s \min_{j=1,\ldots,r} \left\{ s \min_{j=1,\ldots,r} \frac{\nu_j - m_j + \frac{3}{2}}{\mu_j + \frac{3}{2}}, \tilde{q}_{\nu,p} \right\} \).

Then for every function \( f \in S_\Omega \), the function \( F = C(f) \) belongs to \( A^p_q(T_\Omega) \), and moreover,

\[
\|F\|_{A^p_q(T_\Omega)} \lesssim \|f\|_{B^p_q(T_\Omega)}.
\]

**Lemma 6.4.** Let \( 1 \leq p, \quad s < \infty \) and assume that \( (36) \) holds for some number \( \delta > 0 \) and some vector \( \mu = (\mu_1, \cdots, \mu_r) \in \mathbb{R}^r \). Then for every function \( f \in \mathcal{D}_\Omega \) and \( y \in \Omega \), the function \( F(\cdot + iy) = \mathcal{F}^{-1}(\hat{f} e^{-|y|^2}) \) belongs to \( L^p(V) \). Moreover,

\[
\|F(\cdot + iy)\|_p \lesssim Q^{-\frac{p}{2}}(y) \|f\|_{B^p_q(V)}
\]

with constants independent of \( f \) or \( y \in \Omega \).
In the proofs, we denote \( \{ \chi_j \} \) a family of functions defined as \( \hat{\chi}_j(\xi) := \hat{\chi}(g_j^{-1}(\xi)) \) from an arbitrary \( \hat{\chi} \in C_c^\infty(B_4(e)) \) so that \( 0 \leq \hat{\chi} \leq 1 \) and \( \hat{\chi} \) is identically 1 in \( B_2(e) \).

We shall use the following estimate (formula (3.47) of [2]): there exist two positive numbers \( C \) and \( \gamma \) such that

\[
\| F^{-1}(\hat{\chi}_j e^{-|y|}) \|_1 \leq C e^{-(g_j y / e)}
\]

**Proof of Lemma 6.4.** By homogeneity (see [10, Proposition 3.19]), it is sufficient to prove (37) when \( y = \eta e \), for some fixed \( \eta > 0 \) to be chosen below. Let us denote \( \hat{\gamma} = \hat{f} e^{-\eta(e)} \), so that \( g = \sum_j g \ast \psi_j \in S_\Omega \). Applying (36) to the sequence \( \{ f_j = g \ast \psi_j \} \) and using the Young inequality, we obtain

\[
\| F(\cdot + i \eta e) \| = \| g \| = \| \sum_j g \ast \psi_j \| \leq \left[ \sum_j (Q_j)^{-\mu}(\xi_j) e^{\delta(\xi_j | e)} \| (g \ast \psi_j \|_p^s \right]^{1/2}
\]

\[
\leq \left[ \sum_j (Q_j)^{-\mu}(\xi_j) e^{\delta(\xi_j | e)} \| F^{-1}(\hat{\psi}_j e^{-\eta(e)}) \|_p^s \right]^{1/2}
\]

\[
\leq \left[ \sum_j (Q_j)^{-\mu}(\xi_j) e^{\delta(\xi_j | e)} \| f \ast \psi_j \|_p^s \| F^{-1}(e^{-\eta(e)} \hat{\chi}_j) \|_1^s \right]^{1/2}.
\]

Now, \( \| F^{-1}(e^{-\eta(e)} \hat{\chi}_j) \|_1 \) is bounded by a constant times \( e^{-\gamma \eta(\xi_j | e)} \) by formula (38). Therefore,

\[
\| F(\cdot + i \eta e) \| \leq \left[ \sum_j (Q_j)^{-\mu}(\xi_j) e^{\delta(\xi_j | e) - \gamma \eta(\xi_j | e)} \| f \ast \psi_j \|_p^s \right]^{1/2}
\]

We only need to choose \( \eta \) larger than \( \delta / \gamma \). \( \square \)

We now conclude the proof of Theorem 6.3. Given \( f \in D_\Omega \) and \( F := C f \), Lemma 6.4 applied to \( F^{-1}(\hat{f} e^{-|y|}) \) gives us

\[
\| F(\cdot + i 2y) \| \leq Q^{-\mu}(y) \left[ \sum_j (Q_j)^{-\mu}(\xi_j) \| F^{-1}(\hat{\psi}_j e^{-|y|}) \|_p^s \right]^{1/2}
\]

\[
\leq Q^{-\mu}(y) \left[ \sum_j (Q_j)^{-\mu}(\xi_j) e^{-\gamma(j | y)} \| f \ast \psi_j \|_p^s \right]^{1/2},
\]
where we have used formula (38) and Young’s inequality again. Thus

\[
I := \int_{\Omega} \|F(\cdot + i2y)\|_{p}^{\alpha} Q^{\nu - \frac{n}{2}}(y)dy \\
\leq \int_{\Omega} Q^{-\frac{m}{s}}(y) \left[ \sum_{j} (Q^{*})^{-\mu}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \|f \ast \psi_j\|_{p}^{\alpha} \right]^{\frac{2}{\alpha}} Q^{\nu - \frac{n}{2}}(y)dy.
\]

When \(q/s \leq 1\) i.e. \(q \leq s\), then

\[
I \leq \int_{\Omega} Q^{-\frac{m}{s}}(y) \sum_{j} (Q^{*})^{-\mu}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \|f \ast \psi_j\|_{p}^{\alpha} Q^{\nu - \frac{n}{2}}(y)dy \\
\leq \sum_{j} (Q^{*})^{-\mu}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} \int_{\Omega} e^{-\gamma(\xi_j)\|y\|} Q^{-\frac{m}{s} + \nu - \frac{n}{2}}(y)dy \\
\leq \sum_{j} (Q^{*})^{-\mu}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} = \sum_{j} (Q^{*})^{-\nu}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha}
\]

provided \(-\mu_j \frac{q}{s} + \nu_j > \frac{m_j}{2}, j = 1, \ldots, r\) thanks to Lemma 3.1 This leads to condition (i).

Assume now that \(q/s > 1\) i.e. \(q > s\). Let \(\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{R}^r\) be a vector to be chosen below. We have

\[
I \leq \int_{\Omega} Q^{-\frac{m}{s}}(y) \left[ \sum_{j} (Q^{*})^{-\mu}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} (Q^{*})^{\beta}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \right]^{\frac{2}{\alpha}} Q^{\nu - \frac{n}{2}}(y)dy
\]

Then by Hölder’s inequality,

\[
(39) \quad \left[ \sum_{j} (Q^{*})^{-\mu}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} (Q^{*})^{\beta}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \right]^{\frac{2}{\alpha}} \leq \\
\left[ \sum_{j} (Q^{*})^{-\frac{\mu}{2} + \beta}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} e^{-\gamma(\xi_j)\|y\|} \right] \left[ \sum_{j} (Q^{*})^{\beta}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \right]^{(\frac{2}{\alpha})/(\frac{2}{\alpha})'}
\]

But

\[
\sum_{j} (Q^{*})^{\beta}(\xi_j) e^{-\gamma(\xi_j)\|y\|} \sim \int_{\Omega^{*}} e^{-\gamma(\xi_j)\|y\|} (Q^{*})^{\beta}(\xi_j) \frac{1}{\gamma}(\xi_j)e^{-\gamma(\xi_j)\|y\|}d\xi \sim Q^{-\beta(\frac{2}{\alpha})'}(y)
\]

provided \(\beta_j \left(\frac{q}{s}\right) > \frac{n_j}{2}, j = 1, \ldots, r\) thanks to Lemma 3.1. It follows that

\[
I \leq \int_{\Omega} Q^{-\frac{m}{s} - \beta(\frac{2}{\alpha})'}(y) \left[ \sum_{j} (Q^{*})^{-\frac{\mu}{2} + \beta}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} e^{-\gamma(\xi_j)\|y\|} \right] Q^{\nu - \frac{n}{2}}(y)dy \\
\leq \sum_{j} (Q^{*})^{-\frac{\mu}{2} + \beta}(\xi_j) \|f \ast \psi_j\|_{p}^{\alpha} \int_{\Omega} e^{-\gamma(\xi_j)\|y\|} Q^{-\frac{\mu}{2} + \nu - \frac{n}{2}}(y)dy
\]
The last integral converges if \(-\frac{q}{s}(\mu_j + \beta_j) + \nu_j > \frac{m_j}{2}, j = 1, \ldots, r\) thanks to Lemma 3.1. It follows that

\[
I \lesssim \sum_j (Q^*)^{-\nu}(\xi_j) \|f \ast \psi_j\|_p^q
\]

whenever \(\beta_j \left(\frac{q}{s}\right) > \frac{n_j}{2}, j = 1, \ldots, r\) and \(-\frac{q}{s}(\mu_j + \beta_j) + \nu_j > \frac{m_j}{2}, j = 1, \ldots, r\). That is we will conclude with \(I := \int_\Omega \|F(\cdot + i2y)\|_p^q Q^{\nu}(y)dy \lesssim \|f\|_{B^q_p}\) if we can choose real numbers \(\beta_j\) so that

\[
\frac{n_j}{2} \left(1 - \frac{1}{q/s}\right) < \beta_j < \frac{1}{q/s} \left(\nu_j - \frac{m_j}{2}\right) - \mu_j, j = 1, \ldots, r.
\]

Solving for \(q/s\) we get

\[
\frac{q}{s} < \min_{j=1,\ldots,r} \frac{r_j - \frac{m_j}{2} + \frac{n_j}{2}}{\mu_j + \frac{n_j}{2}}
\]

which gives condition (ii).

From Theorems 6.1 and 6.2, we deduce the following corollary for tube domains over symmetric cones.

**Corollary 6.5.** Let \(\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r\) such that \(\nu_j > \frac{m_j}{2}, j = 1, \cdots, r, 1 < p < p_0\) and \(1 < s < \infty\). Assume further that there is a positive number \(\delta\) and a vector \(\mu = (\mu_1, \cdots, \mu_r) \in \mathbb{R}^r\) with \(\mu_j > 0, j = 1, \cdots, r\) and a constant \(C = C(\mu, \delta) > 0\) such that

\[
\left\|\sum_j f_j\right\|_p \leq C \left[\sum_j (Q^*)^{-\mu}(\xi_j)e^{\delta(|\xi|e)}\|f_j\|_p^s\right]^{\frac{1}{2}}
\]

holds for every finite sequence \(\{f_j\} \in L^p(V)\) with \(\text{Supp } \hat{f}_j \in B_2(\xi_j)\). We assume that for the index \(q\), we are in one of the following two situations.

(a) If \(q'(p) < s\), then there are two cases:

(i) \(q'(p) < q \leq s\) and \(q < \min\left\{s \min_{j=1,\cdots,r} \frac{r_j - \frac{m_j}{2}}{\mu_j}, q_{\nu,p}\right\}\);

(ii) \(s < q < \min\left\{s \min_{j=1,\cdots,r} \frac{r_j - \frac{m_j}{2} + \frac{n_j}{2}}{\mu_j + \frac{n_j}{2}}, q_{\nu,p}\right\}\).

(b) If \(q'(p) \geq s\), then

\[
q'(p) < q < \min\left\{s \min_{j=1,\cdots,r} \frac{r_j - \frac{m_j}{2} + \frac{n_j}{2}}{\mu_j + \frac{n_j}{2}}, q_{\nu,p}\right\}\).

Then the Bergman projector \(P_\nu\) admits a bounded extension from \(L^q_{p,q}(T_\Omega)\) to \(A^q_{p,q}(T_\Omega)\).

6.2. The proof of Theorem 2.3. We refer to [2, section 5]. Let \(\Omega = \Lambda_n\) denote the Lorentz cone in \(\mathbb{R}^n, n \geq 3\), defined by

\[
\Lambda_n = \{y = (y_1, y') \in \mathbb{R}^n : \Delta_1(y) > 0, \ \Delta_2(y) > 0\},
\]
with \( \Delta_1(y) = y_1 \) and \( \Delta_2(y) = y_1^2 - |y'|^2 \). The rank of \( \Lambda_n \) is \( r = 2 \). For \( j \geq 1 \), take a maximal \( 2^{-j} \)-separated sequence \( \{ \omega_k^{(j)} \}_{k=1}^{k_j} \) of points of the sphere \( \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1} \), with respect to the Euclidean distance (so that \( k_j \sim 2^{(n-2)} \)). Then define the sets

\[
E_{j,k} = \left\{ (\xi_1, \xi') \in \Lambda_n : 2^{-1} < \tau < 2, \ 2^{-2j-2} < 1 - \frac{\|\xi'\|^2}{\xi_1^2} < 2^{-2j+2} \text{ and } \left| \frac{\xi'}{\xi_1} - \omega_k^{(j)} \right| \leq \delta 2^{-j} \right\}
\]

where the constant \( \delta > 0 \) is suitably chosen \([2]\).

Recall that for \( \Lambda_n \) we have \( r = 2 \), \( m_j = (2-j)d \), \( n_j = (j-1)d \) with \( d = n-2 \). Thus for \( \nu = (\nu_1, \nu_2) \in \mathbb{R}^2 \) such that \( \nu_1 > \frac{n}{2} - 1 \), \( \nu_2 > 0 \), we have

\[
p_\nu = 1 + \frac{\nu_2 + \frac{n}{2}}{\left( \frac{n}{2} - 1 - \nu_2 \right)_+};
\]

\[
q_\nu = 1 + \min_{1 \leq i \leq 2} \frac{\nu_i - \frac{m_i}{2}}{\frac{m_i}{2}} = 1 + \frac{\nu_2}{\nu_2 + \left( 1 - \frac{1}{\nu_1} \left( \frac{n}{2} - 1 \right) \right)};
\]

\[
q'_\nu(p) = \frac{\nu_2 + \frac{n}{2} - 1}{\nu_2 + \left( 1 - \frac{1}{\nu_1} \left( \frac{n}{2} - 1 \right) \right)};
\]

\[
\tilde{q}_{\nu,p} = \min_{1 \leq i \leq 2} \frac{\nu_j + (j-1)d_2}{\left( \frac{n}{2\nu_1} - 1 - (2-j)d_2 \right)_+ + \left( \frac{n}{2\nu_1} - 1 \right)_+} = \frac{\nu_2 + \frac{n}{2} - 1}{\nu_2 + \left( 1 - \frac{1}{\nu_1} \left( \frac{n}{2} - 1 \right) \right)}.
\]

The following theorem is a consequence of \([2] \) Proposition 5.5] and Corollary 6.5 above.

**Theorem 6.6.** Let \( 1 \leq p < p_\nu \), \( 1 \leq s < \infty \). Suppose that for some \( \mu \geq 0 \) there exists a constant \( C_\mu \) such that

\[
\left( \sum_{k=1}^{k_j} f_k \right) \leq C_\mu 2^{2\mu} \left( \sum_{k=1}^{k_j} \|f_k\|_p^s \right)^{\frac{1}{s}} \text{ for all } j \geq 1,
\]

for every sequence \( \{ f_k \} \) satisfying \( \text{Supp } \hat{f}_k \subset E_{j,k} \). We assume that for the index \( q \) we are in one of the following two situations.

(a) If \( q'_\nu(p) < s \), then there are two cases:

   (i) \( q'_\nu(p) < q \leq s \) and \( q < \min \left\{ s \min_{j=1,\ldots,r} \frac{\nu_j - \frac{m_j}{2}}{\mu}, \tilde{q}_{\nu,p} \right\} \);

   (ii) \( s < q < \min \left\{ s \min_{j=1,\ldots,r} \frac{\nu_j - \frac{m_j}{2} + \frac{n_j}{2}}{\mu + \frac{n_j}{2}}, \tilde{q}_{\nu,p} \right\} \).

(b) If \( q'_\nu(p) \geq s \), then

\[
q'_\nu(p) < q < \min \left\{ s \min_{j=1,\ldots,r} \frac{\nu_j - \frac{m_j}{2} + \frac{n_j}{2}}{\mu + \frac{n_j}{2}}, \tilde{q}_{\nu,p} \right\}.
\]
Then $P_\nu$ is bounded in $L^{p,q}_p$.

The following theorem is a consequence of the $l^2$-decoupling theorem recently proved by Bourgain and Demeter [3].

**Theorem 6.7.** The estimate (40) is valid for $s = 2$ and the following values of $p$ and $\mu$:

1. $2 \leq p \leq \frac{2n}{n-2}$ and $\mu = 0$;
2. $p \geq \frac{2n}{n-2}$ and $\mu = \frac{n-2}{2} - \frac{n}{p}$.

We deduce the following corollary.

**Corollary 6.8.** Let $n \geq 3$ and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ such that $\nu_1 > \frac{n}{2} - 1$, $\nu_2 > 0$. The weighted Bergman projector $P_\nu$ is bounded in $L^{p,q}_p(T_{\Lambda_n})$ for the following values of $p, q$ and $\nu$:

1. $2 \leq p \leq \frac{2n}{n-2}$, $p < p_\nu$ and $q_\nu(p) < q < 2q_\nu$;
2. $p_\nu > p \geq \frac{2n}{n-2}$, $q_\nu(p) < q < \min\left\{2^{\nu_1 - \frac{n}{2} + 1} \frac{1}{2 - 1 - \frac{n}{p}}, q_\nu(p)\right\}$ provided $0 < \nu_2 < \frac{n}{2} - 1$;
3. $p_\nu > p > \frac{2n}{n-2}$ and $2 < q < \min\left\{2^{\nu_1 - \frac{n}{2} + 1} \frac{1}{2 - 1 - \frac{n}{p}}, q_\nu(p)\right\}$ provided $\nu_2 \geq \frac{n}{2} - 1$.

**Proof.** We first notice that if $p \geq 2$, the following equivalence holds

\[(41) \quad q_\nu(p) < 2 \quad \text{if and only if} \quad \nu_2 > \left(\frac{n}{2} - 1\right)\left(1 - \frac{2}{p}\right).\]

Also,

\[(42) \quad \tilde{q}_\nu(p) \geq 2q_\nu > 2 \quad \text{for all} \quad 1 \leq p \leq \frac{2n}{n-2} \quad \text{and} \quad \nu_2 > 0.

1) Suppose first that $2 \leq p \leq \frac{2n}{n-2}$ and $\mu = 0$. Then by Theorem 6.7, estimate (40) is satisfied for $s = 2$. We distinguish two cases.

**Case 1.** We suppose that $\nu_2 > \left(\frac{n}{2} - 1\right)\left(1 - \frac{2}{p}\right)$. Then by equation (41), we have $q_\nu(p) < 2$. It follows from Theorem 6.6 that $P_\nu$ is bounded in $L^{p,q}_p(T_{\Lambda_n})$ if

- $q_\nu(p) < q \leq 2$ and $q < \tilde{q}_\nu(p)$
- $2 < q < \min\{2q_\nu, \tilde{q}_\nu(p)\} = 2q_\nu$

and the last equality above follows from equation (42).

**Case 2.** We suppose that $0 < \nu_2 \leq \left(\frac{n}{2} - 1\right)\left(1 - \frac{2}{p}\right)$. Then by (1), we have $q_\nu(p) \geq s = 2$. It follows from Theorem 6.6 that $P_\nu$ is bounded in $L^{p,q}_p(T_{\Lambda_n})$ if $q_\nu(p) < q < \min\{2q_\nu, \tilde{q}_\nu(p)\} = 2q_\nu$ and the last equality above again follows from equation (42). This proves the assertion (1) of the corollary.

2) Observe that $p = \frac{2n}{n-2}$ means that $\mu = 0$. We assume then that $p > \frac{2n}{n-2}$. Notice that

\[p_\nu = 1 + \frac{\nu_2 + \frac{n}{2}}{(\frac{n}{2} - 1 - \nu_2)_+} = \begin{cases} 1 + \frac{\nu_2 + \frac{n}{2}}{\frac{2}{2} - 1 - \nu_2} & \text{if} \quad 0 < \nu_2 < \frac{n}{2} - 1 \\ \infty & \text{elsewhere} \end{cases}.\]
This suggests two cases: $0 < \nu_2 < \frac{n}{2} - 1$ and $\nu_2 \geq \frac{n}{2} - 1$. Also, the case $p_\nu \leq \frac{2n}{n-2}$ is irrelevant since we must have $p < p_\nu$: it would refer to assertion (1) of the corollary. This suggests to replace the first case by $\frac{n-2}{2n} < \nu_2 < \frac{3}{2} - 1$. Furthermore, note that $p > \frac{2n}{n-2}$ is equivalent to $\mu = \frac{n}{2} - 1 - \frac{n}{p} > 0$. Also, $p > \frac{2n}{n-2}$ implies that

$$\frac{n}{2p} - 1 > \frac{2n-2}{4} > 0$$

and so

$$\tilde{q}_{\nu,p} = 2\frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}}.$$

Moreover, we check that $\tilde{q}_{\nu,p} > 2$ in the following two cases:

- $\frac{n-2}{2n} < \nu_2 < \frac{n}{2} - 1$ and $p_\nu > p > \frac{2n}{n-2}$;
- $\nu_2 \geq \frac{n}{2} - 1$ and $p > \frac{2n}{n-2}$.

Case 1. We suppose first that $\frac{n-2}{2n} < \nu_2 < \frac{n}{2} - 1$, $p_\nu > p > \frac{2n}{n-2}$ and $\mu = \frac{n}{2} - 1 - \frac{n}{p}$.

By Theorem 6.7, estimate (40) is satisfied for $s = 2$. We check easily that $\frac{n-2}{2n} < (\frac{n}{2} - 1)(1 - \frac{2}{p})$ whenever $p > \frac{2n}{n-2}$. According to equation (1), we distinguish two subcases:

(i) If $\frac{n-2}{2n} < \nu_2 \leq (\frac{n}{2} - 1) \left(1 - \frac{2}{p}\right)$, then $q'_\nu(p) \geq s = 2$. It follows from Theorem 6.6 that $P_\nu$ is bounded in $L_p^{\nu,q}(T_{\Lambda_n})$ if

$$q'_\nu(p) < q < \min \left\{ 2 \min \left\{ \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}} \right\}, \tilde{q}_{\nu,p} \right\} = \min \left\{ 2 \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \tilde{q}_{\nu,p} \right\}.

(ii) If $(\frac{n}{2} - 1) \left(1 - \frac{2}{p}\right) < \nu_2 < \frac{n}{2} - 1$, then $q'_\nu(p) < s = 2$. It follows from Theorem 6.6 that $P_\nu$ is bounded in $L_p^{\nu,q}(T_{\Lambda_n})$ if

- $q'_\nu(p) < q \leq 2$ and $q < \min \left\{ 2 \min \left\{ \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}} \right\}, \tilde{q}_{\nu,p} \right\} = \min \left\{ 2 \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \tilde{q}_{\nu,p} \right\}$
- $2 < q < \min \left\{ 2 \min \left\{ \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}} \right\}, \tilde{q}_{\nu,p} \right\} = \min \left\{ 2 \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \tilde{q}_{\nu,p} \right\}.

This proves the assertion (2) of the corollary.

Case 2. We suppose that $\nu_2 \geq \frac{n}{2} - 1$, $p_\nu = \infty > p > \frac{2n}{n-2}$ and $\mu = \frac{n}{2} - 1 - \frac{n}{p}$. By Theorem 6.7, estimate (40) is satisfied for $s = 2$. In this case, $q'_\nu(p) < s = 2$ since $\nu_2 \geq \frac{n}{2} - 1$. It follows from Theorem 6.6 that $P_\nu$ is bounded in $L_p^{\nu,q}(T_{\Lambda_n})$ if

- $q'_\nu(p) < q \leq 2$ and $q < \min \left\{ 2 \min \left\{ \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}} \right\}, \tilde{q}_{\nu,p} \right\} = \min \left\{ 2 \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \tilde{q}_{\nu,p} \right\}$
- $2 < q < \min \left\{ 2 \min \left\{ \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \frac{\nu_2 + \frac{n}{2} - 1}{n - 2 - \frac{n}{p}} \right\}, \tilde{q}_{\nu,p} \right\} = \min \left\{ 2 \frac{\nu_1 - \frac{n}{2} + 1}{\nu_2 - 1 - \frac{n}{p}}, \tilde{q}_{\nu,p} \right\}$. This proves the assertion (3) of the corollary.

Assertion (1) of the previous corollary is just assertion (2) of Theorem 2.3. Assertions (1) and (2) of Theorem 2.3 are particular cases of [11] Corollary 1.4 for tube domains over Lorentz cones with $\mu = \nu$. For assertions (3) and (4), for $p_\nu > p > \frac{2n}{n-2}$, we obtain by interpolation the following result which is sharper than assertions (2) and (3) of the previous corollary.
Corollary 6.9. Let \( n \geq 3 \) and \( \nu = (\nu_1, \nu_2) \in \mathbb{R}^2 \) such that \( \nu_1 > \frac{n}{2} - 1, \nu_2 > 0 \). The weighted Bergman projector \( P_\nu \) is bounded in \( L^{p,q}_\nu(T_{\Lambda_n}) \) for the following values of \( p, q \) and \( \nu \).

1. \( \frac{n-1}{2} - 1 - \frac{1}{\nu_2} > p > \frac{2n}{n-2} \) and \( 2 < q < \tilde{q}_{\nu,\nu} \) provided \( \frac{n-2}{2n} < \nu_2 < \frac{n}{2} - 1 \);
2. \( p > \frac{2n}{n-2} \) and \( 2 < q < \tilde{q}_{\nu,\nu} \) provided \( \nu_2 \geq \frac{n}{2} - 1 \).

Proof. The situation is represented in Figure 6. From assertion (1) of Theorem 2.3, we obtain that \( P_\nu \) is bounded in \( L^{p,q}_\nu(T_{\Lambda_n}) \) if

\[
0 \leq \frac{1}{p} \leq 1 \quad \text{and} \quad \frac{1}{q} < \frac{1}{q_\nu}. \]

Combining with assertion (1) of Corollary 6.8, we deduce by interpolation that \( P_\nu \) is bounded in \( L^{p,q}_\nu(T_{\Lambda_n}) \) for

\[
\frac{1}{p_\nu} < \frac{1}{p} < \frac{n-2}{2n} \quad \text{and} \quad \frac{n-2}{2n} < \nu_2 < \frac{n}{2} - 1
\]

(resp. \( \frac{1}{p} < \frac{n-2}{2n} \) and \( \nu_2 \geq \frac{n}{2} - 1 \)),

if the couple \( (\frac{1}{p}, \frac{1}{q}) \) lies in the triangle given by the inequalities

\[
y > \frac{n-2}{n} (-q_\nu x + 1), \quad \frac{1}{2q_\nu} < x < \frac{1}{q_\nu}.
\]

Remind that for such values of \( p \), we have \( \frac{n}{2p} - 1 > 0 \) and so \( \tilde{q}_{\nu,\nu} = 2^{\frac{\nu_2 + \frac{n}{2} - 1}{n-2}} \). It is now easy to conclude that the first inequality in (43) can be written in the form

\[
q < \tilde{q}_{\nu,\nu}.
\]

Remark 6.10. According to Theorem 2.3, the conjecture stated in the introduction of [2] for \( \nu_1 = \nu_2 \) is valid for tube domains over Lorentz cones. More precisely, the weighted Bergman projector \( P_\nu \) is bounded in \( L^{p,q}_\nu \) when the couple \( (\frac{1}{p}, \frac{1}{q}) \) lies in the blank region of Figure 1.1 of [2] depicted below. This result has been proved in the blue region in [4] and in the red region in [2] for \( \nu_1 = \nu_2 \). The result in the blank region is given by Theorem 2.3. In particular, the case \( \nu_1 = \nu_2 = \frac{n}{2} \) and \( p = q \) in Theorem 2.3 corresponds to [7, Theorem 1.2].
Finally, the proof of Theorem 2.4 is just a combination of the Theorem 2.3 for $n = 3$ and Theorem 2.1 for the Pyateckii-Shapiro domain.

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