TWO-STEP ESTIMATION OF ERGODIC LÉVY DRIVEN SDE

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ABSTRACT. We consider high frequency samples from ergodic Lévy driven stochastic differential equation (SDE) with drift coefficient \( a(x, \alpha) \) and scale coefficient \( c(x, \gamma) \) involving unknown parameters \( \alpha \) and \( \gamma \). We suppose that the Lévy measure \( \nu_0 \), has all order moments but is not fully specified. We will prove the joint asymptotic normality of some estimators of \( \alpha, \gamma \) and a class of functional parameter \( \int \varphi(z)\nu_0(\text{dz}) \), which are constructed in a two-step manner; first, we use the Gaussian quasi-likelihood for estimation of \( (\alpha, \gamma) \); and then, for estimating \( \int \varphi(z)\nu_0(\text{dz}) \) we make use of the method of moments based on the Euler-type residual with the previously obtained quasi-likelihood estimator.

1. INTRODUCTION

It is widely recognized that a diffusion model is a typical candidate model to describe the high activity time-varying dynamics. However, especially in the biological, technological and financial application, there do exist many phenomena where driving noise process exhibits highly non-Gaussian behavior. A jump-type Lévy process may serve as a suitable building block in modeling such phenomena. In this paper, we consider a high frequency data \((X_{t_0}, X_{t_1}, \ldots, X_{t_n})\) from the one-dimensional Lévy driven stochastic differential equation (SDE):

\[
dX_t = a(X_t, \alpha)dt + c(X_t, \gamma)\,dJ_t, \quad X_0 = x_0, \tag{1.1}
\]

where:

- \( \alpha = (\alpha_l) \) and \( \gamma = (\gamma_l) \) are unknown finite dimensional parameters and we suppose that each of them are elements of bounded convex domains \( \Theta_\alpha \subset \mathbb{R}^{p_\alpha}, \Theta_\gamma \subset \mathbb{R}^{p_\gamma} \) and we write \( \Theta = \Theta_\alpha \times \Theta_\gamma \) and \( p_\alpha + p_\gamma = p \).
- The functional forms of the drift coefficient \( a : \mathbb{R} \times \Theta_\alpha \to \mathbb{R} \) and the scale coefficient \( c : \mathbb{R} \times \Theta_\gamma \to \mathbb{R} \) are known.
- \( J_t \) is a one-dimensional pure jump Lévy process with Lévy measure \( \nu_0 \).

We denote by \( P_0 \) the true image measure of \( X \) associated with the true value \( \theta_0 \in \Theta \). Note that we do not consider the case of misspecification of the functional form of the coefficients. We suppose that the path of \( X_t \) is not observed continuously but observed discretely at high frequency: we consider the samples \((X_{t_0}, X_{t_1}, \ldots, X_{t_n})\), where \( t_j = t_{j+1} = nh_j \) for some \( h_j > 0 \) which satisfies that

\[
nh_j^2 \to 0 \quad \text{and} \quad nh_j^{1+\epsilon} \to \infty,
\]

for \( n \to \infty \) and some \( \epsilon_0 \in (0, 1) \). The objective of this paper is to estimate \( \theta_0 \) and the functional parameter \( \int \varphi(z)\nu_0(\text{dz}) \) for some function \( \varphi \) in a two-step manner. It is not essential in our results that \( X \) has no Wiener part, but the absence is assumed from the very beginning just for simplicity of the statements; see Remark 3.8 for a brief discussion.

Up to the present, many results about the estimation of the diffusion process (this process corresponds to the case of replacing \( J_t \) with a standard Wiener process in (1.1)) have been established both continuous sampling case and discrete sampling case. In the continuous sampling case, the explicit form of its likelihood is given (see, for example, [13]). Hence we can construct the maximum likelihood estimator of \( \alpha \) and under some conditions, it has consistency and asymptotic normality (for details, see [12] and [17]). In the discrete sampling case, we can not obtain the closed form of its likelihood in general, so that we have to consider another method. Typically, we resort to the quasi-likelihood based on the local Gaussian approximation. By the Itô-Taylor expansion, [10] gives the estimation scheme in the case of \( nh_n \to \infty \) and \( nh_n^q \to 0 \) (\( \forall q \geq 2 \)). [8] shows its local asymptotic normality; he also shows the local asymptotic normality in the non-ergodic case. Needless to say, there are many estimation methods.
for our main results. Section 3 provides our main results: the stochastic expansion
us to bypass simultaneous optimization problem, which may result in high computational load.
still do not presume the closed form of the noise distribution, so that our way of estimation is beneficial in
GQMLE, and next construct the estimator of $J_\nu$ Lévy risk model. The example of moment-fitting estimation of $J_\nu$ from the discretely samples, $(J_{h_1}, J_{2h_1}, \ldots, J_{nh_1})$, are proposed in [6] and [19]. The main claim of [6] says that under some moment
conditions, for a function $\varphi$ vanishing in a neighborhood of the origin it follows that
\[
\sqrt{n}h_n \left( \frac{1}{nh_n} \sum_{j=1}^{n} \varphi(\Delta_j J) - \int \varphi(z) \nu_0(dz) \right) \xrightarrow{L^2} N \left( 0, \int \varphi(z)^2 \nu_0(dz) \right),
\]
where $\Delta_j J = J_{jh_n} - J_{(j-1)h_n}$. However, in the estimation of Lévy driven SDE, we encounter the difficulty, that is, $(J_{h_1}, J_{2h_1}, \ldots, J_{nh_1})$ cannot be observed directly. One may think of utilizing a martingale estimating function for joint estimation of $\theta_0$ and $\int \varphi(z) \nu_0(dz)$. However, we then have to specify what kind of conditional expectation is to be used in an explicit way, which inevitably requires more specific structural assumptions about $\nu_0(dz)$ beyond Assumption 2.2.

Here we will take another route. Previously, [15] used the Gaussian quasi-likelihood, which can apply
to a large class of Lévy processes, making it possible to construct Gaussian quasi maximum likelihood estimators (GQMLE) $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n)$ of the true value $\theta_0 = (\alpha_0, \gamma_0)$ without any specific information about the noise distribution; also, [15, Theorem 2.7] shows that it has consistency and asymptotic normality with rate $\sqrt{n}h_n$. By making use of the GQMLE and the functional-parameter moment fitting, we will propose a two-step procedure for joint estimation of $\theta_0$ and $\int \varphi(z) \nu_0(dz)$: we first estimate $\alpha$ and $\gamma$ by GQMLE, and next construct the estimator of $\int \varphi(z) \nu_0(dz)$ based on Euler-Maruyama approximation. We
still do not presume the closed form of the noise distribution, so that our way of estimation is beneficial in terms of the robustness against noise misspecification. Further the proposed two-step procedure enables us to bypass simultaneous optimization problem, which may result in high computational load.

The organization of this paper is as follows. In Section 2, we will introduce notations and assumptions for our main results. Section 3 provides our main results: the stochastic expansion
\[
\sqrt{n}h_n \left( \frac{1}{nh_n} \sum_{j=1}^{n} \varphi \left( \frac{X_{jh_n} - X_{(j-1)h_n} - a(X_{(j-1)h_n}, \hat{\alpha}_n)}{c(X_{(j-1)h_n}, \hat{\gamma}_n)} \right) - \int \varphi(z) \nu_0(dz) \right)
\]
\[
= \sqrt{n}h_n \left( \frac{1}{nh_n} \sum_{j=1}^{n} \varphi(\Delta_j J) - \int \varphi(z) \nu_0(dz) \right) + \hat{b}_n \sqrt{n}h_n(\hat{\gamma}_n - \gamma_0) + o_p(1),
\]
and the asymptotic normality of our estimators; see (3.2) for the explicit form of $\hat{b}_n$. In particular, the second term of the right-hand side reflects the effect of plugging-in the $\sqrt{n}h_n$-consistent estimator $\hat{\gamma}_n$ into the scale components of the Euler-residual sequence. All the proofs of our main results are presented in Section 5.

2. Notations and Assumptions
2.1. Notations. We denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ a complete filtered probability space on which the process $X$ is defined, the initial variable $X_0$ being $\mathcal{F}_0$-measurable and $J_t$ being $\mathcal{F}_t$-adapted and independent of $X_0$.

For abbreviation, we introduce some notations.
- $E_0[\cdot]$ denotes the expectation operator with respect to $P_0$ and we abbreviate $\int \varphi(z) \nu_0(dz)$ to $\nu_0(\varphi)$.
- For differentiable function $f$, $\partial_x f$ stands for the derivative with respect to any variable $x$ and $\partial f$ represents the vector of the derivatives of the components of $f$.
- $t_j := jh_n$.
- $E^{j-1}[\cdot]$ stands for the conditional expectation with respect to $\mathcal{F}_{t_{j-1}}$.
- $\Delta_j J$ stands for $Z_{t_j} - Z_{t_{j-1}}$ for any process $Z$.
- $\sum_j := \sum_{j=1}^{n}$ and $\sum_{j} := \sum_{j=1}^{n}$.
- $\eta(x, \theta) := a(x, \alpha)c^{-1}(x, \gamma)$ and $M(x, \theta) := \partial_x a(x, \alpha)c^{-2}(x, \gamma)$. 

besides (quasi) maximum likelihood method (see, for example, [12] and [17]). We emphasize that these estimation methods essentially rely on the scaling and finite-moment properties of Wiener process.
• \( f_x := f(X_t, \theta_0) \) for any function \( f \) on \( \mathbb{R} \times \Theta \); e.g. \( a_1(\alpha) = a(X_t, \alpha) \) and \( M(\theta) = M(X_t, \theta) \).

• We will write \( x_n \gtrsim y_n \) when there exists a positive constant \( C \) such that \( x_n \leq C y_n \) for large enough \( n \); \( C \) does not depend on \( n \) and varies line to line.

We define the random functions \( G^\alpha_n(\theta) \in \mathbb{R}^{\theta_n} \) and \( G^\alpha_n(\theta) \in \mathbb{R}^{\theta_n} \) by

\[
G^\alpha_n(\theta) = \frac{1}{n h_n} \sum_j M_{t_j-1}(\theta)(\Delta_j X - h_n a_{t_j-1}(\alpha)),
\]

\[
G^\alpha_n(\theta) = \frac{1}{n h_n} \sum_j \left\{ -\partial_x c^{-2}_{t_j-1}(\gamma) (\Delta_j X - h_n a_{t_j-1}(\alpha))^2 - h_n \frac{\partial_x c^{-2}_{t_j-1}(\gamma)}{c^{-2}_{t_j-1}(\gamma)} \right\},
\]

and the corresponding GQMLE ([15]) by

\[
\hat{\theta}_n := \text{argmin}_{\theta \in \Theta} \left| \langle G^\alpha_n(\theta), G_n^\alpha(\theta) \rangle \right|,
\]

where \( \bar{\Theta} \) denotes the closure of \( \Theta \) and \( | \cdot | \) the Euclidean norm.

We introduce additional notations associated with GQMLE.

- \( \hat{f}_n := f(X_t, \hat{\theta}_n) \) for any function \( f \) on \( \mathbb{R} \times \Theta \); for notational brevity, we also use the notation \( \partial_x \hat{f}_n \) instead of \( \partial_x \hat{f}_n \).

- \( \delta_j := c^{-1}_{t_j-1}(\Delta_j X - h_n a_{t_j-1}) \) and \( \delta_j := c^{-1}_{t_j-1}(\Delta_j X - h_n a_{t_j-1}) \).

- \( \hat{\nu}_n := \sqrt{n h_n}(\hat{\theta}_n - \theta_0) \) and \( \hat{\nu}_n := \sqrt{n h_n}(\hat{\nu}_n - \gamma_0) \).

2.2. Assumptions. For our asymptotic results, we introduce some assumptions.

Assumption 2.1 (Sampling design). \( n h_n^2 \to 0 \) and \( nh_n^{1+\epsilon_0} \to \infty \) for \( \epsilon_0 \in (0, 1) \).

Assumption 2.2 (Moments). We have \( E[J_1] = 0, E[J_2^2] = 1 \) and \( E[|J_1|^q] < \infty \) for all \( q > 0 \).

Although we only assume the moment conditions on \( J_1 \), the first and the third formulae are valid for all \( t > 0 \), see [18, Theorem 25.18] and we have \( E[J_2^2] = 1 \) from the expression of characteristic function of \( J_t \).

Further, by the definition of Lévy measure and the fact that \( E[J_1^q] \) exists if and only if \( \int_{|z| > 0} |z|^q \nu_0(dz) \) (see [18, Theorem 25.3]), we see that \( \int |z|^q \nu_0(dz) < \infty \), for all \( q \geq 2 \) under Assumption 2.2.

Assumption 2.3 (Smoothness).

1. The drift coefficient \( a(\cdot, \alpha_0) \) and the scale coefficient \( c(\cdot, \gamma_0) \) are Lipschitz continuous.

2. For each \( i \in \{0, 1, 2\} \) and \( k \in \{0, 1, \ldots, 5\} \), the following conditions hold:

   - The coefficient \( a(x, \alpha) \) and \( c(x, \gamma) \) have partial derivatives \( \partial_{x} \partial_{\alpha} a(x, \alpha) \) and \( \partial_{x} \partial_{\gamma} c(x, \gamma) \), and all the functions \( \alpha \mapsto \partial_{x} \partial_{\alpha} a(x, \alpha) \) and \( \gamma \mapsto \partial_{x} \partial_{\gamma} c(x, \gamma) \) for each \( x \in \mathbb{R} \) (including \( \alpha \mapsto a(x, \alpha) \) and \( \gamma \mapsto c(x, \gamma) \) themselves) can be continuously extended to the boundary of \( \Theta \).

   - There exists nonnegative constant \( C_{i,k} \) satisfying

\[
\sup_{(x, \alpha, \gamma) \in R \times \Theta, x, \alpha, \gamma} \frac{1}{1 + |x|^{C_{i,k}}} \left\{ |\partial_{x} \partial_{\alpha} a(x, \alpha)| + |\partial_{x} \partial_{\gamma} c(x, \gamma)| + |c^{-1}(x, \gamma)| \right\} < \infty. \tag{2.1}
\]

In this paper, we will assume that \( X \) is exponentially ergodic together with the boundedness of moments of any order. Let \( P_t \) denote the transition probability of \( X \). Given a function \( \rho : \mathbb{R} \to \mathbb{R}^+ \) and a signed measure \( m \) on one-dimensional Borel space, we define

\[
||m||_\rho := \sup \{|m(f)| : f \text{ is } \mathbb{R} \text{-valued, } m \text{-measurable and satisfies } |f| \leq \rho \}.
\]

Assumption 2.4 (Stability).

1. There exists a probability measure \( \pi_0 \) such that for every \( q > 0 \) we can find positive constants \( a \) and \( c \) for which

\[
\sup_{t \in \mathbb{R}^+} e^{at} ||P_t(x, \cdot) - \pi_0(\cdot)|| \leq cg(x), \quad x \in \mathbb{R}, \tag{2.2}
\]

where \( g(x) := 1 + |x|^q \).

2. For all \( q > 0 \), we have

\[
\sup_{t \in \mathbb{R}^+} E_0[|X_t|^q] < \infty.
\]

The condition (2.2) corresponds to the exponential ergodicity when \( g \) is replaced by 1. When some boundedness conditions about coefficients and their derivatives are assumed, moment conditions written in above can be weakened (see [15, Section 5] for easy sufficient conditions for Assumption 2.4).
Let \( G_\infty(\theta) := (G_\infty^\alpha(\theta), G_\infty^\gamma(\gamma)) \in \mathbb{R}^p \) define by
\[
G_\infty^\alpha(\theta) = \int \frac{\partial_\alpha a(x, \alpha)}{c^2(x, \gamma)} (a(x, \alpha) - a(x, \alpha)) \pi_0(dx),
\]
\[
G_\infty^\gamma(\theta) = 2 \int \frac{\partial_\gamma c(x, \gamma)}{c^2(x, \gamma)} (c^2(x, \gamma_0) - c^2(x, \gamma)) \pi_0(dx).
\]

We need to impose some conditions on \( G_\infty(\theta) \) for the consistency of \( \alpha \) and \( \gamma \). The sufficient condition for the consistency of general M(or Z)-estimator is given in [20].

**Assumption 2.5** (Identifiability). There exist nonnegative constants \( \chi_\alpha \) and \( \chi_\gamma \) such that
\[
|G_\infty^\alpha(\theta)| \geq \chi_\alpha |\alpha - \alpha_0|, \quad |G_\infty^\gamma(\gamma)| \geq \chi_\gamma |\gamma - \gamma_0| \quad \text{for all } \theta.
\]

Define \( \mathcal{I}(\theta_0) := \text{diag}(\mathcal{I}^\alpha(\theta_0), \mathcal{I}^\gamma(\theta_0)) \in \mathbb{R}_+ \otimes \mathbb{R}_+ \) by
\[
\mathcal{I}^\alpha(\theta_0) = \int \frac{(\partial_\alpha a(x, \alpha_0))^{\otimes 2}}{c^2(x, \gamma_0)} \pi_0(dx),
\]
\[
\mathcal{I}^\gamma(\theta_0) = 4 \int \frac{(\partial_\gamma c(x, \gamma_0))^{\otimes 2}}{c^2(x, \gamma_0)} \pi_0(dx),
\]
where \( x^{\otimes 2} := xx^T \) for any vector or matrix \( x \) and \( T \) means the transpose. The matrix \( \mathcal{I}(\theta_0) \) plays a role like a Fisher-information like quantity in GQML estimation.

**Assumption 2.6** (Nondegeneracy). \( \mathcal{I}^\alpha(\theta_0) \) and \( \mathcal{I}^\gamma(\theta_0) \) are invertible.

Our estimation of \( \nu_0(\varphi) \) will be based on (1.2). Here we only think of Euclidean space valued \( \varphi \), while treatment of complex \( \varphi \) being completely analogous. In our setting, we only observe high frequency sample \((X_t, X_{2t}, \ldots, X_{nt})_t \), hence we need to approximate \( \Delta, J \) to estimate \( \nu_0(\varphi) \). Let \( \mathcal{A} \) denote the formal infinitesimal generator with respect to Lévy process \( J \), that is,
\[
\mathcal{A}\varphi(x) = \int (\varphi(x + z) - \varphi(x) - \partial_z \varphi(x)z) \nu_0(dz), \quad (2.3)
\]
for any \( \varphi \) such that the integral exists. In what follows we fix a positive integer \( q \). We now define a positive constant \( \rho \) fulfilling that
\[
\rho > (1 - \epsilon_0) \vee \beta,
\]
where \( \epsilon_0 \) is the same as in Assumption 2.1 and \( \beta \) denotes the Blumenthal-Getoor index of \( J \) defined by
\[
\beta = \inf \left\{ \gamma \geq 0; \int_{|z| \leq 1} |z|^{\gamma} \nu_0(dz) \right\}.
\]
Denote by \( \mathcal{K} \) the set of all \( \mathbb{R}^q \)-valued functions on \( \mathbb{R} \) such that its element \( f = (f_k)_{k=1}^q : \mathbb{R} \to \mathbb{R}^q \) satisfies the following conditions:

1. \( f \) is five times differentiable.
2. There exist nonnegative constants \( C_i \) \((0 \leq i \leq 5)\) such that
\[
\limsup_{z \to 0} \left\{ \frac{1}{|z|^p} |f(z)| + \frac{1}{|z|} |\partial f(z)| \right\} < \infty,
\]
\[
\limsup_{z \to \infty} \left\{ \frac{1}{1 + |z|^{C_0}} |f(z)| + \frac{1}{|z|^{1+C_1}} |\partial f(z)| \right\} < \infty,
\]
\[
\sup_{z \in \mathbb{R}} \frac{1}{1 + |z|^i} |\partial^i f(z)| < \infty, \quad i \in \{2, 3, 4, 5\}.
\]

We now impose

**Assumption 2.7** (Moment-fitting function). \( \varphi \in \mathcal{K} \).

Then, according to the definition of Blumenthal-Getoor index and Assumption 2.2 we have \( \nu_0(\varphi) < \infty \).
3. Main results

The Euler-Maruyama approximation says that
\[ X_{t_j} \approx X_{t_{j-1}} + b_n a_{t_{j-1}} + c_{t_{j-1}} \Delta_j J. \]

This suggests that we may formally regard \( \delta_j \) as the estimator of \( \Delta_j J \), and indeed it will turn out to be true under our assumptions. Also, we will see that the Euler residual \( \tilde{\delta}_j \), which is constructed only by \( (X_{t_{h_n}}, X_{2h_n}, \ldots, X_{nh_n}) \), may also serve as an estimator of \( \Delta_j J \) (see the proof of Theorem 3.1).

Let
\[
\tilde{u}_n := \sqrt{nh_n} \left( \frac{1}{nh_n} \sum_j \varphi(\delta_j) - \nu_0(\varphi) \right),
\]
\[
\hat{u}_n := \sqrt{nh_n} \left( \frac{1}{nh_n} \sum_j \varphi(\tilde{\delta}_j) - \nu_0(\varphi) \right).
\]

As was mentioned in the introduction, we know that \( u_n \) is asymptotically normally distributed: \( u_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nu_0(\varphi \otimes 2)) \). Let
\[
\zeta(z) := z \partial \varphi(z).
\]

The next theorem clarifies the effect of using the statistics \( \hat{\delta}_j \) instead of the unobservable variables \( \Delta_j J \).

**Theorem 3.1.** Under Assumptions 2.1-2.7, we have
\[
\tilde{u}_n = u_n + \hat{b}_n[\tilde{w}_n] + o_p(1), \tag{3.1}
\]
where \( \hat{b}_n \in \mathbb{R}^q \otimes \mathbb{R}^{p'} \) is defined by
\[
\hat{b}_n = -\left( \frac{1}{nh_n} \sum_j \zeta(\hat{\delta}_j) \right) \otimes \left( \frac{1}{n} \sum_j \frac{\partial_c \hat{\delta}_{t_{j-1}}}{\hat{\epsilon}_{t_{j-1}}} \right). \tag{3.2}
\]

Building on the stochastic expansion (3.1), we will see that substituting \( \hat{\delta}_j \) into \( \Delta_j J \) leads to the different asymptotic covariance matrix of the estimator of \( \nu_0(\varphi) \). See the comments after Corollary 3.5 for more details.

**Remark 3.2.** Although GQMLE is adopted as the estimator of \( \theta_0 \), (3.1) is valid for any estimator \( \hat{\theta}_n \) which satisfies \( E[|\sqrt{nh_n}(\hat{\theta}_n - \theta_0)|^q] < \infty \) for all \( q > 0 \) (cf. the proof of Theorem 3.1).

We define the estimating function \( G_n(\theta) \) for \( (\theta, \nu_0(\varphi)) \) by
\[
G_n(\theta) = \left( \frac{1}{\sqrt{nh_n}} u_n, G_n^a(\theta), G_n^g(\theta) \right),
\]
where \( G_n^a(\theta) \) and \( G_n^g(\theta) \) are defined in the previous section. Introduce
\[
\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix}
\]

with \( \Sigma_{11} \in \mathbb{R}^{q} \otimes \mathbb{R}^q \), \( \Sigma_{12} = (\Sigma_{12,kl})_k,l \in \mathbb{R}^q \otimes \mathbb{R}^p \) and \( \Sigma_{22} = (\Sigma_{22,kl})_k,l \in \mathbb{R}^p \otimes \mathbb{R}^p \), where
\[
\Sigma_{11} = \nu_0(\varphi \otimes 2),
\]
\[
\Sigma_{12,kl} = \begin{cases}
\int \varphi_k(z) z^l \nu_0(dz) \int \frac{\partial_{a_l} a(x, a_0)}{c(x, \gamma_0)} \pi_0(dx) & (1 \leq l \leq p_a), \\
2 \int \varphi_k(z) z^l \nu_0(dz) \int \frac{\partial_{a_l} c(x, \gamma_0)}{c(x, \gamma_0)} \pi_0(dx) & (p_a + 1 \leq l \leq p), \\
\int \frac{\partial_{a_l} a(x, a_0) \partial_{a_k} a(x, a_0)}{c^2(x, \gamma_0)} \pi_0(dx) & (k,l \in \{1, \ldots, p_a\}),
\end{cases}
\]
\[
\Sigma_{22,kl} = \begin{cases}
4 \int \frac{\partial_{a_l} c(x, \gamma_0) \partial_{a_k} c(x, \gamma_0)}{c^2(x, \gamma_0)} \pi_0(dx) \int z^l \nu_0(dz) & (k,l \in \{p_a + 1, \ldots, p\}), \\
2 \int \frac{\partial_{a_l} a(x, a_0) \partial_{a_k} c(x, \gamma_0)}{c^2(x, \gamma_0)} \pi_0(dx) \int z^l \nu_0(dz) & (k \in \{1, \ldots, p_a\}, l \in \{p_a + 1, \ldots, p\}).
\end{cases}
\]
Theorem 3.3. If Assumptions 2.1-2.5 and Assumption 2.7 hold, and if $\Sigma$ is positive definite, then
\[
\sqrt{n \log n} G_n(\theta_0) \xrightarrow{\mathcal{L}} N_{p+q}(0, \Sigma).
\]

Remark 3.4. The moment convergence of the estimator is crucial for detecting the asymptotic behavior of statistics which can be used, for example, for mean bias correction and analysis of mean squared prediction error; see the references cited in [15]. As for the GQMLE $\theta_n$, under Assumption 2.1-2.6 we can deduce
\[
E[f(\hat{\theta}_n)] \rightarrow \int_{\mathbb{R}^p} f(u) \varphi(u; 0, I(\theta_0)^{-1}) \Sigma_{22}(I(\theta_0)^{-1})^T du,
\]
for every continuous function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ of at most polynomial growth: see [15, Theorem 2.7]. In this paper, we do not go into details of the moment convergence of $f(\hat{\theta}_n)$.

Define the statistics $\hat{\Gamma}_n \in \mathbb{R}^{p+q} \otimes \mathbb{R}^{p+q}$ by
\[
\hat{\Gamma}_n = \begin{pmatrix} I_q & -\hat{B}_n \\ O & -\partial_0(G_n^a, G_n^b)(\hat{\theta}_n) \end{pmatrix},
\]
where $\hat{B}_n = (O \quad \hat{b}_n) \in \mathbb{R}^q \otimes \mathbb{R}^p$. We also define
\[
\hat{\Sigma}_n = \begin{pmatrix} \hat{\Sigma}_{11,n} & \hat{\Sigma}_{12,n} \\ \hat{\Sigma}_{12,n}^T & \hat{\Sigma}_{22,n} \end{pmatrix},
\]
with $\hat{\Sigma}_{11,n} \in \mathbb{R}^q \otimes \mathbb{R}^q$, $(\hat{\Sigma}_{12,n,kl})_{k,l} \in \mathbb{R}^q \otimes \mathbb{R}^p$ and $(\hat{\Sigma}_{22,n,kl})_{k,l} \in \mathbb{R}^p \otimes \mathbb{R}^p$, where
\[
\hat{\Sigma}_{11,n} = \frac{1}{n \log n} \sum_j \varphi_j^2(\hat{\lambda}_j),
\]
\[
\hat{\Sigma}_{12,n,kl} = \begin{cases} \frac{1}{n \log n} \sum_j \varphi_k(\hat{\lambda}_j) \varphi_l(\hat{\lambda}_j) \left( \frac{1}{n \log n} \sum_j \frac{\partial_{\alpha_k} \hat{a}_{t_{i-1,j}}}{\hat{c}_{t_{i,j}}} \right) & (1 \leq l \leq p_n), \\
\frac{2}{n \log n} \sum_j \varphi_k(\hat{\lambda}_j) \sum_j \frac{\partial_{\alpha_k} \hat{c}_{t_{i,j}}}{\partial_{\alpha_{t_{i,j}}}} & (p_n + 1 \leq l \leq p), \\
\end{cases}
\]
\[
\hat{\Sigma}_{22,n,kl} = \begin{cases} \frac{1}{n \log n} \sum_j \frac{\partial_{\alpha_k} \hat{a}_{t_{i-1,j}}}{\partial_{\alpha_{t_{i,j}}}} \sum_j \frac{\partial_{\alpha_{t_{i,j}}} \hat{a}_{t_{i-1,j}}}{\partial_{\alpha_{t_{i-1,j}}}} & (k, l \in \{1, \ldots, p_n\}), \\
4 \left( \frac{1}{n} \sum_j \frac{\partial_{\alpha_k} \hat{c}_{t_{i,j}}}{\partial_{\alpha_{t_{i,j}}}} \sum_j \frac{\partial_{\alpha_{t_{i,j}}} \hat{c}_{t_{i,j}}}{\partial_{\alpha_{t_{i-1,j}}}} \right) & (k, l \in \{p_n + 1, \ldots, p\}), \\
2 \left( \frac{1}{n \log n} \sum_j \frac{\partial_{\alpha_k} \hat{c}_{t_{i,j}}}{\partial_{\alpha_{t_{i-1,j}}} \partial_{\alpha_{t_{i+1,j}}}} \sum_j \frac{\partial_{\alpha_{t_{i-1,j}}} \hat{c}_{t_{i,j}}}{\partial_{\alpha_{t_{i-1,j}}}} \right) & (k \in \{1, \ldots, p_n\}, l \in \{p_n + 1, \ldots, p\}). 
\end{cases}
\]

It will turn out that $\hat{\Sigma}_n$ is a consistent estimator of the asymptotic variance $\Sigma$, which depends on the true value ($\hat{\theta}_n, \varphi(\hat{\theta}_n)$) under our assumption.

By use of Theorem 3.1 and Theorem 3.3, we can derive the asymptotic normality of the statistics $(\hat{u}_n, \hat{v}_n)$ only constructed from the observed data $(X_{1n}, X_{2hn}, \ldots, X_{nhn})$.

Corollary 3.5. Suppose that Assumptions 2.1-2.7 hold and that $\Sigma$ is positive definite. Then $\hat{\Sigma}_n \xrightarrow{\mathcal{L}} \Sigma$ and
\[
\hat{\Sigma}_n^{-1/2} \hat{\Gamma}_n \left( \frac{\hat{u}_n}{\hat{v}_n} \right) \xrightarrow{\mathcal{L}} N(0, I_{p+q}),
\]
where $I_p$ denotes the $p \times p$ identity matrix.

By means of Lemma 5.6 and Lemma 5.9, we can observe that
\[
\hat{b}_n \xrightarrow{p} b_0 := \left( \int \zeta(z) \nu_0(\mathsf{d}z) \otimes \left( \int \frac{\partial_x c(x, \gamma_0)}{c(x, \gamma_0)} \pi_0(\mathsf{d}x) \right) \right).
\]
Put $B_0 := (O \quad b_0) \in \mathbb{R}^q \otimes \mathbb{R}^p$. Under our assumptions, we can deduce that
\[
\hat{\Sigma}_n \xrightarrow{p} \Sigma \quad \text{and} \quad \hat{\Gamma}_n \xrightarrow{p} \Gamma := \begin{pmatrix} I_q & -B_0 \\ O & -I(\theta_0) \end{pmatrix}.
\]
Thus it follows from (3.3) that we have the joint asymptotic normality of our estimators:
\[
\left( \frac{\hat{u}_n}{\hat{v}_n} \right) \xrightarrow{\mathcal{L}} N(0, \Gamma^{-1}\Sigma(\Gamma^{-1})^T).
\]
Remark 3.6. Recall that if \( J_1 \) is the standard Wiener process, then the rate of \( \hat{\gamma}_n - \gamma_0 \) is \( \sqrt{n} \) (see [10]). The case \( \int z^4 \nu_0(dz) = 0 \) corresponds to this. As was noted in [15] (and as trivial from Lemma 5.6 and Lemma 5.7), \( \hat{\alpha}_n \) and \( \hat{\gamma}_n \) are asymptotically orthogonal (hence asymptotically independent) if \( \int z^4 \nu_0(dz) = 0 \). Likewise, the asymptotic independence between \( \hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n) \) and \( (nh_n)^{-1} \sum_j \varphi(\hat{\delta}_j) \) can be easily seen from the expression of \( L' \): in particular, \( (\hat{\alpha}_n, \hat{\gamma}_n) \) is asymptotically independent of \( (nh_n)^{-1} \sum_j \varphi(\hat{\delta}_j) \) if both \( \int \varphi_k(z) \nu_0(dz) \) and \( \int \varphi_k(z) z^4 \nu_0(dz) \) are zero.

Now we assume that the Lévy measure \( \nu_0 \) is parametrized by a parameter \( \xi \in \Theta_\xi \), say \( \nu_\xi \), where \( \Theta_\xi \) is a bounded convex domain in \( \mathbb{R}^q \), and that there exists a true value \( \xi_0 \in \Theta_\xi \). The delta method then leads to the following corollary.

**Corollary 3.7.** If the conditions of Corollary 3.5 hold and the equation \( F(\int \varphi(z) \nu_\xi(dz), \theta) = (\xi, \theta) \) has a \( C^1 \)-solution \( F: \mathbb{R}^q \times \Theta \rightarrow \Theta_\xi \times \Theta \) such that \( \partial F(\nu_\xi(\varphi), \theta_0) \) is invertible, then

\[
\partial \hat{F}_n := \partial F\left( \frac{1}{nh_n} \sum_j \varphi(\hat{\delta}_j), \hat{\theta}_n \right) \overset{P}{\rightarrow} \partial F(\nu_{\xi_0}(\varphi), \theta_0).
\]

Moreover, we have

\[
\{\partial \hat{F}_n \hat{\Sigma}_n^{-1/2}\}^{-1} \sqrt{nh_n}(\hat{\xi}_n - \xi_0, \hat{\theta}_n - \theta_0) \overset{L}{\rightarrow} \mathcal{N}(0, I_{p+q}),
\]

where \( \hat{\xi}_n \) denotes the random vector consisting of the first \( q \) elements of \( F(\frac{1}{nh_n} \sum_j \varphi(\hat{\delta}_j), \hat{\theta}_n) \).

**Remark 3.8.** As a matter of fact, the absence of the Wiener part in the underlying SDE (1.1) is not essential in our results. Consider

\[
dX_t = a(X_t, \alpha)dt + \sigma(X_t, \gamma)dW_t + c(X_t, \gamma)dJ_t,
\]

where \( W \) is an \( (F_t) \)-adapted standard Wiener process independent of \( (X_0, J) \). We first note that the results of [15] still ensures the asymptotic normality of the corresponding GQMLE of \( (\alpha, \gamma) \) at rate \( \sqrt{nh_n} \), in exchange for, in particular, some stringent identifiability condition on the scale parameter \( \gamma \); e.g. if \( b(x, \gamma) = \gamma_1 \) and \( c(x, \gamma) = \gamma_2 \) for \( \gamma = (\gamma_1, \gamma_2) \), then trivially we cannot estimate \( \gamma_1 \) and \( \gamma_2 \) separately by the naive Gaussian quasi-likelihood. Introducing an additional condition, we could deduce Theorem 3.1 with the same expression (3.2) of \( b_n \), except for the trivial change of the form of \( \Sigma \), which stems from the necessary modification of the “one-step” variance in construction of the Gaussian quasi-likelihood corresponding to (3.4), that is, from “\( \Sigma_n^{1/2}(\gamma) \)” to “\( \Sigma(\sigma_1^2(\gamma) + c_1^2(\gamma)) \)”; see Eq. (2.10) and the expression of \( \Sigma_0 \) in [15, pages 1600 and 1601] for details. More specifically, in the derivation of (3.1), which amounts to the stochastic expansions and estimates concerning the terms \( b_n^{(1)} \) and \( b_n^{(2)} \) in the proof of Lemma 5.8, it turned out that the presence of the Wiener part entails an additional condition on the behavior of the second derivative of \( \varphi \) around the origin, in order to make the remainder terms in the Taylor expansion indeed negligible. Also to be mentioned is that the independence between \( W \) and \( J \) is crucial in the computation of the leading-term of (3.1): formally, in applying [7, Lemma 9] we make use of the calculations such as \( E_0^{\frac{-1}{2}}[g(\Delta J) \Delta J W] = 0 \) and \( E_0^{\frac{-1}{2}}[g^2(\Delta J) \Delta J W^2] = h_n E[g^2(\Delta J)] \) to obtain \( \sqrt{nh_n} \sum_{j=1}^{J_1} g(\Delta J) \Delta J W = o_p(1) \) for suitable \( f \) and \( g \) with \( g(0) = 0 \). Building on these observations, the proofs in case of (3.4) go through as in (1.1) without further difficulty, while the precise statement concerning the model (3.4) requires a series of changes of notation. We would like to omit details of the full picture.

4. Numerical experiments

Consider the following one-dimensional Lévy driven SDE:

\[
dX_t = -\alpha X_t dt - \frac{\gamma}{1 + X_t^2} dJ_t, \quad X_0 = 0,
\]

where the true value is \((\alpha_0, \gamma_0) = (0.5, 0.2)\); the driving noise process is the normal inverse Gaussian Lévy process such that \( \mathcal{L}(J_1) = NIG(\delta, 0, \delta t, 0) \) with \( \delta = 1, 5 \), or 10. It is well known that the cumulant function of \( J_1 \) is explicitly given by

\[
\kappa(u) := \log E[\exp(iuJ_1)] = \int (\cos(uz) - 1) \nu_{0, \delta}(dz) = \delta(\delta - \sqrt{\delta^2 + u^2}).
\]
Table 1. The performance of the two-step type estimators with $\delta = 1$ and the true value $(\alpha_0, \gamma_0, \kappa(1), \kappa(3), \kappa(5)) = (0.5, 0.2, -0.4142, -2.1623, -4.0990)$; the mean is given with the standard deviation in parentheses.

| $T_n$ | $h_n$ | $\hat{\alpha}_n$ | $\hat{\gamma}_n$ | $\hat{\kappa}(1)_n$ | $\hat{\kappa}(3)_n$ | $\hat{\kappa}(5)_n$ |
|-------|-------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 10    | 0.05  | 0.6690            | 0.1923            | -0.4455           | -2.2888           | -4.0765           |
|       |       | (0.3912)          | (0.0489)          | (0.0340)          | (0.5594)          | (0.9300)          |
| 50    | 0.025 | 0.5394            | 0.1967            | -0.4243           | -2.1713           | -4.0020           |
|       |       | (0.1396)          | (0.0224)          | (0.0346)          | (0.2801)          | (0.4929)          |
| 100   | 0.01  | 0.5205            | 0.1986            | -0.4198           | -2.1730           | -4.0755           |
|       |       | (0.0994)          | (0.0163)          | (0.0292)          | (0.2087)          | (0.3746)          |

where $\nu_{0,\delta}$ denotes the Lévy measure of $NIG(\delta, 0, \delta, 0)$. Hence it follows that $E[J_1] = 0, E[J_1^2] = 1$, and that $\mathcal{L}(J_1) \xrightarrow{L} N(0, t)$ as $\delta \to \infty$.

In addition to $(\alpha_0, \gamma_0)$, we estimate the value of $\kappa(u)$ for some $u$, so by the symmetry of $\mathcal{L}(J_1)$ we set the moment fitting function $\varphi(x, u) = \cos(ax) - 1$. Note that the SDE model (4.1) satisfies all of our assumptions; see [15, Proposition 5.4] for the stability condition.

Put $T_n = n h_n$. Our simulations were done for $(T_n, h_n) = (10, 0.05), (50, 0.025)$ and $(100, 0.01)$ with respect to each $\delta$. We simulated 1000 independent sample paths for true model with sufficiently small step size by use of Euler scheme and the 1000 estimates $(\hat{\alpha}_n, \hat{\gamma}_n, \kappa(1)_n, \kappa(3)_n, \kappa(5)_n)$, where

$$
\kappa(u)_n := \frac{1}{nh_n} \sum_{j=1}^n \varphi(\delta_j, u),
$$

were calculated for each sample path. For generating sample paths, we used yuima package [2] for R statistical environment [16]. The mean and the standard deviation of $(\hat{\alpha}_n, \hat{\gamma}_n, \kappa(1)_n, \kappa(3)_n, \kappa(5)_n)$ were computed; these are shown in Table 1-3.

From the results, we can observe the following:

- the performance of $\hat{\alpha}_n$ can be affected not by the value of $\delta$ but by the value of $T_n$;
- the performance of $\hat{\gamma}_n$ seems to improve in terms of standard deviation as the value of $\delta$ increases, which can be thought to come from the fact that the asymptotic variance of $\hat{\gamma}_n$ tends to 0 as $\delta \to \infty$ (we have $\int z^4 \nu_{0,\delta}(dz) = 3\delta^{-2}$);
- the performance of $\hat{\kappa}(u)_n$ becomes better for smaller $u$. This is quite natural because by Theorem 3.3 the asymptotic variance of $\kappa(u)_n$ is $\int \varphi(x, u)^2 \nu_{0,\delta}(dx)$. From the half-angle formula $\varphi(x, u)^2 = (\cos(ax) - 1)^2 = -2\varphi(x, u) + \frac{1}{2}\varphi(x, 2u)$, we have

$$
\int \varphi(x, u)^2 \nu_{0,\delta}(dx) = \int \left( -2\varphi(x, u) + \frac{1}{2}\varphi(x, 2u) \right) \nu_{0,\delta}(dx)
$$

$$
= -\frac{3}{2} \delta^2 + \delta \left( 2\sqrt{\delta^2 + u^2} - \frac{1}{2} \sqrt{\delta^2 + 4u^2} \right) =: f(\delta, u).
$$

Since $\partial_u f(\delta, u) = 2\delta(u/\sqrt{\delta^2 + u^2} - u/\sqrt{\delta^2 + 4u^2}) > 0$ for all $\delta > 0$, the asymptotic variance of $\kappa(u)_n$ is increasing in $u$, clarifying better performance of $\kappa(u)_n$ for smaller value of $u$.

In this example, it should be noted that a large value of $u$ brings about large finite-sample bias and variance of the scaled estimators: $\sqrt{nh_n}(\kappa(u)_n - \kappa(u))$, because then both the term $b_n = b_n(u)$ and the $o_p(1)$ term in the right-hand side of (3.1) will become large in an increasing way with the value $|u|$; as seen from the proof, the latter term involves higher-order partial derivatives of $\varphi(x, u)$ with respect to $x$.

5. Proofs

Throughout our proofs, we will often omit “$n$” of the notation $h_n$ and write $E$ instead of $E_0$.

5.1. Preliminary lemmas. We begin with some lemmas.

Lemma 5.1. Suppose that Assumption 2.1 and Assumption 2.2 hold. For all $q \geq 2$, it follows that

$$
\frac{1}{h} E[|J_h|^q] \to \int |z|^q \nu_0(dz).
$$
Proof. Under Assumption 2.2, \( \varphi(z) = |z|^q \) satisfies the condition of [5, Theorem 1]. \( \square \)

**Remark 5.2.** Although the above convergence might not be valid for all \( 0 < q < 2 \), it holds when \( q \geq \beta \), where \( \beta \) denotes the Blumenthal-Getoor index (for details, see [5, Theorem 1], [9, Section 5.2], and [14, Theorem 1]).

From now on we simply write \( f_{j-1}(\theta) = f(X_{t_{j-1}}, \theta), f_{j-1} = f(X_{t_{j-1}}, \theta_0) \) and \( \hat{f}_{j-1} = f(X_{t_{j-1}}, \hat{\theta}_n) \).

**Lemma 5.3.** Let \( f : \mathbb{R} \times \Theta_\alpha \times \Theta_\gamma \to \mathbb{R} \) be a polynomial growth function with respect to \( x \), uniformly in \( \alpha \) and \( \gamma \). If Assumptions 2.1-2.4 are satisfied, then, for all \( p \in \{1, 2 \} \) and \( q \geq 0 \) it follows that

\[
\sup_n \sup_{\theta} E \left[ \left( \frac{1}{nh} \sum_j f_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha))^q \right) \right] < \infty.
\]

Moreover, we have

\[
\sup_n E \left[ \left( \frac{1}{\sqrt{nh}} \sum_j f_{j-1}(\Delta_j X - ha_{j-1}) \right)^q \right] < \infty,
\]

\[
\sup_n E \left[ \left( \frac{1}{\sqrt{nh}} \sum_j \{f_{j-1}(\Delta_j X - ha_{j-1})^2 - hf_{j-1}(\alpha)^2c_{j-1}^2\} \right)^q \right] < \infty.
\]

**Proof.** First, we show the case of \( p = 1 \) and \( q \geq 2 \). By the definition of \( X \), we have

\[
E \left[ \left( \frac{1}{nh} \sum_j f_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha)) \right)^q \right] \geq E \left[ \left( \frac{1}{nh} \sum_j f_{j-1}(\theta) \int_j (a_s - E^{j-1}[a_s])ds \right)^q \right] + E \left[ \left( \frac{1}{nh} \sum_j f_{j-1}(\theta) \int_j (E^{j-1}[a_s] - a_{j-1})ds \right)^q \right] + E \left[ \left( \frac{1}{n} \sum_j f_{j-1}(\theta)(a_{j-1} - a_{j-1}(\alpha)) \right)^q \right] + E \left[ \left( \frac{1}{nh} \sum_j f_{j-1}(\theta) \int c_s dJ_s \right)^q \right].
\]
We will check separately that all terms are finite. From the assumption on $f$ and Jensen’s inequality, we get
\[
E \left[ \left| \frac{1}{n} \sum_j f_{j-1}(\theta)(a_{j-1} - a_{j-1}(\alpha)) \right|^q \right] \leq \frac{1}{n} \sum_j E \left[ |f_{j-1}(\theta)(a_{j-1} - a_{j-1}(\alpha))|^q \right] < \infty.
\]
By Itô’s formula, we have
\[
E^{j-1}[a_s] - a_{j-1} = \int_{t_{j-1}}^s E^{j-1}[\mathcal{A}a_u] du,
\]
where $\mathcal{A}$ denotes the formal infinitesimal generator of $X$, namely, for $f \in C^1(\mathbb{R})$,
\[
\mathcal{A}f(x) = \partial f(x) a(x) + \int (f(x + c(x)z) - f(x) - \partial f(x)c(x)z) \nu_0(dz).
\]
By [15, Lemma 4.5], the definition of $\mathcal{A}$ and the assumptions about coefficients and moments, for a $v \in (0,1)$, we get
\[
\left| E^{j-1}[\mathcal{A}a_u] \right| \leq E^{j-1} \left[ (\partial_x a_u)a_u + \int (a(X_u + c_u z) - a_u - (\partial_x a_u)c_u z) \nu_0(dz) \right]
\leq E^{j-1} \left[ 1 + |X_u|^C + \int |\partial_x^2 a(X_u + v c_u z)(c_u z)^2| \nu_0(dz) \right]
\leq E^{j-1} \left[ 1 + |X_{j-1}|^C \right] \leq E^{j-1} \left[ 1 + |X_{j-1}|^C + |X_{j-1}|^C \right] \leq 1 + |X_{j-1}|^C.
\]
Note that we used the fact that $\int z^q \nu_0(dz) < \infty$ for any $q \geq 2$. Hence it follows that
\[
E \left[ \left| \frac{1}{nh} \sum_j f_{j-1}(\theta) \int_j (E^{j-1}[a_s] - a_{j-1}) ds \right|^q \right] \leq E \left[ \left| \frac{1}{nh} \sum_j |f_{j-1}(\theta)| \left| \int_j (E^{j-1}[a_s] - a_{j-1}) ds \right| \right|^q \right]
\leq E \left[ \left| \frac{1}{nh} \sum_j |f_{j-1}(\theta)| \int_j \int_{t_{j-1}}^s \left| E^{j-1}[\mathcal{A}a_u] \right| duds \right|^q \right]
\leq E \left[ \frac{h}{n} \sum_j |f_{j-1}(\theta)|(1 + |X_{j-1}|^C)^q \right] \lesssim h^q < \infty.
\]
Burkholder’s inequality for martingale difference array yields that
\[
E \left[ \left| \frac{1}{nh} \sum_j f_{j-1}(\theta) \int_j (a_s - E^{j-1}[a_s]) ds \right|^q \right] \lesssim n^{-\frac{1}{2}} \sum_j E \left[ \frac{f_{j-1}(\theta) \int_j (a_s - E^{j-1}[a_s]) ds}{h} \right]^q \lesssim n^{-\frac{1}{2}} \sum_j \sqrt{E \left[ \frac{f_{j-1}(\theta) \int_j (a_s - E^{j-1}[a_s]) ds}{h} \right]^{2q}} \lesssim n^{-\frac{1}{2}} \sum_j \sqrt{\frac{1}{h} E \left[ \int_j (a_s - E^{j-1}[a_s])^{2q} ds \right]} \lesssim n^{-\frac{1}{2}} \sqrt{h} < \infty.
\]
Define the indicator function $\chi_j$ by
\[
\chi_j(s) = \begin{cases} 1 & s \in (t_{j-1}, t_j], \\ 0 & \text{otherwise}. \end{cases}
\]
Using this indicator function and Burkholder’s inequality, we can obtain

$$
E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \int_j c_{as} dJ_s \right]^q = E \left[ \frac{1}{nh} \int_0^{nh} \sum_j f_{j-1}(\theta) c_{as} \chi_j(s) dJ_s \right]^q \\
\lesssim (nh)^{-\frac{q}{2}} \int_0^{nh} E \left[ \sum_j f_{j-1}(\theta) c_{as} \chi_j(s) \right] ds \\
= (nh)^{-\frac{q}{2}} \sum_j E \left[ |f_{j-1}(\theta) c_{as}|^q \right] ds \\
\lesssim (nh)^{-\frac{q}{2}} < \infty. \quad (5.1)
$$

Second, we look at the cases of $p = 2$ and $q \geq 2$. Quite similarly to the above, we have

$$
E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha))^2 \right]^q \\
\lesssim E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \left( \int_j (a_s - E^{j-1}[a_s]) ds \right)^2 \right]^q + E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \left( \int_j (E^{j-1}[a_s] - a_{j-1}) ds \right)^2 \right]^q \\
+ E \left[ \frac{1}{n} \sum_j f_{j-1}(\theta)(a_{j-1} - a_{j-1}(\alpha))^2 \right]^q + E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \left( \int c_{as} dJ_s \right)^2 \right]^q.
$$

In the same way, we get

$$
E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \left( \int_j (E^{j-1}[a_s] - a_{j-1}) ds \right)^2 \right]^q \lesssim h^{3q} < \infty,
$$

$$
E \left[ \frac{1}{n} \sum_j f_{j-1}(\theta)(a_{j-1} - a_{j-1}(\alpha))^2 \right]^q < \infty.
$$

Jensen’s inequality implies that

$$
E \left[ \frac{1}{nh} \sum_j f_{j-1}(\theta) \left( \int_j (a_s - E^{j-1}[a_s]) ds \right)^2 \right]^q \leq E \left[ \frac{h}{n} \sum_j |f_{j-1}(\theta)| \left( \frac{f_{j}(a_s - E^{j-1}[a_s]) ds}{h} \right)^2 \right]^q \\
\leq E \left[ \frac{1}{n} \sum_j |f_{j-1}(\theta)| \left( \int_j |a_s - E^{j-1}[a_s]|^2 ds \right)^q \right] \\
= E \left[ \frac{1}{n} \sum_j |f_{j-1}(\theta)| \left( \int_j |a_s - E^{j-1}[a_s]|^2 \chi_j(s) ds \right)^q \right] \\
\leq h^q E \left[ \int_0^{nh} \sum_j |f_{j-1}(\theta)| |a_s - E^{j-1}[a_s]|^2 \chi_j(s) ds \right]^q \\
\leq h^q \frac{1}{nh} E \left[ \int_0^{nh} \sum_j |f_{j-1}(\theta)| |a_s - E^{j-1}[a_s]|^2 \chi_j(s) ds \right]^q \\
\lesssim h^q \sqrt{T} < \infty.
$$

From Itô’s formula, we get

$$
\left( \int_j c_{as} dJ_s \right)^2 = 2 \int_j \left( \int_{t_j-1}^s c_{u} dJ_u \right) c_{as} dJ_s + \int_j c_{as}^2 z^2 N(ds, dz)
$$
Proof. By the stationarity and independence of increments of Lévy process suppose that Assumptions 2.1, 2.2 and 2.7 hold. Then we have

\[ A \]

Under the assumptions we see that the last two terms are martingale (see [1, Theorem 4.2.3]) and

\[ \text{where} \]

\[ J \]

\[ N(ds, dz) \] (resp. \( \tilde{N}(ds, dz) \)) is the Poisson random measure (resp. compensated Poisson random measure) associated with \( J \). It follows from this decomposition together with a similar estimate to (5.1) that

\[ E \left[ \left( \frac{1}{nh} \sum_{j=1}^{h-n} f_{j-1}(\theta) \left( \int_{j}^{j+1} ds \right) \right)^2 \right] < \infty. \]

If \( \theta = \theta_0 \), we do not have to consider the term containing \( a_{j-1} - a_{j-1}(\alpha) \). Hence Jensen's inequality gives the desired result for all \( q \geq 0 \).

For the sake of the asymptotic normality of \( u_n \), we introduce the function space:

\[ K_1 = \left\{ f = (f_k) : \mathbb{R} \to \mathbb{R}^q \mid f \text{ is of class } C^2, \quad v_0(f) < \infty, \quad \sup_{0 \leq s \leq h_n} E_0 \left[ |A^2 f(J_s)| \right] = O(1), \right. \]

and

\[ \left. \max_{i=0,1} \int_0^{h_n} E_0 \left[ |A^i f(J_{s-} + z) - A^i f(J_{s-})|^2 \right] v_0(dz) ds = O(1) \right\}. \]

Lemma 5.4. If Assumption 2.1 holds and if the function \( f : \mathbb{R} \to \mathbb{R}^q \) fulfills that \( f(0) = \partial f(0) = 0 \) and \( f \in K_1 \), then we have

\[ \frac{1}{n} E[f(J_n)] - v_0(f) = O(h). \]

Proof. By Itô-Taylor expansion, we see that

\[ f(J_n) = f(0) + hA f(0) + \int_0^h \int_0^s A^2 f(J_u) du \, ds \]

\[ \quad + \int_0^h \int_0^s \left( f(J_{s-} + z) - f(J_{s-}) \right) \tilde{N}(du, dz) + \int_0^h \int_0^s \left( A f(J_{u-} + z) - A f(J_{u-}) \right) \tilde{N}(du, dz) \, ds, \]

Under the assumptions we see that the last two terms are martingale (see [1, Theorem 4.2.3]) and \( A f(0) = v_0(f) \), hence the result follows.

Lemma 5.5. Suppose that Assumption 2.1, Assumption 2.2 and Assumption 2.7 hold. Then we have

\[ u_n \xrightarrow{L} \mathcal{N}(0, \Sigma_{11}), \]

Proof. By the stationarity and independence of increments of Lévy process \( J \), we have

\[ u_n = \frac{1}{\theta h} \sum_{j} \left( \varphi(\Delta J) - E J^{-1} \varphi(\Delta J) \right) \]

\[ + \frac{1}{\theta h} \sum_{j} E J^{-1} \varphi(\Delta J) - v_0(\varphi) \] \[ = e_n + f_n, \]

where

\[ e_n = \frac{1}{\theta h} \sum_{j} \left( \varphi(\Delta J) - E \varphi(J) \right) \]

\[ = \frac{1}{\theta h} \sum_{j} e_j, \]

\[ f_n = \frac{1}{\theta} E \varphi(J) - v_0(\varphi) \]

By the previous lemma, it is clear that \( f_n = o(1) \) under \( nh_n^2 \to 0. \) We will prove that \( e_n \xrightarrow{L} \mathcal{N}(0, \Sigma_{11}) \) by applying the martingale central limit theorem [3]. First, we show that \( \varphi_k, \varphi_k\varphi_l \in K_1 \) where \( \varphi_k \) denotes
From Lemma 9, it suffices to show that
\[
\sup_{u \in [0,1]} E \left[ \int 2 \partial^2 \mathcal{A} \varphi (J_s + uz) z^2 \nu_0 (dz) \right].
\]
Recall that by Assumption 2.2 and the definition of Lévy measure, we have \( \int |z|^q \nu_0 (dz) < \infty \) for all \( q \geq 2 \). By means of Assumption 2.7 and dominated convergence theorem it follows that
\[
|\partial^2 \mathcal{A} \varphi (x)| = \left| \partial^2 \left( \int (\varphi (x + z) - \varphi (x) - \partial \varphi (x) z) \nu_0 (dz) \right) \right| \lesssim 1 + |x|^C,
\]
for all \( x \in \mathbb{R} \). Hence we have
\[
\sup_{0 \leq s \leq h_n} E \left[ |\mathcal{A}^2 \varphi (J_s)| \right] < \infty.
\]
Similarly, we can show that
\[
\int_0^h \int E \left[ |\varphi (J_{s-} + z) - \varphi (J_{s-})|^2 \right] \nu_0 (dz) ds < \infty,
\]
\[
\int_0^h \int E \left[ |\mathcal{A} \varphi (J_{s-} + z) - \mathcal{A} \varphi (J_{s-})|^2 \right] \nu_0 (dz) ds < \infty.
\]
Obviously, we have \( E[e_j] = 0 \). The properties of conditional expectation yield that
\[
E[e_j e_j] = E[\varphi_k (J_s) \varphi (J_s)] - E[\varphi_k (J_s)] E[\varphi (J_s)],
\]
Lemma 5.4 leads to \( \frac{1}{nh^2} \sum_j E[e_j e_j] \to \int \varphi_k (z) \varphi (z) \nu_0 (dz) \). From Assumption 2.2, Assumption 2.7 and Lemma 5.1, we obtain \( E||e_j||^4 = O(h) \). Hence we have \( \frac{1}{(nh)^2} \sum_j E||e_j||^4 \to 0 \), namely Lindeberg condition holds. Combining these discussion, we deduce that \( e_n \overset{\mathcal{D}}{\to} N_q (0, \Sigma_{11}) \) as was to be shown, completing the proof.

Define \( G_n (\theta) \in \mathbb{R}^p \) by
\[
\tilde{G}_n (\theta) = (G_n^a (\theta), G_n^\alpha (\theta)),
\]
and it is easy to calculate its derivative \( \partial \tilde{G}_n (\theta) = (\partial \alpha G_n^a (\theta) \partial \alpha G_n^\alpha (\theta)) \in \mathbb{R}^p \otimes \mathbb{R}^p \) as follows:
\[
\partial \alpha G_n^a (\theta) = \frac{1}{nh} \sum_j \left\{ \frac{\partial^2 \alpha a_j^1 (\gamma)}{c_{j-1}^2 (\gamma)} (\Delta_j X - ha_{j-1} (\alpha)) \right\},
\]
\[
\partial \alpha G_n^\alpha (\theta) = \frac{1}{nh} \sum_j \partial \alpha a_j^1 (\gamma) [\partial \gamma c_{j-1}^2 (\gamma)] (\Delta_j X - ha_{j-1} (\alpha)),
\]
\[
\partial \alpha G_n^a (\theta) = \frac{2}{n} \sum_j (\Delta_j X - ha_{j-1} (\alpha)) [\partial \gamma c_{j-1}^2 (\gamma)] (\partial \gamma a_{j-1} (\alpha)),
\]
\[
\partial \alpha G_n^\alpha (\theta) = \frac{1}{nh} \sum_j \left\{ [\partial^2 \alpha c_{j-1}^2 (\gamma)] (\Delta_j X - ha_{j-1} (\alpha))^2 + 2h (\partial \gamma c_{j-1}^2 (\gamma))^2 - c_{j-1} (\gamma) \partial^2 \alpha c_{j-1} (\gamma) \right\}.
\]

**Lemma 5.6.** Under Assumptions 2.1-2.5, it follows that
\[
\sup_{\theta \in \Theta} \left| \tilde{G}_n (\theta) - G_{\infty} (\theta) \right| \overset{p}{\to} 0,
\]
\[
\sqrt{nh} G_n (\theta_0) \overset{\mathcal{D}}{\to} N_p (0, \Sigma_{22}).
\]

**Proof.** For simplicity, we do suppose that \( p_\alpha = p_\gamma = 1 \); the high dimensional case is similar. First, we will show the \( \theta \)-pointwise convergence
\[
\left| \tilde{G}_n (\theta) - G_{\infty} (\theta) \right| \overset{p}{\to} 0.
\]
From [7, Lemma 9], it suffices to that show
\[
\frac{1}{nh} \sum_j E^{j-1} [M_{j-1} (\theta) (\Delta_j X - ha_{j-1} (\alpha))] \overset{p}{\to} G_{\infty}^a (\theta),
\]
\[
\frac{1}{nh} \sum_j E^{j-1} [M_{j-1} (\theta) (\Delta_j X - ha_{j-1} (\alpha))] \overset{p}{\to} G_{\infty}^\alpha (\theta),
\]
\[
\frac{1}{nh} \sum_j E^{j-1} \left[ -\partial_x c_{j-1}^{-2}(\gamma) \right] (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \xrightarrow{P} G_\infty^\alpha(\gamma),
\]
\[
\frac{1}{(nh)^2} \sum_j E^{j-1} \left[ |M_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha))|^2 \right] \xrightarrow{P} 0,
\]
\[
\frac{1}{(nh)^2} \sum_j E^{j-1} \left[ \left[ -\partial_x c_{j-1}^{-2}(\gamma) \right] (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \right]^2 \xrightarrow{P} 0.
\]

By the definition of \( X \), we observe that
\[
\Delta_j X - ha_{j-1}(\alpha) = \int_j (a_s - a_{j-1}) ds + \int_j c_s - dJ_s + h(a_{j-1} - a_{j-1}(\alpha)).
\]
Hence the martingale property of \( \int_1^n c_u - dJ_u \) implies that
\[
E^{j-1} \left[ |M_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha))| \right] = M_{j-1}(\theta) \left\{ h(a_{j-1} - a_{j-1}(\alpha)) + E^{j-1} \left[ \int_j (a_s - a_{j-1}) ds \right] \right\}.
\]
Now, from [15, Lemma 4.5] and sup_{\theta} |M(x, \theta)| \lesssim 1 + |x|^C for some \( C \geq 0 \), we have
\[
E \left[ \frac{1}{nh} \sum_j M_{j-1}(\theta) E^{j-1} \left[ \int_j (a_s - a_{j-1}) ds \right] \right] \leq \frac{1}{nh} \sum_j \sqrt{E[|M_{j-1}(\theta)|^2]} \sqrt{E \left[ E^{j-1} \left[ \int_j (a_s - a_{j-1}) ds \right]^2 \right]} \leq \frac{1}{nh} \sum_j \sqrt{h \int_j E[|a_s - a_{j-1}|^2] ds} \leq \frac{1}{nh} \sum_j \sqrt{h \int_j E[|X_s - X_{j-1}|^2] ds} \lesssim \sqrt{h} = o(1),
\]
so the ergodic theorem gives
\[
\frac{1}{nh} \sum_j E^{j-1} \left[ |M_{j-1}(\theta)(\Delta_j X - ha_{j-1}(\alpha))| \right] \xrightarrow{P} G_\infty^\alpha(\theta).
\]

Similarly, we see that
\[
E^{j-1} \left[ -\partial_x c_{j-1}^{-2}(\gamma) \right] (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)}
\]
\[
= -\partial_x c_{j-1}^{-2}(\gamma) E^{j-1} \left[ \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s + h(a_{j-1} - a_{j-1}(\alpha)) + c_{j-1} \Delta J \right)^2 \right] - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)}
\]
\[
= -\partial_x c_{j-1}^{-2}(\gamma) E^{j-1} \left[ c_{j-1}^2 + 2 \xi_{s,j-1} c_{j-1} \Delta J + c_{j-1}^2(\Delta J)^2 \right] - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)}
\]
where \( \xi_{s,j-1} := \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s + h(a_{j-1} - a_{j-1}(\alpha)) \). Applying [15, Lemma 4.5] and Burkholder’s inequality, we see that
\[
E^{j-1} [\xi_{s,j-1}^2] \lesssim E^{j-1} \left[ \left\| \int_j (a_s - a_{j-1}) ds \right\|^2 \right] + E^{j-1} \left[ \left\| \int_j (c_s - c_{j-1}) dJ_s \right\|^2 \right] + E^{j-1} \left[ |h(a_{j-1} - a_{j-1}(\alpha))|^2 \right] \lesssim h \int_j E^{j-1}[|X_s - X_{j-1}|^2] ds + h^2 (a_{j-1} - a_{j-1}(\alpha))^2 \lesssim h^2 (1 + |X_{j-1}|^C),
\]
and \( E^{j-1}[\xi_{s,j-1}^2(\Delta J)^2] = h c_{j-1}^2 \). Hence we have \( E^{j-1}[\xi_{s,j-1} c_{j-1} \Delta J] \lesssim h^2 (1 + |X_{j-1}|^C) \) by conditional Cauchy-Schwarz’s inequality. It follows that
\[
\frac{1}{nh} \sum_j E^{j-1} \left[ -\partial_x c_{j-1}^{-2}(\gamma) \right] (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_x c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \xrightarrow{P} G_\infty^\alpha(\gamma),
\]
from the ergodic theorem. Above calculation yields that
\[ E^{j-1} |(\Delta_j X - ha_{j-1}(\alpha))^2| \lesssim h(1 + |X_{j-1}|^C), \]
and obviously, this inequality is valid when we replace 2 with for any \( q \geq 2 \). In the same way we can easily see that
\[
E^{j-1} \left[ \left| -\partial_\gamma c_{j-1}^{-2}(\gamma) (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_\gamma c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \right|^2 \right] \lesssim \left| -\partial_\gamma c_{j-1}^{-2}(\gamma) (\Delta_j X - ha_{j-1}(\alpha))^2 + h^2 \frac{\partial_\gamma c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \right|^2 \lesssim h(1 + |X_{j-1}|^C),
\]
so the ergodic theorem gives
\[
\frac{1}{(nh)^2} \sum_j E^{j-1} \left[ |\Delta_j X - ha_{j-1}(\alpha)|^2 \right] \xrightarrow{P_n} 0,
\]
\[
\frac{1}{(nh)^2} \sum_j E^{j-1} \left[ \left| -\partial_\gamma c_{j-1}^{-2}(\gamma) (\Delta_j X - ha_{j-1}(\alpha))^2 - h \frac{\partial_\gamma c_{j-1}^2(\gamma)}{c_{j-1}(\gamma)} \right|^2 \right] \xrightarrow{P_n} 0.
\]
As a result of these computations, we obtain the \( \theta \)-pointwise convergence
\[
\tilde{G}_n(\theta) - G_\infty(\theta) \xrightarrow{P_n} 0. \tag{5.2}
\]
To prove the uniformity of (5.2), it suffices to show the tightness, which is in turn implied by
\[
\sup_n E \left[ \sup_{\theta} \left| \partial_\theta \tilde{G}_n(\theta) \right| \right] < \infty.
\]
In the case of \( p_\alpha = p_\gamma = q = 1 \), we have
\[
\partial_\alpha G_n^\alpha(\theta) = \frac{1}{nh} \sum_j \left\{ \frac{\partial_\alpha a_{j-1}(\alpha)}{c_{j-1}(\gamma)} (\Delta_j X - ha_{j-1}(\alpha)) - h \frac{(\partial_\gamma c_{j-1}(\gamma))^2}{c_{j-1}(\gamma)} \right\},
\]
\[
\partial_\gamma G_n^\alpha(\theta) = -\frac{1}{nh} \sum_j \frac{\partial_\gamma a_{j-1}(\alpha)}{c_{j-1}(\gamma)} (\Delta_j X - ha_{j-1}(\alpha)),
\]
\[
\partial_\alpha G_n^\gamma(\theta) = -\frac{2}{n} \sum_j \frac{\partial_\alpha a_{j-1}(\alpha)}{c_{j-1}(\gamma)} (\Delta_j X - ha_{j-1}(\alpha)),
\]
\[
\partial_\gamma G_n^\gamma(\theta) = \frac{2}{nh} \sum_j \left\{ \frac{\partial_\gamma c_{j-1}(\gamma)c_{j-1}(\gamma) - 3(\partial_\gamma c_{j-1}(\gamma))^2}{c_{j-1}(\gamma)} (\Delta_j X - ha_{j-1}(\alpha))^2
\]
\[
- h \frac{\partial_\gamma^2 c_{j-1}(\gamma)c_{j-1}(\gamma) - (\partial_\gamma c_{j-1}(\gamma))^2}{c_{j-1}(\gamma)} \right\},
\]
and if we impose some regularity conditions on \( a \) and \( c \), we can calculate the high-order derivative of \( \tilde{G}_n(\theta) \) readily. Sobolev’s inequality and Lemma 5.3 imply that for \( q > p \)
\[
E \left[ \sup_{\theta} \left| \partial_\theta \tilde{G}_n(\theta) \right|^q \right] \lesssim E \left[ \left| \partial_\theta \tilde{G}_n(\theta) \right|^q + \left| \partial_\theta \tilde{G}_n(\theta) \right|^q \right] < \infty.
\]
Hence we are able to conclude that \([\tilde{G}_n(\theta) - G_\infty(\theta)]_{\theta \in \Theta} \] is uniformly tight (see, e.g. [11]) so that the continuous mapping theorem yields that \( \sup_{\theta \in \Theta} |\tilde{G}_n(\theta) - G_\infty(\theta)| \xrightarrow{P_n} 0 \). Moreover, the consistency of \( \tilde{\theta} \) immediately follows from [20, Theorem 5.3]. We will observe that
\[
\sqrt{nh}G_n^\alpha(\theta_0) = \frac{1}{\sqrt{nh}} \sum_j \frac{\partial_\alpha a_{j-1}(\alpha)}{c_{j-1}(\gamma)} \Delta_j J + o_p(1),
\]
\[
\sqrt{nh}G_n^\gamma(\theta_0) = \frac{2}{\sqrt{nh}} \sum_j \left\{ \frac{\partial_\gamma c_{j-1}(\gamma)}{c_{j-1}(\gamma)} (\Delta_j J)^2 - h \right\} + o_p(1).
\]
Trivial decomposition leads to
\[
\sqrt{nh}G^n_{\alpha}(\theta_0) = \frac{1}{\sqrt{nh}} \sum_j M_{j-1}(\Delta_j X - h_n a_{j-1})
\]
\[
= \frac{1}{\sqrt{nh}} \sum_j M_{j-1} \int_j (a_s - a_{j-1}) ds
+ \frac{1}{\sqrt{nh}} \sum_j M_{j-1} \int_j (c_s - c_{j-1}) dJ_s
+ \frac{1}{\sqrt{nh}} \sum_j \frac{\partial a_{j-1}}{c_{j-1}} \Delta_j J.
\]
From this, it suffices to show that \(\frac{1}{\sqrt{nh}} \sum_j M_{j-1} \int_j (a_s - a_{j-1}) ds\) and \(\frac{1}{\sqrt{nh}} \sum_j M_{j-1} \int_j (c_s - c_{j-1}) dJ_s\) are \(o_p(1)\). Notice that \(|M_{j-1}| \lesssim (1 + |X_{j-1}|^C)\). As in the proof of Lemma 5.3, we can observe that these terms are \(o_p(1)\). Hence we get
\[
\sqrt{nh}G^n_{\alpha}(\theta_0) = \frac{1}{\sqrt{nh}} \sum_j \frac{\partial a_{j-1}}{c_{j-1}} \Delta_j J + o_p(1).
\]
It is clear that
\[
\sqrt{nh}G^n_{\alpha}(\theta_0) = \frac{1}{\sqrt{nh}} \sum_j \left\{ -\partial_s c_{j-1}^2 \left( (\Delta_j X - h a_{j-1})^2 - h \frac{\partial c_{j-1}}{c_{j-1}} \right) \right\}
= \frac{1}{\sqrt{nh}} \sum_j \left(-\partial_s c_{j-1}^2 \right) \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s \right)^2
+ \frac{2}{\sqrt{nh}} \sum_j \left(-\partial_s c_{j-1}^2 \right) c_{j-1} \Delta_j J \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s \right)
+ \frac{2}{\sqrt{nh}} \sum_j \left( \frac{\partial c_{j-1}}{c_{j-1}} ((\Delta_j J)^2 - h) \right).
\]
By Assumption 2.3, \(\partial_s c_{j-1}^2\) admits a polynomial majorant, so it follows that
\[
E \left[ \left( \frac{1}{\sqrt{nh}} \sum_j \left(-\partial_s c_{j-1}^2 \right) \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s \right)^2 \right) \right] = o(1),
\]
from Lemma 5.3. Similar calculations yield that
\[
E \left[ \left( \frac{1}{\sqrt{nh}} \sum_j \left(-\partial_s c_{j-1}^2 \right) c_{j-1} \Delta_j J \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s \right) \right) \right]
\lesssim \frac{1}{\sqrt{nh}} \sum_j E \left[ \left| \partial_s c_{j-1} \Delta_j J \left( \int_j (a_s - a_{j-1}) ds + \int_j (c_s - c_{j-1}) dJ_s \right) \right| \right]
\lesssim \frac{1}{\sqrt{nh}} \sum_j \sqrt{E[\Delta_j J]^2} \left( E \left[ \left( \int_j (a_s - a_{j-1}) ds \right)^2 \right] + E \left[ \left( \int_j (c_s - c_{j-1}) dJ_s \right)^2 \right] \right).
\]
In the last inequality, we used the independence of increments of \(J\). By Lemma 5.1, we observe that \(\frac{1}{h} E[J_n^2] \to 1\), so we see that
\[
\frac{1}{\sqrt{nh}} \sum_j \sqrt{E[\Delta_j J]^2} \left( E \left[ \left( \int_j (a_s - a_{j-1}) ds \right)^2 \right] + E \left[ \left( \int_j (c_s - c_{j-1}) dJ_s \right)^2 \right] \right)
\lesssim \frac{1}{\sqrt{nh}} \sum_j \sqrt{h \frac{2}{h} + h} \lesssim \sqrt{nh^2} = o(1).
\]
Hence we get
\[
\sqrt{nh}G^n_{\alpha}(\theta_0) = \frac{2}{\sqrt{nh}} \sum_j \left\{ \frac{\partial c_{j-1}}{c_{j-1}} ((\Delta_j J)^2 - h) \right\} + o_p(1).
\]
We define
\[ \sqrt{nh}G_n^\alpha(\theta_0) = \frac{1}{\sqrt{nh}} \sum_j \frac{\partial_\alpha a_j - 1}{c_j - 1} \Delta_j J, \]
\[ \sqrt{nh}G_n^\gamma(\gamma_0) = \frac{2}{\sqrt{nh}} \sum_j \left\{ \frac{\partial_\gamma c_j - 1}{c_j - 1} ((\Delta_j J)^2 - h) \right\}. \]

From Assumption 2.2, we have
\[
\sum_j E_j^{-1} \left[ \frac{\partial_\alpha a_j - 1}{c_j - 1} \Delta_j J \right] = 0,
\]
\[
\sum_j E_j^{-1} \left[ \frac{\partial_\gamma c_j - 1}{c_j - 1} ((\Delta_j J)^2 - h) \right] = 0.
\]

The ergodic theorem and Lemma 5.1 give
\[
\frac{1}{(nh)^2} \sum_j E_j^{-1} \left[ \frac{\partial_\alpha a_j - 1}{c_j - 1} \Delta_j J \right]^4 \leq \frac{1}{(nh)^2} \sum_j \left| \frac{\partial_\alpha a_j - 1}{c_j - 1} \right|^4 E[J_h^4] \lesssim \frac{1}{nh} = o(1),
\]
\[
\frac{1}{(nh)^2} \sum_j E_j^{-1} \left[ \frac{\partial_\gamma c_j - 1}{c_j - 1} ((\Delta_j J)^2 - h) \right]^4 \leq \frac{1}{(nh)^2} \sum_j \left| \frac{\partial_\gamma c_j - 1}{c_j - 1} \right|^4 E \left[ (J_h^2 - h)^4 \right] \lesssim \frac{1}{nh} = o(1),
\]
so the Lindeberg condition holds. Furthermore we get
\[
E_j^{-1} \left[ \frac{\partial_\alpha a_j - 1}{c_j - 1} \partial_\alpha a_j - 1 \right] = \frac{\partial_\alpha a_j - 1}{c_j - 1} E[J_h^2] = \frac{\partial_\alpha a_j - 1}{c_j - 1} h,
\]
\[
E_j^{-1} \left[ \frac{\partial_\gamma c_j - 1}{c_j - 1} \gamma_j - 1 \right] (\Delta_j J)^2 - h = \frac{\partial_\gamma c_j - 1}{c_j - 1} E[(J_h^2 - h)^2] = \frac{\partial_\gamma c_j - 1}{c_j - 1} \left\{ E[J_h^2] + o(h) \right\},
\]
\[
E_j^{-1} \left( \frac{\partial_\gamma c_j - 1}{c_j - 1} \Delta_j J (\Delta_j J)^2 - h \right) = \frac{\partial_\gamma c_j - 1}{c_j - 1} E[J_h^2].
\]

Finally, we apply the ergodic theorem to derive
\[
E_j^{-1} \left[ \frac{\partial_\alpha a_j - 1}{c_j - 1} \partial_\alpha a_j - 1 (\Delta_j J)^2 \right] \to \frac{\partial_\alpha a(x, \alpha_0) \partial_\alpha a(x, \alpha_0)}{c^2(x, \gamma_0)} \pi_0(dx),
\]
\[
E_j^{-1} \left[ \frac{\partial_\gamma c_j - 1}{c_j - 1} \gamma_j - 1 ((\Delta_j J)^2 - h)^2 \right] \to \frac{\partial_\gamma c(x, \gamma_0) \partial_\gamma c(x, \gamma_0)}{c^2(x, \gamma_0)} \pi_0(dx) \int z^4 \nu_0(dz),
\]
\[
E_j^{-1} \left[ \frac{\partial_\gamma c_j - 1}{c_j - 1} \Delta_j J (\Delta_j J)^2 - h \right] \to \frac{\partial_\gamma c(x, \gamma_0) \partial_\gamma c(x, \gamma_0)}{c^2(x, \gamma_0)} \pi_0(dx) \int z^4 \nu_0(dz),
\]
with which the martingale central limit theorem completes the proof. \(\square\)

Applying Taylor’s theorem to \(\tilde{G}_n(\theta_0)\), we get
\[
\tilde{G}_n(\theta_0) = - \int_0^1 \partial_\theta \tilde{G}(\theta + u(\theta - \tilde{\theta})) du \sqrt{nh}(\theta - \theta_0).
\]
Note that by the consistency of \(\alpha\) and \(\gamma\), we can consider \(\tilde{G}_n(\tilde{\theta}) = 0\) a.s., for large enough \(n\).

**Lemma 5.7.** If Assumptions 2.1-2.6 hold, we have
\[
\sup_{|\theta| \leq \epsilon_n} \left| -\partial_\theta \tilde{G}_n(\theta_0 + \theta) - \mathcal{I}(\theta_0) \right| \to 0, \quad \text{where} \quad \epsilon_n \to 0
\]
\[
\sqrt{nh}(\tilde{\theta} - \theta_0) \to \mathcal{N}(0, (\mathcal{I}(\theta_0)^{-1})^T \Sigma_{\gamma \gamma} \mathcal{I}(\theta_0)^{-1}).
\]

**Proof.** We may set \(p_\alpha = p_\gamma = 1\). Define the \(2 \times 2\)-valued matrix \(\mathcal{I}(\theta)\) such that
\[
\mathcal{I}(\theta) = \begin{pmatrix} \mathcal{I}(\alpha, \alpha)(\theta) & \mathcal{I}(\alpha, \gamma)(\theta) \\ 0 & \mathcal{I}(\gamma, \gamma)(\theta) \end{pmatrix},
\]
where \( I^{(a,\alpha)}(\theta), I^{(a,\gamma)}(\theta) \) and \( I^{(\gamma,\gamma)}(\theta) \) are defined by

\[
I^{(a,\alpha)}(\theta) = \int \left\{ \frac{\partial_a^2 a(x,\alpha)}{c^2(x,\gamma)} [a(x,\alpha) - a(x,\alpha_0)] + \frac{\partial_a a(x,\alpha)}{c(x,\gamma)} \right\} \pi_0(dx),
\]

\[
I^{(a,\gamma)}(\theta) = \int \frac{\partial_a a(x,\alpha) \partial_a c(x,\gamma)}{c^3(x,\gamma)} [a(x,\alpha) - a(x,\alpha_0)] \pi_0(dx),
\]

\[
I^{(\gamma,\gamma)}(\theta) = 4 \int \frac{\partial_a c(x,\gamma)}{c^2(x,\gamma)} \pi_0(dx).
\]

As in the previous lemma, we can prove

\[-\partial_{\theta} \hat{G}_n(\theta) \xrightarrow{P_0} I(\theta), \text{ for all } \theta.\]

By Assumption 2.3, it immediately follows that for all \( k \in \{1,2,3,4\} \), \( \partial_{\theta} \hat{G}_n(\theta) \) can be decomposed as

\[
\partial_{\theta} \hat{G}_n(\theta) = \frac{1}{nh} \sum_j \left\{ M_{j-1}^{(1,k)}(\theta)(\Delta J - h\Delta J - \hat{\theta})^2 + M_{j-1}^{(2,k)}(\theta)(\Delta J - \hat{\theta})^2 + h M_{j-1}^{(3,k)}(\theta) \right\},
\]

where \( M_{j-1}^{(1,k)}, M_{j-1}^{(2,k)} \) and \( M_{j-1}^{(3,k)} \) are functions of \( X_{t_{j-1}} \) at most polynomial growth uniformly in \( \theta \). Hence the Sobolev’s inequality implies that \( \left\{ -\partial_{\theta} \hat{G}_n(\theta) - I(\theta) \right\} \) is uniformly tight and the continuous mapping theorem gives

\[
\sup_{|\theta| \leq \epsilon_n} \left| -\partial_{\theta} \hat{G}_n(\theta_0 + \theta) - I(\theta_0) \right| \xrightarrow{P_0} 0, \text{ where } \epsilon_n \to 0.
\]

Further, the continuity of \( I(\theta) \) and the consistency of \( \hat{\theta} \) give

\[-\int_0^1 \partial_{\theta} \hat{G}(\hat{\theta} + u(\theta_0 - \hat{\theta})) du \xrightarrow{P_0} I(\theta_0).\]

Assumption 2.6 ensures that \( \lim_{n \to \infty} P \left\{ \left| -\int_0^1 \partial_{\theta} \hat{G}(\hat{\theta} + u(\theta_0 - \hat{\theta})) du \right| > 0 \right\} = 1 \), hence we can suppose that \( -\int_0^1 \partial_{\theta} \hat{G}(\hat{\theta} + u(\theta_0 - \hat{\theta})) du \) is invertible for all \( n \) large enough. Hence, applying Slutsky’s lemma, we have the desired result.

Obviously, it follows from Lemma 5.7 that \(-\partial_{\theta} \hat{G}_n(\theta)\) can serve as a consistent estimator of \( I(\theta_0) \). In the same way, we could provide a consistent estimator of the asymptotic variance of \( \hat{\theta} \), making it possible to construct a confidence region.

We introduce the following function space:

\[
\mathcal{K}_2 = \left\{ f = (f_k) : \mathbb{R} \to \mathbb{R}^q \right| f \text{ is of class } C^2, \quad \frac{1}{h} \max_{1 \leq j \leq n} E \left[ |\partial f(\delta_j)|^2 \right] = O(1), \quad \frac{1}{h} \max_{1 \leq j \leq n} \sup_{u \in [0,1]} E \left[ |\partial f(\Delta J + u(\delta_j - \Delta J)|^2 \right] = O(1), \quad \text{and} \quad \forall K > 0, \max_{1 \leq j \leq n} \sup_{u \in [0,1]} E \left[ |\partial^2 f(\hat{\delta}_j + u(\delta_j - \hat{\delta}_j)|^K \right] = O(1) \right\}.
\]

By use of this class we can prove:

**Lemma 5.8.** Suppose that Assumptions 2.1-2.5 hold and that \( \varphi \in \mathcal{K}_2 \). Then we have the stochastic expansion:

\[
\sqrt{nh} \left\{ \frac{1}{nh} \sum_{j=1}^n \varphi(\hat{\delta}_j) - \nu_0(\varphi) \right\} = u_n + \frac{1}{nh} \sum_{j=1}^n (\partial \varphi(\delta_j) \otimes \partial \gamma(c_{j-1}) - \delta_j \Delta J) + o_p(1),
\]

where we also have \( \frac{1}{nh} \sum_{j=1}^n (\partial \varphi(\delta_j) \otimes \partial \gamma(c_{j-1}) - \delta_j \Delta J) = O_p(1) \).

**Proof.** First we decompose the left-hand side as

\[
\sqrt{nh} \left\{ \frac{1}{nh} \sum_{j=1}^n \varphi(\hat{\delta}_j) - \nu_0(\varphi) \right\}
\]
\[
\begin{align*}
= \sqrt{nh} \left\{ \frac{1}{nh} \sum_j \left[ \varphi(\hat{\delta}_j) - \varphi(\hat{\delta}_j) \right] \right\} + \sqrt{nh} \left\{ \frac{1}{nh} \sum_j \left[ \varphi(\hat{\delta}_j) - \varphi(\Delta_j J) \right] \right\} + u_n \\
=: b_n^{(1)} + b_n^{(2)} + u_n.
\end{align*}
\]

Let us first prove \( b_n^{(2)} = o_p(1) \). Applying Taylor’s theorem, we see that

\[
b_n^{(2)} = \frac{1}{\sqrt{nh}} \sum_j \left[ \int_0^1 \partial \varphi(\Delta_j J + u(\delta_j - \Delta_j J)) du \right] (\delta_j - \Delta_j J).
\]

By definition of \( \delta_j \), it follows that

\[
\Delta_j J - \delta_j = c_{j-1}^{-1}(\Delta_j J - \Delta_j X - ha_{j-1}) = c_{j-1}^{-1} \left( \int_j (a_{s} - a_{j-1}) ds + \int_j (c_{s} - c_{j-1}) dJ_s \right).
\]

As in the proof of Lemma 5.6, we have

\[
E[|\Delta_j J - \delta_j|^q] \lesssim h^2,
\]

for all \( q \geq 2 \). Applying Cauchy-Schwarz’s inequality we get

\[
E[|b_n^{(2)}|^2] \leq \frac{1}{\sqrt{n}} E \left[ \sum_j \frac{1}{\sqrt{h}} \left| \int_0^1 \partial \varphi(\Delta_j J + u(\delta_j - \Delta_j J)) du \right| |\Delta_j J - \delta_j| \right]^2
\]

\[
\leq \frac{1}{\sqrt{n}} \sum_j \sqrt{\frac{1}{h} E \left[ \left| \int_0^1 \partial \varphi(\Delta_j J + u(\delta_j - \Delta_j J)) du \right|^2 \right] E[|\Delta_j J - \delta_j|^2]}
\]

\[
\leq \frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} \left( \frac{1}{h} \sup_{u \in [0,1]} E \left[ |\partial \varphi(\Delta_j J + u(\delta_j - \Delta_j J))| \right] \right) \sum_j \sqrt{E[|\Delta_j J - \delta_j|^2]}
\]

\[
\lesssim \frac{1}{\sqrt{nh^2}} = o(1).
\]

Next we turn to \( b_n^{(1)} \). By Taylor’s theorem, we have

\[
b_n^{(1)} = \frac{1}{\sqrt{nh}} \sum_j \left[ \partial \varphi(\delta_j)(\hat{\delta}_j - \delta_j) \right] + \frac{1}{2\sqrt{nh}} \sum_j \left[ \int_0^1 \int_0^1 v \partial^2 \varphi(\delta_j + uv(\delta_j - \delta_j)) dv du \right] (\hat{\delta}_j - \delta_j)^2.
\]

For notational convenience, we denote by \( R(x) \) a generic matrix-valued function defined on \( \mathbb{R} \times \Theta \) for which there exists a constant \( C \geq 0 \) such that \( sup_{x} |R(x, \theta)| \leq C(1 + |x|^2) \) for every \( x \); the argument \( \theta \) is omitted from the notation, and the specific form of \( R_{j-1} \) appearing below may vary from line to line. From the definition of \( \delta_j \) and \( \hat{\delta}_j \),

\[
\hat{\delta}_j - \delta_j = c_{j-1}^{-1}(\Delta_j J - ha_{j-1}) - c_{j-1}^{-1}(\hat{\Delta}_j X - ha_{j-1})
\]

\[
= (c^{-1}_{j-1} - c^{-1}_{j-1})\Delta_j J - h(\hat{\eta}_{j-1} - \eta_{j-1}).
\]

Again applying Taylor’s theorem, we obtain

\[
|\hat{\delta}_j - \delta_j|^2 \lesssim \frac{1}{nh} \left[ \left( \sup_{\gamma} |\partial \varphi(c^{-1}_{j-1}(\gamma))| \right) \right]^2 |\hat{w}|^2 |\Delta_j J|^2 + h^2 \left( \sup_{\theta} |\partial \varphi(c_{j-1}^{-1}(\theta))| \right) \right)^2 |\hat{v}|^2\]

\[
\lesssim \frac{1}{nh} (|\hat{w}|^2 |\Delta_j J|^2 + h^2 |\hat{v}|^2) |R_{j-1}|
\]

\[
\lesssim \frac{1}{nh} (|\Delta_j J|^2 + h^2) |R_{j-1}| |\hat{v}|^2.
\]

A similar argument to the proof of Lemma 5.3 gives the estimate \( E[|R_{j-1}|E^{j-1} |\Delta_j X|^q] \lesssim h \) for all \( q \geq 2 \). By means of these estimates and Hölder’s inequality we can deduce that, for sufficiently large \( p \geq 2 \) and sufficiently small \( q > 1 \),

\[
|b_n^{(1,2)}| \lesssim \frac{1}{\sqrt{nh}} \sum_j \left[ \int_0^1 \int_0^1 v \partial^2 \varphi(\delta_j + uv(\hat{\delta}_j - \delta_j)) dv du \right] |\hat{\delta}_j - \delta_j|^2
\]
\[
\begin{align*}
\leq \frac{1}{\sqrt{nh}} \frac{1}{n} |\hat{v}|^2 \sum_j \left| \int_0^1 \int_0^1 v \partial^2 \varphi(\hat{\delta}_j + uv(\hat{\delta}_j - \delta_j)) dv du \right| |(\Delta_j X|^2 + h^2)| R_{j-1}^{-1} \\
\leq \frac{1}{\sqrt{nh}} \frac{1}{n} |\hat{v}|^2 \left( \sum_j \left| \int_0^1 \int_0^1 v \partial^2 \varphi(\hat{\delta}_j + uv(\hat{\delta}_j - \delta_j)) dv du \right|^p \right)^{\frac{1}{p}} \\
\times \left[ \sum_j \left( |R_{j-1}| (|\Delta_j X|^2 + h^2) \right)^{\frac{n-1}{n}} \right] \frac{1}{n} \\
\leq \frac{1}{\sqrt{nh}} \frac{1}{n} \times O_p(1) \times \left\{ \left( \frac{1}{n} \sum_j |\Delta_j X|^2 \right)^{\frac{n-1}{n}} \times O_p(1) + O_p(h^2) \right\} \\
\leq \frac{1}{\sqrt{nh}} h^{\alpha/2 + \frac{n-1}{n}} \times O_p(1) \leq O_p \left( \frac{1}{\sqrt{nh^{1+c\delta}}} \right) = o_p(1).
\end{align*}
\]

As for \( b_{n,1}^{(1,1)} \), we first observe that
\[
\begin{align*}
\hat{c}_{j-1}^{-1} - c_{j-1}^{-1} &= \frac{1}{\sqrt{nh}} \partial^2 \varphi(c_{j-1}^{-1}) \hat{\Delta}_j X + \frac{1}{2nh} \hat{\Delta}_j X T \left[ \int_0^1 \int_0^1 v \partial^2 \varphi(c_{j-1}^{-1}) (\gamma_0 + uv(\gamma - \gamma_0)) dv du \right] \hat{w}.
\end{align*}
\]
In a similar way to the estimate of \( b_{n,1}^{(1,2)} \), it follows from the definition of \( \mathcal{K}_2 \), the tightness of \( (\hat{w}) \), and Cauchy-Schwarz's inequality that
\[
\begin{align*}
\left( \int_0^1 \int_0^1 v \partial^2 \varphi(c_{j-1}^{-1}) (\gamma_0 + uv(\gamma - \gamma_0)) dv du \right) \hat{w} \\
\leq (nh)^{-\frac{3}{2}} \sum_j |\partial^2 \varphi(c_{j-1}^{-1})| |\Delta_j X| |R_{j-1}| \times O_p(1) \\
\leq O_p \left( \frac{1}{\sqrt{nh}} \right) = o_p(1).
\end{align*}
\]

We also have
\[
\left( \int_0^1 \int_0^1 v \partial^2 \varphi(c_{j-1}^{-1}) (\gamma_0 + uv(\gamma - \gamma_0)) dv du \right) \hat{w} \leq \frac{1}{n} \sum_j |\partial^2 \varphi(c_{j-1}^{-1})| \left( \int_0^1 \partial \eta_{j-1} (\theta_0 + u(\theta - \theta_0)) du \right) |\hat{w}| = o_p(1).
\]

We thus get
\[
\begin{align*}
\hat{b}_{n,1}^{(1,1)} &= \left\{ \frac{1}{nh} \int \Delta_j X \left( \partial \varphi(c_{j-1}^{-1}) \otimes \partial \eta_{j-1} \right) \right\} \hat{w} + o_p(1) =: \mu_n [\hat{w}] + o_p(1). \quad (5.6)
\end{align*}
\]

It remains to take a closer look at \( \mu_n \in \mathbb{R}^q \otimes \mathbb{R}^p \). Substitute the expression
\[
\Delta_j X = \int_j a_s ds + \int_j (c_{s-} - c_{j-1}) dJ_s + c_{j-1} \Delta J
\]
into (5.6) and observe that
\[
\begin{align*}
\left| \frac{1}{nh} \int a_s ds \left( \partial \varphi(c_{j-1}^{-1}) \otimes \partial \eta_{j-1} \right) \right| &\leq \frac{1}{n} \sum_j |\partial^2 \varphi(c_{j-1}^{-1})| |R_{j-1}| \left( \frac{1}{h} \int |a_s| ds \right) \\
&\leq \left( \frac{1}{n} \sum_j |\partial^2 \varphi(c_{j-1}^{-1})|^2 \right)^{1/2} \left( \frac{1}{n} \sum_j |R_{j-1}| \left( \frac{1}{h} \int |a_s|^2 ds \right) \right)^{1/2} \\
&\leq O_p(\sqrt{h}),
\end{align*}
\]
and similarly that, by using Burkholder’s inequality (conditional on $\mathcal{F}_{t_{n-1}}$),
\[
\left| \frac{1}{\sqrt{n}} \sum_j \int (c_{s_j} - c_{j-1}) dJ_s \left( \partial \varphi(\delta_j) \otimes \partial_t(c_{j-1}^{-1}) \right) \right|
\leq \frac{1}{n} \sum_j |\partial \varphi(\delta_j)||R_{j-1}| \left( \frac{1}{\sqrt{n}} \int \frac{1}{\sqrt{n}}(c_{s_j} - c_{j-1}) dJ_s \right)
\leq \left( \frac{1}{n} \sum_j |\partial \varphi(\delta_j)|^2 \right)^{1/2} \left( \frac{1}{n} \sum_j |R_{j-1}| \left( \frac{1}{\sqrt{n}} \int \frac{1}{\sqrt{n}}(c_{s_j} - c_{j-1}) dJ_s \right)^2 \right)^{1/2}
\leq O_p(\sqrt{n}).
\]
Therefore $\mu_n = \frac{1}{n} \sum_j (\partial \varphi(\delta_j) \otimes \partial_t(c_{j-1}^{-1})) c_{j-1} \Delta J + o_p(1)$ and we also get
\[
E \left[ \left| \frac{1}{\sqrt{n}} \sum_j (\partial \varphi(\delta_j) \otimes \partial_t(c_{j-1}^{-1})) c_{j-1} \Delta J \right| \right]
\leq \left( \frac{1}{n} \sum_j \frac{1}{\sqrt{n}} E \left[ |\partial \varphi(\delta_j)|^2 \right] \right)^{1/2} \left( \frac{1}{n} \sum_j E \left[ |R_{j-1}| \left( \frac{1}{\sqrt{n}} \int \frac{1}{\sqrt{n}}(c_{s_j} - c_{j-1}) dJ_s \right)^2 \right] \right)^{1/2} = O(1),
\]
hence the proof is complete. \hfill \Box

5.2. Proof of Theorem 3.1. In order to obtain (3.1), we first show that actually $\varphi \in \mathcal{K}_2$ and $\zeta \in \mathcal{K}_1 \cap \mathcal{K}_2$ (recall the notation $\zeta(z) = z \partial \varphi(z)$). As in the proof of Lemma 5.5, it follows that $\zeta \in \mathcal{K}_1$. From the proof of Lemma 5.1 and Lemma 5.8, for all $C \geq 2$, we have
\[
\max_{1 \leq j \leq n} E \left( |\Delta J - \delta_j|^C \right) = O(h^2), \quad \max_{1 \leq j \leq n} E \left( |\Delta J|^C \right) = O(h).
\]
Moreover, \cite[Theorem 2.7]{15} and (5.5) give
\[
E \left[ |\delta_j - \delta_j|^C \right] \lesssim (nh)^{-\frac{C}{2}} E \left( (|\Delta J X|^C + h^C) |R_{j-1}| |\delta_j|^C \right) = O \left( (nh)^{-\frac{C}{2}} h^{1-a} \right),
\]
for any $a \in (0, 1)$. Hence the Chebyshev’s inequality yields that
\[
\max_{1 \leq j \leq n} \sup_{u \in [0, 1]} \left\{ P \left( |\Delta J + u(\delta_j - \Delta J)| > M \right) \lor P \left( |\delta_j + u(\delta_j - \delta_j)| > M \right) \right\} = O(h).
\]
We will use these estimates without notice below. By the condition on $\partial \varphi$, we have
\[
\sup_{u \in [0, 1]} E \left[ |\partial \varphi(\Delta J + u(\delta_j - \Delta J))|^2 \right]
\lesssim E \left[ |\Delta J|^2 + |\delta_j - \Delta J|^2 + |\Delta J|^2(1+C_1) + |\delta_j - \Delta J(1+C_1)| \right] = O(h).
\]
In the same way as above, we also obtain $E \left[ |\partial \varphi(\delta_j)|^2 \right] = O(h)$. By Assumption 2.7, for all $K > 0$, there exists a constant $C \geq 2$ such that
\[
|\partial^2 \varphi(\delta_j + u(\delta_j - \delta_j))| \lesssim 1 + |\delta_j|^C + |\delta_j - \delta_j|^C
\lesssim 1 + |\delta_j - \delta_j|^C + |\Delta J - \delta_j|^C + |\Delta J|^C,
\]
so it is straightforward that
\[
\max_{1 \leq j \leq n} \sup_{u \in [0, 1]} E \left[ \left| \partial^2 \varphi(\delta_j + u(\delta_j - \delta_j)) \right|^K \right]
\lesssim 1 + \max_{1 \leq j \leq n} E \left[ \left| \delta_j - \delta_j \right|^C + |\Delta J - \delta_j|^C + |\Delta J|^C \right] = O(1).
\]
Hence $\varphi \in \mathcal{K}_2$; similarly $\zeta \in \mathcal{K}_2$. 

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Now we have $\delta - \Delta J = c_{j-1}^{-1} \int_j (a_s - a_{j-1}) ds + c_{j-1}^{-1} \int_j (c_s - c_{j-1}) dJ_s$; then, $E[\|\delta - \Delta J\|^2] \lesssim h^2$.

Plugging-in the expression $\partial \varphi(\delta_j) = \partial \varphi(\Delta J) + (\delta_j - \Delta J) \int_0^1 \partial^2 \varphi(\Delta J + u(\delta_j - \Delta J)) du$ and then applying analogous estimates under Assumption 2.7 as before, we can deduce that

$$\left| \frac{1}{nh} \sum_j \left( (\delta_j - \Delta J) \int_0^1 \partial^2 \varphi(\Delta J + u(\delta_j - \Delta J)) du \otimes \partial_j(c_{j-1}^{-1}) \right) c_{j-1} \Delta J \right|$$

$$\leq \left( \frac{1}{nh^2} \sum_j |\delta_j - \Delta J|^2 \right)^{1/2} \times \left( \frac{1}{n} \sum_j \left| \int_0^1 \partial^2 \varphi(\Delta J + u(\delta_j - \Delta J)) du \right|^2 |R_{j-1}|^2 |\Delta J|^2 \right)^{1/2}$$

$$\lesssim o_p(1) \times \left( \frac{1}{n} \sum_j |R_{j-1}|^2 |\Delta J|^2 (1 + |\Delta J - \delta_j|^C + |\Delta J|^C) \right)^{1/2} = o_p(1).$$

It follows from [5, Theorem 1] and Lemma 5.8 that under the present assumptions about $\zeta$ we have $\frac{1}{h} E[\zeta(\Delta J)] = \nu_0(\zeta) + o(1) = \frac{1}{nh} \sum_j \zeta(\delta_j) + o_p(1)$. Therefore,

$$\mu_n = \frac{1}{nh} \sum_j \left\{ \zeta(\Delta J) \otimes \partial_j(c_{j-1}^{-1}) \right\} c_{j-1} + o_p(1)$$

$$= -\frac{1}{h} E[\zeta(\Delta J)] \otimes \left( \frac{1}{n} \sum_j \partial_j(c_{j-1}^{-1}) \right)$$

$$+ \frac{1}{nh} \sum_j \left\{ \zeta(\Delta J) - E[\zeta(\Delta J)] \otimes \partial_j(c_{j-1}^{-1}) \right\} c_{j-1} + o_p(1)$$

$$= -\frac{1}{h} E[\zeta(\Delta J)] \otimes \left( \frac{1}{n} \sum_j \partial_j(c_{j-1}^{-1}) \right) + o_p(1)$$

$$= -\left( \frac{1}{nh} \sum_j \zeta(\delta_j) \right) \otimes \left( \frac{1}{n} \sum_j \partial_j(c_{j-1}^{-1}) \right) + o_p(1),$$

where we used the martingale central limit theorem together with Burkholder’s inequality for the third equality. Thus the proof is complete.

5.3. Proof of Theorem 3.3. From Lemmas 5.5, 5.6 and 5.7, it suffices to show that

$$\frac{1}{nh} \sum_j E^{j-1} \left[ (\varphi_k(\Delta J) - E[\varphi_k(\Delta J)]) \left( \frac{\partial_x a_{j-1}}{c_{j-1}} \Delta J \right) \right] \overset{p}{\to} \int \varphi_k(z) \nu_0(dz) \int \frac{\partial_x a(x, a_0)}{c(x, \gamma_0)} \pi_0(dx),$$

$$\frac{1}{nh} \sum_j E^{j-1} \left[ (\varphi_k(\Delta J) - E[\varphi_k(\Delta J)]) \left( \frac{\partial_x c_{j-1}}{c_{j-1}} ((\Delta J)^2 - h) \right) \right] \overset{p}{\to} \int \varphi_k(z) z^2 \nu_0(dz) \int \frac{\partial_x c(x, \gamma_0)}{c(x, \gamma_0)} \pi_0(dx).$$

Assumption 2.2 yields that

$$E^{j-1} \left[ (\varphi_k(\Delta J) - E[\varphi_k(\Delta J)]) \left( \frac{\partial_x a_{j-1}}{c_{j-1}} \Delta J \right) \right] = \frac{\partial_x a_{j-1}}{c_{j-1}} E[\varphi_k(J_h) J_h].$$

Similarly, we have

$$E^{j-1} \left[ (\varphi_k(\Delta J) - E[\varphi_k(\Delta J)]) \left( \frac{\partial_x c_{j-1}}{c_{j-1}} ((\Delta J)^2 - h) \right) \right] = \frac{\partial_x c_{j-1}}{c_{j-1}} \left\{ E[\varphi_k(J_h) J_h^2] - h E[\varphi_k(J_h)] \right\}. $$

From the proof of Lemma 5.5 we can readily observe that $z\varphi, z^2\varphi \in \mathcal{K}_1$. Hence the ergodic theorem and Lemma 5.4 lead to the desired result.

5.4. Proof of Corollary 3.5. For the construction of asymptotic variance, we define the function space:

$$\mathcal{K}_3 = \left\{ f = (f_k) : \mathbb{R} \to \mathbb{R}^q \mid f \text{ is of class } C^1, \quad \max_{1 \leq j \leq n} \sup_{u \in [0,1]} E \left[ |\partial f(\delta_j + u(\delta_j - \delta_j))^2 \right] = o(1), \quad \frac{1}{nh} \max_{1 \leq j \leq n} \sup_{u \in [0,1]} E \left[ |\partial f(\delta_j + u(\delta_j - \delta_j))^2 \right] = o(1) \right\}.$$
Lemma 5.9. Assume that an \( \mathbb{R}^2 \)-valued or \( \mathbb{R}^q \otimes \mathbb{R}^2 \)-valued function \( f \) is differentiable and there exist nonnegative constant \( D \) such that \( \limsup_{z \to 0} \frac{1}{|z|} |\partial f(z)| < \infty \) and \( \limsup_{z \to 0} \frac{1}{1 + |z|} |\partial f(z)| < \infty \), where \( \epsilon_0 \) is given in Assumption 2.1. Then \( f \in \mathcal{K}_3 \).

Proof. Dividing the events and applying Hölder’s inequality, we have

\[
\frac{1}{nh^2} \max_{1 \leq j \leq n} \sup_{u \in (0,1)} E \left[ |\partial f(\delta_j + u(\delta_j - \hat{\delta}_j))^2 \right]
\]

\[
= \frac{1}{nh^2} \max_{1 \leq j \leq n} \sup_{u \in (0,1)} E \left[ |\partial f(\delta_j + u(\delta_j - \hat{\delta}_j))^2; |\delta_j + u(\delta_j - \hat{\delta}_j)| \leq 1 \right] + \frac{1}{nh^2} \max_{1 \leq j \leq n} \sup_{u \in (0,1)} E \left[ |\partial f(\delta_j + u(\delta_j - \hat{\delta}_j))^2; |\delta_j + u(\delta_j - \hat{\delta}_j)| > 1 \right]
\]

\[
\lesssim \frac{1}{nh^2} \max_{1 \leq j \leq n} E \left[ |\delta_j|^{2(1-\epsilon_0)} + |\delta_j - \hat{\delta}_j|^{2(1-\epsilon_0)} \right]^2 + \frac{1}{nh^2} \max_{1 \leq j \leq n} \left( E \left[ 1 + |\delta_j|^{4\epsilon_0} + |\delta_j - \hat{\delta}_j|^{4\epsilon_0} \right] \right)^2 \left( P \left( |\delta_j| + |\delta_j - \hat{\delta}_j| > 1 \right) \right)^{1-\epsilon_0}
\]

\[
\lesssim \frac{1}{nh^2} h^{1-\epsilon_0} = \frac{1}{nh^{1+\epsilon_0}} = o(1).
\]

The other condition can be verified as well. \( \Box \)

First we note that Assumption 2.7, the mappings \( z \mapsto z^3, z^4, z \varphi(z), z^2 \varphi(z) \) satisfies the conditions of Lemma 5.4 and Lemma 5.9. Let us show that for any \( f \in \mathcal{K}_1 \cap \mathcal{K}_3 \) we have

\[
\frac{1}{nh} \sum_{j} f(\delta_j) - \nu_0(f).
\]

From a similar decomposition to that in the proof of Theorem 3.1, we have

\[
\frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\delta_j) = \left\{ \frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\delta_j) \right\} + \left\{ \frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\Delta J) \right\} + \left\{ \frac{1}{nh} \sum_{j} f(\Delta J) - \nu_0(f) \right\}.
\]

Then Lemma 5.4 implies the last term is \( o_p(1) \). Taylor’s expansion and Hölder’s inequality yield that

\[
\left| \frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\delta_j) \right| = \left| \frac{1}{nh} \sum_{j} \int_{0}^{1} f'(\delta_j + u(\delta_j - \hat{\delta}_j))du(\delta_j - \hat{\delta}_j) \right|
\]

\[
\leq \frac{1}{\sqrt{nh}} \sum_{j} \left[ \int_{0}^{1} f'(\delta_j + u(\delta_j - \hat{\delta}_j))du \left( \sup_{\gamma} |\partial c^{-1}_{\epsilon_0}(\gamma)| \right) |\Delta J| |\hat{\delta}| \right]
\]

\[
+ \frac{1}{\sqrt{nh}} \sum_{j} \left[ \int_{0}^{1} f'(\delta_j + u(\delta_j - \hat{\delta}_j))du \left( \sup_{\theta} |\eta_{j-1}(\theta)| \right) |\hat{\delta}| \right]
\]

\[
\leq \frac{1}{(nh)^{1.5}} \sum_{j} \left[ \int_{0}^{1} f'(\delta_j + u(\delta_j - \hat{\delta}_j))du \right]^{2.5} \times \frac{1}{nh} \sum_{j} \sup_{\gamma} |\partial c^{-1}_{\epsilon_0}(\gamma)|^{2} |\Delta J|^{2} = O_p(1) + o_p\left( \frac{1}{\sqrt{nh}} \right).
\]

Hence, using the conditioning argument together with \( E[|\Delta J|^2] \leq h^2 \) we obtain \( \frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\delta_j) = o_p(1) \). Recall that \( E[|\Delta J - \hat{\delta}_j|^2] \leq h^2 \), from which

\[
\left| \frac{1}{nh} \sum_{j} f(\delta_j) - \frac{1}{nh} \sum_{j} f(\Delta J) \right|
\]

\[
\leq \frac{1}{n} \left[ \int_{0}^{1} f'(\Delta J + u(\delta_j - \Delta J))du \right]^{2} \times \frac{1}{nh^2} \sum_{j} |\Delta J - \delta_j|^2 = o_p(1),
\]

hence (5.7) follows.
Hence the consistency of \( \hat{\Sigma} \) follows from Theorem 3.1, Theorem 3.3 and Slutsky’s lemma. Similar estimates and Slutsky’s lemma lead to \( \hat{\Sigma} \).

Proof of Corollary 3.7. From the result of Theorem 3.1, \( \frac{1}{\pi_n} \sum_{j=1}^{n} \varphi(\hat{\theta}) - \mu(\varphi) = o_p(1) \). Hence the continuity of \( \partial F \) and the invertibility of \( \partial F(\mu(\varphi), \theta_0) \) yield the first result. Finally, [20, Theorem 3.1] leads to the second result.

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