The Solvability Of Magneto-heating Coupling Model With Turbulent Convection Zone And The Flow Fields

Changhui Yao · Yanping Lin · Lixiu Wang · Xuefan Jia

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Abstract In this paper, the magneto-heating coupling model is studied in details, with turbulent convection zone and the flow field involved. Our main work is to analyze the well-posed property of this model with the regularity techniques. For the magnetic field, we consider the space $H_0(curl) \cap H(div)_0$ and for the heat equation, we consider the space $H_0(\Omega)$. Then we present the weak formulation of the coupled magneto-heating model and establish the regularity problem. Using Roth’s method, monotone theories of nonlinear operator, weak convergence theories, we prove that the limits of the solutions from Roth’s method converge to the solutions of the regularity problem with proper initial data. With the help of the spacial regularity technique, we derive the results of the well-posedness of the original problems when the regular parameter $\epsilon \to 0$. Moreover, with additional regularity assumption for both the magnetic field and temperature variable, we prove the uniqueness of the solutions.

Keywords Magneto-heating coupling model · Regularity · Well-posedness · Stability

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Corresponding author: Changhui Yao
School of Mathematics and Statistics, Zhengzhou University, 450001, China. E-mail: chyao@bsec.cc.ac.cn
Yanping Lin
Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong. E-mail: yanping.lin@polyu.edu.hk.
Lixiu Wang
Beijing Computational Science Research Center, Beijing 100193, China. E-mail: lixiuwang@csrc.ac.cn.
Xuefan Jia
School of Mathematics and Statistics, Zhengzhou University, 450001, China. E-mail: chyao@zzu.edu.cn
1 Introduction

It is well known that the manifestation of magnetohydrodynamic dynamo (MHD) processes can be applied to demonstrate large-scale magnetic activities [1]. Assume that the magnetic field \( B \), the electric field \( E \) and the electric current density \( J \) are governed by the Maxwell’s equations and constitutive relations in the magnetohydrodynamic approximation, that is [1],

\[
\begin{align*}
\partial_t B + \nabla \times E &= 0, \quad \nabla \cdot B = 0, \\
\nabla \times B &= \mu J, \quad J = \sigma (E + U \times B),
\end{align*}
\]

where \( \mu \) and \( \sigma \) are the magnetic permeability and the electric conductivity, and \( U \) is the velocity of the fluid.

Large-scale magnetic and flow fields activities can also drive small-scale turbulent flows as well as large-scale global circulations in their interiors [2,3]. Then it is useful to introduce mean-field dynamo theory [4], which describes the large-scale behavior of such fields. The magnetic and velocity fields can be divided into mean fields and deviations (called “fluctuations”), \( B = \bar{B} + b \) and \( U = \bar{U} + u \). The equations (1)-(2) can be averaged by

\[
\begin{align*}
\partial_t \bar{B} + \nabla \times \bar{E} &= 0, \quad \nabla \cdot \bar{B} = 0, \\
\nabla \times \bar{B} &= \mu \bar{J}, \quad \bar{J} = \sigma (\bar{E} + \bar{U} \times \bar{B} + \mathcal{E}),
\end{align*}
\]

where \( \mathcal{E} \) is the mean electromotive force due to fluctuations; it is crucial variable for all mean-field electrodynamics:

\[ \mathcal{E} = u \times b. \]

In order to discuss \( \mathcal{E} \), its mean part \( \bar{U} \) and the fluctuations \( u \) are assumed to be known. Then the fluctuations \( b \) are determined by

\[
\eta \nabla \cdot \nabla b + \nabla \times (\bar{U} \times b + G) - \partial_t b = -\nabla \times (u \times \bar{B}), \quad G = (u \times B) - u \times B. \tag{5}
\]

This equation implies that \( b \) can be considered as a sum \( b^0 + b^B \), where \( b^0 \) is independent of \( \bar{B} \) and \( b^B \) is a linear and homogeneous in \( \bar{B} \). This in turn leads to

\[ \mathcal{E} = \mathcal{E}^0 + \mathcal{E}^B \]

in which \( \mathcal{E}^0 \) is independent of \( \bar{B} \) and \( \mathcal{E}^B \) is a linear and homogeneous in \( \bar{B} \).

For simplicity, we assume that there is no mean motion, and \( u \) corresponds to a homogeneous isotropic turbulence. One can derive the relationship

\[ \mathcal{E} = \alpha \bar{B} - \beta \nabla \times \bar{B}, \tag{7} \]

where the two coefficients, \( \alpha \) and \( \beta \), are independent of position and are determined by \( u \), and \( \eta = \frac{1}{\mu \sigma} \). The term \( \alpha \bar{B} \) describes the \( \alpha \)-effect. Substituting (7) into (3)-(4), one can get

\[
\begin{align*}
\partial_t \bar{B} + \nabla \times ((\eta + \beta) \nabla \times \bar{B}) &= \nabla \times (\alpha \bar{B}) + \nabla \times (\bar{U} \times \bar{B}), \\
\nabla \cdot \bar{B} &= 0. \tag{8}
\end{align*}
\]
Here $\lambda =: \eta + \beta$ is the effective magnetic diffusivity, covering both magnetic diffusion at the microscopic level and the turbulent diffusion, respectively and it is also effected by the temperature. The $\alpha$ term represents the turbulent magnetic helicity. In order to deal with the feedback of the magnetic field on fluid motions (the Lorentz force), we employ a so-called $\alpha$-effect or $\alpha$-quench \[5\] by the form

$$\alpha(\overline{B}) = \frac{\alpha_0 f(x, t)}{1 + (R_m)^n |\overline{B}/B_{eq}|^2}, \quad (10)$$

where $\alpha_0 > 0$ is constant, $0 \leq n \leq 2$, $f(x, t)$ is a model-oriented function, and the $R_m$ dependent quenching expression should be regarded as a simplified steady state expression for the nonlinear dynamo \[6\], $B_{eq}$ is the equipartition magnetic field and can be assumed as a constant. For the convenience, here and later, we still denote $\overline{B}$ by $B$ and simplify (8)-(9) by the following form

with $\theta(x, t)$ denoting the temperature at location $x \in \Omega$ and time $t$.

\[
\begin{align*}
\partial_t B + \nabla \times (\lambda(\theta) \nabla \times B) - \Lambda \nabla (\nabla \cdot B) &= R_\alpha \nabla \times \left( \frac{f(x, t)B}{1 + \gamma |B|^2} \right) \\
&\quad + \nabla \times (U \times B), \quad \text{in } (0, T] \times \Omega, \quad (11) \\
\nabla \cdot B &= 0, \quad \text{in } (0, T] \times \Omega, \quad (12)
\end{align*}
\]

where $\lambda(\theta)$ is bounded and strictly positive i.e. $0 < \lambda_0 \leq \lambda \leq \lambda_M < +\infty$, $\gamma$ is a constant parameter, $R_\alpha$ is a dynamo parameter in connection with the generation process of small scale turbulence. With the boundary condition

$$\lambda(\theta) \nabla \times B \times n = 0, \quad \text{on } \partial \Omega, \quad (13)$$

and the initial data

$$B(x, 0) = B_0(x).$$

The local density of Joule’s heat equation generated by

$$E \cdot J = \sigma(|\nabla \times B|^2 - \nabla \times B \cdot (U \times B) - R_\alpha \nabla \times B \cdot \left( \frac{f(x, t)B}{1 + \gamma |B|^2} \right)).$$

Thus, from Fouriers law and the conservation of energy \[7,8,9\], we see that $\theta(x, t)$ satisfies

$$\begin{align*}
\partial_t \theta - \nabla \cdot (\kappa \nabla \theta) &= \sigma(\theta)(|\nabla \times B|^2 - \nabla \times B \cdot (U \times B) \\
&\quad - R_\alpha \nabla \times B \cdot \left( \frac{f(x, t)B}{1 + \gamma |B|^2} \right)), \quad \text{in } (0, T] \times \Omega, \quad (14)
\end{align*}$$

with the initial data and boundary conditions \[9\]

$$\begin{align*}
\theta(x, 0) &= \theta_0, \quad \text{in } \Omega, \quad (15) \\
\theta &= \theta_0, \quad \text{on } (0, T] \times \Gamma_1, \quad (16) \\
\n-\kappa \frac{\partial \theta}{\partial n} &= \zeta (\theta^4 - \theta_0^4) + \omega (\theta - \theta_0), \quad \text{on } (0, T] \times \Gamma_2, \quad (17)
\end{align*}$$
where $\theta_0 \in L^\infty(\Omega \cup I_1)$ is the background temperature, $\partial \Omega = I_1 \cup I_2$, $\zeta$ is the heat convection coefficient and $\omega$ the radiation coefficient, $\kappa$ is the thermal conductivity and other physical constants such as density and specific heat have been normalized. $n$ is the unit outer normal to $\Omega$. $\theta_0$ and $\kappa$ are reasonable to assume that $\theta_0 \geq \theta_{\text{min}} > 0$, $\kappa \geq \kappa_{\text{min}} > 0$.

Let $\theta = \xi + \theta_0$, we have

$$\Psi(\theta) = \Psi(\xi + \theta_0), \quad \Psi(\theta) - \Psi(\theta_0) = \zeta(\theta^4 - \theta_0^4) + \omega(\theta - \theta_0).$$

We also define

$$q(\xi) := \sigma(\xi + \theta_0),$$

$$K(B) = (|\nabla \times B|^2 - \nabla \times B \cdot (U \times B) - R_{\alpha} \nabla \times B \cdot \left(\frac{f(x,t)B}{1 + \gamma |B|^2}\right)),$$

and

$$Q_T = (0,T] \times \Omega.$$

The phenomenon of magneto-heating has been the main point of interest for many researches. In [13], the authors aim to develop a mathematical model for magnetohydrodynamic flow of biofluids through a hydrophobic micro-channel with periodically contracting and expanding walls under the influence of an axially applied electric field, and different temperature jump factors have also been used to investigate the thermomechanical interactions at the fluid-solid interface. In [14], the authors aim is to investigate the mixed convection flow of an electrically conducting and viscous incompressible fluid past an isothermal vertical surface with Joule heating in the presence of a uniform transverse magnetic field fixed relative to the surface. In [15], they study the coupling of the equations of steady-state magnetohydrodynamics (MHD) with the heat equation when the buoyancy effects due to temperature differences in the flow as well as Joule effect and viscous heating are taken into account, where the existence results of weak solutions are presented under certain conditions on the data and some uniqueness results are derived. In [16], the authors study a coupled system of Maxwell’s equations with nonlinear heat equation while they employ time discretization based on the Rothe’s method to provide energy estimates for discretized system and prove the existence of a weak solutions to this coupled system with controlled Joule heating term.

The most significant differences of our mathematical model compared to models stated in papers mentioned above can be summed into three points:

- The model coupled with turbulent convection zone and the flow fields.
- The nonlinear term concluding $\alpha$-quench.
- The coefficient of magnetic diffusion is temperature dependent and the temperature field is controlled by mixed nonlinear boundaries.

The outline of the paper is as follows:
2 Preliminaries

For any \( p \leq 1 \), let \( L^p(\Omega) \) be the sobolev space with the norm

\[
\|p\|_{L^p(\Omega)} = \left( \int_\Omega |p(x)|^p dx \right)^{1/p}.
\]

For \( p = \infty \), \( L^\infty(\Omega) \) denotes the space of essentially bounded functions with the norm

\[
\|u\|_{L^\infty(\Omega)} = \text{esssup}|u(x)|.
\]

For \( p = 2 \), \( L^2(\Omega) \) denotes the Hilbert space equipped with the inner product and norm

\[
(u, v) = \int_\Omega u(x)v(x)dx, \quad \|u\|_{L^2(\Omega)} = \sqrt{(u, u)}.
\]

Define \( H^m(\Omega) = \{ u \in L^2(\Omega) : D^k u \in L^2(\Omega), |k| \leq m \} \), which is equipped with the following norm and semi-norm

\[
\|u\|_{m, \Omega} = \left( \sum_{|k| \leq m} \|D^k u\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad |u|_{m, \Omega} = \left( \sum_{|k| = m} \|D^k u\|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

The most frequently used spaces in the subsequent analysis are the following two Sobolev spaces:

\[
H(\text{curl}, \Omega) = \{ u \in L^2(\Omega)^3 ; \nabla \times u \in L^2(\Omega) \},
\]

\[
H(\text{div}, \Omega) = \{ u \in L^2(\Omega)^3 ; \nabla \cdot u \in L^2(\Omega) \}
\]

and their subspaces

\[
H_0(\text{curl}, \Omega) = \{ u \in H(\text{curl}, \Omega), u \times n = 0, \text{ on } \partial \Omega \},
\]

\[
H(\text{div}_0, \Omega) = \{ u \in H(\text{div}, \Omega), \nabla \cdot u = 0, \text{ in } \Omega \}
\]

which are the equipped with the inner product

\[
(u, v)_{H(\text{curl}, \Omega)} = (u, v) + (\nabla \times u, \nabla \times v),
\]

\[
(u, v)_{H(\text{div}, \Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v),
\]

and the norm

\[
\|u\|^2_{H(\text{curl}, \Omega)} = \|u\|_{0, \Omega}^2 + \|\nabla \times u\|_{0, \Omega}^2, \quad \|u\|^2_{H(\text{div}, \Omega)} = \|u\|_{0, \Omega}^2 + \|\nabla \cdot u\|_{0, \Omega}^2.
\]

To treat the constraint equation \( \nabla \cdot B = 0 \), we shall need the following subspace

\[
V = H_0(\text{curl}, \Omega) \cap H(\text{div}_0, \Omega)
\]

with the inner product and norm

\[
(u, v)_V = (u, v) + (\nabla \times u, \nabla \times v) + (\nabla \cdot u, \nabla \cdot v), \quad \|u\|^2_V = \|u\|_{0, \Omega}^2 + \|\nabla \times u\|_{0, \Omega}^2 + \|\nabla \cdot u\|_{0, \Omega}^2.
\]
We also need define the functional space for the radiative and conductive heat equation

\[ H^1_0(\Omega) = \{ v \in H^1(\Omega), v|_{r_1} = 0 \}, \]
\[ Y = \{ v \in H^1_0(\Omega) \cap L^2(\Gamma_2), \| v \|_Y := \| v \|_1 + \| v \|_{L^2(\Gamma_2)} \}, \]
\[ W^{\theta, 4}(\text{curl}, \Omega) = \{ u \in L^2(\Omega)^3, \nabla \times u \in L^2(\Omega)^3 \}. \]

The coupling system (11)-(17) can be is equivalent to the following variational problem: Find \( B \in L^2(0,T;\mathcal{Y}) \) and \( \xi \in L^2(0,T;\mathcal{Y}) \) such that for any \( \Phi \in \mathcal{V}, T \in \mathcal{Y} \cap L^\infty(\Omega) \)

\[
(\partial_t B, \Phi) + (\lambda(\xi + \theta_0) \nabla \times B, \nabla \times \Phi) + A(\nabla \cdot B, \nabla \cdot \Phi) = R_\alpha \left( \frac{f(x,t)B}{1 + \frac{\gamma}{|B|^2}}, \nabla \times \Phi \right) + (U \times B, \nabla \times \Phi), \quad \forall \Phi \in \mathcal{V},
\]

\[
(\partial_t \xi, \mathcal{T}) + (\kappa \nabla \xi, \nabla \mathcal{T}) + \leq (\Psi(\xi + \theta_0) - \Psi(\theta_0), \mathcal{T}) \geq \tau_0, \quad \forall \mathcal{T} \in \mathcal{Y} \cap L^\infty(\Omega),
\]

where \( < (\Psi(\xi + \theta_0) - \Psi(\theta_0), \mathcal{T} \geq \tau_0, \int_{\Gamma_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) \mathcal{T} ds \).

In this paper, we consider the well-posedness of the coupling system with the regularity technique. In order to be convenient for the following proofs, we introduce two nonlinear operators defined by: for a given constant \( \tau > 0 \), let \( \mathcal{P} : \mathcal{V} \rightarrow \mathcal{V}' \) and \( \mathcal{L} : \mathcal{Y} \rightarrow \mathcal{Y}' \) such that

\[
< \mathcal{P} A, \Phi > := \frac{1}{\tau} (A, \Phi) + (\lambda(\xi + \theta_0) \nabla \times A, \nabla \times \Phi) + A(\nabla \cdot A, \nabla \cdot \Phi)
\]

\[
- R_\alpha \left( \frac{f(x,t)A}{1 + \gamma |A|^2}, \nabla \times \Phi \right) - (U \times A, \nabla \times \Phi), \forall A, \Phi \in \mathcal{V},
\]

\[
< \mathcal{L} \omega, \mathcal{T} > := \frac{1}{\tau} (\omega, \mathcal{T}) + (\kappa \nabla \omega, \nabla \mathcal{T})
\]

\[
+ < \Psi(\omega + \theta_0) - \Psi(\theta_0), \mathcal{T} \geq \tau_0, \forall \omega, \mathcal{T} \in \mathcal{Y}.
\]

**Lemma 1** There exists a constant \( C_1 \) dependent of \( R_\alpha, \lambda_M, \| f \|_{L^\infty(0,T;L^\infty(\Omega))}, \| U \|_{L^\infty(0,T;L^\infty(\Omega))}, \) \( C_2 \) dependent of \( \xi, \omega, \tau_2, C_3 \) dependent of \( \kappa, \) and parameters \( \tau \) such that

\[
\| \mathcal{P} B \|_{\mathcal{V}'} \leq C_1 \| B \|_{\mathcal{V}}, \quad \| \mathcal{L} \xi \|_{\mathcal{Y}'} \leq C_3 \| \xi \|_1 + C_2 \left( \sum_{j=1}^{\frac{4}{\tau}} \| \xi \|_{L^2(\Gamma_2)} \right).
\]

**Proof** Noting that

\[
\left( \frac{f(x,t)}{1 + \frac{\gamma}{|B|^2}} \right) \leq 1, \lambda(\xi + \theta_0) \leq \lambda_M \text{ and using Cauchy-Schwarz inequality, we have}
\]

\[
< \mathcal{P} B, \Phi > := \frac{1}{\tau} (B, \Phi) + (\lambda \nabla \times B, \nabla \times \Phi) + A(\nabla \cdot B, \nabla \cdot \Phi)
\]

\[
- R_\alpha \left( \frac{f(x,t)B}{1 + \gamma |B|^2}, \nabla \times \Phi \right) - (U \times B, \nabla \times \Phi),
\]

\[
\leq \frac{1}{\tau} \| B \|_0 \| \Phi \|_0 + \lambda_M \| \nabla \times B \|_0 \| \nabla \times \Phi \|_0 + A \| \nabla \cdot B \|_0 \| \nabla \cdot \Phi \|_0
\]

\[
+ R_\alpha \| f(x,t) \|_{L^\infty(0,T;L^\infty(\Omega))} \| B \|_0 \| \nabla \times \Phi \|_0
\]

\[
+ \| U(x,t) \|_{L^\infty(0,T;L^\infty(\Omega))} \| B \|_0 \| \nabla \times \Phi \|_0
\]

\[
\leq C_1 \| B \|_{\mathcal{V}} \| \Phi \|_{\mathcal{V}},
\]

(24)
where $C_1 = \max\{\frac{1}{r}, \lambda_M, R_{\alpha}, \|f(x, t)\|_{L^{\infty}(0, T; L^{\infty}(\Omega))}, \|U(x, t)\|_{L^{\infty}(0, T; L^{\infty}(\Omega))}\}$.

For the function $\theta_0 > 0$, we have

$$
|\Psi(\xi + \theta_0) - \Psi(\theta_0)| = |(\zeta \xi + \theta_0)^3 + \omega)(\xi + \theta_0) - (\zeta \theta_0)^3 + \omega)\theta_0| \\
\leq |\xi|(|\zeta \xi + \theta_0|^3 + 3\zeta^2 \theta_0 + 3\zeta \xi \theta_0^2 + 3\zeta \theta_0^3 + \omega).
$$

Then there exists a constant $C_2$ dependent of $\zeta, \omega, \Gamma_2$ and $\|\theta_0\|_{L^{\infty}(\Omega)}$ such that

$$
|\int_{\Gamma_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) T ds| \\
\leq \|\Psi(\xi + \theta_0) - \Psi(\theta_0)\|_{L^{2}(\Gamma_2)} \|T\|_{L^2(\Gamma_2)} \\
\leq C_2 \|T\|_{L^2(\Gamma_2)} \sum_{j=1}^{4} \|\xi\|^j_{L^j(\Gamma_2)}.
$$

Therefore, there exists a constant $C_3$ dependent of $\tau$ and $\kappa$ so that the boundedness of the nonlinear operator $\mathcal{L}$ can be estimated by

$$
<\mathcal{L}, \xi, \mathcal{T}> \leq \frac{1}{r} \|\xi\|_0 \|\mathcal{T}\|_0 + \kappa \|\xi\|_1 \|\mathcal{T}\|_1 + \sum_{j=1}^{4} \|\xi\|^j_{L^j(\Gamma_2)} \|\mathcal{T}\|_{L^2(\Gamma_2)} \\
\leq \max\{\kappa, \frac{1}{\tau}\} \|\xi\|_1 \|\mathcal{T}\|_1 + C_2 \sum_{j=1}^{4} \|\xi\|^j_{L^j(\Gamma_2)} \|\mathcal{T}\|_{L^2(\Gamma_2)} \\
\leq C_3 \|\xi\|_1 \|\mathcal{T}\|_1 + C_2 \sum_{j=1}^{4} \|\xi\|^j_{L^j(\Gamma_2)} \|\mathcal{T}\|_{L^2(\Gamma_2)}
$$

Lemma 2 There exist a positive constant $C_4$ depending on $\tau, \kappa, \lambda_0, R_{\alpha}, \|f\|_{L^{\infty}(\Omega)}, \|U\|_{L^{\infty}(\Omega)}$ and $C_5$ depending on $\tau, \kappa$ such that

$$
<\mathcal{P} B, B > \geq C_4 \|B\|_{\Omega}^2 <\mathcal{L}, \xi > \geq C_5 \|\xi\|^2_{1} + \frac{\zeta}{8} \|\xi\|^5_{L^5(\Gamma_2)}.
$$

Proof From Young inequality and $\lambda(\xi + \theta_0) \geq \lambda_0$, we have

$$
<\mathcal{P} B, B > = \frac{1}{\tau} (B, B) + (\lambda \nabla \times B, \nabla \times B) + A(\nabla \cdot B, \nabla \cdot B) \\
- R_{\alpha} \frac{f(x, t) \mathcal{B}}{1 + \eta |\mathcal{B}|^2}, \nabla \times B) + (U \times B, \nabla \times B) \\
\geq \frac{1}{\tau} \|B\|^2_0 + \lambda_0 \|\nabla \times B\|_0^2 + A\|\nabla \cdot B\|_0^2 \\
- R_{\alpha} \|f(x, t)\|_{L^{\infty}(\Omega)} \|B\|_0 \|\nabla \times B\|_0 - \|U\|_{L^{\infty}(\Omega)} \|B\|_0 \|\nabla \times B\|_0 \\
\geq \frac{1}{\tau} \|B\|^2_0 + \lambda_0 \|\nabla \times B\|_0^2 + A\|\nabla \cdot B\|_0^2 - \frac{R_{\alpha}\|f\|_{L^{\infty}(\Omega)} \|B\|_0^2}{4\epsilon_1} \\
- \epsilon_1 R_{\alpha}\|f\|_{L^{\infty}(\Omega)} \|\nabla \times B\|_0^2 - \frac{\|U\|_{L^{\infty}(\Omega)} \|B\|_0^2}{4\epsilon_2} - \epsilon_2 \|\nabla \times B\|_0^2.
$$
Lemma 3 For the vector $\mathbf{A}, \mathbf{B}$ and the parameter $\gamma > 0$, there holds

$$\frac{1}{1 + \gamma |\mathbf{A}|^2} - \frac{1}{1 + \gamma |\mathbf{B}|^2} \leq \frac{9}{4} |\mathbf{B} - \mathbf{A}|.$$ 

Proof By calculating, we have

$$\frac{|\mathbf{B}|}{1 + \gamma |\mathbf{B}|^2} - \frac{\mathbf{A}}{1 + \gamma |\mathbf{A}|^2} \leq \frac{|\mathbf{B} - \mathbf{A}| + \gamma |\mathbf{A}||(|\mathbf{A}| - |\mathbf{B}|)(|\mathbf{A}| + |\mathbf{B}|)}{1 + \gamma |\mathbf{B}|^2(1 + \gamma |\mathbf{A}|^2)} \leq \frac{|\mathbf{B} - \mathbf{A}|(1 + 2\gamma |\mathbf{A}|^2 + \gamma |\mathbf{A}||\mathbf{B}|)}{(1 + \gamma |\mathbf{A}|^2)(1 + \gamma |\mathbf{B}|^2)}.$$
By the symmetry, we have
\[
\frac{A}{1 + \gamma|A|^2} - \frac{B}{1 + \gamma|B|^2} \\
\leq |B - A| \left( \frac{(1 + 2\gamma|B|^2 + \gamma|A||B|)}{(1 + \gamma|A|^2)(1 + \gamma|B|^2)} \right)
\]

Therefore, we have
\[
\frac{A}{1 + \gamma|A|^2} - \frac{B}{1 + \gamma|B|^2} \\
\leq |B - A| \left( \frac{(1 + \gamma|A|^2 + \gamma|B|^2 + \gamma|A||B|)}{(1 + \gamma|A|^2)(1 + \gamma|B|^2)} \right)
\]
\[
\leq |B - A| \left( \frac{(1 + \frac{3}{2}\gamma|A|^2) + \frac{1}{2}\gamma|B|^2}{(1 + \gamma|A|^2)(1 + \gamma|B|^2)} \right)
\]
\[
\leq \frac{9}{4} |B - A|.
\]

**Lemma 4** The operator $P$ and $L$ is strictly monotone in the sense that
\[
< PB - PA, B - A > \geq C_6 \|B - A\|_V^2,
\]
(31)
where $C_6$ is taken as $\min \{ 1, R_0 \frac{\|f\|_{L^\infty(\Omega)}}{4\epsilon_3}, A, (\lambda_0 - \epsilon_3 R_0 \|f\|_{L^\infty(\Omega)}) - \epsilon_4 \|U\|_{L^\infty(\Omega)} \}$, And
\[
< Lv - Lu, v - w > \geq C_7 \|v - w\|_1^2 + \frac{\zeta}{8} \|v - w\|_{L^2(I_2)}^2,
\]
(32)
where the constant $C_7$ can be taken as $C_7 = \min(\tau^{-1}, \kappa)$.

**Proof** From the Young inequality and Lemma 3, we have
\[
< PB - PA, B - A >
\]
\[
= \frac{1}{\tau} \left( B - A, B - A \right) + \lambda \left( \nabla \times (B - A), \nabla \times (B - A) \right) + A \left( \nabla \cdot (B - A), \nabla \cdot (B - A) \right)
\]
\[
- R_0 \frac{\|f(x,t, t)\|_{L^\infty(\Omega)}}{1 + \gamma|B|} B - \|f(x,t, t)\|_{L^\infty(\Omega)} A, \nabla \times (B - A) \right) - \left( U \times (B - A), \nabla \times (B - A) \right)
\]
\[
\geq \frac{1}{\tau} \|B - A\|_0^2 + \lambda_0 \|\nabla \times (B - A)\|_0^2 + A \|\nabla \cdot (B - A)\|_0^2 - \frac{9 R_0 \|f\|_{L^\infty(\Omega)}}{16\epsilon_3} \|B - A\|_0^2
\]
\[
- \epsilon_3 R_0 \|f\|_{L^\infty(\Omega)} \|\nabla \times (B - A)\|_0^2 - \frac{\|U\|_{L^\infty(\Omega)}}{4\epsilon_4} \|B - A\|_0^2 - \epsilon_4 \|U\|_{L^\infty(\Omega)} \|\nabla \times (B - A)\|_0^2
\]
\[
= \left( \frac{1}{\tau} - \frac{R_0 \|f\|_{L^\infty(\Omega)}}{4\epsilon_3} \right) \|B - A\|_0^2 + \lambda_0 \|\nabla \times (B - A)\|_0^2 + A \|\nabla \cdot (B - A)\|_0^2
\]
\[
+ (\lambda - \epsilon_3 R_0 \|f\|_{L^\infty(\Omega)} - \epsilon_4 \|U\|_{L^\infty(\Omega)}) \|\nabla \times (B - A)\|_0^2
\]
\[
\geq C_6 \|B - A\|_V^2,
\]
(33)
where $C_0$ is taken as $\min\{\left(\frac{1}{\tau} \left(\frac{9R_o}{\|f\|_{L^\infty(\Omega)}} - \frac{\|U\|_{L^\infty(\Omega)}}{d_0}\right)\right), A, (\lambda_0 - \epsilon_3 R_o)\|f\|_{L^\infty(\Omega)} - \epsilon_4 \|U\|_{L^\infty(\Omega)}\}\}$.

Since $\Psi(\cdot)$ is a monotone function, we have

$$
<\mathcal{L}v - \mathcal{L}w, v - w> = \frac{1}{\tau} \|v - w\|_0^2 + \kappa \|\nabla(v - w)\|_0^2 + \int_{\Gamma_2} (\Psi(v + \theta_0) - \Psi(w + \theta_0))(v - w)ds
$$

$$
\geq \frac{1}{\tau} \|v - w\|_0^2 + \kappa \|\nabla(v - w)\|_0^2 + \frac{\zeta}{8} \|v - w\|_{H^2(\Gamma_2)} + \omega \|v - w\|^2_{H^2(\Gamma_2)}
$$

$$
\geq C_7 \|v - w\|_0^2 + \frac{\zeta}{8} \|v - w\|_{H^2(\Gamma_2)} + \omega \|v - w\|^2_{H^2(\Gamma_2)},
$$

(34)

where $C_7$ is take as the $C_7 = \min(\tau^{-1}, \kappa)$.

**Lemma 5** The nonlinear operator $\mathcal{P} : \mathcal{Y} \rightarrow \mathcal{Y}'$ and $L : \mathcal{Y} \rightarrow \mathcal{Y}'$ is hemi-continuous, that is

$$
\mathcal{S}(s) - \mathcal{S}(s_0) = \left| \mathcal{P}(Q(s)) - \mathcal{P}(Q(s_0)) \right|, \quad \mathcal{Z}(s) = \left| \mathcal{L}(u + sv), w \right|
$$

is continuous on $s \in [0, 1]$, respectively, for any $Q, R, \Phi \in \mathcal{Y}, \forall u, v, w \in \mathcal{Y}$.

**Proof** For convenience, we denote $Q(s) = R + sQ$. For any $s, s_0 \in [0, 1]$, we have

$$
|\mathcal{S}(s) - \mathcal{S}(s_0)| = \left| \mathcal{P}(Q(s)) - \mathcal{P}(Q(s_0)) \right|, \Phi >
$$

$$
= \frac{1}{\tau} \left( Q(s) - Q(s_0), \Phi \right) + \lambda(\nabla \times (Q(s) - Q(s_0)), \nabla \times \Phi) + \lambda(\nabla \cdot (Q(s) - Q(s_0)), \nabla \cdot \Phi) - (R_o \frac{f(x, t)}{1 + \gamma |Q(s)|^2} Q(s) - R_o \frac{f(x, t)}{1 + \gamma |Q(s_0)|^2} Q(s_0), \nabla \times \Phi)
$$

$$
\leq \frac{1}{\tau} \left| Q(s) - Q(s_0), \Phi \right| + \lambda \|\nabla \times (Q(s) - Q(s_0))\|_0\|\nabla \times \Phi\|_0|s - s_0| + \|\nabla \cdot (Q(s) - Q(s_0))\|_0\|\nabla \cdot \Phi\|_0|s - s_0| + \|\nabla \times (Q(s) - Q(s_0))\|_0\|\nabla \times \Phi\|_0|s - s_0| + \|\nabla \cdot (Q(s) - Q(s_0))\|_0\|\nabla \cdot \Phi\|_0|s - s_0|
$$

(35)

where we use $\frac{1}{1 + \gamma |Q(s)|^2} \leq 1, \frac{1}{1 + \gamma |Q(s_0)|^2} \leq 1$. This shows that $\mathcal{S}(s)$ is continuous on $[0, 1]$ for any $Q, R \in \mathcal{Y}$.

We also denote $u(t) = v + sv, \forall u, v \in \mathcal{Y}, t \in [0, 1]$. Then for any $s, s_0 \in [0, 1]$, we have

$$
|\mathcal{Z}(s) - \mathcal{Z}(s_0)| = \left| \mathcal{L}v(s) - \mathcal{L}u(s_0), w \right|
$$

$$
= \frac{1}{\tau} \left| u(s) - u(s_0), w \right| + \kappa(\nabla(u(s) - u(s_0)), \nabla w) + \int_{\Gamma_2} (\Psi(u(t)) - \Psi(u(t_0)))wds
$$

$$
\leq |s - s_0| \left| \frac{1}{\tau} \left| u(s) - u(s_0), w \right| + \kappa(\nabla u(s), \nabla w) + \gamma + \omega \|v\|wds \right|, \quad (36)
$$

which means $\mathcal{Z}(s)$ is continuous on $[0, 1]$ for any $u, v \in \mathcal{Y}$. 


3 The Regularized Problem

We have to notice the test function $\Upsilon \in L^\infty(\Omega)$ in (20), which increases the difficulties deeply when analyzing the well-posedness. In order deal with this problem, the Regularized techniques can be employed: given the small parameter $0 < \epsilon < 1$, find $B \in L^2(0, T; \mathcal{V})$ and $\xi \in L^2(0, T; \mathcal{Y})$ such that

\begin{align}
(\partial_t B, \Phi) + (\lambda (\xi + \theta_0) \nabla \times B, \nabla \times \Phi) + A(\nabla \cdot B, \nabla \cdot \Phi) &= R_\alpha \left( \frac{f(x, t)B_1}{1 + \gamma |B|^2}, \nabla \times \Phi \right) \\
+ (U \times B, \nabla \times \Phi), \quad \forall \Phi \in \mathcal{V},
\end{align}

(37)

\begin{align}
(\partial_t \xi, \Upsilon) + (\kappa \nabla \xi, \nabla \Upsilon) + < (\Psi (\xi + \theta_0) - \Psi (\theta_0)), \Upsilon >_{\mathcal{Y}_2} &= \left( [q(\xi)K(B)]_\epsilon, \Upsilon \right) - (\kappa \nabla \theta_0, \nabla \Upsilon), \forall \Upsilon \in \mathcal{Y},
\end{align}

(38)

where $[D]_\epsilon$ is the cut-off of $D$ defined by

$$[D]_\epsilon = \frac{D}{1 + \epsilon |D|}, \quad \epsilon > 0.$$ 

It is clear that $[D]_\epsilon \in L^\infty(\Omega)$. If $D \in L^p(\Omega)$, the

$$\lim_{\epsilon \to 0} \| [D]_\epsilon - D \|_{L^{p/2}(\Omega)} = 0.$$ 

3.1 Semi-discrete Approximation

We will use Roth’s method [10] to explore the well-posedness of solution of the regularized problem (37)-(38). Let $N$ be a positive integer and let an equidistant partition of $[0, T]$ be given by

$$t_n = n\tau, \; n = 0, 1, 2, \cdots, N, \; \tau = T/N.$$ 

The semi-discrete approximation to (37)-(38) can be formulated by: for $\forall \Phi \in \mathcal{V}, \; \forall \Upsilon \in \mathcal{Y}$, find $B^n \in \mathcal{V}$ and $\xi^n \in \mathcal{Y}$, $1 \leq n \leq N$ with initial data $B^0 = B_0(x), \xi^0 = 0$ such that,

\begin{align}
\frac{(B^n - B^{n-1})}{\tau}, \Phi) + (\lambda (\xi^{n-1} + \theta_0) \nabla \times B^n, \nabla \times \Phi) + A(\nabla \cdot B^n, \nabla \cdot \Phi) &= R_\alpha \left( \frac{f(x, n\tau)B_1}{1 + \gamma |B^{n-1}|^2}, \nabla \times \Phi \right) \\
+ (U \times B^n, \nabla \times \Phi), \quad \forall \Phi \in \mathcal{V},
\end{align}

(39)

\begin{align}
\frac{(\xi^n - \xi^{n-1})}{\tau}, \Upsilon) + (\kappa \nabla \xi^n, \nabla \Upsilon) + < (\Psi (\xi^n + \theta_0) - \Psi (\theta_0)), \Upsilon >_{\mathcal{Y}_2} &= \left( [q(\xi^{n-1})K(B^n)]_\epsilon, \Upsilon \right) - (\kappa \nabla \theta_0, \nabla \Upsilon).
\end{align}

(40)

For convenience, we also denote the difference operator

$$\delta_n w = \frac{w^n - w^{n-1}}{\tau}, \quad \text{in} \quad [t_{n-1}, t_n].$$

Obviously, (39)-(40) can be solved sequentially since (39) is independent of (40) for a given $B^{n-1}$ and (40) can be solved after given by $B^n$ in (39) and $\xi^{n-1}$. 

3.2 Well-posedness of the Nonlinear Magnetic Equation

Let $\tilde{B}_\tau$ and $B_\tau$ denote the piecewise constant and piecewise linear interpolations using the discrete solutions, that is
\[
\tilde{B}_\tau(\cdot, t) = B^n, B_\tau(\cdot, t) = L_n(t)B^n + (1 - L_n(t))B^{n-1},
\]
for any $t \in [t_{n-1}, t_n]$ and $1 \leq n \leq N$ with $L_n(t) = (t - t_{n-1})/\tau$. Obviously, we have
\[
\tilde{B}_\tau \in L^2(0, T; V), B_\tau \in C(0, T; V).
\]
We also denote $\hat{B}_\tau = \tilde{B}_\tau(\cdot, t - \tau) = B^n_{t_{n-1}}, \forall t \in (t_{n-1}, t_n]$.

Let $\tilde{\xi}_\tau$ and $\xi_\tau$ denote the piecewise constant and piecewise linear interpolations using the discrete solutions, that is
\[
\tilde{\xi}_\tau(\cdot, t) = \xi^n, \xi_\tau(\cdot, t) = L_n(t)\xi^n + (1 - L_n(t))\xi^{n-1},
\]
for any $t \in [t_{n-1}, t_n]$ and $1 \leq n \leq N$. We also denote
\[
\hat{\xi}_\tau = \tilde{\xi}_\tau(\cdot, t - \tau) = \xi^{n-1}, \forall t \in (t_{n-1}, t_n].
\]

**Theorem 1** For any $1 \leq n \leq N$ and for given $B^{n-1}$, the weak formula (39) has a unique solution $B^n \in V$. For a given $B^n \in V$ and $\xi^{n-1} \in Y$, the weak formula (40) has a unique solution $\xi^n \in Y$.

**Proof** We rewrite the weak formula (39) as: find $B^n \in V$ such that
\[
\begin{align*}
&\left( \frac{B^n}{\tau}, \Phi \right) + (\lambda(\xi^{n-1} + \theta_0)\nabla \times B^n, \nabla \times \Phi) + \left( \nabla \cdot B^n, \nabla \cdot \Phi \right) \\
&\quad - R_\alpha \left( \frac{f(x, n\tau)B^n}{1 + \gamma|B^{n-1}|^2}, \nabla \times \Phi \right) - (U \times B^n, \nabla \times \Phi) \\
&\quad = \left( \frac{B^{n-1}}{\tau}, \Phi \right), \forall \Phi \in V,
\end{align*}
\]
which is equivalent to an nonlinear operator equation
\[
\mathcal{P}B^n = F_{n-1},
\]
where $F_{n-1} \in V'$ defined by $(F_{n-1}, \Phi) = \left( \frac{B^{n-1}}{\tau}, \Phi \right)$.

We also rewrite (40) as: find $\xi^n \in Y$ such that
\[
\begin{align*}
&\left( \frac{\xi^n}{\tau}, \Upsilon \right) + (\kappa \nabla \xi^n, \nabla \Upsilon) + < (\Psi(\xi^n + \theta_0) - \Psi(\theta_0)), \Upsilon > \geq \gamma_1 \\
&\quad = \left( \frac{\xi^{n-1}}{\tau}, \Upsilon \right) + (q(\xi^{n-1})|K(B^n)|_\epsilon, \Upsilon) - (\kappa \nabla \theta_0, \Upsilon), \forall \Upsilon \in Y,
\end{align*}
\]
which is equivalent to an nonlinear operator equation
\[
\mathcal{L}\xi^n = H_{n-1},
\]
where \( H_{n-1} \in \mathcal{Y}' \) defined by
\[
(H_{n-1}, Y) = (\xi_{n-1}^+, T) + (q(\xi_{n-1})[K(B^n)]_0, T) - (\kappa \nabla \theta_0, T), \forall Y \in \mathcal{Y}.
\]

From Lemma 4 and Lemma 5 we know that \( \mathcal{P} \) and \( \mathcal{L} \) are a bounded, coercive, strictly monotone, and semi-continuous operator on \( \mathcal{V} \).

Lemma 6 From [11, 12], we know that problem (43) has a solution \( B^n \in \mathcal{V} \), and (45) has a solution \( \xi^n \in \mathcal{Y} \).

Now we have to prove the uniqueness of the solution. Let \( B^n, \tilde{B}^n \) be the two solutions of (43). From Lemma 4 we have
\[
0 = \langle \mathcal{P} B^n - \mathcal{P} \tilde{B}^n, B^n - \tilde{B}^n \rangle \geq C_0 \| B^n - \tilde{B}^n \|^2_0.
\]

We can conclude \( B^n = \tilde{B}^n \) in \( \Omega \), which means the uniqueness of the solution of (43). Let \( \xi^n, \tilde{\xi}^n \) be the two solutions of (45). From Lemma 4 we also have
\[
0 = \langle \mathcal{L} \xi^n - \mathcal{L} \tilde{\xi}^n, \xi^n - \tilde{\xi}^n \rangle \geq C_0 \| \xi^n - \tilde{\xi}^n \|^2_0 + C_0 \| \xi^n - \tilde{\xi}^n \|^2_{L^2(\Omega)},
\]

which means the uniqueness of the solution of (45).

**Lemma 6** There exists two positive constants \( C_8 \) and \( C_9 \) dependent of \( R_\alpha \), \( \| f(x, t) \|_{L^\infty(0, T; L^\infty(\Omega))} \), \( \| U \|_{L^\infty(0, T; L^\infty(\Omega))} \) such that
\[
\| B^n \|^2_0 + \sum_{i=1}^n \tau \lambda_0 \| \nabla \times B^i \|^2_0 + \sum_{i=1}^n \tau A \| \nabla \cdot B^i \|^2_0 \leq C_8 \| B^n \|^2_0. \tag{47}
\]
\[
\| B^0 \|_{L^\infty(0, T; L^2(\Omega))} + \sqrt{C_0} \| \nabla \times B^0 \|_{L^2(0, T; L^2(\Omega))} + A \| \nabla \cdot B^0 \|_{L^2(0, T; L^2(\Omega))} \leq C_9 \| B^n \|^2_0. \tag{48}
\]
\[
\| B^0 \|_{L^\infty(0, T; L^2(\Omega))} + \sqrt{C_0} \| \nabla \times B^0 \|_{L^2(0, T; L^2(\Omega))} + A \| \nabla \cdot B^0 \|_{L^2(0, T; L^2(\Omega))} \leq C_9 \| B^n \|^2_0. \tag{49}
\]

**Proof** Taking \( \Phi = B^n \) in (39), we have
\[
(B^n - B^{n-1}, B^n) + \tau \lambda_0(\nabla \times B^n, \nabla \times B^n) + \tau A(\nabla \cdot B^n, \nabla \cdot B^n)
\leq \tau R_\alpha \left( 1 + \frac{f(x, t)}{1 + \gamma \| B^{n-1} \|^2_0} \right) (B^n, \nabla \times B^n) + \tau (U \times B^n, \nabla \times B^n). \tag{50}
\]
Since
\[
2(B^n - B^{n-1}, B^n) \geq \| B^n \|^2_0 - \| B^{n-1} \|^2_0,
\]
summing up (50) from \( i = 1, 2, \ldots, n \), we have
\[
\frac{1}{2}(\| B^n \|^2_0 - \| B^0 \|^2_0) + \sum_{i=1}^n \tau \lambda_0 \| \nabla \times B^i \|^2_0 + \sum_{i=1}^n \tau A \| \nabla \cdot B^i \|^2_0
\leq \sum_{i=1}^n \left( \tau R_\alpha \| f(x, t) \|_{L^\infty(\Omega)} \| B^i \|_0 \| \nabla \times B^i \|_0 \right)
\leq \frac{1}{2}(\| B^n \|^2_0 - \| B^0 \|^2_0) + \sum_{i=1}^n \left( \tau \| U \|_{L^\infty(\Omega)} \| B^i \|_0 \| \nabla \times B^i \|_0 \right)
\[ \begin{align*}
&\leq \sum_{i=1}^{n} \tau R_{\alpha} \| f(x, t) \|_{L^{\infty}(\Omega)} \left( \frac{\|B_i \|_{0}^2}{4\delta_i} + \delta_i \| \nabla \times B_i \|_{0}^2 \right) \\
&+ \sum_{i=1}^{n} \tau \| U \|_{L^{\infty}(\Omega)} \left( \frac{\|B_i \|_{0}^2}{4\delta_i} + \delta_i \| \nabla \times B_i \|_{0}^2 \right) .
\end{align*} \]

Taking \( \delta_i, \delta_i \) such that

\[ \frac{1}{2\tau} - \frac{R_{\alpha} \| f(x, t) \|_{L^{\infty}(\Omega)}}{4\delta_n} - \frac{\| U \|_{L^{\infty}(\Omega)}}{4\delta_n} > 0, \]

\[ \Lambda_0 - R_{\alpha} \| f(x, t) \|_{L^{\infty}(\Omega)} \delta_i - \| U \|_{L^{\infty}(\Omega)} \delta_i > 0, \]

based on Grownall’s inequality, we have

\[ \| B^\alpha \|_0^2 + \sum_{i=1}^{n} \tau \lambda_0 \| \nabla \times B_i \|_0^2 + \sum_{i=1}^{n} \tau A \| \nabla \cdot B_i \|_0^2 \leq C_8 \| B^0 \|_0^2, \]  

where \( C_8 \) is independent of \( n \), which means

\[ \| B \|_{L^{\infty}(0, T, L^2(\Omega))} + \sqrt{\| \nabla \times B \|_{L^2(0, T, L^2(\Omega))}} + A \| \nabla \cdot B \|_{L^2(0, T, L^2(\Omega))} \leq C. \]

Similarly, \((49)\) comes directly from \( (48)\).

**Lemma 7** There exists two positive constants and \( C_9 \) and \( C \) dependent of \( \epsilon, \kappa, \eta_{\text{max}}, \zeta, \| \xi_0 \|_{0}, \| \nabla \theta_0 \|_{0} \) such that

\[ \| \xi^n \|_0^2 + \sum_{i=1}^{n} \tau \kappa \| \nabla \xi^n \|_0^2 + \sum_{i=1}^{n} \tau \omega \| \xi^n \|_{L^2(\Omega_2)}^2 \leq C_9 \| \xi_0 \|_0^2 + \| \nabla \theta_0 \|_{0}. \]  

\[ \| \xi \|_{L^{\infty}(0, T, L^2(\Omega))} + \| \nabla \xi \|_{L^2(0, T, L^2(\Omega))} + \frac{\zeta}{\delta} \| \xi \|_{L^2(0, T, L^2(\Omega_2))} \leq C, \]

\[ \| \xi \|_{L^{\infty}(0, T, L^2(\Omega_2))} + \| \nabla \xi \|_{L^2(0, T, L^2(\Omega_2))} + \frac{\zeta}{\delta} \| \xi \|_{L^2(0, T, L^2(\Omega_2))} \leq C. \]

**Proof** Taking \( T = \xi^n \) in \((40)\), we have

\[ (\xi^n - \xi^{n-1}, \xi^n) + \tau (\kappa \nabla \xi^n, \nabla \xi^n) + \tau < (\Psi(\xi^n + \theta_0) - \Psi(\theta_0), \xi^n)_{\Omega_2} \]

\[ = \tau (q(\xi^{n-1})|\mathcal{K}(B^n)|, \xi^n) - \tau (\kappa \nabla \theta_0, \nabla T). \]  

Firstly, we estimate the right hand of \((55)\). We should notice

\[ \| [\mathcal{K}(B^n)]_c \|_{0} = \frac{|\mathcal{K}(B^n)|}{1 + |\mathcal{K}(B^n)|} \leq \frac{1}{\epsilon}, \]  

which leads to

\[ (q(\xi^{n-1})|\mathcal{K}(B^n)|, \xi^n) \leq \frac{1}{\epsilon} \| \xi^n \|_{0}. \]
And we also have
\[ \tau |(\kappa \nabla \theta_0, \nabla \xi^n)| \leq \kappa \tau \| \nabla \theta_0 \|_0 \| \nabla \xi^n \|_0. \]

Since
\[ 2(\xi^n - \xi^{n-1}, \xi^n) \geq \| \xi^n \|_0^2 - \| \xi^{n-1} \|_0^2, \]
and
\[ (\Psi(\xi^n + \theta_0) - \Psi(\theta_0), \xi^n) \geq \frac{c_1}{2} \| \xi^n \|_{L^2(I,T)}^2 + \omega \| \xi^n \|_{L^2(I,T)}^2 \geq \frac{c_1}{8} \| \xi^n \|_{L^2(I,T)}^2, \]
summing up (55) from \( i = 1, 2, \cdots, n \), we have
\[ \frac{1}{2}(\| \xi^n \|_0^2 - \| \xi^0 \|_0^2) + \sum_{i=1}^{n} \tau \kappa \| \nabla \xi_i \|_0^2 + \sum_{i=1}^{n} \frac{c_1}{8} \| \xi_i \|_{L^2(I,T)}^2 \]
\[ \leq \sum_{i=1}^{n} \left( \frac{\tau}{\kappa} \| \nabla \theta_0 \|_0 \| \nabla \xi_i \|_0 \right) \]
\[ \leq \sum_{i=1}^{n} \left( \frac{\tau^2}{\epsilon^2} + \frac{\delta}{\kappa} \| \nabla \theta_0 \|_0^2 + \frac{\kappa \tau}{\epsilon} \| \nabla \xi_i \|_0^2 \right) \]
(59)

Taking \( \delta_n \) and \( \zeta \) such that \( \delta_n + \kappa \tau \zeta \leq \frac{1}{2} \), and using Growall’s inequality, we have
\[ \| \xi^n \|_0^2 + \sum_{i=1}^{n} \tau \kappa \| \nabla \xi_i \|_0^2 + \sum_{i=1}^{n} \frac{c_1}{8} \| \xi_i \|_{L^2(I,T)}^2 \leq C_0(\| \xi^0 \|_0^2 + \| \nabla \theta_0 \|_0^2), \]
(60)
where \( C_0 \) is independent of \( n \), which means
\[ \| \xi \|_{L^\infty(0,T;V)} + \| \nabla \xi \|_{L^2(0,T;V)} + \frac{c_1}{8} \| \xi \|_{L^2(0,T;L^2(\Omega))} \leq C. \]

Similarly, (55) comes directly from (54).

### 3.3 The Existence of the Solution of the Regularized Problem

From Lemma [6]-Lemma [7] we can see that the two discrete solutions of the Regularized Problem imply the boundedness in \( V \) and \( Y \), respectively. Since both \( V \) and \( Y \) are reflexive, there exist a subsequence of \( B_\tau \) and a subsequence of \( B_\tau \), which have common subscripts and are denoted the same notations such that
\[ B_\tau \rightharpoonup B, \quad B_\tau \rightharpoonup B, \quad \text{in} \quad L^2(0,T;V), \]
where \( B \in L^2(0,T;V) \) and “\( \rightharpoonup \)” denote the weak convergence of the sequences.

Similarly, there exist a subsequence of \( \xi_\tau \) and a subsequence of \( \xi_\tau \) with the same subscripts such that
\[ \xi_\tau \rightharpoonup \xi, \quad \xi_\tau \rightharpoonup \xi, \quad \text{in} \quad L^2(0,T;\mathcal{Y}). \]
Furthermore, based on $L^2(Q_T)$ is embedded compactly into $L^1(Q_T)$, we have
\[ \begin{align*}
\tilde{B}_r &\rightarrow B, \nabla \times \tilde{B}_r \rightarrow \nabla \times B, \quad \xi_r \rightarrow \xi, \quad \text{in} \quad L^1(Q_T), \\
B_r &\rightarrow B, \nabla \times B_r \rightarrow \nabla \times B, \quad \xi_r \rightarrow \xi, \quad \text{in} \quad L^1(Q_T),
\end{align*} \tag{61, 62} \]

where "$\rightarrow$" means strong convergence of the sequences. In this subsection, we shall present the proof that the limits $B, \xi$ solve the regularized problem \[37\text{-}39\]. Without causing the confusions, we always use \{$\tilde{B}_r$\}, \{\$B_r$\}, \{$\xi_r$\}, \{$\xi_r$\} to denote their convergence subsequences in the rest of this section.

**Theorem 2** The limit function $B$ is the weak solution of problem \[37\] with initial data $B(x, 0) = B_0(x)$.

**Proof** Remember that $C_0^\infty(\Omega) \subset V$. For any $v(x, t) = \Phi(x)\phi(t)$ with $\Phi(x) \in C_0^\infty(\Omega)$ and $\phi(t) \in C_0^\infty(0, T)$, from \[37\] after calculating, we have
\[ \int_{Q_T} \frac{\partial B_r}{\partial t} v + \int_{Q_T} \lambda (\xi_r + \theta_0) \nabla \times \tilde{B}_r \nabla \times v + \int_{Q_T} \Lambda \nabla \cdot \tilde{B}_r \cdot \nabla \cdot v = R_0 \int_{Q_T} \frac{f(x, t)}{1 + \gamma |B_r|^2} B_r \cdot \nabla \times v + \int_{Q_T} (U \times \tilde{B}_r) \cdot \nabla \times v. \tag{63} \]

For the first term, we have
\[ \lim_{\tau \rightarrow 0} \int_{Q_T} \frac{\partial B_r}{\partial t} v = - \lim_{\tau \rightarrow 0} \int_{Q_T} B_r \frac{\partial v}{\partial t} = - \int_{Q_T} B \frac{\partial v}{\partial t} = \int_{Q_T} \frac{\partial B}{\partial t} \cdot v. \tag{64} \]

For the second and the third term, since $\nabla \times \tilde{B}_r$ and $\nabla \cdot B_r$ converge to $\nabla \times B$ and $\nabla \cdot B$, respectively, we have
\[ \lim_{\tau \rightarrow 0} \int_{Q_T} \lambda (\xi_r + \theta_0) \nabla \times \tilde{B}_r \cdot \nabla \times v = \int_{Q_T} \lambda (\xi + \theta_0) \nabla \times B \cdot \nabla \times v, \]
\[ \lim_{\tau \rightarrow 0} \int_{Q_T} \Lambda \nabla \cdot \tilde{B}_r \cdot \nabla \cdot v = \int_{Q_T} \Lambda \nabla \cdot B \cdot \nabla \cdot v. \]

Since both $f(x, t)$ and $U$ are Lipschitz continuous and $\frac{1}{1 + \gamma |B_r|^2} \rightarrow \frac{1}{1 + \gamma |B|^2}$ and $U \times \tilde{B}_r \rightarrow U \times B$ strongly in $L^1(Q_T)$, we have
\[ \lim_{\tau \rightarrow 0} \int_{Q_T} \frac{1}{1 + \gamma |B_r|^2} B_r \cdot \nabla \times v = \int_{Q_T} \frac{1}{1 + \gamma |B|^2} B \cdot \nabla \times v, \tag{65} \]
\[ \lim_{\tau \rightarrow 0} \int_{Q_T} U \times \tilde{B}_r \cdot \nabla \times v \rightarrow \int_{Q_T} U \times B \cdot \nabla \times v. \tag{66} \]

From \[64\text{-}66\], we have
\[ \int_{Q_T} \frac{\partial B}{\partial t} \cdot v + \int_{Q_T} \lambda (\xi + \theta_0) \nabla \times B \nabla \times v + \int_{Q_T} \Lambda \nabla \cdot B \cdot \nabla \times v = R_0 \int_{Q_T} \frac{f(x, t)}{1 + \gamma |B|^2} B \cdot \nabla \times v + \int_{Q_T} (U \times B) \cdot \nabla \times v. \tag{67} \]
From (40) we know that for any \( \xi \), the initial condition \( \phi \) is the initial function. The limit function \( L \) is bounded and the strong convergence of \( \hat{\xi} \), we have

\[
\int_{Q_T} \frac{f(x,t)}{1+|B|^2} B \cdot \nabla \times \Phi + \int_{Q_T} (U \times B) \cdot \nabla \times \Phi.
\]

By the density of \( C^0_0 \) in \( V \), the equation (68) holds for any \( \Phi \in V \), too.

**Theorem 3** The limit function \( \xi \) is the weak solution of problem (40) with the initial condition \( \xi(x,0) = 0 \).

**Proof** For any \( \eta(x) = T(x)\phi(t) \) with \( T(x) \in C^0_0(\Omega) \) and \( \phi(t) \in C^0_0(0,T) \), from (40) we know that

\[
\int_{Q_T} \frac{\partial \xi}{\partial t} \eta + \int_{Q_T} \kappa \nabla \xi \cdot \nabla \eta + \int_0^T \int_{\Gamma_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) \eta = \int_{Q_T} q(\xi)[\kappa(\hat{B}_r)]_\xi \eta - \int_{Q_T} \kappa \nabla \theta_0 \cdot \nabla \eta.
\]

Since \( L^2(\Gamma_2) \) is embedded compactly into \( L^4(\Gamma_2) \) and \( \xi_\tau \) converges strongly to \( \xi \) in \( L^4(\Gamma_2) \), we have

\[
\lim_{\tau \to 0} \int_{Q_T} \frac{\partial \xi}{\partial t} \eta = \int_{Q_T} \frac{\partial \xi}{\partial t} \eta, \quad \lim_{\tau \to 0} \int_{Q_T} \kappa \nabla \xi \cdot \nabla \eta \to \int_{Q_T} \kappa \nabla \xi \cdot \nabla \eta.
\]

The right hand of the equation (69) can be considered from the the uniform boundedness and the strong convergence of \( \xi_\tau \) in \( L^2(0,T;Y) \) and the strong convergence of \( \nabla \times B_\tau \to \nabla \times B, U \times B_\tau \to U \times B, \frac{f(x,t)B}{1+|B|^2} \to \frac{f(x,t)B}{1+|B|^2} \) in \( L^1(Q_T) \), we have

\[
\lim_{\tau \to 0} \int_{Q_T} [g(\xi_\tau)K(B_\tau)]_\xi \eta \to \int_{Q_T} [g(\xi)K(B)]_\xi \eta.
\]

From (69)-(72), we can get

\[
\int_{Q_T} \frac{\partial \xi}{\partial t} \eta + \int_{Q_T} \kappa \nabla \xi \cdot \nabla \eta + \int_0^T \int_{\Gamma_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) \eta = \int_{Q_T} [g(\xi)K(B)]_\xi - \int_{Q_T} \kappa \nabla \theta_0 \cdot \nabla \eta.
\]
By the arbitrariness of \( \phi(t) \), it yields
\[
\int_\Omega \frac{\partial \xi}{\partial t} Y + \int_\Omega \kappa \nabla \xi \cdot \nabla Y + \int_{F_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) Y = \int_\Omega q(\xi) \mathcal{K}(B) Y - \int_\Omega \kappa \nabla \theta_0 \cdot \nabla Y, \quad \forall Y \in \mathcal{Y} \cap C^\infty(\Omega). \tag{74}
\]

By the density of \( \mathcal{Y} \cap C^\infty(\Omega) \) in \( \mathcal{Y} \), the above equation (74) holds for any \( Y \in \mathcal{Y} \).

Taking any \( Y \in C^0_0 \) and let \( \eta(t) = (T - t) Y \), we have \( \eta(0) = T \eta, \eta(T) = 0 \).

Using integration by part, we have
\[
T \int_\Omega \xi(0) \cdot Y(x) = - \int_0^T \int_\Omega \frac{\partial}{\partial t} (\xi \cdot \eta) = \int_0^T \int_\Omega \xi \cdot Y - \int_0^T \int_\Omega \frac{\partial \xi}{\partial t} \cdot \eta
\]
\[
= \lim_{\tau \to 0} \int_0^T \int_\Omega [\xi \cdot Y + \int_{F_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) \eta + \int_0^T [\kappa \nabla \xi \cdot \nabla \eta - q(\xi) \mathcal{K}(B) \cdot \eta]]
\]
\[
= \lim_{\tau \to 0} \int_0^T \int_\Omega [\xi \cdot Y - \int_{F_2} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) \eta - \int_0^T \int_\Omega [\kappa \nabla \xi \cdot \nabla \eta - q(\xi) \mathcal{K}(B) \cdot \eta]]
\]
\[
= \lim_{\tau \to 0} \int_0^T \int_\Omega \xi \cdot Y - \int_0^T \int_\Omega \frac{\partial \xi}{\partial t} \cdot \eta = \lim_{\tau \to 0} T \int_\Omega \xi(0) \cdot Y(x) = 0, \forall Y(x) \in C^0_0(\Omega).
\]

Therefore, \( \xi(x, 0) = 0 \), which finish the proof.

3.4 Stability of the Regularized Problem

Now we present the stability estimate of the regularized problem to ensure the well-posedness of the equations (19) - (20).

**Lemma 8** There exists a constant \( C_{10} \) depending of \( \Omega, T, \lambda_0, A, R_\alpha, \|f(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))} \) and \( \|U\|_{L^\infty(0, T; L^\infty(\Omega))} \) such that
\[
\|B\|_{L^\infty(0, T; L^2(\Omega))} + \sqrt{\lambda_0} \|\nabla \times B\|_{L^2(0, T; L^2(\Omega))} + A \|\nabla \cdot B\|_{L^2(0, T; L^2(\Omega))} \leq C_{10} \|B_0\|_0.
\]

**Proof** Taking \( \Phi = B \) in (77) and using \( \lambda(\xi + \theta_0) \geq \lambda_0 > 0 \), we have
\[
\frac{1}{2} \frac{\partial}{\partial t} \|B\|^2_0 + \lambda_0 \|\nabla \times B\|^2_0 + A \|\nabla \cdot B\|^2_0 \leq R_\alpha \left( \frac{f(x, t)B}{1 + \gamma \|B\|^2_0} \nabla \times B \right) + (U \times B, \nabla \times B). \tag{75}
\]

Integrating the above with respect to \( 0 \leq t \leq s \) and using Cauchy inequality and young inequality, we have
\[
\|B\|^2_0 + \int_0^s (\lambda_0 \|\nabla \times B\|^2_0 + A \|\nabla \cdot B\|^2_0)
\]
\[
\leq R_\alpha \|f(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))} \int_0^s \|B\|^2_0 + \int_0^s a_1 \|\nabla \times B\|^2_0
\]
\[
+ \frac{\|U\|_{L^\infty(0, T; L^\infty(\Omega))}}{4a_2} \int_0^s \|B\|^2_0 + \int_0^s a_2 \|\nabla \times B\|^2_0 + \|B_0\|^2_0. \tag{76}
\]
Taking $a_1, a_2$ such that $\lambda_0 - a_1 - a_2 > 0$, by employing the Growall's inequality, we have
\[
\|B\|_{L^\infty([0,T];L^2(\Omega))} + \sqrt{\Delta_0} \|\nabla \times B\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\Lambda} \|\nabla \cdot B\|_{L^2(0,T;L^2(\Omega))} \leq C_10 \|B_0\|_0.
\]

The proof is completed.

**Lemma 9** There exists a constant $C_{11}$ depending on $\Omega, T, \kappa, \lambda, \Lambda, R_\alpha, \|f(x, t)\|_{L^\infty(0,T;L^\infty(\Omega))}$ and $\|U\|_{L^\infty(0,T;L^\infty(\Omega))}$ such that
\[
\|\xi\|_{L^\infty(0,T;L^1(\Omega))} + \|\xi\|_{L^2(0,T;L^2(\Omega))} + \int_0 \|\xi\|_{L^4(\Omega)} \, dt \leq C_{11} \|B_0\|_0.
\]  

**Proof** First we need define a function
\[
h_\rho(s) = \frac{1}{\rho} \text{sign}(s) \min(|s|, \rho).
\]
Obviously, $h_\rho(s)$ is a bounded, absolutely continuous and increasing function in $\mathbb{R}$. For any $t \in \mathcal{Y}$, it can be calculate to get $h_\rho(t) \in \mathcal{Y}$ and
\[
\nabla h_\rho(t) = \chi_\{(s \in \mathcal{Y} | |s| < \rho\} \nabla t, \quad \lim_{\rho \to 0} h_\rho(t) \to \text{sign}(t), \text{ a.e. in } \Omega,
\]
where $\chi_\{(s \in \mathcal{Y} | |s| < \rho\}$ is the characteristic function.

Taking $T = h_\rho(T)$ in (88), we have
\[
(\partial_t \xi, h_\rho(T)) + (\kappa \nabla \xi, \nabla h_\rho(T)) + \langle \Psi(\xi + \theta_0) - \Psi(\theta_0), h_\rho(T) \rangle = (\xi, \kappa \nabla \theta_0, \nabla h_\rho(T)).
\]

We need analyze (80) term by term. From the convergence of (79), we have
\[
\lim_{\rho \to 0} \int_\Omega \frac{\partial \xi}{\partial t} h_\rho(t) = \int_\Omega \frac{\partial \xi}{\partial \xi} \text{sign}(\xi) = \frac{\partial}{\partial \xi} \int_\Omega |\xi|,
\]
\[
\int_\Omega \kappa \nabla \xi \cdot \nabla h_\rho(t) = \frac{1}{\rho} \int_\Omega \chi_\{(s \in \mathcal{Y} | |s| < \rho\} \kappa |\nabla \xi|^2 \geq 0,
\]
\[
\lim_{\rho \to 0} \int_{\Omega} (\Psi(\xi + \theta_0) - \Psi(\theta_0)) h_\rho(\xi) \geq \frac{C}{8} \|\xi\|_{L^4(\Omega)} + \omega \|\xi\|_{L^4(\Omega)}.
\]

and
\[
\int_{\Omega} |q(\xi, \kappa(B), \xi) h_\rho(\xi)| \leq \int_{\Omega} q(\xi, \kappa(B), \xi) \leq C(\|\nabla \times B\|^2_0
\]
\[
+ \frac{R_\alpha \|f(x, t)\|_{L^\infty(0,T;L^\infty(\Omega))} \|B\|_0^2 + a_3 \|\nabla \times B\|^2_0}{4a_4}
\]
\[
+ \|U\|_{L^\infty(0,T;L^\infty(\Omega))} \|B\|_0^2 + a_4 \|\nabla \times B\|^2_0.
\]

From (81)– (84) and Lemma 9, we have
\[
\frac{\partial}{\partial t} \int_\Omega |\xi| + \frac{C}{8} \|\xi\|_{L^4(\Omega)} \leq C(\|B\|_0 + \|\nabla \theta_0\|_0).
\]

By using The Growall’s inequality and integrating the inequality with respect to $0 \leq t \leq s$ for any $s \in [0, T]$, we can finish the proof.
Lemma 10 There exists a constant $C_{12}$ depending only on $\Omega, T, \lambda, A, R_0, \|f(x, t)\|_{L^\infty(0, T, L^\infty(\Omega))}$ and $\|U\|_{L^\infty(0, T, L^\infty(\Omega))}$ such that

$$
\int_{Q_T} \frac{\kappa|\nabla \xi|^2}{(1 + |\xi|^2)} \leq C_{12}(\|B_0\|_0 + \kappa \|\nabla \theta_0\|_0^2) \quad (86)
$$

Proof Defining $h(s) = \text{sign}(s)[1 - (1 + |s|)^{-\frac{1}{2}}], we have \|h(s)\| \leq 1$. For any $\eta \in \mathcal{V},$ there holds $h(\eta) \in \mathcal{V}$ and $\nabla h(\eta) = \frac{\nabla \eta}{2(1 + |\eta|^2)}$.

Let $H(s)$ be the primitive function of $h(s)$ defined by

$$H(s) = \int_0^s h(s') ds' = 2 + |s| - \frac{2}{1 + |s|} \geq 0,$$

which implies

$$
\int_0^T \int_\Omega \frac{\partial \xi}{\partial t} h(\xi) = \int_0^T \int_\Omega H(\xi) = \int_\Omega H(\xi(T)) \geq 0. \quad (87)
$$

Taking $T = h(T)$ in (86), we have

$$(\partial_t \xi, h(T)) + (\kappa \nabla \xi, \nabla h(T)) + (\Psi(\xi + \theta_0) - \Psi(\theta_0), h(T)) \geq r_2
$$

$$= (\eta h(\xi), h(T)) - (\kappa \nabla \theta_0, \nabla h(T)). \quad (88)
$$

From (87), we have

$$
\int_0^T \int_\Omega \kappa \nabla \xi \cdot \nabla h(T) \leq \int_0^T \int_\Omega (\Psi(\xi + \theta_0) - \Psi(\theta_0)) h(T)
$$

$$\leq \int_0^T \int_\Omega [\eta h(\xi)] h(T) - \int_0^T \int_\Omega \kappa \nabla \theta_0 \nabla h(T). \quad (89)
$$

Now we estimate (88) term by term.

$$
\int_0^T \int_\Omega \kappa \nabla \xi \cdot \nabla h(T) = \frac{1}{2} \int_0^T \int_\Omega \frac{\kappa|\nabla \xi|^2}{(1 + |\xi|^2)} \quad (90)
$$

$$(\Psi(\xi + \theta_0) - \Psi(\theta_0)) h(T) \geq [1 - (1 + |\xi|)^{-\frac{1}{2}}] \frac{\xi}{8} (|\xi|^4 + \omega |\xi|) \geq 0. \quad (91)
$$

From (84) and Lemma (8), we have

$$
\int_0^T \int_\Omega [\eta h(\xi)] h(T) \leq \int_0^T \int_\Omega q(\xi)\mathcal{K}(\mathcal{B}) \leq C_{14}\|\mathcal{B}_0\|_0. \quad (92)
$$

$$
\int_0^T \int_\Omega \kappa \nabla \theta_0 \nabla h(T) \leq \|\kappa \theta_0\|_{L^2(\Omega)} \int_{Q_T} \frac{|\nabla \xi|^2}{(1 + |\xi|^2)^{\frac{1}{2}}} \quad (93)
$$

By Young’s inequality in (84), we have

$$
\frac{1}{2} \int_0^T \int_\Omega \frac{\kappa|\nabla \xi|^2}{(1 + |\xi|^2)} \leq C_{13}\|\mathcal{B}_0\|_0 + C_{14}\|\kappa \nabla \theta_0\|_{L^2(\Omega)}, \quad (94)
$$

which implies the estimate of this lemma.
Lemma 11 Assume that $1 \leq q \leq \frac{5}{3}$, there exists a constant $C > 0$ such that
\[
\|\xi\|_{L^{\frac{3}{2}}(Q_T)} + \|\nabla\xi\|_{L^q(Q_T)} \leq C. \tag{95}
\]

Proof Taking $p = \frac{6q}{3-4q}$, $q_1 = \frac{3q}{3-4q}$, by the Cauchy-Schwarz inequality, we have
\[
\int_{Q_T} |\xi(t)|^p = \int_{Q_T} |\xi(t)|^{\frac{q}{2}} |\xi(t)|^{q_1} \leq \left[ \int_{Q_T} |\xi(t)|^{\frac{q}{2}} \right]^\frac{q}{q_1} \left[ \int_{Q_T} |\xi(t)|^{q_1} \right]^{1-\frac{q}{q_1}} \leq \|\xi(t)\|_{L^\frac{q}{2}(Q_T)}^\frac{q}{q_1} \left[ \int_{Q_T} |\xi(t)|^{q_1} \right]^{1-\frac{q}{q_1}}.
\]

By the embedding of $W^{1,q} \hookrightarrow L^q$ and using Poincare's inequality, we have
\[
\int_{Q_T} |\xi|_p \leq C\|\xi\|_{L^{\infty}(0,T;L^1(\Omega))} \|\nabla\xi\|_{L^q(\Omega)}^q \leq C\|\nabla\xi\|_{L^q(\Omega)}^q.
\]

Taking $r = \frac{5-4q}{3}$, then we have $p = \frac{(1+r)q}{2-q}$. By lemma 10 and Cauchy-Schwarz inequality, we have
\[
\int_{Q_T} |\nabla\xi|^q = \int_{Q_T} \frac{|\nabla\xi|^q}{(1 + |\xi|)} \frac{d(1+r)}{2} \leq \left[ \int_{Q_T} \frac{|\nabla\xi|^q}{(1 + |\xi|)} \frac{d(1+r)}{2} \right]^\frac{q}{q_1} \left[ \int_{Q_T} (1 + |\xi|) \frac{d(1+r)}{2} \right]^{1-\frac{q}{q_1}} \leq [1 + \|\xi\|_{L^q(\Omega)}^p] \leq C(1 + \|\nabla\xi\|_{L^q(\Omega)}^{(1-\frac{q}{q_1})}),
\]
which implies $\|\nabla\xi\|_{L^q(\Omega)} \leq C$, and (97) implies $\|\xi\|_{L^q(\Omega)} \leq C$.

4 Well-posedness of the Source Problem

We will prove the well-posedness of the problem (19) - (20), that is, we will investigate the limit of the solution of the regularized problem (37) - (38) as the regularization parameter $\epsilon \to 0$. For convenience, we denote the solutions $(\mathbf{B}_0, \xi)$ by $(\mathbf{B}_0, \xi_0)$. Then the regularized problem (37) - (38) can be represented by: find $\mathbf{B}_\epsilon \in L^2(0,T;\mathcal{V})$ and $\xi_\epsilon \in L^2(0,T;\mathcal{V})$ such that
\[
(\partial_t \mathbf{B}_\epsilon, \Phi) + (\lambda(\xi + \theta_0) \nabla \times \mathbf{B}_\epsilon, \nabla \times \Phi) + A(\nabla \cdot \mathbf{B}_\epsilon, \nabla \cdot \Phi) = R_\alpha \left( \frac{f(x,t)\mathbf{B}_\epsilon}{1 + r|\mathbf{B}_\epsilon|^2}, \nabla \times \Phi \right)
\]
\[
+ (\mathbf{U} \times \mathbf{B}_\epsilon, \nabla \times \Phi) + (\nabla \cdot \mathbf{B}_\epsilon, \nabla \cdot \Phi), \forall \Phi \in \mathcal{V}, \tag{99}
\]
\[
(\partial_t \xi_\epsilon, \Upsilon) + (\kappa \nabla \xi_\epsilon, \nabla \Upsilon) + <(\psi(\xi_\epsilon + \theta_0) - \psi(\theta_0)), \Upsilon > r_2
\]
\[
= (q(\xi_\epsilon) \mathbf{K}(\mathbf{B}_\epsilon))_\epsilon, \Upsilon + (\kappa \nabla \theta_0, \nabla \Upsilon), \forall \Upsilon \in \mathcal{Y}. \tag{100}
\]

From Lemma 8 there exists $\mathbf{B} \in L^2(0,T;\mathcal{V})$ and a sequence $\mathbf{B}_\epsilon$ such that
\[
\mathbf{B}_\epsilon \to \mathbf{B}, \text{ in } L^2(0,T;\mathcal{V}) \quad \text{and} \quad \mathbf{B}_\epsilon \to \mathbf{B}, \nabla \times \mathbf{B}_\epsilon \to \nabla \times \mathbf{B}, \text{ in } L^1(Q_T).
\]

From lemma [11] there exists a $\xi \in W^{1,q}(Q_T)$ and a sequence $\xi_\epsilon$ such that

$$\xi_\epsilon \to \xi \text{ in } W^{1,q}(Q_T), \forall q \in [1, \frac{5}{4}).$$

Since $q(\xi)$ is bounded and Lipschitz continuous, we know that

$$q(\xi_\epsilon) \to q(\xi), \ a.e. \ in \ Q_T.$$

**Theorem 4** Let $B$ be the limit of the approximate solutions $B_\epsilon$ as $\epsilon \to 0$. Then $B$ satisfies the weak formulation (19) together with the initial condition $B(0) = B_0(x)$.

**Proof** The proof is parallel to that of Theorem 2 and we omit the details here.

**Lemma 12** There exists a subsequence of $B_\epsilon$ denoted still by the same notation such that

$$\lim_{\epsilon \to 0} \|q(\xi_\epsilon)\mathcal{K}(B_\epsilon) - q(\xi)\mathcal{K}(B)\|_{L^1(Q_T)} = 0. \quad (101)$$

**Proof** Firstly, we have

$$\lim_{\epsilon \to 0} \int_{Q_T} |q(\xi_\epsilon)\mathcal{K}(B_\epsilon) - q(\xi)\mathcal{K}(B)|$$

$$= \lim_{\epsilon \to 0} \int_{Q_T} |q(\xi_\epsilon)(|\nabla \times B_\epsilon|^2 - |\nabla \times B|^2) + (q(\xi_\epsilon) - q(\xi))|\nabla \times B|^2$$

$$= \lim_{\epsilon \to 0} \int_{Q_T} [q(\xi_\epsilon)(|\nabla \times B_\epsilon| - |\nabla \times B|)(|\nabla \times B_\epsilon| + |\nabla \times B|)$$

$$+ (q(\xi_\epsilon) - q(\xi))|\nabla \times B|^2] = 0 \quad (102)$$

Secondly, we have

$$\lim_{\epsilon \to 0} \int_{Q_T} |q(\xi_\epsilon)\nabla \times B_\epsilon \cdot (U \times B_\epsilon) - q(\xi)\nabla \times B \cdot (U \times B)|$$

$$= \lim_{\epsilon \to 0} \int_{Q_T} (q(\xi_\epsilon) - q(\xi))\nabla \times B_\epsilon \cdot (U \times B_\epsilon) + q(\xi_\epsilon)\nabla \times B_\epsilon \cdot (U \times (B_\epsilon - B))$$

$$+ q(\xi)(|\nabla \times B_\epsilon - \nabla \times B| \cdot (U \times B)$$

$$\leq \lim_{\epsilon \to 0} \int_{Q_T} |q(\xi_\epsilon) - q(\xi)|\nabla \times B_\epsilon \cdot (U \times B_\epsilon)|$$

$$+ \lim_{\epsilon \to 0} \int_{Q_T} |q(\xi)(\nabla \times B_\epsilon - \nabla \times B) \cdot (U \times B)|$$

$$+ \lim_{\epsilon \to 0} \int_{Q_T} |q(\xi)(\nabla \times B_\epsilon - \nabla \times B) \cdot (U \times B)| = 0, \quad (103)$$
Thirdly, we have
\[
\lim_{\epsilon \to 0} R_\alpha \int_{Q_T} |q(\xi) \nabla \times \mathbf{B}_\epsilon - \frac{f(x, t)B}{1 + \gamma |B|^2} - q(\xi) \nabla \times \mathbf{B} \cdot \frac{f(x, t)B}{1 + \gamma |B|^2} | \leq \lim_{\epsilon \to 0} R_\alpha \int_{Q_T} |(q(\xi) - q(\xi)(\nabla \times \mathbf{B}_\epsilon - \nabla \times \mathbf{B})| \leq 0.
\]

From (102)-(104), by using the triangle inequality, (101) can be proved.

**Theorem 5** Let \( \xi \) be the limit of the approximate solutions \( \xi_\epsilon \) as \( \epsilon \to 0 \). Then \( \xi \) satisfies the weak formulation (20) together with the initial condition \( \xi(0) = 0 \).

**Proof** Define the function: for any \( 0 > 0 \)
\[
g_r(s) = \frac{1}{1 + r s^2}, \quad G_r(s) = \int_0^s g_r(s') ds'.
\]

It is easy to see that \( G_r \) is a primitive function of \( g_r \) and it satisfies \( |g_r(s)| \leq 1, |G_r| \leq T \). Since \( \xi_\epsilon \to \xi \) in \( W^{1,6/5}(Q_T) \) when \( \epsilon \to 0 \), it is easy to see \( g_r(\xi_\epsilon) \to g_r(\xi) \) in \( W^{1,6/5}(Q_T) \) and \( G_r(\xi_\epsilon) \to G_r(\xi) \) in \( W^{1,6/5}(Q_T) \). Moreover, \( g_r(\xi_\epsilon) \) and \( G_r(\xi_\epsilon) \) are uniformly bounded with respect to \( \epsilon \), we infer that
\[
G_r(\xi_\epsilon) \to G_r(\xi), \quad g_r(\xi_\epsilon) \to g_r(\xi) \quad \text{in} W^{1,6/5}(Q_T).
\]

For any \( v \in C_0^\infty(0, T; C^\infty(\Omega)) \), let \( T_\epsilon = v g_r(\xi_\epsilon), \quad T = v g_r(\xi) \). Clearly, we have \( \phi_\epsilon \to \phi \) in \( Q_T \). The proof consists of two steps.

From (100), we have
\[
(\partial_t \xi_\epsilon, T_\epsilon) + (\kappa \nabla \xi_\epsilon, \nabla T_\epsilon) + <\Psi(\xi_\epsilon + \theta_0) - \Psi(\theta_0), T_\epsilon >_{r_2}
= ([g(\xi) \mathcal{K}(\mathbf{B}_\epsilon)]_\epsilon, T_\epsilon) - (\kappa \nabla \theta_0, \nabla T_\epsilon), \quad (105)
\]

It is easy to see that
\[
\lim_{\epsilon \to 0} \int_{Q_T} \partial_t \xi_\epsilon T_\epsilon = \lim_{\epsilon \to 0} \int_{Q_T} \frac{\partial G_r(\xi_\epsilon)}{\partial t} v
= - \lim_{\epsilon \to 0} \int_{Q_T} G_r(\xi_\epsilon) \frac{\partial v}{\partial t} = \int_{Q_T} G_r(\xi) \frac{\partial v}{\partial t} = \int_{Q_T} \frac{\partial G_r(\xi)}{\partial t} v \quad (106)
\]

At the same time, since \( \xi_\epsilon \to \xi \) in \( W^{1,2}(Q_T), \forall q \in [1, \frac{5}{4}] \), there holds
\[
\lim_{\epsilon \to 0} \int_{Q_T} \kappa \nabla \xi_\epsilon \cdot \nabla T_\epsilon
\]
\[
\lim_{\epsilon \to 0} \int_{Q_T} \kappa g_r(\xi) \nabla \xi \cdot \nabla v = \lim_{\epsilon \to 0} \int_{Q_T} \frac{4r\kappa \xi^3}{1 + r\xi^4} |\nabla \xi|^2 \\
= \int_{Q_T} \kappa g_r(\xi) \nabla \xi \cdot \nabla v - \int_{Q_T} \frac{4r\kappa \xi^3}{1 + r\xi^4} |\nabla \xi|^2 \\
= \int_{Q_T} \kappa \nabla \xi \cdot \nabla \gamma. \tag{107}
\]

From Lemma 9 there exists a subsequence denoted by the same notation such that \( \xi \to \xi \) in \( L^3([0, T]; \Gamma_2) \). This implies that

\((\xi|\xi + \theta_0|^3 + \omega)g_r(\xi) \to (\xi|\xi + \theta_0|^3 + \omega)g_r(\xi), \text{ a.e. in } (0, T) \times \Gamma_2.\)

The third term of (105) satisfies

\[
\lim_{\epsilon \to 0} \int_0^T \int_{\Gamma_2} \Psi(\xi + \theta_0) \gamma = \int_0^T \int_{\Gamma_2} \Psi(\xi + \theta_0) \gamma. \tag{108}
\]

For the righthand side of (105), by Lemma 12 we have

\[
\lim_{\epsilon \to 0} \left[ \int_{Q_T} [q(\xi)K(B_\gamma)] \gamma + \int_0^T \int_{\Gamma_2} \Psi(\theta_0) \gamma - \int_{Q_T} \kappa \nabla \theta_0 \nabla \gamma \right] \\
= \int_{Q_T} q(\xi)K(B) \gamma + \int_0^T \int_{\Gamma_2} \Psi(\theta_0) \gamma - \int_{Q_T} \kappa \nabla \theta_0 \nabla \gamma \tag{109}
\]

From (108)–(109) and (105), we can get

\[
\int_{Q_T} \frac{\partial \xi}{\partial t} \gamma + \int_{Q_T} \kappa \nabla \xi \cdot \nabla \gamma + \int_0^T \int_{\Gamma_2} \psi(\xi + \theta_0) \gamma \\
= \int_{Q_T} q(\xi)K(B) \gamma + \int_0^T \int_{\Gamma_2} \Psi(\theta_0) \gamma - \int_{Q_T} \kappa \nabla \theta_0 \nabla \gamma. \tag{110}
\]

The initial condition \( \xi(0) = 0 \) can be proved similarly as in the proof of Theorem 3. We do not elaborate on the details here.

For the function \( g_r(\xi) \), we know \( g_r(\xi) \to 1, \text{ a.e. in } Q_T \). Then we have

\[
\lim_{r \to 0} \int_{Q_T} \kappa \nabla \xi \cdot \nabla g_r(\xi) = \lim_{r \to 0} \int_{Q_T} \frac{4r\kappa \xi^3}{1 + r\xi^4} |\nabla \xi|^2 = 0.
\]

We can get

\[
\lim_{r \to 0} \int_{Q_T} \frac{\partial \xi}{\partial t} \gamma + \kappa \nabla \xi \cdot \nabla \gamma \] \[= \int_{Q_T} \frac{\partial \xi}{\partial t} \gamma + \kappa \nabla \xi \cdot \nabla \gamma.\]

Since \( \|\xi\|_{L^4([0, T]; \Gamma_2)} \leq C \), there exists a subsequence such that \( \|\xi\|_{L^4([0, T]; \Gamma_2)} \leq \lim_{\epsilon \to 0} \|\xi\|_{L^4([0, T]; \Gamma_2)} \leq C \). Then we have

\[
\lim_{r \to 0} \int_0^T \int_{\Gamma_2} \Psi(\xi + \theta_0) \gamma = \int_0^T \int_{\Gamma_2} \Psi(\xi + \theta_0) v.
\]
Similarly, the right-hand side converges to the form
\[ \int_{Q_T} q(\xi) \mathcal{K}(B)v + \int_0^T \int_{F_2} \Psi(\theta_0)v - \int_{Q_T} \kappa \nabla \theta_0 \nabla v. \]

Collecting all the above equalities and using (110), we finally get for \( \forall v \in C_0^\infty(0, T; \mathcal{Y} \cap C^\infty(\Omega)) \)
\[ \int_{Q_T} \frac{\partial \xi}{\partial t} v + \int_{Q_T} \kappa \nabla \xi \cdot \nabla v + \int_0^T \int_{F_2} \Psi(\xi + \theta_0)v = \int_{Q_T} q(\xi) \mathcal{K}(B)v + \int_0^T \int_{F_2} \Psi(\theta_0)v - \int_{Q_T} \kappa \nabla \theta_0 \nabla v. \tag{111} \]
By the arbitrariness of \( v \), we conclude (20).

In the following, we present the stability of the solutions of the problem (19)-(20) from Lemma 8-Lemma 11 directly.

**Theorem 6** Let \( B, \xi \) be the limits of \( B_i, \xi_i \), given by (99) and (100), respectively. Then \( (B, \xi) \) solves the weak problem (19)-(20). Furthermore,
\[ \|B\|_{L^2(0,T;V)} + \|\xi\|_{L^\infty(B^{\infty}(\Omega))} + \|\nabla \xi\|_{L^q(Q_T)} \leq C, \quad \forall q \in \left[1, \frac{5}{4}\right], \tag{112} \]
where \( C \) depending on \( \Omega, T, \lambda, \Lambda, R_\alpha, \|f(x,t)\|_{L^\infty(0,T,L^\infty(\Omega))} \) and \( \|U\|_{L^\infty(0,T,L^\infty(\Omega))} \).

In the end, we give the uniqueness analysis of the solutions of the problem (19)-(20).

**Theorem 7** Assume that \( B \in L^\infty(0,T;W^{1,4}(\text{curl}, \Omega)), \xi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)), \lambda, \sigma \) satisfy the Lipschitz continuous, \( U, f \in L^\infty(0,T;L^\infty(\Omega)) \), then the equations (19)-(20) have a unique solution pair \((B, \xi)\).

**Proof** Assume \((B_1, \xi_1)\) and \((B_2, \xi_2)\) are two solutions of (19)-(20), with \( B_i \) stays bounded in \( L^\infty(0,T;W^{1,4}(\text{curl}, \Omega)) \) and \( \xi_i \) stays bounded in \( L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \), for \( i = 1, 2 \). By denoting \( \tilde{B} = B_1 - B_2, \xi = \xi_1 - \xi_2 \) and setting \( \phi = \tilde{B}, \tau = \tilde{\xi} \), we get
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{B}\|_0^2 + \langle \lambda(\theta_1) \nabla \times \tilde{B}, \nabla \times \tilde{B} \rangle + \langle (\lambda(\theta_1) - \lambda(\theta_2)) \nabla \times B_2, \nabla \times \tilde{B} \rangle = R_\alpha \frac{\|f(x,t)B_1\|}{1 + \gamma|B_1|^2} - \frac{\|f(x,t)B_2\|}{1 + \gamma|B_2|^2} \|\nabla \times \tilde{B}\| + \langle U \times \tilde{B}, \nabla \times \tilde{B} \rangle, \tag{113} \]
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{\xi}\|_0^2 - (\kappa \nabla \tilde{\xi}, \nabla \tilde{\xi}) + \langle \Psi(\xi_1 + \theta_0) - \Psi(\xi_2 + \theta_0), \tilde{\xi}\rangle_{F_2} = \langle [g(\xi_1)K(B_1)]_c - [g(\xi_2)K(B_2)]_c, \tilde{\xi} \rangle. \tag{114} \]
For the first error equation, based on the equality that \( \lambda(\theta_1) - \lambda(\theta_2) = \lambda'(\eta)\tilde{\xi} \), with \( \eta \) between \( \theta_1 \) and \( \theta_2 \), we have
\[ -\langle (\lambda(\theta_1) - \lambda(\theta_2)) \nabla \times B_2, \nabla \times \tilde{B} \rangle = -\langle \lambda'(\eta)\tilde{\xi} \nabla \times B_2, \nabla \times \tilde{B} \rangle \leq \|\lambda'(\eta)\|_{L^\infty(\Omega)} \|\tilde{\xi}\|_{L^1(\Omega)} \|\nabla \times B_2\|_{L^1(\Omega)} \|\nabla \times \tilde{B}\|_0 \]
\[ \leq C\|\tilde{\xi}\|_{L^1(\Omega)} \|\nabla \times B_2\|_{L^1(\Omega)} \|\nabla \times \tilde{B}\|_0 \leq C\|\tilde{\xi}\|_{L^1(\Omega)} \|\nabla \times \tilde{B}\|_0, \tag{115} \]
in which the last two steps come from the fact that both $\theta_1$ and $\theta_2$ stay bounded in $L^\infty(0, T; L^\infty(\Omega))$, and $B_2$ stays bounded in $L^\infty(0, T; W^{1,4}(\text{curl}, \Omega))$. The right hand side of (113) could be bounded in a more straightforward way:

\[
R_\alpha\left(\frac{f(x, t)B_1}{1 + \gamma |B_1|^2} - \frac{f(x, t)B_2}{1 + \gamma |B_2|^2} \cdot \nabla \times \bar{B}\right) \leq 2R_\alpha \| f \|_{L^\infty(\Omega)} \| \bar{B} \|_0 \| \nabla \times \bar{B} \|_0 \quad (116)
\]

since

\[
\left| \frac{B_1}{1 + \gamma |B_1|^2} - \frac{B_2}{1 + \gamma |B_2|^2} \right| \leq 2|B_1 - B_2|,
\]

\[
(U \times \bar{B}, \nabla \times \bar{B}) \leq \| U \|_{L^\infty(\Omega)} \| \bar{B} \|_0 \| \nabla \times \bar{B} \|_0. \quad (117)
\]

Therefore, a substitution of (115)-(117) into (113) yields

\[
\frac{1}{2} \frac{d}{dt} \| B \|_0^2 + \lambda_{\text{min}} \| \nabla \times \bar{B} \|_0^2 \leq C(\| \tilde{\xi} \|_{L^4(\Omega)} + \| \bar{B} \|_0) \| \nabla \times \bar{B} \|_0. \quad (118)
\]

For the second error equation (114), a direct calculation shows that

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\xi} \|_0^2 + \kappa_{\text{min}} \| \nabla \tilde{\xi} \|_0^2 + (\Psi(\xi_1 + \theta_0) - \Psi(\xi_2 + \theta_0), \tilde{\xi})_{L^2} \leq \left( (q(\xi_1) - q(\xi_2)) \mathcal{K}(B_2), \tilde{\xi} \right)_{L^2} + \left( q(\xi_1)(|\nabla \times B_1|^2 - |\nabla \times B_2|^2) - |(\nabla \times B_1) \cdot (U \times B_1) - (\nabla \times B_2) \cdot (U \times B_2)| \right)_{L^2} - \left( R_\alpha \nabla \times B_1 \frac{fB_1}{1 + \gamma |B_1|^2} - R_\alpha \nabla \times B_2 \frac{fB_2}{1 + \gamma |B_2|^2}, \tilde{\xi} \right). \quad (119)
\]

The assumption that $B_2$ stays bounded in $L^\infty(0, T; W^{1,4}(\text{curl}, \Omega))$ implies that

\[
\| \mathcal{K}(B_2) \|_{L^\infty(0, T; L^2(\Omega))} \leq C. \quad (120)
\]

This in turn indicates that

\[
\left( (q(\xi_1) - q(\xi_2)) \mathcal{K}(B_2), \tilde{\xi} \right)_{L^2} = \left( q(\eta)\xi \mathcal{K}(B_2), \tilde{\xi} \right)_{L^2}\leq C\| \xi \|_{L^4(\Omega)} \| \mathcal{K}(B_2) \|_{L^2(\Omega)} \| \tilde{\xi} \|_{L^4(\Omega)} \leq C\| \xi \|_{L^4(\Omega)}^2. \quad (121)
\]

Again, the fact that both $\xi_1$ and $\xi_2$ stay bounded in $L^\infty(0, T; L^\infty(\Omega))$ has been used in the derivation. For the second expansion term on the right hand side of (119), we see that

\[
\left( q(\xi_1)(|\nabla \times B_1|^2 - |\nabla \times B_2|^2), \tilde{\xi} \right)_{L^2} = \left( q(\xi_1)(\nabla \times (B_1 + B_2)) \cdot (\nabla \times \bar{B}), \tilde{\xi} \right)_{L^2} \leq C\| \nabla \times B_1 \|_{L^4(\Omega)} + \| \nabla \times B_2 \|_{L^4(\Omega)} \| \nabla \times \bar{B} \|_0 \| \tilde{\xi} \|_{L^4(\Omega)} \| \nabla \times \bar{B} \|_0 \leq C\| \tilde{\xi} \|_{L^4(\Omega)} \| \nabla \times \bar{B} \|_0. \quad (122)
\]
The other terms on the right hand side of (119) could be analyzed in a similar way:

$$-\left( q(\xi_1)(\nabla \times B_2 - (U \times B_1))\right)\xi\right)$$

$$\leq C\|\xi\|_{L^4(D)}(\|\tilde{B}\|_0 + \|\nabla \times \tilde{B}\|_0)$$

$$-R a\left( q(\xi_1)(\nabla \times B_1 \frac{\tilde{B}_1}{1 + \gamma|B_1|^2} - \nabla \times B_2 \frac{\tilde{B}_2}{1 + \gamma|B_2|^2})\right)\xi$$

$$\leq C\|\xi\|_0^2\|\tilde{B}\|_0.$$  

(123)

And also, the estimate for the boundary integral term on the left hand side of (119) is trivial:

$$\langle \psi(\xi_1 + \theta_0) - \psi(\xi_2 + \theta_0), \tilde{\xi}\rangle_{H_\Gamma} \geq 0.$$  

(125)

Subsequently, a substitution of (121)-(125) into (119) results in

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}\|^2 + \kappa_{\min}\|\nabla \tilde{\xi}\|^2 \leq C\|\tilde{\xi}\|_{L^4(D)}(\|\tilde{B}\|_0 + \|\nabla \times \tilde{B}\|_0) + C\|\tilde{\xi}\|_{L^4(D)}^2.$$  

(126)

As a result, a combination of (118) and (126) yields

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}\|^2 + \|\tilde{\xi}\|_{H_\Gamma}^2 + \lambda_{\min}\|\nabla \tilde{\xi}\|^2 + \kappa_{\min}\|\nabla \tilde{\xi}\|^2$$

$$\leq C_1\|\tilde{B}\|_0\|\nabla \tilde{B}\|_0 + C_2\|\xi\|_{L^4(D)}(\|\tilde{B}\|_0 + \|\nabla \times \tilde{B}\|_0) + C_3\|\tilde{\xi}\|_{L^4(D)}^2.$$  

(127)

Furthermore, the following Sobolev inequality (in 3-D) is applied:

$$\|\tilde{\xi}\|_{L^4} \leq C\|\tilde{\xi}\|^{\frac{1}{2}}_{H^\frac{1}{2}} \leq C\|\tilde{\xi}\|_{H^\frac{1}{2}} \leq C(\|\tilde{\xi}\|_0 + \|\tilde{\xi}\|_{H^\frac{1}{2}}^\frac{1}{2} \cdot \|\nabla \tilde{\xi}\|_{H^\frac{1}{2}}^\frac{1}{2})$$  

(128)

so that the following estimates become available:

$$C_1\|\tilde{B}\|_0\|\nabla \tilde{B}\|_0 \leq \frac{C_1^2}{\kappa_{\min}}\|\tilde{B}\|_0^2 + \frac{1}{4}\lambda_{\min}\|\nabla \times \tilde{B}\|_0^2.$$  

(129)

$$C_2\|\tilde{\xi}\|_{L^4(D)}\|\tilde{B}\|_0 \leq C_4(\|\tilde{\xi}\|_0 + \|\tilde{\xi}\|_{H^\frac{1}{2}} \cdot \|\nabla \tilde{\xi}\|_{H^\frac{1}{2}})\|\tilde{B}\|_0$$

$$\leq C_5(\|\tilde{\xi}\|_0^2 + \|\tilde{\xi}\|_{H^\frac{1}{2}}^2) + \frac{1}{4}\lambda_{\min}\|\nabla \times \tilde{B}\|_0^2.$$  

(130)

$$C_6\|\nabla \times \tilde{B}\|_0 \leq C_4(\|\tilde{\xi}\|_0 + \|\tilde{\xi}\|_{H^\frac{1}{2}} \cdot \|\nabla \tilde{\xi}\|_{H^\frac{1}{2}})\|\nabla \times \tilde{B}\|_0$$

$$\leq C_6(\|\tilde{\xi}\|_0^2 + \|\tilde{\xi}\|_{H^\frac{1}{2}}^2) + \frac{1}{4}\lambda_{\min}\|\nabla \times \tilde{B}\|_0^2 + \frac{1}{4}\kappa_{\min}\|\nabla \tilde{\xi}\|_0^2.$$  

(131)

$$C_7\|\nabla \times \tilde{B}\|_0 \leq C_7(\|\tilde{\xi}\|_0 + \|\tilde{\xi}\|_{H^\frac{1}{2}} \cdot \|\nabla \tilde{\xi}\|_{H^\frac{1}{2}})^2$$

$$\leq C_8\|\tilde{\xi}\|_0^2 + \frac{1}{4}\kappa_{\min}\|\nabla \tilde{\xi}\|_0^2.$$  

(132)
in which Young’s inequality has been extensively applied. Going back to (127), we arrive at
\[
\frac{1}{2} \frac{d}{dt}(\|\tilde{B}\|_0^2 + \|\tilde{\xi}\|_0^2) + \frac{1}{2} \lambda_{\text{min}} \|\nabla \times \tilde{B}\|_0^2 + \frac{1}{4} \kappa_{\text{min}} \|\nabla \tilde{\xi}\|_0^2 \\
\leq \left( \frac{C^2}{\kappa_{\text{min}}} + C_5 + C_6 \right) \|\tilde{B}\|_0^2 + \left( C_5 + C_6 + C_8 \right) \|\tilde{\xi}\|_0^2. \quad (133)
\]
Consequently, with an application of Gronwall inequality, and making use of the fact that \(\|\tilde{B}(\cdot,t=0)\|_0 = 0\), \(\|\tilde{\xi}(\cdot,t=0)\|_0 = 0\), we arrive at
\[
\|\tilde{B}(\cdot,t)\|_0 = 0, \quad \|\tilde{\xi}(\cdot,t)\|_0 = 0, \quad \forall t > 0.
\]
This completes the uniqueness proof.

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References

1. Molokov, S., Moreau, R., Moffatt, H. K.,: Magnetohydrodynamics, Springer, Netherlands (2007).
2. Parker, E. N.,: Cosmical Magnetic Fields, Clarendon Press, Oxford, 1979.
3. Cattaneo, F. and Hughes, D. W.,: Nonlinear saturation of the turbulent alpha effect where a large scale field is imposed, Phys. Rev. E, 54, 4532-4535 (1996).
4. Moffatt, H. K.,: Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Press, Cambridge, UK, 1978.
5. Sanchez, S., Fournier, A., Pinheiro, K. J. and Aubert, J.,: A mean-field Babcock-Leighton solar dynamo model with long-term variability, Anais da Academia Brasileira de Ciências, 86:1, 11-26 (2014).
6. Brandenburg, A., and Subramanian, K.,: Astrophysical magnetic fields and nonlinear dynamo theory, Physics Reports, 417, 1-209 (2005).
7. Yin, H. M.,: Existence and regularity of a weak solution to Maxwell’s equations with a thermal effect, Math. Meth. Appl. Sci. 29, 1199-1213 (2006).
8. Metaxas, A.C.,: Foundations of Electroheat, A Unified Approach, Wiley, New York, 1996.
9. Elsayed, M.A. Elbashbeshy, Emam,T.G., and Abdelgaber, K.M.; Effects of thermal radiation and magnetic field on unsteady mixed convection flow and heat transfer over an exponentially stretching surface with suction in the presence of internal heat generation/absorption, Journal of the Egyptian Mathematical Society, 20, 215C222(2012).
10. Kačur, J.,: Method of Rothe in evolution equations, Lecture Notes in Math., Springer, Berlin, 1192, 23-34 (1986).
11. Vainberg, M. M.,: Variational method and method of monotone operators in the theory of nonlinear equations, Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973.
12. Zeidler, E.,: Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators, Springer-Verlag, New York, 1990.
13. Ranjit, N. K. and Shit, G. C., Joule heating effects on electromagnetohydrodynamic flow through a peristaltically induced micro-channel with different zeta potential and wall slip, Phys. A, 482, 458–476 (2017).
14. Hossain, M. A. and Gorla, R. S. R., Joule heating effect on magnetohydrodynamic mixed convection boundary layer flow with variable electrical conductivity, Internat. J. Numer. Methods Heat Fluid Flow, 23(2), 275–288 (2013)

15. Bermúdez, A. and Muñoz-Sola, R. and Vázquez, R., Analysis of two stationary magnetohydrodynamics systems of equations including Joule heating, J. Math. Anal. Appl., 368, 444-468 (2010)

16. Chovan, J. and Slodička, M., Induction hardening of steel with restrained Joule heating and nonlinear law for magnetic induction field: solvability, J. Comput. Appl. Math., 311, 630–644 (2017)