Asymptotic behavior of solutions to perturbed superstable wave equations

I Y Kmit\textsuperscript{1,2} and N A Lyulko\textsuperscript{3,4}
\textsuperscript{1} Department of Mathematics of Humboldt University of Berlin, Berlin, Germany
\textsuperscript{2} Institute for Applied Problems of Mechanics and Mathematics of Ukrainian National Academy of Sciences, Kiev, Ukraine
\textsuperscript{3} Sobolev Institute of Mathematics of Russian Academy of Sciences, Novosibirsk, Russia
\textsuperscript{4} Novosibirsk State University, Novosibirsk, Russia
E-mail: kmit@mathematik.hu-berlin.de, natlyl@mail.ru

Abstract. The paper deals with initial-boundary value problems for the linear wave equation whose solutions stabilize to zero in a finite time. We prove that problems in this class remain exponentially stable in $L^2$ as well as in $C^2$ under small bounded perturbations of the wave operator. To show this for $C^2$, we prove a smoothing result implying that the solutions to the perturbed problems become eventually $C^2$-smooth for any $H^1 \times L^2$-initial data.

1. Motivation and our results
A linear system
\begin{equation}
\frac{d}{dt} x(t) = A(t)x(t), \quad x(t) \in X \quad (0 \leq t \leq \infty),
\end{equation}
on a Banach space $X$ is called \textit{exponentially stable} if there exist positive reals $\gamma$ and $M = M(\gamma)$ such that every solution $x(t)$ satisfies the estimate
\begin{equation}
\|x(t)\| \leq Me^{-\gamma t}\|x(0)\|, \quad t \geq 0,
\end{equation}
where $\|\cdot\|$ denotes the norm in $X$.

The papers [2, 3] address a stronger property of exponentially stable systems, known as superstability. They consider the Cauchy problem for the \textit{autonomous} version of (1). Moreover, $A : X \rightarrow X$ is supposed to be the infinitesimal generator of a strongly continuous semigroup $T(t)$; see [4, 13]. A semigroup $T(t)$ is called \textit{superstable} [2] if its stability index is $-\infty$, that is
\begin{equation*}
\lim_{t \rightarrow \infty} \frac{\log \|T(t)\|}{t} = -\infty.
\end{equation*}
The superstability property implies that the system is exponentially stable and, moreover, the estimate (2) holds for every $\gamma > 0$.

The notion of superstability can be appropriately extended to second-order equations, studied in this paper.

An important subclass of superstable systems consists of the systems whose solutions stabilize to zero after some time. The time of the stabilization is called a \textit{finite time extinction}. The
following simple example shows which type of boundary conditions cause the superstability property for the wave equation in the one-dimensional case, see [2, 14]:

\[ w_{tt} - a^2 w_{xx} = 0, \quad (x, t) \in \Pi, \quad (3) \]

\[ w(0, t) = 0, \quad t \in (0, \infty), \]
\[ (w_t + aw_x)(1, t) = 0, \quad t \in (0, \infty), \quad (4) \]

\[ w(x, 0) = w_0(x), \quad x \in [0, 1], \]
\[ w_t(x, 0) = w_1(x), \quad x \in [0, 1], \quad (5) \]

where \( a \) is arbitrary positive constant and \( \Pi = (0, 1) \times (0, \infty) \). Constructing solutions by the method of characteristics shows that all solutions stabilize to zero for \( t > 2/a \). A class of boundary conditions ensuring the superstability property for the wave operator in the multidimensional case is described in [11]. Such boundary conditions naturally appear, for example, in the scattering theory [8, 9] and in the control theory [12].

A comprehensive review of the available results on asymptotic behavior of solutions to linear and quasi-linear first order hyperbolic problems can be found in [1, 7]. The present paper concerns superstable initial-boundary value problems for the one-dimensional wave equation. Specifically, we consider the problem (3), (5) with the boundary conditions either

\[ w(0, t) = p(w_t + aw_x)(0, t), \quad t \in (0, \infty), \]
\[ (w_t + aw_x)(1, t) = 0, \quad t \in (0, \infty), \quad (6) \]

or

\[ w(1, t) = q(w_t - aw_x)(1, t), \quad t \in (0, \infty), \]
\[ (w_t - aw_x)(0, t) = 0, \quad t \in (0, \infty), \quad (7) \]

where \( p \) and \( q \) are arbitrary constants. Note that the finite time extinction for the problems under consideration equals \( 2/a \) (see Section 3.1). In this paper we will investigate the problem (3), (6), (5) (the same result is true for the problem (3), (6), (5) as well). Along with the equation (3) we will consider its perturbed version, namely

\[ w_{tt} - a^2 w_{xx} + c(x, t)w = 0, \quad (x, t) \in \Pi, \quad (8) \]

where \( c \) is a two times continuously differentiable function such that \( c \) itself and its first order and second order derivatives are bounded on \( \overline{\Pi} \).

Given \( T > 0 \), write \( \Pi_T = (0, 1) \times (0, T) \). By \( H^1(0, 1) \) we denote the space of functions \( u : (0, 1) \rightarrow \mathbb{R} \) such that \( u \in L^2(0, 1) \) and its weak derivative \( u' \in L^2(0, 1) \). From [7] it follows that for any initial functions \( w_0 \in H^1(0, 1) \) and \( w_1 \in L^2(0, 1) \) the problem (6), (6), (5) has a unique \( L^2 \)-generalized solution \( w \in C([0, \infty), L^2(0, 1)) \) (see Section 2 for the definition). For this solution we prove the following perturbation theorem.

**Theorem 1.** Let \( w_0 \in H^1(0, 1) \) and \( w_1 \in L^2(0, 1) \). Then for any \( \gamma > 0 \) there exist \( \varepsilon > 0 \) and \( M = M(\gamma) \) such that, whenever \( \sup_{(x, t) \in \Pi} |c| < \varepsilon \), the \( L^2 \)-generalized solution \( w(x, t) \) to the problem (6), (6), (5) fulfills the bound

\[ ||w(\cdot, t)||_{L^2(0, 1)} \leq Me^{-\gamma t} \max(||w_0||_{H^1(0, 1)}, ||w_1||_{L^2(0, 1)}), \quad t > 0. \quad (9) \]

This means that sufficiently small bounded perturbations of the zero order part of the wave equation (3) preserve the exponential stability in \( L^2(0, 1) \) of the unperturbed problem (3), (6), (5).

**Definition 2.** The problem (6), (6), (5) is called smoothing from \( L^2(0, 1) \) to \( C^2([0, 1]) \) if there is \( T > 0 \) such that, given \( w_0 \in H^1(0, 1) \) and \( w_1 \in L^2(0, 1) \), the \( L^2 \)-generalized solution \( w(x, t) \) to this problem belongs to \( C^2([0, 1]) \) whenever \( t \geq T \).
We are prepared to formulate the following smoothing result.

**Theorem 3.** The problem (6), (6), (5) is smoothing from $L^2(0,1)$ to $C^2([0,1])$ with $T = 6/a$. Moreover, given $w_0 \in H^1(0,1)$ and $w_1 \in L^2(0,1)$, the $L^2$-generalized solution $w(x,t)$ fulfills the bound
\[ \| \partial_x^{\alpha,\beta} w(\cdot,t) \|_{C([0,1])} \leq M_1 e^{\omega t} \max(\|w_0\|_{H^1(0,1)}, \|w_1\|_{L^2(0,1)}), \quad t \geq T, \tag{8} \]
where $\alpha + \beta \leq 2$ and $M_1, \omega$ are constants not depending on $t, w_0$, and $w_1$.

**Corollary 4.** Let $w_0 \in H^1(0,1)$ and $w_1 \in L^2(0,1)$. Then for any $\gamma > 0$ there exist $\varepsilon > 0$ and $M_2 = M_2(\gamma) \geq M$ such that, whenever $\sup_{T>0}(\|c\|_{C^2([\pi,T])}) < \varepsilon$, the $L^2$-generalized solution $w(x,t)$ to the problem (6), (6), (5) fulfills the bound
\[ \| \partial_x^{\alpha,\beta} w(\cdot,t) \|_{C([0,1])} \leq M_2 e^{-\gamma t} \max(\|w_0\|_{H^1(0,1)}, \|w_1\|_{L^2(0,1)}), \quad t > T, \tag{9} \]
where $T = 6/a$, $\alpha + \beta \leq 2$, and $M_2, \gamma$ are constants not depending on $t, w_0$, and $w_1$.

**2. Reduction to first order systems and $L^2$-generalized solutions**

For proving Theorems 1 and 3, we will apply our results from [7], where we study the exponential stability of the solutions to the initial-boundary value problems for the general first order nonautonomous hyperbolic systems, that are small bounded perturbations of the corresponding superstable systems. To this end, let us introduce new variable $u = \partial_t w + a \partial_x w$ and rewrite the perturbed problem as the following 2 × 2 first-order system:
\[
\begin{align*}
\partial_t w + a \partial_x w - u &= 0, \quad \partial_t u - a \partial_x u + c(x,t)w = 0, \tag{10} \\
w(0,0), u(0,0) &= 0, \tag{11} \\
w(x,0) = \varphi_1(x), \quad u(x,0) &= \varphi_2(x), \tag{12}
\end{align*}
\]
where $\varphi_1(x) = \varphi_0(x), \quad \varphi_2(x) = \varphi_1(x) + a \partial_x \varphi_0(x)$.

It is evident that the $C^2$-function $w$ is a classical solution to the problem (6), (6), (5) if and only if the $C^2$-vector function $(w, u) = (w, \partial_t w + a \partial_x w)$ is a classical solution to the problem (10)–(12).

As it follows from [5], for $w_0 \in C^3[0,1]$ and $w_1 \in C^2[0,1]$ satisfying the zero order, the first order and the second order compatibility conditions between (11) and (12) the problem (10)–(12) has a unique classical solution $(w, u) \in C^2([0,1]) \times C^2([0,1])$, where the first component is a classical solution to the problem (6), (6), (5). Note that then the compatibility conditions up to the second order between (6) and (5), see e.g. [10], follow immediately from the equality $u = \partial_t w + a \partial_x w$.

Let us introduce the notion of an $L^2$-generalized solution. Fix arbitrary $w_0 \in H^1(0,1)$ and $w_1 \in L^2(0,1)$ and sequences $\varphi_1^l \in C_0^\infty([0,1])$, $\varphi_2^l \in C_0^\infty([0,1])$ such that $\varphi_i^l \to \varphi_i$ in $L^2(0,1)$, $i = 1, 2$. Note that, due to the fact that $\varphi_i^l$ are compactly supported for all $l \in N$, they satisfy the compatibility conditions up to the second order between (11) and (12). By [5], given $l \in N$, the problem (10)–(12) has a unique classical solution, say $(w^l, u^l)$, belonging to $C^2([0,1]) \times C^2([0,1])$. It is proved that these solutions satisfy the following estimate:
\[ \max(\|w^l(\cdot,t)\|_{L^2(0,1)}, \|u^l(\cdot,t)\|_{L^2(0,1)}) \leq M_3 e^{-At} \max_{i=1,2}(\|\varphi_i\|_{L^2(0,1)}), \quad \text{for all } t > 0, \tag{13} \]
for some constants $M_3, A$, not depending on $t, \varphi_1, \varphi_2$, and $l \in N$. Hence, there exist unique functions $w, u \in C([0,\infty), L^2(0,1))$ such that
\[ \|w(\cdot, \theta) - w^l(\cdot, \theta)\|_{L^2(0,1)} \to 0, \quad \|u(\cdot, \theta) - u^l(\cdot, \theta)\|_{L^2(0,1)} \to 0 \quad \text{as } l \to \infty, \]
uniformly in $\theta$ varying in the range $0 \leq \theta \leq t$, for every $t > 0$. The vector-function $(w, u)$ is called an $L^2$-generalized solution to the problem (10)–(12), while the function $w$ is called an $L^2$-generalized solution to the problem (6), (6), (5). Furthermore, the following estimate is true:

$$\|w(\cdot, t)\|_{L^2(0,1)} \leq M_3 e^{At} \max(\|w_0\|_{H^1(0,1)}, \|w_1\|_{L^2(0,1)}) \quad \text{for all } t > 0,$$

what follows from (13).

3. Proofs of the main results

3.1. Proof of Theorem 1

Consider the problem (10)–(12). Due to the method of characteristics, the classical solution to this problem fulfills the following system of integral equations:

$$w(x, t) = \int_{t-x/a}^{t} u(a(\tau - t) + x, \tau) d\tau + pu(0, t-x/a), \quad t > \frac{x}{a},$$

$$u(x, t) = -\int_{t+(x-1)/a}^{t} [cw](x + a(t-\tau), \tau) d\tau, \quad t + \frac{x}{a} > \frac{1}{a}.$$

Note that if $c = 0$ then the finite time extinction for this problem equals $2/a$.

Letting $t > 4/a$ and substituting (16) into (15), we get an integral equation for $w$, namely

$$w(x, t) = -\int_{t-x/a}^{t} \int_{2\tau - t+(x-1)/a}^{\tau} [cw](a(\tau - \xi) + x, \xi) d\xi d\tau - p \int_{t-(x+1)/a}^{t-x/a} [cw](a(t-\tau) - x, \tau) d\tau. $$

Set $W(t) = \|w(\cdot, t)\|_{L^2(0,1)}$. Then the last equality entails

$$W(t) \leq K \sup_{(x,t) \in \Pi} |c(x, t)| \int_{t-4/a}^{t} W(\tau) d\tau, \quad t > \frac{4}{a},$$

the constant $K$ being independent on $t, w$ but only on $p$. On the other side, the estimate (14) causes the bound

$$W(t) \leq K_1 \max(\|w_0\|_{H^1(0,1)}, \|w_1\|_{L^2(0,1)}), \quad 0 \leq t \leq \frac{4}{a},$$

with the constant $K_1 = M_3 e^{4/a}$. On the account of (17), (18) and [7, Lemma 5.1], for any $\gamma > 0$ there exist $\varepsilon > 0$ and $M = M(\gamma)$ such that, whenever $\sup_{(x,t) \in \Pi} |c(x, t)| < \varepsilon$, for the classical solution $w$ to the perturbed problem (6), (6), (5) the following bound is true:

$$W(t) \leq Me^{-\gamma t} \max(\|w_0\|_{H^1(0,1)}, \|w_1\|_{L^2(0,1)}), \quad t > 0.$$

Finally, the statement of the theorem follows from the definition of an $L^2$-generalized solution to the problem (6), (6), (5).

3.2. Proof of Theorem 3

First consider the problem (10)–(12) where the system (10) is replaced by the corresponding decoupled system, namely

$$\partial_t w + a \partial_x w = 0, \quad \partial_t u - a \partial_x u = 0,$$

$$w(0, t) = pu(0, t), \quad u(1, t) = 0,$$

$$w(x, 0) = \varphi_1(x), \quad u(x, 0) = \varphi_2(x).$$

(19)
This system is easily checked to be superstable. Indeed, for the classical solution \((w, u)\) to (19) the method of characteristics gives the formulas

\[
 w(x, t) = w(0, t - \frac{x}{a}) = pu \left(0, t - \frac{x}{a}\right), \quad t > \frac{x}{a},
\]

\[
 u(x, t) = u \left(1, t + \frac{x - 1}{a}\right), \quad t + \frac{x - 1}{a} > \frac{1}{a}.
\]

Since \(u(1, t) = 0\), we conclude that the finite time extinction for this problem is \(2/a\).

Then the smoothing property of the problem (10)–(12) immediately follows from [6, Theorem 12] and [7, Theorem 2.5]. More specifically, there exists a positive real \(T\) such that for any \(\varphi_1, \varphi_2 \in L^2(0, 1)\) and any \(C^2\)-function \(c\) the \(L^2\)-generalized solution \((w, u)\) to the problem (10)–(12) is two times continuously differentiable for \(t > T\). Moreover, it fulfills the following estimate:

\[
 \max_i (\|\partial_x^\alpha \partial_t^\beta w(\cdot, t)\|_{C[0,1]}, \|\partial_x^\alpha \partial_t^\beta u(\cdot, t)\|_{C[0,1]}) \leq M_1 e^{At} \max_i (\|\varphi_i\|_{L^2(0,1)}), \quad t > T,
\]

where \(\alpha + \beta \leq 2\) and the constants \(M_1, A\) are independent of \(t, \varphi_1, \varphi_2\). This entails that the first component \(w\) of this solution, which is the \(L^2\)-generalized solution to the original problem (6), (6), (5), fulfills Theorem 3, as desired.

Finally, to prove Corollary 4, we follow the argument used to prove [7, Theorem 2.7(ii)]. The estimate (9) will then follow from the estimates (7) and (8).

3.3. Generalization

Our results given by Theorems 1 and 3 can be easily extended to the second order equations involving first order terms, of the following type:

\[
 (\partial_t - a \partial_x + a_1(x, t)) (\partial_t + a \partial_x) w + c(x, t) w = 0,
\]

where the coefficient \(a_1\) is a two times continuously differentiable function such that \(a_1\) itself and its first order and second order derivatives are bounded on \(\Pi\). If the coefficient \(a\) is positive, then in the domain \(\Pi\) we consider the initial-boundary value problem (20), (6), (5); if \(a\) is negative, then we consider the problem (20), (6), (5).

For the problem (20), (6), (5), in the proofs we encounter the following minor changes. Similarly to the above, the non-perturbed problem (20), (6), (5) (with \(c \equiv 0\)) has the finite time extinction \(2/a\), while the first order system corresponding to (20) now reads as

\[
 \partial_t w + a \partial_x w = u, \quad \partial_t u - a \partial_x u + a_1(x, t) u + c(x, t) w = 0
\]

and the boundary and the initial conditions (6) and (5) read as (11), (12). The solution to the first order problem (21), (11), (12) satisfies the following integral system:

\[
 w(x, t) = \int_{t - \frac{x}{a}}^{t} u(a(\tau - t) + x, \tau) d\tau + pu(0, t - \frac{x}{a}), \quad t > \frac{x}{a},
\]

\[
 u(x, t) = e^{\int_{t}^{x/(a - x)} a_1(\xi + \frac{x - \xi}{a}) d\xi} \int_{t + \frac{x}{a}}^{x} [e^{\int_{\eta}^{x/(a - x)} a_1(\xi + \frac{x - \xi}{a}) d\xi} \varphi_1(\eta, \tau)] \varphi_2(\eta + x/(a - \tau)) d\tau, \quad t + \frac{x}{a} > \frac{1}{a}.
\]

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References
[1] Bastin G and Coron J-M 2016 Progress in Nonlinear Differential Equations and Their Applications vol 88 (Basel: Birkhäuser)
[2] Balakrishnan A V 2005 J. Appl. Math. and Comput. 164 321-6
[3] Creutz D, Mazo M Jr and Preda C 2013 Superstability and finite time extinction for $C_0$-semigroups ArXiv e-prints (Preprint 0907.4812)
[4] Hille E, Phillips R 1957 Functional analysis and semi-groups (Providence: American Math. Society)
[5] Kmit I 2008 Intern. J. Dynamic Syst. Differ. Equat. 1 191-5
[6] Kmit I 2011 Appl. Anal. 90 1609-34
[7] Kmit I and Lyulko N 2017 Perturbations of superstably linear hyperbolic systems ArXiv e-prints (Preprint 1605.04703)
[8] Lax P and Phillips R 1967 Scattering theory (New York: Academic Press)
[9] Lax P and Phillips R 1975 Indiana Univ. Math. J. 24 1119-38
[10] Lyulko N 2004 J. Math. Phys., Anal., Geom. 11 169-76
[11] Majda A 1975 Indiana Univ. Math. J. 24 1119-33
[12] Moulay E and Perruquetti W 2006 Advances in variable structure and sliding mode control 334 (Berlin: Springer-Verlag) pp 23-41
[13] Pazy A 1983 Semigroups of operators and applications to partial differential equations (Berlin: Springer-Verlag)
[14] Perrollaz V and Rosier L 2014 SIAM J. on Control and Optimiz. 52 143-63