ON THE GLOBAL SMALL SOLUTION OF 2-D PRANDTL SYSTEM WITH INITIAL DATA IN THE OPTIMAL GEVREY CLASS

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Abstract. Motivated by [8], we prove the global existence and large time behavior of small solutions to 2-D Prandtl system for data with Gevrey 2 regularity in the x variable and Sobolev regularity in the y variable. In particular, we extend the global well-posedness result in [26] for 2-D Prandtl system with analytic data to data with optimal Gevrey regularity in the sense of [11].

Keywords: Prandtl system, Littlewood-Paley theory, Gevrey energy estimate

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1. Introduction

In this paper, we investigate the global existence and the large time behavior of solutions to the following 2-D Prandtl system with small initial data in the Gevrey class:

\[
\begin{cases}
\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x P^E = 0, & (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\
\partial_x u + \partial_y v = 0, \\
(u, v) \big|_{y=0} = 0, \\
\lim_{y \to +\infty} u = U^E(t, x), \\
u |_{t=0} = u_0(x, y),
\end{cases}
\]

(1.1)

where \((u, v)\) stand for the tangential and normal velocities of the boundary flow, and \((U^E(t, x), P^E(t, x))\) designate the trances on the boundary of the tangential velocity and pressure of the outflow, which satisfies Bernoulli’s law

\[
\partial_t U^E + U^E \partial_x U^E + \partial_x P^E = 0.
\]

The system (1.1) is the foundation of the boundary layer theory. It was proposed by Prandtl [30] in 1904 in order to explain the disparity between the boundary conditions verified by ideal fluid and viscous fluid with small viscosity. Heuristically, these boundary layers are of amplitude \(O(1)\) and of thickness \(O(\sqrt{\nu})\) where there is a transition from the interior flow governed by Euler equation to the Navier-Stokes flow with a vanishing viscosity \(\nu > 0\). One may check [10, 25] and references therein for more introductions on boundary layer theory. Especially we refer to [17] for a comprehensive recent survey.

Compared with 2-D Navier-Stokes equations, there is no horizontal diffusion in the \(u\) equation of (1.1), and the normal velocity \(v\) is determined by the divergence free condition of the velocity field \((u, v)\), so that the nonlinear term \(v \partial_y u\) almost behaves like \(-\partial_x u \partial_y u\). Moreover, this term is not skew-symmetric in the Sobolev space \(H^s\). As a consequence, one horizontal derivative is lost in the process of energy estimate. Therefore the question of whether or not the Prandtl system with general data is well-posed in Sobolev spaces is still
open. Physically, the difficulty of the above problem is related to the underlying instabilities of the boundary layer, for instance, the phenomenon of separation (see [7, 32]).

The first well-posedness result of (1.1) goes back to Oleinik [24]. Under the assumption that $u_0(x, y)$ in (1.1) is monotonic in the $y$ variable, she first introduced Crocco transformation and then proved the local existence and uniqueness of classical solutions to (1.1). This result was revisited in [1] and [23] by performing energy estimates in weighted Sobolev spaces. While with the additional “favorable” condition on the pressure, Xin and Zhang [33] obtained the global existence of weak solutions to this system. We remark that such monotonicity assumption excludes the phenomenon of reverse flow.

In general, Gérard-Varet and Dormy [11] proved the ill-posedness in Sobolev spaces for the linearized Prandtl system around non-monotonic shear flows. The nonlinear ill-posedness was also established in [13, 17] in the sense of non-Lipschitz continuity of the flow. Yet for the data which is analytic in both $x$ and $y$ variables, Sammartino and Caffisch [31] established the local well-posedness of (1.1) (see also [19]). The analyticity in $y$ variable was removed by Lombardo, Cannone and Sammartino in [22]. The main argument used in [22, 31] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem.

To relax the analyticity condition for the initial data turns out to be very hard. In the special case when $u_0$ has for each value of $x$ a single non-degenerate critical point in $y$, Gérard-Varet and Masmoudi [12] proved the well-posedness of (1.1) for a class of data with Gevrey class $\frac{1}{4}$ in the $x$ variable. Chen, the second author and Zhang [6] proved the well-posedness of the linearized Prandtl equation around a non-monotonic shear flow in Gevrey class $2 - \theta$ for any $\theta > 0$. Li and Yang [20] extended the result in [12] for data that are small perturbations of a shear flow with a single non-degenerate critical point, which belongs to the Gevrey class 2. Eventually, Dietert and Gérard-Varet [8] removed the structure assumption in [20] and proved the local well-posedness of (1.1) for data with Gevrey 2 regularity in $x$ variable and Sobolev regularity in $y$ variable. This result is optimal in the sense that Gérard-Varet and Dormy [11] constructed a class of solution with the growth like $e^{\sqrt{k}t}$ for the linearized Prandtl system around a non-monotonic shear flow, where $k$ is the tangential frequency.

On the other hand, as pointed out by Grenier, Guo, and Nguyen [14, 15, 16] (see also [9]), in order to make progress towards proving or disproving the inviscid limit of the Navier-Stokes equations, one must understand its behavior on a longer time interval than the one which causes the instability used to prove ill-posedness. Oleinik and Samokhin [25] also asked the following question (see open problem 4 on page 500 of [25]): “It has been shown in Chapter 4 that under certain assumptions the system of nonstationary two-dimensional boundary layer admits one and only one solution in the domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$ either for small enough $T$ and any $X > 0$ or small enough $X$ and any $T > 0$. What are the conditions ensuring the existence and uniqueness of a solution of the nonstationary Prandtl system in the domain $D$ with arbitrary $X$ and $T$?”

Indeed the third author and Zhang first addressed the long time existence for Prandtl system with small analytic data in [34] and an almost global existence result was provided in [18]. Lately, Paicu and the third author [26] succeeded in proving the global existence of small analytic solutions to (1.1). Furthermore, the large time behavior of the solutions have been obtained in [26]. Similar result was obtained for the MHD boundary layer equations in [21].

Motivated by [8, 26], we are going to study the global well-posedness of (1.1) with small initial data which have Gevrey 2 regularity in $x$ variable and Sobolev regularity in $y$ variable.
For a clear presentation, we shall take $U^E = 0$ and $P^E = C$ in the rest of this paper, so that $(u, v)$ solves

$$
\begin{aligned}
\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u &= 0, \\
\partial_x u + \partial_y v &= 0,
\end{aligned}
$$

(1.2)

$$
(u, v)|_{y=0} = 0, \quad \lim_{y \to +\infty} u = 0, \\
u|_{t=0} = u_0(x, y).
$$

We remark that as in [26], we can take $U^E(t, x) = e f(t)$ for $f(t)$ decaying sufficiently fast at infinity.

Whereas due to the divergence free condition of the velocity field $(u, v)$, there exists a potential function $\varphi$ such that $u = \partial_y \varphi$ and $v = -\partial_x \varphi$. Then we deduce from (1.2) (see [26] for the detailed derivation) that

$$
\begin{aligned}
\partial_t \varphi + u \partial_x \varphi + 2 \int_y^\infty (\partial_y u \partial_x \varphi) dy' - \partial_y^2 \varphi &= 0, \\
\varphi|_{y=0} = 0, \\
\varphi|_{t=0} = \varphi_0.
\end{aligned}
$$

To derive the faster decay estimate of the solutions to (1.2), which will be needed to control the Gevrey radius of the solutions, we recall from [26] the good quantity

$$
G = u + \frac{y}{2(t)} \varphi
$$

and

$$
g = \partial_y G = \partial_y u + \frac{y}{2(t)} \varphi,
$$

which solves

$$
\begin{aligned}
\partial_t G - \partial_y^2 G + \langle t \rangle^{-1} G + u \partial_y G + v \partial_y G \\
- \frac{1}{2(t)} v \partial_y (y \varphi) + \frac{y}{t} \int_y^\infty (\partial_y u \partial_x \varphi) \, dy' &= 0, \\
G|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} G(t, x, y) = 0, \\
G|_{t=0} = G_0 = u_0 + \frac{y}{2} \varphi_0.
\end{aligned}
$$

Let $\langle \xi \rangle \equiv (1 + \xi^2)^{\frac{1}{2}}$, we denote

$$
e^{\Phi(t, D_x)} \equiv e^{\delta(t)(D_x)^{\frac{1}{2}}} \quad \text{with} \quad \delta(t) = \delta - \lambda \theta(t) \quad \text{and} \quad f_\psi \equiv e^{\Phi(t, D_x) f},
$$

and the weighted anisotropic Sobolev norm

$$
\|f\|_{H^s_\psi} \equiv \sum_{0 \leq \ell \leq k} \left( \int_0^\infty e^{2\psi(t, y)} \|\partial_y^\ell f(t, y)\|_{H^s_\psi}^2 \, dy \right)^{\frac{1}{2}} \quad \text{with} \quad \psi(t, y) \equiv \frac{y^2}{8(t)}.
$$

The main result of this paper states as follows:

**Theorem 1.1.** Let $u_0 = \partial_y \varphi_0$ satisfy $u_0(x, 0) = 0$ and $\int_0^\infty u_0 \, dy = 0$. Let $G_0 \equiv u_0 + \frac{y}{2} \varphi_0$. For some sufficiently small but fixed $\eta \in (0, \frac{1}{2})$, we denote

$$
E(t) \equiv \|\langle t \rangle^{\frac{1}{2} - \frac{n}{4}} u_0\|_{H^{1,0}_\psi}^2 + \|\langle t \rangle^{\frac{3}{4} - \frac{n}{4}} \partial_y u_0\|^2_{H^{2,0}_\psi} + \|\langle t \rangle^{\frac{5}{4} - \frac{n}{4}} G_0\|_{H^{2,0}_\psi}^2 + \|\langle t \rangle^{\frac{n}{4} - \frac{n}{4}} \partial_y G_0\|_{H^{3,0}_\psi}^2 + \|\langle t \rangle^{\frac{11}{4} - \frac{n}{4}} \partial_y^2 G_0\|_{H^{2,0}_\psi}^2,
$$

and

$$
D(t) \equiv \|\langle t \rangle^{\frac{1}{4} - \frac{n}{4}} \partial_y u_0\|_{H^{1,0}_\psi}^2 + \|\langle t \rangle^{\frac{3}{4} - \frac{n}{4}} \partial_y^2 u_0\|_{H^{2,0}_\psi}^2 + \|\langle t \rangle^{\frac{5}{4} - \frac{n}{4}} \partial_y G_0\|_{H^{3,0}_\psi}^2 + \|\langle t \rangle^{\frac{11}{4} - \frac{n}{4}} \partial_y^2 G_0\|_{H^{2,0}_\psi}^2,
$$

(1.7)
We assume moreover that
\[
\|u_{\Phi}(0)\|^2_{H_{\psi}^{2,0}} + E(0) \leq \varepsilon^2.
\]
Then there exists \(\varepsilon_0, \lambda_0 > 0\) so that for \(\varepsilon \leq \varepsilon_0\) and \(\lambda \geq \lambda_0, \theta(t)\) in (1.5) satisfies \(\sup_{t \in [0,\infty)} \theta(t) \leq \frac{\lambda}{\lambda X}\), and the system (1.2) has a unique global solution \(u\) which satisfies
\[
E(t) + c\eta \int_0^t D(t') \, dt' \leq C\varepsilon^2 \quad \forall \ t \in \mathbb{R}.
\]

Remark 1.1. (1) As in [8], there is a loss on the Gevrey radius of the solution \(u\). Moreover, we need \(u_{\Phi}(0) \in H_{\psi}^{2,0}(\mathbb{R}^2_+)\) in order to guarantee that \(u_{\Phi}(t) \in H_{\psi}^{2,0}(\mathbb{R}^2_+)\). These loss of regularities is due to the instabilities described in [3].

(2) Compared with [26] with analytic data, here our smallness condition can not be prescibed only on the initial data of the good quantity, \(G\). Moreover, the decay of Gevrey solutions has a loss of \(\frac{4}{7}\) in (1.10). We do not know if we could get rid of this loss or not.

(3) The main idea to prove (1.10) is to combine the method in [8], which is based on both a tricky change of unknown and appropriate choice of test function, and the time weighted energy estimate method in [26, 27]. Furthermore, as in [26], we shall use the faster decay estimate of the good quantity, \(G\), in (1.3) to control the Gevrey radius of the solutions, which will be one of the crucial step to construct the global solutions of (1.2) with optimal Gevrey regular data.

Let us end this introduction by the notations that we shall use in this context.

For \(a \lesssim b\), we mean that there is a uniform constant \(C\), which may be different on different lines, such that \(a \leq Cb\). \((a \mid b)_{L^2_+} \overset{\text{def}}{=} \int_{\mathbb{R}_+^2} a(x,y) b(x,y) \, dx \, dy\) stands for the \(L^2\) inner product of \(a, b\) on \(\mathbb{R}_+^2\) and \(L^p_+ = L^p(\mathbb{R}_+^2)\) with \(\mathbb{R}_+^2 \overset{\text{def}}{=} \mathbb{R} \times \mathbb{R}_+\). For \(X\) a Banach space and \(I\) an interval of \(\mathbb{R}\), we denote by \(L^q(I; X)\) the set of measurable functions on \(I\) with values in \(X\), such that \(t \mapsto \|f(t)\|_X\) belongs to \(L^q(I)\). In particular, we denote by \(L^p_T(L^q_0(L^r_0))\) the space \(L^p([0,T]; L^q(\mathbb{R}_x; L^r(\mathbb{R}_y^+)))\).

2. PRELIMINARY

2.1. Littlewood-Paley theory. For the convenience of the readers, we shall collect some basic facts on anisotropic Littlewood-Paley theory in this subsection. Let us first recall from [2] that
\[
(2.1) \quad S_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}) \quad \text{and} \quad \Delta_k^h a = \begin{cases} 
\mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}) & \text{if } k \geq 0; \\
S_k^h a & \text{if } k = -1; \\
0 & \text{if } k \leq -2,
\end{cases}
\]
where and in all that follows, \(\mathcal{F}a\) and \(\hat{a}\) always denote the partial Fourier transform of the distribution \(a\) with respect to \(x\) variable, that is, \(\hat{a}(\xi, y) = \mathcal{F}_{x \to \xi}(a)(\xi, y)\), and \(\varphi(\tau), \varphi(\tau)\) are smooth functions such that
\[
\text{Supp } \varphi \subset \{ \tau \in \mathbb{R} / \ 3 \leq |\tau| \leq \frac{8}{3} \} \quad \text{and} \quad \forall \tau > 0, \ \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\tau) = 1,
\]
\[
\text{Supp } \chi \subset \{ \tau \in \mathbb{R} / \ |\tau| \leq \frac{4}{3} \} \quad \text{and} \quad \chi(\tau) + \sum_{k \geq 0} \varphi(2^{-k}\tau) = 1.
\]
To deal with the estimate concerning the product of two distributions, we shall frequently use Bony’s decomposition (see [3]) in the horizontal variable:

\[ f g = T_f^h g + T_f^h f + R^h(f, g), \]

where

\[ T_f^h \overset{\text{def}}{=} \sum_k S_k^h f \Delta_k^h g, \quad R^h(f, g) \overset{\text{def}}{=} \sum_k \Delta_k^h f \Delta_k^h g \quad \text{with} \quad \Delta_k^h f \overset{\text{def}}{=} \sum_{|k-k'| \leq 1} \Delta_k^h f. \]

Finally we recall the definitions of norms to anisotropic Sobolev space. For \( s \in \mathbb{R}, k \in \mathbb{N}, \) we denote

\[ \| g \|_{H^s_k} \overset{\text{def}}{=} \left( \int_{\mathbb{R}} \left( 1 + |\xi|^2 \right)^s |\hat{g}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \]

and

\[ \| f \|_{H^{s,k}} \overset{\text{def}}{=} \sum_{0 \leq \ell \leq k} \left( \int_0^\infty \int_{\mathbb{R}} \left( 1 + |\xi|^2 \right)^s |\partial_\xi^\ell \hat{f}(\xi, y)|^2 \, d\xi \, dy \right)^{\frac{1}{2}}. \]

### 2.2. Para-product related estimates

In what follows, we shall always assume that \( t < T^* \) with \( T^* \) being determined by

\[ T^* \overset{\text{def}}{=} \sup \left\{ t > 0, \quad \theta(t) < \frac{\delta}{2a} \right\}. \]

So that by virtue of (1.5), for any \( t < T^* \), \( \delta(t) \geq \frac{\delta}{2} \) and there holds the following convex inequality

\[ \Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta) \quad \forall \ \xi, \eta \in \mathbb{R}, \]

with \( \Phi(t, \xi) \overset{\text{def}}{=} \delta(t) |\xi|^\frac{1}{2}. \)

Then along the same line to the proof of the classical product laws in [2], which corresponds to the cases when \( \Phi(t, \xi) = 0 \), we have

**Lemma 2.1.** Let \( s \in \mathbb{R} \) and \((T_f^h)^* \) be the adjoint operator of \( T_f^h \). If \( \sigma > \frac{1}{2} \), we have

\[ \|(T_f^h g)\|_{H^s_k} \leq C \|f\|_{H^s_k} \|g\|_{H^s_k}, \]

\[ \|(T_f^h)^* g\|_{H^s_k} \leq C \|f\|_{H^s_k} \|g\|_{H^s_k}. \]

If \( s_1 + s_2 = s + \frac{1}{2} > 0 \), there holds

\[ \|(R^h(f, g))\|_{H^s_k} \leq C \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \]

**Lemma 2.2.** Let \( s > 0 \) and \( \sigma > \frac{3}{2} \). Then one has

\[ \|(T_a^h T_b^h - T_{ab}^h) f\|_{H^s_k} \leq C \|\alpha f\|_{H^{s_1}} \|b f\|_{H^{s_2}} \|f\|_{H^{s_1}}, \]

\[ \|\langle D_x \rangle^s (T_a^h f)\|_{L^2_k} \leq C \|\alpha f\|_{H^{s_1}} \|f\|_{H^{s_1}}, \]

\[ \|\langle T_a^h - (T_h^h)^* \rangle f\|_{H^s_k} \leq C \|\alpha f\|_{H^{s_1}} \|f\|_{H^{s_1}}, \]

where \( \langle D_x \rangle^s \) denotes the Fourier multiplier with symbol \( (1 + |\xi|^2)^s \). We also have

\[ \|\langle T_a^h T_b^h \rangle f\|_{H^s_k} \leq C \|\alpha f\|_{H^{s_1}} \|b f\|_{H^{s_2}} \|f\|_{H^{s_1}}. \]

**Lemma 2.3.** Let \( \Phi(t, \xi) \overset{\text{def}}{=} \delta(t) |\xi|^\frac{1}{2}. \) Let \( s \in \mathbb{R} \) and \( \sigma > \frac{3}{2} \). Then one has

\[ \|(T_a^h \partial_x f) - T_a^h \partial_x f\|_{H^s_k} \leq C \delta(t) \|\alpha f\|_{H^{s_1}} \|f\|_{H^{s_1}}. \]
Lemma 2.4. Let
\[ m(t, \xi, \eta) \overset{\text{def}}{=} (e^{\Phi(t, \xi)} - e^{\Phi(t, \eta)}) e^{-\Phi(t, \xi-\eta) - \Phi(t, \eta)} \] with \( \Phi(t, \xi) = \delta(t) \langle \xi \rangle^\frac{1}{2} \). It is easy to observe that
\[ e^{\Phi(t, \xi)} - e^{\Phi(t, \eta)} = \delta(t) \int_0^1 e^{\delta(t) \left( \alpha \langle \xi \rangle^\frac{1}{2} + (1-\alpha) \langle \eta \rangle^\frac{1}{2} \right)} d\alpha \left( \langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2} \right), \]
and for any \( \alpha \in [0, 1] \),
\[ \alpha \langle \xi \rangle^\frac{1}{2} + (1-\alpha) \langle \eta \rangle^\frac{1}{2} \leq \alpha \langle \xi - \eta \rangle^\frac{1}{2} + \langle \eta \rangle^\frac{1}{2}. \]
So that
\[ |m(t, \xi, \eta)| \leq \delta(t) \frac{\langle |\xi| - |\eta| \rangle}{\langle |\xi| + |\eta| \rangle^2} \leq \delta(t) \frac{|\xi - \eta|}{\langle |\xi| + |\eta| \rangle^2}. \]

In view of (2.8), we find
\[ |\int_R m(t, \xi, \eta) S_{k-1}^h a_\Phi(\xi - \eta) i\eta \Delta_k f_\Phi(\eta) d\eta| \leq \delta(t) \int_R |\xi - \eta| |S_{k-1}^h a_\Phi(\xi - \eta)| |\eta|^\frac{1}{2} \Delta_k f_\Phi(\eta) d\eta, \]
from which we infer
\[ \left\| \int_R m(t, \xi, \eta) S_{k-1}^h a_\Phi(\xi - \eta) i\eta \Delta_k f_\Phi(\eta) d\eta \right\|_{L^2} \leq \delta(t) \left( \frac{1}{2} \right) \|S_{k-1}^h a_\Phi \|_{L^1} \|\Delta_k f_\Phi\|_{L^2} \]
\[ \leq \delta(t) \alpha_0 \|a_\Phi\|_{H^s} \|f_\Phi\|_{H^s}. \]

Lemma 2.4. Let \( s \in \mathbb{R}, \sigma > \frac{s}{2} \) and \( D_x = \frac{1}{2} \partial_x \). Let \( \Phi(t, \xi) = \delta(t) \langle \xi \rangle^\frac{1}{2} \), and \( \Lambda(x) = \xi(1 + \xi^2)^{-\frac{1}{2}} \). Then if \( 0 < \delta(t) \leq L \), one has
\[ \| (T^h a \partial_x f_\Phi) - T^h a \partial_x f - \frac{\delta(t)}{2} T^h \partial_x \Lambda(D) \partial_x f_\Phi \|_{H^s} \leq C \|a_\Phi\|_{H^s} \|f_\Phi\|_{H^s}. \]

Proof. Similar to (2.6), we write
\[ \mathcal{F} \left( (T^h a \partial_x f_\Phi) - T^h a \partial_x f - \frac{\delta(t)}{2} T^h \partial_x \Lambda(D) \partial_x f_\Phi \right)(t, \xi) \]
\[ = \sum_{k \geq 0} \int_R \mathcal{M}(t, \xi, \eta) S_{k-1}^h a_\Phi(\xi - \eta) i\eta \Delta_k f_\Phi(\eta) d\eta \]
with
\[ \mathcal{M}(t, \xi, \eta) \overset{\text{def}}{=} \left( e^{\Phi(t, \xi)} - e^{\Phi(t, \eta)} - \frac{\delta(t)}{2} \Lambda(\xi - \eta) e^{\Phi(t, \eta)} \right) e^{-\Phi(t, \xi-\eta) - \Phi(t, \eta)}. \]

Notice that \( \Phi(t, \xi) = \delta(t) \langle \xi \rangle^\frac{1}{2} \), we get, by applying Taylor’s expansion, that
\[ e^{\Phi(t, \xi)} - e^{\Phi(t, \eta)} = \delta(t) e^{\Phi(t, \xi)} \left( \langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2} \right) \]
\[ + \frac{\delta^2(t)}{2} \int_0^1 (1-\alpha) e^{\delta(t) \left( \alpha \langle \xi \rangle^\frac{1}{2} + (1-\alpha) \langle \eta \rangle^\frac{1}{2} \right)} d\alpha \left( \langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2} \right)^2, \]
and
\[
\langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2} = \frac{\eta}{2} (1 + \eta^2)^{-\frac{1}{2}} (\xi - \eta) + \frac{1}{2} \int_0^1 (1 - \alpha) \Lambda' (\alpha \xi + (1 - \alpha) \eta) \, d\alpha (\xi - \eta)^2,
\]
from which and (2.7), we infer
\[
|\mathcal{M}(t, \xi, \eta)| \leq \frac{\delta(t)}{2} \int_0^1 (1 - \alpha) |\Lambda' (\alpha \xi + (1 - \alpha) \eta)| \, d\alpha (\eta - \xi)^2 + \frac{\delta^2(t)}{2} (\langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2})^2
\leq C_L \left( \int_0^1 (1 - \alpha) |\Lambda' (\alpha \xi + (1 - \alpha) \eta)| \, d\alpha (\eta - \xi)^2 + (\langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2})^2 \right).
\]
Observing that if |\xi - \eta| \leq \frac{|\eta|}{2}, we have |\eta + \alpha (\xi - \eta)| \geq |\eta| - |\xi - \eta| \geq \frac{|\eta|}{2} so that
\[
\int_0^1 (1 - \alpha) |\Lambda' (\alpha \xi + (1 - \alpha) \eta)| \, d\alpha \leq C \int_0^1 \frac{1}{(1 + |\eta + \alpha (\xi - \eta)|)^\frac{3}{2}} \, d\alpha \leq C |\eta|^{-\frac{1}{2}}.
\]
While if |\xi - \eta| \geq \frac{|\eta|}{2}, we have
\[
\int_0^1 (1 - \alpha) |\Lambda' (\alpha \xi + (1 - \alpha) \eta)| \, d\alpha \leq C \int_0^1 \frac{1}{(1 + |\eta + \alpha (\xi - \eta)|)^\frac{3}{2}} \, d\alpha \\
\leq \frac{1}{|\xi - \eta|} \int_0^{\xi - \eta} \frac{1}{(1 + |\eta + \tau|)^\frac{3}{2}} \, d\tau \leq C |\eta|^{-1},
\]
which together with the fact
\[
|\langle \xi \rangle^\frac{1}{2} - \langle \eta \rangle^\frac{1}{2}| \leq \frac{(\eta) - (\xi)}{(\xi)^\frac{1}{2} + (\eta)^\frac{1}{2}} \leq C |\eta|^{-\frac{1}{2}}|\eta - \xi|,
\]
ensures that
\[
(2.11) \quad |\mathcal{M}(t, \xi, \eta)| \leq C_L |\eta|^{-1} (\eta - \xi)^2.
\]
Thanks to (2.11), we obtain
\[
\left\| \int_\mathbb{R} \mathcal{M}(t, \cdot, \eta) \check{S}_{k-1} \hat{a}_\phi (\cdot - \eta) \Delta_k^h \hat{f}_\phi (\eta) \, d\eta \right\|_{L^2_h} \leq C_L \left\| \int_\mathbb{R} \left| \check{S}_{k-1} \hat{a}_\phi (\cdot - \eta) \right| \left| \Delta_k^h \hat{f}_\phi (\eta) \right| \, d\eta \right\|_{L^2_h}
\leq C_L \left\| \hat{a}_\phi \right\|_{L^2_h} \left\| \Delta_k^h \hat{f}_\phi \right\|_{L^2_h}
\leq C_L c_k 2^{-k} \left\| \hat{a}_\phi \right\|_{H^0} \left\| \hat{f}_\phi \right\|_{H^1},
\]
from which the support properties to terms in (2.11) we complete the proof of (2.9). \(\square\)

2.3. Two extended lemmas of [26]. To derive the decay estimates of the solution to (1.2), let us recall the following Lemmas:

**Lemma 2.5.** Let \( \Psi(t, y) \defeq \frac{y^2}{8(t)} \) and \( d \) be a nonnegative integer. Let \( u \) be a smooth enough function on \( \mathbb{R}^d \times \mathbb{R}_+ \) which decays to zero sufficiently fast as \( y \) approaching to \( +\infty \). Then
one has

\[ (2.12a) \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} |\partial_y u(X, y)|^2 e^{2\Psi} \, dX \, dy \geq \frac{1}{2(t)} \int_{\mathbb{R}^d \times \mathbb{R}^+} |u(X, y)|^2 e^{2\Psi} \, dX \, dy, \]

\[ (2.12b) \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} |\partial_y u(X, y)|^2 e^{2\Psi} \, dX \, dy \geq \frac{\kappa}{2(t)} \int_{\mathbb{R}^d \times \mathbb{R}^+} |u(X, y)|^2 e^{2\Psi} \, dX \, dy + \frac{\kappa(1 - \kappa)}{4} \int_{\mathbb{R}^d \times \mathbb{R}^+} |y| u(X, y)|^2 e^{2\Psi} \, dX \, dy, \]

for any \( \kappa > 0 \).

**Proof.** \((2.12a)\) was presented in Lemma 3.3 of [18] and Lemma 3.1 of [26]. Let us now modify its proof to prove \((2.12b)\). Indeed we get, by using integration by parts, that

\[ \int_{\mathbb{R}^+} u^2(X, y)e^{\frac{y^2}{4(t)}} \, dy = \int_{\mathbb{R}^+} (\partial_y y) u^2(X, y)e^{\frac{y^2}{4(t)}} \, dy \]

\[ = - \frac{1}{2} \int_{\mathbb{R}^+} y u(X, y) \partial_y u(X, y) e^{\frac{y^2}{4(t)}} \, dy - \frac{1}{2(t)} \int_{\mathbb{R}^+} y^2 u^2(X, y)e^{\frac{y^2}{4(t)}} \, dy. \]

By integrating the above inequality over \( \mathbb{R}^d \) with respect to the \( X \) variables, we find

\[ \int_{\mathbb{R}^d \times \mathbb{R}^+} u^2 e^{\frac{y^2}{4(t)}} \, dX \, dy + \frac{1}{2(t)} \int_{\mathbb{R}^d \times \mathbb{R}^+} y^2 u^2 e^{\frac{y^2}{4(t)}} \, dX \, dy = -2 \int_{\mathbb{R}^d \times \mathbb{R}^+} y u \partial_y u e^{\frac{y^2}{4(t)}} \, dX \, dy \]

\[ \leq \frac{\kappa}{2(t)} \int_{\mathbb{R}^d \times \mathbb{R}^+} y^2 u^2 e^{\frac{y^2}{4(t)}} \, dX \, dy + \frac{2}{\kappa(t)} \int_{\mathbb{R}^d \times \mathbb{R}^+} (\partial_y u)^2 e^{\frac{y^2}{4(t)}} \, dX \, dy. \]

This leads to \((2.12b)\). \( \square \)

**Lemma 2.6.** Let \( G \) be defined by \((1.3)\) and \( \Psi(t, y) \equiv \frac{y^2}{4(t)} \). Let \( \varphi \) and \( u \) be smooth enough functions which decay to zero sufficiently fast as \( y \) approaching to \( \infty \) on \([0, T]\). Then, for any \( \gamma \in (0, 1) \) and \( t \leq T \), one has

\[ (2.13a) \quad \| e^{\gamma \Psi} \Delta^h_k u \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k G \Phi(t) \|_{L^2}; \]

\[ (2.13b) \quad \| e^{\gamma \Psi} \Delta^h_k \partial_y u \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial_y G \Phi(t) \|_{L^2}; \]

\[ (2.13c) \quad \| e^{\gamma \Psi} \Delta^h_k \partial^2_y u \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial^2_y G \Phi(t) \|_{L^2}; \]

\[ (2.13d) \quad \| e^{\gamma \Psi} \Delta^h_k \partial^3_y u \Phi(t) \|_{L^\infty(L^2_k)} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial^3_y G \Phi(t) \|_{L^\infty(L^2_k)}; \]

\[ (2.13e) \quad \| \partial_y e^{\gamma \Psi} \Delta^h_k \partial_y (y \varphi) \Phi(t) \|_{L^2} \lesssim \| \partial_y e^{\gamma \Psi} \Delta^h_k \partial_y (y \varphi) \Phi(t) \|_{L^2} + \| \partial_y e^{\gamma \Psi} \Delta^h_k \partial^2_y (y \varphi) \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial_y (y \varphi) \Phi(t) \|_{L^2}; \]

\[ (2.13f) \quad \| \partial_y e^{\gamma \Psi} \Delta^h_k \partial^3_y (y \varphi) \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial^3_y G \Phi(t) \|_{L^2}. \]

**Proof.** The estimates \((2.13a)\) have been presented in Lemma 3.2 of [26] except \((2.13b), (2.13d), (2.13f)\) and

\[ (2.14) \quad \| \partial_y e^{\gamma \Psi} \Delta^h_k \partial^2_y (y \varphi) \Phi(t) \|_{L^2} \lesssim \| e^{\gamma \Psi} \Delta^h_k \partial^2_y G \Phi(t) \|_{L^2}. \]
The following fact will be used frequently: for any $\kappa \in (0, 1]$ and function $f$, which decays to zero sufficiently fast as $y$ approaching to $\infty$, it holds that

$$
\|e^{\kappa y} f\|_{L^2_y} \leq C \langle t \rangle^{\frac{1}{2}} \|e^{\kappa y} \partial_y f\|_{L^2_y},
$$

(2.15)

$$
\|e^{\kappa y} f\|_{L^\infty_y} \leq C \langle t \rangle^{\frac{1}{2}} \|e^{\kappa y} \partial_y f\|_{L^\infty_y}.
$$

(2.16)

Indeed we observe that

$$
\|e^{\kappa y} f\|_{L^2_y} = \|e^{\kappa y} \int_y^\infty \partial_y f \, dy\|_{L^2_y} \leq \| \int_y^\infty e^{-\kappa(y-y')^2} e^{\kappa y} \partial_y f \, dy\|_{L^2_y}.
$$

Applying Young’s inequality leads to (2.15). So does (2.16).

On the other hand, notice that $u = \partial_y \varphi$ and $\varphi|_{y=0} = 0$, we deduce from (1.3) that

$$
\varphi(t, x, y) = e^{-y^2/4t} \int_0^y e^{y' \phi} G(t, x, y') \, dy',
$$

which implies

$$
\varphi(t, x, y) = -\frac{y^2}{2t} G + \partial_y G(t, y) + \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) e^{-y^2/4t} \int_0^y e^{y' \phi} G(t, x, y') \, dy',
$$

(2.17)

and

$$
\varphi(t, x, y) = -\frac{y^2}{2t} G + \partial_y G(t, y) + \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) e^{-y^2/4t} \int_0^y e^{y' \phi} G(t, x, y') \, dy',
$$

(2.18a)

$$
\partial_y u = -\frac{y^2}{2t} G + \partial_y G + \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) e^{-y^2/4t} \int_0^y e^{y' \phi} G(t, x, y') \, dy',
$$

(2.18b)

and

$$
\partial_y^2 u = -\frac{y^2}{2t} \partial_y G - \frac{1}{2t} G + \partial_y^2 G + \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) \partial_y \varphi - \frac{1}{2t} \partial_y (1 - \frac{y^2}{2t}) \partial_y \varphi,
$$

(2.19a)

$$
\partial_y^2 \varphi = \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) \partial_y \varphi + G + y \partial_y G,
$$

(2.19b)

and

$$
\partial_y^2 \varphi = \left(-\frac{1}{2} + \frac{y^2}{4t^2}\right) \partial_y \varphi + G + y \partial_y G.
$$

(2.19c)

Let us prove (2.13c) first. Indeed it follows from (2.15) that

$$
\langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 \partial_y G\phi\|_{L^2_y} \leq \langle t \rangle^{-1} \|e^{-(1-\gamma) y} y\|_{L^\infty_y} \|e^{\gamma \phi} \Delta_h^1 \partial_y G\phi\|_{L^2_y} \leq C \|e^y \Delta_h^1 \partial_y G\phi\|_{L^2_y}.
$$

Along the same line, due to $u = \partial_y \varphi$, we deduce from (2.13a) and (2.15) that

$$
\langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 G\phi\|_{L^2_y} \leq \langle t \rangle^{-\frac{1}{2}} \|e^{\gamma y} \Delta_h^1 \partial_y G\phi\|_{L^2_y} \leq C \|e^y \Delta_h^1 \partial_y G\phi\|_{L^2_y},
$$

$$
\langle t \rangle^{-2} \|e^{\gamma y} \Delta_h^1 \varphi\|_{L^2_y} \leq \langle t \rangle^{-\frac{1}{2}} \|e^{\gamma y} \Delta_h^1 \partial_y \varphi\|_{L^2_y} \leq \langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 \partial_y \varphi\|_{L^2_y},
$$

$$
\langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 \partial_y \varphi\|_{L^2_y} \leq C \|e^y \Delta_h^1 \partial_y \varphi\|_{L^2_y},
$$

and

$$
\langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 \partial_y \varphi\|_{L^2_y} \leq C \langle t \rangle^{-1} \|e^{\gamma y} \Delta_h^1 \partial_y \varphi\|_{L^2_y} \leq C \|e^y \Delta_h^1 \partial_y \varphi\|_{L^2_y},
$$

for $\gamma_1 \in (\gamma, 1)$.

By summarizing the above estimates and using (2.19a), we achieve (2.13c). Estimate (2.13d) can be proved similarly by using (2.16).
On the other hand, for any $\gamma \in (\gamma, 1)$, we get, by applying (2.15) and (2.13b), that
\[
\langle t \rangle^{-\frac{1}{2}} \| e^{\gamma \Psi} y \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| t \|^{\gamma - \gamma_{1}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}}.
\]
Along the same line, we have
\[
\| e^{\gamma \Psi} (1 - \frac{y^{2}}{2 \langle t \rangle}) \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| t \|^{\gamma - \gamma_{1}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}}.
\]
Finally, it is easy to observe that
\[
\| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| t \|^{\gamma - \gamma_{1}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}}.
\]
In view of (2.19b), by summarizing the above estimates, we achieve
\[
\langle t \rangle^{-\frac{1}{2}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}} \lesssim \langle t \rangle^{-\frac{1}{2}} \| t \|^{\gamma - \gamma_{1}} \| e^{\gamma \Psi} \Delta_{h}^{\frac{1}{2}} \varphi \|_{L^{2}_{t}}.
\]
The second term in (2.14) and estimate (2.13b) can be proved similarly, we omit the details here. This completes the proof of the lemma. \hfill \square

3. Sketch of the proof to Theorem 1.1

In this section, we shall sketch the proof of Theorem 1.1. Let us start with the new formulation of the problem.

3.1. New formulation of the problem. By applying Bony’s decomposition (2.2) in the horizontal variable, we write
\[
\begin{align*}
    u \partial_{x} u &= T_{u}^{h} \partial_{x} u + T_{u}^{h} \partial_{x} u + R^{h}(u, \partial_{x} u), \\
    v \partial_{y} u &= T_{v}^{h} \partial_{y} u + T_{v}^{h} \partial_{y} u + R^{h}(v, \partial_{y} u).
\end{align*}
\]
Applying the operator: $e^{\Phi(t, D_{x})}$, to the above equalities yields
\[
\begin{align*}
    (u \partial_{x} u) \Phi &= T_{u}^{h} \partial_{x} u \Phi + \frac{\delta(t)}{2} \partial_{D_{x}}^{h} \Lambda(D_{x}) \partial_{x} u \Phi + (T_{u}^{h} \partial^{h} u + R^{h}(u, \partial_{x} u)) \Phi + f_{1}, \\
    (v \partial_{y} u) \Phi &= T_{v}^{h} \partial_{y} u \Phi + T_{v}^{h} \partial_{y} u \Phi + \frac{\delta(t)}{2} \partial_{y}^{h} \partial_{D_{x}}^{h} \Lambda(D_{x}) \partial_{y} u \Phi + R^{h}(v, \partial_{y} u) \Phi + f_{2},
\end{align*}
\]
where the remainder terms $f_{1}, f_{2}$ are defined by
\[
\begin{align*}
    f_{1} &\overset{\text{def}}{=} (T_{u}^{h} \partial_{x} u) \Phi - T_{u}^{h} \partial_{x} u \Phi - \frac{\delta(t)}{2} \partial_{D_{x}}^{h} \Lambda(D_{x}) \partial_{x} u \Phi, \\
    f_{2} &\overset{\text{def}}{=} (T_{v}^{h} \partial_{y} u) \Phi - T_{v}^{h} \partial_{y} u \Phi - \frac{\delta(t)}{2} \partial_{y}^{h} \partial_{D_{x}}^{h} \Lambda(D_{x}) \partial_{y} u \Phi + (T_{v}^{h} \partial_{y} u) \Phi - T_{v}^{h} \partial_{y} u \Phi.
\end{align*}
\]
Here and in the sequel, we always denote $D_{x} \overset{\text{def}}{=} \frac{i}{4} \partial_{x}$. 
In view of (3.1), we get, by applying the operator \( e^{\Phi(t,D_x)} \) to the system (1.2), that

\[
\begin{aligned}
&\partial_t u + \lambda \dot{\theta}(D_x) \frac{1}{2} u + T^h_u \partial_x u + T^h_v \partial_y u + T^h_v \partial_y v + f(t), \\
&+ \frac{\delta(t)}{2} T^h_{u,D_x} \Lambda(D_x) \partial_x u + T^h_{u} \partial_{y,D_x} \Lambda(D_v) v - \partial_y^2 u = f,
\end{aligned}
\]

(3.3)

where the source term \( f \) is given by

\[
f = -f_1 - f_2 - f_3 \quad \text{with} \quad f_3 \overset{\text{def}}{=} \left( T^h_{u} u + R^h(u, \partial_x u) + R^h_v(v, \partial_y u) \right) \Phi.
\]

To construct the global small solution of (1.2) with Gevrey 2 regularity in \( x \) variable, we need first to modify the definitions of the auxiliary functions \( H, \phi \), which are first introduced by Dietert and Gérard-Varet in [8]. In order to do it, let \( \Psi(t,y) = \frac{u^2}{8|y|} \) and \( h(t) \) be a positive and non-decreasing smooth function, we define the operator

\[
\mathcal{L} \overset{\text{def}}{=} \partial_t + \lambda \dot{\theta}(D_x) \frac{1}{2} + T^h_u \partial_x + T^h_v \partial_y + \frac{\delta(t)}{2} T^h_{D_x} \Lambda(D) \partial_x - \partial_y^2,
\]

and its adjoint operator in \( L^2(h(t)e^{2\Psi}) \) is given by

\[
\mathcal{L}^* \overset{\text{def}}{=} - \partial_t - \frac{h'}{h} + \frac{y^2}{4h^2} + \lambda \dot{\theta}(D_x) \frac{1}{2} - \partial_x (T^h_u)^*,
\]

(3.6)

\[
- \left( \partial_y + \frac{y}{2h(t)} \right) \Lambda(D) \partial_x (T^h_{D_x} u)^* - \left( \partial_y + \frac{y}{2h(t)} \right)^2.
\]

We now introduce the function \( H \) via

\[
\begin{aligned}
&\mathcal{L} \int_y^\infty H dz = \dot{\theta}(t) \int_y^\infty u dz,
\end{aligned}
\]

(3.7)

\[
\begin{aligned}
&\partial_y H|_{y=0} = 0, \quad H|_{y \to +\infty} = 0, \quad H|_{t=0} = 0,
\end{aligned}
\]

and \( \phi \) by

\[
\begin{aligned}
&\mathcal{L}^* \phi = \dot{\theta}(t) H, \\
&\phi|_{y=0} = 0, \quad \phi|_{y \to +\infty} = 0, \quad \phi|_{t=0} = 0.
\end{aligned}
\]

(3.8)

The existence of \( H \) and \( \phi \) will be guaranteed by the \textit{a priori} estimates presented in Proposition 3.1.

We have the following remark concerning the definitions of \( H \) and \( \phi \):

1. The reason why we need the function \( h(t) \) in the adjoint operator \( \mathcal{L}^* \) given by (3.6) is for the purpose of deriving the decay estimates of the solutions \( H, \phi \) and \( u \). We notice that \( h(t) = 1 \) in the corresponding definition in [8].
2. In (3.7) and (3.8), we define \( H \) and \( \phi \) via \( \int_y^\infty H dz \) and \( \int_y^\infty \phi dz \) (instead of \( \int_0^y H dz \) and \( \int_0^y \phi dz \) as in [8]) is to make the solutions \( H \) and \( \phi \) decay faster as \( y \) approaching to \( \infty \).
We have the additional $\dot{\theta}(t)$ before $\int_y^\infty u_\Phi \, dz$ in (3.7). and $H$ in (3.8) is for the purpose of controlling the gevrey radius of the solutions $H$ and $\phi$ simultaneously with their norms. This idea goes back to [4] where Chemin introduced a tool to make analytical type estimates and controlling the size of the analytic radius simultaneously to classical Navier-Stokes system. It was used in the context of anisotropic Navier-Stokes system [5] (see also [21, 26, 27, 28, 29, 34]), which implies the global well-posedness of three dimensional Navier-Stokes system with a class of “ill prepared data”, which is slowly varying in the vertical variable, namely of the form $\varepsilon x_3$, and the $B^{-1}_{\infty, \infty}(\mathbb{R}^3)$ norm of which blows up as the small parameter goes to zero.

Applying $\partial_y$ to the first equation of (3.7) yields

$$
\dot{\theta}(t) u_\Phi = \mathcal{L}H - T^h_{\partial_y u} \partial_x \int_y^\infty H dz + T^h_{\partial_y u} H - \frac{\delta(t)}{2} T^h_{\partial_y D_x u} \Lambda(D_x) \partial_x \int_y^\infty H dz,
$$

Whereas by taking $\partial_x$ to the first equation of (3.7) and using the fact $v = \int_y^\infty \partial_x udz$, we find

$$
\dot{\theta}(t) v_\Phi = \mathcal{L} \partial_x \int_y^\infty H dz + T^h_{\partial_x u} \partial_x \int_y^\infty H dz
- T^h_{\partial_x u} H + \frac{\delta(t)}{2} T^h_{\partial_x D_x u} \Lambda(D_x) \partial_x \int_y^\infty H dz.
$$

3.2. Ansatz. We shall use the conjugate method to derive the Weighted Gevrey estimates of $H$ and $\phi$ in Section 4. In the process of the estimates, we shall use the formulas (3.9) and (3.10), and thus the time derivative of $\dot{\theta}(t)$ gets involved in. This makes the evolution equations for $\theta(t)$ as in all the previous references like in [26], which is defined via

$$
\begin{cases}
\dot{\theta}(t) = \langle t \rangle^{-\beta} \| e^{\psi \partial_y G_\Phi(t)} \|_{H^{\frac{1}{2}, 0}}, \\
\theta|_{t=0} = 0.
\end{cases}
$$

(3.11)

to be impossible here.

Fortunately, we observe that what we really used in [26] is the time decay estimate of $\| e^{\psi \partial_y G_\Phi(t)} \|_{H^{\frac{1}{2}, 0}}$. Motivated by this observation, for some $\beta > 1$ which will be fixed in Subsection 3.4, we define

$$
\begin{cases}
\dot{\theta}(t) = \varepsilon^\frac{3}{2} \langle t \rangle^{-\beta}, \\
\theta(0) = 0.
\end{cases}
$$

(3.12)

Let $G$ and $T^*$ be determined respectively by (1.3) and (2.3). Let $\varepsilon > 0$, $\gamma_0 \in \left(1, \frac{3}{4}\right)$ and a large enough constant $C$, we define

$$
T^* \overset{\text{def}}{=} \sup \left\{ t < T^*, \| G_\Phi(t) \|_{H^{4, 0}} + \langle t \rangle^{\frac{3}{2}} \| \partial_y G_\Phi(t) \|_{H^{3, 0}} \\
+ \langle t \rangle \| \partial_y^2 G(t) \|_{H^{3, 0}} + \langle t \rangle^{\frac{5}{2}} \| \partial_y^3 G(t) \|_{H^{2, 0}} \leq C \varepsilon \langle t \rangle^{-\gamma_0} \right\}.
$$

(3.13)

3.3. The key a priori estimates. In this subsection, we shall present the a priori estimates used in the proof of Theorem 1.1.

In Section 4, we shall prove the following proposition concerning the a priori estimates of $H$ and $\phi$. 
Proposition 3.1. Let $H$ and $\phi$ be smooth enough solutions of (3.17) and (3.18) respectively, which decay to zero sufficiently fast as $y$ approaching to $\infty$. Then there exist $c_0, \varepsilon_1 \in (0, 1)$ and $\lambda_1 > 0$ so that for $\varepsilon \leq \varepsilon_1$, $\eta \geq \varepsilon^{\frac{3}{4}}$, and $\lambda \geq \lambda_1$, and for any $T < T^*$, there hold

\[
\left\| \langle t \rangle^{\frac{1-n}{4}} \phi(t) \right\|^2_{H_x^{\frac{12}{4}, 0}} + \lambda \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \phi(t) \right) dt \\
+ c_0 \eta \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \partial_y \phi(t) \right) dt \leq \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \phi(t) \right) dt,
\]

and

\[
\left\| \langle t \rangle^{\frac{1-n}{4}} H(T) \right\|^2_{H_x^{\frac{12}{4}, 0}} + \lambda \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} H(t) \right) dt \\
+ c_0 \eta \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \partial_y H(t) \right) dt \leq \left\| \langle t \rangle^{\frac{1-n}{4}} yH(t) \right\|^2_{H_x^{\frac{12}{4}, 0}} + \left\| \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right\|^2_{H_x^{\frac{12}{4}, 0}} dt.
\]

With Proposition 3.1 motivated by [8] and [26], we shall derive the a priori decaying estimates of the solution $u$ to (1.2) in Subsection 5.1. This will be the most crucial ingredient used in the proof of Theorem 1.1. In order to close the estimate in (3.16) below, we shall derive the Gavery estimate of $\partial_y u$ in Subsection 5.2. The main result states as follows:

Proposition 3.2. Let $u$ be smooth enough solution of (1.2) which decay to zero sufficiently fast as $y$ approaching to $\infty$. Then there exist $\varepsilon_2 \in (0, 1)$ and $\lambda_2 > 0$ so that for $\varepsilon \leq \varepsilon_2$, $\eta \geq \varepsilon^{\frac{1}{2}}$, $\lambda \geq \lambda_2$ and for any $T < T^*$, there hold

\[
\left\| \langle t \rangle^{\frac{1-n}{4}} u \right\|^2_{H_x^{\frac{12}{4}, 0}} + \lambda \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} u(t) \right) dt \\
+ \frac{3}{4} \eta \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \partial_y u(t) \right) dt \leq \left\| \langle t \rangle^{\frac{1-n}{4}} u \right\|^2_{H_x^{\frac{12}{4}, 0}} + \langle t \rangle \partial_y u \right\|^2_{H_x^{\frac{12}{4}, 0}} dt
\]

(3.16)

\[
\left\| \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right\|^2_{H_x^{\frac{12}{4}, 0}} dt \leq \left\| \langle t \rangle^{\frac{1-n}{4}} yH \right\|^2_{H_x^{\frac{12}{4}, 0}} dt + \left\| \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right\|^2_{H_x^{\frac{12}{4}, 0}} dt,
\]

and

\[
\left\| \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right\|^2_{H_x^{\frac{12}{4}, 0}} + \lambda \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right) dt \\
+ (1 - \frac{\eta}{8}) \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} \partial_y u \right) dt \\
\leq \left\| \langle t \rangle^{\frac{1-n}{4}} yH \right\|^2_{H_x^{\frac{12}{4}, 0}} dt + C \eta^{-1} (1 + \lambda \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2} - \eta}) \int_0^T \langle t \rangle \left( \langle t \rangle^{\frac{1-n}{4}} H \right) dt.
\]

(3.17)
In Subsection 6.1, we shall derive the Gevery estimate of $G$. While the Gevery estimate of $\partial_y G$ will be presented in Subsection 6.2. These estimates are crucial to control the Gevery radius to the solution of (1.2). More precisely,

**Proposition 3.3.** Let $G$ be determined by (1.3). Then if $\lambda \geq 2C\varepsilon^{\frac{1}{2}}(1 + \eta^{-1}\varepsilon)$ and $\beta \leq \gamma_0 - \frac{1}{12}$, one has

$$
\| \langle T \rangle^{\frac{n-2}{2}} G(T) \|^2_{H^{3,0}_\Psi} + \lambda \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} G(t) \|^2_{H^{4,0}_\Psi} dt
$$

\[+ \frac{3}{4} \eta \int_0^T \| \langle t \rangle^{\frac{n-2}{2}} \partial_y G(t) \|^2_{H^{3,0}_\Psi} dt\]

\[\leq \|G(0)\|^2_{H^{1,0}_\Psi} + C\varepsilon^{\frac{1}{2}} \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} u(t) \|^2_{H^{3,0}_\Psi} dt,$$

and

$$
\| \langle T \rangle^{\frac{n-2}{2}} \partial_y G(T) \|^2_{H^{3,0}_\Psi} + \lambda \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} \partial_y G(t) \|^2_{H^{4,0}_\Psi} dt
$$

\[+ (1 - \frac{\eta}{4}) \int_0^T \| \langle t \rangle^{\frac{n-2}{2}} + \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} \partial_y G(t) \|^2_{H^{3,0}_\Psi} dt\]

$$\leq C\eta^{-1}(\|G(0)\|^2_{H^{1,0}_\Psi} + \| \partial_y G(0) \|^2_{H^{1,0}_\Psi}) + C\eta^{-1} \varepsilon^{\frac{1}{2}} \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} u(t) \|^2_{H^{3,0}_\Psi} dt. $$

In Section 7 we shall derive the Sobolev estimates of $\partial_y G$ and $\partial^3_y G$.

**Proposition 3.4.** Let $G$ be determined by (1.3). Then if $\lambda \geq 2C\varepsilon^{\frac{1}{2}}(1 + \eta^{-1}\varepsilon)$ and $\beta \leq \gamma_0 - \frac{1}{12}$, one has

$$
\| \langle T \rangle^{\frac{n-2}{2}} \partial_y^2 G(T) \|^2_{H^{3,0}_\Psi} + (1 - \frac{\eta}{4}) \int_0^T \| \langle t \rangle^{\frac{n-2}{2}} \partial_y^2 G(t) \|^2_{H^{3,0}_\Psi} dt
$$

$$\leq C\eta^{-1} \varepsilon^{\frac{1}{2}} \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} u(t) \|^2_{H^{3,0}_\Psi} dt
$$

$$+ C\eta^{-1}(\|G(0)\|^2_{H^{1,0}_\Psi} + \| \partial_y G(0) \|^2_{H^{1,0}_\Psi} + \| \partial_y^2 G(0) \|^2_{H^{1,0}_\Psi}),$$

and if moreover $\eta \geq \varepsilon^2$,

$$
\| \langle t \rangle^{\frac{n-2}{2}} \partial_y^3 G(T) \|^2_{H^{2,0}_\Psi} + (1 - \frac{\eta}{4}) \int_0^T \| \langle t \rangle^{\frac{n-2}{2}} \partial_y^3 G(t) \|^2_{H^{2,0}_\Psi} dt
$$

$$\leq C\eta^{-1} \varepsilon^{\frac{1}{2}} \int_0^T \dot{\theta}(t) \| \langle t \rangle^{\frac{n-2}{2}} u(t) \|^2_{H^{2,0}_\Psi} dt
$$

$$+ C\eta^{-1}(\|G(0)\|^2_{H^{1,0}_\Psi} + \| \partial_y G(0) \|^2_{H^{1,0}_\Psi} + \| \partial_y^2 G(0) \|^2_{H^{1,0}_\Psi} + \| \partial_y^3 G(0) \|^2_{H^{1,0}_\Psi}).$$

### 3.4 Proof of Theorem 1.1

**Proof of Theorem 1.1.** Given initial data satisfying (1.9), we deduce from [8] that (1.2) has a unique solution on $[0, \mathcal{T}]$. We are going to prove that $\mathcal{T} = \infty$ if $\varepsilon$ in (1.9) is sufficiently small.
Otherwise, if \( T < \infty \), we take small enough \( \eta \in \left( 0, \frac{1}{T} \right) \), which satisfies \( \eta \geq C \varepsilon^\frac{1}{4} \), and take \( \gamma_0 = \frac{5}{4} - \eta \) in (3.13). Precisely, let \( T^* \) be determined by (2.3), we denote

\[
T^*_\eta \overset{\text{def}}{=} \sup \left\{ t < T^*, \right. \\
\left. \| G_\phi(t) \|_{H^0_{\Phi}} + \langle t \rangle^{\frac{1}{2}} \| \partial_y G_\phi(t) \|_{H_{\Phi}^{3,0}} \\
+ \langle t \rangle \| \partial_y^2 G(t) \|_{H^1_{\Phi}} + \langle t \rangle^{\frac{3}{2}} \| \partial_y^3 G(t) \|_{H^2_{\Phi}} \leq C_1 \varepsilon(t)^{-\frac{1}{4} + \eta} \right\},
\]

where the constant \( C_1 \) will be determined later on.

Let \( E(t) \) and \( D(t) \) be determined respectively by (1.7) and (1.8). Let \( H \) and \( \phi \) be smooth solutions of (3.7) and (3.8) respectively, which decay to zero sufficiently fast as \( y \) approaching to \( \infty \). For \( t \leq T^*_\eta \), we introduce

\[
(3.23a) \quad \mathcal{E}(t) \overset{\text{def}}{=} \left\| \langle t \rangle^{\frac{1}{4}} \phi \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{1}{4}} H \right\|_{H_{\Phi}^{1,0}}^2 + E(t),
\]

\[
(3.23b) \quad \mathcal{D}(t) \overset{\text{def}}{=} \left\| \langle t \rangle^{\frac{1}{4}} \partial_y \phi \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{1}{4}} \partial_y H \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{3}{4}} y H \right\|_{H_{\Phi}^{1,0}}^2 + D(t),
\]

\[
(3.23c) \quad \mathcal{G}(t) \overset{\text{def}}{=} \left\| \langle t \rangle^{\frac{1}{4}} \phi \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{1}{4}} \partial_y \phi \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{3}{4}} y H \right\|_{H_{\Phi}^{1,0}}^2 + \left\| \langle t \rangle^{\frac{1}{4}} \partial_y H \right\|_{H_{\Phi}^{1,0}}^2,
\]

Then taking \( \gamma_0 = \frac{5}{4} - \eta \) and \( h(t) = \langle t \rangle^{\frac{1}{4}} \) in (5.7) leads to

\[
(3.24) \quad \varepsilon \int_0^T \langle t \rangle^{-\eta - \frac{1}{4}} \left\| \langle t \rangle^{\frac{1}{4}} u_\phi \right\|_{H_{\Phi}^{1,0}}^2 \, dt \leq \eta \left\| \langle T \rangle^{\frac{1}{4}} u_\phi(T) \right\|_{H_{\Phi}^{1,0}}^2 \]

\[
+ C \eta^{-1} \varepsilon \left\langle \langle T \rangle^{\frac{1}{4}} H(T) \right\|_{H_{\Phi}^{1,0}}^2 + \eta \varepsilon^\frac{1}{4} \int_0^T \mathcal{D}(t) \, dt
\]

\[
+ C \left( 1 + \lambda \varepsilon + \varepsilon^\frac{1}{4} \eta^{-1} \right) \int_0^T \hat{\theta}(t) \mathcal{G}(t) \, dt.
\]

By inserting the above estimate into (3.13), and summing up the estimates from (3.14) to (3.21), we conclude that for \( \beta \leq \frac{7}{6} - \eta \), \( \varepsilon \leq \min(\varepsilon_1, \varepsilon_2) \), and \( \lambda \geq \max(\lambda_1, \lambda_2) \)

\[
(3.25) \quad \mathcal{E}(T) + \lambda \int_0^T \hat{\theta}(t) \mathcal{G}(t) \, dt + c_0 \eta \int_0^T \mathcal{D}(t) \, dt \leq C_\eta \left( \| u_\phi(0) \|_{H_{\Phi}^{1,0}}^2 + E(0) \right)
\]

\[
+ C \left( 1 + \lambda \varepsilon + \eta^{-1} \left( 1 + \varepsilon^\frac{1}{4} \eta^{-1} \right) \right) \int_0^T \hat{\theta}(t) \mathcal{G}(t) \, dt + \left( \varepsilon^\frac{1}{4} + \varepsilon^\frac{1}{4} \eta^{-1} \right) \int_0^T \mathcal{D}(t) \, dt.
\]

By taking \( \varepsilon \leq \min(\varepsilon_1, \varepsilon_2, \varepsilon_3) \) with \( \varepsilon_3 \overset{\text{def}}{=} \min\left( \left( \frac{c_0 \eta}{4} \right)^2 \eta^4, \frac{1}{(2\varepsilon)^4} \right) \), and \( \lambda \geq \lambda_0 \overset{\text{def}}{=} \max(\lambda_1, \lambda_2, \lambda_3) \) with \( \lambda_3 \overset{\text{def}}{=} 2C \left( 1 + \eta^{-1} \left( 1 + \varepsilon^\frac{1}{4} \eta^{-1} \right) \right) \), we deduce (1.9) and (3.25) that

\[
(3.26) \quad \mathcal{E}(T) + \frac{c_0 \eta}{2} \int_0^T \mathcal{D}(t) \, dt \leq C_\eta \left( \| u_\phi(0) \|_{H_{\Phi}^{1,0}}^2 + E(0) \right) \leq C_\eta \varepsilon^2 \leq \frac{C_\eta^2}{4} \varepsilon^2, \quad \forall \, T \leq T^*_\eta,
\]

by taking \( C_1 \overset{\text{def}}{=} 2\sqrt{C_\eta} \) in (3.22).
On the other hand, by choosing $\beta = \frac{7}{4} - \eta > 1$, which satisfies all the assumptions from Propositions 3.1-3.4, we deduce from (3.12) that

$$
\theta(t) = \int_0^t \theta(s) ds = \varepsilon^\frac{1}{2} \int_0^t s^{-\beta} ds \leq C \varepsilon^\frac{1}{2} \leq \frac{\delta}{4\lambda}, \quad \forall \ t \leq T^*_\eta,
$$

for $\varepsilon \leq \varepsilon_4$.

Therefore for $\varepsilon \leq \varepsilon_0 \overset{\text{def}}{=} \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and $\lambda \geq \lambda_0$, (3.26) and (3.27) contradicts with (2.3) and (3.22). So that we get, by applying a standard continuous argument, that $T^* = T^*_\eta = \infty$. This completes the proof of Theorem 1.1. □

4. The Gevrey Estimates of $H$ and $\phi$.

Let $(u, v)$ be a smooth enough solution of (1.2) on $[0, T^*)$ with $T^*$ being determined by (2.3). Let $H$ and $\phi$ be determined respectively by (3.4) and (3.8) on $[0, T^*)$. The goal of this section is to present the a priori estimates of $H$ and $\phi$.

Indeed thanks to (3.13), for $t \leq T^*$, we deduce from Lemma 2.6 that for any $\gamma \in (0, 1)$

$$
\|u_\Phi(t)\|_{H_{\gamma, \psi}^4} + \langle t \rangle^\frac{1}{2} \|\partial_y u_\Phi(t)\|_{H_{\gamma, \psi}^3} \leq C \varepsilon(t)^{-\gamma_0}.
$$

Furthermore, we observe that

**Lemma 4.1.** Let $(u, v)$ be a smooth enough solution of (1.2) on $[0, T^*)$. Then for any $\gamma \in (0, 1)$ and $t \leq T^*$, we have

$$
\|u_\Phi(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{1}{4}\right)} \quad \text{and} \quad \|v_\Phi(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{1}{4}\right)},
$$

(4.1a)

$$
\|\partial_y u(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{1}{4}\right)} \quad \text{and} \quad \|\partial_y^2 u(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{5}{4}\right)},
$$

(4.1b)

with $\|f\|_{L_{x, y}^\infty} \overset{\text{def}}{=} \|e^\psi f\|_{L_{x, y}^\infty}$.

**Proof.** It follows from (4.1) and $u = -\int_y^\infty \partial_y u \ dz$ that

$$
\|u_\Phi(t)\|_{L_{x, y}^\infty(H^1_0)} \leq \sup_{y > 0} \left\{ e^{\psi y} \left( \int_y^\infty e^{-2\gamma y} \ dz \right)^\frac{1}{2} \right\} \left( \int_0^\infty e^{2\gamma y} \|\partial_y u_\Phi(t, \cdot, y)\|_{H^1_0}^2 \ dy \right)^\frac{1}{2}
$$

$$
\leq C(t)^{\frac{1}{2}} \|\partial_y u_\Phi(t)\|_{H_{\gamma, \psi}^3} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{1}{4}\right)}.
$$

Similarly as $\partial_x u + \partial_y v = 0$, we deduce from (4.1) that

$$
\|v_\Phi\|_{L_{x, y}^\infty(H^1_0)} = \|\partial_x u_\Phi\|_{L_{x, y}^\infty(H^1_0)}
$$

$$
\leq \sup_{y > 0} \left\{ e^{\psi y} \left( \int_y^\infty e^{-2\gamma y} \ dz \right)^\frac{1}{2} \right\} \left( \int_0^\infty e^{2\gamma y} \|u_\Phi(t, \cdot, y)\|_{H^1_0}^2 \ dy \right)^\frac{1}{2}
$$

$$
\leq C(t)^{\frac{1}{4}} \|u_\Phi(t)\|_{H_{\gamma, \psi}^4} \leq C \varepsilon(t)^{-\left(\gamma_0 + \frac{1}{4}\right)}.
$$

This leads to (4.2a).

The first inequality in (4.2b) can be derived along the same line. For the second one, thanks to Sobolev embedding theorem, for any $\gamma_1 \in (\gamma, 1)$, we deduce from (2.13d) and (3.13) that

$$
\|\partial_y^2 u(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \|\partial_y^2 G(t)\|_{L_{x, y}^\infty(H^1_0)} \leq C \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{1, 0}} \leq C \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{1, 0}} \left( \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{3, 0}} + \langle t \rangle^{-\frac{1}{2}} \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{5, 0}} \right)
$$

$$
\leq C \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{3, 0}} \langle t \rangle^{-\frac{1}{2}} \|\partial_y^2 G(t)\|_{H_{\gamma_1, \psi}^{5, 0}} \leq C \varepsilon(t)^{-\gamma_1 \frac{5}{4}}.
$$
This finishes the proof of Lemma 4.1.

4.1. The Estimate of $\phi$. We shall present the a priori estimate of $\phi$ in this subsection. The main result states as follows:

**Proposition 4.1.** Let $\phi$ be a smooth enough solution of (4.3) on $[0,T^*]$, which decays to zero sufficiently fast as $y$ approaching to $+\infty$. Let $h(t)$ be a non-negative and non-decreasing function on $[0,T]$ with $T \leq T^*$. Then if $\beta \leq \gamma_0 + \frac{1}{4}$, for any sufficiently small $\eta > 0$ and $t < T$, one has

\[
\|\sqrt{\eta} \phi(t)\|_{H^2_{\psi,0}}^2 + \int_t^T \|\sqrt{\eta} \phi(t')\|_{H^2_{\psi,0}}^2 \, dt' + (1 - \eta) \int_t^T \|\sqrt{\eta} \partial_y \phi(t')\|_{H^2_{\psi,0}}^2 \, dt' + 2 \left( \lambda - C(\eta^{-1} e^{-\frac{\phi}{\eta}} + 1) \right) \int_t^T \dot{\theta}(t') \|\sqrt{\eta} \phi(t')\|_{H^2_{\psi,0}}^2 \, dt' \leq \int_t^T \dot{\theta}(t') \|\sqrt{\eta} H(t')\|_{H^2_{\psi,0}}^2 \, dt'.
\]

**Proof.** By taking $H^2_{\psi,0}$ inner product of (4.3) with $h_\phi$, we find

\[
h \langle \mathcal{L}^* \phi, \phi \rangle_{H^2_{\psi,0}} = \dot{h} \langle H, \phi \rangle_{H^2_{\psi,0}}.
\]

Here and in all that follows, we always denote

\[
\langle a, b \rangle_{H^2_{\psi,0}} \overset{\text{def}}{=} \text{Re} \int_0^\infty \int_R e^{2\Psi} \langle D_x \rangle^\sigma a \langle D_x \rangle^\sigma b \, dx \, dy.
\]

It is easy to observe that

\[
\int_t^T |\dot{h} \langle H, \phi \rangle_{H^2_{\psi,0}}| \, dt' \leq \frac{1}{2} \int_t^T \dot{\theta}(t') \|\sqrt{\eta} \phi\|_{H^2_{\psi,0}}^2 \, dt' + \frac{1}{2} \int_t^T \dot{\theta}(t') \|\sqrt{\eta} H\|_{H^2_{\psi,0}}^2 \, dt'.
\]

While in view of (3.6), we get, by using integrating by parts, that

\[
\begin{align*}
\dot{h} \langle \mathcal{L}^* \phi, \phi \rangle_{H^2_{\psi,0}} &= -\frac{1}{2} \frac{d}{dt} \|\sqrt{\eta} \phi(t)\|_{H^2_{\psi,0}}^2 + \frac{1}{2} \frac{d}{dt} \|\sqrt{\eta} \phi(t)\|_{H^2_{\psi,0}}^2 \\
&\quad + \frac{1}{2} \int_R \left( \partial_x^2 \langle D_x \rangle^\sigma \right) \langle D_x \rangle^\sigma \, dx dy + \lambda \dot{\theta}(t) \|\sqrt{\eta} \phi\|_{H^2_{\psi,0}}^2 \\
&\quad + \|\sqrt{\eta} \partial_y \phi\|_{H^2_{\psi,0}}^2 + \frac{h(t)}{2(t)} \int_R y e^{2\Psi} \langle D_x \rangle^\sigma \langle D_x \rangle^\sigma \partial_y \phi \, dx \, dy - h \langle \partial_y (T_h^u)^* \phi, \phi \rangle_{H^2_{\psi,0}} \\
&\quad - h \langle (\partial_y + \frac{y}{2(t)}) (T_v^h)^* \phi, \phi \rangle_{H^2_{\psi,0}} - \delta(t) h(t) \langle \Lambda(D) \partial_x (T_h^u)^* \phi, \phi \rangle_{H^2_{\psi,0}}.
\end{align*}
\]

A direct computation gives

\[
\frac{1}{2} \left( \partial_x (e^{2\Psi}) + \frac{y^2}{2(t)} e^{2\Psi} \right) = \frac{y^2}{8(t^2)} e^{2\Psi},
\]

and applying Hölder inequality yields

\[
\frac{h}{2(t)} \left| \int_R y e^{2\Psi} \langle D_x \rangle^\sigma \langle D_x \rangle^\sigma \partial_y \phi \, dx \, dy \right| \leq \frac{1}{2} \|\sqrt{\eta} \partial_y \phi\|_{H^2_{\psi,0}}^2 + \frac{1}{8(t^2)} \int_R |\sqrt{\eta} y^2 e^{2\Psi} \langle D_x \rangle^\sigma \phi|^2 dx \, dy.
\]
As a result, it comes out
\[
\frac{1}{2} \int_{\mathbb{R}^2_+} \left( \partial_t (e^{2\Psi}) + \frac{y^2}{2(t)^2} e^{2\Psi} \right) |\sqrt{\hbar} \langle D_x \partial_y \rangle \phi|^2 dx dy
\]
\[
+ \frac{\hbar}{2(t)} \int_{\mathbb{R}^2_+} ye^{2\Psi} \langle D_x \partial_y \rangle \phi |\langle D_x \partial_y \rangle \phi|^2 dx dy \geq - \frac{1}{2} \| \sqrt{\hbar} \partial_y \phi \|^2_{H_{\Psi}^{29,0}}.
\]

Next let us turn to the estimate of the remaining terms in (4.5).

• The estimate of \( h \langle \partial_x (T_u^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \). By using a standard commutator’s argument, we write
\[
\langle \partial_x (T_u^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} = - \frac{1}{2} \left( \langle [\langle D_x \rangle^{29}, (T_u^h)^* \rangle \phi, \langle D_x \rangle^{29} \partial_x \phi \rangle_{L_{\Psi}^2} \right)
\]
\[
+ \langle \langle D_x \rangle^{29} \phi, \langle D_x \rangle^{29} \partial_x (T_u^h - T_u^h) \phi \rangle_{L_{\Psi}^2} + \langle \langle D_x \rangle^{29} \phi, \{ T_u^h, \langle D_x \rangle^{29} \partial_x \} \phi \rangle_{L_{\Psi}^2},
\]
which implies
\[
\left| \langle \partial_x (T_u^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \right| \leq C \| \phi \|_{H_{\Psi}^{29,0}} \left( \| \langle [\langle D_x \rangle^{29}, (T_u^h)^* \rangle \phi \|_{H_{\Psi}^{1,0}}^{29,0} + \|\langle (T_u^h)^* - T_u^h \rangle \phi \|_{H_{\Psi}^{34,0}}^{29,0} + \| T_u^h, \langle D_x \rangle^{29} \partial_x \phi \|_{L_{\Psi}^2} \right).
\]

It follows from Lemma 2.2 that
\[
\| \langle [\langle D_x \rangle^{29}, (T_u^h)^* \rangle \phi \|_{H_{\Psi}^{1,0}} \leq C \| u \|_{L^\infty(H_{\Psi}^{34,0})} \| \phi \|_{H_{\Psi}^{29,0}}^{29,0},
\]
\[
\| (T_u^h)^* - T_u^h \phi \|_{H_{\Psi}^{34,0}}^{29,0} \leq C \| u \|_{L^\infty(H_{\Psi}^{34,0})} \| \phi \|_{H_{\Psi}^{29,0}}^{29,0},
\]
\[
\| T_u^h, \langle D_x \rangle^{29} \partial_x \phi \|_{L_{\Psi}^2} \leq C \| u \|_{L^\infty(H_{\Psi}^{34,0})} \| \phi \|_{H_{\Psi}^{29,0}}^{29,0},
\]
which imply that
\[
\left| \langle \partial_x (T_u^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \right| \leq C \left( \| u \|_{L^\infty(H_{\Psi}^{34,0})} + \| u \|_{L^\infty(H_{\Psi}^{34,0})} \right) \| \phi \|_{H_{\Psi}^{29,0}}^{29,0}.
\]

Here and in all that follows, we always denote \( \sigma_+ \) to be a constant which is slightly bigger than \( \sigma \).

Then thanks to (3.12) and (4.2a), we obtain
\[
\hbar \| \langle \partial_y + \frac{y}{2(t)} \rangle (T_v^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \leq C \left( \varepsilon \frac{1}{2} (t)^{-\gamma_0 - \frac{1}{2}} + \varepsilon \frac{1}{2} (t)^{\beta - 2\gamma_0 - \frac{1}{2}} \right) \| \sqrt{\hbar} \phi \|_{H_{\Psi}^{29,0}}^{29,0}.
\]

• The estimate of \( h \langle (\partial_y + \frac{y}{2(t)}) (T_v^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \). By using integration by parts and boundary condition \( \phi|_{y=0} = 0 \), \( \phi|_{y \to +\infty} = 0 \), and using (4.2a), we find
\[
h \| \langle (\partial_y + \frac{y}{2(t)}) (T_v^h)^* \phi, \phi \rangle_{H_{\Psi}^{29,0}} \| = h \| \langle (T_v^h)^* \phi, \partial_y \phi \rangle_{H_{\Psi}^{29,0}} \|
\]
\[
\leq C \| u \|_{L^\infty(H_{\Psi}^{34,0})} \| \sqrt{\hbar} \phi \|_{H_{\Psi}^{29,0}}^{29,0} \| \sqrt{\hbar} \partial_y \phi \|_{H_{\Psi}^{29,0}}^{29,0}
\]
\[
\leq C \varepsilon t^{-\gamma_0 + \frac{1}{2}} \| \sqrt{\hbar} \phi \|_{H_{\Psi}^{29,0}}^{29,0} \| \sqrt{\hbar} \partial_y \phi \|_{H_{\Psi}^{29,0}}^{29,0}.
\]
Applying Young’s inequality gives
\[ h\left| \langle \partial_y \frac{y}{2(t)} (T_{\psi}^h)^* \varphi, \varphi \rangle_{H_{\psi}^{2\alpha,0}} \right| \leq C \eta^{-1} \varepsilon^{\frac{3}{2}} (t)^{\beta-2\gamma_0 + \frac{3}{2}} \vartheta(t) \sqrt{\eta^2} \left| \varphi \right|_{H_{\psi}^{2\alpha,0}}^2 + \frac{\eta}{2} \eta \left| \sqrt{\eta} \vartheta \right|_{H_{\psi}^{2\alpha,0}}^2, \]

- The estimate of \( \frac{\delta(t)}{2} \left| \langle \Lambda(D) \partial_x (T_{\psi}^h)^* \varphi, \varphi \rangle_{H_{\psi}^{2\alpha,0}} \right| \) follows from Lemma 2.1, (3.12) and (4.2a) and (4.9).

\[
\frac{\delta(t) h(t)}{2} \left| \langle \Lambda(D) \partial_x (T_{\psi}^h)^* \varphi, \varphi \rangle_{H_{\psi}^{2\alpha,0}} \right| \leq C \left| u \right|_{L^\infty(H_{\psi}^{2\alpha,0})} \left| \sqrt{\eta} \varphi \right|_{H_{\psi}^{2\alpha,0}}^2.
\]

Notice that \( \varphi |_{t=T} = 0 \), by inserting the above estimates into (4.3) over \([t,T]\) and then integrating the resulting inequality, we arrive at
\[
\frac{1}{2} \left| \sqrt{\eta} \varphi(t) \right|_{H_{\psi}^{2\alpha,0}}^2 - \frac{1}{2} \int_t^T \left| \sqrt{\eta} \varphi(t') \right|_{H_{\psi}^{2\alpha,0}}^2 dt'
+ \lambda \int_t^T \vartheta(t') \left| \sqrt{\eta} \varphi(t') \right|_{H_{\psi}^{2\alpha,0}}^2 dt' + \frac{1 - \eta}{2} \int_t^T \left| \sqrt{\eta} \varphi(t') \right|_{H_{\psi}^{2\alpha,0}}^2 dt'
\leq \frac{1}{2} \int_t^T \vartheta(t') \left| \sqrt{\eta} H(t') \right|_{H_{\psi}^{2\alpha,0}}^2 dt' + C (\eta^{-1} \varepsilon^{\frac{3}{2}} + 1) \int_t^T \vartheta(t') \left| \sqrt{\eta} \varphi(t') \right|_{H_{\psi}^{2\alpha,0}}^2 dt',
\]

if \( \beta \) in (3.12) satisfying \( \beta \leq \gamma_0 + \frac{1}{2} \). This leads to (4.3), and we complete the proof of Proposition 4.1. \( \square \)

In particular, if we take \( h(t) = \langle t \rangle \) in (4.3), we obtain

**Corollary 4.1.** Under the assumptions of Proposition 4.1, there exist \( \varepsilon_0 \) and \( \lambda_1 \) so that for \( \varepsilon \leq \varepsilon_0, \eta \geq 2 \varepsilon^{\frac{3}{2}} \) and \( \lambda \geq \lambda_1, (3.14) \) holds.

**Proof.** Indeed it follows from Lemma 2.5 that
\[
\frac{1}{2} \left| \langle \varphi \rangle_{H_{\psi}^{2\alpha,0}} \right| \leq \left| \langle \frac{1 - \eta}{2} \partial_y \varphi(t) \rangle_{H_{\psi}^{2\alpha,0}} \right|^2,
\]
so that
\[
\frac{1 - \eta}{2} \int_t^T \langle \varphi \rangle_{H_{\psi}^{2\alpha,0}}^2 dt' + (1 - \eta) \int_t^T \langle \varphi \rangle_{H_{\psi}^{2\alpha,0}}^2 dt' \geq 0.
\]

Then by taking \( h(t) = \langle t \rangle^{1+\frac{\alpha}{2}} \) in (4.3) and using the above inequality, we achieve
\[
\langle \varphi \rangle_{H_{\psi}^{2\alpha,0}}^2 + 2 \left( \lambda - C (\eta^{-1} \varepsilon^{\frac{3}{2}} + 1) \right) \int_t^T \vartheta(t') \langle \varphi \rangle_{H_{\psi}^{2\alpha,0}}^2 dt' \leq \int_t^T \vartheta(t') \langle \varphi \rangle_{H_{\psi}^{2\alpha,0}}^2 dt',
\]

Then for \( \varepsilon \) small enough and \( \eta \geq 2 \varepsilon^{\frac{3}{2}} \), we have \( \eta^{-1} \varepsilon^{\frac{3}{2}} + 1 \leq \frac{3}{2} \). Then taking \( \lambda \geq \lambda_1 \) in (4.9) leads to (3.14). \( \square \)
4.2. The estimate of $H$. In this subsection, we shall present the a priori estimate of $H$. Before proceeding, we first derive the estimate of source term $f$ in (3.3).

**Lemma 4.2.** Let $f$ be given by (3.4). Then for any $s > 0$, there holds

$$
\|f\|_{H^s_{\psi}} \leq C(t)^{\frac{s}{2}+1}\|\partial_y G \Phi H_{\psi}^{\frac{s}{2}}\|_{\psi} + \|G \Phi H_{\psi}^{\frac{s}{2}}\|_{\psi}.
$$

**Proof.** Recall from (3.4) that $f = -f_1 - f_2 - f_3$ with $f_i$ $(i = 1, 2, 3)$ being given respectively by (3.2) and (3.4). We first observe from the proof of Lemma 4.1 that for any $\gamma \in (0, 1)$ and $\sigma \in \mathbb{R}$

$$
\|e^{\gamma} u(t)\|_{L^\infty(H^s_\psi)} \leq C(t)^{\frac{s}{2}}\|\partial_y G(t)\|_{H^0_{\psi}}\text{ and } \|e^{\gamma} v(t)\|_{L^\infty(H^s_\psi)} \leq C(t)^{\frac{s}{2}}\|v(t)\|_{H^s_{\psi}}.
$$

Thanks to (3.11), we deduce from Lemma 2.1 that

$$
\|f_1\|_{H^s_{\psi}} \leq C(t)^{\frac{s}{2}}\|\partial_y u(t)\|_{H^0_{\psi}}^2 + \|u(t)\|_{H^0_{\psi}}^2
$$

Similarly due to $v = \int_0^\infty \partial_x u \, dz$, it follows from Lemma 2.3, Lemma 2.4 and Lemma 2.6 that

$$
\|f_2\|_{H^s_{\psi}} \leq C(t)^{\frac{s}{2}}\|\partial_y u(t)\|_{H^0_{\psi}}^2 + \|v(t)\|_{H^0_{\psi}}^2 + \|u(t)\|_{H^0_{\psi}}^2
$$

Finally by applying Lemma 2.1 and Lemma 2.6 we arrive at

$$
\|f_3\|_{H^s_{\psi}} \leq C(t)^{\frac{s}{2}}\|\partial_x u(t)\|_{H^0_{\psi}}^2 + \|v(t)\|_{H^0_{\psi}}^2 + \|u(t)\|_{H^0_{\psi}}^2
$$

By summarizing the above estimates, we conclude the proof of (4.10).

Next let us turn to the estimates of $H$.

**Proposition 4.2.** Let $H$ be a smooth enough solution of (3.1) on $[0, T^*)$ which decays to zero as $y$ approaching $\infty$. Then if $\beta \leq \min\{\gamma_0 + \frac{1}{4}, \frac{1}{2}(\gamma_0 + \frac{3}{4})\}$ and $\lambda \leq C$, for any $T \in [0, T^*)$, we have

$$
\|\sqrt{h}H(T)\|_{H^0_{\psi}}^2 + 2(\lambda(1 - C\varepsilon) - C(1 + \eta^{-1}\varepsilon)\varepsilon)^{\frac{1}{4}} \int_0^T \hat{\theta}(t)\|\sqrt{h}H(t)\|_{H^0_{\psi}}^2 dt
$$

$$
- \int_0^T \|\sqrt{h}H(t)\|_{H^0_{\psi}}^2 dt + (1 - \frac{\eta}{4}) \int_0^T \|\sqrt{h}\partial_y H(t)\|_{H^0_{\psi}}^2 dt
$$

(4.12)

$$
\leq C\left(1 + \lambda \varepsilon \right) \int_0^T \hat{\theta}(t)(\|\sqrt{h}H(t)\|_{H^0_{\psi}}^2 + \|\sqrt{h}\partial_y H(t)\|_{H^0_{\psi}}^2) dt + (\|\sqrt{h}(0)\|_{H^0_{\psi}}^2 + \|\sqrt{h}(0)\|_{H^0_{\psi}}^2)
$$

$$
+ C\varepsilon \int_0^T (t)^{-\gamma_0 - \frac{4}{3}}\|\sqrt{h}u(t)\|_{H^0_{\psi}}^2 + \|\sqrt{h}\partial_y u(t)\|_{H^0_{\psi}}^2) dt.
$$

**Proof.** In view of (3.3) and (3.5), we write

$$
Lu + T_{\partial_y u}(\partial_y uH_{\psi}) + \frac{\delta(t)}{2} T_{\partial_y D_x u}(D_x) = f.
$$
By taking $H_{\Psi}^{27,0}$ inner product of the above equation with $h(t)\phi$ and integrating the resulting equality over $[0, T]$, we find

$$\int_0^T h(t)\langle \mathcal{L}u_{\Phi} + T_{\partial y}^h u_{\nu} + \frac{\delta(t)}{2} T_{\partial y}^h D_x \Lambda(D_x) u_{\Phi}, \phi \rangle \, dt = \int_0^T h(t)\langle f, \phi \rangle \, dt.$$  

We first get, by applying Lemma 4.2 with $s = 6$, that

$$\int_0^T h(t)\langle f, \phi \rangle \, dt \leq \int_0^T ||\sqrt{h}f||_{H_{\Psi}^{0,0}} ||\sqrt{h}\phi||_{H_{\Psi}^{27,0}} \, dt \leq C \int_0^T \dot{\theta}(t) \frac{1}{2} \langle t \rangle^{-2\gamma_0} \langle t \rangle^{-1} ||\sqrt{h}u_{\Phi}||_{H_{\Psi}^{6,0}}^2 + ||\sqrt{h}\partial_y u_{\Phi}||_{H_{\Psi}^{11,0}}^2 \, dt,$$

from which, (3.13) and (3.12), we infer

$$\int_0^T h(t)\langle f, \phi \rangle \, dt \leq C \int_0^T \dot{\theta}(t) ||\sqrt{h}\phi||_{H_{\Psi}^{15,0}}^2 \, dt + \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \langle t \rangle^{-1} ||\sqrt{h}u_{\Phi}||_{H_{\Psi}^{2,0}}^2 + ||\sqrt{h}\partial_y u_{\Phi}||_{H_{\Psi}^{5,0}}^2 \, dt.$$

Due to $\beta \leq \gamma_0 + \frac{1}{4}$ and $\gamma_0 > 1$, we achieve

$$\int_0^T h(t)\langle f, \phi \rangle \, dt \leq C \int_0^T \dot{\theta}(t) ||\sqrt{h}\phi||_{H_{\Psi}^{27,0}}^2 \, dt + \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \langle t \rangle^{-1} ||\sqrt{h}u_{\Phi}||_{H_{\Psi}^{2,0}}^2 + ||\sqrt{h}\partial_y u_{\Phi}||_{H_{\Psi}^{5,0}}^2 \, dt.$$

On the other hand, thanks to (3.8), we get, by using integration by parts, that

$$\int_0^T h(Lu_{\Phi}, \phi) \, dt = \int_0^T h(u_{\Phi}, L^* \phi) \, dt - h(0)\langle u_{\Phi}(0), \phi(0) \rangle \, dt$$

$$= \int_0^T h(\dot{u}_{\Phi}, H) \, dt - h(0)\langle u_{\Phi}(0), \phi(0) \rangle \, dt.$$

Substituting (3.9) into the above equality yields

$$\int_0^T h(Lu_{\Phi}, \phi) \, dt = \int_0^T \left( \langle LH, H \rangle_{H_{\Psi}^{27,0}} - \langle T_{\partial y}^h \partial_x \Lambda(D_x) u_{\Phi}, H \rangle_{H_{\Psi}^{25,0}} \right) \, dt$$

$$- \frac{\delta(t)}{2} \langle T_{\partial y}^h D_x \Lambda(D_x) \partial_y u_{\Phi}, H \rangle_{H_{\Psi}^{25,0}}$$

$$+ \langle T_{\partial y}^h H, H \rangle_{H_{\Psi}^{25,0}} \, dt - h(0)\langle u_{\Phi}(0), \phi(0) \rangle \, dt.$$
Along the same line, by applying (4.10) and then (3.8), we find

\[
\int_0^T h(T_h^x u, \phi) \partial x_{\mathcal{H}_\Psi^0} dt = \int_0^T \frac{h}{\theta} \left( \langle T_h^x u L \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt
\]

\[
- \langle T_h^x u^2 \phi \rangle_{\mathcal{H}_\Psi^0} + \frac{\delta(t)}{2} \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt
\]

\[
+ \frac{1}{2} \delta(t) \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt
\]

\[
- \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} + \frac{\delta(t)}{2} \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt,
\]

and

\[
\frac{1}{2} \int_0^T h(t) \delta(t) \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} dt
\]

\[
- \frac{1}{2} \int_0^T \frac{\delta(t)}{2\theta} h(t) \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} dt
\]

\[
+ \frac{1}{2} \delta(t) \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt
\]

\[
= \frac{1}{2} \int_0^T \frac{\delta(t)}{2\theta} h(t) \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} dt
\]

\[
- \frac{1}{2} \int_0^T \frac{\delta(t)}{2\theta} h(t) \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} dt
\]

\[
+ \frac{1}{2} \delta(t) \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt.
\]

The most trouble terms above are the second and third terms in (4.14) which are fortunately canceled respectively by the first term in (4.15) and (4.16). By summarizing (4.11), (4.15) and (4.16), we arrive at

\[
\int_0^T h(L u + T_h^x u \phi + \frac{\delta(t)}{2} T_h^x u \partial_x \Lambda(D_x) \partial x, \phi) \partial x_{\mathcal{H}_\Psi^0} dt
\]

\[
= \int_0^T h(L H, H) \partial x_{\mathcal{H}_\Psi^0} dt + \frac{\delta(t)}{2\theta} \left( \langle T_h^x u \partial_x \Lambda(D_x) \partial x \int_y^\infty H dz, \phi \rangle_{\mathcal{H}_\Psi^0} \right) dt,
\]
Lemma 4.3. Let $\beta \leq \gamma_0 + \frac{1}{4}$. Then for any small positive constant $\eta$, one has

\[
E_1 \geq \frac{1}{2} \| \sqrt{\nu} H(t) \|_{H_{\psi}^{27,0}}^2 + \left( \lambda - C(1 + \eta)^{-1} \right) \int_0^T \dot{\theta}(t) \| \sqrt{\nu} H(t) \|_{H_{\psi}^{7,0}}^2 dt \tag{4.18}
\]

By virtue of (4.13), to prove (4.12), we are left with the estimates of $E_i$ ($i = 1, \cdots, 13$), which we present below.

**Lemma 4.4.** Let $\beta \leq \min\{\gamma_0 + \frac{1}{4}, \frac{1}{2} (\gamma_0 + \frac{5}{4}) \}$. Then one has

\[
\sum_{i=4}^6 |E_i| + |E_{10}| \leq C \left(1 + \lambda \epsilon^\frac{1}{2} \right) \int_0^T \dot{\theta}(t) \left( \| \sqrt{\nu} H(t') \|_{H_{\psi}^{27,0}}^2 + \| \sqrt{\nu} \phi(t) \|_{H_{\psi}^{\frac{15}{2},0}}^2 \right) dt, \tag{4.19a}
\]

\[
\sum_{i=7}^9 |E_i| + \sum_{i=11}^{13} |E_i| \leq C \epsilon \int_0^T \dot{\theta}(t) \left( \| \sqrt{\nu} H(t) \|_{H_{\psi}^{27,0}}^2 + \| \sqrt{\nu} \phi(t) \|_{H_{\psi}^{\frac{15}{2},0}}^2 \right) dt. \tag{4.19b}
\]

Let us admit the above lemmas for the time being, and continue our proof of Proposition 4.2.

Thanks to Lemmas 4.3 and 4.4, it remains to handle the estimates of $E_2$ and $E_3$ in (4.17). Indeed we get, by applying Lemma 2.2 (4.17a) and (3.12), that

\[
|E_2| \leq \int_0^T \| \partial_y v \|_{L^\infty_t(H_{h_1}^{\frac{1}{2}})} \| \sqrt{\nu} H \|_{H_{\psi}^{27,0}}^2 dt \leq C \epsilon \int_0^T (t)^{-\gamma_0 + \frac{1}{4}} \| \sqrt{\nu} H(t) \|_{H_{\psi}^{27,0}}^2 dt 
\]

which together the fact: $\beta \leq \gamma_0 + \frac{1}{4}$, ensures that

\[
|E_2| \leq C \epsilon \int_0^T \dot{\theta}(t) \| \sqrt{\nu} H(t) \|_{H_{\psi}^{27,0}}^2 dt. \tag{4.20}
\]
Finally by using Hölder inequality, we have

\begin{equation}
\frac{1}{2} \| \sqrt{H}(T) \|_{H_{x,y}^{2 \alpha \sigma}}^2 + \left( \lambda - C(1 + \eta^{-1} \varepsilon) \right) \frac{1}{2} \int_0^T \frac{1}{2} \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \ dt - \frac{1}{2} \int_0^T \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \ dt + \left( \frac{1}{2} - \frac{\eta}{8} \right) \int_0^T \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \ dt \leq C \left( 1 + \lambda \varepsilon \right) \int_0^T \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 + \left( 1 + \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \right) \ dt + \frac{1}{2} \| \sqrt{H}(0) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \right) dt, \end{equation}

from which, we deduce (4.12). This ends the proof of Proposition 4.2. \( \square \)

Let us now present the proof of Lemma 4.3. The proof Lemma 4.4 involves tedious calculation, and we postpone it in the Appendix A.

\textbf{Proof of Lemma 4.3.} Notice that \( 2 \partial_t \Psi + (\partial_y \Psi)^2 \leq 0 \), we get, by using integrating by parts, that

\begin{equation}
\langle (\partial_t H - \partial_y^2 H) e^{2 \Psi H} \rangle_{L_\infty^1} \geq \frac{1}{2} \frac{d}{dt} \| H(t) \|_{H_{x,y}^{-2 \alpha \sigma}}^2 + \frac{1}{2} \| \partial_y H \|_{H_{x,y}^{-2 \alpha \sigma}}^2.
\end{equation}

Then in view of (3.22), we find

\begin{align*}
E_1 & \geq \int_0^T \left( \frac{1}{2} \frac{d}{dt} \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \right) - \frac{h}{2h} \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \\
& \quad + \lambda \frac{1}{h} \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 + \frac{1}{2} \| \sqrt{H}(t) \|_{H_{x,y}^{2 \alpha \sigma}}^2 \left( T_h \partial_x H, H \right)_{H_{x,y}^{2 \alpha \sigma}} - \frac{\delta(t)}{2} \| T_h \partial_x H, H \|_{H_{x,y}^{2 \alpha \sigma}}^2 \ dt.
\end{align*}

It follows from a similar derivation of (4.17) and (4.2), (3.12) that

\begin{align*}
h \left| \left( T_h \partial_x H, H \right)_{H_{x,y}^{2 \alpha \sigma}} \right| & \leq C \left( \| u \|_{L_\infty^\infty(H_{x,y}^{1/2} +)} + \| u \|_{L_\infty^\infty(H_{x,y}^{3/2} +)} \right) \| \sqrt{H} \|_{H_{x,y}^{2 \alpha \sigma}}^2 \\
& \leq C \left( \varepsilon \frac{1}{2} \right)^{3/2} \left( t^{3/2} - 4 \right) + \frac{3}{2} \left( t^{3/2} - 4 \right) \| \sqrt{H} \|_{H_{x,y}^{2 \alpha \sigma}}^2.
\end{align*}

Applying Lemma 2.1 gives

\begin{align*}
h \left| \left( T_h \partial_y H, H \right)_{H_{x,y}^{2 \alpha \sigma}} \right| & \leq C \left( \| u \|_{L_\infty^\infty(H_{x,y}^{1/2} +)} + \| \sqrt{H} \|_{H_{x,y}^{2 \alpha \sigma}}^2 \right) \| \sqrt{H} \|_{H_{x,y}^{2 \alpha \sigma}}^2 \\
& \leq C \left( \varepsilon \frac{1}{2} \right)^{3/2} \left( t^{3/2} - 4 \right) + \frac{3}{2} \| \sqrt{H} \|_{H_{x,y}^{2 \alpha \sigma}}^2.
\end{align*}
While it follows from Lemma 2.1 and (4.2a) that
\[
\frac{\delta(t)}{2} h \left| \int_{D_{x_0}} A(D_x) \partial_x H, H \right|_{H^\infty} \leq C \delta \|u\|_{L^\infty} \left( H^{3/4} \right) \left\| \sqrt{H} \right\|_{H^\infty}^2 \\
\leq C \varepsilon^{3/4} (t)^{\beta - \gamma_0 - \frac{1}{4}} \partial_t \left( \sqrt{H} \right) \|\sqrt{H}\|_{H^\infty}^2.
\]

By summarizing the above estimates and using the fact that \(\beta \leq \gamma_0 + \frac{1}{4}\), we complete the proof of (4.18).

Taking \(h(t) = (t)^{\frac{3}{10}}\) in (4.12) gives rise to

**Corollary 4.2.** Under the assumption of Proposition 4.2 there exist \(c_0, \varepsilon_1 \in (0, 1)\) and \(\lambda_1 > 0\) so that for \(\varepsilon \leq \varepsilon_1, \eta \geq \varepsilon \frac{3}{7}\), and \(\lambda \geq \lambda_1\), there holds (3.15).

**Proof.** By taking \(h(t) = (t)^{\frac{3}{10}}\) in (4.12) and using (4.8), we obtain for \(\lambda \varepsilon \frac{3}{7} \leq C\)

\[
\left\| \left< \int_0^T \frac{1}{-H^{2/3}} H(T) \right\|^2_{H^\infty} \right\|_{H^\infty}^2 + 2 \left( \lambda - C (1 + \eta^{-1} \varepsilon) \frac{3}{7} \right) \int_0^T \hat{\theta}(t) \left< \int_0^T \frac{1}{-H^{2/3}} H(t) \right\|^2_{H^\infty} \right\| dt \\
+ \frac{3}{4} \eta \int_0^T \left< \int_0^T \frac{1}{-H^{2/3}} \partial_y H(t') \right\|^2_{H^\infty} \right\| dt' \\
\leq \left\| u_{\Phi}(0) \right\|^2_{H^\infty} + \left\| \phi(0) \right\|^2_{H^\infty} + C \int_0^T \hat{\theta}(t) \left< \int_0^T \frac{1}{-H^{2/3}} \phi(t) \right\|^2_{H^\infty} dt \\
+ C \varepsilon^{\frac{1}{2}} \int_0^T \left( \left< \int_0^T (t)^{-\gamma_0 - \frac{1}{4}} \|\sqrt{H} u_{\Phi}\|^2_{H^\infty} + \|\sqrt{H} \partial_y u_{\Phi}\|^2_{H^\infty} \right) \right) dt.
\]

Observing from Lemma 2.5 that

\[
\frac{3}{4} \eta \int_0^T \left< \int_0^T \frac{1}{-H^{2/3}} \partial_y H(t') \right\|^2_{H^\infty} \right\| dt' \geq c_0 \eta \int_0^T \left< \int_0^T \frac{1}{-H^{2/3}} y H(t) \right\|^2_{H^\infty} \right\| dt.
\]

And for \(\lambda \geq \max(C, \lambda_1)\), we deduce from Corollary 4.1 that

\[
\left\| \phi(0) \right\|^2_{H^\infty} + C \int_0^T \hat{\theta}(t) \left< \int_0^T \frac{1}{-H^{2/3}} \phi(t) \right\|^2_{H^\infty} dt \leq \int_0^T \hat{\theta}(t) \left< \int_0^T \frac{1}{-H^{2/3}} H(t) \right\|^2_{H^\infty} dt.
\]

By inserting the above two inequalities into (4.23) and taking \(\varepsilon \leq \varepsilon_1, \eta \geq \varepsilon \frac{3}{7}\), and \(\lambda \geq \lambda_1 = \max(2(1 + C), \lambda_1)\), we obtain (3.15).

4.3. **The proof of Proposition 3.1.** By combining Corollaries 4.1 with 4.2 we conclude the proof of Proposition 3.1.

5. **The Gevrey estimates of \(u_{\Phi}\) and \(\partial_y u_{\Phi}\).**

5.1. **The Gevrey estimate of \(u_{\Phi}\).** Motivated by [8] concerning the local well-posedness of Prandtl system with initial data in the optimal Gevery regularity and also [26] concerning the decay of the global analytical solutions to (1.1), we shall present the a priori time-decay estimates for \(u_{\Phi}\) in this subsection. The main result states as follows:
Proposition 5.1. Let $h(t)$ be a non-negative and non-decreasing function on $[0, T]$ with $T \leq T^*$. Then if $\beta \leq \min \left\{ \frac{1}{4}(\gamma_0 + \frac{5}{4}), \frac{3}{4} \gamma_0 + \frac{1}{2}, \frac{5}{4} \gamma_0 + \frac{1}{2}, \frac{3}{4} \gamma_0 \right\}$, for any $\eta \in (0, \eta_1)$ with $\eta_1$ being sufficiently small, there holds

\[
(1 - \eta)\|\sqrt{h}u_{\Phi}(T)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 + 2(\lambda - C(1 + \varepsilon^{-\frac{2}{3}})) \int_0^T \dot{\theta}(t)\|\sqrt{h}u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
- \int_0^T \|\sqrt{h}'u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt + (1 - \eta) \int_0^T \|\sqrt{h}\partial_y u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
\leq C\|\sqrt{h}(0)u_{\Phi}(0)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 + C\eta^{-1} \varepsilon\|\sqrt{h}H(T)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 \\
+ \eta\varepsilon^{\frac{2}{3}} \int_0^T \|\sqrt{h}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt + \eta\varepsilon^{\frac{1}{3}} \int_0^T \|\sqrt{h}H\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
+ C(1 + \lambda\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}}) \int_0^T \dot{\theta}(t)\left(\|\sqrt{h}H\|_{H^{\frac{1}{2},0}_{\Psi}}^2 + \|\sqrt{h}(t)^\frac{1}{2}\partial_y u_{\Phi}\|_{H^{\frac{1}{2},0}_{\Psi}}^2\right) dt.
\]  

(5.1)

Proof. We start the proof of Proposition 5.1 by the following lemma:

Lemma 5.1. Let $h(t)$ be a non-negative and non-decreasing function on $[0, T]$ with $T \leq T^*$. Then if $\beta \leq \gamma_0 + \frac{1}{4}$, for any sufficiently small $\zeta > 0$, one has

\[
\|\sqrt{h}u_{\Phi}(T)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 + 2\lambda (1 + \varepsilon^{-\frac{2}{3}}) \int_0^T \dot{\theta}(t)\|\sqrt{h}u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
- \int_0^T \|\sqrt{h}'u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt + (1 - \frac{\zeta}{4}) \int_0^T \|\sqrt{h}\partial_y u_{\Phi}(t)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
\leq \|\sqrt{h}(0)u_{\Phi}(0)\|_{H^{\frac{1}{2},0}_{\Psi}}^2 + C\varepsilon \int_0^T \|\sqrt{h}H\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt.
\]  

(5.2)

Let us postpone the proof of Lemma 5.1 till we finish the proof of Proposition 5.1.

With (5.2), to prove (5.1), it remains to handle the estimate of $\varepsilon \int_0^T (t)^{-(\frac{1}{2} + \frac{1}{4})}\|\sqrt{h}u_{\Phi}\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt$. As a matter of fact, in view of (3.9), we write

\[
\varepsilon \int_0^T (t)^{-(\frac{1}{2} - \frac{1}{4})}\|\sqrt{h}u_{\Phi}\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt = \varepsilon \int_0^T (t)^{-(\frac{1}{2} - \frac{1}{4})}\|\sqrt{h}u_{\Phi}\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
= \int_0^T \varepsilon(t)^{-(\frac{1}{2} - \frac{1}{4})}\|\sqrt{h}\|_{H^{\frac{1}{2},0}_{\Psi}}^2 dt \\
+ \frac{\delta(t)}{2} \left(\int_y H dz, u_{\Phi}\right)_{H^{\frac{1}{2},0}_{\Psi}} \\
\defeq B_1 + \cdots + B_4.
\]  

(5.3)

The estimate of $B_1$ relies on the following lemma, the proof of which will be postponed in Appendix B.
Lemma 5.2. Let $\beta \leq \min\{\frac{1}{2}(\gamma_0 + \frac{5}{2}), \frac{2}{3}\gamma_0 + \frac{1}{2}, \gamma_0 + \frac{1}{4}, \frac{2}{3}\gamma_0\}$, then one has

\[
|B_1| \leq \zeta \|\sqrt{h}u_\Phi(T)\|_{H^2_{\psi,0}}^2 + C\zeta^{-1} \varepsilon \|\sqrt{h}H(T)\|_{H^2_{\psi,0}}^2 \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \|u_\Phi\|_{H^2_{\psi,0}}^2 dt \\
+ \zeta \varepsilon \frac{1}{2} \int_0^T \|\sqrt{h}\partial_y u_\Phi\|_{H^2_{\psi,0}}^2 dt + \zeta \varepsilon \frac{1}{4} \int_0^T \|\sqrt{h}(t)\frac{1}{2} \partial_y u_\Phi\|_{H^2_{\psi,0}}^2 dt \\
+ \zeta \varepsilon \frac{1}{2} \int_0^T \|\frac{y}{(t)}\sqrt{h}H\|_{H^2_{\psi,0}}^2 dt + C(1 + \lambda \varepsilon + \varepsilon \zeta^{-1}) \times \\
\times \int_0^T \dot{\theta}(t) \left(\|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}}^2 + \|\sqrt{h}H\|_{H^2_{\psi,0}}^2 + \|\sqrt{h}(t)\frac{1}{2} \partial_y u_\Phi\|_{H^2_{\psi,0}}^2\right) dt.
\]

(5.4)

We now turn to the estimates of the remaining terms in (5.3).

- The estimate of $B_2$. We observe that for any $s \in \mathbb{R}$,

\[
\|e^s \int_y^{+\infty} H dz\|_{L^p(H^s)} \leq \| \int_y^{+\infty} e^{-\frac{(y-z)^2}{8s^2}} e^{-\frac{x^2}{8s^2}} \|H(\cdot, z)\|_{H^s} dz\|_{L^p}
\]

(5.5)

Then we get, by applying first Lemma 2.1 and (5.5) and then (4.1), that

\[
|B_2| \leq C\varepsilon \frac{1}{2} \int_0^T \langle t \rangle^{\beta - \gamma_0} \|\partial_y u\|_{L^2(H_{\psi,0}^{\frac{1}{4}})} \|\sqrt{h}\partial_y H\|_{H^2_{\psi,0}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}} dt
\]

\[
\leq C\varepsilon \frac{3}{4} \int_0^T \langle t \rangle^{\beta - 2\gamma_0 - \frac{1}{4}} \|\sqrt{h}H\|_{H^2_{\psi,0}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}} dt.
\]

For any $\zeta > 0$, by applying Young's inequality, we obtain

\[
|B_2| \leq C\varepsilon \frac{3}{2} \zeta^{-1} \int_0^T \dot{\theta}(t) \|\sqrt{h}H\|_{H^2_{\psi,0}}^2 dt + \frac{\zeta}{3} \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}}^2 dt,
\]

if $\beta \leq \gamma_0 + \frac{1}{2}$.

- The estimate of $B_3$. Similar to the estimate of $B_2$, thanks to (4.2a), we deduce

\[
|B_3| \leq C\varepsilon \frac{1}{2} \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{1}{4}} \|\partial_y v\|_{L^2(H_{\psi,0}^{\frac{1}{4}})} \|\sqrt{h}H\|_{H^2_{\psi,0}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}} dt
\]

\[
\leq C\varepsilon \frac{3}{4} \int_0^T \langle t \rangle^{\beta - 2\gamma_0 - \frac{1}{4}} \|\sqrt{h}H\|_{H^2_{\psi,0}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}} dt
\]

\[
\leq C\varepsilon \frac{3}{4} \zeta^{-1} \int_0^T \dot{\theta}(t) \|\sqrt{h}H\|_{H^2_{\psi,0}}^2 dt + \frac{\zeta}{3} \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}}^2 dt,
\]

if $\beta \leq \gamma_0 + \frac{1}{2}$.

- The estimate of $B_4$. Similar to the estimate of $B_2$, we have

\[
|B_4| \leq C\varepsilon \frac{3}{2} \zeta^{-1} \int_0^T \dot{\theta}(t) \|\sqrt{h}H\|_{H^2_{\psi,0}}^2 dt + \frac{\zeta}{3} \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \|\sqrt{h}u_\Phi\|_{H^2_{\psi,0}}^2 dt,
\]

if $\beta \leq \gamma_0 + \frac{1}{4}$. 

By summarizing the estimates of $B_2, B_3, B_4$, we achieve

\begin{equation}
|B_2| + |B_3| + |B_4| \leq C \varepsilon^2 \zeta^{-1} \int_0^T \dot{\theta}(t) \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{2}}} \, dt + \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{2}}} \, dt,
\end{equation}

if $\beta \leq \gamma_0 + \frac{1}{4}$.

By inserting the estimates (5.4) and (5.6) into (5.3) and taking $\zeta$ to be sufficiently small, we arrive at

\begin{equation}
\varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{2}}} \, dt \leq 2 \zeta \|\sqrt{H}u_\Phi(T)\|^2_{H_{\psi,0}^{\frac{11}{2}}} + C \zeta^{-1} \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{2}}} + 2 \zeta \varepsilon \int_0^T \|\dot{\sqrt{H}}\|_{H_{\psi,0}^{\frac{11}{2}}} \, dt + 2 \varepsilon \int_0^T \|\dot{\sqrt{H}}\|_{H_{\psi,0}^{\frac{11}{2}}} \, dt + C(1 + \lambda \varepsilon + \varepsilon \zeta^{-1}) \times \\
\int_0^T \dot{\theta}(t) \left( \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{2}}} + \|\sqrt{H}\|^2_{H_{\psi,0}^{\frac{11}{2}}} + \|\dot{\sqrt{H}}\|_{H_{\psi,0}^{\frac{11}{2}}} \right) \, dt.
\end{equation}

By substituting (5.7) into (5.2) and taking $\zeta$ to be sufficiently small, we achieve (5.1) with $\eta = \frac{1}{2}$. This completes the proof of Proposition 5.1. □

Proposition 5.1 has been proved provided with the proof of Lemma 5.1, which we present now.

**Proof of Lemma 5.1.** Thanks to (4.22), we get, by taking $H_{\psi,0}^{\frac{11}{2}}$ inner product of (3.3) with $hu_\Phi$ and using integrating by parts, that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\sqrt{H}u_\Phi(t)\|^2_{H_{\psi,0}^{\frac{11}{0}}} - \frac{1}{2} \|\sqrt{H}u_\Phi(t)\|^2_{H_{\psi,0}^{\frac{11}{0}}} + \lambda \dot{\theta}(t) \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{0}}} + \frac{1}{2} \|\sqrt{H}\|_{H_{\psi,0}^{\frac{11}{0}}} \left( T_{D_xu}A(D_x)\partial_xu_\Phi \right)_{H_{\psi,0}^{\frac{11}{0}}} + \delta(t) \frac{1}{2} \|\langle T_{D_xu}A(D_x)\partial_xu_\Phi \rangle_{H_{\psi,0}^{\frac{11}{0}}} + \|f, u_\Phi\|_{H_{\psi,0}^{\frac{11}{0}}} \right) \defeq A_1 + \cdots + A_6.
\end{equation}

We first deduce from a similar argument of (4.6) that

\begin{align*}
A_1 & \leq \frac{h}{2} \left( \|\langle D_x \|_{L^2_{\psi}^{\frac{11}{0}}} u_\Phi, [T_{u}^{\frac{11}{2}}, (T_{u}^{\frac{11}{2}})^*] \partial_xu_\Phi \rangle_{L^2_{\psi}^{\frac{11}{0}}} + \|\langle D_x \|_{L^2_{\psi}^{\frac{11}{0}}} u_\Phi, (T_{u}^{\frac{11}{2}} - T_{u}^{\frac{11}{2}}) \partial_xu_\Phi \rangle_{L^2_{\psi}^{\frac{11}{0}}} \right),
\end{align*}

from which, Lemma 2.2 and (4.23), we infer

\begin{align*}
A_1 & \leq C \left( \|u\|_{L^2_{\psi}^{\frac{11}{0}}} + \|u\|^2_{L^\infty_{\psi}(H_{\psi,0}^{\frac{11}{0}})} \right) \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{0}}} + C \varepsilon \frac{\dot{\theta}(t)}{2} \|\sqrt{H}u_\Phi\|^2_{H_{\psi,0}^{\frac{11}{0}}} \leq C \varepsilon \frac{\dot{\theta}(t)}{2} \|u\|_{L^2_{\psi}^{\frac{11}{0}}},
\end{align*}

if $\beta \leq \gamma_0 + \frac{1}{4}$. 

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While for any $\zeta > 0$, it follows from Lemma 2.1 and (4.2) that if $\beta \leq \gamma_0 + \frac{1}{2}$,

$$A_2 \leq C \|v\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}\partial_y u\|_{H_h^{1/4},0} \|\sqrt{h}u\|_{H_h^{1/4},0}$$

$$\leq C \zeta^-\|v\|^2(t)-2\gamma_0\|\sqrt{h}u\|^2_{H_h^{1/4},0} + \frac{\zeta}{32} \|\sqrt{h}\partial_y u\|^2_{H_h^{1/4},0},$$

and

$$A_3 \leq C \|u\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}u\|^2_{H_h^{1/4},0} \leq C \zeta^-\|v\|^2(t)-\gamma_0\|\sqrt{h}u\|^2_{H_h^{1/4},0}.$$

On the other hand, due to $v = \int_0^\infty \partial_x u dz$, we observe from (5.5) that

$$\|v_\Phi\|_{L_\infty^\varepsilon(H_h^{1/2})} \leq C \langle t \rangle^\frac{1}{2} \|u_\Phi\|_{H_h^{1/4},0} \forall s \in \mathbb{R} \text{ and } \gamma \in [0, 1].$$

So that we deduce from Lemma 2.1 and 4.1 that

$$A_4 \leq C \|\partial_y u\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}v_\Phi\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}u_\Phi\|_{H_h^{4/5},0}$$

$$\leq C \zeta^{-}(\gamma_0+\frac{1}{2}) \|\sqrt{h}u_\Phi\|^2_{H_h^{4/5},0},$$

and

$$A_5 \leq C \|D_x \partial_y u\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}v_\Phi\|_{L_\infty^\varepsilon(H_h^{1/2})} \|\sqrt{h}u_\Phi\|_{H_h^{4/5},0}$$

$$\leq C \zeta^{-}(\gamma_0+\frac{1}{2}) \|\sqrt{h}u_\Phi\|^2_{H_h^{4/5},0}.$$
By substituting the above estimates into (5.8) and integrating the resulting inequality over \([0, T]\) for any \(T < T^*\), we achieve

\[
\frac{1}{2} \| \sqrt{h} u \Phi(T) \|_{H^s_\psi,0}^2 - \frac{1}{2} \int_0^T \| \sqrt{h} u \Phi(t) \|_{H^s_\psi,0}^2 + \lambda \int_0^T \dot{\theta}(t) \| \sqrt{h} u \Phi(t) \|_{H^s_\psi,0}^2 dt + \left( \frac{1}{2} - \frac{\zeta}{8} \right) \int_0^T \| \sqrt{h} \partial_y u \Phi(t) \|_{H^s_\psi,0}^2 dt' \\
\leq \| \sqrt{h}(0) u \Phi(0) \|_{H^s_\psi,0}^2 + C \varepsilon \int_0^T \langle t \rangle^{-1/2} \| \sqrt{h} u \Phi \|_{H^s_\psi,0}^2 dt + C (\xi^{-1} + 1) \varepsilon \int_0^T \dot{\theta}(t) \| \sqrt{h} u \Phi(t) \|_{H^s_\psi,0}^2 dt,
\]

if \(\beta \leq \gamma_0 + \frac{1}{4}\). This leads to (5.2). We thus complete the proof of Lemma 5.1.

5.2. The estimate of \(\partial_y u \Phi\). The purpose of this section is to derive the Gevery estimate of \(\partial_y u\). We first get, by taking \(\partial_y\) to the \(u\) equation of (1.2), that

\[
\partial_t \partial_y u + u \partial_x \partial_y u + v \partial_y^2 u - \partial_y^3 u = 0.
\]

Moreover, by virtue of the \(u\) equation of (1.2) and the boundary conditions that \(u|_{y=0} = v|_{y=0} = 0\), we obtain

\[
\partial_y^2 u|_{y=0} = 0.
\]

While by applying Bony’s decomposition to \(u \partial_x \partial_y u\) and \(v \partial_y^2 u\) in (5.11) in the horizontal variable, we write

\[
\begin{align*}
u \partial_y^2 u &= T_{y}^h \partial_y^2 u + R^h(y, \partial_y u), \\
u \partial_y^2 u &= T_{y}^h \partial_y^2 u + R^h(y, \partial_y^2 u).
\end{align*}
\]

Then inserting the above equalities into (5.11) and applying the operator \(e^{\Phi(t, D_x)}\) to resulting equality gives rise to

\[
\partial_t \partial_y u \Phi + T_{y}^h \partial_y \partial_y u \Phi + T_{y}^h \partial_y^2 u \Phi - \partial_y^3 u \Phi = \mathcal{J}, \quad \text{with}
\]

\[
\mathcal{J} = - \left( T_{y}^h \partial_y u \Phi + T_{y}^h \partial_y^2 u \Phi - (T_{y}^h \partial_y^2 u) \Phi - (T_{y}^h \partial_y u) \Phi \right)
\]

The main result states as follows:

**Proposition 5.2.** Let \(u\) be a smooth enough solution of (1.2) which decays to zero sufficiently fast as \(y\) approaching to \(\infty\). Let \(h(t)\) be a non-negative and non-decreasing function on \([0, T]\) with \(T \leq T^*\). Then if \(\beta \leq \gamma_0 + \frac{1}{4}\), one has

\[
\| \sqrt{h} \partial_y u \Phi(T) \|_{H^s_\psi,0}^2 + 2(\lambda - C \varepsilon \xi^2 (1 + \eta^{-1} \varepsilon)) \int_0^T \dot{\theta}(t) \| \sqrt{h} \partial_y u \Phi(t) \|_{H^s_\psi,0}^2 dt + \left( \frac{1}{2} - \frac{\zeta}{8} \right) \int_0^T \| \sqrt{h} \partial_y u \Phi(t) \|_{H^s_\psi,0}^2 dt' \\
\leq \| \sqrt{h}(0) e^{\Phi(D_x)} \|_{H^s_\psi,0}^2 + C \varepsilon \xi^2 (1 + \eta^{-1} \varepsilon) \int_0^T \dot{\theta}(t) \| \sqrt{h} \partial_y u \Phi(t) \|_{H^s_\psi,0}^2 dt,
\]

where \(\mathcal{J} = 0\).
Proof. Again thanks to (4.22), by taking $H^{5,0}_\psi$ inner product of (5.13) with $\partial_y u_\Phi$, we find
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2 - \frac{1}{2} \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2 + \lambda \dot{h}(t) \| \sqrt{h} \partial_y u_\Phi \|_{H^{21,0}_\psi}^2 + \frac{1}{2} \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2 \leq \langle \langle T^h_u \partial_x \partial_y u_\Phi, \sqrt{h} \partial_y u_\Phi \rangle \rangle_{H^{5,0}_\psi} + \langle \langle \partial_y u_\Phi, \sqrt{h} \partial_y u_\Phi \rangle \rangle_{H^{5,0}_\psi} + \langle \langle \partial_y u_\Phi, \sqrt{h} \partial_y u_\Phi \rangle \rangle_{H^{5,0}_\psi} \quad \text{def} = D_1 + D_2 + D_3.
\]

Let us now handle term by term in (5.15). We first get, by a similar derivation of (4.7),
\[
D_1 \leq C \left( \| u \|_{L^\infty(H^{3/4}_\psi)} + \| u \|_{L^\infty(H^{5/4}_\psi)} \right) \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2,
\]
which together with (3.12), (4.2a) and $\beta \leq \gamma_0 + \frac{1}{4}$ implies that
\[
D_1 \leq C \left( \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 - \frac{1}{4}} + \varepsilon \frac{1}{2} (t)^{\beta - 2\gamma_0 - \frac{1}{4}} \right) \dot{\theta}(t) \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2 \leq C \varepsilon \frac{1}{2} \dot{\theta}(t) \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi}^2.
\]

Whereas it follows from Lemma 2.1, 4.1 and (5.9) that
\[
D_2 \leq C \| \partial^2_y u \|_{H^{5/4}_\psi} \| \sqrt{h} u_\Phi \|_{L^\infty(H^{3/4}_\psi)} \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 - \frac{1}{4}} \dot{\theta}(t) \left( \| \sqrt{h}(t)^{-\frac{1}{2}} u_\Phi \|_{H^{12/5}_\psi} + \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi} \right) \leq C \varepsilon \frac{1}{2} \dot{\theta}(t) \left( \| \sqrt{h}(t)^{-\frac{1}{2}} u_\Phi \|_{H^{12/5}_\psi} + \| \sqrt{h} \partial_y u_\Phi \|_{H^{5,0}_\psi} \right),
\]
if $\beta \leq \gamma_0 + \frac{1}{4}$.

To estimate $D_3$, we first get, by applying Lemma 2.1 and (4.2b), that
\[
\| \sqrt{h} (T^h_u \partial_y u_\Phi) \|_{H^{12/5}_\psi} \leq C \| \partial_x \partial_y u_\Phi \|_{L^\infty(H^{3/4}_\psi)} \| \sqrt{h}(t)^{-\frac{1}{2}} u_\Phi \|_{H^{12/5}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 - \frac{1}{4}} \dot{\theta}(t) \| \sqrt{h}(t)^{-\frac{1}{2}} u_\Phi \|_{H^{12/5}_\psi},
\]
and
\[
\| \sqrt{h} (T^h_u \partial^2_y u_\Phi) \|_{H^{12/5}_\psi} \leq C \| u_\Phi \|_{L^\infty(H^{3/4}_\psi)} \| \sqrt{h} \partial^2_y u_\Phi \|_{H^{12/5}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 + \frac{1}{4}} \dot{\theta}(t) \| \sqrt{h} \partial^2_y u_\Phi \|_{H^{12/5}_\psi}.
\]

Along the same line, we have
\[
\| \sqrt{h} (T^h_u \partial_x \partial_y u) \|_{H^{12/5}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 - \frac{1}{4}} \dot{\theta}(t) \| \sqrt{h}(t)^{-\frac{1}{2}} u_\Phi \|_{H^{12/5}_\psi},
\]

\[
\| \sqrt{h} (T^h_v \partial^2_y u_\Phi) \|_{H^{12/5}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 + \frac{1}{4}} \dot{\theta}(t) \| \sqrt{h} \partial_y u_\Phi \|_{H^{12/5}_\psi}.
\]

While we deduce from Lemma 2.2 and (4.2a) that
\[
\| \sqrt{h} (T^h_u \partial_x \partial_y u_\Phi - \sqrt{h} T^h_u \partial_x \partial_y u_\Phi) \|_{H^{12/5}_\psi} \leq C \| u_\Phi \|_{L^\infty(H^{3/4}_\psi)} \| \sqrt{h} \partial_y u_\Phi \|_{H^{12/5}_\psi} \leq C \varepsilon \frac{1}{2} (t)^{\beta - \gamma_0 - \frac{1}{4}} \dot{\theta}(t) \| \sqrt{h} \partial_y u_\Phi \|_{H^{12/5}_\psi}.
\]
Whereas it follows from Lemma 2.24 (4.1) and (5.9) that
\[
\| \sqrt{h} (T_{\partial_y u}^{-1} v) - T_{\partial_y u}^{-1} v_{\Phi} \|_{H^{1/4}, 0} \leq C \| \partial_y u_{\Phi} \|_{H^{1/4}, 0} \| v_{\Phi} \|_{L^\infty_{\lambda, \gamma}(H^3)} \leq C \varepsilon \beta - \gamma_0 (t) \| \sqrt{h}(t) \|^{1/2} u_{\Phi} \|_{H^{1/4}, 0}.
\]

Therefore, if \( \beta \leq \gamma_0 + \frac{1}{4} \), in view of the definition of \( \beta \) given by (5.13), we achieve
\[
\| \sqrt{h} \|_{H^{1/4}, 0} \leq C \varepsilon \beta (t) \left( \| \sqrt{h}(t) \|^{1/2} u_{\Phi} \|_{H^{1/4}, 0} + \| \sqrt{h} \partial_y u_{\Phi} \|_{H^{1/4}, 0} \right) + \frac{2}{3} \beta (t) \| \sqrt{h} \partial_y u_{\Phi} \|_{H^{1/4}, 0},
\]
which implies that for any \( \eta > 0 \)
\[
D_3 \leq \| \sqrt{h} \|_{H^{1/4}, 0} \| \sqrt{h} \partial_y u_{\Phi} \|_{H^{1/4}, 0} \leq C \varepsilon \beta (1 + \eta^{-1} \varepsilon) \beta (t) \left( \| \sqrt{h}(t) \|^{1/2} u_{\Phi} \|_{H^{1/4}, 0} + \| \sqrt{h} \partial_y u_{\Phi} \|_{H^{1/4}, 0} \right) + \frac{2}{3} \beta (t) \| \sqrt{h} \partial_y u_{\Phi} \|_{H^{1/4}, 0}.
\]

By inserting the estimates (5.14), (5.17) and (5.18) into (5.13), we obtain (5.14). This completes the proof of Proposition 5.2. \( \square \)

5.3. The proof of Proposition 3.2

Now we are in a position to complete the proof of Proposition 3.2.

Proof of Proposition 3.2. Taking \( h(t) = \langle t \rangle^{1+\eta} \) in (5.1), (5.16) can be proved along the same line to that of Corollary 4.2.

While taking \( h = \langle t \rangle^{1+\eta} \) in (5.14) leads to
\[
\| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} + 2 \lambda (1 + \varepsilon \eta^{-1}) \int_0^T \hat{\theta}(t) \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
+ \left( 1 - \frac{\eta}{16} \right) \int_0^T \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
\leq \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} + C \varepsilon \beta (1 + \eta^{-1}) \int_0^T \hat{\theta}(t) \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
+ \frac{3 - \eta}{2} \int_0^T \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt.
\]

Yet it follows from (3.16) that
\[
\lambda \int_0^T \hat{\theta}(t) \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt + \int_0^T \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
\leq C \eta^{-1} \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} + C \eta^{-1} \varepsilon \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
+ \varepsilon \frac{1}{4} \int_0^T \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt + \varepsilon \frac{1}{4} \int_0^T \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} dt
\]
\[
+ C \eta^{-1} (1 + \varepsilon \eta^{-1}) \int_0^T \hat{\theta}(t) \left( \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} + \| \langle t \rangle \frac{\partial_y u_{\Phi}(T)}{\| H^{1/4}, 0} \right) dt.
\]

So that by taking
\[
\lambda \geq \lambda_2 \text{ def } C \left( \eta^{-1} (1 + \varepsilon \eta^{-1}) + 2 \varepsilon^2 \right) (1 + \eta^{-1}) \quad \text{and} \quad \varepsilon \leq \varepsilon_2 \quad \text{so that} \quad C \eta^{-1} \leq \frac{1}{4},
\]
we conclude the proof of (3.17) by inserting the above estimate into (5.19). This finished the proof of the proof of Proposition 3.2

6. The Gevery estimates of \(G\) and \(\partial_y G\)

6.1. The Gevery estimates of \(G\). The purpose of this Subsection is to present the decay-in-time estimate for the good quantity, \(G\), given by (1.3). As in the estimate of \(u\) and \(\partial_y u\), we get, by first applying Bony’s decomposition for the horizontal variable to \(u\partial_x G, v\partial_y G, v\partial_y (y\varphi)\) and \(\partial_y u\partial_x \varphi\), and then applying the operator, \(e^\Phi(t.D_x)\), to the equation (1.4), that

\[
\partial_t G\Phi + \lambda \theta(t)\lambda^{1/2}G\Phi - \partial_y^2 G\Phi + \langle t \rangle^{-1} G\Phi + T_h^h \partial_y G\Phi \\
+ T_h^h \partial_y Gv\Phi - \frac{1}{2\langle t \rangle} T_h^h \partial_y(y\varphi) \partial_y v\Phi + \frac{y}{\langle t \rangle} \int_y^\infty T_h^h \partial_y u v\Phi \, dy' = F,
\]

where

\[
-F \overset{\text{def}}{=} (T_u^h \partial_x G) \Phi - T_u^h \partial_x G\Phi + (T_{\partial_x G}^h u) \Phi + (R^h(u, \partial_x G)) \Phi \\
+ (T_{\partial_y G}^h v) \Phi - T_{\partial_y G}^h v\Phi \Phi + (T_v^h \partial_y G) \Phi + (R^h(v, \partial_y G)) \Phi,
\]

\[
-\frac{1}{2\langle t \rangle} \left( (T_u^h \partial_y(y\varphi) v) \Phi - T_u^h \partial_y(y\varphi) v\Phi \right) - \frac{1}{2\langle t \rangle} \left( T_v^h \partial_y(y\varphi) + R^h(\partial_y(y\varphi), v) \right) \Phi \\
+ \frac{y}{\langle t \rangle} \int_y^\infty \left( T_u^h \partial_y u v \Phi - T_{\partial_y u}^h v\Phi \right) \, dy' + \frac{y}{\langle t \rangle} \int_y^\infty \left( T_v^h \partial_y u + R^h(v, \partial_y u) \right) \Phi \, dy'.
\]

Let us first present the estimate for \(F\).

**Lemma 6.1.** It holds that

\[
\|F\|_{L^\infty H^{\frac{1}{2}}_\psi} \leq C \varepsilon \left( \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|G\|_{H^{\frac{1}{2}}_\psi} + \langle t \rangle^{-\gamma_0 + \frac{1}{2}} \|\partial_y G\|_{H^{\frac{1}{2}}_\psi} \right).
\]

**Proof.** Applying Lemma 2.3 and (4.2a) gives

\[
\|(T_u^h \partial_x G) \Phi - T_u^h \partial_x G\Phi\|_{L^\infty H^{\frac{1}{2}}_\psi} \leq C \|u\|_{L^\infty H^{\frac{3}{2}}_\psi} \|G\|_{H^{\frac{1}{2}}_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|G\|_{H^{\frac{1}{2}}_\psi}.
\]

While it follows from (3.9) and Lemma 2.6 that

\[
\|v\|_{L^\infty \psi (H^s_e)} \leq C \langle t \rangle^{\frac{1}{2}} \|u\|_{H^{s+1,0}_\psi} \leq C \langle t \rangle^{\frac{1}{2}} \|G\|_{H^{s+1,0}_\psi} \quad \forall \ s \in \mathbb{R},
\]

which together with Lemma 2.3 and (3.13) ensures that

\[
\|(T_{\partial_y G}^h v) \Phi - T_{\partial_y G}^h v\Phi\|_{L^\infty H^{\frac{1}{2}}_\psi} \leq C \|\partial_y G\|_{H^{\frac{3}{2}}_\psi} \|v\|_{L^\infty \psi (H^s_e)} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|G\|_{H^{\frac{1}{2}}_\psi}.
\]

Notice from (2.13c) and (3.13) that

\[
\langle t \rangle^{-1} \|\partial_y(y\varphi)\|_{H^{\frac{3}{2}}_\psi} \leq C \|\partial_y G\|_{H^{\frac{3}{2}}_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}},
\]

so that applying Lemma 2.3 and (6.4) yields

\[
\langle t \rangle^{-1} \|(T_{\partial_y(y\varphi)}^h v) \Phi - T_{\partial_y(y\varphi)}^h v\Phi\|_{L^\infty H^{\frac{1}{2}}_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \|G\|_{H^{\frac{1}{2}}_\psi}.
\]
Whereas applying Lemma 2.3 (4.11) and (6.3) gives

\[ \| \int_y^\infty ((T^h_{\partial_y}v) - T^h_{\partial_y}(v)\phi) \, dy' \|_{L^\infty(H^{1/8}_h,\omega)} \leq \| \int_y^\infty \| \partial_y u\phi \|_{H^{1/8}_h} \| v\phi \|_{H^{1/8}_h} \, dy' \|_{L^\infty} \]

\[ \leq C \| \partial_y u\phi \|_{L^2(\omega)} \| e^{\frac{2\Psi}{\gamma}} v\phi \|_{L^\infty(H^{1/8}_h,\omega)} \left( \int_y^\infty e^{-3\Psi} \, dy' \right)^{1/2} \]

\[ \leq C\varepsilon(t)^{-\gamma_0} e^{-\frac{3}{2}\Psi} \| G\phi \|_{H^{1/8}_h,\omega}, \]

from which, we infer

\[ \| y(t)^{-1} \int_y^\infty ((T^h_{\partial_y}v) - T^h_{\partial_y}(v)\phi) \, dy' \|_{H^{1/8}_h,\omega} \leq C\varepsilon(t)^{-\gamma_0} e^{-\frac{1}{2}\Psi} \| G\phi \|_{H^{1/8}_h,\omega}. \]

On the other hand, by applying Lemma 2.3 (5.13) and (6.23), we find

\[ \| (T^h_{\partial_y}G)\phi + (R^h(u, \partial_x G))\phi \|_{H^{1/8}_h,\omega} \leq C \| \partial_x G\phi \|_{L^\infty(H^{1/8}_h,\omega)} \| u\phi \|_{H^{1/8}_h,\omega} \]

\[ \leq C\varepsilon(t)^{-\gamma_0} \| G\phi \|_{H^{1/8}_h,\omega}, \]

\[ \| (T^h_{\partial_y}G)\phi + (R^h(v, \partial_y G))\phi \|_{H^{1/8}_h,\omega} \leq C \| v\phi \|_{L^\infty(H^{1/8}_h,\omega)} \| \partial_y G\phi \|_{H^{1/8}_h,\omega} \]

\[ \leq C\varepsilon(t)^{-\gamma_0} \| \partial_y G\phi \|_{H^{1/8}_h,\omega}, \]

\[ \| (t)^{-1} (T^h_{\partial_y}(y\phi) + R^h(\partial_y(y\phi), v))\phi \|_{H^{1/8}_h,\omega} \leq C \| v\phi \|_{L^\infty(H^{1/8}_h,\omega)} \| (t)^{-1} \partial_y(y\phi)\phi \|_{H^{1/8}_h,\omega} \]

\[ \leq C\varepsilon(t)^{-\gamma_0} \| \partial_y G\phi \|_{H^{1/8}_h,\omega}, \]

and

\[ \| \int_y^\infty (T^h_{v\partial_y}u + R^h(v, \partial_y u))\phi \, dy' \|_{L^\infty(H^{1/8}_h,\omega)} \leq \| \int_y^\infty \| \partial_y u\phi \|_{H^{1/8}_h} \| v\phi \|_{H^{1/8}_h} \, dy' \|_{L^\infty} \]

\[ \leq C \| \partial_y u\phi \|_{L^2(\omega)} \| e^{\frac{2\Psi}{\gamma}} v\phi \|_{L^\infty(H^{1/8}_h,\omega)} \left( \int_y^\infty e^{-3\Psi} \, dy' \right)^{1/2} \]

\[ \leq C\varepsilon(t)^{-\gamma_0} \| \partial_y G\phi \|_{H^{1/8}_h,\omega}, \]

from which, we infer

\[ \| y(t)^{-1} \int_y^\infty (T^h_{v\partial_y}u + R^h(v, \partial_y u)\phi) \, dy' \|_{H^{1/8}_h,\omega} \leq C\varepsilon(t)^{-\gamma_0} \| \partial_y G\phi \|_{H^{1/8}_h,\omega}. \]

In view of (6.2), we conclude the proof of (6.3) by summarizing the above estimates. \( \square \)
Proposition 6.1. Let \( h(t) \) be a non-negative and non-decreasing function on \([0, T]\) with \( T \leq T^* \). Let \( G \) be determined by \([2.3]\). Then if \( \beta \leq \gamma_0 - \frac{1}{12} \), for any \( \eta > 0 \), we have

\[
\frac{1}{2} \| \sqrt{h} G(t) \|_{H^4_\psi}^2 + \frac{1}{2} \int_0^T \| \sqrt{h} G(t) \|_{H^4_\psi}^2 dt + \int_0^T \langle t \rangle^{-1} \| \sqrt{h} G(t) \|_{H^4_\psi}^2 dt
\]

\[
+ \left( \lambda - C(1 + \eta^{-1}\varepsilon)^{\frac{3}{2}} \right) \int_0^T \langle t \rangle \| \sqrt{h} G(t) \|_{H^4_\psi}^2 dt
\]

\[
+ \left( \frac{1}{2} - \frac{\eta}{8} \right) \int_0^T \| \sqrt{h} \partial_y G(t) \|_{H^2_\psi}^2 dt
\]

\[
\leq \frac{1}{2} \| \sqrt{h}(0) G(0) \|_{H^4_\psi}^2 + C \varepsilon \frac{3}{2} \int_0^T \hat{\theta}(t) \| \sqrt{h}(t) \|_{H^4_\psi}^2 dt.
\]

Proof. Thanks to \((6.22)\), we first get, by taking \( H^4_\psi \) inner product of \((6.1)\) with \( hG \), that

\[
\frac{d}{dt} \| \sqrt{h} G(t) \|_{H^4_\psi}^2 + \| \sqrt{h} \partial_y G(t) \|_{H^2_\psi}^2 \]

\[
+ \lambda \hat{\theta}(t) \| \sqrt{h} G(t) \|_{H^4_\psi}^2 + \frac{1}{2} \| \sqrt{h} \partial_y G(t) \|_{H^2_\psi}^2
\]

\[
\leq h \langle T^h \partial_y G(t) \rangle_{H^4_\psi} + h \langle T^h \partial_y \partial_y G(t) \rangle_{H^4_\psi} + \frac{h}{2(t)} | \langle T^h \partial_y \partial_y u \rangle_{H^4_\psi} | + h \langle F(t) \rangle_{H^4_\psi} \]

\[
+ h \left( \frac{d}{dt} \right) \int_y \langle T \partial_y \partial_y G(t) dy \rangle_{H^4_\psi} + h \langle F(t) \rangle_{H^4_\psi} \]

Next, let us handle the estimates of \( I_i, i = 1, \ldots, 5 \), term by term.

- The estimate \( I_1 \). It follows from a similar derivation of \((4.11)\) that

\[
I_1 \leq C \left( \| u \|_{L^\infty_\psi(H^2_\psi)} + \| u \|_{L^\infty_\psi(H^2_\psi)} \right) \| \sqrt{h} G(t) \|_{H^4_\psi}^2
\]

\[
\leq C \varepsilon \frac{3}{2} \hat{\theta}(t) \| \sqrt{h} G(t) \|_{H^4_\psi}^2,
\]

if \( \beta \leq \gamma_0 + \frac{1}{4} \).

- The estimate \( I_2 \). By applying Lemma \([2.3]\) and \((6.4)\), we find

\[
I_2 \leq C \sqrt{h} \| \partial_y G \|_{L^2_\psi(H^2_\psi)} \| \partial_y \partial_y G \|_{L^\infty_\psi(H^2_\psi)} \| \sqrt{h} G \|_{H^4_\psi}^2
\]

\[
\leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2} + \frac{\beta}{4}} \| \sqrt{h} u \|_{H^4_\psi} \| \sqrt{h} \partial_y \partial_y G(t) \|_{H^4_\psi}.
\]

Yet it follows from interpolation inequality in the anisotropic Sobolev norm and Lemma \([2.6]\) that

\[
\| u \|_{H^4_\psi} \leq C \| u \|_{H^\frac{1}{4}_\psi} \| u \|_{H^\frac{1}{4}_\psi} \| u \|_{H^\frac{1}{4}_\psi} \leq C \langle t \rangle^{\frac{1}{4}} \| G \|_{H^\frac{1}{4}_\psi} \| \langle t \rangle^{-1} u \|_{H^\frac{1}{4}_\psi}^{\frac{1}{2}} \| \sqrt{h} G \|_{H^4_\psi}.
\]

As a result, it comes out

\[
I_2 \leq C \varepsilon \langle t \rangle^{-\gamma_0 + \frac{1}{12}} \| \sqrt{h} \langle t \rangle^{-1} u \|_{H^\frac{1}{4}_\psi} \| \sqrt{h} G \|_{H^4_\psi}^2
\]

\[
\leq C \varepsilon \frac{1}{2} \langle t \rangle^\beta - \gamma_0 + \frac{1}{12} \hat{\theta}(t) \left( \| \sqrt{h} \langle t \rangle^{-1} u \|_{H^4_\psi}^2 + \| \sqrt{h} G \|_{H^4_\psi}^2 \right).
\]
The estimate $I_3$. Applying Lemmas 2.1 and (4.1) gives

$$I_3 \leq C \| \langle t \rangle^{-1} \partial_y (y \varphi) \| L^2_{v, \frac{1}{2} \Psi} (H^4_{\hbar} \cap \frac{1}{2} \Psi) \| \sqrt{\hbar} u \| L^\infty_{v, \frac{1}{2} \Psi} (H^\frac{15}{4} \cap \frac{1}{4} \Psi) \| \sqrt{\hbar} G \| L^\infty_{v, \frac{1}{2} \Psi} (H^\frac{15}{4} \cap \frac{1}{4} \Psi),$$

from which and the estimate of $I_2$, we obtain

$$I_3 \leq C \varepsilon \langle t \rangle^{\frac{1}{2} (t) - \gamma_0 + \frac{1}{12} \hat{\theta} (t) \left( \| \sqrt{\hbar} (t) \langle t \rangle^{-1} u \|_{H^4_{\hbar} \cap \frac{1}{4} \Psi}^2 + \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}^2 \right).$$

Estimate $I_4$. It follows from Lemmas 2.1 and (4.1) that for any $s \in \mathbb{R}$

$$\left\| \int_0^\infty T_{\partial_y u} v \right\|_{L^\infty (H^s)} \leq \left\| \int_0^\infty \| \partial_y u \|_{H^1_{\hbar}} v \right\|_{L^\infty} \left( \int_0^\infty e^{-3 \Psi} \right)^{\frac{1}{2}}
$$

(6.9)

$$\leq C \| \partial_y u \|_{L^2_{v, \frac{1}{2} \Psi} (H^4_{\hbar} \cap \frac{1}{4} \Psi)} e^{\Psi} \| v \|_{L^\infty (H^s_{\hbar} \cap \frac{1}{4} \Psi)} \left( \int_0^\infty e^{-3 \Psi} \right)^{\frac{1}{2}}
$$

$$\leq C \varepsilon \langle t \rangle^{-\gamma_0} e^{\frac{1}{2} \Psi} \| u \|_{H^{1+s} \cap \frac{1}{4} \Psi},$$

from which and (6.8), we infer

$$I_4 \leq \hbar \| \frac{y}{\langle t \rangle} \| \int_0^\infty T_{\partial_y u} v \right\|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi} \| G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}
$$

$$\leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \| \frac{y}{\langle t \rangle} \|_{L^2_v} \| \sqrt{\hbar} u \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi} \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}
$$

$$\leq C \varepsilon \langle t \rangle^{\frac{1}{2} (t) - \gamma_0 + \frac{1}{12} \hat{\theta} (t) \left( \| \sqrt{\hbar} (t) \langle t \rangle^{-1} u \|_{H^4_{\hbar} \cap \frac{1}{4} \Psi}^2 + \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}^2 \right).$$

The estimate $I_5$. By applying Lemma (6.1) we find

$$I_5 \leq \| \sqrt{\hbar} F \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi} \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}
$$

$$\leq C \varepsilon \langle \langle t \rangle \rangle^{-\gamma_0 - \frac{1}{4}} \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi} + \langle \langle t \rangle \rangle^{-\gamma_0 + \frac{1}{2} \hat{\theta} (t) \left( \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}^2 \right) \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}}.$$

Then for any $\eta > 0$, we get, by applying Young’s inequality, that

$$I_5 \leq C \langle t \rangle^{\frac{1}{2} (t) - \gamma_0 + \frac{1}{4} + \eta^{-1} \varepsilon \langle t \rangle^{\beta - 2 \gamma_0 + \frac{1}{2}} \hat{\theta} (t) \left( \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}^2 \right) \| \sqrt{\hbar} G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}} + \eta \left( \| \sqrt{\hbar} \partial_y G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}^2 \right).$$

By inserting the above estimates into (6.1) and then integrating the resulting inequality over $[0, T]$, we conclude the proof of (6.6) in case that $\beta \leq \gamma_0 - \frac{1}{12}$.

6.2. The Gevery estimate of $\partial_y G$. Due to the loss of derivative in the $G$ equation (1.4), we shall derive the estimate of $\| \partial_y G \|_{H^\frac{1}{2} \cap \frac{1}{4} \Psi}$ instead of $\| \partial_y G \|_{H^\frac{15}{4} \cap \frac{1}{4} \Psi}$ as the estimate of $G \phi$ in the previous section. In order to do so, we first get, by applying $\partial_y$ to (6.1), that

$$\partial_y \partial_y G + \lambda \hat{\theta} (t) (D_x) \frac{1}{\langle t \rangle} \partial_y G \phi - \partial_y G + \langle \langle t \rangle \rangle^{-1} \partial_y G \phi$$

(6.10)

$$+ \partial_y \left( T_u \partial_x G \phi + T_{\partial_x G \phi} - \frac{1}{2} \langle \langle t \rangle \rangle \partial_y (y \varphi) v \phi + \frac{y}{\langle t \rangle} \int_{y} T_{\partial_y u} v \phi \right) = \partial_y F,$$

with $F$ being given by (6.2).
Moreover, in view of the \( G \) equation of \((3.4)\) and the boundary conditions that \( u\big|_{y=0} = v\big|_{y=0} = G\big|_{y=0} = 0 \), we deduce that

\[
(6.11) \quad \partial_y^2 G\big|_{y=0} = 0.
\]

**Proposition 6.2.** Let \( h(t) \) be a non-negative and non-decreasing function on \([0,T]\) with \( T \leq T^* \). Let \( G \) be determined by \((3.3)\). Then if \( \beta \leq 2\gamma_0 - \frac{1}{2} \), for any \( \eta > 0 \), we have

\[
\frac{1}{2} \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 + \frac{1}{2} \int_0^T \left( \langle t \rangle^{-1} \right) \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 \, dt - \frac{1}{2} \int_0^T \left\| \sqrt{h} \partial_y^2 G(t) \right\|_{H^3_\psi}^2 \, dt
\]

\[
+ \left( \lambda - C\eta^{-1} \frac{3}{2} \right) \int_0^T \frac{1}{2} (t) \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 \, dt + (1 - \frac{\eta}{8}) \int_0^T \left\| \sqrt{h} \partial_y^2 G(t) \right\|_{H^3_\psi}^2 \, dt
\]

\[
\leq \frac{1}{2} \left\| \sqrt{h} (0) \partial_y G(0) \right\|_{H^3_\psi}^2 + C\eta^{-1} \frac{3}{2} \int_0^T \frac{1}{2} \left( \sqrt{h}(t) \right)^{-\frac{1}{2}} G(t) \, dt.
\]

**Proof.** Notice that

\[
\partial_y^2 G\big|_{y=0} = F\big|_{y=0} = 0,
\]

we get, by taking \( H^3_\psi \) inner product of \((6.10)\) with \( h \partial_y G \) and using integration by parts, that

\[
\frac{1}{2} \frac{d}{dt} \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 - \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 + \langle t \rangle^{-1} \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2
\]

\[
+ \lambda \left( \sqrt{h}(t) \right) \left\| \sqrt{h} \partial_y G(t) \right\|_{H^3_\psi}^2 + \frac{1}{2} \left\| \sqrt{h} \partial_y^2 G(t) \right\|_{H^3_\psi}^2
\]

\[
\leq h \left\| \langle T^y_u \partial_y G, \partial_y^2 G \rangle \right\|_{H^3_\psi}^2 + h \left\| \langle T^y_v \partial_y G, \partial_y^2 G \rangle \right\|_{H^3_\psi}^2 + \frac{h}{2} \left( \langle T^y_v \partial_y G, \partial_y^2 G \rangle \right)_{H^3_\psi}^2
\]

\[
+ h \left( \langle T^y_v \partial_y G, \partial_y^2 G \rangle \right)_{H^3_\psi}^2 + h \left( \langle F, \partial_y^2 G \rangle \right)_{H^3_\psi}^2 \defeq II_1 + \cdots + II_5.
\]

Next let us deal with the estimates of \( II_i \), \( i = 1, \ldots, 5 \), term by term.

- **The estimate of \( II_1 \).** It follows from Lemma \((2.1)\) and \((4.2a)\) that

\[
\left\| \langle T^y_u \partial_y G, \partial_y^2 G \rangle \right\|_{H^3_\psi}^2 \leq C \left\| u \right\|_{L^\infty(H^\frac{1}{2}_u)} \left\| G \right\|_{H^4_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \left\| G \right\|_{H^4_\psi},
\]

so that for any \( \eta > 0 \), one has

\[
II_1 \leq \left\| \sqrt{h} \partial_y G \right\|_{H^3_\psi} \left\| \sqrt{h} \right\|_{H^3_\psi}^2 \left\| G \right\|_{H^4_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \left\| G \right\|_{H^4_\psi}.
\]

- **The estimate of \( II_2 \).** Applying Lemma \((2.1)\) \((3.13)\) and \((5.9)\) gives

\[
\left\| T^y_v \partial_y G \right\|_{H^3_\psi} \leq C \left\| \partial_y G \right\|_{L^\infty(H^\frac{1}{2} \psi)} \left\| \partial_y G \right\|_{L^\infty(H^\frac{1}{2} \psi)} \left\| G \right\|_{H^4_\psi}
\]

\[
\leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \left\| u \right\|_{H^4_\psi} \leq C \varepsilon \langle t \rangle^{-\gamma_0 - \frac{1}{2}} \left\| G \right\|_{H^4_\psi}.
\]
Then we get, by a similar estimate of $II_1$, that
\[
II_2 \leq \|\sqrt{h} T \partial_x G \Phi \|_{H^3_\psi} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi} \\
\leq \frac{\eta}{40} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}^2 + C \eta^{-1} \varepsilon^{\frac{1}{2}} \langle t \rangle^{\beta - 2\gamma_0 + \frac{1}{2}} \dot{\theta}(t) \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi}^2.
\]

- The estimate of $II_3$. Thanks to (6.5), we deduce by a similar estimate of $II_2$ that
\[
II_3 \leq \frac{\eta}{40} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}^2 + C \eta^{-1} \varepsilon^{\frac{1}{2}} \langle t \rangle^{\beta - 2\gamma_0 + \frac{1}{2}} \dot{\theta}(t) \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi}^2.
\]

- The estimate of $II_4$. Applying (6.9) with $s = 3$ gives
\[
II_4 \leq h \left\| \frac{y}{\langle t \rangle^{\frac{1}{2}}} \int_{y_t}^\infty \partial_{\nu_{\Phi}} d\langle y \rangle_{H^3_\psi} \right\|_{H^3_\psi} \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}.
\]
Applying Young's inequality yields
\[
II_4 \leq \frac{\eta}{40} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}^2 + C \eta^{-1} \varepsilon^{\frac{1}{2}} \langle t \rangle^{\beta - 2\gamma_0 + \frac{1}{2}} \dot{\theta}(t) \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi}^2.
\]

- The estimate of $II_5$. We get, by applying Lemma 6.1 that
\[
II_5 \leq \|F\|_{H^3_\psi} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi} \leq C \varepsilon \left( \langle t \rangle^{-\gamma_0 + \frac{1}{2}} \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi} + \langle t \rangle^{-\gamma_0 + \frac{1}{2}} \|\sqrt{h} \partial_y G \Phi \|_{H^3_\psi} \right) \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}.
\]
Applying Young's inequality gives rise to
\[
II_5 \leq \frac{\eta}{40} \|\sqrt{h} \partial_y^2 G \Phi \|_{H^3_\psi}^2 + C \eta^{-1} \varepsilon^{\frac{1}{2}} \langle t \rangle^{\beta - 2\gamma_0 + \frac{1}{2}} \dot{\theta}(t) \left( \|\sqrt{h} \langle t \rangle^{-\frac{1}{2}} G \Phi \|_{H^4_\psi}^2 + \|\sqrt{h} \partial_y G \Phi \|_{H^3_\psi}^2 \right).
\]
By substituting the above estimates into (6.13) and integrating the resulting inequality over $[0, T]$ leads to (6.12) in case that $\beta \leq 2\gamma_0 - \frac{1}{2}$. This completes the proof of Proposition 6.2.

6.3. The proof of Proposition 3.3

Proof of Proposition 3.3. We first deduce from Lemma 2.5 that
\[
\frac{5 - \eta}{4} \langle t \rangle^{-\frac{5}{4}} \|G \Phi(t)\|_{H^4_\psi}^2 \leq \langle t \rangle^{-2} \langle t \rangle^{-\frac{5}{4}} \|G \Phi(t)\|_{H^4_\psi}^2 + \left( \frac{1}{2} - \frac{\eta}{2} \right) \langle t \rangle^{-\frac{5}{4}} \|\partial_y G \Phi(t)\|_{H^3_\psi}^2.
\]
Then taking $h = \langle t \rangle^{-\frac{5}{4}}$ in (6.6) leads to
\[
\|\langle t \rangle^{-\frac{5}{4}} G \Phi(T)\|_{H^4_\psi}^2 + 2 \left( \lambda - C \varepsilon^{\frac{1}{2}} (1 + \eta^{-1} \varepsilon) \right) \int_0^T \dot{\theta}(t) \langle t \rangle^{-\frac{5}{4}} G \Phi(t) \|_{H^4_\psi}^2 \, dt \\
+ \frac{3}{4} \eta \int_0^T \|\langle t \rangle^{-\frac{5}{4}} \partial_y G \Phi(t)\|_{H^3_\psi}^2 \, dt \\
\leq \|G \Phi(0)\|_{H^4_\psi}^2 + C \varepsilon^{\frac{1}{2}} \int_0^T \dot{\theta}(t) \langle t \rangle^{-\frac{5}{4}} u \|_{H^{\frac{13}{8}}_\psi}^2 \, dt.
\]
By taking $\lambda \geq 2C \varepsilon^{\frac{1}{2}} (1 + \eta^{-1} \varepsilon)$ in (6.14), we achieve (3.18).
While by taking \( h(t) = \langle t \rangle \frac{\tau_n \eta}{2} \) in (6.11), we find
\[
\| (T) \frac{\tau_n \eta}{2} \partial_y G_\Phi(T) \|^2_{H_y^3,0} + 2(\lambda - C \eta^{-1} \varepsilon \frac{1}{2}) \int_0^T \dot{\theta}(t) \| \langle t \rangle \frac{\tau_n \eta}{2} \partial_y G_\Phi(t) \|^2_{H_y^3,0} dt
\]
(6.15)
\[+ (1 - \frac{\eta}{2}) \int_0^T \| \langle t \rangle \frac{\tau_n \eta}{2} \partial_y^2 G_\Phi(t) \|^2_{H_y^3,0} dt \leq \| \partial_y G_\Phi(0) \|^2_{H_y^3,0} + C \eta^{-1} \varepsilon \frac{1}{2} \int_0^T \dot{\theta}(t) \| \langle t \rangle \frac{\tau_n \eta}{2} \partial_y G_\Phi(t) \|^2_{H_y^3,0} dt.
\]
Yet if \( \lambda \geq 2C \varepsilon \frac{1}{2} (1 + \varepsilon \eta^{-1}) \), we deduce from (3.18) that
\[
C \eta^{-1} \varepsilon \frac{1}{2} \int_0^T \dot{\theta}(t) \| \langle t \rangle \frac{\tau_n \eta}{2} \partial_y G_\Phi(t) \|^2_{H_y^3,0} dt + C \int_0^T \| \langle t \rangle \frac{\tau_n \eta}{2} \partial_y G_\Phi(t') \|^2_{H_y^3,0} dt
\]
\[\leq C \eta^{-1} \| G_\Phi(0) \|^2_{H_y^3,0} + C \eta^{-1} \varepsilon \frac{1}{2} \int_0^T \dot{\theta}(t) \| \langle t \rangle \frac{\tau_n \eta}{2} u \|^2_{H_y^3,0} dt,
\]
Substituting the above estimate into (6.15) leads to (3.19). This completes the proof of Proposition 3.3.

7. The Sobolev norm estimates of \( \partial_y^2 G \) and \( \partial_y^3 G \)

7.1. The Sobolev norm estimates of \( \partial_y^2 G \). In this subsection, we shall deal with the Sobolev norm estimate of \( \partial_y^2 G \). We first get, by taking \( \partial_y^2 \) on (1.4), that
\[
\partial_t \partial_y^2 G - \partial_y^4 G + \langle t \rangle^{-1} \partial_y^2 G
\]
(7.1)
\[+ \partial_y^2 \left( u \partial_x G + v \partial_y G - \frac{1}{2} \langle t \rangle^{-1} v \partial_y (y \varphi) + \frac{y}{\langle t \rangle} \int_y^{\infty} (\partial_y u \partial_x \varphi) dy' \right) = 0.
\]
It is easy to observe that
\[
\partial_y \left( u \partial_x G + v \partial_y G - \frac{1}{2} \langle t \rangle^{-1} v \partial_y (y \varphi) + \frac{y}{\langle t \rangle} \int_y^{\infty} (\partial_y u \partial_x \varphi) dy' \right)
\]
(7.2)
\[= u \partial_x \partial_y G + v \partial_y^2 G + \partial_y u \partial_x G + \partial_y v (\partial_y G - \frac{1}{2} \langle t \rangle \partial_y (y \varphi))
\]
\[- \frac{1}{2} \langle t \rangle \partial_y^2 (y \varphi) - \frac{y}{\langle t \rangle} (\partial_y u \partial_x \varphi) + \frac{1}{\langle t \rangle} \int_y^{\infty} (\partial_y u \partial_x \varphi) dy' = \delta .
\]

We first present the estimate of \( \delta \).

Lemma 7.1. Let \( \delta \) be given by (7.2). Then for \( t \leq T^* \), one has
\[
\| \delta \|_{H_y^3,0} \leq C \varepsilon \langle t \rangle^{-7/4} \left( \langle t \rangle^{-1} \partial_y G_\Phi \|_{H_y^3,0} + \langle t \rangle^{-1} \| G_\Phi \|_{H_y^3,0} \right).
\]

Proof. We shall frequently use the following classical law of product in Sobolev space:
\[
\| a b \|_{H_h^s} \leq C \| a \|_{H_h^s} \| b \|_{H_h^s} \quad \forall \ s > \frac{1}{2}.
\]
Moreover, due to \( \Phi(t, \xi) \geq \frac{\delta}{2} \) for any \( k \in \mathbb{N} \) and \( t \leq T^* \), we have
\[
\| a \|^2_{H_h^{k+s}} = \int_{\mathbb{R}} \langle \xi \rangle^{2(k+s)} | \hat{a}(\xi) |^2 d \xi \leq C \int_{\mathbb{R}} e^{2\Phi(t, \xi)} \langle \xi \rangle^{2s} | \hat{a}(\xi) |^2 d \xi = C \| a \|^2_{H_h^s}.
\]
It follows from a similar derivation of (1.2a) that

\[(7.6) \quad \|u(t)\|_{L^\infty(H^0_\psi)} \leq C\langle t \rangle^{\frac{1}{2}} \|\partial_y G(t)\|_{H^{3,0}_\psi} \quad \forall \sigma \in \mathbb{R},\]

from which, (7.4), (7.5) and (3.13), we infer

\[
\|u\partial_x \partial_y G\|_{H^{3,0}_\psi} \leq C\|u\|_{L^\infty(H^0_\psi)} \|\partial_x \partial_y G\|_{H^{3,0}_\psi} \\
\leq C\langle t \rangle^{\frac{1}{2}} \|\partial_y G\|_{H^{3,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|\partial_y G\|_{H^{3,0}_\psi}.
\]

Similarly we deduce from (6.3) and (3.13) that

\[
\|u\partial_y^2 G\|_{H^{3,0}_\psi} \leq C\|v\|_{L^\infty(H^0_\psi)} \|\partial_y^2 G\|_{H^{3,0}_\psi} \\
\leq C\langle t \rangle^{\frac{1}{2}} \|G\|_{H^{4,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|G\|_{H^{4,0}_\psi}.
\]

Whereas it follows from (2.13a) and (6.3) that

\[
\|\langle t \rangle^{-1}u\partial_y^2(y\varphi)\|_{H^{3,0}_\psi} \leq C\|v\|_{L^\infty(L^{\frac{1}{2}}_\psi)} \|\langle t \rangle^{-1} \partial_y^2(y\varphi)\|_{H^{3,0}_\psi} \\
\leq C\langle t \rangle^{-\frac{3}{2}} \|G\|_{H^{4,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|G\|_{H^{4,0}_\psi}.
\]

Due to \(\partial_y v = -\partial_x u\), we get, by applying (3.13) and (7.4), (7.5), (2.13a), that

\[
\|\partial_y v \partial_y G\|_{H^{3,0}_\psi} \leq C\|\partial_x u\|_{L^\infty(H^0_\psi)} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\langle t \rangle^{\frac{1}{2}} \|\partial_y G\|_{H^{3,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \\
\leq C\langle t \rangle^{\frac{1}{2}} \|\partial_y G\|_{H^{3,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|\partial_y G\|_{H^{3,0}_\psi},
\]

and

\[
\langle t \rangle^{-1} \|\partial_y v \partial_y (y\varphi)\|_{H^{3,0}_\psi} \leq C\|\partial_x u\|_{L^\infty(L^{\frac{1}{2}}_\psi)} \|\langle t \rangle^{-1} \partial_y (y\varphi)\|_{H^{3,0}_\psi} \\
\leq C\langle t \rangle^{\frac{1}{2}} \|\partial_y G\|_{H^{3,0}_\psi} \|\partial_y G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|\partial_y G\|_{H^{3,0}_\psi}.
\]

While it follows from (1.2b) that

\[
\|\partial_y u \partial_x G\|_{H^{3,0}_\psi} \leq C\|\partial_y u\|_{L^\infty(H^0_\psi)} \|\partial_x G\|_{H^{3,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|G\|_{H^{4,0}_\psi}.
\]

Finally notice that \(v = \partial_x \varphi\), we deduce from (6.3) that

\[
\|\frac{y}{\langle t \rangle} (\partial_y u \partial_x \varphi)\|_{H^{3,0}_\psi} \leq C\|\frac{y}{\langle t \rangle} \partial_y u\|_{H^{3,0}_\psi} \|v\|_{L^\infty(L^{\frac{1}{2}}_\psi)} \\
\leq C\langle t \rangle^{-\frac{3}{2}} \|\partial_y u\|_{H^{3,0}_\psi} \|G\|_{H^{4,0}_\psi} \leq C\varepsilon \langle t \rangle^{-\frac{7}{2}} \|G\|_{H^{4,0}_\psi},
\]

and

\[
\|\frac{1}{\langle t \rangle} \int_y^\infty (\partial_y u \partial_x \varphi) \ dy\|_{H^{3,0}_\psi} \leq C\langle t \rangle^{-\frac{3}{2}} \|e^{-\frac{1}{2}t} \|L^2_\psi \|\partial_y u\|_{H^{3,0}_\psi} \|v\|_{L^\infty(L^{\frac{1}{2}}_\psi)} \\
\leq C\langle t \rangle^{-\frac{7}{2}} \|G\|_{H^{4,0}_\psi}.
\]

In view of (7.2), we conclude the proof of (7.11) by summarizing the above estimates.
Proposition 7.1. Let $h(t)$ be a non-negative and non-decreasing function on $[0, T]$ with $T \leq T^*$. Then under the assumption that $\beta \leq 2\gamma_0 - \frac{1}{2}$, for any $\eta > 0$, there exists $C$ so that

$$||\sqrt{h}\partial_y^3 G(T)||_{H^3_y}^2 + 2\int_0^T \langle t \rangle^{-\frac{1}{2}} ||\sqrt{h}\partial_y^3 G(t)||_{H^3_y}^2 \, dt + \left(1 - \frac{\eta}{4}\right) \int_0^T ||\sqrt{h}\partial_y^3 G(t)||_{H^3_y}^2 \, dt$$

(7.7)

$$\leq ||\sqrt{h}(0)\partial_y^3 G(0)||_{H^3_y}^2 + \int_0^T ||\sqrt{h'}\partial_y^3 G(t)||_{H^3_y}^2 \, dt$$

$$+ C\eta^{-\frac{1}{2}} \int_0^T \theta(t) \left(||\sqrt{h}(t)^{\frac{1}{2}}\partial_y G\Phi||_{H^3_y}^2 + ||\sqrt{h}(t)^{-\frac{1}{2}}G\Phi||_{H^3_y}^2\right) \, dt.$$  

Proof. Thanks to (7.2) and (6.11), we get, by taking $H_y^{3,0}$ inner product of (7.1) with $h\partial_y^3 G$ and using integration by parts, that

$$\frac{1}{2} \frac{d \langle t \rangle^{-\frac{1}{2}} ||\sqrt{h}\partial_y^3 G(t)||_{H^3_y}^2}{dt} - \frac{1}{2} \langle t \rangle^{-\frac{1}{2}} ||\sqrt{h'}\partial_y^3 G(t)||_{H^3_y}^2 + \langle t \rangle^{-1} ||\sqrt{h} \partial_y^3 G(t)||_{H^3_y}^2$$

(7.8)

$$+ \frac{1}{2} ||\sqrt{h} \partial_y^3 G||_{H^3_y}^2 \leq \langle t \rangle \langle \xi \rangle \langle \partial_y G \rangle_{H^3_y}.$$

Yet it follows from Lemma 7.11 that

$$h \langle \xi \rangle \langle \partial_y G \rangle_{H^3_y} \leq ||\sqrt{h} \xi||_{H^3_y}^2 ||\sqrt{h'} \partial_y^3 G||_{H^3_y}$$

$$\leq \frac{\eta}{8} ||\sqrt{h}\partial_y^3 G||_{H^3_y}^2 + C\eta^{-\frac{1}{2}} \langle t \rangle^{\beta-2\gamma_0+\frac{1}{2}} \theta(t) \left(||\sqrt{h}(t)^{-\frac{1}{2}}\partial_y G\Phi||_{H^3_y}^2 + ||\sqrt{h}(t)^{-1}G\Phi||_{H^3_y}^2\right).$$

By substituting the above inequality into (7.8), we conclude the proof of (7.7). □

7.2. The estimates of $\partial_y^3 G$. To get the decay estimate of $||\partial_y^2 u||_{L^\infty (H^{1,0}_y)}$, which has been used in the proof of Lemma 4.3 (see (A.3)), according to the proof of Lemma 4.1, we need the estimates of $||\partial_y^3 G\Phi||_{H^{3,0}_y}$. To do it, we shall present the estimates of $||\partial_y^3 G||_{H^{3,0}_y}$ in this subsection (compared with $||\partial_y^2 G||_{H^{3,0}_y}$).

We first get, by taking $\partial_y$ to (7.1), that

$$\partial_t \partial_y^3 G - \partial_y^3 G + \langle t \rangle^{\beta-1} \partial_y^3 G + \partial_y \xi = 0.$$  

(7.9)

It is easy to observe from (7.2) that

$$\partial_y \xi = u \partial_x \partial_y^2 G + v \partial_y^3 G + 2\partial_y u \partial_x \partial_y G + \partial_y^2 u \partial_x G + \partial_y^3 v (\partial_y G)$$

$$- \frac{1}{2\langle t \rangle} \partial_y (y\varphi) + \partial_y v (2\partial_y^2 G - \frac{1}{\langle t \rangle} \partial_y^3 (y\varphi)) - \frac{1}{2\langle t \rangle} v \partial_y^3 (y\varphi)$$

$$- \frac{2}{\langle t \rangle} (\partial_y u \partial_x \varphi) - \frac{y}{\langle t \rangle} (\partial_y^2 u \partial_x \varphi + \partial_y u \partial_x \partial_y \varphi).$$

(7.10)

We first present the estimate of $\partial_y \xi$.

Lemma 7.2. Let $\partial_y \xi$ be given by (7.10). Then for $t < T^*$, one has

$$||\partial_y \xi||_{H^{3,0}_y} \leq C\varepsilon(t)^{\gamma_0+\frac{1}{2}} \left(||\langle t \rangle^{-\frac{1}{2}} \partial_y^2 G||_{H^{3,0}_y} + ||\langle t \rangle^{-1} \partial_y G\Phi||_{H^{3,0}_y} + ||\langle t \rangle^{-\frac{3}{2}} G\Phi||_{H^{3,0}_y}\right).$$

Proof. The proof of this lemma follows the same line as that of Lemma 7.1. We leave the technical details to the interested readers. □
Proposition 7.2. Let \( h(t) \) be a non-negative and non-decreasing function on \([0, T]\) with \( T \leq T^* \). Then under the assumption that \( \beta \leq 2\gamma_0 - \frac{1}{2} \), for any \( \eta > 0 \), there exists \( C \) so that

\[
\|\sqrt{h}\partial_y^2 G(T)\|_{H^{2,0}_\psi}^2 + 2 \int_0^T \langle t \rangle^{-1}\|\sqrt{h}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt \\
+ \left( 1 - \frac{\eta}{4} \right) \int_0^T \|\sqrt{h}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt \leq \|\sqrt{h}(0)\partial_y^2 G(0)\|_{H^{2,0}_\psi}^2 \\
+ \int_0^T \|\sqrt{h}\partial^2_y G(t)\|_{H^{2,0}_\psi}^2 dt + C\eta^{-1}\epsilon^2 \int_0^T \|\sqrt{h}(t)^{-1}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt \\
+ C\eta^{-1}\epsilon^2 \int_0^T \hat{\theta}(t) \left( \|\sqrt{h}(t)^{-1}\partial_y G\|_{H^{3,0}_\psi}^2 + \|\sqrt{h}(t)^{-1/2} G\|_{H^{4,0}_\psi}^2 \right) dt.
\]

(7.12)

Proof. Thanks to equation (7.1) and \( \partial_y^2 G|_{y=0} = 0 \), we find

\[
(7.13)
(-\partial_y^4 G + \partial_y \frac{\delta G}{\delta y})|_{y=0} = 0.
\]

Then thanks to (4.22), we get, by taking \( H^{2,0}_\psi \) inner product of (7.9) with \( h\partial_y^2 G \) and using integration by parts, that

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{h}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 - \frac{1}{2} \|\sqrt{h}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 + \langle t \rangle^{-1}\|\sqrt{h}\partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 \\
+ \frac{1}{2} \|\sqrt{h}\partial_y^4 G\|_{H^{2,0}_\psi}^2 \leq h|\langle \partial_y \Psi, \partial^4_y G \rangle|_{H^{2,0}_\psi}^2.
\]

(7.14)

Yet it follows from Lemma 7.2 that

\[
h|\langle \partial_y \Psi, \partial^4_y G \rangle|_{H^{2,0}_\psi}^2 \leq \|\sqrt{h}\partial_y \Psi\|_{H^{2,0}_\psi}^2 \|\sqrt{h}\partial_y G\|_{H^{2,0}_\psi}^2 \\
\leq \frac{\eta}{8} \|\sqrt{h}\partial_y^4 G\|_{H^{2,0}_\psi}^2 + C\eta^{-1}\epsilon^2 \langle t \rangle^{-2\gamma_0 + \frac{1}{2}} \|\sqrt{h}(t)^{-1}\partial_y^2 G\|_{H^{3,0}_\psi}^2 \\
+ C\eta^{-1}\epsilon^2 \langle t \rangle^{\beta - 2\gamma_0 + \frac{1}{2}} \hat{\theta}(t) \left( \|\sqrt{h}(t)^{-1}\partial_y G\|_{H^{3,0}_\psi}^2 + \|\sqrt{h}(t)^{-1/2} G\|_{H^{4,0}_\psi}^2 \right).
\]

By substituting the above inequality into (7.14) and integrating the resulting inequality over \([0, T]\), we conclude the proof of (7.12). \( \square \)

7.3. The proof of Proposition 3.4

Proof of Proposition 3.4. Taking \( h(t) = \langle t \rangle^{\frac{2}{2-\gamma}} \) in (7.7) leads to

\[
\|\langle t \rangle^{\frac{2-\gamma}{2}} \partial_y^2 G(T)\|_{H^{2,0}_\psi}^2 + \left( 1 - \frac{\eta}{4} \right) \int_0^T \|\langle t \rangle^{\frac{2-\gamma}{2}} \partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt \\
\leq \|\partial_y^2 G(0)\|_{H^{4,0}_\psi}^2 + \frac{9 - \eta}{2} \int_0^T \|\langle t \rangle^{\frac{5-\gamma}{4}} \partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt \\
+ C\eta^{-1}\epsilon^2 \int_0^T \hat{\theta}(t) \left( \|\langle t \rangle^{\frac{5-\gamma}{4}} \partial_y G\|_{H^{3,0}_\psi}^2 + \|\langle t \rangle^{\frac{3-\gamma}{4}} G\|_{H^{4,0}_\psi}^2 \right) dt.
\]

While it follows from Proposition 3.3 that if \( \lambda \geq C\epsilon^2 (1 + \epsilon^{-1}) \)

\[
C\eta^{-1}\epsilon^2 \int_0^T \hat{\theta}(t) \left( \|\langle t \rangle^{\frac{5-\gamma}{4}} \partial_y G\|_{H^{3,0}_\psi}^2 + \|\langle t \rangle^{\frac{3-\gamma}{4}} G\|_{H^{4,0}_\psi}^2 \right) dt + C \int_0^T \|\langle t \rangle^{\frac{3-\gamma}{2}} \partial_y^2 G(t)\|_{H^{2,0}_\psi}^2 dt' \\
\leq C\eta^{-1}\|\langle G(0)\|_{H^{4,0}_\psi}^2 + \|\partial_y G(0)\|_{H^{4,0}_\psi}^2 \right) + C\eta^{-1}\epsilon^2 \int_0^T \hat{\theta}(t) \|\langle t \rangle^{\frac{1-\gamma}{2}} u\|_{H^{2,0}_\psi}^2 dt',
\]

(7.15)
which leads to (3.20).

While taking $h(t) = \langle t \rangle^\frac{11-n}{4}$ in (7.12) leads to

$$
\| \langle t \rangle^\frac{11-n}{4} \partial^2_y G(t) \|^2_{H^3_\psi} + \left(1 - \frac{\eta}{4}\right) \int_0^T \| \langle t \rangle^\frac{11-n}{4} \partial^3_y G(t) \|^2_{H^2_\psi} dt \\
\leq \| \partial^2_y G(0) \|^2_{H^2_\psi} + \frac{11 - \eta}{2} \int_0^T \| \langle t \rangle^\frac{9-n}{4} \partial^2_y G(t) \|^2_{H^3_\psi} dt \\
+ C\eta^{-\frac{3}{2}} \int_0^T \| \langle t \rangle^\frac{7-n}{4} \partial^2_y G(t) \|^2_{H^3_\psi} dt \\
+ C\eta^{-\frac{1}{2}} \int_0^T \| \langle t \rangle^\frac{3-n}{4} \partial^2_y G(t) \|^2_{H^3_\psi} dt
$$

(7.15)

While if $\lambda \geq C\varepsilon^\frac{1}{2} (1 + \varepsilon \eta^{-1})$ and $\eta \geq \varepsilon^2$, it follows from (3.15), (3.19) and (3.20) that

$$
C\eta^{-\frac{3}{2}} \int_0^T \partial(t) \| \langle t \rangle^\frac{7-n}{4} \partial_y G(t) \|^2_{H^3_\psi} dt + C\eta^\frac{3}{2} \int_0^T \| \langle t \rangle^\frac{7-n}{4} \partial_y G(t) \|^2_{H^3_\psi} dt
$$

$$
\leq C\eta^{-\frac{1}{2}} \| G(0) \|^2_{H^1_\psi} + \| \partial_y G(0) \|^2_{H^3_\psi} + \| \partial^2_y G(0) \|^2_{H^3_\psi} \\
+ C\eta^{-\frac{3}{2}} \int_0^T \| \langle t \rangle^\frac{3-n}{4} u(t) \|^2_{H^2_\psi} dt.
$$

By inserting the above estimate into (7.15), we achieve (3.21). This completes the proof of Proposition 3.3.

APPENDIX A. THE PROOF OF LEMMA 4.4

The goal of this section is to present the proof of Lemma 4.4.

**Proof of Lemma 4.4.** We decompose the proof of this lemma into the following steps:

- **The estimate of $E_4$.** We first observe from (3.12) that
  $$
  \partial_t \left( \frac{1}{\theta} \right) = \beta e^{-\frac{1}{2}} \langle t \rangle^{\beta - 1}.
  $$

Then in view of (4.17), we get, by applying (5.5) and (4.1), that

$$
|E_4| \leq \beta e^{-\frac{1}{2}} \int_0^T \langle t \rangle^{\beta - \frac{1}{2}} \| \partial_y u \|_{L^2(\mathcal{H}^1_\psi)} \| \sqrt{\theta} H \|_{H^7_\psi} \| \sqrt{\theta} \phi \|_{H^{15}_\psi} dt
$$

$$
\leq C \int_0^T \langle t \rangle^{2\beta - (\gamma_0 + \frac{3}{4})} \| \sqrt{\theta} H(t) \|^2_{H^7_\psi} + \| \sqrt{\theta} \phi(t) \|^2_{H^{15}_\psi} dt,
$$

which together with the fact: $\beta \leq \frac{1}{2} \left( \gamma_0 + \frac{5}{4} \right)$, ensures that

(A.1) $$
|E_4| \leq C \int_0^T \theta(t) \left( \| \sqrt{\theta} H(t) \|^2_{H^7_\psi} + \| \sqrt{\theta} \phi(t) \|^2_{H^{15}_\psi} \right) dt.
$$

- **The estimate of $E_5$.** In view of (1.5), we have
  $$
  \partial_t \left( \frac{\delta(t)}{\theta(t)} \right) = \delta(t) \partial_t \left( \frac{1}{\theta(t)} \right) - \lambda.
  $$


We get, by a similar derivation of (A.1), that
\[
\left| \int_0^T h(t) \delta(t) \partial t \left( \frac{1}{\theta} \right) \langle T_{\partial_y D_x u}^h \Lambda(D_x) \partial_x \int_y^\infty H dz, \phi \rangle \right| H^{2\gamma,0}_\Psi dt \leq C \int_0^T \hat{\theta}(t) (\| \sqrt{u} H(t) \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t) \|^2_{H^{\frac{15}{2},0}_\Psi}) dt.
\]
While it follows from (4.1), and (5.5) that
\[
\lambda \int_0^T h \langle T_{\partial_y D_x u}^h \Lambda(D_x) \partial_x \int_y^\infty H dz, \phi \rangle H^{2\gamma,0}_\Psi dt \leq C \lambda \int_0^T \| \partial_y u \|_{L^2_0(H^{\frac{1}{2}}_\Psi)} \langle t \hat{\theta}(t) (\| \sqrt{u} H(t) \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t) \|^2_{H^{\frac{15}{2},0}_\Psi}) dt \]
\[
\leq C \varepsilon \lambda \int_0^T \langle t \hat{\theta}(t) (\| \sqrt{u} H(t) \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t) \|^2_{H^{\frac{15}{2},0}_\Psi}) dt,
\]
which together with (3.12) and \( \beta \leq \gamma + \frac{1}{4} \) ensures that
\[
\lambda \int_0^T h \langle T_{\partial_y D_x u}^h \Lambda(D_x) \partial_x \int_y^\infty H dz, \phi \rangle H^{2\gamma,0}_\Psi dt \leq C \varepsilon \lambda \int_0^T \langle t \hat{\theta}(t) (\| \sqrt{u} H(t) \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t) \|^2_{H^{\frac{15}{2},0}_\Psi}) dt \]
\[
\leq C \varepsilon \lambda \int_0^T \langle t \hat{\theta}(t) (\| \sqrt{u} H(t') \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t') \|^2_{H^{\frac{15}{2},0}_\Psi}) dt.
\]
Overall, if \( \beta \leq \min \{ \beta + \frac{1}{4}, \frac{1}{2} (\gamma + \frac{5}{4}) \} \), we deduce from that
\[
(E_5) \leq C (1 + \varepsilon^2 \lambda) \int_0^T \hat{\theta}(t) (\| \sqrt{u} H(t) \|^2_{H^{\gamma,0}_\Psi} + \| \sqrt{v} \phi(t) \|^2_{H^{\frac{15}{2},0}_\Psi}) dt.
\]
• The estimate of \( E_6 \). In view of (3.3), we get, by a straight calculation, that
\[
\[ T_{\partial_y u}^h \mathcal{L} f = - T_{\partial_y u}^h \partial_y u - 2 T_{\partial_y u}^h f + \lambda \hat{\theta}[T_{\partial_y u}^h \langle D_x \rangle \frac{1}{2} f] + [T_{\partial_y u}^h, T_{\partial_y u}^h \partial_x] f + [T_{\partial_y u}^h, T_{\partial_y u}^h \partial_y] f + \frac{\delta(t)}{2} [T_{\partial_y u}^h, T_{\partial_y u}^h \Lambda(D) \partial_x] f - 2 T_{\partial_y u}^h \partial_y f.
\]
Yet notice from (4.2) that
\[
\partial_t \partial_y u - \partial^3_y u = - \partial_y (u \partial_x u + v \partial_y u) = - u \partial_x \partial_y u - v \partial^2_y u,
\]
from which and (5.5), we infer
\[
\| T_{\partial_y D_x u - \partial_y D_x u}^h \partial_x \int_y^\infty H dz \|_{H^{6,0}_\Psi} \leq C(t) \frac{1}{2} \| (u \partial_x \partial_y u - v \partial^2_y u) \|_{L^2_0(H^{\frac{1}{2}}_\Psi)} \| H \|_{H^{\gamma,0}_\Psi}
\]
\[
\leq C(t) \frac{1}{2} \left( \| u \|_{L^\infty(H^{\frac{1}{2}}_\Psi)} \| \partial_y u \|_{H^{\frac{1}{2},0}_\Psi} + \| v \|_{L^\infty(H^{\frac{1}{2}}_\Psi)} \| \partial^2_y u \|_{H^{\frac{1}{2},0}_\Psi} \right) \| H \|_{H^{\gamma,0}_\Psi},
\]
which together with (4.1), (4.2a) and (3.12) ensures that
\[
\| T_{\partial_y D_x u - \partial_y D_x u}^h \partial_x \int_y^\infty H dz \|_{H^{6,0}_\Psi} \leq C \varepsilon^2 \| t \|_{H^{\gamma,0}_\Psi} - \frac{1}{2} \| \hat{\theta}(t) \|_{H^{\gamma,0}_\Psi}
\]
\[
\leq C \varepsilon^2 \| t \|_{H^{\gamma,0}_\Psi} - \frac{1}{2} \| \hat{\theta}(t) \|_{H^{\gamma,0}_\Psi}.
\]
Whereas it follows from Lemma 2.2 and (5.5), that

\[
\| [T^h_{\partial_y u}; T^h v] \partial_x \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \leq C(t)^{\frac{1}{2}} \| \partial_y u \|_{L^2_\infty(H^{4,1})} \| u \|_{L^\infty(H^{4,1})} \| H \|_{H^{7,0}_\Psi} \\
\leq C \varepsilon^2(t)^{-2\gamma_0 - \frac{1}{2}} \| H \|_{H^{7,0}_\Psi} \\
\leq C \varepsilon(t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi}.
\]

Along the same line, due to \( \partial_y v = -\partial_x u \), Lemma 4.1 and (4.11), we have

\[
\| [T^h_{\partial_y u}; T^h v] \partial_y \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \leq \| [T^h_{\partial_y u}; T^h v] \partial_x H \|_{H^{6,0}_\Psi} + \| T^h_{\partial_y u} \partial_y v \partial_x \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \\
\leq C(t)^{\frac{1}{2}} \left( \| \partial_y u \|_{L^2_\infty(H^{4,1})} \| v \|_{L^\infty(H^{4,1})} + \| \partial_y u \|_{L^2_\infty(H^{4,1})} \| u \|_{L^\infty(H^{4,1})} \right) \| H \|_{H^{7,0}_\Psi}, \\
\leq C \varepsilon^2(t)^{-2\gamma_0 - \frac{1}{2}} \| H \|_{H^{7,0}_\Psi} \\
\leq C \varepsilon(t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi},
\]

and

\[
\frac{\delta(t)}{2} \| [T^h_{\partial_y u}; T^h_{\partial_x u} A(D)] \partial_x \partial_x \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \leq C(t)^{\frac{1}{2}} \| \partial_y u \|_{L^2_\infty(H^{4,1})} \| u \|_{L^\infty(H^{4,1})} \| H \|_{H^{7,0}_\Psi} \\
\leq C \varepsilon(t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi},
\]

and

\[
\| T^h_{\partial_x^2 u} \partial_x H \|_{H^{6,0}_\Psi} \leq C \| \partial_x^2 u \|_{L^\infty(H^{4,1})} \| H \|_{H^{7,0}_\Psi} \leq C \varepsilon(t)^{-\gamma_0 - \frac{1}{2}} \| H \|_{H^{7,0}_\Psi} \\
\leq C(t)^{2\beta - \gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi}.
\]

By summarizing the above estimates, we get for \( \beta \leq \min\{\gamma_0 + \frac{1}{4}, \frac{1}{2}(\gamma_0 + \frac{5}{4})\} \) that

\[
\| [T^h_{\partial_y u}; L - \lambda \hat{\theta} (D_x)^\beta] \partial_x \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \leq C \hat{\theta}^2(t) \| H \|_{H^{7,0}_\Psi}. 
\]

On the other hand, we deduce from Lemma 2.2 (4.11) and (5.5) that

\[
\lambda \hat{\theta} \| [T^h_{\partial_y u}; (D_x)^\beta] \partial_x \int_y^\infty H \, dz \|_{H^{\alpha_0}_\Psi} \leq C \lambda \hat{\theta}(t)^{\frac{1}{2}} \| \partial_y u \|_{L^2_\infty(H^{4,1})} \| H \|_{H^{\alpha_0}_\Psi} \\
\leq C \varepsilon \lambda(t)^{-\gamma_0 - \frac{1}{2}} \| H \|_{H^{\alpha_0}_\Psi} \\
\leq C \varepsilon(t)^{\frac{1}{2} \lambda(t)^{\beta - \gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi}}.
\]

Therefore, for \( \beta \leq \min\{\gamma_0 + \frac{1}{4}, \frac{1}{2}(\gamma_0 + \frac{5}{4})\} \), we get, by summing up (A.4) and (A.5), that

\[
|E_0| \leq \int_0^T \frac{\lambda(t)}{\hat{\theta}(t)} \| [T^h_{\partial_y u}; L - \lambda \hat{\theta} (D_x)^\beta] \partial_x \int_y^\infty H \, dz \|_{H^{6,0}_\Psi} \| \phi \|_{H^{\alpha_0}_\Psi}^2 dt + \lambda \int_0^T \frac{\hat{\theta}(t)}{\lambda(t)} \| [T^h_{\partial_y u}; (D_x)^\beta] \partial_x \int_y^\infty H \, dz \|_{H^{\alpha_0}_\Psi}^2 \| \phi \|_{H^{\alpha_0}_\Psi}^2 dt \\
\leq C(1 + \varepsilon(t)^{\frac{1}{2}} \lambda(t)^{\beta - \gamma_0 - \frac{1}{2}} \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi} + \| \hat{\theta}^2(t) \|_{H^{7,0}_\Psi}^2) dt.
\]
\[ E_{10} \text{ shares the above estimate.} \]

- The estimates of \( E_7, E_8 \) and \( E_9 \). By applying Lemma 2.1 and (5.5), we obtain

\[
|E_7| \leq C \int_0^T \frac{h}{\theta} (t)^{\frac{1}{2}} \|\partial_y u\|_{L^6(H^\frac{1}{2}_x)} \|\partial_x v\|_{L^6(H^\frac{1}{2}_x)} \|H\|_{H^7,0} \|\phi\|_{\dot{H}^{\frac{15}{2},0}} dt,
\]

which together with (4.1) and (4.2a) ensures that

\[
|E_7| \leq C \varepsilon \int_0^T (t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt
\]

\[
\leq C \varepsilon \int_0^T \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt,
\]

if \( \beta \leq \gamma_0 + \frac{1}{4} \).

Similarly, if \( \beta \leq \gamma_0 + \frac{1}{4} \), we have

\[
|E_8| \leq C \int_0^T \frac{h}{\theta} (t)^{\frac{1}{2}} \|\partial_y u\|_{L^6(H^\frac{1}{2}_x)} \|\partial_x v\|_{L^6(H^\frac{1}{2}_x)} \|H\|_{H^7,0} \|\phi\|_{\dot{H}^\frac{15}{2},0} dt
\]

\[
\leq C \varepsilon \int_0^T (t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt
\]

\[
\leq C \varepsilon \int_0^T \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt,
\]

and

\[
|E_9| \leq C \int_0^T \frac{\delta h}{\theta} (t)^{\frac{1}{2}} \|u\|_{L^6(H^\frac{1}{2}_x)} \|\partial_y u\|_{L^6(H^\frac{1}{2}_x)} \|H\|_{H^7,0} \|\phi\|_{\dot{H}^{\frac{15}{2},0}} dt
\]

\[
\leq C \varepsilon \int_0^T (t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt
\]

\[
\leq C \varepsilon \int_0^T \hat{\theta}(t) \left( \|\sqrt{\theta} H(t)\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt.
\]

As a result, it comes out

\[
(A.7) \quad \sum_{i=7}^{9} |E_i| \leq C \varepsilon \int_0^T \hat{\theta}(t) \left( \|\sqrt{\theta} H(t')\|_{H^7,0}^2 + \|\sqrt{\theta} \phi(t)\|_{\dot{H}^{\frac{15}{2},0}}^2 \right) dt.
\]

\[ \sum_{i=11}^{13} |E_i| \text{ shares the above estimate.} \]

By summing up the estimates (A.1), (A.2) and (A.6), we obtain (4.19a), and (A.7) implies (4.19d). This concludes the proof of Lemma 4.4. \( \square \)
Appendix B. The proof of Lemma 5.2

Proof of Lemma 5.2. Let us first compute $\mathcal{L}^* u_\Phi$. Indeed we deduce from (3.6) and the equation of $u_\Phi$ in (3.3) that

$$\mathcal{L}^* u_\Phi = 2\lambda \dot{h} (D_x) \frac{1}{2} u_\Phi + (T^h u \partial_x - \partial_x (T^h u)^*) u_\Phi + (T^h \partial_y - (\partial_y + \frac{y}{2(t)}) (T^h v)^*) u_\Phi$$

$$+ \frac{\delta(t)}{2} T^h D_{x,u} \Lambda(D_x) \partial_x - \Lambda(D_x) \partial_x (T^h D_{x,u})^* u_\Phi + T^h \partial_y v_\Phi$$

$$+ \frac{\delta(t)}{2} T^h D_{x,u} \Lambda(D_x) v_\Phi - \left( \partial_y^2 + (\partial_y + \frac{y}{2(t)})^2 - \frac{y^2}{4(t)^2} \right) u_\Phi - f,$$

and

$$\left( \partial_y^2 + (\partial_y + \frac{y}{2(t)})^2 - \frac{y^2}{4(t)^2} \right) u_\Phi = 2\partial_y^2 u_\Phi + \frac{y}{t} \partial_y u_\Phi + \frac{1}{2(t)} u_\Phi.$$

Then due to $u|_{y=0} = u|_{y \to +\infty} = 0$, we get, by using integration by parts, that

(B.1)

$$B_1 = \varepsilon \int_0^T \langle (t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0} \dot{\theta}(t) \rangle \int_0^T \tilde{h}(H, \mathcal{L}^* u_\Phi)_{H^6_\Psi} dt = - \varepsilon \int_0^T \partial_t \left( \langle \frac{(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0}}{\theta(t)} \rangle \right) \dot{h}(H, u_\Phi)_{H^6_\Psi} dt$$

$$+ \varepsilon \int_0^T \dot{h}(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0} \langle H(t), u_\Phi(t) \rangle_{H^6_\Psi} |_{t=0}^{t=T} dt$$

$$= \varepsilon \int_0^T \langle (t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0} \dot{h}(H, u_\Phi(t))_{H^6_\Psi} dt - \varepsilon \int_0^T \partial_t \left( \langle \frac{(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0}}{\theta(t)} \rangle \right) \dot{h}(H, u_\Phi)_{H^6_\Psi} dt$$

$$+ 2\varepsilon \lambda \int_0^T \langle \frac{(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0} \dot{h}(H, \langle D_x \rangle^{\frac{1}{2}} u_\Phi)_{H^6_\Psi} dt + \varepsilon \int_0^T \frac{(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0} \dot{h}}{\theta(t)} \rangle \dot{h}(H, u_\Phi)_{H^6_\Psi} dt$$

$$\times \left( {\langle H, (T^h u \partial_x - \partial_x (T^h u)^*) u_\Phi \rangle_{H^6_\Psi}} + \langle H, (T^h \partial_y - (\partial_y + \frac{y}{2(t)}) (T^h v)^*) u_\Phi \rangle_{H^6_\Psi} \right)$$

$$- \langle H, \frac{y}{2(t)} \partial_y v_\Phi \rangle_{H^6_\Psi} + \frac{\delta(t)}{2} \langle H, (T^h D_{x,u} \Lambda(D_x) \partial_x - \Lambda(D_x) \partial_x (T^h D_{x,u})^* u_\Phi \rangle_{H^6_\Psi}$$

$$+ \langle H, T^h \partial_y v_\Phi \rangle_{H^6_\Psi} + \frac{\delta(t)}{2} \langle H, T^h \partial_y D_{x,u} \Lambda(D_x) v_\Phi \rangle_{H^6_\Psi} - 2\langle H, \partial_y^2 u_\Phi \rangle_{H^6_\Psi}$$

$$- \langle H, \frac{y}{t} \partial_y u_\Phi \rangle_{H^6_\Psi} - \frac{1}{2(t)} \langle H, u_\Phi \rangle_{H^6_\Psi} - \langle H, f \rangle_{H^6_\Psi} \rangle dt \overset{\text{def}}{=} B_1^1 + \cdots + B_1^{13}.$$

Next let us handle the estimates of $B_1^i, i = 1, \cdots, 13$, term by term.

- The estimate of $B_1^1$. For any $\zeta > 0$, by applying Young’s inequality, we get for $\beta \leq \gamma_0 + \frac{1}{4}$ that

$$|B_1^1| \leq \zeta \|\sqrt{h} u_\Phi(T)\|_{H^{\frac{1}{2},0}_\Psi}^2 + C \zeta^{-1} \langle T \rangle^{2\beta - 2\gamma_0 - \frac{1}{2}} \|\sqrt{h} H(T)\|_{H^{\frac{1}{2},0}_\Psi}^2$$

$$\leq \zeta \|\sqrt{h} u_\Phi(T)\|_{H^{\frac{1}{2},0}_\Psi}^2 + C \zeta^{-1} \langle \sqrt{h} H(T)\|_{H^{\frac{1}{2},0}_\Psi}^2.$$

- The estimate of $B_1^2$. In view of (3.12), we have

$$\varepsilon \partial_t \left( \langle \frac{(t)^{-\frac{1}{4}} \varepsilon^{-\gamma_0}}{\theta(t)} \right) = (\beta - \gamma_0 - \frac{1}{4}) \varepsilon \langle (t)^{2\beta - 2\gamma_0 - \frac{1}{2}} \rangle \dot{\theta}(t)^{2\beta - 2\gamma_0 - \frac{1}{2}}.$$
Applying Hölder inequality and using $\beta \leq \frac{1}{2}(\gamma_0 + \frac{5}{4})$ yields

$$|B^1_1| \leq C \int_0^T (t)^{2\beta - \gamma_0 - \frac{7}{4}} \hat{\theta}(t) \left( \|\sqrt{\nu} u\|_{H^{\frac{3}{2}}_{\psi}}^2 + \|\sqrt{\nu} H\|_{H^{\frac{3}{2}}_{\psi}}^2 \right) dt$$

$$\leq C \int_0^T \hat{\theta}(t) \left( \|\sqrt{\nu} u\|_{H^{\frac{3}{2}}_{\psi}}^2 + \|\sqrt{\nu} H\|_{H^{\frac{3}{2}}_{\psi}}^2 \right) dt.$$

- The estimate of $B^3_1$. Due to $\beta \leq \gamma_0 + \frac{1}{4}$, we get, by using Young’s inequality, that

$$|B^3_1| \leq C \varepsilon^{\frac{1}{2}} \lambda \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{1}{4}} \hat{\theta}(t) \|\sqrt{\nu} u\|_{H^{\frac{3}{2}}_{\psi}} \|\sqrt{\nu} H\|_{H^{\frac{3}{2}}_{\psi}} dt$$

$$\leq C \varepsilon^{\frac{1}{2}} \lambda \int_0^T \hat{\theta}(t) \left( \|\sqrt{\nu} u\|_{H^{\frac{3}{2}}_{\psi}}^2 + \|\sqrt{\nu} H\|_{H^{\frac{3}{2}}_{\psi}}^2 \right) dt.$$

- The estimate of $B^4_1$. As $\beta \leq \gamma_0 + \frac{1}{4}$, we get, by applying Lemma 2.1, that

$$|B^4_1| \leq C \varepsilon^{\frac{1}{2}} \int_0^T \|H\|_{H^{\frac{5}{2}}_{\Psi}} \|\sqrt{\nu} u\|_{H^{\frac{5}{2}}_{\Psi}} \|H\|_{H^{\frac{5}{2}}_{\Psi}} dt$$

$$\leq C \varepsilon^{\frac{1}{2}} \lambda \int_0^T \hat{\theta}(t) \|\sqrt{\nu} u\|_{H^{\frac{5}{2}}_{\Psi}}^2 dt + \frac{\zeta}{6} \varepsilon \int_0^T \langle t \rangle^{-\gamma_0 - \frac{1}{4}} \|u\|_{H^{\frac{5}{2}}_{\Psi}}^2 dt.$$
The estimate of $B_1^0$. We first observe from Lemma 2.6 and (3.13) that
\[ \left\| \frac{y}{2(t)} \psi \right\|_{L^\infty(H^\frac{1}{2} \psi)} = \left\| \frac{y}{2(t)} \int_0^\infty \partial_y u \, dz \right\|_{L^\infty(H^\frac{1}{2} \psi)} \leq C \left\| \frac{y}{2(t)} e^{-\frac{3}{2} \psi} \right\|_{L^\infty} \| u \|_{H^\frac{1}{2} \psi} \leq C(t)^{-\frac{1}{4}} \| G \|_{H^\frac{1}{2} \psi} \leq C \varepsilon \langle t \rangle^{-\frac{3}{4}}, \]
from which, Lemma [2.1] and $\beta \leq \gamma_0 + \frac{1}{4}$, we infer
\[ |B_1^0| \leq C \varepsilon^{\frac{3}{4}} \int_0^T \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi} \left\| \frac{y}{2(t)} \psi \right\|_{L^\infty(H^\frac{1}{2} \psi)} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi} \, dt \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi}^2 \, dt + \frac{\varepsilon}{6} \int_0^T \langle t \rangle^{-\frac{3}{4}} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt. \]

The estimate of $B_1^1$. Similar to the estimate of $B_1^0$, we have
\[ |B_1^1| \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi}^2 \, dt + \frac{\varepsilon}{6} \int_0^T \langle t \rangle^{-\frac{3}{4}} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt. \]

The estimates of $B_1^2$, $B_1^3$. It follows from Lemma [2.1], (4.1), (5.9) and $\beta \leq \gamma_0 + \frac{1}{4}$ that
\[ |B_1^2| \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{3}{4} \psi} \left\| \sqrt{\theta} u^2 \right\|_{L^\infty(H^\frac{1}{2} \psi)} \, dt \leq C \varepsilon^{\frac{3}{4}} \int_0^T \langle t \rangle^{-\frac{3}{4}} \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{3}{4} \psi} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi} \, dt \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{3}{4} \psi}^2 \, dt + \frac{\varepsilon}{6} \int_0^T \langle t \rangle^{-\frac{3}{4}} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt. \]

Along the same line, one has
\[ |B_1^3| \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{3}{4} \psi}^2 \, dt + \frac{\varepsilon}{6} \int_0^T \langle t \rangle^{-\frac{3}{4}} \left\| \sqrt{\theta} u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt. \]

The estimate of $B_1^{10}$. Applying Hölder inequality gives
\[ |B_1^{10}| \leq \varepsilon^{\frac{3}{4}} \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{3}{4}} \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi} \left\| \sqrt{\theta} (t)^{\frac{3}{4}} \partial_y u^2 \right\|_{H^\frac{1}{2} \psi} \, dt \leq C \varepsilon^{\frac{3}{4}} \int_0^T \langle t \rangle^{\beta - 2\gamma_0 - \frac{3}{4}} \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi}^2 \, dt + \varepsilon^{\frac{3}{4}} \int_0^T \left\| \sqrt{\theta} (t)^{\frac{3}{4}} \partial_y u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt \leq C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi}^2 \, dt + \varepsilon^{\frac{3}{4}} \int_0^T \left\| \sqrt{\theta} (t)^{\frac{3}{4}} \partial_y u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt, \]
if $\beta \leq \frac{3}{4} \gamma_0 + \frac{1}{2}$.

The estimate of $B_1^{11}$. Similar to the estimate of $B_1^{10}$, we have
\[ |B_1^{11}| \leq \varepsilon^{\frac{3}{4}} \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{3}{4}} \left\| \frac{y}{\langle t \rangle} \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi} \left\| \sqrt{\theta} (t)^{\frac{3}{4}} \partial_y u^2 \right\|_{H^\frac{1}{2} \psi} \, dt \leq \varepsilon^{\frac{3}{4}} \int_0^T \left\| \frac{y}{\langle t \rangle} \sqrt{\theta} H(t) \right\|_{H^\frac{1}{2} \psi}^2 \, dt + C \varepsilon^{\frac{3}{4}} \int_0^T \hat{\theta}(t) \left\| \sqrt{\theta} (t)^{\frac{3}{4}} \partial_y u^2 \right\|_{H^\frac{1}{2} \psi}^2 \, dt, \]
if $\beta \leq \frac{3}{4} \gamma_0 + \frac{1}{2}$. 
Estimate of $B_1^{12}$. It is easy to observe that if $\beta \leq \frac{1}{5} (\gamma_0 + \frac{\zeta}{2})$,

$$|B_1^{12}| \leq \varepsilon^2 \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{2}{5}} \| \sqrt{h} H \|_{H^{1/2}_\psi,0} \| \sqrt{T} f \|_{H^{5/2}_\psi,0} \, dt \leq \int_0^T \dot{\theta}(t) \left( \| \sqrt{h} H \|_{H^{1/2}_\psi,0}^2 + \| \sqrt{T} f \|_{H^{5/2}_\psi,0}^2 \right) \, dt.$$

The estimate of $B_1^{13}$. Finally applying Lemma 4.2 for $s = 5$ gives

$$|B_1^{13}| \leq C \varepsilon \int_0^T \langle t \rangle^{\beta - \gamma_0 - \frac{2}{5}} \| \sqrt{h} H \|_{H^{1/2}_\psi,0} \| \sqrt{T} f \|_{H^{5/2}_\psi,0} \, dt \leq C \varepsilon \int_0^T \langle t \rangle^{\beta - \gamma_0} \| \sqrt{h} H \|_{H^{1/2}_\psi,0} \left( \| \partial_y G_F \|_{H^{5/2}_\psi,0} + \| G_F \|_{H^{1/2}_\psi,0} \right) \, dt.$$

So that by virtue of (3.13), for any $\zeta > 0$, we get, by applying Young’s inequality, that

$$|B_1^{13}| \leq C \varepsilon \int_0^T \langle t \rangle^{\beta - \gamma_0} \| \sqrt{h} H \|_{H^{1/2}_\psi,0} \| \sqrt{T} f \|_{H^{5/2}_\psi,0} \, dt + \zeta \left( \varepsilon^2 \int_0^T \| \sqrt{T} f \|_{H^{5/2}_\psi,0}^2 \, dt \right) \leq C \left( \varepsilon + \zeta \left( \varepsilon^2 \int_0^T \| \sqrt{h} H \|_{H^{1/2}_\psi,0}^2 \, dt \right) \right) \, dt.$$

If $\beta \leq \min \{ \gamma_0 + \frac{1}{5}, \frac{\zeta}{2} \}$.

By substituting the above estimates into (3.11), we conclude the proof (5.4).

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