REFINING THE ABEL–JACOBI MAPS

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Abstract. Given a smooth projective variety $X$ over a field $k$ of characteristic zero, we consider the composition of the de Rham cohomology cycle class map over $k$ from the Chow group $CH^q(X \times_k K)$, where $K$ is the field of fractions of henselization $A^h$ of the local ring of a smooth closed point of a variety over the field $k$ with an appropriate projection:

$$CH^q(X \times_k K) \rightarrow \bigoplus_{p=1}^q gr_F^{q-p} N^{q-p} H_{dR/k}^{2q-p}(X) \otimes_k \Omega^p_{A^h/k, \text{closed}},$$

where $F^*$ and $N^*$ are the Hodge and the coniveau filtrations on the de Rham cohomology, respectively. The classical Abel–Jacobi map corresponds to the composition of this homomorphism with the projection to the summand $p = 1$.

This homomorphism is not injective, however, its composition with the embedding into the space

$$\bigoplus_{p=1}^q gr_F^{q-p} N^{q-p} H_{dR/k}^{2q-p}(X) \otimes_k \lim_{\leftarrow M} d(\Omega^{p-1}_{A_M/k}),$$

where $A_M = A^h/m^M$ and $m$ is the maximal ideal, is dominant for any $q$ for which the inverse Lefschetz operator $H^{2 \dim X-q}(X)(\dim X) \sim H^q(X)(q)$ is induced by a correspondence.

As it is shown by Mumford, if there is a holomorphic 2-form on a smooth complex projective surface $S$ then the group of 0-cycles of degree 0 on $S$ is not presentable as a quotient of an algebraic group, “the group $CH_0(S)^0$ is infinite-dimensional”. It is clear from the argument that the “arithmetics” of the complex numbers is pretty much involved.

Given, say, a complex variety and having in mind the Mumford’s obstruction to describing the Chow groups in geometrical terms and that the arithmetics of the huge field of complex numbers is deeply hidden, we replace the field of the complex numbers with the field of fractions of henselization of a smooth closed point of a variety over a “small” subfield of $\mathbb{C}$.

As it is well-known, the Picard group of a smooth projective variety is an extension of a finitely generated group by an abelian variety. We remark that the group of points of the abelian variety over the field of fractions of an algebra of formal power series is an extension of the “small” group of points of the abelian variety over the field of constants by the maximal ideal in the algebra of formal power series tensored with the tangent space of the abelian variety, i.e., the $H^1(\cdot, \mathcal{O})$-group of the original variety.

In general, suppose $X$ is a smooth projective variety over a field $k$ of characteristic zero and $K$ is the field of fractions of henselization $A^h$ of the local ring of a smooth closed point of a variety over the field $k$. Then in Corollary 2.1 a natural, continuous in a certain sense homomorphism from the group of cycles of codimension $q$ on $X \times_k K$, refining the Abel–Jacobi map:

$$CH^q(X \times_k K) \rightarrow \bigoplus_{p=1}^q gr_F^{q-p} N^{q-p} H_{dR/k}^{2q-p}(X) \otimes_k \Omega^p_{A^h/k, \text{closed}},$$

where $F^*$ and $N^*$ are the Hodge and the coniveau filtrations on the de Rham cohomology, respectively, and $\Omega^p_{A^h/k}$ is the space of differential $p$-forms on $A^h$. The classical Abel–Jacobi map corresponds to the composition of this homomorphism with the projection to the summand $p = 1$. A version of this map has first been studied by V. Srinivas in [Sr].
This homomorphism is by no means injective. However, its appearance reminds the conjectural formula \( \text{Ext}_{\mathcal{M}_X}^n(\mathbb{Q}, H^{2n-q}(\omega_X)(q)) \) for the successive quotients of the Chow group \( CH^q(X)_\mathbb{Q} \) with respect to a conjectural filtration on it (cf. \([B]\)).

Partially motivated by calculations of infinitesimal deformations of the Chow groups by Bloch (cf. \([B]\), Lecture 6) and Stienstra \([S]\), one could hope its composition with the canonical embedding into the space

\[
\bigoplus_{p=1}^q \gr_F q^{-p} N^{q-p} H^{2q-p}_{dR/k}(X) \otimes_k \lim_{\leftarrow M} d(\Omega_{A_M/k}^{p-1}),
\]

where \( A_M = A^h/\mathfrak{m}^M \) and \( \mathfrak{m} \) is the maximal ideal, is dominant.

It is shown in Proposition 2.1 together with the concluding remark on p.\([B]\) that whenever the inverse Lefschetz operator \( H^{2\dim X-q}(X)(\dim X) \rightrightarrows H^q(X)(q) \) is induced by a self-correspondence on \( X \) (e.g., for \( q = \dim X \), and conjecturally, for arbitrary \( q \)) the composition is actually dominant.

I am grateful to Alexander Beilinson for inspiring discussions on related topics several years ago and pointing out a mistake in a previous version of this note.

1. A CYCLE CLASS \( \text{cl} \cdot \)

Let \( X \) be a smooth \( n \)-dimensional projective variety over an algebraically closed field \( k \) of characteristic 0. Fix an auxiliary smooth projective variety \( Y \) over \( k \) with the field of rational functions \( K \).

Using the Poincaré duality \( H^{2q}_{dR/k}(X) = \text{Hom}_{k}(H^{2n-2q}_{dR/k}(X), k) \), one can define the class map \( \text{cl}_{dR} : CH^q(X) \rightarrow H^{2q}_{dR/k}(X) \) on the classes of irreducible subvarieties as \([Z] \mapsto (\omega \mapsto i_Z^*\omega)\), where \( i_Z : \tilde{Z} \rightarrow X \) is a desingularization of \( Z \) and \( H^{2n-2q}_{dR/k}(\tilde{Z}) \) is canonically identified with \( k \) via the trace isomorphism.

**Lemma 1.1.** Composition of the class map \( \text{cl}_{dR} \) with the Künneth isomorphism induces a homomorphism \( CH^q(X \times_k Y) \rightarrow \bigoplus_{r=0}^q N^rH^{q+r}_{dR/k}(X) \otimes_k H^{q-r}_{dR/k}(Y) \). that factors through

\[
CH^q(X \times_k Y) \rightarrow \bigoplus_{r=0}^q N^rH^{q+r}_{dR/k}(X) \otimes_k H^{q-r}_{dR/k}(Y) \oplus \bigoplus_{r=1}^q H^{q-r}_{dR/k}(X) \otimes_k N^rH^{q+r}_{dR/k}(Y). \tag{2}
\]

**Proof.** We need to show that for any subvariety \( Z \subset X \times_k Y \) of codimension \( q \) and its desingularization \( \tilde{Z} \) composition

\[
H^{2n-q+r}_{dR/k}(X) \xrightarrow{pr_X} H^{2n-q+r}_{dR/k}(\tilde{Z}) \xrightarrow{pr_Y} H^{q+r}_{dR/k}(Y)
\]

factors through \( H^{2n-q+r}_{dR/k}(X) \rightarrow N^rH^{q+r}_{dR/k}(Y) \). Note, that thanks to the weak Lefschetz theorem, \( H^{2n-q+r}_{dR/k}(X) = N^{n-q+r}H^{2n-q+r}_{dR/k}(X) \), so our composition factors through

\[
H^{2n-q+r}_{dR/k}(X) \xrightarrow{pr_X} N^{n-q+r}H^{2n-q+r}_{dR/k}(\tilde{Z}),
\]

and, since \( Z \) is of relative dimension \( n - q \) over \( Y \), we have \( N^{n-q+r}H^{2n-q+r}_{dR/k}(\tilde{Z}) \xrightarrow{pr_Y} N^rH^{q+r}_{dR/k}(Y) \). \( \square \)

**Lemma 1.2.** For any smooth \( n \)-projective varieties \( X \) and \( Y \) over \( k \) there is a natural homomorphism

\[
CH^q(X \times_k Y) \rightarrow \bigoplus_{p=0}^q H^{q+p}_F N^r H^{q+r}_{dR/k}(X) \otimes_k \Gamma(X, \Omega_{Y/k}^{q-r}).
\]

**Proof.** Suppose \( D \) is an effective irreducible divisor on \( Y \). Then for its desingularization \( \tilde{D} \) the composition of the inclusion map \( CH^{q-1}(X \times_k \tilde{D}) \rightarrow CH^q(X \times_k Y) \) with the class map \( \mathcal{F} \) commutes
with composition of the corresponding class with $Y$ replaced by $\tilde{D}$ with the Gysin map and factors through the subspace $\bigoplus_{r=2-q}^{q-2} H_{dR/k}^{q+r}(X) \otimes_k N^1 H_{dR/k}^{q-r}(Y)$, i.e., the following diagram commutes

$$CH^{q-1}(X \times_k \tilde{D}) \quad \xrightarrow{id \otimes \text{Gysin}} \quad \bigoplus_{r=2-q}^{q-2} H_{dR/k}^{q+r}(X) \otimes_k N^1 H_{dR/k}^{q-r}(Y) \quad \subseteq \quad H_{dR/k}^{2q}(X \times_k Y)$$

Since one has the exact localization sequence

$$\bigoplus_{D : \text{divisors on } Y} CH^q(X \times_k D) \longrightarrow CH^q(X \times_k Y) \longrightarrow CH^q(X \times_k k(Y)) \longrightarrow 0,$$

the homomorphism (2) leads to a well-defined homomorphism from the Chow group $CH^q(X \times_k k(Y))$ to the quotient of the right hand side of (4) by the subspace $\bigoplus_{r=1}^{q} H_{dR/k}^{q-r}(X) \otimes_k N^1 H_{dR/k}^{q+r}(Y)$, i.e.,

$$CH^q(X \times_k k(Y)) \longrightarrow \bigoplus_{r=0}^{q} N^r H_{dR/k}^{q+r}(X) \otimes_k \frac{H_{dR/k}^{q-r}(Y)}{N^1 H_{dR/k}^{q+r}(Y)}.$$

In fact, the space $H_{dR/k}^{q-r}(Y)/N^1 H_{dR/k}^{q-r}(Y)$ is a birational invariant of $Y$ and can be considered as a subspace of $H_{dR/k}^{q-r}(k(Y)/k)$, or as a subspace of $\Omega_{dR/k}^{q-r}(Y/k)$.

Also, from the Künneth decomposition

$$F^q H_{dR/k}^p(X \times_k Y) = \sum_{s=0}^{q} \sum_{t=0}^{p} F^s H_{dR/k}^t(X) \otimes_k F^{-s} H_{dR/k}^{-t}(Y)$$

one sees that any class in $F^q$ and simultaneously in the space

$$F^r H_{dR/k}^{q+r}(X) \otimes_k H_{dR/k}^{q-r}(Y)$$

but not in the subspace $F^{r+1} H_{dR/k}^{q+r}(X) \otimes_k H_{dR/k}^{q-r}(Y)$

belongs to $F^r H_{dR/k}^{q+r}(X) \otimes_k F^{q-r} H_{dR/k}^{q-r}(Y)$. Therefore, we have a homomorphism

$$CH^q(X \times_k k(Y)) \longrightarrow \bigoplus_{r=0}^{q} qr^* N^r H_{dR/k}^{q+r}(X) \otimes_k \Gamma(Y, \Omega_{dR/k}^{q-r}). \quad \Box$$

At this point, we may replace the field of rational functions on $Y$ with an arbitrary field extension $K \subset k$ to obtain the following class map

$$cl_q : CH^q(X \times_k K) \longrightarrow \bigoplus_{r=0}^{q} qr^* N^r H_{dR/k}^{q+r}(X) \otimes_k \Omega_{K/k}^{q-r}. \quad (3)$$

2. Invariants of cycles over fraction fields of henselizations

For any pair of rings $B \subseteq C$ denote by $\Omega^\bullet_{C/B}$ the exterior $C$-algebra of the module $\Omega^1_{C/B}$ of differentials on $C$ relative to $B$.

**Lemma 2.1.** Let $k$ be a field of characteristic zero and $A$ be a local ring of a closed regular point of a variety over $k$ with the fraction field $K$. Let $X$ be a smooth projective variety and $\sigma : k(X) \hookrightarrow K$ an embedding of fields. Obviously, one can extend $\sigma : k(X) \hookrightarrow K$ to a homomorphism $\Omega^\bullet_{k(X)/k} \hookrightarrow \Omega^\bullet_{K/k}$.
of differential $k$-algebras in a unique way. Then the restriction of this homomorphism $\Gamma(X, \Omega^*_{X/k}) \to \Omega^*_{K/k}$ factors through an embedding $\Gamma(X, \Omega^*_{X/k}) \to \Omega^*_{A/k} \subset \Omega^*_{K/k}$. 

Proof. Since $K = k(Y)$ is the function field of a smooth projective variety $Y$, we may suppose that $\sigma$ is induced by a morphism $Y \to X$. Then we may find a smooth projective variety $Y'$ birational to $Y$ with $A$ as the local ring of $Y'$ at a (closed) point $y$. We are done, since 

$$\sigma : \Gamma(X, \Omega^*_{X/k}) \to \Gamma(Y, \Omega^*_{Y/k}) = \Gamma(Y', \Omega^*_{Y'/k}) \subset \Omega^*_{\mathcal{O}_{Y',y/k}}.$$ 

This lemma together with the homomorphism $cl_q$ from (3) enables us to define the homomorphism $\mathcal{D}_q$.

Corollary 2.1. For any smooth projective variety $X$ over $k$ and the field of fractions $K$ of $A^b$ the map

$$\bigoplus_{q \geq 0} CH^q(X \times_k K) \to \bigoplus_{0 \leq p \leq q} gr^{q-p} N^{q-p} H^{2q-p}_{dR/k}(X) \otimes_k \Omega^p_{A^b/k, \text{closed}}$$

is a homomorphism of the graded rings. \hfill \Box

Lemma 2.2. Suppose that for a local domain $A \supset k$ with the fraction field $K$, an integer $q \geq 0$, for any integer $n$ and any smooth $n$-dimensional projective variety $X$ over $k$ and any integer $M > 0$

$$CH_q(X \times_k K) \to H^{n-q}(X, \mathcal{O}_X) \otimes_k d(\Omega^{n-q-1}_{A_M/k}),$$

where $A_M = A/m^n_A$. Then the map

$$CH_q(X \times_k K) \to \bigoplus_{p=1}^{n-q} gr^{n-p-q} N^{n-p-q} H^{2(n-q)-p}_{dR/k}(X) \otimes_k d(\Omega^{p-1}_{A_M/k})$$

is surjective for any integer $M > 0$. \hfill \Box

Proof. We proceed by induction on dimension of $X$. The case dim $X \leq q + 1$ is trivial.

Denote by $\tilde{D}$ a desingularization of divisor $D$ on $X$. Then Lemma follows from the commutativity of the diagram

$$CH_q(X \times_k K) \to \bigoplus_{p=1}^{n-q} gr^{n-p-q} N^{n-p-q} H^{2(n-q)-p}_{dR/k}(X) \otimes_k d(\Omega^{p-1}_{A_M/k})$$

Gysin\text{id}

$$\bigoplus_{D \in \mathcal{X}} CH_q(\tilde{D} \times_k K) \quad \text{onto} \quad \bigoplus_{p=1}^{n-q-1} gr^{n-p-q-1} N^{n-p-q-1} H^{2(n-q)-p}_{dR/k}(\tilde{D}) \otimes_k d(\Omega^{p-1}_{A_M/k})$$

where the surjectivity of the Gysin maps $\bigoplus_{D \in \mathcal{X}} N^{*-1} H^{*-2}(\tilde{D}) \to N^* H^*(X)$ should be used. \hfill \Box

\footnote{To check the surjectivity of the Gysin maps one uses alternative description of the coniveau filtration due to Grothendieck: $N^* H^b(X) = \text{Im}(\oplus \cap_{Z \in \mathcal{X}} H^{b-2a}(Z) \xrightarrow{\text{Gysin}} H^b(X))$, cf. [18], formula (10.7) or (9.17), and a correction in footnote on p.300 of [17].}
Lemma 2.3. Let $R$ be an integral domain, and $\|f_{jk}\|$ be an invertible $g \times g$-matrix with entries in the algebra $R[[t]]$ of power series in one variable $t$.

Then there exists a unique collection of power series $\phi_l \in tR[[t]]$ for $1 \leq l \leq g$ such that

$$\sum_{j=1}^{g} f_{ij}(\phi_j) \frac{d\phi_j(t)}{dt} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\phi_l = \phi_l(t) \in tR[[t]]$. We need to solve the system

$$\begin{cases} \sum_{l=1}^{g} f_{1l}(\phi_l(t)) \phi'_l(t) = 1 \\ \sum_{l=1}^{g} f_{2l}(\phi_l(t)) \phi'_l(t) = 0 \\ \ldots \\ \sum_{l=1}^{g} f_{gl}(\phi_l(t)) \phi'_l(t) = 0 \end{cases}$$

and claim that there is a unique solution of this system. This is equivalent to saying that there is a unique collection of power series $\phi_l^{(j)}(0)$. We prove the latter by induction on $j$, the case $j = 0$ being trivial. Taking $j$th derivative at $t = 0$ leads to the system

$$\begin{cases} \sum_{l=1}^{g} f_{1l}(0) \phi'_l^{(j+1)}(0) = \langle \text{polynomial in derivatives of } \phi' \text{s at 0 of orders } \leq j \rangle \\ \sum_{l=1}^{g} f_{2l}(0) \phi'_l^{(j+1)}(0) = \langle \text{polynomial in derivatives of } \phi' \text{s at 0 of orders } \leq j \rangle \\ \ldots \\ \sum_{l=1}^{g} f_{gl}(0) \phi'_l^{(j+1)}(0) = \langle \text{polynomial in derivatives of } \phi' \text{s at 0 of orders } \leq j \rangle \end{cases} \tag{4}$$

Since the $g \times g$-matrix $\|f_{jl}\|$ is invertible, the system (4) has a unique solution $(\phi_1^{(j+1)}(0), \ldots, \phi_g^{(j+1)}(0))$. \hfill \Box

Lemma 2.4. Let $U$ be a variety over an algebraically closed field $k$ and $f_1, \ldots, f_g \in \mathcal{O}(U)$ be a collection of regular functions on $U$, linear independent over the field $k$.

Then there exist such a collection of points $p_1, \ldots, p_g$ that the $g \times g$-matrix $\|f_j(p_k)\|$ is invertible together with its first-row minors.

Proof. We proceed by induction on $g$, the case $g = 1$ being trivial.

The matrix $\|f_j(p_k)\|$ is not invertible for any collection of points $p_1, \ldots, p_g$ means that the function

$$\det \|f_j(p_k)\| : U \times \cdots \times U \rightarrow k$$

is identically zero.

Decomposing the determinant via the first row, we get

$$\det \|f_k(p_l)\| = \sum_{j=1}^{g} (-1)^{j-1} \det \|f_k(p_l)\|_{k \neq j, l > 1} f_j(p_1),$$

where, under the induction assumption, $\det \|f_k(p_l)\|_{k \neq j, l > 1}$ is a non-zero function in $p_2, \ldots, p_g$.

After fixing sufficiently general $p_2, \ldots, p_g$ we get $\det \|f_j(p_k)\| = \sum_{j=1}^{g} a_j f_j(p_1)$, where $a_j$ are non-zero elements of $k$, and therefore, we get the $k$-linear dependence of the functions $f_1, \ldots, f_g$. \hfill \Box

Proposition 2.1. Let $k$ be a field of characteristic zero, $N > 0$ an integer, $K$ the algebraic closure of the field of rational functions in the fraction field of $k[[t_1, \ldots, t_N]]$.

Then for any smooth $n$-dimensional projective variety $X$ over $k$ and an integer $M > 0$ there is a natural surjection

$$CH_0(X \times_k K) \rightarrow \bigoplus_{q=1}^{n} H^n(X, \Omega_{X/k}^{n-q} \otimes_k d(\Omega_{A_M/k}^{q-1}), \tag{5}$$

where $A_M = k[t_1, \ldots, t_N]/(t_1, \ldots, t_N)^M$.\hfill 5
Proof. Since surjectivity is preserved by taking Galois invariants, we may suppose that the field \( k \) is algebraically closed. By Lemma 2.2 it is enough to check that the composition of \( (\mathbb{A}^n) \) with the projection to \( H^n(X, \mathcal{O}_X) \otimes_k d(\Omega^n_{\mathcal{A}_M/k}) \) is surjective.

We fix a basis \( \{\omega_0, \ldots, \omega_N\} \) of the space \( \Gamma(X, \Omega^n_{X/k}) \) and choose functions \( x_1, \ldots, x_n \) algebraically independent over \( k \). Set \( f_j = \omega_j/\omega_0 \).

Fix an open subset \( U \) where \( (x_1, \ldots, x_n) : U \rightarrow \mathbb{A}^n_k \) is an étale morphism and the functions \( f_j \) are regular. Choose points \( p_1, \ldots, p_g \) as in Lemma 2.3. This guarantees that the matrix \( \|f_i(p_j)\| \) is invertible.

Then for each point \( p_j \) we embed the completion of \( \mathcal{O}(U) \) at \( p_j \) into the algebra \( k[[t_1, \ldots, t_n]] \) by sending \( x_i \) to \( x_i(p_j) + t_i \). Denote by \( f_{ij} \) the image of \( f_i \) under this embedding.

Note, that the matrix \( \|f_{ij}\| \) with entries in \( k[[t_1, \ldots, t_n]] \) is invertible together with its first-row minors, since its reduction modulo the maximal ideal in \( k[[t_1, \ldots, t_n]] \) is the matrix \( \|f_i(p_j)\| \).

Finally, we consider \( f_{ij} \) as elements of \( R[[t_1]] \) with \( R = k[[t_2, \ldots, t_n]] \), and apply Lemma 2.3 to show that there exists a collection of non-zero \((\partial \phi_i/\partial t_1(0, \ldots, 0) \neq 0)\) formal series \( \phi_i(t_1, \ldots, t_n) \in t_1k[[t_1, \ldots, t_n]] \) such that

\[
\sum_i f_{ij}(\phi_i(t_1, \ldots, t_n), t_2, \ldots, t_n)dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n = \begin{cases} dt_1 \wedge \cdots \wedge dt_n & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Remark. It follows from the standard conjecture on algebraicity of the inverse Lefschetz operator \( H^{2\dim X-q}(X)(\dim X) \sim H^q(X)(q) \) that the map \( (\mathbb{I}) \) is dominant.

Proof. Thanks to Lemma 2.2 we only need to check that for any element \( \beta \) of the \( k \)-vector space \( H^q(X, \mathcal{O}_X) \otimes_k d(\Omega^{n-1}_{\mathcal{A}_M/k}) \) there exists a cycle \( \alpha \in CH^q(X \times_k K) \) with the image \( \beta \) under the composition of the map \( (\mathbb{I}) \) with the corresponding projection.

Fix a smooth \( q \)-dimensional plane section \( W \) of \( X \). Denote by \( \overline{\beta} \) the image of \( \beta \) under the isomorphism

\[
H^q(X, \mathcal{O}_X) \otimes_k d(\Omega^{n-1}_{\mathcal{A}_M/k}) \xrightarrow{\cup[W] \otimes \text{id}} H^n(X, \Omega^n_{X/k}) \otimes_k d(\Omega^{q-1}_{\mathcal{A}_M/k}).
\]

Due to Proposition 2.3, there is a \( 0 \)-cycle \( \overline{\sigma} \) with the image \( \overline{\beta} \) under the map \( (\mathbb{I}) = (\mathbb{I}) \). Suppose that \( \overline{\sigma} \) is, in fact, defined over a field \( k(Y) \subset K \) finitely generated over \( k \), and \( Y \) is its smooth projective model over \( k \). Let \( \tilde{\alpha} \in CH^n(X \times_k Y) \) correspond to the \( 0 \)-cycle \( \overline{\sigma} \in CH_0(X \times_k K) \). Assuming that the inverse Lefschetz operator \( H^{2n-q}(X)(n) \sim H^q(X)(q) \) is induced by a correspondence \( [L] \in CH^n(X \times_k X) \), we set \( \tilde{\alpha} = [L \times \Delta_Y](\tilde{\alpha}) \).

The image of the cycle \( \tilde{\alpha} \) in \( CH^q(X \times_k K) \) gives us the desired cycle \( \alpha \).

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\square
\]

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