Zeros of Chromatic and Flow Polynomials of Graphs

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Abstract

We survey results and conjectures concerning the zero distribution of chromatic and flow polynomials of graphs, and characteristic polynomials of matroids.

1 Introduction

The study of chromatic polynomials of graphs was initiated by Birkhoff [7] in 1912 and followed by Whitney [41, 42] in 1932. Inspired by the 4-Colour Conjecture, Birkhoff and Lewis [8], obtained results concerning the distribution of the real zeros of chromatic polynomials of planar graphs and made the stronger conjecture that chromatic polynomials of planar graphs have no real zeros greater than or equal to four. Their hope was that results from analysis and algebra could be used to prove their stronger conjecture and hence deduce that the 4-colour conjecture was true. This has not yet occurred: indeed the 4-colour conjecture is now a theorem [1, 2, 23], but the stronger conjecture of Birkhoff and Lewis remains unsolved. Nevertheless, many beautiful results have been obtained concerning the zero distribution of chromatic polynomials both on the real line and in the complex plane, and many other intriguing problems remain open. I will summarise these in Section 2 below.
Flow polynomials were introduced by Tutte in [33] as dual polynomials to chromatic polynomials. This duality holds only for planar graphs, however, and the zero distributions of chromatic and flow polynomials for non-planar graphs seem to be quite different. There are many similarities between the zero distribution of flow polynomials for general graphs and for planar graphs. This is not the case for chromatic polynomials. One possible reason for this is that chromatic polynomials of planar graphs are strongly influenced by the fact that the cycle space of a planar graph has a basis consisting of a set of circuits which cover every edge at most twice (obtained by taking the boundaries of all but one of the faces). The dual property, that the cocycle space has a basis consisting of a set of cocircuits which cover every edge at most twice holds for all graphs, and can be obtained by taking the stars centred on all but one of the vertices. Flow polynomials have received much less attention than chromatic polynomials in the literature. I will summarise what little is known about their zero distribution in Section 3.

Characteristic polynomials of matroids provide a common generalization of chromatic and flow polynomials of graphs. I will describe how some of the results from Sections 2 and 3 can be extended to this more general setting in Section 4.

All graphs considered are finite and may contain loops and multiple edges. We shall refer to graphs without loops and multiple edges as simple graphs. We use $K_n$ and $K_{m,n}$ to denote the complete graph on $n$ vertices and the complete bipartite graph with vertex sets of sizes $m$ and $n$, respectively.

A graph $G$ is said to be non-separable if $G$ is connected and $G - v$ is connected for all $v \in V(G)$. A subgraph $H$ of $G$ is a component, respectively block, of $G$ if it is a maximal connected, respectively non-separable, subgraph of $G$. The graph $G$ is $k$-connected if $|V(G)| \geq k + 1$ and $G - U$ is connected for all $U \subseteq V(G)$ with $|U| < k$. Given an edge $e$ in $G$, we use $G - e$ to denote the graph obtained from $G$ by deleting $e$, and $G/e$ to denote the graph obtained from $G$ by contracting $e$. We say that $e$ is a bridge of $G$ if $G - e$ has more components than $G$.

2 Chromatic Polynomials

Let $t$ be a positive integer and $G$ be a graph. A proper $t$-colouring of $G$ is an assignment of $t$ colours to the vertices of $G$ in such a way that adjacent
vertices of $G$ receive different colours. If $G$ has a loop then $G$ does not have a proper $t$-colouring for any $t$. If $G$ is loopless, then the **chromatic number**, $\chi(G)$ is the minimum value of $t$ for which $G$ has a proper $t$-colouring. We use $P(G,t)$ to denote the **chromatic polynomial** of $G$. This is defined for integer values of $t \geq 1$ as the number of distinct $t$-colourings of $G$. (Hence $P(G,t) \equiv 0$ if $G$ has a loop.) The fact that $P(G,t)$ is a polynomial in $t$ and many other properties of $P(G,t)$ can be deduced from the following elementary lemma.

**Lemma 1** Let $G$ be a graph and $e$ be an edge of $G$. Then

$$P(G,t) = P(G - e, t) - P(G/e, t).$$

Lemma 1 also gives rise to a recursive procedure for calculating chromatic polynomials. This calculation can often be simplified using:

**Lemma 2** Let $G$ be a graph and $G_1$ and $G_2$ be subgraphs of $G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 \cong K_r$. Then

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{t(t-1)...(t-r+1)}.$$  

Note that it is NP-hard to determine the chromatic number of a graph. Since the chromatic number of $G$ is one greater than than the smallest positive integer root of $P(G,t)$, it is also NP-hard to determine $P(G,t)$.

I will refer to the zeros of $P(G,t)$ as **chromatic roots** of $G$. The distribution of chromatic roots either on the real line or in the complex plane is of interest to graph theorists principally because of its implication for the integer chromatic roots and hence the chromatic number of a graph. The distribution of chromatic roots in the complex plane is also of independent interest to physicists because of its relevance to phase transitions via the Potts model partition function, see for example [27, Section 1].

### 2.1 General Graphs

Since graphs can have arbitrarily large chromatic numbers (for example $K_n$ has chromatic number $n$) they can have arbitrarily large integer chromatic roots. It follows, from the definition of $P(G,t)$, however, that a graph $G$
can have no integer chromatic root in the interval \([\chi(G), \infty)\). One might hope that this result can be extended to real chromatic roots but this is not the case. Woodall [43] has shown that complete bipartite graphs (which have chromatic number two) have arbitrarily large chromatic roots. More precisely, he showed that if \(n\) is large enough compared to \(m\), then \(K_{m,n}\) has chromatic roots arbitrarily close to \(i\), for all integers \(i\), \(2 \leq i \leq m/2\).

On the other hand, the following results due to Tutte, Woodall and the author imply that the only real numbers in \((-\infty, \frac{32}{27}]\) which are chromatic roots are 0 and 1.

**Theorem 3** Let \(G\) be a loopless graph with \(n\) vertices, \(c\) components, and \(b\) blocks which are not isolated vertices. Then:

(a) \(P(G,t)\) is non-zero with sign \((-1)^n\) for \(t \in (-\infty, 0)\) [37];

(b) \(P(G,t)\) has a zero of multiplicity \(c\) at \(t = 0\) [37];

(c) \(P(G,t)\) is non-zero with sign \((-1)^{n+c}\) for \(t \in (0,1)\), [37];

(d) \(P(G,t)\) has a zero of multiplicity \(b\) at \(t = 1\), [43];

(e) \(P(G,t)\) is non-zero with sign \((-1)^{n+c+b}\) for \(t \in (1, \frac{32}{27}] \approx (1, 1.185]\), [13];

The next result, due to Thomassen and Sokal, indicates that it is not possible to extend Theorem 3 beyond the real interval \((-\infty, \frac{32}{27}]\) or into any region of the complex plane.

**Theorem 4** (a) The real chromatic roots of graphs are dense everywhere in \([\frac{32}{27}, \infty)\) [31]

(b) The complex chromatic roots of graphs are dense everywhere in the complex plane. [28].

### 2.2 3-connected graphs

We can construct 2-connected graphs with real chromatic roots in \((1,2)\) as follows. Let \(G\) be a 2-connected graph with an odd number of vertices. It follows from Theorem 3(c),(d) that \(P(G,1) = 0\) and that \(P(G,t)\) becomes negative when \(t\) increases from 1. If, in addition, we assume that \(G\) is bipartite, then \(P(G,2) = 2\) and hence \(P(G,t)\) must have a zero in \((1,2)\). We
may combine these graphs with any other graph using Lemma 2, with \( r = 2 \), to construct 2-connected non-bipartite graphs with chromatic roots in \((1, 2)\). It is conceivable, however, that Theorem 3 can be extended if we add the hypothesis that \( G \) is 3-connected. (The same construction cannot be used to construct 3-connected non-bipartite examples because \( K_r \) is not bipartite when \( r = 3 \).)

**Conjecture 5** Let \( G \) be a loopless 3-connected graph.

(a) \( P(G, t) \) is non-zero with sign \((-1)^n\) for \( t \in (1, \alpha) \), where \( \alpha \approx 1.781 \) is the chromatic root of \( K_{3,4} \) in \((1, 2)\).

(b) If \( G \) is not a bipartite graph with an odd number of vertices, then \( P(G, t) \) is non-zero with sign \((-1)^n\) for \( t \in (1, 2) \).

Conjecture 5(b) is a slight strengthening of a conjecture given in [13].

The proof technique used in proving Theorem 3 is inductive, using Lemmas 1 and 2. The main difficulty in extending this technique to prove Conjecture 5, is that edge deletion and contraction may result in a graph of connectivity 2, to which Lemma 2 cannot be directly applied.

There seems to be no hope of extending Theorem 3 into the complex plane by adding a connectivity hypothesis since Sokal’s result Theorem 4(b) holds for chromatic roots of \( k \)-connected graphs for any fixed \( k \).

### 2.3 Hamiltonian graphs

Thomassen [30] suggested that the family of hamiltonian graphs may be another family for which Theorem 3 can be extended. He made the following attractive conjecture:

**Conjecture 6** [30] If \( G \) is hamiltonian and loopless then \( P(G, t) \) is non-zero with sign \((-1)^n\) for \( t \in (1, 2) \).

As evidence in favour of this conjecture, Thomassen showed in [32] that the zero free interval of \((1, \frac{32}{27}]\) can be extended for graphs with a Hamilton path.

**Theorem 7** Let \( G \) be a loopless graph with \( n \) vertices and \( b \) blocks. If \( G \) has a Hamilton path, then \( P(G, t) \) is non-zero with sign \((-1)^{n+b+1}\) for \( t \in (1, \gamma) \) where \( \gamma \approx 1.296 \) is the real root of \((2 - t)^3 = 4(t - 1)^2\).
Thomassen [32] also constructed a sequence of graphs with Hamilton paths and with chromatic roots converging to $\gamma$.

We can use Lemmas 1 and 2 to show that a smallest counterexample to Conjecture 6 would be 3-connected, see [30]. Thus Conjecture 6 would follow from Conjecture 5(b). Conjecture 6 would also follow from an affirmative answer to the following intriguing problem posed by Thomassen, using Lemmas 1 and 2, see [30].

**Problem 8** [30] Does every hamiltonian graph of minimum degree at least three have an edge $e$ such that both $G - e$ and $G/e$ are hamiltonian?

### 2.4 Planar Graphs

As mentioned above, chromatic polynomials were introduced by Birkhoff and Lewis [8] in 1946 as a means of attacking the four colour conjecture. Heawood [12] had already proved that the chromatic number of any planar graph is at most five. One of their main results in [8] is the following nice generalization of Heawood’s theorem.

**Theorem 9** Let $G$ be a loopless planar graph. Then $P(G, t) > 0$ for $t \in [5, \infty)$.

Birkhoff and Lewis [8] conjecture that a similar extension of the 4-colour theorem holds.

**Conjecture 10** Let $G$ be a loopless planar graph. Then $P(G, t) > 0$ for $t \in [4, \infty)$.

One might hope that Theorem 3 could also be extended into the real interval $(\frac{32}{27}, 4)$ for the special case of planar graphs. The following result and conjecture of Thomassen [31] indicate that this is probably not the case.

**Theorem 11** The real chromatic roots of planar graphs are dense everywhere in $[\frac{32}{27}, 3]$.

**Conjecture 12** The real chromatic roots of planar graphs are dense everywhere in $[\frac{32}{27}, 4]$. 

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One may also hope that the complex plane may contain zero-free regions for chromatic polynomials of planar graphs. The following result of Sokal [28] limits the location of any such region.

**Theorem 13** The complex chromatic roots of planar graphs are dense everywhere in the complex plane with the possible exception of the disc \(|t - 1| < 1\).

The planar graphs which Sokal uses to prove Theorem 13 are simply two vertices joined by \(p\) internally disjoint paths of length \(d\). He shows that as \(p\) and \(d\) vary, the complex chromatic roots of graphs in this family are dense everywhere outside \(|t - 1| < 1\). Graphs in the family also have roots inside the disc \(|t - 1| < 1\), but it is still an open problem to determine whether the complex roots of planar graphs are dense everywhere in the complex plane. Theorem 13 shows that Theorem 9 cannot be extended to the complex plane.

Much less is known about complex chromatic roots of planar graphs of connectivity greater than 2. Read and Tutte [22] have shown that the ‘bipyramids’ \(C_n + \bar{K}_2\) are examples of 4-connected plane triangulations with complex chromatic roots of unbounded modulus, and Jacobsen, Salas, and Sokal [16] have constructed 4-connected plane graphs which have complex chromatic roots with real part greater than 4. Thus there is no obvious extension of Conjecture 10 to the complex plane, even for 4-connected planar graphs.

### 2.5 Plane Triangulations

A **plane triangulation** is a loopless plane graph in which all faces have size three. They have a special significance in the study of chromatic roots of planar graphs; for example, we can use Lemma 1 to show that Conjecture 10 is true if and only if it is true for plane triangulations. We shall see below that the chromatic roots of plane triangulations are much better behaved than those of planar graphs in general.

Birkhoff and Lewis, and Woodall, have shown that Theorem 3 can be extended beyond \((-\infty, \frac{32}{27}]\) for plane triangulations.

**Theorem 14** Let \(G\) be a 3-connected plane triangulation with \(n\) vertices. Then:

(a) \(P(G, t)\) is non-zero with sign \((-1)^n\) for \(t \in (1, 2)\) [8];
(b) $P(G, t)$ has a simple zero at $t = 2$ [43];

(c) $P(G, t)$ is non-zero with sign $(-1)^{n+1}$ for $t \in (2, \delta)$, where $\delta \approx 2.546$ is the chromatic root of the octahedron in $(2, 3)$ [44].

One may construct infinite families of 3-connected plane triangulations with a chromatic root at $\delta$ by combining the octahedron with an arbitrary plane triangulation using Lemma 2 with $r = 3$. In particular, if we iteratively combine copies of the octahedron with itself, we obtain an infinite family of plane triangulations whose only chromatic root in $(2, 3)$ is $\delta$. On the other hand, the following beautiful conjectures of Woodall [46] suggest that it may be possible to extend Theorem 14 for plane triangulations of connectivity greater than three.

**Conjecture 15** Let $G$ be a plane triangulation.

(a) If $G$ is 4-connected then $P(G, t)$ has at most one zero in $(2, \tau^2)$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\tau^2 \approx 2.61803$.

(b) If $G$ is 5-connected then $P(G, t)$ has exactly one zero in $(2, \theta)$, and no zeros in $(\theta, 3)$, where $\theta \approx 2.61819$ is the chromatic root of the icosahedron in $(2, 3)$.

**Conjecture 16**

(a) For all $\epsilon > 0$, there exist only finitely many 4-connected plane triangulations with a chromatic root in $(2, \tau^2 - \epsilon)$.

(b) For all $\epsilon > 0$, there exist only finitely many 5-connected plane triangulations with a chromatic root in $(\tau^2 + \epsilon, 3)$.

The following intriguing result of Tutte [35, 36] shows that $\tau^2$ has a special significance for chromatic polynomials of plane triangulations.

**Theorem 17** Let $G$ be a plane triangulation with $n$ vertices. Then $0 \neq |P(G, \tau^2)| \leq \tau^{5-n}$.

Note however that Tutte's result should not be seen as evidence that plane triangulations have chromatic roots close to $\tau^2$: we have seen above that there exists an infinite family of 3-connected plane triangulations whose only chromatic root in $(2, 3)$ is $\delta \approx 2.546$. 

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The number $\tau^2$ is an element of an infinite sequence of real numbers, called the Beraha numbers, which seem to have a special significance for chromatic polynomials of plane triangulations, see [37, 5]. For each integer $r \geq 2$, let $b_r = 2 + 2 \cos \frac{2\pi}{r}$. Thus $b_2 = 0, b_3 = 1, b_4 = 2, b_5 = \tau^2, b_6 = 3$, and $\lim_{r \to \infty} b_r = 4$.

**Conjecture 18** (Beraha [3], see [18, 14.6]) There exists a plane triangulation with a real chromatic root in $(b_r - \epsilon, b_r + \epsilon)$ for all $r \geq 2$ and all $\epsilon > 0$.

This conjecture would imply that there exist plane triangulations with real chromatic roots arbitrarily close to 4. Note that it is mistakenly stated in [18] that Beraha and Kahane [4] have shown that this is true: in fact they construct plane triangulations with complex chromatic roots arbitrarily close to 4. Indeed it is an open problem to determine the supremum of the real chromatic roots of plane triangulations. (Lemma 1 can be used to show that this will be equal to the supremum over all planar graphs, and hence Conjecture 12 would also imply that the supremum is 4.) The largest real chromatic root of a plane triangulation that I know of, 3.8267..., is a root of a plane triangulation found by D.R. Woodall [46].

One could also consider the supremum of the real chromatic roots of planar bipartite graphs. Salas and Sokal conjecture that this is $\tau^2$.

**Conjecture 19** [24] Let $G$ be a planar bipartite graph. Then $P(G, t) > 0$ for $t \in [\tau^2, \infty)$.

They constructed families of planar bipartite graphs with chromatic roots tending to $\tau^2$ in [24].

### 2.6 Graphs of Bounded Degree

Let $G$ be a connected graph and $\Delta(G)$ denote the maximum degree of $G$. Brooks [10] has shown that the chromatic number of $G$ is at most $\Delta(G) + 1$ with equality if and only if $G$ is a complete graph or an odd circuit. Brenti, Royle and Wagner [9] asked whether some form of Brooks’s theorem could be true for the complex chromatic roots of $G$. More precisely, they asked whether there exists a real function $f$ such that all complex chromatic roots of $G$ lie in the disc $|t| < f(\Delta(G))$. (This is, in fact, equivalent to an earlier
conjecture for regular graphs due to Biggs, Damerell and Sands [6].) The conjectures have recently been verified by Sokal [27] using an exciting new proof technique.

**Theorem 20** Let \( G \) be a graph. Then \(|P(G,t)| > 0\) for all complex \( t \) with \(|t| \geq C\Delta(G)\), where \( C \approx 7.9639 \).

It is conceivable that Sokal’s theorem is valid for some \( C < 2 \), although there are examples which show that we must take \( C > 1 \), see [27, Section 7], and hence Brooks’s theorem cannot be extended to a disc in the complex plane. It is possible however that the following extension of Brooks’s theorem is valid.

**Conjecture 21** [29] Let \( G \) be a graph. Then \(|P(G,t)| > 0\) for all complex \( t \) with \( \Re(t) > \Delta(G) \).

A graph \( G \) is said to be **d-degenerate** if every subgraph of \( G \) has a vertex of degree at most \( d \). Let \( D(G) \) denote the minimum value of \( d \) for which \( G \) is \( d \)-degenerate. Clearly \( D(G) \leq \Delta(G) \). It is well known that \( \chi(G) \leq D(G) + 1 \). It is not possible, however, to extend this result on integer chromatic roots to the real line, let alone the complex plane, since Thomassen [31] has constructed 2-degenerate graphs with arbitrarily large real chromatic roots.

Let \( G \) and \( H \) be graphs. We say that \( H \) is a **minor** of \( G \) if \( H \) can be obtained from \( G \) by a sequence of edge deletions and contractions. A **simple minor** of \( G \) is a minor which contains no loops or multiple edges. If we modify the definition of \( d \)-degenerate by replacing the condition that “every subgraph of \( G \) has a vertex of degree at most \( d \)” by the condition that “every simple minor of \( G \) has a vertex of degree at most \( d' \)” then Woodall [45] has shown that we do obtain a bound on the size of the real chromatic roots of \( G \).

**Theorem 22** Let \( G \) be a graph. Suppose that every simple minor of \( G \) has a vertex of degree at most \( d \). Then \( P(G,t) > 0 \) for all real \( t \in (d, \infty) \).

Since every simple planar graph has a vertex of degree at most five, his result implies Theorem 9 for all \( t > 5 \). Note that Theorem 22 cannot be extended to a disc in the complex plane since even Theorem 9 does not extend to a disc in the complex plane.
The following conjecture of Sokal, which was inspired by a remark of Shrock and Tsai [26, p220], suggests a possible way to extend Theorem 20 to a larger family of graphs. Let $\Lambda(G)$ denote the maximum number of edge-disjoint paths joining any pair of vertices of $G$. Thus $\Lambda(G) \leq \Delta(G)$. Furthermore, it can be seen that $D(G) \leq \Lambda(G)$ and hence $\chi(G) \leq \Lambda(G) + 1$, see [15].

Conjecture 23 [27, Section 7] There exists a constant $C$ such that $|P(G, t)| > 0$ for all complex $t$ with $|t| \geq C\Lambda(G)$.

3 Flow Polynomials

Let $\Gamma$ be an additive abelian group and $G$ be a graph. Suppose we construct a digraph $\vec{G}$ by giving the edges of $G$ an arbitrary orientation. For $U \subseteq V(G)$ and $\bar{U} = V(G) - U$, let $E^+(U)$ be the set of arcs from $U$ to $\bar{U}$ in $\vec{G}$ and $E^-(U) = E^+(\bar{U})$. Let $f : E(\vec{G}) \to \Gamma$ and put $f^+(U) = \sum_{e \in E^+(U)} f(e)$ and $f^-(U) = \sum_{e \in E^-(U)} f(e)$. For $v \in V(G)$ let $f^+(\{v\}) = f^+(\{v\})$ and $f^-(\{v\}) = f^-(\{v\})$. Then $f$ is a $\Gamma$-flow for $G$, with respect to $\vec{G}$, if $f^+(v) = f^-(v)$ for all $v \in V(G)$. If in addition, $f(e) \neq 0$ for all $e \in E(G)$, then we say that $f$ is a nowhere-zero $\Gamma$-flow of $G$. It can be seen that the condition $f^+(v) = f^-(v)$ for all $v \in V(G)$ is equivalent to the apparently stronger condition that $f^+(U) = f^-(U)$ for all $U \subseteq V(G)$. Thus, if $G$ has a nowhere zero $\Gamma$-flow, then $G$ is bridgeless. Since reversing the orientation on an edge $e$ of $\vec{G}$ is equivalent to replacing $f(e)$ by $-f(e)$, the number of distinct nowhere-zero $\Gamma$-flows for $G$ is independent of the chosen orientation $\vec{G}$ of $G$.

A nowhere-zero $t$-flow of $G$ is a nowhere-zero $\mathbb{Z}$-flow, $f$, such that $|f(e)| \leq t - 1$ for all $e \in E(G)$. Tutte [33] has shown that $G$ has a nowhere-zero $t$-flow if and only if $G$ has a nowhere-zero $\mathbb{Z}_t$-flow. Furthermore, the number of distinct nowhere-zero $\Gamma$-flows of $G$ is the same for all abelian groups $\Gamma$ of the same order. Note, however, that the number of nowhere-zero $\mathbb{Z}_t$-flows of $G$ may differ from the number of nowhere-zero $t$-flows of $G$. Nowhere-zero flows were introduced by Tutte [33] as a dual concept to proper colourings. He showed that a connected plane graph $G$ has a proper $t$-colouring if and only if its planar dual $G^*$ has a nowhere-zero $t$-flow. The two concepts differ for non-planar graphs, however. Indeed, whereas there
exist loopless graphs which are not \( t \)-colourable for arbitrarily large integers \( t \), the same is not true for bridgeless graphs and nowhere-zero \( t \)-flows.

**Theorem 24** Let \( G \) be a bridgeless graph. Then:

(a) \( G \) has a nowhere-zero 6-flow, (Seymour [25]);

(b) if \( G \) has no 3-edge cuts then \( G \) has a nowhere-zero 4-flow, (Jaeger [17]).

Tutte conjectures that both of these results can be strengthened.

**Conjecture 25** Let \( G \) be a bridgeless graph. Then

(a) \( G \) has a nowhere-zero 5-flow [34];

(b) if \( G \) has no 3-edge cuts then \( G \) has a nowhere-zero 3-flow.

Conjecture 25(b) was stated by Tutte in 1972.

Following Tutte [34] we define the flow polynomial \( F(G, t) \) of \( G \) as the number of distinct nowhere-zero \( \mathbb{Z}_t \)-flows of \( G \) for any positive integer \( t \). Thus \( F(G, t) \equiv 1 \) if \( E(G) = \emptyset \) and \( F(G, t) \equiv 0 \) if \( G \) has a bridge. We shall refer to the zeros of \( F(G, t) \) as flow roots of \( G \).

By the above remarks, \( F(G, t) \) is independent of the chosen orientation of \( G \), and remains the same if we replace \( \mathbb{Z}_t \) by any other abelian group of order \( t \). Note also that since the existence of a nowhere-zero \( t \)-flow for \( G \) implies the existence of a nowhere-zero \((t + 1)\)-flow by definition, and is equivalent to the existence of a nowhere-zero \( \mathbb{Z}_t \)-flow as noted above, we may deduce that if \( F(G, t_0) \neq 0 \) for some positive integer \( t_0 \), then \( F(G, t) \neq 0 \) for all integers \( t \geq t_0 \).

We could also consider the polynomial \( I(G, t) \) defined to be the number of distinct nowhere-zero \( t \)-flows of \( G \). Kochol [19] gives relationships between \( I(G, t) \) and \( F(G, t) \), but these seem to be the only results on \( I(G, t) \) in the literature. Attention has concentrated on \( F(G, t) \) because it is dual to \( P(G, t) \) for plane graphs. Let \( G \) be a connected plane graph and \( G^* \) be its planar dual. There is a surjection from the \( t \)-vertex-colourings of \( G^* \) to the nowhere-zero \( \mathbb{Z}_t \)-flows of \( G \), such that each nowhere-zero \( \mathbb{Z}_t \)-flow of \( G \) has exactly \( t \) pre-images, see [34]. Thus

\[
F(G, t) = t^{-1} P(G^*, t).
\]
We may use this identity to restate the results and conjectures on chromatic roots of plane graphs in subsections 2.4 and 2.5 in terms of flow roots of plane graphs. In the remainder of this section we shall survey results and conjectures on flow roots of graphs which are not necessarily planar.

We first state some fundamental recurrence relations for flow polynomials.

Lemma 26 Let $G$ be a graph and $e$ be an edge of $G$.

(a) If $e$ is a loop, then $F(G, t) = (t - 1)F(G - e, t)$.

(b) If $e$ is not a loop, then $F(G, t) = F(G/e, t) - F(G - e, t)$.

Lemma 27 Let $G$ be a graph, $v$ be a vertex of $G$, and $G_1$ and $G_2$ be edge-disjoint subgraphs of $G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{v\}$. Then

$$F(G, t) = F(G_1, t)F(G_2, t).$$

Lemma 28 Let $G$ be a 2-connected graph, $v$ be a vertex of $G$, $e$ be an edge of $G$, and $H_1$ and $H_2$ be edge-disjoint subgraphs of $G$ such that $H_1 \cup H_2 = G - e$ and $H_1 \cap H_2 = \{v\}$. Let $G_1$ be obtained from $G$ by contracting $E(H_2)$, and define $G_2$ analogously. Then

$$F(G, t) = \frac{F(G_1, t)F(G_2, t)}{(t - 1)}.\]$$

Lemma 29 Let $G$ be a graph, $S$ be a 3-edge-cut of $G$, and $H_1$ and $H_2$ be the components of $G - S$. Let $G_1$ be obtained from $G$ by contracting $E(H_2)$, and define $G_2$ analogously. Then

$$F(G, t) = \frac{F(G_1, t)F(G_2, t)}{(t - 1)(t - 2)}.\]$$

3.1 General Graphs

Wakelin [38] obtained the following analogue of Theorem 3 for flow roots of graphs.

Theorem 30 Let $G$ be a bridgeless graph with $n$ vertices, $m$ edges, $b$ blocks, and no isolated vertices. Then:
(a) \( F(G,t) \) is non-zero with sign \( (-1)^{m-n+1} \) for \( t \in (-\infty, 1) \);

(b) \( F(G,t) \) has a zero of multiplicity \( b \) at \( t = 1 \);

(c) \( F(G,t) \) is non-zero with sign \( (-1)^{m-n+b+1} \) for \( t \in (1, \frac{32}{27}) \).

Using planar duality, we can restate Theorems 11 and 13 in terms of flow roots to deduce that there is no obvious way to extend Theorem 30 beyond the real interval \((-\infty, \frac{32}{27})\), or into the complex plane.

**Theorem 31**  
(a) The real flow roots of planar graphs are dense everywhere in \([\frac{32}{27}, 3]\).

(b) The complex flow roots of planar graphs are dense everywhere in the complex plane with the possible exception of the disc \(|t-1| < 1\).

We can construct 2-connected graphs with real flow roots in \((1, 2)\) as follows. Let \( G \) be a 2-connected graph with \( n \) vertices and \( m \) edges and suppose that \( m-n \) is odd. It follows from Theorem 30(b),(c) that \( F(G,1) = 0 \) and that \( F(G,t) \) becomes negative when \( t \) increases from 1. If, in addition, we assume that \( G \) is Eulerian, then \( F(G,2) = 1 \) and hence \( F(G,t) \) must have a zero in \((1, 2)\). We may combine these graphs with any other graph using Lemma 28 to construct non-Eulerian 2-connected graphs with flow roots in \((1, 2)\). It is conceivable, however, that Theorem 30 can be extended if we add the hypothesis that \( G \) is 3-connected.

**Conjecture 32** Let \( G \) be a 3-connected graph with \( n \) vertices, and \( m \) edges. Then:

(a) \( F(G,t) \) is non-zero with sign \( (-1)^{m-n} \) for \( t \in (1, \phi) \), where \( \phi \approx 1.749 \) is the flow root of \( K_5 \) in \((1, 2)\).

(b) If \( G \) is not an Eulerian graph with \( m-n \) odd, then \( F(G,t) \) is non-zero with sign \( (-1)^{m-n} \) for \( t \in (1, 2) \).

Welsh [40] has conjectured that the dual form of Conjecture 10 holds for all graphs (with the slight relaxation that 4 is allowed to be a flow root).

**Conjecture 33** Let \( G \) be a bridgeless graph. Then \( F(G,t) > 0 \) for all real \( t \in (4, \infty) \).
It is not even known whether this conjecture is true if 4 is replaced by any large constant $C$. Note however that there seems to be no obvious analogue of Conjecture 33 for the complex plane: by applying planar duality to the remark given at the end of subsection 2.4, we may deduce that there exist 3-connected plane cubic graphs with complex flow roots of unbounded modulus, and that there exist 3-connected plane graphs with complex flow roots whose real part is greater than 4.

### 3.2 Cubic Graphs

A graph $G$ is **cubic** if all its vertices have degree three. They have a special significance in the study of flow roots of graphs; for example, we can use Lemma 26 to show that Conjectures 25(a) and 33 are true if and only if they are true for cubic graphs. We shall see below that the flow roots of cubic graphs are much better behaved than those of graphs in general. Note that, using planar duality, we can restate the results and conjectures on chromatic roots of plane triangulations in terms of flow roots of planar cubic graphs. I have recently obtained the following extension of (the dual form of) Theorem 14 to cubic graphs which are not necessarily planar.

**Theorem 34** [14] Let $G$ be a 3-connected cubic graph with $n$ vertices, and $m$ edges. Then

(a) $F(G, t)$ is non-zero with sign $(-1)^{m-n}$ for $t \in (1, 2)$;

(b) $F(G, t)$ has a zero of multiplicity 1 at $t = 2$;

(c) $F(G, t)$ is non-zero with sign $(-1)^{m-n+1}$ for $t \in (2, \delta)$, where $\delta \approx 2.546$ is the flow root of the cube in $(2, 3)$.

It seems likely that the non-zero interval given in Theorem 34(c) may be extended beyond $\delta$ for cubic graphs of ‘higher connectivity’. We use the following concept to give a measure of ‘4-connectivity’ in cubic graphs. A graph $G$ is **cyclically $k$-connected** if, whenever we can express $G$ as $G = G_1 \cup G_2$, where $E(G_1) \cap E(G_2) = \emptyset$ and $G_1$ and $G_2$ both contain circuits, we must have $|V(G_1) \cap V(G_2)| \geq k$. (Cyclic $k$-connectivity is the dual concept to $k$-connectivity in plane graphs.) Using this concept we may pose the following strengthenings of Conjectures 15(a) and 16(a).
**Conjecture 35** Let $G$ be a cyclically-4-connected cubic graph. Then $F(G, t)$ has at most one zero in $(2, \tau^2)$.

**Conjecture 36** For all $\epsilon > 0$, there exist only finitely many cyclically-4-connected cubic graphs with a flow root in $(2, \tau^2 - \epsilon)$.

### 4 Characteristic Polynomials of Matroids

The reader may have wondered whether there is a more general framework in which the duality between chromatic and flow polynomials of plane graphs can be extended. Matroids provide us with such a framework. I refer the reader to Oxley [21] or Welsh [39] for matroid definitions not explicitly stated in this article.

The **characteristic polynomial** $C(M, t)$ of a matroid $M = (E, r)$ is the polynomial in $t$ defined by

$$C(M, t) = \sum_{A \subseteq E} (-1)^{|A|} t^{r(E) - r(A)}.$$

We can define the dual matroid $M^*$ for any matroid $M$. We can also associate a pair of dual matroids to every graph $G$, the cycle matroid $M_G$ and the cocycle matroid $M^*_G$. Then $C(M_G, t) = t^{-c} P(G, t)$, where $c$ is the number of components of $G$, and $C(M^*_G, t) = F(G, t)$. Furthermore, when $G$ is a connected plane graph with planar dual $G^*$, the cycle matroid of $G^*$ is equal to the cocycle matroid of $G$. This gives the above mentioned identity $F(G, t) = t^{-1} P(G^*, t)$.

The integer chromatic roots and flow roots of a graph both occur as sequences of consecutive integers. Examples are given in [39, p254] to show that this basic property may not hold for the integer zeros of the characteristic polynomial of a matroid. Nevertheless, some of the above mentioned results on real and complex chromatic roots do extend to matroids. The following result from [11] gives a common generalization of Theorems 3 and 30.

**Theorem 37** Let $M$ be a loopless matroid with rank $r$ and $b$ components. Then:

(a) $C(M, t)$ is non-zero with sign $(-1)^r$ for $t \in (-\infty, 1)$;
(b) $C(M, t)$ has a zero of multiplicity $b$ at $t = 1$;

(c) $C(M, t)$ is non-zero with sign $(-1)^{r+b}$ for $t \in (1, \frac{32}{27})$.

Let $M$ and $N$ be matroids. We say that $N$ is a minor of $M$ if $N$ can be obtained from $M$ by a sequence of deletions and contractions. A simple minor of $M$ is a minor which contains no loops or circuits of length two.

Oxley [20] has shown that if every cocircuit of $M$ has size at most $d$ then $C(M, t) > 0$ for all real $t \in (d, \infty)$. His proof, which uses induction on $|E(M)|$, can be used to prove a stronger inductive statement which extends Theorem 22.

**Theorem 38** Let $M$ be a matroid. Suppose that every simple minor of $M$ has a cocircuit of size at most $d$. Then $C(M, t) > 0$ for all real $t \in (d, \infty)$.

Applying this result to the special case of cographic matroids, we obtain:

**Corollary 39** Let $G$ be a bridgeless graph. Suppose that every 3-edge-connected minor of $G$ has a circuit of length at most $d$. Then $F(G, t) > 0$ for all real $t \in (d, \infty)$.

Since every 3-connected graph $G$ has a circuit of length at most $2 \log_2 |V(G)|$, this gives:

**Corollary 40** Let $G$ be a bridgeless graph on $n$ vertices. Then $F(G, t) > 0$ for all real $t \in (2 \log_2 n, \infty)$.

It is also possible that Theorem 20 and Conjecture 23 can be extended to binary matroids. Given a binary matroid $M$, let $\Lambda(M) = \min_B \max_{C \in B} \{|C|\}$ where the minimum is taken over all bases $B$ of the cocycle space of $M$. (It can be seen that if $M$ is the cycle matroid of a graph $G$ then $\Lambda(M) = \Lambda(G)$, see [15].)

**Conjecture 41** [15] There exists a constant $D$ such that for all loopless binary matroids $M$ and all complex numbers $t$ with $|t| \geq D\Lambda(M)$, we have $C(M, t) \neq 0$.

Applying this conjecture when $M$ is the cographic matroid of a graph $G$ we obtain a conjecture for flow roots of graphs. Let $\Lambda^*(G) = \min_B \max_{C \in B} \{|C|\}$, where the minimum is taken over all bases $B$ of the cycle space of $G$.

**Conjecture 42** [15] There exists a constant $D$ such that for all bridgeless graphs $G$ and all complex numbers $t$ with $|t| \geq D\Lambda^*(G)$, we have $F(M, t) \neq 0$. 17
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