COMPACTIFIED QUANTUM FIELDS. IS THERE LIFE BEYOND THE CUT-OFF SCALE?

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ABSTRACT. A consistent definition of high dimensional compactified quantum field theory without breaking the Kaluza–Klein tower is proposed. It is possible in the limit when the size of compact dimensions is of the order of the cut off. This limit is nontrivial and depends on the geometry of compact dimensions. Possible consequences are discussed for the scalar model.

INTRODUCTION

The Nature at high energies is supposed to be described by some string models (see e.g. [1]). In the local approximation the strings are known to be described at tree level field models. Strings are formulated in high-dimensional space-time, therefore the respective field models are also high-dimensional. The low energy observable world, however, is low-dimensional and is described by local quantum field theories. Compactification provides a mechanism that lower the effective dimensionality of the model in low energy limit. In 80’s it was shown (see [2]) that at classical level compactification may lead to low-dimensional ($D = 4$) unified models through Kaluza–Klein mechanism.

Thus, one is justified to expect the compactification to appear due to quantum corrections to the high-dimensional field model. The main problem the quantum treatment is faced with is one of non-renormalisability proper for the high-dimensional field models.

The possible one-loop counterterms arising in high-dimensional quantum field models were computed many years ago [3], and are known indeed to lead to non-renormalisability of the effective action at any finite size of the compact dimensions. The model become worse as the number of compact dimensions increase.

The non-renormalisability in high dimensions is an UV feature and it persists also in compactified models (unless the compactification size reach the UV region!).

A way to avoid the problem with non-renormalisability would be consideration of a larger model, i.e. of the string theory [4] in the case when the high dimensional model under consideration is the effective field theory for such a string model, in this case to absorb the non-renormalisable divergences one has to consider nonlocal modes, thus leaving the framework of local field theory.

The careful analysis here can, probably, be done but the field theoretical approach is simpler and, we believe, more instructive. We believe that the quantum

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\[ ^1 \text{In Refs. [4], the problem of the supersymmetry breaking and coupling renormalisation induced by compactification was studied in the framework of string theory. It was shown that for some compactifications the the low dimensional models feel extra dimensions at energies } \sim 1 \text{ TeV.} \]
field theory approach starts to be applicable when the size of the compact dimensions is close enough to zero. Quantitatively this may happen when the size of compact dimensions is of the order of UV cutoff (or even higher), i.e. when the contribution of the order of size of compact dimensions is negligible.

Compactification can be realised in different ways the simplest one being when the high-dimensional space-time is represented as a product of a compact space with (low-dimensional) Minkowski space-time. Compactification to circle of the scalar field was considered in our previous work [5].

Spectrum of compactified models consists of propagating part which correspond to modes which are “constant” in the compact directions, they form the field content of the low-dimensional model. Beyond the propagating part there is an infinite number of massive Kaluza–Klein (KK) modes. Their masses are of the order of inverse specific size of the compact dimensions. When this size is small, the respective masses are large and KK modes cannot propagate in low dimensions, but due to interaction with propagating modes their quantum fluctuations can influence the dynamics of the latter even at low energies.

The aim of the actual paper is to investigate which is the condition of existence of compactified model as a renormalisable low-dimensional effective quantum field theory, and which is the effect of the high-dimensions to this theory. These days phenomenological aspects of the compactified high dimensional models are intensively studied in the framework of GUT building, (for a review see e.g. [6]), although the nature of compactified models as QFT in these investigations is not yet clear. We hope that actual study clarifies some of these points.

Our aim is integration of KK modes in order to get the low dimensional effective action for remaining propagating fields. The non-renormalisability problem here takes the form of divergence of the sums over the KK contributions in some Feynman diagrams although the loop contribution of each KK field corresponding to this diagram taken separately is renormalisable and vanishes due in the limit of the small compactification size. These divergencies, however, are regularised by the same UV regularisation we use to compute each KK diagram. Combining cutoff removing with the zero limit of the compactification size one may get some finite contribution to effective action from the KK modes. It may seem that in this limit the model is trivially reduced to a low dimensional one which “forgot” everything about the higher dimensions. We show that even there the low dimensional effective model has strong dependence on the geometry of compact dimensions.

Such a combined limiting procedure means that the (physical) size of compact dimensions is in UV region. In other words the validity of QFT approach is limited to energies below the inverse compactification size. Also, we consider that at energies where the QFT approach is applicable gravity is in its classical regime. Otherwise, one is thrown from the local QFT approach due to non-renormalisability of gravitational interactions.

The plan of the paper is as follows. We consider scalar model with $\frac{\lambda}{4!}\phi^4$ interaction in a high-dimensional space-time. First we compactify the space-time on the product of low-dimensional Minkowski space-time and a compact Riemannian manifold. After that we integrate over KK and high energy modes of propagating field i.e. find their one loop-contribution. Then we find the renormalisability conditions for obtained model, and, finally analyse the dependence of the low dimensional model on the compactification geometry.
1. Compactified Model Renormalisation

In what follows we will consider the model of self-interacting scalar field $\phi$ on a $(D+p)$-dimensional space-time manifold $\mathcal{W}_{D+p}$. It is described by the classical action,

$$S_{(D+p)} = \int_{\mathcal{W}_{D+p}} d^{D+p}x \left\{ \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \right\},$$

where capital roman letters $M,N = 0,1,\ldots,D+p-1$, are associate with indices of high-dimensional space-time manifold, metric signature is chosen to be $+ - - \cdots -$. We assume potential $V(\phi)$ to be renormalisable in $D$-dimensions. Thus if one compactifies this model to $D=4$ the potential $V(\phi)$ should be at most quartic in $\phi$. For definiteness we will consider that,

$$V(\phi) = \frac{\lambda}{4!} \phi^4.$$  

Let the manifold $\mathcal{W}_{D+p}$ on which the fields are defined be represented as a product $\mathcal{W}_{D+p} = \mathcal{M}_D \times \mathcal{K}_p$, where the $\mathcal{M}_D$ is $D$-dimensional Minkowski space-time and $\mathcal{K}_p$ is a compact Riemannian manifold. In what follows greek indices $\mu, \nu, \ldots$ enumerate coordinates $x^\mu$ of $\mathcal{M}_D$ and latin indices $i,j, \ldots$, respectively, coordinates $y^i$ of $\mathcal{K}_p$.

Kinetic term of the action (1) can be split in the Minkowski and $\mathcal{K}_p$-part according to the decomposition $\mathcal{W}_{D+p} = \mathcal{M}_D \times \mathcal{K}_p$ as follows,

$$\frac{1}{2} \partial_M \phi \partial^M \phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \phi \Delta \phi,$$

where $\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is kinetic term on the Minkowski space-time $\mathcal{M}_D$, and

$$\Delta = \frac{1}{\sqrt{\det g}} \partial_i g^{ij} \sqrt{\det g} \partial_j,$$

is the scalar Laplace operator on $\mathcal{K}_p$.

Since $\mathcal{K}_p$ is compact the operator $\Delta$ has discrete spectrum with orthonormal eigenfunctions

$$\Delta \psi_n = -\mu_n^2 \psi_n,$$

$$\int_{\mathcal{K}_p} dy \psi_n^* \psi_m = \delta_{nm},$$

where $dy \equiv \sqrt{\det(g)}d^p y$ is the invariant measure on $\mathcal{K}_p$ and $n,m$ span a $p$-dimensional (generally irregular) lattice $\Gamma$. Complex conjugate of an eigenfunction is an eigenfunction again,

$$\psi_n^* = \psi_{n^*},$$

with $\mu_{n^*}^2 = \mu_n^2$.

Let us note that zero modes of $\Delta$ are locally constant functions. The number of independent locally constant functions is equal to the number of connected components of $\mathcal{K}_p$. This represents well known fact from the cohomology theory (see e.g. [7]), as well as from the properties of scalar Laplace operator. In any case, eigenfunctions of $\Delta$ can be chosen in such a way to have support on a single connected component of $\mathcal{K}_p$. 


One can decompose $\phi(x, y)$ in terms of eigenfunctions of $\Delta$,

$$\phi(x, y) = \sum_{n \in \Gamma} \phi_n(x) \psi_n(y),$$

where, $$\phi_n(x) = \int_{K_p} dy \psi^* (y) \cdot \phi(x, y).$$

In terms of this decomposition the classical action (1) looks as follows

$$S_D = \int_{\mathcal{M}_D} dx \left\{ \frac{1}{2} \sum_n \left( \partial_\mu \phi_n^* \partial^\mu \phi_n - (m_n^2 + \mu_n^2) |\phi_n|^2 \right) + \int_{K_p} dy V(\phi) |_{\phi = \sum \phi_n \psi_n} \right\}.$$

To obtain eq. (10) we used orthonormality of the spectrum of $\Delta$. In the case of disconnected $K_p$ the action is split in independent non-interacting parts corresponding to each connected component. Thus, without loss of generality from here on we can assume that $K_p$ is connected.

In what follows we will consider compactifications with small compact size, in this case one can neglect $m_n^2$ in comparison with nonzero $\mu_n^2$ in eq. (10).

If one drops out the terms with nonzero $\mu_n^2$ one gets classically compactified (or dimensionally reduced) $D$-dimensional model. It is given by the following action,

$$S_D = \int_{\mathcal{M}_D} dx \left\{ \frac{1}{2} \left( \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right) - \bar{\lambda} \frac{1}{4!} \varphi^4 \right\},$$

where $\varphi$ is the zero mode component of $\phi$, and coupling $\bar{\lambda}$ is given by rescaling, $\bar{\lambda} = \lambda V_p^{-1}$, $V_p = \int_{K_p} dy$ is the volume of $K_p$.

Action (11) plays the role of the bare action which will get corrections from KK fields in (10). As we mentioned in the Introduction, to be fully consequent we have to add also the contribution of high energy modes of $\varphi$. The last could be obtained by introducing an extra Pauli–Villars (PV) regularisation for $\varphi$-loops (see e.g. [8]) with cutoff mass $M \sim (V_p)^{-\frac{1}{D}}$ and computing one-loop contribution from the regulator(s). For our purposes will suffice three PV fields with the action,

$$S_{PV} = \sum_{r=1}^3 \int_{\mathcal{M}_D} dx \left\{ \frac{1}{2} \left( \partial_\mu \phi_r \partial^\mu \phi_r - M_r^2 \phi_r^2 \right) - \bar{\lambda} \frac{1}{4!} \phi_r^4 \right\},$$

where the PV masses $M_r$ satisfy,

$$\sum_r (-1)^r M_r^2 = -m^2.$$  

Grassmann parity of $\phi_r$ is chosen to be $(-1)^r$.

The effective action we want to compute is defined as follows,

$$e^{iS_{eff}(\varphi)} = \int d\phi_M \prod_{n \neq 0} d\phi_n \ e^{iS_D(\varphi, \phi_n) + iS_{PV}(\varphi, \phi_r)}.$$

The coupling $\lambda$ in (1) gives rise to an infinite number of couplings in compactified action (10),

$$\frac{\lambda}{4!} \varphi^4 \rightarrow \frac{1}{4!} \lambda^{n_1 \ldots n_4} \phi_{n_1} \phi_{n_2} \phi_{n_3} \phi_{n_4} \equiv \lambda \int_{K_p} dy \psi_{n_1} \psi_{n_2} \psi_{n_3} \psi_{n_4},$$

where

$$\lambda^{n_1 \ldots n_4} = \lambda \int_{K_p} dy \psi_{n_1} \psi_{n_2} \psi_{n_3} \psi_{n_4}.$$
where, in general, $\lambda^{n_1\cdots n_l}$ are all nonzero, but for a smooth manifold $K_p$ they decay faster than any power of $n = \|n_1 + \cdots + n_4\|$ (where $\| \|$ is some properly defined norm on the lattice $\Gamma$). We will not discuss further this property, moreover in one-loop approximation it does not play any role.

In fact, one-loop computations require only a small part of couplings $\lambda^{n_1\cdots n_l}$. Since for nonzero $\mu^2_n$ fields $\phi_n$ are non-propagating the one-loop diagrams contain only interactions quadratic in KK modes $\phi_n$, $n \neq 0$,

$$V_{(2)} = \frac{\bar{\lambda}}{4} \nu^2 |\phi_n|^2. \quad (16)$$

To compute the one-loop effective action we also need the propagators of the fields. They look as follows,

$$D_{n}^{-1}(p) = p^2 - \mu^2_n + i\epsilon, \quad \text{for } n \neq 0, \quad (17)$$
$$D^{-1}_r(p) = p^2 - M^2_r + i\epsilon, \quad \text{for PV regulator}, \quad (18)$$

where $i\epsilon$ stands for the causal pole prescription. Let us note that both KK interaction term (16) and KK propagator (17) is of the same form as interaction term in (12) and propagator for PV fields (18). This allows one to include KK and PV fields in the same set. From now on let the index $n$ span both KK and PV fields.

Consider diagrams with KK/PV fields running in the loop and with external $\phi$ legs.

In what follows we use dimensional regularisation for KK-fields and PV regulators. General one-loop diagram with $2N$ (truncated) external legs has $N$ vertices and looks as follows,

$$(19) \quad G(k_1, \ldots, k_N) =$$
$$\frac{\bar{\lambda}^N}{4^N} \sum_n \int \frac{d^D p}{(2\pi)^D} D_{n r}^{-1}(p + k_1) \ldots D_{n r}^{-1}(p + k_1 + \cdots + k_N),$$

where integration is performed over $D$-dimensional Minkowski momentum space. As usual in dimensional regularisation scheme coupling $\bar{\lambda}(D) = \lambda_0 \kappa^{4-D}$, where $\lambda_0$ is the dimensionless coupling and $\kappa$ is the mass unity. Summation in (19) and on is performed through the eigenvalue lattice of $\Delta$ where zero term is substituted by PV regulator. Cutoff removing is obtained when $D \to D$.

The formal divergence index of this diagram is $\omega_G = D - 2N$. This index, however, is computed termwise and does not reflect possible divergencies due to the summation over $n$. From above it follows the termwise divergencies described by $\omega_G$ are the same for KK and PV modes. They are due to large momenta in low dimensional directions and have low dimensional nature. From the other hand, divergencies due to summation over KK modes correspond to large momenta in compactified directions and, thus they have the “high-dimensional nature”.

Let us perform a change of the integration variable, $p \to \frac{p}{\mu_n}$ for each $n$. This is legitimate since all integrals are regularised.

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2An alternative choice would be Wick rotation to Euclidean space-time as was used in [5].
After this the integral (19) looks as follows,

\begin{equation}
G(k_1, \ldots, k_N) = \frac{\lambda^N}{4^N} \sum_n (\mu_n^2)^{-(N - \frac{d}{2})} \int \frac{d^D p}{(2\pi)^D} D(p + k_i^0) \ldots D(p + k_N^0 + \cdots + k_N^0),
\end{equation}

where $k_i^0 = \frac{q_i}{\mu_n}$.

One can see that all dependence on $n$ in integrand in (20) resides in $k_i^0$. All “external momenta” $k_i^0$ are proportional to a vanishing factor $\frac{1}{\mu_n}$. So, one can safely expand Feynman integral (20) in powers of $k_i$. This yields,

\begin{equation}
G(k_1, \ldots, k_N) = \frac{\lambda^N}{4^N} \left\{ I_{(0)}(\hat{D}, m) \sum_n (\mu_n^2)^{-(N - \frac{d}{2})} + I_{(2)}(\hat{D}, \bar{D}) \sum_n (\mu_n^2)^{-(N + \frac{d}{2} - \frac{D}{2})} + \ldots + I_{(r)}(\hat{D}, \bar{D}) \sum_n (\mu_n^2)^{-(N + \frac{d}{2} - \frac{D}{2})} + \ldots \right\},
\end{equation}

where integrals $I_{(r)}$ are defined as follows

\begin{equation}
I_{(r)}(\hat{D}) = \frac{1}{r!} \left. q^i_{\mu_1} \ldots q^r_{\mu_r} \frac{\partial^r}{\partial t_{\mu_1} \ldots \partial t_{\mu_r}} \right|_{t_1 = 0} \int \frac{d^D p}{(2\pi)^D} D(p + t_1) \ldots D(p + t_N),
\end{equation}

we introduced notations $q^i = \sum_{j=1}^N k_j$. Divergence degree of each $I_{(r)}$ is $\omega_{G,r} = D - (2N + r)$. Let us note that, starting from some $r \geq D - 2N$ the integrals $I_r$ are finite, therefore, expansion in $k_i^0$ is well defined.

One can see that expression (21) has the standard form of a high energy mode contribution, except for masses which are substituted by the series over powers of $\mu_n^2$.

These series define well-known $\zeta$-function of the Laplace operator, $\zeta(s)$,

\begin{equation}
\zeta\Delta(s) = \text{tr}(\hat{\Delta})^{-s} = \sum_n (\mu_n^2)^{-s},
\end{equation}

where the prime means that summation is performed over the nonzero eigenvalues $\mu_n^2$ (PV masses not included). These series are convergent for Re $s > p_d/2$, but can be analytically continued to other values of $s$.

Using definition (23) one can rewrite eq. (21) in the following form,

\begin{equation}
G(k_1, \ldots, k_N) = \frac{\lambda}{4^N} \left\{ I(\hat{D}) \left( \zeta\Delta \left( N - \frac{\hat{D}}{2} \right) + \sum_r (-1)^r M_r^\hat{D} - 2N \right) + I_{(2)}(\hat{D}, \bar{D}) \left( \zeta\Delta \left( N + \frac{2 - \hat{D}}{2} \right) + \sum_r (-1)^r M_r^\hat{D} - 2(N + 1) \right) + \ldots + I_{(r)}(\hat{D}, \bar{D}) \left( \zeta\Delta \left( N + \frac{r - \hat{D}}{2} \right) + \sum_r (-1)^r M_r^\hat{D} - 2(N + r) \right) + \ldots \right\}.
\end{equation}
Expression in the r.h.s. of eq (24) can diverge in the limit $\tilde{D} \to D$ due to the following two factors. First one is singularity of some $I(r)(\tilde{D})$ and second one is singularity of $\zeta_\Delta(s)$.

Singularities of $I(r)$ does not depend on index $n$. Therefore, the renormalisability of the low dimensional reduced model implies possibility to eliminate also singularities of this type due to KK modes by a modification of already existing low dimensional counter-terms. Let us note that $I(r)$ is proportional to the factor,

$$I_r(\tilde{D}) \sim \Gamma \left( N + \frac{r-D}{2} \right),$$

which is singular for $N + \frac{r-D}{2} = 0, -1, \ldots, \frac{D}{2}$, when $D$ is even. At the same time $\zeta_\Delta \left( N + \frac{r-D}{2} \right)$ in this points is regular (see [9]). Thus, the counterterms required to cancel this type of singularities have the form

$$\Delta Z_{N,r}^I = \left( m^{2N+(r-D)} - \kappa^{2N+(r-D)} \zeta_\Delta \left( N + \frac{r-D}{2} \right) \right) \Delta Z_{N,r},$$

where $\Delta Z_{N,r}$ is the low dimensional (reduced) model counterterm and $\kappa$ is the “mass unity” of dimensional regularisation.

Consider now divergencies due to singularities of $\zeta$-function. It is known, that $\zeta_\Delta(s)$ has a meromorphic extension to the entire complex plane except for isolated simple poles on the real axis at $s = p/2 - n$, $n = 0, 1, \ldots, \left[ \frac{p}{2} \right] - 1$. Residues of $\zeta_\Delta(s)$ at this points can be expressed in terms of the heat kernel invariants $a_{2n}(\Delta)$ of the Laplace operator,

$$\text{Res}_{s=\frac{p}{2} - n} \zeta_\Delta(s) = \frac{a_{2n}(\Delta)}{\Gamma \left( \frac{p}{2} - n \right)}.$$  

One can observe that $\zeta$-function variable is related to the external divergence index as $s = -\frac{1}{2} \omega_{G,r}$. Thus, eq. (27) lead to singular contributions corresponding to several diagrams with negative divergence index. In low dimensional theory such diagrams are convergent and there are no terms in the bar action (11) to allow absorption of the divergent part corresponding to such diagrams. In order to avoid the non-renormalisability one have to get rid of such contribution.

To do this consider invariants $a_{2n}(\Delta)$. They arise as quotients in expansion of the diagonal part of the Heat Kernel at $t = 0$,

$$K(t; y, y) = \sum_{n=0}^{\infty} t^{-\frac{n}{2}} a_n(\Delta, y),$$

where

$$a_n(\Delta) = \int_{K_y} dy a_n(\Delta, y).$$

For a Riemannian manifold they can be computed in terms of integrals of local quantities (metric, spin connection and curvature tensor). The first three nontrivial

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3We use notations of the Gilkey’s book [9].
invariants look as follows \[9\],

\[
a_0(\Delta) = \frac{V_p}{(4\pi)^{p/2}},
\]

\[
a_2(\Delta) = -\frac{1}{6(4\pi)^{p/2}} \int_{K_p} dy \, R,
\]

\[
a_4(\Delta) = \frac{1}{360(4\pi)^{p/2}} \int_{K_p} dy \left( -12\Delta R + 5R^2 - 2R_{ij} R^{ijkl} + 2R_{ijkl} R^{ijkl} \right),
\]

where \(R_{ijkl}\) and \(R\) are respectively Riemannian and scalar curvature. Invariants with odd index vanish, \(a_{2n+1} = 0\). Higher invariants depend on higher powers of curvature and its covariant derivatives.

Let define \(\ell \equiv (V_p)^{\frac{1}{p}}\) to be the characteristic size of \(K_p\). From the definition of \(\zeta\)-function one can see that at some fixed point \(s = s_0\),

\[
\zeta(s_0) \to L^{2s_0} \zeta(s_0)
\]

under global rescaling of the metric \(g_{ij} \to L^2 g_{ij}\). Therefore, at \(s_0\), \(\zeta(s_0) \sim \ell^{2s_0}\).

If one takes the limit \(\ell \to 0\) prior to cutoff removing all Feynman diagram contributions with \(\omega_{G,r} < 0\) vanish. Since in this limit appears also divergent terms this limit should be combined with cutoff removing in such a way that,

\[
\lim_{\tilde{D} \to D} \lim_{\ell \to 0} \zeta \left( N + \frac{r - \tilde{D}}{2} \right) = 0, \quad \text{for } N + \frac{r - D}{2} > 0.
\]

Condition \(32\) provides that limit \(\ell \to 0\) is reached faster than cutoff is removed which guarantee that after the cutoff removing no term with \(\omega_{G,r} < 0\) will survive (c.f. eq \(24\)). Thus, the only contribution which survives in this limit is given by the terms which renormalise the bar action \(11\).

From eq. \(32\) it follows that cutoff removing \(\tilde{D} \to D\), and the limit \(\ell \to 0\) must combine in a way satisfying,

\[
\tilde{D} \to D,
\]

\[
\frac{(\kappa \ell)^2}{(D - \tilde{D})} \to 0,
\]

where \(\kappa\) is the dimensional regularisation “mass unity”. In what follows we will consider that \(\ell\) decays as

\[
\ell^2 \sim \kappa^{-2} \epsilon^{1+\alpha}, \quad 0 < \alpha < 1,
\]

where \(\epsilon = 4 - \tilde{D}\). This is sufficient to satisfy \(34\) without too fast decay of \(\ell\).

Physically, such a limiting procedure means that the size of compact dimensions should be below the scale of low dimensional effective quantum field theory or physically zero. In other words, the quantum field theoretical approach is limited to energies much smaller than the inverse size of compact dimensions. This is required to make sense of compactified model as a quantum field one.

As a result the effective theory is described by the same action \(11\) but with mass \(m\) and coupling \(\bar{\lambda}\) substituted by renormalised quantities. In general one can

\[\text{From here on we drop the subscript } \Delta \text{ in the notation of } \zeta\text{-function}\]
have also the field renormalisation of $\phi$, although in one-loop approximation of $\phi^4$-model it is not the case.

Let us consider non-vanishing terms in more details, assuming that $D = 4$ and potential is given by (3). These terms correspond to integrals with $\omega_{G, r} \geq 0$ and are low-dimensionally UV divergent. As we already mentioned these terms produce only renormalisation of the bar action (11). There are only two diagrams which contribute in one-loop approach in our model. They are, respectively, two-point function,

\begin{equation}
G_2 = -\frac{\bar{\lambda}}{4} \phi^2 \sum_n \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - \mu_n^2 + i\epsilon}, \quad \omega_2 = 2,
\end{equation}

and the zeroth term in $k$ expansion of $G_4(k),

\begin{equation}
G_{4,0} = \frac{\bar{\lambda}}{16} \phi^4 \int \frac{d^D p}{(2\pi)^D} \left( \frac{1}{p^2 - \mu_n^2 + i\epsilon} \right)^2, \quad \omega_{4,0} = 0.
\end{equation}

Integrals $G_2$ and $G_{4,0}$ lead respectively, to mass $m^2$ and coupling $\bar{\lambda}$ renormalisation. Computation of regularised $G_2$ and $G_{4,0}$ yields,

\begin{equation}
G_2 = -\frac{\bar{\lambda}}{4(2\sqrt{\pi})^D} \phi^2 \left( \zeta(1 - \tilde{D}/2) + \sum_r (-1)^r M_r^{\tilde{D}-2} \right) \Gamma(1 - \tilde{D}/2),
\end{equation}

\begin{equation}
G_{4,0} = \frac{\bar{\lambda}^2}{16(2\sqrt{\pi})^D} \phi^4 \left( \zeta(2 - \tilde{D}/2) + \sum_r (-1)^r M_r^{2\tilde{D}-4} \right) \Gamma(2 - \tilde{D}/2).
\end{equation}

When $\ell$ and $2\epsilon = 4 - \tilde{D}$ are sent to zero according to (33) and (34), functions $G_2$ and $G_4$ look like,

\begin{equation}
G_2 = \frac{\lambda_0}{4(2\sqrt{\pi})^D} (2\sqrt{\pi})^{2\epsilon} \times
\left( \ell^{-2}(\kappa \ell)^{2\epsilon} \zeta_0(-1 + \epsilon) + \sum_r (-1)^r M_r^{2}(M_r/\kappa)^{-2\epsilon} \right) \Gamma(-1 + \epsilon),
\end{equation}

\begin{equation}
G_{4,0} = \frac{\lambda_0^2}{4(2\sqrt{\pi})^D} (2\sqrt{\pi})^{2\epsilon} \left( (\kappa \ell)^{2\epsilon} \zeta_0(\epsilon) + \sum_r (-1)^r (M_r/\kappa)^{-2\epsilon} \right) \Gamma(\epsilon),
\end{equation}

where the “dimensionless” $\zeta$-function $\zeta_0(s)$ is defined as $\zeta$-function for the unity volume scaled manifold $K^0_p$, $\text{vol} K^0_p = 1$, i.e. one obtained from $K_p$ by rescaling of the metric $g_{ij}^0 = \ell^{-2} g_{ij}$. We used (31) to separate the dimensional factor from the $\zeta$-function.
In the limit (33), (34) diagram $G_2$ produces counterterm leading to renormalisation of the mass,

\[ \delta m^2 \equiv \Delta Z_{2,0} = \frac{\lambda_0}{4(2\sqrt{\pi})^2} \ell^{-2} \left( -\frac{1}{\epsilon} (\zeta(0)(-1) - m^2\ell^2) + \right. \]
\[ + ((\zeta_0(-1) - \xi(2))(-1 + \gamma) - \zeta''_0(-1) - 2\log(2\sqrt{\pi}\kappa\ell)\zeta_0(-1)) \]
\[ + \left( \zeta_0(-1) \left( -\frac{1}{12} \pi^2 - \frac{1}{2} \gamma^2 - 1 + \gamma \right) \right. \]
\[ + \left. (\xi(2) + \zeta'_0(-1) + 2\log(2\sqrt{\pi}\kappa\ell)\zeta_0(-1))(-1 + \gamma) \right. \]
\[ - \frac{1}{2} \zeta''_0(-1) - 2(\log(2\sqrt{\pi}\kappa\ell))^2 \zeta_0(-1) - 2\log(2\sqrt{\pi}\kappa\ell)\zeta_0(-1) \epsilon \right) \]
\[ + \text{finite and vanishing terms}, \]

where $\gamma$ is the Euler constant, $\xi(2) = (2\ell^2 \sum_r (-1)^r M_r^2 \log M_r)$. Analogously, $G_4$ leads to renormalisation of the coupling,

\[ \delta \bar{\lambda} \equiv \bar{\lambda} \Delta Z_{4,0} = \frac{\lambda^2}{4(2\sqrt{\pi})^4} \left( \frac{1}{\epsilon} (-1 + \zeta_0(0)) + \xi_0(0) + 2\zeta_0(0) \log(2\sqrt{\pi}\kappa\ell) + \zeta'_0(0) + (1 - \zeta_0(0))\gamma \right) \]
\[ + \text{vanishing terms}, \]

where $\xi_0(0) = 2 \sum_r (-1)^r \log M_r / \kappa$.

As we expected the contribution of the KK and PV fields reduces in the limit $\ell \to 0$ to mass and coupling renormalisation which at first look seems trivial. In fact, at energies $\sim \kappa \ll \ell^{-1}$ the low dimensional effective model we obtained does not feel directly the fluctuations along $\mathcal{K}_p$, however, as one can see from eqs. (42) and (43) the coupling/mass renormalisation is sensible to geometry of $\mathcal{K}_p$ through $\zeta$-function and $\ell$ present in the counterterms. Thus, if the shape of $\mathcal{K}_p$ is varied the effect of such variation consists in finite counterterms renormalising respectively the coupling and the mass of propagating field $\varphi$. This flow affects renormalised physical quantities as well, provided fluctuations of $\varphi$ with momenta below $\ell^{-1}$ are taken into consideration. This is because the geometry dependent counterterms cannot be cancelled by constant ones.

Flow of the parameters under the variation of the compact size $\ell$ can be computed using renormalisation group methods. If one renormalises the parameters at some value of the compact size $\ell_0$ and than vary it one encounters some flow of the renormalised parameters (renormalisation is $\ell$-independent), $m_R = m_R(\ell)$ and $\bar{\lambda}_R = \bar{\lambda}_R(\ell)$. By dimensional arguments it is clear that this flow is governed by the renormalisation group. For example, for the renormalised coupling one can write,

\[ -\ell \frac{\partial \bar{\lambda}_R}{\partial \ell} = \beta(\bar{\lambda}_R), \]

where $\beta(\bar{\lambda})$ is the Calan-Simanzik beta function, which in the one-loop approximation is given by

\[ \beta(\bar{\lambda}) = \kappa \lambda \frac{\partial Z_{4,0}}{\partial \kappa} = \frac{\lambda^2}{2(2\sqrt{\pi})^4} (\zeta(0) - 1) \]

here we used that at $s = 0$, $\zeta_0(0) = \zeta(0)$. 

Solving eq. (44) yields for the coupling flow,

\[
\lambda(\ell) = \frac{\lambda(\ell_0)}{1 + (a_p - 1)\frac{\lambda(\ell_0)}{\pi^2} \log \frac{\ell}{\ell_0}},
\]

where \(\lambda(\ell)\) and \(\lambda(\ell_0)\) are renormalised couplings computed when the size of \(\mathcal{K}_p\) is, respectively, \(\ell\) and \(\ell_0\), we also used eq. (27) to substitute \(\zeta(0)\) by its value \(a_p(\Delta)\).

**Discussions**

In the actual paper we considered compactifications of high-dimensional quantum field models. We have shown, that a renormalisable compactified model can be defined when the size of compact dimensions is of the order of cutoff scale of the low dimensional model. In dimensional regularisation scheme this corresponds to vanishing of the size of compact dimensions when the cutoff dimension \(\tilde{D}\) approaches the physical value \(D\).

In condition (33), (34) we required that the compact size decays faster than cutoff is removed. This condition arose from the requirement not to have nonrenormalisable terms in the low dimensional effective theory. In fact, one can try to slightly relax this condition by requiring,

\[
\frac{(\kappa\ell)^2}{\epsilon} \rightarrow \text{finite},
\]

rather than zero. Our analysis in this case still remains applicable, but resulting model will not be a renormalisable one.

Indeed, condition (47) still leads to absence of nonrenormalisable infinite contributions to the effective action, but in this case beyond the mass/coupling renormalisation one would have new finite terms in the effective action of the following types \((\partial^2 \varphi)^2\) and \(\varphi^6\) (in general also terms like \((\partial^2 \varphi)^2\) may appear, however in one-loop they are absent). These are, so called, IR irrelevant operators so, if the energy scale is lowered they are believed to decouple then one recovers the old situation. Due to the presence of higher derivative terms this this non-renormalisability could be interpreted as effective switch-on of additional degrees of freedom from the compact dimensions.

Having our results at hand one can draw the following qualitative physical picture. At energies much lower than the size of compact dimensions one has a renormalisable low dimensional QFT whose parameters depend on the geometry of compact dimensions. When the energy is increased new degrees of freedom appear, due to the non-renormalisable interactions and higher order terms at energy \(\kappa \sim \ell^{-1}\).

An alternative approach to define a renormalisable compact model may consist in compactifications to non-commutative compact manifolds \([10]\). Fields on non-commutative spaces \([12]\) are now under intensive study in connection with matrix models, \([11]\). So far, we considered the scalar field model. However, generalisation to other models both ordinary and supersymmetric is more or less straightforward.

In fact one can consider more general compactifications, when manifold \(W_{D+p}\) is a fibre bundle \(W_{D+p} = M_D \times K_p\), where Minkowski space \(M_D\) is the base and \(K_p\) is the fibre. In actual paper one may consider that compactification is of the later type but the fiber \(K_p\) depends on Minkowski space time-point adiabatically.

In the actual paper we considered compactifications of high-dimensional quantum field models.
The last means that for large enough space-time regions this fibre bundle should look almost as a direct product.

Before concluding let us make a few comments. First, let us note that coupling flow (46) differs from the standard flow one would have in the $\phi^4$ model by a factor $(1 - a_p)$ in the denominator. This factor in general changes the rate of the flow and in the case when $a_p \leq 1$ even its character. For $p = 2, 4$ e.g. the respective $a_p$ are given by second and third eqs. (30).

In actual paper we neglected contributions of the order $O(m^2\ell^2)$. These can be taken into consideration by substituting $\Delta$ by $\Delta + m^2$ in eq. (5) and (6), which results in using $\zeta$-function of $\Delta + m^2$ rather than one of $\Delta$. All analysis in this case remain the same except a slight modification of heat kernel invariants (30).

Renormalisation group flow (46) makes sense only when theory is well-defined also beyond the one loop. Although our analysis heavily used specific one-loop tools, such as Pauli–Villars regularisation and zeta function sums this analysis can be extended beyond one-loop by some technical modifications, and, we believe, the present conclusions to remain valid.

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