A UNIVERSAL RIGID ABELIAN TENSOR CATEGORY

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Abstract. We prove that any rigid additive symmetric monoidal category can be mapped to a rigid abelian symmetric monoidal category in a universal way. This sheds a new light on abelian ⊗-envelopes and on motivic conjectures such as Grothendieck’s standard conjecture D and Voevodsky’s smash nilpotence conjecture.

Introduction

Recently there has been a lot of activity on embedding a given rigid additive symmetric monoidal category into an abelian one in a universal way: we can quote on the one hand Coulembier’s monoidal abelian envelopes from [7] and his subsequent works alone and with coauthors [8, 9], on the other O’Sullivan’s super Tannakian hulls [25]. Different but closely related is the work of Schäppi [30]. We may also quote that of Delpeuch in the non-additive context [12, App. B.3].

Both Coulembier and O’Sullivan impose the condition that their envelopes represent faithful monoidal functors (for very good reasons, see introduction to Section 6). In this paper, we take a different approach which relaxes this restriction, based on Freyd’s free abelian category on a given additive category [15]. Freyd’s construction associates to an additive category \( \mathcal{C} \) an abelian category \( \text{Ab}(\mathcal{C}) \) and a (fully faithful) additive functor \( \iota_{\mathcal{C}} : \mathcal{C} \to \text{Ab}(\mathcal{C}) \) such that any additive functor from \( \mathcal{C} \) to an abelian category \( \mathcal{A} \) extends uniquely to an exact functor from \( \text{Ab}(\mathcal{C}) \) to \( \mathcal{A} \). Suppose that \( \mathcal{C} \) is symmetric monoidal. In [5], the authors provided \( \text{Ab}(\mathcal{C}) \) with a right exact, symmetric monoidal structure such that \( \iota_{\mathcal{C}} \) is strong monoidal and universal for strong monoidal functors to abelian categories provided with a right exact symmetric monoidal structure (with a technical condition, see loc. cit., Prop. 1.10).

Suppose now that \( \mathcal{C} \) is rigid. We construct a localisation \( T(\mathcal{C}) \) of \( \text{Ab}(\mathcal{C}) \) which is rigid and such that the induced functor \( \mathcal{C} \to T(\mathcal{C}) \) is, this time, universal for strong additive symmetric monoidal (not necessarily faithful) functors from \( \mathcal{C} \) to rigid symmetric monoidal abelian categories.

2020 Mathematics Subject Classification. 18D15, 14C15.
Key words and phrases. Rigid category, envelope, motive.
categories: see Theorem 5.1. When \( \mathcal{C} \) admits a \( \otimes \)-envelope, the latter turns out to be a localisation of \( T(\mathcal{C}) \) (Proposition 6.2).

The novel thing here is that the ring of endomorphisms \( Z(T(\mathcal{C})) \) of the unit object of \( T(\mathcal{C}) \) need not be a field even if this is the case for \( \mathcal{C} \): if \( \mathcal{C} \) is the category of representations of an affine group scheme \( G \) over a field of characteristic 0, then \( Z(T(\mathcal{C})) \) is a field if and only if \( G \) is proreductive, see Example 7.10. This example is the only one where \( T(\mathcal{C}) \) can be computed so far (and only for certain \( G \)'s), besides Example 5.5.

Our motivating example was the one where \( \mathcal{C} \) is the \( \mathbb{Q} \)-linear category \( \mathcal{M}_{\text{rat}}(k) \) of Chow motives over a field \( k \) (motives with \( \mathbb{Q} \) coefficients modulo rational equivalence). Write \( T(k) \) for \( T(\mathcal{M}_{\text{rat}}(k)) \). By Jannsen’s work \([17]\), the category \( \mathcal{M}_{\text{num}}(k) \) of motives modulo numerical equivalence is abelian semi-simple, whence a canonical \( \mathbb{Q} \)-linear exact \( \otimes \)-functor \( T(k) \to \mathcal{M}_{\text{num}}(k) \). We prove in Theorem 8.1 that this functor is a Serre localisation: it is an equivalence of categories if and only if \( Z(T(k)) \) is a field. In particular, the latter condition implies Grothendieck’s standard conjecture D \([20\, 3.6 \,(i)]\): homological and numerical equivalences agree for any Weil cohomology over \( k \).

In Proposition 8.4, we give a related consequence of the existence of \( T(k) \) on Schäppi’s category mentioned earlier.

Conversely, Voevodsky’s conjecture \([31\, \text{Conj. 4.2}]\), predicting that smash nilpotent and numerical equivalences agree, implies that \( Z(T(k)) \) is a field: see Proposition 8.5. To summarise the situation, we have the following chain of implications, noting that the canonical functor \( \mathcal{M}_{\text{rat}}(k) \to T(k) \) factors through \( \mathcal{M}_{\text{tnil}}(k) \) by Theorem 5.1, where \( \text{tnil} \) is smash-nilpotent equivalence:

**Theorem.** *Voevodsky's conjecture \( \iff \) \( Z(T(k)) \) is a field and \( \mathcal{M}_{\text{tnil}}(k) \to T(k) \) is faithful \( \iff \) \( Z(T(k)) \) is a field \( \iff \) \( T(k) \to \mathcal{M}_{\text{num}}(k) \) \( \Rightarrow \) the standard conjecture D.*

If one wants to stay away from any conjecture, one can argue that \( T(k) \) gives in some sense an answer to Grothendieck’s quest for a universal abelian category representing all cohomology theories on smooth projective varieties. However, Grothendieck really thought of *Weil cohomologies*, and \( T(k) \) does not carry *a priori* a grading: we plan to solve this issue in a further work, by adjoining such grading universally (and unconditionally).

1. **Notation and Terminology**

An additive, symmetric, monoidal, unital category (with bilinear tensor product) will be briefly called a \( \otimes \)-category. A \( \otimes \)-functor between
⊗-categories is a strong symmetric, monoidal, unital additive functor. See [29] or [11] for the background.

**Notation 1.1.** For any ⊗-category \( \mathcal{C} \), we write \( Z(\mathcal{C}) \) for \( \text{End}_{\mathcal{C}}(1) \).

Recall that the ring \( Z(\mathcal{C}) \) is commutative [29, I.1.3.3.1] and that \( \mathcal{C} \) is a \( Z(\mathcal{C}) \)-linear category; it coincides with the ring of [16, III.5.d] (cf. [29, I.2.5.2]). If \( F : \mathcal{C} \to \mathcal{D} \) is a ⊗-functor, we write \( Z(F) : Z(\mathcal{C}) \to Z(\mathcal{D}) \) for the induced ring homomorphism.

**Notation 1.2.**

a) \( \text{Add}^\otimes \): the 2-category of ⊗-categories, ⊗-functors and (additive) ⊗-natural isomorphisms.

b) \( \text{Ex}^\otimes \): the 2-category of abelian ⊗-categories, exact ⊗-functors and (additive) ⊗-natural isomorphisms.

c) \( \text{Add}^{\text{rig}} \) and \( \text{Ex}^{\text{rig}} \): their 1-full and 2-full sub-2-categories of rigid categories.

We have the following basic lemma:

**Lemma 1.3.** For \( \mathcal{C} \in \text{Add}^\otimes \), let \( \mathcal{C}^{\text{rig}} \) denote its strictly full subcategory of dualisable objects. Then \( \mathcal{C}^{\text{rig}} \) is closed under direct sums and direct summands, and \( F(\mathcal{C}^{\text{rig}}) \subset \mathcal{D}^{\text{rig}} \) for any ⊗-functor \( F : \mathcal{C} \to \mathcal{D} \).

**Proof.** Both points follow from [14, Th. 1.3].

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2. Reminders and complements on rigid abelian ⊗-categories

Let \( \mathcal{A} \) be a rigid abelian ⊗-category.

**Lemma 2.1.** The ⊗-structure of \( \mathcal{A} \) is exact.

This is [11, Prop. 1.16].

**Proposition 2.2.** a) If \( U \) is a subobject of \( 1 \), then \( 1 = U \oplus U^\perp \) where \( U^\perp = \text{Ker}(1 \to U^\vee) \), and \( U \otimes U = U \).

b) There is a bijective correspondence between subobjects of \( 1 \), idempotents \( e \in Z(\mathcal{A}) \) and decompositions \( \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \) in which an object is in \( \mathcal{A}_1 \) (resp. in \( \mathcal{A}_2 \)) if \( e \) (resp. \( 1 - e \)) acts as the identity morphism on it.

**Proof.** a) The first fact is [11, Prop. 1.17], and the second one is contained in its proof. b) is [11, Rem. 1.18].

**Lemma 2.3.** Let \( \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \). Any ⊗-functor \( F : \mathcal{A} \to \mathcal{B} \) such that \( Z(\mathcal{B}) \) is a field vanishes either on \( \mathcal{A}_1 \) or \( \mathcal{A}_2 \).
Proof. Indeed, the idempotent corresponding to this decomposition in Proposition 2.2 b) must go to 0 or 1 in \( \mathbb{Z}(\mathcal{B}) \).

Proposition 2.4. Suppose that \( \mathbb{Z}(\mathcal{A}) \) is a field. Then a \( \otimes \)-functor from \( \mathcal{A} \) to a (nonzero) abelian \( \otimes \)-category \( \mathcal{B} \) is faithful if it is exact. The converse is true if \( \mathbb{Z}(\mathcal{B}) \) is a field.

Proof. “If” is [11, Prop. 1.19], and “only if” is [8, Th. 2.4.1] (see also [25, Lemma 10.7] in the case of super Tannakian categories).

Proposition 2.5. a) if \( \mathbb{Z}(\mathcal{A}) \) is a field, \( \mathcal{A} \) is integral: for two morphisms \( f, g \), \( f \otimes g = 0 \) implies \( f = 0 \) or \( g = 0 \).

b) In general, \( \mathcal{A} \) is reduced: for a morphism \( f \), \( f \otimes^2 = 0 \) implies \( f = 0 \).

In particular, the ring \( \mathbb{Z}(\mathcal{A}) \) is reduced.

c) \( \mathcal{A} \) is fractionally closed in the sense of [25, Sect. 3, bot. p. 6]. In particular, \( \mathbb{Z}(\mathcal{A}) \) is its own total ring of quotients.

d) If \( \mathbb{Z}(\mathcal{A}) \) is a domain then it is a field.

e) \( \mathbb{1} \) is Noetherian (equivalently Artinian) if and only if \( \mathbb{Z}(\mathcal{A}) \) is, and then \( \mathcal{A} \) is equivalent to a product \( \prod_{i \in I} \mathcal{A}_i \), where \( I \) is finite and \( \mathbb{Z}(\mathcal{A}_i) \) is a field for each \( i \).

f) \( \mathbb{Z}(\mathcal{A}) \) is absolutely flat (= von Neumann regular).

Proof. a) is [18, Rem. 2.10]. b) was suggested to the second author by Peter O’Sullivan. By rigidity, we may assume that the domain of \( f \) is \( \mathbb{1} \). Then \( f \) factors through a monomorphism \( \text{Coker}(f) \hookrightarrow A \), where \( A \) is the codomain of \( f \). If \( f \neq 0 \), \( \text{Coker}(f) \neq 0 \); but this is a subobject of \( \mathbb{1} \) by Proposition 2.2 a), hence \( \text{Coker}(f) \otimes^2 = \text{Coker}(f) \neq 0 \) by the same Proposition, and \( f \otimes f \neq 0 \) by Lemma 2.1. c) is [25, Lemma 3.1].

d) is special case of c), but we give a direct proof: if \( \mathbb{Z}(\mathcal{A}) \) is not a field, it contains an idempotent by Proposition 2.2 b) so it cannot be a domain. e) follows easily from the same proposition. The consequences of b) and c) on \( \mathbb{Z}(\mathcal{A}) \) follow from the fact that, in this ring, composition and tensor product coincide, see [29, I.1.3.3.1]. For f), see [19, Prop. 3.2].

3. Complements on abelian \( \otimes \)-categories

In this section, \( \mathcal{A} \) is an abelian (not necessarily rigid) \( \otimes \)-category. The following proposition completes the proof of [5, Prop. 1.13 (3)], removing its hypothesis of right exactness of the Homs.

Proposition 3.1. If the tensor structure of \( \mathcal{A} \) is exact, then the full subcategory \( \mathcal{A}_{\text{rig}} \) of dualisable objects is closed under kernels and cokernels. In particular, \( \mathcal{A}_{\text{rig}} \) is abelian and the inclusion functor \( \mathcal{A}_{\text{rig}} \hookrightarrow \mathcal{A} \) is exact.
Proof. Let $A' \to A \to A'' \to 0$ be exact, with $A', A \in \mathcal{A}_{\text{rig}}$, and write $K = \text{Ker}(A' \to A'')$. For any $B, C \in \mathcal{A}$, we have a commutative diagram of exact sequences

$0 \to A(A'' \otimes B, C) \longrightarrow A(A \otimes B, C) \longrightarrow A(A' \otimes B, C)$

due to the exactness of $\otimes$. It induces an isomorphism $A(A'' \otimes B, C) \cong A(B, K \otimes C)$ natural in $B$ and $C$, showing that $K$ is right dual to $A''$. But then, it is also left dual since $\otimes$ is symmetric (see [6]).

For an exact sequence $0 \to A' \to A \to A''$ with $A, A'' \in \mathcal{A}_{\text{rig}}$, we argue similarly by changing the side of the tensor products in the Hom groups. □

**Proposition 3.2.** Suppose $\mathcal{A}$ essentially small with a right exact tensor structure. Let $\mathcal{I} \subseteq \mathcal{A}$ be a Serre subcategory containing the objects $K(f, A) := \text{Ker}(1_A \otimes f)$ for all monomorphisms $f$ and stable under (external) tensor products. Then $\mathcal{A}/\mathcal{I}$ inherits a tensor structure, which is exact. If $\mathcal{I}$ is minimal for those properties, $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is initial for exact $\otimes$-functors from $\mathcal{A}$ to an abelian $\otimes$-category with exact tensor structure.

Proof. Let $\Sigma$ be the multiplicative system associated to $\mathcal{I}$, i.e. $s \in \Sigma \iff \text{Ker} s, \text{Coker} s \in \mathcal{I}$. The first point means that $\Sigma$ is stable under tensor product with identities on the left and on the right, so, say, on the left by symmetry. We proceed as follows:

1) Let $s : C \hookrightarrow D$ be a monomorphism in $\Sigma$, and let $A \in \mathcal{A}$. The exact sequence

$0 \to K(s, A) \to A \otimes C \xrightarrow{1_A \otimes s} A \otimes D \to A \otimes \text{Coker} s \to 0$

shows that $1_A \otimes s \in \Sigma$.

2) Let $s : C \twoheadrightarrow D$ be an epimorphism in $\Sigma$, and let $A \in \mathcal{A}$. The exact sequence

$A \otimes \text{Ker} s \to A \otimes C \xrightarrow{1_A \otimes s} A \otimes D \to 0$

shows that $1_A \otimes s \in \Sigma$.

3) Any $s \in \Sigma$ can be written $t \circ u$, with $u$ epi, $t$ mono and $t, u \in \Sigma$. This, with 2) and 3), completes the proof of the first point.

Let us now prove exactness of the tensor product. Let $A \in \mathcal{A}/\mathcal{I}$ and let $(\star) : 0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence in $\mathcal{A}/\mathcal{I}$ (recall that $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ have the same objects). By [16, p. 368, Cor. 1],
is isomorphic to a short exact sequence of $A$; to show that $A \otimes (\ast)$ is exact, we may therefore assume that $f$ and $g$ come from morphisms of $A$. Since the projection functor $A \rightarrow A/I$ is exact, we already have right exactness. But $1_A \otimes f$ is a monomorphism in $A/I$ by definition of $I$, so the proof is complete.

Note that a minimal $I$ exists: any intersection of Serre subcategories (resp. closed under external tensor product) is a Serre subcategory (resp. is closed under external tensor product). The initiality is now clear, since any exact $\otimes$-functor from $A$ to an abelian $\otimes$-category sends all objects $K(f, A)$ to $0$. ☐

**Remark 3.3.** Let $I_0$ be the minimal Serre subcategory as in Proposition 3.2, and let $I' \subseteq I_0$ denotes the smallest Serre subcategory of $A$ containing the objects $K(f, A)$ (no $\otimes$-ideal condition). It is tempting to try and show that $I' = I_0$. At least, $B \otimes K(f, A) \in I'$ for any $A, B \in A$ and any monomorphism $f : C \hookrightarrow D$, as follows from the exact sequence

$$0 \rightarrow K(g, B) \rightarrow B \otimes K(f, A) \rightarrow K(f, B \otimes A)$$

where $g$ is the monomorphism $K(f, A) \hookrightarrow A \otimes C$. But this does not seem sufficient: since $\otimes$ is right exact, $C \otimes I'$ is stable under quotients and extensions, but maybe not under kernels.

**Remark 3.4.** In general, $Z(A/I)$ need not be a field even if $Z(A)$ is: see Example 7.10 below. Here are two things one can say:

a) We have

$$Z(A/I) = \lim_{\substack{\rightarrow \rightarrow}} A'(A', 1/A'')$$

where $A'$ (resp. $A''$) runs through subobjects of $1$ such that $1/A' \in I$ (resp. $A'' \in I$): this translates the definition of morphisms in $A/I$ as in [16, III.1].

b) $1_{A/I}$ is irreducible $\iff$ any subobject of $1_A$ belongs to $I$ $\iff$ any quotient of $1_A$ belongs to $I$: this follows from [16, p. 368, Cor. 1] which was already used in the proof of Proposition 3.2. In particular, $1_A$ irreducible $\Rightarrow 1_{A/I}$ irreducible.

**Proposition 3.5.** Let $A \in \text{Ex}^\otimes$ be essentially small and let $I$ be a Serre subcategory stable under external tensor product. If the tensor structure of $A$ is exact, there is a unique tensor structure on $A/I$ such that $p : A \rightarrow A/I$ is a $\otimes$-functor, and this tensor structure is exact. If $A$ is rigid, so is $A/I$, and $Z(A) \rightarrow Z(A/I)$ is surjective.
Proof. The first two points are special cases of Proposition 3.2, since all objects $K(f, A)$ are 0 by assumption. The rigidity of $\mathcal{A}/\mathcal{I}$ follows from Lemma 1.3, since $p$ is surjective. The last point follows from (3.1) and Proposition 2.2, which implies that $Z(\mathcal{A}) \to \mathcal{A}(A', 1/A'')$ is (split) surjective for all $A', A'' \subseteq 1$. □

4. Freyd’s universal construction

Recall ([15], see also [21, 2.10], [27, Th. 4.1 and Cor. 4.2], [5, §1]) that, for any additive category $\mathcal{C}$, there is an abelian category $\text{Ab}(\mathcal{C})$ and an additive functor $\iota_{\mathcal{C}}: \mathcal{C} \to \text{Ab}(\mathcal{C})$ such that any additive functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category, extends through $\iota_{\mathcal{C}}$ to an exact functor $\text{Ab}(\mathcal{C}) \to \mathcal{A}$, unique up to unique equivalence of categories. The functor $\iota_{\mathcal{C}}$ is fully faithful.

Lemma 4.1. a) The category $\mathcal{C}$ generates $\text{Ab}(\mathcal{C})$ as an abelian category in the following sense: if $\mathcal{A} \subseteq \text{Ab}(\mathcal{C})$ is a strictly full abelian subcategory such that the inclusion functor is exact and $\mathcal{A}$ contains $\iota_{\mathcal{C}}(\mathcal{C})$, then $\mathcal{A} = \text{Ab}(\mathcal{C})$. In particular, any object of $\text{Ab}(\mathcal{C})$ is a subquotient of $\iota_{\mathcal{C}}(C)$ for some $C \in \mathcal{C}$.

b) $Z(\mathcal{C}) \xrightarrow{\sim} Z(\text{Ab}(\mathcal{C}))$.

Proof. a) is [27, Lemma 4.12], and b) is obvious by full faithfulness. In a) “In particular” holds because any subobject and any quotient of a subquotient is a subquotient. □

Definition 4.2. An abelian category $\mathcal{A}$ is split if $\mathcal{A} \xrightarrow{\iota_{\mathcal{A}}} \text{Ab}(\mathcal{A})$ is an equivalence of categories.

Proposition 4.3. For an abelian category $\mathcal{A}$, the following are equivalent:

1. $\mathcal{A}$ is split.
2. Every additive functor from $\mathcal{A}$ to an abelian category is exact.
3. Every short exact sequence splits.
4. Every object is projective.
5. Every object is injective.

This holds in particular if $\mathcal{A}$ is semisimple\(^1\).

If $\mathcal{A}$ is split, any full pseudo-abelian subcategory of $\mathcal{A}$ is a Serre subcategory (in particular, abelian) and is split, and any Serre localisation of $\mathcal{A}$ is split. The same holds when replacing “split” by “semisimple”.

\(^1\)Here we adopt the terminology of [2, 2.1.1]: a preadditive category $\mathcal{A}$ is semisimple if every left $\mathcal{A}$-module is a direct sum of simple objects. If $\mathcal{A}$ is abelian, this means [2, A.2.10 (10)] that every object of $\mathcal{A}$ is a finite direct sum of simple objects.
Proof. The only possibly nonobvious point is (2) ⇒ (4): use the Hom functor.

If $C$ has a $\otimes$-structure, then $\text{Ab}(C)$ inherits a right exact $\otimes$-structure for which $\iota_C$ is a $\otimes$-functor [5, Prop. 1.8].

**Proposition 4.4.** Let $I \subseteq \text{Ab}(C)$ be minimal in Proposition 3.2, and write $T(C) = \text{Ab}(C)/I$. Then the composition $C \xrightarrow{\iota_C} \text{Ab}(C) \to T(C)$ is 2-universal for $\otimes$-functors from $C$ to abelian $\otimes$-categories with exact tensor structure (with respect to exact $\otimes$-functors). In particular, $C$ generates $T(C)$ in the same sense as in Lemma 4.1 a).

**Proof.** Let $F : C \to A$ be a $\otimes$-functor, with $A \in \text{Ex}^\otimes$. If the tensor structure of $A$ is exact, $F$ factors uniquely through an exact $\otimes$-functor $\tilde{F} : \text{Ab}(C) \to A$. Indeed, we apply [5, Prop. 1.10]: by the exactness of the tensor product, we may take $A^\flat = A$ in loc. cit. so its hypothesis is trivially verified. Then $\tilde{F}$ factors uniquely through $T(C)$ by Proposition 3.2. $\square$

**Proposition 4.5.** Let $A \in \text{Ex}^{\text{rig}}$. Then $A$ is split if and only if $1$ is projective.

**Proof.** If $1$ is projective, the functor $A \mapsto A(B, A) \simeq A(1, B^\vee \otimes A)$ is right exact, since $\otimes$ is exact by Lemma 2.1. $\square$

5. Main theorem

**Theorem 5.1.** Let $C$ be a rigid additive $\otimes$-category. Then the 2-functor $A \mapsto \text{Add}^\otimes(C, A)$ from $\text{Ex}^{\text{rig}}$ to $\text{Cat}$ is 2-representable by the category $T(C)$ of Proposition 4.4. Moreover, the obvious functors

$$T(C) \to T(C/\sqrt{0}) \to T((C/\sqrt{0})_{fr}) \to T(((C/\sqrt{0})_{fr})^\natural)$$

are equivalences of categories, where $\sqrt{0}$ is the $\otimes$-ideal of $\otimes$-nilpotent morphisms [2, Def. 7.4.1], $D_{fr}$ (resp. $D^\natural$) is the fractional closure of a $\otimes$-category $D$ [25, p. 8] (resp. its pseudo-abelian envelope).

**Proof.** Let $T(C)_{rig}$ be the strictly full subcategory of dualisable objects: by Proposition 3.1, it is abelian and the full embedding $T(C)_{rig} \to T(C)$ is exact. Since $C$ is rigid, its image in $T(C)$ lands into $T(C)_{rig}$ by Lemma 1.3; therefore, $T(C)_{rig} = T(C)$ by Proposition 4.4. Let now $A \in \text{Ex}^{\text{rig}}$. Its tensor structure is exact by Lemma 2.1, hence, again by Proposition 4.4, any $\otimes$-functor $C \to A$ factors through $T(C)$, uniquely up to unique $\otimes$-equivalence.
In the last claim, the first equivalence follows from Proposition 2.5 b) and the second (resp. third) one from the fact that rigid abelian \( \otimes \)-categories are fractionally closed [25, Lemma 3.1] (resp. that abelian categories are pseudo-abelian).

\[ \square \]

**Corollary 5.2.** If \( \mathcal{C} \) is abelian in Theorem 5.1, the canonical functor \( \mathcal{C} \to T(\mathcal{C}) \) has an exact \( \otimes \)-retraction \( \sigma_\mathcal{C} \). If moreover \( \mathcal{C} \) is split, this functor is an equivalence of \( \otimes \)-categories.

**Proof.** The first claim follows from the universal property of \( T(\mathcal{C}) \) applied to \( \text{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C} \). The second one follows from the definition of split (Definition 4.2).

\[ \square \]

**Remarks 5.3.**

a) One can presumably extend Theorem 5.1 to not necessarily rigid additive \( \otimes \)-categories by using [12, App. B.3].

b) For \( \mathcal{C} \) as in Theorem 5.1, \( \text{Ab}(\mathcal{C}) \) is rigid if and only if its tensor structure is exact. Necessity follows from Lemma 2.1; if conversely the tensor structure of \( \text{Ab}(\mathcal{C}) \) is exact, then \( \text{Ab}(\mathcal{C}) \xrightarrow{\sim} T(\mathcal{C}) \) which is rigid by (the proof of) Theorem 5.1.

c) The category \( (\mathcal{C}/\sqrt{0}_h)^2 \) of Theorem 5.1 is reduced, fractionally closed and pseudo-abelian.\(^2\)

**Example 5.4.** Suppose that, in \( \mathcal{C} \), there exists a nilpotent endomorphism with nonzero trace. Then \( T(\mathcal{C}) = 0 \) because, in rigid abelian \( \otimes \)-categories, any nilpotent endomorphism has trace 0 [2, Prop. 7.3.3]. An example where this happens is given in [10, §5.8].

**Example 5.5.**\(^3\) Let \( R^+ \) be the additive completion of a commutative ring \( R \) considered as a preadditive category (the objects of \( R^+ \) are \( R^n \)), provided with its canonical \( \otimes \)-structure. By [22, Prop. 5], there exists a ring homomorphism \( R \to R^{\text{abs}} \) which is universal for homomorphisms from \( R \) to absolutely flat rings. We claim that \( T(R^+) = R^{\text{abs}}\text{-mod} \), where the latter is the (abelian) rigid \( \otimes \)-category of finitely presented (equivalently, finitely generated projective) \( R^{\text{abs}} \)-modules. If \( R \) is absolutely flat, this follows from [28, Prop. 10.2.38] and Remark 5.3 b), the first reference showing that \( \text{Ab}(R^+) \xrightarrow{\sim} R\text{-mod} \) in this case. In general, we use the fact that \( Z(\mathcal{A}) \) is absolutely flat for any \( \mathcal{A} \in \text{Ex}^{\text{rig}} \), Proposition 2.5 f). For such \( \mathcal{A} \), a \( \otimes \)-functor \( F : R^+ \to \mathcal{A} \) amounts to an \( R \)-module structure on \( 1_\mathcal{A} \). Thus \( Z(\mathcal{A}) \) is an \( R^{\text{abs}} \)-algebra, so \( F \) factors uniquely through \( (R^{\text{abs}})^+ \), hence through \( R^{\text{abs}}\text{-mod} \) as promised.

\(^2\)We thank Peter O’Sullivan for confirming that the fractional closure of a reduced \( \otimes \)-category is reduced.

\(^3\)This example was found independently by P. O’Sullivan [26].
6. Comparisons

6.1. **Abelian \(\otimes\)-envelopes.** Let \(\text{Add}^\text{rig}\) be the sub-2-category of \(\text{Add}^\text{rig}\) restricted to the \(C\)'s such that \(Z(C)\) is a field and to *faithful* functors. Similarly, let \(\text{Ex}^\text{rig}\) be the 1-full, 2-full sub-2-category of \(\text{Ex}^\text{rig}\) determined by those \(A \in \text{Ex}^\text{rig}\) such that \(Z(A)\) is a field; note that in \(\text{Ex}^\text{rig}\) all exact \(\otimes\)-functors are automatically faithful by Proposition 2.4, so \(\text{Ex}^\text{rig}\) is contained in \(\text{Add}^\text{rig}\). Coulembier as well as O’Sullivan consider the universal property of Theorem 5.1 restricted to these 2-categories. This has an advantage and a drawback:

- By Proposition 2.4, a solution to this universal problem is automatically an envelope (an idempotent construction).
- Such a solution is much more difficult to construct (when it exists, see Example 5.4).

Nevertheless, both authors provide a solution in special cases, by very different methods.

To formalise things, let us set up a definition:

**Definition 6.1.** Let \(C \in \text{Add}^\text{rig}\). An *abelian \(\otimes\)-envelope* of \(C\) is a category \(E(C) \in \text{Ex}^\text{rig}\) which 2-represents the 2-functor

\[
A \mapsto \text{Add}^\text{rig}(C, A)
\]

from \(\text{Ex}^\text{rig}\) to \(\text{Cat}\).

Note that \(Z(E(C))\) must be a field extension of \(Z(C)\) by definition. If \(C \to E(C)\) is also full then we have

\[
Z(E(C)) = Z(C).
\]

**Proposition 6.2.** a) \(E(C)\) exists if and only if

(i) there exists a faithful \(\otimes\)-functor \(F: C \to A\) with \(A \in \text{Ex}^\text{rig}\);

(ii) the (Serre) kernel \(S\) of the induced functor \(T(C) \to A\) does not depend on \((A, F)\).

In this case, \(C\) is integral and \(C \to T(C)\) is faithful.

b) Any abelian \(C\) is its own abelian \(\otimes\)-envelope.

In particular, if \(E(C)\) exists we have a canonical localisation \(\otimes\)-functor

\[
T(C) \to E(C).
\]

**Proof.** a) Conditions (i) and (ii) are obviously necessary. Conversely, assume (i) and (ii). Then \(E(C) = T(C)/S\) is rigid by Proposition 3.5. Moreover, \(Z(E(C))\) is a subring of the field \(Z(A)\) and therefore it is a field by Proposition 2.5 d). Thus \(E(C) \in \text{Ex}^\text{rig}\). Clearly, any functor \(F\)
as in (i) factors uniquely through \( E(\mathcal{C}) \), so \( E(\mathcal{C}) \) satisfies the universal property. Finally, the induced functor \( \mathcal{C} \to E(\mathcal{C}) \) is faithful because its composition with \( E(\mathcal{C}) \to \mathcal{A} \) is faithful for \( \mathcal{A} \) as in (i).

The integrality of \( \mathcal{C} \) follows from Proposition 2.5 a) and the faithfulness is obvious.

b) As explained above, this follows from Proposition 2.4.

The last remark follows from the proof of a). \( \square \)

Suppose that \( E(\mathcal{C}) \) exists. Then (6.2) is an equivalence if and only if any additive \( \otimes \)-functor from \( \mathcal{C} \in \text{Add}^{\text{rig}} \) to \( \mathcal{A} \in \text{Ex}^{\text{rig}} \) is faithful. For example, the category of representations of an affine group scheme \( G \) is abelian hence its own envelope by Proposition 6.2 b), but (6.2) is not an equivalence if \( G \) is not proreductive by Example 7.10 below.

**Proposition 6.3.** a) If \( E(\mathcal{C}) \) exists, (6.2) is an equivalence of categories if and only if \( Z(T(\mathcal{C})) \) is a field.

b) Suppose that \( Z(T(\mathcal{C})) \) is a field. Then \( T(\mathcal{C}) \) is the abelian \( \otimes \)-envelope of \( \mathcal{C}/I \), where \( I \) is the (additive) kernel of \( \mathcal{C} \to T(\mathcal{C}) \). In particular, \( T(\mathcal{C}) \) is an abelian \( \otimes \)-envelope of \( \mathcal{C} \) if and only if \( \mathcal{C} \to T(\mathcal{C}) \) is faithful.

**Proof.** All this follows once again from Proposition 2.4. \( \square \)

### 6.2. Coulembier’s work.

Coulembier’s condition for an envelope [7, Def. 1.3.4] is Definition 6.1 plus the requirement that (6.1) holds. He proves:

**Theorem 6.4 ([7, Th. A]).** Let \( \mathcal{C} \in \text{Add}^{\text{rig}} \). Then \( E(\mathcal{C}) \) exists, with property (6.1), provided every morphism \( f \) in \( \mathcal{C} \) is split by a strongly faithful object in \( \mathcal{C} \).

Here, \( X \in \mathcal{C} \) is strongly faithful if \( X \otimes - : \mathcal{C} \to \mathcal{C} \) reflects all kernels and cokernels in \( \mathcal{C} \), and a morphism \( f : X \to Y \) in \( \mathcal{C} \) is split if there exists \( g : Y \to X \) such that \( f \circ g \circ f = f \). “Split by \( X \)” means that \( 1_X \otimes f \) is split.

See [7, Th. 4.1.1 (a)] for another sheaf-theoretic sufficient condition.

### 6.3. O’Sullivan’s work.

Let \( \mathcal{C} \in \text{Add}^{\text{rig}} \) be essentially small, \( \mathbb{Q} \)-linear, integral (see Proposition 2.5 a)) and Schur-finite. (In particular, the integrality hypothesis implies that \( Z(\mathcal{C}) \) is a field.) According to [25, Def. 10.2], we call such a category pseudo-Tannakian, and super Tannakian if it is abelian.

Write \( \text{Add}^{\text{rig}}_{\text{f}} \) for the 1-full and 2-full subcategory of \( \text{Add}^{\text{rig}} \) formed of pseudo-Tannakian categories, and \( \text{Ex}^{\text{rig}}_{\text{f}} \) or the 1-full and 2-full subcategory of \( \text{Ex}^{\text{rig}} \) formed of super Tannakian categories. We have:
Theorem 6.5. For any $\mathcal{C} \in \text{Add}^{\text{rig}}$, the 2-functor

$$\mathcal{A} \mapsto \text{Add}^{\text{rig}}(\mathcal{C}, \mathcal{A})$$

from $\text{Ex}^{\text{rig}}$ to $\text{Cat}$ is 2-representable.

Proof. This is [25, Lemma 10.7 and Th. 10.10].

Remark 6.6. Let $ST(\mathcal{C})$ be the solution of the above universal problem. It can be proven that $ST(\mathcal{C}) = E(\mathcal{C})$. See [19, Thm. 5.5.]

Remarks 6.7. a) As pointed out by O’Sullivan, any Kimura category verifying the conditions of Theorem 6.4 is semi-simple.

b) As far as we know, and in spite of Propositions 6.2 and 6.3, Theorem 5.1 does not imply either Theorem 6.4 or Theorem 6.5 in any obvious way!

7. The split quotient of $T(\mathcal{C})$

7.1. The ideal $\mathcal{N}$. Let $\mathcal{C} \in \text{Add}^{\text{rig}}$. Recall from [2, 7.1] the $\otimes$-ideal $\mathcal{N}_\mathcal{C} \subseteq \mathcal{C}$ of morphisms universally of trace 0: for $A, B \in \mathcal{C}$,

$$\mathcal{N}_\mathcal{C}(A, B) = \{ f \in \mathcal{C}(A, B) \mid \text{tr}(gf) = 0 \ \forall g \in \mathcal{C}(B, A) \}$$

where $\text{tr}$ is the categorical trace.

Lemma 7.1. For any split $\mathcal{A} \in \text{Ex}^{\text{rig}}$ (Definition 4.2), we have $\mathcal{N}_\mathcal{A} = 0$. Conversely, if $\mathcal{N}_\mathcal{A} = 0$ and $Z(\mathcal{A})$ is Noetherian, then $\mathcal{A}$ is split.

Proof. By [2, 6.1.5], it suffices to show that $\mathcal{N}_\mathcal{A}(1, A) = 0$ for any $A \in \mathcal{A}$. Let $f : 1 \to A$ be a nonzero morphism. If $U = \text{Im} f$, $U$ is injective (Proposition 4.3 (5)), hence the induced monomorphism $U \to A$ has a retraction. Since $U$ is a direct summand of $1$, this retraction yields a morphism $g : A \to 1$ such that $gf \neq 0$.

For the converse, it suffices by Proposition 4.5 to show that $1$ is projective. Let $f : A \to 1$ be an epimorphism, with $A \in \mathcal{A}$. Since $\mathcal{N}(A, 1) = 0$, there is $g : 1 \to A$ such that $fg \neq 0$. Let $U \neq 0$ be the image of $fg$. Replace $f$ by $f_1 = (1 - e)f$, where $e$ is the idempotent with image $U$ in the decomposition $1 = U \oplus U^\perp$ of Proposition 2.2 a). Since $f$ was epi, $f_1$ is epi on $U^\perp$, hence nonzero if $U^\perp \neq 0$. Using $\mathcal{N}(A, U^\perp) = 0$, we find $g_1 : U^\perp \to A$ such that $g_1f_1 \neq 0$. Iterating, we get a strictly increasing sequence of subobjects of $1$, which must stop at $1$ at a finite step. Collecting everything, we get a section of $f$. □
Lemma 7.2. Let \( F : \mathcal{C} \to \mathcal{D} \) be a full \( \otimes \)-functor with \( \mathcal{C}, \mathcal{D} \in \text{Add}^{\text{rig}} \). Let \( f \) be a morphism of \( \mathcal{C} \). Then \( f \in \mathcal{N}_\mathcal{C} \Rightarrow F(f) \in \mathcal{N}_\mathcal{D} \), and the converse is true if \( Z(F) : Z(\mathcal{C}) \to Z(\mathcal{D}) \) is injective. Under this condition, the induced full \( \otimes \)-functor

\[
\mathcal{C}/\mathcal{N}_\mathcal{C} \to \mathcal{D}/\mathcal{N}_\mathcal{D}
\]

is faithful.

Proof. Since

\[
\text{tr}(F(f)) = F(\text{tr}(f))
\]

for any \( f \in \mathcal{C} \), this lemma is trivial. \( \square \)

Proposition 7.3. Let \( \mathcal{C} \in \text{Add}^{\text{rig}} \) be such that \( Z(\mathcal{C}) \) is a field. Then the following conditions are equivalent:

(i) \((\mathcal{C}/\mathcal{N}_\mathcal{C})^2\) is abelian.

(ii) \((\mathcal{C}/\mathcal{N}_\mathcal{C})^2\) is (abelian and) split.

(iii) There exists \( F : \mathcal{C} \to \mathcal{D} \) as in Lemma 7.2, with \( \mathcal{D} \) split.

Proof. (i) \(\Rightarrow\) (ii) follows from the second part of Lemma 7.1, since \( Z((\mathcal{C}/\mathcal{N}_\mathcal{C})^2) = Z(\mathcal{C}) \) is a field. (ii) \(\Rightarrow\) (iii) is trivial. If (iii) holds, the hypotheses imply that \( Z(F) \) is injective and \( \mathcal{N}_\mathcal{D} = 0 \) (by the first part of Lemma 7.1), thus Lemma 7.2 gives us a fully faithful \( \otimes \)-functor \( \mathcal{C}/\mathcal{N}_\mathcal{C} \to \mathcal{D} \). Since \( \mathcal{D} \) is pseudo-abelian, it extends to a full embedding \( (\mathcal{C}/\mathcal{N}_\mathcal{C})^2 \hookrightarrow \mathcal{D} \). Finally, \((\mathcal{C}/\mathcal{N}_\mathcal{C})^2\) is abelian and split thanks to Proposition 4.3. So (iii) \(\Rightarrow\) (ii). \( \square \)

Remark 7.4. Proposition 7.3 means that \( \mathcal{C} \to (\mathcal{C}/\mathcal{N}_\mathcal{C})^2 \) is universal with respect to full \( \otimes \)-functors to split abelian \( \otimes \)-categories – assuming such functors exist, which fails e.g. in Example 5.4. The situation is parallel to that of Proposition 6.2. When \( \mathcal{C} \) is the category of pure motives over a field (see Section 8 below), we recover the classical fact that the Hodge conjecture or the Tate conjecture (plus semi-simplicity) implies the standard conjecture \( \mathcal{D} \).

7.2. A splitting. In this subsection, we assume that \((\mathcal{C}/\mathcal{N}_\mathcal{C})^2\) is abelian split. We write \( S(\mathcal{C}) \) for this category.

Example 7.5. By Lemma 1.3 and [3, Th. 1 a)], the above hypothesis is satisfied provided \( K = Z(\mathcal{C}) \) is a field and there exists an extension \( L/K \) and a \( K \)-linear \( \otimes \)-functor \( H : \mathcal{C} \to \mathcal{V} \) to a nonzero rigid \( L \)-linear abelian \( \otimes \)-category \( \mathcal{V} \) in which Hom groups have finite \( L \)-dimension. In this case, \( S(\mathcal{C}) \) is even semisimple.
Write $\pi$ for the $\otimes$-functor $C \to S(C)$. By Theorem 5.1, $\pi$ induces an exact $\otimes$-functor

\begin{equation}
\bar{\pi} : T(C) \to S(C).
\end{equation}

Let $T_0(C)$ be the quotient $T(C)/\text{Ker} \bar{\pi}$ and let

$$\bar{\pi}_0 : T_0(C) \to S(C)$$

be the induced faithful exact $\otimes$-functor. Note that $T_0(C)$ is still rigid by Proposition 3.5.

**Theorem 7.6.** $\bar{\pi}_0$ is an equivalence of categories.

We first prove that $\bar{\pi}_0$ is full. By rigidity, this is equivalent to

**Lemma 7.7.** The injection

$$T_0(C)(1, X) \xrightarrow{\bar{\pi}_0} S(C)(1, \bar{\pi}_0(X))$$

is surjective for any $X \in T_0(C)$.

**Proof.** Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $T_0(C)$. We then get the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & T_0(C)(1, X') \\
\downarrow \bar{\pi}_0 & & \downarrow \bar{\pi}_0 \\
S(C)(1, \bar{\pi}_0(X')) & \to & S(C)(1, \bar{\pi}_0(X)) \\
\downarrow \bar{\pi}_0 & & \downarrow \bar{\pi}_0 \\
0 & \to & S(C)(1, \bar{\pi}_0(X''))
\end{array}
\]

with exact rows and vertical injections. The bottom exact row follows from the exactness of $\bar{\pi}_0$ and the splitness of $S(C)$, granting that $S(C)(1, -)$ is right exact. If the middle vertical arrow is an epimorphism then the left-most one is by diagram chase; moreover, the right-most is an epimorphism. Therefore, the statement is stable under passing to subquotients: since it is true for objects coming from $C$, it is true in general by Proposition 4.4.

\[\square\]

To conclude the proof of Theorem 7.6, we show that $\bar{\pi}_0$ is essentially surjective. Let $Y = (\pi(C), e) \in S(C)$, where $C \in C$ and $e$ is an idempotent of $\text{End}_{C/N_0}(\pi(C))$. By the full faithfulness of $\bar{\pi}_0$, $e$ lifts to an idempotent of $p(C)$, where $p : C \to T_0(C)$ is the canonical functor.

\[\square\]

**Corollary 7.8.** Let $F : C \to A$ be a $\otimes$-functor with $A \in \text{Ex}^{rig}$. Assume that $Z(C)$ is a field. If $F$ is full and $A$ is split, then the functor

$$\bar{F} : T(C) \to A$$

induced from Theorem 5.1 factors through $S(C)$ and $\text{Ker} \bar{F} = \text{Ker} \bar{\pi}$.
Proof. This follows from Remark 7.4 and Theorem 7.6. \qed

Corollary 7.9. If $Z(C)$ is a field, consider the following conditions:

(i) The functor $\bar{\pi}$ of (7.1) is an equivalence of $\otimes$-categories.
(ii) $Z(T(C))$ is a field.
(iii) $C \to T(C)$ is faithful.
(iv) $N = 0$.

Then (i) $\iff$ (ii), and (i) + (iii) $\iff$ (iv).

Proof. (i) $\iff$ (ii) is obvious in view of Proposition 2.4. The implication (i) + (iii) $\implies$ (iv) is also trivial. Assume (iv). Then $C^\natural$ is abelian and split. By Corollary 5.2, the functor $C^\natural \to T(C^\natural)$ is an equivalence of $\otimes$-categories. By Theorem 5.1, $T(C) \to T(C^\natural)$ is also an equivalence of $\otimes$-categories. This implies (i), and obviously (iii) as well. \qed

Example 7.10. Suppose $\text{char } K = 0$. Let $C = \text{Rep}_K(G)$ where $G$ is an affine $K$-group. In Example 7.5, we may take $L = K$, $V = \text{Vec}_K$ and for $H$ the forgetful functor. Here $C$ and $C/N$ are abelian. The functor $\bar{\pi}$ from (7.1) and the $\otimes$-retraction $\rho_C$ of Corollary 5.2 yield an exact $\otimes$-functor

(7.2) $T(C) \to S(C) \times C$.

If $G$ is proreductive, then $N \neq 0$, $C = T(C) = S(C)$ and (7.2) factors through the diagonal functor. On the other hand, if $G$ is not proreductive, then $N \neq 0$ and $\bar{\pi}$ does not factor through the retraction $\rho_C$.

If $G = \mathbb{G}_a$ (or more generally if its prounipotent radical is $\mathbb{G}_a$, as in [2, App. C]), O'Sullivan has proven that (7.2) is an equivalence [26]. Can one compute $T(C)$ in more complicated cases?

8. APPLICATION TO MOTIVIC CONJECTURES

8.1. The standard conjecture D.

Theorem 8.1. Let $M_{\text{rat}}(k)$ be the $\mathbb{Q}$-linear category of Chow motives over a field $k$.

a) Let $\pi : M_{\text{rat}}(k) \to M_{\text{num}}(k)$ be the canonical functor to the $\mathbb{Q}$-linear abelian category of motives modulo numerical equivalence. The functor $\pi$ induces a $\otimes$-functor

(8.1) $\bar{\pi} : T(M_{\text{rat}}(k)) \to M_{\text{num}}(k)$

which is a Serre localisation.

b) Let $F : M_{\text{rat}}(k) \to A$ be a $\otimes$-functor where $A$ is abelian and rigid. If $A$ is split and $F$ is full, then $F$ factors through numerical equivalence.

c) We have: $\bar{\pi}$ is an equivalence $\iff$ $Z(T(M_{\text{rat}}(k))$ is a field.
Proof. Note that $Z(M_{\text{rat}}(k)) = \mathbb{Q}$.

a) In Example 7.5, take $\mathcal{C} = M_{\text{rat}}(k)$ and for $H$ a Weil cohomology. Here, $\mathcal{V} = \text{Vec}_L^\pm$, the abelian $\otimes$-category of finite-dimensional $\mathbb{Z}/2$-graded $L$-vector spaces, where $L$ is the field of coefficients of $H$; the commutativity constraint is given by the Koszul rule. We apply Theorem 7.6. The category $S(\mathcal{C}) = (\mathcal{C}/\mathcal{N})^\sharp$ is the category $M_{\text{num}}(k)$ ([17], which inspired Example 7.5).

b) follow from Corollary 7.8.

d) follows from Corollary 7.9. 

Corollary 8.2. If $Z(T(M_{\text{rat}}(k)))$ is a field, the standard conjecture D is true.

In Theorem 8.1, we can replace rational equivalence by any adequate equivalence relation which is coarser than the homological equivalence given by a Weil cohomology $H$ as in the proof of a). For example, this homological equivalence itself yields the category of motives $M_H(k)$ together with the faithful functor $H : M_H(k) \to \text{Vec}_L^\pm$. Applying Corollary 7.9, we find:

Theorem 8.3. The standard conjecture D holds for the Weil cohomology $H$ if and only if $Z(T(M_H(k)))$ is a field.

In [30], Schäppi constructs a graded-Tannakian category $M_H(k)$ along with a $\otimes$-functor $S : M_H(k) \to M_H(k)$ lifting $H$, and proves that $S$ is an equivalence of categories under the standard conjecture D. From Theorem 5.1, we obtain a $\otimes$-functor

$$\bar{S} : T(M_H(k)) \to M_H(k)$$

extending $S$. Using $\bar{S}$, we get:

Proposition 8.4. The standard conjecture D holds for $H$ if and only if $M_H(k)$ is split (e.g. semi-simple) and $S$ is full.

Proof. If: by [30, Th. 3.2.1 (i)], the standard conjecture D implies that $S$ is an equivalence of categories. Since it also implies that $M_H(k)$ is abelian semi-simple, we conclude. Only if: by Remark 7.4, the hypothesis implies that $\bar{S}$ factors through $S(M_H(k))$; this in turn implies that $H$ factors through numerical equivalence.

(See [30, Prop. 3.3.4] for a consequence of the semi-simplicity of $M_H(k)$, assumed alone.)
8.2. **Voevodsky’s conjecture.** Let \( k \) be a field. Recall that an algebraic cycle \( z \) on a smooth projective variety \( X \) is smash-nilpotent if \( z \times n \) is rationally equivalent to \( 0 \) on \( X^n \) for \( n \gg 0 \). This defines an adequate equivalence relation \( \text{tnil} \). In [31, Conj. 4.2], Voevodsky conjectured that 

\[
\mathcal{M}_{\text{tnil}}(k) \xrightarrow{\sim} M_{\text{num}}(k).
\]

By Theorem 5.1, we have \( T(M_{\text{rat}}(k)) \xrightarrow{\sim} T(M_{\text{tnil}}(k)) \). Let us write \( T(k) \) for this rigid abelian \( \otimes \)-category.

**Proposition 8.5.** Voevodsky’s conjecture is equivalent to the following two statements put together:

1. \( Z(T(k)) \) is a field;
2. the functor \( M_{\text{tnil}}(k) \to T(k) \) is faithful.

**Proof.** This follows from Corollary 7.9 applied to \( C = \mathcal{M}_{\text{tnil}}(k) \), noting that this category is pseudo-abelian by definition. \( \square \)

8.3. **Motives over a base.** Let \( S \) be a nonempty, connected, separated regular excellent scheme of finite Krull dimension. We then have the Deninger-Murre–O’Sullivan rigid \( \otimes \)-category of relative Chow motives over \( S \) [13], [24, §5.1]. For coherence with the classical definition of motives, we shall restrict to the thick subcategory of O’Sullivan’s category defined by motives of smooth projective \( S \)-schemes (O’Sullivan considers more generally smooth proper \( S \)-schemes): we denote this category by \( \mathcal{M}_{\text{rat}}(S) \). We have \( Z(\mathcal{M}_{\text{rat}}(S)) = \mathbb{Q} \).

Let \( K \) be the function field of \( S \). If \( j : \text{Spec} \, K \to S \) is the corresponding inclusion, we have the restriction \( \otimes \)-functor

\[
j^* : \mathcal{M}_{\text{rat}}(S) \to \mathcal{M}_{\text{rat}}(K).
\]

We write \( \mathcal{M}_{\text{rat}}(K, S) \) for its essential image (motives with good reduction relatively to \( S \)).

**Theorem 8.6.** The functor \( j^* \) is full and its kernel is smash-nilpotent. It induces an equivalence of categories

\[
T(S) \xrightarrow{\sim} T(K, S)
\]

where \( T(S) := T(\mathcal{M}_{\text{rat}}(S)) \) and \( T(K, S) := T(\mathcal{M}_{\text{rat}}(K, S)) \), and a full embedding

\[
\mathcal{M}_{\text{num}}(S) \hookrightarrow \mathcal{M}_{\text{num}}(K)
\]

where \( \mathcal{M}_{\text{num}}(S) := (\mathcal{M}_{\text{rat}}(S)/N)^\natural \).

**Proof.** The first assertions are [24, Prop. 5.1.1]. The equivalence of categories then follows from Theorem 5.1, while the full embedding follows from Lemma 7.2. \( \square \)
Let now $i : Z \hookrightarrow S$ be a closed subscheme of $S$, also connected and regular, with function field $L$; we have a pull-back functor $i^* : \mathcal{M}_{\text{rat}}(S) \to \mathcal{M}_{\text{rat}}(Z)$. Theorem 8.6 then yields a “specialisation” functor

$$i^! : T(K, S) \to T(L, Z).$$

Since $i^*$ is not full, it does not a priori induce a functor $\mathcal{M}_{\text{num}}(S) \to \mathcal{M}_{\text{num}}(Z)$.

Using a Weil cohomology $H$ verifying the smooth and proper base change theorem (e.g. $l$-adic cohomology for a prime $l$ invertible on $S$) and using the monoidal section theorem, we can construct as in [1, Th. 11] a “specialisation” $\otimes$-functor $\mathcal{M}_{\text{num}}(S) \to \mathcal{M}_{\text{num}}(Z)$, depending a priori on $H$; we leave details to the interested reader.

Acknowledgements. We would like to thank Y. André for his kind interest in this work and for pointing out Example 7.10 at an early stage, and J. Wildeshaus for the reference to Prop. 5.1.1 in [24]. The first author acknowledges K. Coulembier for explaining his works [7] and [8]. The second author thanks P. O’Sullivan for several exchanges on his work [25] and for communicating [26]. Finally, the first author thanks Institut de Mathématiques de Jussieu-Paris Rive Gauche for hospitality and both authors thank CNRS for support.

References

[1] Y. André & B. Kahn: Construction inconditionnelle de groupes de Galois motiviques, *C. R. Acad. Sci. Paris* 334 (2002) 989–994.

[2] Y. André & B. Kahn: Nilpotence, radicaux et structures monoïdales (with an appendix by P. O’Sullivan), *Rend. Sem. Mat. Univ. Padova* 108 (2002), 107–201.

[3] Y. André & B. Kahn: Erratum: nilpotence, radicaux et structures monoïdales, *Rend. Sem. Mat. Univ. Padova* 113 (2005), 125–128.

[4] L. Barbieri-Viale & M. Prest: Definable categories and $\mathbb{T}$-motives, *Rend. Sem. Mat. Univ. Padova* 139 (2018) 205-224.

[5] L. Barbieri-Viale, A. Huber-Klawitter & M. Prest: Tensor structure for Nori motives, *Pacific Journal of Math.* 306 No. 1 (2020) 1–30.

[6] A. Bruguières: Théorie tannakienne non commutative, *Comm. in Alg.* 22 (1994), 5817–5860.

[7] K. Coulembier: Monoidal abelian envelopes, *Compositio Math.* 157 (2021), 1584–1609.

[8] K. Coulembier, P. Etingof, V. Ostrik & B. Pauwels: Monoidal Abelian Envelopes with a quotient property, preprint, 2021, https://arxiv.org/abs/2103.00094.

[9] K. Coulembier: Homological kernels of monoidal functors, preprint, 2021, https://arxiv.org/abs/2107.02374.

[10] P. Deligne: La catégorie des représentations du groupe symétrique $S_t$, lorsque $t$ n’est pas un entier naturel, in *Algebraic groups and homogeneous spaces* (Tata
[11] P. Deligne & J. Milne: Tannakian categories, in Hodge cycles, motives, and Shimura varieties, Lect. Notes in Math. 900, Springer, 1982, 101–228.

[12] A. Delpeuch: Autonomization of Monoidal Categories, Proceedings Applied Category Theory 2019, Electron. Proc. Theor. Comput. Sci. (EPTCS) 323, 2020, pp. 24–43, https://arxiv.org/abs/1411.3827.

[13] C. Deninger & J. P. Murre: Motivic decomposition of abelian schemes and the Fourier transform, J. reine angew. Math. 422 (1991), 201–219.

[14] A. Dold & D. Puppe: Duality, trace, and transfer, in Proceedings of the International Conference on Geometric Topology (Warsaw, 1978), PWN, Warsaw, 1980, 81–102.

[15] P. Freyd: Representations in abelian categories, in Proc. Conf. Categorical Algebra (La Jolla, CA, 1965), edited by S. Eilenberg et al., Springer, 1966, 95–120.

[16] P. Gabriel: Des catégories abéliennes, Bull. SMF 90 (1962), 323–448.

[17] U. Jannsen: Motives, numerical equivalence and semi-simplicity, Invent. Math. 107 (1992), 447–452.

[18] B. Kahn: Exactness and faithfulness of monoidal functors, preprint, 2021, https://arxiv.org/abs/2110.09381.

[19] B. Kahn: Universal rigid abelian tensor categories and Schur finiteness, preprint, 2021.

[20] S. Kleiman: Algebraic cycles and the Weil conjectures, in Dix expositions sur la cohomologie des schémas, Adv. Stud. Pure Math., 3, Masson–North-Holland, Amsterdam, 1968, 359–386.

[21] H. Krause: Functors on locally finitely presented additive categories, Coll. Math. 75 (1998), 105–132.

[22] J.-P. Olivier, Anneaux absolument plats universels et épimorphismes à buts réduits, Sém. Samuel, Algèbre commutative 2 (1967), 1–12.

[23] P. O'Sullivan: The generalised Jacobson-Morosov theorem, Mem. Amer. Math. Soc. 207 (2010), no. 973.

[24] P. O'Sullivan: Algebraic cycles on an abelian variety, J. reine angew. Math. 654 (2011), 1–81.

[25] P. O'Sullivan: Super Tannakian hulls, preprint (2020), https://arxiv.org/abs/2012.15703.

[26] P. O'Sullivan: Absolute flatness and universal rigid abelian tensor categories, manuscript, 2021.

[27] M. Prest: Definable additive categories: purity and model theory, Mem. AMS. 987. AMS, 2011.

[28] M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121. Cambridge University Press, 2009.

[29] N. Saavedra Rivano, Catégories tannakiennes, Lect. Notes in Math. 265, Springer, 1972.

[30] D. Schäppi: Graded-Tannakian categories of motives, preprint, 2020, https://arxiv.org/abs/2001.08567.

[31] V. Voevodsky: A nilpotence theorem for cycles algebraically equivalent to zero, IMRN 4 (1995), 187–199.
