Darboux transformations of supersymmetric Heisenberg magnet model

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Abstract

The Darboux transformation of supersymmetric Heisenberg magnet model is investigated. We calculate the exact superfield multisoliton solutions of the HM model by iteration of Darboux transformation. Explicit soliton solutions are also constructed for the SU(2) case.

1. Introduction

Supersymmetry (SUSY) is a symmetry that relates the particles having different statistical properties and spins or we can say that it is a symmetry in which we treat equally with the fermions and bosons. Mathematically, it is applied by introducing anti-commuting variables called the Grassman variables along with the usual commuting variables. The purpose of introducing supersymmetry in integrable systems of classical and quantum theories is to calculate the fermionic extensions of known integrable systems and also develop understanding about their geometric structures, interactions and physical content in the form of soliton solutions.

During the past few years, there has been great interest in the field of classical and quantum integrability of Heisenberg magnet (HM) model [3–12]. The integrability of the HM model based upon SU(2) through the inverse scattering method is studied in [3, 4] and its SU(n) is reported in [6]. The HM model based upon Hermitian symmetric spaces has been investigated in [9–12]. The Darboux transformation of the generalized Heisenberg magnet GHM model and its soliton solutions in terms of quasideterminants were presented in [13].

In this paper, we extend the earlier results obtained in [13] for the case of supersymmetric GHM model. We propose the supersymmetric generalization of the HM model and express the superfield Lax representation of the system. Further, we investigate the Darboux transformation by introducing a superfield Darboux matrix for HM model and calculate the superfield multisoliton solutions of the supersymmetric HM model. We then present the Darboux transformation on the component form. In the last section, we take the system based on Lie group SU(N) and derive the SU(2) based explicit solutions. In order to construct the SUSY field equations, we have to extend the bosonic system having spacetime variables (x, t) to one with super-spacetime variables (x, t, θ, ξ) where θ and ξ are anti-commuting Grassmann variables.

The Hamiltonian of the supersymmetric Heisenberg magnet model is defined as

\[
\mathcal{H} = \frac{1}{2} \text{Tr}((D_x U)^T (D_x U)), \tag{1.1}
\]

where ‘T’ is the transpose and \( U(x, t, \theta, \xi) \) is superfield matrix which takes values in the Lie group \( \mathcal{GL}(n) \) in the Lie algebra \( \mathfrak{g} \). The equation of motion can be written as

\[ D^\dagger D - \theta D^\dagger Q + Q D^\dagger - \theta D^\dagger Q + Q D^\dagger = 0, \]

\[ D^\dagger D - \xi D^\dagger Q + Q D^\dagger - \xi D^\dagger Q + Q D^\dagger = 0. \]

The theory that combines fermions and bosons.

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2. Note that here \( \mathcal{GL}(n) \) is a general linear group over the field of real numbers therefore it is a real Lie group of dimensions \( n^2 \).
\[ D_\lambda \mathcal{U} = \{ \mathcal{H}, D_\lambda \mathcal{U} \}, \quad \text{(1.2)} \]

where the bracket \{,\} represents an anti-commutator. By substituting equation (1.1) in (1.2), we get
\[ D_\lambda \mathcal{U} = D_\lambda [\mathcal{U}, D_\lambda \mathcal{U}], \quad \text{(1.3)} \]

where \([,\) is the commutator. The superfield \( \mathcal{U}(x, t, \theta, \xi) \) is corresponding to \( \mathcal{G} \) transformation is \( \mathcal{U} = GTG^{-1} \), where \( T \) is the \( N \times N \) constant matrix. The superfield equation of motion (1.3) can also be expressed as the zero-curvature condition
\[ \begin{bmatrix} D_x - \frac{i}{1 - \lambda} \mathcal{U} & D_t - \frac{ic^2}{(1 - \lambda)^2} \mathcal{U} + \frac{i}{1 - \lambda} [\mathcal{U}, D_x \mathcal{U}] \end{bmatrix} = 0. \quad \text{(1.4)} \]

The above zero-curvature condition (1.4) is equivalent to the compatibility condition of the following supersymmetric Lax pair is given as
\[ D_x \mathcal{V} = \frac{i}{1 - \lambda} \mathcal{U} \mathcal{V}, \quad \text{(1.5)} \]
\[ D_t \mathcal{V} = \left( \frac{c^2}{(1 - \lambda)^2} \mathcal{U} + \frac{1}{1 - \lambda} [\mathcal{U}, D_x \mathcal{U}] \right) \mathcal{V}, \quad \text{(1.6)} \]

where \( \lambda \) is the real (or complex) parameter, \( c \in \mathbb{R} \text{ (or } \mathbb{C}) \) and \( \mathcal{V}(x, t, \theta, \xi) \) is the matrix superfield.

### 2. Darboux transformation of supersymmetric HM model and multisoliton solutions

In this section, we construct the Darboux transformation for the supersymmetric HM model. In order to construct the multisoliton solution of the SUSY HM model, we define the Darboux transformation on the superfields that provides the generalization of Darboux transformation to the supersymmetric case. In our case, the Darboux transformation is defined by \( N \times N \) superfield matrix \( \mathcal{D}(x, t, \theta, \xi, \lambda) \) called the superfield Darboux matrix (for detail discussion on the Darboux transformation we can see [14–26]).

Now, let us define the superfield matrix \( \mathcal{D}(x, t, \theta, \xi; \lambda) \), such that superfield \( \mathcal{V} \) of the Lax pair transforms as
\[ \mathcal{V}[1] = \mathcal{D}(\lambda) \mathcal{V}, \quad \text{(2.1)} \]
where \( \mathcal{D}(x, t, \lambda) \) is the Darboux matrix which is written as
\[ \mathcal{D}(x, t, \lambda) = \lambda I - \mathcal{S}(x, t), \quad \text{(2.2)} \]

where \( \mathcal{S} \) is the \( N \times N \) matrix superfield, \( \lambda \) is spectral parameter and \( I \) be the \( N \times N \) identity matrix. Here it is noted that in order to calculate the superfield Darboux matrix we have to only find the value of superfield \( \mathcal{S} \) because superfield Darboux matrix is linear in \( \lambda \). The new solution \( \mathcal{V}[1] \) satisfies the Lax pairs are given as
\[ D_\lambda \mathcal{V}[1] = \frac{i}{1 - \lambda} \mathcal{U}[1] \mathcal{V}[1], \quad \text{(2.3)} \]
\[ D_t \mathcal{V}[1] = \left( \frac{c^2}{(1 - \lambda)^2} \mathcal{U}[1] + \frac{1}{1 - \lambda} [\mathcal{U}[1], D_x \mathcal{U}[1]] \right) \mathcal{V}[1]. \quad \text{(2.4)} \]

Now substituting (2.1) and (2.2) in (2.3) and (2.4) respectively, we get the following Darboux transformation on superfield \( \mathcal{U} \), i.e:
\[ \mathcal{U}[1] = \mathcal{U} - i D_\lambda \mathcal{S}, \quad \text{(2.5)} \]

and the superfield \( \mathcal{S} \) is required to satisfy the following conditions
\[ D_\lambda \mathcal{S}(I - \mathcal{S}) = i[\mathcal{U}, \mathcal{S}], \quad \text{(2.6)} \]
\[ D_t \mathcal{S}(I - \mathcal{S})^2 = i[c^2 \mathcal{U} + [\mathcal{U}, D_\lambda \mathcal{U}], \mathcal{S}] + i[\mathcal{S}[\mathcal{U}, D_\lambda \mathcal{U}], \mathcal{S}] - i[\mathcal{U}, D_\lambda \mathcal{U}] \mathcal{S}^2. \quad \text{(2.7)} \]

We can solve the above equations (2.6), (2.7) to calculate the explicit solution for the superfield matrix function \( \mathcal{S} \). An explicit expression for \( \mathcal{S} \) can be found as follows.

Let us construct a matrix superfield from the particular column superfield solutions of the superfield Lax pair at a particular values of spectral parameter \( \lambda_i \). Take \( N \) distinct real (or complex) constant parameters \( \lambda_1, \lambda_2, \ldots, \lambda_N \) also \( \lambda_i = \pm 1; i = 1, 2, \ldots, N \). Also take \( N \) column vectors \( |1\rangle, \ldots, |N\rangle \) such that
\[ \mathcal{M} = (\mathcal{V}(\lambda_1)|1\rangle, \ldots, \mathcal{V}(\lambda_N)|N\rangle) = (|\Theta_1\rangle, \ldots, |\Theta_N\rangle), \quad \text{(2.8)} \]
be an $N \times N$ invertible superfield matrix. Each column $|\Theta_i\rangle = V(\lambda_i)|i\rangle$ in $\mathcal{M}$ is a column solution of Lax pair (1.5) and (1.6), when $\lambda = \lambda_i$, i.e.

$$D_{\lambda}|\Theta_i\rangle = \frac{i}{1 - \lambda_i}U|\Theta_i\rangle,$$

(2.9)

$$D_{\lambda}|\Theta_i\rangle = \frac{ie^2}{(1 - \lambda_i)^2}U + \frac{1}{1 - \lambda_i}[U, D_{\lambda}U]|\Theta_i\rangle,$$

(2.10)

where $i = 1, 2, \ldots, N$. Assuming $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, the $N \times N$ superfield matrix generalization of the super Lax pair (2.9), (2.10) can be written as

$$D_{\lambda}\mathcal{M} = iU\mathcal{M}(I - \Lambda)^{-1},$$

(2.11)

$$D_{\lambda}\mathcal{M} = i e^2U\mathcal{M}(I - \Lambda)^{-2} + i[U, D_{\lambda}U]\mathcal{M}(I - \Lambda)^{-1}.$$

(2.12)

Now the superfield matrix $S$ is selected such that when it is inserted in equation (2.2), then it will give the Darboux transformation. So the choice of superfield matrix $S$ is as:

$$S = \mathcal{M}\mathcal{M}^{-1},$$

(2.13)

We can now check whether the equation (2.13) be the solution of equation (2.6) and (2.7) by apply $D_{\lambda}$ and $D_{\lambda}$ on equation (2.13). Which shows that the choice is the good one. We can also say that if $(V, U)$ is the solution of super Lax pair (1.5) and (1.6) and the matrix $S$ is defined by equation (2.13), then $(\mathcal{V}[1], U[1])$ defined by (2.1) and (2.5) respectively, are also the solutions of the same super Lax pair. Therefore, we can write

$$\mathcal{V}[1] = (\mathcal{M} - \mathcal{M}\mathcal{M}^{-1})\mathcal{V},$$

(2.14)

$$\mathcal{U}[1] = (\mathcal{M} - S)\mathcal{U}(\mathcal{M} - S)^{-1},$$

(2.15)

as the required Darboux transformation on the solution $\mathcal{V}$ to the super Lax pair. Now we will express the superfield multisoliton solutions of the supersymmetric Heisenberg magnet model in terms of quasideterminants over a noncommutative ring of matrices whose entries are Grassmann even superfields. We can write (2.1) as

$$\mathcal{V}[1] = \mathcal{D}(\lambda)\mathcal{V} = (\mathcal{M} - S)\mathcal{V},$$

$$= (\mathcal{M} - \mathcal{M}\mathcal{M}^{-1})\mathcal{V} = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}(I - \Lambda) & \mathcal{M} \end{vmatrix},$$

(2.16)

and the superfield matrix $U$ can be expressed as

$$\mathcal{U}[1] = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}(I - \Lambda) & \mathcal{M} \end{vmatrix}^{-1}.$$

(2.17)

Now we write the $K$-fold Darboux transformation on the superfield $\mathcal{V}$ as

$$\mathcal{V}[K + 1] = \prod_{k=1}^{K} (\mathcal{M} - S[K - k + 1])\mathcal{V} = \prod_{k=1}^{K} \begin{vmatrix} \mathcal{M}[K - k + 1] & I \\ \mathcal{M}[K - k + 1] \Lambda_{K - k + 1} & \mathcal{M} \end{vmatrix} \mathcal{V},$$

$$= \lambda \mathcal{V}[K] - \mathcal{M}[K]\Lambda_{K}\mathcal{M}[K]^{-1}\mathcal{V}[K],$$

$$= \begin{vmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_K \\ \mathcal{M}_1 \Lambda_1 & \mathcal{M}_2 \Lambda_2 & \cdots & \mathcal{M}_K \Lambda_K \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_1 \Lambda_1^K & \mathcal{M}_2 \Lambda_2^K & \cdots & \mathcal{M}_K \Lambda_K^K \end{vmatrix} \mathcal{V},$$

(2.18)

where for each $k = 1, 2, 3, \ldots, K$. We have defined $N \times N$ an invertible matrix superfield $\mathcal{M}_k$ be the matrix superfield solution of the Lax pair (1.5), (1.6) at $\Lambda = \Lambda_k$. Similarly the $K$ times iteration of the Darboux transformation gives the expression
\[ U[K+1] = (I - S[K]) \ldots (I - S[2])(I - S[1])U(I - S[1])(I - S[2]) \ldots (I - S[K])^{-1}, \]
\[ = \prod_{k=1}^{K} \left[ M[K-k+1]^{-1} \right] \]
\[ \times U \times \prod_{k=1}^{K} \left[ M[K-k+1]^{-1} \right] , \]
\[ = \begin{vmatrix} M_1 & M_2 & \ldots & M_K \\ M_1(I - \lambda_1) & M_2(I - \lambda_2) & \ldots & M_K(I - \lambda_K) \\ \vdots & \vdots & \ddots & \vdots \\ M_1(I - \lambda_1)^K & M_2(I - \lambda_2)^K & \ldots & M_K(I - \lambda_K)^K \end{vmatrix} \times U \]
\[ = \begin{vmatrix} M_1 & M_2 & \ldots & M_K \\ M_1(I - \lambda_1) & M_2(I - \lambda_2) & \ldots & M_K(I - \lambda_K) \\ \vdots & \vdots & \ddots & \vdots \\ M_1(I - \lambda_1)^K & M_2(I - \lambda_2)^K & \ldots & M_K(I - \lambda_K)^K \end{vmatrix} \times U \]
\[ = \begin{vmatrix} M_1 & M_2 & \ldots & M_K \\ M_1(I - \lambda_1) & M_2(I - \lambda_2) & \ldots & M_K(I - \lambda_K) \\ \vdots & \vdots & \ddots & \vdots \\ M_1(I - \lambda_1)^K & M_2(I - \lambda_2)^K & \ldots & M_K(I - \lambda_K)^K \end{vmatrix} \times U \]
\[ = \begin{vmatrix} M_1 & M_2 & \ldots & M_K \\ M_1(I - \lambda_1) & M_2(I - \lambda_2) & \ldots & M_K(I - \lambda_K) \\ \vdots & \vdots & \ddots & \vdots \\ M_1(I - \lambda_1)^K & M_2(I - \lambda_2)^K & \ldots & M_K(I - \lambda_K)^K \end{vmatrix} \times U \]
Equation (2.19) gives us the desired expression of \( K \)th superfield component of the SUSY HM model which is expressed in terms of the quasideterminants involving the particular solutions of the superfield linear problem.

3. Supersymmetric HM model in component form

In this section, we calculate the component fermionic and bosonic expressions by expanding the superfields \( U(x, t, \theta, \xi) \) and \( V(x, t, \theta, \xi) \). The component expansion form of \( U(x, t, \theta, \xi) \) and \( V(x, t, \theta, \xi) \) can be written as
\[ U = \psi_+ + \theta U - \frac{i}{2} \xi \{ \psi_+ , \psi_- \} - i \theta \xi \left( \partial_\theta \psi_+ - [U, \psi_+] - \frac{i}{2} [\psi^2_+, \psi_-] \right) , \]
where \( \psi_+ , \psi_- \) are the Majorana spinors such that \( \psi_+^g = g^{-1}\psi_+^l g \), \( F(x) \) is the auxiliary field further which will be eliminated and \( U, V \) are the bosonic components of the superfields \( U, V \) respectively. Now substituting the expressions (3.1) and (3.2) in the SUSY Lax pair (1.5), (1.6) and substituting \( h_+ = \psi^2_+ \Leftrightarrow h_+^a = \frac{1}{2} f^{abc} \psi_+^b \psi_+^c \), we get the component expansion
\[ \partial_\theta V(x, t, \theta) = \frac{1}{1 - \lambda} UV(x, t) - \frac{1}{(1 - \lambda)^2} h_+^a V(x, t) , \]
\[ \partial_\theta V(x, t, \theta) = \frac{c^2}{(1 - \lambda)^2} (h_+^a V - UV) - \frac{1}{1 - \lambda} [U, h_+] V , \]
\[ \partial_\theta \psi_+ - \frac{1}{2} [U, \psi_+] + \frac{i}{4} [h_+, \psi_+] = 0 , \]
\[ \partial_\theta \psi_- - \frac{1}{2} \psi_+ [U, h_+] - \frac{i}{2} [U, h_+] = 0 . \]
Now the compatibility condition for the Lax pairs (3.3), (3.4) gives the expressions
\[ \partial_\theta U = \partial_\theta [U, \partial_\theta U] + i \partial_\theta h_+ , \]
\[ 2 \kappa^2 U_x = [U, [U, h_+]] - 2 c^2 \partial_\theta h_+ , \]
where the bosonic component is \( U = U - ih_+ \) and the expression (3.7) is the equation of motion for the bosonic limit.

3.1. Darboux transformation on component SUSY HM model

In this section, we have to define the Darboux transformation on component fields of the superfields for the explicit soliton solutions of the supersymmetric Heisenberg magnet model. In order to do this, we expand all the superfields then, collect different coefficient’s of \( \lambda \)’s, The Darboux transformation on the component fields of the supersymmetric Heisenberg magnet model can be defined by a Darboux matrix \( D \) as
\[ D(\lambda) = M - S , \]
where $D$ and $S$ are the leading bosonic components of the superfields matrix $\mathcal{D}$ and $\mathcal{S}$, respectively. The matrix $D$ must be defined such that the solution $V$ of the Lax pair (3.3), (3.4) transforms as

$$
\tilde{V} = D(\lambda)V,
$$

with suitable $\tilde{U}$, $\tilde{h}_+$ satisfying the equations (3.5) and (3.6). This implies that $\tilde{V}$, $\tilde{U}$, $\tilde{h}_+$ are required to satisfy the transformed Lax pair

$$
\begin{align*}
\partial_x \tilde{V}(x, t) &= \frac{1}{1 - \lambda} \tilde{U} \tilde{V}(x, t) - i \frac{1}{(1 - \lambda)^2} \tilde{h}_+ \tilde{V}(x, t), \\
\partial_t \tilde{V}(x, t) &= \frac{c^2}{(1 - \lambda)^2} (\tilde{h}_+ \tilde{V} - \tilde{U} \tilde{V}) - \frac{1}{1 - \lambda} [\tilde{U}, \tilde{h}_+] \tilde{V}.
\end{align*}
$$

(3.11)

Now following the previous section’s argument. Let

$$
M = (V(\lambda_i)[i], \ldots, V(\lambda_N)[N]) = ([m_1], \ldots, [m_N]),
$$

(3.12)

be an invertible $N \times N$ matrix, which represents the leading bosonic component of the matrix superfield $\mathcal{M}$ with each column $[m_i] = V(\lambda_i)[i]$ in $M$ representing a column solution of the Lax pair (3.3), (3.4) when $\lambda = \lambda_i$, i.e.

$$
\partial_\lambda M = UM(I - \Lambda)^{-1} - i\hbar_+ M(I - \Lambda)^{-2},
$$

$$
\partial_t M = ic^2\hbar_+ M(I - \Lambda)^{-2} - c^2UM(I - \Lambda)^{-2} - i[U, h_+] M(I - \Lambda)^{-1}.
$$

(3.13)

The choice of matrix $S$ is given as

$$
S = M\Lambda M^{-1}.
$$

(3.14)

Now substituting (3.11) into equations (3.11) and using (3.3), (3.4), we get the Darboux transformation on $h_+$, $U, \psi_+$ as

$$
\begin{align*}
\tilde{h}_+ &= (I - S)h_+(I - S)^{-1}, \\
\tilde{U} &= (I - S)U(I - S)^{-1}, \\
\tilde{\psi}_+ &= (I - S)\psi_+(I - S)^{-1},
\end{align*}
$$

(3.15)

where the conditions on $S$ are given by

$$
\partial_\lambda S(I - S) = [U, S] - i[h_+, S],
$$

(3.16)

and

$$
[i\hbar_+ - c^2U - i[U, h_+], S] = \delta S[U, h_+]S - i[U, h_+]S^2 + \partial_t S(I - S)^2.
$$

(3.17)

Our next step is to verify that whether the equation (3.14) is solution of equations (3.16), (3.17). For this we operate $\partial_\lambda, \partial_t$ on (3.14), we get

$$
\partial_\lambda S = i\hbar_+ - U + (I - S)U(I - S)^{-1} - i(I - S)h_+(I - S)^{-1},
$$

(3.18)

$$
\begin{align*}
\partial_t S &= (i\hbar_+ - c^2U - i[U, h_+])S - c^2UM(I - \Lambda)^{-2} - i[U, h_+]M(I - \Lambda)^{-1}\Lambda S^{-1} \\
&\quad - M\Lambda M^{-1}(i\hbar_+ - c^2U - i[U, h_+])M(I - \Lambda)^{-1}\theta^{-1}.
\end{align*}
$$

(3.19)

Equation (3.18) and (3.19) are similar to equations (3.16) and (3.17) respectively. This shows that the choice of matrix $S$ satisfies all the conditions imposed by the covariance of Lax pair under Darboux transformation. We can write these results in terms of quasideterminants. Note that equations (3.15) are different in essence from the results of [20] because the leading bosonic component of our spin function $U$ does not contain a separate fermionic part.

4. The explicit solutions of the SU(2) system

Before starting with SU(2) case, we first discuss the generalized HM model based on SU(N). The matrix $U$ takes values in the Lie algebra $SU(N)$ i.e. we can write $U$ as

$$
U = U^a T^a, \quad a = 1, 2, \ldots, n^2,
$$

(4.1)

where the generator $T^a$ of the group $SU(N)$ are the anti-Hermitian $N \times N$ matrices satisfying the condition $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. These generators satisfy the following commutation relation

$$
[T^a, T^b] = f^{abc} T^c,
$$

(4.2)

where $f^{abc}$ are the structure constants. For any $X \in SU(N)$, we write

$$
X = X^a T^a, \quad \text{and} \quad U^a = -2 \text{Tr}(U T^a).
$$

(4.3)
Since $U \in SU(N)$ therefore we must have
\[ U^T = -U, \quad \text{and} \quad \operatorname{Tr} U = 0. \] 
(4.4)
Similarly the Darboux transformed solution will also satisfy the following equations
\[ U^T [1] = -U [1], \quad \text{and} \quad \operatorname{Tr} U [1] = 0. \] 
(4.5)
The matrix $M$ therefore satisfies the following conditions
\[ M^T = -M, \quad \text{and} \quad \operatorname{Tr} M = 0. \] 
(4.6)
The seed solution will have the form
\[
U = \begin{pmatrix}
  u_1 & \cdots & 0 \\
  0 & \ddots & \vdots \\
  0 & \cdots & u_n
\end{pmatrix},
\]
\[
ih_+ = \begin{pmatrix}
  h_1 & 0 & \cdots \\
  0 & \ddots & \vdots \\
  0 & \cdots & h_n
\end{pmatrix},
\]
(4.7)
(4.8)
where $\delta$ is some non-zero real parameter. Also we have
\[
V(\lambda) = \begin{pmatrix}
  w_p(\lambda) & O \\
  O & w_{n-p}(\lambda)
\end{pmatrix},
\]
(4.9)
where
\[
w_p(\lambda) = \begin{pmatrix}
  e^{i\nu_p(\lambda)} & O \\
  0 & e^{i\nu_p(\lambda)}
\end{pmatrix},
\]
(4.10)
also
\[
w_{n-p}(\lambda) = \begin{pmatrix}
  e^{i\nu_{n-p}(\lambda)} & O \\
  0 & e^{i\nu_{n-p}(\lambda)}
\end{pmatrix},
\]
(4.11)
and
\[
w_i(\lambda) = \exp\left( i \frac{\xi}{1 - \lambda_i} + \frac{4}{(1 - \lambda_i)^2} t \right) \exp\left( \frac{\delta \xi}{(1 - \lambda_i)^2} \right).
\]
(4.12)
Now for the $SU(2)$ case equations (4.7), (4.8), (4.9) and (4.12) becomes
\[
U = \begin{pmatrix}
  i & 0 \\
  0 & -i
\end{pmatrix},
\]
(4.13)
\[
ih_+ = \begin{pmatrix}
  i\delta & 0 \\
  0 & -i\delta
\end{pmatrix}.
\]
(4.14)
The corresponding solution of the $V(\lambda)$ can be written as
\[
V [1](\lambda) = \begin{pmatrix}
  w(\lambda) & 0 \\
  0 & w^{-1}(\lambda)
\end{pmatrix}.
\]
(4.15)
Note that in equation (4.15), we used
\[
w(\lambda) = \exp\left( i \frac{\xi}{1 - \lambda} + \frac{4}{(1 - \lambda)^2} t \right) \exp\left( \frac{\delta \xi}{(1 - \lambda)^2} \right).
\]
(4.16)
Therefore in this sense $U$, $h_+$ and $V$ establish the seed solution for the Darboux transformation. Let us take $\lambda_0 = \mu$ and $\lambda_0 = \bar{\mu}$. The matrix $M$ and constant matrix $\Lambda$ are given as
\[
M = \begin{pmatrix}
w(\mu) & w(\bar{\mu}) \\
-w^{-1}(\mu) & w^{-1}(\bar{\mu})
\end{pmatrix},
\]
(4.17)
\[
\Lambda = \begin{pmatrix}
\mu & 0 \\
0 & \bar{\mu}
\end{pmatrix}.
\]
(4.18)
The direct calculations give us
\[
\det M = \exp(u + r) + \exp(-(u + r)) > 0,
\]
\[
S = MAM^{-1},
\]
\[
= \frac{1}{e^{(u+\tau')} + e^{-(u+\tau')}} \left( \mu e^{(u+\tau')} + \bar{\mu} e^{-(u+\tau')} \right) \left( \mu - \bar{\mu} \right) e^{i(u+\tau')}
\]
(4.19)

where
\[
u(x, t) = i \left( \frac{1}{1 - \mu} - \frac{1}{1 - \bar{\mu}} \right) x + 4 i \left( \frac{1}{(1 - \mu)^2} - \frac{1}{(1 - \bar{\mu})^2} \right) t,
\]
\[
u'(x, t) = \left( \frac{1}{1 - \mu} + \frac{1}{1 - \bar{\mu}} \right) x + 4 \left( \frac{1}{(1 - \mu)^2} + \frac{1}{(1 - \bar{\mu})^2} \right) t,
\]
\[
\begin{align*}
r(x) &= i \left( \frac{1}{(1 - \mu)^2} - \frac{1}{(1 - \bar{\mu})^2} \right) \delta x, \\
r'(x) &= \left( \frac{1}{(1 - \mu)^2} + \frac{1}{(1 - \bar{\mu})^2} \right) \delta x.
\end{align*}
\]
(4.20)

Let us consider \( \mu = e^{i\theta} \) and simplifying, we have
\[
S = \begin{bmatrix}
\cos \theta + i \sin \theta \tanh a & -i (\sin \theta \sech a) e^{ia'} \\
-i (\sin \theta \sech a) e^{-ia'} & \cos \theta - i \sin \theta \tanh a
\end{bmatrix},
\]
(4.21)

where
\[
a(x, t) = i \left( \frac{1}{1 - \mu} - \frac{1}{1 - \bar{\mu}} \right) x + 4 i \left( \frac{1}{(1 - \mu)^2} - \frac{1}{(1 - \bar{\mu})^2} \right) t
\]
\[
+ i \left( \frac{1}{(1 - \mu)^2} - \frac{1}{(1 - \bar{\mu})^2} \right) \delta x,
\]
\[
a'(x, t) = \left( \frac{1}{1 - \mu} + \frac{1}{1 - \bar{\mu}} \right) x + 4 \left( \frac{1}{(1 - \mu)^2} + \frac{1}{(1 - \bar{\mu})^2} \right) t
\]
\[
+ \left( \frac{1}{(1 - \mu)^2} + \frac{1}{(1 - \bar{\mu})^2} \right) \delta x.
\]
(4.22)

By using (4.21) in (3.15), we have the following Darboux transformed matrices
\[
U[1] = \begin{pmatrix}
iU_1 & U_2 \\
-U_2 & -iU_1
\end{pmatrix},
\]
(4.23)
\[
h_+[1] = \begin{pmatrix}
i\delta U_1 & \delta U_2 \\
-\delta U_2 & -i\delta U_1
\end{pmatrix},
\]
(4.24)

where
\[
U_1 = 1 - (1 + \cos \theta) \sech^2 a, \\
U_2 \equiv \delta U_2 = -i e^{ia'} [i \sin \theta + (1 + \cos \theta) \tanh a] \sech a.
\]
(4.25)

From the arbitrary seed solution, we can generate the new solutions. In the asymptotic limit \( t \to \pm \infty, \ u, \ r \to \pm \infty \), the solution (4.23) reduces to the bosonic model. When the fermions are set to zero then the results we obtained here for the supersymmetric Heisenberg model becomes reduce to the results which are presented in \([13] \). By using the Darboux transformation we get the explicit solutions for the \( U, h_+ \) of the SUSY HM model. When we substitute (4.25) into (4.23) and (4.24), we get that \( \text{Tr}U[1] = \text{Tr} h_+[1] = 0 \). Therefore \( U[1] \) and \( h_+[1] \) satisfies the additional constraints for \( g \in SU(N) \).

5. Conclusions

In this paper, we have composed the supersymmetric Heisenberg magnet model and calculated the multisoliton solutions with the help of Darboux transformation. We started from the superspace formalism and expressed the Lax pair of the HM model in the form of component fields, then expressed the solutions in the form of quasideterminants. We have also explicitly calculated the expressions of multisolitons for the case of \( SU(2) \) for the given model. When we set the fermions equal to zero, these results reduced to the bosonic model. The work can be extended by analyzing the soliton solutions obtained. We can also bilinearize the system. Our next goal is...
to find the Poisson bracket algebra for the SUSY HM model. We will then extend it to Dirac brackets of the system. It may lead to some interesting results.

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