EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 AND ADDITIONAL INTRINSIC CHARACTERISTICS IN A HOMOGENEOUS MAGNETIC FIELD

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With the use of the general covariant matrix 10-dimensional Petiau – Duffin – Kemmer formalism in cylindrical coordinates exact solutions of the quantum-mechanical equation for a particle with spin 1 in the presence of an external homogeneous magnetic field are constructed. Three linearly independent types of solutions are separated; in each case the formula for the energy levels has been found. Within similar technique for the quantum-mechanical equation for a particle with spin 1 and additional intrinsic electromagnetic characteristics – polarizability, exact solutions are found in the presence of an external homogeneous magnetic field.

Key words: spin 1, tetrad formalism, magnetic field, quantum mechanics, exact solutions, intrinsic electromagnetic structure, differential equations

1 Introduction

The problem of a quantum-mechanical particle in an external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger’s) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli’s and relativistic Dirac’s) with spin 1/2 (the first investigation were \[1, 2, 3, 4\]). In the case of spin 1 particle, the most popular quantum-mechanical problem is the Coulomb one \[4\].

In the first part of the paper (Sections 1 – 3), exact solutions for an ordinary vector particle will be constructed. In the second part (Sections 4 – 6), the exact solutions for a particle with spin 1 and an additional intrinsic electromagnetic parameter (polarizability) will be constructed explicitly as well. In principle, these results provide us with a possibility for experimental testing of this characteristics – polarizability of the spin 1 particle.

To treat the problem for an ordinary vector particle we take the matrix Petiau – Duffin – Kemmer approach extended to a general covariant form on the basis of the tetrad formalism (recent consideration and references see e.g., in \[5, 6\]). The main equation in the tetrad form reads \[6\]

\[
\left[ i \beta^\alpha(x) \left( \partial_\alpha + B_\alpha - i \frac{e}{\hbar} A_\alpha \right) - \frac{Mc}{\hbar} \right] \Psi(x) = 0 ,
\]

\[
\beta^\alpha(x) = \beta^\alpha \ e^a_{(\alpha)}(x) , \quad B_\alpha(x) = \frac{1}{2} J^{ab} e^\beta_{(a)} \nabla_\alpha e_{(b)\beta} ;
\]

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$e^a_{(a)}(x)$ is a tetrad, $J^{ab}$ stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note $Mc/\hbar$ as $M$). The homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ corresponds to 4-potential $A^a = \left( 0, \frac{1}{2} \vec{B} \times \vec{r} \right)$; in the cylindric coordinates, and the last is given by

$$dS^2 = c^2 dt^2 - dr^2 - d\phi^2 - dz^2, \quad A_\phi = -Br^2/2.$$

Choosing a diagonal cylindric tetrad

$$e^a_{(0)} = (1, 0, 0, 0), \quad e^a_{(1)} = (0, 1, 0, 0), \quad e^a_{(2)} = (0, 0, \frac{1}{r}, 0), \quad e^a_{(3)} = (0, 0, 0, 1),$$

after simple calculations, the main equation (1) reduces to the form

$$\left[ i\beta^0 \partial_0 + i\beta^1 \partial_r + i\frac{\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M \right] \Psi = 0.$$

For brevity we will note $(eB/2\hbar)$ as $B$. It is better to choose the matrices $\beta^a$ in the so-called cyclic form, where the generator $J^{12}$ has a diagonal structure. These matrices are given in [6].

2 Separation of variables

With the use of a special substitution (it corresponds to diagonalization of the third projections of momentum $P_3$ and angular momentum $J_3$ for a particle with spin 1, specified to the cylindric tetrad basis)

$$\Psi = e^{-iet} e^{im\phi} e^{iz} \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ E \\ H \end{vmatrix} ,$$

the main equation reads

$$\left[ \epsilon\beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \right] \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ E \\ H \end{vmatrix} = 0 ;$$

after calculations we arrive at the radial system of 10 equations

$$-\hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 = M \Phi_0 ,$$
$$-i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 + i\epsilon E_2 = M \Phi_1 ,$$
$$i\hat{a}_m H_2 + i\epsilon E_1 - k H_1 = M \Phi_2 ,$$
$$-i\hat{b}_m H_2 + i\epsilon E_3 + k H_3 = M \Phi_3 ,$$

$$\hat{a}_m \Phi_0 - \epsilon_1 \Phi_1 = ME_1 , \quad -i\hat{a}_m \Phi_2 + k \Phi_1 = MH_1 ,$$
$$\hat{b}_m \Phi_0 - \epsilon_3 \Phi_3 = ME_3 , \quad i\hat{b}_m \Phi_2 - k \Phi_3 = MH_3 ,$$
$$-\epsilon_2 \Phi_2 - ik \Phi_0 = ME_2 , \quad i\hat{b}_{m-1} \Phi_1 - i\hat{a}_{m+1} \Phi_3 = MH_2 ,$$

(6)
where special abbreviations were used for first order differential operators

\[
\frac{1}{\sqrt{2}} \left( \frac{d}{dr} + \frac{m + Br^2}{r} \right) = \hat{a}_m, \quad \frac{1}{\sqrt{2}} \left( -\frac{d}{dr} + \frac{m + Br^2}{r} \right) = \hat{b}_m.
\]

From (8) – (9) it follows 4 equations for the components \( \Phi_a \)

\[
\begin{align*}
( -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2) \Phi_0 &- \epsilon k \Phi_2 \\
+ i\epsilon ( \hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3 ) &= 0 , \\
( -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 ) \Phi_2 &+ \epsilon k \Phi_0 \\
- i\epsilon ( \hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3 ) &= 0 , \\
( -\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2 ) \Phi_1 &+ \hat{a}_m \hat{a}_{m+1} \Phi_3 \\
+ i\epsilon \hat{a}_m \Phi_0 + i k \hat{a}_m \Phi_2 &= 0 , \\
( -\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2 ) \Phi_3 &+ \hat{b}_m \hat{b}_{m-1} \Phi_1 + \\
+ i\epsilon \hat{b}_m \Phi_0 + i k \hat{b}_m \Phi_2 &= 0 ; \\
\end{align*}
\]

(7)

3 General analysis of the radial equations

Eqs. (7) can be transformed to the form

\[
\begin{align*}
[ -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2 ] (k \Phi_0 + \epsilon \Phi_2) &= 0 , \\
[ -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2 - M^2 ] (\epsilon \Phi_0 + k \Phi_2) \\
&= (\epsilon^2 - k^2)( \epsilon \Phi_0 + k \Phi_2) - (i\hat{b}_{m-1} \Phi_1 + i \hat{a}_{m+1} \Phi_3) ; \\
( -\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2 ) \Phi_1 &+ \hat{a}_m \hat{a}_{m+1} \Phi_3 + i\epsilon \hat{a}_m \Phi_0 + i k \hat{a}_m \Phi_2 = 0 , \\
( -\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2 ) \Phi_3 &+ \hat{b}_m \hat{b}_{m-1} \Phi_1 + i\epsilon \hat{b}_m \Phi_0 + i k \hat{b}_m \Phi_2 = 0 .
\end{align*}
\]

(8)

(9)

Let us introduce new variables

\[
F(r) = k \Phi_0(r) + \epsilon \Phi_2(r) , \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r) ,
\]

then Eqs. (8) – (9) read

\[
\begin{align*}
[ -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2 ] F &= 0 , \\
[ -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - M^2 ] G \\
&= -(\epsilon^2 - k^2) (i \hat{b}_{m-1} \Phi_1 + i \hat{a}_{m+1} \Phi_3) , \\
( -\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2 ) \Phi_1 &+ \hat{a}_m \hat{a}_{m+1} \Phi_3 + i\epsilon \hat{a}_m \Phi_0 + i k \hat{a}_m \Phi_2 = 0 , \\
( -\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2 ) \Phi_3 &+ \hat{b}_m \hat{b}_{m-1} \Phi_1 + i \hat{b}_m \Phi_0 + i k \hat{b}_m \Phi_2 = 0 .
\end{align*}
\]

(10)

(11)

(12)

For equations (12), let us multiply the first one (from the left) by \( \hat{b}_{m-1} \) and the second one by the \( \hat{a}_{m+1} \), that results in

\[
\begin{align*}
-\hat{b}_{m-1} \hat{a}_m (\hat{b}_{m-1} \Phi_1) &+ (\epsilon^2 - k^2 - M^2)(\hat{b}_{m-1} \Phi_1) \\
+ \bar{b}_{m-1} \hat{a}_m (\hat{a}_{m+1} \Phi_3) &+ \bar{b}_{m-1} \hat{a}_m G = 0 , \\
-\hat{a}_{m+1} \hat{b}_m (\hat{a}_{m+1} \Phi_3) &+ (\epsilon^2 - M^2 - k^2)(\hat{a}_{m+1} \Phi_3) \\
+ \bar{a}_{m+1} \bar{b}_m (\hat{b}_{m-1} \Phi_1) &+ i \hat{a}_{m+1} \hat{b}_m G = 0 .
\end{align*}
\]

(13)
Again, let us introduce two new field variables

$$\hat{b}_{m-1} \Phi_1 = Z_1, \quad \hat{a}_{m+1} \Phi_3 = Z_3; \quad (14)$$

Eqs. (13) read as follows

$$- \hat{b}_{m-1} \hat{a}_m Z_1 + (\epsilon^2 - k^2 - M^2) Z_1 + \hat{b}_{m-1} \hat{a}_m Z_3 + i \hat{b}_{m-1} \hat{a}_m G = 0, \quad (15)$$

With the help of new functions $f(r), g(r)$

$$Z_1 = \frac{f + g}{2}, \quad Z_3 = \frac{f - g}{2}, \quad Z_1 + Z_3 = f, \quad Z_1 - Z_3 = g; \quad (16)$$

the system (15) is transformed to the following form

$$- \hat{b}_{m-1} \hat{a}_m g + (\epsilon^2 - k^2 - M^2) \frac{f + g}{2} + i \hat{b}_{m-1} \hat{a}_m G = 0, \quad (17)$$

Combining these equations we get

$$[- \hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2 - M^2] g + i (\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m) G = 0, \quad (18)$$

In turn, Eqs. (11) can be presented as

$$(- \hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2) F = 0, \quad (19)$$

Further, with the use of identities

$$- \hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m = \Delta, \quad - \hat{b}_{m-1} \hat{a}_m + \hat{a}_{m+1} \hat{b}_m = 2B \quad (20)$$

Eqs. (19) and (18) can be written as follows

$$\Delta F + (\Delta + \epsilon^2 - M^2 - k^2) F = 0, \quad (21)$$

With the help of the second equation, from the forth one it follows

$$f = - i G + \frac{2B}{M^2} g. \quad (22)$$
Now, one excludes the function $f$ in the second equation in (21) and gets

$$ (\Delta + \epsilon^2 - k^2 - M^2) G = -i(\epsilon^2 - k^2) \frac{2B}{M^2} g. \quad (23) $$

Thus, the general problem is reduced to the system of four equations

$$(\Delta + \epsilon^2 - M^2 - k^2) F = 0,$$

$$f = -i G + \frac{2B}{M^2} g,$$

$$(\Delta + \epsilon^2 - k^2 - M^2) g = 2iB G,$$

$$(\Delta + \epsilon^2 - k^2 - M^2) G = -2iB \frac{\epsilon^2 - k^2}{M^2} g. \quad (24)$$

The structure of this system allows to separate an evident linearly independent solution as follows

$$f(r) = 0, \quad g(r) = 0, \quad H(r) = 0,$$

$$F(r) \neq 0, \quad (\Delta - k^2 - M^2 + \epsilon^2) F = 0.$$  \quad (25)

Corresponding functions and energy spectrum are known. We are to solve the system of two last equations in (24); in matrix form it reads (let $\gamma = (\epsilon^2 - k^2)/M^2$)

$$\begin{pmatrix}
\Delta + \epsilon^2 - M^2 - k^2
\end{pmatrix}
\begin{pmatrix}
g \\
G
\end{pmatrix}
= \begin{pmatrix}
0 & 2iB \\
-2iB\gamma & 0
\end{pmatrix}
\begin{pmatrix}
g \\
G
\end{pmatrix}. \quad (26)$$

Let us construct the transformation changing the matrix on the right to a diagonal form

$$\begin{pmatrix}
\Delta + \epsilon^2 - M^2 - k^2
\end{pmatrix}
\begin{pmatrix}
g' \\
G'
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
g' \\
G'
\end{pmatrix},$$

$$\begin{pmatrix}
g' \\
G'
\end{pmatrix}
= S
\begin{pmatrix}
g \\
G
\end{pmatrix}, \quad S = \begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{pmatrix}. \quad (27)$$

The problem reduces to linear systems

$$\begin{cases}
-\lambda_1 \ s_{11} - 2iB\gamma \ s_{12} = 0, \\
2iB \ s_{11} - \lambda_2 \ s_{12} = 0,
\end{cases} \quad \begin{cases}
-\lambda_2 \ s_{21} - 2iB\gamma \ s_{22} = 0, \\
2iB \ s_{21} - \lambda_2 \ s_{22} = 0.
\end{cases}$$

The values of $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1 = +2B\sqrt{\gamma}, \quad \lambda_2 = -2B\sqrt{\gamma},$$

$$i \ s_{11} - \sqrt{\gamma} \ s_{12} = 0, \quad i \ s_{21} + \sqrt{\gamma} \ s_{22} = 0,$$

$$s_{12} = 1, \ s_{22} = 1, \quad S = \begin{pmatrix}
-i \sqrt{\gamma} & 1 \\
+i \sqrt{\gamma} & 1
\end{pmatrix}. \quad (28)$$

In the new (primed) basis, Eqs. (26) take the form of two separated differential equations

$$\left( \Delta + \epsilon^2 - k^2 - M^2 - 2B \sqrt{\gamma} \right) g' = 0,$$

$$\left( \Delta + \epsilon^2 - k^2 - M^2 + 2B \sqrt{\gamma} \right) G' = 0; \quad (29)$$
Recalling the meaning of $\Delta$, let us specify the second order differential equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{(m + Br)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 ,$$

$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma} , \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} . \quad (30)$$

It is convenient to introduce a new variable $x = Br^2$, then Eq. (30) reads

$$x \frac{d^2 \varphi}{dx^2} + \frac{d \varphi}{dx} - \left( \frac{m^2}{4x} + \frac{x}{4} + \frac{m^2 - \lambda^2}{4B} \right) \varphi = 0 . \quad (31)$$

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[ \frac{A^2 - m^2/4}{x} + (C^2 - \frac{1}{4})x - 2AC - C - \frac{m^2 - \lambda^2}{4B} \right] f = 0 .$$

When $A, C$ are taken as $A = + | m | / 2 , \quad C = +1/2$ the previous equation becomes simpler

$$x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left( A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) R = 0 ,$$

which is of confluent hypergeometric type

$$x Y'' + (\gamma - x)Y' - \alpha Y = 0 ,$$

$$\alpha = \frac{| m |}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} , \quad \gamma = | m | + 1 .$$

To obtain polynomials we must impose an additional condition $\alpha = -n$; which provides us with the following quantization rule for $\lambda^2$

$$\lambda^2 = 4B \left( n + \frac{1}{2} + \frac{| m | + m}{2} \right) . \quad (32)$$

Thus, we have arrived at two formulas for the energy

$$\sqrt{\epsilon^2 - k^2} = +B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} = \frac{M}{M} ,$$

$$\sqrt{\epsilon^2 - k^2} = -B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} = \frac{M}{M} . \quad (33)$$

In turn, the energy spectrum for the case $\lambda^2$ is given by

$$\epsilon^2 = M^2 + k^2 + \lambda^2$$

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and the corresponding energy spectra are found.

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3For definiteness let us consider $B$ to be positive, which does not affect the generality of the analysis. So, to infinite values of $r$ correspond infinite and positive values of $x$. 
4 On a spin 1 particle with intrinsic structure – polarizability

In [7–13], it was described a generalized equation for spin 1 particle possessing in addition to electric charge the special electromagnetic characteristics named polarizability. In the frame of the first order relativistic wave equations, such a particle requires a 15-dimensional wave function, consisting of a 4-vector \( \Phi_a(x) \), 4-tensor \( \Phi_{ab}(x) \), and subsidiary scalar and 4-vector fields, \( C(x) \) and \( C_a(x) \).

To treat the problem we take the matrix approach in the theory of the generalized \( S = 1 \) particle extended to a general covariant form on the base of tetrad formalism (recent consideration, notation and list of references see in [15, 16]). The use of cylindric tetrad permits to take account of the cylindric symmetry of the problem. The main equation in tetrad form is [12]

\[
\left[ \Gamma^0 \partial_0 + \Gamma^1 \partial_r + \frac{1}{r} \Gamma^2 (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + \Gamma^3 \partial_2 - M \right] \Psi = 0 .
\]

(35)

It is better to choose the matrices \( \beta^a \) in the so-called cyclic form, where the generator \( J^{12} \) has a diagonal structure. These matrices \( \Gamma^a \) are given in [6].

5 Separation of variables

With the use of special substitution

\[
\Psi = \{ C, C_0, \tilde{C}, \Phi_0, \bar{\Phi}, \tilde{E}, \bar{H} \} , \quad C(x) = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} C(r) ,
\]

\[
C_0 = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} C_0(r) , \quad \tilde{C} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{pmatrix} C_1(r) \\ C_2(r) \\ C_3(r) \end{pmatrix} ,
\]

\[
\Phi_0 = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \Phi_0(r) , \quad \bar{\Phi} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{pmatrix} \Phi_1(r) \\ \Phi_2(r) \\ \Phi_3(r) \end{pmatrix} ,
\]

\[
\tilde{E} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{pmatrix} E_1(r) \\ E_2(r) \\ E_3(r) \end{pmatrix} , \quad \bar{H} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{pmatrix} H_1(r) \\ H_2(r) \\ H_3(r) \end{pmatrix} ,
\]

(36)

after calculations we arrive at the radial system of 15 equations

\[
-i\epsilon C_0 - \hat{b}_{m-1} C_1 - \hat{a}_{m+1} C_3 - ik C_2 = MC , \quad (37)
\]

\[
-i\epsilon E_1 - \hat{a}_{m+1} E_3 - ik E_2 = MC_0 ,
\]

\[
i\epsilon E_1 + i\hat{a}_m H_2 - ik H_1 = MC_1 ,
\]

\[
i\epsilon E_2 - i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 = MC_2 ,
\]

\[
i\epsilon E_3 - i\hat{b}_m H_2 + k H_3 = MC_3 ,
\]

(38)

\[
-i\epsilon \sigma C - \hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 = MC_0 ,
\]

\[
i\epsilon E_1 - \sigma \hat{a}_m C + i\hat{a}_m H_2 - k H_1 = MC_1 ,
\]

\[
i\epsilon E_2 - i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 + i k \sigma C = MC_2 ,
\]

\[
i\epsilon E_3 - \sigma \hat{b}_m C - i\hat{b}_m H_2 + k H_3 = MC_3 ,
\]

(39)
\[-i \epsilon \Phi_1 + \hat{a}_m \Phi_0 = M E_1, \quad -i \epsilon \Phi_2 - i k \Phi_0 = M E_2, \]
\[-i \epsilon \Phi_3 + \hat{b}_m \Phi_0 = M E_3, \quad -i \hat{a}_m \Phi_2 + k \Phi_1 = M H_1, \]
\[\hat{b}_{m-1} \Phi_1 - i \hat{a}_{m+1} \Phi_3 = M H_2, \quad i \hat{b}_m \Phi_2 - k \Phi_3 = M H_3. \quad (40)\]

6 Solution of the radial system

With the use of (38), Eqs. (39) give
\[
C_0 = \Phi_0 + i \frac{\epsilon \sigma}{M} C, \quad C_1 = \Phi_1 + \frac{\sigma}{M} \hat{a}_m C, \\
C_2 = \Phi_2 - i \frac{k \sigma}{M} C, \quad C_3 = \Phi_3 + \frac{\sigma}{M} \hat{b}_m C. \quad (41)
\]

Substituting these formulas for \(C_a\) into (37)
\[
-i \epsilon (\Phi_0 + i \frac{\epsilon \sigma}{M} C) - \hat{b}_{m-1} (\Phi_1 + \frac{\sigma}{M} \hat{a}_m C) - i \hat{a}_m (\Phi_3 + \frac{\sigma}{M} \hat{b}_m C) - i k (\Phi_2 - i \frac{k \sigma}{M} C) = M C,
\]
we further get
\[
M(\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) = -i M (\epsilon \Phi_0 + k \Phi_2) + \sigma(-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2)C - M^2 C. \quad (42)
\]

This equation will be required below.

Note that Eqs. (39) and (40) include the main field variables, 4-vector and 4-tensor, and also the scalar \(C\) obeying Eq. (42)
\[
-i \epsilon \sigma C - \hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - i k E_2 = M \Phi_0, \\
i \epsilon E_1 - \sigma \hat{a}_m C + i \hat{a}_m H_2 - k H_1 = M \Phi_1, \\
i \epsilon E_2 - i \hat{b}_{m-1} H_1 + \hat{a}_{m+1} H_3 + i k \sigma C = M \Phi_2, \\
i \epsilon E_3 - \sigma \hat{b}_m C - i \hat{b}_m H_2 + k H_3 = M \Phi_3, \quad (43)
\]
\[-i \epsilon \Phi_1 + \hat{a}_m \Phi_0 = M E_1, \quad -i \epsilon \Phi_2 - i k \Phi_0 = M E_2, \\
-i \epsilon \Phi_3 + \hat{b}_m \Phi_0 = M E_3, \quad -i \hat{a}_m \Phi_2 + k \Phi_1 = M H_1, \\
i \hat{b}_{m-1} \Phi_1 - i \hat{a}_{m+1} \Phi_3 = M H_2, \quad i \hat{b}_m \Phi_2 - k \Phi_3 = M H_3. \quad (44)\]

By means of (44), we are to eliminate tensor components in (43). Then we obtain two equations
\[
-i \epsilon \sigma M C + (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - k^2 - M^2) \Phi_0 + i \epsilon (\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3 + i k \Phi_2) = 0, \quad (45)
\]
\[
\epsilon k M C + (\epsilon^2 - \hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - M^2) \Phi_2 - i k (i \epsilon \Phi_0 + \hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) = 0. \quad (46)
\]
Multiplying the first one (45) by +ik, and the second one (46) by \(i\epsilon\), and summing the results, we get

\[
(-\dot{b}_{m-1} \dot{a}_m - \dot{a}_{m+1} \dot{b}_m - k^2 - M^2 + \epsilon^2) (k \Phi_0 + \epsilon \Phi_2) = 0 .
\] (47)

In the same manner, combining Eqs. (45)–(46) with other coefficients, we arrive at

\[
(-\dot{b}_{m-1} \dot{a}_m - \dot{a}_{m+1} \dot{b}_m - M^2) (\epsilon \Phi_0 + k\Phi_2)
\]

\[
= -i(\epsilon^2 - k^2) \left( \dot{b}_{m-1} \Phi_1 + \dot{a}_{m+1} \Phi_3 \right) + i\sigma M (\epsilon^2 - k^2) C .
\] (48)

Thus, two second order equations have been found

\[
(-\dot{b}_{m-1} \dot{a}_m - \dot{a}_{m+1} \dot{b}_m - k^2 - M^2 + \epsilon^2) (k \Phi_0 + \epsilon \Phi_2) = 0 ,
\] (49)

\[
(-\dot{b}_{m-1} \dot{a}_m - \dot{a}_{m+1} \dot{b}_m - M^2) (\epsilon \Phi_0 + k\Phi_2)
\]

\[
= -i(\epsilon^2 - k^2) \left( \dot{b}_{m-1} \Phi_1 + \dot{a}_{m+1} \Phi_3 \right) + i\sigma M (\epsilon^2 - k^2) C .
\] (50)

Now, let us turn to equations in (43), containing functions \(m\Phi_1\) and \(m\Phi_3\):

\[
(-\dot{a}_m \dot{b}_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1
\]

\[
+ \dot{a}_m \dot{a}_{m+1} \Phi_3 + i\dot{a}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \dot{a}_m C = 0 ;
\] (51)

and

\[
(-\dot{b}_m \dot{a}_{m+1} + \epsilon^2 - k^2 - M^2) \Phi_3
\]

\[
+ \dot{b}_m \dot{b}_{m-1} \Phi_1 + i\dot{b}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \dot{b}_m C = 0 ;
\] (52)

In two last equations, (51) and (52), multiplying the the first one by \(\dot{b}_{m-1}\) (from the left) and the second one by \(\dot{a}_{m+1}\) (from the left), we produce

\[
(-\dot{b}_{m-1} \dot{a}_m + \epsilon^2 - k^2 - M^2) \dot{b}_{m-1} \Phi_1
\]

\[
+ \dot{b}_{m-1} \dot{a}_m \dot{a}_{m+1} \Phi_3 + i\dot{b}_{m-1} \dot{a}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \dot{b}_{m-1} \dot{a}_m C = 0 ,
\] (53)

\[
(-\dot{a}_{m+1} \dot{b}_m + \epsilon^2 - k^2 - M^2) \dot{a}_{m+1} \Phi_3
\]

\[
+ \dot{a}_{m+1} \dot{b}_m \dot{b}_{m-1} \Phi_1 + i\dot{a}_{m+1} \dot{b}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \dot{a}_{m+1} \dot{b}_m C = 0 .
\] (54)

It is better to introduce new field variables

\[
F(r) = k \Phi_0 + \epsilon \Phi_2 , \quad G(r) = \epsilon \Phi_0 + k \Phi_2 ,
\]

\[
\dot{b}_{m-1} \Phi_1 = Z_1 , \quad \dot{a}_{m+1} \Phi_3 = Z_3 ;
\] (55)

then the system (53)–(54) reads

\[
(-\dot{b}_{m-1} \dot{a}_m + \epsilon^2 - k^2 - M^2) Z_1
\]

\[
+ \dot{b}_{m-1} \dot{a}_m Z_3 + i\dot{b}_{m-1} \dot{a}_m G - M\sigma \dot{b}_{m-1} \dot{a}_m C = 0 ,
\] (56)

\[
(-\dot{a}_{m+1} \dot{b}_m + \epsilon^2 - k^2 - M^2) Z_3
\]

\[
+ \dot{a}_{m+1} \dot{b}_m Z_1 + i\dot{a}_{m+1} \dot{b}_m G - M\sigma \dot{a}_{m+1} \dot{b}_m C = 0 .
\] (57)
Again, it is convenient to define new variables \( f(r), \ g(r) \):

\[
Z_1 = \frac{f + g}{2}, \quad Z_3 = \frac{f - g}{2}, \quad Z_1 + Z_3 = f, \quad Z_1 - Z_3 = g;
\]

then Eqs. (56) – (57) give

\[
\begin{align*}
\left(-\hat{b}_{m-1} \hat{a}_m + \epsilon^2 - k^2 - M^2\right) \frac{f + g}{2} \\
+ \hat{b}_{m-1} \hat{a}_m \frac{f - g}{2} + i \hat{b}_{m-1} \hat{a}_m G - M\sigma \hat{b}_{m-1} \hat{a}_m C = 0,
\end{align*}
\]

\[
\begin{align*}
\left(-\hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2 - M^2\right) \frac{f - g}{2} \\
+ \hat{a}_{m+1} \hat{b}_m \frac{f + g}{2} + i \hat{a}_{m+1} \hat{b}_m G - M\sigma \hat{a}_{m+1} \hat{b}_m C = 0,
\end{align*}
\]

After simple manipulation, from two last equations it follows that

\[
\begin{align*}
\left[ -\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2 - M^2 \right] g \\
+ (-\hat{b}_{m-1} \hat{a}_m + \hat{a}_{m+1} \hat{b}_m) (-iG + M\sigma C) = 0, \\
(\hat{b}_{m-1} \hat{a}_m + \hat{a}_{m+1} \hat{b}_m) g + (\epsilon^2 - k^2 - M^2)f \\
+ (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m) (-iG + M\sigma C) = 0.
\end{align*}
\]

With the use of identities

\[
-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m = \Delta, \quad -\hat{b}_{m-1} \hat{a}_m + \hat{a}_{m+1} \hat{b}_m = 2B
\]

Eqs. (61) can be written as

\[
\begin{align*}
\left[ \Delta + \epsilon^2 - k^2 - M^2 \right] g + 2B (-iG + M\sigma C) = 0, \\
2B g + (\epsilon^2 - k^2 - M^2)f + \Delta (-iG + M\sigma C) = 0.
\end{align*}
\]

In turn, Eqs. (47), (48) will read (in the new variables)

\[
\begin{align*}
(\Delta - k^2 - M^2 + \epsilon^2) F = 0, \\
(\Delta - M^2) G = -i(\epsilon^2 - k^2) f + i\sigma M (\epsilon^2 - k^2) C.
\end{align*}
\]

Let us collect results together

\[
\begin{align*}
\left(\Delta - k^2 - M^2 + \epsilon^2\right) F = 0, \\
(\Delta - M^2) G = -i(\epsilon^2 - k^2) f + i\sigma M (\epsilon^2 - k^2) C, \\
(\Delta + \epsilon^2 - k^2 - M^2) g + 2B (-iG + M\sigma C) = 0, \\
2B g + (\epsilon^2 - k^2 - M^2)f + \Delta (-iG + M\sigma C) = 0.
\end{align*}
\]

It is possible to eliminate the function \( C(r) \) in the above equation. To show how it can be done, let us turn to a couple of equations in (39), containing the terms \( M \Phi_1, M \Phi_3 \), and find
the combination

\[ \hat{b}_{m-1} M \Phi_1 + \hat{a}_{m+1} \Phi_3 = i \epsilon \hat{b}_{m-1} E_1 - \sigma \hat{b}_{m-1} \hat{a}_m C + i \hat{a}_{m+1} \hat{a}_m \hat{H}_2 - k \hat{b}_{m-1} \hat{H}_1 + i \epsilon \hat{a}_{m+1} E_3 - \sigma \hat{a}_{m+1} b_m C - i \hat{a}_{m+1} \hat{b}_m \hat{H}_2 + k \hat{a}_{m+1} \hat{H}_3 \]

from whence, with the help of the first and third equations in (39) in the form

\[ (\hat{b}_{m-1} E_1 + \hat{a}_{m+1} E_3) = -i \epsilon \sigma C - i k E_2 - M \Phi_0 , \]
\[ (\hat{b}_{m-1} \hat{H}_1 - \hat{a}_{m+1} \hat{H}_3) = \epsilon E_2 + k \sigma C + i M \Phi_2 , \]

we obtain

\[ \hat{b}_{m-1} M \Phi_1 + \hat{a}_{m+1} \Phi_3 = i \epsilon (-i \epsilon \sigma C - i k E_2 - M \Phi_0) - \sigma (\hat{b}_{m-1} \hat{a}_m + \hat{a}_{m+1} \hat{b}_m) C + i (\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m) \hat{H}_2 - k (\epsilon E_2 + k \sigma C + i M \Phi_2) . \]

From this, after evident calculation, we arrive at

\[ \hat{b}_{m-1} M \Phi_1 + \hat{a}_{m+1} \Phi_3 = -i M (\epsilon \Phi_0 + k \Phi_2) + \sigma (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2) C - 2iB \hat{H}_2 \]

Comparing (42) and (68), we conclude that there exists a linear relation

\[ 2iB \hat{H}_2(r) = M^2 C(r) . \]

Due to Eq. (40), it holds

\[ i \hat{b}_{m-1} \Phi_1 - i \hat{a}_{m+1} \Phi_3 = M \hat{H}_2 \quad \implies \quad ig = M \hat{H}_2 ; \]

therefore, the function \( C(r) \) is expressed through \( g(r) \):

\[ C(r) = -\frac{2B}{M^2} g(r) \]

The system (64) – (67), after excluding \( C(r) \), takes the form

\[ (\Delta - k^2 - M^2 + \epsilon^2) F = 0 , \]
\[ (\Delta - M^2) G = -i(\epsilon^2 - k^2) f - i \sigma (\epsilon^2 - k^2) \frac{2B}{M^2} g , \]
\[ \left[ \Delta + \epsilon^2 - k^2 - M^2 - \sigma \left( \frac{2B}{M} \right)^2 \right] g - 2iB G = 0 , \]
\[ 2B g + (\epsilon^2 - k^2 - M^2) f + \Delta (-iG - \sigma \left( \frac{2B}{M} \right) g) = 0 . \]
The eigenvalues \( \lambda \) the problem is reduced to a couple of linear systems corresponding functions and the energy spectrum are known (also see below). We are to solve the system of three last equations in (73) – (75). With the help of (74) Eq. (75) takes the form of the linear relation

\[ \Delta G = M^2 G - i(\epsilon^2 - k^2) f - i\sigma (\epsilon^2 - k^2) \frac{2B}{M^2} g, \]
\[ \Delta g = -[\epsilon^2 - k^2 - M^2 - \sigma(\frac{2B}{M})^2] g - 2iBG, \]

Eq. (75) takes the form of the linear relation

\[ M^2 f = i \left( -M^2 + \frac{4B^2}{M^2} \right) G + 2B \left( 1 - \sigma - \frac{\sigma^2 4B^2}{M^2} \right) g. \]  

Now, returning to Eqs. (73)–(74), after excluding the function \( f \) and using the notation

\[ \gamma = \frac{\epsilon^2 - k^2}{M^2}, \quad \beta = \frac{4B^2}{M^2}, \quad \alpha = \gamma \rho, \quad \rho = 1 - \frac{4B^2\sigma^2}{M^4}; \]

we arrive at two equations

\[ (\Delta + \epsilon^2 - k^2 - M^2) g = \beta g(r) + 2iBG(r), \]
\[ (\Delta + \epsilon^2 - k^2 - M^2) G = -2iB\alpha g(r) + \beta \gamma G(r). \]  

In matrix form they read

\[ (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g(r) \\ G(r) \end{vmatrix} = \begin{vmatrix} \beta & 2iB \\ -2iB\alpha & \beta \gamma \end{vmatrix} \begin{vmatrix} g(r) \\ G(r) \end{vmatrix}. \]  

Let us construct the transformation changing the matrix on the right to a diagonal form

\[ (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix}, \]

\[ \begin{vmatrix} g' \\ G' \end{vmatrix} = S \begin{vmatrix} g \\ G \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}. \]  

the problem is reduced to a couple of linear systems

\[ \begin{cases} (\beta - \lambda_1) s_{11} - 2iB\alpha s_{12} = 0, \\ 2iB s_{11} + (\beta \gamma - \lambda_1) s_{12} = 0, \end{cases} \]
\[ \begin{cases} (\beta - \lambda_2) s_{21} - 2iB\alpha s_{22} = 0, \\ 2iB s_{21} + (\beta \gamma - \lambda_2) s_{22} = 0. \end{cases} \]  

The eigenvalues \( \lambda_1, \lambda_2 \) are

\[ \lambda_1 = \frac{\beta (1 + \gamma) + \sqrt{\beta^2(1 - \gamma)^2 + 16B^2\rho\gamma}}{2}, \]
\[ \lambda_2 = \frac{\beta (1 + \gamma) - \sqrt{\beta^2(1 - \gamma)^2 + 16B^2\rho\gamma}}{2}. \]  

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let it be
\[ s_{12} = 1, \quad s_{22} = 1, \quad s_{11} = \frac{\lambda_1 - \beta \gamma}{2iB}, \quad s_{21} = \frac{\lambda_2 - \beta \gamma}{2iB}, \]
\[ g' = \frac{\lambda_1 - \beta \gamma}{2iB} g + G, \quad G' = \frac{\lambda_2 - \beta \gamma}{2iB} g + G; \quad (82) \]

In the new (primed) basis, Eqs. (80) take the form of two separated differential equations
\[ \left( \Delta + \epsilon^2 - k^2 - M^2 - \lambda_1 \right) g' = 0, \]
\[ \left( \Delta + \epsilon^2 - k^2 - M^2 - \lambda'_2 \right) G' = 0; \quad (83) \]

Recalling the meaning of \( \Delta \), let us specify the second order equation
\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0. \quad (84) \]

This equation was examined above. We obtain two possibilities for the energy spectrum:
\[ g' \neq 0, \quad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda_1, \]
\[ G' \neq 0, \quad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda'_2. \quad (85) \]

Both Eqs. (85) can be written as
\[ M^2 \gamma = \lambda^2 + M^2 + \frac{\beta(1 + \gamma) \pm \sqrt{\beta^2(1 - \gamma)^2 + 16B^2 \rho \gamma}}{2}. \quad (86) \]

It is convenient to introduce new variable \( x = \gamma - 1 \), and also with the help of
\[ \beta = \frac{4B^2}{M^2}, \quad \rho = 1 - \frac{4B^2 \sigma^2}{M^4} = 1 - \frac{\beta^2}{4B^2}, \quad 16\rho B^2 = 16B^2 - 4\beta^2 \]
to eliminate the parameter \( \rho \):
\[ (2M^2 - \beta) x - 2(\lambda^2 + \beta) = \pm \sqrt{\beta^2 x^2 + (16B^2 - 4\beta^2)(x + 1)}. \quad (87) \]

Thus, we get the second order equation
\[ M^2(M^2 - \beta)x^2 - \left[ (\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right] x \]
\[ + (\lambda^2 + \beta)^2 - (4B^2 - \beta^2) = 0; \quad (88) \]
its solutions read
\[ \epsilon^2 - M^2 - k^2 = \frac{1}{2(M^2 - \beta)} \left\{ \left[ (\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right] \right. \]
\[ \pm \left[ (\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right]^2 \]
\[ - 4M^2(M^2 - \beta)(\lambda^2 + \beta)^2 - (4B^2 - \beta^2) \right\}. \quad (89) \]
Note, that the case (76) gives the following spectrum $\epsilon^2 = M^2 + k^2 + \lambda^2$, so for these solutions the polarizability does not manifest itself in a magnetic field.

Thus, on the base of general covariant formalism in the vector particle with polarizability, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and corresponding energy spectra are found.

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References

[1] I.I. Rabi, Das freie Electron in Homogenen Magnetfeld nach der Diraschen Theorie, Ztshr. Phys. 49, 507–511 (1928).

[2] L. Landau, Diamagnetismus der Metalle, Ztshr. Phys. 64, 629–637 (1930).

[3] M.S. Plesset, Relativistic wave mechanics of the electron deflected by magnetic field, Phys. Rev. 12, 1728–1731 (1931).

[4] I.E. Tamm, Motion of a meson in electromagnetic fields. Collection of papers. Vol. 2. Moskow, Nauka, 1975, 95–99 (in Russian).

[5] A.A. Bogush, V.V. Kisel, N.G. Tokarevskaya, V.M. Red’kov, Duffin–Kemmer–Petiau formalism reexamined: non-relativistic approximation for spin 0 and spin 1 particles in a Riemannian space-time, Annales de la Fondation Louis de Broglie, 32, 355–381 (2007).

[6] V.M. Red’kov, Fields in Riemannian space and the Lorentz group, Publishing House “Belarusian Science”, Minsk (2009).

[7] F.I. Fedorov, V.A. Pletyukhov, Wave equations with multiple representations of the Lorentz group. Intejer spin, Vesti AN BSSR. ser. fiz.-mat. 6, 81–88 (1969).

[8] V.A. Pletyukhov, F.I. Fedorov, Wave equations with multiple representations for a particle with spin 0, Vesti AN BSSR. ser. fiz.-mat. 2, 79–85 (1970).

[9] V.A. Pletyukhov, F.I. Fedorov, Wave equations with multiple representations for a particle with spin 1, Vesti AN BSSR. ser. fiz.-mat. 3, 84–92 (1970).

[10] F.I. Fedorov, V.A. Pletyukhovm Wave equations with multiple representations of the Lorentz group. Half-intejer spin, Vesti AN BSSR. ser. fiz.-mat. 3, 78–83 (1970).

[11] V.V. Kisel, Electrical polarizanility of a particle with spin 1 in the theor of relativistic wave equations, Vesti AN BSSR. ser. fiz.-mat. 3, 73–78 (1982).

[12] A.A. Bogush, V.V. Kisel, Description of a free particle by means of deferent wave equations. Doklady AN BSSR. 28, 702–705 (1984).
[13] A.A. Bogush, V.V. Kisel, F.I. Fedorov, *On interpretation of additional components of the wave functions in presence of electromagnetic interaction*. Doklady AN SSSR. **277**, 343–346 (1984).

[14] V.V. Kisel, *On a scalar particle with polarizability in external magnetic field*. Vesti BDPU **2**, 166–170 (2001).

[15] V.V. Kisel, N.G. Tokarevskaya, V.M. Red’kov, *On the theory of vector particles with extended set of representations of the Lorentz group in flat and Riemannian spaces*, Preprint 730, 25 pages, Institute of Physics NAS of Belarus, Minsk (2001).

[16] V.V. Kisel, N.G. Tokarevskaya, A.A. Bogush, V.M. Red’kov, *Spin 1 Particle in a 15-Component Formalism, Interaction with Electromagnetic and Gravitational Fields*, 36 pages. [hep-th/0309132](http://arxiv.org/abs/hep-th/0309132)