POSSIBLY EVERY REAL FUNCTION IS CONTINUOUS ON A NON-MEAGRE SET

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§0 Introduction

By Abraham, Rubin and Shelah [ARSh 153] it is consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some uncountable set (and more ...). We may consider strengthening this statement, by demanding the subset on which the function is continuous to be large in a stronger sense. In [Fe94, Problem AR(b)], David Fremlin asked exactly this, namely, is it consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some set which is non-meagre (i.e. not countable union of nowhere dense sets). We answer it (positively) here. We use $\omega^2$ for reals, and for $B \subseteq \omega^2$ we say $f : B \rightarrow \omega^2$ is continuous if $f(\eta_0) = \eta_1, n < \omega$ implies that for some $m$ we have

$$\forall \eta \mid m \subset \eta \in B \Rightarrow \eta_1 \mid n \subset f(\eta),$$

where for sequences $\nu, \eta, \nu \subset \eta$ means “$\eta$ (properly) extends $\nu$”.

The non-meagre sets here are of cardinality $\aleph_1$ and we may wonder whether we can get them of higher cardinality. From the point of view of those who asked the original question, maybe this does not add much (but I think it does make for much more interesting partition relations). Another interesting generalization, asked by Heinrich von Weizsäcker (see [Fe94, Problem AR(a)]), is:

$$(*)$$ is it consistent to have that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some non-null set?

We may ask about 2-place functions; Sierpinski colouring implies we cannot ask for one colour, but we may want the consistency of:

$$(**)$$ for every 2-place function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ there is a non-meagre set (non-null) $A \subseteq \mathbb{R}$ and 2-place continuous functions $f_0, f_1 : A \times A \rightarrow \mathbb{R}$ such that

$$\forall x, y \in A (f(x, y) = f_0(x, y) \lor f(x, y) = f_1(x, y))$$

(similarly for larger $n$ or all $n$ simultaneously). It is not clear to us how much this interests non-logicians, but to us it seems to be the right question and we shall deal with $$(**)$$ and the cardinality in Rabus Shelah [RaSh:585].

We can also generalize the proof replacing $\aleph_0$ by $\mu = \mu^{<\mu}$ (as done here). On consistent partition relations see [Sh 276], [Sh 288], [Sh 546] and [RaSh:585]. Another related theorem, proved in [Sh 481], is the consistency of

$$(***)$$ if $B_1$ is a $2^{\aleph_0}$-c.c. Boolean algebra and $B_2$ is a c.c.c. Boolean algebra, then $B_1 \ast B_2$ is a $2^{\aleph_0}$-c.c. Boolean algebra.

Originally the proof goes through “meagre preserving” and iterations as in section 2 of [Sh 276], but the proof was cumbersome, had flaws and proper version was not manufactured. Here we present a simpler though a “degenerated” variant (i.e. we use only the forcing we have to and so we get $2^{\aleph_0} = \aleph_2$ rather than $2^{\aleph_0} = \aleph_3$).

We thank Andrzej Roslanowski and Zoran Spasojević for much help in proof-reading.
§1 Continuity on a Non-Meagre Set

1.1 Definition. 1) Cohen$\mu(\alpha) = \{ f : f$ a partial function from $\alpha$ to $\{0, 1\}$ of cardinality $< \mu\}$ ordered by inclusion.
2) $\text{Cohen}_\mu = \text{Cohen}_\mu(\mu)$.
3) A $\mu$-Cohen forcing $P$ means $\text{Cohen}_\mu(\alpha)$ for some ordinal $\alpha$ (or at least the set $\{p \in P : P \upharpoonright \{q : q \geq p\}$ is equivalent to some $\text{Cohen}_\mu(\alpha)\}$ is a dense subset of $P$).

1.2 Definition. A set $A \subseteq \mu^2$ is $\mu$-meagre if it is the union of $\leq \mu$ nowhere dense subsets of $\mu^2$.

1.3 Main Lemma. Let $\mu$ be a regular cardinal such that $\mu = \mu^{< \mu}$. For $\ell < 2$ and $\alpha < \mu^+$, let $Q_{\alpha, \ell}$ be $(\mu^>, \alpha)$ (i.e. Cohen$\mu$ forcing) and for $I \subseteq \mu^+ \times 2$ let $P_I$ be the product $\prod_{i \in I} Q_i$ with $< \mu$-support and $P = P_{\mu^+ \times 2}$.

Let $\eta_{\alpha, \ell}$ be the $Q_{\alpha, \ell}$-name of the generic function from $\mu$ to $2 = \{0, 1\}$. In $V^P = V[\langle \eta_{\alpha, \ell} : \alpha < \mu^+, \ell < 2 \rangle]$ let $R$ be a forcing notion defined by:

$$R = \left\{ (u_0, v, \bar{\nu}) : (a) \quad u_0 \text{ is a subset of } \mu^+ \text{ of cardinality } < \mu, \right.$$ 

(b) $v$ is a subset of $\mu^>2$ of cardinality $< \mu$

(c) if $\rho \in v$ then $\ell g(\rho)$ is a successor ordinal and $\rho(\ell g(\rho) - 1) = 1$,

(d) $\bar{\nu} = \langle \nu_\rho : \rho \in v \rangle$ with $\nu_\rho \in \mu^>2$,

(e) if $\rho_1 \triangleleft \rho_2$ are from $v$ then $\nu_{\rho_1} \triangleleft \nu_{\rho_2}$ (hence they are not equal),

(f) for every $\rho \in v$ there is $\alpha \in u_0$ such that $\rho \triangleleft \eta_{\alpha, 0}$,

(g) if $\alpha \in u_0, \rho \triangleleft \eta_{\alpha, 0}$ and $\rho \in v$ then $\nu_\rho \triangleleft \eta_{\alpha, 1}$.$\right\}$. 

The order is natural:

$(u_0, v, \bar{\nu}) \leq (u_0', v', \bar{\nu}')$ if and only if $u_0 \subseteq u_0'$, $v \subseteq v'$, and $\bar{\nu} = \bar{\nu}' \upharpoonright v$ (and both are in $R$).

Let $\mathcal{U}$ be an $R$-name such that

$$\| R \mathcal{U} \| = \bigcup \{ u_0 : \text{there are } v, \bar{\nu} \text{ such that } (u_0, v, \bar{\nu}) \in G_R \},$$

where $G_R$ is the canonical $R$-name for the generic filter on $R$.

Then:
(a) In $V^{P \ast R}$, the function $\eta_{\alpha,0} \rightarrow \eta_{\alpha,1}$ from $\{\eta_{\alpha,0} : \alpha \in \mathcal{U}\}$ to $\{\eta_{\alpha,1} : \alpha \in \mathcal{U}\}$ is continuous.
(b) In $V^{P \ast R}$, the set $\{\eta_{\alpha,0} : \alpha \in \mathcal{U}\}$ is a non-$\mu$-meagre subset of $\mu^+$, in fact everywhere non-meagre (i.e. its intersection with any non-empty open set is non-meagre).
(c) $P \ast R$ is a $\mu$-Cohen forcing, moreover $P_{\alpha \times 2} \vdash P$ (i.e. $P_{\alpha \times 2}$ is a complete suborder of $P$) and $P \ast R / P_{\alpha \times 2}$ is a $\mu$-Cohen forcing for every $\alpha < \mu^+$, $\alpha \geq \mu$.

1.3A Remark. We can get slightly more than continuity for $\mu > \aleph_0$ (a slight change of the forcing notion), namely, for some club $W \subseteq \mu$, for every $i \in W$ we have that $\eta_{\alpha,0} \downarrow i$ determines $\eta_{\alpha,1} \downarrow i$ (and so can get this in the theorem).

Proof. For $\alpha \leq \mu^+, \alpha \geq \mu$ let

$$A'_{\alpha} = \left\{(p, r) : (\alpha) \ p \in P_{\alpha \times 2} \text{ and } \begin{array}{l} p = \langle \eta^p_{\gamma, \ell} : (\gamma, \ell) \in u^p \times 2 \rangle \text{ where } \eta^p_{\gamma, \ell} \in \mu^+, u^p \in [\alpha]^{\mu^+}, \\
(\beta) \ r = (u_0, v, \tilde{v}) \text{ satisfies clauses } (a), (b), (c), (d), (e) \text{ of the definition of } R \text{ and } \\
(f)' \ \text{ for every } \rho \in v \text{ there is } \beta \in u^p \text{ such that } \\
\beta \in u_0 \text{ and } \rho \leq \eta^p_{\beta, 0}, \\
(g)' \ \text{ if } \beta \in u_0 \text{ and } \rho \in v \text{ and } \rho \leq \eta^p_{\beta, 0}, \\
\text{ then } \nu_\rho \leq \eta^p_{\beta, 1}, \\
(\gamma) \ u_0 \subseteq u^p \end{array} \right\}$$

and partially order $A'_{\alpha}$ in the natural way.

Also let

$$A_\alpha = \left\{(p, r) : p, r \text{ satisfy clauses } (\alpha), (\beta), (\gamma) \text{ above and } \begin{array}{l} (\delta) \ \text{ if } \beta \in u_0 \text{ and } \rho \in v \text{ then } -(\eta^p_{\beta, 0} \prec \rho), \\
(\varepsilon) \ \text{ for every } \beta \in u_0 \text{ there is } \gamma \in u_0 \cap \mu \text{ such that } \\
(\eta^p_{\beta, 0}, \eta^p_{\beta, 1}) = (\eta^p_{\gamma, 0}, \eta^p_{\gamma, 1}) \end{array} \right\}.$$

Note that in $A'_{\alpha}$ any increasing sequence of length $< \mu$ has a least upper bound, i.e. if a sequence $\langle q_i : i < \delta \rangle \subseteq A'_{\alpha}$ is increasing, $\delta < \mu$ then there is a condition $q \in A'_{\alpha}$ such that $q_i \leq q$ for all $i < \delta$ and if $\bigwedge_{i < \delta} q_i \leq q' \in A'_{\alpha}$ then $q \leq q'$.

Now:

(A) $A_\alpha$ is a dense subset of $A'_{\alpha}$. If $(p, r) \in A_\alpha$ and $\mu \leq \alpha \leq \mu^+$ then $(p, r) \in P \ast R$ (i.e. $p \vdash P \check{r} \in R^\alpha$).
Moreover: that: (note that clause (A)
and we shall find (p', r') ∈ A, such that  
( p', r') ≤ (p', r). Let β* := sup(wρ ∩ µ) + 1 
and ε = sup{ηg(ρ) + 1 : ρ ∈ ϵ} and let Ω be the sequence of zeros of length ε and 
let wρ = {γ, ℓ : γ < |wρ|} be an enumeration. 
Let wρ' = wρ ∪ {β* + ζ : ζ < |wρ|} and
\[ \eta^{p'}_{γ, ℓ} = \begin{cases} 
\eta^p_{γ, ℓ} \bar{O} \varepsilon & \text{if } γ \in w^p, ℓ = 0, \\
\eta^p_{γ, ℓ} \bar{O} & \text{if } γ \in w^p, ℓ = 1, \\
\eta^p_{γ, ℓ} & \text{if } γ = β* + ℓ, \gamma, ℓ = 0, \\
\eta^p_{γ, ℓ} & \text{if } γ = β* + ℓ, \gamma, ℓ = 1.
\end{cases} \]

Note: if ρ ∈ v then ρ ⊆ η\text{∗}Ω, ∴ ρ ⊆ η (because by clause (c) in the definition of R, 
the last element in ρ is one).]

(B) \[ A^+ \] is a dense subset of \[ P * R. \]

Why? Since \( P \) is \( µ \)-closed, for each \( (p, r) \in P * R \) we can find \( p_1 \in P \) and \( r \) such 
that \( p \leq p_1 \in P \) and \( p_1 ∥_P \) “r = r”. Next we choose \( p_2 \) such that \( p_1 ≤ p_2 \in P \) 
and clauses (A)–(E) in the definition of \( A^+ \) hold. So \( (p, r) ≤ (p_2, r) \) (in \( P * R \) and 
\( (p_2, r) \in A^+ \).

If \( µ ≤ α ≤ β ≤ µ^+ \) and \( (p, r) ∈ A_β \) then we let \( (p, r) \vert α = (p \vert α, r \vert α) \) where 
\( p \vert α = (η^p_{γ, ℓ} : (γ, ℓ) ∈ (w^p \cap α) \times 2) \) and \( r \vert α = (u^0_0 \cap α, v^r, \bar{v}^r) \). It is easy to check that:

(C) If \( µ ≤ α ≤ β ≤ µ^+ \) and \( (p, r) ∈ A_β \) then \( (p, r) \vert α = A_β \) and \( (p, r) \vert α = A_α \) 
(note that clause (f)’ of the definition of \( A_α \) follows from clause (ε) in the definition 
of \( A_α \)).

(D) If \( µ ≤ α ≤ β ≤ µ^+ \) and \( (p_1, r_1) ∈ A_β \) and \( (p_1, r_1) \vert α = (p_2, r_2) ∈ A_α \) then 
\( (p_1, r_1), (p_2, r_2) \) are compatible (in \( A_β \)); moreover, they have the natural 
least upper bound 
\[ \left( p_2 \cup p^\ast, (u^0_1 \cup u^0_2, v^{r_2}, \bar{v}^{r_2}) \right) \]
in \( A_β \), where \( p^\ast \in P_{β×2} \) is such that \( w^p = w^p_1 \setminus α \), and for \( γ ∈ w^p \)
\[ \eta^p_{γ, ℓ} = \begin{cases} 
\eta^p_{γ, ℓ} & \text{if } \text{either } γ ∉ u^0_1 \text{ or } ℓ = 0, \\
\eta^p_{γ, ℓ_1} \cup \{v^{p_2} : ρ ⊆ n^p_{γ, ℓ_0} \} & \text{if } γ ∈ u^0_1 \text{ and } ℓ = 1.
\end{cases} \]

Moreover:

\( (D) \) if \( (p_{1,i,1}, r_{1,i}) : i ≤ δ \) is an increasing continuous sequence of conditions 
in \( A^\prime_δ \) such that \( δ < µ \) and \( (p_{1,i,1}, r_{1,i}) ∈ A_δ \) for each non-limit \( i ≤ δ \) and 
\( (p_{1,i,1}, r_{1,i}) \vert α ≤ (p_2, r_2) \) for \( i < δ \) 
then \( (p_{1,δ,1}, r_{1,δ}) \) and \( (p_{2,1}, r_{2,1}) \) are compatible in \( A^\prime_δ \) (and \( (p_{1,δ,1}, r_{1,δ}) \vert α ≤ (p_2, r_2). \)

Hence 
\( (D) \) if \( µ ≤ α ≤ β ≤ µ^+ \) then \( A_α ⊆ A_β \), moreover \( A_α \natural A_β \) (by (D)) and the 
quotient \( A_β/A_α \) is \( µ \)-closed (i.e. increasing chains of length < \( µ \) have upper 
bound).
Remark. For better understanding note that if $Q_0$ is adding a $\mu$-Souslin tree, $Q_1$ is the forcing determined by this tree, then the composition $Q_0 \ast Q_1$ is not even $\mu$-closed. The point is that increasing sequences in $Q_0 \ast Q_1$ of length $< \mu$ have an upper bound, but not necessarily least upper bound.

Now (note the support)

(E) the sequence $(A_\alpha : \alpha \in [\mu, \mu^+])$ is increasing, and for limit $\alpha \in (\mu, \mu^+)$, $A_\alpha$ is the inverse limit of $(A_\beta : \beta < \alpha)$ if $\mathrm{cf}(\alpha) < \mu$, and direct limit of it if $\mathrm{cf}(\alpha) \geq \mu$. Hence $\mu \leq \alpha < \beta \leq \mu^+ \Rightarrow A_\beta/A_\alpha$ is a $\mu$-Cohen forcing (remember that atomless $\mu$-closed forcing notion of size $\mu$ is $\mu$-Cohen).

It is also clear that

(F) $P_{\alpha \times 2} \not\leq P_{\beta \times 2}$ if $\alpha < \beta \leq \mu^+$.

Lastly note that

(G) for $\alpha \in [\mu, \mu^+]$, $A_\alpha/P_{\alpha \times 2}$ is $\mu$-closed and hence it is $\mu$-Cohen.

From the desirable conclusions, we have gotten clause (c) (by combining clauses (E) and (G)).

(H) If $(p_1, r_1) \in A_{\mu^+}, \beta^* \in \mu^+ \setminus u^{p_1}$ and $\nu^* \in \nu^{>2}$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \beta^* \in u^{r_2} \quad \text{and} \quad \nu^* \# \eta_{\beta^*,0}^{p^2}.$$

[Why? Let $\beta^* \in \mu^+ \setminus u^{p_1}, \beta^{**} \in \mu \setminus (u^{p_1} \cup \{\beta^*\})$ and let]

$\nu^{\otimes} = \bigcup \{\nu^r_\rho : \rho \leq \nu^* \text{ and } \rho \in v^r_1\} \quad \text{and} \quad \gamma = \sup\{\ell g(\nu) + 1 : \nu \in v^r_1 \cup \{\nu^*\}\}.$

Now define $(p_2, r_2)$ as follows:

$$u^{p_2} = u^{p_1} \cup \{\beta^*, \beta^{**}\}$$

$$\eta_{\beta^*,0}^{p_2} = \begin{cases} 
\eta_{\beta^*,0}^{p_1} & \text{if } \beta \in u^{p_1}, \ell < 2, \\
\nu^{\otimes} & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\
\nu^* & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 1,
\end{cases}$$

and $r_2 = (u^{r_2}_0, v^{r_2}, \nu^{r_2})$ is defined by $u^{r_2}_0 = u^{r_1}_0 \cup \{\beta^*, \beta^{**}\}$, $v^{r_2} = v^{r_1}$, $\nu^{r_2} = \nu^{r_2}$. Easily $(p_2, r_2)$ is as required.]

(I) If $(p_1, r_1) \in A_{\mu^+}, \beta^* \in u^{r_1}_0$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \eta_{\beta^*,0}^{p_2} \in v^{r_2} \quad \text{and} \quad \eta_{\beta^*+1}^{p_2} = (\nu_{\beta^*,0}^{p_2})^{r_2}.$$

[Why? By clause (A) above it is enough to find such $(p_2, r_2)$ in $A_\beta$. Take $\beta^{**} \in u^{r_1} \cap \mu$ such that $(\eta_{\beta^{**},0}^{p_1}, \eta_{\beta^{**}+1}^{p_1}) = (\eta_{\beta^*,0}^{p_1}, \eta_{\beta^*+1}^{p_1})$, and let]

$$\gamma = \sup\{\ell g(\nu) + 1 : \nu \in v^{r_1} \cup \{\eta_{\beta^*,\ell}^0 : \beta \in u^{p_1}, \ell < 2\}\}.$$
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Define condition \((p_2, r_2)\) by: \(u^{p_2} = u^{p_1}\), and
\[
\eta^{p_2}_{\beta, \ell} = \begin{cases}
\eta^{p_1}_{\beta, 0} \cdot \tilde{0}_{\gamma}^{-1} \langle 1 \rangle & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\
\eta^{p_1}_{\beta, 1} \cdot \langle 1 \rangle & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 1, \\
\eta^{p_1}_{\beta, \ell} \cdot \tilde{0}_{\ell+1} & \text{if } \beta \in u^{p_1} \setminus \{\beta^*, \beta^{**}\}, \ell < 2,
\end{cases}
\]
\[
u^{p_2}_{\rho} = \begin{cases}
u^{p_1}_{\rho} & \text{if } \rho \in v^{p_1}, \\
\eta^{p_1}_{\beta, 1} \cdot \langle 1 \rangle & \text{if } \rho = \eta^{p_1}_{\beta, 0} \cdot \tilde{0}_{\gamma}^{-1} \langle 1 \rangle.
\end{cases}
\]

It is easy to check that \((p_2, r_2) \in A_{\mu^+}\) is as required.]

Due to clause \((I)\) above we immediately get assertion \((a)\) of 1.3, i.e. the function \(\eta \mapsto \bigcup \{v^{p_1}_p : \rho < \eta, \rho \in \bigcup_{r \in G_R} v^r\}\) is trivially continuous and its range is contained in \(\mu^2\), but clause \((I)\) implies that for each \(\alpha \in U\) the image of \(\eta_{\alpha, 0}\) is \(\eta_{\alpha, 1} \in \mu^2\) (and not a proper initial segment of it).

For clause \((b)\) assume that
\[(*) \nu^* \in \mu^2 \text{ and } \langle B_i : i < \mu \rangle \text{ is a sequence of } (P \ast R)\text{-names},
\]
\(\forces_{P \ast R} \text{"} B_i \subseteq \mu^2 \text{ is nowhere dense and (for simplicity) with no isolated points".}\)

It suffices to prove
\(\forces_{P \ast R} \text{"} \{\eta_{\alpha, 0} : \alpha \in U \text{ and } \nu^* < \eta_{\alpha, 0}\} \notin \bigcup_{i < \mu} B_i."\)

Let \(T_i = \{\eta \mid \gamma : \eta \in B_i \text{ and } \gamma < \mu\}\), so \(\forces_{P \ast R} \text{"} T_i \subseteq \mu^2 \text{ is a nowhere dense subtree and } (\forall \eta \in \mu^2)(\eta \in B_i \Rightarrow \bigwedge_{\gamma < \mu} \eta \mid \gamma \in T_i)"\), where "\(T \subseteq \mu^2 \text{ is a nowhere dense subtree" means: } \langle \rangle \in T, (\forall \eta \in T)(\forall \gamma < \ell g(\eta))(\eta \mid \gamma \in T) \text{ and for every } \eta \in T \text{ and } \gamma < \mu \text{ there are two } \prec\text{-incomparable sequences } \nu_0, \nu_1 \text{ such that}\)
\(\eta < \nu_0 \in T \text{ and } \eta < \nu_1 \in \mu^2 \setminus T \text{ and } \ell g(\nu_0) \geq \gamma \text{ and } \ell g(\nu_1) \geq \gamma.\)

We can find \(\alpha < \mu^+\) such that \(\alpha > \mu\) and for every \(i < \mu, T_i\) is an \(A_\alpha\)-name (remember \(\mu^\prec \mu = \mu\)). Let \((p_0, r_0) \in P \ast R\). By the statement \((B)\) above there is \((p_1, r_1)\) such that \((p_1, r_0) \leq (p_1, r_1) \in A_{\mu^+}\). We can easily find \((p_2, r_2)\) such that \((p_1, r_1) \leq (p_2, r_2) \in A_{\mu^+}\) and \(\beta^* \in (\alpha, \mu^+)\) such that \(\beta^* \in u^{p_1}_{i_2}\) and \(\nu^* < \eta^{p_2}_{i_0, 0} \in v^{p_2}_{i_2}, \)
\((\nu^{p_2}_{i_0, 0})_{r_2} = \eta^{p_2}_{i, 1} \langle 1 \rangle\) (apply clauses \((H)\) and \((I)\)).

It suffices to prove that for each \(i < \mu, (p_2, r_2) \forces \text{"} \eta^{p_2}_{i, 0} \notin B_i \text{"}.\) So let \(i < \mu\) and \((p_2, r_2) \leq (p_3, r_3) \in A_{\mu^+}.\) Since \((p_3, r_3) \upharpoonright \alpha \in A_\alpha, T_i\) is an \(A_\alpha\)-name and \(\forces_{A_\alpha} \text{"} T_i \subseteq \mu^2 \text{ is nowhere dense"}, there is a condition \((p_4, r_4) \in A_\alpha\) such that \((p_3, r_3) \upharpoonright \alpha \leq (p_4, r_4)\) and for some \(\nu^{**}\) we have
\(\eta^{p_4}_{i, 0} < \nu^{**} \in \mu^2\) and \((p_4, r_4) \forces_{A_\alpha} \text{"} \nu^{**} \notin T_i \text{".}\)
Lastly we define \((p_5, r_5) \in A'_\mu^+\) such that \((p_5, r_5)\) is above \((p_3, r_3)\) and above \((p_4, r_4)\) and \((p_5, r_5) \Vdash "u^{**} < \eta_{\beta^*, 0}"\). So let

\[ u^{p_5} = u^{p_3} \cup u^{p_4}, \]

\[ \eta_{\beta, \ell}^{p_5} = \begin{cases} 
\eta_{\beta, \ell}^{p_4} & \text{if } \beta \in u^{p_4}, \\
\eta_{\beta, \ell}^{p_3} & \text{if } \beta \in u^{p_3} \setminus (u^{p_4} \cup \{\beta^*\}), \\
\nu^{**} & \text{if } \beta = \beta^*, \ell = 0, \\
\bigcup \{\nu_{\rho}^{r_4} : \rho \in v^{r_4} \text{ and } \rho \leq \nu^{**}\} & \text{if } \beta = \beta^*, \ell = 1,
\end{cases} \]

\[ r_5 = (u^{r_4} \cup u^{r_3}, v^{r_4}, \bar{\nu}^{r_4}). \]

By clause (A) there is \((p_6, r_6)\) such that \((p_5, r_5) \leq (p_6, r_6) \in A_\mu^+\). This clearly suffices. Thus we finish proving clause (b) of the conclusion of 1.3.

Together we get all the required conclusions. \(\square_{1.3}\)

1.4 Theorem. Assume that \(\mu = \mu^{<\mu}, 2^\mu = \mu^+, 2^{\mu^+} = \mu^{+2}\) and \(S\) is a stationary subset of

\[ S^{\mu^{+2}}_{\mu^+} =: \{\delta < \mu^{+2} : \text{cf}(\delta) = \mu^+\}. \]

Suppose that \(\diamond_S\) holds.

Then there is a \(\mu\)-closed, \(\mu^+\)-c.c. forcing notion \(P\) of cardinality \(\mu^{+2}\) such that, in \(V^P\),

\((*)\) for every function \(f : \mu^2 \to \mu^2\), there is a non \(\mu\)-meagre set \(A \subseteq \mu^2\) such that \(f \upharpoonright A\) is continuous.

Proof. Without loss of generality we may also assume that \(S^{\mu^{+2}}_{\mu^+} \setminus S\) is stationary.

For \(\alpha \leq \mu^{+2}\) let \(\mathcal{A}_\alpha\) be the family of sequences \(\bar{Q} = \langle P_i, Q_i : i < \alpha\rangle\) such that

(a) \(\bar{Q}\) is iteration with support \(< \mu\) and \(P_\alpha\) is the limit,
(b) for \(i \in \alpha \cap \mu^{+2} \setminus S\), \(Q_i = Q_i\) is \((\mu^2 > 2, <)\) (i.e. a \(\mu\)-Cohen forcing notion) and we denote by \(\eta_{\bar{Q}}^i_{i+1}\) the name of a Cohen generic element of \(\mu^2\) adjoined by \(Q_i\),
(c) if \(\beta_1 < \beta_2 \leq \alpha, \beta_2 < \mu^{+2}, \beta_1 \notin S\) then \(P_{\beta_2}/P_{\beta_1}\) and \(P_{\beta_2}/(P_{\beta_1} * Q_{\beta_1})\) are \(\mu\)-Cohen forcing notions (and for clarity we fix representation of \(P_{\beta_2}\) as \(P_{\beta_1} \times Q_{\beta_1} \times \prod \{Q_{j_{\beta_1, \beta_2}} : j < j(\beta_1, \beta_2)\}\) with support \(< \mu\)),
(d) we may assume that each \(Q_i\) (for \(i < \alpha\)) is \(\mu\)-closed, \(\mu^+\)-cc and of cardinality \(\leq \mu^+\).
Note that by clause (c), each $P_\beta$ is a $\mu$-closed forcing notion satisfying the $\mu^+$-cc. Let $\mathcal{R} = \bigcup_{\alpha < \mu^+} \mathcal{R}_\alpha$.

There is a natural ordering of $\mathcal{R}$:

$$\bar{Q}^1 \leq \bar{Q}^2 \quad \text{if and only if} \quad \bar{Q}^1 = \bar{Q}^2 \upharpoonright \ell g(\bar{Q}^1).$$

This partial order is $\mu^+$-complete and every (strictly) increasing chain of length $\mu^+$ has a limit in $\mathcal{R}_{\mu^+}$.

Note that by clause (c) we have that

$$(\diamondsuit) \text{ if } \bar{Q} \in \mathcal{R}_\eta, \beta < \alpha, \beta \notin S \text{ and } \models_{P_\beta} "B \subseteq \mu \text{ is not } \mu \text{-meagre}"$$

then $\models_{P_\alpha} "B \subseteq \mu \text{ is not } \mu \text{-meagre}"$.

Because of this, and by $\diamondsuit_S$, it is enough to show that

$$(\otimes) \text{ if } \bar{Q} \in \mathcal{R}_{\mu^+}, P = P_{\mu^+} = \lim(\bar{Q}), \text{ and } f \text{ is a } P\text{-name of a function from } \mu \text{ to } \mu^2 \text{ then for some club } E \text{ of } \mu^+, \text{ for every } \delta \in S \cap E \text{ we have}$$

$$(\alpha) \quad i < \delta \Rightarrow f(\eta_{i+1}^\bar{Q}) \text{ is a } P_\delta\text{-name},$$

$$(\beta) \quad \text{there is a } \bar{Q}^{\delta+1} \in \mathcal{R}_{\delta+1} \text{ such that } \bar{Q}^{\delta+1} \upharpoonright \delta = \bar{Q} \upharpoonright \delta \text{ and}$$

$$\models_{\lim(\bar{Q}^{\delta+1})} "\text{for some } A \subseteq \delta, |A| = \mu^+, \text{ we have that} \quad (i) \quad \{\eta_{i+1}^\bar{Q} : i \in A\} \text{ is not meagre,}$$

$$\quad (ii) \quad \text{the function } \{(\eta_{i+1}^\bar{Q}, f(\eta_{i+1}^\bar{Q})) : i \in A\}$$

$$\text{is continuous}".$$  

(Here: $\eta_{i+1}^\bar{Q}$ is a $P_\delta$-name (as $i + 1 < \delta$) and also $f(\eta_{i+1}^\bar{Q})$ is a $P_\delta$-name (by clause $(\alpha)$)).

**Proof of $(\otimes)$**. For each $\delta \in S^{\mu^+}_{\mu^+} \setminus S$, $f(\eta_{\delta+1}^\bar{Q})$ is a $P_{\mu+2}$-name of a member of $\mu^2$, so for some $\gamma_\delta > \delta + 1$, it is a $P_{\gamma_\delta}$-name and we can demand that $\gamma_\delta$ is a successor ordinal. As

$$P_{\gamma_\delta} = P_\delta \times Q_\delta \times \prod_{j < \delta(\delta, \gamma_\delta)} Q_{\delta}^{\delta, \gamma_\delta}$$

(with support $< \mu$), we find a set $\omega_{\delta} \subseteq j(\delta, \gamma_\delta)$ of cardinality $\leq \mu$ such that $f(\eta_{\delta+1}^\bar{Q})$ is a $P_\delta \times Q_\delta \times \prod_{j \in \omega_{\delta}} Q_{\delta}^{\delta, \gamma_\delta}$-name. Also as $\text{cf}(\delta) = \mu^+$ there is $\beta_\delta < \delta$ such that $f(\eta_{\delta+1}^\bar{Q})$ is a $P_{\beta_\delta} \times Q_\delta \times \prod_{j \in \omega_{\delta}} Q_{\delta}^{\delta, \gamma_\delta}$-name and we can demand that $\beta_\delta$ is a successor ordinal.

By Fodor’s lemma (as $2^\mu = \mu^+$) for some stationary $S_1 \subseteq S^{\mu^+}_{\mu^+} \setminus S$ (and $\zeta < \mu^+$, $\beta^* < \mu^+$) we have that

$$(\alpha) \quad \delta \in S_1 \Rightarrow \beta_\delta = \beta^*$$

$$(\beta) \quad \delta \in S_1 \Rightarrow \text{otp}(\omega_{\delta}) = \zeta$$
Clearly \( Q^\delta = \prod_{j \in w_3} Q_j^{\delta_j, \gamma_j} \) is isomorphic to Cohen forcing.

Lastly, let
\[
E = \left\{ \delta < \mu^+ : \delta = \sup(S_1 \cap \delta) \text{ and } \delta_1 \in S_1 \cap \delta \Rightarrow \gamma_{\delta_1} < \delta \right\}.
\]

Clearly the set \( E \) is a club of \( \mu^+ \). We are going to show that for each \( \delta^* \in E \cap S \) there is \( Q^{\delta^*+1} \) as required in (\( \bigcirc \)).

Let \( \delta^* \in E \cap S \). We can find an increasing continuous sequence \( \langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle \) with limit \( \delta^* \) and a sequence \( \langle \delta_\varepsilon : \varepsilon < \mu^+ \rangle \) such that \( \beta_0 = \delta^*, \beta_\varepsilon \) a successor ordinal if \( \varepsilon \) is a non-limit, \( \beta_\varepsilon \not\in S, \delta_\varepsilon \in S_1, \) and \( \beta_\varepsilon < \delta_\varepsilon < \gamma_{\delta_\varepsilon} < \beta_{\varepsilon+1} \). Let \( \eta_{\varepsilon,0}^* \) and \( \eta_{\varepsilon,1}^* \) be \( P_\varepsilon/P_{\beta_0} \)-names such that \( \eta_{\varepsilon,0}^* \) is \( \hat{Q}_{\delta_\varepsilon+1} \) and \( \eta_{\varepsilon,1}^* \) is a name for the Cohen subset of \( \mu \) added by \( Q^{\delta^*_\varepsilon} \).

Now let
\[
R_0^\varepsilon =: \prod_{\delta \in w_3} \{Q_{\delta_\varepsilon} : \varepsilon < \mu^+ \} \cup \{Q^{\delta^*_\varepsilon} : \varepsilon < \mu^+ \}
\]
and note that \( R_0^\varepsilon \ll P_\varepsilon/P_{\beta_\varepsilon} \). The forcing notion \( R_0^\varepsilon \) is naturally isomorphic to \( P \) from 1.3 with \( \eta_{\varepsilon,\ell}^* \) corresponding to \( \eta_{\varepsilon,\ell} \). Moreover the quotient \( P_\varepsilon/(P_{\beta^*_\varepsilon} \times R_0^\varepsilon) \) is a \( \mu \)-Cohen forcing notion. [Why? For each \( \varepsilon < \mu^+, \delta_\varepsilon, \gamma_{\delta_\varepsilon}, \beta_\varepsilon \not\in S \) and hence we may write \( P_{\beta_\varepsilon+1} \) as
\[
P_{\beta_\varepsilon} \times Q_{\beta_\varepsilon} \times \prod_{j < \beta_\varepsilon, \delta_\varepsilon} Q_j^{\delta_\varepsilon, \delta_\varepsilon} \times Q_{\delta_\varepsilon} \times \prod_{j < \beta_\varepsilon, \gamma_{\delta_\varepsilon}} Q_j^{\delta^*_\varepsilon, \gamma_{\delta_\varepsilon}} \times Q_{\gamma_{\delta_\varepsilon}} \times \prod_{j < j(\delta_\varepsilon, \beta_{\varepsilon+1})} Q_j^{\gamma_{\delta_\varepsilon}, \beta_{\varepsilon+1}}.
\]

But
\[
\prod_{j < j(\delta_\varepsilon, \gamma_{\delta_\varepsilon})} Q_j^{\delta^*_\varepsilon, \gamma_{\delta_\varepsilon}} = Q^{\delta^*_\varepsilon} \times \prod_{j < j(\delta_\varepsilon, \beta_{\varepsilon+1})} Q_j^{\delta^*_\varepsilon, \beta_{\varepsilon+1}}.
\]

So we may represent \( P_{\beta_\varepsilon+1} \) as
\[
P_{\beta_\varepsilon+1} = P_{\beta_\varepsilon} \times Q_{\beta_\varepsilon} \times Q^{\delta^*_\varepsilon} \times R^{\varepsilon}_\varepsilon,
\]
where \( R^{\varepsilon}_\varepsilon \) is a product (with support \( < \mu \)) of \( \mu \)-Cohen forcing notions. Now, since the sequence \( \langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle \) is increasing and continuous with limit \( \delta^* \) we may write
\[
P_\varepsilon = P_{\beta^*_\varepsilon} \times \prod_{\varepsilon < \mu^+} Q_{\delta_\varepsilon} \times \prod_{\varepsilon < \mu^+} Q^{\delta^*_\varepsilon} \times \prod_{\varepsilon < \mu^+} R^{\varepsilon}_\varepsilon
\]
(all products with \( < \mu \) support).]
2.1 Observation. Haim Judah asked us whether adding Cohen reals is enough.

Let $Q_{δ^*}$ be $R[\langle \eta_{ε,0}^*: ε < δ^* \rangle]$ (from 1.3), $Q^{δ^*+1} = (Q^* | δ^*) ◦ (P_δ, Q_{δ^*})$.

To check that $Q^{δ^*+1}$ is as desired we use 1.3. Thus $P_{δ^*}/P_δ$ (for $β < δ, β \notin S$) is $μ$-Cohen by 1.3(c) and because, as said above, $P_{δ^*}/(P_{δ^*} × R_{δ^*})$ is $μ$-Cohen. It follows from 1.3(a) that the function $η_{ζ,0}^* \mapsto η_{ζ,1}^*$ (for $ε \in U$) is continuous and hence, as the names $f(η_{δ,0}^*)$ are isomorphic, the function $η_{δ,1}^* \mapsto f(η_{δ,0}^*)$ (for $ε \in U$) is continuous (it is obviously Borel, which actually suffices, but in fact, as the $η_{ζ,0}^*$ are Cohen over the definition of $f$ it is continuous). Finally, by 1.3(b), $\models P_{δ+1} \{ η_{δ,1}^* : ε \in U \}$ is not meagre”.

This finishes the proof. □

1.5 Remark: 1. Alternatively, one may force the iteration instead of using $\hat{Q}_S$.

2. The forcing notion from 1.4 is constructed like “$λ$-free not free (Boolean) algebra of cardinality $λ$”, with free interpreted as $μ$-Cohen.

§2 Final remarks

Haim Judah asked us whether adding Cohen reals is enough.

2.1 Observation. If $P$ is the forcing notion for adding $2^{κ_0}$ Cohen real, then in $V^P$ there is $f : R \rightarrow R$ which is not continuous on any uncountable set.

Proof. Let $λ = 2^{κ_0}, P = \{ f : f$ is a finite function from $λ$ to $\{0, 1\}\}$. For any set $w \subseteq λ$ let $P_w = \{ f \in P : \text{Dom}(f) \subseteq w\}$. It is known that $P_w \otimes P = P_λ$ and for any $P$-name $η$ of a real (i.e. $\models P \ "η \in ω^2\”$) for some countable $w \subseteq λ, η$ is a $P_w$-name.

For every ordinal $α < λ$ let $r_α \in ω^2$ be defined by:

$$r_α(n) = ℓ \text{ if and only if } \text{ for some } f \in G_P, f(ωα + n) = ℓ.$$

Clearly $r_α$ is a Cohen real. Let $\langle s_ζ : ζ < λ \rangle$ list all $P$-names of reals, so for some countable $w_ζ \subseteq λ, s_ζ$ is a $P_{w_ζ}$-name. Without loss of generality

$$α + ω = β + ω \ & α \in w_ζ \Rightarrow β \in w_ζ.$$

We now define by induction on $ζ < λ$ an ordinal $α_ζ < λ$ as follows:

$α_ζ$ is the first ordinal $α$ such that $α \notin \{α_ε : ε < ζ\} ∪ \{ β : β + ωkβ \in w_ζ \text{ for some } k \in ω\}$.

Let $G \subseteq P$ be a generic filter over $V$. In $V[G]$ we define a function $F : ω^2 \rightarrow ω^2$ as follows:

$$\text{for } s \in ω^2 \text{ let } γ(s) = \min\{ζ : s = s_ζ[G]\} \text{ and } F(s) = r_{α_γ(ε)}.$$
Suppose now that, in $V[G]$, a set $A \subseteq \omega^2$ is uncountable and $F \upharpoonright A$ is continuous. Thus we have a Borel function $F' : \omega^2 \rightarrow \omega^2$ such that $F \upharpoonright A = F' \upharpoonright A$. The function $F'$ is coded by a single real $s \in \omega^2$, $\gamma(s) = \zeta$.

Let $p_0 \in G, A, F, F', s$ and $\zeta < \lambda$ be such that

$$p_0 \parallel p \ "A, F, F', s = s\zeta, \zeta = \gamma(s) \) are as above".$$ 

The set $\{ \xi < \lambda : [\omega\alpha\xi, \omega\alpha\xi + \omega) \cap w_\xi \neq \emptyset \}$ is countable, so we find $\xi < \lambda$ and $p_1 \geq p_0, p_1 \in G$ such that

$$p_1 \parallel p \ "s\xi \in A and \gamma(s\xi) = \xi"$$

and $[\omega\alpha\xi, \omega\alpha\xi + \omega) \cap w_\xi = \emptyset$. Now

$$p_1 \parallel p \ "F'(s\xi) = F(s\xi) = \tau\alpha\xi".$$ 

But we can compute $F'(s\xi)$ in $V[G \cap P_{w_\xi \cup w_\xi}]$ and (by the choice of $\xi$) $\tau\alpha\xi$ is Cohen over $V[G \cap P_{w_\xi \cup w_\xi}]$ (remember the choice of $\alpha\xi$’s), which is a contradiction. $\Box_{2.1}$

2.2 Remark. Instead of forcing it suffices to assume the existence of a Luzin set in a strong sense.

* * *

We call $A \subseteq \omega^2$ a Luzin set if it is uncountable and non-meagre and any uncountable $B \subseteq A$ is non-meagre.

2.3 Definition. A forcing notion $Q$ is Luzin-preserving if it satisfies the c.c.c. and

(∗) for any Luzin set $A \subseteq \omega^2$ and a sequence $\langle p_\eta : \eta \in A \rangle$ of members of $Q$, we have that

(∗∗) for some $\eta \in A$

$$p_\eta \parallel Q \ "for some \rho \in \omega^{>2}, (\forall \nu)[\rho \leq \nu \in \omega^{>2} \Rightarrow for uncountably many \eta_1 \in A, \nu \triangleleft \eta_1 & p_\eta \in G_Q]".$$ 

2.4 Claim. 0) Assume there are Luzin sets, then in Definition 2.3 “it satisfies the c.c.c.” follows from (∗) of 2.3.

1) A forcing notion $Q$ is Luzin-preserving iff it satisfies the c.c.c. and for any Luzin $A \subseteq \omega^2$ we have, $\parallel_Q "A \ is \ Luzin"$.

2) Being Luzin-preserving is preserved by composition.

3) If $\langle P_i : i < \delta \rangle$ is an increasing continuous sequence of Luzin-preserving forcing notions then $\bigcup_{i<\delta} P_i$ is a Luzin-preserving forcing notions.

4) The statement “a forcing notion $P$ is Luzin preserving” depend on $P$ only up to equivalence of forcing notions.

5) In Definition 2.3 we can weaken the demand to:

(∗’) for some $\eta \in A$ there is $q$ such that $p_\eta \leq q \in Q$ and $q \parallel Q "...".$
Proof. 0) Should be clear.
1) \( \Rightarrow \) (the "only if" implication).
By definition, \( Q \) satisfies the c.c.c. Assume \( A \) is Luzin but \( p \models "A \) is not Luzin".
Then for some \( Q \)-names \( \eta_i \) (for \( i < \omega_1 \)), we have

\[
p \models "\eta_i \in A \) for \( i < \omega_1, \bigwedge_{i < j} \eta_i \neq \eta_j \) and \( \{ \eta_i : i < \omega_1 \} \) is not meagre".
\]

Thus changing the \( \eta_i \)'s we can get

\[
(\oplus) \ p \models "\eta_i \in A \) for \( i < \omega_1, \bigwedge_{i < j} \eta_i \neq \eta_j \) and \( \{ \eta_i : i < \omega_1 \} \) is nowhere dense".
\]

Let \( p \leq p_i \in Q, p_i \models "\eta_i = \nu_i" \) so \( \nu_i \in (\omega_2)^V \) and necessarily \( \nu_i \in A \). For no \( \nu \) the set \( \{ i : \nu_i = \nu \} \) is uncountable as then \( Q \) fails the c.c.c. So without loss of

\[
generality i \neq j \Rightarrow \nu_i \neq \nu_j. \ Let A' = \{ \nu_i : i < \omega_1 \}. \ It is an uncountable subset of A and hence it is Luzin. Let \( p_\nu = p_i \) if \( \nu = \nu_i \). Apply (\#) of Definition 2.3 to \( \langle p_\nu : i < \omega_1 \rangle \) and get contradiction to (\oplus) above.
\]

\( \Leftarrow \) (the "if" implication). Of course \( Q \) satisfies the c.c.c. Let \( \langle p_\eta : \eta \in A \rangle \) be given,

\( A \subseteq \omega_2 \) a Luzin set and \( p_\eta \in Q \) (for \( \eta \in A \)) and we should show that (\**\) of

Definition 2.3 holds. Then \( A = \{ \eta \in A : p_\eta \in G_Q \} \) is a \( Q \)-name of a subset of \( A \).

It is known that for all but countably many \( \eta \in A \), we have

\[
p_\eta \models "A \subseteq A \) is uncountable"
\]

so that (by the assumption of the implication we are proving)

\[
p_\eta \models "A \) is non-meagre, hence nowhere meagre above some \( \nu_\eta \in \omega>2".
\]

We have really finished, but we can elaborate. As we can replace \( A \) by \( A \backslash B \) for

any countable \( B \subseteq A \). Without loss of generality this holds for every \( \eta \in A \). Now for \( \eta \in A \) and \( p'_\eta, q_\eta \leq p'_\eta \in P \) there are \( \nu_\eta^* \) and \( q_\eta, p'_\eta \leq q_\eta \in Q \) such that

\[
q_\eta \models "\nu_\eta = \nu_\eta^*". \ So for all \( \eta \in A \) we have
\]

\[
q_\eta \models "\{ \nu : \nu \in A \) \subseteq p_\nu \in G_Q \) is uncountable and everywhere non-meagre above \( \nu_\eta^* ".
\]

As this holds for any \( p'_\eta, q_\eta \leq p'_\eta \in P \) we are done.
2) Follow by 2.4(1) (and as c.c.c. is preserved).
3) Like the preservation of c.c.c.
4) Easy (e.g. use 2.4(1)).
5) Left to the reader. \( \square \)
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