1 Introduction

Presently, there are known two different approaches to the problem of constructing of a relativistic quantum theory suitable to describe the scattering processes. According to the first – canonical – approach one has to fix a finite set of (fundamental) fields and the Lagrangian that satisfies certain conditions (locality, hermiticity, symmetry, renormalizability, etc.). To obtain the quantum version, one has to carry out the procedure of canonical quantization. This allows one to construct the Fock space of asymptotic states and to calculate the Green functions and the S-matrix.

The second approach (first suggested in papers [1]; we will call it as the S-matrix approach) is less known and practically was not discussed in literature (see however [2], [3] and chapters 2 – 5 of the monograph [4]). In this approach the structure of the Fock space of asymptotic states is postulated initially. The field operators are constructed according to the symmetry properties of the corresponding one-particle states. The Hamiltonian of the system (built out of these fields) is also postulated as the operator in the interaction picture.

The elements of the S-matrix are calculated according to Dyson formula ($f$, $i$ – the final and initial states respectively):

$$S_{fi} = \langle f|T_{\text{w}}\exp \left\{ -i \int H_{\text{int}} dx \right\} |i\rangle.$$  (1)

The symbol $T_{\text{w}}$ in this formula denotes Wick’s (manifestly covariant) T-product. The non-covariant terms in the Hamiltonian and in propagators are to be discarded (see [4]). In the case of effective theories only discussed below this does not lead to any uncertainties.

Each of these two approaches has its advantages and shortcomings. However, the comparative analysis is not our goal here. We use the S-matrix approach just because the canonical quantization of the theories with Lagrangians containing high (second and higher) powers of the field time derivatives is the problem the solution of which is presently
unknown. At the same time, the effective theories, which are the main subject of our study, contain all the powers of first (and higher) field derivatives by the very construction.

We use the slightly improved version of the definition of the effective theory (originally given in \[5\]): the theory is called as effective if the corresponding Hamiltonian (in the interaction picture) contains all the local terms consistent with the requirements of a given algebraic symmetry. In the operator sense this construction is not very well defined. However, we are only interested in the \(S\)-matrix elements calculated (only on the mass shell!) with the help of the expansion based on the formula \(1\). In the papers \([6], [7]\) it was shown that for those objects it is possible to formulate simple correctness conditions for the expressions calculated in the given order of loop expansion. In the zeroth order (tree graphs) of the renormalized perturbation theory these conditions lead to reasonable restrictions on the values of physical parameters of a theory.

The effective theories, by the very construction, show the property of multiplicative renormalizability (in the case of absence of anomalies). However, usually they are not considered in the textbooks on renormalization theory (see for example \([8]\)). The reason is that to fix the physical content of such a theory one needs to impose an infinite set of renormalization prescriptions (conditions). It is absolutely unrealistic if we have no guiding principle limiting the freedom of this step. Preliminary results of our research (see \([6], [7]\)) show that the analyticity-type restrictions could play the role of such a principle. Moreover, they seem to be natural from the point of view of correctness of perturbative scheme. For example, one could obtain the bootstrap conditions (widely discussed in connection with dual models – see \([9]\)) from the properly formulated requirements of meromorphy and polynomial boundedness of the tree-level amplitudes constructed in the framework of renormalized perturbation theory.

In this paper we illustrate the techniques of derivation of the bootstrap equations from the condition that the function of two complex variables is meromorphic and polynomially bounded in different domains. However, before starting the analysis of the examples we would like to outline the reasons why these conditions (automatically fulfilled in conventional field theories) turn out to be fruitful in the case of effective theory. Three following sections serve for this purpose.

2 Preliminary notes

First of all we remind the meaning of some notions and terms used below. We specify the definitions given in \([4]\) and introduce the notion of minimal parametrization of the effective theory. We use these terms because they are suitable for the work with on-shell matrix elements in terms of complex analysis.

Let us emphasize that in what follows it is assumed that the masses of the particles with spin \(J > 1/2\) are nonzero. This assumption is purely a technical one, but at present we cannot proceed without it. It does not lead to any limitations when we are describing the strong interaction of hadrons.
All the combinations of coupling constants that do not appear in the expressions for the renormalized $S$-matrix elements of the $L$-th order in loop expansion are called as the redundant parameters of the order $L$. These combinations can appear in the expressions for Green functions, but the corresponding contributions prove to be irrelevant after renormalization, passing to the mass shell and multiplying by the wave functions. The example of the redundant parameter is given by the gauge fixing constant in the renormalizable vector models. Another example is the wave function renormalization constant $[4]$.

All the independent combinations of coupling constants that appear in the expressions for renormalized matrix elements of the $L$-th order of the loop expansion are called as the essential parameters of the order $L$.

Consider now the elementary (pointlike) vertex with $n$ legs carrying the momenta $p_1, p_2, \ldots, p_n$. In general, it can be written as follows

$$ V(p_1, \ldots, p_n) = \sum_{a=1}^{M+N} T^{(a)} F_a, $$

where \(\{T^{(a)}\}\) is the full set of independent tensor structures ($M$ of them being minimal and $N$ – nonminimal; see the definitions below), and $F_a$ are the functions of invariant kinematical variables (the total number of those variables is equal to 3 when $n = 3$ and $4n - 10$ when $n > 3$; we are working in $D = 3 + 1$ dimensions). It is convenient to choose these variables as follows:

$$ \pi, \nu \equiv [\pi_1, \ldots, \pi_n; \nu_1, \ldots, \nu_{3n-10}], $$

where

$$ \pi_i \equiv p_i^2 - m_i^2. $$

The concrete choice of the rest $3n - 10$ variables $\nu_i$ (independent linear combinations of the momentum scalar products) is not important for the present.

The vertex (2) is the element of the system of Feynman rules of the effective theory under consideration. It describes the contribution of a term in the Hamiltonian which is constructed from $n$ field operators.

It is clear that the functions $F_a$ polynomially depend on the variables (3); the polynomial coefficients are the combinations of coupling constants. It would be premature to discuss the convergency conditions for the series of vertices with different number of derivatives. Our purpose is to obtain the well-defined expressions for the $S$-matrix elements of the given order; to obtain them one needs to take into account not only the vertices of the type (2), but also the contributions of all possible graphs with $n$ legs. In what follows we would only take care about the correctness of the expressions that appear as the result of infinite summation of graphs.

The contribution of the vertex (2) to the matrix element describing the process with $n$ external particles can be obtained by passing to the mass shell $\pi_i = 0, (i = 1, \ldots, n)$ and multiplying by the relevant wave functions. Hence, those combinations of coupling constants which form the coefficients at nonzero powers of $\pi_i$ in the series for $F_a$, do not
appear in the contribution of the corresponding pointlike vertex to the $S$-matrix element in question. The same is true for the combinations of coupling constants which appear in expressions for those $F_a$ that are the coefficients at tensor structures (called as nonminimal) resulting in zero after passing to the mass shell and multiplying by the relevant wave functions.

It is easy to understand that the discussion above is also relevant to the vertices with arbitrary number of “bubbles” (self-closed lines) and “tadpoles” (a line that starts in a vertex and ends by one or several “bubbles”). We treat such vertices as pointlike.

Thus, after the vertex (2) is put on the mass shell and multiplied by the relevant wave functions, it takes the form:

$$V(p_1, \ldots, p_n) \sim \sum_{a=1}^{M} T^{(a)} F_a(\pi = 0, \nu).$$

Here the set of $T^{(a)} (a = 1, \ldots, M)$ contains only minimal tensor structures and does not contain any nonminimal ones.

The line $p_k$ of the vertex $V(p_1, \ldots, p_n)$ is called as minimal, if the explicit form of the expression (2) does not change its appearance when the $k$-th particle is put on its mass shell $\pi_k = 0$ and the vertex is multiplied by the relevant wave function $u(p_k)$.

We call the vertex $V(p_1, \ldots, p_n)$ as minimal, if all its lines are minimal. Minimal vertex can always be presented in the form (5). Finally, we call the propagator of a particle with mass $m$ and spin $J$ as minimal, if its numerator is just a spin sum written in a covariant form (on the mass shell $q^2 = m^2$!) and considered as a function of four independent components of momentum.

Each $S$-matrix graph of the effective theory can be rewritten in terms of minimal elements: vertices (of different orders) and propagators (see [10]; the complete proof will be published later). This means that the full set of essential parameters includes only masses and those combinations of coupling constants which define the minimal parametrization of the vertices of different orders. Thus to obtain the finite $S$-matrix it is sufficient to impose only those normalization prescriptions which are necessary to fix the finite parts of the minimal parameters. However the results of the papers [6], [7] show that this set of prescriptions is also excessive. The necessity to observe the bootstrap restrictions (see below) results in the fact that only a part of minimal parameters could be treated as independent.

The renormalization prescriptions should be only imposed on the set of independent constants of the theory. This means that there are two (equivalent) methods to impose them. The first method consists in finding the explicit solution of bootstrap conditions (see below) results in the fact that only a part of minimal parameters could be treated as independent.

The renormalization prescriptions should be only imposed on the set of independent constants of the theory. This means that there are two (equivalent) methods to impose them. The first method consists in finding the explicit solution of bootstrap conditions (see below) results in the fact that only a part of minimal parameters could be treated as independent.

---

4It is this point were our suggestion on the absence of massless particles of higher ($J > 1/2$) spins happens important.

5The explicit expressions of minimal parameters in terms of the coupling constants certainly depend on the perturbation order.

6Such a solution certainly exists: the example is provided by conventionally renormalizable theories.
imposes the renormalization prescriptions for this very set. The second method is to impose an arbitrary set of prescriptions on all the minimal parameters and then to use the bootstrap equations as the binding limitations on the possible structure of this set. We use the second method (just because we cannot find the explicit solution of the bootstrap conditions); so we would like to discuss it in more detail.

The renormalization prescriptions, irrespectively to their explicit form, should be satisfied at every fixed order. In particular this is true with respect to the lowest (tree-level) order. This means that each relation between the minimal parameters of zeroth order (recall that in the framework of the renormalized perturbation theory this is the relation between the physical values of the parameters!) should be treated as the relation between the prescriptions. In other words, if there exist some connections between the minimal parameters of zeroth order (this is exactly the case in the mathematically reasonable effective theory) then one cannot impose arbitrary renormalization prescriptions. This statement would be a triviality if we were discussing the restrictions due to some symmetry (group) limitations. But in the case of effective theories these restrictions (bootstrap equations) arise from the certain requirement of localizability, discussed in the following section.

3 Localizable effective theories

In case of ordinary renormalizable theory every term of the loop expansion of the S-matrix based on the expression (1) is well defined (the regularization is implied). Not much could be said about the convergency of this expansion, but this does not create a problem at arbitrary finite order. The situation is quite different in the case of effective theory. The Hamiltonian contains the terms with many derivatives (of arbitrary high degree and order) hence the infinite power series appear in the expressions for matrix elements already at the tree level. In other words, in this case the Hamiltonian is not a local operator, and one has to exercise caution when working with it. That is why in what follows we will limit ourselves with the special class of effective theories.

We will only consider the Hamiltonians from the class of localizable ones. One can intuitively give an idea on localizability considering the simple example from electrostatics. The interaction Hamiltonian of point charge with the extended charge is nonlocal. Nevertheless, under certain conditions (well separated systems) it can be localized (rewritten in the form of a convergent infinite series of local terms) with the help of multipole expansion.

The localizability requirements could be briefly formulated in the implicit form. The explicit formulation would take too much space but not much illuminate the general idea. The main idea is suggested by the quasiparticle method well known in the non-relativistic quantum mechanics.

We call the Hamiltonian as localizable if the tree-level amplitudes formally obtained from it could be reproduced in the framework of well defined tree approximation of a certain extended effective theory containing auxiliary fields that correspond to the particles (with masses $M_i$), unstable with respect to decays into the states of initial theory.
An important note: in the instability condition
\[ M_i \geq m + \mu \] \hspace{1cm} (6)
small letters denote the physical particle masses, i.e. the masses of asymptotic states of the initial theory Hamiltonian (see [4]). The quantities \( M_i \) have the meaning of mass parameters of the extended Hamiltonian. They define the position of the poles of bare propagators. Their treatment, thus, depends on the renormalization scheme that is being used (see [12], [13]). However the detailed study of this point lies beyond the scope of this paper. We use the term “mass” in both senses, because this does not lead to misunderstanding.

The words “well defined” are to be understood in the sense that the formal tree level series of the extended theory should be summable in all the domains of definition. Besides, the tree level series of the initial theory should converge at least in a small domain \( D \) – otherwise, the comparison would happen impossible. In this domain the tree approximations for all the amplitudes describing scattering and creation of stable particles in both theories should coincide identically. In other words, in the sector of stable particles the tree approximation of the extended theory is just the analytic continuation of the tree approximation of the initial theory. This is precisely the essence of the extension idea. The condition (6) is necessary to ensure the coincidence of asymptotic states in both theories. In the extended theory the loop corrections lead to non-stability of the particles described by auxiliary fields and the asymptotic space (the space of stable states) becomes the same as that in the initial theory. It is well known (see [14]) that in the theory with unstable particles the \( S \)-matrix constructed in accordance with the formal Feynman rules turns out to be a unitary operator on the space of stable states.

Before transforming this philological description of localizable theories into the definite limitations on the values of coupling constants we need to make several preliminary remarks.

The requirement of localizability mirrors the natural wish to work with the series, each term of which is a well defined function of momenta. And – by the very construction – the general structure of (1) provides a guarantee of covariance, causality, unitarity and crossing symmetry of the \( S \)-matrix.

Up to present the only known tool producing the series with the desired structure is the formalism of the quantum field theory with the Hamiltonian containing the finite number of local interaction terms. It is essential that in this approach the tree level amplitude of every process happens to be a rational function of each pair energy (with all other variables fixed). It is easy to see that in the case of the theory containing an infinite number of scalar fields \( \phi_i \) (\( i = 1, 2, \ldots \)) with masses \( m_i \) and the renormalizable type of interaction
\[ H_{int} = g_{ijk} \phi_i \phi_j \phi_k + \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l, \]
no changes are needed in the general scheme of quantum theory. It is sufficient that the matrices of coupling constants \( g_{ijk}, \lambda_{ijkl} \) and the masses \( m_i^2 \) satisfy the conditions ensuring the convergence of series at every given order of loop expansion. When this condition is applied, the tree-level amplitudes of all processes (not only of those describing the scattering
and production of stable states) happen to be meromorphic functions in each pair energy. Aside from this one should require (see [7], [10]) that these functions are polynomially bounded (in the sense of contour asymptotics – see for example [15]) at zero momentum transfer. This condition is necessary if we would like to construct the loop graphs by means of closing the external lines in the corresponding tree graphs, without being anxious about the order of operations.

In principle, the situation in the initial effective theory will not differ from that described above if the analytically continued (from the postulated convergency domain $D$) amplitudes turn out to be polynomially bounded meromorphic functions (not arbitrary; see below) of each pair energy. The meromorphy properties make us hope that these functions could be reproduced in the framework of the extended effective theory, containing the auxiliary fields with suitable masses. Of course this is not always possible because not every meromorphic function could be obtained as a tree level amplitude of some hypothetical field theory. Thus the requirement of localizability is to be interpreted as the conditions of the existence of extended effective theory of the most general form. The only limitation is that the mass spectrum of this latter theory must satisfy the non-stability condition (6).

The localizability requirement leads to certain conditions for the $S$-matrix elements, we call them as the analyticity conditions. The special term is used because we want to avoid the necessity of stressing the formal difference between the Hamiltonians of the extended and initial theories. The extended theory is introduced just because we have no tool allowing us to work beyond the frames of Dyson’s perturbation theory.

In principle, the restrictions on the coupling constants of the extended theory can be transformed into the desired conditions of localizability restricting the possible set of couplings in the initial Hamiltonian. For this it is sufficient to compare the expansions of the scattering amplitudes calculated in both theories in the domain $D$.

## 4 Analyticity conditions

First of all we need to consider the extended theory and formulate the analyticity conditions. However, this problem is rather complicated. Here we are going to take only a first step: we will formulate the (necessary) analyticity conditions for the amplitudes of binary processes. This case is relatively simple because the kinematics is completely described by two independent variables.

It is convenient to introduce three equivalent sets:

$$(x, \nu_x) , \quad x = (s, t, u) ,$$

where $(s, t, u)$ stand for conventional Mandelstam variables and

$$\nu_s = t - u , \quad \nu_t = u - s , \quad \nu_u = s - t .$$

The tree-level amplitude $M(s, t, u)$ of an arbitrary binary process with scalar particles (the generalization for the case of arbitrary spins does not lead to any particular difficulties)
constructed in accordance with Feynman rules takes a form of the following formal series:

\[ M(s, t, u) = \sum_{i,j,k=0}^{\infty} a_{ijk} s^i t^j u^k + \sum_{R_s} \frac{N_s(s, t, u)}{s - M_{R_s}^2} + \sum_{R_t} \frac{N_t(s, t, u)}{t - M_{R_t}^2} + \sum_{R_u} \frac{N_u(s, t, u)}{u - M_{R_u}^2}. \]  

(9)

The summation should be carried out over all kinematically allowed resonances \( R_x \) (\( M_R \) stand for the corresponding resonance masses) in each channel, and also (in the first sum) over all four-particle vertices. In turn, the numerators \( N_x(s, t, u) \) take a form of (formal, maybe infinite) sums

\[ N_x(s, t, u) = \sum_{i,j,k=0}^{\infty} b_{ijk}^{(x)} s^i t^j u^k. \]

Numerical matrices \( a_{ijk} \) and \( b_{ijk}^{(x)} \) are the functions of the coupling constants of the (extended) Hamiltonian.

The series (9) should be summable in order to make sense and to be used for constructing the next orders of the loop expansion. The result must be a meromorphic function in each of the Mandelstam variables. To provide the possibility of constructing loops and carrying out the renormalization procedure, this function necessarily should be polynomially bounded in \( \nu_x \) at \( x = 0 \) and, by continuity, in the small vicinity of this value.

With the help of definitions (8), one can rewrite the formal series (9) in three different forms:

\[ M(s, t, u) = \sum_{i=0}^{\infty} \alpha_i(s) s^i + \sum_{R_t} \frac{\rho^{(st)}(s)}{(\sigma - 2M_R^2) - s + \nu_s} + \sum_{R_u} \frac{\rho^{(su)}(s)}{(\sigma - 2M_R^2) - s - \nu_u}; \]  

(10)

\[ M(s, t, u) = \sum_{i=0}^{\infty} \alpha_i(t) t^i + \sum_{R_u} \frac{\rho^{(ut)}(t)}{(\sigma - 2M_R^2) - t + \nu_u} + \sum_{R_s} \frac{\rho^{(tu)}(t)}{(\sigma - 2M_R^2) - t - \nu_t}; \]  

(11)

\[ M(s, t, u) = \sum_{i=0}^{\infty} \alpha_i(u) u^i + \sum_{R_u} \frac{\rho^{(us)}(u)}{(\sigma - 2M_R^2) - u + \nu_u} + \sum_{R_t} \frac{\rho^{(ut)}(u)}{(\sigma - 2M_R^2) - u - \nu_t}. \]  

(12)

Here \( \sigma \) stands for the sum of squares of the external particle masses. The right hand sides of the expressions (10) – (12) are written in terms of the natural coordinate systems in the corresponding layers

\[ B_x\{ x \in \mathbb{R}, x \sim 0; \nu_x \in \mathbb{C}, |\nu_x| < \infty \}, x = (s, t, u). \]

(13)

This form of notations is convenient for the constructive formulating of the analyticity conditions. Notice, that each pair of layers (13) has a nonempty intersection:

\[ D_s = B_t \cap B_u, \quad D_t = B_u \cap B_s, \quad D_u = B_s \cap B_t \]

(14)

One can convince himself that this is the case when considering the loop graph as the integral of the product of a tree graph by a relevant propagator. The integral is to be taken at zero momentum transfer between the legs that are being closed [10].
(for example, \( t, u \sim 0 \) in \( D_s \), etc.). Hence the summability conditions in every layer should be adjusted in such a way that in the domain

\[
R = B_s \cup B_t \cup B_u
\]  
(15)

they define the unique meromorphic function. The system of bootstrap equations is an algebraic form of these matching conditions.

Taking into account the quoted above general considerations we formulate the analyticity conditions for the amplitudes of binary processes as follows\(^8\) \([7], [10]\). \textit{The tree level amplitudes must be meromorphic functions in each pair energy} \( s_{ij} \in \mathbb{C} \) \textit{at arbitrary fixed value of the second independent variable}. \textit{In every layer} \([13]\), \textit{containing the zero value hyperplane of one of the momentum transfers} \( x \), \textit{they must be polynomially bounded functions of the corresponding variable} \( \nu_x \). \textit{The bounding polynomial degree} \( N \) \textit{may depend on the quantum numbers characterizing the process}.

This formulation of analyticity conditions might seem unnecessarily complicated, especially, if one takes into account that the domain \((15)\) is only a part of the full complex space of two variables describing the process. We use it because of two reasons. First, the results of the papers \([6], [7] \) and \([10]\) show that it leads to reasonable physical consequences. Second, even in the case under consideration (binary processes) the corresponding systems of bootstrap equations for the minimal parameters of the extended theory turn out to be very complicated. In this case an imprudent attempt to formulate more general requirements without sufficient physical and mathematical motivation could lead to inconsistency.

The examples that we analyze in the following sections illustrate the structure and techniques of derivation of bootstrap equations. However, before starting their consideration we would like to make a short review of the Cauchy form method.

5 \textbf{The Cauchy forms}

We are going to use the method (known from the complex analysis; see, e.g., \([15]\)), which allows one to present the polynomially bounded meromorphic function of one complex variable as a uniformly converging series of pole contributions (in what follows we call such representations as the Cauchy forms or Cauchy expansions). The possibility to work in the layer

\[
B_x\{ x \in \mathbb{R}, \ x \in (a, b); \ z \in \mathbb{C}, \ |z| < \infty \}
\]  
(16)

(not only in the plane \( x = \text{const} \)) is provided by the natural modification of the method (see \([6], [7]\)): all the coefficients are considered to be smooth (real-analytic) functions of the parameter \( x \).

First of all we need to specify the definition of the bounding polynomial degree — this turns out to be important for the analysis of effective theories. We suppose that the reader is familiar with the notion of the system of contours \( C_n \), that appear in the definition of the

\(^8\)Let us emphasize that here we only discuss the scalar amplitudes (the functions \( F^{(a)} \) in \([5]\))
polynomially bounded meromorphic function of one complex variable (see [15]). When we work in the layer (16) we assume the smooth dependence of this system on the parameter $x$. The meromorphic function $f(x, z)$ defined in the layer (16) is called as polynomially bounded with the degree $N$ (or, simply $N$-bounded), if $N$ is the minimal integer such that for all $x \in (a, b)$

$$\left| \frac{f(x, z)}{z^{N+1}} \right|_{z \to C} \xrightarrow{m \to \infty} 0.$$  

(17)

The Cauchy form allows one to present the $N$-bounded in the layer (16) function $f(x, z)$ as the uniformly converging series of pole contributions. In the case, most interesting for our further purposes, when all the poles are simple and there is no pole at $z = 0$, it looks as follows:

$$f(x, z) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(x, 0) z^n + \sum_{i=1}^{\infty} \left\{ \frac{r_i(x)}{z - p_i(x)} - h_i^{(N)}(x, z) \right\}.$$  

(18)

Here $p_i(x)$ and $r_i(x)$ stand for the position of $i$-th pole and the corresponding residue. The poles are numbered such that $|p_i(x)| \leq |p_{i+1}(x)|$.

The correcting polynomials $h_i^{(N)}(x, z)$, ensuring the convergence of the series look as follows:

$$h_i^{(N)}(x, z) \equiv - \frac{r_i(x)}{p_i(x)} \sum_{n=0}^{N} \left( \frac{z}{p_i(x)} \right)^n \equiv \sum_{n=0}^{N} h_{i,n}(x) z^n.$$  

(19)

It is not difficult to show [7], that for the $N-$bounded function $f(x, z)$, represented by (18) in the layer (16), certain “collapsing” conditions are valid: the correcting polynomial degrees of order higher than $N$ converge themselves to the values of appropriate derivatives:

$$\sum_{i=1}^{\infty} h_{i,N+k}(x) = \frac{1}{(N+k)!} f^{(N+k)}(x, 0), \quad x \in (a, b) \quad k = 1, 2, \ldots.$$  

(20)

So, if in (18) one uses some $M > N$ instead of $N$ (thus eq. 17 holds), the Cauchy expansion is still correct but can be reduced to the one with $N$: the superfluous degrees of correcting polynomials just cancel higher order terms in the first sum of (18). These conditions help us to puzzle out the system of bootstrap equations; the corresponding example is given below.

The form (18) is the main tool used in [7], [10] to derive the bootstrap equations.

6 Bootstrap equations: a simple example

Let us consider the simple example to illuminate the general scheme discussed in the previous sections. It will allow us to show explicitly how to obtain the bootstrap equations for the parameters of a rational function of two variables, restricted by the corresponding
analyticity conditions. This example obtains an explicit solution and makes the terminology more transparent.

Consider the rational function of two complex variables $F(x, y)$. Let’s demand (this is an analog of analyticity requirements) that in the layer

$$B_y \{ x \in \mathbb{C}; \ y \in \mathbb{R}, \ y \in (-\eta, +\eta) \}$$

(21)
it has the single pole (in $x$), and in the layer

$$B_x \{ y \in \mathbb{C}; \ x \in \mathbb{R}, \ x \in (-\xi, +\xi) \}$$

(22)
also a single pole (in $y$). Asymptotics is considered to be decreasing in each layer (in terms of section 5 this function is 0-bounded in each layer). The question that we are trying to answer is: what is the structure of the set of the essential parameters describing this function?

In this case the essential parameters are just the coefficients $f_{ij}$ of the expansion

$$F(x, y) = \sum_{i,j=1}^{\infty} f_{ij} x^i y^j.$$  

(23)
The posed above question can be phrased in a more concrete way: how many independent combinations can be fixed arbitrarily and what are these combinations? Or, in terms of field theory: how many independent renormalization prescriptions is it necessary to impose in order to fix the amplitude $F(x, y)$ in the unique way, and what is the explicit form of those prescriptions?

In the layer (21) $F(x, y)$ can be represented as follows:

$$F(x, y) = \frac{\rho(y)}{x - \pi(y)}, \quad (x, y) \in B_y.$$  

(24)
The functions $\rho(y)$ and $\pi(y)$ are considered to be smooth in the vicinity of the origin:

$$\pi(y) = \sum_{i=0}^{\infty} \pi_i y^i, \quad \rho(y) = \sum_{i=0}^{\infty} \rho_i y^i.$$  

(25)
By analogy, in the layer (22):

$$F(x, y) = \frac{r(x)}{y - p(x)}, \quad (x, y) \in B_x,$$

(26)
where

$$p(x) = \sum_{i=0}^{\infty} p_i x^i, \quad r(x) = \sum_{i=0}^{\infty} r_i x^i.$$  

(27)
In the intersection domain $B_x \cap B_y \equiv D_{xy}$ we obtain:

$$\frac{r(x)}{y - p(x)} = \frac{\rho(y)}{x - \pi(y)}, \quad (x, y) \in D_{xy} \{ x \in (-\xi, +\xi), \ y \in (-\eta, +\eta) \}.$$  

(28)
Substituting (25) and (27) into (28), we obtain an infinite system of conditions on the coefficients $p_k, r_k, \pi_k, \rho_k$:

$$r_{i+1}\pi_0 - p_{i+1}\rho_0 = r_i, \quad \rho_{i+1}p_0 - \pi_{i+1}r_0 = \rho_i, \quad r_{i+1}\rho_{j+1} = \rho_{i+1}\pi_{j+1} \quad i, j = 0, 1, ... \ (29)$$

This system provides an example of what is called in [7] as the bootstrap equations. Once solved, it permits to express the parameters $p_i, r_i$ in terms of $\pi_i, \rho_i$. It also gives an answer to the question if it is possible to carry out the analytic continuation from one layer to another. This is an infinite system of equations with respect to $2 \times \infty$ (formal notation!) unknown parameters, which we need to reexpress the function $F(x, y)$ in the layer (22) in terms of the parameters defining it in the layer (21). In general, it is very difficult to find the solutions of such systems and even to show solvability. Fortunately, in this simple example it turns out possible to give the explicit form of the solution. This exercise is really useful because it gives an idea of the “power” of bootstrap restrictions.

After we rewrite (28) as

$$r(x)\{x - \pi(y)\} = \rho(y)\{y - p(x)\} \ , \quad (30)$$

take the derivatives $\partial^2_{x y}$, and separate the variables, we obtain:

$$\frac{r'(x)}{p'(x)} = \frac{\rho'(y)}{\pi'(y)} \equiv a \ , \quad (31)$$

where primes mean the derivative with respect to the corresponding variable, and $a$ is the separation parameter. From (31) we obtain

$$r(x) = ap(x) + b \ , \quad \rho(y) = a\pi(y) + c \ , \quad (32)$$

where $b$ and $c$ – new constants. Finally, substituting (32) into (30) and separating the variables once more, we find

$$p(x) = \frac{d - bx}{c + ax} \ , \quad \pi(y) = \frac{d - cy}{b + ay} \ , \quad (33)$$

were $d$ – another separation parameter. The formulae (33) together with (32), (21) and (26) give the exhaustive solution to the problem in question (it is easy to check that the exceptional cases provide us nothing). The important property of this solution is that it contains only 4 arbitrary parameters! This means that the infinite system (29) only turns out to be consistent if the function $F(x, y)$ defined in the layer (21) belongs to the four-parametric family

$$F(x, y) = \frac{ad + bc}{-d + axy + bx + cy} \ . \quad (34)$$

This is the only case when there exists the analytic continuation of this function from $B_y$ into $B_x$ with the desired properties. It is clear that in this case this continuation is unique.
The direct analysis of the system (29) would lead to the same conclusion. It turns out possible in this simple example. Unfortunately, the regular method of solving the infinite-dimension algebraic systems is not known, except several trivial cases.

With the help of (34), one can express the essential parameters

\[ f_{ij} = f_{ij}(a, b, c, d) \]

in terms of “fundamental constants” \((a, b, c, d)\). Then one can choose four arbitrary coefficients \(f_k (k = 1, 2, 3, 4)\), that allow the inversion

\[ a = a(f_1, \ldots, f_4), \ldots, \quad d = d(f_1, \ldots, f_4), \]

and impose arbitrary “renormalization conditions” for these four quantities. The renormalization of all other essential parameters should respect the conditions (29).

Thus, now we can answer the question posed in the beginning of this section. To fix the amplitude \(F(x, y)\) uniquely it is sufficient to impose four renormalization prescriptions fixing the “fundamental” constants \(a, b, c, d\).

This example explains the prudence with which we have formulated the analyticity conditions in section (4). If, in addition to these conditions, one would impose supplementary analyticity conditions (for example, in the layer

\[ C_x\{y \in \mathbb{C}; \quad x \in \mathbb{R}, \quad x \in (1 - \xi, 1 + \xi)\} \]

with arbitrary number of poles and arbitrary asymptotic behavior in this layer) then, except the lucky chance, he would fall in a contradiction.

It is interesting to note that if we modify the problem and demand that the function \(F(x, y)\) has one pole in the layer \(B_x\), as in the previous case, but is 1-bounded (in place of 0-bounded) in this layer, we would obtain a solution that depends also on 4 parameters. This solution, however, will be found among the exceptional cases.

### 7 Cauchy forms for the string amplitude

The example, considered in section (6), was too simple, and the method that was applied to solve it could hardly be useful in the case of effective theories where the number of poles is infinite. In this section we will show how to obtain the bootstrap conditions for the function with infinite number of poles. For this we use the techniques of Cauchy forms. Of course, we are not able to show the explicit solution of those conditions. However, we will show that even in the case when the function \(F(x, y)\) is given explicitly (i.e. the set of minimal parameters is known) the bootstrap conditions can be used as a source of non-trivial relations connecting these parameters with each other. This very property was used in [7], [10] to obtain the restrictions on the physical characteristics of pion-kaon and pion-nucleon scattering processes.

\(^9\)Or four arbitrary combinations.
As an illustrative example we have chosen the Euler $B$-function (or, to be more precise, the so-called Lovelace amplitude [16], which differs by a factor). This choice is explained by several reasons\(^{10}\). First of all it is easy to follow the details of calculations, because all the necessary identities are widely known. Second, though there is known a great number of summation formulae for Pochhammer symbols, we obtain (using a very simple and extremely elegant method) an infinite sequence of identities that could hardly be deduced with the help of traditional methods. Third, Euler’s $B$-function plays an important role in dual models and in string theory (see [9]). That is why our choice is justified from the physical point of view. Finally, the last argument in favor of our choice is that the numerical test of the corresponding bootstrap relations allows us to understand qualitatively the structure of the criteria that are necessary to evaluate the rapidity of convergence. This point becomes very important when one tries to compare various theoretical predictions (sum rules) with the experimental data.

Let us consider the simple (string-like) model for the scattering amplitude that is constructed – in accordance with idea of Veneziano [17] – out of $B$-function without a tachyon:

$$A(s, t) = (-s - t)B(\frac{1}{2} - s, \frac{1}{2} - t) = \frac{\Gamma(\frac{1}{2} - s)\Gamma(\frac{1}{2} - t)}{\Gamma(-s - t)}. \tag{35}$$

It has the following specific points (hyperplanes) \((m, n = 0, 1, 2, ...):\)

- **Zero hyperplanes**: \(s + t = n.\)
- **Pole hyperplanes in \(s\) (\(t\) fixed, \(s + t \neq m\)): \(s = \frac{1}{2} + n.\)
- **Pole hyperplanes in \(t\) (\(s\) fixed, \(s + t \neq m\)): \(t = \frac{1}{2} + n.\)
- Three series of ambiguity points located at the intersections of the zero hyperplanes with the hyperplanes of poles in any variable. They have the following coordinates\(^{11}\.\)
  - Series \(A^{++}\): \(s = +\frac{2m+1}{2}, \ t = +\frac{2n+1}{2};\)
  - Series \(A^{+-}\): \(s = +\frac{2m+1}{2}, \ t = -\frac{2n+1}{2}, \ (m \geq n).\)
  - Series \(A^{-+}\): \(s = -\frac{2m+1}{2}, \ t = +\frac{2n+1}{2}, \ (m \leq n).\)

Let us consider the behavior of the amplitude \(A(s, t)\) in the layers \(B_t\{t \in \mathbb{R}, \ s \in \mathbb{C}, \ |s| < \infty\}\) with \(t \neq k + 1/2,\) were \(k\) – integer. The only singularities of the amplitude in such layers are the poles in variable \(s.\)

Notice that, starting from some \(n,\) there is always a zero between the two poles of \(A(s, t).\) For the contours \(C_n\) on the complex plane \(s\) we have chosen the system of circles (with the center at the coordinate origin) passing through zeroes of the amplitude. It could be shown that everywhere on this system of contours, except the narrow sector in the vicinity of the real positive axis, the amplitude \(A(s, t)\) has the Regge type asymptotics \((\sim s^{\frac{1}{2}+t}).\) In the vicinity of real axis the asymptotics is controlled by the presence of zero.

\(^{10}\)We also discussed this function in a slightly different context in [7], Sec. 4.

\(^{11}\)One can find the corresponding plot in [7].
In the terminology of Sec. 5 in the layers\textsuperscript{12}

\[ B_t \{ t \in (n - 1/2, n + 1/2); \ n = 0, 1, \ldots \} \]

the amplitude \( A(s, t) \) is the \( n \)-bounded function of the complex variable \( s \) and of one real parameter \( t \).

In the layers

\[ B_t \{ t \in (-n - 1/2, -n + 1/2); \ n = 1, 2, \ldots \} \]

it has a decreasing asymptotics.

Residues of \( A(s, t) \) at the poles in \( s \) are the same in all the layers \( B_t \), because the pole positions do not depend on \( t \). In the case when the point under consideration is not the ambiguity one, we have

\[ r_n(t) \equiv \text{Res}_{s=n+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} - s\right)\Gamma\left(\frac{1}{2} - t\right)}{\Gamma(-s - t)} = \frac{1}{n!} \left(\frac{1}{2} + t\right) \cdots \left(\frac{1}{2} + t + n\right) \equiv \frac{1}{n!} \left(t + \frac{1}{2}\right)_{(n+1)}, \quad (36) \]

were \( \left(t + \frac{1}{2}\right)_{(n+1)} \) stands for the so-called Pochhammer symbol (shifted factorial).

For example, let us construct the Cauchy expansion of \( A(s, t) \) when \( t \in (-3/2, -1/2) \). In this layer \( A(s, t) \) grows not faster then \( s^{0-\epsilon}, \ \epsilon > 0 \), and there is no need in correcting polynomials. Here the Cauchy expansion looks as follows:

\[ A(s, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(t + \frac{1}{2})_{(n+1)}}{(s - n - \frac{1}{2})}, \quad t \in (-3/2, -1/2). \quad (37) \]

Notice, that because the asymptotic becomes “softer” at large negative \( t \), this expansion is also valid at every \( t < -1/2 \) (except the values corresponding to the coordinates of ambiguity points \( t = -(2k + 1)/2 \) \( (k = 0, 1, \ldots) \), where the expansion makes no sense\textsuperscript{13}).

In the layer \( B_t \{ t \in (-1/2, 1/2) \} \) the amplitude \( A(s, t) \) grows slower than a linear function of \( s \), and thus in our Cauchy expansion we have to account for the correcting polynomials of \( 0- \)th degree. Thus we obtain the following expansion:

\[ A(s, t) = A(0, t) + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(t + \frac{1}{2})_{(n+1)}}{s - n - \frac{1}{2}} + \frac{(t + \frac{1}{2})_{(n+1)}}{n + \frac{1}{2}}\right), \quad t \in (-1/2, 1/2). \quad (38) \]

The techniques of the Cauchy forms allows us to represent the meromorphic function of two complex variables as a converging series of pole (in one variable) contributions; the convergence being uniform in both variables. This will give us a possibility to obtain two types of conditions on \( A(s, t) \): the collapse conditions of superfluous degrees of correcting polynomials (see \textsuperscript{20}), and the bootstrap equations.

\textsuperscript{12}When treating the example with \( B \)-function we use the natural shortened notations for the layers.

\textsuperscript{13}In what follows we do not mention this condition.
8 Collapse conditions and bootstrap for Pochhammer symbols.

The collapse conditions (20) on the regular part of the amplitude appear when we pass from the layer where the amplitude has an increasing asymptotics to another one, where the asymptotic regime is weaker. The expansion (38) for $A(s, t)$ in the layer $B_t \{ t \in (-1/2, 1/2) \}$ is also valid for $t < -1/2$. When $t$ crosses the boundary value $t = -1/2$, corresponding to the change of asymptotic regime, the series of correcting polynomials can be summed independently:

$$\sum_{n=0}^{\infty} h_n^{[0]}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{(t + \frac{1}{2})(n+1)}{n + \frac{1}{2}} \right) = - \frac{\Gamma(\frac{1}{2} - t)\Gamma(\frac{1}{2})}{\Gamma(-t)} = -A(0, t), \quad t < -\frac{1}{2},$$

and the expansion (38) coincides with (37).

The bootstrap equations arise naturally from the requirement that the Cauchy expansion in one variable in some layer should coincide with the expansion in the cross-conjugated variable (i.e. in the perpendicular layer) in the domain of intersection of these two layers. For example, the expansion (38) is valid for the amplitude $A(s, t)$ in the layer $B_t \{ t \in (-1/2, 1/2) \}$. A similar expansion can be written in the layer $B_s \{ s \in (-1/2, 1/2) \}$:

$$A(s, t) = A(s, 0) + \sum_{n=0}^{\infty} \left( \frac{\rho_n(s)}{t-n-\frac{1}{2}} + \frac{\rho_n(s)}{n+\frac{1}{2}} \right), \quad (s \sim 0, t \sim 0) \quad (39)$$

where $\rho_n(s) \equiv \frac{1}{n!} \left( s + \frac{1}{2} \right)_{(n+1)}$. These two expansions should coincide in the square formed by the intersection of two layers. Hence in the domain $(s \sim 0, t \sim 0)$ the following condition must be valid:

$$A(0, t) = A(s, 0) + \sum_{n=0}^{\infty} \left( \frac{\rho_n(s)}{t-n-\frac{1}{2}} + \frac{\rho_n(s)}{n+\frac{1}{2}} \right) - \sum_{n=0}^{\infty} \left( \frac{r_n(t)}{s-n-\frac{1}{2}} + \frac{r_n(t)}{n+\frac{1}{2}} \right) \equiv A(s, 0) + \Psi(s, t). \quad (40)$$

In some vicinity of the point $(0, 0)$ the function $\Psi(s, t)$ is analytic because the corresponding series converge uniformly; so it is completely determined by the coefficients of its Taylor expansion at this point. Let us differentiate both parts of the equation (40) with respect to $t$:

$$\frac{\partial A(0, t)}{\partial t} = \frac{\partial \Psi(s, t)}{\partial t}, \quad (s \sim 0, \ t \sim 0).$$

The left hand side of this equality only depends on one variable $t$. This means that the dependence of the right hand side on the second variable is purely fictitious. So one can assign to $s$ any arbitrary value from the domain $s \sim 0$ to compute the $\frac{\partial \Psi(s, t)}{\partial t}$. This allows us to determine the regular part of the amplitude up to one arbitrary constant $A(0, 0)$.

These considerations allow us to rewrite (40) in the form of two conditions on the regular part of the amplitude plus an infinite system of consistency conditions:

$$\frac{\partial A(0, t)}{\partial t} = \frac{\partial \Psi(s, t)}{\partial t} \big|_{s=0}, \quad (t \sim 0) \quad (41)$$
\[
\frac{\partial A(s, 0)}{\partial s} = -\frac{\partial \Psi(s, t)}{\partial s} \bigg|_{t=0}, \quad (s \sim 0)
\]  \hspace{1cm} (42)

\[
\frac{\partial^{k+p+2}}{\partial s^{k+1} \partial t^{p+1}} \Psi(s, t) \big|_{s=0, t=0} = 0, \quad \forall \ k, p = 0, 1, ... .
\]  \hspace{1cm} (43)

The consistency conditions express the fact that, in some vicinity of the point (0, 0), the derivative of \(\Psi(s, t)\) with respect to any variable does not depend on the cross-conjugated variable.

Notice, that in this example the full symmetry between the variables \(s\) and \(t\) allows us to limit our analysis of consistency conditions to the case \(k > p\).

Such systems of conditions are called as bootstrap equations. They represent nontrivial relations between the resonance parameters (pole positions and residue values) of the function under consideration. In the present example the pole position does not depend on the cross-channel variable. In this case the system of bootstrap equations leads to an infinite set of relations for the values of residues (Pochhammer symbols).

For example, let us consider the identity for the Pochhammer symbols, following from (43) with \(k = 1, p = 0\):

\[
\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \rho_n^{(2)}(s)}{(t - n - \frac{1}{2})^2} - \sum_{n=0}^{\infty} \frac{(-1)^n 2! \ r_n^{(1)}(s)}{(s - n - \frac{1}{2})^3} \right\} \bigg|_{s=0, t=0} = 0.
\]  \hspace{1cm} (44)

One can easily show, that the following equalities are valid for the arbitrary order derivative of the residue:

\[
r_n^{(p)}(t) = 0, \quad (p > n + 1);
\]

\[
r_n^{(p)}(t) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1-1} \ldots \sum_{i_p=0}^{i_{p-1}-1} \frac{p! \ r_n(t)}{(\frac{1}{2} + i_1 + t)...(\frac{1}{2} + i_p + t)}, \quad (p \leq n + 1).
\]

This allows us to rewrite (44) in the following way:

\[
\sum_{n=0}^{\infty} \left\{ \frac{1}{(n + \frac{1}{2})^3} \sum_{i_1=0}^{n} \frac{1}{n!} \frac{(\frac{1}{2})(n+1)}{\left(\frac{1}{2} + i_1\right)} \right\} - \sum_{n=1}^{\infty} \left\{ \frac{1}{(n + \frac{1}{2})^2} \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1-1} \frac{1}{n!} \frac{(\frac{1}{2})(n+1)}{\left(\frac{1}{2} + j_1\right)(\frac{1}{2} + j_2)} \right\} = 0. \]  \hspace{1cm} (45)

One can observe, that already the first consistency condition provides us with a highly non-trivial identity for the Pochhammer symbols. The subsequent conditions lead to even more complicated identities. These identities mirror the special properties of residues, which ensure the existence of the solution of the bootstrap system. And we know it exist - the solution is the Pochhammer symbols themselves, that is why the relations above hold.
9 Bootstrap system is overdetermined

It is evident that the system of bootstrap equations is most probably badly overdetermined. And the question that arises immediately is: how to pick out a full subsystem or to tell what equations are evidently unnecessary (related)?

In addition to the system of bootstrap equations we have at our disposal the collapse conditions of superfluous correcting polynomials in the layers with softer asymptotic behavior. Now we will show how with the help of collapse conditions and bootstrap in one layer it is possible to obtain some of the bootstrap conditions in another layer.

In the three intersecting layers: $B_t\{t \in (-1/2, 1/2)\}$, $B_t\{t \in (-3/2, -1/2)\}$ and $B_s\{s \in (-3/2, -1/2)\}$ the following Cauchy-expansions for the amplitude $A(s, t)$ are valid:

$$A(s, t) = A(0, t) + \sum_{n=0}^{\infty} \left( \frac{r_n(t)}{s - n - 1/2} + \frac{r_n(t)}{n + 1/2} \right), \quad B_t\{t \in (-1/2, 1/2)\};$$  \hspace{1cm} (46)

$$A(s, t) = \sum_{n=0}^{\infty} \frac{r_n(t)}{s - n - 1/2}, \quad B_t\{t \in (-3/2, -1/2)\};$$  \hspace{1cm} (47)

$$A(s, t) = \sum_{n=0}^{\infty} \frac{\rho_n(s)}{t - n - 1/2}, \quad B_s\{s \in (-3/2, -1/2)\}. \hspace{1cm} (48)$$

We demand that the corresponding expansions in the intersection domains of each of two layers $B_t$ with the layer $B_s$ should represent the same meromorphic function. This allows us to obtain the bootstrap conditions

$$A(0, t) = \sum_{n=0}^{\infty} \frac{\rho_n(s)}{t - n - 1/2} - \sum_{n=0}^{\infty} \left( \frac{r_n(t)}{s - n - 1/2} + \frac{r_n(t)}{n + 1/2} \right) \equiv \Psi_1(s, t), \quad (s \sim -1, \ t \sim 0). \hspace{1cm} (49)$$

$$0 = \sum_{n=0}^{\infty} \frac{\rho_n(s)}{t - n - 1/2} - \sum_{n=0}^{\infty} \frac{r_n(t)}{s - n - 1/2} \equiv \Psi_2(s, t), \quad (s \sim -1, \ t \sim -1). \hspace{1cm} (50)$$

Using the same argumentation as in the previous section, we rewrite (49) as the system of the following conditions:

$$A(0, t) = \Psi_1(-1, t),$$

$$\left. \frac{\partial^{k+p+1}}{\partial s^{k+1} \partial t^p} \Psi_1(s, t) \right|_{s=-1, \ t=0} = 0, \quad \forall \ k, p = 0, 1, ... .$$

The first one of them gives the explicit expression for the regular part of the amplitude, while the second mirrors the independence of $\Psi_1(s, t)$ of the argument $s$.

The condition (50) could be rewritten in the similar form:

$$\left. \frac{\partial^{k+p}}{\partial s^k \partial t^p} \Psi_2(s, t) \right|_{s=-1, \ t=-1} = 0, \quad \forall \ k, p = 0, 1, ... . \hspace{1cm} (51)$$
The explicit expression for $A(0, t)$, which is valid in the vicinity of $t = 0$, could be without any obstruction analytically continued to the domain of negative $t$, where the corresponding series converge very well.

The expansion is also valid at $t < -\frac{1}{2}$. From the requirement that in this case it should coincide with (17), we obtain the collapse condition:

$$A(0, t) = -\sum_{n=0}^{\infty} \frac{r_n(t)}{n + \frac{1}{2}}, \quad t < -\frac{1}{2}.$$  

This expression for the regular part of the amplitude should coincide with the one obtained from (19) by the analytic continuation to the domain $t < -\frac{1}{2}$. In particular this means that

$$\Psi_1(-1, t) = -\sum_{n=0}^{\infty} \frac{r_n(t)}{n + \frac{1}{2}}, \quad (s \sim -1, \ t \sim -1).$$

As usual, let us rewrite this as a condition on the coefficients of power series expansion in the vicinity of $(s = -1, \ t = -1)$:

$$\frac{\partial^p}{\partial t^p} \Psi_1(-1, t) \bigg|_{t=-1} = -\sum_{n=0}^{\infty} \frac{r_n^{(p)}(t)}{n + \frac{1}{2}} \bigg|_{t=-1}.$$  

Using the explicit expression for $\Psi_1$, we obtain:

$$\left\{ \sum_{n=0}^{\infty} \frac{p!}{(t - n + \frac{1}{2})^{p+1}} \rho_n(-1) - \sum_{n=0}^{\infty} \left( \frac{r_n^{(p)}(t)}{1 - n - \frac{1}{2}} + \frac{r_n^{(p)}(t)}{n + \frac{1}{2}} \right) \right\} \bigg|_{t=-1} = -\sum_{n=0}^{\infty} \frac{r_n^{(p)}(t)}{n + \frac{1}{2}} \bigg|_{t=-1}.$$  

After collecting the similar terms one will find out that this condition coincides with (51) for all $p$ when $k = 0$.

Thus the example of three layers strengthens our confidence that the system consisting of bootstrap equations and collapse conditions is overdetermined. A share of information about bootstrap in lower layers is contained in the bootstrap equations in upper layers and also in the collapse conditions for transitions from upper to the lower layers.

### 10 Numerical test of the convergence rapidity

The numerical test of the convergence rapidity of the series (45) could be of great interest. This is because to check the theoretical predictions of the dual models one often saturates the quite similar to (45) identities (the so-called sum rules) with a finite number of resonances.

Unfortunately the up-to-date information on the hadron spectrum is far from being exhaustive (especially in the region $M > 2$ GeV). This means that only those sum rules that converge sufficiently rapidly could undergo the experimental verification. That is why it would be extremely instructive to learn how to pick these identities out of the
infinite system of bootstrap and collapse conditions. In this section, by way of treating the example of the string amplitude discussed in the Sec. 7, we suggest a possible approach to this problem in the realistic situation.

Let us carry out the numerical test of the system of identities for residues $r_n(t)$ and $\rho_n(s)$, obtained from the consistency conditions (43):

\[
\left\{ \sum_{n=0}^{\infty} \frac{(-1)^{p+1}(p+1)! \rho_n^{(k+1)}(s)}{(t-n-\frac{1}{2})^{p+2}} \right\}_{s=0} - \left\{ \sum_{n=0}^{\infty} \frac{(-1)^{k+1}(k+1)! r_n^{(p+1)}(s)}{(s-n-\frac{1}{2})^{k+2}} \right\}_{s=0} = 0, \quad (52)
\]

where $k, p = 0, 1, ..., k > p$. The condition $k > p$ originates from the symmetry of spectrum in $s$ and $t$.

First of all we need to define a quantity that would allow us to characterize the precision of saturation of the sum rule (52) after one takes into account the finite number of items. This could be done in the standard way, but in the current example the procedure of calculation could be sufficiently facilitated. However this point needs some comments because this is not always possible in the realistic situations encountered in the field theory.

In the expression (52) we deal with the difference of two absolutely convergent numerical series. Taking into account the symmetry of spectrum, it looks natural to consider the difference of the contributions from the $t$- and $s$-channel poles at every step of the computation. For the few first poles this contribution has the definite sign (positive in the case $k > p$) but, starting from some number $N_+(k, p)$, depending on $k$ and $p$, the sign of this difference changes. Thus the convergence of this series to zero is provided by the negative contribution of the large number of distant poles. It compensates gradually the positive contribution of the first few poles. As the convergence characteristics we chose the ratio:

\[
D \equiv \frac{\Delta S(N)}{S_+},
\]

where $\Delta S(N)$ is the discrepancy that remains after one considers $2N$ poles ($N$ in the $s$-channel and $N$ in the $t$-channel), and $S_+$ – is the sum of all positive contributions (that correspond to the finite number of initial terms of the series of differences)\(^{14}\). With the help of $D$ we can describe the convergence rapidity of the series (52).

It is sufficient to consider a small number of poles to reduce significantly the relative discrepancy in the rapidly converging sum rules.

The dependence of the relative discrepancy on the number of poles taken into account for three first sum rules from the system (52) is shown on the picture.

\(^{14}\)Of course it only makes sense for $N > N_+$.
The sum rule with $k = 2, p = 1$ converges sufficiently fast. After one takes into account 100 first poles\textsuperscript{15} the relative discrepancy equals approximately 8\% (1\% accuracy could be attained after accounting for 700 poles). Thus from this condition we could obtain a sufficiently good relation between the residues in the first 100 poles. The identities with $k = 1, p = 0$ and $k = 2, p = 0$ do not suite for this purpose. Sum rule with $k = 1, p = 0$ converges much slower: the consideration of 3000 terms gives 18\% discrepancy; and to reduce it to 9\%, one needs to take account of more than 22000 terms. The sum rule with $k = 2, p = 0$ converges even more slowly.

The sum rules with best convergence are those with $p = k - 1$ at large values of $k$. One can expect that for large $k$ these sum rules would be saturated rapidly.

It should be taken into account that $r_n^{(p)}(t) = 0$ for $p > n + 1$, hence the first poles do not contribute. Thus these sum rules could serve as a source of relatively precise relations between the parameters of several resonances with $n > p$.

The considered example allows one to understand in a qualitative way the main properties of the constructions arising from the bootstrap conditions. In the realistic situation only the parameters of the few lightest resonances are known. The more astonishing is that, as shown in\textsuperscript{6, 7} and\textsuperscript{10}, some of the bootstrap restrictions are well saturated by the available experimental data and provide the theoretical explanation to some phenomenological relations. This circumstance leads to the idea that the $B$-function gives a reasonable description only for the “tail” of the resonance spectrum. The parameters of the lower states are governed mostly by the dynamical properties such as chiral symmetry.

\section{Conclusion}

\textsuperscript{15}In each channel.
The examples considered above show that the method of Cauchy forms is a useful tool for deriving the relations between the parameters of the polynomially bounded meromorphic functions of two complex variables. It is easy to understand that, after the corresponding formulation of the analyticity conditions, this method could be in principle applied to the case of more variables. But the practical advantage of this approach to the study of inelastic amplitudes could hardly be notable. Even in the case of the simplest inelastic process \(2 \rightarrow 3\) one needs 5 independent variables. This leads to extremely bulky expressions. The more powerful techniques is necessary that would allow us to compactify the notations.

The case of binary processes is interesting because it allows one to obtain the relations between the spectrum parameters following from the correctness requirements of the perturbative scheme of the \(S\)-matrix calculation. As we already mentioned, many of those relations happen to be in excellent agreement with the experimental data. This shows that even such a complicated construction as effective field theory could be successfully applied to the data analysis.

**Acknowledgements**

We are grateful to A. Andrianov, A. Vasiliev, M. Vyazovski, M. Polyakov, V. Sukhanov, H. Nielsen, V. Cheianov and J. Schechter for fruitful discussions of various problems associated with the concept of the effective field theory. This work was supported by INTAS (2000, project 587), RFBR (grant 01-02-17152), Ministry of Education of Russia (E00-3.3-208) by the program “Universities of Russia” (UR.02.01.001) and Meltzers Høyskolefond, Studentprosjektstipend 2002.

**References**

[1] S. Weinberg. Phys. Rev. **133**, B1318 (1964); *ibid* **134**, B882 (1964); *ibid* **135**, B1049 (1964); *ibid* **138**, B988 (1965); *ibid* **181**, 1893 (1969).

[2] Y. V. Novozhilov. *Introduction to the Theory of Elementary Particles.* (“Nauka”, Moscow, 1972.)

[3] J. B. Rumer, A. I. Fet. *Group Theory and Quantum Fields.* (“Nauka”, Moscow, 1977.) (in Russian)

[4] S. Weinberg. *The Quantum Theory of Fields.* (Cambridge University Press, Cambridge, 2000). vv. 1-3.

[5] S. Weinberg, Physica A **96**, 327 (1979).

[6] V. Vereshagin, Phys. Rev. D**55**, 5349 (1997).
[7] A. Vereshagin and V. Vereshagin, Phys. Rev. D\textbf{59}, 016002 (2000).

[8] J. C. Collins. \textit{Renormalization.}  
(Cambridge University Press, Cambridge, 1984).

[9] P. H. Frampton. \textit{Dual Resonance Models and Superstrings.}  
(World Scientific Publ., 1986.)

[10] A. Vereshagin, $\pi N$ Newsletter, \textbf{16}, 426 (2002).

[11] S. Weinberg, Phys. Rev. \textbf{130}, 776 (1963); \textit{ibid} \textbf{131}, 440 (1963);  
\textit{ibid} \textbf{133}, B232 (1964);  
M. Scadron and S. Weinberg, Phys. Rev. \textbf{133}, B1589 (1964);  
M. Scadron, S. Weinberg, and J. Wright, Phys. Rev. \textbf{135}, B202 (1964).

[12] M. L. Nekrasov, Preprint IHEP 2001-9; e-archive hep-ph/0102283 (2001).

[13] B. A. Kniehl and A. Sirlin, Phys. Letters, \textbf{B530}, 129 (2002).

[14] M. Veltman, Physica, \textbf{29}, 186 (1963).

[15] B. V. Shabat. \textit{Introduction to the complex analysis.} (“Nauka”, Moscow, 1969). (in Russian)

[16] C. Lovelace, Phys. Letters, \textbf{28B}, 264 (1968).

[17] G. Veneziano, Nuovo Cimento A \textbf{57}, 190 (1968).