Quantum $sl_n$ Toda field theories

L.Bonora, V.Bonservizi
International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2, 34014 Trieste, Italy
and INFN, Sezione di Trieste.

Abstract
We quantize $sl_n$ Toda field theories in a periodic lattice. We find the quantum exchange algebra in the diagonal monodromy (Bloch wave) basis in the case of the defining representation. In the $sl_3$ case we extend the analysis also to the second fundamental representation. We clarify, in particular, the relation of Jimbo and Rosso’s quantum $R$ matrix with the quantum $R$ matrix in the Bloch wave basis.
1 Introduction

This paper deals with the quantization of Toda field theories based on a finite Lie algebra in a periodic lattice.

The reason for the interest in Toda field theories is twofold: on the one hand these theories, which we recall are characterized by a W-algebra symmetry, underlie a vast set of conformal field theories, in particular the W minimal models [1], [2]; on the other hand they define the so-called W-gravities, generalizations of the Liouville theory that might relate to the most recent results from matrix models and topological gravity.

While the quantum $sl_2$ Toda theory, i.e. the Liouville theory, has been analyzed in a number of papers, the attempts to do the same for a general Toda field theory are fewer and much less complete.

Motivated by the renewed interest in Toda theories, in this paper we want to analyse in detail the quantization of the $sl_n$, and in particular the $sl_3$, Toda field theory. The main difficulty in a continuum context is that we have to regularize path-ordered exponentials, this being done usually by introducing a normal ordering. However the two orderings do not quite match. Here we overcome this difficulty by regularizing the theory with a lattice cutoff (as suggested by [3]).

The purpose of our paper is to derive the exchange algebra in the Bloch wave basis, a crucial intermediate step in the calculation of conformal blocks with diagonal monodromy. We follow the treatment of ref.[4] (see also [5]). I.e. we start discretizing the chiral and antichiral Drinfel’d-Sokolov linear systems associated with the Toda theory. In this way we introduce first the exchange algebra in the $\sigma$ basis. The exchange algebra in this basis is expressed in terms of the quantum $R$ matrix of Jimbo and Rosso [6]. At this point we are very close to the usual Coulomb gas derivation of conformal blocks (with non-diagonal monodromy). But instead of proceeding to the calculation of the conformal blocks, we diagonalize first the monodromy and pass from the $\sigma$ basis to the $\psi$ basis (Bloch wave basis) through an operator-valued matrix transformation. This passage guarantees not only periodicity but also locality in the form explained below. The exchange algebra in the $\psi$ basis is expressed in terms of an $R$ matrix whose entries are in general functions of zero mode operators. We will show that, up to a diagonal matrix transformation, this coincides with the $R$ matrix found in ref.[9].

As a bonus of our treatment we are able to answer a question raised in ref.[7] (see also [8]): what is the relation between the $R$ matrix of Drinfel’d, Jimbo and Rosso and the $R$ matrix that appear in the Bloch wave basis? The answer is: they coincide up to the transformation that allows us to pass from the $\sigma$ to the $\psi$ basis.

The paper is organized as follows. In section 1 we review the formulation of general classical Toda field theories both in the continuum and on the lattice. In section 3 we first review the recipe for quantizing Toda field theories on the lattice, found in ref.[4], then we apply it to the $sl_3$ Toda field theory in the fundamental representations, we find the quantum exchange algebra in the Bloch wave basis and verify periodicity and locality. Section 4 is devoted to a
comparison of our results with the exchange algebra obtained previously in ref. [2], [5] and [8]: the results coincide (up to normalization problems). In section 5 we extend the same quantization procedure to the case of the defining representation of $sl_n$.

2 Classical Toda Field Theories

2.1 In the continuum...

Let $G$ be a simple finite dimensional Lie algebra of rank $n$. We choose a Cartan subalgebra (CSA) with an orthonormal basis $\{H_i\}$. The Toda field equations are

$$\partial_{x^+} \partial_{x^-} \Phi = \frac{1}{2} \sum_{\alpha \text{ simple}} e^{2\alpha(\Phi)} H_\alpha,$$  \hspace{1cm} (1)

where $x^\pm = x \pm t$ and the $x$ coordinate lies on a circle; $\Phi$ is valued in the CSA and $H_\alpha = [E_\alpha, E_{-\alpha}]$, for any simple root $\alpha$. We showed in [10], [11] that the solutions of (1) can be obtained from the chirally split Drinfel’d-Sokolov linear systems [12],

$$\partial_{x^+} Q_+ - (P - \mathcal{E}_+) Q_+ = 0$$  \hspace{1cm} (2)

$$\partial_{x^-} Q_- + Q_- (\bar{P} - \mathcal{E}_-) = 0,$$  \hspace{1cm} (3)

where $P$ and $\bar{P}$ are periodic fields which take values in the CSA and

$$\mathcal{E}_+ = \sum_{\alpha \text{ simple}} E_\alpha,$$

$$\mathcal{E}_- = \sum_{\alpha \text{ simple}} E_{-\alpha}.$$

$P$ and $\bar{P}$ have the Poisson brackets

$$\{P(x) \otimes P(y)\} = -(\partial_x - \partial_y) \delta(x - y) t_0$$  \hspace{1cm} (4)

$$\{P(x) \otimes \bar{P}(y)\} = 0$$  \hspace{1cm} (5)

$$\{\bar{P}(x) \otimes \bar{P}(y)\} = (\partial_x - \partial_y) \delta(x - y) t_0,$$  \hspace{1cm} (6)

where

$$t_0 = \sum_i H_i \otimes H_i.$$
From the solution $Q_+(x)$ and $Q_-(x)$ of eqs. (2) and (3) normalized by $Q_+(0) = 1$, $Q_-(0) = 1$, and for any highest weight vector $|\Lambda^{(r)}\rangle$ of $\mathcal{G}$, we define a basis $\sigma$, $\bar{\sigma}$

$$\sigma^{(r)}(x) = \langle \Lambda^{(r)} | Q_+(x)$$

$$\bar{\sigma}^{(r)}(x) = Q_-(x) | \Lambda^{(r)} \rangle$$

and Bloch wave bases $\psi$, $\bar{\psi}$

$$\psi^{(r)}(x) = \sigma^{(r)}(x) \rho, \quad \bar{\psi}^{(r)}(x) = \bar{\rho} \bar{\sigma}^{(r)}(x).$$

Here $g$ and $\bar{g}$ diagonalize the monodromy matrices

$$S = Q_+(2\pi), \quad \bar{S} = Q_-(2\pi)$$

i.e.

$$S = g\kappa g^{-1}, \quad \kappa = e^{2\pi P_0}$$

$$\bar{S} = \bar{g}^{-1}\bar{\kappa}\bar{g}, \quad \bar{\kappa} = e^{-2\pi \bar{P}_0},$$

$P_0$ and $\bar{P}_0$ being the zero modes of $P$ and $\bar{P}$. The matrices $\rho$ and $\bar{\rho}$ contain the exponentiated conjugate operators of $P_0$ and $\bar{P}_0$, respectively. Finally the solutions of the equations (1), are reconstructed by means of

$$e^{-2\Lambda^{(r)}(\Phi)(x_+,x_-)} = \psi^{(r)}(x_+) \bar{\psi}^{(r)}(x_-),$$

where $\Lambda^{(r)}$ is the weight corresponding to $|\Lambda^{(r)}\rangle$. It is possible to prove that these solutions are periodic and local, provided

$$\kappa\bar{\kappa} = 1,$$

which amounts to the condition $P_0 = \bar{P}_0$.

2.2 ...and on the lattice.

The formulation of Toda field theories on a (periodic) lattice with $N$ sites runs parallel to the construction of the above section. We summarize in the following the recipe found in [4]. We limit ourselves to one chirality, as the other chirality has completely parallel formulas. Moreover we set $t = 0$.

We replace $Q_+(x)$ with a discrete matrix $Q_n$ which satisfies the Poisson bracket

$$\{Q_n \otimes Q_m\} = Q_n \otimes Q_m \left\{ \theta(n-m) \left[ -r + Q_m^{-1} \otimes Q_m^{-1}(r-t_0)Q_m \otimes Q_m \right] \\
+ \theta(m-n) \left[ -r + Q_n^{-1} \otimes Q_n^{-1}(r+t_0)Q_n \otimes Q_n \right] \right\}. $$

(12)
If we write

\[ L_n = Q_n Q_{n-1}^{-1} \]

and assume

\[ L_n = 1 + \Delta(P_n - E_+) + O(\Delta^2) , \]

where \( \Delta \) is the lattice spacing, we obtain eq.(4) in the limit \( \Delta \to 0 \).

Next we define the monodromy matrix \( S \) by means of

\[ Q_{N+n} = Q_n S . \]

This must satisfy the following Poisson brackets

\[
\{ Q_n \otimes S \} = Q_n \otimes S \left( -r + Q_n^{-1} \otimes Q_n^{-1} \cdot (r + t_0) \cdot Q_n \otimes Q_n - 1 \otimes S^{-1} \cdot t_0 \cdot 1 \otimes S \right) \tag{13}
\]

\[
\{ S \otimes Q \} = S \otimes S \left( -r + S^{-1} \otimes S^{-1} \cdot r \cdot S \otimes S + S^{-1} \otimes 1 \cdot t_0 \cdot S \otimes 1 - 1 \otimes S^{-1} \cdot t_0 \cdot S \otimes 1 \right) . \tag{14}
\]

In eqs.(12), (13) and (14) the matrix \( r \) is either one of the following classical \( r \) matrices

\[
r^+ = t_0 + 2 \sum_{\alpha \text{ positive}} \frac{E_\alpha \otimes E_{-\alpha}}{E_\alpha, E_{-\alpha}}
\]

\[
r^- = -t_0 - 2 \sum_{\alpha \text{ positive}} \frac{E_{-\alpha} \otimes E_\alpha}{E_{-\alpha}, E_\alpha}
\]

The matrix \( \rho \) must satisfy

\[
\{ Q_n \otimes \rho \} = -\alpha Q_n \otimes \rho \cdot t_0 \tag{15}
\]

\[
\{ S \otimes \rho \} = -\alpha S \otimes \rho \cdot t_0 - \beta t_0 \cdot S \otimes \rho \tag{16}
\]

\[
\{ \rho \otimes \rho \} = 0 , \tag{17}
\]

where \( \alpha \) and \( \beta \) are two arbitrary constants such that \( \alpha + \beta = 1 \). Since the final results expressed in the Bloch basis do not depend on the value of these constants, we will choose throughout the paper

\[ \alpha = 0 \]

Finally, given any highest weight vector \( |\Lambda^{(r)}\rangle \), we can construct the discrete \( \sigma \) basis

\[ \sigma_n = \langle \Lambda^{(r)} | Q_n \]

and, then, the \( \psi \) basis, and verify periodicity and locality of \( \psi_n^{(r)} \bar{\psi}_n^{(r)} \).
2.3 The $sl_3$ case

In the $sl_3$ case we will consider essentially the two fundamental representations $3$ and $3^*$. The conventions we choose are the following. Let $e_{ij}$ represent the $3 \times 3$ matrix with elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Then, the orthonormal basis for the CSA in the $3$ is

$$H_1 = \frac{1}{\sqrt{2}}(e_{11} - e_{22}), \quad H_2 = \frac{1}{\sqrt{6}}(e_{11} + e_{22} - 2e_{33})$$

In the $3^*$ representation we have

$$H_1^* = \frac{1}{\sqrt{2}}(e_{22} - e_{33}), \quad H_2^* = \frac{1}{\sqrt{6}}(2e_{11} - e_{22} - e_{33}).$$

It is now easy to construct the classical $r$ matrices and see, in particular, that

$$(r^\pm)(3,3) = (r^\pm)(3^*,3^*), \quad (r^\pm)(3,3^*) = (r^\pm)(3^*,3).$$

In general we will append a $^*$ to any object in the $3$ representation to denote the corresponding object in the $3^*$.

It is perhaps worth writing down the quadratic algebra that can be obtained from the exchange algebra for the following composed objects

$$W^{(i)}_n = \epsilon^{ijkl}\sigma^1_{n+j}\sigma^2_{n+k}\sigma^3_{n+l}; \quad i,j,k,l = 0, 1, 2, 3,$$

where $\epsilon$ is the completely antisymmetric symbol. In the $3$ representation we get

$$\begin{align*}
\{W^{(1)}_n, W^{(1)}_m\} &= -\frac{1}{3}W^{(1)}_nW^{(1)}_m(\delta_{n,m-3} - \delta_{n,m-1} + \delta_{n,m+1} - \delta_{n,m+3}) + \\
&\quad + W^{(2)}_nW^{(3)}_m\delta_{n,m-1} - W^{(3)}_nW^{(2)}_m\delta_{n,m+1} + \\
\{W^{(1)}_n, W^{(2)}_m\} &= -\frac{1}{3}W^{(1)}_nW^{(2)}_m(\delta_{n,m-3} + \delta_{n,m-2} - 2\delta_{n,m-1}) + \\
&\quad + \delta_{n,m} - \delta_{n,m+1} + \delta_{n,m+2} - \delta_{n,m+3} + \\
&\quad + W^{(3)}_nW^{(3)}_m\delta_{n,m-1} - W^{(3)}_nW^{(3)}_m\delta_{n,m+2} + \\
\{W^{(1)}_n, W^{(3)}_m\} &= -\frac{1}{3}W^{(1)}_nW^{(3)}_m(\delta_{n,m-3} + \delta_{n,m-2} - \delta_{n,m-1} - \delta_{n,m+2}) + \\
\{W^{(2)}_n, W^{(2)}_m\} &= -\frac{1}{3}W^{(2)}_nW^{(2)}_m(\delta_{n,m-3} - \delta_{n,m-1} + \delta_{n,m+1} - \delta_{n,m+3}) + \\
&\quad - W^{(3)}_nW^{(1)}_m\delta_{n,m-1} + W^{(1)}_nW^{(3)}_m\delta_{n,m+1} + \\
\{W^{(2)}_n, W^{(3)}_m\} &= -\frac{1}{3}W^{(2)}_nW^{(3)}_m(\delta_{n,m-3} + \delta_{n,m} - \delta_{n,m+1} - \delta_{n,m+2}) + \\
\{W^{(3)}_n, W^{(3)}_m\} &= -\frac{1}{3}W^{(3)}_nW^{(3)}_m(\delta_{n,m-2} + \delta_{n,m-1} - \delta_{n,m+1} - \delta_{n,m+2})
\end{align*}$$

(20)
We remark that if we construct analogous $W_n$’s in the $\psi_n$ basis, the above algebra does not change.

An analogous quadratic algebra was obtained for the $sl_2$ Toda field theory on the lattice in refs. [3]. In this case it does not seem to be possible to construct rational combinations of the $W^{(i)}_n$, which play the role of discrete generators of the Virasoro and $W_3$ algebra like the $S_n$ of [3].

3 Quantum theory on the lattice.

3.1 General formulas

Here we summarize the recipe for quantization obtained in [4]. For any matrix $X$ we use the notation

$$X_1 = X \otimes I, \quad X_2 = I \otimes X.$$ 

The quantum analogue of $Q_n$ must satisfy:

$$R_{12} Q_{1n} Q_{2n} = Q_{2n} Q_{1n} R_{12},$$

$$Q_{1n} Q_{1m}^{-1} A_{12} R_{12} Q_{1m} Q_{2m} = Q_{2m} Q_{1n} R_{12}, \quad n > m,$$ 

with $n, m < N$ and

$$A_{12} = e^{i\bar{h}t_0}$$

$$R_{12} = 1 - i\bar{h}r_{12} + O(\bar{h}^2)$$

$R_{12}$ must satisfy the quantum Yang-Baxter equation. Hereafter we denote $R_{12}^\pm$ the two solutions of the Yang-Baxter equation whose classical limits are $r_{12}^\pm$, respectively. We have $R_{12}^{-} = (R_{12}^{+})^{-1}$.

In a periodic lattice with $N$ sites we introduce the quantum monodromy matrix via

$$Q_{n+N} = Q_n BS,$$

where

$$B = e^{\frac{\bar{h}}{2} \sum_i \mu_i^2}.$$ 

Here, $S$ must satisfy

$$R_{12} S_1 A_{12} S_2 = S_2 A_{12} S_1 R_{12}$$

$$A_{12} R_{12} Q_{1n} Q_{2n} = Q_{1n} S_1^{-1} Q_{2n} A_{12} S_1 R_{12}.$$ 

6
Finally, the quantum matrix $\rho \in \exp(\mathcal{H})$ has the following properties

$$\rho_1 \rho_2 = \rho_2 \rho_1$$  \hspace{1cm} (25)

$$A_{12} S_1 \rho_2 A_{12} = \rho_2 S_1$$  \hspace{1cm} (26)

$$Q_{1n}\rho_2 = \rho_2 Q_{1n}.$$  \hspace{1cm} (27)

In the above equation we understood the representation labels. They can be supplied in an obvious way.

Let us define now the quantum $\sigma$ basis

$$\sigma_n^{(r)} = \langle \Lambda^{(r)} | Q_n \rangle.$$  \hspace{1cm} (28)

From eq.(22) we obtain the exchange algebra

$$\sigma_{1n} \sigma_{2m}^{(r)} = \sigma_{2m} \sigma_{1n}^{(r)} (R_{12}^+)^{(r,r')}_{ij,kl}, \quad n > m$$

$$\sigma_{1n}^{(r)} \sigma_{2m}^{(r')} = \sigma_{2m}^{(r')} \sigma_{1n}^{(r)} (R_{12}^-)^{(r,r')}_{ij,kl}, \quad n < m.$$  \hspace{1cm} (29)

From eq.(21) we can also obtain the exchange algebra for $n=m$.

### 3.2 Quantum $sl_3$ Toda field theory on the lattice

#### 3.2.1 The quantum exchange algebra

The purpose of this subsection is to use the general formulas of the previous subsection to define the Block wave basis $\psi_n^{(r)}$ and find the relevant exchange algebra in the $sl_3$ case.

We start with the quantum $R$ matrix to be inserted in the above formulas. From the universal $R$ matrix for $sl_n$ as given in [6] we obtain

$$\begin{align*}
(R_{12}^+)^{(3,3)}_{ij,kl} &= (R_{12}^+)^{(3^*,3^*)}_{ij,kl} = \\
&= \begin{cases}
q^\frac{1}{2} & i=j=k=l \\
q^{-\frac{1}{2}} & i=k \neq j=l \\
q^{-\frac{1}{2}} x & i=l \neq j=k \\
0 & \text{otherwise}
\end{cases} \\
(R_{12}^-)^{(3,3)}_{ij,kl} &= (R_{12}^-)^{(3^*,3^*)}_{ij,kl} = \\
&= \begin{cases}
q^\frac{1}{2} & i=j, \ k=l, \ i+j \neq 3 \\
q^{-\frac{1}{2}} & i=k, \ j=l, \ i+j=3 \\
q^{-\frac{1}{2}} x & i=l=1, \ j=3, \ k=2 \ \text{and} \ \ i=j=2, \ k=3, \ l=1 \\
-q^{\frac{1}{2}} x & i=l=1, \ j=k=3 \\
0 & \text{otherwise}
\end{cases}
\end{align*}$$  \hspace{1cm} (30)
where $q = e^{i\hbar}$, while, here and hereafter, we denote $x = q - q^{-1}$. Using these equations we can easily derive the exchange algebra in the $\sigma$ basis, which will not be written down explicitly here.

As a first step in the direction of the $\psi$ basis, we have to diagonalize the upper triangular monodromy matrices $S$ and $S^*$ in the two fundamental representations

$$S = \begin{pmatrix} A & D & F \\ 0 & B & E \\ 0 & 0 & C \end{pmatrix}, \quad S^* = \begin{pmatrix} A^* & D^* & F^* \\ 0 & B^* & E^* \\ 0 & 0 & C^* \end{pmatrix}.$$  

Eqs.(23) and (24) allow us to compute the commutation relations of the entries of $S$ and $S^*$ among themselves and with the component of the $\sigma_n$'s. One finds that the diagonal elements $A, B, C$ and $A^*, B^*, C^*$ commute with everything (except $\rho$, see below) while, for example,

$$DF = q^{-1} FD \quad EF = q FE \quad ED = q^{-1} DE + x BF$$  

and

$$D\sigma_n^1 = q^{-1} \sigma_n^1 D + x \sigma_n^2 A \quad D\sigma_n^2 = q \sigma_n^2 D$$

$$E\sigma_n^1 = \sigma_n^1 \quad E\sigma_n^2 = q^{-1} \sigma_n^2 E + x \sigma_n^3 B$$

$$F\sigma_n^1 = q^{-1} \sigma_n^1 F + x \sigma_n^3 A \quad F\sigma_n^2 = \sigma_n^2 F + x \sigma_n^3 D$$

$$D\sigma_n^3 = \sigma_n^3 D$$

$$E\sigma_n^3 = q \sigma_n^3 E$$

$$F\sigma_n^3 = q \sigma_n^3 F.$$  

We will not write down here the remaining relations, except for

$$AC^* = A^* C = BB^*.$$  

This relation explains why we can express the exchange algebras in terms of zero modes $A, B, C$ only (see Appendix).

Next we diagonalize $S$ and $S^*$ with upper triangular matrices $g$ and $g^*$, respectively, which have unit entries in the main diagonals. That is to say

$$S = g\kappa g^{-1}, \quad S^* = g^*\kappa^*(g^*)^{-1},$$

$\kappa$ and $\kappa^*$ being diagonal matrices, whose main diagonals coincide with the main diagonal of $S$ and $S^*$, i.e. $A, B, C$ and $A^*, B^*, C^*$, respectively. It is immediate to compute the commutators of the entries of $g$ and $g^*$ with all the operators introduced so far.
Next we introduce the matrix $\rho$ and the analogous $\rho^*$.

Analogously for the conjugate variables to the zero modes

$$\rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 \end{pmatrix}, \quad \rho^* = \begin{pmatrix} \rho_1^* & \rho_2^* \\ \rho_3^* \end{pmatrix}.$$  (35)

Due to eqs. (25, 27) and (26), $\rho_i$ and $\rho_i^*$ with $i = 1, 2, 3$ commute among themselves and with all the operators we introduced so far, except for the elements of $S$ and $S^*$. For these we have

$$S_{ij}\rho_k = \rho_k S_{ij}q^{\frac{65_i_{-k-2}}{3}}, \quad S_{ij}\rho_k^* = \rho_k^* S_{ij}q^{\frac{2-65_i_{+k+4}}{3}}.$$  (36)

and two more equations which can be obtained by “starring” these two (with the understanding that this formal * operation is involutive).

After introducing the complete set of operators of the theory, we can draw some immediate useful conclusions. First, $ABC$ and $A^* B^* C^*$ belong to the center of the theory. Second, $\rho_2\rho_2^*$, $\rho_1\rho_3^*$ and $\rho_3\rho_1^*$ also commute with everything else. Therefore, we can and will henceforth impose

$$ABC = 1 = A^* B^* C^*$$  (37)

and

$$\rho_2\rho_2^* = \rho_1\rho_3^* = \rho_3\rho_1^*.$$  (38)

This will allow us to simplify many formulas. In particular, (37) allows us to parametrize the zero modes as follows:

$$\begin{align*}
A &= q^{\frac{N-1}{4}} e^{2\pi(\frac{1}{\sqrt{2}}p_{01} + \frac{1}{\sqrt{6}}p_{02})}, & A^* &= q^{\frac{N-1}{4}} e^{2\pi(\sqrt{7}p_{02})}, \\
B &= q^{\frac{N-1}{4}} e^{2\pi(-\frac{1}{\sqrt{2}}p_{01} + \frac{1}{\sqrt{6}}p_{02})}, & B^* &= q^{\frac{N-1}{4}} e^{2\pi(\frac{1}{\sqrt{2}}p_{01} - \frac{1}{\sqrt{6}}p_{02})}, \\
C &= q^{\frac{N-1}{4}} e^{2\pi(-\sqrt{7}p_{02})}, & C^* &= q^{\frac{N-1}{4}} e^{2\pi(\frac{1}{\sqrt{2}}p_{01} - \frac{1}{\sqrt{6}}p_{02})},
\end{align*}$$  (39)

in agreement with the conventions chosen for the CSA basis and with our quantization procedure.

Now what remains to be done is to define the quantum Block wave basis

$$\psi_n = \sigma_n g \rho, \quad \psi_n^* = \sigma_n^* g^* \rho^*$$  (40)

and compute its exchange algebra. The calculation is long but uneventful and the result has the form

$$\psi_1n\psi_2m = \psi_2m\psi_{1m}(R_{12}^\pm(p_0))^{(3,3)}, \quad \begin{cases} + & n > m \\ - & n < m \end{cases}$$  (41)
where the argument \( p_0 \) is to remember the dependence on the zero modes. The entries of the (zero mode dependent) quantum \( R \) matrix\(^1\) in the Bloch wave basis are written down explicitly in Appendix A.1 and A.2.

This completes our proof about the relation between the quantum \( R \) matrix of Jimbo and Rosso and the quantum \( R \) matrix in the Bloch wave basis. Such a relation cannot be envisaged as an (operator-valued) similarity transformation since, for example in (41), \( g_l \rho_l \) does not commute with \( \psi_{2m} \). We can only say that the relation is specified by the operator-valued change of basis (40).

In order to discuss periodicity and locality, one has to repeat everything for the discretization of the antichiral half, in order to calculate the exchange algebra of

\[
\tilde{\psi}_n = \tilde{\rho} g \tilde{\sigma}_n, \quad \tilde{\psi}_n^* = \tilde{\rho}^* g^* \tilde{\sigma}_n^*
\]

(43)

The result of this calculation can also be found in Appendix A.1 and A.2.

### 3.2.2 Periodicity and locality

In analogy with the continuum case, we define

\[
e^{-\varphi_n} = \psi_n \tilde{\psi}_n, \quad e^{-\varphi_n^*} = \psi_n^* \tilde{\psi}_n^*.
\]

We find

\[
e^{-\varphi_{n+N}} = q^{\frac{4}{3}} \left( A \tilde{A} \psi_n^1 \tilde{\psi}_n^1 + B \tilde{B} \psi_n^2 \tilde{\psi}_n^2 + C \tilde{C} \psi_n^3 \tilde{\psi}_n^3 \right)
\]

\[
e^{-\varphi_{n+N}} = q^{\frac{4}{3}} \left( A^* \tilde{A}^* \psi_n^1 \tilde{\psi}_n^1 + B^* \tilde{B}^* \psi_n^2 \tilde{\psi}_n^2 + C^* \tilde{C}^* \psi_n^3 \tilde{\psi}_n^3 \right).
\]

Since \( A \tilde{A}, B \tilde{B}, C \tilde{C}, A^* \tilde{A}^*, B^* \tilde{B}^* , C^* \tilde{C}^* \) commute with all the operators of the theory, we can project out of the full Hilbert space \( \mathcal{H} \) the subspace \( \mathcal{H}_0 \) where

\[
A \tilde{A} = B \tilde{B} = C \tilde{C} = A^* \tilde{A}^* = B^* \tilde{B}^* = C^* \tilde{C}^* = q^{-\frac{4}{3}}.
\]

(44)

In \( \mathcal{H}_0 \) both \( e^{-\varphi_n} \) and \( e^{-\varphi_n^*} \) are periodic.

To prove locality, we compute

\[
[e^{-\varphi_n}, e^{-\varphi_m}] = x \frac{B \tilde{B} - A \tilde{A}}{(B - A)(B - A)} \left( \psi_n^1 \psi_m^1 \tilde{\psi}_n^1 \tilde{\psi}_m^1 - \psi_n^2 \psi_m^2 \tilde{\psi}_n^2 \tilde{\psi}_m^2 \right) +
\]

\[
+ x \frac{C \tilde{C} - A \tilde{A}}{(C - A)(C - A)} \left( \psi_n^1 \psi_m^1 \tilde{\psi}_n^3 \tilde{\psi}_m^3 - \psi_n^3 \psi_m^3 \tilde{\psi}_n^3 \tilde{\psi}_m^3 \right) +
\]

\[
+ x \frac{C \tilde{C} - B \tilde{B}}{(C - B)(C - B)} \left( \psi_n^2 \psi_m^2 \tilde{\psi}_n^2 \tilde{\psi}_m^2 - \psi_n^3 \psi_m^3 \tilde{\psi}_n^3 \tilde{\psi}_m^3 \right).
\]

\(^1\)Here we enlarge the notion of \( R \) matrix. Indeed the exchange matrices of eqs. (11) and (12) are not solutions of the YBE’s, but of a modified version of them.
Therefore the commutator vanishes in the subspace $\mathcal{H}_0$. The same conclusion holds if we consider $[e^{-\varphi_n^*}, e^{-\varphi_m^*}]$. Next let us consider $[e^{-\varphi_n^*}, e^{-\varphi_m^*}]$. This commutator is more complicated than the previous ones. However it can be proven that it is a combination of terms each of which factorize out either $BB - AA$ or $CC - AA$ or $CC - BB$. Therefore we again can conclude that, in $\mathcal{H}_0$,

$$[e^{-\varphi_n^*}, e^{-\varphi_m^*}] = 0$$

This completes the derivation of our result as far as the $sl_3$ Toda field theory is concerned. It is perhaps useful spending a few words to give the reader the coordinates of this result in the prospect of evaluating correlation functions. The lattice analogues of the conformal blocks are given by expressions like (we consider here only the 3 representation)

$$\langle \theta_\infty | \psi_{n_1} \psi_{n_2} \cdots \psi_{n_k} | \theta_0 \rangle$$

and

$$\langle \tilde{\theta}_\infty | \tilde{\psi}_{n_1} \tilde{\psi}_{n_2} \cdots \tilde{\psi}_{n_k} | \tilde{\theta}_0 \rangle$$

where the $\theta$ states tend to the corresponding conformal vacua in the continuum limit. Putting together the two halves, we can compute

$$\langle e^{-\varphi(x_1)} e^{-\varphi(x_2)} \cdots e^{-\varphi(x_k)} \rangle,$$

where $e^{-\varphi(x)}$ is the continuum limit of $e^{-\varphi_n}$. Single-valuedness and locality of (45) is then guaranteed by the condition (44).

4 Comparison with previous results.

In the previous section we computed the exchange algebra for the $sl_3$ Toda field theory in a periodic lattice. Since this algebra does not depend on the lattice spacing, we can immediately translate it into a continuous language by the simple replacements

$$\psi_n^i \rightarrow \psi^i(x), \quad \psi_n^{*i} \rightarrow \psi^{*i}(x), \quad \text{etc.,} \quad \theta(n-m) \rightarrow \theta(x-y).$$

In this way we can compare our results with those of ref.[2] (see also [3] and, for the specific case of $sl_3$, [4]). There, following a different approach, the $sl_{l+1}$ exchange algebra in the Bloch wave basis for the defining representation was calculated to be

$$\phi_j(\sigma)\phi_j(\sigma') = e^{-i\hbar l/l+1} \phi_j(\sigma')\phi_j(\sigma),$$

$$\phi_j(\sigma)\phi_k(\sigma') = e^{i\hbar l/l+1} \frac{\sin(h(w_{jk}+1)}{\sin(hw_{jk})} \phi_k(\sigma')\phi_j(\sigma) + \epsilon = \text{sign}(\sigma - \sigma')$$

$$+ e^{i\hbar l/l+1} \frac{\sin(hw_{jk})}{\sin(hw_{jk})} \phi_j(\sigma')\phi_k(\sigma),$$

(46)
where
\[ \varpi_{jk} = (\lambda_k - \lambda_j) \cdot \varpi \]
\[ \lambda_i = \text{weights of the defining representation} \]
\[ \tilde{\varpi} = -\frac{i}{2} \sqrt{\frac{2\pi}{h}} \Lambda_0 \]
\[ \Lambda_0 = \sum_{i=1}^{n} \Lambda_i \tilde{p}_0^i \]
\[ \Lambda_i = \text{fundamental weights} \]

Here, the zero modes \( \tilde{p}_0^i \) \( (i = 1, \ldots, l) \) correspond to the weight space basis consisting of the simple root system \( \{\alpha_i\}_{i=1,\ldots,l} \) (that is to say, to the basis \( \{h_i\}_{i=1,\ldots,l} \) of the CSA, \( h_i = e_{ii} - e_{i+1,i+1} \)). Considering as an example the case of \( sl_3 \), the \( \tilde{p}_{0,1} \) are related to the zero modes introduced in eqs.\((33)\) by the linear transformation
\[ \tilde{p}_0^1 = \sqrt{\frac{4\pi}{h}} (\sqrt{2}p_0^1) \quad \tilde{p}_0^2 = \sqrt{\frac{4\pi}{h}} (-\frac{1}{\sqrt{2}}p_0^1 + \sqrt{\frac{3}{2}}p_0^2). \] (47)

Taking into account such a rotation, together with the usual identification \( q = e^{-ih} \), we obtain the relations
\[ (q - q^{-1}) \frac{A}{B-A} = -\frac{\sin(h)}{\sin(h\varpi_{12})} e^{-ih\varpi_{12}}, \quad (q - q^{-1}) \frac{B}{B-A} = \frac{\sin(h)}{\sin(h\varpi_{21})} e^{-ih\varpi_{21}}, \]
\[ (q - q^{-1}) \frac{C}{C-B} = -\frac{\sin(h)}{\sin(h\varpi_{23})} e^{-ih\varpi_{23}}, \quad (q - q^{-1}) \frac{C}{C-A} = \frac{\sin(h)}{\sin(h\varpi_{32})} e^{-ih\varpi_{32}}, \] (48)

This allows us to identify the off-diagonal coefficients of the operator algebra \((12)\) with the corresponding elements of the zero mode dependent \( R \) matrices of eq.\((11)\) (see Appendix A.1). As for the diagonal entries, it is important to remark that they coincide only up to a change in the normalization of the Bloch wave basis. Indeed, in order to reproduce the operator algebra of Appendix A.1, the vertex fields \( \phi_i, \ i = 1, 2, 3 \), should be multiplied by the factors
\[ c_1(\tilde{p}_0) = [\sin(h\varpi_{13})]^a [\sin(h\varpi_{12})]^{1-a}, \]
\[ c_2(\tilde{p}_0) = [\sin(h(\varpi_{12} + 1))]^a [\sin(h\varpi_{23})]^{1-a}, \] (49)
\[ c_3(\tilde{p}_0) = [\sin(h(\varpi_{23} + 1))]^a [\sin(h(\varpi_{13} + 1))]^{1-a}, \]
respectively, where \( a \) is an arbitrary parameter. Since the \( c_i \)'s depend only on the zero modes, this operation does not modify the monodromy behaviour of the fields, which still constitute a Bloch wave basis. After this transformation, while the off-diagonal elements remain unchanged, the diagonal ones take the expressions\(^2\)
\[ R_{12}^{21}(\tilde{p}_0) \rightarrow \frac{\sin(h\varpi_{12})}{\sin(h(\varpi_{12} + 1))} R_{12}^{21}(\tilde{p}_0) = e^{ih\pi/3} = q^{-\frac{3}{4}}, \]

\(^2\)Following ref.\([3]\), we use for the \( R \) matrix elements the notation \( \phi_i\phi_j = \sum_{kl} R_{ij}^{kl} \phi_k \phi_l \).
while the entries

\[ R_{23}^{12}(\tilde{p}_0) \rightarrow \frac{\sin(h\varphi_{23})}{\sin(h\varphi_{23}+1)} \quad R_{23}^{32}(\tilde{p}_0) = e^{ihv/3} = q^{-\frac{1}{3}}, \]

\[ R_{13}^{31}(\tilde{p}_0) \rightarrow \frac{\sin(h\varphi_{13})}{\sin(h\varphi_{13}+1)} \quad R_{13}^{31}(\tilde{p}_0) = e^{ihv/3} = q^{-\frac{1}{3}}, \]

\[ R_{21}^{12}(\tilde{p}_0) \rightarrow \frac{\sin(h\varphi_{12}+1)}{\sin(h\varphi_{12})} \quad R_{21}^{12}(\tilde{p}_0) = \]

\[ = e^{ihv/3} \left[ \cos^2(h) - \sin^2(h) \frac{\cos^2(h\varphi_{12})}{\sin^2(h\varphi_{12})} \right] = q^{-\frac{1}{3}} \left[ 1 - (q - q^{-1})^2 \frac{AB}{(B-A)^2} \right], \]

\[ R_{32}^{23}(\tilde{p}_0) \rightarrow \frac{\sin(h\varphi_{23}+1)}{\sin(h\varphi_{23})} \quad R_{32}^{23}(\tilde{p}_0) = \]

\[ = e^{ihv/3} \left[ \cos^2(h) - \sin^2(h) \frac{\cos^2(h\varphi_{23})}{\sin^2(h\varphi_{23})} \right] = q^{-\frac{1}{3}} \left[ 1 - (q - q^{-1})^2 \frac{BC}{(C-B)^2} \right], \]

\[ R_{31}^{13}(\tilde{p}_0) \rightarrow \frac{\sin(h\varphi_{13}+1)}{\sin(h\varphi_{13})} \quad R_{31}^{13}(\tilde{p}_0) = \]

\[ = e^{ihv/3} \left[ \cos^2(h) - \sin^2(h) \frac{\cos^2(h\varphi_{13})}{\sin^2(h\varphi_{13})} \right] = q^{-\frac{1}{3}} \left[ 1 - (q - q^{-1})^2 \frac{AC}{(C-A)^2} \right], \]

while the entries \( R_{ii}^{ii} \) do not transform.

The same can be repeated when, instead of the representations 3 and 3, we have 3* and 3* or 3 and 3*.

In conclusion, we have shown that the \( R \) matrices in the Bloch wave basis of \([2],[7],[9]\) are the same as the ones we exhibit in Appendix, except for the renormalization pointed out above. It should however be added that only by virtue of such a change of normalization can the locality property of the previous subsection be fulfilled.

5 The \( sl_n \) case

It is easy to generalize the above results to the \( sl_n \) case, at least as far as the defining representation is concerned. For the Cartan subalgebra we choose the following orthonormal basis

\[ H_k = \frac{1}{\sqrt{k(k+1)}} \sum_{i=1}^{k} i(e_{ii} - e_{i+1,i+1}), \quad k = 1, \ldots, n - 1 \]

The quantum \( R \) matrix in defining representation of \( sl_n \) is (see \([3]\))

\[ R_{ij,kl} = \begin{cases} 
q^{\frac{1}{n}} & i = j = k = l \\
q^{-\frac{1}{n}} & i = k, \ j = l, \ i \neq j \\
q^{-\frac{1}{n}x} & i = l < j = k \\
0 & \text{otherwise}
\end{cases} \]

where \( x = q - q^{-1} \), as above. Let us denote by \( A_i \) and \( \bar{A}_i \), \( i = 1, \ldots, n \) the diagonal elements \( S_{ii} \) and \( \bar{S}_{ii} \) of the monodromy matrices \( S \) and \( \bar{S} \), respectively. The exchange algebra in the Bloch wave basis is,
for $n > m$

$$
\psi^i_n \psi^j_m = q^\frac{m-n}{n} \psi^i_m \psi^j_n
$$

$$
\psi^i_n \psi^j_m = q^{-1} \left[ \psi^j_m \psi^i_n + x \frac{A_i}{A_j - A_i} \psi^i_m \psi^j_n \right], \quad i < j
$$

$$
\psi^i_n \psi^j_m = q^{-1} \left[ (1 - x^2 \frac{A_i A_j}{(A_i - A_j)^2}) \psi^j_m \psi^i_n + x \frac{A_i}{A_j - A_i} \psi^i_m \psi^j_n \right], \quad i > j
$$

for $n < m$

$$
\psi^i_n \psi^j_m = q^{-1} \psi^i_m \psi^j_n
$$

$$
\psi^i_n \psi^j_m = q^{-1} \left[ \psi^j_m \psi^i_n + x \frac{A_i}{A_j - A_i} \psi^i_m \psi^j_n \right], \quad i < j
$$

$$
\psi^i_n \psi^j_m = q^{-1} \left[ (1 - x^2 \frac{A_i A_j}{(A_i - A_j)^2}) \psi^j_m \psi^i_n + x \frac{A_i}{A_j - A_i} \psi^i_m \psi^j_n \right], \quad i > j.
$$

Likewise we can write down the exchange algebra for $\bar{\psi}_n$ and construct $\psi_n \bar{\psi}_n$. Periodicity of these objects is guaranteed in the subspace $H_0$ of the total Hilbert space $H$ where the conditions

$$
A_i \bar{A}_i = q^\frac{n+1}{n}
$$

are satisfied. As for locality, we find

$$
[\psi_n \bar{\psi}_m, \psi_n \bar{\psi}_m] = x \sum_{i<j} \frac{A_j \bar{A}_j - A_i \bar{A}_i}{(A_j - A_i)(A_j - A_i)} \left( \psi^i_n \psi^j_m \bar{\psi}^i_m \bar{\psi}^j_m - \psi^i_m \psi^j_n \bar{\psi}^i_n \bar{\psi}^j_n \right).
$$

So, also in this general case, the condition (52) guarantees locality as well.
Appendix

A.1 The $\psi \psi$ exchange algebra

In the defining representation we find (recall that $x = q - q^{-1}$)

i) case $n > m$

\[
\psi_n^1 \psi_m^1 = q^{\frac{2}{3}} \psi_m^1 \psi_n^1
\]

\[
\psi_n^1 \psi_m^2 = q^{\frac{1}{3}} \psi_m^2 \psi_n^1 - q^{\frac{1}{3}} x \frac{A}{B-A} \psi_m^1 \psi_n^2
\]

\[
\psi_n^1 \psi_m^3 = q^{\frac{1}{3}} \psi_m^3 \psi_n^1 - q^{\frac{1}{3}} x \frac{A}{C-A} \psi_m^1 \psi_n^3
\]

\[
\psi_n^2 \psi_m^1 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{AB}{(B-A)^2} \right] \psi_m^1 \psi_n^2 + q^{-\frac{1}{3}} x \frac{B}{B-A} \psi_m^2 \psi_n^1
\]

\[
\psi_n^2 \psi_m^2 = q^{\frac{2}{3}} \psi_m^2 \psi_n^2
\]

\[
\psi_n^2 \psi_m^3 = q^{\frac{1}{3}} \psi_m^3 \psi_n^2 - q^{\frac{1}{3}} x \frac{B}{C-B} \psi_m^2 \psi_n^3
\]

\[
\psi_n^3 \psi_m^1 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{AC}{(C-A)^2} \right] \psi_m^1 \psi_n^3 + q^{-\frac{1}{3}} x \frac{C}{C-A} \psi_m^3 \psi_n^1
\]

\[
\psi_n^3 \psi_m^2 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{BC}{(C-B)^2} \right] \psi_m^2 \psi_n^3 + q^{-\frac{1}{3}} x \frac{C}{C-B} \psi_m^3 \psi_n^2
\]

\[
\psi_n^3 \psi_m^3 = q^{\frac{2}{3}} \psi_m^3 \psi_n^3
\]

ii) case $n = m$

\[
\psi_n^1 \psi_n^2 = \frac{B-A}{qB-q^{-1}A} \psi_n^2 \psi_n^1
\]

\[
\psi_n^2 \psi_n^3 = \frac{C-B}{qC-q^{-1}B} \psi_n^3 \psi_n^2
\]

\[
\psi_n^1 \psi_n^3 = \frac{C-A}{qC-q^{-1}A} \psi_n^3 \psi_n^1
\]

iii) case $n < m$

\[
\psi_n^1 \psi_m^1 = q^{-\frac{2}{3}} \psi_m^1 \psi_n^1
\]
\[ \psi_n^1 \psi_m^2 = q^{\frac{1}{3}} \psi_m^1 \psi_n^1 - q^{\frac{1}{3}} x \frac{B}{B-A} \psi_m^1 \psi_n^2 \]

\[ \psi_n^1 \psi_m^3 = q^{\frac{1}{3}} \psi_m^3 \psi_n^1 - q^{\frac{1}{3}} x \frac{C}{C-A} \psi_m^1 \psi_n^3 \]

\[ \psi_n^2 \psi_m^1 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{AB}{(B-A)^2} \right] \psi_m^1 \psi_n^2 + q^{\frac{1}{3}} x \frac{A}{B-A} \psi_m^2 \psi_n^1 \]

\[ \psi_n^2 \psi_m^2 = q^{-\frac{2}{3}} \psi_m^2 \psi_n^2 \]

\[ \psi_n^2 \psi_m^3 = q^{\frac{1}{3}} \psi_m^3 \psi_n^2 - q^{\frac{1}{3}} x \frac{C}{C-B} \psi_m^2 \psi_n^3 \]

\[ \psi_n^3 \psi_m^1 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{AC}{(C-A)^2} \right] \psi_m^1 \psi_n^3 + q^{\frac{1}{3}} x \frac{A}{C-A} \psi_m^3 \psi_n^1 \]

\[ \psi_n^3 \psi_m^2 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{BC}{(C-B)^2} \right] \psi_m^2 \psi_n^3 + q^{\frac{1}{3}} x \frac{B}{C-B} \psi_m^3 \psi_n^2 \]

\[ \psi_n^3 \psi_m^3 = q^{-\frac{2}{3}} \psi_m^3 \psi_n^3 \]

The exchange algebra \( \psi^* \psi^* \) can be obtained from this by simply starring it, i.e. replacing everywhere \( \psi, A, B, C \) by \( \psi^*, A^*, B^* \) and \( C^* \), respectively.

The antichiral algebra \( \bar{\psi} \bar{\psi} \) can be obtained from the above following the recipe: in order to get the exchange relation of \( \bar{\psi}_n^i \bar{\psi}_m^j \), take the exchange relation of \( \psi_n^j \psi_m^i \) and bar everything including \( A, B \) and \( C \). For example, for \( n > m \), we get

\[ \bar{\psi}_n^1 \bar{\psi}_m^2 = q^{\frac{1}{3}} \left[ 1 - x^2 \frac{\bar{A} \bar{B}}{(\bar{B} - \bar{A})^2} \right] \bar{\psi}_m^2 \bar{\psi}_n^1 + q^{\frac{1}{3}} x \frac{\bar{A}}{\bar{B} - \bar{A}} \bar{\psi}_m^1 \bar{\psi}_n^2 \]

Of course the algebra \( \bar{\psi}^* \bar{\psi}^* \) is obtained by simply “starring” the algebra \( \bar{\psi} \bar{\psi} \).

A.2 The \( \psi \, \psi^* \) exchange algebra

This algebra is given by

i) case \( n > m \)

\[ \psi_n^1 \psi_m^1 = q^{\frac{1}{3}} \psi_m^1 \psi_n^1 \]

\[ \psi_n^1 \psi_m^2 = q^{\frac{1}{3}} \psi_m^2 \psi_n^1 \]

\[ \psi_n^1 \psi_m^3 = q^{-\frac{2}{3}} \psi_m^3 \psi_n^1 - q^{-\frac{2}{3}} x \frac{A}{B-A} \psi_m^2 \psi_n^2 + q^{-\frac{2}{3}} x \frac{C-q^{-1}B}{C-B} \psi_m^1 \psi_n^3 \]
\[\psi_n^2 \psi_m^1 = q^{\frac{1}{3}} \psi_m^1 \psi_n^2\]

\[\psi_n^2 \psi_m^2 = q^{-\frac{2}{3}} x^2 \frac{B}{B-A} \psi_m^3 \psi_n^1 + q^{-\frac{2}{3}} \left[1 - x^2 \frac{AB}{(B-A)^2}\right] \psi_m^2 \psi_n^2 + q^{-\frac{2}{3}} x^2 \frac{C-CB}{C-B} \frac{q^{B} - q^{A}}{B-A} \psi_m^1 \psi_n^3\]

\[\psi_n^2 \psi_m^3 = q^{\frac{1}{4}} \psi_m^3 \psi_n^2\]

\[\psi_n^3 \psi_m^1 = -q^{-\frac{2}{3}} x^2 \frac{C}{C-A} \frac{q^{C} - q^{B}}{C-B} \psi_m^3 \psi_n^1 + q^{-\frac{2}{3}} x^2 \frac{C}{C-B} \frac{q^{B} - q^{A}}{B-A} \psi_m^3 \psi_n^2 + q^{-\frac{2}{3}} \left[1 - x^2 \frac{AC}{C-A}\right] \left[1 - x^2 \frac{BC}{B-C}\right] \psi_m^1 \psi_n^3\]

\[\psi_n^3 \psi_m^2 = q^{\frac{1}{4}} \psi_m^2 \psi_n^3\]

\[\psi_n^3 \psi_m^3 = q^{\frac{1}{4}} \psi_m^3 \psi_n^3\]

## iii) case \(n < m\)

\[\psi_n^1 \psi_m^1 = q^{-\frac{1}{3}} \psi_m^1 \psi_n^1\]

\[\psi_n^1 \psi_m^2 = q^{-\frac{1}{3}} \psi_m^2 \psi_n^1\]

\[\psi_n^1 \psi_m^3 = q^{\frac{2}{3}} \psi_m^3 \psi_n^1 + q^{\frac{2}{3}} x^2 \frac{B}{B-A} \psi_m^2 \psi_n^2 + q^{\frac{2}{3}} x^2 \frac{C-CB}{C-B} \psi_m^1 \psi_n^3\]

\[\psi_n^2 \psi_m^1 = q^{\frac{1}{3}} \psi_m^1 \psi_n^2\]

\[\psi_n^2 \psi_m^2 = q^{\frac{2}{3}} x^2 \frac{A}{B-A} \psi_m^3 \psi_n^1 + q^{\frac{2}{3}} \left[1 - x^2 \frac{AB}{(B-A)^2}\right] \psi_m^2 \psi_n^2 + q^{\frac{2}{3}} x^2 \frac{C-CB}{C-B} \frac{q^{B} - q^{A}}{B-A} \psi_m^1 \psi_n^3\]

\[\psi_n^2 \psi_m^3 = q^{\frac{1}{4}} \psi_m^3 \psi_n^2\]
\[ \psi_3^3 \psi_1^4 = -q \frac{x}{C-A} \frac{q^{-1} C - q B}{C-B} \psi_3^3 \psi_1^4 + \]
\[ + q \frac{x}{C-B} \frac{q B - q^{-1} A}{B-A} \frac{q^{-1} C - q A}{C-A} \psi_3^3 \psi_1^4 + \]
\[ + q \frac{1}{2} \left[ 1 - x^2 \frac{B C}{(C-A)^2} \right] \left[ 1 - x^2 \frac{AC}{(C-B)^2} \right] \psi_m^1 \psi_n^3 \]
\[ \psi_3^3 \psi_2^2 = q^{-\frac{1}{3}} \psi_m^2 \psi_n^3 \]
\[ \psi_3^3 \psi_3^3 = q^{-\frac{1}{3}} \psi_m^3 \psi_n^3 \]

The exchange algebra for \( n = m \) is the same as for \( n > m \) with the RHS multiplied by \( q^{-\frac{1}{3}} \).

By “starring” this algebra we obtain the algebra \( \psi^* \psi \).

The corresponding antichiral algebra \( \bar{\psi} \bar{\psi}^* \) can be obtained according to the recipe: to get \( \bar{\psi}_n^i \bar{\psi}_m^j \), take \( \psi_m^j \psi_n^i \) and bar everything including \( A, B \) and \( C \).

From this one can write down the algebra \( \bar{\psi}^* \bar{\psi} \).

Acknowledgements One of us (L.B.) would like to thank the Instituto de Fisica Teorica – UNESP of S.Paulo for the kind hospitality extended to him during the completion of this paper.

References

[1] V.A.Fateev and S.L.Lukyanov, Int.Jour.Mod.Phys. A3 (1988), 507, A7 (1992), 853.
[2] J.-L.Gervais and A.Bilal, Nucl.Phys. B314 (1989) 646, B318 (1989) 579.
[3] L.D.Faddeev and L.Takhtadjan, Springer Lecture Notes in Physics, Vol.246 (1986) 166.
A.Volkov, Theor.Math.Phys. 74 (1988) 135. O.Babelon, Phys.Lett. B238 (1990) 234.
[4] O.Babelon and L.Bonora, Phys.Lett. B253 (1991) 365.
[5] O.Babelon, Comm.Math.Phys. 139 (1991) 619.
[6] M.Jimbo, Comm.Math.Phys. 102 (1986) 537. M.Rosso, Comm.Math.Phys. 124 (1989), 307.
[7] E.Cremmer and J.-L.Gervais, The quantum group structure associated with non-linearly extended Virasoro algebras, preprint, LPTENS 89/19
[8] J.Balog, L.Dabrowski and L.Fehér, Phys.Lett. B244 (1990) 227, B257 (1991) 74.
[9] J.-L.Gervais and Rostand, Nucl.Phys. B346 (1990), 473.
[10] O.Babelon, L.Bonora and F.Toppan, Comm.Math.Phys. 140 (1991) 93.

[11] E.Aldrovandi, L.Bonora, V.Bonservizi and R.Paunov, in preparation.

[12] V.G.Drinfel’d and V.V.Sokolov, J.Sov.Math. 30 (1984) 1975.