Frobenius Groups and Retract Rationality

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Abstract. Let $k$ be any field, $G$ be a finite group acting on the rational function field $k(x_g : g \in G)$ by $h \cdot x_g = x_{hg}$ for any $h, g \in G$. Define $k(G) = k(x_g : g \in G)^G$. Noether’s problem asks whether $k(G)$ is rational (= purely transcendental) over $k$. A weaker notion, retract rationality introduced by Saltman, is also very useful for the study of Noether’s problem. We prove that, if $G$ is a Frobenius group with abelian Frobenius kernel, then $k(G)$ is retract $k$-rational for any field $k$ satisfying some mild conditions. As an application, we show that, for any algebraic number field $k$, for any Frobenius group $G$ with Frobenius complement isomorphic to $SL_2(\mathbb{F}_5)$, there is a Galois extension field $K$ over $k$ whose Galois group is isomorphic to $G$, i.e. the inverse Galois problem is valid for the pair $(G, k)$. The same result is true for any non-solvable Frobenius group if $k(\zeta_8)$ is a cyclic extension of $k$.

2010 Mathematics Subject Classification. Primary 13A50, 14E08, 12F12, 20J06.
Keywords and phrases. Noether’s problem, the inverse Galois problem, retract rationality, Frobenius groups.
Partially supported by National Center for Theoretic Sciences (Taipei Office).
§1. Introduction

Let $k$ be any field and $G$ be a finite group acting on the rational function field $k(x_g : g \in G)$ by $k$-automorphisms so that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g : g \in G)^G$. Noether’s problem asks whether $k(G)$ is rational (= purely transcendental) over $k$. It is related to the inverse Galois problem, to the existence of generic $G$-Galois extensions over $k$, and to the existence of versal $G$-torsors over $k$-rational field extensions \cite[33.1, p. 86]{Sw2}. Noether’s problem for abelian groups was studied by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan’s paper for a survey of this problem \cite{Sw2}.

On the other hand, just a handful of results about Noether’s problem for non-abelian groups are obtained.

Before stating the main results of this paper, we recall a notion introduced by Saltman, retract rationality, which is weaker than rationality, but is still very useful.

**Definition 1.1** Let $k \subset K$ be an extension of fields. $K$ is rational over $k$ (for short, $k$-rational) if $K$ is purely transcendental over $k$. $K$ is stably $k$-rational if $K(y_1, \ldots, y_m)$ is rational over $k$ for some $y_1, \ldots, y_m$ are algebraically independent over $K$.

When $k$ is an infinite field, $K$ is said to be retract $k$-rational if there is a $k$-algebra $A$ contained in $K$ such that (i) $K$ is the quotient field of $A$, (ii) there exist a non-zero polynomial $f \in k[X_1, \ldots, X_n]$ (where $k[X_1, \ldots, X_n]$ is the polynomial ring) and $k$-algebra morphisms $\varphi : A \to k[X_1, \ldots, X_n][1/f]$ and $\psi : k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$. See \cite{Sa2, Ka4} for details.

It is not difficult to see that “$k$-rational” $\Rightarrow$ “stably $k$-rational” $\Rightarrow$ “retract $k$-rational”. Note that $k(G)$ is retract $k$-rational is equivalent to the existence of a generic $G$-Galois extension over $k$: \cite{Sa1, Sa2, Ka4, Theorem 1.2].

The following result ensures that $k(G)$ is retract $k$-rational is as good as Noether’s problem for solving the inverse Galois problem. The reader is referred to Serre’s monograph \cite{Se} for Hilbert’s irreducibility theorem and the inverse Galois problem.

**Theorem 1.2** Let $G$ be a finite group, and $k$ be an infinite field such that Hilbert’s irreducibility theorem is valid (e.g. $k$ is an algebraic number field). If $k(G)$ is retract $k$-rational, then there is a Galois extension field $K$ over $k$ whose Galois group is isomorphic to $G$.

**Proof.** Write $E = k(x_g : g \in G)$ and $F = E^G = k(G)$. Since $k(G)$ is retract $k$-rational, there is $k$-algebra $A$ in $F$ such that the quotient field of $A$ is $F$, and the $k$-algebra morphisms $\varphi : A \to k[X_1, \ldots, X_n][1/f]$ and $\psi : k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$.

Let $B$ be the integral closure of $A$ in $E$. Write $E = F(\theta)$ for some primitive element $\theta$. We may assume that $\theta \in B$ and $B$ is a Galois extension of $A$ with group $G$ (see \cite[Section 3]{Sw2} for details).
Since $A[\theta]$ and $B$ have the same quotient field, there exist elements $s, t \in A$ such that $A[\theta][1/s] = B[1/t]$ by [Sw1] Lemma 8. Further localization will ensure that $A[\theta] = B$. In short, we have a Galois extension $B$ over $A$ and the $k$-algebra morphisms $\varphi: A \to k[X_1, \ldots, X_n][1/f]$ and $\psi: k[X_1, \ldots, X_n][1/f] \to A$ such that $B = A[\theta]$.

Let $g(T) \in A[T]$ be the minimum polynomial of $\theta$ over $F$ where $T$ is a variable which is algebraic independent over $k(X_1, \ldots, X_n)$ (and over $F$ via the injective morphism $\varphi$). Write $g(T) = T^m + a_1 T^{m-1} + \cdots + a_m$ for some elements $a_1, \ldots, a_m \in A$ with $m = |G|$. Note that $g(T)$ is an irreducible polynomial in $F[T]$. Define $h(T) = T^m + \varphi(a_1) T^{m-1} + \cdots + \varphi(a_m)$. We claim that $h(T) \in k(X_1, \ldots, X_n)[T]$ is an irreducible polynomial over $k(X_1, \ldots, X_n)$; otherwise, applying the morphism $\psi$ would lead to a contradiction.

Now we may apply Hilbert’s irreducibility theorem. There is a $k$-specialization $\lambda: k[X_1, \ldots, X_n][1/f] \to k$ such that $T^m + \lambda(\varphi(a_1)) T^{m-1} + \cdots + \lambda(\varphi(a_m))$ is irreducible in $k[T]$. Thus $\lambda \circ \varphi: A \to k$ is the $k$-specialization we need. The remaining proof is similar to [Sw2] Section 3.

Now we turn to Frobenius groups.

**Definition 1.3** ([Is] pages 181–182) A finite group $G$ is called a Frobenius group if $G = N \rtimes G_0$ where $N$ and $G_0$ are non-trivial subgroups of $G$ satisfying (i) $N$ is a normal subgroup of $G$, and (ii) the action of $G_0$ on $N$ is fixed point free, i.e. for any $x \in N \setminus \{1\}$, any $g \in G_0 \setminus \{1\}$, $g x g^{-1} \neq x$.

In other words, a group $G$ is a Frobenius group if and only if the celebrated Frobenius Theorem in representation theory is valid for $G$ [Ro, Theorem 8.5.5, page 243]. In this situation, the normal subgroup $N$ is called the Frobenius kernel of $G$, and the subgroup $G_0$ (or any of its conjugates in $G$) is called a Frobenius complement of $G$.

A group is called a Frobenius complement (resp. a Frobenius kernel) if it is a Frobenius complement (resp. a Frobenius kernel) of some Frobenius group.

**Theorem 1.4** ([Ro, page 299; Is, Chapter 6]) Let $G = N \rtimes G_0$ be a Frobenius group with kernel $N$ and complement $G_0$.

1. (John Thompson) $N$ is a nilpotent group.
2. (Burnside) The $p$-Sylow subgroups of $G_0$ are cyclic if $p \geq 3$, while the 2-Sylow subgroups of $G_0$ are cyclic or generalized quaternion.

Frobenius groups and Frobenius complements are ubiquitous in various mathematical areas (see Section 2). We will coin some names for these groups.

**Definition 1.5** Following the terminology used by Suzuki [Su], a finite group $G$ is called a $Z$-group if all of its Sylow subgroups are cyclic. Imitating this usage, we called a finite $G$ a $GZ$-group if the $p$-Sylow subgroups of $G$ are cyclic for $p \geq 3$ and the 2-Sylow subgroups of $G$ are cyclic or generalized quaternion.

From Theorem 1.4, we find that the Frobenius complements are $GZ$-groups.

The purpose of this paper is to study the rationality problems of Frobenius groups and Frobenius complements. But let us discuss an example first.
Example 1.6 Let \( p, q \) be distinct odd prime numbers. Define \( G = \langle \sigma, \tau : \sigma^p = \tau^q = 1, \tau \sigma \tau^{-1} = \sigma^r \rangle \) where \( 2 \leq r \leq p - 1 \) and \( r^q \equiv 1 \pmod{p} \). Then \( G \) is a Frobenius group and \( G \cong C_p \rtimes C_q \). By Theorem 3.9 if \( \mathbb{Z}[\zeta_q] \) is a unique factorization domain, then \( \mathbb{C}(C_p \rtimes C_q) \) is rational over \( \mathbb{C} \). However, when \( \mathbb{Z}[\zeta_q] \) is not a unique factorization domain, it is still unknown whether \( \mathbb{C}(C_p \rtimes C_q) \) is rational or not.

Because of the above example, we will study the retract rationality of \( k(G) \) where \( k \) is a field satisfying mild assumptions, and \( G \) is a \( GZ \)-group, a Frobenius complement or a Frobenius group with abelian kernel. We will not aim at the rationality problem for these groups in this paper. The rationality problem of \( \mathbb{Q}(G) \) where \( G = (\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z})^\times \) was indeed explored by Samson Breuer [Br] (when \( p \) is a prime number with \( p \leq 19 \)); we will study the rationality problem for these groups in a separate paper.

The retract rationality for a Frobenius group with abelian kernel is a consequence of that for \( GZ \)-groups (for the details about the results for \( GZ \)-groups, see Section 4). We list our results for Frobenius groups as follows.

**Theorem 1.7** Let \( G \) be a solvable Frobenius group of exponent \( e = 2^u3^l n \) (where \( u, l \geq 0, 2 \nmid n, 3 \nmid n \)), and \( k \) be an infinite field with \( \text{char } k \neq 2, 3 \), and \( \zeta_{2^u'}, \zeta_{3^l} \in k \) where \( u' = \max\{u, 3\} \). If the Frobenius kernel of \( G \) is an abelian group, then \( k(G) \) is retract \( k \)-rational.

**Theorem 1.8** Let \( G \) be a non-solvable Frobenius group. If \( k \) is an infinite field with \( \text{char } k = 2 \) or \( \text{char } k = 0 \) such that \( k(\zeta_8) \) is a cyclic extension of \( k \), then \( k(G) \) is retract \( k \)-rational.

Note that, in Theorem 1.8, the assumption that the Frobenius kernel is abelian is unnecessary because of Theorem 2.1 and Theorem 2.9. A criterion of retract rationality when the Frobenius complement is a \( Z \)-group and the Frobenius kernel is abelian is given in Theorem 4.2.

**Example 1.9** Let \( p \) be a prime number with \( p \equiv 1 \pmod{8} \). Choose an integer \( r \) such that \( 2 \leq r \leq p - 1 \) and \( r^q \equiv 1 \pmod{p} \). Define \( G = \langle \sigma, \tau : \sigma^p = \tau^8 = 1, \tau \sigma \tau^{-1} = \sigma^r \rangle \). Then \( G \) is a Frobenius group and \( G \cong C_p \rtimes C_8 \). We claim that \( \mathbb{Q}(G) \) is not retract \( \mathbb{Q} \)-rational. Otherwise, \( \mathbb{Q}(C_8) \) would be retract \( \mathbb{Q} \)-rational by Theorem 3.1. But this is impossible by Theorem 3.2.

The above example illustrates that, in Theorem 1.7, the assumption that \( \zeta_8 \in k \) is crucial. A similar situation happens to Theorem 1.8; see Example 4.12.

Applying Theorem 1.7 and Theorem 1.8 together with Theorem 1.2 we deduce results of the inverse Galois problem. We record only results for non-solvable Frobenius groups.

**Theorem 1.10** Let \( G \) a non-solvable Frobenius group. If \( k \) is an algebraic number field such that \( k(\zeta_8) \) is a cyclic extension of \( k \), then there is a Galois extension field \( K \) over \( k \) whose Galois group is isomorphic to \( G \).
According to Suzuki’s classification (see Theorem 2.8), there are two types of non-solvable Frobenius groups. The result in Theorem 1.8 for the type (1) group in Theorem 2.8 may be sharpened as follows.

Theorem 1.11 Let $k$ be a field with $\text{char } k = 0$, $G$ be a Frobenius group whose Frobenius complement is isomorphic to $G_1 \times G_2$ where $G_1$ is a $Z$-group and $G_2 \cong SL_2(\mathbb{F}_5)$. Then $k(G)$ is retract $k$-rational.

A corollary of the above theorem is the following.

Theorem 1.12 Let $k$ be an algebraic number field, $G$ be a Frobenius group whose Frobenius complement is isomorphic to $G_1 \times G_2$ where $G_1$ is a $Z$-group and $G_2 \cong SL_2(\mathbb{F}_5)$. Then there is a Galois extension field $K$ over $k$ whose Galois group is isomorphic to $G$.

We note that it is possible to solve the inverse Galois problem in Theorem 1.12 by other methods using [Fe; Mc; ILF, page 55, Theorem 3.12] or [MM, page 326, Theorem 8.1].

The proof of Theorem 1.7, Theorem 1.8 and Theorem 1.11 will be given at the end of Section 4. We use Saltman’s result (Theorem 3.1) and the structure theorems of Frobenius groups as a method of reduction process. The structure theorems of Frobenius groups were investigated by Zassenhaus, Suzuki, Wolf and other people [Za1; Su; Wo]. We summarize these results in Section 2. However, to prove the retract rationality is rather tricky. Sometimes we should prove the stronger result of rationality in order to prove the retract rationality; thus we are reduced to solving Noether’s problem for the corresponding groups. For example, Theorem 4.3 proves that $k(G_1)$ and $k(G_2)$ are $k$-rational where $G_1 \cong Q_8 \rtimes C_3'$, $G_2$ is defined by the group extension $1 \to G_1 \to G_2 \to C_2 \to 1$ ($Q_8$ is the quaternion group of order 8) and $k$ is a field containing $\zeta_e$ with $\text{exp}(G_2) = e$. Also see the proof of Theorem 4.10 and Theorem 4.11.

In Section 5, some remarks about the unramified Brauer groups for Frobenius groups will be given; here we don’t assume the Frobenius kernels are abelian.

Acknowledgments. I should like to thank I. M. Isaacs, E. Khukhro and Victor Mazurov for providing many helpful messages about Frobenius groups.

Standing notations. In discussing retract rationality, we always assume that the ground field is infinite (see Definition 1.1). For emphasis, recall $k(G) = k(x_g : g \in G)^G$, which is defined in the first paragraph of this section.

We denote by $\zeta_n$ a primitive $n$-th root of unity in some extension field of the ground field $k$. When we write $\zeta_n \in k$ or $\text{char } k \nmid n$, it is understood that either $\text{char } k = 0$ or $\text{char } k = p > 0$ with $p \nmid n$.

All the groups in this paper are finite groups. $C_n$ denotes the cyclic group of order $n$. The exponent of a group $G$, $\exp(G)$, is defined as $\exp(G) = \text{lcm}\{\text{ord}(g) : g \in G\}$. We denote $\mathbb{F}_p$ the finite field with $p$ elements.
§2. Structure theorems of Frobenius groups

We recall some standard results of Frobenius groups in this section.

**Theorem 2.1** ([Is, Lemma 6.1 and Theorem 6.3]) Let $G = N \rtimes G_0$ be a Frobenius group with kernel $N$ and complement $G_0$. Then

1. $\gcd\{|N|, |G_0|\} = 1$, and
2. if $|G_0|$ is even, then $N$ is abelian.

**Definition 2.2** ([Wo, page 160]) Let $p$ and $q$ be prime numbers (the situation $p = q$ is not excluded). We say that a finite group $G$ satisfies the $pq$-condition if every subgroup of order $pq$ in $G$ is cyclic.

**Theorem 2.3** ([Wo, Theorem 5.3.2, page 161; CE, Theorem 11.6, page 262]) Let $G$ be a finite group. Then the following conditions are equivalent,

(i) $G$ satisfies the $p^2$-condition for all prime numbers $p$;
(ii) $G$ is a $GZ$-group;
(iii) every abelian subgroup of $G$ is cyclic;
(iv) $G$ has periodic cohomology, i.e. there is some integer $d \neq 0$, some element $u \in \hat{H}^d(G, \mathbb{Z})$ such that the cup product map $u \cup - : \hat{H}^n(G, \mathbb{Z}) \to \hat{H}^{n+d}(G, \mathbb{Z})$ is an isomorphism for all integers $n$ (where $\hat{H}^n(G, \mathbb{Z})$ is the Tate cohomology).

Note that the equivalence of (ii), (iii) and (iv) of the above theorem was due to Artin and Tate [CE, p.232; Mi, p.627; Su, p.688]. The $GZ$-groups arise naturally in algebraic topology and differential geometry ([CE] pages 357–358; Wo; Jo]. For example, if $G$ is a finite group acting on a homology sphere without fixed points, then (i) P. A. Smith shows that all abelian groups of $G$ are cyclic (and therefore $G$ is a $GZ$-group by Theorem 2.3), and (ii) Milnor shows that there is at most one element of order 2, which is necessary to lie in the center of $G$ [Mi]. The complete determination of such groups was solved by Madsen, Thomas and Wall in 1976 (see [Bro, page 158]).

**Theorem 2.4** (Burnside [Ro, page 281, 10.1.10]) Let $G$ be a finite group. Then $G$ is a $Z$-group if and only if it is a split meta-cyclic group, i.e. $G = \langle \sigma, \tau : \sigma^m = \tau^n, \tau \sigma \tau^{-1} = \sigma^r \rangle$ where $m, n \geq 1, r^n \equiv 1 \pmod{m}$, $\gcd\{m, n(r - 1)\} = 1$.

In the above theorem, note that cyclic groups arise when $m = 1$.

**Theorem 2.5** (Burnside [Is, Theorem 6.9, page 188]) If $G$ is a Frobenius complement, then $G$ satisfies the $pq$-condition for all prime numbers $p$ and $q$ (the possibility $p = q$ is allowed).

**Theorem 2.6** (Zassenhaus [Za1, Za2, Maz, Me, Wo, Theorem 6.3.1, page 195]) Let $G$ be a Frobenius complement. If $G$ is a perfect group (i.e. $G$ satisfies the condition $G = [G, G]$), then $G$ is isomorphic to $SL_2(\mathbb{F}_5)$. 6
Theorem 2.7 (Zassenhaus [Za], [Wo] page 179, Theorem 6.1.11; Jo, pages 164–166) Let \( G \) be a finite solvable group. Then \( G \) is a \( GZ \)-group if and only if \( G \) is isomorphic to one of the following groups.

(I) \( G = \langle \sigma, \tau : \sigma^m = \tau^n, \tau \sigma \tau^{-1} = \sigma^r \rangle \) where \( m, n \geq 1, r^n \equiv 1 \pmod m, \gcd\{m, n(r - 1)\} = 1 \);

(II) \( G = \langle \sigma, \tau, \lambda : \sigma^m = \tau^n = 1, \lambda^2 = \tau^{n/2}, \tau \sigma \tau^{-1} = \sigma^r, \lambda \sigma \lambda^{-1} = \sigma^l, \lambda \tau \lambda^{-1} = \tau^k \rangle \) where \( m, n \geq 1, n = 2^u v \) with \( u \geq 2, 2 \nmid v, r^n \equiv 1 \pmod m, \gcd\{m, n(r - 1)\} = 1, l^2 \equiv r^{k-1} \equiv 1 \pmod m, k \equiv -1 \pmod{2^u}, k^2 \equiv 1 \pmod{n} \);

(III) \( G = \langle \sigma, \tau, \lambda, \rho : \sigma^m = \tau^n = \lambda^4 = 1, \lambda^2 = \rho^2 = (\lambda \rho)^2, \tau \sigma \tau^{-1} = \sigma^r, \lambda \sigma \lambda^{-1} = \sigma, \rho \sigma \rho^{-1} = \sigma, \tau \lambda \tau^{-1} = \rho, \tau \rho \tau^{-1} = \lambda \rho \rangle \) where \( m, n \geq 1, r^n \equiv 1 \pmod m, \gcd\{m, n(r - 1)\} = 1, n \equiv 1 \pmod{2^2} \) and \( n \equiv 0 \pmod{3} \);

(IV) \( G = \langle \sigma, \tau, \lambda, \rho, \nu : \sigma^m = \tau^n = \lambda^4 = 1, \lambda^2 = \rho^2 = (\lambda \rho)^2 = \nu^2, \tau \sigma \tau^{-1} = \sigma^r, \lambda \sigma \lambda^{-1} = \sigma, \rho \sigma \rho^{-1} = \sigma, \tau \lambda \tau^{-1} = \rho, \tau \rho \tau^{-1} = \lambda \rho, \nu \lambda \nu^{-1} = \rho \nu, \nu \rho \nu^{-1} = \rho^{-1}, \nu \sigma \nu^{-1} = \sigma^t, \nu \tau \nu^{-1} = \tau^k \rangle \) where \( m, n \geq 1, r^n \equiv r^{k-1} \equiv t^2 \equiv 1 \pmod m, \gcd\{m, n(r - 1)\} = 1, n \equiv 1 \pmod{2^2}, n \equiv k + 1 \equiv 0 \pmod{3}, k^2 \equiv 1 \pmod{n} \).

Theorem 2.8 (Suzuki [Su] Theorem E; Wo, pages 197–198; Jo, pages 169–170) Let \( G \) be a finite non-solvable group. Then \( G \) is a \( GZ \)-group if and only if \( G \) is isomorphic to one of the following groups.

(I) \( G = H \times SL_2(\mathbb{F}_p) \) where \( H \) is a \( Z \)-group, \( p \) is a prime number \( \geq 5 \) and \( \gcd\{|H|, |SL_2(\mathbb{F}_p)|\} = 1 \);

(II) \( G = \langle \sigma, \tau, \lambda, L \rangle \) where \( L \) is isomorphic to \( SL_2(\mathbb{F}_p) \) with \( p \) being a prime number \( \geq 5 \) and with the relations

\[
\sigma^m = \tau^n = \lambda^4 = 1, \quad \tau \sigma \tau^{-1} = \sigma^r, \quad \lambda \sigma \lambda^{-1} = \sigma^{-1}, \quad \tau \lambda = \lambda \tau, \quad \lambda^2 = \varepsilon \in L, \quad \forall \rho \in L, \quad \sigma \rho = \rho \sigma, \quad \tau \rho = \rho \tau, \quad \lambda \rho \lambda^{-1} = \theta(\rho)
\]

where \( \varepsilon \in L \) is the element corresponding to \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in SL_2(\mathbb{F}_p) \) under the isomorphism \( L \simeq SL_2(\mathbb{F}_p) \) and \( \theta \) is the automorphism on \( L \) induced from the automorphism \( \theta_0 \) on \( SL_2(\mathbb{F}_p) \) defined by

\[
\theta_0 : SL_2(\mathbb{F}_p) \to SL_2(\mathbb{F}_p), \quad \rho \mapsto \left( \begin{array}{cc} 0 & -1 \\ \omega & 0 \end{array} \right) \rho \left( \begin{array}{cc} 0 & -1 \\ \omega & 0 \end{array} \right)^{-1}
\]

where \( \omega \) is a generator of the multiplicative group \( \mathbb{F}_p \setminus \{0\} \) and \( r^n \equiv 1 \pmod m, \gcd\{m, n(r - 1)\} = \gcd\{mn, p(p^2 - 1)\} = 1 \).

Theorem 2.9 Let \( G_0 \) be a non-solvable Frobenius complement. Then \( G_0 \) is isomorphic to one of the groups (I) or (II) in Theorem 2.8 with \( p = 5 \).
Proof. By Theorem 1.4, $G_0$ is a non-solvable $GZ$-group. Hence we may apply Theorem 2.8. It remains to show that $p = 5$.

Since $L$ is a subgroup of $G_0$, $L$ is also a Frobenius complement (reason: If $N \rtimes G_0$ is a Frobenius group, then $N \rtimes L$ is also a Frobenius group by definition.).

Since $L \simeq SL_2(\mathbb{F}_p)$ is a perfect group (see, for example, [Ro, page 74, 3.2.13]), by Theorem 2.6 it is necessary that $p = 5$.

Frobenius complements appear as the groups satisfying all the $pq$ conditions in the list of the above Theorem 2.7, Theorem 2.8 and Theorem 2.9. Victor Mazurov informed us many of his works about Frobenius groups; most of them were written in Russian under the name V. D. Mazurov. For example, if $G$ is a sovable group satisfying all the $pq$ conditions in the list of Theorem 2.7, then $G$ is a Frobenius complement. It is known that $SL_2(\mathbb{F}_5)$ is a Frobenius complement [Hu, page 500]; Mazurov had another proof of it. However, it is still unknown whether non-solvable groups in the list of Theorem 2.8 and Theorem 2.9 satisfying all the $pq$ conditions are eligible Frobenius complements. The following result is implicit in Zassenhaus’s paper [Za1] and is contained in one of Mazurov’s Russian papers.

Theorem 2.10 Let $G$ be a finite group. Then $G$ is a Frobenius complement if and only if the subgroup of $G$ generated by all elements of (various) prime orders is isomorphic to $C_n \times H$ where $n$ is a square-free integer, $\gcd\{n, |H|\} = 1$ with $H = \{1\}, SL_2(\mathbb{F}_3)$, or $SL_2(\mathbb{F}_5)$.

§3. Preliminaries

We recall several known results of rationality problems in this section, which will be used later.

Theorem 3.1 (Saltman [Sa1, Theorem 3.1 and Theorem 3.5; Ka4, Theorem 3.5]) Let $k$ be an infinite field, $G = N \rtimes G_0$ where $N$ is a normal subgroup of $G$ with $G_0$ acting on $N$.

(1) If $k(G)$ is retract $k$-rational, so is $k(G_0)$.

(2) Assume furthermore that $N$ is abelian and $\gcd\{|N|, |G_0|\} = 1$. If both $k(N)$ and $k(G_0)$ are retract $k$-rational, so is $k(G)$.

Theorem 3.2 ([Sa2, Theorem 4.12; Ka4, Theorem 3.7]) Let $k$ be an infinite field and $G$ be a finite abelian group of exponent $e = 2^r s$ with $2 \nmid s$. Then $k(G)$ is retract $k$-rational if and only if $\text{char } k = 2$ or $k(\zeta_{2^r})$ is a cyclic extension of $k$.

Theorem 3.3 Let $G_1$ and $G_2$ be finite groups.
(1) (Saltman [Sa1, Theorem 1.5]) If \( k \) is an infinite field and both \( k(G_1) \) and \( k(G_2) \) are retract \( k \)-rational, then \( k(G_1 \times G_2) \) is also retract \( k \)-rational.

(2) (Kang and Plans [KP, Theorem 1.3]) For any field \( k \), if both \( k(G_1) \) and \( k(G_2) \) are \( k \)-rational, then \( k(G_1 \times G_2) \) is also \( k \)-rational.

**Theorem 3.4** (Ahmad, Hajja and Kang [AHK, Theorem 3.1]) Let \( L \) be any field, \( L(x) \) be the rational function field in one variable over \( L \), and \( G \) be a finite group acting on \( L(x) \). Suppose that, for any \( \sigma \in G \), \( \sigma(L) \subset L \) and \( \sigma(x) = a_\sigma \cdot x + b_\sigma \) where \( a_\sigma, b_\sigma \in L \) and \( a_\sigma \neq 0 \). Then \( L(x)^G = L^G(f) \) for some polynomial \( f \in L[x] \). In fact, if \( m = \min \{ \deg g(x) : g(x) \in L[x]^G \setminus L^G \} \), any polynomial \( f \in L[x]^G \) with \( \deg f = m \) satisfies the property \( L(x)^G = L^G(f) \).

**Theorem 3.5** (Hajja and Kang [HK, Theorem 1]) Let \( G \) be a finite group acting on \( L(x_1, \ldots, x_n) \), the rational function field in \( n \) variables over a field \( L \). Suppose that

(i) for any \( \sigma \in G \), \( \sigma(L) \subset L \),

(ii) the restriction of the action of \( G \) to \( L \) is faithful,

(iii) for any \( \sigma \in G \),

\[
\begin{pmatrix}
\sigma(x_1) \\
\sigma(x_2) \\
\vdots \\
\sigma(x_n)
\end{pmatrix} = A(\sigma) \cdot \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} + B(\sigma)
\]

where \( A(\sigma) \in GL_n(L) \) and \( B(\sigma) \) is an \( n \times 1 \) matrix over \( L \).

Then there exist elements \( z_1, \ldots, z_n \in L(x_1, \ldots, x_n) \) so that \( L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n) \) and \( \sigma(z_i) = z_i \) for any \( \sigma \in G \), any \( 1 \leq i \leq n \).

**Theorem 3.6** (Kang and Plans [KP, Theorem 1.1]) Let \( k \) be a field with \( \text{char} \ k = p > 0 \) and \( \tilde{G} \) be a group defined by \( 1 \to \mathbb{Z}/p\mathbb{Z} \to \tilde{G} \to G \to 1 \) where \( G \) is a finite group. Then \( k(\tilde{G}) \) is rational over \( k(G) \). In particular, if \( G \) is a \( p \)-group and \( k \) is a field with \( \text{char} \ k = p > 0 \), then \( k(G) \) is \( k \)-rational (Kuniyoshi’s Theorem).

**Theorem 3.7** (Fischer [Sw2, Theorem 6.1]) Let \( G \) be a finite abelian group of exponent \( e \), and let \( k \) be a field containing a primitive \( e \)-th root of unity. Then \( k(G) \) is rational over \( k \).

**Theorem 3.8** (Hajja [Ha]) Let \( G \) be a finite group acting on the rational function field \( k(x, y) \) by monomial \( k \)-automorphisms, i.e. for any \( \sigma \in G \), \( \sigma(x) = \alpha \cdot x^a y^b \), \( \sigma(y) = \beta \cdot x^c y^d \) where \( \alpha, \beta \in k \setminus \{0\} \), \( a, b, c, d \in \mathbb{Z} \) and \( \alpha, \beta, a, b, c, d \) are parameters depending on \( \sigma \). Then \( k(x, y)^G \) is \( k \)-rational.

**Theorem 3.9** (Kang [Ka2, Corollary 3.2]) Let \( k \) be a field and \( G \) be a finite group. Assume that (i) \( \tilde{G} \) contains an abelian normal subgroup \( H \) so that \( G/H \) is cyclic of order \( n \), (ii) \( \mathbb{Z}[\mathbb{Z}_n] \) is a unique factorization domain, and (iii) \( \zeta_{e'} \in k \) where \( e' = \text{lcm}\{\text{ord}(\tau), \text{exp}(H)\} \) and \( \tau \) is an element of \( G \) whose image generates \( G/H \). Then \( k(G) \) is \( k \)-rational.
§4. Main results

Lemma 4.1 Let $G$ be a group isomorphic to the group (I) in Theorem 2.7. Suppose that $\exp(G) = 2^s$ with $2 \nmid s$. If $k$ is an infinite field such that either $\text{char } k = 2$ or $k(\zeta_{2^s})$ is a cyclic extension of $k$, then $k(G)$ is retract $k$-rational.

Proof. Note that $G = G_1 \times G_2$ where $G_1 \simeq C_m$, $G_2 \simeq C_n$ with $\gcd\{m, n\} = 1$. By Theorem 3.2 both $k(G_1)$ and $k(G_2)$ are retract $k$-rational. Apply Theorem 3.1. Done.

Theorem 4.2 Let $G = N \rtimes G_0$ be a Frobenius group with kernel $N$ and complement $G_0$. Let $\exp(G) = 2^s$ where $2 \nmid s$. Assume that (i) $G_0$ is a $Z$-group, and (ii) $N$ is an abelian group. If $k$ is an infinite field such that either $\text{char } k = 2$ or $k(\zeta_{2^s})$ is a cyclic extension of $k$, then $k(G)$ is retract $k$-rational.

Proof. By Theorem 2.4, $G_0$ is a group isomorphic to the group (I) in Theorem 2.7. Moreover, $\gcd\{|N|, |G_0|\} = 1$ by Theorem 2.1.

Apply Theorem 3.2 and Lemma 4.1. Both $k(N)$ and $k(G_0)$ are retract $k$-rational. Thus $k(G)$ is also retract $k$-rational by Theorem 3.1. ■

Lemma 4.3 Let $G$ be a group isomorphic to the group (II) in Theorem 2.7 with the parameters $m$, $n$, $r$, $\ldots$ defined there such that $n = 2^u v$ (where $u \geq 2, 2 \nmid v$). If $k$ is an infinite satisfying that either $\text{char } k = 2$ or $\zeta_{2^u} \in k$ (when $\text{char } k \neq 2$), then $k(G)$ is retract $k$-rational.

Proof. Define $G_1 = \langle \sigma \rangle \subset G$, $G_2 = \langle \tau, \lambda \rangle$. Then $G \simeq G_1 \times G_2$ with $G_1 \triangleleft G$.

We claim that $m$ is an odd integer. Otherwise, the subgroup $\langle \sigma^{m/2}, \lambda^2 \rangle$ is isomorphic to $C_2 \times C_2$. This is impossible because $G$ is a $GZ$-group.

Thus $\gcd\{|G_1|, |G_2|\} = 1$ and $k(G_1)$ is retract $k$-rational by Theorem 3.2. If $k(G_2)$ is retract $k$-rational, then $G(G_2)$ is retract $k$-rational by Theorem 3.1.

Define $\tau_1 = \tau^{2^u}$, $\tau_2 = \tau^v$, $H_1 = \langle \tau_1 \rangle$, $H_2 = \langle \tau_2, \lambda \rangle$. Then $G_2 \simeq H_1 \times H_2$ and $H_2$ is a 2-group, while $2 \nmid |H_1|$. By Theorem 3.1, if $k(H_2)$ is retract $k$-rational, then $k(G_2)$ is also retract $k$-rational.

Note that $H_2 = \langle \tau_2, \lambda \rangle$ is a 2-group with $\tau_2^{2^u} = \lambda^4 = 1$, $\lambda \tau_2 \lambda^{-1} = \tau_2^k$ and $\lambda^2 = \tau_2^{2^{u-1}}$.

If $\text{char } k \neq 2$ and $\zeta_{2^u} \in k$, by Theorem 3.9, $k(H_2)$ is $k$-rational. Hence $k(H_2)$ is retract $k$-rational.

If $\text{char } k = 2$, $k(H_2)$ is $k$-rational by Kuniyoshi’s Theorem (see Theorem 3.6). Hence the result.

Theorem 4.4 Define two finite groups $G_1$ and $G_2$ by

$$G_1 = \langle \tau, \lambda, \rho : \tau^3 = \lambda^4 = 1, \lambda^2 = \rho^2 = (\lambda \rho)^2, \tau \lambda \tau^{-1} = \rho, \tau \rho \tau^{-1} = \lambda \rho \rangle$$

where $l \geq 1$, and

$$G_2 = \langle \nu, \lambda : \lambda^2 = \nu^2, \nu \lambda \nu^{-1} = \rho \lambda, \nu \rho \nu^{-1} = \rho^{-1}, \nu \tau \nu^{-1} = \tau^k \rangle$$

where $l \geq 1$. Then $k(G_1)$ is retract $k$-rational.

Proof. By Theorem 2.4, $G_0$ is a group isomorphic to the group (I) in Theorem 2.7. Moreover, $\gcd\{|N|, |G_0|\} = 1$ by Theorem 2.1.

Apply Theorem 3.2 and Lemma 4.1. Both $k(N)$ and $k(G_0)$ are retract $k$-rational. Thus $k(G)$ is also retract $k$-rational by Theorem 3.1. ■
where \( k + 1 \equiv 0 \pmod{3} \) and \( k^2 \equiv 1 \pmod{3^l} \).

Let \( k \) be a field with \( \zeta \in k \) where \( \exp(G_2) = e \). Then both \( k(G_1) \) and \( k(G_2) \) are \( k \)-rational.

Remark. Let \( G_1 \) and \( G_2 \) be the groups defined above. When \( l = 1 \) (recall \( \tau^3 = 1 \)), it can be shown that \( G_1 \cong \tilde{A}_4 \cong SL_2(\mathbb{F}_3) \) and \( G_2 \cong \tilde{S}_4 \) (see Definition 4.8); the rationality problem of \( k(SL_2(\mathbb{F}_3)) \) and \( k(\tilde{S}_4) \) has been solved in [Ri] and [KZ, Theorem 1.4] respectively. When \( l \geq 2 \) and \( k \) is a field with \( \text{char } k = 2 \), we don’t know whether \( k(G_1) \) and \( k(G_2) \) are \( k \)-rational or not.

Proof. Note that \( \langle \lambda, \rho \rangle \cong \{\pm 1, \pm i, \pm j, \pm k\} \) the quaternion group of order 8. Thus the 2-Sylow subgroups of \( G_1 \) and \( G_2 \) are quaternion group and the generalized quaternion group of order 16 respectively. It follows that \( \exp(G_1) = 4 \cdot 3^l \) and \( \exp(G_2) = 8 \cdot 3^l \).

Case 1. The group \( G_1 \) and char \( k \neq 2, 3 \) with \( \zeta_3, \zeta_8 \in k \).
Write \( \zeta = \zeta_3 \). Define \( \eta = \zeta_8 \) with \( \eta^2 = \sqrt{-1} \). Define a representation of \( G_1 \) by

\[
\Phi : G_1 \rightarrow GL_2(k) \\
\lambda \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \\
\rho \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\tau \mapsto \frac{\zeta}{\sqrt{2}} \begin{pmatrix} -\eta & \eta \\ \eta^3 & \eta^3 \end{pmatrix}.
\]

\( \Phi \) is a faithful irreducible representation of \( G_1 \). Let \( k \cdot x_1 \oplus k \cdot x_2 \) be its representation space. We can embed \( k \cdot x_1 \oplus k \cdot x_2 \) into the regular representation space \( \oplus_{g \in G_1} k \cdot x_g \). Thus the \( G_1 \)-field \( k(x_1, x_2) \) can be embedded into the \( G_1 \)-field \( k(x_g : g \in G_1) \). Applying Theorem 3.5, we get \( k(G_1) = k(x_g : g \in G_1)^{G_1} = k(x_1, x_2)^{G_1}(u_1, \ldots, u_t) \) where \( t = |G_1| - 2 = 8 \cdot 3^l - 2 \) and \( u_1, \ldots, u_t \) are elements fixed by \( G_1 \).

Define \( x = x_1/x_2 \). Then \( k(x_1, x_2) = k(x, x_2) \) and, for any \( g \in G_1, g \cdot x \in k(x), g(x_2) = \alpha_g x_2 \) for some \( \alpha_g \in k(x) \). Applying Theorem 3.4, we get \( k(x, x_2)^{G_1} = k(x)^{G_1}(f) \) for some element \( f \) fixed by \( G_1 \). By Lüroth’s Theorem, \( k(x)^{G_1} \) is \( k \)-rational. Hence \( k(G_1) \) is \( k \)-rational.

Case 2. The group \( G_2 \) and char \( k \neq 2, 3 \) with \( \zeta_3, \zeta_8 \in k \).
Since \( 3^l \mid k^2 - 1 \) and \( 3 \mid k + 1 \), it follows that \( 3^l \mid k + 1 \), i.e. \( \nu \tau \nu^{-1} = \tau^{-1} \).
Step 1. As before, write $\zeta = \zeta_3$. Define a representation of $G_2$ by

$$
\Psi : G_2 \to GL_4(k)
$$

$$
\begin{array}{ccc}
\lambda & \mapsto & \begin{pmatrix}
\sqrt{-1} & 0 \\
0 & -\sqrt{-1} \\
\sqrt{-1} & 0 \\
0 & \sqrt{-1}
\end{pmatrix} \\
\rho & \mapsto & \begin{pmatrix}
0 & -1 \\
1 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix} \\
\tau & \mapsto & \begin{pmatrix}
-\frac{1-\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\
-\frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \\
1 & 0 \\
0 & \zeta^{-1}
\end{pmatrix} \\
\nu & \mapsto & \begin{pmatrix}
\frac{\sqrt{-1}}{2} & \frac{\sqrt{-1}}{2} \\
\frac{-\sqrt{-1}}{2} & \frac{-\sqrt{-1}}{2} \\
0 & 1 \\
1 & 0
\end{pmatrix}
\end{array}
$$

It is not difficult to see that $\Psi$ is a faithful representation of $G_2$. Let $\oplus_{1\leq i \leq 4} k \cdot x_i$ be its representation space. Thus we may embed $\oplus_{1\leq i \leq 4} k \cdot x_i$ into the regular representation space $\oplus_{g \in G_2} k \cdot x_g$. By the same method as in Case 1, we embed the $G_2$-field $k(x_1, x_2, x_3, x_4)$ into the $G_2$-field $k(x_g : g \in G_2)$ and apply Theorem 3.5. We find that $k(G_2)$ is rational over $k(x_1, x_2, x_3, x_4)^{G_2}$.

Step 2. Define $y_1 = x_1/x_2$, $y_2 = x_3/x_4$. Then $k(x_1, x_2, x_3, x_4) = k(y_1, y_2, x_2, x_4)$ and, for all $g \in G_2$, $g(y_1), g(y_2) \in k(y_1, y_2)$, $g(x_2) = \alpha_g x_2$, $g(x_4) = \beta_g x_4$ for some $\alpha_g, \beta_g \in k(y_1, y_2)$. Applying Theorem 3.4 twice, we find that $k(y_1, y_2)^{G_2} = k(y_1, y_2)^{G_2}(v_1, v_2)$ for some elements $v_1, v_2$ fixed by $G_2$.

Note that $\tau(x_1) = [(1-\sqrt{-1})x_1/2] + [(1+\sqrt{-1})x_1/2]$, $\tau(x_2) = [(1+\sqrt{-1})x_1/2] + [(-1 + \sqrt{-1})x_2/2]$, etc. Thus the action of $G_2$ on $k(y_1, y_2)$ is given by

$$
\begin{array}{ccc}
\lambda : y_1 & \mapsto & -y_1, \\
\rho : y_1 & \mapsto & -1/y_1, \\
\tau : y_1 & \mapsto & (-y_1 + \sqrt{-1})/(y_1 + \sqrt{-1}), \\
\nu : y_1 & \mapsto & (y_1 + 1)/(y_1 - 1), \\
\end{array}
$$

Define $y_3 = y_1^2 + (1/y_1^2)$. Then $k(y_1, y_2)^{(\lambda, \rho)} = k(y_2, y_3)$ and

$$
\tau : y_3 \mapsto (2y_3 - 12)/(y_3 + 2), \quad \nu : y_3 \mapsto (2y_3 + 12)/(y_3 - 2).
$$

Note that $\tau^3(y_3) = y_3$. Define $y_4 = y_2^{y_3-1}$. Then $k(y_2, y_3)^{(\tau^3)} = k(y_3, y_4)$. 

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The action of \((\tau, \nu)\) on \(k(y_3, y_4)\) is given by

\[
\begin{align*}
\tau : y_3 &\mapsto (2y_3 - 12)/(y_3 + 2), \quad y_4 \mapsto \omega' y_4 \\
\nu : y_3 &\mapsto (2y_3 + 12)/(y_3 - 2), \quad y_4 \mapsto 1/y_4
\end{align*}
\]

where \(\omega' = \zeta^{2:3^{k-1}}\) and therefore \(\omega'\) is a primitive 3rd root of unity.

Step 3. Note that, for a \(k\)-automorphism \(\sigma\) on \(k(x)\) with \(\sigma(x) = (ax + b)/(cx + d)\) where \(a, b, c, d \in k\) and \(ad - bc \neq 0\), if \(y = (ax + b)/(\gamma x + \delta)\) with \(\alpha, \beta, \gamma, \delta \in k\) and \(\alpha \delta - \beta \gamma \neq 0\), then \(\sigma(y) = (Ay + B)/(Cy + D)\) where \(A, B, C, D\) are defined by

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\]

In particular, if

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
tw & 0 \\
0 & t
\end{pmatrix}
\]

where \(ad - bc \neq 0\), \(\alpha \delta - \beta \gamma \neq 0\) and \(\sigma(x) = (ax + b)/(cx + d)\), then \(\sigma(y) = wy\) if \(y\) is defined as \(y = (ax + \beta)/(\gamma x + \delta)\).

Step 4. Return to the actions of \(\tau\) and \(\nu\) on \(y_3\). Since \(\tau^3(y_3) = 1\), it follows that the order of the matrix \(\begin{pmatrix} 2 & -12 \\ 1 & 2 \end{pmatrix} \in PGL_2(k)\) is 3. Regard this matrix as a \(2 \times 2\) matrix over \(k\). Its characteristic polynomial is \((X - 2)^2 + 12 = (X - 2 - 2\sqrt{-3})(X - 2 + 2\sqrt{-3})\). Since \(\zeta_3 \in k\) (because \(\zeta_3^2 \in k\)), it follows the eigenvalues \(2 \pm \sqrt{-3} \in k\). Hence this matrix can be diagonalized over \(k\). In other words, there is an invertible \(2 \times 2\) matrix \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(k)\) such that

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
2 & -12 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}^{-1}
= \begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}.
\]

Since the order of \(\begin{pmatrix} 2 & -12 \\ 1 & 2 \end{pmatrix} \in PGL_2(k)\) is 3, it is necessary that \(a/b = \omega\) is a primitive 3rd root of unity (where \(\omega' = \omega\) or \(\omega^2\)).

Apply the result of Step 3. Define \(y_5 = (\alpha y_3 + \beta)/(\gamma y_3 + \delta)\). Then \(\tau(y_5) = \omega y_5\).

Note that \(\nu(y_5) = (Ay_5 + B)/(Cy_5 + D)\) for some \(A, B, C, D \in k\) with \(AD - BC \neq 0\).

Because \(\nu \tau\nu^{-1} = \tau^{-1}\), we find \(\tau \nu \tau = \nu\) and \(\tau \nu \tau(y_5) = \nu(y_5)\). Plugging in the formula \(\nu(y_5) = (Ay_5 + B)/(Cy_5 + D)\), we get the identity

\[
(\omega^2 Ay_5 + \omega B)/(\omega Cy_5 + D) = (Ay_5 + B)/(Cy_5 + D).
\]

The above identity implies that \(A = D = 0\). In other words, \(\nu(y_5) = d/y_5\) for some \(d \in k \setminus \{0\}\).

In summary, we have \(k(y_3, y_4) = k(y_4, y_5)\) and

\[
\begin{align*}
\tau : y_4 &\mapsto \omega' y_4, \quad y_5 \mapsto \omega y_5, \\
\nu : y_4 &\mapsto 1/y_4, \quad y_5 \mapsto d/y_5.
\end{align*}
\]

By Theorem 3.8, \(k(y_4, y_5)^{\tau, \nu}\) is \(k\)-rational. Hence \(k(G_2)\) is \(k\)-rational. 

\[\blacksquare\]
Theorem 4.5 Let $G$ be a group isomorphic to the group (III) in Theorem 2.7 with the parameters $m$, $n$, $r$, ... defined there. Write $n = 3^n' w$ with $l \geq 1$ and $3 \nmid n'$. If $k$ is an infinite field satisfying that either $\text{char } k = 2$ or $\zeta_3 \in k$ (when $\text{char } k \neq 2, 3$), then $k(G)$ is retract $k$-rational.

Proof. Step 1. Note that the parameter $m$ is an odd integer as in the proof of Lemma 4.3.

Define $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau, \lambda, \rho \rangle$. Then $G \simeq H_1 \rtimes H_2$, $H_1$ is an abelian normal subgroup of $G$, and $\gcd\{ |H_1|, |H_2| \} = 1$.

Since $|H_1| = m$ is odd, $k(H_1)$ is retract $k$-rational by Theorem 3.2. We will prove that $k(H_2)$ is also retract $k$-rational. Once it is proved, we may apply Theorem 3.1 to conclude that $k(G)$ is retract $k$-rational.

Step 2. Note that $n = 3^n w'$ and $n$ is odd by assumption. Define $\tau_1 = \tau^w \varepsilon$ and $\tau_2 = \tau^w$. Then $H_2 = \langle \tau_1, \tau_2, \lambda, \rho \rangle$ with $\tau_1^2 = \tau_2^2 = 1$ and $\tau_1 \lambda \tau_1^{-1} = \lambda$, $\tau_1 \rho \tau_1^{-1} = \rho$.

Define $H_3 = \langle \tau_1 \rangle$, $H_4 = \langle \tau_2, \lambda, \rho \rangle$. Then $H_2 \simeq H_3 \rtimes H_4$, $H_3$ is an abelian normal subgroup of $H_2$, and $\gcd\{ |H_3|, |H_4| \} = 1$. $k(H_3)$ is retract $k$-rational by Theorem 3.2. We will prove that $k(H_4)$ is retract $k$-rational in the next step. Once it is proved, we obtain that $k(H_2)$ is retract $k$-rational by Theorem 3.1.

Step 3. We will show that $k(H_4)$ is retract $k$-rational.

Note that $H_4$ is isomorphic to the group $G_1$ in Theorem 4.4. In fact, if $n' \equiv 1$ (mod 3), then $\tau_2 \lambda \tau_2^{-1} = \rho$, $\tau_2 \rho \tau_2^{-1} = \lambda \rho$; thus $H_4$ is identical to $G_1$. If $n' \equiv 2$ (mod 3), then $\tau_2 \lambda \tau_2^{-1} = \lambda \rho$, $\tau_2 \rho \tau_2^{-1} = \rho$; replace the generators $\lambda$, $\rho$ by $\lambda$, $\lambda \rho$ and we find $H_4 \simeq G_1$.

Case 1 char $k \neq 2, 3$ and $\zeta_3 \varepsilon, \zeta_8 \in k$.

By Theorem 4.4 $k(G_1)$ is $k$-rational. Thus $k(H_4)$ is $k$-rational. In particular, it is retract $k$-rational.

Case 2 char $k = 2$.

Consider the group extension $1 \to \langle \lambda^2 \rangle \to H_4 \to H_5 \to 1$. Applying Theorem 3.1 we find that $k(H_4)$ is rational over $k(H_5)$.

Note that $H_5 = \langle \bar{\lambda}, \bar{\rho}, \bar{\tau}_2 \rangle \simeq V \times C_3 \varepsilon$ where $V = \langle \bar{\lambda}, \bar{\rho} \rangle \simeq C_2 \times C_2$. Since both $k(V)$ and $k(C_3 \varepsilon)$ are retract $k$-rational Theorem 3.2 we find that $k(H_5)$ is also retract $k$-rational.

Since retract rationality is stable under rational extension [Sa2, Proposition 3.6; Ka4, Lemma 3.4], we conclude that $k(H_4)$ is also retract $k$-rational. 

Theorem 4.6 Let $G$ be a group isomorphic to the group (IV) in Theorem 2.7 with the parameters $m$, $n$, $r$, ... defined there. Write $n = 3^n w'$ with $l \geq 1$ and $3 \nmid n'$. If $k$ is an infinite field with $\text{char } k \neq 2, 3$ and $\zeta_3 \varepsilon, \zeta_8 \in k$, then $k(G)$ is retract $k$-rational. In other words, if $\exp(G) = 2^{u} 3^{v} t$ where $u, l \geq 1$, $2 \nmid t$, $3 \nmid t$, then $k(G)$ is retract $k$-rational provided that $\zeta_2^{u}, \zeta_3 \varepsilon \in k$.

Proof. The proof is similar to that of Theorem 4.5.

Choose $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau, \lambda, \rho, \nu \rangle$. Then $G \simeq H_1 \rtimes H_2$. To prove that $k(G)$ is retract $k$-rational, it suffices to show that so is $k(H_2)$. 

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As $n = 3^i n'$, define $\tau_1 = \tau^{3^i}$, $\tau_2 = \tau^{n'}$. Define $H_2 = \langle \tau_1 \rangle$, $H_4 = \langle \tau_2, \lambda, \rho, \nu \rangle$. Then $H_2 \simeq H_3 \rtimes H_4$. To prove that $k(H_2)$ is retract $k$-rational, it suffices to show that so is $k(H_4)$.

Since $H_4$ is isomorphic to the group $G_2$ in Theorem 4.4 and $\exp(G_2) = 8 \cdot 3^i$, it follows that $k(H_4)$ is $k$-rational by Theorem 4.4. Hence $k(H_4)$ is retract $k$-rational.

For final assertion note that, if $\exp(G) = 2^u 3^i t$, then $\exp(G_2) = 2^u 3^i$.

**Theorem 4.7** Let $G$ be a solvable $GZ$-group of exponent $2^u 3^i t$ where $u, l \geq 0$, $2 \nmid t$, $3 \nmid t$. If $k$ is an infinite field such that $\text{char} k \neq 2, 3$, $\zeta_{3^i} \in k$ and $\zeta_{2^u}$ (where $u' = \max\{3, u\}$), then $k(G)$ is retract $k$-rational.

**Proof.** By Theorem 2.7, $G$ is isomorphic to the groups (I)–(IV). Apply Lemma 4.1, Lemma 4.3, Theorem 4.5 and Theorem 4.6. ■

**Definition 4.8** For $n \geq 4$, there are two inequivalent non-split central extensions of $S_n$ by $\mathbb{Z}/2\mathbb{Z}$. We follow the notations of Serre [GMS, pages 58, 88, 90]. The non-split central extension $1 \to \{\pm 1\} \to \tilde{S}_n \to S_n \to 1$ defines a double cover $\tilde{S}_n$ of $S_n$ in which the transposition and the product of two disjoint transpositions in $S_n$ lift to elements of order 4 of $\tilde{S}_n$. The non-split central extension $1 \to \{\pm 1\} \to \tilde{S}_n \to S_n \to 1$ defines a double cover $\tilde{S}_n$ of $S_n$ in which a transposition in $S_n$ lifts to an element of order 2 of $\tilde{S}_n$, but a product of two disjoint transpositions lifts to an element of order 4. The non-split central extension $1 \to \{\pm 1\} \to \tilde{A}_n \to A_n \to 1$ defines the (unique) double cover $\tilde{A}_n$ of $A_n$ [Se, page 88].

**Lemma 4.9**

1. $SL_2(\mathbb{F}_5) \simeq \tilde{A}_5$.

2. Let $G$ be the group (II) in Theorem 2.8 with $p = 5$. Let $G_+$ be the subgroup of $G$ defined by $G_+ = \langle \lambda, L \rangle$. Then $G_+ \simeq \tilde{S}_5$.

**Proof.** Step 1. Note that $1 \to \{\pm 1\} \to \tilde{A}_5 \to A_5 \to 1$ is the unique non-split extension of $A_5$ by $\{\pm 1\}$ [Kar, page 94; Se, page 88].

Since $PSL_2(\mathbb{F}_5)$ is a simple group of order 60, we find that $PSL_2(\mathbb{F}_5) \simeq A_5$. Hence the group extension $1 \to \{\pm 1\} \to SL_2(\mathbb{F}_5) \to PSL_2(\mathbb{F}_5) \simeq A_5 \to 1$ gives a Schur covering group of $A_5$. By the uniqueness, we conclude that $SL_2(\mathbb{F}_5) \simeq A_5$.

Step 2. The binary icosahedral group $H$ is defined in [Sp, page 93] as follows. For a field $k$ with $\text{char} k \neq 2, 5$ and $\zeta_5 \in k$, $H$ is the subgroup of $GL_2(k)$ defined by $H = \langle a, b, c \rangle$ where

\[
a = -\begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c = \frac{1}{\zeta^2 - \zeta^{-2}} \begin{pmatrix} \zeta + \zeta^{-1} & 1 \\ 1 & -(\zeta + \zeta^{-1}) \end{pmatrix}
\]

with $\zeta = \zeta_5$.

Also by [Sp, page 93], $H$ may be presented by exhibiting generators and relations as $H = \langle a, b, c \rangle$ where $Z(H) = \langle \varepsilon \rangle$ with $\varepsilon = a^5$ and

\[
\varepsilon^2 = 1, \quad a^5 = b^2 = c^2 = \varepsilon, \quad bab^{-1} = a^{-1}, \quad bcb^{-1} = \varepsilon c, \quad cac = acba, \quad ca^2c = a^{-2}ca^{-2}.
\]
It follows that $H = \{a^i, ba^i, a^i ca^j, a^i cba^j : 0 \leq i \leq 9, 0 \leq j \leq 4\}$ is a group of order 120.

Note that the group homomorphism
\[
\pi : H \to A_5 \\
a \mapsto (1\ 2\ 3\ 4\ 5) \\
b \mapsto (1\ 4)(2\ 3) \\
c \mapsto (1\ 3)(2\ 4)
\]
defines a central extension $1 \to \{\pm 1\} \to H \to A_5 \to 1$. Hence $H \cong \tilde{A}_5$.

Step 3. By Step 1 and Step 2, $SL_2(\mathbb{F}_5) \cong \tilde{A}_5 \cong H$. We will exhibit an isomorphism from $SL_2(\mathbb{F}_5)$ onto $H$.

Define $A, B, C \in SL_2(\mathbb{F}_5)$ by
\[
A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 2 & -2 \end{pmatrix}.
\]

It is not difficult to verify that $SL_2(\mathbb{F}_5) = \langle A, B, C \rangle$ and the group homomorphism $\varphi : SL_2(\mathbb{F}_5) \to H$ defined by $\varphi(A) = a, \varphi(B) = b, \varphi(C) = c$ is an isomorphism.

Step 4. We will study the automorphism $\vartheta : SL_2(\mathbb{F}_5) \to SL_2(\mathbb{F}_5)$ defined in Theorem 2.8. Choose $\omega = 2 \in \mathbb{F}_5$. Then $\theta$ is given by
\[
\rho \mapsto \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}^{-1}
\]
for any $\rho \in SL_2(\mathbb{F}_5)$. It follows that $\theta(A) = -A^3CA, \theta(B) = -C, \theta(C) = -B$.

Step 5. Now we turn to the group $G_+ = \langle \lambda, L \rangle$ of this theorem.

Because $L \cong SL_2(\mathbb{F}_5) \cong H$, we may use the presentation of $H$ in Step 2 for a presentation of $L$, i.e. we write $L = \langle a, b, c \rangle$ with the relations given there. It follows that $\theta(a) = \varepsilon a^3ca, \theta(b) = \varepsilon c, \theta(c) = \varepsilon b$ by Step 4. Define a group $G_-$ by $1 \to \{1, \varepsilon\} \to G_+ \to G_- \to 1$. Then $G_- = \langle \bar{a}, \bar{b}, \bar{c}, \bar{\lambda} \rangle$ with the relations induced from $a, b, c$ and the relation $\bar{\lambda} \bar{g} \bar{\lambda}^{-1} = \theta(\bar{g})$ for any $\bar{g} \in \langle \bar{a}, \bar{b}, \bar{c} \rangle$.

It is easy to check the following map is a well-defined group homomorphism
\[
\psi : G_- \to S_5 \\
\bar{a} \mapsto (1\ 2\ 3\ 4\ 5) \\
\bar{b} \mapsto (1\ 4)(2\ 3) \\
\bar{c} \mapsto (1\ 3)(2\ 4) \\
\bar{\lambda} \mapsto (3\ 4).
\]

Moreover, $\psi$ is an onto map. Since $|G_-| = 120$, it follows that $\psi$ is an isomorphism.

Thus we get a group extension $1 \to \{1, \varepsilon\} \to G_+ \to G_- \cong S_5 \to 1$. Since $\lambda \in G_+$ is mapped to $(3\ 4) \in S_5$ and $\lambda$ is an element of order 4, it follows that $G_+ \cong \hat{S}_5$ by Definition 4.8. ■
Theorem 4.10 Let \( G = G_1 \times G_2 \) be a \( GZ \)-group where \( G_1 \) is a \( Z \)-group, \( G_2 \approx SL_2(\mathbb{F}_5) \), i.e. \( G \) is the group \((I)\) in Theorem 2.8 with \( p = 5 \).

Assume that \( k \) is a field satisfying at least one of the following conditions: \( \text{char } k = 0 \), \( \text{char } k = 2 \), or \( \text{char } k = l > 0 \) with \( l \equiv \pm 1 \pmod{5} \). Then \( k(G_2) \) is \( k \)-rational. If it is assumed furthermore that \( k \) is an infinite field, then \( k(G) \) is retract \( k \)-rational.

Proof. Consider \( k(G_2) \) first.

Note that \( G_2 \approx SL_2(\mathbb{F}_5) \approx \tilde{A}_5 \) by Lemma 4.9.

Suppose \( \text{char } k = 0 \). By [Pf] Theorem 14, \( \mathbb{Q}(\tilde{A}_5) \) is rational. Thus for any field \( k \) with \( \text{char } k = 0 \), \( k(\tilde{A}_5) \) is also \( k \)-rational.

If \( \text{char } k = 2 \), consider the central extension \( 1 \to \{\pm 1\} \to \tilde{A}_5 \to A_5 \to 1 \). Applying Theorem 3.6, we find that \( k(\tilde{A}_5) \) is rational over \( k(A_5) \). By Maeda’s Theorem [Ma], \( k(A_5) \) is \( k \)-rational for any field. Hence \( k(\tilde{A}_5) \) is \( k \)-rational.

Finally consider the case when \( k \) is field with \( \text{char } k = l > 0 \) with \( l^2 \equiv 1 \pmod{5} \).

By [Hu] page 500, there is a faithful representation \( \Phi : \tilde{A}_5 \cong SL_2(\mathbb{F}_5) \to GL_2(k) \) (in case \( l \equiv 1 \pmod{5} \), we may use the representation in Step 2 of the proof of Lemma 4.9). Let \( k \cdot x_1 \oplus k \cdot x_2 \) be its representation space. We can embed \( k \cdot x_1 \oplus k \cdot x_2 \) into the regular representation space \( \bigoplus_{g \in \tilde{A}_5} k \cdot x_g \). Thus the \( \tilde{A}_5 \)-field \( k(x_1, x_2) \) can be embedded into \( k(x_g : g \in \tilde{A}_5) \). Applying Theorem 3.5, we find that \( k(\tilde{A}_5) = k(x_g : g \in \tilde{A}_5) \) is rational over \( k(x_1, x_2)^{\tilde{A}_5} \). Write \( x = x_1/x_2 \). We get \( k(x_1, x_2) = k(x, x_2) \). The proof is similar to the last paragraph of Case 1 in the proof of Theorem 4.4. We get \( k(x_1, x_2)^{\tilde{A}_5} = k(x, x_2)^{\tilde{A}_5} \) is \( k \)-rational.

Now consider \( k(G) \). Note that \( \gcd(|G_1|, |G_2|) = 1 \). Thus \( |G_1| \) is an odd integer, \( k(G_1) \) is retract \( k \)-rational by Lemma 4.1 (with the aid of Theorem 2.3). Using the result that \( k(G_2) \) is \( k \)-rational, we conclude that \( k(G) \) is retract \( k \)-rational by Theorem 3.3. \(\blacksquare\)

Theorem 4.11 Let \( G \) be the group \((II)\) in Theorem 2.8 with \( p = 5 \). Define a subgroup \( G_+ \) of \( G \) by \( G_+ = \langle \lambda, L \rangle \). If \( k \) is a field with \( \text{char } k = 2 \) or \( \text{char } k = 0 \) such that \( k(\zeta_8) \) is a cyclic extension of \( k \), then \( k(G_+) \) is \( k \)-rational. If it is assumed furthermore that \( k \) is an infinite field, then \( k(G) \) is retract \( k \)-rational.

Proof. By Lemma 4.9 \( G_+ \cong \tilde{S}_5 \). We will show that \( k(G_+) \) is \( k \)-rational.

If \( \text{char } k = 0 \) and \( k(\zeta_8) \) is cyclic over \( k \), then \( k(G_+) \) is \( k \)-rational by [KZ, Theorem 1.4].

If \( \text{char } k = 2 \), consider the group extension \( 1 \to \{\pm 1\} \to \tilde{S}_5 \cong G_+ \to S_5 \to 1 \). By Theorem 3.6, \( k(\tilde{S}_5) \) is rational over \( k(S_5) \). But \( k(S_5) \) is \( k \)-rational. Hence so is \( k(\tilde{S}_5) \).

Now we will show that \( k(G) \) is retract \( k \)-rational. The proof is similar to that of Theorem 4.5.

Define \( H_1 = \langle \sigma \rangle \), \( H_2 = \langle \tau, \lambda, L \rangle \). We obtain \( G \cong H_1 \rtimes H_2 \). Since \( k(H_1) \) is retract \( k \)-rational by Theorem 3.2, we find that \( k(G) \) is retract \( k \)-rational if and only if \( k(H_2) \) is retract \( k \)-rational by Theorem 3.1.
Define $H_{3} = \langle \tau \rangle$ and $G_{+} = \langle \lambda, L \rangle$. Then $H_{2} \simeq H_{3} \rtimes G_{+}$. Using the same arguments, we find that $k(H_{2})$ is retract $k$-rational if and only if so is $k(G_{+})$.

Under the assumption that (i) char $k = 0$ with $k(\zeta_{8})$ cyclic over $k$, or (ii) char $k = 2$ and $k$ is infinite, it is clear that $k(G_{+})$ is retract $k$-rational because it is $k$-rational. ■

Example 4.12 Let $G$ and $G_{+}$ be the same as in Theorem 4.11 and $k$ be a field with char $k = 0$. Checking the proof of Theorem 4.11 we find that $k(G)$ is retract $k$-rational if and only so is $k(G_{+})$.

By Serre’s Theorem, $Q(\hat{S}_{5})$ is not retract $Q$-rational [GMS, Example 33.27, page 90; KZ, Theorem 1.2]. In particular, $Q(G)$ is not retract $Q$-rational.

Let $\hat{G}$ be a Frobenius group with Frobenius complement $G$ defined above. We claim that $Q(\hat{G})$ is not retract $Q$-rational.

Suppose that $Q(\hat{G})$ is retract $Q$-rational. By Theorem 3.1 we find that $Q(G)$ would be retract $Q$-rational, which is a contradiction.

This example shows that the assumption $k(\zeta_{8})/k$ being cyclic is crucial in Theorem 1.8.

By the same method, it is possible to find a solvable Frobenius group $\hat{G}$ whose Frobenius complement is the group (III) or (IV) in Theorem 2.7 such that $Q(\hat{G})$ is not retract $Q$-rational. For, the 2-Sylow subgroup of $\hat{G}$ is a generalized quaternion group of order $\geq 16$. Apply Serre’s Theorem that $Q(G_{16})$ is not retract $Q$-rational (where $G_{16}$ is the generalized quaternion group of order 16) [GMS Theorem 34.7, page 92; Ka4, Section 1]. The details are omitted.

Now we give the proof for results in Section 1.

Proof of Theorem 1.7 Write $G = N \rtimes G_{0}$ where $N$ is abelian and $G_{0}$ is solvable. By Theorem 1.4 and Theorem 2.7, $G_{0}$ is isomorphic to the groups (I)–(IV) listed in Theorem 2.7. Thus we may apply Theorem 4.7 to show that $k(G_{0})$ is retract $k$-rational.

As to $k(N)$, $k(N)$ is retract $k$-rational by Theorem 3.2 if $|N|$ is odd. If $|N|$ is even and the exponent of $N$ is $2^{u}n'$ (where $2 \nmid n'$), then the exponent of $G$ is $2^{u}m$ (where $2 \nmid m$). Since $\zeta_{2^{u}} \in k$, it follows that $k(N)$ is also retract $k$-rational by Theorem 3.2.

Applying Theorem 3.1 we find the $k(G)$ is retract $k$-rational. ■

Proof of Theorem 1.8 Write $G = N \rtimes G_{0}$ where $G_{0}$ is non-solvable. By Theorem 1.3, Theorem 2.8 and Theorem 2.9, $G_{0}$ is isomorphic to the groups (I) or (II) listed in Theorem 2.8 with $p = 5$. Applying Theorem 4.10 and Theorem 4.11 we find that $k(G_{0})$ is retract $k$-rational.

By Theorem 2.1, $N$ is of odd order and is abelian, because $|G_{0}|$ is even and $\gcd(|N|, |G_{0}|) = 1$. By Theorem 3.2, $k(N)$ is retract $k$-rational.

By Theorem 3.1 we find that $k(G)$ is retract $k$-rational. ■

Proof of Theorem 1.11 The proof is similar to that of Theorem 1.8. This time we apply only Theorem 4.10 because we are working on the groups (I) in Theorem 2.8. Hence the result.
§5. Remarks about Bogomolov multipliers

First of all, let us recall the notions of unramified Brauer groups and Bogomolov multipliers.

Let $k \subset K$ be an extension of fields. The unramified Brauer group of $K$ over $k$, denoted by $Br_{v,k}(K)$ was defined as $Br_{v,k}(K) = \bigcap_R \text{Image} \{ Br(R) \to Br(K) \}$ where $Br(R) \to Br(K)$ is the natural map of Brauer groups and $R$ runs over all the discrete valuation rings $R$ such that $k \subset R \subset K$ and $K$ is the quotient field of $R$.

By [Sa3], $Br_{v,k}(K)$ is an obstruction for $K$ to be $k$-rational. In particular, if $k$ is an algebraically closed field and $K$ is retract $k$-rational, then $Br_{v,k}(K) = 0$. The following result shows that $Br_{v,k}(k(G))$ depends only on the group $G$.

**Theorem 5.1** (Bogomolov, Saltman [Bo; Sa4, Theorem 12]) Let $G$ be a finite group, $k$ be an algebraically closed field with $\gcd\{|G|, \text{char} k\} = 1$. Then $Br_{v,k}(k(G))$ is isomorphic to the group $B_0(G)$ defined by

$$B_0(G) = \bigcap_A \text{Ker} \{ \text{res}_C^A : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z}) \}$$

where $A$ runs over all the bicyclic subgroups of $G$ (a group $A$ is called bicyclic if $A$ is either a cyclic group or a direct product of two cyclic groups).

Thus, if $C(G)$ is retract $C$-rational, then $B_0(G) = 0$.

By Theorem 1.7 and Theorem 1.8, $C(G)$ is retract $C$-rational for any Frobenius group $G$ with abelian Frobenius kernel. In particular, $B_0(G) = 0$, a phenomenon observed by Moravec [Mo, Corollary 6.6]. In fact, our result of the retract rationality of $C(G)$ also implies $H^q_{nr,C}(C(G), \mathbb{Q}/\mathbb{Z}) = 0$ for all $q \geq 3$ where $H^q_{nr,C}(C(G), \mathbb{Q}/\mathbb{Z})$ are the higher unramified cohomology groups defined by Colliot-Thélène and Ojanguren [CTO].

We remark that the assumption of abelian Frobenius kernels can be waived by the following lemma.

**Lemma 5.2** Let $G$ be a Frobenius group with Frobenius kernel $N$. If $B_0(N_p) = 0$ for all Sylow subgroups $N_p$ of $N$, then $B_0(G) = 0$.

**Proof.** Let $\text{res} : B_0(G) \to B_0(N)$ be the map induced from the restriction map $\text{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(N, \mathbb{Q}/\mathbb{Z})$. It is shown in the proof of [Mo, Corollary 6.6] that $\text{res} : B_0(G) \to B_0(N)$ is injective. Since $B_0(N) \to \bigoplus_p B_0(N_p)$ is also injective where $p$ runs over all the prime divisors of $|N|$, thus $B_0(G) \to \bigoplus_p B_0(N_p)$ is injective. Done. ■

From Lemma 5.2, it is important to know the situations for which $B_0(H) = 0$ (where $H$ is a $p$-group). The following example is a partial answer to it.

**Example 5.3** If $H$ is a $p$-group, we list several sufficient conditions to ensure that $B_0(H) = 0$.
First, if $H$ is abelian, then $\mathbb{C}(H)$ is $\mathbb{C}$-rational by Theorem 3.7. Hence $B_0(H) = 0$.

For any $p$-group $H$, if $H$ is meta-cyclic or $H$ contains a cyclic subgroup of index $p^2$, then $\mathbb{C}(H)$ is $\mathbb{C}$-rational by [Ka1, Ka3]. Hence $B_0(H) = 0$.

If $H$ has an abelian normal subgroup $H_0$ such that $H/H_0$ is a cyclic group, then $B_0(H) = 0$ by [Bo, Ka4, Theorem 5.10].

If $H = H_0 \rtimes B$ where $H_0$ is abelian normal in $H$ and $B$ is bicyclic, then $B_0(H) = 0$ by [Ba, HKK, Theorem 4.2].

If $H$ is a 2-group of order $\leq 32$, then $\mathbb{C}(H)$ is $\mathbb{C}$-rational by [CK, CHKP]. Hence $B_0(H) = 0$. If $H$ is a group of order 64 and $H$ is not in the 13rd isoclinism family, then $B_0(H) = 0$ [CHKK, HKK, Theorem 1.14]. (For the numbering of the isoclinism families, see [HKK, Section 1] for details.)

If $p$ is an odd prime number and $H$ is a $p$-group of order $\leq p^4$, then $\mathbb{C}(H)$ is $\mathbb{C}$-rational by [CK]. Thus $B_0(H) = 0$. If $H$ is a group of order $p^5$ and $H$ is not in the 10th isoclinism family, then $B_0(H) = 0$ [HKK].
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