Second-order quark number susceptibility of deconfined QCD matter in presence of magnetic field

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Considering the strong field approximation we compute the hard thermal loop pressure at finite temperature and chemical potential of hot and dense deconfined QCD matter in lowest Landau level in one-loop order. We consider anisotropic pressure in presence of strong magnetic field i.e., longitudinal and transverse pressure along parallel and perpendicular to the magnetic field direction. As a first effort we compute and discuss the anisotropic quark number susceptibility of deconfined QCD matter in lowest Landau level. The longitudinal quark number susceptibility is found to increase with temperature whereas that of transverse one decreases with the temperature. We also compute quark number susceptibility in weak field approximation. The thermomagnetic correction is very marginal in weak field approximation.

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I. INTRODUCTION

Fluctuations of the conserved quantum numbers like baryon number, electric charge, strangeness number have been proposed as the probe of the hot and dense matter created in high energy heavy-ion collisions. However if one collects all the charged particle in heavy-ion collision then the net charge will be conserved and there will be no fluctuation. But all the particles can not be collected by any detector [1]. One should consider grand canonical ensemble for the case of real detector. An isolated system does not fluctuate because it is at thermodynamic limit. But if we consider small portion of a system which is small enough to consider the rest of the system as bath and is large enough to ignore the quantum fluctuations then one can calculate the fluctuation of conserved quantities like baryon number using grand canonical ensemble [2]. These fluctuations can be measured experimentally [1–3]. Several lattice calculations are there which calculate fluctuation and correlation of the conserved quantities [4–8]. The fluctuation of the conserved quantum numbers can be used to determine the degrees of freedom of the system [2]. Second and fourth order quark number susceptibilities in thermal medium have been calculated using Hard Thermal Loop (HTL) approximation [9–14], pQCD [15–17]. Ref. [18] calculates the second-order quark number susceptibility (QNS) considering different quark masses for u, d and s quark.

On the other hand, recent findings show that magnetic field of the order of $10^{18}$ Gauss can be created at the center of the fire ball by the charged spectator particles in non-central heavy-ion collisions [19, 20]. The time varying magnetic field is created in a direction perpendicular to the reaction plane [21–25] and its strength depends on the impact parameter. The strength of the magnetic field decreases after few fm/$c$ of the collision [19]. Several activities are under way to study the properties of strongly interacting matter in presence of magnetic field. Effects like magnetic catalysis [21, 26, 27], inverse magnetic catalysis [28–31], chiral magnetic effect [32, 33] in presence of magnetic field in non-central heavy-ion collision have been reported. Furthermore, various thermodynamic quantities [34, 35], transport coefficients [36, 37], dilepton production rate [38–43], photon production rate [44, 45] and damping of photon [46] of magnetised QCD matter have been obtained.

Here for simplicity, we consider strong ($gT < T < \sqrt{|eB|}$) and weak ($\sqrt{|q_f B|} < m_{th} \sim gT < T$) magnetic field with two different scale hierarchies. As a first effort in this article we, using the one-loop HTL pressure of quarks and gluons at finite quark chemical potential in presence of magnetic field, calculate the second-order QNS of deconfined QCD matter in this two scale hierarchies.
The paper is organized as follows: in Sec. II we present the setup to calculate second-order QNS. In Subsec. III A, one-loop HTL free-energy of quark in presence of strong magnetic field at finite temperature and chemical potential is calculated. The gauge boson free-energy in presence of strong magnetic field is obtained in Subsec. III B. We discuss in Subsec. III C the anisotropic pressure and second-order QNS of QCD matter in a strong field approximation. Considering one-loop HTL pressure quark-gluon plasma in weak field approximation [35], we also calculate and discuss the second-order QNS in the presence of weak magnetic field in Sec. IV. We conclude in Sec. V.

II. SETUP

Here we consider the deconfined QCD matter as grand canonical ensemble. The free-energy of the system can be written as

\[ F(T, V, \mu) = u - Ts - \mu n \]  

where \( \mu \) is the quark chemical potential, \( n \) number density and \( s \) is the entropy density. The pressure of the system is given as

\[ P = -F. \]  

However, we consider the system to be anisotropic in presence of strong magnetic field and the free-energy of the system is defined in Eq. (28).

The second-order QNS is defined as

\[ \chi = -\frac{\partial^2 F}{\partial \mu^2} \bigg|_{\mu=0} = \frac{\partial^2 P}{\partial \mu^2} \bigg|_{\mu=0} = \frac{\partial n}{\partial \mu} \bigg|_{\mu=0} \]  

which is the measure of the variance or the fluctuation of the net quark number. One can find out the covariance of two conserved quantities when the quark flavors have different chemical potential. Alternatively, one can work with other basis according to the system e.g., net baryon number \( B \), net charge \( Q \) and strangeness number \( S \) or \( B, Q \) and third component of isospin \( I_3 \). In our case we take strangeness and charge chemical potential to be zero. Moreover, we have considered same chemical potential for all flavors which results in zero off-diagonal quark number susceptibilities. Thus the net second order baryon number susceptibility is related to the second-order QNS as

\[ \chi_B = \frac{1}{3} \chi. \]

The strength of the magnetic field produced in non-central heavy-ion collision can be up to \((10-20)m_\pi^2\) at the time of collision [47]. However, it decreases very fast being inversely proportional
to the square of time [44, 48]. But if one considers finite electric conductivity of the medium, then the magnetic field strength will not die out very fast [49–51]. We consider two different cases with strong and weak magnetic field in this article.

III. STRONG MAGNETIC FIELD

In this section we consider strong field scale hierarchy $gT < T < \sqrt{eB}$. In presence of magnetic field, the energy of charged fermion becomes $E_n = \sqrt{k_3^2 + m_f^2 + 2nq_f B}$ where $k_3$ is the momentum of fermion along the magnetic field direction, $m_f$ is the mass of the fermion and the Landau level, $n$, can vary from 0 to $\infty$. The transverse momentum of fermion becomes quantised. It can be shown that at very high magnetic field, the contribution from all the Landau levels except the lowest Landau level can be ignored [39]. Consequently, the dynamics becomes $(1 + 1)$ dimensional when one considers only lowest Landau level (LLL). The general structures of quark and gluon self-energy in presence of magnetic field have been formulated in Ref. [34] at finite temperature but for zero quark chemical potential. Here we extend it for the case of non-zero quark chemical potential. In the presence of strong magnetic field, the general structure of quark self-energy can be written as [34]

$$\Sigma(p_0, p_3) = a\hat{\gamma} + b\gamma_5\hat{\gamma} + c\gamma_5\hat{\gamma} + d\gamma_5\hat{\gamma},$$

(4)

where the rest frame of heat bath velocity $u_\mu = (1, 0, 0, 0)$ and the direction of magnetic field $n_\mu = (0, 0, 0, 1)$. Now, the various form factors can be obtained as

$$a = \frac{1}{4} \text{Tr}[\Sigma\hat{\gamma}],$$

(5)

$$b = -\frac{1}{4} \text{Tr}[\Sigma\gamma_5\hat{\gamma}],$$

(6)

$$c = \frac{1}{4} \text{Tr}[\gamma_5\Sigma\hat{\gamma}],$$

(7)

$$d = -\frac{1}{4} \text{Tr}[\gamma_5\Sigma\gamma_5\hat{\gamma}].$$

(8)

The form factors are calculated up to $O(\mu^4)$ in Appendix A as

$$a = -d = c_1 \left[ \frac{p_0}{p_0^2 - p_3^2} c_2 + \frac{(p_0^2 + p_3^2)}{2(p_0^2 - p_3^2)^2} c_3 \right],$$

(9)

$$b = -c = -c_1 \left[ \frac{p_3}{p_0^2 - p_3^2} c_2 + \frac{p_0 p_3}{(p_0^2 - p_3^2)^2} c_3 \right],$$

(10)

where $c_1, c_2$ and $c_3$ are defined in Eqs. (A16).
A. One-loop quark free-energy in the presence of a strongly magnetized medium

Here we calculate the quark free-energy within HTL approximation using the form factors of quark self-energy in (9) and (10). The quark free-energy can be written as

\[ F_q = -d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln \left( \det \left[ S^{-1}_{\text{eff}}(p_0, p_3) \right] \right), \tag{11} \]

where \( d_F = N_c N_f \).

Inverse of the effective fermion propagator can be written as

\[ S^{-1}_{\text{eff}} = \not{P} + \Sigma = (p_0 + a)\not{\gamma} + (b - p_3)\not{\gamma} + c\gamma_5 \not{\gamma} + d\gamma_5 \not{\gamma} \]

\[ = (p_0 + a)\gamma^0 + (b - p_3)\gamma^3 + c\gamma_5 \gamma^0 + d\gamma_5 \gamma^3. \tag{12} \]

Now we evaluate the determinant as

\[ \det [S^{-1}_{\text{eff}}] = \left( b + c - p_3 \right)^2 - \left( a + d + p_0 \right)^2 \left( -b + c + p_3 \right)^2 - \left( a - d + p_0 \right)^2 \]

\[ = \left( p_0^2 - p_3^2 \right) \left( p_0 + 2a \right)^2 - \left( p_3 - 2b \right)^2 \]

\[ = P_0^2 \left( P_0^2 + 4ap_0 + 4bp_3 + 4a^2 - 4b^2 \right) \]

\[ = P_0^4 \left( 1 + \frac{4a^2 - 4b^2 + 4ap_0 + 4bp_3}{P_0^2} \right), \tag{13} \]

where we have used \( d = -a \) and \( c = -b \).

So Eq. (11) becomes

\[ F_q = -d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln \left( P_0^4 \left( 1 + \frac{4a^2 - 4b^2 + 4ap_0 + 4bp_3}{P_0^2} \right) \right) \]

\[ = -2d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln (-P_0^2) - d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + \frac{4a^2 - 4b^2 + 4ap_0 + 4bp_3}{P_0^2} \right] \]

\[ = F_q^{\text{ideal}} + F_q', \tag{14} \]

where the free-energy of free quarks in presence of magnetic field

\[ F_q^{\text{ideal}} = -2d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln (-P_0^2) = -2d_F \sum_{f} \frac{q_f B}{(2\pi)^2} \sum_{\{p_0\}} dp_3 \ln (-P_0^2) \]

\[ = -d_F \sum_{f} q_f B T^2 \left( 1 + 12\hat{\mu}^2 \right), \tag{15} \]

where \( \hat{\mu} = \mu/2\pi T \).

\[ F_q' = -d_F \sum_{\{p_0\}} \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + \frac{4a^2 - 4b^2 + 4ap_0 + 4bp_3}{P_0^2} \right] \]
\[ \text{where we have kept terms up to } \mathcal{O}(g^4) \text{ to obtain the analytic expression of free-energy. The expansion made above is valid for } g^2(q_f B/T^2) < 1, \text{ which can be realized as } (q_f B)/T^2 \gtrsim 1 \text{ and } g \ll 1.\]

As in the strong field approximation, the fermion is considered to be in LLL. So Eq. (16) becomes,

\[ F_q' = -d_F \sum_j \frac{q_f B T^2}{(2\pi)^2} \sum_p dp_3 \left[ \frac{4(a p_0 + b p_3)}{P_i^2} + \frac{4(a^2 P_i^2 - b^2 P_i^2 - 2a^2 p_0^2 - 2b^2 p_3^2 - 4a b p_0 p_3)}{P_i^4} \right] + \mathcal{O}(g^6), \tag{16} \]

The sum-integrals are calculated in Appendix B and the expression for the quark free-energy up to \( \mathcal{O}(g^4) \) is obtained by adding individual contributions as

\[ F_q = F_q^{\text{ideal}} + F_q' = -d_F \sum_j \frac{q_f B T^2}{6} \left( 1 + 12 \mu^2 \right) + 4d_F \sum_j \frac{g^2 C_F(q_f B)^2}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^2 \left[ -\frac{1}{2\epsilon} \left( \ln 2 - \frac{7}{8\pi^2 T^2} \zeta(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) - \frac{1}{2} \left( 3\gamma_E + 4 \ln 2 - \ln \pi \right) \left( \ln 2 - \frac{7}{8\pi^2 T^2} \zeta(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) \right] \times \left( \ln 2 - \frac{7}{8\pi^2 T^2} \zeta(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) \right] + \mathcal{O}(\mu^6). \tag{18} \]

The renormalized quark free-energy is given as

\[ F_q^r = -d_F \sum_j \frac{q_f B T^2}{6} \left( 1 + 12 \hat{\mu}^2 \right) + 4d_F \sum_j \frac{g^2 C_F(q_f B)^2}{(2\pi)^4} \left[ -\frac{1}{2} \left( \ln 2 + \ln \hat{\Lambda} \right) \left( 2 \ln 2 - 7 \zeta(3) \hat{\mu}^2 \right) + 31 \zeta(5) \hat{\mu}^4 \right] \times \left( \ln 2 - 7 \ln 2 \zeta(3) \hat{\mu}^2 + \frac{49}{4} \frac{\zeta(3)}{T^2} \hat{\mu}^4 \right) + \frac{g^2 C_F(q_f B)^2}{4\pi^2} \frac{7 \zeta(3)}{8\pi^2 T^2} \times \left( \ln 2 - 7 \ln 2 \zeta(3) \hat{\mu}^2 + \frac{49}{4} \frac{\zeta(3)}{T^2} \hat{\mu}^4 \right) \times \frac{31 \zeta(5) \hat{\mu}^4}{256 \pi^4 T^4} \tag{19} \]

where \( \hat{\Lambda} = \Lambda/2\pi T \) and \( \hat{\mu} = \mu/2\pi T \).
B. Gauge boson free-energy in a strongly magnetized medium

The general structure of gauge boson self-energy can be written from Ref. [52] as

$$\Pi^{\mu\nu} = \alpha B^{\mu\nu} + \beta R^{\mu\nu} + \gamma Q^{\mu\nu} + \delta N^{\mu\nu},$$

where the form factors can be calculated for non-zero quark chemical potential as

$$\alpha = \frac{m_D^2}{\bar{u}^2} \left[ 1 - T_P(p_0, p) \right] - \sum_f \frac{(\delta m^2_{D,f})_s}{\bar{u}^2} e^{-\frac{\bar{p}_F}{2q_f B}} \frac{p^2}{p_0^2 - p_3^2},$$

$$\beta = \frac{m_D^2}{2} \left[ \frac{p_0^2}{p^2} - \frac{p^2}{p^2} T_P(p_0, p) \right],$$

$$\gamma = \frac{m_D^2}{2} \left[ \frac{p_0^2}{p^2} - \frac{p^2}{p^2} T_P(p_0, p) \right] + \sum_f \frac{(\delta m^2_{D,f})_s}{\bar{u}^2} e^{-\frac{\bar{p}_F}{2q_f B}} \frac{p^2}{p_0^2 - p_3^2},$$

$$\delta = \sum_f (\delta m^2_{D,f})_s \sqrt{\frac{\bar{n}^2}{\bar{u}^2}} e^{-\frac{\bar{p}_F}{2eB}} \frac{p_0 p_3}{p_0^2 - p_3^2},$$

where $\bar{u}^2 = -p^2/P^2$, $\bar{n}^2 = -p^2_3/p^2$ and

$$T_P(p_0, p) = \frac{p_0}{2p} \ln \frac{p_0 + p}{p_0 + p}.$$

The thermal and magnetic correction of the Debye screening mass is given as

$$m^2_D = \frac{g^2 N_c T^2}{3},$$

$$\langle \delta m^2_{D,f} \rangle_s = \frac{g^2 |q_f B|}{2\pi T} \int_{-\infty}^{\infty} \frac{dk_3}{4\pi} \left[ n_F(k_3 + \mu) \left\{ 1 - n_F(k_3 + \mu) \right\} + n_F(k_3 - \mu) \left\{ 1 - n_F(k_3 - \mu) \right\} \right]$$

$$+ \frac{g^2 |q_f B|}{4\pi^2},$$

$$\langle m^2_D \rangle = m^2_D + \sum_f (\delta m^2_{D,f})_s = m^2_D + (\delta m^2_D)_s.$$
\[ + \frac{2N_f \pi T^2}{9} \left( \frac{g_F^2}{4\pi^2} \right)^2 \sum_f q_f B \left( 3\zeta'(-1) - \left( -1 + 3 \ln \Lambda \right) \right) + \left( N_f^2 + \sum_{f_1, f_2} q_{f_1} B_{f_2} \right) \times \frac{g^4 T^4}{32} \left( -12 \zeta'(4) \frac{\pi^4}{\pi^4} + \frac{2}{15} \left( \ln \frac{\Lambda}{2} + \gamma_E + \ln 4\pi \right) - \frac{17}{75} \right) - \frac{1}{2} \left( \frac{g^2}{4\pi^2} \right)^2 \times \sum_{f_1, f_2} q_{f_1} B_{f_2} B \left( 4 - 4(\ln 2 - 1)(\ln \Lambda + \gamma_E) - \frac{\pi^2}{3} + 2(\ln 2 - 2) \ln 2 \right) - \frac{C_{AN_f g^4 T^4}}{36} \times \left( 1 - 2 \frac{\zeta'(-1)}{\zeta(-1)} - 2 \ln \frac{\Lambda}{2} \right) - \sum_f C_{A g^4 T^2 q_f B} \left( \pi^2 - 4 + 12 \ln \frac{\Lambda}{2} - 2 \ln 2 \left( 6\gamma_E + 4 + 3 \ln 2 - 6 \ln \frac{\Lambda}{2} \right) + 12\gamma_E \right) \right] - \frac{d_A (m_D^2)^3 T^3}{12\pi}. \tag{27} \]

C. Longitudinal and Transverse Pressure and corresponding Susceptibilities

Free-energy density of the quark-gluon plasma is given by

\[ F = u - T s - \mu n - eB \cdot M, \tag{28} \]

where \( u \) is total the energy density and magnetization per unit volume is given by

\[ M = - \frac{\partial F}{\partial (eB)}. \tag{29} \]

The pressure becomes anisotropic \([34, 53]\) due to the magnetization acquired by the system in presence of strong magnetic field which results in two different pressure along parallel and perpendicular to the magnetic field direction. The longitudinal pressure is given as

\[ P_z = -F = -(F_q^z + F_g^z). \tag{30} \]

and transverse pressure is given as

\[ P_{\perp} = -F - eB \cdot M. \tag{31} \]

One gets two different second-order QNS, namely, along the longitudinal \((\chi_z)\) and transverse \((\chi_{\perp})\) direction in the presence of the strong magnetic field. The longitudinal second-order QNS can be obtained as

\[ \chi_z = \frac{\partial^2 P_z}{\partial \mu^2} \bigg|_{\mu=0}, \tag{32} \]

whereas the transverse one can be obtained as

\[ \chi_{\perp} = \frac{\partial^2 P_{\perp}}{\partial \mu^2} \bigg|_{\mu=0}. \tag{33} \]
The pressure of non-interacting quark-gluon gas in the presence of strong magnetic field is given as

\[ P_{sf} = \sum_f N_c N_f q_f B \frac{T^2}{6} (1 + 12\mu^2) + (N_c^2 - 1) \frac{\pi^2 T^4}{45}. \]  

(34)

The second-order diagonal QNS for the ideal quark gluon plasma is given as

\[ \chi_{sf} = \sum_f N_c N_f q_f B \frac{\pi}{T^2}. \]  

(35)

In the left panel of Fig. 1 the variation of the longitudinal second-order QNS with temperature is displayed for two values of magnetic field strength. For a given magnetic field strength the longitudinal second-order QNS is found to increase with temperature and approaches the free field value at high temperature. On the other hand for a given temperature the longitudinal second-order QNS decreases with increase of the magnetic field strength as shown in the right panel of Fig. 1 for two different temperatures.

In the left panel of Fig. 2 the variation of transverse second-order QNS with temperature is displayed for two values of magnetic field strength. It is found that the transverse second-order QNS decreases with increase of the magnetic field strength as shown in the right panel of Fig. 1 for two different temperatures.

In the left panel of Fig. 2 the variation of transverse second-order QNS with temperature is displayed for two values of magnetic field strength. It is found that the transverse second-order QNS decreases with increase of the magnetic field strength as shown in the right panel of Fig. 1 for two different temperatures.

This is an indication that the system may shrink in the transverse direction.
FIG. 2: Variation of the transverse part of the second-order QNS scaled with that of free field value in presence of strong magnetic field with temperature (left panel) and magnetic field (right panel) strength for $N_f = 3$.

For a given temperature the transverse second-order QNS is found to increase with the increase of the magnetic field strength as shown in the right panel of Fig. 2 for two different temperatures. This behaviour is in contrary to that of longitudinal one.

IV. WEAK MAGNETIC FIELD

In this section we consider magnetic field strength to be the lowest among all the scales $T, m_{\text{th}}$ as $\sqrt{q_f B} < m_{\text{th}} \sim gT < T$. The HTL one-loop free energy for the deconfined QCD matter has been calculated upto $O(g^4)$ in Ref. [35]. The total renormalized free-energy in presence of weak magnetic field is sum of renormalized quark and gluon free-energy and can be written [35] as

$$F = F_q^r + F_g^r,$$

(36)

where the renormalized quark free-energy is

$$F_q^r = N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120\bar{\mu}^2}{7} + \frac{240\bar{\mu}^4}{7} \right) + \frac{g^2 C_F T^4}{48} (1 + 4\bar{\mu}^2) (1 + 12\bar{\mu}^2) 
+ \frac{g^4 C_F^2 T^4}{768\pi^2} (1 + 4\bar{\mu}^2)^2 (\pi^2 - 6) + \frac{g^4 C_F^2}{27N_f} M_B^4 \left( 12 \ln \frac{\Lambda}{2} - 6\gamma(z) + \frac{36\zeta(3)}{\pi^2} + 2 - \frac{72}{\pi^2} \right) \right].$$

(37)
$M_{B,f}$ is the thermomagnetic mass for quark flavor $f$ in presence of weak magnetic field and $M_B$ represents flavor summed thermomagnetic quark mass as

$$M_B = \sum_f M_{B,f}^2 = \sum_f q_f B \frac{1}{16\pi^2} \left[ -\frac{1}{4} \mathcal{N}(z) - \frac{\pi T}{2m_f} - \frac{\gamma_E}{2} \right].$$

(38)

$\mathcal{N}(z)$ in Eq. (37) is abbreviated as

$$\mathcal{N}(z) \equiv \Psi(z) + \Psi(z^*),$$

(39)

with $\Psi(z)$ is the digamma function

$$\Psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)},$$

(40)

and $z = 1/2 - i\mu$. At small chemical potential, $\mathcal{N}(z)$ can be expanded as

$$\mathcal{N}(z) = -2\gamma_E - 4\ln 2 + 14\zeta(3)\mu^2 - 62\zeta(5)\mu^4 + 254\zeta(7)\mu^6 + O(\mu^8).$$

(41)

In addition to the renormalized quark free-energy in Eq (37), the renormalized gluon free-energy is given as

$$\frac{F_g}{d_A} = -\frac{\pi^2 T^4}{45} \left[ 1 - \frac{15}{2} \tilde{m}_D^2 + 30(\tilde{m}_D^2)^3 + \frac{45}{8} \tilde{m}_D^2 \left( 2\ln \frac{\hat{A}}{2} - 7 + 2\gamma_E + \frac{2\pi^2}{3} \right) \right]$$

$$- \pi^2 T^4 \tilde{m}_D^2 \delta \tilde{m}_D^2 \left( \gamma_E + \ln \hat{A} \right) + \sum_f g^2(q_f B)^2 T^2 \left[ \frac{4.97 + 2\ln \frac{\hat{A}}{2}}{m_f^4} \right]$$

$$+ 3\tilde{m}_D^2 \left\{ 2(1 - \ln 2)\ln^2 \frac{\hat{A}}{2} + 2 \left( \frac{7}{2} - \frac{\pi^2}{6} - \ln^2(2) - 2\gamma_E(\ln 2 - 1) \right) \ln \frac{\hat{A}}{2} + 4.73 \right\}$$

$$- \sum_f g^2(q_f B)^2 \frac{\pi T}{(12\pi)^2 32m_f} \left\{ \frac{3}{4} \ln^2 \frac{\hat{A}}{2} + 2\ln \frac{\hat{A}}{2} \left( \frac{21}{8} + 3\zeta'(-1) \right) + \frac{7}{4} \ln 2 \right\}$$

$$+ \frac{3}{4} \tilde{m}_D^2 \left[ 2\ln \frac{\hat{A}}{2} \left( 5\pi^2 - \frac{609}{10} + \frac{114\ln 2}{5} \right) + 2\ln \frac{\hat{A}}{2} \left( 30\zeta(3) - \frac{5779}{75} + \frac{121}{6} \pi^2 + \frac{114}{5} \ln^2(2) \right) \right]$$

$$+ \frac{468}{25} \ln 2 + \gamma_E \left[ 10\pi^2 - \frac{609}{5} + \frac{228}{5} \ln 2 \right] + 106.477 \right\}$$

$$+ \frac{8}{3\pi} \left\{ (3\ln 2 - 4) \ln \frac{\hat{A}}{2} - 3.92 \right\}$$

$$+ 3\tilde{m}_D^2 \left[ \frac{1}{20} \ln^2 \frac{\hat{A}}{2} \left( 11 + 5\pi^2 - 92\ln 2 \right) + 2\ln \frac{\hat{A}}{2} \left( \frac{3}{4} \zeta(3) + \frac{1557}{200} - \frac{\pi^2}{3} - \frac{23}{10} \ln^2(2) \right) \right]$$

$$- \frac{168}{25} \ln 2 + \gamma_E \left[ 11 \frac{\pi^2}{4} - \frac{23}{5} \ln 2 \right] - 1.86 \right\},$$

(42)

where $\tilde{m}_D^w = m_D^w / 2\pi T, \tilde{m}_D = m_D / 2\pi T, \delta \tilde{m}_D = \delta m_D / 2\pi T$ and $m_D^w$ represents the Debye mass in weak magnetic field approximation and is obtained as

$$<m_D^w>^2 \approx \frac{g^2 T^2}{3} \left[ \left( N_c + \frac{N_f}{2} \right) + 6N_f \hat{\mu}^2 \right]$$
\[ + \sum_f \frac{g^2(q_f B)^2}{12 \pi^2 T^2} \sum_{l=1}^{\infty} (-1)^{l+1} l^2 \cosh (2l \pi \hat{\mu}) \frac{m_f}{T} K_0 \left( \frac{m_f l}{T} \right) + O[(q_f B)^4] \]
\[ = m_f^2 + \delta m_f^2. \] (43)

Considering the expression of free energy vis-a-vis pressure we calculate the second-order QNS in weak field limit by using Eq. (3). The second-order QNS of free quarks and gluons in thermal medium is given as
\[ \chi_f = \frac{1}{3} N_c N_f T^2. \] (44)

The left panel of Fig. 3 shows the variation of the scaled second-order QNS with the temperature at different values of the magnetic field strength. The weak field effect appears as a correction to the thermal medium, the weak field second-order QNS is not very much different than that of thermal medium. It is found to increase with temperature and approaches the free field value at high enough temperature. The magnetic field effect on the second-order QNS is visible at low temperature. The value of second-order QNS slowly decreases as one increases the magnetic field strength as shown in the right panel of Fig. 3.

V. CONCLUSION

We consider a hot and dense deconfined QCD matter in the presence of the background strong and weak magnetic field within HTL approximation. The quarks are directly affected by magnetic
field whereas gluons are affected via quark loop in the gluon self-energy. In the strong field approximation we assume quarks are in lowest Landau level. We compute the one-loop HTL pressure in the presence of finite temperature and chemical potential in the lowest Landau level within the strong field approximation. Various divergent terms are eliminated by choosing appropriate counterterms in the $\overline{\text{MS}}$ renormalization scheme. The presence of magnetization causes the system to be anisotropic, and one obtains two different pressures in directions parallel and perpendicular to the magnetic field. Both the longitudinal and transverse pressures are computed analytically by calculating the magnetization of the system. We then compute both the longitudinal and transverse second-order QNS in the strong field approximation. For a given magnetic field strength, the longitudinal second-order QNS increases with temperature and approaches to the non-interacting value at high enough temperature. For a given temperature the longitudinal second-order QNS is found to decrease with increase of magnetic field strength. In contrast the transverse second-order QNS is found to decrease with temperature and increase with the increase of magnetic field. Further, in weak field approximation we consider one-loop HTL pressure of hot and dense QCD matter of Ref. [35] and compute the second-order QNS. The thermomagnetic correction is found to be marginal and slowly varies with magnetic field. Our calculation can be compared with future lattice QCD calculation.

VI. ACKNOWLEDGEMENT

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Appendix A: Calculation of the quark self-energy form factors

1. Calculation of the form factors $a$ and $d$

We calculate the form factor $a$ from Eq. (5) as
\[
a = \frac{1}{4} \text{Tr}[\Sigma\gamma^\mu] = -2g^2C_F \sum_{\{K\}} e^{-\frac{x^2}{4T^2}} \left[ \frac{k_0}{K^2(K-P)i} + \frac{(k-p)^2}{K^2(K-P)i} \right] \\
= -2g^2C_F \int \frac{d^3k}{(2\pi)^3} e^{-\frac{x^2}{4T^2}} \left[ T_2 + (k-p)^2 T_4 \right] \\
= -2g^2C_F \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \left[ \frac{q_f B}{4\pi} T_2 + \frac{q_f B}{4\pi} (p^2 + q_f B) T_4 \right]
\]
\[ T_2 = \sum_{\{k_0\}} \frac{k_0}{K_n^2(K - P)^2}, \]
\[ T_4 = \sum_{\{k_0\}} \frac{k_0}{K_n^2(K - P)^4} = -\frac{1}{2k_3} \frac{\partial T_2}{\partial p_3}. \] (A2)

Here we also note that in LLL, \( p_\perp = 0 \). Now \( T_2 \) can be calculated as
\[ T_2 = \sum_{\{k_0\}} \frac{k_0}{K_n^2(K - P)^2}, \]
\[ = -\frac{1}{4k_3} \left[ \frac{n_B(k_3) + n_F(k_3 - \mu)}{p_0 + p_3} + \frac{n_B(k_3) + n_F(k_3 + \mu)}{p_0 - p_3} \right]. \] (A3)

We note that
\[ \int_{-\infty}^{\infty} \frac{dk_3}{k_3} n_\mp^*(k_3) = \int_{0}^{\infty} \frac{dk_3}{k_3} \left( n_\mp^* + n_F \right), \]
\[ \int_{-\infty}^{\infty} \frac{dk_3}{k_3^2} n_\mp(k_3) = \pm \int_{0}^{\infty} \frac{dk_3}{k_3^2} \left( n_\mp^* - n_F \right). \] (A4)

Hence,
\[ \int_{-\infty}^{\infty} \frac{dk_3}{k_3} T_2 \approx -\int_{0}^{\infty} \frac{dk_3}{4k_3} \left[ \frac{2n_B(k_3) + n_F(k_3 + \mu) + n_F(k_3 - \mu)}{p_0 + p_3} \right. \]
\[ + \left. \frac{2n_B(k_3) + n_F(k_3 + \mu) + n_F(k_3 - \mu)}{p_0 - p_3} \right] \]
\[ = \int_{0}^{\infty} \frac{dk_3}{4k_3} \left( \frac{2n_B(k_3) + n_F(k_3 + \mu) + n_F(k_3 - \mu)}{p_0 + p_3} \right) \frac{(-2p_0)}{p_0^2 - p_3^2} \]
\[ = \frac{p_0}{p_0^2 - p_3^2} \ln 2 - \frac{p_0}{p_0^2 - p_3^2} \left( \frac{7\mu^2}{8\pi^2} T^2 \zeta(3) - \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) + O[\mu^5]. \] (A5)

and
\[ \int_{-\infty}^{\infty} \frac{dk_3}{k_3} q_f B T_4 = -\frac{q_f B}{2} \frac{\partial}{\partial p_3} \int_{-\infty}^{\infty} \frac{dk_3}{k_3} T_2 \]
\[ = \frac{q_f B}{2} \frac{\partial}{\partial p_3} \int_{-\infty}^{\infty} \frac{dk_3}{k_3^2} \left[ \frac{n_B(k_3) + n_F(k_3 - \mu)}{p_0 + p_3} + \frac{n_B(k_3) + n_F(k_3 + \mu)}{p_0 - p_3} \right] \]
\[ \approx \frac{q_f B}{4} \frac{\partial}{\partial p_3} \int_{0}^{\infty} \frac{dk_3}{k_3^2} \left( n_F(k_3 + \mu) - n_F(k_3 - \mu) \right) \frac{p_3}{p_0^2 - p_3^2} \]
\[ = -\frac{q_f B}{2(p_0^2 - p_3^2)} \left( \frac{7\mu}{T_4} \zeta'(-2) + \frac{31}{6} \frac{\mu^3}{T_4^3} \zeta'(-4) \right) + O[\mu^5]. \] (A6)
So the form factor $a = -d$ up to $O(\mu^4)$ can be written as

$$a = -d = -\frac{g^2 C_F(q_f B)}{4\pi^2} \left[ \frac{p_0}{p_0^2 - p_3^2} \left( \frac{1}{8\pi^2} \ln 2 - \frac{7}{24} \frac{\mu^2}{T^2} \zeta(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) - q_f B \frac{(p_0^2 + p_3^2)}{2(p_0^2 - p_3^2)^2} \right] \times \left( \frac{7}{24} \frac{\mu^2}{T^2} \zeta'(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right) \right]. \quad (A7)$$

2. Calculation of quark form factor $b$ and $c$

Similarly one can calculate $b$ from Eq. (6) as

$$b = -\frac{1}{4} \text{Tr}[\Sigma'] = 2g^2 C_F \sum_{\{K\}} e^{-\frac{k_3^2}{2q_f^2}} \left[ \frac{k_3}{K^2_0(K - P)^2} + (k - p)^2 \frac{k_3}{K^2_0(K - P)^2} \right]$$

$$= 2g^2 C_F \int \frac{d^4k}{(2\pi)^4} e^{-\frac{k_3^2}{2q_f^2}} k_3 \left[ T_1 + (k - p)^2 T_3 \right]$$

$$= 2g^2 C_F \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} k_3 \left[ \frac{q_f B}{4\pi} T_1 + \frac{q_f B}{4\pi} (q_f B) T_3 \right]$$

$$= \frac{g^2 C_F(q_f B)}{4\pi^2} \int_{-\infty}^{\infty} dk_3 \ k_3 \left[ T_1 + q_f B \ T_3 \right], \quad (A8)$$

where

$$T_1 = \sum_{\{k_0\}} \frac{1}{K^2_0(K - P)^2}, \quad (A9)$$

$$T_3 = \sum_{\{k_0\}} \frac{1}{K^2_0(K - P)^2} = -\frac{1}{2k_3} \frac{\partial T_1}{\partial p_3}. \quad (A10)$$

After doing the Matsubara sum, Eq. (A9) becomes

$$T_1 = \frac{1}{4k_3^2} \left[ \frac{n_B(k_3) + n_F(k_3 - \mu)}{p_0 + p_3} - \frac{n_B(k_3) + n_F(k_3 + \mu)}{p_0 - p_3} \right], \quad (A11)$$

Hence,

$$\int_{-\infty}^{\infty} dk_3 \ k_3 T_1 \approx \int_{0}^{\infty} \frac{dk_3}{4k_3} \left[ \frac{2n_B(k_3) + n_F(k_3 + \mu) + n_F(k_3 - \mu)}{p_0 + p_3} \right]$$

$$= \int_{0}^{\infty} \frac{dk_3}{4k_3} \left[ \frac{2n_B(k_3) + n_F(k_3 + \mu) + n_F(k_3 - \mu)}{p_0 - p_3} \right]$$

$$= \frac{p_3}{p_0^2 - p_3^2} \left[ \ln 2 - \frac{7}{8\pi^2} \frac{\mu^2}{T^2} \zeta(3) + \frac{31}{24} \frac{\mu^4}{T^4} \zeta'(-4) \right] + O(\mu^6). \quad (A12)$$
and
\[ \int_{-\infty}^{\infty} dk_3 \, k_3 \, q_f B \, T_3 = -\frac{q_f B}{2} \frac{\partial}{\partial p_3} \int_{-\infty}^{\infty} dk_3 \, T_1 \]
\[ = -\frac{q_f B}{2} \frac{\partial}{\partial p_3} \int_{-\infty}^{\infty} dk_3 \left[ \frac{n_B(k_3) + n_F(k_3 - \mu)}{p_0 + p_3} - \frac{n_B(k_3) + n_F(k_3 + \mu)}{p_0 - p_3} \right] \]
\[ \approx \frac{q_f B}{4} \frac{\partial}{\partial p_3} \int_{0}^{\infty} dk_3 \left( \frac{n_F(k_3 + \mu) - n_F(k_3 - \mu)}{k_3^2} \right) \frac{p_0}{p_0^2 - p_3^2} \]
\[ = -\frac{q_f B}{(p_0^2 - p_3^2)^2} \left( \frac{7\mu}{2T^2} \zeta'(-2) + \frac{31\mu^3}{6T^4} \zeta'(-4) \right) + \mathcal{O}[\mu^5]. \quad (A13) \]

The form factor \( b = -c \) is obtained up to \( \mathcal{O}[\mu^4] \) as
\[ b = -c = \frac{g^2 C_F(q_f B)}{4\pi^2} \left[ \frac{p_3}{p_0^2 - p_3^2} \left( \ln 2 - \frac{7}{8\pi^2} \frac{\mu^2}{T^2} \zeta(3) + \frac{31\mu^4}{24T^4} \zeta'(-4) \right) \right. \]
\[ - q_f B \frac{p_0 p_3}{(p_0^2 - p_3^2)^2} \left( \frac{7\mu}{2T^2} \zeta'(-2) + \frac{31\mu^3}{6T^4} \zeta'(-4) \right) \]. \quad (A14) \]

Eqs. (A7) and (A14) can also be rewritten in compact form as
\[ a = -d = c_1 \left[ \frac{p_0}{P_{ii}^2} c_2 + \frac{1}{2} \left( \frac{1}{P_{ii}^2} + \frac{2p_3^2}{P_{ii}^4} \right) c_3 \right], \]
\[ b = -c = -c_1 \left[ \frac{p_3}{P_{ii}^2} c_2 + \frac{p_0 p_3}{P_{ii}^4} c_3 \right], \quad (A15) \]

with
\[ c_1 = -\frac{g^2 C_F(q_f B)}{4\pi^2} \]
\[ c_2 = \left( \ln 2 - \frac{7}{8\pi^2} \frac{\mu^2}{T^2} \zeta(3) + \frac{31\mu^4}{24T^4} \zeta'[-4] \right) \]
\[ c_3 = -q_f B \left( \frac{7\mu}{2T^2} \zeta'[-2] + \frac{31\mu^3}{6T^4} \zeta'[-4] \right). \quad (A16) \]

**Appendix B: One-loop sum-integrals for quark free-energy**

Eq. (17) can be rewritten as
\[ F'_q = -4d_F \sum \int \frac{q_f B}{(2\pi)^2} \sum \int dp_3 \left[ \frac{a p_0}{P_{ii}^2} + \frac{b p_3}{P_{ii}^2} - \frac{a^2}{P_{ii}^2} - \frac{b^2}{P_{ii}^2} - \frac{2a^2 p_3^2}{P_{ii}^4} - \frac{2b^2 p_3^2}{P_{ii}^4} - \frac{4ab p_0 p_3}{P_{ii}^4} \right] \quad (B1) \]

The various sum-integrals in Eq. B1 can be written using Eq. (A15) as
\[ \sum \int \frac{a p_0}{P_{ii}^2} = c_1 \sum \int \left[ c_2 \left( \frac{1}{P_{ii}^2} + \frac{p_3^2}{P_{ii}^4} \right) + \frac{c_3}{2} \left( \frac{p_0}{P_{ii}^2} + \frac{2p_0 p_3^2}{P_{ii}^4} \right) \right], \quad (B2) \]
\[
\sum_{\{p_0\}} \frac{b_{p_3}}{P_{p_3}^2} = -c_1 \sum_{\{p_0\}} \left[ \frac{p_{p_3}^2}{P_{p_3}^2} c_2 + \frac{p_0 p_{p_3}^2}{P_{p_3}^4} p_{p_3} c_3 \right],
\]

(B3)

\[
\sum_{\{p_0\}} \frac{a^2}{P_{p_3}^2} = c_1^3 \sum_{\{p_0\}} \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{1}{P_{p_3}^6} + \frac{4 p_{p_3}^2}{P_{p_3}^8} + \frac{4 p_{p_3}^4}{P_{p_3}^{10}} \right) c_2 c_3 + c_2^2 \left( \frac{p_0}{P_{p_3}^6} + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} \right) \right],
\]

(B4)

\[
\sum_{\{p_0\}} \frac{b^2}{P_{p_3}^2} = c_1^3 \sum_{\{p_0\}} \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{3 p_{p_3}^2}{P_{p_3}^8} + \frac{3 p_{p_3}^4}{P_{p_3}^{10}} + \frac{4 p_{p_3}^6}{P_{p_3}^{12}} \right) c_2 c_3 + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} c_3 \right],
\]

(B5)

\[
\sum_{\{p_0\}} \frac{a^2 p_{p_3}^2}{P_{p_3}^4} = c_1^3 \sum_{\{p_0\}} \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{3 p_{p_3}^2}{P_{p_3}^8} + \frac{3 p_{p_3}^4}{P_{p_3}^{10}} + \frac{4 p_{p_3}^6}{P_{p_3}^{12}} \right) c_2 c_3 + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} c_3 \right],
\]

(B6)

\[
\sum_{\{p_0\}} \frac{b^2 p_{p_3}^2}{P_{p_3}^4} = c_1^3 \sum_{\{p_0\}} \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{3 p_{p_3}^2}{P_{p_3}^8} + \frac{3 p_{p_3}^4}{P_{p_3}^{10}} + \frac{4 p_{p_3}^6}{P_{p_3}^{12}} \right) c_2 c_3 + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} c_3 \right],
\]

(B7)

\[
\sum_{\{p_0\}} \frac{a b p_{p_3}^3}{P_{p_3}^4} = -c_1^2 \sum_{\{p_0\}} \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{3 p_{p_3}^2}{P_{p_3}^8} + \frac{3 p_{p_3}^4}{P_{p_3}^{10}} + \frac{4 p_{p_3}^6}{P_{p_3}^{12}} \right) c_2 c_3 + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} c_3 \right],
\]

(B8)

which leads to

\[
F' = -4 d_F \sum f \frac{q_f B}{(2 \pi)^2} \sum_{\{p_0\}} \left[ c_1 \left[ c_2 \left( \frac{1}{P_{p_3}^2} + \frac{p_{p_3}^2}{P_{p_3}^4} \right) + \frac{c_3}{2} \left( \frac{p_0}{P_{p_3}^4} + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^6} \right) \right] - c_1 \left[ \frac{p_{p_3}^2}{P_{p_3}^4} c_2 + \frac{p_0 p_{p_3}^2}{P_{p_3}^6} c_3 \right] \right]
\]

\[
- c_1^2 \left[ \left( \frac{1}{P_{p_3}^4} + \frac{p_{p_3}^2}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{1}{P_{p_3}^6} + \frac{4 p_{p_3}^2}{P_{p_3}^8} + \frac{4 p_{p_3}^4}{P_{p_3}^{10}} \right) c_2 c_3 + c_2^2 \left( \frac{p_0}{P_{p_3}^6} + \frac{2 p_0 p_{p_3}^2}{P_{p_3}^8} \right) \right]
\]

\[
+ \left( \frac{p_{p_3}^2}{P_{p_3}^4} + \frac{p_{p_3}^4}{P_{p_3}^6} \right) c_2 c_3 \right] - 2 c_1^2 \left[ \left( \frac{p_{p_3}^2}{P_{p_3}^4} + \frac{p_{p_3}^4}{P_{p_3}^6} \right) c_2^2 + \frac{1}{4} \left( \frac{p_{p_3}^6}{P_{p_3}^8} + \frac{3 p_{p_3}^8}{P_{p_3}^{10}} + \frac{4 p_{p_3}^{10}}{P_{p_3}^{12}} \right) c_2 c_3 \right]
\]

\[
+ \left( \frac{p_{p_3}^2}{P_{p_3}^4} + \frac{p_{p_3}^4}{P_{p_3}^6} \right) c_2 c_3 + \frac{1}{2} \left( \frac{p_{p_3}^6}{P_{p_3}^8} + \frac{3 p_{p_3}^8}{P_{p_3}^{10}} + \frac{2 p_{p_3}^{10}}{P_{p_3}^{12}} \right) c_3 \right]
\]

\[
= -4 d_F \sum f \frac{q_f B}{(2 \pi)^2} \sum_{\{p_0\}} \left[ c_1 c_2 \left( \frac{1}{P_{p_3}^2} + \frac{c_3}{P_{p_3}^4} + \frac{p_0}{P_{p_3}^4} \right) - c_1^2 \left( \frac{p_{p_3}^2}{P_{p_3}^4} c_2 + \frac{p_0 p_{p_3}^2}{P_{p_3}^6} c_3 \right) \right]
\]

(B9)

One can calculate

\[
\sum_{\{p_0\}} \frac{1}{P_{p_3}^2} = -\frac{1}{2 p_3} \left( 1 - 2 n_F(p_3) \right),
\]

(B10)

\[
\sum_{\{p_0\}} \frac{1}{P_{p_3}^4} = \frac{1}{2 p_3} \frac{\partial}{\partial p_3} \left( \sum_{\{p_0\}} \frac{1}{P_{p_3}^2} \right) \approx \frac{1}{2 p_3} \frac{\partial}{\partial p_3} \left( n_F(p_3) \right) = \frac{1}{2 p_3} \left[ \beta \frac{\partial n_F(p_3)}{p_3} - n_F(p_3) \right].
\]

(B11)

Now we perform the sum-integrals in Eq. (B9) as

\[
\sum_{\{p_0\}} \frac{1}{P_{p_3}^2} = \left( \frac{e^{\gamma} \Lambda^2}{4 \pi} \right) \int_{-\infty}^{\infty} d^{1-2} p_3 n_F(p_3) \frac{p_3}{p_3}.
\]
Using the above sum-integrals in Eq. (B9) \( F \) becomes, 

\[
\sum_{\{p_0\}} \frac{1}{p_i^4} = \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{2\epsilon} - \frac{1}{2} (3\gamma_E + 4\ln 2 - \ln \pi) + O(\epsilon) \right].
\]

\[
\sum_{\{p_0\}} \frac{1}{p_i^3} = \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{93\zeta(5)}{128\pi^4 T^4} + O(\epsilon) \right],
\]

\[
\sum_{\{p_0\}} \frac{p_3^2}{p_i^5} = \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{31\zeta(5)}{256\pi^4 T^4} + O(\epsilon) \right].
\]

Using the above sum-integrals in Eq. (B9) \( F' \) up to \( O(g^4) \) becomes,

\[
F'_q = -4d_F \sum_f \frac{q_f B}{(2\pi)^2} \sum_{\{p_0\}} \left[ c_1 c_2 \frac{1}{p_i^2} - c_1^2 c_2^2 \frac{1}{p_i^4} - \frac{c_1^2 c_3^2}{4} \frac{1}{p_i^2} - \frac{c_1^2 c_3}{2} \frac{p_3^2}{p_i^8} \right] 
\]

\[
= 4d_F \sum_f \frac{g^2 C_F(q_f B)}{(4\pi^2)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ c_2 \left( - \frac{1}{2\epsilon} - \frac{1}{2} (3\gamma_E + 4\ln 2 - \ln \pi) \right) + c_1 c_2 \left( \frac{7\zeta(-2)}{2 T^2} \right) \right] 
\]

\[
+ c_1 c_3 \left( \frac{31\zeta(5)}{256\pi^4 T^4} \right) 
\]

\[
= 4d_F \sum_f \frac{g^2 C_F(q_f B)^2}{(4\pi^2)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ - \frac{1}{2\epsilon} \ln 2 - \frac{7}{8\pi^2 T^2} \zeta(3) + \frac{31\mu^4}{24T^4} \zeta'(-4) \right] 
\]

\[
- \frac{1}{2} \left( 3\gamma_E + 4\ln 2 - \ln \pi \right) \left( \ln 2 - \frac{7}{8\pi^2 T^2} \zeta(3) + \frac{31\mu^4}{24T^4} \zeta'(-4) \right) - \frac{g^2 C_F(q_f B)}{4\pi^2} \left( \frac{7\zeta(-2)}{2 T^2} \right) \right) 
\]

\[
\times \left( \ln 2 - \frac{7\zeta(3)}{8\pi^2 T^2} \mu^2 + \frac{7\zeta(3)}{8\pi^2 T^2} \mu^4 + \frac{31\zeta(-4) \ln 2}{12T^4} \mu^4 \right) - \frac{g^2 C_F(q_f B)}{4\pi^2} \left( \frac{31\zeta(5)}{256\pi^4 T^4} \right) 
\]

\[
\times \left( \frac{49(q_f B)^2}{T^4} \zeta'(-2) \mu^2 + 217(q_f B)^2 \mu^4 \right) + O(\mu^6). \quad \text{(B16)}
\]

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