On \( \Lambda \)-positioning of an arc between two parallel support lines

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Abstract

We show that a rectifiable plane arc \( g \) has two parallel support lines and a triple of consecutive points \( g(r), g(s), g(t), \ r < s < t \), so that \( g(s) \) lies on one line, while \( g(r) \) and \( g(t) \) lie on the other. If the arc is simple, such a pair of lines is unique. \(^1\)

Introduction. In the articles [1, 3] we had to use the result in [2]: *If a convex set covers any simple polygonal unit arc, it covers any unit arc.* In [1, 3] the requirement on an arc to be simple and polygonal was used only in Theorem 5.1 of [1] establishing that *any simple polygonal arc assumes a so-called \( \Lambda \)-configuration* (Figure 1). Two proofs of Theorem 5.1 exist for simple arcs: one by Y. M. (Geometry Seminar, UIUC, 2009) and the other by R. Alexander, J. E. Wetzel, W. Wichiramala in their recently submitted paper "The \( \Lambda \)-property of a simple arc". In this note we prove Theorem 5.1 of [1] omitting both requirements: simple and polygonal.

Given a parametrization \( g(s), \ s \in [0, 1] \), for points \( p = g(s_1), \ q = g(s_2) \) with \( s_1 < s_2 \), we say that \( p \) precedes \( q \) and write it as \( p \prec q \). Points \( p_1, p_2, p_3 \) form a *triple of consecutive points* if either

\[
(1) \quad p_1 \prec p_2 \prec p_3 \ \text{or} \ \ p_3 \prec p_2 \prec p_1.
\]

We are seeking a pair of parallel support lines with and a triple of consecutive points \( p_1, p_2, p_3 \) such that \( p_1, p_3 \) lie on one line and \( p_2 \) lies on the other.

\[ \text{Figure 1. } \Lambda \text{-configuration.} \]

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This is clearly true when \( g \) is closed or is a straight segment.

**Lines \( l(\theta) \), walls \( W(\theta) \) and touch points in Figure 2.** Let \( g(s) : [0,1] \to \mathbb{R}^2 \) be an open rectifiable plane arc in the horizontal \((x,y)\)-plane, whose thickness \( h \) is positive. We denote by \( l(\theta) \) a counter-clockwise oriented support line for \( g \) in the direction of polar angle \( \theta \) and assume that \( g \) lies above \( l(0) \).

Let \( G(s) = (g(s), s) : [0,1] \to \mathbb{R}^3 \times [0,1] \) be a simple lift of \( g \) to \( \mathbb{R}^3 \) with \( G(0) = A, G(1) = Z \). Given a direction \( \theta \), let us denote by \( W(\theta) \) the support plane (wall) to \( G \) through \( l(\theta) \) orthogonal to the \((x,y)\)-plane. The touch points \( G \cap W(\theta) \) are denoted by \( I_k \). Note that a set of touch points of a wall \( W(\theta) \) is its compact subset. The wall \( W(\theta) \) has its lowest most left \( I_1(\theta) \) and highest most right \( C(\theta) \) touch points with respect to the counter-clockwise orientation of the support line \( l(\theta) \). Let \( v(\theta) \) be the vertical unit segment through \( C(\pi+\theta) \).

**Rotation of a support wall as a map.** Note that for each \( \theta \in [0,2\pi) \), there is one support line \( l(\theta) \) and thus one support wall \( W(\theta) \). Referring to a
rotation of a support line around \( g \) as \( \theta \) changes from 0 to \( 2\pi \), we think of two maps from \([0, 2\pi)\) : one is the map to the set of all lines in the \((x, y)\)-plane and the other is the map to all planes in the \((x, y, z)\)-space. The images of such maps are the set of oriented support lines \( l(\theta) \) of \( g \) and the set of support walls \( W(\theta) \) through \( l(\theta) \). Geometrically the first map is represented by a line moving in a plane so that it coincides with \( l(\theta) \) for each \( \theta \), while the second map is represented by a plane moving in the space so that it coincides with \( W(\theta) \) for each \( \theta \).

**Local stability of a non-\( \Lambda \)-configuration.** A subarc of \( g \) between points \( X \) and \( Y \) is denoted by \( \tilde{XY} \).

**Lemma 1.** The following set \( \Theta = \{ \theta \geq 0 : \ C(\theta) \prec C(\theta + \pi) \} \) is a half-open interval. That is if \( \theta \in \Theta \), then there exist \( \delta \) so that \( \theta + \varepsilon \in \Theta \) for any \( \varepsilon < \delta \).

**Proof.** To keep our proof transparent, we assume that \( v(\theta) \) has only finitely many touch points and we will use a particular configuration of Figure 2. Let points \( P, Q \in \tilde{I_4D} \) be so that

\[
(2) \quad I_4(\theta) \prec P \prec Q \prec C(\theta + \pi) \quad \text{and} \quad \text{length}(\tilde{I_4P}) = \text{length}(\tilde{QD}) = \frac{1}{3}h.
\]

Denote by \( R_v(\theta) \subset W(\theta) \) a half-plane of points to the right of \( v(\theta) \). Let

\[
\sigma(\theta) = \min \left\{ \text{dist}(\tilde{PZ}, R_v(\theta)), \text{dist}(\tilde{AQ}, R_v(\pi + \theta)) \right\}.
\]

Then \( \sigma > 0 \) and \( \text{dist}(\tilde{PZ}, v(\theta)) \) and \( \text{dist}(\tilde{AQ}, v(\theta + \pi)) \) are \( \geq \sigma \). We take any

\[
\varepsilon < \delta = \frac{\sigma(\theta)}{88 \text{ diameter}(G)}.
\]

The obstacles to the counter-clockwise rotation of the walls by \( \varepsilon \) could be only subarcs \( \tilde{QZ} \) and \( \tilde{AP} \). Therefore, \( C(\theta + \varepsilon) \in \tilde{AP} \) while \( D(\theta + \varepsilon) \in \tilde{QZ} \) and hence by (2), \( C(\theta + \varepsilon) \prec D(\theta + \varepsilon) \). ♦

**Theorem 1.** Let \( g \) be an open rectifiable arc with a thickness \( h > 0 \). Then there exist support lines \( l(\theta) \) and \( l(\pi + \theta) \) containing a triple of consecutive points \( p_1, p_2, p_3 \) of \( g \) with the lone middle point \( p_2 \).
**Proof.** We may assume that $A \prec C(0), C(\pi) \prec Z$. (Figure 2). If a triple of the theorem exists in the strip between $l(0)$ and $l(\pi)$ then $C(\pi) \prec C(0)$ (Figure 1). Otherwise,

\[(3) \quad C(0) \prec C(\pi).\]

By Lemma 1 this property is locally stable and so if $\Theta$ is the set given by this lemma and $\theta_T = \text{lub}(\Theta)$, then $\theta_T \notin \Theta$. That is $C(\theta_T + \pi) \prec C(\theta_T)$. However, limits of most right touch points in $\Theta$ preserve this property of $\Theta$:

\[(4) \quad \lim_{\theta \uparrow \theta_T} C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi),\]

because they are separated in distance by $h$. Indeed,

**Small rotations around $G$ by a small angle $\psi$.** Configurations of local behavior near a touch point are given in Figure 3:

$I(\theta+\psi)=I(\theta)$ in rotations without an obstacle, dist$(I(\theta+\psi),I(\theta))$ is small in rotations around smooth convex arcs.

Figure 3

Thus one or both $C(\theta_T)$ or $C(\theta_T + \pi)$ are not limits in (4). Suppose that $C(\theta_T)$ is not equal to $\lim_{\theta \uparrow \theta_T} C(\theta_T)$ Then the triples $p_1 \prec p_2 \prec p_3$ satisfying the theorem are either

\[\lim_{\theta \uparrow \theta_T} C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi) \prec C(\theta_T)\]

or

\[C(\theta_T + \pi) \prec C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi).\]

**An assumption that there were no $\theta_T \leq \pi$ leads to a contradiction.** Suppose that $\theta_T > \pi$, that is $C(\pi) \prec C(\pi + \pi)$. On the other hand, after a rotation by $\pi$, we arrive to the initial position of $g$ between the same parallel support lines. In this configuration, the highest most right touch point on
the wall $W(\pi)$ is a successor to such point on the wall $W(\pi + \pi) = W(0)$ and (3) is true: $C(\pi + \pi) = C(0) \prec C(\pi)$. That is a contradiction. ■

**Corollary** (W. Wichirimala). If $g$ is simple, then the pair of support lines with $g$ positioned as in Figure 4 is unique.

![Figure 4](image)

**Proof.** In Figure 4, the point $C$ divides the curve into two curves $L_1$ and $L_2$ where each point on $L_1$ is parametrically precedes each point on $L_2$. Consider a different pair of two parallel support lines. All touch points on one, say, the left line belong to $L_1$, therefore none of them can be parametrically between two touch points on the right line belonging to $L_2$. ♦

**References**

[1] P. Coulton and Y. Movshovich, Besicovitch triangles cover unit arcs, *Geom. Dedicata* 123 (2006), 79-88.

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[3] Y. Movshovich, Besicovitch triangles extended, *Geometriae Dedicata* 159(1) (2012), 99-107.