Optimal Impulse Control of Dynamical Systems with Functional Constraints

Alexey Piunovskiy
Department of Mathematical Sciences, University of Liverpool, L69 7ZL, UK.
piunov@liv.ac.uk

Yi Zhang
Department of Mathematical Sciences, University of Liverpool, L69 7ZL, UK.
yi.zhang@liv.ac.uk

Abstract

This paper considers a constrained optimal impulse control problem of dynamical systems generated by a flow. Under quite general and natural conditions, we prove the existence of an optimal stationary policy. This is done by making use of the tools of Markov decision processes. Two linear programming approaches are established and justified. In absence of constraints, we show that these two linear programming approaches are dual to the dynamic programming method with the optimality equations in the integral and differential form, respectively.

Keywords: Dynamical System, Optimal Control, Impulse Control, Total Cost, Constraints, Randomized Strategy, Markov Decision Process, Linear Programming, Duality.

AMS 2000 subject classification: Primary 49N25; Secondary 90C40.

1 Introduction

Impulse control of dynamical systems attracts attention of many researchers, see [2, 6, 13, 14, 17] to mention the most relevant and the most recent works. The underlying system can be described in terms of an ordinary differential equation [2, 6, 13, 14], or by a fixed flow in an abstract Borel space [17]. Impulse/intervention means the instantaneous change of the state of the system. In all the cited works, the target was to optimize the single objective functional, usually having the shape of the integral with respect to time of the running cost and the impulse costs. The popular method of attack to such problems is dynamic programming [2, 17]; in [13, 14], versions of the Pontryagin maximum principle were used.

The impulse control problem considered in this paper can be described as follows. In the absence of impulses, the dynamical system evolves according to \( \phi(x_0, t) \), where \( x_0 \in X \) is a fixed initial state, \( X \) is a Borel space, and \( \phi \) is a (measurable) mapping from \( X \times [0, \infty) \) to \( X \) such that \( \phi(x, 0) = x \) and \( \phi(x, t + s) = \phi(\phi(x, t), s) \) for each \( x \in X \) and \( s, t \geq 0 \). The latter property is called the semigroup property, and the mapping \( \phi \) is also often referred to as a flow. If an impulse \( a \in A \), with \( A \) being a Borel space, is applied when the system state is \( x \), then it instantaneously results in a post-impulse state given by \( l(x, a) \), where \( l \) is a (measurable) mapping from \( X \times A \) to \( X \). There are \( J+1 \in \{1, 2, \ldots, \} \) objective functionals corresponding to the pairs of functions \( \{C_g^j, C_I^j\}_{j=0}^J \), where \( C_g^j(x) \) is the running cost rate at the state \( x \in X \), and \( C_I^j(x, a) \) is the cost incurred when the impulse \( a \in A \) is applied at the state \( x \in X \). The \( j \)th objective function is obtained as the sum of total impulse cost and running cost over the infinite time horizon \([0, \infty)\) with respect to the pair \( \{C_g^j, C_I^j\} \). The decision maker decides
when to apply impulses and what impulses to select so as to minimize one objective function subject to the constraints that the other cost functions are not too big. The more detailed formulation of the concerned impulse control problem is given in Section 2.

We would view the consecutive occasions of applying an impulse as a sequence of decision epochs. More precisely, the first decision epoch occurs at the initial time 0, where the decision maker selects a pair \((t, a) \in [0, \infty) \times A\), where \(t\) is the time to wait until an impulse is applied, and the impulse to be applied then is \(a\). Then the next decision epoch occurs at \(t\) after this impulse is applied, where decision maker selects another pair, and so on. In this way, the impulse control problem can be viewed as an MDP (Markov Decision Process) with the action space \([0, \infty) \times A\), and investigated using the tools of MDPs. For impulse control problems, which are with a single objective and thus unconstrained, this method was pursued in [17], where the dynamic programming equations, both in the integral form and in a very general differential form, were established and discussed in detail.

In this paper, we consider the impulse control problem of dynamical systems with multiple objectives. For optimal control problems with functional constraints, dynamic programming is not convenient, and another method, the convex analytic approach, also known as the linear programming approach, proved to be effective, e.g., for MDPs, see [9, 16]. In a nutshell, this approach, if justified, reduces the original optimal control problem to a linear program with the same (optimal) value, and one can retrieve an optimal control policy for the original problem from an optimal solution to the induced linear program. For impulse control problem of dynamical systems, to the best of our knowledge, the only work concerned with this method is [6], which dealt with an unconstrained problem for a specific model, and obtained partial results mostly related to the second linear programming approach in our paper. Moreover, the generalization of the results in [6] to the constrained case was problematic. More detailed discussions and comparisons with [6] are given below.

The key step in establishing the induced linear program is in the investigations of the so-called occupation measures and their characterizations. There are two possible definitions of occupation measures for the impulse control problem considered in this paper. The first possible definition comes from the occupation measures of the MDP corresponding to the impulse control problem. Based on results in the recent work [9] for MDPs, we establish a linear programming approach for the impulse control problem, and obtain, under very general and natural conditions, the existence of an optimal (possibly randomized) stationary policy. This linear programming approach will be referred to as the first because it is based on the first definition of occupation measures. Here the problematic issue is that while the MDP model in [9] is required to be semicontinuous, the induced MDP from our impulse control problem does not have a continuous transition kernel. To get over this difficulty, we use the following trick: the state space can be extended in such a way that one can introduce a suitable topology on it, with respect to which, the resulting MDP model becomes semicontinuous. (In fact, this trick also facilitates the development of our second linear programming approach.) This is the first contribution of this paper.

The first linear programming approach seems not considered in the previous literature. In case of a single objective, we show that this method is related (and dual) to the dynamic programming equation in the integral form. In this connection, the second possible definition of occupation measures, which we call the aggregated occupation measures, is in some sense dual to the dynamic programming equation in the differential form. It turns out that the task of establishing the linear programming approach based on aggregated occupation measures is far not simple and much more challenging as compared to that of the first linear programming approach, because at least to say, now one does not have tools from MDPs for it. The second and major contribution of this work lies in establishing and justifying the second linear programming approach.

In terms of the previous literature, the second linear program is most closely related to, albeit different from the one formulated in the interesting work [5], where a specific impulse control problem with a single objective in a different setup was considered, and only partial results were obtained.
To get an idea of our second linear program and its relation with the one that follows from the idea of [6], let us consider and describe roughly the following simple but non-trivial optimal impulse control problem with a single objective in the state space $[x_0, \infty] \subseteq (-\infty, \infty)$:

$$
\begin{align*}
\frac{dx}{dt} &= f(x)dt + dw(t), \quad x(0) = x_0; \\
V(w) &= \int_0^\infty h(x(t))dt + \int_0^\infty dw(t) \to \inf_w
\end{align*}
$$

(1)

where $x_0$ is a fixed initial state,

$$
w(t) := \sum_{j=1}^{\infty} \left( \sum_{i=1}^{j-1} a_i \right) I\{T_{j-1} \leq t < T_j\};
$$

and the impulse control strategy $w$, represented by $\{T_j, a_j\}_{j=1}^\infty$ with $a_j \in A = [a, \overline{a}]$ for some finite constants $a, \overline{a}$, can be arbitrary. The measurable functions $f > 0$ and $h > 0$ are assumed to be smooth and regular enough, such that $\int_0^\infty h(x(t))dt$ is well defined and finite for all $w$. (This is the case when e.g., as $x \to \infty$, $h(x)$ approaches zero rapidly enough, and the function $f$ is separated from zero.)

Although the work [6] is about the finite horizon $[0, T]$, their constructions can be generalized for the infinite horizon $[0, \infty)$. Namely, the following linear program on the space of the so called occupation measures $\mu$ and $\omega$ can be formulated for problem (1):

$$
\begin{align*}
V_M(\mu, \omega) &= \int_{[x_0, \infty)} h(y)\mu(dy) + \int_{[x_0, \infty)} \omega(dy) \to \inf_{\mu, \omega} \\
\text{subject to} \\
0 &= v(x_0) + \int_{[x_0, \infty)} \frac{dv(y)}{dy}f(y)\mu(dy) + \int_{[x_0, \infty)} \frac{dv(y)}{dy}\omega(dy),
\end{align*}
$$

(2)

where the constraints must be satisfied by all test functions $v$, which are continuously differentiable on $[x_0, \infty)$ and $\lim_{y \to \infty} v(y) = 0$. Here $\mu$ is $\sigma$-finite and $\omega$ is finite. For the case of a finite horizon, the version of this linear program was formulated in [6]. However, it was left as an open problem whether the optimal value of this linear program coincides with the value of the original impulse control problem.

Next, let us formulate our second linear program for problem (1). To this end, some notations are needed. Let us introduce

$$
V^c = \{x = (y, t) : y \in [x_0, \infty), \ t \in [0, D(y)]\},
$$

which is the extended state space, which includes the time elapsed since the previous impulse; $D(y)$ comes from the relations

$$
dz = f(z) \ dt; \ z(0) = x_0; \ z(D(y)) = y.
$$

Note that if $z(0) \in [x_0, \infty)$, then $z(t) = y$ implies that $t$ cannot be bigger than $D(y)$. Moreover, in terms of the notations in this paper, $A = [a, \overline{a}]$ is the action (impulse) space, and $l((y, t), a) = (y+a, 0)$. Now, our second linear program for problem (1), which is in the space of aggregated occupation measures $\eta$ on $V^c \times (A \cup \{\square\})$ can be written as follows:

$$
\begin{align*}
V(\eta) &= \int_{V^c} h(y)\eta(dy \times dt \times \square) + \int_{V^c \times A} a \eta(dx \times da) \to \inf_\eta \\
\text{subject to} \\
0 &= w(x_0) + \int_{V^c} \chi w(x)\eta(dx \times \square) + \int_{V^c \times A} [w(y+a, 0) - w(y, t)]\eta(dy \times dt \times da),
\end{align*}
$$

(3)
where the test functions $w(x) = w(y, t)$ are measurable, bounded and absolutely continuous along the two-dimensional flow $\Phi$: on $[x_0, \infty)$ we have the flow $\phi$ generated by the differential equation $dz = f(z) \, dt$, and on the time scale $[0, \infty)$ the flow is trivial: $\phi^{\text{time}}(s, t) = s + t$. Moreover, $\lim_{t \to \infty} w(\Phi(x, t)) = 0$, and the functions $w$ are either negative and increasing, or positive and decreasing along the flow $\Phi$. $\chi w$ is the “derivative” of $w$ along the flow $\Phi$: $w(\Phi(x, t)) = w(x) + \int_{[0,t]} \chi w(\Phi(x, s)) \, ds$.

Trivial calculations show that all the corresponding expressions in (2) and (3) coincide for the common test functions. Those are continuously differentiable functions $v(y)$ on $[x_0, \infty)$, monotonically increasing or decreasing, and such that $\lim_{y \to \infty} v(y) = 0$. The correspondence between the measures $(\omega, \mu)$ and $\eta$ is as follows:

$$\omega(\Gamma) = \int_{\Gamma} \left[ \int_{V^c \times A} I\{y < z < y + a\} \eta(dy \times dt \times da) \right] \, dz;$$

$$\mu(\Gamma) = \int_{(\Gamma \times [0,\infty)) \cap V^c} \eta(dx \times \square), \quad \Gamma \in B([x_0, \infty)).$$

This indicates the relation between our second linear program with the one that would follow from the idea in (6). However, what was not obtained in (6), we established the equivalence between problems (4) and (5), in the sense that the minimal values of the objectives coincide, and indicated how to obtain an optimal policy for problem (4) from an optimal solution to problem (5). See Corollary 5.1 below for more details. It seems that for impulse control problems of dynamical systems, this challenging issue is dealt with for the first time in the present paper.

Besides, the underlying dynamical systems in this paper are more general than in (6). For example, we deal with more general objective functions: in (6), they were affine in the (impulse) controls. Moreover, our impulses lead to the new state $l(x, a)$, which is not necessarily equal to $x + a$ as compared to (6); and so on. But the main difference is that here we consider multiple objectives by incorporating into problem (4) functional constraints of the type

$$V_j(w) = \int_0^\infty h_j(x(t)) \, dt + \int_0^\infty g_j \, dw(t) \leq d_j, \quad j = 1, 2, \ldots, J.$$
open interval \((a, \infty)\). Expressions like “positive, negative, increasing, decreasing” are understood in the non-strict sense, like “nonnegative” etc. For \(I \subset \mathbb{R}, \tau \in \mathbb{R}, \tau + I := \{\tau + x : x \in I\}\) is the shifted set. \(\mathbb{I}\{\cdot\}\) is the indicator function; \(\delta_y(dx)\) is the Dirac measure at the point \(y\). For \(b, c \in [-\infty, +\infty], b^+ := \max\{b, 0\}, b^- := -\min\{b, 0\}, b \land c := \min\{b, c\}, b \lor c := \max\{b, c\}\).

## 2 Problem Statement

We will deal with a control model defined through the following elements.

- \(X\) is the state space, which is a Borel space, i.e., a topological space homeomorphic to a Borel subset, endowed with the restricted topology, of a complete separable metrizable (i.e., Polish) space. See [3, Chap.7].
- \(\phi(\cdot, \cdot) : X \times \mathbb{R}_+^0 \to X\) is the measurable flow possessing the semigroup property \(\phi(x, t + s) = \phi(\phi(x, s), t)\) for all \(x \in X\) and \((t, s) \in (\mathbb{R}_+^0)^2\); \(\phi(x, 0) = x\) for all \(x \in X\). Between the impulses, the state changes according to the flow.
- \(A\) is the action space, again a Borel subset of a complete separable metric space with metric \(\rho_A\) and the Borel \(\sigma\)-algebra.
- \(l(\cdot, \cdot) : X \times A \to X\) is the mapping describing the new state after the corresponding action/impulse is applied.
- For each \(j = 0, 1, \ldots, J\), where and below \(J\) is a fixed natural number, \(C_j^g(\cdot) : X \to \mathbb{R}_+^0\) is a (gradual) cost rate.
- For each \(j = 0, 1, \ldots, J\), \(C_j^I(\cdot, \cdot) : X \times A \to \mathbb{R}_+^0\) is a cost function associated with the actions/impulses applied in the corresponding states.

All the mappings \(\phi, l, \{C_j^g\}_{j=0}^J\) and \(\{C_j^I\}_{j=0}^J\) are assumed to be measurable.

Let \(X_\Delta = X \cup \{\Delta\}\), where \(\Delta\) is an isolated artificial point describing the case that the controlled process is over and no future costs will appear. The dynamics (trajectory) of the system can be represented as one of the following sequences

\[
x_0 \to (\theta_1, a_1) \to x_1 \to (\theta_2, a_2) \to \ldots; \quad \theta_i < +\infty \text{ for all } i \in \mathbb{N},
\]

or

\[
x_0 \to (\theta_1, a_1) \to \ldots \to x_n \to (+\infty, a_{n+1}) \to \Delta \to (\theta_{n+2}, a_{n+2}) \to \Delta \to \ldots,
\]

where \(x_0 \in X\) is the initial state of the controlled process and \(\theta_i < +\infty\) for all \(i = 1, 2, \ldots, n\). For the state \(x_{i-1} \in X, i \in \mathbb{N}\), the pair \((\theta_i, a_i) \in \mathbb{R}_+^0 \times A\) is the control at the step \(i\): after \(\theta_i\) time units, the impulsive action \(a_i\) will be applied leading to the new state

\[
x_i = \begin{cases} l(\phi(x_{i-1}, \theta_i), a_i), & \text{if } \theta_i < +\infty; \\ \Delta, & \text{if } \theta_i = +\infty. \end{cases}
\]

The state \(\Delta\) will appear forever, after it appeared for the first time, i.e., it is absorbing.

After each impulsive action, if \(\theta_1, \theta_2, \ldots, \theta_{i-1} < +\infty\), the decision maker has in hand complete information about the history, that is, the sequence

\[
x_0, (\theta_1, a_1), x_1, \ldots, (\theta_{i-1}, a_{i-1}), x_{i-1}.
\]

The next control \((\theta_i, a_i)\) is based on this information and we allow the pair \((\theta_i, a_i)\) to be randomized.
For each $j = 0, 1, \ldots, J$, the cost on the coming interval of the length $\theta_i$ equals
\[ C_j^\delta(\phi(x_{i-1}, u))du + I\{\theta_i < +\infty\} C_j^\delta(\phi(x_{i-1}, \theta_i), a_i), \]
the last term being absent if $\theta_i = +\infty$. The next state $x_i$ is given by formula (5).

In the space of all the trajectories \[8\]
\[ \Omega = \bigcup_{n=1}^\infty [X \times (([0,1) \times A) \times X]^n \times \{\{t \leq \ell\} \times (\mathbb{R}_+ \times A) \times \{\ell\})^\infty] \]
we fix the natural $\sigma$-algebra $\mathcal{F}$. Finite sequences
\[ h_i = (x_0, (\theta_1, a_1), x_1, (\theta_2, a_2), \ldots, x_i) \]
will be called (finite) histories; $i = 0, 1, 2, \ldots$, and the space of all such histories will be denoted as $H_i$; $\mathcal{F}_i := \mathcal{B}(H_i)$ is the restriction of $\mathcal{F}$ to $H_i$. Capital letters $X_i, T_i, \Theta_i, A_i$ and $H_i$ denote the corresponding functions of $\omega \in \Omega$, i.e., random elements.

**Definition 2.1** A control strategy $\pi = \{\pi_i\}_{i=1}^\infty$ is a sequence of stochastic kernels $\pi_i$ on $[0,1) \times A$ given $H_{i-1}$. A (non-stationary) Markov strategy is defined by stochastic kernels $\{\pi_n(d\theta \times da|x_{n-1})\}_{n=1}^\infty$.

A control strategy is called stationary, and denoted as $\bar{\pi}$, if there is a stochastic kernel $\bar{\pi}$ on $[0,1) \times A$ given $X$ such that $\pi_n(d\theta \times da|x_{n-1}) = \bar{\pi}(d\theta \times da|x_i)$ for all $i = 1, 2, \ldots$. A control strategy is called stationary deterministic and denoted as $(\varphi_0, \varphi_a)$, if, for all $i = 1, 2, \ldots$, $\pi_i(d\theta \times da|h_{i-1}) = \delta_{\varphi_0(x_{i-1})}(d\theta)\delta_{\varphi_a(x_{i-1})}(da)$, where $\varphi_0: X_\Delta \rightarrow \mathbb{R}_+$ and $\varphi_a: X_\Delta \rightarrow A$ are measurable mappings.

If the initial distribution $\nu$ on $X$ and a strategy $\pi$ are fixed, then there is a unique probability measure $P_{\nu}^\pi(\cdot)$ on $\Omega$ satisfying the following conditions:
\[ P_{\nu}^\pi(X_0 \in \Gamma_X) = \nu(\Gamma_X) \]
for all $\Gamma_X \in \mathcal{B}(X_\Delta)$, for all $i \in \mathbb{N}$, $\Gamma \in \mathcal{B}(\mathbb{R}_+ \times A)$, $\Gamma_X \in \mathcal{B}(X_\Delta)$,
\[ P_{\nu}^\pi((\Theta_i, A_i) \in \Gamma|H_{i-1}) = \pi_i(\Gamma|H_{i-1}); \]
\[ P_{\nu}^\pi(X_i \in \Gamma_X|H_{i-1}, (\Theta_i, A_i)) = \left\{ \begin{array}{ll} \delta_{\phi(X_{i-1}, \Theta_i), A_i}(\Gamma_X), & \text{if } X_{i-1} \in X, \Theta_i < +\infty; \\ \delta_{\Delta}(\Gamma_X), & \text{otherwise} \end{array} \right. \]
For details, see the Ionescu-Tulcea Theorem [3, Prop.7.28]. When the initial distribution $\nu$ on $X$ is a Dirac measure concentrated on a singleton, say $\{x_0\}$, we write $P_{\nu}^\pi$ as $P_{x_0}^\pi$. The mathematical expectation with respect to $P_{\nu}^\pi$ and $P_{x_0}^\pi$ is denoted as $E_{\nu}^\pi$ and $E_{x_0}^\pi$, respectively.

Let us introduce the notation
\[ V_j(\nu, \pi) := E_{\nu}^\pi \left[ \sum_{i=1}^\infty I\{X_{i-1} \neq \Delta\} \int_{(0,\Theta_i)} C_j^\delta(\phi(X_{i-1}, u))du + I\{\Theta_i < +\infty\} C_j^\delta(\phi(X_{i-1}, \Theta_i), A_i) \right] \]
for each initial distribution $\nu$, strategy $\pi$ and $j = 0, 1, \ldots, J$. Again, when $\nu = \delta_{x_0}$, we write $V_j(x_0, \pi)$ for $V_j(\nu, \pi)$.

The constrained optimal control problem under study is the following one:
\[ \text{Minimize with respect to } \pi \quad V_0(x_0, \pi) \]
\[ \text{subject to } \quad V_j(x_0, \pi) \leq d_j, \quad j = 1, 2, \ldots, J. \]

Here and below, we take $x_0 \in X$ as a fixed point, and $\{d_j\}_{j=1}^J$ as fixed constraint constants.

**Definition 2.2** A strategy $\pi$ is called feasible if it satisfies all the constraint inequalities in problem [8]. A feasible strategy $\pi^*$ is called optimal if, for all feasible strategies $\pi$, $V_0(x_0, \pi^*) \leq V_0(x_0, \pi)$. 
3 MDP Approach and the First Linear Programming Method

In this section, by using the relevant results of MDPs, we show the existence of a stationary optimal strategy for the impulsive control problem (8) and justify the first linear programming method for it, see linear program (15)-(18).

Clearly, the control model presented in Section 2, from the formal viewpoint, is a specific constrained MDP, which is defined by the following elements. The state space is

\[ X_{\Delta} := X \cup \{\Delta\}, \]

where the state \( \Delta \notin X \) is an isolated point. The action space is

\[ B := \mathbb{R}_0^+ \times A, \]

which is endowed with the product topology and the corresponding Borel \( \sigma \)-algebra. The transition kernel is defined by

\[
Q(dy|x, (\theta, a)) := \begin{cases} 
\delta_{l(\phi(x, \theta), a)}(dy), & \text{if } x \neq \Delta, \ \theta \neq +\infty; \\
\delta_{\Delta}(dy), & \text{otherwise},
\end{cases}
\]

The cost functions are given by

\[
\tilde{C}_j(x, (\theta, a)) = \mathbb{I}\{x \neq \Delta\} \left\{ \int_{[0, \theta]} C^g_j(\phi(x, u))du + \mathbb{I}\{\theta < +\infty\}C^I_j(\phi(x, \theta), a) \right\}, \quad j = 0, 1, \ldots, J,
\]

and the constraint constants are denoted by \( d_j \in \mathbb{R}_0^+, \ j = 1, 2, \ldots, J \). Here \( J \) is the number of constraints.

The first statement concerns the solvability of problem (8). Regarding this, it is a trivial situation if either problem (8) has no feasible strategy: in that case problem (8) is not solvable; or all the feasible strategies \( \pi \) are with infinite value, i.e., \( V_0(x_0, \pi) = \infty \): in that case any feasible strategy is optimal.

We therefore assume the following regarding the consistency of problem (8).

**Condition 3.1** There exists some feasible strategy \( \pi \) such that \( V_0(x_0, \pi) < \infty \).

We also need to assume the following compactness-continuity conditions for the solvability of problem (8), under which, we actually show the existence of a stationary optimal strategy, and develop the linear programming approach for obtaining it.

**Condition 3.2**

(a) The space \( A \) is compact, and \( +\infty \) is the one-point compactification of the positive real line \( \mathbb{R}_0^+ \).

(b) The mapping \( (x, a) \in X \times A \rightarrow l(x, a) \) is continuous.

(c) The mapping \( (x, \theta) \in X \times \mathbb{R}_0^+ \rightarrow \phi(x, \theta) \) is continuous.

(d) For each \( j = 0, 1, \ldots, J \), the function \( (x, a) \in X \times A \rightarrow C^l_j(x, a) \) is lower semicontinuous.

(e) For each \( j = 0, 1, \ldots, J \), the function \( x \in X \rightarrow C^g_j(x) \) is lower semicontinuous.

**Theorem 3.1** Suppose Conditions 3.1 and 3.2 are satisfied. Then there exists an optimal stationary strategy for problem (8).
According to [9, Thm.4.1], for a constrained total cost MDP with Borel state space \( X_\Delta \), Borel action space \( B \), transition probability \( Q \), and positive cost functions \( \{\tilde{C}_j\}_{j=0}^J \), if the model is semicontinuous, then, provided that there exists a feasible strategy with finite value, there is an optimal stationary strategy. Here the model is called semicontinuous if its action space \( B \) is compact, \( \{\tilde{C}_j\}_{j=0}^J \) are all lower semicontinuous, and \( Q \) is continuous, i.e., for each bounded continuous function \( f \) on \( X_\Delta \), 
\[
\int_{X_\Delta} f(y)Q(dy|x,a) \quad \text{is continuous on} \quad X_\Delta \times B.
\]

Since problem (8) can be formulated as such an MDP problem, the statement of the current theorem would follow from [9, Thm.4.1], if the corresponding MDP model described earlier in this subsection is semicontinuous. However, for this MDP, while its action space \( B \) is compact, and its cost functions are lower semicontinuous (as verified in the proof of [17, Thm.1]), its transition probability \( Q \) is in general not continuous, because, if \( x \neq \Delta \), \( \theta_n \in \mathbb{R}_+^0 \) and \( \theta_n \to +\infty \) then the transition probabilities \( Q(dy|x,(\theta_n,a)) \) do not converge to \( \delta_\Delta(dy) \) in the standard weak topology on the space of Borel probability measures on \( X_\Delta \) (generated by bounded continuous functions). Although the conditions in [9] are not all satisfied, some relevant results survive, by checking the proofs therein. Below in the proof of the above theorem, we firstly state the corresponding versions of those results, and then show the existence of an optimal strategy for problem (8) by considering an auxiliary MDP model, which is semicontinuous whereas the original MDP model is not.

**Proof of Theorem 5.1.** The proof proceeds in the following two steps. Firstly, we explain that if there is an optimal strategy for the MDP problem (8), then there is an optimal stationary strategy for it. Secondly, the existence of an optimal strategy for the MDP problem (8) is shown by considering an auxiliary MDP.

Consider the MDP for problem (8). We shall often refer to this MDP as the original model, since we will introduce another auxiliary MDP model below. For each strategy \( \pi \), define its occupation measure (in the original MDP model) \( \mu^\pi \) on \( X_\Delta \times B \) by

\[
\mu^\pi(\Gamma_1 \times \Gamma_2) := E_{x_0}^{\pi} \left[ \sum_{n=0}^\infty \mathbb{1}(\{X_n, B_{n+1}\} \in \Gamma_1 \times \Gamma_2) \right], \quad \forall \Gamma_1 \in B(X_\Delta), \Gamma_2 \in B(B). \tag{9}
\]

Now \( V_j(x_0, \pi) = \int_{X_\Delta \times B} \tilde{C}_j(x,b) \mu^\pi(dx \times db) \).

For each feasible strategy \( \pi \), its occupation measure \( \mu^\pi = \mu \) is feasible in the following linear program:

Minimize over all measures \( \mu \) on \( X_\Delta \times B \) :  

\[
\int_{X_\Delta \times B} \tilde{C}_0(x,b) \mu(dx \times db) \tag{10}
\]

subject to :  

\[
\mu(dx \times B) = \delta_{x_0}(dx) + \int_{X_\Delta \times B} Q(dx|y,b) \mu(dy \times db); 
\]

\[
\int_{X_\Delta \times B} \tilde{C}_j(x,b) \mu(dx \times db) \leq d_j, \quad j = 1,2, \ldots, J.
\]

In the opposite direction, if one solves the above linear program for an optimal solution \( \mu^* \), then one can produce an optimal stationary strategy \( \tilde{\pi}^* \) as follows. Let

\[
V := \left\{ x \in X_\Delta : \inf_{\pi} E_x^\pi \left[ \sum_{n=0}^\infty \sum_{j=0}^J \tilde{C}_j(X_n, B_{n+1}) \right] = 0 \right\}, \tag{11}
\]

where \( \{B_n\}_{n=1}^\infty \) is the action process of the original MDP model of problem (8). Note that \( \Delta \in V \). We also mention that by [17, Thm.1], the function \( x \in X_\Delta \rightarrow \inf_{\pi} E_x^\pi \left[ \sum_{n=0}^\infty \sum_{j=0}^J \tilde{C}_j(X_n, B_{n+1}) \right] \) is positive lower semicontinuous, and there is a measurable mapping \( f^\pi \) from \( X_\Delta \) to \( B \) such that
\[
\inf \mathbb{E}_x^T \left[ \sum_{n=0}^{\infty} \sum_{j=0}^{J} \tilde{C}_j (X_n, B_{n+1}) \right] = E_x^T \left[ \sum_{n=0}^{\infty} \sum_{j=0}^{J} \tilde{C}_j (X_n, B_{n+1}) \right]
\]
for each \( x \in \mathbf{X}_\Delta \). In particular, the set \( V \) is a closed subset of \( \mathbf{X}_\Delta \). Throughout this paper, we fix this measurable mapping \( f^* \). It can be shown as in \cite[Prop.3.2]{9} that \( Q(V^c|x, f^*(x)) = 0 \) for each \( x \in V \).

A feasible measure \( \mu \) in linear program \cite{10} satisfying
\[
\int_{\mathbf{X}_\Delta \times \mathbf{B}} \tilde{C}_0 (x, b) \mu(dx \times db) < \infty
\]
is called to be of finite value. It can be shown as in \cite[Thm.3.2]{9} that each feasible measure \( \mu \) with finite value is \( \sigma \)-finite on \( V^c = \mathbf{X}_\Delta \setminus V \), so that there is a stochastic kernel \( \varphi_\mu \) on \( \mathbf{B} \) given \( V^c \) satisfying \( \tilde{\varphi}_\mu(db|x) \mu(dx \times \mathbf{B}) = \mu(dx \times \mathbf{B}) \) on \( V^c \times \mathbf{B} \), see \cite[Appendix 4]{10} or \cite{12}. It can be shown as in the proof of \cite[Thm.3.3, Cor.3.1]{9} that the stationary strategy \( \tilde{\pi}_\mu \) defined by
\[
\tilde{\pi}_\mu(db|x) := \mathbb{I}\{ x \in V^c \} \varphi_\mu(db|x) + \mathbb{I}\{ x \in V \} \delta_{f^*(x)}(db)
\]
(c.f. \cite[Def.3.1]{9}) dominates the given feasible measure \( \mu \) with finite value in the sense \( V_j (x_0, \tilde{\pi}_\mu) \leq \int_{\mathbf{X}_\Delta \times \mathbf{B}} \tilde{C}_j (x, b) \mu(dx \times db) \) for each \( j = 0, 1, \ldots, J \). Moreover, due to \cite[Thm.3.3]{9}, \( \mu \tilde{\pi}_\mu (\Gamma \times \mathbf{B}) \leq \mu(\Gamma \times \mathbf{B}) \) for each \( \Gamma \in \mathcal{B}(V^c) \). Consequently, we have the following observations, which will be used in the rest of this paper without special reference.

- If one obtains a solution \( \mu^* \) to linear program \cite{10} (such a solution is necessarily with finite value in view of Condition \cite{3.1}), then the stationary strategy \( \tilde{\pi}_{\mu^*} \) defined above with \( \mu \) being replaced by \( \mu^* \) is the required optimal stationary strategy for problem \cite{8}. Incidentally, we mention that for this optimal solution \( \mu^* \), it holds that \( \int_{\mathbf{X}_\Delta \times \mathbf{B}} \tilde{C}_0 (x, b) \mu^*(dx \times db) = V_0 (x_0, \tilde{\pi}_{\mu^*}) \). Nevertheless, in general, \( \mu^* \) and \( \mu \tilde{\pi}_\mu \) may be different, and in particular, it can happen that \( \int_{\mathbf{X}_\Delta \times \mathbf{B}} \tilde{C}_j (x, b) \mu^*(dx \times db) > V_j (x_0, \tilde{\pi}_{\mu^*}) \) for some \( j = 1, 2, \ldots, J \).

- An optimal stationary strategy for the MDP problem \cite{8} exists, as soon as there exists an optimal strategy.

For the future reference, we show that it suffices to solve a simpler linear program for an optimal stationary strategy. Consider the following linear program on the space of \( \sigma \)-finite measures on \( V^c \times \mathbf{B} \):
\[
\text{Minimize over } \sigma\text{-finite measures } \mu \text{ on } V^c \times \mathbf{B} : \int_{V^c \times \mathbf{B}} \tilde{C}_0 (x, b) \mu(dx \times db)
\]
subject to:
\[
\mu(dx \times \mathbf{B}) = \delta_{x_0} (dx) + \int_{V^c \times \mathbf{B}} Q(dx|y, b) \mu(dy \times db);
\]
\[
\int_{V^c \times \mathbf{B}} \tilde{C}_j (x, b) \mu(dx \times db) \leq d_j, \ j = 1, 2, \ldots, J,
\]
where the first constraint holds on \( \mathcal{B}(V^c) \), or to put it more simply, say, on \( V^c \). This problem is consistent for the facts stated in the previous paragraph. For the optimal solution \( \mu^* \) to linear program \cite{10}, we consider the associated occupation measure \( \mu V^c_{\mu^*} \), which is also optimal for problem \cite{10}, as explained in the above. Here \( \mu V^c_{\mu^*} \) is an optimal stationary strategy satisfying \( \tilde{\mu}_V^c (db|x) = \delta_{f^*(x)}(db) \) for each \( x \in V \) (c.f. \cite{12}). Then the restriction of \( \mu V^c_{\mu^*} \) on \( V^c \times \mathbf{B} \) defined by
\[
\mu V^c_{\mu^*} (dx \times db) := \mu V^c_{\mu^*} (dx \times db \cap V^c \times \mathbf{B})
\]
is a feasible solution to problem \cite{13}. Indeed, \( \mu V^c_{\mu^*} (dx \times db) = \mu V^c_{\mu^*} (dx \times \mathbf{B}) \tilde{\mu}_{\mu^*} (db|x) \), which actually is valid for all stationary strategies, and \( \sum_{j=0}^{J} \tilde{C}_j (x, f^*(x)) = 0 \) and \( Q(V^c|x, f^*(x)) = 0 \) for each \( x \in V \) by the definition of the set \( V \), as mentioned earlier. On the other hand, if \( \mu \) is an optimal solution to
problem (13), then the stationary strategy defined by (12) generates a feasible measure for problem (10), which is necessarily optimal, for otherwise, it would contradict the relation mentioned earlier. This follows from the fact that $\mu(dx \times B) \geq \mu^\wedge_\rho(dx \times B)$ on $V^c$, which can be shown as in the proof of [9 Thm.3.3]. In greater detail, suppose for contradiction that the feasible measure $\mu^\wedge_\rho$ is not optimal for linear program (10). Then there is some feasible measure $\gamma$ in linear program (10) such that

$$\int_{X \times B} \tilde{C}_0(x,b) \gamma(dx \times db) < \int_{X \times B} \tilde{C}_0(x,b) \mu^\wedge_\rho(dx \times db).$$

Then

$$\int_{X \times B} \tilde{C}_0(x,b) \mu^\wedge_\rho(dx \times db) = \int_{V^c \times B} \tilde{C}_0(x,b) \mu^\wedge_\rho(dx \times db) \geq \int_{X \times B} \tilde{C}_0(x,b) \mu(dx \times db)$$

$$\geq \int_{X \times B} \tilde{C}_0(x,b) \gamma(dx \times db) \geq \int_{X \times B} \tilde{C}_0(x,b) \mu(dx \times db)$$

where the first equality is by the definition of $\mu^\wedge_\rho$, c.f., (12), the first inequality holds as $\mu$ is optimal for linear program (13) by assumption, and $\mu^\wedge_\rho$ is feasible for linear program (13) by the earlier explanation in this paragraph, and the strict inequality is by the assumption. This yields the desired contradiction. In the discussions in the next section, we will concentrate on linear program (13) instead of (10).

Next, let us show the existence of an optimal strategy. Let $\rho_X$ be a compatible metric on $X$. Since $X$ is a Borel space, according to the lemma of Urysohn, see [3 Prop.7.2], it is without loss of generality to assume that $\rho_X(x,y) \leq 2$ for each $x, y \in X$. We consider the MDP model with the state space

$$\tilde{X} := \mathbb{R}^0_+ \times X \cup \{(\infty, \Delta)\},$$

and action space $B$. Here, we introduce the following metric on $\tilde{X}$: for each $(x_1, s_1), (x_2, s_2) \in \tilde{X}$,

$$\rho_{\tilde{X}}((s_1, x_1), (s_2, x_2)) := \rho_X(x_1, x_2)(1 - g(s_1) \vee g(s_2)) + |g(s_1) - g(s_2)|,$$

where $g(s) := \frac{1}{1 + s}$ defines a one-to-one correspondence between $\mathbb{R}^0_+$ and $[0, 1]$, accepting $g(0) := 0$ and $g(\infty) := 1$. Note that in the previous definition, if $x_i = \Delta$ for some $i = 1, 2$, then $\rho_{\tilde{X}}(x_1, x_2)$ is undefined, but $1 - g(s_1) \vee g(s_2) = 0$; in this case we formally regard $\rho_{\tilde{X}}(x_1, x_2)(1 - g(s_1) \vee g(s_2)) = \rho_{\tilde{X}}(x_1, x_2)0 := 0$. This convention should be kept in mind below in this proof.

Let us firstly verify that $\rho_{\tilde{X}}$ is indeed a metric on $\tilde{X}$. It is clearly positive and symmetric. For notational convenience, we write $g(s) = \tilde{s}$ for all $s \in \mathbb{R}^0_+$. If $\rho_{\tilde{X}}((s_1, x_1), (s_2, x_2)) = 0$, then necessarily $\tilde{s}_i = \tilde{s}_2$. If $\tilde{s}_1 = \tilde{s}_2 < 1$, then $\rho_{\tilde{X}}(x_1, x_2) = 0$, i.e., $(s_1, x_1) = (s_2, x_2)$. If $\tilde{s}_1 = \tilde{s}_2 = 1$, then $(\tilde{s}_1, x_1) = (\tilde{s}_2, x_1) = (\infty, \Delta)$, as required. It suffices to verify the triangle inequality. Let $(s_i, x_i) \in \tilde{X}$, $i = 1, 2, 3$, be arbitrarily fixed. In case of $\tilde{s}_1 \leq \tilde{s}_2 \leq \tilde{s}_3$,

$$\rho_{\tilde{X}}((s_1, x_1), (s_3, x_3)) + \rho_{\tilde{X}}((s_3, x_3), (s_2, x_2)) - \rho_{\tilde{X}}((s_1, x_1), (s_2, x_2)) = \rho_X(x_1, x_3)(1 - \tilde{s}_3) + \tilde{s}_3 - \tilde{s}_1 + \rho_X(x_3, x_2)(1 - \tilde{s}_3) + \tilde{s}_3 - \tilde{s}_2 - \rho_X(x_1, x_3)(1 - \tilde{s}_2) - \tilde{s}_2 + \tilde{s}_1$$

$$= \rho_X(x_1, x_3)(1 - \tilde{s}_3) + \rho_X(x_3, x_2)(1 - \tilde{s}_3) + 2(\tilde{s}_3 - \tilde{s}_2) - \rho_X(x_1, x_2)(1 - \tilde{s}_2)$$

$$\geq (1 - \tilde{s}_3)\rho_X(x_1, x_2) - \rho_X(x_1, x_2)(1 - \tilde{s}_2) + 2(\tilde{s}_3 - \tilde{s}_2)$$

$$= \rho_X(x_1, x_2)(\tilde{s}_2 - \tilde{s}_3) + 2(\tilde{s}_3 - \tilde{s}_2) \geq 0.$$
Finally, in case $\tilde{s}_2 \leq \tilde{s}_3 \leq \tilde{s}_1$,
\begin{align*}
\rho_{\tilde{X}}((s_1, x_1), (s_3, x_3)) + \rho_{\tilde{X}}((s_3, x_3), (s_2, x_2)) - \rho_{\tilde{X}}((s_1, x_1), (s_2, x_2)) \\
= \rho_{\tilde{X}}(x_1, x_3)(1 - \tilde{s}_1 + \tilde{s}_3 + \rho_{\tilde{X}}(x_3, x_2)(1 - \tilde{s}_3) + \tilde{s}_3 - \tilde{s}_2 - \rho_{\tilde{X}}(x_1, x_2)(1 - \tilde{s}_1) - \tilde{s}_1 + \tilde{s}_2 \\
= (\rho_{\tilde{X}}(x_1, x_3) - \rho_{\tilde{X}}(x_1, x_2))(1 - \tilde{s}_1) + \rho_{\tilde{X}}(x_3, x_2)(1 - \tilde{s}_3) \\
\geq (\rho_{\tilde{X}}(x_1, x_3) - \rho_{\tilde{X}}(x_1, x_2))(1 - \tilde{s}_1) + \rho_{\tilde{X}}(x_3, x_2)(1 - \tilde{s}_1) \geq 0,
\end{align*}
as required. The claimed relation holds for all the other possible orders of $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ by symmetry.

With this metric, $(s_n, x_n) \to (s, x) \in \mathbb{R}_+^n \times \tilde{X}$ if and only if $s_n \to s$ and $x_n \to x$, whereas if $(s, x) = (\infty, \Delta)$, then the convergence takes place if and only if $s_n \to \infty$ in the Euclidean norm.

Next we show that $\tilde{X}$ is a Borel subset of $\mathbb{X}$. Let $\mathbb{X}$ be a Borel subset of a Polish space $[X]$. Then

$$[\tilde{X}] := \mathbb{R}_+^n \times [X] \cup \{(\infty, \Delta)\},$$
endowed with $\rho_{[\tilde{X}]}$ defined as above with $\mathbb{X}$ being replaced by $[X]$, is a Polish space. Indeed, for the separability, the union of $\{(\infty, \Delta)\}$ and the countable dense subset of $\mathbb{R}_+^n \times [X]$ (endowed with product topology) provides a required dense subset of $[\tilde{X}]$. For the completeness, consider a Cauchy sequence $\{(s_n, x_n)\}_{n=0}^{\infty} \subseteq [\tilde{X}]$. Then $\{s_n\}_{n=0}^{\infty}$ is Cauchy in $[0, 1]$, and so it converges to some $s \in [0, 1]$. If $s = 1$, then $s_n \to \infty$ and so $(s_n, x_n) \to (\infty, \Delta)$ in the metric space $[\tilde{X}]$. If $s \neq 1$, then $s_n \to s \neq \infty$, and $x_n \to x$ for some $x \in [X]$ as $[X]$ is Polish. In either case, Cauchy sequences in $[\tilde{X}]$ converge, as required.

Finally, observe that $\tilde{X}$ is a Borel subset of $[\tilde{X}]$. Indeed, $\mathbb{R}_+^n \times [X]$ is an open subset of the Polish space $[\tilde{X}]$, and the relative Borel $\sigma$-algebra on $\mathbb{R}_+^n \times [X]$ is the same as the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}_+^n \times [X])$ with respect to the relative topology, which is the product topology on $\mathbb{R}_+^n \times [X]$. Consequently, $\mathbb{R}_+^n \times [X]$ is a Borel subset of $\mathbb{R}_+^n \times [X]$ and thus $\tilde{X}$ is a Borel subset of $[\tilde{X}]$.

Note that $\mathcal{B}([\tilde{X}])$ is generated by $\mathcal{B}(\mathbb{R}_+^n \times [X])$ and $\{(\infty, \Delta)\}$. Consequently, we legitimately define the transition probability $\tilde{Q}$ on $\mathbb{X}$ given $\mathbb{X} \times \mathcal{B}$ by

$$\tilde{Q}(\Gamma_1 \times \Gamma_2|(s, x), (\theta, a)) := \delta_{s+\theta}(\Gamma_1)Q(\Gamma_2|x, (\theta, a))$$

for each $\Gamma_1 \in \mathcal{B}(\mathbb{R}_+^n)$ and $\Gamma_2 \in \mathcal{B}(\mathbb{X})$, and $\tilde{Q}(\{(\infty, \Delta)\}|(s, x), (\theta, a)) := \mathbb{I}\{\theta + s = \infty\}$. The cost functions are $\tilde{C}_j((s, x), (\theta, a)) := \tilde{C}_j(x, (\theta, a))$ for each $(s, x) \in \tilde{X}$ and $(\theta, a) \in \mathcal{B}$, and $j = 0, 1, \ldots, J$.

In this MDP model, which we call the hat model, there is only one extra component compared to the original MDP model: the first coordinate of the state process records the consecutive time moments of impulses, and the second coordinate records the corresponding state after the impulses. Therefore, we take the initial distribution for the hat MDP model as $\delta_0(dt) \times \delta_{x_0}(dx)$. Since the consecutive time moments of impulses can be calculated by summing up the times between consecutive impulses, i.e., the first coordinate of the action in the original MDP model, any strategy in the hat model admits an equivalent strategy in the original model. In particular, if there is an optimal strategy in the hat model, the original MDP admits an optimal strategy, too.

The reason for introducing this hat MDP model is that, under the conditions of the statement, it is semi-continuous, and consequently, according to the paragraph preceding this proof, there is an optimal strategy for it as well as the original MDP. This would complete the proof of this statement. It remains to verify that the hat model is lower semicontinuous as follows. The action space $\mathcal{B}$ is compact trivially. For the continuity of $\tilde{Q}$, consider a bounded continuous function $f$ on $\tilde{X}$. Let $(s_n, x_n) \to (s, x)$, and $(\theta_n, a_n) \to (\theta, a)$. Then consider

$$\int_X f(t, y)\tilde{Q}(dt \times dy|(s_n, x_n), (\theta_n, a_n)) = f(s_n + \theta_n, l(\phi(x_n, \theta_n), a_n))\mathbb{I}\{s_n + \theta_n < \infty\}$$

$$+ \mathbb{I}\{s_n + \theta_n = \infty\}f(\infty, \Delta),$$

as required. The claimed relation holds for all the other possible orders of $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ by symmetry.
If \( s + \theta < \infty \), then \( s_\theta, \theta_\theta < \infty \), and \( x_n \in X \) for all large enough \( n \geq 1 \), and the right hand side of the above equality converges to \( f(s + \theta, l(\phi(x, \theta), a)) = \int_X f(t, y)\bar{Q}(dt \times dy)(s, x, (\theta, a)) \) according to Condition 3.2(b,c). If \( s = \infty \) or \( \theta = \infty \), then \( s_\theta + \theta_\theta \to \infty \), and hence \( f(s_\theta + \theta_\theta, l(\phi(x, \theta, \theta, a), a_n)) \to f(\infty, \Delta) \) according to the definition of topology on \( X \). Thus, the right hand side of (14) still converges to \( f(\infty, \Delta) = \int_X f(t, y)\bar{Q}(dt \times dy)(s, x, (\theta, a)) \), as required. For the lower semicontinuity of \( \hat{C}_j \), where \( j = 0, 1, \ldots, J \) is fixed, consider \( (s_n, x_n) \to (s, x) \), and \( (\theta_n, a_n) \to (\theta, a) \). If \( (s, x) \in \mathbb{R}_+^J \times X \), then \( (s_n, x_n) \in \mathbb{R}_+^J \times X \) for all large enough \( n \geq 1 \), and so

\[
\lim_{n \to \infty} \hat{C}_j((s_n, x_n), (\theta_n, a_n)) = \lim_{n \to \infty} \left\{ \int_{[0, \theta_n]} C_j^d(\phi(x_n, u))du + \mathbb{I} \{ \theta_n < +\infty \} C_j^d(\phi(x_n, \theta_n), a_n) \right\} \\
\geq \int_{[0, \theta]} C_j^d(\phi(x, u))du + \mathbb{I} \{ \theta < +\infty \} C_j^d(\phi(x, \theta), a) = \hat{C}_j((s, x), (\theta, a)),
\]

where the inequality holds because the function \( (\theta, x) \in \mathbb{R}_+^J \times X \to \int_{[0, \theta]} C_j^d(\phi(x, u))du + \mathbb{I} \{ \theta < +\infty \} C_j^d(\phi(x, \theta), a) \) is lower semicontinuous under Condition 3.2(c,d,e), as was proved in the proof of [12] Thm.1, see equation (7) therein. If \( (s, x) = (\infty, \Delta) \), then \( \lim_{n \to \infty} \hat{C}_j((s_n, x_n), (\theta_n, a_n)) \geq 0 = \hat{C}_j((\infty, \Delta), (\theta, a)) \) still holds. The lower semicontinuity of \( \hat{C}_j \) is now seen.

To complete the proof, one can refer to [9] or [13] for the existence of an optimal strategy for the hat MDP model, because it is semicontinuous. According to the previous discussions, this induces an optimal strategy for the original MDP model, from which we deduce the existence of an optimal stationary strategy for the original MDP model as well as for problem (8), recall the paragraph at the beginning of this proof. Note also that linear programs (10) and (13) are solvable. \( \square \)

Theorem 3.4 is about the solvability of problem (8). Its proof also reveals how to obtain an optimal stationary strategy. Namely, under the conditions of Theorem 3.1 first obtain the set \( V \) (11) by solving the unconstrained problem

\[
\text{Minimize over } \pi: \mathbb{E}_x^c \left[ \sum_{n=0}^\infty \sum_{j=0}^J \tilde{C}_j(X_n, B_{n+1}) \right]
\]

with a deterministic stationary strategy \( f^* \). After that, solve the linear program (13) for the optimal solution \( \mu^* \). Then obtain a stochastic kernel \( \varphi_* \) on \( \mathbb{R}_+^J \times \mathbb{A} \) from \( V^c \) by disintegration: \( \mu^*(dx \times d\theta \times da) = \mu^*(dx \times \mathbb{R}_+^J \times \mathbb{A}) \varphi_*(d\theta \times da|x) \), and the stationary strategy

\[
\pi_* \varphi_* (d\theta \times da|x) := \mathbb{I} \{ x \in V^c \} \varphi_*(d\theta \times da|x) + \mathbb{I} \{ x \in V \} \delta_{f^*(x)} (d\theta \times da)
\]

for each \( x \in X_\Delta \) is optimal for problem (8).

**Condition 3.3** There exists \( \delta > 0 \) such that \( \sum_{j=0}^J C_j^d(x, a) \geq \delta \) for all \( (x, a) \in X \times \mathbb{A} \).

Below, we assume that Conditions 3.3, 3.2 and 3.3 are satisfied.

According to the definition of the set \( V \), see (11), and the mapping \( f^* \), under Condition 3.3 if \( x \in V \), then \( f^*(x) = (\infty, a) \) because otherwise the expectation \( \mathbb{E}_x^c \left[ \sum_{n=0}^\infty \sum_{j=0}^J \tilde{C}_j(X_n, B_{n+1}) \right] \) would have been bigger than \( \delta > 0 \). This means that, if \( x \in V \cap X \), then necessarily for the optimality no impulses are applied in the future on the whole trajectory \( \phi(x, u) \), \( u \in \mathbb{R}_+^J \). Note also that \( \sum_{j=0}^J C_j^d(\phi(x, t)) \geq 0 \) for almost all \( t \in \mathbb{R}_+^J \), for each \( x \in V \cup X \); consequently, \( \phi(x, t) \in V \cup X \) and \( C_j^d(\phi(x, t)) = 0 \) for almost all \( t \in \mathbb{R}_+^J \) (in fact, for all \( t \) due to Conditions 3.2(c,e)), for each \( x \in V \cap X \), for all \( j = 0, 1, \ldots, J \). Moreover, Condition 3.3 implies that one can actually focus on the finite
measures $\mu$ on $V^c \times \mathbb{R}_0^+ \times A$ in the linear program (13). The reason is that an infinite occupation measure $\tilde{\mu}$ on $V^c \times \mathbb{R}_0^+ \times A$ would be either infeasible or with infinite value. Consequently it is suboptimal in view of Condition 3.1. The more detailed explanation is as follows:

$$\int_{V^c \times \mathbb{R}_0^+ \times A} \left( \sum_{j=0}^J \tilde{C}_j(x, (\theta, a)) \right) \mu^\pi(dx \times d\theta \times da) \geq \delta E_{\pi \circ \tilde{\nu}} \left[ \sum_{n=0}^{\infty} \mathbb{I}\{X_n \in V^c, \Theta_{n+1} < \infty\} \right]$$

subject to:

$$\mu(\Gamma \times \mathbb{R}_0^+ \times A) = \delta_{x_0}(\Gamma) + \int_{V^c \times \mathbb{R}_0^+ \times A} \mathbb{I}\{I(\phi(y, \theta), a) \in \Gamma\} \mathbb{I}\{\phi(y, \theta) \in V^c\} \mu(dy \times d\theta \times da),$$

$$\forall \Gamma \in \mathcal{B}(V^c),$$

$$\int_{V^c \times \mathbb{R}_0^+ \times A} \left( \int_{[0, \theta]} C_j^\phi(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} du + \mathbb{I}\{\theta < +\infty\} \mathbb{I}\{\phi(x, \theta) \in V^c\} C_j^\theta(\phi(x, \theta), a) \right) \mu(dx \times d\theta \times da) \leq d_j, \quad \forall j = 1, 2, \ldots, J. \quad (17)$$

The terms $\mathbb{I}\{\phi(x, u) \in V^c\}$ in (16) and (18) appear legitimately because, as explained above, $C_j^\phi(\phi(x, u)) = 0$ for almost all $u \in \mathbb{R}_0^+$, for each $x \in V$, for all $j = 0, 1, \ldots, J$.

As for the terms $\mathbb{I}\{I(\phi(y, \theta), a) \in \Gamma\}$ in (15), (17) and (18), note that, for the optimal stationary strategy $\pi^\mu_\ast$, one should have

$$\pi^\mu_\ast(\{\theta : \phi(x, \theta) \in V\} \times A|x) = 0$$

for almost all $x$ with respect to $\mu^\pi_\ast (dx \times B)$; otherwise, under $\pi^\mu_\ast$ impulses will be applied in the states $\phi(x, \theta) \in V$ with positive probability, which is not optimal. Thus

$$\int_{V^c \times \mathbb{R}_0^+} \mathbb{I}\{\phi(x, \theta) \in V\} \mu^\pi(dx \times d\theta \times A) = \int_{V^c} \int_{\mathbb{R}_0^+} \mathbb{I}\{\phi(x, \theta) \in V\} \pi^\mu_\ast(d\theta \times A|x) \mu^\pi(dx \times B) = 0,$$

and we do not miss the desired optimal solution of the linear program (13) if we insert the terms under consideration in (15), (17) and (18). The minimal value of linear program (15)–(18) is finite by Condition 3.1. Linear program (15)–(18) has an optimal solution because $\pi^\mu_\ast$ does program (13).

Note that the case $x_0 \in V \cap X$ is trivial: $\mu = 0$ on $V^c \times \mathbb{R}_0^+ \times A$ and the stationary deterministic strategy $f^\ast$ from the proof of Theorem 3.1 is optimal. Below, we assume that $x_0 \in V^c$. As was established in the proof of Theorem 3.1 under Conditions 3.1 and 3.2 the set $V$ is closed in $X_\Delta$. With some abuse of notation, we denote the closed set $V \cap X$ as $V$ as well, just to avoid new notations. As was explained, for each $x \in V$, $\phi(x, t) \in V$ for all $t \in \mathbb{R}_0^+$. 

13
4 Aggregated Occupation Measures and Modified Linear Program

In this section, we introduce the aggregated occupation measures based on the measures in the first linear program \([16] - [18]\), and provide a candidate characterizing relation satisfied by them. We also modify the first linear program \([16] - [18]\) and rewrite it partially in terms of aggregated occupation measures. The results in this section also serve the analysis in the next section, where a second linear programming method, which is completely in terms of aggregated occupation measures will be provided.

Firstly, note that, under Condition 3.3, the objective functionals \([16]\) and \([18]\) can be represented in terms of other measures, in some sense easier than the measures \(\mu\).

In the linear program \([16] - [18]\), for each \(j = 0, 1, \ldots, J\), the integral

\[
\int_{V^c \times \mathbb{R}_+^0} \left\{ \int_{[0, \theta]} C^g_j(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} du \right\} \mu(dx \times d\theta \times da)
\]

can be rewritten in the following way. Since the measures \(\mu\) are finite on \(V^c \times \mathbb{R}_+^0 \times A\), they have the form

\[
\mu(dx \times d\theta \times da) = p_T(d\theta|x, a)p_A(da|x)\mu(dx \times \mathbb{R}_+^0 \times A),
\]

where \(p_T(\cdot)\) and \(p_A(\cdot)\) are stochastic kernels on \(\mathbb{R}_+^0\) and \(A\) correspondingly: see [3, Prop.7.27]. The dependence of \(p_T\) and \(p_A\) on \(\mu\) is not explicitly indicated. Hence, using the Tonelli Theorem [11 Thm.11.28] (remember, \(C^g_j(\cdot) \geq 0\)), we obtain

\[
\int_{V^c \times \mathbb{R}_+^0} \left\{ \int_{[0, \theta]} C^g_j(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} du \right\} \mu(dx \times d\theta \times da)
\]

\[
= \int_{V^c} \int_{\mathbb{R}_+^0} \left\{ \int_{[0, \theta]} C^g_j(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} du \right\} p_T(d\theta|x, a)p_A(da|x)\mu(dx \times \mathbb{R}_+^0 \times A)
\]

\[
= \int_{V^c} \int_{\mathbb{R}_+^0} \int_{[u, \infty]} C^g_j(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} p_T(d\theta|x, a)\, dx \, dp_A(da|x)\mu(dx \times \mathbb{R}_+^0 \times A)
\]

\[
= \int_{V^c} \int_{\mathbb{R}_+^0} C^g_j(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} p_T([u, \infty]|x, a)\, dx \, dp_A(da|x)\mu(dx \times \mathbb{R}_+^0 \times A).
\]

After we introduce the following measure on \(V^c\)

\[
\eta(dy \times \Box) := \int_{V^c} \int_{\mathbb{R}_+^0} \delta_{\phi(x,u)}(dy) \mathbb{I}\{\phi(x, u) \in V^c\} \left( \int_A p_T([u, \infty]|x, a)\, dp_A(da|x) \right) \, dx \, \mu(dx \times \mathbb{R}_+^0 \times A),
\]

the last integral equals simply

\[
\int_{V^c} C^g_j(y) \eta(dy \times \Box).
\]

Here the point \(\Box \notin A\), introduced for the notational convenience in the future, plays no role.

Similarly to the above, for each \(j = 0, 1, \ldots, J\),

\[
\int_{V^c \times \mathbb{R}_+^0 \times A} \left\{ \mathbb{I}\{\theta < +\infty\} \mathbb{I}\{\phi(x, \theta) \in V^c\} C^f_j(\phi(x, \theta), a) \right\} \mu(dx \times d\theta \times da)
\]

\[
= \int_{V^c} \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi(x, \theta) \in V^c\} C^f_j(\phi(x, \theta), a)p_T(d\theta|x, a)p_A(da|x)\mu(dx \times \mathbb{R}_+^0 \times A)
\]

\[
= \int_{V^c \times A} C^f_j(y, a) \eta(dy \times da),
\]
where
\[
\eta(dy \times da) := \int_{V^c} \int_{\mathbb{R}^+_0} \delta_{\phi(x, \theta)}(dy) I\{\phi(x, \theta) \in V^c\} p_T(d\theta|x, a)p_A(da|x) \mu(dx \times \mathbb{R}^+_0 \times \mathbf{A})
\]
\[
= \int_{V^c} \int_{\mathbb{R}^+_0} \delta_{\phi(x, \theta)}(dy) I\{\phi(x, \theta) \in V^c\} \mu(dx \times d\theta \times da)
\]
(21)
is a finite measure on \(V^c \times \mathbf{A}\), since the measure \(\mu\) is finite.

Let us introduce the modified action space \(\mathbf{A}_{\square} := \mathbf{A} \cup \{\square\}\), where \(\square\) is the additional isolated point mentioned above, the cost functions
\[
C_j(x, a) := \left\{ \begin{array}{ll}
C_j^g(x), & \text{if } a = \square; \\
C_j^l(x, a), & \text{if } a \in \mathbf{A}
\end{array} \right.
\]
on \(V^c \times \mathbf{A}_{\square}\), and the following combined measure \(\eta\).

**Definition 4.1** Given the measure \(\mu\) in linear program (13)-(15), the measure on \(V^c \times \mathbf{A}_{\square}\) defined by
\[
\eta(\Gamma_X \times \Gamma_A) := \eta(\Gamma_X \times (\Gamma_A \cap \mathbf{A})) + \eta(\Gamma_X \times \{\square\}) I\{\square \in \Gamma_A\},
\]
(22)
where the measures \(\eta(dy \times \square)\) on \(V^c\) and \(\eta(dy \times da)\) on \(V^c \times \mathbf{A}\) were introduced in (21) and (22), is called an aggregated occupation measure (induced by \(\mu\)).

Now, under Condition 3.3 the linear program (13)-(15) can be rewritten in the following way:
\[
\text{Minimize } \int_{V^c \times \mathbf{A}_{\square}} C_0(y, a) \eta(dy \times da) \text{ over finite measures } \mu \text{ on } V^c \times \mathbb{R}^+_0 \times \mathbf{A}
\]
(23)
subject to:
\[
\int_{V^c \times \mathbf{A}_{\square}} C_j(y, a) \eta(dy \times da) \leq d_j, \quad \forall j = 1, 2, \ldots, J;
\]
\[
\eta(\Gamma_X \times \square) = \int_{V^c} \int_{\mathbb{R}^+_0} \delta_{\phi(x, \theta)}(\Gamma_X) \left( \int_{\mathbf{A}} p_T([u, \infty]|x, a)p_A(da|x) \right) \mu(dx \times \mathbb{R}^+_0 \times \mathbf{A})
\]
\[
= \int_{\mathbb{R}^+_0} \left\{ \int_{V^c \times \mathbf{A}} \delta_{\phi(x, \theta)}(\Gamma_X) \mu(dx \times [u, \infty] \times da) \right\} du, \quad \Gamma_X \in \mathcal{B}(V^c);
\]
(24)
\[
\eta(\Gamma_X \times \Gamma_A) = \int_{V^c} \int_{\mathbb{R}^+_0} \delta_{\phi(x, \theta)}(\Gamma_X) p_T(d\theta|x, a)p_A(\Gamma_A|x) \mu(dx \times \mathbb{R}^+_0 \times \mathbf{A})
\]
\[
= \int_{V^c \times \mathbb{R}^+_0} \delta_{\phi(x, \theta)}(\Gamma_X) \mu(dx \times d\theta \times \Gamma_A), \quad \Gamma_X \in \mathcal{B}(V^c), \Gamma_A \in \mathcal{B}(\mathbf{A});
\]
(25)
\[
\mu(\Gamma \times \mathbb{R}^+_0 \times \mathbf{A}) = \delta_{x_0}(\Gamma) + \int_{V^c \times \mathbb{R}^+_0 \times \mathbf{A}} I\{l(\phi(y, \theta), a) \in \Gamma\} I\{\phi(y, \theta) \in V^c\} \mu(dy \times d\theta \times da),
\]
(26)
\[
\Gamma \in \mathcal{B}(V^c).
\]
In (24), the Tonelli Theorem was used. Below, the Fubini-Tonelli Theorem is used without explicit references.

It was shown in the proof of Theorem 3.1 that the optimal solution exists and the minimal value \(\mu^*\) is finite. As soon as the solution \(\mu^*\) is found, the optimal control strategy is defined by formula (12). The target is to formulate the linear program without references to measures \(\mu\). The new linear program, in terms of aggregated occupation measures only, is given in Corollary 6.1.

We will need the following simple lemma concerning Markov strategies.
Lemma 4.1 Let $\pi$ be a Markov strategy defined by stochastic kernels $\pi_i(d\theta \times da|x) = p^i_T(d\theta|x)p^i_A(da|x, \theta)$ and $\eta$ be the corresponding aggregated occupation measure (22) coming from the occupation measure $\mu^n$ as in (9). Introduce the (partial) aggregated occupation measures

$$\eta^n(\Gamma_X \times \Gamma_A) := \eta^n(\Gamma_X \times (\Gamma_A \cap A)) + \eta^n(\Gamma_X \times \Box) \ll \{\Box \in \Gamma_A\}$$
onumber

on $V^c \times A$, defined recursively:

$$\eta^0(\Gamma_X \times \Gamma_A) \equiv 0;$$

$$\eta^{i+1}(\Gamma_X \times \Box) = \eta^i(\Gamma_X \times \Box) + \int_{V^c} \int_{\mathbb{R}^0_+} \delta_{d(x,w)}(\Gamma_X)p^{i+1}_T([u, \infty]|x)du \nu^i(dx), \Gamma_X \in \mathcal{B}(V^c);$$

$$\eta^{i+1}(\Gamma_X \times \Gamma_A) = \eta^i(\Gamma_X \times \Gamma_A) + \int_{V^c} \int_{\mathbb{R}^0_+} \delta_{d(x,\theta)}(\Gamma_X)p^{i+1}_A(\Gamma_A|x,\theta)p^{i+1}_T(\theta|dx)\nu^i(dx), \Gamma_X \in \mathcal{B}(V^c), \Gamma_A \in \mathcal{B}(A),$$

where $\nu^i(dx) = P_{x_0}^\pi(X_i \in dx), i \geq 0$.

Then $\eta^n \uparrow \eta$ on $V^c \times A$ set-wise as $n \to \infty$.

The proofs of this and most other lemmas are presented in the Appendix.

Definition 4.2 $W$ is the space of measurable bounded functions $w$ on $X$, absolutely continuous, negative and increasing or positive and decreasing along the flow $\phi$ and satisfying conditions:

1. $w(y) = 0$ for all $y \in V$ and
2. $\lim_{t \to \infty} w(\phi(x,t)) = 0$ for all $x \in V^c$ such that $\phi(x,t) \in V^c$ for all $t \in \mathbb{R}^0_+$.

Throughout this paper, $\chi w$ denotes a function as in Lemma A.1 (see Appendix). Without loss of generality, one can assume for each negative (or positive) function $w \in W$ that the function $\chi w$ is positive (or negative), i.e., in [23] one can put $g(\cdot) \equiv 0$. Note that below we consider only such measures $\zeta$ on $V^c$ that the value of the integral $\int_{V^c} \chi w(x)\zeta(dx)$ does not depend on the function $g$ in [54].

Theorem 4.1 Let Conditions 3.1, 3.2 and 3.3 be fulfilled. Suppose a finite measure $\mu$ on $V^c \times \mathbb{R}^0_+ \times A$ satisfies equation (17). Then the aggregated occupation measure $\eta$ given by (22) (recall (19), (20) and (27)) satisfies equation

$$0 = w(x_0) + \int_{V^c} \chi w(x)\eta(dx \times \Box) - \int_{V^c} w(x)\eta(dx \times A) + \int_{V^c \times A} w(l(x,a))\eta(dx \times da)$$

(27)

for all functions $w \in W$. All particular integrals in (27) are finite; the measure $\eta(dx \times da)$ is finite on $V^c \times A$.

Proof. Note that, for each function $w \in W$, for each fixed $x \in V^c$, the function $w(\phi(x,\cdot))$ is bounded on $\mathbb{R}^0_+$.

According to Lemma A.1, for each fixed $x \in V^c$,

$$w(\phi(x,\theta)) = w(x) + \int_{[0,\theta]} \chi w(\phi(x,s))ds,$$

where the function $\chi w$ is given by (54). After we integrate this equation over $\mathbb{R}^0_+$ with respect to the measure

$$\int_A \mathbb{I}\{\phi(x,\theta) \in V^c\}p_T(d\theta|x,a)p_A(da|x)\mu(dx \times \mathbb{R}^0_+ \times A),$$

16
on $V^c \times \mathbb{R}_+^0$, where the stochastic kernels $p_T$ and $p_A$ are as in [19], we obtain the equality

$$\int_{V^c} w(y)\eta(dy \times A) = \int_{V^c} \int_{\mathbb{R}_+^0} w(\phi(x, \theta)) \mathbb{I}\{\phi(x, \theta) \in V^c\} \hat{p}(d\theta|x) \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$= \int_{V^c} w(x) \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi(x, \theta) \in V^c\} \hat{p}(d\theta|x) \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$+ \int_{V^c} \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi(x, \theta) \in V^c\} \int_{[0, \theta]} \chi w(\phi(x, s)) ds \int_{\mathbb{R}_+^0} \hat{p}(d\theta|x) \mu(dx \times \mathbb{R}_+^0 \times A),$$

where $\hat{p}(d\theta|x) := \int_A p_T(d\theta|x, a)p_A(da|x)$. Note that all the integrals here are finite because the function $w(\cdot)$ is bounded and the measures $\mu$ and $\eta(dy \times A)$ are finite. For each $x \in V^c$, let us denote

$$\theta^*(x) := \inf\{\theta \in \mathbb{R}_+^0 : \phi(x, \theta) \in V\}. \tag{28}$$

As usual, $\inf \emptyset := +\infty$. Since the flow $\phi$ is continuous, the function $\theta^*(\cdot)$ is measurable: see [8, Lemma 27.1] or [11] Prop.1.5, p.154. Besides, $\theta^*(x) > 0$ because the set $V^c$ is open and the set $V$ is closed.

Since the set $V$ is closed and the flow $\phi$ is continuous, $\phi(x, \theta^*(x)) \in V \cap X$ and the infimum is attained, if $\theta^*(x) < +\infty$, and, as mentioned after Condition 3.3, $\phi(x, s) \in V$ for all $s \geq \theta^*(x)$. Therefore,

$$\int_{V^c} w(y)\eta(dy \times A) = \int_{V^c} w(x)\int_{\mathbb{R}_+^0} \mathbb{I}(\theta^*(x), \infty] dx \times \mathbb{R}_+^0 \times A)$$

$$- \int_{V^c} w(x)\hat{p}(\theta^*(x), \infty] \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$+ \int_{V^c} \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi(x, \theta) \in V^c\} \int_{[0, \theta]} \chi w(\phi(x, s)) ds \int_{\mathbb{R}_+^0} \hat{p}(d\theta|x) \mu(dx \times \mathbb{R}_+^0 \times A).$$

Recall, the measure $\eta(dy \times A)$ is finite and the function $\chi w(\phi(x, s))$ is integrable on $[0, \theta]$ with $\theta < \infty$. After we apply the Tonelli Theorem [11] Thm.11.28 to the last integral, we obtain:

$$\int_{V^c} w(y)\eta(dy \times A) = \int_{V^c} w(x)\mu(dx \times \mathbb{R}_+^0 \times A) - \int_{V^c} w(x)\hat{p}(\theta^*(x), \infty] \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$+ \int_{V^c} \int_{\mathbb{R}_+^0} \int_{[s, \infty)} \mathbb{I}\{\phi(x, \theta) \in V^c\} \chi w(\phi(x, s)) \hat{p}(d\theta|x) ds \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$= \int_{V^c} w(x)\mu(dx \times \mathbb{R}_+^0 \times A) - \int_{V^c} w(x)\hat{p}(\theta^*(x), \infty] \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$+ \int_{V^c} \chi w(\phi(x, s))\mathbb{I}\{\phi(x, s) \in V^c\} \hat{p}(s, \theta^*(x)] ds \mu(dx \times \mathbb{R}_+^0 \times A).$$

The artificially added term $\mathbb{I}\{\phi(x, s) \in V^c\}$ does not change anything as $\chi w(\phi(x, s)) = 0$ for $\phi(x, s) \in V$. (See [14], where, in our case, $V \subseteq D$ and $W(y) = 0$ for all $y \in V$.) Now

$$\int_{V^c} w(y)\eta(dy \times A) = \int_{V^c} w(x)\mu(dx \times \mathbb{R}_+^0 \times A) - \int_{V^c} w(x)\hat{p}(\theta^*(x), \infty] \mu(dx \times \mathbb{R}_+^0 \times A)$$

$$+ \int_{V^c} \chi w(y)\eta(dy \times \emptyset) - \int_{V^c} \int_{\mathbb{R}_+^0} \chi w(\phi(x, s))\hat{p}(\theta^*(x), \infty] ds \mu(dx \times \mathbb{R}_+^0 \times A).$$
All the integrals here are finite because of the following. Since in any case, whether \( \theta^*(x) \) is finite or not,
\[
\lim_{t \to \infty} w(\phi(x,t)) = w(x) + \int_{\mathbb{R}^+_0} \chi w(\phi(x,s))ds = 0,
\]
we conclude that
\[
\int_{V^c} w(x) \hat{p}([\theta^*(x), \infty]|x) \mu(dx \times \bar{\mathbb{R}}^+_0 \times \mathbf{A}) + \int_{V^c} \int_{\mathbb{R}^+_0} \chi w(\phi(x,s)) \hat{p}([\theta^*(x), \infty]|x) ds \mu(dx \times \bar{\mathbb{R}}^+_0 \times \mathbf{A}) = 0.
\]
Here and above, all the integrals are finite.

Finally,
\[
\int_{V^c} w(y)\eta(dy \times \mathbf{A}) = \int_{V^c} w(x)\mu(dx \times \bar{\mathbb{R}}^+_0 \times \mathbf{A}) + \int_{V^c} \chi w(y)\eta(dy \times \mathbf{A})
\]
\[
= w(x_0) + \int_{V^c \times \mathbb{R}^+_0 \times \mathbf{A}} w((\phi(y,\theta),a))\mathbb{I}\{\phi(y,\theta) \in V^c\} \mu(dy \times d\theta \times da)
\]
\[
+ \int_{V^c} \chi w(y)\eta(dy \times \mathbf{A})
\]
by (17), and the required formula (27) follows from the definition (21).

The finiteness of the measure \( \eta(dx \times da) \) is obvious and was mentioned earlier. \( \square \)

5 The Second Linear Programming Method

In this section, we will present a second linear programming method for the impulsive control problem §3. To this end, we will consider the impulsive control problem §3 in the extended model, where we extend the state with the extra coordinate, which records the time since the most recent impulse. It will be clear that the impulsive control problem in the extended model is equivalent to the one in the original model. We would apply the results established earlier to the impulse control problem in the extended model. For this reason, it would be convenient if we denote the enlarged state space \( \tilde{\mathbf{X}} \), and thus will denote the original state space \( \hat{\mathbf{X}} \). Accordingly, the flow will be denoted by \( \tilde{\phi} \) in the model with extended state space and by \( \hat{\phi} \) in the original model. More precisely, the following notation will be in use.

Definition 5.1 Let \( \tilde{\mathbf{X}} \) be the original state space with the flow \( \tilde{\phi} \) satisfying the requirements formulated earlier in Section §2 with \( \mathbf{X} \) being replaced by \( \hat{\mathbf{X}} \) therein. Then we put
\[
\mathbf{X} := \{(\tilde{y},t) \subset \tilde{\mathbf{X}} \times \bar{\mathbb{R}}^+_0 : \tilde{y} = \tilde{\phi}(\tilde{x},t) \text{ for some } \tilde{x} \in \tilde{\mathbf{X}}\}.
\]

In \( \mathbb{R}^+_0 \), the standard Euclidean topology is fixed. The product space \( \tilde{\mathbf{X}} \times \mathbb{R}^+_0 \) is equipped with the product topology [\( \mathbb{R}^+_0 \) §2.14] which is metrizable. The topology on \( \mathbf{X} \) is the restriction of the product topology on \( \tilde{\mathbf{X}} \times \mathbb{R}^+_0 \) on it. We endow \( \mathbf{X} \) with its Borel \( \sigma \)-algebra, which is the restriction of the Borel \( \sigma \)-algebra \( \mathcal{B}(\tilde{\mathbf{X}} \times \mathbb{R}^+_0) \) on \( \tilde{\mathbf{X}} \), see [3 Lem.7.4]. The flow is modified in the obvious way:
\[
\tilde{\phi}((\tilde{x},t),u) := (\tilde{\phi}(\tilde{x},u),t + u).
\]

The mappings \( C^f_j \) and \( C^g_j \) do not depend on the component \( t \) of the state \( x = (\tilde{x},t) \) so that
\[
C^f_j((\tilde{x},t),a) := \begin{cases} C^g_j(\tilde{x}), & \text{if } a = \Box; \\
C^f_j(\tilde{x},a), & \text{if } a \in \mathbf{A}; \end{cases}
\]
\[
C^g_j((\tilde{x},t),a) := C^g_j(\tilde{x},a).
\]

18
l(x,a) = l(\tilde{x},t), \ a = (\tilde{l}(\tilde{x},a),0),$ where $\tilde{l}$ is the original mapping describing the state after an impulse: after each impulse the t component goes down to zero. The initial state $x_0 = (\tilde{x}_0,0)$ has the time component zero.

Rigorously speaking, for the above definition to be consistent with those in Section 2 we need $X$ in the above definition to be a Borel space. This will be guaranteed under some further conditions imposed below, see Lemma 5.2.

Recall that Theorem 1.1 provides a relation, see (27), which is satisfied by the aggregated occupation measures. The next and indeed more demanding step is to show that, if equation (27) holds for a measure $\eta$ with $\eta(V^c \times A) < \infty$, for all functions $w \in \mathcal{W}$, then the following assertion is valid. There exists a finite measure $\mu$ on $V^c \times \mathbb{R}^0_+ \times A$ satisfying equation (17) such that the corresponding aggregated occupation measure $\tilde{\eta}$ (22), defined using (19), (20), (21), is set-wise majorized by the measure $\eta$. In fact, that measure $\mu$ comes from a Markov strategy $\pi$ according to (9): see Theorem 5.1 below. This will give rise to the second linear programming method, see Corollary 5.1 and the discussions following it. We implement this plan in the rest of this section as well as in the next section.

**Definition 5.2** We call the orbit of a point $\tilde{x} \in \tilde{X}$ the following subset of $X$:

$$\tilde{x} \mathcal{X} = \{ (\tilde{\phi}(\tilde{x},t), t) : t \in \mathbb{R}^0_+ \} = \{ \phi((\tilde{x},0), t) : t \in \mathbb{R}^0_+ \}.$$ 

**Lemma 5.1** The following assertions hold.

(i) The flow $\phi$ has no cycles.

(ii) Suppose the original flow $\tilde{\phi}$ is continuous, so that $\phi$ is continuous, too. Then every orbit is a closed set in $\tilde{X} \times \mathbb{R}^0_+$.

**Condition 5.1** Two different orbits do not intersect, i.e., for any two distinct points $\tilde{x}_1 \neq \tilde{x}_2 \in \tilde{X}$, $\tilde{x}_1 \mathcal{X} \cap \tilde{x}_2 \mathcal{X} = \emptyset$.

**Definition 5.3** Under Condition 5.1 for each $y = (\tilde{y}, t) \in X$, we introduce $h(y)$ equal to the point $\tilde{x}^0 \in \tilde{X}$ such that $\tilde{y} = \tilde{\phi}(\tilde{x}^0, t)$ and put $\tau_y = t$.

Note that the mapping $h : X \to \tilde{X}$ is well defined: if, for $y = (\tilde{y}, t) \in X$, for two points $\tilde{x}_1^0 \neq \tilde{x}_2^0$ from $\tilde{X}$, $\tilde{y} = \tilde{\phi}(\tilde{x}_1^0, t) = \tilde{\phi}(\tilde{x}_2^0, t)$, then the different orbits $\tilde{x}_1^0 \mathcal{X}$ and $\tilde{x}_2^0 \mathcal{X}$ are not disjoint having the common point $y$.

All the introduced notations are illustrated on Figure 1.

If Condition 5.1 is satisfied, one can define the flow $\tilde{\phi}$ in the reverse time. For each $y = (\tilde{y}, t) \in X$ we say that $\tilde{\phi}(\tilde{y}, -t) = h(y)$ and, for all $u \in [0,t]$, we put $\tilde{\phi}(\tilde{y}, -u) := \tilde{\phi}(h(y), t-u)$. The semigroup property here takes the form $\phi(x, t+s) = \phi(\phi(x, s), t)$ for $s$ and $t$ satisfying $s \geq -\tau_x$, $t+s \geq -\tau_x$. In general, the flow $\phi$ in the reverse time is not continuous. Similarly, in general, the set $X$ is analytic, see [12, §12.5], [13, Def.7.16], as the image of the continuous mapping $(\tilde{x}^0, t) \to (\tilde{\phi}(\tilde{x}^0, t), t)$ from $\tilde{X} \times \mathbb{R}^0_+$ to $\tilde{X} \times \mathbb{R}^0_+$, and needs not be Borel. The space $X$ is a Borel space, see Lemma 5.2 if we assume that the divergence of the original trajectories in $\tilde{X}$, in the reverse time, is bounded by some function of time. Or, roughly speaking, the speed of moving along the flow $\tilde{\phi}$ from $h((\tilde{y}, t))$ to $\tilde{y}$ is bounded. The precise condition is as follows.

**Condition 5.2** Condition 5.1 is satisfied; the original flow $\tilde{\phi}$ is continuous, and thus so is the flow $\phi$; and there exists a $(0, \infty)$-valued function $d$ on $\mathbb{R}^0_+$, bounded on every finite interval $[0,T]$ and such that for all $y_1 = (\tilde{y}_1,t_1)$, $y_2 = (\tilde{y}_2,t_2) \in X$,

$$\tilde{\rho}(h(y_1), h(y_2)) = \rho(\tilde{\phi}(\tilde{y}_1, -t_1), \tilde{\phi}(\tilde{y}_2, -t_2)) \leq d(t_1) \vee d(t_2) \rho(y_1,y_2),$$

where $\rho$ and $\tilde{\rho}$ denote the compatible metrics on $X$ and $\tilde{X}$, respectively.
Figure 1: Flows \( \tilde{\phi} \) and \( \phi \). The grey area is \( \tilde{V}_c^\nu \): outside it \( \nu \equiv 0 \).

**Lemma 5.2** Suppose Condition 5.2 is satisfied. Then the mapping \( h : X \to \tilde{X} \), introduced in Definition 5.3, is continuous, the original flow \( \tilde{\phi} \) in the reverse time is continuous, the mapping \( F : \tilde{X} \times \mathbb{R}_0^+ \to X \) defined as

\[
F(x_0, t) := (\tilde{\phi}(x_0, t), t) = \phi((x_0, 0), t), \quad \text{with} \quad F^{-1}(y) = (h(y), \tau_y),
\]

is a homeomorphism, and the set \( X \) is a Borel space.

As was mentioned above, the modified flow \( \phi \) is obviously continuous if \( \tilde{\phi} \) was so. Conditions 3.1, 3.2 and 3.3 remain fulfilled if and only if they were so for the original model, i.e., the model with the original state space \( \tilde{X} \). Now suppose Conditions 3.1, 3.2 and 3.3 are satisfied, so that Theorems 3.1 and 4.1 remain valid for the current model with the extended state space. Histories \( h_i \) contain only elements \( x_k \) of the form \( x_k = (\tilde{x}_k, 0) \); hence there is a one-to-one correspondence between the control strategies in the original “tilde” model and in the extended model with the state space \( X \). The set \( V \) defined as in (11) for the “tilde” model gives rise to the corresponding set

\[
V := \{ y = (\tilde{y}, t) \in X : \tilde{y} \in \tilde{V} \cap \tilde{X} \} \cup \{ \Delta \}
\]

with the complement in \( X \) being \( V^c \). For \( \tilde{\nu} \in \tilde{V}^c \),

\[
\tilde{\nu}^*(\tilde{z}) := \inf\{ \theta \in \mathbb{R}_0^+ : \phi((\tilde{z}, 0), \theta) \in V \} = \inf\{ \theta \in \mathbb{R}_0^+ : \tilde{\phi}(\tilde{z}, \theta) \in \tilde{V} \}.
\]

(cf [28]; as was shown, the function \( \tilde{\nu}^*(\cdot) \) is measurable and \( \tilde{\nu}^*(\tilde{z}) > 0 \) because the set \( \tilde{V}^c \) is open.)

Definition 4.2 of the space \( W \) remains the same, but for the extended state space \( X \), the flow \( \phi \) and the sets \( V \) and \( V^c \). The next remark is in its position now.

**Remark 5.1** Suppose Condition 5.2 is satisfied. If Conditions 3.1, 3.2 and 3.3 are satisfied, so that the sets \( \tilde{V} \) and \( \tilde{V}^c \) are well defined, then \( F \) provides a homeomorphism between the set

\[
D := \{ (\tilde{x}_0, t) : \tilde{x}_0 \in \tilde{V}^c, \ 0 \leq t < \tilde{\nu}^*(\tilde{x}_0) \} = \{ (\tilde{x}_0, t) : \tilde{\phi}(\tilde{x}_0, t) \in \tilde{V}^c \} \quad (29)
\]

and \( V^c \). Indeed, \( F(\tilde{x}_0, t) \in V^c \) if and only if the pair \( (\tilde{x}_0, t) \) belongs to the set \( D \) defined in (29). See Figure 3. Recall that \( \phi(\tilde{x}_0, t) \in V \) for all \( t \in \mathbb{R}_0^+ \) if \( \tilde{x}_0 \in \tilde{V} \).
We underline that the points \((\bar{x}^0, t) \in D \) and \((\bar{y}, t) \in X\) have different meanings, although the components \(\bar{x}^0, \bar{y} \in X\) and \(t \in \mathbb{R}^+\) look the same. That is the reason to equip the first coordinates of points in \(D\) with the upper index 0, to make them look different from the points in \(X\).

**Definition 5.4** Suppose Conditions 5.1, 5.2, 5.3, and 5.2 are satisfied. If \(\zeta\) is a measure on \(V^c\), then \(\zeta\) is the image of \(\zeta\) under the mapping \(F\):

\[
\zeta(\Gamma) = \zeta(F(\Gamma)), \quad \Gamma \in \mathcal{B}(D).
\]

In case the measure \(\zeta\) is finite, we, with slight but convenient abuse of notations, introduce \(\zeta(\Gamma) := \zeta(\Gamma \times \mathbb{R}^+_0)\) for \(\Gamma \in \mathcal{B}(V^c)\), the marginal of \(\zeta\) on \(V^c\), and \(\zeta(dt|\bar{x}^0)\), the stochastic kernel from \(V^c\) to \(\mathbb{R}^+_0\) such that

\[
\zeta(d\bar{x}^0 \times dt) = \zeta(d\bar{x}^0)\zeta(dt|\bar{x}^0),
\]

see [5] Cor. 7.27.2 or [12] Prop. D.8. Below, we sometimes simplify the notations and, given \(\zeta\), denote \(\zeta(dt|\bar{x}^0)\) as \(\zeta(dt|\bar{x}^0)\). Note that \(\zeta([0, \bar{\theta}^*(\bar{x}^0)])|\bar{x}^0) = 1\) for \(\zeta\)-almost all \(\bar{x}^0 \in V^c\). Clearly,

\[
\zeta_1 \subseteq \zeta_2 \text{ set-wise} \iff \zeta_1 \leq \zeta_2 \text{ set-wise}.
\]

If the measure \(\zeta\) is zero outside the set \(\bar{V}^c \times \{0\}\), then \(\zeta(\Gamma) = 0\) for all measurable subsets \(\Gamma \subseteq D \cap \{\bar{x}^0, t) \in X \times \mathbb{R}^+_0 : t > 0\}\), \(\zeta(\Gamma) = \zeta(\bar{x}^0 \times \{0\})\) for all \(\Gamma \in \mathcal{B}(V^c)\), and \(\zeta(dt|\bar{x}^0) = \delta_0(dt)\) for \(\zeta\)-almost all \(\bar{x}^0 \in \bar{V}^c\).

**Definition 5.5** Under Conditions 5.1, 5.2, 5.3, and 5.2 a measure \(\zeta\) on \(V^c\) is called normal if there exist a finite measure \(L\) on \(\bar{V}^c\) and a bounded measurable function \(g(\bar{x}^0, u) : \bar{V}^c \times \mathbb{R}^+_0 \to \mathbb{R}^+_0\) such that

\[
\zeta(d\bar{x}^0 \times du) = g(\bar{x}^0, u)du L(d\bar{x}^0).
\]

Clearly, every normal measure is \(\sigma\)-finite. Similarly, a normal measure on some orbit is understood.

**Remark 5.2** Suppose Conditions 5.1, 5.2, 5.3, and 5.2 are satisfied.

(a) The restriction to \(V^c \times (\mathbb{R}^+_0 \times A)\) of the measure \(\mu^\pi\) defined in (27) is concentrated on \(\bar{V}^c \times \{0\} \times \mathcal{B} = \{\bar{x}, 0) : \bar{x} \in \bar{V}^c\} \times \mathcal{B}\) because, for all \(n = 0, 1, \ldots\), if \(X_n \neq \Delta\), then \(X_n = (\bar{x}, 0)\) when \(\bar{x}\) remains in \(X^c\). Therefore, in problems (17), (18) and (26), (27), and (28), one needs to consider only such finite measures \(\mu\).

(b) Consider a finite measure \(\mu\) on \(V^c \times \mathbb{R}^+_0 \times A\) satisfying (26) and concentrated on \(\bar{V}^c \times \{0\} \times \mathbb{R}^+_0 \times A\). Then for the aggregated occupation measure \(\eta\) induced by \(\mu\), \(\eta(dy \times \square)\) (see (26)) is obviously a normal measure because

\[
\eta(\Gamma) = \eta(F(\Gamma)) = \int_{\mathbb{R}^+_0} \int_{\bar{V}^c} \delta_{(\bar{x}^0, 0)}(F(\Gamma)) \int_A p_T([u, \infty])|\bar{x}^0, 0, a)p_A(da)\mu(d\bar{x}^0 \times \{0\} \times \mathbb{R}^+_0 \times A)\]

\[
= \int_{\mathbb{R}^+_0} \int_{\bar{V}^c} \chi_{\{\bar{x}^0, u) \in \Gamma\}} \int_A p_T([u, \infty])|\bar{x}^0, 0, a)p_A(da)\mu(d\bar{x}^0 \times \{0\} \times \mathbb{R}^+_0 \times A),
\]

where the second equality is by the definition of the aggregated occupation measure \(\eta\), and the third equality is by the definition of the mapping \(F\).
Suppose Conditions 3.1, 3.2, 3.3 and 5.2 are satisfied. For a given measure $\zeta$ on $V^c$ and a function $w \in W$, the value of the integral $\int_{V^c} \chi w(x) \zeta(dx)$ depends on the selection of the function $g(\cdot)$ in (54), see Definition 4.2. We would like to get rid of this dependence. This is the case if the measure $\zeta$ is normal, or more generally, weakly normal, defined as follows.

**Definition 5.6** Suppose Conditions 3.1, 3.2, 3.3 and 5.2 are satisfied. Then a measure $\zeta$ on $V^c$ is called weakly normal if it satisfies the following two conditions.

(i) 
\[ \zeta\{(x = (\tilde{x}, u) \in V^c : \tilde{x} \in \tilde{V}^c, 0 \leq u < t}\} < \infty \ \forall \ t \in \mathbb{R}_+. \]
Equivalently,
\[ \zeta(\Gamma_t) < \infty \ \forall \ t \in \mathbb{R}_+, \]
where 
\[ \Gamma_t := \{(\tilde{x}^0, u) : \tilde{x}^0 \in \tilde{V}^c; \ 0 \leq u < t \wedge \tilde{\theta}^*(\tilde{x}^0)\} \in \mathcal{B}(D). \] (30)

(ii) For each $t \in \mathbb{R}_+$, the stochastic kernel $\zeta_t(du|\tilde{x}^0)$, coming from the decomposition 
\[ \hat{\zeta}_t(dx^0 \times du) = \hat{\zeta}_t(dx^0)\zeta_t(du|\tilde{x}^0) \]

of the finite measure $\hat{\zeta}_t$, which is the restriction of $\tilde{\zeta}$ on $\Gamma_t$, is absolutely continuous with respect to the Lebesgue measure for $\tilde{\zeta}_t$-almost all $\tilde{x}^0$: $\zeta_t(du|\tilde{x}^0) = G_t(\tilde{x}^0, u)du$. Here $\tilde{\zeta}_t$ is the marginal of $\tilde{\zeta}_t$ on $\tilde{V}^c$.

**Lemma 5.3** Suppose Conditions 3.1, 3.2, 3.3 and 5.2 are satisfied. Then every normal measure $\zeta$ on $V^c$ is weakly normal.

Now, if Conditions 3.1, 3.2, 3.3 and 5.2 are satisfied, then, for each weakly normal measure $\zeta$ and $w \in W$, independently on the selection of $g$ in (54), the integral $\int_{V^c} \chi w(x) \zeta(dx)$ is uniquely defined, being equal to the limit of the monotone sequence of integrals 
\[ \int_{\Gamma_t} \chi w(\phi((\tilde{x}^0, 0), u))\zeta(dx^0 \times du) = \int_{\tilde{V}^c} \int_{[0,1] \times \tilde{\theta}^*(\tilde{x}^0)} \chi w(\phi((\tilde{x}^0, 0), u))G_t(\tilde{x}^0, u)du \hat{\zeta}_t(dx^0) \]
as $t \uparrow \infty$, where $\Gamma_t$ is defined by (30). For the justification, see Item (b) of Lemma A.2 and remember that $\chi w(\phi((\tilde{x}^0, u)))$ is positive (or negative) for all $u$, and $\Gamma_{t_2} \supseteq \Gamma_{t_1}$ if $t_2 > t_1$.

For two weakly normal measures $\zeta^1$ and $\zeta^2$ and the corresponding measures $\tilde{\zeta}^1$ and $\tilde{\zeta}^2$, in case $\zeta^1 \geq \zeta^2$ set-wise, we define the difference $\tilde{\zeta}^1 - \tilde{\zeta}^2$ as
\[ (\tilde{\zeta}^1 - \tilde{\zeta}^2)(\Gamma) := \lim_{t \to \infty} (\tilde{\zeta}^1(\Gamma \cap \Gamma_t) - \tilde{\zeta}^2(\Gamma \cap \Gamma_t)), \ \forall \ \Gamma \in \mathcal{B}(D), \]
where $\Gamma_t$ is defined by (30). Then, the difference $\zeta^1 - \zeta^2$ is defined by
\[ (\zeta^1 - \zeta^2)(\Gamma) := (\tilde{\zeta}^1 - \tilde{\zeta}^2)(F^{-1}(\Gamma)), \ \forall \ \Gamma \in \mathcal{B}(V^c). \]
Now the differences $\tilde{\zeta}^1 - \tilde{\zeta}^2$ and $\zeta^1 - \zeta^2$ are (positive) measures.

**Lemma 5.4** Suppose Conditions 3.1, 3.2, 3.3 and 5.2 are satisfied.
Remark 5.3

By checking its proof, one can see that the statement of Theorem 5.1 remains true if for \( \eta \) the measure in the next section.

This is why a more strict definition of normal measures is in use, which also serves the duality results as far as linear program (31) is concerned, it is desirable to have it in a smaller space of measures.

To be more precise, if the finite measure \( \mu \) on \( V^c \times A_\square \), coming from the occupation measure \( \eta \) such that the optimal solution \( \bar{\eta} \) as in (4), inequalities \( \bar{\eta}(\Gamma) \leq \eta(\Gamma) \) hold for all \( \Gamma \in B(V^c \times A_\square) \).

The statement of Theorem 5.1 can be further strengthened as in the next remark.

Remark 5.3 By checking its proof, one can see that the statement of Theorem 5.1 remains true if for the measure \( \eta \), \( \eta(dx \times \square) \) is weakly normal. Consequently, Corollary 5.1 remains true if one considers linear program (31) therein in such a space of measures \( \eta \) that \( \eta(dx \times \square) \) are weakly normal. However, as far as linear program (31) is concerned, it is desirable to have it in a smaller space of measures. This is why a more strict definition of normal measures is in use, which also serves the duality results in the next section.

The proofs of Theorem 5.1 and of the next corollary are postponed to the next section.

Corollary 5.1. Let Conditions 5.1, 5.2, and 5.3 be satisfied. Then linear program (23)-(26) is equivalent to the following one:

\[
\begin{align*}
\text{Minimize over the measures } \eta & \text{ on } V^c \times A_\square \\
\int_{V^c \times A_\square} C_0(x,a)\eta(dx \times da) & \\
\text{subject to} & \\
(i) & \eta(V^c \times A) < \infty \text{ and the measure } \eta(dx \times \square) \text{ on } V^c \text{ is normal.} \\
(ii) & \text{Equation (27) is satisfied for all functions } w \in W: \\
& 0 = w(x_0) + \int_{V^c} \chi w(x)\eta(dx \times \square) - \int_{V^c} w(x)\eta(dx \times A) + \int_{V^c \times A} w(l(x,a))\eta(dx \times da). \\
(iii) & \int_{V^c \times A_\square} C_j(x,a)\eta(dx \times da) \leq d_j, \forall j = 1,2,\ldots,J.
\end{align*}
\]

To be more precise, if the finite measure \( \mu^* \) on \( V^c \times \bar{R}_+ \times A \) solves linear program (23)-(26), then the measure \( \eta^* \) on \( V^c \times A_\square \), given by (25) and (26) solves linear program (31). Conversely, if the measure \( \eta^* \) on \( V^c \times A_\square \) solves linear program (31), then, for the Markov strategy \( \pi^* \) as in Theorem 5.1, the corresponding occupation measure \( \mu^* \) on \( V^c \times \bar{R}_+ \times A \), defined in (4), solves linear program (23)-(26).

As soon as the optimal solution \( \eta^* \) to the linear program (31) formulated above is obtained, the Markov strategy \( \pi^* \) from Theorem 5.1 solves the original optimal impulsive control problem (8). Recall that linear program (23)-(26) has an optimal solution; hence the linear program (31) is also solvable. Note also that, having in hand the Markov strategy \( \pi^* \), one can compute the corresponding occupation measure \( \mu^* \), (9), and after that the stationary strategy (12) also solves the optimal impulsive control problem (8).
6 Proof of Theorem 5.1 and Corollary 5.1

The proofs will be based on a series of lemmas.

Lemma 6.1 Suppose Conditions 3.1 3.2 3.3 and 5.2 are satisfied. Suppose an orbit
\[ \mathcal{X} \cap V^c = \{ \phi((\tilde{x},0),t) : t \in [0,\tilde{\theta}^*(\tilde{x})) \} \]
is fixed and \( p^* \) is a probability measure on \( \mathbb{R}_+^0 \) such that \( p^*([\tilde{\theta}^*(\tilde{x}),\infty)) = 0 \).
Then the measures \( \eta^*_\mathcal{X} \) and \( \eta^*_A \) on \( \mathcal{X} \cap V^c \), defined as
\[ \eta^*_\mathcal{X}(\Gamma) := \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi((\tilde{x},0),u) \in \Gamma\} p^*(\{u,\infty\}) du = \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi((\tilde{x},0),u) \in \Gamma\} (1 - p^*(\{0,u\}) du, \]
\[ \eta^*_A(\Gamma) := \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi((\tilde{x},0),u) \in \Gamma\} p^*(du), \quad \Gamma \in B(\mathcal{X} \cap V^c), \]
satisfy equation
\[ 0 = w((\tilde{x},0)) + \int_{\mathcal{X} \cap V^c} \chi w(x) \eta^*_\mathcal{X}(dx) - \int_{\mathcal{X} \cap V^c} w(x) \eta^*_A(dx) \quad (32) \]
for all functions \( w \in \mathcal{W} \). The measure \( \eta^*_\mathcal{X} \) is finite, and the measure \( \eta^*_A \) is normal on the orbit.

Proof. The properties of the measures \( \eta^*_\mathcal{X} \) and \( \eta^*_A \) formulated in the last sentence of this lemma are obvious, c.f. the reasoning in Item (b) of Remark 5.2. Now let \( w \in \mathcal{W} \) be fixed. We verify the rest of the statement of this lemma by distinguishing the following two cases.

(i) Suppose that \( u^* := \inf\{u \in \mathbb{R}_+^0 : p^*([0,u]) = 1\} \geq \tilde{\theta}^*(\tilde{x}) \). The expression
\[ I := w((\tilde{x},0)) + \int_{\mathcal{X} \cap V^c} \chi w(x) \eta^*_\mathcal{X}(dx) - \int_{\mathcal{X} \cap V^c} w(x) \eta^*_A(dx) \]
is well defined because the measure \( \eta^*_\mathcal{X} \) is normal, the integral \( \int_{\mathcal{X} \cap V^c} \chi w(x) \eta^*_\mathcal{X}(dx) \) is positive or negative, the function \( w \) is bounded and the measure \( \eta^*_A \) is finite. According to Lemma A.2(c),
\[ I = w((\tilde{x},0)) + \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} \chi w(\phi((\tilde{x},0),t)) [1 - p^*([0,t])] dt - \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} w(\phi((\tilde{x},0),t)) p^*(dt) \]
\[ = - \left[ \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} \chi w(\phi((\tilde{x},0),t)) p^*([0,t]) dt + \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} w(\phi((\tilde{x},0),t)) p^*(dt) \right]. \]
The last equality is by Lemma A.1 and Definition 1.2 of the space \( \mathcal{W} \):
\[ w((\tilde{x},0)) + \lim_{T \to \tilde{\theta}^*(\tilde{x})} \int_{\{0,T\}} \chi w(\phi((\tilde{x},0),t)) dt = \lim_{T \to \tilde{\theta}^*(\tilde{x})} w(\phi((\tilde{x},0),T)) = 0. \]
We apply the Tonelli Theorem [11 Thm.11.28] to the first integral in the square brackets and again use Lemma A.1,
\[ \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} \chi w(\phi((\tilde{x},0),t)) \int_{\{0,t\}} p^*(du) dt = \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} \int_{\{u,\tilde{\theta}^*(\tilde{x})\}} \chi w(\phi((\tilde{x},0),t)) dt p^*(du) \]
\[ = \int_{\{0,\tilde{\theta}^*(\tilde{x})\}} \int_{\{u,\tilde{\theta}^*(\tilde{x})\}} [-w(\phi((\tilde{x},0),u))] p^*(du). \]
Thus \( I = 0 \).
(ii) Suppose that \( u^* := \inf \{ u \in \mathbb{R}_+^0 : p^*([0, u]) = 1 \} < \tilde{\theta}^*(\tilde{x}) \). Since measures \( \tilde{\eta}_A \) and \( \tilde{\eta}_C \) both equal zero on the set \( \{ \phi((\tilde{x}, 0), t) : t > u^* \} \), it is sufficient to show that
\[
I := w((\tilde{x}, 0)) + \int_{[0, u^*]} \chi w(\phi((\tilde{x}, 0), t))[1 - p^*([0, t])] dt
- \int_{[0, u^*]} w(\phi((\tilde{x}, 0), t))p^* (dt) - w(\phi((\tilde{x}, 0), u^*))[1 - p^*([0, u^*])].
\]

In the last term, \( [1 - p^*([0, u^*])] = p^*([u^*, \infty)) \). Since
\[
w((\tilde{x}, 0)) + \int_{[0, u^*]} \chi w(\phi((\tilde{x}, 0), t)) dt - w(\phi((\tilde{x}, 0), u^*)) = 0
\]
(see Lemma A.1), after we subtract this equality from \( I \), we obtain
\[
I = -\int_{[0, u^*]} \chi w(\phi((\tilde{x}, 0), t))p^*([0, t]) dt - \int_{[0, u^*]} w(\phi((\tilde{x}, 0), t))p^* (dt) + w(\phi((\tilde{x}, 0), u^*))p^*([0, u^*]).
\]

Finally, apply the Tonelli Theorem [1] Thm.11.28 to the first term and again use Lemma A.1
\[
\int_{[0, u^*]} \int_{[0, t]} \chi w(\phi((\tilde{x}, 0), t)) p^* (du) dt = \int_{[0, u^*]} \int_{[u, u^*]} \chi w(\phi((\tilde{x}, 0), t)) dt p^* (du)
= \int_{[0, u^*]} [w(\phi((\tilde{x}, 0), u^*)) - w(\phi((\tilde{x}, 0), u))] p^* (du)
= w(\phi((\tilde{x}, 0), u^*)) p^*([0, u^*]) - \int_{[0, u^*]} w(\phi((\tilde{x}, 0), u)) p^* (du).
\]

Therefore, \( I = 0 \).

The proof is completed. \( \square \)

**Lemma 6.2** Let Conditions \( \mathcal{A}, \mathcal{B}, \mathcal{B}' \) and \( \mathcal{B}'' \) be satisfied. Suppose \( \nu \) is a finite measure on \( V^c \) such that \( \nu(V^c \times \{ t : t > 0 \}) = 0 \), or say more rigorously, \( \nu(V^c \cap (V^c \times \{ t : t > 0 \})) = 0 \), \( \tilde{\eta} \) is a finite measure on \( V^c \times \mathbb{A} \), \( \tilde{\eta}_C \) is a normal (or weakly normal) measure on \( V^c \) and \( \tilde{\eta}_A \) is a finite measure on \( V^c \) which satisfy equation
\[
0 = \int_{V^c} w(x) \nu(dx) + \int_{V^c} \chi w(x) \tilde{\eta}_C(dx) - \int_{V^c} w(x) \tilde{\eta}_A(dx) + \int_{V^c \times \mathbb{A}} w(t(x, a)) \tilde{\eta}(dx \times da) \tag{33}
\]
for all functions \( w \in \mathcal{W} \). Then there is a stochastic kernel \( \tilde{p}(dt|x) \) on \( \mathbb{R}_+^0 \) given \( V^c \) such that, for \( \theta^* \) as in (33), \( \tilde{p}(\theta^*(x), \infty)|x) = 0 \) for \( \nu \)-almost all \( x \in V^c \) and the measures

\[
\tilde{\eta}_A^w(\Gamma) := \int_{V^c} \int_{\mathbb{R}_+^0} \mathbb{1}\{\phi(x, u) \in \Gamma\} \tilde{p}(du|x) \nu(dx)
\]

and

\[
\tilde{\eta}_\Gamma^w(\Gamma) := \int_{V^c} \int_{\mathbb{R}_+^0} \mathbb{1}\{\phi(x, u) \in \Gamma\} \tilde{p}(|u, \infty||x)du \nu(dx), \quad \Gamma \in \mathcal{B}(V^c)
\]

satisfy equation

\[
0 = \int_{V^c} w(x) \nu(dx) + \int_{V^c} \chi w(x) \tilde{\eta}_\Gamma^w(dx) - \int_{V^c} w(x) \tilde{\eta}_A^w(dx)
\]

(34)

for all functions \( w \in \mathcal{W} \). Moreover, the set functions \( \tilde{\eta}_\Gamma(\Gamma) - \tilde{\eta}_\Gamma(\Gamma) \) and \( \tilde{\eta}_A(\Gamma) - \tilde{\eta}_A(\Gamma) \) on \( \mathcal{B}(V^c) \) are again normal (or weakly normal) and finite measures, correspondingly.

**Proof.** (i) First of all, we underline that, for each \( \Gamma \in \mathcal{B}(V^c) \), the function \( \mathbb{1}\{\phi(x, u) \in \Gamma\} \) is measurable since the flow \( \phi \) is continuous. Below, \( \hat{\nu}(\Gamma) := \nu(\Gamma \times \{0\}) \) for \( \Gamma \in \mathcal{B}(V^c) \).

According to Definition 5.4, we introduce the finite measure \( \hat{\eta}_A(dx) \) and stochastic kernel \( \hat{\eta}_A(dt|x) \) coming from \( \hat{\eta}_A(dx) \). Next, introduce the finite measure

\[
K := \hat{\nu} + \hat{\eta}_A
\]

on \( \tilde{V}^c \) and the Radon-Nikodym derivatives

\[
n(x^0) := \frac{d\hat{\nu}}{dK}(x^0), \quad \text{and} \quad a(x^0) := \frac{d\hat{\eta}_A}{dK}(x^0).
\]

Below, we fix one specific version of the derivative \( n \) and of the derivative \( a \). On the set

\[
\tilde{V}_\nu^c := \{x^0 \in \tilde{V}^c : n(x^0) > 0\},
\]

we have

\[
\hat{\eta}_A(\tilde{\Gamma}) = \int_{\tilde{\Gamma}} a(x^0) K(dx^0) = \int_{\tilde{\Gamma}} \frac{a(x^0)}{n(x^0)} \hat{\nu}(d\tilde{x}^0)
\]

for all \( \tilde{\Gamma} \in \mathcal{B}(\tilde{V}_\nu^c) \). See Figure 1.

Since the function \( \mathbb{1}\{u \leq t\} \) of \( (u, t) \) is measurable, the integral \( \int_{\mathbb{R}_+^0} \mathbb{1}\{u \leq t\} \hat{\eta}_A(du|x^0) \) is a measurable function of \( (x^0, t) \) [3, Prop.7.29], and hence the function

\[
G(x^0, t) := \hat{\eta}_A([0, t]|x^0) \frac{a(x^0)}{n(x^0)} = \int_{\mathbb{R}_+^0} \mathbb{1}\{u \leq t\} \hat{\eta}_A(du|x^0) \frac{a(x^0)}{n(x^0)}, \quad x^0 \in \tilde{V}_\nu^c, \quad t \in \mathbb{R}_+
\]

is measurable. For all \( x^0 \in \tilde{V}_\nu^c \), the function \( G(x^0, \cdot) \) clearly increases and is right-continuous: it is constant for \( t \geq \theta^*(x^0) \) and, if \( t_i \downarrow t \in [0, \theta^*(x^0)] \) then \( \tilde{\eta}_A([0, t_i]|x^0) \downarrow \tilde{\eta}_A([0, t]|x^0) \).

Let us introduce the function

\[
u^*(x^0) := \inf\{t \in \mathbb{R}_+^0 : G(x^0, t) \geq 1\} \in \mathbb{R}_+^0, \quad x^0 \in \tilde{V}_\nu^c.
\]

This infimum is clearly attained and \( G(x^0, \nu^*(x^0) -) \leq 1 \). To show that the function \( \nu^*(\cdot) \) is measurable, it is sufficient to notice that \( \nu^* \) is the solution to the one-step MDP with the state space \( \tilde{V}_\nu^c \), admissible compact action spaces

\[
T(x^0) := [\nu^*(x^0), \infty] \subset \mathbb{R}_+^0
\]

26
and the cost \( c(\tilde{x}^0, t) = t \) [12, Prop.D.5]. (The multifunction \( \tilde{x}^0 \to T(\tilde{x}^0) \) is measurable because the graph

\[
Gr = \{(\tilde{x}^0, t) : \tilde{x}^0 \in \tilde{V}^c, t \in T(\tilde{x}^0)\} = \{(\tilde{x}^0, t) \in \tilde{V}^c \times \bar{R}_+^0 : G(\tilde{x}^0, t) \geq 1\}
\]

is measurable [12, Prop.D.4].) Note also that if \( u^*(\tilde{x}^0) > \tilde{\theta}^*(\tilde{x}^0) \) then \( u^*(\tilde{x}^0) = \infty \). Figure 3 can serve as an illustration.

For \( \tilde{x}^0 \in \tilde{V}^c \), we put

\[
\tilde{p}(I(\tilde{x}^0, 0)) := \tilde{\eta}_A(I \cap [0, u^*(\tilde{x}^0) \land \tilde{\theta}^*(\tilde{x}^0)]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} + \mathbb{I}\{u^*(\tilde{x}^0) < \tilde{\theta}^*(\tilde{x}^0)\} \mathbb{I}\{u^*(\tilde{x}^0) \in I\} \left[ 1 - \tilde{\eta}_A([0, u^*(\tilde{x}^0)]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \right]
\]

for all \( I \in \mathcal{B}(\bar{R}^0_+) \), and \( \tilde{p}(\{\infty\})(\tilde{x}^0, 0) := 1 - \tilde{p}(\bar{R}^0_+|\tilde{x}^0, 0)) \).

For all other points \( x \in V^c \), we put \( \tilde{p}(\{\infty\}|x) = 1 \) and \( \tilde{p}(I|x) \equiv 0 \) for \( I \in \mathcal{B}(\bar{R}^0_+) \). Clearly, \( \tilde{p}(\theta^*(x), \infty)|x) = 0 \) for \( \nu \)-almost all \( x \in V^c \). The possible shapes of the distribution function \( \tilde{p}([0, t]|(\tilde{x}^0, 0)) \) are shown on Figure 2.

![Figure 2: Graphs of the function \( \tilde{p}([0, t]|(\tilde{x}^0, 0)) \), see also Figure 3](image)

In case a), \( \tilde{p}([0, t]|(\tilde{x}^0, 0)) = \tilde{\eta}_A([0, t]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \) for all \( t \in \bar{R}^0_+ \). This holds for all \( \tilde{x}^0 \in \tilde{V}^c \), where

\[
\tilde{p}(I(\tilde{x}^0, 0)) := \tilde{\eta}_A(I \cap [0, u^*(\tilde{x}^0)]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} + \mathbb{I}\{u^*(\tilde{x}^0) < \tilde{\theta}^*(\tilde{x}^0)\} \mathbb{I}\{u^*(\tilde{x}^0) \in I\} \left[ 1 - \tilde{\eta}_A([0, u^*(\tilde{x}^0)]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \right]
\]

for all \( I \in \mathcal{B}(\bar{R}^0_+) \), and \( \tilde{p}(\{\infty\})(\tilde{x}^0, 0) := 1 - \tilde{p}(\bar{R}^0_+|\tilde{x}^0, 0)) \).

(ii) Let us prove that equation (44) holds. Since \( \nu(V^c \times \{ t : t > 0 \}) = 0 \),

\[
\tilde{\eta}'_A(\Gamma) = \int_{V^c} \tilde{\eta}_A^c(\Gamma(\tilde{x}^0, 0)) \tilde{\nu}(d\tilde{x}^0); \quad \tilde{\eta}'_c = \int_{V^c} \tilde{\eta}_c^c(\Gamma(\tilde{x}^0, 0)) \tilde{\nu}(d\tilde{x}^0)
\]

for all \( \Gamma \in \mathcal{B}(V^c) \), where

\[
\tilde{\eta}_A^c(\Gamma(\tilde{x}^0, 0)) := \int_{\bar{R}^0_+} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{p}(du|(\tilde{x}^0, 0));
\]

\[
\tilde{\eta}_c^c(\Gamma(\tilde{x}^0, 0)) := \int_{\bar{R}^0_+} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{p}(\infty|((\tilde{x}^0, 0))) du.
\]
The introduced measures $\eta_A^*$ and $\eta_\nu^c$ are concentrated on $\mathbb{P}_0\mathcal{X} \cap V^c$ for each $\tilde{x}^0 \in \tilde{V}^c$. By the way, $\eta_A^*$ and $\eta_\nu^c$ are measurable kernels because the flow $\phi$ is continuous and $\tilde{p}$ is a (measurable) stochastic kernel. Now

$$
\int_{V^c} w(x)\nu(dx) + \int_{V^c} \chi w(x)\eta_\nu^c(dx) - \int_{V^c} w(x)\tilde{\eta}_A^*(dx)
\quad = \int_{V^c} w((\tilde{x}^0, 0))\tilde{\nu}(d\tilde{x}^0) + \int_{V^c} \int_{\tilde{x}^0, \chi \cap V^c} \chi w(x)\eta_\nu^c(dx|\tilde{x}^0, 0)\tilde{\nu}(d\tilde{x}^0)
\quad - \int_{V^c} \int_{\tilde{x}^0, \chi \cap V^c} w(x)\eta_A^*(dx|\tilde{x}^0, 0)\tilde{\nu}(d\tilde{x}^0)
\quad = \int_{V^c} \left[w((\tilde{x}^0, 0)) + \int_{\tilde{x}^0, \chi \cap V^c} \chi w(x)\eta_\nu^c(dx|\tilde{x}^0, 0) - \int_{\tilde{x}^0, \chi \cap V^c} w(x)\eta_A^*(dx|\tilde{x}^0, 0)\right] \tilde{\nu}(d\tilde{x}^0).
$$

The re-arrangement is legal because the function $w$ is bounded, the measure $\tilde{\eta}_\nu^c$ is normal, and the measures $\tilde{\nu}$ and $\eta_A^*(dx|\tilde{x}^0, 0)$ are finite (for all $\tilde{x}^0 \in \tilde{V}^c$). Equation (34) follows from Lemma 6.1. (iii) Let us show that $\tilde{\eta}_A - \tilde{\eta}_A^*$ is a finite measure. In case $u^*(\tilde{x}^0) < \theta^*(\tilde{x}^0)$,

$$
\tilde{p}(\{u^*(\tilde{x}^0)\}|(\tilde{x}^0, 0)) = 1 - \tilde{\eta}_A([0, u^*(\tilde{x}^0)]|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \leq \tilde{\eta}_A(\{u^*(\tilde{x}^0)\}|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)}
$$

because

$$
\tilde{\eta}_A([0, u^*(\tilde{x}^0)]|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} = G(\tilde{x}^0, u^*(\tilde{x}^0)) \geq 1.
$$

Therefore, in any case

$$
\tilde{p}(I|(\tilde{x}^0, 0)) \leq \tilde{\eta}_A(I|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)}
$$

for all $I \in \mathcal{B}(\mathbb{R}_+^0)$ and for all $\tilde{x}^0 \in \tilde{V}^c$. Now, for each measurable subset $\Gamma \subset V^c$,

$$
\tilde{\eta}_A(\Gamma) = \int_{\tilde{V}^c} \int_{\mathbb{R}_+^0} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{p}(du|(\tilde{x}^0, 0))\tilde{\nu}(d\tilde{x}^0)
\quad = \int_{\tilde{V}^c} \int_{[0, \theta^*(\tilde{x}^0))} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{p}(du|(\tilde{x}^0, 0))\tilde{\nu}(d\tilde{x}^0)
$$

because $\nu(\tilde{V}^c \times \{t: t > 0\}) = 0$ (see the comments in Definition 5.4). Therefore, for each measurable subset $\Gamma$ of $V^c$,

$$
\tilde{\eta}_A(\Gamma) \leq \int_{\tilde{V}^c} \int_{[0, \theta^*(\tilde{x}^0))} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{\eta}_A(du|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \tilde{\nu}(d\tilde{x}^0)
\quad = \int_{\tilde{V}^c} \int_{[0, \theta^*(\tilde{x}^0))} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{\eta}_A(du|\tilde{x}^0)\frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} n(\tilde{x}^0) K(d\tilde{x}^0)
\quad = \int_{\tilde{V}^c} \int_{[0, \theta^*(\tilde{x}^0))} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{\eta}_A(du|\tilde{x}^0)\tilde{\eta}_A(d\tilde{x}^0)
\quad \leq \int_{\tilde{V}^c} \int_{[0, \theta^*(\tilde{x}^0))} \mathbb{I}\{\phi((\tilde{x}^0, 0), u) \in \Gamma\} \tilde{\eta}_A(du|\tilde{x}^0)\tilde{\eta}_A(d\tilde{x}^0)
\quad = \int_{\tilde{V}^c} \mathbb{I}\{y \in \Gamma\} \tilde{\eta}_A(dy) = \tilde{\eta}_A(\Gamma).
$$

The last but one equality is by Lemma A.2(a). Hence, $\tilde{\eta}_A - \tilde{\eta}_A^*$ is a finite measure.
(iv) Let us show that $\tilde{\eta}_0 \geq \tilde{\eta}'_0$ set-wise. Recall that the measure $\tilde{\eta}'_0$ is normal. It is convenient to consider the images $\tilde{\eta}_0$ and $\tilde{\eta}$ of the measures $\tilde{\eta}_0$ and $\tilde{\eta}(\cdot \times A)$ as in Definition 5.4 and, similarly, for $\Gamma \in B(D)$, we introduce

$$\tilde{\eta}_0(\Gamma) := \tilde{\eta}_0(F(\Gamma)).$$

Now, according to Lemma A.2(a,b), equation (33) takes the form:

$$0 = \int_{\tilde{V}_c} w((\tilde{x}^0, 0)) \tilde{\phi}(d\tilde{x}^0) - \int_{\tilde{V}_c} \chi w(\phi((\tilde{x}^0, 0), u))\tilde{\eta}_0(d\tilde{x}^0 \times du)$$

and the stochastic kernel $\tilde{\eta}$ equal to the value of $\tilde{\eta}(\cdot \times A)$ as in Definition 5.4, and similarly, for $\Gamma \in B(D)$, we introduce

$$\tilde{\eta}_0(\Gamma) := \tilde{\eta}_0(F(\Gamma)).$$

Now, according to Lemma A.2(a,b), equation (33) takes the form:

$$0 = \int_{\tilde{V}_c} w((\tilde{x}^0, 0)) \tilde{\phi}(d\tilde{x}^0) - \int_{\tilde{V}_c} \chi w(\phi((\tilde{x}^0, 0), u))\tilde{\eta}_0(d\tilde{x}^0 \times du)$$

and the stochastic kernel $\tilde{\eta}(da|y)$ comes from the decomposition

$$\tilde{\eta}(dy \times da) = \tilde{\eta}(da|y)\tilde{\eta}(dy \times A).$$

According to [41 V.1;Thm.1.5.6], it suffices to show that the value of the measure $\tilde{\eta}_0$ is greater or equal to the value of $\tilde{\eta}_0$ on each set of the form

$$Y_{T_1, T_2, \tilde{\Gamma}} := \{(\tilde{x}^0, u) : \tilde{x}^0 \in \tilde{\Gamma}, T_1 \leq u < T_2 \land \tilde{\phi}^*(\tilde{x}^0)\}, \quad \tilde{\Gamma} \in B(\tilde{V}_c), \quad 0 \leq T_1 < T_2 < \infty.$$

See Figure 3 and also Figure 2 for illustration.

Note that, in case $\tilde{\Gamma} \subset \tilde{V}_c \setminus \tilde{V}_c^c$, $\tilde{\eta}_0(Y_{T_1, T_2, \tilde{\Gamma}}) = 0$ and hence $\tilde{\eta}_0(Y_{T_1, T_2, \tilde{\Gamma}}) - \tilde{\eta}_0(Y_{T_1, T_2, \tilde{\Gamma}}) \geq 0$ for all $T_1, T_2$ because $\nu((\tilde{V}_c \setminus \tilde{V}_c^c) \times \{0\}) = 0$. Therefore, below in this proof, we assume that $\tilde{\Gamma} \subset \tilde{V}_c^c$.

**Figure 3:** Space $D = \{(\tilde{x}^0, t) : \tilde{\phi}(\tilde{x}^0, t) \in \tilde{V}_c\}$ and “rectangle” $Y_{T_1, T_2, \tilde{\Gamma}}$. The points $\tilde{x}_1^0, \tilde{x}_2^0, \tilde{x}_3^0$ belong to $\tilde{V}_c^c \subset \tilde{V}_c$. The dashed area is the part of $Y_{T_1, T_2, \tilde{\Gamma}}$ where $\tilde{\eta}_0$ might be positive.

To use equality (35) to calculate $\tilde{\eta}_0(Y_{T_1, T_2, \tilde{\Gamma}})$, we put

$$\chi w_{T_1, T_2, \tilde{\Gamma}}(\phi((\tilde{x}^0, 0), u)) := -\mathbb{I}\{(\tilde{x}^0, u) \in Y_{T_1, T_2, \tilde{\Gamma}}\},$$
and consider the following positive function decreasing along the flow:

\[
    w_{T_1, T_2, \tilde{\Gamma}}(\tilde{y}, t) = w_{T_1, T_2, \tilde{\Gamma}}((\tilde{y}, t)) := \begin{cases} 
    T_2 \land \tilde{\theta}^*(h(\tilde{y})) - T_1 \land \tilde{\theta}^*(h(\tilde{y})), & \text{if } 0 \leq t \leq T_1; \\
    T_2 \land \tilde{\theta}^*(h(\tilde{y})) - t, & \text{if } T_1 < t \leq T_2 \land \tilde{\theta}^*(h(\tilde{y})); \\
    0, & \text{if } t > T_2 \land \tilde{\theta}^*(h(\tilde{y})).
    \end{cases}
\]

See Figure 4.

Figure 4: Graph of the function 

\[
    w_{T_1, T_2, \tilde{\Gamma}}(\tilde{y}, t) = w_{T_1, T_2, \tilde{\Gamma}}((\tilde{\phi}(\tilde{x}^0, t), t)) \text{ for a fixed value of } h(\tilde{y}) = \tilde{x}^0 \in \tilde{\Gamma} \text{ and } \tilde{\theta}^*(h(\tilde{y})) > T_1.
\]

One can easily see that, for all \(0 \leq T_1 < T_2 < \infty, \tilde{\Gamma} \in \mathcal{B}(\tilde{V}^c), w_{T_1, T_2, \tilde{\Gamma}} \in \mathcal{W}.\) Now

\[
    w_{T_1, T_2, \tilde{\Gamma}}((\tilde{x}^0, 0)) = \mathbb{I}\{\tilde{x}^0 \in \tilde{\Gamma}\} \left( T_2 \land \tilde{\theta}^*(\tilde{x}^0) - T_1 \land \tilde{\theta}^*(\tilde{x}^0) \right); \\
    w_{T_1, T_2, \tilde{\Gamma}}(\tilde{\phi}((\tilde{x}^0, 0), u)) = \mathbb{I}\{\tilde{x}^0 \in \tilde{\Gamma}\} \times \begin{cases} 
    T_2 \land \tilde{\theta}^*(\tilde{x}^0) - T_1 \land \tilde{\theta}^*(\tilde{x}^0), & \text{if } u \leq T_1; \\
    T_2 \land \tilde{\theta}^*(\tilde{x}^0) - u, & \text{if } T_1 < u \leq T_2 \land \tilde{\theta}^*(\tilde{x}^0); \\
    0, & \text{if } u > T_2 \land \tilde{\theta}^*(\tilde{x}^0),
    \end{cases}
\]

and expression \((36)\) takes the form

\[
    w_{T_1, T_2, \tilde{\Gamma}}(y) = w_{T_1, T_2, \tilde{\Gamma}}((\tilde{y}, t)) := \int_{\mathbb{A}} w_{T_1, T_2, \tilde{\Gamma}}((\tilde{l}(\tilde{y}, a), 0)) \tilde{\eta}^A(da|y)
    \]

\[
    = \mathbb{I}\{\tilde{l}(\tilde{y}, a) \in \tilde{\Gamma}\} \int_{\mathbb{A}} \left[ T_2 \land \tilde{\theta}^*(\tilde{l}(\tilde{y}, a)) - T_1 \land \tilde{\theta}^*(\tilde{l}(\tilde{y}, a)) \right] \tilde{\eta}^A(da|y).
\]

This function is positive for all \(T_1, T_2, \tilde{\Gamma}\) and for \(\tilde{\eta}(\cdot \times \mathbb{A})\)-almost all \(y \in V^c.\)

From equality \((35)\), using the expression \((\tilde{\eta}_A(d\tilde{x}^0) = \frac{a(\tilde{x}^0)}{m(\tilde{x}^0)} \tilde{\nu}(d\tilde{x}^0),\) we have for

\[
    \tilde{\Gamma}_\tilde{\theta} := \tilde{\Gamma} \cap \{\tilde{x}^0 : \tilde{\theta}^*(\tilde{x}^0) \geq T_1\}.
\]
\[ \hat{\eta}_\circ(Y_{T_1,T_2,\hat{\Gamma}}) = \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \dot{\nu}(\hat{x}^0) - \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A([0,T_1]|\hat{x}^0) \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) \\
- \int_{\hat{\Gamma}_\theta} \int_{(T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - u \right] \tilde{\eta}_A(du|\hat{x}^0) \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) \\
+ \int_D w_A^{T_1,T_2,\hat{\Gamma}}(\phi((\hat{x}^0,0),u)) \tilde{\eta}(\hat{x}^0 \times du) .
\]

For the last but one integral, note that \( \tilde{\eta}_A(\{\tilde{\theta}^*(\hat{x}^0)\}|\hat{x}^0) = 0 \) for \( K \)-almost all \( \hat{x}^0 \). The corresponding integrals over \( \hat{\Gamma} \setminus \hat{\Gamma}_\theta \) equal zero and hence are omitted; the last term above, denoted below as \( J(\hat{\Gamma}) \), is positive. According to Lemma [A,3] for \( K \)-almost all \( \hat{x}^0 \in \hat{\Gamma}_\theta ,
\[
\int_{(T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))} \left[ u - T_2 \land \tilde{\theta}^*(\hat{x}^0) \right] \tilde{\eta}_A(du|\hat{x}^0) = \int_{(T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))} [u - T_1] \tilde{\eta}_A(du|\hat{x}^0) \\
- \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A((T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))|\hat{x}^0) \\
= \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A((T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))|\hat{x}^0) - \int_{(0,T_2 \land \tilde{\theta}^*(\hat{x}^0)) - T_1} \tilde{\eta}_A((T_1,T_1 + s)|\hat{x}^0)ds \\
- \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A((T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))|\hat{x}^0),
\]

so that

\[
\hat{\eta}_\circ(Y_{T_1,T_2,\hat{\Gamma}}) = \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \dot{\nu}(\hat{x}^0) - \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A([0,T_1]|\hat{x}^0) \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) \\
+ \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A([T_1]|\hat{x}^0) \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) \\
- \int_{\hat{\Gamma}_\theta} \int_{(0,T_2 \land \tilde{\theta}^*(\hat{x}^0)) - T_1} \tilde{\eta}_A((T_1,T_1 + s)|\hat{x}^0)ds \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) + J(\hat{\Gamma}) \\
= \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \dot{\nu}(\hat{x}^0) - \int_{\hat{\Gamma}_\theta} \left[ T_2 \land \tilde{\theta}^*(\hat{x}^0) - T_1 \right] \tilde{\eta}_A([0,T_1]|\hat{x}^0) \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) \\
- \int_{\hat{\Gamma}_\theta} \int_{(T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))} \tilde{\eta}_A((T_1,u)|\hat{x}^0)du \frac{a(\hat{x}^0)}{n(\hat{x}^0)} \dot{\nu}(\hat{x}^0) + J(\hat{\Gamma}).
\]

According to the definitions of the measures \( \hat{\eta}_\circ \) and \( \hat{\eta}_\circ \),

\[
\hat{\eta}_\circ(F(Y_{T_1,T_2,\hat{\Gamma}})) = \int_{\hat{\Gamma}_\theta} \int_{[0,\hat{x}^0)} \mathbb{I}\{ (\hat{x}^0, u) \in Y_{T_1,T_2,\hat{\Gamma}) \} \left(1 - \tilde{\eta}_A([0, u]|(\hat{x}^0, 0)) \right) \dot{\nu}(\hat{x}^0) \\
- \int_{\hat{\Gamma}_\theta} \int_{[T_1,T_2 \land \tilde{\theta}^*(\hat{x}^0))} \mathbb{I}\{ \hat{x}^0 \in \hat{\Gamma} \} \left(1 - \tilde{\eta}_A([0, u]|(\hat{x}^0, 0)) \right) \dot{\nu}(\hat{x}^0) \]

(39)

The difference

\[
\hat{\eta}_\circ(Y_{T_1,T_2,\hat{\Gamma} \setminus \hat{\Gamma}_\theta}) - \hat{\eta}_\circ(Y_{T_1,T_2,\hat{\Gamma} \setminus \hat{\Gamma}_\theta}) = J(\hat{\Gamma} \setminus \hat{\Gamma}_\theta) \geq 0
\]

31
because \( \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_s}) = 0 \). Below, we split the main set \( \tilde{\Gamma}_\theta \) into three measurable subsets:

\[
\tilde{\Gamma}_1 := \tilde{\Gamma}_\theta \cap \{ \tilde{x}^0 : u^*(\tilde{x}^0) < T_1 \}, \\
\tilde{\Gamma}_2 := \tilde{\Gamma}_\theta \cap \{ \tilde{x}^0 : u^*(\tilde{x}^0) \geq T_2 \land \tilde{\theta}^*(\tilde{x}^0) \}, \\
\text{and } \tilde{\Gamma}_3 := \tilde{\Gamma}_\theta \cap \{ \tilde{x}^0 : T_1 \leq u^*(\tilde{x}^0) < T_2 \land \tilde{\theta}^*(\tilde{x}^0) \}.
\]

For each \( \tilde{x}^0 \in \tilde{\Gamma}_1 \), \( \tilde{p}([0,u])((\tilde{x}^0,0)) = 1 \) for all \( u \in [T_1,T_2 \land \tilde{\theta}^*(\tilde{x}^0)) \). Hence, according to \( \text{(39)} \), \( \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_1}) = 0 \) and

\[
\tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_1}) - \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_1}) \geq 0.
\]

For each \( \tilde{x}^0 \in \tilde{\Gamma}_2 \) (see the point \( \tilde{x}^0_1 \) on Figure 3),

\[
\tilde{p}([0,T_1])((\tilde{x}^0,0)) = \tilde{\eta}_A([0,T_1]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \text{ and } \tilde{p}([T_1,u])((\tilde{x}^0,0)) = \tilde{\eta}_A([T_1,u]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)}
\]

for all \( u \in (T_1,T_2 \land \tilde{\theta}^*(\tilde{x}^0)) \). Therefore, by \( \text{(38)} \) and \( \text{(39)} \),

\[
\tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_2}) - \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_2}) = \mathbf{J}(\tilde{\Gamma}_2) \geq 0.
\]

For the set \( \tilde{\Gamma}_3 \) (the typical points in \( \tilde{\Gamma}_3 \) are \( \tilde{x}^0_2 \) and \( \tilde{x}^0_3 \) on Figure 3), we compute \( \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_3}) \) and \( \tilde{\eta}^0(Y_{T_1,T_2,\hat{\Gamma}_3}) \) using the representation \( Y_{T_1,T_2,\hat{\Gamma}_3} = Y^1 \cup Y^2 \), where

\[
Y^1 := \{ (\tilde{x}^0,u) : \tilde{x}^0 \in \tilde{\Gamma}_3, T_1 \leq u < u^*(\tilde{x}^0) \}; \quad Y^2 := \{ (\tilde{x}^0,u) : \tilde{x}^0 \in \tilde{\Gamma}_3, u^*(\tilde{x}^0) \leq u < T_2 \land \tilde{\theta}^*(\tilde{x}^0) \}.
\]

To compute \( \tilde{\eta}^0(Y^1) \), we introduce the function

\[
w(y) = w((\hat{g},t)) = 1\{ h(y) \in \tilde{\Gamma}_3 \} \times \begin{cases} u^*(h(y)) - T_1, & \text{if } t \leq T_1; \\ u^*(h(y)) - t, & \text{if } T_1 < t \leq u^*(h(y)); \\ 0, & \text{if } t > u^*(h(y)) \end{cases}
\]

(cf \( \text{(54)} \)). Calculations similar to those presented above, lead to the following version of expression \( \text{(38)} \):

\[
\tilde{\eta}^0(Y^1) = \int_{\tilde{\Gamma}_3} [u^*(\tilde{x}^0) - T_1] \tilde{\nu}(d\tilde{x}^0) - \int_{\tilde{\Gamma}_3} [u^*(\tilde{x}^0) - T_1] \tilde{\eta}_A([0,T_1]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \tilde{\nu}(d\tilde{x}^0)
\]

\[
- \int_{\tilde{\Gamma}_3} \int_{(T_1,T_2,\tilde{x}^0)} \tilde{\eta}_A([T_1,u]|\tilde{x}^0)du \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \tilde{\nu}(d\tilde{x}^0) + \mathbf{J}^1.
\]

The last term is similar to \( \mathbf{J}(\tilde{\Gamma}) \), its calculation is based on the function similar to \( w^A_{T_1,T_2,\hat{\Gamma}} \): one only has to replace \( \hat{\Gamma} \) with \( \tilde{\Gamma}_3 \) and \( \tilde{\theta}^*(\cdot) \) with \( u^*(\cdot) \). Like previously, \( \mathbf{J}^1 \geq 0 \). Again, similarly to \( \text{(39)} \), we have

\[
\tilde{\eta}^0(Y^1) = \int_{\tilde{\Gamma}_3} [u^*(\tilde{x}^0) - T_1] \tilde{\nu}(d\tilde{x}^0) - \int_{\tilde{\Gamma}_3} [u^*(\tilde{x}^0) - T_1] \tilde{p}([0,T_1]|(\tilde{x}^0,0)) \tilde{\nu}(d\tilde{x}^0)
\]

\[
- \int_{\tilde{\Gamma}_3} \int_{(T_1,u^*(\tilde{x}^0))} \tilde{p}([T_1,u]|(\tilde{x}^0,0))du \tilde{\nu}(d\tilde{x}^0)
\]

and, like in the case of \( \tilde{\Gamma}_2 \), for each \( \tilde{x}^0 \in \tilde{\Gamma}_3 \)

\[
\tilde{p}([0,T_1]|(\tilde{x}^0,0)) = \tilde{\eta}_A([0,T_1]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)} \text{ and } \tilde{p}([T_1,u]|(\tilde{x}^0,0)) = \tilde{\eta}_A([T_1,u]|\tilde{x}^0) \frac{a(\tilde{x}^0)}{n(\tilde{x}^0)}
\]

32
for all \( u \in (T_1, u^*(\tilde{x}^0)] \). Therefore, 
\[
\tilde{\eta}_C(Y^1) - \tilde{\eta}_C^*(Y^1) = J^1 \geq 0.
\]
Finally, similarly to (39),
\[
\tilde{\eta}_C(Y^2) = \int_{\Gamma_3} \int_{u^*(\tilde{x}^0),T_2>A(\tilde{x}^0)} (1 - \tilde{p}([0, u]|(\tilde{x}^0, 0))) du \tilde{\nu}(d\tilde{x}^0) = 0
\]
because for each \( \tilde{x}^0 \in \tilde{\Gamma}_3, \tilde{p}([0, u]|(\tilde{x}^0, 0)) = 1 \) for all \( u > u^*(\tilde{x}^0) \). Hence,
\[
\tilde{\eta}_C(Y^2) - \tilde{\eta}_C^*(Y^2) \geq 0.
\]

To summarise, \( \tilde{\eta}_C(Y_{T_1,T_2,T_3}) - \tilde{\eta}_C^*(Y_{T_1,T_2,T_3}) \geq 0 \) and \( \tilde{\eta}_C(Y_{T_1,T_2,T_3}) - \tilde{\eta}_C^*(Y_{T_1,T_2,T_3}) \geq 0 \) for all \( \tilde{\Gamma} \in B(V^c) \) and \( 0 \leq T_1 < T_2 < \infty \).

Therefore, \( \tilde{\eta}_C \geq \tilde{\eta}_C^* \) set-wise on \( D \). Since the measures \( \tilde{\eta}_C \) and \( \tilde{\eta}_C^* \) are both normal (or weakly normal), we see that the difference \( \tilde{\eta}_C - \tilde{\eta}_C^* \) is a normal (or weakly normal) measure on \( V^c \); see the reasoning in Item (b) of Remark 5.2 and Lemma 5.4. The proof is completed.

**Proof of Theorem 5.1**

When \( x \in V \), we fix \( \pi_i(d\theta \times da) := \delta_{f^*(x)}(d\theta \times da) \), where the mapping \( f^* \) is as in the proof of Theorem 3.1. Recall that \( f^*(x) = (\infty, a) \), and the value of \( a \in A \) plays no role. Below, for two finite or normal measures \( \zeta^1 \) and \( \zeta^2 \) on \( V^c \), the inequality \( \zeta^1(dx) \leq \zeta^2(dx) \) is understood set-wise. The same concerns measures on \( V^c \times A \).

Let \( p_A^i(da|x) \) be the stochastic kernel on \( A \) given \( V^c \) coming from the decomposition \( \eta(dx \times da) = p_A^i(da|x)\eta(dx \times A) \). For all \( i \geq 1 \), we put
\[
p_A^i(da|x, \theta) = p_A^i(da|\phi(x, \theta))
\]
for \( x \in V^c, \theta < \theta^*(x) \), and \( p_A^i(da|x, \theta) \) is an arbitrarily fixed stochastic kernel on \( A \) for \( x \in V^c, \theta \geq \theta^*(x) \).

We will prove by induction the following statement.

For each \( i \geq 1 \), there is a stochastic kernel \( \pi_i \) on \( B = \mathbb{R}^0_+ \times A \) given \( V^c \), having the form
\[
\pi_i(d\theta \times da|x) = p_A^i(d\theta|x)p_A^i(da|x, \theta),
\]
such that, for the sequence \( \{\pi_i\}_{i=1}^n \), the following assertions are fulfilled.

(i) The (partial) aggregated occupation measures \( \{\tilde{\eta}^n\}_{i=0}^n \), defined as in Lemma 4.1, exhibit the following properties:
\[
\tilde{\eta}^n(dx \times \square) \leq \eta(dx \times \square) \quad \text{and} \quad \tilde{\eta}^n(dx \times da) = \eta(dx \times A)p_A^i(da|x) \leq \eta(dx \times A)p_A^i(da|x) = \eta(dx \times da).
\]

(ii) The measure \( \nu^n(dx) := P^n_{x_0}(X_n \in dx) \) on \( V^c \) is such that, for each function \( w \in W \),
\[
0 = \int_{V^c} w(x)\nu^n(dx) + \int_{V^c} \chi_w(x)[\eta - \tilde{\eta}^n](dx \times \square) - \int_{V^c} w(x)[\eta - \tilde{\eta}^n](dx \times A)
+ \int_{V^c \times A} w(l(x,a))[\eta - \tilde{\eta}^n](dx \times da), \tag{40}
\]
and all the integrals here are finite. Note that \( \nu^n \) is uniquely defined by the finite sequence \( \{\pi_i\}_{i=1}^n \); see (7).
When \( n = 0 \), \( \tilde{\eta}^0(dy \times \square) \equiv 0 \), \( \tilde{\eta}^0(dy \times da) \equiv 0 \), and \( \nu^0(dx) = \delta_{x_0}(dx) \). Assertions (i) and (ii) are obviously fulfilled.

Suppose assertions (i) and (ii) hold true for \( i = 0, 1, 2, \ldots, n \geq 0 \). Note that \( \nu^n(\mathcal{V}^c \times \{ t : t > 0 \}) = 0 \) (see Definition 5.1). We apply Lemma 5.2 to the measures \( \nu := \nu^n \), \( \tilde{\eta} := (\eta - \tilde{\eta}^n)(dx \times \square) \), \( \tilde{\eta}_A := (\eta - \tilde{\eta}^n)(dx \times A) \), and \( \tilde{\eta} := (\eta - \tilde{\eta}^n)(dx \times da) \) satisfying equation (40). All of them are finite, maybe apart from \( \tilde{\eta} \) which is normal. As a result, we have the stochastic kernel \( \tilde{p}(dt|x) \) on \( \mathbb{R}_0^+ \) given \( \mathcal{V}^c \) and the measures

\[
\tilde{\eta}_A(dx) \leq (\eta - \tilde{\eta}^n)(dx \times A) \quad \text{and} \quad \tilde{\eta}_\square(dx) \leq (\eta - \tilde{\eta}^n)(dx \times \square)
\]  

(41)
on \( \mathcal{V}^c \), which satisfy equation (44):

\[
0 = \int_{\mathcal{V}^c} w(x)\nu^n(dx) + \int_{\mathcal{V}^c} \chi w(x)\tilde{\eta}_A(dx) - \int_{\mathcal{V}^c} w(x)\tilde{\eta}_A(dx), \quad w \in \mathcal{W}.
\]  

(42)
All the integrals here are finite.
For \( x \in \mathcal{V}^c \), we put

\[
p^n_{T+1}(d\theta|x) := \tilde{p}(d\theta|x).
\]
All the kernels \( \{\pi_i\}_{i=1}^n \) were built on the previous steps of the induction. According to the definition of the measure \( \tilde{\eta}^{n+1} \),

\[
\tilde{\eta}^{n+1}(\Gamma \times \square) = \tilde{\eta}^n(\Gamma \times \square) + \int_{\mathcal{V}^c} \int_{\mathbb{R}_0^+} \delta_{\phi(x,\omega)}(\Gamma)\tilde{p}(\eta, \infty)(x)du \nu^n(dx)
\]

\[
= \tilde{\eta}^n(\Gamma \times \square) + \tilde{\eta}^{0}(\Gamma), \quad \Gamma \in \mathcal{B}(\mathcal{V}^c);
\]

\[
\tilde{\eta}^{n+1}(\Gamma \times A) = \tilde{\eta}^n(\Gamma \times A) + \int_{\mathcal{V}^c} \int_{\mathbb{R}_0^+} \delta_{\phi(x,\omega)}(\Gamma)\tilde{p}(du|x)\nu^n(dx) = \tilde{\eta}^n(\Gamma \times A) + \tilde{\eta}_A^{n+1}(\Gamma), \quad \Gamma \in \mathcal{B}(\mathcal{V}^c).
\]  

(43)
(44)
Inequalities are valid according to the basic properties of the measures \( \tilde{\eta}^{0} \) and \( \tilde{\eta}^{n} \) presented in (41). Recall that

\[
\tilde{\eta}^{n+1}(\Gamma_X \times \Gamma_A) = \tilde{\eta}^n(\Gamma_X \times \Gamma_A) + \int_{\mathcal{V}^c} \int_{\mathbb{R}_0^+} \delta_{\phi(x,\theta)}(\Gamma_X)p^n_{T+1}(\Gamma_A|x, \theta)p^n_{T+1}(d\theta|x)\nu^n(dx),
\]

\[
\Gamma_X \in \mathcal{B}(\mathcal{V}^c), \quad \Gamma_A \in \mathcal{B}(A).
\]
Since \( \nu^n(\mathcal{V}^c \times \{ t : t > 0 \}) = 0 \), the last term equals

\[
I := \int_{\mathcal{V}^c} \int_{(0,\tilde{x}_0]} \delta_{\phi((\tilde{x}_0,0),\eta)}(\Gamma_X)p^n_A(\Gamma_A|\phi((\tilde{x}_0,0),\eta))\tilde{p}(d\theta|\tilde{x}_0)\nu^n(d\tilde{x}_0),
\]  

(45)
where \( \tilde{\nu}^n(\Gamma) := \nu^n(\{ (\tilde{x}_0,0), \tilde{x}_0 \in \Gamma \}) \). According to Lemma 5.2 for all \( \Gamma \in \mathcal{B}(D) \) and for the mapping \( F \) as in Lemma 5.2

\[
\tilde{\eta}_A^n(F(\Gamma)) = \int_{\mathcal{V}^c} \int_{(0,\tilde{x}_0]} \mathbb{1}\{ \phi((\tilde{x}_0,0),u) \in \{ y = \phi((\tilde{x}_0,0),t) : (\tilde{x}_0,t) \in \Gamma \} \} \tilde{p}(du|\tilde{x}_0)\tilde{\nu}^n(d\tilde{x}_0)
\]

\[
= \int_{\mathcal{V}^c} \int_{(0,\tilde{x}_0]} \mathbb{1}\{ (\tilde{x}_0,u) \in \Gamma \}\tilde{p}(du|\tilde{x}_0)\tilde{\nu}^n(d\tilde{x}_0).
\]

(46)
Therefore, for each $\Gamma \in \mathcal{B}(V^c)$,
\[
I = \int_{V^c} \delta_x(\Gamma_X) p'_A(\Gamma_A| x) \eta'_A(dx) = \int_{\Gamma_X} p'_A(\Gamma_A| x) \eta'_A(dx),
\]
meaning that
\[
\tilde{\eta}^{n+1}(dx \times da) = \tilde{\eta}^n(dx \times da) + p'_A(da|x) \eta'_A(dx) = \tilde{\eta}^n(dx \times A)p'_A(da|x) + \eta'_A(dx)p'_A(da|x)
\]
\[
= \tilde{\eta}^{n+1}(dx \times A)p'_A(da|x)
\]
\[
\leq \tilde{\eta}^n(dx \times A)p'_A(da|x) + [\tilde{\eta} - \tilde{\eta}^n](dx \times A)p'_A(da|x) = \eta(dx \times A)p'_A(da|x).
\]

The second equality is by the inductions supposition, the third equality follows from (44), and the inequality is according to the basic property (41) of the measure $\tilde{\eta}'_A$.

Property (i) for $n + 1$ is established.

For the proof of Item (ii), note that, by (40) at $n$, (42), (43), and (44), we have equation
\[
0 = \int_{V^c} \chi w(x)[\eta - \tilde{\eta}^{n+1}](dx \times \square) - \int_{V^c} w(x)[\eta - \tilde{\eta}^{n+1}](dx \times A)
\]
\[
+ \int_{V^c \times A} w(l(x, a))[\eta - \tilde{\eta}^{n+1}](dx \times da) + \int_{V^c \times A} w(l(x, a))[\tilde{\eta}^{n+1} - \tilde{\eta}^n](dx \times da)
\]
valid for all functions $w \in W$, and all the integrals here are finite. Since the stochastic kernel $p'_A(da|x)$ is the same in the decompositions $\tilde{\eta}^n(dx \times da) = \tilde{\eta}^n(dx \times A)p'_A(da|x)$ and $\tilde{\eta}^{n+1}(dx \times da) = \tilde{\eta}^{n+1}(dx \times A)p'_A(da|x)$, the last integral, according to (44), equals
\[
\int_{V^c} \int_A w(l(x, a))p'_A(da|x)\eta'_A(dx),
\]
i.e., the function $w$ is integrated with respect to the measure
\[
m(\Gamma) = \int_{V^c} \int_A \delta_{l(x, a)}(\Gamma)p'_A(da|x)\eta'_A(dx), \quad \Gamma \in \mathcal{B}(V^c),
\]
and it remains to show that this measure coincides with $\nu^{n+1}$ on $V^c$.

From equation (40), we have for all $\Gamma \in \mathcal{B}(V^c)$:
\[
m(\Gamma) = \int_{V^c} \int_{[0, \tilde{\eta}^n(z^0)]) \int_A \delta_{l(\phi((z^0,0),u),a)}(\Gamma)p'_A(da|\phi((z^0,0),u))p^{n+1}_T(du|(z^0,0))\tilde{\eta}^n(dx^0),
\]
and, keeping in mind that $\nu^n(\tilde{V}^c \times \{ t : t > 0 \}) = 0$, we have from (7):
\[
\nu^{n+1}(\Gamma) = \int_{V^c} \int_{[0, \tilde{\eta}^n(z^0)]) \int_A \delta_{l(\phi((z^0,0),\theta),a)}(\Gamma)p^{n+1}_A(da|(z^0,0),(z^0,0),\theta)p^{n+1}_T(du|(z^0,0))\tilde{\eta}^n(dx^0) = m(\Gamma)
\]
for all $\Gamma \in \mathcal{B}(V^c)$ by the definition of the stochastic kernel $p^{n+1}_A$.

The proof of the induction statement for $n + 1$ is completed.

According to Lemma 4.1 for the constructed Markov strategy $\pi = \{ \pi_i \}_{i=1}^\infty$ and for the corresponding aggregated occupation measure $\tilde{\eta}$, we have the convergence $\tilde{\eta}^n \uparrow \tilde{\eta}$ set-wise as $n \to \infty$. Since $\tilde{\eta}^n \leq \eta$ set-wise on $V^c \times A$, the desired set-wise inequality $\tilde{\eta} \leq \eta$ follows.

Proof of Corollary 5.1. We denote by $Val(23)$ and $Val(31)$ the minimal values of linear programs (23)–(20) and (31), respectively.
Suppose the finite measure \( \mu^* \) on \( V^c \times \bar{\mathbb{R}}^0_+ \times A \) solves linear program (23)-(26). Then the measure \( \eta^* \) on \( V^c \times A_D \) given by (24) and (25) satisfies equation (27) according to Theorem 5.1. Conditions (i) and (iii) are also fulfilled by \( \eta^* \). Thus
\[
\infty > Val(23) = \int_{V^c \times A_D} C_0(x, a) \eta^*(dx \times da) \geq Val(31).
\]
In case the last inequality is strict, there exists a feasible solution \( \eta \) to linear program (31) satisfying inequality
\[
\int_{V^c \times A_D} C_0(x, a) \eta(dx \times da) < Val(23).
\]
According to Theorem 5.1, there is a Markov control strategy \( \pi \) such that, for the aggregated occupation measure \( \tilde{\eta} \) coming from the occupation measure \( \mu^* \), inequalities \( \tilde{\eta}(\Gamma) \leq \eta(\Gamma) \) hold for all \( \Gamma \in \mathcal{B}(V^c \times A_D) \). Therefore, since \( C_j \geq 0 \) for all \( j = 0, 1, \ldots, J \), all the conditions in linear program (23)-(26) are satisfied for \( \mu^* \) and
\[
\int_{V^c \times A_D} C_0(x, a) \eta(dx \times da) \leq \int_{V^c \times A_D} C_0(x, a) \eta(dx \times da) < Val(23) < \infty.
\]
The measure \( \mu^* \) cannot take infinite value as explained above linear program (15)-(17). We obtained a contradiction to the optimality of the measure \( \mu^* \). Hence, \( Val(23) = Val(31) \), and the measure \( \eta^* \) solves linear program (31).

Suppose now that the measure \( \eta^* \) on \( V^c \times A_D \) solves linear program (31) and consider the Markov strategy \( \pi^* \) as in Theorem 5.1. The corresponding occupation measure \( \mu^* \) is feasible in linear program (23)-(26). More detailed reasoning is similar to that presented above. Therefore, for the aggregated occupation measure \( \tilde{\eta} \) coming from \( \mu^* \), we have relations
\[
Val(23) \leq \int_{V^c \times A_D} C_0(x, a) \tilde{\eta}(dx \times da) \leq \int_{V^c \times A_D} C_0(x, a) \eta^*(dx \times da) = Val(31).
\]
But we have shown that \( Val(31) = Val(23) \), (recall that an optimal solution \( \mu^* \) exists by Theorem 3.1), so that
\[
\int_{V^c \times A_D} C_0(x, a) \tilde{\eta}(dx \times da) = Val(23)
\]
meaning that the measure \( \mu^* \) solves linear program (23). The proof is completed.

### 7 Duality

In this section, we assume that there are no constraints \( (J = 0) \) and investigate the programs dual to program (15)-(17) and to program (31). The target is to show that the linear programming method developed in the current paper is in some sense dual to the dynamic programming approach developed in [17].

Consider program (15)-(17) assuming that Conditions 3.1, 3.2, and 3.3 are satisfied.

Let \( \mathcal{M}_1 \) be the space of finite measures \( \mu \) on \( V^c \times \bar{\mathbb{R}}^0_+ \times A \), \( \mathcal{U}_1 \) be the space of bounded measurable functions \( u \) on \( V^c \), and
\[
K_1(\mu, u) := \int_{V^c \times \bar{\mathbb{R}}^0_+ \times A} C'(x, \theta, a) \mu(dx \times d\theta \times da) + u(x) + \int_{V^c \times \bar{\mathbb{R}}^0_+ \times A} u(y)Q'(dy|x, \theta, a) \mu(dx \times d\theta \times da) - \int_{V^c} u(x) \mu(dx \times \bar{\mathbb{R}}^0_+ \times A).
\]
be the $(-\infty, +\infty]$-valued function on $M_1 \times U_1$, where

$$C'(x, \theta, a) := \int_{[0, \theta]} C^0_0(\phi(x, u))I\{\phi(x, u) \in V^c\}du + \mathbb{I}\{\theta < +\infty\}\mathbb{I}\{\phi(x, \theta) \in V^c\}C_1^0(\phi(x, \theta), a)$$

is an $\bar{\mathbb{R}}^0_+$-valued function, and

$$Q'(dy|x, \theta, a) := \mathbb{I}\{l(\phi(y, \theta), a) \in dy\}\mathbb{I}\{\phi(y, \theta) \in V^c\}$$

is a substochastic kernel. Now the “primal” program (15)-(17) can be rewritten as

$$\sup_{u \in U_1} K_1(\mu, u) \rightarrow \inf_{\mu \in M_1}$$

Indeed, for each measure $\mu \in M_1$, if it does not satisfy equality (17), then $\sup_{u \in U_1} K_1(\mu, u) = +\infty$. The primal program has an optimal solution $\mu^* \in M_1$, as was established in Section 3.

The “dual” program (see problems (1.5) and (1.6) in [18])

$$\inf_{\mu \in M_1} K_1(\mu, u) \rightarrow \sup_{u \in U_1}$$

can be rewritten as

Maximize $u(x_0)$ over $u \in U_1$ \\
subject to $C'(x, \theta, a) + \int_{V^c} u(y)Q'(dy|x, \theta, a) - u(x) \geq 0$ \\
for all $(x, \theta, a) \in V^c \times \bar{\mathbb{R}}^0_+ \times A$.

Indeed, for the functions $u \in U_1$ satisfying (not satisfying) the presented inequality in program (17), $\inf_{\mu \in M_1} K_1(\mu, u) = K_1(0, u) = u(x_0)$ (inf $\mu \in M_1, K_1(\mu, u) = -\infty$). Here 0 is the zero measure. Throughout this section,

$$x_0 \in V^c,$$

as adopted at the end of Section 3.

**Proposition 7.1** Suppose Conditions 3.1, 3.2, and 3.3 are satisfied, and

$$\sup_{x \in V^c} \int_{[0, \infty)} C^0_0(\phi(x, u))du < \infty.$$ 

Then the solution to the (dual) program (17) is provided by the Bellman function $u^* \in U_1$, which is the unique bounded solution to the integral Bellman equation

$$\inf_{(\theta, a) \in \bar{\mathbb{R}}^0_+ \times A} \left\{ C'(x, \theta, a) + \int_{V^c} u(y)Q'(dy|x, \theta, a) \right\} - u(x) = 0, \text{ for all } x \in V^c;$$

the values of the primal program (15)-(17) and the dual program (17) coincide, and $(\mu^*, u^*)$ is a saddle-point of $K_1$:

$$K_1(\mu^*, u) \leq K_1(\mu^*, u^*) \leq K_1(\mu, u^*) \text{ for all } \mu \in M_1, u \in U_1.$$
Proof. Under the imposed conditions, the Bellman equation (48) has a unique bounded positive lower semicontinuous solution \( u^* \), and \( \inf_{\pi} V_0(x, \pi) = u^*(x_0) \): see [17] Thm.1, Prop.1. Note that \( \inf_{x \in V} V_0(x, \pi) \equiv 0 \) for \( x \in V \). According to the proof of Theorem 5.1 the value of the primal program (10)-(17) (which has an optimal solution \( \mu^* \)), equivalent to (13), also coincides with \( \inf_{\pi} V_0(x_0, \pi) \). Since, in any case,

\[
\sup_{u \in \mathcal{U}_1} \inf_{\mu \in \mathcal{M}_1} K_1(\mu, u) \leq \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}_1} K_1(\mu, u) = \inf_{\pi} V_0(x_0, \pi),
\]

and the function \( u^* \) is feasible in program (17), we conclude that

\[
u^*(x_0) = \inf_{\mu \in \mathcal{M}_1} K_1(\mu, u^*) = \sup_{u \in \mathcal{U}_1} \inf_{\mu \in \mathcal{M}_1} K_1(\mu, u) = \inf_{\pi} V_0(x_0, \pi),
\]

so that, the function \( u^* \) solves program (17).

We have showed that the values of the primal and dual programs coincide:

\[
\inf_{\mu \in \mathcal{M}_1} K_1(\mu, u^*) = \inf_{\pi} V_0(x_0, \pi) = \sup_{u \in \mathcal{U}_1} K_1(\mu^*, u).
\]

Hence, the common value equals \( K_1(\mu^*, u^*) \) because

\[
\inf_{\mu \in \mathcal{M}_1} K_1(\mu, u^*) \leq K_1(\mu^*, u^*) \leq \sup_{u \in \mathcal{U}_1} K_1(\mu^*, u),
\]

and, in fact, we have equalities in the latter expression. Therefore,

\[
K_1(\mu^*, u) \leq \sup_{u \in \mathcal{U}_1} K_1(\mu^*, u) = K_1(\mu^*, u^*) = \inf_{\mu \in \mathcal{M}_1} K_1(\mu, u^*) \leq K_1(\mu^*, u^*) \quad \text{for all } \mu \in \mathcal{M}_1, \ u \in \mathcal{U}_1.
\]

Next, we turn to program (31). Assume that the (extended) state space \( \mathbf{X} \) is as in Definition 5.1 and Conditions 5.1.2, 5.1.3, and 5.2 are satisfied. Then program (31) has an optimal solution \( \eta^* \) as mentioned at the end of Section 5.

Assume additionally that

\[
\sup_{x \in V} \int_{[0, \infty)} C_0^0(\phi(x, u)) du < \infty.
\]

Let \( \mathcal{M}_2 \) be the space of measures \( \eta \) on \( V^c \times \mathbf{A}_0 \) satisfying the requirement (i) in Corollary 5.1. \( \mathcal{U}_2 \) be the space of finite linear combinations \( u \) of the functions from \( \mathcal{W} \), and

\[
K_2(\eta, u) := \int_{V^c \times \mathbf{A}} C_0^0(x, a)\eta(dx \times da) + \int_{V^c} C_0'(y)\eta(dy \times \square) + u(x_0) + \int_{V^c} \chi u(y) \eta(dy \times \square) - \int_{V^c} u(x) \eta(dx \times \mathbf{A}) + \int_{V^c \times \square} u(l(x, a)) \eta(dx \times da)
\]

be the \([-\infty, +\infty]\)-valued function on \( \mathcal{M}_2 \times \mathcal{U}_2 \). The integral \( \int_{V^c} \chi u(x) \eta(dx \times \square) \) in (50) is calculated, as usual, separately for \( (\chi u)^+ \) and \( (\chi u)^- \) with the convention \( +\infty - \infty := +\infty \). Now the “primal” program (31) can be rewritten as

\[
\sup_{u \in \mathcal{U}_2} K_2(\eta, u) \rightarrow \inf_{\eta \in \mathcal{M}_2}.
\]

Indeed, if the requirement (ii) in program (31) is satisfied by \( \eta \), then

\[
\forall \ u \in \mathcal{U}_2 \quad K_2(\eta, u) = \sup_{u \in \mathcal{U}_2} K_2(\eta, u) = \int_{V^c \times \mathbf{A}_0} C_0(y, a)\eta(dy \times da);
\]

38
otherwise, \( \sup_{u \in \mathcal{U}_2} K_2(\eta, u) = +\infty \).

The “dual” program

\[
\inf_{\eta \in \mathcal{M}_2} K_2(\eta, u) \to \sup_{u \in \mathcal{U}_2}
\]

can be rewritten as

Maximize \( u(x_0) \) over \( u \in \mathcal{U}_2 \)

subject to

\( \chi u(\phi(x, t)) + C_0^g(\phi(x, t)) \geq 0 \) for all \( x \in V^c \)

and

\( \text{for almost all } t \text{ such that } \phi(x, t) \in V^c \)

\[ C_0^l(x, a) + u(l(x, a)) - u(x) \geq 0 \text{ for all } (x, a) \in V^c \times A. \]

The explanations are as follows. Note that any point \( y \in V^c \) can be represented as \( \phi(x, t) \) with \( x \in V^c \) and \( t \in \mathbb{R}^0_+ \) such that \( \phi(x, t) \in V^c \), and remember that the measure \( d\eta(dx \times \Box) \) is normal for each \( \eta \in \mathcal{M}_2 \). Note also that, in view of (49), \( \int_{V^c} C_0^g(y)\eta(dy \times \Box) < \infty \) for each normal measure \( \eta(dy \times \Box) \).

Therefore, if a function \( u \in \mathcal{U}_2 \) does not satisfy the constraints presented in program (51), then, one can make \( K_2(\eta, u) \) arbitrarily close to \( -\infty \) by choosing appropriately a corresponding measure \( \eta \in \mathcal{M}_2 \), so that \( \inf_{\eta \in \mathcal{M}_2} K_2(\eta, u) = -\infty \). Otherwise, if those constraints are satisfied, then

\[
\inf_{\eta \in \mathcal{M}_2} K_2(\eta, u) = K_2(0, u) = u(x_0).
\]

Here 0 is the zero measure.

Proposition 7.2 below shows that, under certain conditions, the solution to the dual program (51) is provided by the Bellman function (solution to the integral Bellman equation (48)) \( u^* \), which also satisfies the following Bellman equation in the differential form, or say for brevity, differential Bellman equation, see [17]:

\[
\min \left\{ \chi u(\phi(x, t)) + C_0^g(\phi(x, t)); \inf_{a \in A} \{C_0^l(\phi(x, t), a) + u(l(\phi(x, t), a)) - u(\phi(x, t))\} \right\} = 0
\]

for all \( x \in V^c \), for almost all \( t \text{ such that } \phi(x, t) \in V^c \).

**Proposition 7.2** Suppose the following holds true.

(a) Conditions 3.1, 3.2, 3.3, and 5.2 are satisfied, and in addition, (49) holds.

(b) For each \( x \in V^c \),

\[
\int_{[0, \infty)} \left( \int_{[t, \infty)} C_0^g(\phi(x, u))du \right) dt < \infty.
\]

(c) The solution \( u^* \) to the integral Bellman equation (48) is absolutely continuous along the flow \( \phi \).

Then \( u^* \) provides a solution to the (dual) program (51) and satisfies also the differential Bellman equation (52); the values of the primal program (37) and the dual program (51) coincide, and \( (\eta^*, u^*) \) is saddle-point of \( K_2 \):

\[
K_2(\eta^*, u) \leq K_2(\eta^*, u^*) \leq K_2(\eta, u^*) \text{ for all } \eta \in \mathcal{M}_2, \ u \in \mathcal{U}_2.
\]
Proof. Under assertion (a), the Bellman equation (48) has a unique bounded positive lower semicontinuous solution $u^*$, and $\inf_\pi V_0(x_0, \pi) = u^*(x_0)$: see [17, Thm.1, Prop.1]. Note that $\inf_\pi V_0(x, \pi) \equiv 0$ for $x \in V$. According to Corollary 5.1 and to the proof of Theorem 5.1 the value of the primal program (31), equivalent to program (15)-(17) and to program (13), also coincides with $\inf_\pi V_0(x_0, \pi)$.

Let us show that $u^* \in \mathcal{U}_2$. Under the assumptions (a), (b) and (c) in the statement of this proposition, [17, Thm.2] is applicable, and implies that

$$\forall x \in X, \chi u^*(\phi(x, t)) + C_0^\phi(\phi(x, t)) \geq 0$$

for almost all $t \in \mathbb{R}^+_0$.

Therefore,

$$\forall x \in X, \ 0 \leq (\chi u^* - (\phi(x, t))) \leq C_0^\phi(\phi(x, t))$$

for almost all $t \in \mathbb{R}^+_0$

and

$$\sup_{x \in X} \int_{[0, \infty)} (\chi u^*)^- (\phi(x, t)) dt \leq \sup_{x \in X} \int_{[0, \infty)} C_0^\phi(\phi(x, t)) dt < \infty$$

meaning that the positive function

$$w^-(x) := \int_{[0, \infty)} (\chi u^*)^- (\phi(x, u)) du,$$

which decreases along the flow $\phi$, is (uniformly) bounded on $X$.

As explained after Condition 3.3 $C_0^\phi(\phi(x, t)) = 0$ and $\phi(x, t) \in V$ for all $x \in V$, $t \in \mathbb{R}^+_0$; hence $w^-(x) = 0$ for all $x \in V$. In case $x \in V^c$ and $\phi(x, t) \in V^c$ for all $t \in \mathbb{R}^+_0$,

$$\lim_{t \to \infty} w^-(\phi(x, t)) \leq \lim_{t \to \infty} \int_{[0, \infty)} C_0^\phi(\phi(x, t), u) du = \lim_{t \to \infty} \int_{[t, \infty)} C_0^\phi(\phi(x, r)) dr = 0$$

meaning that the positive function $w^-$ satisfies $\lim_{t \to \infty} w^-(\phi(x, t)) = 0$. Hence, $w^- \in \mathcal{W}$.

We know that $u^*(y) = 0$ for all $y \in V$ by the definition (11). Assumption (b) in the statement of this proposition implies that

$$\forall x \in X, \int_{[0, \infty)} u^*(\phi(x, t)) dt < \infty,$$

so that

$$\lim_{t \to \infty} u^*(\phi(x, t)) = 0$$

for all $x \in X$.

As a result, we obtain that the positive function

$$w^+(x) := \int_{[0, \infty)} (\chi u^*)^+ (\phi(x, u)) du = u^*(x) + w^-(x),$$

which is decreasing along the flow $\phi$, belongs to $\mathcal{W}$. (Obviously, the both function $w^+$ and $w^-$ are bounded and absolutely continuous along the flow $\phi$.)

Therefore, $u^* = w^+ - w^- \in \mathcal{U}_2$.

According to [17, Thm.2], the function $u^*$ is feasible in program (51) and also satisfies the differential Bellman equation (52). Since, in any case,

$$\sup_{u \in \mathcal{U}_2} \inf_{\eta \in \mathcal{M}_2} K_2(\eta, u) \leq \inf_{\eta \in \mathcal{M}_2} \sup_{u \in \mathcal{U}_2} K_2(\eta, u) = \inf_\pi V_0(x_0, \pi),$$

and the function $u^*$ is feasible in program (51), we conclude that

$$u^*(x_0) = \inf_{\eta \in \mathcal{M}_2} K_2(\eta, u^*) = \sup_{u \in \mathcal{U}_2} \inf_{\eta \in \mathcal{M}_2} K_2(\eta, u) = \inf_\pi V_0(x_0, \pi),$$

so that, the function $u^*$ solves program (51).

The proof of the last assertion in this proposition is identical to the end of the proof of Proposition [7.4].
8 Conclusion

To sum up, we developed two linear programming approaches to impulsively controlled dynamical systems with constraints, and under quite general and natural conditions, we showed the existence of an optimal control policy. In the absence of constraints, the two linear programs proposed here are in correspondence with the dynamical programming equations in the integral and differential form, which were the objects investigated in [17]. It looks that the similar technique can be successful for solving optimal gradual-impulsive control problems for piecewise deterministic processes with functional constraints. We leave this for future research.

9 Acknowledgement

This research was supported by the Royal Society International Exchanges award IE160503. We would like to thank Professor Alexander Plakhov from University of Aveiro (Portugal) and Institute for Information Transmission Problems (Russia) for his initial participation in this work and for providing Lemma A.1.

Appendix

Lemma A.1 and its proof presented below are similar to Lemma 2.2 in [7], where the authors assumed that $E$ was a subset of an Euclidean space.

Let $E$ be an arbitrary set and $\phi : E \times [0, \infty) \to E$ be a flow in $E$. The semigroup property of the flow is assumed to be satisfied, $\phi(x, t+s) = \phi(\phi(x, t), s)$ for all $x \in E$ and all $t \geq 0$ and $s \geq 0$.

**Definition A.1** A function $w : E \to \mathbb{R}$ is said to be absolutely continuous along the flow if for all $x \in E$ the function $t \mapsto w(\phi(x, t))$, $t \in [0, \infty)$ is absolutely continuous. It is called increasing (decreasing) along the flow if so is the function $t \mapsto w(\phi(x, t))$, $t \in [0, \infty)$ for all $x \in E$.

**Lemma A.1** Suppose function $w$ is absolutely continuous along the flow $\phi$. Then the following assertions are valid.

(a) There exists a function $\chi^w : E \to \mathbb{R}$ such that, for any $x \in E$, the function $\chi^w(\phi(x, s))$ is Lebesgue integrable with respect to $s$ on any finite interval $[0, t] \subset [0, \infty)$ and

\[
 w(\phi(x, t)) - w(x) = \int_{[0,t]} \chi^w(\phi(x, s)) \, ds
\]

for all $x \in E$ and $t \geq 0$.

(b) If, additionally, $E$ is a measurable space (that is, is equipped with a $\sigma$-algebra of subsets), $w$ is measurable, and the functions $\phi(\cdot, t) : E \to E$ are measurable for all $t \geq 0$, then the function $\chi^w$ satisfying (a) can be chosen measurable.

**Proof.** We provide one common proof for (a) and (b) underlining the measurability properties as soon as they appear.

Define the functions

\[
 \overline{W}(x) = \lim_{n \to \infty} \frac{w(\phi(x, \frac{1}{n})) - w(x)}{1/n}, \quad \underline{W}(x) = \lim_{n \to \infty} \frac{w(\phi(x, \frac{1}{n})) - w(x)}{1/n}
\]

and the set $D = \{ x \in E : \overline{W}(x) = \underline{W}(x) \neq \pm \infty \}$. Let us additionally define the function $W : D \to \mathbb{R}$ by $W(x) = \overline{W}(x)$; that is, $W(x)$ coincides with the limit $\lim_{n \to \infty} n [w(\phi(x, \frac{1}{n})) - w(x)]$, if it exists and is finite.
If \( w \) and \( \phi(\cdot, t) \) are measurable, then \( w(\phi(x, \frac{1}{n})) \) is also measurable. Hence the functions \( \overline{W} \) and \( \underline{W} \) are measurable as the upper and lower limits of the sequence of measurable functions \( n [w(\phi(x, 1/n)) - w(x)] \). Consequently, the set \( D \) is also measurable.

Define the function \( \chi w \) on \( E \) by

\[
\chi w(x) = \begin{cases} 
W(x), & \text{if } x \in D; \\
g(x), & \text{otherwise},
\end{cases}
\]

(54)

where \( g \) is any function. In the measurable case we take \( g \) to be measurable and readily get that \( \chi w \) is also measurable.

Since \( w \) is absolutely continuous along the flow then for any \( x \in E \) there exists a subset of full measure \( T_x \subset \mathbb{R}_+^n \setminus \{0\} \) such that the derivative \( \frac{d}{dt} w(\phi(x, t)) \) exists and is finite for all values \( t \in T_x \). For any such value (let it now be denoted by \( s \in T_x \)) we can write down the following (below we denote \( x' = \phi(x, s) \) and use the semigroup property of the flow)

\[
\frac{d w(\phi(x, t))}{d t} \bigg|_{t=s} = \lim_{\varepsilon \to 0} \frac{w(\phi(x, s + \varepsilon)) - w(\phi(x, s))}{\varepsilon} = \lim_{n \to \infty} \frac{w(\phi(x', \frac{1}{n})) - w(x')}{1/n}.
\]

The latter value exists and is finite, and therefore coincides with \( W(x') \). This argument also shows that \( \phi(x, T_x) \subset D \).

Since \( w \) is absolutely continuous along the flow, one can write down

\[
w(\phi(x, t)) - w(x) = \int_{[0, t] \cap T_x} \frac{d w(\phi(x, \tau))}{d \tau} \big|_{\tau=s} \, ds = \int_{[0, t] \cap T_x} W(\phi(x, s)) \, ds.
\]

Now taking into account that \( [0, t] \setminus T_x \) has Lebesgue measure zero and \( \chi w \) is an extension of \( W \) to \( E \), we conclude that the latter integral coincides with \( \int_{[0, t]} \chi w(\phi(x, s)) \, ds \), and so, formula (53) is proved.

\[\square\]

**Proof of Lemma 4.1**. We will need the (partial) occupation measure on \( V^c \times \mathbb{R}_+^n \times \mathbb{A} \)

\[
\mu^n(dx \times d\theta \times da) := E_{\mathcal{F}_0} \left[ \sum_{i=1}^{n} \mathbb{I}\{X_{i-1} \times dx, \Theta_i \times d\theta, A_i \times da\} \right], \quad n = 0, 1, 2, \ldots.
\]

Clearly, \( \mu^n \uparrow \mu^n \) on \( V^c \times \mathbb{R}_+^n \times \mathbb{A} \) set-wise as \( n \to \infty \). Therefore, according to the definition of the measure \( \eta \), for each positive measurable function \( C^g \) on \( V^c \),

\[
I^n := \int_{V^c \times \mathbb{R}_+^n \times \mathbb{A}} \left\{ \int_{[0, \theta]} C^g(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} \, du \right\} \mu^n(dx \times d\theta \times da)
\]

\[\uparrow\]

\[
\int_{V^c \times \mathbb{R}_+^n \times \mathbb{A}} \left\{ \int_{[0, \theta]} C^g(\phi(x, u)) \mathbb{I}\{\phi(x, u) \in V^c\} \, du \right\} \mu^n(dx \times d\theta \times da) = \int_{V^c} C^g(\eta(dy \times \Box))
\]

and, for each positive measurable function \( C^I \) on \( V^c \times \mathbb{A} \),

\[
J^n := \int_{V^c \times \mathbb{R}_+^n \times \mathbb{A}} \mathbb{I}\{\theta < +\infty\} \mathbb{I}\{\phi(x, \theta) \in V^c\} C^I(\phi(x, \theta), a) \mu^n(dx \times d\theta \times da)
\]

\[\uparrow\]

\[
\int_{V^c \times \mathbb{R}_+^n \times \mathbb{A}} \mathbb{I}\{\theta < +\infty\} \mathbb{I}\{\phi(x, \theta) \in V^c\} C^I(\phi(x, \theta), a) \mu^n(dx \times d\theta \times da)
\]

\[= \int_{V^c \times \mathbb{A}} C^I(y, a) \eta(dy \times da).
\]
We will prove by induction the following assertions:

\[ I^n = \int_{V^c} C^g(y) \eta^n (dy \times \square) \quad \text{and} \quad J^n = \int_{V^c \times A} C^f(y, a) \eta^n(dy \times da). \]

If \( n = 0 \), then \( \mu^0 = 0, \eta^0 = 0, I^0 = 0, \) and \( J^0 = 0. \)

Suppose the above assertions are valid for some \( n \geq 0. \) Then

\[
\begin{align*}
I^{n+1} &= I^n + \int_{V^c} \int_{R_+^d} \int_A \left\{ \int_{[0, \theta]} C^g(\phi(x, u)) \mathbb{1}\{\phi(x, u) \in V^c\} du \right\} p_A^{n+1}(da|x, \theta) p_T^{n+1}(d\theta|x) \nu^n(dx) \\
J^{n+1} &= J^n + \int_{V^c} \int_{R_+^d} \int_A \left\{ \mathbb{1}\{\theta < \infty\} \mathbb{1}\{\phi(x, \theta) \in V^c\} C^f(\phi(x, \theta), a) p_A^{n+1}(da|x, \theta) p_T^{n+1}(d\theta|x) \nu^n(dx) \right\}
\end{align*}
\]

because, according to (1),

\[
\mu^{n+1}(dx \times d\theta \times da) = \mu^n(dx \times d\theta \times da) + p_A^{n+1}(da|x, \theta) p_T^{n+1}(d\theta|x) \nu^n(dx).
\]

Using the Tonelli Theorem [1, Thm.11.28], we obtain:

\[
\begin{align*}
&\int_{V^c} \int_{R_+^d} \int_A \left\{ \int_{[0, \theta]} C^g(\phi(x, u)) \mathbb{1}\{\phi(x, u) \in V^c\} du \right\} p_T^{n+1}(d\theta|x) \nu^n(dx) \\
&= \int_{V^c} \int_{R_+^d} \int_A \left\{ \mathbb{1}\{\phi(x, u) \in V^c\} p_T^{n+1}(d\theta|x) du \right\} \nu^n(dx) \\
&= \int_{V^c} \int_{R_+^d} C^g(\phi(x, u)) \mathbb{1}\{\phi(x, u) \in V^c\} p_T^{n+1}([u, \infty]|x) du \nu^n(dx) \\
&= \int_{V^c} C^g(y) \left\{ \int_{V^c} \int_{R_+^d} \delta_{\phi(x,u)}(dy) p_T^{n+1}([u, \infty]|x) du \right\} \nu^n(dx)
\end{align*}
\]

and, by induction and the definition of the measure \( \eta^{n+1}(\Gamma_X \times \square) \),

\[ I^{n+1} = \int_{V^c} C^g(y) \eta^{n+1}(dy \times \square). \]

Similarly,

\[
\begin{align*}
&\int_{V^c} \int_{R_+^d} \int_A \left\{ \mathbb{1}\{\phi(x, \theta) \in V^c\} C^f(\phi(x, \theta), a) p_A^{n+1}(da|x, \theta) p_T^{n+1}(d\theta|x) \nu^n(dx) \right\} \\
&= \int_{V^c} \int_A C^f(y, a) \left\{ \int_{V^c} \int_{R_+^d} \delta_{\phi(x,\theta)}(dy) p_A^{n+1}(da|x, \theta) p_T^{n+1}(d\theta|x) \right\} \nu^n(dx)
\end{align*}
\]

and, by induction and the definition of the measure \( \eta^{n+1}(\Gamma_X \times \Gamma_A) \),

\[ J^{n+1} = \int_{V^c \times A} C^f(y, a) \eta^{n+1}(dy \times da). \]

Since, for all positive measurable functions \( C^g \) on \( V^c \) and \( C^f \) on \( V^c \times A \),

\[
\int_{V^c} C^g(y) \eta^n(dy \times \square) \uparrow \int_{V^c} C^g(y) \eta(dy \times \square) \quad \text{and} \quad \int_{V^c \times A} C^f(y, a) \eta^n(dy \times da) \uparrow \int_{V^c \times A} C^f(y, a) \eta(dy \times da),
\]

43
we conclude that \( \eta^n \uparrow \eta \) on \( V^c \times A \) set-wise as \( n \to \infty \).

Proof of Lemma 5.1. (i) This is obvious because the second component of the flow (time) has no cycles.

(ii) Let \( (\hat{z}, \tau) \) be a limiting point of \( \hat{z} \mathcal{X} \). Then there exists a sequence of values \( u_n \in \mathbb{R}_+^0 \) such that the corresponding sequence of points \( (\hat{\phi}(\hat{x}, u_n), u_n) \in \hat{z} \mathcal{X} \) converges to \( (\hat{z}, \tau) \). We have \( \lim_{n \to \infty} u_n = \tau \in \mathbb{R}_+^0 \). Further, using the continuity of the flow \( \hat{\phi} \), we have \( \hat{z} = \lim_{n \to \infty} \hat{\phi}(\hat{x}, u_n) = \hat{\phi}(\hat{x}, \tau) \). Thus, \( (\hat{z}, \tau) = (\hat{\phi}(\hat{x}, \tau), \tau) = \phi((\hat{x}, 0), \tau) \in \hat{z} \mathcal{X} \). □

Proof of Lemma 5.2. In this proof, let us denote by \( \rho \) and \( \hat{\rho} \) the compatible metrics on \( \mathcal{X} \) and \( \hat{\mathcal{X}} \). If \( y_n \to y \), where \( y_n = (\hat{y}_n, t_n) \), then the sequence \( \{t_n\}_{n=1}^\infty \) is bounded: \( t, t_n \in [0, T] \) for some \( T < \infty \). Now \( \hat{\rho}(h(y_n), h(y)) \leq \sup_{t \in [0, T]} d(t) \rho(y_n, y) \to 0 \). The proved continuity of the mapping \( h \) and continuity of the original flow \( \hat{\phi} \) immediately imply that the flow \( \hat{\phi} \) in the reverse time is continuous.

The mapping \( F \) is continuous because the flow \( \hat{\phi} \) is continuous. It is a bijection from \( \hat{\mathcal{X}} \times \mathbb{R}_+^0 \) to \( \mathcal{X} \), and the inverse mapping \( F^{-1}(y) = (h(y), \tau_y) \) is continuous, as has been proved above. (For \( y = (\hat{y}, t) \), \( \tau_y = t \) is obviously the continuous function of \( y \).) Thus, \( F \) is a homeomorphism, and \( \mathcal{X} \) is a Borel space, being the homeomorphic image of the Borel space \( \hat{\mathcal{X}} \times \mathbb{R}_+^0 \). □

Proof of Lemma 5.3. Item (i) is obvious, so let us prove Item (ii). Clearly,

\[
\hat{\zeta}(\Gamma_X) = \int_{\Gamma_X} \int_{[0, t \land \hat{\theta}^*(\hat{x}^0))]} g(\hat{x}^0, u) \, du \, L(d\hat{x}^0), \quad \Gamma_X \in \mathcal{B}(V^c).
\]

Now

\[
\int_{\Gamma_X \times \Gamma_R} \hat{\zeta}(d\hat{x}^0 \times d\hat{x}) = \left( \int_{\Gamma_X} \int_{\Gamma_R} \left\{ \int_{[0, t \land \hat{\theta}^*(\hat{x}^0))]} g(\hat{x}^0, v) \, dv > 0 \right\} \frac{g(\hat{x}^0, u) \, du}{\int_{[0, t \land \hat{\theta}^*(\hat{x}^0))]} g(\hat{x}^0, v) \, dv} \right) + \frac{1}{\int_{[0, t \land \hat{\theta}^*(\hat{x}^0))]\hat{\zeta}(d\hat{x}^0)} \hat{\zeta}(d\hat{x}^0).
\]

Recall that \( \hat{\theta}^*(\hat{x}^0) > 0 \). Hence

\[
\int_{[0, t \land \hat{\theta}^*(\hat{x}^0))] \frac{g(\hat{x}^0, u)}{\int_{[0, t \land \hat{\theta}^*(\hat{x}^0)]} g(\hat{x}^0, v) \, dv} \, du + \int_{[0, t \land \hat{\theta}^*(\hat{x}^0)]} \frac{g(\hat{x}^0, v) \, dv}{\hat{\theta}^*(\hat{x}^0)} = \frac{du}{\hat{\theta}^*(\hat{x}^0)}
\]

is the required version of the stochastic kernel \( \hat{\zeta}(du|\hat{x}^0) \).

Lemma A.2 Let Conditions 3.1, 3.2, 3.3 and 5.2 be satisfied.

(a) Suppose \( \zeta \) is a finite measure on \( V^c \), and the measure \( \hat{\zeta}(d\hat{x}^0) \) and the stochastic kernel \( \zeta(dt|\hat{x}^0) \) are as in Definition 5.4. Then, for each bounded measurable function \( g \) on \( V^c \),

\[
\int_{V^c} g(y) \hat{\zeta}(dy) = \int_{V^c} \int_{[0, \hat{\theta}^*(\hat{x}^0)]} g(\hat{\phi}(\hat{x}^0, 0), u) \zeta(du|\hat{x}^0) \hat{\zeta}(d\hat{x}^0).
\]

(b) Suppose \( \zeta \) is a weakly normal measure on \( V^c \), and the measure \( \hat{\zeta}(d\hat{x}^0 \times dt) \) is as in Definition 5.4. Then, for each positive (or negative) measurable function \( g \) on \( V^c \),

\[
\int_{V^c} g(y) \zeta(dy) = \int_{D} g(\hat{\phi}(\hat{x}^0, 0), u) \zeta(d\hat{x}^0 \times du).
\]
(c) Suppose $\zeta$ is a weakly normal measure (or a finite measure) on the orbit
\[ \mathcal{Z}\mathcal{X} \cap V^c = \{ \phi((\tilde{z},0),t) : t \in [0,\tilde{\theta}^* (\tilde{z})) \} \]
and
\[ m(I) := \zeta(\{ \phi((\tilde{z},0),t) : t \in I \}) \]
is the $\sigma$-finite (or finite) measure on $[0,\tilde{\theta}^*(\tilde{z}))$. Then, for each positive or negative measurable function $g$ on $\mathcal{Z}\mathcal{X} \cap V^c$,
\[ \int_{\mathcal{Z}\mathcal{X} \cap V^c} g(y)\zeta(dy) = \int_{[0,\tilde{\theta}^*(\tilde{z}))} g(\phi((\tilde{z},0),t)) m(dt). \]

**Proof.** (a) It is sufficient to check the required formula for $g(y) = 1\{y \in Y\}$, where $Y \in \mathcal{B}(V^c)$. According to the definition of the mappings $F$ and $F^{-1}$,
\[ (\tilde{x}^0,u) \in F^{-1}(Y) \iff F(\tilde{x}^0,u) \in Y \iff (\phi(\tilde{x}^0,u),u) = \phi((\tilde{x}^0,0),u) \in Y. \]
Hence
\[ \int_{V^c} g(y)\zeta(dy) = \zeta(Y) = \tilde{\zeta}(F^{-1}(Y)) = \int_{\mathcal{V}_c \times \mathbb{R}^+_c} 1\{(\tilde{x}^0,u) \in F^{-1}(Y)\} \tilde{\zeta}(d\tilde{x}^0 \times du) \]
\[ = \int_{\mathcal{V}_c} \int_{\mathbb{R}^+_c} 1\{\phi((\tilde{x}^0,0),u) \in Y\} \zeta(du|\tilde{x}^0) \tilde{\zeta}(d\tilde{x}^0). \]
But, for $u \geq \tilde{\theta}^*(\tilde{x}^0)$, $\phi((\tilde{x}^0,0),u) \in V$ and thus $\phi((\tilde{x}^0,0),u)$ cannot belong to $Y$. The desired formula
\[ \int_{V^c} g(y)\zeta(dy) = \int_{\mathcal{V}_c} \int_{[0,\tilde{\theta}^*(\tilde{x}))} 1\{\phi((\tilde{x}^0,0),u) \in Y\} \zeta(du|\tilde{x}^0) \tilde{\zeta}(d\tilde{x}^0) \]
is proved.

(b) The required formula is justified after we represent the function $g$ as $g(x) = \sum_{t=1}^\infty g_t(x)$ with $g_t((\tilde{x},u)) = 1\{u \in [t-1,t)\} g((\tilde{x},u))$ and use the statement (a) separately for all $g_t$, where one can legitimately use the (finite) restriction of $\zeta$ to the set \( \{ x = (\tilde{x},u) \in V^c : t-1 \leq u < t \} \).

(c) Without loss of generality, we assume that $\zeta(\mathcal{Z}\mathcal{X} \cap V^c) > 0$.
If $\tilde{\theta}^*(\tilde{z}) < \infty$ then the measure $\zeta$ is finite and can be extended to $V^c$ by putting $\zeta(V^c \setminus \mathcal{Z}\mathcal{X}) := 0$. Now
\[ \tilde{\zeta}(d\tilde{x}^0 \times dt) = m(dt)\delta_{\tilde{z}}(d\tilde{x}^0); \]
\[ \tilde{\zeta}(d\tilde{x}^0) = \zeta(\mathcal{Z}\mathcal{X} \cap V^c) \delta_{\tilde{z}}(d\tilde{x}^0); \]
\[ \zeta(dt|\tilde{x}^0) = \begin{cases} m(dt)/\zeta(\mathcal{Z}\mathcal{X} \cap V^c), & \text{if } \tilde{x}^0 = \tilde{z}; \\ \text{arbitrarily fixed probability measure,} & \text{if } \tilde{x}^0 \neq \tilde{z}, \end{cases} \]
and the required equality follows from Item (a).
Suppose $\tilde{\theta}^*(\tilde{z}) = \infty$, so that $\mathcal{Z}\mathcal{X} \cap V^c = \mathcal{Z}\mathcal{X}$. It is sufficient to check the required formula for $g_t(y) = 1\{y \in Y_t\}$, where
\[ Y_t = \{ \phi((\tilde{z},0),u) : u \in I_t \in \mathcal{B}([t-1,t)) \}, \quad t = 1,2,\ldots \]
Recall that, according to Lemma 5.2, the mapping $F(t) = \phi((\tilde{z},0),t)$ is a homeomorphism between $\mathbb{R}^+_c$ and $\mathcal{Z}\mathcal{X}$: all different subsets $I_t \in \mathcal{B}([t-1,t))$ produce all possible subsets $Y_t \in \mathcal{B}(\{ \phi((\tilde{z},0),u) : u \in [t-1,t) \})$.  

45
Now
\[ \int_{\mathcal{N}} g_t(y) \zeta(dy) = \zeta(Y_t); \]
\[ \int_{\mathbb{R}_+^4} g_t(\phi(\bar{z}, 0), u)m(du) = \int_{\mathbb{R}_+^4} \mathbb{I}\{\phi(\bar{z}, 0), u) \in Y_t\}m(du) = m(I_t), \]
and \( m(I_t) = \zeta(Y_t) \) by the definition of the measure \( m \).

**Proof of Lemma 5.4.** (a) If
\[ \tilde{\zeta}^1(d\bar{x}^0 \times du) = g^1(x^0, u)du \quad \text{and} \quad \tilde{\zeta}^2(d\bar{x}^0 \times du) = g^2(x^0, u)du \]
then we put \( L := L^1 + L^2 \) and
\[ g(x^0, u) := \left[ g^1(x^0, u) \frac{dL^1}{dL}(x^0) - g^2(x^0, u) \frac{dL^2}{dL}(x^0) \right] \mathbb{I}\left\{ \left. g^1(x^0, u) \frac{dL^1}{dL}(x^0) - g^2(x^0, u) \frac{dL^2}{dL}(x^0) \right| 0 \right. \geq 0 \}
Now, for \( \zeta := \tilde{\zeta}^1 - \tilde{\zeta}^2 \), we have, for each \( t \in \mathbb{R}_+ \) and \( \Gamma_t \) defined by (30),
\[ \tilde{\zeta}(d\bar{x}^0 \times du) = \tilde{\zeta}^1(d\bar{x}^0 \times du) - \tilde{\zeta}^2(d\bar{x}^0 \times du) = g(x^0, u)du \quad L(d\bar{x}^0), \]
because, for each \( t \in \mathbb{R}_+ \), the set
\[ \left\{ (x^0, u) \in \Gamma_t : g^1(x^0, u) \frac{dL^1}{dL}(x^0) - g^2(x^0, u) \frac{dL^2}{dL}(x^0) < 0 \right\} \]
is null with respect to the measure \( L(d\bar{x}^0) \times du \).

(b) Item (i) of Definition 6.6 is obvious. For Item (ii), we fix \( t \in \mathbb{R}_+ \) and consider the measures
\[ \tilde{\zeta}^i_t(d\bar{x}^0 \times du) = \tilde{\zeta}^i_t(dx^0)G_t^i(x^0, u)du, \quad i = 1, 2. \]
Below, \( \zeta_t := \zeta^1_t - \zeta^2_t \), so \( \hat{\zeta}_t = \hat{\zeta}^1_t - \hat{\zeta}^2_t \) and \( \check{\zeta}_t = \check{\zeta}^1_t - \check{\zeta}^2_t \). Remember that \( \zeta^1_t \geq \zeta^2_t \); hence \( \hat{\zeta}^1_t \geq \hat{\zeta}^2_t \) and there exist Radon-Nikodym derivatives
\[ \frac{d\hat{\zeta}^2_t}{d\zeta^1_t}(x^0) \quad \text{and} \quad \frac{d\check{\zeta}^2_t}{d\zeta^1_t}(x^0). \]
Now
\[ \hat{\zeta}_t(d\bar{x}^0 \times du) = \left[ \mathbb{I}\left\{ \frac{d\hat{\zeta}_t}{d\zeta^1_t}(x^0) > 0 \right\} + \mathbb{I}\left\{ \frac{d\check{\zeta}_t}{d\zeta^1_t}(x^0) \geq 0 \right\} \right] \left[ G^1_t(x^0, u) - G^2_t(x^0, u) \frac{d\hat{\zeta}^2_t}{d\zeta^1_t}(x^0) \right] du \quad \check{\zeta}^1_t(dx^0). \]
The \( \hat{\zeta}_t \)-measure of the set \( \{ \bar{x}^0 \in V \cup \mathbb{R}^c : \frac{d\hat{\zeta}_t}{d\zeta^1_t}(x^0) \leq 0 \} \) is zero, because so is the \( \hat{\zeta}^1 \)-measure, and one can put \( \zeta_t(du|x^0) \) arbitrarily on it, e.g.,
\[ \zeta_t(du|x^0) := \frac{du}{t \land \theta^*(x^0)}. \]
Recall that \( \theta^*(x^0) > 0 \). Therefore,
\[ \check{\zeta}_t(d\bar{x}^0 \times du) = \left\{ \mathbb{I}\left\{ \frac{d\hat{\zeta}_t}{d\zeta^1_t}(x^0) > 0 \right\} \left[ G^1_t(x^0, u) - G^2_t(x^0, u) \frac{d\hat{\zeta}^2_t}{d\zeta^1_t}(x^0) \right] \frac{du}{t \land \theta^*(x^0)} \right\} \check{\zeta}_t(dx^0) \]
\[ + \mathbb{I}\left\{ \frac{d\check{\zeta}_t}{d\zeta^1_t}(x^0) \leq 0 \right\} \frac{du}{t \land \theta^*(x^0)} \check{\zeta}_t(dx^0) \]
\[
\dot{\zeta}(d\tilde{x}^0 \times du) = G_t(\tilde{x}^0, u)du \dot{\zeta}(d\tilde{x}^0),
\]
where
\[
G_t(\tilde{x}^0, u) := \mathbb{I}\left(\frac{d\dot{\zeta}_t}{d\xi_t}(\tilde{x}^0) > 0\right) \left[ G_1^1(\tilde{x}^0, u) - G_2^2(\tilde{x}^0, u) \frac{d^2\zeta_t}{d\xi_t^2}(\tilde{x}^0) \right] + \mathbb{I}\left(\frac{d\dot{\zeta}_t}{d\xi_t}(\tilde{x}^0) \leq 0\right) \frac{1}{t \wedge \theta^*(\tilde{x}^0)}.
\]

\[\tag{56} \]

\textbf{Lemma A.3} Suppose \( m \) is a finite measure on \( \mathbb{R} \). Then, for each \( \tau, t \in \mathbb{R} \),
\[
 tm([\tau, \tau + t]) = \int_{[0,t]} m([\tau, \tau + s])ds + \int_{[\tau, \tau + t]} (s - \tau) \, dm(s)
\]
and
\[
 tm([\tau, \tau + t]) = \int_{[0,t]} m([\tau, \tau + s])ds + \int_{[\tau, \tau + t]} (s - \tau) \, dm(s).
\]

\textbf{Proof.} For all cadlag (i.e., right-continuous, having left limits) real-valued functions \( U \) and \( V \) on \( \mathbb{R} \) with finite variation (on finite intervals),
\[
U(t_2)V(t_2) = U(t_1)V(t_1) + \int_{(t_1, t_2]} U(s-) \, dV(s) + \int_{(t_1, t_2]} V(s-) \, dU(s) \quad (55)
\]
for any \(-\infty < t_1 < t_2 < \infty \). (See \[\text{5.} \text{Appendix A4,§2} \].) Equivalently, in the symmetric form:
\[
U(t_2)V(t_2) = U(t_1)V(t_1) + \int_{(t_1, t_2]} U(s-) \, dV(s) + \int_{(t_1, t_2]} V(s-) \, dU(s) + \sum_{u \in (t_1, t_2]} \Delta U_u \, \Delta V_u.
\]

Introduce cadlag functions of finite variation (on finite intervals):
\[
U(s) := m([\tau, s]) \quad \text{and} \quad V(s) := s - \tau, \quad s \in \mathbb{R}.
\]

Then, according to the above formulae,
\[
 tm([\tau, \tau + t]) = \int_{[\tau, \tau + t]} m([\tau, s])ds + \int_{[\tau, \tau + t]} (s - \tau) \, dm(s) = \int_{[0,t]} m([\tau, \tau + s])ds + \int_{[\tau, \tau + t]} (s - \tau) \, dm(s). \tag{56}
\]
For the last equality to be proved, it is sufficient to consider a sequence \( t_i \uparrow t > 0 \) and pass to the limit in \[\text{56} \]. The case \( t \leq 0 \) is trivial. \[\square\]

\textbf{References}

[1] Aliprantis, Ch. and Border, K.C. (2006). \textit{Infinite Dimensional Analysis}. Springer-Verlag, New York.

[2] Avrachenkov, K., Habachi, O., Piunovskiy, A. and Zhang, Y. (2015). Infinite horizon optimal impulsive control with applications to Internet congestion control. \textit{Intern. J. of Control} \textbf{88}, 703–716.

[3] Bertsekas, D. and Shreve, S. (1978). \textit{Stochastic Optimal Control}. Academic Press, New York.
[4] Bogachev, V.I. (2007). *Measure Theory (Volumes 1 and 2)*. Springer-Verlag, Berlin.

[5] Bremaud, P. (1981). *Point Processes and Queues*. Springer-Verlag, New York.

[6] Clays, M., Arzelier, D., Henrion, D. and Lasserre, J. (2014). Measures and LMIs for impulsive nonlinear optimal control. *IEEE Trans. on Automatic Control*, 59, 1374–1379.

[7] Costa, O. and Dufour, F. (2013). *Continuous Average Control of Piecewise Deterministic Markov Processes*. Springer Briefs in Mathematics.

[8] Davis, M.H.A. (1993). *Markov Models and Optimization*. Chapman and Hall / CRC, Boca Raton.

[9] Dufour, F., Horiguchi, M. and Piunovskiy, A. (2012). The expected total cost criterion for Markov decision processes under constraints: a convex analytic approach. *Adv. Appl. Probab.* 44, 774–793.

[10] Dynkin, E. and Yushkevich, A. (1979). *Controlled Markov Processes*. Springer, Berlin.

[11] Ethier, S. and Kurtz, T. (1986). *Markov Processes*. Wiley, New York.

[12] Hernández-Lerma, O. and Lasserre, J. (1996). *Discrete-Time Markov Control Processes*, Springer-Verlag, New York.

[13] Hou, S.H. and Wong, K.H. (2011). Optimal impulsive control problem with application to human immunodeficiency virus treatment, *J. Optim. Theory Appl.* 151, 385–401.

[14] Leander, R., Lenhart, S. and Protopopescu, V. (2015). Optimal control of continuous systems with impulse controls, *Optim. Control Appl. Meth.* 36, 535–549.

[15] Menaldi, J. L. and Robin, M. (2018). On some ergodic impulse control problems with constraint. *SIAM J. Control Optim.* 56, 2690-2711.

[16] Piunovskiy, A. (1997). *Optimal Control of Random Sequences in Problems with Constraints*, Kluwer, Dordrecht.

[17] Piunovskiy, A., Plakhov, A., Torres, D. and Zhang, Y. (2019). Optimal impulse control of dynamical systems. *SIAM J. Control Optim.* 57, 2720–2752.

[18] Rockafellar, R.T. (1974). *Conjugate Duality and Optimization*. SIAM, Philadelphia.

[19] Schäl, M. (1975). On dynamic programming: compactness of the space of policies. *Stoch. Proc. Appl.* 3, 345–364.

[20] Taylor, M.E. (2006). *Measure Theory and Integration*. American Mathematical Society, Providence.