Local indices within a mathematical framework for urban water distribution systems

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Abstract: In this paper, a set of local energetic indices for an urban water distribution system is proposed, together with a mathematical organization of the matter which is also able to incorporate many of the well known (global) indicators. In fact, arranged such incoming indices in vectors and drawing on some notions of linear algebra and vector calculus, they are placed at the starting point of a mathematical framework that seems never used before for engineering-hydraulic purposes. It is then told how such a study and local treatment of a water network can be very useful, accurate and rich in information, and can be considered as a natural and simple theoretical development of a common global vision generally pursued so far: in particular, it is shown how it is possible to recover from the drawn layout, through suitable formulas, some of the most known and mainly used energetic parameters in a very clear and effective way.

Subjects: Engineering Mathematics; Water Engineering; Hydraulic Engineering

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For an urban water supply system, many (global) indicators, that summarize in a single value important characteristics of the whole network, are today available (for instance, the so-called resilience indices). In this paper, an inedit mathematical framework presents some local network indices as the building bricks that go to make up some of the most known and widely used global indicators.
1. Introduction

In the last decades water supply systems have been extensively studied under many aspects and a widespread research feature, even today, is to introduce new indices that could somehow measure, or summarize in a concrete number, some important properties and characteristics of the whole distribution network. In our opinion, it would be very interesting and fruitful to associate to the study of networks from a global point of view, a local analysis which considers single nodes, neighborhoods of them, and which investigates in depth each point of the whole water system; this would also give a further impulse to contemporary research on the subject.

An additional motivation that supports the convenience of investigating local methods for water supply systems is represented by some recent new issues and new trends arisen in research in the last years: for example, the strong requirement to distrectualize, or even to sectorize, large-looped networks for many different reasons (see, e.g. (Di Nardo & Di Natale, 2011; Di Nardo, Di Natale, & Santonasraso, 2014; Di Nardo, Di Natale, & Santonastaso, 2012; Di Nardo, Di Natale, Santonastaso, Tzatchkov, & Alcocer-Yamanaka, 2015, 2014; Galdiero, De Paola, Fontana, Giugni, & Savic, 2016; Maiolo & Pantusa, 2015, 2016, 2018; Marques, Cunha, Savic, & Giustolisi, 2017)), is clamoring for the need to have effective and powerful tools that allow a local study both of different types of networks, both of networks that differ greatly within them.

It is obvious that global indices are often very far from a local interpretation of a real network: this happens simply because they are designed for other purposes, and the defects of adherence, sometimes very evident, with the local realities are expected and do not detract from their usefulness. In other words, global indices represent a sort of “mean value” of the characteristics of thousands of nodes, but it is possible that no one of them actually approaches such value.

Although global indicators are not meant for local use, sometimes, when are required parameters that interpret the system locally, technicians try to calculate the same indicators in a small portion of the system. Often however, such an idea proves difficult to apply because a small portion of a large network strongly interconnected (e.g. a quarter), in general, is very different from a small autonomous system (for example, water flows between two quarters can even reverse depending on the time). And moreover, also in small quarters such type of computation can give values that are not found in most of the nodes (for instance, it is sufficient just a steep slope for a considerably variation of pressures at a distance of few hundred meters, even in presence of regulators).

A local treatment (or better a local-global treatment) of water networks hence needs more tools and more powerful ones, than the simply local adaptation of a global index. So, a setting that allows to go in-depth into the local analysis, without losing anything, indeed incorporating as much as possible the known tools of global type, is in our opinion the beginning of an important challenge for the next years.

The main aim of this paper is, therefore, to provide some basic mathematical tools and a rigorous formal setting and organization that can serve as a starting point and allow, in the next future, a deep study and a productive investigation of local properties and local behavior of water distribution networks.

Let us now give some quick reference to some research topics similar to those addressed here for urban water networks, emphasizing how this paper can be framed in a much broader vein that relies on mathematical modeling. For example, many different indices have recently been introduced to assess the vulnerability of infrastructures (see (Agathokleous, Christodoulou, & Christodoulou, 2017; Gheisi, Forsyth, & Naser, 2016; Maiolo et al., 2019; Ostfeld, 2004; Pinto, Varum, Bentes, & Agarwal, 2010; Shuang, Zhang, & Yuan, 2014; Soldi, Candelieri, & Archetti, 2015) and the references therein), the sustainability of water resources (see (Böhringer & Jochem, 2007; Maiolo, Carini, Capano, Piro, & Abdul Aziz, 2017, Maiolo, Martirano, Morrone,
Pantusa, 2006; Maiolo & Pantusa, 2016, 2017; Mori & Christodoulou, 2011) and various types of hydropotable risk (see (Davis & Janke, 2011; Ezell, 2007; Loucks, 2002; Maiolo & Pantusa, 2015)) including those linked to terrorist attacks (see (Maiolo & Pantusa, 2018)) which should attract today much more attention in many countries. Moreover, interesting models also use a wide range of mathematical tools like fractals (see (Di Nardo, Di Natale, Giudicianni, Greco, & Santonasraso, 2017; Veltri, Veltri, & Maiolo, 1996) and the references therein), graph theory (see (Di Nardo & Di Natale, 2011; Di Nardo et al., 2014; Ferrari, Savic, & Becciu, 2014; Herrera, Abraham, & Stoianov, 1685–1699)) or others, and a special mention is certainly due to the various optimization techniques widely employed in the field (see, e.g. (Carini, Maiolo, Pantusa, Chiaravalloti, & Capano, 2017; D’Ambrosio, Lodi, Wiese, & Bragalli, 2015; Galdiero et al., 2016; Jung & Kim, 2018; Labadie, 2004; Maiolo & Pantusa, 2016; Marques et al., 2017; Vasan & Simonovic, 2010; Veltri, Maiolo, & Morosini, 1994; Zheng, Simpson, & Zecchin, 2011)) in relation to different issues as climatic changes (see (Cervarolo, Mendicino, & Senatore, 2012; Maiolo, Mendicino, Pantusa, & Senatore, 2017; Moss et al., 2010)), water heating and solar energy (see (Bahrami et al., 2015; Hoseinzadeh & Azadi, 2017; Hoseinzadeh, Moafi, Shirkhani, & Chamkha, 2019; Yousef Nezhad & Hoseinzadeh, 2017)), etc.

Being this paper halfway between an engineering and an applied mathematics article, we briefly explain the organization of the contents as is usually done in the latter.

Section 2 introduces the general setting and recalls some of the most used energetic indices of a urban water supply system; Section 3, instead, recalls some concepts of linear algebra and mathematical analysis used in the following to develop the set-up, as the Hadamard product for vectors and matrices, $L^p$-norms and relative spaces over $\mathbb{R}^n$, etc. Section 4 is the central part of the paper; local measures are introduced for each node of the network and collected in vectors (see in particular (24)), then it is built a mathematical framework which provides the tools to handle and operate with such high-dimensional vectors. Moreover, expressing and recovering through our machinery the well-known global indices, it is shown how such approach is easy and natural to use for both local and global analysis on water networks. Section 5 is devoted to give some simple numerical examples and, finally, Section 6 handles the conclusions.

Some final remarks on notations: $\rho$ and $\gamma = g \rho$ denote the density and the specific weight of water, respectively, instead various $h$ or $H$ are used to denote the head and will be better defined in loco or appropriate references will be given.

2. Energetic parameters and global resilience indices

This section introduces the working setting and gives motivations for a mathematical excursus in the next one and for the introduction of local indices in Section 4. In particular, in (5) and (7) below two similar definitions of the resilience index are recalled, which will then be rewritten in a new way in (28)–(31) of Section 4.

Let $r$ be the number of reservoirs of the considered drinking water supply system and $Q_k$, $H_k$ be the discharge and the head, respectively, of the $k$-th reservoir ($k = 1, \ldots, r$). The total available power $P_{\text{tot}}$ at the entrance in the water network can be represented as (see also (Todini, 2000), (Di Nardo et al., 2012))

$$P_{\text{tot}} = \gamma \sum_{k=1}^{r} Q_k H_k. \quad (1)$$

Let $n$ be the number of nodes (and, in general, of all output points) in the water distribution network. Since the solution for a given network is unique, the total available power $P_{\text{tot}}$ determines the output power at each node $i = 1, \ldots, n$ and the network energetic equation (or balance)
\[ P_{\text{tot}} = P_0 + P_E, \quad (2) \]

where \( P_0 \) is the amount of dissipated power in all the tracts of the system and \( P_E = \gamma \sum_{i=1}^{n} q_i h_i \) is the total power delivered to the output nodes \( i = 1, \ldots, n \) in terms of discharge \( q_i \) and head \( h_i \). By convenience, we also denote the delivered power at the \( i \)-th node by \( p_i \), hence \( p_i = \gamma q_i h_i \).

For each node \( i \) in the distribution network, \( p_i^* = \gamma q_i^* h_i^* \) denotes the minimal power that must be delivered to the \( i \)-th node, where \( q_i^* \) and \( h_i^* \) refer, respectively, to the related project discharge and head \( (i = 1, \ldots, n) \). Therefore, the global minimum output power of the distribution network, necessary to satisfy the project constraints at each node \( i = 1, \ldots, n \), can be computed as

\[ P_{E \text{ min}} = \gamma \sum_{i=1}^{n} q_i^* h_i^*. \quad (3) \]

Hence

\[ P_{D \text{ max}} = P_{\text{tot}} - P_{E \text{ min}} = \gamma \left( \sum_{k=1}^{r} Q_k H_k - \sum_{i=1}^{n} q_i^* h_i^* \right) \quad (4) \]

is the maximum dissipable power in order to satisfy the minimal needs of the nodes \( i = 1, \ldots, n \).

The ratio between the dissipated power and the maximum dissipable one, has an important and widespread use in many fields of engineering (in fact it often appears in relation to the general type of power in several different contexts). In this framework from such ratio descends the resilience index \( I_R \) of the water network, as defined and used in (Di Nardo & Di Natale, 2010, 2011; Di Nardo et al., 2014, 2015), i.e.

\[ I_R = 1 - \frac{P_D}{P_{D \text{ max}}} = \frac{\sum_{i=1}^{n} (q_i h_i - q_i^* h_i^*)}{\sum_{k=1}^{r} Q_k H_k - \sum_{i=1}^{n} q_i^* h_i^*}. \quad (5) \]

We notice that such an index contains a little difference from the original resilience index \( I_r \), introduced by Todini in (Todini, 2000): posing

\[ p_D = P_{\text{tot}} - \gamma \sum_{i=1}^{n} q_i^* h_i^*. \quad (6) \]

\( I_r \) is defined as

\[ I_r = 1 - \frac{p_D}{p_{D \text{ max}}} = \frac{\sum_{i=1}^{n} q_i^* (h_i - h_i^*)}{\sum_{k=1}^{r} Q_k H_k - \sum_{i=1}^{n} q_i^* h_i^*}. \quad (7) \]

**Remark 2.1.** Note that if we assume, as usual when there are no deficiencies, \( 0 \leq q_i^* \leq q_i \) and \( 0 \leq h_i^* \leq h_i \), then

\[ 0 \leq I_r \leq I_R \leq 1 \]

and in general \( I_r \neq I_R \). In this regard, we better clarify also for the future, that this article does not want to discuss or compare in any way the physical usefulness of indices defined by other authors because it primarily follows a mathematical project (cf., for instance, Example 4.2).

**Remark 2.2.** In case of a water network with the presence of \( p \) pumps, the total power expressed in (1) has to be modified as follows
\[ P_{\text{tot}} = \gamma \sum_{k=1}^{L} Q_k H_k + \sum_{j=1}^{P_j} P_j \]  

where \( P_j \) is the power added into the network by the \( j \)-th pump. In this paper, we deal only with water distribution systems without pumps, because in presence of pumps everything can be easily adapted just by replacing (1) with (8) everywhere (see also (Shin et al., 2018; Todini, 2000)). For instance, in this section, it affects (2), (4)–(7), and an example in the waited context will be given in Remark 4.3.

3. A quick overview of some linear algebra and vector calculus

In this section, some basic definitions and notations of linear algebra and mathematical analysis are recalled. Extensive references will be given during the discussion of the topics.

Let \( \tilde{x} = (x_1, x_2, \ldots, x_n) \), \( \tilde{y} = (y_1, y_2, \ldots, y_n) \) be any two elements of \( \mathbb{R}^n \); the Hadamard product \( \tilde{x} \circ \tilde{y} \) is the entrywise product of the two vectors, i.e.

\[ \tilde{x} \circ \tilde{y} := (x_1 \cdot y_1, x_2 \cdot y_2, \ldots, x_n \cdot y_n). \]

More in general, the Hadamard product is defined for matrices of the same size with elements in some ring; for more details the interested reader can see (Horn & Johnson, 2012).

For any real number \( p \geq 1 \) is defined a norm on \( \mathbb{R}^n \) denoted by \( \| \cdot \|_p \) and called the \( L^p \)-norm, or briefly the \( p \)-norm, in \( \mathbb{R}^n \) by posing

\[ \| \tilde{x} \|_p = \| (x_1, x_2, \ldots, x_n) \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \]  

for any \( n \)-tuple \( \tilde{x} = (x_1, x_2, \ldots, x_n) \) belonging to \( \mathbb{R}^n \).

An important special case of (9) is obtained for \( p = 1 \): the resulting norm, also known as the taxicab or the Manhattan norm, is given by the sum of the absolute value of the components of the vector, i.e.

\[ \| \tilde{x} \|_1 = \| (x_1, x_2, \ldots, x_n) \|_1 = \sum_{i=1}^{n} |x_i| \]

for all \( \tilde{x} \in \mathbb{R}^n \). As a remark, the \( L^1 \)-norm of the Hadamard product of \( \tilde{x} \) by \( \tilde{y} \), i.e. \( \| \tilde{x} \circ \tilde{y} \|_1 \), is different from the usual scalar product of \( \tilde{x} \) and \( \tilde{y} \) which is denoted by

\[ \tilde{x} \cdot \tilde{y} := \sum_{i=1}^{n} x_i \cdot y_i. \]  

In particular, the following relation

\[ |\tilde{x} \cdot \tilde{y}| \leq \| \tilde{x} \circ \tilde{y} \|_1 \]  

is true for all \( \tilde{x}, \tilde{y} \in \mathbb{R}^n \), where equality holds in (11) if and only if \( x_i \cdot y_i \geq 0 \) for every \( i = 1, \ldots, n \).

Another important special case of (9) is achieved when \( p = 2 \), for which it is recovered the ordinary Euclidean norm

\[ \| \tilde{x} \|_2 = \| (x_1, x_2, \ldots, x_n) \|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}. \]

A further norm, equally interesting for the applications in the next section, is the infinity norm or maximum norm defined by posing
for all $\mathbf{x} \in \mathbb{R}^n$. There exist many properties, relations, and a rich theory on the norms $L^p$ and on the so-called $L^p$ spaces, for which the reader is encouraged to consult the easily comprehensible references (Hunter & Nachtergaele, 2001; Rudin, 1987), or any of the several introductory texts existing on the subject. Here we just point out that, for any fixed $n$-tuple $\mathbf{x} \in \mathbb{R}^n$, it is simple to see as the following inequalities hold
\[
\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p, \quad \|\mathbf{x}\|_p \leq n^{1/p} \cdot \|\mathbf{x}\|_\infty
\]  \tag{13}
whenever $1 \leq p \leq q \leq \infty$, and
\[
\|\mathbf{x}\|_p \to \|\mathbf{x}\|_\infty \quad \text{if} \quad p \to \infty,
\]
i.e. the $p$-norm approaches the infinity norm as $p$ approaches $\infty$. In conclusion, this means that it is available a whole family of decreasing norms $\mathcal{N} = \{\|\cdot\|_p : p \in [1, \infty]\}$ which vary “continuously with $p$”. Moreover, as important remark, all the norms $\|\cdot\|_p$ in $\mathcal{N}$ (even, in case, those with non-integer $p$) lend themselves very well to be used in physics and engineering because they fully agree with physical equations and dimensional homogeneity (recall, in particular, the definition in (9)), as will be seen in some examples in the next section.

Finally, we also recall some easy but useful notations and definitions. For any $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ we pose
\[
\mathbf{x} \vee \mathbf{y} := (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}) \in \mathbb{R}^n
\]  \tag{14}
and
\[
\mathbf{x} \wedge \mathbf{y} := (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}) \in \mathbb{R}^n.
\]
Now the positive part of $\mathbf{x}$ is defined as the $n$-th tuple $\mathbf{x}^+ \in \mathbb{R}^n$ obtained by replacing the negative entries of $\mathbf{x}$ with zero, that is
\[
\mathbf{x}^+ := \mathbf{x} \vee (0, 0, \ldots, 0) = (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\}),
\]  \tag{15}
and, similarly, the negative part of $\mathbf{x}$ is defined as the $n$-tuple $\mathbf{x}^- \in \mathbb{R}^n$ given by
\[
\mathbf{x}^- := (-\mathbf{x}) \vee (0, 0, \ldots, 0) = (\max\{-x_1, 0\}, \max\{-x_2, 0\}, \ldots, \max\{-x_n, 0\})
\]
\[
= -(\min\{x_1, 0\}, \min\{x_2, 0\}, \ldots, \min\{x_n, 0\}) = -(-\mathbf{x} \wedge (0, 0, \ldots, 0)).
\]  \tag{16}
Since $\mathbf{x}^+$ and $\mathbf{x}^-$ have only non-negative entries, the vectors $\mathbf{x}$ and $(|x_1|, |x_2|, \ldots, |x_n|)$ can be decomposed, as usual in mathematics, respectively in
\[
\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^- \quad \text{and} \quad (|x_1|, |x_2|, \ldots, |x_n|) = \mathbf{x}^+ + \mathbf{x}^-.
\]  \tag{17}

4. Local indices for a water distribution network
Recalling the notations of Section 2, $h_i$ and $h^+_i$ can be decomposed for any $i = 1, \ldots, n$ as sums
\[
h_i = h_i^0 + z_i \quad \text{and} \quad h^+_i = h^+_i + z_i,
\]  \tag{18}
where $z_i$ is the elevation head, $h_i$ the pressure head and $h^+_i$ the design pressure head, i.e. the minimal pressure head request, at the node $i$, respectively. Hence, consequently,
\[
p_i = \gamma q_i(h_i + z_i) \quad \text{and} \quad p^+_i = \gamma q_i(h^+_i + z_i).
\]  \tag{19}
Now, for each node \( i = 1, \ldots, n \) (and, in general for every output point of the network, if there are any), two indices \( q_i^i \) and \( h_i^i \), called, respectively, the local discharge surplus index and the local pressure head surplus index at the node \( i \), are introduced and defined as

\[
q_i^i := \frac{q_i}{q_i^i} - 1 = \frac{q_i - q_i^i}{q_i^i}, \quad h_i^i := \frac{h_i}{h_i^i} - 1 = \frac{h_i - h_i^i}{h_i^i}.
\]

(20)

In the same way, it is natural to define the local piezometric head surplus index and the local power surplus index at the node \( i \) as

\[
h_i^s := \frac{h_i}{h_i^s} - 1 = \frac{h_i - h_i^s}{h_i^s}, \quad p_i^s := \frac{p_i}{p_i^s} - 1 = \frac{p_i h_i - q_i h_i^s}{q_i^s h_i^s},
\]

(21)

and moreover, it can be easily observed that the following relations hold

\[
j_i = h_i - h_i^s = h_i^s + \left(1 + \frac{q_i^s}{h_i^s}ight)^{-1},
\]

(22)

\[
j_i = q_i^s \cdot h_i^s + q_i^s + h_i^s
\]

(23)

for every \( i = 1, \ldots, n \).

The dimensionless indices \( q_i^i, h_i^i, h_i^s, p_i^s \) have clear meanings. For example, the higher \( p_i^s \) is, the more surplus of power is obtained at a node \( i \), and it is negative if and only if there is a deficiency of power at the node in question, so that the minimal needed power request \( \gamma q_i^s h_i^s \) is not fully delivered. As a further example, the index \( h_i^s \) is 1 when the pressure head surplus is equal to the minimum pressure need of the node (i.e. there is a 100% extra pressure head).

**Remark 4.1.** We want to notice a subtle difference among the local indices defined above: while \( h_i^s \) and \( q_i^i \) are independent from the choice of the zero height level (denoted, e.g. by \( z \) ), \( h_i^s \) and \( p_i^s \) depends on it. In particular, for \( h_i^s \) and \( h_i^s \), this follows the analogous difference existing between \( h_i \), \( h_i^s \) and \( h_i \), \( h_i^s \) on the other hand. However, in many uses of the last indices nothing changes from the choice of \( z \) (for instance, to evaluate the head difference between two nodes), and in several others the dependence of the results from \( z \) is negligible in real computations, sometimes by choosing a suitable value of \( z \) (for instance, the altitude of the lowest node in the network). In any case, we are not interested to discuss such kind of issues (although very interesting) in this paper and, in particular, we will not do it in relation to the indices recalled here but introduced by other authors.

All local indices defined in (20) and (21) are collected in suitable \( n \)-tuples

\[
\vec{q}_i^s = (q_1^i, \ldots, q_n^i), \quad \vec{h}_i^s = (h_1^i, h_2^i, \ldots, h_n^i), \quad \vec{p}_i^s = (p_1^i, p_2^i, \ldots, p_n^i),
\]

(24)

and are called the local discharge surplus vector, the local pressure head surplus vector, the local piezometric head surplus vector and the local power surplus vector of the network, respectively. In addition to the local surplus vectors defined in (24), which represent a central point of this paper, we moreover pose, by convenience,

\[
\vec{q}_i = (q_1, q_2, \ldots, q_n), \quad \vec{h}_i = (h_1^i, h_2^i, \ldots, h_n^i), \quad \vec{p}_i = (p_1^i, p_2^i, \ldots, p_n^i), \quad \vec{z} = (z_1, z_2, \ldots, z_n),
\]

(25)

and we can call \( \vec{q}_i \) the minimal discharge request vector (or the constraint discharge vector) of the network, and similarly for \( \vec{h}_i, \vec{p}_i, \vec{z} \).
It is important to note also the difference between (24) and (25): $\mathbf{q}^s$, $\mathbf{h}^s$, $\mathbf{h}^r$ and $\mathbf{p}^s$ are vectors of $\mathbb{R}^n$ (i.e. their components are dimensionless ratios), while the ones in (25) are tuples of physical quantities with their own dimensions. However, we extend in the obvious way also to such vectors the Hadamard product, the scalar product, the norm $\|\cdot\|_1$, the positive part, etc., previously defined in Section 3. For instance, using such tools, we can write

$$\mathbf{p}^r = \gamma \mathbf{q}^s \circ \mathbf{h}^r$$

and, moreover, we can state the identities (22) and (23) for all indices at the same time, as in the following

$$\mathbf{h}^r = \left( \left(1 + \frac{z_1^i}{h_1^i} \right)^{-1}, \ldots, \left(1 + \frac{z_n^i}{h_n^i} \right)^{-1} \right) \circ \mathbf{h}^r,$$  \hspace{1cm} (26)

$$\mathbf{p}^r = \mathbf{q}^s \circ \mathbf{h}^r + \mathbf{q}^s + \mathbf{h}^r.$$  \hspace{1cm} (27)

Equations (22), (23), (26) and (27) show the link between the defined indices, expressed in scalar or vector form, respectively; they also prove to be very useful formulas in explicit calculations.

The first two vectors in (24), depicting the ordered set of data $\{q^s_i : i = 1, \ldots, n\}$ and $\{h^s_i : i = 1, \ldots, n\}$, respectively, are in our opinion the most representative, useful and, at the same time, synthetic tools that can be attached to a water distribution network from the energetic point of view but not only. Moreover, because their local character, they carry a great amount of information on the whole network point by point, and they seem very adapt and effective for conducting both local (for example, around a given node or in a small region of the network) and global analyses on the network. In fact, many known global indices of the network can be retrieved starting from the local indices above and in particular from the vectors $\mathbf{q}^s$ and $\mathbf{h}^r$, as the following examples show.

**Example 4.2 (The resilience index)**. The resilience indices $I_R$ and $I_r$ are often read as the same index or are confused with each other, but this is not surprising looking at the Equations (5) and (7). Instead, in terms of the local surplus vectors introduced in (24), it is possible to express the resilience index $I_R$ as follows

$$I_R = \frac{\mathbf{p}^r \cdot \tilde{\mathbf{p}}}{\mathbf{q}^s \cdot \mathbf{h} - \mathbf{q}^s \cdot \mathbf{h}^r},$$  \hspace{1cm} (28)

and the $I_r$ index as

$$I_r = \frac{\mathbf{h}^r \cdot \mathbf{p}^r}{\mathbf{q}^s \cdot \mathbf{h} - \mathbf{q}^s \cdot \mathbf{h}^r}.$$  \hspace{1cm} (29)

Equations (28) and (29) are enlightening to interpret the meaning and the mutual differences between the two resilience indices $I_R$ and $I_r$, from several points of view, but, as previously said, this paper often adopts only the mathematical one. Through it, the former is the mean weighted value of the components of the minimal power request vector $\mathbf{p}^r$ with the local power surplus vector $\tilde{\mathbf{p}}$ as weights vector (normalized through the factor $P_{D,\text{max}}$), while the second is the mean value of the same vector $\mathbf{p}^r$ but with the local piezometric head surplus vector $\mathbf{h}^r$ as weight vector. Hence, $I_R$ is to $\tilde{\mathbf{p}}$ as $I_r$ is to $\mathbf{h}^r$, and if they are written as

$$\text{matrix}_{I_R} = \tilde{\mathbf{p}} \cdot \frac{\mathbf{q}^s \circ \mathbf{h}^r}{\mathbf{q}^s \cdot \mathbf{h} - \mathbf{q}^s \cdot \mathbf{h}^r}$$  \hspace{1cm} (30)

and

$$I_r = \mathbf{h}^r \cdot \frac{\mathbf{q}^s \circ \mathbf{h}^r}{\mathbf{q}^s \cdot \mathbf{h} - \mathbf{q}^s \cdot \mathbf{h}^r}.$$  \hspace{1cm} (31)
meaning and differences are perhaps even more evident as well as readability better.

Remark 4.3. In case of a system with pumps (see Remark 2.2) it is easy to adapt discussions and formulas of this section consequently. For instance, setting $\mathbf{P} = (P_1, P_2, \ldots, P_p)$ where $P_j$ is the power addition by the $j$-th pump (see (8)), then the resilience index $R_I$ can be written as (cf. (28) and (30))

$$R_I = \frac{\mathbf{P} \circ \mathbf{H}}{\mathbf{Q} \circ \mathbf{H} + \gamma^{-1} \mathbf{P} \circ \mathbf{I}}$$

where $\mathbf{I}$ denotes the vector $(1, 1, \ldots, 1) \in \mathbb{R}^p$.

Example 4.4 (Other resilience indices). In addition to the resilience indices $R_I$ and $I$, seen in Section 2, there are several other indices obtained by modifying or improving that of Todini (7), or based on completely different parameters.

(i) An example is provided by the modified resilience index (MrI)\(^2\) as called and defined by Jayaram and Srinivasan in (Jayaram & Srinivasan, 2008)

$$\text{matrixMrI} = \frac{\sum_{i=1}^{n} q_i^j (h^i_j - h^i_j)}{\sum_{i=1}^{n} q_i^h h^i_j} = \frac{P_{\text{tot}} - P_{\text{D}}}{P_{\text{tot}} \min} - 1 \quad (32)$$

(see also (6) and (3)). Using our framework, (32) can be written as follows

$$\text{matrixMrI} = \frac{\mathbf{H}^i \circ \mathbf{P}^j}{\mathbf{Q} \circ \mathbf{H}} = \frac{\mathbf{H}^i \circ \mathbf{H}^i}{\mathbf{Q} \circ \mathbf{H}}, \quad (33)$$

which well clarifies the central role of the vector $\mathbf{H}^i$ for the modified resilience index as for the original one (see (29) and (30)). In addition, the last formula in (33) is particularly suggestive and mathematically elegant, with the only difference between numerator and denominator of the fraction, consisting in two different products.

(ii) Another resilience index is introduced by Prasad and Park in (Prasad & Park, 2004). Considering the given network as an (algebraic) graph with $n$ vertices (or nodes) and $m$ edges (i.e. pipes connections between two different nodes), let $m(i)$ be the number of pipes outgoing from the $i$-th node, $i = 1, 2, \ldots, n$ (hence, obviously, $\sum_{i=1}^{n} m(i) = 2m - r^2$). Let $d_{ij}$ be the diameter of the $j$-th pipe outgoing from the $i$-th node, where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m(i)\}$. The network resilience index (NrI) is a reformulation of Todini’s index $I$, which wants to take into account the effects of surplus power and loop reliability; it is defined in (Prasad & Park, 2004) as

$$\text{NrI} := \frac{\sum_{i=1}^{n} C_i q_i^j (h^i_j - h^i_j)}{\sum_{k=1}^{n} Q_k h^i_k - \sum_{i=1}^{n} q_i^h h^i_j}, \quad (34)$$

where

$$C_i := \frac{\sum_{j=1}^{m(i)} d_{ij}}{m(i) \max\{d_{i1}, d_{i2}, \ldots, d_{im(i)}\}}$$

measures the “uniformity” of pipe diameters at the node $i$. To employ our mathematical framework to the network resilience index, we pose for all $i = 1, 2, \ldots, n$

$$\bar{d}_i := (d_{i1}, d_{i2}, \ldots, d_{im(i)}) \in \mathbb{R}^{m(i)} \quad (35)$$

and
\[ \bar{C} := \left( \frac{\| \bar{d}_1 \|_1 / \| \bar{d}_1 \|_{\infty}}{m(1)}, \frac{\| \bar{d}_2 \|_1 / \| \bar{d}_2 \|_{\infty}}{m(2)}, \ldots, \frac{\| \bar{d}_n \|_1 / \| \bar{d}_n \|_{\infty}}{m(n)} \right) \in \mathbb{R}^n. \] (36)

The \( i \)-th component of the vector \( \bar{C} \) can be read as the ratio between the two extreme \( p \)-norms, the greatest and the lowest for \( p \in [1, \infty] \) (see (13)), of the vector \( \bar{d}_i \), divided by the number of pipes connecting the node \( i \). Therefore, in conclusion we can write the network resilience index as

\[ \text{NrI} = \left( \frac{\bar{C} \circ \bar{h}}{\bar{p} \circ \bar{q}} \right) \circ \frac{\bar{p} \circ \bar{h}}{\bar{q} \circ \bar{h}}. \] (37)

Comparing (37) and (31), it is natural to interpret the network resilience index as obtained from \( I_r \) through a (suitable Hadamard) left product by the vector \( \bar{C} \).

**Example 4.5 (MRI and NRI).** This example is also intended to suggest how a rich and structured mathematical framework can act, in our opinion, as a catalyst for new ideas and a systematic development of many subjects in engineering.

The MrI index by Jayaram and Srinivasan and the NrI index by Prasad and Parkin are recalled in (32) and (34), respectively, as originally formulated by their authors. Then, they are reintroduced through our framework in a different way (see (33) and (37)) which emphasizes and better clarifies their inner structure so that, from a simple comparison of (33), (37), (31) and (30), it arises naturally the introduction and the study of two alternative indices,

\[ \text{MRI} = \bar{p}^{\circ} \circ \frac{\bar{q} \circ \bar{h}}{\bar{q} \circ \bar{q}} \] (38)

and

\[ \text{NRI} = \left( \frac{\bar{C} \circ \bar{p}}{\bar{q} \circ \bar{h}} \right) \circ \frac{\bar{p} \circ \bar{h}}{\bar{q} \circ \bar{q} \circ \bar{h}}. \] (39)

which implement the resilience index \( I_R \) as MrI and NrI do with \( I_r \).

This is a simple but clear instance of the mathematical systematicity mentioned at the beginning of this example, which, inter alia, is a powerful and widespread tool in science for not leaving unexplored potentially fruitful research fields.

As regards instead the indices (38) and (39), in our opinion it is interesting (but obviously not possible here) a thorough investigation and probably an experimental verification of the real performances related to the indices MRI and NRI, especially when compared with the well-known ones, MrI and NrI.

**Example 4.6 (Diameters, means and weighted means).** Since many hydraulic indices, likeNrI and NRI just seen above, involve the diameters of pipes, we take the opportunity to write some averages using the language of Section 3. The purpose of this elementary example is to illustrate how simple and natural it is to apply such a language, each when the case arises, on any set of measures or data.

Let \( n, m, m(i) \) and \( d_{ij} \) be as in Example 4.4 (ii) and denote by \( l_{ij} \) the length of the pipe with diameter \( d_{ij} \), i.e. the length of the \( j \)-th pipe outgoing from the \( i \)-th node, where \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m(i)\} \). In analogy with (35), we pose

\[ \bar{l}_i := (l_{i1}, l_{i2}, \ldots, l_{im(i)}) \in \mathbb{R}^{m(i)}. \]
for all \( i = 1, 2, \ldots, n \). Simultaneously with the local order of the pipes focused on the nodes, we suppose given, by convenience, a global order of the set of all pipes of the network \( E = \{ p_j \mid j = 1, \ldots, m \} \) (or edges of the graph, in the algebraic language). Moreover, for any \( j = 1, \ldots, m \) let \( D_j \) and \( L_j \) be the diameter and the length of the \( j \)-th pipe \( p_j \), respectively, and let us collect these data in the following two vectors

\[
\vec{D} := (D_1, D_2, \ldots, D_m) \quad \text{and} \quad \vec{L} := (L_1, L_2, \ldots, L_m).
\]

Then, the (global) mean diameter \( D_{\text{Mean}} \) and the (global) weighted mean diameter \( D_{\text{WMean}} \) of the network can be written as

\[
D_{\text{Mean}} = \frac{\| \vec{D} \|_1}{m} = \frac{\sum_{j=1}^{m} D_j}{m} \quad \text{and} \quad D_{\text{WMean}} = \frac{\| \vec{D} \circ \vec{L} \|_1}{\| \vec{L} \|_1} = \frac{\sum_{j=1}^{m} D_j L_j}{\sum_{j=1}^{m} L_j},
\]

respectively. On the other hand, as regard local means, the (local) mean diameter \( d_{\text{Mean}}^i \) and the (local) weighted mean diameter \( d_{\text{WMean}}^i \) relative to the \( i \)-th node of the network can be expressed as

\[
d_{\text{Mean}}^i = \frac{\| \vec{d}_i \|_1}{m(i)} = \frac{\sum_{j=1}^{m(i)} d_{ij}}{m(i)} \quad \text{and} \quad d_{\text{WMean}}^i = \frac{\| \vec{d}_i \circ \vec{l}_i \|_1}{\| \vec{l}_i \|_1} = \frac{\sum_{j=1}^{m(i)} d_{ij} l_{ij}}{\sum_{j=1}^{m(i)} l_{ij}},
\]

for all \( i = 1, 2, \ldots, n \).

It is well known and often witnessed by numerous historical scientific events of the last centuries, that a rich and well-structured mathematical framework is like a fertile ground for the achievement of application results and for the development of scientific theories in general. For instance, even in such a simple setting as that of the present example, the equations in (42), written in this way, seem to suggest to consider a second mean of the overall the nodes of the network. More precisely, we introduce two values \( d_{\text{AV}} \) and \( d_{\text{AVW}} \), called the average of the local mean diameters and the average of the local weighted mean diameters, respectively, and given by

\[
d_{\text{AV}} = \frac{\| (d_{\text{1,Mean}}, \ldots, d_{n,\text{Mean}}) \|_1}{n} = \frac{\sum_{i=1}^{n} d_{\text{Mean}}^i}{n},
\]

and

\[
d_{\text{AVW}} = \frac{\| (d_{\text{1,\text{WMean}}, \ldots, d_{n,\text{\text{WMean}}} \|_1}{n} = \frac{\sum_{i=1}^{n} d_{\text{WMean}}^i}{n}.
\]

If at first glance it may seem to have recovered the means already computed in (41), we point out that they are almost always quite different from each other (it can be easily seen by some trivial computation comparing (41), (43) and (44), or see also (??)). These differences between the two pairs, if properly elaborated, could probably give rise to some index concerning the homogeneity of the network.

**Example 4.7 (The failure index).** As further example it can be considered the failure index \( I_f \) presented in (Todini, 2000) as

\[
I_f = \frac{\sum_{i=1}^{n} I_{f_i}}{\sum_{i=1}^{n} q_i h_i^*}, \quad \text{where} \quad I_{f_i} = \begin{cases} 0 & \text{if } h_i \geq h_i^* \\ h_i^* (h_i^* - h_i) & \text{if } h_i < h_i^* \end{cases}
\]
Using the proposed mathematical framework and the previously defined local indices, we obtain

\[ I_f = \frac{\left\| \left[ \frac{1}{\Delta} - I \right] \right\|_{1}}{\| \Delta \|_{1}}, \]

\[ = \frac{\left\| \max \left\{ -h_i^* p_i ; 0 \right\}, ... \max \left\{ -h_i^* p_i , 0 \right\} \right\|_{1}}{\| \Delta \|_{1}}, \]

\[ = \frac{\sum_{i=1}^{n} \max \left\{ -h_i^* p_i , 0 \right\}}{\sum_{i=1}^{n} p_i^*}, \tag{45} \]

where, in the last equality, we used \( p_i^* \geq 0 \) for all \( i \).

**Example 4.8 (Pressure indices).** The mean pressure surplus (MPS) and the mean pressure deficit (MPD) of a network, as used and defined for example in (Di Nardo et al., 2015), are given by

\[ \text{MPS} = \frac{\sum_{i=1}^{n} \alpha_i q_i^*}{\sum_{i=1}^{n} q_i}, \text{ where } \alpha_i = \begin{cases} 0 & \text{if } h_i \leq h_i^* \\ h_i - h_i^* & \text{if } h_i > h_i^* \end{cases} \]

and

\[ \text{MPD} = \frac{\sum_{i=1}^{n} \beta_i q_i^*}{\sum_{i=1}^{n} q_i}, \text{ where } \beta_i = \begin{cases} 0 & \text{if } h_i \geq h_i^* \\ h_i^* - h_i & \text{if } h_i < h_i^* \end{cases} \]

(see (18) for the definitions of \( h_i \) and \( h_i^* \)). In the suggested setting they can be easily written in a very effective way as follows

\[ \text{MPS} = + \frac{\left\| \left[ (G - \Delta) + q^* \right] \right\|_{1}}{\| q^* \|_{1}}, \text{ where } \sum_{i=1}^{n} \max \left\{ h_i - h_i^* ; 0 \right\} q_i^* 

\[ = \frac{\sum_{i=1}^{n} \max \left\{ h_i - h_i^* ; 0 \right\} q_i^*}{\sum_{i=1}^{n} q_i}, \tag{46} \]

and

\[ \text{MPD} = - \frac{\left\| \left[ (G - \Delta) - q^* \right] \right\|_{1}}{\| q^* \|_{1}}, \text{ where } \sum_{i=1}^{n} \max \left\{ h_i - h_i^* , 0 \right\} q_i^* 

\[ = \frac{\sum_{i=1}^{n} \max \left\{ h_i - h_i^* , 0 \right\} q_i^*}{\sum_{i=1}^{n} q_i}, \tag{47} \]

where we assumed \( q_i^* \geq 0 \ \forall i = 1, \ldots, n \) in both equations and we posed \( \mathbf{h} = (h_1, h_2, \ldots, h_n) \).

To evaluate the absolute discrepancy between the pressure head and the design pressure, it is often used the *mean squared deviation from the design pressure* (MSDP) which can be written as follows (see (Di Nardo et al., 2015) or also (Di Nardo & Di Natale, 2010, 2011))

\[ \text{MSDP} = \frac{\| \mathbf{h} - \mathbf{h}^* \|_2}{n^{1/2}} = \frac{\sqrt{\sum_{i=1}^{n} (h_i - h_i^*)^2}}{n}. \]

Various other indices exist for pressure, but it is preferable not to exceed by adding others. It can just be noted that the maximum node pressure \( h_M \) and the mean node pressure \( h_m \) are trivially written as

\[ h_M = \| \mathbf{h} \|_{\infty} \quad \text{and} \quad h_m = \| \mathbf{h} \|_1 \]

respectively, and the standard deviation of node pressure \( \text{hSD} \) by
\[ h_{SD} = \frac{\| h - h_m \cdot 1_n \|_2}{n^{1/2}} = \sqrt{\frac{\sum_{i=1}^{n} (h_i - h_m)^2}{n}}, \]

where \( 1_n \) denotes the vector \((1, 1, \ldots, 1) \in \mathbb{R}^n\).

**Example 4.9 (Flow deficit index).** As last example we consider the flow deficit index and, in particular, among the many possible definitions existing in literature, we refer to the ones in (Di Nardo et al., 2014, 2014; Di Nardo, Di Natale, Santonastaso, & Venticinque, 2013). For any \( i = 1, \ldots, n \) let \( q_{a,i} \) be the actual flow demand delivered at the node \( i \) (see (Di Nardo et al., 2013) for details) and, keeping the notations used before, the flow deficit index \( I_{fd} \) is defined as follows

\[ I_{fd} = \frac{\sum_{i=1}^{n} \alpha_i q_i^*}{\sum_{i=1}^{n} q_i^*}, \quad \text{where} \quad \alpha_i = \begin{cases} 1 & \text{if } q_{a,i} > q_i^* \\ \frac{q_{a,i}}{q_i^*} & \text{if } 0 \leq q_{a,i} \leq q_i^* \\ 0 & \text{if } q_{a,i} < 0 \end{cases}. \quad (48) \]

Now, setting \( \bar{q}_a := (q_{a,1}, \ldots, q_{a,n}) \), then the flow deficit index can be written through the following expression

\[ I_{fd} = \frac{\| \bar{q}_a \wedge \bar{q} \|_1}{\| \bar{q} \|_1} = \frac{\sum_{i=1}^{n} \min\{q_{a,i}, q_i^*\}}{\sum_{i=1}^{n} q_i^*}, \]

which, in addition to being placed in a mathematical framework with numerous potentialities, is also expressed in a much clearer and more manageable form than (48).

In literature there exists a multitude of indicators similar or related with the previous ones and, in addition, also a wide number of other different indices derived from the energy, mechanics, topology and the so-called “entropy” of the water network; many of them can be revisited and implemented as well, by adopting the presented framework and a local approach as has been done with the vectors introduced in (24).

**5. Numerical examples on a simple network**

A further important advantage of the setting given in the previous sections is of a computational nature, and is well represented by the possibility of using the described mathematical tools and the suitable elementary “building bricks”, to easily implement new (or old) indices and hydraulic functions within computational software based on linear algebra (in primis, for example, MATLAB), on symbolic calculus (e.g. Wolfram Mathematica), or other software of a strictly hydraulic nature like EPANET. But before a thorough investigation of this kind, a broad study and application in tangible cases of what has been theorized in this paper can be very useful and advantageous.

In this section, it will be presented some first numerical examples and explicit calculations that directly apply much of what seen so far. Since real water networks produce very large vectors, it is convenient, indeed it is better, to consider small networks to focus attention on the mathematical formalization.

A useful network for this purpose could, therefore, be the one originally proposed by Alperovits and Shamir in (Alperovits & Shamir, 1977) which, although very simple, is very widespread in literature examples usually under the name of “two-loop network”. It is schematically represented in Figure 1 and here, according to the original data (see, e.g. (Alperovits & Shamir, 1977; Todini, 2000) until the recent (Dini & Tabesh, 2016)), many and full details will be given of what theoretically exposed in Section 4 for general supply systems. Each pipe of the network is 1000 m long and the nodal discharge \( q_i \) is taken to be equal to the respective project discharge \( q_i^* \), for all
Using the notations of Section 4, the altimetric data, the mentioned flow rates and the minimum pressure at the nodes can be written as

\[ \tilde{z} = (150\text{m}, 160\text{m}, 155\text{m}, 150\text{m}, 165\text{m}, 160\text{m}) \]

\[ \tilde{q} = \bar{q} = (100\text{m}^3/\text{h}, 100\text{m}^3/\text{h}, 120\text{m}^3/\text{h}, 270\text{m}^3/\text{h}, 330\text{m}^3/\text{h}, 200\text{m}^3/\text{h}) \]

\[ \bar{h}^* = (30\text{m}, 30\text{m}, 30\text{m}, 30\text{m}, 30\text{m}) \]

\[ \bar{h}^* = (1.1, 1.2, 2.7, 3.3, 2) \cdot 10^2 \text{m}^3/\text{h}, \]

\[ \bar{h}^* = (30\text{m}, 30\text{m}, 30\text{m}, 30\text{m}, 30\text{m}) \]

\[ \bar{h}^* = (3, 3, 3, 3, 3) \cdot 10\text{m}, \]

while for the reservoir \( Q = 1120\text{m}^3/\text{h} \) and \( H = 210\text{m} \) coincides with the geodetic head. From (49) it immediately follows

\[ \bar{h}^* = (180\text{m}, 190\text{m}, 185\text{m}, 180\text{m}, 195\text{m}, 190\text{m}) \]

\[ \bar{h}^* = (18, 19, 18.5, 18, 19.5, 19) \cdot 10\text{m}, \]

\[ \bar{h}^* = (18, 19, 22.2, 48.6, 64.35, 38) \cdot 10^3 \gamma \text{m}^4/\text{h}, \]

where \( \gamma \) is the specific weight of water as recalled in the Introduction. Now, by using the Hazen-Williams formula with a roughness coefficient equal to 130 and the software EPANET 2.0.12 with the following diameters vector expressed in inches

\[ \bar{D} = (D_1, \ldots, D_8) = (18, 10, 16, 4, 16, 10, 10, 1) \]

(see (40) and the “optimal cost solution” (Todini, 2000 in table 3), the pressure results are

\[ \bar{h} = (53.25 \text{m}, 30.46 \text{m}, 43.45 \text{m}, 33.8 \text{m}, 30.44 \text{m}, 30.55 \text{m}) \]

and consequently

\[ \bar{h} = (203.25 \text{m}, 190.46 \text{m}, 198.45 \text{m}, 183.8 \text{m}, 195.44 \text{m}, 190.55 \text{m}). \]

Therefore, through easy computations, the local surplus vectors defined in (24) are
\( \bar{q}_i = (0.0, 0.0, 0.0, 0.0) \),
\( \bar{b}_i \approx (0.775, 0.0153, 0.4483, 0.1266, 0.0146, 0.0183) \),
\( \tilde{h}_i \approx (0.1292, 0.0024, 0.0727, 0.0211, 0.0023, 0.0029) \),
\( \tilde{\rho}_i = (\text{since } \bar{q}_i = 0) = \tilde{h}_i \).

(54)

As regards the resilience index, the current assumption, \( q_i = q_i \forall i \), causes the coincidence of the two values \( I_q \) and \( I_r \). Equations (28) and (29) can be in fact written in the same way as follows

\[
I_q = I_r = \frac{\tilde{h}_i \cdot \tilde{\rho}_i}{\gamma (\bar{Q} \cdot \bar{H} - q_i \cdot \tilde{h}_i)}
\]

\[
= \frac{(0.1292, 0.0024, 0.0727, 0.0211, 0.0023, 0.0029) \cdot (18.19, 22.2, \ldots, 10^3 m^3/h})}{\gamma (1120 m^3/h) \cdot 210 m - (1.1, 1.2, \ldots, 10^2 m^3/h) \cdot (18.19, 18.5, 19.5, 19.9) 10m}
\]

\[
= \frac{5268.805}{25050} \approx 0.21033^6.
\]

(55)

The same happens for the modified resilience indices MrI and MRI (see (33) and (38)),

\[
\text{MRI} = \text{MrI} = \frac{\tilde{h}_i \cdot \tilde{\rho}_i}{\gamma (\bar{Q} \cdot \bar{H} - q_i \cdot \tilde{h}_i)}
\]

\[
= \frac{(0.1292, 0.0024, 0.0727, 0.0211, 0.0023, 0.0029) \cdot (18.19, 22.2, \ldots, 10^3 m^3/h})}{\gamma (1120 m^3/h) \cdot 210 m - (1.1, 1.2, \ldots, 10^2 m^3/h) \cdot (18.19, 18.5, 19.5, 19.9) 10m}
\]

\[
= \frac{5268.805}{210150} \approx 0.02507
\]

and for the network resilience indices NrI and NRI (see (37) and (39), respectively), whose computation requires first to determine (using inches for diameters)

\[
\bar{d}_1 = (18.10, 16), \quad \bar{d}_2 = (10.10), \quad \bar{d}_3 = (16.4, 16),
\]
\[
\bar{d}_4 = (4.10, 1.1), \quad \bar{d}_5 = (16.10), \quad \bar{d}_6 = (10.1).
\]

\[
\bar{c} = \left( \frac{14/18}{12}, \frac{20/10}{8}, \frac{36/16}{2}, \frac{15/10}{2}, \frac{26/16}{3}, \frac{11/10}{2} \right)
\]

then

\[
\text{NRI} = \text{NrI} = \frac{\left( \tilde{c} \cdot \tilde{h}_i \right) \cdot \tilde{\rho}_i}{\gamma (\bar{Q} \cdot \bar{H} - q_i \cdot \tilde{h}_i)}
\]

\[
= \frac{(\frac{14}{18}, \frac{20}{10}, \frac{36}{16}, \frac{15}{10}, \frac{26}{16}, \frac{11}{10}) \cdot (0.1292, 0.0024, 0.0727, 0.0211, 0.0023, 0.0029) \cdot (18.19, 22.2, \ldots, 10^3 m^3/h})}{\gamma (1.1, 1.2, \ldots, 10^2 m^3/h) \cdot (18.19, 18.5, 19.5, 19.9) 10m}
\]

\[
= \frac{23350.754}{25050} \approx 0.93217
\]

Although all the pipes have the same length and, therefore, the vector \( \bar{L} \) in (40) is rather trivial and does not affect anything, the mean diameters in (41) and (43) are different (see the discussion after Eq. (44)):

\[
D_{\text{Mean}} = 10.625 \text{ in} \neq d_{\text{Av}} \approx 9.556 \text{ in}.
\]

(56)

The failure index and the mean pressure deficit (see Eq. (45) and (47)) are trivially zero and the flow deficit index (see Eq. (48)) is 1. For the mean pressure surplus and the other measures in Example 4.7, an immediate computation gives

\[
\text{MPS} = \left\| (\tilde{h} - \tilde{h}_i) \cdot \bar{q} \right\| \approx 4.702 \text{ m}, \quad \text{MSDP} = \left\| \tilde{h} - \tilde{h}_i \right\| \approx 11.0801 \text{ m}.
\]

\[
h_m = \left\| \frac{\bar{c}}{n} \right\| \approx 36.992 \text{ m}, \quad \text{and} \quad h_{3D} = \left\| \frac{\bar{c} - h_m}{n} \right\| \approx 7.6676 \text{ m}.
\]
Obviously, changing some of the data in (49) or (51), a whole set of different results will be obtained. For example, choosing as diameters those proposed for the “solution A” in (Todini, 2000), i.e. \( \bar{D} = (18, 16, 14, 6, 14, 1, 14, 10) \), then

\[
\begin{align*}
\bar{h}^s &= (0.775, 0.3397, 0.446, 0.5397, 0.033, 0.045), \\
\bar{p}^s &= \bar{h}^s = (0.1292, 0.0536, 0.0723, 0.0899, 0.0051, 0.0071), \\
I_R &= I_r = \frac{h^s \cdot \bar{p}^s}{\gamma (Q \cdot \bar{H} - \bar{q} \cdot \bar{h})} \\
&= \frac{0.1292, 0.0536, 0.0723, 0.0899, 0.0051, 0.0071 \cdot (18, 19.22.2, 48.6.64.35.38) \cdot 10^3 m^4/h}{1120(m^3/h) \cdot 210m - (1.1,1.2,2.7,3.3,2)10^3(m^3/h) \cdot (18,19.18.5,18.19.5,19)10m} \\
&= \frac{9916.185}{25050} \approx 0.39586,
\end{align*}
\]

and so on.

The last few lines of the section are dedicated to a further example of a graphic nature obtained by considering the Kang and Lansey network (see (Kang & Lansey, 2012)). It is a network with 935 nodes, 1274 pipelines and a single source (reservoir) to which a lifting system is associated. The average total demand is 177l/s, but in peak conditions, the network will have a total consumption of about 310l/s with a minimum allowable constant pressure of 28m (40psi). In Figure 2 it is shown the local pressure head surplus index \( h^s_i \).

After the computation of the index \( h^s_i \) for each node of the network, two extreme values were excluded and the remaining range was divided into three sections. The related nodes have been colored green for the segment corresponding to the highest values, yellow and red.

Starting from this type of data, some projects that aim to use local indices and combinations of them within objective functions are also in progress. From the first evidences, in particular, they play a clear indispensable role above all in the case in which the minimum requests are not constant but vary from node to node or from area to area within the network.
6. Conclusions

In this paper, a series of indices, computed locally for each node of the network and arranged in vectors, are introduced as work tools of local-global analysis on the water system. The most important are those defined in (20) and (21), which are proposed as the main and very powerful descriptors of a water network from the energy point of view. But to use effectively or just to manage any set of local indicators organized in vectors of very large dimension, there is a need of some mathematical tools like the ones exposed in Section 3 in theoretical terms, as a part of a language to apply to the engineering of water distribution systems. The resulting mathematical framework gives a double opportunity: to be able to conduct both a detailed local analysis on the network, and to retrieve, at the same time and in a handy and particularly effective way, some well-known global indices even viewed from a broad perspective. In addition to this, another important benefit is to have a base setting that allows to use in a hydraulic-engineering context various easily applicable mathematical results and properties (the reader considers, for instance, the many inequalities that emerge from the norms and the $L^p$-spaces).

As examples of the described proposal, we revisited through this lens various resilience indices, the failure index, some others concerning the pressure surplus, deficit, deviation, etc., and lastly the flow deficit index. It is preferable, instead, to mention Example 4.5 separately: there an alternative version of the MRI index by Jayaram and Srinivasan and of the NRI index by Prasad and Parkin is theoretically deduced as a further working example of the proposed framework.

Finally, numerical computations have also been displayed in detail in the case of very simple networks.

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Notes
1. The notation $\sum p$ should not be confused with $\sum p$, the usual symbol for the identity $p \times p$ matrix.
2. Such index is usually written [MRI], but we prefer a slightly different notation for reasons that will be clear in Example 4.5. The same applies to the NRI index discussed in part (ii) of this example.
3. Bringing to the left-hand side, it is known in graph theory as the degree sum formula.
4. Note the strong explicative character of the proposed tools and in particular, the symmetry between the middle term of (46) and (47): two “+” signs for the surplus of the mean pressure become, in a natural way, two “−” signs for the deficit.
5. $\gamma = \rho \cdot g = 10^3 \left( \text{kg/m}^3 \right) \cdot 9.81 \text{m/s}^2 = 9810 \text{N/m}^3$.
6. The small divergence with the value 0.22 obtained in [Todini, 2000] in table 3 is probably attributable, beyond the made approximations, to the differences in the software.
7. It is important to observe as such difference is not (only) due to the special pipe 1 connecting node 1 to the reservoir, but to the irregularity of the algebraic underlying graph. In fact, removing pipe 1 from the network, the result would be $D_{\text{Mean}} \approx 9.571$ in $\alpha = 4.056 \approx 9.278$ in.

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