Chasles’ mystic tetrahedron

A L Kheyfets (Kheyfets)
Department of Engineering and Computer Graphics, South Ural State University, 76, Lenin Avenue, Chelyabinsk, 454080, Russian Federation
E-mail: heifets@yandex.ru

Abstract. The results of the experimental study and testing of the theorem by M. Chasles are given. The theorem shows the peculiarities of the intersection of an arbitrary tetrahedron with an arbitrary quadratic surface (second-order surface). The multiple variants of the intersection stipulated in the theorem have been considered. It has been experimentally proven that for the majority of these variants the four intersecting lines constructed according to the algorithm of this theorem belong to the surface of a single one-sheeted hyperboloid. For two of the variants, the results that differ from the theorem have been received. The experiments implied building and studying 3D computer models obtained using the AutoCAD and SolidWorks packages. All the variants of the quadratic surface at different mutual position with respect to the tetrahedron have been considered. The methods of holding the experiments have been considered in detail. The proof of the theorem for one of its variants has been obtained and presented.

1. Introduction
Chasles’ theorem [1] reveals the theoretical peculiarities of intersection of polyhedrons with second-order surfaces (Quadric). Chasles did not give the proof of his theorem, but he rather demonstrated its multiple manifestations, which still amaze and arouse admiration even today. Chasles drew an analogy between his theorem and Pascal’s theorem [2], which was also not proven by its author, thus conjuring mystical feelings about it among the contemporaries [3] (“Pascal’s mystic hexagon”). Chasles’ theorem causes the same feeling today, that is why we have suggested a term of “Chasles’ mystic tetrahedron”.

The principal wording of Chasles’ theorem is as follows. Let us consider an arbitrary tetrahedron and an arbitrary quadratic surface. All the tetrahedron edges cross the quadric surface, each of them – in two points. For each of the tetrahedron apexes, let us choose the nearest three points located on the edges converging in this apex. Let us take them as the vertices of a triangle. We draw a line along which this triangle intersects with the face of the tetrahedron opposite the vertex. Let us repeat this construction for each of the four tetrahedron apexes. The theorem states that the obtained four straight lines are the guide lines of a certain one-sheeted hyperboloid (Hyper).

It is known that Hyper is determined by three skew lines [4-7]. By determining four lines belonging to one Hyper, Chasles’ theorem reveals a previously unknown interrelation between the intersection of the tetrahedron with the quadratic surface and one-sheeted hyperboloid. As it will be shown below, this interrelation can be explained by the manifestation of Pascal’s theorem. We believe that in the revealing of these interrelations lies the scientific novelty of Chasles’ theorem.

There exist very few works and research studies on the topic of Chasles’ theorem. This could probably be explained by the complexity in the theorem’s apprehension since its author did not
provide any illustrations and gave only a reference [8]. Over the past years a single article has been
published [9], along with a brief report in work [10]. This entitles us to speak of Chasles’ theorem as
of an unjustifiably neglected one. The relevance of the work on studying Chasles’ theorem is related to
its scientific novelty, development of the methods of computer 3D geometrical modeling, and the use
of polyhedrons and quadratic surfaces in architectural design.

Work [9] gives the proof of Chasles’ theorem for an ellipsoid. Probably, based on projective
transformations, it could also be translated to the rest of the Quadric. However, such an approach
makes the understanding of the theorem more difficult.

Our work’s objective is to create and study the algorithms of building illustrative 3D models
implementing Chasles’ theorem.

Chasles gave five main variants of the mutual position of the tetrahedron and the quadratic surface,
in which his theorem is manifested. This work presents the research results for three of the variants.
Part of the examples are given in work [11]. The remaining variants have also been studied, and the
results will be published in the nearest future.

The construction and studies have been performed in the AutoCAD package [12] along with using
a number of programs in AutoLisp. These are the programs of building Hyper using three directrices
[13], for building of conics using the set of five parameters (points and tangents) [14]. The research
studies on the accuracy of 3D constructs based on the Pascal’s hexagon have been taken into account
[15]. The algorithms building of quadric surfaces were used [16], as well as the parametrization in the
SolidWorks package [17].

2. Tetrahedron edges cross the quadratic surface twice
In the variant under consideration each tetrahedron edge must cross the Quadric in two points.
Multiple variants of such position of the tetrahedron and the Quadric are possible. The edges may be
crossing the Quadric on their present segment or on continuation. All or part of the apexes are located
inside or outside the Quadric. The variant, in which all the apexes are located outside the quadric
surface, and the points of the edges’ intersection are located between the apexes, was also considered
in works [9,11]. The variant of the position of the tetrahedron and all its apexes inside the Quadric
was covered in work [11].

Let us consider an example of the combined location of the tetrahedron apexes intersecting with a
triaxial ellipsoid, see Figure 1(a). Apexes A, C are located outside the Quadric, and apexes B, D –
inside the Quadric.

Let us consider the tetrahedron apex A. Let us find the points of intersection of the edges
converging in this apex and unite them into triangle sA, see Figure 1(b). For this apex the opposing
face is BCD. Let us determine the line a = sA ∩ (BCD). Then we repeat this construction for the
remaining tetrahedron apexes B, C, D: b = sB ∩ (ACD); c = sC ∩ (ABD); d = sD ∩ (ABC).
According to Chasles’ theorem, the obtained lines a, b, c, d are the guide lines of the single Hyper.

Let us verify this conclusion experimentally. First, we need to make sure that lines a, b, c, d belong
to the skew lines. Next, out of these lines, we choose three arbitrary ones, for example: a, b, c. With
the use of the Lisp program [13] we build Hyper, for which these lines are the guide lines. We
determine the distance between the fourth line d to the surface of Hyper. The distance does not exceed
0.01% of the diameter of the neck or diameters of bases of Hyper. This has allowed to come to a high-
accuracy conclusion that line d belongs to the surface of Hyper. Therefore, the experiment has
confirmed Chasles’ theorem.

As an illustration of it, see Figure 1(c), the points of the intersection between the lines a–d and
bases of the Hyper are marked.
Figure 1. Tetrahedron edges cross the quadric surface: $a$ – model; $b$ – triangular sections; $c$ – control hyperboloid; $d$ – proof of the theorem; $e$ – Pascal’s hexagon.

The second method of verification is based on the existence of two families of straight lines on the surface of Hyper: guides and generators. All generators cross all the guides [5-8]. We experimentally verified the existence of such a line $m$ that intersects all lines $a$, $b$, $c$, $d$. The experiment has been performed in the SolidWorks package, using the 3D parametrization [17]. Many lines $m$ have been found. If we consider these lines as a family of generators, we come to a conclusion that lines $a$, $b$, $c$, $d$ belong to the family of the guides of the single Hyper. This also confirms the conclusion of Chasles’ theorem.

3. Proof of Chasles’ theorem
Work [9], on the example of an ellipsoid, gives proof for the 1st variant, in which all the apexes are located outside the quadric surface. Work [11] repeats this proof for a paraboloid. Let us consider the proof of Chasles’ theorem for variant 1 with the combined location of the tetrahedron apexes using the example of its intersection with a triaxial ellipsoid, see Figure 1.

Let us consider one of the tetrahedron faces, for example, $ABC$, see Figure 1(d). We build the intersection of this face with the quadric surface. Thus, we obtain the conic, which acts as the ellipse $e$ for the chosen face. Through points 1–6 in the intersection of the face with the quadric surface, we build the Pascal’s hexagon, see Figure 1(e). We present one of its possible variants, determined by the disposition of points 1–6. We find points $P1 = (1-2) \cap (4-5)$, $P2 = (2-3) \cap (5-6)$, $P3 = (3-4) \cap (6-1)$. According to Pascal’s theorem, these points are located on one line.

If we take into account that $P1 \subset (1-2)$, then, $P1 \subset sA$. Also $P1 \subset (4-5)$ and $P1 \subset (BCD)$. That is why $P1$ belongs to the line in the intersection of triangles $sA$ and $BCD$. But line $a$ is the line of intersection of these triangles, therefore ($\Rightarrow$) $P1 \subset a$.

We can summarize the same for points $P2$, $P3$:

- $P2 \subset (5-6)$, $P2 \subset sB$; $P2 \subset (2-3)$, $P2 \subset (ACD) \Rightarrow P2 \subset b (sB \cap ACD)$;
- $P3 \subset (3-4)$, $P3 \subset sC$; $P3 \subset (6-1)$, $P3 \subset (ABD) \Rightarrow P3 \subset c (sC \cap ABD)$.

Since points $P1$, $P2$, $P3$ belong to the line, this line crosses lines $a$, $b$, $c$. 
We can also see that lines $d$ and line are located in one plane – this is the plane of the chosen face $ABC$. Therefore, these lines intersect in a certain point $P4 \in \text{line}$.

Thus, the line crosses all lines $a,b,c,d$. Hence (see the second method of verification above), lines $a,b,c,d$ are the guides of a certain single $\text{Hyper}$. The theorem is proved.

The constructed line crosses the guides of the $\text{Hyper}$. That is why it belongs to the family of generatrix lines of this $\text{Hyper}$ and also belongs to its surface, see Figure 1(c).

The proof of the theorem can be repeated for any quadratic surface, any tetrahedron face, and different variants of the Pascal’s hexagon set by the disposition of points 1-6. For the chosen variant of the quadratic surface, face and hexagon under consideration, the construction has proved to be the most illustrative one.

4. Additional examples
Models have been built for all types of the $\text{Quadric}$. In all the Models the confirmation of Chasles’ theorem has been obtained.

For a hyperbolic paraboloid ($\text{Hypar}$), see Figure 2(a), and a two-sheet hyperboloid, see Figure 2(c), some of the edges are crossing the quadric surface on their present segment, and some – on the continuation (the combination variant). For one-sheeted hyperboloid, see Figure 2(b), and cylinder, see Figure 2(d), all the edges are crossing at the present segment.

![Figure 2. Examples with various quadric surface: a – hyperbolic paraboloid; b – one-sheeted hyperboloid; c – two-sheet hyperboloid; d – cylinder.](image)

Other models with multiple variants of the tetrahedron edges crossing the quadric surface have also been obtained. The exception is the $\text{Hypar}$, for which it has only become possible to build a model with a combination crossing variant.

The model for a two-sheet hyperboloid is performed through the intersection both with two sheets, see Figure 2(c), and with one sheet.

5. All the tetrahedron apexes belong to the quadric
In this variant all the tetrahedron apexes are located on the $\text{Quadric}$. Chasles considered this variant as a passage to the limit during the approximation of the tetrahedron vertices to the $\text{Quadric}$, in the course of which the triangular sections transform into the tangent planes to the $\text{Quadric}$.

Let us consider an example, in which the tetrahedron apexes $ABCD$ are located in a sphere, see Figure 3(a). In these apexes we build tangent planes to the sphere as the planes being perpendicular to the lines which connect these points to the sphere’s center $S$. These planes are shown as sectors $sA...sD$, see Figure 3(b). We build the intercepts of the interception lines between the tangent planes
and the tetrahedron faces opposite to the apexes. For example, line \( a = sA \cap (BCD) \). Using three lines, which were arbitrarily selected from the four \( a, b, c, d \), we construct \( \text{Hyper} \), see Figure 3(c). Next, we verify that the fourth line also belongs to the \( \text{Hyper} \) surface, what is proof of Chasles’ theorem.

**Figure 3.** Tetrahedron inscribed in a sphere: a – model; b – tangent planes; c – control \( \text{Hyper} \).

In the considered example a control \( \text{Hyper} \) has been obtained with a narrow neck and low height. Such \( \text{Hypers} \) also appeared in other experiments. In this case the \( \text{Hyper} \) has been scaled together with the guides, setting different scaling along the axes to obtain an illustrative model, see Figure 3(c).

In work [11] we provided a model, in which the tetrahedron apexes are located on the surface of HypAr. The theorem conclusions were also confirmed.

**Figure 4.** Tetrahedron edges are tangent to sphere (a) and paraboloid (b).

6. **Tetrahedron edges are tangent to the quadric (wireframe tetrahedron)**

This variant has a limited manifestation since a Wireframe tetrahedron can only be built for convex quadric (sphere, ellipsoid, paraboloid, two-sheet hyperboloid).

Let us consider the building of a Wireframe tetrahedron for a sphere, see Figure 4(a). It is known [18] that for a sphere the following condition must be met: the sums of the lengths of the skew edges Wireframe-tetrahedron are equal, that is for a \( ABCD \) tetrahedron it should read as \( AB + CD = AC + BD = AD + CD = S \). For example, by taking \( S = 140 \), we obtain the following values \( AB = 70; BC = 50; AC = 60; AD = 90; CD = 70; BD = 80 \). Next, we build a \( ABC \) face. The vertex D is located at the intersection of three auxiliary spheres with radii AD, BD, and CD. We build inscribed circles on two faces. Then we build perpendiculars from the centers of the circles, and find the center of the inscribed sphere in their intersection [19]. After that we determine the radius of this sphere and six tangent points 1–6.

For other **Quadric** (ellipsoid, paraboloid, see Figure 4(b), and two-sheet hyperboloid, see Figure 5(a), the indicated property of the Wireframe tetrahedron sides does not hold true. That is why we
have been performing the building in SolidWorks package using 3d parametrization. We have been building the Quadric and six lines tangent to the Quadric. We obtained the frame tetrahedron by uniting the apexes of the lines. We also marked the tangent points. In order to set the tangent planes, in the tangent points we built the lines perpendicular to the Quadric.

Chasles’ theorem features two paragraphs for frame tetrahedrons: paragraph 3 and paragraph 7. Various building algorithms are available under these paragraphs. Let us consider them on an example of a two-sheet hyperboloid, see Figure 5(a).

According to paragraph 3, the tangent points belonging to the edges, which run from one and the same apex, should be united into triangular sectors. There are four of such sectors (as per the number of the apexes). We find the lines of intersection of each sector with the face opposite to the apex of this sector. We repeat that for each apex. For example, for apex B we build sector (3-4-5) and line \( b = (3-4-5) \cap (ACD) \), see Figure 5(b). As a result, for all the apexes we obtain lines \( a, b, c, d \), see Figure 5(c). The experiment has shown that these lines belong to the single plane (are coplanar).

According to paragraph 7 of Chasles’ theorem, in the tangent points belonging to the edges of the single face, the tangent planes to the Quadric have been built. There are three of such planes for each face (as per the number of the edges of a triangular face). Let us consider further building on an example of \( ABC \) face, see Figure 5(d). The tangent points of the edges of this face are 1, 3, 5. The tangent planes to the Quadric in these points are shown as rectangles \( s_1, s_3, s_5 \). Next, we find apex \( V \) of the trihedral angle generated by these tangent planes. We connect this apex by to the tetrahedron apex \( D \), which is opposite to the face under consideration. We have obtained line \( d \). We repeat the building for each face and obtain four lines \( a, b, c, d \), see Figure 5(e). The performed building has shown that these lines intersect in one point \( K \), while there are no coplanar threes of lines among them.

Figure 5. Tetrahedron edges are tangent to two-sheet hyperboloid: a – model; b – building of intercept \( b \) for apex \( B \); c – four coplanar intercepts; d – building of intercept \( d \) for face \( ABC \); e – four intersecting intercepts.
Thus, for wireframe tetrahedrons we obtain either four coplanar lines, or four noncoplanar lines with a common point of intersection. Our results differ from the conclusions by Chasles, who believed that the four obtained lines belonged to the surface of a single hyperboloid. Though at the same time, he pointed to an analogy to Brianchon’s theorem [1,2], where the diagonals of a hexagon circumscribed around a conic intersect in one point.

7. Conclusion
Illustrative and geometrically accurate [15] models have been obtained for all the variants of Chasles’ theorem. Some of them have been provided in this work, others were covered in work [11], and the rest of them are considered for the next publication. Performing of complex 3D geometric building has been required for these models.

The models have confirmed the conclusions of Chasles’ theorem as applied to the main variants of the theorem. However, with regard to a wireframe tetrahedron, we have obtained a conclusion different from that of Chasles.

The proof of Chasles’ theorem was found only for the first and second variants of the theorem. There arises a task to find proofs for all the variants, or find a multi-purpose proof of the theorem. Obtaining the proof is a relevant task. The lack of proof highlights the “mystic” properties of Chasles’ tetrahedron.

Chasles considered his theorem as a 3D analog to Pascal’s theorem. We believe that the performed experiments give us ground to assume that Chasles’ theorem can prove to be as valuable for 3D geometrical modeling as Pascal’s theorem is for projective geometry.

The models obtained as a result of the experiments could be used as teaching examples for those mastering 3D modeling. We plan on including them into a new course on the theoretical basics of 3D geometrical computer modeling. The course is intended for students of engineering specialties and is an alternative to the course of descriptive geometry [20].

References
[1] Chasles M 1883 Historical review of the origin and development of geometric methods vol. 2 Note XXXII, History of geometry (Moscow ) p 428
[2] Chetverukhin N F 1961 Projective geometry (Moscow) p 360
[3] Chasles M 1883 Historical review of the origin and development of geometric methods vol. 1, History of geometry (Moscow) p 311
[4] Frolov S A 2016 Descriptive geometry (Moscow) p 285
[5] Peklich V A 2007 Descriptive geometry (Moscow) p 272
[6] Chetverukhin N F 1963 Descriptive geometry (Moscow) p 420
[7] Ivanov G S 1998 Theoretical Foundations of Descriptive Geometry (Moscow) p 158
[8] Chasles Géométrie de situation. Démonstration de quelques propriétés du triangle, de l'angle trièdre et du tétraèdre, considérés par rapport aux lignes et surfaces du second ordre Annales de mathématiques pures et appliquées (1828-1829) 19 65–85. http://www.numdam.org/volume/AMPA_1828-1829__19/
[9] Morducai-Boltovskoy D D 1953 Three-dimensional and four-dimensional analog of Pascal's theorem (Advances in mathematical Sciences V VIII № 2 (54) pp 135-138 http://mi.mathnet.ru/umn8192
[10] Prasolov V V and Tikhomirov V M 2007 Geometry (Moscow) p 328
[11] Kheifetc A L 2020 Chasles’ Theorem as a 3D Analogue of Pascal’s Theorem V International Conference on Information Technologies in Engineering Education (Inforino ) (Moscow) pp 1–5. DOI:10.1109/Inforino48376.2020.9111690
[12] Kheifetc A, Loginovskiy A, Butorina I and Vasileva V 2018 Engineering 3D computer graphics: tutorial and workshop for academic undergraduate (Moscow: Yurayt) p 602
[13] Kheyfets A L and Loginovskiy A N 2008 3d-models of ruled surfaces with three rectilinear guides Bulletin of the South Ural State University Series "Construction Engineering and
Architecture” 7 25(125) (Cheljabinsk) pp 51–56. http://dspace.susu.ru/xmlui/bitstream/handle/0001.74/615/10.pdf?sequence=1&isAllowed=y

[14] Korotkii V A and Kheyfets A L 2005 3D modeling of conics in AutoCAD (Actual issues of graphic education of youth) Materials of the VI All-Russian scientific-methodical conference (Rybinsk: RGTA) pp 102–105

[15] Kheyfets A L 2016 Geometrical accuracy of computer algorithms for constructive problems. The problem of the quality of the graphic preparation of students in technical high school: tradition and innovation (Perm: PNRPU) I 3 367–87. http://dgng.pstu.ru/conf2016/papers/74/

[16] Kheyfets A L 2008 3D-model of ruled surface with three curvilinear guides 11 th International Conference 3IA’2008, the International Conference in Computer Graphics and Artificial Intelligence, 30-31 May 2008, Athens. Greece Eurographics. Organised by: XLIM Laboratory University of Limoges pp 223–227

[17] Kheifetc (Kheyfets) A L 2016 3d Models and Algorithms for computer-based parameterization for the decision of tasks of constructive Geometry (at some historical Examples) Bulletin of the South Ural State University. Series "Computer technologies, automatic control & radioelectronics" (Cheljabinsk: SUSU) 16(2) 24–42. https://vestnik.susu.ru/ctcr/article/view/4909

[18] Wireframe tetrahedron. https://studbooks.net/2395470/matematika_himiya_fizika/karkasnye_tetraedry

[19] Half inscribed sphere. http://www.myshared.ru/slide/546522/

[20] Kheifetc A L 2015 Descriptive geometry as a factor limiting the development of geometric modeling Proceedings of the V International scientific and practical Internet-conference (Perm: PGTU) pp 292–325. http://dgng.pstu.ru/conf2015/papers/72/, http://dgng.pstu.ru/conf2015/news/44/