INSTANTANEOUS BETHE–SALPETER EQUATION: UTMOST ANALYTIC APPROACH

Wolfgang LUCHA
Institut für Hochenergiephysik, Österreichische Akademie der Wissenschaften, Nikolsdorfgasse 18, A-1050 Wien, Austria

Khin MAUNG MAUNG
Department of Physics, Hampton University, Hampton, VA 23668

Franz F. SCHÖBERL
Institut für Theoretische Physik, Universität Wien, Boltzmannasse 5, A-1090 Wien, Austria

Abstract

The Bethe–Salpeter formalism in the instantaneous approximation for the interaction kernel entering into the Bethe–Salpeter equation represents a reasonable framework for the description of bound states within relativistic quantum field theory. In contrast to its further simplifications (like, for instance, the so-called reduced Salpeter equation), it allows also the consideration of bound states composed of “light” constituents. Every eigenvalue equation with solutions in some linear space may be (approximately) solved by conversion into an equivalent matrix eigenvalue problem. We demonstrate that the matrices arising in these representations of the instantaneous Bethe–Salpeter equation may be found, at least for a wide class of interactions, in an entirely algebraic manner. The advantages of having the involved matrices explicitly, i.e., not “contaminated” by errors induced by numerical computations, at one’s disposal are obvious: problems like, for instance, questions of the stability of eigenvalues may be analyzed more rigorously; furthermore, for small matrix sizes the eigenvalues may even be calculated analytically.

PACS numbers: 11.10.St, 03.65.Ge

* E-mail address: wolfgang.lucha@oeaw.ac.at
† E-mail address: maung@jlab.org
‡ E-mail address: franz.schoeberl@univie.ac.at
1 Introduction

The Bethe–Salpeter formalism [1, 2] is generally accepted to represent the appropriate framework for the description of bound states within relativistic quantum field theory. Within this formalism, a bound state is described by its “Bethe–Salpeter amplitude,” which is defined as the (time-ordered) product of the field operators of the bound-state constituents between the vacuum and the bound state. In principle, this bound-state amplitude should be found as solution of the (homogeneous) Bethe–Salpeter equation.

However, apart from very few special cases—like the famous Wick–Cutkosky model which describes the interaction of two scalar particles by exchange of a massless scalar particle—the Bethe–Salpeter equation turns out to be practically not tractable. One of the main reasons for this fact is the appearance of timelike variables in the equation of motion. Consequently people usually consider some three-dimensional reduction of the Bethe–Salpeter equation. The most popular among these three-dimensional reductions is based on the assumption that the interaction between the bound-state constituents is instantaneous in the center-of-momentum frame of the bound state; the result of this is called the “Salpeter equation” or “instantaneous Bethe–Salpeter equation” [2]. This equation may be formulated as eigenvalue problem for the mass $M$ of the bound state. Its dynamical quantity is the “Salpeter amplitude,” obtained from the Bethe–Salpeter amplitude by equating the time variables of the involved bound-state constituents.

By expanding the Salpeter amplitude into some convenient set of basis matrices in Dirac space [3], Lagaë [4, 5] managed to reduce the full instantaneous Bethe–Salpeter equation for fermion–antifermion bound states to a set of coupled equations for radial wave functions. His easy-to-handle formalism immediately stimulated renewed interest in the full Salpeter equation as a manageable tool for describing mesonic bound states [6, 7, 8, 9, 10, 11]. The first question to be answered in this context refers to the Lorentz structure of the confining Bethe–Salpeter kernel. The most important outcome of these studies is the following:

- Lorentz-scalar confinement turns out (1) to face serious problems with respect to the stability of the resulting solutions [6, 7, 11] and (2) to be in conflict with the phenomenologically dictated linearity of the mesonic Regge trajectories [5, 7, 11].

- Time-component Lorentz-vector confinement, on the other hand, does not share either of these two problems, that is, it yields (1) stable solutions [6, 7, 11] and (2) linear Regge trajectories for the bound states of quark–antiquark systems [7, 11].

The square of the norm of the amplitudes obtained as solutions of the instantaneous Bethe–Salpeter equation is not positive definite; in order to stay in the physical sector of the theory, in all these analyses one has to retain only bound states of positive norm squared but to reject all bound states of zero or negative norm squared. In this sector, all mass eigenvalues $M$ of the instantaneous Bethe–Salpeter equation are guaranteed to be real [4, 6].

A standard technique for determining the solutions of a given eigenvalue equation is the conversion of this equation into the equivalent matrix eigenvalue problem. (For the instantaneous Bethe–Salpeter equation, this route has been followed in Refs. [4, 6, 11].) We show here that, for a suitable choice of basis states in the space of solutions [12], the matrix representation of the instantaneous Bethe–Salpeter equation may be explicitly given in analytical form. Among others, this might prove to be advantageous for future discussions of the stability of solutions of the instantaneous Bethe–Salpeter equation.
2 The Instantaneous Bethe–Salpeter Equation

It goes without saying that we would like to demonstrate our procedure for solving the instantaneous Bethe–Salpeter equation at the simplest conceivable example. First of all, let us agree on two simplifying assumptions:

- The (full) fermion propagators $S_i(p_i, m_i), i = 1, 2$, entering in the Bethe–Salpeter equation may be approximated by the corresponding free propagators, given by
  \[ S_{i,0}^{-1}(p_i, m_i) = -i (\gamma_\mu p_\mu^i - m_i), \quad i = 1, 2. \]

  In the above free-propagator approximation, the masses $m_i, i = 1, 2$, of the two fermionic bound-state constituents are then interpreted as some effective ones.

- The two bound-state constituents have equal masses $m$, that is, $m_1 = m_2 = m$. (The generalization to the case of unequal masses of the bound-state constituents is, of course, straightforward [6].)

Moreover, we illustrate the developed technique by investigating mesons carrying the pion’s quantum numbers, treated as bound states of a quark–antiquark pair [13]. Thus, we set up the instantaneous Bethe–Salpeter equation for $^1S_0$ bound states of massless constituents with confining interaction kernels of time-component Lorentz-vector type.

2.1 Pseudoscalar Bound States of Massless Constituents

We consider fermion–antifermion bound states of total spin $J$, with parity and charge-conjugation quantum numbers $P$ and $C$, expressed in terms of $J$, given by $P = (-1)^{J+1}$ and $C = (-1)^J$, respectively. In usual spectroscopic notation, these states are denoted by $^1J_J$. The general expansion of the Salpeter amplitude $\chi$ in terms of a complete set of Dirac matrices performed in Ref. [4] involves precisely eight independent components. For the bound states under consideration, only two of these independent components are relevant. We call the corresponding wave functions in momentum space $\Psi_1$ and $\Psi_2$. In the notation of Lagaë [4], these wave functions are labelled $L_1$ and $L_2$, respectively.

For two particles of equal masses $m$ and internal momentum $k$ constituting a bound state with the quantum numbers specified above, the Salpeter amplitude $\chi$ describing this bound state in its center-of-momentum frame reads in momentum space

\[ \chi(k) = \left[ \Psi_1(k) \frac{m - \gamma \cdot k}{E(k)} + \Psi_2(k) \gamma^0 \right] \gamma_5, \]

with the abbreviation

\[ E(k) \equiv \sqrt{k^2 + m^2}, \quad k \equiv |k|. \]

The norm $\|\chi\|$ of the Salpeter amplitude $\chi$ may be calculated from Eq. (2.9) of Ref. [4] (or from Eq. (9) of Ref. [11]). For the above amplitude, the squared norm is given by

\[ \|\chi\|^2 = 4 \int \frac{d^3k}{(2\pi)^3} [\Psi_1^*(k) \Psi_2(k) + \Psi_2^*(k) \Psi_1(k)], \]

in accordance with Eq. (4.13) of Ref. [4], and with Eq. (18) of Ref. [11].

For simplicity, we focus our interest to quark–antiquark bound states of spin $J = 0$; these are bound states with the spin-parity-charge conjugation assignment $J^{PC} = 0^-$. It is these pseudoscalar quark–antiquark bound states which will be investigated here.
In view of the findings recalled in the Introduction, let us settle on the investigation of the instantaneous Bethe–Salpeter equation for some time-component Lorentz vector interaction between the bound-state constituents, that is, on a Bethe–Salpeter kernel of the Dirac structure $\gamma^0 \otimes \gamma^0$. For this case, the radial wave functions $\Psi_1(k)$ and $\Psi_2(k)$, obtained from $\Psi_1(k)$ and $\Psi_2(k)$ by factorizing off the spherical harmonics involving the angular variables, satisfy the following system of equations—which is equivalent to the instantaneous Bethe–Salpeter equation and which may be deduced, e.g., from Eq. (5.7) or Eq. (5.9) of Ref. [4], or from Eq. (A1) of Ref. [11]:

\[ 2E(k)\Psi_2(k) + \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} V_0(k, k') \Psi_2(k') = M\Psi_1(k), \]

\[ 2E(k)\Psi_1(k) + \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} \left[ \frac{m}{E(k)} V_0(k, k') - \frac{k}{E(k')} V_1(k, k') \right] \Psi_1(k') = M\Psi_2(k), \]

where, in terms of some interaction potential $V(r)$ in configuration space,

\[ V_L(k, k') \equiv 8\pi \int_0^\infty dr r^2 V(r) j_L(k r) j_L(k' r), \quad L = 0, 1, 2, \ldots. \]

Here, $j_n(z)$ ($n = 0, \pm 1, \pm 2, \ldots$) are the spherical Bessel functions of the first kind.

For a vanishing mass of the two bound-state constituents (that is, for $m = 0$), this set of equations simplifies to

\[ 2k \Psi_2(k) + \frac{1}{(2\pi)^2} \int_0^\infty dk' k'^2 V_0(k, k') \Psi_2(k') = M\Psi_1(k), \]

\[ 2k \Psi_1(k) + \frac{1}{(2\pi)^2} \int_0^\infty dk' k'^2 V_1(k, k') \Psi_1(k') = M\Psi_2(k). \]

The above set of equations constitutes the starting point for the present investigation.

### 2.2 Analytical Solution by Series Expansion

The particular structure of the instantaneous Bethe–Salpeter equation allows us to extract its solutions by the following simple procedure. Express one of the components $\Psi_1, \Psi_2$ of the Salpeter amplitude $\chi$ from one of the coupled equations in the system by the other component. Insert this expression into the other equation of this system, in order to obtain an eigenvalue equation for the other component with eigenvalue $M^2$:

\[ M^2 \Psi_2(k) = 4k^2 \Psi_2(k) \]

\[ + \quad 2k \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} V_0(k, k') \Psi_2(k') \]

\[ + \quad 2 \int_0^\infty \frac{dk' k'^3}{(2\pi)^2} V_1(k, k') \Psi_2(k') \]

\[ + \quad \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} V_1(k, k') \int_0^\infty \frac{dk'' k''^2}{(2\pi)^2} V_0(k', k'') \Psi_2(k''). \]
Clearly, all our results should be independent of which component has been selected in the first place. For the present analysis, we have chosen to express $\Psi_1$ in terms of $\Psi_2$.

An approximate solution to the above eigenvalue equation may be very easily found by expanding the “Salpeter components” over a suitable set of basis functions \[\mathcal{L}\]. To this end, we introduce complete orthonormal systems \{\{\phi_i\}, i = 0, 1, 2, \ldots\} as bases for the Hilbert space $L_2(R^+)$ of (with the weight function $\nu(x) = x^2$) square-integrable functions $f(x)$ on the positive real line $R^+$; these basis vectors $|\phi_i\rangle$ satisfy $\langle \phi_i | \phi_j \rangle = \delta_{ij}$. The different sets of basis vectors are distinguished by some parameter $\ell = 0, 1, 2, \ldots$. The representations of any basis vector $|\phi_i\rangle$ in configuration and momentum space are related to the corresponding bases for the Hilbert space $L_2(R^3)$ of all square-integrable functions on the three-dimensional space $R^3$ by factorizing off the spherical harmonics $Y_{\ell m}(\Omega)$ for angular momentum $\ell = 0, 1, 2, \ldots$ and its projection $m = -\ell, -\ell + 1, \ldots, +\ell$ which describe the dependence on the angular variables summarized by the solid angle $\Omega$. The parameter $\ell$ is identified with the angular momentum of the $R^3$ basis functions. For a given value of $\ell$, the $R^+$ basis functions in configuration and momentum space are called $\phi_i^{(\ell)}(r)$ and $\phi_i^{(\ell)}(p)$, respectively. They satisfy the orthonormalization conditions

$$\int_0^{\infty} dr r^2 \phi_i^{(\ell)}(r) \phi_j^{(\ell)}(r) = \int_0^\infty dp p^2 \phi_i^{(\ell)}(p) \phi_j^{(\ell)}(p) = \delta_{ij} \, , \quad i, j = 0, 1, 2, \ldots .$$

In order to introduce an additional degree of freedom in the search for the solutions of the instantaneous Bethe–Salpeter equation, we allow the basis functions to depend on some positive real variational parameter $\mu$ (with the dimension of mass). Our choice of radial basis functions, involving, in their configuration-space representation $\phi_i^{(\ell)}(r)$, the generalized Laguerre polynomials \[\mathcal{L}\], is summarized in due detail in Appendix \[\mathcal{L}\]. A particular feature of this choice is our phase convention: all configuration-space basis functions $\phi_i^{(\ell)}(r)$ and, for $\ell$ even, the momentum-space basis functions $\phi_i^{(\ell)}(p)$ are real. Starting with expansions over the radial basis functions $\phi_i^{(0)}(p)$ corresponding to $\ell = 0$, the solution of the instantaneous Bethe–Salpeter equation then simply amounts to the diagonalization of the—a priori, infinite-dimensional—matrix (with eigenvalues $M^2$)

$$\mathcal{M}_{ij} = A_{ij} + B_{ij} + C_{ij} + D_{ij} \, ,$$

with the abbreviations

$$A_{ij} \equiv 4 \int_0^\infty dk k^4 \phi_i^{(0)}(k) \phi_j^{(0)}(k) \, ,$$

$$B_{ij} \equiv 2 \int_0^\infty dk k^3 \phi_i^{(0)}(k) \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} V_0(k, k') \phi_j^{(0)}(k') \, ,$$

$$C_{ij} \equiv 2 \int_0^\infty dk k^2 \phi_i^{(0)}(k) \int_0^\infty \frac{dk' k'^3}{(2\pi)^2} V_1(k, k') \phi_j^{(0)}(k') \, ,$$

$$D_{ij} \equiv \int_0^\infty dk k^2 \phi_i^{(0)}(k) \int_0^\infty \frac{dk' k'^2}{(2\pi)^2} V_1(k, k') \int_0^\infty \frac{dk'' k''^2}{(2\pi)^2} V_0(k', k'') \phi_j^{(0)}(k'') \, .$$

In actual calculations, an infinite-dimensional matrix, arising from a countably infinite number of basis states $|\phi_i\rangle$, is not manageable. One has to be content with a truncation of the matrix $\mathcal{M}_{ij}$ to a, say, $d \times d$ matrix. Moreover, in the expansions encountered in the intermediate steps only the first, say, $N + 1$ basis vectors can be taken into account.
The (straightforward) evaluation of the matrix $\mathcal{M}_{ij}$ is briefly sketched in Appendix B:

$$
\mathcal{M}_{ij} = 4 I^{(2)}_{ij}(\mu) + 2 \sum_{r=0}^{N} b_{ri}(\mu) V^{(0)}_{rj}(\mu) + 2 \sum_{r=0}^{N} \sum_{s=0}^{N} c_{rs} V^{(1)}_{rs}(\mu)
$$

$$
+ \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} c_{ri} c_{st} V^{(1)}_{sr}(\mu) V^{(0)}_{ij}(\mu).
$$

Let us present the various quantities entering in this result in their order of appearance. The (real and symmetric) matrix $I^{(2)}_{ij}(\mu)$ may be easily read off from the result (23) for the integral $I^{(n)}_{ij}(\mu)$ defined by Eq. (22) for the case $n = 2$:

$$
I^{(2)}_{ij}(\mu) = \frac{4 \mu^2}{\pi \sqrt{(i+1)(i+2)(j+1)(j+2)}} \times \sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} \binom{i+2}{i-r} \frac{1}{j+2} \binom{j+2}{j-s} (r+1)(s+1)
$$

$$
\times \left[ \sum_{k=0}^{[r-s]} \frac{\Gamma(\frac{1}{2}(k+3)) \Gamma(\frac{1}{2}(r+s+1+|r-s|-k))}{\Gamma(\frac{1}{2}(r+s+4+|r-s|))} \cos \left( \frac{k \pi}{2} \right) \right]^{2}.
$$

Note the scaling behaviour of this expression with respect to the variational parameter $\mu$:

$$
I^{(2)}_{ij}(\mu) = \mu^2 I^{(2)}_{ij}(1).
$$

Explicitly, the matrix $I^{(2)}(\mu) \equiv (I^{(2)}_{ij}(\mu))$ reads

$$
I^{(2)}(\mu) \equiv (I^{(2)}_{ij}(\mu)) = \mu^2 \begin{pmatrix}
1 & \frac{2}{\sqrt{3}} & \cdots \\
\frac{2}{\sqrt{3}} & 7 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}.
$$

According to Eq. (23), the real and symmetric matrix $b_{ij}(\mu)$ is identical to the integral $I^{(n)}_{ij}(\mu)$ defined by Eq. (22) for the case $n = 1$; consequently, it may be read off from the result (23) for $n = 1$:

$$
b_{ij}(\mu) = \frac{4 \mu}{\pi \sqrt{(i+1)(i+2)(j+1)(j+2)}} \times \sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} \binom{i+2}{i-r} \frac{1}{j+2} \binom{j+2}{j-s} (r+1)(s+1)
$$

$$
\times \left[ \sum_{k=0}^{[r-s]} \frac{\Gamma(\frac{1}{2}(k+2)) \Gamma(\frac{1}{2}(r+s+2+|r-s|-k))}{\Gamma(\frac{1}{2}(r+s+4+|r-s|))} \cos \left( \frac{k \pi}{2} \right) \right]^{2}.
$$
Note the scaling behaviour of this expression with respect to the variational parameter \( \mu \):

\[
b_{ij}(\mu) = \mu b_{ij}(1)
\]

Explicitly, the matrix \( b(\mu) \equiv (b_{ij}(\mu)) \) reads

\[
b(\mu) = \frac{8 \mu}{3\pi} \begin{pmatrix}
1  & \frac{1}{\sqrt{3}} & \cdots \\
1 & \frac{7}{5} & \\
\vdots & \vdots & \\
\end{pmatrix}.
\]

According to Eq. (33), the (purely imaginary) matrix \( c_{ij} \) is nothing else but the integral \( J^{(n)}_{ij}(\mu) \) defined by Eq. (24) specified to the case \( n = 0 \); consequently, it may be read off from the result (25) for \( n = 0 \) (here, because of the singularity of \( \Gamma(0) \), some care has to be taken when performing the limit \( n \to 0 \) in the algebraic expression (25) for \( J^{(n)}_{ij}(\mu) \):

\[
c_{ij} = \frac{8}{\pi \sqrt{(i+1)(i+2)(i+3)(i+4)(j+1)(j+2)}} \times 
\sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} (r+1)(r+2)(r+3)(s+1) \left( \frac{i+4}{i-r} \right) \left( \frac{j+2}{j-s} \right) 
\times \left\{ \frac{1}{r+2} \left[ -\Psi\left(\frac{1}{2}(4+r+s+|r-s|)\right) + \Psi(4+r+s) \right] 
+ \sum_{k=2}^{[r-s]} \left( \frac{|r-s|}{k} \right) \frac{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}(4+r+s+|r-s|-2)\right)}{\Gamma\left(\frac{k}{2}(4+r+s+|r-s|)\right)} \cos\left(\frac{k\pi}{2}\right) 
- \sum_{k=2}^{4+r+s} \left( \frac{4+r+s}{k} \right) \frac{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}(8+2r+2s-k)\right)}{\Gamma(4+r+s)} \cos\left(\frac{k\pi}{2}\right) \right\} 
- \frac{1}{r+3} \left[ -\Psi\left(\frac{1}{2}(5+r+s+|1+r-s|)\right) + \Psi(5+r+s) \right] 
+ \sum_{k=2}^{[1+r-s]} \left( \frac{1+r-s}{k} \right) \frac{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}(5+r+s+1+r-s-k)\right)}{\Gamma\left(\frac{k}{2}(5+r+s+1+r-s)\right)} \cos\left(\frac{k\pi}{2}\right) 
- \sum_{k=2}^{5+r+s} \left( \frac{5+r+s}{k} \right) \frac{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}(10+2r+2s-k)\right)}{\Gamma(5+r+s)} \cos\left(\frac{k\pi}{2}\right) \right\},
\]

where \( \Psi(z) \) denotes the logarithmic derivative of the gamma function, the “digamma function.” Note that the quantities \( c_{ij} \) are independent of the variational parameter \( \mu \). Explicitly, the matrix \( c \equiv (c_{ij}) \) reads

\[
c \equiv (c_{ij}) = \frac{16}{3\pi} \begin{pmatrix}
1  & \frac{1}{\sqrt{3}} & \cdots \\
\frac{1}{\sqrt{3}} & \frac{1}{15} & \\
\frac{1}{15} & \frac{19}{15\sqrt{5}} & \\
\vdots & \vdots & \\
\end{pmatrix}.
\]
Similarly, according to Eq. (34), the (purely imaginary) matrix $d_{ij}(\mu)$ is identical to the integral $J_{ij}^{(n)}(\mu)$ defined by Eq. (24) specified to the case $n = 1$; consequently, it may be read off from the result (25) for $n = 1$:

$$d_{ij}(\mu) = i 8 \mu \pi \sqrt{(i + 1)(i + 2)(i + 3)(i + 4)(j + 1)(j + 2)} \sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} (r+1)(r+2)(r+3)(s+1) \left( \frac{i+4}{i-r} \right) \left( \frac{j+2}{j-s} \right) \times \left\{ \frac{1}{r+2} \sum_{k=0}^{[r-s]} \left( \frac{|r-s|}{k} \right) \frac{\Gamma(\frac{1}{2}(k+1)) \Gamma(\frac{1}{2}(3+r+s+|r-s|)-k)}{\Gamma(\frac{1}{2}(4+r+s+|r-s|))} \cos\left(\frac{k\pi}{2}\right) \right. \\
- \sum_{k=0}^{4+r+s} \left( \frac{4+r+s}{k} \right) \frac{\Gamma(\frac{1}{2}(k+1)) \Gamma(\frac{1}{2}(7+2r+2s-k))}{\Gamma(4+r+s)} \cos\left(\frac{k\pi}{2}\right) \\
- \frac{1}{r+3} \sum_{k=0}^{[1+r-s]} \left( \frac{|1+r-s|}{k} \right) \frac{\Gamma(\frac{1}{2}(k+1)) \Gamma(\frac{1}{2}(5+r+s+|1+r-s|-k))}{\Gamma(\frac{1}{2}(5+r+s+|1+r-s|))} \cos\left(\frac{k\pi}{2}\right) \left. \\
- \sum_{k=0}^{5+r+s} \left( \frac{5+r+s}{k} \right) \frac{\Gamma(\frac{1}{2}(k+1)) \Gamma(\frac{1}{2}(9+2r+2s-k))}{\Gamma(5+r+s)} \cos\left(\frac{k\pi}{2}\right) \right\}.$$

Note the scaling behaviour of this expression with respect to the variational parameter $\mu$:

$$d_{ij}(\mu) = \mu d_{ij}(1).$$

Explicitly, the matrix $d(\mu) \equiv (d_{ij}(\mu))$ reads

$$d(\mu) \equiv (d_{ij}(\mu)) = \frac{i \mu}{2} \begin{pmatrix} \sqrt{3} & 1 & \cdots \\ \sqrt{3} & \sqrt{5} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Finally, the real and symmetric matrix $V_{ij}^{(\ell)}(\mu)$ of expectation values of the interaction potential $V(r)$ with respect to the basis states $|\phi_i\rangle$ (characterized by a particular value of $\ell$) is defined by

$$V_{ij}^{(\ell)}(\mu) \equiv \langle \phi_i | V(r) | \phi_j \rangle = \int_0^\infty dr r^2 V(r) \phi_i^{(\ell)}(r) \phi_j^{(\ell)}(r).$$

For power-law potentials, that is, for interaction potentials $V(r)$ of the power-law form

$$V(r) = \sum_n a_n r^{b_n},$$

$$\sum_{n} a_n b_n,$$
with (arbitrary) real constants \(a_n\) and \(b_n\), these matrix elements \(\langle \phi_i | V(r) | \phi_j \rangle\) are easily worked out algebraically \([13, 14, 17]\). The algebraic expression for the general case can be found in Sec. 4 of Ref. \([13]\), Sec. 3.10 of Ref. \([14]\), or Sec. 2.8.1 of Ref. \([17]\).

The simplest model for a confining interaction between all bound-state constituents is provided by a linear potential

\[
V(r) = \lambda r, \quad \lambda > 0,
\]

whence

\[
V_L(k, k') = 8\pi \lambda \int_{0}^{\infty} dr^3 j_L(k r) j_L(k' r), \quad L = 0, 1, 2, \ldots .
\]

In this case, the above-mentioned general expression for the expectation values \(V_{ij}^{(0)}(\mu)\) of the interaction potential \(V(r)\), taken with respect to our basis states \(|\phi_i\rangle\), simplifies to

\[
V_{ij}^{(0)}(\mu, \lambda) = \frac{\lambda}{2 \mu} V_{ij}^{(0)}(1, 1).
\]

The explicit expression of this potential matrix \(V^{(0)}(\mu, \lambda)\) reads, for instance, for \(\ell = 0\),

\[
V^{(0)}(\mu, \lambda) \equiv \left( V_{ij}^{(0)}(\mu, \lambda) \right) = \frac{\lambda}{2 \mu} \begin{pmatrix}
3 & -\sqrt{3} & \cdots \\
-\sqrt{3} & 5 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

and, for \(\ell = 1\),

\[
V^{(1)}(\mu, \lambda) \equiv \left( V_{ij}^{(1)}(\mu, \lambda) \right) = \frac{\lambda}{2 \mu} \begin{pmatrix}
5 & -\sqrt{5} & \cdots \\
-\sqrt{5} & 7 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}.
\]

Taking into account, for the special case of a linear potential, the above-mentioned scaling behaviour of all the expressions entering in our main result for the matrix \(M_{ij}\), we finally arrive at

\[
M_{ij} = 4 \mu^2 I_{ij}^{(2)}(1) + 2 \lambda \sum_{r=0}^{N} b_{ri}(1) V_{rij}^{(0)}(1, 1) + 2 \lambda \sum_{r=0}^{N} \sum_{s=0}^{N} c_{ri}^* c_{sj} V_{rjs}^{(0)}(1, 1) V_{ij}^{(1)}(1, 1)
\]

\[
+ \frac{\lambda^2}{\mu^2} \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} c_{ri}^* c_{st} V_{rst}^{(1)}(1, 1) V_{ij}^{(0)}(1, 1).
\]

Clearly, this structure of the matrix \(M_{ij}\), with the squared masses of the bound states as its eigenvalues, has to emerge already on dimensional grounds: the mass dimension of the variational parameter \(\mu\) is 1 while the mass dimension of the slope \(\lambda\) of the linear potential (10) is 2; the potential \(V(r)\) enters into the expression (4) for the matrix \(M_{ij}\) in, at most, second order.
3 Results

The dependence of the matrix $\mathcal{M}_{ij}$ on the variational parameter $\mu$, as apparent from Eq. (12), is

$$\mathcal{M}_{ij} = \mu^2 E_{ij} + F_{ij} + \frac{1}{\mu^2} G_{ij},$$

with

$$E_{ij} \equiv 4 I_{ij}^{(2)}(1),$$
$$F_{ij} \equiv 2 \lambda \sum_{r=0}^{N} b_{ri}(1) V_{lj}^{(0)}(1,1) + 2 \lambda \sum_{r=0}^{N} \sum_{s=0}^{N} c_{ri}^* d_{sj}(1) V_{lj}^{(1)}(1,1),$$
$$G_{ij} \equiv \lambda^2 \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} c_{ri}^* c_{st} V_{sr}^{(1)}(1,1) V_{lj}^{(0)}(1,1).$$

Any solution method relying on truncated expansions over some complete set(s) of basis vectors only makes sense if the approximate result obtained in this way converges to the exact solution of the problem under study for increasing numbers of basis vectors taken into account in the expansions. Assuming this convergence to hold, a reasonable estimate of the uncertainties induced by the truncation of the expansion series to finite numbers of basis vectors can be given by the inspection of the lowest-dimensional case. For $i = j = 0$, we find, for the first elements of the matrices $I^{(2)}(1), b(1), c,$ and $d(1),$

$$I_{00}^{(2)}(1) = 1, \quad b_{00}(1) = \frac{8}{3\pi}, \quad c_{00} = \frac{1}{3} \frac{16}{\sqrt{3\pi}}, \quad d_{00}(1) = \frac{1}{2} \sqrt{\frac{3}{2}},$$

and, for the expectation values $V^{(\ell)}(1,1), \ell = 0, 1,$ of the linear potential,

$$V_{00}^{(0)}(1,1) = \frac{3}{2}, \quad V_{00}^{(1)}(1,1) = \frac{5}{2}.$$ 

This implies, for $N = 0$, for the quantities $E \equiv E_{00}, F \equiv F_{00},$ and $G \equiv G_{00},$

$$E = 4, \quad F = \frac{64}{3\pi} \lambda, \quad G = \frac{320}{9\pi^2} \lambda^2.$$ 

Minimizing

$$\mathcal{M}_{00} = \mu^2 E + F + \frac{1}{\mu^2} G$$

with respect to the variational parameter $\mu$, yields

$$\mathcal{M}_{00} = 2 \sqrt{E G} + F.$$ 

The above values of $E, F,$ and $G$ then yield, for the square of the bound-state mass $M$,

$$M^2 = \frac{32}{3\pi} \left(2 + \sqrt{5}\right) \lambda.$$

(14)

For a linear interaction potential like the one used in this investigation, it would be natural to present the numerical results for the masses $M$ of the bound states in terms of the dimensionless quantities $M/\sqrt{\lambda}$, as has been done, e.g., in Ref. [3]. However, in order to facilitate the comparison of our findings with the ones quoted in Ref. [11], we present our results for the particular, more or less reasonable [13] value of the slope $\lambda$ of the linear interaction potential (10) adopted in Ref. [11], viz., for $\lambda = 0.2$ GeV$^2$. For this slope we obtain, for the “one-dimensional” result (14) for the bound-state mass $M$, $M = 1.69604$ GeV.
In general, the approximate values of the bound-state masses $M$, calculated within the present approach as the square roots of the eigenvalues of the matrix $M_{ij}$, depend on the variational parameter $\mu$. In our almost entirely algebraic method of solving the instantaneous Bethe–Salpeter equation, this dependence on $\mu$ has been made manifest in form of Eq. (13). However, any solution technique relying on some series expansions can be regarded as reasonable only if it exhibits stability with respect to the increase in the number of basis states, in the sense that increasing the number of basis states—in our case, increasing the size $d$ of the matrix $M_{ij}$ or the number $N$ of basis states taken into account in the series expansions performed in the intermediate steps—reduces the dependence of the results, the bound-state masses $M$, on the variational parameter $\mu$.

Figure 1 shows (in perfect agreement with Fig. 1 of Ref. [11]) the dependence of the masses $M$ of the three lowest-lying bound states, i.e., the first three radial excitations, extracted from Eq. (13) for a linear interaction potential (10) with slope $\lambda = 0.2$ GeV$^2$, on the variational parameter $\mu$. When increasing the number of basis states (by varying the matrix size $d$ from $d = 5$ to $d = 15$) the formation of regions where the eigenvalues $M$ become independent of $\mu$ should be observed; this feature is indeed found in Fig. 1.

For the ground state, the region of $\mu$-independence corresponding to the matrix size $d = 5$ is centered somewhere around $\mu = 1$ GeV. For this value of $\mu$, the ground-state mass $M$, calculated for a matrix size $d = 15$ and $N = 49$ in the intermediate-step series expansions, is $M = 1.656$ GeV, which reveals that the rough approximation (14) of the above one-dimensional analysis is not too bad (at least for the situation studied here).

In order to pave the way for a future quantitative discussion of the accuracy of our method as well as for a numerical comparison with the findings of different approaches, we list in Table 1 the masses $M$ of the three lowest-lying bound states for several values of the variational parameter $\mu$ in the neighborhood of the above “region of stability.”
Table 1: Eigenvalues $M$, in units of GeV, of the instantaneous Bethe–Salpeter equation for two massless (i.e., $m = 0$) spin-$\frac{1}{2}$ fermions experiencing an interaction described by a linear potential with slope $\lambda = 0.2$ GeV$^2$ and forming bound states of radial quantum number $n_r = 0, 1, 2$ and spin-parity-charge conjugation assignment $J^{PC} = 0^{--}$ (called $1^1S_0$, $2^1S_0$, and $3^1S_0$ in the usual spectroscopic notation) as functions of the variational parameter $\mu$, for matrix size $d = 15$ and $N = 49$ in the intermediate series expansions.

| $\mu$ [GeV] | $1^1S_0$ | $2^1S_0$ | $3^1S_0$ |
|-------------|---------|---------|---------|
| 0.05        | 1.807   | 3.253   | 5.392   |
| 0.1         | 1.673   | 2.360   | 3.173   |
| 0.2         | 1.656   | 2.283   | 2.812   |
| 0.5         | 1.656   | 2.280   | 2.770   |
| 1           | 1.656   | 2.281   | 2.771   |
| 1.2         | 1.657   | 2.281   | 2.771   |
| 2           | 1.658   | 2.284   | 2.774   |
| 3           | 1.658   | 2.287   | 2.843   |
| 4           | 1.662   | 2.393   | 3.165   |
| 5           | 1.702   | 2.617   | 3.614   |

In our procedure, the series expansion of the radial Salpeter component $\Psi_2(k)$ with respect to the (primary) system $\{\phi_i^{(0)}(k), i = 0, 1, 2, \ldots\}$ of radial basis functions reads

$$\Psi_2(k) = \sum_{i=0}^{d-1} e_i \phi_i^{(0)}(k).$$

(15)

The expansion coefficients $e_i$ in this series are obtained, in the course of diagonalization of the matrix $M_{ij}$ introduced in Eq. (4), as the eigenvectors of $M_{ij}$.

From the first of the two relations constituting the instantaneous Bethe–Salpeter equation for massless constituents, Eq. (3), the radial Salpeter component $\Psi_1(k)$ is then obtained in the form

$$\Psi_1(k) = \sum_{i=0}^{d-1} e_i \psi_i(k),$$

(16)

where—in order to match the structure of the first of Eqs. (3)—we took the liberty to define

$$\psi_i(k) := \frac{1}{M} \left[ 2 k \phi_i^{(0)}(k) + \sum_{j=0}^{N} V_{ji}^{(0)}(\mu) \phi_j^{(0)}(k) \right], \quad i = 0, 1, 2, \ldots.$$

The corresponding representations of the above Salpeter components $\Psi_1(k)$ and $\Psi_2(k)$ in configuration space, denoted by $\Psi_1(r)$ and $\Psi_2(r)$, are calculated by a Fourier–Bessel transformation (for $J = 0$):

$$\Psi_1(r) = \sum_{i=0}^{d-1} e_i \psi_i(r),$$

(17)

$$\Psi_2(r) = \sum_{i=0}^{d-1} e_i \phi_i^{(0)}(r),$$

(18)
with
\[
\psi_i(r) = \frac{1}{M} \left[ 2 \sum_{j=0}^{N} b_{ij}(\mu) \phi_j^{(0)}(r) + V(r) \phi_i^{(0)}(r) \right], \quad i = 0, 1, 2, \ldots .
\]

Clearly, the relevant objects are the matrix elements calculated from these amplitudes.

Figure 2 shows (similarly to Fig. 1) in reasonable agreement with Fig. 4 of Ref. [11], as judged by the eye) the behaviour of the radial Salpeter components \( \Psi_1(r) \) and \( \Psi_2(r) \) in configuration space for the ground state of the physical system under consideration. According to Sec. 2.1, the normalization of the radial Salpeter components \( \Psi_1(k) \) and \( \Psi_2(k) \) in momentum space is determined by the norm \( \| \chi \| \) of the Salpeter amplitude \( \chi \):
\[
\| \chi \|^2 = \frac{4}{(2\pi)^3} \int_0^\infty dk k^2 \left[ \Psi_1^*(k) \Psi_2(k) + \Psi_2^*(k) \Psi_1(k) \right].
\]

In the present approach (summarized, as far as the Salpeter amplitude \( \chi \) is concerned, in Eqs. (14) and (13) or Eqs. (17) and (18), respectively) the normalization of the radial component \( \Psi_2 \) is identical in configuration and momentum space:
\[
\int_0^\infty dr r^2 |\Psi_2(r)|^2 = \int_0^\infty dk k^2 |\Psi_2(k)|^2 = \sum_{i=0}^{d-1} |c_i|^2.
\]

For the plot presented in Fig. 2 we have chosen this normalization to be equal to unity. The normalization of the radial component \( \Psi_1 \) then follows from Eq. (17) or Eq. (16). Once the desired value of the norm \( \| \chi \| \) of the Salpeter amplitude \( \chi \) has been fixed, it is an easy task to determine the common normalization factor of \( \Psi_1 \) and \( \Psi_2 \) such that the Salpeter amplitude involving these wave functions satisfies its normalization condition.

Figure 2: Radial “Salpeter components” \( \Psi_1(r) \) and \( \Psi_2(r) \) in configuration space (in units of \( \text{GeV}^{3/2} \)) for the lowest-lying (positive-norm) \( J^{PC} = 0^{-+} \) bound state of two massless spin-\( \frac{1}{2} \) constituents obtained by the instantaneous Bethe–Salpeter equation, with a time-component Lorentz-vector confining interaction kernel involving a linear potential \( V(r) = \lambda r \) with slope \( \lambda = 0.2 \ \text{GeV}^2 \), as a function of the relative distance \( r \) (in units of \( \text{GeV}^{-1} \)) of the bound-state constituents, for a matrix size \( d = 50 \) and \( N = 49 \) in the intermediate-step series expansions; the lower and upper curves represent the Salpeter components \( \Psi_1(r) \) and \( \Psi_2(r) \), respectively.
4 Summary, Conclusions, and Outlook

The present investigation has been devoted to the explicit construction of some matrix representation of the Bethe–Salpeter equation in the instantaneous approximation for the involved interaction kernel. As usual, this conversion to a matrix equation has been achieved by systematic expansions of the solutions and certain intermediate quantities in terms of suitably chosen sets of basis states in the Hilbert space under consideration. A very important feature of our particular choice of basis states is that these states may be represented equally well in configuration and in momentum space. As a consequence of this, matrix elements of operators may be evaluated in the (with respect to the given operator) more convenient representation space: the matrix elements of the relativistic kinetic-energy operator (which, because of the troublesome square root, is nonlocal in configuration space) can be evaluated in momentum space (where it is represented by a simple multiplication operator which, of course, will be reminiscent of the square root); on the other hand, the matrix elements of the interaction-potential operator (which is, in general, nonlocal in momentum space) can be evaluated in configuration space (that is, the environment where the interaction potentials are more frequently formulated). For the wide class of power-law potentials, the algebraic expression of the latter matrix elements can be easily deduced from the general result given in any of Refs. [15, 16, 17].

Merely for the sake of illustration of our technique, we adopted the linear potential as some kind of toy model for quark confinement.

Our main result is the formulation of the instantaneous Bethe–Salpeter equation as a matrix eigenvalue problem with explicitly known matrices. We notice several highly welcome features of this approach, which render it a very efficient technique of solution:

1. Because of the scaling behaviour of the involved quantities pointed out in Sec. 2.2 at every instance, all dependence on both the variational parameter $\mu$ and on the coupling constants entering into the interaction potential $V(r)$ (like, in the case of a linear potential, the slope $\lambda$) may be factored out from the matrix elements. As a consequence, only matrices evaluated for unit values of the parameters are needed in our explicit formulation of the eigenvalue problem. For a given number of basis states taken into account, these matrices have to be calculated only once. This set of matrices includes the “moment” integrals $I^{(2)}_{ij}(1)$, all the (non-square) matrices of expansion coefficients $b_{ij}(1)$, $c_{ij}$, and $d_{ij}(1)$, as well as the expectation values $V^{(\ell)}_{ij}(1, 1)$, $\ell = 0, 1$, of the linear interaction potential (in our special case).

2. The only numerical operation required in the course of our method of solution is the diagonalization of the matrix representing the instantaneous Bethe–Salpeter equation. For the size of this matrix representation small enough, more precisely, for $d \leq 4$, even this diagonalization may be performed analytically. Because of an adequate choice of basis, already the case $d = 1$ yields a remarkably good result.

Thus, our analytic approach offers a viable way of solving the Bethe–Salpeter equation.

Finally, we feel obliged to address the questions of the significance of the developed technique and to discuss prospects for an actual application of the formalism presented here to the description of hadrons—in particular, of mesons—as bound states of quarks within the quantum field theory of the strong interactions, quantum chromodynamics. A quantitative discussion of these problems is certainly beyond the scope of the present approach. Consequently, we’ll content ourselves with the addition of a few modest and merely qualitative comments.
• In order to justify at all the consideration of the instantaneous approximation to
the Bethe–Salpeter formalism, one should be able to answer the question: Which
physical systems may be reliably described by the instantaneous Bethe–Salpeter
equation? In other words, for which physical systems is the neglect of retardation
effects meaningful? Any sound quantitative statement about this problem would
require a comparison of the predictions for the bound-state masses of, on the one
hand, the full Bethe–Salpeter equation and, on the other hand, the instantaneous
Bethe–Salpeter equation. (This analysis would be similar to the one performed in
Ref. [18] for the region of validity of the reduced Salpeter equation.) However, in
order to carry out a study of this kind, one must have at one’s disposal a reliable
method for extracting information from the full Bethe–Salpeter equation; this is,
at present, unfortunately not the case.

• In the hopefully pedagogical example chosen for the introduction of our method,
we confined ourselves to the (somewhat simpler) case of bound-state constituents
with vanishing masses. Clearly, the next step in this game has to be the extension
of this method to the case of bound-state constituents with nonvanishing masses.
The generalization to the massive case is more or less straightforward (and can be
done for an arbitrary Lorentz structure of the Bethe–Salpeter interaction kernel).
Of course, the nonvanishing values of the bound-state constituents’ masses entail
a more complicated structure of the Bethe–Salpeter equation and, consequently,
a higher degree of complexity of all resulting expressions. From our point of view,
this fact renders the results for the general case not suitable for the introductory
presentation of the main ideas. Accordingly, the results generalizing this study to
massive bound-state constituents have been reserved for a separate publication,
see Ref. [19]; our conclusions there are that the formalism developed here may be
applied with due modifications to the same class of potentials as considered here.

• One long-standing problem in meson phenomenology is related to the question
of the Lorentz structure of the confining Bethe–Salpeter interaction kernel. For the
two reasons explained in the Introduction—namely, stability of the solutions and
linearity of the mesonic Regge trajectories—we decided to focus our attention to
a confining interaction kernel of time-component Lorentz-vector type. However,
as is now well known [5, 7, 11], a time-component Lorentz-vector confining kernel
leads to a (from the phenomenological point of view) wrong sign of the spin–orbit
coupling and can thus not be considered a realistic model for the confinement of
the colour degrees of freedom present in quantum chromodynamics. Needless to
say, our formalism can accommodate an arbitrary Dirac structure of the kernel.
For the most popular choices of the Lorentz nature of the Bethe–Salpeter kernel,
in particular, for an interaction kernel of Lorentz-scalar (1 ⊗ 1), time-component
Lorentz-vector (γ^0 ⊗ γ^0), and full Lorentz-vector (γ^μ ⊗ γ^μ) type, the explicit form
of the instantaneous Bethe–Salpeter equation may be read off from Refs. [4, 11].

• Any realistic quark model for light mesons requires the inclusion of a mechanism
for the spontaneous breaking of chiral symmetry. Spontaneous chiral symmetry
breaking will manifest itself in the structure of the quark propagators; the latter
will deviate in their form from the form relevant for the free case. In principle, in
our formalism the inclusion of spontaneous chiral symmetry breaking in this way
should always be possible by application of the appropriate expansions in terms
of the basis states to the expressions involving the (non-free) quark form factors.
A proper embedding of spontaneous chiral symmetry breaking should manage to reconcile completely the Bethe–Salpeter formalism with the Nambu–Goldstone theorem, which demands the existence of a massless Nambu–Goldstone boson for every spontaneously broken global symmetry of the theory. This would allow for the interpretation of all light pseudoscalar mesons $\pi$, $K$, $\eta$ as (pseudo-) Goldstone bosons in the Bethe–Salpeter framework. (For a time-component Lorentz-vector Bethe–Salpeter kernel, this mechanism has been shown to operate in the desired way in Ref. [5]. This proves that the existence of massive bound states of massless constituents is definitely no inherent feature of the Bethe–Salpeter formalism.)

Thus, we may be optimistic that our technique for solving the Bethe–Salpeter equation in the instantaneous approximation by explicit construction of a matrix representation of this eigenvalue equation will eventually become an important tool for the description of hadrons as bound states of quarks from first principles, quantum chromodynamics.

**Acknowledgements**

One of us (W. L.) would like to thank Michael Beyer for encouraging discussions at the “International Symposium on ‘Quarks in Hadrons and Nuclei’” at Oberwölz, Austria. Moreover, we would like to thank Martin Olsson for several discussions of the results of the recent study of the instantaneous Bethe–Salpeter equation by him and coworkers. One of us (K. M. M.) would like to thank the Erwin Schrödinger International Institute for Mathematical Physics, where part of this work was done, for hospitality and would like to acknowledge also the support by the NSF under grant no. HRD-9633750.
A The “Generalized Laguerre” Basis

Our choice of the basis states \( |\phi_i\rangle \) is defined by their configuration-space representation

\[
\phi_i^{(\ell)}(r) = \sqrt{\frac{(2\mu)^{2\ell+3} i!}{\Gamma(2\ell + i + 3)}} r^\ell \exp(-\mu r) L_i^{(2\ell+2)}(2\mu r), \quad i = 0, 1, 2, \ldots ,
\]

which involves the generalized Laguerre polynomials \( L_i^{(\gamma)}(x) \) (for the parameter \( \gamma \)): the latter quantities are orthogonal polynomials which are defined by the power series \[14\]

\[
L_i^{(\gamma)}(x) = \sum_{t=0}^\infty (-1)^t \binom{i + \gamma}{i - t} \frac{x^t}{t!}, \quad i = 0, 1, 2, \ldots ,
\]

and which are orthonormalized, with the weight function \( x^\gamma \exp(-x) \), according to \[14\]

\[
\int_0^\infty dx \ x^\gamma \exp(-x) L_i^{(\gamma)}(x) L_j^{(\gamma)}(x) = \frac{\Gamma(\gamma + i + 1)}{i!} \delta_{ij}, \quad i, j = 0, 1, 2, \ldots .
\]

The requirement of normalizability of the Hilbert-space basis states \( |\phi_i\rangle \) imposes on the variational parameter \( \mu \) the constraint

\[\mu > 0.\]

In this case, the configuration-space basis functions \( \phi_i^{(\ell)}(r) \), defined by Eq. \[19\], satisfy the orthonormalization condition

\[
\int_0^\infty dr \ r^2 \phi_i^{(\ell)}(r) \phi_j^{(\ell)}(r) = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots .
\]

Note: the configuration-space representation of our basis states \( |\phi_i\rangle \) is chosen to be real.

The momentum-space representation of these basis states, \( \phi_i^{(\ell)}(p) \), obtained by Fourier transformation of Eq. \[19\], reads

\[
\phi_i^{(\ell)}(p) = \sqrt{\frac{(2\mu)^{2\ell+3} i!}{\Gamma(2\ell + i + 3)}} \frac{(-i)^\ell p^\ell}{2^{\ell+1/2} \Gamma(\ell + \frac{3}{2})} \times \sum_{t=0}^i \binom{i + 2\ell + 2}{i - t} \frac{\Gamma(2\ell + t + 3)(2\mu)^t}{(p^2 + \mu^2)^{(2\ell+3)/2}} \times F\left(\frac{2\ell + t + 3}{2}, -\frac{1 + t}{2}; \ell + \frac{3}{2}; -\frac{p^2}{p^2 + \mu^2}\right), \quad i = 0, 1, 2, \ldots ,
\]

with the hypergeometric series \( F \), defined, in terms of the gamma function \( \Gamma \), by \[14\]

\[
F(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(u)\Gamma(v)} \sum_{n=0}^\infty \frac{\Gamma(u+n)\Gamma(v+n)z^n}{\Gamma(w+n)n!}.
\]

The momentum-space basis functions \( \phi_i^{(\ell)}(p) \) satisfy the orthonormalization condition

\[
\int_0^\infty dp \ p^2 \phi_i^{(\ell)}(p) \phi_j^{(\ell)}(p) = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots .
\]

The availability of the Fourier transform of our basis functions \( \phi_i^{(\ell)}(r) \) in analytic form represents the main advantage of our choice \[19\]. Note that the momentum-space basis functions are real for \( \ell = 0 \), as well as for all even values of \( \ell \):

\[
\phi_i^{(\ell)}(p) = \phi_i^{(\ell)}(p) \quad \text{for } \ell = 0, 2, 4, \ldots .
\]
B Evaluation of the Matrix $M_{ij}$

B.1 Preliminaries

In the course of evaluating the various matrices $A_{ij}, \ldots, D_{ij}$ summing up to the matrix $M_{ij}$ according to Eq. (4), we occasionally encounter (“moment”) integrals of the type

$$I^{(n)}_{ij}(\mu) \equiv \int_0^\infty dk \ k^{2+n} \ \phi_i^{(0)}(k) \ \phi_j^{(0)}(k), \quad n = 0, 1, 2, \ldots .$$

(22)

According to the above definition, all the matrices $I^{(n)}_{ij}(\mu)$ are both real and symmetric under permutation of the indices $i$ and $j$. Taking advantage of a somewhat simplified expression for our momentum-space basis functions $\phi_i^{(\ell)}(p)$ valid in the case $\ell = 0$ [15], viz.,

$$\phi_i^{(0)}(p) = \sqrt{\frac{i!}{\mu \pi \Gamma(i+3)}} \frac{4}{p} \sum_{t=0}^i (-2)^t \left( \begin{array}{c} i+2 \\ i-t \\ \end{array} \right) \left( \begin{array}{c} j+2 \\ j-s \\ \end{array} \right) (r+1) (s+1) \times \left( 1 + \frac{p^2}{\mu^2} \right)^{-(t+2)/2} \sin \left( (t+2) \arctan \frac{p}{\mu} \right),$$

it is rather straightforward to find the (exact) analytic expression for the integrals (22):

$$I^{(n)}_{ij}(\mu) = \frac{4 \mu^n}{\pi \sqrt{(i+1) (i+2) (j+1) (j+2)}} \times \sum_{r=0}^i \sum_{s=0}^j (-2)^{r+s} \left( \begin{array}{c} i+2 \\ i-r \\ \end{array} \right) \left( \begin{array}{c} j+2 \\ j-s \\ \end{array} \right) (r+1) (s+1) \times \sum_{k=0}^{r-s} \left( \begin{array}{c} r-s \\ k \\ \end{array} \right) \frac{\Gamma \left( \frac{1}{2} (k+n+1) \right) \Gamma \left( \frac{1}{2} (r+s+3+|r-s|-n-k) \right)}{\Gamma \left( \frac{1}{2} (r+s+4+|r-s|) \right)} \cos \left( \frac{k \pi}{2} \right) - \sum_{k=0}^{r+s+4} \left( \begin{array}{c} r+s+4 \\ k \\ \end{array} \right) \frac{\Gamma \left( \frac{1}{2} (k+n+1) \right) \Gamma \left( \frac{1}{2} (2r+2s+7-n-k) \right)}{\Gamma (r+s+4)} \cos \left( \frac{k \pi}{2} \right).$$

Furthermore, we will need the (exact) analytic expressions of integrals of the kind

$$J^{(n)}_{ij}(\mu) \equiv \int_0^\infty dk k^{2+n} \ \phi_i^{*(1)}(k) \ \phi_j^{(0)}(k), \quad n = 0, 1, 2, \ldots .$$

(24)

In order to get rid of the (difficult-to-handle) hypergeometric series $F$ entering in the momentum-space basis functions $\phi_i^{(1)}(p)$, we employ a suitable recursion formula [14]:

$$\phi_i^{(1)}(p) = -i \sqrt{\frac{\mu^5}{\pi (i+1) (i+2) (i+3) (i+4) p^2}} \times \sum_{t=0}^i (-2)^t \left( \begin{array}{c} i+4 \\ i-t \\ \end{array} \right) \frac{(t+3)! \mu^t}{(p^2 + \mu^2)^{(t+3)/2}} \times \left[ \frac{\sqrt{p^2 + \mu^2}}{t+2} \sin \left( (t+2) \arctan \frac{p}{\mu} \right) - \frac{\mu}{t+3} \sin \left( (t+3) \arctan \frac{p}{\mu} \right) \right].$$
Note: since \( \phi^{(1)}_i(p) \) is purely imaginary, all these matrices \( J^{(n)}_{ij}(\mu) \) are purely imaginary. Again it is a tedious but straightforward task to work out the integrals (24) explicitly:

\[
J^{(n)}_{ij}(\mu) = i \frac{8 \mu^n}{\pi \sqrt{(i + 1) (i + 2)} (i + 3) (i + 4) (j + 1) (j + 2)}
\times \sum_{r=0}^{j} \sum_{s=0}^{j} (-2)^{r+s} (r + 1) (r + 2) (r + 3) (s + 1) \left( \begin{array}{c} i + 4 \\ i - r \end{array} \right) \left( \begin{array}{c} j + 2 \\ j - s \end{array} \right)
\times \left\{ \begin{array}{c} 1 \\ \frac{1}{r + 2} \end{array} \right\} \sum_{k=0}^{r-s} \left( \begin{array}{c} |r - s| \\ k \end{array} \right) \frac{\Gamma(\frac{1}{2} (n + k)) \Gamma(\frac{1}{2} (4 + r + s + |r - s| - n - k))}{\Gamma(\frac{1}{2} (4 + r + s + |r - s|))} \cos \left( \frac{k \pi}{2} \right)
\times \cos \left( \frac{k \pi}{2} \right)
\times \frac{\Gamma(\frac{1}{2} (n + k)) \Gamma(\frac{1}{2} (5 + r + s + |1 + r - s| - n - k))}{\Gamma(\frac{1}{2} (5 + r + s + |1 + r - s|))} \cos \left( \frac{k \pi}{2} \right)
\times \sum_{k=0}^{5+r+s} \left( \begin{array}{c} 5 + r + s \\ k \end{array} \right) \frac{\Gamma(\frac{1}{2} (n + k)) \Gamma(\frac{1}{2} (10 + 2 r + 2 s - n - k))}{\Gamma(5 + r + s)} \cos \left( \frac{k \pi}{2} \right) \right\}.
\]

The radial basis functions \( \phi^{(\ell)}_i(r) \) and \( \phi^{(\ell)}_i(p) \) in configuration and momentum space given in Eqs. (19) and (20), respectively, are derived from the respective basis functions on the (full) three-dimensional Euclidean space \( R^3 \) by factorizing off the dependence on the angular variables. This dependence is described by the spherical harmonics \( Y_{\ell m}(\Omega) \) for angular momentum \( \ell = 0, 1, 2, \ldots \) and its projection \( m = -\ell, -\ell + 1, \ldots, +\ell \), which depend on the solid angle \( \Omega \) and are orthonormalized according to

\[
\int d\Omega \, Y_{\ell m}(\Omega) Y_{\ell' m'}(\Omega) = \delta_{\ell \ell'} \delta_{m m'}.
\]

Since the two sets of basis functions on \( R^3 \) are related by a Fourier transformation, the corresponding radial basis functions \( \phi^{(\ell)}_i(r) \) and \( \phi^{(\ell)}_i(p) \), \( i = 0, 1, 2, \ldots \), are related by

\[
\phi^{(\ell)}_i(r) = i^\ell \sqrt{\frac{2}{\pi}} \int_0^\infty dp \, p^{\ell} j_\ell(p r) \phi^{(\ell)}_i(p),
\]

\[
\phi^{(\ell)}_i(p) = (-i)^\ell \sqrt{\frac{2}{\pi}} \int_0^\infty dr \, r^{\ell} j_\ell(p r) \phi^{(\ell)}_i(r),
\]

which may be easily seen with the help of the well-known expansion of the plane waves in terms of spherical harmonics \( Y_{\ell m} \) in configuration \( (\Omega_\chi) \) and momentum \( (\Omega_\mu) \) space,

\[
\exp(i \mathbf{p} \cdot \mathbf{x}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(p r) Y_{\ell m}^*(\Omega_\mu) Y_{\ell m}(\Omega_\chi).
\]
B.2 Evaluation of $A_{ij}$

Expressed in terms of the integral $I_{ij}(n)(\mu)$ introduced in Eq. (22), the term $A_{ij}$ in the matrix $M_{ij}$, defined in Eq. (5), is simply given by

$$A_{ij} = 4 I_{ij}^{(2)}(\mu) .$$

B.3 Evaluation of $B_{ij}$

The term $B_{ij}$ in the matrix $M_{ij}$, Eq. (6), becomes, for some interaction potential $V(r)$, which enters in this term linearly in form of Eq. (2),

$$B_{ij} = 4 \pi \int_0^\infty dr \int_0^\infty dk^{3} \frac{\phi_{i}^{(0)}(k) j_0(k r) \int_0^{\infty}dk' k'^{2} j_0(k' r) \phi_{j}^{(0)}(k')}{0} .$$

In order to be able to apply the Fourier–Bessel relation (26), we expand the expression $k \phi_{i}^{(0)}(k)$ in terms of the momentum-space basis functions $\phi_{i}^{(0)}(k)$:

$$k \phi_{i}^{(0)}(k) = \sum_{j=0}^{N} b_{ji} \phi_{j}^{(0)}(k) . \quad (28)$$

As a consequence of the orthonormality (21) of the (momentum-space) basis functions $\phi_{i}^{(1)}(p)$, the expansion coefficients $b_{ij}$ may be expressed in terms of the integral $I_{ij}^{(1)}(\mu)$ defined in Eq. (22):

$$b_{ij} = \int_0^{\infty} dk^{3} \frac{\phi_{i}^{(0)}(k) \phi_{j}^{(0)}(k)}{0} \equiv I_{ij}^{(1)}(\mu) . \quad (29)$$

Inserting the expansion (28) involving the coefficients $b_{ij}$, applying the Fourier–Bessel relation (26), and remembering the definition (4) of the expectation values $V_{ij}^{(t)}$ of the interaction potential $V(r)$ under consideration, the term $B_{ij}$ reduces to the—at least, for all power-law potentials algebraic—expression

$$B_{ij} = 2 \sum_{r=0}^{N} b_{ri}(\mu) V_{rj}^{(0)}(\mu) . \quad (30)$$

B.4 Evaluation of $C_{ij}$

The term $C_{ij}$ in the matrix $M_{ij}$, Eq. (7), becomes, for some interaction potential $V(r)$, which enters in this term linearly in form of Eq. (5),

$$C_{ij} = 4 \pi \int_0^\infty dr \int_0^\infty dk^{2} \frac{\phi_{i}^{(0)}(k) j_1(k r) \int_0^{\infty}dk' k'^{3} j_1(k' r) \phi_{j}^{(0)}(k')}{0} .$$

In order to be able to apply the Fourier–Bessel relation (26), we are forced to expand both the momentum-space basis functions $\phi_{i}^{(0)}(k)$ and the expression $k \phi_{i}^{(0)}(k)$ in terms of the momentum-space basis functions $\phi_{i}^{(1)}(k)$:

$$\phi_{i}^{(0)}(k) = \sum_{j=0}^{N} c_{ji} \phi_{j}^{(1)}(k) , \quad (31)$$

$$k \phi_{i}^{(0)}(k) = \sum_{j=0}^{N} d_{ji} \phi_{j}^{(1)}(k) . \quad (32)$$
As a consequence of the orthonormality (21) of the (momentum-space) basis functions \( \phi_i^{(\ell)}(p) \), the expansion coefficients \( c_{ij} \) and \( d_{ij} \) may be expressed in terms of the integrals \( J_{ij}^{(n)}(\mu) \), \( n = 0, 1 \), defined in Eq. (24):

\[
c_{ij} = \int_0^\infty dk \, k^2 \, \phi_i^{(1)}(k) \, \phi_j^{(0)}(k) = J_{ij}^{(0)}(\mu) ,
\]

\[
d_{ij} = \int_0^\infty dk \, k^3 \, \phi_i^{(1)}(k) \, \phi_j^{(0)}(k) = J_{ij}^{(1)}(\mu) .
\]

Inserting the expansions (31) and (32) involving the expectation values \( V_{ij}^{(\ell)} \) of the interaction potential \( V(r) \) under consideration, the term \( C_{ij} \) reduces to the—at least, for all power-law potentials algebraic—expression

\[
C_{ij} = 2 \sum_{r=0}^N \sum_{s=0}^N c_{ri}^* d_{sj}(\mu) V_{rs}^{(1)}(\mu) .
\]

### B.5 Evaluation of \( D_{ij} \)

The term \( D_{ij} \) in the matrix \( M_{ij} \), Eq. (8), becomes, for some interaction potential \( V(r) \), which enters in this term quadratically in form of Eq. (2),

\[
D_{ij} = \frac{4}{\pi^2} \int_0^\infty dr \, r^2 \, V(r) \int_0^\infty dr' \, r'^2 \, V(r') \int_0^\infty dk \, k^2 \, \phi_i^{(0)}(k) \, j_1(kr) \times \int_0^\infty dk' \, k'^2 \, j_1(k' r) \, j_0(k' r') \int_0^\infty dk'' \, k''^2 \, j_0(k'' r') \, \phi_j^{(0)}(k'') .
\]

Adopting the expansion (31) of \( \phi_i^{(0)}(k) \) in terms of \( \phi_i^{(1)}(k) \)—which introduces again the coefficients \( c_{ij} \)—it is clearly no problem to apply twice the Fourier–Bessel relation (24):

\[
D_{ij} = \frac{i}{2} \sum_{s=0}^N c_{si}^* \int_0^\infty dr \, r^2 \, V(r) \phi_s^{(1)}(r) \int_0^\infty dr' \, r'^2 \, V(r') \phi_j^{(0)}(r') \int_0^\infty dk \, k^2 \, j_1(kr) \, j_0(kr') .
\]

However, in order to be able to apply the Fourier–Bessel relation (27), we are forced to expand the expressions \( V(r) \phi_i^{(\ell)}(r) \), \( \ell = 0, 1 \), over the appropriate configuration-space basis functions \( \phi_i^{(\ell)}(r) \), \( \ell = 0, 1 \), respectively. The corresponding expansion coefficients are the expectation values \( V_{ij}^{(\ell)}(\mu) \) of the interaction potential \( V(r) \), defined by Eq. (9):

\[
V(r) \phi_i^{(\ell)}(r) = \sum_{j=0}^N V_{ji}^{(\ell)}(\mu) \phi_j^{(\ell)}(r) , \quad \ell = 0, 1 .
\]

Inserting these expansions involving the expectation values \( V_{ij}^{(\ell)}(\mu) \) of the interaction potential \( V(r) \) in Eq. (9), applying the Fourier–Bessel relation (27), and remembering the orthonormalization relation (21) satisfied by the momentum-space basis functions \( \phi_i^{(\ell)}(p) \), the term \( D_{ij} \) reduces to the—at least, for all power-law potentials algebraic—expression

\[
D_{ij} = \sum_{r=0}^N \sum_{s=0}^N \sum_{t=0}^N c_{ri}^* c_{st} V_{sr}^{(1)}(\mu) V_{ij}^{(0)}(\mu) .
\]
B.6 Relations Among the Expansion Coefficients

The expansion coefficients $b_{ij}, c_{ij}, d_{ij}$ defined by Eqs. (28), (31), (32), respectively, are, of course, not independent; they satisfy the (only in the limit $N \to \infty$, exact) relations

\[
\sum_{r=0}^{N} c_{ri}^{*} c_{rj} = \delta_{ij} , \tag{37}
\]

\[
\sum_{r=0}^{N} c_{ri}^{*} d_{rj}(\mu) = \sum_{r=0}^{N} d_{ri}^{*}(\mu) c_{rj} = I_{ij}^{(1)}(\mu) \equiv b_{ij}(\mu) , \tag{38}
\]

\[
\sum_{r=0}^{N} d_{ri}^{*}(\mu) d_{rj}(\mu) = I_{ij}^{(2)}(\mu) . \tag{39}
\]

Clearly, relation (37) expresses only the unitarity (in the limit $N \to \infty$) of the matrix $c \equiv (c_{ij})$ which represents, according to its definition (31), the transformation from the basis $\{\phi_{i}^{(1)}(p), i = 0, 1, 2, \ldots\}$ to the basis $\{\phi_{i}^{(0)}(p), i = 0, 1, 2, \ldots\}$ of the space $L_2(R^+)$.

B.7 Accuracy Considerations

The existence of relations like the ones in Eqs. (37), (38), and (39) opens a possibility to investigate systematically the errors induced by the truncations of the expansion series in the terms $B_{ij}, C_{ij},$ and $D_{ij}$. For instance, for $d = 15$ (i.e., $15 \times 15$ matrices) and for $N = 49$ (i.e., a truncation to the first 50 basis states), the relative error is less than 3%.

Furthermore, taking advantage of the fact that the momentum-space basis function $\phi_{0}^{(0)}(p)$ may be cast into the simple form

\[
\phi_{0}^{(0)}(p) = 4 \sqrt{\frac{2}{\pi}} \frac{\mu^{5/2}}{(p^2 + \mu^2)^2} ,
\]

the first entries of the terms $B_{ij}$ and $C_{ij}$ defined in Eqs. (8) and (9), respectively, can be evaluated, for appropriate choices of the interaction potential $V(r)$, analytically. These explicit results may be compared with the corresponding approximate values obtained from the series expansions (30) and (35), respectively. For the linear potential (10), the numerical value of the term $B_{00}$ is found to be

\[
\frac{4\pi}{\lambda} B_{00} \equiv \frac{2}{\pi \lambda} \int_{0}^{\infty} dk k^3 \phi_{0}^{(0)}(k) \int_{0}^{\infty} dk' k'^2 V_0(k, k') \phi_{0}^{(0)}(k') = \frac{64}{3} = 21.3 .
\]

This has to be compared with the series expansion

\[
\frac{8\pi}{\lambda} \sum_{r=0}^{N} b_{r0}(\mu) V_{r0}^{(0)}(\mu) = \frac{8\pi}{\lambda} \sum_{r=0}^{N} b_{r0}(1) V_{r0}^{(0)}(1) ,
\]

which yields, for, e.g., $N = 49$, that is, taking into account the first 50 basis functions,

\[
\frac{8\pi}{\lambda} \sum_{r=0}^{49} b_{r0}(1) V_{r0}^{(0)}(1) = 21.33333333333334562 .
\]

Similarly, for a constant potential, i.e., $V(r) = \lambda$, the numerical value of the term $C_{00}$ is found to be

\[
\frac{4\pi}{\lambda \mu} C_{00} \equiv \frac{2}{\pi \lambda \mu} \int_{0}^{\infty} dk k^3 \phi_{0}^{(0)}(k) \int_{0}^{\infty} dk' k'^3 V_1(k, k') \phi_{0}^{(0)}(k') = \frac{64}{3} = 21.3 .
\]
This has to be compared with the series expansion
\[
\frac{8\pi}{\lambda \mu} \sum_{r=0}^{N} \sum_{s=0}^{N} c_{r0}^* d_{s0}(\mu) V_{rs}^{(1)}(\mu) = 8\pi \sum_{r=0}^{N} c_{r0}^* d_{r0}(1),
\]
which yields, for, e.g., \(N = 49\), that is, taking into account the first 50 basis functions,
\[
8\pi \sum_{r=0}^{49} c_{r0}^* d_{r0}(1) = 21.333307054.
\]
Beyond doubt, this accuracy of the series expansions should suffice for our purposes.

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