A note on intermittency for the fractional heat equation

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Abstract
The goal of the present note is to study intermittency properties for the solution to the fractional heat equation

\[ \frac{\partial u}{\partial t}(t, x) = -(-\Delta)^{\beta/2}u(t, x) + u(t, x)W(t, x), \quad t > 0, x \in \mathbb{R}^d \]

with initial condition bounded above and below, where \( \beta \in (0, 2] \) and the noise \( W \) behaves in time like a fractional Brownian motion of index \( H > 1/2 \), and has a spatial covariance given by the Riesz kernel of index \( \alpha \in (0, d) \). As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is \( \alpha < \beta \).

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1 Introduction
In this article we consider the fractional heat equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= -(-\Delta)^{\beta/2}u(t, x) + u(t, x)W(t, x), \quad t > 0, x \in \mathbb{R}^d \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

where \( \beta \in (0, 2] \), \( (-\Delta)^{\beta/2} \) denotes the fractional power of the Laplacian, and \( u_0 \) is a deterministic function such that

\[ a \leq u_0(x) \leq b \quad \text{for all} \quad x \in \mathbb{R}^d \]
for some constants \( b \geq a > 0 \). We let \( W = \{W(\varphi) ; \varphi \in \mathcal{H}\} \) be a zero-mean Gaussian process with covariance
\[
E(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}.
\]
Here \( \mathcal{H} \) is a Hilbert space defined as the completion of the space \( C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \) of infinitely differentiable functions with compact support on \( \mathbb{R}_+ \times \mathbb{R}^d \), with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) defined by:
\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_{\mathcal{H}} \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \varphi(t, x) \psi(s, y) |t - s|^{2H - 2} |x - y|^{-\alpha} \, dt \, ds \, dy,
\]
where \( \alpha_{\mathcal{H}} = H(2H - 1), \) \( H \in (1/2, 1) \) and \( \alpha \in (0, d) \). We denote by \( \dot{W} \) the formal derivative of \( W \). The noise \( W \) is spatially homogeneous with spatial covariance given by the Riesz kernel \( f(x) = |x|^{-\alpha} \) and behaves in time like a fractional Brownian motion of index \( H \). We refer to [2, 3, 5] for more details.

Let \( G(t, x) \) be the fundamental solution of \( \frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0 \) and
\[
w(t, x) = \int_{\mathbb{R}^d} u_0(y) G(t, x - y) \, dy
\]
be the solution of the equation \( \frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0 \) with initial condition \( u(0, x) = u_0(x) \). Note that
\[
G(t, \cdot) \text{ is the density of } X_t \tag{4}
\]
where \( X = (X_t)_{t \geq 0} \) is a symmetric Lévy process with values in \( \mathbb{R}^d \). If \( \beta = 2 \), then \( X \) coincides with a Brownian motion \( B = (B_t)_{t \geq 0} \) in \( \mathbb{R}^d \) with variance 2. If \( \beta < 2 \), then \( X \) is a \( \beta \)-stable Lévy process given by \( X_t = B_{\delta_t} \), where \( (S_t)_{t \geq 0} \) is a \((\beta/2)\)-stable subordinator with Lévy measure
\[
\nu(dx) = \frac{\beta/2}{\Gamma(1 - \beta/2)} x^{-\beta/2 - 1} 1_{\{x > 0\}} dx.
\]
Due to [2] and [4], it follows that for all \( t > 0 \) and \( x \in \mathbb{R}^d \),
\[
a \leq w(t, x) \leq b. \tag{5}
\]

There is a rich literature dedicated to the case \( H = 1/2 \), when the noise \( W \) is white in time. We refer to [3, 12] for some general properties, and to [11, 8, 7] for intermittency properties of the solution to the heat equation with this type of noise. Different methods have to be used for \( H > 1/2 \), since in this case the noise is not a semi-martingale in time.

In the present article, we follow the approach of [13, 5] for defining the concept of solution. We say that a process \( u = \{u(t, x) ; t \geq 0, x \in \mathbb{R}^d\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\) is a mild solution of (11) if it is square-integrable, adapted with respect to the filtration induced by \( W \), and satisfies:
\[
u(dx) = \frac{\beta/2}{\Gamma(1 - \beta/2)} x^{-\beta/2 - 1} 1_{\{x > 0\}} dx.
\]
where the stochastic integral is interpreted as the divergence operator of \( W \) (see (15)). Using Malliavin calculus techniques, it can be shown that the mild solution (if it exists) is unique and has the Wiener chaos decomposition:

\[
u(t, x) = \sum_{n \geq 0} I_n(f_n(\cdot, t, x))
\]

where \( I_n \) denotes the multiple Wiener integral (with respect to \( W \)) of order \( n \), and the kernel \( f_n(\cdot, t, x) \) is given by:

\[
f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = G(t - t_n, x - x_n) \cdots G(t_2 - t_1, x_2 - x_1) w(t_1, x_1) 1_{0 < t_1 < \ldots < t_n < t}
\]

(see page 303 of [13]). By convention, \( f_0(t, x) = w(t, x) \) and \( I_0 \) is the identity map on \( \mathbb{R} \).

The necessary and sufficient condition for the existence of the mild solution is that the series in (6) converges in \( L^2(\Omega) \), i.e.

\[
S(t, x) := \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t, x) < \infty,
\]

where

\[
\alpha_n(t, x) = n! E[I_n(f_n(\cdot, t, x))]^2 = (n!)^2 \| \hat{f}_n(\cdot, t, x) \|^2_{\mathfrak{H}^n}
\]

and \( \hat{f}_n(\cdot, t, x) \) is the symmetrization of \( f_n(\cdot, t, x) \) in the \( n \) variables \( (t_1, x_1), \ldots, (t_n, x_n) \). If the solution \( u \) exists, then \( E[u(t, x)]^2 = S(t, x) \). We refer to Section 4.1 of [13] and Section 2 of [5] for the details. Note that if \( u_0(x) = u_0 \) for all \( x \in \mathbb{R}^d \), then the law of \( u(t, x) \) does not depend on \( x \), and hence \( \alpha_n(t, x) = \alpha_n(t) \).

The goal of the present work is to give an upper bound for the \( p \)-th moment of the solution of (1) (for \( p \geq 2 \)), and a lower bound for its second moment. In particular, this will show that, if \( u_0(x) \) does not depend on \( x \), then the solution \( u \) of (1) is weakly \( \rho \)-intermittent, in a sense which has been recently introduced in [4], i.e. \( \gamma_\rho(2) > 0 \) and \( \gamma_\rho(p) < \infty \) for all \( p \geq 2 \), where

\[
\gamma_\rho(p) = \limsup_{t \to \infty} \frac{1}{t^p} \log E[u(t, x)]^p
\]

is a modified Lyapunov exponent (which does not depend on \( x \)), and

\[
\rho = \frac{2H\beta - \alpha}{\beta - \alpha}.
\]

As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is \( \alpha < \beta \). Note that this condition is equivalent to

\[
I_\beta(\mu) := \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{\beta/2} \mu(d\xi) < \infty
\]

with \( \mu(d\xi) = c_{\alpha, d}|\xi|^{-d+\alpha} d\xi \), which is encountered in the study of equations with white noise in time. When \( \beta = 2 \), (9) is called Dalang’s condition (see [4]).
2 The result

The goal of the present article is to prove the following result.

**Theorem 2.1.** The necessary and sufficient condition for equation (1) to have a mild solution is \(\alpha < \beta\). If the solution \(u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}\) exists, then for any \(p \geq 2\), for any \(x \in \mathbb{R}^d\) and for any \(t > 0\) such that \(pt^{2H-\alpha/\beta} > t_1\)

\[ E|u(t,x)|p \leq b^p \exp(C_1 p^{(2\beta-\alpha)/(\beta-\alpha)} t^p) \]

and for any \(x \in \mathbb{R}^d\) and for any \(t > t_2\),

\[ E|u(t,x)|2 \geq a^2 \exp(C_2 t^p), \]

where \(p\) is given by (3), \(a, b\) are the constants given by (2), and \(t_1, t_2, C_1, C_2\) are some positive constants depending on \(d, \alpha, \beta\) and \(H\).

Before giving the proof, we recall from [5] that

\[ \alpha_n(t, x) = \alpha^n_H \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(t, s) dtds \quad (10) \]

where

\[ \psi_n(t, s) = \int_{\mathbb{R}d} \prod_{j=1}^n |x_j - y_j|^{-\alpha} f_n(t_1, x_1, \ldots, t_n, x_n, t, x) f_n(s_1, y_1, \ldots, s_n, y_n, t, x) dx dy \]

and we denote \(t = (t_1, \ldots, t_n)\), \(s = (s_1, \ldots, s_n)\) with \(t_i, s_i \in [0, t]\) and \(x = (x_1, \ldots, x_n)\), \(y = (y_1, \ldots, y_n)\) with \(x_i, y_i \in \mathbb{R}^d\).

Note that the Fourier transform of \(G(t, \cdot)\) is given by:

\[ \mathcal{F}G(t, \cdot)(\xi) := \int_{\mathbb{R}d} e^{-i\xi \cdot x} G(t, x) dx = \exp(-|\xi|^d), \quad \xi \in \mathbb{R}^d \quad (11) \]

where \(|\cdot|\) denotes the Euclidean norm in \(\mathbb{R}^d\). Recall that for any \(\varphi, \psi \in L^1(\mathbb{R}^d)\),

\[ \int_{\mathbb{R}d} \int_{\mathbb{R}d} \varphi(x) \psi(y) |x - y|^{-\alpha} dx dy = c_{\alpha,d} \int_{\mathbb{R}d} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} |\xi|^{-d+\alpha} d\xi \quad (12) \]

where \(\mathcal{F}\varphi\) is the Fourier transform of \(\varphi\), \(c_{\alpha,d} = (2\pi)^{-d} C_{\alpha,d}\) and \(C_{\alpha,d}\) is the constant given by (21) (see Appendix A). This identity can be extended to functions \(\varphi, \psi \in L^1(\mathbb{R}^{nd})\):

\[ \int_{\mathbb{R}^{nd}} \int_{\mathbb{R}^{nd}} \varphi(x) \psi(y) \prod_{j=1}^n |x_j - y_j|^{-\alpha} dx dy = c_{\alpha,d}^{nd} \int_{\mathbb{R}^{nd}} \mathcal{F}\varphi(\xi_1, \ldots, \xi_n) \mathcal{F}\psi(\xi_1, \ldots, \xi_n) \prod_{j=1}^n |\xi_j|^{-d+\alpha} d\xi_1 \ldots d\xi_n. \quad (13) \]

We will use the following elementary inequality.
Lemma 2.2. For any $t > 0$ and $\eta \in \mathbb{R}^d$

$$
\int_{\mathbb{R}^d} e^{-t|\xi|^\beta} |\xi - \eta|^{-d+\alpha} d\xi \leq K_{d,\alpha,\beta} t^{-\alpha/\beta}
$$

where

$$K_{d,\alpha,\beta} := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^{\beta}}|\xi|^{-d+\alpha} d\xi.
$$

Proof: Using the change of variable $z = t^{1/\beta}(\eta - \xi)$, we have:

$$
\int_{\mathbb{R}^d} e^{-t|\xi|^\beta} |\xi - \eta|^{-d+\alpha} d\xi = t^{-\alpha/\beta} \int_{\mathbb{R}^d} e^{-|z-t^{1/\beta}|^{\beta}/\beta} |z|^{-d+\alpha} dz.
$$

The result follows using the inequality $e^{-z} \leq 1/(1 + x)$ for $x > 0$. □

Proof of Theorem 2.1 Step 1. (Sufficiency and upper bound for the second moment) Suppose that $\alpha < \beta$. We will prove that the series (7) converges, by providing upper bounds for $\psi_n(t, s)$ and $\alpha_n(t, x)$.

By the Cauchy-Schwarz inequality, $\psi_n(t, s) \leq \psi_n(t, t) 1/2 \psi_n(s, s) 1/2$. So it is enough to consider the case $t = s$. Let $u_j = t_{\mu(j+1)} - t_{\mu(j)}$ where $\mu$ is a permutation of $\{1, \ldots, n\}$ such that $t_{\mu(1)} < \ldots < t_{\mu(n)}$ and $t_{\mu(n+1)} = t$. Using (5), (11) and (13), and arguing as in the proof of Lemma 3.2 of [3], we obtain:

$$
\psi_n(t, t) \leq b^2 c_{\alpha,d} \sum_{\eta_1} d\eta_1 \exp(-u_1 |\eta_1|^{\beta}) |\eta_1|^{-d+\alpha} \sum_{\eta_2} d\eta_2 \exp(-u_2 |\eta_2|^{\beta}) |\eta_2 - \eta_1|^{-d+\alpha}
$$

$$
\ldots \sum_{\eta_n} d\eta_n \exp(-u_n |\eta_n|^{\beta}) |\eta_n - \eta_{n-1}|^{-d+\alpha}.
$$

By Lemma 2.2 it follows that:

$$
\psi_n(t, t) \leq b^2 c_{\alpha,d} K_{d,\alpha,\beta}^n (u_1 \ldots u_n)^{-\alpha/\beta}.
$$

By inequality (20) (Appendix A), $K_{d,\alpha,\beta} \leq c_\beta I_{d,\alpha,\beta}$, where $c_\beta = 2^{3/2} - 1$ and

$$
I_{d,\alpha,\beta} := \int_{\mathbb{R}^d} \left( \frac{\beta/2}{1 + |\xi|^{\beta}} \right)^{\beta/2} |\xi|^{-d+\alpha} d\xi = \frac{(2\pi)^d c_d \Gamma((\beta - \alpha)/2) \Gamma(\alpha/2)}{2 \Gamma(\beta/2)}
$$

(see relation (21) and Remark A.3 Appendix A). Hence,

$$
\psi_n(t, s) \leq b^2 C_{d,\alpha,\beta}^n |\beta(t)\beta(s)|^{-\alpha/2(\beta)}
$$

where $\beta(t) = u_1 \ldots u_n$, $\beta(s)$ is defined similarly, and $C_{d,\alpha,\beta} > 0$ is a constant depending on $d, \alpha, \beta$. Similarly to the proof of Proposition 3.5 of [3], we have:

$$
\alpha_n (t, x) \leq b^2 C_{d,\alpha,\beta, H}^n (n!)^{\alpha/\beta} t^{n(2H - \alpha/\beta)},
$$

(14)

where $C_{d,\alpha,\beta, H} > 0$ is a constant depending on $d, \alpha, \beta, H$. Since $\alpha < \beta$, it follows that the series (7) converges and

$$
E|u(t, x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n (t, x) \leq b^2 \sum_{n \geq 0} \frac{C_{d,\alpha,\beta, H}^n (n!)^{\alpha/\beta}}{(n!)^{1-\alpha/\beta}} t^{n(2H - \alpha/\beta)} \leq b^2 \exp(C_0 t^\nu),
$$
for all $t > t_0$, where $C_0 > 0$ and $t_0 > 0$ are constants depending in $d, \alpha, \beta, H$.

We used the fact that for any $a > 0$ and $x > 0$,

$$\sum_{n \geq 0} \frac{x^n}{(n!)^a} \leq \exp(c_0 x^{1/a}) \quad \text{for all} \quad x > x_0,$$

where $x_0 > 0$ and $c_0 > 0$ are some constants depending on $a$.

**Step 2. (Upper bound for the $p$-the moment)** Note that $u(t, x) = \sum_{n \geq 0} J_n(t, x)$ in $L^2(\Omega)$, where $J_n(t, x)$ lies in the $n$-th order Wiener chaos $H_n$ associated to the Gaussian process $W$ (see [15]). Hence,

$$E|u(t, x)|^2 = \sum_{n \geq 0} E|J_n(t, x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t, x).$$

We denote by $\| \cdot \|_p$ the $L^p(\Omega)$-norm. We use the fact that for a fixed Wiener chaos $H_n$, the $\| \cdot \|_p$ are equivalent, for all $p \geq 2$ (see the last line of page 62 of [15] with $q = p$ and $p = 2$). Hence,

$$\|J_n(t, x)\|_p \leq (p - 1)^{n/2}\|J_n(t, x)\|_2 = (p - 1)^{n/2} \left( \frac{1}{n!} \alpha_n(t, x) \right)^{1/2} \leq b{(p - 1)C_d,\alpha,\beta,H}^{n/2} \left( \frac{1}{(n!)^{(\beta-\alpha)/(2\beta)}} \right) p^{(2\beta(\beta-\alpha))/(2\beta)}$$

using (14) for the last inequality. Using Minkowski’s inequality for integrals (see Appendix A.1 of [16]) and inequality (15), we obtain that:

$$\|u(t, x)\|_p \leq \sum_{n \geq 0} \|J_n(t, x)\|_p \leq b \exp(C_1 (p - 1)^{\beta/(\beta-\alpha)} t^p)$$

if $pt^{2H-\alpha/\beta} > t_1$, where the constants $C_1 > 0$ and $t_1 > 0$ depend on $d, \alpha, \beta, H$.

**Step 3. (Necessity and lower bound for the second moment)** Suppose that equation (1) has a mild solution $u$, i.e. the series (7) converges. In particular,

$$\infty > \alpha_1(t, x) \geq a^2 \alpha_H \int_{[0,t]^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |r - s|^{2H-s} |y - z|^{-\alpha} G(s, y) G(r, z) dydzdrds$$

$$= a^2 \alpha_H c_{\alpha,d} \int_{\mathbb{R}^d} \left( \int_0^t \int_0^t |r - s|^{2H-2e-(r+s)} |\xi|^\beta drds \right) |\xi|^{-d+\alpha} d\xi$$

$$\geq a^2 \alpha_H c_{\alpha,d} c_H \int_{\mathbb{R}^d} \left( \frac{1}{1/|\xi| + |\xi|^{\beta}} \right)^{2H} |\xi|^{-d+\alpha} d\xi,$$

where we used (12) for the equality and Theorem 3.1 of [2] for the last inequality. From here, we infer that

$$\alpha < 2H\beta.$$

In particular, this implies that $\alpha < 2\beta$. 
Note that one can replace \( \psi_n(t, s) \) by \( \psi_n(te - t, te - s) \) in the definition \([1]\) of \( \alpha_n(t, x) \), where \( e = (1, \ldots, 1) \in \mathbb{R}^n \). By Lemma 2.2 of \([1]\), we have:

\[
\psi_n(te - t, te - s) = E \left[ w(t - t^*, x + X^1_{t^*}) w(t - s^*, x + X^2_s) \prod_{j=1}^n |X^1_{t_j} - X^2_{s_j}|^{-\alpha} \right],
\]

where \( t^* = \max\{t_1, \ldots, t_n\} \), \( s^* = \max\{s_1, \ldots, s_n\} \) and \( X^1, X^2 \) are two independent copies of the Lévy process \( X \) mentioned in the Introduction. (Lemma 2.2 of \([1]\) was proved for \( \beta = 2 \). The same proof is valid for \( \beta < 2 \).

Due to \((5)\), it follows that

\[
a^2M_n(t) \leq \alpha_n(t, x) \leq b^2M_n(t)
\]

(17)

where

\[
M_n(t) := E \left[ \alpha_H^2 \int_{0}^{t} \prod_{j=1}^{n} |t_j - s_j|^{2H - 2} \prod_{j=1}^{n} |X^1_{t_j} - X^2_{s_j}|^{-\alpha} dt ds \right] = E(L(t)^n)
\]

and \( L(t) \) is a random variable defined by:

\[
L(t) := \alpha_H \int_{0}^{t} \int_{0}^{t} |r - s|^{2H - 2} |X^1_r - X^2_s|^{-\alpha} dr ds.
\]

To prove that \( L(t) \) is finite a.s., we show that its mean is finite. Note that \( X^1_r - X^2_s \overset{d}{=} X_{r+s} = (r + s)^{1/\beta} X_1 \), and hence

\[
E[L(t)] = \alpha_H C_{d,\alpha,\beta} \int_{0}^{t} \int_{0}^{t} |r - s|^{2H - 2} (r + s)^{-\alpha/\beta} dr ds,
\]

where

\[
C_{d,\alpha,\beta} := E[X_1]^{-\alpha} = \frac{c_d C_{\alpha,d}}{\beta} \Gamma(\alpha / \beta).
\]

(The negative moment of the \( \beta \)-stable random variable \( X_1 \) can be computed similarly to \([27]\), Appendix A.) Due to \((10)\), it follows that \( E[L(t)] < \infty \).

By \((17)\), we have:

\[
a^2E(e^{L(t)}) \leq E|u(t, x)|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t, x) \leq b^2E(e^{L(t)}).
\]

(18)

We consider also the random variable

\[
\zeta(t) := \int_{0}^{t} \int_{0}^{t} |X^1_r - X^2_s|^{-\alpha} dr ds.
\]

Since \( |r - s|^{2H - 2} \geq (2t)^{2H - 2} \) for any \( r, s \in [0, t] \), \( L(t) \geq \beta_H t^{2H - 2} \zeta(t) \), where \( \beta_H = \alpha_H 2^{2H - 2} \). Hence \( \zeta(t) \) is finite a.s.
By the self-similarity (of index $1/\beta$) of the processes $X^1$ and $X^2$, it follows that for any $t > 0$ and $c > 0$,

$$\zeta(t) \overset{d}{=} c^{(2\beta-\alpha)/\beta} \zeta(t/c).$$

In particular, for $c = t^{-(2H-2\beta)/(2\beta-\alpha)}$, we obtain that

$$t^{2H-2} \zeta(t) \overset{d}{=} \zeta(t^\delta), \quad \text{with} \quad \delta = \frac{2H\beta - \alpha}{2\beta - \alpha}$$

and for $c = t$, we obtain that $\zeta(t) \overset{d}{=} t^{(2\beta-\alpha)/\beta} \zeta(1)$. Hence,

$$E(e^{L(t)}) \geq E(e^{\beta_H t^{2H-2} \zeta(t)}) = E(e^{\beta_H \zeta(t^\delta)}). \quad (19)$$

The asymptotic behavior of the moments of $\zeta(t)$ was investigated in [6], under the condition $\alpha < 2\beta$. More precisely, under this condition, by relation (2.3) of [6], we know that:

$$\lim_{n \to \infty} \frac{1}{n} \log \left\{ \frac{1}{(n!)^{\alpha/\beta}} E[\zeta(1)^n] \right\} = \log \left( \frac{2\beta}{2\beta - \alpha} \right)^{(2\beta-\alpha)/\beta} + \log \gamma,$$

where $\gamma > 0$ is a constant depending on $d, \alpha, \beta$. Hence, there exists some $n_1 \geq 1$ such that for all $n \geq n_1$, $E[\zeta(1)^n] \geq c^n (n!)^{\alpha/\beta}$, where $c > 0$ is a constant depending on $d, \alpha, \beta$. Consequently, for any $t > 0$,

$$E[\zeta(t)^n] \geq c^n t^{(2\beta-\alpha)/\beta} (n!)^{\alpha/\beta} \quad \text{for all } n \geq n_1.$$

Hence, for any $\theta > 0$,

$$E(e^{\theta \zeta(t)}) = \sum_{n \geq 0} \frac{1}{n!} \theta^n E[\zeta(t)^n] \geq \sum_{n \geq n_1} \frac{1}{(n!)^{1-\alpha/\beta}} \theta^n c^n t^{n(2\beta-\alpha)/\beta}. \quad (20)$$

Using (18), (19) and (20), we obtain that:

$$\infty > E[u(t,x)]^2 \geq a^2 E(e^{L(t)}) \geq a^2 E(e^{\beta_H \zeta(t^\delta)}) \geq a^2 \sum_{n \geq n_1} \frac{\beta_H c^n t^{n(2\beta-\alpha)/\beta}}{(n!)^{1-\alpha/\beta}}.$$

*This implies that* $\alpha < \beta$. *For any* $x > 0$ *and* $h \in (0,1)$, *we note that*

$$E_h(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^h} \geq \left( \sum_{n \geq 0} \frac{(x^{1/h})^n}{n!} \right)^h = \exp(hx^{1/h}).$$

We denote $x_t = \theta c t^{(2\beta-\alpha)/\beta}$ and $h = 1 - \alpha/\beta$. Writing the last sum in (20) as the sum for all terms $n \geq 0$, minus the sum $S_t$ with terms $n \leq n_1$, we see that for all $\theta > 0$, and for all $t \geq t_0$,

$$E(e^{\theta \zeta(t)}) \geq E_h(x_t) - S_t \geq \exp(hx_t^{1/h}) - S_t \geq \frac{1}{2} \exp(hx_t^{1/h})$$

$$\geq \exp(c_0 \theta^{2/\beta}(\beta-\alpha) t^{(2\beta-\alpha)/\beta}).$$
where \( c_0 = \hbar c^{1/h} \) and \( t_0 > 0 \) is a constant depending on \( \theta, \alpha, \beta \). Using this last inequality with \( \theta = \beta_H \) and \( t^d \) instead of \( t \), we obtain that:

\[
E|u(t, x)|^2 \geq a^2 E \left( e^{\beta_H (t^d)} \right) \geq a^2 \exp(C_2 t^d),
\]

where \( C_2 = c_0 \beta_H^{\beta/(\beta - \alpha)} \) depends on \( d, \alpha, \beta, H \). □

### A Some useful identities

In this section, we give a result which was used in the proof of Theorem 2.1 for finding an upper bound for \( \psi_n(t, t) \). This result may be known, but we were not able to find a reference. We state it in a general context.

Following Definition 5.1 of [14], we say that a function \( f : \mathbb{R}^d \to [0, \infty] \) is a kernel of positive type if it is locally integrable and its Fourier transform in \( S'(\mathbb{R}^d) \) is a function \( g \) which is non-negative almost everywhere. Here we denote by \( S'(\mathbb{R}^d) \) the dual of the space \( S(\mathbb{R}^d) \) of rapidly decreasing, infinitely differentiable functions on \( \mathbb{R}^d \).

The Riesz kernel defined by \( f(x) = |x|^{-\alpha} \) for \( x \in \mathbb{R}^d \setminus \{0\} \) and \( f(0) = \infty \) (with \( \alpha \in (0, d) \)), is a kernel of positive type. Its Fourier transform in \( S'(\mathbb{R}^d) \) is given by \( g(\xi) = C_{\alpha,d}|\xi|^{-(d-\alpha)} \) where

\[
C_{\alpha,d} = \pi^{-d/2} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}
\]

(see Lemma 1, page 117 of [14]).

Let \( f \) be a continuous symmetric kernel of positive type such that \( f(x) < \infty \) if and only if \( x \neq 0 \). By Lemma 5.6 of [14], for any Borel probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \), we have:

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \mu(dx) \nu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}_\mu(\xi) \mathcal{F}_\nu(\xi) g(\xi) d\xi,
\]

where \( \mathcal{F}_\mu, \mathcal{F}_\nu \) denote the Fourier transforms of \( \mu, \nu \). In particular, if \( \mu(dx) = \varphi(x) dx \) and \( \nu(dy) = \psi(y) dy \) for some density functions \( \varphi, \psi \) in \( \mathbb{R}^d \), then

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \varphi(x) \psi(y) dx dy = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}_\varphi(\xi) \mathcal{F}_\psi(\xi) g(\xi) d\xi.
\]

This relation holds for arbitrary non-negative functions \( \varphi, \psi \in L^1(\mathbb{R}^d) \). (To see this, we consider the normalized functions \( \varphi/\|\varphi\|_1 \) and \( \psi/\|\psi\|_1 \), where \( \| \cdot \|_1 \) denotes the \( L^1(\mathbb{R}^d) \)-norm.) Using the decomposition \( \varphi = \varphi^+ - \varphi^- \) with non-negative functions \( \varphi^+, \varphi^- \), we see that (22) holds for any functions \( \varphi, \psi \in L^1(\mathbb{R}^d) \). In fact, (22) holds for any functions \( \varphi, \psi \in L^1_c(\mathbb{R}^d) \), replacing \( \psi(y) \) by its conjugate \( \psi^*(y) \) on the left-hand side. (To see this, we write \( \varphi = \varphi_1 + i\varphi_2 \) where \( \varphi_1, \varphi_2 \) are the real and imaginary parts of \( \varphi \).)
We consider the Bessel kernel (in \(\mathbb{R}^d\)) of order \(\beta > 0\):

\[
G_{d,\beta}(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty u^{\beta/2-1}e^{-u} \frac{1}{(4\pi u)^{d/2}} e^{-|x|^2/(4u)} du.
\]

Note that \(G_{d,\beta}\) is a density function (see Remark A.3 below) and

\[
\mathcal{F}G_{d,\beta}(\xi) = \left(\frac{1}{1 + |\xi|^2}\right)^{\beta/2}, \quad \xi \in \mathbb{R}^d.
\]  \(23\)

Moreover, \(G_{d,\alpha} * G_{d,\beta} = G_{d,\alpha+\beta}\) for any \(\alpha, \beta > 0\) (see pages 130-135 of \[16\]).

The following result is an extension of relations (3.4) and (3.5) of \[10\] to the case of arbitrary \(\beta > 0\).

**Lemma A.1.** Let \(f\) be a continuous symmetric kernel of positive type such that \(f(x) < \infty\) if and only if \(x \neq 0\). Let \(\mu(d\xi) = (2\pi)^{-d}g(\xi)d\xi\), where \(g\) is the Fourier transform of \(f\) in \(S'(\mathbb{R}^d)\). Let \(\beta > 0\) be arbitrary. Then

\[
\int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx = \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2}\right)^{\beta/2} \mu(d\xi) := I_\beta(\mu).
\]  \(24\)

If \(I_\beta(\mu) < \infty\), then, for any \(a \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} e^{ia \cdot x}G_{d,\beta}(x)f(x)dx = \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi - a|^2}\right)^{\beta/2} \mu(d\xi).
\]  \(25\)

**Proof:** Relation (24) follows from (22) with \(\phi = \psi = G_{d,\beta/2}\). On the left-hand side (LHS), we use the fact that \(G_{d,\beta/2} * G_{d,\beta/2} = G_{d,\beta}\). On the right-hand side (RHS), we use (23) (with \(\beta/2\) instead of \(\beta\)).

To prove (25), we apply (22) to the complex-valued functions:

\[
\varphi(x) = \psi(x) = e^{ia \cdot x}G_{d,\beta/2}(x).
\]

The term on the LHS is

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ia \cdot (x-y)}G_{d,\beta/2}(x)G_{d,\beta/2}(y)f(x-y)dxdy = \int_{\mathbb{R}^d} e^{ia \cdot x}f(x)G_{d,\beta}(x)dx,
\]

using Fubini’s theorem. The application of Fubini’s theorem is justified since

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ia \cdot (x-y)}G_{d,\beta/2}(x)G_{d,\beta/2}(y)f(x-y)dxdy = \int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx < \infty.
\]

For the term on the RHS, we use the fact that

\[
\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi - a) \cdot x}G_{d,\beta/2}(x)dx = \mathcal{F}G_{d,\beta/2}(\xi - a) = \left(\frac{1}{1 + |\xi - a|^2}\right)^{\beta/4}.
\]

\(\Box\)
Corollary A.2. Let \((f, \mu)\) be as in Lemma A.1 and \(\beta > 0\) be arbitrary. Assume that \(I_\beta(\mu) < \infty\). Then

\[
\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi - a|^\beta} \right)^{\beta/2} \mu(d\xi) = I_\beta(\mu).
\]

Consequently,

\[
\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - a|^\beta} \mu(d\xi) \leq c_\beta I_\beta(\mu),
\]

(26)

where \(c_\beta = 2^{\beta/2 - 1}\).

Proof: The fact that \(I_\beta(\mu)\) is smaller than the supremum is obvious. To prove the other inequality, we take absolute values on both sides of (25) and we use the fact that \(|\int \cdots| \leq \int |\cdots|\). For the last statement, we use the fact that \((1 + |\xi - a|^2)^{\beta/2} \leq c_\beta(1 + |\xi - a|^{\beta})\). □

Remark A.3. The Bessel kernel \(G_{d,\beta}(x)\) arises in statistics as the density of the random vector \(X\) given by the following hierarchical model:

\[
X | U = u \sim N_d(0, 2uI) \quad U \sim \text{Gamma}(\beta/2, 1)
\]

where \(N_d(0, 2uI)\) denotes the \(d\)-dimensional normal distribution with covariance matrix \(2uI\), \(I\) being the identity matrix. Hence, the term on the LHS of (24) is

\[
\int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx = E[f(X)] = \frac{1}{\Gamma(\beta/2)} \int_0^\infty u^{\beta/2 - 1} e^{-u} E[f(X)|U = u]du.
\]

This can be computed explicitly if \(f(x) = |x|^{-\alpha}\) with \(\alpha \in (0, d)\). First, note that if \(Z \sim N_d(0, 2tI)\), then its negative moment of order \(-\alpha\) is:

\[
E(|Z|^{-\alpha}) = \int_{\mathbb{R}^d} |x|^{-\alpha} \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d + \alpha} e^{-t|\xi|^2} d\xi.
\]

(27)

where \(c_d = 2\pi^{d/2}/\Gamma(d/2)\) is the surface area of the unit sphere in \(\mathbb{R}^d\). To see this, we use the fact that \(\mathcal{F}f(\xi) = C_{\alpha,d} |\xi|^{-d + \alpha} d\xi\) in \(S'(\mathbb{R}^d)\). Hence,

\[
E(|Z|^{-\alpha}) = \int_{\mathbb{R}^d} |x|^{-\alpha} \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d + \alpha} e^{-t|\xi|^2} d\xi
\]

and (27) follows by passing to the polar coordinates. We obtain that

\[
\int_{\mathbb{R}^d} G_{d,\beta}|x|^{-\alpha} dx = \frac{C_{\alpha,d} \Gamma((\beta - \alpha)/2)}{2\Gamma(\beta/2)} \int_0^\infty u^{(\beta - \alpha)/2 - 1} e^{-u} du = \frac{C_{\alpha,d} \Gamma((\beta - \alpha)/2) \Gamma(\alpha/2)}{2\Gamma(\beta/2)}.
\]

(Note that the integral is finite if and only if \(\alpha < \beta\).)
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