“Hard-scattering” approach to very hindered magnetic-dipole transitions in quarkonium

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For a class of hindered magnetic dipole (M1) transition processes, such as $\Upsilon(3S) \rightarrow \eta_b + \gamma$ (the discovery channel of the $\eta_b$ meson), the emitted photon is rather energetic so that the traditional approaches based on multipole expansion may be invalidated. We propose that a “hard-scattering” picture, somewhat analogous to the pion electromagnetic form factor at large momentum transfer, may be more plausible to describe such types of transition processes. We work out a simple factorization formula at lowest order in the strong coupling constant, which involves convolution of the Schrödinger wave functions of quarkonia with a perturbatively calculable part induced by exchange of one semihard gluon between quark and antiquark. This formula, without any freely adjustable parameters, is found to agree with the measured rate of $\Upsilon(3S) \rightarrow \eta_b + \gamma$ rather well, and can also reasonably account for other recently measured hindered M1 transition rates. The branching fractions of $\Upsilon(4S) \rightarrow \eta_b^{(0)} + \gamma$ are also predicted.

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As a century-old subject, electromagnetic (EM) transitions have been extensively studied in the fields of atomic, nuclear and elementary particle physics. EM transition is of considerable interests in heavy quarkonium physics from both experimental and theoretical aspects \cite{1}. Experimentally, it proves to be a powerful tool to discover new quarkonium states that cannot be directly produced in $e^+e^-$ annihilation into a virtual photon. A very recent example is that the long-sought bottomonium ground state, the $\eta_b$ meson, was finally seen by BABAR collaboration in the magnetic dipole (M1) transition process $\Upsilon(3S) \rightarrow \eta_b\gamma$ \cite{2}. Theoretically, it provides a useful means to probe the internal structure, and the interplay between different dynamic scales in quarkonium.

The standard textbook treatment of EM transitions is based on the concept of multipole expansion \cite{1}, by assuming the emitted photon to be ultrasoft, i.e. $k^0 \sim mv^2$, where $m$ is heavy quark mass, $v$ denotes the typical velocity of the quark inside a quarkonium. Consequently, the long wave length of photon cannot resolve the geometrical details of quarkonium. Obviously, the multipole expansion method is valid provided that $kr \ll 1$, where $r \sim 1/mv$ is the typical radius of a quarkonium.

One of the great theoretical undertakings is to understand EM transitions in a situation where the multipole expansion may break down. It is difficult to find such a situation in atomic system, since the typical atomic energy spacings are always of order $mv^2 \sim m\alpha^2$ (where $\alpha \approx 1/137$ is the fine structure constant). By contrast, in the realm of QCD, the linearly-rising inter-quark strong force can host rather highly excited quarkonium states, thus one may encounter EM transitions in quarkonium with energetic photon. The aim of this work is to offer a new perspective to tackle such situation. For definiteness, in this work we will concentrate on the hindered magnetic-dipole (M1) transition (i.e., two quarkonium states with the same orbital angular momentum but with different spin and principal quantum numbers). Such study is of practical importance, because it will help one to better understand the process $\Upsilon(3S) \rightarrow \eta_b\gamma$, where the photon carries a momentum as large as 1 GeV and multipole expansion may cease to be a decent method.

One usually assumes that the $M1$ transition can proceed without gluon exchange between $Q$ and $\overline{Q}$. In the nonrelativistic limit, the transition rate between two $S$-wave quarkonia is usually described by the well-known formula \cite{1}:

$$\Gamma[n^3S_1 \rightarrow n'^1S_0 + \gamma] = \frac{4}{3} \alpha e_Q^2 \frac{k^3}{m^2} \int_0^{\infty} dr \, r^2 \, R_{n0}(r) J_0 \left( \frac{kr}{2} \right) R_{n0}(r) \, r^2 \, dr,$$

where $e_Q$ is the fractional electric charge of $Q$, $k$ is the photon momentum viewed in the rest frame of $n^3S_1$ state, and $R_{n0}(r)$ stands for the radial Schrödinger wave function of quarkonium of principal quantum number $n$ and orbital angular momentum $l$. The spherical Bessel function $j_0 \left( \frac{kr}{2} \right)$ ($j_0(x) \equiv \sin(x) / x$) takes into account the so-called finite-size effect (equivalently, resum-

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\textsuperscript{1} The literal meaning of this term is to expand the electromagnetic field $A^\mu(t, \mathbf{R} \mp \frac{\mathbf{r}}{2})$ around $\mathbf{R}$, the center-of-mass coordinate of $Q\overline{Q}$ pair, in powers of the relative coordinate $\mathbf{r}$, and the expansion parameter is essentially $kr$, $k$ denoting the photon momentum. Some authors prefer to dubbing it as long wave-length approximation. These two terms are equivalent in this work.
We present a new attempt to analyze very hindered \( M_1 \) transitions. When \( k \) is expected to be ultrasoft, it is then legitimate to expand this function, and the leading contribution to the hindered transition vanishes due to orthogonality of wave functions. For several observed hindered transition processes, Eq. (1) usually yields predictions a few times smaller than the measured values.

It is widely believed that hindered \( M_1 \) transitions are very sensitive to the relativistic corrections to the hard mode already makes crucial contribution at tree level. Impressive progress has been made in calculating the transition rate typified by \( \Upsilon(3S) \to \eta_b \gamma \). The key observation is very simple: in such situation, it is more appropriate to count the radiated photon as semihard \((k^\mu \sim mv, \text{often called soft in NRQCD terminology})\), rather than ultrasoft. As a consequence, we should and must give up the notion of multipole expansion. We further make a key assumption: the leading contribution to such a very hindered transition is described by Fig. (1).

The underlying rationale is that, in order for the spectator antiquark to join the final quarkonium state with a significant probability, a semihard gluon must be exchanged between \( Q \) and \( \bar{Q} \) to exert a kick on it. It may be worth digressing into pion EM form factor temporarily. At large momentum transfer, there exists a well-known factorization theorem for this case [6]:

\[
F_\pi(Q^2) = \int_0^1 \int_0^1 dx dy \phi_\pi(x) T(x,y,Q) \phi_\pi(y) + \cdots, \tag{2}
\]

where \( \phi_\pi \) implies the nonperturbative light-cone distribution amplitude amplitude of a pion, and \( T \) refers to the hard-scattering part, which can be computed in perturbation theory. The lowest-order contribution to \( T \) is also depicted by Fig. (1). It is generally believed that, at large \( Q^2 \), this hard-scattering picture is physically more plausible than the so-called Feynman mechanism (without exchange of a hard gluon).

We plan to derive a factorization formula analogous to (2). In our process, the analogous “hard-scattering” part is obtained by integrating out the semihard mode. We will assume this part is also perturbatively calculable, crucially because \( mv \gg \Lambda_{\text{QCD}} \), which seems legitimate for the \( \Upsilon \), presumably even for the \( \psi \) family. Quite naturally, we expect that the counterpart of \( \phi_\pi(x) \) in our nonrelativistic problem will be the Schrödinger wave function of quarkonium.

In passing we highlight the very different role played by the semihard mode in this work and in Ref. [4]. In the latter case when photon is treated as ultrasoft, the semihard mode can only appear in loop. In contrast to the potential mode \((p^0 \sim mv^2, p \sim mv)\), it does not make contribution when descending from NRQCD onto potential NRQCD [4]. According to our scheme, however, the semihard mode already makes crucial contribution at tree level. It is the very mode that we attempt to integrate out perturbatively, in order to fulfill the intended factorization.

This said, let us turn to the derivation of the very hindered \( ^3S_1 \to ^1S_0 \) radiative transition rate. We will perform the calculation in a covariant fashion at the level of QCD. Since the hard \((p^\mu \sim m)\) quanta decouple in this process, it is also feasible, perhaps more illuminating, to directly start from NRQCD. We first note that parity and Lorentz invariance constrain the transition amplitude to be the form

\[
M[\bar{n} \, ^3S_1(P) \to \eta_b \, ^1S_0 (P') + \gamma(k)] = \mathcal{A} \epsilon_{\mu\nu\alpha\beta} P^\mu \varepsilon_{[\alpha \, ^3S_1]} k^\nu \varepsilon^\beta_\gamma , \tag{3}
\]
where $\varepsilon_{[\alpha^3S_1]}$ and $\varepsilon_\gamma$ represent the polarization vectors of the initial quarkonium and the photon, respectively. At the rest frame of the initial state, as we will always work in, the Lorentz structure becomes $\varepsilon_{[\alpha^3S_1]} \cdot \mathbf{k} \times \varepsilon_\gamma^*$, clearly corresponding to the M1 transition. The scalar coefficient $\mathcal{A}$ encodes all the nontrivial dynamics, and we will proceed to deduce its explicit form.

We begin with the parton process $Q(p)\overline{Q}(\bar{p}) \to Q(p')\overline{Q}(\bar{p}') + \gamma(k)$, as indicated in Fig. 1. We assign the momentum carried by each constituent as

$$p = \frac{P}{2} + q, \quad \bar{p} = \frac{P}{2} - q;$$

$$p' = \frac{P'}{2} + q', \quad \bar{p}' = \frac{P'}{2} - q',$$

where $q$ and $q'$ are relative momenta inside each pair, which satisfy $P \cdot q = P' \cdot q' = 0$. The invariant mass of the pairs are $P^2 = 4E_q^2$ and $P'^2 = 4E_{q'}^2$, and the Lorentz scalars $E_q = \sqrt{m^2 - q^2}$, $E_{q'} = \sqrt{m^2 - q'^2}$, which guarantees that each (anti)quark stays on their mass-shell. Note in the rest frame of $P^{(i)}$, $q^{(i)}$ becomes purely space-like.

The quark propagator in Fig. 1(a) can be expanded:

$$\frac{1}{(p' + k)^2 - m^2} = \frac{1}{k \cdot P' + 2k \cdot q'} \approx \frac{1}{(\frac{k}{2} \cdot P + \frac{2k \cdot q'}{(k \cdot P)^2}) + \cdots},$$

because $k \cdot q' \sim m^2 v^2 \ll k \cdot P' = k \cdot P \sim m^2 v$. We have neglected the small $q'^0$ component induced by the recoil of $P'$, as well as the Lorentz boost effect on $q'$, which are higher order corrections. The quark propagator in Fig. 1(b) can be expanded in a similar fashion. Note this expansion is also legitimate when $k$ is ultrasoft.

Fig. 1(a) and b) share a common gluon propagator:

$$\frac{1}{(\frac{k}{2} + q' - q)^2 + i\epsilon} \approx \frac{-1}{(q' - q)^2 + k \cdot (q' - q) - i\epsilon}.$$  

Here we retain the $i\epsilon$ term explicitly, for the momentum integration to be properly evaluated. The two terms in the denominator are of comparable size, so $k$ cannot be further expanded. If $k$ is nevertheless counted as ultrasoft, the second term can be treated as a perturbation. Note our situation is in drastic contrast to the ordinary NRQCD calculation for hard exclusive processes. In that case, there is always a hard scale, and the leading Surviving $O(q^{(i)})$ contribution can only be obtained by retaining the second term in the expanded quark propagator (i), while neglecting $q^{(i)}$ terms altogether in elsewhere of the amplitude. After some efforts, we can read off the reduced amplitude:

$$\mathcal{A} = \frac{2\alpha_e^2 Q^2 C_F}{(k \cdot P)^2} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 q'}{(2\pi)^4} \phi_{n'0}^*(q') \times T(q - q') \phi_{n0}(q),$$

where $C_F = \frac{4}{3}$, and the prefactor 2 indicates that two undrawn diagrams make equal contributions as Fig. 1(a) + b), owing to charge conjugation symmetry. $\phi_{n'0}$ signifies the momentum-space Schrödinger wave function, and the “hard-scattering” kernel is

$$T(q) = -\frac{k \cdot q}{q^2 + k \cdot q - i\epsilon}.$$  

Eq. 9 is the desired factorization formula in momentum space.

It would be more convenient to work with the familiar spatial wave functions. Thanks to the fact that the “hard-scattering” part depends only on the difference between the relative momenta of two quarkonia, $q' - q$, and not on $q$ or $q'$ separately, upon Fourier transformations, one can arrive at a compact expression in the position space via contour integral:

$$\mathcal{A} = \frac{4eC_F\alpha_s}{M_n} \varepsilon_{nn'},$$

$$\varepsilon_{nn'} = \int_0^{\infty} dr r^2 R_{n'0}^*(r) \mathcal{T}(r) R_{n0}(r),$$

where $R(r)$ appearing in the overlap integral $\varepsilon_{nn'}$ is the radial wave function. We have used the relation $k \cdot P = k M_n$ and $M_n$ is the mass of the initial-state quarkonium. The dimensionless kernel $\mathcal{T}(r)$ is obtained by Fourier

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2 This is somewhat analogous to the hard exclusive process $\eta_b \to J/\psi J/\psi$, where the amplitude also vanishes at the lowest order in charm quark relative velocity.
transforming $T(q)$ and integrating over solid angle:

$$\mathcal{T}(r) = e^{\frac{kr}{M_n r}} \left[ j_0 \left( \frac{kr}{2} \right) - \frac{2}{kr} j_1 \left( \frac{kr}{2} \right) + i j_1 \left( \frac{kr}{2} \right) \right],$$

where $j_1(x) = \frac{\sin x}{x} - \frac{\cos x}{x}$. It may be worth reminding that the above combination of spherical Bessel functions in the bracket resembles the conventional electric-dipole (E1) transition formula with finite-size effect incorporated. Notice $\mathcal{T}(r)$ develops an imaginary part, as $kr \ll 1$. Using $j_1(x) \sim \frac{x}{2} + \frac{x^2}{4}$ at small $x$, one finds $\mathcal{T}(r) \to \frac{1}{3m} + i \frac{k}{3m}$ as $kr \to 0$. It turns out that, the real part might be identified with, up to a constant, the $O(\alpha_s)$ matching coefficient $V_s^2 (r \times r \times B) / m^2$ in Sec. IIIC of [4] (note there arises some subtle issue regarding gauge invariance). Since the imaginary part becomes $r$-independent in the long wavelength limit, as expected, it does not contribute to the hindered $M1$ transition.

Finally we can express the transition width as

$$\Gamma [n^3 S_1 \to n^1 S_0 + \gamma] = \left| \frac{k^3}{12\pi} \cdot \mathcal{A} \right|^2 = \frac{16}{3} \alpha^2 \frac{k^3}{M_n^2} C^2 \alpha_0^2 |\mathcal{E}_{mn}|^2,$$

where we have averaged upon spin of the initial $^3S_1$ state and sum over two transverse polarizations of the photon.

Eq. (10) is the key formula of this work, which looks quite simple. In evaluating the overlap integral $\mathcal{E}_{mn'}$, the input wave functions are obtained by solving Schrödinger equation with the widely-used Cornell potential model [5] and Buchmuller-Tye (BT) potential model [6]. Parameters in both potential models are tuned such that the $bb$ and $cc$ spectroscopy below open flavor threshold are successfully reproduced. The only freely adjustable parameter seems to be the strong coupling constant, $\alpha_s(\mu)$. However, the choice of the renormalization scale $\mu$ is by no means arbitrary. On physical ground, it should be fixed around the typical value of the quark 3-momentum in quarkonium, which is about 1.2 GeV for $bb$ system, and 0.9 GeV for $cc$ system [7]. Therefore, with $\alpha_s$ fixed, our formalism becomes rather predictive, and, readily falsifiable.

In Table II we have tabulated various predictions to hindered $M1$ transitions of $n^3 S_1 \to n^1 S_0$. We also present the numerical results for $\mathcal{E}_{mn'}$, and reassuringly, the contribution from $\text{Im} \mathcal{T}(r)$ is indeed insignificant. As one can tell, the agreement between our predictions, especially from the Cornell potential model, and the measurement for the transition rate of $\Upsilon(3S) \to \eta_\gamma \gamma$, is strikingly successful. Curiously, for other hindered $M1$ transitions, where the photon is not that energetic so that the multipole expansion method may still apply, our formalism again appears to make a decent account of the measured transition rates, agrees typically within $2 - 3 \sigma$. It seems fair to conclude that our simple factorization formula has passed quite nontrivial tests. Given the fact that there is almost no free parameters in (10), we feel encouraged that our formalism has captured at least some correct and relevant ingredients. We hope future measurements of $\Upsilon(4S) \to \eta_\gamma \gamma$ can further test our mechanism.

It might be tempting to seek simplified expression for the overlap integral $\mathcal{E}_{mn'}$, by exploiting some hierarchy multiply between different $bb$ energy levels. Higher radial excitation, say, $\Upsilon(3S)$, is known to have considerably larger radius than $\eta_b$. An intuitive guess is that $\mathcal{E}_{31}$ may not be necessarily sensitive to the full profile of $P_0(r)$, instead may only sensitive to its value at small distance (about the radius of $\eta_b$). If this were true, one could pull $R_{30}(r)$ outside of the integral, and approximate it by its value at the origin. The transition rate predicted this way turns out to be about two orders of magnitude greater than the measured one! If we play the same game for $R_{30}(r)$, the result would be about ten times larger than the data. The failure of these approximations may be understood from the empirical fact that, in the Cornell or BT potential models, the average moment of quark in different $bb$ energy levels is more or less equal. As a result, there seems no ground to neglect $q$ or $q'$ in the “hard-scattering” kernel in (10).

For the $M1$ transition from $n^1 S_0$ to $n^3 S_1$, one needs multiply (10) by a statistical factor of 3. Various partial widths for $\eta_b(nS) \to \Upsilon \gamma$ are about 10 eV, and that for $\eta_b(2S) \to J/\psi \gamma$ is about 1 keV. These bottomonium transitions may be accessible in high-energy hadron collider experiments such as CERN Large Hadron Collider (LHC), and BESIII program may provide a chance to look for this charmonium hindered transition.

As in any factorization framework, we expect that the factorization formula (10) is perturbatively improvable. It will be a major progress to calculate the next-to-leading order correction to the “hard-scattering” kernel. To achieve this, it might prove easier to reformulate our derivation in the context of NRQCD. It would also be interesting to implement relativistic corrections to (10).

Obviously our strategy needs not to be confined to hindered $M1$ transitions only. It should be applicable whenever the radiated photon cannot be viewed as ultrasoft and multipole expansion breaks down. It will be interesting to work out the corresponding factorization formula for $E1$ transitions such as $\chi_{bJ}(2P) \to \Upsilon \gamma$. It would be
also interesting to generalize this “hard-scattering” formalism to explore the hadronic transition processes such as Υ(3S,4S) → Υ + ππ.

Table I: Measured and predicted branching fractions of various hindered M1 transition processes n$^3S_1 \rightarrow n' ^1S_0 + \gamma$ for bottomonium and charmonium. The photon momentum $k$ is determined by physical kinematics. The total widths of Υ(nS) and ψ(2S) states, as well as all the quarkonium masses, are taken from PDG08 compilation \cite{PDG08}, except $\eta_b$ mass is taken to be 9389 MeV \cite{etabmass}, and $\eta_c(2S)$ mass taken as 9997 MeV \cite{etacmass}. For Υ(2S)→$\gamma\eta_b$, we use the preliminary BABAR result \cite{babar_eta_b}; for ψ(2S)→$\gamma\eta_c$, we quote the latest CLEO measurement \cite{cleo_eta_c}, instead of the world average value given in \cite{PDG08}. We have taken $\alpha_s(\mu) = 0.43$ and 0.59 for $\mu = 1.2$ and 0.9 GeV, respectively.

| Decay modes | $k$ (MeV) | $\mathcal{B}$ (Exp.) | $\alpha_s$ | $\mathcal{B}_{\text{pre}} \times 10^{-2}$ | $\mathcal{B}$ (Our predictions) |
|-------------|-----------|-----------------------|------------|------------------------------------------|---------------------------------|
| $\Upsilon(2S) \rightarrow \gamma\eta_b$ | 614 | $(4.2 \pm 1.4) \times 10^{-4}$ | 0.43 | $3.7e^{+2.0}_{-2.7}$ | 1.4 \times 10^{-4} |
| $\Upsilon(3S) \rightarrow \gamma\eta_b$ | 921 | $(4.8 \pm 1.3) \times 10^{-4}$ | 0.43 | $2.7e^{+2.6}_{-3.5}$ | 3.7 \times 10^{-4} |
| $\Upsilon(4S) \rightarrow \gamma\eta_b$ | 1123 | – | 0.43 | $2.2e^{+2.8}_{-3.7}$ | 4.3 \times 10^{-7} |
| $\Upsilon(4S) \rightarrow \gamma\eta_b(2S)$ | 566 | – | 0.43 | $1.7e^{+2.2}_{-3.2}$ | $3.2 \times 10^{-8}$ |
| $\psi(2S) \rightarrow \gamma\eta_c$ | 638 | $(4.3 \pm 0.6) \times 10^{-3}$ | 0.59 | $6.4e^{+9.7}_{-12.9}$ | $2.7 \times 10^{-3}$ |

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References

[1] N. Brambilla et al. [Quarkonium Working Group], arXiv:hep-ph/0412158
[2] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 101, 071801 (2008).
[3] S. Godfrey and J. L. Rosner, Phys. Rev. D 64, 074011 (2001) [Erratum-ibid. D 65, 039001 (2002)], and many theoretical references therein.
[4] N. Brambilla, Y. Jia and A. Vairo, Phys. Rev. D 73, 054005 (2006).
[5] J. J. Dudek, R. G. Edwards and D. G. Richards, Phys. Rev. D 73, 074507 (2006).
[6] G. Li and Q. Zhao, Phys. Lett. B 670, 55 (2008).
[7] G. P. Lepage and S. J. Brodsky, Phys. Rev. D 22, 2157 (1980).
[8] G. T. Bodwin and A. Petrelli, Phys. Rev. D 66, 094011 (2002).
[9] Y. Jia, Phys. Rev. D 78, 054003 (2008).
[10] E. Eichten et al., Phys. Rev. D 17, 3090 (1978) [Erratum-ibid. D 21, 313 (1980)]; Phys. Rev. D 21, 203 (1980).
[11] W. Buchmuller and S. H. H. Tye, Phys. Rev. D 24, 132 (1981).
[12] E. Braaten, arXiv:hep-ph/9702225
[13] C. Amsler et al. [Particle Data Group], Phys. Lett. B 667, 1 (2008).
[14] P. Kim, talk given at 6th International Workshop on Heavy Quarkonia, Nara, Japan, Dec. 2-5, 2008.
[15] R. E. Mitchell et al. [CLEO Collaboration], Phys. Rev. Lett. 102, 011801 (2009).