Covariant Anomalies, Horizons and Hawking Radiation

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Abstract:

Hawking radiation is obtained from anomalies resulting from a breaking of diffeomorphism symmetry near the event horizon of a black hole. Such anomalies, manifested as a nonconservation of the energy momentum tensor, occur in two different forms – covariant and consistent. The crucial role of covariant anomalies near the horizon is revealed since this is the only input required to obtain the Hawking flux, thereby highlighting the universality of this effect. A brief description to apply this method to obtain thermodynamic entities like entropy or temperature is provided.

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1 Introduction

On quantising matter fields in a background spacetime containing an event horizon – for instance a black hole – Hawking radiation is obtained, a result that is classically unattainable. Ever since Hawking’s initial paper [1], there have been several derivations [2, 3, 4], each with its own advantages or disadvantages. While such derivations seem to reinforce Hawking’s original conclusion, none is completely general or truly clinching.

In this essay I will discuss a new approach to the Hawking effect that is based solely on properties near the event horizon. The method is essentially linked to the existence of gauge and gravitational (diffeomorphism) anomalies near the event horizon, as exemplified in recent approaches [5, 6, 7, 8].

An anomaly signifies the breakdown of some classical symmetry due to the process of quantization. The ubiquitous role of anomalies in explaining various physical phenomena is well known [9] and its connection with the Hawking effect dates back to a seminal paper by Christensen and Fulling [3]. However, central role ascribed to conformal symmetry is somewhat conceptually unpleasant since general relativity is not conformally symmetric. Rather, the crucial symmetry in general relativity is the invariance under general coordinate transformations (diffeomorphism symmetry).

A couple of years back Wilczek and collaborators [5, 6] advanced a new approach to the Hawking flux where diffeomorphism symmetry plays a significant factor. They observe that, near the horizon, black hole dynamics is effectively described by a two dimensional chiral theory that breaks diffeomorphism symmetry. Requiring that the complete theory with contributions near the horizon, outside the horizon and inside the horizon be anomaly free, a condition is obtained from which the Hawking flux is identified.

To put the above considerations in a proper perspective, it is important to realise that there are two types of chiral anomalies – covariant and consistent. Those satisfying the Wess Zumino consistency condition are called consistent while those transforming covariantly under the appropriate symmetry transformation are called covariant. The covariant and consistent structures are different and related by local counterterms [9]. In fact this mismatch between the covariant and consistent currents (or e.m. tensors) is the germ of the anomaly. For an anomaly free theory the covariant and consistent expressions for the currents (or e.m. tensors) are identical.
2 Covariant anomaly and Hawking flux

The anomaly method [5, 6] raises several issues, both technically and conceptually. The flux is obtained there from the consistent expression but the boundary condition involves the covariant form. Note that the flux is measured at infinity where there is no anomaly, so that covariant and consistent structures are identical. Hence if the anomaly method is viable the flux should equally well be obtainable from the covariant expression. Apart from being conceptually clean, since only covariant expressions are involved, it also entails considerable technical simplification. Shifts between covariant and consistent expressions through counterterms, as is mandatory in [5, 6], are avoided. How this is done is now illustrated following Banerjee and Kulkarni [7, 8].

Consider a metric
\[ ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\Omega^2, \]
where \( d\Omega^2 \) is the line element on \( S^{(d-2)} \), which admits an event horizon at \( f(r_h) = 0 \). As explained, the effective theory near the horizon is chiral and two dimensional with the metric
\[ ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2. \]
Here the covariant diffeomorphism anomaly is given by
\[ \nabla_\mu T^\mu_{t(H)} = \frac{1}{96\pi \sqrt{-g}} \varepsilon^{\nu\rho} \nabla_\rho R, \]
where \( T^\mu_{t(H)} \) is the covariant e.m. tensor near the horizon, \( \varepsilon_{\nu\rho} \) is the numerical antisymmetric tensor and \( R \) is the Ricci scalar. Moreover \( \sqrt{-g} = 1 \). In a static background the anomaly is time like and one finds, for \( \nu = t \) [7],
\[ \nabla_\mu T^\mu_{t(H)} = \partial_t T^r_{t(H)} = \partial_r A^r_t; \quad A^r_t = \frac{1}{96\pi}[ff'' - \frac{1}{2}f'^2] \] (1)

This yields the solution,
\[ T^r_{t(H)} = C + \int^r_{r_h} dr \partial_r A^r_t = A^r_t(r) - A^r_t(r_h). \] (2)

The integration constant \( C \) is set to zero by imposing the boundary condition that the e.m. tensor vanishes at the horizon \( T^r_{t(H)}(r_h) = 0 \). Such a condition is equivalent to the regularity condition used in obtaining Hawking fluxes in Unruh vacuum [8].

Initially ignoring the ingoing modes, the e.m. tensor is now written as a sum of two regions, near and outside the horizon. Taking the divergence of this tensor and using (1) naturally yields a Wess Zumino term that is interpreted as a contribution from the classically ignored ingoing modes. The redefined e.m. tensor, after including this contribution, is anomaly free provided \( T^r_{t(0)} - T^r_{t(H)} + A^r_t = 0 \). Using (1) and (2) the Hawking energy momentum flux is obtained,
\[ T^r_{(0)} = -A^r_t(r_h) = \frac{1}{192\pi}f'^2(r_h) = \frac{\pi}{12\beta^2} \] (3)
where the final result is expressed in terms of the inverse Hawking temperature $\beta$ by using $k = \frac{2\pi}{\beta} = \frac{f'(r_h)}{2}$. This is just the desired flux from blackbody radiation.

### 3 New approach – universality of Hawking radiation

As seen above the reformulation of the anomaly method [5, 6] in terms of covariant expressions only [7, 8] is simple and straightforward, ironing out technical complexities and conceptual issues. However certain problems still persist. The universality of Hawking radiation requires that the flux be determinable only from information at the horizon. The point is that, apart from the anomalous Ward identity at the horizon, the normal Ward identity outside the horizon is also required. Furthermore, it is also necessary to interpret an additional Wess-Zumino term as a contribution from the (classically irrelevant) ingoing modes. The question is whether it is possible to derive the flux just from the information of the chiral anomaly at the horizon. This is indeed so.

On observing the structure of the anomaly in (1) and imposing asymptotic flatness of the metric ($f(r \to \infty) = 1, f'(\infty) = f''(\infty) = \ldots = 0$) it is found that $A_t^r$ vanishes in the $r \to \infty$ limit. Hence the anomaly also vanishes in this limit. This implies that the $r \to \infty$ limit of $T_{t(t)}^r$ would correspond to the anomaly free expression associated with $T_{t(0)}^r$. Consequently the Hawking flux, which is measured at infinity, will be given by,

$$T_{t(0)}^r = T_{t(t)}^r(r \to \infty) = -A_t^r(r_h) = \frac{\pi}{12\beta^2}$$

which follows from (2) on using $A_t^r(r \to \infty) = 0$. This reproduces (3).

The Hawking flux is therefore obtained solely from information at the horizon with the minimum of effort. Inclusion of gauge fields, as required in the case of charged black holes, poses no problems. Also, higher spin fluxes can be computed in an identical way leading to the full Hawking spectrum.

An important open issue in this context is the computation of thermodynamic entities like entropy or temperature of a black hole. Arguments will be given to show that this should be feasible. The present method has similarities with conformal field theory techniques [10] used to study black hole thermodynamics. There the entropy is computed by the Cardy formula [11] which requires both the central charge and the conserved charge. Something analogous can be done here. In the conformal field theory approach, the central charge (or conformal anomaly) is obtained from the central extension
of the Virasoro algebra by redefining the usual diffeomorphism generators by ‘stretched horizon’ constraints [12]. Likewise, the conserved charge is obtained from a boundary term needed to make the redefined generators differentiable. A central extension is naturally obtained in the present analysis without the need of any redefinitions, due to the chiral anomaly near the horizon. This has already been computed in the literature [9], though in a different context. Also, the conserved charges, which are roughly analogous to energies, can be computable, as outlined in [8] from two dimensional effective actions subjected to the boundary conditions used here.

It appears that different manifestations of the anomaly provide a unifying picture that illuminates the universality of the Hawking effect.

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