Basic Canonical Brackets Without Canonical Conjugate Momenta: Supersymmetric Harmonic Oscillator

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Abstract: We exploit the ideas of spin-statistics theorem, normal-ordering and the key concepts behind the symmetry principles to derive the canonical (anti)commutators for the case of a one \((0 + 1)\)-dimensional \((1D)\) supersymmetric \((SUSY)\) harmonic oscillator without taking the help of the mathematical definition of the canonical conjugate momenta with respect to the bosonic and fermionic variables of this toy model for the Hodge theory (where the continuous and discrete symmetries of the theory provide the physical realizations of the de Rham cohomological operators of differential geometry). In our present endeavor, it is the full set of continuous symmetries and their corresponding generators that lead to the derivation of basic (anti)commutators amongst the creation and annihilation operators that appear in the normal mode expansions of the dynamical variables of our theory.

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1 Introduction

The standard method of canonical quantization scheme invokes primarily three basic ideas. First and foremost, we distinguish between the fermionic and bosonic variables/fields of the theory by using the celebrated spin-statistics theorem (which dictates the existence of (anti)commutators at the quantum level for such variables/fields). Second, we define the canonical conjugate momenta for the above variables/fields and define the (graded) Poisson brackets at the classical level which are promoted to the (anti)commutators at the quantum level. Finally, if the variables/fields allow normal mode expansions due to their equations of motion, we express the above (anti)commutators in terms of the creation and annihilation operators (of the normal mode expansions). These basic (anti)commutators at the variables/fields level get translated into the basic (anti)commutators amongst the creation/annihilation operators. Normal-ordering is required to make physical sense out of the physical quantities (e.g., Hamiltonian, conserved charge, etc.) when they are expressed in terms of the creation and annihilation operators of the normal mode expansions.

In our present endeavor, we shall exploit the ideas of spin-statistics theorem and normal-ordering but we shall not use the purely mathematical definition of the canonical conjugate momenta in the quantization of a one (0 + 1)-dimensional (1D) SUSY harmonic oscillator (SUSY-HO) in terms of its creation and annihilation operators. Instead, we shall utilize the ideas of symmetry principles (i.e. continuous symmetries and their generators) to obtain the basic (anti)commutators of this SUSY system (which has been proven to be a model for the Hodge theory in our earlier work [1]). The central claim of our present investigation is the observation that the quantization of a class of theories can be performed without the mathematical definition of the canonical conjugate momenta. These set of theories belong to the models which are tractable physical examples of Hodge theory.

At the field theoretic level, it has been shown that the 2D free Abelian 1-form gauge theory [2,3] and its interacting version (where there is a coupling between 2D Abelian gauge field with Dirac fields [4]) are tractable physical models for the Hodge theory. The common features of all the above cited models for the Hodge theories is the observation that their discrete and continuous symmetry transformations provide the physical realizations of the de Rham cohomological operators of differential geometry. In our recent works [5,6], we have demonstrated that the covariant canonical quantization of these 2D gauge theories can be performed without the definition of canonical conjugate momenta. In fact, we have shown that the symmetries of these theories, within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism, are good enough to lead to the derivation of the basic (anti)commutators where inputs from the spin-statistics theorem and normal-ordering are required. The noteworthy point is the observation that the definition of the canonical conjugate momenta is not required for the derivation of the basic (anti)commutators amongst the creation and annihilation operators of such a specific class of theories.

In our present investigation, we have chosen the model of 1D SUSY-HO because it contains bosonic as well as fermionic variables which require (anti)commutators for their quantization (at the quantum level). Moreover, as we know, the range and reach of physics behind the system of a harmonic oscillator is very wide as it encompasses in its folds the theoretical ideas from classical mechanics to field theory to string theory. Thus, it is a very good proposition to say something novel about such a widely applicable system of
theoretical physics. We establish, in our present endeavor, that there is no need of the mathematical definition of canonical conjugate momenta for the quantization of SUSY-HO as its symmetries are good enough to entail upon the existence of basic canonical brackets at the level of creation and annihilation operators.

Our present investigation has been motivated by the following key and decisive factors. First, the definition of the canonical conjugate momenta is purely mathematical in nature. Thus, any physical alternative to it is a welcome sign as far as the richness of ideas in the realm of theoretical physics is concerned. Second, it is useful to derive the basic canonical (anti)commutators from another method than the usual canonical method because it would enrich the tools and techniques in the realm of theoretical physics. Third, to put our ideas and experiences in the context of 2D free as well as interacting Abelian gauge theories [5,6] on firmer-footings, it is essential to prove the sanctity of these ideas in the context of other examples of Hodge theory. Our present endeavor is an attempt in that direction. Finally, our method of derivation of the canonical (anti)commutators is based on symmetry considerations. Thus, even though algebraically more involved, our method of derivation is physically more appealing as far as the concepts in theoretical physics are concerned.

The contents of our present endeavor are organized as follows. In Sec. 2, we very briefly mention about the usual $\mathcal{N} = 2$ SUSY symmetries of the SUSY-HO and the bosonic symmetry that emerges from their anticommutators. Our Sec. 3 captures the usual method of canonical quantization scheme to derive the non-vanishing basic canonical brackets. Our Sec. 4 is devoted to the derivation of basic brackets from the first of the $\mathcal{N} = 2$ SUSY symmetries. In Sec. 5, we derive the (anti)commutators that results in from the other $\mathcal{N} = 2$ SUSY transformations. Sec. 6 contains the basic (anti)commutators that emerge from the bosonic symmetry transformations. Finally, we make some concluding remarks and point out a few future directions for further investigations in our Sec. 7.

2 Preliminaries: Continuous Symmetries and Conserved Charges as Generators

We begin with the following Lagrangian for the 1D SUSY-HO with mass $m = 1$ and natural frequency $\omega$ (see, e.g. [7,1] for details)

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 + i \bar{\psi} \dot{\psi} - \omega \bar{\psi} \psi,$$

where $\dot{x} = (dx/dt)$, $\dot{\psi} = (d\psi/dt)$ are the generalized “velocities” for the bosonic and fermionic variables $x$ and $\psi$, respectively. For the bosonic variable $x$, there are two fermionic ($\vec{\psi} = 0$, $\vec{\psi} = 0$, $\psi \bar{\psi} + \bar{\psi} \psi = 0$) variables in the $\mathcal{N} = 2$ SUSY theory (at the classical level). The latters are the superpartners of the former. In other words, as we shall see in Eq. (2) below), under the $\mathcal{N} = 2$ SUSY transformations, the bosonic and fermionic variables transform to one-another obeying the basic principles of SUSY.

The above Lagrangian respects the following $\mathcal{N} = 2$ SUSY symmetries

$$s_1 x = \psi, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = i (\dot{x} + i \omega x),
$$

$$s_2 x = \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = i (\dot{x} - i \omega x), \quad (2)$$

3
which are nilpotent (i.e. \( s_1^2 = 0, s_2^2 = 0 \)) of order two on the on-shell (\( \dot{\psi} + i \omega \psi = 0, \dot{\bar{\psi}} - i \omega \bar{\psi} = 0 \)). The above symmetry transformations are generated by the following conserved (\( \dot{Q} = \dot{\bar{Q}} = 0 \)) charges (\( Q \) and \( \bar{Q} \)), namely;

\[
Q = (\dot{x} + i \omega x) \psi, \quad \bar{Q} = \bar{\psi} (\dot{x} - i \omega x),
\]

which are also nilpotent (\( Q^2 = \bar{Q}^2 = 0 \)) of order two and they are conserved. The latter property can be checked by using the Euler-Lagrange (EL) equations of motion

\[
\ddot{x} + \omega^2 x = 0, \quad \dot{\psi} + i \omega \psi = 0, \quad \dot{\bar{\psi}} - i \omega \bar{\psi} = 0, \quad \ddot{\psi} + \omega^2 \psi = 0, \quad \ddot{\bar{\psi}} + \omega^2 \bar{\psi} = 0,
\]

which emerge from the Lagrangian (1) of our present SUSY theory from the least action principle where we demand that \( \delta S = 0 \) (for the action integral \( S = \int dt L \)). It should be emphasized, at this stage, that the Noether theorem does not invoke the definition of the canonical conjugate momenta. Rather, it is derived from the action principle where the physical system follows the trajectory that is described by the EL equations of motion.

The anticommutator (i.e. \( s_\omega = \{s_1, s_2\} \)) leads to the definition of a bosonic symmetry \((s_\omega)\) in the theory. Under this transformation, the variables change as:

\[
s_\omega x = \{s_1, s_2\} x = \dot{x}, \quad s_\omega \psi = \{s_1, s_2\} \psi = \dot{\psi}, \quad s_\omega \bar{\psi} = \{s_1, s_2\} \bar{\psi} = \dot{\bar{\psi}},
\]

modulo a factor of \( 2i \). The above transformations demonstrate the validity and existence of \( \mathcal{N} = 2 \) SUSY symmetries in our 1D theory because the anticommutator of two SUSY transformations is equivalent to a time-translation. Under the above transformations, the Lagrangian transforms \((s_\omega L = \frac{d}{dt} [L])\) to its own time-derivative. Thus, the generator of transformations (5) is nothing but the Hamiltonian of our theory\(^*\), namely;

\[
Q_\omega = \frac{1}{2} \ddot{x}^2 + \frac{1}{2} \omega^2 x^2 + \omega \dot{\bar{\psi}} \psi \equiv \frac{p_x^2}{2} + \frac{\omega^2 x^2}{2} + \omega \dot{\bar{\psi}} \psi \equiv H,
\]

where \( p_x = \dot{x} \) is the momentum corresponding to the bosonic variable \( x \) and \( H \) is the canonical Hamiltonian. The latter can be also derived by the Legendre transformations:

\[
H = \dot{x} \Pi_x + \dot{\psi} \Pi_{\bar{\psi}} + \dot{\bar{\psi}} \Pi_{\psi} - L \equiv \frac{p_x^2}{2} + \frac{\omega^2 x^2}{2} + \omega \dot{\bar{\psi}} \psi,
\]

where \( \Pi_x = p_x = \dot{x}, \Pi_\psi = -i \dot{\bar{\psi}}, \Pi_{\bar{\psi}} = 0 \) are the canonical conjugate momenta where we have used the idea of left-derivative with respect to the fermionic variables (i.e. \( \Pi_\psi = \partial L / \partial \dot{\psi}, \Pi_{\bar{\psi}} = \partial L / \partial \dot{\bar{\psi}} \)). This is the reason that there is a negative sign in \( \Pi_\psi \).

\(^*\)It is quite elementary to check that the transformations \((s_1, s_2, s_\omega)\) satisfy the algebra: \( s_1^2 = s_2^2 = 0, s_\omega = \{s_1, s_2\}, [s_\omega, s_1] = 0, [s_\omega, s_2] = 0 \) in their operator form. This algebra is also mimicked by the charges \((Q, \bar{Q}, H)\) which is nothing but the \( \mathcal{N} = 2 \) SUSY quantum mechanical algebra \( sl(1/1) \) in its simplest form. In our earlier works [1, 11], these algebras are also shown to be identified with the algebra of de Rham cohomological operators of differential geometry.
3 Basic Brackets: Standard Canonical Method

We begin with the following mode expansions for the variable $x(t), \psi(t)$ and $\tilde{\psi}(t)$:

$$x(t) = \frac{1}{\sqrt{2\omega}} [a e^{-i\omega t} + a^\dagger e^{i\omega t}], \quad \psi(t) = be^{-i\omega t}, \quad \tilde{\psi}(t) = b^\dagger e^{i\omega t},$$

(8)

which satisfy the EL equations of motion (4). It is the validity of the equations of motion ($\dot{\psi} + i\omega \psi = 0, \dot{\tilde{\psi}} - i\omega \tilde{\psi} = 0$) which forces us to choose the solutions for $\psi$ and $\tilde{\psi}$ as given in (8). In the above, the time-independent dagger and non-dagger operators are the creation and annihilation operators. The following standard canonical (anti)commutators:

$$[x, x] = [\Pi_x, \Pi_x] = 0, \quad [x, \Pi_x] = i \equiv [x, p_x], \quad \Pi_x = p_x,$$

$$\{\psi, \dot{\psi}\} = 0, \quad \{\psi, \dot{\psi}\} = 0, \quad \{\psi, \Pi_\psi\} = i \implies \{\psi, \dot{\psi}\} = -1,$$

(9)

can be expressed in terms of the creation and annihilation operators as

$$[a, a] = [a^\dagger, a^\dagger] = 0, \quad [a, a^\dagger] = 1, \quad \{b, b\} = \{b^\dagger, b^\dagger\} = 0 \quad \{b, b^\dagger\} = -1,$$

(10)

if we exploit the mode expansions given in (8) and use them in the canonical brackets (9). In other words, the basic (anti)commutators of (9) and (10) are equivalent and they imply each-others in a clear-cut fashion. It should be noted that all the rest of the commutators (e.g. $[x, \psi] = 0, [\Pi_x, \psi] = 0, [x, \tilde{\psi}] = 0$) are zero. These, in turn, imply that $[a, b] = 0, [a^\dagger, b] = 0, [a, b^\dagger] = 0, [a^\dagger, b^\dagger] = 0$, etc.

In other words, we have the following explicit (anti)commutators

$$[x(t), x(t)] = [\Pi_x, \Pi_x] = 0 \quad \implies [a, a] = 0, \quad [a^\dagger, a^\dagger] = 0,$$

$$[x(t), \Pi_x(t)] = [x(t), \dot{x}(t)] = i \quad \implies [a, a^\dagger] = 1,$$

$$\{\psi(t), \Pi_\psi(t)\} = \{\psi(t), \dot{\psi}(t)\} = -1 \implies \{b, b^\dagger\} = -1,$$

$$\{\psi(t), \dot{\psi}(t)\} = \{\tilde{\psi}(t), \dot{\tilde{\psi}}(t)\} = 0 \quad \implies \{b, b\} = \{b^\dagger, b^\dagger\} = 0,$$

(11)

which are the basic brackets of our present theory in terms of the creation and annihilation operators. All the other possible commutators (e.g. $[a, b] = 0, [a^\dagger, b^\dagger] = 0$, etc.) are zero in our present theory. In our forthcoming sections, we shall derive these brackets from symmetry properties without using the definition of canonical conjugate momenta ($\Pi_x = p_x = \dot{x}$ and $\Pi_\psi = \partial L/\partial \dot{\psi} = -i \psi, \Pi_\psi = 0$). It may be worthwhile to mention that the latter definitions are mathematical and there is almost no physical intuition in it.

We end this section with the remark that we have used here the definition of the canonical conjugate momenta and spin-statistic theorem to obtain the basic brackets at the variable level (cf. (9)). When we express these brackets in terms of the mode expansions, we end up with the basic (anti)commutators amongst the creation and annihilation operators. Thus, the (anti)commutators at the variable level are equivalent to the (anti)commutators at the level of creation/annihilation operators. We note that, in our present derivation, there has been no need of normal-ordering at any arbitrary stage. However, this becomes essential when we deal with the Hamiltonian formalism and express it (and other relevant quantities) in terms of the creation/annihilation operators.
4 Basic Brackets from the First Transformation of the $\mathcal{N} = 2$ SUSY Symmetries: Symmetry Principles

To derive the basic canonical brackets amongst the creation and annihilation operators, from the symmetry transformations ($s_1$), first of all, we note that:

$$s_1 \Phi = -i [\Phi, Q]_\pm, \quad \Phi = x, \psi, \bar{\psi},$$

(12)

where the ± signs, as subscripts on the square bracket, denote the (anti)commutator for the generic variable $\Phi$ being (fermionic) bosonic in nature. Here we have already used the concept of spin-statistics theorem. It can be explicitly checked that the conserved charge $Q = (\dot{x} + i\omega x)\psi$ can be expressed in terms of creation and annihilation operators, using the mode expressions (8), as

$$Q = \frac{+2i\omega}{\sqrt{2}\omega} a^\dagger b,$$

(13)

where we have used $\dot{x} = -i\omega/\sqrt{2}\omega(a e^{-i\omega t} - a^\dagger e^{i\omega t})$. The above charge automatically appears in the normal-ordered form. Thus, there is no need of the application of normal-ordering. This far, we have used two ideas of the standard method of quantization. These are the spin-statistics theorem and normal-ordering.

We discuss here the derivation of the (anti)commutators from the symmetry considerations (and the principles involved in these transformations in the language of generators). To begin with, first of all, we focus on the following transformations

$$s_1 x = -i [x, Q] = \psi.$$  

(14)

Using the normal mode expression (8) and the expression for $Q$ from (13), we obtain the following basic brackets (i.e. (anti)commutators):

$$[a, b] = [a^\dagger, b] = [a^\dagger, a^\dagger] = 0, \quad [a, a^\dagger] = 1$$

(15)

where we have compared the coefficients of $e^{-i\omega t}$ and $e^{+i\omega t}$ from the l.h.s. and r.h.s. Obviously, the r.h.s. (i.e $\psi = be^{-i\omega t}$) contains only $e^{-i\omega t}$ but the l.h.s. contains both the exponentials $e^{-i\omega t}$ as well as $e^{+i\omega t}$. Thus, it is clear that the coefficient of $e^{+i\omega t}$, from the l.h.s., should be zero. Some of the basic brackets of (15) have been derived from this input. Moreover, we have also used the basic tricks of the (anti)commutators involving composite operators (so that $[a, a^\dagger b] = [a, a^\dagger] b + a^\dagger [a, b]$, etc.).

Now, we concentrate on the transformations $s_1 \psi = 0$ and $s_1 \bar{\psi} = i(\dot{x} + i\omega x)$. These can be written in terms of the generator as

$$s_1 \psi = -i \{\psi, Q\} = 0, \quad s_1 \bar{\psi} = -i \{\bar{\psi}, Q\} = i(\dot{x} + i\omega x).$$

(16)

Plugging in the mode expansions from (8) and expression for $Q$ from (13), we obtain the following basic brackets (i.e. (anti)commutators)

$$[a^\dagger, b] = 0, \quad \{b, b\} = 0,$$

(17)
from the transformation $s_1 \psi = 0$. The other transformation $s_1 \bar{\psi} = -i \{\bar{\psi}, Q\}$ leads to the following basic brackets:

$$[a^\dagger, b^\dagger] = 0, \quad \{b, b^\dagger\} = -1. \tag{18}$$

Ultimately, we note that we have derived the following basic brackets from the transformations $s_1$ on the generic variable $\Phi$ (i.e. $s_1 \Phi = -i[\Phi, Q]_\pm$ for $\Phi = x, \psi, \bar{\psi}$), namely;

$$[a, b] = 0, \quad [a^\dagger, b] = 0, \quad [a^\dagger, a^\dagger] = 0, \quad \{b, b\} = 0, \quad [a^\dagger, b^\dagger] = 0, \quad \{b, b^\dagger\} = -1, \quad [a, a^\dagger] = +1, \tag{19}$$

which are seven in number. Out of these, the non-vanishing brackets are merely two (i.e $\{b, b^\dagger\} = -1, [a, a^\dagger] = +1$). We also note that the use of Eq. (12) leads to the derivation of all possible brackets. We note that the symmetry considerations in (14) do not yield the bracket $\{b^\dagger, b^\dagger\} = 0$ which is required for the precise quantization.

5 Basic Brackets from the Other Transformation of the $\mathcal{N} = 2$ SUSY Symmetries: Symmetry Principles

We dwell a bit on the nilpotent transformations $s_2$ and concentrate on the transformation of the bosonic variable $x$ as:

$$s_2 x = -i [x, \bar{Q}] = \bar{\psi}, \tag{20}$$

where the conserved charge $Q = \bar{\psi}(\dot{x} - i \omega x)$ can be expressed in terms of the mode expansions in (8) as:

$$\bar{Q} = -\frac{2i \omega}{\sqrt{2}} b^\dagger a. \tag{21}$$

The substitution the mode expansions of (8) in (20) leads the emergence of the following basic brackets from transformations (20), namely;

$$[a, b^\dagger] = [a, a] = [a^\dagger, b^\dagger] = 0, \quad [a, a^\dagger] = 1, \tag{22}$$

where we have equated the coefficients of $e^{-i \omega t}$ and $e^{+i \omega t}$ from the l.h.s and r.h.s. Next, the trivial transformations $s_2 \bar{\psi} = -i \{\bar{\psi}, Q\} = 0$ yields the derivation of $[a, b^\dagger] = 0, \{b^\dagger, b^\dagger\} = 0$ when we use the basic tricks of anticommutators with composite operators. Finally, the transformations

$$s_2 \psi = -i \{\psi, Q\} = i (\dot{x} - i \omega x), \tag{23}$$

generates the basic brackets that are as follows:

$$[a, b] = 0, \quad \{b, b^\dagger\} = -1. \tag{24}$$

In the above derivation, we have compared the coefficients of $e^{-i \omega t}$ and $e^{+i \omega t}$ from the l.h.s. and r.h.s. and used the basic tricks of (anti)commutators with composite operators.
(e.g. \([b, b^\dagger]a = \{b, b^\dagger\}a - b^\dagger[b, a]\)). Finally, we observe that \(s_2 \Phi = -i [\Phi, \bar{Q}]_\pm\) (with \(\Phi = x, \psi, \bar{\psi}\)) leads to the following basic (anti)commutation relations amongst the creation and annihilation operators:

\[
\begin{align*}
[a, b^\dagger] & = [a, a] = [a^\dagger, b^\dagger] = [a, b] = 0, \\
\{b^\dagger, b^\dagger\} & = 0, \quad [a, a^\dagger] = +1, \quad \{b, b^\dagger\} = -1. \quad (25)
\end{align*}
\]

A careful and close observation at (19) and (25) demonstrates that we have already derived all the (non-)vanishing brackets amongst the creation and annihilation operators (i.e. \(a, a^\dagger, b, b^\dagger\)) The non-vanishing brackets are \([a, a^\dagger] = +1\) and \(\{b, b^\dagger\} = -1\) which are consistent with the ones derived in Sec. 3. We lay emphasis on the fact that the symmetry transformations (20) do not produce the bracket \(\{b, b\} = 0\).

6 Basic Brackets from the Bosonic symmetry

In Sec. 2, we have seen that the anticommutator of the nilpotent \(\mathcal{N} = 2\) SUSY transformations produces a bosonic symmetry transformation \((s_\omega)\). Under this symmetry transformations, we have the conserved (i.e. \(Q_\omega = 0\)) charge \(Q_\omega = \frac{p_x^2}{2} + \frac{\omega^2 x^2}{2} + \omega \bar{\psi} \psi\) that can be expressed in terms of the mode expansions (8) as

\[
Q_\omega = H = \frac{\omega}{2} (a a^\dagger + a^\dagger a) + \omega b^\dagger b \equiv \omega (a^\dagger a + b^\dagger b),
\]

where we have used the normal-ordering to make physical sense out of \(Q_\omega\). This expression would be used in the derivation of the transformations \(s_\omega \Phi = \pm i [\Phi, Q_\omega]_\pm\) where \(\Phi = x, \psi, \bar{\psi}\) and \(Q_\omega = H\) that is given in Sec. 2 as well as (26).

To obtain the basic (anti)commutators, first of all, we focus on the transformation of the bosonic variable \(x\). This can be written as

\[
s_\omega x = -i [x, Q_\omega] = \dot{x}. \quad (27)
\]

The l.h.s. and r.h.s. of the above expression can be written in terms of the creation and annihilation operators and exponentials. The comparison of the coefficients of \(e^{-i \omega t}\) and \(e^{+i \omega t}\) (from the l.h.s. and r.h.s.) yields the following basic commutator relations:

\[
[a, b^\dagger] = [a, b] = [a, a] = [a^\dagger, b^\dagger] = [a^\dagger, b] = [a^\dagger, a^\dagger] = 0, \quad [a, a^\dagger] = 1. \quad (28)
\]

Thus, we note that the non-vanishing bracket is \([a, a^\dagger] = 1\). Let us now concentrate on the transformations:

\[
s_\omega \psi = i [\psi, Q_\omega] = \dot{\psi}, \quad s_\omega \bar{\psi} = i [\bar{\psi}, Q_\omega] = \ddot{\bar{\psi}}. \quad (29)
\]

Plugging in the mode expansions (8) and comparing the coefficients of \(e^{-i \omega t}\) and \(e^{+i \omega t}\) from l.h.s and r.h.s, we obtain

\[
[a, b] = [a^\dagger, b] = \{b, b\} = 0, \quad \{b, b^\dagger\} = -1,
\]

\[
[a, b^\dagger] = [a^\dagger, b^\dagger] = \{b^\dagger, b^\dagger\} = 0, \quad \{b, b^\dagger\} = -1, \quad (30)
\]

from the transformations \(s_\omega \psi = \dot{\psi}\) and \(s_\omega \bar{\psi} = \ddot{\bar{\psi}}\), respectively. Thus, we have obtained all the basic (anti)commutators of our theory where the non-vanishing brackets are \([a, a^\dagger] = 1\) and \(\{b, b^\dagger\} = -1\) which are equivalent to the canonical brackets \([x, \Pi_x] = i\), \(\{\psi, \bar{\psi}\} = -1\) at the level of the variables of our theory.


7 Conclusions

In our present investigation, we have established that, for the models of the Hodge theory, the canonical quantization conditions can be achieved by exploiting the spin-statistics theorem, normal-ordering and symmetry principles. All these ideas are very nicely backed and bolstered by the physical arguments and insights. For these models, the mathematical definition of the canonical conjugate momenta, corresponding the dynamical variables, are not required. We have corroborated the above statements for the case of 1D SUSY-HO where we have derived the basic (anti)commutators amongst the creation/annihilation operators by exploiting the virtues of symmetry principles and have not used the definition of canonical conjugate momenta (corresponding to the dynamical variables) in the central results of our present endeavor. For the sake of comparison, we have derived these brackets from the standard canonical quantization scheme as well (cf. Sec. 3) so that the sanctity of our results could be clearly and firmly established.

Earlier attempts are present in literature where the alternative methods of the derivation of basic brackets have been performed. For instance, by exploiting the global spacetime Poincaré group and its generators, it has been shown (in the standard book on quantum field theory [8]) that the canonical brackets can be derived for the bosonic fields and their creation/annihilation operators. However, in our present investigation and earlier works [5,6], we have exploited only the internal symmetries of a given theory. Similarly, in a very nice piece of work by Wigner [9], it is the importance of the EL equations of motion that has led to the derivation of basic canonical brackets where, once again, only the bosonic variables/fields have been taken into consideration. Unlike our present work which is based on the symmetry principle, this attempt [9] is not based on such considerations.

We lay emphasis on the observation that the \( \mathcal{N} = 2 \) SUSY transformations \( s_1 \) and \( s_2 \) (and their generators in \( Q \) and \( \bar{Q} \)) do not produce all the (anti)commutators of the theory. As pointed out after (19) and (25), the brackets \( \{b^\dagger, b^\dagger\} = 0 \) and \( \{b, b\} = 0 \) are not produced by the pairs \( (s_1, Q) \) and \( (s_2, \bar{Q}) \), respectively. However, the transformations \( s_\omega \) (generated by \( Q_\omega \)) produce all the appropriate (anti)commutators as is illustrated in (30). Thus, we observed that the results, produced by \( Q \) and \( \bar{Q} \) together, emerge automatically by using \( Q_\omega = H \). There is a deeper mathematical reason behind it. It has been shown in [1] that the set \( (Q, \bar{Q}, Q_\omega \equiv H) \) provides the physical realizations of the de Rham cohomological operators of differential geometry where \( Q_\omega \equiv H \) is identified with the Laplacian operator (which is equal to the anticommutator of the exterior and co-exterior derivatives). The latter are identified with the \( Q \) and \( \bar{Q} \) in the language of the symmetry generators. Thus, it is clear that the consequences that emerge from \( Q_\omega \) would be equivalent to the results obtained by \( Q \) and \( \bar{Q} \) separately and independently (see, e.g. [1], [11] for details).

We have proven that the 1D model of a rigid rotor [10], \( \mathcal{N} = 2 \) SUSY quantum mechanical model with an arbitrary superpotential [11], \( \mathcal{N} = 2 \) SUSY model for the motion of a charged particle under influence of a magnetic field [12], free 4D Abelian 2-form and 6D Abelian 3-form gauge theories [13-15], etc., are models for the Hodge theory. For all these models, we can perform the canonical quantization without taking recourse to the mathematical definition of the canonical conjugate momenta. It would be nice future en-

\footnote{This idea, which is used in the standard canonical quantization scheme, has no physical backing.}
deavor for us to accomplish this goal in a clear-cut fashion. These are the problem we are involved with at present and the results would be reported in our future publications [16].

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