Vector field models of inflation and dark energy

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Abstract. We consider several new classes of viable vector field alternatives to the inflaton and quintessence scalar fields. Spatial vector fields are shown to be compatible with the cosmological anisotropy bounds if only slightly displaced from the potential minimum while dominant, or if driving an anisotropic expansion with nearly vanishing quadrupole today. The Bianchi I model with a spatial field and an isotropic fluid is studied as a dynamical system, and scaling solutions of several types are found. On the other hand, time-like fields are automatically compatible with large-scale isotropy. We show that they can be dynamically important if non-minimal gravity couplings are taken into account. We reconstruct as an example a vector–Gauss–Bonnet model which generates the concordance model acceleration at late times and supports an inflationary epoch at high curvatures. The evolution of the vortical perturbations in such models is computed.

Keywords: dark energy theory, inflation, physics of the early universe
1. Introduction

Most inflationary and dark energy models [1] are based on scalar fields [2]. They, as conventional cosmological sources, are fully characterized by their equation of state and speed of sound. For a more general phenomenology, one should also take into account possible imperfect properties of a fluid [3, 4]. Indeed, it has been proposed that dark energy might have such a characteristic [5], and the possibilities for distinguishing this observationally have been examined recently [6, 7]. In [8] we contemplated the possibility of such a significant anisotropy: that the expansion rate would be better described as a direction-dependent quantity within the Bianchi I model. The phenomenological set-up there could describe viscous fluids, Yang–Mills fields, strings or cosmology with non-commutative infrared properties. It was found that significant anisotropy may be allowed in some cases, and that these models have the potential for explaining the anomalous cosmological observations [9, 10]. In the future, optimistically, one could expect to detect signatures of possible early anisotropies in the polarization, and of possible late anisotropies in the supernova luminosity–distance relationship data.

In the present study we take a complementary approach and rather than working within a framework of phenomenological parametrization, we study vector field theories with explicitly written Lagrangians as our starting point. In particular, we look for (1) solutions which may be compatible with symmetries of cosmology, and then (2) which of these solutions might be used to model inflation or dark energy. The resulting models can then be divided into two classes, those with space-like and those with time-like field components. Both cases then seem to violate Lorentz invariance. However, this violation is already introduced when choosing the cosmological background, not when adding the vector. The cosmological metric picks up a direction for time. The vector fields that we consider are then particular solutions of a Lorentz-invariant theory compatible with the
broken space–time symmetry of the metric. The conceptual difference from the aether [11] and such theories arises from the unit-norm requirement, which is achieved by adding a norm-fixing term into the Lagrangian that then forces the field to be always time-like, even in vacuum. In the present study we do not impose the unit-norm requirement. See [12]–[14] as regards spontaneous Lorentz violation in string theory and gravity.

Space-like vector fields in cosmology have been used to model inflation by Ford and others [15,16]. We will therefore focus in the present study more on the dark energy era, where the qualitative differences are that matter cannot be neglected and that the cosmologies do not have to isotropize. For completeness we also include a coupling to matter. In particular, we work within the Bianchi I model, and look for scaling solutions. This approach differs from previous studies which have been confined to the case of the exact Friedmann–Lemaître–Robertson–Walker (FLRW) symmetry. Space-like fields are compatible with this symmetry only in the case of a so called triad, which has three identical vector fields for each spatial direction [17]. This can be used to model dark energy [17]–[19]. Inflation has been also considered in the case where a large amount of randomly oriented vector fields result in an average FLRW space–time [20]. We note also that a vector field with quintessence can be considered [21], and that adding non-linear terms to the electrodynamic Lagrangian could cause the universe to accelerate [22].

Time-like fields have attracted a lot of interest in cosmology [23]–[31]. If the field is canonical and minimally coupled, the equation of motion for its time component becomes trivial. Therefore one has to either couple the field non-minimally or add non-canonical terms (these modifications are equivalent in some cases). Within modified gravity models attempting to eliminate dark matter, vector field components have been promoted to a central role due to their ability to mimic the observed gravitational lensing phenomena from dark matter [25,32,33].

Recently non-minimally coupled time-like vector fields have been taken into consideration as dark energy candidates [34]–[36]. Even if the vector field does not drive the inflationary expansion, it might have a role in slowing the expansion rate or on the generating perturbations [23], [37]–[40]. Recently the observed anomalies in the CMB have inspired a number of studies concerning perturbations in an anisotropic background [41]–[44], generation of structure during anisotropic inflation [45]–[51] and on the formation of structure in the presence of anisotropic sources [52,53]. In the present study we focus on role of the vector fields as driving the background expansion, especially the accelerating expansion associated with dark energy and inflation. The evolution of the spin-1 type perturbation in such a case is determined and we find that these modes are decaying.

We study cases where the field is coupled to gravity non-minimally. These vector–tensor couplings can cause the field to effectively break the usual energy conditions. Therefore they could be applied to model non-singular cosmologies and to construct wormholes [54]. If the photon has such couplings during inflation, large-scale magnetic fields could be generated [55]–[57]. In the present universe, the vector–tensor theories can be severely constrained by solar system experiments [58,59]. We consider in particular two classes of these theories: with a coupling to the Ricci scalar and to the Gauss–Bonnet invariant. The latter has special interest, since the Gauss–Bonnet term has theoretically and phenomenologically desirable properties. In particular, its presence in the action does not immediately imply the existence of ghosts and severe conflicts with observational data on gravitation. Our results include that this model could generate both the inflation and
the present acceleration of the universe. The outline of the paper is simple: in section 2 we derive our results concerning space-like models, in section 3 those about time-like models, and in section 4 we conclude.

2. Space-like fields

2.1. Minimally coupled vectors

Consider the vector field action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A^2) \right], \tag{1} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( A^2 = A_\mu A^\mu \). The field equations are

\[ G_{\mu\nu} = 8\pi G \left( T_{\mu\nu}^m + T^A_{\mu\nu} \right), \tag{2} \]

where the energy–momentum tensor of the vector field follows by variation with respect to the metric

\[ T^A_{\mu\nu} = F_{\mu\alpha} F^\alpha_{\nu} + 2V'(A^2)A_\mu A_\nu - \left( \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + V(A^2) \right) g_{\mu\nu}. \tag{3} \]

The equations of motion for the four components of the vector can be written as

\[ \nabla_\mu F^{\mu\nu} - 2A^\nu V'(A^2) = 0. \tag{4} \]

These follow either by setting the divergence of (3) to zero or setting the variation of (1) with respect to the field to zero.

We want to study the consequences of such matter sources in a Bianchi I type universe. Such is a generalization of the FLRW case with three different expansion rates. Thus we allow for a possible anisotropy. Then the metric may be written as

\[ ds^2 = -dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \]

\[ = -dt^2 + \sum_{i=1}^{3} (a_i x_i)^2 \],

where we have introduced an obvious labelling of the spatial coordinates and the metric components with italic letters. Following this convention, we define \( A_\mu = (\phi, A_i) \). Then the components of the energy–momentum tensor may then be written as

\[ T^A_{00} = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{a_i^2} \dot{A}_i^2 + V(A^2) + 2V'(A^2)\phi^2, \tag{7} \]

\[ T^A_{0i} = 2V'(A^2)\phi A_i, \tag{8} \]

\[ T^A_{ij} = -\dot{A}_i \dot{A}_j + 2V'(A^2)A_i A_j + a_ia_j \left( \frac{1}{2} \sum_{k=1}^{3} \frac{1}{a_k^2} \dot{A}_k^2 - V(A^2) \right) \delta_{ij}. \tag{9} \]

Because the symmetries of the metric (6) do not allow a velocity field \( T_{0i} \), one immediately sees that we are restricted to considering either a purely time-like vector, \( A_\mu = (\phi, 0) \), or a space-like vector with \( \phi = 0 \). However, the former case reduces to triviality since the
equation of motion (4) for the non-zero vector component states that \( V'(A^2)\phi = 0 \) (since one has \( \nabla_\mu F^{\mu\nu} = 0 \)). Therefore we will consider the space-like vector fields. For them, the off-diagonal components of \( T_{ik} \) should vanish. In principle, one could then consider non-trivial (space-like) vector fields which satisfy
\[
-A_i A_j + 2V'(A^2)A_i A_j = 0.
\]
Then the evolution of the vector field, \( A_i(t) \), would be determined \textit{a priori} without reference to the other matter (which then might or might not be consistently added into the system). Such possibilities seem contrived and we exclude them from our present considerations. Thus, we have found that only space-like vector fields parallel to one of the principal axis could be allowed. For the FLRW metric all the diagonal components of \( T_{ij} \) should be equal, which then requires three vector fields, one in each coordinate direction, of exactly equal magnitudes, as the triads of Armendariz-Picon [17]. With the metric equation (6), the only constraint is that every vector field should be along a coordinate axis. Any system of these vector fields is thus \textit{required} to have an anisotropic equation of state, unless it reduces to the very special triad case. Let us consider the case \( A_1 = A_3 = 0, A_2 = B \). We then define the additional dimensionless parameters
\[
X \equiv \frac{\kappa \dot{B}}{bH}, \quad Y \equiv \frac{\kappa^2 V}{H^2}, \quad Y_1 = 2\kappa^2 \frac{V'B^2}{b^2 H^2},
\]
(10)
where \( H \) is the average expansion rate
\[
H \equiv \frac{1}{3} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right).
\]
(11)
It is also useful to define the fractional difference between expansion rates as
\[
R \equiv \frac{1}{H} \left( \frac{\dot{a}}{a} - \frac{\dot{b}}{b} \right).
\]
(12)
This is a simple quantification of anisotropy with immediate generalization to the invariant shear. We can then make contact with the parametrization of [8,60] by noting that
\[
w = \frac{X^2 - 2Y}{X^2 + 2Y}, \quad \delta = \frac{-X^2 + Y_1}{(3/2)X^2 + 3Y},
\]
(13)
where \( w \) is the equation of state along the \( x \) direction, and \( \delta \) is the difference of the equations of state along \( y \) and \( x \) directions. Generalization to multiple vector fields is straightforward.

2.2. Scaling solutions with and without matter couplings

In this subsection we consider space-like cosmological vector fields. Similar anisotropic inflation has been considered in the early universe [15,16]. Therefore we focus in the present study more on the dark energy era, where the qualitative differences are that matter cannot be neglected and that the cosmologies do not have to isotropize. Our main aim is to find scaling solutions. There are several reasons for being interested in scaling solutions. (1) Simply, if they are attractors, they may describe the realistic dynamics of the field. In any case they tell us about the phase structure of a given model. (2) Scaling
solutions have also the possibility to alleviate the coincidence problem. If dark energy and dark matter had similar (even within an order of magnitude or two) energy densities in the past, it seems less discomforting that the energy densities are about equal just today. However, it may still remain unexplained why the transition to acceleration began just at small redshifts—this is even if the dark matter and dark energy were to continue scaling in the future, i.e. they would always have (roughly) equal magnitudes. Still, the coincidence problem is good motivation for studying scaling solutions [61]–[65]. (3) In addition, the possibility of early dark energy, which the scaling solutions automatically incorporate, is phenomenologically interesting [66]. A model predicting the presence of a dark energy component in the earlier universe can already be strongly constrained, and it may feature new signatures which might be useful in distinguishing between alternative models of acceleration. Scaling solutions are known to exist in FLRW universes for specific forms of scalar field Lagrangians and these have been applied in attempts to address the coincidence problem. Such a property could be shared by space-like vector fields in the Bianchi I background. For completeness we also include a coupling to matter. However, the presence of a coupling is not a necessary condition for this, as will be seen below.

Consider the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A^2) + q(A^2)L_m \right],$$

(14)

where we have included a general matter Lagrangian $L_m$ and the coupling function $q(A^2)$. To proceed, we write down the complete evolution equations using the variables (10). The amount of matter can be now described by a generalization of the usual fractional energy density,

$$\Omega_m \equiv \frac{8\pi G \rho_m}{3H^2}.$$  

(15)

The Friedmann equation may then be written in the form

$$3 = 3\Omega_m + \frac{1}{2} X^2 + Y + \frac{1}{3} R^2.$$  

(16)

Our time variable is chosen to be a generalization of the e-fold number. The prime will denote a derivative with respect to that,

$$y' \equiv \frac{3}{d\log abc}.$$  

(17)

It is convenient to derive an average effective equation of state, which then describes the overall expansion rate of the universe:

$$w_{\text{eff}} \equiv -\frac{2}{3} \frac{H'}{H} - 1.$$  

(18)

Finally, we define two dimensionless variables,

$$\beta \equiv \frac{\kappa B}{b}, \quad \hat{q} \equiv \frac{3}{\kappa^2} q(A^2) \beta,$$  

(19)

which are the comoving field $\beta$ and the coupling function $\hat{q}$. The $\kappa = 1/(8\pi G)$. 

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Table 1. Scaling solutions for space-like vector fields and matter. The last column on the right shows the shear \( R \) in the dust filled universe. All of these solutions exhibit anisotropy.

| Solution               | Equations | \( w_{\text{eff}} \) | \( w \)      | \( \delta \) | Existence         | \( R \)   |
|------------------------|-----------|----------------------|--------------|------------|-------------------|----------|
| Comoving field         | (24)      | -1                   | \( w(\beta,\ldots) \) | \( w(\beta,\ldots) \) | Needs coupling | \( 4\dot{q}^2/3 \) |
| Potential driven       | (25)      | \( w_{\text{m}} \) -1 | 1/3          | Large field | \( 3/2 - 9/4n \) |
| Kinetically driven     | (26)      | \( w_{\text{m}} \) 1 | -2/3         | Not for dust |                  |          |

The system of equations then becomes

\[
X' = - \left( \frac{H'}{H} + 2 + \frac{2}{3} R \right) X - \frac{Y_1}{\beta} - 2\dot{q} (1 - 3 w_{\text{m}}) \Omega_m, \tag{20}
\]

\[
Y' = -2 \frac{H'}{H} Y + \frac{Y_1}{\beta} \left[ X - \beta \left( 1 - \frac{2}{3} R \right) \right], \tag{21}
\]

\[
R' = - \left( 3 + \frac{H'}{H} \right) R + X^2 - Y_1, \tag{22}
\]

\[
\Omega_m' = - \Omega_m \left[ 2 \frac{H'}{H} + 3 (1 + w_{\text{m}}) - \frac{2}{3} X \dot{q} (1 - 3 w_{\text{m}}) \right]. \tag{23}
\]

For a conventional scaling solution one should have \( X' = Y' = \Omega_m' = 0 \). It is easy to see that this makes sense only if also \( R' = 0 \). Then the derivative of the average Hubble rate is a constant, \( 2H'/H = -3(1 + w_{\text{eff}}) \), which is related to the effective (average) equation of state for the universe, \( w_{\text{eff}} \). Since also \( w \) and \( w + \delta \) now both correspond to the dark energy equations of state, \( \delta \) and thus \( Y_1 \) must be a constant as well. One deduces that this could be satisfied only if either

\[
1) \beta' = 0, \quad 2) Y \sim (A^2)^n \quad \text{or} \quad 3) Y = Y_1 = 0.
\]

Below in table 1 we list the results for each case, but before that we also derive and discuss these in more detail.

- In the case (1) that the comoving field stays constant, the square bracket term in equation (21) vanishes (since, as one can readily check, the derivative of the comoving field is proportional to the square bracket term). One uses this fact to relate the comoving field to \( X \) and \( R \). Moreover, now either \( Y = 0 \) or \( H' = 0 \) because equation (21) must be satisfied. The former possibility of vanishing potential goes under case (3) which we consider below. The latter possibility would correspond to the case \( w_{\text{eff}} = -1 \), which in this regard resembles a de Sitter-like space. However, since equation (22) sets now

\[
R = \frac{X^2 - Y_1}{3},
\]

the solution is anisotropic unless \( X^2 = Y_1 \). From equation (23) one sees that for a scaling solution with \( \Omega_m \neq 0 \), one would have to force a coupling to keep the average...
Hubble rate constant. Since \( \dot{q} \) is a function of the comoving field, the coupling term is now a constant. Plugging this into equation (20) only gives us an equivalent of the Hubble equation as a consistency check. Therefore we have a whole set of solutions, which can be parametrized by the constants \( Y_1 \) and \( \dot{q} \). We summarize these solutions as

\[
\begin{align*}
    w_{\text{eff}} &= -1, \\
    X &= \frac{9(1 + w_m)}{2\dot{q}(1 - 3w_m)}, \\
    Y &= \left\{ 648\dot{q}^3 (1 - 3w_m)^3 Y_1 + \beta \left[ -6561(1 + w_m)(1 + w_m)^3 \\
                      &+ 162\dot{q}^2(1 - 3w_m)^2(1 + w_m)(9 - 27w_m + 4w_m Y_1) \\
                      &- 16\dot{q}^4 (1 - 3w_m)^4(9(Y_1^2 - 81)) \right] \right\} \frac{1}{432\beta (\dot{q} - 3\dot{q}w_m)^4},
\end{align*}
\]

\( R = \frac{4\dot{q}^2(1 - 3w_m)^2}{3}, \)

\( \Omega_m = \frac{\beta(1 + w_m)(4\dot{q}^2(1 - 3w_m)^2(Y_1 - 9) - 81(1 + w_m)^2) + 4\dot{q}^3(3w_m - 1)^3 Y_1}{8\beta (\dot{q} - 3\dot{q}w_m)^4}. \)

The expression for the most general case is complicated. However, one notices that this scaling solution is unlikely to be able to describe a realistic era between the inflation and dark energy domination, since it implies that \( H \) is equal to a constant, and requires a coupling to matter, which can easily be problematic.\(^1\)

- In the power-law potential case (2) one notes that \( Y' = 0 \) is consistent with equation (21) only if \( X = 0 \) (unless we go back into either of the two other cases). Then \( w = -1 \) and \( \delta = 2n/3 \). The coupling then does not affect the matter scaling since the field is constant, and equation (23) tells us that \( w_{\text{eff}} = w_m \), since we would like to consider a scaling solution with non-zero \( \Omega_m \). The constancy of \( R \) requires, from equation (22), that

\[
R = \frac{4nY}{w_m - 1}.
\]

Plugging this back into equation (21) we get that

\[
Y = 9(w_m - 1) \frac{2n - 3(1 + w_m)}{16n^2}.
\]

This should of course be positive, imposing a constraint \( n < 3/2 \) if \( w_m = 0 \). The amount of matter that one then finds from the Friedmann equation (16) is given by

\[
\Omega_m = \frac{3}{8} \left( 2 - \frac{3(1 + w_m)}{n^2} + \frac{3 + w_m}{n} \right).
\]

This may be positive for \( w_m = 0 \) when \( n < -(3 + \sqrt{33})/4 \approx -2.19 \) or \( n > (-3 + \sqrt{33})/4 \approx 0.69 \). Thus we have found that there exists a scaling solution for a

\(^1\) A possible solution to this issue would be to use chameleon vector fields [67]–[69].
Figure 1. Convergence to the scaling attractor equation (25) and the transition to the accelerating era for minimally coupled space-like vector fields. The potential is of the form $V(x) = V_+ x^n + V_- x^{-n}$. In the LHS figure $n = 3$ and in the RHS figure $n = 10$. The solid (black) lines are the ratios $U$ of the vector to the total density, the dashed (red) lines are the average total equations of state $w_{\text{eff}}$, and the dash-dotted (blue) lines are the expansion normalized shears $R$. The properties of the scaling attractor depend only on $n$. The transition to acceleration depends on the scale $V_+$ which we have tuned such that the vector dominance is taking place near the present time $(x = 0)$.

vector coupled to matter, which may be described as

$$
\begin{align*}
    w_{\text{eff}} & = w_m, \\
    X & = 0, \\
    Y & = \frac{9(w_m - 1)(2n - 3(1 + w_m))}{16n^2}, \\
    R & = \frac{-(9 - 6n + 9w_m)}{4n}, \\
    \Omega_m & = \frac{3(2n^2 - 3(1 + w_m) + n(3 + w_m))}{8n^2}.
\end{align*}
$$

(25)

However, since we should have $X' = X = 0$, one may be concerned with equation (20). We mean that the kinetic contribution to the total energy should stay negligible, and thus terms in the RHS equation (20) should somehow cancel. One option would be to introduce a coupling which cancels the effect of the potential that would otherwise set the kinetic contribution $X$ varying: $\beta \hat{q} = -nY / (1 - 3w_m) \Omega_m$. This could be consistent only for a coupling of the specific form $q(A^2) \sim \log(A^2)$. However, from a practical point of view we rather notice that $Y_1 / \beta$ can be approximately neglected if the field has run to large enough values. We have found that this approximate scaling behaviour does indeed arise naturally and is an attractor for the system for $n < -2.2$. These considerations can be confirmed by numerical computation. In figure 1 we show some models featuring the scaling solution equation (25).

- Let us finally consider the case (3) that the potential can be neglected. We see directly that for such a solution $w = 1$ and $\delta = -2/3$. Again equation (23) gives the effective equation of state, $w_{\text{eff}} = w_m - 2\hat{q}X(1 - 3w_m)/9$, which now may be influenced by the
coupling. Then equation (22) gives

\[ R = \frac{6X^2}{9 + 2\hat{q}X - 3w_m(3 + 2\hat{q}X)}. \]

Inserting \( \Omega_m = 1 - X^2/6 - R^2/9 \) in equation (20) would now give an expression for \( X \), but this is generally quite complicated. Let us first look at the case where coupling vanishes. Then

\[ w_{\text{eff}} = w_m, \]
\[ \Omega_m = \frac{3(3 - w_m)}{8}, \]
\[ R = \frac{3(-1 + 3w_m)}{4}, \]
\[ X = \pm \frac{3}{2\sqrt{2}} \sqrt{-1 + (4 - 3w_m)w_m}. \]

The case of immediate concern to us is a background of dust, \( w_m = 0 \), but there this solution does not exist since \( X \) should of course be real. Thus we are led to consider a case where the field is non-minimally coupled to dust, \( w_m = 0, \hat{q} \neq 0 \). This requires that the coupling function \( \hat{q} \) is constant, which is possible, if \( \beta \) is not a constant, only for the special form of the coupling \( q(A^2) \sim \sqrt{(A^2)} \). We again dismiss this possibility.

We focus on the most relevant case, where dust dominates and there are no non-minimal couplings. We show that the scaling solution (25) is an attractor. For that purpose, we eliminate the \( H'/H \) and \( \Omega_m \) terms from the system (20)–(23) and get

\[ X' = \frac{X}{12} \left( -6 - 8R + 2R^2 + X^2 - 6Y + 4nY \right) + O\left( \frac{1}{\beta} \right), \]
\[ Y' = \frac{Y}{6} \left( 18 + 2R^2 + X^2 - 6Y + 4n(-3 + 2R + Y) \right) + O\left( \frac{1}{\beta} \right), \]
\[ R' + \frac{1}{12} \left( 2R^3 + 12(X^2 - 2nY) + R(-18 + X^2 - 6Y + 4nY) \right) + O\left( \frac{1}{\beta} \right). \]

We consider then areas where \( \beta \) is large and terms proportional to its inverse become negligible. Then consider the expansion

\[ X = \bar{X} + \epsilon x + O(\epsilon^2) = (\bar{R}, \bar{X}, \bar{Y}) + \epsilon(x, y, z) + O(\epsilon^2), \]

where background marked with bars is given by the solution (25). Then \( \dot{X} = 0 \) at the background order, and to first order in \( \epsilon \) we get

\[ \dot{x} = Mx, \]

where the three eigenvalues of the matrix \( M \) are given by

\[ [M_1, M_2, M_3] = \left[ \frac{3}{2} \left( \frac{1}{n} - 1 \right), -\frac{3}{4} \left( 1 + \frac{1}{|n|}\sqrt{(4n - 3)(n^2 + n - 3)} \right), \right. \]
\[ -\frac{3}{4} \left( 1 - \frac{1}{|n|}\sqrt{(4n - 3)(n^2 + n - 3)} \right]. \]
The real parts of the eigenvalues are always positive for $n < -\frac{1}{4}(3 + \sqrt{33})$, so the point is a local sink. This confirms that the scaling solution is an attractor given that the exponent $n$ is negative enough.

We have thus systematically considered every possible route towards a scaling solution of a cosmological (space-like) vector field which might be coupled to matter. Several possibilities were noticed, and of particular relevance is that the scaling solution seems to be equation (25), which is valid in the limit of large enough field values. We should however note that several kinds of different behaviours, anisotropic or not, with matter domination $\Omega_m = 1$, or with vanishing matter contribution $\Omega_m = 0$, were left this time without explicit treatment, since we were only looking for a scaling universe. In addition, more interesting solutions probably exist, in which the vector field tracks the background matter (in analogy to the scalar field tracker quintessence) without the tracking being exact (which is precisely what we mean by a scaling solution). We summarize the scaling solutions in table 1.

2.3. Examples of models satisfying the anisotropy constraints

In previous study [52] it was shown that the magnitude of the quadrupole resulting from anisotropic background expansion is given by

$$Q_2 = \frac{2}{5\sqrt{3}} e_2^2,$$

where $e_2^2$ is the eccentricity,

$$e_2^2 = \left(\frac{a_2}{b_2}\right)^2 - 1,$$

which tells how ellipsoidal the last scattering surface (denoted by star) appears to us (the scale factors normalized to unity today). The constraint on the magnitude of quadrupole is then $Q_2 \lesssim 2.72 \times 10^{-5}$. However, it is in principle possible for arbitrarily anisotropic expansion to escape detection from the CMB as long as the expansion rates evolve in such a way that $e_2$ remains zero. In other words, the quadrupole vanishes, if each scale factor has expanded—no matter how anisotropically—by the same amount since the last scattering.

This is demonstrated explicitly with examples in figure 2. In the left-hand side (LHS) there is an example of an exponential potential featuring large anisotropies at late times after the acceleration has begun. Although the anisotropy in such a case has been perturbatively small until today, such models could not be straightforwardly described within the standard approach of perturbing a FLRW metric.

Another example, in the right-hand side (RHS) of figure 2, shows that the skewness of the universe, as defined by equation (12), could be large, while the eccentricity, as defined by equation (34), might well vanish just at the present. The latter is related to the integral $\log(a/b) = \int R \, dx$, which might even be zero on average, if the shear $R$ happens to oscillate suitably. This could naturally happen, for instance, if dark energy is a vector field slightly displaced from the minimum of its potential. The model depicted in the RHS of figure 2 exhibits this property and is constructed as a simple, minimally coupled and one-component vector field introduced in the previous section 2.1.
Figure 2. Two types of scenarios with a minimally coupled vector field that satisfy the CMBR quadrupole constraint. The solid (black) lines are the vector field density fraction $U = \rho_A/\left(\rho_m + \rho_A\right)$, the dashed (red) lines are the effective total equations of state of the universe $w_{\text{eff}}$, and the dash–dotted (blue) lines describe the evolution of eccentricity, $E = 500e^2$. In the LHS figure, the displacement of the field from the minimum of its potential is important only at late times and only then do significant anisotropies begin to form. The potential here is an exponential, $V(x) \sim e^{-x}$. In the RHS figure, the potential is a double power law, $V(x) \sim x^{-4} + V_\perp x$. The dynamics of the field is such that though there are significant anisotropies, the eccentricity at the present is nearly zero. Note that we have chosen both the models in such a way that they feature a recently begun acceleration. Thus these models are viable though finely tuned.

The vector field models of inflation and dark energy

a fine-tuned model, but proves that there exist expansion histories exhibiting very large anisotropies with dynamics that leave the quadrupole amplitude, equation (33), nearly or completely unchanged.

The quadrupole constraint is applied here to constrain the overall amount of anisotropy, and we remark that the detailed anisotropy patterns which ensue within these constraints remain uncovered. The general evolution equations for the overdensities in the presence of anisotropic stresses have been presented [52], and one may observe that when the anisotropy is very small (as is forced by observations) the crucial modification comes from the anisotropic stress of the matter source. It is this quantity which sets all the perturbation modes direction dependent. This will have an impact on the CMB spectra, in such a way that the correlations between adjacent $\ell$ modes become non-zero. Also unlike in the usual statistically isotropic case, different $m$-modes do not decouple. The detailed computation of anisotropies from the direction-dependent inhomogeneities [52] is left for forthcoming studies.

To summarize, we constructed two classes of viable space-like vector dark energy models, described by the following features: (1) the field evolves only very recently, and thus the impact of anisotropy on observations is strongly suppressed, or (2) there is earlier anisotropy, but in such a way that its effect cancels today. Both cases require some fine-tuning. Note however, that the quadrupole constraint suffices for estimating the amount of fine-tuning. The reason is that only the quadrupole is of the order of the background anisotropy in Bianchi I models. If the quadrupole is not too large, the higher multipoles are of the observed order $10^{-5}$ [52]. However, at the few lowest multipoles at least, there
can appear some statistically anisotropic features. Thus we have found explicit viable mechanisms for generating such anomalies in the late universe.

For completeness, one may add that an approximate isotropy may also be achieved by considering (3) ‘a triad’ of identical fields [17] or possibly (4) a huge number of randomly oriented fields [20]. In addition, let us note the case (1) was mentioned already by Ford in the context of inflation: a very slowly rolling field resembles of course a cosmological constant.

3. Time-like fields

3.1. Vector field coupled to the curvature

One might also allow non-minimal couplings of the vector and gravity. Couplings of a massless Maxwell vector field have been recently considered in an $f(R)$ model that could feature both the inflation and late-time acceleration of the universe, where the non-minimal coupling of the Maxwell field could then generate large-scale magnetic fields [56, 70]. Here we allow also a potential for the vector field, and do not couple the kinetic term of the field but add an interaction with the Ricci scalar $R$ and the field $A_μ$,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \frac{1}{8\pi G} + \omega(A^2) \right) R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A^2) + L_m \right] . \quad (35)$$

This is a special case of a vector–tensor theory [11, 25] satisfying the following three conditions

1. the Lagrangian density is a 4-scalar,
2. the resulting theory is metric and
3. there are no derivatives higher than second in the resulting field equations [58]. The most general theory would allow also a coupling of the form $A^μ A^ν R_{μν}$.

In section 3.2 we however present a new model, which is of a quadratic form in the curvature invariants, but also seems to satisfy these requirements (depending on how we interpret condition (2)).

The contribution of the coupling term $ω$ to the field equations can be presented as an effective energy–momentum tensor,

$$G_{\mu\nu} = 8\pi G \left( T^m_{\mu\nu} + T^A_{\mu\nu} + T^ω_{\mu\nu} \right) , \quad (36)$$

where $T^A_{\mu\nu}$ is given by equation (7) and $T^ω_{\mu\nu}$ reads

$$T^ω_{\mu\nu} = -ωG_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) ω - ω' A_μ A_ν R . \quad (37)$$

To write the components explicitly for the line element (6), it is useful to introduce the notation $A_μ = (φ, a_i A_i)$ for the field. We then find

$$T^ω_{00} = -6H \left( \mathbf{A} \cdot \dot{\mathbf{A}} - \ddot{φ} \right) \omega' - \frac{1}{2} (9H^2 - \mathbf{H} \cdot \mathbf{H}) \omega - φ^2 ω' R, \quad (38)$$

$$T^ω_{0i} = -φ A_i ω' R, \quad (39)$$

$$T^ω_{ij} = 2a_i a_j \left[ \left( \mathbf{A} \cdot \dot{\mathbf{A}} - \ddot{φ} \right) (3H - H_i) + \dot{\mathbf{A}} \cdot \dot{\mathbf{A}} + \ddot{\mathbf{A}} \cdot \ddot{\mathbf{A}} - \ddot{φ} \dot{φ} \right] \omega' δ_{ij}$$

$$- a_i a_j \left( 3H + \frac{1}{2} H^2 + \frac{1}{2} \mathbf{H} \cdot \mathbf{H} - H_i - 3HH_i \right) δ_{ij} \omega$$

$$+ 4a^2 \left( \mathbf{A} \cdot \dot{\mathbf{A}} \right) δ_{ij} \omega'' - A_i A_j ω' R , \quad (40)$$

2 Requiring positivity of the free-field energies of both the metric and the vector may impose further constraints on the form of $ω$.

3 Which in fact equals the kinetic term $A^α A^ν R_{αν} - (\nabla_α A^α)^2 - \nabla_α A^α \nabla_ν A^ν$, as is seen by using the geometric identity $[\nabla_μ, \nabla_ν] A^α = R^α_{\betaμν} A^β$ after a partial integration of the action.
where \( R = 9H^2 + 6\dot{H} + H \cdot H \), and we have used the obvious notation for the 3-vectors \( \mathbf{H}, \mathbf{H} = (H_1, H_2, H_3) \) and similarly for \( \Lambda \).

The equation of motion for the time component of the field is
\[
\phi \left( 2V'(A^2) - \omega(A^2)R \right) = 0. \tag{41}
\]
The \( G_{0i} \) component of the Einstein equations is identically satisfied if either the space-like or the time-like vector components vanish. The condition \( G_{ij} = 0 \) then gives
\[
-\dot{A}_i \dot{A}_j + (2V'(A^2) - \omega(A^2)R) A_i A_j = 0, \tag{42}
\]
which translates into condition \( \dot{A}_i \dot{A}_j = 0 \), which should be satisfied for all \( i \neq j \). Given this condition and \( \phi = 0 \), an arbitrary vector field is compatible with the Bianchi I metric. This excludes the FLRW metric.

In the remainder of this section we consider the special case of the FLRW universe including a (solely) time-like field. Similar vector models have been recently considered by Böhmer and Harko [34] to successfully model the late acceleration of the universe while satisfying the stringent solar system constraints. Recently Jimenez and Maroto [36] have also noted that a vector-like dark energy could avoid some of the fine-tuning problems present in scalar field quintessence type models. They showed that a time-like vector field with a non-standard kinetic term could explain the SNIa data even without a potential.

A time-like field does break the Lorentz invariance as well as the spatial fields considered in the previous subsection, since picking up the time coordinate introduces a preferred frame. However, in an FLRW universe one can treat the vector as an isotropic field whose time component resembles a non-standard scalar field (there will be interesting changes when perturbations are taken into account).

Summing the contributions from equations (8) and (38) the energy density is
\[
\rho_\phi = 6H \phi \dot{\phi} - 3H^2 \omega + V, \tag{43}
\]
where we have used the equation of motion \( 2V'' = \omega' R \) to eliminate the last term in equation (38). One notes that this solution does not exist for the case of a massive vector field \( V(x) = \frac{1}{2} m^2 x \) linearly coupled to the curvature, \( \omega(x) = \omega_0 x \), except in the special case of constant curvature. However, for the more general case of non-linear coupling and/or non-linear potential such dynamical solutions do exist. The pressure of the field follows from equations (9) and (40),
\[
p_\phi = -V - 2 \left( 2H \dot{\phi} \phi - \dot{\phi}^2 - \ddot{\phi} \phi \right) \omega' + 4 \left( \dot{\phi} \phi \right)^2 \omega'' + \left( \frac{2\ddot{a}}{a} + H^2 \right) \omega. \tag{44}
\]
One may check that the sum \( \rho_\phi + p_\phi \) satisfies the consistency relation
\[
\rho_\phi + p_\phi = -\dot{\rho}_\phi / (3H),
\]
which follows from the covariant conservation of matter that applies now since the \( L_m \) is minimally coupled [71]. An interpretation is that the vector only changes the geometry. The conservation law may be written as
\[
\rho_\phi + p_\phi = \ddot{\omega} + H^3 \left( \frac{\omega}{H^2} \right)^\prime. \tag{45}
\]
Let us first look at de Sitter solutions assuming that we may neglect $\rho_m$. We then set the Hubble parameter to a constant, $H_0$. Firstly, one notes that an equation of motion is identically satisfied if the field is constant. Then the coupling follows from (45) inserted into the Friedmann equation as

$$\omega_0 = V/(3H_0^2) - (8\pi G)^{-1}.\]

Then, consistently, equation (44) gives $p_\phi = -3H_0^2/(8\pi G)$. This holds independently of the potential. One is thus able to mimic a cosmological constant with a time-like vector coupling, without having to have any potential present. Secondly, one may also mimic a constant term even if the field is rolling. Then, however, the equation of motion gives the non-trivial constraint

$$V' = 6H_0^2\omega'$$

and one needs a potential

$$V = 6H_0^2\omega + V_0.$$ 

If we for instance assume that the coupling has a power-law form,

$$\omega(\phi) = \omega_1 \phi^{\alpha} + \omega_0,$$

then equation (45) tells us that the field grows (or decreases, if $\alpha \gg -1$) exponentially with time, $\phi \sim e^{-H_0 t/\alpha}$. From equation (43) one may then deduce that if $\omega_1 = \alpha/2$ and if $\omega_0 = 0$, then $V_0 \neq 0$. Similar results have been obtained in the unit-norm scalar–tensor–vector inflation model studied in [38], where it was noticed that a vector energy may stay constant even if the field is rolling down a potential, and on the other hand that such field can drive inflation even without a potential.

Finally we will turn to a late-time universe where $\rho_m$ should be taken into account too. We look for a model which reproduces the background expansion of the $\Lambda$CDM model. The sum (45) should again vanish. The derivative of the Hubble rate would be given by

$$\dot{H} = \frac{1}{2}(\tilde{\Lambda} - 3H^2),$$

where $\tilde{\Lambda} = \Lambda/(8\pi G)$. Considering then the coupling as a function of the Hubble rate, $\omega = \dot{\omega}(H)$, we find that its evolution equation

$$\frac{1}{2}(\tilde{\Lambda} - 3H^2)\ddot{\omega}(H) - 4H\dot{\omega}'(H) + 2\dot{\omega}(H) = 0 \tag{46}$$

is of the Legendre form and thus has solutions in terms of the Legendre functions. Therefore, there seems to be no simple form for the curvature-coupling function of time-like vector field mimicking the $\Lambda$CDM expansion. From the (Legendre type) solution of equation (46) one could then deduce, using the equation of motion, $2V' = \omega' R$, the form of the potential. But since these analytic forms would be unappealing, we do not write them down here. The bottom line is that there exist solutions with non-trivial vector field dynamics, with the energy density equation (43) residing in the field which however remains a constant. The model can therefore share an identical background with the $\Lambda$CDM model at late times in addition to possibly driving an inflationary period at early times. In the following we will see that with another kind of curvature coupling, this can be achieved with polynomial forms for the coupling and potential functions. Such might appear more naturally in the low energy corrections to the gravitational action. We leave the possible generation of magnetic fields in this model to future studies.

3.2. A vector–Gauss–Bonnet model

The Gauss–Bonnet topological invariant

$$R_{GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

can have dynamical consequences in four dimensions if it interacts with the matter sector or has
self-interactions. Dark energy cosmologies with self-interactions of $R^2_{\text{GB}}$ and with scalar couplings have been considered previously [26], [72]–[74]. Here we introduce a possible coupling to a vector and write the action as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - f(A^2)R^2_{\text{GB}} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - V(A^2) + L_m \right].$$ \hspace{1cm} (47)

In analogy with the discussions in the previous subsections, we now find an effective energy–momentum tensor

$$T^f_{\mu\nu} = -8 \left[ f_{,\alpha\beta} R_{\mu}^{\alpha\beta} + \Box f R_{\mu\nu} - 2 f_{,\alpha(\mu} R^{\alpha}_{\nu)} + \frac{1}{2} f_{\mu\nu} R \right]$$

$$- 4 \left[ 2 f_{,\alpha\beta} R^{\alpha\beta} - \Box f R \right] g_{\mu\nu} + 2 f' A_\alpha A_\nu R^2_{\text{GB}}. \hspace{1cm} (48)$$

To ease notation, we define $g^* \equiv -\dot{a}b\dot{c}$. The determinant of the metric is in our Bianchi I background $g = -abc$. We then have $R^2_{\text{GB}} = 8g^*/g$. The components of the tensor (48) can be written as

$$T^f_{00} = 48 f' g^* \left( \Lambda \cdot \dot{\Lambda} - \phi\dot{\phi}\right) + 16 f' \ddot{g}^* g^2, \hspace{1cm} (49)$$

$$T^f_{ai} = 16 f' \ddot{g}^* \Lambda_i a_i \phi, \hspace{1cm} (50)$$

$$T^f_{ij} = -16 a_i a_j \left[ \frac{1}{2} f' g^* a_i \left( \Lambda \cdot \dot{\Lambda} - \phi^2 \right) \right] \delta_{ij} + 2 f' a_i a_j g^* \left( \Lambda \cdot \dot{\Lambda} - \phi\dot{\phi}\right)^2 \delta_{ij}. \hspace{1cm} (51)$$

Again the coexistence of space and time vector components is not generally possible, but in Bianchi I space–times one may allow one space component to evolve simultaneously with the vector field.

Here we specialize to what may be the simplest possible model, a time-like vector field in FLRW space–time. Picking up the relevant terms from equations (8) and (49), we get an energy density

$$\rho_\phi = -48 H^3 \dot{\phi}\dot{f}' + V, \hspace{1cm} (52)$$

where we have used the equation of motion $V' = -f' R^2_{\text{GB}}$ to drop a couple of terms. Again one might note that this solution does not exist for the case of a massive vector field $V(x) = \frac{1}{2} m^2 x$ linearly coupled to the Gauss–Bonnet invariant, $f(x) = f_0 x$ except in the special case of constant $R^2_{\text{GB}}$. However, for the more general case of non-linear coupling and/or non-linear potential such dynamical solutions do exist. The pressure of the field can be read from equations (9) and (51),

$$p_\phi = 16 \left[ 2H \left( \dot{H} + H^2 \right) \dot{\phi} + H^2 (\dot{\phi}^2 + \ddot{\phi}\phi) \right] f' - 32H^2 (\dot{\phi}\phi)^2 f'' - V. \hspace{1cm} (53)$$

One can check that the sum

$$\rho_\phi + p_\phi = 16H \left[ (2\dot{H} - H^2) \dot{J} - \frac{1}{2} H \dot{f} \right]$$

satisfies the consistency relation

$$\rho_\phi + p_\phi = -\dot{\rho}_\phi/(3H).$$
In the remainder of this subsection we will attempt reconstructions of a coupling and the potential in our action (47). For reconstruction schemes in other string-inspired cosmologies taking into account (scalar) couplings of the $R_g^2$GB term see [75]–[77]. The de Sitter solutions in a vacuum would be relevant for the early inflation. Now it is clear that if the field is constant, one would need a potential to achieve inflation. This is because as a topological invariant, the Gauss–Bonnet term cannot contribute to the dynamics if the coupling is a constant. However, if the field rolls down its potential, the coupling affects the dynamics, though possibly having a constant energy density with a potential and thus yielding a de Sitter space as a solution, analogously to the Ricci-coupled case of the previous subsection. Let us pursue this first.

The vanishing of the sum (54) requires
\[ f(t) = f_0 + f_1 e^{2H_0 t}, \]
where $f_0$, $f_1$ and $H_0$ are constants. Since $f_0$ is classically irrelevant, we drop it. Then integrating the equation of motion gives us
\[ V(t) = V_0 - 24H_0^4 f_1 e^{-2H_0 t}. \]
Here the constant $V_0$ corresponds effectively to a cosmological constant and we drop it also. If then we insert our result into the Friedmann equation, $H^2 = \frac{8\pi G \rho_\phi}{3}$, where the energy density $\rho_\phi$ is given by equation (52), we obtain an equation for the field as a function of time. The result is
\[ \phi(t) = \frac{1}{4} + \frac{1}{256\pi G H_0^2 f_1} e^{2H_0 t}. \]
We then find that the potential and the coupling may have a simple inverse form,
\[ V(A^2) = \frac{3H_0^2}{8\pi G (1 - 4|A|)}, \quad f(A^2) = \frac{1}{64\pi G H_0^2 (4|A| - 1)}. \]
There thus exist exact de Sitter solutions when the field is evolving. In fact the field is exponentially increasing as a function of the cosmic time $t$. Therefore additional terms, say positive powers of $|A|$, could naturally terminate the de Sitter phase and end inflation. Hence this could give rise to a possible graceful exit from inflation.

Next we aim to find a vector–Gauss–Bonnet model which produces, at late times, exactly the same background evolution as the ΛCDM model. Since the vector then mimics a constant, the sum (54) should vanish. We want to have $\dot{H} = (C - 3H^2)$, where the constant $C = 8\pi \lambda$ is proportional to the corresponding cosmological constant $\lambda$. If we again think of the coupling as a function of the Hubble rate $\dot{f} = g(H)$, we can solve it as
\[ (C - 4H^2)g(H) - \frac{1}{4}(C - 3H^2)g'(H) = 0 \Rightarrow \frac{dg}{g} = 4 \left( 1 - \frac{H^2}{C - 3H^2} \right) \frac{dH}{H} \]
\[ \Rightarrow g(H) \sim H^4 \left( C - 3H^2 \right)^{2/3}. \]
Furthermore, using the ΛCDM Hubble rate
\[ 3H = (3H_0^2 - C)e^{-3x} + C, \]
integrating once more we get the coupling as a function of $x$:

$$f'(x) \sim H^3 \left( C - 3H^2 \right)^{-2/3} \sim e^{-2x} \left( 1 + \tilde{C} e^{-3x} \right)$$

$$\Rightarrow f(x) = f_0 + f_1 \left( 220 e^{-2x} + 264 \tilde{C} e^{-5x} + 165 \tilde{C}^2 e^{-8x} + \tilde{C}^3 e^{-11x} \right), \quad (58)$$

where $f_0$ and $f_1$ are integration constants and $\tilde{C} = 3H_0^2/C - 1$. The constant $f_0$ is dynamically irrelevant and we drop it. Our model has two free functions: the potential $V$ and the coupling $f$. The solution equation (58) tells us how the coupling has to evolve as the scale factor expands like in $\Lambda$CDM cosmology. How this corresponds to the evolution of the field $\phi$ depends on the functional form of $f(A^2)$. We will choose its form as a polynomial in such a way that $\phi \sim e^{-x/2}$:

$$f(A^2) \sim \left[ 220(-A^2) + 264 \tilde{C}(-A^2)^{5/2} + 165 \tilde{C}^2(-A^2)^4 + \tilde{C}^3(-A^2)^{11/2} \right]. \quad (59)$$

Then, from the equation of motion $V' = f'G$ we find that

$$V'(A^2) \sim \left[ -2 + \tilde{C}(-A^2)^{3/2} \left[ 1 + \tilde{C}(-A^2)^{3/2} \right] \left[ 40 + 120 \tilde{C}(-A^2)^{3/2} + 120 \tilde{C}^2(-A^2)^4 + \tilde{C}^3(-A^2)^{9/2} \right] \right], \quad (60)$$

yielding the form for the potential as

$$V(A^2) \sim A^2 \left[ 220 + 264 \tilde{C}(-A^2)^{3/2} + 165 \tilde{C}^2(-A^2)^4 + \tilde{C}^3(-A^2)^{9/2} \right], \quad (61)$$

where we have set an integration constant to zero since it would correspond to a cosmological constant. We have thus found that a time component of a vector field coupled to the Gauss–Bonnet invariant may generate (1) an early inflation epoch and (2) the $\Lambda$CDM background expansion without invocation of a cosmological constant or any scalar fields. This may happen when both the coupling and the potential are of a polynomial form, equations (59), (61) in the case of dark energy and equation (56) in the case of inflation. These results seem quite interesting, keeping in mind that both vector fields and the Gauss–Bonnet term have crucial roles in fundamental theories. Suppose a theory gives corrections to the action in a form of power-law expansion of the field strength $|A|$. Our results then imply that the negative powers could cause the early inflation and the positive powers could be responsible for the present acceleration of the universe.

### 3.3. Vortical perturbations

The conventional decomposition of the metric separates perturbations into scalar, vector and tensor quantities according to their transformation properties under the spatial rotation group. Here we consider the vector type (i.e. vortical) perturbations, and to avoid confusion we call them spin-1 perturbations. In flat cosmology the metric, taking into account the spin-1 perturbations, can be specified without loss of generality by the line element

$$ds^2 = a^2(\tau) \left[ -d\tau^2 + B \cdot d\vec{x} \cdot d\tau + d\vec{x} \cdot d\vec{x} \right]. \quad (62)$$

We use here the conformal time $\tau = \int dt/a$. Generally the spatial sections would also have inhomogeneities, but we employ a gauge where they are flat. The shift vector $B$
then fully characterizes the inhomogeneities. It is transverse, $\nabla \cdot B = 0$. The vector field we write as

$$A_\mu = a(\tau)[\varphi, L],$$

(63)

where the transverse vector $L$ is a perturbation. The conformal background field $\varphi = \phi(\tau)/a(\tau)$ is introduced for convenience. It follows that

$$A^\mu = \frac{1}{a(\tau)} \left[ -\varphi, L + \frac{1}{2} B \right],$$

(64)

and that $A^2 = -\varphi^2$, which is understandable since $A^2$ is of spin-0 type. We can also write up to linear order that

$$a^2 R = 6(\dot{H} + H^2), \quad a^4 R_{GB}^2 = 24H^2 H',$$

(65)

where $H \equiv a'/a$ is the conformal Hubble factor and the prime denotes the derivative with respect to conformal time. The equation of motion for $L$ then reads

$$L'' + 2H L' + \left[ k^2 + (H' + H^2)(1 + 6\omega') + \frac{48}{a^2} f' H^2 H' + 2a^2 V' \right] L = 0.$$  

(66)

In this gauge the metric perturbations decouple from the evolution equation of the spin-1 type perturbation of the vector field. A wave-like propagation appears, unlike in the unit-norm models where the equation of motion instead becomes a first-order differential equation, whose solutions are diluting [78]. We are then able to deduce various properties of this perturbation. It is interesting to note that each component propagates like a scalar field fluctuation in the spatially flat gauge living in an effective potential given by the square bracket (excluding the gradient term). However, for the specific models considered in the last subsection the background equations motions reduce this to

$$L'' + 2H L' + [k^2 + (H' + H^2)] L = 0.$$  

(67)

Thus the evolution of the modes is independent of the detailed form of the coupling functions, and depends only on their effect to the background. As canonical scalar fields, the components of $L$ propagate with light speed at small scales. This important fact establishes that the spin-1 type perturbations are stable at small scales, quantum mechanically consistent (i.e. not ghost modes\(^4\)) and causal. For these models we find that the modes decay during inflation also outside the horizon, though at a slower rate than vector modes do usually, $L \sim a^{-1+\sqrt{m}/4}$. However, under an expansion with $w_{\text{eff}} > 1/3$ the superhorizon vector modes become tachyonic. This kind of effect could amplify the vorticity seeds during reheating. For radiation domination a constant mode exists. Given a regular primordial spectrum, vortical can result in a large $B$-mode polarization signal at small scales $\ell \sim 2000$, which is distinguishable from the effects of tensor and gravitational lensing on polarization [79], and therefore possibly detectable with Planck. However, such effects might be pronounced in the present models.

\(^4\) To see this, the equation of motion itself is not enough. One has to also check that the overall sign of the vector Lagrangian in the action corresponds to the right sign of the Hamiltonian, as now happens to be the case.
4. Conclusions and discussion

We investigated field theories where vector fields mediate non-gravitational interactions and drive the accelerated expansion of our universe, during both the early inflation and the present dark energy domination. The motivation for this investigation comes from the frequent appearance of vector fields and possible Lorentz violations in fundamental theories, from the need to test the robustness of the basic assumptions of cosmology, and from the hints of statistical anisotropy of our universe that several observations seem to suggest.

In this study we considered the question of whether such vector fields could have an important cosmological role instead of the more popular hypothetical scalar fields and their exotic variants. A usual reason for excluding this otherwise reasonable possibility is that vector fields are incompatible with the established isotropy of the universe. However, space-like fields could feature only small anisotropy, which some anomalous observations do indeed hint could exist in the universe. Secondly, time-like fields do not necessarily break isotropy, but they have non-trivial dynamics only if non-minimally interacting with gravity. To summarize our results, we found for space-like fields that:

- there exist for space-like fields (anisotropic) scaling solutions even without couplings to matter;
- the cosmological bounds are satisfied in models of possibly large anisotropy if (1) the anisotropic era began just recently or (2) the eccentricity cancels today.

We learnt about (time-like) vector–tensor models that:

- with non-minimal couplings to gravity, dynamical solutions for cosmological vector fields become naturally available;
- in particular, there are models with coupling to the Ricci or to the Gauss–Bonnet scalar that incorporate inflation and dark energy.

In general, the amount of fine-tuning in vector dark energy models is similar to the amount of fine-tuning in scalar field models. As we have shown that there exist scaling attractors, the initial conditions for the fields and their velocities do not matter. One is then left to explain the energy scales involved. Just like with scalar fields, by redefinitions and zero-point shiftings one may choose the numbers that go into the Lagrangian, but the field turns out to be extremely light. Consider, as an example, the exponential potential, $V(A^2) = V_0 e^{\lambda (A^2 - A_0^2)}$. If the potential energy makes a significant contribution to the expansion rate today, the Friedmann equation tells us that the potential is of the order $V \sim H_0^2 / \kappa$. However, the potential scale $V_0$ can be set to whatever seems natural by choosing suitably the parameter $A_0^2$. The mass of the associated particle $m_A \sim \sqrt{V} \sim 10^{-33}$ eV if the slope $\lambda$ is roughly of order 1. For the power-law potential, $V(A^2) = V_0 (\kappa A^2)^n$, one finds the exact relation

$$m_A = \sqrt{-n} \left( \frac{V}{V_0} \right)^{1/2 - 1/2n} \sqrt{\kappa V_0},$$

(68)

where $V$ is the value of the potential at a given time. If the time is now and the scale of the potential is of the order of the Planck mass, then the mass of the field is the Planck
mass suppressed by a factor $10^{-50(1-1/n)}$, which for large inverse powers can be quite tiny. These considerations hold also even if the potential vanishes [36], since one may interpret the mass as effectively arising from the non-minimal coupling. For the vector inflatons, one may find TeV scale masses and potentials.

Thus, several cases of observationally viable and theoretically motivated vector fields in either FLRW or in a Bianchi I background were found. Those include models where again scaling solutions do exist for (space-like) vectors, with and without couplings to matter, and with given forms of the vector field potentials. With matter and the vector field present, there are three kinds of scaling solutions, which were summarized in table 1. Allowing anisotropy of the dark energy equation of state could be useful in the quest for a realistic description of the present cosmological acceleration, since the exact symmetries of the FLRW metric exclude from the beginning for example these space-like vector fields.

We also considered vector fields non-minimally coupled to gravity. We showed that a time-like vector coupled to the Ricci scalar or to the Gauss–Bonnet invariant may generate both the inflation and the ΛCDM expansion. In the vector–Gauss–Bonnet model, the form of the potential and the coupling which mimic the concordance model background expansion can be simple polynomial functions. Negative powers may drive inflation and positive powers the late-time acceleration. Amusingly, both the time-like field and the Gauss–Bonnet term become trivial if the coupling is set to zero. Indeed, we have presented the first inflation models driven by time-like vector field. This may have relevance to the problem of definition of the initial vacuum state of the universe. We considered the evolution of spin-1 type perturbations showing that it is both causal and stable.

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