ON THE $A_\infty$-FORMALITY CONJECTURE

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Abstract. It is proved that the associative differential graded algebra of (polynomial) polyvector fields on a vector space (may be infinite-dimensional) is quasi-isomorphic to the corresponding cohomological Hochschild complex of (polynomial) functions on this vector space as an associative differential graded algebra. This result is an $A_\infty$-version of the Formality conjecture of Maxim Kontsevich [K].

1. Configuration spaces and their compactifications

Let $n, m$ be non-negative integers satisfying the inequality $n + m \geq 1$. We define a configuration space $Conf_{n,m}$ which will play the crucial role in the construction of the $A_\infty$-quasi-isomorphism $F: T_{\text{poly}}^\bullet \to D_{\text{poly}}^\bullet$ in Section 2. We set

$$Conf_{n,m} = \{(p_1, \ldots, p_n; q_1, \ldots, q_m) \mid p_i \in \mathbb{R}_{<0}, q_j \in \mathbb{R}_{>0} \text{ and } p_1 < \ldots < p_n, q_1 > \ldots > q_m\}.$$

$Conf_{n,m}$ is a smooth manifold of dimension $n + m$. The group $G(1) = \{t \mapsto at, a > 0, a \in \mathbb{R}\}$ acts on the space $Conf_{n,m}$. The quotient space $C_{n,m} = Conf_{n,m}/G(1)$ is a manifold of dimension $n + m - 1$. We are going to describe a compactification $\overline{C}_{n,m}$ of the space $C_{n,m}$, which is a manifold with corners in the sense of Maxim Kontsevich [K], Sect. 5.2. We defined also a manifold $C_n, n \geq 2$:

$$Conf_n = \{(p_1, \ldots, p_n) \mid p_i \in \mathbb{R}, p_1 < \ldots < p_n\},$$

$G(2) = \{t \mapsto at + b, a, b \in \mathbb{R}, a > 0\}$ and $C_n = Conf_n/G(2)$, dim$C_n = n - 2$.

The construction of the compactification $\overline{C}_n$ is analogous to [K], Sect. 5.2. The strata $C_T$ are labeled by trees $T$ and

$$\overline{C}_n = \bigcup_{\text{labeled trees } T \in V_T \setminus \{\text{leaves}\}} \prod_{v \in V_T} C_{\text{Star}(v)}$$

(here $V_T$ is set of all the vertices of the tree $T$ and $\text{Star}(v)$ is the number of edges starting at the vertex $v$).

In the case of $C_{n,m}$ we have several possibilities:

(i) points $p_{n_1}, \ldots, p_n$ ($n_1 \leq n$) and $q_{m_1}, \ldots, q_m$ ($m_1 \leq m$) are close to each other and to $0 \in \mathbb{R}$; then the corresponding stratum is the product $C_{n_1-1,m_1-1} \times C_{n-n_1+1,m-m_1+1}$;

(ii) points $p_{n_1}, \ldots, p_{n_2}$ ($n_2 - n_1 \geq 1$) are close to each other and far from $0 \in \mathbb{R}$; then the corresponding stratum is $C_{n_2-n_1+1} \times C_{n-n_2+n_1,m}$; analogously in the case when points $q_{m_1}, \ldots, q_{m_2}$ ($m_2 - m_1 \geq 1$) are close to each other;

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(iii) points \( p_{m_1}, \ldots, p_n (n - n_1 \geq 0) \) are close to each other and to 0 \( \in \mathbb{R} \), then the corresponding stratum is \( C_{n-1,m} \times C_{n-m+1,0} \); analogously for the points \( q_{m_1}, \ldots, q_m (m - m_1 \geq 0) \).

These are all the strata of codimension 1. It is easy to describe all other strata. The strata “of the first level” are obtained when there exists several groups of the points which are close to each other. When we are “looking through a magnifying glass” on these strata we obtain strata of “second level,” and so on.

2. \( A_\infty \)-Formality Conjecture for \( A = \mathbb{C}[x_1, \ldots, x_d] \)

2.1. Let \( T_{\text{poly}}^*(\mathbb{R}^d) \) be the associative super-commutative algebra of polyvector fields on \( \mathbb{R}^d \) (with polynomial coefficients), and let \( D_{\text{poly}}^*(\mathbb{R}^d) \) be the Hochschild cohomological complex of the algebra \( \mathbb{C}[x_1, \ldots, x_d] \); we consider \( D_{\text{poly}}^*(\mathbb{R}^d) \) as an associative DG algebra with the product

\[
(\Theta_1 \cdot \Theta_2)(f_1, \ldots, f_{k+l}) = \Theta_1(f_1, \ldots, f_k) \cdot \Theta_2(f_{k+1}, \ldots, f_{k+l}).
\]

The \( A_\infty \)-Formality conjecture in this case states, that associative DG algebra \( D_{\text{poly}}^*(\mathbb{R}^d) \) is quasi-isomorphic (as an associative algebra) to its cohomology. It is well-known result (Hochschild–Konstant–Rosenberg Theorem), that \( H^*(D_{\text{poly}}^*(\mathbb{R}^d)) = T_{\text{poly}}^*(\mathbb{R}^d) \), and the map \( \phi_{\text{HKR}}: T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d) \)

\[
\phi_{\text{HKR}}(\xi_1 \wedge \cdots \wedge \xi_k)(f_1, \ldots, f_k) = \frac{1}{k!} \text{Alt} \xi_1(f_1) \cdot \cdots \cdot \xi_k(f_k)
\]

is an quasi-isomorphism of the complexes (here \( \xi_1, \ldots, \xi_k \) are vector fields and \( f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_d] \)).

Moreover, the inducing map \( T_{\text{poly}}^*(\mathbb{R}^d) \to H^*(D_{\text{poly}}^*(\mathbb{R}^d)) \) is an isomorphism of algebras (see, for example, [KSh], Sect. 3).

We want to construct an \( A_\infty \)-morphism \( \mathcal{F}: T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d) \), which first component coincides with the map \( \phi_{\text{HKR}} \). It means, that we want to find maps

\[
\mathcal{F}_1 = \phi_{\text{HKR}}: T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d),
\]

\[
\mathcal{F}_2: \otimes^2 T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d)[-1],
\]

\[
\mathcal{F}_3: \otimes^3 T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d)[-2],
\]

\[
\mathcal{F}_n: \otimes^n T_{\text{poly}}^*(\mathbb{R}^d) \to D_{\text{poly}}^*(\mathbb{R}^d)[-n],
\]

\[
\vdots
\]

such that for any \( n = 1, 2, \ldots \) and for homogeneous polyvector fields \( \gamma_1, \ldots, \gamma_n \) one have:

\[
d\mathcal{F}_n(\gamma_1 \otimes \cdots \otimes \gamma_n) - \sum_{k,l \geq 1} \pm \mathcal{F}_k(\gamma_1 \otimes \cdots \otimes \gamma_k) \cdot \mathcal{F}_l(\gamma_{k+1}, \ldots, \gamma_n) -
\]

\[
- \sum_{i=1, \ldots, n-1} \pm \mathcal{F}_{n-1}(\gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_i \otimes \gamma_{i+2} \otimes \cdots \otimes \gamma_n) = 0.
\]

2.2. Let us recall the construction with graphs from [K], Sect. 6.3.

We consider oriented graphs with \( n \) vertices of the first type and \( m \) vertices of the second type, the edges start at the vertices of the first type, and these are no loops. The vertices of the first type are labeled by the symbols \( \{1, \ldots, n\} \), and the vertices of the second type are labeled by the symbols \( \{\overline{1}, \ldots, \overline{m}\} \).
For any vertex $k$ of the first type we denote by $\text{Star}(k)$ the set of edges starting at the vertex $k$.

Any such a graph $\Gamma$ defines a map
\[
U_\Gamma : \otimes^n T_{\text{poly}}^\bullet(\mathbb{R}^d) \to D_{\text{poly}}^\bullet(\mathbb{R}^d)[1 - n + l]
\]
where the number of edges of $\Gamma$ is equal to $n + m - 1 + l$. This map is defined as follows.

First of all, this map has the unique nonzero component, it is $T_\#\text{Star}(1) \otimes \ldots \otimes T_\#\text{Star}(n)$.

If $\gamma_i \in T_\#\text{Star}(i)(\mathbb{R}^d)$, we are going to define the function
\[
\Phi = U_\Gamma(\gamma_1 \otimes \ldots \otimes \gamma_n)(f_1 \otimes \ldots \otimes f_m).
\]

Let $E_\Gamma$ be the set of the edges of the graph $\Gamma$.

The formula for $\Phi$ is the sum over all configurations of indices running from 1 to $d$, labeled by $E_\Gamma$:
\[
\Phi = \sum_{I : E_\Gamma \to \{1, \ldots, d\}} \Phi_I,
\]
where $\Phi_I$ is the product over all $n + m$ vertices of $\Gamma$ of certain partial derivatives of functions $f_j$ and of coefficients of $\gamma_i$.

Namely, for each vertex $i$, $1 \leq i \leq n$ of the first type we associate function $\Psi_i$ on $\mathbb{R}^d$ which is a coefficient of the polyvector field $\gamma$:
\[
\Psi_i = \langle \gamma_i, dx^{I(e_i^1)} \otimes \ldots \otimes dx^{I(e_i^{k_i})} \rangle
\]
(here $k_i = \#\text{Star}(i)$, and the edges from $\text{Star}(i)$ are labeled by the symbols $(e_i^1, \ldots, e_i^{k_i})$).

For each vertex $j$ of second type the associated function $\Psi_j$ is defined as $f_j$.

Now, at each vertex of the graph $\Gamma$ we put a function on $\mathbb{R}^d$ (i.e. $\Psi_i$ or $\Psi_j$).

Also, on edges of the graph $\Gamma$ there are indices $I(e)$ which label coordinates in $\mathbb{R}^d$. In the next step we put into each vertex $v$ instead of function $\Psi_v$ its partial derivative
\[
\left( \prod_{e \in E_\Gamma, e = (v, v')} \partial_{I(e)} \right) \Psi_v,
\]
and then take the product over all vertices of $\Gamma$. The result is by definition the summand $\Phi_I$.

2.3. Definition. The set $G_{n,m}$ is the set of all the oriented graphs with $n$ vertices of the first type, $m$ vertices of the second type, $n + m - 1$ edges, and all the edges start at vertices of the first type and end at the vertices of the second type.

The following are typical pictures:

![Figure 1](image-url)
**Definition** (the weight $W_\Gamma$). Let $\varphi(x)$ be any function (defined for $x < 0$) such that is derivative $\varphi'(x)$ is a function with a compact support, and such that $\int \varphi'(x)dx = 1$ (see Fig. 2).

For any pair of points $p_i, q_j$ such that $p_i < 0$, $q_j > 0$ we define $\varphi(p_i, q_j)$ as $\varphi\left(\frac{p_i}{q_j}\right)$.

Any edge $e$ of $\Gamma$ defines the 1-form $d_{DR} \varphi_e$ on $\overline{C}_{n,m}$. By definition,

$$W_\Gamma = \int_{\overline{C}_{n,m}} \wedge_{e \in E_\Gamma} d_{DR} \varphi_e.$$  

**Theorem.** The maps $F_1, F_2, F_3, \ldots$ ($F_i: \otimes^i T^\bullet_{\text{poly}}(\mathbb{R}^d) \to D^\bullet_{\text{poly}}[1-i]$), where $F_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \times U_\Gamma$, defines an $A_\infty$-morphism.

**Proof.** As in the $L_\infty$-case in [K], the proof is just an application of the Stokes formula.

Let us denote by $(F)$ the left-hand side of the formula (2). One can write $(F)$ as a linear combination

$$\sum_{\Gamma} C_\Gamma U_\Gamma(\gamma_1 \otimes \ldots \otimes \gamma_n)(f_1 \otimes \ldots \otimes f_m)$$

where $\Gamma$ has $n$ vertices of the first type, $m$ vertices of the second type, and $n + m - 2$ edges. We want to check that $C_\Gamma$ vanishes for each $\Gamma$. The idea (as in [K], Sect. 6.4) is to identify $C_\Gamma$ with the integral over the boundary $\partial \overline{C}_{n,m}$ of the closed differential form $\wedge_{e \in E_\Gamma} d\varphi_e$. We have:

$$\int_{\partial \overline{C}_{n,m}} \wedge_{e \in E_\Gamma} d\varphi_e = \int_{\overline{C}_{n,m}} d(\wedge_{e \in E_\Gamma} d\varphi_e) = 0$$

On the other hand,

$$\int_{\partial \overline{C}_{n,m}} \wedge_{e \in E_\Gamma} d\varphi_e$$

is an integral over all the strata of codimension 1. All the strata of codimension 1 were listed in the end of Section 1.

The case (i) corresponds to the second summand in (2). The stratum (ii) has nonzero contribution in the integral $\int_{\overline{C}_{n,m}} \wedge_{e \in E_\Gamma} d\varphi_e$ only in the case $\dim C_{n_2-n_1+1} = 0$, i.e. it is the case where two neighbour points of the first type (of the second type) are close to each other and far from 0; this case corresponds to the third (resp., first) summand in (2). The stratum (iii) has a nonzero contribution in the integral only in the case $n_1 = n$ ($m_1 = m$), it is the contribution to the second (first) summand of (2). 

\[\square\]
2.4. It is easy to see that $\mathcal{F}_1 = \varphi_{HKR}$ (see Section 1) and therefore the $A_\infty$-map $T_{poly}^\bullet (\mathbb{R}^d) \to D_{poly}^\bullet (\mathbb{R}^d)$ we have constructed is an $A_\infty$-quasi-isomorphism. It follows from the general theory, that this fact is equivalent to the statement, that the DG algebras $T_{poly}^\bullet (\mathbb{R}^d)$ and $D_{poly}^\bullet (\mathbb{R}^d)$ are quasi-isomorphic in the derived category of associative DG algebras.

Note also that all the integrals $W_T$ can be easily calculated, in the difference with the case of the $L_\infty$-Formality conjecture [K].

2.5. The remarkable difference from the case of $L_\infty$-Formality ([K]) is that our formulas make sense in the case of the algebra of polynomials of infinite number of variables, $A = \mathbb{C}[x_1, x_2, x_3, \ldots]$. The proof is the same. Also, these formulas defines an $A_\infty$-quasi-isomorphism for any super-algebra $A$.

References

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