Hamilton-Jacobi theory, Symmetries and 
Coisotropic Reduction

Manuel de León †, David Martín de Diego‡ and Miguel Vaquero †

Instituto de Ciencias Matemáticas, ICMAT,
c\ Nicolás Cabrera, n 13-15, Campus Cantoblanco, UAM,
28049 Madrid, Spain

Abstract

Reduction theory has played a major role in the study of Hamiltonian systems. On the other hand, the Hamilton-Jacobi theory is one of the main tools to integrate the dynamics of certain Hamiltonian problems and a topic of research on its own. Moreover, the construction of several symplectic integrators rely on approximations of a complete solution of the Hamilton-Jacobi equation. The natural question that we address in this paper is how these two topics (reduction and Hamilton-Jacobi theory) fit together. We obtain a reduction and reconstruction procedure for the Hamilton-Jacobi equation with symmetries, even in a generalized sense to be clarified below. Several applications and relations to other reduction of the Hamilton-Jacobi theory are shown in the last section of the paper. It is remarkable that as by-product we obtain a generalization of the Ge-Marsden reduction procedure ([18]) and the results in [17]. Quite surprisingly, the classical ansatzs available in the literature to solve the Hamilton-Jacobi equation (see [2, 19]) are also particular instances of our framework.

Keywords: Hamilton-Jacobi theory, reduction, momentum mapping, symmetries, lagrangian submanifolds.

†mdeleon@icmat.es
‡david.martin@icmat.es
¶miguel.vaquero@icmat.es

arXiv:1509.00419v1 [math-ph] 1 Sep 2015
1 Introduction

The Hamilton-Jacobi theory is today a well-known theory by mathematicians and physicist. The equations

1. $H(q^i, \frac{\partial S}{\partial q}(q^i)) = E,$
2. $\frac{\partial S}{\partial t} + H(t, q^i, \frac{\partial S}{\partial q}(t, q^i)) = E$

appear in any classical mechanics book, like [1, 19]. The Hamilton-Jacobi theory is connected to geometric optics and to classical and quantum mechanics in several intriguing ways. In geometric optics it establishes the link between particles and waves through the characteristic function, [21]. Hamilton and Jacobi extended this duality (wave-particle) to classical mechanics, where a solution of the Hamilton-Jacobi equation allows the reduction of the number of equations of motion by half, and a complete solution of the Hamilton-Jacobi equation allows us to make a change of variables that makes the integration of
Hamilton’s equations trivial (usually called “a transformation to equilibrium”). A detailed account of these topics can be found in [1, 3]. Recently the Hamilton-Jacobi theory has also been extended to the non-holonomic setting, [6, 10, 22, 11].

The Hamilton-Jacobi theory and the theory of generating functions also gave rise to families of symplectic numerical integrators which over long times are clearly superior to other methods (see [7, 12, 18]). Extending those integrators to the Lie-Poisson setting motivated the beginning of the reduction of the Hamilton-Jacobi theory in [17, 18], by Z. Ge and J.E. Marsden. After their approach, several works appeared along the same lines, [4, 8, 26, 27, 30]. Although we are not dealing with numerical methods, getting a deeper understanding of those results motivated this work to some extent. Moreover, a general setting to develop numerical methods based on the Hamilton-Jacobi theory for (integrable) Poisson manifolds will appear elsewhere [14]. The importance of the development of such geometric-Poisson integrators is beyond any doubt, taking into account the success of their symplectic analogues.

On the other hand, reduction theory is still nowadays an important topic of research. Since Jacobi’s elimination of the node, and its formalization through the Meyer-Marsden-Weinstein reduction, the usefulness of the theory is widely known. A complete reference for Hamiltonian reduction is [25].

The present paper studies how to apply reduction theory to simplify the Hamilton-Jacobi equation via the coisotropic reduction of lagrangian submanifolds (see [33]). We combine the aforementioned coisotropic reduction and cotangent bundle reduction to obtain the reduced Hamilton-Jacobi equation, which turns out to be an algebraic-PDE equation. As mentioned above, previous attempts to obtain a reduction of the (complete solutions of the) Hamilton-Jacobi equation were carried out by Ge and Marsden in [18] in order to provide a setting to develop Lie-Poisson integrators. Nonetheless, they only work out the details in the case where the configuration manifold is a Lie group, although they claim that the general procedure can be obtained. The main difference between their and our approach is that while Ge and Marsden reduce the generating function, say $S$, we focus on the corresponding lagrangian submanifold, say $\text{Im}(dS)$, that allows us to obtain a more general setting of wide applicability. For instance, generating functions which are not of type I, in the language of [19] can be treated using our approach, while this seems not to be the case for the previous settings. Of course, Ge’s framework can be obtained from our results in a straightforward fashion as will be shown in the last section, where we also deal with some examples, like a two particles Calogero-Moser system. Finally, although we did not include it here, the results by H. Wang in [32] are a particular case of our framework as well. The use of generating families to obtain lagrangian submanifolds ([9]) is another interesting topic not treated here that fits into our work.

The paper is organized as follows. In Section 2 we provide the necessary preliminaries and we establish the notation and conventions that we follow during the rest of the paper. In Section 3 we introduce the Hamilton-Jacobi equation and the announced reduction and reconstruction procedure. In Section 4 we
show that the “two reduced dynamics” are related in the expected way. Section 5 is devoted to applications and examples. We also include two appendices about adjoint bundles and magnetic terms to make the paper self-contained.

2 Preliminaries

In this paper all manifolds and mappings are supposed to be infinitely differentiable \((C^\infty)\). Given a map \(f : M \to N\) between manifolds \(M\) and \(N\), we will use the notation \(Tf\) to denote the tangent map \((Tf : TM \to TN)\), and \(Tf(p)\), where \(p\) is a point on \(M\), to denote the tangent map at that point \(Tf(p) : T_pM \to T_{f(p)}N\). Given a vector field on the manifold \(M\), say \(X\), the evaluation of that vector field at a point \(p \in M\) will read \(X(p)\). The flow of the vector fields under consideration will be assumed to be defined globally, although our results hold for locally defined flows with the obvious modifications. Along this paper \(G\) will be a connected Lie group and \(\mathfrak{g}\) the corresponding Lie algebra. We will make use of \(Ad^*\) to represent the Coadjoint action on the dual of \(\mathfrak{g}\) given by

\[
Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*
\]

\[
(g, \mu) \mapsto Ad_g^*(\mu) = \mu \circ TR_g \circ TL_g^{-1},
\]

where \(L_g(h) = g \cdot h\) and \(R_g(h) = h \cdot g\) are the left and right multiplication on the group \(G\). Notice that the Coadjoint action is a left action. Given \(\mu \in \mathfrak{g}^*\), \(Orb_{Ad^*}(\mu)\) denotes the orbit by the Coadjoint action through \(\mu\).

2.1 Lifted actions to \(TQ\) and \(T^*Q\)

Let \(G\) be a connected Lie group acting freely and properly on a manifold \(Q\) by a left action \(\Phi\)

\[
\Phi : G \times Q \to Q
\]

\[
(g, q) \mapsto \Phi(g, p) = g \cdot q
\]

Given \(g \in G\), we denote by \(\Phi_g : Q \to Q\) the diffeomorphism defined by \(\Phi_g(q) = \Phi(g, q) = g \cdot q\). Recall that under these conditions the quotient \(Q/G\) can be endowed with a manifold structure such that the canonical projection \(\pi : Q \to Q/G\) is a \(G\)-principal bundle. The action \(\Phi\) introduced above can be lifted to actions on the tangent and cotangent bundles, \(\Phi^T\) and \(\Phi^{T^*}\) respectively. We briefly recall here their definitions.

- **Lifted action on \(TQ\).** We introduce the action \(\Phi^T : G \times TQ \to TQ\) such that \(\Phi^T_g : TQ \to TQ\) is defined by

\[
\Phi^T_g(v_q) = T\Phi_g(q)(v_q) \in T_{gq}Q \quad \text{for } v_q \in T_qQ.
\]

- **Lifted action on \(T^*Q\).** Analogously, we introduce the following action
\(\Phi^T : G \times T^* Q \to T^* Q\) such that \(\Phi^T_g : T^* Q \to T^* Q\) is defined by

\[
\Phi^T_g(\alpha_q) = (T\Phi_g^{-1})^* (gq)(\alpha_q) \in T^*_q Q \quad \text{for } \alpha_q \in T^*_q Q.
\]

Both actions can be easily checked to be free and proper. If \(\alpha_q \in T^* Q\), we will denote the orbit through \(\alpha_q\) by \(\text{Orb}(\alpha_q)\).

### 2.2 Momentum Mapping

As is well-known, there exists a \(G\)-equivariant momentum mapping for the above action on \(T^* Q\) with respect to its canonical symplectic form, from now on denoted by \(\omega_Q\). This momentum map is given by \(J : T^* Q \to \mathfrak{g}^\ast\) where \(J(\alpha_q)\) is such that \(J(\alpha_q)(\xi) = \alpha_q(\xi_Q(q))\) for \(\xi \in \mathfrak{g}\). Here \(\xi_Q\) is the vector field on \(Q\) determined via the action \(\Phi\), called the infinitesimal generator. The integral curve of \(\xi_Q\) passing through \(q \in Q\) is just \(t \to \exp(t\xi)(q)\).

Given \(\xi \in \mathfrak{g}\), we denote by \(J_\xi : T^* Q \to \mathbb{R}\) the real function obtained by the pairing between \(\mathfrak{g}\) and \(\mathfrak{g}^\ast\), \(J_\xi(\alpha_q) = (J(\alpha_q), \xi)\). By the definition of momentum mapping we have \(\xi_{T^* Q} = X_{J_\xi}\), where \(\xi_{T^* Q}\) is the fundamental vector field generated by \(\xi\) via the action \(\Phi^T\). Indeed, we have \(\imath_{\xi_{T^* Q}} \omega_Q = \omega_Q = dJ_\xi\) and \(X_{J_\xi}\) is the vector field satisfying \(\imath_{X_{J_\xi}} \omega_Q = dJ_\xi\).

The next proposition, combined with the fact that \(\Phi^T\) is free and \(G\) connected, ensures that for a connected Lie group every \(\mu \in \mathfrak{g}^\ast\) is a regular value and so \(J^{-1}(\mu)\) is a submanifold. In fact, the next proposition characterizes regular values of momentum mappings taking into account the infinitesimal behavior of the symmetries. We define \(\mathfrak{g}_p = \{\xi \in \mathfrak{g} \text{ such that } \xi_Q(p) = 0\}\).

**Proposition 1** *(Marsden et al. [25])* Let \((M, \Omega)\) be a symplectic manifold and \(G\) a Lie group which acts by symplectomorphism with equivariant momentum map \(J\). An element \(\mu \in \mathfrak{g}^\ast\) is a regular value of \(J\) iff \(\mathfrak{g}_p = \{0\}\) for all \(p \in J^{-1}(\mu)\).

**Proof:** Let \(p \in J^{-1}(\mu)\) and assume that \(\mathfrak{g}_p = 0\), then we will show that \(TJ(p)\) is surjective. This is equivalent to proving that the anihilator of \(\text{Im}(TJ(p))\) is \(\{0\}\). Assume that \(\xi \in \mathfrak{g}\) is such that the natural pairing \(\langle TJ(p)(X), \xi \rangle = 0\) for all for all \(X \in T_pS\). That means that \(TJ(p)(X)(\xi) = 0\) or that \(\Omega(X(p), \xi_M(p)) = 0\) for all \(X \in T_pS\). Since \(\Omega\) is non-degenerate that means that \(\xi_M(p) = 0\) and so \(\xi \in \mathfrak{g}_p\), and therefore \(\xi = 0\) by hypothesis. Reversing the computation the converse easily follows.

\(\square\)

**Remark 1** In the case that concerns us, namely \((T^* Q, \omega_Q)\) with the action \(\Phi^T\), the previous theorem says that \(J^{-1}(\mu)\) is always a submanifold of \(T^* Q\).

Now we introduce \(G\)-invariant lagrangian submanifolds and the main results about them. The main results of this paper will be direct applications of these results.
Definition 2 Assume as above that the triple \((T^*Q, \omega_Q, h)\) is endowed with a hamiltonian action \(\Phi\). A \(G\)-invariant lagrangian submanifold is a lagrangian submanifold \(L\) in \(T^*Q\) such that for all \(g \in G\) we have \(\Phi^g(L) = L\).

We give a characterization of \(G\)-invariant lagrangian submanifolds in terms of equivariant momentum mappings. The next result should be considered as a generalization of the Hamilton-Jacobi theory, an explanation for this claim will be given in Remark 2. More detailed results in this direction are given in [15].

Lemma 3 Under the previous assumptions, let \(L \subset T^*Q\) be a lagrangian submanifold of \((T^*Q, \omega_Q)\). Then \(J\) is constant along \(L\) if and only if \(L\) is \(G\)-invariant.

Proof: Let be \(\alpha_q \in L\) and \(X \in T_{\alpha_q}L\), then
\[
dJ(\xi)(X) = (i_{\xi T^*Q} \omega_Q)(\alpha_q)(X) = \omega_Q(\alpha_q)(\xi_{T^*Q}(\alpha_q), X).
\]
(1)

Now, notice that \(\xi_{T^*Q}(\alpha_q)\) is tangent vector at \(t = 0\) to the curve \(\exp(t\xi)(\alpha_q)\).

Since \(\exp(t\xi)(\alpha_q)\) is contained in the orbit of \(\alpha_q \in L\), and \(\text{Orb}(\alpha_q) \subset L\) since \(L\) is \(G\)-invariant (that is, \(G \cdot L \subset L\)), we deduce that \(\xi_{T^*Q}(\alpha_q) \in T_{\alpha_q}L\). Therefore, (1) vanishes since \(L\) is lagrangian. Finally, since \(J_\xi\) is constant along \(L\), we have \(J_\xi(\alpha_q) = c\xi\) for all \(\alpha_q \in L\) and for all \(\xi \in \mathfrak{g}\) and thus, \(J(\alpha_q) = \mu\) for all \(\alpha_q \in L\) (such that \(\mu(\xi) = c\xi\)). Reversing the computations we obtain the other implication.

\[\square\]

Remark 2 Notice that the Hamilton-Jacobi theory itself is a particular case of the theorem above, which should be considered as a generalization of that theory. Let us clarify this assertion, we have a hamiltonian system \((T^*Q, \omega_Q, h)\), with the associated hamiltonian vector field \(X_h \in \mathfrak{X}(T^*Q)\); we denote the flow of \(X_h\) by \(\Psi^h : \mathbb{R} \times T^*Q \rightarrow T^*Q\); recall that the flow is just an \(\mathbb{R}\) action on \(T^*Q\). By Liouville’s Theorem this action is hamiltonian and it is easy to see that the hamiltonian \(h\) is a momentum map for that action. If we seek a \(\mathbb{R}\)-invariant lagrangian submanifold, say \(L\), then, by Lemma 3 \(h|_L = E\), where \(E\) is a constant. Moreover, assume that \(L = \text{Im}(dS)\) where \(S : Q \rightarrow \mathbb{R}\) is a real function, then we recover the classical Hamilton-Jacobi equation
\[
H(q^i, \frac{\partial S}{\partial q^i}(q^i)) = E.
\]

The time-dependent and complete solutions cases of the Hamilton-Jacobi equation follow by an analogous construction.

Remark 3 Most of the results in [17] can be recovered from Lemma 3. Indeed, there the author claims that there is a deep connection between the symmetry
of a symplectic difference scheme and the preservation of first integrals. For instance, in [17] p. 378, the following theorem is stated

**Theorem** A symplectic difference scheme preserves a function \( f \) up to a constant

\[ f \circ D_h = f + c \]

iff the scheme is invariant under the phase flow of \( f \).

Assume a hamiltonian system \((M, \Omega, H)\). Then a symplectic scheme (following [17], p. 377), after fixing Darboux coordinates on \( M \), is a rule which assigns to every hamiltonian function a symplectic map depending smoothly on a parameter \( \tau \), called the time step. A symplectic difference scheme is denoted in [17] by \( D^\tau_h \). This means that a symplectic difference scheme is just a lagrangian submanifold

\[ (D^\tau_h) = L \subset M \times M, \]

with the symplectic structure \( \Omega = \pi_1^*\Omega - \pi_2^*\Omega \) on \( M \times M \), where \( \pi_i : M \times M \to M \) are the corresponding projections over the \( i \)-factor. Consider now \( \tilde{f} = \pi_1^*f - \pi_2^*f \), it is obvious that \( f \) is preserved by \( D^\tau_h \) iff \( \tilde{f} \) is constant along graph \((D^\tau_h) = L \). By definition \( X_f = (X_f, X_f) \) and a straightforward application of Lemma 3, taking into account that \( \tilde{f} \) is the moment of the action give by the flow of \( X_f \), gives that \( \tilde{f} \) is constant along \( L \) iff \( L \) is invariant under the flow of \( X_f \). Recalling that the first statement is equivalent to \( f \) being preserved by the symplectic scheme and the second claim is equivalent to saying that the scheme is invariant under the phase flow we recover the main “principle” of [17].

**Remark 4** Assume that we are in the hypothesis of the above lemma. If \( J(L) = \{ \mu \}, \mu \in g^* \), then we deduce that \( \mu \) is a fixed point for the Coadjoint action \( Ad^* : G \to Aut(g^*) \). Indeed, remember that \( J \) is \( G \)-equivariant, that is, the following diagram is commutative

\[ \begin{array}{ccc}
T^*Q & \xrightarrow{J} & g^* \\
\Phi_g^{T^*} \downarrow & & \downarrow \text{Ad}_{g^{-1}}^* \\
T^*Q & \xrightarrow{J} & g^*
\end{array} \]

Then \( Ad_{g^{-1}}^*(\mu) = Ad_{g^{-1}}^*J(\alpha_q) = J(\Phi_g^{T^*}(\alpha_q)) = \mu \), for all \( g \in G \).

**Lemma 4** If \( \mu \) is such that \( G_\mu = G \), then \( J^{-1}(\mu) \) is a coisotropic submanifold \((G_\mu \text{ denoted the isotropy group with respect to the Coadjoint action})\).

**Proof:** Given any point \( \alpha_q \in J^{-1}(\mu) \), we have
\[ T_{\alpha_q}J^{-1}(\mu) = \ker(TJ(\alpha_q)) = \{ X \in T_{\alpha_q}(T^*Q) \text{ such that } TJ(\alpha_q)(X) = 0 \} = \{ X \in T_{\alpha_q}(T^*Q) \text{ such that } TJ(\alpha_q)(X)(\xi) = 0 \text{ for all } \xi \in \mathfrak{g} \} = \{ X \in T_{\alpha_q}(T^*Q) \text{ such that } \omega(\alpha_q)(X, \xi_{T^*Q}) = 0 \text{ for all } \xi \in \mathfrak{g} \} = (TORb(\alpha_q))^{\perp} \]

since \( \xi_{T^*Q}(\alpha_q) \) generates the orbit through \( \alpha_q \). Therefore, we have \( (T_{\alpha_q}J^{-1}(\mu))^{\perp} = T_{\alpha_q}ORb(\alpha_q) \) for all \( \alpha_q \in J^{-1}(\mu) \). But \( J \) is \( G \)-equivariant and \( G = G_\mu \), thus \( J(\Phi_{g}^T(\alpha_q)) = Ad_{g}^*J(\alpha_q) = Ad_{g}^*\mu = \mu \) and so \( \Phi_{g}^T(\alpha_q) \in J^{-1}(\mu) \). Then, \( ORb(\alpha_q) \subset J^{-1}(\mu) \), and thus \( T_{\alpha_q}ORb(\alpha_q) \subset T_{\alpha_q}J^{-1}(\mu) \). Consequently, we have \( (T_{\alpha_q}J^{-1}(\mu))^{\perp} = T_{\alpha_q}ORb(\alpha_q) \subset T_{\alpha_q}(J^{-1}(\mu)) \) and we conclude that \( J^{-1}(\mu) \) is coisotropic.

The next result is a well-known theorem in symplectic geometry, see [23, 33].

It will allow us to carry out our reduction procedure in a straightforward way.

**Theorem 5 (Coisotropic Reduction)** Let \((M, \omega)\) be a symplectic manifold, \( C \subset M \) a coisotropic submanifold and \( C/\sim \) the quotient space of \( C \) by the characteristic distribution \( D = \ker(\omega|_C) \); we shall denote by \( \pi : C \to C/\sim \) the canonical projection and by \( \omega_C \) the natural projection of \( \omega \) to \( C/\sim \) (notice that \((C/\sim, \omega_C)\) is again a symplectic manifold, assuming that it is again a manifold). Assume that \( L \subset M \) is a lagrangian submanifold such that \( L \cap C \) has clean intersection, then \( \pi(L \cap C) \) is a lagrangian submanifold of \((C/\sim, \omega_C)\).

The following diagram illustrates the above situation

\[
\begin{array}{ccc}
L \cap C & \xrightarrow{i_{L \cap C}} & C \\
\downarrow \pi & & \downarrow \pi \\
\pi(L \cap C) & \xrightarrow{i_{\pi(L \cap C)}} & C/\sim
\end{array}
\]

We can apply this theorem to the situation described before. Indeed, given \( \mu \in \mathfrak{g}^* \) such that it is a fixed point of the coadjoint action (i.e. \( Ad^*_g(\mu) = \{ \mu \} \) for all \( g \in G \)), then we have the following diagram, since by Lemma 4 we know
that $J^{-1}(\mu)$ is coisotropic:

$$J^{-1}(\mu) \xrightarrow{i_{J^{-1}(\mu)}} T^*Q$$

$$\pi' \downarrow \downarrow T^*Q$$

$$J^{-1}(\mu)/\ker(\omega_{Q/J^{-1}(\mu)})$$

But $\ker(\omega_{Q/J^{-1}(\mu)})(\alpha_q) = (T_{\alpha_q}J^{-1}(\mu))^\perp = T_{\alpha_q}\text{Orb}(\alpha_q)$ for all $\alpha_q \in J^{-1}(\mu)$, and since $G = G_\mu$, we can see that $J^{-1}(\mu)/\ker(\omega_{Q/J^{-1}(\mu)}) = J^{-1}(\mu)/G$. But this is just the symplectic reduction of $T^*Q$ according to the Marsden-Weinstein reduction theorem, see [24].

3 The Hamilton-Jacobi Equation

3.1 Generalized Solutions

Along this section $h : T^*Q \to \mathbb{R}$ will be a hamiltonian function. We are going to use the previous results to carry out our reduction of the Hamilton-Jacobi equation. By Hamilton-Jacobi equation we mean

1. The time-independent Hamilton-Jacobi equation:

$$h(q^i, \frac{\partial S}{\partial q^i}(q^i)) = E.$$ 

2. The time-dependent Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + h(t, q^i, \frac{\partial S}{\partial q^i}(t, q^i)) = E.$$ 

3. A complete solution of the Hamilton-Jacobi equation: that is, a real-valued function $S(t, q^i, \alpha^i)$ depending on as many parameters ($\alpha^i$) as the dimension of the configuration manifold, such that

(a) For every (fixed) value of the parameters ($\alpha^i$), $S(t, q^i, \alpha^i)$ satisfies the time-dependent Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial t}(t, q^i, \alpha^i) + h(t, q^i, \frac{\partial S}{\partial q^i}(t, q^i, \alpha^i)) = 0.$$ 

(b) The non-degeneracy condition: consider the matrix whose components $(i, j)$ are given by $\frac{\partial^2 S}{\partial q^i \partial \alpha^j}$, that we denote by $(\frac{\partial^2 S}{\partial q^i \partial \alpha^j})$, then

$$\det(\frac{\partial^2 S}{\partial q^i \partial \alpha^j}) \neq 0.$$
We define below the concept of generalized solution, which is a generalization of a solution of the time-independent Hamilton-Jacobi equation (see [5]), and we develop our theory for this case. Analogous procedures hold for the time-independent Hamilton-Jacobi equation and for the complete solutions cases, as both settings can be (almost) considered as particular cases of the time-independent Hamilton-Jacobi theory. Along the examples section, sections 5.2.1 and 5.3, we will make this claim explicit.

**Definition 6** We say that a submanifold $L \subset T^*Q$ is a solution of the (time-independent) Hamilton-Jacobi problem for $h$, if:

- $L$ is a lagrangian submanifold of $T^*Q$.
- $h$ is constant along $L$.

A solution $L$ of the Hamilton-Jacobi equation for $h$ is horizontal if $L = \text{Im}(\gamma)$, being $\gamma$ a 1-form on $Q$.

**Remark 5** Let us describe with more detail the case of horizontal solutions, that is, when $L = \text{Im}(\gamma)$, $\gamma$ a 1-form on $Q$. Recall that $\text{Im}(\gamma)$ is lagrangian if and only if $\gamma$ is closed, so locally

$$\gamma = dS.$$ 

Therefore, the condition $h_{\text{Im}(\gamma)} = \text{cte}$, can be equivalently written as

$$h \circ \gamma = \text{cte}$$

or

$$h(q, \frac{\partial S}{\partial q}) = \text{cte}$$

which is the usual form of the Hamilton-Jacobi equation. This fact justifies the definition above.

**Remark 6** Notice that the fact that a horizontal lagrangian submanifold $L$ is $G$-invariant does not imply that its generating function is invariant too. In fact, its generating function will be invariant iff $J(L) = 0$. Since $J \circ dS = \mu$, then $dS(q)(\xi_q(q)) = J \circ dS(q)(\xi_{T^*Q}(q)) = \mu(\xi)$, which only vanish for all $\xi \in g$ if $\mu = 0$. Here the advantages of dealing with lagrangian submanifolds instead of functions are already manifest, as there are $G$-invariant lagrangian manifolds whose generating function is not $G$-invariant, see Section 5. Notice that invariance of the generating function has been assumed in [17, 18].

### 3.2 Invariant $G$-solutions

We assume now that a Lie group $G$ acts on $Q$ such that the action is free and proper. Given $\mu \in g$, then $J^{-1}(\mu)$ is a submanifold of $T^*Q$. We can summarize
the situation in the following diagram:

\[
\begin{array}{c}
J^{-1}(\mu) \quad \xrightarrow{\pi} \quad T^*Q \quad \xrightarrow{\pi} \quad \mathfrak{g}^* \\
\pi_Q \quad \xrightarrow{\pi} \quad Q \quad \xrightarrow{\pi_G} \quad T^*Q/G \\
\xrightarrow{p} \quad Q/G,
\end{array}
\]

where \(\pi_Q, \pi, \pi_G\) and \(p\) are the canonical projections. We will use \(\pi'\) for the projection

\[
\pi' = \pi_{|J^{-1}(\mu)} : J^{-1}(\mu) \to T^*Q/G.
\]

As we know, \(T^*Q/G\) has a Poisson structure induced by the canonical symplectic structure on \(T^*Q\), such that.

\[
\pi : T^*Q \to T^*Q/G
\]

is a Poisson morphism (see [25] for the details). The next proposition shows the symplectic structure of the leaves of the characteristic distribution of the Poisson structure of \(T^*Q/G\).

**Proposition 7** (Marsden et al. [23, 25, 31]) The symplectic leaves of \(T^*Q/G\) are just the quotient spaces \((J^{-1}(\text{Orb}^{Ad}(\mu))/G)\).

### 3.3 Reduction and Reconstruction

Assume now that \(\mu\) is a fixed point for the Coadjoint action, i.e. \(\text{Orb}^{Ad}(\mu) = \{\mu\}\). Then \(J^{-1}(\mu)/G\) is a symplectic leaf of \(T^*Q/G\). Assume now that \(L \subset J^{-1}(\mu)\) is a lagrangian submanifold; since \(J^{-1}(\mu)\) is a coisotropic submanifold of \((T^*Q, \omega_Q)\), we deduce that \(\pi(L)\) is a lagrangian submanifold of the quotient \(J^{-1}(\mu)/G\) by applying the Coisotropic Reduction Theorem. Obviously, the condition of clean intersection is trivially satisfied.

In reference [25] it is shown that \(J^{-1}(\mu)/G\) is diffeomorphic to the cotangent bundle \(T^*(Q/G)\). Moreover, considering the symplectic structure \(\omega_\mu\) on \(J^{-1}(\mu)/G\) given by the Marsden-Weinstein reduction procedure, the two manifolds are symplectomorphic, where on \(T^*(Q/G)\) we are considering the symplectic structure given by the canonical one plus a magnetic term \(\omega_{Q/G} + B_\mu\) see (Appendix B). Combining the last two paragraphs we can see \(\pi(L)\) as a lagrangian submanifold of a cotangent bundle with a modified symplectic structure. We
Therefore, we have
\[ T^*Q/G = T^*(Q/G) \times_{Q/G} \tilde{g}^* \] (2)
(see [25] for a detailed discussion of this splitting) where \( \tilde{g}^* \) denotes the adjoint bundle to \( \pi_Q : Q \to Q/G \) via the Coadjoint representation, \( \tilde{g}^* = Q \times G \tilde{g}^* \) (see [25] and Appendix A for a description of this bundle). The identification (2) is given by

\[ \Psi : T^*Q/G \to T^*(Q/G) \times_{Q/G} \tilde{g}^* \]

\[ [\alpha_q] \to [\Psi(\alpha_q)] = [(\alpha_q \circ h, J(\alpha_q))], \]

where \( h \) represents the horizontal lift \( T_{\pi_G(q)}(Q/G) \to T_qQ \) of the connection \( A \). Therefore, we have

\[ T_{\pi_G(q)}(Q/G) \xrightarrow{h} T_qQ \]

\[ \xrightarrow{\alpha_q \circ h} \quad \xrightarrow{\alpha_q} \quad \xrightarrow{\mathbb{R}.} \]

If \( \alpha_q \in J^{-1}(\mu) \) then \( J(\alpha_q) = \mu \), and \( \Psi([\alpha_q]) = (\alpha_q \circ h, J(\alpha_q) = \mu) \), so that \( J^{-1}(\mu)/G \) can be identified with \( T^*(Q/G) \).

\[ \Psi(J^{-1}(\mu)/G) = T^*(Q/G) \times_{Q/G} (Q \times \{\mu\}/G) \equiv T^*(Q/G) \]

**Remark 7** Notice that \( \dim(Q) = n \), and then \( \dim(J^{-1}(\mu)) = 2m - k \) where \( \dim(G) = k \). Thus, \( \dim(J^{-1}(\mu)/G) = 2n - k - k = 2(n - k) \) and \( \dim(T^*(Q/G)) = 2(n - k) \).

Notice that \( J^{-1}(\mu)/G \) and \( T^*(Q/G) \) are not only diffeomorphic, moreover, it is possible to show that they are symplectomorphic, while \( J^{-1}(\mu)/G \) is considered as a symplectic leaf of \( T^*Q/G \) and \( T^*(Q/G) \) is equipped with the canonical symplectic structure modified by a magnetic term (it is explained in the cited paper, [25], and the magnetic term \( \beta_\mu \) comes from the connection \( A, \omega_{Q/G} + \beta_\mu \)). If \( \mu = 0 \), then the magnetic term vanishes and we have the canonical symplectic structure \( \omega_{Q/G} \).

Next, we consider a \( G \)-invariant hamiltonian \( h \) on \( T^*Q \). Then we have

\[ T^*Q \xrightarrow{h} \mathbb{R} \]

\[ T^*Q/G \]

\[ \xrightarrow{\pi} \quad \xrightarrow{k_G} \]
where \( h_G \circ \pi = h \) is the natural projection of \( h \). Consider the mapping \( \Psi \) defined above

\[
\begin{array}{c}
T^*Q \\
\pi \\
\downarrow h_G \\
T^*Q/G \\
\Psi^{-1} \\
\downarrow \Psi \\
T^*(Q/G) \times_{Q/G} \mathfrak{g}^* \\
\end{array}
\]

and define \( \tilde{h} : T^*(Q/G) \to \mathbb{R} \) by \( \tilde{h}_\mu(\tilde{\alpha}_\tilde{q}) = \tilde{h}(\tilde{\alpha}_\tilde{q}, [q, \mu]) \), where \( \tilde{\alpha}_\tilde{q} \in T^*_\tilde{q}(Q/G) \), \( \tilde{q} = [q] \in Q/G, \mu \in \mathfrak{g}^* \). Assume that \( \tilde{L} \) is \( G \)-invariant solution of the Hamilton-Jacobi equation for \( \tilde{h} \) and define

\[
\tilde{L} = \Psi(\pi(L)) \in T^*(Q/G) \times_{Q/G} \mathfrak{g}^*.
\]

As we have proved before \( \tilde{L} \subset T^*(Q/G) \) is a lagrangian submanifold with respect to \( \omega_{Q/G} + \beta_\mu \). Using the previous results we can prove that a \( G \)-invariant solution for the Hamilton-Jacobi problem for \( \tilde{h} \) projects onto a solution of the Hamilton-Jacobi for \( h_\mu \). In addition, if \( L \) is horizontal then \( \tilde{L} \) is horizontal.

**Proposition 8 (Reduction)** Given \( L \) a \( G \)-invariant solution of the Hamilton-Jacobi equation, then \( \tilde{L} \) is a solution of the Hamilton-Jacobi equation for \( h_\mu \) \((\mu = J(L))\). Moreover, if \( L \) is horizontal, then \( \tilde{L} \) is horizontal.

**Proof:** Recall that since \( L \) is \( G \)-invariant then \( J(L) = \mu \). As we have seen before, \( \tilde{L} = \pi(L) \) is a lagrangian submanifold of \( J^{-1}(\mu)/G \). Now, we take \( \mu \in \mathfrak{g}^* \) and since it is a regular value of \( J \), then \( J^{-1}(\mu) \) is a submanifold of \( T^*Q \). Since in our case, \( L \) is \( G \)-invariant lagrangian submanifold then \( J \) is constant along \( L \), say \( J(L) = \mu \). Recall that \( \mu \in \mathfrak{g}^* \) is a fixed point for \( Ad^* \) if and only if \( G_\mu = G \), and in this case \( J^{-1}(\mu) \) is coisotropic. Therefore, we have that \( \mu \) is such that \( G_\mu = G \). This happens for instance if \( G \) is abelian. \( L = \pi(L) \) is a lagrangian submanifold of \( J^{-1}(\mu)/G \), but this is a symplectic leaf with symplectic structure \( \omega_{Q/G} + \beta_\mu \), when we are using the natural identification via \( \Psi \) and considering a fixed connection \( A \) in \( Q \to Q/G \) to obtain the corresponding decomposition. In addition, if \( \tilde{\alpha}_\tilde{q} \in \tilde{L} \), then \( \tilde{h}_\mu(\tilde{\alpha}_\tilde{q}) = h(\Psi^{-1}(\tilde{\alpha}_\tilde{q}), \mu) \). Therefore, \( \tilde{h}_\mu \) is constant along \( \tilde{L} \). Assume now that \( L \) is horizontal, so \( L = \text{Im}(\gamma) \), for a 1-form \( \gamma \) on \( Q \) such that \( d\gamma = 0 \). Since \( \gamma \) takes values into \( J^{-1}(\mu) \) and is \( G \)-invariant, then \( \gamma \) induces a mapping

\[
Q \xrightarrow{J^{-1}(\mu) \subset T^*Q/G} \Psi \xrightarrow{\gamma} T^*(Q/G)
\]

which is \( G \)-invariant. So it induces a new mapping \( \tilde{\gamma}_\mu : Q/G \to T^*(Q/G) \) such that \( \text{Im}(\tilde{\gamma}_\mu) = \tilde{L} \).
We also prove a reconstruction theorem. With this theorem at hand, once a reduced solution is found it can be lifted to find a solution of the original unreduced problem.

**Proposition 9 (Reconstruction)** Assume that \( \tilde{L} \) is a lagrangian submanifold of \( (T^*(Q/G), \omega_{Q/G} + \beta_\mu) \) for some \( \mu \in \mathfrak{g}^* \) which is a fixed point of the Coadjoint action. Assume that \( h_\mu \) is the reduced hamiltonian defined as above and that \( \tilde{L} \) is a Hamilton-Jacobi solution for \( h_\mu \). Using the diffeomorphism

\[
T^*Q/G \xrightarrow{\Psi^{-1}} T^*(Q/G) \times_{Q/G} \mathfrak{g}^*
\]

we define \( \tilde{L} \) by

\[
\tilde{L} = \{ (\tilde{\alpha}_q, [\mu]_\eta) \in T^*(Q/G) \times_{Q/G} \mathfrak{g}^* \text{ such that } \tilde{\alpha}_q \in \tilde{L} \}
\]

and take

\[
L = \pi^{-1}(\tilde{L}).
\]

Then

1. \( L \) is \( G \)-invariant and lagrangian with respect to the canonical symplectic structure of the cotangent bundle, \( \omega_Q \), and a solution for the Hamilton-Jacobi problem given by \( h \).

2. If \( \tilde{h} \) is horizontal, then \( L \) is horizontal too.

**Proof:** Since \( \tilde{L} \) is a Lagrangian submanifold of \( (T^*(Q/G), \omega_{Q/G} + \beta_\mu) \) and \( \Psi|_{J^{-1}(\mu)/G} \) is a symplectomorphism, then \( \overline{L} = \Psi^{-1}(\tilde{L}) \) is a lagrangian submanifold of the symplectic leaf \( J^{-1}(\mu)/G \). Since \( \pi: T^*Q \to T^*Q/G \) is a submersion then \( \pi^{-1}(\overline{L}) \) is an immersed submanifold of dimension \( \dim(\pi^{-1}(\overline{L})) = \dim(L) + \dim(G) \), and since

\[
\dim(L) = \dim(\tilde{L}) = 1/2 \cdot \dim(T^*(Q/G))
\]

\[
= 1/2 \cdot 2 \cdot (\dim(Q) - \dim(G)) = \dim(Q) - \dim(G),
\]

then \( \dim(\pi^{-1}(\overline{L})) = (\dim(Q) - \dim(G)) + \dim(G) = \dim(Q) \) which is half the dimension of \( T^*Q \); so we only have to show that \( \pi^{-1}(\overline{L}) \) is an isotropic submanifold. Notice that since \( J^{-1}(\mu)/G_\mu = J^{-1}(\mu)/G \), then the symplectic structure on \( J^{-1}(\mu)/G \) (denoted by \( \omega_\mu \)) is the one obtained by the Marsden-Weinstein reduction theorem, which is characterized by the equation \( i_\mu^*\omega_Q = \pi^*\omega_\mu \) where \( i_\mu: J^{-1}(\mu) \to T^*Q \) is the inclusion and \( \omega_Q \) the canonical symplectic structure on \( T^*Q \). Since \( \pi^{-1}(\overline{L}) \subset J^{-1}(\mu) \), it is easy to see that

\[
(\omega_Q)|_{\pi^{-1}(\overline{L})} = (\pi^*\omega_\mu)|_{\pi^{-1}(\overline{L})} = 0,
\]

\[\square\]
and we can conclude that \( \pi^{-1}(L) \) is a lagrangian submanifold. The fact that 
\[ h|_{\pi^{-1}(T)} = E, \]
where \( E \) is a constant, follows from the identity
\[ h|_{\pi^{-1}(L)} = (\hat{h}_\mu)|_{\tilde{L}} \]
and thus the result holds. \( \square \)

**Remark 8** It is clear that, by Propositions 8 and 9, we have a bijection between
\[ G \]-invariant solutions of Hamilton-Jacobi problem for \( h \) and solutions of the
Hamilton-Jacobi equation for \( \tilde{h}_\mu \) where \( \mu \) is a fixed point of the Coadjoint action.

\[
\{ \text{G-invariant solutions of HJ} \} \xrightarrow{\text{one to one}} \{ \text{reduced solutions of HJ} \}
\]

**Remark 9** In the symplectic manifold \((T^*Q, \omega_Q)\), given a 1-form \( \gamma \) on \( Q \) its
image is an horizontal lagrangian submanifold if and only if \( d\gamma = 0 \). In that case
that lagrangian submanifold is locally given by a generating function \( L = \text{Im}(dS) \).
Given the symplectic manifold \((T^*(Q/G), \omega_{Q/G} + \beta_\mu)\) it is natural to ask which
is the analogous condition to \( d\gamma = 0 \). In [25] one can check that \( B_\mu \) is actually
the pullback of a 2-form on the base \( Q/G \) so \( B_\mu = \pi_{Q/G}^* \beta_\mu \). So given a 1-form
on \( Q/G \), say \( \gamma \), its image is lagrangian for the modified structure if and only if
0 = \( \gamma^*(\omega_{Q/G} + \pi_{Q/G}^* \beta_\mu) = d\gamma + \beta_\mu \) or equivalently \( d\gamma = -\beta_\mu \). In that case, it is no
possible in general to find a generating function, and instead one PDE, we have
a system of algebraic-PDE equations.

## 4 Reduction of H-J equation and reduction of
dynamics

Assume that we have a hamiltonian system \((T^*Q, \omega_Q, h)\) and let \( \gamma \) be a 1-
form which is a solution of the Hamilton-Jacobi equation for \( h \). Then we can
construct the projected vector field \( X^\gamma_h \) by
\[
X^\gamma_h = T\pi_Q \circ X_h \circ \gamma.
\]

A basic result in the Hamilton-Jacobi theory (see [1]) is that \( X^\gamma_h \) and \( X_h \) are
\( \gamma \)-related. If we assume that we are in the conditions of the previous sections,
that is, we have a free and proper action \( \Phi : G \times Q \to Q \) and all the constructions
previously introduced follow, we get a new (reduced) hamiltonian system
\((T^*(Q/G), \omega_{Q/G} + B_\mu, \hat{h}_\mu)\) and a solution \( \hat{\gamma} \) of the corresponding Hamilton-
Jacobi theory. As before, we can define the projected vector field for the reduced
system
\[
X^\hat{\gamma}_{\hat{h}_\mu} = T\pi_Q \circ X_{\hat{h}_\mu} \circ \hat{\gamma}.
\]
and therefore the current situation is the one described in the diagram below

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{J^{-1}(\mu)} & J^{-1}(\mu)/G \\
\downarrow & \downarrow & \downarrow \\
Q & \xrightarrow{\pi_Q} & T^*(Q/G) \\
\downarrow & & \\
Q/G & \xleftarrow{\pi_{Q/G}} & \end{array}
\]

We point out the vector fields and the manifolds on which they are defined:

\[
\begin{array}{ccc}
T^*Q, X_h & \xrightarrow{\gamma} & T^*(Q/G), X_{\tilde{h}_\mu} \\
\downarrow & & \downarrow \\
Q, X_\gamma & \xrightarrow{\pi_Q} & Q/G, X_{\tilde{\gamma}_\mu} \\
\end{array}
\]

The relation between the dynamics on \(T^*Q\) and \(J^{-1}(\mu)/G\) (recall that we are identifying this space with \(T^*(Q/G)\)) is well-known. There are reconstruction procedures to integrate the vector field \(X_h\) after integrating the vector field \(X_{\tilde{h}_\mu}\). So we have

\[
\begin{array}{ccc}
T^*Q, X_h & \xrightarrow{projection} & T^*(Q/G), X_{\tilde{h}_\mu} \\
\downarrow & & \downarrow \\
Q, X_\gamma & \xrightarrow{reconstruction} & Q/G, X_{\tilde{\gamma}_\mu} \\
\end{array}
\]

Figure 1: Relations between vector fields

Moreover, since \(\tilde{\gamma} \circ \pi_G = \pi \circ \gamma\) we can conclude that the vector field \(X_\gamma\) projects onto \(X_{\tilde{\gamma}_\mu}\) via \(\pi_G\); and so, \(X_\gamma\) is \(G\)-invariant.

We recall now the basic reconstruction procedure to integrate the vector field \(X_h\) via the integration of \(X_{\tilde{h}_\mu}\) in order to complete the diagram in Figure 1. Let \(c : (a, b) \subset \mathbb{R} \to Q/G\) be an integral curve of \(X_{\tilde{h}_\mu}\) and consider a curve \(d(t) : (a, b) \to Q\) such that \(\pi_{Q/G} \circ d = c\); for instance, since we made the previous constructions using a connection on the principal bundle \(\pi_G : Q \to Q/G\) then \(d\) can be taken as the horizontal lift of \(c\). Next, consider the connection 1-form, that we will denote also by \(A : TQ \to \mathfrak{g}\), and assume that we have a curve \(g : (a, b) \to G\) such that \(\frac{d}{dt}g(t) = A(X_h(d(t)) - \frac{d}{dt}d(t))\) where we are using the identification \(TG = G \times \mathfrak{g}\) given by the left trivialization. It is easy to check that then \(g(t) \cdot d(t)\) is an integral curve of \(X_\gamma\)?
5 Examples

It is our believe that the theory above described has wide applicability in concrete situations. Here we present some examples but we would like to stress that much more involved settings fall in our setting.

5.1 Lie groups

Let \( G \) be a Lie group and \( T^*G \) its cotangent bundle. Using left trivialization we have the identification \( T^*G \cong G \times \mathfrak{g}^* \). Since \( T^*G/G \cong G \times \mathfrak{g}^*/G \cong \mathfrak{g}^* \) then, to find a \( G \)-invariant solution \( L \) of the Hamilton-Jacobi problem is equivalent to finding an element \( \mu \in \mathfrak{g}^* \) such that \( \text{Ad}_{g}^* \mu(g) = \mu(g) \) for all \( g \in G \). Given such \( \mu \) we can construct \( L \subset G \times \mathfrak{g}^* \) given by \( L = G \times \{ \mu \} \). It is easy to see that a 1-form defined in this way is closed, \( G \)-invariant and satisfies \( H/\text{div}G \times \{ \mu \} = \tilde{H}(\mu) \). Therefore we obtain a characterization of the closed \( G \)-invariant 1-forms on a Lie group.

5.2 The trivial case: \( Q = M \times G \)

Assume now that we have \( Q = M \times G \) and we are considering the action

\[
\Phi : G \times (M \times G) \rightarrow (M \times G)
\]

\[
(g, (m, h)) \rightarrow (m, g \cdot h).
\]

If we trivialize \( T^*G = G \times \mathfrak{g}^* \) via the left action, then the lifted action, \( \Phi^{T^*} \) is given by

\[
\Phi^{T^*} : G \times (T^*M \times G \times \mathfrak{g}^*) \rightarrow (T^*M \times G \times \mathfrak{g}^*)
\]

\[
(g, (\alpha_m, h, \mu)) \rightarrow (\alpha_m, g \cdot h, \mu).
\]

The momentum map is given by \( J(\alpha_m, g, \mu) = \text{Ad}_{g}^{-1} \mu \). If we have the hamiltonian system \( (T^*(M \times G), H, \Omega_{M \times G}) \) (with \( H \) assumed \( G \)-invariant), given \( \mu \) such that \( G_{\mu} = G \) then \( J^{-1}(\mu)/G \cong T^*M \) and \( \tilde{H}_{\mu}(\alpha_m) = H(\alpha, g, \mu) \) where by the \( G \)-invariance of \( H \) the element \( g \) is arbitrary. In this case, the reduced system is equivalent to the hamiltonian system given by \( (T^*M, \tilde{H}_{\mu}, \Omega_M) \). Assume that \( S_M : T^*M \rightarrow \mathbb{R} \) is the generating function of \( \tilde{L} \), a horizontal lagrangian submanifold which solves the Hamilton-Jacobi problem. On the other hand it is easy to see that \( \mu \) viewed as a section of the projection onto \( G \), \( G \times \mathfrak{g}^* \rightarrow G \), is a closed 1-form and so there exists \( S_G : G \rightarrow \mathbb{R} \) such that \( \text{Im}(dS_G) = (g, \mu) \).

Let us denote by \( S_{M \times G} \) the generating function of the corresponding lagrangian submanifold \( \tilde{L} \) obtained by reconstruction from \( \tilde{L} \), then we have.

**Lemma 10** The generating functions are related by

\[
S_{M \times G} = S_M + S_G + c
\]

where \( c \) is a constant on each connected component.
Proof: Given $\xi \in \mathfrak{g}$, since $\text{Im}(dS_{M \times G}) \subset J^{-1}(\mu)$ then $dS(\xi_{M \times G}) = \mu(\xi) = d(S_M + S_G)(\xi_{M \times G}) = dS_G(\xi_{M \times G})$. Given $X \in T_mM$ the analogous computations holds and the result follows.

5.2.1 Time-dependent H-J solution for time-independent systems

An immediate application of the previous result is the obtainment of the classical relation between time-dependent and time independent solutions of the Hamilton-Jacobi equation. This is a very classical ansatz that follows from our results.

Let be $H : T^*Q \rightarrow \mathbb{R}$ and consider the corresponding hamiltonian $H^R = H \circ p_{T^*Q} + \epsilon : T^*(\mathbb{R} \times Q) \rightarrow \mathbb{R}$, where $p_{T^*Q} : T^*(\mathbb{R} \times Q) \rightarrow T^*Q$ is the projection onto $T^*Q$ and $\epsilon$ denotes the time conjugate momentum. We can introduce the action given by translation in time

$$\Phi : \mathbb{R} \times (\mathbb{R} \times Q) \rightarrow (\mathbb{R} \times Q)$$

$$(r, (t, q)) \rightarrow (t + r, q).$$

The corresponding lifted action is

$$\Phi^{T^*} : \mathbb{R} \times T^*(\mathbb{R} \times Q) \rightarrow T^*(\mathbb{R} \times Q)$$

$$(r, (t, e, \alpha_q)) \rightarrow (t + r, e, \alpha_q).$$

The momentum map is just

$$J(t, e, \alpha_q) = e.$$

If $E \in \mathbb{R} \cong \mathbb{R}^n$ then $\mathbb{R}E = \mathbb{R}$ since the group is abelian and $J^{-1}(E) \cong \mathbb{R} \times T^*Q$ and $\mathbb{R} \times T^*Q/\mathbb{R} \cong T^*Q$. Summarizing, we have that $(J^{-1}(E)/G, \overline{H}^R_E, \overline{\Omega})$ is given by $(T^*Q, H, \Omega_Q)$ and, if we denote by $S$ the generating function of $L$ and by $W$ the generating function of $\overline{L}$, then we obtain $S_R = t \cdot E$ and $S_Q = W$ and we recover

$$S = t \cdot E + W$$

5.3 Complete Solutions

This subsection is devoted to applying the previous results to what is usually called a complete solution of the Hamilton-Jacobi equation. The knowledge of a complete solution of the Hamilton-Jacobi equation is equivalent to integrating the Hamilton’s equations of motion (see [3]). Before getting into our results, we sketch in this subsection the classical results. They are local and written in a coordinate dependent way, but the global, geometric aspects of the theory are easier to understand after taking a look at the classical theory. We restrict ourselves to the time-independent case but the results can be easily extended to the time-dependent setting.

Let $h(q^i, p_i)$ be a hamiltonian on the phase space $(q^i, p_i), i = 1, \ldots, n$. By a complete solution of the Hamilton-Jacobi equation for $h$ we mean the following.
Definition 11 A complete solution of the Hamilton-Jacobi equation for the hamiltonian \( h(q^i, p_i, i = 1, \ldots, n) \) is a real-valued function \( S(t, q^i, \alpha^i), i = 1, \ldots, n \), such that

1. For every (fixed) value of the parameters \( (\alpha^i) \), \( S(t, q^i, \alpha^i) \) satisfies the Hamilton-Jacobi equation,
\[
\frac{\partial S}{\partial t} + h(t, q^i, \frac{\partial S}{\partial q^i}(t, q^i)) = 0.
\]

2. The non-degeneracy condition: consider the matrix with component \( i, j \) given by \( \frac{\partial^2 S}{\partial q^i \partial \alpha^j} \), that we denote by \( (\frac{\partial^2 S}{\partial q^i \partial \alpha^j}) \), then
\[
\det \left( \frac{\partial^2 S}{\partial q^i \partial \alpha^j} \right) \neq 0.
\]

Then, we can define (at least locally by the implicit function theorem) the following implicit, time-dependent transformation, from the \( (t, q^i, p_i) \)-space to the \( (t, \alpha^i, \beta^i) \)-space:
\[
\frac{\partial S}{\partial q^i}(t, q^i, \alpha^i) = p_i, \quad \frac{\partial S}{\partial \alpha^i}(t, q^i, \alpha^i) = \beta_i. \tag{3}
\]

A computation shows that this transformation sends the system to equilibrium, i.e., Hamilton’s equations become now
\[
\frac{d\alpha^i}{dt}(t) = 0, \quad \frac{d\beta^i}{dt}(t) = 0, \tag{4}
\]
see [1, 3].

We give now a geometric interpretation of the previous procedure. The function \( S \) can be interpreted as a function on the product manifold \( \mathbb{R} \times Q \times Q \) and so \( \text{Im} \ (dS) \) is a lagrangian submanifold in \( T^*(\mathbb{R} \times Q \times Q) \) (notice that we are thinking about the \( (q^i) \) as coordinates on the first \( Q \), and \( (\alpha^i) \) as coordinates on the second factor \( Q \)). On the other hand, consider the projections \( \pi_I : T^* (\mathbb{R} \times Q \times Q) \to \mathbb{R} \times T^* Q, I = 1, 2 \), defined by \( \pi_I(t, e, \alpha^1, \alpha^2) = (t, (-1)^{I+1} \alpha^I) \). With these geometric tools, the non-degeneracy condition is equivalent to saying that \( \pi_I|\text{Im} \ (dS) \) is a local diffeomorphism for \( I = 1, 2 \). We assume here for simplicity that it is a global diffeomorphism, so we can consider the mapping \( \pi_2|\text{Im} \ (dS) \circ (\pi_1|\text{Im} \ (dS))^{-1} : T^*(\mathbb{R} \times Q) \to T^*(\mathbb{R} \times Q) \). This mapping can be easily checked to be the global description of the change of variables introduced above. The Hamilton-Jacobi equation, can be understood as the fact that \( dS^* h^\text{ext} = 0 \), where \( h^\text{ext} = \pi_1^* h + e \). The diagram below helps to have a global picture of the procedure:

In the precedent setting all the information is given by the lagrangian manifold defined by \( \text{Im} \ (dS) \), so we can introduce a generalized solution to the Hamilton-Jacobi equation as follows.
Definition 12 A Lagrangian submanifold $L$ in $T^* (\mathbb{R} \times Q \times Q)$ is a complete solution of the Hamilton-Jacobi equation if

1. $L \subset (h^{ext})^{-1}(e)$.

2. The restriction of $\pi_1$ to $L$ is a diffeomorphism. From now on we will refer to this property as the non-degeneracy condition.

Remark 10 When $L$ is given by $\text{Im} \left( dS \right)$ we say that $S$ is a generating function for the transformation induced by $L$. This type of generating functions are usually called in the literature type I generating functions, see [19]. It is remarkable that our theory deals with the Lagrangian submanifolds instead of their generating functions, so our theory is applicable to other types of generating functions. This does not happen in previous approaches to reduction of the Hamilton-Jacobi theory.

Under the previous conditions we are still able to define the symplectomorphism that solves Hamilton’s equations. We can now apply our reduction procedure. Assume that we have an action $\Phi : G \times Q \to Q$, such that $\Phi^{T^*}$ leaves the Hamiltonian invariant. We consider the diagonal action

$$
\Phi_0 : \quad G \times \mathbb{R} \times Q \times Q \to \mathbb{R} \times Q \times Q
$$

$$(g, (t, q^1, q^2)) \to (t, \Phi(g, q^1), \Phi(g, q^2)).$$

It is easy to see that $\Phi_0^{T^*}$ leaves $h^{ext}$ invariant and the corresponding momentum mapping is $J_0 = J \circ \pi_1 - J \circ \pi_2$, where $J$ is the momentum mapping corresponding to the action $\Phi$. Then, we can look for $G$-invariant complete solutions. After applying our reduction method, we obtain the (reduced) cotangent manifold $T^* (\mathbb{R} \times Q \times Q)$. All the reduction theory for Hamilton-Jacobi applies in this manner to complete solutions in a straightforward way.
Remark 11 A simple computation, following the arrows in Figure 2, shows that $G$-invariant lagrangian submanifolds in $T^*(\mathbb{R} \times Q \times Q)$ which satisfy the non-degeneracy condition induce time-dependent $G$-equivariant symplectic automorphisms on $T^*Q$.

Remark 12 If our hamiltonian comes from a regular lagrangian, $L$, then there is always a (local) $G$-invariant solution which lives in $J^{-1}_0(0)$, just the one given by the action functional

$$S(t,q,\bar{q}) = \int_c L(\dot{c}) \, dt,$$

where $c : [a,b] \to Q$ is the curve satisfying the Euler-Lagrange equations and verifying $c(a) = q$ and $c(b) = \bar{q}$. Under the previous assumptions, that curve exists for $q$ and $\bar{q}$ close enough.

Remark 13 There is a very important lagrangian submanifold, the one given by the flow. Let $\Psi^h_t$ be the flow of the hamiltonian vector field $X_h$. Then we have the lagrangian submanifold in $T^*(\mathbb{R} \times Q \times Q)$

$$L = \{(t,h(t,\alpha_q),\alpha_q,-\Psi^h_t(\alpha_q)) \text{ such that } t \in \mathbb{R}, \alpha_q \in T^*Q\}$$

At the end, the Hamilton-Jacobi theory is about finding a generating function for this lagrangian submanifold. If $G$ is a symmetry of the hamiltonian then $L$ is $G$-invariant and lives in the 0-level set, as a consequence of the conservation of the momentum mapping. This lagrangian submanifold can locally be obtained by type II generating function (see next section).

Remark 14 Observe that $T^*(\mathbb{R} \times Q \times Q)$ has a well-known geometric structure; indeed, it is the cotangent bundle of the gauge groupoid $\mathbb{R} \times Q \times Q$. It suggest that the geometric structure behind all this theory is the symplectic groupoid structure. Moreover, following this pattern we were able to develop a Hamilton-Jacobi theory for certain Poisson manifolds that will appear in a forthcoming paper [14].

Remark 15 The reduced lagrangian submanifold $\hat{L} \subset T^*(\mathbb{R} \times \frac{Q \times Q}{G})$ induces a (Poisson) transformation $\mathbb{R} \times T^*Q/G \to \mathbb{R} \times T^*Q/G$, using the source and the target of the groupoid structure, in the same way we have used the projections $\pi_I$ above. The Poisson structure considered on $\mathbb{R} \times T^*Q/G$ is the product of the 0 Poisson structure on $\mathbb{R}$ and the natural Poisson structure induced on $T^*Q/G$ by the quotient of the symplectic structure on $T^*Q$. In the case $Q = G$, the source and the target are the left and right momentum mappings $J^L$ and $J^R$, and in the pair groupoid case the projections $\pi_I$. This reinforces the idea that symplectic groupoids play an essential role in this theory.

Remark 16 As a by-product, we obtain all the results related to the reduction of the Hamilton-Jacobi theory of reference [18]. The previous discussion specializes to Lie groups, $Q = G$ and then the reduced space $T^*(\mathbb{R} \times \frac{Q \times Q}{G}) = T^*(\mathbb{R} \times \frac{\mathbb{R} \times Q}{G})$ can be identified with $T^*(\mathbb{R} \times G)$ to recover the theory in Ge and Marsden, [18].
### 5.3.1 Other types of generating function

The goal of this section is to show how our results can be applied to other types of generating functions; in this way we recover some classical results about cyclic coordinates. We chose the so-called type II generating functions, but since our theory is valid for any lagrangian submanifold it can be used to deal with any type of generating functions. This type II generating functions are very important, because they can generate the identity transformation and all the “nearby” canonical transformations. We introduce below the classical situation, we assume that $Q = \mathbb{R}^n$ and so $T^*Q = \mathbb{R}^{2n}$ and consider global coordinates $(q^i, p_i), i = 1, \ldots, n$. Doubling these coordinates we get a coordinate system for $T^*(\mathbb{R} \times Q \times Q) = \mathbb{R}^{4n}$, say $(q^i, p_i, \alpha_i, \beta_i)$, and we obtain coordinates $(t, e, q^i, p_i, \alpha_i, \beta_i)$ on $T^*(\mathbb{R} \times Q \times Q)$. Given a function $S(t, q^i, \beta_i)$ it is easy to check that the submanifold given by

\[ L = \{(t, \frac{\partial S}{\partial t}(t, q^i, \beta_i), q^i, \frac{\partial S}{\partial q^i}(t, q^i, \beta_i), \frac{\partial S}{\partial \beta_i}(t, q^i, \beta_i), -\beta_i) \mid t, q^i, \beta_i \in \mathbb{R}\} \]

is lagrangian.

**Remark 17** A more detailed explanation about the construction of this submanifold can be found in [14].

Following the same pattern than above, such generating function gives a time-dependent canonical transformation, given implicitly by

\[ \frac{\partial S}{\partial q^i}(t, q^i, \beta_i) = p_i, \quad \frac{\partial S}{\partial \beta_i}(t, q^i, \beta_i) = \alpha^i. \tag{5} \]

as long as $\det(\frac{\partial^2 S}{\partial q^i \partial \beta_j}) \neq 0$.

Now, our reduction procedure can be applied to the lagrangian submanifold $L$ in a straightforward way. We work out here the details in the case of a time-independent hamiltonian with one cyclic variable in order to recover some results present in the literature, the cases with more than one cyclic variables are obvious. Assume that $h(q^i, p_i)$ does not depend on $t$ and $q^i$, i.e. $q^i$ is a cyclic variable. We are looking for a type II solution of the Hamilton-Jacobi equation for $h$, that is, $S(t, q^i, \beta_i)$ such that

1. The Hamilton-Jacobi equation $\frac{\partial S}{\partial t} + h(q^i, \frac{\partial S}{\partial q^i}) = E$, where $E$ is a real constant.

2. Non-degeneracy condition, $\det(\frac{\partial^2 S}{\partial q^i \partial \beta_j}) \neq 0$.

Using Section 5.2.1 we assume $S(t, q^i, \beta_i) = t \cdot E + W(q^i, \beta_i)$, where $W$ should satisfy

\[ h(q^i, \frac{\partial W}{\partial q^i}(q^i, \beta_i)) = F \tag{6} \]
for some constant $F$ and the non-degeneracy condition. Notice that such function $W$ gives a lagrangian submanifold in $\mathbb{R}^{4n}$ by

$$L_1 = \{(q^i, \frac{\partial W}{\partial q^i}(q^i, \beta_i), \frac{\partial W}{\partial \beta_i}(q^i, \beta_i), -\beta_i) \text{ such that } q^i, \beta_i \in \mathbb{R}\}.$$ 

In order to solve (6) we use the theory previously developed. Notice that $q^1$ is a cyclic variable if and only if the hamiltonian is invariant by the $\mathbb{R}$ action given by $(r, (q^1, \ldots, q^i, \ldots, q^n)) = (q^1 + r, \ldots, q^i, \ldots, q^n)$, which has an associated momentum mapping given by $J(q^1, p_i) = p_1$. The corresponding diagonal action is given by

$$(r, (q^i, p_i, \alpha^i, \beta_i) = (q^1 + r, \ldots, q^i, \ldots, q^n, p_i, \alpha^1 + r, \ldots, \alpha^i, \ldots, \alpha^n, \beta_i))$$

with momentum mapping

$$J(q^1, p_i, \alpha^i, \beta_i) = p_1 + \beta_1.$$ 

So, if we are looking for a lagrangian submanifold $L_1$ living in the 0 level set of the momentum mapping, that is natural regarding the previous remarks, we should impose

$$\frac{\partial W}{\partial q^1} - \beta_1 = 0$$

which implies, by simple integration, that

$$W = q^1 \beta_1 + V(q^i, \beta_j), \quad i = 2, \ldots, n; \quad j = 1, \ldots, n,$$

where the important observation here is that $V$ does not depend on the cyclic variable $q^1$. In this way, we have reduced the number of independent variables by one, this could simplify drastically the Hamilton-Jacobi equation. Here we have recovered the classical ansatz for cyclic variables, see [2, 19], from our geometric interpretation of the Hamilton-Jacobi theory in a straightforward way.

**Remark 18** In the case of more than one cyclic variables an analogous result holds, there

$$W = q^l \beta_l + V(q^i, \beta_j), \quad i = k, \ldots, n; \quad j = 1, \ldots, n,$$

where $l = 1, \ldots k$ are the cyclic variables.

We show how to obtain a complete solution, using this method, of the Hamilton-Jacobi equation for a heavy-top with to equal moments of inertia. The hamiltonian is given by

$$h(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi) = 1/2 \left(\frac{p_\theta^2}{I} + \frac{(p_\phi - p_\psi \cos(\theta))^2}{I \sin^2(\theta)} + \frac{p_\psi^2}{J} \right) + mgl \cos(\theta),$$

where $I, J$ are the moments of inertia, $m$ the mass, $g$ the acceleration of gravity. Using the constructions above

$$S = t \cdot E + W(\theta, \phi, \psi, \beta_1, \beta_2, \beta_3)$$

23
and
\[ W(\theta, \phi, \psi, \beta_1, \beta_2, \beta_3) = \phi \cdot \beta_2 + \psi \cdot \beta_3 + V(\theta, \beta_1, \beta_2, \beta_3). \]

Taking into account all this expressions, we get for the Hamilton-Jacobi equation
\[
\frac{1}{2} \left( \frac{\partial W^2}{\partial \theta} \frac{1}{I} + \frac{(\beta_2 - \beta_3 \cos(\theta))^2}{I \sin^2(\theta)} + \frac{\beta_3^2}{J} \right) + mgl \cos(\theta) = F.
\]

From here it is immediate to integrate the equation and to chose a solution that is non-degenerate; notice that the only unknown is \( \frac{\partial W}{\partial \theta} \) and so by simple integration we can achieve the solution. Although this result was well-known classically, our point here is that it fits directly within our setting. Compare our results with [2], p. 315.

5.4 Calogero-Moser system

We would like to treat another concrete application. Here we deal with a Calogero-Moser system of two particles. Although simple, this system illustrates how our method works. Consider the hamiltonian
\[
H : \ T^* \mathbb{R}^2 \to \mathbb{R} \quad (q^1, q^2, p_1, p_2) \to H(q^1, q^2, p_1, p_2) = 1/2 \ (p_1^2 + p_2^2) + 1/(q^1 - q^2)^2.
\]

In this example \( Q = \mathbb{R}^2 \) and the action
\[
\Phi : \ \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \quad (r, (q^1, q^2, p_1, p_2)) \to (r + q^1, r + q^2)
\]
is a symmetry of the system, i.e., the hamiltonian is invariant under the corresponding lifted action, \( \Phi^T \). We are now looking for a solution of the equation
\[
H(q^1, q^2, \frac{\partial W}{\partial q^1}, \frac{\partial W}{\partial q^2}) = E
\]
which in this case becomes
\[
1/2 \left( \frac{\partial W^2}{\partial q^1} + \frac{\partial W^2}{\partial q^2} \right) + 1/(q^1 - q^2)^2 = 0.
\]

We are looking for solutions in \( J^{-1}(0) \), where
\[
J : \ T^* \mathbb{R}^2 \to \mathbb{R} \quad (q^1, q^2, p_1, p_2) \to J(q^1, q^2, p_1, p_2) = p_1 + p_2
\]
Then, \( J^{-1}(0) = \{(q^1, q^2, p_1, p_2) \text{ such that } p_2 = -p_1\} \), and thus coordinates on \( J^{-1}(0) \) are given by \( (q^1, q^2, p) \to (q^1, q^2, p, -p) \). In the same way, \( J^{-1}(0)/\mathbb{R} \) is \( \mathbb{R}^2 \), with coordinates \( (q, p) \) and the natural projection \( \pi : J^{-1}(0) \to J^{-1}(0)/\mathbb{R} \) reads \( \pi(q^1, q^2, p) = (q = q^1 - q^2, p) \). Some abuse of notation is made, but there is
no room for confusion. Since $H$ is $\mathbb{R}$-invariant there is a reduced hamiltonian, $\overline{H} : J^{-1}(0)/\mathbb{R} \cong \mathbb{R}^2 \to \mathbb{R}$, such that $\overline{H}(q,p) = p^2 + 1/q$. Now the reduced Hamilton-Jacobi equation is just an ODE

$$\overline{H}\left(q, \frac{\partial W}{\partial q}\right) = E,$$

and the reduced Hamilton-Jacobi equation can be integrated looking for a primitive

$$\overline{W} = \int \left(E - \frac{1}{q^2}\right)^{1/2}$$

for the values of $q$ where it makes sense. That can be checked to be

$$\overline{W}(q) = (\sqrt{Eq^2} - 1 - \arctan\left(\frac{1}{\sqrt{Eq^2} - 1}\right)).$$

Then, the reconstruction procedure gives us

$$W(q^1, q^2) = \overline{W}(q^1 - q^2) = (\sqrt{E(q^1 - q^2)^2} - 1 - \arctan\left(\frac{1}{\sqrt{E(q^1 - q^2)^2} - 1}\right))$$

which is defined when $q^1 - q^2 > \sqrt{E}$.

6 Conclusions and Future Research

In this paper we developed a complete theory of reduction and reconstruction of the Hamilton-Jacobi equation for hamiltonian systems with symmetry. The symmetry is supposed to be the lifted action of an action on the configuration manifold, $Q$. Our theory is explained for the time-independent and time dependent Hamilton-Jacobi equations, moreover, complete solutions are also considered. We showed that our theory unifies and extends previous approaches by Ge and Marsden and we can recover in a straightforward way the classical ansatz used in the literature to deal with cyclic variables and time-independent hamiltonians. The results in [17] are also particular instances of our approach. On the other hand, one of the main points of our theory is that we link reduction theory with symplectic groupoids. That link was started in [16] but our approach is quite different and will appear elsewhere ([14]) with some applications to (Poisson) numerical methods. Some open problem related to this work are:

1. Relate our theory to the theory of generating functions in [28, 29]. The theory developed there relies on generating function, so it seems that our theory should be the natural framework to deal with this kind of theories. Connections with the Poincaré generating function would be also very interesting.
2. Develop an analogous theory for general symmetries. Although quite useful out setting only deals at this moment with cotangent lifts of symmetries, to develop an analogous theory for any kind of symmetries should provide means to integrate more general systems. The results in [13] could be of some help in this regard.

3. Construction of geometric integrators from complete solutions of the Hamilton-Jacobi equation. Complete solutions of the Hamilton-Jacobi equations are sometimes hard to find, but they can be approximated in order to find numerical methods that preserve the underlying geometry. This procedure is well-known in the symplectic case, see [12, 20]. Our setting is useful in order to develop the analogous Poisson integrators in the situations treated here. Related work will appear in [14].

A Principal bundles and adjoint bundles

Consider the $G$-principal bundle

$$
\pi : Q \to Q/G
$$

with the action on the left

$$
\Phi : G \times Q \to Q
$$

and $F$ a manifold endowed with a left action

$$
\rho : G \times F \to F.
$$

We shall construct the fiber bundle $Q \times_G F$. Let $Q \times F$ be the product manifold and we introduce the action

$$
G \times (Q \times F) : \quad \longrightarrow \quad Q \times F
$$

$$(g,(q,f)) \quad \rightarrow \quad (\Phi(g,q),\rho(g,f)).$$

The quotient space of $Q \times F$ by this action is called the fiber bundle over the base $Q/G$ with standard fiber $F$ and structure group $G$, which is associated with the principal bundle $Q$ and it is denoted by $Q \times_G F$. We introduce now the differentiable structure of this bundle. The mapping

$$
\tilde{\pi} : \quad Q \times F \quad \longrightarrow \quad Q/G
$$

$$(g,f) \quad \rightarrow \quad \pi(g)$$

induces a mapping $\tilde{\pi} : Q \times_G F \to Q/G$. Since for each $x \in Q/G$ there exists a neighborhood $U$ such that $\pi^{-1}(U) \equiv U \times G$, it can be easily seen that there is an isomorphism $\tilde{\pi}^{-1}(U) \equiv U \times F$. Therefore we can introduce a differentiable structure on $Q \times_G F$ by the requirement that $\tilde{\pi}^{-1}(U)$ is an open submanifold of $Q \times_G F$ diffeomorphic with $U \times F$ under the isomorphism $\tilde{\pi}^{-1}(U) \equiv U \times F$. Then, it follows that $\tilde{\pi}$ is a differentiable mapping.
We now specialize the previous construction to the case when \( F = \mathfrak{g}^* \) and the action is given by

\[
\hat{\rho} : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*
\]

\[
(g, \mu) \rightarrow \hat{\rho}(g, \mu) = Ad_{g^{-1}}(\mu)
\]

The corresponding bundle obtained using the action \( \hat{\rho} \) will be denoted by \( \hat{\mathfrak{g}}^* \).

## B Magnetic Terms

Let \( Q \) be a manifold, \( \Phi : G \times Q \rightarrow Q \) a free and proper action, \( \Phi^{T^*} \) the cotangent-lifted action and \( J : T^*Q \rightarrow \mathfrak{g} \) the corresponding momentum mapping. Recall that \( Q/G \) is endowed with a structure of differentiable manifold and \( \pi_G : Q \rightarrow Q/G \) is a principal bundle.

We will prove that \( J^{-1}(0)/G \) is symplectomorphic to \( (T^*(Q/G), \omega_{Q/G}) \). To see that, consider the codifferential of the mapping \( \pi_G, T\pi_G^* : T^*(Q/G) \rightarrow T^*Q \) and compose with the natural projection over the quotient \( p : T^*Q \rightarrow T^*Q/G \). Then the mapping \( p \circ T\pi_G \) is easily seen to give the desired identification when restricted to its image, which is \( J^{-1}(0)/G \). Notice that although the codifferential \( T^*\pi_G \) is not a mapping (it is multi-valued), the composition does become an identification.

\( J^{-1}(\mu)/G_\mu \) is known to be symplectomorphic to \( (T^*(Q/G), \omega_{Q/G} + B_\mu) \) when \( G_\mu = G \) and where \( B_\mu \) is a magnetic term. To prove that, take \( \alpha_\mu \) a 1-form on \( Q \) such that

1. \( \alpha_\mu \) is \( G \)-invariant by \( \Phi^{T^*} \).
2. \( J \circ \alpha_\mu = \mu. \)

Then we have the shift by \( \alpha_\mu \) given by \( shift : T^*Q \rightarrow T^*Q \), such that \( shift(\alpha_q) = \alpha_q - \alpha_\mu \). This mapping is \( G \)-equivariant and \( shift^*(\omega_{Q/G}) = \omega_{Q/G} - d\alpha_\mu \). Moreover, this map satisfies \( shift(J^{-1}(\mu)) = J^{-1}(0) \) and thus, by \( G \)-equivariance, \( shift(G^{-1}(\mu)/G) = J^{-1}(0)/G \), where \( G^{-1} \) is the mapping induced on the quotient. The right hand side of the last equality is identified with \( T^*(Q/G) \) but since \( shift \) is not a symplectomorphism between the canonical symplectic structures of cotangent bundles, the form \( \omega_{Q/G} \) must be modified. Since \( J \circ \alpha_\mu = \mu \), then \( \alpha_\mu(\xi_Q) = \mu(\xi) \) is a constant function on \( Q \). We deduce that

\[
i_\xi_Q d\alpha_\mu = \mathcal{L}_\xi_Q d\alpha_\mu - d(\alpha_\mu(\xi_Q)) = 0
\]

and so there exists a unique 2-form on \( Q/G \) such that \( \pi_G^*\beta_\mu = d\alpha_\mu \). It is not hard to see now that \( J^{-1}(\mu)/G \) with the symplectic structure provided by the Marden-Weinstein reduction is symplectomorphic to the cotangent bundle \( T^*(Q/G) \) with the symplectic structure given by \( \omega_{Q/G} + \pi_G^*\beta_\mu \).

**Remark 19** In the constructions of our paper, we used a connection from the beginning. With a connection at hand, the construction of the form \( \alpha_\mu \) is just the composition of the connection 1-form (which is a \( \mathfrak{g} \)-valued 1-form) and \( \mu \).
Acknowledgments

This work has been partially supported by MINECO MTM 2013-42-870-P and the ICMAT Severo Ochoa project SEV-2011-0087. M. Vaquero wishes to thank MINECO for a FPI-PhD Position. We wish to thank D. Iglesias-Ponte and J.C. Marrero for several useful discussions.

References

[1] Abraham, R., and Marsden, J. E. Foundations of mechanics. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[2] Ardemà, M. Analytical Dynamics: Theory and Applications. Kluwer Academic/ Plenum Publishers, New York, 2005.

[3] Arnol’d, V. I. Mathematical methods of classical mechanics, vol. 60 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1993. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[4] Benzel, S., Ge, Z., and Scovel, C. Elementary construction of higher-order Lie-Poisson integrators. Phys. Lett. A 174, 3 (1993), 229–232.

[5] Cardin, F. On the geometrical Cauchy problem for the Hamilton-Jacobi equation. Nuovo Cimento B (11) 104, 5 (1989), 525–544.

[6] Cariñena, J. F., Gràcia, X., Marmo, G., Martín, E., Muñoz-Lecanda, M. G., and Román-Roy, N. Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems. Int. J. Geom. Methods Mod. Phys. 7, 3 (2010), 431–445.

[7] Channell, P. J., and Scovel, C. Symplectic integration of Hamiltonian systems. Nonlinearity 3, 2 (1990), 231–259.

[8] Channell, P. J., and Scovel, J. C. Integrators for Lie-Poisson dynamical systems. Phys. D 50, 1 (1991), 80–88.

[9] Chaperon, M. On generating families. In The Floer memorial volume, vol. 133 of Progr. Math. Birkhäuser, Basel, 1995, pp. 283–296.

[10] de León, M., Marrero, J. C., and Martín de Diego, D. Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics. J. Geom. Mech. 2, 2 (2010), 159–198.

[11] de León, M., Martín de Diego, D., and Vaquero, M. A Hamilton-Jacobi theory on Poisson manifolds. J. Geom. Mech. 6, 1 (2014), 121–140.
REFERENCES

[12] Feng, K., and Qin, M. *Symplectic geometric algorithms for Hamiltonian systems*. Zhejiang Science and Technology Publishing House, Hangzhou; Springer, Heidelberg, 2010. Translated and revised from the Chinese original, With a foreword by Feng Duan.

[13] Fernandes, R. L., Ortega, J.-P., and Ratiu, T. S. The momentum map in Poisson geometry. *Amer. J. Math. 131*, 5 (2009), 1261–1310.

[14] Ferraro, S., de León, M., Marrero, J. C., Martín de Diego, D., and Vaquero, M. On the geometry of the Hamilton-Jacobi equation. Preprint, 2015.

[15] García-Toraño Andrés, E., Guzmán, E., Marrero, J. C., and Mestdag, T. Reduced dynamics and Lagrangian submanifolds of symplectic manifolds. *J. Phys. A 47*, 22 (2014), 225203, 24.

[16] Ge, Z. Generating functions, Hamilton-Jacobi equations and symplectic groupoids on Poisson manifolds. *Indiana Univ. Math. J. 39*, 3 (1990), 859–876.

[17] Ge, Z. Equivariant symplectic difference schemes and generating functions. *Phys. D 49*, 3 (1991), 376–386.

[18] Ge, Z., and Marsden, J. E. Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators. *Phys. Lett. A 133*, 3 (1988), 134–139.

[19] Goldstein, H. *Classical mechanics*, second ed. Addison-Wesley Publishing Co., Reading, Mass., 1980. Addison-Wesley Series in Physics.

[20] Hairer, E., Lubich, C., and Wanner, G. *Geometric numerical integration*, vol. 31 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2010. Structure-preserving algorithms for ordinary differential equations, Reprint of the second (2006) edition.

[21] Holm, D. D. *Geometric mechanics. Part I*, second ed. Imperial College Press, London, 2011. Dynamics and symmetry.

[22] Iglesias-Ponte, D., de León, M., and Martín de Diego, D. Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems. *J. Phys. A 41*, 1 (2008), 015205, 14.

[23] Kostant, B. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*. Springer, Berlin, 1970, pp. 87–208. Lecture Notes in Math., Vol. 170.

[24] Marsden, J., and Weinstein, A. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys. 5*, 1 (1974), 121–130.

[25] Marsden, J. E., Mishler, G., Ortega, J.-P., Perlmutter, M., and Ratiu, T. S. *Hamiltonian reduction by stages*, vol. 1913 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
REFERENCES

[26] McLachlan, R. I., and Scovel, C. Equivariant constrained symplectic integration. *J. Nonlinear Sci.* 5, 3 (1995), 233–256.

[27] McLachlan, R. I., and Scovel, C. A survey of open problems in symplectic integration. In *Integration algorithms and classical mechanics (Toronto, ON, 1993)*, vol. 10 of *Fields Inst. Commun.* Amer. Math. Soc., Providence, RI, 1996, pp. 151–180.

[28] Meyer, K. R. Generic bifurcation of periodic points. *Trans. Amer. Math. Soc.* 149 (1970), 95–107.

[29] Meyer, K. R. Equivariant generating functions and periodic points. In *New trends for Hamiltonian systems and celestial mechanics (Cocoyoc, 1994)*, vol. 8 of *Adv. Ser. Nonlinear Dynam.* World Sci. Publ., River Edge, NJ, 1996, pp. 289–299.

[30] Scovel, C., and Weinstein, A. Finite-dimensional Lie-Poisson approximations to Vlasov-Poisson equations. *Comm. Pure Appl. Math.* 47, 5 (1994), 683–709.

[31] Souriau, J.-M. *Structure of dynamical systems*, vol. 149 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997. A symplectic view of physics. Translated from the French by C. H. Cushman-de Vries, Translation edited and with a preface by R. H. Cushman and G. M. Tuynman.

[32] Wang, H. Symmetric reduction and Hamilton-Jacobi equation of rigid spacecraft with a rotor. *J. Geom. Symmetry Phys.* 32 (2013), 87–111.

[33] Weinstein, A. *Lectures on symplectic manifolds*. American Mathematical Society, Providence, R.I., 1977. Expository lectures from the CBMS Regional Conference held at the University of North Carolina, March 8–12, 1976, Regional Conference Series in Mathematics, No. 29.