APPROXIMATION THEOREMS FOR CONTROLLABILITY PROBLEM GOVERNED BY FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this manuscript, we discuss the optimal control problem for a nonlinear system governed by the fractional differential equation in a separable Hilbert space $X$. We utilize the fixed point technique and $\eta$-resolvent family to present the existence of control for the fractional system. The optimal pair is obtained as the limit of the optimal pair sequence of the unconstrained problem. Further, we derive some approximation results, which guarantee the convergence of the numerical method to optimal pair sequence. Finally, the main results are validated with the aid of an example.

1. Introduction. The fractional calculus (calculus of integrals and derivatives of arbitrary order) has gained considerable popularity and importance during the past three decades or so, due mainly to its applications in the diverse fields of science and engineering such as viscoelasticity, biological population models, electrochemistry, economics, biology and control theory. The fractional calculus has been recognized as one of the best tools to describe the allometric scaling laws, long-range interactions and long memory processes. For fundamental theory on fractional differential equations, see [7, 13, 21, 23, 24]. The fundamental concepts of controllability have an important practical and theoretical value, it determines the conditions, for which the optimal control is generally possible for a system. Normally, the goal of a controllability problem is to steers or transfers the state variable from any initial state to any given target state by selecting a suitable control among available options. For

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more recent works on controllability, one may see [6, 10, 14, 17, 18, 19, 20, 25, 29, 31]. Klamka [14] presented a survey paper on the controllability of dynamical systems. The theory of optimal control plays a crucial role in the applications of mathematics. In time-optimal control problem, we are looking for a control function which not only steers or transfer the state from initial state to given target state but also does it with minimum cost in the given time. The concept of optimal control and controllability arise naturally in diverse fields such as control of electric bulk power systems, chemical process control, reactor control, aerospace engineering, quantum systems theory, crystal growth and vascular surgery. Wang and Zhou [30] discussed the existence of optimal controls for the fractional differential equations. Agarwal et al. [1] presented a survey paper on controllability, optimal control and optimal feedback control for several different kinds of fractional evolution equations. Knowles [15] presented the numerical approximation for a parabolic time optimal control problem via piecewise linear splines. Cao [4] discussed the numerical approximations of controllability problem by optimal control problem for linear parabolic differential equation. Diaz and Ramos [8] considered the numerical experiments regarding the distributed control of semilinear parabolic equations. Joshi and Kumar [12] discussed the approximation for controllability of linear parabolic differential equation which is of the form

\[
\begin{cases}
\frac{\partial x}{\partial t} + Ax = v & \text{in } B, \\
x = 0 & \text{on } \Delta, \\
x(0) = x_0 & \text{on } \Theta,
\end{cases}
\]

where \( \Theta \) is a bounded domain in \( \mathbb{R}^k \) with smooth boundary \( \partial \Theta \) and \( B = (0, b) \times \Theta, \Delta = (0, b) \times \partial \Theta, b > 0, A \) is the elliptic operator of second order. Kumar et al. [16] studied the approximation theorems for controllability of parabolic differential equation which is of the form

\[
\begin{cases}
\frac{\partial x(t)}{\partial t} + Ax(t) = v(t) + h(t, x(t)), & t \in (0, b], \\
x(0) = x_0,
\end{cases}
\]

where \( h(t, x) \) is a nonlinear function defined form \([0, b) \times X \) into \( X \).

Let \( X \) and \( U \) be two separable Hilbert spaces, \( Z = L^2[0, b ; X] \) and \( Y = L^2[0, b ; U] \) be the function spaces, defined on \( J = [0, b], 0 \leq b < \infty \). Motivated by the above-mentioned facts and works described in the papers [15, 16], in this manuscript, we consider a control system governed by nonlinear fractional differential equation which is of the form

\[
\begin{cases}
\notag cD^\eta_t z(t) = Az(t) + \mathcal{F}(t, z(t)) + u(t), & t \in (0, b], \\
z(0) = z_0,
\end{cases}
\]

where the state function \( z : [0, b] \to X \), control function \( u \in Y, \notag cD^\eta_t \) is the Caputo derivative of order \( 1/2 < \eta < 1, A : D(A) \subset X \to X \) is the generator of an \( \eta \)-resolvent family \( \{S\eta(t) : t \geq 0\} \). The function \( \mathcal{F} : [0, b] \times X \to X \) is satisfying some suitable conditions which will be specified later. For simplicity, we assume that \( U = X \).

A strong motivation for investigating this class of equation comes mainly from two compelling reasons: the differential models with the fractional derivative providing an excellent instrument for the description of memory and hereditary properties
have recently been proved a valuable tools in the modeling of many physical phenomena. The fractional order models of the real system are always more adequate than the classical integer order models since the description of some systems is more accurate when the fractional order derivative is used. Therefore it is natural to extend the concept of optimal control problem for a system governed by the fractional differential equation. For such type of control problems, we obtained the results by using the technique of the paper [16]. However, the analysis is different as we have used the η-resolvent family, fractional calculus and first-order approximation method for the computation of Caputo’s fractional derivative.

The outline of this manuscript is as follows. Section 2 introduces preliminary facts which will be used in subsequent sections. In section 3, we discussed the existence of control for the proposed system. Section 4 is devoted to the investigation of existence and convergence of optimal control for the fractional system and section 5 deals with the approximation theorems. In section 6, we discuss a numerical example to illustrate the validation of the obtained results.

2. Preliminaries. In this section, we give some notions and certain important preliminaries that will be used in the subsequent sections.

Definition 2.1. [26] The Caputo derivative of order α for a function \( f : (0, \infty) \to \mathbb{R} \) can be written as

\[
\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) \, ds, \quad t > 0,
\]

where \( m - 1 < \alpha < m, \, m \in \mathbb{N} \) and \( f(t) \in C^m(0, \infty) \).

Definition 2.2. [9] A closed and linear operator \( A \) is called a sectorial operator with domain \( D(A) \) in \( X \) if there are constants \( N > 0, \, \lambda_1 \in \mathbb{R}, \, \theta_1 \in [\pi/2, \pi] \) such that the subsequent conditions are satisfied

1. \( \rho(A) \subset \Sigma(\theta_1, \lambda_1) = \{ \lambda_2 \in \mathbb{C} : \lambda_1 \neq \lambda_2, \, |\arg(\lambda_2 - \lambda_1)| < \theta_1 \} \),
2. \( |R(\lambda_2, A)| \leq N/|\lambda_2 - \lambda_1|, \, \lambda_2 \in \Sigma(\theta_1, \lambda_1) \),

where \( \rho(A) \) is a resolvent set of \( A \).

Definition 2.3. [3] Let \( A : D(A) \subset X \to X \) be a closed and linear operator. The operator \( A \) is called the generator of η-resolvent family if there exist \( \lambda_1 \geq 0 \) and a strongly continuous operator \( S_\eta : \mathbb{R}_+ \to L(X) \), where \( L(X) \) is a Banach space of all bounded linear operators from \( X \) into \( X \), such that \( \{ \lambda_2^\eta : \text{Re} \lambda_2 > \lambda_1 \} \subset \rho(A) \) and

\[
(\lambda_2^\eta I - A)^{-1} z = \int_0^\infty e^{-\lambda_2^\eta t} S_\eta(t) z dt, \quad \text{Re} \lambda_2 > \lambda_1, \, z \in X.
\]

In this case, \( \{ S_\eta(t) : t \geq 0 \} \) is called the η-resolvent family generated by \( A \).

Definition 2.4. [5] Let \( A : D(A) \subset X \to X \) be a linear closed operator and \( \eta > 0 \). The operator \( A \) is called the generator of an solution operator if there exist \( \lambda_1 \geq 0 \) and a strongly continuous operator \( T_\eta : \mathbb{R}_+ \to L(X) \), such that \( \{ \lambda_2^\eta : \text{Re} \lambda_2 > \lambda_1 \} \subset \rho(A) \) and

\[
\lambda_2^{-\eta} (\lambda_2^\eta I - A)^{-1} z = \int_0^\infty e^{-\lambda_2^\eta t} T_\eta(t) z dt, \quad \text{Re} \lambda_2 > \lambda_1, \, z \in X.
\]

In this case, \( \{ T_\eta(t) : t \geq 0 \} \) is called the solution operator generated by \( A \).
Definition 2.5. [5] By the mild solution of the fractional system (3), we mean that the continuous function \( z(t) \in X \), which satisfies the following integral equation

\[
z(t) = T_\eta(t)z_0 + \int_0^t S_\eta(t-s)\mathcal{F}(s,z(s))ds + \int_0^t S_\eta(t-s)u(s)ds,
\]
(7)

where \( T_\eta(t) = \frac{1}{2\pi i} \int_{\mathcal{B}_r} e^{\lambda t} \lambda_2^{\eta-1} (\lambda^2 - \mathcal{A})^{-1} d\lambda \) and \( S_\eta(t) = \frac{1}{2\pi i} \int_{\mathcal{B}_r} e^{\lambda t}(\lambda^2 - \mathcal{A})^{-1} d\lambda \), here, \( \mathcal{B}_r \) denotes the Bromwich path.

For initial data \( z_0 \), control \( u, b > 0 \), we define

\[
\mathcal{R}(b,z_0,u) = \{ z(b) : z \text{ is a mild solution of the system } (3) \text{ with control } u \in \mathcal{Y} \},
\]
(8)

the set \( \mathcal{R}(b,z_0,u) \) is called set of reachable state.

Definition 2.6. The fractional system (3) is said to be exactly controllable on the interval \([0,b]\) if for all \( z_0, z_1 \in X \), there exists \( u \in \mathcal{Y} \) such that corresponding mild solution of the fractional system (3) satisfying \( z(0) = z_0 \), also satisfies \( z(b) = z_1 \).

For a control \( u \), we arrive at

\[
z_b = z(b) = T_\eta(t)z_0 + \int_0^b S_\eta(b-s)[\mathcal{F}(s,z(s)) + u(s)]ds.
\]
(9)

Now, we define \( \mathcal{N} : \mathcal{Y} \to \mathcal{Y}, \mathcal{K} : \mathcal{Y} \to X \) and \( \mathcal{W} : \mathcal{Y} \to \mathcal{Y} \) respectively, as

\[
(\mathcal{N}z)(t) = \mathcal{F}(t,z(t)), \quad \mathcal{K}u = \int_0^b S_\eta(b-s)u(s)ds \text{ and } z = \mathcal{W}u,
\]

where \( \mathcal{N} \) is a non-linear bounded and continuous operator known as the Nemytskii operator [11], \( \mathcal{K} \) is a bounded linear operator, \( \mathcal{W} \) is a solution operator. The operator equation corresponding to (9) is

\[
\varphi = \mathcal{K}u + \mathcal{N}\mathcal{W}u,
\]

where \( \varphi = z_b - T_\eta(b)z_0 \). Now, we define a new set \( \mathcal{U}_{sx} \subset \mathcal{Y} \) such that

\[
\mathcal{U}_{sx} = \{ u \in \mathcal{Y} : \mathcal{K}u + \mathcal{N}\mathcal{W}u = \varphi \},
\]

the above set \( \mathcal{U}_{sx} \) contain all admissible exact controls of the fractional system (3).

We can state our minimization problem as follows

Problem 2.1. Find \( \overline{\pi} \in \mathcal{U}_{sx} \subset \mathcal{Y} \) such that

\[
\mathcal{J}(\overline{\pi}) = \inf_{u \in \mathcal{U}_{sx}} \left[ \mathcal{J}(u) = \frac{1}{2} \| u \|_Y^2 \right].
\]
(10)

Definition 2.7. Let the constraint problem (10) has a solution \( \overline{\pi} \in \mathcal{U}_{sx} \subset \mathcal{Y} \) with \( \pi \in \mathcal{Y} \) as the corresponding mild solution of the fractional system (3), then \( (\overline{\pi}, \pi) \) is called the optimal pair of the constraint problem (10).

Definition 2.8. Let \( \mathcal{E}_n \) and \( \mathcal{E} \) be non-empty subsets of a Hilbert space \( X \). If \( \text{dist}(\mathcal{E}_n, \mathcal{E}) \to 0 \) as \( n \to 0 \), then \( \mathcal{E}_n \) converges to a subset \( \mathcal{E} \).

Property 2.1. Let \( \mathcal{E}_n \) and \( \mathcal{E} \) be non-empty subsets of a Hilbert space \( X \). The sequence \( \mathcal{E}_n \) converges to a subset \( \mathcal{E} \) if

1. for all \( \varphi \in \mathcal{E} \), there exists a sequence \( \{ \varphi_n \} \), \( \varphi_n \in \mathcal{E}_n \), such that \( \varphi = \lim_{n \to \infty} \varphi_n \),
2. for a sequence \( \{ \varphi_k \} \), \( \varphi_k \in \mathcal{E}_{nk} \) (where \( \mathcal{E}_{nk} \) is a subsequence of \( \mathcal{E}_n \) ), such that \( \lim_{k \to \infty} \varphi_k = \varphi \), then \( \varphi \in \mathcal{E} \).

In order to establish the main results, we assume that the following hypotheses
3. Existence of control. In this section, we prove the existence of control for the fractional system (3). Now, we consider the following linearized fractional system which indexed by a fixed $x \in Y$

\[
\begin{aligned}
\begin{cases}
\,^cD_t^\eta z_x(t) = A z_x(t) + F(t, x(t)) + u_x(t), \, t \in (0, b], \\
z_x(0) = z_0.
\end{cases}
\end{aligned}
\]

(11)

The mild solution given by

\[
z_x(t) = T_\eta(t)z_0 + \int_0^t S_\eta(t-s)F(s, x(s))ds + \int_0^t S_\eta(t-s)u_x(s)ds.
\]

(12)

Hence,

\[
\varphi = \int_0^b S_\eta(b-s)F(s, x(s))ds + \int_0^b S_\eta(b-s)u_x(s)ds,
\]

where $\varphi = z_x(b) - T_\eta(b)z_0$. Next, the operator equation corresponding to above integral equation is given by

\[
K u_x = \varphi - KN x,
\]

(13)

for each fixed $x$ and we choose the feedback control $u_x$ as follows

\[
u_x = K^{-1}[\varphi - KNx].
\]

(14)

Now, we define the operator $Q : Y \to Y$, which assign the mild solution of fractional system (3) (given by (12)) corresponding to each fixed $x$

\[
Q x(t) = T_\eta(t)z_0 + \int_0^t S_\eta(t-s)[F(s, x(s)) + K^{-1}(\varphi - KNx)(s)]ds.
\]

(15)
Set \( \mathcal{G}(s, x(s)) = \mathcal{F}(s, x(s)) + \mathcal{K}^{-1}(\varphi - \mathcal{K}N x)(s) \), then
\[
\mathcal{Q}x(t) = \mathcal{T}_\eta(t)z_0 + \int_0^t \mathcal{S}_\eta(t-s)\mathcal{G}(s, x(s))ds.
\]
Moreover, \( \mathcal{G}(s, x(s)) \) satisfy the following condition
\[
\|\mathcal{G}(s, x_1(s)) - \mathcal{G}(s, x_2(s))\|_X \leq \|\mathcal{F}(s, x_1) - \mathcal{F}(s, x_2)\|_X + \|N x_1 - N x_2\|_X
\]
\[
= 2\|\mathcal{F}(s, x_1) - \mathcal{F}(s, x_2)\|_X
\]
\[
\leq 2\alpha\|x_1 - x_2\|_X.
\]
Thus \( \mathcal{G}(s, x(s)) \) is lipschitz continuous with Lipschitz constant \( \mu = 2\alpha \).

We are now in a position to state and prove the main theorem of this manuscript.

**Theorem 3.1.** Assume that the hypotheses \([H_1] - [H_3]\) hold and the operators \( \mathcal{Q} \) and \( \mathcal{M} = \mathcal{K}^{-1}(\varphi - \mathcal{K}N) \) are Lipschitz continuous. Then the operator \( \mathcal{Q}^m \) is a contraction map on \( Y \). Moreover, for any arbitrary \( x_0 \), the iterative sequence \( \{x_k\} \), defined as
\[ x_{k+1} = \mathcal{Q}^m x_k, \ k = 0, 1, \ldots, \] converges to \( \mathcal{p} \), which is a solution of the fractional system (3). Further, \( u_k = \mathcal{M}x_k \) is such that \( u_k \) converges to \( \mathcal{u} = \mathcal{M}\mathcal{p} \in \mathcal{U}_x \) and fractional system (3) is exact controllable on \([0, b]\).

**Proof.** Let \( x_1, x_2 \in Y \), then
\[
\|\mathcal{Q}x_1 - \mathcal{Q}x_2(t)\|_X \leq \int_0^t \|\mathcal{S}_\eta(t-s)\|_X \|\mathcal{G}(s, x_1(s)) - \mathcal{G}(s, x_2(s))\|_X ds
\]
\[
\leq M_2\mu \int_0^t (t-s)^{n-1}\|x_1 - x_2\|_X ds.
\]
By Hölder’s inequality, we obtain
\[
\|\mathcal{Q}x_1 - \mathcal{Q}x_2\|_Y \leq \frac{M_2\mu b^n}{\sqrt{2n(2n-1)}}\|x_1 - x_2\|_Y.
\]
Proceeding inductively, we obtain that there exists a constant
\[
\delta_m = \left[ B\left(\eta, 0 + \frac{1}{2}\right) : B\left(\eta, \eta + \frac{1}{2}\right) \cdots B\left(\eta, (m-1)\eta + \frac{1}{2}\right) \right] \frac{(M_2\mu b^n)^m}{\sqrt{2mn(2n-1)}},
\]
such that
\[
\|\mathcal{Q}^m x_1 - \mathcal{Q}^m x_2\|_Y \leq \delta_m\|x_1 - x_2\|_Y.
\]
Here, \( B(\cdot, \cdot) \) is a beta function. Next, we can choose \( m \) large enough (not dependent of \( M_2, \mu \) and \( b \)) such that \( \delta_m < 1 \) and by using Banach fixed point theorem, we obtain a fixed point, say, \( \mathcal{p} \) of the operator \( \mathcal{Q}^m \), which is also unique fixed point of the operator \( \mathcal{Q} \). Since \( \mathcal{M} \) is continuous and \( x_k \to \mathcal{p} \), it implies that \( u_k = \mathcal{M}x_k \to \mathcal{M}\mathcal{p} = \mathcal{p} \). Next, we show that \( \mathcal{p} \) is the unique mild solution of the fractional system (3) corresponding to control \( \mathcal{u} \). As \( x_k \to \mathcal{p} \) and \( \mathcal{p} \) is a unique fixed point of \( \mathcal{Q} \), from (15), we obtain
\[
\mathcal{Q}x_k(t) = \mathcal{T}_\eta(t)z_0 + \int_0^t \mathcal{S}_\eta(t-s)\mathcal{F}(s, x_k(s))ds + \int_0^t \mathcal{S}_\eta(t-s)u_k(s)ds,
\]
and hence, as \( k \to \infty \)
\[
\mathcal{p} = \mathcal{T}_\eta(t)z_0 + \int_0^t \mathcal{S}_\eta(t-s)\mathcal{F}(s, \mathcal{p}(s))ds + \int_0^t \mathcal{S}_\eta(t-s)\mathcal{u}(s)ds.
\]
Hence, $\pi$ is a unique mild solution of the fractional system (3) with control $\pi$ and we have $\pi = \mathcal{W}\pi$. Next, we show that $\pi \in \mathcal{U}_{ex}$, form (13), we obtain

$$Ku_k = \varphi - \mathcal{K}\mathcal{N}x_k.$$ 

The continuity of $\mathcal{N}$, $\mathcal{K}$ along with fact $x_k \to \pi$ and $u_k \to \pi$ implies that

$$K\pi = \varphi - \mathcal{K}\mathcal{N}\pi = \varphi - \mathcal{K}\mathcal{N}\mathcal{W}\pi.$$ 

Therefore,

$$\varphi = K\pi + \mathcal{K}\mathcal{N}\pi,$$

and $\pi \in \mathcal{U}_{ex}$. Hence, fractional system (3) is exact controllable and the set $\mathcal{U}_{ex}$ is non-empty.

**Remark 3.1.** The pair $(\pi, \pi)$ is need not to be optimal and hence, the constraint problem (10) remain unanswered.

**Remark 3.2.** We now change our strategy to obtain the optimal pair of the constrained problem through a sequence of optimal pairs of the unconstrained problems. For this purpose now, we define the unconstrained cost functional $J_n : Y \to \mathbb{R}$ as

$$J_n(u) = \frac{1}{2}\|u\|_Y^2 + \frac{\gamma_n}{2}\|\mathcal{K}u + \mathcal{K}\mathcal{N}\mathcal{W}u - \varphi\|_X^2,$$

for each $n \in \mathbb{N}$, where $\gamma_n$ is a sequence of positive number, which is strictly increasing.

**Problem 3.1.** Find $\pi_n \in Y$ such that

$$J_n(\pi_n) = \inf_{u \in Y} \left[J_n(u) = \frac{1}{2}\|u\|_Y^2 + \frac{\gamma_n}{2}\|\mathcal{K}u + \mathcal{K}\mathcal{N}\mathcal{W}u - \varphi\|_X^2\right].$$

**Definition 3.2.** Let the unconstrained problem (16) has a solution $\pi_n \in Y$ with $\pi_n \in Y$ as the corresponding mild solution of the fractional system (3), then $\pi_n$, $\pi_n$ is called the optimal pair of the problem (16).

The constraint problem (10) has an optimal pair $(\pi, \pi)$, which is attain as limit of $(\pi_n, \pi_n)$, where $(\pi_n, \pi_n)$ is an optimal pair sequence of the problem (16).

4. **Convergence of optimal control.** In this section, we prove the convergence of the optimal pair sequence.

**Theorem 4.1.** Assume that the hypotheses $[H_1] - [H_4]$ hold, then the unconstrained problem (16) has an optimal pair $(\pi_n, \pi_n)$ such that $\pi_n \in Y$ minimizes the functional $J_n(u)$ and $\pi_n$ solves the fractional system (3) with control $\pi_n$.

**Proof.** Let $u_m^n$ converges weakly to $\pi_n$ in $Y$, then we obtain

$$\limsup_{m \to \infty} J_n(u_m^n) \leq \liminf_{m \to \infty} \left[\frac{1}{2}\|u_m^n\|_Y^2 + \frac{\gamma_n}{2}\|\mathcal{K}u_m^n + \mathcal{K}\mathcal{N}\mathcal{W}u_m^n - \varphi\|_X^2\right].$$

Since $\mathcal{W}$ is completely continuous and $\mathcal{K}$ is continuous, this implies that $\mathcal{K}u_m^n$ converges weakly to $\mathcal{K}\pi_n$, $\mathcal{K}\mathcal{N}\mathcal{W}u_m^n - \varphi \to \mathcal{K}\mathcal{N}\mathcal{W}\pi_n - \varphi$ and $\langle \mathcal{K}u_m^n, \mathcal{K}\mathcal{N}\mathcal{W}u_m^n - \varphi \rangle \to$
\langle \kappa \pi_n, KNW \pi_n - \varphi \rangle \) and the norm is weakly lower semi-continuous functional. Using the both facts, we obtain
\[
\liminf_{m \to \infty} J_n(u_m) \geq \frac{1}{2} \|u_m\|^2_Y + \frac{\gamma_n}{2} \|\kappa \pi_n + KNW \pi_n - \varphi\|^2_X = J_n(\pi_n),
\]
this implies that the functional \(J_n\) is weakly lower semi-continuity functional. Let the functional \(J_n\) has an minimizing sequence \(\{u_m\}\). Since \(J_n\) is coercive and the sequence \(\{u_m\}\) is bounded in the space \(Y\), then there exists a subsequence, relabeled as \(u_m\), which converges weakly to \(\pi_n\) in the space \(Y\) and \(J_n\) is weakly lower semi-continuous, then we arrive at
\[
\inf_{u \in Y} J_n(u) = \liminf_{m \to \infty} J_n(u_m) = \liminf_{m \to \infty} J_n(u_m) \geq J_n(\pi_n), \tag{17}
\]
hence, we obtain
\[
J_n(\pi_n) = \inf_{u \in Y} J_n(u).
\]
Since \(W\) is completely continuous and \(\pi_m = W u_m\), \(u_m\) converges weakly to \(\pi_n\), this implies that \(\pi_m \to \pi_n\), where \(\pi_n = W \pi_n\). Hence, unconstrained problem \(16\) has an optimal pair \((\pi_n, \pi_n)\). \(\square\)

**Lemma 4.2.** \([12]\) Let \(\bar{u} \in U_{ex}\) and functional \(J_n(u)\) has an minimizer \(\pi_n \in Y\) in \(Y\) with \(\{\gamma_n\}\) as a sequence of positive number, which is strictly increasing. Then the following inequality hold

1. \(J_n(\pi_n) \leq J_{n+1}(u_{n+1})\),
2. \(J(\pi_n) \leq J(\pi_{n+1})\),
3. \(J(\pi_n) \leq J_n(\pi_n) \leq J(\bar{u})\).

Note that Lemma 4.2 \(A_3\), implies that \(\{\pi_n\}\) is uniformly bounded. Below, we now state and prove one of the important theorems of this section.

**Theorem 4.3.** Assume that the hypotheses \([H_1] - [H_4]\) hold, then the unconstrained problem \(16\) has an optimal pair \((\pi_n, \pi_n)\). As \(n \to \infty\), there exists a subsequence of \((\pi_n, \pi_n)\), which converges to \((\bar{u}, \bar{u})\), where \((\bar{u}, \bar{u})\) is optimal pair of the problem \(10\).

**Proof.** Theorem 4.1 implies that the unconstrained problem \(16\) has an optimal pair. As \(U_{ex} \neq \emptyset\), from Lemma 4.2, we obtain
\[
J_n(\pi_n) \leq J_{n+1}(u_{n+1}) \leq J(\bar{u}).
\]
Therefore \(\{J_n(\pi_n)\}\) is a monotone and bounded sequence, hence, the sequence has a limit and by Lemma 4.2, we have
\[
J(\pi_n) \leq J(u_{n+1}) \leq J(\bar{u}).
\]
Thus \(\{J(\pi_n)\}\) is a convergent sequence and hence, \(\|\kappa \pi_n + KNW \pi_n - \varphi\|^2\) be the difference of two convergent sequence, which is also convergent sequence, this implies that
\[
\|\kappa \pi_n + KNW \pi_n - \varphi\|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]
Since \(W\) is a complete continuous operator, \(K\) is a weakly linear continuous operator and the sequence \(\pi_n\) is a bounded in the space \(Y\), as \(n \to \infty\), there exists a subsequence of \(\pi_n\) which converges weakly to \(\pi\) in \(Y\), this implies that \(\kappa \pi_n + KNW \pi_n\) converges weakly to \(\kappa \pi + KNW \pi\). Hence, \(\kappa \pi + KNW \pi = \varphi \Rightarrow \pi \in U_{ex}\). Since norm is weak lower semicontinuity, we obtain
\[
\|\pi\| \leq \liminf_{n \to \infty} \|\pi_n\| \leq \limsup_{n \to \infty} \|\pi_n\| \leq \|\pi\|.
\]
Hence, we obtain
\[ \lim_{n \to \infty} \|u_n\| = \|\overline{u}\|. \] (18)
Combining above equation together with weak convergence of \(u_n\) to \(\overline{u}\), we obtain
\[ \lim_{n \to \infty} u_n = \overline{u} \text{ in } Y. \]
Next, we show that \(z_n\) converges to \(z\) in the space \(Y\). From (7), we have
\[ \|z_n(t) - z(t)\|_X \leq \int_0^t \|S_\eta(t-s)\|_X \|F(s, \overline{z}_n(s)) - F(s, z(s))\|_X ds \]
\[ + \int_0^t \|S_\eta(t-s)\|_X \|u_n(s) - u(s)\|_X ds \]
\[ \leq M_2 \alpha \int_0^t (t-s)^{\eta-1} \|z_n - z\|_X ds \]
\[ + M_2 \int_0^t (t-s)^{\eta-1} \|u_n - u\|_X ds. \]
By applying the generalized Gronwall inequality \([32]\), we obtain
\[ \|z_n(t) - z(t)\|_X \leq M_2 E_\eta(M_2 \alpha \Gamma(\eta)b^\eta) \left[ \frac{t^{2\eta-1}}{2\eta-1} \right] \frac{1}{2} \|u_n - u\|_Y, \]
Hence,
\[ \|z_n - z\|_Y \leq M_2 E_\eta(M_2 \alpha \Gamma(\eta)b^\eta) \frac{b^\eta}{\sqrt{2\eta(2\eta-1)}} \|u_n - u\|_Y, \]
which means that \(z_n \to z\) in the space \(Y\) as \(n \to \infty\). Hence prove. \(\Box\)

**Remark 4.1.** Above Theorem 4.3 shows the existence of solution of the unconstrained problem (16), which may not be unique. So based on the Theorem 4.3, we give the following result and by using the Property (2.1), we prove the convergence of set.

**Theorem 4.4.** Assume that the hypotheses \([H_1] - [H_4]\) hold. Let constrained problem (10) has a set \(E\) of solutions and also, the unconstrained problem (16) has a set \(E_n\) of minimizers. Then \(E_n\) converges to \(E\) as \(n \to \infty\).

**Lemma 4.5.** [16] Assume that the hypotheses \([H_5] - [H_6]\) hold, then the solution operator \(\mathcal{W}\) is Fréchet differentiable with derivative given by
\[ \mathcal{W}'(u) = [1 - LN'(z)]^{-1} \mathcal{L} \text{ and } z = \mathcal{W}u. \]
Now, we define a operator \(\mathcal{H}\) such that \(\mathcal{H}u = Ku + K\mathcal{N}\mathcal{W}u\) and also establishing some essential property of functional \(\mathcal{J}_\gamma\), which arising form the derivative of the \(\mathcal{J}_\gamma\), which is of the form
\[ \mathcal{J}_\gamma(u) = \frac{1}{2} \|u\|_Y^2 + \frac{\gamma}{2} \|\mathcal{H}u - \varphi\|_X^2, \]
where \(\varphi\) is a fixed element of \(X\).
Remark 4.2. The operator $\mathcal{H}$ is sum of complete nonlinear continuous operator $\mathcal{W}$ and linear continuous operator $\mathcal{K}$, hence, $\mathcal{H}$ is weakly continuous.

Problem 4.1. Find $u \in Y$ such that
\[
\mathcal{J}_\gamma(u) = \inf_{u_1 \in Y} \left[ \mathcal{J}_\gamma(u_1) = \frac{1}{2} \| u_1 \|^2_Y + \frac{\gamma}{2} \| \mathcal{H} u_1 - \varphi \|^2_X \right].
\] (19)

Lemma 4.6. The functional $\mathcal{J}_\gamma$ has a critical point, which is given by the solution of the following equation
\[
u + \gamma(\mathcal{H}'(u))^* (\mathcal{H} u - \varphi) = 0,
\] (20)
where $\mathcal{H}'(u) = \mathcal{K} + \mathcal{K} \mathcal{N}'(z) \mathcal{W}'(u)$, $(\mathcal{H}'(u))^*$ is the adjoint of $\mathcal{H}'(u)$ and $z = \mathcal{W} u$.

Proof. If $\mathcal{J}(u) = \frac{1}{2} \| u \|^2_Y$, then we obtain $\mathcal{J}'(u) = u$.

Let
\[
g(u) = \frac{1}{2} \| \mathcal{H} u - \varphi \|^2 = \frac{1}{2} (\mathcal{H} u - \varphi, \mathcal{H} u - \varphi).
\]
By using the chain rule, we get
\[
g'(u) h = \langle \mathcal{H} u - \varphi, \mathcal{H}'(u) h \rangle = \langle (\mathcal{H}'(u))^* (\mathcal{H} u - \varphi), h \rangle.
\]

We obtain
\[
g'(u) = (\mathcal{H}'(u))^* (\mathcal{H} u - \varphi).
\]
Hence, we have
\[
\mathcal{J}'_\gamma(u) = u + \gamma (\mathcal{H}'(u))^* (\mathcal{H} u - \varphi).
\]
If the functional $\mathcal{J}_\gamma$ has a critical point $u$, then
\[
u + \gamma (\mathcal{H}'(u))^* (\mathcal{H} u - \varphi) = 0.
\]
This concludes the proof.

Next, we assume that the functional $\mathcal{J}_\gamma$ has a unique minimizer, which is critical point of the $\mathcal{J}_\gamma$. Now, we denote $(\mathcal{H}'(u))^*$ by $\mathcal{C}(u)$
\[
u + \gamma \mathcal{C}(u) (\mathcal{H} u - \varphi) = 0.
\]
From definition of $\mathcal{H}'(u)$, we obtain that $(\mathcal{H}'(u))^*$ is a linear and bounded operator for each $u \in Y$. The operator $\mathcal{C}(u)$ satisfies the following lemma.

Lemma 4.7. If $u_n$ converges weakly to $u$, then $\mathcal{C}(u_n) x$ converges to $\mathcal{C}(u) x$ and $\mathcal{C}^*(u_n) x$ converges to $\mathcal{C}^*(u) x$ for all $x \in Y$.

Proof. From definition of $\mathcal{C}(u)$, we have
\[
\mathcal{C}(u) x = \mathcal{K}^* x + [\mathcal{W}'(u)]^* [\mathcal{N}'(z)]^* \mathcal{K}^* x \quad \text{and} \quad \mathcal{C}(u_n) x = \mathcal{K}^* x + [\mathcal{W}'(u_n)]^* [\mathcal{N}'(z_n)]^* \mathcal{K}^* x,
\]
where $z = \mathcal{W} u$ and $z_n = \mathcal{W} u_n$. If $u_n$ converges weakly to $u$ and $\mathcal{W}$ is complete continuity this implies that $z_n \rightarrow z$. Hence, $z_n(t) \rightarrow z(t)$ for all $t$ almost every in $X$. By using the Lebesgue convergence theorem, we can conclude that
\[
\mathcal{N}'(z_n) q \rightarrow \mathcal{N}'(z) q \quad \text{for all} \quad q \in Y
\]
and hence, we have
\[
[\mathcal{N}'(z_n)]^* q \rightarrow [\mathcal{N}'(z)]^* q \quad \text{for all} \quad q \in Y.
\]
Note that
\[
[\mathcal{N}'(z_n)]^* \mathcal{K}^* x = y_n \quad \text{and} \quad [\mathcal{N}'(z)]^* \mathcal{K}^* x = y.
\]
Hence,

\[ y_n \to y \text{ in } Y. \]

Now, we define

\[ ([I - \mathcal{L}\mathcal{N}(z)]^{-1})y = z \text{ and } ([I - \mathcal{L}\mathcal{N}(z_n)]^{-1})y_n = z_n. \]

Therefore, we can write

\[ y = [I - \mathcal{L}\mathcal{N}(z)]^* z \text{ and } y_n = [I - \mathcal{L}\mathcal{N}(z_n)]^* z_n. \]

In integral form,

\[ z(t) = y(t) + [\mathcal{N}(z)]^* M_2 \int_t^b (s-t)^\eta-1 z(s)ds, \]

and

\[ z_n(t) = y_n(t) + [\mathcal{N}(z_n)]^* M_2 \int_t^b (s-t)^\eta-1 z_n(s)ds. \]

Taking difference, we obtain

\[
\|z_n(t) - z(t)\|_X \leq \|y_n(t) - y(t)\|_X + \|\mathcal{N}(z_n)^* \omega(t) - \mathcal{N}(z)^* \omega(t)\|_X \\
+ M_2 \|\mathcal{N}(z_n)^* \| \int_t^b (s-t)^\eta-1 \|z_n(s) - z(s)\|_X ds,
\]

where \( \omega(t) = \int_t^b (s-t)^\eta-1 z(s)ds \). By generalized Gronwall inequality, we obtain

\[
\|z_n(t) - z(t)\|_X \leq \int_0^b (\|y_n(s) - y(s)\|_X + \|\mathcal{N}(z_n)^* \omega(s) - \mathcal{N}(z)^* \omega(s)\|_X)ds \\
\times E_n(M_2 \|\mathcal{N}(z_n)^* \| \Gamma(\eta)b^\eta).
\]

Using Hölder’s inequality and \( c_1 = E_n(M_2 \|\mathcal{N}(z_n)^* \| \Gamma(\eta)b^\eta) \) is bounded by \( \sigma \), we obtain

\[
\|z_n - z\|_Y \leq \sigma_1 (\|y_n - y\|_Y + \|\mathcal{N}(z_n)^* \omega - \mathcal{N}(z)^* \omega\|_Y).
\]

Here \( \sigma_1 \) depends on parameters \( b \) and \( \sigma \). Since \( y_n \to y \) and \( \mathcal{N}(z_n)^* \omega \to \mathcal{N}(z)^* \omega \) in the space \( Y \), then we obtain

\[ z_n \to z \text{ in } Y. \]

Hence, we obtain

\[ \mathcal{W}(z_n)^* y_n = ([I - \mathcal{L}\mathcal{N}(z_n)]^{-1})^* y_n \to ([I - \mathcal{L}\mathcal{N}(z)]^{-1})^* y = \mathcal{W}(z)^* y. \]

Therefore, we obtain

\[ \mathcal{C}(u_n)x \to \mathcal{C}(u)x, \forall x \in Y. \]

Similarly, we have

\[ \mathcal{C}^*(u_n)x \to \mathcal{C}^*(u)x, \forall x \in Y, \]

and the lemma is proved.
5. Approximation theorems. In this section, we give the approximation theorems for the fractional system. Consider a family \( \{X_m\} \), which contain the finite dimensional subspace of space \( X \) such that

\[
X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_{m-1} \subset X_m \subset \cdots \subset X \text{ with } \bigcup_{m=1}^{\infty} X_m = X.
\]

Let \( X_m = \text{span}\{\mu_1, \mu_2, \ldots, \mu_m\} \), where \( \{\mu_j\}_{j=1}^{\infty} \) is the set of orthogonal eigenfunction corresponding to eigenvalues \( \{\lambda_j\}_{j=1}^{\infty} \) of \( A \) and assume that \( \{\mu_j\}_{j=1}^{\infty} \) is basis of space \( X \). Also, consider a family \( Y_m = L^2[0,b; X_m] \) of subspace of the space \( Y = L^2[0,b; X] \) such that

\[
Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_{m-1} \subset Y_m \subset \cdots \subset Y \text{ with } \bigcup_{m=1}^{\infty} Y_m = Y.
\]

Define projections \( P_m : X \to X_m \) and \( \tilde{P}_m : Y \to Y_m \), respectively as

\[
P_m[z(t)] = \sum_{j=1}^{m} \rho_j \mu_j \text{ and } \tilde{P}_m z = P_m z(t),
\]

then both projections \( P_m \) and \( \tilde{P}_m \) generates the approximating operators \( C_m \) and \( \mathcal{H}_m \), which is defined by

\[
C_m(u) = \tilde{P}_m C(u) \text{ and } \mathcal{H}_m(u) = P_m \mathcal{H}(u).
\]

Now, we can restate our minimization problem as follows

**Problem 5.1.** Find \( u_m \in Y_m \) such that

\[
\mathcal{J}_{\gamma,m}(u_m) = \inf_{u \in Y_m} \left[ \mathcal{J}_{\gamma,m}(u) = \frac{1}{2} \| \tilde{P}_m u \|^2_{Y_m} + \frac{\gamma}{2} \| P_m \mathcal{H} u - P_m \varphi \|^2_{X_m} \right]. \tag{21}
\]

The minimization problem (21) has a solution \( u_m \), which is critical point of functional \( \mathcal{J}_{\gamma,m} \), hence it satisfies the following equation in \( Y_m \)

\[
u_m + \gamma C_m(u_m) \mathcal{H}_m u_m = \gamma C_m(u_m) \varphi_m,
\]

where \( \varphi_m = P_m \varphi \).

Using the above scheme, the fractional system (3) is approximated by system which is of the form

\[
\begin{align*}
\frac{\partial^n \rho_j}{\partial t^n} &= \lambda_j \rho_j + \tilde{F}_j \rho_j + \xi_j, \quad 1 \leq j \leq m, \\
\rho_j(0) &= \rho_{j_0},
\end{align*}
\]  

(23)

where \( A \mu_j = \lambda_j \mu_j \), \( u(t) = \sum_{j=1}^{\infty} \xi_j \mu_j \), \( \tilde{F}_j = \langle F(\sum_{i=1}^{\infty} \xi_i \mu_i), \mu_j \rangle \), \( z(t) = \sum_{j=1}^{\infty} \rho_j \mu_j \) and \( \rho_{j_0} = (z_0, \mu_j) \). Assume that \( z_m = \sum_{j=1}^{m} \rho_j \mu_j \) solves the fractional system (23) corresponding to each \( \xi_j \), \( 1 \leq j \leq m \) and \( u_m = \sum_{j=1}^{m} \xi_j \mu_j \) solves the equation (21) and \( \tilde{F}_j = \langle F(\sum_{i=1}^{m} \xi_i \mu_i), \mu_j \rangle \).

**Definition 5.1.** Assume that \( u_m = \sum_{j=1}^{m} \xi_j \mu_j \) solves the equation (21). Suppose \( z_m = \sum_{j=1}^{m} \rho_j \mu_j \), where \( \rho_j \), \( 1 \leq j \leq m \), solves the fractional system (23) with each \( \xi_j \), \( 1 \leq j \leq m \). For every \( m \), we have \((u_m, z_m)\) as approximating optimal pair of the problem (21).

**Theorem 5.2.** Assume that the hypotheses \([H_5] \sim [H_6]\) are hold with \( U_{ex} \neq \emptyset \). Let the problem (21) has a solution \( u_m \), then \( u_m \) is uniformly bounded in the space \( Y_m \). Suppose that the functional \( \mathcal{J}_\gamma \) has a unique minimizer in the space \( Y \), which is unique critical point of functional \( \mathcal{J}_\gamma \), then the problem (21) has an optimal pair.
\((u_m, z_m)\), which converges to \((\bar{u}, \bar{z})\) with \(u_m\) converges weakly to \(\bar{u}\) and \(z_m \to \bar{z}\), where \((\bar{u}, \bar{z})\) is optimal pair of the problem (19).

**Proof.** Let \(\bar{u} \in U_{ex}\), then we have \(H\bar{u} = \varphi\), this implies 
\[P_m H\bar{u} = P_m \varphi.\]

Now, we define \(u_m = \tilde{P}_m\bar{u}\), then
\[
\frac{1}{2} \|u_m\|^2 \leq J_{\gamma,m}(u_m) = \frac{1}{2} \|\tilde{P}_m\bar{u}\|^2 + \frac{\gamma}{2} \|P_m Hu_m - P_m \varphi\|^2 \leq \frac{1}{2} \|\tilde{P}_m\|^2 \|\bar{u}\|^2 + \frac{\gamma}{2} \|P_m\|^2 \|H u_m - H\bar{u}\|^2.
\]

Since \(H\) is continuous and \(u_m \to \bar{u}\), we get \(H u_m \to H\bar{u}\). Hence, the right side are bounded, therefore the sequence \(u_m\) is uniformly bounded, there exists a subsequence, relabeled as \(u_m\), which is converges weakly to \(\bar{u}\), then we have
\[u_m + \gamma C_m(u_m)\mathcal{H} u_m = \gamma C_m(u_m)\varphi_m.\]

**Step 1.** For all \(w \in X\), we have 
\[
\langle H u_m - H\bar{u}, w \rangle = \langle P_m H u_m - H\bar{u}, w \rangle = \langle H u_m - H\bar{u}, P_m w - w \rangle + \langle H\bar{u}, P_m w - w \rangle.
\]

Since \(H u = K u + KN\mathcal{W} u\),
\[
\langle H u_m - H\bar{u}, w \rangle = \langle K u_m + KN\mathcal{W} u_m - K\bar{u} - KN\mathcal{W}\bar{u}, P_m w \rangle \\
+ \langle H\bar{u}, P_m w - w \rangle \\
= \langle K u_m - K\bar{u}, P_m w - w \rangle + \langle KN\mathcal{W} u_m - KN\mathcal{W}\bar{u}, P_m w \rangle \\
+ \langle K u_m - K\bar{u}, w \rangle + \langle H\bar{u}, P_m w - w \rangle \\
\leq \|K u_m - K\bar{u}\| \|P_m w - w\| + \|K\mathcal{W} u_m - K\mathcal{W}\bar{u}\| \|P_m w\| \\
+ \|K u_m - K\bar{u}, w\| + \|H\bar{u}\| \|P_m w - w\|.
\]

Since \(K\) is continuous and \(u_m\) converges weakly to \(\bar{u}\), this implies that \(K u_m\) converges weakly to \(K\bar{u}\). Also, continuity of \(K\) and \(N\) along with the complete continuity of \(\mathcal{W}\) yields \(K\mathcal{W} u_m \to K\mathcal{W}\bar{u}\), \(P_m w \to w\), hence it is bounded, implies that
\[
\langle H u_m - H\bar{u}, w \rangle \leq \|K u_m - K\bar{u}\| \|P_m w - w\| + \|K\mathcal{W} u_m - K\mathcal{W}\bar{u}\| \|P_m w\| \\
+ \|K u_m - K\bar{u}, w\| + \|H\bar{u}\| \|P_m w - w\| \\
\to 0 \text{ as } m \to \infty,
\]
this implies that \(H u_m\) converges weakly to \(H\bar{u}\).

**Step 2.** Next, we show that \(C_m(u_m)\mathcal{H} u_m\) converges weakly to \(C(\bar{u})\mathcal{H}\bar{u}\). For \(w \in Y\), we arrive at
\[
\langle C_m(u_m)\mathcal{H} u_m - C(\bar{u})\mathcal{H}\bar{u}, w \rangle \\
= \langle C(u_m)\mathcal{H} u_m - C(\bar{u})\mathcal{H} u_m, \tilde{P}_m w \rangle \\
+ \langle C(u_m)\mathcal{H} u_m - C(\bar{u})\mathcal{H}\bar{u}, \tilde{P}_m w \rangle \\
+ \langle (\tilde{P}_m - I)C(\bar{u})\mathcal{H}\bar{u}, w \rangle \\
= \langle \mathcal{H} u_m, C^*(u_m) w - C^*(\bar{u}) w \rangle \\
+ \langle \mathcal{H} u_m - \mathcal{H}\bar{u}, C^*(\bar{u}) w \rangle \\
+ \langle \mathcal{H} u_m - \mathcal{H}\bar{u}, (C^*(u_m) - C^*(\bar{u}))(\tilde{P}_m w - w) \rangle \\
+ \langle C(\bar{u})\mathcal{H}\bar{u}, \tilde{P}_m w - w \rangle + \langle \mathcal{H} u_m - \mathcal{H}\bar{u}, C^*(\bar{u})(\tilde{P}_m w - w) \rangle.
\]
where \( \implies \) implies that \( J \).

As the functional \( C \) is completely continuous, this implies that \( C \) is uniform bounded and \( P_m \) converges weakly to \( H(\pi) \).

**Step 3.** We show that \( C_m(u_m) \varphi_m \to C(\pi) \varphi \) where \( w \in Y \).

\[
\langle C_m(u_m)H_mu_m - C(\pi)H\pi, w \rangle
\leq \|H_mu_m\|\|C^*(u_m)w - C^*(\pi)w\| + \langle H_mu_m - H\pi, C^*(\pi)w \rangle
+ \|H_mu_m\|\|C^*(u_m) - C^*(\pi)\|\|P_mw - w\| + \|C(\pi)\|\|H\pi\|
\times \|P_mw - w\| + \|H_mu_m - H\pi\|\|C^*(\pi)\|\|P_mw - w\|
\to 0 \text{ as } m \to \infty,
\]

this implies that \( C_m(u_m)H_mu_m \) converges weakly to \( C(\pi)H\pi \) for \( w \in Y \).

As \( C(u_m) \) is uniform bounded and \( P_m \varphi \to \varphi \), \( C(u_m) \varphi \to C(\pi) \varphi \).

\[
\langle C_m(u_m)\varphi_m - C(\pi)\varphi, w \rangle
= \langle P_mC(u_m)P_m\varphi - C(\pi)\varphi, w \rangle
= \langle C(u_m)(P_m\varphi - \varphi), P_mw \rangle + \langle C(u_m)\varphi - C(\pi)\varphi, P_mw \rangle
+ \langle C(\pi)\varphi, (P_m - I)w \rangle
\leq \|C(u_m)\|\|P_m\varphi - \varphi\|\|P_mw\|
+ \|C(u_m)\varphi - C(\pi)\varphi\|\|P_mw\| + \|C(\pi)\|\|\varphi\|\|P_mw - w\|
\to 0 \text{ as } m \to \infty,
\]

this implies that weak continuity of \( C_m(u_m)\varphi_m \) to \( C(\pi)\varphi \).

Since \( C_m(u_m)\varphi_m \) converges weakly to \( C(\pi)\varphi \), \( C_m(u_m)H_mu_m \) converges weakly to \( C(\pi)H\pi \) and \( u_m \) converges weakly to \( \pi \), it follows that

\[
\pi + \gamma C(\pi)H\pi = C(\pi)\varphi.
\]

As the functional \( J_\gamma \) has a unique minimizer, hence \( u_m \) converges to \( \pi \), i.e. \( \pi \) is the critical point as well as unique minimizer of \( J_\gamma \).

Since \( W \) is completely continuous, \( z_m = W u_m \) and \( u_m \) converges weakly to \( \pi \), this implies that

\[
z_m \to \pi,
\]

where \( \pi = W\pi \). Hence proved.

Now, in the next step, we discretize the \( t \) and get the finite-dimensional subspace \( Y_k^m \) of each fixed \( Y_m \), which is of the form

\[
Y_k^m = \{ z^k_m \in P : z^k_m[t_{\nu-1}, t_\nu] = z^\nu_m, t_0 = 0, t_k = b, t_\nu = \nu \theta, \theta = b/k, 1 \leq \nu \leq k \}.
\]

Here, \( P \) is the space of polynomials, which is piecewise constant. \( Y_k^m \) satisfies the following property

\[
Y_1^m \subset Y_2^m \subset \cdots \subset Y_k^m \subset \cdots \subset Y_m \text{ with } \bigcup_{k=1}^\infty Y_k^m = Y_m.
\]
Define the orthogonal projection
\[ Q^k_m : Y_m \to Y^k_m, \]
this induced the operators
\[ \mathcal{C}^k_m(u) = Q^k_m \mathcal{C}_m(u) \text{ and } \mathcal{H}^k_m(u) = Q^k_m \mathcal{H}_m(u). \]
Now, the approximation scheme of the Caputo derivative (4) is given by the following expression
\[ \frac{\partial^\nu \rho^\nu_j}{\partial t^\nu} = c D^\nu_j \rho^\nu_j + O(\theta), \] (24)
with
\[ c D^\nu_j \rho^\nu_j \cong \sigma_{\eta, \theta} \sum_{p=1}^\nu \omega^\eta_p (\rho^\nu_j - p^\nu_j - p^\nu_p), \] (25)
where
\[ \sigma_{\eta, \theta} = \frac{1}{\Gamma(1-\eta)(1-\eta)\theta^\nu} \text{ and } \omega^\eta_p = p^{1-\eta} - (p-1)^{1-\eta}. \] (26)
For more details on approximation scheme for the Caputo derivative see [22, 28].

Now, C-N FDM with the formula (25) is used to estimate the Caputo derivative to solve the system (23). Using the equation (25) the restriction of the exact solution to the grid points centered at \( (z_j, t_\nu) = (jh, \nu \theta)(\text{where } h = 1/m), \) in equation (23), satisfies for \( j = 1, 2, \ldots, m \)
\[ \sigma_{\eta, \theta} \sum_{p=1}^\nu \omega^\eta_p (\rho^\nu_j - p^\nu_j - p^\nu_p) + O(\theta) = \frac{\lambda_j}{2} (\rho^\nu_j - p^\nu_j - p^\nu_p) + \frac{1}{2} \{ \tilde{\varphi}_j^\nu + \tilde{\varphi}_j^{\nu-1} \} + \frac{1}{2} \{ \tilde{\varphi}_j^\nu + \tilde{\varphi}_j^{\nu-1} \}, \]
\[ \sigma_{\eta, \theta} \omega^\eta_1 (\rho^\nu_j - p^\nu_j - p^\nu_1) = - \sum_{p=2}^\nu \omega^\eta_p (\rho^\nu_j - p^\nu_j - p^\nu_p) + \frac{\lambda_j}{2} (\rho^\nu_j - p^\nu_j - p^\nu_1) \]
\[ + \frac{1}{2} (\tilde{\varphi}_j^\nu + \tilde{\varphi}_j^{\nu-1}) + \frac{1}{2} (\tilde{\varphi}_j^\nu + \tilde{\varphi}_j^{\nu-1}). \] (27)
with boundary conditions \( \rho_j(0) = \rho_{j0} = (z_0, \mu_j), \) where \( 1 \leq j \leq m, \) \( 1 \leq \nu \leq k \) and \( \rho^\nu_j = \rho_j(\nu \theta), \) \( \xi^\nu_j = \xi_j(\nu \theta). \) For a given \( \xi^\nu_j, \) iteratively defines \( \rho^\nu_j \) and hence, for \( 1 \leq j \leq m, \) the approximation \( \rho^\nu_j(t) \) to \( \rho_j(t) \) is known. This gives a approximation \( z_m(t) = \sum_{j=1}^m \rho^\nu_j \mu_j \) to \( z_m(t) = \sum_{j=1}^m \rho_j \mu_j, \) similarly, we obtain the approximation \( u^k_m \) to \( u_m. \)

**Problem 5.2.** Find \( u^k_m \in Y^k_m \) such that
\[ J^k_{\gamma, m}(u^k_m) = \inf_{u_m \in Y^k_m} \left[ J^k_{\gamma, m}(u_m) = \frac{1}{2} \| Q^k_m u_m \|^2_{Y^k_m} + \frac{\gamma}{2} \| Q^k_m \mathcal{H}_m u_m - \varphi_m \|^2_{X^k_m} \right] \] (28)
The minimization problem (28) has a solution \( u^k_m, \) which is critical point of \( J^k_{\gamma, m}, \) hence it satisfies the following operator equation in the space \( Y^k_m \)
\[ u^k_m + \gamma C^k_m(u^k_m) \mathcal{H}_m u^k_m = \gamma C^k_m(u^k_m) \varphi_m. \] (29)
Definition 5.3. Let \( u_k^m = \sum_{j=1}^{m} \xi_j^k \mu_j \) be the solution of the problem (28) and let \( z_m^k = \sum_{j=1}^{m} \rho_j^k \mu_j \), where \( \rho_j \) solves the equation (27) corresponding to \( \xi_j \), \( 1 \leq j \leq m \), \( 1 \leq \nu \leq k \), \( \rho_j^{\nu} = \rho_j(\nu \theta) \), \( \xi_j^{\nu} = \xi_j(\nu \theta) \). For each \( m \), \( (u_k^m, z_k^m) \) is the approximating pair of the problem (28) in the space \( Y_k^m \).

Now, we can apply the same methods as in the above Theorem 5.2 to obtain the following result. So we omit details of the proof. Below, we now state one of the important theorems of this section.

Theorem 5.4. Let the problem (28) has a solution \( u_k^m \). Assume that the assumptions \([H_5] - [H_6] \) hold. Then approximating pair \( (u_k^m, z_k^m) \) converges to \( (u_m, z_m) \) in the space \( Y_m \) with \( u_k^m \) converges weakly to \( u_m \) and \( z_k^m \) converges to \( z_m \) in the space \( Y_m \).

6. Example. We take the following problem

\[
\begin{aligned}
\frac{\partial^2 z(t, \epsilon)}{\partial t^2} &= \frac{\partial^2 z(t, \epsilon)}{\partial \epsilon^2} + F(t, z(t, \epsilon)) + u(t, \epsilon), \quad \epsilon \in (0, \pi), \quad t \in (0, 3), \\
\frac{\partial z(t, 0)}{\partial \epsilon} &= 0 = z(t, \pi), \quad t \in [0, 3], \\
z(0, \epsilon) &= z_0, \quad \epsilon \in [0, \pi],
\end{aligned}
\]

where \( F(t, z(t, \epsilon)) = t + z(t, \epsilon) \), \( \eta = 0.75 \), \( b = 3 \).

\[\text{Figure 1.} \quad \text{Comparison between } z_b \text{ and } z(b)\]

Let \( X = L^2[0, \pi] \) and define the operator \( A : D(A) \subset X \to X \) by \( A \omega = \omega'' \) with the domain \( D(A) = \{ \omega \in X : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in X, \omega(0) = 0 = \omega(\pi) \} \), then

\[
A \omega = \sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in D(A),
\]

where \( \omega_n(\epsilon) = \sqrt{2/\pi} \sin(n \epsilon), \quad n = 1, 2, \ldots \), is the orthogonal set of eigenvectors of \( A \). It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( \{S(t)\}_{t \geq 0} \) in \( X \) and is given by

\[
S(t) \omega = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \omega, \omega_n \rangle \omega_n.
\]
From these expressions, it follows that $\{S(t)\}_{t \geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda_2, A) = (\lambda_2 - A)^{-1}$ is a compact operator for all $\lambda_2 \in \rho(A)$, i.e., $A \in \mathcal{A}^0(\theta_0, \lambda_0)$. By [27], we can see $M_1 = M_2 = 1$.

The function $F(t, z(t, \epsilon)) = t + z(t, \epsilon)$ is Lipschitz continuous with Lipschitz constant $\alpha = 1$. Set $m = 34$, $z_0 = 0$ and $z_b = 3 \sin(\epsilon)$. Moreover, $z_b \in \mathcal{R}(3, 0)$ and $z(t, \epsilon) = t \sin(\epsilon)$ is an exact solution of the fractional system (30) with control function $u(t, \epsilon) = [4^{1/4}/\Gamma(0.25)] \sin(\epsilon) - t$ and $z(3, \epsilon) = 3 \sin(\epsilon)$. We obtain $\mu = 2\alpha = 2$ and

$$\delta_{34} = \left[ B\left(\eta, 0 + \frac{1}{2}\right) \cdot B\left(\eta, \eta + \frac{1}{2}\right) \cdots \cdot B\left(\eta, (34 - 1)\eta + \frac{1}{2}\right) \right] \frac{(M_2 \mu b^n)^{34}}{\sqrt{68\eta(2\eta - 1)}} = 0.5727 < 1.$$

By the Theorem 3.1, we obtain that the set $\mathcal{U}_{ex}$ is non-empty for the system (30).

Here, we choose $m, k$. Using Matrix Optimization Algorithm (MOA) (see, [12]), we
computed $u^k_m$, $z^k_m$ by applying the fractional Crank-Nicolson scheme (25). In Fig. 1., we plot the graph of the approximate state at time $b = 3$ with the given final state $z_b = 3 \sin(\epsilon)$. The surface of the approximated optimal control $\bar{u}$ is given in Fig. 2 and Fig. 3 which shows the surface of the numerical solution $\tau$ corresponding to optimal control $\bar{u}$ with $\eta = 0.75$.

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