CHARACTERIZING CONTINUITY BY PRESERVING COMPACTNESS AND CONNECTEDNESS

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Abstract. Let us call a function $f$ from a space $X$ into a space $Y$ preserving if the image of every compact subspace of $X$ is compact in $Y$ and the image of every connected subspace of $X$ is connected in $Y$. By elementary theorems a continuous function is always preserving. Evelyn R. McMillan [6] proved in 1970 that if $X$ is Hausdorff, locally connected and Fréchet, $Y$ is Hausdorff, then the converse is also true: any preserving function $f : X \to Y$ is continuous. The main result of this paper is that if $X$ is any product of connected linearly ordered spaces (e.g. if $X = \mathbb{R}^\kappa$) and $f : X \to Y$ is a preserving function into a regular space $Y$, then $f$ is continuous.

Let us call a function $f$ from a space $X$ into a space $Y$ preserving if the image of every compact subspace of $X$ is compact in $Y$ and the image of every connected subspace of $X$ is connected in $Y$. By elementary theorems a continuous function is always preserving. Quite a few authors noticed—mostly independently from each other—that the converse is also true for real functions: a preserving function $f : \mathbb{R} \to \mathbb{R}$ is continuous. (The first paper we know of is [9] from 1926!) Whyburn proved [13] that a preserving function from a space $X$ into a Hausdorff space is always continuous at a first countability and local connectivity point of $X$.

Evelyn R. McMillan [6] proved in 1970 that if $X$ is Hausdorff, locally connected and Fréchet, moreover $Y$ is Hausdorff, then any preserving function $f : X \to Y$ is continuous. This is quite a significant and deep result that is surprisingly little known.

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We shall use the notation $Pr(X,T_i)$ ($i = 1, 2, 3$ or $3\frac{1}{2}$) to denote the following statement: Every preserving function from the topological space $X$ into any $T_i$ space is continuous.

The organization of the paper is as follows: In §1 we give some basic definitions and then treat our results that are closely connected to McMillan’s theorem. §2 treats several important technical theorems that enable us to conclude that certain preserving functions are continuous. In §3 we prove that certain product spaces $X$ satisfy $Pr(X,T_3)$; in particular, any preserving function from a product of connected linearly ordered spaces into a regular space is continuous. In §4 we discuss some results concerning the continuity of preserving functions defined on compact or sequential spaces. Finally, §5 treats the relation $Pr(X,T_1)$.

Our terminology is standard. Undefined terms can be found in [2] or in [3].

1. Around McMillan’s theorem

Our first theorem implies that (at least among Tychonoff spaces) local connectivity of $X$ is a necessary condition for $Pr(X,T_3\frac{1}{2})$. For metric spaces this result was proved by Klee and Utz[5].

**Theorem 1.1.** If the Tychonoff space $X$ is not locally connected at a point $p \in X$, then there exists a preserving function $f$ from $X$ into the interval $[0,1]$ which is not continuous at $p$.

**Proof.** Suppose $X$ is not locally connected at the point $p$, then there is an open neighbourhood $U$ of $p$ such that if $K$ denotes the component of $p$ in $U$, then $K$ is not a neighbourhood of $p$. As $X$ is Tychonoff, there exists an open neighbourhood $V \subset U$ of $p$ and a continuous function $f: X \to [0,1]$ such that $f$ is identically $0$ on $V$ and identically $1$ outside of $U$. Select another continuous function $g: X \to [0,1]$ such that $g(p) = 1$ and $g$ is identically $0$ outside $V$. Put now

$$f(x) = \begin{cases} f(x) + g(x) & \text{if } x \in K; \\ f(x) & \text{otherwise.} \end{cases}$$

($f: X \to [0,1]$ because $f|V = \{0\}$ and $g|X - V = \{0\}$.) The function $f$ is not continuous at $p$ because $f(p) = 1$ and in every neighbourhood of $p$ there is a point from $V - K$ mapped to $0$.

On the other hand, we claim that $f$ is preserving. Indeed, let $C \subset X$ be compact. The restriction of $f$ to the closed set $F = (X-V) \cup K$ is evidently continuous (here $f = \overline{f} + g$), hence $C' = f(C \cap F)$ is compact. But $f$ is $0$ on $V - K$ and so $f(C)$ is either $C'$ or $C' \cup \{0\}$, thus $f(C)$ is clearly compact.

Let now $C$ be any connected subset of $X$. The function $f$ is continuous on $X - K$ (here $f = \overline{f}$), hence $C \cap K \neq \emptyset$ can be assumed. Similarly, $f$ is continuous on $X - V$ ($f(x) = \overline{f}(x)$ for $x \in X - V$), hence we can suppose that $C$ also meets $V$. If $C \subset U$, then necessarily $C \subset K$ because $C$ is
connected and meets the component $K$ of $U$. As $f$ is continuous on $K$, the
image $f(C)$ is then connected.

Thus it remains to check only the case in which $C$ meets both $V$ and
$X - U$. In this case $\overline{f}(C)$ is the whole interval $[0, 1]$. But $f$ and $\overline{f}$ are equal
at each point where $\overline{f}$ does not vanish, thus $f(C)$ contains the interval $[0, 1]$
and so it is connected. □

It is not a coincidence that the target space in Theorem 1.1 is the interval
$[0, 1]$, because of the following result:

**Lemma 1.2.** Suppose $f : X \to Y$ is a preserving function into a Tychonoff
space $Y$ and $f$ is not continuous at the point $p \in X$. Then there exists a
preserving function $h : X \to [0, 1]$ which is also not continuous at $p$.

*Proof.* Since $f$ is not continuous at $p$, there exists a closed set $F \subset Y$
such that $f(p) \notin F$ but $p$ is an accumulation point of $f^{-1}(F)$. Choose a
continuous function $g : Y \to [0, 1]$ such that $g(f(p)) = 0$ and $g$ is identically 1
on $F$. Then the composite function $h(x) = f(g(x))$ has the stated properties. □

The following Lemmas will be often used in the sequel. They all state
simple properties of preserving functions.

**Lemma 1.3.** If $f : X \to Y$ is a compactness preserving function, $Y$
is Hausdorff, $M \subset X$ with $\overline{M}$ compact and $f(M)$ infinite then for every accu-
imation point $x$ of $f(M)$ there is an accumulation point $x$ of $M$ such that
$f(x) = y$, i.e. $f(M)' \subset f(M')$.

*Proof.* Let $N = M - f^{-1}(y)$ then $f(N) = f(M) - \{y\}$ and so we have
$y \in \overline{f(N)} - f(N)$. But $f(N)$ is also compact, hence closed in $Y$ and so
$y \in \overline{f(N)} - f(N)$ as well. Thus there is an $x \in \overline{N} - N$ such that $f(x) = y$
and then $x$ is as required. □

We shall often use the following immediate consequence of this lemma:

**Lemma 1.3’ (E. R. McMillan [3]).** If $f : X \to Y$ is a compactness preserving
function, $Y$ is Hausdorff, $\{x_n : n < \omega\} \subset X$ converges to $x \in X$ then either
$\{f(x_n) : n < \omega\}$, converges to $f(x)$ or the set $\{f(x_n) : n < \omega\}$ is finite □

Actually, to prove Lemma 1.3’ we do not need the full force of the assu-
ption that $f$ is compactness preserving. It suffices to assume that the
image of a convergent sequence together with its limit is compact, in other
words: the image of a topological copy of $\omega + 1$ is compact. For almost all of
our results given below only this very restricted special case of compactness
preservation is needed.

**Lemma 1.4** [3]. If $f : X \to Y$ preserves connectedness, $Y$ is a $T_1$-space
and $C \subset X$ is a connected set, then $f(C) \subset f(C)$.

*Proof.* If $x \in C$ then $C \cup \{x\}$ is connected. Thus $f(C \cup \{x\}) = f(C) \cup \{f(x)\}$
is also connected and hence $f(x) \in f(C)$. □
The next lemma will also play a crucial role in some theorems of the paper. A weaker form of it appears in [1].

**Definition 1.5.** We shall say that \( f : X \to Y \) is **locally constant** at the point \( x \in X \) if there is a neighbourhood \( U \) of \( x \) such that \( f \) is constant on \( U \).

**Lemma 1.6.** Let \( f \) be a connectivity preserving function from a locally connected space \( X \) into a \( T_1 \)-space \( Y \). If \( F \subset Y \) is closed and \( p \in f^{-1}(F) - f^{-1}(F) \) then \( p \) is also in the closure of the set \( \{ x \in f^{-1}(F) : f \text{ is not locally constant at } x \} \).

**Proof.** Let \( G \) be a connected open neighbourhood of \( p \) and \( C \) be a component of the non-empty subspace \( G \cap f^{-1}(F) \). Then \( C \) has a boundary point \( x \) in the connected subspace \( G \) because \( \emptyset \neq C \neq G \). Clearly, \( f(x) \in F \) by Lemma 1.4. If \( V \subset G \) is any connected neighbourhood of \( x \) then \( V \cup C \) is connected and \( V - C \neq \emptyset \) because \( x \) is a boundary point of \( C \) hence \( V \) is not contained in \( f^{-1}(F) \), so \( f \) is not locally constant at \( x \). \( \square \)

**Lemma 1.7.** Let \( f : X \to Y \) be a connectivity preserving function into the \( T_1 \)-space \( Y \). Suppose that \( X \) is locally connected at the point \( p \in X \) and \( f \) is not locally constant at \( p \). Then \( f(U) \cap V \) is infinite for every neighbourhood \( U \) of \( p \) and for every neighbourhood \( V \) of \( f(p) \).

**Proof.** Choose any connected neighbourhood \( U \) of \( x \); then \( f(U) \) is connected and has at least two points. Thus if \( V \) is any open subset of \( Y \) containing \( f(p) \) then \( f(U) \cap V \) can not be finite because otherwise \( f(p) \) would be an isolated point of the non-singleton connected set \( f(U) \). \( \square \)

The following proof of McMillan’s theorem is based upon the same ideas as her original proof, although, we think, it is much simpler. We include it here mainly to make the paper self-contained.

**Theorem 1.8** (E. R. McMillan [1]). If \( X \) is a locally connected and Fréchet Hausdorff space, then \( Pr(X, T_2) \) holds.

**Proof.** Assume \( Y = T_2 \) and \( f : X \to Y \) is preserving but not continuous at the point \( p \in X \). Then by Lemma 1.3 there is a sequence \( x_n \to p \) such that \( f(x_n) = y \neq f(p) \) for all \( n < \omega \). Using Lemma 1.6 with \( F = \{ y \} \) we can also assume that \( f \) is not locally constant at the points \( x_n \).

As \( Y = T_2 \), there is an open set \( V \subset Y \) such that \( y \in V \) but \( f(p) \notin V \). By Lemma 1.7 the image of every neighbourhood of each point \( x_n \) contains infinitely many points (different from \( y \)) from \( V \).

Now we select recursively sequences \( \{ x_n^m : k < \omega \} \) converging to \( x_n \) for all \( n < \omega \). Suppose \( n < \omega \) and the points \( x_k^m \) are already defined for \( m < n \) and \( k < \omega \) so that \( f(x_k^m) \neq y \). Then \( x_n \) is in the closure of the set

\[
f^{-1}(V - (\{ f(x_k^m) : m, k < n \} \cup \{ y \})) \]

hence, as $X$ is Fréchet, the new sequence $\{x^n_k : k < \omega\}$ converging to $x_n$ can be chosen from this set.

Since the sequence $\{x^n_k : k < \omega\}$ converges to $x_n$ and $\{x_n : n < \omega\}$ converges to the (Fréchet) point $p$, there is also a sequence $\{x^{n_l}_k : l < \omega\}$ converging to $p$. But then the sequence $\{n_l : l < \omega\}$ must tend to infinity so, by passing to a subsequence if necessary, we can assume that $n_{l+1} > \max(n_l, k_l)$ for all $l < \omega$. However, the sequence $\{f(x^{n_l}_k) : l < \omega\}$ does not converge to $f(p)$ because $f(p) \notin \{f(x^{n_l}_k) : l < \omega\} \subset \nabla$, while the points $f(x^{n_l}_k)$ are all distinct, contradicting Lemma 1.3. □

We could prove the following local version of McMillan’s theorem:

**Theorem 1.9.** If $X$ is a locally connected Hausdorff space, $p$ is a Fréchet point of $X$ and $f$ is a preserving function from $X$ into a Tychonoff space $Y$, then $f$ is continuous at $p$.

**Proof.** By Lemma 1.2 it suffices to prove this in the case when $Y$ is the interval $[0,1]$. Assume, indirectly, that $f$ is not continuous at $p$ then, since $p$ is a Fréchet point and by Lemma 1.4, we can choose a sequence $x_n \rightarrow p$ and a $y \in [0,1]$ with $y \neq f(p)$ such that $f(x_n) = y$ and $f$ is not locally constant at $x_n$ for all $n < \omega$.

For each $n$ choose a neighbourhood $U_n$ of $x_n$ with $p \notin \overline{U}_n$ and put $A_n = \{x \in U_n : 0 < |f(x) - y| < 1/n\}$. For any connected neighbourhood $W$ of $x_n$ its image $f(W)$ is a non-singleton interval containing $y$, hence the local connectivity of $X$ implies that $x_n \in \overline{A}_n$ for all $n < \omega$ and so $p$ belongs to the closure of $\bigcup \{A_n : n < \omega\}$. As $p$ is a Fréchet point, there is a sequence $z_k \in A_{n_k}$ converging to $p$. But $n_k$ necessarily tends to infinity because $p \notin \overline{A}_n \subset \overline{U}_n$ for each $n < \omega$, hence $f(z_k) \rightarrow y \neq f(p)$, contradicting Lemma 1.3. □

Indeed, the set $\{f(z_k) : k < \omega\}$ is infinite because $f(z_k) \neq y$.

Theorem 1.9 is not a full local version of Theorem 1.8 because local connectivity is assumed in it globally for $X$. This leads to the following natural question:

**Problem 1.10.** Let $X$ be a Hausdorff space, $f$ be a preserving function from $X$ into a Tychonoff space $Y$ and let $X$ be locally connected and Fréchet at the point $p \in X$. Is it true then that $f$ is continuous at $p$?

We do not know the answer to this problem, however we can prove some partial affirmative results.

**Definition 1.11 ([1]).** A point $x$ of a space $X$ is called an $(\alpha_4)$ point if for any sequence $\{A_n : n < \omega\}$ of countably infinite sets with $A_n \rightarrow x$ for each $n < \omega$ there is a countably infinite set $B \rightarrow x$ such that $\{n < \omega : A_n \cap B \neq \emptyset\}$ is infinite.

An $(\alpha_4)$ and Fréchet point will be called an $(\alpha_4)$-F point in $X$. 
Theorem 1.12. Let $f$ be a preserving function from a topological space $X$ into a Hausdorff space $Y$ and let $p$ be a point of local connectivity and an $(\alpha_4)$-F point in $X$. Then $f$ is continuous at $p$.

Proof. Assume not. Then by the Lemma 1.3 there is a point $y \in Y$ such that $y \neq f(p)$ but $p$ is in the closure of $f^{-1}(y)$. Choose an open neighbourhood $V$ of $y$ in $Y$ with $f(p) \notin V$. By Lemma 1.7 and Lemma 1.3 we can recursively choose pairwise distinct points $y_n \in V$ such that $p$ is in the closure of $f^{-1}(y_n)$ for all $n \in \omega$. As the point $p$ is an $(\alpha_4)$-F point in $X$, there is a “diagonal” sequence $\{x_m : m \in M\}$ converging to $p$, where $f(x_m) = y_m$ and $M \subset \omega$ is infinite, contradicting Lemma 1.3. □

Theorem 1.13. Let $f$ be a preserving function from a topological space $X$ into a Tychonoff space $Y$ and let $p$ be a Fréchet point of local connectivity of $X$ with character $\leq 2^\omega$. Then $f$ is continuous at $p$.

Proof. Assume not, $f$ is discontinuous at the point $p \in X$. By Lemmas 1.2 and 1.3 we can suppose that $Y = [0, 1]$, $f(p) = 0$ and every neighbourhood of $p$ is mapped onto the whole interval $[0, 1]$. Let $U$ be a neighbourhood-base of $p$ of size $\leq 2^\omega$ and choose for each $U \in U$ a point $x_U \in U$ such that $f(x_U) \in [1/2, 1]$ and the points $f(x_U)$ are all distinct. Put $A = \{x_U : U \in U\}$, then $p \in \overline{A}$ and so there exists a sequence $\{x_n : n < \omega\} \subset A$ converging to $p$, contradicting Lemma 1.3. □

There is a variant of this result in which the assumption that $Y$ be Tychonoff is relaxed to $T_3$, however the assumption on the character of the point $p$ is more stringent. Its proof will make use of the following (probably well-known) lemma:

Lemma 1.14. Let $Z$ be an infinite connected regular space, then any non-empty open subset $G$ of $Z$ is uncountable.

Proof. Choose a point $z \in G$ and an open proper subset $V$ of $G$ with $z \in V \subset \overline{V} \subset G$. If $G$ would be countable then, as a countable regular space, $G$ would be Tychonoff, and so there would be a continuous function $f : G \to [0, 1]$ such that $f(z) = 1$ and $f$ is identically zero on $G - V$. Extend $f$ to a function $\overline{f} : Z \to [0, 1]$ by putting $\overline{f}(y) = 0$ if $y \in Z - G$. Then $\overline{f}$ is continuous and hence $\overline{f}(Z)$ is also connected. Consequently we have $f(G) = \overline{f}(Z) = [0, 1]$ implying that $|G| \geq |[0, 1]| > \omega$, and so contradicting that $G$ is countable. □

Theorem 1.15. Let $f$ be a preserving function from a topological space $X$ into a regular space $Y$ and let $p \in X$ be a Fréchet point of local connectivity with character $\leq \omega_1$. Then $f$ is continuous at $p$.

Proof. Assume $f$ is discontinuous at the point $p \in X$. As $p$ is a a Fréchet point, there is a sequence $x_n \to p$ such that $f(x_n)$ does not converge to
f(p). Taking a subsequence if necessary, we can suppose by Lemma 1.3 that \( f(x_n) = y \neq f(p) \) for all \( n < \omega \).

Choose now an open neighbourhood \( V \) of the point \( y \in Y \) with \( f(p) \notin V \). Let \( U \) be a neighbourhood base of the point \( p \) in \( X \) such that \( |U| \leq \omega_1 \) and the elements of \( U \) are connected.

Choose now points \( x_U \) from the sets \( U \in U \) such that \( f(x_U) \in V \) and the points \( f(x_U) \) are all distinct. This can be accomplished by an easy transfinite recursion because for each \( U \in U \) the set \( f(U) \) is connected and infinite, hence \( f(U) \cap V \) is uncountable by the previous lemma. Put \( A = \{ x_U : U \in U \} \). Then \( p \in A \) and so there exists a sequence \( \{ y_n : n < \omega \} \subset A \) converging to \( p \), contradicting Lemma 1.3. \( \square \)

We shall now consider some further topological properties and prove several results about them saying that preserving functions are sequentially continuous. Since in a Fréchet point sequential continuity implies continuity, these results are clearly relevant to McMillan’s theorem. Their real significance, however, will only become clear in the following two sections.

**Definition 1.16.** A point \( x \) in a topological space \( X \) is called a sequentially connectible (in short: SC) point, if \( x_n \in X, x_n \to x \) implies that there are an infinite subsequence \( \langle x_{n_k} : k < \omega \rangle \) and a sequence \( \langle C_k : k < \omega \rangle \) consisting of connected subsets of \( X \) such that \( \{ x_{n_k} \} \subset C_k \) for all \( k < \omega \), (i.e. \( C_k \) “connects” \( x_{n_k} \) with \( x \), this explains the terminology,) moreover \( C_k \to x \), i.e. every neighbourhood of the point \( x \) contains all but finitely many \( C_k \)'s. A space \( X \) is called an SC space if all its points are SC points.

**Remark 1.17.** It is clear that the SC property is closely related to local connectivity. Let us say that a point \( x \) in space \( X \) is a strong local connectivity point if it has a neighbourhood base \( B \) such that the intersection of an arbitrary (non-empty) subfamily of \( B \) is connected. For example, local connectivity points of countable character or any point of a connected linearly ordered space has this property.

We claim that if \( x \) is a strong local connectivity point of \( X \) then \( x \) is an SC point in \( X \). Indeed, assume that \( x_n \to x \) and for every \( n \in \omega \) let \( C_n \) denote the intersection of all those members of \( B \) which contain both points \( x_n \) and \( x \). (As the sequence \( \{ x_n : n < \omega \} \) converges to \( x \), we can suppose that some element \( B_0 \in B \) contains all the \( x_n \)'s.) Then \( \{ x, x_n \} \subset C_n \), moreover \( C_n \to x \). Indeed, the latter holds because if \( x \in B \in B \) then, by definition, \( x_n \in B \) implies \( C_n \subset B \). \( \square \)

The SC property does not imply local connectivity. (If every convergent sequence is eventually constant then the space is trivially SC.) However, the following simple lemma shows that if there are “many” convergent sequences then such an implication is valid.

**Lemma 1.18.** Let \( x \) be a both Fréchet and SC point in a space \( X \). Then \( x \) is also a point of local connectivity in \( X \).
Proof. Let \( G \) be any open set containing \( x \) and set
\[
H = \bigcup \{ C : x \in C \subset G \text{ and } C \text{ is connected} \}.
\]
We claim that (the obviously connected) set \( H \) is a neighbourhood of \( x \).
Indeed, otherwise, as \( x \) is a Fréchet point, we could choose a sequence \( x_n \to x \) from the set \( G - H \) while for every point \( y \in G - H \) no connected set containing both \( x \) and \( y \) is a subset of \( G \), contradicting the SC property of \( x \). \)

If SC holds globally, i.e. in an SC space, then in the above result the Fréchet property can be replaced with a weaker property that will turn out to play a very important role in the sequel.

**Definition 1.19.** A point \( p \) in a topological space \( X \) is called an s point if for every family \( A \) of subsets of \( X \) such that \( p \in \bigcup A \) but \( p \notin A \) for all \( A \in A \) there is a sequence \( \langle (x_n, A_n) : n < \omega \rangle \) such that \( x_n \in A_n \in A \), the sets \( A_n \) are pairwise distinct and \( \{x_n\} \) converges to some point \( x \in X \) (that may be different from \( p \)).

A Fréchet point is evidently an s point, moreover any point that has a sequentially compact neighbourhood is also an s point. Other examples of s points will be seen later.

**Theorem 1.20.** Any s point in a \( T_3 \) and SC space is a point of local connectivity.

**Proof.** Let \( p \) be an s point in the regular SC space \( X \) and let \( G \) be an open neighbourhood of \( p \). We have to prove that the component \( K_0 \) of the point \( p \) in \( G \) is a neighbourhood of \( p \). Assume this is false and choose an open set \( U \) such that \( p \in U \subset \overline{U} \subset G \).

Put
\[
A = \{ K \cap \overline{U} : K \text{ is a component of } G, \ K \neq K_0 \}.
\]

Then \( p \in \bigcup A \) and \( p \notin A \) for \( A \in A \) (because a component of \( G \) is relatively closed in \( G \)), hence, by the definition of an s point, there exists a sequence \( \langle (x_n, A_n) : n < \omega \rangle \) such that \( x_n \in A_n \in A \), \( x_n \to x \) for some \( x \in X \) and if \( A_n = K_n \cap \overline{U} \) then the components \( K_n \) are distinct. Note that \( x \in \overline{U} \subset G \). As distinct components are disjoint, we can assume that \( x \notin K_n \) for all \( n < \omega \). As \( x \) is an SC point, there are a connected set \( C \) and some \( n < \omega \) such that \( \{x, x_n\} \subset C \subset G \). However, this is impossible, because then \( K_n \cup C \) would be a connected set in \( G \) larger then the component \( K_n \).

The significance of the SC property in our study of continuity properties of preserving functions is revealed by the following result.

**Theorem 1.21.** A preserving function \( f : X \to Y \) into a Hausdorff space \( Y \) is sequentially continuous at each SC point of \( X \).
Proof. Let \( x \in X \) be an SC point and assume that \( x_n \to x \) but \( f(x_n) \) does not converge to \( f(x) \) for a sequence \( \{x_n : n < \omega \} \) in \( X \). We can assume by Lemma \ref{lemma:1.3} that \( f(x_n) = y \neq f(x) \) for all \( n < \omega \). Choose an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( f(x) \notin V \).

As \( x \) is an SC point in \( X \), we can also assume that there is a sequence of connected sets \( C_n \) such that \( C_n \to x \) and \( x, x_n \in C_n \) for \( n < \omega \). We can now define a sequence \( z_n \in C_n \) such that \( f(z_n) \in V \) and the points \( f(z_n) \) are all distinct. Indeed, assume \( n < \omega \) and the points \( z_i \) are already defined for \( i < n \) in this way. As \( f(C_n) \) is connected and \( f(C_n) \cap V \) is a non-empty open proper subset, this intersection \( f(C_n) \cap V \) is not closed and hence is infinite. Consequently there exists a point \( z_n \in C_n \) with \( f(z_n) \in f(C_n) \cap V \) and \( \{f(z_i) : i < n\} \). But then the sequence \( \{z_n\} \) contradicts Lemma \ref{lemma:1.3}. \( \square \)

**Corollary 1.22.** Let \( f \) be a preserving function from a topological space \( X \) into a Hausdorff space \( Y \) and let \( p \) be both an SC point and a Fréchet-point in \( X \). Then \( f \) is continuous at \( p \). \( \square \)

The following example (due to E. R. McMillan \[6\]) yields a locally connected \( SC \) space with a discontinuous preserving function. (Compare this with Theorem \[1.21\].)

**Example 1.23.** Take \( \omega _1 \) copies of the interval \([0, 1]\) and identify the 0 points. We get in this way a “hedgehog” \( X = \{0\} \cup \{R_\xi : \xi \in \omega _1\} \), where the spikes \( R_\xi = (0, 1] \times \{\xi\} \) are disjoint copies of the half closed interval \((0, 1]\). A basic neighbourhood of a point \( x \in R_\xi \) is an open interval around \( x \) in \( R_\xi \). A basic neighbourhood of 0 is a set of the form \( \{0\} \cup \bigcup \{J_\xi : \xi < \omega _1\} \), where each \( J_\xi \) is a non-empty initial interval of \( R_\xi \) and \( J_\xi = R_\xi \) holds for all but countably many ordinals \( \xi \).

It is easy to see that in this way we get a locally connected Tychonoff \( SC \) space \( X \). The function \( f : X \to [0, 1] \) defined by \( f((x, \xi)) = x, f(0) = 0 \) is preserving but not continuous at the point 0 because every neighbourhood of 0 is mapped onto \([0, 1]\). \( \square \)

The next example is locally connected, hereditary Lindelof, \( T_\delta \), of countable tightness and with a preserving function given on it that is not even sequentially continuous. It follows that this space is not an \( SC \) space.

**Example 1.24.** The underlying set of our space \( X \) consists of a point \( p \), of a sequence of points \( p_n \) for \( n < \omega \) and countably many arcs \( \{I(n, m) : m < \omega \} \) with disjoint interiors connecting the points \( p_n \) and \( p_{n+1} \) for every \( n < \omega \).

If \( x \) is an inner point of some arc, then its basic neighbourhoods are the open intervals around it on the arc. The basic neighbourhoods of a point \( p_n \) are the unions of initial (or final) segments of the arcs containing \( p_n \). Finally, basic neighbourhoods of \( p \) are the sets which contain all but finitely many \( p_n \)'s together with their basic neighbourhoods and for any two consecutive \( p_n \)'s in the set all but finitely many of the arcs \( I(n, m) \). Note that the
subspace $X - \{p\}$ can be realised as a subspace of the plane, hence it is easy
to check that $X$ has the above stated properties.

Now let $f : X \to [0,1]$ be defined as follows: $f(p) = 0$, $f(p_n) = 0$ if $n$ is
even, $f(p_n) = 1$ if $n$ is odd, and $f$ is continuous on each arc $I(n,m)$. Then
$f$ is not sequentially continuous at $p$ as shown by the sequence $\{p_n : n \text{ odd }\}$
converging to $p$, but it is preserving. Indeed, an infinite sequence whose
members are from the interiors of different arcs is closed discrete and so
a compact subset of $X$ can meet only finitely many open arcs. It follows
then that its $f$-image is the union of finitely many compact subsets of $[0,1]$. Moreover, if a connected set contains both $p$ and some other point, then it
also contains an arc $I(n,m)$, and thus its image is the whole $[0,1]$. □

In the rest of this section we shall consider a slight weakening of the se-
quential continuity property that comes up naturally in the proof of McMil-
lan’s theorem or Theorems 2.3 and 2.4 below.

**Definition 1.25.** A function $f : X \to Y$ is said to be *weakly sequentially
continuous* at the point $x$ if $f(x_n) \to f(x)$ whenever $x_n \to x$ in $X$ and $f$
is not locally constant at $x_n$ for all $n < \omega$.

We shall give below two types of points in which preserving functions turn
out to be weakly sequentially continuous. In the next section then these will
be used to yield “real” continuity of preserving functions on some interesting
classes of spaces.

**Definition 1.26.** A point $x$ in a space $X$ is called an *inflatable point*
if $x_n \to x$ with $x_n \neq x$ for all $n < \omega$ implies that there are a subsequence
$\{x_{n_k} : k < \omega\}$ and neighbourhoods $U_k$ of $x_{n_k}$ for $k < \omega$ such that $U_k \to x$
(i.e. every neighbourhood of $x$ contains all but finitely many $U_k$’s). The
space $X$ will be called inflatable if all its points are.

It is obvious that any generalized ordered space is inflatable.

**Theorem 1.27.** Any preserving function $f : X \to Y$ from a locally con-
nected space $X$ into a $T_2$-space $Y$ is weakly sequentially continuous at an
inflatable point $x$.

**Proof.** Assume, indirectly, that $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$,
while $f$ is not locally constant at $x_n$ for all $n < \omega$. By Lemma 1.3’ we can
assume that $f(x_n) = y \neq f(x)$ for all $n < \omega$. Choose an open neighbourhood
$V \subset Y$ of $y$ such that $f(x) \not\in V$. As $x$ is inflatable, we may also assume
to have open sets $U_n$ with $x_n \in U_n$ such that $U_n \to x$. Using Lemma 1.7
we can recursively choose points $z_n \in U_n$ with distinct $f$-images such that
$f(z_n) \in V$ for all $n < \omega$, contradicting Lemma 1.3’ again. □

The other property is both a weakening of the Fréchet property and a
variation on the $s$ property.

**Definition 1.28.** We call a point $x$ in a space $X$ a *set-Fréchet* point if
whenever $A = \bigcup\{A_n : n < \omega\}$ with $x \in \overline{A}$ but $x \not\in \overline{A_n}$ for all $n < \omega$ then
there is a sequence \( \{x_n\} \subset A \) such that \( x_n \to x \). Of course, a space is set-Fréchet if all its points are.

**Theorem 1.29.** Let \( f \) be a preserving function from a locally connected \( T_2 \)-space \( X \) into the interval \([0,1]\). Then \( f \) is weakly sequentially continuous at every set-Fréchet point \( x \) of \( X \).

**Proof.** Assume indirectly that \( x_n \to x \) but \( f(x_n) \) does not converge to \( f(x) \), moreover \( f \) is not locally constant at each \( x_n \) for \( n < \omega \). By Lemma 1.3', we can assume that \( f(x) = 1 \) and \( f(x_n) = 0 \) for all \( n < \omega \). Note that then for any connected neighbourhood \( G \) of any point \( x_n \) the image \( f(G) \) is a proper interval containing 0. For every \( n < \omega \) choose an open sets \( U_n \) such that \( x_n \in U_n \) and \( x \not\in \overline{U_n} \) and put \( A_n = \{z \in U_n : 0 < f(z) < 1/n + 1\} \). By Lemma 1.3' the conditions in the definition of a set-Fréchet point are satisfied for the sets \( A_n \) so there is a sequence of points \( z_n \in A = \bigcup \{A_n : n < \omega\} \) converging to \( x \). It is immediate that \( f(z_n) \) converges to \( 0 \neq 1 = f(x) \) while the set \( \{f(z_n) : n < \omega\} \) is infinite because \( f(z_n) \neq 0 \) for all \( n \), contradicting Lemma 1.3'.

\[ \square \]

2. From sequential continuity to continuity

The aim of this section is to prove that if a locally connected space \( X \) fulfills one of the “convergence-type” conditions of the first section (i.e. \( X \) is an \( SC \)-space or it is inflatable or set-Fréchet) and \( f : X \to Y \) is a preserving function then, assuming in addition appropriate separation axioms for \( X \) and \( Y \), \( f \) is continuous at every \( s \)-point of \( X \). The proofs of these theorems, just like their formulations, are very similar.

**Theorem 2.1.** A preserving function \( f : X \to Y \) from a locally connected \( SC \)-space \( X \) into a regular space \( Y \) is continuous at every \( s \)-point of \( X \).

**Proof.** Assume indirectly that \( f \) is not continuous at the \( s \)-point \( p \in X \). Then there exists a closed set \( F \subset Y \) such that \( p \in \overline{f^{-1}(F)} \) but \( f(p) \not\in F \). Choose an open set \( V \subset Y \) such that \( F \subset V \) and \( f(p) \not\in \overline{V} \).

Let \( \mathcal{K} \) be the family of the components of \( f^{-1}(\overline{V}) \) and put \( \mathcal{A} = \{K \cap f^{-1}(F) : K \in \mathcal{K}\} \). As \( p \not\in \overline{K} \) for \( K \in \mathcal{K} \) by Lemma 1.4 and \( p \in \overline{f^{-1}(F)} \), the conditions given in the definition of an \( s \) point are fulfilled for the family \( \mathcal{A} \). Thus there is a sequence \( \{x_n : n < \omega\} \subset f^{-1}(F) \) such that \( x_n \to x \) for some \( x \in X \) and if \( K_n \) is the component of \( x_n \) in \( f^{-1}(\overline{V}) \), then \( K_m \neq K_n \) for \( m \neq n \).

As the components \( K_n \) are pairwise disjoint, we can suppose that \( x \not\in K_n \) for all \( n < \omega \). It follows that if \( C \) is a connected set which contains both \( x \) and some \( x_n \) then \( C \not\supset f^{-1}(\overline{V}) \), because otherwise \( K_n \cup C \) would be a connected subset of \( f^{-1}(\overline{V}) \) strictly larger than the component \( K_n \), a contradiction. Hence, using that \( x \) is an \( SC \)-point, we may assume to have a sequence \( C_n \) of connected sets such that \( C_n \to x \) and \( C_n \not\subset f^{-1}(\overline{V}) \).

We can choose points \( z_n \in C_n - f^{-1}(\overline{V}) \) for all \( n < \omega \), then \( z_n \to x \) and
f(z_n) \notin \overline{V}. But by Theorem [1.24] f is sequentially continuous, consequently \( f(x) = \lim f(z_n) \in Y - V \). On the other hand, \( f(x_n) \in F \) for \( n < \omega \) and so, using again the sequential continuity of f at the point x, we get that \( f(x) = \lim f(x_n) \in F \), a contradiction. \( \Box \)

The proofs of the other two analogous theorems for inflatable and set-Fréchet spaces, respectively, make essential use of the following lemma:

**Lemma 2.2.** Assume that \( f : X \to Y \) is a preserving and weakly sequentially continuous function from a locally connected \( T_3 \)-space X into a \( T_3 \)-space Y and \( f \) is not continuous at some s-point of X. Then there are two sets \( F \subset V \subset Y \), \( F \) closed and \( V \) open in Y and a convergent sequence \( x_n \to x \) in X such that for all \( n < \omega \) we have \( x_n \neq x \), \( f(x_n) \in F \) but \( f(U) \notin \overline{V} \) whenever \( U \) is a neighbourhood of \( x_n \). It follows that \( f \) is not locally constant at the points \( x_n \) and \( x \).

**Proof.** Assume \( f \) is not continuous at the s-point \( p \in X \), then there exists a closed set \( F \subset Y \) such that \( p \in f^{-1}(F) \) but \( f(p) \notin F \). Choose an open set \( V \subset Y \) such that \( F \subset V \) and \( f(p) \notin \overline{V} \).

Put \( B = \{ x \in f^{-1}(F) : f \text{ is not locally constant at } x \} \), then \( p \in \overline{B} \) by Lemma [1.6]. Now let \( H = \{ x \in f^{-1}(F) : f(U) \notin \overline{V} \text{ for every neighbourhood } U \text{ of } x \} \), clearly \( H \subset B \). We assert that \( p \in \overline{H} \) as well. To show this, fix a closed neighbourhood \( W \) of \( p \). Let \( K \) be the family of the components of \( f^{-1}(\overline{V}) \) and put \( \mathcal{A} = \{ K \cap B \cap W : K \in \mathcal{K} \} \). Since \( p \notin \overline{K} \) for \( K \in \mathcal{K} \) by Lemma 1.4, the conditions in the definition of an s-point are satisfied for \( p \) and \( \mathcal{A} \). Thus there is a sequence \( \{ y_n : n < \omega \} \subset B \cap W \) such that \( y_n \to y \) for some \( y \in X \) and if \( K_n \) is the component of \( y_n \) in \( f^{-1}(\overline{V}) \), then \( K_m \neq K_n \) if \( m \neq n \).

We claim that \( y \in W \cap H \). As \( W \) is closed, trivially \( y \in W \). The sets \( K_n \) are disjoint so we can assume that \( y \notin K_n \) for all \( n < \omega \). Using that \( f \) is weakly sequentially continuous and \( y_n \in B \), we get that \( f(y) = \lim f(y_n) \in F \), hence \( y \in B \). We have yet to show that \( f(U) \notin \overline{V} \) if \( U \) is any neighbourhood of \( y \). By local connectivity, we can suppose that \( U \) is connected. But the connected set \( U \) meets (infinitely many) distinct components of \( f^{-1}(\overline{V}) \), so indeed \( U \notin f^{-1}(\overline{V}) \).

Now applying the s-point property of \( p \) to the family \( \mathcal{A} = \{ \{ x \} : x \in H \} \) there is an infinite sequence of points \( x_n \in H \) converging to some point \( x \), completing the proof. \( \Box \)

**Theorem 2.3.** A preserving function from a locally connected and inflatable \( T_3 \) space \( X \) into a \( T_3 \) space \( Y \) is continuous at every s-point of \( X \).

**Proof.** Assume, indirectly, that the preserving function \( f : X \to Y \) is not continuous at an s-point of \( X \). By Theorem [1.27] \( f \) is weakly sequentially continuous, hence we can apply the preceding lemma and choose appropriate
sets $F$, $V$ in $Y$ and points $x_n$ and $x$ in $X$. As $X$ is inflatable and locally connected, we can also assume that there is a sequence of connected open sets $U_n$ such that $U_n \to x$ and $x_n \in U_n$ for $n < \omega$. Then for all $n$ we have $f(U_n) - V \neq \emptyset$, hence by Lemma 1.14 these sets are uncountable. Consequently we can recursively select another sequence of points $y_n \in U_n$ (and so converging to $x$) such that $f(y_n) \in Y - V$ and the $f(y_n)$’s are pairwise distinct. But then the sequence $f(y_n)$ does not converge to $f(x)$ because $f(x) = \lim f(x_n) \in F \subset V$ by the weak sequential continuity of $f$, contradicting Lemma 1.3 again. □

Corollary 2.4. If $X$ is locally compact, locally connected, and monotonically normal (in particular if $X$ is a locally connected LOTS) then $Pr(X, T_3)$ holds.

Proof. See [4, theorem 3.12] for a proof that a (locally) compact, monotonically normal space is both inflatable and radial. Consequently, it is also locally sequentially compact, and thus an $s$-space. □

Theorem 2.5. A preserving function from a locally connected and set-Fréchet $T_3$ space $X$ into a $T_{3\frac{1}{2}}$ space is continuous at every $s$-point of $X$.

Proof. By Lemma 1.2, it suffices to prove this for preserving functions $f : X \to [0, 1]$. Assume, indirectly, that the function $f$ is not continuous at some $s$-point $p \in X$. Then, by Theorem 1.29, $f$ is weakly sequentially continuous, hence we may again apply Lemma 2.2 to choose appropriate sets $F$ and $V$ in $Y = [0, 1]$ and points $x_n$ and $x$ in $X$. There is a continuous function $g : [0, 1] \to [0, 1]$ which is identically 1 on $F$ and 0 on $[0, 1] - V$. Then the composite function $h = gf : X \to [0, 1]$ is also preserving, $h(x_n) = 1$ for all $n$, and the $h$-image of any neighbourhood of a point $x_n$ is the whole interval $[0, 1]$.

Let us choose open sets $G_n$ for $n < \omega$ such that $x_n \in G_n$ and $x \notin \overline{G_n}$. If $A_n = G_n \cap f^{-1}((0, \frac{1}{n}))$ and $A = \bigcup A_n$, then $x \notin \overline{A}$ but $x \in \overline{A}$. So, as $X$ is a set-Fréchet space, there is a sequence of points $y_n \in A$ converging to $x$. But then the set $\{h(y_n) : n < \omega\}$ is infinite and $h(y_n) \to 0 \neq 1 = f(x)$, contradicting Lemma 1.3 □

Corollary 2.6. If $X$ is a locally connected, locally countably compact, and set-Fréchet $T_3$-space then $Pr(X, T_{3\frac{1}{2}})$ holds.

Proof. It is enough to note that a countably compact set-Fréchet space is also sequentially compact and so an $s$-space. □

3. Some product theorems

The aim of this section is to prove that an arbitrary product $X$ of certain “good” spaces has the property $Pr(X, T_3)$. To achieve this, we shall make use of Theorem 2.1. It is well-known that the product of connected and
locally connected spaces is locally connected, moreover a very similar argument (based on the productivity of connectedness) implies that the product of connected $SC$ spaces is $SC$. Hence two of the assumptions of Theorem 2.1 are productive if the factors are connected. Nothing like this can be expected, however, about the third assumption of Theorem 2.1, namely the $s$-property. To make up for this, we are going to consider a stronger property that is at least countably productive, and use this stronger property to establish what we want, first for $\Sigma$-products and then for arbitrary products. Now, this stronger property will require the existence of a winning strategy for player $I$ in the following game.

**Definition 3.1.** Fix a space $X$ and a point $p \in X$. The game $G(X, p)$ is played by two players $I$ and $II$ in $\omega$ rounds. In the $n$-th round first $I$ chooses a neighbourhood $U_n$ of $p$ and then $II$ chooses a point $x_n \in U_n$. $I$ wins if the produced sequence $\{x_n : n < \omega\}$ has a convergent subsequence, otherwise $II$ wins.

We shall say that $X$ is *winnable at the point $p$* if $I$ has a winning strategy in the game $G(X, p)$. $X$ is *winnable* if $G(X, p)$ is winnable for all $p \in X$.

Note that, formally, a winning strategy for $I$ is a map $\sigma : X^{<\omega} \to \mathcal{V}(p)$, where $\mathcal{V}(p)$ is the family of neighbourhoods of $p$, such that if $\langle x_n : n < \omega \rangle$ is a sequence played following $\sigma$, i.e. $x_n \in \sigma(x_0, x_1, ..., x_{n-1})$ for all $n < \omega$, then $\langle x_n : n < \omega \rangle$ has a convergent subsequence.

**Lemma 3.2.** A winnable point $p$ of a space $X$ is always an s point.

**Proof.** Let $\sigma$ be a winning strategy of $I$ in the game $G(X, p)$ and fix a family $\mathcal{A}$ of subsets of $X$ with $p \in \bigcup \mathcal{A}$ but $p \not\in \overline{A}$ for all $A \in \mathcal{A}$. Let us play the game $G(X, p)$ in such a way that $I$ follows $\sigma$ and assume that the first $n$ rounds of the game have been played with the points $x_i \in U_i$ and the distinct sets $A_i \in \mathcal{A}$ with $x_i \in A_i$ chosen by player $II$ for $i < n$. Let $U_n = \sigma(x_0, x_1, ..., x_{n-1})$ be the next winning move of $I$, then $II$ can choose a set $A_n \in \mathcal{A}$ with $A_n \cap (U_n - \bigcup_{i<n} A_i) \neq \emptyset$ and then pick $x_n \in A_n \cap (U_n - \bigcup_{i<n} A_i)$ as his next move. But player $I$ wins hence, a suitable subsequence of $\{x_n : n < \omega\}$, whose members were picked from distinct elements of $\mathcal{A}$, will converge. 

In order to prove the desired product theorem for winnable spaces we shall consider monotone strategies for player $I$. A strategy $\sigma$ of player $I$ is said to be monotone if for every subsequence $\langle x_{i_0}, ..., x_{i_{r-1}} \rangle$ of a sequence $\langle x_0, x_1, ..., x_{n-1} \rangle$ of points in $X$ we have

$$\sigma(x_0, x_1, ..., x_{n-1}) \subset \sigma(x_{i_0}, ..., x_{i_{r-1}}).$$

**Lemma 3.3.** If player $I$ has a winning strategy in the game $G(X, p)$ then he also has a monotone winning strategy.

**Proof.** Let $\sigma$ be a winning strategy for player $I$; we define a new strategy $\sigma_0$ as follows : for any sequence $s = \langle x_0, x_1, ..., x_{n-1} \rangle$ put

$$\sigma_0(s) = \bigcap \{\sigma(x_{i_0}, ..., x_{i_{r-1}}) : 0 \leq i_0 < i_1 < ... < i_{r-1} < n\}.$$
The function $\sigma_0$ is clearly a monotone winning strategy for $\mathbf{I}$. \hfill \Box

It is easy to see that if $\mathbf{I}$ plays using the monotone strategy $\sigma_0$ then any infinite subsequence of the sequence chosen by player $\mathbf{II}$ is also a win for $\mathbf{I}$, i.e. has a convergent subsequence. Another important property of a monotone winning strategy $\sigma_0$ is the following: If $\langle x_n : n < \omega \rangle$ is a sequence of points in $X$ such that we have $x_n \in \sigma_0(x_0, x_1, \ldots, x_{n-1})$ only for $n \geq m$ for some fixed $m < \omega$ then this is still a winning sequence for $\mathbf{I}$. Indeed, this holds because for every $n \geq m$ we have

$$x_n \in \sigma_0(x_0, x_1, \ldots, x_{n-1}) \subseteq \sigma_0(x_m, x_{m+1}, \ldots, x_{n-1})$$

by monotonicity, hence the “tail” sequence $\langle x_n : m \leq n < \omega \rangle$ is produced by a play of the game where $\mathbf{I}$ follows the strategy $\sigma_0$.

For a family of spaces $\{X_s : s \in S\}$ and a fixed point $p$ (called the base point) of the product $X = \prod \{X_s : s \in S\}$ let $T(x)$ denote the support of the point $x$ in $X$: this is the set $\{s \in S : x(s) \neq p(s)\}$. Then $\Sigma(p)$ (or simply $\Sigma$ if this does not lead to misunderstanding) denotes the $\Sigma$-product with base point $p$: it is the subspace of $X$ of the points with countable support, i.e.

$$\Sigma = \{x \in X : |T(x)| \leq \omega\}.$$

In the proof of the next result we shall use two lemmas. The first one is an easy combinatorial fact:

**Lemma 3.4.** Let $\langle H_k : k < \omega \rangle$ be a sequence of countable sets, then for every $n < \omega$ there is a finite set $F_n$ depending only on the first $n$ many sets $\langle H_k : k < n \rangle$ such that $F_n \subseteq \bigcup_{i<n} H_i$, $F_n \subseteq F_{n+1}$ and $\bigcup F_n = \bigcup H_n$.

**Proof.** Fix an enumeration $H_i = \{x(i,j) : j < \omega\}$ of the set $H_i$ for all $i < \omega$ and then let $F_n = \{x(i,j) : i, j < n\}$. \hfill \Box

The second lemma is about the sequences which play a crucial role in the games $G(X, p)$. Let us call a sequence $\{x_n\}$ in the space $X$ good if every infinite subsequence of it has a convergent subsequence.

**Lemma 3.5.** If $x_n \in X = \prod \{X_i : i < \omega\}$ for $n < \omega$ and $\{x_n(i) : n < \omega\}$ is a good sequence in $X_i$ for all $i \in \omega$ then $\{x_n\}$ is a good sequence in $X$.

**Proof.** We shall prove that if $N$ is any infinite subset of $\omega$ then there is in $X$ a convergent subsequence of $\{x_n : n \in N\}$.

We can choose by recursion on $k < \omega$ infinite sets $N_k$ such that $N_{k+1} \subseteq N_k \subseteq N$ and $\{x_n(k) : n \in N_k\}$ converges to a point $x(k)$ in $X_k$. Then there is a diagonal sequence $\{n_k : k < \omega\}$ such that $n_k \in N_k$ and $n_k < n_{k+1}$ for all $k < \omega$. The sequence $\{n_k : k < \omega\}$ is eventually contained in $N_i$, hence $x_{n_k}(i) \rightarrow x(i)$ in $X_i$, for all $i < \omega$. It follows that $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n : n \in N\}$ in $X$. \hfill \Box

The following result says a little more than that winnability is a countably productive property.
Lemma 3.6. Let \( p \in X = \prod \{ X_s : s \in S \} \) and suppose that \( G(X_s, p(s)) \) is winnable for every \( s \in S \). Then \( G(\Sigma(p), p) \) is also winnable.

Proof. We have to construct a winning strategy \( \sigma \) for player I in the game \( G(\Sigma(p), p) \). By Lemma 3.3, we can fix a monotone winning strategy \( \sigma_s \) of I in the game \( G(X_s, p(s)) \) for each \( s \in S \). Given a sequence \( \langle x_0, ..., x_{n-1} \rangle \in [\Sigma(p)]^{\omega} \), let \( H_i \) denote the support of \( x_i \) and \( F_n \) be the finite set assigned to the sequence \( \{ H_i : i < n \} \) as in Lemma 3.4. Now, if \( \pi_s \) is the projection from the product \( X \) onto the factor \( X_s \) for \( s \in S \) then set

\[
\sigma(x_i : i < n) = \bigcap\{ \pi_s^{-1}(\sigma_s(x_i(s) : i < n)) : s \in F_n \}.
\]

Now let \( \langle x_n : n < \omega \rangle \) be a sequence of points in \( \Sigma(p) \) produced by a play of the game \( G(\Sigma(p), p) \) in which I followed the strategy \( \sigma \). Then for every \( s \in H = \bigcup H_n \) the sequence \( \langle x_n(s) : n < \omega \rangle \) is a win for player I in the game \( G(X_s, p(s)) \) because there is an \( m < \omega \) with \( s \in F_m \) and then \( x_n(s) \in \sigma_s(x_i(s) : i < n) \) is valid for all \( n \geq m \). Consequently, by Lemma 3.3, the sequence \( \langle x_n : H : n < \omega \rangle \) has a convergent subsequence in \( \prod \{ X_s : s \in H \} \), while for \( s \in S - H \) we have \( x_n(s) = p(s) \) for all \( n < \omega \), and so \( \langle x_n : n < \omega \rangle \) indeed has a convergent subsequence in \( \Sigma(p) \).

The following two statements both easily follow from the fact that any product of connected spaces is connected.

Lemma 3.7. A \( \Sigma \)-product of connected \( SC \) spaces is also an \( SC \) space. \( \square \)

Lemma 3.8. A \( \Sigma \)-product of connected and locally connected spaces is also locally connected. \( \square \)

We now have all the necessary ingredients needed to prove our main product theorem.

Theorem 3.9. Let \( f : X = \prod \{ X_s : s \in S \} \to Y \) be a preserving function from a product of connected and locally connected \( SC \) spaces into a regular space \( Y \). If \( p \in X \) and \( G(X_s, p(s)) \) is winnable for all \( s \in S \) then \( f \) is continuous at the point \( p \).

Proof. Let \( \Sigma \) denote the sigma-product with base point \( p \). Then, by Lemma 3.6, \( G(\Sigma, p) \) is winnable and so \( p \) is an \( s \) point in \( \Sigma \). Moreover, by Lemmas 3.7 and 3.8, \( \Sigma \) is also a locally connected \( SC \) space. Hence Theorem 2.1 implies that the restriction of the function \( f \) to the subspace \( \Sigma \) of \( X \) is continuous at \( p \).

To prove that \( f \) is also continuous at the point \( p \) in \( X \), fix a neighbourhood \( V \) of \( f(p) \) in \( Y \). As the restriction \( f|\Sigma \) is continuous at \( p \), there is an elementary neighbourhood \( U \) of \( p \) in the product space \( X \) such that \( f(U \cap \Sigma) \subset V \). Since the factors \( X_s \) are connected and locally connected, we can assume that \( U \) and \( U \cap \Sigma \) are also connected and hence, by Lemma 1.4, we have

\[
f(U) \subset f(U \cap \Sigma) \subset f(U \cap \Sigma) \subset V.
\]
The regularity of $Y$ then implies that $f$ is continuous at $p$.

**Corollary 3.10.** $Pr(X, T_3)$ holds whenever $X$ is any product of connected and locally connected, winnable SC spaces. In particular, if $X = \prod \{X_s : s \in S\}$ where each factor $X_s$ is either a connected linearly ordered space (with the order topology) or a connected and locally connected first countable space then $Pr(X, T_3)$ is valid.

For the proof of the next Corollary we need a general fact about the relations $Pr(X, T_i)$.

**Lemma 3.11.** If $q : X \to Y$ is a quotient mapping of $X$ onto $Y$ then, for any $i$, $Pr(X, T_i)$ implies $Pr(Y, T_i)$.

**Proof.** Let $f : Y \to Z$ be a preserving function into the $T_i$ space $Z$. The function $fq : X \to Z$, as the composition of a continuous (and so preserving) and of a preserving function is also preserving, hence, by $Pr(X, T_i)$, it is continuous. But then $f$ is continuous because $q$ is quotient.

**Corollary 3.12.** Let $X = \prod \{X_s : s \in S\}$ where all the spaces $X_s$ are compact, connected, locally connected, and monotonically normal. Then $Pr(X, T_2)$ holds.

**Proof.** It follows from the recent solution by Mary Ellen Rudin of Nikiel’s conjecture [10], combined with results of L.B.Treybig [11] or J.Nikiel [7], that every compact, connected, locally connected, monotonically normal space is the continuous image of a compact, connected, linearly ordered space. Hence our space $X$ is the continuous image of a product of compact, connected, linearly ordered spaces. But any $T_2$ continuous image of a compact $T_2$ space is a quotient image (and $T_3$), hence Corollary 3.10 and Lemma 3.11 imply our claim.

Comparing this result with Corollary 2.4, the following question is raised naturally.

**Problem 3.13.** Let $X$ be a product of locally compact, connected and locally connected monotonically normal spaces. Is then $Pr(X, T_3)$ true?

The following is result is mentioned here mainly as a curiosity.

**Corollary 3.14.** Let $X = \prod \{X_s : s \in S\}$ be a product of linearly ordered and/or first countable spaces. Then the following are equivalent:

a) $Pr(X, T_3)$;
b) $X$ is locally connected;
c) the spaces $X_s$ are all locally connected and all but finitely many of them are also connected.

**Proof.** a)$\Rightarrow$b) : Lemma 1.1

b)$\Rightarrow$c) : [2, 6.3.4]

c)$\Rightarrow$a) : Corollary 3.10.
Remark 3.15. E. R. McMillan raised the following question in [4]: does $\text{Pr}(X, T_2)$ imply that $X$ is a $k$-space? We do not know the (probably negative) answer to this question, however we do know that the answer is negative if $T_2$ is replaced by $T_3$ in it. Indeed, for instance $\mathbb{R}^\omega$ is not a $k$-space (see e.g. [2, exercise 3.3.E]), but, by Corollary 3.10, $\text{Pr}(\mathbb{R}^\omega, T_3)$ is valid.

4. The sequential and the compact cases

The two examples given in section 1 of (locally connected) spaces on which there are non-continuous preserving functions both lack the properties of (i) sequentiality and (ii) compactness. Here (i) arises naturally as a weakening of the Fréchet property from McMillan’s theorem while the significance of (ii) needs no explanation. This leads us naturally to the following problem.

Problem 4.1. Assume that the locally connected space $X$ is (i) sequential and/or (ii) compact. Is then $\text{Pr}(X, T_2)$ (or $\text{Pr}(X, T_3)$) true?

The answer in case (i) turns out to be positive if we assume the $\text{SC}$ property instead of local connectivity. The following result reveals why local connectivity need not be assumed in it. (Compare this also with Lemma 1.18.)

Theorem 4.2. Any sequential $\text{SC}$ space $X$ is locally connected.

Proof. We have to prove that if $K$ is a component of an open set $G \subset X$ then $K$ is open. Assume not, then $X - K$ is not closed, hence as $X$ is sequential there is a sequence $\{x_n\} \subset X - K$ such that $x_n \to x \in K$. Since $X$ is an $\text{SC}$ space and $G$ is a neighbourhood of $x$ there is a connected set $C \subset G$ such that $\{x, x_n\} \subset C$ for some $n < \omega$. But this is impossible because then the connected set $K \cup C \subset G$ would be larger than the component $K$ of $G$. □

Now we give the above promised partial solution to Problem 4.1 in case (i), i.e. for sequential spaces.

Theorem 4.3. If $X$ is a sequential $\text{SC}$ space then $\text{Pr}(X, T_2)$ holds.

Proof. Let $f : X \to Y$ be a preserving map into a $T_2$ space $Y$. Since $X$ is sequential it suffices to show that the function $f$ is sequentially continuous but this immediate from Theorem 1.21. □

Our next result implies that a counterexample to Problem 4.1 (in either case) can not be very simple in the sense that discontinuity of a preserving function cannot occur only at a single point (as it does in both examples 1.23 and 1.24). In order to prepare this result we first introduce a topological property that generalizes both sequentiality and (even countable) compactness.

Definition 4.4. $X$ is called a countably $k$ space if for any set $A \subset X$ that is not closed in $X$ there is a countably compact subspace $C$ of $X$ such that $A \cap C$ is not closed in $C$. 
This condition means that the topology of $X$ is determined by its countably compact subspaces. All countably compact and all $k$ (hence also all sequential) spaces are countably $k$. It is easy to see that the countably $k$ property is always inherited by closed subspaces and for regular spaces by open subspaces as well.

**Theorem 4.5.** Let $X$ be countably $k$ and locally connected and $Y$ be $T_3$, moreover let $f : X \to Y$ be a preserving function. Then the set of points of discontinuity of $f$ is not a singleton. So if $X$ is also $T_3$ then the discontinuity set of $f$ is dense in itself.

**Proof.** Assume, indirectly, that $f$ is not continuous at $p \in X$ but it is continuous at all other points of $X$. Then we can choose a closed set $F \subset Y$ with $A = f^{-1}(F)$ not closed. Evidently, then $\overline{A} - A = \{p\}$. As $A$ is not closed in $X$ and $X$ is countably $k$, there is a countably compact set $C$ in $X$ such that $A \cap C$ is not closed in $C$; clearly then $p \in C$ and $p$ is in the closure of $A \cap C$.

Let $V \subset Y$ be an open set with $F \subset V$ and $f(p) \notin \overline{V}$ and put $H = f^{-1}(V)$. Then $H$ is open in $X$ and contains the set $A$. For every component $L$ of $H$ we have $p \notin \overline{L}$ by $f(p) \notin \overline{L} \subset \overline{V}$ and Lemma 1.4, hence $p \in \overline{A \cap C}$ implies that there are infinitely many components of $H$ that meet $A \cap C$. Thus we may choose a sequence $\{L_n : n < \omega\}$ of distinct such components with points $x_n \in L_n \cap A \cap C$.

We claim that $x_n \to p$. As the $x_n$’s are chosen from the countably compact set $C$, it is enough to prove for this that if $x \neq p$ then $x$ is not an accumulation point of the sequence $\{x_n\}$. If $x \notin \overline{A} = A \cup \{p\}$ this is obvious so we may assume that $x \in A \subset H$. But then, as $H$ is open and $X$ is locally connected, the connected component of $x$ in $H$ is a neighbourhood of $x$ that contains at most one of the points $x_n$.

The point $f(p)$ is not in the closure of the set $\{f(x_n) : n < \omega\} \subset F$, hence, by Lemma 1.3 we can suppose that $f(x_n) = y \neq f(p)$ for each $n < \omega$.

Now we choose a sequence of neighbourhoods $V_n$ of $y$ in $Y$ with $V_0 = V$ and $\overline{V}_{n+1} \subset V_n$ for all $n < \omega$ and then put $U_n = f^{-1}(V_n)$. Clearly $U_n$ is open in $X$ and $U_0 = H$, hence, as was noted above, the closure of any connected set contained in $U_0$ contains at most one of the points $x_n$.

Next, let $K_n$ be the component of $x_n$ in $U_n$ for $n < \omega$. We claim that every boundary point of $K_n$ is mapped by $f$ to a boundary point of $V_n$, i.e.

$$f(Fr K_n) \subset Fr V_n.$$ 

Indeed, $f(\overline{K_n}) \subset \overline{V_n}$ by Lemma 1.4 (or by continuity at all points distinct from $p$). Moreover, we have

$$Fr K_n = \overline{K_n} - K_n \subset Fr U_n = \overline{U_n} - U_n$$

because $K_n, U_n$ are open and $K_n$ is a component of $U_n$, and $f(Fr U_n) \cap V_n = \emptyset$ because $U_n = f^{-1}(V_n)$. Thus indeed $f(Fr K_n) \subset \overline{V_n} - V_n = Fr V_n$.

Every connected neighbourhood of $p$ meets all but finitely many $K_n$’s hence also $Fr K_n$, consequently $K = \bigcup \{Fr K_n : n < \omega\}$ is not closed. So
there exists a countably compact set $D$ with $D \cap K$ not closed in $D$. But the sets $FrK_n$ are closed, thus $D$ must meet infinitely many of them, i.e. the set

$$N = \{ n < \omega : D \cap FrK_n \neq \emptyset \}$$

is infinite. Let us choose a point $z_n$ from each nonempty $D \cap K_n$.

Again we claim that the only accumulation point of the sequence $\{z_n : n \in N\}$ is $p$. Indeed, if $x \neq p$ would be such an accumulation point, then $f(z_n) \in V_n \subset V_1$ for all $n \in N - \{0\}$ would also imply $f(x) \in V_1 \subset V_0$. By continuity and local connectivity at $x$ then there is a connected neighbourhood $W$ of $x$ with $f(W) \subset V_0$. But then the set

$$W \cup \bigcup \{K_n : W \cap K_n \neq \emptyset\}$$

would be a connected subset of $U_0$ that has $p$ in its closure, a contradiction.

Consequently, the sequence $\{z_n : n \in N\}$ must converge to $p$, while $\{f(z_n) : n \in N\} \subset V$ does not converge to $f(p)$, contradicting Lemma 1.3 because $f(z_n) \in FrV_n$ for all $n \in N$ and the boundaries $FrV_n$ are pairwise disjoint, hence the $f(z_n)$’s are pairwise distinct.

The last statement of the theorem now follows easily because an isolated discontinuity of $f$ yields an open subspace of $X$ on which the restriction of $f$ has a single point of discontinuity, although if $X$ is $T_3$ then any open subspace of $X$ is both countably $k$ and locally connected.

Noting that Problem 4.1 really comprises three different questions, and having shown above that, in a certain sense, it seems to be hard to find counterexamples to any of these, we now turn our attention to the case in which both (i) and (ii) are assumed. In this case we can provide a positive answer, at least consistently and with the extra assumption that the cellularity of the space in question is “not too large”. In fact, what we can prove is that if $2^\omega < 2^p$ then any locally connected, compact $T_2$ space $X$ that is sequential and does not contain a cellular family of size $p$ satisfies $Pr(X, T_2)$. Of course, here $p$ stands for the well-known cardinal invariant of the continuum whose definition is recalled below.

A set $H \subset \omega$ is called a pseudo intersection of the family $A \subset [\omega]^{\omega}$ if $H$ is almost contained in every member of $A$, i.e. $H - A$ is finite for each $A \in A$. Then $p$ is the minimal cardinal $\kappa$ such that there exists a family $A \subset [\omega]^{\omega}$ of size $\kappa$ which has the finite intersection property but does not have an infinite pseudo intersection. (Here the finite intersection property means that any finite subfamily of $A$ has infinite intersection.)

It is well-known (see e.g. [12]) that the cardinal $p$ is regular, $\omega_1 \leq p \leq 2^\omega$ and $2^\kappa = 2^\omega$ for $\kappa < p$. The condition “$2^p > 2^\omega$” of our result is satisfied if $p = 2^\omega$ (hence Martin’s axiom implies it), but it is also true if $2^{\omega_1} > 2^\omega$.

Now, our promised consistency result on compact sequential spaces will be a corollary of a ZFC result of somewhat technical nature. Before formulating this, however, we shall prove two lemmas that may have some independent interest in themselves.
Lemma 4.6. Let $X$ be a compact $T_2$ space of countable tightness and $f : X \to [0,1]$ be a compactness preserving map of $X$ into the unit interval. If $x \in X$ is a point in $X$ and $[a,b]$ is a subinterval of $[0,1]$ such that for every neighbourhood $U$ of $x$ we have $[a,b] \subset f(U)$ then for any $G_{<p}$ set $H$ containing $x$ we also have $[a,b] \subset f(H)$.

Proof. Without loss of generality we may assume that $H$ is closed. Now the proof will proceed by induction on $\kappa$ where $\omega \leq \kappa < p$ and $H$ is a (closed) $G_\kappa$ set, or equivalently, the character $\chi(H,X) = \kappa$. If $\kappa = \omega$ then we can write $H = \bigcap\{G_n : n < \omega\}$ with $G_n$ open and $G_{n+1} \subset G_n$ for all $n < \omega$. Fix a countable dense subset $\{c_n : n < \omega\}$ of $[a,b]$ and then pick $x_n \in G_n$ with $f(x_n) = c_n$, this is possible by assumption. Note that then every accumulation point of the set $M = \{x_n : n < \omega\}$ is in $H$, hence by Lemma 1.3 we have $[a,b] = f(M) \subset f(M') \subset f(H)$.

Next, if $\omega < \kappa < p$ then we have $x \in H = \bigcap\{S_\xi : \xi < \kappa\}$, where $S_\xi \supset S_\eta$ if $\xi < \eta$ and the $S_\xi$ are closed sets of character $< \kappa$. By induction, we have $f(S_\xi) \supset [a,b]$ for all $\xi < \kappa$, and we have to prove that $f(H) \supset [a,b]$ as well. In fact, it suffices to show that $f(H) \cap [a,b] \neq \emptyset$ because applying this to all (non-singleton) subintervals of $[a,b]$ we actually get that $f(H) \cap [a,b]$ is dense in $[a,b]$ while $f(H)$ is also compact, hence closed.

We do this indirectly; assume $f(H) \cap [a,b] = \emptyset$ then we can choose points $x_\xi \in S_\xi - H$ for all $\xi < \kappa$ such that the images $f(x_\xi) \in [a,b]$ are all distinct. Let $\hat{x}$ be a complete accumulation point of the set $\{x_\xi : \xi < \kappa\}$. Then $\hat{x} \in H$ and $t(X) = \omega$ implies that there is a countable subset $A \subset \{x_\xi : \xi < \kappa\}$ such that $\hat{x} \in \overline{A} - A$. Choose now a neighbourhood base $B$ of $H$ in $X$ of size $\kappa < p$. The family $\{A \cap B : B \in \mathcal{B}\} \subset [A]^{<p}$ has the finite intersection property hence it has an infinite pseudo intersection $P \subset A$, i. e. the set $P - B$ is finite for each $B \in \mathcal{B}$. This implies that every accumulation point of $P$ is contained in $H$. But $\overline{P}$ is compact, hence by Lemma 1.3 we have $\emptyset \neq f(P) \subset f(P') \cap [a,b] \subset f(H) \cap [a,b]$, which is a contradiction. \hfill \qed

Before we give the other lemma, let us recall that for any space $X$ we use $\mathcal{c}(X)$ to denote the smallest cardinal $\kappa$ such that $X$ does not contain $\kappa$ disjoint open sets.

Lemma 4.7. Let $f : X \to Y$ be a connectivity preserving map from a locally connected space $X$ into a $T_2$ space $Y$. Then for every $x \in X$ with $\chi(f(x),Y) < \mathcal{c}(X)$ there is a $G_{<\mathcal{c}(X)}$ set $H$ in $X$ such that $x \in H$ and if $z \in H$ is any point of continuity of $f$ then $f(z) = f(x)$.

Proof. Let $\kappa = \mathcal{c}(X)$ and fix a neighbourhood base $\mathcal{V}$ of the point $f(x)$ in $Y$ with $|\mathcal{V}| < \kappa$. For every $V \in \mathcal{V}$ let us then set $G_V = \bigcup\{G : G$ is open in $X$ and $f(G) \cap V = \emptyset\}.$
For every component $K$ of the open set $G_V$ we have $f(x) \notin \overline{f(K)}$ and therefore $x \not \in K$ by Lemma 1.4. Moreover the components of $G_V$ form a cellular family because $X$ is locally connected, hence their number is less than $\kappa$. Consequently,

$$H_V = \bigcap \{X - K : K \text{ is a component of } G_V\}$$

is a $G_{<\kappa}$ set with $x \in H_V$ and $H_V \cap G_V = \emptyset$.

The cardinal $\kappa$ is regular (see e.g. [3, 4.1]), hence $H = \bigcap \{H_V : V \in \mathcal{V}\}$ is also a $G_{<\kappa}$ set that contains the point $x$. Now, suppose that $z$ is a point of continuity of $f$ with $f(z) \neq f(x)$. Then there is a basic neighbourhood $V \in \mathcal{V}$ of $f(x)$ and a neighbourhood $W$ of $f(z)$ with $V \cap W = \emptyset$, and there is an open neighbourhood $U$ of $x$ in $X$ with $f(U) \subset W$. Then, by definition, we have $z \in U \subset G_V$, hence $z \not \in H_V \supset H$.

**Theorem 4.8.** Let $X$ be a locally connected compact $T_2$ space of countable tightness. If, in addition, we also have $|X| < 2^p$ and $\check{c}(X) \leq p$ then $Pr(X,T_2)$ holds.

**Proof.** Using Lemma 1.2 it suffices to show that any preserving function $f : X \to [0,1]$ is continuous. To this end, first note that if $f$ is not continuous at a point $x \in X$ then the oscillation of $f$ at $x$ is positive, hence, by local connectivity at $x$ and because $f$ is preserving there are $0 \leq a < b \leq 1$ such that $f(U) \supset [a,b]$ holds for every neighbourhood $U$ of $x$. Consequently, by Lemma 1.4 we also have $f(H) \supset [a,b]$ whenever $H$ is any $G_{<p}$ set containing the point $x$. In particular, this implies that if the singleton $\{x\}$ is a $G_{<p}$ set (equivalently, if the character of $x$ in $X$ is less than $p$) then $f$ is continuous at $x$.

On the other hand, by Lemma 1.4, for every point $x \in X$ there is a closed $G_{<p}$ set $H_x$ with $x \in H_x$ such that for any point of continuity $z \in H_x$ of $f$ we have $f(z) = f(x)$. We claim that $f$ is constant on every such set $H_x$ and then, by the above, $f$ is continuous at every point $x \in X$.

For this it suffices to show that $f$ has a point of continuity in every (non-empty) closed $G_{<p}$ set $H$. Indeed, for any point $y \in H_x$ then the intersection $H_x \cap H_y$ contains a point of continuity $z$ for which $f(x) = f(z) = f(y)$ must hold. By the Čech-Pospíšil theorem (see e.g. [3, 3.16]) and by $|H| < 2^p$ there is a point $z \in H$ with $\chi(z,H) < p$ and so $\chi(z,X) < p$ as well, for $H$ is a $G_{<p}$ set in $X$. But we have seen above that then $z$ is indeed a point of continuity of $f$.

**Theorem 4.9.** Assume that $2^\omega < 2^p$ and $X$ is a locally connected and sequential compact $T_2$ space with $\check{c}(X) \leq p$. Then $Pr(X,T_2)$ holds.

**Proof.** By a slight strengthening of some well-known results of Shapirovski (see e.g. [3, 2.37 and 3.14]), for any compact $T_2$ space $X$ we have both $\pi \chi(X) \leq t(X)$ and

$$d(X) \leq \pi \chi(X)^{<\check{c}(X)}.$$
Consequently, for our space \( X \) we have
\[
d(X) \leq \omega^{<\psi} = 2^\omega
\]
and so by sequentiality \(|X| \leq 2^\omega\) as well. But this shows that all the conditions of Theorem 4.8 are satisfied by our space \( X \).

To conclude, let us emphasize again that Lemma 1.3, i.e. the full force of compactness preservation, as opposed to just the preservation of the compactness of convergent sequences, was only used in this section (cf. the remark made after 1.3’).

### 5. The relation \( Pr(X, T_1) \)

The main aim of this section is to prove that if \( Pr(X, T_1) \) holds and \( X \) is \( T_3 \) then \( X \) is discrete. Note the striking contrast between \( Pr(X, T_1) \) and \( Pr(X, T_2) \): the latter holds for a large class of (non-discrete) spaces (see Theorem 1.8 or Corollary 3.11).

Let us recall that the cofinite topology on an underlying set \( X \) is the coarsest \( T_1 \) topology on \( X \); the open sets are the empty set and the complements of the finite subsets of \( X \). It is not hard to see that such a space is hereditarily compact and any infinite subset in it is connected. Let us start with a result that gives several different characterizations of \( T_1 \) spaces \( X \) that satisfy \( Pr(X, T_1) \).

**Theorem 5.1.** For a \( T_1 \) space \( X \) the following conditions are equivalent:

a) If \( Y \) is \( T_1 \) and \( f : X \to Y \) is a connectedness preserving function then \( f \) is continuous.

b) If \( Y \) is \( T_1 \) and \( f : X \to Y \) is a preserving function then \( f \) is continuous (i.e. \( Pr(X, T_1) \) holds).

c) If \( Y \) has the cofinite topology and \( f : X \to Y \) is a preserving function then \( f \) is continuous.

d) If \( A \subset X \) is not closed then there exists a connected set \( H \subset X \) such that \( H \cap A \neq \emptyset \neq H - A \) and \( H - A \) is finite.

**Proof.** a) \( \Rightarrow \) b) and b) \( \Rightarrow \) c) are obvious.

c) \( \Rightarrow \) d) Assume that \( A \subset X \) is not closed. Let \( Y \) denote the space with the cofinite topology on the underlying set of \( X \). Choose a point \( a_0 \in A \). (\( A \) is not closed so it is not empty, either.) Define now the function \( f : X \to Y \) by
\[
f(x) = \begin{cases} a_0, & \text{if } x \in A, \\ x, & \text{otherwise.} \end{cases}
\]

Then \( f \) is not continuous because the inverse image of the closed set \( \{a_0\} \) is the non-closed set \( A \) hence \( ,\text{by c), } f \) is not preserving. As an arbitrary subset of \( Y \) is compact, \( f \) preserves compactness, so there is a connected set \( H \subset X \) such that \( f(H) \) is not connected. It follows that \( H \) is infinite and \( f(H) \) is finite but not a singleton. As \( f \) is the identity map on \( X - A \), the
set $H - A$ is finite and so $H \cap A \neq \emptyset$. Finally, $H \subset A$ is impossible because $f(H)$ is not a singleton.

d) $\Rightarrow$ a) Assume $f : X \to Y$ is not continuous for a $T_1$-space $Y$, hence there is a closed set $F \subset X$ such that $A = f^{-1}(F)$ is not closed in $X$. By d), there is a connected set $H$ such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. But then $f(H)$ is not connected because it is the disjoint union of two non-empty relatively closed sets, namely of $f(H) \cap F$ and of the finite set $f(H) - F$. Consequently, $f$ does not preserve connectedness. \hfill $\square$

**Corollary 5.2.** If $Pr(X,T_1)$ holds for a $T_1$-space $X$ then every closed subspace of $X$ is the topological sum of its components.

*Proof.* Let $K$ be a component of the closed subset $F \subset X$. It is enough to prove that $K$ is relatively open in $F$. Assume this is false; then $A = F - K$ is not closed in $X$, and thus, by condition d) of Theorem 5.1, there is a connected set $H$ in $X$ such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. Then $H - F$ is also finite, consequently $H \subset F$ because $H$ is connected and $F$ is closed. Thus $H$ is a connected subset of $F$ which meets the component $K$ of $F$, contradicting that $H \cap A \neq \emptyset$. \hfill $\Box$

**Corollary 5.3.** If $Pr(X,T_1)$ holds for a $T_3$-space $X$ then every closed subspace of $X$ is locally connected.

*Proof.* By 5.2 it is enough to prove that if every closed subset of a regular space $X$ is the topological sum of its components then $X$ is locally connected.

Let $U$ be a closed neighbourhood of a point $x \in X$. By our assumption if $K$ denotes the component of $x$ in $U$ then $K$ is open in $U$, hence $K \subset U$ is a connected neighbourhood of $x$ in $X$. As the closed neighbourhoods of a point form a neighbourhood-base of the point in a regular space, $X$ is locally connected. \hfill $\square$

**Theorem 5.4.** If $Pr(X,T_1)$ holds for a $T_3$-space $X$ then $X$ is discrete.

By Corollary 5.3 it is enough to prove the following result that, we think, is interesting in itself:

**Theorem 5.5.** If $X$ is $T_3$ and every regular closed subspace of $X$ is locally connected then $X$ is discrete.

*Proof.* We can assume without any loss of generality that $X$ is connected. Suppose, indirectly, that $x$ is a non-isolated point in $X$. By regularity, there is a sequence of non-empty open sets $\{G_n : n < \omega\}$ such that $x \notin G_n$ and $\overline{G}_n \subset G_{n+1}$ for all $n < \omega$. Then the open set $G = \bigcup G_n$ can not be also closed in the connected space $X$, so there is a point $p \in \overline{G} - G$.

Put $U_0 = G_0$ and $U_n = G_n - \overline{G}_{n-1}$ for $0 < n < \omega$. If $H_0 = \bigcup \{U_n : n \text{ is even}\}$ and $H_1 = \bigcup \{U_n : n \text{ is odd}\}$, we claim that then $\overline{G} = \overline{H}_0 \cup \overline{H}_1 = \overline{H_0} \cup \overline{H_1}$. Indeed, if $z \in \overline{G}$ and $W$ is any open neighbourhood $W$ of $z$ then
there is a $k < \omega$ such that $W \cap G_k \neq \emptyset$. Clearly, if $n$ is the least such $k$ then we have $\emptyset \neq W \cap U_n$ as well, and so $x \in \overline{H}_0 \cup \overline{H}_1$.

Consequently, $p \in \overline{H}_0$ or $p \in \overline{H}_1$; assume e. g. that $p \in \overline{H}_0$. We shall show that then $\overline{H}_0$ is not locally connected at $p$, although it is a regular closed set, arriving at a contradiction.

Indeed, let $U$ be any neighbourhood of $p$ in $\overline{H}_0$ and fix an even number $n < \omega$ with $U_n \cap U = \emptyset$. Then $U \cap U_{n+1} \subset \overline{H}_0 \cap H_1 = \emptyset$ implies $U \subset \overline{G}_n \cup (X \setminus G_{n+1})$, where $\overline{G}_n$ and $X \setminus G_{n+1}$ are disjoint closed sets both meeting $U$, hence $U$ is disconnected. \( \square \)

With a little more effort it can also be shown that for any non-isolated point $p$ in a $T_3$ space $X$ there is a regular closed set $H$ in $X$ with $p \in H$ such that $p$ is not a local connectivity point in $H$.

We do not know if every $T_2$ space $X$ with $\text{Pr}(X,T_1)$ has to be discrete. Also, the following $T_2$ version of Theorem 5.5 seems to be open: If $X$ is $T_2$ and all closed subspaces of $X$ are locally connected then $X$ has to be discrete. Note that if $X$ has the cofinite topology then it is hereditarily locally connected and satisfies $\text{Pr}(X,T_1)$.

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