The world according to Rényi: thermodynamics of fractal systems

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Abstract. We discuss a basic thermodynamic properties of systems with multifractal structure. This is possible by extending the notion of Gibbs–Shannon's entropy into more general framework - Rényi's information entropy. We show a connection of Rényi's parameter $q$ with the multifractal singularity spectrum $f(\alpha)$ and clarify a relationship with the Tsallis–Havrda–Charvat entropy. Finally, we generalize Hagedorn's statistical theory and apply it to high–energy particle collisions.

INTRODUCTION

One of the fundamental observations of information theory is that the most general functional form for the mean transmitted information (i.e., information measure) is that of Rényi. Although Rényi's information measure offers perhaps the most general and conceptually cleanest setting for the entropy, it has not found so far as much applicability as its Shannon's counterpart. To clarify the position of Rényi's entropy in physics, we resort to systems with a multifractal structure. Such systems are very important and highly diverse, including phase transitions, turbulent flow of fluids, irregularities in heartbeat, population dynamics, chemical reactions, plasma physics, and most recently the motion of groups and clusters of stars. We shall argue that for the aforementioned the Rényi parameter is connected via a Legendre transformation with the multifractal singularity spectrum. To put some flesh on bones we generalize Hagedorn's statistical theory and subsequently apply to a differential cross section in high–energy scattering experiments. More thorough investigation will be published elsewhere.

RÉNYI'S ENTROPY

Motivation

From information theory follows that the most general information entropy is that of Rényi [1]. In discrete cases where the probability distribution $\mathcal{P} = \{p_n\}$ the Rényi entropy is defined as

$$I_q(\mathcal{P}) = \frac{1}{(1-q)} \log_2 \left( \sum_{k=1}^{n} p_k^q \right).$$
On the other hand, in continuous probability cases a renormalization is needed - with an arbitrary precision of measurement comes infinity of information. If \( f(x) \) is a probability density, say in the interval \([a, b]\) one may define the integrated probability
\[
p_{nk} = \int_{k/n}^{(k+1)/n} f(x) dx,
\]
then [1]
\[
I_q(f) \equiv \lim_{n \to \infty} \left( I_q(p_{nk}) - \log_2 n \right) = \frac{1}{(1-q)} \log_2 \left( \int_a^b f^q(x) dx \right). \tag{1}
\]
Eq.(1) might be generalized to any Lebesgue or Hausdorff measurable sets [2].

In the former context a natural question arises; how comes that there are other information entropies apart from Shannon’s one. To understand this we should go to information theory. The latter asserts that the amount of information received by learning that an event of probability \( p \) took place (in bit units) is \( I(p) = -\log_2(p) \). In general, if the possible outcomes of an experiment are \( A_1, A_2, \ldots, A_n \) with corresponding probabilities \( p_1, p_2, \ldots, p_n \), and \( A_k \) conveys \( I_k \) bits, then the mean conveyed information is
\[
I(P) = \sum_{k=1}^{n} p_k I_k.
\]

However, the linear averaging is only a specific case of a more general mean! It has been recognized by A.Kolmogorov [3] and M.Nagumo [4] that the most general mean compatible with postulates of probability theory gives the entropy
\[
I_g(P) = g^{-1} \left( \sum_{k=1}^{n} p_k g(I_k) \right),
\]
where \( g \) is an arbitrary invertible function.

Applying the postulate of additivity of independent information one obtains only two possible classes of \( g \) [1], namely \( g(x) = cx + d \) which implies \( I(P) = -\sum_{k=1}^{n} p_k \log_2(p_k) \) (i.e., Shannon’s information measure) and \( g(x) = c2^{(1-q)x} + d \) which implies directly Rényi’s information measure \( I_q(P) \).

Among the basic properties of Rényi’s entropy we may mention; positivity \( (I_q \geq 0) \), for \( q \leq 1 \) Rényi’s entropy is concave but for \( q > 1 \) is not pure convex nor pure concave and, in addition, when \( q \) is continued to 1, Rényi’s entropy equals Shannon’s one, i.e., \( \lim_{q \to 1} I_q = I \).

In physics Rényi’s entropy has been sporadically, albeit successfully applied in various non–equilibrium dynamical systems, e.g., fully developed turbulence \( (q \) directly relates with Reynolds number), percolating clusters \( (q \) directly describes \( p_c \) ), etc.. It also provides a consistent mathematical setting for Tsallis–Havrda–Charvat (THC) entropy and, as we shall see, it is a correct entropy for (multi)fractal systems.
Connection with Tsallis–Havrda–Charvat entropy

THC entropy introduced originally by J.H.Havrda and F.Charvat [5] and later applied to physical problems by C.Tsallis [6] is currently fruitfully used in many statistical systems; 3–dimensional fully developed hydrodynamic turbulence, 2–dimensional turbulence in pure electron plasma, Hamiltonian systems with long–range interactions, granular systems, systems with strange non–chaotic attractors, peculiar velocities in galactic clusters, etc.. Its form reads

\[ S_q = \frac{1}{(1-q)} \left[ \sum_{k=1}^{n} (p_k)^q - 1 \right], \quad q > 0. \]

Among important properties of THC entropy we can mention positivity \((S_q \geq 0)\), concavity in \(P\), gibbsian limit: \(\lim_{q \to 1} S_q = I\), and peculiar non–extensive behavior

\[ S_q(A + B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B), \]

for two independent events \(A\) and \(B\).

Rényi’s entropy vs. THC entropy

To find a connection between Rényi and THC entropies we utilize the expansion of \(\log_2 (1 + x)\). Then we may write

\[ I_q = \frac{1}{(1-q)} \log_2 \left[ (1-q)S_q + 1 \right] = \frac{1}{k} S_q - \frac{1}{2k} (1-q)S_q^2 + O \left[ (1-q)^2 S_q^3 \right], \quad (2) \]

with the scale factor \(k = \ln 2\). So \(I_q \approx S_q\), provided

\[ \frac{1}{|1-q|} \left[ \sum_{l} (p_l)^q - 1 \right]^2 \ll 1. \quad (3) \]

Condition (3) is fulfilled in numerous ways. For instance, when \(q \approx 1, q \gg 1\), for systems with large deviations or for rare events systems [7].

At this stage some comments are in order. First of all we see from (2) that THC entropy and Rényi’s entropy are monotonic functions of each other and, as a result, both are extremized by the same \(P\). However, while Rényi’s entropy is additive, THC entropy is not, so it appears that the additivity property is not important for entropies required for extremizing purposes. Thus from thermodynamic point of view both entropies give the same predictions (see e.g.,[8])!

Secondly, as we show in [7], Rényi’s entropy provides a consistent renormalization prescription for a continuous THC entropy, we find that

\[ S_q(f) \equiv \lim_{n \to \infty} \left( \frac{S_q(f)}{n^{1-q}} - \frac{S_q(1/n)}{n^{1-q}} \right) = \frac{1}{(1-q)} \int_a^b dx f(x) \left( f^{q-1}(x) - 1 \right). \]
Brief introduction into (multi)fractal sets

Fractals

Let us begin to illustrate the basic features of fractals sets on a simple example - triadic Koch curve (TKC). The latter is defined iteratively in the following way: in 0th iteration \( n = 0 \) we start with a straight line - initiator - with length \( r_0 = a \). In the following step \( n = 1 \) we raise an equilateral triangle over the middle third of initiator. The result is generator. Its four straight line segments \( N_1 = 4 \) have length \( r_1 = a / 3 \) and total length \( L[a/3] = 4a/3 \). The construction of the Koch curve proceeds by replacing each segment of initiator with generator, i.e., for \( n = 2 \), \( r_2 = (1/3)^2 a \), \( L[(1/3)^2 a] = (4/3)^2 a \) and \( N_2 = 16 \), etc. So when \( n = k \) we have \( r_k = (1/3)^k a \), \( L[(1/3)^k a] = (4/3)^k a \) and \( N_k = 4^k \). Note that the length \( L \) diverges as \( k \to \infty \! 

Can we define somehow a finite length for the triadic Koch curve? The answer is yes, provided we extend the notion of euclidean dimension. To see this let us note that for ordinary smooth curves the approximative length is \( L[r] \sim N(r) r \), and as \( r \) goes to zero \( L[r] \) approaches the finite limit - length; \( L = \lim_{r \to 0} N(r)r \).

Generalization to any \( D \)-dimensional volumes is then natural: \( V = \lim_{r \to 0} N(r)r^D \). However, in order to get \( V \) finite, the following scaling must apply

\[
N(r) \sim \frac{c}{r^D} \Leftrightarrow \log N(r) \sim c + D \log \frac{1}{r},
\]

and so

\[
\lim_{r \to 0} \frac{\log N(r)}{\log \frac{1}{r}} = D. \tag{4}
\]

As the LHS of (4) is well defined for wider class of sets than just usual metric spaces one may accept it as a generalized definition of dimension. The latter is usually called the Hausdorff–Besicovitch or fractal dimension. It should be stressed that \( D \) in (4) is not necessarily integer - price which is paid for the finiteness of the volume.

Thus, for instance, in the case of TKC the fractal dimension is

\[
D = \lim_{r \to 0} \frac{\log N(r)}{\log \frac{1}{r}} = \lim_{n \to \infty} \frac{\log 4^n}{\log \frac{1}{r_G}} = \frac{\log 4}{\log 3} = 1.26\ldots.
\]

One may often write (e.g., for strictly self–similar fractals), after \( n \) iterations \( N = N_G^n \) (\( N_G \) is the number of pieces of the generator) \( r = a r_G^n \) (\( r_G \) is the length of the segments of the generator). In such cases the fractal dimension follows from a simple analysis:

\[
\lim_{r \to 0} N(r)r^D = \lim_{n \to \infty} \left(N_G r_G^D\right)^n = \text{const.} \Rightarrow D = \frac{\log N_G}{\log \frac{1}{r_G}}. \tag{5}
\]

Relation (5) allows to recover fairly simply some standard results; e.g., the well known triadic Cantor dust \((r_G = 1/3, N_G = 2)\) has \( D = \log 2/\log 3 \).
**Multifractals**

Multifractals are related to the study of a distribution of physical or other quantities on a generic support (be it or not fractal) and thus provide a move from the geometry of sets as such to geometric properties of distributions. An intuitive picture about an inner structure of multifractals is obtained by introducing the $f(\alpha)$ spectrum [9]. To elucidate the latter let us suppose that some support (usually a subset of a metric space) is covered by probability of a certain phenomenon. If we pave the support with boxes of size $l$ and denote the integrated probability in the $i$th box as $p_i$, we may define the scaling exponent $\alpha_i$ by $p_i(l) \sim l^{\alpha_i}$. The factor $\alpha_i$ is called Lipshitz–Hölder exponent. Counting boxes $dN(\alpha, l)$ in which $p_i$ has $\alpha_i \in (\alpha, \alpha + d\alpha)$, then the singularity spectrum $f(\alpha)$ is defined as $dN(\alpha, l) = n(\alpha)l^{-f(\alpha)}d\alpha$ (the proportionality function $n(\alpha)$ is $l$ independent). Accordingly, we may view a multifractal as the ensemble of intertwined (mono)fractals each with its own fractal dimension $f(\alpha_i)$. It is thus suggestive to define the “partition function”

$$Z(q) = \sum_i p_i^q = \int d\alpha' n(\alpha') l^{-f(\alpha')} l^{q\alpha'}.$$  \hspace{1cm}(6)

In the small $l$ limit the partition function (6) scales as $Z(q) \sim l^\tau$, where

$$\tau(q) = \min_\alpha (q\alpha - f(\alpha)), \quad f'(\alpha(q)) = q.$$  \hspace{1cm}(7)

Eq.(7) represents defining relations for the Legendre transformation.

**Rényi’s entropy - entropy of self–similar systems**

Let us now turn to the question whether there is any connection of Rényi’s entropy with (multi)fractal systems. At present it seems to us that there are two such connections.

**a) Formal connection - generalized dimensions**

Generalized dimensions are defined as:

$$D_q = \lim_{l \to 0} \left( \frac{1}{(q-1) \log l} \log Z(q) \right) = -\lim_{l \to 0} I_q(l).$$

For example, $D_0$ is the usual fractal dimension - dimension of the support, $D_1$ is known as information dimension and $D_2$ is correlation dimension. $D_0, D_1$ and $D_2$ are usually sufficient to describe simple fractals (e.g., strictly self similar ones). However, in general all $D_q$ are necessary to pinpoint fractals uniquely. This is typical e.g., for strange attractors [11]! The situation is somehow analogous to statistical physics when the whole tower of correlation function equations (BBGKY hierarchy) is needed to get the full information on density matrix.
b) Direct physical connection

We will show now that from the maximal entropy (MaxEnt) point of view, extremizing the Gibbs–Shannon entropy on fractals is equivalent to extremizing directly Rényi’s entropy without invoking the underlying fractal structure explicitly.

Let us have a multifractal with a measure \( p(x) \). Shannon’s entropy for the corresponding process is 
\[
I = - \sum p_k \log_2 p_k.
\]
The Billingsley theorem then states [10] that there is an intimate connection between Shannon’s entropy and the Hausdorff dimension of the measure theoretic support \( M \) of \( p(x) \) (i.e., the infimum of the dimensions of all sets on which \( p(x) \) lives). Namely,
\[
d_h(M) = - \lim_{N \to \infty} \frac{1}{\log_2 N} \sum_k p_k \log_2 p_k \sim \frac{1}{\log_2 \varepsilon} \sum_k \mu_k(\varepsilon) \log_2 \mu_k(\varepsilon),
\]
with the cutoff scale \( \varepsilon \sim 1/N \). In this connection it is useful to introduce a one-parametric family of normalized measures \( \mu(q) \) (escort or zooming distributions)
\[
\mu_i(q,l) = \left[ \frac{p_i(l)}{\sum_j p_j(l)} \right]^q.
\]

It is important to notice that the parameter \( q \) provides a microscope for exploring different regions of the singular measure. Indeed, for \( q > 1 \), \( \mu(q) \) amplifies the more singular regions of \( p \), while for \( q < 1 \), \( \mu(q) \) accentuates the less singular ones. So one may zoom into any required regions of fractality. The corresponding “zooming” entropy
\[
\tilde{I}(q) = - \sum \mu_k \log_2 \mu_k, \text{ and } d_h \text{ of the measure theoretic support of } \mu(q) \text{ then reads}
\]
\[
f(q) = - \lim_{N \to \infty} \frac{1}{\log_2 N} \sum_k \mu_k \log_2 \mu_k \sim \frac{1}{\log_2 \varepsilon} \sum_k \mu_k(\varepsilon) \log_2 \mu_k(\varepsilon). \tag{8}
\]

In addition, the average value of the singularity exponent \( \alpha_i = \log_2 (p_i) / \log_2 \varepsilon \) with respect to \( \mu(q) \) is
\[
\alpha(q) = \frac{\sum_k \mu_k(\varepsilon) \log_2 p_k(\varepsilon)}{\log_2 \varepsilon}. \tag{9}
\]
Eqs.\( (8) \) and \( (9) \) establish a relationship between a Hausdorff dimension \( f(q) \) and an average singularity exponent \( \alpha(q) \) via functional dependence on the parameter \( q \). Note that \( f = q\alpha - \tau, \alpha = d\tau/dq \) is precisely the Legendre transformation. Thus Shannon’s entropy on a multifractal with a given \( f(\alpha) \)
\[
- \sum_k p_k \log_2 p_k \bigg|_{f(\alpha)} = - \sum_k \mu_k \log_2 \mu_k \bigg|_{\alpha(q)} = -q\alpha(q) \log_2 \varepsilon + (1 - q) I_q(\mathcal{P}). \tag{10}
\]

So as long as we fix the “fractality” condition, Shannon’s entropy \( I \) turns out to be (up to an additive constant) Rényi’s entropy. Thus, from MaxEnt point of view \( I|_{f(\alpha)} \) and \( I_q \) are completely interchangeable\(^1\).

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\(^1\) In [7] we show that (1) together with \( \varepsilon \to 0 \) imply that \( I(f)|_{f(\alpha)} = (1 - q) I_q(f) \).
GENERALIZED HAGEDORN’S STATISTICAL THEORY

Hagedorn’s statistical theory is applicable whenever the density of quantum states grows exponentially with temperature, i.e., when

$$v(E) \propto \exp[\beta_H E] .$$

(11)

Assumption (11) applies for example to (quantized) string theory [12], to cosmic string theory [13], or to high–energy particle collisions [14].

Except a proliferation of the states near $$T_H = 1/\beta_H$$, the state space acquires an approximately multifractal structure which is exact at critical point [14, 15]. This suggests that Rényi’s rather than Gibbs entropy could better grasp vital features near $$T_H$$.

Differential cross section in high–energy scattering experiments

Recent experiments suggest that Hagedorn’s theory is not adequate for generic high–energy scattering experiments. For instance, in $$e^+e^-$$ processes Hagedorn’s description is satisfactory provided CMS energies are small ($$E < 10 Gev$$) but it fails at large energies. Hagedorn’s approach at $$E > 10 GeV$$ predicts an exponential decay of differential cross sections while experiments observe a power–law behavior [16].

It addition, the latest applications of THC entropy in the context of high energy collisions [17], high energy cosmic ray physics [18] or a Focker–Planck equation treatment of charmed quarks in quark–gluon plasma [19] fit far better with experimental data than predictions based on the Gibbs–Hagedorn approach.

Hagedorn’s theory basically predicts that the differential cross section in high–energy collisions should be

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} = cp_T \int_0^\infty dx e^{-\beta \sqrt{x^2 + \mu^2}}, \mu = \sqrt{p_T^2 + m^2},$$

which for large $$p_T$$ asymptotically behaves as

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} \sim p_T^{3/2} e^{-\beta p_T} .$$

Yet, for $$ep$$ high–energy collisions the results are best fitted by a power law [16];

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} \sim (1 + const \ p_T)^{-\gamma}, \ \gamma = 5.8 \pm 0.5 .$$

(12)

Whereas ordinary thermodynamics (and hence Hagedorn’s theory itself) is derived by extremizing Gibbs–Shannon entropy we extremize now $$I_q$$ instead. Following Hagedorn [14] we arrive at the probability density of transverse momenta $$\rho(p_T)$$ [7];

$$\rho(p_T) \propto u \int_0^\infty dx \left(1 + (q - 1)\beta \sqrt{x^2 + u^2 + m_\rho^2}\right)^{-\frac{q}{q-1}},$$

2 Actually Rényi’s statistics also modifies $$T_H$$. Usually $$T_R \leq T_H$$
where $u = \beta p_T$ and $m_{\beta} = \beta m_0$. Using the fact that $\rho(p_T) \propto \sigma^{-1} d\sigma/dp_T$ we have

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} \sim c \sqrt{2(q-1)} B \left( \frac{1}{2}, \frac{q}{q-1} - \frac{1}{2} \right) u^{3/2} (1 + (q - 1)u)^{-\frac{q}{q+1} + \frac{1}{2}}. \quad (13)$$

Formula (13) agrees well with the fit (12), provided one suitably determines parameters $c, q, T$. This in turn may specify the multifractal structure of the state space.

Finally note that (13) has its maximum at $p_T = T \left( \frac{3}{3-q} \right)$, on the other hand, the experimentally measured cross sections, e.g., in relativistic heavy ion–collisions have maximum at roughly the same value of $p_T \approx 180\text{MeV}$, so $T_R \approx (1 - q/3)p_T \approx 105\text{MeV} < T_H = 158\text{MeV}$.

**OUTLOOK AND OPEN QUESTIONS**

Because of a build–in predisposition to account for self–similar systems Rényi’s entropy naturally aspires to be an effective tool to describe phase transitions. It is thus a challenging task to find a closer connection with such typical tools of critical–phenomena physics as are conformal and renormalization groups.

In addition, witnessing an encouraging agreement of THC non–extensive statistics predictions with the cosmic ray experiments [18] or heavy–ion collisions [20] we may naturally ask whether there is a physical grounding for Rényi’s entropy in similar (non–equilibrium) systems. Work along those lines is currently in progress.

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