Counting and localizing defective nodes by Boolean network tomography

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Abstract

Identifying defective items in larger sets is a main problem with many applications in real life situations. We consider the problem of localizing defective nodes in networks through an approach based on boolean network tomography (BNT), which is grounded on inferring informations from the boolean outcomes of end-to-end measurements paths. Identifiability conditions on the set of paths which guarantee discovering or counting unambiguously the defective nodes are of course very relevant. We investigate old and introduce new identifiability conditions contributing this problem both from a theoretical and applied perspective. (1) What is the precise trade-off between number of nodes and number of paths such that at most $k$ nodes can be identified unambiguously? The answer is known only for $k = 1$ and we answer the question for any $k$, setting a problem implicitly left open in previous works. (2) We study upper and lower bounds on the number of unambiguously identifiable nodes, introducing new identifiability conditions which strictly imply and are strictly implied by unambiguous identifiability; (3) We use these new conditions on one side to design algorithmic heuristics to count defective nodes in a fine-grained way, on the other side to prove the first complexity hardness results on the problem of identifying defective nodes in networks via BNT. (4) We introduce a random model where we study lower bounds on the number of unambiguously identifiable defective nodes and we use this model to estimate that number on real networks by a maximum likelihood estimate approach.

1 Introduction

Identifying a subset of defective items out of a much larger set of items is a problem that found numerous application in a variety of situations such as medical screening, network reliability, DNA screening, streaming algorithms. Network Tomography is a general inference technique based on end-to-end measurements aimed to extract internal network characteristics such as link delays and link loss rates but also defective items. In this paper, we consider Boolean Network Tomography (BNT) where the outcome of the measurements is a boolean value. Duffield, who as first introduced boolean network tomography [3] to identify network failure components, proposed an inference algorithm based on BNT to identify sets of failure links. The BNT approach was later studied also to identify node failures in networks [2,6,8,11–13].

In the case of identifying failure nodes, the BNT approach deals with extracting as much accurate as possible information on the number and the positions of the corrupted nodes from the solutions $\vec{x}$ of a boolean system $P\vec{x} = \vec{b}$, where $P$ is the incidence matrix of the $m$ measurement paths over the $n$ nodes and $\vec{b}$ is the $m$-vector of the boolean outcomes of the measurement paths (see Figure 1). The challenge of localizing failure nodes is that different sets of failure nodes can produce the same measurement along the paths and so are indistinguishable from each other using the measurements. This leads to pose the following question: given the set of paths $P$ what is the maximal set of defective nodes we can hope to identify unambiguously? Identifiability conditions on the matrix $P$ under which failure nodes can be localized unambiguously (or also counted accurately) from the solution of the system $P\vec{x} = \vec{b}$ are of course of the utmost interest. In this paper we study old and introduce new of these conditions, contributing:
1. to understand the combinatorics and the complexity of the theoretical problem of unambiguously identify failure node sets under the BNT approach, and,

2. to devise new algorithms and heuristics to count or localize as more precisely as possible failure nodes in networks.

1.1 Previous work

The condition introduced as $k$-identifiability (for $\mathcal{P}$) states that any two distinct node sets of size at most $k$ can be separated by at least a path in $\mathcal{P}$. $k$-identifiability initially introduced for link failure detection [10,14], was later studied with success also for node failure detection [2,6,8,11–13]. If this condition is true for a set of measurements paths $\mathcal{P}$ it ensures that if there are at most $k$ failure nodes in $\mathcal{P}$ then these nodes can be identified unambiguously. Hence the optimization problem of computing the maximal $k \leq n$ such that a set $\mathcal{P}$ is $k$-identifiable ($k$-ID), i.e. admits the $k$-identifiability property is very relevant to the problem of node failure localization. We refer to this maximal value as $\mu(\mathcal{P})$ (it was called $\Omega(\mathcal{P})$ in [11]).

As observed in [11,12] $k$-identifiability can be scaled to each single node yet preserving the property for the whole set of paths. A node $u$ is $k$-ID if any two sets of size at most $k$ differing on $u$ are separated by at least a path in $\mathcal{P}$ (see Definition 2.2). Hence understanding the combinatorics of the set $\text{ID}_k(\mathcal{P})$ of the $k$-identifiable nodes in $\mathcal{P}$ and study upper and lower bounds for $|\text{ID}_k(\mathcal{P})|$ is of great importance to develop algorithms to maximize the identification of failure nodes in real networks.

Both definitions of $k$-identifiability were largely investigated. In [12] they started this study quantifying the capability of failure localization through (1) the maximum number of failures such that failures within a given node set can be localized unambiguously, and (2) the largest node set failures can be uniquely localized under a given bound on the total number of failures. These measures where used to evaluate the impact of maximum identifiability on various parameters of the network (underlying the set of paths) like the topology, the number of monitor and the probing mechanisms.

In the work [6,7] we studied $k$-identifiability from the topological point of view of the graph underlying $\mathcal{P}$. We were proving tight bounds on the maximum identifiability that can be reached in the case of topologies like trees, grids and hypergrids and under embeddings on directed graphs. Our results culminated with a heuristic to design networks with a high degree of identifiability or to modify a network to boost identifiability. In the work [8] we were employing Menger’s theorem establishing a precise relation of $\mu(\mathcal{P})$ with the vertex connectivity of the graph underlying $\mathcal{P}$. We generalize results in [6] to Line-of-Sight Networks and started the study of identifiability conditions on random graphs and random regular graphs.

Monitor placement can be in fact relevant to improve identifiability of failure nodes. The works [2,11] considered the problem of optimizing the capability of identifying network failures through different monitoring schemes giving upper bounds on the maximum number of identifiable nodes, given the number of monitoring paths, the routing scheme and the maximum path length. In [2] in particular studied upper bounds on the set of $|\text{ID}_1(\mathcal{P})|$ and as in...
In this work we introduce new identifiability measures and deepen the study of \( k \)-identifiability obtaining several new results and new heuristics to test networks against the number and position of failing nodes. From a theoretical perspective our contributions are the following:

1. We set a question implicitly left open in some previous works \([2],[10]\) about the limits of upper bounds on identifiability of node failures via Boolean network tomography. What are the precise tradeoffs between number of nodes \( n \) and number of paths \( m \) of \( \mathcal{P} \) such that \( \mathcal{P} \) is no longer \( k \)-identifiable, that is \( \mu(\mathcal{P}) < k \)? The answer is known only for \( k = 1 \) where the tradeoff \( n \geq 2^m - 1 \) implies \( \mu(\mathcal{P}) < 1 \), is obtained by a straightforward counting argument (see Lemma 3.1 and [2]).

Using the notion of regular union-free families, we answer to the problem for any \( 2 \leq k \leq n \), showing that \( n \geq 2^{\frac{1}{k-1}(m+k-1)^{1+\epsilon}} \) implies \( \mu(\mathcal{P}) < k \), for any \( \epsilon > 0 \).

2. We introduce two new identifiability notions, namely, \( k \)-separability (\( k \)-SEP) and \( k \)-distinguishability (\( k \)-DIS). Analogously to identifiability we define these notions on nodes and we consider the corresponding node sets \( \text{SEP}_k(\mathcal{P}) \) and \( \text{DIS}_k(\mathcal{P}) \). These conditions provide significant upper and lower bounds to identifiability: namely we prove that for all \( k \leq n \), \( k \)-SEP implies \( k \)-ID and \( k \)-ID implies \( k \)-DIS, both strictly. Hence \( \text{SEP}_k(\mathcal{P}) \subseteq \text{ID}_k(\mathcal{P}) \subseteq \text{DIS}_k(\mathcal{P}) \). We use these measures to get upper and lower bounds for \( |\text{ID}_k(\mathcal{P})| \) and \( \mu(\mathcal{P}) \), to study the computational complexity of identifiability conditions and to estimate the number of \( k \)-identifiable nodes through a random model. Namely:

3. We prove that the problem of deciding the non \( k \)-separability (hence the non \( k \)-identifiability) of a given node in \( \mathcal{P} \) is polynomial time reducible to the minimum hitting set problem (MHS). Furthermore we prove that the optimization problem of finding the minimal \( k \) such that a given node is not \( k \)-separable in \( \mathcal{P} \) is NP-complete. To our knowledge these are the first known hardness results of identifiability problems arising from boolean network tomography.

4. We introduce and study a random model for \( \mathcal{P} \) based on the binomial distribution and we estimate lower bounds on the number of \( k \)-identifiable nodes \( |\text{ID}_k(\mathcal{P})| \) in this model by analyzing the number of \( k \)-separable nodes in \( \mathcal{P} \).

5. We use node distinguishability to study upper bounds on the number of \( k \)-identifiable nodes parameterizing the search of such nodes in terms of specific subset of nodes and specific subset of paths in \( \mathcal{P} \). We introduce the relation (Definition 7.1) \( u \)-equal \( W \) modulo \( \mathcal{P} \), where \( u \) is a node, \( W \) a set of nodes and \( \mathcal{P} \) a family of paths in \( \mathcal{P} \) that characterizes non-distinguishability of \( u \) restricted to the set \( W \) with respect to \( \mathcal{P} \). A recursive construction (Definition 7.3 of \( \tau_k \)) built on the previous relation allows to upper bound efficiently the number of \( k \)-identifiable nodes in a fine-grained way.

From a more applied perspective our results have the following consequences and applications.

1. The result in item [1] can be used as an estimate of upper bounds on the number of \( k \)-identifiable nodes in \( \mathcal{P} \). As [2] use the result for \( k = 1 \) to prove that \( |\text{ID}_1(\mathcal{P})| \leq \min(n, 2^m - 1) \) (see Theorem 3.2), our bound proves the general statement that for all \( 2 \leq k \leq n \), \( |\text{ID}_k(\mathcal{P})| \leq \min\{n, 2^{\frac{1}{k-1}(m+k-1)^{1+\epsilon}}\} \) (Theorem 3.9). Our bound can also be used as a black-box in algorithms and heuristics aimed at approximating the number of identifiable \( (\text{[11],[13]}\) nodes which use the bound for \( k = 1 \). For instance the ICE heuristic of [2], that creates a set of paths \( \mathcal{P} \) reaching a certain value of \( \mu(\mathcal{P}) \), is generating paths according to the result for \( k = 1 \).

2. The fact that the MHS problem is reducible to the non-separability problem (item [5]) suggests the idea of using the minimal hypergraph transversal (instead of a minimum hitting set) to lower bound the number of separable nodes (hence identifiable nodes) in \( \mathcal{P} \). Given an order of the variables a minimum hypergraph transversal in
a set-system can be efficiently computed. We propose two algorithms based on the hypergraph transversal (Simple-SEP and Decr-SEP). In particular in the second algorithm we use a new idea which partition the set of nodes of $P$ in family of subsets of nodes called 0-decreasing which allow to apply in a more efficient way the hypergraph transversal heuristic (Decr-SEP).

3. We employ the random model in item 2 to approximately counting the number of $k$-identifiable nodes on concrete networks using an approach based on the maximum likelihood estimate for binomial distributions. Our experimental results indicate that a lower bound for the number of $k$-identifiable nodes of a real network can be computed very accurately using a relatively simple random model based on the binomial distributions and computing the probability that a node is $k$-separable in this model. We then consider a real set of measurement paths $\bar{P}$ as it was a random experiment, we plug in the MLE estimates on $\bar{P}$ in the probability formula of the random model to estimate the cardinality of the sets $\text{SEP}_k(\bar{P})$.

4. We use the definition of $\mathbb{E}_k$ to upper bound the number of $k$-identifiable nodes in $P$ according to specific families of subset of nodes and subset of paths. As we show in Section 7 this can be used to compute approximations of the value of $\mu(P)$ and $|\text{ID}_k(P)|$ which are efficiently computable (Algorithm 1b–DIS$_k$).

The paper is organized as follows: first we give the preliminary definitions on boolean network tomography and identifiability, showing the connection with unambiguous identification of failure nodes. In Section 3 we study the tradeoffs between number of nodes and number of paths. In Section 4 we give the definitions of $k$-separability and $k$-distinguishability and prove the relation with identifiability. In Section 5 we introduce the random model and we show how to count $k$-separable nodes (hence lower bounds on $k$-identifiable nodes) on real networks through a maximum likelihood estimate method. In Section 6 we present the results on the computational complexity of $k$-identifiability and we introduce two algorithms based on hypergraph transversal to count identifiable nodes. In Section 7 we introduce the definition of $\mathbb{E}_k$ and a corresponding method (based on distinguishability) to compute upper bounds on identifiable nodes in a fine-grained way, when the set of paths is obtained by taking all the paths in a graph from a set of sources to a set of target nodes.

2 Preliminary definitions

Let $n, k \in \mathbb{N}$ and $k \leq n$. $\binom{n}{k}$ is the set of subsets of $[n]$ of size $k$. $\binom{[n]}{k}$ be the set of subsets of $[n]$ of size at most $k$. $2^A$ is the set of subsets of the set $A$. $A \oplus B$ is the symmetric difference between $A$ and $B$. $\overline{A}$ denotes the complement of $A$.

Let $n$ and $m$ be positive integers. We encode a set of $m$ paths over nodes in $[n]$ as a collection $P$ of $n$ distinct $m$-bit vectors such that $0 \notin P$, i.e. the $m$-bit zero vector is not in $P$ (this condition means that each node in $[n]$ is used in at least a path).

We can view $P$ in three different ways: as a boolean $m \times n$-matrix, as a collection of $n m$-bit vectors and as a collection of $m n$-bit vectors. For a node $u \in [n]$, $e_u$ is then the $m$-bit vector whose $p$-th coordinate indicates whether the node $u$ is in the $p$-th path or not.

We use also $P$ in a graph notation as follows: if $u \in [n]$ is a node, then $P(u)$ identifies the set of all paths touching $u$, in other words the set $\{p \in [m] : P[p, u] = 1\}$. If $U \subseteq [n]$ is a set of nodes, $P(U)$ denotes the set of paths in $[m]$ touching at least a node in $U$, i.e. $P(U) = \bigcup_{u \in U} P(u)$.

2.1 Identifiability

Let $P$ be a set of $m$ paths over $n$ nodes. We consider the following definition [13].

**Definition 2.1.** $P$ is $k$-identifiable if for all $U, W \subseteq [n]$ such that $|U|, |W| \leq k$ and $U \neq W$, it holds that $P(U) \neq P(W)$.

Notice that in terms of the column-vector notation, the previous definition says that for all distinct sets $U, W \subseteq [n]$ of size at most $k$, $\bigvee_{u \in U} e_u \oplus \bigvee_{w \in W} e_w \neq 0$. 

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The definition of $k$-identifiability can be equivalently given for nodes $u \in [n]$ as follows (see also [11]).

**Definition 2.2.** ($k$-identifiable nodes) A node $u \in [n]$ is $k$-identifiable with respect to $P$, if for all $U, W \subseteq [n]$ of size at most $k$ and such that $U \cap \{u\} \neq W \cap \{u\}$, it holds that $P(U) \neq P(W)$.

$\text{ID}_k(P)$ denotes the set of $k$-identifiable nodes in $P$. From the definitions a second of thought allows to see that $k$-identifiability implies $k'$-identifiability for $k' < k$. Hence

**Lemma 2.3.** Let $P$ be a set of $m$ paths over $n$ nodes. Then $\text{ID}_k(P) \subseteq \text{ID}_{k'}(P)$ for $k' \leq k \leq n$.

Furthermore scaling to identifiability of nodes does not affect the main property of $k$-identifiability (which we see below). Next theorem is proved in [11] (Theorem 4).

**Theorem 2.4.** ( [11]) Let $P$ be a set of $m$ paths over $n$ nodes. $P$ is $k$-identifiable if and only if every node in $[n]$ is $k$-identifiable with respect to $P$.

We denote by $\mu(P)$ the maximal $k \leq n$ s.t. $P$ is $k$-identifiable.

Let us motivate our definitions in the context of the approach of boolean network tomography to detect failure nodes in networks. Assume to have a set $P$ of $m$ end-to-end paths over $n$ nodes $P$. A binary measurement $\mathcal{M}$ along a path $p \in [m]$ is obtained by sending a message through $p$ and recording the outcome $\mathcal{M}(p)$, a bit, which identifies (in the case $\mathcal{M}(p) = 1$) that some node in $p$ is failing, or (in the case $\mathcal{M}(p) = 0$) that no node is failing along the path $p$.

We claim that if $P$ is $k$-identifiable, then under any binary measurement $\mathcal{M}$, we can uniquely localize in $P$ up to $k$ failing nodes. Given a binary measurement $\mathcal{M}$ over $P$, let $\text{fail}_\mathcal{M}(P) = \{p \in [m] | \mathcal{M}(p) = 1\}$.

**Definition 2.5.** (Unique failure) Let $P$ be a set of $m$ paths over $n$ nodes and $\mathcal{M}$ a binary measurement on $P$. A set of nodes $W \subseteq [n]$ is failing in $P$ if $P(W) \subseteq \text{fail}_\mathcal{M}(P)$. $W$ is uniquely failing if it is failing and furthermore $P(W) \subseteq \text{fail}_\mathcal{M}(P)$, i.e. on any path not touching $W$ the measurement is not failing.

**Theorem 2.6.** Let $P$ be a set of $m$ paths over $n$ nodes. If $\mu(P) \geq k$, then there is exactly one set of nodes of size at most $k$ that is uniquely failing in $P$.

**Proof.** Let $\mathcal{M}$ be a measurements over $P$. Assume by contradiction that there are two distinct sets $U$ and $W$ of size at most $k$ which are both uniquely failing in $P$ under $\mathcal{M}$. Since $U \neq W$, and $P$ is $k$-identifiable, then there is either a $p \in P(U) \setminus P(W)$ or a $p \in P(W) \setminus P(U)$. Assume wlog the former. Since $U$ is failing, then $\mathcal{M}(p) = 0$. But since $W$ is uniquely failing and $\mathcal{M}(p) = 1$, then $p \in \text{fail}_\mathcal{M}(P)$ and hence $\mathcal{M}(p) = 1$. Contradiction.

## 3 Upper bounds on $\mu(P)$ by counting

In this subsection we show that under what bounds on the number of paths $m$ in $P$, we have that $\mu(P) < k$. We start by showing under what conditions on $m$, $\mu(P) < 1$.

Notice that to prove that $\mu(P) < 1$, by Definition 2.1 it is sufficient to find two distinct nodes $u, w \in [n]$ such that $c_u \oplus c_w = 0$, that is for all $p \in [m] : c_u[p] = c_w[p]$. $\mu(P) < 1$ will follow from an easy information theoretic bound on sets of $m$-vectors.

**Lemma 3.1.** Let $P$ be a set of $m$ paths built on $n$ nodes. If $m < \log_2(n+1)$, then $\mu(P) < 1$.

**Proof.** $P$ is a collection of $n$ $m$-bit strings. There are at most $2^m - 1$ different such strings ($0 \notin P$). Hence whenever $n > 2^m - 1$ there are two elements $u \neq w \in [n]$ such that $c_u = c_w$, which means $c_u \oplus c_w = 0$.

Corollary IV.1 in [2] can be obtained by previous observation immediately.

**Theorem 3.2.** ( [2]) Let $P$ be a set of $m$ paths over $n$ nodes. Then $|\text{ID}_1(P)| \leq \min\{n, 2^m - 1\}$.

**Proof.** $|\text{ID}_1(P)| \leq n$ since it is a set of nodes. Assume that $n > 2^m - 1$, hence by previous Lemma 3.1 $\mu(P) = 0$, hence there are at least two nodes $u_1 \neq u_2$ not 1-identifiable. Hence $|\text{ID}_1(P)| \leq 2^m - 1$.

We will prove a similar results for $\mu(P) < k$ for a generic $k \leq n$.}


3.1 Union-free families and upper bounds for $k$-identifiability

A hypergraph $\mathcal{F}$ on the set $[m]$ is a family of distinct subsets of $[m]$, called edges of $\mathcal{F}$. If each edge is of fixed size $r \leq m$, then $\mathcal{F}$ is said to be $r$-regular, i.e., $\mathcal{F} \subseteq \binom{[m]}{r}$.

**Definition 3.3.** For a positive integer $k$, $\mathcal{F}$ is called $k$-union-free if for any two distinct subsets of edges $A, B \subseteq \mathcal{F}$, with $1 \leq |A|, |B| \leq k$, it holds that $\cup_{A \in A} A \neq \cup_{B \in B} B$.

Union-free regular hypergraphs are investigated in extremal combinatorics [5]. It is immediate to see that a set $P$ of $m$ paths over $n$ nodes defines a hypergraph $\mathcal{F}_P$ on the set $[m]$ in the following way: for $i \in [n]$ let $A_i = \{ j \in [m] : \epsilon_i[j] = 1 \}$ and define $\mathcal{F}_P = \{ A_1, \ldots, A_n \}$. Given a $U \subseteq [n]$, consider the subset of $\mathcal{F}_P, U = \{ A_i \in \mathcal{F}_P | i \in U \}$. Observe that then $\mathcal{P}(U) = \bigcup_{A \in U} A$. Hence immediately by definition of $k$-identifiability and that of $k$-union-freeness it follows that:

**Lemma 3.4.** If $\mathcal{P}$ is a set of $m$ paths over $n$ nodes and $\mu(\mathcal{P}) \geq k$, then $\mathcal{F}_\mathcal{P}$ is $k$-union free.

$\mathcal{F}_P$ is not necessary a regular hypergraph. For $r \in [m]$ let $\mathcal{F}_P(r) = \{ A \in \mathcal{F}_P | |A| = r \}$. Notice that each $\mathcal{F}_P(r)$ is now a $r$-regular hypergraph on $[m]$. Moreover the family of the $\mathcal{F}_P(r)$’s partitions $\mathcal{F}_P$ and hence $|\mathcal{F}_P| = \sum_{r \in [m]} |\mathcal{F}_P(r)|$. Since $|\mathcal{F}_P| = n$, it follows that:

**Lemma 3.5.** $\sum_{r \in [m]} |\mathcal{F}_P(r)| = n$

Furthermore notice that if $\mathcal{F}_P$ is $k$-union free then such it will be $\mathcal{F}_P(r)$ for each $r \in [m]$.

Let $m > r, k \in [m]$ with $k \geq 2$, and let $f(k, r, m)$ denote the maximum cardinality of a $k$-union-free $r$-regular hypergraph over $[m]$.

**Theorem 3.6 ([6,15]).** $\Omega(m^{\frac{k}{r-1}}) \leq f(k, r, m) \leq O(m^{\frac{1}{\frac{k}{r-1}}})$.

Let $m_0 \in \mathbb{N}$ and $C$ be the constant such that for all $m \geq m_0, f(k, r, m) \leq Cm^{\frac{1}{\frac{k}{r-1}}}$.

**Theorem 3.7.** Let $m$ be an integer such that $m \geq m_0$. Let $\mathcal{P}$ be a set of $m$ paths over $n$ nodes. If $n > \sum_{r \in [m]} Cm^{\frac{1}{\frac{k}{r-1}}}$, then $\mu(\mathcal{P}) < k$.

**Proof.** Assume by contradiction that $n > \sum_{r \in [m]} Cm^{\frac{1}{\frac{k}{r-1}}}$ and $\mu(\mathcal{P}) \geq k$. By Lemma 3.4 $\mathcal{F}_P$ is $k$-union free. Hence (see observation after Lemma 3.5) for each $r \in [m], \mathcal{F}_P(r)$ is a $r$-regular $k$-union free hypergraph and hence by previous theorem $|\mathcal{F}_P(r)| \leq Cm^{\frac{1}{\frac{k}{r-1}}}$ The $\mathcal{F}_P(r)$ partition $\mathcal{F}_P$ and by Lemma 3.5 we have $n = \sum_{r \in [m]} |\mathcal{F}_P(r)| \leq \sum_{r \in [m]} Cm^{\frac{1}{\frac{k}{r-1}}}$. $\square$

**Corollary 3.8.** Let $\mathcal{P}$ be a set of $m$ paths over $n$ nodes and $2 \leq k \leq m$. If $m < \frac{1}{\sqrt{\frac{(k-1)}{k}}} (\log_2{n-D}) - (k-1)$, for some $\epsilon > 0$ and where $D = \log_2{C}$, then $\mu(\mathcal{P}) < k$.

**Proof.** Assume for the moment that $m$ divides $k - 1$. We prove that if $m < \frac{1}{\sqrt{\frac{(k-1)}{k}}} (\log_2{n-D}) - (k-1)$, then

$$n > Cmm^{\frac{1}{\frac{k}{r-1}}}.$$  \hspace{1cm} (1)

This immediately implies $n > \sum_{r \in [m]} Cm^{\frac{1}{\frac{k}{r-1}}}$, since $\sum_{r \in [m]} Cm^{\frac{1}{\frac{k}{r-1}}} \leq Cmm^{\frac{1}{\frac{k}{r-1}}}$. Equation (1) follows by the following implications:
Proof.

Let \( m < \sqrt[k - 1]{\frac{k - 1}{k} (\log_2 n - D)} \) \tag{2}
\[
m^{1+\varepsilon} < \frac{k - 1}{k} (\log_2 n - D) \tag{3}
\]
\[
\frac{k}{k - 1} m^{1+\varepsilon} < \log_2 n - D \tag{4}
\]
\[
\log C + \frac{k}{k - 1} m \log m < \log n \tag{5}
\]
\[
\log C + \frac{m}{k - 1} \log m < \log n \tag{6}
\]
\[
C^m m^{-\varepsilon} < n \tag{7}
\]
Equation \(7\) follows from Equation \(5\) since \( K = \frac{k}{k - 1} > 1 \) and \( K m \log m > \log m + \frac{m \log m}{k - 1} \) for all \( m \).

If \( m \) does not divide \((k - 1)\), let \( a < (k - 1) \) be the smallest non-negative integer such that \( m + a \) divides \( k - 1 \). Hence \( m + a < m + (k - 1) \). Let \( \tilde{m} = m + a \). Since \( m < \sqrt[k - 1]{\frac{(k - 1)}{k} (\log_2 n - D) - (k - 1)} \), then \( \tilde{m} < \sqrt[k - 1]{\frac{(k - 1)}{k} (\log_2 n - D)} \). Hence the previous argument proves that \( n > C^m \tilde{m}^{-\varepsilon} \). Since \( m < \tilde{m} \), then \( n > C^m \tilde{m}^{-\varepsilon} \). Now by Theorem 3.7, this implies that \( \mu(P) < k \). \(\square\)

Theorem 3.9. Let \( P \) be a set of \( m \) paths over \( n \) nodes. Then for all \( k \leq n \), \( |ID_k(P)| \leq \min \{ n, 2^{\frac{k(m + k - 2)^2}{k + 1}} \} \).

Proof. Notice that if \( n \geq 2^{\sqrt[k - 1]{\frac{(k - 1)}{k} (\log_2 n - D) - (k - 1)}} \), then \( m < \sqrt[k - 1]{\frac{(k - 1)}{k} (\log_2 n - D) - (k - 1)} \). Hence Corollary 3.8 and the same proof of Theorem 3.2 imply the claim. \(\square\)

4 Refining identifiability: separability and distinguishability

We introduce two new definitions approximating identifiability from above and from below that we are going to use to prove upper and lower bounds on the number of \( k \)-identifiable nodes.

**Definition 4.1.** (\( k \)-separable nodes) A node \( u \in [n] \) is \( k \)-separable in \( P \), if for all \( U \subseteq [n] \) of size at most \( k \) and such that \( u \notin U \), it holds that there is a path \( p \in P(u) \setminus P(U) \), i.e. there is at least a path passing through \( u \) but not touching any node of \( U \).

We say that \( P \) is \( k \)-separable if each node \( u \in [n] \) is \( k \)-separable. \( k \)-separability is a stronger notion than \( k \)-identifiability as captured by the following lemma.

**Lemma 4.2.** If \( u \) is \( k \)-separable in \( P \), then \( u \) is \( k \)-identifiable in \( P \).

Proof. Let \( U \) and \( W \) be distinct subset of \([n]\) of size at most \( k \). Then there exists a \( u \) such that \( U \cap \{ u \} \neq W \cap \{ u \} \) and then either \( u \in U \setminus W \) or \( u \in W \setminus U \). Assume wlog the former. Then \( u \notin W \). \( u \) is \( k \)-separable in \( P \), there is a path \( p \in P(u) \setminus P(W) \). Since \( u \in U \), then \( p \in P(U) \setminus P(W) \) and then \( P(U) \neq P(W) \). \(\square\)

Notice that opposite direction is not true as we argue: assume that \( P \) is \( k \)-identifiable and that \( u \notin W \) for \( W \) a set of at most \( k \) nodes. The \( k \)-identifiability of \( P \) implies that \( P(u) \neq P(W) \), yet this condition alone does not guarantee that the path separating \( \{ u \} \) from \( W \), pass through \( u \) and not touching \( W \).

**Definition 4.3.** (\( k \)-distinguishable nodes) A node \( u \in [n] \) is \( k \)-distinguishable in \( P \), if for all \( U \subseteq [n] \) of size at most \( k \) and such that \( u \notin U \), it holds \( P(u) \neq P(U) \).

We say that \( P \) is \( k \)-distinguishable if each node \( u \in [n] \) is \( k \)-distinguishable.

**Lemma 4.4.** If \( u \) is \( k \)-identifiable in \( P \), then \( u \) is \( k \)-distinguishable in \( P \).
Proof. Assume that \( u \in [n] \) is \( k \)-ID in \( \mathcal{P} \). Let \( W \subseteq [n] \) of size at most \( k \) such that \( u \notin W \). We want to prove that \( \mathcal{P}(u) \neq \mathcal{P}(W) \). By \( k \)-ID of \( u \) we know that for all \( U' \) and \( W' \) in \([n]\) of size at most \( k \), such that \( U' \cap \{u\} \neq W' \cap \{u\} \), it holds that \( \mathcal{P}(U') \neq \mathcal{P}(W') \). Fix \( U' = \{u\} \) and \( W' = W \). Since \( u \notin W \), then \( U' \cap \{u\} \neq W' \cap \{u\} \), hence \( \mathcal{P}(u) \neq \mathcal{P}(W) \), as required.

Notice that the opposite direction is not necessary true: indeed if \( u \in U \setminus W \), knowing that \( \mathcal{P}(u) \neq \mathcal{P}(W) \) it is not sufficient to conclude \( \mathcal{P}(U) \neq \mathcal{P}(W) \), exactly in those case when \( \mathcal{P}(u) \neq \mathcal{P}(W) \) is witnessed by a path in \( \mathcal{P}(W) \setminus \mathcal{P}(u) \), which can touch other nodes in \( U \) but not \( u \).

We denote by \( \text{ID}_k(\mathcal{P}), \text{SEP}_k(\mathcal{P}), \text{DIS}_k(\mathcal{P}) \) the set of nodes which are respectively \( k \)-identifiable, \( k \)-separable and \( k \)-distinguishable in \( \mathcal{P} \). And we use to say respectively that \( u \) is \( k \)-ID, \( k \)-SEP and \( k \)-DIS in \( \mathcal{P} \).

By previous Lemmas and discussion it holds that

Lemma 4.5. For all \( k \in [n] \), \( |\text{SEP}_k(\mathcal{P})| \leq |\text{ID}_k(\mathcal{P})| \leq |\text{DIS}_k(\mathcal{P})| \).

Furthermore since the three properties are clearly antimonotone, it holds that:

Lemma 4.6. For all \( k \in [n] \), \( |\text{ID}_k(\mathcal{P})| \leq |\text{ID}_{k-1}(\mathcal{P})| \), \( |\text{SEP}_k(\mathcal{P})| \leq |\text{SEP}_{k-1}(\mathcal{P})| \), \( |\text{DIS}_k(\mathcal{P})| \leq |\text{DIS}_{k-1}(\mathcal{P})| \).

We denote by \( \sigma(\mathcal{P}) \) (respectively \( \delta(\mathcal{P}) \)) the maximal \( k \leq n \) s.t. \( \mathcal{P} \) is \( k \)-separable (respectively \( k \)-distinguishable). Hence we have \( \delta(\mathcal{P}) \leq \mu(\mathcal{P}) \leq \sigma(\mathcal{P}) \).

5 Lower bounds on \( \mu(\mathcal{P}) \) by a random model

To study lower bounds on \( \text{ID}_k(\mathcal{P}) \) (or on \( \mu(\mathcal{P}) \)) for real set of paths we introduce a simple random model. We are given \( m \) and \( n \) natural numbers and \( n \) real numbers \( \lambda_i \in [0,1] \). The random set \( \mathcal{P} \) of \( m \) paths over \( n \) nodes is obtained by taking independently \( n \) binary strings of length \( m \) such that the \( i \)-th string is distributed according to the binomial distributions \( \text{Bin}(m, \lambda_i) \). That means that node \( i \in [n] \) will be present on each path with probability \( \lambda_i \) and absent with probability \( (1 - \lambda_i) \).

Our approach to estimate \( |\text{ID}_k(\mathcal{P})| \) is the following:

1. by Lemma 4.5 \( |\text{SEP}_k(\mathcal{P})| \leq |\text{ID}_k(\mathcal{P})| \).
2. For \( u \in [n] \) we obtain \( \nu_{n,m,\lambda}(u) = \text{Pr}[u \in \text{SEP}_k(\mathcal{P})] \).
3. Given a real set \( \hat{\mathcal{P}} \) of \( M \) paths on \( N \) nodes, we consider \( \hat{\mathcal{P}} \) to be a random experiment and from \( \hat{\mathcal{P}} \) we compute a maximum likelihood estimate \( \lambda_i \) for each of the \( \lambda_i \).
4. We estimate \( |\text{SEP}_k(\hat{\mathcal{P}})| = \sum_{u \in [N]} \nu_{N,M,\hat{\lambda}}(u) \).

Let \( u \in [n] \) and \( W \subseteq [n] \setminus \{u\} \). Let us say that \( (u,W) \) is GOOD if there is a path \( p \in [m] \) such that \( p \in \mathcal{P}(u) \setminus \mathcal{P}(W) \). \( (u,W) \) is BAD if it is not GOOD.

Lemma 5.1. Let \( u \in [n] \) and \( W \subseteq [n] \setminus \{u\} \). \( \text{Pr}[(u,W) \text{ BAD}] = (1 - \lambda_u \prod_{w \in W} (1 - \lambda_w))^m \).

Proof. \( (u,W) \) is BAD if and only if for all \( p \in [m] \) : \( p(u) \rightarrow p(W) \). Then \( \text{Pr}[(u,W) \text{ BAD}] = \left( \text{Pr}[(p(u) \rightarrow p(W))] \right)^m \). The condition \( p(u) \rightarrow p(W) \) is the same as \( \neg p(u) \lor \bigvee_{w \in W} p(w) \) which is the same as \( (p(u) \land \bigwedge_{w \in W} \neg p(w)) \lor (1 - \lambda_w) \). Hence the claim.

Let \( k \leq n \) and let \( S(k) = \binom{[n]-1}{k} \).

Theorem 5.2. Let \( n, m, k \in \mathbb{N} \), \( u \in [n] \), and \( k \leq n \). \( \text{Pr}[u \in \text{SEP}_k(\mathcal{P})] = \prod_{W \in S(k)} \left( 1 - (1 - \lambda_u \prod_{w \in W} (1 - \lambda_w))^m \right) \).
Proof. Observe that \( \Pr[u \in \text{SEP}_k(P)] = \Pr[u \text{ is } k-\text{SEP in } P] = \Pr[\exists W \text{ with } u \notin W \text{ and } |W| \leq k : (u, W) \text{ GOOD}] \). By previous Lemma \( \Pr[(u, W) \text{ GOOD}] = 1 - (1 - \lambda_u \prod_{w \in W} (1 - \lambda_w))^m \). Hence the theorem follows. \( \square \)

Assume we have a set \( \hat{P} \) of \( m \) paths over \( n \) nodes. We consider \( \hat{P} \) as a random experiment. The standard approach to compute an MLE estimate \( \hat{\lambda}_i \) of the \( \lambda_i \) in the case of binomial distribution is to compute \( \hat{\lambda}_i \) as the zero of the polynomial obtained by the prime derivative of the function expressing the probability that the node \( i \) touches \( N_i \) paths in \( \hat{P} \).

Let \( p_i = \Pr[\text{node } i \text{ touches } N_i \text{ paths in } P] \). Since in \( P \) the column \( i \) is distributed accordingly to the Bin\((m, \lambda_i)\), then \( p_i = \binom{m}{N_i} \lambda_i^{N_i} (1 - \lambda_i)^{m-N_i} \). We study \( \frac{d}{d\lambda_i} p_i \) and compute \( \hat{\lambda}_i \) by setting \( \frac{d}{d\lambda_i} p_i = 0 \). It is easy to see that this happen for \( \hat{\lambda}_i = \frac{N_i}{m} \).

5.1 Experiments

Let \( \nu_{n,m,\hat{\lambda}}(u) = \Pr[u \in \text{SEP}_k(P)] \). Assume to have a real set of \( M \) paths \( \hat{P} \) over \( N \) nodes. From \( \hat{P} \) we extract the \( \hat{\lambda}_i \) for all \( i \in [N] \) and we then estimate \( \text{SEP}_k(\hat{P}) \) as \( \chi(\hat{P}, k, \hat{\lambda}) = \sum_{u \in [N]} \nu_{N,M,\hat{\lambda}}(u) \), using the closed formula in Theorem 5.2.

In Figure 2 and 3 we consider two graphs from the Internet topology Zoo (ClaraNet and BTEurope) and we consider set of measurement paths \( P \) obtained from these networks by taking all the different paths starting in source and ending in a target node (green nodes are source and red nodes are target). In the second table in each Figure ee
consider the subset polynomial time. Assume Theorem 6.1. Consider the optimization problem of H.

It is easy to see that $\chi(\bar{P}, k, \hat{\lambda}) \geq \chi_2(\bar{P}, k, \hat{\lambda}_{\max})$.

\section{Complexity of $k$-identifiability and the minimum hitting set}

Consider the optimization problem Minimum Hitting Set, MHS, that given an hypergraph (a set-system) $\mathcal{H} = (V,E)$, where $E \subseteq 2^V$, asks to find the smallest $V' \subseteq V$ such that for all $e \in E$, $V' \cap e \neq \emptyset$. MHS is a notorious NP-complete problem, i.e. there exits a set of nodes $W$ of size $|W| \leq k$, such that $\mathcal{P}(u) \subseteq \mathcal{P}(W)$.

\textbf{Theorem 6.1.} Assume MHS is solvable in polynomial time, then, deciding whether $u$ is not $k$-SEP in $\mathcal{P}$ is solvable in polynomial time.

\textbf{Proof.} Consider the subset $T(u)$ of $[n]$ of those nodes touching at least a path in $\mathcal{P}(u)$. Let $Y$ be the vector of dimension $|\mathcal{P}(u)|$ defined in the $j$-th coordinate as follows:

$$Y[j] = \bigvee_{v \in T(u)} \mathcal{P}[j, v] \quad j \in \mathcal{P}(u)$$

$Y$ has no 0-coordinate. For otherwise there is a path in $\mathcal{P}$ only touching $u$. Hence $Y$ has all 1-coordinates. We consider the set-system $\mathcal{H}$ obtained from $\mathcal{P}$ by restricting the columns to $T(u)$ and the rows to $\mathcal{P}(u)$. Let $W$ be the smallest subset of $T(u)$ provided by MHS and covering all $\mathcal{P}(u)$. Hence $u$ is not $|W|$-SEP, since $\mathcal{P}(u) \subseteq \mathcal{P}(W)$.

The optimality of the bound is an immediate consequence of the optimality of MHS. There is no set $Z$ of $[n]$ smaller than $U$ such that $\mathcal{P}(u) \subseteq \mathcal{P}(Z)$, since of course $Z \subseteq T(u)$ and, by optimality of MHS, $Z$ it cannot be smaller than $U$.

The problem of finding a minimal transversal in an hypergraph is a simplification of MHS (see below) which can be decided efficiently. Our reduction hence suggests to implement an algorithm on concrete example of paths where we find the smallest transversal instead of the minimal hitting set.

Let us recall the following definitions from hypergraph transversal problem $[4]$.

\textbf{Definition 6.2.} Let $\mathcal{H} = (V,E)$ be an hypergraph. A set $T \subseteq V$ is called a transversal of $\mathcal{H}$ if it meets all the edges of $\mathcal{H}$, i.e. if $\forall e \in E : T \cap e \neq \emptyset$. A transversal $T$ is called minimal if no proper subset $T'$ of $T$ is a transversal.

It is possible to find in time $O(|V||E|)$ a minimal transversal of $\mathcal{H}$ by the following algorithm (see also $[4]$). If $E = \emptyset$, then every subset of $V$ is a transversal of $\mathcal{H}$, hence the minimal one is $\emptyset$. If $E \neq \emptyset$, let $V = \{v_1, \ldots, v_n\}$. Then define:

$$V_0 = V$$

$$V_{i+1} = \begin{cases} 
V_i \\ V_i \setminus \{v_i\} & \text{is not a transversal of } \mathcal{H} \\
V_i \setminus \{v_i\} & \text{is a transversal of } \mathcal{H}
\end{cases}$$

Hence $V_n$ is a minimal transversal of $\mathcal{H}$. Notice however that $V_n$ it is not necessarily the smallest (by cardinality) transversal of $\mathcal{H}$. In fact this last problem is the MHS problem which is NP-hard.

\footnote{This is because we need to use the $\lambda_w$ for all $w \in W$.}

\footnote{It is easy to see that $\chi(\bar{P}, k, \hat{\lambda}) \geq \chi_2(\bar{P}, k, \hat{\lambda}_{\max})$.}
Let us call $HT$ be a procedure that implements the previous algorithm on a given $H(V,E)$, and given an order on $V$, and outputs a minimal transversal of $H$.

The proof of Theorem 6.1 suggests an algorithm to compute an upper bound on the $k$-separability of a node $u$ in $P$, where instead of computing the minimum hitting set we compute a minimal transversal using $HT$ on any order of the variables.

Consider the following sets: for all $i \in P(u)$, let $Z(v) = \{ i \in [n] \mid P[i,v] = 0 \}$, for all $v \in [n]$ and $V' = \{ v \in [n] \mid P(u) \cap Z(v) = i \}$. Let $I = \{ i_1, \ldots, i_N \} \subseteq P(u)$ be the set of indices of the $V_{i_j} \neq \emptyset$. We say that $V_{i_N}, \ldots, V_{i_1}$ is a 0-decreasing sequence since, by definition, $Z(v) > Z(w)$ whenever $v \in V_i$, $w \in V_j$ and $i < j$.

### Algorithm 1: Algorithm Simple-SEP

**Data:** $P, u$

**Result:** $(W, s)$ s.t. $P(u) \subseteq P(W)$ and $|W| = s$

1. $W = HT([n], P(u))$
2. return $(W, |W|)$

### Algorithm 2: Decr-SEP

**Data:** $P, u$

**Result:** $(W, s)$ s.t. $P(u) \subseteq P(W)$ and $|W| = s$

1. Compute all $V_i$’s
2. Compute $I$
3. $P_0[u] = P[u]$
4. for $l = 0 \ldots N$ do
5.   $k = N - l$
6.   for $j \in P(u)$ do
7.     if $j \in P[I[u]$ then
8.       $\hat{Y}_{ik}[j] = \vee_{v \in V_{i_j} P_i[j,u]$
9.     else
10.    $\hat{Y}_{ik}[j] = 0$
11. end
12. end
13. $\hat{V}_k = HT(V_{ik}, P[I[u])$
14. $Z_{ik} = 0$-coordinates of $Y_{ik}$
15. $P_{i+1}(u) = P_i(u) \cap Z_{ik}$
16. end
17. $Y = HT(I, \cup_{i \in I} \hat{Y}_i)$
18. $W = \cup_{i \in Y} \hat{V}_i$
19. return $(W, |W|)$

The algorithm starts by computing the set $V_i$ and the set of indices $I$ of such sets which are not empty. The main observations on the algorithm are the following:

- that $V_{i_N}, \ldots, V_{i_1}$ is a 0-decreasing sequence. At each step we try to cover only the paths in $P(u)$ not already covered before. This is the reason why in line 15 we restrict only to 0-coordinates in $Z_{ik}$. The vectors $Y_i$ are also defined accordingly. Only the coordinates in $P_i(u)$ are important since the rest are already covered by some previous $V_i$. That is the reason why in line 10 we define to be 0 the $Y$ vector in all the coordinates not in $P_i(u)$.
• Another observation is that at each step \( l \) we want to save the minimal set of nodes \( \hat{V}_{i_{N-l}} \), sufficient to cover all the 1’s in \( P_{i_{l+1}} \). This is the meaning of the call to \( \text{HT} \) in line 13.

• finally, when we have done with analyzing all the family of the sets \( V_i \)'s, \( P(u) \) is covered by the union of the \( Y \) vectors (this is by an argument similar to that of Theorem 6.1). But it is sufficient to have the minimal subset of this family for covering all \( P(u) \). To this end we perform a final call to \( \text{HT} \) on input the set-system, \( (I, \bigcup_{i \in I} Y_i) \) in line 17.

### 6.1 NP-Completeness

Consider the following optimization problem MIN-NOT-SEP (MNS).

**Input:** A Boolean \( m \times n \) matrix \( P \), an element \( u \in [n] \);

**Output:** \( k \) such that \( u \) is not \( k \)-SEP and \( u \) is \( k'-\text{SEP} \) for all \( k' < k \).

**Theorem 6.3.** MNS is NP-complete.

**Proof.** To see that MNS is in NP we can use the reduction in Theorem 6.1 which is in fact proving that \( MNS \leq_p MHS \).

Since MHS is in \( NP \) [1], then MNS is in \( NP \).

To prove the NP-hardness of MNS we show the opposite reduction, i.e. that \( MHS \leq_p MNS \). Hence the result follows by the NP-hardeness of MHS [1]. Let \( \mathcal{H} = (V, E) \) be an instance of MHS. We define an instance of MNS as follows:

• The set of nodes of \( P \) is \( V \cup \{u\} \).

• The set of paths of \( P \) is \( E \);

• \( P(u) = E \);

Since a minimal hitting set \( W \) is touching all edges in \( E \), that means that \( P(W) = E = P(u) \). Hence \( u \) is not \( |W| \)-SEP. Moreover since it is minimal, then for any subset \( W' \) of \( [n] \) of size smaller that \( |W| \) there is an edge \( e \in E \) not in \( W' \). That means that \( e \in P(v) \setminus P(W) \), that is \( u \) is \( k' \)-SEP in \( P \) for any \( k' < |W| \).

On the opposite direction, assume that \( W \subseteq [n] - \{u\} \) is witnessing that \( u \) is not \( |W| \)-SEP but is \( k' \)-SEP for any \( k' < |W| \), then \( W \) is clearly a minimal hitting set in \( \mathcal{H} \).

### 7 Localizing failure nodes in real networks

In this section we study some heuristics to compute as more precisely as possible the number of \( k \)-identifiable nodes in set of measurements paths defined on concrete networks, that is the the set of all paths from between monitor nodes. According to Section 4 we study upper bounds on the number of \( k \)-distinguishable nodes.

To upper bound \( |\text{DIS}_k| \) we lower bound the number of node which are not distinguishable in \( P \). In fact we will localize specific sets of nodes which we can guarantee to be not \( k \)-distinguishable.

Let \( P \) be given and let \( u \in V \). We let \( \mathcal{W}_{k}(u) \) be a subset of \( ([n] \setminus \{u\}) \). This should be meant as (a method to generate) a collection of subset of at most \( k \) nodes in \( V - \{u\} \) as function of the node \( u \). An example can be: the subsets of \( [n] \) made by at most \( k \) nodes which are at distance at most \( d \) from \( u \). For any \( v \in \mathcal{W}(u) \), let \( P(u, v) \subseteq P(u) \cap P(v) \). This should be meant as (a method to generate) a subset of all paths touching both nodes \( u \) and \( v \).

**Definition 7.1.** Let \( \mathcal{W} \) and \( P \) be given for \( P \). We say that \( u \in [n] \) and \( W \in \mathcal{W}_k(u) \) are \( k \)-equal modulo \( P \) in \( P \) if

1. \( \exists w, w' \in W \) such that \( P(u) \setminus P(u, w) \subseteq P(w') \), and
2. \( \forall w \in W, P(w) \setminus P(u, w) \subseteq P(u) \).

Let

\[
E_{V,k}[\mathcal{W}, P] := \{ u \in V : \text{there is a } W \in \mathcal{W}(u) \text{ s.t. } u \text{ and } W \text{ are } k \text{-equal modulo } P \}
\]
Lemma 7.2. For all $V \subseteq [n]$, $E_{V,k}[\mathcal{W}, \mathcal{P}] \subseteq \overline{\mathrm{DIS}_k}(\mathcal{P})$.

Proof. Let $u \in E_{V,k}[\mathcal{W}, \mathcal{P}]$ we have to find a $W \in \binom{V}{\leq k}$ with $u \notin W$ such that $\mathcal{P}(u) = \mathcal{P}(W)$. Fix as $W$ the one in $\mathcal{W}(u)$ given by the the definition of $E_{V,k}[\mathcal{W}, \mathcal{P}]$. We first argue that $\mathcal{P}(u) \subseteq \mathcal{P}(W)$. By Definition 7.1 case (1) we know that there are $w, w' \in W$ such that $\mathcal{P}(u) - \mathcal{P}(u, w) \subseteq \mathcal{P}(u')$. Consider a $p \in \mathcal{P}(u)$. If $p \in \mathcal{P}(u, w)$, then $p \in \mathcal{P}(w)$ and hence $p \in \mathcal{P}(W)$. If $p \notin \mathcal{P}(u, w)$, then $p \in \mathcal{P}(u) \setminus \mathcal{P}(u, w)$ and then by Definition 7.1 case (1) $p \in \mathcal{P}(W')$ and hence in $\mathcal{P}(W)$.

Let $q \in \mathcal{P}(W)$, then $q \in \mathcal{P}(w)$ for some $w \in W$. If $q \in \mathcal{P}(u, w)$, then $q \in \mathcal{P}(u)$. If $q \notin \mathcal{P}(u, w)$, then $q \in \mathcal{P}(w) \setminus \mathcal{P}(u, w)$ and then, by Definition 7.1, $q \in \mathcal{P}(u)$. By Lemma 7.2, nodes in $E_{[n],k}[\mathcal{W}, \mathcal{P}]$ are not $k$-distinguishable and, for the anti-monotonicity, are not $(k + 1)$-, $(k + 2)$-... $n$-distinguishable.

We now study how to upper bound the number of $k$-distinguishable nodes in $\overline{\mathcal{P}}$ given a specific definition of $\mathcal{W}$ and $\mathcal{P}$. Consider the following family of vertices in $\overline{\mathcal{P}}$:

$$
\left\{ \begin{array}{l}
V_1 = [n] \\
V_k = [n] - \bigcup_{j=1}^{n-1} E_{V_{j,k}}[\mathcal{W}, \mathcal{P}] \\
k > 1
\end{array} \right.
$$

Definition 7.3. Let $k \leq n$. $\tau_k := |E_{V_{1,k}}[\mathcal{W}, \mathcal{P}]|$

Theorem 7.4. $|\overline{\mathrm{DIS}_k}(\mathcal{P})| \leq n - \sum_{j=1}^{k} \tau_j$.

Proof. We abbreviate $E_{V_{j,k}}[\mathcal{W}, \mathcal{P}]$ with $E_{V_{j,k}}$. First we claim that $|\bigcup_{j \leq k} E_{V_{j,k}}| \leq \sum_{j=1}^{k} \tau_j$. This is because for all $k \leq n$, if $u \in E_{V_{1,k}}$, then $u \notin \bigcup_{j \leq k-1} E_{V_{j,k}}$, by definition of $E_{V_{1,k}}$.

Further we claim that $E_{V_{1,k}} \subseteq \overline{\mathrm{DIS}_k}(\mathcal{P}) \setminus \bigcup_{j \leq k-1} E_{V_{j,k}}$.

Indeed by Lemma 7.2, $E_{V_{1,k}} \subseteq \overline{\mathrm{DIS}_k}(\mathcal{P})$ and again by definition of $E_{V_{1,k}}$, if $u \in E_{V_{1,k}}$, then $u \notin \bigcup_{j \leq k-1} E_{V_{j,k}}$. Therefore:

$$
|\overline{\mathrm{DIS}_k}(\mathcal{P})| \geq |E_{V_{1,k}}| + \left| \bigcup_{j \leq k-1} E_{V_{j,k}} \right|.
$$

By definition of $\tau_k$ it follows that $|\overline{\mathrm{DIS}_k}(\mathcal{P})| \geq \tau_k + \sum_{j=1}^{k-1} \tau_j$, and hence that $|\overline{\mathrm{DIS}_k}(\mathcal{P})| \leq n - \sum_{j=1}^{k} \tau_j$.

Notice that the proof of the theorem is constructive and is counting well-defined nodes in the network, so that nodes can also precisely be localized.

### 7.1 Examples of applications

We show how to use previous results to localize and upper bound the number of $k$-identifiable nodes on real set of measurements paths. The estimate will depend on what set $\mathcal{W}(u)$ we consider for any node $u$ and on what set of paths $\mathcal{P}$ we are going to test path not distinguishability. However once we have fixed $\mathcal{W}$ and $\mathcal{P}$ the algorithm we run is always the same and reflects the discussion in the previous subsection (See Algorithm 1b b–DIS$_k$).

Our method can be applied on a network given as a graph once we have decided the set of measurements paths. Every possible way of choosing $\mathcal{W}$, and $\mathcal{P}$ is giving a way to count nodes which are no distinguishable. We can therefore thinking of applying the method restricting for each node $u$ the nodes we are checking be not distinguishable and the effective paths we are going to to consider. We consider here three potential examples. We will add details on experiments and we will talk about other cases in the final version of the paper.
Algorithm 3: lb-DIS$_k$: Counting $k$-SEP nodes

Data: $\mathcal{P}$

Result: number of $k$-SEP nodes

1. for $u \in [n]$ do
2.  Compute $\mathcal{W}(u)$;
3.  for $w \in \mathcal{W}(u)$ do
4.    Compute $\mathcal{P}(u, w)$
5.  end
6. end
7. $V = [n], i = 1, \tau = 0$;
8. while $i \leq k$ do
9.  Compute $E_{V,i}\{\mathcal{W}, \mathcal{P}\}$;
10. $\tau = \tau + |E_{V,i}\{\mathcal{W}, \mathcal{P}\}|$;
11. $V = V - E_{V,i}\{\mathcal{W}, \mathcal{P}\}$;
12. $i = i + 1$;
13. end
14. return $n - \tau$;

7.1.1 Neighbours

For any given $u \in [n]$, let $\mathcal{W}_k(u) = \binom{N(u)}{\leq k}$, where $N(u)$ are the neighbours of $u$ and consider for all $v \in N(u)$, the $\mathcal{P}^N(u)$ of the paths touching both $u$ and its neighbours $v$.

7.1.2 Nodes at a fixed distance $d$

For any given $u \in [n]$, let $N_d(u) = \{v \in V : d(u, v) = d, d \geq 1\}$ and $\mathcal{W}_k(u) = \binom{N_d(u)}{\leq k}$. For $v \in N_d(u)$, the $\mathcal{P}^d$ of the paths touching both $u$ and $v$.

7.1.3 Shortest paths

In this case we consider as set $\mathcal{W}_k(u) = \binom{V - \{u\}}{\leq k}$, and for all $v \in V \setminus \{u\}$, $\mathcal{P}$ is the set of shortest paths from $u$ to $v$.

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