The boundary of the chronology violating set

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Abstract

A sufficient definition for the future and past boundaries of the chronology violating region is given. In comparison to previous studies, this work does not assume that the complement of the chronology violating set is globally hyperbolic. The boundary of the chronology violating set is studied and several propositions are obtained which confirm the reasonability of the definition. Some singularity theorems related to chronology violation are considered. Among the other results we prove that compactly generated horizons are compactly constructed.

Keywords: time machines, singularity theorems, null hypersurface

1. Introduction

This work is devoted to the study of the boundary of the chronology violating region of spacetime. This boundary shares some features that are reminiscent of the Cauchy horizon $H(S)$ of an achronal hypersurface. In fact, in some cases in which the chronology violation can be removed by passing to a suitable covering (that 'counts' the numbers of times a timelike curve crosses a hypersurface $S$), this equivalence can be made manifest [17]. Unfortunately, in the general case there is no such correspondence and the boundary of the chronology violating set must be studied for its own sake.

As today there are few results concerning the boundary of the chronology violating set $\mathcal{C}$, and indeed the very definition of future and past boundary seems to be missing in the literature. In some cases, as done by Thorne [30] and Hawking [9], the character and properties of the boundary $\partial \mathcal{C}$ are obtained by assuming a globally hyperbolic complement $M \setminus \mathcal{C}$, and by identifying the future boundary of the chronology violating set with the past Cauchy horizon of such chronological complement. Of course, this definition is not completely satisfactory as it is well posed only for spacetimes which admit a globally hyperbolic chronological region.
The development of closed timelike curves in spacetimes which admit non-compact partial Cauchy hypersurfaces is fairly well understood. Hawking argued [9] that if the closed timelike curves originated by the actions of an advanced civilization, then the generators of the Cauchy horizon, followed in the past direction, would enter the space region were those actions took place. This fact should be expected since the generators themselves represent the flow of the information which signals the fact that a causality violation took place. Without a breaking of the spacetime continuum those generators would have to enter a compact region, namely the future Cauchy horizon would have to be compactly generated. However, Hawking showed that compactly generated horizons cannot form and some well known gaps in the proof, connected with a tacit differentiability assumption on the horizon, have been recently solved [24], thus confirming the validity of chronology protection at the classical level. Similar issues connected with Tipler’s analysis [31] have also been clarified [24].

It has been shown that Hawking’s no go theorem on the formation of timelike curves can be circumvented either by relaxing the assumption on the compact generation of the horizon or by admitting violation of the null energy condition, see [6, 8, 25–28] (we shall say more on this in section 3). The reader is warned that on this topic some imprecise or misleading statements can be repeatedly found in the literature; the most relevant example is given by Hawking’s claims [9, section 3] that (a) ‘absence of closed null geodesics’ on compact Cauchy horizons would be unstable, that is, the least perturbation of the metric would cause the horizon to contain closed null geodesics; (b) ‘presence of closed null geodesics’ would be stable. At present there is no convincing proof for these claims, and some studies seem to suggest different conclusions [5, 22].

In this work we study the boundary of the chronology violating set without making restrictive assumptions and we eventually obtain a definition of its future and past parts. As it happens for the concept of Cauchy horizon, the results of this work could prove useful for the study of singularities under chronology violation. Indeed, we shall argue that a deeper understanding of this boundary could clarify the mutual relationship between chronology violation and geodesic incompleteness (i.e. singularities).

Let us recall that a spacetime \((M, g)\) is a connected, time-oriented Lorentzian manifold of arbitrary dimension \(n + 1 \geq 2\), where \(g \in C^k, k \geq 3\), has signature \((- , +, \cdots, +)\). As a matter of notation, the boundary of a set is denoted with a dot. In some cases in which this notation could be ambiguous the dot is replaced by the symbol \(\partial\). The subset symbol \(\subset\) is reflexive, i.e. \(X \subset X\). A set is achronal if no timelike curve joins two of its points. If \(S\) is a closed achronal set, the Cauchy development \(D^+(S)\) is the set of those \(p \in M\) such that every future inextendible causal curve ending at \(p\) intersects \(S\). The Cauchy horizon is \(H^+(S) = D^+(S) \setminus I^-(D^+(S))\).

Let us also recall that a future lightlike ray is a future inextendible achronal causal curve, in particular it is a lightlike geodesic. Past lightlike rays are defined analogously. A lightlike line is an achronal inextendible causal curve, hence a lightlike geodesic without conjugate points. In this work, unless otherwise specified, all the curves will be future directed, thus, for instance, a past lightlike ray ends at its endpoint.

The condition of absence of lightlike lines is implied under the null genericity and the null convergence conditions by null completeness (as these three conditions together imply the existence of conjugate points on any null geodesic [1, 10]). Therefore, in the study of singularity theorems it is often a good strategy to assume the absence of lightlike lines and to look for contradictions.

The chronology violating region \(C = \{x : x \ll x\}\) is the set formed by those points through which passes at least one closed timelike curve. The relation \(x \sim y\) if \(x \ll y\) and \(y \ll x\) is an equivalence relation in \(C\) and, as it is well known since the work by Carter, it
splits the chronology violating region into (open) equivalence classes denoted in square bracket, \([x] = I^+(x) \cap I^-(x)\). Two points belonging to the same class have the same chronological future and the same chronological past.

2. The boundary of a chronology violating class

In this work we are going to study the boundary of a generic chronology violating class since the boundary of the chronology violating region can be recovered from those. In this respect the following result [18, theorem 4.5] is worth recalling.

**Theorem 2.1.** Let \([x]\) and \([y]\) be the boundaries of the distinct chronology violating classes \([x]\) and \([y]\). Through every point of \([x] \cap [y]\) (a set which may be empty) there passes a lightlike line entirely contained in \([x] \cup [y]\). Thus, a spacetime without lightlike lines has chronology violating set components having disjoint closures.

For the proof of the next lemma see [14, proposition 2], or the proof of [20, theorem 12].

**Lemma 2.2.** Let \([r]\) be a chronology violating class. If \( p \in \dot{r}\) then through \( p \) passes a future lightlike ray contained in \([r]\) or a past lightlike ray contained in \([r]\) (and possibly both).

**Definition 2.3.** Let \([r]\) be a chronology violating class. The set \( R_p([r]) \) is that subset of \([r]\) which consists of the points \( p \) through which passes a future lightlike ray contained in \([r]\). The set \( R_p([r]) \) is defined analogously.

**Lemma 2.4.** The sets \( R_p([r]) \) and \( R_p([r]) \) are closed and \( \dot{r} = R_p([r]) \cup R_p([r]) \).

**Proof.** It is a consequence of the fact that a sequence of future lightlike rays \( s_n \) of starting points \( x_n \rightarrow x \) has as limit curve a future lightlike ray of starting point \( x \) [18], and analogously in the past case. Clearly, by lemma 2.2, \([r] = R_p([r]) \cup R_p([r]) \).}

Note that it can be \( R_p([r]) \cap R_p([r]) = \emptyset \) (see figure 1).

A set \( F \) is a said to be a future set if \( \dot{I}^+(F) \subset F \). A future set is open iff \( \dot{I}^+(F) = F \). If \( F \) is future then \( J^+(F) \subset F \) which implies that the closure \( \bar{F} \) is future. Analogous definitions and results hold for past sets, in particular \( F \) is a future set iff \( M \setminus F \) is a past set. The boundary of a future set is an achronal boundary [1].

The achronal boundary \( \partial I^-([r]) \) will be particulary important in what follows. The proof of the next result is rather standard.

**Proposition 2.5.** Through every point \( p \) of the achronal boundary \( \partial I^-([r]) \) starts a (possibly non-unique) future lightlike ray contained in \( \partial I^-([r]) \). Furthermore, if a causal curve connects two distinct points \( x \) and \( y \) of \( \partial I^-([r]) \) then the causal curve is contained in \( \partial I^-([r]) \) and coincides with a segment of future lightlike ray contained in \( \partial I^-([r]) \).

**Proof.** Let \( \sigma_n \) be a timelike curve connecting \( p_n \in \dot{I}^-([r]) \) to \( r \), with \( p_n \rightarrow p \). By the limit curve theorem [18] either there is a continuous causal curve connecting \( p \) to \( r \), which is impossible because \( p \not\in \dot{I}^-([r]) \) or there is a future inextendible continuous causal curve \( \sigma \).
contained in \( I^-([r]) \). No point of this curve can be contained in \( I^-([r]) \) otherwise \( p \in I^-([r]) \) thus \( \sigma \in \partial I^-([r]) \). Since \( \partial I^-([r]) \) is achronal \( \sigma \) is a lightlike ray.

If the causal curve \( \gamma \) connects \( x \) to \( y \) then between \( x \) and \( y \) no point of it can belong to \( I^-([r]) \) otherwise \( x \in I^-([r]) \), a contradiction. Let \( z \in \gamma \setminus \{ y \} \), and take \( z' \ll y \) and since \( I^+ \) is open and \( y \in \partial I^-([r]) \) we have \( z' \in I^-([r]) \). Taking the limit \( z' \rightarrow z \) we obtain \( z \in I^-(\{ r \}) \) thus \( z \in \partial I^-([r]) \). The causal curve obtained by joining \( \gamma \) with the lightlike ray starting from \( y \) must be achronal as it is contained in \( \partial I^-([r]) \) and thus it is a lightlike ray.

\[ \square \]

**Lemma 2.6.** Let \([ r \) be a chronology violating class then \( I^-(r) = I^-([r]) = I^-([r]) \) and the following sets coincide:

(i) \( [r] \setminus I^-([r]) \),
(ii) \([r] \setminus I^-([r]) \),
(iii) \( R_f([r]) \setminus I^-([r]) \),
(iv) \([r] \cap \partial I^-([r]) \).

**Proof.** The inclusion \( I^-([r]) \subset I^-([r]) \) is obvious. The other direction follows immediately from the fact that \( I^+ \) is open.

(i) \( \iff \) (ii) \( \iff \) (iii). \( R_f([r]) \setminus I^-([r]) \subset [r] \setminus I^-([r]) \) is trivial, \( [r] \setminus I^-([r]) \subset [r] \setminus I^-([r]) \) follows from \( [r] \subset I^-([r]) \), and it remains to prove \( [r] \setminus I^-([r]) \subset R_f([r]) \setminus I^-([r]) \). Let \( p \in [r] \setminus I^-([r]) \), there is a sequence \( p_n \in [r] \), \( p_n \rightarrow p \). Since \( p_n \in [r] \) there are timelike curves \( \sigma_n \) entirely contained in \([r] \) which connect \( p_n \) to \( r \). By the limit curve theorem there is either (a) a limit continuous causal curve connecting \( p \) to \( r \), in which case as \([r] \) is open, \( p \in I^-([r]) \), a contradiction, or (b) a limit future inextendible continuous causal curve \( \sigma \) starting from \( p \) and contained in \([r] \). Actually \( \sigma \) is contained in \([r] \) otherwise \( p \in I^-([r]) \), a contradiction. Moreover, \( \sigma \) is a future lightlike ray, otherwise there would be \( q \in [r] \cap \sigma \), \( p \ll q \) and as \( I^+ \) is open \( p \in I^-([r]) \), a contradiction.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Minkowski 1 + 1 spacetime with four spacelike segments removed. The interior of the sides with the same label, \( A \) or \( B \), have been identified. The shaded region is the only chronology violating class \([ r \) and \( \sigma \) is an example of closed timelike curve. The points \( x, y, z \) belong to \( B_f([r]) \cap B_p([r]) \) but \( x, z \in \text{edge}(B_f([r])) \) while \( y \not\in \text{edge}(B_f([r])) \). In particular, not all the generators of \( B_f([r]) \) have past endpoint in \( \text{edge}(B_f([r])) \) even if they leave \( B_f([r]) \) in the past direction. The inextendible geodesic \( \gamma \) is contained in the boundary \([ r \) but is not achronal.
(ii) $\Leftrightarrow$ (iv). Let $p \in [r] \setminus I^+(\{r\})$ and let $x \ll p$. Since $I^+$ is open and $p \in \overline{r}$, $x \in I^+(\{r\})$, and taking the limit $x \to p$ we obtain $p \in \overline{I^+(\{r\})}$. But $p \not\in I^+(\{r\})$ thus $p \in \partial I^+(\{r\})$ and hence $[r] \setminus I^+(\{r\}) \subset [r] \cap \partial I^+(\{r\})$. For the converse note that if $p \in [r] \cap \partial I^+(\{r\})$ then $p \not\in I^+(\{r\})$ hence $p \in [r] \setminus I^+(\{r\})$.

Let us define the sets

$$B_I(\{r\}) := \overline{[r]} \setminus I^+(\{r\}), \quad \text{and} \quad B_p(\{r\}) := \overline{[r]} \setminus I^+(\{r\}).$$

Observe that (iv) establishes that $B_I(\{r\})$ is a subset of the achronal boundary $\partial I^+(\{r\})$ and similarly, $B_p(\{r\})$ is a subset of the achronal boundary $\partial I^+(\{r\})$.

**Definition 2.7.** By *generator* of the achronal set $A$ we mean a lightlike ray contained in $A$.

We do not impose that the generator be a maximally extended lightlike ray contained in $A$. In other words, as a matter of terminology, if $\sigma : [0, b) \to M$ is a generator of $A$ then $\sigma : [a, b) \to M$, $0 \leq a < b$, is also a generator of $A$.

**Lemma 2.8.** The set $B_I(\{r\})$ is closed, achronal, and generated by future lightlike rays. Analogously, the set $B_p(\{r\})$ is closed, achronal, and generated by past lightlike rays.

**Proof.** Let us give the proof for $B_I(\{r\})$, the proof for the other case being analogous. The closure of $B_I(\{r\})$ is immediate from the definition. Let $p \in B_I(\{r\})$, as $p \in R_I(\{r\})$ there is a future lightlike ray starting from $p$ entirely contained in $[r]$ and hence in $R_I(\{r\})$. Moreover, no point of this ray can belong to $I^+(\{r\})$ otherwise $p$ would belong to $I^+(\{r\})$. We conclude that the whole ray is contained in $B_I(\{r\})$.

Let us come to the proof of achronality. Assume by contradiction that there is a timelike curve $\sigma : [0, 1] \to M$ whose endpoints $p = \sigma(0)$ and $q = \sigma(1)$ belong to $B_I(\{r\})$. There cannot be a value of $t \in (0, 1)$ such that $\sigma(t) \in [r]$ otherwise as $I^+$ is open, and $p, q \in [r]$, we would have $r \ll \sigma(t) \ll r$, that is $\sigma(t) \in [r]$, in contradiction with $\sigma(t) \not\in [r]$. Thus either $\sigma((0, 1))$ is contained in $[r]$ or it is contained in $M \setminus \overline{r}$. The former case would imply $p \in I^+(\{r\})$, a contradiction. In the latter case it is possible to find $z \in \sigma((0, 1)) \cap M \setminus \overline{r}$, and as $p \ll z \ll q$ and $I^+$ is open, $r \ll z \ll r$, a contradiction.

**Proposition 2.9.** Let $[r]$ be a chronology violating class, then $I^+(\{r\}) \cap [r] \subset B_I(\{r\})$ and $I^+(\{r\}) \cap [r] \subset B_p(\{r\})$. Moreover, if $p \in I^+(\{r\}) \cap R_p(\{r\})$ or $I^+(\{r\}) \cap R_I(\{r\})$ then through $p$ passes an inextendible lightlike geodesic contained in $[r]$.

**Proof.** Let us prove the former inclusion, the latter being analogous.

Let $q \in I^+(\{r\}) \cap [r]$, we have only to prove that $q \not\in I^+(\{r\})$. If it were $q \in I^+(\{r\})$ then $r \ll q \ll r$, a contradiction.

Let us come to the last statement. As $p \in R_p(\{r\})$ there is a past lightlike ray $\eta$ contained in $[r]$ ending at $p$. As $p \in I^+(\{r\}) \cap [r] \subset R_I(\{r\})$, there is a future lightlike ray $\sigma$ passing through $p$ and contained in $[r]$. This ray is the continuation of the past lightlike ray $\eta$. Indeed, assume that they do not join smoothly at $p$. Take a point $x \in I^+(\{r\}) \cap [r] \setminus \{p\}$ (recall that $I^+$ is open), so that, because of the corner at $p$, $\sigma \setminus \{p\} \subset I^+(x)$. Again, since $I^+(x)$ is open and $\sigma \subset [r]$ we have $x \ll r$, thus since $r \ll x$, we conclude $x \ll x$ which is impossible as $x \in \eta \subset [r]$. We have therefore obtained a lightlike geodesic $\gamma = \sigma \circ \eta$ passing through $p$ entirely contained in $[r]$.

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Corollary 2.10. The following identity holds: \([r] = \bar{B}_p([r]) \cup B_p([r]).\)

**Proof.** In a direction the inclusion is obvious, thus since \(B_p([r]) = [r] \setminus I^+(r)\) and \(\bar{B}_p([r]) = [r] \setminus \bar{I}^+(r)\) we have only to prove that if \(p \in [r]\) then \(p \not\in \bar{I}^+(r)\) or \(p \not\in I^+(r)\). Indeed, if \(p\) belongs to both sets \(r \ll p \ll r\), a contradiction. 

Note that it can be \(B_p([r]) \cap B_f([r]) \not= \emptyset\) (see figure 1). The previous results justify the following definition

**Definition 2.11.** The sets \(B_f([r])\) and \(B_p([r])\) are respectively the future and the past boundaries of the chronology violating class \([r]\).

The previous and the next results will prove the reasonableness and the good behavior of these definitions.

**Proposition 2.12.** Let \([r]\) be a chronology violating class then \(I^+(B_f([r])) \cap [r] = \emptyset\). Moreover, if \(B_f([r]) \not= \emptyset\) then \(I^+(B_f([r])) = I^-(r)\). Analogous statements hold in the past case.

**Proof.** If there were a \(p \in B_f([r])\) such that \(I^+(p) \cap [r] = \emptyset\) then \(p \in \bar{I}^-([r])\), a contradiction.

In a direction, \(I^-(B_f([r])) \subseteq I^-(\bar{I}^-(r)) = I^-(r)\). In the other direction, assume \(I^-(r) \cap B_f([r]) = \emptyset\), then there is \(q \in I^-(B_f([r])) \cap [r]\), hence \(I^-(r) = I^-(q) \subseteq I^-(B_f([r]))\).

The alternative \(I^+(r) \cap B_f([r]) = \emptyset\) cannot hold, indeed under this assumption no point of \(I^+(r)\) would stay outside \([r]\) as this would imply that \(I^+(r) \cap [r] = \emptyset\) and hence because of \(I^+(r) \cap [r] \subseteq B_f([r]), I^+(r) \cap B_f([r]) = \emptyset\). Thus the case \(I^+(r) \cap B_f([r]) = \emptyset\) leads to \(I^+(r) \subseteq [r]\) and hence \(I^+(r) = [r]\), i.e. \([r]\) is a future set. As \(B_f([r]) \subseteq [r]\), and \(B_f([r]) = \emptyset\) taken \(x \in B_f([r])\) by the property of future sets [1, proposition 3.7], \(I^+(x) \subseteq [r]\) hence \(x \in I^-(r)\) in contradiction with the definition of \(B_f([r])\).

**Proposition 2.13.** Let \([r]\) be a chronology violating class then \(B_f([r]) = [r]\) if and only if \(B_p([r]) = \emptyset\). Analogously, \(B_p([r]) = [r]\) if and only if \(B_f([r]) = \emptyset\).

**Proof.** The direction \(B_p([r]) = \emptyset \Rightarrow B_f([r]) = [r]\) follows from \([r] = B_p([r]) \cup B_f([r])\). For the converse, assume \(B_f([r]) = [r]\) and that, by contradiction, \(p \in \bar{B}_p([r])\) (hence \(p \in B_p([r]) \cap B_f([r]))\), then \(I^+(p)\) has no point in \([r]\) otherwise \(p \in I^+(r)\) and hence \(p \not\in B_p([r])\), a contradiction. Thus if \(p \in B_p([r])\) then \(I^+(p) \cap [r] = \emptyset\). Take \(q \ll p\), as \(I^+\) is open and \(p \in [r]\) there is a timelike curve joining \(q\) to \(r\). This curve intersects \([r]\) at some point \(x\), thus \(x \in [r] \cap I^+(r)\), and \(x \not\in B_f([r])\), a contradiction. We conclude that \(B_p([r]) = \emptyset\). The proof of the time reversed case is analogous.

The definition of the edge of an achronal set can be found in [10, section 6.5] or [1, definition 14.27].

**Definition 2.14.** Given an achronal set \(S\) the edge of \(S\), edge\((S)\), is the set of points \(q \in S\) such that for every open set \(U \ni q\) there are \(p \in I^-q, U\), \(r \in I^+q, U\), necessarily not
belonging to $S$, such that there is a timelike curve in $U$ connecting $p$ to $r$ which does not intersect $S$.

It is useful to recall that edge($S$) is closed and $\bar{S} \setminus S \subset \text{edge}(S) \subset \bar{S}$.

**Proposition 2.15.** \[\text{edge}(B_f([r])) = \text{edge}(B_p([r]))\].

**Proof.** Let $q \in \text{edge}(B_f([r]))$ then for every neighborhood $U \ni q$ there are $x, y \in U$, $x \ll q \ll y$ and a timelike curve $\sigma$ not intersecting $B_f([r])$ connecting $x$ to $y$ entirely contained in $U$. The point $y$ cannot belong to $\bar{U}$ for otherwise $q \in I^+(r)$ and hence $q \notin B_f([r])$ (recall that the edge of an achronal closed set belongs to the same set), a contradiction. Every intersection point of $\sigma$ with $[r]$ does not belong to $B_f([r])$, and hence belongs to $B_p([r])$. There cannot be more than one intersection point otherwise if $z_1 \ll z_2$ are any two intersection points, $z_2 \in I^+(r) \subset I^+([r])$ thus $z_2$ cannot belong to $B_p([r])$, a contradiction. Moreover, $\sigma$ cannot enter $[r]$ otherwise, by the same argument, the next intersection point with $[r]$ would not belong to $B_p([r])$, a contradiction. Let us exclude the possibility of just one intersection point between $\sigma \setminus \{x\}$ and $[r]$. The intersection point would belong to $B_p([r]) \subset \partial I^+([r])$ but not to $B_f([r]) = \overline{\partial I^+(r)}$, thus it would belong to $[r] \cap I^-(r) \subset I^+(r)$. Thus $\sigma$ enters $[r]$ after the intersection point, a case that we have already excluded. We conclude that $\sigma \setminus \{x\} \subset M \setminus \overline{\partial F}$ with possibly $x \in B_p([r])$. However, we can redefine $x$ by slightly shortening $\sigma$ so that we can assume $\sigma \subset M \setminus \overline{\partial F}$. It remains to prove that $q \in B_p([r]),$ from which it follows, as $\sigma$ does not intersect $B_f([r]), q \in \text{edge}(B_p([r]))$. Assume by contradiction, $q \notin B_p([r]),$ so that $q \in I^+[r] = I^+(r)$. Since the previous analysis can be repeated for every $U \ni q,$ we can find a sequence $x_n \notin \overline{\partial F}, x_n \rightarrow q,$ $x_n \ll q.$ As $I^+(r)$ is open we can assume $x_n \gg r,$ but since $x_n \ll q$ and $q \in [r],\, x_n \ll r$, thus $x_n \in [r], a$ contradiction. We conclude that $\text{edge}(B_f([r])) \subset \text{edge}(B_p([r]))$ and the other inclusion is proved similarly. \[\square\]

From the previous proposition it follows that $\text{edge}(B_f([r])) \subset B_f([r]) \cap B_p([r]),$ however, the reverse inclusion does not hold in general (see figure 1). Contrary to what happens with Cauchy horizons the generators of the boundary do not need to reach its edge.

**Proposition 2.16.** The identities $B_f([r]) \cap I^+([r]) = B_f([r]) \setminus B_p([r]) = \partial I^+([r]) \cap I^+([r])$ hold. If $p \in B_f([r]) \setminus B_p([r])$ and $\gamma : [1, 0] \rightarrow M$ is a timelike curve such that $\gamma(0) = p$ then there is $\epsilon$, $0 < \epsilon < 1,$ such that $\gamma(-\epsilon, 0) \subset [r].$ An analogous past version also holds.

**Proof.** The first identity follows from the chain of equalities, $B_f([r]) \setminus B_p([r]) = [r] \setminus [r] \setminus [r] \setminus [r] = B_f([r]) \cap I^+([r]).$ For the second identity, the inclusion $B_f([r]) \setminus I^+(r) \subset \partial I^+(r) \cap I^+(r)$ is obvious. For the converse, let $p \in \partial I^+(r) \cap I^+(r)$ and let $q \ll p$ sufficiently close to $p$ that $q \in I^+[r]$. Since $q \ll p$, we have $q \in I^+[r]$ thus $q \in [r]$ and taking the limit $q \rightarrow p$ we obtain $p \in \overline{\partial F}$, but $p \notin I^+(r) \supset [r]$ thus $p \in [r] \cap I^+(r) \supset I^+(r) = B_f([r]) \cap I^+(r)$.

Let us come to the last statement. Since $I^+(r)$ is open and $p \in I^+(r)$ there is some $\epsilon > 0$ such that $\gamma(1-\epsilon) \subset I^+(r)$. But $\gamma((-\epsilon, 0)) \subset I^+(p) \subset I^+(r)$ because $p \in \overline{\partial F}$, thus $\gamma((-\epsilon, 0)) \subset [r].$ \[\square\]
Since $\text{edge}(B_f([r])) \subset B_f([r]) \cap B_p([r])$ the previous result implies the inclusion $B_f([r]) \cap I^+(r) \subset B_f([r]) \setminus \text{edge}(B_f([r]))$.

**Proposition 2.17.** $B_f([r]) \setminus \text{edge}(B_f([r]))$ is an open set (in the induced topology) of the achronal boundary $\partial I^-(r)$. An analogous past version also holds.

**Proof.** Let $B = \partial I^-(r)$ and let $q \in B_f([r]) \setminus \text{edge}(B_f([r]))$. We want to prove that there is a neighborhood $U \ni q$ such that $U \cap B_f([r]) = U \cap B$. By contradiction assume not, then for every causally convex neighborhood $U \ni q$ and $x, y \in U, x \ll q \ll y$, we consider the neighborhood of $q$, $I^+(x) \cap I^-(y)$. By assumption this neighborhood contains some point $z \in B \setminus B_f([r])$. The timelike curve $\xi \subset U$ joining $x$ to $z$ and then $z$ to $y$ does not intersect $B_f([r])$. Indeed, $x, y \notin B_f([r])$ as $q \in B_f([r])$ and $B_f([r])$ is achronal. The curve $\xi$ cannot intersect $B_f([r])$ between $x$ and $z$ because, as $z \in B$, and $B_f([r]) \subset B$ it would imply that $B$ is not achronal. Analogously, $\xi$ cannot intersect $B_f([r])$ between $z$ and $y$ because, as $z \in B$, and $B_f([r]) \subset B$ it would imply that $B$ is not achronal. Since every point admits arbitrarily small causally convex neighborhoods we have proved $q \in \text{edge}(B_f([r]))$ a contradiction. □

Figure 1 shows that $B_f([r]) \setminus \text{edge}(B_f([r]))$ can be different from $B_f([r]) \cap I^+(r)$. A non-trivial problem consists in establishing if $B_f([r])$ can be defined as $\partial [r] \cap I^+(r)$. The answer is affirmative and shows in particular that no point of $\text{edge}(B_f([r]))$ is isolated from $B_f([r]) \setminus \text{edge}(B_f([r]))$ or from $B_f([r]) \cap I^+(r)$.

In the next theorem $\text{Int}_{\partial I^-(r)}$ denotes the interior with respect to the topology induced on the achronal boundary $\partial I^-(r)$.

**Theorem 2.18.** The identities $B_f([r]) = [r] \cap I^+(r) = B_f([r]) \setminus \text{edge}(B_f([r]))$ and

$$\text{Int}_{\partial I^-(r)} B_f([r]) = B_f([r]) \setminus \text{edge}(B_f([r]))$$

hold. Analogous past versions also hold.

**Proof.** Let us prove the identity $B_f([r]) = [r] \cap I^+(r)$. Since $[r] \cap I^+(r) \subset B_f([r])$ one direction is obvious. For the other direction, let $p \in B_f([r])$. By lemma 2.6 (iv) $B_f([r]) \subset \partial I^+(r)$. Since $\partial I^-(r)$ is an achronal boundary it is possible to introduce in a neighborhood $O$ of $p$ coordinates $\{x^0, x^1, \ldots, x^n\}$ such that $\partial / \partial x^0$ is timelike and the timelike ‘vertical’ curves $\{x^i = \text{cnst.} \ (i = 1, \ldots, n)\}$ intersect $\partial I^-(r)$ exactly once. Furthermore in these coordinates the achronal boundary $O \cap \partial I^-(r)$ is expressed as the graph of a function $x^0(\{x^i \ (i \neq 0)\})$ which is Lipschitz [10, proposition 6.3.1]. Let $p_n \in [r] \cap O$ be a sequence such that $p_n \rightarrow p$. The timelike vertical curve $\sigma$ passing through $p_n$ intersects $\partial I^-(r)$ at some point $q_n$ different from $p_n$ because $p_n \in I^-(r)$. It cannot be $q_n \ll p_n$ otherwise $q_n \in I^+(r)$ while $q_n \in \partial I^-(r)$, a contradiction. Thus we have just $p_n \ll q_n$. Since $q_n \in \partial I^-(r)$ and $I^+$ is open for every $\supseteq q_n$ there is some point $q'_n \in U \cap I^+(p_n)$ $\cap I^+(r) \subset U \cap I^+(r) \cap I^+(r)$ which implies $q'_n \in [r]$ and since $U$ is arbitrary $q_n \in [r]$. Furthermore, we have $q_n \in I^+(p_n) = I^+(r)$, $q_n \in r \cap I^+(r)$, and the continuity of the graphing function $x^0(\{x\})$ of the achronal boundary implies $q_n \rightarrow p$, that is $p \in \partial I^-(r)$.

The identity $B_f([r]) = B_f([r]) \setminus \text{edge}(B_f([r]))$ follows from $B_f([r]) = [r] \cap I^+(r)$ using the inclusion $[r] \cap I^+(r) \subset B_f([r]) \setminus \text{edge}(B_f([r])) \subset B_f([r])$ proved in proposition 2.17.
Coming to the last identity, the inclusion
\[ \text{Int}_{\partial I^-([r])} B_{I_f}([r]) \supset B_{I_f}([r]) \setminus \text{edge}(B_{I_f}([r])) \]
is a rephrasing of proposition 2.17. Suppose that the reverse inclusion does not hold, then there is \( p \in \text{edge}(B_{I_f}([r])) \) and an open neighborhood \( U \ni p \), such that \( U \cap \partial I^-([r]) \subset B_{I_f}([r]) \). However, this is impossible because taking \( r \ll p \ll q \), \( q, r \in U \), they must be connected by a timelike curve contained in \( U \) which does not intersect \( B_{I_f}([r]) \), but since \( \partial I^-([r]) \) is edgeless and \( p \in \partial I^-([r]) \), this curve intersects \( \partial I^-([r]) \) at some point inside \( U \) thus belonging to \( B_{I_f}([r]) \), a contradiction.

**Corollary 2.19.** If \( \text{edge}(B_{I_f}([r])) = \emptyset \) then \( B_{I_f}([r]) \) is a connected component of \( \partial I^-([r]) \).

**Proof.** By theorem 2.18 \( \text{Int}_{\partial I^-([r])} B_{I_f}([r]) = B_{I_f}([r]) \), thus \( B_{I_f}([r]) \) is an open and closed subset of \( \partial I^-([r]) \) in the induced topology from which the thesis follows.

Theorem 2.18 proves that \([r] \) is like a shell, the boundary \([r] \) is obtained by gluing the two \( n \)-dimensional topological submanifolds \( B_{I_f}([r]) \setminus \text{edge}(B_{I_f}([r])) \) and \( B_{p_r}([r]) \setminus \text{edge}(B_{I_f}([r])) \) along their rims. Furthermore, these submanifolds can touch in some points in their interior. Nevertheless, as the next result proves, this touching region has vanishing interior.

**Proposition 2.20.** The following identity holds
\[ \text{Int}_{\partial I^-([r])} (B_{I_f}([r]) \cap B_{p_r}([r])) = \emptyset. \]

**Proof.** Let \( p \in B_{I_f}([r]) \cap B_{p_r}([r]) \), since \( p \in B_{p_r}([r]) = \overline{[r] \cap I^+(r)} \) there is a sequence \( p_n \in [r] \cap I^+(r) \) such that \( p_n \to p \), but \( p_n \in B_{I_f}([r]) \subset [r] \) and \( p_n \in I^+(r) \) thus \( p_n \not\in B_{p_r}([r]) \) hence \( p_n \in B_{I_f}([r]) \setminus B_{p_r}([r]) \) which proves the thesis.

The next example proves that \( \text{edge}(B_{I_f}([r])) \) is not necessarily acausal and that in fact \( \text{edge}(B_{I_f}([r])) \) could be generated by inextendible lightlike lines (see figure 2).

**Example 2.21.** Let \( M = \mathbb{R} \times \mathbb{R}^2 \) be endowed with the metric
\[
\text{d}s^2 = -2(\cos \alpha(r) \text{d}t - \sin \alpha(r) r \text{d} \varphi)(\sin \alpha(r) \text{d}t + \cos \alpha(r) r \text{d} \varphi) + \text{d}r^2,
\]
where \( (r, \varphi) \) are polar coordinates on \( \mathbb{R}^2 \), and \( \alpha: [0, +\infty) \to [0, \pi/4] \) is such that \( \alpha(0) = \pi/4 \) and \( \alpha = 0 \) (only) for \( r = 1 \), an \( \text{d} \alpha / \text{d}r (0) = 0 \). This metric can be obtained from the usual Minkowski \( 1 + 2 \) metric by tilting the cones of an angle \( \pi/4 - \alpha(r) \) in the positive \( \varphi \) direction. The cones become tangent to the slices \( t = \text{const} \) at \( r = 1 \) and then begin to tilt up again. As a result \( t \) is a semi-time function, in the sense that \( x \ll y \Rightarrow t(x) < t(y) \). The curves \( t = \text{const}, \ r = 1 \), are closed lightlike curves and since they are achronal they are lightlike lines.

The metric can be written in the Kaluza–Klein reduction form
\[
\text{d}s^2 = r^2 \sin 2\alpha \left( \text{d} \varphi - \frac{1}{r \tan 2\alpha} \text{d}t \right)^2 + \left[ -\frac{1}{\sin 2\alpha} \text{d}t^2 + \text{d}r^2 \right].
\]
If we focus on sets that are rotationally invariant the causal sets corresponding to those are obtained just considering the metric in square brackets rather than the full metric. This is a general feature of spacelike dimensional reduction, and rests on the fact that the horizontal lift
of a causal curve on the base is a causal curve in the full spacetime and the projection of a causal curve of the full spacetime is a causal curve on the base. Furthermore, for what concerns causality the metric in square brackets can be multiplied by a conformal factor so that in the end the causality is determined by the metric \( a^{-1} + tr ds^2 + 2d^2 \).

The idea is to consider the disk \( \{(x, t(x) = 0, r(x) \leq 1) \} \), represented in the reduced spacetime by the segment \([0, 1]\) and define \( C^\pm = \{y: t(y) = \pm k\} \cap D^\pm(S) \). For reasons of symmetry \( C^\pm \) is a, possibly empty, disk but for \( k \) sufficiently small \( C^\pm \) has non-vanishing radius. The fact that the causality can be reduced to that of a two-dimensional spacetime, and the fact that in two-dimensional spacetime the geodesics do not have conjugate points [lemma 10.45] implies the identity \( J^+(C^+) \cap J^+(S) = D^+(S) \). Indeed both rotationally invariant sets have a boundary described by the equation \( t(r) = \int_R^1 \sqrt{\sin 2\alpha} \ dr' \). In particular the radius \( R \) of \( C^\pm \) satisfies \( k = \int_R^1 \sqrt{\sin 2\alpha} \ dr' \).

Our spacetime is constructed by removing \( C^+ \) and \( C^- \) and by identifying the interior of the lower side of the former set with the interior of the upper side of the latter set. In this way we get a chronology violating class \([r]\) such that \( \text{edge}(B_f([r])) \) is the rim \( \gamma \) of \( S \), hence a closed achronal geodesic. In this example the generators of \( B_f([r]) \) are past inextendible lightlike geodesic which accumulate on edge \( B_f([r]) \) without reaching it.

Let us investigate the causal convexity of the chronology violating set and its boundaries.

**Proposition 2.22.** Let \( r \in C \), the set \([r]\) is causally convex and the set \([\overline{r}]\) is chronologically convex. Moreover, the sets \( B_f([r]) \setminus B_p([r]) \) and \( B_p([r]) \setminus B_f([r]) \) are causally convex. Finally, if \([\overline{r}]\) is not causally convex then there is an inextendible lightlike geodesic without...
conjugate points which intersects both $B_p([r])$ and $B_f([r])$, in fact it is a generator for these sets for a suitably restricted domain of definition.

**Proof.** Let $x \leq y \leq z$ with $x, z \in [r]$. We know that $r \ll x$ and $z \ll r$ thus $r \ll y \ll r$, that is $y \in [r]$, which proves that $[r]$ is causally convex.

Let $x \leq y \leq z$ with $x, z \in [r]$. Since $I^+$ is open there are $x', z' \in [r]$ such that $x' \ll y \ll z'$ thus $y \in [r]$, which proves that $[r]$ is chronologically convex.

Let us come to the last statement. Let $x \leq y \leq z$ with $x, z \in [r]$. If $x \ll y$ or $y \ll z$ then it is easy to construct a timelike curve connecting $x$ to $z$ which passes arbitrarily close to $y$. Since this timelike curve is necessarily contained in $[r]$ (because $x, z \in [r]$ and $I^+$ is open) we get $y \in [r]$. We can therefore assume that $x$ is connected to $y$ by an achronal lightlike geodesic and analogously for the pair $y, z$. If the two geodesic segments do not join smoothly it is possible again to construct, using the smoothing of the corner argument, a timelike curve which connects $x$ to $z$ which passes arbitrarily close to $y$. We can therefore consider the case in which $x$ and $y$ are connected by a lightlike geodesic segment $\gamma$ passing through $y$.

Let us consider the case $x, z \in B_f([r]) \subset \partial P$ where $P = I^{-}(r)$. Since for every past set $J^-(\bar{P}) \subset \bar{P}$ and $\gamma$ does not enter $P$ (otherwise $x \in I^-(r)$) a contradiction we have $\gamma \subset \partial P = \partial I^-(r)$. Let $y' \in \gamma \setminus \{x, z\}$.

If $x \in B_f([r]) \setminus B_p([r])$ then $x \in I^+(r)$ and it is possible to find a timelike curve $\sigma$ connecting $r$ to $z$ passing arbitrarily close to $y'$. Since $I^+$ is open, $\sigma \setminus \{z\} \subset I^+(r)$ thus $\sigma \setminus \{z\} \subset [r]$ and $y' \in [r]$. Together with $\gamma \subset \partial I^-(r)$ this fact implies $\gamma \subset B_f([r])$ and in particular $y \in B_f([r])$. (This case proves also that $B_f([r]) \setminus B_p([r])$ is causally convex indeed it cannot be $y \in B_f([r])$ as $x \in I^+(r)$ and thus $y \in I^+(r)$.)

If $x \in B_f([r]) \cap B_p([r])$ let $\sigma$ be the past lightlike ray contained in $B_f([r])$ ending at $x$. If $\sigma$ does not join smoothly with $\gamma$ then $\gamma \setminus \{x\} \subset I^+(r)$ thus $\gamma \setminus \{x\} \subset \partial I^{-}(r) \cap I^+(r) = B_f([r]) \setminus B_p([r])$, in particular $y \in B_f([r])$. If $\sigma$ joins smoothly with $\gamma$ let us consider a future inextendible lightlike ray $\eta$ starting from $z$ and contained in $B_f([r])$. If $\eta$ does not join smoothly $\gamma$ then $\gamma \setminus \{z\} \subset I^{-}(r)$ which is impossible since $x \in B_f([r])$.

Thus we are left with the case in which $\gamma$ can be extended to an inextendible lightlike geodesic which in the past direction becomes a generator of $B_p([r])$ (coincident with $\sigma$) and in the future direction becomes a generator of $B_f([r])$ (coincident with $\eta$).

The case $x, z \in B_f([r])$ leads to time dual results and we are left only with the cases (i) $x \in B_f([r]) \setminus B_f([r])$, $z \in B_f([r]) \setminus B_p([r])$, and (ii) $x \in B_f([r]) \setminus B_p([r])$, $z \in B_f([r]) \setminus B_f([r])$. The case (ii) cannot apply because $z \in I^-(r)$ which would imply $x \in I^+(r)$ a contradiction with $x \in B_f([r])$. In case (i) let $\sigma$ be the past inextendible lightlike ray contained in $B_f([r])$ ending at $x$ and let $\eta$ be the future inextendible lightlike ray contained in $B_f([r])$ ending at $z$. If $\sigma$ does not join smoothly with $\gamma$ then $\gamma \setminus \{x\} \subset I^+(r)$ and it is possible to find a timelike curve $\alpha$ connecting $r$ to $z$ passing arbitrarily close to $y$. Since $I^+$ is open, $\alpha \setminus \{z\} \subset I^+(r)$ and $y \in [r]$. Analogously, if $\eta$ does not join smoothly with $\gamma$ then $\gamma \setminus \{z\} \subset I^+(r)$ and it is possible to extend an inextendible lightlike geodesic which in the past direction becomes a generator of $B_p([r])$ (coincident with $\alpha$) and in the future direction becomes a generator of $B_f([r])$ (coincident with $\eta$).

If this geodesic contains a pair of conjugate points then by taking a small timelike variation [10, proposition 4.5.12], every curve of the variation belongs to the chronology violating set and hence $y$ belongs to the closure of the chronology violating set. Thus if $y \notin [r]$ the constructed inextendible geodesic has no pair of conjugate points. \qed
The set $B_f([r]) \cap B_p([r])$ is not necessarily causally convex, see figure 3.

If we follow a generator of $B_f([r])$ in the past direction we may suspect that as long as the geodesic stays in $[r]$ its points belong to $B_f([r])$. This is false as figure 1 shows, however, if the geodesic does not enter $B_f([r])$ then it is true as the next proposition proves.

Proposition 2.23. Let $\gamma : [0, b) \to M$, $0 < b$, be a causal curve which is a generator of $B_f([r])$ if restricted to the domain $[a, b)$, $0 \leq a < b$. If $\gamma([0, a)) \subset [r]$ and $\gamma(0) \in B_f([r])$ then $\gamma : [0, b) \to M$, is a generator of $B_f([r])$.

Proof. Let $t \in (0, b)$, it cannot be $\gamma(t) \notin B_f([r]) \cap B_f([r]) \subset I^-(r)$ for in this case $\gamma(0) \in I^-(r)$, a contradiction. Thus $\gamma \subset B_f([r])$ and $\gamma$ is necessarily achronal as $B_f([r])$ is achronal, i.e. $\gamma$ is a generator.

If we follow a generator of $B_f([r])$ in the past direction we may suspect that the first exit point from $[r]$ (if there is any) should belong to edge($B_f([r])$). This is not generically so as as the next proposition and example show. Ultimately the generators do not end on the edge as it happens for Cauchy horizons because in the present case the inextendible direction of the generators moves away from the edge while in the Cauchy horizon case it moves towards the edge.

Proposition 2.24. Let $\gamma : (-a, b) \to M$, $0 < a, b$, be a causal curve which is a generator of $B_f([r])$ if restricted to the domain $[0, b)$, and assume that for every $\epsilon > 0$, $\gamma((-\epsilon, 0))$ contains some point in $M \setminus \overline{T}$. Then $\gamma(0) \in B_f([r]) \cap B_p([r])$.

Proof. As $\gamma$ is causal, $\gamma \subset T([r])$. It cannot be $\gamma(0) \in B_f([r]) \cap B_p([r])$ for in this case for sufficiently small $\epsilon$, $\gamma((-\epsilon, 0)) \subset I^+(r)$, so that either $\gamma((-\epsilon, 0)) \subset I^-(r)$ and hence
\(\gamma((-\epsilon, 0)) \subseteq [r]\) a contradiction, or \(\epsilon\) could have been chosen so small that 
\(\gamma((-\epsilon, 0)) \subseteq \partial I^+([r])\) in which case \(\gamma((-\epsilon, 0)) \subseteq B_f([r]) \setminus B_p([r])\) which would contradict the assumption that \(\gamma(0)\) is the first exit point in the past direction. We conclude that \(\gamma(0) \in B_f([r]) \cap B_p([r])\).

\[\square\]

**Example 2.25.** We construct an example which proves that a generator of \(B_f([r])\) can have starting point belonging to \((B_f([r]) \cap B_p([r])) \setminus \text{edge}(B_f([r]))\) and immediately escape \(B_f([r])\) if prolonged in the past direction.

Consider \(1 + 2\) Minkowski spacetime of coordinates \((t, x, y)\) and identify the hyperplanes \(t = -2\) and \(t = 2\), so that the spacetime \(N\) between the two slices becomes totally vicious. Next remove from \(t = -1\) an ellipse (including the interior) whose minor axis is 2 and whose major axis is 4. Do the same on the slice \(t = 1\) but let the new ellipse be rotated of \(\pi/2\) radians with respect to the former. A point belonging to both \(B_f([r]) \cap B_p([r])\) is \(q = (0, 0, 0)\); \(q\) does not belong to \(\text{edge}(B_f([r]))\) and it is easy to check that the two generators starting from \(q\) of \(B_f([r])\) escape \(B_f([r])\) if prolonged in the past direction.

### 2.1. Differentiability of \([r]\)

Let us consider the issue of the differentiability of \([r]\). We regard this set as the union of the two \(n\)-dimensional topological submanifolds \(B_f([r]) \setminus \text{edge}(B_f([r]))\) and \(B_p([r]) \setminus \text{edge}(B_f([r]))\), thus we focus first on the differentiability of \(B_p([r]) \setminus \text{edge}(B_f([r]))\).

The differentiability of topological hypersurfaces generated by past inextendible lightlike geodesics has been studied in [2–4, 24]. This analysis was carried out having in mind Cauchy horizons but, as it is clarified with [3, theorem 2.3], the results hold in general. Points at which the generators leave the hypersurface in the future direction are called future endpoints. The quoted works prove that at non-future endpoints the hypersurface is \(C^4\), at future endpoints at which ends only one generator the hypersurface is still \(C^4\) and at future endpoints at which ends more than one generator the hypersurface is non-differentiable. Therefore these results hold unchanged for \(B_p([r]) \setminus \text{edge}(B_f([r]))\), and a time dual version holds for \(B_f([r]) \setminus \text{edge}(B_f([r]))\). A better way to apply them is by considering \(B_p([r])\) as a subset of \(\partial I^+([r])\) which is also generated by past inextendible lightlike geodesics. From that we can infer that \(B_p([r])\) is non-differentiable at \(p \in \text{edge}(B_f([r]))\) if and only if \(p\) admits more that one generator of \(B_p([r])\) ending at it.

Furthermore, Chruściel and Galloway [4] have given an example of Cauchy horizon which is non-differentiable in a dense set. They first constructed [4, theorem 1.1] a compact set \(C = \mathbb{R}^2 \setminus K \subset \mathbb{R}^3\) having a connected Lipschitz boundary such that on the spacetime \(M = (-1, 1) \times \mathbb{R}^2\), endowed with the usual Minkowski metric, \(E^+([0] \times C)\) was non-differentiable on a dense set.

We construct an example of spacetime in which \([r]\) is non-differentiable on a dense set as follows. We remove from the just constructed spacetime the sets \([0] \times C\) and \([1/2] \times C\) and we identify the interior of the upper-side of \([0] \times C\) with the interior of lower-side of \([1/2] \times C\). This operation introduces closed timelike curves and the boundary of the chronology violating region is a subset of what, before the removal of the sets, was \(E^+([0] \times C) \cup E^-([1/2] \times C)\). As such \([r]\) is non-differentiable on a dense set.

We say that \(B_f\) is *compactly generated* if there is a compact set \(K\) such that its (future inextendible) generators enter \(K\). For the notions of the next theorem not previous introduced we refer the reader to [24]. Observe that for the study of the development of time machines one is interested in the time dual version involving \(B_p\).
Theorem 2.26. Assume that the null convergence condition holds. If $B_f$ is compactly generated and its generators are future complete then it is compact, $C^3$, and generated by inextendible lightlike geodesics. Actually smooth if the metric is smooth, and analytic if the metric is analytic. Moreover, $B_f$ has zero Euler characteristic, it is generated by future complete lightlike lines and on $B_f$

$$\theta = \sigma^2 = \text{Ric}(n, n) = 0, \quad b = \tau = \mathcal{R} = \mathcal{C} = 0.$$  

In other words, denoting with $n$ a lightlike tangent field to $B_f$, for every $X \in TB_f$, $\nabla_X n \propto n$ and $R(X, n)n \propto n$, that is, the second fundamental form vanishes on $B_f$ and the null genericity condition is violated everywhere on $B_f$. In 2+1 spacetime dimensions either $B_f$ is a torus or a Klein bottle where the latter case is excluded if the spacetime is time orientable.

If it is know that if $B_f = H^+(S)$ for some partial Cauchy surface $S$ (e.g. see next section) then the condition on the geodesic completeness of $B_f$ can be dropped, for in this case one can use directly [24, theorem 18]. We stress once again [24] that physically speaking it is incorrect to demand the validity of the null genericity condition on a compact set as done by some authors [31], thus its violation does not imply that the spacetime is unphysical.

Observe that $B_f$ belongs to the boundary of the chronology violating region, so if it is not compactly generated then the chronology violating region propagates to the boundary of spacetime. Thus this theorem establishes that either the formation of closed timelike curves happens as in the theorem, with a compact smooth $B_p$ with all its mentioned nice properties, or such CTC formation either violates energy conditions, extends to the boundary, or generates (geodesic) singularities.

Proof. The proof coincides with that of [24, theorem 18] where this time the completeness of the future hypersurface must be assumed while there it was a consequence of it being a Cauchy horizon. If the spacetime dimension is three $B_f$ is two-dimensional and the only compact closed surfaces with Euler characteristic zero are the Klein bottle and the torus. If the spacetime is orientable then $B_f$ is orientable thus the Klein bottle can be excluded.

In three spacetime dimensions one could obtain other interesting results by applying the Schwartz–Poincaré–Bendixson theorem to $B_f$. In fact, observe that $B_f$ is at least $C^1$ thus its tangent vector field is $C^2$ as required by SPB’s theorem. We conclude that under the assumption of theorem 2.26 $B_f$ contains a closed causal curve (fountain) which is a minimal invariant set for the future (and past) lightlike flow on the horizon or the whole torus is itself a minimal invariant set. We recall that a minimal invariant sets is a closed minimal set which is left invariant by the future flow on $B_f$. The concept makes sense on any imprisoned causal curve. Any minimal invariant set is generated by lightlike lines [11, 19].

Any lightlike geodesic is no more lightlike if we open the light cones. However, in some cases the lightlike geodesic is stable in the sense that it gets simply moved aside, while in other cases it is unstable as it disappears completely. The next result does not assume null completeness but, rather, it relates it to the concept of stability.

Theorem 2.27. Suppose that $B_f([r])$ is compactly generated, then any geodesic on $B_f \setminus B_p$ belonging to one of its minimal invariant sets is future complete or it is unstable.

A better understanding of this theorem can be obtained from the proof.
Proof. Suppose that this geodesic $\gamma$ is future incomplete, then there is a small timelike variation towards the future of $\gamma$ which brings this curve to a timelike curve $\eta$ such that $\eta$ accumulates in the future to the same points to which accumulates $\gamma$, and hence accumulates on $\gamma$ itself [23, theorem 2.1] [13]. This fact implies that it is possible to construct a closed timelike curve $\sigma$ in $I^-(\gamma) \cap U$ where $U$ is any neighborhood of $\gamma$. That is, $\gamma$ is in the past boundary of a chronology violating class $[q]$ and on the future boundary of another chronology violating class $[r]$ (the two classes are different otherwise $\gamma$ would belong to a chronology violating class, just take a timelike curve moving from $[r]$ to $[q] = [r]$ passing through a point of $\gamma$). Thus by opening slightly the cones, $\gamma$ disappears but it cannot be recreated anywhere else since the two distinct classes join in a single class, thus showing that the previous configuration was unstable. 

On might ask whether the violation of chronology near a point of $B_f([r])$ is a local or global phenomenon. The next result shows that if a minimal invariant set generator is incomplete in the past direction, then closed timelike curves can be found in any neighborhood of the generator.

**Theorem 2.28.** Let $B_f([r])$ be compactly generated and let $\gamma$ be a lightlike geodesic on $B_f \setminus B_p$ belonging to one of its minimal invariant sets. Then either $\gamma$ is past complete or for every neighborhood $U \supset \gamma$ there is a closed timelike curve in $U \cap I^-(\gamma) \cap [r]$.

The proofs goes similarly to that of the previous theorem but reworked in the past direction.

2.2. The coincidence with previous definitions of boundary

The next result shows that, provided the chronal region is globally hyperbolic, the past Cauchy horizon of a suitable hypersurface is the future boundary of the chronology violating set. This result relates our definition of boundary with the more restrictive one given in some other papers [30].

**Proposition 2.29.** Let $[r]$ be a chronology violating class and assume that the manifold $N = M \setminus I^-([r])$ with the induced metric is globally hyperbolic, then for every Cauchy hypersurface $S$ of $N$, $S$ is edgeless in $M$ and $H^+(S) = \partial I^-([r])$. Moreover, if $M = I^+(|[r])$ then $H^+(S) = [r] = B_f([r])$.

**Proof.** Since $S$ is a (acausal) Cauchy hypersurface for $N$, $\text{Int}D(S) = N$, thus $\partial D(S) \subset \bar{N} = \partial I^-([r])$. The set $S$ has no edge in $N$, moreover, it has no edge also in $M$. Indeed, let $q \in \text{edge}(S)$, then as $S$ is closed in $N$, $q \in \partial I^-([r])$. But $I^+(q)$ is an open set that cannot intersect the past set $I^-(|[r])$, thus $I^+(q) \subset N$, moreover no inextendible timelike curve starting from $q$ (e.g. a geodesic) can intersect $S$ for otherwise $S$ would not be achronal. But since such curve would be inextendible in $N$ this would contradict the fact that $S$ is a Cauchy hypersurface. Thus edge$(S) = \emptyset$.

Note that $\partial D(S) = H^+(S) \cup H^-(S)$, thus $H^-(S) \subset \partial I^-([r])$. For the converse note that if $p \in \partial I^-([r])$, $I^+(p)$ is an open set that cannot intersect $I^-([r])$, thus $I^+(p) \subset N$. Note that $p \in I^+(S)$ for otherwise a future inextendible timelike curve issued from $p$ would not intersect $S$, still when regarded as an inextendible curve in $N$ this empty intersection would contradict the fact that $S$ is a Cauchy hypersurface. Since $p \in I^+(S)$ the points in $I^+(p) \cap I^+(S)$ necessarily belong to $D^-(S)$ thus $p \in D^-(S)$ and moreover $p$ does not belong to $\text{Int}D^-(S)$.
because the points in \( I^-(\{r\}) \) clearly do not belong to \( D^-(S) \), as the future inextendible timelike curves issued from there may enter the chronology violating set \( \{r\} \) and remain there confined. Thus \( p \in H^+(S) \).

By the previous result if \( M = I^+(\{r\}) \) then \( I^-(\{r\}) = \{r\} \) and \( B_\gamma(\{r\}) = \{r\} \).

3. Relationship between compact generation and compact construction

We have recalled that theorems on the non-existence of time machines are based on the observation that any creation of a region of chronology violation would lead to a Cauchy horizon which is compactly generated, namely, such that the generators followed in the past direction enter and get imprisoned in a compact set \( K \). The idea is that the information on the production of closed timelike curves would propagate on spacetime along the generators of the horizon, so followed in the backward direction those generators have to enter the compact space region where the advanced civilization produced the timelike curves in the first place.

This is the argument which is used to justify the assumption of ‘compact generation of the horizon’ in connection to the creation of time machines. It must be remarked that the generators being confined to the horizon cannot reach the Cauchy hypersurface, however, they do intersect the world tube of the compact region in which they are past imprisoned. In this sense the term ‘space region’ used in the previous paragraph is appropriate. Nevertheless, Amos Ori in a series of papers [27, 28] has criticized the previous argument maintaining that the assumption of local time machine creation would have to be expressed by the following concept, which he terms compact construction.

**Definition 3.1.** The Cauchy horizon \( H^+(S) \) is **compactly constructed** if there is a compact set \( S_0 \subset S \) such that \( H^+(S_0) \cap H^+(S) \) contains almost closed causal curves.

Here \( S_0 \) represents the region were the actions of the advanced civilization leading to the formation of closed timelike curves took place.

Actually, Ori uses ‘closed causal curves’ in place of ‘almost closed causal curves’ in the above definition. The difference does not seem to be important: the idea is that almost closed (and possibly closed) causal curves would signal the development of closed timelike curves just behind the horizon. Ori shows that a compactly constructed time machine can be initiated with no violation of energy conditions [28].

The relative strength of compact generation and compact construction has remained open so far. One could suspect ‘compact construction’ to be a weaker property than ‘compact generation’, for the latter with its strength prevents the formation of time machines. In fact we are able to prove

**Theorem 3.2.** Let \( S \) be a closed acausal hypersurface without edge (partial Cauchy hypersurface). If \( H^+(S) \) is compactly generated then it is compactly constructed.

**Proof.** Let \( K \) be the imprisoning compact, we can assume that \( K \subset H^+(S) \), otherwise replace \( K \) with \( K \cap H^+(S) \). Let \( C \subset I^+(S) \) be another compact set, chosen so that \( K \subset \text{Int } C \). Let us prove that \( S_0 := J^-(C \cap D^+(S)) \cap S \) is compact. Suppose not, then there is a sequence of past inextendible casual curves \( \gamma_n \) with future endpoint \( q_n \subset C \cap D^+(S) \) which intersect \( S \) at \( q_n \) with \( q_n \to \infty \), meaning by this that the sequence \( q_n \) escapes every compact subset of \( S \).

Following \( \gamma_n \) in the future direction let \( r_n \subset \partial C \cap D^+(S) \) be the first point in \( C \) and let \( \eta_n := \gamma_n|_{q_n \to r_n} \) be the portion of \( \gamma_n \) not in \( C \) saved for \( r_n \). Let \( r \subset \partial C \cap D^+(S) \) be an
accumulation point of \( r_n \). By the limit curve theorem [18] there is a past inextendible causal curve \( \eta \) with future endpoint \( r \) which does not intersect \( S \) (if it were to intersect it at some \( y \in S \) then a subsequence \( q_n \) would converge to \( y \) which is impossible since every subsequence escapes all compact sets). Being \( \eta \) the limit of curves contained in the closed set \( M \setminus \text{Int} C \) it is also contained in this closed set and so does not intersect \( K \). Observe that it is a causal curve which cannot enter \( D^+(S) \) for otherwise it would be forced to reach \( S \), thus it is entirely contained in \( H^+(S) \). This fact proves that \( r \in H^+(S) \). Since the horizon is achronal \( \eta \) is a lightlike geodesic, that is a generator (lightlike geodesics on the horizon cannot cross for it is easy to see that it would contradict achronality). This is a contradiction with compact generation since we have shown that \( \eta \) does not intersect \( K \subset \text{Int} C \) where every generator should enter. The contradiction proves that \( S_0 \) is compact. Let \( x \in K \), and consider a sequence \( x_k \rightarrow x \), \( x_k \in I^-(x) \). As a consequence, \( x_k \in D^+(S) \). For sufficiently large \( k \), \( x_k \in C \) which implies that \( x_k \in D^+(S_0) \) and consequently, \( x \in D^+(S_0) \) which implies \( x \in H^+(S_0) \). We have shown that \( K \subset H^+(S_0) \) where \( K \) contains almost closed causal curves since it contains a minimal invariant set [19].

\[ \square \]

4. The case \( I^+(\{r\}) = M \) and a singularity theorem

Hawking has suggested that the laws of physics prevent the formation of closed timelike curves in spacetime [9] (the chronology protection conjecture). According to this conjecture the effects preventing the formation of closed timelike curves could be quantistic in nature, in fact Hawking claims that the divergence of the stress energy tensor at the boundary of the chronology violating set would be a feature of this prevention mechanism.

Despite some work aimed at proving the chronology protection conjecture its present status remains quite unclear with some papers supporting it and other papers suggesting its failure [12, 15, 16, 31, 32]. Some people think that in order to solve the problem of the chronology protection conjecture a full theory of quantum gravity would be required [7, 9].

A weak form of chronology protection would forbid the formation of closed timelike curves without denying the possibility that closed timelike curves could have been present since the very beginning of the Universe. For this reason it is important to study spacetimes that originate causally from a chronology violating region \( \{r\} \), namely \( I^+(\{r\}) = M \).

**Proposition 4.1.** There is at most one chronology violating class \( \{r\} \) with the property \( I^+(\{r\}) = M \).

**Proof.** Let \( \{x\} \) be a second chronology violating class such that \( I^+(\{x\}) = M \) then \( x \ll r \) and, since \( I^+(\{r\}) = M \), \( r \ll x \) thus \( \{x\} = \{r\} \). \[ \square \]

**Proposition 4.2.** Let \( \{r\} \) be a chronology violating class such that \( I^+(\{r\}) = M \), then \( \{r\} = B_f(\{r\}) \), \( J^-(\{\bar{r}\}) = \{\bar{r}\} \) and \( I^-(\{\bar{r}\}) = \{r\} \).

**Proof.** Since \( I^+(\{r\}) \cap \{r\} \subset B_f(\{r\}) \) we have \( \{r\} \subset B_f(\{r\}) \) and hence the first equality. For the second equality the inclusion \( \{\bar{r}\} \subset J^-(\{\bar{r}\}) \) is obvious. For the other direction assume by contradiction, \( p \in J^-(\{\bar{r}\}) \setminus \{\bar{r}\} \). Since \( p \in M = I^+(r) \) there is a timelike curve joining \( r \) to \( p \) and a causal curve joining \( p \) to \( \bar{r} \). By making a small variation starting near \( p \) we get a timelike curve from \( r \) to \( \bar{r} \), and hence equivalently, from \( r \) to \( r \) passing arbitrarily close to \( p \), thus \( p \in \{\bar{r}\} \), a contradiction.

For the last equality it suffices to take the interior of the second one. \[ \square \]
Proposition 4.3. Let \( [r] \) be a chronology violating class such that \( I^+(\{r\}) = M \). A past or future inextendible achronal causal curve on \( M \) is either entirely contained in \( M \setminus [r] \) or in \([r]\).

Proof. Let \( \gamma \) be a past inextendible achronal causal curve which passes through a point \( p \in M \setminus [r] \). Let us follow it to the past of \( p \). If it intersects \([r]\) at some point \( q \) then it cannot be tangent to a generator \( \eta \) of \( B_q(\{r\}) \) at \( q \), for otherwise it would coincide with that generator to the future of \( q \) and hence would be entirely contained in \( B_q(\{r\}) \subset [r] \), a contradiction with \( p \in M \setminus [r] \). However, if it makes a corner with \( \eta \) then any point \( q' \in \gamma \) to the past of \( q \) would belong to \( I^-(\{r\}) = [r] \), which is impossible since a lightlike line cannot intersect the chronology violating region.

Let \( \gamma \) be a future inextendible achronal causal curve which passes through a point \( p \in M \setminus [r] \). Then it cannot intersect \([r]\) because \( J^+(\{r\}) = [r] \).

The interesting fact is that \( M \setminus [r] \) must admit a time function, provided null geodesic completeness and other reasonable physical conditions are satisfied (see theorem 4.4). For more details on these conditions see [10]. It can be read as a singularity theorem: under fairly reasonable physical conditions if the spacetime outside the chronology violating region does not admit a time function then the spacetime is geodesically singular.

Theorem 4.4 is a non-trivial generalization over the main theorem contained in [20]. Note that null geodesic completeness is required only on those geodesics intersecting \( M \setminus [r] \). These geodesics cannot be tangent to some geodesic generating the boundary \([r]\), because since this boundary is generated by future lightlike rays contained in \([r]\) (proposition 4.2) the geodesic would have to be contained in \([r]\), a contradiction.

Theorem 4.4. Let \((M, g)\) be a spacetime which admits no chronology violating class but possibly for the one, denoted \([r]\), which generates the whole Universe, i.e. \( I^+(\{r\}) = M \). Assume that the spacetime satisfies the null convergence condition and the null genericity condition on the lightlike inextendible geodesics which are entirely contained in \( M \setminus [r] \), and suppose that these lightlike geodesics are complete. Then the spacetime \( M \setminus [r] \) is stably causal and hence admits a time function.

Proof. Consider the spacetime \( N = M \setminus [r] \) with the induced metric \( g_N \), and denote with \( J_N^\pm \) its causal relation. This spacetime is clearly chronological and in fact strongly causal. Indeed, if strong causality would fail at \( p \in N \) then there would be sequences \( p_n, q_n \to p \), and causal curves \( \sigma_n \) of endpoints \( p_n, q_n \), entirely contained in \( N \), but all escaping and reentering some neighborhood of \( p \). By an application of the limit curve theorem [1, 18] on the spacetime \( M \) there would be an inextendible continuous causal curve \( \sigma \) passing through \( p \) and contained in \( N \) to which a reparametrized subsequence \( \sigma_n \) converges uniformly on compact subsets (\( \sigma \) can possibly be closed). The curve \( \sigma \) must be achronal otherwise one would easily construct a closed timelike curve intersecting \( N \) (a piece of this curve would be a segment of some \( \sigma_n \) thus intersecting \( N \)). Thus \( \sigma \) is a lightlike line and hence, by lemma 4.3, it is entirely contained in \( N \). By assumption \( \sigma \) is complete thus by null genericity and null convergence it has conjugate points, which is in contradiction with it being achronal. The contradiction proves that \((N, g_N)\) is strongly causal.

The next step is to prove that \( J_N^+ \) is transitive. In this case \( N \) would be causally easy [21] and hence stably causal (thus admitting time functions). Suppose \((x, y) \in J_N^+ \) and \((y, z) \in J_N^+ \). The transitivity of \( J_N^+ \) is proved as done in [20, theorem 5], observing that
the limit curve passing through $y$ constructed in that proof, necessarily contained in $\bar{N}$, is either achronal and hence, by lemma 4.3, entirely contained in $N$, which allows to apply that original argument, or non-achronal. In the latter case that argument of proof shows that $(x, z) \in \mathcal{T}$. Let us recall that $\mathcal{T} = \mathcal{T}$, thus there are neighborhoods $U$ and $V$ such that any timelike curve connecting $U \ni x, U \subset N$ to $V \ni z, V \subset N$ must stay in $N$, because otherwise there would be some $w \in \mathcal{P}$ such that $x' \leq w$, with $x' \in U$. This is impossible because by proposition 4.2, $J^-(\mathcal{P}) \subset \mathcal{P}$. Thus $(x, z) \in \mathcal{T}_N = \mathcal{T}_N$.

In a different work [22] I have argued, using entropic and homogeneity arguments, that our spacetime could indeed have been causally preceded by a region of chronology violation. In this picture the null hypersurface $[\mathcal{r}]$ would be generated by achronal inextendible lightlike geodesics, and would replace the usual big bang (which is usually taken as a spacelike hypersurface in the spacetime completion). Since $[\mathcal{r}]$ would be generated by lightlike lines a rigidity mechanism would take place and several components of the Weyl tensor would vanish at the boundary (because the Weyl tensor causes focusing [10]). This fact is in accordance with Penrose’s expectations on the beginning of the Universe [29] (the Weyl tensor hypothesis) according to which, in order to solve the entropic problem of cosmology, the Weyl tensor must be small at the beginning of the Universe.

5. Conclusions

We have studied the boundary of the chronology violating set, defining its future and past parts and proving the reasonability of the definition. For instance, we have shown that the edges of these parts coincide and that the full boundary is obtained by gluing the future and past parts along their edges. We have shown that our definitions are compatible with a previous definition in the domain of applicability of the latter. We have studied other properties of these boundaries, including causal convexity, differentiability and smoothness under energy conditions. Theorem 2.26 clarified the connection with singularities. We have also proved that compactly generated horizons are compactly constructed. This results did not use the definition of chronological boundary but it is relevant in order to clarify no-go theorems on the creation of time machines. Finally, we have considered the circumstance in which there is just one chronology violating region at the beginning of the Universe, proving that under reasonable energy and genericity conditions either there is a time function outside it or the spacetime is singular.

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