POLYNOMIAL TIME LOGIC:
INABILITY TO EXPRESS
SH634

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§1 The polynomial time logic presented
[We present this logic from a paper of Blass, Gurevich, Shelah [BGSh 533];
to compute what you can compute from a model in polynomial time without
arbitrary choices (like ordering the model).]

§2 The general elimination of quantifiers and proof it’s non-expressive
[We define a criterion for showing the logic cannot say too complicated
things on some model using a family of partial automorphism (rather than
real automorphisms) and prove that it works. This is a relative of the
Ehrenfeucht-Fraisse games, and more recent pebble games.]

§3 The canonical example
[We deal with random enough graphs and conclude that they satisfy the 0-1
law so proving the logic cannot express two strong properties.]

§4 Closing comments
[We present a variant of the criterion (the existence of a simple $k$-system).
We then define a logic which naturally expresses it. We comment on defining
$N_I[M]$ for ordinals.]
We present below the choiceless polynomial time logic, introduced under the name $\tilde{CPT}$ time in Blass Gurevich Shelah [BGSh 533]. Knowledge of [BGSh 533] which is phrased with ASM (abstract state machine) is not required except when we explain how the definitions fit in 1.3(4). See on more relevant works there. The aim of this logic is to capture statements on a (finite) model $M$ in polynomial time and space without additional choices but with no additional bound on the depth, so its being this logic is a thesis. So we are not allowed to use a linear order on $M$, but if $P^M$ has $\log_2(\|M\|)$ elements we are allowed to list all subsets of $P^M$, and if e.g. $(\|P^M\|)! \leq \|M\|$ we can list the permutations of $P^M$. Formally for a given $M$, we consider the elements of $M$ as urelements, and build inductively $N_t = N_t[M]$, with $N_0 = M, N_{n+1} \subseteq N_t[M] \cup P(N_t[M])$ but the definition is uniform and $N_t[M]$ should not be too large (i.e. has a polynomial bound) and the process stops.

Informally, we start with a model $M$ with each element an atom=urelement, we successively define $N_t[M], t$ running on the stages of the “computations”; to $N_{t+1}[M]$ we add few families of subsets of $N_t$, each of those defined by a $\psi(-,a)$-formula for some $a$ from $N_t[M]$, and we update few relations or functions, by defining them from those of the previous stage. Those are coded by $c_{t,\ell}$. We may then check if a target condition holds, then finishing. Note that each stage increases the size of $N_t[M]$ at most by a (fix) power, but in $\|M\|$ steps we can arrive to a model of size $2^{\|M\|}$. So we shall have a timing function $t$ in $\|M\|$, normal polynomial, so when we have wasted too many resources (e.g. $\|N_t[M]\| + t$) our time is up whether we got an answer or not.

More formally

1.1 Definition. 1) For a model $M$, with vocabulary $\tau = \tau_0, \tau$ finite and $\notin$ not in $\tau$, let $\tau^+ = \tau_0 \cup \{\in\}$ considering the elements of $M$ as atoms = urelements, we define $V_t[M]$ by induction of $t : V_0[M] = (M, \in \mid M)$ with $\in \mid M$ being empty (as we consider the members of $M$ as atoms = “urelements”). Next $V_{t+1}[M]$ is the model with universe $V_t[M] \cup \{a : a \subseteq V_t[M]\}$ (so we assume $a \subseteq V_t[M] \Rightarrow a \notin M$ by “urelements”) with the predicates and individual constants and function symbols of $\tau$ interpreted as in $M$ (so function symbols in $\tau$ are interpreted as partial functions) and $\in^{V_{t+1}[M]}$ is $\in \mid V_{t+1}[M]$. 2) We say $\bar{\gamma} = (\bar{\psi}, \bar{\varphi}, \bar{c})$ is an inductive scheme for the language $\mathcal{L}_{t,0}(\tau^+)$ (where $\mathcal{L}_{t,0}$ is first order logic) if: letting $m_0 = \ell g(\bar{\psi}), m_1 = \ell g(\bar{\varphi})$ we have

(a) $\bar{\gamma} = (\mathcal{P}_t : \ell < m_1)$ is a sequence of unary¹ predicates not in $\tau^+$ (b) $\bar{\psi} = (\psi_t : \ell < m_0), \psi_t = \psi_t(x; \bar{y}_t$) is first order in the vocabulary $\tau_2 = \tau_2[N] = \tau_1 \cup \{\mathcal{P}_t : \ell < \ell g(\bar{c})\}$ (c) $\bar{\varphi} = (\varphi_t : \ell < m_1), \varphi_t = \varphi_t(x)$ is first order in the vocabulary $\tau_2$. 3) For $M, \tau = \tau_0, \tau_1, \bar{\gamma} = (\bar{\psi}, \bar{\varphi}, \bar{c})$ and $\tau_2$ as above, we define by induction on $t$ a submodel $N_t = N_t[M] = N_{\bar{\gamma},t}[M]$ of $V_t[M]$ and $\bar{c}_t = (c_{t,\ell} : \ell < m_1)$ with

¹but they will usually be interpreted as singletons, i.e. acts as individual constants but for the crucial “times” $t = 0, 1$ not necessarily
1.2 Definition. Let $T : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ and $\mathcal{L}^* \subseteq \mathcal{L}_{\text{fo}}$ be a logic (w.l.o.g. $\mathcal{L}_{\text{card}}$ usually) and let $\tau$ be a vocabulary. If $T$ is constantly $\infty$ we may write $\infty$.

We define for $i = 1, 2, 3, 4$ the logic $\mathcal{L}^*_T[\mathcal{L}^*](\tau)$, for all of those logics the set of sentences for a vocabulary $\tau$ is a subset of $\Theta = \Theta_{\tau} = \{\theta_{\tau, c, t} : (\mathcal{L}_{\text{card}} \cup \{\text{Th}\}) \text{ an inductive scheme for } \mathcal{L}^*(\tau), \chi \in \mathcal{L}^*(\tau) \text{ and } t \in \mathbb{T}\}$, (equal if not said otherwise), and for most of them we define the stopping time $t_{\infty}(M, \bar{Y}, T)$ or $t_{\infty}(M, \bar{Y})$ (if $T$ does not matter). The satisfaction relation for $\mathcal{L}^*_T[\mathcal{L}^*](\tau)$ is denoted by $\models$. Also we write $\theta_{\tau, c, t}$ if $T$ does not matter. We may let $\text{Dom}(T)$ be the set of relevant structures.
Case 1: For $\iota = 1$.

\[ t_1[M, \Upsilon] = \min\{t : t \geq 2 \text{ and } c_{t,0}[M, \Upsilon] = c_{t,2}[M, \Upsilon]\} \]

(If there is no such $t \in \mathbb{N}$ we let it be $\infty$ (i.e. $\omega$ for set theorists), we could also use “undefined”).

\[ M \models \theta_{T,\chi} \text{ iff } N_t[M, \Upsilon] \models \chi \text{ for } t = t_1[M, \Upsilon]. \]

Case 2: For $\iota = 2$.

\[ t_2[M, \Upsilon, t] = \min\{t : \|N_{t+1}[M, \Upsilon]\| + (t + 1) > t(\|M\|) \text{ or } c_{t,0}[M, \Upsilon] = c_{t,1}[M, \Upsilon]\} \]

and

(a) if $t = t_2[M, \Upsilon, t] < t(\|M\|)$, then the truth value of $\theta_{T,\chi, t}$ in $M$ is true or false iff $N_t[M, \Upsilon] \models \chi$ or $N_t[M, \Upsilon] \models \neg \chi$ respectively and we write $M \models \theta_{T,\chi, t}$ or $M \models \neg \theta_{T,\chi, t}$ respectively.

(b) if $t_2[M, \Upsilon, t] = t(\|M\|)$ we say “$M \models \theta_{T,\chi, t}$” is undefined and we say “the truth value of $\theta_{T,\chi, t}$ in $M$” is undefined.

Case 3: $\iota = 3$.

We restrict ourselves to standard $\Upsilon$ and let

\[ t_3[M, \Upsilon, t] = \min\{t : \|N_{t+1}[M, \Upsilon]\| > t(\|M\|)\} \]

and define $\models$ as in Case 2.

Case 4: $\iota = 4$.

We restrict ourselves to standard $\Upsilon$ and

\[ t_4[M, \Upsilon, t] = \sup\{t : \text{for some } k < m_0 \text{ we have } \mathcal{P}_{t+1,k}[M, \Upsilon]\] has $> t(\|M\|)$ members\}

(so it can be $\infty$; i.e. $\omega$ for set theorists, but we can guarantee $\mathcal{P}_{t,0} = \{0, \ldots, t-1\}$ so that this never happens) and define $\models$ as in Case 2.

We can replace in clause (b), $c_0 = c_1$ (i.e. $(\forall x)(P_0(x) \equiv P_1(x)) \land (\exists x)P_0(x)$) by a sentence $\chi$.

1.3 Discussion: 1) The most smooth variant for our purpose is where $\mathcal{L}^* = \mathcal{L}_{\text{card},T}$ or $\mathcal{L}^* = \mathcal{L}_{\text{card},T}$ and $\iota = 4$. From considering the motivation the most natural $T$ is $\{n^m : m < \omega\}$, and $\iota = 3$.

2) For $\iota = 1, 2, 3$ some properties of $M$ can be “incidentally” expressed by the logic, as the stopping time gives us some information on concerning cardinality can be expressed. This suggests preferring the option $\models$ undefined in case (b) rather than false.

3) If you like set theory, you can let $t$ be any ordinal; but this is a side issue here; see §4.

Implicit in 1.2 (and alternative to 1.2) is
1.4 Definition. Let $M$, $\Upsilon$ be as in Definition 1.2 be given.
1) We say $(N, \bar{c})$ is an $M$-candidate or $(M, \Upsilon)$-candidate
   
   (a) $N$ is a transitive submodel of $\bigcup_t V_t[M]$
   
   (b) $\bar{c} = \langle c_\ell : \ell < m_1 \rangle$, $c_\ell \in N$ (or is undefined, so is really a unary relation on $N$ of cardinality $\leq 1$).

2) We say $(N', \bar{c}')$ is the $(\Upsilon, t)$-successor of $(N, \bar{c})$ if $N', \bar{c}'$ are defined as in Definition 1.2, but

   $|N'| = |N| \cup \bigcup_{\ell < m_1} \mathcal{P}_\ell[N, \Upsilon]$

   $\mathcal{P}_\ell[N, \Upsilon] = A_\ell$ is $\{ \{ a : (N, \bar{c}) \models \psi_\ell(a, \bar{b}) \} : \bar{b} \in i^t(g(t))N \}$ if this family has $\leq t(M)$ members. $A_\ell$ is empty otherwise.

3) We define $N_t[M, \Upsilon, t]$ and $\bar{c}_t[M, \Upsilon, t]$ by induction on $t$ as follows:

   for $t = 0$ it is $M$ ($c_{t, \ell}$ undefined)
   
   for $t + 1$, $(N_{t+1}, \bar{c}_{t+1})$ is the $(\Upsilon, t)$-successor of $(N_t, \bar{c}_t)$, see below.

1.5 Claim. 1) if $(N, \bar{c})$ is an $(M, \Upsilon)$-candidate, it has at most one $(\Upsilon, t)$-successor.
2) The various definition fits, so we can use 1.4.

1.6 Discussion: How do we translate between the definitions above and [BGSh 533]?

   (i) an infinite structure $I$ there corresponds to a $\tau$-model here
   
   (ii) a state $A$ there corresponds to a model of the form $N = N_t[M, \Upsilon]$ in 1.2 and $(N, \bar{c})$ in 1.4
   
   (iii) dynamic function there corresponds to $\mathcal{P}_\ell$ (that is $c_{t, \ell}$) here
   
   (iv) an object $x$ is active at $A$ in 5.1 there it corresponds to $x \in N$
   
   (v) a program 4.6 there corresponds to an $\Upsilon$ in 1.1(2) here (mainly the first order formulas used);
   
   (vi) the counting function in 4.7 there corresponds to the cardinality quantifier (1.1(6)) here
   
   (vii) the polynomial functions $p, q$ in 5.1 there corresponds to $t \in T$ here.

Note that the $c_{t+1, \ell}$ can in the usual set theory manner be actually 7-place function from $N_t$ to $N_t$ or 7-place relation on $N_t$ or be the universe of $N_t$. Understanding this to interpret the successor step there to here we need that all parts of the program are expressible in $L_{f.o.}$ (or $L_{\text{card}}$). For the other direction we need to show f.o. operations can be expressed by the programs of ASM there (see 6.1 there), no problem (and not needed to show our results solved problems there).
In Blass Gurevich Shelah [BGSh 533] we deal with the case of equality and permutations here we are using partial isomorphisms. It seems a reasonably precise way, so we shall later, hopefully, get a kind of inverse.

2.1 Discussion: Our aim is to have a family $F$ of partial automorphisms as in Ehrenfeucht Fraisse games (or Karp), of the model $M$ we analyze, not total automorphism (as in [BGSh 533]) - too restrictive. But it has to be lifted to the $N_t$'s. But their domains (and ranges) can contain an element of high rank. So we should not lose anything when we get up on $t$. The solution is $I \subseteq \{ A : A \subseteq M \}$ closed downward and $F$ (really $\langle F_\ell : \ell < m_1 \rangle$), a family of partial automorphisms of $M$. So every $x \in N$ will have a support $A \in I$ and for $f \in F$, its action on $A$ determines its action on $x$, $G(f)(x)$ in the section notation. It is not unreasonable to demand that there is a minimal one, still it is somewhat restrictive (or we have to add imaginary elements as in [Sh:a] or [Sh:c], not a very appetizing choice).

But how come we in stage $t + 1$ succeed to add “all sets $X = X_{i,b}$” definable by $\psi_1(x,\vec{b}), \vec{b} \in \varphi(b)N_t$?

The parameter $b_1, \ldots, b_m$ each has a support say $A_1, \ldots, A_m$ all in $I$, so we have enough mappings in the family, the new set has in a sense support $A = \bigcup_{\ell=1}^m A_\ell$, in the sense that suitable partial mappings do, if $y$ has support $B$ ($BR y$ in this section notation) $A \cup B \subseteq \text{Dom}(f), f \upharpoonright A = \text{id}_A$ the mapping $f$ induces in $N$, map $y$ to a member of $B$.

But we are not allowed to increase the possible support and $A$ though a kind of support is probably too large: $I$ is not closed under union. But, if we add $X = X_{i,\vec{b}}$ we have to add all similar $X' = X_{i,\vec{b}'}$. So our strategy is to say no to looking for a support $A' \in I$. So fixing $A'$ we like that if $f \in F, f \upharpoonright A' = \text{id}_{A'}, A \subseteq \text{Dom}(f)$, then $f \upharpoonright A$ induces a mapping of $X_{i,\vec{b}}$ to some $X_{i,\vec{b}'}$, which we like to demand that will be equal thus justifying the statement “$A'$ supports $X$.”

How? We use our bound on the size of the computation. So we need a dichotomy: either there is such $A' \in I$ or the number $X_{i,\vec{b}'}$ defined by $\psi_1(x,\vec{b}')$ is too large!!

On this dichotomy hangs the proof.

However, we do not like to state this as a condition on $N_t$ rather on $M$. We do not “know” how $\psi_1(x,\vec{b}')$ will act but for any possible $A'$ this induces an equivalence relation on the images of $A$ ($F$ has to be large enough).

Actually, we can ignore $f$ and develop set theory of elements demanding support in $F$. So we break the proof to definition and claims.

We consider three variants of the logic: usual variant to make preservation clear, and the case with the cardinality quantifier.

We could use one $F$ but we use $\langle F_\ell : \ell \leq m_{\text{eq}}(*) \rangle$. Actually, for much of the treatment only $F_0$ count.

Discussion: Each $a \in N$ will have support $A \in I$. Now should we in $(N, \vec{c})$ add the support of each $\vec{c}_k$ or this will be included? No! The $\vec{c}_i$ will have support $\emptyset$. 
2.2 The Main Definition. 1) We say $\mathcal{Y} = (M, I, \mathcal{F})$ is a $k$-system if $\bar{m}^* = (m_{qd}(\ast), m_{tv}(\ast))$ and:

(A) $I$ is a family of subsets of $|M|$ (the universe of $M$) closed under subsets and each singleton belongs to it

[hint: intended as first approximation to the possible supports of the partial automorphism of $M$, the model of course; the intention is that $M$ is finite]

Let $I[m] = \{ \bigcup_{t=1}^{m} A_t : A_t \in I \}$

(B) $\mathcal{F}$ is a family of partial automorphisms of $M$ such that $A \subseteq \text{Dom}(f)$ & $A \in I \Rightarrow f''(A) \in I; \mathcal{F}$ closed under inverse (i.e. $f \in \mathcal{F}_\ell \Rightarrow f^{-1} \in \mathcal{F}_\ell$) and composition and restriction

(C) if $f \in \mathcal{F}$ then $\text{Dom}(f)$ is the union of $\leq k$ members of $I$

(D) if $f \in \mathcal{F}$ and $A_1, \ldots, A_{k-1}, A_k \in I$ and $\ell \in \{1, \ldots, k-1\} \Rightarrow A_\ell \subseteq \text{Dom}(f)$, then for some $g \in \mathcal{F}$ we have

$$f \upharpoonright \bigcup_{\ell=1}^{k-1} A_\ell \subseteq g$$

$$A_k \subseteq \text{Dom}(g)$$

(E) if $(\alpha)$ then $(\beta)_1 \lor (\beta)_2$ where

(\alpha) $h$ is a function from some $[m] = \{1, \ldots, m\}$, $\text{Rang}(h)$ belongs to $I[s], E$ is an equivalence relation on $H_h = \{ h' : \text{for some } f \in \mathcal{F}_0, \text{Rang}(h) \subseteq \text{Dom}(f) \text{ and } h' = f \circ h \}$ satisfying

$$(*) \text{ if } h_1, h_2, h_3, h_4 \in H_h \text{ and there is } f \in \mathcal{F} \text{ such that } \text{ (Rang} h_1) \cup (\text{Rang} h_3) \subseteq \text{Dom}(f) \text{ and } f \circ h_1 = h_3, f \circ h_2 = h_2 \text{ then } (h_1 Eh_2 \Leftrightarrow h_3 Eh_4)$$

(\beta)_1 there is $u \subseteq [m]$ such that $\text{Rang}(h \upharpoonright u) \in I$ and $(\forall h_1, h_2 \in H_h)(h_1 \upharpoonright u = h_2 \upharpoonright u \rightarrow h_1 Eh_2)$

(\beta)_2 the number of $E$-equivalence classes is $> t(M)$

[Hint: this is how from “there is some not too large number” we get “there is a support in $I$”].

Note that without loss of generality $h$ is one-to-one.

2) We say $(M, I, \mathcal{F})$ is a $(t, \bar{m})$-system if $\bar{m} = (k, s)$ and it is a $(t, s)$-dichotomical $k$-system; we may also say $(t, k, s)$-system.
2.3 Definition. 1) We say \( \mathcal{Y} = (M, I, \mathcal{F}) \) is a super s-system if it is an \( \bar{m}^\ast \)-system and in addition

\((E)^{\dagger}\) In clause (E), fixing some \( A^\ast \in [s] \) as a set of parameters, the number of equivalence classes is preserved; that is, if for \( \ell = 1, 2 \) we have \( A_\ell \in [s] \) and for some \( m \) and \( h_\ell : [m] \to A \) with range \( I \) and \( \mathcal{E}_\mathcal{Y}(A_\ell, h_\ell) \) (see below) and \( f \in \mathcal{F} \) such that \( A_1 \cup \text{Rang}(h_1) \subseteq \text{Dom}(f) \) we have: \( f \) maps \( A_1 \) into \( A_2, f \circ h_1 = h_2 \) and \( f \) maps \( E_1 \) to \( E_2 \) in the natural way, then the number of \( E_1 \)-equivalence classes is equal to the number of \( E_2 \)-equivalence classes.

2) Let \( \mathcal{E}_\mathcal{Y}(A, h) \) be defined for \( A \in [s] \) and \( h \) a function from some \( [m] \) into \( M \) with \( \text{Rang}(h) \) belonging to \( I, A \subseteq \text{Rang}(h) \) as follows:

\( \mathcal{E}_\mathcal{Y}(A, h) \) is the set of equivalence relations \( E \) on

\[ H_{A,h} = \{ h' : \text{for some } f \in \mathcal{F}, f \upharpoonright A = \text{id}, \text{Rang}(h) \subseteq \text{Dom}(f) \text{ and } h' = f \circ h \}. \]

such that the parallel (\( * \)) of clause (E) of Definition 2.2(2) holds. (So \( H_h \) is replaced by \( H_{h,A} \).

Let \( \mathcal{E}_\mathcal{Y}(A) = \bigcup_h \mathcal{E}_\mathcal{Y}(A, h) \).

3) \( \mathcal{Y} = (M, I, \mathcal{F}) \) is a super \((t, (k, s))\)-system if \( \mathcal{Y} \) is a super \( k \)-system and is \((t, s)\)-dichotomical.

2.4 Observation: If \( f \in \mathcal{F} \) maps \( A_1 \) to \( A_2 \), then it induces a natural extension \( \hat{f} \) of \( f \) mapping also \( \mathcal{E}_\mathcal{Y}(A_1) \) onto \( \mathcal{E}_\mathcal{Y}(A_2) \).

2.5 Definition. 1) Let \( \mathcal{Y} = (M, I, \mathcal{F}) \) be an \( k \)-system, \( M \) a \( \tau \)-model, \( \Upsilon \) an inductive scheme for \( \mathcal{L}_{f_0}(\tau^+) \) and \( \bar{m}^\ast = \bar{m}(\Upsilon) \).

We say that \( \mathcal{Z} = (N, \bar{c}, G, R) = (N^X, \bar{c}^X, G^X, R^X) \) is a \( \Upsilon \)-lifting of \( \mathcal{Y} \) if

(a) \((N, \bar{c})\) is an \((M, \Upsilon)\)-candidate so \( N \) is a transitive submodel of set theory with \( M \) as its set of urelements (see 1.4(1))

(b) \( G \) is a function with domain \( \mathcal{F} \)

(c) for \( f \in \mathcal{F}, f \subseteq G(f) \) and \( G(f) \) is a partial automorphism of \( N \)

(d) if \( f \in \mathcal{F}, g \in \mathcal{F}, f \subseteq g \) then \( G(f) \subseteq G(g) \)

(e) \( R \) is a two-place relation written \( xRy \) such that

\[ xRy \Rightarrow x \in I \& y \in N \]

[intention: \( x \) is a support of \( y \)]

(f) if \( ARy \) and \( f \in \mathcal{F}, A \subseteq \text{Dom}(f) \), then \( y \in \text{Dom}(G(f)) \) and \( f \upharpoonright A = \text{id} \Rightarrow G(f)(y) = y \)

(g) \((\forall y \in N)(\exists A \in I)ARy \)

(h) if \( A \in I \) and \( A \subseteq \text{Dom}(f), y \in \text{Dom}(G(f)) \) then

\[ ARy \Leftrightarrow f''(A)R(G(f)(y)) \]

(i) for \( f \in \mathcal{F}, G(f^{-1}) = (G(f))^{-1} \)

(j) for \( f_1, f_2 \in \mathcal{F}, f = f_2 \circ f_1 \) we have \( G(f) \subseteq G(f_2) \circ G(f_1) \)

(k) if \( c_\ell \in \text{Dom}(f) \) and \( f \in \mathcal{F} \) then \( f(c_\ell) = c_\ell \).
2.6 Definition. For $\mathcal{Y} = (M, I, \mathcal{F})$, a $k$-system the 0 — $\Upsilon$-lifting is $(M, \bar{c}, G, R)$ where

(a) $G$ is the identity on $\mathcal{F}$
(b) $ARy \iff A \subseteq I \& y \in A$
(c) each $c_i$ undefined.

2.7 Fact: The 0 — $\Upsilon$-lifting (in Definition 2.6) exists and is a lifting.

2.8 Definition. Let $\mathcal{Y} = (M, I, \mathcal{F})$ an $k$-system and $\mathcal{Z} = (N, \bar{c}, G, R)$ be a $\Upsilon$-lifting.
1) We say $X$ is good or $(\mathcal{Y}, \mathcal{Z})$-good if

(a) $X$ a subset of $N$
(b) for some $A \in I$ we have $A$ supports $X$ (for our $\mathcal{Y}$ and $\mathcal{Z}$) which means:
   \begin{itemize}
   \item $f \in \mathcal{F}$, $BRy$ (so $B \subseteq N$, $y \in N$)
   \item and $A \cup B \subseteq \text{Dom}(f)$, and $f \upharpoonright A = \text{id}_A$
   \item then $y \in X \iff G(f)(y) \in X$
   \end{itemize}
   (note: $G(f)(y)$ is well defined by clause (g) of 2.5)
(c) $X \notin N$.

2) Let $\mathcal{P} = \mathcal{P}_{\mathcal{Y}, \mathcal{Z}}$ be the family of good subsets of $N$, let $\mathcal{R}$ be the two-place relation defined by: $A\mathcal{R}X$ iff $A$ supports $X$, i.e. (b) of part (1) holds.
3) For $f \in \mathcal{F}$ and for good $X$ such that $A\mathcal{R}X, A \in I$ when $A \subseteq \text{Dom}(f)$ we let

(a) $(G^+(f))(X) = \{y \in N : \text{for some } g \in \mathcal{F} \text{ and } y' \in X \text{ we have } f \upharpoonright A \subseteq g \text{ and } G(g)(y') = y\}$ as $k \geq 3$, if the result is good
(b) $G^+(f) \upharpoonright N = f$.

We can prove this, see 2.9 below.
4) We define $E = E_{\mathcal{Y}, \mathcal{Z}}$ as the following relation: $X_1EX_2 \iff X_1, X_2$ are good subsets of $N^2$ and for some $f \in \mathcal{F}$ we have $(G^+(f))(X_1) = X_2$; this is an equivalence relation as $k \geq 2$.
5) $\mathcal{Z}' = (N', \bar{c}', G', R')$ is a successor of $\mathcal{Z}$ if both are $\bar{m}$-systems and:

(a) $N \subseteq N' \subseteq N \cup \mathcal{P}_{\mathcal{Y}, \mathcal{Z}}$
(b) $X_1EX_2, X_1 \in \mathcal{P}_{\mathcal{Y}, \mathcal{Z}}, X_2 \in \mathcal{P}_{\mathcal{Y}, \mathcal{Z}} \Rightarrow [X_1 \in N' \leftrightarrow X_2 \in N']$
(c) $G'(f)$ is defined as $G^+(f)$ from part (3)
(d) $R'$ is $R \cup [\mathcal{R} \upharpoonright (I \times N')]$.

6) $\mathcal{Z}' = (N', \bar{c}', G', R')$ is a full $t$-successor of $\mathcal{Z}$ if above $N' = N \cup \{x \in \mathcal{P}_{\mathcal{Y}, \mathcal{Z}} : |X/\mathcal{E}_{\mathcal{Y}, \mathcal{Z}}| \leq t(M)\}$. If we omit $t$ we mean $t(N) = \infty$.
7) $\mathcal{Z}' = (N', \bar{c}', B', R')$ is the true $(\Upsilon, t)$-successor of $\mathcal{Z}$ if above:

(a) $(N, \bar{c})$ is the $(\Upsilon, t)$-successor of $(N, \bar{c})$, (see Definition 1.4(2)).

We now prove that Definition 2.8 is O.K. The functions defined are functions with the right domain and range. The $E$'s are equivalence relations. This is included in the proof of 2.9.
2.9 Claim. Assume $\mathcal{Y}$ is a $(t, m^*)$-system (see Definition 2.2(2)) and $\mathcal{Z}, m^*$ are as in Definition 2.8.
1) The true successor $\mathcal{Z}'$ of $\mathcal{Z}$ if exists is a successor of $\mathcal{Z}$.
2) A successor $\mathcal{Z}'$ of $\mathcal{Z}$ is a $\mathcal{Y}$-lifting of $(\mathcal{Y}, m^*)$.
3) A full successor of $\mathcal{Z}$ exists.

Proof. 1) Trivial.
2) We check the clauses in Definition 2.5.

Clause (a): as $N$ is transitive with $M$ its set of urelements, and $x \in N' \setminus N \Rightarrow x \in \mathcal{P}_N \Rightarrow x \subseteq N$ also $N'$ is transitive with $M$ its set of urelements.

Clause (b): So we have defined $G'$ above.

Clause (c): Let $f \in \mathcal{F}, G'(f) = f'$ and let $x, y \in N'$ belongs to the domain of $f'$ and we should prove $N' \models y \in x \iff N' \models f'(y) \in f'(x)$ (we shall do more toward having clause (g) later).

If $x \in N$, then $f'(x)$ is necessarily in $N$, hence $N' \models y \in x \Rightarrow y \in N$ and $N' \models z \in f'(x) \Rightarrow z \in N$, so as $f' : N = f$ we are done. So we can assume $x \in N' \setminus N$, so $x$ is a good subset of $N$, so for some $A_0 \in I, A_0Rx$. We define

$$z : = \{ b \in N : \text{for some } g \in \mathcal{F} \text{ and } b' \in x \text{ we have } f \uparrow A \subseteq g \text{ and } G(g)(b') = t \}. $$

We need the following

$$(*)_1 \ z \text{ is a good subset of } N \text{ with } A_1 = f'''(A_0) \text{ a support of } z$$

$$(*)_2 \ x = \{ b' \in N : \text{ for some } g \in \mathcal{F} \text{ and } b \in z \text{ we have } f^{-1} \uparrow (f'''(A_0)) \subseteq g \text{ and } G(g)(b) = b' \}$$

$$(*)_3 \ z \text{ does not belong to } N$$

$$(*)_4 \ z = f'(x)$$

$$(*)_5 \text{ if } B \text{ is another support of } x, \text{ then } z' = z \text{ when } z' = \{ b \in N : \text{ for some } g \in \mathcal{F} \text{ and } a \in x \text{ we have } f \uparrow A \subseteq g \text{ and } G(g)(a) = b \}.$$

Proof of $(*)_1$. Suppose $a, b \in N$ and $g_1 \in \mathcal{F}$ and $A_1 \subseteq \text{ Dom}(g_1)$ and $g_1 \uparrow A_1$ is the identity and $a \in \text{ Dom}[G(g_1)]$ and $b = G(g_1)(a)$. Now we should prove that $a \in z \iff b \in z$. It is enough to prove $\Rightarrow$ as applying it to $g_1^{-1}$ we get the other implication. As $t = G(g_1)(s)$ necessarily by (i) of 2.5 for some support $B_1$ of $s, B_1 \subseteq \text{ Dom}(g_1)$.

If $a \in z$ then by the definition of $z$ we can find $g \in \mathcal{F}$ and $a' \in X$ such that $\text{id}_A \subseteq g$ and $G(g)(a') = a$. There is $B_2 \in I$ such that $B_2Ra'$ and $B_2 \subseteq \text{ Dom}(g)$. As $A, B_1, B_2 \in I$ by clause (D) of Definition 7 without loss of generality $B_1 \subseteq \text{ Dom}(g) \rightarrow \text{ see (2.1) undefined}$

(as $3 \leq m_{eq}(*)$, see Definition 1.1).
Let \( g' = g_1 \circ g \), so \( A \cup B_1 \subseteq \text{Dom}(g') \), \( g' \mid A = g_1 \mid A = \text{id}_A \) hence \( a' \in \text{Dom}(G(g')) \) and so \( G(g')(a') = G(g_1)(G(g)(a')) = G(g_1)(a) = b \). (See Definition 2.5, clause (j).)

So \( g', a' \) witness \( b \in z \) as required.

**Proof of \((\ast)_2\).**

Similar.

**Proof of \((\ast)_3\).** If \( z \in N \) there is \( A^* \in I \) such that \( A^* R z \).

Now there is \( f_1 \in \mathcal{F}, f \upharpoonright A \subseteq f_1 \) such that \( A^* \subseteq \text{Rang}(f_1) \), \( f_1 \in \mathcal{F} \). So \( z_1 = G(f_1^{-1})(z) \) is well defined and as in \((\ast)_2\) the proof of \((\ast)_1\) we can check that \( \{ b \in N : b \in z_1 \} = x \); contradiction to \( \langle x \notin N \rangle \).

**Proof of \((\ast)_4\).**

Should be clear.

**Proof of \((\ast)_5\).**

Should be clear.

Clause (d): Check.

Clauses (i),(j),(k): Check.

Clause (e): See Definition of \( R' \).

Clause (f): Included in the proof of clause (d).

Clause (g): Check.

Clause (h):

The new case if: \( A \subseteq \text{Dom}(f) \), \( A \in I \), \( X \in \text{Dom}(G'(f)) \), \( X \in N' \setminus N \). Then \( A R' X \iff f''(A) R'(G'(f))(X) \) by clause (j) it is enough to prove the implication \( \Rightarrow \).

There is \( A_1 \in I \) such that \( A_1 R' X \). If \( \neg A R' X \) we can find \( g \in \mathcal{F}, g \upharpoonright A = \text{id}_A \), and \( z_0 \in \text{Dom} G(f) \), \( z_1 = G(f)(z_0) \), such that \( z_0 \in X \equiv z_1 \notin X \). We can find \( B_0 \in I \) such that \( A_0 R z_0 \) and \( B_1 \subseteq \text{Dom} g \) and let \( A_1 =: g''(A_0) \). We can find \( f_1, f \upharpoonright A \subseteq f_1, B_0 \subseteq \text{Dom}(f_1), f_1 \in \mathcal{F} \). Let \( B'_0 = f_1(B_0) \).

Now chase arrows.

3) Straight.

\[ \square_{2.9} \]

2.10 Claim. Assume

(a) \( \Upsilon \) is an inductive scheme for \( \mathcal{L}_{\text{la}}(\tau^+) \), \( \bar{m}^* = \bar{m}^*(\Upsilon) \)

(b) \( k, s \) satisfy?

(c) \( M \) is a finite \( \tau \)-model, \( t \in T \)

(d) \( \mathcal{Y} = (M, I, \mathcal{F}) \) is a \((t, k, s)\)-system.
Then

(a) if \( N_t[M, \bar{\Upsilon}, \bar{t}], \bar{c}_t[M, \bar{\Upsilon}, \bar{t}] \) are well defined, then for some \( \mathcal{F} = (N, \bar{c}, R, G) \) a \( \Upsilon \)-lifting we have \((N, \bar{c}, \mathcal{F})(N_t, \bar{c}_t)\).

2.11 Definition. 1) We say that \( \mathcal{H} \) is a \( k \)-witness to the equivalence of \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) if

(a) for \( \ell = 1, 2 \) we have \( \mathcal{Y}_\ell = (M_\ell, I_\ell, \mathcal{F}_\ell) \) is a \( k \)-system
(b) \( \mathcal{Y} \) is a family of partial isomorphisms from \( M_1 \) into \( M_2 \)
(c) for every \( g \in \mathcal{H} \), we have \( \text{Dom}(g) \subseteq I_1 \), \( \text{Rang}(g) \subseteq I_2 \)
(d) if \( g \in \mathcal{H} \) and \( f_1 \in \mathcal{F}_1 \) then \( g \circ f \in \mathcal{H} \)
(e) if \( g \in \mathcal{H} \) and \( f_2 \in \mathcal{F}_2 \) then \( f_2 \circ g \in \mathcal{H} \)
(f) if \( g \in \mathcal{H} \) and \( A \in I_1[k - 1] \) and \( B \in I_1 \), then for some \( g_1 \in \mathcal{H} \) we have \( g \upharpoonright A \subseteq g_1 \) and \( B \subseteq \text{Dom}(g_1) \)
(g) if \( g \in \mathcal{H} \) and \( A \in I_2[k - 1] \) and \( B \in I_2 \), then for some \( g_1 \in \mathcal{H} \) we have \( g^{-1} \upharpoonright A \subseteq g_1 \) and \( B \subseteq \text{Rang}(g_1) \).

2) We say that \( \mathcal{H} \) is a \((k, s)\)-witness to the equivalence of \((\mathcal{Y}_1, t_1)\) and \((\mathcal{Y}_2, t_2)\) if

(i) \( \mathcal{Y}_\ell \) is \((t, k, s)\)-system, i.e. \((t_\ell, s)\)-dichotomical \( k \)-system for \( \ell = 1, 2 \)
(ii) \( \mathcal{H} \) is a witness to the \( k \)-equivalence of \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \)
(iii) each \( g \in \mathcal{G} \) preserved the possibility chosen in the definition or \((t, s)\)-dichotomical.

3) We say that \( \mathcal{H} \) is a super \((k, s)\)-witness to the equivalence of \((\mathcal{Y}_1, t_1)\) and \((\mathcal{Y}_2, t_2)\) if

(i) \( \mathcal{Y}_\ell \) is a super \((t_\ell, k, s)\)-system
(ii) \( \mathcal{H} \) is a witness to the \( k \)-equivalence of \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \)
(iii) each \( g \in \mathcal{H} \) preserve the cardinalities involved in the definition of super.

2.12 Main Conclusion. Assume

(a) \( \mathcal{Y}_\ell = (M_\ell, I_\ell, \mathcal{F}_\ell) \) is a \((t, k, s)\)-system for \( \ell = 1, 2 \) and \( \tau(M_\ell) = \tau \)
(b) \( \mathcal{H} \) is a \((k, s)\)-witness to the equivalence of \((\mathcal{Y}_1, t_1)\) and \((\mathcal{Y}_2, t_2)\)
(c) \( \chi \in \mathcal{L}_{\bar{\Upsilon}, \tau} \), i.e. a first order sentence in the vocabulary \( \tau^+ = \tau \cup \{\in\} \),
and every subformula of \( \chi \) has at most \( k \)-free variables
(d) \( \Upsilon \) is an inductive scheme not too complicated relative to \( k, s \).

Then

(a) the truth value of \( \theta_{\Upsilon, \chi, t_\ell} \) in \( M_1 \) and \( \theta_{\Upsilon, \chi, t_2} \) in \( M_2 \) are equal except possibly when: for some \( \ell \in \{1, 2\} \) we have the truth value of \( \theta_{\Upsilon, \chi, t_\ell} \) in \( M_\ell \) is undefined whereas that of \( \theta_{\Upsilon, \chi, t_{3-\ell}} \) in \( M_{3-\ell} \) is well defined and \( t[M_\ell, \Upsilon, t_\ell] < t[M_{3-\ell}, \Upsilon, t_{3-\ell}] \)
(b) for any \( t \) if \( N^\ell = N_t[M_\ell, \Upsilon, t_\ell] \) is well defined for \( \ell = 1, 2 \), then for every sentence \( \chi \) such that every subformula has at most \( k \)-free variables, we have \( N^1 \models \psi \iff N^2 \models \psi \).
Proof. Straight.

2.13 Claim. In 2.10, 2.11 we can allow in the $\psi_i$ and in the $\varphi$ the cardinality quantifier provided that $\mathcal{Y}$’s are super.

Proof. Clearer than 2.10, 2.11.

2.14 Discussion We consider now some variants.
1) We can define a natural equivalence of two $\vec{m}^*$-systems. Again the case with cardinality quantifiers is clearer.
   This makes the proof of applications slightly different.
2) We have to consider the stopping times. If $L^* = L_{\text{car},T}$ or $L_{\text{card},T}$ this is natural, (and they are stronger logics than the earlier variants). If we still would like to analyze in particular for the others, we should be careful how much information can be gotten by the time.
3) We can omit the $c_\ell$’s if the models are rich enough by a first order formula. In $N_{i+1}$ reconstruct the sequence $\langle (N_i, \vec{c}_\ell) : \ell \leq s \rangle$ (see §4).
§3 The canonical example

We apply §2 to the canonical example: random enough graph.

3.1 Definition. Let \( \tau \) be a fixed vocabulary consisting of predicates only. We say \( M \) is a \((t,k)\)-random \( \tau \)-model if every quantifier free 1-type over \( A \subseteq M, |A| < k \) (not explicitly inconsistent) is realized in \( M \) by at least \( s(\|M\|) \) elements. If \( t = \infty \) we may write \( k \)-random.

3.2 Definition. \( T_{\text{pol}} \) is \( \{ f_q : q \in \mathbb{Q}, q > 0 \} \) where \( f_q : \omega \to \omega \) is \( f_q(n) = n^q \) or the least integer \( \geq n^q \), more exactly.

3.3 Claim. Assume

(a) \( M_\ell \) is \((s_\ell,k)\)-random \( \tau \)-model for \( \ell = 1, 2 \)

(b) \( 3s \leq k \)

(c) \( 2^{(2^r)(\tau)} < t_\ell(M_\ell) < s_\ell(M_\ell) - s \)

(d) \( \Upsilon \) is an inductive scheme for \( \mathcal{L}_{\omega_1,\omega}(\tau^+) \), \( \chi \) a sentence in \( \mathcal{L}_{\omega_1,\omega}(\tau^+) \) and \( m_{f_\ell}(\Upsilon) \leq s \) and each subformula of any formula in \( \Upsilon \) or \( \chi \) has at most \( s \) free variables.

Then the truth values of \( \theta_{\Upsilon,\chi,\ell_1} \) in \( M_1 \) and \( \theta_{\Upsilon,\chi,\ell_2} \) in \( M_2 \) are equal except the case in 2.11.

Proof. Let \( \ell = 1, 2 \). We let \( I_\ell = \{ A \subseteq M_\ell : |A| \leq q \} \) and

\[ \mathcal{F}_\ell = \{ f : f \text{ is a partial automorphism of } M_\ell \}
\text{ and } \text{Dom}(f) \text{ has at } \leq q \ell \text{ elements} \]

(*) \( \mathcal{F}_\ell = (M_\ell, I_\ell, \mathcal{F}_\ell) \) is a \( k \)-system

[why? the least obvious clause in Definition 2.2(1) is clause (D) which holds by Definition 3.1 above.]

(*) \( \mathcal{F}_\ell = (M_\ell, I_\ell, \mathcal{F}_\ell) \) is \((t_\ell,s)\)-dichotomical

[why? let \( m \in \mathbb{N} \) and let \( E \) be an equivalence relation on the set of \( h : [m] \to M_\ell \) satisfying (*) of Definition 2.2(2). Without loss of generality \( h \) is one-to-one, so necessarily \( m \leq s \). As \( s \leq (qk)/2 \) there is a quantifier free formula \( \varphi(\bar{x}, \bar{y}) \in \mathcal{L}(\tau), \ell \varphi(\bar{x}) = \ell \varphi(\bar{y}) = m \) such that \( h_1 E h_2 \) iff \( M_1 \models \varphi((h_1(i) : i \in [m]), (h_2[i] : i \in [m])]. \]

Can there be \( \tilde{a}_0, \tilde{a}_1 \in m(M_\ell) \) realizing the same quantifier free type (say \( p(\bar{x}) \)) (over the empty set) which are not \( E \)-equivalent? If so we can find \( \tilde{a}_2 \in m(M_\ell) \) realizing the same quantifier free type \( p(\bar{x}) \) and disjoint to \( \tilde{a}_0 \tilde{a}_1 \) (use “\( M_\ell \) is \((3s)\)-random”), so without loss of generality \( \tilde{a}_0, \tilde{a}_1 \) are disjoint. Now we ask “are there disjoint \( \tilde{b}_0, \tilde{b}_1 \in m(M_1) \) realizing \( p(\bar{x}) \) which are \( E \)-equivalent? If yes, we easily get a contradiction to “\( E \) an equivalence relation”. So easily there are at least \( s(M_\ell) - m \) pairwise disjoint sequences realizing \( p(\bar{x}) \). Moreover, easily by transitivity for some
We replace “quantifiers” by “quantifier and counting” if we add: and the two sets to $M\upharpoonright u$ realize the same quantifier free type over $M$.

3.8 Claim. 1) We can in 3.3 weaken the demand “$\bar{M}_\ell$ is $(s_\ell,k)$-random $\tau$-model” to

(a) $M_\ell$ has $k$-elimination of quantifier

(b) if $\varphi(\bar{x},\bar{y})$ is a quantifier free formula defining an equivalence relation
and $\ell g(\bar{x}) = \ell g(\bar{y}) \leq s$ then the number of classes is $\geq t_\ell(M_\ell)$ or each equivalence class is definable by a quantifier free type.

3.9 Remark. Parallel claims hold for the logic with the cardinality quantifier.
3.10 Claim. Choiceless polynomial time + counting logic does not capture polynomial time.

Proof. Use 2.12 on the question: $|P^M| \geq \|M\|/2$, similarly to 3.3 with $\tau = \{P\}$. 
§4 Closing comments

We may consider

4.1 Definition. 1) A context is \((K, \mathcal{I})\) such that

(a) \(K\) be a class of models with vocabulary (= the set of predicates) \(\tau\)
(b) \(\mathcal{I}\) is a function such that
(c) \(\text{Dom}(\mathcal{I}) = K\)
(d) \(\mathcal{I}(M)\) is a family of subsets of \(K\), whose union is \(|M|\), and closed under subsets
(e) \(\mathcal{I}\) is preserved by isomorphisms.

2) In 1) let \(\text{Seq}^\ast_I(M) = \{\bar{a} : \bar{a} a sequence of members of \(M\) of length \(\alpha\), \(\text{Rang}(\bar{a}) \in \mathcal{I}(M)\}\).

3) We define a logic \(\mathcal{L}\). For \(k < \omega\) and \(\alpha < \omega\) or just \(\alpha\) an ordinal let us define the formulas in \(\mathcal{L}_{k,\alpha}\) by induction on \(\alpha\), each formula \(\varphi\) has the form \(\varphi(\bar{x}_0, \ldots, \bar{x}_{k_1 - 1})\), \(k_1 \leq k\), where the \(\bar{x}_\ell\)'s are pairwise disjoint (finite) sequences of variables and every variable appearing freely in \(\varphi\) appear in one of those sequences (so any formula is coupled with such \((\bar{x}_0, \ldots, \bar{x}_{k_1 - 1})\), probably some not actually appearing) (we may restrict to \(\bar{x}_\ell\) finite).

\(\alpha = 0\): quantifier free formula; i.e. any Boolean combination of atomic ones (with the right variables, of course).

\(\alpha + 1\): \(\alpha\) non-limit \(\varphi(\bar{x}_0, \ldots, \bar{x}_{k_1 - 1})\) is a Boolean combination of formulas of the form \((\exists \bar{y})\psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k_2 - 2}}, \bar{y})\) where \(k_2 \leq k\), \(\psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k_2 - 2}}, \bar{y}) \in \mathcal{L}_{k,\alpha}\).

\(\alpha\) limit: \(\mathcal{L}_{k,\alpha} = \bigcup_{\beta < \alpha} \mathcal{L}_{k,\beta}\).

\(\alpha + 1, \alpha\) limit: \(\mathcal{L}_{k,\alpha + 1}\) is the set of Boolean combinations of members of \(\mathcal{L}_{k,\alpha}\) of the right variables.

Let \(\mathcal{L}_k = \bigcup_{\alpha} \mathcal{L}_{k,\alpha}\), \(\mathcal{L}_* = \bigcup_{k < \omega} \mathcal{L}_k, \mathcal{L}_{k,\alpha} = \bigcup_{\beta < \alpha} \mathcal{L}_{k,\beta}\).

4) We now define a satisfaction relation \(M \models \varphi(\bar{a}_0, \ldots, \bar{a}_{k_1 - 1})\) where \(k_1 \leq k\)
(depending on \(\mathcal{I}\)).

I.e. we define by induction on \(\alpha\), for \(\varphi(\bar{x}_0, \ldots, \bar{x}_{k_1 - 1}) \in \mathcal{L}_{k,\alpha}\), \(\bar{a}_\ell \in \text{Seq}_{I}^{\mathcal{I}}(\bar{a}_\ell)(M)\), when does \(M \models \varphi(\bar{a}_0, \ldots, \bar{a}_{k_1 - 1})\) and when \(M \models \neg \varphi(\bar{a}_0, \ldots, \bar{a}_{k_1 - 1})\). This is done naturally, in particular \(M \models (\exists \bar{y})\varphi(\bar{a}_0, \ldots, \bar{a}_{k_2 - 1}, \bar{y})\) iff for some \(\bar{b} \in \text{Seq}_{I}^{\mathcal{I}}(\bar{b})(M)\), (so \(\text{Rang} \bar{b} \in \mathcal{I}(M)\)) we have \(M \models \varphi(\bar{a}_0, \ldots, \bar{a}_{k_2 - 1}, \bar{b})\).

4.2 Discussion: We may replace \(M\) by \(M^+\), adding elements coding each \(A \in \mathcal{I}(M)\), with decoding by functions, still this requires infinitely many functions, we need to actually code any sequence listing each \(A \in \mathcal{I}(M)\).

Still this framework seems to work more smoothly for its purposes.
4.3 Observation: In the framework of Definition 4.1, \( M_1 \equiv_{\mathcal{F}} M_2 \) if and only if there is a family \( \mathcal{F} \) witnessing it which means

(a) \( f \in \mathcal{F} \Rightarrow f \) is a partial isomorphism from \( M_1 \) to \( M_2 \)
(b) \( f \in \mathcal{F} \Rightarrow \text{Dom}(f) \) is the union of \( \leq k \) members of \( \mathcal{F}(M_1) \)
(c) \( f \in \mathcal{F} \Rightarrow \text{Rang}(f) \) is the union of \( \leq k \) members of \( \mathcal{F}(M_2) \)
(d) if \( f \in \mathcal{F}, A_1, \ldots, A_{k-1}, A_k \in \mathcal{F}(M_1), \ell < k \Rightarrow A_\ell \subseteq \text{Dom}(f) \), then for some \( g \in \mathcal{F}, \bigcup_{\ell=1}^{k-1} A_\ell \subseteq \text{Dom}(g) \),

(e) like (d) for \( f^{-1}, M_2, M_1 \).

4.4 Discussion: 1) In §2 we can define \( N_t[M] \equiv N_{\mathcal{T},t}[M,t] \) for every ordinal \( t \), and so \( V_{\mathcal{T}}[M,t] = \cup\{ N_{\mathcal{T},\alpha}[M,t] : \alpha \text{ an ordinal} \} \), see below. Now as in the case \( i = 4 \), the analysis in §2 works for this but it is not clear if we can get any interesting things.

Can this give interesting proofs of consistency for set theory with no choice but with urelement?

4.5 Definition. 1) We say \( \mathcal{Y} \) (from Definition 1.1) is pure if \( m_1[\mathcal{Y}] = 0 \) so no \( c_\ell \).
2) For pure \( \mathcal{Y} \), let “\( \mathcal{Z} \) is the full \( t \)-successor of order \( t \) of \( \mathcal{Z} \)” as in 2.8 iterating \( t \) times, noting now the full \( t \)-successor is unique allowing \( t \) to be an ordinal and for limit ordinal \( t \) take just the union.

4.6 Fact: For any \( \mathcal{Y} \) we can find \( \mathcal{Y}' \) which is equivalent if we use in Definition 1.2 the case \( i = 4 \) (well when \( t(M_\ell) \) always is \( \geq 2 \)). In fact, we can reconstruct the sequence of \( \langle c_{t,\ell} : t' < t \rangle \) in \( N_t \).

4.7 The Main Definition. 1) We say \( \mathcal{Y} = (M,I,\mathcal{F}) \) is a \( k \)-system if \( \hat{m}^* = (m_{qd}(\ast),m_{iv}(\ast)) \) and:

(A) \( I \) is a family of subsets of \( |M| \) (the universe of \( M \)) closed under subsets and each singleton belongs to it

[Hint: intended as first approximation to the possible supports of the partial automorphism of \( M \), the model of course; the intention is that \( M \) is finite]

Let \( I[m] = \{ \bigcup_{\ell=1}^{m} A_\ell : A_\ell \in I \} \)

(B) \( \mathcal{F} \) is a family of partial automorphisms of \( M \) such that \( A \subseteq \text{Dom}(f) \) \& \( A \in I \Rightarrow f''(A) \in I ; \mathcal{F} \) closed under inverse (i.e. \( f \in \mathcal{F}_t \Rightarrow f^{-1} \in \mathcal{F}_t \)) and composition and restriction

(C) if \( f \in \mathcal{F} \) then \( \text{Dom}(f) \) is the union of \( \leq k \) members of \( I \)

(D) if \( f \in \mathcal{F} \) and \( A_1, \ldots, A_{k-1}, A_k \in I \) and \( \ell \in \{1, \ldots, k-1\} \Rightarrow A_\ell \subseteq \text{Dom}(f) \), then for some \( g \in \mathcal{F} \) we have

\[ f \upharpoonright \bigcup_{\ell=1}^{k-1} A_\ell \subseteq g \]
A_k \subseteq \text{Dom}(g)

(E) if (\alpha) then (\beta)_1 \lor (\beta)_2 where

(\alpha) h is a function from some [m] = \{1, \ldots, m\}, \text{Rang}(h) \text{ belongs to } I[s], E is an equivalence relation on \text{H}_h = \{h': \text{for some } f \in \mathcal{F}_0, \text{Rang}(h) \subseteq \text{Dom}(f) \text{ and } h' = f \circ h\} satisfying

(\ast) \text{if } h_1, h_2, h_3, h_4 \in \text{H}_h \text{ and there is } f \in \mathcal{F} \text{ such that}

(Rang h_1) \cup (Rang h_3) \subseteq \text{Dom}(f) \text{ and }

f \circ h_1 = h_3, f \circ h_2 = h_2

then (h_1 Eh_2 \Leftrightarrow h_3 Eh_4)

(\beta)_1 \text{ there is } u \subseteq [m] \text{ such that}

\text{Rang}(\bar{h} | u) \in I \text{ and }

(\forall h_1, h_2 \in \text{H}_h)(h_1 | u = h_2 | u \rightarrow h_1 Eh_2)

(\beta)_2 \text{ the number of } E\text{-equivalence classes is } > t(M)

[hint: this is how from “there is some not too large number” we get “there is a support in I”].

We can connect this to §2 as follows.

4.8 Claim. Assume \mathcal{Y}_0 = (M, I, \mathcal{F}), \mathcal{L}^* = (N, \bar{c}, G), \bar{m}^* are as in 2.2, 2.5. Then

(a) if \varphi(\bar{x}) \in \mathcal{L}^{\text{m}_{\text{qf}}(*)}_{\text{m}_{\text{qf}}(*)} \text{ so } \ell g(\bar{x}) \leq \text{m}_{\text{qf}}(\bar{x}), \text{ and every subformula of } \varphi(\bar{x}) \text{ has } \leq \text{m}_{\text{qf}}(*) \text{ free variable, } \bar{a} \in \ell g(\bar{x}) N, \bigwedge_{\ell < \ell g(\bar{a})} A_\ell R a_\ell \text{ and } \bigcup_{\ell < \ell g(\bar{a})} A_\ell \subseteq \text{Dom}(f) \text{ and } f \in \mathcal{F}_0 \text{ then }

N \models \varphi[\ldots, a_\ell, \ldots] \equiv \varphi[\ldots, G(f)(a_\ell), \ldots]

(b) if \varphi(\bar{x}) has quantifier depth \text{ k and } \leq \text{m}_{\text{qf}}(*) + \text{m}_{\text{qf}}(*) - k \text{ free variables, } f \in \mathcal{F}_{\text{m}_{\text{qf}}(*)} \text{ and } \ldots, a_\ell, \ldots, \in \text{Dom}(G(f)) \text{ then }

N \models \varphi[\ldots, a_\ell, \ldots] \equiv \varphi[\ldots, G(f)(a_\ell), \ldots].

Proof. Straight.

4.9 Conclusion 1) Assume \mathcal{Y}_0 \text{ is super (see 2.3). Then we can define } R_t, G_t \text{ for every } t (N_t \text{ the “computation” in time } t) \text{ such that }

(M, \bar{c}_0, G_0, R_0) \text{ is } 0\text{-lifting }

(N_{t+1}, \bar{c}_{t+1}, R_{t+1}, \langle G_{t+1} \rangle) \text{ is a lifting, successor of } (N_t, \bar{c}_t, G_t, R_t).

2) So the formula the \bar{\varphi} defines is preserved by } f \in \mathcal{F}_0.

Proof. Straight.
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