STANDING WAVES OF FIXED PERIOD FOR \( n + 1 \) VORTEX FILAMENTS

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Abstract. The \( n + 1 \) vortex filament problem has explicit solutions consisting of \( n \) parallel filaments of equal circulation in the form of nested polygons uniformly rotating around a central filament which has circulation of opposite sign. We show that when the relation between temporal and spatial periods is fixed at certain rational numbers, these configurations have an infinite number of homographic time dependent standing wave patterns that bifurcate from these uniformly rotating central configurations.

Introduction

In reference [16], a model system of equations was derived for the interaction of near-parallel vortex filaments. The model considers vortex filaments in \( \mathbb{R}^3 \) to be coordinatized by curves \( (u_j(t, s), s) \in \mathbb{C} \times \mathbb{R} \) for \( j = 0, \ldots, n \) that describe the positions of \( n + 1 \) vertically oriented vortex filaments. Different aspects of this problem have been investigated in [3, 4, 8, 11, 12, 13, 17] and references therein. In this article we study central configurations of \( n + 1 \) vortex filaments with \( n \) filaments of equal circulation and one filament of opposite circulation.

Let \( u_j(t, s) \) for \( j = 1, \ldots, n \) be the positions of the \( n \) filaments of circulation 1 and \( u_0(t, s) \) the filament of circulation \(-\kappa\) with \( \kappa > 0 \). A homographic standing wave of the \( n + 1 \) vortex filament problem with fixed period is a solution of the form

\[
(1) \quad u_j(t, s) = ae^{i\omega t} (a_j + a_j u(t/q, s)),
\]

where \( \omega = -a^{-2} \) is real, \( q \) is an integer and \( u(t, s) \) is a complex 2\( \pi \)-periodic function in \( t \) and \( s \).

The complex numbers \( a_j \in \mathbb{C} \) for \( j = 0, \ldots, n \) lie in a central configuration with \( a_0 = 0 \). That is, the complex numbers \( a_j \) satisfy

\[
(2) \quad 0 = \sum_{i=1}^{n} \frac{a_i}{|a_i|^2}, \quad -a_j = \sum_{i=1(i\neq j)}^{n} \frac{a_j - a_i}{|a_j - a_i|^2} - \kappa \frac{a_j}{|a_j|^2},
\]

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Walter Craig is deceased.
for \( j = 1, ..., n \). There are many configurations that satisfy (2), for example in the form of nested polygons. In particular, an explicit solution of (2) is given by the regular polygon

\[
a_j = (\kappa - (n - 1)/2)^{-1/2} e^{ij\zeta}, \quad \zeta = 2\pi/n,
\]

if \( \kappa > (n - 1)/2 \).

Setting \( u = 0 \) in equation (1) corresponds to the family of homographic solutions for which \( n \) straight parallel filaments rotate around the central filament with uniform frequency \( \omega \) and amplitude \( a \). The standing waves of the title of this article correspond to non-trivial \( 2\pi \)-periodic solutions \( u(t, s) \) of the equation \( Lu + g(u) = 0 \), where \( L \) is the linear operator

\[
(3) \quad L(\omega)u := -(i/q) \partial_t u - \partial_s^2 u + \omega (u + \bar{u}),
\]

and \( g \) is an analytic nonlinearity describing the horizontal vortex filament interaction. Our goal is to construct standing wave solutions that bifurcate from the initial configuration \( u = 0 \), for which the frequency \( \omega \) is the bifurcation parameter. The solution given by (1) with a \( 2\pi \)-periodic function \( u \) has fixed spatial period \( s \in [0, 2\pi) \) and temporal period \( t \in [0, 2\pi q) \) in a frame of reference that is rotating with frequency \( \omega \), i.e. the solution is periodic or quasiperiodic with the two temporal frequencies \( \omega \) and \( 1/q \) when observed in a stationary reference frame. The main theorem is as follows.

**Theorem 1.** Let \( q \) be an integer. For each \( k_0 \in \mathbb{N} \), there is a local continuum of \( 2\pi \)-periodic solutions \( u \) bifurcating from the unperturbed configuration with \( u = 0 \) and initial frequency

\[
(4) \quad \omega_0 = -\frac{1}{q} \left( 1 - \frac{1}{2k_0^2 q} \right).
\]

The local bifurcation \( (u, \omega) \) consists of standing waves satisfying the estimates

\[
u(t, s) = b \left[ \cos j_0 t + i \left( 1 - k_0^{-2}/q \right) \sin j_0 t \right] \cos k_0 s + O(b^2),
\]

with \( \omega = \omega_0 + O(b^2) \) and \( j_0 = qk_0^2 - 1 \), where \( b \in [0, b_0] \) gives a local parameterization of the bifurcation curve. Furthermore, these solutions satisfy the following symmetries

\[
u(t, s) = \bar{u}(-t, s) = u(t, -s).
\]

Therefore, for any central configuration \( a_j \) satisfying (2), the previous theorem gives homographic solutions of the form (1). The periodic solutions \( u \) are special in that the ratio of their temporal and spatial periods are rational. In reference [8] we studied the case of irrational ratios, which is a small divisor problem for a nonlinear partial differential equation which requires techniques related to KAM theory even for the case of constructions of periodic solutions. Our approach is parallel to that of the semilinear wave and beam equation in one dimension, where time periodic solutions with rational periods (free vibrations) were shown to exist in [1, 2, 14, 15, 18],
and later for irrational periods in [5, 11]. On the other hand, time periodic solutions bifurcating from stationary solutions with irrational periods is a small divisor problem, for which constructions of solutions by Nash-Moser methods came much later in [6, 7, 9], and references therein.

In the present analysis the ratio of the periodic solution is rational and the small divisor problem does not occur. The key element of the proof consists on the fact that for special temporal frequencies, given by $1/q$, the Schrödinger operator $L(\omega)$, when restricted to the orthogonal complement of the null space, has a bounded inverse in the set of frequencies $\omega_0 \in \left(-\frac{1}{q}, 0\right)$. Unlike in semilinear wave and beam equations, our equation is a genuine Hamiltonian PDE represented by a Schrödinger operator $L(\omega)$ which does not have the regularity that is usually obtained in other equations, i.e. our result can be obtained only in a narrow set of parameters where $L(\omega_0)$ has a nontrivial kernel. This is also the case of the counter-rotating vortex filament pair studied in [11], but this is the first time that periodic solutions without small divisors are obtained in a genuine non-linear Hamiltonian PDE using this method.

In section 1, we set up a Lyapunov-Schmidt reduction to prove the existence of standing waves. In section 2 we solve the range equation for $\omega_0 \in \left(-\frac{1}{q}, 0\right)$ using the contracting mapping theorem. In section 3 we use the symmetries of the problem to solve the bifurcation equation by means of the Crandall-Rabinowitz theorem.
1. Setting the problem

From [16] the system of model equations for the dynamics of \( n + 1 \) near-parallel vortex filaments, with circulations \( \Gamma_0 = -\kappa \) and \( \Gamma_j = 1 \) for \( j = 1, \ldots, n \), is given by

\[
\frac{\partial u_j}{\partial t} = i \left( \Gamma_j \partial_{ss} u_j + \sum_{i=0 \ (i \neq j)}^{n} \frac{\Gamma_j u_j - u_i}{|u_j - u_i|^2} \right), \quad j = 0, \ldots, n. \tag{5}
\]

Homographic solutions of the \( n + 1 \) filaments are particular solutions of the form

\[
u_j(t, s) = w(t, s) a_j,
\]

where \( w(t, s) \) is a complex valued function and where \( a_j \)'s are complex numbers satisfying the condition of a central configuration. In this class of solutions the shape of the intersections of the filaments with a horizontal complex plane is homographic with the shape of their intersection with any other horizontal plane \( \{x_3 = c\} \) for any \( c \) and at any time \( t \).

For a general central configuration

\[
a_j = \sum_{i=0 \ (i \neq j)}^{n} \Gamma_i \frac{a_j - a_i}{|a_j - a_i|^2}, \quad j = 0, \ldots, n, \tag{6}
\]

homographic solutions satisfy the system of equations [5] if \( w(t, s) \) solves the system of equations

\[
a_j \partial_t w(t, s) = i \left( \Gamma_j a_j \partial_{ss} w(t, s) - \frac{w(t, s)}{|w(t, s)|^2} a_j \right), \quad j = 0, \ldots, n.
\]

In the particular case that \( a_0 = 0 \) in the central configuration, the condition for the configuration \( a_j \) becomes [2] and the system of equations is satisfied by solutions of the simple equation,

\[
\partial_t w = i \left( \partial_{ss} w - \frac{w}{|w|^2} \right). \tag{7}
\]

Therefore, \( u_j(t, s) = w(t, s) a_j \) is an homographic solution of the vortex filament problem if the configuration \( a_j \) satisfies [2] and \( w \) is a solution of the equation [7].

A particular solution of [2] is given by a regular polygon \( a_j = r e^{ij\xi} \) with radius \( r = (\kappa - (n - 1)/2)^{-1/2} \) if \( \kappa > (n - 1)/2 \), because

\[
\sum_{i=1 \ (i \neq j)}^{n} \frac{a_j - a_i}{|a_j - a_i|^2} - \kappa \frac{a_j}{|a_j|^2} = - \left( \kappa - \frac{n - 1}{2} \right) \frac{a_j}{r^2} = -a_j.
\]

Also, there are other solutions of [2] corresponding to nested polygons.

Equation [7] has the set of solution \( w = ae^{i\omega t} \) with

\[
\omega = -a^{-2} < 0,
\]
that corresponds to \( n \) vortex filaments uniformly rotating in the central configuration \( a_j \) with amplitude \( a \) and frequency \( \omega \). We look for bifurcation of solutions of the equation (7) of the form

\[
    w(t, s) = ae^{i\omega t} (1 + u(t/q, s)),
\]

where \( q \) is an integer and \( u(t, s) \) is \( 2\pi \)-periodic in \( t \) and \( s \). This is a solution that has fixed temporal and spatial periodicity when viewed in a coordinate frame rotating about the \( x_3 \)-axis with frequency \( \omega \). When \( u = 0 \) the solution corresponds to \( n \) vortex filaments uniformly rotating in the central configuration \( a_j \). The equation (7) for a perturbation from this configuration is

\[
    (i/q) \partial_t u = -u_{ss} + \omega (u + \bar{u}) + g(u, \bar{u}),
\]

where the nonlinearity \( g \) is given by

\[
    g(u, \bar{u}) = \omega \frac{\bar{u}^2}{1 + \bar{u}} = \omega \sum_{n=2}^{\infty} (-1)^n \bar{u}^n.
\]

In order to simplify the analysis of symmetries, the equation is represented in real coordinates \( u(\tau, s) = (x(\tau, s), y(\tau, s)) \in \mathbb{R}^2 \), i.e., the equation is equivalent to

\[
    Lu + g(u) = 0,
\]

where \( g(u) = O \left( |u|^2 \right) \) is analytic for \( |(x, y)| < 1 \) and \( L \) is the linear operator

\[
    Lu := -(1/q) J \partial_t u - \partial_s^2 u + \omega (I + R) u,
\]

where \( R = \text{diag}(1, -1) \).

We define the Hilbert space \( L^2(\mathbb{T}^2; \mathbb{R}^2) \), with the inner product

\[
    \langle u_1, u_2 \rangle = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} u_1 \cdot u_2 \, dt \, ds.
\]

A function \( u \in L^2 \) can be written in a Fourier basis as

\[
    u = \sum_{(j,k) \in \mathbb{Z}^2} u_{j,k} e^{i(j \tau + k s)}, \quad u_{j,k} = \bar{u}_{-j,-k} \in \mathbb{C}^2.
\]

The Sobolev space \( H^s \) is the usual subspace of functions in \( L^2 \) with bounded norm

\[
    \|u\|^2_{H^s} = \sum_{(j,k) \in \mathbb{Z}^2} |u_{j,k}|^2 \left( j^2 + k^2 + 1 \right)^s.
\]

This space has the Banach algebra property for \( s > 1 \),

\[
    \|uv\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^s}.
\]

The Banach algebra property implies that the nonlinear operator \( g(u) = O(|u|^2_{H^s}) \) is well defined and continuous for \( \|u\|_{H^s} < 1 \).

The linear operator \( L : D(L) \rightarrow H^s \) is continuous when the domain

\[
    D(L) = \{ u \in H^s : Lu \in H^s \},
\]
is completed under the graph norm
\[ \|u\|_L^2 = \|Lu\|_{H^s}^2 + \|u\|_{H^s}^2. \]

In Fourier basis, the operator \( L : D(L) \to H^s \) is given by
\[
Lu = \sum_{(j,k) \in \mathbb{Z}^2} M_{j,k} u_{j,k} e^{i(jt+ks)},
\]
where
\[
M_{j,k} = \begin{pmatrix}
k^2 + 2\omega & i(j/q) \\
-i(j/q) & k^2
\end{pmatrix}.
\]

Then, the eigenvalues and eigenvectors of \( L \) are
\[
\lambda_{j,k,l} = k^2 + \omega + l\sqrt{(j/q)^2 + \omega^2},
\]
\[
e_{j,k,l} = \begin{pmatrix}
-\omega - l\sqrt{(j/q)^2 + \omega^2} \\
i(j/q)
\end{pmatrix},
\]
for \((j,k,l) \in \mathbb{Z}^2 \times \mathbb{Z}_2\), where \( \mathbb{Z}_2 = \{1, -1\} \) is a group under the product.

The eigenvalue \( \lambda_{j,k,1} \) always is positive, and \( \lambda_{j,k,-1}(\omega_0) = 0 \) if
\[
\omega_0 = \left( (j/qk)^2 - k^2 \right) / 2 < 0.
\]

Given that \( L(\omega_0) \) has a nontrivial kernel, we expect bifurcation of solutions of \( L(\omega)u + g(u) = 0 \) as \( \omega \) crosses \( \omega_0 \).

**Definition 2.** We define \( N \) as the subset of all lattice points corresponding to zero eigenvalues,
\[ N(\omega_0) = \{(j,k,1) \in \mathbb{Z}^2 \times \mathbb{Z}_2 : \lambda_{j,k,-1}(\omega_0) = 0\}. \]

By definition we have that the kernel of \( L(\omega_0) \) is generated by eigenfunctions \( e_{j,k,l}e^{i(jt+ks)} \) with \((j,k,l) \in N\). Notice that additional sites to \((\pm j_0, \pm k_0, -1)\) may be present in \( N(\omega_0) \) due to resonances. The Lyapunov-Schmidt reduction separates the kernel and the range equations using the projections
\[
Qu = \sum_{(j,k,l) \in N} u_{j,k,l} e_{j,k,l} e^{i(jt+ks)}, \quad Pu = (I - Q)u.
\]

Setting
\[
u = v + w, \quad v = Qu, \quad w = Pu,
\]
the equation \( Lu + g(u) = 0 \) is equivalent to the kernel equation
\[
QLQv + Qg(v + w) = 0,
\]
and the range equation
\[
PLPw + Pg(v + w) = 0.
\]
In this section, the range equation is solved as a fixed point \( w(\omega, v) \in H^s \) of the operator
\[
Kw = -(PLP)^{-1} g(w + v, \omega) .
\]
The local solution \( w = w(\omega, v) \) is provided by an application of the contraction mapping theorem, where we only need to prove that \( (PLP)^{-1} : PH^s \rightarrow PH^s \) is well defined and bounded. For this, we will establish bound estimates in the eigenvalues \( \lambda_{j,k,l} \).

For \( l = 1 \), we clearly have
\[
\lambda_{j,k,1} = k^2 + \omega + \sqrt{(j/q)^2 + \omega^2} \gtrsim k^2 + |j| .
\]

For \( l = -1 \), we have the following estimate,

**Lemma 3.** For \( 2\varepsilon < |\omega| < 1/q - 2\varepsilon \), we have
\[
|\lambda_{j,k,-1}(\omega_0)| \geq \varepsilon \text{ for } (j, k, l) \in N^c .
\]

**Proof.** In the case \( |j|/q \neq k^2 \), the inequality \( k^2 - |j|/q \geq 1/q \) holds and
\[
|\lambda_{j,k,-1}(\omega_0)| \geq |k^2 - |j|/q| - \left| |j|/q + \omega - \sqrt{(j/q)^2 + \omega^2} \right| .
\]
Since \( \lim_{x \to \infty} \left( x + \omega - \sqrt{x^2 + \omega^2} \right) = \omega \), then
\[
\left| |j|/q + \omega - \sqrt{(j/q)^2 + \omega^2} \right| < |\omega| + \varepsilon ,
\]
for \( |k| + |j| \geq M \) with \( M \) big enough. Therefore,
\[
|\lambda_{j,k,-1}(\omega_0)| \geq |k^2 - |j|/q| - |\omega| - \varepsilon \geq \frac{1}{q} - |\omega| - \varepsilon \geq \varepsilon .
\]

In the case \( |j|/q = k^2 \), then
\[
|\lambda_{j,k,-1}(\omega_0)| = \left| k^2 + \omega - \sqrt{k^2 + \omega^2} \right| \geq |\omega| - \varepsilon \geq \varepsilon .
\]
for \( |k| \) big enough. In both cases we have that \( |\lambda_{j,k,-1}(\omega)| \geq \varepsilon \) if \( |k| + |j| \geq M \) with \( M \) big enough. We conclude that the estimate holds except by a finite number of points \((j, k) \in \mathbb{Z}^2\). Therefore, there is a constant \( c \) such that the estimate \( |\lambda_{j,k,-1}(\omega)| \geq c\varepsilon \) holds for all \((j, k, -1) \in N^c\). \( \square \)

From the previous estimates we have that \( (PLP)^{-1} \) is a bounded operator with
\[
\left\| (PLP)^{-1} w \right\|_{H^s} \lesssim \varepsilon^{-1} \|w\|_{H^s} .
\]

**Proposition 4.** Assume \( 2\varepsilon < |\omega| < 1/q - 2\varepsilon \). There is a unique continuous solution \( w(v, \omega) \in H^s \) of the range equation defined for \((v, \omega)\) in a small neighborhood of \((0, \omega) \in \ker L(a_0) \times \mathbb{R} \) such that
\[
\|w(v, \omega)\|_{H^s} \lesssim \varepsilon^{-1} \|v\|^2 ,
\]
for small \( \varepsilon \).
Proof. By the Banach algebra property of $H^s$, the operator
\[
g(w) = \mathcal{O}(\|w\|_{H^s}^2) : B_\rho \to H^s
\]
is well defined in the domain $B_\rho = \{w \in H^s : \|w\|_{H^s} < \rho\}$ for $\rho < 1$. We can choose a small enough $\varepsilon$ such that the hypothesis of the previous lemma hold true. Therefore,
\[
Kw = -(PLP)^{-1} g(w + v, \omega) = \mathcal{O}(\varepsilon^{-1} \|w\|_{H^s}^2)
\]
is well defined and continuous. Moreover, it is a contraction for $\rho$ of order $\rho = \mathcal{O}(\varepsilon)$. By the contraction mapping theorem, there is a unique continuous fixed point $w(v, \omega) \in B_\rho$. The estimate $\|w(v, \omega)\|_{H^s} \leq \varepsilon^{-1} \|v\|^2$ is obtained from
\[
\|Kw\|_{H^s} \lesssim \varepsilon^{-1} (\|w\|_{H^s}^2 + \|v\|^2).
\]
□

Remark 5. Since $(PLP)^{-1}$ is continuous but not compact, we do not automatically obtain the regularity of the solutions by bootstrapping arguments. Instead, the regularity is obtained using the Sobolev embedding $H^s \subset C^2$ for $s \geq 3$.

3. The bifurcation equation

Proposition 6. For $k_0 \in \mathbb{N}$, we define
\[
\omega_0 = -\frac{1}{q} \left( 1 - \frac{1}{2qk_0^2} \right), \quad j_0 = qk_0^2 - 1.
\]
For these frequencies we have $\omega_0 \in (-1/q, 0)$ and
\[
N(\omega_0) = \{(0, 0, -1), (\pm j_0, \pm k_0, -1)\}.
\]

Proof. Since $\lambda_{j,k,-1} = k^2 + \omega - \sqrt{(j/q)^2 + \omega^2}$, then $\lambda_{j,0,-1}(\omega) = 0$ only if $j = 0$. For $k_0 \in \mathbb{N}^+$, the condition $\lambda_{j_0,k_0,-1}(\omega_0) = 0$ is satisfied only if
\[
\omega_0 = \left( (j_0/qk_0)^2 - k_0^2 \right) / 2.
\]
In addition, the condition $\omega_0 \in (-1/q, 0)$ holds if an only if the lattice point $(j_0, k_0) \in \mathbb{N}^2$ satisfies $j_0 = qk_0^2 - 1$. In this case
\[
\omega_0 = \frac{1}{2} \left( \left( k_0 - \frac{1}{qk_0} \right)^2 - k_0^2 \right) = -\frac{1}{q} \left( 1 - \frac{1}{2qk_0^2} \right),
\]
then the frequency $\omega_0$ is determined uniquely for each point $(j_0, k_0) \in \mathbb{N}^2$ because $\omega_0$ is decreasing in $k_0$. Therefore, we have that $(0,0,-1)$ and $(\pm j_0, \pm k_0, -1)$ are the only elements in $N(\omega_0)$.
□
Since $\ker L(\omega_0)$ has dimension 5 for $\omega_0 \in (-1/q, 0)$, we need to reduce the bifurcation equation to a subspace of dimension one in order to apply the Crandall-Rabinowitz theorem. This is attained by exploiting the equivariance of the system under the action of the group $G = O(2) \times O(2)$ given by

$$\rho(\sigma)u(t, s) = u(t + \sigma, s + \sigma),$$

for the abelian components, and for the reflections,

$$\rho(\kappa_1)u(t, s) = Ru(-t, s), \quad \rho(\kappa_2)u(t, s) = u(t, -s),$$

where $R = \text{diag}(1, -1)$. By the uniqueness of $w(v, \omega)$, the bifurcation equation has the same equivariant properties as the differential equation. This property is used in the following proposition to reduce the bifurcation equation to a subspace of dimension one.

**Proposition 7.** The bifurcation equation has a local continuum of $2\pi$-periodic solution bifurcating from $(v, \omega) = (0, \omega_0)$ with estimates

$$v(t, s) = b \left( \cos j_0 t \left( 1 - k_0^{-2}/q \right) \sin j_0 t \right) \cos k_0 s + O(b_0^2), \quad \omega = \omega_0 + O(b_0^2),$$

where $b \in [0, b_0]$ gives a parameterization of the local bifurcation, and symmetries

$$v(t, s) = Rv(-t, s) = v(t, -s) = v(t + \pi/j_0, s + \pi/k_0).$$

**Proof.** In Fourier components

$$v = \sum_{(j, k, l) \in \mathbb{N}} u_{j, k, l} e^{i(jt + ks)} \quad u_{j, k, l} = \bar{u}_{-j, -k, l},$$

the action of the abelian part of the group $G$ is given by

$$\rho(\varphi)u_{j, k, l} = e^{ij\varphi} u_{j, k, l}, \quad \rho(\theta)u_{j, k, l} = e^{ik\theta} u_{j, k, l}.$$

Since

$$e_{j, k, -1} = \left( -\omega_0 - \sqrt{(j/q)^2 + \omega_0^2} \right) = \left( \begin{array}{c} k_0^2 \\ i(j/q) \end{array} \right),$$

then $Re_{j, k, -1} = e^{-j, -k, -1}$ and $e_{j, k, -1} = e_{j, -k, -1}$. Therefore, we have

$$\rho(\kappa_1)v = \sum_{(j, k, l) \in \mathbb{N}} u_{j, k, l} e_{-j, -k, -1} e^{i(-j + ks)} = \sum_{(j, k, l) \in \mathbb{N}} u_{-j, -k, l} e_{j, k, -1} e^{i(jt + ks)},$$

and

$$\rho(\kappa_2)v = \sum_{(j, k, l) \in \mathbb{N}} u_{j, k, l} e_{j, k, -1} e^{i(jt + ks)} = \sum_{(j, k, l) \in \mathbb{N}} u_{j, -k, l} e_{j, k, -1} e^{i(jt + ks)}.$$
The irreducible representations under the action of $O(2) \times O(2)$ correspond to the subspaces

$$(u_{j_0,k_0,-1}, u_{j_0,-k_0,-1}) \in \mathbb{C}^2.$$ 

The linear operator $L$ is diagonal in these irreducible representations with eigenvalue $\lambda_{j_0,k_0,-1}$ of complex multiplicity two. The group

$$S = \langle \kappa_1, \kappa_2, (\pi/j_0, \pi/k_0) \rangle$$

has fixed point space $(u_{j,k,-1}, u_{j,-k,-1}) = (b, b)$ for $b \in \mathbb{R}$ in this representation. By setting

$$\ker L^S(\omega_0) := \ker L(\omega_0) \cap \text{Fix}(S),$$

the bifurcation equation

$$QLQw + Qg(v + w(v, \omega)) : \ker L^S(\omega_0) \times \mathbb{R} \to \ker L^S(\omega_0)$$

is well defined by the equivariance properties. Moreover, since $u_{0,0,-1}$ is not fixed by the subgroup $S$, then $\ker L^S(\omega_0)$ is generated by the simple eigenfunction

$$\sum_{j=\pm j_0, k=\pm k_0} e_{j,k,-1} e^{i(jt+ks)} = 4 \left( \frac{k_0^2 \cos j_0 t}{j_0/q \sin j_0 t} \right) \cos k_0 s.$$ 

Since $\ker L^S(\omega_0)$ has dimension one, the local bifurcation for $\omega$ close to $\omega_0$ follows from the Crandall-Rabinowitz theorem applied to the bifurcation equation (20). It is only necessary to verify that $\partial_\omega L(\omega)f$ is not in the range of $L$ for $f \in \ker L^S(\omega_0)$, which follows from the fact that

$$\partial_\omega L(\omega)f = (I + R) f.$$ 

The estimates $\omega = \omega_0 + O(b)$ and

$$v(t, s) = b \left( \frac{\cos j_0 t}{1 - k_0^{-2}/q} \sin j_0 t \right) \cos k_0 s + O(b^2)$$

are consequence of the Crandall-Rabinowitz estimates. Moreover, the $S^1$-action of the element $\varphi = \pi$ in the kernel generated is given by $\rho(\varphi) = -1$. This symmetry implies that the bifurcation equation is odd and $\omega = \omega_0 + O(b^2)$. 

The main theorem follows from this proposition and the fact that $u = v + w(v, \omega)$ with

$$\|w(v, \omega)\|_H^s = O \left( \|v\|^2 \right) = O \left( b^2 \right).$$

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