GENERIC SINGULARITIES OF NILPOTENT ORBIT CLOSURES

1. Introduction

1.1. Generic singularities of nilpotent orbit closures. Let \( G \) be a connected, simple algebraic group of adjoint type over the complex numbers \( \mathbb{C} \), with Lie algebra \( \mathfrak{g} \). A nilpotent orbit \( O \) in \( \mathfrak{g} \) is the orbit of a nilpotent element under the adjoint action of \( G \). Its closure \( \overline{O} \) is a union of finitely many nilpotent orbits. The partial order on nilpotent orbits is defined to be the closure ordering.

We are interested in the singularities of \( \overline{O} \) at points of maximal orbits of its singular locus. Such singularities are known as the generic singularities of \( \overline{O} \). Kraft and Procesi determined the generic singularities in the classical types, while Brieskorn and Slodowy determined the generic singularities of the whole nilpotent cone \( N \) for \( \mathfrak{g} \) of any type. The goal of this paper is to determine the generic singularities of \( \overline{O} \) when \( \mathfrak{g} \) is of exceptional type.

In fact, the singular locus of \( \overline{O} \) coincides with the boundary of \( O \) in \( \overline{O} \), as was shown by Namikawa using results of Kaledin [Nam], [Kal06]. This result also follows from the main theorem in this paper in the exceptional types and from Kraft and Procesi’s work in the classical types [KP81], [KP82]. Therefore to study generic singularities of \( \overline{O} \), it suffices to consider each maximal orbit \( O' \) in the boundary of \( O \) in \( \overline{O} \). We call such an \( O' \) a minimal degeneration of \( O \).

The local geometry of \( \overline{O} \) at a point \( e \in O' \) is determined by the intersection of \( \overline{O} \) with a transverse slice in \( \mathfrak{g} \) to \( O' \setminus e \). Such a transverse slice in \( \mathfrak{g} \) always exist and is provided by the affine space \( S_e = e + \mathfrak{g}' \), known as the Slodowy slice. Here, \( e \) and \( f \) are the nilpotent parts of an \( sl_2 \)-triple and \( \mathfrak{g}' \) is the centralizer of \( f \) in \( \mathfrak{g} \). The local geometry of \( \overline{O} \) at a point \( e \) is therefore encoded in \( S_{O, e} = \overline{O} \cap S_e \), which we call a nilpotent Slodowy slice. If \( O' \) is a minimal degeneration of \( O \), then \( S_{O, e} \) has an isolated singularity at \( e \). The generic singularities of \( \overline{O} \) can therefore be determined by studying the various \( S_{O, e} \), as \( O' \) runs over all minimal degenerations and \( e \in O' \). The isomorphism type of the variety \( S_{O, e} \) is independent of the choice of \( e \).

The main theorem of this paper is a classification of \( S_{O, e} \) up to algebraic isomorphism for each minimal degeneration \( O' \) of \( O \) in the exceptional types. In a few cases, however, we are only able to determine the normalization of \( S_{O, e} \) and in a few others, we have determined \( S_{O, e} \) only up to local analytical isomorphism.
1.2. Symplectic varieties. Recall from [Bea00] that a symplectic variety is a normal variety $W$ with a holomorphic symplectic form $\omega$ on its smooth locus such that for any resolution $\pi : Z \to W$, the pull-back $\pi^* \omega$ extends to a regular 2-form on $Z$. By a result of Namikawa [Nam01], a normal variety is symplectic if and only if its singularities are rational Gorenstein and its smooth part carries a holomorphic symplectic form.

The normalization of a nilpotent orbit $\mathfrak{O}$ is a symplectic variety: it is well-known that $\mathfrak{O}$ admits a holomorphic non-degenerate closed 2-form (see [CM93, Ch. 1.4]) and by work of Hinich [Hin91] and Panyushev [Pan91], the normalization of $\mathfrak{O}$ has only rational Gorenstein singularities. Hence the normalization of $\mathfrak{O}$ is a symplectic variety.

For a minimal degeneration $\mathfrak{O}'$ of $\mathfrak{O}$ and $e \in \mathfrak{O}'$, the nilpotent Slodowy slice $\mathfrak{S}_{\mathfrak{O},e}$ has an isolated singularity at $e$. Since the normalization of $\mathfrak{O}$ has rational Gorenstein singularities, the normalization $\tilde{\mathfrak{S}}_{\mathfrak{O},e}$ of $\mathfrak{S}_{\mathfrak{O},e}$ also has rational Gorenstein singularities. Its smooth locus admits a symplectic form (see [GG02]), which is this restriction of the symplectic form on $\mathfrak{O}$. Thus by the aforementioned result of Namikawa, $\tilde{\mathfrak{S}}_{\mathfrak{O},e}$ is also a symplectic variety.

The term symplectic singularity refers to a singularity of a symplectic variety. A better understanding of isolated symplectic singularities could shed light on the long-standing conjecture (e.g. [LeB95]) that a Fano contact manifold is homogeneous. The importance of finding new examples of isolated symplectic singularities was stressed in [Bea00]. It is therefore of interest to determine such singularities, as a means to find new examples of isolated symplectic singularities. Our study of the isolated symplectic singularity $\tilde{\mathfrak{S}}_{\mathfrak{O},e}$ at $e$ contributes to this program.

1.3. Simple surface singularities and their symmetries.

1.3.1. Definition. Let $\Gamma$ be a finite subgroup of $SL_2(\mathbb{C}) \cong Sp_2(\mathbb{C})$. Then $\Gamma$ acts on $\mathbb{C}^2$ and the quotient variety $\mathbb{C}^2/\Gamma$ is an affine symplectic variety with an isolated singularity at the image of $0$. This variety is known as a simple surface singularity and also as a rational double point, a du Val singularity, or a Kleinian singularity.

The set of such $\Gamma$, up to conjugacy in $SL_2(\mathbb{C})$, are in bijection with the simply-laced, simple Lie algebras over $\mathbb{C}$. The bijection is obtained via the exceptional fiber of a minimal resolution of $\mathbb{C}^2/\Gamma$. The exceptional fiber (that is, the inverse image of $0$) is a union of projective lines which intersect transversely. The dual graph of the resolution is given by one vertex for each projective line in the exceptional fiber and an edge joining two vertices when the corresponding projective lines intersect. The dual graph is always a connected, simply-laced Dynkin diagram, which defines the Lie algebra attached to $\mathbb{C}^2/\Gamma$. Hence one refers to a simple surface singularity with one of the capital letters $A_k, D_k (k \geq 4), E_6, E_7, E_8$, according to the associated simple Lie algebra.

In dimension two, an isolated symplectic singularity is equivalent to a simple surface singularity, that is, it is locally analytically isomorphic to some $\mathbb{C}^2/\Gamma$ (cf. [Bea00, Section 2.1]). More generally, if $\Gamma \subseteq Sp_{2n}(\mathbb{C})$ is a finite subgroup whose non-trivial elements have no non-zero fixed points on $\mathbb{C}^{2n}$, then the quotient $\mathbb{C}^{2n}/\Gamma$ is an isolated symplectic singularity.

1.3.2. Symmetries of simple surface singularities. Any automorphism of the simple surface singularity $X = \mathbb{C}^2/\Gamma$ fixes $0 \in X$ and induces a permutation of the projective lines in the exceptional fiber of a minimal resolution. Hence it gives rise to a graph automorphism of the dual graph $\Delta$ of $X$. Let $Aut(\Delta)$ be the group of graph automorphisms of $\Delta$. Then $Aut(\Delta) = 1$ when $\mathfrak{g}$ is $A_1$, $E_7$, or $E_8$; $Aut(\Delta) = \mathfrak{E}_3$ when $\mathfrak{g}$ is $D_4$; and otherwise, $Aut(\Delta) = \mathfrak{E}_2$.

We now address the question of when the action of $Aut(\Delta)$ on the dual graph $\Delta$ of $X$ comes from an algebraic action on $X$ (cf. [Slo84, III.6]). When $X$ is of type $A_{2k-1} (k \geq 2)$, $D_{k+1} (k \geq 3)$, or $E_8$, then $Aut(\Delta)$ comes from an algebraic action on $X$. In fact, the action is induced from a subgroup $\Gamma' \subset SL_2(\mathbb{C})$ containing $\Gamma$ as a normal subgroup. Namely, there exists such a $\Gamma'$ with $\Gamma' / \Gamma \cong Aut(\Delta)$ and the induced action of $\Gamma' / \Gamma$ on the dual graph of $X$ coincides with the action of $Aut(\Delta)$ on $\Delta$ via this isomorphism. Such a $\Gamma'$ is unique. The result also holds for any subgroup of $Aut(\Delta)$, which is relevant only for the $D_4$ case.
Generic singularities of nilpotent orbit closures

Slodowy denotes the pair \((X, K)\) consisting of \(X\) together with the induced action of \(K = \Gamma'/\Gamma\) on \(X\) by

\[
\begin{align*}
B_k, & \quad \text{when } X = A_{2k-1} \text{ and } K = \mathbb{S}_2, \\
C_4, & \quad \text{when } X = D_{k+1} \text{ and } K = \mathbb{S}_2, \\
F_4, & \quad \text{when } X = E_6 \text{ and } K = \mathbb{S}_2, \\
G_2, & \quad \text{when } X = D_4 \text{ and } K = \mathbb{S}_3.
\end{align*}
\]

The reasons for this notation will become clear shortly. We also refer to corresponding pairs \((\Delta, K)\), where \(\Delta\) is the dual graph and \(K\) is a subgroup of \(\text{Aut}(\Delta)\), in the same way. The symmetry of the cyclic group of order 3 when \(X = D_4\) is not considered.

When \(X = A_{2k}\), the symmetry of \(X\) did not arise in Slodowy’s work. It does, however, make an appearance in this paper. In this case \(\text{Aut}(\Delta) = \mathbb{S}_2\), but the action on the dual graph does not lift to an action on \(X\). Instead, there is a cyclic group \(\sigma\) of order 4 acting on \(X\), with \(\sigma\) acting by non-trivial involution on \(\Delta\), but \(\sigma^2\) acts non-trivially on \(X\). This cyclic action is induced from a \(\Gamma' \subset \text{SL}_2(\mathbb{C})\) corresponding to \(D_{2k+3}\). We define the symmetry of \(X\) to be the induced action of \(\Gamma'\) on \(X\) and denote it by \(A_{2k}^\ast\). Only the singularities \(A_1^\ast\) and \(A_1^\circ\) will appear in the sequel, and then only when \(\mathfrak{g}\) is of type \(E_7\) or \(E_8\).

1.4. The regular nilpotent orbit.

1.4.1. Generic singularities of the nilpotent cone. The problem of describing the generic singularities of the nilpotent cone \(\mathcal{N}\) of \(\mathfrak{g}\) was carried out by Brieskorn [Bri71] and Slodowy [Slo80] in confirming a conjecture of Grothendieck. In their setting \(\mathcal{O}\) is the regular nilpotent orbit and so \(\overline{\mathcal{O}}\) equals \(\mathcal{N}\), and there is only one minimal degeneration, at the subregular nilpotent orbit \(\mathcal{O}'\). Slodowy’s result from [Slo80, IV.8.3] is that when \(e \in \mathcal{O}'\), \(S_{\mathcal{O}',e}\) is algebraically isomorphic to a simple surface singularity of the form \(\mathbb{C}^2/\Gamma\). Moreover, as in [Bri71], when the Dynkin diagram of \(\mathfrak{g}\) is simply-laced, the Lie algebra associated to this simple surface singularity is \(\mathfrak{a}\). On the other hand, when \(\mathfrak{g}\) is not simply-laced, the singularity \(S_{\mathcal{O}',e}\) is determined from the list in [§1.3.2]. For example, if \(\mathfrak{g}\) is of type \(B_1\), then \(S_{\mathcal{O}',e}\) is a type \(A_{2k-1}\) singularity. This explains the notation in the list in [§1.3.2]. Next we explain an intrinsic realization of the symmetry of \(S_{\mathcal{O}',e}\) when \(\mathfrak{g}\) is not simply-laced.

1.4.2. Intrinsic symmetry action on the slice. Let \(\Delta\) be the Dynkin diagram of \(\mathfrak{g}\) and \(K \subset \text{Aut}(\Delta)\) be a subgroup. The group \(\text{Aut}(\Delta)\) is trivial if and only if \(\mathfrak{g}\) is simply-laced. The action of \(K\) on \(\Delta\) can be lifted to an action on \(\mathfrak{g}\) as in [OV91, Chapter 4.3]; namely, fix a canonical system of generators of \(\mathfrak{g}\). Then there is a subgroup \(\hat{K} \subset \text{Aut}(\mathfrak{g})\), isomorphic to \(K\), which permutes the canonical system of generators, and whose induced action on \(\Delta\) coincides with \(K\). Any two choices of systems of generators define conjugate subgroups of \(\text{Aut}(\mathfrak{g})\). The automorphisms in \(\hat{K}\) are called \(\text{diagram automorphisms of } \mathfrak{g}\).

Now given \(\mathfrak{g}\) we can associate a pair \((\mathfrak{g}_e, \hat{K})\) where \(\mathfrak{g}_e\) is a simple, simply-laced Lie algebra with Dynkin diagram \(\Delta_e\), and \(\hat{K} \subset \text{Aut}(\mathfrak{g}_e)\) is a lifting of some \(K \subset \text{Aut}(\mathfrak{g}_e)\) and \(\hat{K} \cong (\mathfrak{g}_e)^\Lambda\). If \(\mathfrak{g}\) is already simply-laced, then \(\mathfrak{g} = \mathfrak{g}_e\) and \(\hat{K} = 1\). If \(\mathfrak{g}\) is not simply-laced, then the pair \((\Delta_e, K)\) appears in the list in [§1.3.2] according to the type of \(\mathfrak{g}\).

Let \(\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})\) be the subalgebra of \(\mathfrak{g}\) generated by \(e\) and \(f\). The centralizer of \(\mathfrak{s}\) in \(G\) is denoted \(C(\mathfrak{s})\). Then \(C(\mathfrak{s})\) acts on \(S_{\mathcal{O}',e}\) for any nilpotent orbit \(\mathcal{O}'\), fixing the point \(e\). Also the component group \(A(e)\) of the centralizer of \(G\) in \(e\) is isomorphic to component group of \(C(\mathfrak{s})\) (see [§1.2] for more details). When \(e\) is in the subregular nilpotent orbit, Slodowy observed that \(C(\mathfrak{s})\) is a semidirect product of its connected component \(C(\mathfrak{s})^\circ\) and a subgroup \(H \cong A(e)\). Moreover \(H\) is well-defined up to conjugacy in \(C(\mathfrak{s})\). This is immediate except when \(\mathfrak{g}\) is type \(B_k\). Also, \(A(e) \cong K\). In particular, \(A(e)\) is trivial if \(\mathfrak{g}\) is simply-laced since \(G\) is adjoint.

There is a second Lie algebra associated to \(\mathfrak{g}\): let \(\mathfrak{l}^2 \mathfrak{g}\) be the Langlands dual Lie algebra of \(\mathfrak{g}\). Now let \((\mathbb{L}^2 \Delta), \mathbb{L}^2 K)\) be the pair attached to \(\mathfrak{l}^2 \mathfrak{g}\). We have \(A(e) \cong H \cong K \cong \mathbb{L}^2 K\). Then Slodowy’s classification
and symmetry result can be summarized as follows: the pair \((S_{O,e}, H)\), of \(S_{O,e}\) together with the action of \(H\), corresponds to the pair \(((\Delta),^*K)\) \([\text{Slo}80\ IV.8.4]\).

1.5. **The other nilpotent orbits in Lie algebras of classical type.** Kraft and Procesi described the generic singularities of nilpotent orbit closures for all the classical groups, up to smooth equivalence (see \([3.2]\) for the definition of smooth equivalence) \([\text{KP}81], [\text{KP}82]\).

1.5.1. **Minimal singularities.** Let \(O_{min}\) be the minimal (non-zero) nilpotent orbit in a simple Lie algebra \(g\). Then \(\overline{O}_{min}\) has an isolated symplectic singularity at \(0 \in \overline{O}_{min}\). Following Kraft and Procesi \([\text{KP}82\ 14.3]\), we refer to \(\overline{O}_{min}\) by the lower case letters for the ambient simple Lie algebra: \(a_k, b_k, c_k, d_k(k \geq 4), g_2, f_4, e_6, e_7, e_8\). The equivalence classes of these singularities, under smooth equivalence, are called **minimal singularities**.

1.5.2. **Generic singularities in the classical types.** The results of Kraft and Procesi for Lie algebras of classical type can be summarized as follows: an irreducible component of a generic singularity is either a simple surface singularity or a minimal singularity, up to smooth equivalence. Moreover, when a generic singularity is not irreducible, then it is smoothly equivalent to a union of two simple surface singularities of type \(A_{2k-1}\) meeting transversely in the singular point. This is denoted \(2A_{2k-1}\). In more detail:

**Theorem 1.1.** \([\text{KP}81], [\text{KP}82]\) Assume \(O'\) is a minimal degeneration of \(O\) in a simple complex Lie algebra of classical type. Let \(e \in O'\). Then

\((a)\) If the codimension of \(O'\) in \(\overline{O}\) is two, then \(S_{O,e}\) is smoothly equivalent to a simple surface singularity of type \(A_k, D_k,\) or \(2A_{2k-1}\). The latter two singularities do not occur for \(sl_n(\mathbb{C})\), and the singularity \(A_k\) for \(k\) even does not occur in the classical Lie algebras besides \(sl_n(\mathbb{C})\).

\((b)\) If the codimension is greater than two, then \(S_{O,e}\) is smoothly equivalent to \(a_k, b_k, c_k,\) or \(d_k\). The latter three singularities do not occur for \(sl_n(\mathbb{C})\).

1.6. **The case of type \(G_2\).** The case of a Lie algebra of type \(G_2\) has been studied by Levasseur-Smith \([\text{LS}88]\) and Kraft \([\text{Kra}89]\). Levasseur-Smith showed that the closure of the nilpotent orbit \(\tilde{A}_1\) of dimension 8 is not normal and that its non-normal locus coincides with its singular locus (and hence equals the closure of the minimal nilpotent orbit). Kraft gave another proof that this orbit is not normal and showed it has bijective normalization. Kraft also showed that the closure of the subregular orbit has singularity of type \(A_1\) in the \(\tilde{A}_1\) orbit.

1.7. **Main results.** We now summarize the main results of the paper describing the classification of generic singularities in the exceptional Lie algebras. Here, \(O'\) is a minimal degeneration of \(O\) and \(e \in O'\) and \(S_{O,e}\) is the nilpotent Slodowy slice defined from an \(st\)-triple through \(e\).

1.7.1. **Overview.** Most generic singularities are like those in the classical types: the irreducible components will be either simple surface singularities or minimal singularities. But some new features occur in the exceptional groups. There is more complicated branching and several new types of singularities occur. Among these are three singularities whose irreducible components are not normal (one of these already occurs in \(G_2\)), and three additional singularities of dimension four.

A key observation (\([3.4]\)) is that all irreducible components of \(S_{O,e}\) are isomorphic since the action of \(C^e(\mathfrak{g})\) is transitive on irreducible components. This result is not true in general when \(O'\) is not a minimal degeneration.

For most minimal degenerations, we determine the isomorphism type of \(S_{O,e}\), a stronger result than classifying the singularity up to smooth equivalence. In ten of the remaining cases, all in \(E_6\) (\([10.3]\)), we can only determine the isomorphism type of \(S_{O,e}\) up to normalization and in four cases we only know the result up to smooth equivalence (\([12]\)). It is possible to use the results here to go back and establish that Kraft and Procesi’s results in Theorem \([3]\) hold up to algebraic isomorphism (rather than smooth equivalence), but we defer the details to a later paper.
We also calculate the symmetry action on $S_{O,e}$ induced from $A(e)$, as Slodowy did when $O$ is regular orbit. This involves extending Slodowy’s result on the splitting of $C(s)$ and introducing the notion of symmetry on a minimal singularity. Again, it is possible to carry out this program for the classical groups, but we also defer the details to a later paper.

1.7.2. Symmetry of a minimal singularity. Let $g$ be a simple, simply-laced Lie algebra with Dynkin diagram $\Delta$. As in §4.4, let $K \subset \text{Aut}(g)$ be a subgroup of diagram automorphisms lifting the subgroup $K \subset \text{Aut}(\Delta)$. We call a pair $(\mathfrak{g}_{\text{min}}, K)$, consisting of $\mathfrak{g}_{\text{min}}$ with the action of $K$, a symmetry of a minimal singularity. We write these pairs as $a_k^+$, $d_k^+(k \geq 4)$, $d_k^+(\text{for the action of the full automorphism group})$, and $e_k^+$. As in the surface cases, $|K| = 3$ in $D_4$ does not arise.

1.7.3. Intrinsic symmetry action on a slice: general case. In §6.1 it is shown that the splitting of $C(s)$ that Slodowy observed for the subregular orbit holds in general, with four exceptions. Namely, there exists a subgroup $H \subset C(s)$ such that $C(s) \cong C(s)^+ \rtimes H$. So in particular $H \cong C(s)/C(s)^+ \cong A(e)$. The choice of splitting is in general no longer unique up to conjugacy in $C(s)$, but if we choose $H$ to represent diagram automorphisms of the semisimple part of $e(s)$, then the image of $H$ in $\text{Aut}(e(s))$ is unique up to conjugacy in $\text{Aut}(e(s))$. The four exceptions to the splitting of $C(s)$ have $|A(e)| = 2$, but the best possible result is that there exists $H \subset C(s)$, cyclic of order 4, with $C(s) = C(s)^+ \circ H$ [Som98, §3.4].

Next, imitating §4.4, we describe the action of $H$ on $S_{O,e}$. The four cases where $C(s)$ does not split give rise to the symmetries which include $A_2^r$ and $A_2^r$. Three of these four cases (when $O'$ has type $A_4 + A_3$ in $E_7$ and $E_6$ or type $E_6(a_1) + A_4$ in $E_8$) are well-known: under the Springer correspondence, their Weyl group representations lead to unexpected phenomena (see, for example, [Car93, pg. 373]). The phenomena observed here for these three orbits is directly related to the fact that $A(e) = E_8$ acts without fixed points on the irreducible components of the Springer fiber over $e$. It is not clear why the fourth orbit (of type $D_7(a_2)$ in $E_8$) appears in the same company as these three orbits.

1.7.4. Unexpected singularities. In the Lie algebras of exceptional type, we have identified six varieties that arise as some $S_{O,e}$, which are neither simple surface singularities nor minimal singularities. There may be additional cases depending on whether certain $S_{O,e}$ are normal or not.

The first three cases are all non-normal.

The variety $m$. Let $V(i)$ denote the irreducible representation of highest weight $i$ of $SL_2(C)$. Consider the linear representation of $SL_2(C)$ on $V = V(2) \oplus V(3)$. Let $v \in V$ be a highest weight vector for a Borel subgroup of $SL_2(C)$ with non-zero projection on each summand. Then the variety $m$ is defined to be the closure in $V$ of the orbit through $v$. This is a two-dimensional variety with an isolated singularity at zero. It is not normal, but has smooth normalization, equal to the affine plane $A^2$. The normalization map is given by

$$
\phi_{2,3}: \mathbb{C}^2 \to \mathbb{C}^3 = \mathbb{C}^3 \oplus \mathbb{C}^4 \cong V, (s, t) \mapsto (s^2, st, t^2, s^3, s^2t, st^2, t^3),
$$

where the right side is written with respect to an appropriate basis of weight vectors. Alternatively, $m = \text{Spec } R$ where $R$ is the subring of $\mathbb{C}[s, t]$ generated by all homogeneous polynomials of degree 2 or higher.

The first case where $m$ appears is for the minimal degeneration $(O, O') = (\tilde{A}_1, A_1)$ in $G_2$. This singularity appears at least once in each exceptional Lie algebra, always for two non-special orbits which lie in the same special piece.

The variety $m'$. This is a four-dimensional analogue of $m$, with $SL_2(C)$ replaced by $Sp_4(C)$. Consider the linear representation of $Sp_4(C)$ on $V = V(2\omega_1) \oplus V(3\omega_1)$ where $V(\omega_1)$ is the defining 4-dimensional representation of $Sp_4(C)$, so $V(2\omega_1)$ is the adjoint representation and $V(3\omega_2)$ is equal to the symmetric cube of $V(\omega_1)$, a representation of dimension 20. Let $v \in V$ be a highest weight vector for a Borel subgroup of $Sp_4(C)$ with non-zero projection on each summand. Then the variety $m'$ is defined to be the closure in $V$ of the $Sp_4(C)$-orbit through $v$. This is a four-dimensional variety with an isolated singularity at zero. It is not normal, but has smooth normalization, equal to $A^4$. Alternatively, $m' = \text{Spec } R$, where
$R$ is the subring of $\mathbb{C}[s,t,u,v]$ generated by all homogeneous polynomials of degree 2 or higher. We will show that $m'$ occurs exactly once as $\mathcal{S}_{\mathcal{O},e}$, for the minimal degeneration $(\mathcal{O}, \mathcal{O}') = (A_3 + 2A_1, 2A_2 + 2A_1)$ in $E_6$.

The variety $\mu$. The coordinate ring of the simple surface singularity $A_3$ is $R = \mathbb{C}[s,t,u,v]$, as a hypersurface in $\mathbb{C}^4$. We define the variety $\mu$ by $\mu = \text{Spec } R'$ where $R' = \mathbb{C}[s,t,u,v]$. This variety is non-normal and its normalization is isomorphic to $A_3$ via the inclusion of $R'$ in $R$. Using the methods of $\S 3$ the normalization of $\mathcal{S}_{\mathcal{O},e}$ for $(\mathcal{O}, \mathcal{O}') = (D_5(a_1), E_6(b_0))$ in $E_6$ is shown to be isomorphic to $A_3$ with an order two symmetry arising from $A_3$. We will prove in [FJLSb] that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to $\mu$. The closure of $\mathcal{O}$ was known to be non-normal, but our result establishes that it is non-normal in codimension 2.

There are also three additional unexpected singularities, each of dimension 4 and normal.

The degeneration $(2A_2 + A_1, A_2 + 2A_1)$ in $E_6$. Let $\zeta = e^{2\pi i}$ and let $\Gamma$ be the cyclic subgroup of $\text{Sp}_4(\mathbb{C})$ of order three generated by the diagonal matrix with respective eigenvalues $\zeta, \zeta^{-1}$. Then $\mathbb{C}^4/\Gamma$ has an isolated singularity at 0, and we denote this variety by $\tau$. We show in $\S 12.2$ that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to $\tau$ for the minimal degeneration $(2A_2 + A_1, A_2 + 2A_1)$ in $E_6$.

The degeneration $(A_4 + A_1, A_3 + A_2 + A_1)$ in $E_7$. Let $\mathfrak{e}_2$ be the cyclic group of order two acting on $\mathfrak{sl}_6(\mathbb{C})$ via an outer involution. All such involutions are conjugate in $\text{Aut}(\mathfrak{sl}_6(\mathbb{C}))$. The quotient $a_2/\mathfrak{e}_2$ has an isolated singularity at 0 since there are no minimal nilpotent elements in $\mathfrak{sl}_6(\mathbb{C})$ which are fixed by an outer involution. We will prove in [FJLSb] that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to $a_2/\mathfrak{e}_2$ for the minimal degeneration $(A_4 + A_1, A_3 + A_2 + A_1)$ in $E_7$.

The degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ in $E_8$. Let $\Delta$ be a dihedral group of order 10, acting on $\mathbb{C}^4$ via the sum of the reflection representation and its dual. Then it turns out that the blow-up of $\mathbb{C}^4/\Delta$ at its singular locus has an isolated singularity at a point lying over 0. We denote this blow-up by $\chi$. We show in $\S 12.3$ that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to $\chi$ for the minimal degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ in $E_8$.

1.7.5. Statement of main theorem. The main result is the determination of the generic singularities of nilpotent orbit closures in a simple Lie algebra of exceptional type, up to normalization for ten cases in $E_8$. The graphs at the end of the paper list the precise results.

**Theorem 1.2.** Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ in a simple Lie algebra of exceptional type. Let $e \in \mathcal{O}'$. Taking into consideration the intrinsic symmetry of $A(e)$, we have

(a) If the codimension of $\mathcal{O}'$ in $\mathcal{O}$ is two, then, with one exception, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a simple surface singularity of type $A - G$ or to one of the following

$$A_1^+, 2A_1, 3A_1, 3C_2, 3C_3, 3C_5, 4G_2, 5G_2, 10G_2, \text{ or } m,$$

up to normalization for ten cases in $E_7$ or $E_8$. Here, $kX_n$ denotes $k$ copies of $X_n$ meeting pairwise transversally at the common singular point. In the one remaining case, the singularity is smoothly equivalent to $\mu$.

(b) If the codimension is greater than two, then, with three exceptions, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a minimal singularity of type $a - g$ or to one of the following types:

$$a_1^+, a_2^+, a_3^+, 2a_2, d_4^+, e_6^+, 2g_2, \text{ or } m',$$

where the branched cases $2a_2$ and $2g_2$ denote two minimal singularities meeting transversally at the common singular point. The singularities for the three remaining cases are smoothly equivalent to $\tau, a_2/\mathfrak{e}_2$, and $\chi$, respectively.

In the statement of the theorem, we have recorded the induced symmetry of $A(e)$ relative to the stabilizer in $A(e)$ of an irreducible component of $\mathcal{S}_{\mathcal{O},e}$. See $\S 6.3$ for a complete statement.
1.8. **Brief description of methods.** The first method is an adaptation of arguments used in [KP81], where the closure of the minimal nilpotent orbit in the Lie algebra of $C(s)$ is related to $S_{O_{\min}}$. This method is successful in dealing with most higher codimension cases and when the singularity is $kA$, $or m$. In short, then, the Cartan-Killing type of $C(s)$ turns out to be sufficient to classify $S_{O_{\min}}$ in these cases. The second approach, on the other hand, is more apt for the surface cases. The idea is to use the fact that the normalization of a transverse slice is a simple surface singularity and then obtain a minimal resolution of the singularity by restricting the Q-factorial terminalizations of the nilpotent orbit closure to the transverse slice. Then we can apply a formula of Borho-MacPherson to compute the number of projective lines in the exceptional fiber and the action of $A(e)$ on the projective lines.

These two methods are sufficient to describe $S_{O_{\min}}$ for all minimal degenerations except when $\mu$, $\tau$, $\tau$, or $a_2/\mathbb{E}_2$ occur, perhaps up to normalization in a few of the surface singularity cases. The two cases where $\chi$ or $\tau$ occur are dealt with separately in §2. The two cases where $\mu$ or $a_2/\mathbb{E}_2$ occur are deferred to subsequent papers.

1.9. **Some consequences.**

1.9.1. **Isolated symplectic singularities coming from nilpotent orbits.** Examples of isolated symplectic singularities include $\mathcal{O}_{\min}$ and quotient singularities $\mathbb{C}^{2n}/\Gamma$, where $\Gamma \subset \text{Sp}_{2n}(\mathbb{C})$ is a finite subgroup acting freely on $\mathbb{C}^{2n} \setminus \{0\}$. It was established in [Bea00] that an isolated symplectic singularity with smooth projective tangent cone is locally analytically isomorphic to some $\mathcal{O}_{\min}$. It turns out that all of the isolated symplectic singularities we obtain, with one exception, are finite quotients of $\mathcal{O}_{\min}$ or $\mathbb{C}^{2n}$. It seems very likely that the singularity $\chi$ described above is not equivalent to a singularity of this form.

Another byproduct is the discovery of examples of symplectic contractions to an affine variety whose generic positive-dimensional fiber is of type $A_2$ and with a non-trivial monodromy action. These examples correspond to minimal degenerations with singularity $A^+_2$. The orbits $\mathcal{O}$ in these cases have closures which admit a Springer resolution. Examples include the even orbits $A_4 + A_2$ in $E_7$ and $E_6(b_0)$ in $E_8$. In [Wie03], a symplectic contraction to a projective variety of the same type is constructed. As far as we know, our examples are the first affine examples that have been constructed. This disproves Conjecture 4.2 in [AW].

1.9.2. **Representation theory.** The topology and geometry of the nilpotent cone $\mathcal{N}$ have played an important role in representation theory centered around Springer’s construction of Weyl group representations and the resulting Springer correspondence (e.g., [Pr76], [Lus81], [BM81], [Jos84], [Lus90]). The second author of the present paper defined a modular version of Springer’s correspondence [Jut07] to the effect that the modular representation theory of the Weyl group of $g$ is encoded in the geometry of $\mathcal{N}$. In particular, its decomposition matrix is a part of the decomposition matrix for equivariant perverse sheaves on $\mathcal{N}$. The connection with decomposition numbers makes it desirable to be able to compute the stalks of intersection cohomology complexes with modular coefficients. In this setting the Lusztig-Shoji algorithm to compute Green functions is not available and one has to use other methods, such as Deligne’s construction which is general, but hard to use in practice. To actually compute modular stalks it is necessary to have a good understanding of the geometry. The case of a minimal degeneration is the most tractable.

The decomposition matrices of the exceptional Weyl groups are known, so here we are not trying to use the geometry to obtain new information in modular representation theory. However, it is interesting to see how the reappearance of certain singularities in different nilpotent cones leads to equalities (or more complicated relationships) between parts of decomposition matrices. In the $GL_n$ case, the row and column removal rule for nilpotent singularities of $[KP81]$ gives a geometric explanation for a similar rule for decomposition matrices of symmetric groups ([Jam81], [Jut07]).

It would also be interesting to investigate whether the equivalences of singularities that we obtain in exceptional nilpotent cones have some significance for studying primitive ideals in finite $W$-algebras (see the survey article [Los10]).
1.9.3. Special pieces. For a special nilpotent orbit \( O \), the special piece \( \mathcal{P}(O) \) containing \( O \) is the union of all nilpotent orbits \( \sigma' \subset \sigma \) which are not contained in \( \mathfrak{g} \), for any special nilpotent orbit \( O' \subset \mathfrak{g} \). This is a locally-closed subvariety of \( \sigma \) and it is rationally smooth (see [Lus97] and the references therein). To explain rational smoothness geometrically, Lusztig conjectured in [Lus97] that every special piece is a finite quotient of a smooth variety. This conjecture is known for classical types by [KP89], but for exceptional types it is still open.

Each special piece contains a unique minimal orbit under the closure ordering. Motivated by the aforementioned conjecture of Lusztig, we studied the transverse slice for any Lie algebra. Then a nilpotent Slodowy slice in \( \mathcal{P}(O) \) to the minimal orbit in \( \mathcal{P}(O) \) is isomorphic to

\[
(h_n \oplus h_n')^k/\mathfrak{S}_{n+1}
\]

where \( h_n \) is the \( n \)-dimensional reflection representation of the symmetric group \( \mathfrak{S}_{n+1} \) and \( k \) and \( n \) are uniquely determined integers.

This theorem also includes the Lie algebras of classical types where \( n = 1 \), but \( k \) can be arbitrarily large. In the exceptional types Theorem 1.2 covers the cases where \( \mathcal{P}(O) \) consists of two orbits, in which case \( n = k = 1 \) (that is, the slice is isomorphic to the \( A_1 \) simple surface singularity). This leaves only those special pieces containing more than two orbits. Some of these remaining cases can be handled quickly with the same techniques, but others require more difficult calculations.

1.9.4. Normality of nilpotent orbit closures. By work of Kraft and Procesi [KP82], together with the remaining cases covered in [Som05], in classical Lie algebras the failure of \( \sigma \) to be normal is explained by branching for a minimal degeneration, and then only with two branches. In exceptional Lie algebras, the question of which nilpotent orbit closures are normal has not been completely solved in \( E_7 \) or \( E_8 \), but in [Bro98a, Section 7.8], a list of non-normal nilpotent orbit closures is given, which is expected to be the complete list.

Our analysis sheds some new light on the normality question. The occurrence of \( m \), \( m' \), and \( \mu \) at a minimal degeneration of \( \sigma \) gives a new geometric explanation for why \( \sigma \) is not normal. Previously the only geometric explanation for the failure of normality was branching and the appearance of the non-normal singularity in the closure of the \( A_1 \) orbit in \( G_2 \), which was known to be unibranch (see [Kra89]).

We also establish: (1) for many \( \sigma \) known to be non-normal that \( \sigma \) is normal at points in some minimal degeneration; and (2) for many \( \sigma \) that are expected to be normal that \( \sigma \) is indeed normal at points in all of its minimal degenerations. So we are able to make a contribution to determining the non-normal locus of \( \sigma \). Examples of the above phenomena are given starting in §7.3. Along these same lines, we also note that a consequence of Theorem 1.3 is that the special pieces are normal, a question studied by Achar and Sage in [AS09].

1.9.5. Duality. An intriguing result from [KP83] for \( g = \mathfrak{sl}_n(\mathbb{C}) \) is the following: a simple surface singularity of type \( A_k \) is always interchanged with a minimal singularity of type \( a_k \) under the order-reversing involution on the set of nilpotent orbits in \( g \).

This result leads to a generalization in the other Lie algebras, both classical and exceptional, by restricting to the set of special nilpotent orbits, which are reversed under the Lusztig-Spaltenstein involution. For a minimal degeneration of one special orbit to another, in most cases a simple surface singularity is interchanged with a singularity corresponding to the closure of the minimal special nilpotent orbit of dual type. There are a number of complicating factors outside of \( \mathfrak{sl}_n(\mathbb{C}) \), related to Lusztig’s canonical quotient and the existence of multiple branches. The duality can also be formulated as one from special orbits in \( g \) to those in \( ^*g \), the more natural setting for Lusztig-Spaltenstein duality.

Numerical evidence for such a duality was discovered by Lusztig in the classical groups using the tables in [KPS82]. The duality is already hinted at by Slodowy’s result for the regular nilpotent orbit.
in §1.4.2, which requires passing from $g$ to $\mathfrak{t}_g$. In a subsequent article [FJLS04] we will give a more complete account of the phenomenon of duality for degenerations between special orbits.

1.10. **Notation.** $G$ will be a connected, simple algebraic group of adjoint type over the complex numbers $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, and $\mathcal{O}$ and $\mathcal{O}'$ will be nilpotent $\mathfrak{Ad}(G)$-orbits in $\mathfrak{g}$. We use the notation in [Car93, p. 401-407] to refer to nilpotent orbits. For $x \in \mathfrak{g}$, $\mathcal{O}_x$ refers to the orbit $\mathfrak{Ad}(\mathfrak{g}(x))$, also written $G \cdot x$. For $x \in G$ or $\mathfrak{g}$ we denote by $G^r$ (resp. $\mathfrak{g}^r$) the centralizer of $x$ in $G$ (resp. $\mathfrak{g}$). Similar notation applies to other algebraic groups which arise, often as subgroups of $G$.

Generally, $e$ is a nilpotent element in an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra $s$ with standard basis $e, h, f$. The centralizer of $s$ in $\mathfrak{g}$ is $c(s)$ and its centralizer in $G$ is $C(s)$. If $e_0 \in c(s)$ is a nilpotent element, we use $s_0$ for an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra in $c(s)$ with standard basis $e_0, h_0, f_0$. Usually $\mathcal{O}'$ is a nilpotent orbit in $\mathcal{O}$ with $\mathcal{O}' \neq \mathcal{O}$ and $e \in \mathcal{O}'$. We write $(\mathcal{O}, \mathcal{O}')$ for such a pair of nilpotent orbits. Often, but not always, $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$. The nilpotent Slodowy slice $\mathfrak{s} \cap (e + \mathfrak{g}')$ is denoted $\mathfrak{s}_{\mathcal{O}, e}$.

The field of fractions of an integral domain $Y$ will be denoted $\text{Frac}(A)$. The symmetric group on $n$ letters is denoted $\Sigma_n$. Where we refer to explicit elements of $g$, we use GAP’s structure constants.

1.11. **Acknowledgements.** The authors thank Miles Reid, Dmitrii Rumynin, and Anthony Henderson for helpful conversations. B. Fu was supported by NSFC 11225106 and 11321101 and the KIAS Scholar Program. P. Levy was supported in part by Engineering and Physical Sciences Research Council grant EP/K022997/1. E. Sommers was supported by NSA grant H98230-11-1-0173 and through a National Science Foundation Independent Research and Development plan. Computer calculations were carried out in GAP and Python. Research visits were also supported by the Centre Nationale de Recherches Scientifiques and the AMSS of Chinese Academy of Sciences.

2. **Transverse slices**

2.1. **Smooth equivalence.** To study singularities it is useful to introduce the notion of smooth equivalence. Given two varieties $X$ and $Y$ and two points $x \in X$ and $y \in Y$, the singularity of $X$ at $x$ is **smoothly equivalent** to the singularity of $Y$ at $y$ if there exists a variety $Z$, a point $z \in Z$ and morphisms $\varphi : Z \to X$ and $\psi : Z \to Y$ which are smooth at $z$ and such that $\varphi(z) = x$ and $\psi(z) = y$ (see [Hes76, 1.7]). This defines an equivalence relation on pointed varieties $(X, x)$ and the equivalence class of $(X, x)$ will be denoted $\text{Sing}(X, x)$. As in [KPS1], §2.1, two singularities $(X, x)$ and $(Y, y)$ with $\dim Y = \dim X + r$ are equivalent if and only if $(X \times \mathcal{O}'(x, 0))$ is locally algebraically isomorphic to $(Y, y)$.

Let $\mathcal{O}'$ and $\mathcal{O}$ be nilpotent orbits in $\mathfrak{g}$ with $\mathcal{O}' \subset \mathcal{O}$. Let $e \in \mathcal{O}'$. As discussed in the Introduction, the goal of this paper is to study the local geometry of $\mathcal{O}$ at $e$ in the exceptional Lie algebras. This leads us to study the equivalence class of $(\mathcal{O}, e)$ under smooth equivalence. The equivalence class of the singularity $(\mathcal{O}, e)$ will be denoted $\text{Sing}(\mathcal{O}, \mathcal{O}')$ since the equivalence class is independent of the choice of element in $\mathcal{O}' = \mathcal{O}_e$. We will primarily be interested in the case when $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$ (§1.1), but this is not assumed unless stated otherwise.

2.2. **Transverse slices.** The main tool in studying $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is the transverse slice. [Slo80, III.5.1] and [KPS2, §12] are references for what follows.

Let $X$ be a variety on which $G$ acts, and let $x \in X$. A **transverse slice** in $X$ to $G \cdot x$ at $x$ is a locally closed subvariety $S$ of $X$ with $x \in S$ such that the morphism

$$G \times S \to X, (g, s) \mapsto g \cdot s$$

is smooth at $(1, x)$ and such that the dimension of $S$ is minimal subject to these requirements. It is immediate that $\text{Sing}(X, x) = \text{Sing}(S, x)$. If $X$ is a vector space then it is easy to construct such a transverse slice as $x + u$ where $u$ is a vector space complement to $T_x(G \cdot x) = [g, x]$ in $X$. More generally, this also suffices to construct a transverse slice to a $G$-stable subvariety $Y \subset X$ containing $x$ by taking the intersection $(x + u) \cap Y$. In such a case $\text{codim}_Y (G \cdot x) = \dim_x ((x + u) \cap Y)$.
These observations are especially helpful for nilpotent orbits in the adjoint representation, where there is a natural choice of transverse slice. As before, pick \( e \in \mathcal{O}' \). Then there exists \( h, f \in \mathfrak{g} \) so that \( \{e, h, f\} \subset \mathfrak{g} \) is an \( \mathfrak{sl}_2 \)-triple. Then by the representation theory of \( \mathfrak{sl}_2(\mathbb{C}) \), we have \([e, \mathfrak{g}] \oplus \mathfrak{g}' = \mathfrak{g}\). The affine space

\[
S_e = e + \mathfrak{g}'
\]

is a transverse slice of \( \mathfrak{g} \) at \( e \), called the \textit{Slo\''dowy slice}. The variety

\[
\mathcal{S}_{\mathcal{O}, e} := S_e \cap \mathfrak{g}'
\]

is then a transverse slice of \( \mathfrak{g} \) to \( \mathcal{O}' \) at the point \( e \), which we call a nilpotent Slo\''dowy slice. In this setting

\[
\text{codim}_{\mathfrak{g}}(\mathcal{O}') = \dim \mathcal{S}_{\mathcal{O}, e}.
\]

Since any two \( \mathfrak{sl}_2 \)-triples for \( e \) are conjugate by an element of \( G^0 \), the isomorphism type of \( \mathcal{S}_{\mathcal{O}, e} \) is independent of the choice of \( \mathfrak{sl}_2 \)-triple. Moreover, the isomorphism type of \( \mathcal{S}_{\mathcal{O}, e} \) is independent of the choice of \( e \in \mathcal{O}' \). By focusing on \( \mathcal{S}_{\mathcal{O}, e} \) we reduce the study of \( \text{Sing}(\mathcal{O}, \mathcal{O}') \) to the study of the singularity of \( \mathcal{S}_{\mathcal{O}, e} \) at \( e \). In fact, most of our results will be concerned with determining the isomorphism type of \( \mathcal{S}_{\mathcal{O}, e} \).

2.3. Group actions on \( \mathcal{S}_{\mathcal{O}, e} \). An important feature of the transverse slice \( \mathcal{S}_{\mathcal{O}, e} \) is that it carries the action of two commuting algebraic groups, which both fix \( e \). Let \( s \) be the subalgebra spanned by \( \{e, h, f\} \) and \( C(s) \) the centralizer of \( s \) in \( G \). Then \( C(s) \) is a maximal reductive subgroup of \( G^0 \) and \( C(s) \) acts on \( \mathcal{S}_{\mathcal{O}, e} \), fixing \( e \).

The second group which acts is \( C^* \). Since \([h, f] = -2f, \text{ad} \ h \) preserves the subspace \( \mathfrak{g}' \) and by \( \mathfrak{sl}_2 \)-theory all of its eigenvalues are nonpositive integers. Set

\[
\mathfrak{g}'(i) = \{x \in \mathfrak{g}' : [h, x] = ix\}
\]

for \( i \leq 0 \). The special case \( \mathfrak{g}'(0) \) is simply \( \mathfrak{c}(s) \), the centralizer of \( s \) in \( \mathfrak{g} \), which coincides with \( \text{Lie}(C(s)) \).

Define \( \phi : SL_2(\mathbb{C}) \to G \) such that the image of \( d\phi \) is equal to \( s \), with \( d\phi \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) = e \) and \( d\phi \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) = h \). Set \( \gamma(t) = \phi \left( \begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix} \right) \) for \( t \in C^* \). On the one hand, \( \text{Ad} \gamma(t) \) preserves \( \mathfrak{g} \) and so does the scalar action of \( C^* \) on \( \mathfrak{g} \) since \( \mathfrak{g} \) is conical in \( \mathfrak{g} \). On the other hand, for \( x_i \in \mathfrak{g}'(-i) \) and \( t \in C^* \),

\[
\text{Ad} \gamma(t)(e + x_0 + x_1 + \ldots + x_m) = t^{-2}e + x_0 + tx_1 + \ldots + t^m x_m.
\]

Composing this action with the scalar action of \( t^2 \) on \( \mathfrak{g} \), gives an action of \( t \in C^* \) on \( e + \mathfrak{g}' \) by

\[
t \cdot (e + x_0 + x_1 + \ldots) = e + t^2 x_0 + t^3 x_1 + \ldots,
\]

which preserves \( \mathcal{S}_{\mathcal{O}, e} = \mathfrak{g}' \cap \mathcal{S}_e \). The \( C^* \)-action fixes \( e \) and commutes with the action of \( C(s) \), since \( C(s) \) commutes with \( \text{ad} \ h \) and so preserves each \( \mathfrak{g}'(i) \). Thus \( C(s) \times C^* \) acts on \( \mathcal{S}_{\mathcal{O}, e} \).

2.4. Branching and component group action. The \( C(s) \times C^* \)-action on \( \mathcal{S}_{\mathcal{O}, e} \) has consequences for the irreducible components of \( \mathcal{S}_{\mathcal{O}, e} \).

An irreducible variety \( X \) is \textit{unibranch} at \( x \) if the normalization \( \pi : (\tilde{X}, \tilde{x}) \to (X, x) \) of \((X, x)\) is locally a homeomorphism at \( x \). Since the \( C^* \)-action on \( \mathcal{S}_{\mathcal{O}, e} \) in (2.2) is attracting to \( e \), \( \mathcal{S}_{\mathcal{O}, e} \) is connected and its irreducible components are unibranch (see [CG10], proof of Proposition 3.7.15). Consequently the number of irreducible components of \( \mathcal{S}_{\mathcal{O}, e} \) is equal to the number of branches of \( \mathfrak{g}' \) at \( e \). The latter can be determined from the tables of Green functions in [BS84, Sho80], as discussed in [BS84, 5(E)–(F)].

The identity component \( C(s)^0 \) of \( C(s) \), being connected, preserves each irreducible component of \( \mathcal{S}_{\mathcal{O}, e} \), hence there is a natural action of \( C(s)/C(s)^0 \) on the irreducible components of \( \mathcal{S}_{\mathcal{O}, e} \). The finite group \( C(s)/C(s)^0 \) is isomorphic to the component group \( A(e) := G^0/(G^0)^0 \) of \( G^0 \) via \( C(s) \to G^0 \to G^0/(G^0)^0 \). Since any two \( \mathfrak{sl}_2 \)-triples containing \( e \) are conjugate by an element of \( (G^0)^0 \), this gives a well-defined action of \( A(e) \) on the set of irreducible components of \( \mathcal{S}_{\mathcal{O}, e} \). Moreover, as noted in \textit{op. cit.}, the permutation representation of \( A(e) \) on the branches of \( \mathfrak{g}' \) at \( e \), and hence on the irreducible components of \( \mathcal{S}_{\mathcal{O}, e} \), can be computed. For a minimal degeneration, the situation is particularly nice. We observe by looking at the tables in [BS84, Sho80] that
Proposition 2.1. When $O'$ is a minimal degeneration of $O$ in an exceptional Lie algebra, the action of $A(e)$ on the set of irreducible components of $S_{O,e}$ is transitive. In particular, the irreducible components of $S_{O,e}$ are mutually isomorphic.

The proposition also holds in the classical types, which can be deduced using the results in [KPS2]. In §6.2 we will discuss the full symmetry action on $S_{O,e}$ induced from $A(e)$.

2.5. Passing to a reductive subalgebra. Often the isomorphism type of $S_{O,e}$ can be determined by knowing the isomorphism type of a nilpotent Slodowy slice in a reductive subalgebra of $\mathfrak{g}$. The following results are variants of [KPS2, Cor 13.3].

Let $O$ and $O'$ be nilpotent $G$-orbits in $\mathfrak{g}$ with $O' \subset O$. Here, $O'$ is not assumed to be a minimal degeneration of $O$. Let $M \subset G$ be a closed reductive subgroup and set $m = \text{Lie}(M)$. Let $e \in m \cap O'$ and $x \in m \cap O$ and suppose that $M \cdot e \subset M \cdot x$. Since $m$ is reductive, we may assume the $sl_2$-subalgebra $s$ containing $e$ lies in $m$. Let $S_{M,x,e}$ be the nilpotent Slodowy slice $M \cdot x \cap (e + m')$ in $m$. Of course, $S_{M,x,e} \subset S_{O,e}$.

Lemma 2.2. Suppose that $\text{codim}(M \cdot e) = \text{codim}(O')$ and $S_{M,x,e}$ is equidimensional. Then $S_{M,x,e}$ is a union of irreducible components of $S_{O,e}$. Moreover if $\overline{O}$ is unibranch at $e$ or if the number of branches of $\overline{O}$ at $e$ equals the number of branches of $M \cdot x$ at $e$, then $S_{M,x,e} = S_{O,e}$.

Proof. The first statement follows from (2.1) and the fact that $S_{M,x,e} \subset S_{O,e}$. The second statement follows from the fact that the irreducible components of $S_{O,e}$ and $S_{M,x,e}$ are unibranch (§2.4).

There are two main situations where we apply Lemma 2.2:

1. when $m$ is the centralizer of a semisimple element of $G$; or
2. when $m = s \oplus c(s)$.

Consider the case when $m = s \oplus c(s)$. Then $x$ can be written as $x = x' + e_0$ with $x' \in s$ and $e_0 \in c(s)$. Since $x$ is nilpotent, both $x'$ and $e_0$ are nilpotent. Since $M \cdot e \subset M \cdot x$, $x'$ belongs to the nonzero $M$-orbit through $e$, so without loss of generality $x' = e$. The Slodowy slice in $m$ to $e$ is then $e + c(s)$. Therefore the nilpotent Slodowy slice $S_{M,x,e}$ in $m$ is

$$S_{M,x,e} = e + C(s) \cdot e_0.$$

It follows that $e + C(s) \cdot e_0$ is an irreducible component of $S_{M,x,e}$ and thus $C(s)$ acts transitively on the irreducible components of $S_{M,x,e}$. Hence $S_{M,x,e}$ is equidimensional. A reformulation of Lemma 2.2 in this setting is the following:

Corollary 2.3. Let $(O,O')$ be a pair of nilpotent orbits with $O' \subset O$ and $e \in O'$. Suppose there exists $x \in O$ such that $x = e + e_0$ with $e_0 \in c(s)$ nilpotent. If $\dim C(s) \cdot e_0 = \text{codim}(O')$, then $e + C(s) \cdot e_0$ is a union of irreducible components of $S_{O,e}$. Moreover, if the number of branches of $\overline{O}$ at $e$ equals the number of irreducible components of $C(s) \cdot e_0$, then

$$e + C(s) \cdot e_0 = S_{O,e}.$$

In this case, the natural map from $C(s) \cdot e_0$ to $S_{O,e}$ is a $C(s)$-equivariant isomorphism.

2.6. A generalization of Corollary 2.3. Corollary 2.3 is a tool for showing that many $S_{O,e}$ are isomorphic to closures of nilpotent orbits in $c(s)$. Sometimes we need a more general result.

As before $(O,O')$ is a pair of nilpotent orbits with $O' \subset O$ and $e \in O'$. Suppose that there exists $x \in O$ of the form

$$x = e + e_0 + x'$$

with $e_0 \in c(s)$ and $x' \in \oplus_{i \leq -1} \mathfrak{g}(i)$. In particular $x \in S_{O,e}$. Assume further that $e_0$ is nilpotent. The next lemma gives a condition for the closure of the $C(s) \cdot (e_0 + x')$ in $\mathfrak{g}$ to be isomorphic to $S_{O,e}$.

Let

$$Z = C(s) \cdot (e_0 + x').$$
By construction $Z$ is equidimensional with irreducible components permuted transitively by $C(s)/C(s)^\circ$. Certainly $Z \subset g'$ and so $e + Z \subset S_e = e + g'$. Let $i : g' \to S_e$ be the map $z \to e + z$. Since $i$ is a $C(s)$-equivariant isomorphism, $V(g') \cdot z = e + Z$. Certainly $V(g') \cdot z \subset \overline{O}$ and so $e + Z \subset S_{O,e}$. The proof of Lemma 2.2 extends to the following general version of Corollary 2.3.

**Lemma 2.4.** Suppose that $\dim Z = \text{codim}_{\overline{O}}(O)$, then $e + Z$ is a union of irreducible components of $S_{O,e}$. Moreover, if the number of branches of $\overline{O}$ at $e$ equals the number of irreducible components of $Z$, then

$$e + Z = S_{O,e}.$$ 

and the map $i$ restricts to an isomorphism of $C(s)$-varieties $i' : Z \to S_{O,e}$.

The case when $x' = 0$ is just Corollary 2.3. In all the cases encountered in this paper, it turns out that $c(s)^\circ \subset c(s)^\circ$ so that $\dim C(s) \cdot e_0 = \dim Z$ and the condition in the Lemma that must be checked is $\dim C(s) \cdot e_0 = \text{codim}_{\overline{O}}(O)$.

Lemma 2.4 is a tool for determining $S_{O,e}$ for some minimal degenerations not handled by the $x' = 0$ case of Corollary 2.3 (see §3.3 and §3.4). Both Corollary 2.3 and the more general Lemma 2.4 also apply in many cases when $e_0$ is not in the minimal nilpotent orbit of $c(s)$.

## 3. Statement of the main results

In this section we state the main propositions which underlie Theorem 1.2. The propositions give more precise information about $S_{O,e}$. We defer discussion of the intrinsic symmetry action until §4.3. The notation from the previous sections remains in effect. That is, $O'$ and $O$ are nilpotent orbits, with $e \in O'$ and $O' \subset \overline{O}$; $s$ is an $\mathfrak{sl}_2$-subalgebra containing the standard triple $(e, h, f)$; $c(s)$ is the centralizer of $s$ in $G$ and is equal to the Lie algebra of $C(s)$, the centralizer of $s$ in $G$; and $S_{O,e}$ is the nilpotent Slodowy slice $\overline{O} \cap S_e$ where $S_e = e + g'$. Throughout this section $O'$ is always a minimal degeneration of $O$.

### 3.1. Most of the codimension 4 or greater cases

First, we consider the minimal degenerations where Corollary 2.3 applies.

**Proposition 3.1.** Let $O'$ be a minimal degeneration of $O$ of codimension at least 4. Assume that $c(s)$ has a simple summand different from $\mathfrak{sl}_2(\mathbb{C})$ and that $O'$ is neither of type $D_4(a_1) + A_2$ nor $2A_2 + 2A_1$ in $E_8$. Then there exists a minimal nilpotent element $e_0 \in c(s)$ such that $e + e_0 \in O$ and

$$S_{O,e} \cong C(s) \cdot e_0$$

under the natural $C(s)$-equivariant inclusion.

**Proof.** The proof is case-by-case until we exhaust all such minimal degenerations. We first locate a minimal nilpotent element $e_0 \in c(s)$ such that $e + e_0 \in O$ and then check that $\dim S_{O,e} = \dim C(s) \cdot e_0$. This is carried out using the ideas in §1. The checking is greatly simplified by Corollary 2.3 in most cases. Since $O'$ is a minimal degeneration of $O$, $A(c) = C(s)/C^0(s)$ acts transitively on the irreducible components of $S_{O,e}$ by Proposition 2.7. At the same time, $C(s)/C^0(s)$ acts transitively on the irreducible components of $C(s) \cdot e_0$. Hence the full statement of Corollary 2.3 applies. □

In the cases cover by the previous proposition, $S_{O,e}$ is a minimal singularity or a union of minimal singularities meeting transversely at $e$.

### 3.2. Surface cases

The case of a minimal degeneration of codimension two is summarized by the following proposition.

**Proposition 3.2.** Let $O'$ be a minimal degeneration of $O$ of codimension 2. Then there exists a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that the normalization $S_{O,e}$ of $S_{O,e}$ is isomorphic to a disjoint union of $k$ copies of $X$ where $X = \mathbb{C}^2/\Gamma$.
This is proved in §3.3 where techniques for determining $\Gamma$ and $k$ are given. For most cases in Proposition 3.2 we can show that the irreducible components of $S_{O,e}$ are normal either by using that $\overline{\mathcal{O}}$ is normal, by using Lemma 2.2 to move to a smaller subalgebra where the slice is known to be normal, or by doing an explicit computation (often with the help of Lemma 2.4). In the surface case, we found only two ways that an irreducible component of $S_{O,e}$ fails to be normal:

- When $\Gamma = 1$, we show below that $S_{O,e} \cong m$ (§1.7.4). This happens for several different minimal degenerations.
- When $(O,O') = (D_r(a_1), E_n(b_n))$, we have $\Gamma \cong \mathbb{Z}/4$, but $S_{O,e}$ is not normal. Instead, $S_{O,e}$ is smoothly equivalent to $\mu$ (§1.7.4).

A handful of cases are left unresolved up to normalization.

There is a more precise statement when $|\Gamma| = 1$ or 2. In these cases there always exists $x \in \mathcal{O}$ satisfying Lemma 2.4. As a result, $S_{O,e}$ is either isomorphic to $m$ in the former case or to $kA_1$ in the latter. Let $\pi_{0,1} : S_e \to g'(0) \oplus g'(-1)$ be the natural projection, which is $C(s) \times C^*$-equivariant.

**Proposition 3.3.** Let $O'$ be a minimal degeneration of $O$ of codimension 2, with $|\Gamma| = 1$ or 2. Then $e(s)$ contains a simple factor $\lambda \cong s_\lambda(\mathbb{C})$ and there exists

$$x = e + e_0 + x_1 + x^+ \in \mathcal{O}$$

with $e_0 \in \lambda$ nonzero nilpotent, $x_1 \in g'(-1)$, and $x^+ \in \bigoplus_{i \leq -2} g'i$. Furthermore the restriction of $\pi_{0,1}$ gives a $C(s) \times C^*$-equivariant isomorphism

$$S_{O,e} \cong C(s)\cdot(e_0 + x_1).$$

If $|\Gamma| = 1$, then $C(s)\cdot(e_0 + x_1) \cong m$. If $|\Gamma| = 2$, then $x_1 = 0$ and $C(s)\cdot e_0 \cong kA_1$ for some $k$.

**Proof.** This is again case-by-case. The hard part is showing the existence of $x = e + e_0 + x_1 + x^+ \in \mathcal{O}$. We give some general techniques starting in §4.4, but sometimes have to resort to explicit computer calculations. Once such an $x$ is found, necessarily $\dim C(s)\cdot e_0 = 2$, so that $\dim S_{O,e} = \dim Z$, from which the full statement of Lemma 2.4 holds. The rest of the proposition follows from §4.5.

### 3.3. Remaining cases.
There are 6 minimal degenerations $(O,O')$ not covered by Propositions 3.1 and 3.2, all with $\dim S_{O,e} = 4$:

1. $(A_3 + 2A_2, 2A_2 + 2A_1)$ in $E_8$. Here, $S_{O,e} \cong m'$ (§1.7.4).
2. $(A_4 + A_1, D_4(a_1) + A_2)$ and $(2A_3, D_4(a_1) + A_2)$ in $E_8$. For these both, $S_{O,e} \cong a_2$.
3. $(2A_2 + A_1, A_2 + 2A_1)$ in $E_6$
4. $(A_4 + A_1, A_3 + A_2 + A_1)$ in $E_7$
5. $(A_4 + A_1, A_3 + A_2 + A_1)$ in $E_8$

Cases (3)-(5) share the property that $e(s)$ contains a factor isomorphic to $s_{A_2}(\mathbb{C})$, but $C(s)$ does not act transitively on the smooth part of $S_{O,e}$, in contrast to the other four-dimensional or higher cases covered by Proposition 3.1. They do share a common feature amongst each other: they are covers of slices in smaller groups. Cases (3) and (5) are handled in §4.2 and (4) is deferred to [FJLSb].

Cases (1) and (2) fit into a slight variant of Proposition 3.3.

**Proposition 3.4.** Let $(O,O')$ be one of the minimal degenerations in Case (1) or (2). Then $e(s)$ contains a simple factor $\lambda \cong s_{A_2}$ or $s_{A_3}$, respectively, and there exists

$$x = e + e_0 + x_1 + x^+ \in \mathcal{O}$$

with $e_0 \in \lambda$ minimal nonzero nilpotent, $x_1 \in g'(-1)$, and $x^+ \in \bigoplus_{i \leq -2} g'i$. Furthermore the restriction of $\pi_{0,1}$ gives a $C(s) \times C^*$-equivariant isomorphism

$$S_{O,e} \cong C(s)\cdot(e_0 + x_1).$$

In Case (1), $C(s)\cdot(e_0 + x_1) \cong m'$ and in Case (2), $x_1 = 0$ and $C(s)\cdot e_0 \cong a_2$.

**Proof.** The proof is along the same lines as Proposition 3.3. See Table 10.
Remark 3.5. Propositions 3.1 and 3.2 also hold in the classical groups, as does Proposition 3.3 (the singularity \( m \) does not appear). In the classical groups all irreducible components of \( \mathcal{O}_s \) are normal by Kraft and Procesi’s work. It follows that smooth equivalence in Kraft and Procesi’s Theorem 1.1 can be replaced by algebraic isomorphism.

Remark 3.6. In Propositions 3.1, 3.3, and 3.4, \( C(s) \) has two orbits on \( \mathcal{O}_{s,e} \), namely, \( e \) and its smooth complement.

Remark 3.7. In the settings of Propositions 3.1, 3.3, and 3.4, the minimal nilpotent orbits in the summands of \( c(s) \) closely control \( \mathcal{O}_{s,e} \). It is not the case, however, that every summand of \( c(s) \) contributes to this story. For example, when \( e \in \mathcal{O}' = 3A_1 \) in \( E_6 \), the centralizer \( c(s) \) has type \( A_2 + A_1 \). If \( e_0 \) belongs to the minimal nilpotent orbit in the simple summand of type \( A_2 \), then \( \mathcal{O}' \) is not a minimal degeneration of \( \mathcal{O}_{s,e} \). We are not able to explain these missing summands.

Remark 3.8. The isomorphism in Proposition 3.2 is compatible with the natural \( C^* \)-actions on both sides. On \( \tilde{\mathcal{O}}_{s,e} \), the \( C^* \)-action is the one induced from \( \mathcal{O}_{s,e} \) on \( C^*/\Gamma \) it is the one coming from the central torus in \( GL_2(C) \). This follows from Proposition 3.2.

Proposition 3.3 shows that the minimal degenerations of type \( A_1 \) share the feature with the larger codimension minimal degenerations that the singularity is controlled by a summand(s) of \( c(s) \). We have observed that the minimal degenerations of type \( A_k \) (or the geometrically equivalent \( B_k \)) are also related to \( c(s) \): in all these cases \( C(s) \) contains a central torus. The singularity \( A_k \) described as \( C^*/\Gamma \) (where \( \Gamma \) is cyclic of order \( k+1 \) in \( SL_2 \)) is acted upon by the two-dimensional torus of \( GL_2(C) \). This suggests that when \( \mathcal{O}_{s,e} \) is a surface singularity of type \( A_k \), then the isomorphism in Proposition 3.2 extends to one compatible with the action of a central torus in \( C(s) \), in addition to the \( C^* \)-action from \( \mathcal{O}_{s,e} \). Along these lines, the absence of a central torus in \( C(s) \) for \( e = E_6(b_6) \), where \( \mathcal{O}_{s,e} \) is of type \( A_3 \), led us to suspect that \( \mathcal{O}_{s,e} \neq \tilde{\mathcal{O}}_{s,e} \) and then to the \( \mu \) singularity.

4. Tools for establishing Propositions 3.1, 3.3, and 3.4

In this section we give a way to verify Corollary 2.3 and Lemma 2.4 so that the remaining details in the proofs of Propositions 3.1, 3.3, and 3.4 can be checked. The first step is locating nilpotent elements in \( c(s) \) relative to the embedding of \( c(s) \) in \( g \).

4.1. Locating nilpotent elements in \( c(s) \). We want to be able to find nilpotent \( e_0 \in c(s) \) and then compute \( e + e_0 \) in order to verify cases where Corollary 2.3 hold.

First, if \( e_0 \in c(s) \), then \( e_0 \) centralizes the semisimple element \( h \in s \). Hence \( e_0 \in g^h \), which is a Levi subalgebra of \( g \). Assume \( h \) lies in a chosen Cartan subalgebra \( \mathfrak{h} \subset g \) and is dominant for a chosen Borel subalgebra \( \mathfrak{b} \subset g \) containing \( \mathfrak{h} \). The type of the Levi subalgebra \( g^h \) can then be read off from the weighted Dynkin diagram for \( h \); the Dynkin diagram for the semisimple part of \( g^h \) corresponds to the zeros of the diagram.

Therefore in order to locate a nilpotent element in \( c(s) \), we first choose a nilpotent element \( e_0 \in g^h \); the \( G^h \)-orbits of such elements are known by Dynkin’s and Bala and Carter’s results [Car93]. In particular we can compute the semisimple element \( h_0 \in g^h \cap h \) of an \( sl_2 \)-subalgebra \( s_0 \) through \( e_0 \) in \( g^h \).

Next, we compute \( h + h_0 \) and see whether it corresponds to a nilpotent orbit in \( g \): for if \( e \) and \( e_0 \) commute (or some conjugate of \( e_0 \) under \( G^h \)), then \( h + h_0 \) will be the semisimple element in an \( sl_2 \)-subalgebra through the nilpotent element \( e + e_0 \). Together with knowledge of the Cartan-Killing type of the reductive Lie algebra \( c(s) \subset g^h \) (see [Car93]), this search usually suffices to locate the nilpotent orbit through \( e_0 \) in \( g \) for nilpotent elements \( e_0 \in c(s) \) and the resulting nilpotent orbit through \( e + e_0 \). In particular we carried out this approach for all the minimal nilpotent \( C(s) \)-orbits in \( c(s) \). In a few cases we needed to employ ad hoc methods.

Two special situations are worth mentioning.
4.1.1. One special situation is when \( e_0 \) is minimal in \( g \), that is, of Bala-Carter type \( A_1 \). Then \( \epsilon(e_0) \) is itself a Levi subalgebra of \( g \) and can be computed directly from the extended Dynkin diagram of \( g \). Of course \( e \in \epsilon(e_0) \). Consequently it is easy to locate all \( e \) which have \( e_0 \in \epsilon(e) \) when \( e_0 \) is of type \( A_1 \) in \( g \).

We will see in Corollary 4.2 that in this setting \( x = e + e_0 \) always satisfies Corollary 2.3. Moreover the Bala-Carter type of \( x \) in \( g \) is easy to determine: if we know the type of \( e \) in the Levi subalgebra \( \epsilon(e_0) \), call it \( X \), then \( x \) has generalized Bala-Carter type \( X + A_1 \). Then the usual Bala-Carter type can be looked up in \([Som98]\) or in Dynkin’s seminal paper \([Dyn52]\).

For example, in \( E_8 \) when \( e_0 \) is of type \( A_1 \), then \( \epsilon(e) \) is of Cartan-Killing type \( E_7 \). Any nilpotent element \( e \) in a Levi subalgebra of type \( E_7 \) will have a conjugate of \( e_0 \) in \( \epsilon(e) \). If, for instance, \( e \) is a regular nilpotent element, then \( e + e_0 \) has generalized Bala-Carter type \( E_7 + A_1 \), which is the same as \( E_8(e_0) \).

There is another way to determine \( e + e_0 \) when \( e_0 \) is minimal in \( g \). It has the advantage of locating the simple summand of \( \epsilon(e) \) in which \( e_0 \) lies. As above, assume \( h \) is dominant relative to \( b \). Since \( e_0 \in g^h \) has type \( A_1 \), the semisimple element \( h_0 \in h \) is equal to the coroot of a long root \( \theta \) for \( g^h \). Therefore, \( \alpha(h_0) \geq -2 \) for any root of \( g \) and equality holds if and only \( \alpha = -\theta \). Now choose \( h_0 \) dominant in \( g^h \) (relative to \( b \in g^h \)). Then \( \alpha(h_0) \geq -1 \) for all simple roots \( \alpha \) of \( g \) since \( -\theta \) is a negative root. Moreover \( \alpha(h_0) = -1 \) only if \( \alpha \) is not a simple root for \( g^h \). In that case \( \alpha(h) \geq 1 \) since the simple roots of \( g^h \) correspond to the zeros of the weighted Dynkin diagram for \( h \). This shows that \( \alpha(h_0 + h) \geq 0 \) for all simple roots \( \alpha \) of \( g \) and thus \( h + h_0 \) yields the weighted Dynkin diagram for \( e + e_0 \) without having to conjugate by an element of the Weyl group.

For example, let \( e \) belong to the orbit \( E_7(e_0) \), which has weighted Dynkin diagram
\[
\begin{array}{ccccccccc}
2 & 1 & 0 & 1 & 0 & 0 & 2
\end{array}
\]

Then \( g^h \) has type \( 4A_1 \) and \( \epsilon(e) \) must have type \( A_1 \), since \( \epsilon(e) \) has rank one and contains \( e_0 \), a nonzero nilpotent element. We want to know in which summand of \( g^h \) the element \( e_0 \) lies and what is \( e + e_0 \).

The diagram for \( h_0 \) relative to \( g \), and dominant for \( g^h \), is either:
\[
\begin{array}{cccccccc}
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 2
\end{array}
\]

Only the second choice leads to a weighted Dynkin diagram for \( h + h_0 \), namely for \( D_4(a_1) \). Hence we know the type of \( e + e_0 \) and the embedding of \( \epsilon(e) \) in \( g^h \).

4.1.2. The other special situation occurs when \( \epsilon(e) \) has rank 1.

Let \( t \) be a minimal Levi subalgebra containing \( e \). Then \( t \) has semisimple rank equal to the rank of \( g \) minus one. Assume that \( t \) is a standard Levi subalgebra. Let \( \alpha_1 \) be the simple root of \( g \) which is not a simple root of \( t \). For nonzero \( e_0 \in \epsilon(e) \), the corresponding \( h_0 \) centralizes \( t \) and hence lies in the one-dimensional subalgebra of \( h \) spanned by the coweight \( \omega_\alpha \) for \( \alpha \). Since the values in any weighted Dynkin diagram are 0, 1, or 2, if \( h_0 \) is dominant, then \( h_0 \) must be either \( \omega_\alpha \) or \( 2\omega_\alpha \).

For example, let \( e \) be of type \( A_2 \) in \( E_8 \), which has \( \epsilon(e) \) of type \( A_1 \). The weighted Dynkin diagram of a nonzero \( h_0 \in \epsilon(e) \) must either be
\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Both of these are actual weighted Dynkin diagrams in \( E_8 \), but only the orbit for the first diagram, which corresponds to \( 4A_1 \), meets \( g^h \), which has semisimple type exactly \( 4A_1 \). Therefore a nonzero nilpotent element in \( \epsilon(e) \) has type \( 4A_1 \) in \( g \).

4.2. Establishing the dimension condition in Corollary 2.3. Once a nilpotent \( e_0 \in \epsilon(e) \) is located, as in the previous section, with corresponding semisimple element \( h_0 \in \epsilon(e) \), we can compute \( h + h_0 \) and check by hand whether the dimension condition
\[
\dim(C(e) \cdot e_0) = \text{codim}(O_e)
\]
holds for the orbit \( O \) through \( e + e_0 \). If it holds, then the first hypothesis in Corollary 2.3 is true with \( O' = O_e \) and \( x = e + e_0 \).
As before, let \( s_0 \) be an \( sl_2 \)-subalgebra in \( c(s) \) with standard basis \( \{ e_0, h_0, f_0 \} \). Clearly, \( s \) and \( s_0 \) commute. There is a statement equivalent to the dimension condition (4.1) in terms of the decomposition of \( g \) into irreducible subrepresentations for \( s \oplus s_0 \cong sl_2(C) \oplus sl_2(C) \).

Let \( V_{m,n} \) denote an irreducible representation of \( s \oplus s_0 \) with \( h \in s \) acting by \( m \) and \( h_0 \in s_0 \) acting by \( n \) on a highest weight vector \( u \in V_{m,n} \) annihilated by both \( e \) and \( e_0 \). The eigenvalues of \( h + h_0 \) on \( V_{m,n} \) are either all even if \( m \) and \( n \) have the same parity or all odd if \( m \) and \( n \) have opposite parities. In the former case the quantity

\[
\min(m, n) + 1
\]

is equal to the dimension of the 0-eigenspace of \( h + h_0 \); in the latter case, it is equal to the dimension of the 1-eigenspace of \( h + h_0 \). This is analogous to what occurs in the proof of the Clebsch-Gordan formula.

Let

\[
(4.2)
\]

be a decomposition into irreducible subrepresentations \( V_{m_i,n_i} \cong V_{m_i,n_i} \) for the action of \( s \oplus s_0 \). The relationship between (4.1) and this decomposition in (4.2) is the following:

**Proposition 4.1.** Let \( O \) be the orbit through \( e + e_0 \). The equation

\[
\dim(C(s) \cdot e_0) = \text{codim}(O_e)
\]

holds if and only if

\[
(4.3)
\]

\( m_i \geq n_i \) whenever \( m_i > 0 \).

**Proof.** By \( sl_2(C) \)-theory, the sum of the dimensions of the 0-eigenspace and the 1-eigenspace for \( \text{ad}(h + h_0) \) on \( g \) equals the dimension of the centralizer of \( x = e + e_0 \) in \( g \). It therefore follows that

\[
\dim g^x = \sum_{i=1}^{N} (\min(m_i, n_i) + 1).
\]

At the same time

\[
\dim g^x = \sum_{i=1}^{N} (n_i + 1)
\]

since the kernel of \( \text{ad}(e) \) on \( V_{m_i,n_i} \) is isomorphic to \( V(n_i) \). Here, \( V(n_i) \) is an irreducible representation of \( sl_2(C) \) of highest weight \( n_i \), hence of dimension \( n_i + 1 \). Putting the two formulas together, the codimension of \( O_e \) in \( g \) is equal to

\[
\sum_{i=1}^{N} (n_i - \min(m_i, n_i)) - \sum_{i=1}^{N} (\min(m_i, n_i)).
\]

It is also necessary to compute \( \dim c(s)^0 \). Since \( s_0 \subset c(s) \) and \( c(s) \) is exactly \( \text{ker ad e} \cap \text{ker ad} h \), it follows that \( c(s) \) coincides with the sum of all \( V^{(i)}_{m_i,n_i} \) where \( m_i = 0 \). The centralizer \( c(s)^0 \) is then the span of the highest weight vectors of these \( V^{(i)}_{0,n_i} \) and hence its dimension is given by the number of these subrepresentations. That is,

\[
\dim c(s)^0 = \# \{ 1 \leq i \leq N : m_i = 0 \}.
\]

Thus

\[
\dim C(s) \cdot e_0 = \dim c(s) - \dim c(s)^0 = \sum_{m_i \geq 0} (n_i + 1) - \sum_{m_i \geq 0} 1 = \sum_{m_i \geq 0} n_i.
\]

The equality of \( \dim C(s) \cdot e_0 \) and the codimension of \( O_e \) in \( g \) is therefore equivalent to \( \min(m_i, n_i) = n_i \) for all \( i \) with \( m_i \neq 0 \).

It follows from the proof that if \( J = \{ i \mid n_i > m_i > 0 \} \), then

\[
(4.4)
\]

\[
\dim S_{O_e} - \dim C(s) \cdot e_0 = \sum_{i \in J} n_i - m_i.
\]

The element \( e_0 \in g \) is called height 2 if all the eigenvalues of \( \text{ad} h_0 \) on \( g \) are at most 2, and \( e \) is called even if all the eigenvalues of \( \text{ad} h \) on \( g \) are even.
Corollary 4.2. Suppose that either (1) $e_0$ belongs to the minimal nilpotent orbit in $\mathfrak{g}$, or (2) $e_0$ is of height 2 in $\mathfrak{g}$ and $e$ is even. Then

$$\dim(C(\mathfrak{s}) \cdot e_0) = \text{codim}(\mathcal{O}_e).$$

Proof. If $e_0$ belongs to the minimal nilpotent orbit of $\mathfrak{g}$, then $e_0$ is of height two and the 2-eigenspace of $ad h_0$ is spanned by $e_0$. This is the case since $h_0$ is conjugate to the coroot of the highest root. But since $z_0 \subset C(s)$, it follows that $z_0 \cong V_{n,2}$ is the unique subrepresentation of $\mathfrak{g}$ isomorphic to $V_{m,n}$ with $n \geq 2$. Therefore all other $V_{m,n}^{\text{ad}}$ must have $n_1 = 0$ or $n_1 = 1$ and so condition (4.3) holds.

Next assume the second hypothesis. Since $e$ is even, all $V_{m,n}^{\text{ad}}$ with $m_1 > 0$ satisfy $m_1 \geq 2$. Since $e_0$ is of height two, $n_1 \leq 2$ and thus condition (1.3) is true and hence also (1.1). □

4.3. Calculations in the proof of Proposition 3.1. First, we consider the cases where $e_0$ is a minimal nilpotent element in $\mathfrak{g}$ and compute $e + e_0$. Then Corollary 4.2 ensures that (4.2) holds. The degeneration $(\mathcal{O}_{e + e_0}, \mathcal{O}_e)$ turns out always to be a minimal degeneration so the explanation in the proof of Proposition 4.1 shows that $\mathcal{O}_{e} \cong C(\mathfrak{s}) \cdot e_0$ for $\mathcal{O} = O_{e + e_0}$. The results are recorded in Tables 1, 3, 6, and 9 for each of the exceptional groups $F_4$, $E_6$, $E_7$, and $E_8$, respectively. We also consider all other cases where $e_0$ is a minimal nilpotent element in $\mathfrak{g}(s)$ and check whether or not (4.2) holds for $e + e_0$. In the cases when it does hold, the degeneration $(\mathcal{O}_{e + e_0}, \mathcal{O}_e)$ turns out again always to be a minimal degeneration and thus $\mathcal{O}_{e} \cong C(\mathfrak{s}) \cdot e_0$. The results are recorded in the first lines of Tables 2, 4, 7, and 10. These two sets of calculations include all the minimal degenerations covered by Proposition 3.1. They also include those cases in Proposition 3.3 where $|\Gamma| = 2$ and $x^\Gamma = 0$.

4.4. Establishing Lemma 2.4. There are some remaining examples where $e_0$ is minimal nonzero nilpotent in $\mathfrak{g}(s)$, but (1.1) does not hold for $e + e_0$. Many of these lead to a situation where the more general Lemma 2.4 holds for some $x' \neq 0$. Such examples suffice to handle the remaining cases in the proof of Proposition 3.3 and the two cases covered by Proposition 3.4.

Let $e_0 \in \mathfrak{g}(s)$ be nilpotent, but not necessarily minimal nilpotent, and suppose that the dimension condition (1.1) does not hold for $e + e_0$. Sometimes it may happen that Lemma 2.4 holds for a nilpotent orbit $\mathcal{O}$ with $\mathcal{O}_e \subset \mathcal{O} \subset \mathcal{O}_{e + e_0}$. That is, it may be possible to locate $x' \in \oplus_{i \leq -1} g'^{(i)}$ with

$$x = e + e_0 + x' \in \mathcal{O}$$

and so that the dimension condition

$$\dim C(\mathfrak{s}) \cdot (e_0 + x') = \dim \mathcal{O}_{e_0}$$

holds. We now discuss how such an $x \in \mathcal{O}_{e_0}$ might be located.

4.4.1. A smaller slice result. Let $y = e + e_0$, which is nilpotent with corresponding semisimple element $h_y = h + h_0$. Write $g_j$ for the $j$-eigenspace of $ad h_0$ on $\mathfrak{g}$. The centralizer $\mathcal{G} := G^{(y)}$ has Lie algebra $\mathfrak{g}_0$ and $\mathcal{G}$ acts on each $g_j$. Then $y \in g_2$ and the $\mathcal{G}$-orbit through $y$ is the unique dense orbit. Now $e \in g_2$ since

$$[h_y, e] = [h + h_0, e] = 2e + 0 = 2e.$$

We want to find a transverse slice for the $\mathcal{G}$-action on $g_2$ to the $\mathcal{G}$-orbit through $e$. In fact, since $g_2$ is a direct sum of $ad h$-eigenspaces, the decomposition $\mathfrak{g} = \text{Im} ad e \oplus \ker ad f$ restricts to a decomposition

$$g_2 = [e, \mathfrak{g}_0] \oplus (g_2 \cap \ker ad f).$$

Therefore, setting $\mathcal{S}^{(2)} = e + (g_2 \cap \ker ad f)$, it follows that the affine space $\mathcal{S}^{(2)}$ is a transverse slice of $g_2$ at $e$ with respect to the $\mathcal{G}$-action. Consequently, every $\mathcal{G}$-orbit in $g_2$ containing $e$ in its closure meets $\mathcal{S}^{(2)}$.

Let $\mathfrak{g}(r, s)$ denote the subspace of $\mathfrak{g}$ where $ad h$ has eigenvalue $r$ and $ad h_0$ has eigenvalue $s$. Define $g^{r}(r, s) = \mathfrak{g}(r, s) \cap \ker ad f$. Then

$$g_2 \cap \ker ad f = \bigoplus_{r > 0} g^{r}(-r, r + 2).$$

Next, we relate this decomposition to the decomposition (1.2) of $\mathfrak{g}$ under $s \oplus \mathfrak{g}_0$. Let $\mathcal{E} = \{ i \mid n_i > m_i > 0 \text{ and } n_i - m_i \text{ even} \}$
where \((m_i, n_i)\) are defined in \([4.2]\). Then \(\mathcal{E} \subset \mathcal{J}\) and \(\mathcal{E}\) is empty if Proposition \([4.3]\) is true. For each \(i \in \mathcal{E}\), let \(v_i\) be a nonzero vector in the one-dimensional space \(V^{(i)}_{m_i, n_i} \cap g(-m_i, m_i + 2)\). Then \(v_i\) is a lowest weight vector for \(s\), but not in general a highest weight vector for \(s_0\). The set \(\{v_i \mid i \in \mathcal{E}\}\) is then a basis for \[\bigoplus_{r \geq 1} g'(-r, r + 2)\]

since each vector in \(g'(-r, r + 2)\) lies in some subrepresentation of type \(V_{r, s}\) with \(r + 2 \leq s\) and \(s - r\) even. The subspace \(g'(0, 2)\) is just the 2-eigenspace of \(ad h_0\) in \(\mathfrak{e}(s)\), which coincides with \(\mathfrak{e}(s) \cap g(0, 2)\). It contains \(e_0\). A consequence of the above observations is the following

Lemma 4.3. Let \(x \in \mathfrak{g}_2\). If \(e \in \mathfrak{g} \cdot x\), then some \(\mathfrak{g}\)-conjugate of \(x\) can be expressed as
\[e + aw + \sum_{i \in \mathcal{E}} b_i v_i\]
where \(w \in \mathfrak{e}(s) \cap g(0, 2)\) and \(a, b_i \in \mathbb{C}\).

The cases in which we are interested are those where \(\dim(C(s) \cdot e_0) = \dim S_{\mathcal{E}, x}\). In all these cases, it turns out that \([4.7]\) holds with \(a \neq 0\), \(w = e_0\), and \([e_0, v_i] = 0\) for all \(i \in \mathcal{E}\) with \(b_i \neq 0\). In particular \([e_0, x] = 0\). Hence the \(v_i\)'s turn out to be highest weight vectors for \(s_0\) in \(V^{(i)}_{m_i, n_i}\), which forces \(n_i = m_i + 2\). When \(e_0 \in \mathfrak{e}(s)\) is minimal the fact that \(w = e_0\) is clear since then \(\mathfrak{e}(s) \cap g(0, 2)\) is spanned by \(e_0\). But in general we do not have a conceptual proof of why \(a \neq 0\), \(w = e_0\), or \([e_0, x] = 0\). However, if we assume that \(w = e_0\), \(a \neq 0\), and \(\dim(C(s) \cdot e_0) = \dim S_{\mathcal{E}, x}\), then the following result implies that \([e_0, x] = 0\).

Lemma 4.4. Let \(x \in \mathfrak{g}_2\) with \(e \in \mathfrak{g} \cdot x\) satisfying \([4.7]\) with \(w = e_0\) and \(a \neq 0\). If \(\dim(C(s) \cdot e_0) = \dim S_{\mathcal{E}, x}\), then \([e_0, x] = 0\). Equivalently, for those \(i \in \mathcal{E}\) with \(b_i \neq 0\), we have \(n_i = m_i + 2\).

Proof. By assumption, after rescaling, we may write \(x = e + e_0 + \sum v_i\). Consider the natural map \(C(s)\)-equivariant map \(\pi : S_{\mathcal{E}, x} \to \mathfrak{e}(s)\). Then \(\pi\) is surjective onto \(C(s) \cdot e_0\). The dimension assumption and the \(C(s)\)-equivariance of \(\pi\) implies that \(\pi\) is finite over each point in the dense orbit \(C(s) \cdot e_0\). Now if \(g\) is in the unipotent subgroup \(U \subset C(s)\) corresponding to the line through \(e_0\), then \(g \cdot x = e + e_0 + \sum g \cdot v_i\). Thus \(U \cdot x\) lies in the fiber of \(\pi\) over \(e_0\). Therefore \(U \cdot x\) is finite and hence a single point since \(U\) is irreducible. It follows that \(U \cdot v_i = v_i\) and hence \([e_0, v_i] = 0\) for all \(i\) with \(b_i \neq 0\) and so \([e_0, x] = 0\).

4.4.2. Applying Lemma \([4.2]\). In order to apply Lemma \([4.3]\) for some \(x \in \mathfrak{g}\) with \(\mathfrak{g}_x \subset \mathfrak{g}_y \subset \mathfrak{g}_y\), we need to check two things, after possibly replacing \(x\) by a conjugate:

\begin{enumerate}
\item \(x \in \mathfrak{g}_2\)
\item \(e \in \mathfrak{g} \cdot x\)
\end{enumerate}

The first condition can often be shown as follows. Let \(s_x\) be an \(s_x\)-subalgebra through some conjugate of \(x\) with standard semisimple element \(h^x \in \mathfrak{h}\). In all cases we are interested in, there exists nilpotent \(c^x_0 \in \mathfrak{e}(s_x)\) with semisimple element \(h^x_0 \in \mathfrak{h}\), such that \(h^x + h^x_0 = h^x\), after possibly replacing \(x\) again by a conjugate. Then just as in \([4.6]\), \(x \in \mathfrak{g}_2\) and the first condition holds.

We may further assume that \(h_y\) is dominant with respect to the Borel subalgebra \(\mathfrak{b} \subset \mathfrak{g}\) and \(h^x\) is dominant for the corresponding Borel subalgebra \(\mathfrak{b}_y\) of \(g^y\). Then since \([h^x, x] = 2x\) and \([h^y, x] = 2x\), if follows that \(x\) belongs to
\[I_x := \mathfrak{g}_2 \cap \bigcup_{i \geq 2} g(h^x + i),\]
where \(g(h^x + i)\) are the eigenspaces for \(ad h^x\). This subspace is preserved by the action of \(\mathfrak{b}_y\). Thus \(\mathfrak{g} \cdot I_x = \mathfrak{g} \cdot x\). We can carry out a similar process for \(e\) and obtain a subspace \(I_e \subset \mathfrak{g}_2\), with \(\mathfrak{g} \cdot I_e = \mathfrak{g} \cdot e\). Then if \(I_e \subset I_x\), it necessarily follows that \(\mathfrak{g} \cdot I_e \subset \mathfrak{g} \cdot I_x\)

and the second condition holds. For the cases we are interested in, this approach will suffice to check the condition in the Lemma.
4.4.3. An example in \( C_2 \). Let \( g = sp(6) \). Nilpotent orbits in \( g \) can also be parametrized by the Jordan partition for any element in the orbit, viewed as a \( 6 \times 6 \)-matrix. Pick \( e \in g \) with partition \([2,2,2]\) (so \( e \) is of type \( A_1 + \tilde{A}_1 \)). Then \( \mathfrak{e}(e) \) is isomorphic to \( sL_2(\mathbb{C}) \cong sL_2(3) \). Let \( e_0 \in \mathfrak{e}(e) \) be a nonzero nilpotent element. Then viewed as an element in \( g \), \( e_0 \) has partition \([3,3] \). We complete \( e \) and \( e_0 \) to commuting \( sL_2(\mathbb{C}) \)-subalgebras \( s_1 \) and \( s_2 \), respectively, with standard semisimple elements \( h \) and \( h_0 \), respectively. Let \( y = e + e_0 \). Then the eigenvalues of \( h_y := h + h_0 \) acting on the natural six-dimensional representation of \( g \) are \( \{3,1,1,-1,-1,-3\} \) and \( y \) has partition \([4,2] \), which corresponds to the subregular nilpotent orbit in \( g \). Moreover, the decomposition of \( g \) under \( s \oplus s_0 \) in \([4,4,2]\) is \( g \cong V(0,2) \oplus V(2,0) \oplus V(2,4) \). Therefore \((m,n) := (m_0,n_0) = (2,4) \) is a 2-dimensional subspace of \( g_2 \). Similarly we can pick \( h^* \) and \( h^*_0 \) to have weighted diagrams

\[
\begin{array}{ccc}
0 & 2 & 0 \\
2 & 2 & 2
\end{array}
\]

respectively. Then \( h_y = h^* + h^*_0 \) and thus without loss of generality we can assume \( x \in \mathfrak{g}_2 \). Moreover, \( I_x \) is a 3-dimensional subspace of \( \mathfrak{g}_2 \) containing \( I \) and therefore the two conditions above hold. Hence by Lemma \([4.3]\), \( x = e + ae_0 + bv \), after perhaps replacing \( x \) by a conjugate.

Now \( b \neq 0 \) since otherwise \( x \) would be conjugate to \( e + e_0 \), which is not the case. We also want to eliminate the case where \( a = 0 \). Since \([e_0,v] = 0 \), the stabilizer of \( x \) in \( \mathfrak{e}(e) \) is the line through \( e_0 \) and so \( \dim C(e) \cdot x = 2 \). Also \( C(e) \) is normal, hence unbranched at \( e \). Thus by Lemma \([2.4]\), \( S_{C,e} = C(e) \cdot x \). Therefore if \( a = 0 \), \( S_{C,e} \) would be isomorphic to the closure of the \( SL_2(\mathbb{C}) \)-orbit in \( V(4) \) through the highest weight. But the latter variety is normal, but not Gorenstein (cf. \([FZ03\), Theorem 2.19]). Since \( S_{C,e} \) is normal and Gorenstein \([3.2]\), we deduce that \( a \neq 0 \). Now it is easy to see that \( S_{C,e} \cong C(e) \cdot e_0 \), the nilpotent cone in \( sL_2(\mathbb{C}) \) (or see Lemma \([4.3]\)). Thus the singularity \((C,e)\) is an \( A_1 \)-singularity, as was already known from \([KPS82] \). This is an example where Proposition \([3.3]\) holds with \( x^* \neq 0 \). The result also holds for the orbit \([4,1,1]\) by the same kind of argument.

4.4.4. Example: \((\tilde{A}_1,A_1)\) in type \( G_2 \). Let \( g \) be of type \( G_2 \) and let \( e \) be a minimal nilpotent element. Then \( \mathfrak{e}(e) \cong sL_2(3) \). Let \( e_0 \) be a minimal nilpotent element in \( \mathfrak{e}(e) \). Then \( e_0 \) has type \( \tilde{A}_1 \) in \( g \) and \( y = e + e_0 \) has type \( G_2(a_1) \). The decomposition of \( g \) in \([4.2]\) is \( V(0,2) \oplus V(2,0) \oplus V(1,3) \). Therefore \((m,n) := (m_0,n_0) = (1,3) \) for the unique \( i \in e \) and \((m_1,n_1) = (1,3) \) does not hold for \( e + e_0 \). Fix nonzero \( v = v_1 \in V(1,3) \) satisfying \([e_0,v] = 0 \) and \([f,v] = 0 \). Between \( \overline{\mathfrak{g}}_x \) and \( \overline{\mathfrak{g}}_e \) there is a unique \( G \)-orbit \( \mathfrak{g} \), the one of type \( \tilde{A}_1 \). We will show how that there exits \( x \in \mathfrak{g} \) satisfying \( x = e + ae_0 + bv \).

Choose \( h_y \) so that its weighted diagram is the usual weighted Dynkin diagram \( \begin{array}{ccc}0 & 2 & 0 \end{array} \) and \( h^* \) and \( h^*_0 \) to have weighted diagrams \( \begin{array}{ccc}0 & 1 & 0 \end{array} \) and \( \begin{array}{ccc}2 & -1 & 0 \end{array} \), respectively. Then \( h_y = h^* + h^*_0 \) and thus we may assume \( x \in g_2 \). Similarly, let \( h \) and \( h_0 \) have weighted diagrams \( \begin{array}{ccc}1 & 0 & 0 \end{array} \) and \( \begin{array}{ccc}2 & -3 & 0 \end{array} \), respectively. Then \( I_x \) is one-dimensional and \( I_x \subset I_x \). The two conditions above are met, so indeed there exists \( x \in \mathfrak{g} \) with \( x = e + ae_0 + bv \) by Lemma \([4.3]\).

Now \( b \neq 0 \) since \( x \) and \( e + e_0 \) are not in the same \( G \)-orbit. If \( a = 0 \), then since \( \overline{\mathfrak{g}} \) is unibranch at \( e \), we would have that \( S_{C,e} = C(e) \cdot x \) is isomorphic to the closure of \( SL_2(\mathbb{C}) \)-orbit of a highest weight vector in \( V(3) \). As in the previous example, the latter is normal, but not Gorenstein, a contradiction of \([4.2]\). Since \( e_0 \in V(2) \) and \( v \in V(3) \) are highest weight vectors for \( SL_2(\mathbb{C}) \) (relative to \( e_0 \)), and both \( a \) and \( b \) are nonzero, \( S_{C,e} \cong m \), by the definition of \( m \).
4.5. A lemma for Propositions 3.3 and 3.4. Recall that $V(i)$ denotes an irreducible module for $SL_2$ with highest weight $i \geq 0$. Consider the $SL_2$-representation
\[ V_T = V(i_1) \oplus V(i_2) \oplus \ldots \oplus V(i_r) \]
where $\mathcal{I} := \{i_1, i_2, \ldots, i_r\}$. Let $v_j$ be a highest weight vector in $V(i_j)$.

Define $X_\mathcal{I}$ to be the closure of the orbit
\[ SL_2 \cdot (v_1 + v_2 + \ldots + v_r) \]
in $V_\mathcal{I}$. If any of the natural numbers in $\mathcal{I}$ are repeated, then it is possible to pass to a subrepresentation $V_\mathcal{J}$ of $V_\mathcal{I}$ where $\mathcal{J}$ contains one copy of each distinct number in $\mathcal{I}$. It is clear that $X_\mathcal{J} \cong X_\mathcal{I}$. Therefore it is enough to consider the case of distinct $i_j$’s.

Let $\mathcal{I}_{\min}$ be the minimal generating set of the monoid in $\mathbb{N}$ generated by $i_1, \ldots, i_r$.

**Lemma 4.5.** Let $\pi$ be the $SL_2$-linear projection of $V_\mathcal{I}$ onto $V_{\mathcal{I}_{\min}}$.

a) Then $\pi$ restricts to an $SL_2$-equivariant isomorphism of $X_\mathcal{I}$ onto $X_{\mathcal{I}_{\min}}$.

b) Let $d$ be the greatest common divisor of $i_1, \ldots, i_r$. Then $X_{\langle d \rangle}$ is the normalization of $X_\mathcal{I}$.

**Proof.** Since the elements of $\mathcal{I}$ are distinct, $V_\mathcal{I}$ is isomorphic to a subrepresentation of the symmetric algebra $S^r \mathbb{C}^2$ on $V(1) \cong \mathbb{C}^2$. After fixing $f \in \mathbb{C}^2$, we may identify $v_j \in V(i_j)$ with $f^{i_j} \in S^r \mathbb{C}^2$. Then $SL_2 \cdot (v_1 + v_2 + \ldots + v_r)$ corresponds to the set of elements $\sum_{i \in \mathcal{I}} L^i$ where $L \in \mathbb{C}^2$ is nonzero. It follows that $X_\mathcal{I}$ identifies equivariantly with $\sum_{i \in \mathcal{I}} L^i$ for $L \in \mathbb{C}^2$. It follows that the restriction of $\pi$, which maps $X_\mathcal{I}$ to $X_{\mathcal{I}_{\min}}$, corresponds to the map
\[ \sum_{i \in \mathcal{I}} L^i \to \sum_{i \in \mathcal{I}_{\min}} L^i. \]
This is an isomorphism since $L^i$ for $i \notin \mathcal{I}_{\min}$ depends polynomially on those $L^j$ for $j \in \mathcal{I}_{\min}$.

For part (b) the map $L^d \to \sum_{i \in \mathcal{I}} (L^i)^{i/d}$ from $X_{\langle d \rangle}$ onto $X_\mathcal{I}$ is regular and clearly surjective. It is also injective since $d$ is the greatest common divisor of $i_1, \ldots, i_r$. Thus the map restricts to an isomorphism between the open orbits, hence is birational. Since $X_{\langle d \rangle}$ is normal (being a quotient of $\mathbb{C}^2$ by a cyclic group), Zariski’s main theorem implies that $X_{\langle d \rangle}$ is the normalization of $X_\mathcal{I}$. \(\square\)

An analogous result holds for any simple $G$. Let $V(\mu)$ denote the irreducible highest weight module for $G$ with weight $\mu$. Let $\lambda$ be a dominant weight for $G$ relative to a Borel subgroup $B$ and consider $V_\mathcal{I} = V(\nu_1) \oplus V(\nu_2) \oplus \ldots \oplus V(\nu_r)$. Define $v_j$ and $X_\mathcal{I}$ in analogy with the $SL_2$-case and the conclusions of the lemma go through for this situation. The $X_\mathcal{I}$, when normal, are spherical varieties for the Borel subgroup opposite to $B$. Hence Lemma 1.17 and its generalization to simple $G$ follow from known results about spherical varieties (see [Brill, Thm 1.4]).

In all the cases where we apply Lemma 4.5 or its generalization, it turns out that $i_1 = 2$ and either the $i_j$’s are all even or $i_2 = 3$. In the former case when $G = SL_2(\mathbb{C})$, the lemma implies that $X_\mathcal{I}$ identifies with the minimal nilpotent orbit closure in $sl_2(\mathbb{C})$, hence is isomorphic to the $A_1$ singularity. In the latter case for $G = SL_2(\mathbb{C})$ or $Sp_4(\mathbb{C})$, the lemma implies that $X_\mathcal{I}$ is not normal, $X_\mathcal{I}$ identifies with either $m$ or $m'$, respectively, and $X_\mathcal{I}$ has normalization $\mathbb{C}^2$ or $\mathbb{C}^4$, respectively.

4.6. Finding $v_i$ for $i \in \mathcal{I}$. We sometimes need to do explicit computations to verify (4.7) or to show that $w = e_0$ and $a \neq 0$, especially for degenerations which are not minimal (e.g., starting with $\mathcal{I}_7$). In these cases there arises the need for an analog of Lemma 2.4. Here we describe a way to find $v_i$ for $i \in \mathcal{I}$ which frequently leads to an isomorphism of $\mathcal{Z} = C(s) \cdot (e_0 + x')$ with $C(s) \cdot e_0$ in Lemma 2.4, when such an isomorphism exists.

Write $g(h; j)$ for the $j$-eigenspace of $ad h$. Since $c(s) \subset g^h = g(h; 1)$, the $g(h; j)$ are $c(s)$-modules. The $g^h \oplus g(h; 1)$ isomorphic as $c(s)$-modules to $g^f$. In particular for $j \geq 0$,
\[ g^f(-2j) \cong (ad f)^j(g^h) \cap g^f \text{ and } g^f(-2j - 1) \cong (ad f)^j(g(h; -1)) \cap g^f. \]
as $c(s)$-modules.
Take, as usual, $e_0 \in c(\mathfrak{s})$ and $y = e + e_0$. If $\mathcal{E} \neq \emptyset$, then the following method suffices in most cases to locate $v_i \in \mathcal{E}$ when $m_i$ is even. Suppose that $\mathfrak{g}^h$ contains a simple factor of type $\mathfrak{sl}_N$. Then setting $X = e_0$, we may consider the powers $X^r \subset \mathfrak{sl}_N \subset \mathfrak{g}^h$. Of course $[X, X^r] = 0$ in $\mathfrak{sl}_N \subset \mathfrak{g}^h$ and hence in $\mathfrak{g}$, which means that $X^r$, if nonzero, is a highest weight vector for $\mathfrak{s}_0$ relative to $e_0$. Moreover in $\mathfrak{sl}_N$ the identity $[h_0, X^r] = 2rX^r$ holds because $[h_0, X] = h_0X - Xh_0 = 2X$; hence this also holds in $\mathfrak{g}$. Assume $X^r$ is not zero. Then for some largest $j$,

$$X^r_j := (ad f)^jX^r \in \mathfrak{g}^f(-2j)$$

is nonzero. Since $x$ and $\mathfrak{s}_0$ commute, $X^r_j$ also commutes with $e_0$, so in fact $X^r_j$ is a nonzero element of $\mathfrak{g}(-2j; 2r) \cap \mathfrak{g}^f \cap \mathfrak{g}^{r_0}$.

Now let $(m_i, n_i)$ for $i \in \mathcal{E}$ with $m_i$ even. In the cases of interest $n_i = m_i + 2$, as suggested by Lemma 4.4, and we we often find that

$$v_i = (ad f)^{m_i}(X^{m_i+1})$$

Moreover, if $x$ in (4.7) is a linear combination of these $v_i$‘s and $e_0$, then it follows that $\mathfrak{C}(\mathfrak{s}) \cdot x \cong \mathfrak{C}(\mathfrak{s}) \cdot e_0$ via the natural projection (see §3.2) since $\mathfrak{C}(\mathfrak{s})$-commutes with taking powers. This phenomenon appears more generally when $e_0$ is regular nilpotent in $\mathfrak{g}^h$ (or some simple factor of $\mathfrak{g}^h$); then any element in $\mathfrak{g}^h \cap \mathfrak{g}^{r_0}$ is polynomial function on $\mathfrak{g}$ evaluated at $e_0$.

4.6.1. Example in $\mathcal{C}_3$ (continued). Continuing with Example 4.4.3 to find an explicit $v$, let $X = e_0$. Here $\mathfrak{g}^h$ has semisimple part $\mathfrak{s}_0(\mathbb{C})$. Then if $X = e_0$, $X^2 \neq 0$ since $X$ is regular in $\mathfrak{s}_0(\mathbb{C})$ and $X^2 \notin \mathfrak{s}_0$. Since $[h, X^2] = 4X^2$, it must be that $X^3 \in V(2, 4)$ and so $(ad f)(X^2)$ is in the span of $v$. Consequently, $\mathcal{S}_{0, v}$ identifies with the closure of the $\mathfrak{C}(\mathfrak{s})$-orbit of $(X, X^2) \in \mathfrak{C}(\mathfrak{s}) \oplus V(4) = \mathfrak{s}_0(\mathbb{C})$ where $X \in \mathfrak{C}(\mathfrak{s})$ is nilpotent. The latter is isomorphic, as noted before, to the nilpotent cone in $\mathfrak{s}_0(\mathbb{C})$.

4.7. Calculations in the proof of Propositions 3.3 and 3.4. We pick up where §4.3 left off. That is, $e_0$ is a minimal nilpotent element in $c(\mathfrak{s})$ such that the dimension condition (1.4) does not hold for $e + e_0$. For such $e$ and $e + e_0$, we look for nilpotent orbits $O$ with $\mathfrak{O}_o \subset \overline{O} \subset \overline{\mathfrak{C}}^{e+e_0}$ such that $\mathfrak{O}_o$ is a minimal degeneration of $O$ and $\dim(C(\mathfrak{s}) \cdot e_0) = \codim(\mathfrak{C}(\mathfrak{s}) \cdot e_0)$. Then $\mathfrak{O} = \overline{O}$ and $\mathfrak{O}$ are candidate orbits for Lemma 2.4 to hold. We proceed as in §4.4.1 and §4.4.2, but sometimes have to carry out ad hoc computer calculations. When we find candidates where Lemma 2.4 holds, then Lemma 1.3 and its generalization turn out always to be enough to describe $\mathcal{S}_{0, v}$. Comparing with the surface cases treated in §3, we find that all the cases in Propositions 3.3 and 3.4 are now handled. The results are recorded in the latter lines of Tables 3, 4, 6, 7 and 11 where $\mathcal{E} \neq \emptyset$. As noted in Remark 3.7 there are some $e_0$, minimal in $c(\mathfrak{s})$, that do not contribute to describing any minimal degeneration.

5. Geometric method for surface singularities

In this section we consider a minimal degeneration $\mathfrak{C}'$ of $\mathfrak{O}$ such that $\mathfrak{C}'$ is of codimension 2 in $\overline{\mathfrak{O}}$. Let $e \in \mathfrak{C}'$. We show that the normalization of each irreducible component of $\mathcal{S}_{0, e}$ is isomorphic to $\mathfrak{C}'/\Gamma$ for some finite subgroup $\Gamma \subset \text{SL}_2(\mathbb{C})$. Our method allows us to determine the group $\Gamma$, hence we determine $\mathcal{S}_{0, e}$ up to normalization. As mentioned in §3.2 we can often use results on normality of nilpotent orbit closures or other methods (e.g. Lemma 2.2) to decide whether the irreducible components of $\mathcal{S}_{0, e}$ are normal. Sometimes we have to state our results up to normalization.

5.1. Two-dimensional Slodowy transverse slices. Recall that a contracting $\mathfrak{C}'$-action on a variety $\mathfrak{X}$ is a $\mathfrak{C}'$-action on $\mathfrak{X}$ with a unique fixed point $o \in \mathfrak{X}$ such that for any $x \in \mathfrak{X}$, we have $\lim_{t \to o} \lambda t x = o$. Recall from [Bea00] that a symplectic variety is a normal variety $\mathfrak{W}$ with a holomorphic symplectic form $\omega$ on its smooth locus such that for any resolution $\pi : Z \to \mathfrak{W}$, the pull-back $\pi^* \omega$ extends to a regular 2-form on $Z$. For a nilpotent orbit, we write $\overline{O}$ for the normalization of $\overline{\mathfrak{O}}$.

Lemma 5.1. The normalization $\overline{S}_{0, e}$ of $\mathcal{S}_{0, e}$ is an affine normal variety with each irreducible component having at most an isolated symplectic singularity and endowed with a contracting $\mathfrak{C}'$-action.
Proof. As $\mathcal{O}$ has rational Gorenstein singularities by [Hin91] and [Pan91], $\mathcal{S}_{\mathcal{O},e}$ has only rational Gorenstein singularities. On the other hand, there exists a symplectic form on its smooth locus, hence $\mathcal{S}_{\mathcal{O},e}$ has only symplectic singularities by [Nam01] (Theorem 6). By construction, the contracting $C^*$-action on $\mathcal{S}_{\mathcal{O},e}$ in (2.2) has positive weights, hence it lifts to a contracting $C^*$-action on $\mathcal{S}_{\mathcal{O},e}$. □

The two-dimensional symplectic singularities are exactly rational double points (cf. [Bea00, Section 2.1]). The following is immediate from [FZ03, Lemma 2.6].

**Proposition 5.2.** Let $X$ be an affine irreducible surface with an isolated rational double point at $o$. If there exists a contracting $C^*$-action on $X$, then $X$ is isomorphic to $C^2/\Gamma$ for some finite subgroup $\Gamma \subset SL_2(\mathbb{C})$.

Note that by Proposition 2.1, the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are mutually isomorphic. As an immediate corollary, we get

**Corollary 5.3.** Let $\mathcal{S}_{\mathcal{O},e}$ be a two-dimensional nilpotent Slodowy slice. Then there exists a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that each irreducible component of the normalization $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to $C^2/\Gamma$.

Hence to determine $\mathcal{S}_{\mathcal{O},e}$, we only need to determine the subgroup $\Gamma$. In the following, we shall describe a way to construct the minimal resolution of $\mathcal{S}_{\mathcal{O},e}$. Then the configuration of exceptional $\mathbb{P}^1$’s in the minimal resolution will determine $\Gamma$.

### 5.2. $\mathbb{Q}$-factorial terminalization for nilpotent orbit closures.

A general reference for minimal model program in algebraic geometry is [Mat02]. Here we recall some basic definitions.

Let $X$ be a normal variety. A Weil divisor $D$ on $X$ is called $\mathbb{Q}$-Cartier if $ND$ is a Cartier divisor for some non-zero integer $N$. We say that $X$ is $\mathbb{Q}$-Gorenstein if its canonical divisor $K_X$ is $\mathbb{Q}$-Cartier. The variety $X$ is called $\mathbb{Q}$-factorial if every Weil divisor on $X$ is $\mathbb{Q}$-Cartier. A $\mathbb{Q}$-Gorenstein variety $X$ is said to have terminal singularities if there exists a resolution $\pi : Z \to X$ such that $K_Z = \pi^* K_X + \sum a_i E_i$ with $a_i > 0$ for all $i$, where $E_i, i = 1, \cdots, k$ are the irreducible components of the exceptional divisor of $\pi$. A $\mathbb{Q}$-factorial terminalization of a $\mathbb{Q}$-Gorenstein variety $X$ is a projective birational morphism $\pi : Z \to X$ such that $K_Z = \pi^* K_X$ and $Z$ is $\mathbb{Q}$-factorial with only terminal singularities.

It is well-known that two-dimensional terminal singularities are necessarily smooth (cf. Theorem 4.6-5 [Mat02]), hence a normal variety $X$ with only terminal singularities is smooth in codimension 2, that is, $\text{codim}_{X} \text{Sing}(X) \geq 3$.

For the normalization of the closure of a nilpotent orbit, one way to obtain its $\mathbb{Q}$-factorial terminalization is by the following method. Consider a parabolic subgroup $Q$ in $G$. Let $L$ be a Levi subgroup of $Q$. For a nilpotent element $x \in \text{Lie}(L)$, we denote by $\mathcal{O}_x^L$ its orbit under $L$ in $\text{Lie}(L)$. Let $n(q)$ be the nilradical of $\text{Lie}(Q)$. Then the natural map $p : G \times \mathbb{Q}(n(q) + \mathcal{O}_x^L) \to \mathcal{O}_x$ has image equal to $\mathcal{O}$ for some nilpotent orbit $\mathcal{O}$ and $p$ is called a generalized Springer map for $\mathcal{O}$. Then $\mathcal{O}$ is said to be induced from $(L, \mathcal{O}_x^L)$ [LS79]. When $t = 0$, then $\mathcal{O}$ is called the Richardson orbit for $Q$ and $G \times \mathbb{Q} n(q)$ identifies with the cotangent bundle $\mathcal{T}^*(G/Q)$. If $p$ is birational and the normalization of $\mathcal{O}$ is $\mathbb{Q}$-factorial terminal, then the normalization of $p$ gives a $\mathbb{Q}$-factorial terminalization of $\mathcal{O}$, the normalization of $\mathcal{O}$. In [Ful10], it was proved in confirming a conjecture of Namikawa that for a nilpotent orbit $\mathcal{O}$ in an exceptional Lie algebra, either $\mathcal{O}$ is $\mathbb{Q}$-factorial terminal or every $\mathbb{Q}$-factorial terminalization of $\mathcal{O}$ is given by a generalized Springer map.

### 5.3. Minimal resolutions of two-dimensional nilpotent Slodowy slices.

We now use the generalized Springer maps to construct a minimal resolution of $\mathcal{S}_{\mathcal{O},e}$ when $\mathcal{S}_{\mathcal{O},e}$ is two-dimensional.

Recall from [Ful10] that in a simple Lie algebra of exceptional type, $\mathcal{O}$ has only terminal singularities if and only if $\mathcal{O}$ is either a rigid orbit or it belongs to the following list: $2A_1, A_2 + A_1, A_2 + 2A_1$ in $E_6$; $A_2 + A_1, A_1 + A_2, A_1 + 2A_1$ in $E_7$; $A_4 + A_1, A_4 + 2A_1, 2A_1$ in $E_8$.

First consider the case where $\mathcal{O}$ has only terminal singularities. Then $\mathcal{O}$ is smooth in codimension two by the previous subsection. This implies that the singularities of $\mathcal{O}$ along $\mathcal{O}$ are smoothable by
its normalization. In other words, $\mathcal{S}_{O,e}$ is smooth, which is then isomorphic to $\mathbb{C}^2$ by Proposition 5.2 and we are done.

**Example 5.4.** Consider again the minimal degeneration $(\tilde{A}_1, A_1)$ in $G_2$ from §4.4.4. As $O = O_{\tilde{A}_1}$ is a rigid orbit, its normalization has $\mathbb{Q}$-factorial terminal singularities by [Fu10]. In particular, the singular locus of $\mathcal{O}$ has codimension at least 4. Since the orbit $A_1$ is codimension two in $\mathcal{O}$, this implies that $\mathcal{O}$ is non-normal and $\mathcal{S}_{O,e} \cong \mathbb{C}^2$ for $e \in \text{Sing}(\mathcal{O})$, which is consistent with the description $\mathcal{S}_{O,e} \cong \mathbb{C}^2$ in §4.4.

Next, assume that the normalization $\mathcal{O}$ is not terminal. Then by [BM83], $O$ is an induced orbit and $\mathcal{O}$ admits a $\mathbb{Q}$-factorial terminalization $\pi : Z \to \mathcal{O}$ given by the normalization of a generalized Springer map. We denote by $U$ the open subset $\mathcal{O} \cup \mathcal{O}_e$ of $\mathcal{O}$ and $\nu : \mathcal{O} \to U$ the normalization map. As $Z$ has only terminal singularities, it is smooth in codimension two. As $\pi$ is $G$-equivariant and $\mathcal{O}_e \subset \mathcal{O}$ is of codimension two, we get that $\pi(\text{Sing}(Z)) \cap \nu^{-1}(\mathcal{O}_e) = \emptyset$. We deduce that $V := \nu^{-1}(U)$ is smooth. In particular, we obtain a symplectic resolution $\pi|_V : V \to \mathcal{O}$. By restriction, we get a resolution $\pi : \pi^{-1}(\mathcal{O}_e) \to \mathcal{S}_{O,e}$, which is a symplectic, hence minimal, resolution.

Let $y \in \nu^{-1}(e)$. If we know: (1) the number of $\mathbb{P}^1$'s in $\pi^{-1}(y)$ and in $\pi^{-1}(\nu^{-1}(e))$; and (2) the action of $A(e)$ on the $\mathbb{P}^1$'s in $\pi^{-1}(\nu^{-1}(e))$, then in most cases we can determine the configuration of $\mathbb{P}^1$'s in $\pi^{-1}(\nu^{-1}(e))$, and hence in $\pi^{-1}(y)$, and therefore determine $\mathcal{S}_{O,e}$. We next introduce some methods to compute this information.

### 5.4. The method of Borho-MacPherson.

Let $W$ be the Weyl group of $G$. The Springer correspondence assigns to any irreducible $W$-module a unique pair $(\mathcal{O}, \phi)$ consisting of a nilpotent orbit $O$ in $\mathfrak{g}$ and an irreducible representation $\phi$ of the component group $A(x)$ where $x \in \mathcal{O}$. The corresponding irreducible $W$-module will be denoted by $\rho_{\mathcal{O}, \phi}$.

Let $W(L)$ denote the Weyl group of $L$, viewed as a subgroup of $W$. Let $B_e$ denote the Springer fiber over $x$ for the resolution of the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$ and let $B^L_e$ be the Springer fiber of $t$ for the group $L$. If $O^L_e$ is the orbit of $L$ through the nilpotent element $t \in \text{Lie}(L)$, we denote by $\rho_{O^L_e, \phi}$ the $W(L)$-module corresponding to the pair $(O^L_e, \phi)$ via the Springer correspondence for $L$.

**Lemma 5.5.** Let $Z = G \times O (n(q) + O^L_e)$. Let $p : Z \to \mathcal{O}$ be the generalized Springer map. Let $\mathcal{O} \subset \mathcal{O}$ be a nilpotent orbit of codimension $2d$. Assume that $Z$ is rationally smooth at all points of $p^{-1}(e)$ for $e \in \mathcal{O}$. Then the number of irreducible components of $p^{-1}(e)$ of dimension $d$ is given by the formula

$$\deg \rho_{O^L_e, \phi} = \sum_{\phi \in \text{Irr}(A(e))} \deg \phi \cdot \deg \rho_{O^L_e, \phi}$$

where the sum is over the irreducible representations $\phi$ of $A(e)$ appearing in the Springer correspondence for $G$.

**Proof.** By [BM83], Thm. 3.3], we have $\text{Hom}(p^{-1}(e)) \cong H^0(\mathfrak{g}) \cong H^0(B^L_e \mathcal{N}^L(e, 1))$, where the right hand side denotes the $\rho_{O^L_e, \phi}$-isotypical component of the restriction of $H^0(B_e)$ to $W(L)$. Recall that $H^0(B_e) = \bigoplus_{\phi} \rho_{O^L_e, \phi} \oplus \phi$, which gives

$$H^0(p^{-1}(e)) \cdot H^0(B^L_e) = \deg \rho_{O^L_e, \phi} \sum_{\phi} \deg \phi \cdot \text{Res}^W W(L) \rho_{O^L_e, \phi} \rho_{O^L_e, \phi}$$

where $H^0(X)$ denotes the dimension of $H^0(X)$.}

Now the component group $A(e)$ acts on the left-hand side of

$$H^0(p^{-1}(e)) \otimes H^0(B^L_e) \cong H^0(B^L_e \mathcal{N}^L(e, 1))$$

where it acts trivially on $H^0(B^L_e)$. It also acts on the right-hand side since the $A(e)$ action commutes with the $W$ and hence $W(L)$ action. Note that the action of $A(e)$ is compatible with the isomorphism (see Corollary 3.5 [BM83]). This gives the following
Corollary 5.6. The permutation action of $A(e)$ on the irreducible components of dimension $d$ of $p^{-1}(e)$ gives rise to the linear representation

$$\bigoplus_{\phi \in \text{Irr} A(e)} \deg \rho_{(1,1)}[\text{Res}^W_L \rho_{(e,0)} : \rho_{(1,1)}] \phi$$

In particular the number of orbits of $A(e)$ on the irreducible components of $p^{-1}(e)$ of dimension $d$ equals the multiplicity of the trivial representation of $A(e)$, that is,

$$\deg \rho_{(1,1)}^L[\text{Res}^W_L \rho_{(e,0)} : \rho_{(1,1)}].$$

Example 5.7. Let $g$ of type $F_4$. Let $\mathcal{O}$ be the nilpotent orbit of type $B_4$ and $\mathcal{O}'$ of type $F_4(a_2)$. Then $\mathcal{O}' \subset \mathcal{O}$ is codimension two. Since $\mathcal{O}$ is even, its weighted Dynkin diagram shows that $\mathcal{O}$ is Richardson for the parabolic subgroup $Q$ with Levi subgroup $L$ of semisimple type $A_4$. This gives rise to the generalized Springer map $p : G \times^Q \mathfrak{n}(q) \to \mathcal{O}$ as in Lemma 5.4, with $t = 0$. The map $p$ is birational because $e$ is even. Since $\mathcal{O}$ is normal and $p$ is birational, the restriction of $p$ gives a minimal resolution of $\tilde{S}_{\mathcal{O},e} = S_{\mathcal{O},e}$ where $e \in \mathcal{O}'$ as in Lemma 5.5.

Now $A(e) = \mathfrak{s}_4$. Since $t = 0$, the representation $\rho_{(1,1)}^L$ is the sign representation of $W(L)$. By the Springer correspondence for $F_4$, $\rho_{(e,2[2]3)} = \rho_{(1,1)}^L$, $\rho_{(e,2[3]2)} = \rho_{(1,1)}^L$, $\rho_{(0,0)} = \rho_{(1,1)}^L$ and $\rho_{(0,4)} = \phi_{12.4}$ (see [Car93, pg. 428]). The multiplicity of the sign representation in the restriction of $\rho_{(e,2[2]3)}$ to $W(L)$ is 1 and in the restriction of $\rho_{(0,4)}$ is 2 and it is zero otherwise. By Lemma 5.4, the number of $\mathfrak{p}^i$’s in $p^{-1}(e)$ is $1 \cdot 2 + 2 \cdot 1 = 4$ and by Corollary 5.4, the group $A(e)$ fixes one component and permutes the remaining three components transitively. Consequently the dual graph of $S_{\mathcal{O},e} = \tilde{S}_{\mathcal{O},e}$ is the Dynkin diagram of type $D_4$ and $A(e)$ acts on the dual graph via the unique quotient of $A(e)$ isomorphic to $\mathfrak{s}_4$. Hence the singularity is $G_2$.

The fact that the dual graph is $D_4$ could also be obtained by restricting to a maximal subalgebra of type $B_4$ (Lemma 5.5). In this way we would only need to know that the degeneration in $F_4$ is unibranch, instead of the stronger statement that $\mathcal{O}$ is normal.

5.5. Orbital varieties and the exceptional divisor of $\pi$. The next lemma can sometimes be used to simplify computations. Its proof follows from that of [Fu10, Lemma 4.3].

Lemma 5.8. Let $X$ be an affine variety with rational singularities such that $\text{Pic}(X_{nm}) = 0$. Then for any resolution $\pi : Z \to X$, the number of irreducible components in $\text{Exc}(\pi)$ equals $b_2(Z)$. Here, $\text{Exc}(\pi)$ denotes the exceptional divisor of $\pi$.

Let us consider a $\mathbb{Q}$-factorial terminalization of a nilpotent orbit closure $\pi : Z \to \mathcal{O}$. Since $\text{codim}_X \text{Sing}(Z) \geq 3$, we get a resolution $\pi_0 : Z \setminus \text{Sing}(Z) \to \mathcal{O}$, which shares the same configuration of exceptional divisors as $\pi$. If $\text{Pic}(\mathcal{O})$ is finite, then the previous lemma applies to $\pi_0$, which gives the number of irreducible components of $\text{Exc}(\pi_0)$, hence of $\text{Exc}(\pi)$. If $\mathcal{O} \setminus \mathcal{O}$ consists of several irreducible components $\cup_i \mathcal{O}_i$ of codimension 2, then $b_2(Z)$ gives the sum of number of irreducible components of $\pi^{-1}(\mathcal{O}_i)$.

From [Fu10, Prop. 4.4.4] it follows that $\text{Pic}(\mathcal{O})$ is always finite unless $\mathcal{O}$ is one of the following orbits in $E_6$: $2A_1, A_2 + A_1, A_2 + 2A_1, A_2 + A_1, A_3 + A_1, A_4, A_2 + A_1, D_5$. Even for this list of orbits in $E_6$, the number of irreducible components has been explicitly computed in the proof of [Fu10, Prop. 4.4.4].

It is especially convenient to apply the lemma when $\mathcal{O}$ is the Richardson orbit for a parabolic subgroup $Q$ and the corresponding generalized Springer map $\pi : G \times^Q \mathfrak{n}(q) \to \mathcal{O}$ is birational. Then we can take $Z = G \times^Q \mathfrak{n}(q)$ above and so $b_2(G \times^Q \mathfrak{n}(q)) = b_2(G/P)$ equals the rank of $G$ minus the semisimple rank of a Levi subgroup of $Q$. Moreover in this situation the irreducible components of $\text{Exc}(\pi)$ have a description in terms of the orbital varieties for the $\mathcal{O}_i$’s.

Recall that an orbital variety for $\mathcal{O}_i$ is an irreducible component of $\tilde{\mathcal{O}} \cap \mathfrak{n}$ where $\mathfrak{n} := \mathfrak{n}(b)$ is the nilradical of the Borel subalgebra $b$. It is known that each orbital variety has dimension $\frac{1}{2} \dim \mathcal{O}_i$. Let $X$ be an orbital variety for $\mathcal{O}_i$, which is contained in $\mathfrak{n}(q)$. Then $X$ is codimension one in $\mathfrak{n}(q)$ since $\mathcal{O}_i$ is codimension two in $\tilde{\mathcal{O}}$ and $\dim \mathfrak{n}(q) = \frac{1}{2} \dim \mathcal{O}_i$. Moreover $X$ is stable under the action of the connected group $Q$ since $X \subset Q \cdot X \subset \tilde{\mathcal{O}} \cap \mathfrak{n}$ and $X$ is maximal irreducible in $\tilde{\mathcal{O}} \cap \mathfrak{n}$.

Let $\pi_X$ be the restriction of $\pi$ to $G \times^Q X$. The image of $\pi_X$ is $\tilde{\mathcal{O}}$, since $X$ is irreducible and $Q$ is a parabolic. By dimension considerations, $\pi_X^{-1}(\tilde{\mathcal{O}}) = G \times^Q X$ is an irreducible component of $\text{Exc}(\pi)$.
Conversely, any irreducible component of $\operatorname{Exc}(\pi)$ equals $G \times^\theta Y$ for some irreducible component of $\overline{O} \cap \mathfrak{n}(q)$. Now $\dim Y$ can only equal $\dim \mathfrak{n}(q) - 1$ or $\dim \mathfrak{n}(q) - 2$ since $\ker \pi_Y = \overline{O}$. In the former case, $Y$ is an orbital variety of $X$ contained in $\mathfrak{n}(q)$. In the latter case, $\pi_Y^{-1}(e_i)$ is finite for $e_i \in \mathcal{O}$, contradicting the fact that the irreducible components of $\operatorname{Exc}(\pi)$ are exactly the $G \times^\theta X$ where $X$ is an orbital variety of some $\mathcal{O}$ lying in $\mathfrak{n}(q)$.

Next, the map $G \times^\theta X \to G \times^\phi X$ has connected fibers isomorphic to $Q/B$. It follows from Spa82 that the $\mathbb{P}^1$’s in $\pi_X^{-1}(e_i)$ are permuted transitively under the induced action of $A(e_i)$ since the analogous statement holds for the irreducible components of $p_X^{-1}(e_i)$ where $p_X : G \times^\theta X \to \mathcal{N}$. Consequently, if $m_i$ equals the number of $(A(e_i))$-orbits on $\pi^{-1}(e_i)$, then $\sum m_i = b_2(G/P)$. See, for example, [Wie03, Thm 1.3] for a more general setting where this phenomenon occurs.

**Example 5.9.** Consider the minimal degeneration where $\mathcal{O}$ has type $\widetilde{A}_2$ and $\mathcal{O}'$ has type $A_1 + \widetilde{A}_1$ in $E_8$. The codimension of $\mathcal{O}$ in $\overline{\mathcal{O}}$ is two. The orbit $\mathcal{O}$ is even and so it is Richardson for the parabolic subgroup $Q$ whose Levi subgroup has type $B_3$. Moreover, the map $\pi : Z := G \times^\theta \mathfrak{n}(q) \to \overline{\mathcal{O}}$ is birational, hence a minimal resolution. The hypotheses of Lemma 5.8 hold. Since $b_2(Z) = 1$, there is no other minimal degeneration of $\mathcal{O}$, there must be exactly one irreducible component in $\pi^{-1}(\mathcal{O}')$. Since $A(e) = 1$ for $e \in \mathcal{O}'$, there is only one irreducible component in $\pi^{-1}(e)$. Hence the singularity is of type $A_1$ since $\overline{\mathcal{O}}$ is normal.

**5.6. Three remaining cases.** There are three cases where the information in Lemma 5.5 and Corollary 5.6 is not sufficient to determine a minimal surface degeneration, even up to normalization. They are $(E_6(a_1), D_5)$ in $E_6$, $(E_7(a_1), E_7(a_2))$ in $E_7$, and $(E_8(a_1), E_8(a_2))$ in $E_8$. In this section we give an ad hoc way to determine the singularity.

In each of the three cases, the larger orbit $\mathcal{O}$ is the subregular nilpotent orbit and so $\overline{\mathcal{O}}$ is normal. Since $\mathfrak{g}$ is simply-laced, $A(z)$ is trivial for $z \in \mathcal{O}$. Hence for any parabolic subgroup $Q$ with Levi factor $A_1$ the map $\pi : G \times^\theta \mathfrak{n}(q) \to \overline{\mathcal{O}}$ is a minimal resolution. Moreover in each case the smaller orbit $\mathcal{O}'$ is the unique maximal orbit in $\overline{\mathcal{O}}$. Since $A(e) = 1$ for $e \in \mathcal{O}'$, there are rank$(q) - 1$ $\mathbb{P}^1$’s in $\pi^{-1}(e)$ by 5.5.3. At the same time, this uniqueness means that $\mathcal{O}'$ is the Richardson orbit for any parabolic $Q'$ with Levi factor $A_1 \times A_1$. In other words, $\mathfrak{n}(q')$ is an orbital variety for $\mathcal{O}'$. Hence if we fix $Q$ corresponding to a simple root $\alpha$, then we find an orbital variety $\mathfrak{n}(q') \subset \mathfrak{n}(q)$ for $\mathcal{O}'$ for each simple root $\beta$ not connected to $\alpha$ in the Dynkin diagram. Since $A(e)$ is trivial, each of these $\mathfrak{n}(q')$ gives rise to a unique $\mathbb{P}^1$ in $\pi^{-1}(e)$. By looking in the Levi subalgebra corresponding to the simple roots not connected to $\alpha$, it is possible to determine the intersection pattern of these $\mathbb{P}^1$’s.

5.6.1. The case of $(E_6(a_1), D_5)$ in $E_6$. There are 5 $\mathbb{P}^1$’s in $\pi^{-1}(e)$. The singularity could only be $A_5$ or $D_5$ since $\overline{\mathcal{O}}$ is normal. If we choose $\alpha$ so that the remaining simple roots form a root system of type $A_5$, then there are 4 orbital varieties of the form $\mathfrak{n}(q')$ in $\mathfrak{n}(q)$. The 4 $\mathbb{P}^1$’s have intersection diagram of type $A_2 + A_2$. This could only happen for a dual graph of type $A_5$, so $\mathfrak{S}_{\mathcal{O},e} \cong A_5$.

5.6.2. The case of $(E_7(a_1), E_7(a_2))$ in $E_7$. There are 6 $\mathbb{P}^1$’s in $\pi^{-1}(e)$. The singularity could only be $A_6$, $D_6$, or $E_6$ since $\overline{\mathcal{O}}$ is normal. If we choose $\alpha$ so that the remaining simple roots form a system of type $D_6$. Then there are 5 orbital varieties of the form $\mathfrak{n}(q')$ in $\mathfrak{n}(q)$. Then the 5 $\mathbb{P}^1$’s have intersection diagram of type $D_6$. This eliminates $A_6$ as a possibility. If we choose $\alpha$ so that the remaining simple roots form a system of type $A_6$, then there are 5 orbital varieties of the form $\mathfrak{n}(q')$ in $\mathfrak{n}(q)$ and the corresponding 5 $\mathbb{P}^1$’s have intersection diagram of type $A_3 + A_3$. This eliminates $E_6$, hence $\mathfrak{S}_{\mathcal{O},e} \cong D_6$.

5.6.3. The case of $(E_8(a_1), E_8(a_2))$ in $E_8$. There are 7 $\mathbb{P}^1$’s in $\pi^{-1}(e)$. The singularity could only be $A_7$, $D_7$, or $E_7$ since $\overline{\mathcal{O}}$ is normal. If we choose $\alpha$ so that the remaining simple roots form a system of type $E_7$, then there are 6 orbital varieties of the form $\mathfrak{n}(q')$ in $\mathfrak{n}(q)$. The corresponding 6 $\mathbb{P}^1$’s have intersection diagram of type $E_6$. Hence $\mathfrak{S}_{\mathcal{O},e} \cong E_7$.

6. On the splitting of $C(s)$ and intrinsic symmetry action
6.1. The splitting of \(C(s)\). In this section we establish the splitting on \(C(s)\) discussed in §4.7.3. Namely, we determine when

\[ C(s) \cong C(s)^e \times H \]

for some \(H \subset C(s)\). Necessarily \(H \cong A(e)\). We continue to assume that \(G\) is of adjoint type.

In the classical groups, \(C(s)\) is a product of orthogonal groups and a connected group, possibly up to a quotient by a central subgroup of order two. Since the result holds for any orthogonal group, it holds for \(C(s)\).

Let \(C \subset A(e)\) be a conjugacy class. There exists \(s \in C(s)\) whose image \(\bar{s}\) in \(A(e)\) lies in \(C\) such that the order of \(s\) equals the order of \(\bar{s}\), except when \(e\) belongs to one of the following four orbits:

\[
A_4 + A_1 \text{ in } E_7; A_4 + A_1, D_4(a_2), E_6(a_1) + A_1 \text{ in } E_8.
\]

For these four orbits, which all have \(A(e) = E_8\), the best result is an \(s\) of order 4 to represent the non-trivial \(C\) in \(A(e)\) [Som98, §3.4]. Hence the splitting holds for all other orbits with \(A(e) = E_8\).

This leaves the cases where \(A(e) = E_6, G_2,\) or \(F_4\). If \(e\) is distinguished, meaning \(C(s)^e = 1\), there is nothing to check. This leaves a handful of cases where \(A(e) = E_6\) and \(e\) is not distinguished. The first such case is \(e = D_4(a_1)\) in \(E_6\), which we now explain.

6.1.1. \(E_6\) cases. Let \(G\) be of type \(E_6\) and \(s \in G\) be an involution with \(G^e\) of semisimple type \(A_5 + A_1\). Then there exist \(\bar{e} \in g^e\) nilpotent of type \(2A_2\). Let \(\bar{s} \subset g^e\) be an \(s\)-triple through \(\bar{e}\). Then \(\bar{c}(\bar{s})\) has type \(G_2\). Now \(g^e \cap \mathcal{C}(\bar{s})\) is easy to compute inside of \(A_5 + A_1\); it is a semisimple subalgebra of type \(A_5\). Let \(\bar{e}_0\) be regular nilpotent in \(g^e \cap \mathcal{C}(\bar{s})\). Then \(\bar{e}_0\) is in the subregular nilpotent orbit in \(\mathcal{C}(\bar{s})\). Clearly \(s\) belongs to the centralizer of \(\bar{e}_0\) in \(\mathcal{C}(\bar{s})\), which is a finite group \(H \cong E_6\), from the case of the subregular orbit in \(G_2\). Next, a calculation in \(A_5 + A_1:\) shows that \(\bar{e} + \bar{e}_0\) has type \(A_3 + A_2\). From this we conclude that \(e = \bar{e} + \bar{e}_0\) belong to the nilpotent orbit \(D_4(a_1)\) in \(E_6\) and \(s\) represents an involution in \(A(e)\) [Som98, §4].

A similar argument works if \(s \in G\) is an element of order 3 with \(G^e\) of semisimple type \(3A_2\). Therefore the centralizer \(H \cong E_6\) of \(\bar{e}_0\) in \(\mathcal{C}(\bar{s})\) also centralizes \(\bar{e} + \bar{e}_0\) and the image of \(H\) in \(A(e)\) is all of \(A(e)\). This proves the splitting for \(e = D_4(a_1)\) in \(E_6\). The same procedure works for the other \(E_6\) cases.

6.1.2. We have shown

**Proposition 6.1.** There exists \(H \subset C(s)\) such that

\[ C(s) \cong C(s)^e \times H, \]

except when \(e\) belongs to one of the four orbits in (6.1). For those four cases, \(A(e) = G_2\) and

\[ C(s) = C(s)^e \cdot H \]

where \(H \subset C(s)\) is cyclic of order 4.

While the above splitting is unique up to conjugacy in \(C(s)\) in the subregular case (§4.4.2), this is not the case in general, as the next example shows.

**Example 6.2.** Let \(e = A_2\) in \(g = E_8\). Then \(\mathcal{C}(s)\) has type \(E_8\) and \(A(e) = G_2\). The generalized Bala-Carter notation for the non-trivial class \(\mathcal{C}\) in \(A(e)\) is \(4A_1\). From this it follows that both conjugacy classes of involution in \(G\) represent \(C\). For one choice of involution \(s_1 \in C(s)\) lifting \(C, g^e\) has type \(D_4\). The partition of \(e\) in \(g^e\) is \([2]^3\), so the reductive centralizer of \(e\) in \(g^e\) is \(sp_{12}\). For the other another choice \(s_2 \in C(s)\) lifting \(C, g^e\) has type \(E_7 + A_1\) and \(e\) corresponds to \((3A_1)^3 + A_1\). Hence the reductive centralizer of \(e\) in \(g^e\) is of type \(F_4\). Consequently, there are two choices of splitting in Proposition 6.1 that are not only non-conjugate under \(C(s)\), but also in \(Aut(c(s))\).

Although the choice of splitting in Proposition 6.1 is not unique up to conjugacy in \(C(s)\) or even \(Aut(c(s))\), we can restrict the choice of \(H\) further so that the image of \(H\) in \(Aut(c(s))\) will be well-defined up to conjugacy in \(Aut(c(s))\). Let \(c(s)^{**}\) be the semisimple summand of \(c(s)\). Let

\[ a : C(s) \rightarrow Aut(c(s)^{**}) \]
be the natural map. Then Im $a = \text{Int}(\langle c(s) \rangle) \times K$ for some subgroup of diagram automorphisms $\langle c(s) \rangle$. By a case-by-case check, $H$ in the Proposition 6.1 can be chosen so that $H$ maps onto $K$ via $a$. Then the image of $H$ in $\text{Aut}(\langle c(s) \rangle)$ is well-defined up to conjugacy in $\text{Aut}(\langle c(s) \rangle)$. In the above example, $H = \langle s_2 \rangle$ since the induced automorphism is a diagram automorphism.

6.2. Computing the intrinsic symmetry. Having chosen $H$ with $a(H) = K$ as above, we can determine the action of $H$ on $\mathcal{S}_{O,e}$. Here, we restrict to the exceptional groups and to a minimal degeneration $O'$ of $O$, with $e \in O'$. We summarize the possibilities and record the action of $H$ on $\mathcal{S}_{O,e}$ in the graphs at the end of the paper.

6.2.1. Minimal singularities: $A(e) = \mathfrak{e}_3$ cases. Let $\mathcal{S}_{O,e}$ be an irreducible minimal singularity admitting an involution as in §1.7.2. If $|H| = 2$, then it turns out that $H$ realizes this involution. There is one case of this kind when $|H| = 4$, when $e = A_4 + A_4$ in $E_8$ and $\mathcal{S}_{O,e} \cong a_2$. Let $H = \langle s \rangle$. Then $s \in H$ realizes the involution on $\mathcal{S}_{O,e}$ and $s^2$ acts trivially on $\mathcal{S}_{O,e}$. We will still refer to this singularity with induced symmetry by $a_2^+$.

If $\mathcal{S}_{O,e}$ is a reducible minimal singularity, then it is turns out that $\mathcal{S}_{O,e}$ has exactly two irreducible components and $H$ interchanges the two components. The only three cases which occur are the singularities with symmetry action $[2A_1]^+, [2A_2]^+$, and $[2A_2]^+$.

6.2.2. Minimal singularities: $A(e) = \mathfrak{e}_3$ cases. If $\mathcal{S}_{O,e}$ is the unique irreducible minimal singularity admitting an action of $\mathfrak{e}_3$ as in §1.7.2, then $H$ realizes the full symmetry $d_1^+$. This only occurs once, in $E_8$.

If $\mathcal{S}_{O,e}$ is a reducible minimal singularity, then $\mathcal{S}_{O,e}$ turns out to have 3 irreducible components and $H$ acts by permuting transitively the three components. In other words, the stabilizer of a component acts trivially on the component. All of these cases are of the form $3A_1$ and the singularity with symmetry action is denoted $[3A_1]^{++}$.

6.2.3. Simple surface singularities: $A(e) = \mathfrak{e}_2$ cases. If $\mathcal{S}_{O,e}$ is an irreducible simple surface singularity admitting an involution as in §1.3.2 (or in the case of $A_2$ and $A_4$, admitting the appropriate cyclic action of order 4), then $H$ realizes this symmetry. To show this, we first checked that $A(e)$ has the appropriate action on the dual graph of a minimal resolution in Corollary 1.1 and Theorem 1.2 in [Cat87]. Then since $C(s)$ acts symplectically on $\mathcal{S}_{O,e}$, Corollary 1.1 and Theorem 1.2 in [Cat87] imply that $H$ corresponds to the $\Gamma \subset \text{SL}_2(\mathbb{C})$ which defines the symmetry involution.

The only reducible surface singularities with $A(e) = \mathfrak{e}_2$ are those with $\mathcal{S}_{O,e} \cong 2A_1$, hence covered previously.

6.2.4. Simple surface singularities: $A(e) = \mathfrak{e}_3$ cases. If $\mathcal{S}_{O,e}$ is an irreducible simple surface singularity admitting an $\mathfrak{e}_3$ action as in §1.3.2, then $H$ realizes the symmetry action and so $\mathcal{S}_{O,e} \cong G_2$.

An unusual situation occurs for the minimal degeneration $(D_7(a_1), E_8(b_0))$. Here, $A(e) = \mathfrak{e}_3$, but $\mathcal{S}_{O,e}$ only admits a two-fold symmetry, compatible with its normalization $\tilde{\mathcal{S}}_{O,e}$ which is $A_3$. Here, $\Gamma \subset \text{SL}_2(\mathbb{C})$ corresponding to $\tilde{\mathcal{S}}_{O,e}$ is cyclic of order 4. The normal cyclic subgroup of $H \cong \mathfrak{e}_3$ is generated by an element $s$ with $g^s$ of type $E_6 + A_2$ and hence $s$ acts without fixed point on the orbit $D_7(a_1)$ since the latter orbit does not meet the subalgebra $E_6 + A_2$. On the other hand, using Corollary 1.1 we see that $A(e)$ induces the involution on the dual graph of a minimal resolution of $\tilde{\mathcal{S}}_{O,e}$. Since $C(s)$ acts symplectically on $\mathcal{S}_{O,e}$ and $\tilde{\mathcal{S}}_{O,e}$, Corollary 1.1 and Theorem 1.2 in [Cat87] imply that $H$ acts on $\tilde{\mathcal{S}}_{O,e} = C^2/\Gamma$ via the action of $\Gamma \subset \text{SL}_2(\mathbb{C})$, the binary dihedral of order 24 containing $\Gamma$ as normal subgroup.

If $\mathcal{S}_{O,e}$ is a reducible surface singularity, then $\mathcal{S}_{O,e}$ is isomorphic to $3C_5, 3C_6, 3(C_5)$, or the previously covered $[3A_1]^{++}$. We have omitted the superscript in $3C_2$, etc. The notation means that $H$ permutes the three components transitively and the stabilizer of any component is order 2, which acts by the indicated symmetry. The notation $(C_5)$ refers to the fact that we do not know if an irreducible component is normal.
6.2.5. Simple surface singularities: \(A(e) = \mathfrak{s}_4\) case. This only occurs in \(F_4\). One degeneration has \(S_{O_6} \cong G_2\) (see §6.4). Here, the Klein 4-group in \(H\) acts trivially on \(S_{O_6}\) and the quotient action realizes the full symmetry of \(\mathfrak{s}_4\) on \(S_{O_6}\). This follows either from the list of possible symplectic automorphisms of \(S_{O_6}\) or from a direct calculation that the Klein 4-group in \(H\) fixes \(S_{O_6}\) pointwise. The other degeneration has \(S_{O_6} \cong 4G_2\) (see §6.5). Here, \(H\) permutes the four components transitively and the stabilizer of any component is an \(\mathfrak{s}_3\), which acts by the indicated symmetry.

6.2.6. Simple surface singularities: \(A(e) = \mathfrak{s}_5\) case. This only occurs in \(E_6\). One degeneration has \(S_{O_8} \cong 10G_2\). Here, \(H\) permutes the ten components transitively and the stabilizer of any component is a Young subgroup \(\mathfrak{s}_5 \times \mathfrak{s}_2\). The \(\mathfrak{s}_2\) factor acts trivially on the given component and the \(\mathfrak{s}_5\) factor acts by the indicated symmetry.

The other degeneration has \(S_{O_8} \cong 5G_2\). Here, \(H\) permutes the five components transitively and the stabilizer of any component is \(\mathfrak{s}_4\). The \(\mathfrak{s}_4\) factor acts on the given component as in the \(F_4\) case above.

Remark 6.3. Not every non-trivial \(A(e)\) contributes a symmetry on a \(S_{O_8}\). For example, \(e = C_3(a_1)\) in \(F_4\). The only degeneration above \(O\) has \(S_{O_8} \cong A_1\). Here, \(H\) acts trivially on the \(S_{O_8}\), reflecting the fact that \(SL_2(\mathbb{C})\) has no outer automorphisms. Indeed, \(C(s)\) is a direct product of \(C(s)^e\) and \(H\).

7. Results for \(F_4\)

7.1. Proving Proposition 3.3. Here we record the details for establishing Proposition 3.3 for \(g\) of type \(F_4\) as outlined in §4.3. First, we enumerate the \(G\)-orbits of those \(e\) such that \(\mathfrak{c}(s)\) has nontrivial intersection with the minimal nilpotent orbit in \(g\). To that end, let \(e_0 \in \mathfrak{g}\) be minimal nilpotent and recall that \(\mathfrak{s}_0\) is an \(\mathfrak{sl}(\mathbb{C})\)-subalgebra through \(e_0\). The centralizer \(\mathfrak{c}(\mathfrak{s}_0)\) is a simple subalgebra of type \(C_3\), equal to the semisimple part of a Levi subalgebra of \(g\). The relevant nonzero nilpotent elements \(e \in \mathfrak{c}(\mathfrak{s}_0)\) are therefore those in the \(G\)-orbits

\[
A_1, \tilde{A}_1, A_1 + A_1, \tilde{A}_1, B_2, C_3(a_1) \text{ and } C_3
\]

and hence Corollary 4.2 applies to these elements. The computation of \(e + e_0 \in \mathfrak{o}\) proceeds as in §4.1. The results are in Table 4. In the cases where \(\mathfrak{c}(s)\) is not simple, we use boldface font in Table 4 to indicate those simple factors whose minimal nilpotent orbit is of type \(A_1\) in \(g\).

**Table 1.** \(F_4\): \(e_0 \in \mathfrak{c}(s)\) of type \(A_1\) in \(g\)

| \(e\) | \(e + e_0 \in \mathfrak{o}\) | \(\mathfrak{c}(s)\) | Isomorphism type of \(S_{O_8}\) |
|-------|-----------------|-----------------|----------------------------------|
| \(A_1\) | \(2A_1 = A_1\) | \(C_3\) | \(c_3\) |
| \(A_1\) | \(A_1 + A_1\) | \(A_3\) | \(a_1^2\) |
| \(A_1 + A_1\) | \(2A_1 + A_1 = A_2\) | \(A_1 + A_1\) | \(A_1\) |
| \(A_2\) | \(A_2 + A_1\) | \(G_2\) | \(g_2\) |
| \(B_2\) | \(B_2 + A_1 = C_3(a_1)\) | \(2A_1\) | \([2A_1]^+\) |
| \(C_3(a_1)\) | \(C_3(a_1) + A_1 = F_4(a_1)\) | \(A_1\) | \(A_1\) |
| \(C_3\) | \(C_3 + A_1 = F_4(a_2)\) | \(A_1\) | \(A_1\) |

In the first two lines of Table 4 we record the remaining cases where Corollary 4.3 holds for a minimal nilpotent element \(e_0 \in \mathfrak{c}(s)\). This completes the verification of Proposition 3.1 for \(F_4\).

7.2. Verifying the remaining cases where Proposition 3.3 holds. In the previous section there were several minimal degenerations with \(S_{O_8}\) isomorphic to an \(A_1\) singularity. These cases are the ones with \(x^+ = 0\) in Propositions 3.3. Next, we find the cases in the proposition with \(x^+ \neq 0\). There are no other cases since by §7.3 these are the only surface minimal degenerations with \(|\Gamma| = 1\) or 2.

We use the method in §1.4, with some simplifications afforded by passing to a subalgebra as in Proposition 2.2. The values of \((m_i, n_i)\) for \(i \in \mathcal{E}\) are listed in Table 3. Boldface is used for those \((m_i, n_i)\)
where the corresponding coefficient $b_i \neq 0$ in $[4.7]$. Recall that it always turns out that $w = e_0$ and the coefficient $a$ of $e_0$ is nonzero in $[4.7]$.

7.2.1. $\mathfrak{a}_1 + \mathfrak{a}_1$. For $e$ of type $\mathfrak{a}_1 + \mathfrak{a}_1$, $\mathfrak{c}(s)$ is semisimple of type $\mathfrak{a}_1 + \mathfrak{a}_1$. The minimal nilpotent orbit in one simple factor is minimal in $\mathfrak{g}$ and this was handled earlier. The other simple factor has minimal nilpotent orbit which is of type $\mathfrak{a}_2$ in $\mathfrak{g}$. Let $e_0 \in \mathfrak{c}(s)$ be a representative of this orbit. The sum $y = e + e_0$ is of type $C_{5}(a_1)$ and $(m, n) = (2, 4)$ for the unique element in $\mathfrak{e}^*$.

Let $t$ be a Levi subalgebra of $\mathfrak{g}$ of type $C_1$. We can assume $e, e_0 \in t$. By Example $4.4.3$, there exists $x \in t$ of type $\mathfrak{a}_2$ satisfying $[4.7]$ with $a, b \neq 0$ and $w = e_0$. Since $\text{codim}_{\mathfrak{g}}(\mathfrak{c}(s)) = 2$ in $F_{s_1}$, $x$ satisfies Lemma 7.2.4, not just in $t$, but also in $F_{s_1}$. It follows that $S_{\mathfrak{c}_2, e} \cong A_1$ in $F_{s_2}$.

7.2.2. $\mathfrak{a}_2 + \mathfrak{a}_1$. For $e$ of type $\mathfrak{a}_2 + \mathfrak{a}_1$, $\mathfrak{c}(s)$ is of type $\mathfrak{a}_2 + \mathfrak{a}_1$ in $\mathfrak{g}$. Let $e_0 \in \mathfrak{c}(s)$ be a representative of this orbit. The sum $y = e + e_0$ is of type $F_{5}(a_2)$ and $(m, n) = (1, 3)$ for the unique element in $\mathfrak{e}^*$. Let $x$ be in the $C_{5}(a_1)$ orbit. Then $\mathfrak{c}_1 \subset \mathfrak{c}_2 \subset \mathfrak{c}_3$. We check that $I_2 \subset I_3 \subset I_2$ as in $[4.1.2]$. It follows as in Example 4.4.4 that $S_{\mathfrak{c}_2, e} \cong m$. This result can also be deduced from Lemma 2.2 by working in the subalgebra $s' \oplus \mathfrak{c}(s')$, where $s'$ is the $sl_2$-subalgebra through an element of type $\mathfrak{a}_1$ and then using Example 4.4.4 since $\mathfrak{c}(s')$ is of type $G_2$.

7.2.3. $\mathfrak{a}_2 + \mathfrak{a}_1$. For $e$ of type $\mathfrak{a}_2 + \mathfrak{a}_1$, $\mathfrak{c}(s) \cong sl_2$. The minimal nilpotent orbit in $\mathfrak{c}(s)$ also has type $\mathfrak{a}_2 + \mathfrak{a}_1$ in $\mathfrak{g}$. Let $e_0 \in \mathfrak{c}(s)$ be a representative of this orbit. The sum $y = e + e_0$ is of type $F_{5}(a_2)$ and $(m_1, n_1) = (1, 3)$ and $(m_2, n_2) = (2, 4)$ for the two elements in $\mathfrak{e}^*$. Indeed the decomposition of $\mathfrak{g}$ in $[4.2]$ is $V(0, 2) \oplus V(1, 3) \oplus V(2, 0) \oplus V(2, 4) \oplus V(3, 1) \oplus V(4, 2)$. For nilpotent $x \in \mathfrak{g}$ of type $B_2$ or of type $\mathfrak{a}_2 + \mathfrak{a}_1$, $\mathfrak{c}(s) \subset \mathfrak{c}_1 \subset \mathfrak{c}_2$ and we checked that the two conditions in $\mathfrak{c}_1$ and thus $\mathfrak{c}_2$, hold for each orbit.

Now if $x \in \mathfrak{c}$ is of type $B_2$, then $\mathfrak{c}$ meets the reductive subalgebra $t$ of type $B_1$ in $F_{s_1}$, where it has partition $[4, 4, 1]$ in $\mathfrak{so}(4)$. Also, $\mathfrak{c}_1$ meets $t$ in the orbit $[5, 3, 1]$. Assume that $s, s_0$ are in $t$. Calculating $\mathfrak{e}$ for $e$ and $e_0$ relative to $t$, one finds that only $(m, n) = (2, 4)$ occurs (reflecting the fact that $e = [3, 3, 3]$ is even in $t$). Thus $v_1 \notin t$ and hence $b_i = 0$. It follows that $a, b_2 \neq 0$ as in Example 4.4.3. Consequently $S_{\mathfrak{c}_2, e} \cong A_1$.

On the other hand for $x$ in $\mathfrak{a}_2 + \mathfrak{a}_1$, we have to carry out an explicit computation. Choose $e := e_{1000} + e_{0100} + e_{0010}$, $f := 2f_{1000} + 2f_{0100} + f_{0010}$.

Then $e_0 \in \mathfrak{c}(s)$ can be chosen to be $e_0 = 2e_{0122} + e_{1220} + e_{1121}$, $f_0 = f_{0122} + 2f_{1220} + f_{1121}$

Since $\mathfrak{g}(-1, 3)$ and $\mathfrak{g}(-2, 4)$ are 1-dimensional, it is easy to locate $v_1$ and $v_2$. Relative to these choices for $e, f, e_0, f_0$, we find that $v_1 = e_{1221} \in \mathfrak{g}(-1, 3)$, $v_2 = e_{1242} \in \mathfrak{g}(-2, 4)$.
Using GAP and de Graaf’s package, we found that the following two elements of the Slodowy slice $e + g'$

$$x_\pm := e + e_0 \pm \sqrt{\frac{-3}{27}} e_{1231} + \frac{2}{3} e_{1242}$$

are of type $A_2 + A_1$. Since $\dim S_{O, e} = 2 = \dim C(e) \cdot e_0$, Lemma 2.4 holds. Then Lemma 3.3 implies that $S_{O, e} \cong m$. This completes the proof of Proposition 3.3 for $F_4$.

7.3. Remaining surface singularities. This section summarizes the calculations of the singularities of the minimal degenerations of dimension two, using the methods in §5. Up until this point we have not needed to know whether a nilpotent orbit has normal closure to determine the singularity type of a minimal degeneration. Knowing the branching was sufficient. Indeed, the closure of the orbit $B_2$ is non-normal, but it was shown above that it is normal at points in the orbit $A_2 + A_1$ since the singularity is $A_1$. Similarly for the orbit $A_2$. The remaining non-normal orbit closures, of which there are three [Bro98b], are detected through a minimal degeneration: either the closure is branched at a minimal degeneration (as for $C_4$) or is isomorphic to $m$ at a minimal degeneration (as for $C_3(a_1)$ and for $A_2 + A_1$).

In what follows we use the fact that the orbit $F_4(a_1)$ has normal closure [Bro98b] to classify the type of its minimal degeneration. This is the only case where we need to know whether the closure is normal or not in order to resolve the type of a minimal degeneration in $F_4$.

(1) $(\mathcal{O}, \mathcal{O'}) = (F_4(a_1), F_4(a_2))$. The even orbit $F_4(a_1)$ is Richardson for the parabolic subgroup $Q$ with Levi factor of type $A_1$ and the resulting map $p : G \times^Q n(q) \to \mathfrak{g}$ is birational, hence a minimal resolution. The hypotheses of Lemma 5.8 hold and $b_2(G/Q) = 3$. Since $\mathcal{O}$ is the unique orbit of codimension two in $\mathfrak{g}$, it follows from §5.5 that there are 3 orbits of $A(e) = \mathcal{S}_2$ on the irreducible components of $p^{-1}(e)$. On the other hand, there are a total of four irreducible components of $p^{-1}(e)$ by §5.4. Thus the singularity must be $C_3$, given that $\mathfrak{g}$ is normal.

(2) $(\mathcal{O}, \mathcal{O'}) = (C_3, F_4(a_3))$. The orbit $\mathcal{O}$ is Richardson for the parabolic subgroup $Q$ with Levi factor of type $A_2$. The map $p : G \times^Q n(q) \to \mathfrak{g}$ is birational, hence a minimal resolution, since $A(e) = 1$ for $e \in \mathcal{O}$. If $e \in \mathcal{O'}$, then $A(e) = \mathcal{S}_4$. By Lemma 5.5 and Corollary 5.6, there are 16 irreducible components in $p^{-1}(e)$ with two orbits under $A(e)$. The number of orbits can also be deduced from §5.3. Looking at the possibilities for the dual graph, it is clear that $\mathfrak{g}$ is non-normal and the normalization map $\nu : \mathfrak{g} \to \mathfrak{g}$ restricts to a degree 4 map over $\mathcal{O'}$. This also follows from [Sho80] (§2.4). By Corollary 5.6, there is a fixed component of $\pi^{-1}(y)$ under the $A(e)$-action for $y = \nu^{-1}(e)$. This implies that the singularity of $\mathfrak{g}$ at $y$ is $G_2$. We show in §7.4 that $S_{O, e}$ is isomorphic to $4G_2$. In other words, the irreducible components of $S_{O, e}$ are normal and hence each is isomorphic to $G_2$.

(3) $(\mathcal{O}, \mathcal{O'}) = (B_3, F_4(a_3))$. The singularity is $G_2$ by the example in §7.4.

7.4. Singularity $(C_3, F_4(a_3))$ is $4G_2$. Left unresolved by the previous discussion is whether an irreducible component of the nilpotent Slodowy slice is normal for the minimal degeneration $(C_3, F_4(a_3))$. In this section we prove it is. We also prove an independently interesting fact: the nilpotent Slodowy slice of $C_3$ at $A_2$ contains an irreducible component isomorphic to the nilpotent cone $\mathcal{N}_{G_2}$ in $G_2$.

Let $e$ be in the orbit $A_2$. Then $e(s)$ is simple of type $G_2$. Let $e_0$ be a regular nilpotent element in $e(s)$. Then $y = e + e_0$ lies in the orbit $F_4(a_2)$. The decomposition of $g$ in (4.3) is

$$V(0, 2) \oplus V(0, 10) \oplus V(2, 0) \oplus V(4, 6),$$

so $\mathfrak{g}$ has a single element, with $(m, n) = (4, 6)$.

Let $\mathcal{O}$ be the orbit $C_3$. Then $\mathcal{O}, \subset \mathfrak{g} \subset \mathfrak{g}$, and $\dim S_{O, e} = \dim C(e) \cdot e_0$. We will show that there exists $x \in \mathcal{O}$ satisfying the first part of Lemma 2.4 and therefore $S_{O, e}$ contains an irreducible component isomorphic to $C(e) \cdot e_0$, which is the nilcone of $e(s)$. We set

$$e = e_{0010} + e_{0001}, \quad f = 2f_{0010} + 2f_{0001}, \quad h = [e, f].$$
and 

\[ e_0 = e_{0111} - e_{0120} + e_{1000}. \]

The space \( g(4,6) \) is 1-dimensional, spanned by \( e_{1200} \), so \( v = v_1 = e_{1200} \). This is also a highest weight vector for the full action of \( C(s) \) on \( g(-4) \), which is the 7-dimensional irreducible representation of \( G_2 \). We computed in GAP that there is an \( x \in O \) with \( x = e + e_0 - \frac{1}{4} v \), which establishes (4.7).

Next we show that \( C(s) \cdot (e_0 - 4v) \cong C(s) \cdot e_0 \) and relate the choice of \( v \) to the discussion in §4.1. Here \( g^h \cong so_7 \oplus \mathbb{C} \) and \( so_7 \) decomposes under \( c(s) \) into \( c(s) \cong V(\omega_2) \), where the latter is the irreducible 7-dimensional representation of \( c(s) \). Now \( ad f \) annihilates \( c(s) \), while \( (ad f)^2 \) carries the \( V(\omega_2) \) summand onto \( g(-4) \cong V(\omega_2) \). Let \( X = e_0 \in c(s) \subset so_7 \subset sl_7 \). Then \( X^3 \in so_7 \). It is nonzero since \( X \) is regular in \( so_7 \).

Hence, \( (ad f)^2(X^3) \) is in the span of \( v \). This argument gives a version of Lemma 4.3 for the closure of the \( C(s) \)-orbit \( (e_0, v) \) in \( c(s) \oplus V(\omega_2) \cong so_7 \): namely, the closure is exactly the set of elements \( (X, X', X) \) where \( X \in c(s) \) is nilpotent. Hence, there is a natural \( C(s) \)-equivariant isomorphism of this orbit closure with the closure of \( C(s) \cdot e_0 \), the nilpotent cone \( \mathcal{N}_{G_2} \) in \( G_2 \).

Thus \( S_{O,e} \) contains an irreducible component isomorphic to \( \mathcal{N}_{G_2} \). An element \( x' \) in the orbit \( F_{2}(\alpha_3) \) with \( x' \in S_{O,e} \) corresponds to an element in the subregular nilpotent orbit in \( \mathcal{N}_{G_2} \) under this identification. Thus \( S_{O,e}' \) contains a component isomorphic to the simple surface singularity \( D_4 \). Incorporating the symmetry of \( A(x') = \mathbb{E}_4 \), we have \( S_{O,e'} \cong 4G_2 \).

**Remark 7.1.** There are two branches of the \( C_3 \) orbit closure in a neighborhood of \( e \). These two branches are not conjugate under the action of \( G^* \), which shows that Proposition 2.1 does not generally hold for degenerations which are not minimal. The second branch of \( C_3 \) at \( e \) splits into three separate branches in a neighborhood of a point in the orbit \( F_{2}(\alpha_3) \).

8. **Results for \( E_6 \)**

8.1. **Proving Proposition 3.1.** In Table 3 are the cases where Corollary 4.2 holds for \( e_0 \) in the minimal orbit of \( E_6 \). Here \( c(\beta_0) \) is the semisimple part of a Levi subalgebra and has type \( A_4 \). The relevant nonzero nilpotent \( G \)-orbits are those that have nontrivial intersection with \( c(\beta_0) \). They appear in the table for \( e \).

### Table 3. \( E_6 \): \( e_0 \in c(s) \) of type \( A_4 \) in \( g \)

| \( e \) | \( e + e_0 \in \mathcal{O} \) | \( c(s) \) | Isomorphism type of \( S_{O,e} \) |
|---|---|---|---|
| \( 2A_1 \) | \( 2A_1 \) | \( A_5 \) | \( a_5 \) |
| \( 2A_1 \) | \( 3A_4 \) | \( B_4 + T_1 \) | \( b_3 \) |
| \( 3A_1 \) | \( 4A_1 = A_2 \) | \( A_2 + A_1 \) | \( A_1 \) |
| \( A_2 \) | \( A_2 + A_1 \) | \( 2A_2 \) | \( [a_2] \) |
| \( A_2 + A_1 \) | \( A_2 + 2A_1 \) | \( A_2 + T_1 \) | \( a_2 \) |
| \( 2A_2 \) | \( 2A_2 + A_1 \) | \( G_2 \) | \( g_2 \) |
| \( A_1 \) | \( A_3 + A_1 \) | \( B_4 + T_1 \) | \( b_2 \) |
| \( A_3 + A_1 \) | \( A_3 + 2A_1 = D_4(\alpha_3) \) | \( A_1 + T_1 \) | \( A_1 \) |
| \( A_4 \) | \( A_4 + A_1 \) | \( A_1 + T_1 \) | \( A_1 \) |
| \( A_5 \) | \( A_5 + A_1 = E_6(\alpha_3) \) | \( A_1 \) | \( A_1 \) |

In the first line of Table 3 is the remaining case where Corollary 4.3 holds for a minimal nilpotent element \( e_0 \in c(s) \). This completes the verification of Proposition 3.1 for \( E_6 \).

8.2. **The remaining cases where Proposition 3.3 holds.** There are only two cases that remain to be checked. Both cases can be handled by passing to a subalgebra of type \( F_4 \), as in Lemma 2.2. The values of \( (m_i, n_i) \) for \( i \in \mathcal{E} \) are listed in Table 4. **Boldface** is used for those \( (m_i, n_i) \) where the corresponding coefficient \( b_i \neq 0 \) in (4.7).
### Table 4. $E_6$: Remaining relevant cases with $e_0$ minimal in $\iota(\mathfrak{s})$

| $e$ | $e_0$ | $e + e_0$ | $\mathcal{O}$ | $e(\mathfrak{s})$ | $(m_i, n_i)$ for $i \in \mathcal{E}$ | Isomorphism type of $\mathcal{S}_{0, e}$ |
|-----|------|---------|------------|----------------|-------------------------------|--------------------------------|
| $D_4$ | $2A_1$ | $D_5(a_1)$ | $D_3(a_1)$ | $A_2$ | $\emptyset$ | $a_2$ |
| $A_2 + 2A_1$ | $A_2 + 2A_1$ | $D_4(a_1)$ | $A_3$ | $A_1 + T_1$ | $(1, 3), (1, 3), (2, 4)$ | $A_1$ |
| $2A_2 + A_1$ | $3A_1$ | $D_4(a_1)$ | $A_3 + A_1$ | $A_1$ | $(1, 3)$ | $m$ |

8.2.1. $e$ in $A_2 + 2A_1$. We can choose $e$ in $\mathfrak{t}$, the Lie algebra of type $F_4$ fixed under outer involution of $E_6$. The result in $F_4$ for $(B_2, A_2 + A_1)$ implies the result for $(A_3, A_2 + 2A_1)$ in $E_6$ by Lemma 2.2. The result can also be deduced from Lemma 2.2 by working in a Levi subalgebra of type $D_5$.

8.2.2. $e$ in $2A_2 + A_1$. We can again choose $e$ in $\mathfrak{t}$ of type $F_4$. The result in $F_4$ for $(C_5(a_1), A_2 + A_1)$ implies the result for $(A_3, A_2 + 2A_1)$ in $E_6$. This result can also be deduced from Lemma 2.2 by working in the subalgebra $\mathfrak{s}' \oplus e(\mathfrak{s})$, where $\mathfrak{s}'$ is an $\mathfrak{sl}_2$-subalgebra through an element of type $2A_2$ and then using Example 1.4.4 since $\iota(\mathfrak{s}')$ is of type $G_2$.

8.3. Remaining surface singularities. The results are listed in Table 7. In the first four entries of the table we have the fact that the larger orbit has normal closure $\mathfrak{so}(6,2)$. The entry for $(E_6(a_1), D_5)$ is from [5.6.1]. The entry for $(A_1, D_4(a_1))$ is $\mathcal{S}_{0, e}$ since the irreducible components are isomorphic and one of them is isomorphic to $C_2$ from Table 1.3. Alternatively, it follows from working in the Levi subalgebra $D_5$ (2.3). The entry for $(D_4, D_4(a_1))$ is also clear from working in the Levi subalgebra $D_5$. The degenerations $(E_6(a_1), D_4(a_1))$ and $(2A_2, A_2 + A_1)$ are both $A_2$ from Table 1.3. Note that $2A_2$ is unibranch at $A_2 + A_1$, but its closure is not normal.

### Table 5. Surface singularities using $5\mathbb{P}^1 S$ $E_6$

| Degeneration | Induced from | $5\mathbb{P}^1 S$ | $A(\mathfrak{s})$ | $\sharp$ orbits of $A(\mathfrak{s})$ | $\mathcal{S}_{0, e}$ |
|--------------|--------------|-----------------|----------------|-----------------|----------------|
| $(E_6(a_1), D_5)$ | $(A_1, 0)$ | 5 | 1 | | $A_5$ |
| $(D_5, E_6(a_1))$ | $(2A_1, 0)$ | 4 | $S_2$ | 3 | $C_3$ |
| $(D_4(a_1), A_4 + A_1)$ | $(A_2 + A_1, 0)$ | 2 | 1 | | $A_2$ |
| $(A_5, A_4 + A_1)$ | $(D_4, 32^{21})$ | 2 | 1 | | $A_2$ |
| $(D_4, D_4(a_1))$ | $(2A_2, 0)$ | 4 | $S_3$ | 2 | $G_2$ |
| $(A_4, D_4(a_1))$ | $(A_2, 0)$ | 9 | $S_3$ | 2 | $3C_2$ |

8.4. The degeneration $(2A_2 + A_1, A_2 + 2A_1)$. See §12.

9. Results for $E_7$

9.1. Proving Proposition 3.3. In Table 6 are the cases where Corollary 1.2 holds for $e_0$ in the minimal orbit of $E_7$. Here $\iota(s_0)$ is the semisimple part of a Levi subalgebra and has type $D_5$. The relevant nonzero nilpotent $G$-orbits are those that have nontrivial intersection with $\iota(s_0)$. They appear in the table for $e$. In the first several lines of Table 6 are the remaining cases where Corollary 2.3 holds for a minimal nilpotent element $e_0 \in \iota(\mathfrak{s})$. This completes the verification of Proposition 3.1 for $E_7$.

9.2. Verifying the remaining cases where Proposition 3.3 holds. The nine remaining cases, involving $e$ from six different $G$-orbits, are listed in Table 5. As before, the values of $(m_i, n_i)$ are listed in boldface for those $(m_i, n_i)$ for $i \in \mathcal{E}$ where the corresponding coefficient $b_i \neq 0$ in [17].

The cases where $e$ is type $A_2 + 2A_1$ and $2A_2 + A_1$ follow by restricting to a subalgebra of type $E_6$ and using Lemma 2.2. The case where $e$ is type $A_3 + A_1$ proceeds as in Example 1.4.4 or by restricting to a subalgebra $\mathfrak{s}' \oplus \iota(\mathfrak{s}')$, where $\mathfrak{s}'$ is an $\mathfrak{sl}_2$-subalgebra through an element of type $A_2'$. The two cases where
### Table 6. \( E_7 \): \( e_0 \in c(s) \) of type \( A_1 \) in \( \mathfrak{g} \)

| \( e \) | \( e + e_0 \in \mathcal{O} \) | \( c(s) \) | Isomorphism type of \( S_{\mathcal{O},e} \) |
|------|----------------|------|---------------------|
| \( A_1 \) | 2\( A_1 \) | 2\( A_1 \) | \( D_6 \) | \( d_6 \) |
| 2\( A_1 \) | (3\( A_1 \))'' | \( B_4 + A_1 \) | \( A_1 \) |
| (3\( A_1 \))' | \( B_4 + A_1 \) | \( b_4 \) |
| (3\( A_1 \))'' | 4\( A_1 \) | \( F_4 \) | \( f_4 \) |
| (3\( A_1 \))' | 4\( A_1 \) | \( C_3 + A_1 \) | \( c_3 \) |
| \( A_2 \) | \( A_2 + A_1 \) | \( A_5 \) | \( a^5 \) |
| 4\( A_1 \) | 5\( A_1 = A_2 + A_1 \) | \( C_3 \) | \( c_3 \) |
| \( A_2 + A_1 \) | \( A_2 + 2A_1 \) | \( A_3 + T \) | \( a^2 \) |
| \( A_2 + 2A_1 \) | \( A_2 + 3A_1 \) | \( A_1 + A_1 + A_1 \) | \( A_1 \) |
| \( A_3 \) | (3\( A_1 \))'' | \( B_3 + A_1 \) | \( A_1 \) |
| 2\( A_2 \) | 2\( A_2 + A_1 \) | \( G_2 + A_1 \) | \( g_2 \) |
| (3\( A_1 \))' | \( A_3 + 2A_1 \) | \( A_1 + A_1 + A_1 \) | \( A_1 \) |
| (3\( A_1 \))'' | \( A_3 + 2A_1 = D_4(a_1) \) | \( A_1 + A_1 + A_1 \) | \( A_1 \) |
| \( A_3 + A_1 \)'' | \( A_3 + 2A_1 \) | \( B_3 \) | \( b_3 \) |
| \( D_4(a_1) \) | \( D_4(a_1) + A_1 \) | \( 3A_1 \) | \( [3A_1]^{++} \) |
| \( A_3 + 2A_1 \) | \( A_3 + 3A_1 = D_4(a_1) + A_1 \) | \( A_1 + A_1 \) | \( A_1 \) |
| \( D_4 \) | \( D_4 + A_1 \) | \( C_3 \) | \( c_3 \) |
| \( D_4(a_1) + A_1 \) | \( D_4(a_1) + 2A_1 = A_3 + A_2 \) | \( 2A_1 \) | \( [2A_1]^+ \) |
| \( A_3 + A_2 \) | \( A_3 + A_2 + A_1 \) | \( A_1 + T \) | \( A_1 \) |
| \( A_4 \) | \( A_4 + A_1 \) | \( A_2 + T \) | \( a^2 \) |
| \( D_4 + A_1 \) | \( D_4 + 2A_1 = D_3(a_1) \) | \( B_2 \) | \( b_2 \) |
| (3\( A_1 \))'' | \( A_5 + A_1 \) | \( G_2 \) | \( g_2 \) |
| \( D_5(a_1) \) | \( D_5(a_1) + A_1 \) | \( A_1 + T \) | \( A_1 \) |
| \( (A_5)^{'} \) | \( (A_5 + A_1)^{'} = E_6(a_3) \) | \( A_1 + A_1 \) | \( A_1 \) |
| \( D_6(a_2) \) | \( D_6(a_2) + A_1 = E_7(a_5) \) | \( A_1 \) | \( A_1 \) |
| \( D_6 \) | \( D_6 + A_1 \) | \( A_1 + A_1 \) | \( A_1 \) |
| \( D_6 \) | \( D_6 + A_1 = E_7(a_4) \) | \( A_1 \) | \( A_1 \) |

\( \epsilon \) is type \( D_5(a_1) + A_1 \) are similar to Example 4.4.3. The three minimal degenerations lying above the orbit \( A_1 + A_2 \) and the one above the orbit \( A_1 + A_2 + A_1 \) require an explicit computer calculation, similar to §7.2.3, whose details are omitted.

#### 9.3. Remaining surface singularities.

The results using §8 are collected in Table 8. We have used the fact that \( E_7(a_1), E_7(a_3), E_7(a_5), E_6(a_3) \) have normal closure [Bro98a, Section 7.8]. The method from [Som03] can be used to show \( D_6 \) has normal closure. The entry for \( (E_7(a_1), E_7(a_2)) \) is from §7.6.2. For the three degenerations above \( E_7(a_3) \), the irreducible components are normal (see §9.4).

The remaining six minimal degenerations are unibranch, but either the larger orbit has non-normal closure or it is not known whether the larger orbit has normal closure. In all cases we are able to determine that the slice is normal and hence fully determine the singularity. The corresponding action of \( A(\epsilon) \) is determined using §1. The degeneration \( (D_6, E_6(a_3)) \) is \( C_5 \) and \( (D_5(a_1), A_4 + A_1) \) is \( A^2_1 \) by restriction to \( E_6 \), see Table 12. The other four degenerations follow from Table 13.

#### 9.4. Additional calculations in \( E_7 \).
34

#### Table 7. $E_7$: Remaining relevant cases with $e_0$ minimal in $c(s)$

| $e$       | $e_0$ | $e + e_0$ | $\mathcal{O}$ | $c(s)$ | $(m, n_i)$ for $i \in \mathcal{E}$ | Isomorphism type of $S_{\mathcal{O}, e}$ |
|-----------|-------|-----------|----------------|--------|---------------------------------|---------------------------------|
| $A_2 + 2A_1$ | $2A_1$ | $A_2$ | $A_2$ | $A_1 + A_1 + A_1$ | $\emptyset$ | $A_1$ |
| $A_2 + 3A_1$ | $2A_1$ | $2A_2 + A_1$ | $2A_2 + A_1$ | $G_2$ | $\emptyset$ | $G_2$ |
| $2A_2$ | $(3A_1)^*$ | $(A_3 + A_1)^*$ | $(A_3 + A_1)^*$ | $G_2 + A_1$ | $\emptyset$ | $A_1$ |
| $(A_3)^*$ | $(3A_1)^*$ | $D_6(a_2)$ | $D_6(a_2)$ | $A_1 + A_1$ | $\emptyset$ | $A_2$ |
| $D_5$ | $2A_1$ | $D_6(a_1)$ | $D_6(a_1)$ | $A_1 + A_1$ | $\emptyset$ | $A_1$ |
| $D_5 + A_1$ | $2A_1$ | $E_7(a_4)$ | $E_7(a_4)$ | $A_1$ | $\emptyset$ | $A_1$ |
| $A_6$ | $A_2 + 3A_1$ | $E_7(a_4)$ | $E_7(a_4)$ | $A_1$ | $\emptyset$ | $A_1$ |
| $E_6(a_1)$ | $(3A_4)^*$ | $E_7(a_4)$ | $E_7(a_4)$ | $A_1$ | $\emptyset$ | $A_1$ |
| $E_6$ | $(3A_4)^*$ | $E_7(a_2)$ | $E_7(a_2)$ | $A_1$ | $\emptyset$ | $A_1$ |

| $A_2 + 2A_1$ | $A_2 + 2A_1$ | $D_6(a_1)$ | $A_1$ | $A_1 + A_1 + A_1$ | $(1, 3)^*, (2, 4)$ | $A_1$ |
| $2A_2 + A_1$ | $(3A_1)^*$ | $D_6(a_1)$ | $(A_3 + A_1)^*$ | $A_1 + A_1$ | $(1, 3)$ | $m$ |
| $A_3 + A_2 + A_1$ | $(3A_1)^*$ | $E_7(a_3)$ | $E_7(a_3)$ | $A_1 + A_1$ | $(2, 4), (2, 8), (4, 6)$ | $A_1$ |
| $A_4 + A_2$ | $(3A_1)^*$ | $E_7(a_3)$ | $E_7(a_3)$ | $A_1 + A_1$ | $(2, 4), (4, 6)$ | $A_1$ |
| $A_5 + A_1$ | $(3A_1)^*$ | $E_7(a_3)$ | $E_7(a_3)$ | $A_1 + A_1$ | $(2, 4), (4, 6)$ | $A_1$ |
| $D_5(a_1) + A_1$ | $(3A_1)^*$ | $E_7(a_3)$ | $E_7(a_3)$ | $A_1 + A_1$ | $(2, 4)$ | $A_1$ |

9.4.1. \(\text{Sing}(A_6, E_7(a_4)) = G_2\) and \(\text{Sing}(D_5 + A_1, E_7(a_5)) = G_2\). We will show that the degenerations \((A_6, A_7)^*\) and \(D_5 + A_1, A_7^*\) are both isomorphic to the nilpotent cone of a Lie algebra of type $G_2$, from which the result will follow. The proof parallels $\S$ 4.6 except that here the singularities are unibranch.

Let $e$ be in the orbit $A_7^*$. Then $c(s)$ is a simple Lie algebra of type $G_2$. Let $e_0$ be a regular nilpotent element in $c(s)$. Then $y = e + e_0$ lies in the orbit $E_7(a_3)$ and $(m, n) = (4, 6)$ for the unique element in $\mathcal{E}$. Here, the simple part of $g^b$ is $s_0s$. Still, we can choose $v = (ad f)^2(X^3)$ with $X = e_0 \in c(s) \subset s_0s \subset s_0$. Using GAP we showed that there an element in the orbit $A_6$ equal to $e + e_0 + be$ for a unique $b \neq 0$, and also for an element in the orbit $D_5 + A_1$. The rest of the proof in $\S$ 4.6 applies to give the result.

9.4.2. \(\text{Sing}(D_6(a_1), E_7(a_5)) = 3C_3\). We will show that the degeneration \((D_6(a_1), D_4)\) has one branch which is equivalent to the whole nilpotent cone of $s_0$. (There are two branches of $D_6(a_1)$ above $D_4$.) Let $e$ be in the orbit $D_4$. Then $c(s) \cong s_0$. Let $e_0$ be a regular nilpotent element in $c(s)$. Then $y = e + e_0$ also

---

#### Table 8. Surface singularities using $\S 4.5$: $E_7$

| Degeneration | Induced from | $\mathbb{P}^1$ | $A(e)$ | $\sharp$ orbits of $A(e)$ | $S_{\mathcal{O}, e}$ |
|--------------|--------------|---------------|--------|--------------------------|-------------------|
| $(E_7(a_1), E_7(a_2))$ | $(A_1, 0)$ | 6 | 1 | $D_6$ |
| $(E_7(a_2), E_7(a_3))$ | $(2A_1, 0)$ | 5 | $S_2$ | 4 | $C_4$ |
| $(E_7(a_3), E_7(a_4))$ | $(3A_4)^*, 0$ | 5 | $S_2$ | 3 | $B_3$ |
| $(E_6, E_7(a_1))$ | $(3A_4)^*, 0$ | 6 | $S_2$ | 4 | $F_4$ |
| $(E_6(a_1), E_7(a_4))$ | $(A_4, 0)$ | 4 | $S_3$ | 3 | $C_3$ |
| $(D_6, E_7(a_1))$ | $(D_3, 32^1)^+$ | 4 | $S_2$ | 3 | $C_3$ |
| $(A_6, E_7(a_1))$ | $(A_2 + 3A_1, 0)$ | 4 | $S_3$ | 2 | $G_2$ |
| $(D_6(a_1), E_7(a_5))$ | $(2A_2, 0)$ | 4 | $S_3$ | 2 | $G_2$ |
| $(D_6(a_1), E_7(a_5))$ | $(A_3, 0)$ | 12 | $S_3$ | 3 | $3C_3$ |
lies in the orbit $E_7(a_4)$ and $\mathfrak{g}$ decomposes in \text{[1 2]} as
\[ V(0, 10) \oplus V(0, 6) \oplus V(0, 2) \oplus V(2, 0) \oplus V(6, 4) \oplus V(6, 8) \oplus V(10, 0), \]
reflecting that $\mathfrak{c}(s)$ decomposes under $s_0$ as $V(10) \oplus V(6) \oplus V(2)$ and $\mathfrak{g}'(-6)$ decomposes under $s_0$ as $V(4) \oplus V(8)$, which is 14-dimensional and as a representation of $\mathfrak{c}(s)$ is $V(\omega_2)$. Also $(m, n) = (6, 8)$ for the unique element in $\mathfrak{s}$.

The semisimple part of $\mathfrak{g}^e$ is isomorphic to $\mathfrak{s}l_6$. If we take $X = e_0$, then $X^4 \in \mathfrak{s}l_6$ is nonzero since $X$ is regular. It cannot be in $\mathfrak{s}l_6$ since only odd powers of $X$ are. It satisfies $[h_0, X^4] = 8X^4$ and so it must be a highest weight vector in $V(8)$ for $s_0$ with respect to $e_0$. Hence we can choose $v = (ad f)^8(X^4)$ (§4.4).

We checked using GAP that there is an $x$ in the orbit $D_4(a_1)$ with
\[ x = e + e_0 + be, \]
with $b \neq 0$. Since the elements $C(s) \cdot (e_0 + be)$ consist of pairs $(X, X^4) \in \mathfrak{s}p_6 \oplus V(\omega_2) \cong \mathfrak{s}l_6$ with $X \in \mathfrak{s}p_6$ nilpotent, the result follows. It then follows that one branch of $(D_4(a_1), E_7(a_1))$ is isomorphic to $C_3$, hence the singularity is $3C_3$.

9.4.3. The degeneration $(A_4 + A_1, A_5 + A_2 + A_1)$. In the Appendix of \text{[FJLSb]}, we shall prove that the degeneration $(A_4 + A_1, A_5 + A_2 + A_1)$ has singularity $a_2/S_2$, where the action of $S_2$ is given by $A \mapsto -A'$, where $A \in a_2$ is a $3 \times 3$ nilpotent matrix of rank 1.

10. Results for $E_8$

10.1. Proving Proposition 3.1. In Table 1 are the cases where Corollary 4.2 holds for $e_0$ in the minimal orbit of $E_8$. The centralizer $\mathfrak{c}(s_0)$ is the semisimple part of a Levi subalgebra of type $E_7$. The nonzero nilpotent $G$-orbits meeting $\mathfrak{c}(s)$ are those which appear in the table.

In the first several lines of Table 10 are the remaining case where Corollary 3.3 holds for a minimal nilpotent element $e_0 \in \mathfrak{c}(s)$. This completes the verification of Proposition 3.1 for $E_8$.

10.2. Verifying the remaining cases where Propositions 3.3 and 3.4 hold. The results are listed in the latter lines of Table 10.

10.2.1. The degeneration $(A_3 + 2A_1, 2A_2 + 2A_1)$ is $m'$. Here $e$ is in the orbit $2A_2 + 2A_1$ and $\mathfrak{c}(s) \cong \mathfrak{s}p_4$. Let $e_0$ be in the minimal nilpotent orbit of $\mathfrak{c}(s)$. In this case $x$ has one element with $(m, n) = (1, 3)$.

Consider the Levi subalgebra $\mathfrak{l}$ of type $E_8 + A_1$. Then without loss of generality $e \in \mathfrak{l}$ (with nonzero component on the $A_1$ factor) and $e_0 \in \mathfrak{l}$ (contained in the $E_8$ factor). The result in §4.2.2 for $E_8$ shows that there is an $x$ in the orbit $\mathfrak{O}$ of type $A_3 + 2A_1$ (in $E_8$) with $x = e + e_0 + v$ for a choice of $v$ corresponding to $(1, 3)$. If $\mathfrak{c}(s) \cong V(2\omega_1)$, then $v$ is a highest weight vector for a $\mathfrak{c}(s)$-module $V(3\omega_1)$. Hence $\mathfrak{S}_{\mathfrak{c},v} \cong m'$.

10.2.2. The degenerations $(A_4 + A_1, D_4(a_1) + A_2)$ and $(2A_3, D_4(a_1) + A_2)$ are $a_2^+$.

We can work in the maximal subalgebra of type $D_8$ and imitate §4.4.3 to show that $x = e + e_0 + be$ for a unique $b \neq 0$, for an $x$ in either the orbit $A_4 + A_1$ or the orbit $2A_3$. Since (4.7) holds in $D_8$, it holds in $\mathfrak{g}$. The results follows from §4.3.

10.2.3. The degenerations $(A_5 + A_1, A_4 + A_3)$ and $(D_5(a_1) + A_2, A_4 + A_3)$ are $m$. In both of these cases $e$ is the orbit $A_4 + A_3$, for which $\mathfrak{c}(s) \cong \mathfrak{s}l_6$. This case is a more complicated version of §7.2.3 in $F_4$.

Using the information in \text{[LT11]} p. 146 (adjusted for sign differences in GAP), let
\[ e = -(4f_{a_1} + 6f_{a_2} + 6f_{a_4} + 4f_{a_5} + 3f_{a_6} + 4f_{a_7} + 3f_{a_8}), \quad f = \sum_{i \neq 5} e_{a_i} \quad \text{and} \quad h = [e, f]. \]

A nilpositive element in $\mathfrak{c}(s)$ is
\[ e_0 = 2e_{112211} - e_{1232110} + 2e_{0122211} + e_{1322100} + e_{122210} + e_{1222111} \]
| $e$   | $e + e_0 \in \mathcal{O}$ | $c(\hat{s})$ | Isomorphism type of $\mathcal{S}_{O,*}$ |
|-------|--------------------------|-------------|--------------------------------------|
| $A_1$ | $2A_1$                   | $E_7$       | $e_7$                                |
| $2A_1$| $3A_1$                   | $B_6$       | $b_6$                                |
| $3A_1$| $4A_1$                   | $F_4 + A_1$ | $f_4$                                |
| $A_2$ | $2A_2 + A_1$             | $E_6$       | $e_6^+$                              |
| $4A_1$| $5A_1 = A_2 + A_1$       | $C_4$       | $c_4$                                |
| $A_2 + A_1$| $A_2 + 2A_1$           | $A_1$       | $a_1^+$                              |
| $A_2 + 2A_1$| $A_2 + 3A_1$         | $B_3 + A_1$ | $b_3$                                |
| $A_2 + 3A_1$| $A_2 + 4A_1 = 2A_2$   | $G_2 + A_1$ | $A_1$                                |
| $A_3$ | $A_3 + A_1$              | $B_6$       | $b_6^+$                              |
| $2A_2$| $2A_2 + A_1$             |             | $2G_2$                               |
| $2A_2 + A_1$| $2A_2 + 2A_1$         | $G_2 + A_1$ | $g_2$                                |
| $A_3 + A_1$| $A_3 + 2A_1$           | $B_3 + A_1$ | $b_3$                                |
| ($A_3 + 2A_1)^{++} = D_4(a_1)$ |             |             |                                       |
| $A_3 + 2A_1$| $A_3 + 3A_1 = D_4(a_1) + A_1$ | $B_2 + A_1$ | $b_2$                                |
| $D_4(a_1)$| $D_4(a_1) + A_1$        | $D_4$       | $d_4^{++}$                           |
| $D_4(a_1) + A_1$| $D_4(a_1) + 2A_1 = A_3 + A_2$ | $3A_1$     | $[3A_1]^{++}$                        |
| $A_3 + A_2$| $A_3 + A_2 + A_1$      | $B_2 + T_1$ | $b_2$                                |
| $A_3 + A_2 + A_1$| $A_3 + 2A_1 = D_4(a_1) + A_2$ | $A_1 + A_1$ | $A_1$                                |
| $A_4$ | $A_4 + A_1$              | $A_4$       | $A_4$                                |
| $D_5$ | $D_5 + A_1$              | $F_4$       | $f_4$                                |
| $D_4 + A_1$| $D_4 + 2A_1 = D_5(a_1)$ | $C_5$       | $c_3$                                |
| $A_4 + A_1$| $A_4 + 2A_1$           | $A_3 + T_1$ | $a_2^+$                              |
| $2A_3$| $2A_3 + A_1 = A_4 + A_1$ | $B_3$       | $b_2$                                |
| $D_5(a_1)$| $D_5(a_1) + A_1$        | $A_3$       | $a_3$                                |
| $A_4 + A_2$| $A_4 + A_2 + A_1$      | $A_4 + A_1$ | $A_1$                                |
| $D_5(a_1) + A_1$| $D_5(a_1) + 2A_1 = D_4 + A_2$ | $A_4 + A_1$ | $A_1$                                |
| $A_5$ | $A_5 + A_1$              | $G_2 + A_1$ | $g_2$                                |
| $A_5 + A_1$| $A_5 + A_1 = E_6(a_3)$ | $G_2 + A_1$ | $g_2$                                |
| $A_6$ | $A_6 + A_1$              | $A_6 + A_1$ | $A_1$                                |
| $A_6 + A_1$| $A_6 + A_1 = E_7(a_5)$ | $G_2$       | $g_2$                                |
| $D_6$ | $D_6 + A_1$              | $B_3$       | $b_3$                                |
| $E_6(a_5)$| $E_7(a_5) + A_1 = E_6(a_5)$ | $A_1$     | $A_1$                                |
| $D_6 + A_1$| $D_6 + 2A_1 = D_6(a_1)$ | $A_1 + A_1$ | $A_1$                                |
| $D_6(a_1)$| $D_6(a_1) + A_1 = E_7(a_4)$ | $2A_1$ | $[2A_1]^{++}$                        |
| $A_7$ | $A_7 + A_1$              | $A_7 + A_1$ | $A_1$                                |
| $E_6(a_4)$| $E_7(a_4) + A_1 = D_6 + A_2$ | $A_1$     | $A_1$                                |
| $E_6(a_1)$| $E_6(a_1) + A_1$        | $A_2$       | $a_2^+$                              |
| $D_6$ | $D_6 + A_1 = E_7(a_3)$  | $B_3$       | $b_3$                                |
| $E_6$ | $E_6 + A_1$              | $G_2$       | $g_2$                                |
| $E_7(a_3)$| $E_7(a_3) + A_1 = D_7(a_1)$ | $A_1$     | $A_1$                                |
| $E_7(a_2)$| $E_7(a_2) + A_1 = E_8(b_4)$ | $A_1$ | $A_1$                                |
| $E_7(a_1)$| $E_7(a_1) + A_1 = E_8(b_4)$ | $A_1$ | $A_1$                                |
| $E_7$ | $E_7 + A_1 = E_8(a_4)$  | $A_1$       | $A_1$                                |
Table 10. $E_8$: Remaining relevant cases with $e_0$ minimal in $\epsilon(s)$

| $\epsilon$ | $e_0$ | $\epsilon + e_0$ | $O$ | $\epsilon(s)$ | $(m_i, n_i)$ for $i \in \mathcal{E}$ | Isomorphism type of $s_{O, i}$ |
|------------|-------|------------------|-----|---------------|--------------------------------|-------------------------------|
| $D_4 + A_2$ | 2A1   | $D_5(a_1) + A_2$ | $D_5(a_1) + A_2$ | $A_2$ | $\emptyset$ | $A_2^2$ |
| $A_6$       | $A_2 + 3A_1$ | $E_7(a_8)$      | $E_7(a_8)$ | $A_1 + A_1$ | $\emptyset$ | $A_1$ |
| $A_4 + 2A_1$| 2A1   | $A_4 + A_2$     | $A_4 + A_2$ | $A_1 + T_1$ | $\emptyset$ | $A_1$ |
| $A_6 + A_1$ | $A_2 + 3A_1$ | $D_5 + A_2$    | $D_5 + A_2$ | $A_1$ | $\emptyset$ | $A_1$ |
| $A_7$       | 4A1   | $E_8(b_8)$      | $E_8(b_8)$ | $A_1$ | $\emptyset$ | $A_1$ |
| $D_t$       | 2A1   | $E_8(a_8)$      | $E_8(a_8)$ | $A_1$ | $\emptyset$ | $A_1$ |
| $A_2 + 2A_1$| $A_2 + 2A_1$ | $D_5(a_1)$    | $A_2$ | $B_3 + A_4$ | $(1,3)^\times (2,4)$ | $A_1$ |
| $2A_2 + A_1$| 3A1   | $D_4(a_1)$     | $D_4(a_1)$ | $A_1 + A_1$ | $G_2 + A_4$ | $(1,3)$ | $m$ |
| $2A_2 + 2A_1$| 3A1 | $D_4(a_1) + A_1$ | $A_1 + 2A_1$ | $B_2$ | $(1,3)$ | $m'$ |
| $A_3 + A_2 + A_1$ | $A_1 + A_2$ | $E_7(a_7)$ | $E_7(a_7)$ | $A_1 + A_1$ | $(1,2,4,2,8,4,6)$ | $A_1$ |
| $D_4(a_1) + A_2$ | 2A1 + 2A1 | $A_4 + 2A_1$ | $A_4 + 2A_1$ | 2A3 | $A_2$ | $(2,4)$ | $a_2^2$ |
| $A_4 + A_2$ | $A_3 + A_3 + A_1$ | $E_7(a_7)$ | $E_7(a_7)$ | $A_3$ | $A_1 + A_1$ | $(2,4,4,6)$ | $A_1$ |
| $D_5(a_1) + A_1$ | 2A2 | $E_7(a_7)$ | $E_7(a_7)$ | $A_2$ | $A_1 + A_1$ | $(2,4)$ | $a_2^2$ |
| $A_4 + A_2 + A_1$ | $A_3 + A_3 + A_1$ | $E_7(a_7)$ | $E_7(a_7)$ | $A_3$ | $A_1 + A_1$ | $(2,4,4,6)$ | $A_1$ |
| $A_4 + A_3$ | 2A2 + 2A1 | $E_8(a_8)$ | $E_8(a_8)$ | $A_3 + A_1$ | $A_1$ | $(1,3,2,4,3,5)$ | $m$ |
| $A_5 + A_1$ | 3A1 | $E_7(a_7)$ | $E_7(a_7)$ | $D_5(a_1) + A_2$ | $A_1 + A_1$ | $(1,3,2,4,3,5)$ | $m$ |
| $D_5(a_1) + A_2$ | 2A1 + A1 | $E_7(a_7)$ | $E_7(a_7)$ | $D_5(a_1) + A_2$ | $A_1 + A_1$ | $(1,3,2,4,3,5)$ | $m$ |
| $E_8(a_8) + A_1$ | 3A1 | $E_7(a_7)$ | $E_7(a_7)$ | $A_3$ | $A_1 + A_1$ | $(1,3,2,4,3,5)$ | $m$ |

embedded in an $s_t$-triple $(e_0, h_0, s_0)$ for $\epsilon(s)$. Then $\mathcal{E} = \{(1,3), (2,4), (3,5)\}$. The spaces $g\langle -1 \rangle$, $g\langle -2 \rangle$ and $g\langle -3 \rangle$ contain highest weight modules for $\epsilon(s)$ with respective highest weights 3, 4 and 5, and highest weight vectors:

$$v_1 = e_{2443210} - e_{1343211} + e_{1243221} - e_{1233321}; \quad v_2 = e_{2454321} + e_{2354321}; \quad v_3 = e_{2465432}.$$

We checked in GAP that

$$e + e_0 + 3v_2 \pm (2v_1 + 4v_3) \in D_5(a_1) + A_2,$$

and

$$e + e_0 - \frac{7}{6}v_2 \pm \sqrt{\frac{8}{27}}(v_1 - \frac{19}{3}v_3) \in A_5 + A_1.$$

By Lemma [4.3](a), in both cases the nilpotent Slodowy slice is isomorphic to $m$.

10.2.4. The case where $e = A_4 + A_2 + A_1$ also needs to be handled by an explicit computation in GAP; we omit the details. The other remaining cases follow by restricting to some subalgebra.

10.3. Remaining surface singularities. The results using [5] are collected in Table 11. We use the fact that $E_6$, $E_8(a_1)$, $E_8(a_2)$, $E_8(a_3)$, $E_8(a_4)$ have normal closure [Bro98a, Section 7.8]. The method from [Som03] can be used to show $E_7$, $E_8(b_4)$, and $E_8(a_5)$ have normal closure. The entry for $E_8(a_6)$ is from [Som03]. That each irreducible component of $(D_5(a_1), E_8(a_7))$ and $(A_6, E_8(a_7))$ is $G_2$ follows from the fact that the degeneration $(E_6, D_4)$ contains a branch isomorphic to the nilpotent cone in $F_3$, and then from the results in type $F_4$. There are ten cases which are determined only up to normalization.
In four of these cases, the orbit closure is known to be non-normal: \((E_6(a_1), E_6(b_1)), (E_7(a_3), E_6(a_3) + A_1), (D_4(a_2), D_5 + A_1), (D_6, D_6 + A_2)\). The latter three are unibranched. The orbit closures, and hence the slices, for the other six are expected to be normal. We use \((\gamma)\) to denote a singularity with normalization \(\gamma\).

The remaining nine surface degenerations are unibranch, but either the larger orbit has non-normal closure or it is not known whether the larger orbit has normal closure. In these cases we are able to show that the slice is normal and hence fully determine the singularity. The action of \(A(e)\) is determined using \(\S\). The degeneration \((D_6, E_6(a_3))\) is \(C_5\) and the degeneration \((D_6(a_4), A_4 + A_1)\) is \(A_2^7\), both by restriction to \(E_6\) (see Table 13). The other degenerations follow from Table 13.

### Table 11. Surface singularities using \(\S\) in \(E_6\)

| Degeneration | Induced from | \(\mathbb{P}^1\) | \(A(e)\) | \(\S\) orbits of \(A(e)\) |
|--------------|--------------|-----------------|----------|----------------------------|
| \((E_6(a_1), E_6(a_2))\) | \((A_1, 0)\) | 7 | 1 | \(E_7\) |
| \((E_6(a_2), E_6(a_3))\) | \((2A_1, 0)\) | 7 | \(S_2\) | 6 | \(C_5\) |
| \((E_6(a_3), E_6(a_4))\) | \((3A_1, 0)\) | 6 | \(S_2\) | 4 | \(F_4\) |
| \((E_6(a_4), E_6(b_1))\) | \((4A_1, 0)\) | 5 | \(S_2\) | 4 | \(C_4\) |
| \((E_6(a_5), E_6(b_5))\) | \((A_2 + 3A_1, 0)\) | 4 | \(S_3\) | 2 | \(G_2\) |
| \((E_7(a_1), E_6(b_3))\) | \((A_3, 0)\) | 18 | \(S_4\) | 5 | \(3C_3\) |
| \((E_6(a_6), E_6(a_6))\) | \((2A_3 + A_1, 0)\) | 4 | \(S_4\) | 2 | \(G_2\) |
| \((E_7(a_3), E_6(a_1) + A_1)\) | \((D_5, 3^2 2^2 1^2)\) | 4 | \(S_2\) | 2 | \(A_1^7\) |
| \((E_7(a_2), D_5 + A_2)\) | \((2A_2, 0)\) | 3 | \(S_2\) | 2 | \(C_2\) |
| \((E_7, E_6(b_1))\) | \((D_5, 32^2 1)\) | 6 | \(S_2\) | 4 | \(F_4\) |
| \((D_7, E_6(a_1))\) | \((D_4 + A_2, 32^2 1 + 0)\) | 4 | \(S_3\) | 2 | \(G_2\) |
| \((E_6(b_4), E_6(a_5))\) | \((A_2 + 2A_1, 0)\) | 4 | \(S_4\) | 3 | \(C_3\) |
| \((E_7(a_2), D_7(a_1))\) | \((D_5, 32^2 1^2)\) | 5 | \(S_4\) | 3 | \(B_3\) |
| \((D_7(a_1), E_6(b_6))\) | \((A_2 + A_2, 0)\) | 3 | \(S_5\) | 2 | \(C_2 = \mu\) |
| \((E_6 + A_1, E_6(b_6))\) | \((E_6, 2A_2 + A_1)\) | 4 | \(S_3\) | 2 | \(G_2\) |
| \((A_2 + D_7(a_2))\) | \((D_6 + A_2, 32^2 1^2 + 0)\) | 2 | \(S_2\) | 1 | \(A_2\) |
| \((E_6(a_1) + A_1, D_7(a_2))\) | \((E_7, A_4 + A_1)\) | 2 | \(S_2\) | 1 | \(A_2\) |
| \((D_6, D_6 + A_2)\) | \((D_6, 32^2 1)\) | 3 | \(S_2\) | 2 | \(C_2\) |
| \((D_6(a_1), E_6(a_2))\) | \((A_3, 0)\) | 40 | \(S_5\) | 2 | \(10G_2\) |
| \((A_6, E_6(a_2))\) | \((D_4 + A_2, 0)\) | 20 | \(S_4\) | 2 | \(5G_2\) |

### Table 12. Some surface cases where Lemma 2.2 holds

| \(g\) | \(e\) | \(x \in O\) | subalgebra | \(\S_{O,e}\) |
|-------|-------|------------|------------|--------------|
| \(E_7, E_6\) | \(E_6(a_3)\) | \(D_6\) | \(E_6\) | \(C_5\) |
| \(E_7, E_6\) | \(A_1 + A_1\) | \(D_6(a_1)\) | \(E_6\) | \(A_2^7\) |
| \(E_7, E_6\) | \(A_3 + A_2\) | \(A_4\) | \(D_6\) | \(C_2\) |

### 11. Slices related to entire nilcones

The main goal of the paper was to describe the nilpotent Slodowy slice \(S_{O,e}\) for a minimal degeneration. Many of the same ideas can be used to show that \(S_{O,e}\) has a familiar description when the degeneration is not minimal. In particular, there are many cases where \(S_{O,e}\) is isomorphic to the closure of a non-minimal orbit in a nilcone for a subalgebra of \(g\) or is isomorphic to a slice between two orbits in such a nilcone. Rather than listing all these cases here, we write down some cases where \(S_{O,e}\), or one of its irreducible components, is isomorphic to an entire nilcone. Some of these were used
to show in the surface case that $S_{\mathcal{O}^e}$, or an irreducible component of $S_{\mathcal{O}^e}$, is normal. These examples are relevant for the duality discussed in the $[9, 5]$ to be explored in future work.

11.1. **Exceptional groups.** The results are listed in Table 13. The notation $\mathcal{N}_X$ refers to the nilcone in the Lie algebra of type $X$. The proofs use Lemma 2.4, usually for $x' \neq 0$, and often require a computer calculation.

### Table 13. Slices containing a smaller nilcone

| $\mathfrak{g}$ | Degeneration and nilcone |
|---------------|--------------------------|
| $F_4$         | $(B_3, A_2) = N_G\gamma$ |
|               | $(C_3, A_2) \supset N_G\gamma$ |
| $E_6$         | $(E_6(a_3), D_4) = N_G\gamma$ |
|               | $(A_4, A_3) \supset N_G\gamma$ |
|               | $(2A_2, A_2) = [2N_{a_1}]$ |
| $E_7$         | $(E_7(a_3), D_5) = N_G\gamma$ |
|               | $(D_6(a_4), D_4) \supset N_G\gamma$ |
|               | $(E_7(a_5), A_4') = N_G\gamma$ |
|               | $(A_6, A_4') = N_G\gamma$ |
|               | $(D_6 + A_1, A_4') = N_G\gamma$ |
|               | $(A_4 + A_2, A_4) = N_{G\gamma}$ |
|               | $(D_4 + A_3, 3A_1) = N_G\gamma$ |
|               | $(D_4, 2A_2) = N_G\gamma$ |
|               | $(D_4(a_1) + A_1, (A_3 + A_1)^+) = N_G\gamma$ |
|               | $(A_6, A_3) \supset N_{G\gamma}$ |
| $E_8$         | $(E_8(a_5), E_6) = N_G\gamma$ |
|               | $(E_8(a_5), D_7) = N_G\gamma$ |
|               | $(E_6, D_4) \supset N_{E_7}$ |
|               | $(D_8 + A_1, A_5) \supset N_G\gamma$ |
|               | $(A_6, E_6(a_3)) \supset N_G\gamma$ |
|               | $(D_8(a_1), E_6(a_1)) \supset N_G\gamma$ |
|               | $(E_8(b_3), E_6(a_1)) = N_{G\gamma}$ |
|               | $(A_4, A_3 + 2A_1) \supset N_G\gamma$ |
|               | $(D_4, 2A_2) = 2N_G\gamma$ |

11.2. **Slices isomorphic to entire nilcones: two slices in $\mathfrak{sl}_N$.** These two examples are special cases of isomorphisms discovered by Henderson [Hen] using Maffei's work on quiver varieties [Maf05]. Here we give direct proofs that fit into the framework of Lemma 2.4 and 4.4. We are grateful to Henderson for bringing these examples to our attention.

11.2.1. **First Slice.** It is slightly more convenient to work in $\mathfrak{g} = \mathfrak{gl}_{nk}$. Assume $n \geq 2$ and $k \geq 1$. Consider the nilpotent orbit $\mathcal{O}'$ with partition $[n^g]$. Write $k = p(n + 1) + q$ with $0 \leq q < n + 1$, which gives $kn = (pn + q - 1)(n + 1) + (n + 1 - q)$ for maximally dividing $kn$ by $n + 1$. Let $\mathcal{O}$ be the nilpotent orbit with partition $[(n + 1)^{p(n + 1) - q}, n + 1 - q]$, which is a partition of $kn$. Then $\mathcal{O}' \subset \mathcal{O}$ by the dominance order for partitions. Moreover, $x \in \mathcal{O}'$ implies $x^{n+1} = 0$ and $\mathcal{O}$ is maximal for nilpotent orbits in $\mathfrak{gl}_{nk}$ with this property.

**Proposition 11.1.** [Hen, Corollary 9.5] Let $e \in \mathcal{O}'$. The variety $S_{\mathcal{O}^e}$ is isomorphic to

$$Y := \{ Y \in \mathfrak{gl}_k \mid Y^{n+1} = 0 \}.$$
In particular, $S_{O,e}$ is isomorphic to the closure of the nilpotent orbit in $\mathfrak{gl}_k$ with partition $[(n+1)^r,q]$, which is the whole nilcone when $k \leq n+1$.

**Proof.** Let $I_k$ be the $k \times k$ identity matrix. Define $e = (e_{ij})$, $h = (h_{ij})$, and $f = (f_{ij})$ to be $n \times n$-block matrices, with blocks of size $k \times k$, as follows:

$$e_{ij} = \begin{cases} j(n-j)I_k & i = j + 1 \\ 0 & \text{else} \end{cases}, \quad h_{ij} = \begin{cases} (2i-n-1)I_k & i = j \\ 0 & \text{else} \end{cases}, \quad f_{ij} = \begin{cases} I_k & j = i + 1 \\ 0 & \text{else} \end{cases}$$

The Jordan type of $e$ and $f$ is $[n^k]$, and so $e, f \in \mathcal{O}'$. The elements $(e, h, f)$ are a standard basis of an $\mathfrak{sl}_2$-subalgebra $s$, as in the $k = 1$ case. Also, as in the $k = 1$ case, the centralizer $g' \ell$ consists of $n \times n$-block matrices $Z = (z_{ij})$ of the form

$$z_{ij} = \begin{cases} Y_{j-i} & j \geq i \\ 0 & \text{otherwise} \end{cases}$$

for any choice of $Y_0, Y_1, \ldots, Y_{n-1} \in \mathfrak{gl}_k$. We abbreviate this matrix by $Z(\{Y_i\})$. In particular, $e(s) \cong \mathfrak{gl}_k$ consists of the matrices of the form $Z(\{Y, 0, \ldots, 0\})$.

We are interested in

$$S_{O,e} := S_e \cap \overline{S} = S_e \cap \{X \in \mathfrak{g} : X^{n+1} = 0\},$$

where as before $S_e = e + g' \ell$. Let $M = e + Z(\{Y_i\}) \in S_e$. Set $Y_0 = -\frac{1}{2}Y$ for a fixed matrix $Y$ for reasons that will become clear shortly. Since $M^{n+1} = 0$, we can find constraints on the entries of $M^{n+1}$. The $(n,1)$-entry of $M^{n+1}$ is equal to $rY_1 + sY_0^2$ where $r$ is a sum of products of the coefficient in $e$, hence nonzero. Thus $rY_1 + sY_0^2 = 0$ and $Y_1$ is proportional to $Y^2$. Given this fact, the $(n,2)$-entry of $M^{n+1}$ is equal to $r'Y_2 + s'Y_0^2$ where $r'$ is nonzero. Hence $Y_2$ is proportional to $Y^3$, and so on. In this way, we conclude that $Y_i = e_iY^{i+1}$ for all $i = 0, 1, 2, \ldots, n-1$, where the $c_i \in \mathbb{C}$ are uniquely determined constants (which depend on $n$, but not $k$). Consequently $M \in S_{O,e}$ takes the form $e + Z(\{c_iY^{i+1}\})$ for some $Y$. We were not able to find a general formula for the $c_i$'s, but in all cases that we computed, the $c_i$'s were nonzero, which we expect to be true in general.

Now let $T^n + \sum_{i=1}^{n-1} b_i T^{n-i} \in \mathbb{C}[T]$ be the characteristic polynomial for the $n \times n$-matrix $e + Z(\{c_i, I_i\})$ in the $k = 1$ case. A direct computation with block matrices then shows that $p(T) := T^n + \sum_{i=1}^{n-1} b_i T^{n-i}$ is the characteristic polynomial of $M$, viewing $M$ as an $n \times n$-matrix over the commutative ring $\mathbb{C}[Y]$, where $Y$ acts by simultaneous multiplication on each of the block entries of $M$. By the Cayley-Hamilton Theorem over $\mathbb{C}[Y]$, it follows that $p(M) = 0$. In fact, $p(T)$ is the minimal polynomial of $M$ over $\mathbb{C}[Y]$. Indeed, for $1 \leq i \leq n-1$, the $i$-th block lower diagonal of $M^i$ consists of non-zero scalar matrices while everything below that diagonal is zero. Thus $M$ cannot satisfy a polynomial of degree less than $n$ over $\mathbb{C}[Y]$.

The next step is to show that $Y^{n+1}$ must be the zero matrix. Since $p(M) = 0$,

$$0 = Mp(M) - b_1 Yp(M) = \sum_{i=2}^{n} (b_i - b_{i+1}) Y^i M^{n-i+1} - b_{n+1} Y^{n+1}.$$ 

Since the minimal polynomial of $M$ over $\mathbb{C}[Y]$ has degree $n$, it follows that $(b_i - b_{i+1}) Y^i = 0$ for $i = 2, \ldots, n$ and $b_{n+1} Y^{n+1} = 0$. Note that $b_i = 1$ by taking the trace of $M$ since $c_0 = -\frac{1}{2}$. Now if $Y^{n+1} \neq 0$, then recursively $b_i = b_{i+1} = b_{n+1} = 0$, a contradiction. Similarly, if $Y^i = 0$ and $Y^{i-1} \neq 0$ for some $\ell \leq n+1$, then $b_i = b_{i+1} = 1$ for $i = 1, 2, \ldots, \ell - 1$. We conclude that all elements in $S_{O,e}$ take the form $e + Z(\{c_i Y^i\})$ where $Y^{n+1} = 0$. Hence $S_{O,e}$ is isomorphic to a subvariety of $Y$ via restriction of the natural projection $\pi : S_e \twoheadrightarrow e(s)$ by the argument in [1.7]. Now $S_{O,e}$ and $Y$ both have dimension $p^2 n^2 + 2 p q n + p^2 n + q^2 - q$, and the latter variety is irreducible; hence $\pi$ gives an isomorphism of $S_{O,e}$ onto $Y$.

An interesting consequence is the following: since the $c_i$'s, and hence the $b_i$'s, are independent of $k$, choosing $k > n$, we deduce that all $b_i = 1$, an interesting fact in its own right. \hfill \Box

**Remark 11.2.** Fix $e_0 \in e(s)$ in the orbit $[(n+1)^p, q]$. In the language of [1.4], the vectors $v_i = Z(\{0, \ldots, 0, Y^{i+1}, 0, \ldots, 0\})$ have weight $(i, i+2)$ under $s \oplus s_0$, hence lie in $\mathcal{E}$. The proof shows that [1.5]...
holds for \( e_0 = v_0 \) together with the \( v_i \)'s for \( i \geq 1 \) with the coefficients \( c_i \) in the proof. As mentioned above, we do not know whether \( c_i \neq 0 \) in general, except for small values of \( i \).

11.2.2. Second Slice. Next, let \( \mathcal{O} \) be the orbit in \( \mathfrak{gl}_{n,k} \) with partition \( [(n + k - 1, (n - 1)^{k-1}] \). Then again \( e \in \mathfrak{O} \). The elements in \( \mathcal{O} \) correspond to matrices which are nilpotent and which have rank\( (M') = k(n - i) \) for \( i = 1, 2, \ldots, n-1 \).

**Proposition 11.3.** \([\text{Hen}] \) Corollary 9.3 The variety \( S_{\mathcal{O},e} \) is isomorphic to the nilcone in \( \mathfrak{gl}_n \).

**Proof.** Up to smooth equivalence, this result is a consequence of \([\text{KP81}]\), by cancellation of the first \( n-1 \) columns of the partitions for \( \mathcal{O} \) and \( \mathcal{O}' \). Here, we show that, in fact, \( S_{\mathcal{O},e} \cong \mathcal{N}_{A_{k-1}} \), which also follows from \([\text{Hen}] \) Corollary 9.3.

Keep the notation from the proof of the previous proposition. Let \( M \in S_n \) satisfying the rank conditions rank(\( M' \)) = \( k(n - i) \) for \( i = 1, 2, \ldots, n-1 \). The last rank condition is rank\( (M^{n-1}) = k \). The bottom, left \( 2 \times 2 \)-submatrix of \( M^{n-1} \) consists of \( \left( Y_{ij}, Y_{ij} \right) \), with each of \( r, s, t \) positive, since the coefficients of \( e \) are positive. Multiply the last row by \( tY_0 \) and subtract it from the second-to-last row to zero out the \((n-1,1)\)-entry. Then since rank\( (U_k) = k \), it follows that for rank\( (M^{n-1}) = k \) to hold, necessarily the second-to-last row must be identically zero. In particular, the \((n-1,2)\)-entry is zero, that is, \( Y_1 \) is a scalar multiple of \( Y_2 \). Continuing in this way for the smaller powers of \( M \), we conclude that \( Y_i = c_i Y_0 \) for some \( c_i \in \mathbb{C} \), as in the previous proposition.

Next a direct computation shows that \( M^{n+k-1} \) has entry \((n,1)\) which is a scalar multiple of \( Y_k \) and all other entries are scalar multiples of \( Y_m \) for \( m > k \). If any of these scalar multiples are nonzero, then since \( M^{n+k-1} = 0 \), it follows that \( Y_0 \) is nilpotent, whence \( Y_0^k = 0 \) since \( Y_0 \in \mathfrak{gl}_n \). These multiples are independent of \( k \). The \( k = 1 \) case implies that the entries in \( M^{n+k-1} \) cannot all be zero unless all \( c_i = 0 \) since \( e \) is the only nilpotent element in \( S_n \). We have therefore shown that \( S_{\mathcal{O},e} \) is contained in a variety isomorphic to the nilcone in \( \mathfrak{gl}_n \). By dimension reasons, this must be an equality as in the previous proof. \( \square \)

11.2.3. Example. An example of the first proposition is the degeneration \([2^3] < [3^2] \) and of the second proposition is the degeneration \([3^2] < [4,1^2] \), both in \( \mathfrak{sl}_6 \). Both slices are isomorphic to the nilcone of \( \mathfrak{sl}_6 \). In this setting, the common intermediate orbit \([3,2,1]\) corresponds to the minimal nilpotent orbit in \( \mathfrak{sl}_4 \). Upon restriction to \( \mathfrak{sp}(6) \), the slice becomes isomorphic to the nilcone in \( \mathfrak{so}(3) \), which is of type \( A_1 \). This gives another proof of \([4,4,3]\), one which does not require knowing that either \([3^2] \) or \([4,1^2] \) have closures which are unibranch at \([2^3] \).

12. The remaining unexpected singularities

The singularity \( \mu \) and \( a_2/S_2 \) will be discussed in subsequent work. Here we discuss the minimal degenerations of type \( \tau \) and \( \chi \), respectively:

\[
(2A_2 + A_1, A_2 + 2A_1) \text{ in } E_6 \\
(A_4 + A_3, A_4 + A_2 + A_1) \text{ in } E_8
\]

These two cases, as well as \( a_4/S_2 \), are of dimension 4. All three carry an action of \( SL_2(\mathbb{C}) \) from \( C(\mathfrak{s}) \) but, unlike the other dimension 4 (or greater) cases, this action cannot be transitive on the smooth part of \( S_{\mathcal{O},e} \).

12.1. Preliminaries. Let \( (\mathcal{O}, \mathcal{O}') \) be either \((2A_2 + A_1, A_2 + 2A_1) \) in type \( E_6 \) or \((A_4 + A_3, A_4 + A_2 + A_1) \) in type \( E_8 \). We tackle these cases by choosing a suitable orbit \( \mathcal{O}'' < \mathcal{O}' \): for the first case we choose \( \mathcal{O}'' \) to be the orbit labelled \( A_2 \), while for the second we choose \( \mathcal{O}' \) to be \( A_4 \). Let \( (h, e, f) = s \) be an \( sl_2 \)-subalgebra with \( e \in \mathcal{O}' \). In both cases we can choose a nilpotent element \( e_0 \in \mathcal{O}(s) \) such that \( e + e_0 \in \mathcal{O}' \) and \( S_{\mathcal{O}',e} = e + C(s) \cdot e_0 \). Then we look for a representative of \( \mathcal{O} \) of the form \( x = e + x_0 + x_1 + \ldots + x_m \) with \( x_i \in \mathfrak{g}'(-i) \) and \( x_0 \) a nilpotent element of \( \mathcal{O}(s) \) whose \( C(s) \)-orbit lies one step above that of \( e_0 \) in the closure order. Having found such a representative, the following lemma allows us to relate the singularity of \((\mathcal{O}, \mathcal{O}') \) to the singularity of the pair \((x_0, e_0) \) in \( C(s) \).
Lemma 12.1. Let $\mathcal{O}' \subseteq \mathcal{O}$ be nilpotent orbits in $\mathfrak{g}$ and let $\langle h, e, f \rangle$ be an $\mathfrak{sl}_2$-subalgebra in $\mathfrak{g}$ with $e \in \mathcal{O}'$. Let $g' = \sum_{i < 0} g'(i)$ be the $(\mathfrak{ad} h)$-grading of $g'$. Let $\{h_0, e_0, f_0\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{c}(s)$ such that $e + e_0 \in \mathcal{O}'$. Suppose that $\dim C(s) \cdot e_0 + \dim \mathcal{O}' = \dim \mathcal{O}$. Then

$$S_{e+e_0} = e + e_0 + c(s)^{j_0} \oplus \sum_{i < 0} g'(i)$$

is a transverse slice to $\mathcal{O}_{e+e_0}$ at $e + e_0$.

Proof. For ease of notation, let $Z = C(s)$ and $j = \mathfrak{c}(s)$. Let $t = \dim G \cdot e$ and $s = \dim Z \cdot e_0$. Then by assumption, $\dim G \cdot (e + e_0) = s + t$. Choose a basis $\{u_i : 1 \leq i \leq t\}$ for $[f, g]$ with each $u_i$ belonging to one of the spaces $g(m_i)$. Then $\langle e + e_0, u_i \rangle = \langle e, u_i \rangle + \langle e_0, u_i \rangle$ with highest degree term $\langle e, u_i \rangle \in g(m_i + 2)$. Since the terms $\langle e, u_i \rangle$ are linearly independent, so is the set $\{\langle e + e_0, u_i \rangle : 1 \leq i \leq t\}$. Now let $\langle v_j : 1 \leq j \leq s\rangle$ be a basis for $[f_0, j]$. Then $\langle e + e_0, v_j \rangle = \langle e_0, v_j \rangle$ and therefore the set $\{\langle e + e_0, u_i \rangle, \langle e_0, v_j \rangle : 1 \leq i \leq t, 1 \leq j \leq s\}$ is a basis for $\langle e + e_0, g \rangle$. Let $g(\leq r) = \sum_{1 \leq r \leq r} g(\gamma^r)$. Then for $r \neq 0$, $g(\leq r)/g(\leq (r - 1)) \cong g(r)$ is spanned by those $\langle e + e_0, u_i \rangle$ such that $m_i = r - 2$, along with $g(r)$. Furthermore, $g(\leq 0)/g(\leq (-1)) \cong g(0)$ is spanned by those $\langle e + e_0, u_i \rangle$ with $m_i = -2$, together with $j = \langle e + e_0, j \rangle + j^{10}$. Thus $j^{10} \oplus \sum_{i < 0} g'(i)$ is a complementary subspace to $\langle e + e_0, g \rangle$ in $g$, and we are done. \qed

Having found a representative of $\mathcal{O}$ of the form $e + x_0 + x_1 + \ldots + x_m$, our approach then consists of the following series of steps:

1. Describe the (closure of the) set of conjugates of $x_0$ which are in $e_0 + j^{10}$.
2. For each such conjugate $y_0$ of $x_0$ found in step 1, find an element $z \in Z$ such that $\operatorname{Ad} z(x_0) = y_0$.
3. With $z$ as in step 2, determine the values of $z \cdot x_1$, $z \cdot x_2$ etc.

In the case of $(2A_2 + A_1, A_2 + 2A_1)$ it turns out that we can provide a purely conceptual proof that there exists a representative of $\mathcal{O}$ of the desired form.

12.2. $(2A_2 + A_1, A_2 + 2A_1)$ in $\mathfrak{e}_0$. Let $s = \langle h, e, f \rangle$ where:

$$e = e_{\alpha_2} + e_{\alpha_1 \alpha_2 \alpha_3}, \quad f = 2f_{\alpha_2} + 2f_{\alpha_1 \alpha_2 \alpha_3}, \quad h = [e, f]$$

and let $Z = C(s)$, $j = \mathfrak{c}(s)$. Then $j$ is isomorphic to $\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$, with basis of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_2, \alpha_3\}$. Let $t_1$ be the simple Lie subalgebra of $Z$ with basis of simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ and let $t_2$ be the subalgebra with basis of simple roots $\{\alpha_3, \alpha_2, \alpha_1\}$, so that $j = t_1 \oplus t_2$. Similarly, $Z^* = L_1 \times L_2 \cong \mathfrak{sl}_3 \times \mathfrak{sl}_3$, where $\operatorname{Lie}(L_1) = t_1$ and $\operatorname{Lie}(L_2) = t_2$; on the other hand, $Z/Z^*$ is cyclic of order 2, generated by an element $y \in N_G(T)$ which satisfies $\operatorname{Ad} g(y)(t_1) = t_2$ and $\operatorname{Ad} g(y)(t_2) = t_1$.

We recall [LT11, p. 81] that $g'(-2) = \mathbb{C} f \oplus V \oplus W$ where $V$ is isomorphic to the tensor product of the natural representation of $L_1$ with the dual of the natural representation of $L_2$, and $W \cong V^*$. The only other non-trivial space $g'(-i)$ is $g'(-4)$, which is one-dimensional, spanned by $f_a$ where $\alpha$ is the highest positive root. We have the following highest weight vectors: $v_1 = f_{01210}$ is a highest weight vector in $V$ and $w_1 = f_{02321}$ is a highest weight vector in $W$. In respect of the $Z^*$-action, we can identify $V$ (resp. $W$) with the space of $3 \times 3$ matrices, on which $\langle g, h \rangle \in L_1 \times L_2$ acts via: $\langle g, h \rangle \cdot M = g M h^{-1}$ (resp. $\langle g, h \rangle \cdot M = h M g^{-1}$). In what follows we will systematically make this identification: then $v_1$ and $w_1$ are both given by the matrix with 1 in the top right entry, and zero everywhere else.

Let $e_1 = e_{\alpha_1 + \alpha_2}, e_2 = e_{\alpha_3 + \alpha_2}$, $\tilde{e}_1 = e_{\alpha_3}, \tilde{e}_2 = e_{\alpha_3 + \alpha_2}, \tilde{h}_1 = 2h_{\alpha_3}, \tilde{h}_2 = 2h_{\alpha_3} + 2h_{\alpha_2}$. Then $e_1$ (resp. $e_2$) is in the minimal nilpotent orbit in $t_1$ (resp. $t_2$) and $\tilde{e}_1$ (resp. $\tilde{e}_2$) is in the regular nilpotent orbit in $t_1$ (resp. $t_2$).

Lemma 12.2. There exists an element of $S_n \cap \mathcal{O}$ of the form

$$e + \tilde{e}_1 + \tilde{e}_2 + \xi v_1 + \eta w_1$$

where $\xi \eta \neq 0$. Thus $S_{O,n}$ is the closure of a single $Z^*$-orbit.
Proof. We verified the lemma by computer and found \( \xi = \eta = 3 \). It is also possible to give a conceptual proof.

We remark that it is straightforward to check that the connected centralizer in \( Z^* \) of \( \bar{e}_1 + \bar{e}_2 \) acts trivially on \( \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1 \). This is a necessary condition for the \( Z^* \)-orbit through \( f + \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1 \) to have dimension 12. After scaling, we assume that \( \xi = \eta = 1 \). For later use we note the following easy consequence.

Lemma 12.3. The \( Z^* \)-orbit through \( e + \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1 \) is of dimension 12 for any \( \xi, \eta \). Thus \( e + \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1 \in \text{Ad} Z^*(e + \bar{e}_1 + \bar{e}_2 + v_1 + w_1) \) if and only if \((\xi, \eta) \in \{(1,1), (\omega, \omega^{-1}), (\omega^{-1}, \omega)\}\).

Proof. The first statement follows from the fact that \( (Z^* \xi^j \omega^k)^* \) acts trivially on \( v_1 + w_1 \). For the second part, if \( e + \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1 \in \text{Ad} Z^*(e + \bar{e}_1 + \bar{e}_2 + v_1 + w_1) \) then by dimensional considerations, \( \text{Ad} Z^*(e + \bar{e}_1 + \bar{e}_2 + \xi v_1 + \eta w_1) = \text{Ad} Z^*(e + \bar{e}_1 + \bar{e}_2 + v_1 + w_1) \). Thus \( \xi v_1 + \eta w_1 \in \text{Ad}(Z^* \xi^j \omega^k)(v_1 + w_1) \), from which the statement follows.

To determine the singularity type of the degeneration \((2A_2 + A_1, A_2 + 2A_1)\), we will first consider an element of \( j \) of type \((A_1, A_1)\).

Lemma 12.4. Let 
\[
e_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

be a (minimal) \( sl_2 \)-triple in \( sl_3 \). The intersection of the Slodowy slice \( e_0 + sl_2/f_0 \) with the nilpotent cone in \( sl_3 \) is the set of elements of the form:
\[
X_{st} := \begin{pmatrix} \frac{1}{2} st & 0 & 1 \\ s^3 & -st & 0 \\ -\frac{1}{2} s^2 t^2 & t^3 & \frac{1}{2} st \end{pmatrix}
\]
for \( s, t \in \mathbb{C} \).

Proof. To see that any such element is nilpotent, we simply have to check that the determinant, the trace and the sum of all \( 2 \times 2 \) minors are zero. Then equality of dimensions (and unbranchness) shows us that all nilpotent elements of the Slodowy slice must have the form \( X_{st} \) for some \( s, t \).

Let \( \omega = e^{2\pi i/3} \). We remark that \( X_{st} = X_{\omega, \omega^{-1}1s, \omega} = X_{\omega^{-1}, \omega} \).

Lemma 12.5. Let \( \bar{e}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). If \( s \neq 0 \) then \( g_{st}\bar{e}_0 g_{st}^{-1} = X_{st} \) where
\[
g_{st} := \begin{pmatrix} -t & -1/s & 0 \\ -s^2 & 0 & 0 \\ st^2/2 & -t/2 & -1/s \end{pmatrix}
\]
Moreover, \( g_{st} \) is of determinant 1 and \( g_{st}^{-1} = \begin{pmatrix} 0 & 1/s & 0 \\ -s & t/s & 0 \\ s^2 t^2/4 & -t^2 & -s \end{pmatrix} \).

Proof. Once again, this is a straightforward matrix calculation. It is easy to check that \( \det g_{st} = 1 \), that \( g_{st}^{-1} \) is as in the statement of the Lemma and that the columns \( c_1, c_2, c_3 \) of \( g_{st} \) satisfy \( X_{st} c_1 = 0 \), \( X_{st} c_i = c_{i-1} \) for \( i \geq 2 \). The result follows.

We are almost ready to describe the singularity \((2A_2 + A_1, A_2 + 2A_1)\). Let \( f_1 = f_{\omega_1 + \omega_0}, f_2 = f_{\omega_2 + \omega_0}, \)
\( h_1 = h_{\omega_1 + \omega_3}, h_2 = h_{\omega_2 + \omega_0} \) so that \( \{h_1, e_1, f_1\} \) and \( \{h_2, e_2, f_2\} \) are \( sl_2 \)-triples. By Lemma 12.3, the affine linear space \( S_{\omega + \omega_0} = e + e_0 + \xi^4 + \omega + \omega'(-2) + \omega''(-4) \) is transverse to \( \mathcal{O}' \), and hence \( \text{Sing}(\mathcal{O}, \mathcal{O}') \) can be determined by calculating the intersection \( S_{\omega + \omega_0} := S_{\omega + \omega_0} \cap \mathcal{O}' \).
Theorem 12.6. The intersection of $\mathbb{Z}^\vee \cdot (e + \delta_1 + \delta_2 + v + w)$ with $S'_{\gamma+0}$ is the set of all elements of the form:

$$e + \left( X_{e^i}, X_{u^j}, \begin{pmatrix} -\frac{1}{2}tu^2v & tu & \frac{1}{2}s^2v & sv \\ -\frac{1}{2}s^2tu^2 & t^2u & sv & su^2 \\ s^2u & \frac{1}{2}s^2tu^2 & \frac{1}{2}yt^2u^2 & -\frac{1}{2}s^2u \\ \end{pmatrix} \right) \in e + l_1 \oplus l_2 \oplus V \oplus W$$

where $s,t,u,v \in \mathbb{C}$.

Proof. Let $U = \mathbb{Z}^\vee \cdot (f + \delta_1 + \delta_2 + v + w)$. By Lemmas 12.2 and 12.3, the restriction $p$ to $U$ of the $Z^\vee$-equivariant projection onto $\mathbb{C}$ is a dominant map $W \to N(1)$, and if $z$ is a regular nilpotent element of $\mathfrak{g}$ then $p^{-1}(z)$ consists of three elements $z + v + w, z + \omega v + \omega^{-1} w, z + \omega^{-1} v + \omega w$, for some $v \in V, w \in W$. Suppose $s,u \neq 0$. To determine $p^{-1}(X_{e^i}, X_{u^j})$ we use Lemma 12.5: identifying $Z^\vee$ with $SL_3 \times SL_3$, we have:

$$Ad(g_{\sigma},g_{\omega})(\delta_1,\delta_2,v_1,w_1) = (X_{x_1},X_{x_2},g_{s_1}v_{g_{s_1}},g_{s_2}w_{g_{s_2}})$$

It follows that $Z^\vee \cdot (\delta_1,\delta_2,v_1,w_1)$ contains all elements of the form

$$\begin{pmatrix} X_{x_1}, X_{x_2}, \begin{pmatrix} -\frac{1}{2}tu^2v & tu & \frac{1}{2}s^2v & sv \\ -\frac{1}{2}s^2tu^2 & t^2u & sv & su^2 \\ s^2u & \frac{1}{2}s^2tu^2 & \frac{1}{2}yt^2u^2 & -\frac{1}{2}s^2u \\ \end{pmatrix} \end{pmatrix}$$

where $s,u \neq 0$. In particular, $S'_{\gamma+0} \cap U$ contains a subset of dimension four; but $\dim S'_{\gamma+0} \cap U \leq 4$ since this is the codimension of $\mathfrak{g}^\vee$ in $\mathfrak{g}$. Moreover, $\mathfrak{g}$ is unibranch at $\mathfrak{g}$. So $S'_{\gamma+0} \cap U$ is irreducible of dimension four, and in particular is the closure of the set of elements of the form $e + (X_{x_1},X_{x_2},g_{s_1}v_{g_{s_1}},g_{s_2}w_{g_{s_2}})$ where $s,u \neq 0$. The closure of this set is evidently the set of elements described in the Theorem.

Let $\omega = e^{2\pi i/3}$ and let $\Gamma$ be the subgroup of $Sp_2$, generated by $diag(\omega,\omega^{-1},\omega^{-1})$.

Corollary 12.7. The transverse slice singularity associated to the degeneration $(2A_2 + A_1, A_2 + 2A_1)$ is isomorphic to $\mathbb{C}^4/\Gamma$.

Proof. By Thm. 12.6 and the preceding lemma, $Sing(2A_2 + A_1, A_2 + 2A_1)$ is equivalent to the affine variety with coordinate ring $\mathbb{C}[s,t,u,v]$ which is exactly the coordinate ring of the singularity $m$.

We note some interesting consequences of the above description:
- the closed subset given by setting $u = v = 0$ has coordinate ring $\mathbb{C}[s^3,t^3,s,t]$, that is, it is a simple singularity of type $2D_2$.
- the closed subset given by setting $s = t = 0$ has coordinate ring $\mathbb{C}[s^3,t^3,s^2,t^2,s^2,t^2]$, which is exactly the coordinate ring of the singularity $m$. This follows from taking fixed points in $g$ under an appropriate outer involution of $g$.
- the closed subset given by setting $s = t = 0$ has coordinate ring $\mathbb{C}[s^3,t^3,s^2,t^2,s^2,t^2]$, which is exactly the coordinate ring of the singularity $m$. This follows from taking fixed points in $g$ under an appropriate outer involution of $g$.

We note some interesting consequences of the above description:
- the closed subset given by setting $u = v = 0$ has coordinate ring $\mathbb{C}[s^3,t^3,s,t]$, that is, it is a simple singularity of type $2D_2$.
- the closed subset given by setting $s = t = 0$ has coordinate ring $\mathbb{C}[s^3,t^3,s^2,t^2,s^2,t^2]$, which is exactly the coordinate ring of the singularity $m$. This follows from taking fixed points in $g$ under an appropriate outer involution of $g$.
i \geq 1), whence the $G_i$ satisfy: $2AG_i - CG_{i-1} + BG_{i+1} = 0$ for $1 \leq i \leq 4$. (These equations are also satisfied by the $F_i$.)

Let $Y = \text{Spec}(\mathbb{C}[A, B, C, G_0, \ldots, G_5])$. We claim that $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is equivalent to $Y$.

**Remark 12.8.** a) The singularity $Y$ can be obtained by blowing up $V/\Delta$ at its singular locus, as follows. It is not hard to show that the ideal of elements of $\mathbb{C}[V]^\Delta$ which vanish at the singular points is generated by $D$ and $DG_0, \ldots, DG_5$. Thus the blowup of $V/\Delta$ can be described as the subset of $\Delta^9 \times \mathbb{P}^6$ which is the closure of the set of elements of the form $(A, B, C, F_0, \ldots, F_5, \Omega : DG_0 : \ldots : DG_5)$ with at least one of $D, DG_0, \ldots, DG_5 \neq 0$. Clearly, the affine open subset given by $D \neq 0$ has affine coordinates $A, B, C, F_i, G_i$, and hence is isomorphic to $Y$. An immediate consequence of this description is that $Y$ is birational to $V/\Delta$.

b) It can be shown that the ideal of relations satisfied by $A, B, C, G_0, \ldots, G_5$ is generated by the expressions $2AG_i + BG_{i+1} - CG_{i-1} = 0$ together with ten identities of the form $G_i G_{i+1} G_{i-1} - p(A, B, C) = 0$, where $p$ is a cubic polynomial. For example, $G_i G_{i+1} - G_{i+2} G_{i-1} = \frac{i-1}{i+1} AB^2 C^i$ for $i \leq 3$ and $G_i G_{i+1} - G_{i+2} G_{i-1} = (\frac{-1}{i+1}) AB^2 C^i$ for $i \leq 2$.

c) It can be shown that all of the remaining affine open subsets of the blow-up given by $DG_i \neq 0$ are smooth, in fact are isomorphic to $\Delta^9$. For example, the open subset given by $DG_0 \neq 0$ is the affine variety with coordinate ring $R = \mathbb{C}[A, B, C, F_0, \ldots, F_5, 1/G_1, G_1/G_0, \ldots, G_5/G_0]$. It is an easy calculation (using the identities for the $G_i$ mentioned above) to check that this ring is generated by $B, F_0, 1/G_0$ and $G_1/G_0$, hence by dimensions is a polynomial ring of rank four. Thus the point of $Y$ corresponding to the maximal ideal $(A, B, C, G_i)$ is the unique singular point of the blow-up of $C^4/\Delta$. This justifies the more succinct description of $\text{Sing}(\mathcal{O}, \mathcal{O}')$ given in the introduction.

d) In general, a blow-up of a symplectic singularity is not a symplectic singularity. In our case, $\mathcal{O}$ inherits a symplectic structure from that of $\mathcal{O}_0$, and so (subject to our claim) $Y$ is a symplectic singularity. More generally, it can be shown that the blow-up (at the singular locus) of the quotient of $\mathcal{O}$ by any dihedral group (with $\mathcal{O}$ identified with two copies of its reflection representation) is a symplectic singularity.

Let $(\mathcal{O}', \mathcal{O}, \mathcal{O}) = (A_1, A_2 + A_3 + A_4 + A_5 + A_6)$ and let $e \in \mathcal{O}'$. Specifically, let $e = e_{12} + e_{45} + e_{36} + e_{36}$, $f = 4f_{31} + 6f_{32} + 6f_{34} + 4f_{35}$, $h = \{e, f\}$ and let $s = (h, e, f)$. Then $j : s(s) = s(\alpha)$, with basis of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and $Z := C(\alpha)$ is isomorphic to the semidirect product of $\mathfrak{sl}_2(\mathbb{C})$ by an outer involution. Let $s \in e + \mathfrak{g}'$. Similarly to the previous subsection, it is easily checked that $s_{0} \cap \mathfrak{g}' = e + Z^* \cap \mathfrak{g}_0$ and $s_{0} \cap \mathfrak{g}' = e + Z^* \cap \mathfrak{g}_0$ where $e_0$ is a nilpotent element of $Z$ of partition type $[3, 2]$. Once more, we introduce the slice $S_{s+e_0} : = e + e_0 + y_0 \oplus \sum_{i < 0} g_i(i)$, which is a transverse slice to the orbit $\mathcal{O}'$ by Lemma 12.1.

**Lemma 12.9.** The intersection $S_{s+e_0} \cap \mathcal{O}$ is isomorphic to $Y$.

For the purposes of our discussion we will identify $Z^*$ (resp. $\mathfrak{g}'$) with $\mathfrak{sl}_3$ (resp. $\mathfrak{s}_3$) via the basis of simple roots $\{\beta_1, \beta_2, \beta_3, \beta_4\}$. Let $W$ be the natural module for $Z^*$. The $Z^*$-module structure of $\mathfrak{g}'$ includes the following spaces:

$\mathfrak{g}'(-2) \cong W \oplus W^* \oplus C$, $\mathfrak{g}'(-4) \cong \Lambda^2(W) \oplus \Lambda^2(W^*) \oplus C$

Precisely, we have the following highest weight vectors (relative to the basis of positive roots $\{\beta_i\}$ given above):

$w_1 = 3e_{0111111} - 2e_{0111111} \in W$, $u_1 = 2e_{1354321} - 3e_{1234321} \in W^*$,

$y_1 = e_{1233321} \in \Lambda^2(W)$, $z_1 = e_{0122221} \in \Lambda^2(W^*)$.

Then it is easy to verify computationally that

$x = e + e_{12} + e_{36} + e_{36} = w_1 + u_1 + 10y_1 - 10z_1 \in \mathcal{O}.$
Remark 12.10. The above argument proves that the degeneration \([A_4 + A_3, A_4]_{q_8}\) is equivalent to the spectrum of the ring of regular functions on the universal cover of the regular nilpotent orbit in \(s_{\mathfrak{t}}\), cf. \([\text{Gra}92]\).

Moreover, \(S_{\mathfrak{t} + q_0} \cap \mathfrak{g}\) is equal to the closure of the set of conjugates \(z \cdot \tilde{M}_0, z \in Z\) such that \(\text{Ad} z(M_0) \in e_0 + \mathfrak{t}_0\). We can state this more precisely as follows:

**Lemma 12.11.** The intersection \(S_{\mathfrak{t} + q_0} \cap \mathfrak{g}\) is isomorphic to the closure of the set of all \((M, w_1', w_1' \wedge w_2', w_2' \wedge w_3', w_3' \wedge w_4', w_4' \wedge w_5', w_5' \wedge w_6', w_6' \wedge w_7', w_7' \wedge w_8', w_8' ) \in (e_0 + \mathfrak{t}_0) \times (W \oplus \Lambda^3(W) \oplus \Lambda^4(W))\) of \(\mathfrak{c}^5\) such that there is a basis \(\{w_1', \ldots, w_8'\}\) for \(W\) with \(w_1' \wedge \ldots \wedge w_8' = 1\) and \(M w_i' = w_{i-1}' (i \geq 2), M w_8' = 0\).

**Proof.** This follows from the above discussion. \(\square\)

Concretely, we can determine the intersection \(S_{\mathfrak{t} + q_0} \cap \mathfrak{g}\) in the following way: if \(M \in e_0 + \mathfrak{t}_0\) is nilpotent then generically \(M\) is regular and therefore there exists a basis \(B = \{w_1', \ldots, w_8'\}\) of \(\mathfrak{c}^5\) such that \(M w_i' = 0\) and \(M w_i' = w_{i-1}'\) for \(i \geq 2\). After scaling, we may assume that \(g_8 = (w_1' w_2' w_3' w_4' w_5' w_6' w_7' w_8')\) has determinant one. Then the tuple \((M, w_1', \ldots, w_8', w_1' \wedge \ldots \wedge w_8')\) with \(M w_i' = w_{i-1}' (i \geq 2), M w_8' = 0\).

We may assume that

\[
e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and then \(e_0 + \mathfrak{t}_0\) consists of all matrices of the form

\[
\begin{pmatrix}
2a & b & c & d & g \\
0 & -3a & h & k & l \\
2 & 0 & 2a & b & c \\
0 & 1 & 0 & -3a & h \\
0 & 0 & 2 & 0 & 2a
\end{pmatrix}
\]

where \(a, b, c, d, g, h, k, l \in \mathbb{C}\). For the purposes of our calculation, we consider the matrices in \(e_0 + \mathfrak{t}_0\) of the form:

\[
M = \begin{pmatrix}
2a & b & c - 6a^2 & d - 2ab & 40a^3 - 10ac - \frac{4}{3}bh \\
0 & -3a & h & 9a^2 - 4c & l - 2ah \\
2 & 0 & 2a & b & c - 6a^2 \\
0 & 1 & 0 & -3a & h \\
0 & 0 & 2 & 0 & 2a
\end{pmatrix}
\]

A straightforward computer verification confirms that such any matrix satisfies \(\text{Tr} M^2 = \text{Tr} M^3 = 0\), and that the conditions \(\text{Tr} M^4 = 0\) and \(\text{Tr} M^5 = 0\) are expressed in terms of the coordinates \(a, b, c, d, h, l\) as:

\[
dh + bl + \frac{8}{3}c^2 = a(9bh - 216a^3 + 72ac), \quad dl = c(9bh - 216a^3 + 48ac)
\]
Since every irreducible component of the set of \((a, b, c, d, h, l)\) satisfying these two equations has dimension at least four, it follows that the set of matrices given by the coordinates satisfying (12.1) is equal to the set of nilpotent elements of \(e_0 + \mathfrak{z}^0\) (and is therefore irreducible).

It is easy to verify that the rational functions \(a = A/6, b = -G_0/3, c = -BC/16, d = BG_1/4, h = G_3/3, l = CG_4/4\) in \(\mathbb{C}(p, q, s, t)\) satisfy (12.2). Since \(A, BC, G_0, G_3, BG_1, CG_4\) are regular functions on \(Y\), we have therefore constructed a morphism from \(Y\) to \(N(j) \cap (e_0 + \mathfrak{z}^0)\), corresponding to the inclusion \(\mathbb{C}[A, BC; G_0, G_3, BG_1, CG_4] \subset \mathbb{C}[Y]\). In fact, this morphism corresponds to quotienting \(Y\) by the action of a group of order five, as follows: let \(\rho\) be the automorphism of order five of \(V\) which sends \((p, q, s, t)\) to \((\zeta p, \zeta^{-1} q, s, \zeta^{-1} t)\). Then \(\rho\) normalizes \(\Gamma\) and has an induced action on \(Y\) satisfying \(\mathbb{C}[Y]^{\rho} = \mathbb{C}[A, BC, G_0, G_3, BG_1, CG_4]\). The invariants \(B^2G_2\) and \(C^2G_3\) are contained in this ring, since \(BG_4 = CG_0 - 2AG_1\) and \(CG_3 = 2AG_4 + BG_5\). It follows that the coordinates \(a = A/6\) etc. given above define an isomorphism from \(Y/\rho\) to \(N(st_b) \cap (e_0 + \mathfrak{z}^0)\).

**Remark 12.12.** The above discussion indicates an interesting way to view the singularity \(([5], [3, 2])\) in \(st_5\), as an affine open subset of the blow up of the quotient of \(\mathbb{C}^4\) by a group of order 50. Indeed, the group generated by \(\Gamma\) and \(\rho\) is isomorphic to the complex reflection group \(G(5, 1, 2)\), acting on \(\mathbb{C}^4 = U \oplus U^*\) where \(U\) is the defining representation for \(G(5, 1, 2)\). Blowing up the quotient at the set of orbits of points of the form \((u, u)\), and restricting to the affine open subset given by \(D \neq 0\), one obtains the variety \(Y/\rho\).

We will first give an ad hoc justification that \(S_{\Delta+\epsilon_0}\) is as claimed, and then a more rigorous proof. Fix a matrix \(M\) as above with coordinates \(a = A/6\), etc which we think of as depending on the point \((A, B, C, G_0, \ldots, G_7) \in Y\). The space of (column) vectors in \(W\) which are annihilated by \(M\) is generically of dimension one, spanned by

\[
w_i' = \begin{pmatrix} -\frac{1}{6}G_0G_4 + \frac{1}{3}A^2B + \frac{1}{12}B^2C \\ -\frac{1}{6}BG_5 \\ -\frac{1}{6}AB \\ -G_4 \\ B \end{pmatrix}
\]

Similarly, the space of (row) vectors in \(W^*\) which are annihilated by \(M\) is also generically of dimension one, spanned by \(u_i' = (C, -G_1, -\frac{1}{2}AC, \frac{1}{6}CG_0 - \frac{1}{2}AG_1, \frac{1}{6}A^2C + \frac{1}{12}BC^2 + \frac{1}{2}G_4G_5)\). It follows that if \(z \in Z^* = st_5\) is such that \(Ad_z(M_0) = M\), then \(zw_i\) is a scalar multiple of \(w_i'\), and \(w_i'\) is also a scalar multiple of \(w_i'\). Our more rigorous argument below will (essentially) consist of showing that these scalars, up to multiplication by a fifth root of unity, are independent of \(p, q, s, t\). Thus the ring of regular functions on \(S_{\Delta+\epsilon_0}\) also contains elements which naturally correspond to \(B, C, G_1\) and \(G_2\). To continue along this line, we would have to find a vector \(w_i'' \in W^*\) such that \(Mw_i'' = w_i''\), and similarly for \(u_i'\). Then it turns out that the coordinates of \(w_i'' \wedge w_i''\) and \(w_i'' \wedge w_i''\) are contained in \(\mathbb{C}[Y]\), and include scalar multiples of \(G_2\) and \(G_3\). Thus one obtains a morphism \(\varphi : Y \to S_{\Delta+\epsilon_0}\), which (since all of the generators \(A, B, C, G\), appear somewhere in the coordinates describing \(\varphi\)) is evidently a closed immersion, hence an isomorphism by equality of dimensions and reducedness.

For a more careful analysis, we note that finding a basis \(\{w_1', \ldots, w_5\}\) for \(\mathbb{C}^5\) such that \(Mw_i' = w_i'_{\perp}\), for \(i \geq 2\) and \(Mw_1' = 0\) is essentially equivalent to finding an element \(w_5' \in \mathbb{C}^3\) such that \(M^4w_5' \neq 0\). Moreover, any transformation of the form \(w_i' \mapsto w_i' + \alpha w_i' + \beta w_i' + \gamma w_i' + \delta w_i'\) preserves the elements \(w_i', w_i' \wedge w_i', w_i' \wedge w_i' \wedge w_i', w_i' \wedge w_i' \wedge w_i' \wedge w_i'\). Thus, to find \(z \cdot M_0\) where \(Ad_z(M_0) = M\), it suffices to choose an element \(w_5'\) such that \(M^4w_5' \neq 0\), and then to multiply \(w_i'\) by an appropriate scalar such that \(\det(w_i' w_i' w_i' w_i' w_i' w_i' w_i') = 1\). For this purpose, we first choose

\[
w_i' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \text{ then } w_4' = \begin{pmatrix} \frac{1}{2}A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_5' = \begin{pmatrix} -\frac{1}{6}A^2 - \frac{1}{2}BC \\ \frac{2}{5}G_5 \\ \frac{1}{2}A \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad w_2'' = \begin{pmatrix} \frac{1}{2}A^3 + \frac{1}{24}ABC + \frac{1}{6}G_0G_5 \\ CG_4 - \frac{1}{2}AG_1 \\ -\frac{1}{6}G_5 \\ 2G_5 \\ 4A \end{pmatrix}
\]
and finally

\[
\begin{bmatrix}
A^4 + \frac{1}{18} A^2 B C + \frac{1}{10} B^2 C^2 + AG_0 G_5 + \frac{1}{14} B G_1 G_5 - \frac{1}{14} C G_0 G_4 \\
\frac{1}{4} A C G_4 + \frac{1}{6} B C G_5 \\
\frac{1}{4} A B C \\
C G_4 \\
-BC
\end{bmatrix}
= -C
\begin{bmatrix}
-\frac{1}{4} G_0 G_4 + \frac{1}{4} A^2 B + \frac{1}{14} B^2 C \\
-\frac{1}{4} C G_5 - \frac{1}{14} B G_5 \\
-\frac{1}{4} A B \\
-\frac{1}{4} G_4 \\
B
\end{bmatrix}
\]

Then one can show that the determinant of the matrix \((w'_i w'_j w'_k w'_l w'_m)\) is \(-C^5\). Thus we replace each of \(w'_i, 1 \leq i \leq 5\) by \(-w'_i/C\), which is well-defined whenever \(C \neq 0\). In other words, whenever \(C \neq 0\) we can construct a matrix \(g_B = (w''_i w''_j w''_k w''_l w''_m)\) of determinant 1 such that \(g_B M_0 g_B^{-1} = M\).

It is now a routine computer calculation to verify that, relative to the obvious basis for \(\Lambda^4(W)\), we have

\[
w''_i \wedge w''_j = 
\begin{bmatrix}
\frac{1}{18} G_0 G_2^2 - \frac{1}{14} C G_0 + \frac{1}{14} A C^2 G_1 - \frac{1}{14} A^2 C G_2 + \frac{1}{18} A^4 G_1 \\
\frac{1}{10} A^2 B^2 + \frac{1}{10} B^2 C - \frac{1}{14} A G_0 G_3 - \frac{1}{14} B G_0 G_4 \\
-\frac{1}{14} A^2 G_3 - \frac{1}{10} A B G_4 - \frac{1}{10} B^2 G_5 \\
\frac{1}{6} A B G_5 + \frac{1}{4} G_0 G_3 \\
-\frac{1}{6} C (A G_2 + 2 B G_3) \\
\frac{1}{2} G_3 G_5 - G_4^2 \\
A G_3 + B G_4 \\
\frac{1}{2} A G_4 + B B G_4 \\
-\frac{1}{B^2} \\
G_5
\end{bmatrix}
\]

and similarly

\[
w''_i \wedge w''_j \wedge w''_k = 
\begin{bmatrix}
-\frac{1}{18} A^3 G_2 + \frac{1}{10} A^4 C G_1 + \frac{1}{10} A B C G_2 - \frac{1}{10} B C^2 G_1 - \frac{1}{18} G_0 G_2 G_4 \\
-\frac{1}{10} A G_2 G_4 + \frac{1}{10} B G_2 G_5 - \frac{1}{10} B G_4^2 \\
\frac{1}{5} A^2 G_2 + \frac{1}{14} A B G_3 + \frac{1}{14} B C G_2 - \frac{1}{14} C^2 G_0 \\
-\frac{1}{14} A B G_3 + \frac{1}{14} B^2 G_4 - \frac{1}{14} A^2 G_2 \\
-\frac{1}{10} B^3 + \frac{1}{10} G_0 G_2 \\
\frac{1}{6} A G_2 + B B G_4 \\
\frac{1}{10} G_3 G_5 - \frac{1}{12} A C^2 \\
A G_2 + B G_3 \\
\frac{1}{2} C^2 \\
-2 G_2
\end{bmatrix}
\]

Finally, it is straightforward to show using the identification of \(\Lambda^4(W)\) with \(W^*\) that

\[
w''_i \wedge w''_j \wedge w''_k \wedge w''_l = (-C, G_1, \frac{1}{6} A C, \frac{1}{6} C G_0 - \frac{1}{2} A G_3, \frac{1}{6} A^2 C + \frac{1}{12} B C^2 + \frac{1}{6} G_1 G_5).
\]

What these computations amount to is the following:

**Lemma 12.13.** There is a morphism from the open subset of \(Y\) given by \(C \neq 0\) to \(S_{\Delta, e+e_0} \cap S_0\), given by the matrix \(g_B\) act on \(M_0\). Moreover, this morphism extends to an isomorphism from \(Y\) to the closure \(S_{\Delta, e+e_0}\).

**Proof.** The first part follows from the above discussion, since \(g_B\) has coordinates in the localized ring \(\mathbb{C}[Y]_C\). But we can see by our calculations that in fact, the coordinates of \(M\) and \(w''_i, \ldots, w''_l \wedge w''_k \wedge w''_j \wedge w''_l\) all lie in \(\mathbb{C}[Y]\). Thus we can extend the morphism to a morphism \(\varphi\) from \(Y\) to \(S_{\Delta, e+e_0}\). Since each of the generators \(A, B, C, G_0, \ldots, G_5\) appears (up to multiplication by a scalar) as a coordinate of the map \(\varphi\), it follows that \(\varphi\) is a closed immersion, and hence by dimensions, irreducibility and reducedness, is an isomorphism onto \(S_{\Delta, e+e_0}\). \(\square\)
13. Graphs

Capital letters are used to denote simple singularities and lower-case letters to denote singularities of closures of minimal nilpotent orbits. The notation $m$, $m'$, $\mu$, $\chi$, $a_2/\Gamma_2$ and $\tau$ are explained in §1.7.4. The intrinsic symmetry action induced from $A(e)$ is explained in §6 and the notation is explained in §6.2. We use $(Y)$ to denote a singularity with normalization $Y$. 

\[ 
\begin{array}{c}
G_2 \\
| G_2 \\
G_2(a_1) \\
| A_1 \\
\tilde{A}_1 \\
| m \\
A_1 \\
| g_2 \\
0
\end{array}
\]

\[ 
\begin{array}{c}
F_4 \\
| F_4 \\
F_4(a_1) \\
| C_3 \\
A_1 \\
C_3(a_2) \\
B_3 \\
G_2 \\
F_4(a_3) \\
| A_1 \\
C_3(a_1) \\
m \\
\tilde{A}_2 + A_1 \\
| A_1 \\
\tilde{A}_2 \\
A_1 + \tilde{A}_1 \\
| a_1^+ \\
A_2 \\
A_1 + \tilde{A}_1 \\
| a_1^+ \\
\tilde{A}_1 \\
| c_3 \\
A_1 \\
| f_4 \\
0
\end{array}
\]
REFERENCES

[AS09] P. N. Achar and D. S. Sage. Perverse coherent sheaves and the geometry of special pieces in the unipotent variety. Adv. Math., 220(4):1265–1296, 2009.

[AW] M. Andr`eeva and J. A. Wisniewski. 4-dimensional symplectic contractions. arXiv:1101.4884.

[Bea00] A. Beaville. Symplectic singularities. Invent. Math., 139(3):541–549, 2000.

[BM81] W. Borho and R. MacPherson. Repr´esentations des groupes de Weyl et homologie d’intersection pour les vari´et´es nilpotentes. C. R. Acad. Sci. Paris S´er. I Math., 292(15):707–710, 1981.

[BM83] W. Borho and R. MacPherson. Partial resolutions of nilpotent varieties. In Analysis and topology on singular spaces, II, III (Luminy, 1981), volume 101 of Ast´erisque, pages 23–74. Soc. Math. France, Paris, 1983.

[Bri71] E. Brieskorn. Singular elements of semi-simple algebraic groups. In Actes du Congr`es International des Math´ematiciens (Nice, 1970), Tome 2, pages 279–284. Gauthier-Villars, Paris, 1971.

[Bri10] M. Brion. Introduction to actions of algebraic groups. In Hamiltonian Actions: invariants et classification, volume 1 of Les cours du CIRM, pages 1–22, 2010.

[Bro98a] A. Broer. Decomposition varieties in semisimple Lie algebras. In Invent. Math.

[Bro98b] A. Broer. Normal nilpotent varieties in Invent. Math.

[Bri10] M. Brion. Introduction to actions of algebraic groups. In Hamiltonian Actions: invariants et classification, volume 1 of Les cours du CIRM, pages 1–22, 2010.

[BS84] W. M. Beynon and N. Spaltenstein. Green functions of finite Chevalley groups of type $G_2$. J. Algebra, 88(2):584–614, 1984.

[Car93] R. W. Carter. Finite groups of Lie type. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Cat87] F. Catanese. Automorphisms of rational double points and moduli spaces of surfaces of general type. Compositio Math., 61(1):81–102, 1987.

[CG10] N. Chriss and V. Ginzburg. Representation theory and complex geometry. Modern Birkh¨auser Classics. Birkh"auser Boston Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[CM93] D. H. Collingwood and W. M. McGovern. Nilpotent Orbits in Semisimple Lie Algebras. Van Nostrand Reinhold Co, New York, 1993.

[Dyn52] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. Mat. Sbornik N.S., 30(72):349–402 (3 plates), 1952.

[FJLSa] B. Fu, D. Juteau, P. Levy, and E. Sommers. Duality for generic singularities of nilpotent orbits. In preparation.

[FJLSb] B. Fu, D. Juteau, P. Levy, and E. Sommers. Geometry of special pieces in nilpotent orbits. In preparation.

[Fu10] B. Fu. On Q-factorial terminalizations of nilpotent orbits. J. Math. Pures Appl. (9), 93(6):623–635, 2010.

[FZ03] H. Flenner and M. Zaidenberg. Rational curves and rational singularities. Math. Z., 244(3):549–575, 2003.

[GG02] W. L. Gan and V. Ginzburg. Quantization of Slodowy slices. Inventiones Math.

[Gra92] W. A. Graham. Functions on the universal cover of the principal nilpotent orbit. Invent. Math., 108(1):15–27, 1992.

[Hen] A. Henderson. Singularities of nilpotent orbit closures. arXiv:1408.3888, to appear in proceedings of the 5th Japanese-Australian Workshop on Real and Complex Singularities.

[His76] W. Hesselink. Singularities in the nilpotent scheme of a classical group. Trans. Amer. Math. Soc., 222:1–32, 1976.

[Hin91] V. Hinich. On the singularities of nilpotent orbits. Israel J. Math., 73(3):297–308, 1991.

[Jam81] G. D. James. On the decomposition matrices of the symmetric groups III. Journal of Algebra, 71:115–122, 1981.

[Jos84] A. Joseph. On the variety of a highest weight module. J. Algebra, 88(1):238–278, 1984.

[Jut07] D. Juteau. Modular Springer correspondence and decomposition numbers. PhD thesis, Universit´e Paris 7 – Denis-Diderot, 2007.

[Kal06] D. Kaledin. Symplectic singularities from the Poisson point of view. J. Reine Angew. Math., 600:135–156, 2006.

[KP81] H. Kraft and C. Procesi. Minimal singularities in $G_L$. Invent. Math., 62(3):503–515, 1981.

[KP82] H. Kraft and C. Procesi. On the geometry of conjugacy classes in classical groups. Comment. Math. Helv., 57(4):539–602, 1982.

[KP89] H. Kraft and C. Procesi. A special decomposition of the nilpotent cone of a classical lie algebra. Ast´erisque, 173–174:271–279, 1989.

[Kra89] H. Kraft. Closures of conjugacy classes in $G_2$. J. Algebra, 126(2):454–465, 1989.

[LeB95] C. LeBrun. Fano manifolds, contact structures, and quaternionic geometry. Internat. J. Math., 6(3):419–437, 1995.

[Los10] I. Losev. Finite W-algebras. In Proceedings of the International Congress of Mathematicians. Volume III, pages 1281–1307. Hindustan Book Agency, New Delhi, 2010.

[LS88] T. Levasseur and S. P. Smith. Primitive ideals and nilpotent orbits in type $G_2$. J. Algebra, 114(1):81–105, 1988.
54

Baohua Fu
Hua Loo-Keng Key Laboratory of Mathematics and AMSS, Chinese Academy of Sciences, 55 ZhongGuanCun East Road, Beijing, 100190, P. R. China
E-mail address: bhfu@math.ac.cn

Daniel Juteau
LMNO, Universite de Caen Basse-Normandie, CNRS, BP 5186, 14032 Caen Cedex, France
E-mail address: daniel.juteau@math.unicaen.fr

Paul Levy
Department of Mathematics and Statistics Fylde College, Lancaster University Lancaster LA1 4YF, United Kingdom
E-mail address: p.d.levy@lancaster.ac.uk

Eric Sommers
Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-4515, USA
E-mail address: esommers@math.umass.edu