Stochastic Bifurcation of Pathwise Random Almost Periodic and Almost Automorphic Solutions for Random Dynamical Systems

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Abstract

In this paper, we introduce concepts of pathwise random almost periodic and almost automorphic solutions for dynamical systems generated by non-autonomous stochastic equations. These solutions are pathwise stochastic analogues of deterministic dynamical systems. The existence and bifurcation of random periodic (random almost periodic, random almost automorphic) solutions have been established for a one-dimensional stochastic equation with multiplicative noise.

Key words. Pullback attractor; random periodic solution; random almost periodic solution; random automorphic solution; stochastic bifurcation.

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1 Introduction

This paper is concerned with almost periodic and almost automorphic dynamics of random dynamical systems associated with stochastic differential equations driven by time-dependent deterministic forcing. We will first define pathwise random almost periodic solutions and almost automorphic solutions for such systems, which are special cases of random complete solutions and random complete quasi-solutions. We then study existence, stochastic pitchfork and transcritical bifurcation of these types of solutions for one-dimensional non-autonomous stochastic equations.

Almost periodic and almost automorphic solutions of deterministic differential equations have been extensively studied by many experts, see, e.g., [17, 22, 23, 25, 27, 28, 29, 30, 31, 35, 36, 38, 39]...
and the references therein. In particular, the \( \omega \)-limit sets of such solutions have been investigated in \[23, 27, 28, 29, 30, 31, 35\]. However, as far as the author is aware, it seems that there is no result available in the literature on existence and stability of \textit{pathwise} random almost periodic or almost automorphic solutions for stochastic equations. The first goal of the present paper is to introduce these concepts for random dynamical systems generated by non-autonomous stochastic equations. Roughly speaking, a pathwise random almost periodic (almost automorphic) solution is a random complete quasi-solution which is pathwise almost periodic (almost automorphic) (see Definitions 2.1 and 2.2 below). It is worth mentioning that a pathwise random almost periodic (almost automorphic) solution is actually \textit{not} a solution of the system for a fixed sample path, and it is just a complete quasi-solution in the sense of Definition 2.1. In this paper, in addition to existence of pathwise random periodic (almost periodic, almost automorphic) solutions, we will also study the stability and bifurcation of these solutions. More precisely, we will investigate stochastic pitchfork bifurcation of the one-dimensional non-autonomous equation

\[
\frac{dx}{dt} = \lambda x - \beta(t)x^3 + \gamma(t,x) + \delta x \circ d\omega, \tag{1.1}
\]

and transcritical bifurcation of the equation

\[
\frac{dx}{dt} = \lambda x - \beta(t)x^2 + \gamma(t,x) + \delta x \circ d\omega, \tag{1.2}
\]

where \( \lambda \) and \( \delta \) are constants, \( \beta : \mathbb{R} \rightarrow \mathbb{R} \) is positive, and \( \gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies some growth conditions. The stochastic equations (1.1) and (1.2) are understood in the sense of Stratonovich integration.

In the deterministic case (i.e., \( \delta = 0 \)), these equations are classical examples for demonstrating pitchfork and transcritical bifurcation of fixed points. In the stochastic case with constant \( \beta = 1 \) and \( \gamma = 0 \), the stochastic bifurcation of stationary solutions and invariant measures of (1.1)-(1.2) has been studied in [1, 2]. In the real noise case, the same problem was discussed in [37]. When \( \beta = 1 \) and \( \gamma \) is time-independent, the bifurcation of stationary solutions of (1.1)-(1.2) was examined in [1, 3]. For the bifurcation of stationary solutions of (1.1) with additive noise, we refer the reader to [14]. It seems that the bifurcation problem of (1.1) and (1.2) has not been studied in the literature when \( \beta \) and \( \gamma \) are time-dependent. The purpose of this paper is to investigate this problem and explore bifurcation of pathwise random complete solutions including random periodic (almost periodic, almost automorphic) solutions. Actually, for time-dependent \( \beta \) and \( \gamma \) satisfying certain conditions, we prove the pathwise complete quasi-solutions of (1.1) undergo a stochastic
pitchfork bifurcation at $\lambda = 0$: for $\lambda \leq 0$, the zero solution is the unique random complete quasi-solution of (1.1) which is pullback asymptotically stable in $\mathbb{R}$; for $\lambda > 0$, the zero solution is unstable and two more tempered random complete quasi-solutions $x_\lambda^+ > 0$ and $x_\lambda^- < 0$ bifurcate from zero, i.e.,

$$\lim_{\lambda \to 0} x_\lambda^\pm(\tau, \omega) = 0, \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$ 

The tempered random attractor $\mathcal{A}_\lambda$ of (1.1) is trivial for $\lambda \leq 0$, and is given by $\mathcal{A}_\lambda = \{ [x^-_\lambda(\tau, \omega), x^+_\lambda(\tau, \omega)] : \tau \in \mathbb{R}, \omega \in \Omega \}$. If, in addition, $\beta$ and $\gamma$ are both $T$-periodic in time for some $T > 0$, then $x^-_\lambda$ and $x^+_\lambda$ are also $T$-periodic. In this case, we obtain pitchfork bifurcation of pathwise random periodic solutions of (1.1). It seems that the bifurcation of almost periodic and almost automorphic solutions is much more involved. Nonetheless, for $\gamma = 0$, we will prove if $\beta$ is almost periodic (almost automorphic), then so are $x^-_\lambda$ and $x^+_\lambda$. As a consequence, we obtain stochastic pitchfork bifurcation of pathwise random almost periodic (almost automorphic) solutions of (1.1) in this case. By similar arguments, we will establish stochastic transcritical bifurcation of pathwise random complete quasi-solutions of (1.2) at $\lambda = 0$. If $\gamma = 0$ and $\beta$ is periodic (almost periodic, almost automorphic), we further establish the transcritical bifurcation of random periodic (almost periodic, almost automorphic) solutions of (1.2) (see Corollary 4.2 and Theorem 4.3).

This paper is organized as follows. In the next section, we introduce the concepts of pathwise random almost periodic (random almost automorphic) solutions for random dynamical systems generated by differential equations driven simultaneously by non-autonomous deterministic and stochastic forcing. We will also review some results regarding pullback attractors. In the last two sections, we prove stochastic pitchfork bifurcation and transcritical bifurcation for equations (1.1) and (1.2), respectively.

## 2 Preliminaries

In this section, we introduce concepts of pathwise random almost periodic and almost automorphic solutions for random dynamical systems generated by differential equations driven simultaneously by non-autonomous deterministic and stochastic forcing. We also review some known results regarding random attractors for non-autonomous stochastic equations. The reader is further referred to [5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 18, 19, 21, 24] for autonomous random attractors and to [4, 20, 26, 32] for deterministic attractors.

Let $(X, d)$ be a complete separable metric space and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be a metric dynamical
system as in [1]. Given a subset $A$ of $X$, the neighborhood of $A$ with radius $r > 0$ is denoted by $N_r(A)$. A mapping $\Phi: \mathbb{R}^+ \times \Omega \times X \to X$ is called a continuous cocycle on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mu, \{\theta_t\}_{t \in \mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (i)-(iv) are satisfied:

(i) $\Phi(\cdot, \tau, \cdot): \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on $X$;

(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(iv) $\Phi(t, \tau, \omega, \cdot): X \to X$ is continuous.

Such $\Phi$ is called a continuous periodic cocycle with period $T$ if $\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot)$ for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Let $D$ be a collection of some families of nonempty subsets of $X$:

$$D = \{ D = \{ D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega \} \}.$$  
(2.1)

We now define $D$-complete solutions for $\Phi$.

**Definition 2.1.** Let $D$ be a collection of families of nonempty subsets of $X$ given by (2.1).

(i) A mapping $\psi: \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a complete orbit (solution) of $\Phi$ if for every $t \in \mathbb{R}^+$, $s, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\Phi(t, \tau + s, \theta_s \omega, \psi(s, \tau, \omega)) = \psi(t + s, \tau, \omega).$$

If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ such that $\psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t \omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $\psi$ is called a $D$-complete orbit (solution) of $\Phi$.

(ii) A mapping $\xi: \mathbb{R} \times \Omega \to X$ is called a complete quasi-solution of $\Phi$ if for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\Phi(t, \tau, \omega, \xi(\tau, \omega)) = \xi(\tau + t, \theta_t \omega).$$

If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ such that $\xi(\tau, \omega)$ belongs to $D(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $\xi$ is called a $D$-complete quasi-solution of $\Phi$.

**Definition 2.2.** Let $\xi: \mathbb{R} \times \Omega \to X$ be a mapping.

(i) $\xi$ is called a random periodic function with period $T$ if $\xi(\tau + T, \omega) = \xi(\tau, \omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$. 

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(ii) $\xi$ is called a random almost periodic function if for every $\omega \in \Omega$ and $\varepsilon > 0$, there exists $l = l(\omega, \varepsilon) > 0$ such that every interval of length $l$ contains a number $t_0$ such that

$$|\xi(\tau + t_0, \omega) - \xi(\tau, \omega)| < \varepsilon, \quad \text{for all } \tau \in \mathbb{R}. $$

(iii) $\xi$ is called a random almost automorphic function if for every $\omega \in \Omega$ and every sequence $\{\tau_n\}_{n=1}^{\infty}$, there exist a subsequence $\{\tau_{nm}\}_{m=1}^{\infty}$ of $\{\tau_n\}_{n=1}^{\infty}$ and a map $\zeta^\omega : \mathbb{R} \to X$ such that for all $\tau \in \mathbb{R}$,

$$\lim_{m \to \infty} \xi(\tau + \tau_{nm}, \omega) = \zeta^\omega(\tau) \quad \text{and} \quad \lim_{m \to \infty} \zeta^\omega(\tau - \tau_{nm}) = \xi(\tau, \omega).$$

If $\xi$ is a complete quasi-solution of $\Phi$ and is also a random periodic (random almost periodic, random almost automorphic) function, then $\xi$ is called a random periodic (random almost periodic, random almost automorphic) solution of $\Phi$.

Notice that pathwise random periodic solution was introduced in [40]. We here further extend the concepts of deterministic almost periodic and almost automorphic solutions to the stochastic case.

**Definition 2.3.** Let $x_0 \in X$ and $E$ be a subset of $X$. Then $x_0$ is called a fixed point of $\Phi$ if $\Phi(t, \tau, \omega, x_0) = x_0$ for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. A fixed point $x_0$ is said to be pullback Lyapunov stable in $E$ if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon > 0$, there exists $\delta = \delta(\tau, \omega, \varepsilon) > 0$ such that for all $t \in \mathbb{R}^+$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, \mathcal{N}\delta(x_0) \cap E) \subseteq \mathcal{N}\varepsilon(x_0) \cap E.$$  

If $x_0$ is not pullback Lyapunov stable in $E$, then $x_0$ is said to be pullback Lyapunov unstable in $E$. A fixed point $x_0$ is said to be pullback asymptotically stable in $E$ if it is pullback Lyapunov stable in $E$ and for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $x \in E$,

$$\lim_{t \to \infty} \Phi(t, \tau - t, \theta_{-t}\omega, x) = x_0.$$  

**Definition 2.4.** Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of $X$. We say $D$ is tempered in $X$ with respect to $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ if there exists $x_0 \in X$ such that for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{ct}d(D(\tau + t, \theta_t\omega), x_0) = 0.$$  

**Definition 2.5.** Let $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ with $\mathcal{D}$ given by (2.1). Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor of $\Phi$ if for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, 

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(i) $\mathcal{A}$ is measurable and $\mathcal{A}(\tau, \omega)$ is compact.

(ii) $\mathcal{A}$ is invariant: $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t \omega)$.

(iii) $\mathcal{A}$ attracts every member $B \in \mathcal{D}$,

\[
\lim_{t \to \infty} d(\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), \mathcal{A}(\tau, \omega)) = 0.
\]

If, further, there exists $T > 0$ such that

\[
\mathcal{A}(\tau + T, \omega) = \mathcal{A}(\tau, \omega), \quad \forall \tau \in \mathbb{R}, \forall \omega \in \Omega,
\]

then $\mathcal{A}$ is called a periodic attractor with period $T$.

We recall the following result from [33, 34]. Similar results can be found in [5, 13, 18, 24].

**Proposition 2.6.** Let $\mathcal{D}$ be an inclusion closed collection of some families of nonempty subsets of $X$, and $\Phi$ be a continuous cocycle on $X$ over $\mathbb{R}$ and $(\Omega, F, P, \{\theta_t\}_{t \in \mathbb{R}})$. Then $\Phi$ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $X$ and $\Phi$ has a closed measurable $\mathcal{D}$-pullback absorbing set $K$ in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is characterized by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

\[
\mathcal{A}(\tau, \omega) = \{\psi(0, \tau, \omega) : \psi \text{ is a } \mathcal{D}\text{-complete solution of } \Phi\}
\]

\[
= \{\xi(\tau, \omega) : \xi \text{ is a } \mathcal{D}\text{-complete quasi-solution of } \Phi\}.
\]

### 3 Pitchfork bifurcation of stochastic equations

In this section, we discuss pitchfork bifurcation of the following one-dimensional stochastic equation with deterministic non-autonomous forcing:

\[
\frac{dx}{dt} = \lambda x - \beta(t)x^3 + \gamma(t, x) + \delta x \circ \frac{d\omega}{dt}, \quad x(\tau) = x_\tau,
\]

(3.1)

where $\tau \in \mathbb{R}$, $t > \tau$, $x \in \mathbb{R}$, $\lambda$ and $\delta$ are constants with $\delta > 0$. The function $\beta : \mathbb{R} \to \mathbb{R}$ in (3.1) is smooth and positive. In addition, we assume there exist $\beta_1 \geq \beta_0 > 0$ such that

\[
\beta_0 \leq \beta(t) \leq \beta_1 \quad \text{for all } t \in \mathbb{R}.
\]

(3.2)
The function $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ in (3.1) is smooth and there exist two nonnegative numbers $c_1$ and $c_2$ with $c_1 \leq c_2 < \beta_0$ such that

$$c_1 x^4 \leq \gamma(t, x) x \leq c_2 x^4 \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}.$$  \hfill (3.3)

Note that condition (3.3) implies that $\gamma(t, 0) = 0$ for all $t \in \mathbb{R}$. Therefore, $x = 0$ is a fixed point of equation (3.1). The stochastic equation (3.1) is understood in the sense of Stratonovich integration with $\omega$ being a two-sided real-valued Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact open topology of $\Omega$, and $P$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$ which is given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega \text{ and } t \in \mathbb{R}.$$ 

It follows from [1] that $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a parametric dynamical system and there exists a $\theta_t$-invariant set $\hat{\Omega} \subseteq \Omega$ of full $P$ measure such that for each $\omega \in \hat{\Omega}$,

$$\frac{\omega(t)}{t} \rightarrow 0 \quad \text{as } t \rightarrow \pm \infty,$$  \hfill (3.4)

and

$$\int_{-\infty}^0 e^{2\delta \omega(s)} ds = \infty \quad \text{and} \quad \int_0^\infty e^{2\delta \omega(s)} ds = \infty.$$  \hfill (3.5)

In the sequel, we only consider $\hat{\Omega}$ rather than $\Omega$, and hence we will write $\hat{\Omega}$ as $\Omega$ for convenience.

Under conditions (3.2) and (3.3), one can prove as in [1] that the stochastic equation (3.1) has a unique measurable solution $x$ for a given initial value. Moreover, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $x_{\tau} \in \mathbb{R}$, the solution $x(\cdot, \tau, \omega, x_{\tau}) \in C([\tau, \infty), \mathbb{R})$ and is continuous in $x_{\tau}$. Therefore, one can define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{F} \times \mathbb{R} \rightarrow \mathbb{R}$ for equation (3.1). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $x_{\tau} \in \mathbb{R}$, let

$$\Phi(t, \tau, \omega, x_{\tau}) = x(t + \tau, \tau, \omega, x_{\tau}).$$  \hfill (3.6)

By (3.1), one can check that for every $t \geq 0$, $\tau \geq 0$, $r \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t + \tau, r, \omega, \cdot) = \Phi(t, \tau + t, r, \theta_r \omega, \cdot) \circ \Phi(t, r, \omega, \cdot).$$

Since the solution of (3.1) is measurable in $\omega \in \Omega$ and continuous in initial data, we find that $\Phi$ given by (3.6) is a continuous cocycle on $\mathbb{R}$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. We will study the dynamics of $\Phi$ in this section.
Given a bounded nonempty subset \( I \) of \( \mathbb{R} \), we write \( \| I \| = \sup \{|x| : x \in I\} \). Let \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \) be a family of bounded nonempty subsets of \( \mathbb{R} \). Recall that \( D \) is tempered if for every \( c > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{t \to -\infty} e^{ct} \| D(\tau + t, \theta \omega) \| = 0. \tag{3.7}
\]
Denote by \( \mathcal{D} \) the collection of all tempered families of bounded nonempty subsets of \( \mathbb{R} \), i.e.,
\[
\mathcal{D} = \{ D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (3.7)} \}. \tag{3.8}
\]
In the next subsection, we consider the bifurcation problem of (3.1) when \( \gamma \) is absent. In this case, the stochastic equation (3.1) is exactly solvable which makes it possible for one to completely determine its dynamics. We will show the random complete quasi-solutions of (3.1) undergo a pitchfork bifurcation when \( \lambda \) crosses zero from below. When \( \beta \) is periodic (almost periodic, almost automorphic), we show the random periodic (random almost periodic, random almost automorphic) solutions have similar bifurcation scenarios. We finally investigate pitchfork bifurcation of (3.1) with \( \gamma \) satisfying (3.3).

### 3.1 Pitchfork bifurcation of a typical non-autonomous stochastic equation

This subsection is devoted to pitchfork bifurcation of (3.1) without \( \gamma \). In other words, we consider the following non-autonomous stochastic equation:
\[
\frac{dx}{dt} = \lambda x - \beta(t)x^3 + \delta x \circ d\omega, \quad x(\tau) = x_\tau, \quad t > \tau. \tag{3.9}
\]
As in the deterministic case, equation (3.9) is exactly solvable. To find a solution of (3.9), one may introduce a new variable \( y = x^{-2} \) for \( x \not= 0 \). Then \( y \) satisfies
\[
\frac{dy}{dt} + 2\lambda y = 2\beta(t) - 2\delta y \circ d\omega, \quad y(\tau) = y_\tau = x^{-2}_\tau, \quad t > \tau. \tag{3.10}
\]
For every \( t, \tau \in \mathbb{R} \) with \( t \geq \tau \), \( \omega \in \Omega \) and \( x_\tau \in \mathbb{R} \), by (3.10) we get
\[
y(t, \tau, \omega, y_\tau) = e^{2\lambda(t-\tau)+2\delta(\omega(\tau)-\omega(t))} y_\tau + 2 \int_{\tau}^{t} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} \beta(r) dr.
\]
Therefore, the solution \( x \) of (3.9) is given by, for every \( t, \tau \in \mathbb{R} \) with \( t \geq \tau \), \( \omega \in \Omega \) and \( x_\tau \in \mathbb{R} \),
\[
x(t, \tau, \omega, x_\tau) = \frac{x_\tau}{\sqrt{e^{2\lambda(\tau-t)+2\delta(\omega(\tau)-\omega(t))} + 2x_\tau^2 \int_{\tau}^{t} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} \beta(r) dr}}. \tag{3.11}
\]
It follows from (3.11) that, for each \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( x_0 \in \mathbb{R} \),

\[
x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{x_0}{\sqrt{e^{-2\lambda t + 2\delta \omega(-t)} + 2x_0^2 \int_{\tau-t}^{\tau} e^{2\lambda(r-\tau) + 2\delta \omega(r-\tau)} \beta(r) dr}}. \tag{3.12}
\]

By (3.4) and (3.12) we get, for every \( \lambda > 0 \) and \( x_0 > 0 \),

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{1}{\sqrt{2 \int_{-\infty}^{\tau} e^{2\lambda(r-\tau) + 2\delta \omega(r-\tau)} \beta(r) dr}}. \tag{3.13}
\]

That is, for every \( \lambda > 0 \) and \( x_0 > 0 \), we have

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{-1}{\sqrt{2 \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} \beta(r + \tau) dr}}. \tag{3.14}
\]

Note that the right-hand side of (3.13) is well defined in terms of (3.2) and (3.4). Similarly, by (3.12) we obtain, for every \( \lambda > 0 \) and \( x_0 < 0 \),

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{-1}{\sqrt{2 \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} \beta(r + \tau) dr}}. \tag{3.15}
\]

It is evident that for each \( \lambda > 0 \) and \( \tau \in \mathbb{R} \), both \( x^+_\lambda(\tau, \cdot) \) and \( x^-_\lambda(\tau, \cdot) \) are measurable. We next prove that \( x^+_\lambda \) and \( x^-_\lambda \) are random complete quasi-solutions of equation (3.1).

**Lemma 3.1.** Suppose (3.2) holds. Then for every \( \lambda > 0 \), \( x^+_\lambda \) and \( x^-_\lambda \) given by (3.15) are tempered random complete quasi-solutions of equation (3.1). Moreover, \( x^+_\lambda \) is the only complete quasi-solution in \((0, \infty)\) with tempered reciprocal, and \( x^-_\lambda \) is the only complete quasi-solution in \((-\infty, 0)\) with tempered reciprocal.

**Proof.** We first prove \( x^+_\lambda \) and \( x^-_\lambda \) are random complete quasi-solutions. Given \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we need to show

\[
\Phi(t, \tau, \omega, x^+_\lambda(\tau, \omega)) = x^+_\lambda(\tau + t, \theta t \omega). \tag{3.16}
\]

By (3.11) we find that, for each \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
x(t + \tau, \tau, \theta_{-\tau} \omega, x^+_\lambda(\tau, \omega))
\]

\[
\]
which implies that for all \( r \),

\[
\sqrt{e^{-2M-28\omega(t)}} \left( x^+(\tau, \omega) \right)^2 = \frac{x^+(\tau, \omega)}{\sqrt{e^{-2M-28\omega(t)}}} + 2 \int_{t-\tau}^{t+\tau} e^{2\lambda(r-t-\tau)+2\delta(\omega(r-\tau)\omega(t))} \beta(r) \, dr
\]

It follows from (3.6) and (3.17) that for each \( t \),

\[
\sqrt{e^{-2M-28\omega(t)}} \left( x^+(\tau, \omega) \right)^2 = 1 + 2 \int_{t-\tau}^{t+\tau} e^{2\lambda(r-t-\tau)+2\delta(\omega(r-\tau)\omega(t))} \beta(r) \, dr
\]

Similarly, we can also verify that for each \( t \),

\[
\sqrt{e^{-2M-28\omega(t)}} \left( x^-\lambda(\tau, \omega) \right)^2 = \frac{1}{\sqrt{e^{-2M-28\omega(t)}}} = x^-\lambda(\tau + t, \theta t \omega).
\]

It follows from (3.6) and (3.17) that for each \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\Phi(t, \tau, \omega, x^\pm\lambda(\tau, \omega)) = x^\pm\lambda(\tau + t, \theta t \omega).
\]

Similarly, we can also verify that for each \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\Phi(t, \tau, \omega, x^-\lambda(\tau, \omega)) = x^-\lambda(\tau + t, \theta t \omega).
\]

By (3.18) and (3.19), we get (3.16), and hence both \( x^+\lambda \) and \( x^-\lambda \) are random complete quasi-solutions of equation (3.1).

We now prove that \( x^+\lambda \) and \( x^-\lambda \) are tempered. Given \( c > 0 \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), by (3.2) and (3.15) we get

\[
e^{ct}|x^\pm\lambda(\tau + t, \theta t \omega)| \leq \frac{e^{ct}}{\sqrt{2 \beta_0 \int_{-\infty}^{0} e^{2\lambda r+2\delta(\omega(r)\omega(t))} \, dr}} \leq \frac{e^{ct}}{\sqrt{2 \beta_0 \int_{-\infty}^{0} e^{2\lambda r+2\delta(\omega(r+t)\omega(t))} \, dr}}
\]

Let \( \varepsilon = \frac{1}{2c} \min\{\lambda, \frac{1}{2}c\} \). By (3.4) we find that for each \( \omega \in \Omega \), there exists \( T = T(\omega) < 0 \) such that for all \( t \leq T \),

\[
\varepsilon t \leq \omega(t) \leq -\varepsilon t,
\]

which implies that for all \( r \leq 0 \) and \( t \leq T \),

\[
\varepsilon r + \varepsilon t \leq \omega(r + t) \leq -\varepsilon r - \varepsilon t.
\]
It follows from \((3.21)-(3.22)\) that, for all \(r \leq 0\) and \(t \leq T\),
\[
2\lambda r + 2\delta (\omega (r + t) - \omega (t)) \geq 2\lambda r + 2\delta \varepsilon t + 4\delta \varepsilon t \geq 3\lambda r + ct.
\]
Therefore, we get, for all \(t \leq T\),
\[
\int_{-\infty}^{0} e^{2\lambda r + 2\delta (\omega (r + t) - \omega (t))} dr \geq \int_{-\infty}^{0} e^{3\lambda r + ct} dr \geq \frac{e^{ct}}{3\lambda}.
\] \(3.23\)

By \((3.20)\) and \((3.23)\) we obtain, for every \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),
\[
\limsup_{t \to -\infty} e^{ct} |x_{-\lambda}^+ (\tau + t, \theta_{-t}\omega)| \leq \limsup_{t \to -\infty} \sqrt{\frac{3\lambda}{2\beta_0}} e^\frac{1}{2} e^{\frac{ct}{2}} = 0.
\]

This shows that \(x_{-\lambda}^+\) and \(x_{-\lambda}^-\) are tempered. Similarly, by \((3.21)\) and \((3.22)\), one can verify \(\frac{1}{x_{+\lambda}^-}\) and \(\frac{1}{x_{-\lambda}^+}\) are also tempered.

Next, we prove that \(x_{+\lambda}^\pm\) is the only complete quasi-solution in \((0, \infty)\) with tempered reciprocal. Suppose \(\xi\) is an arbitrary complete quasi-solution in \((0, \infty)\) such that \(\xi^{-1}\) is tempered. By definition we have, for all \(t \in \mathbb{R}^+\), \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),
\[
\Phi(t, \tau - t, \theta_{-t}\omega, \xi (\tau - t, \theta_{-t}\omega)) = \xi(\tau, \omega).
\] \(3.24\)

On the other hand, by \((3.12)\) we get, for all \(t \in \mathbb{R}\) and \(\omega \in \Omega\),
\[
x(\tau, \tau - t, \theta_{-t}\omega, \xi (\tau - t, \theta_{-t}\omega)) = \frac{1}{\sqrt{e^{-2\lambda T + 2\delta \omega (-t)} \xi^{-2} (\tau - t, \theta_{-t}\omega) + 2 \int_{-\tau}^{t} e^{2\lambda (r - T) + 2\delta \omega (r - T)} \beta (r) dr}}.
\] \(3.25\)

Since \(\xi^{-1}\) is tempered, by \((3.4)\) and \((3.25)\), we obtain, for all \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),
\[
\lim_{t \to \infty} \Phi(t, \tau - t, \theta_{-t}\omega, \xi (\tau - t, \theta_{-t}\omega)) = \lim_{t \to \infty} x(\tau, \tau - t, \theta_{-t}\omega, \xi (\tau - t, \theta_{-t}\omega)) = x_{+\lambda}^+ (\tau, \omega),
\]
which together with \((3.24)\) gives \(\xi(\tau, \omega) = x_{+\lambda}^\pm (\tau, \omega)\), as desired. The uniqueness of \(x_{-\lambda}^-\) in \((-\infty, 0)\) can be proved by a similar approach. The details are omitted.

We now discuss the stability of the zero solution of \((3.9)\).

**Lemma 3.2.** Suppose \(3.2\) holds. Then the zero solution of equation \((3.9)\) is pullback asymptotically stable in \(\mathbb{R}\) if \(\lambda \leq 0\); and pullback Lyapunov unstable in \(\mathbb{R}\) if \(\lambda > 0\).
Proof. Case (i): $\lambda < 0$. In this case, we need to prove the asymptotic stability of $x = 0$. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon > 0$, we must find a positive number $\eta = \eta(\tau, \omega, \varepsilon)$ such that for every $t \geq 0$ and $x_0 \in (-\eta, \eta)$,

$$|\Phi(t, \tau - t, \theta_{-\tau}\omega, x_0)| < \varepsilon.$$ (3.26)

By (3.4) we see that there exists $T = T(\omega) > 0$ such that for all $t \geq T$,

$$\frac{\lambda}{29} t \leq \omega(-t) \leq -\frac{\lambda}{29} t.$$ (3.27)

By (3.27) we get

$$e^{-\lambda t} + \delta \omega(\tau) \geq \frac{1}{2}, \text{ for all } t \geq T.$$ (3.28)

On the other hand, by the continuity of $\omega$, there exists a positive number $c_0 = c_0(\omega)$ such that

$$e^{-\lambda t} + \delta \omega(\tau) \geq c_0, \text{ for all } t \in [0, T].$$ (3.29)

Let $\eta = \min\{\varepsilon, \varepsilon c_0\}$ with $c_0$ as in (3.29). Then for every $t \geq 0$ and $x_0 \in (-\eta, \eta)$, it follows from (3.12) and (3.27) that

$$\varepsilon e^{-\lambda t} + \delta \omega(\tau) \geq \eta > |x_0|,$$

which implies that for every $t \geq 0$ and $x_0 \in (-\eta, \eta),$

$$e^{-2\lambda t + 2\delta \omega(\tau)} x_0^2 + 2 \int_{\tau-t}^{T} e^{2\lambda(r-\tau) + 2\delta \omega(r-\tau)} \beta(r) dr > \varepsilon^{-2}. \quad (3.30)$$

By (3.12) and (3.30) we get, for every $t \geq 0$ and $x_0 \in (-\eta, \eta),$

$$|x(\tau, \tau - t, \theta_{-\tau}\omega, x_0)| < \varepsilon.$$

Therefore, (3.26) is satisfied and thus $x = 0$ is pullback Lyapunov stable in $\mathbb{R}$. Note that for every $x_0 \in \mathbb{R}$, from (3.12) and (3.27) we have

$$\lim_{t \to \infty} \sup |x(\tau, \tau - t, \theta_{-\tau}\omega, x_0)| \leq \lim_{t \to \infty} \sup e^{\lambda t - \delta \omega(-t)} |x_0| = 0,$$

which along with (3.26) and Definition 2.3 shows that $x = 0$ is pullback asymptotically stable in $\mathbb{R}$ for $\lambda < 0$.

Case (ii): $\lambda = 0$. In this case, by (3.2) and (3.5) we get for every $\tau \in \mathbb{R}$ and $\omega \in \Omega,$

$$\lim_{t \to \infty} \int_{\tau-t}^{T} 2e^{2\delta \omega(r-\tau)} \beta(r) dr \geq 2\beta_0 \lim_{t \to \infty} \int_{-t}^{0} e^{2\delta \omega(r)} dr = \infty,$$
which implies that for given $\varepsilon > 0$, there exists $T = T(\omega) > 0$ such that for all $t \geq T$,
\[
2 \int_{T-t}^T e^{2\delta \omega(r-\tau)} \beta(r) dr > \varepsilon^{-2}.
\]
(3.31)

Let $\eta = \varepsilon c_0$ where $c_0$ is the positive number in (3.29) with $\lambda = 0$. Then for every $t \in [0, T]$ and $x_0 \in (-\eta, \eta)$, we have
\[
e^{2\delta \omega(-t)} x_0^{-2} > \varepsilon^{-2}.
\]
(3.32)

It follows from (3.31)-(3.32) that for every $t \geq 0$ and $x_0 \in (-\eta, \eta)$,
\[
e^{2\delta \omega(-t)} x_0^{-2} + 2 \int_{-t}^{T-t} e^{2\delta \omega(r-\tau)} \beta(r) dr > \varepsilon^{-2},
\]
which along with (3.12) shows that for every $t \geq 0$ and $x_0 \in (-\eta, \eta)$,
\[
|x(\tau, \tau - t, \theta - \tau, x_0)| < \varepsilon.
\]

Therefore $x = 0$ is pullback Lyapunov stable in $\mathbb{R}$ for $\lambda = 0$. On the other hand, by (3.2) and (3.5) we get, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[
\limsup_{t \to \infty} |x(\tau, \tau - t, \theta - \tau, x_0)| \leq \limsup_{t \to \infty} \frac{1}{\sqrt{2\beta_0 \int_{-t}^{T-t} e^{2\delta \omega(r)} dr}} = 0.
\]

This indicates that $x = 0$ is pullback asymptotically stable in $\mathbb{R}$ for $\lambda = 0$.

Case (iii): $\lambda > 0$. Note that for every $\lambda > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $x^+_{\lambda}(\tau, \omega)$ and $x^-_{\lambda}(\tau, \omega)$ given by (3.15) are nonzero. In addition, by (3.13)-(3.14), we know that every solution $x(\tau, \tau - t, \theta - \tau, x_0)$ with $x_0 \neq 0$ converges either to $x^+_{\lambda}(\tau, \omega)$ or $x^-_{\lambda}(\tau, \omega)$ as $t \to \infty$. Therefore, we conclude that the zero solution of (3.9) is not pullback stable in $\mathbb{R}$ for $\lambda > 0$.

We are now ready to discuss pitchfork bifurcation of random complete quasi-solutions of (3.9).

**Theorem 3.3.** Suppose (3.2) holds. Then the random complete quasi-solutions of (3.9) undergo a stochastic pitchfork bifurcation at $\lambda = 0$. More precisely:

(i) If $\lambda \leq 0$, then $x = 0$ is the unique random complete quasi-solution of (3.9) which is pullback asymptotically stable in $\mathbb{R}$. In this case, the equation has a trivial $\mathcal{D}$-pullback attractor $\mathcal{A} = \{ \mathcal{A}(\tau, \omega) = \{0\} : \tau \in \mathbb{R}, \omega \in \Omega \}$.

(ii) If $\lambda > 0$, then the zero solution loses its stability and the equation has two more tempered random complete quasi-solutions $x^+_{\lambda} > 0$ and $x^-_{\lambda} < 0$ such that
\[
\lim_{\lambda \to 0} x^+_{\lambda}(\tau, \omega) = 0, \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.
\]
(3.33)
Moreover, $x^+_\lambda$ and $x^-_\lambda$ are the only complete quasi-solutions with tempered reciprocals in $(0, \infty)$ and $(-\infty, 0)$, respectively. In this case, equation (3.9) has a $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) = [x^+_{\lambda}(\tau, \omega), x^-_{\lambda}(\tau, \omega)] : \tau \in \mathbb{R}, \omega \in \Omega\}$, $x^+_\lambda$ and $x^-_\lambda$ pullback attracts every compact subset of $(0, \infty)$ and $(-\infty, 0)$, respectively.

Proof. We first verify (3.33). By (3.2), (3.5) and Fatou’s lemma we find that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\liminf_{\lambda \to 0} \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} \beta(r + \tau) dr \geq \liminf_{\lambda \to 0} \int_{-\infty}^{0} \beta_0 e^{2\lambda r + 2\delta \omega(r)} dr \geq \beta_0 \int_{-\infty}^{0} e^{2\delta \omega(r)} dr = \infty,$$

which along with (3.15) yields (3.33). Note that the rest of this theorem is an immediate consequence of (3.12), Lemmas 3.1 and 3.2. The details are omitted here.

Next, we consider pitchfork bifurcation of random periodic, random almost periodic and random almost automorphic solutions of (3.9). Let $\beta : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $T > 0$. Then by (3.15) we see that for each $\lambda > 0$ and $\omega \in \Omega$, both $x^+_\lambda(\cdot, \omega)$ and $x^-_\lambda(\cdot, \omega)$ are $T$-periodic. In other words, $x^+_\lambda$ and $x^-_\lambda$ are random periodic solutions of (3.9) in this case. Applying Theorem 3.3, we immediately get pitchfork bifurcation of random periodic solutions for (3.9). In the almost periodic case, we need the following lemma.

**Lemma 3.4.** Suppose (3.2) holds and $\beta : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then for every $\lambda > 0$, the complete quasi-solutions $x^+_\lambda$ given by (3.15) are also almost periodic.

**Proof.** Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$g(\tau, \omega) = \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} \beta(r + \tau) dr. \quad (3.34)$$

We first show that $g$ given by (3.34) is a random almost periodic function. Since $\beta$ is almost periodic, given $\varepsilon > 0$, there exists $l = l(\omega, \varepsilon) > 0$ such that every interval of length $l$ contains a $t_0$ such that

$$|\beta(t + t_0) - \beta(t)| < \frac{\varepsilon}{\int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} dr}, \quad \text{for all } t \in \mathbb{R}. \quad (3.35)$$

By (3.34)-(3.35) we obtain, for all $\tau \in \mathbb{R}$,

$$|g(\tau + t_0, \omega) - g(\tau, \omega)| \leq \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} |\beta(r + t_0) - \beta(r + \tau)| dr.$$
\[
\int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} dr \leq \varepsilon,
\]
(3.36)
which shows that \( g(\cdot, \omega) \) is almost periodic for every fixed \( \omega \in \Omega \). By (3.2) we have, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
0 < \beta_0 \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} dr \leq g(\tau, \omega) \leq \beta_1 \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} dr.
\]
(3.37)
It follows from (3.36)-(3.37) that for each fixed \( \omega \in \Omega \), \( g(\cdot, \omega) \) is almost periodic. Then the almost periodicity of \( x^\pm(\cdot, \omega) \) follows from (3.15) immediately, and this completes the proof.

Analogously, for the almost automorphic case, we have the following results.

**Lemma 3.5.** Suppose (3.2) holds and \( \beta : \mathbb{R} \rightarrow \mathbb{R} \) is almost automorphic. Then for every \( \lambda > 0 \), the complete quasi-solutions \( x^\pm \) given by (3.15) are also almost automorphic.

**Proof.** Let \( \{\tau_n\}_{n=1}^\infty \) be a sequence of numbers. Since \( \beta \) is almost automorphic, there exists a subsequence \( \{\tau_{n_m}\}_{m=1}^\infty \) of \( \{\tau_n\}_{n=1}^\infty \) and a function \( h : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( t \in \mathbb{R} \),
\[
\lim_{m \rightarrow \infty} \beta(t + \tau_{n_m}) = h(t) \quad \text{and} \quad \lim_{m \rightarrow \infty} h(t - \tau_{n_m}) = \beta(t).
\]
(3.38)
By (3.2) and (3.38) we have
\[
0 < \beta_0 \leq h(t) \leq \beta_1 \quad \text{for all} \quad t \in \mathbb{R}.
\]
(3.39)
Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), denote by
\[
H(\tau, \omega) = \frac{1}{\sqrt{2} \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} h(r + \tau) dr}.
\]
(3.40)
Note that the right-hand side of (3.40) is well defined due to (3.39). By (3.38), (3.40) and the Lebesgue dominated convergence theorem, we get, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{m \rightarrow \infty} x^+_{\lambda}(\tau + \tau_{n_m}, \omega) = \lim_{m \rightarrow \infty} \frac{1}{\sqrt{2} \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} \beta(r + \tau + \tau_{n_m}) dr} = H(\tau, \omega),
\]
(3.41)
and
\[
\lim_{m \rightarrow \infty} H(\tau - \tau_{n_m}, \omega) = \lim_{m \rightarrow \infty} \frac{1}{\sqrt{2} \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} h(r + \tau - \tau_{n_m}) dr} = x^+_{\lambda}(\tau, \omega).
\]
(3.42)
By (3.41) and (3.42) we find that \( x^+_{\lambda} \) is a random complete quasi-solution of (3.9). By a similar argument, one can verify \( x^-_{\lambda} \) is also a random complete solution. This completes the proof. \( \square \)
As a consequence of Theorem 3.3, Lemmas 3.4 and 3.5, we get the following pitchfork bifurcation of random periodic (almost periodic, almost automorphic) solutions of (3.9).

**Theorem 3.6.** Suppose (3.2) holds and $\beta : \mathbb{R} \to \mathbb{R}$ is periodic (almost periodic, almost automorphic). Then the random periodic (almost periodic, almost automorphic) solutions of (3.9) undergo a stochastic pitchfork bifurcation at $\lambda = 0$. More precisely:

(i) If $\lambda \leq 0$, then $x = 0$ is the unique random periodic (almost periodic, almost automorphic) solution of (3.9) which is pullback asymptotically stable in $\mathbb{R}$. In this case, the equation has a trivial $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) = \{0\} : \tau \in \mathbb{R}, \omega \in \Omega\}$.

(ii) If $\lambda > 0$, then the zero solution loses its stability and the equation has two more random periodic (almost periodic, almost automorphic) solutions $x_+^\lambda > 0$ and $x_-^\lambda < 0$ such that

$$\lim_{\lambda \to 0} x_\pm^\lambda(\tau, \omega) = 0, \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$  \hfill (3.43)

In this case, equation (3.9) has a $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) = [x_-^\lambda(\tau, \omega), x_+^\lambda(\tau, \omega)] : \tau \in \mathbb{R}, \omega \in \Omega\}$. Moreover, $x_+^\lambda$ and $x_-^\lambda$ pullback attracts every compact subset of $(0, \infty)$ and $(-\infty, 0)$, respectively.

**Proof.** Since $\beta$ is periodic (almost periodic, almost automorphic), by Lemmas 3.4 and 3.5 we know that for every $\lambda > 0$, the random complete quasi-solutions $x_+^\lambda$ and $x_-^\lambda$ given by (3.15) are periodic (almost periodic, almost automorphic). Then, by Theorem 3.3 we conclude the proof. \hfill $\square$

### 3.2 Pitchfork bifurcation of a general non-autonomous stochastic equation

In this subsection, we discuss pitchfork bifurcation of the stochastic equation (3.1) with a non-linearity $\gamma$ satisfying (3.3). We first establish existence of $\mathcal{D}$-pullback attractors for a generalized system and then construct random complete quasi-solutions. The comparison principle will play an important role in our arguments.

Given $\tau \in \mathbb{R}$, consider the non-autonomous stochastic equation defined for $t > \tau$:

$$\frac{dx}{dt} = f(t, x) + g(t) + \delta x \circ \frac{d\omega}{dt}, \quad x(\tau) = x_\tau,$$  \hfill (3.44)

where $\delta > 0$, $g : \mathbb{R} \to \mathbb{R}$ is a function, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity satisfying

$$f(t, 0) = 0, \quad f(t, x) \leq -\nu x^2 + h(t)|x|, \quad \text{for all } t, x \in \mathbb{R},$$  \hfill (3.45)
for some fixed \( \nu > 0 \) and \( h : \mathbb{R} \to \mathbb{R} \). By (3.45) we see \( h(t) \geq 0 \) for all \( t \in \mathbb{R} \). In the sequel, we assume that \( g, h \in L^1_{\text{loc}}(\mathbb{R}) \) and there exists \( \alpha \in (0, \nu) \) such that

\[
\int_{-\infty}^{\tau} e^{\alpha t} (|g(t)| + |h(t)|) dt < \infty, \quad \text{for all } \tau \in \mathbb{R}.
\]

This condition will be used to construct pullback absorbing sets for (3.44). To ensure existence of tempered pullback attractors, we further require the following condition for \( g \) and \( h \): for every \( c > 0 \) and \( \tau \in \mathbb{R} \),

\[
\lim_{s \to -\infty} e^{(c-\alpha)s} \int_{-\infty}^{s+\tau} e^{\alpha t} (|g(t)| + |h(t)|) dt = 0.
\]

Note that condition (3.47) is stronger than (3.46), and both conditions do not require \( g \) and \( h \) to be bounded as \( t \to \pm \infty \). Based on (3.45), we may associate a linear system with (3.44). Given \( \tau \in \mathbb{R} \) and \( y_{\tau} \in \mathbb{R} \), consider

\[
\frac{dy}{dt} = -\nu y + |g(t)| + h(t) + \delta y \circ d\omega \circ dt, \quad y(\tau) = y_{\tau}.
\]

By the comparison principle, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), if \( y_{\tau} \geq 0 \), then \( y(t, \tau, \omega, y_{\tau}) \geq 0 \) for all \( t \geq \tau \). This along with (3.45) implies that the solution \( y(t, \tau, \omega, y_{\tau}) \) of the linear equation (3.48) is a super-solution of (3.44) provided \( y_{\tau} \) is nonnegative. Therefore, we are able to control solutions of (3.44) by (3.48) based on the comparison principle. Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the solution of the linear equation (3.48) is given by

\[
y(t, \tau, \omega, y_{\tau}) = e^{\nu(t-\tau)-\delta(\omega(t)-\omega(\tau))} y_{\tau} + \int_{\tau}^{t} e^{\nu(s-t)-\delta(\omega(s)-\omega(t))} (|g(s)| + h(s)) ds.
\]

Let \( \Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R} \) be a mapping given by, for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( y_{\tau} \in \mathbb{R} \),

\[
\Psi(t, \tau, \omega, y_{\tau}) = y(t + \tau, \tau, \theta_{-\tau} \omega, y_{\tau}).
\]

By (3.49) and (3.50), one can check that \( \Psi \) is a continuous cocycle on \( \mathbb{R} \) over \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \).

Next, we show \( \Psi \) has a unique tempered complete quasi-solution which pullback attracts every tempered sets.

**Lemma 3.7.** Suppose (3.46) and (3.47) hold. Then \( \Psi \) associated with (3.48) has a unique tempered complete quasi-solution \( \xi \) given by, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\xi(\tau, \omega) = \int_{-\infty}^{0} e^{\nu s - \delta \omega(s)} (|g(s + \tau)| + h(s + \tau)) ds.
\]

Moreover, \( \Psi \) has a \( \mathcal{D} \)-pullback attractor given by \( \mathcal{A} = \{\mathcal{A}(\tau, \omega) = \{\xi(\tau, \omega)\} : \tau \in \mathbb{R}, \omega \in \Omega\} \). If, in addition, \( g \) and \( h \) are periodic functions with period \( T > 0 \), then \( \xi \) is also \( T \)-periodic.
Proof. First, by (3.4) and (3.46), we can verify the integral on the right-hand side of (3.51) is well defined for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). If \( g \) and \( h \) are \( T \)-periodic, by (3.51), we see that \( \xi(\tau + T, \omega) = \xi(\tau, \omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), and hence \( \xi \) is \( T \)-periodic.

We now prove \( \xi \) is tempered. Given \( c_0 > 0 \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), by (3.51) we have

\[
e^{c_0\tau}|\xi(\tau + \tau, \theta, \omega)| = e^{c_0\tau} \int_{-\infty}^{0} e^{\nu s + \delta \omega(r) - \delta \omega(s + r)} (|g(s + \tau + r)| + h(s + \tau + r)) ds. \tag{3.52}
\]

Let \( \varepsilon = \frac{1}{4} \min\{\nu - \alpha, \frac{1}{4}c_0\} \). By (3.4) we find that for every \( \omega \in \Omega \), there exists \( T = T(\omega) < 0 \) such that for all \( s \leq 0 \) and \( r \leq T \),

\[
\varepsilon r \leq \omega(r) \leq -\varepsilon r \quad \text{and} \quad \varepsilon(r + s) \leq \omega(r + s) \leq -\varepsilon(r + s). \tag{3.53}
\]

It follows from (3.47), (3.52) and (3.53) that

\[
\limsup_{r \to -\infty} e^{c_0\tau}|\xi(\tau + r, \theta, \omega)| \leq \limsup_{r \to -\infty} e^{\frac{1}{2}c_0\tau} \int_{-\infty}^{0} e^{\alpha s}(|g(s + \tau + r)| + h(s + \tau + r)) ds
\]

\[
\leq e^{-\alpha \tau} \limsup_{r \to -\infty} e^{\frac{1}{2}(\varepsilon_0 - \alpha) r} \int_{-\infty}^{r + T} e^{\alpha t}(|g(t)| + h(t)) dt = 0,
\]

and hence \( \xi \) is tempered. Next, we prove \( \xi \) is a random complete quasi-solution of \( \Psi \). By (3.49) and (3.51) we find that for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
y(t + \tau, \tau, \theta_{-\tau} \omega, \xi(\tau, \omega)) = e^{-\nu t + \delta \omega(t)} \xi(\tau, \omega) + \int_{\tau}^{(t+\tau)} e^{-\nu(s-t) - \delta(\omega(s) - \omega(t))} (|g(s)| + h(s)) ds
\]

\[
= \int_{-\infty}^{0} e^{\nu(s-t) - \delta(\omega(s) - \omega(t))} (|g(s + t)| + h(s + t)) ds + \int_{-t}^{0} e^{-\nu s - \delta(\omega(s + t) - \omega(t))} (|g(s + t + \tau)| + h(s + t + \tau)) ds.
\tag{3.54}
\]

By (3.51) and (3.54) we get, for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\Psi(t, \tau, \omega, \xi(\tau, \omega)) = \xi(\tau + t, \theta_t \omega).
\]

This shows that \( \xi \) is a random complete quasi-solution of (3.48).

We now prove the attraction property of \( \xi \) in \( \mathcal{D} \). Recall that \( \mathcal{D} \) is the collection of all tempered families given by (3.8). Let \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) and \( y_{\tau - t} \in D(\tau - t, \theta_{-t} \omega) \). From (3.39) we have, for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
y(\tau, \tau - t, \theta_{-\tau} \omega, y_{\tau - t}) = e^{-\nu \tau - \delta \omega(-\tau)} y_{\tau - t} + \int_{\tau - t}^{\tau} e^{\nu(s-\tau) - \delta \omega(s-\tau)} (|g(s)| + h(s)) ds
\]

\[
= \int_{-\infty}^{0} e^{\nu(s-\tau) - \delta(\omega(s) - \omega(\tau))} (|g(s + \tau)| + h(s + \tau)) ds + \int_{-t}^{0} e^{-\nu s - \delta(\omega(s + \tau) - \omega(\tau))} (|g(s + \tau + \tau)| + h(s + \tau + \tau)) ds.
\]
\[ e^{-\nu t - \delta t} y_{\tau-t} + \int_{-t}^{0} e^{\nu s - \delta t} (|g(s + \tau)| + h(s + \tau)) ds. \]  

(3.55)

By (3.4) we obtain

\[
\limsup_{t \to \infty} e^{-\nu t - \delta t} |y_{\tau-t}| \leq \limsup_{t \to \infty} e^{-\nu t - \delta t} \| D(\tau - t, \theta_{-t}\omega) \| = 0.
\]

which along with (3.50), (3.51) and (3.55) imply that for every \( D \in \mathcal{D}, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{t \to \infty} d(\Psi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), \xi(\tau, \omega)) = 0.
\]

(3.56)

Note that (3.56) implies \( \xi \) pullback attracts every tempered family of subsets of \( \mathbb{R} \), and hence \( \{ \xi(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a \( \mathcal{D} \)-pullback attractor of \( \Psi \).

Taking an arbitrary tempered complete quasi-solution \( \zeta \) of (3.48), we now prove \( \zeta = \xi \). Since \( \zeta \) is a complete quasi-solution, we have, for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\Psi(t, \tau - t, \theta_{-t}\omega, \zeta(\tau - t, \theta_{-t}\omega)) = \zeta(\tau, \omega).
\]

(3.57)

Since \( \xi \) is tempered, by (3.56) and (3.57) we get \( \zeta(\tau, \omega) = \xi(\tau, \omega) \) for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). This implies the uniqueness of tempered complete quasi-solutions of (3.48), and thus completes the proof.

We now prove existence of \( \mathcal{D} \)-pullback attractors for equation (3.44).

**Theorem 3.8.** Suppose (3.45) and (3.46) - (3.47) hold. Then \( \Phi \) associated with (3.44) has a unique \( \mathcal{D} \)-pullback attractor \( A \in \mathcal{D} \) which is characterized by, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
A(\tau, \omega) = \{ \xi(\tau, \omega) : \xi \text{ is a } \mathcal{D} \text{-complete quasi-solution of } \Phi \}.
\]

(3.58)

**Proof.** Let \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), \( D \in \mathcal{D} \) and \( x_{\tau-t} \in D(\tau - t, \theta_{-t}\omega) \). By the comparison principle, we find the solution \( x \) of (3.44) satisfies

\[
|x(\tau, \tau - t, \theta_{-t}\omega, x_{\tau-t})| \leq y(\tau, \tau - t, \theta_{-t}\omega, |x_{\tau-t}|),
\]

where \( y \) is the solution of the linear equation (3.48). Then by (3.49) we have

\[
|x(\tau, \tau - t, \theta_{-t}\omega, x_{\tau-t})| \leq e^{-\nu t - \delta t} |x_{\tau-t}| + \int_{-t}^{0} e^{\nu s - \delta t} (|g(s + \tau)| + h(s + \tau)) ds.
\]

(3.59)

Since \( x_{\tau-t} \in D(\tau - t, \theta_{-t}\omega) \) and \( D \in \mathcal{D} \), by (3.4) we get

\[
\limsup_{t \to \infty} e^{-\nu t - \delta t} |x_{\tau-t}| \leq \limsup_{t \to \infty} e^{-\nu t - \delta t} \| D(\tau - t, \theta_{-t}\omega) \| = 0.
\]

(3.60)
It follows from (3.51) and (3.59)-(3.60) that, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\limsup_{t \to \infty} |x(\tau - t, \theta_{-\tau} \omega, x_{\tau - t})| \leq \xi(\tau, \omega),
\]

(3.61)

where \( \xi \) is the complete quasi-solution of (3.48) given by (3.51). On the other hand, by (3.59) and (3.60), there exists \( T = T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \),

\[
|\Phi(t, \tau - t, \theta_{-\tau} \omega, x_{\tau - t})| = |x(\tau - t, \theta_{-\tau} \omega, x_{\tau - t})| \leq 2 \xi(\tau, \omega).
\]

(3.62)

Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), define \( K(\tau, \omega) = [-2\xi(\tau, \omega), 2\xi(\tau, \omega)] \). Since \( \xi \) is measurable and tempered, by (3.62) we find \( K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \) is a \( D \)-pullback absorbing set of \( \Phi \). Since \( K \) is compact, by Proposition 2.6 \( \Phi \) has a unique \( D \)-pullback attractor \( A \) which is characterized by (3.58).

We next further characterize the structures of the tempered attractor of equation (3.44).

**Theorem 3.9.** Suppose (3.45) and (3.46)-(3.47) hold. Then the cocycle \( \Phi \) associated with (3.44) has two tempered complete quasi-solutions \( x^* \) and \( x^* \) such that \( A = \{x^*(\tau, \omega), x^*(\tau, \omega) \}, \tau \in \mathbb{R}, \omega \in \Omega \} \) is the unique \( D \)-pullback attractor of \( \Phi \).

**Proof.** Let \( t_1 \geq t_2 > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \). By the comparison principle, for the solution \( x \) of (3.44) we have

\[
x(\tau - t_2, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)) \leq y(\tau - t_2, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)),
\]

(3.63)

where \( y \) is the solution of (3.48) and \( \xi \) is given by (3.51). Since \( \xi \) is a complete quasi-solution of (3.48), we get

\[
y(\tau - t_2, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)) = \Psi(t_1 - t_2, \tau - t_1, \theta_{-t_1} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)) = \xi(\tau - t_2, \theta_{-t_2} \omega).
\]

(3.64)

By (3.63)-(3.64) we obtain

\[
x(\tau - t_2, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)) \leq \xi(\tau - t_2, \theta_{-t_2} \omega).
\]

(3.65)

By (3.65) and the comparison principle, we get

\[
x(\tau - t_2, \theta_{-\tau} \omega, x(\tau - t_2, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega))) \leq x(\tau, \tau - t_2, \theta_{-\tau} \omega, \xi(\tau - t_2, \theta_{-t_2} \omega)),
\]

(3.66)

We next further characterize the structures of the tempered attractor of equation (3.44).
which implies that for all $t_1 \geq t_2 > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$x(\tau, \tau - t_1, \theta_{-\tau} \omega, \xi(\tau - t_1, \theta_{-t_1} \omega)) \leq x(\tau, \tau - t_2, \theta_{-\tau} \omega, \xi(\tau - t_2, \theta_{-t_2} \omega)).$$  \hspace{1cm} (3.66)$$

By (3.66) we find that $x(\tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega))$ is monotone in $t \in \mathbb{R}^+$ for each fixed $\tau$ and $\omega$. Since $\xi$ is tempered, by (3.62) we see $x(\tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega))$ is bounded in $t \in \mathbb{R}^+$. Therefore, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $x^*(\tau, \omega) \in \mathbb{R}$ such that

$$\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega)) = x^*(\tau, \omega).$$  \hspace{1cm} (3.67)$$

By the attraction property of $\mathcal{A}$ of the $D$-pullback attractor of (3.44), we have $x^*(\tau, \omega) \in \mathcal{A}(\tau, \omega)$ for every $\tau$ and $\omega$. By (3.61) and (3.67) we get $|x^*(\tau, \omega)| \leq \xi(\tau, \omega)$, and hence $x^*$ is tempered. By a similar argument, we can show there exists $x_*^*(\tau, \omega) \in \mathcal{A}(\tau, \omega)$ with $|x_*^*(\tau, \omega)| \leq \xi(\tau, \omega)$ such that

$$\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, -\xi(\tau - t, \theta_{-t} \omega)) = x_*^*(\tau, \omega).$$  \hspace{1cm} (3.68)$$

Note that $x_*$ is tempered. By (3.67)-(3.68) and the comparison principle, we have $x_*^*(\tau, \omega) \leq x^*(\tau, \omega)$. Note that (3.61) implies

$$\mathcal{A}(\tau, \omega) \subseteq [-\xi(\tau, \omega), \xi(\tau, \omega)], \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$  \hspace{1cm} (3.69)$$

Based on (3.69) we will prove

$$\mathcal{A}(\tau, \omega) \subseteq [x_*^*(\tau, \omega), x^*(\tau, \omega)], \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$  \hspace{1cm} (3.70)$$

Let $x_0 \in \mathcal{A}(\tau, \omega)$ and $t_n \to \infty$. By the invariance of $\mathcal{A}$, there exists $x_{0,n} \in \mathcal{A}(\tau - t_n, \theta_{-t_n} \omega)$ for every $n$ such that $x_0 = x(\tau, \tau - t_n, \theta_{-\tau} \omega, x_{0,n})$. Since $x_{0,n} \in \mathcal{A}(\tau - t_n, \theta_{-t_n} \omega)$, by (3.69) we have $|x_{0,n}| \leq \xi(\tau - t_n, \theta_{-t_n} \omega)$. Then by the comparison principle we get

$$x_0 = x(\tau, \tau - t_n, \theta_{-\tau} \omega, x_{0,n}) \leq x(\tau, \tau - t_n, \theta_{-\tau} \omega, \xi(\tau - t_n, \theta_{-t_n} \omega)).$$

Letting $n \to \infty$, by (3.67) we get $x_0 \leq x^*(\tau, \omega)$. Similarly, by (3.68) one can verify $x_0 \geq x_*^*(\tau, \omega)$. Thus (3.70) follows. Before proving the converse of (3.70), we first prove $x^*$ and $x_*$ are complete quasi-solutions of (3.44). By (3.67) and the continuity of solutions in initial data, we get for every $s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$x(s + \tau, \tau, \theta_{-\tau} \omega, x^*(\tau, \omega)) = \lim_{t \to \infty} x(s + \tau, \tau, \theta_{-\tau} \omega, x(s + \tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega))))$$

$$= \lim_{t \to \infty} x(s + \tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega)) = \lim_{r \to \infty} x(s + \tau, s + \tau - r, \theta_{-\tau} \omega, \xi(s + \tau - r, \theta_{s-r} \omega)).$$
\[
\lim_{t \to \infty} x(s + \tau, s + \tau - t, \theta_{-\tau}s, \xi(s + \tau - t, \theta_{-\tau}s)) = x^*(\tau + s, \theta_s),
\]
where the last limit is obtained by (3.67). By (3.71) we get for every \(s \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \(\omega \in \Omega\),

\[
\Phi(s, \tau, \omega, x^*(\tau, \omega)) = x^*(\tau + s, \theta_s),
\]
and hence \(x^*\) is a complete quasi-solution of \(\Phi\). Similarly, one can check that \(x_s\) is a complete
quasi-solution of \(\Phi\), i.e., for every \(s \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \(\omega \in \Omega\),

\[
\Phi(s, \tau, \omega, x_s(\tau, \omega)) = x_s(\tau + s, \theta_s).
\]
Finally, we prove the converse of (3.70), i.e.,

\[
[x_s(\tau, \omega), x^*(\tau, \omega)] \subseteq A(\tau, \omega), \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.
\]

Given \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(z \in [x_s(\tau, \omega), x^*(\tau, \omega)]\), by the comparison principle, we find that \(x(t + \tau, \tau, \theta_{-\tau}\omega, z)\) is defined for all \(t \in \mathbb{R}\) and

\[
x(t + \tau, \tau, \theta_{-\tau}\omega, x_s(\tau, \omega)) \leq x(t + \tau, \tau, \theta_{-\tau}\omega, z) \leq x(t + \tau, \tau, \theta_{-\tau}\omega, x^*(\tau, \omega)),
\]
that is,

\[
\Phi(t, \tau, \omega, x_s(\tau, \omega)) \leq x(t + \tau, \tau, \theta_{-\tau}\omega, z) \leq \Phi(t, \tau, \omega, x^*(\tau, \omega)).
\]
This along with (3.72)-(3.73) shows that

\[
x_s(\tau + t, \theta_{t}\omega) \leq x(t + \tau, \tau, \theta_{-\tau}\omega, z) \leq x^*(\tau + t, \theta_{t}\omega).
\]
Since \(x^*\) and \(x_s\) are tempered, by (3.75) we know that \(\psi(t, \tau, \omega) = x(t + \tau, \tau, \theta_{-\tau}\omega, z)\) is a \(D\)-complete solution of \(\Phi\). Therefore, by (3.58) we find that \(z = \psi(0, \tau, \omega) \in A(\tau, \omega)\), which yields (3.74). It follows from (3.70) and (3.74) that

\[
A(\tau, \omega) = [x_s(\tau, \omega), x^*(\tau, \omega)], \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.
\]
By (3.72)-(3.73) and (3.76) we conclude the proof. \(\square\)

In what follows, we discuss pitchfork bifurcation of complete quasi-solutions of of (3.1) as \(\lambda\) crosses zero from below. Let

\[
f(t, x) = \lambda x - \beta(t)x^3 + \gamma(t, x), \quad t \in \mathbb{R} \text{ and } x \in \mathbb{R}.
\]
By (3.2) and (3.3) we have, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(t,x)x \leq \lambda x^2 - \beta_0 x^4 + c_2 x^4 \leq -x^2 + \left((\lambda + 1)|x| - (\beta_0 - c_2)|x|^3\right)|x|. \tag{3.78}$$

Since $\beta_0 > c_2$, by Young’s inequality, there exists a positive number $c$ such that

$$|\lambda + 1||x| \leq \frac{1}{2}(\beta_0 - c_2)|x|^3 + c,$$

which along with (3.78) implies that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(t,x)x \leq -x^2 + c|x|.$$ 

Therefore, $f$ given by (3.77) satisfies condition (3.45) with $\nu = 1$ and $h(t) = c$ for all $t \in \mathbb{R}$. Let $g(t) = 0$ for all $t \in \mathbb{R}$. Then $g$ and $h$ satisfy (3.46) and (3.47) for every $\alpha > 0$. In this case, $\xi$ as defined by (3.51) becomes

$$\xi(\tau,\omega) = c \int_{-\infty}^{0} e^{s-\delta\omega(s)}ds, \quad \tau \in \mathbb{R} \text{ and } \omega \in \Omega. \tag{3.79}$$

It is easy to check that $\xi$ given by (3.79) has a tempered reciprocal, i.e., for every $c_0 > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to \infty} e^{-c_0 t}\xi^{-1}(\tau - t, \theta_{-t}\omega) = 0. \tag{3.80}$$

By Theorem 3.9 we find that for each $\lambda \in \mathbb{R}$, equation (3.1) has a unique $D$-pullback attractor $A_\lambda \in D$ such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$A_\lambda(\tau,\omega) = [x^-_\lambda(\tau,\omega), \ x^+_\lambda(\tau,\omega)], \tag{3.81}$$

where $x^+_\lambda$ and $x^-_\lambda$ are tempered complete quasi-solutions of (3.1) given by

$$x^+_\lambda(\tau,\omega) = \lim_{t \to \infty} x(\tau, \tau - t, \theta_{-t}\omega, \xi(\tau - t, \theta_{-t}\omega)), \tag{3.82}$$

and

$$x^-_\lambda(\tau,\omega) = \lim_{t \to \infty} x(\tau, \tau - t, \theta_{-t}\omega, -\xi(\tau - t, \theta_{-t}\omega)), \tag{3.83}$$

with $\xi$ being defined by (3.79). Note that (3.82) and (3.83) follow from (3.67) and (3.68) by replacing $x^*$ and $x_*$ by $x^+_\lambda$ and $x^-_\lambda$, respectively. By the comparison principle, we find from (3.82)-(3.83) that $x^+_\lambda(\tau,\omega) \geq 0$ and $x^-_\lambda(\tau,\omega) \leq 0$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Actually, $x^+_\lambda(\tau,\omega) > 0$ and $x^-_\lambda(\tau,\omega) < 0$ as demonstrated below.
Lemma 3.10. Suppose (3.2) and (3.3) hold. Then for every \( \lambda \in \mathbb{R} \), the tempered complete quasi-solutions \( x^+_{\lambda} \) and \( x^-_{\lambda} \) in (3.81) satisfy, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), \( x^+_{\lambda}(\tau, \omega) > 0 \), \( x^-_{\lambda}(\tau, \omega) < 0 \) and

\[
\frac{1}{\sqrt{2(\beta_1 - c_1) \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega r} dr}} \leq |x^+_{\lambda}(\tau, \omega)| \leq \frac{1}{\sqrt{2(\beta_0 - c_2) \int_{-\infty}^{0} e^{2\lambda r + 2\delta \omega(r)} dr}}. \tag{3.84}
\]

If \( \lambda > 0 \), then the zero solution of (3.1) is unstable in \( \mathbb{R} \).

Proof. For \( x > 0 \) we introduce a new variable \( z = x^{-2} \). By (3.1) we find that \( z \) satisfies,

\[
\frac{dz}{dt} = -2\lambda z + 2\beta(t) - 2z^\frac{3}{2}\gamma(t, z^{-\frac{1}{2}}) - 2\delta z \circ \frac{d\omega}{dt}, \quad z(\tau) = z_\tau. \tag{3.85}
\]

By (3.3) we have

\[
-2c_2 \leq -2z^\frac{3}{2}\gamma(t, z^{-\frac{1}{2}}) \leq -2c_1, \quad \text{for all } z > 0. \tag{3.86}
\]

Consider the linear equations for \( t > \tau \) with \( \tau \in \mathbb{R} \),

\[
\frac{du}{dt} = -2\lambda u + 2\beta(t) - 2c_1 - 2\delta u \circ \frac{d\omega}{dt}, \quad u(\tau) = u_\tau, \tag{3.87}
\]

and

\[
\frac{dv}{dt} = -2\lambda v + 2\beta(t) - 2c_2 - 2\delta v \circ \frac{d\omega}{dt}, \quad v(\tau) = v_\tau. \tag{3.88}
\]

Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the solutions \( u \) and \( v \) of (3.87) and (3.88) are given by

\[
u(t, \tau, \omega, u_\tau) = e^{2\lambda(t-\tau)+2\delta(\omega(\tau)-\omega(t))} u_\tau + 2 \int_{\tau}^{t} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} (\beta(r) - c_1) dr,
\]

and

\[
v(t, \tau, \omega, v_\tau) = e^{2\lambda(t-\tau)+2\delta(\omega(\tau)-\omega(t))} v_\tau + 2 \int_{\tau}^{t} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} (\beta(r) - c_2) dr.
\]

Therefore, for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we have

\[
u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}) = e^{-2\lambda t+2\delta(\omega(\tau)-\omega(t))} u_{\tau-t} + 2 \int_{\tau-t}^{\tau} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} (\beta(r) - c_1) dr, \tag{3.89}
\]

and

\[
v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}) = e^{-2\lambda t+2\delta(\omega(\tau)-\omega(t))} v_{\tau-t} + 2 \int_{\tau-t}^{\tau} e^{2\lambda(r-\tau)+2\delta(\omega(r)-\omega(t))} (\beta(r) - c_2) dr. \tag{3.90}
\]

By (3.86) we see that \( u \) and \( v \) are super- and sub-solutions of (3.85), respectively. Since \( x = z^{-\frac{1}{2}} \) for \( x > 0 \), we get, for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( x_{\tau-t} > 0 \),

\[
\frac{1}{\sqrt{u(\tau, \tau - t, \theta_{-\tau} \omega, x_{\tau-t}^{-2})}} \leq x(\tau, \tau - t, \theta_{-\tau} \omega, x_{\tau-t}^{-2}) \leq \frac{1}{\sqrt{v(\tau, \tau - t, \theta_{-\tau} \omega, x_{\tau-t}^{-2})}}. \tag{3.91}
\]

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Similarly, for \( x_{\tau-t} < 0 \), one can verify that \(-x\) satisfies (3.91). So for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( x_{\tau-t} \neq 0 \), we have
\[
\frac{1}{\sqrt{u(\tau, \tau-t, \theta_{-\tau} \omega, x_{\tau-t}^{-2})}} \leq |x(\tau, \tau-t, \theta_{-\tau} \omega, x_{\tau-t})| \leq \frac{1}{\sqrt{v(\tau, \tau-t, \theta_{-\tau} \omega, x_{\tau-t}^{-2})}},
\]
from which we get, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\frac{1}{\sqrt{u(\tau, \tau-t, \theta_{-\tau} \omega, \xi^{-2}(\tau-t, \theta_{-\tau} \omega))}} \leq |x(\tau, \tau-t, \theta_{-\tau} \omega, \pm \xi(\tau-t, \theta_{-\tau} \omega))| \leq \frac{1}{\sqrt{v(\tau, \tau-t, \theta_{-\tau} \omega, \xi^{-2}(\tau-t, \theta_{-\tau} \omega))}},
\]
where \( \xi \) is given by (3.79). Letting \( t \to \infty \), by (3.4), (3.80), (3.82)-(3.83) and (3.89)-(3.90) we obtain from (3.2) that, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\frac{1}{\sqrt{2 \int_{-\infty}^{\infty} e^{2\lambda r + 2\delta \omega r} (\beta(r + \tau) - c_1) dr}} \leq |x^+(\tau, \omega)| \leq \frac{1}{\sqrt{2 \int_{-\infty}^{\infty} e^{2\lambda r + 2\delta \omega r} (\beta(r + \tau) - c_2) dr}}.
\]
Therefore, by (3.2) we have, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\frac{1}{\sqrt{2(\beta_1 - c_1) \int_{-\infty}^{\infty} e^{2\lambda r + 2\delta \omega r} dr}} \leq |x^+(\tau, \omega)| \leq \frac{1}{\sqrt{2(\beta_2 - c_2) \int_{-\infty}^{\infty} e^{2\lambda r + 2\delta \omega r} dr}},
\]
which implies \( x^+(\tau, \omega) > 0 \) and \( x^-(\tau, \omega) < 0 \). On the other hand, for every \( x_0 \neq 0 \), by (3.89) and (3.92) we get
\[
\liminf_{t \to \infty} |x(\tau, \tau-t, \theta_{-\tau} \omega, x_0)| \geq \frac{1}{\sqrt{2(\beta_1 - c_1) \int_{-\infty}^{\infty} e^{2\lambda r + 2\delta \omega r} dr}}.
\]
By (3.95), the zero solution of (3.1) is unstable in \( \mathbb{R} \). Thus, by (3.94) we conclude the proof. \( \square \)

We now present pitchfork bifurcation of random complete quasi-solutions of (3.1).

**Theorem 3.11.** Suppose (3.2) and (3.3) hold. Then the random complete quasi-solutions of (3.1) undergo a stochastic pitchfork bifurcation at \( \lambda = 0 \). More precisely:

(i) If \( \lambda \leq 0 \), then \( x = 0 \) is the unique random complete quasi-solution of (3.1) which is pullback asymptotically stable in \( \mathbb{R} \). In this case, the equation has a trivial \( D \)-pullback attractor \( A_\lambda = \{A_\lambda(\tau, \omega) = \{0\} : \tau \in \mathbb{R}, \omega \in \Omega\} \).

(ii) If \( \lambda > 0 \), then the zero solution loses its stability and the equation has two more tempered random complete quasi-solutions \( x^+_\lambda > 0 \) and \( x^-_\lambda < 0 \) such that
\[
\lim_{\lambda \to 0} x^\pm_\lambda(\tau, \omega) = 0, \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.
\]
In this case, equation (3.9) has a $\mathcal{D}$-pullback attractor $A_\lambda = \{A_\lambda(\tau, \omega) = [x^-_\lambda(\tau, \omega), x^+_\lambda(\tau, \omega)] : \tau \in \mathbb{R}, \omega \in \Omega\}$.

**Proof.** (i) If $\lambda \leq 0$, by (3.5) and (3.81) we have, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$|x^\pm_\lambda(\tau, \omega)| \leq \frac{1}{\sqrt{2(\beta_0 - c_2) \int_{-\infty}^{0} e^{2\delta_{\omega}(r)} dr}} = 0,$$

and hence $x^\pm_\lambda(\tau, \omega) = 0$. In this case, by (3.81) we see that zero is the only complete quasi-solution of (3.1) which is pullback asymptotically stable. In addition, $A(\tau, \omega) = \{0\}$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

(ii) If $\lambda > 0$, by Lemma 3.10 we know that $x = 0$ is unstable. Moreover, by (3.84), (3.97) and Fatou’s lemma, we have

$$\limsup_{\lambda \to 0} |x^\pm_\lambda(\tau, \omega)| \leq \limsup_{\lambda \to 0} \frac{1}{\sqrt{2(\beta_0 - c_2) \int_{-\infty}^{0} e^{2\lambda r + 2\delta_{\omega}(r)} dr}} \leq \frac{1}{\sqrt{2(\beta_0 - c_2) \int_{-\infty}^{0} e^{2\delta_{\omega}(r)} dr}} = 0,$$

which implies (3.96) and thus completes the proof. \qed

As a consequence of Theorem 3.11, we have the following pitchfork bifurcation of random periodic solutions of (3.1).

**Theorem 3.12.** Let $T$ be a positive number such that $\beta(t + T) = \beta(t)$ and $\gamma(t + T, x) = \gamma(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. If (3.2) and (3.3) hold, then random periodic solutions of (3.1) undergo a stochastic pitchfork bifurcation at $\lambda = 0$. More precisely:

(i) If $\lambda \leq 0$, then $x = 0$ is the unique random periodic solution of (3.9) which is pullback asymptotically stable in $\mathbb{R}$. In this case, the equation has a trivial $\mathcal{D}$-pullback attractor.

(ii) If $\lambda > 0$, then the zero solution loses its stability and the equation has two more random periodic solutions $x^+_\lambda > 0$ and $x^-_\lambda < 0$ such that

$$\lim_{\lambda \to 0} x^\pm_\lambda(\tau, \omega) = 0, \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega. \quad (3.98)$$

In this case, equation (3.9) has a $\mathcal{D}$-pullback attractor $A = \{[x^-_\lambda(\tau, \omega), x^+_\lambda(\tau, \omega)] : \tau \in \mathbb{R}, \omega \in \Omega\}$.

**Proof.** By Theorem 3.11, we only need to show that for each $\lambda > 0$, the tempered complete quasi-solutions $x^+_\lambda$ and $x^-_\lambda$ in (3.81) are $T$-periodic. Note that $x^+_\lambda$ and $x^-_\lambda$ are defined by (3.67) and (3.68).
with $x^*$ and $x_*$ being replaced by $x_+^*$ and $x_-^*$, respectively. In the present case, by Lemma 3.7 we find that $\xi$ given by (3.51) is $T$-periodic. Then, by (3.67) and the periodicity of $\beta$ and $\gamma$, we get for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$x_+^+(\tau + T, \omega) = \lim_{t \to \infty} x(\tau + T, \tau + T - t, \theta_{-\tau} - T \omega, \xi(\tau + T - t, \theta_{-t} \omega)) = \lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, \xi(\tau - t, \theta_{-t} \omega)) = x_+^*(\tau, \omega),$$

which shows that $x_+^*$ is $T$-periodic. Similarly, one can verify that $x_-^*$ is also $T$-periodic. The details are omitted.

### 4 Transcritical bifurcation of stochastic equations

In this section, we discuss transcritical bifurcation of the one-dimensional non-autonomous stochastic equation given by

$$\frac{dx}{dt} = \lambda x - \beta(t)x^2 + \gamma(t, x) + \delta x \circ d\omega, \quad x(\tau) = x_\tau, \quad t > \tau, \quad (4.1)$$

where $\lambda$, $\delta$ and $\beta$ are the same as in (3.1); particularly, $\beta$ satisfies (3.2). However, in the present case, we assume the smooth function $\gamma$ satisfies the following condition: there exist two nonnegative numbers $c_1$ and $c_2$ with $c_1 \leq c_2 < \beta_0$ such that

$$c_1 x^2 \leq \gamma(t, x) \leq c_2 x^2 \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}. \quad (4.2)$$

By (4.2) we have $\gamma(t, 0) = 0$ for all $t \in \mathbb{R}$, and hence $x = 0$ is a fixed point of (4.1). We will first discuss transcritical bifurcation of (4.1) when $\gamma$ is zero and then consider the case when $\gamma$ satisfies (4.2). We will also study transcritical bifurcation of random periodic (random almost periodic, random almost automorphic) solutions of (4.1).

When $\gamma$ is absent, equation (4.1) reduces to

$$\frac{dx}{dt} = \lambda x - \beta(t)x^2 + \delta x \circ d\omega, \quad x(\tau) = x_\tau, \quad t > \tau. \quad (4.3)$$

This equation is exactly solvable and for every $t, \tau \in \mathbb{R}$ with $t \geq \tau$, $\omega \in \Omega$ and $x_\tau \in \mathbb{R}$, the solution is given by

$$x(t, \tau, \omega, x_\tau) = \frac{x_\tau}{e^{\lambda(\tau-t)+\delta(\omega(\tau)-\omega(t))} + x_\tau \int_\tau^t e^{\lambda(r-t)+\delta(\omega(r)-\omega(t))} \beta(r) \, dr}. \quad (4.4)$$
It follows from (4.4) that if \( x_0 > 0 \), then the solution \( x(t, \tau, \omega, x_0) \) is defined for all \( t \geq \tau \). Similarly, if \( x_0 < 0 \), then the solution \( x(t, \tau, \omega, x_0) \) is defined for all \( t \leq \tau \). Based on this fact, we will be able to study the dynamics of (4.3) for positive initial data as \( t \to \infty \) as well as the dynamics for negative initial data as \( t \to -\infty \). In the pullback sense, this allows us to explore the dynamics of solutions with positive initial data as \( \tau \to -\infty \) or with negative initial data as \( \tau \to \infty \). By (4.4) we get that, for each \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( x_0 \in \mathbb{R} \),

\[
x(t, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{x_0}{e^{-\lambda t + \delta \omega (-t)} + x_0 \int_{\tau-t}^{\tau} e^{\lambda (r-\tau) + \delta \omega (r-\tau)} \beta(r) dr} \tag{4.5}
\]

By (3.4) and (4.5) we obtain, for every \( \lambda > 0 \) and \( x_0 > 0 \),

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{1}{\int_{-\infty}^{0} e^{\lambda r + \delta \omega (r)} \beta(r + \tau) dr}. \tag{4.6}
\]

Analogously, by (4.5) we obtain, for every \( \lambda < 0 \) and \( x_0 < 0 \),

\[
\lim_{t \to -\infty} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) = \frac{-1}{\int_{0}^{\infty} e^{\lambda r + \delta \omega (r)} \beta(r + \tau) dr}. \tag{4.7}
\]

Given \( \lambda \in \mathbb{R} \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we set

\[
x_\lambda(\tau, \omega) = \begin{cases} 
\left( \int_{-\infty}^{0} e^{\lambda r + \delta \omega (r)} \beta(r + \tau) dr \right)^{-1} & \text{if } \lambda > 0; \\
- \left( \int_{0}^{\infty} e^{\lambda r + \delta \omega (r)} \beta(r + \tau) dr \right)^{-1} & \text{if } \lambda < 0.
\end{cases} \tag{4.8}
\]

By (4.8) we see that for every fixed \( \tau \in \mathbb{R} \), \( x_\lambda(\tau, \cdot) \) is measurable. By an argument similar to Lemma 3.1 one can verify that \( x_\lambda \) is a tempered complete quasi-solution of (4.3).

**Theorem 4.1.** If (3.2) holds, then the random complete quasi-solutions of (4.3) undergo a stochastic transcritical bifurcation at \( \lambda = 0 \). More precisely:

(i) If \( \lambda < 0 \), then (4.3) has two random complete quasi-solutions \( x = 0 \) and \( x = x_\lambda \) given by (4.8). The zero solution is asymptotically stable in \((0, \infty)\) and pullback attracts every compact subset \( K \) of \((0, \infty)\), i.e.,

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, K) = 0. \tag{4.9}
\]

Moreover, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we have \( x_\lambda(\tau, \omega) < 0 \) and

\[
\lim_{\lambda \to 0} x_\lambda(\tau, \omega) = 0. \tag{4.10}
\]
(ii) If $\lambda > 0$, then \(4.3\) has two random complete quasi-solutions \(x = 0\) and \(x = x_\lambda\) given \(4.8\). The zero solution is unstable in \((0, \infty)\) and \(x_\lambda\) pullback attracts every compact subset \(K\) of \((0, \infty)\), i.e.,

\[
\lim_{t \to \infty} x(\tau, \tau - t, \theta_{-\tau} \omega, K) = x_\lambda(\tau, \omega). \tag{4.11}
\]

Moreover, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have $x_\lambda(\tau, \omega) > 0$ and

\[
\lim_{\lambda \to 0} x_\lambda(\tau, \omega) = 0. \tag{4.12}
\]

**Proof.** (i) Let $K$ be a compact subset of $(0, \infty)$. Then by \(4.5\) we get

\[
\limsup_{t \to \infty} \sup_{x_0 \in K} x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) \leq \limsup_{t \to \infty} \frac{1}{\int_{-t}^{0} e^{\lambda r + \delta \omega(r)} \beta(r + \tau) dr}. \tag{4.13}
\]

On the other hand, by \(3.2\) and \(3.4\) we get, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

\[
\lim_{t \to \infty} \frac{1}{\int_{-t}^{0} e^{\lambda r + \delta \omega(r)} \beta(r + \tau) dr} = 0. \tag{4.14}
\]

By \(4.13\)-\(4.14\) we obtain \(4.9\). The asymptotic stability of $x = 0$ in $(0, \infty)$ and the convergence \(4.10\) can be proved by a argument similar to Theorem 3.11.

(ii) Let $K = [a, b]$ with $a > 0$. By \(4.5\) we have, for all $x_0 \in K$,

\[
x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) \geq \frac{1}{e^{-\lambda t} + \int_{-t}^{0} e^{\lambda r + \delta \omega(r)} \beta(r + \tau) dr}, \tag{4.15}
\]

and

\[
x(\tau, \tau - t, \theta_{-\tau} \omega, x_0) \leq \frac{1}{e^{-\lambda t} + \int_{-t}^{0} e^{\lambda r + \delta \omega(r)} \beta(r + \tau) dr}. \tag{4.16}
\]

Since $\lambda > 0$, by \(3.2\) and \(3.4\) we find that for all $\tau \in \mathbb{R}$ and $\omega \in \omega$, the right-hand sides of \(4.15\) and \(4.16\) converge to $x_\lambda(\tau, \omega)$ as $t \to \infty$, which implies \(4.11\). Note that the instability of $x = 0$ in $(0, \infty)$ is implied by \(4.11\). We then conclude the proof. \(\square\)

By \(4.8\) we see that if $\beta$ is a periodic function with period $T > 0$, then so is $x_\lambda(\cdot, \omega)$ for all $\omega \in \Omega$. By an argument similar to Lemmas 3.3 and 3.5 one can prove $x_\lambda(\cdot, \omega)$ is almost periodic (almost automorphic) provided $\beta$ is almost periodic (almost automorphic). Based on this fact, we have the following results from Theorem 4.1.
Corollary 4.2. Suppose (3.2) holds and \( \beta : \mathbb{R} \to \mathbb{R} \) is periodic (almost periodic, almost automorphic). Then the random periodic (almost periodic, almost automorphic) solutions of (4.3) undergo a stochastic transcritical bifurcation at \( \lambda = 0 \). More precisely:

(i) If \( \lambda < 0 \), then (4.3) has two random periodic (almost periodic, almost automorphic) solutions \( x = 0 \) and \( x = x_\lambda \) given by (4.8). The zero solution is asymptotically stable in \((0, \infty)\) and (4.9) - (4.10) are fulfilled.

(ii) If \( \lambda > 0 \), then (4.3) has two random periodic (almost periodic, almost automorphic) solutions \( x = 0 \) and \( x = x_\lambda \) given by (4.8). The zero solution is unstable in \((0, \infty)\) and (4.11) - (4.12) are fulfilled.

Next, we consider bifurcation of (4.1) with \( \gamma \) satisfying (4.2). In this case, we can associate two exactly solvable systems with (4.1). Given \( t, \tau \in \mathbb{R} \) with \( t > \tau \), consider

\[
\frac{dx}{dt} = \lambda x - (\beta(t) - c_2)x^2 + \delta x \circ \frac{d\omega}{dt}, \quad x(\tau) = x_\tau, \quad t > \tau, \tag{4.17}
\]

and

\[
\frac{dx}{dt} = \lambda x - (\beta(t) - c_1)x^2 + \delta x \circ \frac{d\omega}{dt}, \quad x(\tau) = x_\tau, \quad t > \tau. \tag{4.18}
\]

By (4.2) we find that the solutions of (4.17) and (4.18) are super- and sub-solutions of (4.1), respectively. The random complete quasi-solutions of (4.17) and (4.18) can be studied as equation (4.3). Then by the comparison principle and the arguments discussed in the previous section, we can obtain transcritical bifurcation for (4.1). We here just present the results and will not repeat the details in this case.

Theorem 4.3. If (3.2) and (4.2) hold, then the random complete quasi-solutions of (4.1) undergo a stochastic transcritical bifurcation at \( \lambda = 0 \). More precisely:

(i) If \( \lambda < 0 \), then (4.1) has two random complete quasi-solutions \( x = 0 \) and \( x = x_\lambda \) with \( x_\lambda(\tau, \omega) < 0 \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). The zero solution is asymptotically stable in \((0, \infty)\) and (4.9) - (4.10) are fulfilled.

(ii) If \( \lambda > 0 \), then (4.1) has two random complete quasi-solutions \( x = 0 \) and \( x = x_\lambda \) with \( x_\lambda(\tau, \omega) > 0 \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). The zero solution is unstable in \((0, \infty)\) and (4.11) - (4.12) are fulfilled.

If, in addition, \( \beta \) is a periodic function, then so is \( x_\lambda \) for \( \lambda \neq 0 \).
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