Cusps and Codes

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Abstract

We study a construction, which produces surfaces $Y \subset \mathbb{P}^3(\mathbb{C})$ with cusps. For example we obtain surfaces of degree six with 18, 24 or 27 three-divisible cusps. For sextic surfaces in a particular family of up to 30 cusps the codes of these sets of cusps are determined explicitly.

0 Introduction

The aim of this note is to give examples of algebraic surfaces in $\mathbb{P}^3(\mathbb{C})$ with cusps, and to determine the code of the set of these cusps.

Recall that a cusp (= singularity $A_2$) is a surface singularity given in (local analytic) coordinates $x, y$ and $z$ centered at the singularity by an equation

$$xy - z^3 = 0.$$ 

It is resolved by introducing two $(-2)$-curves. Let $Y \subset \mathbb{P}^3$ be an algebraic surface with $n$ cusps $P_1, \ldots, P_n \in Y$. Let $\pi: X \to Y$ be its minimal desingularisation with $E'_\nu, E''_\nu$ the two $(-2)$-curves over $P_\nu$. The code of this set of cusps is the kernel of the $\mathbb{F}_3$-linear morphism

$$\mathbb{F}_3^n \to H^2(X, \mathbb{F}_3), \quad (i_1, \ldots, i_n) \mapsto \sum_{\nu=1}^n i_\nu [E'_\nu - E''_\nu].$$

A word $(i_1, \ldots, i_n)$ belongs to this code if and only if the class of the divisor $\sum_{\nu} i_\nu (E'_\nu - E''_\nu)$ is divisible by 3 in $NS(X)$. Or equivalently: There is a cyclic triple cover of $Y$ branched precisely over the points $P_\nu$ with $i_\nu \neq 0$. Such a set of cusps is called 3-divisible [B, T].

From

$$\left( \sum_{\nu=1}^n i_\nu [E'_\nu - E''_\nu] \right)^2 = -6 \cdot \text{number of } \nu \text{ with } i_\nu \neq 0$$

it easily follows that the number $n$ of cusps in a 3-divisible set is a multiple of 3. For the maximal number of 3-divisible cusps on a surface $Y$ of given degree $d$ there seems to be no better upper bound than the famous bounds of Miaoka [M]

$$n \leq \frac{1}{4} d(d - 1)^2$$

Supported by the DFG Schwerpunktprogramm ”Global methods in complex geometry”. The second author is supported by a Fellowship of the Foundation for Polish Science and KBN Grant No. 2 P03A 016 25.

2000 Mathematics Subject Classification. 14J25, 14J17.
or Varchenko [V] for the maximal number of cusps, 3-divisible or not. S.-L. Tan [T] shows that a surface of degree \(d\) with \(3 \leq d \leq 5\) can have only \(n\) three-divisible cusps with

\[
\begin{array}{|c|c|c|c|}
\hline
\frac{d}{n} & 3 & 4 & 5 \\
\hline
3 & 6 & 12, 15, 18 \\
\hline
\end{array}
\]

In the accompanying note [BR] we show that the minimal number of cusps in a 3-divisible set on a sextic surface is 18. It is not known whether there is a quintic surface with a 3-divisible set of 18 cusps, nor seems it to be known for which numbers \(n, 18 \leq n \leq 36\), there is a sextic surface with \(n\) three-divisible cusps. Here we construct sextic surfaces with \(n = 18, 24\) and 27 three-divisible cusps.

**Notations and conventions:** All varieties will be defined over the base-field \(\mathbb{C}\). As coefficient field for cohomology we usually use the field \(\mathbb{F}_3\) with three elements. For brevity we denote it by \(\mathbb{F}\).

1 Constructions

1.1 The direct construction

The basic idea for constructing surfaces \(Y \subset \mathbb{P}_3\) with cusps is very simple: just globalization of the local equation \(xy - z^3 = 0\). This means the following: For fixed degree \(d = 6, 9, \ldots\), divisible by 3, take polynomials \(s_1, \ldots, s_k\) and \(s\) of degrees

\[
\text{deg}(s_1) = d_1, \ldots, \text{deg}(s_k) = d_k, \quad d_1 + \ldots + d_k = d, \quad \text{deg}(s) = d/3,
\]

and consider the surface \(Y \subset \mathbb{P}_3\) of degree \(d\) given by the equation

\[
s_1 \cdot \ldots \cdot s_k - s^3 = 0.
\]

Let \(S_1, \ldots, S_k, S \subset \mathbb{P}_3\) be the surfaces with equation \(s_1 = 0, \ldots, s_k = 0, s = 0\). A Bertini-type-argument shows that for general choice of these polynomials

- any three surfaces \(S_i, S_j, S\) meet transversally in \(d_i \cdot d_j \cdot d/3\) points \(P_\nu\), which then are cusps on the surface \(Y\);
- no four surfaces \(S_i, S_j, S_m, S\) meet;
- the surface \(Y\) is smooth away from the cusps \(P_\nu\) at the intersections \(S_i \cap S_j \cap S\).

The simplest examples of surfaces \(Y\) with \(n\) cusps obtained in this way are for \(d = 6\)

| \(d_1, \ldots, d_k\) | 1, 5 | 2, 4 | 3, 3 | 1, 1, 4 | 1, 2, 3 | 2, 2, 2 | 1, 1, 1, 3 | 1, 1, 2, 2 | 1, 1, 1, 1, 2 | 1, 1, 1, 1, 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 10 | 16 | 18 | 18 | 22 | 24 | 24 | 26 | 28 | 30 |

and for \(d = 9\)

| \(d_1, \ldots, d_k\) | 1, 8 | 2, 7 | 1, 1, 7 | 3, 6 | 4, 5 | 1, 2, 6 | 1, 1, 1, 6 | 1, 3, 5 | 1, 4, 4 | 2, 2, 5 | 1, 1, 2, 5 | 2, 3, 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 24 | 42 | 45 | 54 | 60 | 60 | 63 | 69 | 72 | 72 | 75 | 78 |
1.2 The residual construction

The construction from section 1.1 produces only surfaces of degrees $d$ divisible by 3. However in [BR] we observe:

- If $Y \subset \mathbb{P}_3$ is a quartic surface with 6 three-divisible cusps, then there is a residual quadric $R$,
- if $Y \subset \mathbb{P}_3$ is a quintic surface with 12 three-divisible cusps, then there is a residual plane $R$,

such that $Y \cup R$ has an equation $s_1 \cdot s_2 - s^3 = 0$ with $\deg(s_1) = \deg(s_2) = 3, \deg(s) = 2$. Here we use a residual surface $R$ to construct surfaces with a set of 3-divisible cusps in degrees $d$ not necessarily divisible by 3. The Bertini-type arguments for this 'residual construction' unfortunately are more sophisticated than the argument needed in section 1.1. So we restrict to the simplest case of the construction, which probably can be generalized in several ways. We start with

- a smooth residual surface $R : r = 0$ of degree $b$;
- $k$ auxiliary polynomials $r_1, ..., r_k$ of degrees $c_1, ..., c_k$ with $c_1 + ... + c_k =: c$ such that the curves $R_i : r = r_i = 0$ on $R$ are smooth and intersect transversally;
- surfaces $S_i : s_i = 0$ with $s_i = r_i^3 + r \cdot t_i$ of degrees $d_i = 3c_i$;
- a degree-$c$ surface $S : s = 0$ with $s = r_1 \cdot ... \cdot r_k + r \cdot t$.

Here we assume

$$d_i \geq b, \quad c \geq b.$$ 

Then always

$$s_1 \cdot ... \cdot s_k - s^3 = r \cdot (t_1 \cdot r_2^3 \cdot ... \cdot r_k^3 + ... + r_1^3 \cdot ... \cdot r_{k-1}^3 \cdot t_k - 3 \cdot r_1^2 \cdot ... \cdot r_k^2 \cdot t) \mod r^2$$

vanishes on $R$. We are interested in the surface $Y$ defined by the polynomial

$$f := \frac{1}{r}(s_1 \cdot ... \cdot s_k - s^3)$$

of degree $d = 3c - b$ when the polynomials $t_i, t$ of degrees $d_i - b$ and $c - b$ are chosen generally.

1) Each polynomial $s_i$ vanishes along the curve $R_i$ to the first order, except for points, where $t_i = 0$. For general choice of $t_i$ this does not happen at an intersection $R_i \cap R_j$. And $s_i$ vanishes
Theorem 1.1

The residual construction with the degrees taken as above gives surfaces to the second order at points on \( R \), where \( r_i = t_i = 0 \). So \( f \) vanishes on \( R_i \) only at points with \( r_i = r_j = 0 \) or \( r_i = t_i = 0 \), and there to the first order. This shows that \( Y \) is smooth wherever it meets any curve \( R_i \).

2) The polynomials

\[
R = (t_1 \cdot r_2^3 \cdot \ldots \cdot r_k^3 + \ldots + r_1^3 \cdot r_2^3 \cdot \ldots \cdot r_{k-1}^3 \cdot t_k - 3 \cdot r_1^2 \cdot \ldots \cdot r_k^2 \cdot t)|R, \quad t_1, \ldots, t_k, t \text{ varying}
\]

form a linear system with base locus consisting of the finitely many points \( r = r_i = r_j = 0 \). At these points \( df \neq 0 \) by 1). So for general choice of \( t_1, \ldots, t_k, t \) the surface \( f = 0 \) is smooth at its intersection with \( R \).

3) The polynomials \( s = r_1 \cdot \ldots \cdot r_k + r \cdot t \) with \( t \) varying form a linear system with base locus \( R_1 \cup \ldots \cup R_k \). If \( t \) does not vanish at any intersection \( R_i \cap R_j \) the there \( ds \neq 0 \). So for general choice of \( t \) the surface \( S \) is smooth.

4) For each \( i \) the polynomials \( s_i = r_i^3 + r \cdot t_i \) form a linear system with base locus \( R_i \). So for general choice of \( t_i \) the surface \( S_i : s_i = 0 \) as well as the curve \( C_i : s = s_i = 0 \) is smooth outside of \( R \). And for general choice of \( t_i \) and \( t_j \) the surfaces \( S_i, S_j, S \) intersect transversally outside of \( R \).

5) The polynomials

\[
(r_1^3 + r \cdot t_1) \cdot s_2 \cdot \ldots \cdot s_k - s^3 = r_1^3 \cdot s_2 \cdot \ldots \cdot s_k - s^3 + t_1 \cdot r \cdot s_2 \cdot \ldots \cdot s_k, \quad t_1 \text{ varying}
\]

form a linear system with base locus \( Y \cap R \), where \( Y \) is smooth by 2), and the curves \( s = s_2 = 0, \ldots, s = s_k = 0 \). So for general choice of \( t_1 \) the surface \( s_1 \cdot \ldots \cdot s_k - s^3 \) is smooth outside of \( S \).

6) On \( S_i \cap S \) we have

\[
d(s_1 \cdot \ldots \cdot s_k - s^3) = \sum_i s_i \cdot \prod_{j \neq i} s_j.
\]

By 4) this differential is \( \neq 0 \) outside of \( R \) and away from the points in \( S_i \cap S_j; j \neq i \).

Statements 1)-6) show that the surface \( Y : f = 0 \) is smooth away from the points in \( S_i \cap S_j \cap S = 0 \). By 4) the three surfaces \( S_i, S_j, S \) intersect transversally in these points, hence these points are cusps on \( Y \).

Lemma 1.1 In each point \( P \in S_i \cap S_j \cap R \) we have the intersection number

\[
i_P(S_i, S_j, S) = 6.
\]

Proof. By 4) we may take \( x = r_i, y = r_j, z = r \) as local coordinates. Since \( S : xyg(x, y, z) + zt = 0 \) with \( g(0, 0, 0) \neq 0 \) is smooth, we may eliminate \( z = -xyg/t \). The intersection number then is the intersection number of the two curves

\[
s_i = x \cdot (x^2 - \frac{t_ig}{t}y) = 0, \quad s_j = y \cdot (y^2 - \frac{t_ig}{t}x) = 0.
\]

Altogether we found:

Theorem 1.1 The residual construction with the degrees taken as above gives surfaces \( Y \) of degree \( d = 3c - b \) which are smooth but for

\[
n_{i,j} = 3c_i \cdot 3c_j \cdot c - 6 \cdot c_i \cdot c_j \cdot b = 3 \cdot c_i \cdot c_j \cdot (d - b)
\]

cusps on each curve \( S_i \cap S_j \).
In the following table we give some examples.

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| $d_1$ | $d_2$ | $d_3$ | $c$ | $b$ | $d$ | $n_{1,2}$ | $n_{1,3}$ | $n_{2,3}$ |
| 3  | 3  | 2  | 2  | 4  | 6  | -  | -  |   |
| 3  | 3  | 2  | 1  | 5  | 12 | -  | -  |   |
| 3  | 6  | 3  | 3  | 6  | 18 | -  | -  |   |
| 3  | 6  | 3  | 2  | 7  | 30 | -  | -  |   |
| 3  | 6  | 3  | 1  | 8  | 42 | -  | -  |   |
| 3  | 3  | 3  | 3  | 6  | 9  | 9  | 9  |   |
| 3  | 3  | 3  | 2  | 7  | 15 | 15 | 15 |   |
| 3  | 3  | 3  | 1  | 8  | 21 | 21 | 21 |   |

1.3 Codes

Denote by $C_i = R_i \cup B_i$ the curve $s_i = s = 0$. For general choice of the polynomials above, $B_i$ is smooth away from $R$. We have $S_i \cap Y = B_i$, both the surfaces $S_i$ and $Y$ touching along $B_i$ to the third order. Let $\pi : X \to Y$ be the minimal resolution of $Y$, obtained e.g. by blowing up $\mathbb{P}_3$ in the cusps $P_\nu$ of $Y$. Over each cusp $P_\nu$ it introduces two $(-2)$--curves $E'_\nu$ and $E''_\nu$. If $P_\nu \in B_i \cap B_j$, then the proper transform $D_i$ of $B_i$ in $X$ meets (transversally) exactly one of the two exceptional curves over $P_\nu$, while the proper transform $D_j$ of $B_j$ meets the other exceptional curve.

By abuse of notation we put

$$O_X(m) := \pi^*O_Y(m), \quad m \in \mathbb{Z}.$$  

Let $P_\nu$ be the cusps of $Y$ in $B_i$. Let $E'_\nu \subset X$ be the $(-2)$-curve over $P_\nu$ meeting $D_i$ while $E''_\nu$ meets some curve $D_j$, $j \neq i$. Then

$$O_X(d_i) \sim 3D_i + \sum_{\nu} (i'_\nu E'_\nu + i''_\nu E''_\nu)$$

with integers $i'_\nu, i''_\nu \geq 0$. Here $(O_X(1).E'_\mu) = (O_X(1).E''_\mu) = 0$ implies

$$(3D_i + \sum (i'_\nu E'_\nu + i''_\nu E''_\nu)).E'_\mu = 3 - 2i'_\mu + i''_\mu = 0,$$

$$(3D_i + \sum (i'_\nu E'_\nu + i''_\nu E''_\nu)).E''_\mu = i'_\mu - 2i''_\mu = 0.$$  

For $P_\nu \in B_i$ we find

$$i'_\nu = 2i''_\nu, \quad i''_\nu = 1, i'_\nu = 2,$$

and the class

$$O_X(d_i) - \sum_{P_\nu \in C_i} (2E'_\nu + E''_\nu) \sim 3D_i \in NS(X)$$

is divisible by 3.
Definition 1.1 The code of $Y$ is the kernel of the $\mathbb{F}$-linear morphism

$$\mathbb{F}^n \rightarrow H^2(X, \mathbb{F}), \quad (i_1, \ldots, i_n) \mapsto \sum_{\nu} i_{\nu}[E'_\nu - E''_{\nu}].$$

The extended code $[E]$ is the kernel of

$$\mathbb{F}^{n+1} \rightarrow H^2(X, \mathbb{F}), \quad (i_0, i_1, \ldots, i_n) \mapsto \mathcal{O}_X(i_0) + \sum_{\nu} i_{\nu}[E'_\nu - E''_{\nu}].$$

To avoid a clumsy notation we put

$$e_{\nu} := [E'_\nu - E''_{\nu}] = [-2E'_\nu + E''_{\nu}] \mod 3,$$

where the exceptional curves are ordered such that

$$3D_i \sim \mathcal{O}_X(d_i) + \sum_{P_{\nu} \in C_i \cap C_j, j < i} -e_{\nu} + \sum_{P_{\nu} \in C_i \cap C_j, j > i} e_{\nu}.$$

For each $i$ we then obtain a word

$$w_i = (d_i \mod 3, -1, \ldots, -1, 1, \ldots, 1)_{j < i, j > i}$$

in the extended code.

2 Sextics

Here we determine the codes of the sextic surfaces $Y$ given by the construction in section 1.1. We fix a partition $d_1, \ldots, d_k$ of 6. An equation

$$f := s_1 \cdot \ldots \cdot s_k - s^3 = 0, \quad \deg(s_i) = d_i, \quad \deg(s) = 2$$

will be called an equation of type $d_1, \ldots, d_k$. It will be called admissible, if

- Any three surfaces $S_i, S_j, S$ meet transversally at $2 \cdot d_i \cdot d_j$ points. These points then are cusps on $Y$.
- There are no other singularities on $Y$ but these cusps.

A Bertini-Argument shows that for each partition of 6 admissible equations of this type exist. Then they are dense in the family of all equations of a given type. All equations of a fixed type form an irreducible family. Hence all admissible equations of given type form a connected family. Each path $f_t, 0 \leq t \leq 1$, with $Y_0 : f_0 = 0, Y_1 : f_1 = 0$ defines an isomorphism $H^2(X_0, \mathbb{F}) \rightarrow H^2(X_1, \mathbb{F})$ inducing an isomorphism of codes. This shows
Proposition 2.1 All sextic surfaces with admissible equation of a fixed type have isomorphic codes.

Definition 2.1 The proper code $C_{d_1,...,d_k}$, resp. the extended code $E_{d_1,...,d_k}$ of type $d_1,...,d_k$ is the proper, resp. extended code of all the surfaces having an admissible equation of type $d_1,...,d_k$.

To understand the codes of our sextic surfaces we use

Proposition 2.2 a) Each word in the proper code has a weight $n \geq 18$.

b) If the partition $c_1,...,c_l$ of 6 is a sub-partition of $d_1,...,d_k$, then the code of type $d_1,...,d_k$ is isomorphic to a subcode of the code of type $c_1,...,c_l$.

c) All codes of our surfaces admit an involution, which arises from interchanging the cusps in pairs.

Proof. a) The lower bound $n \geq 18$ for $n$ three-divisible cusps on a sextic surface is proven in [BR, thm.1.1].

b) It suffices to prove the assertion for $c_1,...,c_{k+1} = d_1,...,d_{k-1}, c_k, c_{k+1}$ with $d_k = c_k + c_{k+1}$. So fix some admissible equation $s_1 \cdot \ldots \cdot s_k = s^3$ of type $d_1,...,d_k$ and an admissible equation $s_1 \cdot \ldots s_{k-1} \cdot t_k \cdot t_{k+1} = s^3$ of type $d_1,...,d_{k-1}, c_k, c_{k+1}$. Consider the one-parameter-family of surfaces with equation

$$s_1 \cdot \ldots \cdot s_{k-1} \cdot (\lambda s_k + (1 - \lambda) \cdot t_k \cdot t_{k+1}) = s^3, \quad \lambda \in \mathbb{C}.$$

Define the surfaces $T_k : t_k = 0$ and $T_{k+1} : t_{k+1} = 0$. For all $\lambda \neq 0$, but finitely many, the equation is admissible of type $d_1,...,d_k$. For all $i,j \leq k-1$ the cusps $P_i \in S_i \cap S_j \cap S$ coincide. And for $i \leq k-1$ the cusps $P_i$ on $S_i \cap S_k \cap S$ converge for $\lambda \to 0$ to cusps on $S_i \cap T_k \cap S$ or $S_i \cap T_{k+1} \cap S$. This shows that the code of type $d_1,...,d_k$ is the subcode of the code of type $d_1,...,d_{k-1}, c_k, c_{k+1}$ which consists of the words assigning the value 0 to the cusps in $T_k \cap T_{k+1} \cap S$.

c) Fix the quadric $S : s := x_2 x_3 = 0$ admitting the involution

$$I : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_3 : x_2),$$

which interchanges the two planes of $S$. Choose $f_1(x_0, x_1, x_2),...,f_k(x_0, x_1, x_2)$ of degrees $i_1,...,i_k$ such that all the curves $C_i : f_i = 0$ on the plane $x_3 = 0$ are smooth and intersect transversally, not on the line $x_2 = x_3 = 0$. Then put $s_i(x_0,...,x_3) := f_i(x_0, x_1, x_2 + x_3), i = 1,...,k$. The polynomials $s_i$ are $I$-invariant and define surfaces $S_i : s_i = 0$ meeting transversally on $S$. The surface $s_1 \cdot \ldots \cdot s_k - s^3$ has cusps in the points on $S$ where $s_i = s_j = 0$. Elsewhere it is smooth along its intersection with $S$. So by Bertini, for general $\lambda$ the $I$-invariant surface of equation $\lambda \cdot s_1 \cdot \ldots \cdot s_k = s^3$ is smooth but for these points on $S$. \hfill \Box

In the following table we give words, which by 1.3 belong to the extended code. A pair $ij$ in the first row stands for the cusps in the intersection $S_i \cap S_j$. For each type the number below $ij$ is the number of cusps in the corresponding intersection. The words are specified by their values taken at these cusps.
| type  | \(i_0\) | 12 | 13 | 23 | 14 | 24 | 34 | 15 | 25 | 35 | 45 | 16 | 26 | 36 | 46 | 56 |
|-------|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1,5   | \(w\)  | 1  | 10 |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 2,4   | \(w\)  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 3,3   | \(w\)  | 0  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,1,4 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,2,3 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 2,2,2 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,1,1,3 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,1,2,2 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,1,1,1,2 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1,1,1,1,1,1 | \(w_1\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_2\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_3\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_4\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | \(w_5\) | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |

All words given in this table are linearly independent in \(E_{d_1...d_k}\).

### 2.1 The case 1,1,1,1,1,1

We fix an involution \(I\) as in prop. 2.2 c) and order the cusps \(P_\nu\) such that \(I\) interchanges \(P_\nu\) and \(P_{15+\nu}\) for \(\nu = 1, ..., 15\). This induces an involution on \(\{1, ..., 30\}\) and on \(\mathbb{F}^{30}\). The code
$C_{1,1,1,1,1,1} \subset \mathbb{F}^{30}$ is invariant under this involution. So it splits as a direct sum $C^+ \oplus C^-$ with $C^+$ consisting of invariant words and $C^-$ of anti-invariant ones.

The symmetric group $\Sigma_6$ acts on the set of six planes $S_i$ by permutations. Such a permutation can be realized by a path in the space of all admissible equations. This shows that there is a $\Sigma_6$-action on the code.

**Proposition 2.3** The actions of $I$ and $\Sigma_6$ can be chosen such that they commute.

**Proof.** As in prop. 2.2 consider the quadric $S : x_2 \cdot x_3 = 0$ and the involution $I : x_2 \leftrightarrow x_3$. Let $P_1, \ldots, P_{15}$ be the cusps in the plane $x_2 = 0$ and let $P_{16}, \ldots, P_{30}$ be their images in the plane $x_3 = 0$. So $I(P_\nu) = P_{15+\nu}$ for $\nu = 1, \ldots, 15$. Following a path in the space of admissible equations does not interchange cusps between the two planes. So $I\sigma(\nu) = \sigma I(\nu)$ for all $\nu$ and $\sigma \in \Sigma_6$. This is the assertion.

The cusps $P_1, \ldots, P_{15}$ can be relabelled $P_{i,j}$, $i < j \leq 6$ such that $P_{i,j} \in S_i \cap S_j$. This induces an identification $\mathbb{F}^{15} = \Lambda^2(\mathbb{F}^6) \subset \mathbb{F}^{30}$. Let $e_1, \ldots, e_6$ be canonical generators for $\mathbb{F}^6$. Each word $w_i \in E_{1,1,1,1,1,1}$ induces the word

$$u_i := \sum_{k=1}^6 e_i \wedge e_k \in E_{111111} \cap \Lambda^2(\mathbb{F}^6).$$

For $i < j$ the word

$$u_{i,j} := u_i - u_j = (e_i - e_j) \wedge (e_1 + \ldots + e_6) = 2e_i \wedge e_j + (e_i - e_j) \wedge \sum_{k \neq i,j} e_k$$

belongs to $C^+ \cap \Lambda^2(\mathbb{F}^6)$.

Let $e = e_1 + \ldots + e_6 \in \mathbb{F}^6$ and $U \subset \mathbb{F}^6$ be the hyperplane of $u = \sum_i u_i e_i$ with $\sum u_i = 0$. Notice $e \in U$.

**Lemma 2.1** The words $u_{i,j} \in \Lambda^2(\mathbb{F}^6)$ generate the subspace $e \wedge U \subset \Lambda^2(\mathbb{F}^6)$.

**Proof.** Obviously the words $u_{i,j}$ belong to $e \wedge U$. This subspace has dimension 4. And it is easy to see that the four words $u_{1,2}, \ldots, u_{1,5}$ are linearly independent. \qed

We now put

$$B^+ := C^+ \cap \Lambda^2(\mathbb{F}^6), \quad B^- := C^- \cap \Lambda^2(\mathbb{F}^6).$$

Our main technical result is

**Proposition 2.4** a) $B^+ \subset \Lambda^2(\mathbb{F}^6)$ coincides with $e \wedge U$. b) $B^- \subset B^+$.

This implies our main result:

**Theorem 2.1** The code $E_{111111}$ is generated by the words $w_1, \ldots, w_5$ from the table at the beginning of section 2.
Proof. Recall $C = C^+ \oplus C^-$. Now $C^+ = \{(w, w), w \in B^+\}$. By prop. 2.4 a) $B^+$ is generated by the words $u_{i,j}$ and therefore $C^+$ is generated by the $w_i - w_j$. And $C^- = \{(w, -w), w \in B^-\}$. Since $B^- \subset B^+$ by prop. 2.4 b), for each word $(w, -w) \in C^-$ there is a word $(w, w) \in C^+$. The word $(w, w) - (w, -w) \in C_{111111}$ has weight $\leq 15 < 18$. So prop. 2.2 a) implies $w = 0$. We found $C^- = 0$ and $C_{111111} = C^+$. Then $E_{111111}$ is generated by the words $w_i$. 

Each code $B^+, B^- \subset \Lambda^2(\mathbb{F}^6)$ has the following two properties:

1) All its words have length 0, 9, 12, or 15.

2) It is invariant under the action of $\Sigma_6$ induced from the permutations of coordinates in $\mathbb{F}^6$.

Proposition 2.4 therefore follows from

**Proposition 2.5** Let $V \subset \Lambda^2(\mathbb{F}^6)$ be some code with the properties above. Then $V \subset e \wedge U$.

Proof. We abbreviate $e_{i,j} = e_i \wedge e_j = -e_{ji}$ for $i < j$. We proceed in several steps.

**Step 1:** Let $v = \sum_{i<j}v_{ij}e_{ij} \in V$ with some $v_{ij} = 0$. Then $v_{ik} = v_{jk}$ for all $k = 1, 2, 3, 4, 5, 6$.

Proof. Write

$$v = \sum_{k \neq i,j} (v_{ik}e_{ik} + v_{jk}e_{jk}) + \sum_{k,l \neq i,j} v_{kl}e_{kl}.$$ 

The involution $(i, j) \in \Sigma_6$ maps $v$ onto

$$(i, j)v = \sum_{k \neq i,j} (v_{jk}e_{ik} + v_{ik}e_{jk}) + \sum_{k,l \neq i,j} v_{kl}e_{kl}.$$ 

So

$$v - (i, j)v = \sum_{k \neq i,j} (v_{ik} - v_{jk})(e_{ik} - e_{jk})$$

has length $\leq 8$. By property 1) this word is 0 and $v_{ik} = v_{jk}$ for all $k$. 

**Step 2:** Given $v = \sum v_{ij}e_{ij} \in V$ consider $u = v_{12}u_{12} + v_{34}u_{34} + v_{56}u_{56} \in e \wedge U$. Then $v' = v + u$ has the coefficients $v'_{12} = v'_{34} = v'_{56} = 0$. By step 1

$$v' = a \cdot (e_1 + e_2) \wedge (e_3 + e_4) + b \cdot (e_1 + e_2) \wedge (e_5 + e_6) + c \cdot (e_3 + e_4) \wedge (e_5 + e_6), \quad a, b, c \in \mathbb{F}.$$

The assertion follows, if we show $v' \in e \wedge U$.

**Step 3:** We simplify the notation putting $v' = v$ and have to show $v \in e \wedge U$. We apply symmetries from $\Sigma_6$ to obtain

\[ \begin{align*} 
(13)(24)v &= -a \cdot (e_1 + e_2) \wedge (e_3 + e_4) + c \cdot (e_1 + e_2) \wedge (e_4 + e_5) + b \cdot (e_3 + e_4) \wedge (e_4 + e_5), \\
(15)(26)v &= -c \cdot (e_1 + e_2) \wedge (e_3 + e_4) - b \cdot (e_1 + e_2) \wedge (e_4 + e_5) - a \cdot (e_3 + e_4) \wedge (e_4 + e_5). 
\end{align*} \]

From $\text{weight}(v + (13)(24)v) \leq 8$ we conclude that $b = -c$ and from $\text{weight}(v + (15)(26)v) \leq 8$ that $a = c$. The word

$$v = a \cdot ((e_1 + e_2) \wedge (e_3 + e_4) - (e_3 + e_4) \wedge (e_5 + e_6) + (e_3 + e_4) \wedge (e_5 + e_6))$$

$$= a \cdot (e_3 + e_4 - e_1 - e_2) \wedge e$$

therefore belongs to $e \wedge U$. 

\[ \square \]
2.2 The other cases

By prop. 2.2 b) there are inclusions of extended codes. The following table shows these inclusions, the bottom row giving the dimensions of the codes generated by the words $w_i$ from the table at the beginning of section 2:

\[
\begin{array}{ccc}
33 & \rightarrow & 15 \\
& \rightarrow & 123 \\
& \rightarrow & 1113 \\
& \rightarrow & 11112 \\
& \rightarrow & 111111 \\
\end{array}
\]

\[
\begin{array}{ccc}
24 & \rightarrow & 114 \\
& \rightarrow & 1122 \\
& \rightarrow & 222 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

With each inclusion in this table new cusps appear, and the bigger code contains a word taking non-zero values at the new cusps. So the bigger code has a larger dimension than the included code. But the right-hand code $E_{111111}$ has dimension 5 by theorem 2.1. This proves

**Theorem 2.2** For all sextic surfaces constructed by the method 1.1 the extended code is generated by the words given in the table at the beginning of section 2. Its dimension is the number in the table above.

As a consequence we have:

**Theorem 2.3** The two types 3,3 and 1,1,4 of surfaces with 18 cusps, as well as the two types 2,2,2 and 1,1,1,3 of surfaces with 24 cusps differ by the dimensions of their extended codes. So they cannot belong to the same connected family of surfaces with their number of cusps, nor can they be degenerations of each other.

2.3 27 cusps

We apply the residual construction of section 1.2 for $c_1 = c_2 = c_3 = 1, b = 3$ to obtain a sextic surface $Y$ with nine cusps on each curve $S_i \cap S_j$. By section 1.3 the proper code contains two words

\[
\begin{array}{cccccc}
0 & 1 & 1 \\
2 & 0 & 1 \\
\end{array}
\]

and the word $w_1 + w_2$ of weight 27.

We finish with a quite explicit example. Take as residual cubic the Fermat cubic defined by

\[r := x_0^3 + x_1^3 + x_2^3 + x_3^3\]
and put
\[ s_i := x_i^3 + \lambda_i \cdot r, \quad i = 1, 2, 3, \quad s = x_1 \cdot x_2 \cdot x_3. \]

This leads to the sextic defined by
\[
\begin{align*}
  f &= \frac{1}{r}(x_1^3 + \lambda_1 r)(x_2^3 + \lambda_2 r)(x_3^3 + \lambda_3 r) - x_1^2 \cdot x_2^3 \cdot x_3^3 \\
  &= \lambda_1 \cdot x_2^3 x_3^3 + \lambda_2 \cdot x_1^3 x_3^3 + \lambda_3 \cdot x_1^3 x_2^3 + (\lambda_1 \lambda_2 \cdot x_3^3 + \lambda_1 \lambda_3 \cdot x_2^3 + \lambda_2 \lambda_3 \cdot x_1^3) \cdot r + \lambda_1 \lambda_2 \lambda_3 \cdot r^2.
\end{align*}
\]

Here the surface \( S : s = 0 \) of course is not smooth, however it can be checked by direct computation that for general choice of the coefficients \( \lambda_i \) the sextic surface is smooth, but for 27-cusps on the coordinate planes, like the nine points in the intersection \( x_1 = s_2 = s_3 = 0 \).

As far as we know, this number 27 is the largest number of a set of 3-divisible cusps on a sextic surface so far observed.

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