SUPERSYMMETRIC BLACK HOLES

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ABSTRACT

The effective action of $N = 2$, $d = 4$ supergravity is shown to acquire no quantum corrections in background metrics admitting super-covariantly constant spinors. In particular, these metrics include the Robinson-Bertotti metric (product of two 2-dimensional spaces of constant curvature) with all 8 supersymmetries unbroken. Another example is a set of arbitrary number of extreme Reissner-Nordström black holes. These black holes break 4 of 8 supersymmetries, leaving the other 4 unbroken.

We have found manifestly supersymmetric black holes, which are non-trivial solutions of the flatness condition ${\mathcal D}^2 = 0$ of the corresponding (shortened) superspace. Their bosonic part describes a set of extreme Reissner-Nordström black holes. The

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super black hole solutions are exact even when all quantum supergravity corrections are taken into account.
1 Introduction

Despite all successes in quantum gravity in dimensions $d = 2, 3$, development of this theory in $d \geq 4$ is still a problem. One of the main difficulties is the uncontrollable accumulation of divergent quantum corrections in each new order of perturbation theory.

The purpose of this paper is to find some results in $d = 4$ quantum gravity, which remain valid with an account taken of all orders of perturbation theory. With this purpose we investigate some effective quantum actions of 4-dimensional supergravity theories in very specific backgrounds, which admit supercovariantly constant spinors [1], [2].

The main result of our investigation is rather surprising: The effective action of $d = 4$ $N = 2$ supergravity has no quantum corrections in the background of arbitrary number of extreme Reissner-Nordström ($\mathcal{RN}$) black holes in neutral equilibrium. In some sense, to be defined later, the manifestly supersymmetric version of the extreme $\mathcal{RN}$ black holes provides an alternative to the trivial flat superspace.

We will start with a discussion of some earlier results on the vanishing of quantum corrections to the effective actions in supergravity theories. In $N = 1, 2, 3$ supergravities the locally supersymmetric on shell effective action is given in terms of the following chiral superfields:

\begin{align*}
W_{ABC}(x, \theta_i), & \quad \bar{W}_{A'B'C'}(x, \bar{\theta}_i), \quad i = 1 \\
W_{AB}(x, \theta_i), & \quad \bar{W}_{A'B'}(x, \bar{\theta}_i), \quad i = 1, 2 \\
W_A(x, \theta_i), & \quad \bar{W}_{A'}(x, \bar{\theta}_i), \quad i = 1, 2, 3
\end{align*}

(1)

Consider for example quantum corrections to $N = 1$ supergravity [3], [4]. The superfields are

\begin{align*}
W_{ABC}(x, \theta) & = \Psi_{ABC}(x) + C_{ABCD} \theta^D + ... \\
W_{A'B'C'}(x, \theta) & = \Psi_{A'B'C'}(x) + \bar{C}_{A'B'C'D'} \bar{\theta}^{D'} + ... ,
\end{align*}

(2)

where $\Psi, \bar{\Psi}$ are the gravitino field strength spinors and $C, \bar{C}$ are the Weyl spinors of the space-time. Each locally supersymmetric term in the effective
action depends both on \(W\) and \(\bar{W}\) and their covariant derivatives. For example, the troublesome 3-loop counterterm is

\[
S_{3\text{-loop}} = \int d^4x \, d^4\theta \, \det E \, \ W_{ABC} \bar{W}_{AB'C'} \bar{W}'_{AB'C'} .
\]  

(3)

It is a supersymmetrized square of the Bell-Robinson tensor. This term, as well as any other term of the effective quantum action, vanishes in the super-self-dual background

\[
W_{ABC} = 0, \quad \bar{W}'_{AB'C'} \neq 0,
\]  

(4)

or in the super-anti-self-dual background

\[
W_{ABC} \neq 0, \quad \bar{W}'_{AB'C'} = 0.
\]  

(5)

Such non-trivial backgrounds exist only in space-time with Euclidean signature. Indeed, in Minkowski space

\[
W_{ABC} \Rightarrow W_{\text{real}} + i W_{\text{im}}, \quad \bar{W}'_{AB'C'} \Rightarrow W_{\text{real}} - i W_{\text{im}}.
\]  

(6)

Therefore \(W\) and \(\bar{W}\) cannot vanish separately, only together, in which case the background is trivial. With Euclidean signature it is possible to have a vanishing right-handed spinor \(W_{ABC}(x, \theta)\) and a non-vanishing left-handed spinor \(\bar{W}_{AB'C'}(x, \bar{\theta})\) (or opposite). This half-flat superspace, where the left-handed gravitino \(\bar{\Psi}_{A'B'C'}(x)\) lives in the space with only left-handed curvature \(\bar{C}_{A'B'C'D'}(x)\), is the background where \(N = 1\) supergravity effective action has no quantum corrections (up to the above-mentioned topological terms).

In \(N = 2\) supergravity we have

\[
W_{AB}(x, \theta) = F_{AB}(x) + \Psi^i_{ABC}(x) \theta^C_i + C_{ABCD} \theta^C_i \theta^D_j \epsilon^{ij} + ... ,
\]  

(7)

where \(F_{AB}\) is the Maxwell field strength spinor. In the Euclidean half-flat superspace, where

\[
W_{AB} = 0, \quad \bar{W}_{AB'} \neq 0,
\]  

(8)

\(^3\)The only exceptions are the one-loop topological divergences proportional to \(W^2\) or \(\bar{W}^2\), related to the one-loop anomalies.
there are no quantum corrections to the effective action (up to topological terms). For \( N = 3 \) the half-flat superspace is given by \( W_A = 0, \bar{W}_A \neq 0 \).

To summarize, some examples of non-trivial background field configurations in supergravity, which receive no radiative corrections, have been known for more than 10 years \[3\], \[4\]. They all require Euclidean signature of space-time.

\section{Absence of Quantum Corrections in Robinson-Bertotti background}

The special role of the Robinson-Bertotti metric in the context of the solitons in supergravity was explained in lectures by Gibbons \[1\]. His proposal was to consider the Robinson-Bertotti (\( \mathcal{RB} \)) metric as an alternative, maximally supersymmetric, vacuum state. The extreme Reissner-Nordström metric spatially interpolates between this vacuum and the trivial flat one, as one expects from a soliton.

In what follows we are going to prove a non-renormalization theorem for the effective action of \( d = 4, N = 2 \) supergravity in the \( \mathcal{RB} \) background.

The \( \mathcal{RB} \) metric is known to be one particular example of a class of metrics, admitting super-covariantly constant spinors \[1\], \[2\], which are called Israel-Wilson-Perjes (\( \mathcal{IWP} \)) metrics \[6\]. It is also the special metric in this class, which does not break any of the 8 supersymmetries of \( d = 4, N = 2 \) supergravity; all other \( \mathcal{IWP} \) metrics break at least half of the supersymmetries.

In general relativity the \( \mathcal{RB} \) metric is known as the conformally flat solution of the Einstein-Maxwell system with anisotropic electromagnetic field \[6\]. It describes the product of two 2-dimensional spaces of constant curvature:

\[
\begin{align*}
 ds^2 &= \frac{2d\zeta d\bar{\zeta}}{[1 + \alpha \zeta \bar{\zeta}]^2} - \frac{2dudv}{[1 + \alpha uv]^2}, \quad \alpha = \text{const} .
\end{align*}
\]

(9)

The metric can also be written in the form

\[
\begin{align*}
 ds^2 &= (1 - \lambda y^2)dx^2 + (1 - \lambda y^2)^{-1}dy^2 + (1 + \lambda z^2)^{-1}dz^2 - (1 + \lambda z^2)dt^2 .
\end{align*}
\]

(10)
The corresponding Maxwell field $F_{ab}$ is constant (as well as the curvature tensor) and can be written as

\begin{align*}
F_{12}^{RB} &= \sqrt{2\lambda} \sin \beta, \\
F_{34}^{RB} &= \sqrt{2\lambda} \cos \beta, \\
\lambda &= \text{const}, \\
\beta &= \text{const}.
\end{align*}

(11)

The property of the $RB$ metric which is of crucial importance for our analysis is the conformal flatness of this metric, i.e. the vanishing of the Weyl tensor.

\[ C_{abcd}^{RB} = 0 \implies C_{ABCD}^{RB} = C_{A'B'C'D'}^{RB} = 0. \]

(12)

The curvature spinor, corresponding to the traceless Ricci tensor, satisfies Einstein’s equation

\[ R^{RB}_{A'B'AB} = F_{AB}^{RB} F_{A'B'}^{RB}. \]

(13)

The generic term in the effective quantum action of $N = 2$ supergravity is given by

\[ \Gamma \sim \int d^4x d^4N \theta \det A(W_{AB}, \bar{W}_{A'B'}, D_{CC}W, D_{CC}\bar{W}, \ldots), \]

(14)

where $A$ can either be a local or non-local function in $x$ of the superfields $W, \bar{W}$ and their covariant derivatives, the superfield $W$ being given in eq. (7).

For trivial flat superspace $W = \bar{W} = 0$, since there are no Maxwell, gravitino or Weyl curvatures in the flat superspace. Therefore the path integral of $d = 4, N = 2$ supergravity has no quantum corrections in the flat superspace.

Consider now the superfield $W$, containing $F_{AB}^{RB}$ of the $RB$ solution, given in eq. (11), and recall that there is no gravitino nor Weyl spinors in this background. The superfield $W, \bar{W}$ is a constant superfield but it does not vanish as it would be the case for the trivial flat superspace. It has only the first component in the expansion in $\theta$ i.e. it does not dependend on $\theta$ at all.

\[ W_{AB}^{RB} = F_{AB}^{RB}, \quad \bar{W}_{A'B'}^{RB} = \bar{F}_{A'B'}^{RB}. \]

(15)
Now we only have to look for the terms in eq. (14) which depend on \( W, \bar{W} \) but not on their covariant derivatives in bosonic or fermionic directions, since these derivatives are zero for the \( \mathcal{RB} \) solution:

\[
\int d^4x d^4N \theta \det E A(W^{\mathcal{RB}}_{AB}, \bar{W}^{\mathcal{RB}}_{A'B'}) .
\]

(16)

If \( A \) is a local function of \( W, \bar{W} \), this expression takes the form

\[
A(W^{\mathcal{RB}}_{AB}, \bar{W}^{\mathcal{RB}}_{A'B'}) \int d^4x d^4N \theta \det E = A \int dV = 0 ,
\]

(17)

since the invariant volume of the real superspace vanishes \( \square \) in \( N = 2 \) supergravity. For non-local functions we end up with the integral over the volume of the full superspace of certain functions of \( x \). Those integrals are also equal to zero, according to \( \square \).

Thus, we have proved that there are no quantum corrections to the effective action of \( N = 2 \) supergravity in the Robinson-Bertotti background. The basic difference with trivially flat superspace is the fact that the superfield, in terms of which the on shell quantum corrections are expressed, is not zero, but is a constant superfield. However, all supersymmetric invariants vanish as in the case of a flat superspace with vanishing superfield.

### 3 Absence of Quantum Corrections in the Extreme Black Hole Background

The proof of the non-renormalization theorem for the \( \mathcal{RB} \) background was almost trivial due to conformal flatness of this metric and because the Maxwell field is constant. These properties are not present for general metrics admitting super-covariantly constant spinors. In general relativity they are known as conformal-stationary class of Einstein-Maxwell fields with conformally flat 3-dimensional space. This class of metrics has been found by Neugebauer, Perjes, Israel and Wilson \( \square \):

\[
\begin{align*}
 ds^2 &= (V\bar{V})^{-1}(dt + A dx)^2 - (V\bar{V}) (dx)^2 \\
 \nabla \times \mathbf{A} &= -i(\bar{V} \nabla V - V \nabla \bar{V}), \quad \nabla^2 V = 0, \quad V \neq 0 ,
\end{align*}
\]

(18)
where $\nabla^2$ is the flat space Laplacian in $\mathbf{x}$. For real $V$ this metric reduces to the Majumdar-Papapetrou solutions [3], which, according to Hartle and Hawking [4], are the only regular black hole solutions in this class. They describe an arbitrary number $n$ of extreme $\mathcal{R}\mathcal{N}$ black holes with gravitational attraction balanced by electrostatic repulsion:

$$V = \tilde{V} = 1 + \sum_{s=1}^{n} \frac{M_s}{|\mathbf{x} - \mathbf{x}_s|}.$$  

(19)

It has been found by Gibbons and Hull [1], and in the most general form by Tod [2], that these metrics admit super-covariantly constant spinors of $N = 2$ supergravity. We will reformulate here the results of [1], [2] for the special case of pure $N = 2$ supergravity.\footnote{Tod’s parameter $Q$, related to dust density is equal to zero in our theory since there is no dust in pure supergravity.}

We have found that in the treatment of super-covariantly constant spinors of $N = 2$ supergravity it is very helpful to use the original Penrose notation [8]. We introduce a standard spinor basis, or dyad, $o^A, \iota^A$, and we define\footnote{Our spinorial indices take values 0,1 and $0', 1'$. The Greek letters chosen by Penrose for the basis: $\omicron, \iota$ (omicron and iota) visually resemble these numbers. The use of the equations $\epsilon^A_0 = o^A, \epsilon^A_1 = \iota^A$ and many others is particularly simple in this notation.}

$$\epsilon_{AB} o^A \iota^B = o_A \iota^A = V, \quad \epsilon_{A'B'} o^{A'} \iota^{B'} = o_{A'} \iota^{A'} = \tilde{V},$$

(20)

Only when $V = \tilde{V} = 1$ the dyad is a spin frame. However, it is possible to work with a dyad which is not normalized to unity. Associated with any spinor basis of the manifold is a null tetrad $l^a, n^a, m^a, \bar{m}^a$ defined by

$$l^a = o^A o^A', \quad n^a = \iota^A \iota^A', \quad m^a = o^A \iota^A', \quad \bar{m}^a = \iota^A o^A',$$

(21)

and satisfying the following conditions:

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0,$$

$$l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0,$$

$$l^a n_a = -m^a \bar{m}_a = V \tilde{V}.$$

(22)
We will require that these spinors are super-covariantly constant \[1\], \[2\].

\[
\nabla_{AA'} o_B + F_{AB} \iota_{A'} = 0 , \\
\nabla_{AA'} \iota_{B'} - F_{A'B'} o_A = 0 .
\] (23)

From now on we will limit ourselves to the case of black holes only (real $V$), postponing the treatment of more general metrics to a future publication. In particular, basically the same techniques can be used to study eq. (20) with complex $V$, i.e. when $\nabla \times A$ in eq. (18) in the definition of the IWP metric is non-vanishing. Other interesting examples of metrics admitting super-covariantly constant spinors are plane-wave space-times for which $V = 0$ in eq. (18). They also will be considered in a separate publication.

We use that

\[
o^A = V^{-1} K_{AA'} \iota^A , \quad \iota^A = -V^{-1} K_{AA'} o^A ,
\] (24)

where $K_a$ is the Killing vector

\[
K_{AA'} = (l + n)_{AA'} .
\] (25)

Eqs. (23) can be rewritten as follows

\[
\nabla_{AA'} o_B \equiv \nabla_{AA'} o_B - V^{-1} K_{AA'} F_{BC} o^C = 0 , \\
\nabla_{AA'} \iota_{B'} \equiv \nabla_{AA'} \iota_{B'} - V^{-1} K_{AA'} F_{B'C'} \iota^{C'} = 0 .
\] (26)

The hatted derivatives have the standard meaning \[8\] of derivatives in the conformally rescaled metric

\[
\hat{g}_{ab} = V^2 g_{ab} , \quad \hat{\nabla}_a = V^{-1} \nabla_a V , \\
\nabla_{AA'} o_B = \nabla_{AA'} o_B - \nabla_{BA'} o_A .
\] (27)

If the null tetrads are expressed according to eq. (21) through super-covariantly constant omicron and iota satisfying eqs. (23), one gets the following equations for differential forms \[6\]

\[
\hat{d}m \equiv dm = 0 ,
\]

\[6\]The wedge product symbol is omitted for simplicity, when multiplying forms.
\[ \hat{d}m \equiv d\bar{m} = 0 , \]
\[ \hat{d}(l - n) \equiv d(l - n) = 0 , \]
\[ \hat{d}K \equiv dK - 2\Upsilon K = 0 , \]
\[ \hat{d}A \equiv dA - \Upsilon A = 0 , \]
\[ \hat{d}\Upsilon \equiv d\Upsilon = 0 , \]
\[ \hat{d}w_{ab} - w^{ac}w^{b}_c \equiv dw_{ab} - w^{ac}w^{b}_c = 0 , \]
(28)

where the null tetrad and Maxwell curvature forms are defined as

\[ m = dx^a m_a , \quad \bar{m} = dx^a \bar{m}_a , \quad l = dx^a l_a , \quad n = dx^a n_a , \quad K = l + n , \quad F = dA , \]
(29)

and we have introduced the Lorentz connection form \( w_{ab} \) and the Weyl connection form \( \Upsilon \).

These equations can be solved as follows:

\[ m = 2^{-\frac{1}{2}} (dx + idy) , \]
\[ \bar{m} = 2^{-\frac{1}{2}} (dx - idy) , \]
\[ l - n = \sqrt{2} dz , \]
\[ K = \sqrt{2} V^2 dt , \]
\[ A = V dt , \]
\[ \Upsilon = V^{-1} dV , \]
\[ w_{AB} = V^{-1} (\iota_A d\iota_B - o_A d\iota_B + \frac{1}{2} \epsilon_{AB} dV) . \]
(30)

This leads to the Papapetrou-Majumdar metrics

\[ ds^2 = V^2 dt^2 - V^{-2} dx^2 , \quad F = dV \wedge dt , \]
(31)

where the flat-space Laplacian in \( x, y, z \) of \( V \) is zero and \( F \) may still be subject to some dual rotation. These coordinates \( x, y, z \) are called comoving coordinates and \( t \) is defined as \( K^a \nabla_a = \sqrt{2} \frac{\partial}{\partial t} \).

To calculate the curvature of the manifold we act with \( \nabla_{CC'} \) on equations \ref{eq:23}.

\[ R = 0 , \]
\[ R_{A'B'AB} = F_{AB} \bar{F}_{A'B'} , \]
\[ C_{ABCD} = \nabla_{AB'} F_{CD} V^{-1} K_B^{B'} , \]
\[ \bar{C}_{A'B'C'D'} = \nabla_{A'B} \bar{F}_{C'D'} V^{-1} K_B^{B'} . \]  
(32)

Now we have enough information to investigate the effective quantum action in the black hole background with the properties:
\[ F_{AB}(x) = V^{-2} K_A^{A'} \nabla_{A'B} V , \]
\[ \Psi_{iABC}(x) = 0 , \]
\[ C_{ABCD} \equiv C_{ABCD} - \nabla_{AB'} F_{CD} V^{-1} K_B^{B'} = 0 , \]
\[ C_{ABCD}^+ \equiv C_{ABCD} + \nabla_{AB'} F_{CD} V^{-1} K_B^{B'} \neq 0 , \]  
(33)

and the conjugate ones can be easily derived from eqs. (33).

The basic on shell superfield \( W \) in the real basis is given by
\[ W_{AB}(x, \theta, \bar{\theta}) = F_{AB}(x) + \Psi^i_{iABC}(x) \theta^C_i + C_{ABCD} \theta^C_i \theta^D_j \epsilon^{ij} + \nabla_{CD'} F_{AB} \theta^C_i \bar{\theta}^{D'j} + \ldots . \]  
(34)

The first component of this superfield is neither zero, as in flat superspace, nor a constant, as in \( R\mathcal{B} \) case. The second component is zero, we have just a bosonic background. To analyze the second component we first have to change variables. Instead of working with independent unconstrained 8 fermionic coordinates \( \theta^A_i, \bar{\theta}^{B'i} \) of the real \( N = 2 \) superspace, for the black holes we need the following 16 coordinates, satisfying 8 constraints:
\[ \theta_{A'i}^\pm \equiv \theta_{A'i} \pm E_{AA'} \epsilon_{ji} \theta_{A'j}^\pm , \]
\[ \bar{\theta}^{A'i} \equiv \theta_{A'i} \pm E_{A'A} \epsilon_{ij} \theta_{Aj} , \]
\[ \theta_{A'i}^\pm = \pm E_{AA'} \epsilon_{ij} \theta_{A'j}^\pm . \]  
(35)

Here \( E \) is a normalized Killing vector,
\[ E_{A'A} = V^{-1} K_A^{A'} , \]
\[ E_{AA'} E^{A'B} = \delta_A^B , \]
\[ \epsilon_{ij} \epsilon^{kj} = \delta_i^k . \]  
(36)

\(^7\)The equations presented above for black holes can be derived also for the \( R\mathcal{B} \) metric. There will be a second set of covariantly constant spinors, defined by eqs. (23) with opposite sign in front of \( F \) and \( \bar{F} \) for the second set of omicron and iota. The second Killing vector will be built from the second set of these spinors. Both \( C^+ \) and \( C^- \) are equal to zero for \( R\mathcal{B} \) as a consequence of all those equations.
In terms of these coordinates, whose supersymmetry variation is
\[\epsilon_{\pm A i} \equiv \delta \theta_{A i} = \epsilon_{A A'} \epsilon_{j i} \bar{\epsilon}^{A' j},\]
\[\bar{\epsilon}_{A' i \pm} \equiv \delta \bar{\theta}_{A' i} = \bar{\epsilon}^{A' i} \pm E^{A' A} \epsilon_{i j} \epsilon_{A j},\] (37)
the supersymmetry breaking and the shortening of the unbroken superspace related to extreme $\mathcal{R}N$ black holes can be understood. The supersymmetric transformation of the gravitino field strength is
\[\delta \Psi_{A B C i} = C_{A B C D}^+ \epsilon_i^{D+} + C_{A B C D}^- \epsilon_i^{D-}.\] (38)
It can be made zero under two conditions. The first is
\[C_{A B C D}^- \equiv C_{A B C D} - \nabla_{A'B'} F_{C D} V^{-1} K_{B'} = 0,\] (39)
which is satisfied for the black holes according to eqs. (33). This condition is the property of extreme black holes that some combination of curvature and Maxwell fields vanish. It is an integrability condition for the existence of supercovariant spinors $\epsilon_{A-}^i$. The second condition is
\[\epsilon_{A+}^i = \delta \theta_{A+}^i = 0,\] (40)
and requires the breaking of 4 supersymmetries. It also indicates that after the change of coordinates, given by eqs. (35), there are 4 independent combinations of fermionic coordinates. They are given by 8 coordinates $\theta_{A-}^i$, $\bar{\theta}_{A'i-}$, constrained by 4 conditions
\[\theta_{A-}^i = -E^{A'}_{A} \epsilon_{j i} \bar{\theta}^{A'j}.\] (41)
These combinations are still unbroken coordinates of the superspace, since $\epsilon_{A-}^i$ in eq. (38) can take arbitrary values and the variation of gravitino nevertheless vanishes.

At this point it is appropriate to explain the difference between Robinson-Bertotti solution and black holes from the point of view of supersymmetry. Both metrics belong to the general class of Israel-Wilson-Perjes metrics, admitting super-covariantly constant spinors. For $\mathcal{RB}$ both combinations, $C^+$ and $C^-$, which define the supersymmetry transformation of the gravitino
field strength, vanish, since the gravitational Weyl tensor and the derivative of a Maxwell tensor vanish separately. Therefore there are no restrictions on supersymmetry variations of all 8 coordinates of the superspace, i.e. both $\epsilon^A_i+$ and $\epsilon^A_i-$ are arbitrary and nevertheless the supersymmetry variation of gravitino field strength is zero in the $\mathcal{RB}$ background.

We have to analyze the structure of quantum corrections before using the properties of the black hole background. However, we will work with variables which are natural for this problem, like Weyl-Maxwell spinors $C^\pm$, $\bar{C}^\pm$, given in eqs. (33) and fermionic coordinates of the superspace, given in eqs. (35), considering the vector $E_{AA'}$ as some arbitrary one. When the quantum corrections are calculated in an arbitrary Lorentz-covariant background, there is no dependence on any such vector, of course. We have introduced this dependence through our choice of variables, and it should be absent in terms of the original variables after integration over fermionic variables.

In the black hole background all terms which depends on $C^-$ or $\bar{C}^-$ will vanish. The crucial question is: Are there terms which depend only on $C^+$, $\bar{C}^+$ and do not depend either on $C^-$ or on $\bar{C}^-$? The answer is no, they do not exist. The point is that under the $\epsilon^A_i$ transformations the non-vanishing combinations of curvature and Maxwell fields $C^+$, $\bar{C}^+$ do transform. However, the $\epsilon^A_i$ variation coming from any term containing fermions will have the combinations $C^-\epsilon^C_i$, according to eq. (38). Any term with $C^+$ or $\bar{C}^+$ dependence but without $C^-$ or $\bar{C}^-$ dependence will not satisfy the $\epsilon^A_i-$supersymmetry requirements. To illustrate this general statement consider again the 3-loop counterterm. The following combinations can be expected.

$$ (C^+)^2(\bar{C}^+)^2, \quad (C^-)^2(\bar{C}^-)^2, \quad (C^+C^-)(\bar{C}^+\bar{C}^-), \quad \text{etc.} \quad (42) $$

Only the first combination does not vanish in the black hole background. Let us show that it will not appear in the effective quantum action. The straightforward calculation is to check the dependence of each of these terms on the vector $E_{AA'}$ by substituting expressions for $C^\pm$ from eqs. (33). The term $(C^+)^2(\bar{C}^+)^2$, which is forbidden by the above mentioned supersymmetry arguments, does depend on $E_{AA'}$, as opposed to the third term in (42), which is allowed by the $\epsilon^A_i-$supersymmetry and can be shown to be $E_{AA'}$-
independent.

\[ (C^+C^-)(\bar{C}^+\bar{C}^-) = \{(C_{ABCD} + \nabla_{AA'}F_{CD}E^A_{B'})(C^{ABCD} - \nabla^{A'}_B F^{CD}E^{B'})\}(C^+C^-) \]
\[ = \{(C_{ABCD})^2 - (\nabla_{AA'}F_{CD})^2\}\{(\bar{C}_{A'B'C'D'})^2 - (\nabla_{A'A'}\bar{F}_{C'D'})^2\}\]  

Thus, all terms in the on shell effective quantum action of \( N = 2, d = 4 \) supergravity, which are locally supersymmetric and Lorentz invariant, vanish in the extreme multi black hole background.

4 Black Holes as a Flat Superspace

Our approach to the black hole superspace was inspired by the group manifold approach to \( N = 2 \) supergravity [10]. The superspace [10] is formulated in terms of the superspace 1-forms \( E^a, \Psi^i_A, \Psi_{A'}^i, A \) associated to the supergravity physical fields and the spin connection \( w^{ab} \). We are interested only in on shell curvatures associated with \( d = 4, N = 2 \) super-Poincaré algebra with central charge. They are defined in terms of the following differential operator:

\[ \mathcal{D} = d + A^M T_M , \]  

where \( T_M \) are the generators of the super-Poincaré group

\[ [T_M, T_N] = f_{MN}^L T_L , \]  

and \( A^M \) are connection forms \( E^a, \Psi^i_A, \Psi_{A'}^i, A_{ij}, w^{ab} \) related to \( P_a, Q_A^i, \bar{Q}^A_{,i}, M^{ij}, M_{ab} \) generators of super-Poincare group (translation, 8 supersymmetries, central charge and Lorentz generator). Thus, the operator \( \mathcal{D} \) is

\[ \mathcal{D} = d + E^a P_a + \Psi^i_A Q_a^i + \Psi_{A'}^i Q_{A'}^{Ai} + A_{ij} M^{ij} + w^{ab} M_{ab} . \]  

The curvature of this superspace is defined as

\[ \mathcal{D}^2 = R^M T_M , \]
\[ R^M = dA^M + f^M_{LN} A^L A^N . \]
The set of curvatures includes

\[ R^a = dE^a - w^{ab}E_b - i\bar{\Psi}^i\gamma^a\Psi_i \equiv DE^a - i\bar{\Psi}^i\gamma^a\Psi_i , \]

\[ \rho_{Ai} = d\Psi_{Ai} - w^{ab}(\sigma_{ab}\Psi_i)_A \equiv D\Psi_{Ai} , \]

\[ \rho_{A'} = d\Psi_{A'} - w^{ab}(\sigma_{ab}\Psi_i)_{A'} \equiv D\Psi_{A'} , \]

\[ F = dA + (\epsilon^{AB}\epsilon^{ij}\Psi_{Ai}\Psi_{Bj} + h.c.) , \]

\[ R^{ab} = dw^{ab} - w^{a}_{c}w^{cb} . \] (48)

The nilpotency of the operator \( \mathcal{D} \) is the requirement that all curvatures are equal to zero \( \mathcal{D}^2 = R^MT_M = 0 \), or in detail

\[ R^a = 0 , \]

\[ \rho_{Ai} = 0 , \]

\[ \rho_{A'} = 0 , \]

\[ F = 0 , \]

\[ R^{ab} = 0 . \] (49)

Equation (49) has a trivial solution describing a flat \( d = 4, N = 2 \) superspace with 4 bosonic, 8 fermionic coordinates and pure fermionic central charge form:

\[ E^a = dx^a - i\sum_i \bar{\theta}\gamma^a d\theta , \]

\[ \Psi_{Ai} = d\theta_{Ai} , \]

\[ \Psi_{A'} = d\bar{\theta}_{A'} , \]

\[ A = \epsilon^{AB}\epsilon^{ij}\theta_{Ai}d\bar{\theta}_{Bj} + h.c. , \]

\[ w^{ab} = 0 . \] (50)

Now we will build a non-trivial flat superspace for the black holes. The variables which are natural for the black hole superspace are not Lorentz covariant objects, like in eq. (49), but Lorentz invariant ones, like in eqs. (28), (29), (21). The bosonic forms are \( m, \bar{m}, l - n, K = l + n \) and \( A \). In addition we introduce 8 fermionic forms satisfying 4 constraints, in accordance with the shortening of a black hole superspace. These forms are
required to be super-covariantly constant:

\[
\nabla_{AA} \Psi_B^i + F_{AB} \epsilon^{ij} \Psi_{A'j} = 0 ,
\Psi_A^i = -E_A^A \epsilon^{ij} \Psi_{A'}^j ,
\]

(51)

where \( E_A^A \) is the normalized Killing vector, defined in eqs. (36), (25), (20).

We choose the fermionic forms of the black hole superspace to be

\[
\Psi_A^1 = o_A^i \psi_1^i + i_A^i \psi_2^i ,
\Psi_A^2 = o_A^i \psi_2^{\dagger} - i_A^i \psi_1^{\dagger} ,
\Psi_{A'}^1 = o_A^{\dagger} \psi_1^{\dagger} + i_A^{\dagger} \psi_2^{\dagger} ,
\Psi_{A'}^2 = o_A^{\dagger} \psi_2 - i_A^{\dagger} \psi_1 .
\]

(52)

Omicron and iota in equations (51) satisfy eqs. (23). Our 4 independent Lorentz invariant fermionic forms \( \psi^i, \psi^{\dagger} \) satisfy the following equations:

\[
\hat{d} \psi^i = d \psi^i = 0 ,
\hat{d} \psi^{\dagger} = d \psi^{\dagger} = 0 .
\]

(53)

They have a simple solution:

\[
\psi^i = d \theta^i ,
\psi^{\dagger} = d \theta^{\dagger} .
\]

(54)

Thus, our space contains 4 fermionic coordinates in addition to the 4 bosonic ones.

The differential operator \( \mathcal{D} \) for the black hole superspace is defined as follows:

\[
\mathcal{D} = d + K P_K + m P_m + \bar{m} P_{\bar{m}} + (l - n) P_{l-n} + \psi^i Q_i + \psi^{\dagger} Q_i^\dagger + A_{ij} M^{ij} + w^{ab} M_{ab} + \Upsilon W ,
\]

(55)

where \( P_K, P_m, P_{\bar{m}}, P_{l-n} \) are the translation operators in the null tetrad basis. \( Q_i, Q_i^{\dagger} \) are the 4 supersymmetry generators, \( M^{ij} = \epsilon^{ij} M \), \( M_{ab} \) are generators of central charge and Lorentz symmetry and by \( W \) we have denoted the generator of conformal transformations.
The manifestly supersymmetric generalization of the bosonic multi black hole can be obtained by solving the flatness condition \( D^2 \equiv (dA^M + f^M_{NL} A^N A^L)T_M = 0 \) of the black hole superspace with the set of curvatures \( R^M \) required to vanish:

\[
\begin{align*}
\sum & \equiv 0 , \\
& \sum = 0 , \\
& \sum = 0 , \\
& d(l - n) = 0 , \\
& dK - 2\gamma K + 2\sqrt{2}iV^2\psi^2 = 0 , \\
& d\psi_i = 0 , \\
& d\psi^{i\dagger} = 0 , \\
& dA - \gamma A + 2iV\psi^2 = 0 , \\
& dw^{ab} - w^{ac}w^b_c = 0 , \\
& d\gamma = 0 , \\
& \psi^2 \equiv \psi_i\psi^{i\dagger}.
\end{align*}
\]

These equations can be solved as follows:

\[
\begin{align*}
m & = 2^{-\frac{1}{2}}(dx + idy) , \\
\bar{m} & = 2^{-\frac{1}{2}}(dx - idy) , \\
l - n & = \sqrt{2}dz , \\
K & = \sqrt{2}V^2(dt - i\theta_i d\theta^{i\dagger} + i\theta^{i\dagger} d\theta_i) , \\
\psi_i & = d\theta_i , \\
\psi^{i\dagger} & = d\theta^{i\dagger} , \\
A & = V(dt - i\theta_i d\theta^{i\dagger} + i\theta^{i\dagger} d\theta_i) , \\
w_{AB} & = V^{-1}(\iota_Ad\theta_B - o_Ad\iota_B + \frac{1}{2}\epsilon_{AB}dV) , \\
\gamma & = V^{-1}dV ,
\end{align*}
\]

the flat-space Laplacian in \( x, y, z \) of \( V \) is zero. In particular, \( V \) can be chosen in the form \([14]\), and in this case eqs. \([57]\) define a manifestly supersymmetric multi black hole: a flat superspace with 4 bosonic and 4 fermionic coordinates, and

\[\text{Our notation here corresponds to the one in [8]. The corresponding reference with supersymmetry, matching the notation of [8], is [11].}\]
the flatness condition being defined as the vanishing of all curvatures (56) in this superspace. This superspace, when considered at $\theta_i = \theta_i^{\dagger} = d\theta_i = d\theta_i^{\dagger} = 0$, coincides with the set of extreme Reissner-Nordström black holes in the usual 4-dimensional bosonic space.

Thus, to find an extreme black hole we were solving not the classical Einstein-Maxwell equations but the flatness condition of the superspace. The solution of the classical Einstein-Maxwell equations defines the Ricci tensor in terms of a quadratic combination of Maxwell tensors, the Riemann-Christoffel curvature tensor is non-vanishing. All on shell quantum gravity corrections are expressed in terms of the non-vanishing Riemann-Christoffel curvature tensor in the case of non-extreme as well as extreme $\mathcal{RN}$ black holes. In both cases it is impossible to handle quantum corrections without additional information.

The additional information indeed exists for the extreme $\mathcal{RN}$ black holes. It is possible to investigate the effective quantum action of $d = 4$, $N = 2$ supergravity, taking into account the fundamental fact that the bosonic extreme $\mathcal{RN}$ black holes do not break 4 of the original 8 supersymmetries. The investigation shows that all on shell quantum corrections, which are locally supersymmetric and Lorentz invariant, vanish in the extreme $\mathcal{RN}$ multi black hole background.

Moreover, the 4 above mentioned supersymmetries can be made manifest. A shortened superspace exists (55), whose flatness condition (56) is solved by a supersymmetric black hole solution (57). All curvatures of this superspace vanish, and there are no geometrical building blocks for quantum corrections. In this sense the supersymmetric black holes represent an alternative to the trivial flat superspace (46), (50), which also has no quantum corrections because all curvatures of the superspace vanish (49).

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