Convection Inside Nanofluid Cavity with Mixed Partially Boundary Conditions

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Abstract: In recent decades, research utilizing numerical schemes dealing with fluid and nanoparticle interaction has been relatively intensive. It is known that CuO nanofluid with a volume fraction of 0.1 and a special thermal boundary condition with heat supplied to part of the wall increases the average Nusselt number for different aspect ratios ranges and for high Rayleigh numbers. Due to its simplicity, stability, accuracy, efficiency, and ease of parallelization, we use the thermal single relaxation time Bhatnagar-Gross-Krook (SRT BGK) mesoscopic approach D2Q9 scheme lattice Boltzmann method in order to solve the coupled Navier–Stokes equations. Convection of CuO nanofluid in a square enclosure with a moderate Rayleigh number of 10^5 and with new boundary conditions is highlighted. After a successful validation with a simple partial Dirichlet boundary condition, this paper extends the study to deal with linear and sinusoidal thermal boundary conditions applied to part of the wall.

Keywords: nanofluid; heat transfer; mixed boundary condition; lattice Boltzmann method

1. Introduction

Throughout last two decades, many studies have focused on enhancement of macro-scale heat transfer in cavities machined into mechanical components, electronics cooling, communications, and auto-computing industries [1–25]. It is a crucial goal in various large engineering applications. Techniques for enhancing heat transfer in such components has received significant consideration in scientific field of research. In the present study, the lattice Boltzmann method (LBM) is handled to achieve a flow problem’s simulation. LBM is an alternative numerical technique for existing CFD approaches. This approach intends to simulate fluids as collection of particles that undergo two steps, namely collision and propagation over a discrete grid (lattice mesh), successively. Various lattice Boltzmann prototypes and models have been suggested for incompressible NSE (Navier–Stokes equations). More technical details about LBM applications can be found in [11–17,25]. In the literature, researchers have studied heat transfer enhancement by adding nanofluids [1–10]. The leading objective of this reading deals with a new trend that associates the advantages of nanofluids and the suitable choice of boundaries conditions that can enhance fluid flow and heat transfer inside engineering cavities.

2. Numerical Approach and Governing Equations

Figures 1 and 2 depict the LB model and propagation step in a D2Q9 model. The problem schematic is plotted in Figure 3.
We consider an incompressible Newtonian fluid and nanofluids with a steady and laminar stream behavior inside the cavity (Figure 3). It is supposed that no slip occurs between nanoparticles (NPs) and the BF (base fluid) (water), which are at TE (thermal equilibrium). We deal with same size spherical NPs with no radiation heat transfer between their surfaces. Density varies based on Boussinesq approximation. Thermo-physical (TH) properties of the BF (base fluid) and copper nanoparticles (NPs) are tabulated in the following table (Table 1).
Governing equations (continuum, momentum (x and y velocities), and energy equations) for a laminar and steady state (SS) free convective stream of NF (nanofluid) in a 2D enclosure are \[10\]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\rho_{nf}} \frac{\partial p}{\partial x} + \nu_{nf} \nabla^2 u \quad (2)
\]

\[
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{\rho_{nf}} \frac{\partial p}{\partial y} + \nu_{nf} \nabla^2 v + (\rho\beta)_{nf} \gamma(T - T_c) \quad (3)
\]

\[
\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = \alpha_{nf} \nabla^2 T \quad (4)
\]

Then the used non-dimensional parameters are:

\[
U = \frac{uL}{\alpha_f}, \quad X = \frac{x}{L}, \quad V = \frac{vL}{\alpha_f}, \quad Y = \frac{y}{L} \quad (5)
\]

Pressure, temperature, Prandtl number and Rayleigh number are defined, respectively, as:

\[
P = \frac{pL^2}{\rho_{nf}\alpha_f^2}\theta = \frac{T - T_c}{T_h - T_c}, \quad Pr = \frac{\nu_f}{\alpha_f}, \quad Ra = \frac{8\beta_f\Delta T L^3}{\nu_f\alpha_f} \quad (6)
\]

The dimensionless forms of the set of governing equations (Equations (1)–(4)) are:

\[
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (7)
\]

\[
U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial p}{\partial X} + \frac{\nu_{nf}}{\alpha_f} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) \quad (8)
\]

\[
U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{\partial p}{\partial Y} + \frac{\nu_{nf}}{\alpha_f} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \frac{(\rho\beta)_{nf}}{\rho_{nf}\beta_f} Ra Pr \theta \quad (9)
\]

\[
U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial Y} = \frac{\alpha_{nf}}{\alpha_f} \left( \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} \right) \quad (10)
\]

The thermo-physical properties of the NF nanofluid (specific heat capacity, effective density, thermal expansion coefficient, heat conduction coefficient, thermal diffusivity, and dynamic viscosity) are written, respectively, as:

\[
(\rho cp)_{nf} = (1 - \phi)(\rho cp)_f + \phi(\rho cp)_p \quad (11)
\]

\[
\rho_{nf} = \phi \rho_p + (1 - \phi) \rho_f \quad (12)
\]

\[
(\rho\beta)_{nf} = \phi (\rho\beta)_p + (1 - \phi) (\rho\beta)_f \quad (13)
\]
Based on nanoparticle thermal coefficient $k_p$ and base fluid thermal coefficient $k_f$, Maxwell equation [10] is used to calculate $k_{nf}$:

$$
\frac{k_{nf}}{k_f} = \frac{2\phi(k_p - k_f) + (k_p + 2k_f)}{2\phi(k_f - k_p) + (k_p + 2k_f)}
$$  \hspace{1cm} (14)

$$
\alpha_{nf} = k_{nf} / (\rho cp)_{nf}
$$  \hspace{1cm} (15)

The Brinkman model [10] is:

$$
\mu_{nf} = \frac{1}{(1 - \phi)^{2.5}}
$$  \hspace{1cm} (16)

In the above equations $\beta$ is the thermal expansion coefficient, $\alpha$ is the thermal diffusivity, $\rho$ is the density, $cp$ is the specific heat (at constant pressure) and $\phi$ is the solid volume fraction.

3. LBM Approach for Free Convection

In the last decade, LBM has become a successful attractive numerical mesoscopic for many engineering heat-mass transfer associated with fluid flow simulations. The technical approach is based on two distribution functions (DF): $f_i$ for the flow field and $g_i$ for the temperature.

The flow field [11–22] in the presence of an external force $F_i$ is defined by distribution function (DF) $f_i$ for the flow field, which can be given as:

$$
f_i(\vec{r} + \vec{c}_i\Delta t, t + \Delta t) = \frac{\Delta t}{\tau_v} \left[ f_i(\vec{r}, t) - f_i(0)(\vec{r}, t) \right] + f_i(\vec{r}, t) + \Delta t \vec{c}_i F_i
$$  \hspace{1cm} (17)

where the LEDF (local equilibrium distribution function) flow field is:

$$
f_i^{eq} = w_i \rho \left( 1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \left[ \frac{(\vec{c}_i \cdot \vec{u})^2}{2c_s^4} - \frac{\vec{c}_i \cdot \vec{u}}{2c_s^2} \right] \right)
$$  \hspace{1cm} (18)

Distribution function (DF) $g_i$ for the temperature ($\theta$) field can be written as:

$$
g_i(\vec{r} + \vec{c}_i\Delta t, t + \Delta) = \left( 1 - \left( \frac{\Delta t}{\tau_T} \right) \right) g_i(\vec{r}, t) + \left( \frac{\Delta t}{\tau_T} \right) g_i^{eq}(\vec{r}, t)
$$  \hspace{1cm} (19)

The local equilibrium distribution function (LEDF) associated with the thermal field is computed as:

$$
g_i^{eq} = w_i \rho e \left( 1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} \right)
$$  \hspace{1cm} (20)

We consider a unity lattice time step $\Delta t$ and nine discrete particle velocity vectors $\vec{c}_i$, expressed as (Figure 1):

$$
\vec{c}_0 = (0, 0)
$$  \hspace{1cm} (21)

$$
\vec{c}_b = (\cos(\phi_b), \sin(\phi_b))c \text{ for } \phi_b = (b - 1)\pi/2 \ b = 1 - 4
$$  \hspace{1cm} (22)

$$
\vec{c}_b = \sqrt{2}(\cos(\phi_b), \sin(\phi_b))c \text{ for } \phi_b = (b - 5)\pi/2 + \pi \ b = 5 - 8
$$  \hspace{1cm} (23)

For the momentum equation, the relaxation time for the flow field $\tau_v$:

$$
\tau_v = 0.5 + \left[ 3v/c_s^2 \Delta t \right]
$$  \hspace{1cm} (24)

where $v$ is the kinematic viscosity.
Defining \( \alpha \) as the thermal diffusivity of the scalar field, the relaxation-time for the temperature \( \tau_T \) is computed as:

\[
\tau_T = 0.5 + \left( \frac{3\alpha}{c^2\Delta t} \right)
\]  

(25)

The external force term \( F_i \) (for free convection) in Equation (17) is given by:

\[
F_i = \frac{\overrightarrow{w_i}}{c_i^2} F \cdot \overrightarrow{e_i}
\]  

(26)

We consider a unity lattice speed \( c = \Delta x / \Delta t \) related to sound lattice speed as \( c = 3c_s \), and \( \Delta x \) is the lattice-space. For simplicity, in this work, we neglect the inconsistency due to the fact that the viscosity \( \nu \) has the expression \( \tau_f R_T \) in the heat dissipation, whereas it maintains the expression \( R_T (\tau_f - 0.5) \) in Navier–Stokes equations. This inconsistency may be eliminated using a second order strategy when integrating the Boltzmann equation [23,24].

The \( D_2Q_9 \) model is used for weighting factor \( w_b \) for temperature and flow, calculated with the following equations:

\[
w_b = \frac{4}{9} \text{ for } b = 1 - 4
\]  

(27)

\[
w_b = \frac{1}{9} \text{ for } b = 5 - 8
\]  

(28)

The velocity \( u \), macroscopic density \( \rho \), and temperature \( T \) are computed, respectively, as:

\[
u(r,t) = \sum_i c_i f_i(r,t) / \sum_i f_i(r,t)
\]  

(30)

\[
\rho(r,t) = \sum_i f_i(r,t)
\]  

(31)

Because temperature is associated with the internal energy \( e \) deduced from the state equation (\( e = R_T \)), temperature is written as:

\[
T(r,t) = \rho(r,t)e = \sum_i g_i(r,t)
\]  

(32)

Based on Boussinesq approximation, the incompressible regime of free convection imposes a small characteristic velocity compared to the fluid speed of sound, and the Mach number should not exceed \( Ma = 0.3 \).

With a known number of lattices in vertical direction \( m \), Rayleigh number \( Ra \), Prandtl number, and \( Ma \) (Mach number), the thermal diffusivity and viscosity are computed from:

\[
v_f = c_s m Ma Pr^{0.5} Ra^{-0.5}
\]  

(33)

The local Nusselt number is expressed as [10]:

\[
Nu = \frac{Lh}{k_f}
\]  

(34)

where the heat transfer coefficient is:

\[
h = \frac{q_w}{T_h - T_c}
\]  

(35)

The thermal conductivity of the NF (nanofluid) is computed as:

\[
k_{nf} = - \frac{q_w}{\partial T / \partial x}
\]  

(36)
The local Nusselt number along the LW (left wall) can be written as:

$$ Nu = -k_n f \frac{\partial \theta}{\partial X} \bigg|_{X=0} $$

(37)

The average Nusselt number along the heat-source is deduced as:

$$ \overline{Nu} = -\frac{1}{L} \int_0^L k_n f \frac{\partial \theta}{\partial X} \bigg|_{X=0} $$

(38)

4. Validation

In this paper, the numerical procedure used to resolve the dimensionless equations is instituted on the mesoscopic approach LBM (lattice Boltzmann method). The D_2Q_9 lattices have been applied in both dynamic and thermal cases. The numerical approach is implemented using a FORTRAN software. A grid independence test is made using the LBM with different sets of grids, and good agreement with the literature is found for (251 × 251) grids. Aiming to verify the precision of the current numerical technique, validation findings are compared with reference [9].

For validation, the same geometry (Figure 3) used in the literature is highlighted. A numerical simulation is performed for a flow induced by buoyancy force, which is suggested to a partially constant heated enclosure, which is filled with copper and water nano-fluid with the mesoscopic approach model LBM. The Rayleigh number is taken as $Ra = 10^5$, while the Prandtl number is $Pr = 6.2$. These evaluations are offered obviously in Figure 4 in terms of local Nusselt number variation alongside the heated left boundary (Cu-H2O nanofluid) for different volume fraction ($\phi$) at $Ra = 10^5$. Figure 4 demonstrates a very good concordance between the current results and references’ ones [9].

![Figure 4](image-url)

Figure 4. Variation of the left local Nusselt number variation alongside the heated left boundary (Cu-H2O nanofluid) for different volume fractions ($\phi$) at $Ra = 10^5$ [9].

First, aiming to highlight the effect of the solid volume fraction on the dynamic and the thermal flow, we increased the volume fraction ($\phi$) from 0 to 0.2 and chose the same geometry as reference [9]. Figure 5 illustrates the effects of the solid volume fraction on
non-dimensional temperature in axial midline versus x position at a moderately high Rayleigh number ($Ra = 10^5$). As shown in this figure, the non-dimensional temperature in the axial midline versus x position increases as the solid volume fraction becomes higher.

![Non-dimensional temperature in axial midline versus x position for different volume fractions ($\phi$) for $Ra = 10^5$.](image)

Figure 5. Non-dimensional temperature in axial midline versus x position for different volume fractions ($\phi$) for $Ra = 10^5$.

Additionally, Figures 6 and 7 display the steady state non-dimensional dynamic and thermal profiles behaviors for five different volume fractions ($\phi$) at $Ra = 10^5$, where non-dimensional isotherms, streamlines, pressure, horizontal velocities, vertical velocities, and velocity magnitudes are enhanced when $\phi$ from 0 to 0.6.

Next, left local Nusselt evolution versus vertical wall for different volume fraction ($\phi$) in case 1 and at $Ra = 10^5$ is represented in Figure 8. As shown in this plot, it is found that the volume fraction is a parameter that can potentially increase thermal behavior inside the cavity.

We now examine the explicit effect of boundary condition choice on dynamic and thermal behavior of the present engineering problem. So, after applying a simple partial Dirichlet Boundary Condition (BC) (case 1), we aim, in this work, to study the effect of two new benchmarks of left partial boundary condition on the heat transfer and fluid flow behaviors. Linear (case 2) and sinusoidal (case 3) heated partial left boundary conditions are studied for selective parameters of Prandtl number, Rayleigh number, and solid volume fraction. It is to be noted that, for case 3, we consider a sinusoidal excitation, where the phase shift $\phi_0 = 0$. 

Figure 6. Steady state non-dimensional isotherm, streamline, and pressure profile behavior for central partially left boundary conditions (case 1: Dirichlet constant boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Figure 7. Steady state non-dimensional X-velocity, Y-velocity, and velocity magnitude profile behavior for central partially left boundary conditions (case 1: Dirichlet constant boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Figure 8. Variation of left local Nusselt number along the left boundary (Cu-water nanofluid) for different volume fractions ($\phi$) (case 1) at $Ra = 10^5$.

For constant, linear, and sinusoidal boundary conditions (case 1: Dirichlet constant BC, case 2: linear BC, case 3: sinusoidal BC), the variation of local Nusselt number along the central partially heated left boundary in the presence of copper-water nanofluid is illustrated in Figure 9 at $Ra = 10^5$ and volume fraction $\phi = 0$.

Figure 9. Variation of left local Nusselt number along the central partially heated left boundary (Cu-water nanofluid) for different left wall BCs (Boundary Conditions) (case 1: Dirichlet constant BC, case 2: linear BC, case 3: sinusoidal BC) at $Ra = 10^5$ and volume fraction $\phi = 0$. 
It can be perceived that the higher maximum of the left local Nusselt number is reached for a sinusoidal excitation. Moreover, the maximum of the left local Nusselt in the Dirichlet boundary increases slightly compared to linear boundary case. The obtained numerical results approve that a sinusoidal central partially heated left boundary enhances the rate of heat transfer, which is an aim sought in different widespread physical and engineering applications.

A quantitative study is presented in Figure 10, where steady state dynamic and thermal profiles behavior for the three cases of the partially thermalized left boundary conditions at volume fraction $\phi = 0$ and $Ra = 10^5$ are highlighted. We notice that isotherms, streamlines, horizontal velocities, vertical velocities, velocity magnitudes, and pressures present the maximum values for the case of the sinusoidal central partially heated left boundary and the minimum values for the linear ones, which is in good concordance with previous results in Figure 9.
In this section, we introduce the constant linear boundary conditions (case 2: linear BC) via the variation of the left local Nusselt number along the central partially heated left boundary in the presence of copper-water nanofluid, which is illustrated in Figure 3 at $Ra = 10^5$ for five values of the volume fraction ($\phi = 0$). Based on the Figure 11, we notice that the heat transfer rate is also enhanced for the maximum value of the volume fraction imposed in the cavity. The quantitative study is highlighted in Figures 12 and 13 where steady state (SS) dynamic and thermal profiles performance of the heated partially left boundary for different sketch of volume fraction at $Ra = 10^5$ is shown. Isotherms, streamlines, and pressure are shown in Figure 12, and horizontal velocities, vertical velocities, and velocity magnitudes are shown in Figure 13. An evolution in the thermal and the dynamic profiles can be easily observed, which play a noticeable role in the enhancement of the physics of the flow inside the enclosure, depending on the volume fraction value.

![Figure 10. Steady state non-dimensional dynamic and thermal profiles behavior for different partially left boundary conditions at volume fraction $\phi$ and $Ra = 10^5$.](image)

![Figure 11. Variation of left local Nusselt number along the left boundary (Cu-water nanofluid) for different volume fraction $\phi$ (case 2: linear boundary condition) at $Ra = 10^5$.](image)
Figure 12. Steady state non-dimensional isotherm, streamline, and pressure profile behavior for central partially left boundary conditions (case 2: linear boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Figure 13. Steady state non-dimensional X-velocity, Y-velocity, and velocity magnitude profiles behavior for central partially left boundary conditions (case 2: linear boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Next, Figure 14 shows the effect of the applied constant sinusoidal boundary condition on the flow and the heat rate in the cavity where the variation of local Nusselt number along the central partially heated left boundary in the presence of copper-water nanofluid is increased, for $Ra = 10^5$ at different volume fractions ($\phi$).

![Figure 14](image)

**Figure 14.** Variation of left local Nusselt number along the left boundary (Cu-water nanofluid) for different volume fractions ($\phi$) (case 3: Sinusoidal boundary condition) at $Ra = 10^5$.

For a better understanding of the dynamic and thermal physics of the flow inside the cavity in this case test, steady state isotherms, streamlines, and pressure are analyzed in Figure 15. Additionally, Figure 16 depicts the evolution of horizontal velocities, vertical velocities, and velocity magnitude in the enclosure filled with copper-water nanofluid and subjected to sinusoidal central partially heated left boundary thermal excitation.

It is very pertinent to scrutinize the dynamic and thermal behavior inside a given cavity for a local Nusselt number, isotherms, streamlines, pressure, horizontal velocities, vertical velocities, and velocity magnitude with a given parametric study.

Further to the aforementioned analysis, it is pertinent to investigate the variation of left local Nusselt number along the left boundary of the Cu-water nanofluid cavity for volume fraction $\phi = 0.1$ (Figure 17) and volume fraction $\phi = 0.6$ (Figure 18) in the third case (sinusoidal boundary condition) at $Ra = 10^5$. 
Figure 15. Steady state isotherm, streamline, and pressure profile behavior for central partially left boundary conditions (case 3: sinusoidal boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Figure 16. Steady state X-velocity, Y-velocity, and velocity magnitude profiles behavior for central partially left boundary conditions (case 3: sinusoidal boundary condition) at different volume fractions ($\phi$) for $Ra = 10^5$. 
Figure 17. Variation of local Nusselt number along the central partially heated left boundary (Cu-water nanofluid) for different left wall Bcs (boundary conditions) (case 1: Dirichlet constant BC, case 2: linear BC, case 3: sinusoidal boundary condition) at $Ra = 10^5$ at volume fraction $\phi = 0.1$.

Figure 18. Variation of local Nusselt number along the central partially heated left boundary (Cu-water nanofluid) for different left wall Bcs (boundary conditions) (case 1: Dirichlet constant boundary condition, case 2: linear boundary condition, case 3: Sinusoidal boundary condition) at $Ra = 10^5$ and volume fraction $\phi = 0.6$. 
As result, more heat enhancement is obtained with a maximum volume fraction value for a sinusoidal central partially heated left boundary thermal excitation.

5. Conclusions
The current work offered the numerical simulations of free convection heat transfer in a partially central heated left wall enclosure with a D\textsubscript{2}Q\textsuperscript{9} mesoscopic approach LBM. The remaining partial sides of the left wall are adiabatic. The cavity is filled with a copper water nanofluid with a cold right wall and adiabatic bottom and top wall. The present study is validated by extensive research and good validation is reached. It is shown that the left local Nusselt number can be enhanced by the partially sinusoidal thermal boundary condition, where its maximum becomes fluctuant comparing to Dirichlet and linear cases. Besides quantitative results through isotherms, streamlines, horizontal velocities, vertical velocities, velocity magnitude, and pressure profiles illustrate maximum values for the sinusoidal central partially heated left boundary. The focal conclusion that the rate of heat transfer is very enhanced when a sinusoidal excitation is applied on the sinusoidal central partially heated left side. A given numerical result will serve as future processing simulating in different widespread physical and engineering applications.

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