Diagonalization of replicated transfer matrices for disordered Ising spin systems

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Abstract. We present an alternative procedure for solving the eigenvalue problem of replicated transfer matrices describing disordered spin systems with (random) 1D nearest neighbor bonds and/or random fields, possibly in combination with (random) long range bonds. Our method is based on transforming the original eigenvalue problem for a $2^n \times 2^n$ matrix (where $n \to 0$) into an eigenvalue problem for integral operators. We first develop our formalism for the Ising chain with random bonds and fields, where we recover known results. We then apply our methods to models of spins which interact simultaneously via a one-dimensional ring and via more complex long-range connectivity structures, e.g. $1 + \infty$ dimensional neural networks and ‘small world’ magnets. Numerical simulations confirm our predictions satisfactorily.

PACS numbers: 75.10.Nr, 05.20.-y, 64.60.Cn

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1. Introduction

The replica formalism, see e.g. [1], has proven to be a very powerful tool in the study of both statics and dynamics of disordered systems. In statics the presence of frozen disorder in the Hamiltonian makes a direct equilibrium statistical mechanical analysis very difficult. Instead, one starts from the key assumption (supported by numerical, experimental and sometimes analytical evidence) that in the thermodynamic limit the free energy per degree of freedom in such systems is self averaging, i.e. identical to its disorder average for any given realization of the disorder, with probability one. This property allows one to focus on the evaluation of the disorder-averaged free energy per degree of freedom, which for a disordered system of $N$ interacting discrete spins $\sigma_i$ ($i = 1 \ldots N$) in equilibrium at inverse temperature $\beta = T^{-1}$ is calculated using the following identity:

$$\bar{f} = -\lim_{N \to \infty} \frac{1}{\beta N} \lim_{n \to 0} \frac{1}{n} \log Z^n = \sum_{\sigma} e^{-\beta H(\sigma)}$$

Here $\sigma = (\sigma_1, \ldots, \sigma_N)$, $H(\sigma)$ is the Hamiltonian, and \(\bar{\ldots}\) denotes an average over the disorder variables of the model under consideration. The replica method involves changing the order of the limits $n \to 0$ and $N \to \infty$, and subsequently writing the $n$-th
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moment of the partition function $Z$ in terms of $n$ copies (or replicas) of the original system. The disorder average then converts the original problem of $n$ independent but disordered systems into a new problem for $n$ coupled but disorder-free ones. In the limit $N \to \infty$ this new non-disordered problem can often be solved with conventional methods, e.g. saddle point integration. The limit $n \to 0$ has to be taken in the final result. This procedure has over the years been applied with great success to many families of mostly range-free or mean-field models.

Application of the replica formalism to finite dimensional spin models with disordered bonds and/or fields leads to the notion of replicated transfer matrices [2, 3]. For disordered one dimensional Ising spin chains, for instance, the replica method effectively replaces an expression for the free energy in terms of products of $2 \times 2$ random transfer matrices (see e.g. [4] for transfer matrix methods) by an expression for an $n$-replicated chain without disorder but with a more complicated $2^n \times 2^n$ transfer matrix which couples the $n$ replicas at each site of the chain. In the thermodynamic limit one first has to find the largest eigenvalue of this replicated transfer matrix, and subsequently find its analytic continuation for $n \to 0$. It was shown in [2] that for the one dimensional Ising model with random bonds and fields this procedure yields the results obtained earlier by other techniques, see e.g. [5, 6, 7]. Moreover, it was found that the smaller eigenvalues of the replicated transfer matrix contain information about disorder-averaged two-spin connected correlation functions.

In this paper we show how the replicated transfer matrix of disordered Ising models can be diagonalized, by using a particular form for the eigenvectors which transforms the original eigenvalue problem into an eigenvalue problem for integral operators. We believe our method to have a number of possible advantages. It appears more direct and explicit than existing approaches, it can be generalized in a straightforward manner to situations with RSB (which could for instance be induced by super-imposed long-range bonds), and it does not rely on the limit $n \to 0$ being taken (so that it can also be used for finite $n$ replica calculations describing models where the disorder is not truly frozen but evolving on very large time scales, in the sense of [10, 11, 12, 13]).

We first apply our ideas to Ising chains with random bonds and fields, where we can compare the results obtained with our method to those obtained earlier by others. Furthermore, mathematically one may express replicated transfer matrices of models which are not purely one-dimensional (due to super-imposed long range bonds) in terms of those corresponding to random field chains, with the statistics of the random fields mediating the mean-field effect of the long range bonds on a given site. We apply our equations to two examples of such models with one-dimensional and long-range bonds: the $1+\infty$ attractor neural networks of [15, 16], and the ‘small-world’ ferromagnet of [17], and show how one can use our methods to calculate various thermodynamic quantities.
2. Definitions

In this paper we will deal with disordered Ising spin systems in thermal equilibrium, of size $N$ and with microscopic states written as $\mathbf{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$. More specifically, we will analyze the following three models, by diagonalizing the replicated transfer matrices which they generate: the disordered Ising chain (DIC) as in [3,6], the $(1+\infty)$–dimensional attractor neural network (ANN) as in [13,16] and the ‘small world’ ferromagnet (SWM) of [17], which are defined by the Hamiltonians

\[ H_{\text{DIC}}(\mathbf{\sigma}) = -\sum_i J_i \sigma_i \sigma_{i+1} - \sum_i \theta_i \sigma_i \]  
\[ H_{\text{ANN}}(\mathbf{\sigma}) = -J_s \sum_i \sigma_i \sigma_{i+1} (\mathbf{\xi}_i \cdot \mathbf{\xi}_{i+1}) - \frac{J_N}{N} \sum_i \sigma_i \sigma_j (\mathbf{\xi}_i \cdot \mathbf{\xi}_j) \]  
\[ H_{\text{SWM}}(\mathbf{\sigma}) = -J_0 \sum_i \sigma_i \sigma_{i+1} - \frac{J}{c} \sum_{i<j} \mathbf{c}_{ij} \sigma_i \sigma_j \]  

In (2) we have a 1D spin chain with independently identically distributed random bonds and fields $\{J_i, \theta_i\}$ at each site, drawn from some joint distribution $p(J, \theta)$. We will abbreviate $\int \text{d}J \text{d}\theta$ $p(J, \theta)f(\mathbf{\theta}) = \langle f(J, \mathbf{\theta}) \rangle_{J, \theta}$. In (3) we have both 1D and long-range random bonds, but their values are not independent. The short- and long-range bonds take the values $J_s(\mathbf{\xi}_i \cdot \mathbf{\xi}_{i+1})$ and $J_N^{-1}(\mathbf{\xi}_i \cdot \mathbf{\xi}_j)$, respectively, where the binary vectors $\mathbf{\xi}_i = (\xi_1^i, \ldots, \xi_p^i)$ represent stored data and are drawn randomly and independently from $\{-1, 1\}^p$ (with uniform probabilities). Finally, in (4) we have uniform 1D ferromagnetic bonds of strength $J_0$, and the randomness is solely in the realization of the long range bonds. The latter are also ferromagnetic, of strength $J/c$ if present, but constitute a finitely connected Poissonian graph defined by dilution variables $c_{ij}$ which for each pair $(i,j)$ are drawn independently from $p(c_{ij}) = \frac{\xi_i}{N} \delta_{c_{ij},1} + (1 - \frac{\xi_i}{N}) \delta_{c_{ij},0}$. The average connectivity $c$ will remain finite in the thermodynamic limit. In all three cases (2,3,4) the 1D short-range interactions are defined periodically.

At this stage, let us briefly recall from [2][3] how a replicated transfer matrix emerges for the disordered Ising chain (2) upon applying the replica identity [11]. Here one finds, with $\alpha = 1 \ldots n$ and with the short-hand $\mathbf{\sigma}_i = (\sigma_1^i, \ldots, \sigma_n^i) \in \{-1, 1\}^n$,

\[ \overline{Z^n} = \sum_{\mathbf{\sigma}_1 \ldots \mathbf{\sigma}_N} \prod_i \left\langle e^{\beta J \sum_a \sigma_{i+1}^a \sigma_{i}^a + \beta \theta \sum_a \sigma_{i}^a} \right\rangle_{J, \theta} = \text{tr}(T^n_N) \]  

with a $2^n \times 2^n$ matrix $T_n$ whose entries are given by

\[ T_n(\mathbf{\sigma}, \mathbf{\sigma}') = \left\langle e^{\beta J \sum_a \sigma_{i}^a \sigma_{i}^a + \beta \theta \sum_a \sigma_{i}^a} \right\rangle_{J, \theta} \]  

One can thus find the disorder-averaged free energy per spin in the usual manner, via [11], by determining the largest eigenvalue of the replicated transfer matrix $T_n$ for integer $n$. The difficulty lies in the requirement to find an analytic expression for this eigenvalue for arbitrary integer $n$ (in contrast to non-disordered chains, where the dimension is fixed from the start and usually small, and where direct methods can therefore be employed such as calculating the characteristic polynomial of the matrix and finding its zeros).
We will first develop our diagonalization method for the simplest case, viz. the chain (2), and subsequently show that it can also serve to generate the solution of the other two models (3,4), which involve both short- and long range bonds, by writing the transfer matrices of the latter two models again in the form (6), but with suitably defined distributions $p(J, \theta)$ of local bonds and fields.

3. Construction and properties of eigenvectors

3.1. A detour: the Ising chain without disorder

Let us first turn to the simplest possible case: the 1D Ising chain with bonds $J$ and uniform fields $\theta$ (without disorder), where we just have the familiar transfer matrix

$$T(\sigma, \sigma'; \theta, J) = e^{\beta J \sigma \sigma' + \beta \theta \sigma}$$

Diagonalizing (7) is of course trivial [4]. Here, however, we seek a method which does not require knowledge of the characteristic polynomial of the matrix, so that it can be generalized to replicated transfer matrices with arbitrary $n$. To this end we introduce the two vectors $u_0[x], u_1[x, \mu] \in \mathbb{R}^2$, parametrized by $x, \mu \in \mathbb{R}$, and with components

$$u_0(\sigma; x) = e^{\beta x \sigma} \quad u_1(\sigma; x, \mu) = e^{\beta x \sigma} (\sigma - \mu)$$

Inserting the candidates (8) into the eigenvalue equation $\sum_\sigma T(\sigma, \sigma'; \theta, J) u(\sigma') = \lambda u(\sigma)$, and using the general identity $f(\sigma) = e^{\beta [B + A \sigma]}$ where $A = 1/2 \beta \log [f(1)/f(-1)]$ and $B = 1/2 \beta \log [f(1)f(-1)]$, for $\sigma \in \{-1, 1\}$, leads to the following eigenvalue equations:

$$e^{\beta B(J,x) + \beta [\theta + A(J,x)]} = \lambda_0 e^{\beta x \sigma}$$

$$e^{\beta B(J,x) + \beta [\theta + A(J,x)]} A'(J, x) \left( \sigma - \frac{\mu - B'(J, x)}{A'(J, x)} \right) = \lambda_1 e^{\beta x \sigma} (\sigma - \mu)$$

where

$$A(J, x) = \frac{1}{\beta} \arctanh[\tanh(\beta J) \tanh(\beta x)]$$

$$B(J, x) = \frac{1}{2 \beta} \log[4 \cosh(\beta(J + x)) \cosh(\beta(J - x))]$$

with partial derivatives $A'(J, x) = \partial_x A(J, x) = 1/2 [\tanh(\beta J + \beta x) + \tanh(\beta J - \beta x)]$ and $B'(J, x) = \partial_x B(J, x) = 1/2 [\tanh(\beta J + \beta x) - \tanh(\beta J - \beta x)]$, respectively. We conclude from (8) that if $x^*$ is the solution of the algebraic equation $x = \theta + A(J, x)$, then $u_0[x^*]$ is an eigenvector with eigenvalue $\lambda_0 = e^{\beta B(J,x^*)}$. This (unique) solution, which can be viewed as the stable fixed point of the iterative map $x_{i+1} = \theta + A(J, x_i)$, is given by

$$x^* = \frac{1}{2} (J + \theta) + \frac{1}{2 \beta} \log \left[ e^{\beta J \sinh(\beta \theta)} + \sqrt{e^{2\beta J} \sinh^2(\beta \theta) + e^{-2\beta J}} \right]$$

Inserting (13) into our expression for $\lambda_0$ then reproduces the familiar result for the largest eigenvalue of the transfer matrix of the Ising chain with uniform fields and bonds

$$\lambda_0 = e^{\beta B(J,x^*)} = e^{\beta J \cosh(\beta \theta)} + \sqrt{e^{2\beta J} \sinh^2(\beta \theta) + e^{-2\beta J}}$$
Similarly we see that if \( \mu^* = \frac{B(J,x^*)}{1-A(J,x^*)} \), with \( x^* \) as defined before, then also \( u_1[x^*, \mu^*] \) is an eigenvector with eigenvalue \( \lambda_1 = e^{\beta B(J,x^*)} A(J, x^*) \). Insertion of (13) leads to the familiar expression for the second eigenvalue of (7):
\[
\lambda_1 = e^{\beta J} \cosh(\beta \theta) - \sqrt{e^{2\beta J} \sinh^2(\beta \theta) + e^{-2\beta J}}
\] (15)

It turns out that \( \mu^* \) gives the average magnetization at each site:
\[
\mu^* = \frac{\tanh(\beta x^*)[1 + \tanh(\beta J)]}{1 + \tanh(\beta J) \tanh^2(\beta x^*)} = \frac{\sinh(\beta \theta)}{\sqrt{\sinh^2(\beta \theta) + e^{-4\beta J}}} = \langle \sigma \rangle
\] (16)

Note that our expression for (7) is not symmetric (although one could easily write the partition sum in terms of a symmetric transfer matrix), hence we have to distinguish between left and right eigenvectors; so far only right eigenvectors have been calculated.

We can find the left eigenvectors \( v \) via similar ansatz to (8):
\[
v_0(\sigma; y) = e^{\beta y \sigma} \quad v_1(\sigma; y, \nu) = e^{\beta y \sigma}(\sigma - \nu)
\] (17)

Insertion into the left eigenvalue equation \( \sum_{\sigma'} v(\sigma') T(\sigma', \sigma; \theta, J) = \lambda v(\sigma) \) then reveals that the two vectors \( v_0[y^*] \) and \( v_1[y^*, \nu^*] \) are left eigenvectors, where \( y^* \) is the solution of \( y^* = A(J, y^* + \theta) \) and \( \nu^* = \frac{B(J,y^*+\theta)}{1-A(J,y^*+\theta)} \). The associated eigenvalues are \( \lambda_0 = e^{\beta B(J,y^*+\theta)} \) and \( \lambda_1 = e^{\beta B(J,y^*+\theta)} A(J, y^* + \theta) \). The fixed point \( y^* \) of the map \( y_{i+1} = A(J, y_i + \theta) \) is again unique, and is given by:
\[
y^* = \frac{1}{2}(J - \theta) + \frac{1}{2\beta} \log \left[ e^{\beta J} \sinh(\beta \theta) + \sqrt{e^{2\beta J} \sinh^2(\beta \theta) + e^{-2\beta J}} \right]
\] (18)

Obviously \( x^* = y^* + \theta \), so left and right eigenvalues are identical and \( \nu^* = \mu^* = \langle \sigma \rangle \).

Furthermore, upon using the simple relation \( \tanh(\beta x^* + \beta y^*) = \langle \sigma \rangle \) it is clear that left and right eigenvectors corresponding to different eigenvalues are orthogonal:
\[
\sum_{\sigma} v_0(\sigma; y^*) u_1(\sigma; x^*, \mu^*) = 2 \cosh(\beta x^* + \beta y^*)[\tanh(\beta x^* + \beta y^*) - \mu^*] = 0
\]
\[
\sum_{\sigma} v_1(\sigma; y^*, \mu^*) u_0(\sigma; x^*) = 2 \cosh(\beta x^* + \beta y^*)[\tanh(\beta x^* + \beta y^*) - \nu^*] = 0
\]

Finally, to normalize our eigenvectors we require the constants
\[
D_0(x^*, y^*) = \sum_{\sigma} v_0(\sigma; y^*) u_0(\sigma; x^*) = 2 \cosh(\beta x^* + \beta y^*)
\] (19)
\[
D_1(x^*, y^*) = \sum_{\sigma} v_1(\sigma; y^*, \mu^*) u_1(\sigma; x^*, \mu^*) = 2 \cosh(\beta x^* + \beta y^*) \left[ 1 - (\mu^*)^2 \right]
\] (20)

3.2. Uncoupled replicated chains

As a intermediate step from the the diagonalization of (7) for the simple Ising chain to diagonalization of (6) for disordered chains, let us now inspect replicated transfer matrices with uncoupled replicas, viz. (6) but with \( \delta \)-distributed bonds and fields:
\[
T_n(\sigma, \sigma'; \theta, J) = e^{\beta J} \sum_\alpha \sigma_\alpha \sigma_\alpha' + \beta \theta \sum_\alpha \sigma_\alpha
\] (21)

without an average over \( \{ \theta, J \} \). This matrix is just the \( n \)-fold Kronecker product of (7), so its left- and right eigenvectors are simply (Kronecker) products of (17) and (8),
respectively. Each eigenvector is characterized by an index set \( \{ \rho \} \subseteq \{ 1, \ldots, n \} \) of size \( \rho \in \{ 0, \ldots, n \} \), indicating those indices \( \alpha \) for which we select \( u_1[x^+] \) as opposed to \( u_0[x^+] \) (and similarly for left eigenvectors), and with \( \{ 0 \} = \emptyset \). The left- and right eigenvectors of (21) can thus be written as

\[
v_{(\rho)}(\sigma; y^*, \mu^*) = \prod_{\alpha \in \{ \rho \}} v_1(\sigma_\alpha; y^*, \mu^*) \prod_{\alpha \notin \{ \rho \}} v_0(\sigma_\alpha; y^*) \tag{22}\]

\[
u_{(\rho)}(\sigma; x^*, \mu^*) = \prod_{\alpha \in \{ \rho \}} u_1(\sigma_\alpha; x^*, \mu^*) \prod_{\alpha \notin \{ \rho \}} u_0(\sigma_\alpha; x^*) \tag{23}\]

For each \( \rho \in \{ 0, \ldots, n \} \) there are \( \binom{n}{\rho} \) different index subsets, giving us the required total number of \( 2^n \) eigenvectors. The associated eigenvalues follow easily, since here all spin summations factorize over replicas:

\[
\sum_{\rho} T_n(\sigma, \sigma'; \theta, J) u_{(\rho)}(\sigma'; x^*, \mu^*) = \prod_{\alpha \in \{ \rho \}} \sum T(\sigma_\alpha, \sigma'_\alpha; \theta, J) u_0(\sigma'_\alpha; x^*) \prod_{\alpha \notin \{ \rho \}} T(\sigma_\alpha, \sigma'_\alpha; \theta, J) u_0(\sigma'_\alpha; x^*) = \lambda_1^n \lambda_0^{n-\rho} \prod_{\alpha \in \{ \rho \}} \sigma_\alpha \tag{24}\]

\[
\lambda_1^n \lambda_0^{n-\rho} \prod_{\alpha \in \{ \rho \}} \sigma_\alpha = \lambda_1^n \lambda_0^{n-\rho} U_{(\rho)}(\sigma; x^*, \mu^*) \tag{25}\]

Hence (21) has \( n + 1 \) different eigenvalues \( \lambda_\rho(n) = \lambda_1^n \lambda_0^{n-\rho} \), each with multiplicity \( \binom{n}{\rho} \).

Since \( \lambda_0 > \lambda_1 \), we also have the ordering relation \( \lambda_0(n) > \lambda_1(n) > \ldots > \lambda_n(n) \). We can furthermore see that right and left eigenvectors satisfy the orthogonality relations

\[
u_{(\rho)}[y^*, \mu^*] \cdot u_{(\rho')}[x^*, \mu^*] = D^*_{\rho}(x^*, y^*) \delta_{\rho \rho'} \prod_{k=1}^\rho \delta_{\alpha_k \alpha'_k} \tag{24}\]

\[
D^*_{\rho}(x^*, y^*) = 2 \cosh^n(\beta x^* + \beta y^*) \left[ 1 - (\mu^*)^2 \right]^\rho \tag{25}\]

where \( \{ \rho \} = \{ \alpha_1, \ldots, \alpha_\rho \} \) and \( \{ \rho' \} = \{ \alpha_1', \ldots, \alpha'_\rho \} \), and where the factor \( \prod_{k=1}^\rho \delta_{\alpha_k \alpha'_k} \) in (24) is defined as unity for \( \rho = 0 \).

### 3.3. Diagonalization for the disordered Ising chain

We now turn the real problem: the diagonalization of (6), which can also be written as

\[
T_n = \langle T_n[\theta, J] \rangle_{\theta, J}. \tag{26}\]

Clearly \( T_n \) shares many properties with \( T_n[\theta, J] \), e.g. invariance under all permutations \( \pi \) of the permutation group \( S_n \) acting on the indices \( \{ 1, \ldots, n \} : \)

\[
T_n(\pi(\sigma), \pi(\sigma')) = T_n(\sigma, \sigma') \quad \text{for every} \quad \pi \in S_n \tag{26}\]

It follows that if \( \mathbf{u} \) is an eigenvector of \( T_n \) with eigenvalue \( \lambda \), then so is \( D_{\pi} \mathbf{u} \) for any \( \pi \in S_n \) where \( D_{\pi} \) denotes the matrix representation of \( \pi \), i.e. \( D_{\pi}(\sigma, \sigma') = \delta_{\pi(\sigma), \sigma'} \).

In the uncoupled case (21) one observes that \( D_{\pi_1} u_{(\rho_1)} = u_{(\pi_1(\rho_1))} \) for every \( \pi \in S_n \); we will make the ansatz that this also holds for the eigenvectors of (26). The result is again a spectrum of \( n + 1 \) different eigenvalues \( \lambda_\rho(n) \) with \( \rho = 0, 1, \ldots, n \), with multiplicity \( \binom{n}{\rho} \) each. These statements reproduce the results in [2], which were derived using the irreducible representations of the replica permutation group. Here we are now
being led to the following general ansatz for the right- and left eigenvectors of $T_n$:

$$u_{(\rho)}(\sigma; P_{\rho}) = \int dx d\mu \, P_{\rho}(x, \mu|n) \, e^{\beta x \sum_{\alpha=1}^{n} \sigma_{\alpha}} \prod_{\alpha \in \{\rho\}} (\sigma_{\alpha} - \mu)$$  \hspace{1cm} (27)

$$v_{(\rho)}(\sigma; Q_{\rho}) = \int dy d\nu \, Q_{\rho}(y, \nu|n) \, e^{\beta y \sum_{\alpha=1}^{n} \sigma_{\alpha}} \prod_{\alpha \in \{\rho\}} (\sigma_{\alpha} - \nu)$$  \hspace{1cm} (28)

with $P_{\rho}$ and $Q_{\rho}$ denoting functions to be determined, by inserting (27) into the right eigenvalue equation $T_n u_{(\rho)}[P_{\rho}] = \lambda_{\rho}(n) u_{(\rho)}[P_{\rho}]$, and (28) into the left eigenvalue equation $v_{(\rho)}[Q_{\rho}] T_n = \lambda_{\rho}(n) v_{(\rho)}[Q_{\rho}]$, respectively. Working out the first equation gives, with the definitions (11, 12):

$$\sum_{\sigma'} T_n(\sigma, \sigma') u_{(\rho)}(\sigma'; P_{\rho}) = \int dx' d\mu' \, P_{\rho}(x', \mu'|n) \left\langle \prod_{\alpha \notin \{\rho\}} e^{\beta B(J,x') + \beta[\theta + A(J,x')]\sigma_{\alpha}} \right. \left. \times \prod_{\alpha \in \{\rho\}} e^{\beta B(J,x') + \beta[\theta + A(J,x')]\sigma_{\alpha}} A'(J, x') \left(\sigma_{\alpha} - \frac{\mu' - B'(J, x')}{{A'}'(J, x')}\right) \right\rangle_{J,\theta}$$

Upon inserting suitable integrals over $\delta-$functions, viz. $1 = \int dx \, \delta[x - \theta - A(J, x')]$ and $1 = \int d\mu \, \delta[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}]$, we then find our right eigenvalue equation taking the form

$$\int dx d\mu \left[ \int dx' d\mu' \, P_{\rho}(x', \mu'|n) \left\langle e^{n B(J,x')} [A'(J, x')]^{\rho} \delta[x - \theta - A(J, x')] \delta[\mu - \frac{\mu' - B'(J, x')}{{A'}'(J, x')}} \right\rangle_{J,\theta} \right]$$

$$\times \left\langle e^{\beta B \sum_{\alpha=1}^{n} \sigma_{\alpha}} \prod_{\alpha \notin \{\rho\}} (\sigma_{\alpha} - \mu) \right\rangle_{J,\theta} = \lambda_{\rho}(n) \int dx d\mu \, P_{\rho}(x, \mu|n) \left[ e^{\beta B \sum_{\alpha=1}^{n} \sigma_{\alpha}} \prod_{\alpha \notin \{\rho\}} (\sigma_{\alpha} - \mu) \right]$$

We conclude from this that the function $P_{\rho}$ must satisfy the following eigenvalue equation:

$$\int dx' d\mu' \, \Lambda^{(P)}_{\rho}(x, \mu, x', \mu'|n) \, P_{\rho}(x', \mu'|n) = \lambda_{\rho}(n) \, P_{\rho}(x, \mu|n)$$  \hspace{1cm} (29)

with the kernel

$$\Lambda^{(P)}_{\rho}(x, \mu, x', \mu'|n) = \left\langle e^{n B(J,x')} [A'(J, x')]^{\rho} \times \delta[x - \theta - A(J, x')] \delta[\mu - \frac{\mu' - B'(J, x')}{{A'}'(J, x')}} \right\rangle_{J,\theta}$$  \hspace{1cm} (30)

Upon repeating the above procedure also for the left eigenvectors (28) we find a similar eigenvalue problem for the functions $Q_{\rho}$, but now with a different kernel $\Lambda^{(Q)}_{\rho}$:

$$\int dy' d\nu' \, \Lambda^{(Q)}_{\rho}(y, \nu, y', \nu'|n) \, Q_{\rho}(y', \nu'|n) = \lambda_{\rho}(n) \, Q_{\rho}(y, \nu|n)$$  \hspace{1cm} (31)

$$\Lambda^{(Q)}_{\rho}(y, \nu, y', \nu'|n) = \left\langle e^{n B(J,y'+\theta)} [A'(J, y') + \theta]^{\rho} \times \delta[y - A(J, y' + \theta)] \delta[\nu - \frac{\nu' - B'(J, y' + \theta)}{{A'}'(J, y' + \theta)}] \right\rangle_{J,\theta}$$  \hspace{1cm} (32)

We have now transformed the problem of diagonalizing the $2^n \times 2^n$ replicated transfer matrix (3) into a problem involving integral operators (30, 32), where the limit $n \to 0$ can be taken.
We note that, at least for the purpose at finding the eigenvalues $\lambda_\rho(n)$, the two eigenvalue problems (29, 31) can be integrated over $\mu$ and $\nu$, respectively, and replaced by a simpler eigenvalue problem for the two single-argument functions $\Phi_\rho(x|n) = \int d\mu P_\rho(x, \mu|n)$ and $\Psi_\rho(y|n) = \int d\nu Q_\rho(y, \nu|n)$:

\[
\int dx' \Lambda_\rho^{(P)}(x, x'|n) \Phi_\rho(x'|n) = \lambda_\rho(n) \Phi_\rho(x|n)
\]

(33)

\[
\int dy' \Lambda_\rho^{(Q)}(y, y'|n) \Psi_\rho(y'|n) = \lambda_\rho(n) \Psi_\rho(y|n)
\]

(34)

with

\[
\Lambda_\rho^{(P)}(x, x'|n) = \langle e^{n\beta(J,x')} A'(J, x') \rangle_J \delta[x - \theta - A(J, x')]
\]

(35)

\[
\Lambda_\rho^{(Q)}(y, y'|n) = \langle e^{n\beta(J,y'+\theta)} A'(J, y' + \theta) \rangle_J \delta[y - A(J, y' + \theta)]
\]

(36)

Once we know the functions $P_\rho$ and $Q_\rho$, the form of the kernels (35, 36) enables us to integrate (29) and (31) over $x$ and $y$, and obtain relatively expressions for the corresponding eigenvalues:

\[
\lambda_\rho(n) = \frac{\int dx \Phi_\rho(x|n) \langle e^{n\beta(J,x)} A'(J, x) \rangle_J}{\int dx \Phi_\rho(x|n)}
\]

(37)

\[
\lambda_\rho(n) = \frac{\int dy \Psi_\rho(y|n) \langle e^{n\beta(J,y+\theta)} A'(J, y + \theta) \rangle_J}{\int dy \Psi_\rho(y|n)}
\]

(38)

Let us quickly inspect special cases. We see that for $\delta$–distributed fields and bonds the integral equations (29, 31) admit the expected solutions $P_\rho(x, \mu|n) = \delta(x - x^*) \delta(\mu - \mu^*)$ and $Q_\rho(y, \nu|n) = \delta(y - y^*) \delta(\nu - \mu^*)$, the eigenvectors (27, 28) reduce to the eigenvectors of (21), and the eigenvalues become $\lambda_\rho(n) = \lambda_0 n^{\rho}$, as they should. Also the special case of a chain without external fields, i.e. $p(J, \theta) = p(J) \delta(\theta)$, can be easily solved analytically. Here $A(J, 0) = B'(J, 0) = 0$ and $A'(J, 0) = \tanh(\beta J)$ for every $J$, which enables us to verify that (29, 31) have the trivial solutions $P_\rho(x, \mu|n) = \delta(x) \delta(\mu)$ and $Q_\rho(y, \nu|n) = \delta(y) \delta(\nu)$. Hence the eigenvectors become

\[
u_{(\rho)}(\sigma) = \sigma_{\rho} = \prod_{\alpha \in \{\rho\}} \sigma_{\alpha}
\]

They satisfy $v_{(\rho)}[Q] \cdot u_{(\rho)}[P] = 2^n \delta_{\rho \rho'} \prod_{k=1}^{\rho} \delta_{\alpha_k \alpha'_k}$. These eigenvectors are in fact common to all matrices of the form $T'(\sigma, \sigma') = T(\sigma \cdot \sigma')$ (34), and our replicated transfer matrix falls in this category when the external fields are zero. The eigenvalues are given by $\lambda_\rho(n) = \langle [2 \cosh(\beta J)]^n \tanh(\beta J) \rangle_J$, and it is clear that the largest corresponds to $\rho = 0$.

### 3.4. Properties of the kernel eigenvalue problems for $n \to 0$

Let us consider in more detail the $n \to 0$ limits of the eigenvalue problems (30) and (32). We first turn to $\rho = 0$. The eigenvectors corresponding to eigenvalue $\lambda_0(0)$ do not depend on $\{\nu, \mu\}$, so upon writing simply $\Phi_0(x|0) = \Phi(x)$ and $\Psi_0(y|0) = \Psi(y)$ we obtain for $\rho = 0$:

\[
\int dx' \Phi(x') \langle \delta[x - \theta - A(J, x')] \rangle_J = \lambda_0(0) \Phi(x)
\]

(39)
If we assume that \( \Phi(x) \neq 0 \) and \( \Psi(y) \neq 0 \), then integration of (39,40) over \( x \) and \( y \), respectively, gives us in both equations \( \lambda_0(0) = 1 \). This, in turn, implies that \( \Phi(x) \) and \( \Psi(y) \) are the stationary distributions of the two random maps

\[
x_{i+1} = \theta_i + A(J, x_i) \quad y_{i+1} = A(J, y_i + \theta_i)
\]

These maps describe the propagation of the fields \( x \) and \( y \) along the chain. The two distributions are connected via the following equations,

\[
\Phi(x) = \int \mathrm{d}y' \Psi(y') \langle \delta[y - A(J, y' + \theta)] \rangle_{J, \theta} = \lambda_0(0) \Psi(y) \quad \Psi(y) = \int \mathrm{d}x' \Phi(x') \langle \delta[x - A(J, x')] \rangle_{J, \theta}
\]

(40)

which can be verified upon substituting into (39,40), using \( \lambda_0(0) = 1 \).

The case \( \rho > 0 \) is more complicated. Here we find the \( n \to 0 \) eigenvalue problems

\[
\int \mathrm{d}x' \mathrm{d}\mu' \left[ [A'(J, x')]^\rho \delta[x - \theta - A(J, x')] \delta \left[ \mu - \frac{\mu' - B'(J, x')}{A'(J, x')} \right] \right]_{J, \theta} P_\rho(x', \mu'|0) = \lambda_\rho(0) P_\rho(x, \mu|0)
\]

(42)

\[
\int \mathrm{d}y' \mathrm{d}\nu' \left[ [A'(J, y' + \theta)]^\rho \delta[y - A(J, y' + \theta)] \delta \left[ \nu - \frac{\nu' - B'(J, y' + \theta)}{A'(J, y' + \theta)} \right] \right]_{J, \theta} Q_\rho(y', \nu'|0) = \lambda_\rho(0) Q_\rho(y, \nu|0)
\]

(43)

As for \( \rho = 0 \) we can show that these equations admit solutions \( P_\rho \) and \( Q_\rho \) which can be interpreted as probability densities. The difference with \( \rho = 0 \), where these distributions are the stationary measures of the random maps of the propagated fields \( \{x, y\} \), is that here the quantities which are propagated are the distributions themselves, via deterministic but nonlinear functional maps:

\[ P_{\rho,i+1} = A_{P,\rho}(P_{\rho,i}) \quad Q_{\rho,i+1} = A_{Q,\rho}(Q_{\rho,i}) \]

where

\[ [A_{P,\rho}(P)](x, \mu) = \int \mathrm{d}x' \mathrm{d}\mu' \left[ \frac{P(x', \mu') [A'(J, x')]^\rho}{\int \mathrm{d}x'' \mathrm{d}\mu'' P(x'', \mu'') [A'(J', x'')]^\rho} \right]_{J', \theta} \times \delta[x - \theta - A(J, x')] \delta[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}] \right]_{J, \theta} \]

(44)

\[ [A_{Q,\rho}(Q)](y, \nu) = \int \mathrm{d}y' \mathrm{d}\nu' \left[ \frac{Q(y', \nu') [A'(J, y' + \theta)]^\rho}{\int \mathrm{d}y'' \mathrm{d}\nu'' Q(y'', \nu'') [A'(J', y'' + \theta')]^\rho} \right]_{J', \theta} \times \delta[y - A(J, y' + \theta)] \delta[\nu - \frac{\nu' - B'(J, y' + \theta)}{A'(J, y' + \theta)}] \right]_{J, \theta} \]

(45)

We see that the defining properties of a probability density, viz. non-negativity and normalization, are preserved by both functional maps. Hence we may indeed view the eigenvalue problems (42,43) as the fixed point equations of the functional maps (44,45).

The eigenvalues are:

\[ \lambda_\rho(0) = \int \mathrm{d}x \Phi_\rho(x|0) [A'(J, x)]^\rho \right]_{J} = \int \mathrm{d}y \Psi_\rho(y|0) [A'(J, y + \theta)]^\rho \right]_{\theta, J} \]

(46)

where \( \Phi_\rho \) and \( \Psi_\rho \) are as before the marginals of \( P_\rho \) and \( Q_\rho \). Moreover, using the property \( A(J, x) < 1 \) for every \( J, x \) we obtain \( \lambda_\rho(0) < \lambda_0(0) = 1 \) for every \( \rho > 1 \). We may also
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generalize equations (41) which give the relation between the solutions of the two \( \rho = 0 \) eigenvalue problems. It straightforward to check by substitution into (42, 43) that for \( \rho > 1 \) we have:

\[
P_\rho(x, \mu|x0) = \int dyν Q_\rho(y, ν|0) (δ(x − θ − y))_0 δ(μ − ν) \tag{47}
\]

\[
Q_\rho(y, ν|0) = \frac{\int dxμ P_\rho(x, μ|x0)([A'(J, x)]^ρ δ[y − A(J, x)]δ[ν − \frac{μ − B'(J, x)}{A(J, x)}])_J}{\int dx Φ_\rho(x|0)([A'(J, x)]^ρ)_J} \tag{48}
\]

3.5. Spectral decompositions

Standard linear algebra guarantees that left- and right eigenvectors corresponding to different eigenvectors are orthogonal. This, given that our eigenvalues \( λ_\rho(n) \) depend only on the size \( \rho \) of the index sets we know that

\[
ρ \neq ρ' : \sum_\sigma u_{\{ρ\}}(\sigma; P_\rho) v_{\{ρ'\}}(\sigma; Q_\rho) = 0 \tag{49}
\]

It follows that we may always use the decomposition

\[
T_n(\sigma, \sigma') = \sum_{ρ=0}^n λ_ρ(n)U_n^{(ρ)}(\sigma, \sigma') \tag{50}
\]

in which the matrices \( U_n^{(ρ)} \) are projection matrices, each formed of linear combinations of \( λ_ρ(n) \)-eigenvectors and each acting only in one of the orthogonal eigenspaces. We note that also \( T_n^k = \sum_{ρ=0}^n λ_ρ(n)U_n^{(ρ)} \) for any integer \( k > 0 \), and that the trace of a projection operator reduces to the dimension of the space which it projects, i.e. \( \text{tr}(U_n^{(ρ)}) = \binom{n}{ρ} \). Since the dimensions of both the \( λ_0(n) \) and the \( λ_n(n) \) eigenspaces are one, the corresponding eigenvectors are pairwise orthogonal and orthogonal to all other eigenvectors, and therefore

\[
U_n^{(0)}(\sigma, \sigma') = \frac{u_{\{0\}}(\sigma) v_{\{0\}}(\sigma')}{D_0(n)} \quad U_n^{(n)}(\sigma, \sigma') = \frac{u_{\{n\}}(\sigma) v_{\{n\}}(\sigma')}{D_n(n)} \tag{51}
\]

with

\[
D_ρ(n) = \sum_\sigma v_{\{ρ\}}(\sigma) u_{\{ρ\}}(\sigma) \nonumber
\]

\[
= \int dxμ P_ρ(x, μ|n) \int dyν Q_ρ(y, ν|n) \nonumber
\]

\[
\times [2 \cosh(βx + βy)]^n [1 + μν − \tanh(βx + βy)[μ + ν]]^{ρ} \tag{52}
\]

We note that \( \lim_{n→0} D_0(n) = 1 \). Expression (50) will prove useful in calculating observables such as magnetizations and correlation functions. If also within each eigenspace characterized by an index set size \( 1 ≤ ρ ≤ n − 1 \) the eigenvectors would be orthogonal (as in chains without disorder, or in the random bond chain without external fields), then we would have \( U_n^{(ρ)}(\sigma, \sigma') = \sum_{\{ρ\}} u_{\{ρ\}}(\sigma) v_{\{ρ\}}(\sigma')/D_ρ(n) \) for all \( ρ \), and hence

\[
T_n(\sigma, \sigma') = \sum_{ρ=0}^n λ_ρ(n) \frac{u_{\{ρ\}}(\sigma) v_{\{ρ\}}(\sigma')}{D_ρ(n)} \tag{53}
\]
4. Applications of the theory: the random field Ising model

As a benchmark test, let us first calculate the free energy and various observables for the random field Ising chain (2) with nearest neighbour bonds of strength $J_0$.

4.1. The free energy

We recall that the free energy is given by:

$$\bar{f} = -\lim_{n \to 0} \frac{1}{n} \lim_{N \to \infty} \frac{1}{\beta N} \log \text{tr}(T_n^N),$$

where $T_n$ is the replicated transfer matrix (6). Assuming that the largest eigenvalue is $\lambda_0(n)$, we may write the trace as:

$$\text{tr}(T_n^N) = \sum_{\rho=0}^{\infty} [\lambda_\rho(n)]^N \text{tr}(U_\rho^N) = \lambda_0^N(n) \left[ 1 + \sum_{\rho=1}^{\infty} \left( \frac{\lambda_\rho(n)}{\lambda_0(n)} \right)^N \right]$$

(54)

Since $\lim_{N \to \infty} (\lambda_\rho(n)/\lambda_0(n))^N = 0$, only the contribution of the largest eigenvalue survives, so that, upon writing $\lambda_0(n) = 1 + \lambda n + O(n^2)$ (for we had already established that $\lambda_0(0) = 1$):

$$\bar{f} = -\frac{1}{\beta} \lim_{n \to 0} \frac{1}{n} \log \lambda_0(n) = -\frac{1}{\beta} \lim_{n \to 0} \frac{1}{n} \log [1 + n\lambda + O(n^2)] = -\frac{\lambda}{\beta}$$

(55)

The $O(n)$ contribution $\lambda$ to $\lambda_0(n)$ can be found upon expanding (57) for small $n$, and is found to be $\lambda = \beta \int dx \Phi(x) B(J_0, x)$. Insertion into (55) gives us

$$\bar{f} = -\frac{1}{2\beta} \int dx \Phi(x) \log 4 \cosh(\beta(J_0 + x)) \cosh(\beta(J_0 - x))$$

(56)

This expression can be converted into a form more familiar from the one dimensional random systems literature [2, 5, 6]. If we define a new random variable $\tilde{x}$ and an associated density $\tilde{\Phi}(\tilde{x})$ via $\tilde{\Phi}(\tilde{x}) = \int dx \Phi(x) \delta[\tilde{x} - e^{2\beta x}]$, we find after some straightforward manipulations that

$$\bar{f} = \langle \theta \rangle_\theta - \frac{1}{\beta} \int d\tilde{x} \tilde{\Phi}(\tilde{x}) \log [e^{\beta J_0} + \tilde{x} e^{-\beta J_0}]$$

(57)

where

$$\tilde{\Phi}(\tilde{x}) = \int d\tilde{x}' \Phi(\tilde{x}') \left\langle \delta \left[ \tilde{x} - e^{2\beta \theta} e^{-\beta J_0} + \tilde{x}' e^{\beta J_0} \right] \right\rangle_\theta$$

(58)

The resulting (correct) expression (57) for the free energy justifies a posteriori our assumption that $\lambda_0(n)$ as generally the largest eigenvalue, and confirms that our ansatz for the associated right- and left eigenvectors, which are seen themselves to be replica symmetric (i.e. $u_\rho(\pi(\sigma)) = u_\rho(\sigma)$ and $v_\rho(\pi(\sigma)) = v_\rho(\sigma)$ for every permutation $\pi \in S_n$), was correct.

4.2. Single site expectation values and their powers

Let us next show how single-site observables of the form $\langle \sigma_i \rangle^\rho$ (integer $\rho$), with brackets denoting a thermal average over the Boltzman measure and $\cdots$ denoting averaging over
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the disorder, can also be calculated. We use the following replica identity

\[
\langle \sigma_i \rangle^\rho = \lim_{n \to 0} \frac{\sum_{\sigma} \sigma_i e^{-\beta H(\sigma)}}{\sum_{\sigma} e^{-\beta H(\sigma)}} \]

\[
= \lim_{n \to 0} \sum_{\{\sigma\}} \sigma_i^{\alpha_1} \ldots \sigma_i^{\alpha_n} \prod_{\alpha=1}^{n} e^{-\beta H(\sigma^\alpha)}
\]

(59)

and define the diagonal \(2^n \times 2^n\) matrix \(S_{\{\rho\}}\) with entries

\[
S_{\{\rho\}}(\sigma, \sigma') = \delta_{\sigma \sigma'} \prod_{\alpha \in \{\rho\}} \sigma_{\alpha}
\]

(60)

Upon using the replicated transfer matrix (6) to evaluate (59), and upon dividing (59) by \(1 = \lim_{n \to 0} Z^n = \lim_{n \to 0} \text{tr}(T^N)\), expression (59) can be written in the form

\[
\langle \sigma_i \rangle^\rho = \lim_{n \to 0} \lim_{N \to \infty} \text{tr}(S_{\{\rho\}} T^N)
\]

For large \(N\) our spectral decomposition (50) now gives us

\[
\langle \sigma_i \rangle^\rho = \lim_{n \to 0} \lim_{N \to \infty} \frac{\text{tr}(S_{\{\rho\}} U_n^{(0)}) + \sum_{\rho'=1}^{n} [\lambda_{\rho'}(n)/\lambda_0(n)]^N \text{tr}(S_{\{\rho\}} U_n^{(\rho'}})}{1 + \sum_{\rho'=1}^{n} [\lambda_{\rho'}(n)/\lambda_0(n)]^N}
\]

\[
= \lim_{n \to 0} \text{tr}(S_{\{\rho\}} U_n^{(0)})
\]

\[
= \lim_{n \to 0} D_0^{-1}(n) \sum_{\sigma} \nu_0(\sigma) \Psi_0(\sigma) \prod_{\alpha \in \{\rho\}} \sigma_{\alpha}
\]

\[
= \lim_{n \to 0} \int dx dy \Phi_0(x|n) \Psi_0(y|n) [2 \cosh(\beta x + \beta y)]^n \tanh(\beta x + \beta y)
\]

(61)

We note that the dependence on the particular realization of the index set \(\{\rho\}\) has disappeared, as it should, leaving only a dependence on the size \(\rho\) of this set. We may now take the limit \(n \to 0\), and find our transparent and appealing final result

\[
\langle \sigma_i \rangle^\rho = \int dx dy \Phi(x) \Psi(y) \tanh(\beta x + \beta y)
\]

(62)

4.3. Multiple-site observables

Finally we apply our methods to the evaluation of disorder-averaged powers of two-spin correlations, of the form \(\langle \sigma_i \sigma_j \rangle^\rho\) with integer \(\rho\). We choose \(j > i\) and start from the identity

\[
\langle \sigma_i \sigma_j \rangle^\rho = \lim_{n \to 0} \left[ \sum_{\sigma} \sigma_i \sigma_j e^{-\beta H(\sigma)} \right]^\rho \left[ \sum_{\sigma} e^{-\beta H(\sigma)} \right]^{n-\rho}
\]

\[
= \lim_{n \to 0} \sum_{\{\sigma\}} \sigma_i^{\alpha_1} \sigma_j^{\alpha_2} \ldots \sigma_i^{\alpha_n} \sigma_j^{\alpha_n} \prod_{\alpha=1}^{n} e^{-\beta H(\sigma^\alpha)}
\]

\[
= \lim_{n \to 0} \frac{\text{tr}(S_{\{\rho\}} T^{j-i} S_{\{\rho\}} T^{N-j+i})}{\text{tr}(T^N)}
\]

(63)
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Our spectral decomposition (50), together with \( \lambda_0(n) = 1 \), enables us to write for \( N \to \infty \):

\[
\langle \sigma, \sigma' \rangle^p = \lim_{n \to 0} \lim_{N \to \infty} \sum_{\rho'' = 0}^{n} \frac{\lambda_{\rho''}(n)}{\lambda_0(n)} \left[ \frac{\lambda_{\rho''}(n)}{\lambda_0(n)} \right]^{j-i} \left[ \frac{\lambda_{\rho''}(n)}{\lambda_0(n)} \right]^{N-j+i} \text{tr}(S_{\rho}(n) U_n^{(\rho'' \rho)} S_{\rho}(n) U_n^{(\rho'' \rho)}) \left( 1 + \sum_{\rho' = 1}^{n} \left[ \frac{\lambda_{\rho'}(n)}{\lambda_0(n)} \right]^N \right) = \lim_{n \to 0} \sum_{\rho' = 0}^{n} \lambda_{\rho'}(0)^{j-i} \text{tr}(S_{\rho}(n) U_n^{(\rho' \rho)} S_{\rho}(n) U_n^{(0 \rho)})(64)
\]

To work out the trace in (64) we write the entries of our projection matrices as follows:

\[
U_n^{(\rho')} (\sigma, \sigma') = \sum_{\{\varsigma\}, \{\varsigma'\}} V_n^{(\rho')} \left[ u_{\{\varsigma\}} (\sigma) v_{\{\varsigma'\}} (\sigma') \right] (65)
\]

We may now write

\[
\text{tr}(S_{\rho}(n) U_n^{(\rho')} S_{\rho}(n) U_n^{(0 \rho)}) = \sum_{\sigma, \sigma'} \left[ v_{\{\varsigma\}} (\sigma) \prod_{\alpha \in \{\rho\}} \sigma_\alpha \right] U_n^{(\rho')} (\sigma, \sigma') \left[ u_{\{\varsigma'\}} (\sigma') \prod_{\alpha \in \{\rho\}} \sigma_\alpha' \right] = \sum_{\{\varsigma\}, \{\varsigma'\}} V_n^{(\rho')} A_{\{\rho\}}^{(\{\varsigma\}, \{\varsigma'\})} A_{\{\rho\}}^{(\{\varsigma'\}, \{\varsigma\})} (66)
\]

with

\[
A_{\{\rho\}}^{(\{\varsigma\}, \{\varsigma'\})} = \sum_{\sigma} v_{\{\varsigma\}} (\sigma) u_{\{\varsigma'\}} (\sigma) \prod_{\alpha \in \{\rho\}} \sigma_\alpha (67)
\]

\[
A_{\{\rho\}}^{(\{\varsigma'\}, \{\varsigma\})} = \sum_{\sigma} u_{\{\varsigma\}} (\sigma) v_{\{\varsigma'\}} (\sigma) \prod_{\alpha \in \{\rho\}} \sigma_\alpha (68)
\]

Our correlations (64) can apparently be written in the simplified form

\[
\langle \sigma, \sigma' \rangle^p = \lim_{n \to 0} \sum_{\rho' = 0}^{n} \lambda_{\rho'}(0)^{j-i} \sum_{\{\varsigma\}, \{\varsigma'\}} V_n^{(\rho')} A_{\{\rho\}}^{(\{\varsigma\}, \{\varsigma'\})} A_{\{\rho\}}^{(\{\varsigma'\}, \{\varsigma\})} (69)
\]

Inserting the eigenvectors (27, 28) into expressions (67, 68) for the coefficients \( A_{\{\rho\}}^{(\{\varsigma\}, \{\varsigma'\})} (n) \) and \( A_{\{\rho\}}^{(\{\varsigma'\}, \{\varsigma\})} (n) \), followed by summation summation over the spin variables, gives

\[
A_{\{\rho\}}^{(\{\varsigma\}, \{\varsigma'\})} (n) = \int dx d\mu \ P_{\rho'}(x, \mu | n) \int dy \ \Psi_0(y | n) \times [2 \cosh(\beta x + \beta y)]^n [1 - \mu \tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} \times [\tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} [\tanh(\beta x + \beta y) - \mu]^{(\rho) \cap \{\varsigma\}} \times \tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} [\tanh(\beta x + \beta y) - \mu]^{(\rho) \cap \{\varsigma\}} (70)
\]

\[
A_{\{\rho\}}^{(\{\varsigma'\}, \{\varsigma\})} (n) = \int dx \ \Phi_{\rho}(x | n) \int dy d\nu \ Q_{\rho'}(y, \nu | n) \times [2 \cosh(\beta x + \beta y)]^n [1 - \nu \tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} \times [\tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} [\tanh(\beta x + \beta y) - \nu]^{(\rho) \cap \{\varsigma\}} \times \tanh(\beta x + \beta y)]^{(\rho) \cap \{\varsigma\}} [\tanh(\beta x + \beta y) - \nu]^{(\rho) \cap \{\varsigma\}} (71)
\]

These quantities no longer depend on the detailed realizations of the index sets, but only on the sizes of these sets and of their intersections. Let us denote the number of
elements in the intersection of \(\{\rho\}\) and \(\{\varsigma\}\) by \(k = |\{\rho\} \cap \{\varsigma\}|, k = 0, \ldots, \min\{\rho, \rho'\}\) (since \(|\{\varsigma\}| = \rho'):

\[
|\{\rho\} \cap \{\varsigma\}| = k, \quad |\{\rho\} \cap \{\varsigma\}| = \rho - k, \quad |\{\rho\} \cap \{\varsigma\}| = \rho' - k
\]  

(72)

with similar definitions in the case of \(\{\varsigma'\}\), defining the variable \(k'\). We may now write (73) as

\[
\frac{\langle \sigma | \sigma \rangle^\rho}{n \to 0} = \lim_{n \to 0} \sum_{\rho' = 0}^{n} \lambda^{\rho'}_{\rho}(0)^{i - \min\{\rho, \rho'\}} \sum_{k, k' = 0}^{\min\{\rho, \rho'\}} A^{(\rho, \rho')}_{\rho, k} A^{(\rho', 0)}_{\rho, k'} \times \sum_{\{\varsigma\}, \{\varsigma'\}} V^{(\rho')}_{\{\varsigma\}, \{\varsigma'\}}
\]

(73)

in which \(A^{(\rho, \rho')}_{\rho, k}\) and \(A^{(\rho', 0)}_{\rho, k}\) denote the \(n \to 0\) limits of (70) and (71), respectively (with the conventions as laid down in (72):

\[
A^{(\rho, \rho')}_{\rho, k} = \int dx d\mu \: P_\rho(x, \mu|0) \int dy \: \Psi_0(y|0) \left[ \tanh(\beta x + \beta y) \right]^{\rho - k}
\]

\[
\times \left[ 1 - \mu \tanh(\beta x + \beta y) \right]^k \left[ \tanh(\beta x + \beta y) - \mu \right]^{\rho' - k}
\]

(74)

\[
A^{(\rho', 0)}_{\rho, k} = \int dx \: \Phi_0(x|0) \int dy d\nu \: Q_{\rho'}(y, \nu|0) \left[ \tanh(\beta x + \beta y) \right]^{\rho - k}
\]

\[
\times \left[ 1 - \nu \tanh(\beta x + \beta y) \right]^k \left[ \tanh(\beta x + \beta y) - \nu \right]^{\rho' - k}
\]

(75)

The rigorous evaluation of the last line in (73) for arbitrary models requires the explicit calculation of the expansion factors \(V^{(\rho')}_{\{\varsigma\}, \{\varsigma'\}}\). Although one can easily write formal expressions for these quantities in terms of the inverse of the matrix of inner products of the eigenvectors within a given eigenspace \(\rho'\), this leads as yet only to expressions in which it is not clear how the limit \(n \to 0\) can be taken.

We can at present only push the evaluation of (73) to its conclusion for those cases where the eigenvectors within each eigenspace are either explicitly orthogonal for any \(n\) (as in chains without disorder, or in the random bond chain without external fields), or become effectively orthogonal in the \(n \to 0\) limit. The latter is very hard to verify or disprove \(a\ priori\), but can serve as an efficient ansatz, to be verified later using numerical simulations. In these cases we are allowed to write simply \(V^{(\rho')}_{\{\varsigma\}, \{\varsigma'\}} = D^{-1}(n)\delta_{\{\varsigma\}, \{\varsigma'\}}\) and find (73) reducing to

\[
\frac{\langle \sigma | \sigma \rangle^\rho}{n \to 0} = \lim_{n \to 0} \left\{ \sum_{\varsigma = 0}^{\rho} \frac{\lambda^{\varsigma}_{\varsigma}(0)^{\varsigma - i}}{D^{\varsigma}_{\varsigma}(0)} \sum_{k = 0}^{\varsigma} \binom{\rho}{k} \binom{n - \rho}{\varsigma - k} A^{(\varsigma, 0)}_{\rho, k} A^{(\varsigma, 0)}_{\rho, k'} \right. \\
+ \sum_{\varsigma > \rho} \frac{\lambda^{\varsigma}_{\varsigma}(0)^{\varsigma - i}}{D^{\varsigma}_{\varsigma}(0)} \sum_{k = 0}^{\rho} \binom{\rho}{k} \binom{n - \rho}{\varsigma - k} A^{(\varsigma, 0)}_{\rho, k} A^{(\varsigma, 0)}_{\rho, k'} \right\}
\]

(76)

It turns out that in (76) only the terms with \(k = \varsigma\) will survive the limit \(n \to 0\). In the special case of non-disordered models, where \(P_\rho(x, \mu|n) = \delta(x - x^*)\delta(\mu - \mu^*)\) and \(Q_{\rho}(y, \nu|n) = \delta(y - y^*)\delta(\nu - \mu^*)\), with \(\mu^* = \tanh(\beta(x^* + y^*))\), we see that \(A^{(\varsigma, 0)}_{\rho, k}\) and \(A^{(\varsigma, 0)}_{\rho, k'}\) vanish unless \(k = \varsigma\). More generally we show in the Appendix that for integer \(\rho\) and \(\ell\):

\[
\rho \geq 1, \: \ell \geq 0 : \quad \lim_{n \to 0} \frac{n - \rho}{\ell} = \delta_{\ell, 0}
\]

(77)
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It follows that the second line of (76) must vanish entirely since there always \( k \leq \rho < \varsigma \), whereas in the first line we retain only the terms with \( k = \varsigma \), so that together with (52):

\[
\langle \sigma_i \sigma_j \rangle^\rho = \sum_{\varsigma = 0}^{\rho} D_{\varsigma}^{-1} \left( \begin{array}{c} \rho \\ \varsigma \end{array} \right) A_{\rho}^{(0, \varsigma)} A_{\varsigma}^{(\varsigma, 0)} \lambda_{\varsigma}(0)^{j-i}
\]

(78)

\[
A_{\rho}^{(0, \varsigma)} = \int dx \mu P_i(x, \mu|0) \int dy \Psi_0(y|0) [1 - \mu \tanh(\beta x + \beta y)]^\varsigma \tanh(\beta x + \beta y)]^{\rho-\varsigma}
\]

(79)

\[
A_{\rho}^{(\varsigma, 0)} = \int dx \Phi_0(x|0) \int dy \nu Q_i(y, \nu|0) [1 - \nu \tanh(\beta x + \beta y)]^\varsigma \tanh(\beta x + \beta y)]^{\rho-\varsigma}
\]

(80)

\[
D_{\rho} = \int dx \mu P_{\rho}(x, \mu|0) \int dy \nu Q_{\rho}(y, \nu|0) [1 + \mu \nu - \tanh(\beta x + \beta y)]^{\rho}
\]

(81)

This concludes our calculations for the random field Ising chain.

4.4. Comparison with simulations

We have tested the predictions (62, 78) for the random field Ising chain with \( p(\theta) = p\delta(\theta - \tilde{\theta}) + (1 - p)\delta(\theta + \tilde{\theta}) \). Objects such as \( \langle \sigma_i \rangle^2 \) or \( \langle \sigma_i \sigma_j \rangle^2 \) were measured by simulating two copies of the system, with identical disorder realizations but each evolving independently according to standard Glauber dynamics towards equilibrium following a randomly chosen microscopic initial state. The results are shown in figure 1. In all simulations the system size was \( N = 20,000 \) spins. We concentrated on the following quantities:

\[
m = \frac{1}{N} \sum_i \langle \sigma_i \rangle, \quad a_1 = \frac{1}{N} \sum_i \langle \sigma_i \sigma_{i+1} \rangle, \quad a_2 = \frac{1}{N} \sum_i \langle \sigma_i \sigma_{i+2} \rangle
\]

(82)

\[
q = \frac{1}{N} \sum_i \langle \sigma_i \rangle^2, \quad r = \frac{1}{N} \sum_i \langle \sigma_i \sigma_{i+1} \rangle^2
\]

(83)

The evaluation of the theoretical predictions (62, 78) involved solving the relevant functional eigenvalue equations numerically. For \( m \) and \( q \), which both follow from (62), one just needs to solve (39) for \( \lambda_0(0) = 1 \), which is straightforward (either by iteration, or using a population dynamics algorithm). The function \( \Psi(y) \) subsequently follows via identity (41). We see in figure 1 that for \( m \) and \( q \) the agreement between theory and experiment is excellent. Figure 2 shows the corresponding shapes of the distribution \( \Phi(x) \) as well as the associated integrated distribution and the distribution \( W(m) \) of single-site magnetizations, which show the by now familiar characteristics of random field Ising models (see e.g. [5, 6, 8, 9]).

For those observables which require evaluation of (78), and therefore numerical solution of the eigenvalue problems (42, 43) for different values of \( \rho \) (which is feasible but extremely demanding in computing time), we have used the approximation consisting of replacing \( P_{\rho}(\ldots) \) and \( Q_{\rho}(\ldots) \) for \( \rho > 0 \) by \( P_0(\ldots) \) and \( Q_0(\ldots) \), respectively. This would formally be allowed only in the non-disordered case (where also the assumed orthogonality of our eigenvectors within eigenspaces is correct), but is seen to give surprisingly accurate results even for those cases where the shape of these distributions is highly non-trivial; see figures 1 and 2.
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Figure 1. Relaxation of observables towards equilibrium at $T = 1$, in two random field Ising chains with identical disorder realizations, of size $N = 20,000$ and with field distribution $p(\theta) = p\delta(\theta - \theta_0) + (1 - p)\delta(\theta + \theta_0)$. Left column: evolution of the magnetization $m = N^{-1} \sum_i \sigma_i$ and the order parameter $q = N^{-1} \sum_i \sigma_i \sigma_i'$. Right column: evolution of the multiple site quantities $a_1 = N^{-1} \sum_i \sigma_i \sigma_{i+1}$, $a_2 = N^{-1} \sum_i \sigma_i \sigma_{i+2}$, and $r = N^{-1} \sum_i \sigma_i \sigma_{i+1} \sigma_i' \sigma_{i+1}'$. Different rows correspond to different control parameters. Top row: weak random fields, with $J_0 = 1$, $\theta_0 = 0.05$, $p = 0.7$, where the theoretical equilibrium predictions are $m \simeq 0.14$, $q \simeq 0.03$, $a_1 \simeq 0.76$, $a_2 \simeq 0.58$, $r \simeq 0.58$. Middle row: intermediate fields, with $J_0 = 0.5$, $\theta_0 = 0.2$, $p = 0.7$, where our theory predicts $m \simeq 0.20$, $q \simeq 0.08$, $a_1 \simeq 0.47$, $a_2 \simeq 0.22$, $r \simeq 0.21$. Bottom row: strong random fields, with $J_0 = 0.2$, $\theta_0 = 2$, $p = 0.5$, where the theory predicts the equilibrium values $m \simeq 0.006$, $q \simeq 0.91$, $a_1 \simeq 0.018$, $a_2 \simeq 0.0003$, $r = 0.84$. In all cases the predictions are indicated by markers at the right of the graphs.
Figure 2. Field distributions corresponding to the data of the previous figure as obtained by numerical solution of our integral eigenvalue equation via a population dynamics algorithm. The rows correspond to again to weak random fields (top row), intermediate random fields (middle row), and strong random fields (bottom row). Left column: The effective field distribution $\Phi(x)$. Middle column: the integrated distribution $\hat{\Phi}(x) = \int_{-\infty}^{x} dz \; \Phi(z)$. Right column: the distribution of single-site magnetizations $W(m) = \int dx dy \; \Phi(x) \Psi(y) \delta[m - \tanh(\beta x + \beta y)]$.

5. Applications of the theory: neural networks and ‘small world’ systems

The theory in section can be applied to any model which involves replicated transfer matrices. Here we demonstrate how it may be used to analyze models which are structurally different from the random field Ising model, in having not only short-range...
5.1. 1 + ∞ dimensional attractor neural networks

We now turn to the attractor neural network described by the Hamiltonian (83), where short range interactions compete with long-range ones. A detailed study of the model, based on the more conventional methods of [5, 6] can be found in [15, 16]; here our objective is only to demonstrate how the present replicated transfer matrix diagonalization formalism can also be put to use in the context of such models. Upon introducing the p overlap order parameters

\[ m_\mu(\sigma) = N^{-1} \sum_i \xi_\mu^i \sigma_i, \]

each of which measure the similarity between the system’s microscopic configuration \( \sigma \) and a given stored pattern, one arrives after some standard manipulations at the following expression for the partition function

\[ Z = \int \text{d}m \, e^{N\left[-\frac{1}{2} \beta J_\ell m^2 + r(m)\right]} \]

(84)

where \( m = (m_1, \ldots, m_p) \), \( m^2 = \sum_\mu m_\mu^2 \), and \( r(m) = \frac{1}{N} \log R(m) \) with

\[ R(m) = \sum_{\sigma_1, \ldots, \sigma_N} e^{\beta J_\ell \sum_i \sigma_i \sigma_{i+1}(\xi_i, \xi_{i+1}) + \beta J_s \sum_i \sigma_i(m \xi_i)} \]

(85)

One may now calculate \( r(m) \) by regarding the random patterns as disorder and use the replica approach to calculate the disorder average. In the thermodynamic limit, \( r(m) \) (which is itself mathematically identical to the free energy per spin of a suitably defined chain) must be identical to its disorder average, with probability one. Therefore we consider

\[ r(m) = \lim_{n \to 0} \frac{1}{n} \lim_{N \to \infty} \frac{1}{N} \log R^n(m) \]

In particular we have

\[ R^n(m) = 2^{-pN} \sum_{\xi_1, \ldots, \xi_n} \sum_{\sigma_1, \ldots, \sigma_N} \prod_i e^{\beta J_\ell (\sigma_i, \sigma_{i+1}(\xi_i, \xi_{i+1}) + \beta J_s (m \xi_i))} \sum_{\sigma_1, \ldots, \sigma_N} \]

where \( \sigma_i = (\sigma_i^1, \ldots, \sigma_i^N) \), and \( T(m) \) is a \( 2^{np} \times 2^{np} \) transfer matrix with entries

\[ T_{\xi, \xi'}(\sigma, \sigma'; m) = 2^{-p} e^{\beta J_\ell (\xi, \xi') + \beta J_s (m \xi_i)} \sum_{\sigma_1, \ldots, \sigma_N} \]

(86)

In order to determine the largest eigenvalue of this replicated transfer matrix we make the by now familiar type of ansatz for the its left and right eigenvector:

\[ v_\xi(\sigma) = \int \text{d}y \, \Psi_\xi(y|n) e^{\beta y \sum_{\sigma_1, \ldots, \sigma_N} \sigma_1} \quad u_\xi(\sigma) = \int \text{d}x \, \Phi_\xi(x|n) e^{\beta x \sum_{\sigma_1, \ldots, \sigma_N} \sigma_1} \]

(87)

Our motivation for this particular choice of the dependence on the pattern vectors \( \xi \) is that for \( p = 1 \) the dependence on the remaining pattern can be transformed away by the gauge transformation \( \sigma_i \to \xi_i \sigma_i \). This would leave a replicated transfer matrix of an Ising chain with constant bonds, where the role of the external field is played by \( J_s m \). Thus for \( p = 1 \) the present eigenvectors must reduce to those as studied in section 3.3 Secondly, the group (87) obviously represents only a subset of all eigenvectors (to
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be precise: the ρ = 0 family, in the language of the previous section). Building the full set is straightforward, but here we restrict ourselves for brevity to the main ones, i.e. those which control the free energy and the single-site observables (the others only play a role when calculating multiple-site observables).

Having introduced our eigenvectors, we proceed as in the random field Ising model, adding \( m \) as a conditioning label wherever needed. We then find in the limit \( n \to 0 \) that \( \lambda(0; m) = 1 \), that our final eigenvalue problems are defined in terms of joint field-pattern distributions:

\[
\Psi_\xi(y|0) = 2^{-p} \sum_\xi^\prime \int dy' \Psi_\xi(y'|0) \delta[y - A(J_s(\xi'; y', x) + J_t(m, \xi'))] \tag{88}
\]

\[
\Phi_\xi(x|0) = 2^{-p} \sum_\xi^\prime \int dx' \Phi_\xi(x'|0) \delta[x - J_e(m, \xi) - A(J_s(\xi'; \xi'), x')] \tag{89}
\]

These distributions are normalized according to \( \int dx \Phi_\xi(x|0) = \int dy \Psi_\xi(y|0) = 1 \) for all \( \xi \). The actual value to be inserted for the vector \( m \) in the above expressions is to be solved from the saddle-point equations which determine the stationary point of the extensive exponent in the partition sum. This equation can simply be written as \( \lambda(0; m) = 1 \), that our final eigenvalue problems are defined in terms of joint field-pattern distributions:

\[
\Psi_\xi(y|0) = 2^{-p} \sum_\xi^\prime \int dy' \Psi_\xi(y'|0) \delta[y - A(J_s(\xi'; y', x) + J_t(m, \xi'))] \tag{88}
\]

\[
\Phi_\xi(x|0) = 2^{-p} \sum_\xi^\prime \int dx' \Phi_\xi(x'|0) \delta[x - J_e(m, \xi) - A(J_s(\xi'; \xi'), x')] \tag{89}
\]

in which \( S^\mu_\{1\} \) is a diagonal \( 2^{np} \times 2^{np} \) matrix with elements:

\[
S^\mu_\{1\} \xi_\xi'(\sigma, \sigma') = \delta_\xi_\xi' \delta_{\sigma, \sigma'} \xi_{\mu} \sigma_1
\]

The \( n = 0 \) eigenvalue problems for \( \Phi_\xi \) and \( \Psi_\xi \) are coupled to the saddle point equations for the ‘mean field’ order parameters. This feature is typical, within the replica formalism, for all models where a one-dimensional structure is embedded in a mean-field (or range-free) architecture, as is the case here.

In order to calculate the free energy we need to know the \( \mathcal{O}(n) \) contribution \( \lambda(m) \) to \( \lambda(n; m) \) (i.e. \( \lambda(n; m) = 1 + n\lambda(m) + \mathcal{O}(n^2) \)). The latter can be expressed in terms of the \( n = 0 \) effective field distributions, and is found to be given by:

\[
\lambda(m) = 2^{-2p} \sum_\xi_\xi' \int dy \Psi_\xi(y|0) \beta B(J_s(\xi'; \xi'), y + J_t(m, \xi'))
\]

Hence

\[
r(m) = \lim_{n \to 0} \frac{1}{n} \lim_{N \to \infty} \frac{1}{N} \log \lambda^N(n; m)
\]

\[
= \lim_{n \to 0} \frac{1}{n} \log[1 + n\lambda(m) + \mathcal{O}(n^2)] = \lambda(m) \tag{91}
\]

Substitution of this result for \( r(m) \) into the partition leads to our final result

\[
f = \frac{1}{2} J_t m^2 - T \lambda(m) \tag{92}
\]
in which \(m\) is given by the solution of (90). The link with the results of [15] is can now be established upon defining a new random variable \(k\), which in [15] represents the ratio of conditioned partition functions, and is subject to a random non-linear map as one builds up the chain iteratively from \(N = 1\) to \(N = \infty\). With the following definition the two solutions (the one in [15] and the one in this paper) become fully identical:

\[
P(k, \xi) = 2^{-p} \int dy \Psi(\xi(y)) \delta[k - e^{-2\beta y}]
\]  

(93)

5.2. ‘Small-world’ ferromagnets

Our final application example is the so-called ‘small-world’ ferromagnet, defined by the Hamiltonian [11]. As in the previous example this model represents a combination of one-

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**Figure 3.** Relaxation of observables towards equilibrium at \(T = J = 1\), in two ‘small world’ ferromagnets with identical realizations of the disorder (i.e. the Poissonian graph), of size \(N = 20,000\). Left column: evolution of the magnetization \(m = N^{-1} \sum \sigma_i\) and the order parameter \(q = N^{-1} \sum \sigma_i \sigma'_i\). Right column: evolution of the multiple site quantities \(a_1 = N^{-1} \sum \sigma_i \sigma_{i+1}\), \(a_2 = N^{-1} \sum \sigma_i \sigma_{i+2}\), and \(r = N^{-1} \sum \sigma_i \sigma_{i+1} \sigma'_i \sigma'_{i+1}\). Different rows correspond to different control parameters. Top row: high Poissonian connectivity, viz. \(J_0 = 0.25\) and \(c = 4\), where the predicted equilibrium values are \(m \simeq 0.75\), \(q \simeq 0.58\), \(a_1 \simeq 0.62\), \(a_2 \simeq 0.57\), \(r \simeq 0.40\). Bottom row: low Poissonian connectivity, viz. \(J_0 = 1\) and \(c = 0.5\), where the theory predicts \(m \simeq 0.88\), \(q \simeq 0.80\), \(a_1 \simeq 0.85\), \(a_2 \simeq 0.81\), \(r \simeq 0.74\). In all cases the predictions are indicated by markers at the right of the graphs.
dimensional short-range interactions and long-range ones. In contrast to the previous example the long-range bonds are not ‘all-to-all’, but represent a finitely connected Poissonnian random graph. This model was studied in more detail in [17], where it was shown that application of the replica formalism generates the following replicated transfer matrix, with $\sigma, \sigma', \tau \in \{-1, 1\}^n$:

$$T(\sigma, \sigma'|P) = e^{\beta J_0 \sigma \cdot \sigma' + c \sum_{\tau} P(\tau) \exp \left[ \frac{\beta J_c}{c} \sigma \cdot \tau \right]}$$

(94)

Here the mean-field order parameter is a function $P(\tau)$, which gives the fraction of sites where the replicated spin $\sigma_i$ equals $\tau$. The saddle-point equations are here found to take the form of an expression for $P(\tau)$ in terms of those eigenvectors of $T$ which correspond to the largest eigenvalue:

$$P(\tau) = \frac{v_0(\tau)u_0(\tau)}{\sum_{\tau'} v_0(\tau')u_0(\tau')}$$

(95)

(assuming this eigenspace to be non-degenerated, similar to our previous models). In this model one expects a replica symmetric solution (RS) to describe the physics correctly,
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which for the order parameter $P(\tau)$ implies the form

$$P(\tau) = \int \! dh \; W(h) \frac{e^{\beta \sum_{\alpha=1}^{n} \tau_{\alpha}}}{2 \cosh(\beta h)^{n}}$$

(96)

Insertion of this RS expression into (94) results in the following replicated transfer matrix:

$$T_{RS}(\sigma, \sigma') = \int \! d\theta \; p(\theta|n) \; e^{\beta J_{0} \sigma \cdot \sigma' + \beta \sum_{\alpha} \sigma_{\alpha}}$$

(97)

$$p(\theta|n) = \sum_{k} \frac{e^{-c^{k}}}{k!} \int \left\{ \prod_{r=1}^{k} \frac{dh_{r} \; W(h_{r}) e^{n\beta B(J/c, h_{r})}}{[2 \cosh(\beta h_{r})]^{n}} \right\} \delta[\theta - \sum_{r} A(J/c, h_{r})]$$

(98)

Again we observe that our replicated transfer matrix may be viewed as equivalent to that of a one-dimensional chain with suitably chosen random fields. The associated ‘distribution’ of these fields represents the overall effect within the system of the sparse Poissonian long range bonds on a given site of the ring. We note that $p(\theta|n)$ is normalized only for $n = 0$.

Having identified the structure of our RS replicated transfer matrix, one may proceed to solve this model using the eigenvectors introduced in section 3.3. As in the Ising chain, this results in a transformation of the eigenvalue problem to integral equations, viz. (29,30) and (31,32), involving now the above field distribution $p(\theta|n)$. In addition the integral eigenvalue equations become coupled with the new distribution $W(h)$ in (96), which may be viewed as the fundamental ‘mean-field’ order parameter in this model. In the limit $n \to 0$ one finds that $W(h)$ is given by

$$W(h) = \int \! dx \! dy \; \Phi(x) \Psi(y) \delta(h - x - y)$$

(99)

In order to find also correlation functions in the present model we return to the previous derivation in section 4 and invoke the identity:

$$\langle \sigma_{i} \sigma_{j} \rangle^{\rho} = \lim_{n \to 0} \sum_{\{\sigma\}} \sigma_{i}^{\alpha_{1}} \sigma_{j}^{\alpha_{2}} ... \sigma_{i}^{\alpha_{n}} \sigma_{j}^{\alpha_{n}} \prod_{\alpha=1}^{n} e^{-\beta H(\sigma_{\alpha})}$$

We find, after some straightforward and by now standard manipulations (viz. averaging over the disorder, insertion of the relevant order parameters, and use of saddle point equations) that correlation functions can be again written in the form

$$\langle \sigma_{i} \sigma_{j} \rangle^{\rho} = \lim_{n \to 0} \frac{\text{tr}(S_{(\rho)} T_{j-i}[P] S_{(\rho)} T_{N-j+i}[P])}{\text{tr}(T_{N}[P])}$$

where $P$ is now given by expression (96). Since the steps which led us earlier for the random field Ising chain to (78) apply again, we may simply use (78) again to find also the correlation functions for the present model. The results of solving the relevant order parameter equations numerically (via population dynamics algorithms) are shown in figure 8, where we show the predicted equilibrium values for the scalar observables (82,83) together with the corresponding measurements in numerical simulations, for comparison. The corresponding effective field distributions are shown in figure 4. As with the random field Ising model, the order parameter functions required for the
calculation of $m$ and $q$ have been calculated using the exact equations, whereas those required for the multiple-site observables $\{a_1, a_2, r\}$ have been solved approximately. This is borne out by figure 3, which indeed shows excellent agreement between theory and simulations for $m$ and $q$ (left column), but deviations for the three quantities that have been calculated in approximation (right column).

6. Discussion

In this paper we have developed new tools for the diagonalization of replicated transfer matrices, which arise upon applying the replica method to disordered models with one-dimensional short-range bonds, possibly in combination with (random) long range ones. Our method was based on mapping the problem of diagonalizing $2^n \times 2^n$ matrices which are invariant under the replica permutation group onto the problem of diagonalizing appropriate $n$-dependent integral operators, in which the limit $n \to 0$ can be much more easily taken, via a suitable ansatz for the eigenvectors. The result, similar to that obtained earlier via more traditional methods, is an integral eigenvalue problem, which is exact in the relevant limits $N \to \infty$ and $n \to 0$, but which has to be solved numerically (using e.g. population dynamics). Given our explicit expressions for the eigenvectors, the route is open to the evaluation of the free energy and several families of disorder-averaged observables, including the magnetization and the spin-glass order parameter, but also multiple-site correlation functions. It should be emphasized, however, that to evaluate the latter types of objects we had to make two simplifying assumptions, for which the only basis as yet is their validity in simpler and thereby verifiable cases.

We have developed our theory in full detail for the random field Ising chain, and we showed subsequently how the solution of other more complicated models can be obtained from this, especially those where short-range bonds are combined with long-range ones and where one effectively ends up with a random field Ising problem embedded within a mean-field calculation. In particular we have worked out our equations and predictions for $1 + \infty$-dimensional recurrent neural networks, and for ‘small world’ ferromagnets.

Possible future applications of the approach presented in this paper would be to the analysis of two-dimensional disordered spin systems, or to models which require finite–$n$ replica calculations (e.g. those where the disorder is not truly frozen, but slowly and stochastically evolving in time), or to situations where one has RSB (broken replica symmetry) in $1 + \infty$ dimensional or ‘small-world’ spin systems. The latter two calculations would not seem to be easily carried out using the more conventional random field methods as in e.g. [5, 6], but would appear to be feasible extensions of the procedures presented here.

Acknowledgment

One of the authors (TN) acknowledges financial support from the State Scholarships Foundation (Greece)
Appendix A. Combinatorial terms in the $n \to 0$ limit

Here we prove identity (A.1). We note that the natural continuation of factorials to non-integer values is via the Gamma function \[ \Gamma(\nu) = \frac{\nu!}{\nu} \] for integer \( \nu \geq 1 \), integer \( \ell > 0 \) and real-valued \( n < 1 \) (so that always \( \ell > n - \rho + 1 \)) we may therefore write

\[
\begin{align*}
\binom{n-\rho}{\ell} &= \frac{1}{\ell!} \lim_{\epsilon \to 0} \frac{\int_{\ell}^{\infty} dx \ x^{n-\rho} e^{-x}}{\int_{\ell}^{\infty} dx \ x^{n-\rho-\ell} e^{-x}} \\
&= \frac{1}{\ell!} \lim_{\epsilon \to 0} \frac{\int_{\ell}^{1} dx \ x^{n-\rho} e^{-x} + O(\epsilon^0)}{\int_{\ell}^{1} dx \ x^{n-\rho-\ell} e^{-x} + O(\epsilon^0)} \\
&= \frac{1}{\ell!} \frac{1 - n - \rho + 1}{n - \rho + 1} \lim_{\epsilon \to 0} \frac{\epsilon^{n-\rho+1} + O(\epsilon^0)}{\epsilon^{n-\rho-\ell+1} + O(\epsilon^0)} \\
&= \frac{1}{\ell!} \frac{1 - n - \rho + 1}{n - \rho + 1} \lim_{\epsilon \to 0} \frac{\epsilon^{\ell} + O(\epsilon^{\ell+\rho-n-1})}{1 + O(\epsilon^{\ell+\rho-n-1})} = 0 \quad (A.1)
\end{align*}
\]

We are left only with the case \( \ell = 0 \), for which the above factorial terms would be equal to one. This proves (A.1).