A fast total variation regularization algorithm for 2D piecewise constant radially symmetric functions

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Abstract. In this paper, we considered total variation regularization (TV regularization) for 2D radially symmetric piecewise constant (RSPC) functions. We obtained a system of equations solving the direct variational problem with the subgradient method. Using the system, we propose a Condat’s type algorithm for computation of an extremal function.

1. Introduction

Total variation (TV) regularization techniques are one of the most known techniques for signal denoising [1-4]. TV algorithms are very attractive for image processing because they are shift-rotation-invariant and preserve well edges [5-14].

Let \( J(u) \) be the following functional:
\[
J(u) = \frac{1}{2} \|u - u_0\|^2 + \lambda \text{TV}(u),
\]
where \( \| \cdot \| \) is the \( L_2 \) norm and all functions belong to \( BV(\Omega) \), i.e. the class of functions of bounded variation on the set \( \Omega \). The expression \( \frac{1}{2} \|u - u_0\|^2 \) is called a fidelity term, while \( \lambda \text{TV}(u) \) is referred to as a regularization term. Here \( u_0 \) is the observed signal (noisy function) distorted by additive noise \( \eta \)
\[
u = u_0 + \eta.
\]

Let us consider the following variational problem:
\[
\argmin_{u \in BV(\Omega)} J(u)
\]
In (1) \( \lambda > 0 \) is the regularization parameter that determines the balance between goodness of fit to the measured data and the amount of regularization done to the measured data \( u_0 \) in order to produce \( u \). TV regularization looks for an approximation \( u \) to the noisy function \( u_0 \), which has minimal TV, but with no particular bias toward a sharp or smooth solution. The observed signal \( u_0 \) and the regularization parameter \( \lambda \) determine the sharpness or smoothness of the restored signal. Larger values of \( \lambda \) result in more regularization and less goodness of fit of \( u \) to the noisy signal \( u_0 \).
Davies and Kovac [15] considered the problem as non-parametric regression with emphasis on controlling the number of local extrema, and in particular introduced the taut string method. Condat [16] proposed a direct fast algorithm for searching the exact solutions to the one-dimensional TV regularization problem for discrete functions. The system of equations describing an extremal function was derived by solving the dual variational problem. In general, the complexity of the algorithm is \( O(N^2) \), where \( N \) is the number of discrete function samples; however, for a wide class of common functions the complexity can be reduced to \( O(N) \). The usage of the Condat’s method with the taut string method leads to a clear geometric description of the extremal function [17].

In this paper, we consider TV regularization problem for two-dimensional piecewise constant radially symmetric functions. We obtain a system of equations, describing an extremal function, solving the direct variational problem by the subgradient technique. With the help of the system, we propose a Condat’s type algorithm for constructions an extremal function. A clear geometrical interpretation of the extremal function with the taut string approach [15] is provided. In the paper, a new variant of the “tube” arising in the taut string approach (with varying size of the vertical cross-sections and “horizontal steps”) is introduced. The complexity of the proposed algorithm and the complexity of the original Condat’s algorithm are same.

Computer simulation illustrates well a geometrical relationship between extremal functions and the “tubes” for 2D radially symmetric piecewise constant case.

2. TV regularization as a finite-dimensional problem

Let \( u_0 \) from the problem (1)-(3) be a radially symmetric piecewise constant (RSPC) function. Then the variational problem (3) has a unique solution \( u_* \), and \( u_* \) also is a RSPC function [18-22]. Moreover, the functions \( u_* \) and \( u_0 \) have the same structure in terms of areas of constancy. This allows reducing the variational problem (3) to a finite-dimensional problem.

A RSPC function in \( \mathbb{R}^2 \) defines a set of rings on a disc \( \Omega \) with the radius \( R \). Let \( 0 = r_0 < r_1 < \cdots < r_n = R \) be radial partitions on the disc, and \( \Delta_i \) be a line segment (closed interval) \( [r_{i-1}; r_i] \), \( i = 1, \ldots, n \). The partition \( \delta = \{ \Delta_i \} \) consists of a set of such segments. The segment \( [r_{i-1}; r_i] \) defines the ring \( R_i \), \( i = 1, \ldots, n \) as a region enclosed by the two boundaries circles (with radii \( r_{i-1} \) and \( r_i \)); here the first ring \( R_1 \) is a disc with the radius \( r_1 \). A function \( u(r) \) is referred to as piecewise constant with respect to partition \( \delta \), if \( u(r) = u^i \), where \( r \in \Delta_i \), and \( u^i \) is a constant. Let us denote by \( PC(\delta) \) a class of piecewise constant functions with respect to partition \( \delta \).

Let a function \( u \) belong to \( PC(\delta) \). It means that \( u(x) = u^i \), \( x \in R_i \), \( i = 1, \ldots, n \), i.e. \( u = \{ u^1, \ldots, u^n \} \). Therefore, for a given collection of the radii \( r = \{ r^1, \ldots, r^n \} \), the functional space \( PC(\delta) \) can be identified with the space \( \mathbb{R}^n \). Let \( l_i \) be the perimeter of a circle with the radius \( r_i \), \( S_i \) be the area of the ring \( R_i \), \( i = 1, \ldots, n \). For functions from \( PC(\delta) \) the functional \( J(u) \) takes the following form [14]:

\[
J(u) = \frac{1}{2} \sum_{i=1}^{n} (u^i - u_0^i)^2 S_i + \lambda \sum_{i=1}^{n-1} |u^{i+1} - u^i| l_i, \tag{4}
\]

where

\[
S_i = \pi ((r^i)^2 - (r^{i-1})^2), \quad i = 1, \ldots, n, \quad r^0 = 0, \quad l_i = 2\pi r^i, \quad i = 1, \ldots, n-1.
\]

The variational problem (3) can be rewritten as

\[
J(u) = \frac{1}{2} \| u - u_0 \|^2 + \lambda TV(u) \to \min_{u \in PC(\delta)}, \tag{5}
\]

where

\[
\| u \|^2 = \frac{1}{2} \sum_{i=1}^{n} (u^i)^2 S_i \quad \text{and} \quad TV(u) = \sum_{i=1}^{n-1} |u^{i+1} - u^i| l_i.
\]

Since \( \| u - u_0 \|^2 \) is strictly convex, and the functional \( TV(u) \) is convex, there exists a unique solution of (5) [17]. For the extremal (minimum) function \( u_* \), the subgradient \( \nabla J(u_*) \) satisfies the following condition:

\[
0 \in \nabla J(u_*). \tag{6}
\]

Remark. Recall that the subgradient \( \nabla f(x) \) of the function \( f(x) = |x| \) is
2.1. Computation of the subgradient

Consider subgradient $\nabla f(x)$:

$$\nabla f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ [-1; 1], & \text{if } x = 0 \end{cases}$$

The subgradients $\frac{1}{2} \nabla (u^i - u_0^i)^2 S_i$, $i = 1, \ldots, n$, can be written as

$$\frac{1}{2} \nabla (u^i - u_0^i)^2 S_i = (0, \ldots, 0, (u^i - u_0^i) S_i, 0, \ldots, 0),$$

$$\sum_{i=1}^{n} \frac{1}{2} \nabla (u^i - u_0^i)^2 S_i = ((u^1 - u_0^1) S_1, (u^2 - u_0^2) S_2, \ldots, (u^n - u_0^n) S_n).$$

The subgradients $\nabla |u^{i+1} - u^i| l_i$, $i = 1, \ldots, n - 1$, can be written as

$$\nabla |u^2 - u^1| l_1 = \begin{cases} (-1, 1, 0, \ldots, 0) l_1, & \text{if } u^2 > u^1 \\ (1, -1, 0, \ldots, 0) l_1, & \text{if } u^2 < u^1 \end{cases}$$

$$\nabla |u^3 - u^2| l_2 = \begin{cases} ((\delta^1, -\delta^1, 0, \ldots, 0) l_1 | \delta^1 \in [-1; 1]), & \text{if } u^2 = u^1 \\ (0, -1, 0, \ldots, 0) l_2, & \text{if } u^3 > u^2 \end{cases}$$

$$\nabla |u^n - u^{n-1}| l_{n-1} = \begin{cases} (0, \ldots, 0, -1, 1) l_{n-1}, & \text{if } u^n > u^{n-1} \\ (0, \ldots, 0, 1, -1) l_{n-1}, & \text{if } u^n < u^{n-1} \end{cases}$$

From these expressions we get

$$\nabla |u^{i+1} - u^i| l_i = \{ (\delta^1 l_1, \delta^2 l_2, \delta^3 l_3, \ldots, \delta^{n-1} l_{n-1}, -\delta^{n-2} l_{n-2}, -\delta^{n-1} l_{n-1}) | \delta^i = -1, if \ u^{i+1} > u^i, \ \delta^i = 1, if \ u^{i+1} < u^i, \ \delta^i \in [-1; 1], if \ u^{i+1} = u^i, i = 1, \ldots, n - 1 \}.$$

The parameterization (7) of the subgradient leads to

$$\begin{cases} (\nabla f(u))^1 = (u^1 - u_0^1) S_1 + \lambda \delta^1 l_1 \\ (\nabla f(u))^2 = (u^2 - u_0^2) S_2 + \lambda \delta^2 l_2 - \lambda \delta^1 l_1 \\ (\nabla f(u))^3 = (u^3 - u_0^3) S_3 + \lambda \delta^3 l_3 - \lambda \delta^2 l_2 \end{cases}$$

$$\begin{cases} (\nabla f(u))^{n-1} = (u^n - u_0^n) S_n - 1 + \lambda \delta^{n-1} l_{n-1} - \lambda \delta^{n-2} l_{n-2} \\ (\nabla f(u))^{n} = (u^n - u_0^n) S_n + \lambda \delta^{n-1} l_{n-1} \end{cases}$$

where

$$\delta^i = \begin{cases} -1, & \text{if } u^{i+1} > u^i \\ 1, & \text{if } u^{i+1} < u^i \\ \in [-1; 1], & \text{if } u^{i+1} = u^i \end{cases}$$

Note that (6) means that $(\nabla f(u))^i = 0, i = 1, \ldots, n - 1$ for some values of the parameters $\delta^i$ satisfying (8). So, we get

$$\begin{cases} u_1^1 S_1 = u_0^1 S_1 - \lambda \delta^1 l_1 \\ u_2^1 S_2 = u_0^1 S_2 - \lambda \delta^2 l_2 + \lambda \delta^1 l_1 \\ u_3^1 S_3 = u_0^1 S_3 - \lambda \delta^3 l_3 + \lambda \delta^2 l_2 \end{cases}$$

$$\begin{cases} u_1^{n-1} S_{n-1} = u_0^{n-1} S_{n-1} - \lambda \delta^{n-1} l_{n-1} + \lambda \delta^{n-2} l_{n-2} \\ u_2^{n-1} S_n = u_0^{n-1} S_n + \lambda \delta^{n-1} l_{n-1} \end{cases}$$

Adding the equations in (9) we get
Consider a sequence of cumulative sums $U^1, ..., U^n$ of $u^1, ..., u^n$

\[
\begin{align*}
U^1 &= u^1 \delta^1 l_1 \\
U^2 &= u^2 \delta^2 l_2 \\
U^3 &= u^3 \delta^3 l_3 \\
\vdots \\
U^n &= u^n \delta^n l_n
\end{align*}
\]

So the solving of the problem (5) in terms (10) and (11) is reduced to the solving of the following problem:

\[
\begin{align*}
U^1 &= u^1 \delta^1 l_1 \\
U^2 &= u^2 \delta^2 l_2 \\
U^3 &= u^3 \delta^3 l_3 \\
\vdots \\
U^n &= u^n \delta^n l_n
\end{align*}
\]

with a given discrete function $U_0$ and unknown discrete functions $U$ and $\delta$ satisfying (8).

Note that for any $i = 1, ..., n - 1$ the condition $u^i_{i+1} > u^i_i$ is equivalent to the condition $\frac{u^i_{i+1} - u^i_i}{s_{i+1}} > \frac{u^i_{i+1} - u^i_i}{s_i}$, the condition $u^i_{i+1} < u^i_i$ equals to the condition $\frac{u^i_{i+1} - u^i_i}{s_{i+1}} < \frac{u^i_{i+1} - u^i_i}{s_i}$, the condition $u^i_{i+1} = u^i_i$ is equivalent to the condition $\frac{u^i_{i+1} - u^i_i}{s_{i+1}} = \frac{u^i_{i+1} - u^i_i}{s_i}$. Let us consider additional variables $U^0 = U_0^0 = 0$. Then the set of equations (12) can be rewritten with additional variables as

\[
\begin{align*}
U^0 &= U_0^0 = 0 \\
U^1 &= U_0^1 - \lambda \delta^1 l_1 \\
U^2 &= U_0^2 - \lambda \delta^2 l_2 \\
U^3 &= U_0^3 - \lambda \delta^3 l_3 \\
\vdots \\
U^n &= U_0^n
\end{align*}
\]

where

\[
\delta^i = \begin{cases} 
-1, & \text{if } \frac{u^i_{i+1} - u^i_i}{s_{i+1}} > \frac{u^i_{i+1} - u^i_i}{s_i} \\
1, & \text{if } \frac{u^i_{i+1} - u^i_i}{s_{i+1}} < \frac{u^i_{i+1} - u^i_i}{s_i} \\
1, & \text{if } \frac{u^i_{i+1} - u^i_i}{s_{i+1}} = \frac{u^i_{i+1} - u^i_i}{s_i} 
\end{cases}
\]
2.2. Geometrical interpretation of the conditions (13) and (14)

Consider a sequence of the following cumulative sums: 
\[ x_0, x_i = x_{i-1} + S_i, \quad i = 1, \ldots, n. \]

The values \( U_0^0, U_0^1, \ldots, U_0^n \) of the discrete function \( U_0 \) define a piecewise linear curve with vertices \( (x_i, U_0^i) \), which is the axial line of a tube. The values \( U_1^0, U_1^1 + \lambda l_1, \ldots, U_{n-1}^{n-1} + \lambda l_{n-2}, U_n^n \) form a piecewise linear curve with vertices \( (x_i, U_1^i + \lambda l_i) \), that is an upper boundary of the tube. The values \( U_0^0, U_0^1 - \lambda l_1, \ldots, U_{n-1}^{n-1} - \lambda l_{n-1}, U_n^n \) form a piecewise linear curve with vertices \( (x_i, U_0^i - \lambda l_i) \), that is a lower boundary of the tube.

The values \( U_0, U_1, \ldots, U_n \) of the discrete function \( U \) define a piecewise linear curve with vertices \( (x_i, U^i) \). From (13) and (14) we obtain a simple geometrical interpretation of the extremal discrete function \( U \).

Consider an extremal piecewise linear curve defined by the discrete function \( U \). The extremal curve satisfies to the following conditions:

1. Since \( \delta^i, i = 1, \ldots, n - 1 \), takes values from the interval \([-1; 1]\), the extremal curve (i.e. solution of the problem (5)) entirely belongs to the tube.

2. If the extremal curve intersects the \( i \)-th vertical cross-section \( \{x = x_i\} \) of the tube in an inner point, then the extremal curve is a straight line in the neighborhood of the intersection point.

Remark. From last sentence, we see that the extremal curve has a fracture at the points that belong to the upper or lower boundaries of the tube.

3. Computer simulation

A geometrical illustration of the algorithm is provided here on a model example using notations from paragraph 2.1.

Let \( u_0 = \{u_0^0, \ldots, u_0^n\} \) be a RSPC function with \( \{u_0^0, \ldots, u_0^n\} = \{-20, 100, -40, 20, -120, 90\} \), \( \{r^1, \ldots, r^6\} = \{4, 7, 11, 15, 17, 19\} \). Consider a modification \( \overline{u}_0 \) of the function \( u_0 \) defined as 
\[ \overline{u}_0(r) = u_i^1, \quad x_{i-1} < r < x_i, \quad i = 1, \ldots, n. \]

Let the function \( U_0(x) \) be primitive function for \( \overline{u}_0(x) \), that is,
\[ U_0(x) = \int_0^x \overline{u}_0(r) \, dr. \]

The plot of the function \( U_0 \) is a piecewise linear curve with vertices \( (x_i, U_1^i) \). So, this graph is the axial line of the corresponding tube for the function \( u_0 \). For a given \( \lambda \) we construct upper and lower boundaries of the tube. Figure 1 shows the graph of the function \( \overline{u}_0 \) (upper part) and the tube for \( \lambda = 20 \) (lower part). The upper and lower boundaries of the tube are shown in black, the axial line and the function \( U_0 \) are plotted with red color.

Figure 1. Graph of the function \( \overline{u}_0 \) (upper part) and the tube for \( \lambda = 20 \) (lower part).

Figures 2-7 show the performance of the algorithm for different values of the regularization parameter \( \lambda \). The extremal line curve \( U \) blue-colored is shown in the middle part of the figures. The
modified extremal function $\overline{u}_\gamma$ green-colored is shown in the lower part of the figures. Note that the function $\overline{u}_\gamma$ is the derivative of the function $U$, and the extremal function $u_\gamma$ can be easily obtained from $\overline{u}_\gamma$. For comparison purpose, the modified initial function $\overline{u}_0$ is shown in the upper part of the figures.

![Figure 2](image1.png)

**Figure 2.** Upper part: the function $\overline{u}_0$, middle part: extremal line curve in the tube for $\lambda = 20$, lower part: the modified extremal function $\overline{u}_\gamma$.

![Figure 3](image2.png)

**Figure 3.** Upper part: the function $\overline{u}_0$, middle part: extremal line curve in the tube for $\lambda = 40$, lower part: the modified extremal function $\overline{u}_\gamma$.

In figure 3 it can be seen that the behavior of red and green lines are similar; that is, a small jump on the green line at the point $x_3 \approx 380$ corresponds to a small fracture on the blue line at the same point.

In figure 4 it can be observed the jump on the green line at the point $x_3 \approx 380$ disappeared and the blue line at the point has no fracture anymore.

In figure 5 we see that the green line has negligible jumps at the points $x_1 \approx 50$ and $x_4 \approx 706$. At these points the blue plot is almost straight.

In figure 6 the green line jumps at the points $x_1 \approx 50$ and $x_4 \approx 706$ disappeared. At the same points the blue line is straight.
Figure 4. Upper part: the function $u_0$, middle part: extremal line curve in the tube for $\lambda = 70$, lower part: the modified extremal function $\overline{u}_r$.

Figure 5. Upper part: the function $\overline{u}_0$, middle part: extremal line curve in the tube for $\lambda = 100$, lower part: the modified extremal function $\overline{u}_r$.

Figure 6. Upper part: the function $\overline{u}_0$, middle part: extremal line curve in the tube for $\lambda = 150$, lower part: the modified extremal function $\overline{u}_r$.

Finally, in figure 7 the green line has no jumps, and consequently the blue line is straight.
4. Conclusion

In this paper, we proposed a Condat’s type TV regularization algorithm for computation of an extremal function for two-dimensional radially symmetric piecewise constant functions. A clear geometrical interpretation of the extremal function with a modified taut string approach was provided. Computer simulation results were provided to illustrate the performance of the proposed TV regularization algorithms.

5. References

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