A Matrix Model for Bilayered Quantum Hall Systems

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Abstract

We develop a matrix model to describe bilayered quantum Hall fluids for a series of filling factors. Considering two coupling layers, and starting from a corresponding action, we construct its vacuum configuration at $\nu = q_i K^{-1}_{ij} q_j$, where $K_{ij}$ is a $2 \times 2$ matrix and $q_i$ is a vector. Our model allows us to reproduce several well-known wave functions. We show that the wave function $\Psi_{(m,m,n)}$ constructed years ago by Yoshioka, MacDonald and Girvin for the fractional quantum Hall effect at filling factor $\frac{2}{m+n}$ and in particular $\Psi_{(3,3,1)}$ at filling factor $\frac{1}{2}$ can be obtained from our vacuum configuration. The unpolarized Halperin wave function and especially that for the fractional quantum Hall state at filling factor $\frac{2}{3}$ can also be recovered from our approach. Generalization to more than 2 layers is straightforward.

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1 Introduction

The quantum Hall (QH) effect has bred much beautiful theory. Indeed, Laughlin’s wave functions are good wave functions for describing the fractional quantum Hall effect (FQHE) at filling factor , where is an odd integer. For other filling factors several attempts have been suggested to extend Laughlin’s theory by adopting different approaches and assumptions. In particular, Halperin proposed a family of generalized Laughlin wave functions that could incorporate reversed spins. In fact a candidate for an unpolarized wave function at filling factor was given. Subsequently, Yoshioka, MacDonald and Girvin generalized the Laughlin wave functions to those of the bilayered QH systems and derived that corresponding to the state. Moreover, other theories have been elaborated and led to understand the observed values of , in particular and as well as others.

The first experimental indications for an unpolarized ground-state spin configuration in the FQHE came with the discovery of the state and later the state. More compelling evidence for novel spin phenomena in the FQHE was subsequently reported. On the other hand, it was shown experimentally that multi-layer systems also exhibit the FQHE. In fact, several filling factors have been observed, for instance the state and .

Recently, Susskind proposed a novel method to investigate the FQHE. He claimed that the non-commutative Chern-Simons theory (NCCS) at level is exactly equivalent to Laughlin’s theory at the filling factor . He formulated his approach as a matrix theory similar to that describing D0-branes in string theory. However, Susskind’s theory is an alternative approach to the FQHE which so far has not produced anything new but has just recovered the Laughlin approach by adopting a new formalism. Nevertheless it remains a new way of thinking and is worth studying in the hope that it will bring new results in the future.

Although the proposed matrix model seems to reproduce the basic features of the Laughlin QH droplets, still some problems remain to be solved. Indeed, Susskind’s approach is valid only for infinite matrices and also shows an anomaly for . To solve these problems, Polychronakos introduced a boundary term to Susskind matrix model. He proposed a finite matrix model as a regularized version of the NCCS theory. It allowed him to find a quantum correction to , where is shifted to and the filling factor became . As another consequence, he pointed out that his matrix model is equivalent to the Calogero model.

Sometimes later, observing that the Laughlin wave functions can be mapped onto many-body wave functions of the harmonic oscillator, Hellerman and Van Raamsdonk built a complete minimal basis of wave functions of the theory at arbitrary level and rank , see also. Other investigations about the relation between NCCS and Laughlin fluids can be found in. Subsequently, the Susskind model and its regularized version introduced
by Polychronakos was extended to FQH states that are not of Laughlin type: a multicomponent Chern-Simons approach was introduced \[20\] and another proposal based on the Haldane hierarchy \[21\] was developed \[22, 23\].

Despite the progress in the study of the FQH fluids in the framework of NCCS matrix model, several open questions remain which have not been addressed so far. One of these questions concerns the wave functions that are not of Laughlin type. In fact there are many wave functions, that have been constructed years ago, e.g. by Yoshioka et al., Halperin, · · · , but cannot be recovered by what is developed so far.

In what follows we propose a matrix model to investigate the possibility to obtain two of those wave functions. This can be done by extending the Susskind–Polychronakos model to deal with the QH fluids at the filling factor \[24\]

\[
\nu = q_i K^{-1}_{ij} q_j
\]

where \(K_{ij}\) is an \(N \times N\) matrix and \(q_i\) is a vector. The basic idea is to consider several Susskind–Polychronakos systems, let us say \(M\) systems, with an interaction between them and suppose that all systems possess the same number of particles. In the QHE language, this picture is equivalent to considering multi-layered systems. Without loss of generality, we fix \(M = 2\), but as we will see later our analysis can directly be extended to the generic case \(M \geq 3\).

We start by writing down an appropriate action as a sum of two terms, for the free and the interacting part. Subsequently, we derive the corresponding Hamiltonian, which of course contains an interaction. Using a unitary transformation, we show that this Hamiltonian can be transformed to a diagonalized one. Next, we determine the vacuum configuration that allows us to recover two different states. Indeed, we show that how the Yoshioka–MacDonald–Girvin wave functions at the filling factor \(\nu = \frac{2}{m+n}\) can be obtained from our model and in particular that describing the FQHE at \(\nu = \frac{1}{2}\). Moreover, the unpolarized Halperin wave functions will be derived and especially that corresponding to \(\nu = \frac{2}{5}\) state.

In section 2 we recall briefly the NCCS matrix model description of the Laughlin fluid. In section 3, we propose an action describing a system of two layers, we derive the Gauss law constraint as well as the equations of motion for the different variables. A quantum mechanical analysis will be the subject of next section, where we develop a Hamiltonian that corresponds to the system under consideration. Under rotation, we define a set of matrices of harmonic–oscillator operators to diagonalize the system. In section 5, we build the vacuum configuration that satisfies the constraint. A link with literature will be discussed in the last section where the two wave functions mentioned above will be recovered. We conclude our paper by putting some questions to be investigated in forthcoming works.
2 Chern-Simons matrix model

Starting from the matrix formulation of a two-dimensional system with a large number of electrons in the presence of a perpendicular magnetic field $B$, Susskind \cite{13} showed that the resulting effective theory is a non-commutative $U(1)$ Chern–Simons gauge theory at level $k = B\theta$. As a consequence, he found a relation

$$\rho = \frac{1}{2\pi \theta}$$

(2)

which links the non-commutivity parameter $\theta$ to the density of electrons $\rho$. By using the definition of the filling factor

$$\nu = \frac{2\pi \rho}{B}$$

(3)

in the system of units $(\hbar, e, c)$, it is easily seen that the fraction $\nu$ can be written in terms of the parameter $\theta$ as

$$\nu = \frac{1}{B\theta}.$$  

(4)

This beautiful relation is one of the interesting results obtained recently by Susskind in dealing with the FQH fluids.

Moreover, by exploring the possibility to develop a consistent finite matrix model for the description of the FQH droplet, Polychronakos \cite{14} suggested to include a new field into the Susskind model. The proposed action is given by

$$S = \int dt \frac{B}{2} \text{Tr} \left\{ \epsilon^{ab} \left( \dot{X}_a + i[A_0, X_a] \right) X_b + 2\theta A_0 - \omega X_a^2 \right\} + \psi^\dagger \left( i\dot{\psi} - A_0 \psi \right)$$

(5)

where $X_a$, $a = 1, 2$ are $N \times N$ matrices and $\psi$ is a complex $N$-vector, and $\epsilon^{12} = -\epsilon^{21} = 1, \epsilon^{aa} = 0$. The action is invariant under the gauge group $U(N)$ and the matrix model variables transform as

$$X_a \rightarrow UX_a U^{-1}, \quad \psi \rightarrow U\psi.$$ 

(6)

The equation of motion for $A_0$ leads to the Gauss law constraint

$$G \equiv -iB [X_1, X_2] + \psi\psi^\dagger - B\theta = 0.$$

(7)

The trace of this equation gives

$$\psi^\dagger\psi = NB\theta.$$ 

(8)

Upon quantization the matrix elements of $X_a$ and the components of $\psi$ become operators, obeying the commutation relations

$$\left[ \psi_i, \psi^\dagger_j \right] = \delta_{ij},$$

$$\left[ (X_1)_{ij}, (X_2)_{kl} \right] = \frac{i}{B} \delta_{il} \delta_{jk}.$$ 

(9)
The Hamiltonian can be obtained from (5) as
\[ H = \omega \left( \frac{N^2}{2} + \sum A_{nm}^\dagger A_{mn} \right) \] (10)
where the \( N \times N \) matrix of harmonic-oscillator operators is defined by
\[ A_{nm} = \sqrt{\frac{B}{2}} (X_1 + iX_2)_{nm}. \] (11)
The corresponding wave function is [16]
\[ |k\rangle = \left[ \epsilon_{i_1...i_N} \psi_{i_1}^\dagger (\psi_{i_2}^\dagger A_{i_2}^\dagger) ... (\psi_{i_N}^\dagger A_{i_N}^\dagger) \right]^{k} |0\rangle \] (12)
where the vacuum \(|0\rangle\) is annihilated by \( A \)'s and \( \psi \)'s and \( \epsilon \) is the fully antisymmetric tensor. This is a physical state and therefore satisfies the relation
\[ G|k\rangle = 0. \] (13)
It is similar to the Laughlin wave function [1] at the filling factor
\[ \nu = \frac{1}{k + 1}. \] (14)
Subsequently, one of us and others [22, 23] generalized the above results to any filling factor which can be expressed as
\[ \nu_{k_1k_2} = \frac{1}{k_1} + \frac{1}{k_2} \] (15)
and in particular to level two of the Haldane hierarchy [21]
\[ \nu_{p_1p_2} = \frac{p_2}{p_1p_2 - 1} \] (16)
by setting
\[ k_1 = p_1, \quad k_2 = p_1(p_2 - 1) \] (17)
where \( p_1 \) is an odd and \( p_2 \) is an even integer.

### 3 Two coupling matrices model

We consider two systems with a total number of particles \( M_1 + M_2 \) which interact with each other. Such systems can be seen like two coupling layers \( i \) containing \( M_i \) particles. The appropriate action to describe the FQH fluids of the whole system at filling factor [11], is given by
\[ S = \int dt \sum_j \frac{K_{jj}}{2\theta} \text{Tr} \left\{ \epsilon^{ab} \left( \dot{X}_a^{(j)} + i [A_0, X_a^{(j)}] \right) X_b^{(j)} + 2\theta A_0 - \omega_j \left( X_a^{(j)} \right)^2 \right\} \]
\[ + \psi^{(j)} (i\dot{\psi}^{(j)} - A_0\psi^{(j)}) + \int dt \ K_{12} \left\{ \frac{\omega_{12}}{\theta} \text{Tr} \left( X_a^{(1)} X_a^{(2)} \right) + \psi^{(1)} \psi^{(2)} \right\} \] (18)
which involves two copies of the single-layer action forming the free part. It also contains an interacting part, where the scalar $K_{12}$ is playing the role of a coupling parameter between the layers 1 and 2. The ratio $\frac{K_{12}}{\theta}$ is basically the magnetic field $B$.

It is clear that for $K_{12} = 0$, the total system becomes decoupling. Note that as far as the total action is concerned, the full gauge symmetry is $U(M_1) \times U(M_2)$. The matrix model variables transform under this invariance as

$$X_a^{(i)} \rightarrow UX_a^{(i)}U^{-1}, \quad \psi^{(i)} \rightarrow U\psi^{(i)}. \quad (19)$$

Compared to the original matrix model, there is the potential term

$$V = \sum_j \frac{K_{jj}}{2\theta} \omega_j \text{Tr}(X_a^{(j)})^2 - \frac{K_{12}}{\theta} \omega_{12} \text{Tr}(X_a^{(1)}X_a^{(2)}) \quad (20)$$

analogous to the potential of two coupled harmonic oscillators in two-dimensional space. This provides a Hamiltonian for the theory.

The Gauss law constraint can be obtained by evaluating the equation of motion for $A_0$. In our case it reads

$$G \equiv -iK_{11}\left[ X_1^{(1)}, X_2^{(1)} \right] - iK_{22}\left[ X_1^{(2)}, X_2^{(2)} \right] + \left( \psi^{(1)}\psi^{(1)\dagger} + \psi^{(2)}\psi^{(2)\dagger} - K_{11} - K_{22} \right) = 0 \quad (21)$$

where its trace gives

$$\psi^{(1)\dagger}\psi^{(1)} + \psi^{(2)\dagger}\psi^{(2)} = M_1K_{11} + M_2K_{22}. \quad (22)$$

Other equations of motion can also be calculated. For the $X$’s we get

$$K_{11}\epsilon^{ab}\dot{X}_a^{(1)} + K_{11}\omega_1X_a^{(1)} + K_{12}\omega_{12}X_a^{(2)} = 0$$
$$K_{22}\epsilon^{ab}\dot{X}_a^{(2)} + K_{22}\omega_2X_a^{(2)} + K_{12}\omega_{12}X_a^{(1)} = 0 \quad (23)$$

while for the $\psi$’s we obtain

$$i\dot{\psi}^{(1)\dagger} + K_{12}\psi^{(2)} = 0$$
$$i\dot{\psi}^{(2)\dagger} + K_{12}\psi^{(1)} = 0 \quad (24)$$

Of course the last set of equations shows a difference with respect to the decoupled case. It can be solved by using a unitary transformation.

4 Hamiltonian formalism

Let us now consider the proposed model quantum mechanically. We proceed by determining the total Hamiltonian, which describes the system under consideration. It can be obtained from the relation

$$\mathcal{H} = \dot{X} \frac{\partial L}{\partial \dot{X}} - L \quad (25)$$
where $\frac{\partial L}{\partial \dot{X}}$ defines the conjugate momentum. This leads to a Hamiltonian as the sum of the free and the interacting parts as

$$
\mathcal{H} = \sum_j \frac{K_{jj}}{2\theta} \omega_j \text{Tr} \left( X_a^{(j)} \right)^2 - \frac{K_{12}}{\theta} \omega_{12} \text{Tr} \left( X_a^{(1)} X_a^{(2)} \right)
$$

(26)

which is nothing but the confining potential (20). This means that the kinetic energy is negligible compared to $V$.

It is clear that this form of $\mathcal{H}$ cannot be diagonalized directly. Nevertheless, $\mathcal{H}$ can be transformed to another, factorizing Hamiltonian $\mathcal{H}'$. Probably the best way to do this is to perform a rotation by a mixing angle $\varphi$, of the $X$’s to new matrices

$$
Y_a^{(1)} = X_a^{(1)} \cos \frac{\varphi}{2} - X_a^{(2)} \sin \frac{\varphi}{2}, \\
Y_a^{(2)} = X_a^{(1)} \sin \frac{\varphi}{2} + X_a^{(2)} \cos \frac{\varphi}{2}.
$$

(27)

It can easily be checked that this rotation is a unitary transformation. Inserting (27) into (26), one can show that $\mathcal{H}$ transform to

$$
\mathcal{H}' = \alpha \text{Tr} \left( Y_a^{(1)} \right)^2 + \beta \text{Tr} \left( Y_a^{(2)} \right)^2
$$

(28)

if the rotating angle satisfies the relation

$$
\tan \varphi = \frac{K_{12} \omega_{12}}{K_{11} \omega_1 - K_{22} \omega_2}.
$$

(29)

The parameters $\alpha$ and $\beta$ are given by

$$
\alpha = \frac{1}{\theta} \left( K_{11} \omega_1 \cos^2 \frac{\varphi}{2} + K_{22} \omega_2 \sin^2 \frac{\varphi}{2} - \frac{1}{2} K_{12} \omega_{12} \sin \varphi \right), \\
\beta = \frac{1}{\theta} \left( K_{11} \omega_1 \sin^2 \frac{\varphi}{2} + K_{22} \omega_2 \cos^2 \frac{\varphi}{2} + \frac{1}{2} K_{12} \omega_{12} \sin \varphi \right).
$$

(30)

To diagonalize $\mathcal{H}'$, we define two couples of creation and annihilation matrices of harmonic oscillator operators,

$$
C_{nm}^{(1)} = \sqrt{\frac{\alpha}{2}} (Y_1^{(1)} + i Y_2^{(1)})_{nm}, \\
C_{nm}^{(2)} = \sqrt{\frac{\beta}{2}} (Y_1^{(2)} + i Y_2^{(2)})_{nm}.
$$

(31)

They satisfy the commutation relations

$$
\left[ C_{nm}^{(1)}, C_{n'm'}^{(1)\dagger} \right] = \delta_{nm} \delta_{n'm'} \\
\left[ C_{ij}^{(2)}, C_{i'j'}^{(2)\dagger} \right] = \delta_{ij} \delta_{i'j'},
$$

(32)

while all others commutators vanish. Now $\mathcal{H}'$ can be rewritten as

$$
\mathcal{H}' = \frac{\alpha}{2} \left( 2M_1 + M_1^2 \right) + \frac{\beta}{2} \left( 2M_2 + M_2^2 \right)
$$

(33)

where the number operators

$$
M_1 = \sum_{n,m=1}^{M_1} C_{nm}^{(1)\dagger} C_{nm}^{(1)}, \\
M_2 = \sum_{i,j=1}^{M_2} C_{ij}^{(2)\dagger} C_{ji}^{(2)}
$$

(34)

are counting the $M_1$ and $M_2$ particles. Thus under the unitary transformation the system became decoupling.
5 Ground-state wave functions

To begin we emphasize a difference between the ground state of two coupled harmonic oscillators in terms of the coordinates \( x_i \) and that in terms of their mapped representations \( y_i \). The wave function

\[
\psi_0(y) \sim \exp \left\{ -\alpha y_1^2 - \beta y_2^2 \right\}
\]

is separable in the variables \( y_1 \) and \( y_2 \). However, for the variables \( x_1 \) and \( x_2 \), the wave function (35) reads

\[
\psi_0(x) \sim \exp \left\{ -\alpha \left( x_1 \cos \frac{\varphi}{2} - x_2 \sin \frac{\varphi}{2} \right)^2 - \beta \left( x_1 \sin \frac{\varphi}{2} + x_2 \cos \frac{\varphi}{2} \right)^2 \right\}.
\]

Next, we will see how these ground states can be extended to the matrix model formalism. We begin to determine that for the matrices \( Y \). By transforming the Gauss law constraint to the variables \( Y \), i.e.

\[
\left( K_{11} \cos^2 \frac{\varphi}{2} + K_{22} \sin^2 \frac{\varphi}{2} \right) \left[ Y_1^{(1)} , Y_2^{(1)} \right] + \left( K_{11} \sin^2 \frac{\varphi}{2} + K_{22} \cos^2 \frac{\varphi}{2} \right) \left[ Y_1^{(2)} , Y_2^{(2)} \right] + \frac{1}{2} (K_{11} - K_{22}) \sin \varphi \left\{ \left[ Y_1^{(1)} , Y_2^{(2)} \right] + \left[ Y_1^{(2)} , Y_2^{(1)} \right] \right\} = i\theta \left( K_{11} + K_{22} - \phi^{(1)} \phi^{(1)\dagger} - \phi^{(2)} \phi^{(2)\dagger} \right)
\]

where the Polychronakos fields are also rotated to new fields

\[
\phi^{(1)} = \psi^{(1)} \cos \frac{\varphi}{2} - \psi^{(2)} \sin \frac{\varphi}{2} \]
\[
\phi^{(2)} = \psi^{(1)} \sin \frac{\varphi}{2} + \psi^{(2)} \cos \frac{\varphi}{2}.
\]

For simplicity let us fix \( K_{11} = K_{22} = K \), then (37) becomes

\[
\left[ Y_1^{(1)} , Y_2^{(1)} \right] + \left[ Y_1^{(2)} , Y_2^{(2)} \right] = 2iK\theta \left( 1 - \frac{1}{2K} \phi^{(1)} \phi^{(1)\dagger} - \frac{1}{2K} \phi^{(2)} \phi^{(2)\dagger} \right).
\]

Now it is clear that the ground state is simply a tensor product between those states corresponding to each layer

\[
| K \rangle = \left[ \hat{\epsilon}_{i_1} \cdots \hat{\epsilon}_{i_{M_1}} \phi^{(1)\dagger}_{i_1} \left( \phi^{(1)\dagger} C^{(1)\dagger} \right)_{i_2} \cdots \left( \phi^{(1)\dagger} C^{(1)\dagger} \right)_{i_{M_1}} \right]^K \left[ \hat{\epsilon}_{j_1} \cdots \hat{\epsilon}_{j_{M_2}} \phi^{(2)\dagger}_{j_1} \left( \phi^{(2)\dagger} C^{(2)\dagger} \right)_{j_2} \cdots \left( \phi^{(2)\dagger} C^{(2)\dagger} \right)_{j_{M_2}} \right]^K | 0 \rangle.
\]

The ground state (40) can be mapped in terms of the operators of the matrices \( X \) by expressing the matrices \( C \) of harmonic-oscillator operators in terms of those corresponding to the matrices \( X \). Using (27) one can show that (31) takes the form

\[
C^{(1)}_{nm} = \sqrt{\frac{\alpha}{\beta}} \left( A^{(1)} \cos \frac{\varphi}{2} - A^{(2)} \sin \frac{\varphi}{2} \right)_{nm}
\]
\[
C^{(2)}_{nm} = \sqrt{\frac{\beta}{\alpha}} \left( A^{(1)} \sin \frac{\varphi}{2} + A^{(2)} \cos \frac{\varphi}{2} \right)_{nm}
\]
where the operators

\[ A_{nm}^{(1)} = \sqrt{\frac{\hbar}{2}} \left( X_1^{(1)} + iX_2^{(1)} \right)_{nm} \]
\[ A_{nm}^{(2)} = \sqrt{\frac{\hbar}{2}} \left( X_1^{(2)} + iX_2^{(2)} \right)_{nm} \]  

commute:

\[ \left[ A_{nm}^{(1)}, A_{n'm'}^{(1)*} \right] = \delta_{nm}\delta_{n'm} \]
\[ \left[ A_{ij}^{(2)}, A_{i'j'}^{(2)*} \right] = \delta_{ij}\delta_{i'j}. \]  

Inserting (38) and (41) in (40), we obtain

\[ |K\rangle = \left[ e^{i1 \ldots iM_1} \left( \psi^{(1)*} \cos \frac{\phi}{2} - \psi^{(2)*} \sin \frac{\phi}{2} \right)_{i_1} \right. \ldots \left. \{ (\psi^{(1)*} \cos \frac{\phi}{2} - \psi^{(2)*} \sin \frac{\phi}{2} ) (A^{(1)*} \cos \frac{\phi}{2} - A^{(2)*} \sin \frac{\phi}{2})^{M_1-1} \}^{i_{M_1}} \right] K \left[ e^{j1 \ldots jM_2} \left( \psi^{(1)*} \sin \frac{\phi}{2} + \psi^{(2)*} \cos \frac{\phi}{2} \right)_{j_1} \right. \ldots \left. \{ (\psi^{(1)*} \sin \frac{\phi}{2} + \psi^{(2)*} \cos \frac{\phi}{2} ) (A^{(1)*} \sin \frac{\phi}{2} + A^{(2)*} \cos \frac{\phi}{2})^{M_2-1} \}^{j_{M_2}} \right] K |0\rangle. \]  

In what follows, we proceed without the use of the unitary transformation to construct the wave function \( |\Phi\rangle \) describing the system of \( M_1 + M_2 \) electrons at filling factor \( \Pi \). One has to realize a physical state \( |\Phi\rangle \) that satisfies the Gauss law constraint \( \mathcal{G} |\Phi\rangle = 0 \) and allows us to establish a link with two well-known wave functions. May be the best way to do this is to define two operators

\[ A = A^{(1)} \otimes A^{(2)} \]
\[ \psi = \psi^{(1)} \otimes \psi^{(2)} \]  

where \( \otimes \) is the tensor product. Using these matrices of harmonic-oscillator operators, we build a vacuum configuration

\[ |\Psi\rangle = \left[ e^{i1 \ldots iM_1} \psi_{i_1}^{(1)*} (\psi^{(1)*} A^{(1)*})_{i_2} \ldots (\psi^{(1)*} A^{(1)*})^{M_1-1}_{i_{M_1}} \right]^{K_{11} - K_{12}} \left[ e^{j1 \ldots jM_2} \psi_{j_1}^{(2)*} (\psi^{(2)*} A^{(2)*})_{j_2} \ldots (\psi^{(2)*} A^{(2)*})^{M_2-1}_{j_{M_2}} \right]^{K_{22} - K_{12}} \left[ e^{k1 \ldots kM_1 + M_2} \psi_{k_1}^{\dagger} (\psi^{\dagger} A^{\dagger})_{k_2} \ldots (\psi^{\dagger} A^{\dagger})^{M_1 + M_2 - 1}_{k_{M_1 + M_2}} \right]^{K_{12}} |0\rangle. \]  

which satisfies the Gauss law constraint \( \mathcal{G} |\Phi\rangle = 0 \) and therefore we have

\[ (\psi^{(1)*} \psi^{(1)*} + \psi^{(2)*} \psi^{(2)*}) - M_1 K_{11} - M_2 K_{22} |\Phi\rangle = 0. \]  

Novel about this vacuum configuration is that one can interpret the term

\[ \left[ e^{k1 \ldots kM_1 + M_2} \psi_{k_1}^{\dagger} (\psi^{\dagger} A^{\dagger})_{k_2} \ldots (\psi^{\dagger} A^{\dagger})^{M_1 + M_2 - 1}_{k_{M_1 + M_2}} \right]^{K_{12}} \]  

as
as an inter-layer correlation. In conclusion, our configuration could well be a good ansatz for the ground states of double–layered FQH fluids in the formalism of the NCCS matrix model. This will be clarified in the next section.

6 Link with literature

Here we show how the Yoshioka–MacDonald–Girvin and Halperin wave functions describing, respectively, the double-layer and the unpolarized QH systems can be recovered from our vacuum configuration (47).

Before starting, we note that for any $N$-dimensional vector $\psi^\dagger$ and $N \times N$ matrix $A^\dagger$, the expression of the form

$$F(\psi^\dagger, A^\dagger) = \epsilon^{i_1...i_N} \psi_i^{(1)^\dagger} (\psi_i^{(1)^\dagger} A^{(1)^\dagger})_{i_2} \cdots (\psi_i^{(1)^\dagger} A^{(1)^\dagger})_{i_N}$$

has a one-to-one correspondence to the polynomial

$$f(z) = \epsilon^{i_1...i_N} z_i^0 \cdots z_i^{N-1}. \quad (51)$$

Now our task can be done by defining a new complex variable

$$\zeta_i = \begin{cases} z_i^{(1)} & \text{for } i = 1, ..., N \\ z_i^{(2)} & \text{for } i = N + 1, ..., 2N \end{cases} \quad (52)$$

assuming that the particle numbers are equal, $M_1 = M_2 = N$, and recalling the Vandermonde determinant:

$$\prod_{i<j} (z_i - z_j) = \det(z_i^{N-j}) = \epsilon^{i_1...i_N} z_i^0 \cdots z_i^{N-1}. \quad (53)$$

In terms of the complex coordinates, (47) reads

$$\Psi_{(K_{11}, K_{22}, K_{12})} = \left[ \epsilon^{i_1...i_N} \begin{pmatrix} z_i^{(1)}_1 & \cdots & z_i^{(1)}_{N-1} \end{pmatrix} K_{11} \right] \left[ \epsilon^{j_1...j_N} \begin{pmatrix} z_j^{(2)}_1 & \cdots & z_j^{(2)}_{N-1} \end{pmatrix} K_{22} \right] \left[ \epsilon^{k_1...k_{2N}} \begin{pmatrix} z_k^{0} & \cdots & z_k^{2N-1} \end{pmatrix} K_{12} \right] \Psi_0. \quad (54)$$

It can be written in standard form as

$$\Psi_{(K_{11}, K_{22}, K_{12})} = \prod_{i<j} \left( z_i^{(1)} - z_j^{(1)} \right)^{K_{11}} \prod_{i<j} \left( z_i^{(2)} - z_j^{(2)} \right)^{K_{22}} \prod_{i,j} \left( z_i^{(1)} - z_j^{(2)} \right)^{K_{12}} \Psi_0 \quad (55)$$

and now the inter-layer correlation is

$$\prod_{i,j} \left( z_i^{(1)} - z_j^{(2)} \right)^{K_{12}}. \quad (56)$$

Next, we will give two different applications of (55).
6.1 YMG wave functions

Considering the two layers and treating them as additional degrees of freedom, the \( \nu = \frac{1}{2} \) state was predicted by Yoshioka, MacDonald and Girvin \[4\]. They made a straightforward generalization of the Laughlin wave functions to those with the filling factor

\[
\nu = \frac{2}{m+n}
\]

where \( m \) and \( n \) are integers. This can be obtained from our analysis by taking

\[
K = \begin{pmatrix} m & n \\ n & m \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

leading to the wave function

\[
\Psi_{(m,m,n)} = \prod_{i<j} \left( z^{(1)}_i - z^{(1)}_j \right)^m \prod_{i<j} \left( z^{(2)}_i - z^{(2)}_j \right)^m \prod_{i,j} \left( z^{(1)}_i - z^{(2)}_j \right)^n \Psi_0.
\]  

Choosing \( m = 3 \) and \( n = 1 \), we recover the FQHE \( \nu = \frac{1}{2} \) state corresponding to

\[
\Psi_{(3,3,1)} = \prod_{i<j} \left( z^{(1)}_i - z^{(1)}_j \right)^3 \prod_{i<j} \left( z^{(2)}_i - z^{(2)}_j \right)^3 \prod_{i,j} \left( z^{(1)}_i - z^{(2)}_j \right)^2 \Psi_0.
\]

6.2 Halperin wave functions

Another interesting result can be obtained. In the Halperin picture \[3\] in the context of single-layered unpolarized QH systems, the labels 1 and 2 can be considered as an analogue of spin. Following this idea, our bilayered system can be seen as mixing layers of particles with spin up and spin down.

As a consequence, we obtain for \( m = 3 \) and \( n = 2 \) the unpolarized Halperin wave function with the filling factor \( \nu = \frac{2}{5} \) as

\[
\Psi_{(3,3,2)} = \prod_{i<j} \left( z^{(1)}_i - z^{(1)}_j \right)^3 \prod_{i<j} \left( z^{(2)}_i - z^{(2)}_j \right)^3 \prod_{i,j} \left( z^{(1)}_i - z^{(2)}_j \right)^2 \Psi_0.
\]

This can be seen as a wave function of a system of \( N \) particles with spin parallel and another \( N \) particles with spin antiparallel to the external magnetic field.

7 Conclusion

We have developed a matrix model to describe bilayered QH systems at the filling factor \( \nu = q_i K^{-1} q_j \). The basic idea was to use two coupled harmonic-oscillators in a similar fashion
as done by Susskind and Polychronakos. Our model is a generalization of their model and of course reproduces its basic features by taking the coupling parameter $K_{12}$ to be zero.

Starting from an appropriate action we derived the equations of motion for the different matrix model variables. The corresponding Hamiltonian was obtained as the sum of free and interacting terms. A unitary transformation, more precisely a rotation around an angle $\varphi$, led to a factorizing Hamiltonian.

Next, we have constructed the ground states of the system in two different ways. The first was based on the unitary transformation and from the ground state after rotation we have derived that before rotating the system. The second was performed directly in terms of a combination of the matrices of harmonic-oscillator operators of two layers. The obtained vacuum configuration involved three different quantities where one describes the inter-layer interaction.

Subsequently, we have investigated the link between our second wave function and two others from literature. After projecting the vacuum configuration on the complex plane and using the Vandermonde determinant, we have shown how the Yoshioka–MacDonald–Girvin wave function with the filling factor $\nu = \frac{2}{m+n}$ can be obtained from our model, in particular that corresponding to the $\nu = \frac{1}{2}$ state. Likewise, we have recovered the unpolarized Halperin wave function, especially that for the $\nu = \frac{2}{5}$ state.

The case we have studied is in fact just a particular case of more general FQH states where the fluid droplet is assumed to consist of several coupled branches, say $M$ branches. $M = 1$ is the Laughlin (Susskind–Polychronakos) model, $M = 2$ is the model we have discussed here and $M \geq 3$ is the generic case, which can be seen as a straightforward generalization of our case.

Of course still some important questions remain to be answered e.g. about the fractional charge and statistics of the particles and how to describe them in terms of the proposed model. Another interesting question is related to the link between our model and Calogero and super–Calogero models. We will return to these issues and related matter in future.

We close this section by noting that our model will be investigated in the forthcoming work [26] for the case of a single layer. Basically, we will consider the Laughlin liquids in a confining potential that is not of parabolic type and see how this affects the basic features of the Susskind–Polychronakos model.

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