Finite Temperature Bosonic Closed Strings:
Thermal Duality and the Kosterlitz-Thouless Transition

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Abstract
We elucidate the properties of a gas of free closed bosonic strings in thermal equilibrium. Our starting point is the intensive generating functional of connected one-loop closed vacuum string graphs given by the Polyakov path integral. Invariance of the path integral under modular transformations gives a thermal duality invariant expression for the free energy of free closed strings at finite temperature. The free bosonic string gas exhibits a self-dual Kosterlitz-Thouless phase transition. The thermodynamic potentials of the gas of free bosonic closed strings are shown to exhibit an infinite hierarchy of thermal self-duality relations. Note Added (Sep 2005).

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1 Introduction

A reliable and self-consistent analysis of the thermodynamics of free strings is an essential step prior to the formulation of a microscopic and fully nonperturbative framework at finite string coupling that can cope with the physics of black holes, the formation of spacetime horizons and spacetime singularities, and the earliest evolutionary stages of the Universe at Planckian scales. Surprisingly, although the finite temperature behavior of an ensemble of free strings has been extensively studied since the early days of string theory [2, 4, 5, 7, 8, 9, 10] there are few conclusive results, even in this simplest of cases. A dominant theme in early works is that of a limiting temperature beyond which a gas of free strings is thought to become unstable with an exponentially diverging Helmholtz free energy. The notion of the Hagedorn phase transition, in fact, predates the world-sheet formalism of string theory and was originally proposed within the context of phenomenological dual reggeon models for the hadronic bound state spectrum [1, 3]. The motivation for the suggestion comes from the fact that the asymptotic density of states function for a zero temperature gas of free strings is known to exhibit exponential growth [2, 27]. A simple back-of-the-envelope estimate reviewed in the appendix indicates that, at least for particle-like thermodynamic ensembles, an exponential growth in the density of states function implies an exponentially diverging free energy beyond some characteristic temperature $T_H$. This abrupt change in the behavior of the free energy at $T_H$ has been interpreted as a thermal phase transition [3] and is known in the literature as the Hagedorn phase transition.

This intuition relies, however, on the assumption that the thermal behavior of a free string gas is well-approximated by that of infinitely many point particle modes with a level spectrum of integer-spaced Planck-scale masses. We will show in this paper, and even more conclusively in [16], that the lore of a Hagedorn phase transition in perturbative string theory is untrue. As already emphasized by Polchinski in [5], the free energy of a gas of free strings is not the sum of the free energies of infinitely many point-particle field theories with Planck scale masses. The free energy at one-loop order in string theory is expressed as an integral over the conformally inequivalent classes of a torus parameterized by a complex worldsheet modulus $\tau$ [5, 17], or over the fundamental domain of the modular group. The bosonic string theory has a zero temperature tachyon, as a consequence of which the Polyakov path integral exhibits an infrared divergence. However, modular invariance permits one to formally map the IR divergence into what would appear to be a UV divergence [6, 8, 17] by mapping the fundamental domain to a rectangular strip in the $\tau$ plane. This mapping obscures the physical (infrared) origin of the tachyonic divergence. In the tachyon-free heterotic string gas, it will be straightforward to demonstrate that the original worldsheet representation as an integral over the fundamental domain is perfectly adequate as a path integral expression for the vacuum functional at finite temperature. The path integral expression for the free energy of the heterotic string gas has neither ultraviolet nor infrared divergences at all temperatures starting from zero [16]. Thus, there is no signal of a Hagedorn transition in the one-loop free energy. In the type I open and closed string gas, a similar result follows from the absence of both dilaton tadpoles and tachyons in the open string spectrum [15, 16]. This paper serves as a warm-up to these latter works with a study of the pedagogical case of the closed bosonic string gas.

Of great interest, we will find concrete evidence for the occurrence of a Kosterlitz-Thouless continuous phase transition responsible for the formation of a “long string phase” in the high temperature gas of free open and closed strings [7, 17, 15, 16]. In the bosonic and heterotic closed
string gases the phase transition instead takes the form of a self-duality transition. We will show that the thermodynamic potentials characterizing the duality transition in free string gases can be computed in each case in transparent detail [16]. The thermodynamic potentials satisfy an infinite hierarchy of duality relations at criticality.

Our starting point is the Polyakov path integral representation for the one-loop contribution to the vacuum energy density in string theory [19, 5]. Denoted as $W(\beta)$, it gives the sum over connected random surfaces with the topology of a torus in the Euclidean spacetime $R^{25} \times$ (a compact one-dimensional target space whose volume corresponds to the inverse temperature, $\beta$). The generating functional for connected one-loop vacuum string graphs is an intensive thermodynamic variable on which we impose Euclidean T-duality relations. For the self-dual closed bosonic string, this implies invariance of the vacuum string functional: $W(\beta) \equiv \ln Z(\beta)$, under thermal duality transformations: $\beta \rightarrow \beta_2 \frac{C}{\beta}$. In addition, $W(\beta)$ is required to be invariant under modular transformations as a consequence of its definition as a reparametrization invariant sum over Riemann surfaces [19, 5]. We cannot of course achieve a tachyon-free thermal spectrum for the bosonic string but our computation gives a simple illustration of the general framework of string thermodynamics. The reader will find that it is straightforward to infer the corresponding results in the tachyon-free self-dual heterotic string ensemble [16].

The path integral representation of $W$ will lead directly to the Helmholtz function: $F(\beta) = -T \ln Z(\beta)$, also known as the Helmholtz free energy, and the finite temperature effective potential: $\rho(\beta) = -\frac{T}{V} \ln Z(\beta)$. The internal energy, $U$, pressure, $P$, entropy, $S$, enthalpy, $H$, and specific heat, $C_V$, of the free closed bosonic string gas will be derived by simply taking partial derivatives with respect to temperature, or spatial volume, of the modular and thermal duality invariant vacuum string functional, $\ln Z(\beta)$. We will find that the pressure of the free bosonic string gas vanishes in the absence of string interactions and that the enthalpy consequently equals the internal energy. As a result the Gibbs free energy of the free string gas, $G$, is found to coincide with the Helmholtz free energy, $F$: $G=H-TS=U+PV-TS=U-TS=F$. The thermodynamic potentials in general are found to satisfy an infinite hierarchy of thermal duality relations derived in section 4. Previous analyses of bosonic closed string thermodynamics in the canonical framework either neglect, or explicitly violate, thermal duality [4, 5, 6, 7, 8, 9, 10]. An exception is the thermal duality invariant closed bosonic string spectrum described in the recent text [17]. We should note that thermal duality follows naturally in a closed string theory as a consequence of worldsheet modular invariance.

The plan of this paper is as follows. In section 2, we begin with a brief review of some aspects of finite temperature field theory in the canonical formalism, giving a summary of the main thermodynamic potentials and thermodynamic identities that will be of interest in our discussion. In section 3.1, we obtain both the normalization and phase of the generating functional of one-loop vacuum string graphs at finite temperature. The normalization is obtained following the method developed in [5, 17]. We show that the temperature dependent phases in the string path integral are unambiguously determined by modular invariance or, equivalently, by thermal duality.

We begin by noting an ambiguity in the Euclidean time prescription which becomes apparent when we try to apply it to finite temperature string theory. Surprisingly, this has not been pointed out prior to the appearance of our work [15]. The topology of a compact one-dimensional target space is either that of a circle, $S^1$, with inverse temperature $\beta$ identified as the circumference, $2\pi r_{\text{circ}}$, or, that of an orbifold, $S^1/Z_2$, where the $Z_2$ acts on the circle as a reflection: $X^0 \rightarrow -X^0$, with fixed
points at $X^0=0$ and $X^0=\pi r_{\text{circ}}$, and $\beta$ identified as the interval length, $\pi r_{\text{circ}}$. Being extended objects, free strings can explore the global topology of the space in which they live. Consequently, both the free string mass spectrum, and the free energy of the free string gas, are expected to distinguish between the two possible topologies for Euclidean time. Which gives the correct answer? For finite temperature field theories, as mentioned already in [15, 16] and explained further in section 2, this ambiguity is of little consequence. The reason is as follows. While the effective action functional and, consequently, the free energy of the free string gas are unambiguously normalized, the leading $T$ dependence of the free energy comes only from massless field theory modes. A difference of a factor of two in the normalization is easily absorbed in a rescaling of $\beta$ or, more precisely, the Boltzmann constant, which has been set to unity in our use of natural units. The dependence of the free energy on the Planck scale massive modes is genuinely sensitive to the factor of two because of the additional exponential dependence on $\beta$: $F \simeq \beta^r e^{-\beta M^2}$, where $r$ is some integer characteristic of the free string gas. These massive modes are absent in the low energy field theory limit of the free string gas. Thus, while the topology of Euclidean time is indeed distinguished by the full string theory result the ambiguity is a moot point for the free energy of the massless field theory modes.

The physical consequence of imposing the $Z_2$ twist in Euclidean time is as follows. From Ginsparg’s analysis of the moduli space of $c=1$ conformal field theories, we notice that the Hagedorn radius of the circle compactified bosonic string, $2\pi \alpha'^{1/2}$, coincides with the Kosterlitz-Thouless (K-T) point lying at the intersection of orbifold and circle fixed lines [24]. The Kosterlitz-Thouless point is labelled $r_{\text{circ}}=2\pi \alpha'^{1/2}$ on the fixed line of circle compactifications, but along the orbifold fixed line, it is labelled $r_{\text{orb}}=\pi \alpha'^{1/2}$. This is also the self-dual radius of the orbifold. Thus, approaching the K-T point along the orbifold fixed line we find that the K-T transition occurs at the self-dual temperature of the free string gas: $T_C=\pi \alpha'^{1/2}$. Each of the thermodynamic potentials should undergo a continuous phase transition at the self-dual temperature exhibiting analytic behavior in the inverse temperature. This is precisely what we find when we evaluate the thermodynamic potentials: the internal energy is shown to vanish at $T_C$, and the Gibbs and Helmholtz free energies are minimized at the critical point. States localized at the fixed points of the orbifold: $X^0=0$, $\pi r_{\text{circ}}$, under the $Z_2$ transformation, $X^0\rightarrow-X^0$, contribute a constant term to the Helmholtz free energy and entropy of the bosonic string gas, but are otherwise absent from the internal energy, specific heat, and remaining thermodynamic potentials. The free bosonic string gas is, of course, only of pedagogical value since it contains a zero temperature tachyon in its mass spectrum. This fixed-point entropy of the free bosonic string gas will be found to be absent in the physically meaningful case of the heterotic string gas [16]. We will find in [16] that the general features of the duality transition hold also for the self-dual free heterotic string gas at $T_C$, but without the unphysical fixed point entropy.

Finally, in section 4, we derive the one-loop thermodynamic potentials for the free closed bosonic string gas. We should emphasize that, since they are derived by simply taking partial derivatives with respect to inverse temperature, or volume, of the duality invariant expression for the intensive string vacuum functional, the expressions for the thermodynamic potentials are not invariant under thermal duality transformations. Notice that the analyticity of the potentials as functions of inverse temperature follows naturally as a consequence of the analytic dependence of the amplitudes of perturbative string theory on any continuously-varying background modulus. In appendix A, we give a brief description of the tachyonic and massless modes in the physical state spectrum of the bosonic string theory, distinguishing between circle and orbifold compactifications of Euclidean time. We exhibit the tachyonic instability of each momentum mode below a characteristic critical
temperature, \( T_n \), and the corresponding tachyonic instability for each winding mode beyond a characteristic critical temperature, \( T_w \). For the closed bosonic string, the onset of the leading winding instability, unfortunately, occurs prior to removal of the leading momentum instability. As a consequence, the thermal spectrum is tachyonic over the entire temperature range, starting at \( T=0 \). Thus, the features of the thermal self-duality transition derived in section 4 are to be interpreted as strictly formal relations for the tachyonic bosonic string ensemble. They are a consequence of the Euclidean T-duality invariance of the closed bosonic string theory, a property it shares in common with the self-dual heterotic string theory. It is the tachyon-free free heterotic string gas which will provide us with a physical realization of the thermal self-duality phase transition, as shown in [16]. Appendix A concludes with a brief summary of previous arguments in the literature which mistakenly confuse a tachyonic winding mode instability with a signature for a Hagedorn phase transition in the free string gas. We include this clarification for pedagogical purposes. Appendix B clarifies the precise form of the fixed point contribution in the entropy and Helmholtz free energy. We review the derivation of the orbifold partition function, and the relationship of circle and orbifold fixed lines in the \( c=1 \) moduli space following [24, 17].

## 2 Aspects of Particle and String Thermodynamics

In this paper, we will compute the generating functional of connected one-loop vacuum graphs in a finite temperature closed string theory. We begin by recalling some aspects of the zero temperature computation of the one-loop contribution to the vacuum energy density [5]. Note that the Polyakov path integral at fixed topology, or at fixed order in the string coupling constant, is a sum over connected Riemann surfaces of definite topology [19, 5]. Note that in the functional integral formulation for perturbative string theory, the Polyakov path integral corresponds to \( W = -\ln Z \), where \( Z = \langle \Omega | \Omega \rangle \), and \( | \Omega \rangle \) is the interacting perturbative closed string vacuum at zero temperature. There is no string theory analog of the vacuum-to-vacuum amplitude usually computed in quantum field theory [20]. At zero temperature, and in the absence of background fields, the sum over connected Riemann surfaces of different topology gives the string theory analog of the sum over connected vacuum diagrams in field theory [20, 5]:

\[
W[J]|_{\frac{\delta W}{\delta J}=0} = \Gamma(0) = \text{sum of irreducible connected vacuum graphs} \quad , \tag{1}
\]

Here \( \Gamma \) is the quantum effective action functional or generating functional of one-particle-irreducible vacuum graphs [20]:

\[
W[J] = \Gamma(\phi_{\text{cl}}) + \int d^dx \frac{\delta W[J]}{\delta J(x)} J(x), \quad \frac{\delta W[J]}{\delta J(x)} = \phi_{\text{cl.}} = \langle \Omega | \phi \Omega \rangle , \tag{2}
\]

where \( | \Omega \rangle \) is the interacting vacuum and \( J \) an external source. The one-loop closed string vacuum graph computed in [5] corresponds to \( \Gamma(0) \): in the absence of background fields, \( \phi_{\text{cl.}} = \langle 0 | \phi | 0 \rangle = 0 \). As in perturbative quantum field theory, we calculate using a perturbation series in a small dimensionless coupling, \( g_{\text{closed}} \), which, for the purposes of the one-loop vacuum amplitude, implies that \( | \Omega \rangle \) is to be replaced by the free string vacuum, \( | 0 \rangle \). Thus, the mass spectrum we infer from the factorization limit of the one-loop vacuum graph is the free closed string mass spectrum. This
has important implications for string thermodynamics since it implies that, despite the fact that every perturbative string theory includes perturbative quantum gravity, the thermodynamics of the free string gas is expected to have a self-consistent formulation free of gravitational instabilities, including the Jeans instability: free strings do not gravitate. Loop corrections, on the other hand, are expected to be sensitive to gravitational instabilities.

The effective action functional is an intensive, and dimensionless, thermodynamic variable. It is sometimes convenient to work with the extensive effective potential defined as follows \[20, 5\]:

\[
\Gamma[\phi_{\text{cl}}] = -V_{\text{eff}}(\phi_{\text{cl}}) \cdot \int d^{d+1}x .
\] (3)

The one-loop effective potential in the zero temperature free string vacuum, \(V_{\text{eff}}(0)\), is nothing but the one-loop contribution to the vacuum energy density in flat spacetime, or one-loop cosmological constant, \(\rho_0\). For bosonic string theory, a first principles computation of \(\rho_0\) from the Polyakov path integral was given in \[5\], where the unambiguous normalization of the vacuum energy density in string theory was first noted. The one-loop string path integral is pure gauge as a consequence of two dimensional general coordinate invariance \[19\]. Its normalizability in a spacetime infrared finite string theory follows from the existence of a gauge-invariant world-sheet regulator preserving the Weyl invariance of the measure in the string path integral \[5, 29\].

Note that the formal infrared divergences of the bosonic string path integral as a consequence of the tachyon are being ignored here, in the same spirit as the discussion in the introduction. The reader should interpret these statements in the context of their implications for a tachyon-free, and infrared finite, ground state of the self-dual heterotic string. The world-sheet technicalities are otherwise similar. From a spacetime perspective, a tachyon-free heterotic string theory is both ultraviolet and infrared finite. The vacuum energy density will be both finite and calculable as a function of spacetime moduli, the continuously varying background parameters of a consistent string ground state \[18\]. In a spacetime supersymmetric ground state, the vacuum energy density will of course vanish, but the contributions from spacetime bosonic, and spacetime fermionic, modes are separately finite and calculable. A word about dimensions is in order. Since \(Z\) and \(W\) are both dimensionless, the effective potential or vacuum energy density has dimensions of \(L^{-(d+1)}\) in \(d\) spatial dimensions. We use the natural units \(\hbar=c=k_B=1\), where \(k_B\) is Boltzmann’s constant.

Let us now recall some of the standard wisdom in quantum statistical mechanics and finite temperature field theory \[21, 23\]. Consider the time independent quantum statistical mechanics of an equilibrium ensemble of particles with Hamiltonian, \(H\), at fixed temperature \(T\), and confined to a fixed spatial volume \(V\). The canonical partition function, \(Z=\text{Tr}e^{-\beta H}\), the Helmholtz free energy, \(F\), internal energy, \(U\), pressure, \(P\), entropy, \(S\), and specific heat at constant volume, \(C_V\), of the canonical ensemble are defined by the usual thermodynamic identities:

\[
F = -T\ln Z, \quad U = T^2 \frac{\partial}{\partial T} \ln Z, \quad P = -\left(\frac{\partial F}{\partial V}\right)_T, \quad S = -\left(\frac{\partial F}{\partial T}\right)_V, \quad C_V = T \left(\frac{\partial S}{\partial T}\right)_V .
\] (4)

All thermodynamic functions in quantum statistical mechanics can be derived, in principle, from a knowledge of density matrix elements of the evolution operator, \(e^{-\beta H}\). In finite temperature quantum field theory, the ill-defined nature of the trace requires that we replace these definitions with normalized expectation values of observables in a thermal ensemble. In either case, the evolution
equation satisfied by the density matrix suggests that we introduce a fictitious evolution parameter, \( \tau \), with range, \( 0 < \tau \leq \beta \). It is natural to associate \( \tau \) with an auxiliary imaginary time, \( \tau = -it \).

Thus, we can interpret the equilibrium Greens functions of an interacting \( d \)-dimensional thermal field theory as Greens functions in an interacting \( d+1 \)-dimensional Euclidean field theory, with Euclidean time an interval of size \( \beta \). Notice that “time reversal invariance” is a symmetry with no natural physical analog in the Euclidean time prescription. We will return to this point in the conclusions.

Operators in the Euclidean field theory are defined via the Heisenberg representation, \( \phi(\tau, x) = e^{\tau H_0} \phi(x) e^{-\tau H_0} \), with ensemble averages defined in the non-interacting Fock space. The fundamental object in the thermal field theory is the propagator which must satisfy a periodicity, or aperiodicity, condition in Euclidean time as a consequence of cyclicity of the trace [23]:

\[
G_\beta(\tau, \tau') = Z^{-1}(\beta) \left[ \text{Tr} \ e^{-\beta H} P_\tau(\phi(x, \tau) \phi^\dagger(x', \tau')) \right], \quad G_\beta(0, \tau) = \pm G_\beta(\beta, \tau). \tag{5}
\]

The choice of sign corresponds, respectively, to fields with integer, or half-integer, spin. The thermal Greens function for a free scalar field in \( d \) dimensions can be Fourier expanded in the basis:

\[
G_\beta(\tau, \tau') = \frac{1}{\beta} \sum_n \int \frac{d^d k}{(2\pi)^d} e^{-i(\omega_n \tau + k \cdot x)} G_\beta(\omega_n, k), \quad \omega_n = 2n\pi / \beta, \ n \in \mathbb{Z}. \tag{6}
\]

Conversely, the free fermion is expanded in a Fourier basis with odd frequencies: \( \omega_n = (2n+1)\pi / \beta \), ensuring the aperiodicity of the Greens function in Euclidean time. The Euclidean time prescription can be equivalently stated in terms of the Euclidean functional integral, where we restrict the path integral to Fourier modes that are, respectively, periodic or aperiodic in Euclidean time, for fields having, respectively, bosonic or fermi spacetime statistics [23]:

\[
Z(\beta) = \text{Tr} \ e^{-\beta H} = \int d\Phi \ < \Phi | e^{-\beta H} | \Phi > \equiv N \int_{\text{periodic}} [D\phi] e^{-\int_0^\beta d\tau \int d^d x L(\phi(x, \tau))}. \tag{7}
\]

Thus, the equilibrium behavior of a finite temperature field theory in \( d \) spatial dimensions has a field theoretic formulation in the Euclidean spacetime, \( R^d \times (a \text{ one-dimensional compact target space of volume } \beta) \), where \( \beta \) is the inverse temperature.

There is an ambiguity in the Euclidean time prescription that becomes apparent when we try to apply it to finite temperature string theory. Surprisingly, this has not been pointed out prior to the appearance of our work [15]. The topology of a compact one-dimensional target space is either that of a circle, \( S^1 \), with inverse temperature \( \beta \) identified as the circumference, \( 2\pi r_{\text{circ}} \), or, that of an orbifold, \( S^1 / \mathbb{Z}_2 \), where the \( \mathbb{Z}_2 \) acts on the circle as a reflection: \( X^0 \rightarrow -X^0 \), with fixed points at \( X^0 = 0 \) and \( X^0 = \pi r_{\text{circ}} \), and \( \beta \) identified as the interval length, \( \pi r_{\text{circ}} \). Being extended objects, free strings can explore the global topology of the space in which they live. Consequently, both the free string mass spectrum, and the free energy of the free string gas, are expected to distinguish between the two possible topologies for Euclidean time.

Further, recall that the prescription for finite temperature quantum field theory [21] requires that we sum over the same set of connected vacuum graphs as at zero temperature, but taking care to replace all integrals in frequency space with infinite sums over quantized Matsubara frequencies [23]. Either circle or orbifold topologies for Euclidean time result in an infinite sum over quantized
thermal frequencies, since this is simply a property of any compact target space. Moreover, all of the
modes in the bosonic string spectrum are spacetime bosons, and consequently the string functional
integral is already restricted to modes that are periodic in Euclidean time, quite independent of
topology. The ambiguity in the Euclidean time prescription remains. But as already mentioned
in the Introduction, the choice of orbifold topology has the satisfying physical consequence that the
thermal duality transition can be concretely identified with the Kosterlitz-Thouless continuous phase
transition. In the physically meaningful case of the heterotic string gas, the $Z_2$ action is essential
in eliminating the tachyon while preserving modular invariance and the correct zero temperature
limit [16].

Our starting point for a discussion of closed string thermodynamics in this paper is the world-
sheet path integral representation of the generating functional of connected one-loop vacuum string
graphs, $W= W(\beta)$, in the embedding space $R^d \times S^1 / Z_2$, and with interval length identified as inverse
temperature, $\beta$ [19, 5]:

$$W(\beta) \equiv \ln Z, \quad F(\beta) = -W/\beta, \quad \rho(\beta) = -W/V\beta, \quad U(\beta) = -T^2 \left( \frac{\partial W}{\partial T} \right)_V = \left( \frac{\partial W}{\partial \beta} \right)_V . \tag{8}$$

$F$ is the Helmholtz free energy of the ensemble of free strings, $U$ is the internal energy, and $\rho$ is the
finite temperature effective potential. Notice that $\rho$ is the finite temperature analog of the one-loop
vacuum energy density or cosmological constant, $\rho_0$. The inverse temperature, $\beta$, plays the role of a
spacetime modulus, a continuously-varying background parameter. Thus, finite temperature string
theory describes a one-parameter family of consistent ground states of string theory, characterized
by the $\beta$ dependent effective potential. We emphasize that the thermodynamic potentials of the free
string ensemble are obtained directly from the generating functional of connected vacuum graphs,
$\ln Z(\beta)$, as opposed to $Z(\beta)$. Thus, we never need address the troubling issue of defining the
thermodynamic limit of the canonical ensemble, since we never compute the canonical partition
function for string states directly in the path integral framework.

We should warn the reader that it is conventional in the string theory literature— and we
will stick with this convention, to refer to the modular invariant integrand of the string vacuum
functional, or free energy, as the “partition function”, usually denoted by $Z(\tau, \bar{\tau})$. The reason is that
the $q\bar{q} = e^{-4\pi\tau_2}$ expansion of $Z(\tau, \bar{\tau})$ is correctly interpreted as the level expansion in string theory:
the $n$th power of $q\bar{q}$ in the Taylor series expansion gives the degeneracy of the $n$th mass level in the
string mass spectrum, upon imposition of the level matching condition on physical states: $L_0 = \bar{L}_0$.
Terms in the Taylor expansion with unequal powers of $q$ and $\bar{q}$ do not meet the level matching
condition, and correspond to unphysical states in the mass spectrum. They drop out during the
process of integrating over $\tau_1$, and do not contribute to the free energy of the free string gas. A
small clarification is in order. There is a surprisingly extensive literature on unphysical tachyons
and their role in determining the asymptotics of the function $Z(\tau, \bar{\tau})$ as, for example, in ref. [11].
We should clarify that, in our work, when we refer to a tachyonic instability we mean a physical
tachyonic mode which meets the level matching conditions. Unphysical tachyons do enter the BRST
algebra of a covariant string theory, but we have not found any useful role for these states, even in
the case of the thermal fermionic strings discussed in [16].
3 Effective Potential and Free Energy

Consider an equilibrium ensemble of closed bosonic strings at fixed temperature in $d=25$ spatial dimensions. The strings can be assumed to be confined to a box-regularized spatial volume:

$$V = L^{25}(2\pi\alpha')^{25/2}.$$  \hspace{1cm} (9)

Since the vacuum string functional of interest, $W(\beta) = \ln Z(\beta)$, is an intensive thermodynamic variable, we will find that the precise regularization of the spatial volume is, fortunately, of no consequence. We have defined the inverse temperature to have dimensions of length, or inverse energy, working with the natural units $\hbar = c = 1$, and with Boltzmann’s constant set to unity. The normalized generating functional for connected one-loop vacuum string graphs in the Euclidean embedding space $R^{25} \times S^1 / Z_2$ can be computed as shown in [19, 5, 17, 15]. This directly gives the Helmholtz free energy of the free closed bosonic string gas: $F = -W/V \beta$, the basic thermodynamic potential from which we can derive all of string thermodynamics in the canonical formalism. Our results can be compared with the partial insights gained in the previous works [2, 3, 4, 5, 6, 7, 8, 9, 10]. As emphasized in the introduction, the important differences in our approach come from taking seriously the identification of the Polyakov path integral as the world-sheet representation of $\ln Z(\beta)$ and requiring, in addition, that the generating functional of connected string graphs transform correctly under Euclidean T-duality transformations. For the closed bosonic string gas, this implies a precise invariance of $W(\beta)$ under thermal duality transformations: $\beta \rightarrow \beta_C^2 / \beta$.

3.1 Normalization and Phase of the Effective Potential

We begin with the world-sheet representation of the generating functional of connected one-loop vacuum string graphs in closed bosonic string theory at finite temperature. Since the derivation of the one-loop string path integral for closed bosonic string theory compactified on the spatial orbifold $S^1 / Z_2$ can be found in the references [19, 5, 17], we will simply write down the result [15]:

$$W_{\text{bos.}}(\beta) = \frac{1}{2} \int_F \frac{|d\tau|^2}{4\pi \tau_2^2} \left( \frac{2\pi \tau_2}{23/2} \eta(\tau)|^{-46} \right) \times \left[ \frac{1}{\eta \bar{\eta}} \sum_{n,w \in \mathbb{Z}} q^{n \alpha'^2} q^{w \alpha'^2} + (|\Theta_3 \Theta_4| + |\Theta_2 \Theta_3| + |\Theta_2 \Theta_4|) \right].$$  \hspace{1cm} (10)

$W(\beta)$ is an intensive thermodynamic variable. Dividing out by the volume of Euclidean spacetime gives the dimensionful finite temperature effective potential: $\rho(\beta) = -W/V \beta$. The various factors in $\rho(\beta)$ may be understood as follows. The factor of $\beta = \pi \tau_{\text{circ.}}$ arises from the integration over Euclidean time, the interval length of the orbifold $S^1 / Z_2$. The box-regularized spatial volume arises from integration over the zero modes of the 25 spatial embedding coordinates. The factor of $\frac{1}{2}$ is the $Z_2$ symmetry factor obtained when we project to the subspace of string modes invariant under the reflection, $X^0 \rightarrow -X^0$. The relative signs of the different terms in the square brackets are determined by modular invariance. Here, $p_L(R) = \frac{n x}{2} \pm \frac{w x}{2}$. We can introduce a dimensionless inverse temperature (radius) defining $x \equiv r(2/\alpha')^{1/2}$, with $\beta = \pi (\alpha' / 2)^{1/2} x$. The dimensionless quantized momenta live in a $(1, 1)$ dimensional Lorentzian self-dual lattice [17]:

$$\Lambda^{(1,1)}: \quad \left( \frac{\alpha'}{2} \right)^{1/2} (p_L, p_R) \equiv (l_L, l_R) = \left( \frac{n}{x} + \frac{wx}{2}, \frac{n}{x} - \frac{wx}{2} \right).$$  \hspace{1cm} (11)
The momentum summation is manifestly invariant under thermal duality, $\beta \to \pi^2 \alpha'/\beta$, or $x \to 2/x$, simultaneously interchanging the dummy indices, $n \to w$.

We begin with a simple analysis of the massless states in the physical state spectrum of the orbifold, clarifying the necessity for a $Z_2$ twist in the Euclidean time prescription for the thermal bosonic string. Consider the mass formula for physical states in the untwisted sector of the orbifold:

$$\text{(mass)}^2 = \frac{2}{\alpha'} (-2 + \frac{1}{2} H_L^2 + \frac{1}{2} H_R^2 + N_0 + \tilde{N}_0 + N_\mu + \tilde{N}_\mu) ,$$  \hspace{1cm} (12)

where we are required to project onto the subspace of states that are invariant under the reflection: state with $N_0 + \tilde{N}_0$ equal to an even integer, and lattice vectors invariant under the reflection, $(n, w) \leftrightarrow (-n, -w)$. The level-matching condition for physical states takes the form:

$$2nw + N_0 - \tilde{N}_0 + N_\mu - \tilde{N}_\mu = 0, \quad \mu = 1, \cdots, 25. \hspace{1cm} (13)$$

The zero point energy in the twisted sectors differs by $+\frac{1}{16}$, and the oscillator moding will be half-integral. There are no massless states in the twisted sectors of the orbifold.

Denote a generic state in the bosonic string spectrum by $|\alpha_0, \cdots, \alpha_{-1} \cdots |k, \tilde{k}; (n, w) >$. Notice that the $Z_2$ twist in Euclidean time breaks the $U(1) \times U(1)$ symmetry present at generic radius in the circle compactification, thereby removing the Kaluza-Klein vector states: $\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\mu |k^\mu, 0; (0, 0) >$, and $\alpha_0^\mu \tilde{\alpha}_{-1}^\mu |0, \tilde{k}^\mu; (0, 0) >, \mu = 1, \cdots, 25$, from the massless spectrum. A scalar massless modulus, $\alpha_0^\mu \tilde{\alpha}_{-1}^\mu |0, 0; (0, 0) >$, remains, a marginal perturbation of the world-sheet conformal field theory corresponding to a change of radius (inverse temperature). The remaining massless states in the physical state spectrum are the 25-dimensional graviton, antisymmetric tensor field, and dilaton, of the bosonic closed string theory.

It is instructive to compare our expression in Eq. (10) for the Helmholtz free energy with Eq. (33) of [5], where the high temperature limit of the generating functional of connected one-loop vacuum string graphs was computed in the path integral formulation. The normalized, and modular invariant, measure for moduli in Eq. (10) is exactly as derived in [5, 17]. It corresponds to the world-sheet metric, $ds^2 = |d(\sigma^1 + \tau \sigma^2)|^2$, where $\tau = \tau_1 + i\tau_2$ is the complex modulus for the torus. The integration region, $\mathcal{F}$, is the fundamental domain of the torus [17]. The factor in round brackets is the modular invariant contribution from free string periodic oscillators in 23 non-compact transverse directions. Here, $q = e^{2\pi i \tau}$. As mentioned above, the integration over $\tau_1$ imposes the level matching condition, $L_0 = \tilde{L}_0$, projecting out all states other than the physical states in the mass spectrum of the free closed bosonic string.

The thermal duality invariant contribution to $Z(\tau, \bar{\tau})$ from winding and momentum modes is easily derived in the operator formalism using current algebra methods [24, 25]. The integers $n$ and $w$ label, respectively, momentum and winding modes of the string in Euclidean time. For closed bosonic strings in an embedding space with one compact dimension, the physical state condition reads: $2nw + N_0 - \tilde{N}_0 + N_\mu - \tilde{N}_\mu = 0, \mu = 1, \cdots, 25$. The $Z_2$ orbifold twist implies that in the untwisted sector we project to the subspace of states for which $N_0 + \tilde{N}_0$ is even, keeping only the symmetric linear combinations of the $(+, 0)$ and $(-, 0)$ momentum states, and the $(0, +w)$ and $(0, -w)$ winding states. The zero point energy in the twisted sectors is higher than that in the untwisted sector by the factor $+\frac{1}{16}$. There are no massless modes in the twisted sectors. We note that
the tachyonic states in the twisted sector lead to subleading divergences due to the higher zero point energy. We reiterate that the fixed point contributions are absent in the internal energy and successive partial derivatives with respect to temperature. They do appear in the Helmholtz free energy and entropy of the bosonic string gas, but will be absent in the corresponding expressions for the heterotic string gas.

A Poisson resummation of the factor in square brackets in Eq. (10) puts it in a manifestly modular invariant form suitable for comparison with the path integral derivation in [5]:

\[
[\cdots] = \frac{1}{|\eta|} \left[ (2\tau_2)^{-1/2} \sum_{n,w \in \mathbb{Z}} e^{-\frac{\pi^2}{2\tau_2} |n-w\tau|^2} e^{\pi i m n w^2} + (|\Theta_3 \Theta_4| + |\Theta_2 \Theta_3| + |\Theta_2 \Theta_4|) \right] .
\] (14)

The path integral derivation of the contribution from winding and momentum modes is as follows. In the high temperature limit, we can restrict the \( p^0 \) summation to modes winding in the \( \sigma^2 \) direction. We decompose \( X^0 = y^0 + X^0_{\text{cl}} \), where \( X^0_{\text{cl}} \) is a classical solution to the equations of motion periodic in the \( \sigma^2 \) direction [5]:

\[
X^\mu (\sigma^1, \sigma^2 + 2\pi) = y^\mu (\sigma^1, \sigma^2) + 2n\beta \sigma^2 \delta_0^\mu, \quad S(X, g_{ab}) = S(y, g_{ab}) + \frac{\pi n^2 \tau^2}{2\tau_2}, \quad n \in \mathbb{Z} .
\] (15)

\( S \) is the classical action for the Polyakov string. We have used the fact that in the world-sheet metric defined earlier, \( \sqrt{|g|} g^{22} = 1/\tau_2 \). The extra term in the world-sheet action contributes the exponential: \( \sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2 / \pi \alpha' \tau_2} \), to the expression for the one-loop vacuum energy density [5]. Generalizing to modes which wind in both \( \sigma^1 \) and \( \sigma^2 \) directions, and referring to the full world-sheet metric, we obtain:

\[
X^\mu (\sigma^1 + 2\pi, \sigma^2 + 2\pi) = y^\mu (\sigma^1, \sigma^2) + 2n\beta \sigma^2 \delta_0^\mu + 2w\beta \sigma^1 \delta_0^\mu \\
S(X, g_{ab}) = S(y, g_{ab}) + \frac{\pi x^2}{2\tau_2} \left( n^2 + |\tau|^2 w^2 - 2\tau_1 n w \right)^2 ,
\] (16)

with \( w, n \in \mathbb{Z} \). Performing the Gaussian path integrals as before gives the terms in the exponential, in agreement with both Eq. (14) and Eq. (10). Notice that modular invariance is manifest in Eq. (14) while thermal duality invariance becomes manifest in the form given in Eq. (10). Thermal duality interchanges the low and high temperature phases of string theory, and like modular invariance, it requires the presence of both species of modes in the vacuum energy functional [17].

To summarize, the thermal duality invariant result for the generating functional of connected one-loop vacuum graphs in the embedding space \( R^{25} \times S^1 / \mathbb{Z}_2 \) given in Eq. (10) can be written in path integral form as follows:

\[
W(\beta) = (V/\beta)^{-1} \cdot \sum_{\mathbf{p} \in \mathbb{Z}} (-1)^{\frac{1}{2} (\mathbf{p}^0 - \mathbf{R})^2} \int \frac{[dX][dg_{ab}]}{\text{Vol}(\text{gauge})} e^{-S(X, g_{ab}) - \mu_0} \int d^2 \sigma \sqrt{g} - \mu_0 \int d^2 \sigma \sqrt{g} .
\] (17)

Here, \( \mu_0 \) is the bare world-sheet cosmological constant which renormalizes to zero in the Weyl anomaly free critical string [19, 5]. \( R_g \) is the intrinsic curvature of the world-sheet, and the term proportional to the Euler character will, of course, vanish for surfaces with the topology of a torus. We have written \( W \) in a form which clarifies that multi-loop contributions to the vacuum
energy density are also required to be thermal duality invariant: we sum over momentum sectors weighted by arbitrary temperature dependent phases, subject to the preservation of thermal duality invariance. Within each sector, we perform the path integral in a manifestly reparameterization invariant way preserving Weyl invariance [5]. This is an intriguing observation since, in practice, it may be easier to impose thermal duality on the phase of the multiloop path integral as opposed to modular invariance.

3.2 High Temperature Effective Potential and Holography

The generating functional for connected one-loop vacuum string graphs is invariant under a thermal duality transformation: $W(T) = W(T^2/T)$, with self-dual temperature, $T_c = 1/\pi \alpha'^1/2$. As pointed out by Polchinski [17], we can infer the following thermal duality relation which holds for both the Helmholtz free energy, $F(T) = -T \cdot W(T)$, and the effective potential, $\rho(T) = -T \cdot W(T)/V$:

$$F(T) = \frac{T^2}{T_c^2} F\left(\frac{T_c^2}{T}\right), \quad \rho(T) = \frac{T^2}{T_c^2} \rho\left(\frac{T_c^2}{T}\right). \quad (18)$$

Consider the high temperature limit of this expression:

$$\lim_{T \to \infty} \rho(T) = \lim_{T \to \infty} \frac{T^2}{T_c^2} \rho\left(\frac{T_c^2}{T}\right) = \lim_{(T_c^2/T^2) \to 0} \frac{T^2}{T_c^2} \rho\left(\frac{T_c^2}{T}\right) = \frac{T^2}{T_c^2} \rho_0, \quad (19)$$

where $\rho_0$ is the cosmological constant, or vacuum energy density, at zero temperature. Likewise, at high temperatures, the free energy grows as the square of the temperature. Thus, growth in the number of degrees of freedom at high temperature in the gas of free closed bosonic strings is only as fast as in a two-dimensional field theory. This is significantly slower than the $T^{26}$ growth of the high temperature degrees of freedom in the low energy finite temperature field theory. We comment that the coincidence of the self-dual point of the orbifold fixed line with the Hagedorn point of the bosonic string implies that the holographic relation holds at the Hagedorn point, as originally conjectured in [10].

Notice that the prefactor in the high temperature relation is unambiguous, a consequence of the normalizability of the generating functional of one-loop vacuum string graphs in string theory [5]. The pre-factor, $\rho_0/T_c^2$, is also background dependent: it can be computed as a continuously varying function of the background fields upon compactification to lower spacetime dimension [18]. The relation in Eq. (19) is unambiguous evidence of the holographic nature of perturbative string theory: a reduction in the degrees of freedom of a self-dual closed string theory at high temperatures, or short distances [10]. An identical holographic relation holds for a physical realization of the self-dual gas of free closed strings: the tachyon-free high temperature gas of free heterotic strings with gauge group $SO(16) \times SO(16)$ [15, 16].

4 Kosterlitz-Thouless Self-Duality Phase Transition

In this section we show how to systematically derive the full set of thermodynamic potentials for the canonical ensemble of free strings. We will verify from the analytic behavior of the thermodynamic potentials as functions of inverse temperature that the phase transition in a self-dual gas of free
closed strings is of the Kosterlitz-Thouless type: the infinite hierarchy of thermodynamic potentials exhibits analytic behavior as a function of temperature at the critical point [28]. As a consequence of thermal duality, we will find that the internal energy vanishes precisely at the self-dual temperature. The internal energy is a monotonically increasing function of temperature, crossing from negative values at low temperature to positive values at temperatures above $T_c$. The low temperature phase is dominated by the Kaluza-Klein states. In the high temperature phase, long winding strings become energetically feasible, and are entropically favoured. We remind the reader that the thermal spectrum of the bosonic string is tachyonic at all temperatures starting from zero. In the tachyon-free physical realization of a self-dual free closed string gas, namely the finite temperature heterotic ensemble described in [16], we expect to find that both the Helmholtz and Gibbs free energies are minimized at $T_C$, while the specific heat is positive-definite. These features could be taken as indicators of thermodynamic stability for the self-dual gas of free heterotic strings.

We begin with the modular and thermal duality invariant expression for the generating functional of finite temperature one-loop vacuum graphs in the closed bosonic string:

$$ W(\beta) = \frac{1}{2} \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi \tau^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} \left[ \sum_{n,w} q^{\frac{\alpha'}{2}} \frac{q^{\frac{\alpha'}{2}}}{4} + (|\Theta_3\Theta_4| + |\Theta_2\Theta_3| + |\Theta_2\Theta_4|) \right]. \quad (20) $$

Recall the relation, $\beta = \pi x (\frac{\alpha'}{2})^{1/2}$. Referring to the thermodynamic definitions given in Eq. (4), it is evident that the expressions we will derive for the thermodynamic potentials are not invariant under thermal duality transformations. The expression for the internal energy of the gas of free closed bosonic strings takes the form:

$$ U(\beta) = -\left( \frac{\partial W}{\partial \beta} \right)_V = \frac{1}{2} \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi \tau^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} \frac{4\pi \tau_2}{\beta} \sum_{n,w} \left( \frac{w^2 x^2}{4} - \frac{n^2}{x^2} \right) \cdot q^{\frac{1}{2}\bar{\nu}} q^{\frac{1}{2}\bar{\nu}}. \quad (21) $$

$U(\beta)$ vanishes precisely at the self-dual temperature, $T_c=1/\pi \alpha'^{1/2}$, $x_c=\sqrt{2}$, where the internal energy contributed by winding sectors cancels that contributed by momentum sectors. The internal energy changes sign at $T = T_C$, transitioning from negative values at low temperature to positive values at high temperature. The balance between energy and entropy shifts at the self-dual temperature, as seen from the definition of the Helmholtz function: $F = U - TS$. The positive contribution to the internal energy from a winding sector implies that while the creation of a long winding string is costly in energy, it becomes entropically favoured at high temperature.

A clarification about the zero temperature limit is in order. From Eq. (20), the contributions to the free energy from winding sectors are exponentially damped for small $\beta$. Conversely, the contributions from momentum sectors are exponentially damped at large $\beta$. At the self-dual temperature, winding and momentum sectors contribute equally to $F$. Notice that the asymptotic value of the expression for the free energy in the $\beta \to \infty$ limit includes the fixed point contributions in Eq. (20). In order that the asymptotic value match correctly with the expected zero temperature result: $F_0 = V \rho_0$, the noncompact limit has to be defined with care, suitably resolving the orbifold singularity.

It is easy to demonstrate the analyticity of infinitely many thermodynamic potentials in the vicinity of the critical point. It is convenient to define:

$$ [d\tau] \equiv \frac{1}{2} \left[ \frac{|d\tau|^2}{4\pi \tau_2^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} e^{2\pi i n w \tau_1} \right], \quad y(\tau_2; x) \equiv 2\pi \tau_2 \left( \frac{n^2}{x^2} + \frac{w^2 x^2}{4} \right). \quad (22) $$
Denoting the $m$th partial derivative with respect to $\beta$ at fixed volume by $W_{(m)}$, $y_{(m)}$, and setting $x \equiv \alpha \beta$, we note that the higher derivatives of the generating functional take the simple form:

\begin{align}
W_{(1)} &= \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y}(-y_{(1)}) \\
W_{(2)} &= \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y}(-y_{(2)} + (-y_{(1)})^2) \\
W_{(3)} &= \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y}(-y_{(3)} - y_{(1)}y_{(2)} + (-y_{(1)})^3) \\
\cdots &= \cdots \\
W_{(m)} &= \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y}(-y_{(m)} - \cdots + (-y_{(1)})^m) .
\end{align}

(23)

Referring back to the definition of $y$, it is easy to see that the generating functional and, consequently, the full set of thermodynamic potentials is analytic in $x$. Notice that third and higher derivatives of $y$ are determined by the momentum modes alone:

\[ y_{(m)} = (-1)^m n^2 x^{m+2}, \quad m \geq 3 . \]

(24)

For completeness, we give explicit results for the first few thermodynamic potentials:

\[ F = -\frac{1}{\beta} W_{(0)}, \quad U = -W_{(1)}, \quad S = W_{(0)} - \beta W_{(1)}, \quad C_V = \beta^2 W_{(2)}, \cdots . \]

(25)

Let $2\pi \tau_2 \equiv t$. The entropy is given by the expression:

\[ S = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} \left[ 1 + 2t \left( -\frac{n^2 x^2}{4} + \frac{w^2 x^2}{4} \right) \right] + S_0 , \]

(26)

where $S_0$ denotes the fixed point contributions to the entropy. For the specific heat at constant volume, we have:

\[ C_V = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} \left[ 4t^2 \left( -\frac{n^2 x^2}{4} + \frac{w^2 x^2}{4} \right)^2 - 2t \left( 3 \frac{n^2 x^2}{4} + \frac{w^2 x^2}{4} \right) \right] . \]

(27)

It is easy to check the sign of the specific heat at criticality for an infrared finite self-dual ensemble of closed strings. Since the first term in square brackets will not contribute at $T_C$, a stable thermodynamic ensemble with positive specific heat requires positive Helmholtz free energy. This is precisely as found for the finite temperature heterotic string with gauge group $SO(16) \times SO(16)$ [16]. Due to an excess of spacetime fermions over spacetime bosons in the physical state spectrum, it has positive free energy.

Finally, consider the fixed point contributions in the expressions for both the free energy, and in the entropy. The Helmholtz free energy takes the form:

\[ F(\beta) = -\frac{1}{2} \beta \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi \tau_2^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} \left[ \sum_{n,w} q^{\frac{1}{2} u^2 + \frac{1}{2} v^2} + (|\Theta_3 \Theta_4| + |\Theta_2 \Theta_3| + |\Theta_2 \Theta_4|) \right] . \]

(28)
Given the relations in Eq. (25), for the entropy we have the result:

\[
S(\beta) = \frac{1}{2} \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi \tau_2^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} \sum_{n,w} \left[ 1 + 4\pi \tau_2 \left( -\frac{n^2}{x^2} + \frac{w^2 x^2}{4} \right) \right] q^{\frac{1}{2} \alpha' p^2} L^2 \bar{q}^{\frac{1}{2} \alpha' p^2} R^2 + \frac{1}{2} \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi \tau_2^2} (2\pi \tau_2)^{-23/2} |\eta(\tau)|^{-48} |\Theta_2 \Theta_3| + |\Theta_2 \Theta_4| + |\Theta_3 \Theta_4|.
\]

Can we give a more intuitive interpretation for the fixed point entropy \( S_0 \)? Since it arises from string states localized at the fixed points of the orbifold, this has a plausible microscopic interpretation. Using the relation, \( S_0 = k_B \ln \Omega_0 \), we interpret \( \Omega_0 \) as the thermodynamic probability associated to the orbifold singularity in Euclidean time. Fortunately, this somewhat unphysical fixed point entropy will be absent when we carry out an analogous analysis for the self-dual heterotic string gas [16].

5 Conclusions

This work is a satisfying resolution to two puzzles raised in a seminal work on string thermodynamics [10]. The first is the clash between the back-of-the-envelope argument for a Hagedorn phase transition with an exponentially diverging free energy, faced with the absence of a demonstrable ultraviolet divergence in the one-loop effective potential or Helmholtz free energy [17]. The second is the clash of physical intuition with the notion of a thermal duality invariant free energy—dismissed as nonsense in ref. [10]. We have shown that thermal duality follows naturally as a consequence of modular invariance in a closed string theory. Moreover, it leads to sensible results for the free string ensemble in perfect accord with physical intuition: the thermal duality invariant object is the intensive generating functional for connected one-loop vacuum string graphs, \( \ln Z(\beta) \). The Helmholtz and Gibbs free energies are not thermal duality invariant, and the internal energy of the free string gas is a monotonically increasing function of temperature. Furthermore, as a consequence of modular invariance, in a tachyon-free closed string theory, the free energy of the free string gas does not exhibit a Hagedorn divergence [16].

We have pointed out several errors in the standard wisdom about the Hagedorn transition in the previous literature on this subject [5, 4, 6, 9, 8, 10, 17]. The winding number one instability is not a signature for a Hagedorn divergence in the free energy; the free energy of the bosonic string gas is already divergent below \( T_H \) due to the presence of low temperature tachyonic modes in the pure momentum sector. The oft-used mapping of fundamental domain to strip in the Polyakov path integral confuses a divergence whose physical origin is infrared with what appears to be a divergence of ultraviolet origin. This is permissible as a consequence of modular invariance [17]. However, in the tachyon-free heterotic string gas, it is easy to demonstrate that there is no divergence in the free energy at all temperatures starting from zero [16]. Finally, in section 4, we have exhibited the presence of the Kosterlitz-Thouless continuous phase transition at the Hagedorn point. In the self-dual bosonic and heterotic string gases, this is a self-duality transition. In the type I open and closed string gas, it manifests itself as the transition to a high temperature long string phase as anticipated in previous works [7, 12, 17, 13]. The free energy and all of its derivatives are continuous at the critical point in each case, bosonic, heterotic, or type I. In the tachyon-free heterotic and type I string gases, they are also finite and normalizable functions [15, 16]. The peculiarities of the type II string gases are described in [16].
Our results give a clear analysis of the thermodynamics of the free closed bosonic string gas within the framework of the canonical ensemble. While many of the interesting physics applications of string thermodynamics await the ability to pose questions at finite string coupling, ideally in a nonperturbative framework, it is essential to have a reliable and self-consistent framework that describes the perturbative weak coupling limit. Of profound interest is a better understanding of the thermal duality transition in supersymmetric string theories, and the spontaneous breaking of thermal duality in the strongly coupled heterotic string at low temperature [15]. These are clarified in [16]. Fermionic string theories are replete with the possibility of infrared instabilities at low temperature in the absence of spacetime supersymmetry. This phenomenon is easy to exhibit [4, 10, 15] and its resolution brings in new concepts discussed in [16].

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Note Added (Sep 2005): I should first correct an error, corrected in my papers since Aug 2004. The pressure of the string ensemble is non-vanishing, and equal to the negative of the vacuum energy density. I still find the observation that field theory does not distinguish between circle or orbifold topology for Euclidean time, whereas string theory does, insightful. Notice, from the discussion in Appendix B, that the zero temperature limit of the expression for the string free energy computed with the orbifold ansatze would display extraneous fixed point contributions absent in the original, zero temperature, vacuum energy [5]. The reason is that these extraneous terms lack any dependence on $\beta$, as apparent in Eq. (39). Thus, the analysis in this paper establishes the absence of an ambiguity in the Euclidean time formalism for string statistical mechanics: the finite temperature quantization corresponds to the background $R^{d-1} \times S^1$, where $d$ is the critical dimension of the string theory, and $\beta$ is the circumference of the circle. The use of the world “holography” to describe the $T^2$ growth of the string free energy at high temperatures has puzzled some readers. I simply invoked the term in order to stress the drastic thinning of high temperature degrees of freedom in string theory compared with the $T^d$ growth of the free energy in a $d$-dimensional field theory. Finally, the term “infrared instability” as used in conjunction with fermionic string theories simply refers to the fact that the spontaneous breaking of supersymmetry at finite temperature, consistent with modular invariance, in the type II superstring theories immediately leads to tachyonic modes in the thermal spectrum. The stable endpoint of worldsheet RG flow is the zero temperature, 10D N=2 supersymmetric vacuum.
A: Absence of the Hagedorn Phase Transition

The thermal mass spectrum of the free closed bosonic string gas contains tachyonic physical states at all temperatures starting from zero. Each is an instability of the thermal ensemble and, as a consequence, the duality relations and the expressions derived for the thermodynamic potentials in section 4 are strictly formal statements for the bosonic string; an illustration of the thermodynamics of a self-dual gas of free closed strings. This analysis is, however, easily repeated for the tachyon-free, and infrared finite, self-dual gas of free heterotic strings [16].

Recall the mass formula for physical states in the untwisted sector of the orbifold:
\[
(mass)^2 = \frac{2}{\alpha'} \left( -2 + \frac{1}{2} I_L^2 + \frac{1}{2} I_R^2 + N_0 + \bar{N}_0 + N_\mu + \bar{N}_\mu \right),
\] (29)
where we are required to project onto the subspace of states that are invariant under the reflection: state with \(N_0 + \bar{N}_0\) equal to an even integer, and lattice vectors invariant under the reflection, \((n, w) \leftrightarrow (-n, -w)\). The zero point energy in the twisted sectors differs by \(+\frac{1}{16}\), and the oscillator moding will be half-integral. The level-matching condition for physical states takes the form:
\[
2nw + N_0 + \bar{N}_0 + N_\mu + \bar{N}_\mu = 0, \quad \mu = 1, \ldots, 25.
\] (30)
Both the untwisted and twisted sectors of the \(Z_2\) orbifold contain potentially tachyonic thermal modes. Note that the leading tachyonic instability always occurs in the untwisted sector since it has lower zero point energy. The untwisted and twisted sector tachyons with \(n = w = 0\) have mass squared \(-4/\alpha'\) and \(-15/4\alpha'\), respectively. Notice that these states do not contribute to the internal energy as can be seen in Eq. (21), but are present in the Helmholtz free energy and entropy.

The thermal modes in the untwisted sector with \(N_0 = \bar{N}_0 = 0\), and either \(n\) or \(w\) non-zero, are potentially tachyonic over some part of the temperature range. They contribute in both the Helmholtz function, as well as in the internal energy of the free string gas. Specifically, each momentum mode \((\pm n, 0)\) is tachyonic \(upto\) some critical temperature, \(T_n = 2/\pi n \alpha'^{1/2}\), after which it becomes stable. Conversely, each winding mode \((0, \pm w)\) becomes tachyonic \(beyond\) some critical temperature \(T_w = w/2\pi \alpha'^{1/2}\). Either satisfies level matching and belongs in the physical state spectrum; requiring invariance under the \(Z_2\) twist in Euclidean time implies that we restrict to the symmetric linear combination of the \((+n, 0)\) and \((-n, 0)\) modes, and likewise for the pure winding modes. The masses of the potentially tachyonic physical states are:
\[
(mass)_n^2 = \frac{2}{\alpha'} \left( -2 + \frac{\pi^2 n^2 \alpha'}{2 \beta^2} \right), \quad (mass)_w^2 = \frac{2}{\alpha'} \left( -2 + \frac{w^2 \beta^2}{2 \pi^2 \alpha'} \right).
\] (31)
Notice that both the \((\pm 1, 0)\) and \((0, \pm 1)\) states are tachyonic at the self-dual point, while the \((\pm 2, 0)\) and \((0, \pm 2)\) modes are marginally relevant.

The asymptotic density of string states in the zero temperature spectrum is known to grow exponentially as a function of mass in the spacetime ultraviolet regime. We will approximate the mass degeneracies, \(g_N(m_N)\), at the \(N\)th mass level in the string spectrum with \(N >> 1\), by a smooth mass density function \(g(m)\) [1, 2, 27, 3, 4]. As in the zero temperature spectrum, the asymptotic form of the density of states function for highly excited thermal states exhibits an exponential divergence, characterized once again by the Hagedorn temperature of the bosonic string. The point...
is that the asymptotics of the partition function is identical at zero, or at finite, temperature since
the asymptotics is characterized by the zero point energy of the untwisted sector of the critical
string. The asymptotics in the twisted sector differs but the divergence is sub-leading, since the
twisted sector has higher zero point energy. Consider the asymptotic form of the mass formula for
highly excited thermal states in the untwisted sector of the orbifold:

\begin{equation}
(mass)^2 \simeq \frac{4}{\alpha'} N_{\text{tot}}, \quad N_0 = \tilde{N}_0, \quad N_\mu = \tilde{N}_\mu.
\end{equation}

We have dropped the temperature dependent zero mode contributions in comparison to \(N_0\), for
large \(N_0\). The asymptotic growth of Hardy and Ramanujan’s partition sum is well-known to be
exponential [27, 2]:

\begin{equation}
\prod_{k=1}^{\infty} (1 - x^k)^{-d} \equiv \sum_{N=1}^{\infty} p(N)x^N \quad \lim_{N \to \infty} p(N) = N^{-(d+3)/4}e^{2\pi(d/6)^{1/2}N^{1/2}}.
\end{equation}

At high levels, the masses in the free string spectrum approach a continuum, and the approximation
of a smooth number density as a function of mass, \(g(m)\), is usually made in the string literature
[2, 3, 4, 8, 10, 17]. In fact, by substitution from above, the asymptotic mass density, \(g(m)\), for states
of mass, \(m\), in the untwisted sector of the free bosonic string gas is found to take the form [2]:

\begin{equation}
g(m) \simeq e^{4\pi\alpha'^{1/2}m}.
\end{equation}

Exponential growth in the mass density of a particle-like thermodynamic ensemble usually in-
dicates the existence of a characteristic exponential divergence in the free energy, as first noted by
Hagedorn [1]. The argument is simple. Consider a statistical ensemble of particles described by a
Hamiltonian, \(H\), with eigenspectrum described by an exponentially growing mass density, \(g(m) = e^{bm}\), with \(b\) some characteristic length scale. The Helmholtz free energy of the particle ensemble
can be written as [1, 2]:

\begin{equation}
F(\beta) = V \int_{m_0}^{\infty} dm g(m) e^{-\beta H} = V \int_{m_0}^{\infty} dm e^{-\beta -(\beta - b)m},
\end{equation}

where \(m_0\) is an infrared cutoff on the mass spectrum of the statistical ensemble below which the
exponential form assumed for \(g(m)\) may not hold. The free energy is seen to diverge exponentially
for temperatures \(\beta^{-1} > 1/b\), and the critical value, \(1/b\), is known as the Hagedorn transition
temperature. Thus, exponential growth in the asymptotic density of particle states suggests a phase
transition, or limiting temperature, \(1/\beta_H\), beyond which the free energy diverges exponentially.

Notice that this argument does not hold for the gas of free closed strings. Despite exponential
growth in the asymptotic density of states function in the free string spectrum, there is no cor-
responding exponential divergence in the Helmholtz free energy. The reason, as explained in the
introduction, is that the one-loop contribution to the free energy in string theory is given by an
integral over world-sheet moduli of a modular invariant function, \(Z(\tau, \bar{\tau})\). As a consequence, the
integration over the moduli, \(\tau\), is restricted to the fundamental domain of the one-loop modular
group, which does not include the region of \(\tau\) space for which the density of states function diverges
exponentially. Hence, there is no exponentially diverging free energy at one-loop and, consequently,
no Hagedorn phase transition in string theory. An analogous argument can, in fact, be constructed at any loop in the perturbative expansion for string theory [29].

Previous authors have mistakenly referred to the exponential divergence due to the onset of the tachyonic instability in the winding number one thermal mode as a “Hagedorn phase transition”. As explained in the introduction, this is incorrect. The misleading exponential divergence in [5, 6, 7, 9, 8, 10, 11, 13] of apparently ultraviolet origin, arises as a consequence of a Poisson resummation of the direct exponential divergence in the partition function due to the leading tachyonic winding mode in the spectrum. We emphasize that the mistaken identification of the w=1 tachyonic instability as a signal for a Hagedorn phase transition, ignores the proliferation of low temperature tachyonic modes with \( n > 0, \tilde{N}_0 = 0, N_0 = 0 \), that are also in the physical state spectrum. The free energy is already divergent at temperatures much below \( T_H \). Thus, the winding number one tachyonic instability cannot be interpreted as a Hagedorn phase transition in the free string gas. To verify our claim, in [16] we compute the free energy for a free heterotic string gas without tachyons in the thermal spectrum. The free energy is found to be finite at all temperatures starting from zero including the Hagedorn temperature.

### B: Fixed Point Contributions to the Entropy

As mentioned in the introduction, the expressions we will derive for both the Helmholtz free energy and the entropy function will contain temperature independent contributions from string states localized at the fixed points of the orbifold. Since it is of some interest to understand their explicit form, in this subsection we give a brief review of the relationship between \( Z_2 \) orbifold and circle compactifications at generic radius [17]. Recall the precise equivalence of the partition functions of a free 2d boson compactified on a circle of radius \( 2\pi \alpha_1 / 2 \) or on an interval of length \( \pi \alpha_1 / 2 \), i.e., the partition function on the \( Z_2 \) orbifold of the circle of radius \( \alpha_1 / 2 \). This marks the Kosterlitz-Thouless point: the intersection of the circle and orbifold fixed lines in the moduli space of \( c=1 \) conformal field theories [24, 17]. Let us review the derivation of the fixed point contributions following [24, 17].

The untwisted sector of the orbifold contains the subset of states in the circle compactification invariant under the reflection, \( X^0 \rightarrow -X^0 \), with fixed points at \( X^0 = 0, \pi r_{\text{circ}} \). The untwisted sector states have even oscillator number: \( N_0 + \tilde{N}_0 \), and their momentum vectors lie in the subspace of lattice vectors invariant under: \( (n, w) \rightarrow (-n, -w) \). Thus, for the untwisted sector, we have:

\[
(q\bar{q})^{-1/24} \text{tr}_U \frac{1 + R}{2} q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{2} \left( Z_{\text{circ}}(x) + (q\bar{q})^{1/24} \prod_{m=1}^{\infty} \frac{1 + q^m}{1 + q^m} \right).
\]

(36)

This expression is not modular invariant due to the presence of the second term: the projection to states invariant under the reflection spoils the Lorentzian self-duality property of the lattice vectors summed over in Eq. (36). A modular invariant spectrum of free strings living on the interval requires inclusion of additional twisted sectors in the path integral, where \( X^0 \) satisfies the boundary condition \( X^0(\sigma^2 + 2\pi) = -X^0(\sigma^1) \). This implies the absence of the zero mode, \( n = w = 0 \), and the twisted strings are therefore localized at one of two fixed points, \( X^0 = 0, \pi r_{\text{circ}} \). The anti-periodicity in the boundary condition implies a half-integer mode expansion. The result for the sum of the
twisted sectors is [17]:

\[(q\bar{q})^{1/48}\text{tr}_{\mathcal{T}} \frac{1 + \frac{R}{2}q^{L_0}\bar{q}^{\bar{L}_0}}{2} = \frac{1}{2} \left( (q\bar{q})^{1/48} \prod_{m=1}^{\infty} |1 - q^{m-1/2}|^{-2} + (q\bar{q})^{1/48} \prod_{m=1}^{\infty} |1 + q^{m-1/2}|^{-2} \right) . \tag{37} \]

Combining Eqs. (36) and (37), the final result can be expressed in the form given in Eq. (10) [24] by using the Jacobi triple product identity. The \(\Theta_i(0, \tau)\) are the Jacobi theta functions. Notice the manifest modular invariance of the expression in Eq. (14): under \(\tau \to \tau + 1, \Theta_2 \to -\Theta_2, \Theta_3 \to \Theta_4, \) and vice versa. Under \(\tau \to -1/\tau, \Theta_3 \to (-i\tau)^{1/2}\Theta_3, \Theta_4 \to (-i\tau^2)^{1/2}\Theta_4.\) The absolute magnitudes remove the phases. Thus, the partition function, \(Z(\tau, \bar{\tau})\), for the 24 transverse bosonic degrees of freedom in the critical bosonic string can be written in the form:

\[(2\pi\tau_2)^{-23/2} |\eta(\tau)|^{-46} \left\{ |\eta(\tau)|^{-2} \left[ \sum_{n,n'\in\mathbb{Z}} q^{n^2/\tau_2} \frac{\alpha'\beta}{\bar{q}} \frac{\alpha'\beta}{\bar{q}} + (|\Theta_4\Theta_4| + |\Theta_2\Theta_3| + |\Theta_2\Theta_4|) \right] \right\} = Z_{23,\text{orb}} , \tag{38} \]

as given in Eq. (10). The factor inside curly brackets is the orbifold partition function, and \(Z_{23}\) is the partition function of 23 free bosons with noncompact target space \(R^{23}\). Each factor is separately modular invariant.

Recall that the spectrum of states in the partition function of a free boson are in one-to-one correspondence with the operators in a two dimensional conformal field theory with central charge \(c = 1\) [24]. The spectrum contains a single marginal operator, \(\partial_x X^0 \partial_x X^0\), of conformal dimension \((h, \bar{h}) = (1, 1)\). Perturbation of the free Lagrangian by this operator leaves the spectrum and quantum correlation functions of the conformal field theory unchanged, but for a change in the radius of the target space. In thermal string theory, this modulus is the inverse temperature, the size of the interval in the imaginary time direction. The \(c = 1\) conformal field theories have been classified, and the form of their moduli space is known [24, 17]. The two fixed lines correspond to two-dimensional free boson theories whose target spaces are, respectively, a circle and an orbifold. The fixed lines intersect at the continuum limit of the Kosterlitz-Thouless (KT) point of the X-Y model [28]. The partition function on a circle of radius \(r_{\text{circ}}\) is related to the partition function on the \(Z_{2}\)-orbifold of the circle with identical radius as follows [24]:

\[Z_{\text{orb}}(r_{\text{circ}}) = \frac{1}{2} \left[ Z_{\text{circ}}(r_{\text{circ}}) + 2Z_{\text{circ}}(\alpha'\beta^{1/2}) - Z_{\text{circ}}(2\alpha'\beta^{1/2}) \right] . \tag{39} \]

Points on the circle fixed line correspond to \(S^1\) theories with radius \(r_{\text{circ}}\), while points on the orbifold fixed line correspond to \(S^1/Z_2\) theories with interval length \(\pi r_{\text{circ}}\). Setting \(r_{\text{circ}} = \alpha'\beta^{1/2}\) in Eq. (39), gives the equality of orbifold and circle partition functions at the KT point of the \(c = 1\) moduli space:

\[Z_{\text{orb}} (\alpha'\beta^{1/2}) = Z_{\text{circ}} (\alpha'\beta^{1/2}) = Z_{\text{circ}} (2\alpha'\beta^{1/2}) \equiv Z_{\text{KT}} . \tag{40} \]

The second equality follows from thermal duality of the circle partition function. Thermal duality is a simple consequence of Lorentz invariance together with the invariance of the vacuum energy functional of a closed string theory under T-duality transformations mapping small radii to large radii, \(r \to \alpha'/r\) [17]. Note that either fixed line in the \(c = 1\) moduli space displays an \(r \to \alpha'\beta^{1/2}/r\) duality, with self-dual radius, \(r_c = \alpha'\beta^{1/2}\). Thus, the partition function of the KT model has two equivalent representations: as the circle theory of radius \(2\alpha'\), or as the R-orbifold theory with interval length \(\pi \alpha'\beta^{1/2}\).
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