A coloring algorithm for $4K_1$-free line graphs

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Abstract

Let $L$ be a set of graphs. $\text{Free}(L)$ is the set of graphs that do not contain any graph in $L$ as an induced subgraph. It is known that if $L$ is a set of four-vertex graphs, then the complexity of the coloring problem for $\text{Free}(L)$ is known with three exceptions: $L = \{\text{claw}, 4K_1\}$, $L = \{\text{claw}, 4K_1, \text{co-diamond}\}$, and $L = \{C_4, 4K_1\}$. In this paper, we study the coloring problem for $\text{Free}(\text{claw}, 4K_1)$. We solve the coloring problem for a subclass of $\text{Free}(\text{claw}, 4K_1)$ which contains the class of $4K_1$-free line graphs. Our result implies the chromatic index of a graph with no matching of size four can be computed in polynomial time.

Keywords: Graph coloring, claw, $K_5 - e$, line graph

1 Introduction

Graph coloring is one of the most important problems in graph theory and computer science. Determining the chromatic number of a graph is a NP-hard problem. However, for some graph families the problem can be solved in polynomial time. Let $L$ be a set of graphs. Define $\text{Free}(L)$ to be the class of graphs that do not contain any graph in the list $L$ as an induced subgraph. For example, $\text{Free}(P_4)$ is the class of graphs that do not contain a $P_4$ as an induced subgraph; and $\text{Free}(P_3, \text{co-P}_3)$ is the class of graphs that do not contain an induced subgraph isomorphic to a $P_3$ or the complement of a $P_3$. For a single graph $H$, in [14], it is proved the coloring problem for $\text{Free}(H)$ is polynomial time solvable if $H$ is the $P_4$, or the join of a $P_3$ and $P_1$, and NP-complete for any other graph $H$. This result motivates us to consider the problem of coloring the class $\text{Free}(L)$ when $L$ is a family of four-vertex graphs. As this paper is being written, we discovered that [15] has considered the same problem. We found

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some results already discovered in [15], but we also found a new result which we will present in this paper. To explain this result, we will need to discuss the background of the problem. Let VERTEX COLORING be the problem of determining the chromatic number of a graph. For graphs $G$ and $H$, $G + H$ denotes the disjoint union of $G$ and $H$. For a graph $G$ and an integer $k$, $kG$ denotes the disjoint union of $k$ copies of $G$. Let $P_n$ (respectively, $C_n, K_n$) denote the chordless path (respectively, chordless cycle, clique) on $n$ vertices.

Recall the following result in [14]:

Theorem 1.1 For a single graph $H$, VERTEX COLORING is polynomial time solvable for $\text{Free}(H)$ if $H$ is the $P_4$, or $P_3 + P_1$, and NP-Complete otherwise.

In [19], the following result is established (see [19] for the definition of clique widths. For the purposes of this paper, we need only know the fact that if the clique width of a graph is bounded then so is that of its complement.)

Theorem 1.2 VERTEX COLORING is polynomial time solvable for graphs with bounded clique width.

In [5], the authors study the clique widths of $\text{Free}(F)$ where $F$ is a family of four-vertex graphs. Figure 1 shows all 11 graphs on four vertices with their names. They show there are exactly seven minimal classes with unbounded clique width. These are:

$\mathcal{X}_1 = \text{Free}(\text{claw}, C_4, K_4, \text{diamond})$.
$\mathcal{X}_2 = \text{Free}(\text{co-claw}, 2K_2, 4K_1, \text{co-diamond})$.
$\mathcal{X}_3 = \text{Free}(C_4, \text{co-claw}, \text{paw}, \text{diamond}, K_4)$.
$\mathcal{X}_4 = \text{Free}(2K_2, \text{claw}, \text{co-paw}, \text{co-diamond}, 4K_1)$.
$X_5 = \text{Free}(K_4, 2K_2)$.
$X_6 = \text{Free}(C_4, 2K_2)$.
$X_7 = \text{Free}(C_4, 4K_1)$.

Thus, if $F$ is a set of four-vertex graphs and $F \not\subseteq X_i$ ($i = 1, 2, \ldots, 7$), then VERTEX COLORING is polynomial time solvable for $\text{Free}(F)$.

VERTEX COLORING is NP-Complete for

- $X_1$ due to a result in \[14\] which shows the problem is NP-Complete for $\text{Free}(\text{claw, } C_4)$ and for $\text{Free}(\text{claw, } K_4, \text{diamond})$;
- $X_2$ due to Theorem 6 in \[20\];
- $X_3$ due to a remark (Case 1 in Section 4) in \[14\]: they show the problem is NP-Complete for $\text{Free}(L)$ if every graph in $L$ contains a cycle.

Thus, VERTEX COLORING is NP-Complete for $\text{Free}(L)$ whenever $L \subseteq X_i$, for $i = 1, 2, 3$. Therefore, we only need to examine the problem for classes $X_4, X_5, X_6, X_7$ and their super classes defined by forbidding induced subgraphs with four vertices. In \[15\], a polynomial time algorithm is given for VERTEX COLORING for the class $X_5$. The graphs in $X_6$ have a simple structure \[4\] that implies an easy polynomial time algorithm for the coloring problem. Furthermore, in \[11\] a polynomial time algorithm for VERTEX COLORING for the larger class $\text{Free}(\text{claw, co-P}_5)$ is given. The complexity of VERTEX COLORING for the class $X_7$ is unknown. In \[15\], the authors conjecture that the coloring problem can be solved in polynomial time for $X_7$.

We are interested in the class $X_4$. Let $H$ be a subset of $\{2K_2, \text{claw, co-paw, co-diamond, } 4K_1\}$. We examine the complexity of VERTEX COLORING for $\text{Free}(H)$. We may assume $H$ does not contain a co-paw, for otherwise the problem is polynomial time solvable, by Theorem \[11\]. We may assume $H$ contains the claw, for otherwise $H \subset X_2$, and so VERTEX COLORING is NP-Complete. In \[15\], a polynomial time algorithm is given for VERTEX COLORING for class $\text{Free}(\text{claw, } 2K_2)$. Thus we have $H \subseteq \{\text{claw, co-diamond, } 4K_1\}$. In \[15\], it is proved VERTEX COLORING for $\text{Free}(\text{claw, co-diamond})$ is polynomially equivalent to the same problem for the class $\text{Free}(\text{claw, co-diamond, } 4K_1)$. Thus, if VERTEX COLORING is polynomial time solvable for $\text{Free}(\text{claw, } 4K_1)$, then so is the same problem for $\text{Free}(\text{claw, co-diamond})$. These are two challenging problems. We believe VERTEX COLORING can be solved in polynomial time for $\text{Free}(\text{claw, } 4K_1)$. The purpose of this paper is to solve the problem for a subclass of $\text{Free}(\text{claw, } 4K_1)$: the class of $4K_1$-free line graphs.

Given a graph $G$, the line graph $L(G)$ of $G$ is defined to be the graph whose vertices are the edges of $G$, and two vertices of $L(G)$ are adjacent if their corresponding edges in $G$ are incident. Line graphs cannot contain a claw. In \[4\], a characterization of line graphs by forbidden induced subgraphs is given: A graph is a line graph (of some other graph) if and only if it does not contain a graph in Figure 2 as an induced...
The purpose of this paper is to prove the following theorem (see Figure 2 for the names of the graphs mentioned in our theorem.)

**Theorem 1.3** VERTEX COLORING is polynomial time solvable for Free(claw, 4K₁, 5-wheel, C₅-twin, P₅-twin, K₅ - e).

An *edge coloring* of a graph is an assignment of colors to its edges such that every edge receives one color and two edges receive different colors if they are incident. The *chromatic index* of a graph is the smallest number of colors needed to color its edges. Let \( \Delta(G) \) be the maximum degree of \( G \). Then the chromatic index of a graph \( G \) is at least \( \Delta(G) \). A classic theorem of Vizing [22] states that the chromatic index of a graph \( G \) is at most \( \Delta(G) + 1 \). However, computing the chromatic index of a graph is NP-hard [12]. Let \( G \) be a graph and \( L(G) \) be its line graph. Then the chromatic index of \( G \) is equal to the chromatic number of \( L(G) \). A matching of \( G \) is a set of edges such that no two edges in it are incident; this matching of \( G \) corresponds to a stable set of the line graph \( L(G) \). It follows that our Theorem 1.3 implies the

Figure 2: All minimal forbidden induced subgraphs for line graphs
chromatic index can be computed in polynomial time for graphs without matching of size four.

**Corollary 1.4** There is a polynomial-time algorithm to compute the chromatic index of a graph without a matching of size four.

In Section 2, we discuss the background results needed to prove our main theorem. In Section 3, we establish structural properties of the graphs in $\textit{Free}(\text{claw, 4K}_1, 5\text{-wheel, C}_5\text{-twin, P}_5\text{-twin, K}_5 - e)$. In Section 4, we prove Theorem 1.3. And finally, in Section 5, we discuss open problems related to our work.

## 2 Definitions and background

In this section, we discuss the background of our problem. Let $G$ be a graph. Then $\textit{co-}G$ denotes the complement of $G$. A clique cutset of $G$ is a set of vertices $S$ where $G[S]$, the subgraph of $G$ induced by $S$, is a clique whose removal disconnects $G$. An atom is a connected graph containing no clique cutset. Let $\chi(G)$ denote the chromatic number of a graph $G$. Let $\omega(G)$ denote the number of vertices in a largest clique of $G$. Let $\alpha(G)$ denotes the number of vertices in a largest stable set of $G$. A hole is the graph $C_k$ with $k \geq 5$. For a hole $H$, a vertex $x$ outside $H$ is a $k$-vertex (for $H$) if $x$ has exactly $k$ neighbours in $H$. An anti-hole is the complement of a hole. A hole is odd if it has an odd number of vertices. A graph is Berge if it contains no odd holes and no odd anti-holes. For sets $X, Y$ of vertices, we write $X \overset{0}{\not\subset} Y$ to mean there is no edge between any vertex in $X$ and any vertex in $Y$, and $X \overset{1}{\subset} Y$ to mean there are all edges between $X$ and $Y$. We let $|G|$ denotes the number of vertices of $G$. By claw$(a,b,c,d)$ we denote the claw with vertices $a, b, c, d$ and edges $ab, ac, ad$. We say that a stable set is good if it meets every largest clique of the graph. A set is big if it has at least three elements. For a vertex $x$ of $G$, $N(x)$ denote the set of neighbors of $x$, and $d(x)$ is the degree of $x$, that is, $d(x) = |N(x)|$.

Consider the following procedure to decompose a (connected) graph $G$. If $G$ has a clique cutset $C$, then $G$ can be decomposed into subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ where $V = V_1 \cup V_2$ and $C = V_1 \cap V_2$ (recall that $G[X]$ denotes the subgraph of $G$ induced by $X$ for a subset $X$ of vertices of $V(G)$). Given optimal colourings of $G_1, G_2$, we can obtain an optimal colouring of $G$ by identifying the colouring of $C$ in $G_1$ with that of $C$ in $G_2$. In particular, we have $\chi(G) = \max(\chi(G_1), \chi(G_2))$. If $G_i$ ($i \in \{1, 2\}$) has a clique cutset, then we can recursively decompose $G_i$ in the same way. This decomposition can be represented by a binary tree $T(G)$ whose root is $G$ and the two children of $G$ are $G_1$ and $G_2$, which are in turn the roots of subtrees representing the decompositions of $G_1$ and $G_2$. Each leaf of $T(G)$ corresponds to an atom of $G$. Algorithmic aspects of the clique cutset decomposition are studied in [21] and [23]. In particular, the decomposition tree $T(G)$ can be constructed in
$O(nm)$ time such that the total number atoms is at most $n - 1$ [21] (Here, as usual, $n$ and, respectively, $m$, denote the number of vertices, respectively, edges, of the input graph). Thus, to color a graph $G$ in polynomial time, we only need to color its atoms in polynomial time.

Our result relies on known theorems on perfect claw-free graphs, and we discuss these results now. A graph $G$ is perfect if for each induced subgraph $H$ of $G$, we have $\chi(H) = \omega(H)$. In [17], it is proved claw-free Berge graphs are perfect. Claw-free perfect graphs can be recognized in polynomial time [8], and they can be optimally colored in polynomial time [13]. We note [7] proves that a graph is perfect if and only if it is Berge, solving a long standing conjecture of [2]. Perfect graphs can be recognized in polynomial time [6], and they can be optimally colored in polynomial time [10]. In [8], the following result, crucial to our algorithm, is established.

Lemma 2.1 Let $G$ be a connected claw-free graph with $\alpha(G) \geq 3$. If $G$ contains an odd anti-hole then $G$ contains a $C_5$.

Finally, we note the well known observation that VERTEX COLORING is polynomial time solvable for graphs $G$ with $\alpha(G) = 2$; it is sufficient to find a maximum matching in the complement of $G$.

3 Structural properties

In this section, we establish preliminary results needed to prove Theorem 1.3. For the Claims in this section, we will assume $G = (V, E)$ is a connected graph in Free(claw, $K_5 - e$, 5-wheel, $C_5$-twin, $P_5$-twin, $4K_1$). We begin with the following easy Claim.

Claim 3.1 If $\alpha(G) \leq 4$, then $G$ contains no $C_\ell$, $\ell \geq 8$.

Proof. If $G$ contains $C_\ell$, $\ell \geq 8$, then $G$ contains a $4K_1$. $\square$

Let $C$ be a hole with vertices $1, 2, \ldots, k$ (in the cyclic order) with $k \geq 5$ (with respect to $C$, the vertices $i$ will be taken modulo $k$). Define $Y_i$ be the set of vertices with neighbors $i, i + 1, i + 2, i + 3$. Let $Z_i$ be the set of vertices with neighbors $i, i + 1, i + 3, i + 4$.

Claim 3.2 If $G$ contains a $C_7$, then $|G| \leq 21$.

Proof. Suppose $G$ contains a $C_7$. Then $G$ has no $k$-vertex for $k \in \{0, 1, 2\}$ since $G$ is $4K_1$-free, has no $k$-vertex for $k \in \{5, 6, 7\}$ since $G$ is claw-free, and $G$ has no 3-vertex, or else $G$ contains a claw or a $P_5$-twin. Thus, only 4-vertices may exist and they are one of the two types $Y_i$s, $Z_i$s defined above. Let $y_1$ and $y_2$ be two vertices in
If $y_1y_2 \notin E$ then there is a claw$(i + 3, i + 4, y_1, y_2)$ and if $y_1y_2 \in E$ then there is a $K_5-e$ induced by $\{i, i + 1, i + 2, y_1, y_2\}$; so we have $|Y_i| \leq 1$. Let $z_1$ and $z_2$ be two vertices in $Z_i$, $z_1 \neq z_2$. If $z_1z_2 \notin E$ then there is a claw$(i + 4, i + 5, z_1, z_2)$ and if $z_1z_2 \in E$ then there is a $C_5$-twin induced by $\{i, i + 6, i + 5, i + 4, z_1, z_2\}$. It follows that $G$ contains at most 21 vertices. □

Claim 3.3 If $G$ contains a $C_5$ then $G$ has no $k$-vertex for $k \in \{1, 3, 5\}$.

Proof. Suppose $G$ contains a $C_5$. Then $G$ has no $k$-vertex for $k = 1$ for otherwise $G$ contains a claw, or for $k = 3$ for otherwise $G$ contains a claw or a $C_5$-twin, or for $k = 5$ for otherwise $G$ contains a 5-wheel. □

For the all the Claims below, we will assume $G$ contains a $C_5$. Let the 0-vertex set be denoted by $R$, let $X_i$ be the set of 2-vertices with neighbours $i, i + 1$ and let $Y_i$ be the set of 4-vertices with neighbors $i, i + 1, i + 2, i + 3$. Let $X$ denote the set of all 2-vertices and $Y$ denote the set of all 4-vertices. Every vertex of $G - C_5$ belongs to $X \cup Y \cup R$.

Claim 3.4 We have $|Y_i| \leq 1$ for all $i$.

Proof. Let $y_1$ and $y_2$ be two vertices in $Y_i$, $y_1 \neq y_2$. If $y_1y_2 \notin E$ then there is a claw$(i, i - 1, y_1, y_2)$. If $y_1y_2 \in E$ then there is a $K_5-e$ induced by $\{i, i + 1, i + 2, y_1, y_2\}$. □

From Claim 3.4 we have $|Y| \leq 5$.

Claim 3.5 We have $Y_i \cap Y_{i+1}$ for all $i$.

Proof. Let $y_1$ be the vertex from $Y_i$ and $y_2$ be the vertex from $Y_{i+1}$. If $y_1y_2 \in E$ then there is a $K_5-e$ induced by $\{i + 1, i + 2, i + 3, y_1, y_2\}$. □

For the following two claims, we will let $x_i$ (respectively, $y_i$) denote an arbitrary vertex in $X_i$ (respectively, $Y_i$) for all $i$.

Claim 3.6 $X_i \cap Y_i \cup Y_{i+3}$.

Proof. If $x_iy_i \notin E$, then there is a claw$(i, i + 4, x_i, y_i)$. If $x_iy_{i+3} \notin E$, then there is a claw$(i + 1, i + 2, x_i, y_{i+3})$. □

Claim 3.7 If $X_i \neq \emptyset$, and both $y_i$ and $y_{i+3}$ exist, then $y_iy_{i+3} \in E$. 7
Proof. If \( y_i y_{i+3} \notin E \), then by Claim 3.6 the set \( \{x_i, i, i+1, y_i, y_{i+3}\} \) contains a \( K_5 - e \).
\(\square\)

Claim 3.8 \( X_i \uplus Y_{i+1} \cup Y_{i+2} \cup Y_{i+4} \).

Proof. If \( x_i y_{i+1} \in E \), then there is a claw \( (y_{i+1}, x_i, i+2, i+4) \). If \( x_i y_{i+4} \in E \), then there is a claw \( (y_{i+4}, i+4, i+2, x_i) \). If \( x_i y_{i+2} \in E \), then there is a claw \( (y_{i+2}, i+2, i+4, x_i) \).
\(\square\)

Claim 3.9 We have \( R \uplus Y \).

Proof. If some \( y \in Y \) is adjacent to some \( r \in R \), then there is a claw induced by \( y, r \) and some two neighbours \( a, b \) of \( y \) in the \( C_5 \) with \( ab \notin E \).
\(\square\)

Claim 3.10 If \( R \neq \emptyset \), then \( X \neq \emptyset \).

Proof. Since \( G \) is connected by the assumption of this section, there is a path connecting some vertex in \( R \) to some vertex in the \( C_5 \). By Claim 3.9 this path must contain some vertex in \( X \).
\(\square\)

Claim 3.11 Each \( X_i \) forms a clique for all \( i \).

Proof. Let \( v_1 \) and \( v_2 \) be two vertices in \( X_i \), \( v_1 \neq v_2 \). If \( v_1 v_2 \notin E \) then there is a claw \( (i, i-1, v_1, v_2) \). So \( v_1 v_2 \in E \) and \( X_i \) forms a clique.
\(\square\)

Claim 3.12 The set \( R \) induces a clique.

Proof. Let \( r_1 \) and \( r_2 \) be vertices in \( R \). If \( r_1 r_2 \notin E \) then the set \( \{0, 2, r_1, r_2\} \) induces a \( 4K_1 \).
\(\square\)

Claim 3.13 \( R \uplus X_i \) for all \( i \).

Proof. Let \( x \) be a vertex in \( X_i \) and \( r \) be a vertex in \( R \). If \( rx \notin E \) then there is a \( 4K_1 \) induced by \( \{r, x, i-1, i+2\} \).
\(\square\)

Claim 3.14 If \( R \neq \emptyset \) then \( |X_i| \leq 2 \) for all \( i \).
Claim 3.15 A vertex in $X_i$ cannot have two neighbors in $X_j$ for any two distinct $i$ and $j$.

Proof. Suppose some vertex $x_i \in X_i$ is adjacent to two vertices $a, b \in X_j$. We may assume $j = i+1$, or $j = i+2$. Now, there is a $P_5$-twin induced by $\{i, x_i, a, b, j+1, j+2\}$ if $j = i+1$, or $\{j-1, x_i, a, b, j+1, j+2\}$ if $j = i+2$. □

Claim 3.16 If $X \neq \emptyset$ then $|R| \leq 2$ or $X$ is a clique cutset of $G$.

Proof. Suppose $X \neq \emptyset$ and $|R| \geq 3$. By Claim 3.13, $R \cup X$. By Claim 3.9, $X$ is a cutset separating $R$ from the $C_5$. We may assume $X$ contains two non-adjacent vertices $v_1, v_2$, for otherwise $X$ is a clique cutset and we are done. But now, by Claim 3.12 any three vertices in $R$ together with $v_1, v_2$ induce a $K_5-e$ in $G$. □

Claim 3.17 If $R \neq \emptyset$ then $|G| \leq 22$ or $G$ contains a clique cutset.

Proof. If $|R| \geq 3$ then by Claims 3.16 and 3.10 $X$ is a clique cutset of $G$. So we have $|R| \leq 2$. By Claim 3.14 we have $|X_i| \leq 2$ for $i \in 0, 1, 2, 3, 4$. By Claim 3.4, we have $|Y| \leq 5$. Then $|G| = |R| + |X| + |Y| + |C_5| = 2 + 10 + 5 + 5 = 22$. □

Claim 3.18 A vertex in $X_i$ cannot have two non-adjacent neighbors in $X - X_i$.

Proof. Suppose some vertex $x_i \in X_i$ have non-adjacent neighbors $v_1$ and $v_2$ in $X - X_i$. By Claim 3.11 we have $v_1 \in X_j$, $v_2 \in X_k$, $j \neq k$. If $j = i-1$ and $k = i+1$ then there is a $P_5$-twin induced by $\{v_2, x_i, v_1, i, j, j-1\}$. Now, we may assume $k \not\in \{i-1, i+1\}$, it follows there is a claw$(x_i, v_2, v_1, i+1)$. □

Claim 3.19 Suppose $X_i, X_{i+1}, X_{i+2}$ are each non-empty for some $i$. If $|X_j| \geq 3$ then $|X_i| = |X_{i+1}| = 1$, for $\{j, k, \ell\} = \{i, i+1, i+2\}$.

Proof. Suppose $|X_j| \geq 3$. Suppose some vertex $x_k \in X_k$ is non-adjacent to some vertex $x_{\ell} \in X_{\ell}$. By Claim 3.15, there is a vertex $x_j \in X_j$ that is non-adjacent to both $x_k$ and $x_{\ell}$. But now $\{i+4, x_j, x_k, x_{\ell}\}$ induces a $4K_1$. Thus, we have $X_k \uplus X_{\ell}$. It follows from Claim 3.15 that $|X_k| = |X_{\ell}| = 1$. □
Claim 3.20 Suppose \( R = Y = X_{i+2} = X_{i+4} = \emptyset \) for some \( i \). Then \( \chi(G) = \omega(G) \), and an optimal colouring of \( G \) can be found in polynomial time.

Proof. By induction on the number of vertices. Let \( X_k \) be the set with largest cardinality among the three sets \( X_i, X_{i+1}, X_{i+3} \). If \( |X_k| = 1 \), then \( \omega(G) = 3, |V| \leq 8 \), and a 3-coloring of \( G \) can be trivially found. Similarly, if \( |X_k| = 2 \), then \( \omega(G) = 4 \), and a 4-coloring of \( G \) can be found. Now, we may assume \( |X_k| \geq 3 \). From Claims 3.15 and 3.18 there is a stable set \( S \) containing a vertex in \( X_k \) and some vertices in \( X_j \cup X_\ell \) with \( k \notin \{ j, \ell \} \) that is good, i.e., meets all maximum cliques of \( G \). Now, we can recursively color \( G - S \) with \( \omega(G) - 1 \) colors and then give \( S \) a new color. 

In proving the main theorem of this paper, we will reduce the problem to list coloring a restricted class of graphs. We will now define some new notions. Given a graph \( G \) and a list of colors \( L(v) \) for each vertex of \( v \), an \( L \)-coloring of \( G \) is a (proper) coloring such that each vertex is assigned a color from its list.

Lemma 3.21 Let \( G = (V, E) \) be a graph whose vertices can be partitioned into three cliques \( Q_1, Q_2, Q_3 \) such that

(a) each vertex in \( Q_i \) is adjacent to at most one vertex in \( Q_j \) for all \( i, j \) with \( i \neq j \).

(b) if a vertex in \( Q_i \) is adjacent to vertices \( b \in Q_j, c \in Q_k \), then \( bc \in E \) for distinct \( i, j, k \).

(c) for each \( Q_i \), there is a list \( L_i \) of colors such that
   
   (i) all vertices \( v \in Q_i \) have the same list \( L(v) = L_i \),
   
   (ii) \( |L_i| \geq |Q_i| \),
   
   (iii) each \( L_i \) contains a color \( d_i \) such that the three colors \( d_1, d_2, d_3 \) are all distinct.
   
   (iv) no color appears in all three \( L_i \).

Then \( G \) admits an \( L \)-coloring where \( L = L_1 \cup L_2 \cup L_3 \).

Proof. Let \( |L| \) denotes the number of colors in \( L \). We prove the Lemma by induction on \( |L| \). Below, \( i, j, k \) are distinct indices taken from \( \{1, 2, 3\} \).

If some clique \( Q_i \) is empty, then it is easy to see the Lemma holds. If \( |Q_1| = |Q_2| = |Q_3| = 1 \), then color the only vertex in \( Q_i \) with color \( d_i \), and we are done. Thus, some \( Q_i \) has at least two vertices.

Suppose some \( Q_i \) has exactly one vertex \( v \), and \( d_i \notin L_j \cup L_k \). In this case, color \( v \) with color \( d_i \), remove \( v \) from \( G \). By induction, \( G - v \) admits an \( L \)-coloring, and we are done.

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Suppose $Q_i$ has exactly one vertex, $d_i \in L_j$, and $Q_j$ has at least two vertices. Let $v$ be the only vertex in $Q_i$. Take a vertex $b \in Q_j$ which is not adjacent to $v$; color $v$ and $b$ with color $d_i$, remove $d_i$ from $L_j$ (note that $d_j$ remains in $L_j$). By induction $G - \{v, b\}$ admits an $L$-coloring, and we are done.

Now, suppose $Q_i$ has exactly one vertex $v$. From the above, we may assume $d_i \in Q_j$, and $|Q_j| = 1$. It follows that $|Q_k| \geq 2$ and thus, $d_i \not\in L_k$. Note that we have $\{d_i, d_j\} \subseteq L_j$. Color $v$ with color $d_i$, remove color $d_i$ from $L_j$. By induction $G - v$ admits an $L$-coloring, and we are done.

We may now assume each $Q_i$ has at least two vertices. Suppose there is a color $c \not\in \{d_1, d_2, d_3\}$. If $c$ appears in only one $Q_i$, then assign to a vertex $v \in Q_i$ the color $c$, remove $c$ from $L_i$, and by induction $G - v$ admits an $L$-coloring and we are done. Now, assume $c \in Q_i$ and $c \in Q_j$. Since each of $Q_i$ and $Q_j$ has at least two vertices, there are non adjacent vertices (by condition (a)) $a \in Q_i, b \in Q_j$. Assign to $a$ and $b$ color $c$, remove color $c$ from $L_i$ and $L_j$. By induction, $G - \{a, b\}$ admits an $L$-coloring, and we are done.

So, we may assume every color belongs to $\{d_1, d_2, d_3\}$. It follows that $2 \leq |Q_i| \leq 3$ for any $Q_i$. By condition (iv), we have $|Q_i| < 3$ for any $i$. It follows that $|Q_1| = |Q_2| = |Q_3| = 2$. It is now straightforward to verify that $G$ admits an $L$-coloring. □

**Lemma 3.22** Let $G$ be a graph in Free(claw, $4K_1$, 5-wheel, $K_5 - e$, $P_5$-twin, $C_5$-twin) with a $C_5$. Suppose $R = X_{i+2} = X_{i+4} = \emptyset$ for some $i$. If $\omega(G) \geq 5$ then $\chi(G) = \omega(G)$ and an optimal coloring of $G$ can be found in polynomial time.

**Proof.** Suppose $\omega(G) = 5$. For simplicity, we may assume $i = 1$, i.e., the sets $X_3$ and $X_5$ are empty. Each of the sets $X_i$ has at most three vertices for $i = 1, 2, 4$ (since $X_i \cup \{i, i+1\}$ induces a clique by Claim [3.11]). For each vertex $i$ in the $C_5$, color vertex $i$ with color $c(i) = i$. Color the vertex $y_i$ (if it exists) with color $c(y_i) = i - 1$ with the subscript $i$ taken modulo 5. Now, we need to extend this coloring to the set $X$. For $X_i$, we will construct a list $L_i$ of feasible colors which are compatible with the already colored vertices in $C_5 \cup Y$. Each vertex in $X_i$ will have the list $L_i$. Consider a non-empty set $X_i$ ($i \in \{1, 2, 4\}$). If $|X_i| = 1$, then set $L_i = \{c(i+3)\}$ (which is the color of vertex $i+3$). If $|X_i| = 2$, then the vertex $y_i$ or the vertex $y_{i+3}$ (or both) does not exist; for otherwise, by Claim [3.7], the induced graph $G[X_i \cup \{i, i+1, y_i, y_{i+3}\}]$ contains a $K_6$, contradicting the assumption that $\omega(G) = 5$. Let $j$ be the subscript such that $y_j$ ($j \in \{i, i+3\}$) does not exist. Let $L_i = \{c(i+3), c(y_j)\}$. If $|X_i| = 3$, then both vertices $y_i, y_{i+3}$ do not exist for the same reason above. Let $L_i = \{c(i+2), c(i+3), c(i+4)\}$. A coloring of the vertices of $X$ using colors from the lists $L_i$ does not conflict with the already assigned coloring of $C_5 \cup Y$. By Lemma [3.21], $X$ admits an $L$-coloring. So we have $\chi(G) = 5$.

Now, we may assume $\omega(G) \geq 6$. We will find a good stable set (recall the definition of good stable sets in Section [2]). Note that if a maximum clique $C$ of $G$ contains some
vertex of $X_i$, then $C$ contains all vertices of $X_i$. Also, since $\omega(G) \geq 6$, a maximum clique of $G$ does not contain a set $X_i$ with $|X_i| = 1$. Now, since $\omega(G) \geq 6$, some set $X_i$ must have at least three vertices. By Claims 3.15 and 3.18, there is a stable set containing a vertex in every $X_j$ ($j \in \{1, 2, 4\}$) with $|X_j| \geq 2$. Such a set $S$ is the desired stable set. Now, we have $\omega(G - S) = \omega(G) - 1$. We recursively color $G - S$ with $\omega(G) - 1$ colors, and then give vertices in $S$ a new color, and we are done. 

Note that the statement of the Lemma 3.22 is false for $\omega(G) = 4$. There are graphs $G$ in $\text{Free}(\text{claw}, 4K_1, 5\text{-wheel}, K_5 - e, P_5\text{-twin}, C_5\text{-twin})$ with $\omega(G) = 4$ and $\chi(G) = 5$. The graph in Figure 3 (a) is such a graph. This is the graph with a $C_5$ (indicated by the outer $C_5$) and all five $y_i$ vertices (indicated by the inner $C_5$s), there are all edges between the outer and inner $C_5$s except for the non-edges denoted by the dotted lines. It is very interesting to note that this graph is the complement of the Petersen graph.

4 Coloring algorithm

In the section, we prove Theorem 1.3. Let $G$ be a graph satisfying the hypothesis of the theorem. From the discussion in Section 2, we may assume $G$ is connected, is an atom, i.e., $G$ contains no clique cutset, and has $\alpha(G) \geq 3$. Furthermore, if $G$ is perfect, then we are done by [13] or [10]. If $G$ contains an odd anti-hole, then by Lemma 2.1 $G$ contains a $C_5$. So, we may assume $G$ contains an odd hole $H$. Since $G$ is $4K_1$-free, $H$ has 5 or 7 vertices. If $H$ has 7 vertices, then by Claim 3.2, $|G|$ is a constant and we are done. So, $H$ is a $C_5$. From this $C_5$, define the sets $X, Y, R$ as before, and we may rely on the Claims in Section 3. We may assume $|G|$ is not a constant. By Claim 3.17, we have $R = \emptyset$.

We may assume $\omega(G) \geq 14$; for otherwise, Ramsey’s theorem [18] shows that $|G|$ is a constant (both $\omega(G)$ and $\alpha(G)$ are constants, so $|G|$ is a constant). If there is
a vertex \( v \) with degree \( d(v) \leq 13 \), then we recursively color \( G - v \) optimally, and then give \( v \) a color not appearing in \( N(v) \); such a color exists because \( \chi(G - v) \geq \omega(G - v) = \omega(G) \geq 14 \). So, we may assume every vertex of \( G \) has degree at least 14.

Suppose some non-empty \( X_i \) has \( |X_i| \leq 2 \). Then for any vertex \( x_i \in X_i \), we have
\[
d(x_i) = |C_5 \cap N(x_i)| + |Y \cap N(x_i)| + |N(x_i) \cap X_i| \leq 2 + 5 + 5 = 12
\]
by Claims 3.15 and 3.4. Thus, if \( X_i \) is non-empty then it is big (has at least three vertices.)

For a vertex \( i \) in the \( C_5 \), we may assume \( X_i \) or \( X_{i-1} \) is big; for otherwise,
\[
d(i) = |C_5 \cap N(i)| + |X_i| + |X_{i+1}| + |Y \cap N(i)| \leq 2 + 2 + 2 + 5 = 11
\]
by Claim 3.4. Thus, at least three sets \( X_i \)'s must be big. It follows from Claim 3.19 that precisely three sets \( X_i \) are big, and they are not consecutive. By Lemma 3.22 we can color \( G \) with \( \omega(G) \) colors in polynomial time. \( \square \)

5 Conclusions and open problems

Let \( L \) be a family of four-vertex graphs. As mentioned in Section 11, the complexity of VERTEX COLORING is known for the class \( \text{Free}(L) \) with three exceptions: \( L = \{\text{claw}, 4K_1\} \), \( L = \{\text{claw}, 4K_1, \text{co-diamond} \} \), and \( L = \{C_4, 4K_1\} \). We believe each of the three problems is polynomial time solvable. In this paper, we studied the problem for \( \text{Free}(\text{claw}, 4K_1) \). We solved the coloring problem for a subclass, the class of \( 4K_1 \)-free line graphs. Our result implies the chromatic index of a graph with no matching of size 4 can be computed in polynomial time. This is an interesting result in its own right. We conclude our paper with the two open problems below.

**Problem 5.1** For each fixed \( k \), is there a polynomial time algorithm to compute the chromatic index of a graph without a matching of size \( k \)?

**Problem 5.2** For each fixed \( k \), is there a polynomial time algorithm to solve VERTEX COLORING for the class \( \text{Free}(\text{claw}, kK_1) \)?

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References

[1] L. W. Beineke, Characterizations of derived graphs, *Journal of Combinatorial Theory* 9:2 (1970) 129-135.

[2] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 10:114 (1961) 88.

[3] C. Berge and V. Chvátal (eds.), *Topics on perfect graphs*. North-Holland, Amsterdam, 1984.

[4] Z. Blázsik, M. Hujter, A. Pluhr and Zs. Tuza, Graphs with no induced $C_4$ and $2K_2$, *Discrete Mathematics* 115 (1993) 51-55.

[5] A. Brandstädt, J. Engelfriet, H.-O. Le, V.V. Lozin, Clique-Width for Four-Vertex Forbidden Subgraphs, *Theory Comput. Syst.* 34 (2006) 561-590.

[6] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, Recognizing Berge graphs, *Combinatorica* 25 (2005), 143–186.

[7] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, *Annals of Mathematics* 164 (2006), pp. 51–229.

[8] V. Chvátal and N. Sbihi, Recognizing claw-free perfect graphs, *Journal of Combinatorial Theory Ser.B* 44 (1988), 154–176.

[9] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*. Academic Press, New York, 1980.

[10] M. Grötschel, L. Lovász and A. Schrijver, Polynomial algorithms for perfect graphs, in [3].

[11] C. T. Hoàng and D. A. Lazzarato, Polynomial-time algorithms for minimum weighted colorings of $(P_5,\overline{P}_5)$-free graphs and related graph classes, *Discrete Applied Mathematics* 186 (2015) 106–111.

[12] I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Computing* 10 (1981), pp. 718-720.

[13] W.-L. Hsu, How to color claw-free perfect graphs, *Annals of Discrete Mathematics* 11 (1981), pp. 189–197.

[14] J. Kratochvil, D. Kral, Zs. Tuza, G.J. Woeginger, Complexity of Coloring Graphs without Forbidden Induced Subgraphs, *WG 01, Lecture Notes in Computer Science*, Vol. 2204, Springer, Berlin, 2001, pp. 254262.
[15] V.V. Lozin and D.S. Malyshev, Vertex coloring of graphs with few obstructions, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.02.015

[16] F. Maffray and M. Preissmann, On the NP-completeness of the k-colorability problem for trianglefree graphs, Discrete Mathematics 162 (1996), 313-317.

[17] K.R. Parthasarathy, and G. Ravindra, The strong perfect graph conjecture is true for $K_{1,3}$-free graphs, Journal of Combinatorial Theory Ser. B 21 (1976), pp. 212-223.

[18] F.P Ramsey, On a problem of formal logic, Proc. London Math. Soc., Ser. 2 30 (1929), pp. 264-286.

[19] M. Rao, MSOL partitioning problems on graphs of bounded treewidth and clique-width, Theoret. Comput. Sci. 377 (2007) 260-267.

[20] D. Schindl, Some new hereditary classes where graph coloring remains NP-hard, Discrete Math. 295 (2005) 197-202.

[21] R. E. Tarjan, Decomposition by clique separators. Discrete Math 55 (1985), pp. 221–232.

[22] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz. 3 (1964) 25-30.

[23] S. H. Whitesides, A method for solving certain graph recognition and optimization problems, with applications to perfect graphs, in [3].