Quadratic Variation Estimation of Hidden Markov Process and Related Problems

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Abstract

The partially observed linear Gaussian system of stochastic differential equations with low noise in observations is considered. A kernel-type estimators are used for estimation of the quadratic variation of the derivative of the limit of the observed process. Then this estimator is used for nonparametric estimation of the integral of the square of volatility of unobservable component. This estimator is also used for construction of substitution estimators in the case where the drift in observable component and the volatility of the state component depend on some unknown parameter. Then this substitution estimator and Fisher-score device allows us to introduce the One-step MLE-process and adaptive Kalman-Bucy filter.

MSC 2000 Classification: 62M02, 62G10, 62G20

Key words: Hidden Markov processes, quadratic variation estimation, nonparametric estimation, parameter estimation, One-step MLE-process, volatility estimation, adaptive filtration.

1 Introduction

This work is devoted to nonparametric estimation and parameter estimation problems by observations of partially observed linear systems with small noise in observations. Such models of observations and more complex (multidimensional, nonlinear) were extensively studied in filtration theory, where
approximate filters were proposed and studied in the asymptotics of small noise. In particular the order of the errors of approximations are obtained for the large diversity of the statement of the problems, see [3], [1], [13], [14] and references there in. The statistical problems for partially observed linear and non linear systems were studied in [4], Chapter 6. Note that in [4] it is supposed that the small noise is in the observation and state equations. The parameter estimation problems in the case of small noise in observations only were considered recently in [8].

The statistical problems below are considered in two steps. First we propose a nonparametric estimator of the quadratic variation of the derivative of the limit of observed process. Then the obtained result is used in nonparametric estimation of the integral of the square of volatility of unobservable component and in parameter estimation problems.

Let us consider the linear two-dimensional partially observed system

\[
\begin{align*}
dY_t &= -a(t)Y_t dt + b(t) dV_t, \quad Y_0 = 0, \quad 0 \leq t \leq T, \\
dX_t &= f(t)Y_t dt + \varepsilon \sigma(t) dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T,
\end{align*}
\]

(1)

where the Wiener processes \(V_t, 0 \leq t \leq T\) and \(W_t, 0 \leq t \leq T\) are supposed to be independent. The solution \(Y^T = (Y_t, 0 \leq t \leq T)\) of the state equation (1) can not be observed directly and we have available the observations \(X^T = (X_t, 0 \leq t \leq T)\) only. Here \(a(\cdot), b(\cdot), f(\cdot)\) and \(\sigma(\cdot)\) are some bounded functions and \(\varepsilon \in (0, 1]\) is small parameter.

Our first goal is to construct consistent (\(\varepsilon \to 0\)) estimator \(\hat{\Psi}_{\tau,\varepsilon}, 0 \leq \tau \leq T\) of the function

\[
\Psi_\tau = \int_0^\tau f(s)^2 b(s)^2 ds,
\]

\(0 < \tau \leq T\). (3)

Then we show that this estimator \(\hat{\Psi}_{\tau,\varepsilon}\) can be useful in the construction of the estimators of the parameters \(\vartheta\) in the case of models (1), (2) with \(f(t) = f(\vartheta, t)\) or \(b(t) = b(\vartheta, t)\).

The construction of the estimators is based on the following properties of the model (1), (2). The observed process \(X^T\) uniformly on \(t\) with probability 1 converges to the random process \(x^T = (x_t, 0 \leq t \leq T)\):

\[
\sup_{0 \leq t \leq T} |X_t - x_t| \leq C \varepsilon \sup_{0 \leq t \leq T} |W_t| \to 0, \quad x_t = \int_0^t f(s) Y_s ds.
\]

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We have
\[ \frac{dx_t}{dt} = f(t) Y_t, \quad x_0 = 0. \]

Let us denote \( N_t = f(t) Y_t \). This process has the stochastic differential
\[ dN_t = [f'(t) - a(t) f(t)] Y_t dt + f(t) b(t) dV_t, \quad N_0 = 0 \]
and therefore \( N^2_\tau \) can be written as
\[ N^2_\tau = 2 \int_0^\tau N_s dN_s + \int_0^\tau f(s)^2 b(s)^2 ds. \]
Hence
\[ \Psi_{\tau} = N^2_\tau - 2 \int_0^\tau N_s dN_s, \quad 0 \leq \tau \leq T. \tag{4} \]

This relation is the basic for the construction of the estimator of \( \Psi_{\tau}, 0 \leq \tau \leq T \), i.e., we propose estimators of \( N^2_\tau \) and of the integral in (4) and obtain the estimator of \( \Psi_{\tau} \).

Note that \( x_t, 0 \leq t \leq T \) is the limit of observed process \( X_t, 0 \leq t \leq T \), its derivative is \( N_t \) and \( \Psi_t \) is quadratic variation of \( N_t \).

Suppose that in the system (1)-(2) the functions \( b(t) = b(\vartheta, t) \) and \( f(t) = f(\vartheta, t) \), where \( \vartheta \in (\alpha, \beta) \) is unknown parameter. The asymptotic properties of the MLE and BE of this parameter were described in the work [8]. It was shown that these estimators are consistent, asymptotically normal and asymptotically efficient. This parametric model of observations has some unusual features. The Fisher information
\[ \int_0^T \left[ \dot{f}(\vartheta, t) m(\vartheta, t) + f(\vartheta, t) \dot{m}(\vartheta, t) \right]^2 \sigma(t)^{-2} dt \longrightarrow 0, \]
where \( m(\vartheta, t) = \mathbb{E}_\vartheta (Y_t | X_s, 0 \leq s \leq t) \), dot means derivation w.r.t. \( \vartheta \) and for all \( t \in (0, T] \) there is a weak convergence
\[ \varepsilon^{-1/2} \left[ \dot{f}(\vartheta, t) m(\vartheta, t) + f(\vartheta, t) \dot{m}(\vartheta, t) \right] \Rightarrow \frac{\dot{S}(\vartheta, t) \sqrt{\sigma(t)}}{\sqrt{2S(\vartheta, t)}} \xi_t. \]

Here \( S(\vartheta, t) = f(\vartheta, t) b(\vartheta, t) \) and \( \xi_t, t \in (0, T] \) are mutually independent Gaussian \( \mathcal{N}(0, 1) \) random variables.
Here we propose a different construction of the estimator of this parameter which is computationally simpler than the MLE. Having estimator $\Psi_{\tau,\varepsilon}$ we consider the case where the function $\Psi_{\tau}(\vartheta), \vartheta \in (\alpha, \beta)$ is monotone and put $\hat{\vartheta}_{\tau,\varepsilon} = \Psi_{\tau}^{-1}(\Psi_{\tau,\varepsilon})$. Then under regularity conditions we show the consistency and describe the limit distribution of this estimator.

Further, having the estimator $\hat{\vartheta}_{\tau,\varepsilon}$ we use Fisher-score device and obtain One-step MLE-process $\vartheta_{t,\varepsilon}^*, \tau < t \leq T$. The asymptotic properties (consistency and asymptotic normality) of the estimator $\vartheta_{t,\varepsilon}^*$ are established and the possibility of the construction of adaptive Kalman-Bucy filtration equations is discussed.

2 Estimation of quadratic variation

The construction of the estimator of $\Phi_{\tau}$ is a slight modification of this procedure.

Introduce the estimators

$$N_{t,\varepsilon} = \frac{1}{\varphi_{\varepsilon}} \int_0^T K \left( \frac{s-t}{\varphi_{\varepsilon}} \right) dX_s,$$

$$\bar{N}_{\tau,\varepsilon} = \frac{1}{\varphi_{\varepsilon}} \int_0^\tau K^*_s \left( \frac{s-\tau}{\varphi_{\varepsilon}} \right) dX_s,$$

$$\hat{\Psi}_{\tau,\varepsilon} = \bar{N}_{\tau,\varepsilon}^2 - 2 \int_0^\tau N_{s,\varepsilon} dN_{s,\varepsilon} = \frac{1}{\varphi_{\varepsilon}^2} \left( \int_0^\tau K_s \left( \frac{s-\tau}{\varphi_{\varepsilon}} \right) dX_s \right)^2$$

$$- \frac{1}{\varphi_{\varepsilon}^3} \int_0^\tau \int_0^\tau K \left( \frac{t-s}{\varphi_{\varepsilon}} \right) dX_t \int_0^\tau K^r \left( \frac{r-s}{\varphi_{\varepsilon}} \right) dX_r ds.$$

where the bandwidth $\varphi_{\varepsilon} \to 0$. The one-sided continuous kernels $K_s(\cdot), K(\cdot)$ satisfy the usual conditions

$$\int_{-1}^0 K_s(u) \, du = 1, \quad \int_{-1}^0 u K_s(u) \, du = 0, \quad K_s(u) = 0, \text{ for } u \notin [-1, 0],$$

$$\int_0^1 K(u) \, du = 1, \quad \int_0^1 u K(u) \, du = 0, \quad K(u) = 0, \text{ for } u \notin [0, 1]$$

and the kernel $K(\cdot)$ is continuously differentiable.
Introduce the random variables

$\hat{W}_\tau \sim \mathcal{N}(0, d_2^*), \quad d_2^* = \int_{-1}^{0} K_*(u)^2 \, du,$

$\hat{V}_\tau \sim \mathcal{N}(0, d_2^*) , \quad d_2^* = \int_{-1}^{0} \int_{-1}^{0} K_*(u) K_*(v) (u \wedge v) \, du \, dv,$

$\hat{Q}_\tau \sim \mathcal{N}(0, \hat{d}_2^2) , \quad \hat{d}_2^2 = \int_{\tau}^{0} f(t)^4 b(t)^4 \, dt,$

$\hat{R}_\tau \sim \mathcal{N}(0, \hat{d}_2^2) , \quad \hat{d}_2^2 = \int_{\tau}^{0} f(t)^2 b(t)^2 \sigma(t)^2 \, dt,$

$Z_\tau = 2f(\tau) Y_\tau \left[ f(\tau) b(\tau) \hat{V}_\tau + \sigma(\tau) \hat{W}_\tau \right] + 2\hat{Q}_\tau + 2\hat{R}_\tau.$

The regularity condition is $A$. The functions $a(\cdot), b(\cdot) \in C(1)[0, \tau]$ and the functions $f(\cdot), \sigma(\cdot) \in C(2)[0, \tau]$.

**Theorem 1.** Let the condition $A$ be fulfilled then we have the convergence

$$\varepsilon^{-1/2} (\Psi_{\tau, \varepsilon} - \Psi_\tau) \xrightarrow{d} Z_\tau,$$

the random variables $\hat{V}_\tau, \hat{W}_\tau, \hat{Q}_\tau, \hat{R}_\tau$ are independent and for any $p > 0$ there exist a constant $C > 0$ such that

$$\varepsilon^{-p/2} \mathbb{E}_{\theta_0} \left| \Psi_{\tau, \varepsilon} - \Psi_\tau \right|^p \leq C.$$

**Proof.** We have

$$\bar{N}_{\tau, \varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^\tau K_* \left( \frac{s - \tau}{\varphi_\varepsilon} \right) f(s) Y_s ds + \varepsilon \int_0^\tau K_* \left( \frac{s - \tau}{\varphi_\varepsilon} \right) \sigma(s) \, dW_s.$$

We change the variable $u = (s - \tau) \varepsilon^{-1/2}$ in the stochastic integral

$$\frac{\varepsilon}{\varphi_\varepsilon} \int_0^\tau K_* \left( \frac{s - \tau}{\varphi_\varepsilon} \right) \sigma(s) \, dW_s = \frac{\varepsilon}{\sqrt{\varphi_\varepsilon}} \int_{-1}^{0} K_*(u) \sigma(\tau + \varphi_\varepsilon u) \, d\hat{W}_{\tau, \varepsilon}(u)$$

$$= \frac{\varepsilon}{\sqrt{\varphi_\varepsilon}} \sigma(\tau) \hat{W}_{\tau, \varepsilon}(1 + O_P(\varphi_\varepsilon)),$$

where we denoted $\hat{W}_{\tau, \varepsilon}(u) = \varphi_\varepsilon^{-1/2} [W_{\tau+\varphi_\varepsilon u} - W_\tau]$,

$$\hat{W}_{\tau, \varepsilon} = \int_{-1}^{0} K_*(u) \, d\hat{W}_{\tau, \varepsilon}(u) \sim \mathcal{N}(0, d_2^*).$$
and used the relations $\sigma (\tau + \varphi u) = \sigma (\tau) (1 + O(\varphi))$ $\tau/\varphi \geq 1$.

For the ordinary integral the same change of variables gives us the representation

$$\frac{1}{\varphi} \int_0^\tau K_s \left( \frac{s - \tau}{\varphi} \right) f(s) Y_s ds = \int_{-1}^0 K_s(u) N_{\tau+\varphi u} du$$

$$= N_\tau + \varphi f' (\tau) Y_\tau \int_{-1}^0 uK_s(u) du (1 + O_p (\sqrt{\varphi}))$$

$$- \varphi f (\tau) a (\tau) Y_\tau \int_{-1}^0 uK_s(u) du (1 + O_p (\sqrt{\varphi}))$$

$$+ \sqrt{\varphi} f (\tau) b (\tau) \int_{-1}^0 K_s(u) \hat{V}_{\tau,\varphi} (u) du (1 + O_p (\sqrt{\varphi}))$$

$$= N_\tau + \sqrt{\varphi} f (\tau) b (\tau) \hat{V}_{\tau,\varphi} (1 + O_p (\sqrt{\varphi})),$$

where we denoted $\hat{V}_{\tau,\varphi} (u) = \varphi^{-1/2} [V_\tau + \varphi u - V_\tau]$, $\hat{V}_{\tau,\varphi} \sim \mathcal{N} (0, d_{ss}^2)$,

$$\hat{V}_{\tau,\varphi} = \int_{-1}^0 K_s(u) \hat{V}_{\tau,\varphi} (u) du,$$

and used relations $f(\tau + \varphi u) = f(\tau) + \varphi uf'(\tau) (1 + o_p (1))$ and

$$Y_{\tau+\varphi u} = Y_\tau - \int_\tau^{\tau+\varphi u} a (s) Y_s ds + \int_\tau^{\tau+\varphi u} b (s) dV_s$$

$$= Y_\tau - \varphi u a (\tau) Y_\tau (1 + o_p (1)) + \sqrt{\varphi} b (\tau) \hat{V}_{\tau,\varphi} (u) (1 + o_p (1)).$$

Hence we obtain the relation

$$\tilde{N}_{\tau,\varphi} - N_\tau = \sqrt{\varphi} f (\tau) b (\tau) \hat{V}_{\tau,\varphi} (1 + o_p (1)) + \frac{\varepsilon}{\sqrt{\varphi}} \sigma (\tau) \hat{W}_{\tau,\varphi} (1 + o_p (1)),$$

and for any $p > 0$

$$\varepsilon^{-p/2} \mathbb{E}_{\theta_0} \left| \tilde{N}_{\tau,\varphi} - N_\tau \right|^p = \mathbb{E}_{\theta_0} \left| f (\tau) b (\tau) \hat{V}_\tau + \sigma (\tau) \hat{W}_\tau \right|^p + O_p (1).$$

where we put $\varphi = \varepsilon$. Remind that the random variables $\hat{V}_{\tau,\varphi}$ and $\hat{W}_{\tau,\varphi}$ are independent and their distributions do not depend on $\varepsilon$.

Moreover, we obtain the limit in distribution

$$\varepsilon^{-1/2} \left( \tilde{N}_{\tau,\varphi} - N_\tau \right) \xrightarrow{\text{d}} f (\tau) b (\tau) \hat{V}_\tau + \sigma (\tau) \hat{W}_\tau \quad (6)$$
where \( \hat{V}_\tau \) and \( \hat{W}_\tau \) are independent random variables with Gaussian distributions \( \mathcal{N}(0, d_{\tau,\epsilon}^2) \) and \( \mathcal{N}(0, d_{\tau,\epsilon}^2) \) respectively.

Using the standard arguments we obtain the similar relations for the estimator \( \bar{N}_{2,\epsilon}^\tau \) of \( N_2^\tau \)

\[
\varepsilon^{-1/2} \left( \bar{N}_{2,\epsilon}^\tau - N_2^\tau \right) = 2 f(\tau) Y_\tau \left[ f(\tau) b(\tau) \hat{V}_{\tau,\epsilon} + \sigma(\tau) \hat{W}_{\tau,\epsilon} \right] + o_p(1),
\]

\[
\varepsilon^{-p/2} E_{\theta_0} |\bar{N}_{2,\epsilon}^\tau - N_2^\tau|^p = |2 f(\tau)|^p E_{\theta_0} |Y_\tau|^p E_{\theta_0} \left| f(\tau) b(\tau) \hat{V}_{\tau,\epsilon} + \sigma(\tau) \hat{W}_{\tau,\epsilon} \right|^p + o_p(1),
\]

and

\[
\varepsilon^{-1/2} \left( \bar{N}_{2,\epsilon}^\tau - N_2^\tau \right) \rightarrow 2 f(\tau) Y_\tau \left[ f(\tau) b(\tau) \hat{V}_{\tau,\epsilon} + \sigma(\tau) \hat{W}_{\tau,\epsilon} \right].
\]

Note that the Gaussian process \( Y_\tau \) and Gaussian variable \( \hat{V}_{t,\epsilon} \) are asymptotically independent and \( E_{\theta_0} Y_\tau \hat{V}_{t,\epsilon} \rightarrow 0 \).

The integral in (\textit{[1]}) we estimate with the help of a different kernel but before we remark that we have the following representation of this integral

\[
\int_0^\tau N_0 dN_t = \int_0^\tau f(t) Y_t^2 \left[ f'(t) - f(t) a(t) \right] dt + \int_0^\tau f(t)^2 Y_t b(t) dV_t. \tag{7}
\]

Therefore we will estimate these two integrals of the right-hand side of this expression. We have

\[
\int_0^\tau N_{t,\epsilon} dN_{t,\epsilon} = -\frac{1}{\varphi_\epsilon^2} \int_0^\tau \int_0^\tau K \left( \frac{s-t}{\varphi_\epsilon} \right) dX_s \int_0^\tau K' \left( \frac{s-t}{\varphi_\epsilon} \right) dX_s \, dt
\]

\[
= \int_0^\tau \left[ \frac{1}{\varphi_\epsilon^2} \int_0^\tau K \left( \frac{s-t}{\varphi_\epsilon} \right) f(s) Y_s ds + \frac{\varphi_\epsilon}{\varphi_\epsilon} \int_0^\tau K \left( \frac{s-t}{\varphi_\epsilon} \right) \sigma(s) dW_s \right]
\]

\[
\times \left[ -\frac{1}{\varphi_\epsilon^2} \int_0^\tau K' \left( \frac{s-t}{\varphi_\epsilon} \right) f(s) Y_s ds - \frac{\varphi_\epsilon}{\varphi_\epsilon^2} \int_0^\tau K' \left( \frac{s-t}{\varphi_\epsilon} \right) \sigma(s) dW_s \right] \, dt.
\]

Let us describe the asymptotics of these integrals. For the first integrals and \( t \in [0, \tau - \varphi_\epsilon] \) we already proved that

\[
\frac{1}{\varphi_\epsilon} \int_0^\tau K \left( \frac{s-t}{\varphi_\epsilon} \right) dX_s = N_t + \sqrt{\varphi_\epsilon} \left[ f(t) b(t) \hat{V}_{t,\epsilon} + \sigma(t) \hat{W}_{t,\epsilon} \right] (1 + o_p(1)),
\]
Here

\[
\begin{align*}
\tilde{V}_{t,\varepsilon} &= \int_0^1 K(u) \left[ \frac{V_{t+\varphi u} - V_t}{\sqrt{\varphi}} \right] du = \int_0^1 K(u) V_{t,\varepsilon}(u) du \sim \mathcal{N}(0, d^2), \\
W_{t,\varepsilon} &= \int_0^1 K(u) \left[ \frac{W_{t+\varphi u} - W_t}{\sqrt{\varphi}} \right] du = \int_0^1 K(u) W_{t,\varepsilon}(u) du \sim \mathcal{N}(0, d^2)
\end{align*}
\]

Note that

\[
\begin{align*}
\mathbb{E}_{\vartheta_0} \tilde{V}_{t,\varepsilon} &= 0, & |\mathbb{E}_{\vartheta_0} \tilde{V}_{t,\varepsilon} \tilde{V}_{t_2,\varepsilon}| &\leq C \mathbb{I}_{|t_1 - t_2| \leq \varphi}, \quad (8) \\
\mathbb{E}_{\vartheta_0} \tilde{W}_{t,\varepsilon} &= 0, & |\mathbb{E}_{\vartheta_0} \tilde{W}_{t_1,\varepsilon} \tilde{W}_{t_2,\varepsilon}| &\leq C \mathbb{I}_{|t_1 - t_2| \leq \varphi}. \quad (9)
\end{align*}
\]

Further, for \( t \in [\varphi, \tau] \)

\[
\begin{align*}
-\frac{1}{\varphi} \int_0^\tau K' \left( \frac{s-t}{\varphi} \right) f(s) Y_s ds &= -\frac{1}{\varphi} \int_0^\tau f(s) Y_s dK \left( \frac{s-t}{\varphi} \right) \\
&= \frac{1}{\varphi} \int_0^\tau K \left( \frac{s-t}{\varphi} \right) f'(s) Y_s ds - \frac{1}{\varphi} \int_0^\tau K \left( \frac{s-t}{\varphi} \right) f(s) a(s) Y_s ds \\
&\quad + \frac{1}{\varphi} \int_0^\tau K \left( \frac{s-t}{\varphi} \right) f(s) b(s) dV_s. \quad (10)
\end{align*}
\]

The first two integrals of the RHS on (8) allow us estimate the ordinary integral in (7) and the last integral will give us the estimator of the stochastic integral in (8).

We can write

\[
\begin{align*}
\frac{1}{\varphi} \int_0^\tau K \left( \frac{s-t}{\varphi} \right) f'(s) Y_s ds &= f'(t) Y_t + \sqrt{\varphi} f'(t) b(t) \tilde{V}_{t,\varepsilon} (1 + o_p(1)), \\
\frac{1}{\varphi} \int_0^\tau K \left( \frac{s-t}{\varphi} \right) f(s) a(s) Y_s ds &= f(t) a(t) Y_t \\
&\quad + \sqrt{\varphi} f(t) a(t) b(t) \tilde{V}_{t,\varepsilon} (1 + o_p(1)).
\end{align*}
\]

To obtain the stochastic integral in (7) we change the order of integration as
follows
\[
\frac{1}{\varphi_\varepsilon} \int_0^\tau f(t) Y_t \int_0^\tau K\left(\frac{s-t}{\varphi_\varepsilon}\right) f(s) b(s) dV_s \, dt
= \frac{1}{\varphi_\varepsilon} \int_0^\tau f(s) b(s) \int_0^\tau K\left(\frac{s-t}{\varphi_\varepsilon}\right) f(t) Y_t \, dV_s
= - \int_0^\tau f(s) b(s) \int_1^0 K(u) f(s - \varphi_\varepsilon u) Y_{s-u} \, du \, dV_s
= \int_0^\tau f(s)^2 b(s) Y_s \, dV_s + \frac{\varphi_\varepsilon}{\varphi_\varepsilon} \int_0^\tau f(s)^2 b(s)^2 V_{s,\varepsilon} \, dV_s + O_p(\varphi_\varepsilon),
\]
where
\[
V_{s,\varepsilon} = \int_0^1 K(u) \left(\frac{V_{s-\varphi_\varepsilon u} - V_s}{\sqrt{\varphi_\varepsilon}}\right) \, du \sim \mathcal{N}(0, d^2).
\]

Remark, that as \(u\) takes non negative values only, the random process \(Y_{s-\varphi_\varepsilon u}\) is non anticipative in the corresponding stochastic integral.

Further
\[
\frac{\varepsilon}{\varphi_\varepsilon^2} \int_0^\tau K'(s-t) \sigma(s) \, dW_s = \frac{\varepsilon}{\varphi_\varepsilon^{3/2}} \sigma(t) \int_0^1 K'(u) \, d\tilde{W}_{t,\varepsilon}(u) \left(1 + O_p(\varphi_\varepsilon)\right)
= \frac{\varepsilon}{\varphi_\varepsilon^{3/2}} \sigma(t) \tilde{W}_{t,\varepsilon} \left(1 + O_p(\varphi_\varepsilon)\right)
\]
where \(\tilde{W}_{t,\varepsilon} \Rightarrow \tilde{W}_t, 0 < t \leq \tau\) and \(\tilde{W}_t, 0 < t \leq \tau\) are independent random variables.

Remark as well that for any continuous function \(h(\cdot)\) we have the limit in probability
\[
\int_0^\tau h(t) Y_t \tilde{W}_{t,\varepsilon} \, dt \rightarrow 0 \quad (11)
\]
because by \(9\)
\[
\mathbb{E}_{\theta_0} \left(\int_0^\tau h(t) Y_t \tilde{W}_{t,\varepsilon} \, dt\right)^2
= \int_0^\tau \int_0^\tau h(t) h(s) \mathbb{E}_{\theta_0}(Y_t Y_s) \mathbb{E}_{\theta_0}(\tilde{W}_{t,\varepsilon} \tilde{W}_{s,\varepsilon}) \, dt \, ds
\leq C \int_0^\tau \int_0^\tau \mathbb{I}_{|t-s| \leq \varphi_\varepsilon} \, dt \, ds \leq C \varphi_\varepsilon.
\]
Consider the normalized integral

\[ \varphi^{-1/2}_\varepsilon \int_0^\tau f(t) \sigma(t) Y_t \tilde{W}_{t,\varepsilon} \, dt = \varphi^{-1/2}_\varepsilon \int_0^\tau f(t) \sigma(t) Y_t \int_0^1 K'(u) \, d\tilde{W}_{t,\varepsilon}(u) \, dt \]

\[ = \varphi^{-1}_\varepsilon \int_0^\tau f(t) \sigma(t) Y_t \int_0^\tau K' \left( \frac{s-t}{\varphi_\varepsilon} \right) \, dW_s \, dt \]

\[ = \varphi^{-1}_\varepsilon \int_0^\tau \int_0^\tau f(t) \sigma(t) Y_t K' \left( \frac{s-t}{\varphi_\varepsilon} \right) \, dt \, dW_s. \]

For the ordinary integral here we have

\[ \frac{1}{\varphi_\varepsilon} \int_0^\tau f(t) \sigma(t) Y_t K' \left( \frac{s-t}{\varphi_\varepsilon} \right) \, dt = - \int_0^\tau f(t) \sigma(t) Y_t \, dK \left( \frac{s-t}{\varphi_\varepsilon} \right) \]

\[ = \int_0^\tau K \left( \frac{s-t}{\varphi_\varepsilon} \right) [f'(t) \sigma(t) + f(t) \sigma'(t)] Y_t \, dt \]

\[ - \int_0^\tau K \left( \frac{s-t}{\varphi_\varepsilon} \right) f(t) \sigma(t) a(t) Y_t \, dt \]

\[ + \int_0^\tau K \left( \frac{s-t}{\varphi_\varepsilon} \right) f(t) \sigma(t) b(t) \, dV_t. \]

These integrals have the following asymptotics

\[ \int_0^\tau K \left( \frac{s-t}{\varphi_\varepsilon} \right) [f'(t) \sigma(t) + f(t) \sigma'(t) - f(t) \sigma(t) a(t)] Y_t \, dt \]

\[ = \varphi_\varepsilon \left[ f'(s) \sigma(s) + f(s) \sigma'(s) - f(s) \sigma(s) a(s) \right] Y_s, (1 + O_p(\sqrt{\varphi_\varepsilon})), \]

\[ \int_0^\tau K \left( \frac{s-t}{\varphi_\varepsilon} \right) f(t) \sigma(t) b(t) \, dV_t = \sqrt{\varphi_\varepsilon} f(s) \sigma(s) b(s) V_{s,\varepsilon} (1 + O_p(\sqrt{\varphi_\varepsilon})). \]

Due to (11) the main contribution in the error in the problem of estimation of the integral in (11) is given by two integrals

\[ Q_{\tau,\varepsilon} = \int_0^\tau f(s)^2 b(s) V_{s,\varepsilon} \, dV_s, \quad R_{\tau,\varepsilon} = \int_0^\tau f(s) \sigma(s) b(s) V_{s,\varepsilon} \, dW_s. \]

As \( V_{s,\varepsilon} \) converges to independent Gaussian random variables we can show that

\[ \int_0^\tau f(s)^4 b(s)^4 V_{s,\varepsilon}^2 \, ds \rightarrow d^2, \quad \int_0^\tau f(s)^4 b(s)^4 \, ds = d^2, \]
and
\[ \int_0^\tau f(s)^2 b(s)^2 \sigma(s)^2 \, dv_{s,\varepsilon} \, ds \rightarrow \sigma^2 \int_0^\tau f(s)^2 b(s)^2 \, dv_{s,\varepsilon} \, ds = \sigma^2. \]
Therefore \( Q_{\tau,\varepsilon} \) and \( R_{\tau,\varepsilon} \) by the central limit theorem are asymptotically normal. Moreover \( Q_{\tau,\varepsilon} \) and \( R_{\tau,\varepsilon} \) are asymptotically independent because \( E_{\theta_0} Q_{\tau,\varepsilon} R_{\tau,\varepsilon} = 0 \).

Therefore we obtained the stochastic representation of the error of estimation \( \varepsilon^{-1/2} (\Psi_{\tau,\varepsilon} - \Psi_\tau) = Z_{\tau,\varepsilon} \) where \( Z_{\tau,\varepsilon} \) can be written as follows
\[ Z_{\tau,\varepsilon} = 2 f(\tau) Y_\tau \left[ f(\tau) b(\tau) \hat{V}_{\tau,\varepsilon} + \sigma(\tau) \hat{W}_{\tau,\varepsilon} \right] + 2 \hat{Q}_{\tau,\varepsilon} + 2 \hat{R}_{\tau,\varepsilon} + o_p(\sqrt{\varepsilon}). \]

(12)

\[ \square \]

3 Nonparametric estimation

Suppose that we have the model of observations (1), (2), where the functions \( a(\cdot), b(\cdot), f(\cdot), \sigma(\cdot) \) are unknown and the condition \( A \) is fulfilled. Consider the problem of estimation of the function \( \Psi_\tau, 0 \leq \tau \leq T \). Then according to the Theorem 1 the random function \( \Psi_{\tau,\varepsilon}, 0 \leq \tau \leq T \) is a consistent estimator of \( \Psi_\tau, 0 \leq \tau \leq T \) and it converges with the rate \( \sqrt{\varepsilon} \). Moreover, we have the limit distribution of the error of estimation. We suppose that the rate of convergence is optimal.

The obtained result allows us to study two other nonparametric estimation problems of the functions
\[ \hat{\Psi}_\tau = \int_0^\tau b(t)^2 \, dt, \quad 0 < \tau \leq T, \]
and
\[ \check{\Psi}_\tau = \int_0^\tau f(t)^2 \, dt, \quad 0 < \tau \leq T. \]

To discuss these problems we first consider a slightly more general statement. Suppose that the function \( g(\cdot) \in C^{(2)}[0, T] \) and introduce the random process \( H_t = f(t) Y_t \) satisfying the equation
\[ H^2_t = 2 \int_0^t H_s dH_s + \int_0^t g(t)^2 f(t)^2 b(t)^2 \, dt. \]
Hence we have
\[ \Psi_\tau \equiv \int_0^\tau g(t)^2 f(t)^2 b(t)^2 \, dt = H_\tau^2 - 2 \int_0^\tau H_\tau \, dH_\tau \]
The statistic
\[ H_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^\tau K\left(\frac{s-t}{\varphi_\varepsilon}\right) g(s) \, dX_s \]
has the representation
\[ H_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^\tau K\left(\frac{s-t}{\varphi_\varepsilon}\right) g(s) f(s) Y_s \, ds + \varepsilon \frac{1}{\varphi_\varepsilon} \int_0^\tau K\left(\frac{s-t}{\varphi_\varepsilon}\right) g(s) \sigma(s) Y_s \, dW_s \]
\[ = g(t) f(t) Y_t + \sqrt{\varphi_\varepsilon g(t)} f(t) b(t) \int_0^1 K(u) V_{t,\varepsilon}(u) \, du + O_p(\varphi_\varepsilon). \]
The proof of the Theorem \[\square\] can be modified to prove that the statistic
\[ \Psi_{\tau,\varepsilon} = H_{\tau,\varepsilon}^2 - 2 \int_0^\tau H_{t,\varepsilon} \, dH_{t,\varepsilon} \]
has the presentation similar to the given in \[\text{[5]}\].

\textbf{Estimation of} \( \hat{\Psi}_{\tau,\varepsilon}, 0 \leq \tau \leq T \).
Suppose that the function \( f(\cdot) \) is known, has two continuous derivatives and \( \inf_{0 \leq t \leq \tau} f(t) > 0 \). The functions \( a(\cdot), b(\cdot), \sigma(\cdot) \) are unknown. Introduce the set
\[ \Theta(L) = \left\{ a(\cdot), b(\cdot), \sigma(\cdot) : \sup_{0 \leq t \leq T} (|a'(t)| + |b'(t)| + |\sigma'(t)| + |\sigma''(t)|) \leq L \right\} \]
and the statistics
\[ \hat{Y}_{\tau,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^\tau K_s \left(\frac{s-\tau}{\varphi_\varepsilon}\right) f(s)^{-1} \, dX_s, \]
\[ Y_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^\tau K \left(\frac{s-t}{\varphi_\varepsilon}\right) f(s)^{-1} \, dX_s, \]
\[ \hat{\Psi}_{\tau,\varepsilon} = \hat{Y}_{\tau,\varepsilon}^2 - 2 \int_0^\tau Y_{t,\varepsilon} \, dY_{t,\varepsilon} \]
\[ = \hat{Y}_{\tau,\varepsilon}^2 + 2 \frac{1}{\varphi_\varepsilon^3} \int_0^\tau \int_0^\tau K' \left(\frac{s-t}{\varphi_\varepsilon}\right) \frac{dX_s}{f(s)} \int_0^\tau K' \left(\frac{r-t}{\varphi_\varepsilon}\right) \frac{dX_r}{f(r)} \, dt, \]
i.e., we put \( g(t) = f(t)^{-1} \).
Proposition 1. Let \( a(\cdot), b(\cdot), \sigma(\cdot) \in \Theta(L) \). Then the estimator \( \hat{\Psi}_{\tau,\varepsilon} \) is consistent, we have convergence in distribution

\[
\varepsilon^{-1/2} \left( \hat{\Psi}_{\tau,\varepsilon} - \hat{\Psi}_\tau \right) \Rightarrow 
\]

and for any \( p > 0 \) and any \( \delta > 0 \) there exists a constant \( C > 0 \) such that

\[
\sup_{a(\cdot), b(\cdot), \sigma(\cdot) \in \Theta(L)} \sup_{\delta \leq \tau \leq T - \delta} \varepsilon^{-p/2} \mathbb{E}_b \left| \hat{\Psi}_{\tau,\varepsilon} - \hat{\Psi}_\tau \right|^p \leq C.
\]

The proof follows from the proof of the Theorem \( \Box \).

Another possibility, of course, is to put \( g(t) = b(t)^{-1} \) (\( b(\cdot) \) is known). Then under condition \( \inf_{0 \leq t \leq \tau} b(t) > 0 \) we obtain consistent estimator of another function \( \overline{\Psi}_\tau, 0 \leq \tau \leq T \) and the proposition similar to the given above can be formulated and proved for this estimator too.

4 Parameter estimation

Consider the partially observed linear system

\[
dY_t = -a(t)Y_t dt + b(\vartheta, t) dV_t, \quad Y_0 = 0, \quad (13) 
\]

\[
dX_t = f(\vartheta, t)Y_t dt + \varepsilon \sigma(t) dW_t, \quad X_0 = 0. \quad (14) 
\]

The process \( X^T = (X_t, 0 \leq t \leq T) \) is observed and the Wiener processes \( V^T = (V_t, 0 \leq t \leq T) \) and \( W^T = (W_t, 0 \leq t \leq T) \) are independent. The functions \( a(\cdot), b(\cdot), f(\cdot) \) and \( \sigma(\cdot) \) are known. The parameter \( \vartheta \in \Theta = (\alpha, \beta) \) is unknown and has to be estimated by observations \( X^T \).

Substitution estimator \( \hat{\vartheta}_\varepsilon \).

We have partially observed linear system \((13), (14)\), where the functions \( f(\cdot), b(\cdot) \) are supposed to be known and the functions \( a(\cdot), \sigma(\cdot) \) are unknown. Fix some \( \tau \) and define the function

\[
\Psi_{\tau}(\vartheta) = \int_0^\tau f(\vartheta, t)^2 b(\vartheta, t)^2 dt, \quad \vartheta \in (\alpha, \beta) = \Theta. 
\]

We have

\[
\hat{\Psi}_{\tau}(\vartheta) = 2 \int_0^\tau \left[ \dot{f}(\vartheta, t) + \dot{b}(\vartheta, t) \right] f(\vartheta, t) b(\vartheta, t) dt. 
\]
Condition B.

\(B_1\). The functions \(\Psi_\tau(\vartheta)\) has two continuous derivatives \(\dot{\Psi}_\tau(\vartheta), \ddot{\Psi}_\tau(\vartheta)\) w.r.t. \(\vartheta\).

\(B_2\). For a given \(\tau\) we have

\[
\inf_{\vartheta \in \Theta} \left| \dot{\Psi}_\tau(\vartheta) \right| > 0. \tag{15}
\]

By condition (15) the function \(\Psi_\tau(\cdot)\) is monotone. Without loss of generality we suppose that it is increasing.

Introduce the notation

\[
\psi_m = \inf_{\vartheta \in \Theta} \Psi_\tau(\vartheta), \quad \psi_M = \sup_{\vartheta \in \Theta} \Psi_\tau(\vartheta), \quad \psi_m = \Psi_\tau(\alpha), \quad \psi_M = \Psi_\tau(\beta),
\]

\[
G(\psi) = \Psi_\tau^{-1}(\psi), \quad \psi_m < \psi < \psi_M, \quad \alpha < G(\psi) < \beta,
\]

\[
\mathbb{B}_m = \left\{ \omega : \dot{\Psi}_{\tau,\varepsilon} \leq \psi_m \right\}, \quad \mathbb{B}_M = \left\{ \omega : \dot{\Psi}_{\tau,\varepsilon} \geq \psi_M \right\},
\]

\[
\mathbb{B} = \left\{ \omega : \psi_m < \dot{\Psi}_{\tau,\varepsilon} < \psi_M \right\}, \quad \eta_\varepsilon = G(\dot{\Psi}_{\tau,\varepsilon}),
\]

\[
Z_\tau(\vartheta) = 2f(\vartheta, \tau)Y_\tau \left[ f(\vartheta, \tau) b(\vartheta, \tau) \dot{V}_\tau + \sigma(\tau) \dot{W}_\tau \right] + 2\hat{Q}_\tau(\vartheta) + 2\hat{R}_\tau(\vartheta).
\]

We suppose that the definitions of \(\hat{Q}_\tau(\vartheta)\) and \(\hat{R}_\tau(\vartheta)\) are clear.

The substitution estimator (SE) is introduced as follows

\[
\tilde{\vartheta}_{\tau,\varepsilon} = \alpha \mathbb{I}_{\mathbb{B}_m} + \eta_\varepsilon \mathbb{I}_{\mathbb{B}} + \beta \mathbb{I}_{\mathbb{B}_M}. \tag{16}
\]

It has the following properties.

Proposition 2. Suppose that the conditions \(A\) and \(B\) are fulfilled, for any (small) \(\nu > 0\) we have \(g(\nu) > 0\). Then the SE \(\tilde{\vartheta}_{\tau,\varepsilon}\) is uniformly consistent, converges in distribution

\[
\frac{\tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \Rightarrow \dot{\Psi}(\vartheta_0)^{-1} Z_\tau(\vartheta_0),
\]

and for any \(p > 0\) there exists a constant \(C = C(p) > 0\) such that

\[
\sup_{\vartheta_0 \in \Theta} \varepsilon^{-p/2} \mathbb{E}_{\vartheta_0} \left| \tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0 \right|^p \leq C. \tag{17}
\]
Proof. Note that from the identity $G(\Psi(\vartheta)) = \vartheta$ we obtain
\[
\frac{d}{d\vartheta} G(\Psi(\vartheta)) = \frac{d}{d\psi} G(\psi) \bigg|_{\psi=\Psi(\vartheta)} \dot{\vartheta} = G'(\Psi(\vartheta)) \dot{\Psi}(\vartheta) = 1,
\]
and $\sup_\psi G'(\psi) \leq \kappa^{-1}$, where $\kappa = \inf_\vartheta \dot{\Psi}(\vartheta)$. Hence for any $\nu > (\vartheta_0 - \alpha) \land (\beta - \vartheta_0) > 0$ we can write
\[
\mathbb{P}_{\vartheta_0} \left( |\tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0| \geq \nu \right) \leq \mathbb{P}_{\vartheta_0} (\mathbb{B}_m) + \mathbb{P}_{\vartheta_0} (\mathbb{B}_M) + \mathbb{P}_{\vartheta_0} (|\eta_\varepsilon - \vartheta_0| \geq \nu, \mathbb{B}).
\]
Using the representation (12) we obtain the estimates
\[
\mathbb{P}_{\vartheta_0} (\mathbb{B}_m) = \mathbb{P}_{\vartheta_0} \left( \dot{\Psi}_{\tau,\varepsilon} \leq \Psi(\alpha) \right) = \mathbb{P}_{\vartheta_0} \left( \dot{\Psi}_{\tau,\varepsilon} \leq \Psi(\vartheta_0) \leq \Psi(\alpha) \right) = \mathbb{P}_{\vartheta_0} \left( \dot{\Psi}(\vartheta_0) - \dot{\Psi}_{\tau,\varepsilon} \geq \Psi(\vartheta_0) - \Psi(\alpha) \right) \leq \mathbb{P}_{\vartheta_0} \left( |\Psi(\vartheta_0) - \dot{\Psi}_{\tau,\varepsilon}| \geq \kappa \nu \right) = \mathbb{P}_{\vartheta_0} \left( |Z_{\tau,\varepsilon}| \geq \frac{\kappa \nu}{\sqrt{\varepsilon}} \right) \leq \left( \frac{\varepsilon}{\kappa^2 \nu^2} \right)^{p/2} \mathbb{E}_{\vartheta_0} |Z_{\tau,\varepsilon}|^p \leq C \left( \frac{\varepsilon}{\kappa^2 \nu^2} \right)^{p/2} \to 0,
\]
and similarly
\[
\mathbb{P}_{\vartheta_0} (\mathbb{B}_M) \leq C \left( \frac{\varepsilon}{\kappa^2 \nu^2} \right)^{p/2} \to 0.
\]
Further
\[
\mathbb{P}_{\vartheta_0} (|\eta_\varepsilon - \vartheta_0| \geq \nu, \mathbb{B}) = \mathbb{P}_{\vartheta_0} \left( \left| G(\dot{\Psi}_{\tau,\varepsilon}) - G(\Psi(\vartheta_0)) \right| \geq \nu, \mathbb{B} \right) \leq \mathbb{P}_{\vartheta_0} \left( \left| \dot{\Psi}_{\tau,\varepsilon} - \Psi(\vartheta_0) \right| \geq \frac{\nu}{G'_M} \right) \leq \mathbb{P}_{\vartheta_0} \left( |Z_{\tau,\varepsilon}| \geq \frac{\nu}{\sqrt{\varepsilon} G'_M} \right) \leq C \left( \frac{\varepsilon}{\kappa^2 \nu^2} \right)^{p/2} \to 0.
\]
Here $G'_M = \kappa^{-1}$. Therefore for any compact $\mathbb{K} \subset \Theta$ we have
\[
\sup_{\vartheta_0 \in \mathbb{K}} \mathbb{P}_{\vartheta_0} (|\tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0| \geq \nu) \leq C \left( \frac{\varepsilon}{\kappa^2 \nu^2} \right)^{p/2} \to 0
\]
and we obtain uniform consistency of the SE $\tilde{\vartheta}_{\tau,\varepsilon}$. 

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We have the relations

\[
\frac{\bar{\vartheta}_{\tau,\varepsilon} - \vartheta_0}{\varepsilon^{1/2}} = \varepsilon^{-1/2} \left( \dot{\Psi}_{\tau,\varepsilon} - \Psi(\vartheta_0) \right) G'(\Psi(\vartheta_0)) (1 + o_p(1))
\]

\[
= \dot{\Psi}(\vartheta_0)^{-1} Z_{\tau,\varepsilon} (1 + o_p(1)) \implies \dot{\Psi}(\vartheta_0)^{-1} Z_\tau(\vartheta_0)
\]

and for any \(p > 0\)

\[
\varepsilon^{-p/2} \mathbb{E}_{\vartheta_0} \left| \bar{\vartheta}_{\tau,\varepsilon} - \vartheta_0 \right|^p \leq \dot{\Psi}(\vartheta_0)^{-p} \mathbb{E}_{\vartheta_0} \left( |Z_{\tau,\varepsilon}|^p (1 + |o_p(1)|) \right) \leq C.
\]

\[\Box\]

**Example 1.**
Suppose that we have the model of observations (13), (14), where \(f(\vartheta, t) = \vartheta f(t), \vartheta \in (\alpha, \beta), \alpha > 0, b(\vartheta, t) = b(t)\) and all corresponding conditions are fulfilled. Then

\[
\bar{\Psi}_\tau(\vartheta) = \vartheta^2 \int_0^\tau f(t)^2 b(t)^2 \, dt
\]

and SE

\[
\bar{\vartheta}_{\tau,\varepsilon} = \sqrt{\bar{\Psi}_{\tau,\varepsilon}} \left( \int_0^\tau f(t)^2 b(t)^2 \, dt \right)^{-1/2}.
\]

This estimator is consistent and has rate of convergence \(\sqrt{\varepsilon}\) (Theorem 1).

**Example 2.**
Consider the model (13), (14), where \(b(\vartheta, t) = \sqrt{h(t)} + \vartheta g(t)\) and \(f(\vartheta, t) = f(t)\). Suppose that the functions \(h(\cdot)\) and \(g(\cdot)\) are positive and \(\alpha > 0\). Then

\[
\bar{\Psi}_\tau(\vartheta) = \int_0^\tau f(t)^2 [h(t) + \vartheta g(t)] \, dt
\]

and

\[
\dot{\vartheta} = \left( \bar{\Psi}_\tau(\vartheta) - \int_0^\tau f(t)^2 h(t) \, dt \right) \left( \int_0^\tau f(t)^2 g(t) \, dt \right)^{-1}.
\]

Hence the SE is

\[
\bar{\vartheta}_{\tau,\varepsilon} = \left( \bar{\Psi}_{\tau,\varepsilon} - \int_0^\tau f(t)^2 h(t) \, dt \right) \left( \int_0^\tau f(t)^2 g(t) \, dt \right)^{-1}.
\]
and this estimator has the properties described in the Theorem 1.

One-step MLE-process $\hat{\vartheta}_{t,\varepsilon}^*, \tau < t \leq T$.

Suppose that we have slightly different partially observed system

\begin{align*}
\text{d}Y_t &= -a(\vartheta, t)Y_t \text{d}t + b(\vartheta, t) \text{d}V_t, \quad Y_0 = 0, \quad (18) \\
\text{d}X_t &= f(\vartheta, t)Y_t \text{d}t + \varepsilon \sigma(t) \text{d}W_t, \quad X_0 = 0. \quad (19)
\end{align*}

As before the process $X^T = (X_t, 0 \leq t \leq T)$ is observable and $Y^T$ is hidden. All functions $a(\cdot), b(\cdot), f(\cdot)$ and $\sigma(\cdot)$ are supposed to be known. The parameter $\vartheta \in \Theta = (\alpha, \beta)$ is unknown and has to be estimated by observations $X^T$.

One way is to use the MLE $\hat{\vartheta}_\varepsilon$ defined by the equation

\begin{equation}
L(\hat{\vartheta}_\varepsilon, X^T) = \sup_{\vartheta \in \Theta} L(\vartheta, X^T),
\end{equation}

where the likelihood ratio function is

\begin{equation}
L(\vartheta, X^T) = \exp \left\{ \int_0^T \frac{M(\vartheta, t)}{\varepsilon^2 \sigma(t)^2} \text{d}X_t - \int_0^T \frac{M(\vartheta, t)^2}{2\varepsilon^2 \sigma(t)^2} \text{d}t \right\}, \quad \vartheta \in \Theta. \quad (21)
\end{equation}

Here $M(\vartheta, t) = f(\vartheta, t)m(\vartheta, t)$ and the conditional expectation $m(\vartheta, t) = \mathbb{E}_\vartheta(Y_t | X_s, 0 \leq s \leq t)$ satisfies the equation of the Kalman-Bucy filtration

\begin{equation}
dm(\vartheta, t) = -a(\vartheta, t)m(\vartheta, t) \text{d}t + \gamma(\vartheta, t)f(\vartheta, t) \left[ \text{d}X_t - f(\vartheta, t)m(\vartheta, t) \text{d}t \right] \quad (22)
\end{equation}

with initial value $m(\vartheta, 0) = 0$. The function $\gamma(\vartheta, t) = \mathbb{E}_\vartheta(Y_t - m(\vartheta, t))^2$ is solution of Riccati equation

\begin{equation}
\frac{\partial \gamma(\vartheta, t)}{\partial t} = -2a(\vartheta, t)\gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} + b(\vartheta, t)^2, \quad (23)
\end{equation}

with initial value $\gamma(\vartheta, 0) = 0$. The asymptotic properties of the MLE for this model of observations were studied in the work [8]. It was shown that this estimator is consistent, asymptotically normal

\begin{align*}
\varepsilon^{-1/2} \left( \hat{\vartheta}_\varepsilon - \vartheta_0 \right) &\Rightarrow \mathcal{N}(0, I(\vartheta_0)^{-1}), \\
I(\vartheta) &= \int_0^T \frac{\dot{S}(\vartheta, t)^2}{2S(\vartheta, t) \sigma(t)} \text{d}t
\end{align*}
and asymptotically efficient. The same asymptotic properties has Bayesian estimator

$$\tilde{\vartheta}_\varepsilon = \frac{\int_\Theta \vartheta p(\vartheta) L(\vartheta, X^T) d\vartheta}{\int_\Theta p(\vartheta) L(\vartheta, X^T) d\vartheta}. \tag{24}$$

Here $p(\vartheta), \vartheta \in \Theta$ is density a priori of the parameter $\vartheta$.

This approach provides the estimators with the nice asymptotic properties, but the calculation of these estimators by expressions (20), (21) and (24) requires numerical solutions of the equations (22) and (23) for all $\vartheta \in \Theta$. To avoid such sometimes difficult numerical calculations we consider approach based on the Fisher-score device and $\tilde{\vartheta}_{\tau,\varepsilon}$ as preliminary one allows to obtain an estimator-process which has good asymptotic properties and can be used for the construction of adaptive filter.

Suppose that for some fixed small value $\tau > 0$ we have $\text{SE} \tilde{\vartheta}_{\tau,\varepsilon}$ constructed by the observations $X^\tau = (X_t, 0 \leq t \leq \tau)$. Introduce notation

$$I^t_\tau (\vartheta) = \int_t^\tau \frac{\hat{S}(\vartheta, s)^2}{2S(\vartheta, s) \sigma(s)} ds, \quad q_\varepsilon(\vartheta, t) = a(\vartheta, t) + \frac{\gamma(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2}$$

$$\vartheta^*_t,\varepsilon = \tilde{\vartheta}_{\tau,\varepsilon} + \frac{1}{I^\tau_\varepsilon(\tilde{\vartheta}_{\tau,\varepsilon})} \int_\tau^t \frac{M(\tilde{\vartheta}_{\tau,\varepsilon}, s)}{\varepsilon \sigma(s)} [dX_s - M(\tilde{\vartheta}_{\tau,\varepsilon}, s) ds], \quad \tau < t \leq T,$$

$$\xi^*_t,\varepsilon = \frac{\vartheta^*_t,\varepsilon - \vartheta_0}{\sqrt{\varepsilon}}, \quad \xi_t = \frac{1}{I^\tau_\varepsilon(\vartheta_0)} \int_\tau^t \frac{\hat{S}(\vartheta_0, s)}{\sqrt{2S(\vartheta_0, s) \sigma(s)}} dw(s), \quad \tau < t \leq T.$$

We have to define the random processes $M(\tilde{\vartheta}_{\tau,\varepsilon}, s) = \hat{f}(\tilde{\vartheta}_{\tau,\varepsilon}, s) \hat{m}(\tilde{\vartheta}_{\tau,\varepsilon}, s)$ and $\hat{M}(\tilde{\vartheta}_{\tau,\varepsilon}, s) = \hat{f}(\tilde{\vartheta}_{\tau,\varepsilon}, s) \hat{m}(\tilde{\vartheta}_{\tau,\varepsilon}, s) + \hat{f}(\tilde{\vartheta}_{\tau,\varepsilon}, s) \hat{m}(\tilde{\vartheta}_{\tau,\varepsilon}, s)$. The solution of the equation (22) we write as follows

$$m(\vartheta, t) = e^{-\int^t_0 q_\varepsilon(\vartheta, v) dv} \int^t_0 e^{\int^s_0 q_\varepsilon(\vartheta, v) dv} \frac{\gamma(\vartheta, s) f(\vartheta, s)}{\varepsilon^2 \sigma(s)^2} dX_s$$

$$= e^{-\int^t_0 q_\varepsilon(\vartheta, v) dv} \int^t_0 D(\vartheta, s) dX_s$$

with obvious notation. We can not put $\tilde{\vartheta}_{\tau,\varepsilon}$ in this integral because the function $D(\tilde{\vartheta}_{\tau,\varepsilon}, s)$ is not integrable in Itô sens. We have equality

$$\int^t_0 D(\vartheta, s) dX_s = D(\vartheta, s) X_t - \int^t_0 X_s D'(\vartheta, s) ds$$
and we put

\[
m (\dot{\vartheta}_{\tau, \varepsilon}, t) = e^{- \int_0^t q (\dot{\vartheta}_{\tau, \varepsilon}, v) \, dv} \left[ D \left( \dot{\vartheta}_{\tau, \varepsilon}, s \right) X_t - \int_0^t X_s D' \left( \dot{\vartheta}_{\tau, \varepsilon}, s \right) \, ds \right].
\]

The similar transformation can be done for calculation of the random process \( \dot{m} (\dot{\vartheta}_{\tau, \varepsilon}, t) \).

**Conditions C.**

\( C_1 \). For any \( t_0 \in (\tau, T] \)

\[
\inf_{\vartheta \in \Theta} I^0_{t_0} (\vartheta) > 0.
\]

\( C_2 \). The functions \( f (\cdot), \sigma (\cdot) \) are separated from zero and the functions \( f (\cdot), b (\cdot) \) have two continuous derivatives w.r.t. \( \vartheta \).

**Proposition 3.** Let the conditions \( A, B, C \) be fulfilled then the One-step MLE-process \( \dot{\vartheta}^*_{t, \varepsilon}, \tau < t \leq T \) is consistent: for any \( \nu > 0 \)

\[
\mathbb{P}_{\vartheta_0} \left( \sup_{t_0 \leq t \leq T} |\dot{\vartheta}^*_{t, \varepsilon} - \vartheta_0| \geq \nu \right) \longrightarrow 0,
\]

the random process \( \xi_{t, \varepsilon}, t_0 \leq t \leq T \) converges in distribution in the measurable space \((C [t_0, T], \mathcal{B})\) to the Gaussian process

\[
\xi_{t, \varepsilon} \Rightarrow \xi, \quad \xi_t \sim \mathcal{N} \left( 0, I^t_{t_0} (\vartheta_0)^{-1} \right)
\]

**Proof.** The proof follows the same steps as the proof of the Proposition 3 in [10]. The only difference is that in [10] the function \( f (\vartheta, t) = f (t) \).

Note that if \( \inf_{\vartheta \in \Theta} \left| \dot{S} (\vartheta, \tau) \right| > 0 \), then the condition \( C_1 \) is fulfilled.

**Adaptive filtration.**

Let us consider one else possibility to use SE \( \tilde{\vartheta}_{\tau, \varepsilon} \) and the corresponding One-step MLE-process \( \dot{\vartheta}^*_{t, \varepsilon}, t_0 \leq t \leq T \). Suppose that we have the partially observed two-dimensional linear stochastic system (18), (19). We can not use the equations (22), (23) because the true value \( \vartheta_0 \) is unknown.
Introduce the equations of adaptive filtration

\[
\begin{align*}
\text{d} \hat{m}(t) &= -q \varepsilon(\vartheta_\varepsilon^*, t) \hat{m}(t) \, dt + \frac{\gamma(\vartheta_\varepsilon^*, t) f(\vartheta_\varepsilon^*, t)}{\varepsilon^2 \sigma(t)^2} \text{d}X_t, \quad \hat{m}(0) = 0, \\
\frac{\partial \gamma(\vartheta, t)}{\partial t} &= -2a(\vartheta, t) \gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} + b(\vartheta, t)^2, \quad \gamma(\vartheta, 0) = 0.
\end{align*}
\]

Here Riccati equation can be solved before the experience. It is possible to give this couple of equations in recurrent form too

\[
\begin{align*}
\text{d} \tilde{m}(t) &= -q \varepsilon(\vartheta_\varepsilon^*, t) \tilde{m}(t) \, dt + \tilde{\gamma}(t) \varepsilon(\vartheta_\varepsilon^*, t) f(\vartheta_\varepsilon^*, t) \, \varepsilon^2 \sigma(t)^2 \text{d}X_t, \quad \tilde{m}(0) = 0, \\
\frac{\partial \tilde{\gamma}(t)}{\partial t} &= -2a(\vartheta_\varepsilon^*, t) \tilde{\gamma}(t) - \frac{\tilde{\gamma}(t)^2 f(\vartheta_\varepsilon^*, t)^2}{\varepsilon^2 \sigma(t)^2} + b(\vartheta_\varepsilon^*, t)^2, \quad \tilde{\gamma}(0) = 0.
\end{align*}
\]

Note that the error of this approximation was described in [10], where it was shown that \( \hat{m}(t) - m(\vartheta_0, t) = \varepsilon O_p(1) \).

## 5 Possible generalizations

It is possible to consider nonlinear partially observed system

\[
\begin{align*}
\text{d} Y_t &= -A(t, Y_t) \, dt + B(t, Y_t) \, dV_t, \quad Y_0 = 0, \\
\text{d} X_t &= F(t, Y_t) \, dt + \varepsilon \sigma(t) \, dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T.
\end{align*}
\]

The derivative \( N_\tau = F(\tau, Y_\tau) \) of the limit of \( X_t \) has quadratic variation

\[
\Psi_\tau = F(\tau, Y_\tau)^2 - 2 \int_0^\tau F(t, Y_t) \text{d}F(t, Y_t) = \int_0^\tau F'(t, Y_t)^2 B(t, Y_t)^2 \, dt.
\]

The same kernel-type estimator of the derivative \( N_\tau \) has the presentation

\[
\tilde{N}_\tau, \varepsilon = \frac{1}{\varphi_\varepsilon} \int_0^\tau K_\varepsilon \left( \frac{s - \tau}{\varphi_\varepsilon} \right) \, dX_s = \int_0^\tau K_\varepsilon(u) F(\tau + \varphi_\varepsilon u, Y_{\tau + \varphi_\varepsilon u}) \, du \\
+ \varepsilon \sqrt{\varphi_\varepsilon} \int_0^\tau K_\varepsilon(u) \sigma(\tau + \varphi_\varepsilon u) \, dW_{\tau + \varphi_\varepsilon u} = F(\tau, Y_\tau) \\
+ \sqrt{\varphi_\varepsilon} F'_y(\tau, Y_\tau) B(\tau, Y_\tau) \int_0^\tau K_\varepsilon(u) \nu_{\tau, \varepsilon}(u) \, du (1 + o(1)) \\
+ \frac{\varepsilon}{\sqrt{\varphi_\varepsilon}} \sigma(\tau) \int_0^\tau K_\varepsilon(u) \, dw_{\tau, \varepsilon}(u) (1 + o(1)),
\]

\[
\text{(1)}
\]
where 
\[ v_{\tau,\varepsilon}(u) = \frac{V_{\tau+\varphi \varepsilon} u - V_{\tau}}{\sqrt{\varphi \varepsilon}}, \quad w_{\tau,\varepsilon}(u) = \frac{W_{\tau+\varphi \varepsilon} u - W_{\tau}}{\sqrt{\varphi \varepsilon}}. \]

The similar estimators 
\[ N_{t,\varepsilon} = \frac{1}{\varphi \varepsilon} \int_{0}^{\tau} K \left( \frac{s - \tau}{\varphi \varepsilon} \right) dX_s, \quad \Psi_{\tau,\varepsilon} = \tilde{N}_{\tau,\varepsilon}^2 - 2 \int_{0}^{\tau} N_{t,\varepsilon} dN_{t,\varepsilon}, \]
allow us under regularity conditions to prove the consistency \( \Psi_{\tau,\varepsilon} \to \Psi_{\tau} \) and weak convergence 
\[ \varepsilon^{-1/2} (\Psi_{\tau,\varepsilon} - \Psi_{\tau}) \Rightarrow Z_{\tau}^* \]
with corresponding random variable \( Z_{\tau}^* \).

The random process \( \Psi_{\tau}, 0 \leq \tau \leq T \) depends on unobservable component \( Y^T \) and the knowledge of \( \Psi_{\tau} \) can not be used directly for the construction of the parameter estimates. That is why we considered much more simple linear model (1)-(2), where the applications to parameter estimation and construction of adaptive Kalman-Bucy filter can be done directly.

If we have the model of observations (13)-(14) and the parameter \( \vartheta \in \Theta \subset \mathbb{R}^d, d > 1 \). Then we can take \( 0 = \tau_0 < \tau_1, \ldots, < \tau_d \), introduce the the system of equations \( \Psi_{\tau_0}^{\tau_1}(\vartheta) = \psi_1, \ldots, \Psi_{\tau_{d-1}}^{\tau_d}(\vartheta) = \psi_d \), where
\[
\Psi_{\tau_k}^{\tau_{k+1}}(\vartheta) = \Psi_{\tau_{k+1}}(\vartheta) - \Psi_{\tau_k}(\vartheta) = \int_{\tau_k}^{\tau_{k+1}} f(\vartheta, t) b(\vartheta, t)^2 dt, \quad k = 0, \ldots, d - 1.
\]

Suppose that this system of equations has a unique solutions for all \( \vartheta \in \Theta \), then we can once more define the corresponding substitution estimator \( \tilde{\vartheta}_{\varepsilon} \) and under regularity conditions to prove the consistency of this estimator and describe its limit distribution.

Acknowledgment. This research was supported by RSF project no 20-61-47043.

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