A Note on Bipartite Subgraphs and Triangle-independent Sets

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Abstract

Let $\alpha_1(G)$ denote the maximum size of an edge set that contains at most one edge from each triangle of $G$. Let $\tau_B(G)$ denote the minimum size of an edge set whose deletion makes $G$ bipartite. It was conjectured by Lehel and independently by Puleo that $\alpha_1(G) + \tau_B(G) \leq n^2/4$ for every $n$-vertex graph $G$. Puleo showed that $\alpha_1(G) + \tau_B(G) \leq 5n^2/16$ for every $n$-vertex graph $G$. In this note, we improve the bound by showing that $\alpha_1(G) + \tau_B(G) \leq 4403n^2/15000$ for every $n$-vertex graph $G$.

Keywords: Bipartite subgraph, Triangle-independent set

1 Introduction

Let $G$ be a simple undirected graph. A triangle-independent set in $G$ is an edge set that contains at most one edge from each triangle of $G$. We let $\alpha_1(G)$ denote the maximum size of a triangle-independent set in $G$. On the other hand, a triangle edge cover in $G$ is an edge set that contains at least one edge from each triangle of $G$. We let $\tau_1(G)$ denote the minimum size of a triangle edge cover in $G$.

Erdős, Gallai, and Tuza made the following conjecture:

Conjecture 1 (Erdős-Gallai-Tuza [10]) For every $n$-vertex graph $G$, $\alpha_1(G) + \tau_1(G) \leq n^2/4$. 

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Note that the equality holds for the graphs $K_n$ and $K_{n/2,n/2}$, where $n$ is even. Indeed, $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$ (by Mantel’s theorem [12]), while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$.

More generally, let $G_1 \lor \ldots \lor G_t$ denote the graph obtained from the disjoint union $G_1 + \ldots + G_t$ by adding all edges between vertices from different $G_i$. Puleo (see [14, 13]) showed that the equality holds for any graph of the form $K_{r_1,r_1} \lor \ldots \lor K_{r_t,r_t}$.

Conjecture 1 was originally stated only for triangular graphs, which are graphs where every edge lies in a triangle (see [10, 17]). However, later it was stated for general graphs (see [8, 17]). It was proved by Puleo [13] that these two forms of the conjecture are equivalent.

A related parameter, denoted by $\tau_B(G)$, is the minimum size of an edge set in $G$ whose deletion makes $G$ bipartite. Clearly $\tau_B(G) \geq \tau_1(G)$. Erdős [6] asked which graphs satisfy $\tau_B(G) = \tau_1(G)$. Bondy, Shen, Thomassé, and Thomassen [3] proved that $\tau_B(G) = \tau_1(G)$ when $\delta(G) \geq 0.85 |V(G)|$, and later Balogh, Keevash, and Sudakov [2] proved that $\tau_B(G) = \tau_1(G)$ when $\delta(G) \geq 0.79 |V(G)|$.

The following conjecture, which is stronger than Conjecture 1, was proposed by Lehel (see [17]) and independently by Puleo [14].

Conjecture 2 ([14]) For every $n$-vertex graph $G$, $\alpha_1(G) + \tau_B(G) \leq n^2/4$.

Puleo [14, 13] obtained many interesting results towards Conjectures 1 and 2. Conjecture 2 was verified for triangle-free graphs and for graphs that have no induced subgraph isomorphic to $K_{4^-}$ (the graph obtained from $K_4$ by deleting an edge) [13]. For general graphs, Puleo [14] showed the following upper bound:

Theorem 1 ([14]) For every $n$-vertex graph $G$, $\alpha_1(G) + \tau_B(G) \leq 5n^2/16$.

The main purpose of this note is to provide an improved bound towards Conjecture 2. We prove that $\alpha_1(G) + \tau_B(G) \leq 4403n^2/15000$ for every $n$-vertex graph $G$. We use ideas from [13], [14], [15], and [11].
We shall use the following notation and terminology. For shorthand, we let $f_B(G) = \alpha_1(G) + \tau_B(G)$. We let $n(G)$, $e(G)$, and $t(G)$ denote the number of vertices, edges, and triangles in $G$, respectively. When there is no confusion involved, we simply write $n$, $e$, and $t$. We let $d(v)$ denote the degree of a vertex $v$, and $\omega(G)$ denote the clique number of $G$. When $S \subseteq V(G)$, we write $G[S]$ for the subgraph of $G$ induced by $S$, $\overline{S}$ for the set $V(G) - S$, and $[S, \overline{S}]$ for the set of all edges with one endpoint in $S$ and the other endpoint in $\overline{S}$. We use the term minimal counterexample to refer to a vertex-minimal counterexample, that is, a graph $G$ such that the property in question holds for every proper induced subgraph of $G$ but does not hold for $G$.

The rest of the paper is organized as follows. In the next section, we investigate the structure of a minimal counterexample to $f_B(G) \leq cn(G)^2$ where $c > 1/4$. We show that the clique number of such a counterexample is bounded by a function of $c$. Thus, to prove that $f_B(G) \leq cn(G)^2$, we only need to prove it for graphs with small clique number. Then in Section 3 we present a quick proof of $f_B(G) \leq 3n(G)^2/10$, which improves Theorem 1. In Section 4 we give some estimates of $\tau_B(G)$ for $K_6$-free graphs. In particular, we show that every $n$-vertex $K_6$-free graph can be made bipartite by deleting at most $17n^2/100$ edges. In Section 5 we prove our main result.

2 $f_B(G)$ and clique number

We need the following lemma from [13].

**Lemma 1** ([13]) Let $G$ be a graph, and let $A$ be a triangle-independent set of edges in $G$. If $S$ is a nonempty proper subset of $V(G)$, then

$$f_B(G) \leq f_B(G[S]) + f_B(G[\overline{S}]) + \frac{1}{2} |[S, \overline{S}]| + |[S, \overline{S}] \cap A|.$$

In [13], Puleo used Lemma 1 to prove some conclusions on the structure of a minimal counterexample $G$ to Conjecture 2. By slightly extending his argument, we show the following:
Lemma 2 For any constant $c > 1/4$, if $G$ is a minimal counterexample to $f_B(G) \leq cn(G)^2$, then $\omega(G) < 1/(4c - 1)$.

Proof. Let $G$ be a minimal counterexample to $f_B(G) \leq cn(G)^2$. We may assume $n(G) \geq 5$, since it is easy to verify that $f_B(G) \leq n(G)^2/4 \leq cn(G)^2$ when $n(G) \leq 4$. Let $K$ be the largest clique in $G$, and let $k = |K| = \omega(G)$. Since $f_B(G) \leq n(G)^2/4 \leq cn(G)^2$ when $G$ is complete, we may assume $1 \leq k \leq n(G) - 1$.

For simplicity, write $n$ for $n(G)$. Let $A$ be any triangle-independent set in $G$, and for every $v \in V(G)$, let $N_A(v) = \{ w \in V(G) : vw \in A \}$. Since $A$ is triangle-independent, $|N_A(v) \cap K| \leq 1$ for each $v \in K$. It follows that $|[K,K] \cap A| \leq n-k$.

By Lemma 1 and the minimality of $G$, we have

$$cn^2 < f_B(G) \leq f_B(G[K]) + f_B(G[\overline{K}]) + \frac{1}{2} |[K,K]| + |[K,K] \cap A| \leq \frac{k^2}{4} + c(n-k)^2 + \frac{1}{2} |[K,K]| + n-k.$$ 

Thus, $|[K,K]| > -(2c + \frac{1}{2})k^2 + 4cnk + 2k - 2n$. However, since $K$ is the largest clique of $G$, $|[K,K]| \leq (n-k)(k-1)$. Hence, we have

$$(n-k)(k-1) > -(2c + \frac{1}{2})k^2 + 4cnk + 2k - 2n.$$ 

The above inequality simplifies to $(\frac{1}{2} - 2c)k^2 + k < (1 - (4c - 1)k)n$. Assume to the contrary that $k \geq 1/(4c - 1)$. Then $(1 - (4c - 1)k)n \leq (1 - (4c - 1)k)k$. It follows that $(\frac{1}{2} - 2c)k^2 + k < (1 - (4c - 1)k)k$. That is, $c < 1/4$, a contradiction.

\[\square\]

3 A first improvement

In this section we present a quick proof of $f_B(G) \leq 3n(G)^2/10$. We first show that the conclusion holds for $K_5$-free graphs, and then use Lemma 2 to prove that it holds for all graphs.
For a graph $G$, let $b(G)$ denote the largest size of a vertex set $B$ such that $B$ induces a bipartite subgraph of $G$. Puleo [14] proved the following bound for $\alpha_1(G)$:

**Lemma 3** ([14]) For every $n$-vertex graph $G$, $\alpha_1(G) \leq nb(G)/4$.

Now we consider $\tau_B(G)$. A well-known result by Erdős [4] says that $\tau_B(G) \leq e/2$ for every graph $G$ with $e$ edges. Puleo [14] proved the following bound for $\tau_B(G)$:

**Lemma 4** ([14]) For every $n$-vertex graph $G$, $\tau_B(G) \leq (n^2 - b(G)^2)/4$.

When $G$ is a $K_5$-free graph, $\tau_B(G)$ can be bounded as follows:

**Lemma 5** For every $n$-vertex $K_5$-free graph $G$,

$$\tau_B(G) \leq \frac{b(G)(n - b(G))}{2} + \frac{3(n - b(G))^2}{16}.$$

**Proof.** Let $B$ denote the vertex set of a largest bipartite induced subgraph of $G$. Since $G[B]$ is $K_5$-free, by Turán’s theorem [16] it has at most $3(n - b(G))^2/8$ edges. Therefore $G[B]$ can be made bipartite by deleting at most $3(n - b(G))^2/16$ edges. The conclusion follows by considering the two different ways to join the partite sets of a largest bipartite subgraph in $G[B]$ with the partite sets of $G[B]$. □

Now we can give the following bound for $f_B(G)$ when $G$ is $K_5$-free.

**Theorem 2** For every $n$-vertex $K_5$-free graph $G$, $f_B(G) \leq 3n^2/10$.

**Proof.** By Lemma 3 and Lemma 5 we have

$$f_B(G) = \alpha_1(G) + \tau_B(G) \leq \frac{nb(G)}{4} + \frac{b(G)(n - b(G))}{2} + \frac{3(n - b(G))^2}{16}$$

$$= \frac{1}{16} (-5b(G)^2 + 6nb(G) + 3n^2)$$

$$= g(b(G)),$$
where $g(x) = (-5x^2 + 6nx + 3n^2)/16$. Since $g(x)$ achieves its maximum at $x = 3n/5$, we have $f_B(G) \leq g(3n/5) = 3n^2/10$. 

By using Theorem 2 and Lemma 2 we show $f_B(G) \leq 3n^2/10$, which improves Theorem 1.

**Theorem 3** *For every $n$-vertex graph $G$, $f_B(G) \leq 3n^2/10$.***

**Proof.** It is easy to verify the conclusion for small $n$. Assume to the contrary that $G$ is a minimal counterexample. By Theorem 2, $\omega(G) \geq 5$. However, by Lemma 2 we have $\omega(G) < 1/(4 \times 3/10 - 1) = 5$, a contradiction. $\square$

4 $\tau_B(G)$ for $K_6$-free graphs

To improve our bound for $f_B(G)$, we consider $\tau_B(G)$ for $K_6$-free graphs. Similar questions have been investigated by various researchers. Erdős [5] conjectured that every $n$-vertex triangle-free graph can be made bipartite by deleting at most $n^2/25$ edges. Erdős, Faudree, Pach and Spencer [9] proved that it is enough to delete $(1/18 - \epsilon)n^2$ edges to make a $n$-vertex triangle-free graph bipartite. Erdős (see e.g., [2]) also conjectured that it is enough to delete at most $(1 + o(1))n^2/9$ edges to make any $n$-vertex $K_4$-free graph bipartite. This was confirmed by Sudakov [15] in the following strong form:

**Theorem 4** ([15]) *Every $n$-vertex $K_4$-free graph can be made bipartite by deleting at most $n^2/9$ edges. Moreover, equality holds if and only if $G$ is a complete 3-partite graph with parts of size $n/3$.***

Furthermore, Sudakov [15] made the following conjecture on $\tau_B(G)$ for $K_r$-free graphs where $r \geq 5$: 
Conjecture 3 (15) Let $G$ be a $n$-vertex $K_r$-free graph where $r \geq 5$. Then

$$\tau_B(G) \leq \begin{cases} 
\frac{r-3}{4(r-1)}n^2, & \text{if } r \text{ is odd} \\
\frac{(r-2)^2}{4(r-1)}n^2, & \text{if } r \text{ is even}
\end{cases}$$

This conjecture seems to be very difficult. The original paper of Sudakov [15] pointed out that some of the ideas there can be used to make a progress on the conjecture for even $r$.

Our focus in this section is to give some estimates on $\tau_B(G)$ for $K_6$-free graphs. We first consider bounds on $\tau_B(G)$ for $K_5$-free graphs, and then use the bounds that we obtain to prove bounds on $\tau_B(G)$ for $K_6$-free graphs. The key ideas that we use come from [15] and [11]. We start with the following well-known fact.

Lemma 6 (see, e.g., Lemma 2.1 of [1]) Let $G$ be a (at most) $2m$-partite graph with $e$ edges. Then $\tau_B(G) \leq (m-1)e/(2m-1)$.

We also need the following theorem from [11], which is a sharpening of Turán’s theorem. It helps us to deal with the case that the graph is dense:

Theorem 5 (11) Every $n$-vertex $K_{p+1}$-free graph $G$ with $e(T_{n,p}) - k$ edges contains a (at most) $p$-partite subgraph with at least $e(G) - k$ edges, where $T_{n,p}$ is the complete $p$-partite graph of order $n$ having the maximum number of edges.

Corollary 1 Let $G$ be a graph on $n$ vertices with $e$ edges.

(a) If $G$ is $K_5$-free, then $\tau_B(G) \leq n^2/4 - e/3$;
(b) If $G$ is $K_6$-free, then $\tau_B(G) \leq 6n^2/25 - e/5$.

Proof. Suppose $G$ is $K_5$-free. Let $H$ be a 4-partite subgraph of $G$ having the maximum number of edges. By Theorem 3 $e(H) \geq 2e - 3n^2/8$. By Lemma 6, $H$ can be made bipartite by deleting at most $e(H)/3$ edges. Thus,

$$\tau_B(G) \leq e - e(H) + \frac{e(H)}{3} = e - \frac{2e(H)}{3} \leq e - \frac{2}{3} \left(2e - \frac{3n^2}{8}\right) = \frac{n^2}{4} - \frac{e}{3}.$$
This proves (a).

Suppose $G$ is $K_6$-free. Let $H$ be a 5-partite subgraph of $G$ having the maximum number of edges. By Theorem 5, $e(H) \geq 2e - 2n^2/5$. By Lemma 6, $H$ can be made bipartite by deleting at most $2e(H)/5$ edges. Thus,

$$
\tau_B(G) \leq e - e(H) + \frac{2e(H)}{5} = e - \frac{3e(H)}{5} \leq e - \frac{3}{5} \left( 2e - \frac{2n^2}{5} \right) = \frac{6n^2}{25} - \frac{e}{5}.
$$

This proves (b). □

**Lemma 7** (see Lemma 2.3 of [15]) Let $G$ be a graph on $n$ vertices with $e$ edges and $t$ triangles. Then $\tau_B(G) \leq e + \left( 6t - \sum_v d^2(v) \right) / n$.

Our next step is to apply some of the ideas and techniques from [15] to prove a bound on $\tau_B(G)$ for $K_5$-free graphs.

**Lemma 8** Let $G$ be a $K_5$-free graph on $n$ vertices with $e$ edges and $t$ triangles. Then $\tau_B(G) \leq e/2 + \left( 2 \sum_v d^2(v) - 27t \right) / (18n)$.

**Proof.** Let $v$ be a vertex of $G$ and let $e_v$ denote the number of edges spanned by the neighborhood $N(v)$. The induced subgraph $G[N(v)]$ is $K_4$-free, since $G$ is $K_5$-free. By Theorem 4, $G[N(v)]$ can be made bipartite by deleting at most $d^2(v)/9$ edges. Let $A$ and $B$ be the resulting partite sets of $G[N(v)]$. We obtain a bipartite subgraph of $G$ by placing the vertices in $G - N(v)$ into the partite sets $A$ and $B$ randomly and independently with probability $1/2$, and deleting all edges within the partite sets. For each edge in $G - G[N(v)]$, it is deleted with probability $1/2$. By linearity of expectation, $\tau_B(G) \leq (e - e_v)/2 + d^2(v)/9$. By averaging over all vertices $v$, we have

$$
\tau_B(G) \leq \frac{e}{2} + \frac{1}{9n} \sum_v d^2(v) - \frac{1}{2n} \sum_v e_v = \frac{e}{2} + \frac{1}{18n} \left( 2 \sum_v d^2(v) - 27t \right),
$$

where we have used the fact that $\sum_v e_v = 3t$. □

Now we can bound $\tau_B(G)$ for $K_5$-free graphs in terms of $n(G)$ only.
Theorem 6 For every $n$-vertex $K_5$-free graph $G$, $\tau_B(G) \leq 29n^2/200$.

Proof. By Lemma 7 and Lemma 8 we have $\tau_B(G) \leq e + \left(6t - \sum_v d^2(v)\right)/n$ and $\tau_B(G) \leq e/2 + (2\sum_v d^2(v) - 27t)/(18n)$. Multiplying the first inequality by $1/5$ and the second inequality by $4/5$, and adding them together, we have 

$$\tau_B(G) \leq \frac{3e}{5} - \frac{1}{9n} \sum_v d^2(v) \leq \frac{3e}{5} - \frac{1}{9n^2} \left(\sum_v d(v)\right)^2 \leq \frac{3e}{5} - \frac{4e^2}{9n^2},$$

where we have used the Cauchy-Schwartz inequality and the fact that $\sum_v d_v = 2e$.

First consider the case when $e \leq 63n^2/200$. Note that the function $g(x) = 3x/5 - 4x^2/9$ is increasing in the interval $x \leq 63/200$. So we have 

$$\tau_B(G) \leq g(e/n^2)n^2 \leq g(63/200)n^2 = \frac{1449n^2}{10000} < \frac{29n^2}{200}.$$

Next consider the case when $e > 63n^2/200$. By Corollary 1 (a) we have 

$$\tau_B(G) \leq \frac{n^2}{4} - \frac{e}{3} < \frac{n^2}{4} - \frac{21n^2}{200} = \frac{29n^2}{200}.$$

$\square$

Remark. Since a $K_5$-free graph has at most $3n^2/8$ edges, it is enough to delete at most $3n^2/16$ edges to make it bipartite. Although the bound in Theorem 6 is better than that, it probably can be improved substantially. Indeed Conjecture 3 says that it suffices to delete $n^2/8$ edges to make a $K_5$-free graph bipartite. It seems that some new ideas or tools are needed to improve the estimate above.

Next we use the bounds we obtained to prove bounds on $\tau_B(G)$ for $K_6$-free graphs. The approach is nearly identical to that used for $K_5$-free graphs.

Lemma 9 Let $G$ be a $K_6$-free graph on $n$ vertices with $e$ edges and $t$ triangles. Then $\tau_B(G) \leq e/2 + (29\sum_v d^2(v) - 300t)/(200n)$. 

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Proof. Let \( v \) be a vertex of \( G \) and let \( e_v \) denote the number of edges spanned by the neighborhood \( N(v) \). The induced subgraph \( G[N(v)] \) is \( K_5 \)-free, since \( G \) is \( K_6 \)-free. By Theorem 6, \( G[N(v)] \) can be made bipartite by deleting at most \( 29d^2(v)/200 \) edges. Thus \( \tau_B(G) \leq (e - e_v)/2 + 29d^2(v)/200 \). By averaging over all vertices \( v \), we have

\[
\tau_B(G) \leq \frac{e}{2} + \frac{29}{200n} \sum_v d^2(v) - \frac{1}{2n} \sum_v e_v = \frac{e}{2} + \frac{1}{200n} \left( 29 \sum_v d^2(v) - 300t \right),
\]

where we have used the fact \( \sum_v e_v = 3t \).

\[\square\]

**Theorem 7** For every \( n \)-vertex \( K_6 \)-free graph \( G \), \( \tau_B(G) \leq 17n^2/100 \).

**Proof.** By Lemma 7 and Lemma 9 we have

\[
\tau_B(G) \leq e + \left( 6t - \sum_v d^2(v) \right)/n \text{ and } \tau_B(G) \leq e/2 + (29 \sum_v d^2(v) - 300t)/(200n).
\]

Multiplying the first inequality by 1/5 and the second inequality by 4/5, and adding them together, we have

\[
\tau_B(G) \leq \frac{3e}{5} - \frac{21}{250n} \sum_v d^2(v)
\]

\[
\leq \frac{3e}{5} - \frac{21}{250n^2} \left( \sum_v d(v) \right)^2
\]

\[
\leq \frac{3e}{5} - \frac{42e^2}{125n^2},
\]

where we have used the Cauchy-Schwartz inequality and the fact that \( \sum_v d_v = 2e \).

First consider the case when \( e \leq 35n^2/100 \). Note that the function \( g(x) = 3x/5 - 42x^2/125 \) is increasing in the interval \( x \leq 35/100 \). So we have

\[
\tau_B(G) \leq g(e/n^2)n^2 \leq g(35/100)n^2 = \frac{4221n^2}{25000} < \frac{17n^2}{100}.
\]

Next consider the case when \( e > 35n^2/100 \). By Corollary 1(b) we have

\[
\tau_B(G) \leq \frac{6n^2}{25} - \frac{e}{5} < \frac{6n^2}{25} - \frac{7n^2}{100} = \frac{17n^2}{100}.
\]

\[\square\]
Remark. The bound above is also probably not tight. Conjecture 3 says that it enough to delete at most $16n^2/100$ edges to make a $K_6$-free graph bipartite. Nevertheless, it still suffices for our purpose.

To prove our main result, we also need bounds on $\tau_B(G)$ for $K_6$-free graphs in terms of $n$, $e$, and $b(G)$. Let $B$ be the vertex set of a largest bipartite induced subgraph of $G$. By a similar argument to that used in Lemma 5, we have that $\tau_B(G) \leq b(G)\frac{(n - b(G))}{2} + 17\frac{n - b(G)^2}{100}$. However this is a very rough estimate. Indeed, if $|[B, \overline{B}]| = b(G)(n - b(G))$, then since $G$ is $K_6$-free, $G[B]$ cannot have many edges and so it could be made bipartite by deleting less than $17(n - b(G))^2/100$ edges. To refine our argument, we need the following lemma.

**Lemma 10** Let $G$ be a $K_6$-free graph on $n$ vertices, and let $S$ be a vertex set that induces a $K_5$-free subgraph of $G$. If $|S| \geq 49n/50$, then $\tau_B(G) \leq 3n^2/20$.

**Proof.** First note that $\tau_B(G) \leq \tau_B(G[S]) + \tau_B(G[\overline{S}]) + \frac{1}{2} |[S, \overline{S}]|$, which follows by considering the two different ways to join the partite sets of a largest bipartite subgraph in $G[S]$ with those of one in $G[\overline{S}]$. Let $s = |S|$. Since $G[S]$ is $K_5$-free, by Theorem 6 we have $\tau_B(G[S]) \leq 29s^2/200$. Since $G[\overline{S}]$ is $K_6$-free, by Theorem 7 we have $\tau_B(G[\overline{S}]) \leq 17(n - s)^2/100$. Thus,

$$
\tau_B(G) \leq \frac{29s^2}{200} \frac{1 + 17(n-s)^2}{100} \frac{2}{2} + s(n-s)
= \frac{1}{200} (-37s^2 + 32ns + 34n^2).
$$

The function $g(s) = (-37s^2 + 32ns + 34n^2)/200$ is decreasing in the interval $s \geq 49n/50$. Thus, if $s \geq 49n/50$, then $\tau_B(G) \leq g(49n/50) = 74563n^2/500000 < 3n^2/20$. □

We finish this section by the following corollary.

**Corollary 2** Let $G$ be a $K_6$-free graph on $n$ vertices with $e$ edges. Then

(a) $\tau_B(G) \leq \max \left( \frac{-7b(G)^2 + 4nb(G) + 3n^2}{20}, \frac{-32b(G)^2 + 15nb(G) + 17n^2}{100} \right)$. 

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(b) \[
\tau_B(G) \leq \frac{e}{3} + \frac{17(n - b(G))^2}{150}.
\]

**Proof.** Let $B$ be the vertex set of a largest bipartite induced subgraph of $G$.

We first prove (a). Note that by considering the two different ways to join the partite sets of a largest bipartite subgraph in $G[B]$ with the partite sets of $G[B]$, we have \[
\tau_B(G) \leq \tau_B(G[B]) + \frac{1}{2} |B, \overline{B}|. \]
There are two possible cases:

If there is a vertex $v \in B$ that has at least \(49(n - b(G))/50\) neighbors in $G[B]$, then since $G$ is $K_6$-free, those neighbors of $v$ in $G[B]$ must induce a $K_5$-free subgraph of $G[B]$. By Lemma 10, $G[B]$ can be made bipartite by deleting at most \(3(n - b(G))^2/20\) edges. Thus, we have

\[
\tau_B(G) \leq \frac{3(n - b(G))^2}{20} + \frac{b(G)(n - b(G))}{2} = \frac{-7b(G)^2 + 4nb(G) + 3n^2}{20}.
\]

If every vertex $v \in B$ has at most \(49(n - b(G))/50\) neighbors in $G[B]$, then \(|B, \overline{B}| < 49b(G)(n - b(G))/50\). Since $G[B]$ is $K_6$-free, by Theorem 7 we have \(\tau_B(G[B]) \leq 17(n - b(G))^2/100\). It follows that

\[
\tau_B(G) \leq \frac{17(n - b(G))^2}{100} + \frac{49b(G)(n - b(G))}{100} = \frac{-32b(G)^2 + 15nb(G) + 17n^2}{100}.
\]

This proves (a).

We next prove (b). Note that we can make $G$ 4-partite by deleting $\tau_B(G[B])$ edges in $G[B]$. By Lemma 10, we can make the resulting 4-partite graph bipartite by deleting at most \((e - \tau_B(G[B]))/3\) edges. It follows that

\[
\tau_B(G) \leq \tau_B(G[B]) + \frac{e - \tau_B(G[B])}{3} \leq \frac{e}{3} + \frac{2\tau_B(G[B])}{3} \leq \frac{e}{3} + \frac{17(n - b(G))^2}{150}.
\]

This proves (b). $\square$
5 Main Result

In this section we prove $f_B(G) \leq 4403n(G)^2/15000$. We first show that the conclusion holds for $K_6$-free graphs, and then use Lemma 2 to prove that it holds for all graphs.

We need the following lemma from [14].

Lemma 11 ([14]) For every graph $G$ on $n$ vertices with $e$ edges, $\alpha_1(G) \leq n^2/2 - e$.

**Theorem 8** For every $n$-vertex $K_6$-free graph $G$, $f_B(G) \leq 4403n^2/15000$.

**Proof.** There are three possible cases:

**Case 1:** $b(G) \leq 49n/100$. By Lemma 3 and Theorem 7, we have

$$f_B(G) = \alpha_1(G) + \tau_B(G) \leq \frac{nb(G)}{4} + \frac{17n^2}{100} \leq \frac{49n^2}{400} + \frac{17n^2}{100} = \frac{117n^2}{400} < \frac{4403n^2}{15000}.$$  

**Case 2:** $49n/100 < b(G) \leq 7n/10$.

By Lemma 3 and Corollary 2 ($b$), we have

$$f_B(G) = \alpha_1(G) + \tau_B(G) \leq \frac{nb(G)}{4} + \frac{17(n - b(G))^2}{150} + \frac{e}{3}. \tag{1}$$

By Lemma 11 and Corollary 2 ($b$), we have

$$f_B(G) = \alpha_1(G) + \tau_B(G) \leq \frac{n^2}{2} - e + \frac{17(n - b(G))^2}{150} + \frac{e}{3}$$

$$= \frac{n^2}{2} + \frac{17(n - b(G))^2}{150} - \frac{2e}{3}. \tag{2}$$

Multiplying inequality (1) by $2/3$ and inequality (2) by $1/3$, and adding them together, we have

$$f_B(G) \leq \frac{nb(G)}{6} + \frac{n^2}{6} + \frac{17(n - b(G))^2}{150} = \frac{1}{150} \left(17b(G)^2 - 9nb(G) + 42n^2\right).$$
Since the function \( g(x) = \frac{(17x^2 - 9nx + 42n^2)}{150} \) is increasing in the interval \( 49n/100 < x \leq 7n/10 \), it follows that \( f_B(G) \leq g\left(\frac{7n}{10}\right) = \frac{4403n^2}{15000} \).

**Case 3:** \( b(G) > 7n/10 \). It is easy to verify that \( \frac{(-7b(G)^2 + 4nb(G) + 3n^2)}{20} \geq \frac{(-32b(G)^2 + 15nb(G) + 17n^2)}{100} \) in this case. So by Corollary 2 (a), we have \( \tau_B(G) \leq \frac{(-7b(G)^2 + 4nb(G) + 3n^2)}{20} \). Again, by Lemma 3 we have \( \alpha_1(G) \leq nb(G)/4 \). Thus,

\[
f_B(G) = \alpha_1(G) + \tau_B(G) \\
\leq \frac{nb(G)}{4} + \frac{-7b(G)^2 + 4nb(G) + 3n^2}{20} \\
= \frac{1}{20} \left(-7b(G)^2 + 9nb(G) + 3n^2\right).
\]

The function \( h(x) = \frac{(-7x^2 + 9nx + 3n^2)}{20} \) is decreasing in the interval \( x \geq 7n/10 \). So in this case we have

\[
f_B(G) \leq h\left(\frac{7n}{10}\right) = \frac{587n^2}{2000} < \frac{4403n^2}{15000}.
\]

\(\square\)

**Theorem 9** For every \( n \)-vertex graph \( G \), \( f_B(G) \leq \frac{4403n^2}{15000} \).

**Proof.** It is easy to verify the conclusion for small \( n \). Now assume to the contrary that \( G \) is a minimal counterexample. By Theorem 8, \( \omega(G) \geq 6 \). However, by Lemma 2 we have \( \omega(G) < 1/(4 \times \frac{4403}{15000} - 1) = \frac{3750}{653} < 6 \), a contradiction. \(\square\)

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