RANDOM ORDERINGS OF THE INTEGERS AND CARD SHUFFLING

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Abstract. In this paper we study random orderings of the integers with a certain invariance property. We describe all such orders in a simple way. We define and represent random shuffles of a countable set of labels and then give an interpretation of these orders in terms of a class of generalized riffle shuffles.

1. Introduction

In Jacka and Warren (1999) we defined deterministic shuffles on (a countable set of labels indexed by) \( \mathbb{N} \). In this paper we define random shuffles on \( \mathbb{N} \) and represent their laws in terms of the laws of pairs of random variables with uniform marginals (Theorem 4.2). A natural subclass of random shuffles are the shuffle imbedding shuffles: those shuffles whose restrictions to \( \{1, \ldots, n\} \) induce a random walk on \( S_n \). Partly in order to study such shuffles, and partly because they are of substantial interest in their own right, we introduce and study the class of \( \mathcal{I} \)-invariant orderings: random orderings of \( \mathbb{Z} \) whose laws are invariant under increasing relabellings. Section 3 is devoted to defining and representing \( \mathcal{I} \)-invariant orderings in terms of quasi-uniform measures (Theorem 3.4).

2. Preliminaries

We denote by \( O \) the class of all strict total orderings of \( \mathbb{Z} \). This inherits a natural measurable structure as a subset of \( 2^{\mathbb{Z} \times \mathbb{Z}} \). We will denote a generic element of \( O \) by \( \preceq \), and write \( m \preceq n \) if \( m \) is less than \( n \) under \( \preceq \).

Given any strictly increasing map \( f : \mathbb{Z} \mapsto \mathbb{Z} \) there is a naturally induced map \( \hat{f} : O \mapsto O \) defined by

\[
m \overset{f}{\preceq} n \quad \text{if and only if} \quad f(m) \preceq f(n),
\]

where we are denoting the image of the ordering \( \preceq \) under \( \hat{f} \) by \( \hat{\preceq} \).

Definition 2.1. A probability measure \( \mathbb{P} \) on \( O \) is said to be \( \mathcal{I} \)-invariant if \( \mathbb{P} \circ \hat{f}^{-1} = \mathbb{P} \), for all strictly increasing \( f \).

Our purpose is to give an explicit description of all such invariant random order relations.

Hirth and Ressel (2000) considered a similar invariance property on random orderings where \( f \) ranges over finite permutations. Such random orderings are called

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exchangeable. Their characterization of the laws of these orderings is very reminiscent of our results on $\mathcal{I}$-invariant orderings. This is related to the well-known fact (see Lemma 3.7 below) that exchangeability of an infinite sequence of random variables is equivalent to an apparently weaker condition involving the action of increasing maps.

The following is a fundamental example that illustrates the connection with riffle shuffles. Suppose that $(Z_i)_{i \in \mathbb{Z}}$ is a doubly infinite sequence of independent, identically distributed random variables taking values in $\{0, 1\}$. Define a random order $\prec$ as follows.

\[ m \prec n \quad \text{if and only if} \quad (Z_m = Z_n \quad \text{and} \quad m < n) \quad \text{or} \quad Z_m < Z_n. \]

In effect we split $\mathbb{Z}$ into two equivalence classes, namely $\{n : Z_n = 0\}$ and $\{n : Z_n = 1\}$; we order the former below the latter, and within each class we preserve the natural order. This is an example of an $\mathcal{I}$-invariant ordering.

Perhaps the most celebrated example of a shuffle is the Gilbert-Shannon-Reeds riffle shuffle. As described in Diaconis (1998) this is obtained by the following recipe. Suppose that our cards are labelled $1, 2, \ldots, n$ and we take $n$ independent random variables $U_1, \ldots, U_n$ each uniformly distributed on $[0, 1]$. Order the cards initially so that card $k$ is above card $l$ whenever $U_k > U_l$. Then reorder the cards according to the values of $2U_k \mod 1$. The random permutation that must be applied to reorder the cards is the GSR shuffle. When we look at the inverse permutation we see that the cards are divided into two subpacks according to whether $U_k$ is smaller or greater than one-half, and within each subpack order is preserved. Thus the inverse of the GSR shuffle is described by (the restriction to $\{1, \ldots, n\}$ of) an $\mathcal{I}$-invariant ordering. In Bayer and Diaconis (1992), this description is used to investigate the speed of mixing of the GSR shuffle. Generalisations based on replacing $u \mapsto 2u \mod 1$ with other maps have been studied by other authors, see for example Lalley (1999). We will see that there is a quite general correspondence to be made between $\mathcal{I}$-invariant orderings, families of riffle shuffles and a class of measure-preserving maps on $[0, 1]$.

3. Describing $\mathcal{I}$-invariant orderings

**Definition 3.1.** A probability measure $\mu$ on $[0, 1]$ is called quasi-uniform if it satisfies

\[ \mu\{x \in [0, 1] : \mu[0, x] \leq x \leq \mu[0, x]\} = 1. \]

**Remark 3.2** It is not hard to show that the set of all quasi-uniform measures is closed with respect to the topology of weak convergence of probability measures on $[0, 1]$.

Such a measure is really quite a simple object, and it may be described as follows.

**Lemma 3.3.** Suppose that $F$ is a closed subset of $[0, 1]$, and $\lambda_F$ is the measure with density $1_F$ with respect to Lebesgue measure. Corresponding to each open component $G_i$ of its complement $F^c$ is a point mass $m_i \delta_{x_i}$, of size $m_i$ equal to the length of the interval $G_i$, situated at position $x_i$, which is either the left or right hand end of $G_i$. Then the measure $\mu$ given by

\[ \mu = \lambda_F + \sum_i m_i \delta_{x_i}, \]

is quasi-uniform. Moreover every quasi-uniform $\mu$ can be decomposed in this fashion, and the decomposition is unique (up to the labelling of the intervals).

We omit the proof which is elementary.
Notice that it is possible for two distinct masses in the above decomposition to be placed at the same point.

Suppose $\mu$ is quasi-uniform then the measure $\mu'$, obtained by inverting the distribution function of $\mu$:

$$\mu'[0, y] = \inf\{x : \mu[0, x] \geq y\},$$

is also quasi-uniform. This corresponds to switching each mass $m_i$ in the decomposition of $\mu$ to being at the opposite end of the interval to which it belongs. If $X$ and $Y$ are random variables on the same probability space with the law of $X$ being $\mu$ and the law of $Y$ being $\mu'$ and so that $X$ and $Y$ are equal or take values at either end of a component of $F^c$ then let us say that such $X$ and $Y$ form a conjugate pair. Notice that their joint law is specified completely by the above description. Such a pair may contain a little more information than either variable separately: whenever they are not equal they together determine an interval of $F^c$.

Now for any quasi-uniform $\mu$ we construct an $\mathcal{I}$-invariant ordering whose law we denote by $P^\mu$. Consider an infinite sequence of independent pairs of random variables $(X_n, Y_n)_{n\in\mathbb{Z}}$. For each $n$ the variables $X_n$ and $Y_n$ form a conjugate pair with $X_n$ distributed according to $\mu$. Now, supposing that $m < n$ (with respect to the natural ordering of the integers) take $m \triangleright n$ if and only if one of the following happens:

$$\begin{cases}
X_m < X_n,

Y_m < Y_n,

X_m = X_n > Y_m = Y_n.
\end{cases}$$

(3.1)

It is easy to check that this works and defines an $\mathcal{I}$-invariant ordering. Moreover the strong law of large numbers implies that

$$\begin{cases}
X_n = \lim_{N \to \infty} \frac{1}{N} \sum_{-N}^{N} 1_{(k < n)}

Y_n = \lim_{N \to \infty} \frac{1}{N} \sum_{n}^{N} 1_{(k < n)},
\end{cases}$$

exist almost surely. Notice that, since $\mu$ can be recovered from $\triangleleft$ as the empirical distribution of the sequence $X_n$, $P^{\mu_1} \neq P^{\mu_2}$ if $\mu_1 \neq \mu_2$. The following theorem says that by taking mixtures of orderings of this form we obtain all possible $\mathcal{I}$-invariant orderings.

**Theorem 3.4.** Suppose that $\triangleleft$ is an $\mathcal{I}$-invariant ordering. Then almost surely, the random variables defined by (3.2) exist, and for any $m$ and $n$ the relation $m \triangleleft n$ holds if and only if (3.1) does.

Moreover the sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ is exchangeable and with probability one it admits an empirical distribution $\mu(X)$ which is quasi-uniform.

Conditional on $\mu(X) = \mu$ the law of $\triangleleft$ is $P^\mu$.

In general any ordering $\triangleleft$ belonging to $\mathcal{O}$ projects to an equivalence relation, $\sim$, on $\mathbb{Z}$ defined by

$$n \sim m \iff \text{there are only finitely many } k \text{ between (with respect to } \triangleleft) \text{ } n \text{ and } m.$$ 

If the ordering is $\mathcal{I}$-invariant then it follows from the above theorem that this partition is exchangeable in the sense studied by Kingman (1982).

The proof of Theorem 3.4 hinges on the elementary observation of the next lemma, which begins to explain the role of quasi-uniform measures.
Lemma 3.5. Suppose that $\prec$ is some fixed ordering. Define, for each $n$,
\[
\overline{X}_n = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1_{(k, \prec n)} \\
\underline{X}_n = \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1_{(k, \prec n)}
\]
Suppose that the measures $\nu^{(N)}$, defined by
\[
\nu^{N}[0, x] = \frac{1}{N} \sum_{k=1}^{N} 1_{(X_k \leq x)} \quad x \in [0, 1],
\]
converge weakly to a probability measure $\nu$ as $N$ tends to infinity. Then
\[
\nu[0, \overline{X}_1] \leq \underline{X}_1 \leq \overline{X}_1 \leq \nu[0, \underline{X}_1].
\]
Proof. It is an easy consequence of the transitivity of $\prec$ that:
\[
X_k < X_1 \Rightarrow k \prec 1 \Rightarrow X_k \leq X_1,
\]
for any $k \in \mathbb{Z}$. Thus
\[
\frac{1}{N} \sum_{k=1}^{N} 1_{(x_k < X_1)} \leq \frac{1}{N} \sum_{k=1}^{N} 1_{(k < \prec n)} \leq \frac{1}{N} \sum_{k=1}^{N} 1_{(x_k \leq X_1)}.
\]
But the left-hand side is $\nu^{(N)}[0, \overline{X}_1]$ while the right-hand side is $\nu^{(N)}[0, \underline{X}_1]$ and by virtue of weak convergence:
\[
\nu[0, \overline{X}_1] \leq \liminf \nu^{(N)}[0, \overline{X}_1] \\
\nu[0, \underline{X}_1] \geq \limsup \nu^{(N)}[0, \underline{X}_1].
\]

Lemma 3.6. Suppose that $\prec$ is an $I$-invariant ordering. Then the family of random variables
\[
(1_{(k, \prec 0)}; k > 0)
\]
is exchangeable.
Proof. It suffices to check that for finite collections of positive integers $j_1 \ldots j_m$ and $k_1 \ldots k_n$ the value of
\[
\mathbb{E} \left[ 1_{(0 < j_1)} \ldots 1_{(0 < j_m)} 1_{(k_1, \prec 0)} \ldots 1_{(k_n, \prec 0)} \right]
\]
depends only on $n$ and $m$. Now replace $1_{(k, \prec 0)}$ by $1 - 1_{(0, \prec k)}$, multiply out and apply $I$-invariance to obtain an expression involving terms: $\mathbb{E}[1_{(0 < j_1)}1_{(0 < j_2)} \ldots 1_{(0 < j_k)}]$ for $m \leq k \leq m + n$. This lemma is actually a special case of the next result, for the proof of which we refer the reader to Aldous(1985).

Lemma 3.7. Suppose that a sequence of random variables $(X_k; k \in \mathbb{Z})$ is $I$-invariant, in the sense that for any increasing function $f: \mathbb{Z} \mapsto \mathbb{Z}$
\[
(X_k; k \in \mathbb{Z}) \overset{\text{law}}{=} (X_{f(k)}; k \in \mathbb{Z})
\]
then in fact $(X_k; k \in \mathbb{Z})$ are exchangeable- the sequence admits with probability one an empirical distribution and conditional on it the random variables are independent and identically distributed.
Proof of theorem 3.4. We begin by observing that the variables \((X_k, Y_k)\) exist by virtue of the exchangeability property of Lemma 3.6 and De Finetti’s Theorem. Moreover the law of the sequence of pairs \((X_k, Y_k)\) is \(\mathcal{I}\)-invariant so we may deduce from Lemma 3.7 that it is, in fact, an exchangeable sequence. It follows from Lemma 3.5 that the empirical distributions for both \(X_k\) and \(Y_k\) must be quasi-invariant.

The next step is to show that the variables \((X_k, Y_k)\) determine the ordering \(\prec\). Divide \(\mathbb{Z}/\{0\}\) into three classes.

\[
U_0 = \{k : \text{either } X_k > X_0 \text{ or } Y_k > Y_0\},
\]
\[
E_0 = \{k : X_k = X_0 \text{ and } Y_k = Y_0\},
\]
\[
B_0 = \{k : \text{either } X_k < X_0 \text{ or } Y_k < Y_0\}.
\]

Notice that, since \(j < k\) implies \(X_k \geq X_j\) and \(Y_k \geq Y_j\), we must have any element of \(U_0\) ordered above any element of \(E_0\) which in turn must be ordered above any element of \(B_0\). Because of the exchangeability of \(X\) and \(Y\), the three classes have limiting sizes:

\[
|U_0| = \lim \frac{1}{N} \sum \limits_{k=1}^{N} 1_{(k \in U_0)} = \lim \frac{1}{N} \sum \limits_{k=-N}^{-1} 1_{(k \in U_0)}
\]

and similarly for \(|E_0|\) and \(|B_0|\). The exchangeability of \(X\) and \(Y\) also implies that if the size \(|E_0|\) of \(E_0\) is zero then it is, in fact, empty. Otherwise the empirical distributions of \(X\) and \(Y\) have atoms at the values of \(X_0\) and \(Y_0\). We claim the restriction of \(\prec\) to \(E_0\) either preserves or reverses the natural order: it then follows that in the former case: \(Y_0 = X_0 - |E_0|\), while in the latter case: \(Y_0 = X_0 + |E_0|\). This then establishes that the ordering \(\prec\) is determined by the sequence \((X_k, Y_k)\) according to (3.1).

To prove the claim of the previous paragraph suppose that \(0 < j < k\), and let \(p\) be the probability \(P(0 < k \prec j, \text{ and } j \in E_0)\). Now

\[
\frac{1}{N-j} \sum \limits_{r=j+1}^{N} 1_{(0 \prec r \prec j, \text{ and } j \in E_0)} = \frac{1}{N-j} \sum \limits_{r=j+1}^{N} \left(1_{(r \prec j)} - 1_{(r \prec 0)}\right),
\]

must converge to \(1_{(0 \prec j \text{ and } j \in E_0)}(Y_j - Y_0) = 0\) in \(L^1\) (by bounded convergence), yet its expectation is \(\frac{1}{N-j}\), for all \(N\), equal to \(p\). This, and similar versions show that if \(r \in E_0\) then the only \(s\) between 0 and \(r\) with respect to \(\prec\) are also between 0 and \(r\) in the natural ordering. But now we may replace 0 by \(t\) in this statement, then by noting that if \(t \in E_0\) then \(E_t = E_0\) we deduce that whenever \(r\) and \(t\) both belong to \(E_0\) then \(s\) being between them with respect to \(\prec\) implies \(s\) is between them with respect to the natural order.

To complete the proof of the theorem condition on the joint empirical measure of \((X, Y)\) to reduce to the iid case. The arguments in the previous step show that each \(X_k\) and \(Y_k\) must form a conjugate pair, and that the conditional distribution of \(\prec\) is \(\mathbb{P}^{\mu}\) where \(\mu\) is the empirical measure of \(X\).

Let us close this section by noting another natural invariance property that one might impose on a random ordering: a probability measure \(\mathbb{P}\) on \(O\) is said to be \(\mathcal{T}\)-invariant if \(\mathbb{P} \circ \hat{f}^{-1} = \mathbb{P}\), for all \(f\) of the form \(f(n) = n + a\) for some \(a \in \mathbb{Z}\).

Problem 3.8. Obtain an explicit description of all \(\mathcal{T}\)-invariant orderings.

\(\mathcal{T}\)-invariant orders have a much richer structure than \(\mathcal{I}\)-invariant ones, as the following example illustrates. Let \(I_n\) be a stationary sequence of \(\{0,1\}\) random variables and construct \(\prec\) as follows.
• If $I_n = I_m$ then $\prec$ agrees with the natural order.
• the upper class $\{I_n = 1\}$ has slipped one place relative to $\{I_n = 0\}$. Thus for example if $n < m$ belong to class 0 and class 1 respectively then $n \prec m$ unless there is no $n < k < m$ with $I_k = 1$- if this happens then $m \prec n$.

Notice that making the random variables $I_n$ independent does not make $\prec \mathcal{I}$-invariant. Something more interesting is happening!

4. Shuffling an infinite set of cards

The first half of this section is based on the (more leisurely) account contained in Jacka and Warren (1999) of what it might mean to shuffle an infinite set of cards. The state of an infinite pack of cards will be represented by an ordering of the natural numbers. The second half of the section considers classes of Markov processes (indexed by discrete time) taking values in the space of such orderings and shows how one such class is naturally associated with the class of $\mathcal{I}$-invariant orderings we have studied in the previous section.

Recall the standard model for shuffling cards: $n$ cards carrying labels 1 through to $n$ each have a distinct position 1 through to $n$ in the pack. We associate the state of the pack with a permutation $\rho$ belonging to the permutation group on $n$ objects $S_n$. If $\rho(k) = m$ then we say that the card carrying label $k$ is in position $m$ in the pack. A completely randomized pack simply means choosing $\rho$ according to the uniform measure on $S_n$. A shuffle $S$ is a possibly random permutation (belonging to $S_n!$) of the positions in the pack. Thus $S(m) = m'$ means that the card that was in position $m$ is moved to position $m'$. Consequently the state of the pack is changed from $\rho$ to $S\rho$. In this way $S$ induces a map $\hat{S}: S_n \mapsto S_n$ defined by $\hat{S}(\rho) = S\rho$. Such an $\hat{S}$ ignores the labelling of the pack. If $r$ (also belonging to $S_n$) is used to change the labels so that the card that now carries the label $k$ is the card that previously carried the label $r(k)$ and we denote by $\hat{r}$ the induced map $\hat{r}(\rho) = \rho r$ then we obtain the commutation relation

$$\hat{S} \circ \hat{r} = \hat{r} \circ \hat{S},$$

for all $r \in S_n$. Moreover any map $\hat{S}$ that commutes with all relabellings is induced by some $S \in S_n$.

We have rather laboured the point in the previous paragraph so as to motivate our model for shuffling an infinite pack of cards. We have seen that for a finite pack the permutation group plays three distinct roles - it describes the state of the pack, it gives rise to shuffles, and it can be used to relabel the pack. We proceed to the description of three different objects that play these roles in the infinite framework. First note that for a finite pack we may also specify the state of the pack by giving an ordering $\triangleleft^{(n)}$ of $\{1, \ldots, n\}$ related to our previous description by means of a permutation $\rho$ via

$$k \triangleleft^{(n)} k' \quad \text{iff} \quad \rho(k) < \rho(k').$$

With this approach we note that we may restrict the ordering $\triangleleft^{(n)}$ to the first $n - 1$ cards to obtain an ordering $\triangleleft^{(n-1)}$. Moreover, if $\triangleleft^{(n)}$ is chosen uniformly then $\triangleleft^{(n-1)}$ is uniformly distributed also. Because of this consistency there is a unique measure $\lambda$ on the space of total orderings of $\mathbb{N}$ so that the restriction $\triangleleft^{(n)}$ of the ordering to $\{1, \ldots, n\}$ is uniform. It is well known how to construct a random ordering distributed according to $\lambda$. Let $U_1, U_2, \ldots$ be an infinite sequence of independent random variables uniformly distributed on $[0, 1]$. Then put:

$$k \triangleleft k' \quad \text{iff} \quad U_k < U_{k'}.$$
It is immediate that the restriction $\preceq^{(n)}$ is uniform whence, by the uniqueness property of the projective limit, $\prec$ has $\lambda$ as its distribution. Notice that, for each $k$,

$$U_k = \lim_{n \to \infty} \frac{1}{n} \sum_{i \leq n, i \neq k} 1_{(i\prec k)} \quad a.s.,$$

and thus we may regard $U_k$ as being the (relative) position of the card carrying label $k$. A slightly different way of thinking about this: equations (4.3) and (4.4) set up a measure isomorphism between the space of total orderings endowed with $\lambda$ and the space $[0,1]^\infty$ endowed with the infinite product of uniform measure on $[0,1]$. It’s often much easier to think about things in the $[0,1]^\infty$ world, as we shall see.

Suppose that $r$ is an arbitrary bijection of $\mathbb{N}$ onto itself and define the induced relabelling $\hat{r}$ via

$$k \prec k' \iff r(k) \prec r(k').$$

It is easy to see that $\hat{r}$ preserves the uniform measure $\lambda$ and in fact this invariance property characterises $\lambda$. It is also true that if we use (4.4) to define a random variable $U_k$ on the space of orderings then

$$U_k \circ \hat{r} = U_{r(k)}$$

almost surely under $\lambda$. This means that the action of a relabelling on the space $[0,1]^\infty$ is just to permute the co-ordinates.

To see equation (4.6), just note that given $r$, for $\lambda$ almost all $\prec$, we may define a new order $\prec'$ by $k \prec' k'$ iff $U_{r(k)} < U_{r(k')}$ which agrees with $\hat{r}$. Some attention should be paid to the null sets here. Equation (4.6) holds except for a null set that depends on $r$. In fact any two orderings that are dense and open (so between any two elements there is a third and there are no minimal or maximal elements) have the same order-type (see Fraenkel (1976)) and so there is some relabelling carrying one to the other. Thus for any $\prec$ there is a choice of $r$ so that (4.6) fails to hold at $\prec$.

Suppose that $\hat{S}$ is a map from $\tilde{O}$, the space of orderings of $\mathbb{N}$, into itself. When is it appropriate to call $\hat{S}$ a shuffle? When the commutation property (4.1) holds $\lambda$ almost surely for each relabelling of the infinite pack as defined in the previous paragraph. In this case there exists a unique function, $S : [0,1] \mapsto [0,1]$, which preserves Lebesgue measure, such that, for each $k$,

$$U_k \circ \hat{S} = S \circ U_k$$

$\lambda$ almost surely. Moreover, each such $S$ corresponds to some $\hat{S}$.

We define a random shuffle as a suitable generalisation of such functions:

**Definition 4.1.** A random shuffle is described by a family of transition kernels $\kappa$ on the space of orderings of $\mathbb{N}$, satisfying the following generalisation of the commutation relation for any $r$: whenever $A$ is a measurable subset of the space of total orderings, and $\hat{r}$ a relabelling,

$$\kappa(\hat{r}(\prec), \hat{r}(A)) = \kappa(\prec, A) \quad \text{for } \lambda \text{ almost all } \prec.$$

Here we have written $\hat{r}(\prec)$ for the ordering $\prec$. As with deterministic shuffles, we can express $\kappa$ using the card positions.

**Theorem 4.2.** Suppose that $\nu$ is a probability measure on $[0,1]^2$ having both marginals uniform on $[0,1]$. Take a sequence of independent pairs of random variables $((U_1, V_1), \ldots, (U_k, V_k), \ldots)$, each pair distributed according to $\nu$. This then determines,
by virtue of (4.3), the joint law of a pair of orderings \((\prec, \prec')\). Take \(\hat{\nu}(\prec, \cdot)\) to be a regular conditional probability for \(\prec'\) given \(\prec\). Then \(\kappa = \hat{\nu}\) satisfies the commutation relation (4.8). Moreover, any \(\kappa\) satisfying the relation (4.8) is a mixture of kernels constructed in this manner.

**Proof.** Suppose that \((U_k, V_k)\) for \(k \geq 1\) form a sequence of independent pairs of random variables, each pair having the distribution \(\nu\) on \([0, 1]^2\). Then the sequence of independent uniform variables \((U_k; k \geq 1)\) gives rise to, with probability one, an ordering \(\prec\) distributed according to \(\lambda\), and similarly \((V_k; k \geq 1)\) gives rise to an ordering \(\prec'\). Fix a relabelling \(r\). Since the ordering \(\hat{r}(\prec)\) corresponds to the sequence of random variables \(\tilde{U}_k = U_{r(k)}\) and similarly the ordering \(\hat{r}(\prec')\) corresponds to the sequence of random variables \(\tilde{V}_k = V_{r(k)}\) we see that:

\[
(\hat{r}(\prec), \hat{r}(\prec')) \overset{\text{law}}{=} (\prec, \prec').
\]

From this it follows that \(\hat{\nu}\), defined as a regular conditional probability for \(\prec'\) given \(\prec\), satisfies (4.8).

To see the last claim of the theorem, suppose that \(\kappa\) satisfies (4.8), and consider a pair of orderings \((\prec, \prec')\) determined as follows. Let \(\prec\) be distributed according to \(\lambda\), and the let the conditional distribution of \(\prec'\) given \(\prec\) be \(\kappa(\prec, \cdot)\). It follows from the invariance of \(\lambda\) under relabellings and (4.8) that, for any relabelling \(r\),

\[
(\hat{r}(\prec), \hat{r}(\prec')) \overset{\text{law}}{=} (\prec, \prec').
\]

Now, as we remarked above, \(\lambda\) is characterized by its invariance under relabellings and so the law of \(\prec'\) must also be \(\lambda\). Let the card positions corresponding to \(\prec\) be \((U_1, \ldots, U_k, \ldots)\), and those corresponding to \(\prec'\) be \((V_1, \ldots, V_k, \ldots)\). Then the sequence of pairs \(((U_1, V_1), \ldots (U_k, V_k), \ldots)\) is exchangeable. So the sequence admits a random empirical measure \(\Xi\) on \([0, 1]^2\). Let the law of \(\Xi\) be

\[
\int \alpha(d\nu)\nu,
\]

the integral being with respect to a probability measure \(\alpha\) on the space of probability measures on \([0, 1]^2\). Applying the strong law of large numbers to each of the sequences \(U_k\) and \(V_k\), we deduce that the marginals of \(\Xi\) are, with probability one, uniform. Thus \(\alpha\) must be supported on the set of measures \(\nu\) having uniform marginals. Finally observe that, since, conditional on \(\Xi = \nu\), the pairs \((U_k, V_k)\) are independent and distributed as \(\nu\),

\[
\kappa(\prec, \cdot) = \int \alpha(d\nu)\hat{\nu}(\prec, \cdot),
\]

for \(\lambda\) almost all \(\prec\). By choosing an appropriate version for \(\hat{\nu}\) we can obtain equality for all \(\prec\).

Notice how this relates to the representation, (4.7), of deterministic shuffles. Corresponding to a measure-preserving \(S : [0, 1] \mapsto [0, 1]\) is the \(\nu\) with

\[
\nu(A \times B) = \int_A 1_B(S(x))dx, \quad \text{for any Borel subsets } A \text{ and } B \text{ of } [0, 1]. 
\]

The above discussion seems to be the end of the story but let us reflect. Shuffling an infinite set of cards was really a two step procedure. First we built the pack as the limit of a consistent family of finite packs, then we discussed appropriate transformations of the limiting object as shuffles. But we could do this differently. In what follows
we consider transformations on the finite packs first, look for some consistency of the resulting processes, and then we pass to the limit.

Suppose that \( (\rho_h^{(n)}; h \geq 0) \) is a random walk on \( S_n \), starting from a uniformly chosen \( \rho_0^{(n)} \). Think of this as describing the state of a pack of \( n \) cards at times \( h = 0, 1, 2, \ldots \). Now let \( m < n \) and imagine that only the cards carrying the labels \( 1, 2, \ldots, m \) are observed. Recall that, via (4.2), \( \rho_h^{(n)} \) determines an ordering \( \langle h \rangle \) and let \( \langle h \rangle_m \) be the restriction of this ordering to \( 1, 2, \ldots, m \). Then, using (4.2) again, we associate with \( \langle h \rangle_m \) a permutation \( \rho_h^{(m)} \) belonging to \( S_m \). Clearly, for each \( h \) we have that \( \rho_h^{(m)} \) is uniformly distributed but it is easy to construct examples so that the process \( (\rho_h^{(m)}; h \geq 0) \) is not a random walk. What are the weakest conditions that must be placed on the jump distribution of \( \rho^{(n)} \) to ensure that it is a random walk? We do not know. But here are two special cases when it works.

Case 1: \( \langle h \rangle_{h+1}^{(m)} \) is conditionally independent of \( \langle h \rangle_h^{(m)} \) given \( \langle h \rangle_h \).

Case 2: \( \langle h \rangle_{h-1}^{(m)} \) is conditionally independent of \( \langle h \rangle_h^{(m)} \) given \( \langle h \rangle_h \).

It is immediate that if case 1 holds then \( (\rho_h^{(m)}; h \geq 0) \) is a random walk, and, of course, case 2 is just case 1 run backwards!

Now what we really want to do is construct an infinite family of random walks \( ((\rho_h^{(n)}; h \geq 0); n \geq 1) \) so that the associated orderings \( \langle h \rangle^{(n)} \) are consistent, that is, \( \langle h \rangle_h^{(m)} \) is the restriction of \( \langle h \rangle_h^{(n)} \) whenever \( m < n \). Such a consistent family of processes determines a limiting process \( (\langle h \rangle; h \geq 0) \) taking values in the space of orderings of \( N \).

Such a process is Markovian with a transition kernel \( \kappa \) which satisfies (4.8).

**Definition 4.3.** We shall call the kernel of such a limiting process a shuffle imbedding shuffle, or SIS.

When we express \( \kappa \) as a mixture:

\[
\kappa(\langle, \cdot \rangle) = \int \alpha(d\nu) \hat{\nu}(\langle, \cdot \rangle),
\]

the measure \( \alpha \) is supported on \( \nu \) having a special form. We investigate this special form next with the help of \( \mathcal{I} \)-invariant orderings.

**Definition 4.4.** Let us say that \( (\langle h \rangle; h \geq 0) \) is of type 1 if case 1 holds for each pair \( m < n \), and let us say it is of type 2 if case 2 holds for each pair \( m < n \). We always assume that \( \langle h \rangle \) is determined according to \( \lambda \). We call the corresponding kernels type 1 and type 2 shuffles.

**Remark 4.5** It is obvious from the preceding discussion that if \( \kappa \) is a type 1 shuffle then \( \kappa(\langle, \{ \langle \rangle : \langle \rangle^{(n)} = \sigma \}\rangle) \) is \( \lambda \) almost surely constant over \( \{ \langle \rangle : \langle \rangle^{(n)} = \rho \} \). We denote the common value by \( \kappa_n(\rho, \sigma) \). It is clear from the preceding comments that \( \kappa \) is determined by the \( (\kappa_n)_{n \geq 1} \). We now show that:

**Proposition 4.6.** Each type 1 shuffle induces (the law of) an \( \mathcal{I} \)-invariant ordering and vice versa.

**Proof.** Suppose that the type 1 shuffle is \( \kappa \). Then we obtain the law of an \( \mathcal{I} \)-invariant ordering \( \langle \rangle \) of \( Z \), which we denote \( P(\kappa) \), as follows. Given integers \( k_1 < k_2 < \ldots < k_n \) and a permutation \( \rho \in S_n \) then the probability that \( \langle \rangle \) orders \( k_i \) so that

\[
k_i \prec k_j \quad \text{iff} \quad \rho(i) < \rho(j)
\]
is the probability that $\rho_{h+1}(\rho_h) - 1 = \rho$. Notice that $\triangleleft$ is defined in such a way as to be automatically $\mathcal{I}$-invariant. In checking that this definition of $\triangleleft$ is meaningful we need the conditional independence asserted by case 1.

Conversely, according to Theorem 3.3, the law, $\mathbb{P}$, of $\triangleleft$ is a mixture:

\begin{equation}
\int \theta(d\mu)\mathbb{P}^\mu,
\end{equation}

for some probability measure $\theta$ on the space of probability measures on $[0,1]$. In fact, $\theta$ is the law of the random empirical measure, $\mu(X)$, of Theorem 3.4 and is supported on the set of quasi-uniform measures. We induce a kernel $K(\mathbb{P})$, in terms of $\theta$, as follows. Let $\mu$ be any quasi-uniform measure and let $X$ and $Y$ be a conjugate pair of random variables with the law of $X$ being $\mu$. Let $U$ be an independent random variable uniformly distributed on $[0,1]$, and let $\nu_\mu$ be the law of the pair $(U,UX + (1-U)Y)$. It is easy to check that the measure $\nu_\mu$ has uniform marginals and so, by Theorem 4.2, there is a corresponding kernel, satisfying (4.8), which we denote by $\hat{\nu}_\mu$. Now define $K(\mathbb{P})$ by

\begin{equation}
\int \theta(d\mu)\hat{\nu}_\mu.
\end{equation}

It remains to check that $K(\mathbb{P})$ is a type 1 shuffle. Recall that $\hat{\nu}_\mu$ is the regular conditional probability law for $\triangleleft'$ given $\triangleleft$, where $\triangleleft$ and $\triangleleft'$ are the orders of $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ respectively and the $(U_n, V_n)_{n \geq 1}$ are iid with common law $\nu_\mu$. The result now follows from the fact (which we leave to the reader to check) that the order of $V_1, \ldots, V_n$ is independent of $(U_k)_{k \geq 1}$, conditional on the order of $U_1, \ldots, U_n$. \hfill $\square$

**Proposition 4.7.** There is a one-to-one correspondence between the laws of $\mathcal{I}$-invariant orderings and the type 1 shuffles. Under this bijection, the law of the ordering

\begin{equation}
\int \theta(d\mu)\mathbb{P}^\mu,
\end{equation}

corresponds to the kernel

\begin{equation}
\int \theta(d\mu)\hat{\nu}_\mu.
\end{equation}

**Proof.** From the proof of Proposition 4.6, all that remains is to establish that the maps $P$ and $K$ satisfy

\begin{equation}
P \circ K = id
\end{equation}

and

\begin{equation}K \circ P = id.
\end{equation}

To establish (4.13), given a type 1 shuffle, $\kappa$, set $\hat{\kappa} = K \circ P(\kappa)$. Recall from Remark 4 that $\kappa$ is characterised by $(\kappa_n(id, \rho); \rho \in S_n, n \geq 1)$ and observe that

$\kappa_n(id, \rho) = \mathbb{P}(<^{(n)}(\rho) = \rho) = \int \theta(d\mu)\mathbb{P}^\mu(<^{(n)}(\rho) = \rho) = \int \theta(d\mu)\hat{\nu}_\mu(id, \rho) = \hat{\kappa}_n(id, \rho),$

where $\mathbb{P} = P(\kappa)$. The proof of (4.13) is similar. \hfill $\square$

**Remark 4.8** The proof of Proposition 4.6 now makes clear the role of quasi-uniform pairs $\mu, \mu'$ in constructing type 1 shuffles. An extremal type 1 shuffle is realised by taking the appropriate quasi-uniform $\mu$, constructing a corresponding sequence of conjugate iid pairs $(X_n, Y_n)$ and then setting $V_n = U_nX_n + (1-U_n)Y_n$, where $U_n$ and $V_n$ are, respectively, the initial and final positions of card $n$. This definition still makes sense even if there are ties in final card positions. For suppose that $V_n = V_m$; notice
that this can only happen if either the corresponding initial positions are the same
and the corresponding conjugate pairs \((X_n, Y_n)\) and \((X_m, Y_m)\) are equal or if the initial
positions take values in \(\{0, 1\}\) and the conjugate pairs lie on adjacent components of
\(G\). In the latter case we resolve the tie by ordering \(m\) above \(n\) if \((X_m, Y_m)\) belongs
to the higher/rightmost component of \(G\). In the former case, we preserve the initial
ordering between \(m\) and \(n\) if \(Y_n = Y_m < X_m = X_n\) and otherwise reverse it (just
as in 3.1). The corresponding kernel, \(\nu_\mu\), is defined on all \(\triangleright \in \bar{\mathcal{O}}\), and
\(\nu_\mu(\triangleright, \cdot)\) is a probability measure on \(\bar{\mathcal{O}}\) for every \(\triangleright\). Under \(\mathbb{P}_\mu\), the law of the restriction of
\(\triangleright\) to \(N\) is equal to \(\nu_\mu(id, \cdot)\).

For type 2 shuffles the story is similar. An \(\mathcal{I}\)-invariant ordering is determined
as follows. For integers \(k_1 < k_2 < \ldots < k_n\) and a permutation \(\rho \in S_n\) then the
probability that \(\triangleright\) orders \(k_i\) so that
\[
(4.15) \quad k_i \triangleright k_j \quad \text{iff} \quad \rho(i) < \rho(j)
\]
is the probability that \(\rho^{(n)}_h(\rho^{(n)}_{h+1})^{-1} = \rho\). The law of this ordering determines the
transition kernel as in the proof of Proposition 4.6. If \(\mu\) is a quasi-uniform measure,
let \(\nu^\mu\) be the measure on \([0, 1]^2\) defined by
\[
(4.16) \quad \nu^\mu(dx, dy) = \nu_\mu(dy, dx),
\]
and let \(\hat{\nu}^\mu\) be the associated kernel.

**Proposition 4.9.** There is a one-to-one correspondence between the laws of \(\mathcal{I}\)-invariant
orderings and type 2 shuffles. Under this bijection, the law of the ordering
\[
\int \theta(d\mu)\mathbb{P}_\mu,
\]
corresponds to the kernel
\[
\int \theta(d\mu)\hat{\nu}^\mu.
\]

**Proof.** The result follows immediately from Propositions 4.6 and 4.7 by time reversal. \(\square\)

It is this result which generalizes the GSR shuffle. In particular, if the Markov
chain corresponds to a measure \(\theta\) which puts all its mass on a single quasi-uniform
measure \(\mu\), and \(\mu\) is purely atomic, then there exists a function \(S: [0, 1] \mapsto [0, 1]\)
such that (4.9) holds for \(\nu^\mu\), and the chain is in fact deterministic and is obtained by
iterating the shuffle \(\hat{S}\) associated by equation (4.7) with \(S\). If \(G\) corresponding to \(\mu\)
is decomposed as
\[
G = \bigcup_n (l_n, L_n) \cup \bigcup_m (r_m, R_m),
\]
where the atoms of \(\mu\) are situated on \(\{l_n; n \geq 1\} \cup \{R_m; m \geq 1\}\) then, at least for
\(x \in G\),
\[
S(x) = \sum_n \frac{L_n - x}{L_n - l_n} 1_{(l_n, L_n)}(x) + \sum_m \frac{x - r_n}{R_n - r_n} 1_{(r_n, R_n)}(x).
\]
Thus, for example, the GSR shuffle corresponds to the quasi-uniform measure with
atoms of size \(\frac{1}{2}\) at \(\frac{1}{2}\) and 1.

We end with the following:
Problem 4.10. Do there exist Markov processes, other than those constructed above, on the space of orderings of $\mathbb{N}$ such that, for each $n$, the restriction of the ordering to $\{1, \ldots, n\}$ evolves as if induced by a random walk on $S(n)$?

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