REGULARIZING EFFECT OF ABSORPTION TERMS IN SINGULAR AND DEGENERATE ELLIPTIC PROBLEMS

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ABSTRACT

In this paper we study the existence and regularity of solutions to the following singular problem

\[
\begin{align*}
-\text{div}(a(x,u)|\nabla u|^{p-2}\nabla u) + |u|^{s-1}u &= \frac{f}{u^\gamma} \quad &\text{in } \Omega \\
u > 0 &\quad &\text{in } \Omega \\
u &= 0 &\quad &\text{on } \partial \Omega
\end{align*}
\]

(0.1)

proving that the lower order term \( u|u|^{s-1} \) has some regularizing effects on the solutions in the case of an elliptic operator with degenerate coercivity.

Keywords Degenerate coercivity, singular non linearity, regularity, entropy solutions, Sobolev spaces.

1 Introduction

Let us consider the following problem

\[
\begin{align*}
-\text{div}(a(x,u)|\nabla u|^{p-2}\nabla u) + |u|^{s-1}u &= h(u)f \quad &\text{in } \Omega \\
u \geq 0 &\quad &\text{in } \Omega \\
u &= 0 &\quad &\text{on } \partial \Omega
\end{align*}
\]

(1.1)

where \( 1 < p < N, \Omega \) is bounded set in \( \mathbb{R}^N \) and \( a : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a carathéodory function such that for a.e. \( x \in \Omega \) and for every \( s \in \mathbb{R} \), we have

\[
a(x,s) \geq \frac{\alpha}{(1+|s|)^\theta}
\]

(1.2)

and

\[
a(x,s) \leq \beta,
\]

(1.3)

for some real positive constants \( \alpha, \beta \) and \( \theta \). Moreover, \( f \) is a non negative \( L^m(\Omega) \) function, with \( m \geq 1 \) and the term \( h : (0, \infty) \rightarrow (0, \infty) \) is continuous and bounded, such that

\[
\exists c, \gamma > 0 \text{ s.t } h(s) \leq \frac{c}{s^\gamma} \quad \forall s \geq 0,
\]

(1.4)

for some real number \( \gamma \) such that \( 0 \leq \gamma < 1 \). Singular problems of this type have been largely studied in the past also for their connection with the theory of non-Newtonian fluids, boundary layer phenomena for viscous fluids and chemical heterogeneous (see for instance \([12], [14]\)).

Let us briefly recall the mathematical framework concerning problem \( (1.1) \) we start with the case \( a(x,u) := a(x), \theta = 0 \) and \( f \) lies just in \( L^1(\Omega) \) has been studied in \([3]\).
Problem (1.1) in the non-singular case \( h(u) = 1 \), (problem (1.5)) the author studied the existence and regularity of weak solution to the elliptic problem with degenerate coercivity (see [17]).

\[
\begin{aligned}
- \text{div} (a(x, u)|\nabla u|^{p-2} \nabla u) + |u|^{\gamma-1} u &= f, \text{ in } \Omega \\
u &= 0 \text{ in } \partial \Omega,
\end{aligned}
\]  

(1.5)

in the case where \( f \in L^m(\Omega) \) with \( m \geq 1 \) and \( \theta > 0 \). If \( p = 2 \), the problem (1.5) have been treated in [9], i.e., in the case of the following problem

\[
\begin{aligned}
- \text{div} (a(x, u)|\nabla u|) + |u|^{\gamma-1} u &= f, \text{ in } \Omega \\
u &= 0 \text{ in } \partial \Omega,
\end{aligned}
\]  

(1.6)

the authors studied the lower order term \( |u|^{\gamma-1} u \) in (1.5) and (1.6) that has the regularizing effects of the solutions in the case where \( f \in L^m(\Omega) \), with \( m \geq 1 \) and \( \theta \geq 0 \). When \( p = 2 \) and the lower-order term does not appear in (1.5), the existence and regularity of solution to problem (1.5) are proved in [3]. The extension of this work to general case is investigated in [11].

Now we turn our attention recalling some results when the authors had added the singular sourcing term. Problems of p-Laplacien type (i.e \( \theta = 0 \)), have been well studied in both the existence and regularity aspects with \( f \) having different summability (see [10]). This framework has been extended to the problems with a lower order, considering

\[
\begin{aligned}
- \Delta u + u^s &= \frac{f}{u^\gamma}, \text{ in } \Omega \\
u &= 0 \text{ in } \Omega, u = 0 \text{ in } \partial \Omega,
\end{aligned}
\]  

(1.7)

with \( f \in L^m(\Omega), m \geq 1, 0 \leq \gamma < 1 \). Existence and regularity was established in [11]. Recently Olivia [15] have proved the existence and regularity of solution to the problem

\[
\begin{aligned}
- \Delta_p u + g(u) &= h(u)f, \text{ in } \Omega \\
u &= 0 \text{ in } \Omega, \\
u &= 0 \text{ in } \partial \Omega,
\end{aligned}
\]  

(1.8)

\( f \) is nonnegative and it belongs to \( f \in L^m(\Omega), m \geq 1 \), for some \( 0 \leq \gamma < 1 \). While \( g(s) \) is continuous, \( g(0) = 0 \) and, as \( s \to \infty \), could act as \( s^\delta \) with \( \delta \geq -1 \), the p-Laplacian operator is \( \Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) \) and \( h \) is continuous, it possibly blows up at the origin and it is bounded at infinity.

In [13], the authors studied the following degenerate elliptic problem with a singular nonlinearity:

\[
\begin{aligned}
- \text{div} (a(x, u, \nabla u)) &= fh(u) \text{ in } \Omega \\
u &= 0 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]  

(1.9)

\( f \) is nonnegative and it belongs to \( f \in L^m(\Omega), m \geq 1 \), and \( h \) satisfied the condition in (1.4). Following this way in this paper, we are interested again in the regularity results. By adding the singular term to the right of (1.5), we investigate the regularity of solutions of problems of kind (1.1) in light of the influence of some lower order terms.

In the study of problem (1.1), there one to two difficulties, the first one is the fact that, due to hypothesis (1.2), the differential operator \( A(u) = \text{div}(a(x, u)|\nabla u|^{p-2} \nabla u) \) is not coercive on \( W_0^{1,p}(\Omega) \), when \( u \) is large (see [17]). Due to the lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (see [13]) cannot be applied. The second difficulty comes from the right-hand side is singular in the variable \( u \). We overcome these difficulties by replacing operator \( A \) by another one defined by means of truncations, and approximating the singular term by non singular one. We will prove in section(3) that these problems admit a bounded \( W_0^{1,p}(\Omega) \) solution \( u_n \), \( n \in \mathbb{N} \) by using Schauder’s fixed point theorem. In section(4) we will get some a priori estimates and convergence results on the sequence of approximating solutions. In the end, we pass to the limit in the approximate problems.

Notations: In the entire paper \( \Omega \) is an open and bounded subset of \( \mathbb{R}^N \), with \( N \geq 1 \), we denote by \( \partial \Omega \) the boundary and by \( |A| \) the Lebesgue measure of a subset \( A \) of \( \mathbb{R}^N \).

For any \( q > 1, q^* = \frac{q}{q-1} \) is the Hölder conjugate exponent of \( q \), while for any \( 1 \leq p < N, p^* = \frac{Np}{N-p} \) is the Sobolev conjugate exponent of \( p \). For fixed \( k > 0 \) we will use of the truncation \( T_K \) defined as \( T_k(s) = \max \left( -k, \min(k, s) \right) \), we will also use the following functions

\[
V_{\delta,k}(s) = \begin{cases} 
1 & s \leq k \\
\frac{s-k}{\delta} & k < s < k + \delta, \\
0 & s \geq k + \delta,
\end{cases}
\]  

(1.10)

and

\[
S_{\delta,k}(s) := 1 - V_{\delta,k}(s).
\]  

(1.11)
For the sake of implicity we will often use the simplified notation
\[ \int f := \int f(x)dx, \]
when referring to integrals when no ambiguity on the variable of integration is possible. If no otherwise specified, we will denote by \( c \) serval constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance \( c \) can depend on \( \Omega, \gamma, N, k, \ldots \) ) but the will never depend on the indexes of the sequences we will often introduce.

\section{Statement of definitions and the main results}

\subsection{Statement of definitions}

In this context we deal with some class of solutions

\textbf{Definition 2.1.} A positive function \( u \) in \( W^{1,p}_0(\Omega) \) is weak solution (1.1) if \( h(u)f \in L^{1,\text{loc}}(\Omega) \), \( |u|^s \in L^1(\Omega) \) and if
\[ \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\nabla \varphi dx + \int_\Omega |u|^{s-1}u\varphi = \int_\Omega f h(u)\varphi \quad \forall \varphi \in C^1_c(\Omega). \] (2.1)

\textbf{Definition 2.2.} A measurable function \( u \) is an entropy solution to problem (1.1) if \( |u|^s \in L^1(\Omega) \), \( h(u)f \in L^{1,\text{loc}}(\Omega) \), \( T_k(u) \in W^{1,p}_0(\Omega) \) for every \( k > 0 \) and
\[ \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\nabla T_k(u - \varphi) dx + \int_\Omega |u|^{s-1}uT_k(u - \varphi) dx \leq \int_\Omega f h(u)T_k(u - \varphi) dx, \] (2.2)
for every \( \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

\textbf{Definition 2.3.} The Marcinkwicz space \( M^s(\Omega) \), \( s > 0 \), consists of all measurable functions \( v : \Omega \rightarrow \mathbb{R} \) that satisfy the following condition: there exists \( c > 0 \) such that
\[ \text{meas}\{|v| \geq k\} \leq \frac{c}{k^s} \quad \text{for all} \quad k > 0. \]

If \( |\Omega| < \infty \) and \( 0 < \epsilon < s - 1 \), we can show that \( L^s(\Omega) \subset M^s(\Omega) \subset L^{s-\epsilon}(\Omega) \).

Let
\[ p_0 := 1 + \frac{(1 + \theta - \gamma)(N - 1)}{N}. \] (2.3)

\subsection{Statement of the main results}

The main results of this paper are stated as follows:

\textbf{Theorem 2.1.} Let \( f \in L^m(\Omega), m > 1, 1 < p < N \). Then
\begin{enumerate}
  \item If \( s \geq \frac{1 + \theta - \gamma}{m - 1} \), then there exists a distributional solution \( u \) to problem (1.1) such that
    \[ u \in W^{1,p}_0(\Omega) \cap L^{ms+\gamma}(\Omega). \]
  \item If \( \frac{1 + \theta - \gamma}{pm - 1} < s < \frac{1 + \theta - \gamma}{m - 1} \), then there exists a distributional solution \( u \) to problem (1.1) such that
    \[ u^{ms+\gamma} \in L^1(\Omega) \quad \text{and} \quad u \in W^{1,\sigma}_0(\Omega), \quad 1 < \sigma = \frac{pm s}{1 + \theta + s - \gamma}. \]
  \item If \( 0 < s \leq \frac{1 + \theta - \gamma}{pm - 1} \), then there exists an entropy solution \( u \) to problem (1.1) such that
    \[ u^{ms+\gamma} \in L^1(\Omega) \quad \text{and} \quad |\nabla u| \in M^{\frac{pm s}{1 + \theta + s - \gamma}}(\Omega). \]
\end{enumerate}

\textbf{Remark 2.1.} If \( p = 2 \) and \( \gamma = 0 \); the result of Theorem 2.1 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [9], Theorem 1.5).

\textbf{Theorem 2.2.} Let \( f \in L^m(\Omega), m > 1, p_0 < p < N \). Then
i) If $0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, then there exists a distributional solution $u$ to problem (1.1) such that $u^{m s + \gamma} \in L^1(\Omega)$ and $u \in W_0^{1,1}L^1(\Omega)$, where $1 < \sigma = \frac{N(p+s(m-1)-1)}{N+s(m-1)-1-\sigma}$. 

ii) If $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, then item(ii) of Theorem 2 holds.

**Remark 2.2.** If $\gamma = 0$: the result of Theorem 2 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [7], Theorem 2.3) and Theorem 2 coincides with (7, Theorem 4).

**Theorem 2.3.** Let $f \in L^1(\Omega)$, $1 < p < N$. Then

- a) If $s > \frac{1+\theta-\gamma}{p-1}$, then there exists a distributional solution $u$ to problem (1.1) such that $u^{s+\gamma} \in L^1(\Omega)$ and $u \in W_0^{1,r}(\Omega) \cap L^{s+\gamma}(\Omega)$, where $1 < r < \frac{p s}{s+1+\theta-\gamma}$.

- b) If $0 < s \leq \frac{1+\theta-\gamma}{p-1}$, then there exists an entropy solution $u$ to problem (1.1) such that $u^{s+\gamma} \in L^1(\Omega)$ and $|\nabla u| \in M^{\frac{p s}{s+1+\theta-\gamma}}$. 

**Remark 2.4.** If $p = 2$ and $\gamma = 0$: the result of Theorem 2 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [9], Theorem 1.4).

**Theorem 2.4.** Let $f \in L^1(\Omega)$, $p_0 < p < N$. Then

1) If $0 < s \leq \frac{N(1-\gamma)+\gamma}{N-1}$, then there exists a distributional solution $u$ to problem (1.1) such that $u \in W_0^{1,r}(\Omega)$, where $1 < r < \frac{N(p-1-\theta)+\gamma}{N-p}$. 

2) If $\frac{N(1-\gamma)+\gamma}{N-1} < s < \frac{N(p-1-\theta)+\gamma}{N-p}$, then item (b) of theorem 2.3 holds.

3) If $s \geq \frac{N(p-1-\theta)+\gamma}{N-p}$, then item (a) of Theorem 2.3 holds.

**Remark 2.5.** In Theorem 2.3 we have $\frac{N(1-\gamma)+\gamma}{N-1} < s < \frac{N(p-1-\theta)+\gamma}{N-p}$ is meaningful, because $p > p_0$, besides $s \leq \frac{N(1-\gamma)+\gamma}{N-1} \Rightarrow s + \gamma \leq r^* \leq r$, for $r \geq 1$. In Theorem 2.2 $s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)} \Rightarrow ms + \gamma \leq \sigma^*$, for $\sigma \geq 1$.

### 3 A priori estimates and Preliminary facts

Let us introduce the following scheme of approximation

\[
\begin{cases}
-\text{div}(a(x,T_n(u_n))|\nabla u_n|^{p-2}|\nabla u_n|) + |u_n|^{s-1}u_n = h_n(u_n)f_n, \text{ in } \Omega,
\end{cases}
\]

where $f_n = T_n(f)$. Moreover, defining $h(0) := \lim_{s \to 0} h(s)$, we set

\[
h_n(s) = \begin{cases}
T_n(h(s)) & \text{for } s > 0, \\
\min(n, h(0)) & \text{otherwise}.
\end{cases}
\]

The right hand side of (3.1) is non negative, that $u_n$ is non negative. The existence of weak solution $u_n \in W_0^{1,p}(\Omega)$ is guaranteed by the following lemma.

**Lemma 3.1.** Problem (3.1) has a non negative solution $u_n$ in $W_0^{1,p}(\Omega)$, such that

\[
\int_{\Omega} |u_n|^{ms+\gamma} dx \leq c \int_{\Omega} |f|^m dx
\]

and the solution $u_n$ satisfies

\[
\int_{\Omega} a(x,T_n(u_n))|\nabla u_n|^{p-2}|\nabla u_n|\varphi dx + \int_{\Omega} |u_n|^{s-1}u_n \varphi = \int_{\Omega} f_n h_n(u_n) \varphi,
\]

where $0 \leq \gamma < 1$ and $\varphi$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. 

regularizing effect of absorption terms in singular and degenerate elliptic problems

**Proof.** This proofs derived from Schauder’s fixed point argument in [16]. For fixed \( n \in \mathbb{N} \) let us define a map\( G : L^p(\Omega) \to L^p(\Omega), \)
such that, for any \( v \) be a function in \( L^p(\Omega) \) gives the weak solution \( w \) to the following problem\( -div(a(x, T_n(w)))|\nabla w|^{p-2}|\nabla w| + |w|^{s-1}w = f_n h_n(v). \) (3.5)The existence of a unique \( w \in W_0^{1,p}(\Omega) \) corresponding to a \( v \in L^p(\Omega) \) follows from the classical result of [11], [13]. Moreover, since the datum \( f_n h_n(v) \) bounded, we have that \( w \in L^\infty(\Omega) \) and there exists a positive constant \( d_1 \), independents of \( v \) and \( w \) (but possibly depending in \( n \)), such that \( ||w||_{L^\infty(\Omega)} \leq d_1 \). Again, thanks to the regularity of the datum \( f_n h_n(v) \), we have can choose \( w \) as test function in the weak formulation (4.21), we have\[ \int_\Omega a(x, T_n(w))|\nabla w|^{p-2}|\nabla w| + \int_\Omega |w|^{s-1}w.w = \int_\Omega f_n h_n(v)w, \] then, it follows from (1.2)\[ \alpha \int_\Omega \frac{|\nabla w|^p}{(1+n)^\alpha} dx \leq n^2 \int_\Omega |w|dx, \]
using the Poincaré inequality we have\[ \int_\Omega \frac{|\nabla w|^p}{(1+n)^\alpha} dx \leq \frac{c_1}{\alpha} n^2 \int_\Omega |\nabla w|dx, \]
then\[ \int_\Omega |\nabla w|^p dx \leq \frac{c_1}{\alpha}(1+n)^{\frac{\alpha}{2}} \frac{1}{\alpha} \int_\Omega |\nabla w|^p dx \leq c(n, \alpha) |\Omega|^\frac{1}{\alpha} \left( \int_\Omega |\nabla w|^p dx \right)^\frac{1}{\alpha}, \] (3.7)we obtain\[ \int_\Omega |\nabla w|^p dx \leq c(n, \alpha) |\Omega|, \]
using the Poincaré inequality on the left hand side\[ ||w||_{L^p(\Omega)} \leq c(n, \alpha, |\Omega|)^\frac{1}{\alpha}, \] (3.8)where \( c(n, \alpha, |\Omega|) \) is a positive constant independent from \( v \), thus, we have that the ball \( S \) of radius \( c(n, \alpha, |\Omega|) \) is invariant for \( G \).

Now, we are going to prove that the map \( G \) is continuous in \( S \). Consider a sequence \( (v_k) \) that converges to \( v \) in \( L^p(\Omega) \). We recall that \( w_k = f_n h_n(v_k) \) are bounded, we have that \( w_k \in L^\infty(\Omega) \) and there exists a positive constant \( d \), independent of \( v_k \) and \( w_k \), such that \( ||w_k||_{L^\infty(\Omega)} \leq d \). Then by dominated convergence theorem\[ ||f_n h_n(v_k) - f_n h_n(v)||_{L^p(\Omega)} \to 0. \]
Hence, by the uniqueness of the weak solution, we can say that \( w_k = G(v_k) \) converges to \( w = G(v) \) in \( L^p(\Omega) \). Thus \( G \) is continuous over \( L^p(\Omega) \).
What finally needs to be checked is that \( G(S) \) is relatively compact in \( L^p(\Omega) \). Let \( v_k \) be a bounded sequence, and let \( w_k = G(v_k) \). Reasoning as to obtain (3.8), we have\[ \int_\Omega |\nabla w|^p dx = \int_\Omega |\nabla G(v_k)|^p dx \leq c(n, \alpha, \gamma), \]
where \( c \) is clearly independent from \( v_k \), so that, \( G(L^p(\Omega)) \) is relatively compact in \( L^p(\Omega) \). Now, applying the Schauder’s fixed point theorem that \( G \) has a fixed point \( u_n \in S \) that is solution to (3.1) in \( W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \)
To show (3.3), we will consider the cases \( m > 1 \) and \( m = 1 \).
Case \( m > 1 \), choosing \( \varphi = |u_n|^{s(m-1)+\gamma} \) in (3.4), we have\[ \int_\Omega |u_n|^{sm+\gamma} dx \leq \int_\Omega |f| |u_n|^{s(m-1)} dx, \]
therefore\[ \int_\Omega |u_n|^{sm+\gamma} \leq c \left( \int_\Omega |f|^m \right)^\frac{1}{m} \left( \int_\Omega |u_n|^{sm+\gamma} \right)^{1-\frac{1}{m}}, \]
which implies (3.3).

Case $m = 1$. Choosing $\varphi = u_n^\gamma$, then
\[
\int_\Omega |u_n|^{s-1} u_n u_n^\gamma dx \leq \int_\Omega \frac{f}{u_n} u_n^\gamma dx \leq f dx,
\]
which the estimate (3.3), as desired.

**Lemma 3.2.** Let $u_n$ be a solution to problem (3.1) and $f \in L^m(\Omega)$ with $m \geq 1$. Then
\[
\int_{\{k < u_n\}} u_n^{sm} \leq \frac{1}{k^{\theta}} \left( \int_{\{k < u_n\}} fu_n^{(m-1)} \right)^\frac{1}{m} \left( \int_{\{k < u_n\}} u_n^{sm} \right)^{1 - \frac{1}{m}}
\]
and $\lim_{|E| \to 0} \int_E u_n^{sm} = 0$ uniformly with respect to $n$.

**Proof.** Let $k > 0$ and $u_n^{s(m-1)} \psi_i$ be a sequence of increasing, positive, uniformly bounded $C^\infty(\Omega)$ functions, such that
\[
\psi_i(s) \to \begin{cases} 1, & s \geq k \\ 0, & 0 \leq s < k \end{cases}
\]
The limit on $i$ gives
\[
\int_{\{k < u_n\}} u_n^{sm} \leq \frac{1}{(k + \frac{1}{n})^\theta} \left( \int_{\{k < u_n\}} fu_n^{(m-1)} \right)^\frac{1}{m} \left( \int_{\{k < u_n\}} u_n^{sm} \right)^{1 - \frac{1}{m}}
\]
by Hölder inequality
\[
\int_{\{k < u_n\}} u_n^{sm} \leq \frac{1}{(k + \frac{1}{n})^\theta} \int_{\{k < u_n\}} f^m
\]
This implies that
\[
\int_E u_n^{sm} \leq k^r |E| + \int_{E \setminus \{u_n > k\}} u_n^{sm} \leq k^r |E| + \frac{1}{k^\theta} \int_{\{u_n > k\}} f^m
\]
since $f \in L^m(\Omega)$ for any given $\varepsilon > 0$, there exists $k_\varepsilon$ such that $\int_{\{u_n > k_\varepsilon\}} |f|^m \leq \varepsilon$. Therefore
\[
\int_E u_n^{sm} \leq k_\varepsilon^{sm} |E| + \frac{\varepsilon}{k^\theta}
\]
and the statement of this lemma is thus proved.

**Lemma 3.3.** Let $u$ be a measurable function in $M^r(\Omega)$, $s > 0$, and suppose that there exists a positive constant $\rho > 0$ such that
\[
\int_\Omega |\nabla T_k(u)|^p dx \leq Ck^\rho \quad \forall k > 0.
\]
Then $|\nabla u| \in M^\frac{r}{p}(\Omega)$.

**Proof.** Let $\lambda$ be a fixed positive real number. For every $k > 0$, we have
\[
\text{meas}\{|\nabla u| > \lambda\} = \text{meas}\{|\nabla u| > \lambda, |u| \leq k\} + \text{meas}\{|\nabla u| > \lambda, |u| > k\} 
\leq \text{meas}\{|\nabla u| > \lambda, |u| \leq k\} + \text{meas}\{|u| > k\}
\]
and
\[
\text{meas}\{|\nabla u| > \lambda, |u| \leq k\} \leq \frac{1}{\lambda^p} \int_\Omega |\nabla T_k(u)|^p dx \leq C \frac{k^\rho}{\lambda^p}.
\]
Since $u \in M^r(\Omega)$, it follows that
\[
\text{meas}\{|\nabla u| > \lambda\} = C \frac{k^\rho}{\lambda^p} + C \frac{1}{k^r}.
\]
and this latter inequality holds for every \( k > 0 \). Minimizing with respect to \( k \), we easily obtain
\[
\text{meas}\{\nabla u > \lambda\} = \frac{C}{\lambda^{\frac{1}{p'}}}.
\]
Thus, \( |\nabla u| \in M_{\frac{1}{p'}}(\Omega) \).

**Lemma 3.4.** Let \( u_n \) be a sequence of measurable functions such that \( T_k(u_n) \) is bounded in \( W_0^{1,p}(\Omega) \) for every \( k > 0 \). Then there exists a measurable function \( u \) such that \( T_k(u) \in W_0^{1,p}(\Omega) \) and, moreover,
\[
T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u \quad \text{a.e. in } \Omega.
\]

**Proof.** Let us prove that \( u_n \rightarrow u \) locally in measure. To begin with, we observe that, for \( t, \varepsilon > 0 \), we have
\[
\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}.
\]
Therefore,
\[
\text{meas}\{|u_n - u_m| > t\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > t\}.
\]
Choosing \( k \) large enough, we obtain
\[
\text{meas}\{|u_n| > k\} < \varepsilon \quad \text{and} \quad \text{meas}\{|u_m| > k\} < \varepsilon.
\]
We can assume that \( \{T_k(u_n)\} \) is a Cauchy sequence in \( L^q(\Omega) \) for every \( q < p' = \frac{Np}{N-p} \). Then
\[
\text{meas}\{|T_k(u_n) - T_k(u_m)| > t\} \leq t^{-q} \int_{\Omega} |T_k(u_n) - T_k(u_m)|^q dx \leq \varepsilon \quad \forall n, m \geq n_0(k,t).
\]
This proves that \( \{u_n\} \) is a Cauchy sequence in measure in \( \Omega \). Therefore, there exists a measurable function \( u \) such that \( u_n \rightarrow u \) in measure. Hence that \( u_n \rightarrow u \) a.e. in \( \Omega \), and so
\[
T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega).
\]

## 4 Proof of the results

This section is devoted to proving theorems cited above. We start with

**Proof of Theorem 2.1.** We separate our proof in three parts, according to the values of \( s \)

**Part I.** Let \( s > \frac{1+\theta-\gamma}{m-1} \). Choosing \( \varphi = (1 + u_n)^{1+\theta} - 1 \) in (3.4), then, the second term is non negative. By (1.2), we have
\[
\alpha \int_{\Omega} |\nabla u_n|^p dx \leq \int_{\Omega} |f||u_n|^{1+\theta-\gamma} dx.
\]
Hölder’s inequality applied to the right-hand side yields
\[
\int_{\Omega} |f||u_n|^{1+\theta-\gamma} dx \leq c \left[ \int_{\Omega} u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx \right]^{1-\frac{\gamma}{\theta}}.
\]
So, we have
\[
\int_{\Omega} |\nabla u_n|^p dx \leq c \left[ \int_{\Omega} u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx \right]^{1-\frac{\gamma}{\theta}}. \tag{4.1}
\]
Since
\[
\frac{m(1 + \theta - \gamma)}{m-1} \leq ms,
\]
we have \( s > \frac{1+\theta-\gamma}{m-1} \). Lemma 3.1 implies that the right-hand side of (4.1) is uniformly bounded, so we have
\[
\int_{\Omega} |\nabla u_n|^p dx \leq c. \tag{4.2}
\]
In order to prove that the limit function \( u \) is a solution of (1.1) in the sense of Definition 2.1 we need to show that we can pass to the limit in the weak formulation of the approximating problems (3.1).

Now we focus on the left hand side of (3.4), by (4.2) we conclude that there exist a subsequence, still indexed by \( n \),
and a measurable function \( u \) in \( W^{1,p}_0(\Omega) \), such that \( u_n \rightharpoonup u \) weakly in \( W^{1,p}_0(\Omega) \) and \( u_n \to u \) a.e. in \( \Omega \). Fatou’s lemma implies \( u \in L^{m+\gamma}(\Omega) \). We see that (see [5], Lemma 5)

\[
\nabla u_n \to \nabla u \text{ a.e. in } \Omega.
\]

(4.3)

Next, we pass to the limit in (3.4). By (4.3), we can easily obtain

\[
|\nabla u_n|^{p-2} |\nabla u_n| \to |\nabla u|^{p-2} |\nabla u| \text{ weakly in } L^p(\Omega).
\]

Moreover,

\[
\int \nabla(x, T_n(u_n)) \nabla \varphi \to \int a(x, u) \nabla \varphi \text{ in } L^p(\Omega).
\]

Consequently, we have

\[
\int_{\Omega} a(x, T_n(u_n)) |\nabla u_n|^{p-2} |\nabla u_n| \nabla \varphi dx \to \int_{\Omega} a(x, u) |\nabla u|^{p-2} |\nabla u| \nabla \varphi dx.
\]

Therefore, we can pass to the limit in the first term of the left-hand side of (3.4). We will show that

\[
|u_n|^{s-1} u_n \to |u|^{s-1} u \text{ in } L^1(\Omega).
\]

(4.4)

We take \( S_{\eta,k}(u_n) \) as a test function in the weak formulation (3.1), we deduce

\[
\int_{\Omega} a(x, T_n(u_n)) |\nabla u_n|^{p-2} |\nabla u_n| \nabla \varphi dx \to \int_{\Omega} a(x, u) |\nabla u|^{p-2} |\nabla u| \nabla \varphi dx.
\]

which, observing that the first term on the left hand side is non negative and taking the limit with respect to \( \eta \to 0 \), implies

\[
\int_{\{u_n \geq k\}} |u_n|^{s-1} u_n dx \leq \sup_{s \in [k, \infty)} [h(s)] \int_{\{u_n \geq k\}} f_n dx,
\]

which, since \( f_n \) converges to \( f \) in \( L^m(\Omega) \), easily implies that \( |u_n|^{s-1} u_n \) is equi-integrable and so it converges to \( |u|^{s-1} u \) in \( L^1(\Omega) \), this concludes (4.4).

The next step we want to pass to the limit in the right hand side of (3.4). Let us take \( 0 \leq \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) as test function in the weak formulation of (3.1), by using the young inequality and the hypotheses in (1.7) and (1.13), we have

\[
\int_{\Omega} h_n(u_n) f_n \varphi = \int_{\Omega} a(x, T_n(u_n)) |\nabla u_n|^{p-2} |\nabla u_n| \nabla \varphi dx + \int_{\Omega} u_n^{s-1} u_n \varphi dx
\]

\[
\leq C||\varphi||_{L^\infty(\Omega)} \beta \int_{\Omega} |\nabla u_n|^{p-2} |\nabla u_n| \nabla \varphi dx + \int_{\Omega} u_n^{s-1} u_n \varphi dx
\]

\[
\leq C||\varphi||_{L^\infty(\Omega)} + \beta \frac{p-1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p dx
\]

\[
\leq C||\varphi||_{L^\infty(\Omega)} + C \left[ \int_{\Omega} |\nabla \varphi|^p dx + \int_{\Omega} |\nabla u_n|^p dx \right],
\]

then

\[
\int_{\Omega} h_n(u_n) f_n \varphi \leq C||\varphi||_{L^\infty(\Omega)} + C||\varphi||_{W^{1,p}_0(\Omega)} + ||u_n||_{W^{1,p}_0(\Omega)}.
\]

(4.5)

From now on, we assume that \( h(s) \) is unbounded as \( s \) tends to 0. An application of the Fatou Lemma in (4.5) with respect to \( n \) gives

\[
\int_{\Omega} h(u) f \varphi \leq c,
\]

(4.6)

where \( c \) does not depend on \( n \).

Hence \( f h(u) \varphi \in L^1(\Omega) \) for any nonnegative \( \varphi \in W^{1,p}_0(\Omega) \). As a consequence, if \( h(s) \) is unbounded as \( s \) tends to 0, we deduce that

\[
\{u = 0\} \subset \{f = 0\}
\]

(4.7)
up to a set of zero Lebesgue measure.

Now, for $\delta > 0$, we split the right hand side of (3.4) as
\[
\int_{\Omega} h_n(u_n)f_n \phi dx = \int_{\{u_n \leq \delta\}} h_n(u_n)f_n \phi dx + \int_{\{u_n > \delta\}} h_n(u_n)f_n \phi dx,
\]
and we pass to limit as $n \to +\infty$ and then $\delta \to 0$, we remark that we need to choose $\delta \neq \{\eta; |u = \eta| > 0\}$, which is at most a countable set, for the second term (4.8) we have
\[
0 \leq h_n(u_n)f_n \phi \chi_{\{u_n > \delta\}} \leq \sup_{s \in [\delta, \infty)} [h(s)] f \phi \in L^1(\Omega),
\]
which precis to apply the Lebesgue Theorem with respect $n$. Hence on has
\[
\lim_{n \to +\infty} \int_{\{u_n > \delta\}} h_n(u_n)f_n \phi dx = \int_{\{u > 0\}} h(u)f \phi dx.
\]
Moreover it follows by (4.6) that
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n > \delta\}} h_n(u_n)f_n \phi dx = \int_{\{u > 0\}} h(u)f \phi dx.
\]

Now in order to get rid of the first term of the right hand side of (4.8), we take $V_\delta(u_n)\phi$ is a test function in the weak formulation of (3.1), where $V_\delta(u_n) := V_{\delta, \phi}(u_n)$ is defined in (1.11) and by Lemma 1.1 contained in [7], we have $V_\delta(u_n)$ belongs to $W_0^{1,p}(\Omega)$, then (recall $V_\delta(u_n) \leq 0$ for $s \geq 0$)
\[
\int_{\{u_n \leq \delta\}} h_n(u_n)f_n \phi dx \leq \int_{\Omega} h_n(u_n)f_n V_\delta(u_n) \phi dx
\]
\[
= \int_{\Omega} a(x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \nabla \phi V_\delta(u_n) dx
\]
\[
- \frac{1}{\delta} \int_{\{\delta < u_n < 2\delta\}} a(x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \phi \nabla u_n dx + \int_{\Omega} |u_n|^{s-1} u_n V_\delta(u_n) \phi dx,
\]
by using (1.2) and (1.3), we have
\[
\int_{\{u_n \leq \delta\}} h_n(u_n)f_n \phi dx \leq \beta \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi V_\delta(u_n) dx
\]
\[
+ \int_{\Omega} |u_n|^{s-1} u_n V_\delta(u_n) \phi dx,
\]
using that $V_\delta$ is bounded we deduce that $|\nabla u_n|^{p-2} \nabla u_n V_\delta(u_n)$ converges to $|\nabla u|^{p-2} \nabla u V_\delta(u)$ weakly in $L^p(\Omega)^N$ as $n$ tends to infinity. This implies that
\[
\lim_{n \to +\infty} \int_{\{u_n \leq \delta\}} h_n(u_n)f_n \phi dx \leq \beta \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi V_\delta(u) dx + \int_{\Omega} |u|^{s-1} u V_\delta(u) \phi dx.
\]
Since $V_\delta(u)$ converges to $\chi_{\{u=0\}}$ a.e in $\Omega$ as $\delta$ tends to 0 and since $u \in W_0^{1,p}(\Omega)$, then $|\nabla u|^{p-2} \nabla u \nabla \phi V_\delta(u)$ converges to 0 a.e. in $\Omega$ as $\delta$ tends to 0. Applying the Lebesgue Theorem on the right hand side of (4.11) we obtain that
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n \leq \delta\}} h_n(u_n)f_n \phi dx \leq \beta \int_{\{u=0\}} |\nabla u|^{p-2} \nabla u \nabla \phi dx + \int_{\{u=0\}} |u|^{s-1} u \phi dx = 0,
\]
by (4.10) and (4.12), we deduce that
\[
\lim_{n \to +\infty} \int_{\Omega} h_n(u_n)f_n \phi dx = \int_{\Omega} h(u)f \phi dx \quad \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).
\]
Moreover, decomposing any $\phi = \phi^+ - \phi^-$, and using that (4.13) is linear in $\phi$, we deduce that (4.13) holds for every $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. 

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We treated $h(s)$ unbounded as $s$ tends to 0, as regards bounded function $h$ the proof is easier and only difference deals with the passage to the limit in the left hand side of (4.13). We can avoid introducing $\delta$ and we can substitute (4.9) with
\[0 \leq f_n h_n(u_n) \varphi \leq f \|h\|_{L^\infty(\Omega)} \varphi.\]

Using the same argument above we have that
\[
\lim_{n \to +\infty} \int_{\Omega} f_n h_n(u_n) \varphi dx = \int_{\Omega} f h(u) \varphi dx,
\]
whence one deduces (4.10). This concludes the proof of part I.

**Part II.** Let
\[\frac{1 + \theta - \gamma}{p m - 1} < s < \frac{1 + \theta - \gamma}{m - 1}.
\]

Choosing $\varphi = (1 + u_n)^{s(m-1)+\gamma} - 1$ in (3.4), we see that the second term is non-negative. Using assumption (1.2), we have
\[
\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + u_n)^{1 + \theta - s(m-1)-\gamma}} dx \leq c \int_{\Omega} |f||u_n|^{s(m-1)} dx.
\]

Now, using Hölder’s inequality in the right-hand side of the previous inequality, we obtain
\[
\int_{\Omega} |f||u_n|^{s(m-1)} dx \leq c \left[ \int_{\Omega} u_n^{ms+\gamma} dx \right]^{1 - \frac{1}{p}} \leq c.
\]

Therefore,
\[
\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + u_n)^{1 + \theta - s(m-1)-\gamma}} dx \leq c.
\]

Let $1 \leq \sigma < p$. Let us write
\[
\int_{\Omega} |\nabla u_n|^\sigma dx = \int_{\Omega} \frac{|\nabla u_n|^\sigma}{(1 + u_n)^{\frac{\sigma}{p}(1 + \theta - s(m-1)-\gamma)}} \cdot \frac{(1 + u_n)^{\frac{\sigma}{p}(1 + \theta - s(m-1)-\gamma)}}{(1 + u_n)^{\frac{\sigma}{p}(1 + \theta - s(m-1)-\gamma)}} dx
\]

Hölder’s inequality implies
\[
\int_{\Omega} \frac{|\nabla u_n|^\sigma}{(1 + u_n)^{\frac{\sigma}{p}(1 + \theta - s(m-1)-\gamma)}} dx \leq c \left[ \int_{\Omega} (1 + u_n)^{\frac{\sigma}{p} s m - \sigma s(m-1)-\gamma} dx \right]^{1 - \frac{s}{p}}.
\]

Therefore, we have
\[
\int_{\Omega} |\nabla u_n|^\sigma dx \leq c \left[ \int_{\Omega} (1 + u_n)^{\frac{\sigma}{p} s m - \sigma s(m-1)-\gamma} dx \right]^{1 - \frac{s}{p}}.
\]

If
\[
\frac{s}{p - \sigma} [1 + \theta - s(m-1)-\gamma] \leq ms \text{ i.e. } \sigma \leq \frac{p m s}{s + 1 + \theta - \gamma},
\]
then Lemma (3.1) implies that the right-hand side of (4.16) is uniformly bounded. The inequality
\[
\frac{1 + \theta - \gamma}{p m - 1} < s \text{ implies } \frac{p m s}{s + 1 + \theta - \gamma} > 1.
\]

Consequently,
\[
\int_{\Omega} |\nabla u_n|^\sigma dx \leq c, \sigma = \frac{p m s}{1 + \theta + s - \gamma}.
\]

Up to a subsequence, there exists a function $u \in W^{1,\sigma}_0(\Omega)$ such that
\[u_n \to u \text{ weakly in } W^{1,\sigma}_0(\Omega) \text{ and } u_n \to u \text{ a.e in } \Omega.
\]

By Lemma 5 (see[5]), we have $\nabla u_n \to \nabla u$ a.e in $\Omega$. Fatou’s Lemma implies $u_n^{s m + \gamma} \in L^1(\Omega)$ we will now pass to the limit in (3.4). We can easily obtain
\[
|\nabla u_n|^{p-2} |\nabla u_n| \to |\nabla u|^{p-2} |\nabla u| \text{ weakly in } L^{\frac{\sigma}{p_m - 1}}(\Omega),
\]
and
\[
a(x, T_n(u_n)) \varphi \to a(x, u) \varphi, \text{ in } L^{\frac{s m + \gamma}{p m - 1}}(\Omega).
\]
Fatou’s Lemma implies that
\[
\int_{\Omega} a(x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx \to \int_{\Omega} a(x, u) |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx.
\]
The remaining two parts in (3.4) are the same as part I.

**PART III.** Let \(0 < s \leq \frac{1+\theta - \gamma}{pm-1}\). It follows from (4.15) that
\[
\int_{\Omega \cap \{|u_n| < k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}} \, dx \leq c,
\]
and consequently
\[
\int_{\Omega} |\nabla T_k(u_n)|^p \, dx = \int_{\Omega \cap \{|u_n| < k\}} |\nabla T_k(u_n)|^p \, dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}.
\] (4.17)

Lemma 3.1 implies the existence of a measurable function \(u\) such that
\(T_k(u_n) \to T_k(u)\) weakly in \(W^{1,p}_0(\Omega)\) and \(a.e.\) in \(\Omega\).

Fatou’s Lemma implies that \(|u|^s \in L^1(\Omega)\). We can pass to the limit in (4.17), to get
\[
\int_{\Omega} |\nabla T_k(u)|^p \, dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}.
\]
Since
\[
s \leq \frac{1+\theta - \gamma}{pm-1} \leq \frac{1+\theta - \gamma}{m-1},
\]
we have \(1+\theta - s(m-1) - \gamma > 0\).

As a result of the Lemma 3.3, we obtain \(|\nabla u| \in M^{\frac{pm}{1+\theta-s}}(\Omega)\). We will show that \(u\) is an entropy solution of (1.1).

Indeed, let us choose
\(T_k(u_n - \varphi), \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega),\)
as a test function in (3.4), then we have
\[
\int_{\Omega} a(x, T_n(T_k(u_n))) |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - \varphi) \, dx + \int_{\Omega} |u_n|^{s-1} u_n T_k(u_n - \varphi) = \int_{\Omega} f_n h(u_n)(T_k(u_n - \varphi)).
\] (4.18)

Let us pass to the limit in (4.18). For the second term on the left-hand side and for the right-hand side, we can use (4.14) to obtain the limit. For the first term on the left-hand side, we will firstly show that \(\nabla T_k(u_n) \to \nabla T_k(u) \) a.e. in \(\Omega\). Let \(\varphi = T_k(u_n) - T_k(u)\) in (3.4), then we obtain
\[
\int_{\Omega} a(x, T_n(T_k(u_n))) |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n) - \nabla T_k(u) \, dx + \int_{\Omega} |u_n|^{s-1} u_n [T_k(u_n) - T_k(u)]
\]
\[
= \int_{\Omega} f_n h_n(u_n)[T_k(u_n) - T_k(u)].
\]
As a consequence, we have
\[
\int_{\Omega} a(x, T_n(T_k(u_n))) [\nabla T_k(u_n)]^{p-2} \nabla T_k(u_n) - [\nabla T_k(u)]^{p-2} \nabla T_k(u) [\nabla T_k(u_n) - \nabla T_k(u)] \, dx
\]
\[
= \int_{\Omega} f_n h_n(u_n)[T_k(u_n) - T_k(u)] \, dx - \int_{\Omega} |u_n|^{s-1} u_n [T_k(u_n) - T_k(u)] \, dx
\]
\[
- \int_{\Omega} a(x, T_n(T_k(u_n))) |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) |\nabla T_k(u_n) - \nabla T_k(u)| \, dx.
\] (4.19)

We are going to show that the three terms of the right-hand side in (4.19) all converge to zero. For the first term, we can use (4.14) to take the limit. As the result of the proof in part one, we obtain
\(u_n^{s-1} u_n \to |u|^{s-1} u \) in \(L^1(\Omega)\).
Therefore, we have
\[ \int_{\Omega} u_n^{-1} u_n [T_k(u_n) - T_k(u)] dx \to 0 \text{ as } n \to \infty. \]

We can easily know the fact that \( a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2} \nabla T_k(u) \in L^p(\Omega) \). Thus, for every measurable set \( E \subset \Omega \), we can write
\[ \int_E |a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-1} dx \to 0 \text{ as } m \text{ meas } E \to 0. \]

Because
\[ a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2} \nabla T_k(u) \to a(x, T_k(u))|\nabla T_k(u)|^{p-2} \nabla T_k(u) \text{ a.e. in } \Omega, \]
we have
\[ a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2} \nabla T_k(u) \to a(x, T_k(u))|\nabla T_k(u)|^{p-2} \nabla T_k(u) \text{ in } L^p(\Omega), \]
by Vitali’s Theorem. By Lemma 3.4 we see that
\[ \nabla T_k(u_n) - \nabla T_k(u) \rightharpoonup 0 \text{ weakly in } L^p(\Omega). \]

Therefore,
\[ \int_{\Omega} a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2} \nabla T_k(u) - \nabla T_k(u_n) dx \to 0 \text{ as } n \to \infty. \]

From the above, we have
\[ \int_{\Omega} a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2} \nabla T_k(u) dx \to 0. \]

As a consequence, Lemma 5 in [5] implies \( \nabla T_k(u_n) \to \nabla T_k(u) \) in \( L^p(\Omega) \). Therefore,
\[ \nabla T_k(u_n) \to \nabla T_k(u) \text{ a.e. in } \Omega. \]

Let \( m = k + |\varphi| \). The first term on the left-hand side in (4.18) can be rewritten as
\[ \int_{\Omega} a(x, T_n(u_n))|\nabla T_m(u)|^{p-2} \nabla T_m(u) \nabla T_k(u_n - \varphi) dx. \]

Since \( \nabla T_m(u_n) \to \nabla T_m(u) \text{ a.e. in } \Omega \), as a result of the Fatou’s Lemma, we have
\[ \liminf_{n \to \infty} \int_{\Omega} a(x, T_n(u_n))|\nabla T_m(u)|^{p-2} \nabla T_m(u) \nabla T_k(u_n - \varphi) dx \]
\[ \geq \int_{\Omega} a(x, u)|\nabla T_m(u)|^{p-2} \nabla T_m(u) \nabla T_k(u - \varphi) dx \]
\[ = \int_{\Omega} a(x, u)|\nabla u|^{p-2} \nabla u \nabla T_k(u - \varphi) dx. \]

So we see that \( u \) is an entropy solution of (1.1).

**Proof of Theorem** We separate our proof in two parts, according to the values of \( s \)

**Part I.** Let \( 0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)} \). Then \( s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)} \), which implies \( ms + \gamma \leq \sigma^* \) for \( \sigma \geq 1 \). As the result of (4.16) and of Sobolev’s embedding theorem, we have
\[ \int_{\Omega} |u_n|^{\sigma^*} dx \leq c \left( \int_{\Omega} (1 + u_n)^{\frac{\sigma}{p-\sigma} \left[ 1 + \theta - s(m-1) - \gamma \right]} dx \right)^{\frac{(p-\sigma)\sigma^*}{p}}. \]

If
\[ \frac{\sigma}{p-\sigma} \left[ 1 + \theta - s(m-1) - \gamma \right] \leq \sigma^* \text{ i.e. } \sigma \leq \frac{N[p + s(m-1) - 1 - \theta + s(m-1) - \gamma]}{N + s(m-1) - 1 - \theta + \gamma}, \]
then, by \( m > 1 \) and \( p > p_0 > 1 + \frac{(N-1)[1+\theta-s(m-1)-\gamma]}{N} \), which implies
\[ \frac{N[p + s(m-1) - 1 - \theta + \gamma]}{N + s(m-1) - 1 - \theta + \gamma} > 1, \]
\[ \int_{\Omega} |u_n|^{\sigma^*} dx \leq c \left( \int_{\Omega} |u_n|^{\sigma^*} dx \right)^{\frac{(p-\sigma)\sigma^*}{p}}. \]
For (4.20), by Young’s inequality with $\epsilon$, we have
\[
\int_\Omega |u_n|^\sigma dx \leq c.
\]
Which, together with (4.16) and $p-s(1+\theta-m-1-\gamma) \leq \sigma^*$, implies
\[
\int_\Omega |\nabla u_n|^\sigma dx \leq c, \quad \sigma \leq \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}.
\]

The remaining proof of this part is the same as part II in Theorem 2.1, we can show that
\[
\int \frac{|\nabla u_n|^p}{(1+u_n)^{1+r-\gamma}} dx \leq \int |f| dx \leq c.
\]

|Part II. Let $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$. Since $p > p_0$, it follows that
\[
\frac{N(1-\gamma)+\gamma}{m(N-1)} > \frac{1+\theta-\gamma}{pm-1},
\]
thus, we can show that $u$ is a distributional solution to problem (1.1).  
Proof of Theorem 2.3. We separate our proof in two parts, according to the values of $s$  
Part a. Let $s > \frac{1+\theta-\gamma}{p-1}$. Choosing $\varphi = (1+u_n)^\gamma -1$ in (3.4), then the second term is non negative. Using assumption (1.2), we can write
\[
\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+r-\gamma}} dx \leq \int |f| dx \leq c. \tag{4.21}
\]
Let $r < p$, writing
\[
\int_\Omega |\nabla u_n|^r dx = \int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+r-\gamma}} (1+u_n)^{\frac{(1+r-\gamma)}{p}} dx.
\]
\[
\leq \left( \int_\Omega \left( \frac{|\nabla u_n|^p}{(1+u_n)^{1+r-\gamma}} \right)^{\frac{r}{p}} \right)^{\frac{p}{r}} \left( \int_\Omega \left( 1+u_n \right)^{\frac{r(1+r-\gamma)}{p-1}} \right) \left( \int_\Omega \left( 1+u_n \right)^{\frac{r(1+r-\gamma)}{s-r}} \right)^{1-\frac{p}{r}} dx.
\]
Thanks to Lemma 3.1 if
\[
\frac{r}{p-r} (1+\theta-\gamma) \leq s, \text{ ie } r < \frac{ps}{1+\theta+s-\gamma}.
\]
Then
\[
s > \frac{1+\theta-\gamma}{p-1} \text{ implies } , \frac{ps}{1+\theta+s-\gamma} > 1.
\]
In that case, the right-hand sides is uniformly bounded and so we have
\[
\int_\Omega |\nabla u_n|^r dx \leq c, \quad r < \frac{ps}{1+\theta+s-\gamma}.
\]
As a consequence, there exists a function $u \in W^{1,r}_{0}(\Omega)$ such that
\[
\begin{align*}
&u_n \rightharpoonup u \text{ weakly in } W^{1,r}_{0}(\Omega) \text{ and } u_n \rightarrow u \text{ a.e in } \Omega.
\end{align*}
\]
Let
\[
g_n = f_n h_n(u_n) - T_n (|u_n|^{p-1} u_n).
\]
Because $g_n$ is bounded in $L^1(\Omega)$, and $u_n$ is a solution of
\[
\begin{cases}
-\text{div}(a(x,T_n(u_n))|\nabla u_n|^{p-2} |\nabla u_n|) = g_n, \\
u_n \in W^{1,p}_{0}(\Omega),
\end{cases}
\]
it follows from Lemma 1 (see [4]), that
\[
\nabla u_n \rightharpoonup \nabla u \text{ a.e in } \Omega. \tag{4.22}
\]
We are going to show that $u$ is a distributional solution to problem (1.1) by passing to the limit in (3.4). We suppose that $\varphi \in C_0^\infty(\Omega)$. Since $|\nabla u_n|^{p-2} |\nabla u_n| \in L^{\frac{p}{p-1}}(\Omega)$ and (4.22) hold, we have
\[
|\nabla u_n|^{p-2} |\nabla u_n| \rightharpoonup |\nabla u|^{p-2} \nabla u \text{ weakly in } L^{\frac{p}{p-1}}(\Omega).
\]
Vitali’s Theorem implies that
\[ a(x, T_n(u_n)) \nabla \varphi \rightarrow a(x, u) \nabla \varphi, \quad \text{in } L^{\frac{1}{1-\gamma}}(\Omega), \]
where \( \left( \frac{r}{p-1} \right)' = \frac{p-1-r}{p-1} \). Therefore, we can pass to the limit in the first term on the left-hand side of (3.4). For the second term on the left-hand side and the first term on the right-hand side in (3.4) we can namely arguing exactly as part I in Theorem 2.1. Therefore, we conclude that \( u \) is a distributional solution to problem (1.1).

**Part b.** Let \( 0 < s \leq \frac{1 + \theta - \gamma}{p - r} \). Let us choose \( T_k(u_n) \) as a test function in (3.4); then the second term is non-negative.

Using assumption (1.2) we can write
\[ \int \nabla T_k(u_n)^p dx \leq c k^{1-\gamma} (1 + k)^\theta \leq c (1 + k)^{1+\gamma + \theta}. \]  
(4.23)

By Lemma 3.4 there exists a function \( u \) such that \( T_k(u) \in W_0^{1,p}(\Omega) \). Moreover,
\[ T_k(u_n) \rightarrow T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \forall k > 0 \quad \text{and} \quad u_n \rightarrow u \quad \text{a.e. in } \Omega. \]

Fatou’s Lemma implies that \( |u|^s \in L^1(\Omega) \). We can pass to the limit in (4.23) to get
\[ \int \nabla T_k(u)^p dx \leq c (1 + k)^{1+\gamma - \theta}. \]

As a result of the Lemma 3.3 we obtain \( |\nabla u| \in M^{\frac{1}{1+\theta-\gamma}}(\Omega) \).

By the same method as in part II of Theorem 2.1 we can show that \( u \) is an entropy solution of (1.1).

**Proof of theorem 2.4.** We separate our proof into three parts, according to the values of \( s \).

**Part 1.** Let \( 0 < s \leq \frac{N(1-\gamma)}{N-1} \). It is obvious that \( s \leq \frac{N(1-\gamma)}{N-1} \) implies \( s + \gamma \leq r^* \). Using (4.24), we obtain
\[ \int \nabla u_n^p dx \leq c. \]

Let \( 1 \leq r < p \), let us write
\[ \int \nabla u_n^r dx = \int \frac{|\nabla u_n|^r}{(1 + u_n)^{\frac{r}{p} + \frac{(1+\gamma - \theta)}{r}}} (1 + u_n)^{\frac{r(1+\gamma - \theta)}{p}} dx. \]

By Hölder’s inequality, we have
\[ \int \nabla u_n^r dx \leq c \left( \int (1 + u_n)^{\frac{r}{p} + \frac{(1+\gamma - \theta)}{r}} dx \right)^{\frac{1}{p}}. \]  
(4.24)

Sobolev’s embedding Theorem implies
\[ \left( \int u_n^* dx \right)^{\frac{1}{r}} \leq c \left( \int \nabla u_n^r dx \right)^{\frac{1}{p}}, \quad r^* = \frac{Nr}{N - r}. \]

Therefore,\[ \left( \int |u_n|^* dx \right)^{\frac{1}{p}} \leq c \left( \int (1 + u_n)^{\frac{r}{p} + \frac{(1+\gamma - \theta)}{r}} dx \right)^{\frac{(p-r)r^*}{p^r}}. \]

Suppose \( \frac{r(1+\gamma - \theta)}{p - r} \leq r^* \), ie, \( \frac{(1+\gamma - \theta)}{p - r} \leq \frac{N}{N - r} \), so that
\[ r \leq \frac{N[p - 1 - \theta + \gamma]}{N - 1 - \theta + \gamma}. \]

Then \( p > p_0 \) implies \( \frac{N[p - 1 - \theta + \gamma]}{N - 1 - \theta + \gamma} > 1. \)

We obtain
\[ \int u_n^* dx \leq c \left( \int (1 + u_n)^r dx \right)^{\frac{(p-r)r^*}{p^r}} \leq c + c \left( \int u_n^* dx \right)^{\frac{(p-r)r^*}{p^r}}. \]
From the above inequality, by Young’s inequality with $\varepsilon$, we see that
\[ \int_{\Omega} |u_n|^r \, dx \leq c. \]
Which together with (4.24) and \( \frac{\varepsilon}{p-r}(\theta + 1 - \gamma) \leq r^* \) implies
\[ \int_{\Omega} |\nabla u_n|^r \, dx \leq c, \quad r < \frac{N[p - 1 - \theta + \gamma]}{N - 1 - \theta + \gamma}. \]
Just as in the proof of part I in the Theorem 2.3 we can conclude that $u$ is a distributional solution of (1.1).

**Part 2.** Let
\[ \frac{N(1 - \gamma) + \gamma}{N - 1} < s < \frac{N(p - \theta - 1) + p\gamma}{N - p}. \]
We can show that $u$ is an entropy solution (1.1) by the same method in part b of Theorem 2.3.

**Part 3.** Let \( s \geq \frac{N(p - \theta - 1) + p\gamma}{N - p} \). Since \( p > p_0 \), this implies
\[ \frac{N(p - \theta - 1) + p\gamma}{N - p} > \frac{1 + \theta - \gamma}{p - 1}. \]
Therefore, the proof of this part is the same as the proof of part a in Theorem 2.3.

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