Some results on the Hardy space $H^1_k$
associated with the Dunkl operators

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Abstract

In the present paper, we investigate in Dunkl analysis, the action
of some fundamental operators on the atomic Hardy space $H^1_k$.

Keywords: Dunkl operators, Dunkl transform, atomic Hardy spaces, Riesz transform, Hardy operator.

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1 Introduction

In the classical complex analysis, the Hardy spaces were introduced by Hardy [10]
to characterize boundary values of analytic functions on the unit disk. The one-
dimensional atomic decomposition of the Hardy spaces is due to Coifman [4] and
its higher-dimensional extension to Latter [11]. Using the atomic decomposition
Coifman and Weiss [5] extended the definition of the Hardy spaces to more general
structures.

In this paper, we consider the harmonic analysis on the Euclidean space $\mathbb{R}^d$
associated to the class of rational Dunkl operators $T_i$, $1 \leq i \leq d$. Introduced by
C. F. Dunkl in [6], these operators are commuting differential-difference operators
related to an arbitrary finite reflection group $G$ and a non negative multiplicity
function $k$. If the parameter $k \equiv 0$, then the $T_i$, $1 \leq i \leq d$ are reduced to the
partial derivatives $\frac{\partial}{\partial x_i}$, $1 \leq i \leq d$. Therefore Dunkl analysis can be viewed as a
generalization of classical Fourier analysis (see next section).

Inspired by the definition of usual atomic Hardy spaces, we call $L^2_k$-atom any
function $a$ satisfying:
There exists a ball $B$ of $\mathbb{R}^d$ such that

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i) \( \text{Supp}(a) \subset B. \)

ii) \( \left( \int_{\mathbb{R}^d} |a(x)|^2 d\nu_k(x) \right)^{\frac{1}{2}} \leq (\nu_k(B))^{-\frac{1}{2}}. \)

iii) \( \int_B a(x) d\nu_k(x) = 0, \)

where \( \nu_k \) is the weighted Lebesgue measure associated to the Dunkl operators defined by

\[
d\nu_k(x) := w_k(x) dx \quad \text{with} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.
\]

\( R_+ \) being a positive root system and \( \langle \cdot, \cdot \rangle \) the standard Euclidean scalar product on \( \mathbb{R}^d \) (see next section).

We denote by \( L^p_k(\mathbb{R}^d), 1 \leq p < +\infty \) the space \( L^p(\mathbb{R}^d, d\nu_k(x)) \). We define the atomic Hardy space \( H^1_k(\mathbb{R}^d) \) as the space of all functions \( f \in L^1_k(\mathbb{R}^d) \cap L^2_k(\mathbb{R}^d) \) which can be written

\[
f = \sum_{j=1}^{+\infty} \lambda_j a_j,
\]

where the series converges in \( L^1_k(\mathbb{R}^d) \cap L^2_k(\mathbb{R}^d) \) with \( a_j \) are \( L^2_k \)-atoms and \( \sum_{j=1}^{+\infty} |\lambda_j| < +\infty \). If \( f \in H^1_k(\mathbb{R}^d) \), we define

\[
\|f\|_{H^1_k} = \inf \left\{ \sum_{j=1}^{+\infty} |\lambda_j| : f = \sum_{j=1}^{+\infty} \lambda_j a_j \right\}.
\]

Here the infimum is taken over all atomic decompositions of \( f \).

The aim of this paper is to investigate the action of some fundamental operators on the atomic Hardy space \( H^1_k(\mathbb{R}^d) \). We use \( \| \cdot \|_{p,k} \) as a shorthand for \( \| \cdot \|_{L^p_k(\mathbb{R}^d)} \).

First, we consider the Riesz transforms which are the operators \( \mathcal{R}^k_j, j = 1, \ldots, d \) (see [3, 16]) defined on \( L^2_k(\mathbb{R}^d) \) by

\[
\mathcal{R}^k_j(f)(x) = \lambda_k \lim_{\epsilon \to 0} \int_{\|y\|>\epsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{2\gamma+d+1}} d\nu_k(y), \quad x \in \mathbb{R}^d
\]

where \( \tau_x \) is the Dunkl translation operator and

\[
\lambda_k = 2^{\gamma+\frac{d}{2}} \frac{\Gamma(\gamma + \frac{d+1}{2})}{\sqrt{\pi}}, \quad \text{with} \quad \gamma = \sum_{\xi \in R_+} k(\xi).
\]

We show that \( \mathcal{R}^k_j \) maps \( H^1_k(\mathbb{R}^d) \) to \( L^1_k(\mathbb{R}^d) \) and for \( f \) in \( H^1_k(\mathbb{R}^d) \), we have

\[
\|\mathcal{R}^k_j(f)\|_{1,k} \leq c \|f\|_{H^1_k}.
\]
Some results on the Hardy space $H^1_k$

It was proved in [3] that the Riesz transform $R^k_j$ is of a weak-type (1,1) and it can be extended to a bounded operator from $L^p_k(\mathbb{R}^d)$ into itself for $1 < p < +\infty$:
\[
\|R^k_j(f)\|_{p,k} \leq c\|f\|_{p,k}, \text{ for } f \in L^p_k(\mathbb{R}^d).
\] (1.1)

Second, we prove that for $f$ in $H^1_k(\mathbb{R}^d)$, the Dunkl transform $F_k$ which enjoys properties similar to those of the classical Fourier transform (see next section), satisfies the inequality
\[
\int_{\mathbb{R}^d} \|y\|^{-2\gamma+d}\|F_k(f)(y)\|d\nu_k(y) \leq c\|f\|_{H^1_k}.
\]

The case $1 < p \leq 2$ was shown for $f$ in $L^p_k(\mathbb{R}^d)$ (see [1], Section 4, Lemma 1) and gives the Hardy-Littlewood-Paley inequality
\[
\left( \int_{\mathbb{R}^d} \|x\|^{2\gamma+d(p-2)}\|F_k(f)(x)\|^p d\nu_k(x) \right)^{\frac{1}{p}} \leq c\|f\|_{p,k}.
\] (1.2)

Finally, we establish that for all $f$ in $H^1_k(\mathbb{R}^d)$, we have
\[
\|H_kf\|_{1,k} \leq 2\|f\|_{H^1_k},
\]
where $H_k$ is a Hardy-type averaging operator given by
\[
H_kf(x) = \frac{1}{\nu_k(B_{\|x\|})} \int_{B_{\|x\|}} f(y)d\nu_k(y),
\]
with $B_{\|x\|} = B(0, \|x\|)$ is the ball of radius $\|x\|$ centered at 0. The case $1 < p < +\infty$ was obtained for $f$ in $L^p_k(\mathbb{R}^d)$ (see [2]) and gives the weighted Hardy inequality
\[
\left( \int_{\mathbb{R}^d} \left( \frac{1}{\nu_k(B_{\|x\|})} \int_{B_{\|x\|}} |f(y)| d\nu_k(y) \right)^p d\nu_k(x) \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \|f\|_{p,k}.
\] (1.2)

This is an extension to the Dunkl theory of the results obtained in the classical harmonic analysis (see [9, 10, 14]).

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with the rational Dunkl operators.

In section 3, we investigate in three subsection, the action respectively of the Riesz transforms $R^k_j$, $j = 1, \ldots, d$, the Dunkl transform $F_k$ and the Hardy-type averaging operator $H_k$, on the atomic Hardy spaces $H^1_k(\mathbb{R}^d)$.

Along this paper, we denote $\langle \cdot, \cdot \rangle$ the usual Euclidean inner product in $\mathbb{R}^d$ as well as its extension to $\mathbb{C}^d \times \mathbb{C}^d$, we write for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x \rangle}$ and we use $c$ to represent a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by
- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}^{\prime}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives.
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.
2 Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to [7, 8, 12] and the surveys [13].

Let $G$ be a finite reflection group on $\mathbb{R}^d$, associated with a root system $R$. For $\alpha \in R$, we denote by $H_\alpha$ the hyperplane orthogonal to $\alpha$. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} \mathbb{H}_\alpha$, we fix a positive subsystem $R_+ = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $G$-invariant. We associate with $k$ the index

$$
\gamma = \sum_{\xi \in R_+} k(\xi) \geq 0,
$$

and a weighted measure $\nu_k$ given by

$$
d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.
$$

Note that $(\mathbb{R}^d, d\nu_k(x))$ is a space of homogeneous type, that is, there exists a constant $c > 0$ such that

$$
u_k(B(x, 2r)) \leq c \nu_k(B(x, r)), \quad \forall x \in \mathbb{R}^d, r > 0, \quad (2.1)
$$

where $B(x, r)$ is the closed ball of radius $r$ centered at $x$ (see [14], Ch.1).

For every $1 \leq p \leq +\infty$, we denote by $L^p_k(\mathbb{R}^d)$ the spaces $L^p(\mathbb{R}^d, d\nu_k(x))$, and $L^p_k(\mathbb{R}^d)^{\text{rad}}$ the subspace of those $f \in L^p_k(\mathbb{R}^d)$ that are radial. We use $\| \|_{p,k}$ as a shorthand for $\| \|_{L^p_k(\mathbb{R}^d)}$.

We introduce the Mehta-type constant $c_k$ by

$$
c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x)dx \right)^{-1}.
$$

By using the homogeneity of degree $2\gamma$ of $w_k$, it was shown in [8] that for a function $f$ in $L^1_k(\mathbb{R}^d)^{\text{rad}}$, there exists a function $F$ on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function $F$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$
\int_{\mathbb{R}^d} f(x) d\nu_k(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} f(ry) w_k(ry)d\sigma(y) \right) r^{d-1}dr
= \int_0^{+\infty} \left( \int_{S^{d-1}} w_k(ry)d\sigma(y) \right) F(r)r^{d-1}dr
= d_k \int_0^{+\infty} F(r)r^{2\gamma+d-1}dr, \quad (2.2)
$$

where $S^{d-1}$ is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$
d_k = \int_{S^{d-1}} w_k(x)d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d-1}\Gamma(\gamma + \frac{d}{2})}.
$$
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In particular,

$$\nu_k(B(0, R)) = \frac{d_k}{2\gamma + d} R^{2\gamma + d}. $$

The Dunkl operators $T_j, \ 1 \leq j \leq d,$ on $\mathbb{R}^d$ associated with the reflection group $G$ and the multiplicity function $k$ are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x \rangle}, \ f \in \mathcal{E}(\mathbb{R}^d), \ x \in \mathbb{R}^d,$$

where $\rho_\alpha$ is the reflection on the hyperplane $\mathbb{H}_\alpha$ and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

**Remark 2.1** In the case $k \equiv 0$, the weighted function $\omega_k \equiv 1$ and the measure $\nu_k$ associated to the Dunkl operators coincide with the Lebesgue measure. The $T_j$ are reduced to the corresponding partial derivatives. Therefore, the Dunkl theory can be viewed as a generalization of the classical Fourier analysis.

For $y \in \mathbb{C}^d$, the system

$$\begin{cases}
T_j u(x, y) = y_j u(x, y), & 1 \leq j \leq d, \\
u(0, y) = 1.
\end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x, y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$.

(See [13]) For all $x \in \mathbb{R}^d$, $y \in \mathbb{C}^d$ and all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d_+$,

$$|\partial_y^\alpha E_k(x, y)| \leq \|x\|_{|\alpha|} \max_{g \in G} e^{Re<\alpha, x, y>},$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. In particular, for all $x, y \in \mathbb{R}^d$,

$$|E_k(-ix, y)| \leq 1 \quad \text{and} \quad \left| \frac{\partial}{\partial y_j} E_k(-ix, y) \right| \leq \|x\|, \ \forall \ j = 1, \ldots, d. \quad (2.3)$$

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) d\nu_k(y), \ x \in \mathbb{R}^d.$$

We list some known properties of this transform:
i) If $f$ is in $L^1_k(\mathbb{R}^d)$, then $\mathcal{F}_k(f)$ is in $C_0(\mathbb{R}^d)$ where $C_0(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ which vanish at infinity and we have
\[ \|\mathcal{F}_k(f)\|_{\infty} \leq \|f\|_{1,k}. \]

ii) The Dunkl transform is an automorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

iii) When both $f$ and $\mathcal{F}_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula
\[ f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d. \]

iv) (Plancherel’s theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric automorphism on $L^2_k(\mathbb{R}^d)$.

K. Trimèche has introduced in [17] the Dunkl translation operator $\tau_x$, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have
\[ \mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y) \mathcal{F}_k(f)(y). \]
Notice that for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$,
\[ \tau_x \text{ is a continuous linear mapping from } \mathcal{E}(\mathbb{R}^d) \text{ into } \mathcal{E}(\mathbb{R}^d). \]

As an operator on $L^2_k(\mathbb{R}^d)$, $\tau_x$ is bounded. According to ([15], Theorem 3.7), the operator $\tau_x$ can be extended to the space of radial functions $L^p_k(\mathbb{R}^d)^{rad}$, $1 \leq p \leq 2$ and we have for a function $f$ in $L^p_k(\mathbb{R}^d)^{rad}$,
\[ \|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}. \]

(see [16]) The Riesz transforms $\mathcal{R}^k_j$, $j = 1...d$, are multiplier operators given by
\[ \mathcal{F}_k(\mathcal{R}^k_j(f))(\xi) = \frac{-i\xi_j}{\|\xi\|} \mathcal{F}_k(f)(\xi), \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d). \quad (2.4) \]

It was proved in [3] that for $f \in L^2_k(\mathbb{R}^d)$ with compact support and for all $x \in \mathbb{R}^d$ such that $g. x \notin supp(f)$, $\forall g \in G$, we have
\[ \mathcal{R}^k_j(f)(x) = \int_{\mathbb{R}^d} K_j(x, y) f(y) d\nu_k(y), \quad (2.5) \]
where the kernel $K_j$, $j = 1...d$, satisfy
\[ \int_{\min g. x - y > 2\|y - y_0\|} |K_j(x, y) - K_j(x, y_0)| d\nu_k(x) \leq c, \quad \text{for } y, y_0 \in \mathbb{R}^d. \quad (2.6) \]

Using the Calderón-Zygmund decomposition, the authors in [3] showed that the Riesz transform $\mathcal{R}^k_j$ is of a weak-type (1,1) and it can be extended to a bounded operator from $L^p_k(\mathbb{R}^d)$ into itself for $1 < p < +\infty$. 
Some results on the Hardy space $H_k^1$

We begin with a remark and an example.

**Remark 3.1** By the definition given in the introduction, if $a$ is an $L^2_{k}$-atom, then

$$
\|a\|_{1,k} = \int_B |a(y)| d\nu_k(y) \leq (\nu_k(B))^{\frac{1}{2}} \|a\|_{2,k} \leq 1.
$$

**Example 3.1** Fix $h \in L^2_{k}(\mathbb{R}^d)$ with $\|h\|_{2,k} = 1$ and let $(B_j)_{j \in \mathbb{N}}$ be a family of balls such that

$$
B_j \subset C_j = \left\{ x \in \mathbb{R}^d; 4^{-\frac{j+1}{2\gamma+d}} \leq \|x\| \leq 4^{-\frac{j}{2\gamma+d}} \right\}.
$$

The average of $h$ over $B_j$ is given by

$$
\text{Avgh} = \frac{1}{\nu_k(B_j)} \int_{B_j} h(x) d\nu_k(x).
$$

Put $a_j = (\nu_k(B_j))^{-\frac{1}{2}} \left( h - \text{Avgh} \right) \chi_{B_j}$, then each $a_j$ is an $L^2_{k}$-atom and the function $f = \sum_{j=1}^{+\infty} (\nu_k(B_j))^{\frac{1}{2}} a_j$ is in $H_k^1(\mathbb{R}^d)$ with

$$
\|f\|_{H_k^1} \leq \sum_{j=1}^{+\infty} (\nu_k(B_j))^{\frac{1}{2}} \leq \frac{\sqrt{d_k}}{\sqrt{2\gamma + d}}.
$$

Indeed, $a_j$ satisfies $\text{supp}(a_j) \subset B_j$, $\int_{B_j} a_j(x) d\nu_k(x) = 0$ and

$$
\|a_j\|_{2,k}^2 = (\nu_k(B_j))^{-1} \int_{B_j} \left| h(x) - \text{Avgh} \right|^2 d\nu_k(x)
$$

$$
= (\nu_k(B_j))^{-1} \int_{B_j} (h(x))^2 d\nu_k(x) + (\nu_k(B_j))(\text{Avgh})^2 - 2\text{Avgh} \int_{B_j} h(x) d\nu_k(x)
$$

$$
\leq (\nu_k(B_j))^{-1} \int_{B_j} (h(x))^2 d\nu_k(x) - (\nu_k(B_j))(\text{Avgh})^2
$$

Then we obtain,

$$
\|f\|_{H_k^1} \leq \sum_{j=1}^{+\infty} (\nu_k(B_j))^{\frac{1}{2}} \leq \sum_{j=1}^{+\infty} \left[ \nu_k(B(0, 4^{-\frac{j+1}{2\gamma+d}})) \right]^{\frac{1}{2}}
$$

$$
= \frac{\sqrt{d_k}}{\sqrt{2\gamma + d}} \sum_{j=1}^{+\infty} 2^{-j} = \frac{\sqrt{d_k}}{\sqrt{2\gamma + d}}.
$$
3.1 Riesz transform on the Hardy space

In this section, we prove that the Riesz transforms are bounded operators from $H^1_1(\mathbb{R}^d)$ to $L^1_1(\mathbb{R}^d)$. Before, we need to prove a useful lemma.

**Lemma 3.1** There exists a constant $c > 0$ such that for all $L^2_1$-atom $a$, we have

$$
\|R_j(a)\|_{1,k} \leq c, \quad 1 \leq j \leq d.
$$

**Proof.** Let $a$ be an $L^2_1$-atom supported in $B = B(y_0, r), y_0 \in \mathbb{R}^d$. If we put $B^* = B(y_0, 4r)$ and $Q = \bigcup_{g \in G} g.B^*$, then we can write

$$
\int_{\mathbb{R}^d} |R_j^k(a)(x)|d\nu_k(x) = \int_Q |R_j^k(a)(x)|d\nu_k(x) + \int_{Q^c} |R_j^k(a)(x)|d\nu_k(x).
$$

From (1.1), the Riesz transform $R_j^k$ is of strong-type $(2, 2)$, then using Hölder’s inequality, we obtain

$$
\int_Q |R_j^k(a)(x)|d\nu_k(x) \leq (\nu_k(Q))^{\frac{1}{2}} \left( \int_Q |R_j^k(a)(x)|^2d\nu_k(x) \right)^{\frac{1}{2}}
$$

$$
\leq c(\nu_k(Q))^{\frac{1}{2}} \left( \int_B |a(x)|^2d\nu_k(x) \right)^{\frac{1}{2}}
$$

$$
\leq c(\nu_k(Q))^{\frac{1}{2}}(\nu_k(B))^{-\frac{1}{2}}.
$$

From (2.1) we have, $\nu_k(Q) \leq \sum_{g \in G} \nu_k(g.B^*) \leq c|G| \nu_k(B)$, where $|G|$ is the order of $G$. This gives that

$$
\int_Q |R_j^k(a)(x)|d\nu_k(x) \leq c. \quad (3.1)
$$

Now, if $x \notin Q$ then $\min_{g \in G} \|g.x - y\| > 2\|y - y_0\|, \forall y \in B$. Using the integral representation (2.5) for the Riesz transform $R_j^k$ and with the condition iii) for an $L^2_1$-atom, we have

$$
\int_{Q^c} |R_j^k(a)(x)|d\nu_k(x) = \int_{Q^c} \left| \int_B \mathcal{K}_j(x, y)a(y)d\nu_k(y) \right|d\nu_k(x)
$$

$$
= \int_{Q^c} \left| \int_B \left( \mathcal{K}_j(x, y) - \mathcal{K}_j(x, y_0) \right)a(y)d\nu_k(y) \right|d\nu_k(x)
$$

$$
\leq \int_B \int_{Q^c} \left| \mathcal{K}_j(x, y) - \mathcal{K}_j(x, y_0) \right|d\nu_k(x)|a(y)|d\nu_k(y).
$$

Since $y \in B$, we obtain from (2.6)

$$
\int_{Q^c} \left| \mathcal{K}_j(x, y) - \mathcal{K}_j(x, y_0) \right|d\nu_k(x)
$$

$$
\leq \int_{\min_{g \in G} \|g.x - y\| > 2\|y - y_0\|} \left| \mathcal{K}_j(x, y) - \mathcal{K}_j(x, y_0) \right|d\nu_k(x) \leq c.
$$
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and by the remark 3.1, we deduce that

$$\int_{Q_c} |\mathcal{R}_j^k(a)(x)||dv_k(x)\leq c\int_B |a(y)||dv_k(y)\leq c. \tag{3.2}$$

Combining (3.1) and (3.2), we conclude that

$$\|\mathcal{R}_j^k(a)\|_{1,k}\leq c.$$  

□

Theorem 3.1 The Riesz transforms $\mathcal{R}_j^k$, $1 \leq j \leq d$ map $H^1_k(\mathbb{R}^d)$ to $L^1_k(\mathbb{R}^d)$ and we have for all $f \in H^1_k(\mathbb{R}^d)$,

$$\|\mathcal{R}_j^k(f)\|_{1,k}\leq c\|f\|_{H^1_k}.$$ 

Proof. Let $f \in H^1_k(\mathbb{R}^d)$, $f = \sum_{i=1}^{+\infty} \lambda_i a_i$. For $\rho > 0$ and $1 \leq j \leq d$, we have

$$\nu_k\left(\left\{\mathcal{R}_j^k(f) - \sum_{i=1}^{+\infty} \lambda_i \mathcal{R}_j^k(a_i) \mid > \rho\right\}\right) \leq \nu_k\left(\left\{\left|\mathcal{R}_j^k(f) - \sum_{i=1}^{N} \lambda_i \mathcal{R}_j^k(a_i)\right| > \frac{\rho}{2}\right\}\right)$$

$$+ \nu_k\left(\left\{\left|\sum_{i=N+1}^{+\infty} \lambda_i \mathcal{R}_j^k(a_i)\right| > \frac{\rho}{2}\right\}\right). \tag{3.3}$$

Since $\mathcal{R}_j^k$ is of weak-type $(1,1)$, we assert that

$$\nu_k\left(\left\{\left|\mathcal{R}_j^k(f) - \sum_{i=1}^{N} \lambda_i \mathcal{R}_j^k(a_i)\right| > \frac{\rho}{2}\right\}\right) \leq \frac{2}{\rho} \left\|f - \sum_{i=1}^{N} \lambda_i a_i\right\|_{1,k},$$

and clearly

$$\nu_k\left(\left\{\left|\sum_{i=N+1}^{+\infty} \lambda_i \mathcal{R}_j^k(a_i)\right| > \frac{\rho}{2}\right\}\right) \leq \frac{2}{\rho} \left\|\sum_{i=N+1}^{+\infty} \lambda_i \mathcal{R}_j^k(a_i)\right\|_{1,k} \leq \frac{2}{\rho} c \sum_{i=N+1}^{+\infty} |\lambda_i|.$$ 

Now using the fact that $\sum_{i=1}^{N} \lambda_i a_i$ converges to $f$ in $L^1_k(\mathbb{R}^d)$ and $\sum_{i=1}^{+\infty} |\lambda_i| < +\infty$, we have both terms in the sum in (3.3) converge to zero as $N \to +\infty$. Hence, we deduce that

$$\nu_k\left(\left\{\mathcal{R}_j^k(f) - \sum_{i=1}^{+\infty} \lambda_i \mathcal{R}_j^k(a_i) > \rho\right\}\right) = 0, \text{ for all } \rho > 0,$$
which gives that $R^k_j(f) = \sum_{i=1}^{+\infty} \lambda_i R^k_j(a_i)$ a.e. Using the lemma 3.1, we can assert that

$$\|R^k_j(f)\|_{1,k} \leq \sum_{i=1}^{+\infty} |\lambda_i| \|R^k_j(a_i)\|_{1,k} \leq c \sum_{i=1}^{+\infty} |\lambda_i|,$$

which gives $\|R^k_j(f)\|_{1,k} \leq c \|f\|_{H^1_k}$. This proves our result. \(\Box\)

**Corollary 3.1** If $f \in H^1_k(\mathbb{R}^d)$ then $\int_{\mathbb{R}^d} f(x) dv_k(x) = 0$.

**Proof.** By Theorem 3.1, if $f \in H^1_k(\mathbb{R}^d)$ then $R^k_j(f) \in L^1_k(\mathbb{R}^d)$, $1 \leq j \leq d$, then $F_k(R^k_j(f))$ is in $C_0(\mathbb{R}^d)$. From (2.3), it follows that $F_k(R^k_j(f))$ is continuous at zero if and only if $F_k(f)(0) = 0$, which gives that $\int_{\mathbb{R}^d} f(x) dv_k(x) = 0$. \(\Box\)

### 3.2 Dunkl transform on the Hardy space

In this section, we study the Dunkl transform on the space $H^1_k(\mathbb{R}^d)$. Before, we need to prove a useful result.

**Lemma 3.2** There exists a positive constant $c > 0$ such that for all $L^2_k$-atom a supported in the ball $B = B(0, R)$, and all $y \in \mathbb{R}^d$, we have

$$|F_k(a)(y)| \leq c R \|y\|.$$

**Proof.** By virtue of the condition iii) for an $L^2_k$-atom, we have for $y \in \mathbb{R}^d$

$$F_k(a)(y) = c_k \int_B [E_k(-ix, y) - 1] a(x) dv_k(x).$$

Using mean value theorem and (2.3), we get for $x \in B$

$$|E_k(-ix, y) - 1| = \left| \sum_{j=1}^{d} y_j \int_0^1 \frac{\partial E_k}{\partial y_j}(-ix, ty) \, dt \right|$$

$$\leq \sum_{j=1}^{d} |y_j| \|x\|$$

$$\leq \sqrt{d} \|y\| \|x\| \leq \sqrt{d} R \|y\|.$$

According to the remark 3.1, we can assert that

$$|F_k(a)(y)| \leq c_k \sqrt{d} R \|y\| \|a\|_{1,k} \leq c R \|y\|,$$

which proves the result. \(\Box\)
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Theorem 3.2 Let $f \in H^1_k(\mathbb{R}^d)$, then we have

$$
\int_{\mathbb{R}^d} \|y\|^{-(2\gamma+d)} |\mathcal{F}_k(f)(y)| \, dv_k(y) \leq c \|f\|_{H^1_k}.
$$

Proof. Let $a$ be an $L^2_k$-atom supported in the ball $B = B(0, R)$. We write

$$
\int_{\mathbb{R}^d} |\mathcal{F}_k(a)(y)| \frac{dv_k(y)}{\|y\|^{2\gamma+d}} = \int_{\|y\| \leq R} |\mathcal{F}_k(a)(y)| \frac{dv_k(y)}{\|y\|^{2\gamma+d}} + \int_{\|y\| > R} |\mathcal{F}_k(a)(y)| \frac{dv_k(y)}{\|y\|^{2\gamma+d}}
$$

Using the lemma 3.2 and (2.2), we can estimate $I_1$ as follows

$$
I_1 \leq c R \int_{\|y\| \leq R} \|y\|^{-(2\gamma+d-1)} dv_k(y)
$$

$$
= c R d_k \int_{0}^{R} r^{-(2\gamma+d-1)} r^{2\gamma+d-1} \, dr
$$

$$
= c d_k.
$$

(3.4)

To estimate $I_2$, we use Hölder’s inequality, Plancherel’s theorem for the Dunkl transform $\mathcal{F}_k$, and (2.2)

$$
I_2 \leq \left( \int_{\|y\| > R} |\mathcal{F}_k(a)(y)|^2 dv_k(y) \right)^{\frac{1}{2}} \left( \int_{\|y\| > R} \|y\|^{-2(2\gamma+d)} dv_k(y) \right)^{\frac{1}{2}}
$$

$$
\leq \|a\|_{2,k} \sqrt{d_k} \left( R^{+\infty} r^{-2(2\gamma+d)} r^{2\gamma+d-1} \right)^{\frac{1}{2}} dr
$$

$$
\leq (\nu_k(B))^{\frac{1}{2}} \frac{\sqrt{d_k}}{\sqrt{2\gamma+d}} \left( \frac{1}{R} \right)^{-\frac{2\gamma+d}{2}}
$$

$$
\leq \frac{\sqrt{2\gamma+d}}{\sqrt{d_k}} R^{-\frac{2\gamma+d}{2}} \frac{\sqrt{d_k}}{\sqrt{2\gamma+d}} \left( \frac{1}{R} \right)^{-\frac{2\gamma+d}{2}}
$$

$$
= 1.
$$

(3.5)

Hence, from (3.3) and (3.4), we obtain

$$
\int_{\mathbb{R}^d} |\mathcal{F}_k(a(y))| \frac{dv_k(y)}{\|y\|^{2\gamma+d}} \leq c.
$$

(3.6)

Since $\mathcal{F}_k$ is a continuous linear mapping from $L^1_k(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$, we deduce that for $f \in H^1_k(\mathbb{R}^d)$, $f = \sum_{j=1}^{+\infty} \lambda_j a_j$, we have $\mathcal{F}_k(f)(y) = \sum_{j=1}^{+\infty} \lambda_j \mathcal{F}_k(a_j)(y)$ a.e. $y \in \mathbb{R}^d$.

Using (3.6), this gives

$$
\int_{\mathbb{R}^d} \|y\|^{-(2\gamma+d)} |\mathcal{F}_k(f(y))| \, dv_k(y) \leq \sum_{j=1}^{+\infty} |\lambda_j| \int_{\mathbb{R}^d} |\mathcal{F}_k(a_j)(y)| \frac{dv_k(y)}{\|y\|^{2\gamma+d}}
$$

$$
\leq c \sum_{j=1}^{+\infty} |\lambda_j|,
$$
thus the proof is completed. \qed

3.3 Hardy inequality for $H^1_k(\mathbb{R}^d)$

In this section, we give a Hardy-type inequality on $H^1_k(\mathbb{R}^d)$. Recall that the Hardy-type averaging operator $H_k$ is given by

$$H_k f(x) = \frac{1}{\nu_k(B_{||x||})} \int_{B_{||x||}} f(y) d\nu_k(y) \quad \text{with} \quad B_{||x||} = B(0, ||x||).$$

In order to establish the result, we need to show the following lemma.

**Lemma 3.3** For every $L^2_k$-atom $a$, we have

$$\|H_k a\|_{1,k} \leq 2.$$

**Proof.** Let $a$ be an $L^2_k$-atom on $\mathbb{R}^d$, supported in the ball $B = B(0, R)$, then $\text{supp}(H_k a) \subset B$. Indeed, if $||x|| > R$ then $B_{||x||} \cap B = B$ and $H_k a = 0$. We have,

\[
\|H_k a\|_{1,k} \leq \int_B \left( \frac{1}{\nu_k(B_{||x||})} \int_{B_{||x||}} |a(y)|^2 d\nu_k(y) \right)^{\frac{1}{2}} d\nu_k(x).
\]

By the property ii) of $a$ and (2.2), we deduce that

\[
\|H_k a\|_{1,k} \leq \left( \frac{d_k}{2\gamma + d} R^{2\gamma + d} \int_B \left( \frac{d_k}{2\gamma + d} ||x||^{2\gamma + d} \right)^{-\frac{1}{2}} d\nu_k(x) \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{d_k}{2\gamma + d} \right)^{-1} R^{-\frac{2\gamma + d}{2}} d_k \int_0^R r^{-\frac{2\gamma + d}{2}} r^{2\gamma + d - 1} dr
\]

\[
= (2\gamma + d) R^{-\frac{2\gamma + d}{2}} \int_0^R r^{-\frac{2\gamma + d}{2} - 1} dr = 2.
\]

\[\square\]

**Remark 3.2** Let $FH^1_k(\mathbb{R}^d)$ be the set of all finite linear combinations of $L^2_k$-atoms. Using the linearity of $H_k$ and the lemma 3.3, it’s easy to see that if $f \in FH^1_k(\mathbb{R}^d)$,

$$f = \sum_{j=1}^N \lambda_j a_j,$$

then $H_k f = \sum_{j=1}^N \lambda_j H_k a_j$ and $\|H_k f\|_{1,k} \leq \sum_{j=1}^N |\lambda_j|$, which implies that $\|H_k f\|_{1,k} \leq 2 \|f\|_{H^1_k}$. 

**Theorem 3.3** For all $f \in H^1_k(\mathbb{R}^d)$, we have

$$\|H_k f\|_{1,k} \leq 2 \|f\|_{H^1_k}.$$
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**Proof.** For $f \in H^1_k(\mathbb{R}^d)$, $f = \sum_{j=1}^{+\infty} \lambda_j a_j$ and a positive integer $N$, we have from the linearity of $\mathcal{H}_k$ and (1.2)

$$\|\mathcal{H}_k f - \sum_{j=1}^{N} \lambda_j \mathcal{H}_k a_j\|_{2,k} = \|\mathcal{H}_k (f - \sum_{j=1}^{N} \lambda_j a_j)\|_{2,k} \leq 2 \|f - \sum_{j=1}^{N} \lambda_j a_j\|_{2,k},$$

which converges to zero as $N \to +\infty$. This implies that $\mathcal{H}_k f = \sum_{j=1}^{+\infty} \lambda_j \mathcal{H}_k a_j$, a.e.

Finally, using Lemma 3.3, we can assert that

$$\|\mathcal{H}_k f\|_{1,k} \leq \sum_{j=1}^{+\infty} |\lambda_j| \|\mathcal{H}_k a_j\|_{1,k} \leq 2 \sum_{j=1}^{+\infty} |\lambda_j|,$$

which gives $\|\mathcal{H}_k f\|_{1,k} \leq 2 \|f\|_{H^1_k}$. This completes the proof. \qed

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