THE RADIUS OF CONVEXITY OF NORMALIZED BESSEL FUNCTIONS OF THE FIRST KIND

ÁRPÁD BARICZ AND RÓBERT SZÁSZ

ABSTRACT. In this paper we determine the radius of convexity for three kind of normalized Bessel functions of the first kind. In the mentioned cases the normalized Bessel functions are starlike-univalent and convex-univalent, respectively, on the determined disks. The key tools in the proofs of the main results are some new Mittag-Leffler expansions for quotients of Bessel functions of the first kind, special properties of the zeros of Bessel functions of the first kind and their derivative, and the fact that the smallest positive zeros of some Dini functions are less than the first positive zero of the Bessel function of the first kind. Moreover, we find the optimal parameters for which these normalized Bessel functions are convex in the open unit disk. In addition, we disprove a conjecture of Baricz and Ponnusamy concerning the convexity of the Bessel function of the first kind.

1. Introduction and Main Results

Let $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote the open disk centered in $z_0$ and of radius $r > 0$, and let $\mathcal{S}$ be the class of analytic and univalent functions defined in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and having the property that $f(0) = f'(0) - 1 = 0$. Recall that a function $f \in \mathcal{S}$ belongs to the class $\mathcal{K}$ of convex functions if maps the unit disk conformally onto $f(D)$, which is a convex domain in $\mathbb{C}$, that is, the domain $f(D) \subset \mathbb{C}$ contains the entire line segment joining any pair of its points. It is well-known that the class of convex functions can be characterized as

$$\mathcal{K} = \left\{ f \in \mathcal{S} \left| \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in D \right. \right\}.$$

Moreover, for $\alpha \in [0, 1)$ we consider also the class of convex functions of order $\alpha$ defined by

$$\mathcal{K}^{(\alpha)} = \left\{ f \in \mathcal{S} \left| \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in D \right. \right\}.$$

Now let us consider the radius of convexity, and the radius of convexity of order $\alpha$ of the function $f$

$$r^c(f) = \sup \left\{ r \in (0, \infty) \left| \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in D(0, r) \right. \right\}$$

and

$$r^{c(\alpha)}(f) = \sup \left\{ r \in (0, \infty) \left| \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in D(0, r) \right. \right\}.$$

We note that $r^{c}(f)$ is in fact the largest radius for which the image domain $f(D(0, r^{c}(f)))$ is a convex domain in $\mathbb{C}$.

The Bessel function of the first kind of order $\nu$ is defined by \cite{TS} p. 217

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left( \frac{z}{2} \right)^{2n + \nu}.$$

In this paper we focus on the following normalized forms

$$f_\nu(z) = (2^{\nu} \Gamma(\nu + 1)J_\nu(z))^{1/2} = z - \frac{1}{4 \nu (\nu + 1)} z^3 + \ldots, \ \nu \neq 0,$$
\[ g_\nu(z) = 2^\nu \Gamma(\nu + 1)z^{1-\nu}J_\nu(z) = z - \frac{1}{4(\nu + 1)}z^3 + \frac{1}{32(\nu + 1)(\nu + 2)}z^5 - \ldots, \]

\[ h_\nu(z) = 2^\nu \Gamma(\nu + 1)z^{1-\frac{\nu}{2}}J_\nu(\sqrt{z}) = z - \frac{1}{4(\nu + 1)}z^2 + \ldots, \]

where \( \nu > -1 \). We note that

\[ f_\nu(z) = \exp \left( \frac{1}{\nu} \log \left( 2^\nu \Gamma(\nu + 1)J_\nu(z) \right) \right), \]

where Log represents the principal branch of the logarithm, and in this paper every multi-valued function is taken with the principal branch. We also mention that the univalency, starlikeness and convexity of other functions involving the Bessel function of the first kind were studied extensively in several papers. We refer to [2, 3, 4, 5, 7, 9, 14, 21, 22] and to the references therein.

In this paper we make a further contribution to the subject by showing the following new sharp results. For related results the interested reader is referred to [10, 11, 16, 19, 24] and to the references therein, and for more details we refer to [5]. Our approach is completely different than of Brown [9, 10, 11], Nehari [17] and Robertson [19]. The key tools in the proofs of our main results are some new Mittag-Leffler type expansions for Bessel functions of the first kind, properties of the zeros of Bessel functions, and the fact that the smallest positive zeros of some Dini function are less than the first positive zero of the Bessel function of the first kind.

Now, we are going to present some other sharp results on the functions \( f_\nu, g_\nu \) and \( h_\nu \). The proofs of these results can be found in section 3.

Theorem 1. If \( \nu > 0 \) and \( \alpha \in [0, 1) \), then the radius of convexity of order \( \alpha \) of the function \( f_\nu \) is the smallest positive root of the equation

\[ 1 + rJ_\nu'(r) + \left( \frac{1}{\nu} - 1 \right) \frac{rJ_\nu'(r)}{J_\nu(r)} = \alpha. \]

Moreover, \( r_0^\nu(f_\nu) < j_\nu^\nu, j_\nu^\nu, j_\nu^\nu, \) where \( j_\nu^\nu, j_\nu^\nu, j_\nu^\nu, \) denote the first positive zeros of \( J_\nu, J_\nu, J_\nu, \) respectively.

Theorem 2. If \( \nu > -1 \) and \( \alpha \in [0, 1) \), then the radius of convexity of order \( \alpha \) of the function \( g_\nu \) is the smallest positive root of the equation

\[ 1 + rJ_{\nu+2}(r) - 3J_{\nu+1}(r) \]

\[ rJ_{\nu}(r) - rJ_{\nu+1}(r) = \alpha. \]

Moreover, we have \( r_0^\nu(g_\nu) < \alpha_\nu, \alpha_\nu, \alpha_\nu, \) where \( \alpha_\nu, \alpha_\nu, \alpha_\nu, \) is the first positive zero of the Dini function \( z \mapsto (1-\nu)J_\nu(z) + zJ_\nu'(z). \)

Theorem 3. If \( \nu > -1 \) and \( \alpha \in [0, 1) \), then the radius of convexity of order \( \alpha \) of the function \( h_\nu \) is the smallest positive root of the equation

\[ 1 + rJ_{\nu+2}(r) - 4J_{\nu+1}(r) \]

\[ \frac{rJ_{\nu}(r)}{2} - rJ_{\nu+1}(r) = \alpha. \]

Moreover, we have \( r_0^\nu(h_\nu) < \beta_\nu, \beta_\nu, \beta_\nu, \) where \( \beta_\nu, \beta_\nu, \beta_\nu, \) is the first positive zero of the Dini function \( z \mapsto (2-\nu)J_\nu(z) + zJ_\nu'(z). \)

The real number

\[ r^*(f) = \sup \left\{ r \in (0, \infty) \left| \Re \left( \frac{z^{f'(z)}}{f(z)} \right) > 0, \quad z \in \mathbb{D}(0, r) \right. \right\} \]

is called the radius of starlikeness of the function \( f \) and it is the largest radius such that \( f(\mathbb{D}(0, r^*(f))) \) is a starlike domain with respect to 0. It is important to mention here that the problem on the radius of starlikeness of the functions \( f_\nu \) and \( g_\nu \) was first studied by Brown [9] and Robertson [19]. The key tool in Brown’s proofs was the fact that the Bessel function of the first kind is a particular solution of the Bessel differential equation. For related results the interested reader is referred to [10, 11, 16, 19, 24] and to the references therein, and for more details we refer to [5]. Our approach is completely different than of Brown [9, 10, 11], Nehari [17] and Robertson [19]. The key tools in the proofs of our main results are some new Mittag-Leffler type expansions for Bessel functions of the first kind, properties of the zeros of Bessel functions, and the fact that the smallest positive zeros of some Dini function are less than the first positive zero of the Bessel function of the first kind.
Moreover, in particular, the function \( h \) is the unique root of the equation
\[
\nu(\nu^2 - 1) J_\nu^2(1) + (1 - \nu)(J_\nu'(1))^2 = \alpha \nu J_\nu(1) J_\nu'(1),
\]
situated in \((\nu^*, \infty)\), where \( \nu^* \approx 0.3901 \ldots \) is the root of the equation \( J'_\nu(1) = 0 \). Moreover, \( f_\nu \) is convex in \( \mathbb{D} \) if and only if \( \nu \geq 1 \).

**Theorem 5.** The function \( g_\nu \) is convex of order \( \alpha \in [0, 1) \) in \( \mathbb{D} \) if and only if \( \nu \geq \nu_0(g_\nu) \), where \( \nu_\alpha(g_\nu) \) is the unique root of the equation
\[
(2\nu + \alpha - 2) J_{\nu+1}(1) = \alpha J_\nu(1),
\]
situated in \([0, \infty)\). In particular, \( g_\nu \) is convex in \( \mathbb{D} \) if and only if \( \nu \geq 1 \).

**Theorem 6.** The function \( h_\nu \) is convex of order \( \alpha \in [0, 1) \) in \( \mathbb{D} \) if and only if \( \nu \geq \nu_0(h_\nu) \), where \( \nu_\alpha(h_\nu) \) is the unique root of the equation
\[
(2\nu + 2\alpha - 4) J_{\nu+1}(1) = (4\alpha - 3) J_\nu(1),
\]
situated in \([0, \infty)\). In particular, \( h_\nu \) is convex if and only if \( \nu \geq \nu_0(h_\nu) \), where \( \nu_0(h_\nu) \approx -0.1438 \ldots \) is the unique root of the equation
\[
(2\nu - 4) J_{\nu+1}(1) + 3J_\nu(1) = 0.
\]
Moreover, in particular, the function \( h_\nu \) is convex of order \( \frac{4}{\pi} \) if and only if \( \nu \geq \frac{\pi}{4} \).

We note that the convex functions does not need to be normalized. In other words, the analytic and univalent function \( f : \mathbb{D} \to \mathbb{C} \) satisfying \( f'(0) \neq 0 \) is said to be convex of order \( \alpha \in [0, 1) \) if and only if
\[
\text{Re} \left( 1 + z \frac{f''(z)}{f'(z)} \right) > \alpha
\]
for all \( z \in \mathbb{D} \). In 1995 Selinger [20] by using the method of differential subordinations proved that the function \( \varphi_\nu : \mathbb{D} \to \mathbb{C} \), defined by
\[
\varphi_\nu(z) = \frac{h_\nu(z)}{z} = 2^\nu \Gamma(\nu + 1) z^{-\frac{\nu}{2}} J_\nu(\sqrt{z}) = 1 - \frac{1}{4(\nu + 1)} z + \ldots,
\]
is convex if \( \nu \geq -\frac{1}{4} \). In 2009 Szász and Kupán [22], by using a completely different approach, improved this result, and proved that \( \varphi_\nu \) is convex in \( \mathbb{D} \) if \( \nu \geq \nu_1 \approx -1.4069 \ldots \), where \( \nu_1 \) is the root of the equation \( 4
\nu^2 + 17\nu + 16 = 0 \). Recently, Baricz and Ponnusamy [7] presented four improvements of the above result, and their best result was the following [7, Theorem 3]: the function \( \varphi_\nu \) is convex in \( \mathbb{D} \) if \( \nu \geq \nu_2 \approx -1.4373 \ldots \), where \( \nu_2 \) is the unique root of the equation \( 2^\nu \Gamma(\nu + 1)(J_{\nu+2}(1) + 2J_{\nu+1}(1)) = 2 \). Moreover, Baricz and Ponnusamy [7] conjectured that \( \varphi_\nu \) is convex in \( \mathbb{D} \) if and only if \( \nu \geq -1.875 \). Now, we are able to disprove this conjecture and to find the radius of convexity of the function \( \varphi_\nu \).

**Theorem 7.** If \( \nu > -2 \) and \( \alpha \in [0, 1) \), then the radius of convexity of order \( \alpha \) of the function \( \varphi_\nu \) is the smallest positive root of the equation
\[
r^\frac{1}{\nu+1} J_{\nu} \left( r^\frac{1}{\nu+1} \right) = \nu.
\]
Moreover, we have \( r_\nu^\alpha(\varphi_\nu) < j_{\nu+1,1} \).

**Theorem 8.** The function \( \varphi_\nu \) is convex of order \( \alpha \in [0, 1) \) in \( \mathbb{D} \) if and only if \( \nu \geq \nu_\alpha(\varphi_\nu) \), where \( \nu_\alpha(\varphi_\nu) \) is the unique root of the equation
\[
(2\nu + 2\alpha) J_{\nu+1}(1) = J_\nu(1),
\]
situated in \((\nu^*, \infty)\), where \( \nu^* \approx -1.7744 \ldots \) is the root of the equation \( J_{\nu+1}(1) = 0 \). In particular, \( \varphi_\nu \) is convex in \( \mathbb{D} \) if and only if \( \nu \geq \nu_0(\varphi_\nu) \), where \( \nu_0(\varphi_\nu) \approx -1.5623 \ldots \) is the unique root of the equation \( J_\nu(1) = 2\nu J_{\nu+1}(1) \), situated in \((\nu^*, \infty)\).
2. Preliminary Results

This section is devoted to present some preliminary results, which will be used to prove the main theorems. Some of these preliminary results are well-known, however, Lemma 4 and 5 are quite new, and may be of independent interest.

Lemma 1. If \( a > b > 0 \), \( z \in \mathbb{C} \) and \( \lambda \in [0, 1] \), then for all \( |z| < b \) we have
\[
\lambda \Re \left( \frac{z}{a - z} \right) - \Re \left( \frac{z}{b - z} \right) \geq \lambda \frac{|z|}{a - |z|} - \frac{|z|}{b - |z|}. \tag{2.1}
\]

Proof. Let us consider the function \( u : [b, \infty) \to \mathbb{R} \), defined by
\[
u \mapsto -\frac{m - m^2}{(t^2 - 2tx + m^2)} - \frac{m}{t - m}.
\]

Simple computations lead to
\[
u' = \frac{2tm^2 - t^2x - xm^2}{(t^2 - 2tx + m^2)^2} + \frac{m}{(t - m)^2},
\]
where \( z = x + iy \) and \( |z| = m \). Since for \( t \geq b \) we have
\[
u'(t) \geq \frac{2tm^2 - t^2x - xm^2}{(t^2 - 2tx + m^2)^2} + \frac{m}{t^2 - 2tx + m^2} = \frac{(m - x)(t + m)^2}{(t^2 - 2tx + m^2)^2} > 0,
\]
it follows that \( u \) is an increasing function, and consequently we get \( u(a) \geq u(b) \), which is equivalent to
\[
\Re \left( \frac{z}{a - z} \right) - \frac{|z|}{a - |z|} \geq \Re \left( \frac{z}{b - z} \right) - \frac{|z|}{b - |z|}. \tag{2.2}
\]

On the other hand, it is known (see [21]) that if \( z \in \mathbb{C} \) and \( \mu \in \mathbb{R} \) such that \( \mu > |z| \), then
\[
\frac{|z|}{\mu - |z|} \geq \Re \left( \frac{z}{\mu - z} \right). \tag{2.3}
\]

This in turn implies that
\[
\lambda \left( \Re \left( \frac{z}{a - z} \right) - \frac{|z|}{a - |z|} \right) \geq \Re \left( \frac{z}{a - z} \right) - \frac{|z|}{a - |z|},
\]
and combining this with (2.2) we get (2.1).

Lemma 2. [23] p. 482 If \( \nu > -1 \) and \( a, b \in \mathbb{R} \), then \( z \mapsto aJ_{\nu}(z) + bzJ'_{\nu}(z) \) has all its zeros real, except the case when \( a/b + \nu < 0 \). In this case it has two purely imaginary zeros beside the real roots. Moreover, if \( \nu > -1 \) and \( a, b \in \mathbb{R} \), such that \( a^2 + b^2 \neq 0 \), then no function of the type \( z \mapsto aJ_{\nu}(z) + bzJ'_{\nu}(z) \) can have a repeated zero other than \( z = 0 \).

Lemma 3. [23] p. 198 If \( \nu > -1 \), then for the Bessel functions of the third kind \( H^{(1)}_{\nu} \) and \( H^{(2)}_{\nu} \) the following asymptotic expansions are valid
\[
H^{(1)}_{\nu}(w) = \left( \frac{2}{\pi w} \right)^{\frac{1}{2}} e^{i(w^{\frac{\nu}{2}} - \frac{\nu \pi}{2} - \frac{\pi}{4})} (1 + \eta_{1,\nu}(w)),
\]
\[
H^{(2)}_{\nu}(w) = \left( \frac{2}{\pi w} \right)^{\frac{1}{2}} e^{-i(w^{\frac{\nu}{2}} - \frac{\nu \pi}{2} - \frac{\pi}{4})} (1 + \eta_{2,\nu}(w)),
\]
where \( \eta_{1,\nu}(w) \) and \( \eta_{2,\nu}(w) \) are \( O(1/w) \) when \( |w| \) is large.

Lemma 4. Let \( z \in \mathbb{C} \), and \( \alpha_{\nu,n} \) be the \( n \)th positive root of the equation \( J_{\nu}(z) - zJ_{\nu+1}(z) = 0 \). If \( \nu > -1 \), then the following development holds
\[
\frac{g''_{\nu}(z)}{g'_{\nu}(z)} \begin{align*}
&= \frac{zJ_{\nu+2}(z) - 3J_{\nu+1}(z)}{J_{\nu}(z) - zJ_{\nu+1}(z)} = - \sum_{n \geq 1} \left( \frac{\alpha_{\nu,n}^2}{z^2} - z^2 \right),
\end{align*} \tag{2.4}
\]
Proof. Let us consider the integral

\[ \frac{1}{2\pi i} \int_{\mathbb{U}} \frac{z}{w(w-z)} \left( wJ_{\nu+2}(w) - 3J_{\nu+1}(w) \right) dw \]

where \( \mathbb{U} \) is the rectangle, whose vertices are \( \pm a \pm bi \), \( a > 0 \), \( b > 0 \) and \( z \) is a point inside the rectangle \( \mathbb{U} \) other than a zero of \( z \mapsto J_\nu(z) - zJ_{\nu+1}(z) \). Suppose that inside of \( \mathbb{U} \) there are \( m \) positive and \( n \) negative roots of \( z \mapsto J_\nu(z) - zJ_{\nu+1}(z) \). Since [18, p. 222] \( (1 - \nu)J_\nu(z) + zJ'_\nu(z) = J_\nu(z) - zJ_{\nu+1}(z) \), according to Lemma 2 the zeros of \( z \mapsto J_\nu(z) - zJ_{\nu+1}(z) \) are simple and real. The point \( w = 0 \) is a removable singularity. The only poles of the above integrand inside the rectangle are \( z, \pm \alpha_{\nu,1}, \pm \alpha_{\nu,2}, \ldots, \pm \alpha_{\nu,m} \). The residue at \( z \) is

\[ \varphi_\nu(z) = \frac{zJ_{\nu+2}(z) - 3J_{\nu+1}(z)}{J_\nu(z) - zJ_{\nu+1}(z)}, \]

while the residues at \( \pm \alpha_{\nu,n} \) are

\[ \frac{z}{\alpha_{\nu,n}(\alpha_{\nu,n} + z)}. \]

Here we used the recurrence relation [18, p. 222]

\[ zJ'_\nu(z) = -zJ_{\nu+1}(z) + \nu J_\nu(z) \]

for \( \nu \) and \( \nu + 1 \). The residue theorem [18, p. 19] implies that

\[ \frac{1}{2\pi i} \int_{\mathbb{U}} \frac{z}{w(w-z)} \left( wJ_{\nu+2}(w) - 3J_{\nu+1}(w) \right) dw \]

\[ = \frac{zJ_{\nu+2}(z) - 3J_{\nu+1}(z)}{J_\nu(z) - zJ_{\nu+1}(z)} + \sum_{n=1}^{m} \frac{z}{\alpha_{\nu,n}(\alpha_{\nu,n} - z)} + \sum_{n=1}^{m} \frac{z}{\alpha_{\nu,n}(\alpha_{\nu,n} + z)}. \]

In what follows we show that \( a \) and \( b \) can be replaced by suitable sequences which increase without limit and in the same time the expression \( \varphi_\nu(w) \) remains bounded if \( w \in \mathbb{U} \). Since the function \( w \mapsto \varphi_\nu(w) \) is an odd function of \( w \), it is sufficient to consider the case \( \text{Re} w > 0 \). Now, we introduce the notations

\[ H^{(j)}_{\nu+k}(w) = \left( \frac{2}{\pi w} \right)^{1/4} e^{(-1)^{j-1}(w-\frac{1}{2}(\nu+k)\pi-\frac{1}{4}\pi)(1+\eta_{k,j,\nu}(w))}, \]

where, as in Lemma 3 for \( k \in \{0, 1, 2\} \) and \( j \in \{1, 2\} \) the expression \( \eta_{k,j,\nu}(w) \) is \( O(1/w) \) when \( |w| \) is large. The relation \( 2J_\nu(w) = H^{(1)}_{\nu}(w) + H^{(2)}_{\nu}(w) \) and Lemma 4 lead to \( \varphi_\nu(w) = p_\nu(w)/q_\nu(w) \), where

\[ p_\nu(w) = (-1)^j e^{2i(w-\frac{1}{2}(\nu+1)\pi-\frac{1}{4}\pi)(1+\eta_{2,1,\nu}(w))} + i(1 + \eta_{2,1,\nu}(w)) \]

\[ - \frac{3}{w} \left[ e^{2i(w-\frac{1}{2}(\nu+1)\pi-\frac{1}{4}\pi)(1+\eta_{1,1,\nu}(w))} + (1 + \eta_{1,1,\nu}(w)) \right] \]

and

\[ q_\nu(w) = \frac{1}{w} \left[ e^{2i(w-\frac{1}{2}(\nu+1)\pi-\frac{1}{4}\pi)(1+\eta_{0,1,\nu}(w))} - i(1 + \eta_{0,1,\nu}(w)) \right] \]

\[ - e^{2i(w-\frac{1}{2}(\nu+1)\pi-\frac{1}{4}\pi)(1+\eta_{1,1,\nu}(w))} + (1 + \eta_{1,1,\nu}(w)). \]

Since \( \varphi_\nu(x+ib) \) tends to \( i \) as \( b \to \infty \) and the convergence is uniform with respect to \( x \in \mathbb{R} \), it follows that \( \varphi_\nu(w) \) is bounded on \( \{x+ib | x \in [0,a]\} \) if \( b \) is large enough. An analogous argument shows that \( \varphi_\nu(w) \) is bounded on the segment \( \{x-ib | x \in [0,a]\} \). It remains to prove that \( \varphi_\nu(w) \) is bounded in the case when \( w \in \{a+iy | y \in [-b,b]\} \). In this case we put \( a = a_m = 2m\pi + \frac{1}{2}(\nu+1) + \frac{1}{4}\pi \) and we get

\[ \lim_{m \to \infty} \frac{p_\nu(2m\pi + \frac{1}{2}(\nu+1) + \frac{3}{4}\pi + iy)}{q_\nu(2m\pi + \frac{1}{2}(\nu+1) + \frac{1}{4}\pi + iy)} \cdot \frac{i(1 + e^{-2y})}{1 + e^{-2y}} = i, \]

and thus it follows that \( \varphi_\nu(w) \) is bounded on the segment \( \{a+iy | y \in [-b,b]\} \), if \( a_m \) is large enough. Consequently \( \varphi_\nu(w) \) is bounded on the perimeter of the rectangle \( \mathbb{U} \) if \( b \) and \( a_m \) tend to infinity. Thus it follows that

\[ \lim_{b \to \infty} \frac{1}{2\pi i} \int_{\mathbb{U}} \frac{z}{w(w-z)} \left( wJ_{\nu+2}(w) - 3J_{\nu+1}(w) \right) dw = 0, \]

which implies that (2.21) is indeed valid.

\( \square \)
The proof of the next lemma is quite similar to that of the proof of Lemma 4. However, for the sake of completeness we have included also in details the proof of the following lemma. As far as we know the results presented in Lemma 4 and 5 are new. These results may be of independent interest and we believe that can be used to obtain some new inequalities for Bessel functions of the first kind of real or complex variable and real order.

**Lemma 5.** Let $z \in \mathbb{C}$, and let $\beta_{\nu, n}$ be the $n$th positive root of $(2 - \nu)J_\nu(z) + zJ'_\nu(z) = 0$. If $\nu > -1$, then the following development holds

$$
\nu(\nu - 2)J_\nu(z^2) + (3 - 2\nu)zJ'_\nu(z^2) = \sum_{n \geq 1} \frac{z}{\beta_{\nu, n}^2 - z^2}.
$$

**Proof.** We prove first the next development formula

$$
\nu(\nu - 2)J_\nu(z) + (3 - 2\nu)zJ'_\nu(z) = \sum_{n \geq 1} \frac{z^2}{\beta_{\nu, n}^2 - z^2}.
$$

The recurrence formula \[ \text{[18 p. 222]} \] $zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$ implies

$$
(2 - \nu)J_\nu(z) + zJ'_\nu(z) = 2J_\nu(z) - zJ_{\nu+1}(z)
$$

and

$$
\nu(\nu - 2)J_\nu(z) + (3 - 2\nu)zJ'_\nu(z) = \frac{z}{2} \left( zJ_{\nu+2}(z) - 4J_{\nu+1}(z) \right)
$$

where $\mathbb{O}$ is the rectangle, whose vertices are $\pm 4 \pm Bi$ ($A > 0$, $B > 0$) and $z$ is a point inside the rectangle $\mathbb{O}$ other than a zero of $2J_\nu(z) - zJ_{\nu+1}(z)$. Moreover, we assume that inside of $\mathbb{O}$ there are $m$ positive and $m$ negative roots of $2J_\nu(z) - zJ_{\nu+1}(z)$. Of course we can make this assumption since \[ \text{[18 p. 222]} \] $(2 - \nu)J_\nu(z) + zJ'_\nu(z) = 2J_\nu(z) - zJ_{\nu+1}(z)$ and according to Lemma 2 the zeros of $2J_\nu(z) - zJ_{\nu+1}(z)$ are simple and real. Observe that the point $w = 0$ is a removable singularity, and the residue theorem \[ \text{[18 p. 19]} \] implies that

$$
\frac{1}{2\pi i} \int_{\mathbb{O}} \frac{z}{w(w - z)} \frac{wJ_{\nu+2}(w) - 4J_{\nu+1}(w)}{2J_\nu(w) - wJ_{\nu+1}(w)} dw
$$

remains bounded if $w \in \mathbb{O}$. Since the above function is an odd function of $w$, it is sufficient to consider the case when $\text{Re} \, w > 0$. As in Lemma 4 we use the notations

$$
H^{(j)}_{\nu+k}(w) = \left( \frac{2}{\pi w} \right)^{\frac{1}{2}} e^{(\nu+1)\eta_j(w) - \frac{1}{4} \pi} (1 + \eta_{k,j}(w))
$$

where $k \in \{0, 1, 2\}$ and $j \in \{1, 2\}$. The relation

$$
2J_\nu(w) = H^{(1)}_{\nu}(w) + H^{(2)}_{\nu}(w)
$$

and Lemma 3 lead to

$$
\frac{wJ_{\nu+2}(w) - 4J_{\nu+1}(w)}{2J_\nu(w) - wJ_{\nu+1}(w)} = \frac{S(w)}{N(w)},
$$

where

$$
S(w) = (-i) e^{2(\nu+1)\pi - \frac{1}{4} \pi} (1 + \eta_{2,1,\nu}(w)) + i (1 + \eta_{2,2,\nu}(w))
$$

$$
- \frac{4}{w} e^{2(\nu+1)\pi - \frac{1}{4} \pi} (1 + \eta_{1,1,\nu}(w)) + (1 + \eta_{1,2,\nu}(w))
$$
and
\[
N(w) = \frac{2}{\nu} \left[ i e^{2i(w - \frac{1}{2} + \frac{1}{3} + \frac{1}{5})} (1 + \eta_{0,1,\nu}(w)) - i (1 + \eta_{0,2,\nu}(w)) \right] \\
- e^{2i(w - \frac{1}{2} + \frac{1}{3} + \frac{1}{5})} (1 + \eta_{1,1,\nu}(w)) + (1 + \eta_{1,2,\nu}(w)).
\]
Since
\[
\lim_{B \to \infty} \frac{S(x + iB)}{N(x + iB)} = i
\]
and the convergence is uniform with respect to \( x \in \mathbb{R} \), it follows that \( S(w)/N(w) \) is bounded on \( \{ x + iB | x \in [0, A] \} \) if \( B \) is large enough. An analogous argument shows that \( S(w)/N(w) \) is bounded on the segment \( \{ x - iB | x \in [0, A] \} \). It remains to prove the boundedness in case \( w \in \{ A + iy | y \in [-B, B] \} \). In this case we put \( A = A_m = 2m\pi + \frac{1}{2} + \frac{1}{4} \) and we get
\[
(2.6) \quad \lim_{m \to \infty} \frac{S(2m\pi + \frac{1}{2} + \frac{1}{4} + iy)}{N(2m\pi + \frac{1}{2} + \frac{1}{4} + iy)} = i
\]
The convergence in (2.6) is uniform on \( y \in [-B, B] \), it follows that \( S(w)/N(w) \) is bounded on the segment \( \{ A_m + iy | y \in [-B, B] \} \), if \( A_m \) is large enough. Consequently \( S(w)/N(w) \) is bounded on the perimeter of the rectangle \( \mathbb{R} \) if \( B \) and \( A_m \) tend to infinity. Thus, it follows that
\[
(2.7) \quad \lim_{B \to \infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z \nu(z + iB) - 4 J_{\nu+1}(w)}{z \nu(z) - 2 J_{\nu+1}(w)} dz = 0.
\]
Now, (2.5) and (2.7) imply
\[
\frac{\nu(\nu - 2) J_{\nu}(z) + (3 - 2\nu) z J'_{\nu}(z) + z^2 J''_{\nu}(z)}{2(2 - \nu) J_{\nu}(z) + 2 z J'_{\nu}(z)} = \frac{z}{2} \frac{z J_{\nu + 2}(z) - 4 J_{\nu + 1}(z)}{2 J_{\nu}(z) - z J_{\nu + 1}(z)} = \sum_{n \geq 1} \frac{z^2}{\beta_{\nu,n}^2 - z^2},
\]
and finally we get
\[
\frac{\nu(\nu - 2) J_{\nu}(z) + (3 - 2\nu) z J'_{\nu}(z) + z^2 J''_{\nu}(z)}{2(2 - \nu) J_{\nu}(z) + 2 z J'_{\nu}(z)} = \sum_{n \geq 1} \frac{z}{\beta_{\nu,n}^2 - z}.
\]
\[\square\]

**Lemma 6.** If \( \nu \geq 0 \), then \( \alpha_{\nu,1} > 1 \), where \( \alpha_{\nu,1} \) denotes the first positive root of the equation \( J_{\nu}(x) - x J_{\nu+1}(x) = 0 \). Similarly, if \( \nu \leq 0 \), then \( \beta_{\nu,1} > 1 \), where \( \beta_{\nu,1} \) denotes the first positive root of the equation \( 2 J_{\nu}(x) - x J_{\nu+1}(x) = 0 \).

**Proof.** We shall use the following integral representation [15, p. 224]
\[
J_{\nu}(x) = \frac{2}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{1} (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt) dt,
\]
which is valid for all \( x \in \mathbb{R} \) and \( \nu > -\frac{1}{2} \). Observe that for \( x \in (0, 1] \) and \( \nu \geq 0 \) we have
\[
J_{\nu}(x) - x J_{\nu+1}(x) = \frac{2}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{1} \left( 1 - \frac{x^2}{2\nu + 1} \frac{x^2t^2}{2\nu + 1} \right) (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt) dt
\]
\[
> \frac{2}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{1} \frac{2\nu}{2\nu + 1} (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt) dt \geq 0,
\]
\[
2 J_{\nu}(x) - x J_{\nu+1}(x) = \frac{2}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{1} \left( 2 - \frac{x^2}{2\nu + 1} \frac{x^2t^2}{2\nu + 1} \right) (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt) dt
\]
\[
> \frac{2}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{1} \frac{4\nu + 1}{2\nu + 1} (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt) dt > 0,
\]
Thus, the smallest positive roots of the transcendental equations in the question, that is, \( J_{\nu}(x) - x J_{\nu+1}(x) = 0 \) and \( 2 J_{\nu}(x) - x J_{\nu+1}(x) = 0 \), must be bigger then one. \[\square\]

**Lemma 7.** [15] p. 196] For \( \nu > -1 \) let \( \gamma_{\nu,n} \) be the \( n \)th positive root of the equation \( \gamma J_{\nu}(z) + z J'_{\nu}(z) = 0 \). If \( \nu + \gamma \geq 0 \), then the function \( \nu \mapsto \gamma_{\nu,n} \) is strictly increasing on \( (-1, \infty) \) for \( n \in \{1, 2, \ldots\} \) fixed.
3. Proofs of the Main Results

In this section our aim is to prove the main results of this paper.

Proof of Theorem 1. Observe that

\[ 1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 + \frac{zJ''_\nu(z)}{J'_\nu(z)} + \left(\frac{1}{\nu} - 1\right) \frac{zJ'_\nu(z)}{J_\nu(z)} \]

Now, recall the following infinite product representations [18, p. 235]

\[ J_\nu(z) = \left(\frac{4}{z^2}\right)^{\nu/2} \prod_{n \geq 1} \left(1 - \frac{z^2}{j^2_{\nu,n}}\right), \quad J'_\nu(z) = \frac{\left(\frac{4}{z^2}\right)^{\nu-1}}{2\Gamma(\nu)} \prod_{n \geq 1} \left(1 - \frac{z^2}{j^2_{\nu,n}}\right), \]

where \( j_{\nu,n} \) and \( j'_{\nu,n} \) are the \( n \)th positive roots of \( J_\nu \) and \( J'_\nu \), respectively. Logarithmic differentiation yields

\[ \frac{zJ'_\nu(z)}{J_\nu(z)} = \nu - \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}, \quad 1 + \frac{zJ''_\nu(z)}{J'_\nu(z)} = \nu - \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}, \]

which implies that

\[ 1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 - \left(\frac{1}{\nu} - 1\right) \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2} - \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}. \]

Now, suppose that \( \nu \in (0, 1] \). By using the inequality (2.21), for all \( z \in \mathbb{D}(0, j'_{\nu,1}) \) we obtain the inequality

\[ \text{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)}\right) \geq 1 - \left(\frac{1}{\nu} - 1\right) \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2} - \sum_{n \geq 1} \frac{2z^2}{j^2_{\nu,n} - z^2}, \]

where \( |z| = r \). Moreover, observe that if we use the inequality (2.21) then we get that the above inequality is also valid when \( \nu > 1 \). Here we used that the zeros \( j_{\nu,n} \) and \( j'_{\nu,n} \) interlace according to the inequalities [18, p. 235]

\[ \nu \leq j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \ldots. \]

Now, the above deduced inequality implies for \( r \in (0, j'_{\nu,1}) \)

\[ \inf_{z \in \mathbb{D}(0,r)} \left\{ \text{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)}\right) \right\} = 1 + \frac{rf''_\nu(r)}{f'_\nu(r)}. \]

On the other hand, the function \( u_\nu : (0, j'_{\nu,1}) \to \mathbb{R} \), defined by

\[ u_\nu(r) = 1 + \frac{rf''_\nu(r)}{f'_\nu(r)}, \]

is strictly decreasing since

\[ u'_\nu(r) = - \left(\frac{1}{\nu} - 1\right) \sum_{n \geq 1} \frac{4rj^2_{\nu,n}}{(j^2_{\nu,n} - r^2)^2} - \sum_{n \geq 1} \frac{4rj^2_{\nu,n}}{(j^2_{\nu,n} - r^2)^2} \]

\[ < \sum_{n \geq 1} \frac{4rj^2_{\nu,n}}{(j^2_{\nu,n} - r^2)^2} - \sum_{n \geq 1} \frac{4rj^2_{\nu,n}}{(j^2_{\nu,n} - r^2)^2} < 0 \]

for \( \nu > 0 \) and \( r \in (0, j'_{\nu,1}) \). Here we used again that the zeros \( j_{\nu,n} \) and \( j'_{\nu,n} \) interlace and for all \( n \in \{1, 2, \ldots\} \), \( \nu > 0 \) and \( r < \sqrt{j_{\nu,1}j'_{\nu,1}} \) we have that

\[ j^2_{\nu,n} - r^2 < j^2_{\nu,n} - j^2_{\nu,n} \]

Observe also that \( \lim_{r \to 0} u_\nu(r) = 1 > \alpha \) and \( \lim_{r \to j'_{\nu,1}} u_\nu(r) = -\infty \), which means that for \( z \in \mathbb{D}(0, r_1) \) we have

\[ \text{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)}\right) > \alpha, \]

if and only if \( r_1 \) is the unique root of

\[ 1 + \frac{rf''_\nu(r)}{f'_\nu(r)} = \alpha, \]

situated in \((0, j'_{\nu,1})\). \( \square \)
The recurrence formula [18, p. 222] 
\[ zJ_\nu(z) \]
In other words, we have
\[ \text{has a unique root} \]

The function \( v_\nu \), \( \nu, \alpha \in \mathbb{D} \), is strictly decreasing and 
\[ \lim_{r \searrow 0} v_\nu(r) = 1, \quad \lim_{r \nearrow \alpha_{\nu,1}} v_\nu(r) = -\infty. \]

Consequently, the equation
\[ 1 + \frac{r g_\nu''(r)}{g_\nu'(r)} = \alpha \]
has a unique root \( r_2 \) in \((0, \alpha_{\nu,1})\), and this equation is equivalent to
\[ 1 + \frac{r J_{\nu+2}(r) - 3J_{\nu+1}(r)}{J_\nu(r) - r J_{\nu+1}(r)} = \alpha. \]

In other words, we have
\[ \text{Re} \left( 1 + \frac{z g_\nu''(z)}{g_\nu'(z)} \right) > \alpha, \quad z \in \mathbb{D}(0, r_2) \quad \text{and} \quad \inf_{z \in \mathbb{D}(0, r_2)} \left\{ \text{Re} \left( 1 + \frac{z g_\nu''(z)}{g_\nu'(z)} \right) \right\} = \alpha. \]

Finally, let us recall that (see [23, p. 597]) when \( \rho + \nu > 0 \) and \( \nu > -1 \) the so-called Dini function \( z \rightarrow z J_\nu'(z) + \rho J_\nu(z) \) has only real zeros and according to Ismail and Muldoon [13, p. 11] we know that the smallest positive zero of the above function is less than \( j_\nu \). This in turn implies that \( \alpha_{\nu,1} < j_\nu \), which completes the proof. \( \square \)

**Proof of Theorem 3** Observe that
\[ \frac{h_\nu''(z)}{h_\nu'(z)} = \frac{\nu(\nu - 1)J_\nu(z) + 2(1 - \nu)z J_\nu'(z) + z^2 J_\nu''(z)}{(1 - \nu)J_\nu(z) + z J_\nu'(z)} \]
and according to Lemma 5 it follows
\[ 1 + \frac{z h_\nu''(z)}{h_\nu'(z)} = 1 - \sum_{n \geq 1} \frac{z}{\alpha_{\nu,n}^2 - z^2}. \]
In order to finish the proof, we need to show that the equation (3.3) has a unique root in $u$.

The equation (3.3) is equivalent to

$$\text{Re} \left( 1 + z \frac{h''_\nu(z)}{h'_\nu(z)} \right) = \text{Re} \left( 1 - \sum_{n \geq 1} \frac{z^2}{\beta^2_{\nu,n} - z^2} \right) \geq \min_{|z| = r} \text{Re} \left( 1 - \sum_{n \geq 1} \frac{z^2}{\beta^2_{\nu,n} - z^2} \right) = \min_{|z| = r} \left( 1 - \sum_{n \geq 1} \frac{z^2}{\beta^2_{\nu,n} - z^2} \right) + \frac{r^2}{2} \frac{r^2 \nu J_{\nu+2}(r^\nu) - 2J_{\nu+1}(r^\nu)}{J_{\nu}(r^\nu) - J_{\nu+1}(r^\nu)}.$$

Consequently, it follows that

$$\inf_{z \in \mathbb{D}(0,r)} \left\{ \text{Re} \left( 1 + z \frac{h''_\nu(z)}{h'_\nu(z)} \right) \right\} = 1 + r \frac{r^2 \nu J_{\nu+2}(r^\nu) - 2J_{\nu+1}(r^\nu)}{2J_{\nu}(r^\nu) - r^2 J_{\nu+1}(r^\nu)}.$$

Now, let $r_3$ be the smallest positive root of the equation

$$\alpha = 1 + r \frac{r^2 \nu J_{\nu+2}(r^\nu) - 2J_{\nu+1}(r^\nu)}{2J_{\nu}(r^\nu) - r^2 J_{\nu+1}(r^\nu)}.$$

In order to finish the proof, we need to show that the equation (3.3) has a unique root in $(0, \beta^2_{\nu,1})$. But, the equation (3.3) is equivalent to

$$w_\nu(r) = 1 - \alpha - \sum_{n \geq 1} \frac{r^2}{\beta^2_{\nu,n} - r^2} = 0,$$

and we have

$$\lim_{r \to 0^+} w_\nu(r) = 1 - \alpha > 0, \quad \lim_{r \to \beta^2_{\nu,1}} w_\nu(r) = -\infty.$$

Now, since the function $w_\nu$ is strictly decreasing on $(0, \beta^2_{\nu,1})$, it follows that the equation $w_\nu(r) = 0$ has a unique root.

Finally, since for $\rho + \nu > 0$ and $\nu > -1$ the function $z \mapsto z J'_\nu(z) + \rho J_\nu(z)$ has only real zeros (see [23, p. 597]) and the smallest positive zero of the above function is less than $j_{\nu,1}$ (see [13, p. 11]), we obtain that $\beta_{\nu,1} < j_{\nu,1}$, which completes the proof.

**Proof of Theorem 4.** According to (3.1) for $z \in \mathbb{D}$ we obtain that

$$\text{Re} \left( 1 + \frac{z J''_\nu(z)}{J'_\nu(z)} \right) \geq 1 - \frac{1}{\nu - 1} \sum_{n \geq 1} \frac{2}{J^2_{\nu,n} - 1} - \frac{2}{J^2_{\nu,n} - 1} \geq 1 - \frac{1}{\nu - 1} \sum_{n \geq 1} \frac{2}{J^2_{\nu,n} - 1} = 1 + \frac{J'_\nu(1)}{J'_\nu(1)} + \frac{1}{\nu - 1} \frac{J'_\nu(1)}{J'_\nu(1)}.$$

Now, consider the function $u : (\nu^*, \infty) \to \mathbb{R}$, defined by

$$u(\nu) = 1 + \frac{J'_\nu(1)}{J'_\nu(1)} + \frac{1}{\nu - 1} \frac{J'_\nu(1)}{J'_\nu(1)} = 1 - \left( \frac{1}{\nu - 1} \right) \sum_{n \geq 1} \frac{2}{j^2_{\nu,n} - 1} - \sum_{n \geq 1} \frac{2}{j^2_{\nu,n} - 1}.$$

We note that this function is well defined since $J_{\nu}(1) > 0$ and $J'_{\nu}(1) > 0$ when $\nu > \nu^*$. By using (2.3) for $\nu > -\frac{1}{2}$ we get

$$J_{\nu}(1) = \frac{2(1)^{\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(t) dt > 0.$$

Moreover, since $\nu \mapsto j_{\nu,n}$ is strictly increasing on $(0, \infty)$ for each $n \in \{1, 2, \ldots\}$ (see [13, p. 236]), it follows that

$$\frac{d}{d\nu} \left( \frac{J'_\nu(1)}{J'_\nu(1)} \right) = \frac{d}{d\nu} \left( \nu - \sum_{n \geq 1} \frac{2}{j^2_{\nu,n} - 1} \right) = 1 + \sum_{n \geq 1} \frac{4j_{\nu,n} \frac{\partial j_{\nu,n}}{\partial \nu}}{(j^2_{\nu,n} - 1)^2} > 0.$$
for \( \nu > 0 \). This means that if \( \nu > \nu' \), then \( J'_{\nu}(1)/J_{\nu}(1) > J'_{\nu'}(1)/J_{\nu'}(1) = 0 \).

Now, in what follows we show that \( u \) is strictly increasing. For this we distinguish two cases. First we consider that \( \nu \in (\nu^*, 1) \). Since the functions \( \nu \mapsto j_{\nu,n} \) and \( \nu \mapsto j'_{\nu,n} \) are strictly increasing on \([0, \infty)\) for each \( n \in \{1, 2, \ldots \} \) (see [18, p. 236]), it follows that the functions \( \nu \mapsto 2/(j^2_{\nu,n} - 1) \) and \( \nu \mapsto 2/(j'^2_{\nu,n} - 1) \) are strictly decreasing on \([0, \infty)\) for each \( n \in \{1, 2, \ldots \} \), and consequently \( u \) is strictly increasing on \((\nu^*, 1)\).

Suppose that \( \nu > 1 \). In this case we have

\[
\begin{align*}
\frac{d}{d\nu} j_{\nu,n} &= 2 j_{\nu,n} \int_0^{\infty} K_0(2j_{\nu,n} \sinh(t))e^{-2\nu t} dt, \\
\frac{d}{d\nu} j'_{\nu,n} &= \frac{2j'_{\nu,n}}{j_{\nu,n} - \nu} \int_0^{\infty} (j^2_{\nu,n} \cosh(2t) - \nu^2) K_0(2j_{\nu,n} \sinh(t))e^{-2\nu t} dt,
\end{align*}
\]

Recall that for any \( n \in \{1, 2, \ldots \} \) the derivative of \( j_{\nu,n} \) and \( j'_{\nu,n} \) with respect to \( \nu \) can be written as [18, p. 236]

\[
\frac{1}{\nu} \sum_{n \geq 1} \left( \frac{2}{j_{\nu,n}^{2}} - 1 \right) = 4 \sum_{n \geq 1} \left( \frac{j_{\nu,n} d j_{\nu,n}}{j_{\nu,n}^{2} - 1} \right)^2 + 4 \sum_{n \geq 1} \left( \frac{j'_{\nu,n} d j'_{\nu,n}}{j_{\nu,n}^{2} - 1} \right)^2.
\]

Observe that for \( \nu > 1 \), \( n \in \{1, 2, \ldots \} \) and \( t > 0 \) we have

\[
\frac{j^2_{\nu,n} \cosh(2t) - \nu^2}{j_{\nu,n}^{2} - \nu^2} > 1 - \frac{1}{\nu},
\]

\[
\frac{j'^2_{\nu,n}}{(j'^2_{\nu,n} - 1)^2} > \frac{j^2_{\nu,n}}{(j^2_{\nu,n} - 1)^2},
\]

since according to (3.2) we have that \( j^2_{\nu,n} > j'^2_{\nu,n} \) and \( j_{\nu,n} j'^2_{\nu,n} > j_{\nu,n} j^2_{\nu,n} > j^2_{\nu,n} j'^2_{\nu,n} > 1 \). On the other hand, we know that \( K_0 \) is strictly decreasing on \((0, \infty)\), and this implies that for each \( \nu > 1 \), \( n \in \{1, 2, \ldots \} \) and \( t > 0 \) we have

\[
K_0(2j_{\nu,n} \sinh(t)) > K_0(2j_{\nu,n} \sinh(t)).
\]

Combining this with (3.6) and (3.7) we obtain

\[
4 \sum_{n \geq 1} \left( \frac{j'_{\nu,n} d j'_{\nu,n}}{j_{\nu,n}^{2} - 1} \right) = \sum_{n \geq 1} \int_0^{\infty} \frac{j^2_{\nu,n} \cosh(2t) - \nu^2}{j_{\nu,n}^{2} - \nu^2} \frac{j'^2_{\nu,n}}{(j'^2_{\nu,n} - 1)^2} K_0(2j_{\nu,n} \sinh(t))e^{-2\nu t} dt
\]

\[
> 8 \sum_{n \geq 1} \int_0^{\infty} \frac{j^2_{\nu,n} \cosh(2t) - \nu^2}{j_{\nu,n}^{2} - \nu^2} \frac{j'^2_{\nu,n}}{(j'^2_{\nu,n} - 1)^2} K_0(2j_{\nu,n} \sinh(t))e^{-2\nu t} dt
\]

\[
> 8 \sum_{n \geq 1} \int_0^{\infty} \left( 1 - \frac{1}{\nu} \right) \frac{j^2_{\nu,n}}{(j^2_{\nu,n} - 1)^2} K_0(2j_{\nu,n} \sinh(t))e^{-2\nu t} dt
\]

\[
= 4 \left( 1 - \frac{1}{\nu} \right) \sum_{n \geq 1} \frac{j_{\nu,n} d j_{\nu,n}}{j_{\nu,n}^{2} - 1}.
\]

which implies that \( u'('(\nu) > 0 \) for \( \nu > 1 \), and thus the function \( u \) is strictly increasing on \((1, \infty)\), and hence on the whole \((\nu^*, \infty)\). Consequently, if \( \nu \geq \nu_\alpha = \nu_\alpha(f_\nu) \), then we get the inequality \( u(\nu) \geq u(\nu_\alpha) \). This in turn implies that \( \nu_\alpha \) is the smallest value having the property that the condition \( \nu \geq \nu_\alpha \) implies that for all \( \nu > \nu_\alpha \) we have

\[
\text{Re} \left( 1 + z \frac{f'_{\nu}(z)}{f_{\nu}(z)} \right) = u(\nu_\alpha) = \alpha.
\]

Thus, we proved that the function \( f_\nu \) is convex of order \( \alpha \in [0, 1) \) in \( \mathbb{D} \) if and only if \( \nu \geq \nu_\alpha(f_\nu) \), where \( \nu_\alpha = \nu_\alpha(f_\nu) \) is the unique root of the equation \( u(\nu) = \alpha \), that is,

\[
1 + \frac{J'_{\nu}(1)}{J_{\nu}(1)} + \left( 1 - \frac{1}{\nu} \right) \frac{J'_{\nu}(1)}{J_{\nu}(1)} = \alpha.
\]

Since \( J_{\nu} \) is a particular solution of the Bessel differential equation (see [18, p. 217]), it follows that

\[
J''_{\nu}(1) + J'_{\nu}(1) + (\nu^2 - 1)J_{\nu}(1) = 0,
\]
Proof of Theorem 5. Let $\nu \geq 0$. According to Lemma 6 we have $\alpha_{\nu,1} > 1$. Taking into account the proof of Theorem 2 when $r = 1$ we get for $z \in \mathbb{D}$

$$\frac{1}{\nu} \frac{g'_{\nu}(z)}{g_{\nu}(z)} = \frac{1}{\nu} \sum_{n \geq 1} \frac{2z^2}{\alpha_{\nu,n} + z^2} \geq \min_{|z|=1} \left(\frac{1}{\nu} \sum_{n \geq 1} \frac{2z^2}{\alpha_{\nu,n} + z^2}\right)$$

$$\geq 1 - \sum_{n \geq 1} \frac{2}{\alpha_{\nu,n} + 1} = 1 + \frac{g''_{\nu}(1)}{g_{\nu}(1)} = 1 + \frac{J_{\nu+2}(1) - 3J_{\nu+1}(1)}{J_{\nu}(1) - J_{\nu+1}(1)}.$$ 

According to Lemma 7, the function $\nu \rightarrow \alpha_{\nu,n}$ is strictly increasing on $[0, \infty)$ for every fixed natural number $n$. Thus, it follows that the function $v : [0, \infty) \rightarrow \mathbb{R}$, defined by

$$v(\nu) = 1 + \frac{J_{\nu+2}(1) - 3J_{\nu+1}(1)}{J_{\nu}(1) - J_{\nu+1}(1)} = 1 - \sum_{n \geq 1} \frac{2}{\alpha_{\nu,n} + 1},$$

is strictly increasing too. Here we used that $J_{\nu}(1) - J_{\nu+1}(1) \neq 0$ when $\nu \geq 0$, since from (3.8) we have

$$J_{\nu}(1) - J_{\nu+1}(1) = \frac{2(\frac{1}{\nu})^\nu}{\sqrt{\pi^1}} (\nu + \frac{1}{2}) \int_0^1 \left(1 - \frac{1}{2\nu + 1} + \frac{t^2}{2\nu + 1}\right) (1 - t^2)^{\nu - \frac{1}{2}\cos(t)} dt > 0.$$

Since the function $v$ is strictly increasing, it follows that if $\nu \geq \nu_{\alpha} = \nu_{\alpha}(g_{\nu})$, then we get the inequality

$$1 + \frac{J_{\nu+2}(1) - 3J_{\nu+1}(1)}{J_{\nu}(1) - J_{\nu+1}(1)} = v(\nu) \geq v(\nu_{\alpha}) = 1 + \frac{J_{\nu_{\alpha}+2}(1) - 3J_{\nu_{\alpha}+1}(1)}{J_{\nu_{\alpha}}(1) - J_{\nu_{\alpha}+1}(1)} = \alpha.$$ 

Now, from (3.9) we get that $\nu_{\alpha}$ is the smallest value having the property that the condition $\nu \geq \nu_{\alpha}$ implies

$$\frac{1}{\nu} \frac{g'_{\nu}(z)}{g_{\nu}(z)} > 1 + \frac{J_{\nu_{\alpha}+2}(1) - 3J_{\nu_{\alpha}+1}(1)}{J_{\nu_{\alpha}}(1) - J_{\nu_{\alpha}+1}(1)} = \alpha \quad \text{for all} \quad z \in \mathbb{D}.$$ 

Thus, we proved that the function $g_{\nu}$ is convex of order $\alpha \in (0, 1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_{\alpha}(g_{\nu})$, where $\nu_{\alpha} = \nu_{\alpha}(g_{\nu})$ is the unique root of the equation

$$1 + \frac{J_{\nu_{\alpha}+2}(1) - 3J_{\nu_{\alpha}+1}(1)}{J_{\nu_{\alpha}}(1) - J_{\nu_{\alpha}+1}(1)} = \alpha$$

situates in $[0, \infty)$. Observe that by using the recurrence relation (13) p. 222

$$J_{\nu}(z) + J_{\nu+2}(z) = \frac{2(\nu + 1)}{z} J_{\nu+1}(z)$$

we get that the above equation is equivalent to $(2\nu + \alpha - 2)J_{\nu+1}(1) = \alpha J_{\nu}(1)$. In particular, $g_{\nu}$ is convex if and only if $\nu \geq \nu_{\alpha} = \nu_{\alpha}(g_{\nu})$, where $\nu_{\alpha} = 1$ is the unique root of the equation $J_{\nu}(1) - 4J_{\nu+1}(1) + J_{\nu+2}(1) = 0$, that is, $(2\nu - 2)J_{\nu+1}(1) = 0$. Here we used that $J_{\nu+1}(1) > 0$ for $\nu > -\frac{3}{2}$, which follows from (3.4). \qed

Proof of Theorem 6. Let $\nu \geq 0$. According to Lemma 6 we have $\beta_{\nu,1} > 1$. Taking into account the proof of Theorem 4 when $r = 1$ we get for $z \in \mathbb{D}$

$$\frac{1}{\nu} \frac{h'_{\nu}(z)}{h_{\nu}(z)} = \frac{1}{\nu} \sum_{n \geq 1} \frac{z^2}{\beta_{\nu,n} + z^2} \geq \min_{|z|=1} \left(\frac{1}{\nu} \sum_{n \geq 1} \frac{z^2}{\beta_{\nu,n} + z^2}\right)$$

$$\geq 1 - \sum_{n \geq 1} \frac{1}{\beta_{\nu,n} + 1} = 1 + \frac{h''_{\nu}(z)}{h'_{\nu}(z)} = 1 + \left(1 + \frac{J_{\nu+2}(1) - 4J_{\nu+1}(1)}{2J_{\nu}(1) - J_{\nu+1}(1)}\right).$$
According to Lemma 7, the function $\nu \mapsto \beta_{\nu,n}$ is strictly increasing on $(-1, \infty)$ for every fixed natural number $n$. Thus, it follows that the function $w : [0, \infty) \to \mathbb{R}$, defined by
\[
w(\nu) = 1 + \frac{1}{2} \frac{J_{\nu+2}(1) - 4J_{\nu+1}(1)}{2J_{\nu}(1) - J_{\nu+1}(1)} = 1 - \sum_{n \geq 1} \frac{1}{\beta_{\nu,n}^2 - 1},
\]
is strictly increasing too. Note that the function $w$ is well defined since from (2.8) for $\nu \geq 0$ we have
\[
2J_{\nu}(1) - J_{\nu+1}(1) = \frac{2 \left( \frac{1}{2} \right)^{\nu}}{\sqrt{\pi} \nu (\nu + 1)} \int_0^\infty \left( 2 - \frac{1}{2\nu + 1} + \frac{t^2}{2\nu + 1} \right) (1 - t^2)^{\nu - \frac{1}{2}} \cos(t) dt > 0.
\]
Now, since $w$ is strictly increasing, it follows that if $\nu \geq \nu_\alpha = \nu_\alpha(h_\nu)$, then we get the inequality
\[
1 + \frac{1}{2} \frac{J_{\nu+2}(1) - 4J_{\nu+1}(1)}{2J_{\nu}(1) - J_{\nu+1}(1)} = w(\nu) \geq w(\nu_\alpha) = 1 + \frac{1}{2} \frac{J_{\nu_\alpha+2}(1) - 4J_{\nu_\alpha+1}(1)}{2J_{\nu_\alpha}(1) - J_{\nu_\alpha+1}(1)} = \alpha.
\]
Now, from (3.11) we get that $\nu_\alpha$ is the smallest value having the property that the condition $\nu \geq \nu_\alpha$ implies
\[
\text{Re} \left( 1 + \frac{h_\alpha''(z)}{h_\nu''(z)} \right) > 1 + \frac{1}{2} \frac{J_{\nu_\alpha+2}(1) - 4J_{\nu_\alpha+1}(1)}{2J_{\nu_\alpha}(1) - J_{\nu_\alpha+1}(1)} = \alpha \quad \text{for all } z \in \mathbb{D}.
\]
Summarizing, we proved that the function $h_\nu$ is convex of order $\alpha \in [0, 1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(h_\nu)$, where $\nu_\alpha = \nu_\alpha(h_\nu)$ is the unique root of the equation
\[
1 + \frac{1}{2} \frac{J_{\nu+2}(1) - 4J_{\nu+1}(1)}{2J_{\nu}(1) - J_{\nu+1}(1)} = \alpha
\]
situated in $[0, \infty)$. By using (3.10) the above equation can be rewritten as
\[
(2\nu + 2\alpha - 4)J_{\nu+1}(1) = (4\alpha - 3)J_{\nu}(1).
\]
In particular, $h_\nu$ is convex if and only if $\nu \geq \nu_0 = \nu_0(h_\nu)$, where $\nu_0 \simeq 0.6688 \ldots$ is the unique root of the equation $4J_{\nu}(1) - 6J_{\nu+1}(1) + J_{\nu+2}(1) = 0$, that is, $2(\nu - 2)J_{\nu+1}(1) + 3J_{\nu}(1) = 0$.

Finally, observe that, in particular, $h_\nu$ is convex of order $\frac{1}{4}$ if and only if $\nu \geq \nu_\frac{1}{4}(h_\nu)$, where $\nu_\frac{1}{4}(h_\nu) = \frac{3}{4}$ is the unique root of the equation $(4\nu - 5)J_{\nu+1}(1) = 0$. Here we used again that according to (3.3) we have $J_{\nu+1}(1) > 0$ for $\nu > \frac{3}{4}$.$\square$

**Proof of Theorem 7.** Observe that
\[
z \frac{\phi''(z)}{\phi'(z)} = \frac{\nu(\nu + 2)J_{\nu}(z) - (2\nu + 1)z \frac{\partial}{\partial z} J_{\nu}'(z) + z J_{\nu}''(z)}{-2\nu J_{\nu}(z) + 2z \frac{\partial}{\partial z} J_{\nu}'(z)}.
\]
Now, by using the recurrence relation (3.8) and the fact that $J_{\nu}$ satisfies [18, p. 217]
\[
z^2 J_{\nu}'''(z) + z J_{\nu}'(z) + (z^2 - \nu^2)J_{\nu}(z) = 0,
\]
we obtain that
\[
1 + z \frac{\phi''(z)}{\phi'(z)} = \frac{z \frac{\partial}{\partial z} J_{\nu}(z) - \nu}{2J_{\nu+1}(z) \frac{\partial}{\partial z} J_{\nu}'(z)} = 1 - \frac{\nu^2}{\left( \frac{\nu}{2} \right)^2 + \nu^2} J_{\nu+1}(z)
\]
Thus, in view of the Mittag-Leffler expansion
\[
\frac{z J_{\nu}'(z)}{J_{\nu}(z)} = \nu - \sum_{n \geq 1} \frac{2z^2}{J_{\nu,n}^2 - z^2},
\]
one has
\[
1 + z \frac{\phi''(z)}{\phi'(z)} = 1 - \sum_{n \geq 1} \frac{z}{J_{\nu,n}^2 - z^2}.
\]
Now, by using (2.3) it follows that for $z \in \mathbb{D}(0, J_{\nu+1,1})$ we have
\[
(3.12) \quad \text{Re} \left( 1 + z \frac{\phi''(z)}{\phi'(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{J_{\nu,n+1,1}^2 - r} = 1 + \frac{r \phi''(r)}{\phi'(r)},
\]
where $r = |z|$. This inequality implies for $r \in (0, J_{\nu+1,1})$
\[
\inf_{z \in \mathbb{D}(0,r)} \left\{ \text{Re} \left( 1 + z \frac{\phi''(z)}{\phi'(z)} \right) \right\} = 1 + \frac{r \phi''(r)}{\phi'(r)}.
\]
On the other hand, the function $\psi_\nu : (0, j_{\nu+1,1}) \to \mathbb{R}$, defined by

$$\psi_\nu(r) = 1 + \frac{\nu \varphi_\nu''(r)}{\varphi_\nu'(r)},$$

is strictly decreasing since

$$\psi_\nu'(r) = -\sum_{n \geq 1} \frac{j_{\nu+1,n}^2}{(j_{\nu+1,n}^2 - r)^2}.$$  

Observe also that $\lim_{r \to 0} \psi_\nu(r) = 1 > \alpha$ and $\lim_{r \to j_{\nu+1,1}} \psi_\nu(r) = -\infty$, which means that for $z \in \mathbb{D}(0, r_4)$ we have

$$\text{Re} \left( 1 + \frac{z \varphi_\nu''(z)}{\varphi_\nu'(z)} \right) > \alpha,$$

if and only if $r_4$ is the unique root of

$$1 + \frac{\nu \varphi_\nu''(r_4)}{\varphi_\nu'(r_4)} = \alpha,$$

situated in $(0, j_{\nu+1,1})$. □

**Proof of Theorem 8** First we prove that $J_{\nu+1}(1) > 0$ if $\nu > \nu^*$. For this we show that $\nu \mapsto \log(J_{\nu}(1))$ is increasing on $(\nu^* + 1, 0)$. This implies that $\nu \mapsto \log(J_{\nu+1}(1))$ is increasing on $(\nu^*, -1)$, and consequently $\nu \mapsto J_{\nu+1}(1)$ is increasing too on $(\nu^*, -1)$. Thus, $J_{\nu+1}(1) > J_{\nu^*+1}(1) = 0$ if $-1 > \nu > \nu^*$. Combining this with (3.4) we obtain that indeed $J_{\nu+1}(1) > 0$ if $\nu > \nu^*$.

By using the infinite product representation [18, p. 235]

$$J_\nu(1) = \frac{\left(\frac{4}{\pi}\right)^\nu}{\Gamma(\nu + 1)} \prod_{n \geq 1} \left(1 - \frac{1}{j_{\nu,n}}\right),$$

it follows that

$$\Omega(\nu) = \frac{d\log J_\nu(1)}{d\nu} = -\log 2 - \psi(\nu + 1) + \sum_{n \geq 1} \frac{2\frac{d}{d\nu} j_{\nu,n}}{j_{\nu,n}(j_{\nu,n}^2 - 1)},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ stands for the digamma function, that is, for the logarithmic derivative of the Euler gamma function. We note that the function $\Omega$ is well defined since for $\nu > \nu^* + 1$ we have $j_{\nu,n} \neq 1$ for each $n \in \{1, 2, \ldots\}$. This is because $1 = j_{\nu^*+1,1} < j_{\nu,1} < j_{\nu,2} < \ldots < j_{\nu,n} < \ldots$ for each $\nu > \nu^* + 1$ and $n \in \{1, 2, \ldots\}$. Now, by using the known inequalities [12, p. 196]

$$\frac{d}{d\nu} j_{\nu,n} > \frac{2}{j_{\nu,n}} \frac{8(\nu + 1)^2}{j_{\nu,n}^3}, \quad \nu > -1, \quad n \in \{1, 2, \ldots\},$$

and [41, p. 374]

$$\psi(x) < \log x - \frac{1}{2x}, \quad x > 0,$$

we obtain

$$\Omega(\nu) > -\log 2 - \psi(\nu + 1) + \sum_{n \geq 1} \frac{2\frac{d}{d\nu} j_{\nu,n}}{j_{\nu,n}(j_{\nu,n}^2 - 1)}$$

$$> -\log 2 - \psi(\nu + 1) + \sum_{n \geq 1} \frac{2}{j_{\nu,n}^3}$$

$$> -\log 2 - \psi(\nu + 1) + 4 \sum_{n \geq 1} \frac{1}{j_{\nu,n}^3} + 16(\nu + 1)^2 \sum_{n \geq 1} \frac{1}{j_{\nu,n}^2}$$

$$= -\log 2 - \psi(\nu + 1) + \frac{3\nu + 5}{4(\nu + 1)^2(\nu + 2)(\nu + 3)}$$

$$> -\log 2 - \log(\nu + 1) - \frac{1}{2(\nu + 1)} + \frac{3\nu + 5}{4(\nu + 1)^2(\nu + 2)(\nu + 3)} > 0$$

for $\nu \in (\nu^* + 1, 0)$. Here we used the Rayleigh sums [23, p. 502]

$$\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{16(\nu + 1)^2(\nu + 2)}, \quad \sum_{n \geq 1} \frac{1}{j_{\nu,n}^3} = \frac{1}{32(\nu + 1)^3(\nu + 2)(\nu + 3)}.$$
and the fact that function \( f : (-1, 0) \to \mathbb{R} \), defined by
\[
f(x) = -\log 2 - \log(x + 1) + \frac{1}{2(x + 1)} + \frac{3x + 5}{4(x + 1)^2(x + 2)(x + 3)}
\]
is decreasing as the sum of three decreasing functions, and thus \( f(x) > f(0) \approx 0.0151 \ldots > 0 \) if \( x \in (-1, 0) \).

We would like to note that there is another way to prove that \( J_{\nu+1}(1) > 0 \) if \( \nu > \nu^* \). Since we have \( 2^{\nu+1}\Gamma(\nu + 2) > 0 \) for \( \nu \in (-2, \infty) \) it follows that \( J_{\nu+1}(1) \) and \( 2^{\nu+1}\Gamma(\nu + 2)J_{\nu+1}(1) \) have the same sign and the same roots in the interval \((-2, \infty)\). On the other hand, we have
\[
h_{\nu+1}(1) = 2^{\nu+1}\Gamma(\nu + 2)J_{\nu+1}(1) = 1 + \sum_{n \geq 1} \frac{(-1)^n}{4^n n!(\nu + 2) \ldots (\nu + n + 1)}.
\]

Let \( \Lambda : (-2, \infty) \to \mathbb{R} \) be the function defined by
\[
\Lambda(\nu) = h_{\nu+1}(1) = 1 + \sum_{n \geq 1} \frac{(-1)^n}{4^n n!(\nu + 2) \ldots (\nu + n + 1)}.
\]

We will show that \( \Lambda \) is strictly increasing. We have
\[
\frac{d\Lambda(\nu)}{d\nu} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{4^n n!} \left[ \frac{1}{(\nu + 2) \ldots (\nu + n + 1)} \sum_{k=1}^{2n+1} \frac{1}{\nu + k + 1} \right] = \sum_{n \geq 0} \frac{1}{4^{2n+2}(2n + 2)!} \left[ \sum_{k=1}^{2n+1} \frac{1}{\nu + k + 1} \right]^2 - \sum_{n \geq 0} \frac{1}{4(2n + 2)(2n + 3)} \sum_{k=1}^{2n+1} \frac{1}{\nu + k + 1}.
\]

The last expression is certainly positive because the next inequalities hold for \( n \in \{1, 2, \ldots\} \) and \( \nu > -2 \)
\[
\sum_{k=2}^{2n+1} \frac{1}{\nu + k + 1} > \frac{1}{4(2n + 2)(2n + 3)} \sum_{k=2}^{2n+1} \frac{1}{\nu + k + 1},
\]
and
\[
\frac{1}{\nu + 2} > \frac{1}{4(2n + 2)(2n + 3)} \nu + 2 > \frac{1}{2\nu + 2} > \frac{1}{4(2n + 2)(2n + 3)} \nu + 2n + 3.
\]

Consequently we have that \( \Lambda \) is a strictly increasing function, and this implies that the equation \( J_{\nu+1}(1) = 0 \) has a unique root in the interval \((-2, \infty)\). This root is \( \nu^* = -1.7744 \ldots \) It follows also that if \( \nu > \nu^* \) then \( \Lambda(\nu) > \Lambda(\nu^*) = 0 \), or equivalently, \( J_{\nu+1}(1) > 0 \) if \( \nu > \nu^* \).

By using \( \text{(3.12)} \) for \( r = 1 \) and the fact that the function \( \psi \) is strictly decreasing, we get that
\[
\Re \left( 1 + z\frac{\psi''(z)}{\psi'(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{1}{J_{\nu+1,n}^{\prime} - 1} = 1 + \frac{\psi''(1)}{\psi'(1)}
\]
for all \( z \in \mathbb{D} \). Since according to \( \text{(3.55)} \) for each \( n \in \{1, 2, \ldots\} \) the function \( \nu \mapsto J_{\nu,n} \) is strictly increasing on \((-1, \infty)\), it follows that the function \( \nu \mapsto J_{\nu+1,n} \) is strictly increasing on \((-2, \infty)\) for each \( n \in \{1, 2, \ldots\} \). Consequently, the function \( \phi : (\nu^*, \infty) \to \mathbb{R} \), defined by
\[
\phi(\nu) = \frac{J_\nu(1)}{2J_{\nu+1}(1)} - \nu = 1 - \sum_{n \geq 1} \frac{1}{J_{\nu+1,n}^{\prime} - 1},
\]
is strictly increasing too. Note that the function \( \phi \) is well defined since for \( \nu > \nu^* \) we have \( J_{\nu+1}(1) \neq 0 \).

Now, using the fact that \( \phi \) is strictly increasing we obtain that if \( \nu \geq \nu_\alpha = \nu_\alpha(\phi_\nu) \), then the next inequality is valid
\[
\frac{J_\nu(1)}{2J_{\nu+1}(1)} - \nu = \phi(\nu) \geq \phi(\nu_\alpha) = \frac{J_{\nu_\alpha}(1)}{2J_{\nu_\alpha+1}(1)} - \nu = \alpha.
\]
Now, from (3.13) we get that $\nu_\alpha$ is the smallest value having the property that the condition $\nu \geq \nu_\alpha$ implies
\[
\Re \left( 1 + z \frac{\varphi_\nu''(z)}{\varphi_\nu'(z)} \right) > \frac{J_\nu(1)}{2J_{\nu+1}(1)} - \nu = \alpha \quad \text{for all } z \in \mathbb{D}.
\]
In other words, we proved that the function $\varphi_\nu$ is convex of order $\alpha \in [0, 1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(\varphi_\nu)$, where $\nu_\alpha = \nu_\alpha(\varphi_\nu)$ is the unique root of the equation
\[
\frac{J_\nu(1)}{2J_{\nu+1}(1)} - \nu = \alpha
\]
situated in $(-2, \infty)$. In particular, $\varphi_\nu$ is convex if and only if $\nu \geq \nu_0(\varphi_\nu)$, where $\nu_0 \simeq -1.5623\ldots$ is the unique root of the equation $J_\nu(1) = 2\nu J_{\nu+1}(1)$. \hfill $\Box$

**Added in proof.** Recently, Baricz et al. [6] presented an alternative proof of Lemma 4 by using the Hadamard theorem concerning the growth order of entire functions. Moreover, Baricz and Szász [8] presented an alternative proof of the convexity of $h_\nu$ by using a result of Shah and Trimble concerning transcendental entire functions.

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**Department of Economics**, **Babeş-Bolyai University**, 400591 Cluj-Napoca, Romania

*E-mail address: bariczocsi@yahoo.com*

**Department of Mathematics and Informatics**, **Sapientia Hungarian University of Transylvania**, 540485 Târgu-Mureş, Romania

*E-mail address: rszass@sms.sapientia.ro*