Evidence for a gravitational Myers effect

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Abstract

An indication for the existence of a collective Myers solution in the non-abelian D0-brane Born-Infeld action is the presence of a tachyonic mode in fluctuations around the standard diagonal background. We show that this computation for non-abelian D0-branes in curved space has the geometric interpretation of computing the eigenvalues of the geodesic deviation operator for $U(N)$-valued coordinates. On general grounds one therefore expects a geometric Myers effect in regions of sufficiently negative curvature. We confirm this by explicit computations for non-abelian D0-branes on a sphere and a hyperboloid. For the former the diagonal solution is stable, but not so for the latter. We conclude by showing that near the horizon of a Schwarzschild black hole one also finds a tachyonic mode in the fluctuation spectrum, signaling the possibility of a near-horizon gravitationally induced Myers effect.

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1 Introduction

Singularities are both the plague and wonder of classical general relativity. String theory, however, or a fully quantum theory of gravity, is expected to be nonsingular. In string theory, some gravitational singularities are known to be resolved by including contributions to the low energy effective action of highly massive stringy fields or branes which become massless near the classical singularity (see e.g. [1, 2, 3]). A second mechanism is that the singularity may be an artifact of the classical approximation and there are quantum effects which modify or shield its problematic nature [4, 5, 6]. If the singularity is naturally associated with a gauge field, the mechanism of shielding/resolving is by now well-understood. Due to the Myers dielectric effect [7], marginally bound strings and/or other extended objects sensitive to the singularity will polarize to a collective higher dimensional extended object. This smudges the pointlike sensitivity, making the singularity effectively invisible.

The Myers effect is a profound discovery and is relevant in many other situations as well, e.g. it is responsible for the stringy exclusion principle for so-called giant gravitons [8] and explains the massive supersymmetric membrane worldvolume action in maximally symmetric pp-wave backgrounds [9]. Due to its versatile nature, it begs the question how general such induced collective behaviour is in the presence of non-zero background fields. The Myers effect was originally discovered for non-abelian D0-branes in Ramond-Ramond gauge-fields. S-duality allows one to straightforwardly conclude that Neveu-Schwarz fields can be responsible for such an effect as well. The question we will address here is whether gravitational curvature itself may induce collective behaviour.\footnote{This question has been addressed earlier in [10]. These papers investigate whether the RR-Myers solution can be molded to a gravitational Myers solution. The gravitational potential alone is insufficient to make the specific RR-Myers collective D0-brane configuration stable, however.}

We will study this question in the same situation where the original Myers effect was discovered: \(N\) marginally bound non-abelian D0-branes. How to couple non-abelian D0-branes to gravity, has been a long-standing question. In [11] two of us gave an algorithm which imposes the constraints of diffeomorphism invariance on the action for D0-branes in curved backgrounds. The gauge field for diffeomorphisms is the graviton, although the requirement of general coordinate invariance is not stringent enough to determine the action uniquely. This algorithm implements diffeomorphism invariance in an indirect way as basepoint invariance between various Riemann normal coordinate systems. In particular we showed that the often used symmetrized trace approximation is correct to linear order in the background field, as first found in [12], though it breaks down at the next order. Moreover, to this same linear order of approximation, one may in fact use any coordinate system and one is not hampered by the normal coordinate requirement. What we will find is that this approximation, which simplifies the computation dramatically, is already sufficient to study the presence or absence of a gravitational Myers effect.

Naturally the details of the final collective state will depend heavily on the particular choice of gravitational background. This makes it difficult to explicitly construct a collective lesser energetic solution which would prove the existence of a geometrically induced Myers effect. We will follow a different road. In section 2 we will review how the presence of a Myers-dielectric
solution is associated with the presence of a tachyonic mode in the off-diagonal fluctuation spectrum around the naive diagonal “individual” solution. In a curved background the “individual” diagonal solution consists of \( N \) D0-branes following geodesics independently. In section 3 we will show that the spectrum of fluctuations around a geodesic solution has a beautiful geometric and physical interpretation. For a single particle/D0-brane the field equation for quadratic fluctuations is the geodesic deviation equation. It computes the gravitational tidal forces pulling or pushing objects together or apart. Testing for a gravitational Myers effect for a collection of D0-branes is thus the same as computing the non-abelian geodesic deviation equation. We perform the test in the weak gravity approximation explicitly in the next section, both in arbitrary and in Riemann normal coordinates. The latter we discuss in light of its relevance for non-abelian D0-branes coupled to gravity beyond the linear approximation. Having set up a general expression for the mass matrix of fluctuations, we test non-abelian D0-branes on the sphere and the hyperboloid. We find that regions of spacetime with negative curvature have a tachyonic mode in the off-diagonal fluctuation spectrum. This indicates that there may indeed exist a purely gravitationally induced Myers effect. To emphasize this point and test it in a more realistic setting we show that an off-diagonal tachyonic mode is also present near the horizon of a Schwarzschild black hole. The evidence for gravitational Myers-like behaviour we find here is tantalizing for its possible consequences and our understanding of string theory as a theory of quantum gravity. We speculate on this and give an outlook on future directions in the conclusion, section 5.

2 The Myers effect: a brief review

The Myers effect is due to a classical instability in the standard D0-brane configuration in the presence of RR-backgrounds. In a flat empty background a marginally bound state of \( N \)-coincident D0-branes has as low energy effective action 10D \( U(N) \) super-Yang-Mills theory reduced to 0+1 dimensions. Ignoring fermions and choosing the gauge \( A_0 = 0 \), it reads

\[
S(X) = \frac{m}{2} \int d\tau \text{Tr} \dot{X}^i \dot{X}^i + \frac{\lambda^2}{4} \int d\tau \text{Tr} \left( [X^i, X^j]^2 \right), \quad i = 1 \ldots 9.
\] (2.1)

The equations of motion have a unique static solution comprised by the set of diagonal matrices \( X^i = \text{diag}(x_\ell^i) \), with \( x_\ell^i \) interpreted as the position of the \( \ell \)'th D0-brane.\(^2\)

Turning on RR-fields, specifically a constant IIA RR 4-form, modifies the D0-brane potential to [7]\(^3\)

\[
V(X) = -\frac{\lambda^2}{4} \text{Tr} \left( [X^i, X^j]^2 \right) - \frac{i}{3} F_{0ijk} \text{Tr} \left( X^i X^j X^k \dot{X}^0 \right).
\] (2.2)

We will choose the physical gauge \( \dot{X}^0 = 1 \), and we will implicitly limit our attention to the sector \( i = 1, 2, 3 \), and ignore the remaining six directions, i.e. we choose only \( F_{0123} \neq 0 \).

\(^2\)To derive the low energy effective action (2.1) from open string theory, the appropriate solution has \( X^i \) proportional to the unit matrix, \( X^i = x^i 1 \). See the second paragraph in the next subsection for a more detailed discussion of this point.

\(^3\)The signs and factors of \( i \) follow from requiring reality and positivity; we choose \( X^i \) to be Hermitian.
This new term destabilizes the old minima of the potential. Eq. (2.2) has extrema at
\[
\frac{\partial V}{\partial X^i} = 0 = -\lambda^2 [X^j, [X^i, X^j]] - \frac{i}{2} F_{0ijk} [X^j, X^k].
\] (2.3)

The potential has “two” qualitatively different minima, corresponding to static solutions to the field equations. The first is the standard solution with \( X^i \) diagonal, i.e. \( N \) independent D0-branes, and vanishing rest energy. The second set of solutions consists of non-abelian solutions
\[
[X^i, X^j] = -\frac{1}{2 \lambda^2} F_{0ijk} X^k \text{ with negative rest energy } \lambda^2 E_{\text{rest}} = \left( \frac{1}{16} - \frac{1}{12} \right) (F_{0ijk} F_{0}^{ijm} \text{Tr}(X^k X^m));
\]
Here the internal confining force from the commutator squared potential, \([X^i, X^j]^2\), is balanced by the externally induced dielectric stretching. Energetically these second solutions are favoured: a cluster of unattached D0-branes let loose, prefers to form a collective dielectric state, rather then each pursuing its own course.

2.1 Evidence for the Myers effect: quadratic fluctuations

The D0-brane configuration described by the second non-abelian solution had already appeared in the literature prior to Myers’ discovery of the polarization term in (2.2) [13]. The polarization term, however, is crucial in ensuring that the solution is energetically favoured. Suppose that one didn’t know that such a stable non-abelian Myers solution existed. Before tackling the difficult problem of finding new lesser energetic solutions to the field equations by brute force, one could deduce that such a solution exists by calculating the spectrum of quadratic fluctuations around the simple commuting background solution. The presence of the second less energetic type of solutions is reflected in the presence of an tachyonic instability in the off-diagonal fluctuations. [14]. Vice versa, if a tachyonic mode is present, this signals, bar run-away behaviour, a non-abelian collective mode solution. For non-abelian D0-branes coupled to gravity this is the road we shall pursue, rather than attempt to find an explicit less energetic solution to the field equations directly.

As a warm-up exercise, let us confirm this philosophy for the original Myers effect. We will calculate the spectrum of quadratic fluctuations for two flat-space D0-branes coupled to a constant RR-four-form background, as in [14]. We will do this starting from the field equation (2.3), rather than the action (2.2). This way we directly obtain the forces experienced by the D0-branes and it will be a convenient point of view when we discuss D0-branes coupled to gravity in the next section.

We should make one remark at the outset. Already for flat space non-abelian D0-branes coupled to RR-fields, the space of solutions and spectrum of fluctuations is very complicated [14]. In fact, the strict low energy background for a cluster of superposed D0-branes, where the fields \( X^i \) are proportional to the unit matrix, \( X^i = x^i \mathbb{1} \), does not have a tachyonic instability which signals the presence of the dielectric solution. This mode is only visible, as we will see, when the diagonal entries of the background solution are different. This is, strictly speaking, in conflict with the use of the non-abelian effective action in eq. (2.1), for off-diagonal modes are massive in this case. However, as long as the masses proportional to the “distance” between the diagonal entries are less than the string length, the action should be a good approximation. Specifically the regime where we may trust the action is when the following inequalities for \( X^i \),
The first equation simply states the regime of validity for the Born-Infeld action: fields must be slowly varying. The second allows the truncation to second order in velocities. The r.h.s. of the third inequality is also a validity bound of the Born-Infeld action: the Born-Infeld action is a Wilsonian effective action with energies and masses below the string scale $\ell_{s}^{-1} = (\alpha')^{-1/2}$. The left-hand side is the bound set by the eleven dimensional Planck length $\ell_{11} = g_{s}^{1/3}\ell_{s}$. In pure field theory terms, it is the bound which validates the use of perturbation theory.

2.1.1 Computation of the spectrum of fluctuations

Consider thus a small variation $\delta X^{i}$ around a diagonal background solution $\bar{X}^{i}$. The change in the field equation is

$$
\frac{\delta}{\delta X^{i}} \bigg|_{X^{i} = \bar{X}^{i}} = \left. -i F_{0ijk}[\delta X^{j}, \bar{X}^{k}] - \lambda^{2}[[\bar{X}^{j}, \delta X^{i}], [\bar{X}^{j}, \delta X^{i}]] \right|_{X^{i} = \bar{X}^{i}} .
$$

(2.7)

Denoting entries of $\bar{X}^{i}$ and the fluctuations $\delta X^{i}$ as

$$
\bar{X}^{i} = \left( \begin{array}{ccc} x_{1}^{i} & 0 \\ 0 & x_{2}^{i} \end{array} \right) , \quad \delta X^{i} = \left( \begin{array}{ccc} a^{i} \\ b^{i} \\ d^{i} \end{array} \right) , \quad a, d \in \mathbb{R}^{d} ,
$$

(2.8)

the “elementary” commutator $[\delta X^{i}, \bar{X}^{j}]$ equals

$$
[\delta X^{i}, \bar{X}^{j}] = \left( \begin{array}{ccc} 0 & b^{i}(x_{2}^{j} - x_{1}^{j}) \\ -i(b \times \Delta)^{i} + b^{i}\lambda^{2}\Delta^{2} & 0 \end{array} \right) .
$$

(2.9)

The Gauss law $[\bar{X}^{i}, X^{i}] = 0$ tells us that only fluctuations transverse to the static diagonal background are dynamical. Physical fluctuations therefore satisfy $[\delta X^{i}, \bar{X}^{i}] = 0$. Substituting we find $\sum_{i} b^{i}(x_{1}^{i} - x_{2}^{i})^{i} = 0$, i.e. physical fluctuations $b^{i}$ are orthogonal to the D0-brane separation $\Delta^{i} \equiv (x_{1}^{i} - x_{2}^{i})$. This implies that the last term in (2.7) vanishes. The remaining terms are easily computed to be

$$
\frac{\delta}{\delta X^{i}} \bigg|_{X^{i} = \bar{X}^{i}} = \left( \begin{array}{ccc} 0 & i(b \times \Delta)^{i} + b^{i}\lambda^{2}\Delta^{2} \\ -i(b \times \Delta)^{i} + b^{i}\lambda^{2}\Delta^{2} & 0 \end{array} \right) .
$$

(2.10)
We have defined \((a \times b)^i = F_{0ijk}a^j b^k\). The quadratic fluctuation matrix is by definition
\[
\frac{\delta V}{\delta X^ib} \bigg|_{X^i = \tilde{X}^i} = M_{ib,ja} \delta X^ja, \quad a = 1, ..., N^2,
\]
where we consider the matrix fields \(X^i\) as column vectors. Rewriting eq. (2.10) as such a mass term, we find
\[
\delta^2 V \bigg|_{X^i = \tilde{X}^i} = \int d\tau \ b^i \left( \delta_{ij} \lambda^2 \Delta^2 + iF_{0ikj} \Delta^k \right) \bar{b}^j + c.c.
\]
The masses are then given by the eigenvalues of this complex \(d \times d = 3 \times 3\) matrix. Setting \(F_{0ijk} = \lambda^2 \rho \epsilon_{ijk}\) and recalling that \(i, j, k\) take values in the range \(i = 1, 2, 3\), one finds the characteristic polynomial
\[
\left| (-\xi + \Delta^2) I_{3 \times 3} + i \rho \epsilon_{ikj} \Delta^k \right| = (-\xi + \Delta^2)^3 - \rho^2 (-\xi + \Delta^2) \Delta^2 = 0,
\]
with solutions\(^4\)
\[
\xi = \Delta^2, \quad \text{degeneracy : 2},
\]
\[
(-\xi + \Delta^2)^2 - \rho^2 \Delta^2 = 0 \Rightarrow \xi = \Delta^2 \pm \rho \Delta, \quad \text{degeneracy : 2(each)}.
\]
For small values of \(\Delta\) and \(\rho\) of order unity, the second set of solutions will have negative eigenvalues which signal the presence of tachyons. These are responsible for the Myers effect. Note that for \(\rho = 0\), we recover that the off-diagonal fluctuations are proportional to the distance between the D0-branes. But also note how for \(\Delta = 0\) all tachyonic modes are absent. As we forewarned, truly superposed D0-branes are in fact marginally stable. If we insist on the validity of perturbation theory, though, \(\Delta\) has a minimal length set by eq. (2.6). In that case Del strictly vanishing is not allowed, and the diagonal solution is unstable.

In summary, turning on an RR-background field exerts a destabilizing force on the naive commuting solution to the D0-brane field equations. The signal of destabilization is the presence of tachyonic modes in the fluctuation spectra of the off-diagonal degrees of freedom. The presence of these modes indicates that there exists a less energetic solution to the field equations, where the internal confining non-abelian potential forces are balanced by the external dielectric repulsing forces. Physically the commuting solution corresponds to a background of \(N\) independent D0-branes, whereas in the non-abelian solution the D0-branes behave collectively as an extended object, a D2-brane.

### 3 D0-branes and Gravity

The question we seek to answer here, is whether the gravitational force can have a similar destabilizing influence on the commuting background solution. Bar runaway solutions, this

\(^4\)This set of solutions eq. (2.14) corresponds to Appendix A of [14]; The three remaining zero modes in the solution presented there are the freedom to set the values \(a^i\) in \(\delta X^i\) of eq. (2.8); for traceless fluctuations as they consider \(d^i\) then equals \(d^i = -a^i\).
would indicate the existence of a similar non-abelian mode, where the $N$ D0-branes behave collectively as an extended object.

To investigate this we need to know the action for non-abelian D0-branes in curved space. How to couple the non-abelian D0-brane action to gravity has been a prominent question since their discovery. The fields $X^i$ describing the transverse coordinates are now $U(N)$-valued and no longer commute. Any term in the action beyond quadratic order needs a specific ordering instruction. In particular the naive introduction of a metric into the kinetic term of the action (2.1),

$$S_{\text{naive,kin}} \sim \frac{m}{2} \int d\tau \text{Tr} G_{ij}(X) \dot{X}^i \dot{X}^j,$$

(3.1)

is ambiguous. In principle the action, and therefore the ordering, is derivable from string-theory amplitude-computations [16]. In practice this is a daunting task, and the general approach has been to construct the action ab initio subject to a set of consistency conditions and constraints [17]. Based on the general properties of such amplitudes, and known physical quantities which the correctly ordered action should reproduce, Douglas put forth a set of axioms, which the action for D0-branes in curved space ought to obey [18]. Concretely they are

- The action must contain a single $U(N)$ trace [19].
- For diagonal matrices the action must reduce to $N$ copies of the first quantized particle action in curved space.
- The moduli-space of static solutions to the field equations must equal $N$-copies of the (spacetime) manifold $M$ modulo the action of the permutation group: $\mathcal{M} = M^N/S_N$.

and

- **Masses of off-diagonal fluctuations around a diagonal background** $X^i = \text{diag}(x^i_\lambda)$, should be proportional to the geodesic distance between the corresponding entries.

Collectively these requirements are known as the axioms of D-geometry.

In [11] two of us derived an algorithm to constrain the ordering of the action using a more fundamental principle. The characteristic feature of any action coupled to gravity is diffeomorphism invariance. The presence of some form of coordinate invariance for matrix-valued coordinates, together with the single trace requirement and that for linearized gravity the ordering would be completely symmetrized (see section 4), were the sole input we used to determine the ordering. The answer does not, surprisingly, appear to be unique despite the very stringent nature of the non-linear diffeomorphism constraints. However, due the fundamental nature of diffeomorphism invariance, an action obtained by this algorithm obeys all the axioms of D-geometry.

These very axioms, however, put us in a bind where the question of a gravitational Myers effect is concerned. In the previous section we showed that the presence of the Myers effect is signaled by tachyonic modes in the quadratic fluctuation spectrum. But the fourth, italicized,
axiom of D-geometry states that the mass-matrix of off-diagonal fluctuations should be proportional to geodesic lengths, which are “strictly positive”. Hence one’s initial reaction is to say that no gravitational Myers effect exists. However, careful consideration of the D-geometry axioms shows that the fourth axiom is only applicable to infinitesimal fluctuations around a static solution to the field equations.

Static solutions are not very natural in a curved background. A first consequence of the gravitational curvature is the exertion of a gravitational force on the cluster of D0-branes, accelerating them away from zero velocity. Generically we expect that each of the D0-branes will independently follow a geodesic. In fact, D-geometry requires that the naive solution for a cluster of D0-branes, each following geodesic flow independently, remains a solution to the non-abelian field equations. It is in the perturbation around such a set of independent geodesics rather than the static solution that we should search for negative eigenvalues in the mass matrix of off-diagonal fluctuations. The D0-branes must therefore have nonzero velocity for the collective behaviour to occur. Hence a Myers effect arises from balancing a stretching induced by the, now non-zero, kinetic term and the confining potential term. This in fact agrees nicely with the RR-background story, in that for gravity the kinetic term is the universal coupling.

And at this moment we have arrived on familiar territory. The “physical” stretching of an extended object in the presence of gravitational curvature is a consequence the gravitational tidal forces. These forces are traditionally derived using what is known as the geodesic deviation equation. Perhaps less known is that the geodesic deviation equation can be derived from the particle action in curved space by considering quadratic fluctuations around a geodesic solution [20]. This puts the procedure outlined above on firm geometric footing. Testing for the Myers effect is equivalent to computing non-abelian geodesic deviation.

3.1 Quadratic fluctuations, geodesic deviation and tidal forces

Let us briefly recall this connection between quadratic fluctuations and the geodesic deviation equation for the standard particle case (see e.g. [20]). The action is the proper distance,

$$S_{\text{part}} = m \int d\tau \sqrt{-g_{ij}\dot{x}^i\dot{x}^j} ,$$

with as field equation the geodesic equation

$$\frac{\partial S}{\partial x^i} = 0 = g_{ij}\ddot{x}^j + \Gamma_{ijk}\dot{x}^j\dot{x}^k = g_{ij}\dot{x}^k\nabla_k\dot{x}^j \equiv g_{ij}\nabla_\tau\dot{x}^j .$$

Consider now the quantity $\nabla_\tau\dot{x}^j$ and vary its dependence on $x^i$ one more time. This can be implemented explicitly by letting $x^i$ depend on some additional “affine” parameter $s$, vary $\nabla_\tau\dot{x}^j$ w.r.t. this parameter $s$, and then extract an overall factor $x^j = \frac{\delta}{\delta s} x^j$. Of course the resulting combination,

$$\delta_s \nabla_\tau\dot{x}^j \equiv \delta_s x^j \nabla_x\dot{x}^j ,$$

is not covariant, but can be made so by adding an improvement term,

$$\delta_s \nabla_\tau\dot{x}^j + \delta_s \Gamma_{kn} \nabla_\tau\dot{x}^n \equiv \nabla_s \nabla_\tau\dot{x}^j .$$
Note, however, that the improvement term is directly proportional to the geodesic equation. Therefore, if we substitute as background value for $x^j$ a solution to the field equation, the improvement term is formally zero and we may add it for free. As a consequence

$$\delta_s \nabla^j \dot{x}^j \Big|_{x^j \ geodesic} = \nabla^j \nabla^j \dot{x}^j \Big|_{x^j \ geodesic}.$$  \hspace{1cm} (3.6)

From elementary algebra it follows that

$$\nabla_s \nabla^j \dot{x}^j = R_{\tau k}^j \dot{x}^k + \nabla_{\tau \dot{x}} \dot{x}^j = R_{\tau k}^j \dot{x}^k + \nabla_{\tau \dot{x}} \dot{x}^j .$$ \hspace{1cm} (3.7)

If we therefore start from the geodesic equation,

$$\nabla_{\tau \dot{x}} \dot{x}^j = 0 .$$ \hspace{1cm} (3.8)

and vary both sides, we find using eq. (3.6) that the r.h.s. of (3.7) must vanish. This is the geodesic deviation equation. It determines the acceleration $(\nabla^2 \tau)$ of an infinitesimally close geodesic $(\dot{x}^j)$ in terms of the background Riemann tensor. It therefore determines how fast infinitesimally close non-interacting particles are pulled apart/together in a gravitational field due to tidal forces.

It is now clear how quadratic fluctuations and the tidal forces are related. The quadratic fluctuation matrix

$$\delta \frac{\partial S}{\partial x^i} = M_{ij} \delta x^j \iff \delta \frac{\partial S}{\partial x^i} = M_{ij} \dot{x}^j = (R_{ikj} \dot{x}^k \dot{x}^\ell - \nabla_{\tau \dot{x}} \dot{x}^j) x^\ell$$ \hspace{1cm} (3.9)

is the geodesic deviation “operator”. The geodesic deviation equation thus computes the kernel of this operator. Hence, one way to interpret a solution $\delta x^j_{\text{sol}}(\tau)$ to the geodesic deviation equation, is that to first order $x^j(\tau)_{\text{geodesic}} + \delta x^j_{\text{sol}}(\tau)$ is then also a solution to the geodesic equation (see [20]). Since the geodesic equation is the field-equation of the proper-distance action, this specific variation $\delta x^j_{\text{sol}}(\tau)$, such that the geodesic equation is still satisfied, must be a zero mode in the spectrum of fluctuations determined by the action. And this is just what eq. (3.9) says.

If the background D0-brane configuration is constant, i.e. $\dot{x}^i = 0$, the “geodesic deviation operator” reduces to $\partial / \partial^2 \tau$. This is the static solution situation appropriate for D-geometry, as discussed above. If the D0-branes have non-zero velocity, however, the “geodesic deviation operator” includes the characteristic extra term proportional to the Riemann tensor. This term encodes the gravitational potential. In field theory terms we can interpret it as a mass term for the fluctuations. In particular positive mass fluctuations correspond to converging geodesics, and negative mass fluctuations correspond to “unstable” diverging geodesics. For non-abelian D0-branes following independent geodesics, these diagonal fluctuations will still be present. New, however, will be the presence of off-diagonal fluctuations, probing the stability of the configuration. If there exists a gravitational Myers effect for non-abelian D0-branes, it is the non-commutative analog of the extra curvature term in the geodesic deviation operator eq. (3.9) which should be responsible for the destabilization.

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5We are using conventions where $R_{\mu \nu \rho \tau} = -\partial_\mu \Gamma_{\tau \nu \rho} + \Gamma^\kappa_{\mu \rho} \Gamma_{\kappa \nu \tau} - (\mu \leftrightarrow \nu)$. 

9
3.1.1 Intuitive instability of D0-branes in a curved background

The computation that should establish the existence of a gravitational Myers effect thus fits beautifully in a geometric framework from the D0-brane perspective. At a fundamental level, D-brane dynamics is governed by open strings; the D-brane is a (dynamical) defect on which open strings can end. The RR-Myers effect has a simple physical interpretation from this point of view. The RR-field polarizes and stretches the charges at the endpoints of the dynamical open string, until balanced by the internal string tension. For gravitational tidal forces one can develop a similar intuitive picture, which argues that in particular backgrounds a gravitational Myers effect should occur. We can model the string by an elastic cord — a massless ideal spring — connecting two infinitesimally separated masses. If the tidal forces, measured by the geodesic deviation equation are attractive, the additional presence of the connecting spring will be qualitatively irrelevant. If the tidal forces are repulsive, however, one expects the naive geodesic system to be unstable. The preferred state would be that where the separation is such that the tidal and spring forces precisely balance each other. Such a toy model can be described by the equation

\[
\frac{d^2}{d\tau^2}d(x(\tau), y(\tau)) + d(x(\tau), y(\tau)) = 0, \tag{3.10}
\]

where \(d(x, y)\) is the geodesic distance between the positions of the two masses, \(x\) and \(y\). This is an effective equation valid in the frame of a specific observer, whose clock measures \(\tau\). Tidal forces are repulsive if the spacetime is negatively curved. Qualitatively we therefore expect negative curvature to be a precursor of the gravitational Myers effect. The exact computation, with which we will now proceed, will indeed bear this out.

4 The geometric Myers effect

As we stated in the introduction, a consistent, though not unique, coupling of non-abelian D0-branes to gravity can in principle be algorithmically derived to any order of precision [11]. The principle behind this algorithm is the requirement of diffeomorphism invariance of the non-abelian D0-brane action. The algorithm implements diffeomorphism invariance in a non-standard way as base-point independence between normal coordinate systems rooted on an arbitrary basepoint \(P\). This paradoxical appearance of preferential coordinates, however, is only necessary if one goes beyond linear order in the curvature. At linear order, the non-abelian nature of the coordinates is only marginally relevant. Within this approximation one may in fact use any coordinate system, as the action, specified with the completely symmetrized ordering, is up to the same linear order of approximation invariant under diffeomorphisms [11].

4.1 Weak curvature: The symmetrized trace approximation

A natural first attempt would then be to conjecture that a Myers-like effect is already visible in this simplified situation, i.e. when the spacetime is only slightly distorted and we may consider the curvature as a perturbation.
As we just explained the advantage of treating the curvature perturbatively is that to first order in a perturbation the action in any coordinate system is simply the linearized symmetrized trace action action, first found by Taylor and van Raamsdonk [12] and confirmed by explicit string amplitude calculations [16].

\[ S_{\mathcal{O}(h)} = \int d\tau \frac{m}{2} \left( \eta_{\mu\nu} \text{Tr} \dot{X}^\mu \dot{X}^\nu + \text{Str}(h_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu) \right) + \frac{\lambda^2}{2} \text{Str}(h_{\mu\nu}(X)\eta_{\alpha\beta}[X^\mu, X^\alpha][X^\nu, X^\beta]) + \ldots \] (4.1)

This is because the action is invariant under linearized diffeomorphisms, which is easily verified. Once we study specific examples, we will therefore be able to avoid the difficult step of having to transform to normal coordinates first. The restriction to weak gravity does create an additional bound on the validity of the action in addition to eqs. (2.4)-(2.6): the moments of deviation from the flat metric, including the zeroth one, which act as coupling constants in the action (4.1), must be small

\[ (\ell_s \partial^n)^n h_{\mu\nu}(y)|_{y=0} \ll 1 \quad \forall \ n \geq 0. \] (4.2)

However, we do not need that \( \ell_s \partial^{n+1} h_{\mu\nu} \ll \partial^n h_{\mu\nu} \).

We are now in a position to compute the mass matrix of quadratic fluctuations around a diagonal background of geodesics. There will be two contributions to the mass matrix, one from the new curvature term in the geodesic deviation operator which originates in the kinetic term, and one from the potential term. Let us focus on the new curvature contribution from the kinetic term first.

### 4.1.1 The kinetic term

A convenient way to write the symmetrized trace expression for the action is by exponentiation

\[ S_{\text{kin}} = \int d\tau \frac{m}{2} \left( \eta_{\mu\nu} \text{Tr} \dot{X}^\mu \dot{X}^\nu + \text{Str}(h_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu) \right) \] (4.3)

where, implicitly, we limit our attention to linear terms in \( g_{\mu\nu}(y) = \eta_{\mu\nu} + h_{\mu\nu}(y) \). To evaluate the exponential, one uses standard functional differentiation,

\[ \partial_{\gamma}\partial_{\tau}\delta(\tau - \tau') \] (4.4)

Varying the fields \( X^i \) twice, the formal expression for quadratic fluctuations around a background solution equals

\[ \delta^2 S_{\text{kin}} = \int d\tau \int_0^1 ds \left( \text{Tr} e^{s X \dot{X} \delta X} \right) \partial_y e^{(1-s) X \dot{X} \delta X} \partial_y g_{\mu\nu}(y) \dot{y}^\mu \dot{y}^\nu \] (4.5)

To evaluate the derivatives we use the identity

\[ \partial_y^n \partial_y m f(y) = \partial_z^n \partial_y f(y + z)|_{z=0} \] (4.6)
to separate derivatives contracted with $\delta X^\mu$ from those in the exponentials.

\[
\delta^2 S_{\text{kin}} = \frac{m}{2} \int d\tau \int_0^1 ds \text{ Tr } e^s \int X \cdot \partial_y \delta X \cdot \partial_z e^{(1-s)} \int X \cdot \partial_y \delta X \cdot \partial_z \left[ g_{\mu
u}(y+z)(\dot{y} + \dot{z})^\mu (\dot{y} + \dot{z})^\nu \right] \bigg|_{z=0} (4.7)
\]

Expanding the argument $g_{\mu
u}(y+z)(\dot{y} + \dot{z})^\mu (\dot{y} + \dot{z})^\nu$ to second order is by now a well known exercise. Using the field equation and following the steps leading to eq. (3.9) we obtain the kernel of the geodesic deviation operator

\[
\delta^2 S_{\text{kin}} = \frac{m}{2} \int d\tau \int_0^1 ds \text{ Tr } e^s \int X \cdot \partial_y \delta X \cdot \partial_z e^{(1-s)} \int X \cdot \partial_y \delta X \cdot \partial_z \left[ R_{\mu\alpha\beta\nu}(y)\dot{y}^\alpha \dot{y}^\beta z^\mu z^\nu + g_{\mu\nu}(y)\nabla_\tau(y)z^\mu \nabla_\tau(y)z^\nu \right] (4.8)
\]

The second term is the kinetic term for the fluctuations. The Christoffel symbols present in the derivatives serve to covariantize the momenta but do not contribute to the mass-matrix. This is evident from a computation in normal coordinates, which will be presented in the next subsection. As presaged above, it is the first term, proportional to the Riemann tensor, which is relevant. Concentrating only on this term,

\[
\delta^2 S_{\text{kin,R}} = \frac{m}{2} \int d\tau \int_0^1 ds \text{ Tr } e^s \int X \cdot \partial_y \delta X \cdot \partial_z e^{(1-s)} \int X \cdot \partial_y \delta X \cdot \partial_z \left[ R_{\mu\alpha\beta\nu}(y)\dot{y}^\alpha \dot{y}^\beta z^\mu z^\nu \right]
\]

\[
= \frac{m}{2} \int d\tau \int_0^1 ds \text{ Tr } e^s \int X \cdot \partial_y \delta X^\mu e^{(1-s)} \int X \cdot \partial_y \delta X^\nu \left[ R_{\mu\nu}(y) \right]
\]

with $R_{\mu\nu}(y) = R_{\mu\alpha\beta\nu}(y)\dot{y}^\alpha \dot{y}^\beta$, we can simplify it further. We are considering fluctuations around a background $X^\mu_{bg}(\tau) = \text{diag}(x^\mu(\tau))$ with each $x^\mu(\tau)$ obeying the standard geodesic equation (3.3). This means that each exponential is in fact a diagonal matrix. Consider for simplicity a two by two (sub)system of D0-branes, and parameterize the exponential and the fluctuations as

\[
e^s \int X \cdot \partial_y = \begin{pmatrix} e^s \int x_1 \cdot \partial_y & 0 \\ 0 & e^s \int x_2 \cdot \partial_y \end{pmatrix}, \quad \delta X^\mu = \begin{pmatrix} a^\mu \\ b^\mu \end{pmatrix}.
\]

Multiplying and taking the trace of these explicit matrices we find that

\[
\delta^2 S_{\text{kin,R}} = m \int d\tau \int_0^1 ds \left[ f_1(s) f_1(1-s) a^\mu a^\nu + f_2(s) f_2(1-s) d^\mu d^\nu + \right. \]

\[
\left. f_1(s) f_2(1-s) b^\mu b^\nu + f_2(s) f_1(1-s) b^\mu b^\nu \right] R_{\mu\nu}(y).
\]

As $f_1(s) f_1(1-s) = f_1(1) = e^s \int X \cdot \partial_y$ we immediately recognize the $a^\mu a^\nu$ term and the $d^\mu d^\nu$ term as the diagonal quadratic fluctuations around the $x_1$ and $x_2$ geodesic respectively. These measure the standard (abelian) geodesic deviation. The parts of interest to us are the new contributions
from the off-diagonal fluctuations $b^i$; these are relevant for the stability of the configuration. As their mass does not depend on $a^i$ and $d^i$, we will set $a^i = 0 = d^i$ in the remainder.

Noting that
\[ f_1(s)f_2(1 - s) = e^{\int x_2 \cdot \partial_y + s \int (x_1 - x_2) \cdot \partial_y}, \]
we can evaluate all the partial derivatives to
\[ f_1(s)f_2(1 - s)R_{\mu\nu}(y) = R_{\mu\nu}(x_2 + s\Delta), \quad \Delta^i \equiv (x^i_1 - x^i_2). \]
As final expression for the contribution to the mass matrix of off-diagonal fluctuations from the kinetic term we obtain
\[ \delta^2 S_{kin,R} = m \int d\tau \int_0^1 ds \left[ b^\nu \bar{b}^\nu (R_{\mu\nu}(x_2 + s\Delta) + R_{\nu\mu}(x_1 - s\Delta)) \right]. \]
Note that the expression looks remarkably like an averaging of the curvature over a string worldsheet.

To eq. (4.14) we have to add the contribution from the potential term.

### 4.1.2 The potential term

To find the quadratic fluctuations of the potential term,
\[ S_{pot} = \frac{\lambda^2}{4} \int d\tau \int_0^1 ds \operatorname{Tr} e^{s \int X \cdot \partial_y [X^\mu, X^\alpha]} e^{(1 - s) \int X \cdot \partial_y [X^\nu, X^\beta]} g_{\mu\nu}(y) g_{\alpha\beta}(y), \]
(again implicitly limiting our attention to only the linear term in $h_{\mu\nu}(y)$) is quite a bit easier. This is because the commutators vanish when evaluated on the diagonal background. So a non-vanishing contribution to the quadratic fluctuations can only occur when we vary one of the two $X^\mu$’s in each of the commutators,
\[ \delta^2 S_{pot} = \lambda^2 \int d\tau \int_0^1 ds \operatorname{Tr} e^{s \int X \cdot \partial_y [\delta X^\mu, X^\alpha] + [X^\mu, \delta X^\alpha]} e^{(1 - s) \int X \cdot \partial_y [\delta X^\nu, X^\beta]} g_{\mu\nu}(y) g_{\alpha\beta}(y). \]
Just as for the RR-case, the Gauss-Law,
\[ \int_0^1 ds [X^\mu, e^{s \int X \cdot \partial_y \dot{X}^\nu} e^{(1 - s) \int X \cdot \partial_y}] g_{\mu\nu}(y) = 0, \]
tells us that fluctuations parallel to the diagonal non-static background are non-dynamical. In addition the fluctuation must be orthogonal to the background velocity; for a $2 \times 2$ system in a non-static diagonal background the orthogonality conditions read (to first approximation in a weak background metric) $b \cdot \Delta = 0$ and $b \cdot \dot{\Delta} = 0$. Dropping crossterms where $\delta X^\nu$ is contracted with $X^\mu$, the remaining physical terms of the potential are
\[ \delta^2 S_{pot} = \lambda^2 \int d\tau \int_0^1 ds \operatorname{Tr} e^{s \int X \cdot \partial_y [\delta X^\mu, X^\alpha]} e^{(1 - s) \int X \cdot \partial_y [\delta X^\nu, X^\beta]} g_{\mu\nu}(y) g_{\alpha\beta}(y). \]
Substituting the background values and explicit parameterization of $X^\mu$ and $\delta X^\mu$ for the two by two system from eqs. (4.10), the “elementary” commutator is the same as in (2.9),

$$
[\delta X^\mu, X^\alpha] = \begin{pmatrix}
0 & -b^\mu \Delta^\alpha \\
\bar{b}^\mu \Delta^\alpha & 0
\end{pmatrix}.
$$

(4.18)

The exponentials are again diagonal and the trace can be evaluated as before. We get

$$
\delta^2 S_{\text{pot}} = \lambda^2 \int d\tau \int_0^1 ds \left[ -f_1(s) f_2(1-s) b^\mu b^\nu - f_2(s) f_1(1-s) b^\nu b^\mu \right] \Delta^\alpha \Delta^\beta g_{\mu \nu}(y) g_{\alpha \beta}(y)
$$

(4.19)

$$
= \lambda^2 \int d\tau \int_0^1 ds \left[ -g_{\mu \nu}(x_2 + s \Delta) g_{\alpha \beta}(x_2 + s \Delta) - g_{\mu \nu}(x_1 - s \Delta) g_{\alpha \beta}(x_1 - s \Delta) \right] b^\mu b^\nu \Delta^\alpha \Delta^\beta .
$$

The final task is to add eqs (4.19) and (4.14), diagonalize the mass matrix and inspect whether one of the eigenvalues is negative. If so, this signals the presence of a gravitational Myers effect.

### 4.1.3 Riemann Normal Coordinates

These results, eqs. (4.14) and (4.19), are valid in any coordinate system to linear order in the metric. As we mentioned repeatedly, this is due to the use of the weak gravity approximation. To extend beyond linear order, while maintaining the determining characteristic of a theory coupled to gravity, diffeomorphism invariance, we found that it is most readily implemented in an indirect way as a base-point independence constraint between Riemann normal coordinate systems [11]. Preferential coordinates in a coordinate invariant theory may sound odd, but it is useful to think of the noncommuting structure as defined in the tangent space (to a (base)point $P$). And normal coordinates are the coordinates of the tangent space at $P$ pulled back to the base manifold. This is also the way one generalizes standard non-commutative geometry, i.e. with central non-commutativity $[x^i, x^j] = \theta^{ij}$, $[x^i, \theta^{jk}] = 0$, to curved Poisson manifolds [21].

Because RNC systems are relevant in this sense to non-abelian D0-brane dynamics, let us also compute the leading contribution to masses of off-diagonal fluctuations around geodesics explicitly in such a coordinate system. Performing geodesic computations in RNC systems has the additional benefit that one has in effect already solved the geodesic equation. By definition geodesics through the basepoint $P$ are “straight” lines, linear in the affine parameter. Again the price to pay is one of range of validity. RNC are valid in patches where geodesics through $P$ do not cross. In practice, normal coordinates are useful in the patch around $P$ as long as the geodesic Taylor expansion converges. In the Riemann normal coordinate system itself this translates into the fact that one may express the metric and any other tensor as a Taylor series around $P$, the origin of the RNC system (see e.g. [23, 24, 25]). This is how RNC systems are generally used: as approximations near $P$. Taylor expanding the action itself, the first correction in distance to the origin occurs at fourth order (by construction) and is proportional to the Riemann tensor. We will make this approximation to fourth order also for the potential. At this level the weak gravity approximation is academic, as in Riemann normal coordinates with Taylor expanded metric, the weak gravity approximation is the same as an expansion in curvatures; i.e. we keep derivatives of the Riemann tensor, but discard higher powers.
To illustrate the advantage of RNC, let us start again from the action. As eq. (4.23) will show the result will be the same as that obtained from substituting RNC in eqs (4.14) and (4.19).

Ignoring the kinetic term, the action in RNC around a point \( p \) and to fourth order is \([11]\),

\[
S = \frac{1}{2} \int d\tau \frac{m}{3} R_{\mu\alpha\beta\nu}(p) \text{Str}(X^\alpha X^\beta \dot{X}^\mu \dot{X}^\nu) + \frac{\lambda^2}{2} \sum_{\mu\nu} g_{\mu\nu}(p) g_{\alpha\beta}(p) \text{Tr}([X^\mu, X^\alpha][X^\nu, X^\beta]).
\]

(4.20)

Note that the Riemann tensor and the metric evaluated at the base-point \( p \) are well-defined “classical” abelian objects. It is only the deviations from the basepoint which are promoted to \( U(N) \) matrices.

Computing the second order variation of the action in the fluctuation \( \delta X \), we find

\[
\delta^2 S = \frac{1}{2} \int d\tau \left[ 2m R_{\beta\alpha\mu\nu}(p) \text{Str}(\delta X^\alpha \delta X^\mu \dot{X}^\beta \dot{X}^\nu) \\
+ 2\lambda^2 \sum_{\mu\nu} g_{\mu\nu}(p) g_{\alpha\beta}(p) \text{Tr}(\delta X^\mu [\delta X^\alpha, [X^\nu, X^\beta]]) \\
+ 2\lambda^2 \sum_{\mu\nu} g_{\mu\nu}(p) g_{\alpha\beta}(p) \text{Tr}(\delta X^\mu [X^\alpha, [\delta X^\nu, X^\beta]]) \\
+ 2\lambda^2 \sum_{\mu\nu} g_{\mu\nu}(p) g_{\alpha\beta}(p) \text{Tr}(\delta X^\mu [X^\alpha, [\delta X^\nu, \delta X^\beta]]) \right].
\]

(4.21)

Specializing to the case of two D0-branes with

\[
\delta X^\mu = \begin{pmatrix} 0 & \bar{b}^\mu \\ \bar{b}^\mu & 0 \end{pmatrix}, \quad X^\mu = \begin{pmatrix} x_1^\mu & 0 \\ 0 & x_2^\mu \end{pmatrix}.
\]

(4.22)

imposing the Gauss law constraint, and recalling the definitions \( \Delta^\mu = x_1^\mu - x_2^\mu \), \( \bar{x}^\mu \equiv \frac{1}{2} (x_1^\mu + x_2^\mu) \), we finally arrive at the expression for the mass matrix

\[
\delta^2 S = -\int d\tau [b^\nu (2\lambda^2 \sum_{\mu\nu} g_{\mu\nu}(p) g_{\alpha\beta}(p) \Delta^\alpha \Delta^\beta \\
- \sum_{\mu\nu} m(\bar{R}_{\alpha\nu\beta}(p) + \bar{R}_{\alpha\mu\beta}(p))(\dot{x}^\alpha \dot{x}^\beta + \frac{1}{12} \dot{\Delta}^\alpha \dot{\Delta}^\beta)]b^\nu] \\
\equiv -\int d\tau b^\nu m_{\nu\mu}(p)b^\mu.
\]

(4.23)

This is indeed nothing but eqs. (4.14) and (4.19) combined after the substitution \( \bar{R}_{\mu\nu}(x) = R_{\mu\alpha\beta\nu}(p)\dot{x}^\alpha \dot{x}^\beta + \mathcal{O}(x^3) \). This computation thus justifies our neglect of the connection terms in the kinetic operator to the mass-matrix of the fluctuations.

\[\text{For higher order terms, it is probably more convenient to use the auxiliary } dN^2 \text{ dimensional tensors defined in [11].}\]
4.2 D0-branes on a sphere.

The equivalence between quadratic fluctuations and the geodesic deviation equation predicts that a tachyonic mode could only be present in negatively curved region of space-time. Here and in the next subsection we will confirm this prediction by computing the mass-matrix of quadratic fluctuations on the canonical examples of positively and negatively curved spacetimes: respectively the sphere and the hyperboloid.

Consider $N$ D0-branes on $\mathbb{R} \times S^2$ and ignore the remaining seven directions. For the two-sphere we chose the metric

$$ds^2 = -d\tau^2 + \left(\frac{d\rho^2}{1 - \rho^2/D^2}\right) + \rho^2 d\phi^2$$

$$= -d\tau^2 + dx^2 + dy^2 + (dx + dy)^2 \left(\frac{(x^2 + y^2)}{D^2 - (x^2 + y^2)}\right). \quad (4.24)$$

The weak gravity approximation is applicable for $\rho \ll D$, the radius of the two-sphere.

As background configuration we will consider for simplicity two D0-branes on geodesics approximately through the origin $\rho = 0$. That is, we choose $\dot{\phi}_i = 0$, $(\ell = 1, 2)$ for both D0-branes. To explicitly satisfy the orthogonality conditions implied by the Gauss law, we set the angular separation of the D0-branes $\Delta^\phi$ to zero: $\Delta^\phi = 0$. This ensures that the fluctuation $b^\phi$ is unconstrained. If the D0-branes are truly coincident in the seven additional dimensions as well, i.e. $\Delta^i = 0$, $i = 1 \ldots 7$, the Gauss law constrains the $b^i$ fluctuation to vanish.\(^7\) The situation we therefore consider is one with two D0-branes moving on the same great arc through the origin, but slightly separated along the direction of the arc.

To compute the contribution from the kinetic term to the mass-matrix we need to evaluate the intermediate quantity $R_{\mu\nu}(x) \equiv R_{\mu\nu\beta\gamma}(x)\dot{x}^\alpha\dot{x}^\beta$ defined below eq. (4.9). The Riemann tensor for a sphere is well known and using that only $\dot{x}^\rho \neq 0$ we find a single nonzero component: \(^8\)

$$R_{\phi\phi}(x) = R_{\phi\rho\rho\phi}(\dot{x}^\rho)^2$$

$$= -\frac{1}{D^2} \left[ g_{\phi\phi}(x)(\dot{x}^\rho)^2 \right] = -\frac{1}{D^2} (x^\rho)^2 (\dot{x}^\rho)^2. \quad (4.25)$$

The mass-matrix contribution from the kinetic term is therefore simply

$$\delta^2 S_{\text{kin},m} = \int d\tau \int_0^1 ds \: b^\phi \bar{b}^\phi \left[ -\frac{m}{D^2} \left( g_{\phi\phi}(x_2 + s\Delta)(\dot{x}_2^\rho + s\dot{\Delta}^\rho)^2 + g_{\phi\phi}(x_1 - s\Delta)(\dot{x}_1^\rho - s\dot{\Delta}^\rho)^2 \right) \right] (4.26)$$

Note that the unphysical fluctuation $b^i$ does not couple to the background curvature.

Making the additional approximation that $\dot{\Delta}^\rho$ is negligible (the D0-branes have approximately the same initial speed away from the origin; their motion only differs in the $\phi$-direction and a small separation in $\rho$), we find ($\bar{x}_1^\rho = \frac{1}{2}(x_1^\rho + x_2^\rho)$)

$$\delta^2 S_{\text{kin}} = \int d\tau \: b^\phi \bar{b}^\phi \left[ -\frac{8m}{D^2} \left( (\bar{x}^\rho)^2 + \frac{1}{12} (\dot{\Delta}^\rho)^2 \right) (\dot{x}^\rho)^2 \right]. \quad (4.27)$$

\(^7\)This is a completely arbitrary choice. The Gauss law orthogonality conditions must hold in the full nine transverse dimensions. One has a lot of freedom in choosing how to satisfy them. E.g. separating the D0-branes also slightly in one of the seven additional dimensions forces a linear combination of the $b^i$ fluctuations to vanish.

\(^8\)In our conventions the Riemann tensor of an $n$-sphere equals $R_{\mu\nu\rho\tau} = D^{-2} (g_{\mu\rho}g_{\nu\tau} - g_{\nu\rho}g_{\mu\tau})$. 

16
The potential term is diagonal and yields
\[
\delta^2 S_{\text{pot}} = \lambda^2 \int d\tau \int_0^1 ds \ b^\phi \tilde{b}^\phi \left[ -8(\tilde{x}^\rho)^2 \Delta^2 - 4(\tilde{x}^\rho)^2 (h_{\rho\rho}(x_1) + h_{\rho\rho}(x_2)) \Delta^\rho \Delta^\rho + O(\Delta^4) \right] \\
+ b^\rho \tilde{b}^\rho \left[ -2\eta_{\rho\rho} \Delta^2 - 2h_{\rho\rho}(\tilde{x}) (\Delta^2 + \eta_{\rho\rho}(\Delta^\rho)^2) + O(\Delta^4) \right]
\] (4.28)

The unphysical $b^\rho$ sector is therefore the same as for the static situation. The masses of the $b^\rho$ fluctuations are purely determined by D-geometry and will be positive. The physical $b^\phi$-sector, however, is affected. Adding the two terms we find the mass for the off-diagonal fluctuation $b^\phi$ to be
\[
m_{b^\phi}^2 = \frac{8m}{D^2} \left[ (\tilde{x}^\rho)^2 + \frac{1}{12} (\Delta^\rho)^2 \right] (\dot{\tilde{x}}^\rho)^2 + 8\lambda^2 \left[ (\tilde{x}^\rho)^2 \Delta^2 + (\tilde{x}^\rho)^2 h_{\rho\rho}(\tilde{x})(\Delta^\rho)^2 \right] + O(\Delta^4) \] (4.29)

which is positive semidefinite. On a sphere, as expected, and by rough generalization on any positively curved patch of spacetime, no gravitationally induced Myers effect occurs.

### 4.2.1 RNC

For those metrics where one knows a Riemann normal coordinate system the computation is much cleaner and more perspicacious. To illustrate its benefits let us repeat the sphere computation starting from RNC. Since the sphere is a homogeneous space the transformation from the standard metric to RNC is easily found and one obtains the metric:

\[
ds^2_{S^2,RNC} = dz_x^2 + dz_y^2 - \frac{1}{3D^2} (z_x dz_y - z_y dz_x)^2 + O(z^3).
\] (4.30)

One easily checks that geodesics through the origin $z^i = 0$ can be written as $z^i = v^i \tau$ as required. Furthermore one immediately reads off the Riemann tensor at the origin $z^i = 0$:

\[
d s^2_{RNC} = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{3} R_{\mu\alpha\beta\nu}(0) x^\alpha x^\beta dx^\mu dx^\nu + \ldots \Rightarrow R_{xyyx}(0) = -\frac{1}{D^2}.
\] (4.31)

The two D0-brane system we are considering has both moving along the same great arc through the origin, but slightly separated along this arc. If we choose the $z^x$-axis as this arc, this implies that $\dot{z}_x^i = 0$, $\Delta y^2 = 0$ and $b^x$ is the unphysical fluctuation.

Substituting this data into equation (4.23), we find:

\[
\delta^2 S = -\int d\tau \left[ b_x \left( 2\lambda^2 \Delta^2 \right) \partial_x \\
+ b_y \left( 2\lambda^2 \Delta^2 + \frac{2m}{D^2} \left( (\dot{\tilde{x}}^x)^2 + \frac{1}{12} (\Delta^x)^2 \right) \right) \partial_y \right].
\] (4.32)

Again the mass-matrix for the unphysical fluctuation $b^x$ is that given by D-geometry and strictly positive definite. For the physical $b^y$ fluctuation the mass matrix is explicitly positive semidefinite, and we see that there is no tachyon in this case. The physical picture explains this in simple terms. There are two forces acting: the force due to the geodesic deviation and the force of open string between the branes. In this positive curvature background the geodesic deviation works in the same direction as the force by the stretched open strings, leading to a stable configuration.
4.3 D0-branes on a hyperboloid

So far we confirmed our supposition that on positively curved patches standard “individual” geodesic behaviour is a stable solution. The real question, however, is whether negatively curved patches do destabilize this solution and therefore hint towards the existence of a gravitational Myers effect. That this is indeed the case, is now easily seen. For two nearly coincident D0-branes on a two-dimensional hyperboloid, the mass-matrix of fluctuations is simply eq. (4.32) with the substitution $D^2 \rightarrow -D^2$. The gravitational tidal force has changed sign and can now counterbalance the attractive string potential. Specifically there is a tachyonic instability in the spectrum iff the D-brane separation is small compared to the velocity times the curvature:

$$2\lambda^2 \Delta^2 \ll \frac{2m}{D^2} \left( (\dot{z}^x)^2 + \frac{1}{12} (\dot{\Delta}^x)^2 \right), \quad (4.33)$$

Note, however, that for large separations the attractive force of the open strings dominates the repelling force of the background geometry, and the tachyon disappears.

To truly test whether we have found a tachyonic instability one question remains. The tachyon appears to be evident for large speeds or small separations. Both extremes, however are outside the range of validity of the action (eqs. (2.4)-(2.6) and (4.2)). Large speeds violate the truncation to second order in derivatives, eq. (2.5), while separations $\Delta$ must be larger than the eleven-dimensional Planck length, eq. (2.6). We need to check whether the inequality (4.33), signaling the instability, can be satisfied within these bounds. Without loss of generality, we may simplify the inequality by approximating $\dot{\Delta}^x \approx 0$. Substituting $\dot{z}^x = v$ with $|v| \ll 1$, expressing $m$ and $\lambda$ in string units $g_s, \ell_s$ and multiplying both sides of (4.33) by $\ell_s^2$, we obtain

$$\frac{\Delta^2}{\ell_s^2} \ll v^2 \frac{\ell_s^2}{D^2} \ll 1. \quad (4.34)$$

In words, the condition for a tachyon to be present is that the separation in string units must be less then the velocity times the curvature in string units. We have added the implicit weak gravity and Born-Infeld condition: both sides must be less than unity. Most importantly, however, the separation $\Delta$ must also be larger than the eleven dimensional Planck length, eq. (2.6):

$$g_s^{1/3} \ell_s \ll \Delta \ll \ell_s. \quad (4.35)$$

The window where eq. (4.34) can be satisfied within the range of eq. (4.35) is when the velocity times the curvature is much larger than the eleven dimensional Planck length:

$$g_s^{2/3} \ll v^2 \frac{\ell_s^2}{D^2}. \quad (4.36)$$

This is easily satisfied for very weak string coupling. The region where the “individual” geodesic solution is unstable thus falls easily within the region of validity of the action. This should convince us that the existence of a lesser energetic collective mode is highly plausible. It appears that a purely gravitational Myers effect exists.
4.4 D0-branes near Schwarzschild black holes

The sphere and the hyperboloid are abstract situations, that are unlikely to occur in nature and do not confront fundamental questions in quantum gravity. Can a gravitational Myers effect occur in physically interesting situations? As primary physical testbed we will study the Schwarzschild black hole. In the weak gravity approximation the $d$-dimensional Schwarzschild metric may be approximated by

$$ds^2 = - \left(1 - \frac{2M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{2M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu + 2M \frac{r^{d-3}}{r^{d-3}} (dt^2 + dr^2) + \mathcal{O}(M^2).$$

(4.37)

The Schwarzschild spacetime confronts us with an extremely important fact. Generically the ambient spacetime will be curved in the timelike/null directions that are not “orthogonal” to the D0-brane. So far, we have implicitly

i) discussed metrics which only have curvature in the directions transverse to the D0-brane and

ii) always chosen the physical gauge for the time-like direction. How do we deal with non-abelian D0-branes in metrics with non-transverse curvature? In a separate article [22] we will show that diffeomorphism invariance allows one to simply extend the range of the index of the fields to include the direction parallel to the worldvolume of the D0-brane on the condition that one chooses the physical gauge $X^0 = \tau \cdot \mathbb{I}$ at the end. This is what one intuitively expects. It also has an important consequence for the computation of the mass-matrix of quadratic fluctuations. As the standard metric choice contains only deviations in the radial and timelike directions, only the off-diagonal modes $b^r$ and $b^t$ could potentially be tachyonic. The physical gauge choice, however, tells us that $b^t$ is pure gauge and unphysical. The mass matrix therefore has only a single interesting entry, that for the off-diagonal mode $b^r$. To ensure that the Gauss law orthogonality constraints do not affect this mode we choose the zero radial separation between the two D0-branes: $\Delta^r = 0$. The other off-diagonal modes $b^i$ have their standard geodesic-distance masses determined by D-geometry subject to the Gauss law, and we may ignore them.

Collecting then the various pieces we find explicitly

$$\delta^2 S_{\text{kin},r} + \delta^2 S_{\text{pot},r} = \int d\tau \int_0^1 ds \ b^r \tilde{b}^r \left[ m R_{rr}(x_2 + s\Delta) + m R_{rr}(x_1 - s\Delta) ight. 

- \eta_{rr} 2\lambda^2 \Delta^2 \left( 1 + h(x_2 + s\Delta) + h(x_1 - s\Delta) \right) \left] , \right. \right.$$ 

(4.39)

where we have used

$$h_{rr} = h_{tt} \equiv h(r) = \frac{2M}{r^{d-3}} ,$$

$$R_{rr} = R_{tttt} \dot{x}^t \dot{x}^t$$

$$= R_{tttt} = \frac{M(d-3)(d-2)}{r^{d-1}} \left( 1 + \mathcal{O} \left( \frac{M}{r^{d-3}} \right) \right).$$

(4.40)

Note that $h(r)$ and $R_{rr}(r)$ are functions only of the radial direction $r$, and that we have chosen $\Delta^r = 0$. This means that we may simply replace the arguments $x_2^r + s\Delta^r$, $x_1^r - s\Delta^r$ of these
functions by the center of mass $\bar{r} = \frac{1}{2}(r_1 + r_2)$. The mass matrix therefore immediately follows from eqs. (4.39) and (4.40) and reads (ignoring higher order corrections)

$$\delta^2 S = \int d\tau \bar{b}' \left[ \frac{2mM(d-3)(d-2)}{\bar{r}^{d-1}} - 2\lambda^2 \Delta^2 \left( 1 + \frac{4mM}{\bar{r}^{d-3}} \right) \right] \bar{b}' .$$

This corresponds to a negative mass squared iff

$$2\lambda^2 \Delta^2 \left( 1 + \frac{4mM}{\bar{r}^{d-3}} \right) \ll \frac{2mM(d-3)(d-2)}{\bar{r}^{d-1}} .$$  \hspace{1cm} (4.41)

Conform physical intuition, we see that close to the black hole the Myers tachyon appears, whereas far-away ($\bar{r} \gg 1$), eq. (4.41) reduces to the regular flat space mass-matrix. We do still need to test that we remain within the regime of validity of the action. In the weak gravity approximation we may approximate the l.h.s. with the leading term. Denote the curvature scale, eq. (4.40), with

$$\frac{(d-3)(d-2)M}{\bar{r}^{d-1}} = \frac{1}{D(\bar{r})^2} .$$  \hspace{1cm} (4.42)

and rewrite all quantities in string units. We see that we obtain a tachyonic mode if

$$\frac{\Delta^2}{\ell_s^2} \ll \frac{\ell_s^2}{D(\bar{r})^2} ;$$

where again both sides must also be less than unity. As repeatedly emphasized, the separation must in addition be larger than the eleven-dimensional Planck length, (2.6):

$$g_1^{1/3} \ell_s \ll \Delta \ll \ell_s .$$  \hspace{1cm} (4.44)

A tachyon will therefore appear if the curvature in string units is large compared to the string coupling

$$g_s^{2/3} \ll \frac{\ell_s^2}{D(\bar{r})^2} .$$

The Schwarzschild curvature potentially induces near-horizon collective Myers behaviour in D0-branes! Note the close similarity with the condition we found for the hyperboloid, eq. (4.36). This makes the claim that the qualitative nature of the local curvature, positive or negative, determines the existence of a tachyonic mode in the mass-matrix of off-diagonal fluctuations. Locally negative curvature patches of spacetime potentially induce a geometric Myers effect.

## 5 Conclusion

We have provided evidence that $N$ nearly superposed D0-branes following independent geodesic motion are unstable in a patch of negatively curved spacetime. This conclusion follows from
a fluctuation analysis of off-diagonal modes around the diagonal solution which has the geometric interpretation of computing non-abelian geodesic deviation. Given this evidence, one is prompted to ask if and what the collective configuration is that the D0-branes fall into. This final non-abelian configuration will depend heavily on the specific properties of the spacetime, and is therefore difficult to compute in general, although some specific highly symmetric cases may be solvable. In the RR-Myers effect the properties of the Ramond-Ramond gauge field provide a handle on the interpretation of the collective lesser energetic configuration as dipole configurations of fundamental sources. Everything couples to gravity, however, and this makes a gravitational Myers effect less transparent and more universal at the same time. Generically supersymmetry will also be absent, which makes it even less clear whether the resultant configuration is a identifiable polarized cloud of specific D-branes and/or strings. On the other hand, the indication that a gravitationally induced Myers effect may occur near the horizon of a Schwarzschild black hole hints at a universal aspect, which through the link between D0-brane mechanics and M-theory could possibly have consequences for our understanding of GR-singularities in string theory. However, we will need to understand the non-commutative non-abelian geometry underlying D0-brane mechanics better to make progress in this direction. One avenue which a possible near-horizon Myers effect suggests to explore is a connection between D0-brane mechanics and the non-commutative shockwave-black hole geometries of ’t Hooft [26].

Whether a gravitational Myers effect truly exists and whether it may contribute to answering fundamental questions in quantum gravity, however, remain questions for the future.

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