Mannheim’s linear potential in conformal gravity

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We study the equations of conformal gravity, as given by Mannheim, in the weak field limit, so that a linear approximation is adequate. Specializing to static fields with spherical symmetry, we obtain a second-order equation for one of the metric functions. We obtain the Green function for this equation, and represent the metric function in the form of integrals over the source. Near a compact source such as the Sun the solution no longer has Schwarzschild form. Using Flanagan’s method of obtaining a conformally invariant metric tensor we attempt to get a solution of Schwarzschild type. We find, however, that the $1/r$ terms disappear altogether. We conclude that a solution of Mannheim type cannot exist for these field equations.

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I. INTRODUCTION

In this paper we will derive solutions, in the weak field limit, of the field equations of conformal gravity as given by Mannheim (see [1], equation (186); this paper will be referred to as PM from now on):

$$4\alpha_g W^{\mu\nu} = 4\alpha_g \left[ W^{(2)} - W^{(1)} \right] = T^{\mu\nu}$$  \hspace{1cm} (1)

Here $W^{\mu\nu}$ is the Weyl tensor, the two separate parts $W^{(1)}$ and $W^{(2)}$ being defined in PM (61) and (108). $\alpha_g$ is a dimensionless coupling constant. (We adopt the notation of Weinberg [2], with units such that $c = \hbar = 1$.)

The energy-momentum tensor, $T^{\mu\nu}$, is derived from an action principle involving a scalar field, $S$ (see PM (61)). Appropriate variation of this action yields $T^{\mu\nu}$ as given in PM (64). In Mannheim’s model, the solutions of the field equations undergo a symmetry breaking transition (SBT) in the early Universe, with $S$ becoming a constant, $S_0$. Making this change in PM (64) we obtain

$$T^{\mu\nu} = -\frac{1}{6} S_0^2 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_\alpha \right) - g^{\mu\nu} \lambda S_0^4 + T^\mu_M$$  \hspace{1cm} (2)

where $T^\mu_M$ is the matter tensor, containing all the usual fermion and boson fields.

We break from Mannheim’s development at this point. The factor $1/6$ in [2] derives from the original, conformally invariant action. A SBT, however, will not necessarily preserve such relations, and we will instead write

$$T^{\mu\nu} = \frac{1}{8\pi G_0} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_\alpha \right) - g^{\mu\nu} \lambda W + T^\mu_M$$  \hspace{1cm} (3)

so that the field equations can be written

$$W^{\mu\nu} = \frac{1}{32\pi\alpha_g G_0} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_\alpha \right) + g^{\mu\nu} \lambda W \frac{\lambda W}{32\pi\alpha_g G_0} = \frac{1}{4\alpha_g} T^\mu_M$$  \hspace{1cm} (4)

Mannheim is constrained to get an effective $G$ that is negative. We, on the other hand, will assume that the SBT results in a positive value for $G_0$. We can then identify $G_0$ with the Newton gravitational constant.

In the rest of this paper we will ignore the term in $\lambda W$. Defining $\eta = -1/(32\pi\alpha_g G_0)$ and $\xi = 1/(4\alpha_g)$, the field equations become

$$W^{\mu\nu} + \eta \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_\alpha \right) = \xi T^\mu_M$$  \hspace{1cm} (5)

$\xi$ is dimensionless, but $\eta$ has dimension $\text{length}^{-2}$, so its magnitude can be written $|\eta| = 1/r_0^2$, where $r_0$ divides lengths into two regimes, in one of which $(r < r_0)$ the Weyl tensor is dominant, and in the other $(r > r_0)$ the Einstein tensor.

We will call this equation the Weyl-Einstein equation, or “W-E equation” for short. In the important special case that $\alpha_g W^{\mu\nu}$ is negligible, or even identically zero, we regain the usual Einstein equations, as given, for example, in Weinberg [2] (16.2.1). In the opposite limit, $\eta \to 0$, we obtain

$$W^{\mu\nu} = \xi T^\mu_M$$  \hspace{1cm} (6)

the Bach equation. Some solutions of this have been obtained by Fiedler and Schimming [3].

We can take the trace of (5), to get

$$R^\alpha_\alpha = 8\pi G_0 T^\alpha_\alpha$$  \hspace{1cm} (7)

which is, of course, the same as we would get from the Einstein equations since $W^{\mu\nu}$ is traceless.

From this Mannheim derives a traceless energy-momentum tensor, PM (65). We shall not use this,
II. STATIC FIELDS WITH SPHERICAL SYMMETRY

We now specialize further, to static fields with spherical symmetry. Like Fiedler and Schimming, but apparently independent of them, Mannheim and Kazanas [4] addressed the problem of the solution of the Bach equation under these conditions. They found that in addition to the usual $1/r$ term of the Schwarzschild solution there was a term $\gamma r$. Mannheim has used this linear potential to obtain a fit to the rotation curves of galaxies; for a recent paper, see [3]. However, the relevant field equation is not the Bach equation, but the W-E equation, for which a linear potential is not a solution. It is therefore not clear what use a linear potential can be in such studies, except as an approximation over a limited range.

The most general form for a static metric with spherical symmetry is given in Weinberg [2], (8.1.6):

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(8)

For weak fields we write $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$, where $a(r)$ and $b(r)$ are assumed small compared to unity, so that only terms linear in $a(r)$ and $b(r)$ need be considered.

In the presence of matter with density $\rho(r)$, the trace equation becomes, with primes denoting differentiation with respect to $r$:

$$\eta R^\alpha_\alpha = -\xi T^\alpha_\alpha = \xi \rho(r)$$

(9)

$$\frac{\eta}{2r^2} (4a - 4rb' + 4ra' - 2r^2 b'') = \xi \rho(r)$$

(10)

$$2(ra)' - (r^2 b')' = -\frac{r^2 \xi \rho(r)}{\eta}$$

(11)

We assume the density is a smooth function, so that $a'(r)$ and $b'(r)$ are both zero at $r = 0$. Then we can integrate out from the origin to $r$ to get

$$2ra - r^2 b' = -\frac{\xi}{4\pi \eta} \int_0^r 4\pi u^2 \rho(u) du$$

$$= -\frac{\xi}{4\pi \eta} m_e(r)$$

(12)

where $m_e(r)$ is the enclosed mass out to $r$.

The $r, r$ component of the W-E equation gives *

$$-r^3 b''' - 2r^2 b'' - r^2 a'' + 2rb' + 2a + 3r^2 \eta (-rb' + a) = 0$$

(13)

Before going further, we can check that the Schwarzschild solution (SS) is a possible solution of (13) and the trace equation, (11). SS is characterized by $A(r) = 1/B(r) = 1 + \beta/r$, i.e. $a(r) = -b(r) = \beta/r$. Substituting these expressions into (13) and (11), we can verify that the equations are satisfied.

In the limit $\alpha_q \to \infty (\eta \to 0)$, the Weyl tensor is everywhere dominant. The trace equations are irrelevant, and (13) admits the solution

$$a(r) = -b(r) = \gamma r,$$

(14)

the Mannheim linear potential.

These are not, however, the only possibilities, and we will now derive a different form for $a(r)$ and $b(r)$. The solution we will obtain does not, of course, guarantee that a corresponding solution exists for the full nonlinear W-E equations. But it does provide a limiting form, for weak fields, of such a solution, if it exists.

We will now transform (13) and the trace equation to get a second-order equation in $a(r)$ only. Differentiating (11):

$$-4a' - 2ra'' + 4rb'' + r^2 b'' + 2b' = \frac{2\xi \rho(r)}{\eta} + \frac{r^2 \xi \rho'(r)}{\eta}$$

(15)

Combining this with (13) we can eliminate $b''(r)$:

$$-3r^2 a'' - 4ra' + 2a + 2r^2 b'' + 4rb' + 3r^2 \eta (-rb' + a) = \frac{2\xi \rho(r)}{\eta} + \frac{r^2 \xi \rho'(r)}{\eta}$$

(16)

We can now use (10) and (12) to eliminate all terms involving $b(r)$, to arrive at

$$-3r^2 a'' + (6 - 3r^2 \eta) a = 3r^2 \frac{\xi}{4\pi} m_e(r) + \frac{r^3 \xi \rho'(r)}{\eta}$$

(17)

We will develop the solution of this equation as an integral over the source density, using a Green function constructed from the related homogeneous equation

$$-3r^2 a'' + (6 - 3r^2 \eta) a = 0$$

(18)

At this point we choose $\alpha_q < 0$, and therefore $\eta < 0$, purely for computational convenience. This will ensure

* For the geometrical calculations we have used GRTensorII, followed by a Maple script to extract the linear terms.
† A few years ago this writer speculated [6] that a second solution of the W-E equation might also satisfy these Schwarzschild conditions. The present paper suggests this idea is mistaken.
that we deal with modified Bessel functions, which have a particularly simple form.

(18) can be written

\[ a'' + \left(-\frac{\nu^2 - 1/4}{r^2} - k^2\right) a = 0 \] (19)

with \( \nu = 3/2 \) and \( k^2 = -\eta > 0 \). Solutions are (see [7] 9.1.49) \( a(r) = r^{1/2} \mathcal{L}_{3/2}(kr) \), where \( \mathcal{L}_\nu \) stands for \( I_\nu \) or \( K_\nu \).

Because in this paper we are looking for solutions analogous to Mannheim’s linear potential, we will assume, tentatively, that the length \( r_0 = 1/k \) is of galactic scale, intermediate between the scale of the Solar System and truly cosmological scales.

From (19) we can derive a Green function by standard methods [8]. If the function is defined on the range \( 0 \) to \( \infty \), with \( u \) the source point and \( t \) the field point:

\[ G(u,t) = \begin{cases} I_{3/2}(u)K_{3/2}(t), & 0 < u < t, \\ K_{3/2}(u)I_{3/2}(t), & t < u < \infty. \end{cases} \] (20)

so that \( a(r) \) can be written as an integral over the source.

\[ a(r) = r^{1/2} \int_0^\infty G(kr,kt) \left( \frac{1}{3r^{1/2}} \right) \times \left[ \frac{3\xi}{4\pi} m_\epsilon(t) + \frac{t^2 \xi \rho'(t)}{\eta} \right] dt \] (21)

It is convenient to define two new functions:

\[ \mathcal{K}_{3/2}(z) = \frac{K_{3/2}(z)}{z^{1/2}} e^z = \frac{\pi}{2} \left( \frac{1}{z^2} + \frac{1}{2z^2} \right) \] (22)

\[ \mathcal{T}_{3/2}(z) = \frac{I_{3/2}(z)}{z^{1/2}} e^{-z} \]

\[ = \sqrt{\frac{2}{\pi}} e^{-z} \left( -\sinh z + \cosh z \right) \]

\[ = \sqrt{\frac{2}{\pi}} \left[ \left( -\frac{1}{2z^2} + \frac{1}{z} \right) + e^{-2z} \left( \frac{1}{2z^2} + \frac{1}{z} \right) \right] \] (23)

In terms of these functions, (21) reads

\[ a(r) = a_<(r) + a_>(r), \quad \text{where} \]

\[ a_<(r) = kr \mathcal{K}_{3/2}(kr) \int_0^r \mathcal{T}_{3/2}(kt)e^{kt-r} \left( \frac{1}{3} \right) \times \left[ \frac{3\xi}{4\pi} m_\epsilon(t) + \frac{t^2 \xi \rho'(t)}{\eta} \right] dt \] (24)

\[ a_>(r) = kr \mathcal{T}_{3/2}(kr) \int_r^\infty \mathcal{K}_{3/2}(kt)e^{k(r-t)} \left( \frac{1}{3} \right) \times \left[ \frac{3\xi}{4\pi} m_\epsilon(t) + \frac{t^2 \xi \rho'(t)}{\eta} \right] dt \] (25)

III. THE GRAVITATIONAL FIELD OF THE SUN

The most immediate application of our formulae is to obtain the analog of the Schwarzschild solution, i.e. the metric functions in the vacuum surrounding a compact massive object such as the Sun. Our second-order equation, (17), reduces in this case to

\[ -3r^2 a'' + (6 - 3r^2 \eta) a = 3r \frac{\xi}{4\pi} m_\epsilon(r) \] (26)

The general solution of this equation is a Particular Integral (PI) plus a Complementary Function (CF). A suitable PI is

\[ a(r) = -\frac{\xi m_\odot}{4\pi \eta r} = \frac{2G_0 m_\odot}{r} \quad \text{PI only} \] (27)

where the CF is proportional to \( r^{1/2} K_{3/2}(kr) \) (contributions from \( I_{3/2} \) can be ruled out because they contain a rising exponential). So we can write

\[ a(r) = -\frac{\xi m_\odot}{4\pi \eta r} + C \left( 1 + \frac{1}{kr} \right) e^{-kr}, \] (28)

where we will use our Green function to determine the constant \( C \). For this we need only study the most singular terms for small \( r \).

Let us first consider the limit of large \( r \). In the vacuum outside the source, \( m_\epsilon(r) = m_\odot \) and \( \rho' = 0 \). The exponentials in the integrals in (24) and (25) will be sharply peaked around \( u = t \). In (22) and (23) we will keep only the leading terms, i.e. we neglect \( 1/z^2 \) in comparison to \( 1/z \), and omit terms in \( \exp(-2z) \) altogether.

\[ a_<(r) = kr \frac{\xi m_\odot}{4\pi} \int_0^r \frac{1}{\sqrt{2\pi}} e^{kt-r} \left( \frac{1}{3} \right) \times \left[ \frac{3\xi}{4\pi} m_\epsilon(t) + \frac{t^2 \xi \rho'(t)}{\eta} \right] dt \]

\[ \approx \left( \frac{1}{2kr} \right) \left[ \frac{\xi m_\odot}{4\pi} \right] \int_0^r e^{kt-r} dt \approx \left( \frac{1}{2kr} \right) \left[ \frac{\xi m_\odot}{4\pi} \right] \int_0^r e^{kt-r} dt \approx \left( \frac{1}{2kr} \right) \left[ \frac{\xi m_\odot}{4\pi} \right] \int_0^r e^{kt-r} dt \]

\[ a_>(r) \] will contribute an identical amount, so

\[ a(r) = -\left( \frac{1}{r} \right) \frac{\xi m_\odot}{4\pi \eta r} = \frac{2G_0 m_\odot}{r} \] (30)

in the limit of large \( r \). As expected, this is just the PI we obtained earlier.

For the limit of small \( r \) we will consider a point in the Solar System, outside the source but with \( r \ll 1/k \). The Green function equations (21) and (22) can be divided into four integrals:

\[ a_<(r) = kr \mathcal{K}_{3/2}(kr) \left[ \mathcal{H}_1 + \mathcal{H}_2 \right], \quad \text{where} \]

\[ \mathcal{H}_1 = \int_0^r \mathcal{T}_{3/2}(kt)e^{kt-r} \left[ \frac{\xi}{4\pi} m_\epsilon(t) \right] dt \]

\[ \mathcal{H}_2 = \int_r^\infty \mathcal{K}_{3/2}(kt)e^{k(r-t)} \left[ \frac{t^2 \xi \rho'(t)}{3\eta} \right] dt \] (31)
and

\[ a_\nu(r) = krT_{\frac{3}{2}}(kr) [\mathcal{H}_3 + \mathcal{H}_4], \]

where

\[ \mathcal{H}_3 = \int_r^\infty \mathcal{K}_{\frac{3}{2}}(kt)e^{kr-t} \left[ \frac{\xi m_\nu(t)}{4\pi} \right] dt \]
\[ \mathcal{H}_4 = \int_r^\infty \mathcal{K}_{\frac{3}{2}}(kt)e^{kr-t} \left[ \frac{t^2 \xi \rho'(t)}{3\eta} \right] dt \]

(32)

The last of these, \( \mathcal{H}_4 \), is clearly zero. The first, \( \mathcal{H}_1 \), can be simplified by writing \( m_\nu(r) = m_\odot \) throughout the range of integration, even inside the Sun; the error incurred this way is smaller by a factor of \( k^2r_s^2 \) than the dominant terms, where \( r_s \) is the radius of the Sun. We then get:

\[ \mathcal{H}_1 = \left[ \frac{\xi m_\odot}{4\pi} \right] e^{-kr} \int_0^r \mathcal{T}_{\frac{3}{2}}(kt)e^{kt} dt \]
\[ = \frac{\xi m_\odot e^{-kr}}{4\pi} \sqrt{\frac{2}{\pi}} \int_0^r \left( -\sinh(kt) + \cosh(kt) \right) \frac{\xi m_\nu e^{-kr}}{kr} \sqrt{\frac{2}{\pi}} \left( \sinh(kt) \right) \]
\[ - 1 \]

(33)

It will be convenient in what follows to divide \( \mathcal{H}_1 \) into two pieces:

\[ \mathcal{H}_1 = \mathcal{H}_5 + \mathcal{H}_6, \]
\[ \mathcal{H}_5 = \frac{\xi m_\odot e^{-kr}}{4\pi k} \sqrt{\frac{2}{\pi}} \left[ \sinh(kt) \right] \]
\[ \mathcal{H}_6 = -\frac{\xi m_\odot e^{-kr}}{4\pi k} \sqrt{\frac{2}{\pi}} \]

(34)

For \( \mathcal{H}_2 \) we get the leading terms by setting \( \mathcal{T}_{\frac{3}{2}}(kt) \) equal to its limiting value for small \( kt,\sqrt{1/(2\pi)}(2kt/3) \):

\[ \mathcal{H}_2 = \sqrt{\frac{1}{2\pi}} \frac{\xi e^{-kr}}{3\eta} \int_0^r \left( \frac{2kt}{3} \right) t^2 \rho'(t) \]
\[ = \sqrt{\frac{1}{2\pi}} \frac{2\xi k e^{-kr}}{9\eta} \left[ t^3 \rho(t) \right]_0^r - \int_0^r 3t^2 \rho(t) dt \]
\[ = \sqrt{\frac{1}{2\pi}} \frac{\xi m_\odot e^{-kr}}{6\pi k} \]

(35)

Finally, \( \mathcal{H}_3 \) evaluates to:

\[ \mathcal{H}_3 = \frac{\xi m_\odot e^{kr}}{4\pi} \sqrt{\frac{\pi}{2}} \int_r^\infty \left( \frac{1}{kt} + \frac{1}{k^2t^2} \right) e^{-kt} dt \]
\[ = \frac{\xi m_\odot}{4\pi k^2 r} \sqrt{\frac{2}{\pi}} \]

(36)

We get the PI of \( a(r) \) from the \( \mathcal{H}_5 \) and \( \mathcal{H}_3 \) terms:

\[ \text{PI} = kr\mathcal{K}_{\frac{3}{2}}(kr) \left( \frac{\xi m_\odot e^{-kr}}{4\pi k} \right) \sqrt{\frac{2}{\pi}} \left[ \sinh(kt) \right] \]
\[ + kr\mathcal{T}_{\frac{3}{2}}(kt) \left( \frac{\xi m_\odot}{4\pi k^2 r} \right) \sqrt{\frac{2}{\pi}} \]

(37)

Using the definitions in (22) and (23) we can reduce this to

\[ \text{PI} = \frac{\xi m_\odot}{4\pi k^2 r} \left[ \xi m_\odot - \frac{\xi m_\odot}{4\pi \eta r} \right] \]

(38)

in agreement with (27) and (30), as expected.

The CF is formed from the \( \mathcal{H}_2 \) and \( \mathcal{H}_6 \) terms:

\[ \text{CF} = kr\mathcal{K}_{\frac{3}{2}}(kr) \left( \frac{\xi m_\odot e^{-kr}}{\pi k} \right) \left( \frac{2}{\pi} \frac{1}{12} - \frac{2}{\pi} \frac{1}{4} \right) \]
\[ = -\frac{\xi m_\odot}{6\pi k} \left( 1 + \frac{1}{kr} \right) e^{-kr} \]

(39)

so altogether

\[ a(r) = \frac{\xi m_\odot}{4\pi k^2 r} - \frac{\xi m_\odot}{6\pi k} \left( 1 + \frac{1}{kr} \right) e^{-kr} \]

(40)

and \( C = -\xi m_\odot/(6\pi k) \), from (28).

We can now get \( b(r) \) by integrating (12):

\[ b'(r) = \frac{2}{r} a(r) - \frac{\xi m_\odot}{4\pi k^2 r^2} \]
\[ = \frac{\xi m_\odot}{4\pi k^2 r^2} - \frac{\xi m_\odot}{3\pi} \left( \frac{1}{kr} + \frac{1}{k^2 r^2} \right) e^{-kr} \]
\[ b(r) = \frac{\xi m_\odot}{4\pi k^2 r} + \frac{\xi m_\odot}{3\pi k} \left( e^{-kr} \right) \]

(41)

FIG. 1: Upper (lower) curve: \( a \) (b) as a function of \( kr \). The vertical scale is arbitrary.

As they stand, these expressions for \( a(r) \) and \( b(r) \) are unacceptable, because the leading terms, for small \( r \), give

\[ a(r) = b(r) = \frac{\xi m_\odot}{12\pi k^2 r} \]

(42)

whereas observations tell us that in the Solar System we have a solution very close to Schwarzschild form, for which \( a(r) = -b(r) \). Fortunately a possible remedy is close at hand, as described in the next section. Before discussing that, however, we should check our solution for signs of the Mannheim linear potential, which should
be evident in the limit $k \to 0$. In taking this limit, we keep $\xi/k^2$ fixed at its value of $8\pi G_0$. Then (41) gives

$$b(r) = \frac{G_0 m_\odot}{6r} \left[ -3 + 4 \left( 1 - kr + \frac{k^2 r^2}{2} + \cdots \right) \right]$$

$$= \frac{G_0 m_\odot}{6r} - \frac{G_0 m_\odot kr}{6} + \frac{G_0 m_\odot k^2 r}{12} + \cdots \quad (43)$$

The Mannheim linear potential has made its appearance, as part of an approximation to a falling exponential.

IV. GETTING TO A SCHWARZSCHILD SOLUTION

Flanagan [9] has pointed out that the effective metric tensor cannot be $g_{\mu\nu}$, which is not conformally invariant, but must be

$$\hat{g}_{\mu\nu} = F^2(r) g_{\mu\nu} \quad (44)$$

where $F(r)$ is some scalar field of conformal weight $-1$. We will follow Flanagan in identifying $F(r)$ with $S(r)/m_0$ where $S(r)$ is Mannheim’s scalar field, and $m_0$ is some convenient scale of mass, which we take to be the numerical value of $S_0$.

After the SBT, $S(r)$ has the form $S_0[1 + s(r)]$, where $s(r)$ represents oscillations about the minimum of the potential. We will assume an equation of motion for $s(r)$:

$$s^{\mu}_{\mu} - k^2 s = -4\pi D \rho(r) \quad (45)$$

This has the static, point-source solution outside the Sun:

$$s(r) = Dm_\odot \frac{e^{-kr}}{r} \quad (46)$$

Flanagan’s field has the form

$$F(r) = 1 + Dm_\odot \frac{e^{-kr}}{r} \quad (47)$$

For $\hat{g}_{\mu\nu}$ to approximate a metric of Schwarzschild type we must have

$$F(r)A(r) = \frac{1}{F(r)B(r)} \quad (48)$$

Expanding to first order, we get

$$s(r) = -\frac{a(x) + b(x)}{4}$$

$$Dm_\odot \frac{e^{-kr}}{r} \approx -\left( \frac{\xi m_\odot}{24\pi k^2 r} \right) e^{-kr} \quad (49)$$

Since $\xi/k^2 = 8\pi G_0$, we have $D = -G_0/3$.

For our new metric functions, $\hat{a}(r)$ and $\hat{b}(r)$, we get

$$\hat{a}(r) = a(r) + 2s(r)$$

$$= \left[ \frac{\xi m_\odot}{4\pi} \right] \left( \frac{1}{k^2 r} \right)$$

$$- \left[ \frac{\xi m_\odot}{6\pi} \right] \left( \frac{1}{k} \right) \left( 1 + \frac{3}{2kr} \right) e^{-kr} \quad (50)$$

$$\hat{b}(r) = b(r) + 2s(r)$$

$$= - \left[ \frac{\xi m_\odot}{4\pi} \right] \left( \frac{1}{k^2 r} \right)$$

$$+ \left[ \frac{\xi m_\odot}{6\pi} \right] \left( \frac{1}{k} \right) \left( \frac{3}{2kr} \right) e^{-kr} \quad (51)$$

FIG. 2: Upper (lower) curve: $\hat{a}$ ($\hat{b}$) as a function of $kr$. The vertical scale is arbitrary.

The surprise here is that the coefficients of the $1/r$ terms for both $\hat{a}$ and $\hat{b}$ turn out to be zero, so that in getting to a Schwarzschild form we are obliged to eliminate the normal gravitational field altogether. We conclude that a solution of Mannheim type (a Schwarzschild potential plus a linear potential, or an approximation to it) cannot exist for the W-E equation.

V. CAN THE W-E EQUATION REPRESENT REALITY?

We have shown that a solution of Mannheim type cannot exist for the W-E equation. This does not mean, however, that the equation is useless. We should simply discard our initial assumption, that $r_0 = 1/k$ is of galactic scale. Indeed, it would be surprising if a SBT resulted in so large a value of $r_0$. More likely would seem to be a value of order $1\text{fm}$ or less. In this case the Einstein equations would be adequate at all scales accessible to experiment.

The W-E equation could still have important theoretical applications, however, because at the highest energies we expect the SBT to be reversed, so that we recover the original conformal form in which all coupling constants are dimensionless. The theory is then potentially renormalizable.
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