Empirical model of magnetic field line spreading in isotropic turbulence with zero mean field

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Abstract. In many cases, the random walk of magnetic field line in isotropic turbulence with zero mean field is appropriated to describe the transport of energetic particles in the interstellar medium of our Galaxy. To understand the transport, our previous work determined the asymptotic field line diffusion coefficient by using Corrsin’s hypothesis and presumed models of field line spreading. Two of those models of field line spreading are the diffusive decorrelation (DD) model and the random ballistic decorrelation (RBD) model. The variances of the field line displacement in, say, the $x$ direction for the DD and RBD models are assumed as $\sigma_x^2 = 2D|\Delta \tau| = 2\left(\frac{\bar{\lambda}}{\sqrt{3}}\right)b|\Delta \tau|$ and $\sigma_x^2 = (1/3)b^2|\Delta \tau|^2$, respectively. $\tau$ is the field line displacement parameter defined as $d\tau = ds/B$, where $s$ is the arc length along the field line and $B$ is the magnitude of the total magnetic field. $D$ is the asymptotic diffusion coefficient, $b$ is the rms magnetic fluctuation and $\bar{\lambda}$ is the ultrascale. Comparing with simulation results, the DD and RBD models predict the $D$ values with $\leq 15\%$ error and $\leq 21\%$ error, respectively. To improve the theoretical model empirically, in this work, we assume that the proper model of the variance is $\sigma_x^2 = \beta (b|\Delta \tau|)^\alpha$, where $\beta$ is a proportional constant and $\alpha$ is an exponent. We extrapolate $\beta$ with the proportional constants of DD and RBD models. Comparing with the simulation result of Kolmogorov turbulence, we obtain $\alpha = 0.8694$. To test validity of the model, we compare the theoretical results of $\sigma_x^2 = \beta (b|\Delta \tau|)^{0.8694}$ (formulated from Kolmogorov turbulence) with simulation results of two other turbulences: IroshnikovKraichnan turbulence and weak turbulence. The theoretical results of the empirical model match the computer simulation results very well (with $\leq 0.9\%$ error).

1. Introduction

In many cases, the magnetic fields in astrophysical plasmas are turbulent. Hence the magnetic field line trajectories are a random walk in space [1, 2]. Since the charged particles basically follow the magnetic field lines, understanding the diffusion of the magnetic field line random walk (FLRW) relates to understanding the transport of the energetic charged particles in the plasmas. For the interstellar turbulence of our Galaxy, fluctuations are of the same order of magnitude as a regular field (mean field) and have reversals in the field orientation [3]. This implies a turbulence with a negligible regular field. Thus isotropic turbulence without a mean field may be an appropriate model for the interstellar turbulence of our Galaxy.

In order to determine the field line diffusion for FLRW in isotropic turbulence without mean field, the previous work [4] used a non-perturbative analytic framework based on Corrsin’s hypothesis [5]. Using Corrsin’s hypothesis, the magnetic field line spreading is needed to be presumed. To describe the magnetic field line spreading, one needs to statistically describe the
results of tracing magnetic field lines. Since a magnetic field line is defined as a curve that is tangent everywhere to the magnetic field, magnetic field lines can be defined with \( \mathbf{d}s \times \mathbf{B} = \mathbf{0} \), where \( \mathbf{d}s \) is the arc length along the field line. For the turbulent magnetic field \( \mathbf{B} = B_0 \mathbf{\hat{z}} + \mathbf{b} \), where \( B_0 \mathbf{\hat{z}} \) is a mean field and \( \mathbf{b} \) is a fluctuation field, the field line trajectory along each coordinate can be traced by

\[
\frac{dx}{ds} = \frac{b_x}{B} \quad \frac{dy}{ds} = \frac{b_y}{B} \quad \frac{dz}{ds} = \frac{B_0 + b_z}{B}.
\]

The diffusion coefficients in terms of \( s \):

\[
\mathcal{D}_x(s) \equiv \frac{1}{2} \frac{d\langle x^2 \rangle}{ds} \quad \mathcal{D}_y(s) \equiv \frac{1}{2} \frac{d\langle y^2 \rangle}{ds} \quad \mathcal{D}_z(s) \equiv \frac{1}{2} \frac{d\langle z^2 \rangle}{ds},
\]

are nonlinear in the fluctuation \( \mathbf{b} \). To avoid the nonlinear problems, instead of \( s \), the field line trajectory has been described as a function of a field line displacement parameter \( \tau \) defined as \( d\tau = ds/B \) \[4, 6\]. With the parameter \( \tau \), one obtains

\[
\frac{dx}{d\tau} = b_x \quad \frac{dy}{d\tau} = b_y \quad \frac{dz}{d\tau} = B_0 + b_z
\]

and

\[
\mathcal{D}_x(\tau) \equiv \frac{1}{2} \frac{d\langle x^2 \rangle}{d\tau} \quad \mathcal{D}_y(\tau) \equiv \frac{1}{2} \frac{d\langle y^2 \rangle}{d\tau} \quad \mathcal{D}_z(\tau) \equiv \frac{1}{2} \frac{d\langle z^2 \rangle}{d\tau}.
\]

To determine the asymptotic field line diffusion coefficient, Sonsrettee et al. \[4\] used three presumed models of field line spreading defined by variances of field line displacement \( \sigma_i^2(\tau) \), where \( i = x, y, z \). The three models are the diffusive decorrelation (DD) model \[7, 8, 9\], the random ballistic decorrelation (RBD) model \[10\] and the ordinary differential equation (ODE) model \[11\]. Note that, in this work, we consider only the FLRW in isotropic turbulence with zero mean field, in which \( \sigma_i^2 = \sigma_R^2 = \sigma_0^2 \). For the DD model, the magnetic field lines are assumed to spread diffusively with \( \sigma_i^2 = 2D|\Delta \tau| \) over the decorrelation scale of random walk. Here, \( D \) is the asymptotic diffusion coefficient. Moreover, Sonsrettee et al. \[4\] found that \( D = (\lambda/\sqrt{3b}) \), where \( \lambda \) is the ultrascale related to the \( k^{-1} \) moment of the power spectrum. Then, for the DD model, \( \sigma_i^2 \) can be rewritten as \( \sigma_i^2 = (2\lambda/\sqrt{3})(b|\Delta \tau|) \). For the RBD model, the magnetic field lines are assumed that they spread ballistically in random directions with \( \sigma_i^2 = (1/3)(b|\Delta \tau|)^2 \) over the decorrelation scale \[4\]. In DD and RBD models, Sonsrettee et al. \[4\] used assumptions about field line behavior over the decorrelation scale to identify \( \sigma^2 \). In ODE model, they used ensemble averaging the solutions of equation (3) to obtain variance. Comparing theories with simulation results, the DD, RBD and ODE models agree with the simulations that the asymptotic diffusion coefficients are Bohm diffusion \( (D \propto b) \). However, they can only roughly predict the simulation \( D \) values (with \% error less than 15, 21 and 25 for DD, RBD and ODE, respectively).

Motivated by \( \sigma_i \) of DD and RBD models, which are proportional to \( b|\Delta \tau| \) and \( (b|\Delta \tau|)^2 \), respectively, in this work, we assume \( \sigma_i^2 = \beta(b|\Delta \tau|)^\alpha \). Here, \( \beta \) is a proportional constant and \( \alpha \) is an exponent. Since both DD and RBD models predict the \( D \) values to be lower than the simulation \[4\], we propose the extrapolating \( \beta \) for the new model in terms of \( \beta \) for DD and RBD models. Comparing the new model with the computer simulation result for Kolmogorov turbulence \[12, 13\], the \( \alpha \) value is determined empirically. To test the validity of the model, we compared the model, which is empirically formulated from Kolmogorov turbulence simulation, with simulation results of two other turbulences: Ioshnikov-Kraichnan \[14, 15\] and weak turbulence \[16, 17\]. In addition to the simulation results from Sonsrettee et al. \[4\], we performed the simulations for Kolmogorov turbulence at selected values of \( b \) to test the Bohm behavior of FLRW.


2. Empirical model for FLRW with zero mean field

2.1. Analytic theory

Here, we consider the FLRW for homogeneous, isotropic turbulence with zero mean field by adapting the methodology explained in [4]. Let \((x, y, z)\) be the field line displacement relative to its position at \(\tau = 0\). By integrating equation (3), the displacement in, say, the \(x\) coordinate of a field line over \(\tau\) can be specified by

\[
x(\tau) = \int_0^\tau b_x[x(\tau'), y(\tau'), z(\tau')]d\tau'.
\]

Then ensemble average of \(x^2\) for homogeneous turbulence is

\[
\langle x^2 \rangle = \int_0^\tau \int_{-\infty}^{\infty} \mathcal{L}_{xx}(\Delta \tau)d\Delta \tau d\tau',
\]

where \(\Delta \tau \equiv \tau'' - \tau'\) and \(\mathcal{L}_{xx}(\Delta \tau)\) is the Lagrangian correlation function of \(b_x\) at two locations along the same magnetic field line separated by \(\Delta \tau\). Since, for the Lagrangian correlation function, the locations (along a magnetic field line) themselves depend on the representation of the magnetic turbulence, we prefer to describe equation (6) in terms of the Eulerian correlation function, \(R_{xx} \equiv \langle b_x(0, 0) b_x(x, z) \rangle\), which is a correlation function of any two points along magnetic field lines (which are not necessary to be the same). According to Corrsins independence hypothesis [5], \(\mathcal{L}_{xx}\) in equation (6) can be expressed approximately through \(R_{xx}\) as

\[
\mathcal{L}_{xx}(\Delta \tau) = \int R_{xx}(\Delta x)P(\Delta x|\Delta \tau)d\Delta x,
\]

where \(P(\Delta x|\Delta \tau) = P(\Delta x|\Delta \tau)P(\Delta y|\Delta \tau)P(\Delta z|\Delta \tau)\) is the distribution of probability to find the displacement \(\Delta x\) of field lines after \(\Delta \tau\). By definition of \(R_{xx}\), one can rewrite \(R_{xx}\) as the inverse Fourier transform of the power spectrum \(S_{xx}(k)\):

\[
R_{xx}(\Delta x) = \int e^{i k \cdot \Delta x} S_{xx}(k)dk.
\]

Since the displacement of magnetic field line is a result of summation of random walks, with regard to the central limit theorem, one can approximate \(P(\Delta x|\Delta \tau)\) to be Gaussian distribution. With the Fourier transform of the Gaussian \(P(\Delta x|\Delta \tau)\),

\[
\int e^{i k \cdot \Delta x} P(\Delta x|\Delta \tau)d\Delta x = e^{-\frac{1}{2}(\sigma_k^2 k_x^2 + \sigma_y^2 k_y^2 + \sigma_z^2 k_z^2)},
\]

we obtain

\[
\langle x^2 \rangle = \int_0^\tau \int_{-\infty}^{\infty} S_{xx}(k) e^{-\frac{1}{2}(\sigma_k^2 k_x^2 + \sigma_y^2 k_y^2 + \sigma_z^2 k_z^2)}d\Delta \tau d\tau' dk.
\]

With isotropy, the variances in each direction are the same: \(\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \sigma^2\). For the field lines spreading with the variance \(\sigma^2 = \beta (b|\Delta \tau|)^\alpha\) in equation (10)

\[
\langle x^2 \rangle = \int_0^\tau \int_{-\infty}^{\infty} S_{xx}(k) e^{-\frac{1}{2}\beta(b|\Delta \tau|)\alpha(k_x^2 + k_y^2 + k_z^2)}d\Delta \tau d\tau' dk.
\]

\[
= 2^{1+\frac{\alpha}{2}} \tau \beta^{-\frac{1}{2}} \Gamma \left[ 1 + \frac{1}{\alpha} \right] \int S_{xx}(k) k^{-\frac{2}{\alpha}} dk,
\]
\[ D_x \equiv \frac{\langle x^2 \rangle}{2\tau} = 2^{1/2}b^{-1}\beta^{-1/2} \Gamma \left[ 1 + \frac{1}{\alpha} \right] \int S_{xx}(k)k^{-2}dk. \] (13)

Of course, for isotropic turbulence, the asymptotic diffusion coefficients in each direction are the same: \( D_x = D_y = D_z = D \). Using the property of the modal energy spectral density for isotropic turbulence \( S(k) \) that is \( \int S(k)dk = 3 \int S_{xx}(k)dk = b^2 \),

\[ D = 2^{1/2}b^{-1}\beta^{-1/2} \Gamma \left[ 1 + \frac{1}{\alpha} \right] \int \frac{S(k)}{3}k^{-2}dk \times \frac{b^2}{\int S(k)dk} = 2^{1/2}b^{2} \beta^{-1} \Gamma \left[ 1 + \frac{1}{\alpha} \right] \int \frac{S(k)}{\lambda} \frac{dk}{\int S(k)dk}. \] (14)

Equation (14) predicts Bohm diffusion (\( D \propto b \)) for isotropic turbulence with zero mean field. This prediction is in agreement with the previous work [4]. Moreover, The asymptotic diffusion coefficient in equation (14) can be applied to DD and RBD models. Setting \( \alpha = 1 \) and \( \beta = 2\lambda/\sqrt{3} \), where \( \lambda \equiv \sqrt{\int S(k)/k^2dk/\int S(k)dk} \) is the ultrascale [4], equation (14) gives \( D \) for DD. For RBD, \( D \) can be obtained by setting \( \alpha = 2 \) and \( \beta = 1/3 \).

2.2. Extrapolating \( \beta \)
In order to determine \( \beta \), we extrapolate \( \beta \) with the proportional constants of DD and RBD models. Firstly, to linearize \( \beta \), we take the log of \( \sigma^2 = \beta (b|\Delta \tau|)^\alpha \):

\[ \ln \sigma^2 = \ln \beta + \alpha \ln b + \alpha \ln |\tau|. \] (15)

Linearly extrapolating \( \ln \beta \) in equation (15) with the proportional constants of DD and RBD models (\( \beta_1 = 2\lambda/\sqrt{3} \) and \( \beta_2 = 1/3 \), respectively), we obtain

\[ \begin{align*}
(\alpha - 1) & = (\ln \beta_1 + \alpha \ln b + \alpha \ln |\tau|) - (\ln \beta_2 + 2 \ln b + 2 \ln |\tau|), \\
2(1) & = (\ln \beta_1 + 2 \ln b + 2 \ln |\tau|) - (\ln \beta_2 + 2 \ln b + 2 \ln |\tau|), \\
\ln \beta & = \ln \beta_1 + (\alpha - 1)(\ln \beta_2 - \ln \beta_1). \end{align*} \] (16)

Using equations (14) and (16), we obtain

\[ D = 2^{1/2}b^{2}\exp \left[ -\frac{1}{\alpha} (\ln \beta_1 + (\alpha - 1)(\ln \beta_2 - \ln \beta_1)) \right] \Gamma \left[ 1 + \frac{1}{\alpha} \right] \frac{\int S(k)/(3b^{2/\alpha}dk)}{\int S(k)dk}. \] (17)

2.3. Empirical prefactor \( \alpha \)
In order to determine the prefactor \( \alpha \) empirically, we compare \( D \) described in equation (17) with a particular simulation result (\( D_0 \)). With \( D_0 = 0.3600 \) from the simulation result for FLRW in Kolmogorov turbulence with \( b = 1 \), obtained by Sonsrettee et al. [4], we numerically solved equation (17), and obtained \( \alpha = 0.8694 \) and the empirical (EMP) model with

\[ \sigma^2 = \sigma_i^2 = 0.6709 (b|\Delta \tau|)^{0.8694}. \] (18)

Here, with the obtained \( \alpha \) and \( \beta \), the asymptotic diffusion coefficients can be calculated by equation (17).
3. Computer simulations
We performed direct computer simulations for 3D fluctuating magnetic field \( b \) following the techniques explained in [4]. Since the magnetic field in wave number space obeys \( k \cdot b(k) = 0 \), for a given \( k \), \( b(k) \) has only two polarizations and can be given by
\[
b(k) = \frac{1}{2} \sqrt{S(k)} \left[ b_1(k) e^{i \phi_1(k)} + i b_2(k) e^{i \phi_2(k)} \right],
\]
(19)
where \( \phi_1 \) and \( \phi_2 \) are independent random phases at every \( k \). \( b_1 \) and \( b_2 \) are unit vectors. The vectors \( b_1 \), \( b_2 \) and \( k \) are perpendicular to each other. Here \( S(k) \) is the 3D-isotropic power spectrum given by
\[
S(k) = C \frac{k^2 \lambda^2}{(1 + k^2 \lambda^2)^{\Gamma/2 + 2}},
\]
(20)
where \( C \) is a normalization constant and \( \lambda \) is the coherence or bendover scale. Note that, at large \( k \), the omnidirectional power spectrum \( E(k) = 4 \pi k^2 S(k) \) satisfies the scaling \( E(k) \propto k^{-\Gamma} \). Here \( \Gamma \) is used to identify the model of the turbulence power spectrum. We used a zero padding technique to generate \( S(k) \) for \( \lambda = 1 \) over a periodic box of size \( L_x = L_y = L_z = 100 \) with \( N_x \times N_y \times N_z = 1024 \times 1024 \times 1024 \) grid points covering the energy containing and the inertial range of turbulence. Inverse Fourier transform is used to transform \( b(k) \) in equation (19) back into \( b(x) \). We solved equation (3) numerically by using a fifth-order Runge-Kutta method with adaptive time stepping regulated by a fourth-order error estimate step to trace the magnetic field lines. We solved equation (3) over a sufficient range of \( \tau \) to obtain asymptotic diffusion coefficients. We traced 500,000 field lines. After every 12,500 field lines, a new representation was generated. For each field line, the mean squared displacement as a function of \( \tau \) was collected to calculate the ensemble average running diffusion coefficients. The asymptotic diffusion coefficient \( D \) is calculated from \( (D_x + D_y + D_z)/3 \).

4. Results
To test the prediction of the EMP model for the Bohm behavior of FLRW, we compared the asymptotic diffusion coefficients \( D \) of the EMP model with those of the DD, RBD, ODE models and simulation for Kolmogorov turbulence as shown in Table 1. Moreover, to test the validity of EMP model to other turbulences, we compared the \( D \) values of the EMP model with those of the other three models and simulation obtained by Sonsrettee et al. [4] as shown in Table 2. In Table 2, \( \Gamma = 3/2, 5/3, 2 \) corresponding to Iroshnikov-Kraichnan [14, 15], Kolmogorov [12, 13] and weak turbulences [16, 17], respectively.

### Table 1. Asymptotic field line diffusion coefficients for Kolmogorov turbulence with \( \lambda = 1 \) (the starred data are obtained by [4]).

| \( b \) | DD   | RBD   | ODE   | EMP   | Simulation |
|-------|------|-------|-------|-------|------------|
| 0.25  | 0.0774 | 0.0729 | 0.1094 | 0.0900 | 0.0900(0) |
| 0.5   | 0.1547 | 0.1458 | 0.2188 | 0.1800 | 0.1797(4) |
| 1     | 0.3094* | 0.2916* | 0.4376* | 0.3600 | 0.3600(5)* |
| 2     | 0.6188 | 0.5831 | 0.8752 | 0.7200 | 0.7186(2) |
Table 2. Asymptotic field line diffusion coefficients with \( b = 1 \) and \( \lambda = 1 \) (the stared data are obtained by [4]).

| Evaluation     | \( \Gamma = 3/2 \) | \( \Gamma = 5/3 \) | \( \Gamma = 2 \) |
|----------------|---------------------|---------------------|---------------------|
| theory:DD      | 0.2881*             | 0.3094*             | 0.3522*             |
| theory:RBD     | 0.2680*             | 0.2916*             | 0.3405*             |
| theory:ODE     | 0.4075*             | 0.4376*             | 0.4981*             |
| theory:EMP     | 0.3381              | 0.3600              | 0.4034              |
| simulation     | 0.3357(9)*          | 0.3600(5)*          | 0.4069(7)*          |

5. Conclusion

In this work, we used Corrsin’s hypothesis and the assumption that \( \sigma^2 = \beta (b |\Delta \tau|)^\alpha \) to determine the asymptotic diffusion coefficient \( D \) for FLRW in isotropic turbulence with zero mean field. We extrapolated \( \beta \) for the EMP model with the proportional constants of the DD and RBD models. Comparing with the simulation result for FLRW in Kolmogorov turbulence with \( b = 1 \) obtained by Sonsrettee et al. [4], we obtained \( \alpha \) for the EMP model. With the obtained \( \alpha \) and \( \beta \), the \( D \) values for EMP model were calculated by equation (17). For the simulation of Kolmogorov turbulence with varying \( b \), the EMP model predicts the Bohm behavior very well (with \( \leq 0.51\% \) error) as shown in Table 1. For the prediction of EMP model to other turbulences, the EMP model predicts the \( D \) values very close to the simulation for Iroshnikov-Kraichnan turbulence and weak turbulence (with \( \leq 0.72\% \) error and \( \leq 0.85\% \) error, respectively) as shown in Table 2.

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