CONSTRUCTION, EXTENSION AND COUPLING OF FRAMES ON FINITE DIMENSIONAL PONTRYAGIN SPACE.

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Abstract. In this paper we extend to finite-dimensional Pontryagin spaces the methods used in [6, 11] to build frames from an adjoint and positive operator. It is proved that any frame in finite dimensional Pontryagin space K is J-orthogonal projection of a frame for a space H such that K ⊂ H. Furthermore, given \{k_n\}_{n=1}^m and \{x_n\}_{n=1}^k frames for K and H respectively, we build a finite-dimensional Pontryagin space ℜ and a frame \{y_n\}_{n=1}^N for ℜ such that K, H ⊂ ℜ and \{k_n\}_{n=1}^m, \{x_n\}_{n=1}^k ⊂ \{y_n\}_{n=1}^N.

Introduction

From its appearance in [12] the theory of frames in Hilbert space has been quickly developed [5, 6, 9, 11, 14, 18], unlike the theory of frames in Krein space which is giving its first steps, [1, 13, 15, 16, 17]. In [13] a family \{k_n\}_{n∈N} is a frame for a Krein space if there exist constants A, B > 0 such that

A\|k\|_J^2 ≤ \sum_{n∈N} |[k, k_n]|^2 ≤ B\|k\|_J^2, \quad ∀k ∈ K,

In [15] and [17] alternative definitions are proposed. The fundamental idea is to use the versatility and flexibility of frames.

In [6] and [11] we find methods to build and extend frames in finite-dimensional Hilbert space. Based on [13] the main purpose of this work is to understand and extend these results to finite-dimensional Krein space called Pontryagin space. It is further proved that if \{k_n\}_{n=1}^m y \{x_n\}_{n=1}^k are frames for the Krein space K y H respectively then it is possible coupling those families, where by coupling it will be understood, find a Krein space ℜ con K, H ⊂ ℜ and a frame \{y_n\}_{n∈N} such that \{k_n\}_{n=1}^m, \{x_n\}_{n=1}^k ⊂ \{y_n\}_{n=1}^N.

To achieve our objective, in section 1 basic aspects of the finite-dimensional Pontryagin space and the theory of frames in finite-dimensional krein space are given. In section 2 we find the main results of this paper, we build finite frames from a positive and adjoint operator in the Pontryagin space (subsection 2.1). We extend frames to bigger spaces (subsection 2.2), and finally we couple two frames keep on the condition of frame. That is, we build a frame for the Pontyagrin space which contains both frames (subsection 2.3).

1. Preliminaries

Definition 1.1 (Krein Spaces). Let K be a vector space on \(\mathbb{C}\). Consider \([\cdot, \cdot] : K \times K \rightarrow \mathbb{C}\), a sesquilinear form. The vector space \((K, [\cdot, \cdot])\) is a Krein space whether \(K = K^+ \oplus K^-\) and \((K^+, [\cdot, \cdot]), (K^-, -[\cdot, \cdot])\) are Hilbert spaces, where \(K^+, K^-\) are orthogonal with respect \([\cdot, \cdot]\).

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On $\mathcal{K}$ define the following scalar product
\[(x_1, x_2) = [x_1^+, x_2^+] - [x_1^-, x_2^-], \quad x_i^+ \in \mathcal{K}^+, \quad x_i^- = x_i^+ + x_i^-.
\]
This scalar product makes $(\mathcal{K}, (\cdot, \cdot))$ a Hilbert space, which is so-called Hilbert space associated to $\mathcal{K}$. Hence, we can take the orthogonal projections on $\mathcal{K}^+$ and $\mathcal{K}^-$ denoted $P^+$ and $P^-$ respectively. The linear bounded operator $J = P^+ - P^-$ is called Fundamental Symmetry, and it satisfies the equality $[x, y] = (Jx, y), \ \forall x, y \in \mathcal{K}$. Equivalently
\[ [x, y]_J = [Jx, y] = (x, y), \quad \text{and denote} \quad \|x\|_J = \sqrt{[x, x]_J} \ \forall x, y \in \mathcal{K}. \quad (1.1)\]

**Definition 1.2.** Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space. Consider $x, y \in \mathcal{K}$, we say that $x$ is orthogonal to $y$ if $[x, y]_J = 0$, and is denoted by $x \perp y$. We say that $x$ is $J$-orthogonal to $y$ if $[x, y] = 0$, and is denoted by $x \perp y$.

The main purpose in this paper is to study frames in $N$-dimensional Krein spaces $\mathcal{K}$. Therefore $\dim \mathcal{K}^+, \dim \mathcal{K}^- \leq N$, these spaces are called Pontryagin spaces. More details see [3][4].

**Definition 1.3.** [3][4] A Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that
\[ 0 < \aleph = \min\{\dim \mathcal{K}^+, \dim \mathcal{K}^-\} < +\infty \quad (1.2)\]
is called a Pontryagin space.

**Definition 1.4.** [3][4] Let $\mathcal{K}$ be a Krein space, and let $V$ be a closed subspace of $\mathcal{K}$. The subspace
\[ V^{[\perp]} = \{x \in \mathcal{K} : [x, y] = 0, \ \forall y \in V\} \quad (1.3)\]
is so-called the $J$-orthogonal complement of $V$ with respect $[\cdot, \cdot]$ (or simply $J$-orthogonal complement of $V$).

**Definition 1.5.** [3][4] A closed subspace $V$ of $\mathcal{K}$ such that $V \cap V^{[\perp]} = \{0\}$ and $V + V^{[\perp]} = \mathcal{K}$, where $V^{[\perp]}$ is given in (1.3) is called projectively complete.

**Proposition 1.6.** [3][4] Let $(\mathcal{K}, [\cdot, \cdot])$ be a Pontryagin space $k$-dimensional. Every closed subspace $V \subset \mathcal{K}$ is projectively complete. Furthermore, is itself a Pontryagin space with dimension $0 \leq k' \leq k$.

**Remark 1.7.** In [3] shown’s that for any closed subspace $V$ its $J$-orthogonal complement $V^{[\perp]}$ and its orthogonal complement $V^\perp$ are closed subspaces connected by the formulas
\[ V^{[\perp]} = JV^\perp, \quad V^\perp = JV^{[\perp]}, \quad (JV)^{[\perp]} = JV^{[\perp]} \quad (1.4)\]

By (1.4) we note that $JV$ is projectively complete if and only if $V$ is projectively complete. In addition, the condition $V \cap V^{[\perp]} = \{0\}$ tells that every $x \in \mathcal{K}$ has an unique $J$-orthogonal projection on $V$, see [3][4].

**Remark 1.8.** Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space. Let $V$ be a closed subspace of $\mathcal{K}$ which is projectively complete. Then, $V = \overline{V} = (V^{[\perp]})^{[\perp]}$, this implies that $(V, [\cdot, \cdot])$ is a Krein space. Hence $V = V^+ \perp V^-$, where $V^+ \subset \mathcal{K}^+$, and $V^- \subset \mathcal{K}^-$. Thus $JV \subset V$.

**Definition 1.9.** [3][4] Let $\mathcal{K}$ be a Krein space. A system vector $\{e_i\}_{i \in I} \subset \mathcal{K}$, where $I$ is an arbitrary set of indices is called a $J$-orthonormalized system if $[e_i, e_j] = \pm \delta_{i,j}$ for all $i, j \in I$, where $\delta_{i,j}$ is the Kronecker delta.

**Example 1.10.** The simplest example of an $J$-orthonormalized system of $\mathcal{K}$ is the union of two arbitrary orthonormalized (in the usual sense) systems from the subspaces $\mathcal{K}^+$ and $\mathcal{K}^-$ respectively.
Definition 1.11. A J-orthonormalized system is said to be maximal if it is not contained in any wider J-orthonormalized system, and to be J-complete if there is no non-zero vector J-orthonormalized to this system.

Definition 1.12. Let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space. An J-orthonormalized basis in \(\mathcal{K}\) is an J-orthonormalized system which is J-complete and maximal in \(\mathcal{K}\).

Theorem 1.13. (i) \(\mathcal{K}\) admits a fundamental decomposition \(\mathcal{K} = \mathcal{K}^+ [\cdot, \cdot] \mathcal{K}^-\) such that the \(e_n\)'s with \([e_n, e_n] = 1\) belong to \(\mathcal{K}^+\), and the \(e_n\)'s with \([e_n, e_n] = -1\) belong to \(\mathcal{K}^-\).

\(\text{(1.10)}\)

Example 1.14 (\(\mathbb{C}^k\) like a Pontryagin space).

\(\mathbb{C}^k = \{(z_1, z_2, \ldots, z_k) : z_n \in \mathbb{C}, \ n = 1, 2, \ldots, k\}\).

On it, we define the indefinite inner product

\(\langle z, k \rangle_{\mathbb{C}^k} := \sum_{n=1}^{k} (-1)^{n-1} z_n w_n\), \quad z, w \in \mathbb{C}^k.\) \(\text{(1.11)}\)

Now, \(\mathbb{C}^k = \mathbb{C}^+_k + \mathbb{C}^-_k\), where

\(\mathbb{C}^+_k := \{z \in \mathbb{C}^k : z = (z_1, 0, z_3, \ldots, z_k), \text{ if } k = 2n - 1 \text{ for some } n \in \mathbb{N}\}\)

\(\mathbb{C}^-_k := \{z \in \mathbb{C}^k : z = (0, z_2, 0, z_4, \ldots, z_k), \text{ if } k = 2n \text{ for some } n \in \mathbb{N}\}\).

Observe that if \(z \in \mathbb{C}^+_k\) and \(w \in \mathbb{C}^k\), then

\(\langle z, z \rangle_{\mathbb{C}^k} = \sum_{n=1}^{k} (-1)^{n-1} |z_n|^2 = \sum_{n \text{ is even}} |z_n|^2 > 0\)

\(\langle w, w \rangle_{\mathbb{C}^k} = \sum_{n=1}^{k} (-1)^{n-1} |w_n|^2 = -\sum_{n \text{ is odd}} |w_n|^2 < 0.\)

On the other hand, if \(z \in \mathbb{C}^+_k\) and \(w \in \mathbb{C}_k\), then

\(\langle z, w \rangle_{\mathbb{C}^k} = \sum_{n=1}^{k} (-1)^{n-1} z_n w_n = 0.\)

Define the linear operator \(\mathcal{J} : \mathbb{C}^k \rightarrow \mathbb{C}\) given by

\(\mathcal{J}z = (z_1, -z_2, z_3, \ldots, (-1)^{k-1} z_k), \quad z = (z_1, z_2, \ldots, z_k) \in \mathbb{C}^k.\)

This operator is self-adjoint, J-self-adjoint with \(\mathcal{J}^2 = \text{id}\), and

\(\langle \mathcal{J}z, w \rangle_{\mathbb{C}^k} = \left(\left(\sum_{n=1}^{k} (-1)^{n-1} z_n w_n\right), \langle z, w \rangle_{\mathbb{C}^k}\right) = \sum_{n=1}^{k} z_n \overline{w_n} = \langle z, w \rangle_{\mathbb{C}^k}.\) \(\text{(1.11)}\)

Hence, \(\mathbb{C}^k = (\mathbb{C}^k, \langle \cdot, \cdot \rangle_{\mathbb{C}^k})\) is a Pontryagin space with fundamental symmetry \(\mathcal{J}\).
Since \((\mathcal{K}, [\cdot, \cdot])\) is a Hilbert space, we can study linear operators acting on Krein spaces. The topological concepts as continuity, closedness operators and spectral theory and so on, refer to the topology induced by the \(J\)-norm given in (1.1). Therefore, we can conclude that some definitions of operator theory in Hilbert spaces are satisfied. The adjoint of an operator \(T\) in Krein spaces \((T^{[\cdot]}\) satisfies \([T(x), y] = [x, T^{[\cdot]}(y)]\), but, we must consider that \(T\) have an adjoint operator in the Hilbert space \((\mathcal{K}, [\cdot, \cdot])\) denoted \((T^{*[\cdot]}\), where \(J\) is the fundamental symmetry in \(\mathcal{K}\), and there is a relation between \(T^{*[\cdot]}\) and \(T^{[\cdot]}\), which is \(T^{[\cdot]} = JT^{*[\cdot]}J\). Moreover, let \(K\) and \(K'\) be Krein spaces with fundamental symmetries \(J_K\) and \(J_{K'}\) respectively, if \(T \in B(K, K')\) then \(T^{[\cdot]}\) is \(J_K J_{K'} T^{*[\cdot]}J_{K'} J_K\). An operator \(T \in B(K)\) is said to be self-adjoint if \(T = T^{[\cdot]}\) and \(J\)-self-adjoint whether \(T = T^{*[\cdot]}\), moreover, a linear operator \(T\) is said to be positive whether \([Tx, x] \geq 0\) for every \(x \in K\). An operator \(T\) is said to be uniformly positive if there exists \(\alpha > 0\) such that \([Tx, x] \geq \alpha \|x\|^2\) for every \(k \in K\). Furthermore, a linear operator \(T\) is said to be invertible when its range and domain are the whole space.

**Proposition 1.15.** Let \((\mathcal{K}, [\cdot, \cdot])\), \((\mathcal{H}, [\cdot, \cdot])\) be Krein spaces with fundamental symmetries \(J_K\), \(J_H\) respectively. The vector space \(\mathbb{R} = \mathcal{K} \times \mathcal{H}\) with sesquilinear form

\[
[\cdot, \cdot]_{\mathbb{R}} := [\cdot, \cdot]_\mathcal{K} + [\cdot, \cdot]_\mathcal{H},
\]

is a Krein space with fundamental symmetry

\[
J_{\mathbb{R}} = (J_K, J_H).
\]

**Proof.** Since \(\mathcal{K} = K^+ \mathcal{K}^–\) and \(\mathcal{H} = H^+ \mathcal{H}^–\), we define

\[
\mathbb{R}^+ = \mathcal{K}^+ \times \mathcal{H}^+, \quad \mathbb{R}^- = \mathcal{K}^– \times \mathcal{H}^–.
\]

Thus, \(\mathbb{R}^+ \subset \mathbb{R}\), and \(\mathbb{R} = \mathbb{R}^+ + \mathbb{R}^–\). Observe that if \((k^+, h^+) \in \mathbb{R}^+\), and \((k^–, h^–) \in \mathbb{R}^–\), then

\[
[(k^+, h^+), (k^–, h^–)]_{\mathbb{R}} = [k^+, k^–]_\mathcal{K} + [h^+, h^–]_\mathcal{H} = 0.
\]

Equivalently, \(\mathbb{R}^+ \mathcal{K} \mathbb{R}^- \). On the other hand, \((\mathbb{R}^+, [\cdot, \cdot]_{\mathbb{R}})\) and \((\mathbb{R}^–, –[\cdot, \cdot]_{\mathbb{R}})\) are Hilbert spaces. In fact, if \(\{x_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \((\mathbb{R}^+, [\cdot, \cdot]_{\mathbb{R}})\), then \(\{x_n\}_{n \in \mathbb{N}}\) converges in \((\mathbb{R}^+, [\cdot, \cdot]_{\mathbb{R}})\) if and only if \(\{P_{\mathcal{K}^+} x_n\}_{n \in \mathbb{N}}\), \(\{P_{\mathcal{H}^+} x_n\}_{n \in \mathbb{N}}\), and \(\{P_{\mathcal{H}^–} x_n\}_{n \in \mathbb{N}}\) converges in \((\mathcal{K}^+, [\cdot, \cdot]_\mathcal{K})\), \((\mathcal{H}^+, [\cdot, \cdot]_\mathcal{H})\), \((\mathcal{H}^–, [\cdot, \cdot]_\mathcal{H})\), respectively. Therefore, \((\mathbb{R}, [\cdot, \cdot]_{\mathbb{R}})\) is a Krein space. Now, consider \(J_{\mathbb{R}} = (J_K, J_H) : \mathbb{R} \rightarrow \mathbb{R}\), it satisfies the following properties

\[
[J(x, y), (a, b)]_{\mathbb{R}} = ([J_K x, J_H y], (a, b)]_{\mathbb{R}} = [J_K x, a]_\mathcal{K} + [J_H y, b]_\mathcal{H} = [x, J_K a]_\mathcal{K} + [y, J_H b]_\mathcal{H} = ([x, y], J(a, b)] = (J^2(x, y), (a, b))_{\mathbb{R}}.
\]

Thus \(J^{[\cdot]} = J\). Also, \(J^2(x, y) = J(J_K x, J_H y) = (J^2_K x, J^2_H y) = (x, y)\), i.e. \(J^2 = id\). \(\square\)

**Remark 1.16.** On the product space \(\mathbb{R}\) given above, we can define the scalar product

\[
[(x, y), (a, b)]_{\mathbb{R}} := [J(x, y), (a, b)]_{\mathbb{R}} = [J_K x, a]_\mathcal{K} + [J_H y, b]_\mathcal{H}.
\]

(1.14)

1.1. Frames In Krein Spaces. This subsection is based in the results about the frame theory in Krein spaces studied in [13]. We used such results in Pontryagin spaces with dimension \(N\).

**Definition 1.17.** Let \(K\) be a Krein space. A countable sequence \(\{x_n\}_{n \in \mathbb{N}} \subset K\) is called a frame for \(K\), if there exist constants \(0 < A \leq B < \infty\) such that

\[
A\|x\|^2 \leq \sum_{n \in \mathbb{N}} \|x_n, x\|^2 \leq B\|x\|^2 \quad \text{for all } x \in K.
\]

(1.15)

**Remark 1.18.** Since we are mostly interested in finite-dimensional spaces, and since one can always fill up a finite frame with zero elements, we assume that \(\mathbb{R} = \{1, \ldots, k\}\).
As in the Hilbert space case, we refer to $A$ and $B$ as frame bounds. The greatest constant $A$ and the smallest constant $B$ satisfying (1.15) are called optimal lower frame bound and optimal upper frame bound, respectively. A frame is tight, if one can choose $A = B$. If a frame ceases to be a frame when an arbitrary element is removed, the frame is said to be exact.

The next theorem shows that frames for a Krein space are essentially the same objects as frames for the associated Hilbert space.

**Theorem 1.19.** [13] Let $K$ be a finite-dimensional Pontryagin space and $\{x_n\}_{n=1}^k$ a sequence in $K$. The following statements are equivalent:

i) $\{x_n\}_{n=1}^k$ is a frame for the Pontryagin space $K$ with frame bounds $A \leq B$.

ii) $\{Jx_n\}_{n=1}^k$ is a frame for the Pontryagin space $K$ with frame bounds $A \leq B$.

iii) $\{x_n\}_{n=1}^k$ is a frame for the Hilbert space $(K, [\cdot, \cdot])$ with frame bounds $A \leq B$.

iv) $\{Jx_n\}_{n=1}^k$ is a frame for the Hilbert space $(K, [\cdot, \cdot])$ with frame bounds $A \leq B$.

**Corollary 1.20.** Let $(K, [\cdot, \cdot])$ be a finite-dimensional Pontryagin space with fundamental symmetry $J$, and $\{x_n\}_{n=1}^k$ be a family of vectors in $K$. Then the following are equivalent:

i) $\{x_n\}_{n=1}^k$ is a frame for the Pontryagin space $K$.

ii) $\{x_n\}_{n=1}^k$ is a frame for the Hilbert space $(K, [\cdot, \cdot])$.

iii) span$\{x_n\}_{n=1}^k = K$.

**Proof.** The equivalence between i and ii is follows from Theorem 1.19, and the equivalence between ii and iii can be found in [11].

**Definition 1.21.** [13] Let $(K, [\cdot, \cdot])$ be a finite-dimensional Pontryagin space with fundamental symmetry $J$ and let $(C^k, [\cdot, \cdot]_{C^k})$ be the Pontryagin space with fundamental symmetry $\tilde{J}$ given in (1.10). Given a frame $\{x_n\}_{n=1}^k$ for $K$, the linear map

$$T : C^k \rightarrow K, \quad T(\alpha_n)_{n=1}^k = \sum_{n=1}^k \alpha_n k_n$$

(1.16)

is called pre-frame operator.

**Remark 1.22.** The pre-frame operator given in (1.16) is defined for more generally Krein spaces, it can be found in [13]. Just remember that $\ell_2(\{1, \ldots, k\})$ can be identified with $C^k$.

**Proposition 1.23.** [13] Let $(K, [\cdot, \cdot])$ be a Krein space. The family $\{x_n\}_{n\in\mathbb{N}}$ is a frame for $K$, if and only if $T$ is well defined (i.e. bounded) and surjective.

**Definition 1.24.** Let $(K, [\cdot, \cdot])$ be a finite-dimensional Pontryagin space. The adjoint of pre-frame operator $T$ is given by

$$T^{[*]}x = \tilde{J}(x_n, x)_{n=1}^k, \quad k \in K.$$

(1.17)

And is so-called analysis operator.

**Definition 1.25.** Let $(K, [\cdot, \cdot])$ be a finite-dimensional Pontryagin space with fundamental symmetry $J$, $(C^k, [\cdot, \cdot]_{C^k})$ a Pontryagin space with fundamental symmetry $\tilde{J}$ given in (1.14) and (1.10), respectively, and $\{k_n\}_{n=1}^k \subset K$ a frame for $K$. The operator

$$S := T\tilde{J}T^{[*]}$$

is called frame operator.
It follows immediately from (1.16), (1.17) and \( J^2 = \text{id} \) that
\[
Sx = \sum_{n \in \mathbb{N}} [x_n, x] x_n, \quad x \in \mathcal{K},
\]
(1.18)
as desired. Moreover, \( S \) is clearly self-adjoint. If \( (\mathbb{C}^k, [\cdot, \cdot]) = (\mathbb{C}^k, \langle \cdot, \cdot \rangle) \), then \( S = TT^* \), exactly as in the Hilbert space case.

2. Main Results

2.1. Construction of frames with a operator on finite-dimensional Pontryagin space.

**Proposition 2.1.** Let \( (\mathcal{K}, [\cdot, \cdot]) \) be a \( N \)-dimensional Pontryagin space with fundamental symmetry \( J \), and \( S_0 \) be a \( J \)-self-adjoint and positive operator with respect to \([\cdot, \cdot]_J\). Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > 0 \) be the eigenvalues of \( S_0 \). Fix \( k \geq N \) and real numbers \( a_1 \geq a_2 \geq \cdots \geq a_k > 0 \). The following are equivalent:

i). For every \( 1 \leq j \leq N \)
\[
\sum_{n=1}^{j} a_n^2 \leq \sum_{n=1}^{j} \lambda_n, \quad \sum_{n=1}^{k} a_n^2 \leq \sum_{n=1}^{N} \lambda_n.
\]

ii). There is a frame \( \{x_n\}_{n=1}^{k} \) for the Hilbert space \( (\mathcal{K}, [\cdot, \cdot]_J) \) with frame operator \( S_0 \) and \( \|x_n\|_J = a_n \), for all \( n = 1, \ldots, k \).

iii). There is a frame \( \{x_n\}_{n=1}^{k} \) for the Pontryagin space \( \mathcal{K} \) with frame operator \( S_0J \) and \( \|x_n\|_J = a_n \), for all \( n = 1, \ldots, k \).

iv). There is a frame \( \{Jx_n\}_{n=1}^{k} \) for the Pontryagin space \( \mathcal{K} \) with frame operator \( JS_0 \) and \( \|x_n\|_J = a_n \), for all \( n = 1, \ldots, k \).

v). There is a frame \( \{Jx_n\}_{n=1}^{k} \) for the Hilbert space \( (\mathcal{K}, [\cdot, \cdot]_J) \) with frame operator \( JS_0J \) and \( \|x_n\|_J = a_n \), for all \( n = 1, \ldots, k \).

**Proof.** The equivalence between i and ii is proved in [6]. By Theorem 1.19 we have the equivalences
\[
\text{ii} \leftrightarrow \text{iii} \leftrightarrow \text{iv} \leftrightarrow \text{v}.
\]
Only we need to find the relationship of its respective frame operator with the operator \( S_0 \). The details is in [13].

Since \( S_0 \) is the frame operator corresponding to the family \( \{x_n\}_{n=1}^{k} \), then
\[
S_1x = \sum_{n=1}^{k} [x, x_n] x_n = \sum_{n=1}^{k} [Jx, x_n] Jx_n = S_0Jx, \quad \forall x \in \mathcal{K}.
\]

\( S_1 = S_0J \) is the frame operator for the frame \( \{x_n\}_{n=1}^{k} \) in the Pontryagin space \( \mathcal{K} \).

\[
S_2x = \sum_{n=1}^{k} [x, Jx_n] Jx_n = J \left( \sum_{n=1}^{k} [x, x_n] Jx_n \right) = JS_0x, \quad \forall x \in \mathcal{K}.
\]

\( S_2 = JS_0 \) is the frame operator for the frame \( \{Jx_n\}_{n=1}^{k} \) in the Pontryagin space \( \mathcal{K} \).

\[
S_3x = \sum_{n=1}^{k} [x, Jx_n] Jx_n = J \left( \sum_{n=1}^{k} [Jx, x_n] Jx_n \right) = JS_0Jx, \quad \forall x \in \mathcal{K}.
\]

\( S_3 = JS_0J \) is the frame operator for the frame \( \{Jx_n\}_{n=1}^{k} \) in the Hilbert space \( (\mathcal{K}, [\cdot, \cdot]_J) \).

\( \square \)
2.2. Extension of frames on finite-dimensional Pontryagin space.
This subsection is based on the study in Hilbert spaces, more details see [11].

Similar Frames in finite-dimensional Pontryagin Spaces.

**Definition 2.2.** Let \((\mathcal{H}, [\cdot, \cdot]_H)\) and \((\mathcal{K}, [\cdot, \cdot]_K)\) be finite-dimensional Pontryagin spaces. Two frames \(\{x_n\}_{n=1}^k\) and \(\{y_n\}_{n=1}^k\) for \(\mathcal{H}\) and \(\mathcal{K}\) respectively, are said to be similar if there exists an invertible operator \(U : \mathcal{H} \to \mathcal{K}\) such that \(Uy_n = x_n\) for \(n \in \{1, \ldots, k\}\). The frames are called unitarily equivalent if we require \(U\) to be a unitary operator from \(\mathcal{H}\) onto \(\mathcal{K}\).

**Proposition 2.3.** Let \((\mathcal{H}, [\cdot, \cdot]_H)\), \((\mathcal{K}, [\cdot, \cdot]_K)\) be a finite-dimensional Pontryagin spaces, and \(\{x_n\}_{n=1}^k\), \(\{y_n\}_{n=1}^k\) be two frames for \(\mathcal{H}\) and \(\mathcal{K}\) respectively. Then they are similar if and only if their analysis operators have the same range.

**Proof.** Let \(T_{\mathcal{H}}^{[\cdot]_\mathcal{H}}\) and \(T_{\mathcal{K}}^{[\cdot]_\mathcal{K}}\) be the analysis operator for \(\{x_n\}_{n=1}^k\) and \(\{y_n\}_{n=1}^k\) respectively.

\[\Rightarrow\] Suppose that \(\{x_n\}_{n=1}^k\) and \(\{y_n\}_{n=1}^k\) are similar, then there is an invertible operator \(U : \mathcal{H}_N \to \mathcal{K}_M\) such that \(UX_n = Y_n\). Hence

\[T_{\mathcal{K}}^{[\cdot]_\mathcal{K}} y = \sum_{j=1}^k (-1)^{j-1}[y, y_j]e_j = \sum_{j=1}^k (-1)^{j-1}[y, UX_j]e_j = \sum_{j=1}^k (-1)^{j-1}[U^*y, x_j]e_j = T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} (U^*y),\]

Now, since \(U^*\) is invertible if and only if \(U\) is, we concluded that

\[T_{\mathcal{K}}^{[\cdot]_\mathcal{K}}(\mathcal{K}) = T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} U^* (\mathcal{K}) = T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} (\mathcal{H}).\]

\[\Leftarrow\] Suppose that \(T_{\mathcal{K}}^{[\cdot]_\mathcal{K}}(\mathcal{K}) = T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} (\mathcal{H}) = V\). We note that \(T_{\mathcal{H}|_V}\) and \(T_{\mathcal{K}|_V}\) are invertible, thus the operator \(G = T_{\mathcal{K}} \left(T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} T_{\mathcal{H}}^{-1} T_{\mathcal{H}}^{[\cdot]_\mathcal{H}}\right) : \mathcal{H} \to \mathcal{K}\) is well defined and it is invertible. Let \(P\) be a J-orthogonal projection of \(\mathbb{C}^k\) on \(V\). If \(y \in V^{[1]_\mathcal{K}}\), then 0 \(\neq [x, y]_{\mathcal{K}}\) for all \(x \in V\). Thus

\[0 = [x, y]_{\mathcal{K}} = [T_{\mathcal{K}}^{[\cdot]_\mathcal{K}} z, y]_{\mathcal{K}} = [z, T_{\mathcal{K}} y]_{\mathcal{K}},\]

Since \(y\) is arbitrary, we conclude that \(T_{\mathcal{K}} \left(V^{[1]_\mathcal{K}}\right) = \{0\}\). Similarly we have that \(T_{\mathcal{H}} \left(V^{[1]_\mathcal{K}}\right) = \{0\}\). Thus, by the definition of pre-frame operator we consider \(v_{m,j} = \delta_{m,j} \in \mathbb{C}^k\), and we get

\[T_{\mathcal{H}} Pv_m = T_{\mathcal{K}} v_m = \sum_{j=1}^k \delta_{m,j} x_m = x_m,\]

on the other hand, it also satisfies that \(y_m = T_{\mathcal{K}} v_m = T_{\mathcal{K}} Pv_m\). Hence,

\[Ux_m = UT_{\mathcal{H}} Pv_m = T_{\mathcal{K}} \left(T_{\mathcal{H}}^{[\cdot]_\mathcal{H}} T_{\mathcal{H}}^{-1} T_{\mathcal{H}}^{[\cdot]_\mathcal{H}}\right) T_{\mathcal{H}} Pv_m = T_{\mathcal{K}} Pv_m = y_m,\]

This implies that the frames are similar. \(\square\)

**Theorem 2.4.** Let \((\mathcal{H}, [\cdot, \cdot]_H)\) be a Pontryagin space, and suppose that \(\{x_n\}_{n=1}^k\) is frame for \(\mathcal{H}\). Then there exists a Pontryagin space \((\mathcal{K}, [\cdot, \cdot]_K)\) with \(\mathcal{H} \subset \mathcal{K}\), and a frame \(\{v_n\}_{n=1}^k\) for \(\mathcal{K}\) such that \(x_n = Pv_n\), where \(P\) is the J-orthogonal projection from \(\mathcal{K}\) onto \(\mathcal{H}\).

**Proof.** Consider \(V = (\text{Rang } T^{[\cdot]_\mathcal{H}})^{[1]_\mathcal{K}} \subset \mathbb{C}^k\), which is a Pontryagin space. Hence by Proposition 1.15 we have that \(\mathcal{K} = \mathcal{H}_N \times V = \mathcal{K} \times \{0\} \oplus \{0\} \times V \simeq \mathcal{K} \oplus V\) is a Pontryagin space with indefinite inner product \([\cdot, \cdot]_\mathcal{K} = [\cdot, \cdot]_H + [\cdot, \cdot]_V\), and fundamental symmetry \(J_\mathcal{K} = J_\mathcal{H} \oplus J\), where \(J\) is the fundamental simmetry of Pontryagin space \(\mathcal{K}\) given to (1.10). Consider \(Q^{[1]_\mathcal{K}} = I - Q\), where
$Q : \mathcal{C}^k \to \text{Rang } T[^s]$ is a $J$-orthogonal projection from $\mathcal{C}^k$ onto $\text{Rang } T[^s]$. The family $\{e_n\}_{n=1}^k$, where $e_n$ is the vector which has a 1 in the $n$-esimo place and zero in the other, is a base $J$-orthonormal from $\mathcal{C}^k$, and by (1.17) it satisfies that $[e_n, e_n]_{\mathcal{C}^k} = (-1)^{n-1}$.

We defined

$$u_n = x_n \oplus Q[^1]e_n, \quad n = 1, 2, \ldots, k. \quad (2.3)$$

Note that $P : \mathcal{K}_M \to \mathcal{H}_N$ is a $J$-orthogonal projection from $\mathcal{K}$ into $\mathcal{H}$, we will have that $Pu_n = x_n$. It is enough to show that the family $\{u_n\}_{n=1}^k$ is a frame for $\mathcal{K}_M$. Indeed, note that by corollary 1.20 the family $\{e_n\}_{k=1}^k$ is a frame for $\mathcal{C}^k$, therefore $\{Qe_n\}_{n=1}^k$ is a frame for $\text{Rang } T[^s]$, (See 13). Given $x \in \text{Rang } T[^s]$, by (1.17) and (1.13) it is obtained that

$$T_Q[^s]e^k(x) = \sum_{n=1}^k (-1)^{n-1} [x, Qe_n]_{\mathcal{C}^k} e_n = \sum_{n=1}^k (-1)^{n-1} [Qx, e_n]_{\mathcal{C}^k} e_n = \sum_{n=1}^k (-1)^{n-1} [x, e_n]_{\mathcal{C}^k} e_n$$

$$= \sum_{n=1}^k [e_n, e_n]_{\mathcal{C}^k} [x, e_n]_{\mathcal{C}^k} = x,$$

Where $T_Q$ is the pre-frame operator of $\{Qe_n\}_{n=1}^k$. That is, $\text{Rang } T_Q[^s]e^k = \text{Rang } T[^s]$, by theorem 2.3 the frames $\{x_n\}_{k=1}^k$ and $\{Qe_n\}_{n=1}^k$ are similar. Let $W$ be the invertible operator such that $WQe_n = x_n$ for $n = 1, \ldots, k$. Therefore, for each $n$ we have that

$$u_n = WQe_n \oplus Q[^1]e_n = U \left( Qe_n \oplus Q[^1]e_n \right) = Ue_n,$$

where $U = W \oplus I$ is an invertible operator from $\mathcal{C}^k$ into $\mathcal{K}_M$. Furthermore, since $\{e_n\}_{n=1}^k$ is a frame for $\mathcal{C}^k$ and the invertible operators preserve frames (ver 13), thus it concludes that $\{u_n\}_{n=1}^k$ is a frame for $\mathcal{K}_M$. \hfill \Box

2.3. Coupling of frames on finite-dimensional Pontryagin space.

**Theorem 2.5.** Let $(\mathcal{K}, [\cdot, \cdot]_\mathcal{K})$, and $(\mathcal{H}, [\cdot, \cdot]_\mathcal{H})$ be Pontryagin spaces. If $\{x_n\}_{n=1}^k$ and $\{y_n\}_{n=1}^k$ are frames for $\mathcal{K}$ and $\mathcal{H}$ respectively, then there is a Pontryagin space $\mathcal{R}$ containing $\mathcal{K}$ and $\mathcal{H}$, a frame $\{z_n\}_{n=1}^k$ for $\mathcal{R}$ such that $P_\mathcal{K}z_n = x_n$ and $P_\mathcal{H}z_n = y_n$, for each $n = 1, \ldots, k$, where $P_\mathcal{K}$ and $P_\mathcal{H}$ are $J$-orthogonal projection from $\mathcal{R}$ onto $\mathcal{K}$ and $\mathcal{H}$ respectively.

**Proof.** By Theorem 2.4 there are Pontryagin spaces $\mathcal{R}_\mathcal{K}$ and $\mathcal{R}_\mathcal{H}$, such that $\mathcal{K} \subset \mathcal{R}_\mathcal{K}, \mathcal{H} \subset \mathcal{R}_\mathcal{H}$, frames $\{u_n\}_{n=1}^k, \{v_n\}_{n=1}^k$ for $\mathcal{R}_\mathcal{K}$ and $\mathcal{R}_\mathcal{H}$ respectively, with $P_1u_n = x_n, P_2v_n = y_n$ for each $n = 1, \ldots, k$. Here $P_1 : \mathcal{R}_\mathcal{K} \to \mathcal{K}, P_2 : \mathcal{R}_\mathcal{H} \to \mathcal{H}$ are $J$-orthogonal projections. Define $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, by Proposition 1.15 is a Pontryagin space with respect to the indefinite inner product $[\cdot, \cdot]_\mathcal{R} = [\cdot, \cdot]_{\mathcal{R}_\mathcal{K}} + [\cdot, \cdot]_{\mathcal{R}_\mathcal{H}}$, where $\mathcal{K} \times \mathcal{H} \subset \mathcal{R}$. If $T_{\mathcal{R}_\mathcal{K}}$ is the pre-frame operator of the family $\{u_n\}_{n=1}^k$ and $T_{\mathcal{R}_\mathcal{H}}$ is the pre-frame operator of the family $\{v_n\}_{n=1}^k$, then, taking $0 < B = \|T_{\mathcal{R}_\mathcal{K}}\| + \|T_{\mathcal{R}_\mathcal{H}}\| < \infty$ and $T_{\mathcal{R}} : \mathcal{C}^k \to \mathcal{R}$, by

$$T(\{e_n\}_{n=1}^k) = \sum_{n=1}^k c_n(u_n, v_n) = \left( \sum_{n=1}^k c_nu_n, \sum_{n=1}^k c_nv_n \right) = \left( T_{\mathcal{R}_\mathcal{K}}(\{e_n\}_{n=1}^k), T_{\mathcal{R}_\mathcal{H}}(\{e_n\}_{n=1}^k) \right),$$
we get

\[
\|T_\mathbb{R}(\{c_n\}_{n=1}^k)\|^2_{J_\mathbb{R}} = \left[\left(T_\mathbb{R}_K(\{c_n\}_{n=1}^k), T_\mathbb{R}_H(\{c_n\}_{n=1}^k)\right), \left(T_\mathbb{R}_K(\{c_n\}_{n=1}^k), T_\mathbb{R}_H(\{c_n\}_{n=1}^k)\right)\right]_{J_\mathbb{R}}
\]

\[
= \left[T_\mathbb{R}_K(\{c_n\}_{n=1}^k), T_\mathbb{R}_K(\{c_n\}_{n=1}^k)\right]_{J_\mathbb{R}_K} + \left[T_\mathbb{R}_H(\{c_n\}_{n=1}^k), T_\mathbb{R}_H(\{c_n\}_{n=1}^k)\right]_{J_\mathbb{R}_H}
\]

\[
= \|T_\mathbb{R}_K(\{c_n\}_{n=1}^k)\|^2_{J_\mathbb{R}_K} + \|T_\mathbb{R}_H(\{c_n\}_{n=1}^k)\|^2_{J_\mathbb{R}_H}
\]

\[
\leq B \|\{c_n\}_{n=1}^k\|^2_{J_\mathbb{R}_K}.
\]

Hence, \(\|T_\mathbb{R}\| \leq \sqrt{B}\). Furthermore, \(T_\mathbb{R}\) is surjective. Thus, by Proposition 1.23 the family \(\{z_n = (u_n,v_n)\}_{n=1}^k\) is a frame for \(\mathbb{R}\). On the other hand, define the linear operator \(P : \mathbb{R} \rightarrow K \times H\) given by the formula \(P(x,y) = (P_1 x, P_2 y)\), note that

\[
[P(x,y),(a,b)]_\mathbb{R} = [(P_1 x, P_2 y), (a,b)]_\mathbb{R} = [P_1 x, a]_{J_K} + [P_2 y, b]_{J_H} = [x, P_1 a]_{J_K} + [y, P_2 b]_{J_H}
\]

Thus, \(P^{[x]} = P\). Now, \(P^2(x,y) = P(P_1 x, P_2 y) = (P_1^2 x, P_2^2 y) = (P_1 x, P_2 y) = P(x,y)\) para todo \((x,y) \in \mathbb{R}\). Esto es, \(P^2 = P\), for which \(P\) is a \(J\)-orthogonal projection from \(\mathbb{R}\) onto \(K \times H\) and satisfies \(Pz_n = (x_n, y_n)\) for each \(n = 1,\ldots,k\). We finish taking \(P_K(x,y) = (P_1 x,0)\) and \(P_H(x,y) = (0, P_2 y)\).

**Proposition 2.6 (Couplers for frame operators).** Let \((K, [\cdot, \cdot]_K)\) and \((H, [\cdot, \cdot]_H)\) be finite-dimensional Pontryagin spaces with fundamental symmetries \(J_K\) and \(J_H\) respectively. Let \(S_K\) and \(S_H\) be \(J\)-self-adjoint and positive operators with respect to \([\cdot, \cdot]_K\), and \([\cdot, \cdot]_H\) resp. Then, there exists a finite-dimensional Pontryagin space \(\mathbb{R}\) with fundamental symmetry \(J\), and a \(J\)-self-adjoint and positive operator \(S_\mathbb{R} : \mathbb{R} \rightarrow \mathbb{R}\), such that:

\[
S_{\mathbb{R}|_K} = S_K, \quad \text{and} \quad S_{\mathbb{R}|_H} = S_H. \tag{2.4}
\]

**Proof.** By Proposition 1.15 \(\mathbb{R} = K \times H\) is a Pontryagin space with indefinite inner product \([x,y], (a,b)]_\mathbb{R} = [x,a]_K + [y,b]_H\) and fundamental symmetry given by the formula \(J_\mathbb{R}(x,y) = (J_K x, J_H y), \forall (x,y) \in \mathbb{R}\). Define \(S_\mathbb{R} : \mathbb{R} \rightarrow \mathbb{R}\) by

\[
S_\mathbb{R}(x,y) = (S_K x, S_H y), \quad (x,y) \in \mathbb{R}. \tag{2.5}
\]

This operator is positive and \(J_\mathbb{R}\)-self-adjoint with respect to \([\cdot, \cdot]_\mathbb{R}\). Furthermore

\[
S_{\mathbb{R}|_K} = (S_K,0), \quad \text{and} \quad S_{\mathbb{R}|_H} = (0, S_H).
\]

**Remark 2.7.** If the operators \(S_K\) y \(S_H\) given at the proposition 2.6 satisfy the conditions at the proposition 2.7 then the operator \(S_\mathbb{R}\) has an associated frame and satisfies the properties given for the theorem 2.4.

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