QUANTUM REAL NUMBERS AND MEASUREMENT

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Abstract. The quantum mechanical measurement problem does not arise in the quantum real number approach to quantum measurements of the first kind. The attributes of individual microscopic systems in the experimental ensemble always have qr-number values so the individual systems can be followed throughout the process. The interaction with an apparatus connects the qr-number value of the quantity to be measured with the qr-number value of an attribute of the apparatus that can be locally approximated by a classical number and subsequently amplified to a recordable output.

1. Introduction

There are broadly two processes in which measurements are used in modern applications of quantum mechanics: the first is to determine the numerical value of a physical attribute of a quantum system, the second is to determine the state of the system[1]. In this paper the first problem is emphasised.

The measurement problem arises in standard quantum theories of both, for a recent discussion see Schlosshauser[2]. The measurement problem has two parts:

- The problem of definite outcomes.
- The problem of the preferred basis.

The first occurs because the measurement of a microscopic system $S$ yields a probability distribution of the values of one or many attributes of $S$. Any prediction can only be verified by experimental data obtained from an ensemble of identically prepared replicas of $S$. In order that the relative frequencies of the various outcomes can be determined, the final ensemble must be such that each outcome is observationally distinct. This basic requirement for determining probabilities is not satisfied in the standard Hilbert space quantum theories.

The second doesn’t arise in the qr-number approach because it doesn’t accept the premise that a wave function provides a complete state of a quantum system. A complete state in the qr-number model is given by an open set of quantum states, see [2,1] and [19].

Date: February 7, 2019.
1.0.1. The standard description. The following is a simple example, from [1] pp 75-78, illustrating the problem of definite outcomes in the standard quantum mechanical description of a measurement. There are two quantum systems: $S$, the carrier of an attribute, represented by the operator $\hat{A}_S$, which is to be measured, and $M$ a measurement apparatus with a pointer represented by an operator $\hat{B}_M$. At time $t_1$, $S$ is in a superposition of $\hat{A}_S$'s eigenstates $\psi_\pm$ (eigenvalues $\lambda_\pm$) while $M$ is in the null eigenstate $\phi_0^M$ of its pointer operator $\hat{B}_M$.

\begin{equation}
\Psi_{S,M}(t_1) = \psi_S \otimes \phi_0^M = (\alpha_+ \psi_S^+ + \alpha_- \psi_S^-) \otimes \phi_0^M.
\end{equation}

$\psi_S^\pm \perp \psi_S^-$ and $|\alpha_+|^2 + |\alpha_-|^2 = 1$ and all wave functions are unit vectors. The aim is to determine the distribution of the values $\lambda_\pm$ of the attribute $\hat{A}_S$ in the prepared system $S$. To this end the relative frequencies, $|\alpha_+|^2$ and $|\alpha_-|^2$ of the outcomes $\lambda_\pm$, are determined from the ensemble of prepared systems.

An interaction between $S$ and $M$ produces an entangled state,

\begin{equation}
\Psi_{S,M}(t_2) = \alpha_+ \psi_S^+ \otimes \phi_M^+ + \alpha_- \psi_S^- \otimes \phi_M^-,
\end{equation}

at the time $t_2 > t_1$. The vectors $\phi_M^+ \perp \phi_M^-$ and are assumed to be macroscopically distinguishable eigenstates of $\hat{B}_M$.

Because the wave-function $\Psi_{S,M}(t_2)$ is an entangled pure state and not a mixed state, it is not possible to ascribe a particular wave-function to $M$. That is, there is no definite outcome associated with the measurement process. If we assume that a pure state describes the state of an ensemble of identical systems, the wave-function $\Psi_{S,M}(t_2)$ describes that of an ensemble of combined $S$ and $M$ systems. It does not parametrise a variety of outcomes and hence does not determine the probabilities of different outcomes. This is the measurement problem.

The pure state $\Psi_{S,M}(t_2)$ cannot evolve unitarily to a mixed state so that the Schrödinger evolution cannot deliver a definite outcome for the measurement problem. This is where the "collapse hypothesis" or "projection postulate" is inserted, [1] §2.3.3, pp 86-91. The standard unitary time evolution of quantum mechanics is replaced by a jump from the pure state, $\rho = \hat{P}_{\Psi_{S,M}(t_2)}$, to the mixed state

\begin{equation}
\rho' = |\alpha_+|^2 \hat{P}_{\psi_S^+} \otimes \hat{P}_{\phi_M^+} + |\alpha_-|^2 \hat{P}_{\psi_S^-} \otimes \hat{P}_{\phi_M^-}.
\end{equation}

Then the mixed state collapses to $\hat{P}_{\psi_S^\pm} \otimes \hat{P}_{\phi_M^\pm}$ with probability $|\alpha_+|^2$ or to $\hat{P}_{\psi_S^\pm} \otimes \hat{P}_{\phi_M^\pm}$ with probability $|\alpha_-|^2$. The justification of these assumptions is decidedly ad hoc and this has always been a contentious area of quantum mechanics and one which has been often taken as a sign of the incompleteness of the theory. Home [1], Chapter 2, has a good discussion of the issues that have arisen.
2. The qr-number description

The qr-number values of the pertinent quantities always have a qr-number values, see \[|\alpha_\pm|\] so their trajectories throughout the experiment can be followed. We assume that both the system $S$, whose properties are to be measured and the measurement system $M$ are particles with non-zero masses, $m_S$ and $m_M$.

In the preparation stage, see \[\S 3\] the quantity $\hat{Q}_S$ to be measured is prepared so that it can be measured. An epistemic condition is prepared for an ensemble of $S$-particles. In the generic example $\hat{Q}_S$ has only two eigenvalues, $\lambda_\pm$ with eigenvectors $\phi_{\lambda_\pm}$. Let $\psi_S(\vec{\alpha}) = \alpha_+ \phi_{\lambda_+} + \alpha_- \phi_{\lambda_-}$, with $\vec{\alpha} = (\alpha_+, \alpha_-) \in \mathbb{C}^2$ and $|\alpha_+|^2 + |\alpha_-|^2 = 1$, be the wavefunction for $S$ that was used in the standard description, \[\S 1.0.1\]. Then lemma 2 of \[\S 3\] shows that the prepared condition is

\[
W_S(\vec{\alpha}) = |\alpha_+|^2 W^+_S + |\alpha_-|^2 W^-_S.
\]

where $W^+_S = \mathcal{N}(\hat{P}_{\phi_{\lambda_+}}, \hat{Q}_S, \delta)$ are designated epistemic conditions on which the qr-number values $q_S|_{W^+_S}$ are measurable with $q_S|_{W^-_S} \approx \lambda_\pm$.

For example, when $\hat{Q}_S$ is $\varepsilon$-sharp collimated in an interval $I_\varepsilon$ centred at the eigenvalue $\lambda_\pm$ on the condition $W^\varepsilon_S$, see definition 1, \[\S 2.2.1\] then $\hat{Q}_S$’s qr-number value $q_S|_{W^\varepsilon_S}$ is well approximated by the eigenvalue $\lambda_\pm$. The coefficient $|\alpha_\pm|^2$ are the relative frequencies of preparing $W^\varepsilon_S$. At the same time the measurement device is prepared in a condition $W^0_M$ so that the pointer variable $\hat{Q}_M$ is $\varepsilon$-sharp collimated in an interval $I_0$ centred at 0.

In the interaction stage the “particles” $S$ and $M$ interact through a von Neumann impulsive interaction, $\hat{H}_I = \gamma \hat{Q}_S \otimes \hat{P}_M$, see \[\S 3.1\] causing a change in the qr-number value of $\hat{Q}_M$ proportional to the qr-number value of $\hat{Q}_S$ which doesn’t change. When the interact lasts from $t_1$ to $t_2$, if the ontic condition of $S$ is $V^\varepsilon_S \subset W^\varepsilon_S$ then

\[
q_M|_{W^\varepsilon_M}(t_2) - q_M|_{W^\varepsilon_M}(t_1) = \kappa_M q_S(V^\varepsilon_S)
\]

where $\kappa_M = \frac{\gamma}{m_M}(t_2 - t_1)$. But $q_S(V^\varepsilon_S) \approx \lambda_\pm|_{V^\varepsilon_S}$ so that the change in the pointer’s reading will be proportional to $\lambda_\pm$. In \[\S 3.3\] we show that $q_M|_{W^\varepsilon_M}(t_2) = q_M|_{W^\varepsilon_M}$ where $W^\varepsilon_M$ is a condition on which $\hat{Q}_M$ is measurable, $W^\varepsilon_M$ depends on whether $W^\varepsilon_S$ was prepared. The problem of definite outcomes does not exist in the qr-number model. The preparation of $S$ ensures that $S$ has an ontic condition that is an open subset of one of the designated epistemic conditions $W^\varepsilon_S$. The outcome for the ensemble is the determination of the relative frequencies, $|\alpha_+|^2$ and $|\alpha_-|^2$. This does not determine the wave-function $\psi_S(\vec{\alpha})$.

The pointer outcome can be amplified, this is discussed in \[\S 3.4\].
2.1. Basics of the qr-number model. The mathematics of the qr-number model, introduced in [14], is built upon a Hilbert space formalism. It uses a spatial topos, defined in [12] and [13], to obtain qr-numbers as the numerical values taken by physical attributes of a quantum system.

In the qr-number model the quantum system always has a complete state, called its condition, given by an open subset of the smooth state space \( \mathcal{E}_S(\mathcal{A}_S) \), defined in \( \S 4.1 \) and all physical attributes retain their qr-number values even when not being observed. The qr-numbers are contextual, the qr-number value of a physical attribute is essentially a function with values in \( \mathbb{R} \) whose domain is the system’s condition.

There are two classes of quantum conditions: (1) the epistemic condition of an ensemble of systems depends upon the experimental setup and (2) the ontic condition of an individual system in the ensemble. Any open subset of \( \mathcal{E}_S(\mathcal{A}_S) \) can be in either class but an ontic condition is always proper open subset of an epistemic condition. The existence of ontic conditions explains the variation in the individual outcomes in an experiment. In general a mixed condition of the form \( \sum_j \lambda_j W_j \) for \( 0 < \lambda_j < 1 \), \( \sum_j \lambda_j = 1 \) and \( W_j \in \mathcal{O}(\mathcal{E}_S(\mathcal{A}_S)) \) is an epistemic condition, each \( \lambda_j \) is interpreted as the probability preparing the ensemble in \( W_j \).

The physical attributes of a system are represented by the elements of an \( \mathcal{O}^\ast \)-algebra \( \mathcal{A}_S \), see [9], of unbounded operators on a dense subset \( \mathcal{D} \) of the system’s Hilbert space \( \mathcal{H}_S \). \( \mathcal{O}^\ast \)-algebras allow us to directly represent physical qualities like energy, momentum and position of a particle. When the system is a massive Galilean relativistic quantum particle it has a trajectory in its qr-number space, see [16] and [17] for some examples. In this paper each \( \mathcal{O}^\ast \)-algebra comes from a unitary representation \( \hat{U} \) of a Lie group \( G \) on \( \mathcal{H}_S \), see [14.1]. The set of \( \mathcal{C}^\infty \)-vectors for \( \hat{U} \), denoted \( \mathcal{D}^\infty(\hat{U}) \), is a dense linear subspace of \( \mathcal{H} \) which is invariant under \( \hat{U}(g) \), \( g \in G \), [19] has more details. The system’s smooth state space, \( \mathcal{E}_S(\mathcal{A}_S) \), is contained in the convex hull of projections \( \mathcal{P} \) onto one-dimensional subspaces spanned by unit vectors \( \phi \in \mathcal{D} \).

2.1.1. Qr-number probabilities. The spectral families of self-adjoint operators are used to define quantum probability measures on \( \mathbb{R} \) in [3]. If \( \hat{P}^A(S) \) is the spectral projection operator of \( \hat{A} \) on the Borel subset \( S \) of \( \mathbb{R} \), then in the standard interpretation \( \mu^\hat{A}_\rho(S) = \text{Tr} \rho \hat{P}^A(S) \) is the probability that when the system is in the state \( \rho \) a measurement of \( \hat{A} \) gives a result in the set \( S \).

If the system has the condition \( U \), the qr-number probability that \( a(U) \) lies in \( S \) is \( \pi^\hat{A}(S)|_U \), the qr-number value of \( \hat{P}^A(S) \) at \( U \).

\(^3\)It is not necessary that all attributes are represented in an \( \mathcal{O}^\ast \)-algebras, in the Stern-Gerlach experiment spin is represented by bounded operators on \( \mathbb{C}^2 \).
If \( U = \nu(\rho_s; \delta) \) for \( \delta \ll 1 \) then, for all Borel sets \( S \), \( \pi^A(S)|_U \approx Tr\rho_s\hat{P}^A(S) = \mu^A_{\rho_s}(S) \), the standard quantum mechanical probability when the system is in the state \( \rho_s \), so that \( |\pi^A(S)|_U - \mu^A_{\rho_s}(S)| < \delta \).

2.2. Measurement in the qr-number model. Measurements are a special class of interactions between two physical systems. The system \( S \) has an attribute, called the measurand, whose value is to be determined. The interaction couples the measurand to a pointer of the measurement apparatus \( M \) whose numerical value can be read. As a result of the interaction the numerical value of \( M \)'s pointer is changed by an amount that depends on the value of the measurand which is deducible from the difference of the pointer values. Both \( S \) and \( M \) are assumed to be quantum systems.

2.2.1. No measurement is exact. The qr-number model accepts that no measurement is exact. In metrology, see [20], any physical measurement is said to have two components: (1) A numerical value (in a specified system of units) giving the best estimate possible of the quantity measured, and (2) a measure of precision associated with this estimated value. The measure of precision is a parameter that characterises the range of values within which the value of the measurand can lie at a specified level of confidence. The best estimate is quantified by a level of confidence parameter \( (1 - \epsilon) \) in the range \([0, 1] \).

The way these parameters are used in the qr-number model is exemplified in [15] by the processes of passing a system \( S \) through a filter. The \( \epsilon \) sharp collimation of the quantity, represented by \( \hat{Q}_S \), in an interval \( I \subset \mathbb{R} \) when the system \( S \) has the condition \( W_S \) gives a standard real number to approximate the qr-number \( q_S|_{W_S} \).

**Definition 1.** For an interval \( I \), of width \( |I| \), if \( W_S \) is the largest convex open set in \( \mathcal{E}_S(A_S) \) such that \( q_S|_{W_S} \subset I \) and \( (q_S^2|_{W_S} - \langle q_S|_{W_S} \rangle^2) \leq \frac{\epsilon}{4}|I|^2 \), then \( \hat{Q}_S \) is \( \epsilon \) sharp collimated in \( I \) on \( W_S \).

Let \( \sigma(\hat{Q}_S) \) be \( \hat{Q}_S \)'s spectrum. If \( W_S \) is the condition on which \( \hat{Q}_S \) is \( \epsilon \) sharp collimated on \( I \) and \( \exists \alpha_0 \in I \cap \sigma(\hat{Q}_S) \), then with precision \( |I|/2 \) and confidence \((1 - \epsilon)\), \( \alpha_0 \) is the measured value of \( \hat{Q}_S \).

2.2.2. Measurement Conditions. Conditions that support determining a value of \( \hat{Q}_S \) in an interval \( I \) are of the form \( \mathcal{N}(P_{\phi_\lambda}, \hat{Q}_S, \delta) = \{ \rho \in \mathcal{E}_S(A_S) : |Tr(\rho\hat{Q}_S - \hat{P}_{\phi_\lambda} \hat{Q}_S)| < \delta \} \) where \( \phi_\lambda \) is an (approximate) eigenstate for some \( \lambda \in \sigma(\hat{Q}_S) \cap I \). In [15] we prove the following.

**Theorem 1.** If \( \lambda \in \sigma(\hat{Q}_S) \), there exists an interval \( I_\lambda \) centred on \( \lambda \) in which \( \hat{Q}_S \) is \( \epsilon \) sharp collimation on \( \mathcal{N}(\hat{P}_{\phi_\lambda}, \hat{Q}_S, \delta) \) for some \( \delta > 0 \).

This results builds upon the assumption of standard quantum theory that the results of measurements are the eigenvalues of the operator which represents the quantity being measured.
A similar result holds for strictly $\epsilon$ sharp collimation, defined in [15] which is used to define qr-number probabilities and to show in [15] that every attribute of $S$ appears to have undergone a Lüders-von Neumann transformation when the collimation is strictly $\epsilon$ sharp.

In order to complete the determination of a measured value for $\hat{Q}_S$ the system $S$ must interact with a measurement system $M$. The interaction connects the $\epsilon$ sharply collimated qr-number value of $\hat{Q}_S$ with a constant qr-number value of the pointer of $M$ which is observable.

3. Preparing for a measurement

There are two ways in which we can describe the preparation of the system $S$ in the generic example of §1. In this experiment there is only one attribute $\hat{Q}_S$ to be measured, it has only two eigenvalues $\{\lambda_\pm\}_{s=\pm}$ whose corresponding eigenvectors $\{\phi_{\lambda_\pm}\}_{s=\pm}$ span a two dimensional subspace $\mathcal{M}_2 \subset \mathcal{H}_S$.

We can use the qr-number model to describe attempts to prepare a state $\hat{P}_{\psi_S}(\bar{\alpha})$ where $\psi_S(\bar{\alpha}) = \alpha_+ \phi_{\lambda_+} + \alpha_- \phi_{\lambda_-}$ is a superposition of $\hat{Q}_S$’s eigenstates $\phi_{\lambda_\pm}$. Let $\Gamma = \{\bar{\alpha} = (\alpha_+, \alpha_-) \in \mathbb{C}^2 : |\alpha_+|^2 + |\alpha_-|^2 = 1\}$. The vectors $\phi_{\lambda_+}$ and $\phi_{\lambda_-}$ are orthonormal and if $\bar{\alpha} \in \Gamma$ then $\psi_S(\bar{\alpha})$ is normalised.

The condition $W_S^{\text{pure}}(\bar{\alpha}) = \mathcal{N}(\hat{P}_{\psi_S}(\bar{\alpha}), \hat{Q}_S, \delta)$ is centred on the state $\hat{P}_{\psi_S}(\bar{\alpha})$. If $W_S^{\text{pure}} = \bigcup_{\bar{\alpha} \in \Gamma} W_S^{\text{pure}}(\bar{\alpha})$ is the prepared epistemic condition of $S$, then $W_S^{\text{pure}}(\bar{\alpha})$ for a particular pair $\bar{\alpha} \in \Gamma$ is an ontic condition.

Alternatively assume that the fraction $|\alpha_+|^2$ of an ensemble is prepared in an epistemic condition $W_S^+ = \mathcal{N}(\hat{P}_{\phi_{\lambda_+}}, \hat{Q}_S, \delta)$, centred on the eigenstate $\hat{P}_{\phi_{\lambda_+}}$, whilst the fraction $|\alpha_-|^2 = 1 - |\alpha_+|^2$ is prepared in the epistemic condition $W_S^- = \mathcal{N}(\hat{P}_{\phi_{\lambda_-}}, \hat{Q}_S, \delta)$, centred on $\hat{P}_{\phi_{\lambda_-}}$. The epistemic condition of this ensemble is $W_S^{\text{mix}}(\bar{\alpha}) = |\alpha_+|^2 W_S^+ + |\alpha_-|^2 W_S^-$. If $\rho_S^{\text{mix}}(\bar{\alpha}) = |\alpha_+|^2 \hat{P}_{\phi_{\lambda_+}} + |\alpha_-|^2 \hat{P}_{\phi_{\lambda_-}}$ then $W_S^{\text{mix}} = \mathcal{N}(\rho_S^{\text{mix}}(\bar{\alpha}), \hat{Q}_S, \delta)$ as $W_S^\pm = \mathcal{N}(\hat{P}_{\phi_{\lambda_{\pm}}}, \hat{Q}_S, \delta)$.

Lemma 2.

\begin{equation}
W_S^{\text{pure}}(\bar{\alpha}) = W_S^{\text{mix}}(\bar{\alpha}) = |\alpha_+|^2 W_S^+ + |\alpha_-|^2 W_S^-.
\end{equation}

so the two ways of preparing $S$ produce the same open subset of states.

$\mathcal{N}(\hat{P}_{\psi_S}(\bar{\alpha}), \hat{Q}_S, \delta) = \mathcal{N}(\rho_S^{\text{mix}}(\bar{\alpha}), \hat{Q}_S, \delta)$ because $\phi_{\lambda_+}$ and $\phi_{\lambda_-}$ are orthogonal eigenvectors of $\hat{Q}_S$, for all $\delta > 0$. Moreover $\mathcal{N}(\hat{P}_{\psi_S}(\bar{\alpha}), \hat{Q}_S, \delta)$ can be decomposed, see [15,2] as

\begin{equation}
\mathcal{N}(\rho_S^{\text{mix}}(\bar{\alpha}), \hat{Q}_S, \delta) = |\alpha_+|^2 \mathcal{N}(\hat{P}_{\phi_{\lambda_+}}, \hat{Q}_S, \delta) + |\alpha_-|^2 \mathcal{N}(\hat{P}_{\phi_{\lambda_-}}, \hat{Q}_S, \delta).
\end{equation}

The coefficients $\{|\alpha_r|^2\}_{r=\pm}$ are the frequency probabilities that when $S$ has the condition $W_S(\bar{\alpha}) = \mathcal{N}(\hat{P}_{\psi_S}(\bar{\alpha}), \hat{Q}_S, \delta)$ the attribute $\hat{Q}_S$ is
located in intervals \( \{ I_\pm \} \), centred on the eigenvalues \( \lambda_\pm \). The qr-number probability for location in \( I_\pm \) is \( |\alpha_\pm|^2 \pi Q S(I_\pm)|W_s^\pm \), \( \{ 2.1.1 \} \) because \( \pi Q S(I_\pm)|W_s(\bar{\alpha}) = |\alpha_\pm|^2 \pi Q S(I_\pm)|W_s^\pm \) as \( \pi Q S(I_\pm)|W_s^- = 0 \).

Therefore, in the fraction \( |\alpha_+|^2 \) of preparation procedures, \( S \) is prepared in an ontic condition \( V_S(\bar{\alpha}) \in \mathcal{O}(W_s^+) \) and, in the fraction \( |\alpha_-|^2 \) of procedures, \( S \) is prepared in an ontic condition \( V_S(\bar{\alpha}) \in \mathcal{O}(W_s^-) \). The goal of the experiment is to determine these relative frequencies.

If \( M \) is prepared in a epistemic condition \( W_M^0 = \mathcal{N}(\hat{P}_{\phi_M}, \hat{Q}_M, \delta) \) where \( \phi_M \) is an eigenstate of the operator \( \hat{Q}_M \) for eigenvalue 0 and \( S \) was prepared \( W_S(\bar{\alpha}) = \mathcal{N}(\hat{P}_{\psi_S(\bar{\alpha})}, \hat{Q}_S, \delta) \). Then the combined system has \( W_{S,M}(\bar{\alpha}, 0) = W_S(\bar{\alpha}) \otimes W_M^0 = \mathcal{N}(\hat{P}_{\psi_S(\bar{\alpha})} \otimes \hat{P}_{\phi_M}, \hat{Q}_S \otimes \hat{Q}_M, 2\delta) \) whose central state \( \hat{P}_{\psi_S,M} \) projects onto the product wave-function of the standard model,

\[ \Psi_{S,M}(t_1) = \psi_S(\bar{\alpha}) \otimes \phi_M = (\alpha_+ \phi_{\lambda_+} + \alpha_- \phi_{\lambda_-}) \otimes \phi_M. \]

However using Lemma 2 the product condition can also be expressed

\[ W_{S,M}(\bar{\alpha}, 0) = \sum_{r=\pm} |\alpha_r|^2 \mathcal{N}(\hat{P}_{\phi_r} \otimes \hat{P}_{\phi_M}, \hat{Q}_S \otimes \hat{Q}_M, 2\delta) \]

If \( S \) is prepared in an ontic condition \( W_S(\bar{\alpha}_+) \subset W_S^+ \) and \( M \) in an ontic condition \( V_M \subset W_M^0 \) then the qr-number values of \( \hat{Q}_S \) and \( \hat{Q}_M \) are

\[ q_S|W_S(\bar{\alpha}_+) \approx \lambda_+ \text{ and } q_S|W_S(\bar{\alpha}_-) \approx \lambda_- \text{ while } q_M|V_M \approx 0. \]

These conditions are such, see theorem \( 2 \) that if \( \hat{Q}_S \) and \( \hat{Q}_M \) were measured at this stage of the experiment, \( \hat{Q}_S \) would register a value \( \lambda_+ \) or \( \lambda_- \) and \( \hat{Q}_M \) would be 0.

Now the prepared systems \( S \) and \( M \) are brought together to interact.

3.1. The coupling interaction. The purpose of this interaction is to couple the qr-number value of the measurand \( \hat{Q}_S \) to that of the pointer \( \hat{Q}_M \) of the measurement apparatus so that a quantitative value can be more easily observed.

The appropriate interactions include the von Neumann impulsive interactions \( 7 \), Zurek’s controlled shifts \( 22 \), as well as Bohm’s approximation for the interaction between a magnetic field and the spin of a particle in the Stern-Gerlach experiment \( 6, 22.6 \), and the electric dipole interaction Hamiltonian used in Haroche’s Schrödinger cat experiment \( 4 \). Each interaction Hamiltonian operator has a similar structure, it features the product of \( S \)’s attribute, which is to be measured, with an attribute of \( M \). For example, if \( S \)’s attribute is a position operator \( \hat{Q}_S \) then \( M \)’s attribute will be a momentum operator \( \hat{P}_M \) which is conjugate to the position operator \( \hat{Q}_M \) for \( M \).

\[ H_I = \gamma \hat{Q}_S \otimes \hat{P}_M \]
As the interaction is assumed to be impulsive and the Hamiltonian has only this interaction term, the equations of motion are linear.

The choice of the attributes depends on the physics, for example in the coupling of fields to charges for the Schrödinger cat experiment, a free electron of charge $q$, mass $m$, position $\vec{X}_e$ and momentum $\vec{P}_e$, is coupled to the field which is described in the Schrödinger picture by the vector potential $\vec{A}(\vec{x})$. If the field is thought of as a quantum system whose spatial locations are labelled by the three components $\{\hat{Q}_j = A_j\}_{j=1}^3$ of its vector potential and its momenta by the three components $\{\hat{P}_j = E_j\}_{j=1}^3$ of its electric field (because $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$). A charge-field interaction term

\begin{equation}
\hat{H}_{int} = -\frac{q}{m} \hat{P}_e \cdot \hat{A}(\vec{x}) = -\frac{q}{m} \hat{P}_e \cdot \hat{Q}^f
\end{equation}

is obtained by neglecting the small magnetic interaction with the electron spin for the Hamiltonian in the Coulomb gauge and neglecting a $\vec{A}^2$ term. This interaction is in the form of equation (11).

3.2. The output. Consider the prototypical von Neumann interaction in which the two systems are assumed to be massive one-dimensional quantum particles and the measurand is $\hat{Q}_S$, [7] pp 443. The qr-number value of the interaction Hamiltonian is, during the period $t_1 < t < t_2$,

\begin{equation}
|h|_W = \gamma q_S |w p_M|_W
\end{equation}

where $W = W_{S,M}(\vec{\alpha}, 0)$ is the prepared product condition. The coupling constant $\gamma$, of dimension $T^{-1}$, is large enough that the kinetic energy can be neglected during the interaction.

The qr-number equations of motion for the position and momentum of $S$ are, see §5,

\begin{equation}
m_S \frac{dq_S|_W}{dt} = \frac{\partial h|_W}{\partial p_S|_W} \quad \text{and} \quad \frac{dp_S|_W}{dt} = -\frac{\partial h|_W}{\partial q_S|_W}
\end{equation}

while those for the position and momentum of $M$ are

\begin{equation}
m_M \frac{dq_M|_W}{dt} = \frac{\partial h|_W}{\partial p_M|_W} \quad \text{and} \quad \frac{dp_M|_W}{dt} = -\frac{\partial h|_W}{\partial q_M|_W}
\end{equation}

If the interaction acts over an infinitesimal period $\tau = (t_2 - t_1)$, the qr-number values of $Q_S$ and $Q_M$ at time $t_2$ will be,

\begin{equation}
q_S|_W(t_2) = q_S|_W(t_1), \quad q_M|_W(t_2) = q_M|_W(t_1) + \kappa_M q_S|_W(t_1),
\end{equation}

where for $K = S, M$, $\kappa_K = \frac{\gamma \tau}{m_K}$ and $m_K$ is the mass.

If $V = V_S \otimes V_M$ with $V_S \in \mathcal{O}(W_S^\pm)$ and $V_M \in \mathcal{O}(W_M^0)$ by equation (11), $q_S|_V(t_1) \approx \lambda_+$ so that $q_M|_V(t_2) \approx 0 + \kappa_M \lambda_+$ and when $V_S \in \mathcal{O}(W_S^0)$ and $V_M \in \mathcal{O}(W_M^0)$ then $q_M|_V(t_2) \approx 0 + \kappa_M \lambda_-$. The standard numbers $\lambda_+$ or $\lambda_-$ are the measured values because the conditions $W_S^\pm$

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2$\vec{\nabla} \cdot \vec{A} = 0$ and the scalar potential is negligibly small.
support \( \varepsilon \)-sharp collimation in intervals centred on \( \lambda_+ \) or \( \lambda_- \) and \( W^0_M \) supports \( \varepsilon \)-sharp collimation in an interval centred on 0.

Thus the difference between the measurement pointer readings is proportional to \(|\lambda_+ - \lambda_-|\) which is observable when the eigenvalues are sufficiently separated. This resolves the problem of definite outcomes in the qr-number approach.

**3.3. How the conditions changed.** It is interesting to see how the conditions changed during a measurement, the analysis is closer to that of standard quantum theory.

Before interacting, at time \( t_1 \), the joint condition was \( W_{S,M}(t_1) = \mathcal{N}(\hat{P}_{\psi_{S,M}}(t_1), \hat{Q}_S \otimes \hat{Q}_M, 2\delta) \) where \( \psi_{S,M}(t_1) = \psi_S(\vec{\alpha}) \otimes \phi_M = (\alpha_+\phi_{\lambda_+} + \alpha_-\phi_{\lambda_-}) \otimes \phi_M \) and \(|\alpha_+|^2 + |\alpha_-|^2 = 1\). The initial qr-numbers values of \( S \)'s attribute \( \hat{Q}_S \) is \( \langle q_S|W_{S,M}(t_1) = q_S|W_S(t_1) \rangle \) as its initial qr-number value.

Using the evolution of the conditions, discussed in §5.2. The solution of the \( \psi \)-number equations of motion are given in equation (16) of §3.2, the first expression equates the qr-number value of \( \hat{Q}_M \) at the end of the interaction to its value at its commencement. Initially \( S \)'s reduced condition \( W_S(t_1) = \mathcal{N}(\hat{P}_{\psi_S(\vec{\alpha})}, \hat{Q}_S, \delta) \) is centred on the state \( \hat{P}_{\psi_S(\vec{\alpha})} \), at the end \( S \)'s reduced condition \( W_S(t_2) = \mathcal{N}(\rho_{S,M}^{mix}, \hat{Q}_S, \delta) \) is centred on the mixed state \( \rho_S^{mix} = |\alpha_+|^2 \hat{P}_{\psi^+_S} + |\alpha_-|^2 \hat{P}_{\psi^-_S} \). As was shown in §3

\begin{equation}
\mathcal{N}(\hat{P}_{\psi_S(\vec{\alpha})}, \hat{Q}_S, \delta) = \mathcal{N}(\rho_S^{mix}9\vec{\alpha}), \hat{Q}_S, \delta).
\end{equation}

\begin{align}
N(\hat{P}_{\psi_{S,M}}(t_1), \hat{Q}_S \otimes \hat{Q}_M, 2\delta) &= N(\rho_{S,M}^{mix}, \hat{Q}_S \otimes \hat{Q}_M, 2\delta). \\
\rho_{S,M}^{mix} &= |\alpha_+|^2 \hat{P}_{\phi_{\lambda_+}} \otimes \hat{\phi}_{\lambda_+} + |\alpha_-|^2 \hat{P}_{\phi_{\lambda_-}} \otimes \hat{\phi}_{\lambda_-}.
\end{align}
so that \( W_S(t_2) = W_S(t_1) \).

The change in \( M \)'s pointer expressed in the second expression of equation (16) has been discussed in §3.2. If \( W_S(t_2) \) and \( W_M(t_2) \) are conditions reduced from \( W_{S,M}(t_2) \) then

\[
q_M|W_M(t_2) - q_M|W_M(t_1) = \kappa_M q_S|W_S(t_1)
\]

shows how definite outcomes are obtained, as \( \hat{Q}_S \) is respectively \( \epsilon \)-sharp collimated in \( I_S^\pm \) centred on \( \lambda_\pm \) on the conditions \( W_S(t_1) \subset W_S^\pm \) while \( \hat{Q}_M \) is \( \epsilon \)-sharp collimated in \( I_M^\pm \) centred on \( \kappa_M \lambda_\pm \) on \( W_M(t_2) \subset W_M^\pm \) and is \( \epsilon \)-sharp collimated in \( I_M^S \) centred on 0 on \( W_M(t_1) \subset W_M^0 \).

If we wish to measure the momentum of a system the prototype would use a von Neumann impulsive interaction whose labels were interchanged as in equation (15) then

\[
h_I|W = \gamma p_S|W q_M|W,
\]

with \( \gamma \) the coupling constant, \( \hat{Q}_M \) is \( \mathcal{M} \)'s position operator and \( \hat{P}_S \) is \( S \)'s momentum operator whose value is to be measured. A similar set of outcomes when \( \hat{P}_S \) is \( \epsilon \)-sharp collimated follows the obvious changes.

3.4. Amplification of the output. Consider a chain of couplings between a sequence of outputs and measurement systems each of which augments the magnitude of the next output. The component of the measurement apparatus that initially interacts with the system \( S \) will be denoted \( M_0 \). The output \( q_{M_0}|W_{M_0}(t_2) \) is the input for a second von Neumann interaction between the attributes \( \hat{Q}_{M_0} \) and \( \hat{P}_{M_0} \) of the first component and attributes \( \hat{Q}_{M_1} \) and \( \hat{P}_{M_1} \) of the next.

For the \( k^{th} \) link in this chain of events, the input is denoted \( q_{M_k-1}|W_{M_k-1} \) and the output is \( q_{M_k}|W_{M_k} \). Here \( W_{M_k-1} \subset W_{M_k-1}^\pm \) so that the interaction at the \( k^{th} \) stage is

\[
h|W_{M_{k-1}} = \gamma q_{M_{k-1}}|W_{M_{k-1}} p_{M_k}|W_{M_{k-1}}
\]

The interaction is assumed to be impulsive and only acting between \( t_{k-1} \) and \( t_k \), then at \( t = t_k \), the qr-number value of \( \hat{Q}_{M_k} \) is

\[
q_{M_k}|W_{M_k}(t_k) - q_{M_k}|W_{M_k}(t_{k-1}) = \kappa_{M_k} q_{M_{k-1}}|W_{M_{k-1}}(t_{k-1})
\]

When the pointer is linked via impulsive interactions to the parts \( \{M_l\}_{l=0}^k \) and \( M_0 = M \), then the location after the \( k^{th} \) interaction is changed by

\[
(\prod_{l=0}^{k} \kappa_{M_l})|q_S|W_S(t_1)|
\]

Thus the output is amplified if each \( \kappa_{M_l} > 1 \).
4. Appendices

4.1. Mathematics of qr-numbers. The qr-number value of a physical quantity depends not only on the operator \( \hat{A} \) that represents it but also on the condition of the system. They differ from standard real numbers that are represented in the qr-number model by globally constant qr-numbers. For a summary of the mathematical structure of the qr-number model, see Corbett [19].

When a system \( \mathcal{S} \) has a Hilbert space \( \mathcal{H}_\mathcal{S} \) that carries a unitary representation \( U \) of a symmetry group \( G \) then its physical attributes are represented by operators that form an \( O^* \)-algebra \( \mathcal{A}_\mathcal{S} \): the representation \( dU \) of the enveloping algebra \( E(\mathcal{G}) \) of the Lie algebra \( \mathcal{G} \) of \( G \) see [9]. The operators have a common domain \( D = D^\infty(U) \), the set of \( C^\infty \)-vectors for the representation \( U \).

Definition 3. The states on \( \mathcal{A}_\mathcal{S} \) are the strongly positive linear functionals on \( \mathcal{A}_\mathcal{S} \) that are normalised to take the value 1 on the unit element \( 1 \) of \( \mathcal{A}_\mathcal{S} \), they form the state space \( \mathcal{E}_\mathcal{S}(\mathcal{A}_\mathcal{S}) \).

\( \mathcal{E}_\mathcal{S}(\mathcal{A}) \) has the weak topology generated by the functions \( a(\cdot) \) where, given \( \hat{A} \in \mathcal{A}_\mathcal{S}, a(\rho) = Tr\hat{A}\rho, \forall \rho \in \mathcal{E}_\mathcal{S}(\mathcal{A}_\mathcal{S}) \). This topology is the weakest that makes all the functions \( a(\cdot) \) continuous. For \( \hat{A} \in \mathcal{A}_\mathcal{S}, \varepsilon > 0 \) and \( \rho_0 \in \mathcal{E}_\mathcal{S}(\mathcal{A}_\mathcal{S}) \), the sets \( \mathcal{N}(\rho_0; \hat{A}; \varepsilon) = \{ \rho : |Tr\hat{A}\rho - Tr\rho_0\hat{A}| < \varepsilon \} \) form an open sub-base for the weak topology on \( \mathcal{E}_\mathcal{S}(\mathcal{A}) \). The basic open subsets are denoted \( \nu(\rho_1; \delta) = \{ \rho : Tr|\rho - \rho_1| < \delta \} \). \( \mathcal{E}_\mathcal{S}(\mathcal{A}_\mathcal{S}) \) is compact in the weak topology [18].

Definition 4. A trace functional on \( \mathcal{A} \) is a functional of the form \( \hat{A} \in \mathcal{A} \mapsto Tr(\hat{B}\hat{A}) \) for some trace class operator \( \hat{B} \).

Theorem 2. [9] Every strongly positive linear functional on \( \mathcal{A} \) is given by a trace functional.

4.1.1. Locally linear qr-numbers. are denoted \( \hat{A}(\mathcal{E}_\mathcal{S}(\mathcal{A})) \).

Definition 5. Let \( U \in \mathcal{O}(\mathcal{E}_\mathcal{S}(\mathcal{A})) \), a function \( f : U \to \mathbb{R} \) is locally linear if each \( \rho \in U \) has an open neighborhood \( U_\rho \subset U \) with an essentially self-adjoint operator \( \hat{A} \in \mathcal{A} \) such that \( f|_{U_\sigma} = a(U_\sigma) \) for every \( \sigma \in U \).

Density: Given any qr-number \( f \) on \( U \in \mathcal{O}(\mathcal{E}_\mathcal{S}(\mathcal{A})) \) and any integer \( n \) there exists an open cover \( \{ U_j \} \) of \( U \) with for each \( j \) a locally linear function \( g_j : U_j \to \mathbb{R} \) such that \( |f|_{U_j} - g_j(U_j)| < \kappa/n \), where \( \kappa < \infty \) has the same physical dimensions as \( f \) and \( g \). This means that every qr-number is a union of locally linear qr-numbers, \( f(U) = \bigcup g_j(U_j) \).

4.1.2. Infinitesimal qr-numbers. The relationship of the qr-number equations of motion with the standard quantum mechanical equations is obtained using infinitesimal qr-numbers. In the following \( \mathcal{A} \) is assumed to be the representation of the enveloping Lie algebra \( dU(E(\mathcal{G})) \) obtained from the unitary representation of the Lie group \( G \).
Infinitesimal qr-numbers are the difference between neighbouring qr-numbers. Two qr-numbers $x$ and $y$ are neighbours if they are not identical but they do not satisfy $x > y \lor x < y$ on any non-empty open subset of $\mathcal{E}_S(A)$. The difference $(x - y)$ between neighbouring numbers is an order theoretical infinitesimal number because there is no open set on which $(x - y) > 0 \lor (x - y) < 0$ is true. Since qr-real numbers do not satisfy trichotomy the difference between neighbouring real numbers is not zero.

For example: if $V_0 = \nu(\rho_0; \delta)$, for $\rho_0 \in \mathcal{E}_S(A)$ and $\delta > 0$, consider a depleted open set $V_0 = V_0 \setminus \{\rho_0\}$. Then $q_j(V_0)$ and $q_j(\tilde{V}_0)$ are neighbouring qr-numbers because

$$q_j(V_0) \neq q_j(\tilde{V}_0)$$

(28) and neither $q_j(V_0) > q_j(\tilde{V}_0)$ nor $q_j(\tilde{V}_0) > q_j(V_0)$ on any open subset of $\mathcal{E}_S(A)$.

In fact, $q_j(V_0) - q_j(\tilde{V}_0) = q_j^0(\rho_0) = Tr\rho_0\hat{Q}_j$. Since the singleton set $\{\rho_0\}$ has empty interior, there is no non-empty open set $W$ on which the difference is non-zero. Thus the expectation values of quantum mechanical operators are order theoretic infinitesimal qr-numbers. They are also algebraic infinitesimal qr-numbers because there is no non-empty open set on which the square is non-zero, for $(q_j(V_0) - q_j(\tilde{V}_0))^2 = q_j(V_0)^2 - q_j(\tilde{V}_0)^2 = (Tr\rho_0\hat{Q}_j)^2$, which is only non-zero at $\rho_0$.

The expectation values $Tr\rho\hat{A}$ are infinitesimal linear qr-numbers for any state $\rho \in \mathcal{E}_S(A)$ and any self adjoint operator $\hat{A}$ in the algebra $A$. They are part of the infinitesimal structure of the qr-number world.

### 4.2. Preparation processes.

During a preparation process a number of quantities are treated successively. One of $\mathcal{S}$’s attributes, represented by the self-adjoint operator $\hat{A}$, is strictly $\epsilon$-sharp collimated in the interval $I = [a_1, a_2]$ when $\mathcal{S}$ has the condition $U$ and immediately afterwards a second attribute, represented by a self-adjoint operator $\hat{B}$, compatible with $\hat{A}$ (that is they strongly commute), is strictly $\epsilon$-sharp collimated in the interval $J = [b_1, b_2]$ when $\mathcal{S}$ has the condition $W$. The qr-number values of $\hat{A}$ and $\hat{B}$ will persist with a probability greater than $(1 - \epsilon)$.[13] §III A. The temporal order in which the qr-number values were prepared does not affect their values. The system ends up in a a condition $U \cap W$. This extends to finite sets $\{A_j\}_{j=1}^n$ of commuting operators in the obvious way. If the attributes, represented by the operators $\{\hat{A}_j\}_{j=1}^n$, are each $\epsilon$ sharp collimated in their respective intervals $\{I_j\}_{j=1}^n$ on conditions $\{W_j\}_{j=1}^n$ then if $\{\alpha_j\}_{j=1}^n$ are the midpoints of the intervals, we can, with precision $|I_j|/2$ and confidence $(1 - \epsilon)$, take $\alpha_j$ to be the classical value of the quantity represented by $\hat{A}_j$ when the system has the condition $\cap_{j=1}^n W_j$. This an epistemic condition, any open subset of $\cap_{j=1}^n W_j$ may be the ontic condition of an individual system in the ensemble.
It can be extended to attributes represented by operators that don’t commute. Heisenberg’s uncertainty relations limit the precision of the simultaneous measurements of the attributes but do not prohibit their measurement, \[15\] §C, Theorem 2. For example, a particle’s position \(\hat{Q}\) and momentum \(\hat{P}\) satisfy \(i[\hat{P}, \hat{Q}] = \hbar\), so that if the particle with the condition \(W\) has both \(\hat{Q}\) and \(\hat{P}\) \(\epsilon\)-sharp collimated in intervals \(I_q\) and \(I_p\) with precisions \(\kappa_q\) and \(\kappa_p\) then \(\kappa_q\kappa_p \geq \frac{\hbar}{2\epsilon}\) and the product of the intervals’ widths satisfy \(|I_q||I_p| \geq \frac{\hbar}{\epsilon}\). The precisions of the measured values are thus restricted by the inequality \(\kappa_q\kappa_p \geq \frac{\hbar}{2\epsilon}\).

4.2.1. *More on \(\epsilon\) sharp collimation.* Recall the definition of \(\epsilon\) sharp collimation,

**Definition 6.** For an interval \(I\), of width \(|I|\), if \(W_S\) is the largest convex open set in \(E_S(A_S)\) such that \(q_S|_W \subset I\) and \((q_S^2|_W - (q_S|_W)^2) \leq \frac{\epsilon^2}{4}|I|^2\) then \(Q_S\) is \(\epsilon\) sharp collimated in \(I\) on \(W_S\).

On the other hand the qr-number value of an attribute, \(\hat{A}\), can be weakly or strongly contained in an interval. Let \(S\) have the condition \(W\), then \(\hat{A}\) lies weakly in an interval \(I_a \subset \mathbb{R}\) if the range of \(a|_W \subseteq I_a\).

Using the qr-number value \(\pi^A(I_a)|_W\) of \(\hat{A}\)’s spectral projection operator \(\hat{P}^A(I_a)\) for \(I_a\), we say that \(a|_W\) lies strongly in \(I_a\) when it lies weakly in \(I_a\) and \((1 - \epsilon) < \pi^A(I_a)|_W \leq \frac{\epsilon}{4}\). \(\hat{A}\) is then said to be \(\epsilon\) sharp located in the interval \(I_a\) on the condition \(W\).

The following result was proven in \[15\].

**Theorem 3.** If \(\hat{A}\) is \(\epsilon\) sharp collimated in \(I_a\) on \(W\), then \(\hat{A}\) is \(\epsilon\) sharp located in \(I_a\) on \(W\).

Strictly \(\epsilon\) sharp collimation is a stronger version of \(\epsilon\) sharp collimation that also uses the spectral projection operator, \(\hat{P}^A(I)\), for \(\hat{A}\) on \(I\). It requires that \(W\) is such that the qr-number \(a|_W\) closely approximates the qr-number value \(pap|_W\) of \(\hat{P}^A(I)\hat{A}\hat{P}^A(I)\).

**Definition 7.** \(\hat{A}\) is strictly \(\epsilon\) sharp collimated in \(I\) on \(W\) if it is \(\epsilon\) sharp collimated on \(W\) and for all \(\rho \in W\), \(\text{Tr} |\rho - \hat{P}^A(I)\rho\hat{P}^A(I)| < \epsilon\).

When the \(O^*\)-algebra \(A\) is the infinitesimal representation \(d\hat{U}\) of the enveloping algebra \(E(G)\) obtained from a unitary representation \(\hat{U}\) of a Lie group \(G\) this suffices because for all \(\rho \in W\)

\[29\]
\(|\text{Tr} \rho(\hat{A} - \hat{P}^A(I)\hat{A}\hat{P}^A(I))| \leq \kappa_m(\hat{A})\text{Tr} |\rho - \hat{P}^A(I)\rho\hat{P}^A(I)|\]

where \(\kappa_m(\hat{A}) = \sup_{\psi \in D^\infty(\hat{U})} \|\hat{A}\hat{U}((1 - \Delta)^m)^{-1}\psi\|/\|\psi\| < \infty\) with \(\Delta = \sum_{i=1}^d x_i^2\) is the Nelson Laplacian in \(E(G)\) with basis \(\{x_1, x_2, \ldots, x_d\}\)

3The qr-number \(\pi^A(I_a)|_W\) can be interpreted\[15\] as the qr-number probability of the system passing through the slit \(I_a\), then \(\epsilon\) sharp location in the interval \(I_a\) requires the qr-number probability to be greater than \((1 - \epsilon)\).
and integer \( m > 0 \). Thus if \( Tr|\rho - \hat{P}^A(I)\rho\hat{P}^A(I)| < \epsilon \) for all \( \rho \in W \) then \( |a|_W - (pap)|_W < \kappa_m(\hat{A})\epsilon \). In [22], §5.5, for \( j = 1, 2, 3 \), it is shown that if \( G \) is the Weyl-Heisenberg group, \( \kappa_1(\hat{Q}_j) = \kappa_1(\hat{P}_j) = \frac{1}{2} \).

The next theorems reveal that when \( \alpha_0 \) is in the spectrum of \( \hat{A} \) the condition for strictly \( \epsilon \) sharp collimation is a basic open set centred on the eigenstate for \( \hat{A} \) at \( \alpha_0 \), they are proven in [15].

**Theorem 4.** If \( \alpha_0 \in \sigma(\hat{A}) \cap I \) and \( \rho_0 = |\psi_0\rangle\langle\psi_0| \) is an eigenstate of \( \hat{A} \) at \( \alpha_0 \) with \( \hat{P}^A(I_a)\rho_0\hat{P}^A(I_a) = \rho_0 \), then \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( \hat{A} \) is strictly \( \epsilon \) sharp collimated in \( I_a \) on \( \nu(\rho_0, \delta) \) and on \( \mathcal{N}(\rho_0, \hat{Q}_S, \frac{\epsilon}{2}) \).

There is an analogous result for the interval \( I_a \) with midpoint \( \alpha_0 \) is in the continuous spectrum of \( \hat{A} \).

**Theorem 5.** If \( \alpha_0 \in \sigma_c(\hat{A}) \), the continuous spectrum of \( \hat{A} \), and \( \rho_0 = |\psi_0\rangle\langle\psi_0| \) is an approximate eigenstate of \( \hat{A} \) at \( \alpha_0 \) at accuracy \( \delta_0 \) and \( \hat{P}^A(I_a)\rho_0\hat{P}^A(I_a) = \rho_0 \), then \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( \hat{A} \) is strictly \( \epsilon \) sharp collimated in \( I_a \) on \( \nu(\rho_0, \delta) \) and on \( \mathcal{N}(\rho_0, \hat{Q}_S, \frac{\epsilon}{2}) \).

5. **QR-Number Equations of Motion for Massive Particles.**

The motion of microscopic particles is governed by equations which have the same form as those for macroscopic particles with qr-numbers replacing standard real numbers, [14].

The laws of motion for a particle of mass \( m > 0 \) are Hamiltonian equations of motion expressed in qr-numbers; \( m\frac{dq_j|U\rangle}{dt} = \frac{\partial h}{\partial p_j|U\rangle} \) and \( \frac{dp_j|U\rangle}{dt} = -\frac{\partial h}{\partial q_j|U\rangle} \), where \( q_j|U\rangle, p_j|U\rangle \) and \( h|U\rangle \) are qr-number values of the \( j \)th components of its position, momentum and of the Hamiltonian at the condition \( U \). Thus, if \( h(q_j|U\rangle(t), p_j|U\rangle(t)) = \sum_{j=1}^{3} \frac{1}{2m} (p_j|U\rangle(t))^2 + V(q_j|U\rangle(t)) \) is the qr-number value of the Hamiltonian

\[
\frac{dq_j|U\rangle(t)}{dt} = \frac{\partial h(q_j|U\rangle(t), p_j|U\rangle(t))}{\partial p_j|U\rangle(t)} = \frac{1}{m} p_j|U\rangle,
\]

\[
\frac{dp_j|U\rangle(t)}{dt} = -\frac{\partial h(q_j|U\rangle(t), p_j|U\rangle(t))}{\partial q_j|U\rangle(t)} = f_j(q_j|U\rangle).
\]

The force has components \( f_j(q_j|U\rangle) = -\frac{\partial V(q_j|U\rangle(t))}{\partial q_j|U\rangle(t)} \).

When \( \hat{A} \in A \) and the time derivative of its qr-number \( a(q_j|U\rangle, p_j|U\rangle) \) is taken along a trajectory of the particle, then

\[
\frac{da}{dt} = [a, h] \equiv \sum (\frac{\partial a}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial h}{\partial q_i}).
\]

If the time \( t \) occurs explicitly in \( a \), \( \frac{da}{dt} \) must be added to \([a, h]\). The bracket \([a, h]\) is the Poisson bracket of the functions \( a(q_j|U\rangle, p_j|U\rangle) \) and \( h(q_j|U\rangle, p_j|U\rangle) \). The qr-number equation \( \frac{da}{dt} = [a, h] \) is the basic dynamical equation for the evolution of the qr-number values of attributes.
5.1. Infinitesimal qr-number equations of motion. In [19], using approximate eigenvectors for numbers in the continuous spectra of the commuting operators \( \{\hat{Q}_j\}_{j=1}^3 \) when the force operators \( \hat{F}_j = f_j(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3) \), for \( j = 1, 2, 3 \), belong to the algebra \( \mathcal{A} \), the standard quantum mechanical equations of motion for a massive particle are obtained from linear infinitesimal qr-number approximations to the qr-number Hamiltonian equations of motion, equations (15) and (16).

When the operators \( \{\hat{Q}_j\}_{j=1}^3 \) have only continuous spectra, for all \( \rho \in \mathcal{E}_S(\mathcal{A}) \) and any \( \epsilon > 0 \),

\[
|\text{Tr}\rho\hat{F}_l - f_l(\text{Tr}\rho\hat{Q}_1, \text{Tr}\rho\hat{Q}_2, \text{Tr}\rho\hat{Q}_3)| < \epsilon, \text{ for } l = 1, 2, 3.
\]

Therefore for all states \( \rho \in \mathcal{E}_S(\mathcal{A}) \), the linear qr-number approximations to the qr-number equations of motion yield the infinitesimal qr-number equations,

\[
\frac{d}{dt}\text{Tr}\rho\hat{Q}_j = \frac{1}{m}\text{Tr}\rho\hat{P}_j \quad \text{and} \quad \frac{d}{dt}\text{Tr}\rho\hat{P}_j = \text{Tr}\rho\hat{F}_j, \quad \text{for } j = 1, 2, 3,
\]

from which Heisenberg’s operator equations follow on the assumption that all the time dependence is carried by the operators. If all the time dependence were carried by the states and we assume that \( \text{Tr}\rho\hat{A} = \text{Tr}\rho\hat{A}_t \) holds for all operators \( \hat{A} \in \mathcal{A}_S \) then it is possible that the time dependence of the states is unitary, \( \hat{A}_t = \hat{U}_t\hat{A}\hat{U}_t^{-1} \). A unitary evolution of the conditions is compatible with the infinitesimal qr-number equations.

In the following the conditions can be ontic or epistemic.

5.2. The evolution of the conditions. The unitary evolution of the states is compatible with the infinitesimal qr-number equations, see [5.1] so that a condition evolves following the unitary evolution of its component states, that is, if \( \rho \rightarrow \rho_i = \hat{U}_i\rho\hat{U}_i^* \) for all \( \rho \in W \) then \( W \rightarrow W_i = \hat{U}_iW\hat{U}_i^* \). Since the open sets \( \{\nu(\rho, \delta)\} \) are basic in the topology on \( \mathcal{E}_S(\mathcal{A}) \), it suffices to show that for any \( \delta > 0 \), \( \nu(\rho, \delta) \rightarrow \nu(\rho_i, \delta) \).

Lemma 8. If \( \rho, \rho' \in \mathcal{E}_S(\mathcal{A}) \) then \( \text{Tr}|\rho - \rho'| = \text{Tr}|\rho - \rho'| \) when \( \forall \rho \in \mathcal{E}_S(\mathcal{A}) \), \( \rho_i = \hat{U}_i\rho\hat{U}_i^* \) for a unitary group \( \{\hat{U}_t; t \in \mathbb{R}\} \), thus if \( \rho_0 \rightarrow \rho_i \) then \( \nu(\rho_0, \delta) \rightarrow \nu(\rho_i, \delta) \) for any \( \rho_0 \) and any \( \delta > 0 \).

The proof uses \( |\hat{U}_i(\rho - \rho')\hat{U}_i^*| = \hat{U}_i|\rho - \rho'|\hat{U}_i^* \) and that the trace is independent of the orthonormal basis used in its evaluation.

6. Conditions for two systems

The combined conditions are product conditions when \( S \) and \( M \) are not interacting. Each system has its own attributes, represented by \( \mathcal{O}^* \)-algebras \( \mathcal{A}_S \) and \( \mathcal{A}_M \), defined on dense subsets \( \mathcal{D}_S \) and \( \mathcal{D}_M \) of their Hilbert spaces \( \mathcal{H}_S \) and \( \mathcal{H}_M \) with smooth state spaces \( \mathcal{E}_S(\mathcal{A}_S) \) and \( \mathcal{E}_S(\mathcal{A}_M) \). The attributes have independent qr-number values.
Definition 9. A condition $W = W_{S,M}$ is a product condition with respect to the decomposition into systems $S$ and $M$ if for every pair of physical attributes, $\hat{A}_S \otimes \hat{I}_M$ of $S$ and $\hat{I}_S \otimes \hat{B}_M$ of $M$, the qr-number value of $\hat{A}_S \otimes \hat{B}_M$ is a product

$$\text{(35)} \quad (a_S \otimes b_M)|_{W_{S,M}} = a_S|_{\tilde{W}_S} b_M|_{\tilde{W}_M}$$

where $\tilde{W}_S$ and $\tilde{W}_M$ are the reduced conditions for $S$ and $M$ respectively.

Before they interact every state of the combined system is a product state so that every condition is a product condition.

If $S$ was prepared in a mixed condition $W_{S}^m = \sum_{j=1}^{N} p_j W_S^j$, with the condition $W_S^j$ occurring with probability $p_j$, while the condition $W_M$ was held fixed for $M$, the ensuing combined condition is still a product condition as

$$\text{(35)} \quad (a_S \otimes b_M)|_{W_{S,M}} = a_S|_{W_{S}^m} b_M|_{W_M}$$

for all $\hat{A}_S$ and $\hat{B}_M$.

On the other hand there are entangled conditions, produced when the systems are interacting.

Definition 10. $W = W_{S,M}$ is an entangled condition if it is not a product condition, i.e., if there is at least one pair of attributes, $\hat{A}_S \otimes \hat{I}_M$ of $S$ and $\hat{I}_S \otimes \hat{B}_M$ of $M$ such that the qr-number value of $\hat{A}_S \otimes \hat{B}_M$ is not a product

$$\text{(36)} \quad (a_S \otimes b_M)|_{W_{S,M}} \neq a_S|_{\tilde{W}_S} b_M|_{\tilde{W}_M}.$$

A product condition for the combined system before the interaction can evolve into an entangled condition during the interaction, in the same way as product states evolve into entangled states.

Since relations that hold between qr-numbers at a condition $W$ hold on all open subsets $V \subset W$, if an epistemic condition $W$ is entangled it has no open subset $V \subset W$ that is a product condition and if $W$ is a product condition then so also is every open subset $V \subset W$.

Finally, a separable mixed condition is prepared if, while preparing a mixed condition for $S$, whenever a $W_S^j$ is prepared for $S$ a companion condition $W_M^j$ is prepared for $M$. Then $W_{S,M}^{\text{sep}} = \sum_{j=1}^{N} p_j W_S^j W_M^j$ so that

$$\text{(37)} \quad (a_S \otimes b_M)|_{W_{S,M}^{\text{sep}}} = \sum_{j=1}^{N} p_j a_S|_{W_S^j} b_M|_{W_M^j}.$$

Such a combined condition is not a product condition nor is it an entangled condition, the outcomes are correlated which is explainable in terms of its preparation at the classical probabilities $p_j$.

6.1. Reduced conditions. For non-identical massive Galilean invariant particles, let $(\mathcal{H}(1,2), \mathcal{A}(1,2), \mathcal{E}_S(\mathcal{A}(1,2)))$ represent a two particle system’s Hilbert space, its algebra of physical attributes, and smooth state space with $\mathcal{H}(1,2) = \mathcal{H}(1) \otimes \mathcal{H}(2)$ and $\mathcal{A}_{1,2} = \mathcal{A}_1 \otimes \mathcal{A}_2$. 
If \( W(1, 2) \) is a two particle condition then, for \( j = 1, 2 \), the reduced single particle conditions \( \tilde{W}(j), \ j = 1, 2 \) are obtained by tracing over an orthonormal basis of the Hilbert space \( \mathcal{H}(k) \) for \( k = 1 \lor 2 \neq j \), a straight forward calculation in \( \mathbb{H} \) yielded

**Proposition 11.** If \( \rho_0(1, 2) = \hat{P}_{\phi L(1)} \otimes \hat{P}_{\phi L(2)} \) is a product state then \( \nu(\rho_0(1, 2); \delta) \) has reduced conditions \( \tilde{W}(1) = \nu(\hat{P}_{\phi L(1)}; \delta) \) and \( \tilde{W}(2) = \nu(\hat{P}_{\phi L(2)}; \delta) \).

For an entangled two particle wave-function \( \Psi(1, 2) = (\alpha \phi^+ (1) \otimes \beta \phi^- (1) \otimes \phi^-(2) + \beta \phi^- (1) \otimes \phi^- (2)) \) with orthogonal single particle wave functions, \( \{\phi_k^+, \phi_k^-\}; k = 1, 2, \) the entangled pure state is \( \hat{P}_{\Psi(1,2)} \) and its reduced states are mixed states, \( \rho(k) = (|\alpha|^2 \hat{P}_{\phi^+(k)} + |\beta|^2 \hat{P}_{\phi^-(k)}) \) for \( k = 1, 2 \).

**Proposition 12.** If \( \rho_0(1, 2) = \hat{P}_{\Psi(1,2)} \) is an entangled state then the condition \( \nu(\rho_0(1, 2); \delta) \) has reduced conditions \( \tilde{W}(k) = |\alpha|^2 \nu(\hat{P}_{\phi^+(k)}; \delta) + |\beta|^2 \nu(\hat{P}_{\phi^-(k)}; \delta) \) for \( k = 1, 2 \).

6.1.1. **When systems interact.** For a wide class of interactions in finite dimensional Hilbert spaces, Durt \([8]\), has shown that quantum states become entangled. There is a similar result for the conditions of two particle systems that holds on Hilbert spaces of arbitrary dimensions.

**Definition 13.** An interaction is separable if its potential function satisfies

\[
V(\vec{q}(1), \vec{p}(1), \vec{q}(2), \vec{p}(2)) = V_1(\vec{q}(1), \vec{p}(1)) + V_2(\vec{q}(2), \vec{p}(2)).
\]

A classical example is the small oscillations of a spherical pendulum, for which the potential energy is \( V(q(1), q(2)) = \frac{1}{2} (q(1)^2 + q(2)^2) \). It provides independent equations of motion for the variables \( q(1) \) and \( q(2) \). A non-separable interaction would produce coupled equations.

**Theorem 6.** For a two particle system the joint condition becomes entangled when the particles interact via a non-separable interaction.

**Proof.** Using Hamiltonian equations, see \( \S \) \([5]\), it is clear that if the particles were prepared in a product condition \( W_0 = \nu(\hat{P}_{\phi(1)} \otimes \hat{P}_{\phi(2)}; \delta) = \nu(\hat{P}_{\phi(1)}; \frac{\delta}{2}) \otimes \nu(\hat{P}_{\phi(2)}; \frac{\delta}{2}) \), with unit vectors \( \phi(j) \in \mathcal{H}(j), \ j = 1, 2 \) and \( 0 < \delta < \frac{1}{2} \), then under a separable potential \( W(t) \) stays a product condition.

When the particles interact via a non-separable potential, the equations of motion for the individual particles are coupled so that after the interaction has ceased \( (q_1 \otimes q_2)|_{W(t)} \neq q_1|_{W(t)} q_2|_{W(t)} \).

For a one dimensional example take an impulsive von Neumann interaction.
Lemma 14. Let \( H_T(q_1|w, p_1|w, q_2|w, p_2|w) = \gamma q_1|w p_2|w \), and \( \kappa = \gamma T \), where \( T \) is the duration of the impulse. Then

\[
q_2|w(T) = q_2|w(0) + \kappa q_1|w(0), \quad q_1|w(T) = q_1|w(0)
\]

so that

\[
(q_1 \otimes q_2)|w_0(T) \neq q_1|w_1(T))q_2|w_2(T)
\]

\( \tilde{W}_1(T), \tilde{W}_2(T) \) are reduced conditions for particles 1 and 2 at time \( T \).

Proof. Since Hamilton’s equations are linear, particles 1 and 2 keep their trajectories whether we use the qr-number equations or Heisenberg’s equations for the operators averaged over open sets of states.

From Heisenberg’s operator equations, \( \dot{Q}_1(T) \otimes \dot{I}_2 = \dot{Q}_1(0) \otimes \dot{I}_2 \) and \( \dot{I}_1 \otimes \dot{Q}_2(T) = \dot{I}_1 \otimes \dot{Q}_2(0) + \kappa \dot{Q}_1(0) \otimes \dot{I}_2 \). Therefore \( (\dot{Q}_1(0) \otimes \dot{I}_2)(\dot{I}_1 \otimes \dot{Q}_2(T)) = \dot{Q}_1(0) \otimes \dot{Q}_2(0) + \kappa \dot{Q}_1(0)^2 \otimes \dot{I}_2 \). Since \( W_0 \) is a product condition, every \( (1, 2) \in W_0 \) is a product state, so that \( Tr\rho(1, 2)\dot{Q}_1(T) \otimes \dot{Q}_2(T) = (Tr\rho(1, 2)\dot{Q}_1(0))(Tr\rho(1, 2)\dot{Q}_2(0)) + \kappa Tr\rho(1, 2)\dot{Q}_1(0)^2 \).

Thus \( (q_1 \otimes q_2)|w(T) = q_1|w_1(0)q_2|w_2(0) + \kappa q_1|w_1(0)q_2|w_2(0) \neq q_1|w_1(0)q_2|w_2(0) \). Therefore the joint condition condition \( W_T \) has become entangled. \( \square \)

In §III of Corbett and Home’s paper [1], the preparation of a two particle entangled state is described using an impulsive von Neumann interaction, \( H = \gamma Q_S \cdot P_M \), and time-dependent coordinate wave functions. Under disjointness assumptions on the supports of the functions \( \psi^+_S = \psi_+(q_S, t_1) \) and \( \psi^-_S = \psi_-(q_S, t_1) \) and assuming that \( \phi_M = \phi_0(q_M, t_1) \) is an approximate eigenfunction of position they obtain an entangled wave function \( \Phi_{S,M}(t_2) = (\alpha \psi^+_S \otimes \phi^+_M + \beta \psi^-_S \otimes \phi^-_M) \), with \( |\alpha|^2 + |\beta|^2 = 1 \) and both \( \psi^+_S \perp \psi^-_S \) and \( \phi^+_M \perp \phi^-_M \). Although the coordinate spaces of \( S \) and \( M \) were assumed to be one dimensional in [1], the argument extends to 3 dimensional coordinate spaces. For \( s = \pm \), the wave-functions \( \phi^s_M \) are given by convolutions, see [10] §0.C,

\[
\phi_s(q_M, t_2) = \int |\psi_s(q_S, t_1)|^2 \phi_0(q_M - \Gamma(t_2)q_S, t_1) dq_S,
\]

where \( \Gamma(t_2) = \gamma(t_2 - t_1) \).

The evolution of the wave function \( \Psi_{S,M}(t_1) = (\alpha \psi^+_S \otimes \phi^+_M + \beta \psi^-_S \otimes \phi^-_M) \) into an entangled wave function \( \Psi_{S,M}(t_2) = (\alpha \psi^+_S \otimes \phi^+_M + \beta \psi^-_S \otimes \phi^-_M) \) leads to the following evolution of the conditions.

Theorem 7. Under the unitary group \( \hat{U}(t) = \exp(i\hat{H}t/\hbar) \), for an impulsive interaction \( \hat{H} = \gamma \hat{Q}_S \cdot \hat{P}_M \), the condition \( \nu(\hat{P}_{\Psi_{S,M}(t_1)}, \delta) \) evolves to \( \nu(\hat{P}_{\Psi_{S,M}(t_2)}, \delta) \) with \( \Psi_{S,M}(t_2) = (\alpha \psi^+_S \otimes \phi^+_M + \beta \psi^-_S \otimes \phi^-_M) \).

4 For the meaning of approximate eigenvector/value see Weyl’s criterion in Reed and Simon [7], pp237 and pp 364 for unbounded self-adjoint operators.
6.2. Decomposing conditions. If the condition is the union of basic open sets, $W = \bigcup_{j=1}^{n} \nu(\rho_j; \epsilon_j)$ with $\epsilon_j > 0$ and if each $\rho_j = \lambda \rho_j^+ + (1 - \lambda) \rho_j^-$ for $0 < \lambda < 1$, then $W$ also has a convex decomposition, $W = \lambda W^+ + (1 - \lambda) W^-$ where $W^\pm = \bigcup_{j=1}^{n} \nu(\rho_j^\pm; \epsilon_j)$.

The proof of this follows from the lemma concerning the decomposition of the basic open sets $\nu(\rho, \delta)$ and the fact that every open set is a union of basic open sets.

**Lemma 15.** If $\rho_+ \neq \rho_-$ are distinct states in $\mathcal{E}_S(A)$ and $\rho_0 = \lambda \rho_+ + (1 - \lambda) \rho_-$ with $0 < \lambda < 1$ then $\nu(\rho_0, \epsilon)$ can be decomposed following the decomposition of the state $\rho_0$: $\nu(\rho_0, \epsilon) = \lambda \nu(\rho_+; \epsilon) + (1 - \lambda) \nu(\rho_-; \epsilon)$.

This true since $\nu$ determines a norm $\| \cdot \|_t$ on the space of trace class operators, so that if $\sigma = \lambda \sigma_+ + (1 - \lambda) \sigma_-$ with $\sigma_s \in \nu(\rho_s; \epsilon)$ for $s = \pm$, then $\sigma \in \nu(\rho_0, \delta)$ as

$$\| \sigma - \rho_0 \|_t \leq \lambda \| \rho_+ - \sigma_+ \|_t + (1 - \lambda) \| \rho_- - \sigma_- \|_t < \epsilon.$$  

Conversely if $\rho_m \in \nu(\rho_0; \epsilon)$ then $\rho_m = \lambda \rho_m^+ + (1 - \lambda) \rho_m^-$ where $\rho_m^+ = \rho_m + (\rho_+ - \rho_0)$ and $\rho_m^- = \rho_m + (\rho_- - \rho_0)$ hence $\rho_m^+ \in \nu(\rho_+; \epsilon)$ and $\rho_m^- \in \nu(\rho_-; \epsilon)$. Therefore $\nu(\rho_0; \epsilon) \subseteq \lambda \nu(\rho_+; \epsilon) + (1 - \lambda) \nu(\rho_-; \epsilon)$.

These results are easily extended to finite convex sums.

Using a similar argument for the sub-basic open sets $\mathcal{N}(\rho_0, \hat{A}, \delta) = \{ \rho \in \mathcal{E}_S(A) : |Tr(\rho \hat{A} - \rho_0 \hat{A})| < \delta \}$,

**Lemma 16.** If $\rho_+ \neq \rho_-$ and $\rho_0 = \lambda \rho_+ + (1 - \lambda) \rho_-$ with $0 < \lambda < 1$ then $\mathcal{N}(\rho_0, \hat{A}, \delta)$ can be decomposed following the decomposition of $\rho_0$: $\mathcal{N}(\rho_0, \hat{A}, \delta) = \lambda \mathcal{N}(\rho_+, \hat{A}, \delta) + (1 - \lambda) \mathcal{N}(\rho_-, \hat{A}, \delta)$.

Applying this to $\mathcal{N}(\rho_0, \hat{A}, \delta)$ when $\rho_0 = \rho_{S, m}^\text{mix} = |\alpha|^2 \hat{P}_{S_+}^{\psi_{S_+}} + |\beta|^2 \hat{P}_{S_+}^{\psi_{S_-}}$, $\lambda = |\alpha|^2$, $(1 - \lambda) = |\beta|^2$, $\rho_+ = \hat{P}_{S_+}^{\psi_{S_+}}$, $\rho_- = \hat{P}_{S_+}^{\psi_{S_-}}$, and $\hat{A} = \hat{Q}_S$ then

$$\mathcal{N}(\rho_{S, m}^\text{mix}, \hat{Q}_S, \delta) = |\alpha|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_+}}, \hat{Q}_S, \delta) + |\beta|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_-}}, \hat{Q}_S, \delta).$$

**Lemma 17.** If $\psi_S = \alpha \psi_{S_+}^\delta + \beta \psi_{S_-}^\delta$, $\psi_{S_+}^\delta$ are orthonormal eigenvectors of $\hat{Q}_S$, $|\alpha|^2 + |\beta|^2 = 1$ and $\delta > 0$ then

$$\mathcal{N}(\hat{P}_{S_+}^{\psi_{S_+}}, \hat{Q}_S, \delta) = |\alpha|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_+}}, \hat{Q}_S, \delta) + |\beta|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_-}}, \hat{Q}_S, \delta).$$

**Proof.** Firstly $\mathcal{N}(\rho_{S, m}^\text{mix}, \hat{Q}_S, \delta) = |\alpha|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_+}}, \hat{Q}_S, \delta) + |\beta|^2 \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_-}}, \hat{Q}_S, \delta)$ was shown above and $Tr\hat{P}_{S_+}^{\psi_{S_+}}\hat{Q}_S = |\alpha|^2 Tr\hat{P}_{S_+}^{\psi_{S_+}}\hat{Q}_S + |\beta|^2 Tr\hat{P}_{S_+}^{\psi_{S_-}}\hat{Q}_S$. If $\rho = |\alpha|^2 \rho_+ + |\beta|^2 \rho_-$ with $\mu_t \in \mathcal{N}(\hat{P}_{S_+}^{\psi_{S_+}}, \hat{Q}_S, \delta)$ for $t = \pm$ then $|Tr(\rho \hat{Q}_S -$
\[ \hat{P}_{\psi S} \hat{Q}_S = |\alpha|^2 \text{Tr}(\rho_+ \hat{Q}_S - \hat{P}_{\psi S} \hat{Q}_S) + |\beta|^2 \text{Tr}(\rho_- \hat{Q}_S - \hat{P}_{\psi S} \hat{Q}_S) < \delta, \]

showing that

\[ |\alpha|^2 \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) + |\beta|^2 \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) \subseteq \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta). \]

On the other hand if \( \rho_c \in \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) \) then \( \rho_c = |\alpha|^2 \rho_c^+ + |\beta|^2 \rho_c^- \) where \( \rho_c^+ = \rho_c + (\hat{P}_{\psi S} - \hat{P}_{\psi S}) \) and \( \rho_c^- = \rho_c + (\hat{P}_{\psi S} - \hat{P}_{\psi S}) \) so that \( \rho_c^+ \in \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) \) and \( \rho_c^- \in \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) \), therefore \( \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) \subseteq |\alpha|^2 \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta) + |\beta|^2 \mathcal{N}(\hat{P}_{\psi S}, \hat{Q}_S, \delta). \]

\[ \square \]

**Acknowledgments**

I wish to thank Professor Dipankar Home for introducing me to the quantum mechanical measurement problem many years ago. Any misinterpretations of the problem are my own doing.

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