OPTIMAL EVALUATIONS FOR THE SÁNDOR-YANG MEAN BY
POWER MEAN

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ABSTRACT. In this paper, we prove that the double inequality

\[ M_p(a, b) < B(a, b) < M_q(a, b) \]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq \frac{4 \log 2}{4 + 2 \log 2 - \pi} = 1.2351 \cdots \) and \( q \geq \frac{4}{3} \). Where \( M_r(a, b) = \left( \frac{a^r + b^r}{2} \right)^{1/r} \) (\( r \neq 0 \)) and \( M_0(a, b) = \sqrt{ab} \) is the \( r \)th power mean, \( B(a, b) = Q(a, b)e^{A(a, b)/T(a, b) - 1} \) is the Sándor-Yang mean, \( A(a, b) = \frac{a + b}{2} \), \( Q(a, b) = \sqrt{\frac{a^2 + b^2}{2}} \) and \( T(a, b) = (a - b)/[2 \text{arctan}(a - b)/(a + b)] \).

1. INTRODUCTION

For \( r \in \mathbb{R} \), the \( r \)th power mean \( M_r(a, b) \) of two distinct positive real numbers \( a \) and \( b \) is defined by

\[
M_r(a, b) = \begin{cases} 
\left( \frac{a^r + b^r}{2} \right)^{1/r}, & r \neq 0, \\
\sqrt{ab}, & r = 0.
\end{cases}
\]

It is well known that \( M_r(a, b) \) is continuous and strictly increasing with respect to \( r \in \mathbb{R} \) for fixed \( a, b > 0 \) with \( a \neq b \). Many classical means are the special cases of the power mean, for example, \( M_{-1}(a, b) = 2ab/(a + b) = H(a, b) \) is the harmonic mean, \( M_0(a, b) = \sqrt{ab} = G(a, b) \) is the geometric mean, \( M_1(a, b) = (a + b)/2 = A(a, b) \) is the arithmetic mean, and \( M_2(a, b) = \sqrt{(a^2 + b^2)/2} = Q(a, b) \) is the quadratic mean. The main properties for the power mean are given in [1].

Let

\[
L(a, b) = \frac{a - b}{\log a - \log b}, \quad I(a, b) = \frac{1}{e} \left( \frac{a}{b} \right)^{1/(a-b)}, \quad P(a, b) = \frac{a-b}{2 \arcsin \left( \frac{a-b}{a+b} \right)},
\]

\[
U(a, b) = \frac{a-b}{\sqrt{2 \arctan \left( \frac{a-b}{\sqrt{a^2-b^2}} \right)}}, \quad T^*(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta,
\]

\[
NS(a, b) = \frac{a-b}{2 \sinh^{-1} \left( \frac{a-b}{a+b} \right)}, \quad X(a, b) = A(a, b)e^{G(a, b)/P(a, b) - 1},
\]

\[
T(a, b) = \frac{a-b}{2 \arctan \left( \frac{a-b}{a+b} \right)}, \quad B(a, b) = Q(a, b)e^{A(a, b)/T(a, b) - 1}.
\]
be respectively the logarithmic mean, identric mean, first Seiffert mean \[2\], Yang mean \[3\], Toader mean \[4\], Neuman-Sándor mean \[5, 6\], Sándor mean \[7\], second Seiffert mean \[8\], Sándor-Yang mean \[3\] of \(a\) and \(b\).

Recently, the sharp bounds for certain bivariate means in terms of the power mean have attracted the attention of many mathematicians. Lin \[9\] proved that the double inequality
\[
M_p(a, b) < L(a, b) < M_q(a, b)
\]
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(p \leq 0\) and \(q \geq 1/3\).

Stolarsky \[10\] and Pittenger \[11\] found that \(M_{2/3}(a, b)\) and \(M_{\log 2}(a, b)\) are respectively the best possible lower and upper power mean bounds for the identric mean \(I(a, b)\). In \[12-15\], the authors proved that the double inequality
\[
M_p(a, b) < T^*(a, b) < M_q(a, b)
\]
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(p \leq 3/2\) and \(q \geq \log 2/\left(\log \pi - \log 2\right)\).

Jagers \[16\], Håstö \[17, 18\], Yang \[19\], and Costin and Toader \[20\] proved that \(p_1 = \log 2/\log \pi\), \(q_1 = 2/3\), \(p_2 = \log 2/\left(\log \pi - \log 2\right)\) and \(q_2 = 5/3\) are the best possible parameters such that the double inequalities
\[
M_{p_1}(a, b) < P(a, b) < M_{q_1}(a, b), \quad M_{p_2}(a, b) < T(a, b) < M_{q_2}(a, b)
\]
hold for all \(a, b > 0\) with \(a \neq b\).

In \[20-25\], the authors proved that the double inequalities
\[
M_{\lambda_1}(a, b) < NS(a, b) < M_{\mu_1}(a, b),
M_{\lambda_2}(a, b) < U(a, b) < M_{\mu_2}(a, b),
M_{\lambda_3}(a, b) < X(a, b) < M_{\mu_3}(a, b)
\]
hold for all \(a, b > 0\) with \(a \neq b\) if and only if \(\lambda_1 \leq \log 2/\log[2\log(1+\sqrt{2})]\), \(\mu_1 \geq 4/3\), \(\lambda_2 \leq 2\log 2/\left(2\log \pi - \log 2\right)\), \(\mu_2 \geq 4/3\), \(\lambda_3 \leq 1/3\) and \(\mu_3 \geq \log 2/(1 + \log 2)\).

Yang et al. \[26\] proved that
\[
(1.3) \quad M_1(a, b) < B(a, b) < M_2(a, b)
\]
for all \(a, b > 0\) with \(a \neq b\).

Motivated by inequality (1.3), it is natural to ask what are the greatest value \(p\) and the least value \(q\) such that the double inequality
\[
M_p(a, b) < B(a, b) < M_q(a, b)
\]
holds for all \(a, b > 0\) with \(a \neq b\)? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.
Lemma 2.1. (See [27, Lemma 7]) Let \( \{a_k\}_{k=0}^{\infty} \) be a nonnegative real sequence with \( a_m > 0 \) and \( \sum_{k=m+1}^{\infty} a_k > 0 \), and
\[
P(t) = \sum_{k=0}^{m} a_k t^k - \sum_{k=m+1}^{\infty} a_k t^k
\]
be a convergent power series on the interval \((0, \infty)\). Then there exists \( t_{m+1} \in (0, \infty) \) such that \( P(t_{m+1}) = 0 \), \( P(t) > 0 \) for \( t \in (0, t_{m+1}) \) and \( P(t) < 0 \) for \( t \in (t_{m+1}, \infty) \).

Lemma 2.2. (See [22, Lemma 6]) The function \( r \to 2^{1/r} M_r(a, b) \) is strictly decreasing and log-convex on \((0, \infty)\) for all \( a, b > 0 \) with \( a \neq b \).

Lemma 2.3. Let \( t > 0 \), \( p \in \mathbb{R} \) and
\[
(2.1) \quad f_1(t, p) = -\arctan(\tanh(t)) + \sinh(t) \cosh(t) - \tanh(pt) \sinh^2(t).
\]
Then the following statements are true:
(i) if \( p \leq 1 \), then \( f_1(t, p) \) is strictly increasing with respect to \( t \) on \((0, \infty)\);
(ii) if \( p \geq 4/3 \), then \( f_1(t, p) \) is strictly decreasing with respect to \( t \) on \((0, \infty)\);
(iii) if \( p \in (1, 4/3) \), then there exists \( t_1 \in (0, \infty) \) such that \( f_1(t, p) \) is strictly increasing with respect to \( t \) on \((0, t_1)\) and strictly decreasing with respect to \( t \) on \((t_1, \infty)\).

Proof. Let
\[
(2.2) \quad u_n(p) = (2 - p)^2 - p^2 + (1 - p)^2 n - 2p,
\]
\[
f_2(t, p) = 4 \sinh^2(t) \cosh^2\left(\frac{p t}{2}\right) - 4p \cosh(t) \sinh^2\left(\frac{t}{2}\right) - \sinh(2t) \sinh(pt).
\]
Then simple computations lead to
\[
(2.3) \quad u_1\left(\frac{4}{3}\right) = 0, \quad u_n\left(\frac{4}{3}\right) = -\frac{4^{2n} - 2^{2n}}{3^{2n}} - \frac{2^{2n} - 8}{3} < 0 \quad (n \geq 2),
\]
\[
(2.4) \quad \frac{\partial f_1(t, p)}{\partial t} = -\frac{1}{\cosh(2t)} + \cosh(2t) - \frac{p \sinh^2(t)}{\cosh^2(pt)} - \tanh(pt) \sinh(2t)
\]
\[
= \frac{f_2(2t, p)}{4 \cosh(2t) \cosh^2(pt)},
\]
\[
(2.5) \quad f_2(t, p) = \cosh[(p - 2)t] - \cosh(pt) + (1 - p) \cosh(2t) + 2p \cosh(t) - p - 1
\]
\[
= \sum_{n=1}^{\infty} \frac{u_n(p)}{(2n)!} 2^{2n},
\]
\[
(2.6) \quad \frac{\partial f_2(t, p)}{\partial p} = 2 \cosh(t) - \cosh(2t) + t \sinh[(p - 2)t] - t \sinh(pt) - 1
\]
\[
= -2[\cosh(t) - 1] \cosh(t) - 2t \sinh(t) \cosh[(p - 1)t] < 0.
\]
(i) If \( p \leq 1 \), then equations (2.5) and (2.6) lead to
\[
(2.7) \quad f_2(t, p) \geq f_2(t, 1) = 2[\cosh(t) - 1] > 0.
\]
Therefore, Lemma 2.3(i) follows easily from (2.4) and (2.7).
(ii) If \( p \geq 4/3 \), then from (2.3), (2.5) and (2.6) we have

\[
 f_2(t, p) \leq f_2 \left( t, \frac{4}{3} \right) = \sum_{n=1}^{\infty} \frac{u_n(4/3)}{(2n)!} t^{2n} < 0. \tag{2.8}
\]

Therefore, Lemma 2.3(ii) follows easily from (2.4) and (2.8).

(iii) If \( p \in (1, 4/3) \), then from (2.4) it is enough to prove that there exists \( t_1 \in (0, \infty) \) such that \( f_2(t, p) > 0 \) for \( t \in (0, t_1) \) and \( f_2(t, p) < 0 \) for \( t \in (t_1, \infty) \).

It follows from (2.2) that

\[
 u_1(p) = 2(4 - 3p) > 0, \quad \lim_{n \to \infty} \frac{u_n(p)}{2^n} = 1 - p < 0, \tag{2.9}
\]

\[
 u_{n+1}(p) - u_n(p) = -(p - 1) \left[ (3 - p)(2 - p)^2 + 3 \times 2^n + (p + 1)p^{2n} \right] < 0 \tag{2.10}
\]

for all \( n \geq 1 \).

Therefore, the desired result follows from (2.5), (2.9), (2.10) and Lemma 2.1. \( \square \)

**Lemma 2.4.** Let \( t > 0, p \in \mathbb{R} \) \( f_1(t, p) \) be defined by (2.1). Then

(i) \( f_1(t, p) > 0 \) for all \( t \in (0, \infty) \) if and only if \( p \leq 1 \);

(ii) \( f_1(t, p) < 0 \) for all \( t \in (0, \infty) \) if and only if \( p \geq 4/3 \);

(iii) there exists \( t_0 \in (0, \infty) \) such that \( f_1(t_0, p) = 0 \), \( f_1(t, p) > 0 \) for \( t \in (0, t_0) \) and \( f_1(t, p) < 0 \) for \( t \in (t_0, \infty) \) if \( p \in (1, 4/3) \).

**Proof.** (i) If \( p \leq 1 \), then Lemma 2.3(i) and (2.1) lead to the conclusion that \( f_1(t, p) > f_1(0, p) = 0 \) for all \( t \in (0, \infty) \).

If \( f_1(t, p) > 0 \) for all \( t \in (0, \infty) \), then \( \lim_{t \to \infty} f_1(t, p) \geq 0 \). We claim that \( p \leq 1 \).

Indeed, if \( p > 1 \), then from (2.1) we have

\[
 \lim_{t \to \infty} f_1(t, p) = \lim_{t \to \infty} \left[ - \arctan(\tanh(t)) + \frac{\sinh(t) \cosh((p - 1)t)}{\cosh(pt)} \right]
\]

\[
 = \lim_{t \to \infty} \left[ - \arctan(\tanh(t)) + \frac{1 - e^{-2t} + e^{-2|p-1|t}}{2} \right]
\]

\[
 = - \pi/4 + 1/2 < 0.
\]

(ii) If \( p \geq 4/3 \), then Lemma 2.3(ii) and (2.1) imply that \( f_1(t, p) < f_1(0, p) = 0 \) for all \( t \in (0, \infty) \).

If \( f_1(t, p) < 0 \) for all \( t \in (0, \infty) \), then we clearly see that

\[
 \lim_{t \to 0} \frac{f_1(t, p)}{t^3} \leq 0. \tag{2.11}
\]

It follows from (2.1), (2.2), (2.4) and (2.5) that

\[
 \lim_{t \to 0} \frac{f_1(t, p)}{t^3} = \lim_{t \to 0} \frac{\partial f_1(t, p)/\partial t}{3t^2} = \lim_{t \to 0} \frac{1}{3 \cosh(2t) \cosh^2(pt)} \times \lim_{t \to 0} \frac{f_2(2t, p)}{(2t)^2}
\]

\[
 = \frac{1}{3} \times \frac{1}{2} u_1(p) = - \left( p - \frac{4}{3} \right).
\]

Inequality (2.11) and equation (2.12) lead to the conclusion that \( p \geq 4/3 \).

(iii) If \( p \in (1, 4/3) \), then from Lemma 2.3(iii) and the facts that \( f_1(0, p) = 0 \) and \( \lim_{t \to \infty} f_1(t, p) = -\pi/4 + 1/2 < 0 \) we clearly see that there exists \( t_0 \in (0, \infty) \) such that \( f_1(t_0, p) = 0 \), \( f_1(t, p) > 0 \) for \( t \in (0, t_0) \) and \( f_1(t, p) < 0 \) for \( t \in (t_0, \infty) \). \( \square \)
Lemma 2.5. Let $t > 0$, $p \in (-\infty, 0) \cup (0, \infty)$ and
\begin{equation}
F(t, p) = \frac{1}{2} \log[\cosh(2t)] + \frac{\arctan(\tanh(t))}{\tanh(t)} - \frac{1}{p} \log[\cosh(pt)] - 1.
\end{equation}

Then
(i) $F(t, p)$ is strictly increasing with respect to $t$ on $(0, \infty)$ if and only if $p \leq 1$;
(ii) $F(t, p)$ is strictly decreasing with respect to $t$ on $(0, \infty)$ if and only if $p \geq 4/3$;
(iii) there exists $t_0 \in (0, \infty)$ such that $f_1(t_0, p) = 0$, $F(t, p)$ is strictly increasing with respect to $t$ on $(0, t_0)$ and strictly decreasing with respect to $t$ on $(t_0, \infty)$, where $f_1(t, p)$ is defined by (2.1).

Proof. It follows from (2.13) that
\begin{equation}
\frac{\partial F(t, p)}{\partial t} = -\frac{\arctan(\tanh(t)) + \sinh(t) \cosh(t) - \tanh(pt) \sinh^2(t)}{\sinh^2(t)} = f_1(t, p).
\end{equation}

Therefore, Lemma 2.5 follows from Lemma 2.4 and (2.14).

\[ \square \]

3. Main Results

Theorem 3.1. The inequality
\begin{equation}
B(a, b) < M_p(a, b)
\end{equation}
holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 4/3$. Moreover, the inequality
\begin{equation}
B(a, b) > \lambda_p M_p(a, b)
\end{equation}
holds for all $a, b > 0$ and $a \neq b$ with the best possible parameter $\lambda_p = e^{\pi/4 - 121/2p-1/2}$ if $p \geq 4/3$.

Proof. Since $B(a, b)$ and $M(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log \sqrt{b/a} > 0$, $p \in \mathbb{R}$ and $p \neq 0$, $f_1(t, p)$ and $F(t, p)$ be defined by (2.1) and (2.13), respectively. Then (1.1), (1.2), (2.1), (2.12), (2.13) and (2.14) lead to
\begin{align*}
M_p(a, b) &= \sqrt{ab} \cosh^{1/p}(pt), \quad T(a, b) = \sqrt{ab} \frac{\sinh(t)}{\arctan[\tanh(t)]}, \\
B(a, b) &= \sqrt{ab} \cosh^{1/2}(2t)e^{\arctan[\tanh(t)]/\tanh(t)-1}, \\
\log[B(a, b)] - \log[M_p(a, b)] &= F(t, p), \\
F(0^+, p) &= 0,
\end{align*}
\begin{align*}
\lim_{t \to 0^+} \frac{F(t, p)}{t^2} &= \lim_{t \to 0^+} \frac{\partial F(t, p)/\partial t}{2t} = \lim_{t \to 0^+} \frac{f_1(t, p)}{2t \sinh^2(t)} = -\frac{1}{2} \left( p - \frac{4}{3} \right), \\
\lim_{t \to \infty} F(t, p) &= \lim_{t \to \infty} \left[ \left( 1 - \frac{|p|}{p} \right) t + \frac{1}{2} \log \left( \frac{1 + e^{-4t}}{2} \right) + \frac{\arctan(\tanh(t))}{\tanh(t)} - \frac{1}{p} \log \left( \frac{1 + e^{-2|p|t}}{2} \right) - 1 \right] \\
&= \frac{1}{4} \pi - \frac{1}{2} \log 2 + \frac{1}{p} \log 2 - 1 = \log(\lambda_p) \quad (p > 0).
\end{align*}
If \( B(a, b) < M_p(a, b) \), then (3.3) and (3.5) lead to \( p \geq 4/3 \).

If \( p \geq 4/3 \), then from (3.4) and (3.6) together Lemma 2.5(ii) we clearly see that
\[
(3.7) \quad \log(\lambda_p) < \lim_{t \to \infty} F(t, p) < F(t, p) < F(0^+, p) = 0
\]
for all \( t > 0 \) with the best possible parameter \( \lambda_p \).

Therefore, the double inequality
\[
\lambda_p, M_p(a, b) < B(a, b) < M_p(a, b)
\]
holds for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameter \( \lambda_p \) follows from (3.3) and (3.7). \( \square \)

Note that
\[
(3.8) \quad \lambda_p, M_p(a, b) = \frac{\sqrt{2}}{2} e^{\pi/4 - 1} \left( 2^{1/p} M_p(a, b) \right), \quad \lim_{p \to \infty} M_p(a, b) = \max \{a, b\}.
\]

Let \( p = 4/3, 3/2, 2, 3, \cdots, \infty \). Then from Lemma 2.2, (3.1), (3.2) and (3.8) together with the monotonicity of the function \( p \to M_p(a, b) \) we get Corollary 3.1.

**Corollary 3.1.** The inequalities
\[
\lambda_\infty \max \{a, b\} < \cdots < \lambda_2 M_2(a, b) < \lambda_3/2 M_3/2(a, b) < \lambda_4/3 M_4/3(a, b)
\]
\[
< B(a, b) < M_4/3(a, b) < M_3/2(a, b) < M_2(a, b) < \cdots < \max \{a, b\}
\]
hold for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameters \( \lambda_\infty = \frac{\sqrt{2}}{2} e^{\pi/4 - 1} = 0.5705 \cdots \), \( \lambda_2 = e^{\pi/4 - 1} = 0.8068 \cdots \), \( \lambda_3/2 = 2^{1/6} e^{\pi/4 - 1} = 0.9056 \cdots \) and \( \lambda_4/3 = 2^{1/4} e^{\pi/4 - 1} = 0.9595 \cdots \).

**Theorem 3.2.** Let \( p_0 = 4 \log 2/(4 + 2 \log 2 - \pi) = 1.2351 \cdots \). Then the inequality
\[
(3.9) \quad B(a, b) > M_p(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq p_0 \).

**Proof.** If \( B(a, b) > M_p(a, b) \), then (3.3) and (3.6) lead to \( p \leq p_0 \).

If \( p = p_0 \), then (3.4), (3.6) and Lemma 2.5(iii) lead to the conclusion that
\[
(3.10) \quad F(0^+, p_0) = \lim_{t \to \infty} F(t, p_0) = 0
\]
and there exists \( t_0 \in (0, \infty) \) such that the function \( t \to F(t, p_0) \) is strictly increasing on \( (0, t_0) \) and strictly decreasing on \( (t_0, \infty) \).

Therefore,
\[
B(a, b) > M_{p_0}(a, b) > M_p(a, b)
\]
for all \( p \leq p_0 \) follows easily from (3.3), (3.10), the piecewise monotonicity of the function \( t \to F(t, p_0) \) and the monotonicity of the function \( p \to M_p(a, b) \). \( \square \)

**Corollary 3.2.** Let \( f_1(t, p), F(t, p) \) and \( \lambda_p \) be defined respectively by (2.1), (2.13) and Theorem 3.1, and \( p_0 = 4 \log 2/(4 + 2 \log 2 - \pi) = 1.2351 \cdots \). Then the inequality
\[
(3.11) \quad B(a, b) < \lambda_p, M_p(a, b)
\]
holds for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameter \( \lambda_p \) if \( p \in (0, 1] \), and the inequality
\[
(3.12) \quad B(a, b) \leq e^{F(t_0, p)} M_p(a, b)
\]
holds for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameter \( e^{F(t_0,p)} \) if \( p \in (1, p_0] \), where \( t_0 \) is the unique solution of the equation \( f_1(t,p) = 0 \) on the interval \((0, \infty)\). In particular, Numerical computations show that \( e^{F(t_0,p_0)} = 1.012 \ldots \).

**Proof.** If \( p \in (0,1] \), then inequality (3.11) holds for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameter \( \lambda_p \) follows from (3.3) and (3.6) together with Lemma 2.5(i).

If \( p \in (1, p_0] \), then inequality (3.12) holds for all \( a, b > 0 \) and \( a \neq b \) with the best possible parameter \( e^{F(t_0,p)} \) follows from (3.3) and Lemma 2.5(iii). \( \square \)

Let \( p \in \mathbb{R}, b > a > 0 \), \( L_p(a,b) = (a^{p+1} + b^{p+1}) / (a^{p} + b^{p}) \) be the \( p \)-th Lehmer mean [28] of \( a \) and \( b \), \( f_1(t,p) \) be defined by (2.1), and \( t = \log \sqrt{b/a} > 0 \). Then \( f_1(t,p) \) can be rewritten as

\[
(3.13) \quad f_1(t,p) = - \arctan(\tanh(t)) + \sin(t) \cosh((p-1)t) - \frac{\sinh(t)}{\cosh(pt)} \frac{\cosh((p-1)t)}{\cosh(pt)}
\]

\[
= \frac{\arctan(\tanh(t)) \cosh((p-1)t)}{\cosh(pt)} \left( \frac{\sinh(t)}{\arctan(\tanh(t))} - \frac{\cosh(pt)}{\cosh((p-1)t)} \right)
\]

\[
= \frac{\arctan(\tanh(t)) \cosh((p-1)t)}{\sqrt{ab} \cosh(pt)} (T(a,b) - L_{p-1}(a,b)).
\]

Lemma 2.4 and (3.13) lead to Corollary 3.3 immediately.

**Corollary 3.3.** (see [29, Theorem 2.2]) The double inequality

\[
L_p(a,b) < T(a,b) < L_q(a,b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq 0 \) and \( q \geq 1/3 \).

**Corollary 3.4.** The double inequality

\[
\lambda L_{1/3}(a,b) < T(a,b) < \mu L_0(a,b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \lambda \leq 2/\pi \) and \( \mu \geq 4/\pi \).

**Proof.** Without loss of generality, we assume that \( b > a > 0 \). Let \( t = \log \sqrt{b/a} > 0 \). Then simple computations lead to

\[
(3.14) \quad \frac{T(a,b)}{L_0(a,b)} = \frac{\sinh(t) \cosh(\frac{t}{2})}{\cosh(\frac{t}{2}) \arctan(\tanh(t))}, \quad \frac{T(a,b)}{L_0(a,b)} = \frac{\sinh(t)}{\cosh(t) \arctan(\tanh(t))}.
\]

\[
(3.15) \quad \lim_{t \to \infty} \frac{\sinh(t) \cosh(\frac{t}{2})}{\cosh(\frac{t}{2}) \arctan(\tanh(t))} = \frac{2}{\pi}, \quad \lim_{t \to \infty} \frac{\sinh(t)}{\cosh(t) \arctan(\tanh(t))} = \frac{4}{\pi}.
\]

The log-convexity of the function \( r \to 2^{1/r} M_r(a,b) \) given by Lemma 2.2 implies that

\[
\left(2^{3/2} M_{5/3}(a,b)\right)^{3/4} \left(2^{3} M_{1/3}(a,b)\right)^{1/4} > 2^{3/4} M_{4/3}(a,b),
\]

which can be rewritten as

\[
(3.16) \quad \frac{2^{8/5}}{\pi} M_{5/3}(a,b) > \frac{2}{\pi} M_{1/3}^{1/3}(a,b) = \frac{2}{\pi} L_{1/3}(a,b).
\]
Yang et al. [30] and Witkowski [31] proved that

$$\frac{2^{8/5}}{\pi} M_{5/3}(a, b) < T(a, b) < \frac{4}{\pi} A(a, b) = \frac{4}{\pi} L_0(a, b).$$

Therefore, Corollary 3.4 follows from (3.14)-(3.17).

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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