A Recent Look at the Quantum Berezinian in the Yangian

\[ Y(\mathfrak{gl}_{m|n}) \]

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Brundan and Kleshchev recently introduced a new family of presentations of the Yangian \( Y(\mathfrak{gl}_n) \) associated to the general linear Lie algebra \( \mathfrak{gl}_n \), and thus provided a fresh approach to its study. In this article, we would like to show how some of their ideas can be fruitfully extended to consider the Yangian \( Y(\mathfrak{gl}_{m|n}) \) associated to the Lie superalgebra \( \mathfrak{gl}_{m|n} \). In particular, we give a new proof of the result by Nazarov that the quantum Berezinian is central.\(^1\)

1 Definition of \( Y(\mathfrak{gl}_{m|n}) \)

The Yangian \( Y(\mathfrak{gl}_{m|n}) \) is defined in [7] to be the \( \mathbb{Z}_2 \)-graded associative algebra over \( \mathbb{C} \) with generators \( t_{ij}^{(r)} \) and certain relations described below. We define the formal power series

\[ t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \ldots, \]

and a matrix

\[ T(u) = \sum_{i,j=1}^{m+n} t_{ij}(u) \otimes E_{ij} (-1)^{\overline{i}(\overline{j}+1)}, \tag{1} \]

where \( E_{ij} \) is the standard elementary matrix and \( \overline{i} \) is the parity of the index \( i \). In analogy with the usual Yangian \( Y(\mathfrak{gl}_n) \) (see for example [2, 3, 4]), the defining relations are then expressed by the matrix product

\[ R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v) \]

where

\[ R(u - v) = 1 - \frac{1}{(u - v)} P_{12} \]

and \( P_{12} \) is the permutation matrix:

\[ P_{12} = \sum_{i,j=1}^{m+n} E_{ij} \otimes E_{ji} (-1)^{\overline{i}}. \]

Then we have the following equivalent form of the defining relations:

\[ [t_{ij}(u), t_{kl}(v)] = (-1)^{\overline{i}j + \overline{k}l + \overline{r}} \frac{(u - v)}{(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u))}. \]

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Throughout this article we will observe the following notation for entries of the inverse matrix of $T(u)$:

$$T(u)^{-1} =: (t'_{ij}(u))_{i,j=1}^n.$$ 

A straightforward calculation yields the following relation in $Y(gl_{m|n})$:

$$[t_{ij}(u), t'_{kl}(v)] = \left(\frac{(-1)^{n+k-l+m}}{u-v}\right) \cdot \left( \delta_{ij} \sum_{s=1}^{m+n} t_{is}(u)t'_{sl}(v) - \delta_{kl} \sum_{s=1}^{m+n} t'_{ks}(v)t_{sj}(u) \right). \quad (2)$$

### 2 Gauss Decomposition of $T(u)$

In [1], the Drinfeld presentation is described in terms of the quasideterminants of Gelfand and Retakh ([3], [4]). In this article we make use of the analogous set of generators of the Yangian $Y(gl_{m|n})$.

**Definition 2.1.** Let $X$ be a square matrix over a ring with identity such that its inverse matrix $X^{-1}$ exists, and such that its $j$th entry is an invertible element of the ring. Then the $ij$th quasideterminant of $X$ is defined by the formula

$$|X|_{ij} = \left((X^{-1})_{ji}\right)^{-1}.$$ 

It is sometimes convenient to adopt the following alternative notation for the quasideterminants:

$$|X|_{ij} =: \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}.$$ 

The matrix $T(u)$ defined in [1] has the following Gauss decomposition in terms of quasideterminants (by Theorem 4.96 in [3]; see §5 in [1]):

$$T(u) = F(u)D(u)E(u)$$

for unique matrices

$$D(u) = \begin{pmatrix} d_1(u) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{m+n}(u) \end{pmatrix},$$

$$E(u) = \begin{pmatrix} \begin{array}{cccc} 1 & e_{12}(u) & \cdots & e_{1,m+n}(u) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 \end{array} \end{pmatrix},$$

$$F(u) = \begin{pmatrix} 1 & \cdots & 0 \\ f_{21}(u) & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1 \end{pmatrix}.$$
where

\[
  d_i(u) = \begin{vmatrix}
    t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i1}(u) & \cdots & t_{i,i-1}(u) & t_{ii}(u)
  \end{vmatrix},
\]

\[
  e_{ij}(u) = d_i(u)^{-1} \begin{vmatrix}
    t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{ij}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i1}(u) & \cdots & t_{i,i-1}(u) & t_{ij}(u)
  \end{vmatrix},
\]

\[
  f_{ji}(u) = d_i(u)^{-1} \begin{vmatrix}
    t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{ji}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i1}(u) & \cdots & t_{i,i-1}(u) & t_{ji}(u)
  \end{vmatrix}.
\]

It is easy to recover each generating series \( t_{ij}(u) \) by multiplying together and taking commutators of \( d_i(u); 1 \leq i \leq m + n \), and \( e_i(u) := e_{i,i+1}(u), f_i(u) = f_{i+1,i}(u); 1 \leq i < m + n \) (see §5 of [1]). Thus the Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \) is generated by the coefficients of the latter.

### 2.1 Some Useful Maps

Here we define some automorphisms of the Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \) and homomorphisms between Yangians, so that we may refer to them in the next section.

Let \( \omega_{m|n} : \mathcal{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{Y}(\mathfrak{gl}_{m|n}) \) be the automorphism defined by

\[
  \omega : T(u) \mapsto T(-u)^{-1}.
\]

Let \( \tau : \mathcal{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{Y}(\mathfrak{gl}_{m|n}) \) be the anti-automorphism defined by

\[
  \tau(t_{ij}(u)) = t_{ji}(u) \times (-1)^{(i-1)(j-1)}.
\]

Let \( \rho_{m|n} : \mathcal{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{Y}(\mathfrak{gl}_{m|n}) \) be the isomorphism defined by

\[
  \rho_{m|n}(t_{ij}(u)) = t_{m+n+1-i,m+n+1-j}(-u).
\]

Let \( \varphi_{m|n} : \mathcal{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{Y}(\mathfrak{gl}_{m+k|n}) \) be the inclusion which sends each generator \( t_{ij}^{(r)} \in \mathcal{Y}(\mathfrak{gl}_{m|n}) \) to the generator \( t_{k+i,k+j}^{(r)} \) in \( \mathcal{Y}(\mathfrak{gl}_{m+k|n}) \).

Finally, let \( \psi_k : \mathcal{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{Y}(\mathfrak{gl}_{m+k|n}) \) be the injective homomorphism defined by

\[
  \psi_k = \omega_{m+k|n} \circ \varphi_{m|n} \circ \omega_{m|n}.
\]

This last homomorphism is useful for studying quasideterminants so we discuss it in some detail with the following remarks.

**Remark 2.1.** We can calculate \( \psi_k(t_{ij}(u)) \) explicitly for any \( 1 \leq i, j \leq m + n \) (see Lemma 4.2 of [1])

\[
  \psi_k(t_{ij}(u)) = \begin{vmatrix}
    t_{11}(u) & \cdots & t_{1,k}(u) & t_{1,k+j}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{k1}(u) & \cdots & t_{kk}(u) & t_{k,k+j}(u) \\
    t_{k+1,1}(u) & \cdots & t_{k+k}(u) & t_{k+k+j}(u)
  \end{vmatrix}.
\]

In particular, this means that for \( k \geq 1 \), we have \( \psi_k(d_1(u)) = d_{k+1}(u), \psi_k(e_1(u)) = e_{k+1}(u), \) and \( \psi_k(f_1(u)) = f_{k+1}(u). \)

Furthermore, by [5], we have for any \( k, l \geq 1 \) that \( \psi_k \circ \psi_l = \psi_{k+l} \), so we may generalise this observation to give for instance \( \psi_k(d_i(u)) = d_{k+l}(u) \).
Remark 2.2. Notice that the map \( \psi_k \) sends \( t_{ij}^{(r)} \in Y(\mathfrak{gl}_{m|n}) \) to the element \( t_{ij}^{(r)} \in Y(\mathfrak{gl}_{m+k|n}) \). Thus the subalgebra \( \psi_k(Y(\mathfrak{gl}_{m|n})) \) is generated by the elements \( \{t_{k+i,j}^{(r)}\}_{i,j=1}^{n} \). Then, by Remark 2.1, all elements of this subalgebra commute with those of the subalgebra generated by the elements \( \{t_{ij}^{(r)}\}_{i,j=1}^{n} \).

By Remark 2.1, this implies in particular that for any \( i, j \geq 1 \), the quasideterminants \( d_i(u) \) and \( d_j(v) \) commute.

3 The Quantum Berezinian

The quantum Berezinian was defined by Nazarov [7] and plays a similar role in the study of the Yangian \( Y(\mathfrak{gl}_{m|n}) \) as the quantum determinant does in the case of the Yangian \( Y(\mathfrak{gl}_n) \) (see [5]).

Definition 3.1. The quantum Berezinian is the following power series with coefficients in the Yangian \( Y(\mathfrak{gl}_{m|n}) \):

\[
b_{m|n}(u) := \sum_{\tau \in S_m} \text{sgn}(\tau) t_{\tau(1)}(u) t_{\tau(2)}(u-1) \cdots t_{\tau(m)}(u-m+1) \\
\times \sum_{\sigma \in S_n} \text{sgn}(\sigma) t_{m+1,m+\sigma(1)}(u-m+1) \cdots t_{n,m+n,\sigma(n)}(u-m+n)
\]

The first part of this expression for \( b_{m|n}(u) \) is quite special and so is given its own notation:

\[
C_m(u) := \sum_{\tau \in S_m} \text{sgn}(\tau) t_{\tau(1)}(u) t_{\tau(2)}(u-1) \cdots t_{\tau(m)}(u-m+1).
\]

It is clear that \( C_m(u) \) is an element of the subalgebra of \( Y(\mathfrak{gl}_{m|n}) \) generated by the set \( \{t_{ij}^{(r)}\}_{1 \leq i, j \leq m; r \geq 0} \). This subalgebra is isomorphic to the Yangian \( Y(\mathfrak{gl}_m) \) associated to the Lie algebra \( \mathfrak{gl}_m \) by the inclusion \( Y(\mathfrak{gl}_m) \rightarrow Y(\mathfrak{gl}_{m|n}) \) which send each generator \( t_{ij}^{(r)} \) in \( Y(\mathfrak{gl}_m) \) to the generator of the same name in \( Y(\mathfrak{gl}_{m|n}) \). Moreover, \( C_m(u) \) is in fact the image under this map of the quantum determinant of the smaller Yangian \( Y(\mathfrak{gl}_m) \) (see [1], [5]). Then it is well known (see Theorem 2.32 in [6]) that we have the alternative expression:

\[
C_m(u) = d_1(u)d_2(u-1) \cdots d_m(u-m+1).
\]

We can extend this observation as follows:

Theorem 1. We have the following alternative expression for the quantum Berezinian:

\[
b_{m|n}(u) = d_1(u)d_2(u-1) \cdots d_m(u-m+1) \\
\times d_{m+1}(u-m+1)^{-1} \cdots d_{m+n}(u-m+n)^{-1}.
\]

Proof. Notice that the second part of the expression for \( b_{m|n}(u) \) in Definition 3.1 is the image under the isomorphism \( \rho_{n|m} \circ \omega_{n|m} : Y(\mathfrak{gl}_{n|m}) \rightarrow Y(\mathfrak{gl}_{m|n}) \) of

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) t_{n,\sigma(n)}(u-m+1) \cdots t_{2,\sigma(2)}(u-m+n-1) t_{1,\sigma(1)}(u-m+n).
\]

(4)

where in this expression [4] we are following the usual convention for denoting generators in the Yangian \( Y(\mathfrak{gl}_{n|m}) \). We recognise (by comparing with (8.3) of [4] for example) that the expression [4] is in fact \( C_n(u-m+n) \), the image of the quantum determinant of \( Y(\mathfrak{gl}_n) \) under the natural inclusion \( Y(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_{n|m}) \). So in order to verify the claim we must calculate the image of \( C_n(u-m+n) \) under this map explicitly in terms of our quasideterminants \( d_i(v) \). Applying Proposition 1.6 of [4], we find that the image of \( d_i(v) \) in \( Y(\mathfrak{gl}_{n|m}) \) is \( (d_{m+n+1-i}(v))^{-1} \) in \( Y(\mathfrak{gl}_{m|n}) \). This gives the desired result. \( \square \)
Similarly, by considering the commutator 
\[ t \]
We rewrite these relations to find
\[ \text{Substituting in the expressions from (6) and (7) then cancelling} \]
\[ f_i(v) \]
\[ \text{for each } i \text{ between 1 and } m + n - 1. \]
We proceed by breaking this problem into three cases.

\textbf{Case 1:} \[ 1 \leq i \leq m - 1. \] In this case, \( e_i(v) \) commutes with \( C_m(u) = d_1(u) \cdots d_m(u - m + 1) \)
by Theorem 7.2 in \[ 1 \]. On the other hand, \( e_i(v) \) is an element of the subalgebra generated by \( \{t_{jk}^{(r)}\}_{1 \leq j, k \leq m} \) and thus by Remark 2.2 commutes with \( d_{m+1}(u - m + s)^{-1} = t'_{m+s,m+s}(u - m + s) \)
for \( 1 \leq s \leq n \). Now we may use the anti-automorphism \( \tau \) to show that \( f_i(v) \) also commutes with the quantum Berezinian in this case because \( \tau(e_i(v)) = f_i(v) \) for \( 1 \leq i \leq m - 1 \).

\textbf{Case 2:} \[ m + 1 \leq i \leq m + n - 1. \] Applying Propositions 1.6 and 1.4 of \[ 4 \] in turn to \( f_i(v) \), we find an alternative expression:
\[ f_i(v) = \begin{bmatrix} t'_{i+1,i}(v) & t'_{i+1,i+2}(v) & \cdots & t'_{i+1,m+n}(v) \\
               t'_{i+2,i}(v) & t'_{i+2,i+2}(v) & \cdots & \vdots \\
               \vdots & \vdots & \ddots & \vdots \\
               t'_{m+n,i}(v) & t'_{m+n,i+2}(v) & \cdots & t'_{m+n,m+n}(v) \end{bmatrix} \begin{bmatrix} t'_{i+1,i+1}(v) & \cdots & t'_{i+1,m+n}(v) \\
                             \vdots & \ddots & \vdots \\
                             \vdots & \vdots & \ddots \\
                             t'_{m+n,i+1}(v) & \cdots & t'_{m+n,m+n}(v) \end{bmatrix}^{-1}. \]

Thus, we find that for \( m + 1 \leq i \leq m + n - 1 \),
\[ \rho_{n|m} \circ \omega_{n|m}(-f_{m+n-i}(v)) = e_i(v), \]
and similarly
\[ \rho_{n|m} \circ \omega_{n|m}(-e_{m+n-i}(v)) = f_i(v). \]
We apply this isomorphism to the results of Case 1 in the Yangian \( Y(g_{l_{n|m}}) \). This shows that \( e_i(v) \) and \( f_i(v) \) commute with the quantum Berezinian in the case where \( m + 1 \leq i \leq m + n - 1 \).

\textbf{Case 3:} \( i = m \). We begin by considering the Yangian \( Y(g_{l_{1|1}}) \). For this algebra we have that the quantum Berezinian is \( b_{11}(u) = d_1(u)d_2(u)^{-1} \) and we would like to show that it commutes with \( e_1(v) \) and \( f_1(v) \). So it will suffice to show
\[ d_1(u)e_1(v)d_2(u) = d_2(u)e_1(v)d_1(u). \]
We have
\[ \begin{bmatrix} t_{11}(u) & t_{12}(u) \\
                t_{21}(u) & t_{22}(u) \end{bmatrix} = \begin{bmatrix} d_1(u) & d_1(u)e_1(v) \\
                             f_1(u)d_2(u) & f_1(u)d_1(u)e_1(u) + d_2(u) \end{bmatrix}. \]

An application of (2) gives
\[ (u - v)[t_{11}(u), t'_{12}(v)] = t_{11}(u)t'_{12}(v) + t_{12}(u)t'_{22}(v). \]
Substituting in the expressions from (6) and (7), then cancelling \( d_2(v) \), this gives
\[ (u - v)[d_1(u), e_1(v)] = d_1(u)(e_1(v) - e_1(u)). \]
Similarly, by considering the commutator \( [t_{12}(u), t'_{22}(v)] \), we derive the relation
\[ (u - v)[d_2(u), e_1(v)] = d_2(u)(e_1(v) - e_1(u)). \]
We rewrite these relations to find
\[ (u - v)e_1(v)d_1(u) = (u - v - 1)d_1(u)e_1(v) + d_1(u)e_1(u), \]
\[ (u - v)e_1(v)d_2(u) = (u - v - 1)d_2(u)e_1(v) + d_2(u)e_1(u), \]
and by considering these expressions we see that (5) holds. Similarly, the quantum Berezinian commutes with $f_1(v)$.

Now we return our attention to the general Yangian $Y(\mathfrak{gl}_{m|n})$. By similar appeals to Remark 2.2 as in the first case, we see that $e_m(v)$ and $f_m(v)$ commute with $d_1(u) \cdots d_{m-1}(u - m + 2)$ and with $d_{m+2}(u - m + 2)^{-1} \cdots d_{m+n}(u - m + n)^{-1}$. So we need only show they commute with $d_m(u - m + 1)d_{m+1}(u - m + 1)^{-1}$. This follows immediately when we apply the homomorphism $\psi_{m-1}$ to the identity (5) in $Y(\mathfrak{gl}_{1|1})$.

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