BIHALOFIT: A NEW FITTING FORMULA OF NON-LINEAR MATTER BISPECTRUM

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\textbf{ABSTRACT}

We provide an accurate fitting formula of the matter bispectrum in the non-linear regime calibrated by high-resolution cosmological N-body simulations for 41 wCDM models around the Planck 2015 best-fit parameters. Our fitting function assumes a similar parameterization as in Halofit for the non-linear matter power spectrum and so is named BIHalofit. The simulation volume is large enough ($> 10 \, \text{Gpc}^3$) to cover almost all measurable triangle configurations of bispectrum in the universe. Our formula can reproduce the matter bispectrum within 10 (15) % accuracy for wavenumber $k < 3 (10) \, h \, \text{Mpc}^{-1}$ at redshifts $z = 0$–3 for the Planck 2015 model. We also provide a fitting formula to correct baryonic effects such as radiative cooling and active galactic nuclei feedback based on the latest hydrodynamical simulation, IllustrisTNG. Further, we show that our new formula provides more accurate predictions for the weak lensing bispectrum than the existing fitting formulas. The formula would be quite useful for current and future weak lensing surveys and cosmic microwave background lensing experiments.

\textit{Subject headings:} gravitational lensing; weak – methods; numerical – cosmology: theory – large-scale structure of universe

1. \textbf{INTRODUCTION}

Observations of cosmic microwave background (CMB) have revealed that the primordial density fluctuations are well described by a Gaussian field (Planck Collaboration 2013). The statistical property of a Gaussian field is fully described by the two-point correlation function, or its Fourier transform, the power spectrum (PS). However, at late times, the density fluctuations become non-Gaussian through gravitational evolution at small scales. Then, we need higher-order statistics to fully characterize the statistical property of a non-Gaussian field and to access cosmological information beyond the two-point statistics. The bispectrum (BS), the Fourier transform of the three-point correlation function, is the leading correction to the commonly used PS.

Weak lensing can map a projected density field via coherent distortion of background galaxies (e.g., Bartelmann & Schneider 2001). Several weak lensing surveys are currently in operation, e.g., the Subaru Hyper Suprime-Cam (HSC)\textsuperscript{3} and the Dark Energy Survey (DESt)\textsuperscript{4} and the Kilo-Degree Survey (KiDS)\textsuperscript{5}. They provide strong constraints on the cosmological parameters such as the matter density $\Omega_m$ and the amplitude of density fluctuations $\sigma_8$ from the cosmic-shear two-point function (e.g., Abbott et al. 2018, van Uitert et al. 2018, Hamana et al. 2019, Hikage et al. 2019). In the 2020s, ground- and space-based missions will start their operations such as Large Synoptic Survey Telescope (LSST)\textsuperscript{6}, Wide Field Infrared Survey Telescope (WFIRST)\textsuperscript{7} and Euclid\textsuperscript{8}.

The weak lensing BS contains additional and complementary information to the PS. The BS is sensitive to smaller-scale and lower-redshift structures than the PS because the BS arises from the non-Gaussianity. A joint analysis of both the spectra breaks parameter degeneracy and thus provides tighter constraints (e.g., Takada & Jain 2004, Kilbinger & Schneider 2005, Sefusatti et al. 2006, Munshi et al. 2011, Kayo & Takada 2013, Byun et al. 2017, Gatti et al. 2019). The BS can be comparable to or more powerful than the PS (Bergé et al. 2010, Sato & Nishimichi 2013, Coulton et al. 2019). Several groups previously measured the three-point cosmic-shear statistics from real data and provided useful constraints (Bernardeau et al. 2002b, Jarvis et al. 2004, Semboloni et al. 2011, Van Waerbeke et al. 2013, Fu et al. 2014, Simon et al. 2015).

CMB lensing is also a promising cosmological probe to measure the density fluctuations at higher redshift...
where \( BS \) model is necessary for the current and forthcoming surveys. Recent CMB experiments have measured lensing signals from temperature and polarization fluctuations (Planck Collaboration 2018b). The CMB lensing-potential PS provides a rich cosmological information which is complementary to that obtained from the galaxy weak lensing (e.g., Planck Collaboration 2018b). The BS and higher-order spectra induced by the non-Gaussian density fluctuations would be important in the future CMB lensing observations. The non-Gaussianity slightly affects the lensing PS (Pratten & Lewis 2016) as well as the CMB temperature and polarization power spectra (Lewis & Pratten 2016 Marozzi et al. 2018). It also contaminates the CMB lensing reconstruction (Bohm et al. 2018; Beck et al. 2018 Fabbiani et al. 2018). On the other hand, future CMB experiments will measure the lensing BS as a useful signal (Namikawa 2016).

In this regard, an accurate model of non-linear BS is highly demanded. Regarding the PS, we should prepare a non-linear model with a few percent accuracy up to \( k = 10 \, \text{h} \, \text{Mpc}^{-1} \) to meet the statistical accuracy requirements of the forthcoming weak-lensing surveys (Huterer & Takada 2002; Hearin et al. 2012). Though Scoccimarro & Couchman (2001) provided a fitting formula of BS calibrated by \( N \)-body simulations and Gil-Marín et al. (2012) improved the formula further, these overestimate the squeezed BS by a factor 2 in the worst cases, compared to latest numerical simulations (Fu et al. 2014; Coulton et al. 2019; Namikawa et al. 2019; Munshi et al. 2019). In this paper, we construct an improved fitting formula of the non-linear BS calibrated by high-resolution cosmological \( N \)-body simulations for \( 41 \, \text{wCDM} \) (cold dark matter and dark energy with a constant equation of state \( w \)) models up to \( k = 30 \, \text{h} \, \text{Mpc}^{-1} \) in the range of \( z = 0 \sim 1 \). We bin both simulation data and theoretical prediction in terms of wavenumbers \((k_1, k_2, k_3)\), which is critically important for the accurate estimation. Our formula also takes into account the baryonic effects by using the public hydrodynamic simulation, the IllustrisTNG suite (Nelson et al. 2019).

2. THEORY

2.1. Basics

The cosmological density contrast is usually described by its Fourier transform, \( \delta(k) \). The matter PS and BS are defined as

\[
P_{1}(k_1) \delta_1(k_1 + k_2) = \langle \delta(k_1) \delta(k_2) \rangle,
\]

\[
B(k_1, k_2, k_3) \delta_3(k_1 + k_2 + k_3) = \langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle,
\]

where \( \delta_3 \) is the Dirac delta function. Here and throughout this paper, we do not explicitly write the redshift dependence in the arguments of the functions, but the following discussion is applicable at an arbitrary redshift.

At the tree level (i.e., the leading order in perturbation theory), the matter BS is given by a product of the linear matter PS, \( P_1(k) \), as (e.g., Bernardeau et al. 2002a)

\[
B_{\text{tree}}(k_1, k_2, k_3) = 2F_2(k_1, k_2)P_1(k_1)P_1(k_2) + 2 \text{perm.}
\]

Here, the last term means two permutations: \((k_1, k_2) \leftrightarrow (k_2, k_3)\) and \((k_3, k_1)\) applied to the wavevectors in the first term. The \( F_2 \) kernel is

\[
F_2(k_1, k_2) = \frac{5}{7} + \frac{1}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \mu_{12} + \frac{2}{7} \mu_{12}^2,
\]

where \( \mu_{12} \) is the cosine of the angle between \( k_1 \) and \( k_2 \), i.e., \( \mu_{12} = k_1 \cdot k_2 / (k_1 k_2) \).

To explore the non-linear regime, beyond the tree level, one usually relies on higher-order perturbation theories (e.g., Scoccimarro et al. 1998; Rampf & Wong 2012; Angulo et al. 2015; Hashimoto et al. 2017; Bose & Taruya 2018; Lazanu et al. 2018; Lazanu & Liguori 2018). However, these can be reliable up to the mildly non-linear regime \((k \lesssim 0.2 \, \text{h} \, \text{Mpc}^{-1})\). Another strategy is employing the halo model (e.g., Cooray & Sheth 2002), which is an analytical model assuming that all the matter is confined to halos. This model can be used for a wide range of scale and redshift, but the current accuracy is about 30\% level (e.g., Lazanu et al. 2016; Bose et al. 2019). The last one is a fitting function calibrated by \( N \)-body simulations for various scales, epochs and cosmological models. This is our approach in this paper.

2.2. Previous fitting formulae

Scoccimarro & Couchman (2001) (SC01) provided a fitting formula for the non-linear BS. Their function is similar to the tree level formula (Eq. 2), but replaces the linear PS with a non-linear model and modifies the \( F_2 \) kernel to enhance the amplitude of the BS at small scale. In low-\( k \) limit, their formula is consistent with the tree level. In high-\( k \) limit, the BS is proportional to \( P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1) \) according to the hyper-extended perturbation theory (Scoccimarro & Frieman 1999). Their modified \( F_2 \) kernel contains 6 free parameters, which are fitted by their \( N \)-body result in a range of \( k < 3 \, \text{h} \, \text{Mpc}^{-1} \) at \( 0 < z < 1 \) for four CDM models. Later, Gil-Marín et al. (2012) (GM12) increased the number of free parameters in \( F_2 \) from 6 to 9 and re-calibrated them from their \( N \)-body simulations in a relatively narrow range of wavenumbers, \( k < 0.4 \, \text{h} \, \text{Mpc}^{-1} \), at \( 0 \leq z \leq 1.5 \) for a single \( \Lambda \text{CDM} \) model.

However, these formulae have several shortcomings as follows. (i) A non-linear PS model must be supplied to their formula as a basic building block, such as Halofit (Smith et al. 2003; Takahashi et al. 2012), the halo model (Mead et al. 2013), or COSMIC EMULATOR (Lawrence et al. 2017). Therefore, the user should prepare the PS fitting formula as well. Furthermore, these PS models still exhibit small but non-negligible discrepancies to each other, typically a few percent level (e.g., Schneider et al. 2016), which degrades the BS accuracy. (ii) As indicated by Namikawa et al. (2019), these models overestimate the squeezed BS (i.e., for the configuration of \( k_1 \approx k_2 \gg k_3 \)). (iii) Their fitting range of \( k \) and \( z \) is narrow. The current weak-lensing surveys measure the correlation function down to arcmin scale, which means that we need a calibration up to \( k = 10 \, \text{h} \, \text{Mpc}^{-1} \). In ad-

\[ (z \approx 1 \sim 3) \]
dition, the CMB lensing probes high-redshift structures (\(z \approx 1-3\)), requiring a calibration up to \(z = 10\). (iv) They did not include the baryonic effects which are important at \(k \gtrsim 1 \text{ h Mpc}^{-1}\).

### 2.3. Our fitting formula

We construct a fitting formula based on the halo model in similar to Halo fit for the non-linear PS [Smith et al. 2003]. The halo model has been frequently used to evaluate the multi-point statistics of non-linear density fields (see also Appendix A for detailed discussion about the halo-model BS). This model assumes that all particles are contained in halos and decomposes BS to three terms: one-, two- and three-halo terms (hereafter denoted as 1h, 2h and 3h terms, respectively). The 1h term describes the correlation in an individual halo, and the 2h (3h) term accounts for the correlation among two (three) different halos. The 1h and 3h terms dominate at small and large scales, respectively. On the other hand, the 2h term is usually subdominant in most of the triangle configurations (except for the squeezed case, see e.g., Valageas & Nishimichi 2011, Valageas et al. 2012). Therefore, we drop this term, but instead enhance the 3h term at intermediate scales to absorb the missed 2h contribution.

The fitting function consists of two terms:

\[
B(k_1, k_2, k_3) = B_{1h}(k_1, k_2, k_3) + B_{3h}(k_1, k_2, k_3),
\]

which approaches the tree level formula in low-\(k\) limit. The function contains 52 free parameters in total which will be fitted by our \(N\)-body data. The explicit form of the fitting function is given in Appendix B.

### 3. SIMULATIONS

We use cosmological dark-matter \(N\)-body data set prepared by the DARK EMULATOR simulations [Nishimichi et al. 2019, hereafter N19]. N19 prepared 101 flat cosmological models (a fiducial ΛCDM with additional 100 \(w\)CDM models) in the range of \(z = 0-1.48\). Their purpose is to construct an emulator for several halo observables such as halo-matter correlation function, halo mass function and halo bias for ongoing weak lensing surveys. Their emulator will be publicly available soon.

#### 3.1. Cosmological models

We use the N19 simulations for 41 flat cosmological models\(^8\). The fiducial ΛCDM model is consistent with the Planck 2015 best-fit [Planck Collaboration 2016]: the matter density \(Ω_m = 1 - Ω_λ = 0.3156\), baryon density \(Ω_b = 0.0492\), Hubble constant \(h = 0.6727\), spectral index \(n_s = 0.9645\), and amplitude of matter density fluctuation at the scale of \(8\text{ h}^{-1}\text{ Mpc} \sigma_8 = 0.831\).

The other \(w\)CDM models have 6 cosmological parameters of \(Ω_b h^2, Ω_{cdm} h^2, Ω_\gamma, n_s, w\) and \(w\). Here, the dark energy equation of state, \(w\), is assumed to be constant, and \(A_\gamma\) is the amplitude of the primordial PS. These parameters are distributed around the fiducial model in the range of \(±5\%\) for \(Ω_b h^2\) and \(n_s\), \(±10\%\) for \(Ω_{cdm} h^2\), and \(±20\%\) for \(Ω_\gamma\), in \(A_\gamma\) and \(w\). N19 sampled the cosmological parameters based on a Latin Hypercube Design (e.g., Heitmann et al. 2009). They prepared 5 subsets containing 20 models each. We use their two subsets. Figure 1 shows the distribution of \(w\) and \(σ_8\) vs. \(Ω_m\). The parameter range is wide enough for the current and future weak-lensing surveys.

These are dark-matter only simulations but the free-streaming dumping by massive neutrinos is accounted for in the initial condition\(^9\). In all the models, the neutrino density is fixed to be \(Ω_ν h^2 = 6.4 \times 10^{-4}\), corresponding to 0.06 eV for the total mass. This \(Ω_ν\) is included in \(Ω_m\).

#### 3.2. \(N\)-body simulations

Our simulation setting is summarized in Table 1. We used four box sizes, \(L = 4, 2, 1, 0.2 \text{ h}^{-1}\text{ Gpc}\) (side lengths of the cubic boxes in comoving scale), to cover a wide range of length scales. Note that the simulation volume is large enough to cover almost all measurable triangle configurations of BS in the real universe. The large-volume simulations (> 10 Gpc\(^3\)) are required for a less noisy measurement of BS at small \(k\). The small-box simulations (\(L = 0.2 \text{ h}^{-1}\text{ Gpc}\)) will be used to check the asymptotic behavior at high \(z\). Here, the simulations with \(L = 1\) and 2 \(h^{-1}\) Gpc at \(z = 0-1.48\) are taken from N19, while the others are newly prepared in this work. The largest- and smallest-box simulations are performed to supplement the dynamic range covered by N19. The number of particles is 4096\(^6\) for \(L = 4 \text{ h}^{-1}\text{ Gpc}\) and 2048\(^3\) for the rest. There are dozens of realizations for the fiducial ΛCDM model while there is a single one for each \(w\)CDM model. The initial matter PS is prepared by

\(^8\) Unfortunately, the particle position data was lost for the rest (60) of the models due to hard-disk trouble.

\(^9\) They computed the linear transfer function at \(z = 0\) with the neutrinos and then multiplied it by the linear growth factor squared without the neutrinos to scale back to the initial redshift.
the public Boltzmann code CAMB (Lewis et al. 2000). The initial particle distribution is prepared based on the second order Lagrangian perturbation theory (2LPT; Crocce et al. 2006; Nishimichi et al. 2009) at redshifts $z_{\text{in}} = 31, 29, 59, 99$ for $L = 4, 2, 1, 0.2 \ h^{-1}\ Gpc$, respectively. We slightly change the initial redshifts for the 40 wCDM models because the initial amplitude is slightly different in each model. We then follow the non-linear gravitational evolution using GADGET2 (Springel 2005). The particle snapshots are dumped at 14 redshifts in $z = 0–10$ (see Table 1 for the exact output redshifts).

In order to measure the density contrast, we assign the N-body particles to $1024^3$ regular grid cells in the box. Then, we perform the fast Fourier transform (FFT) to obtain its Fourier transform, $\delta(k)$. Here, we employ the CIC (cloud-in-cell) interpolation with the interfacing scheme (e.g., Jing 2005; Selhusati et al. 2016). We also employ the folding method (Jenkins et al. 1998) to explore smaller scales by using 1/4 and 1/10 times smaller side lengths for $L = 4$ and 1 $h^{-1}\ Gpc$, respectively. The minimum and maximum wavenumbers with $1024^3$ cells are $k_{\text{min}} = 2\pi / L = 6.3 \times 10^{-3} h \ Mpc^{-1} \ [L / (h^{-1}\ Gpc)]^{-1}$ and $k_{\text{max}} = 512^3 k_{\text{min}} = 3.2 h \ Mpc^{-1} \ [L / (h^{-1}\ Gpc)]^{-1}$, respectively. After performing the folding scheme, both $k_{\text{min}}$ and $k_{\text{max}}$ simply become 4 or 10 times larger. The resultant $k_{\text{max}}$ is given in Table 1

3.3. Power spectrum measurement for accuracy check

We measure the matter PS from the simulations and then compare with the previous fitting formula to check the numerical accuracy. The PS estimator is given by,

$$\hat{P}(k) = \frac{1}{N_{\text{mode}}^{\text{PS}}} \sum_{|k'| < k} \left| \delta(k') \right|^2,$$

where the summation is done in a range of $k - \Delta k/2 < |k'| < k + \Delta k/2$ and $N_{\text{mode}}^{\text{PS}}$ is the number of mode within a fixed bin-width of $\Delta \log_{10} k = 0.1$. Figure 2 plots a PS ratio to the revised Halofit prediction (Smith et al. 2003; Takahashi et al. 2012). We plot average $P(k)$ with $1\sigma$ error measured from the realizations. The results from different boxes nicely agree with each other. As the simulation box becomes larger, the measured PS becomes smaller than the Halofit prediction at large $k$ due to the lack of spatial resolution. Here, we do not subtract the shot noise since its contribution is less than $3\%$ at the scales shown in the figure. The simulations agree with the fitting formula within $5\%$ for $k < 1 h^{-1}\ Mpc$ at $z = 0–10$ and for $k < 10 h^{-1}\ Mpc$ at $z = 0–1.5$.

| Table 1: N-body Simulation Parameters |
|---------------------------------------|
| Cosmological model | Box Size $(h^{-1}\ Gpc)$ | Number of particles | Number of realizations | Maximum wavenumber $(h^{-1}\ Mpc)$ | Output redshifts |
|---------------------|--------------------------|---------------------|-----------------------|--------------------------------|-----------------|
| Planck 2015 CDM    | 4                        | 4096$^3$            | 8                     | 2.85                          | 0.55, 1.03, 1.48 & HighZ |
|                     | 2                        | 2048$^3$            | 15                    | 1.42                          | LowZ            |
|                     | 1                        | 2048$^3$            | 21                    | 28.5                          | LowZ & HighZ    |
|                     | 0.2                      | 2048$^3$            | 10                    | 14.2                          | HighZ           |
| 40 wCDM             | 2                        | 2048$^3$            | 1                     | 1.42                          | LowZ            |
|                     | 1                        | 2048$^3$            | 1                     | 28.5                          | LowZ            |

Note: — In the column of output redshifts, LowZ means ten low redshifts: $z = 0, 0.15, 0.31, 0.42, 0.55, 0.69, 0.85, 1.03, 1.23, 1.48$. HighZ is four high redshifts: $z = 2, 3, 5, 10$. The simulations with $L = 1$ and $2 h^{-1}\ Gpc$ at LowZ are taken from Nishimichi et al. (2019), while the others are newly prepared in this work.

3.4. Bispectrum measurement

**Fig. 2.** — Power spectrum ratio of the simulation to the Halofit prediction for the Planck 2015 best-fit CDM model. Each symbol corresponds to each simulation box-size, $L = 4, 2, 1, 0.2 h^{-1}\ Gpc$ (blue, green, black, gray). Here, the shot noise contribution is less than $3\%$. 

[10] The 2LPT can reduce error of the BS estimate induced by transients from initial conditions to less than $2\%$ at $z \leq 1$ (McCullagh et al. 2016).

[11] FFTW (Fast Fourier Transform in the West) is available at http://www.fftw.org
The BS estimator is

$$\hat{B}(k_1, k_2, k_3) = \frac{1}{N_{\text{triangle}}} \sum_{|k_1'| \in k_1} \sum_{|k_2'| \in k_2} \sum_{|k_3'| \in k_3} \delta(k_1') \delta(k_2') \delta(k_3') \delta^K_{k_1 + k_2 + k_3}$$ (6)

where the summation is done for all the modes in the bin, $|k_i'| \in k_i \ (i = 1, 2, 3)$, $N_{\text{triangle}}$ is the number of triangles, and $\delta^K$ is the Kronecker delta. Throughout this paper, we adopt a constant bin-width in log scale, $\Delta \log_k k = 0.1$. Here, we employ the FFT-based quick estimator to calculate Eq. (6) (e.g., Scoccimarro 2015). Using the identity, $\delta^K_{k_1 + k_2 + k_3} = N_{\text{cell}} \sum_x e^{i(k_1' + k_2' + k_3') x}$, Eq. (6) can be reduced to,

$$\hat{B}(k_1, k_2, k_3) = \frac{1}{N_{\text{triangle}} N_{\text{cell}}} \sum_x \left[ \sum_{|k_1'| \in k_1} \delta(k_1') e^{i k_1' x} \times \sum_{|k_2'| \in k_2} \delta(k_2') e^{i k_2' x} \sum_{|k_3'| \in k_3} \delta(k_3') e^{i k_3' x} \right],$$ (7)

where $x$ is the discrete coordinate on the grid, and $N_{\text{cell}} = 1024^3$ is the total number of cells. The summation over $k_i'$ can be easily evaluated with FFT.

We also measure the shot noise as,

$$\hat{B}_\text{sn}(k_1, k_2, k_3) = \frac{1}{n_p} \left[ \hat{P}(k_1) + \hat{P}(k_2) + \hat{P}(k_3) \right] - \frac{2}{n_p^2},$$

where $n_p$ is the particle number density and $\hat{P}(k)$ is the PS estimator including the shot noise.

For the fiducial model, we calculate the average and standard deviation of BS from among the realizations (see Table I for the exact number of realizations). However, since there is only a single realization in each $w$CDM model, the result has a large scatter. Therefore, we will calibrate the fitting formula mainly for the Planck 2015 model and further investigate the other models only supplementarily to check the dependence on the cosmological parameters.

4. FITTING PROCEDURE AND RESULTS

4.1. Fitting to the N-body results

In this section, we present our fitting procedure to the N-body results. The BS fitting function in Eq. (4) contains 52 parameters. Denoting an array of these parameters as $p = (p_1, p_2, \ldots)$, we perform the standard chi-squared analysis to find the best fit:

$$\chi^2_{\text{sim}}(p) = \sum_{z, c, L} \sum_{k_1, k_2, k_3} W_c W_z W_k \times \left[ \frac{B_{\text{bin}}(k_1, k_2, k_3; p) - B_{\text{sim}}(k_1, k_2, k_3)}{\Delta B_{\text{sim}}(k_1, k_2, k_3) + \epsilon(k_1, k_2, k_3)} \right]^2,$$ (8)

where the summation is done for the 41 cosmological models labeled as $c$, all the redshifts $z = 0–10$, the simulations in different box sizes $L$, and triangles $(k_1, k_2, k_3)$ up to $30 h^{-1} \text{Mpc}$. Here, $B_{\text{bin}}$ is the binned prediction of the fitting formula (will be given in Eq. (9)), $B_{\text{sim}}$ is the simulation result, and $\Delta B_{\text{sim}}$ is the standard deviation estimated from the N-body realizations. Since we have only one realization for each of the 40 $w$CDM models, we assume that the relative standard deviations, $\Delta \ln B_{\text{sim}} (\equiv \Delta B_{\text{sim}}/B_{\text{sim}})$, for these models are the same as those for the Planck 2015. We also include a “softening” term $\epsilon = 0.02 \times B_{\text{sim}}$ to reduce an influence of some data points.

![Fig. 3.](image-url) Triangle configurations included in the calibration from the simulations with $L = 4, 2, 1 Gpc/h$ (blue, green and black) and from the SPT (orange). The thick (thin) arrows correspond to particular triangles: $k_{\text{mid}} = k_{\text{max}}$ is the squeezed, $k_{\text{mid}} = k_{\text{max}}/2$ is the flattened, and $k_{\text{mid}} = k_{\text{max}}$ is the equilateral. The arrows in the upper panel indicate the maximum wavenumbers in the calibration from the simulations (blue, green and black) and from the SPT (orange). The thick (thin) arrows are with (without) the folding scheme. All the points satisfy the conditions a) – c) in section 4.1.
having very small $\Delta B_{\text{sim}}$. We introduce weight factors, $W$, to give a greater importance to the data points (i) at lower redshift ($W_z$) because cosmic shear probes low-$z$ structures, (ii) at larger scale ($W_k$) because the unaccounted baryonic effects can play a role at small scale ($k \gtrsim 1\ h\ Mpc^{-1}$), and (iii) at the fiducial cosmological model ($W_1$).

In this analysis, we include all the triangles $(k_1, k_2, k_3)$ satisfying the following three conditions:

a) The relative standard deviation is smaller than 10% (i.e., $\Delta \ln B_{\text{sim}} < 0.1$).

b) The shot-noise contribution is less than 3%.

c) The larger-box simulation gives smaller $B_{\text{sim}}$ at high $k$ due to the lack of spatial resolution (see also Fig. 2 in the PS case). Therefore, if the deviation is larger than 3% with the small statistical error of $\Delta \ln B_{\text{sim}} < 3\%$, we use the smaller-box result only.

The conditions a) and b) exclude data points at very small $k$ and large $k$, respectively. The condition c) has negligible impact on the data selection. Figure 4 plots triangles $(k_1, k_2, k_3)$ satisfying all the above conditions for $L = 1, 2, 4\ h^{-1} \text{Gpc}$ at $z = 0.55$. In a range of $0.1 \lesssim k/(h\ Mpc^{-1}) \lesssim 2$, we can use the simulation results from all the box-sizes, enabling us to give a reliable fit. This figure clearly shows that the simulations cover almost all the triangles up to $k = 3\ h\ Mpc^{-1}$. The lower panel shows a discontinuity at $k_{\text{max}} \approx 3 (0.8\ h\ Mpc^{-1})$ for $L = 1 (4)\ h^{-1} \text{Gpc}$ where the box size effectively changes from $L$ to $L/10 (L/4)$ due to the folding scheme (see also section 3.2). In the bottom panel, triangles in the lower-right part are missing, which means the calibration does not include very squeezed cases ($k_{\text{max}} \gg k_{\text{min}}$). This is because the maximum value of $k_{\text{max}}/k_{\text{min}}$ is 512 which is determined by the number of FFT grids (102413). The use of the folding method does not change this ratio (512). The number of independent triangular bins calibrated by the simulations in each cosmological model and redshift is about 950 at low $z$ ($0 = 0.55, 1$ and 1.48) and 690 at high $z$ ($= 2, 3$ and 5), respectively.

One should consistently bin both the simulation result and the fitting formula when comparing them because BS is sensitive to the binning especially for the squeezed limit (Sefusatti et al. 2014, Namikawa et al. 2019). Throughout this paper, we use the binned fitting formula:

$$B_{\text{bin}}^\text{sim}(k_1, k_2, k_3) = \frac{1}{N_{\text{triangle}}} \int [d^3k_1] \int [d^3k_2] \int [d^3k_3] B(k_1', k_2', k_3') \delta_D(k_1' + k_2' + k_3'),$$

where $B(k_1', k_2', k_3')$ is the un-binned one and the number of triangles is

$$N_{\text{triangle}} = \int [d^3k_1'] \int [d^3k_2'] \int [d^3k_3'] \delta_D(k_1' + k_2' + k_3').$$

(10)

Here, $k_1$ is the weighted mean wavenumber defined as $k_1 = \int [d^3k_1'] k_1' [d^3k_1']^{-1}$. We show how the binning affects BS in Appendix B. Note that, although the un-binned triangle $(k_1', k_2', k_3')$ satisfies the triangle condition (i.e., $|k_1' - k_2'| < |k_3' - k_2'| < |k_1' + k_2'|$), the bin center $(k_1, k_2, k_3)$ does not always satisfy this.

4.2. Fitting to perturbation theory

Since the simulation result is noisy at large scale, we also consider perturbation theory for the calibration on the linear to quasi-linear scales. Here, we use the 1-loop SPT (standard perturbation theory) which includes the tree level and next-to-leading order terms (e.g., Scoccimarro 1997; Scoccimarro et al. 1998). The chi-square is defined analogously to the previous one in Eq. 8:

$$\chi^2_{\text{spt}}(p) = W_{\text{spt}} \sum_{z, c} \sum_{k_1, k_2, k_3} W_z W_c \left(\frac{B_{\text{spt}}(k_1, k_2, k_3; p) - B_{\text{sim}}(k_1, k_2, k_3)}{\Delta B_{(k_1, k_2, k_3) + c(k_1, k_2, k_3)}}\right)^2,$$

(11)

where $B_{\text{spt}}$ is the SPT prediction. Note that we do not need to consider a binning for $B$ or $B_{\text{sim}}$ in this case. We set $\Delta B = 0.5 |B_{\text{spt}} - B_{\text{tree}}|$ and $c = 0.01 \times B_{\text{opt}}$. We also set $W_{\text{spt}} = 0.08$ so as to give more weight to the calibration from simulation$^{14}$ $W_c = 1 (3 \times 10^{-5})$ for the Planck 2015 (otherwise), and $W_z$ is the same as in section 4.1. We include all the triangles $(k_1, k_2, k_3)$ satisfying that $B_{\text{spt}}$ agrees with $B_{\text{tree}}$ within 5% and up to 0.3 $h\ Mpc^{-1}$, and thus the fitting is restricted to large scales. While we can consider any unbinned triangular configurations for the SPT calibration, we use the same bin central values of $(k_1, k_2, k_3)$ as those for the $L = 4\ h^{-1} \text{Gpc}$ simulation with bin-width $\Delta \log_{10} k = 0.1$. Then, the calibration covers a range of $1.6 \times 10^{-3} \leq k/(h^{-1} \text{Mpc}^{-1}) < 0.3$ at $z = 0-10$. Figure 4 shows all the triangles used in the calibration by SPT (orange diamonds). The average number of triangles in each cosmological model and redshift is about 350.

4.3. Result

We introduce the total $\chi^2$,

$$\chi^2(p) = \chi^2_{\text{sim}}(p) + \chi^2_{\text{spt}}(p),$$

(12)

and then numerically search for the best-fitting parameters $p$ by minimizing $\chi^2$. The resulting best-fit model is presented in Appendix B. We use the downhill simplex routine (amoeba) in Numerical Recipes (Press et al. 2002) to find the minimum.

Figure 11 plots the matter BS using the tree level formula, our fitting formula, SC01, or GM12 compared with the simulation results for the Planck 2015 model.
Accurate fitting formula of matter bispectrum

![Equilateral, flattened, squeezed matter bispectrum comparison](image)

**Fig. 4.**—Matter bispectrum comparison of $N$-body simulations and fitting formulas for the Planck 2015 best-fit ΛCDM model. The curves denote theoretical models: our fit (BiHaloff; solid red), Gil-Marín et al. (2012) (GM12; long-dashed orange), Scoccimarro & Couchman (2001) (SC01; short-dashed pink) and the tree level (dotted purple). The symbols denote the simulation results with various box-sizes: $L = 4, 2, 1 h^{-1}$ Gpc (blue triangle, green triangle, black circle) and $200 h^{-1}$ Mpc (gray diamond). Both theories and simulations are consistently binned with $\Delta \log_{10} k = 0.1$. In the vertical axis, the bispectrum is multiplied by $\propto k^n$ ($n = 1, 2$ or $3$) as denoted above the top panels for a clear presentation. Throughout this paper, $k$ and $B(k_1, k_2, k_3)$ have the units of $h$ Mpc$^{-1}$ and $(h^{-1}$ Mpc$)^3$, respectively.
Fig. 5.— Same as Fig. 4 but ratios to the red curves (BiHalofit).
Fig. 6.— Similar to Fig. 5, but ratios to the tree level, focusing on quasi-linear scales. The dot-dashed orange curves are the binned 1-loop SPT prediction.
Fig. 7.— Bispectrum ratios of the simulations to BiHalofit for all triangles for the *Planck* 2015 model. The horizontal axis is a maximum wave number of $(k_1, k_2, k_3)$. Dozens of points are distributed along vertical axis on each $k_{\text{max}}$. Here, horizontal positions for different $L$ are slightly offset for a clear presentation.
at $z = 0–2$. The four panels, from left to right, correspond to particular triangle configurations: the equilateral (i.e., $k_1 = k_2 = k_3$), flattened ($k_1 = 2k_2 = 2k_3$) and two squeezed cases ($k_1 = k_2 \gg k_3$ with $k_3 = 0.045$ and $0.45\, h\, \text{Mpc}^{-1}$). In this and following figures, the simulation data points satisfy all the conditions a) – c) in section 4.1. Here, for SC01 and GM12, we used the measured PS in their fitting formulae to remove the inaccuracy of the PS appearing in these models. Figure 6 plots ratios of the different models and simulations to our fitting formula. As clearly seen, our fitting formula is in good agreement with the simulations over the scales, redshifts, and triangle shapes. On the other hand, the previous formulae over-predict the squeezed BS, which is consistent with the previous finding by Namikawa et al. (2019). This figure also shows agreement among the simulations performed in different box sizes.

Figure 6 shows BS ratios to the tree-level at quasi non-linear scales. On larger scales, both our simulations and fitting formula are consistent with the tree level prediction. The 1-loop SPT slightly over-predicts the BS at quasi non-linear scales at low $z$ ($<1$) (consistent with Fig. 19 of Lazanu et al. 2016). The SPT is more accurate at higher redshifts. For the flattened case, the SPT slightly suppresses the BS at $k \sim 0.1\, h\, \text{Mpc}^{-1}$ and our model captures this trend. Some data points at

**Fig. 8.**—Same as Fig. 7, but for the 40 $\Lambda$CDM models at $z = 0–1.48$. Each color corresponds to each cosmological model shown in Fig. 1. The cyan, gray and magenta points have different $S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$ values. The plus (cross) symbols are simulations with $L = 1\,(2)\, h^{-1}\, \text{Gpc}$. Note that every data point has intrinsic scatter of 10% level because there is a only single realization in each $\Lambda$CDM model.
$k < 0.1 \, h\, \text{Mpc}^{-1}$ are missing because of a large relative error ($> 10\%$).

Figure 7 shows the BS ratios to our formula for all the triangles satisfying the conditions a) – c) in section 4.1. The $x$-axis is the maximum wavenumber $k_{\text{max}}$ and there are $\sim 1800$ data points in each redshift. Our model agrees with the simulations within $10(15)\%$ accuracy up to $k = 3(10) \, h\, \text{Mpc}^{-1}$ at $z = 0$ – 3. At $z = 5 (10)$, the agreement is $20\%$ level up to $k = 3 \, h\, \text{Mpc}^{-1}$. The r.m.s. deviation is $2.7\, (3.2)\%$ up to $k = 3 \, (10) \, h\, \text{Mpc}^{-1}$ at $z = 0$ – 3. We have confirmed that the accuracy does not depend on the bin width by explicitly testing a narrower bin width, $\Delta \log_{10} k = 0.05$. In this case, the r.m.s. deviation is $2.9\, (3.4)\%$ up to $k = 3 \, (10) \, h\, \text{Mpc}^{-1}$ at $z = 0$ – 3, which is quantitatively consistent with those above.

Figure 5 plots the BS ratios to our model for the 40 wCDM models. In this case, we prepare a single realization for each cosmological model and box size ($L = 1$ and $2 \, h^{-1}\, \text{Gpc}$), and therefore the BS measurement has relatively large scatter of typically $10\%$ level. Here, all the data points satisfy the conditions a) – b) in section 4.1. There are huge number of data points ($\sim 5 \times 10^4$) in each redshift. The r.m.s. deviation is $8.0\, (11.2)\%$ up to $k = 3 \, (10) \, h\, \text{Mpc}^{-1}$ at $z = 0$ – 1.5.

To further investigate the cosmological dependence of the accuracy, we divide the models into three groups based on the values of $S_8 = \sigma_8(\Omega_m/0.3)^{0.5}$. The data points shown in Fig. 5 are color coded in the same manner as in Fig. 4. Our formula shows a good agreement within about $20\%$ for the models with $S_8 = 0.6$ – 1.0. However, the agreement is worse for smaller or larger $S_8$ models, because the amplitude of fluctuations ($\sigma_8$) and the linear growth factor are largely different from the Planck 2015 best-fit model. All as the cosmological models converge to the Einstein–de Sitter at high $z$, the fit is better at higher redshifts.

5. BARYONIC EFFECTS

Our $N$-body simulations do not include baryonic processes such as galactic cooling, star formation, supernovae and active galactic nuclei (AGN) feedbacks. As well recognized, baryons significantly affect the non-linear PS at $k \gtrsim 1 \, h\, \text{Mpc}^{-1}$ (e.g., van Daalen et al. 2011; Semboloni et al. 2013; Osato et al. 2015; Hellwing et al. 2018; Chisari et al. 2018, 2019). In this section, we take into account the baryonic effects on the BS fitting formula using state-of-art hydrodynamic simulations, the IllustrisTNG data set (Nelson et al. 2019; Springel et al. 2018). The simulations follow galaxy formation and evolution by incorporating astrophysical processes with a subgrid model. In the IllustrisTNG project, three sets of simulations in different box sizes are carried out with three mass resolutions in each box-size. Here, we use the highest-resolution one in the largest box (referred to as TNG300-1), in which the box-size is $L = 205 \, h^{-1}\, \text{Mpc}$ ($\sim 300 \, \text{Mpc}$) with $2500^3$ dark matter and baryon particles, respectively. Their cosmological model is based on the Planck 2015 best-fit $\Lambda$CDM (Planck Collaboration 2016). They released particle positions and masses of dark matter and baryons (in the form of gas, star and black hole) at $z = 0$ – 20. The IllustrisTNG team also performed dark-matter only (dmo) runs. A comparison between the simulations with and without baryons enables us to single out the impact of baryons on the matter clustering.

We assign all the particle masses to $10^{23}$ grid cells in order to calculate the density contrast and then measure the BS following the same procedure as in section 3.4. We define a BS ratio of the simulations with baryons ($B_b$) to the dmo run ($B_{\text{dmo}}$) as,

$$R_b(k_1, k_2, k_3) = \frac{B_b(k_1, k_2, k_3)}{B_{\text{dmo}}(k_1, k_2, k_3)}.$$  \hspace{1cm} (13)

We measure this ratio at nine redshifts of $z = 0$ – 5 ($z = 0, 0.2, 0.4, 0.7, 1, 1.5, 2, 3, \text{ and } 5$).

Figure 9 plots the ratios for three triangle configurations at $z = 2$. The simulations with and without baryons have the same seed in their initial conditions, which reduces the sample-variability scatter in the ratio at large scales. The baryons clearly suppress the power due to the AGN feedback at $k \sim 10 \, h\, \text{Mpc}^{-1}$, but strongly enhance the amplitude due to gas cooling at high $k (> 10 \, h\, \text{Mpc}^{-1}$). This trend is consistent with the PS (see also Fig. 10). However, the baryons slightly enhance the amplitude by about $10\%$ at intermediate scale ($k \sim 1$ – $10 \, h\, \text{Mpc}^{-1}$) only for low redshifts ($z < 1$). To our knowledge, this small enhancement is not common in PS.

Figure 10 plots the PS ratio with and without baryons measured from the TNG300-1. The circles (crosses) are ratios of the total matter (dark-matter) PS to the dmo run. The small enhancement at intermediate scales appears in the crosses but can not be seen in the circles. This feature was also mentioned in section 3 of Springel et al. (2018). Therefore, the dark-matter component causes this enhancement. We can see that the enhancement of the dark-matter PS appears at almost the same wavenumbers as in the total-matter BS.

While preparing for this paper, Foreman et al. (2019) posted a paper on arXiv about the baryonic effects on BS measured from hydrodynamic simulations (including the TNG300-1). They also found the same trend and gave a clear explanation of what causes this. After the AGN feedback becomes less effective at late time ($z < 1$), the expelled gas re-accretes to a halo. Then the gas contraction affects the dark matter distribution in the halo. As the BS is more sensitive to the dark matter (not baryons) compared to PS (see their section 3.1.1), this enhancement appears only in BS. The baryonic effects on both PS and BS can be used to discriminate the baryonic models (Semboloni et al. 2013; Foreman et al. 2019).

To incorporate the baryonic effect to our BS model, we also construct a fitting function of the ratio, $R_b$, in Eq. (13), shown as the solid red curves in Fig. 9. Its explicit functional form is given in Appendix C. This can fit the measurement within $7.3\, (5.3)\%$ for $k < 10 \, h\, \text{Mpc}^{-1}$ at low (high) redshift, $z = 0$ – 1 (1.5 – 5). The r.m.s. deviation is $1.9\%$ for $k < 30 \, h\, \text{Mpc}^{-1}$ at $z = 0$ – 2. The number of triangles in this fit is about 760 (6800) in each redshift (all the redshifts of $z = 0$ – 5). The user can easily include the baryonic effects by multiplying this ratio by the BS fitting formula. This approach is the same as

http://www.tng-project.org
Fig. 9.— Ratio of bispectrum with to without baryons, $R_b = B_b/B_{b\text{dom}}$ defined in Eq. (13), measured from the TNG300-1. The filled circles are the total-matter (dark matter and baryons) bispectrum divided by that from the dark-matter only run. The red curves are our fit given in Appendix C.

Fig. 10.— Similar to Fig. 9, but ratio of $P(k)$ measured from the TNG300-1. The filled circles (crosses) are the total-matter (dark matter) $P(k)$ divided by that from the dark-matter only run. The red curves are our fit given in Appendix C.

6. COMPARISON WITH WEAK LENSING SIMULATIONS

With the fitting formula for the matter BS calibrated over a wide wavenumber and redshift range, we can make a prediction for the lensing observable by integrating along the line of sight. In this section, we compare our theoretical prediction with weak lensing BS measured from ray-tracing simulations. Here we consider the convergence BS for CMB lensing (section 6.1) and cosmic shear (section 6.2).

The convergence field is a dimensionless integrated matter density along the line-of-sight direction towards the source. The convergence at an angular position $\theta$ for a source distance $r_s$ is given by (e.g., Bartelmann & Schneider 2001),

$$\kappa(\theta) = \int_0^{r_s} dr W(r, r_s) \delta(r\theta, r; z),$$

with the weight function

$$W(r, r_s) = \frac{3H_0^2\Omega_m}{2\pi^2} \frac{r (r_s - r)}{a(r) r_s},$$

where $r (r_s)$ is the comoving distance (to the source) and $a(r)$ is the scale factor. The convergence BS is

$$B_\kappa(\ell_1, \ell_2, \ell_3) = \int_0^{r_s} dr W^3(r, r_s) \frac{\ell_1 \ell_2 \ell_3}{r^4} B \left( \frac{\ell_1}{r}, \frac{\ell_2}{r}, \frac{\ell_3}{r}; z \right),$$

where $\ell_i (= k_i r)$ is the multipole moment and $B(k_1, k_2, k_3; z)$ is the matter BS at $z$. This formula is derived under the flat-sky and the Born approximations. For a higher source redshift, the Born approximation is less accurate and we need the so-called post-Born corrections (Pratten & Lewis 2010). We include these corrections only for CMB lensing (safely neglected for cosmic shear because its contribution is $O(1\%)$, see Fig. 7 of Pratten & Lewis 2010).

The convergence BS is sensitive to lower-$z$ structures compared to the convergence PS for a given source redshift (e.g., see Fig. 4 of Takada & Jain 2002), because the matter BS (PS) evolves in proportional to the forth power (square) of the linear growth factor in the linear regime. Therefore, they can probe different-$z$ structures in a complimentary manner.

6.1. CMB lensing

Namikawa et al. (2019) recently measured the convergence BS from full-sky light-cone simulations (Takahashi et al. 2017). Here, we compare their measurement with theoretical predictions. Takahashi et al. (2017) ran cosmological N-body simulations to reproduce inhomogeneous mass distribution in the universe...
Fig. 11.— Convergence bispectrum measured from simulation maps for CMB lensing. The black symbols are the averages with the standard deviations measured from 108 full-sky maps (Takahashi et al. 2017, Namikawa et al. 2019). Here, the error bars scale as $[(\text{survey area})/(4\pi)]^{-1/2}$. The solid red, dashed orange and dotted pink curves are theoretical predictions based on BiHalofit, GM12, and SC01, respectively. The middle panels plot relative deviations from the red curves: $\Delta B/B = B_\kappa/B_{B}^{\text{BiHalofit}} - 1$. In the bottom panel, these deviations are further divided by the relative standard deviation $(\sigma/B)$.

from the present to the last scattering surface. The cosmological model is consistent with the WMAP 9yr result (Hinshaw et al. 2013). The authors performed a ray-tracing simulation to calculate light-ray paths deflected by the intervening matter. The light-rays are emitted from the position of an observer (at $z = 0$) and trace back the trajectories up to the last scattering surface (for detailed discussion about the ray-tracing scheme, see Shirasaki et al. 2013). They do not use the Born approximation and therefore it includes the post-Born effects. They provide 108 full-sky convergence maps based on the HEALPix pixelization with $N_{\text{side}} = 8192$ (4096), corresponding to a pixel size of 0.48 (0.96) arcmin (Górski et al. 2005). They confirmed that the convergence PS agrees with theoretical prediction by CAMB with the Halofit PS option within 5% at $\ell \leq 2000$ for the high-resolution maps ($N_{\text{side}} = 8192$).

Figure 11 plots the BS measurement from the 108 maps with $N_{\text{side}} = 8192$ (Namikawa et al. 2019). The curves show theoretical predictions including the post-Born correction adopting the WMAP 9yr cosmological model consistently with the simulations. For a fair comparison, both theoretical predictions and simulation results are binned in the same manner with a bin-width $\Delta = 100$. The error bars are for an ideal full-sky measurement (i.e., the cosmic-variance limit) and scale as $[(\text{survey area})/(4\pi)]^{-1/2} \Delta \ell^{-3/2}$ assuming the Gaussian variance. Overall, our fitting formula provides better predictions for the BS of the CMB lensing than the other existing formulæ. In the equilateral case, the analytic and simulated BS agree within $\sim 10\%$ accuracy at most of the angular scales. The difference is within $0.2\sigma$, as shown in the bottom panel. For the flattened case, the ratio (in the middle panel) is far from unity because the BS is around zero at $\ell \gtrsim 1000$. The discrepancy is about $0.2\sigma$ of the cosmic variance. In the squeezed and isosceles configurations, our fitting formula significantly reduces the discrepancy between the simulation and analytical prediction.

Although the accuracy is surely improved, there still remains noticeable discrepancy between our fitting formula and the simulations. One of the reasons would be the finite thickness of the lens planes employed in the ray-tracing simulations which may affect the simulations at $\ell < 200$ (for the same effect on convergence PS, see Fig. 10 of Takahashi et al. 2017). Another reason would be that the flat-sky formula in Eq. (10) becomes inaccurate for large angular scales (the accuracy of the flat-sky approximation in the cosmic-shear PS is discussed in detail by Kilbinger et al. 2013 Kitching et al. 2017. For
example, in the squeezed limit, the minimum multipole is fixed to be \( \ell_s = 50 \), while the discrepancy is mitigated if we choose a larger \( \ell_s \) (Namikawa et al. 2019). Since reducing the finite thickness of lens planes requires more numerically expensive simulations, we will leave the detailed study for our future work.

6.2. Cosmic shear

Let us now turn to the cosmic-shear signal from measurements of galaxy shapes, which probes lower redshifts compared to the CMB lensing. Sato et al. (2009) ran cosmological N-body simulations and subsequently performed a ray-tracing simulation under the flat-sky approximation. Their field of view is somewhat small \((5 \times 5 \text{deg}^2)\) but the number of weak lensing maps \((1000)\) is sufficient for an accurate measurement of BS. Their cosmological model is consistent with the WMAP 3yr result (Spergel et al. 2007). Figure 12 plots the convergence BS at a source redshift \( z_s = 1 \) measured from the 1000 maps by Kayo et al. (2013). Both theory and simulation are binned in the same manner with \( \Delta \log_{10} \ell = 0.13 \). The simulation result is valid within \( 5\% \) up to \( \ell \approx 4000 \) confirmed by a comparison with low- and high-resolution maps (Sato et al. 2009; Valageas et al. 2012). Our fitting formula agrees with the simulation well within \( 10\% \) level up to \( \ell = 4000 \). The deviation at small scale \((\ell \geq 4000)\) is due to the lack of resolution in the simulation.

7. DISCUSSION

7.1. Systematics in CMB lensing

In CMB lensing measurements, the lensing map is reconstructed through mode-mixing of CMB anisotropies induced by lensing (Hu & Okamoto 2002). Therefore, any other sources of mode-mixing could bias lensing measurements and thus the BS of CMB lensing. For example, masking, inhomogeneous noise, beam and point sources are potential sources of the bias (Hanson et al. 2003; Namikawa et al. 2013). In addition to the instrumental uncertainties, extra-galactic foregrounds such as the thermal Sunyaev-Zel’dovich effect and cosmic infrared background (Osborne et al. 2014; van Engelen et al. 2014; Madhavacheril & Hill 2018) and its lensing (Mishra & Schaan 2019) could lead to the bias in CMB lensing measurements. The calibration uncertainties of the CMB map is also an important systematic because the measured BS depends on the sixth power of the map calibration uncertainties if the quadratic estimator is used for lensing reconstruction. This dependence is stronger than that of the lensing PS which depends on the forth power of these uncertainties. However, the bias from the above instrumental uncertainties and astrophysical sources would be constrained by combining the BS and PS of CMB lensing since these spectra have a different dependence on these uncertainties. A joint analysis of PS and BS would be thus crucial for a robust cosmological analysis in future CMB experiments.

7.2. Intrinsic alignment

The intrinsic alignment (IA) of galaxies is one of major systematics in cosmic shear (for a review, see e.g., Troxel & Ishak 2013; Joachimi et al. 2017). A massive structure near the source galaxy induces a shape distortion due to the tidal force, which becomes a contamination to lensing signal. The IA contamination is about \( 10\% \) level for the cosmic-shear BS (Semboloni et al. 2008). Several authors proposed a method to mitigate or remove this contamination from the signal (Shi et al. 2010; Troxel & Ishak 2012). Combining lensing PS and BS can provide a strong constraint on IA model as well as cosmological parameters.

7.3. Bispectrum covariance

We have so far discussed the modeling of BS, but its covariance is also an important ingredient for a cosmological likelihood analysis. For Gaussian fluctuations, the BS covariance has a simple form given by the PS and the shot noise (Sefusatti et al. 2006). However, in the non-linear regime, one should consider the non-Gaussian and super-sample contributions (e.g., Takada & Hu 2013) and therefore its evaluation becomes much more difficult. In that case, several authors estimated the covariance via perturbation theory (e.g., Sugiyama et al. 2019), the halo model (e.g., Kayo et al. 2013; Rizzato et al. 2018) and an ensemble of simulation mocks (e.g., Sato & Nishimichi 2013; Chan & Blot 2017; Chan et al. 2018; Colavincenzo et al. 2019). In the last approach, the number of mocks should be larger than a number of \( k \)-bins to estimate an unbiased inverse covariance (e.g., Hartlap et al. 2007), and therefore one requires a huge number of mocks \((> 10^{2-3})\). Anyway, we leave this topic for future work.

7.4. Emulator

Several groups recently have been developing a non-linear PS emulator which interpolates simulation results in a wide range of wavenumber, redshift, and cosmological models (Lawrence et al. 2017; Garrison et al. 2018; Nishimichi et al. 2018; Knabenhans et al. 2019; DelRose et al. 2019). We expect that it is probably much more difficult to construct a similar emulator for BS. One reason is that we measure a binned BS while we need an unbinned one (note again that BS is sensitive to binning). Therefore, we cannot simply interpolate the measured quantities. The other reason is that a BS measurement is quite noisy due to the larger sample variance compared to PS, which means many realizations are needed in each cosmological model. This is computationally expensive.

8. CONCLUSION

We have constructed a fitting formula of the matter BS calibrated by high-resolution N-body simulations for 41 \( \Lambda \)CDM models around the Planck 2015 best-fit \( \Lambda \)CDM model. We also include a calibration from perturbation theory at large scale \((k < 0.3 \text{h}\text{Mpc}^{-1})\). Our formula can be used for a wide range of wavenumbers \((up to k = 30 \text{h}\text{Mpc}^{-1})\) and redshifts \((z = 0–10)\). The simulation boxes are large enough \((L = 1, 2, \text{and} 4 \text{h}^{-1}\text{Gpc} \text{on a side length})\) to cover almost all triangles \((k_1, k_2, k_3)\) measured by the forthcoming weak-lensing surveys and CMB lensing experiments. It achieves an accuracy of 10\((15\%\)) up to \( k = 3(10) \text{h}\text{Mpc}^{-1} \) at \( z = 0–3 \) for the Planck 2015 model. The accuracy for the 40 \( \Lambda \)CDM models is about 20\% level for \( k < 3 \text{h}\text{Mpc}^{-1} \) at \( z = 0–1.5 \).
though the simulation data have 10% level intrinsic scatter. The user can easily incorporate the baryonic effects, calibrated from the IllustrisTNG, in the fitting formula. We also confirm that the formula reproduces the weak-lensing convergence BS measured from light-cone simulations.

Recently, larger $\sigma_s \simeq 0.81$ is inferred from the Planck results while cosmic shear and galaxy-galaxy lensing prefer lower $\sigma_s \simeq 0.77$ (the $\sigma_s$ tension, e.g., MacCrann et al. 2018; Abbott et al. 2018; Planck Collaboration 2018). Combining weak lensing PS and BS can improve the $\sigma_s$ constraint by a factor of 1.6–3 (e.g., Takada & Jain 2004; Kavv & Takada 2013) and would give a new clue to solve this issue.

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APPENDIX

A. HALO MODEL

The halo model has been widely used to evaluate the non-linear BS (e.g., Cooray & Hu 2001; Cooray 2002; Valageas & Nishimichi 2011; Kavv et al. 2013; Yamamoto et al. 2017). This model assumes that all matter is confined in halos. The basic halo properties are characterized by the mass function, $dn(M)/dM$, the spherical density profile, $\rho(r; M)$, and the first- and second-order halo biases, $b_{1,2}(M)$, for a given mass $M$. In this model, the matter BS is decomposed into three terms: one- ($1h$), two- ($2h$), and three-halo ($3h$) terms. The $2h$ term fills the gap between the $1h$ and $3h$ terms and gives a minor contribution at intermediate scale except for the squeezed limit. By ignoring this term, the BS is given as

$$B(k_1, k_2, k_3) = B_{1h}^{HM}(k_1, k_2, k_3) + B_{3h}^{HM}(k_1, k_2, k_3).$$  \hspace{1cm} (A1)

The $1h$ and $3h$ terms dominate at small and large scales, respectively. The $1h$ term comes from the density profile of a single halo:

$$B_{1h}^{HM}(k_1, k_2, k_3) = \int dM \frac{dn(M)}{dM} \left( \frac{M}{\rho} \right)^3 u(k_1; M) u(k_2; M) u(k_3; M).$$ \hspace{1cm} (A2)
where $\bar{\rho}$ is the cosmic mean density and $u(k; M)$ is the Fourier transform of the scaled density profile, $\rho(r; M)/M$. The 3h term comes from a spacial correlation among three different halos:

$$B_{3h}^{\text{HM}}(k_1, k_2, k_3) = I_1^3(k_1)I_1^3(k_2)I_1^3(k_3)B_{\text{tree}}(k_1, k_2, k_3) + \left[ I_1^3(k_1)I_1^3(k_2)I_1^3(k_3)P_L(k_1)P_L(k_2) + 2 \text{ perm.} \right]$$

$$= 2 \left[ F_2(k_1, k_2) + I_2^3(k_3) \right] I_1^3(k_1)I_1^3(k_2)I_1^3(k_3)P_L(k_1)P_L(k_2) + 2 \text{ perm.} \quad \text{(A3)}$$

with

$$I_1^3(k) = \int dM \frac{dn(M)}{dM} \frac{M}{\bar{\rho}} b_3(M)u(k; M). \quad \text{(A4)}$$

### B. FITTING FORMULA

Our fitting formula adopts the Halofit parameterization for non-linear PS [Smith et al. 2003]. The dimensionless linear power spectrum is defined as $\Delta^2_P(k) = k^3P_L(k)/(2\pi^2)$. The non-linear scale $k_{\text{NL}}$ is determined via

$$\sigma^2(k_{\text{NL}}^{-1}) = 1, \quad \sigma^2(R) = \int d\ln k \Delta^2_P(k) e^{-k^2R^2}. \quad \text{(B1)}$$

The effective spectral index at $k_{\text{NL}}$ is defined as

$$n_{\text{eff}} + 3 = -\frac{d\ln \sigma^2(R)}{d\ln R} \bigg|_{R=k_{\text{NL}}^{-1}}. \quad \text{(B2)}$$

We also introduce a scaled wavenumber, $q_i$, defined as $q_i = k_i/k_{\text{NL}}$ ($i = 1, 2$ and 3). Note that these quantities, $k_{\text{NL}}$ and $n_{\text{eff}}$, are evaluated for a given redshift. These parameters are identical to those defined in [Smith et al. 2003].

The fitting function is the sum of 1h and 3h terms:

$$B(k_1, k_2, k_3) = B_{1h}(k_1, k_2, k_3) + B_{3h}(k_1, k_2, k_3). \quad \text{(B3)}$$

The 1h term is

$$B_{1h}(k_1, k_2, k_3) = \prod_{i=1}^3 \left[ \frac{1}{\sigma_n q_i^{\alpha_n} + b_n q_i^{\beta_n} + (c_n q_i^{-\gamma_n})} \right]. \quad \text{(B4)}$$

Here, we assume that $B_{1h}$ is a product of identical functions of $q_1, q_2$ and $q_3$ similar to the halo model, $B_{1h}^{\text{HM}}$, in Eq. (A2), which is given by a product of $u(k_i)$. The 3h term is

$$B_{3h}(k_1, k_2, k_3) = 2 \left[ F_2(k_1, k_2) + d_n q_3 \right] I(k_1)I(k_2)I(k_3)P_E(k_1)P_E(k_2) + 2 \text{ perm.}, \quad \text{(B5)}$$

with

$$P_E(k) = \frac{1 + f_n q^2}{1 + g_n q + h_n q^2} P_L(k) + \frac{1}{m_n q^{\mu_n} + n_n q^{\nu_n} + (p_n q)^{\pi_n}}, \quad I(k) = \frac{1}{1 + c_n q}. \quad \text{(B6)}$$

Here, $P_E(k)$ is an “enhanced” PS which is the linear PS with adding a small-scale enhancement. The first (second) term of $P_E$ is similar to the 2h (1h) term in Halofit for the non-linear PS. Similarly, $I(k)$ and $d_n q$ correspond to $I_1^3(k)$ and $I_2^3(k)/[2I_1^3(k)]$, respectively, in the halo model. This 3h term approaches the tree level in the low $k$ limit.

The above fitting parameters $(\alpha_n, \beta_n, \ldots)$ are given as polynomials in terms of $n_{\text{eff}}$ and $\log_{10} \sigma_8$. Here, $\sigma_8$ is the spherical overdensity with a radius of 8 $h^{-1}$ Mpc at redshift $z$ (= $\sigma_8(z = 0)$ multiplied by the linear growth factor). The parameters determining the amplitude (i.e., $a_n, b_n, m_n$ and $n_n$) are given as functions of $\log_{10} \sigma_8$, while the most of others are functions of $n_{\text{eff}}$. Since $B(k_1, k_2, k_3)$ and $P(k)$ have the dimensions of [(Length)$^6$] and [(Length)$^3$], respectively, the parameters $a_n$ and $b_n$ have [(Length)$^{-2}$], $m_n$ and $n_n$ have [(Length)$^{-3}$], and the others are dimensionless. Though [Length] is an arbitrary length unit, one may choose $[h^{-1}$ Mpc] or [Mpc].

The fitting parameters of the 1h term are

$$\log_{10} \alpha_n = -2.167 - 2.944 \log_{10} \sigma_8 - 1.106 (\log_{10} \sigma_8)^2 - 2.865 (\log_{10} \sigma_8)^3 - 0.310 r_1^{\alpha_n},$$

$$\log_{10} \beta_n = -3.428 - 2.681 \log_{10} \sigma_8 + 1.624 (\log_{10} \sigma_8)^2 - 0.095 (\log_{10} \sigma_8)^3,$$

$$\log_{10} \gamma_n = 0.159 - 1.107 n_{\text{eff}},$$

$$\log_{10} \sigma_n = \min \left[ -4.348 - 3.006 n_{\text{eff}} - 0.5745 n_{\text{eff}}^2 + 10^{-0.9+0.2 n_{\text{eff}}} r_2^2, \log_{10} \left( 1 - \frac{2}{3} n_{\text{eff}} \right) \right],$$

$$\log_{10} \beta_n = -1.731 - 2.845 n_{\text{eff}} - 1.4995 n_{\text{eff}}^2 - 0.2811 n_{\text{eff}}^3 + 0.007 r_2,$$

$$\log_{10} \gamma_n = 0.182 + 0.570 n_{\text{eff}}. \quad \text{(B7)}$$
where $r_{1,2}$ are ratios of the minimum ($k_{\text{min}}$), middle ($k_{\text{mid}}$), and maximum ($k_{\text{max}}$) wavenumbers of the triangle:

$$r_1 = \frac{k_{\text{min}}}{k_{\text{max}}}, \quad r_2 = \frac{k_{\text{mid}} + k_{\text{min}} - k_{\text{max}}}{k_{\text{max}}}.$$  \hspace{1cm} (B8)

These $r_{1,2}$ terms effectively include a “halo triaxiality” in the 1h term (Smith et al. 2006): $r_{1,2} \to 0$ for the squeezed case ($k_{\text{min}} \ll k_{\text{mid}} \simeq k_{\text{max}}$), $r_1 (r_2) \to 0.5 \ (0)$ for the flattened ($k_{\text{min}} \simeq k_{\text{mid}} \simeq k_{\text{max}}/2$), and $r_{1,2} \to 1$ for the equilateral ($k_{\text{min}} \simeq k_{\text{mid}} \simeq k_{\text{max}}$). These terms slightly enhance (suppress) the squeezed (equilateral) BS at $z \gtrsim 5 \, h \, \text{Mpc}^{-1}$. We set the maximum $\alpha_n$ value to $\alpha_{n,\text{max}} = 1 - (2/3) n$, where $n_a$ is the spectral index of initial PS) so that the 1h term should be less than the tree-level in the low-$k$ limit.

The parameters of the 3h term are

$$\log_{10} f_n = -10.533 - 16.838 \, n_{\text{eff}} - 9.3048 \, n_{\text{eff}}^2 - 1.8263 \, n_{\text{eff}}^3, $$
$$\log_{10} g_n = 2.787 + 2.405 \, n_{\text{eff}} + 0.4577 \, n_{\text{eff}}^2,$$
$$\log_{10} h_n = -1.118 - 0.394 \, n_{\text{eff}}, $$
$$\log_{10} m_n = -2.605 - 2.434 \log_{10} \sigma_8 + 5.710 \, (\log_{10} \sigma_8)^2,$$
$$\log_{10} n_n = -4.468 - 3.080 \log_{10} \sigma_8 + 1.035 \, (\log_{10} \sigma_8)^2,$$
$$\log_{10} \mu_n = 15.312 + 22.977 \, n_{\text{eff}} + 10.9579 \, n_{\text{eff}}^2 + 1.6586 \, n_{\text{eff}}^3,$$
$$\log_{10} \nu_n = 1.347 + 1.246 \, n_{\text{eff}} + 0.4525 \, n_{\text{eff}}^2,$$
$$\log_{10} p_n = 0.071 - 0.433 \, n_{\text{eff}}, $$
$$\log_{10} q_n = -0.483 + 0.892 \log_{10} \sigma_8 - 0.086 \, n_{\text{eff}}, $$
$$\log_{10} r_n = -0.632 + 0.646 \, n_{\text{eff}}. $$  \hspace{1cm} (B9)

Here, $\Omega_m$ is the matter density parameter at $z$. Since the calibration is done up to $z = 10$, the formula should be switched to the tree-level at $z > 10$.

Figure 14 plots $B_{1h}$ and $B_{3h}$ separately at $z = 0.55$ for the Planck 2015. Here, these are unbinned results. As clearly seen, for the equilateral and flattened cases, the 1h (3h) term dominates at small (large) scale. For the squeezed case, the 1h (3h) term dominates if $k_3$ is in the non-linear (linear) regime.

Figure 14 shows a binning effect on BS. As shown clearly, the squeezed BS is sensitive to the binning. This is because the cosine term in the $F_2$ kernel is very sensitive to the squeezed triangle configuration (for detailed discussion, see section II B of Namikawa et al. 2019).

### C. FITTING TO THE RATIO OF BISPECTRUM WITH TO WITHOUT BARYONS

In this Appendix, we present a fitting function of ratio of BS with to without baryons, $R_b$, defined in Eq. (B9), calibrated with the TNG300-1 simulation (Nelson et al. 2019). We include the triangle configurations $(k_1, k_2, k_3)$ satisfying the following two conditions: a) the number of triangles in the bin is larger than $10^6$ in order to remove noisy data points and b) the shot noise contribution is less than 3%. This fitting range is from $k = 0.03$ to $100 \, h \, \text{Mpc}^{-1}$ at $z = 0 - 5$.

We fit the BS ratio $R_b$ as

$$R_b(k_1, k_2, k_3) = \prod_{i=1}^{3} A_0 \exp \left\{ - \frac{|x_i - \mu_0|}{\sigma_0} \right\}^\alpha - A_1 \exp \left\{ - \frac{(x_i - \mu_1)}{\sigma_1} \right\}^2 + \left\{ \frac{k_1}{k_3} \right\}^{\alpha_2}+1 \right\}^{\beta_2}, \hspace{1cm} (C1)$$

Fig. 13.— One (1h) and three-halo (3h) term contributions to the total matter bispectrum in the fitting formula.

where $r_{1,2}$ are ratios of the minimum ($k_{\text{min}}$), middle ($k_{\text{mid}}$), and maximum ($k_{\text{max}}$) wavenumbers of the triangle:
where $x_i = \log_{10}[k_i/(h \text{ Mpc}^{-1})]$. These fitting parameters are given in terms of the scale factor $a$:

\begin{equation}
\begin{aligned}
A_0 &= 0.068 (a - 0.5)^{0.47} \Theta(a - 0.5), \\
\mu_0 &= 0.018 a + 0.837 a^2, \\
\sigma_0 &= 0.881 \mu_0, \\
\alpha_0 &= 2.346, \\
A_1 &= 1.052 (a - 0.2)^{1.41} \Theta(a - 0.2), \\
\mu_1 &= 0.172 + 3.048 a - 0.675 a^2, \\
\sigma_1 &= (0.494 - 0.039 a) \mu_1, \\
k_* &= 29.90 - 38.73 a + 24.30 a^2, \\
\alpha_2 &= 2.25, \\
\alpha_2 \beta_2 &= \frac{0.563}{(a/0.06)^{0.92} + 1},
\end{aligned}
\end{equation}

where $k_*$ has the unit of $h \text{ Mpc}^{-1}$ and $\Theta(x)$ is the step function; $\Theta(x) = 1$ and 0 for $x \geq 0$ and $x < 0$, respectively. The first term of Eq. (C1) represents the small enhancement at intermediate scale ($k \approx 1 - 10 h \text{ Mpc}^{-1}$) at low $z (< 1$), the second term is the depression at $k \approx 1 h \text{ Mpc}^{-1}$, and the last term is the strong enhancement at high $k (\gtrsim 10 h \text{ Mpc}^{-1})$. The ratio $R_b$ approaches unity in the low-$k$ limit.

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