Research Article

The Fractional View Analysis of Polytropic Gas, Unsteady Flow System

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1.Introduction

In recent years, nonlinear fractional partial differential equations (FPDEs) have attracted researchers because of their useful applications in science and engineering [1–3]. The analysis of exact solutions to these nonlinear PDEs plays a very significant role in the Soliton theory since much of the information are provided on the description of the physical models, in the transmission of electrical signals, as a standard diffusion-wave equation, the transfer of neutrons by nuclear reactor, the theory of random walks, and so on [4–14].

In recent decades, many researchers have used different approaches to analyze the solutions of nonlinear PDEs, such as Laplace transform [15], Akbari–Ganjii’s method [16], homotopy analysis method [17], lattice Boltzmann method [18, 19], volume of fluid method [20, 21], Laplace homotopy analysis method [22, 23], Adomian decomposition technique [24–27], the variational iteration technique [28], Adams–Bashforth–Moulton algorithm [29], homotopy perturbation Sumudu transform method [30], the tanh method [31], the sinh-cosh method [32], finite difference method [33], the homotopy perturbation method [34], and
the Laplace decomposition technique, to handle fractional-order Zakharov–Kuznetsov equations [35].

In the present study, we consider the gas-dynamic equations fractional-order scheme describing the evolution of an ideal gas’s two-dimensional unsteady flow. The polytropic gas in astrophysics is given as follows [36]:

\[
\psi = k\omega^{1+(1/m)},
\]

where \(\psi = (\theta/\phi)\) is the energy density, \(\phi\) is the container volume, \(\theta\) is the total energy of the gas, \(m\) is the polytropic index, and \(k\) is a constant. Degenerate adiabatic gas and electron gas are two instances of such gases. In astrophysics and cosmology, the analysis of polytropic plays a critical role, and these gases can perform like dark energy [37]. Now consider the gas-dynamic equations scheme, which describes the evolution of unstable flow of a perfect gas with fractional derivatives [36, 38]:

\[
\begin{align*}
D^\delta_{\eta}\psi + \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \omega} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} &= 0, \\
D^\delta_{\zeta}\psi + \frac{\partial \psi}{\partial \omega} + \frac{\partial \psi}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} &= 0,
\end{align*}
\]

with initial conditions

\[
\begin{align*}
\mu(\xi, \zeta, 0) &= \alpha(\xi + \zeta), \\
\nu(\xi, \zeta, 0) &= \beta(\xi + \zeta), \\
\omega(\xi, \zeta, 0) &= \gamma(\xi + \zeta), \\
\psi(\xi, \zeta, 0) &= \Phi(\xi + \zeta),
\end{align*}
\]

where \(\mu(\xi, \zeta, \eta)\) and \(\nu(\xi, \zeta, \eta)\) are the velocity components, \(\omega(\xi, \zeta, \eta)\) is the density, \(\psi(\xi, \zeta, \eta)\) is the pressure, and \(r\) is the ratio of the specific heat and it represents the adiabatic index. In past decade, the appropriate analytical results of several distinct types of gas-dynamic equations are achieved using many analytical and numerical methods. Various methods have been solved by a gas-dynamic model such as fractional reduced differential transform method [39], Elzaki transform homotopy perturbation method [40], \(q\)-homotopy analysis method [36], Adomian decomposition method [41], fractional homotopy analysis transform method [38], and natural decomposition method [42].

The variational iteration transform method (VITM) combines the variational iteration method and the Shehu transform. Many researchers commonly used this technique to solve linear and nonlinear models [43–45]. The method provides a reliable and effective procedure for a broad range of science. VIM does not need discretization, linearization, or perturbation. It provides quick convergence and successive approximations of the exact result [46–48]. Various equations solve the variational iteration method with the help of different transformations, such as Kuramoto–Sivashinsky equations [49] and fourth-order parabolic partial differential equation [45].

The ADM is an efficient and accurate technique that was suggested initially to solve analytically frontier physical models [50]. Since then, ADM has been implemented in nonlinear ODEs and PDEs without using perturbation or linearization procedure. The Shehu decomposition method (SDM) is a mixture of ADM and Shehu transform [51–54].

The motivation and novelty of the current research work are to modify the ADM and VIM along with Shehu transformation to investigate the solution of a nonlinear system of nonlinear FPDEs of unsteady flow of polytropic gas-dynamics equations. Besides the nonlinear system of four equations, the given problem’s solution is calculated by an effortless and straightforward procedure. A higher degree of accuracy is achieved with a tiny number of calculations. The fractional-order solutions are achieved with some graphical justifications. The visual representation has confirmed the effectiveness and applicability of the suggested techniques. In the future, the proposed techniques are preferred to solve other nonlinear FPDEs that frequently arise in science and engineering.

### 2. Preliminaries Concepts

#### 2.1. Definition 1.

The Riemann–Liouville fractional integral is given as follows [55, 56]:

\[
I^\delta_{\eta}h(\eta) = \frac{1}{\Gamma(\delta)} \int_{0}^{\eta}(\eta-s)^{\delta-1}h(s)ds, \quad \delta > 0, \eta > 0,
\]

#### 2.2. Definition 2.

The Caputo’s fractional-order derivative of \(h(\eta)\) is defined as follows [55, 56]:

\[
D^\delta_{\eta}h(\eta) = I^{n-\delta}\int_{\eta}^{0}f^n, \quad n-1 < \delta < n, \quad n \in \mathbb{N}
\]

#### 2.3. Definition 3.

The Shehu transformation is the new and modern transformation which is described for exponential-order functions. In set \(A\), we take a function represented as follows [51, 52, 57, 58]:

\[
A = \{\nu(\eta); \exists, \rho_1, \rho_2 > 0, |\nu(\eta)| < Me^{-\eta/\rho_2}, \quad \text{if} \ \eta \in [0, \infty).
\]

The Shehu transform which is given by \(S(.)\) for a function \(\nu(\eta)\) is defined as

\[
S[\nu(\eta)] = V(s,\mu) = \int_{0}^{\infty}e^{-(s/\rho_2)}\nu(\eta)d\eta,
\]

\(\eta > 0, \ s > 0.\)
The Shehu transform of a function \( v(\eta) \) is \( V(s, \mu) \), and then, \( v(\eta) \) is called the inverse of \( V(s, \mu) \), which is given as
\[
S^{-1} \{ V(s, \mu) \} = v(\eta), \text{ for } \eta \geq 0,
\]
\( S^{-1} \) is inverse Shehu transformation.

2.4. Definition 4. Shehu transform for \( n \)th derivatives is given as follows [51, 52, 57, 58]:
\[
S\{v^{(n)}(\eta)\} = \frac{s^n}{u^n} V(s, u) - \sum_{k=0}^{n-1} \left( \frac{s}{u} \right)^{n-k-1} v^{(k)}(0).
\]

2.5. Definition 5. Shehu transform for fractional-order derivatives [51, 52, 57, 58]:
\[
S\{v^\beta(\eta)\} = \frac{s^\beta}{u^\beta} V(s, u) - \sum_{k=0}^{\beta-1} \left( \frac{s}{u} \right)^{\beta-k-1} v^{(k)}(0),
\]
\[ 0 < \beta \leq n. \]

2.6. Definition 6. The Mittag–Leffler function denoted by \( E_\beta(z) \) for \( \beta > 0 \) is defined as
\[
E_\beta(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\beta m + 1)} \delta > 0, \quad z \in \mathbb{C}. \]

3. The Procedure of VITM
This section describes the VITM solution, the system of FPDEs:
\[
\begin{align*}
D^\beta_\eta \mu(\xi, \zeta, \eta) + \mathcal{G}_1(\mu, v) + \mathcal{N}_1(\mu, v) - \mathcal{G}_1(\xi, \zeta, \eta) &= 0, \\
D^\beta_\eta v(\xi, \zeta, \eta) + \mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v) - \mathcal{G}_2(\xi, \zeta, \eta) &= 0,
\end{align*}
\]
with initial conditions
\[
\begin{align*}
\mu(\xi, \zeta, 0) &= g_1(\xi, \zeta), \\
v(\xi, \zeta, 0) &= g_2(\xi, \zeta),
\end{align*}
\]
where \( D^\beta_\eta = \frac{\partial^\beta}{\partial \eta^\beta} \) is the Caputo fractional derivative of order \( \beta \); \( \mathcal{G}_1, \mathcal{G}_2, \mathcal{N}_1, \mathcal{N}_2 \) are linear and nonlinear functions, respectively; and \( \mathcal{G}_1, \mathcal{G}_2 \) are source operators.

The Shehu transformation is applied to equation (1):
\[
\begin{align*}
S \left[ D^\beta_\eta \mu(\xi, \zeta, \eta) \right] + S \left[ \mathcal{G}_1(\mu, v) + \mathcal{N}_1(\mu, v) \right] - S \left[ \mathcal{G}_1(\xi, \zeta, \eta) \right] &= 0, \\
S \left[ D^\beta_\eta v(\xi, \zeta, \eta) \right] + S \left[ \mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v) \right] - S \left[ \mathcal{G}_2(\xi, \zeta, \eta) \right] &= 0.
\end{align*}
\]
Applying the differentiation property of Shehu transform, we have
\[
\begin{align*}
S \left[ \mu(\xi, \zeta, \eta) \right] &= \frac{\sum_{k=0}^{m-1} \frac{s^{\beta-k-1}}{u^{\beta-k}} \frac{\partial^k \mu(\xi, \zeta, \eta)}{\partial \eta^k} \bigg|_{\eta=0}}{u^\beta}, \\
S \left[ v(\xi, \zeta, \eta) \right] &= \frac{\sum_{k=0}^{m-1} \frac{s^{\beta-k-1}}{u^{\beta-k}} \frac{\partial^k v(\xi, \zeta, \eta)}{\partial \eta^k} \bigg|_{\eta=0}}{u^\beta}.
\end{align*}
\]
The iteration method for equation (15) may be utilized to indicate the major iterative scheme requiring the Lagrange multiplier as
\[
\lambda(s) \left[ S \left[ \mathcal{G}_1(\mu, v) + \mathcal{N}_1(\mu, v) \right] - S \left[ \mathcal{G}_1(\xi, \zeta, \eta) \right] \right],
\]
\[
\lambda(s) \left[ S \left[ \mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v) \right] - S \left[ \mathcal{G}_2(\xi, \zeta, \eta) \right] \right].
\]
A Lagrange multiplier
\[
\lambda(s) = \frac{u^\delta}{s^\delta},
\]
using inverse Shehu transformation \( S^{-1} \), and equation (16) can be written as
\[
\mu_{m+1}(\xi, \zeta, \eta)
\]
\[
\begin{align*}
\mu_{m+1}(\xi, \zeta, \eta) &= \mu_m(\xi, \zeta, \eta) - S^{-1} \left[ \frac{s^\delta}{u^\delta} \sum_{k=0}^{m-1} \frac{s^{\beta-k-1}}{u^{\beta-k}} \frac{\partial^k \mu(\xi, \zeta, \eta)}{\partial \eta^k} \bigg|_{\eta=0} \right], \\
\mathcal{G}_1(\mu, v) + \mathcal{N}_1(\mu, v), \\
\mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v), \\
\mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v),
\end{align*}
\]
\[
\begin{align*}
\mu_{m+1}(\xi, \zeta, \eta) &= \mu_m(\xi, \zeta, \eta) - S^{-1} \left[ \frac{s^\delta}{u^\delta} \sum_{k=0}^{m-1} \frac{s^{\beta-k-1}}{u^{\beta-k}} \frac{\partial^k \mu(\xi, \zeta, \eta)}{\partial \eta^k} \bigg|_{\eta=0} \right], \\
\mathcal{G}_1(\mu, v) + \mathcal{N}_1(\mu, v), \\
\mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v), \\
\mathcal{G}_2(\mu, v) + \mathcal{N}_2(\mu, v),
\end{align*}
\]
The initial value can be found as
\begin{equation}
\mu_0(\xi, \zeta, \eta) = S^{-1}\left[ \frac{\delta}{s} \left\{ \sum_{k=0}^{m-1} \frac{\delta^k \mu(\xi, \zeta, \eta)}{\delta^k \eta} \big|_{\eta=0} \right\} \right],
\end{equation}

\begin{equation}
\nu_0(\xi, \zeta, \eta) = S^{-1}\left[ \frac{\delta}{s} \left\{ \sum_{k=0}^{m-1} \frac{\delta^k \nu(\xi, \zeta, \eta)}{\delta^k \eta} \big|_{\eta=0} \right\} \right].
\end{equation}

(19)

The converge of this technique is as follows [59, 60].

4. The Procedure of SDM

In this section, we discuss the SDM solution for system of FPDEs:

\begin{align*}
D_\eta^\delta \mu(\xi, \zeta, \eta) + \mathcal{G}_1(\mu, \nu) + N_1(\mu, \nu) - \mathcal{F}_1(\xi, \zeta, \eta) &= 0, \\
D_\eta^\delta \nu(\xi, \zeta, \eta) + \mathcal{G}_2(\mu, \nu) + N_2(\mu, \nu) - \mathcal{F}_2(\xi, \zeta, \eta) &= 0,
\end{align*}

with initial conditions

\begin{align*}
\mu(\xi, \zeta, 0) &= g_1(\xi, \zeta), \\
\nu(\xi, \zeta, 0) &= g_2(\xi, \zeta),
\end{align*}

(20)

where \( D_\eta^\delta = (\partial^\delta / \partial \eta^\delta) \) is the Caputo fractional derivative of order \( \delta \); \( \mathcal{G}_1, \mathcal{G}_2 \) and \( N_1, N_2 \) are linear and nonlinear functions, respectively; and \( \mathcal{F}_1, \mathcal{F}_2 \) are source functions.

Apply Shehu transform to equation (20):

\begin{align*}
S[D_\eta^\delta \mu(\xi, \zeta, \eta)] + S[\mathcal{G}_1(\mu, \nu) + N_1(\mu, \nu)] \\
- S[\mathcal{F}_1(\xi, \zeta, \eta)] &= 0, \\
S[D_\eta^\delta \nu(\xi, \zeta, \eta)] + S[\mathcal{G}_2(\mu, \nu) + N_2(\mu, \nu)] \\
- S[\mathcal{F}_2(\xi, \zeta, \eta)] &= 0.
\end{align*}

(22)

Applying the differentiation property of Shehu transform, we have

\begin{align*}
S\left[ \sum_{m=0}^{\infty} \mu_m(\xi, \zeta, \eta) \right] &= \frac{\delta}{s} \sum_{k=0}^{m-1} \frac{\delta^k \mu(\xi, \zeta, \eta)}{\delta^k \eta} \big|_{\eta=0} + \frac{\delta}{s} S[\mathcal{F}_1(\xi, \zeta, \eta)] - \frac{\delta}{s} S\left[ \sum_{m=0}^{\infty} \mu_m(\xi, \zeta, \eta) \right], \\
S\left[ \sum_{m=0}^{\infty} \nu_m(\xi, \zeta, \eta) \right] &= \frac{\delta}{s} \sum_{k=0}^{m-1} \frac{\delta^k \nu(\xi, \zeta, \eta)}{\delta^k \eta} \big|_{\eta=0} + \frac{\delta}{s} S[\mathcal{F}_2(\xi, \zeta, \eta)] - \frac{\delta}{s} S\left[ \sum_{m=0}^{\infty} \nu_m(\xi, \zeta, \eta) \right].
\end{align*}

(27)
Applying the inverse Shehu transformation to equation (20), we get
\[
\sum_{m=0}^{\infty} \mu_m (\xi, \zeta, \eta) \left[ \mathcal{G}_1 (\xi, \zeta, \eta) \right] = - \frac{u^\delta}{s^\delta} S \left\{ \mathcal{G}_1 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{B}_m \right\}.
\]
\[
\sum_{m=0}^{\infty} \nu_m (\xi, \zeta, \eta) = S^{-1} \left[ \frac{u^\delta}{s^\delta} \sum_{k=0}^{\infty} \frac{\partial^k \nu (\xi, \zeta, \eta)}{\partial \eta^k} - \frac{u^\delta}{s^\delta} S \mathcal{G}_2 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{B}_m \right] \right].
\]

We define the following terms:
\[
\mu_0 (\xi, \zeta, \eta) = S^{-1} \left[ \frac{u^\delta}{s^\delta} \sum_{k=0}^{\infty} \frac{\partial^k \mu (\xi, \zeta, \eta)}{\partial \eta^k} \right]_{\eta=0} + \frac{u^\delta}{s^\delta} S \mathcal{G}_1 (\xi, \zeta, \eta),
\]
\[
\nu_0 (\xi, \zeta, \eta) = S^{-1} \left[ \frac{u^\delta}{s^\delta} \sum_{k=0}^{\infty} \frac{\partial^k \nu (\xi, \zeta, \eta)}{\partial \eta^k} \right]_{\eta=0} + \frac{u^\delta}{s^\delta} S \mathcal{G}_2 (\xi, \zeta, \eta),
\]
\[
\mu_1 (\xi, \zeta, \eta) = -S^{-1} \left[ \frac{u^\delta}{s^\delta} S \mathcal{G}_1 (\mu_0, \nu_0) + \mathcal{A}_0 \right],
\]
\[
\nu_1 (\xi, \zeta, \eta) = -S^{-1} \left[ \frac{u^\delta}{s^\delta} S \mathcal{G}_2 (\mu_0, \nu_0) + \mathcal{B}_0 \right],
\]
in general for \( m \geq 1 \), and we have
\[
\mu_{m+1} (\xi, \zeta, \eta) = -S^{-1} \left[ \frac{u^\delta}{s^\delta} S \mathcal{G}_1 (\mu_m, \nu_m) + \mathcal{A}_m \right],
\]
\[
\nu_{m+1} (\xi, \zeta, \eta) = -S^{-1} \left[ \frac{u^\delta}{s^\delta} S \mathcal{G}_2 (\mu_m, \nu_m) + \mathcal{B}_m \right].
\]

5. Implementation of the Methods

Example 1. Consider fractional-order system of nonlinear equations of unsteady flow of a polytropic gas [36, 38]:
\[
D_\eta^\delta \mu + \mu \frac{\partial \mu}{\partial \xi} + \nu \frac{\partial \mu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} = 0,
\]
\[
D_\eta^\delta \nu + \mu \frac{\partial \nu}{\partial \xi} + \nu \frac{\partial \nu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} = 0,
\]
\[
D_\eta^\delta \omega + \mu \frac{\partial \omega}{\partial \xi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \zeta} + \frac{\partial \nu}{\partial \zeta} \right) = 0,
\]
\[
D_\eta^\delta \psi + \mu \frac{\partial \psi}{\partial \xi} + \nu \frac{\partial \psi}{\partial \zeta} + \tau \psi \left( \frac{\partial \mu}{\partial \zeta} + \frac{\partial \nu}{\partial \zeta} \right) = 0,
\]
with initial conditions
\[
\mu (\xi, 0, 0) = e^{\xi \zeta},
\]
\[
\nu (\xi, 0, 0) = -1 - e^{\xi \zeta},
\]
\[
\omega (\xi, 0, 0) = e^{\xi \zeta},
\]
\[
\psi (\xi, 0, 0) = c,
\]
where \( c \) is the real constant.

First, SDM is used to solve equation (31).
For this applying Shehu transformation to equation (31),
\[
S \left\{ \frac{\partial \mu}{\partial \eta} \right\} = S \left\{ -\left[ \mu \frac{\partial \mu}{\partial \xi} + \nu \frac{\partial \mu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} \right] \right],
\]
\[
S \left\{ \frac{\partial \nu}{\partial \eta} \right\} = S \left\{ -\left[ \mu \frac{\partial \nu}{\partial \xi} + \nu \frac{\partial \nu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \zeta} \right] \right],
\]
\[
S \left\{ \frac{\partial \omega}{\partial \eta} \right\} = S \left\{ -\left[ \mu \frac{\partial \omega}{\partial \xi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \zeta} + \frac{\partial \nu}{\partial \zeta} \right) \right] \right],
\]
\[
S \left\{ \frac{\partial \psi}{\partial \eta} \right\} = S \left\{ -\left[ \mu \frac{\partial \psi}{\partial \xi} + \nu \frac{\partial \psi}{\partial \zeta} + \tau \psi \left( \frac{\partial \mu}{\partial \zeta} + \frac{\partial \nu}{\partial \zeta} \right) \right] \right].
\]
The above algorithm is reduced to simplified form:

\[
S[\mu(\xi, \zeta, \eta)] = \frac{1}{s} \{ \mu(\xi, \zeta, 0) \}
- \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \mu}{\partial \xi} + \nu \frac{\partial \mu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \xi} \right\} \right],
\]

\[
S[\nu(\xi, \zeta, \eta)] = \frac{1}{s} \{ \nu(\xi, \zeta, 0) \}
- \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \nu}{\partial \xi} + \nu \frac{\partial \nu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \xi} \right\} \right],
\]

\[
S[\omega(\xi, \zeta, \eta)] = \frac{1}{s} \{ \omega(\xi, \zeta, 0) \}
- \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \omega}{\partial \xi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \nu}{\partial \xi} \right) \right\} \right],
\]

\[
S[\psi(\xi, \zeta, \eta)] = \frac{1}{s} \{ \psi(\xi, \zeta, 0) \}
- \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \psi}{\partial \xi} + \nu \frac{\partial \psi}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \nu}{\partial \xi} \right) \right\} \right].
\]  

(34)

Applying inverse Shehu transformation, we get

\[
\mu(\xi, \zeta, \eta) = \mu(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \mu}{\partial \xi} + \nu \frac{\partial \mu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \xi} \right\} \right],
\]

\[
\nu(\xi, \zeta, \eta) = \nu(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \nu}{\partial \xi} + \nu \frac{\partial \nu}{\partial \zeta} + \frac{1}{\omega} \frac{\partial \psi}{\partial \xi} \right\} \right],
\]

\[
\omega(\xi, \zeta, \eta) = \omega(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \omega}{\partial \xi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \nu}{\partial \xi} \right) \right\} \right],
\]

\[
\psi(\xi, \zeta, \eta) = \psi(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left[ \left\{ \mu \frac{\partial \psi}{\partial \xi} + \nu \frac{\partial \psi}{\partial \zeta} + \omega \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \nu}{\partial \xi} \right) \right\} \right].
\]  

(35)

Equation (35) can be written in an operator form as

\[
\mu(\xi, \zeta, \eta) = \mu(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left\{ A_1(\mu, \mu_\xi) + B_1(\nu, \nu_\xi) + C_1(\omega, \psi_\xi) \right\},
\]

\[
\nu(\xi, \zeta, \eta) = \nu(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S \left\{ A_2(\mu, \nu_\xi) + B_2(\nu, \nu_\xi) + C_2(\omega, \psi_\xi) \right\},
\]

\[
\omega(\xi, \zeta, \eta) = \omega(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S A_3(\mu, \omega_\xi) + B_3(\nu, \omega_\xi) + C_3(\omega, \nu_\xi) + D_3(\omega, \nu_\xi),
\]

\[
\psi(\xi, \zeta, \eta) = \psi(\xi, \zeta, 0) - \frac{u^\delta}{s^\delta} S A_4(\mu, \psi_\xi) + B_4(\nu, \psi_\xi) + C_4(\mu, \psi_\xi) + D_4(\nu, \psi_\xi).
\]  

(36)

Assume that the unknown functions \( \mu(\xi, \zeta, \eta) \), \( \nu(\xi, \zeta, \eta) \), \( \omega(\xi, \zeta, \eta) \), and \( \psi(\xi, \zeta, \eta) \) have infinite series solution as follows:

\[
\mu(\xi, \zeta, \eta) = \sum_{m=0}^{\infty} \mu_m(\xi, \zeta, \eta),
\]

\[
\nu(\xi, \zeta, \eta) = \sum_{m=0}^{\infty} \nu_m(\xi, \zeta, \eta),
\]

(37)

\[
\omega(\xi, \zeta, \eta) = \sum_{m=0}^{\infty} \omega_m(\xi, \zeta, \eta),
\]

\[
\psi(\xi, \zeta, \eta) = \sum_{m=0}^{\infty} \psi_m(\xi, \zeta, \eta).
\]

(38)

All forms of nonlinear Adomian polynomials can be defined as
\[
A_1(\mu, \nu) = \mu_0\mu_0 + (\mu_1\mu_0 + \mu_0\mu_0) + \ldots,
\]
\[
B_1(\nu, \mu) = \nu_0\mu_0 + (\nu_1\mu_0 + \nu_0\mu_0) + \ldots,
\]
\[
C_1(\omega, \psi) = \psi_0\psi_0 + \omega_0\psi_0 + \omega_1\psi_0 + \ldots,
\]
\[
A_2(\mu, \nu) = \mu_0\nu_0 + (\mu_1\nu_0 + \mu_0\nu_0) + \ldots,
\]
\[
B_2(\nu, \mu) = \nu_0\nu_0 + (\nu_1\nu_0 + \nu_0\nu_0) + \ldots,
\]
\[
C_2(\omega, \psi) = \psi_0\omega_0 + \omega_0\psi_0 + \omega_1\psi_0 + \ldots,
\]
\[
A_3(\mu, \omega) = \mu_0\omega_0 + (\mu_1\omega_0 + \mu_0\omega_0) + \ldots,
\]
\[
B_3(\omega, \mu) = \omega_0\omega_0 + (\nu_1\omega_0 + \nu_0\omega_0) + \ldots,
\]
\[
C_3(\omega, \gamma) = \gamma_0\omega_0 + (\gamma_1\omega_0 + \gamma_0\omega_0) + \ldots,
\]
\[
D_3(\omega, \gamma) = \gamma_0\omega_0 + (\gamma_1\omega_0 + \gamma_0\omega_0) + \ldots,
\]
\[
A_4(\mu, \psi) = \mu_0\psi_0 + (\mu_1\psi_0 + \mu_0\psi_0) + \ldots,
\]
\[
C_4(\psi, \mu) = \psi_0\mu_0 + (\psi_1\mu_0 + \psi_0\mu_0) + \ldots,
\]
\[
D_4(\psi, \mu) = \psi_0\mu_0 + (\psi_1\mu_0 + \psi_0\mu_0) + \ldots.
\]

The initial sources are
\[
\mu_0(\xi, \zeta, \eta) = e^{\xi\zeta},
\]
\[
\nu_0(\xi, \zeta, \eta) = -1 - e^{\xi\zeta},
\]
\[
\omega_0(\xi, \zeta, \eta) = e^{2\xi\zeta},
\]
\[
\psi_0(\xi, \zeta, 0) = c.
\]
\[
\mu_m(\xi, \zeta, \eta) = e^{\xi\zeta} \frac{\eta^m}{\Gamma(\delta + 1)},
\]
\[
\nu_m(\xi, \zeta, \eta) = -e^{\xi\zeta} \frac{\eta^m}{\Gamma(\delta + 1)},
\]
\[
\omega_m(\xi, \zeta, \eta) = e^{2\xi\zeta} \frac{\eta^m}{\Gamma(3\delta + 1)},
\]
\[
\psi_m(\xi, \zeta, \eta) = 0.
\]

For \( m = 2 \),
\[
\mu_2(\xi, \zeta, \eta) = e^{3\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)},
\]
\[
\nu_2(\xi, \zeta, \eta) = -e^{3\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)},
\]
\[
\omega_2(\xi, \zeta, \eta) = e^{3\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)},
\]
\[
\psi_2(\xi, \zeta, 0) = 0.
\]

For \( m = 3 \),
\[
\mu_3(\xi, \zeta, \eta) = e^{4\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)},
\]
\[
\nu_3(\xi, \zeta, \eta) = -e^{4\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)},
\]
\[
\omega_3(\xi, \zeta, \eta) = e^{4\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)},
\]
\[
\psi_3(\xi, \zeta, 0) = 0.
\]

In general, we have
\[
\mu(\xi, \zeta, \eta) = e^{\xi\zeta} \frac{\eta^m}{\Gamma(\delta + 1)} + e^{\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} + e^{\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)} + \ldots,
\]
\[
\nu(\xi, \zeta, \eta) = -1 - e^{\xi\zeta} \frac{\eta^m}{\Gamma(\delta + 1)} - e^{\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} - e^{\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)} - \ldots,
\]
\[
\omega(\xi, \zeta, \eta) = e^{2\xi\zeta} \frac{\eta^m}{\Gamma(3\delta + 1)} + e^{2\xi\zeta} \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} + e^{2\xi\zeta} \frac{\eta^{3\delta}}{\Gamma(3\delta + 1)} + \ldots,
\]
\[
\psi(\xi, \zeta, 0) = c + 0 + \ldots.
\]
\[
\mu_m(\xi, \zeta, \eta) = e^{\xi + \zeta} \eta^{m\delta} \Gamma(m\delta + 1)
\]

\[
\nu_m(\xi, \zeta, \eta) = -e^{\xi + \zeta} \eta^{m\delta} \Gamma(m\delta + 1).
\]

\[
\omega_m(\xi, \zeta, \eta) = e^{\xi + \zeta} \eta^{m\delta} \Gamma(m\delta + 1).
\]

\[
\psi_m(\xi, \zeta, 0) = 0, \quad m = 1, 2, \ldots
\]

The Approximate Solution by VITM

According to equation (16) and the iteration formulas for system (31), we get

\[
\mu_{m+1}(\xi, \zeta, \eta) = \mu_m(\xi, \zeta, \eta) - N^{-1}\left[\frac{u^{\delta}}{s} S \left\{\frac{\delta^m}{u} \left[\frac{\partial \mu_m}{\partial \eta} + \mu_m \frac{\partial \nu_m}{\partial \zeta} + \nu_m \left(\frac{\partial \mu_m}{\partial \zeta} + \frac{1}{\omega_m} \frac{\partial \psi_m}{\partial \zeta}\right)\right]\right]\right],
\]

\[
\nu_{m+1}(\xi, \zeta, \eta) = \nu_m(\xi, \zeta, \eta) - S^{-1}\left[\frac{u^{\delta}}{s} S \left\{\frac{\delta^m}{u} \left[\frac{\partial \nu_m}{\partial \eta} + \nu_m \frac{\partial \nu_m}{\partial \zeta} + \omega_m \left(\frac{\partial \mu_m}{\partial \zeta} + \frac{1}{\omega_m} \frac{\partial \psi_m}{\partial \zeta}\right)\right]\right]\right],
\]

\[
\omega_{m+1}(\xi, \zeta, \eta) = \omega_m(\xi, \zeta, \eta) - N^{-1}\left[\frac{u^{\delta}}{s} S \left\{\frac{\delta^m}{u} \left[\frac{\partial \omega_m}{\partial \eta} + \mu_m \frac{\partial \omega_m}{\partial \zeta} + \nu_m \left(\frac{\partial \mu_m}{\partial \zeta} + \frac{1}{\omega_m} \frac{\partial \psi_m}{\partial \zeta}\right)\right]\right]\right],
\]

\[
\psi_{m+1}(\xi, \zeta, \eta) = \psi_m(\xi, \zeta, \eta) - S^{-1}\left[\frac{u^{\delta}}{s} S \left\{\frac{\delta^m}{u} \left[\frac{\partial \psi_m}{\partial \eta} + \nu_m \frac{\partial \psi_m}{\partial \zeta} + \omega_m \left(\frac{\partial \mu_m}{\partial \zeta} + \frac{1}{\omega_m} \frac{\partial \psi_m}{\partial \zeta}\right)\right]\right]\right],
\]

where

\[
\mu_0(\xi, \zeta, \eta) = e^{\xi + \zeta},
\]

\[
\nu_0(\xi, \zeta, \eta) = -1 - e^{\xi + \zeta},
\]

\[
\omega_0(\xi, \zeta, \eta) = e^{\xi + \zeta},
\]

\[
\psi_0(\xi, \zeta, 0) = c.
\]

For \(m = 0, 1, 2, \ldots\),
\[ \psi_2 (\xi, \zeta, \eta) = -1 - e^{\xi + \zeta + \eta} \left\{ 1 + \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right\}, \]
\[ \omega_2 (\xi, \zeta, \eta) = e^{\xi + \zeta + \eta} \left\{ 1 + \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right\}, \]
\[ \psi_2 (\xi, \zeta, \eta) = c + 0. \]
\[ \mu_3 (\xi, \zeta, \eta) = \mu_2 (\xi, \zeta, \eta) - \]
\[ N^{-1} \left[ \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right] \]
\[ \omega_3 (\xi, \zeta, \eta) = \omega_2 (\xi, \zeta, \eta) - N \left[ \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right] \]
\[ \psi_3 (\xi, \zeta, \eta) = \psi_2 (\xi, \zeta, \eta) - \]
\[ S^{-1} \left[ \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right] \]
\[ \mu_3 (\xi, \zeta, \eta) = e^{\xi + \zeta + \eta} \left\{ 1 + \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \right\}, \]
\[ \omega_3 (\xi, \zeta, \eta) = \omega_2 (\xi, \zeta, \eta) + \frac{\eta^\delta}{\Gamma (\delta + 1)} + \frac{\eta^{2\delta}}{\Gamma (2\delta + 1)} \]
\[ \psi_3 (\xi, \zeta, \eta) = \psi_2 (\xi, \zeta, \eta) + \]
\[ \sum_{m=0}^{\infty} \psi_m (\xi, \zeta) \]
\[ \omega (\xi, \zeta, \eta) = \sum_{m=0}^{\infty} \omega_m (\xi, \zeta) \]
\[ \psi (\xi, \zeta, 0) = \sum_{m=0}^{\infty} \psi_m (\xi, \zeta) \]
\[ \mu (\xi, \zeta, 0) = \sum_{m=0}^{\infty} \mu_m (\xi, \zeta) \]
\[ \mu (\xi, \zeta, \eta) = e^{\xi + \zeta + \eta}, \]
\[ \psi (\xi, \zeta, \eta) = -1 - e^{\xi + \zeta + \eta}, \]
\[ \omega (\xi, \zeta, \eta) = e^{\xi + \zeta + \eta}, \]
\[ \psi (\xi, \zeta, 0) = c. \]
6. Results Discussion

In this section, we discuss the solution-graphs of fractional-order system of nonlinear equations of unsteady flow of a polytropic gas which has been solved by using SDM and VITM. In Figures 1–4, the solutions $\mu$, $\nu$, $\omega$, and $\psi$ obtained by using SDM and VITM are compared by keeping one variable and other constants. The dotted and line subgraphs are, respectively, denoted the SDM and VITM solutions. It is observed that SDM and VITM solution-graphs are identical and within close contact. In similar way, in Figures 5–7, the three-dimensional graphs for variables $\mu$, $\nu$, and $\psi$ are plotted for Example 1. The identical solution-graphs of the suggested methods are attained and confirmed that the results obtained by two different procedures are identical and verified the applicability of the proposed techniques. In Figures 8–10, the SDM and VITM
solutions are plotted in two dimensions at fractional order $\delta = 0.4, 0.6, 0.8, 1$ for Example 1. The convergence phenomenon of the fractional solutions towards integer solution is observed. The three-dimensional graphs of the fractional-order solutions for Example 1 are represented in Figures 11–13 for variables $\mu, \nu,$ and $\omega$, respectively. In Table 1 and Figure 14, the combined graph for variables $\mu, \nu,$ and $\psi$ is displayed at $\delta = 1$. The solution comparison of the suggested methods, SDM and VITM, is discussed. The suggested techniques have provided the solutions with the desire degree of accuracy with the consideration of very few terms in its series form solutions.
Figure 9: SDM and VITM graph of different value of $\delta$ for $\nu(\xi, \zeta, \eta)$ of Example 1.

Figure 10: SDM and VITM graph of different value of $\delta$ for $\omega(\xi, \zeta, \eta)$ of Example 1.
Figure 11: SDM and VITM 3d graph of different value of $\delta$ for $\mu(\xi, \zeta, \eta)$ of Example 1.

Figure 12: SDM and VITM 3d graph of different value of $\delta$ for $\nu(\xi, \zeta, \eta)$ of Example 1.
Figure 13: SDM and VITM solution graph of different value of $\delta$ for $\omega(\xi, \zeta, \eta)$ of Example 1.

Figure 14: Combine graph of $\mu(\xi, \zeta, \eta)$, $\nu(\xi, \zeta, \eta)$, and $\omega(\xi, \zeta, \eta)$ of Example 1.
7. Conclusion

In this article, the analytical solution of the system of time-fractional partial differential equations of unsteady flow of polytrophic dynamics is investigated by using two different techniques:

(i) The proposed techniques are the mixture of Shehu transformation with Adomian decomposition method and variational iteration method, respectively.

(ii) The obtained solutions of the suggested techniques for both fractional and integer orders are calculated and plotted via two- and three-dimensional graphs.

(iii) A close contact between the actual and the derived results is observed.

(iv) The fractional-order solutions provide various dynamics for a different fractional order of the derivative.

(v) Using analytical solutions, the task can be done rather simple and effective as compared to numerical investigations that need larger calculations.

(vi) After all, the researchers are now able to select the fractional-order problem whose solution is comparatively very close to the experimental results of any physical problem.

(vii) Due to simple and straightforward implementation, the suggested techniques are considered to be preferable to solve other system of FPDEs.

The following abbreviations are used in this article:

**Nomenclature**

ST: Shehu transform  
LT: Laplace transform  
FPDEs: Fractional partial differential equations  
VITM: Variational iteration transform method  
SDM: Shehu decomposition method  
ADM: Adomian decomposition method.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

H. K. conceptualized the study, prepared the methodology, wrote the original draft, and did formal analysis. M. A. analyzed using the software and supervised the study. S. I. validated, investigated, and visualized the study and administrated the project. H. K. and M. A. managed resources and reviewed and edited the document.

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