Gauge symmetries of the teleparallel theory of gravity

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Abstract

We study gauge properties of the general teleparallel theory of gravity, defined in the framework of Poincaré gauge theory. It is found that the general theory is characterized by two kinds of gauge symmetries: a specific gauge symmetry that acts on Lagrange multipliers, and the standard Poincaré gauge symmetry. The canonical generators of these symmetries are explicitly constructed and investigated.

1. Introduction

Modern developments in particle physics suggest the possibility that gravity might be described by a geometric structure different from Riemannian space of general relativity (GR). Such geometric structures appear naturally in gauge theories of gravity, which represent a promising framework for describing gravitational interaction [1]. Here, we focus our attention on the teleparallel description of gravity in the framework of Poincaré gauge theory (PGT) [2, 3, 4]. Basic dynamical variables in PGT are the tetrad field $b^k_{\mu}$ and Lorentz connection $A^i_{j\mu}$, and the corresponding field strengths are geometrically identified with the torsion and the curvature:

$$T_{\mu\nu}^i = \partial_\mu b^i_\nu + A^i_{s\mu} b^s_\nu - (\mu \leftrightarrow \nu),$$

$$R_{\mu\nu}^{ij} = \partial_\mu A_{\nu}^{ij} + A_{s\mu}^{ij} A_{s\nu} - (\mu \leftrightarrow \nu).$$

General geometric structure of PGT corresponds to Riemann-Cartan space $U_4$, defined by metric (or tetrad) and metric compatible connection.

The teleparallel or Weitzenböck geometry $T_i$ is defined as a special limit of PGT by the requirement of vanishing curvature:

$$R_{\mu\nu}^{ij}(A) = 0. \tag{1.1}$$

The teleparallel theory has been one of the most attractive alternatives to GR [5, 6, 7] until the work of Kopczyński [8]. He demonstrated the existence of a hidden gauge symmetry, and concluded that the theory is inconsistent since the torsion tensor is not completely determined by the field equations. Nester improved the arguments of Kopczyński by showing that the unpredictable behaviour of torsion occurs only for some very special solutions [9].

The canonical analysis of the teleparallel formulation of GR, performed in Ref. [10], was aimed at clarifying these unusual properties of the teleparallel theory. In the present paper, we continue this investigation by a detailed study of all gauge symmetries of the general teleparallel theory. The precise form of the gauge generators obtained in this paper

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can be used to introduce the important concepts of energy, momentum and other conserved charges, and understand the related role of asymptotic conditions in the teleparallel theory of gravity [11, 12].

We begin our considerations in Sect. 2 by introducing the basic Lagrangian and Hamiltonian properties of the teleparallel theory. In Sect. 3, we construct the gauge generator of a specific symmetry, called \( \lambda \) symmetry, which is present in any teleparallel theory. Then, in Sect. 4, we study a simple model in order to understand the relation between the \( \lambda \) and Poincaré gauge symmetries. Finally, in Sect. 5, we construct the Poincaré gauge generator for a general teleparallel theory. Section 6 is devoted to concluding remarks.

Our conventions are the same as in Refs. [10, 12]: the Latin indices refer to local Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of both alphabets (\( a, b, c, \ldots; \alpha, \beta, \gamma, \ldots \)) run over 1, 2, 3, and the middle alphabet letters (\( i, j, k, \ldots; \mu, \nu, \lambda, \ldots \)) run over 0, 1, 2, 3; \( \eta_{ij} = (+, -, -, -) \), \( \varepsilon^{ijkl} \) is completely antisymmetric symbol normalized by \( \varepsilon^{0123} = +1 \), \( \delta = \delta(x - x') \); the Hamiltonian \( H \) and its density \( \mathcal{H} \) are related by \( H = \int d^3x \mathcal{H} \).

2. Basic dynamical features of the teleparallel theory

Lagrangian dynamics. Gravitational dynamics of the general teleparallel theory, in the framework of PGT, is described by a class of Lagrangians quadratic in the torsion,

\[
\mathcal{L} = b\mathcal{L}_T + \lambda_{ij}^{\mu\nu} R_{ij}^{\mu\nu}, \\
\mathcal{L}_T = a\left( A T_{ijk} T^{ijk} + B T_{ijk} T^{jik} + C T_k^{jk} \right) \equiv \beta_{ijk}(T) T^{ijk},
\]

where the Lagrange multipliers \( \lambda_{ij}^{\mu\nu} \) ensure the teleparallelism condition (1.1), \( A, B \) and \( C \) are free parameters, \( a = 1/2\kappa \) (\( \kappa = \) Einstein’s gravitational constant), and \( T_k = T^m_{nk} \). Note that, following Ref. [12], \( \lambda_{ij}^{\mu\nu} \) is assumed to be a tensor density rather than a tensor, which leads to a slightly simplified constraint analysis, as compared to Ref. [10].

The physical relevance of the theory (2.1) lies in the fact that there is a one-parameter family of teleparallel theories, defined by the conditions

\( i) \ 2A + B + C = 0, \ C = -1, \)

which passes all the standard gravitational tests [3, 4, 5]; hence, it is empirically indistinguishable from GR. One particularly interesting member of this family is defined by

\( ii) \ 2A - B = 0 \quad (i.e. \ A = 1/4, \ B = 1/2, \ C = -1). \)

Using a well known geometric identity [10], one finds that this choice leads effectively (up to a four-divergence) to the Hilbert-Einstein Lagrangian defined in Riemann space \( V_4 \). We call this theory the teleparallel form of GR, and denote it by \( \text{GR}_\parallel \).

By varying the teleparallel Lagrangian (2.1) with respect to \( b_{i\mu} \), \( A_{ij}^{\mu} \) and \( \lambda_{ij}^{\mu\nu} \), we obtain the following gravitational field equations:

\[
4 \nabla_{\mu} \left( b_{\beta}^{i\mu} \right) + 4b_{\beta}^{\mu\nu} T_{nmi} - h_{i}^{\nu} b \mathcal{L}_T = 0, \\
\nabla_{\mu} \lambda_{ij}^{\mu\nu} + 2b_{\beta[ij]}^{\nu} = 0, \\
R_{ij}^{\mu\nu} = 0.
\]
Note that the field equations (2.2b) satisfy 6 differential identities since the covariant divergence of the left hand side vanishes on account of $R^{ij \mu \nu} = 0$. Hence, the number of independent equations (2.2b) is $24 - 6 = 18$.

The gravitational Lagrangian (2.1) is, by construction, invariant under the local Poincaré transformations:

\[
\begin{align*}
\delta_0 b^k_\mu &= \omega^k_\mu b^s_\mu - \xi^s_\mu b^k_\mu + \xi^s_\rho \partial_\rho b^k_\mu, \\
\delta_0 A^{ij}_\mu &= -\omega^{ij}_\rho A^{ij}_\mu + \omega^j_\rho A^{is}_\mu - \xi^s_\rho A^{ij}_\mu, \\
\delta_0 \lambda^{ij \mu \nu} &= \omega^i_\rho \lambda^{sj \mu \nu} + \omega^j_\rho \lambda^{is \mu \nu} + \xi^\mu_\rho \lambda^{ij \rho \nu} + \xi^\nu_\rho \lambda^{ij \mu \rho} - \partial_\rho (\xi^\rho \lambda^{ij \mu \nu}),
\end{align*}
\]

(2.3)

where $\delta_0 \varphi(x) = \varphi'(x) - \varphi(x)$ is the form variation of $\varphi(x)$. In addition, it is also invariant, up to a four-divergence, under the transformations

\[
\delta_0 \lambda^{ij \mu \nu} = \nabla_\rho \varepsilon^{ij \mu \nu \rho},
\]

(2.4a)

where the gauge parameter $\varepsilon^{ij \mu \nu \rho} = -\varepsilon^{ji \mu \nu \rho}$ is completely antisymmetric in its upper indices, and has $6 \times 4 = 24$ components. The invariance is easily verified by using the Bianchi identity $\nabla_\rho R^{ij \mu \nu} + \text{cyclic} (\mu, \nu, \rho) = 0$. On the other hand, the invariance of the field equation (2.2b) follows directly from $R^{ij \mu \nu} = 0$. The symmetry (2.4a) will be referred to as the $\lambda$ symmetry.

It is useful to observe that the $\lambda$ transformations can be written in the form

\[
\begin{align*}
\delta_0 \lambda^{ij \alpha \beta} &= \nabla_\gamma \varepsilon^{ij \alpha \beta \gamma}, \\
\delta_0 \lambda^{ij \beta \gamma} &= \nabla_\gamma \varepsilon^{ij \beta \gamma}.
\end{align*}
\]

(2.4b)

We shall show in the next section, by canonical methods, that the only independent parameters of the $\lambda$ symmetry (2.4a) are $\varepsilon^{ij \alpha \beta}$; in other words, the six parameters $\varepsilon^{ij \alpha \beta \gamma}$ can be completely discarded. Consequently, the number of independent gauge parameters is $24 - 6 = 18$. They can be used, for instance, to fix $\lambda^{ij \alpha \beta}$, whereupon the independent field equations (2.2b) determine $\lambda^{ij \beta \gamma}$ (at least locally).

It is evident that the Poincaré and $\lambda$ gauge symmetries are always present (sure symmetries), independently of the values of parameters $A, B$, and $C$ in the teleparallel theory (2.1). Moreover, it will become clear from the canonical analysis that there are no other sure gauge symmetries. Specific models, such as GR, may have extra gauge symmetries, present only for some special (critical) values of parameters, but these will not be the subject of our analysis. Our task in the subsequent sections is the construction of the sure gauge generators, describing the Poincaré symmetry and the $\lambda$ symmetry.

**Hamiltonian and constraints.** Gauge symmetries of a dynamical system are best described by the related canonical generators. The program of constructing the gauge generators of the general teleparallel theory (2.1) demands the complete knowledge of the Hamiltonian and the constraints (3)]. However, the Hamiltonian structure of the general theory is missing. Instead, we can use the known Hamiltonian of GR, and construct the generators of the $\lambda$ and Poincaré symmetries in this particular case. We shall see that the form of the results obtained in this manner has a natural extension to the general case. After making this extension, the action of the extended generator on the whole phase space will be explicitly verified. This approach is essentially based on the ideas used in Ref. (14) to construct the Poincaré gauge generator of the general $R + T^2 + R^2$ theory.
We begin by displaying the Hamiltonian and the constraints of GR$_{\parallel}$ [12]. The canonical Hamiltonian is given in Appendix A. The general Hamiltonian dynamics is described by the total Hamiltonian:

$$\mathcal{H}_T = \dot{H}_T + \partial_\alpha \tilde{D}^\alpha,$$

$$\mathcal{H}_T \equiv N\mathcal{H}_{\perp} + N^\alpha \mathcal{H}_\alpha - \frac{1}{2} A_{ij0}^0 \mathcal{H}_{ij} - \lambda_{ij}^\alpha \mathcal{H}_{ij}^\alpha + u^i_0 \pi_i^0 + \frac{1}{2} u_{ij}^0 \pi_{ij}^0 + \frac{1}{4} u_{ij}^\alpha \pi_{ij}^\alpha + (u \cdot \phi), \tag{2.5a}$$

where

$$\mathcal{H}_{\perp} = \mathcal{H}_{\perp} - \frac{1}{4} (\partial \mathcal{H}_{\perp} / \partial A_{ij0}) \pi_{ij0\alpha},$$

$$\mathcal{H}_\alpha = \pi_\beta^i T_{\alpha\beta}^i - b_k^0 \nabla_\beta \pi_k^\alpha + \frac{1}{2} \pi_{ij0\alpha} \nabla_\beta \lambda_{ij}^0,\quad \mathcal{H}_{ij} = 2 \pi_{[i}^j b_{j\alpha]} + \nabla_\alpha \pi_{ij}^\alpha + 2 \pi_k^\alpha \pi_{ij}^0,\quad \mathcal{H}_{ij}^\alpha = R_{ij}^\alpha - \frac{1}{2} \nabla_k \pi_{ij0}\beta, \quad \tilde{D}^\alpha = b_k^0 \pi_k^\alpha + \frac{1}{2} A_{ij0}^i \pi_{ij}^\alpha - \frac{1}{4} \lambda_{ij}^\alpha \pi_{ij0}\beta. \tag{2.5b}$$

The expression for $\mathcal{H}_{\perp}$ is defined in Eq. (A.1b). Note that $\mathcal{H}_\alpha$ and $\mathcal{H}_{ij}$ differ from the corresponding expressions in Ref. [12] by squares of constraints, which is irrelevant for our analysis. The term $(u \cdot \phi) = \frac{1}{2} u^{ik} \phi_{ik}$ in $\mathcal{H}_T$ describes extra primary first class constraints specific to GR$_{\parallel}$, which are given in Eq. (A.2).

The complete dynamical classification of the constraints is given in the following table:

| primary | secondary |
|---------|-----------|
| $\pi_i^0, \pi_{ij}^0, \pi_{ij\alpha\beta}, \phi_{ik}$ | $\mathcal{H}_{ij}^\alpha, \mathcal{H}_{ij0\beta}$ |

where

$$\phi_{ij}^\alpha = \pi_{ij}^\alpha - 4 \lambda_{ij}^0. \tag{2.6}$$

The constraints $\phi_{ij}^\alpha$ and $\pi_{ij0\beta}$ are second class since $\{\phi_{ij}^\alpha, \pi_{kl0}\beta\} \neq 0$. The first class constraints are identified by observing that they appear multiplied by arbitrary multipliers or unphysical variables in the total Hamiltonian (2.5a).

We display here, for later convenience, the part of the Poisson bracket algebra of constraints involving $\mathcal{H}_{ij}^\alpha$:

$$\{\mathcal{H}_{ij}^\alpha, \mathcal{H}_{kl}^j\} = (\delta_k^i \mathcal{H}_{ij}^\alpha + \delta_k^i \mathcal{H}_{kl}^\alpha) \delta - (k \leftrightarrow l),$$

$$\{\mathcal{H}_{ij}^\alpha, \mathcal{H}_{ij}^\gamma\} = \{\mathcal{H}_{ij}^\alpha, \mathcal{H}_{ij}^\gamma\} = \{\mathcal{H}_{ij}^\alpha, \mathcal{H}_{ij0\beta}\} = 0. \tag{2.7}$$

The equations $\{\mathcal{H}_{ij}^\alpha, \mathcal{H}_{ij}^\gamma\} = 0$ hold up to squares of constraints, which are always ignored in an on-shell analysis.

The total Hamiltonian $\mathcal{H}_T$, in contrast to $\dot{H}_T$, does not contain the derivatives of momentum variables. The only components of $\mathcal{H}_T$ that depend on the specific form of the Lagrangian are $\mathcal{H}_{\perp}$ and $\phi_{ij}$. In the next section we shall use the above canonical structure of GR$_{\parallel}$ to construct the generator of the $\lambda$ symmetry (2.4). The result will be generalized to hold for any teleparallel theory of the given class (2.1).
3. The $\lambda$ symmetry

If gauge transformations are given in terms of arbitrary parameters $\varepsilon(t)$ and their first time derivatives $\dot{\varepsilon}(t)$, as is the case with the symmetries of our Lagrangian (2.1), the gauge generators have the form

$$G = \varepsilon(t) G^{(0)} + \dot{\varepsilon}(t) G^{(1)},$$

(3.1a)

where the phase space functions $G^{(0)}$ and $G^{(1)}$ satisfy the conditions [13]

$$G^{(1)} = C_{PFC},$$

$$G^{(0)} + \{G^{(1)}, H_T\} = C_{PFC},$$

$$\{G^{(0)}, H_T\} = C_{PFC},$$

(3.1b)

and $C_{PFC}$ denotes a primary first class (PFC) constraint. These conditions clearly define the procedure for constructing the generator: one starts with an arbitrary PFC constraint $G^{(1)}$, evaluates its Poisson bracket with $H_T$, and defines $G^{(0)}$ in accordance with $\{G^{(0)}, H_T\} = C_{PFC}$.

Now, we are going to construct the gauge generator of the $\lambda$ symmetry (2.4). The only PFC constraint that acts on the Lagrange multipliers $\lambda_{ij}^{\mu\nu}$ is $\pi_{ij}^{\alpha\beta}$. Starting with $\pi_{ij}^{\alpha\beta}$ as our $G^{(1)}$, we look for the generator in the form

$$G_A(\varepsilon) = \frac{1}{4} \dot{\varepsilon}_{ij}^{\alpha\beta} \pi_{ij}^{\alpha\beta} + \frac{1}{4} \varepsilon_{ij}^{\alpha\beta} S_{ij}^{\alpha\beta}.$$  

(3.2a)

The phase space function $S_{ij}^{\alpha\beta}$ is to be found from (3.1b). In the first step, we obtain the $G^{(0)}$ part of the generator up to PFC constraints:

$$S_{ij}^{\alpha\beta} = -4 \bar{\mathcal{H}}_{ij}^{\alpha\beta} + C_{PFC}.$$  

Then, using the algebra of constraints involving $\bar{\mathcal{H}}_{ij}^{\alpha\beta}$ given in Eq. (2.7), and the third condition in (3.1b), we find

$$S_{ij}^{\alpha\beta} = -4 \bar{\mathcal{H}}_{ij}^{\alpha\beta} + 2 A^{[i}_{k0} \pi^{j]k}^{\alpha\beta}.$$  

(3.2b)

This completely defines the generator $G_A(\varepsilon)$ we were looking for.

Applying the generator (3.2) to the fields according to $\delta_0 X = \int d^3 x' \{X, G'\}$, we find

$$\delta_0^A \lambda_{ij}^{0\alpha} = \nabla_\beta \varepsilon_{ij}^{\alpha\beta}, \quad \delta_0^A \lambda_{ij}^{\alpha\beta} = \nabla_0 \varepsilon_{ij}^{\alpha\beta},$$

(3.3)

as the only nontrivial field transformations. This result, however, does not agree with the form of the $\lambda$ symmetry given in Eq. (2.4b), which contains an additional piece, $\nabla_\gamma \varepsilon_{ij}^{\alpha\beta\gamma}$, in the expression for $\delta_0 \lambda_{ij}^{\alpha\beta}$. Since there are no other PFC constraints that could produce the transformation of $\lambda_{ij}^{\alpha\beta}$, the canonical origin of the additional term seems somewhat puzzling.

The solution of the problem is, however, quite simple: if we consider only independent gauge transformations, this term is not needed, as it is not independent of what we already have in (3.3). To prove this statement, consider the following PFC constraint:

$$\Pi_{ij}^{\alpha\beta\gamma} = \nabla_\alpha \pi_{ij}^{\beta\gamma} + \nabla_\gamma \pi_{ij}^{\alpha\beta} + \nabla_\beta \pi_{ij}^{\alpha\gamma}.$$
This constraint is essentially a linear combination of $\pi^{ij}_{\alpha\beta}$; hence, the related gauge generator will not be truly independent of the above general expression (3.2). Further, using the Bianchi identity for $R^{ij}_{\alpha\beta}$, one finds the relation

$$\nabla_\alpha \hat{H}^{ij}_{\beta\gamma} + \nabla_\beta \hat{H}^{ij}_{\gamma\alpha} + \nabla_\gamma \hat{H}^{ij}_{\alpha\beta} = 0,$$

which holds up to squares of constraints. As a consequence, $\Pi^{ij}_{\alpha\beta\gamma}$ commutes with the total Hamiltonian up to PFC constraints, and is, therefore, a correct gauge generator by itself. Hence, we can introduce a new gauge generator,

$$G_B(\varepsilon) = -\frac{1}{4} \varepsilon^{ij}_{\alpha\beta\gamma} \nabla_\alpha \pi^{ij}_{\beta\gamma},$$

(3.4)

where the parameter $\varepsilon^{ij}_{\alpha\beta\gamma}$ is totally antisymmetric with respect to its upper indices. The only nontrivial field transformation produced by this generator is

$$\delta^B_0 \lambda^{ij}_{\alpha\beta} = \nabla_\gamma \varepsilon^{ij}_{\alpha\beta\gamma},$$

and it coincides with the missing term in Eq. (3.3). This concludes the proof that the six parameters $\varepsilon^{ij}_{\alpha\beta\gamma}$ in the $\lambda$ transformations (2.4) can be completely discarded if we are interested only in the independent $\lambda$ transformations.

Although the generator $G_B$ is not truly independent of $G_A$, it is convenient to define

$$G(\varepsilon) \equiv G_A(\varepsilon) + G_B(\varepsilon)$$

(3.5)

as an overcomplete gauge generator, since it automatically generates the covariant Lagrangian form of the $\lambda$ symmetry, Eq. (2.4).

The action of the generator (3.5) on momenta is easily seen to be correct, in the sense that it yields the result in agreement with the defining relations $\pi_A = \partial \mathcal{L}/\partial \dot{\varphi}^A$ (see Appendix B). In particular, the only nontrivial transformation law for the momenta,

$$\delta^0_0 \pi^{ij}_{\alpha} = 4 \nabla_\beta \varepsilon^{ij}_{\alpha\beta},$$

agrees with (2.4b) through the conservation of the primary constraint $\phi^{ij}_{\alpha} \approx 0$.

The above construction is based on using the first class constraints $\pi^{ij}_{\alpha\beta}$, $\hat{H}^{ij}_{\alpha\beta}$, the part of the Poisson bracket algebra involving these constraints, and the second class constraints $\phi^{ij}_{\alpha}$. All these constraints and their properties are independent of the values of parameters in the theory; hence, we can conclude that

$$G(\varepsilon) \text{ is the correct generator of } \lambda \text{ symmetry in the general teleparallel theory.}$$

4. A simple special case

Before we proceed with the construction of the Poincaré gauge generator, let us make a few comments. The $\mathcal{L}_T$ part of our Lagrangian (2.1) is a special case of the general $R + T^2 + R^2$ theory, whose Poincaré gauge generator has already been constructed in the literature [14]. In the teleparallel theory of gravity, this result is to be corrected with the terms stemming from the $\lambda^{\mu
u}_{\alpha\beta} R^{ij}_{\mu\nu}$ part of the Lagrangian. Since the general construction procedure is rather complicated, we give in this section a detailed analysis of the simple special case
The Poisson bracket algebra of the Hamiltonian constraints has the form

\[ (3.1) \]

with the PFC constraint algebra (4.3) and the third condition in (3.1b), we find the complete function \( \tilde{S}_{ij} \).

From the second condition in (3.1b), we obtain \( \tilde{S}_{ij} = -\tilde{H}_{ij} + C_{PFC} \). Then, using the constraint algebra (4.3) and the third condition in (3.1b), we find the complete function \( \tilde{S}_{ij} \) to read:

\[ \tilde{S}_{ij} = -\tilde{H}_{ij} + 2A^s_{\alpha i \beta j} + \lambda_{ij}^{\alpha \beta} \pi_{\alpha \beta} - \eta^s_{\alpha i \beta j} \]

The Lorentz generator. We begin with the PFC constraint \( \pi_{ij}^0 \), and define

\[ G(\omega) = -\frac{1}{2} \omega^{ij} \pi_{ij}^0 - \frac{1}{2} \omega^{ij} \tilde{S}_{ij} . \]

The \( \lambda \) generator. The \( \lambda \) gauge generator is obtained by starting Castellani’s procedure (3.1) with the PFC constraint \( \pi_{ij}^{\alpha \beta} \). All the steps of the construction, the analysis and the final result are the same as in Sect. 3. Thus, \( G(\epsilon) \) gives:

\[ G(\epsilon) = \frac{1}{4} \epsilon_{ij}^{\alpha \beta} \pi_{ij}^{\alpha \beta} + \frac{1}{4} \epsilon_{ij}^{\alpha \beta} S_{ij}^{\alpha \beta} - \frac{1}{4} \epsilon_{ij}^{\alpha \beta} \nabla^\alpha \pi_{ij}^{\beta \gamma} \]

with \( S_{ij}^{\alpha \beta} \) given by Eq. (3.2b). The action on the fields is the same as in Eq. (2.4).

Therefore, the action of the complete generator \( G(\omega) + G(\epsilon) \) gives:

\[ \delta_0 A_{ij}^{\mu \nu} = \omega_{i}^k A_{kj}^{\mu} + \omega_{j}^k A_{ik}^{\mu} - \omega_{ij}^{\mu \nu} \]

\[ \delta_0 \lambda_{ij}^{\mu \nu} = \omega_{i}^k \lambda_{kj}^{\mu \nu} + \omega_{j}^k \lambda_{ik}^{\mu \nu} + \nabla^\alpha \epsilon_{ij}^{\mu \nu \lambda} \]
The transformation laws (4.6) exhaust the gauge symmetries of the simple theory (4.1). Notice, however, that the Lagrangian \( \tilde{\mathcal{L}} \) also possesses the local translational symmetry, which has not been obtained by Castellani’s procedure. If Castellani’s algorithm is an exhaustive one, then the translational symmetry must be somehow hidden in the above result (4.6). In what follows, we shall demonstrate that this is really true, namely that the translational symmetry emerges from a simple redefinition of the gauge parameters in (4.6).

The Poincaré generator. Let us consider the following replacement of the parameters \( \omega_{ij} \) and \( \varepsilon_{ij\mu\nu\lambda} \) in Eq. (4.6):

\[
\begin{align*}
\omega_{ij} &\rightarrow \omega_{ij} + \xi_{i\mu}, \\
\varepsilon_{ij\mu\nu\lambda} &\rightarrow -\left(\xi^\nu \lambda_{ij}^{\nu\lambda} + \xi^\lambda \lambda_{ij}^{\mu\nu}\right).
\end{align*}
\] (4.7)

The resulting on-shell field transformations,

\[
\begin{align*}
\delta_0 A_{ij\mu} &\approx \omega_{i\mu} A_{j\mu} + \omega_{j\mu} A_{i\mu} - \omega_{ij\mu} - \xi_{i\mu},
\delta_0 \lambda_{ij\mu\nu} &\approx \omega_{i\mu} \lambda_{kj\nu} + \omega_{j\mu} \lambda_{ik\nu} + \xi_{\alpha\mu} \lambda_{ij\alpha\nu} + \xi_{\nu\rho} \lambda_{ij\mu\rho} - \partial_\rho \left(\xi_{\rho} \lambda_{ij\mu\nu}\right),
\end{align*}
\]

are the exact Poincaré gauge transformations we expected to find in this theory. As we can see, the local translations are not obtained as independent gauge transformations, but rather emerge as a part of the \( \lambda \) and Lorentz symmetries in (4.6). The corresponding Poincaré generator is obtained by using the replacement (4.7) in the gauge generator \( \tilde{G}(\omega) + \tilde{G}(\varepsilon) \).

Thus, we find:

\[
\tilde{G} = \tilde{G}(\omega) + \tilde{G}(\xi),
\] (4.8a)

where the first term, describing local Lorentz rotations, has the form (4.4), while the second term, describing local translations, is given by

\[
\begin{align*}
\tilde{G}(\xi) = & -\dot{\xi}^0 \left(\frac{1}{2} A_{ij\alpha} \pi_{ij}^{\alpha} + \frac{1}{4} \lambda_{ij}^{\alpha\beta} \pi_{ij}^{\alpha\beta}\right) - \xi^0 \tilde{H}_T \\
& - \dot{\xi}^\alpha \left(\frac{1}{2} A_{ij\alpha} \pi_{ij}^0 + \frac{1}{2} \lambda_{ij}^{0\beta} \pi_{ij}^{0\beta}\right) \\
& - \xi^\alpha \left[\tilde{P}_\alpha - \frac{1}{2} \lambda_{ij}^{\beta\gamma} \partial_\alpha \pi_{ij}^{\beta\gamma} - \frac{1}{2} \partial_\gamma \left(\lambda_{ij}^{\beta\gamma} \pi_{ij}^{\alpha\beta}\right)\right].
\end{align*}
\] (4.8b)

In the above expressions we used the following notation:

\[
\begin{align*}
\tilde{P}_\alpha &\equiv \tilde{H}_\alpha - \frac{1}{2} A_{ij\alpha} \tilde{H}_{ij} + 2 \lambda_{ij}^{0\beta} \tilde{H}_{ij\alpha\beta} + \frac{1}{2} \pi_{ij}^0 \partial_\alpha A_{ij\alpha}, \\
\tilde{H}_\alpha &\equiv \frac{1}{2} \pi_{ij}^{0\beta} \partial_\beta \lambda_{ij}^{0\beta}.
\end{align*}
\] (4.8c)

Notice that the term \( \tilde{H}_\alpha \) in \( \tilde{G}(\xi) \) has the structure of squares of constraints, and therefore, does not contribute to the nontrivial field transformations. Nevertheless, we shall retain it in the generator because it makes the field transformations practically off shell (up to \( R_{ij\alpha\beta} \approx 0 \)). This will help us to straightforwardly find the form of the extension of \( \tilde{G} \) in the general teleparallel theory.

5. Poincaré gauge symmetry

Staring from the Poincaré gauge generator (4.8) of the simple theory (4.1), and comparing it with the earlier results obtained in Ref. [14], it is almost evident how its modification to
include the tetrad sector should be defined. In this section, we are going to prove that the complete Poincaré gauge generator of the general teleparallel theory (2.1) has the form

\[ G = G(\omega) + G(\xi), \]  

(5.1a)

where the first term describes local Lorentz rotations,

\[ G(\omega) = -\frac{1}{2} \dot{\omega}^{ij} \pi_{ij}^0 - \frac{1}{2} \omega^{ij} S_{ij}, \]  

(5.1b)

while the second term describes local translations,

\[ G(\xi) = -\xi^0 \left( b_i^k \pi_k^0 + \frac{1}{2} A_j^0 \dot{\pi}_{ij}^0 + \frac{1}{4} \lambda_{ij}^{\alpha\beta} \pi_{ij}^{\alpha\beta} \right) - \xi^0 \mathcal{P}_0 -\xi^0 \left[ \tilde{\mathcal{P}}_a - \frac{1}{4} \lambda_{ij}^{\alpha\gamma} \partial_\alpha \pi_{ij}^{\beta\gamma} - \frac{1}{2} \partial_\gamma \left( \lambda_{ij}^{\beta\gamma} \pi_{ij}^{\alpha\beta} \right) \right]. \]  

(5.1c)

In the above expressions, we used the following notation:

\[ S_{ij} = -\mathcal{H}_{ij} + 2b_{[i|0}\pi_{j]}^0 + 2A^s_{[i|0}\pi_{sj]}^0 + 2\lambda_{s[i}^{\alpha\beta} \pi_{s]j\alpha\beta}, \]  

\[ \mathcal{P}_0 \equiv \mathcal{H}_T = \mathcal{H}_T - \partial_\alpha \tilde{D}^\alpha, \]  

\[ \tilde{\mathcal{P}}_a = \mathcal{H}_\alpha - \frac{1}{2} A_{ij}^a \mathcal{H}_{ij} + 2\lambda_{ij}^{\alpha\beta} \mathcal{H}_{ij}^{\alpha\beta} + \pi_{k}^{0} \partial_\alpha b^k_0 + \frac{1}{2} \pi_{ij}^{0} \partial_\alpha A_{ij}^0. \]  

(5.1d)

The form of the total Hamiltonian \( \mathcal{H}_T \) is defined by the choice of the Lagrangian: in the case of GR\( _{ij} \), it is determined by Eq. (2.5a), while the general \( \mathcal{H}_T \) is constructed by the principles of Appendix C. The Poincaré generator \( G \) is obtained from the simplified expression \( \tilde{G} \) in Eq. (4.8a) by a natural process of extension, which consists of

- the replacements \( \mathcal{H}_\alpha \rightarrow \tilde{\mathcal{H}}_\alpha, \mathcal{H}_{ij} \rightarrow \mathcal{H}_{ij}, \mathcal{H}_{ij}^{\alpha\beta} \rightarrow \mathcal{H}_{ij}^{\alpha\beta}, \)
- the addition of \( -\omega^{ij} b_i^0 \pi_{j0}^0, -\xi^0 b_i^0 \pi_{k0}^0, -\xi^0 b_i^0 \pi_{k0}^0, -\xi^0 \pi_{k0}^0 \partial_\alpha b^k_0 \), and
- the replacement \( \mathcal{H}_T \rightarrow \tilde{\mathcal{H}}_T. \)

This amounts to completing the Poincaré gauge generator so as to act correctly also in the tetrad sector [14].

The proof that the Poincaré gauge generator has the form (5.1) is realized by showing that \( G \) produces the correct Poincaré gauge transformations on the complete phase space, i.e. on all the fields and momenta.

**Action on the fields.** We now demonstrate that the action of the generator (5.1) on the fields produces the complete Poincaré gauge transformations (2.3).

It is straightforward to verify \( \omega^{ij} \) and \( \xi^0 \) transformations in (2.3). The derivation of \( \xi^0 \) transformations is more subtle. Let us illustrate the procedure on \( \lambda_{ij}^{0\beta} \).

\[ \delta_0 (\xi^0) \lambda_{ij}^{0\beta} = - \left\{ \lambda_{ij}^{0\beta}, \mathcal{P}_0' \right\} = - \left\{ \lambda_{ij}^{0\beta}, \mathcal{H}_T' - \partial_\gamma D^\gamma \right\} \approx -\xi^0 \dot{\lambda}_{ij}^{0\beta} + (\partial_\gamma \xi^0) \lambda_{ij}^{\gamma\beta}. \]

In deriving the last (weak) equality we used the relation \( \int d^3 x' \xi^0 \left\{ \lambda_{ij}^{0\beta}, \mathcal{H}_T' \right\} \approx \xi^0 \dot{\lambda}_{ij}^{0\beta} \), which is based on the fact that \( \mathcal{H}_T \) does not depend on the derivatives of momentum.
variables. For GR$_\parallel$ this is verified by an explicit inspection of $\mathcal{H}_T$, while for the general case we use the arguments of Appendix C.

In a similar way, one can find the (on-shell) transformation rules for the other fields, and they all agree with Eq. (2.3). The calculations are based on using the Hamiltonian equations of motion and the sure constraints $R^{ij}_{\alpha\beta} \approx 0$.

To summarize, the only properties used in the derivation are:

- $\mathcal{H}_T$ does not depend on the derivatives of momenta,
- it governs the time evolution of dynamical variables by $\dot{Q} = \{Q, \mathcal{H}_T\}$,
- $R^{ij}_{\alpha\beta} \approx 0$ are the sure constraints of the theory.

Consequently, the obtained transformation rules of the fields are correct for an arbitrary choice of parameters in the general teleparallel theory (2.1).

**Action on the momenta.** In the next step, we are going to compare the action of the generator (5.1) on the momenta ($\pi^\mu$) with the correct transformation rules for these variables. The correct rules are determined by the defining relation $\pi_A = \partial\mathcal{L}/\partial\dot{\varphi}^A$, and the known transformation laws for the fields. The general formula derived in Appendix B leads to

$$
\delta\pi^\mu = \omega^s_i\pi^s + \xi^\rho_{,\rho}\pi^\rho + \xi^0_{,\gamma}\pi^\gamma
- \xi^0_{,\gamma}\pi^\mu
\approx
\omega^s_i\pi^s + \xi^\rho_{,\rho}\pi^\rho + \xi^0_{,\gamma}\pi^\gamma
- \xi^0_{,\gamma}\pi^\mu
$$

To check if the generator (5.1) produces the above gauge transformations of momenta, we shall use the results of Appendix C, and the relations $\phi^\alpha = \partial\mathcal{L}/\partial\dot{\varphi}^A = -4\lambda^\mu_{ij}^{\alpha}$, which characterize the teleparallel theory (2.1) for any choice of the parameters $A, B, C$.

We begin by noting that the generator (5.1) has the standard form in the tetrad sector; hence, it follows that the transformation law for $\pi^{\mu}$ has the correct form given in Eq. (5.2), as has been shown in Ref. [14].

Since all $\omega^ij$ transformations can be verified straightforwardly, we focus our attention on $\xi^\mu$ transformations. Consider, first, the $\xi^\mu$ transformations of $\pi_{ij}^\alpha$:

$$
\delta_0(\xi^0)\pi_{ij}^\alpha = \int d^3x\{\pi_{ij}^\alpha_{,\alpha} - \langle\xi^0P_0\rangle\} = -\xi^0_{,\gamma}\pi_{ij}^\alpha - 4\xi^0_{,\gamma}\lambda_{ij}^{\alpha\gamma}
= \xi^0_{,\gamma}\pi_{ij}^\alpha - 4\xi^0_{,\gamma}\lambda_{ij}^{\alpha\gamma},
$$

$$
\delta_0(\xi^\gamma)\pi_{ij}^\alpha = \int d^3x\{\langle\xi^\gamma\pi_{ij}^\alpha\rangle - \langle\xi^\gamma\pi_{ij}^\alpha\rangle\}
= \xi^\alpha_{,\gamma}\pi_{ij}^0 - \partial_\gamma(\xi^\gamma\pi_{ij}^\alpha) + \xi^\alpha_{,\gamma}\pi_{ij}^\gamma.
$$

Here, the last (weak) equality is obtained by discarding terms proportional to $\phi_{ij}^\alpha$. Comparing with Eq. (5.4), we find the complete agreement. Note that these transformations, combined with those in Eq. (2.3), lead to $\phi_{ij}^\alpha \approx 0$, as they should.

In a similar manner, one can show that the $\xi^\mu$ transformations of $\pi_{ij}^0$ and $\pi_{ij}^{\mu\nu}$ agree with those displayed in Eq. (5.2). In the process of demonstrating this property we have used only those relations that characterize an arbitrary teleparallel theory. Hence,

*the expression (5.1) is the correct generator of Poincaré gauge transformations for any choice of parameters in the teleparallel theory (2.1).*
6. Concluding remarks

Gauge structure of the general teleparallel theory (2.1) is characterized by some specific features, as compared to the standard PGT [14]. We found two types of sure gauge symmetries, which are always present in the theory, independently of the values of parameters in Eq. (2.1).

The first type is the so-called $\lambda$ symmetry (2.4), with 24 Lagrangian parameters. The related canonical generator (3.2) is based on the PFC constraints $\pi_{ij\alpha\beta}$. The number of independent parameters of the $\lambda$ symmetry is shown to be not 24, but only 18, which clarifies the true dynamical meaning of the covariant Lagrangian symmetry. They can be used to fix 18 Lagrange multipliers, while the remaining 18 can be found using the same number of independent field equations (2.2b).

The second type is the usual Poincaré gauge symmetry (2.3). Although the meaning of this symmetry is well known, the construction of the related gauge generator shows some unusual features. In particular, we have found in the simple model of Sect. 4 that local translations are not obtained as independent gauge transformations, but rather emerge as a part of the Lorentz and $\lambda$ symmetries.

The Hamiltonian analysis of the present paper gives a very clear picture of the general gauge structure of the teleparallel theory, which has been the subject of many discussions in the past [8, 9]. The two gauge symmetries completely describe the gauge structure of the theory (2.1), in the sense that there are no other sure gauge symmetries. The canonical gauge generators obtained here will be very useful in studying the important problem of the conservation laws of energy, momentum and angular momentum [11, 12].

Acknowledgments

This work was partially supported by the Serbian Science Foundation, Yugoslavia.

Appendix A: Canonical description of GR$_\parallel$

In this Appendix, we present some formulas related to the canonical description of GR$_\parallel$. The canonical Hamiltonian density of GR$_\parallel$ can be written in the form [12]

\begin{equation}
\mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \frac{1}{2} A^{ij}_0 \mathcal{H}_{ij} - \lambda_{ij}^{\alpha\beta} R_{ij}^{\alpha\beta} + \partial_\alpha D^\alpha, \tag{A.1a}
\end{equation}

where

\begin{align*}
\mathcal{H}_{ij} &= 2\pi_{ij}^{\alpha} b_{j}^\alpha + \nabla_\alpha \pi_{ij}^\alpha, \\
\mathcal{H}_\alpha &= \pi_i^{\beta} T_{i\alpha\beta} - b^k_\alpha \nabla_\beta \pi_k^{\beta}, \\
\mathcal{H}_\perp &= \frac{1}{2} P_T^2 - J \mathcal{L}_T (T) - n^k \nabla_\beta \pi_k^\beta, \\
D^\alpha &= b^k_0 \pi_k^\alpha + \frac{1}{2} A_{ij}^0 \pi_{ij}^\alpha, \tag{A.1b}
\end{align*}

and

\begin{align*}
P_T^2 &= \frac{1}{2aJ} \left( \pi_{(ik)}^{\hat{n}} \pi_{(k)}^{\hat{n}} \right) - \frac{1}{2} \pi_{mn}^\alpha \pi_{m}^{\hat{n}} \pi_{n}^{\hat{n}}, \\
\mathcal{L}_T (T) &= a \left( \frac{1}{4} T_{m\bar{n}k} T^{m\bar{n}k} + \frac{1}{2} T_{m\bar{n}k} T^{m\bar{n}k} - T_{\bar{m}k} T^{\bar{m}k} \right). \tag{A.1c}
\end{align*}
Here, $\nabla_k = h_k^\mu \nabla_\mu$ is the covariant derivative, $n_k = h_k^0/\sqrt{g^{00}}$ is the unit normal to the hypersurface $x^0 = \text{const}$, the bar over the Latin index is defined by the decomposition

$$V_k = V_{\perp} n_k + V_k, \quad V_{\perp} = n^k V_k,$$

of an arbitrary vector $V_k$, $\hat{\pi}_{ik} = \pi_i^\alpha b_{k\alpha}$, $N$ and $N^\alpha$ are lapse and shift functions, respectively, $N = n_k b_k^0$, $N^\alpha = h^\alpha_0 b^0_k$, and $J$ is defined by $b = N J$. Note that $H_c$ is linear in unphysical variables ($b^0_k, A_{ij}^0, \lambda_{ij}^{\alpha\beta}$).

A specific feature of GR is the existence of the extra PFC constraints $\tilde{\phi}_{ij}$, which appear in the total Hamiltonian (2.5a):

$$\tilde{\phi}_{ik} = \phi_{ik} - \frac{1}{4} a \left( \pi_i^a s_0 B_{ik}^{0a} + \pi_k^s 0a B_{is}^{0a} \right),$$

$$\phi_{ik} = \hat{\pi}_{ik} - \hat{\pi}_{ki} + a \nabla_\alpha B_{ik}^{0\alpha}, \quad B_{ik}^{0\alpha} \equiv \varepsilon^{\alpha\beta\gamma} \varepsilon_{ikmn} b_m^\beta b^n_\gamma. \quad (A.2)$$

### Appendix B: General transformation laws for momenta

To find the correct transformation rules for momentum variables $\pi_A = \partial L/\partial \dot{\varphi}^A$ with respect to the spacetime transformations

$$x'^\mu = x^\mu + \xi^\mu, \quad \varphi'^A(x') = \varphi^A(x) + \delta \varphi^A(x),$$

we assume that the Lagrangian $L(\varphi^A, \partial \varphi^A)$ is a scalar density,

$$\delta_0 L + \partial_\mu (\xi^\mu L) = 0.$$

Then, it follows that the momenta $\pi_A$ transform in the following way:

$$\delta \pi_A = -\pi_B \frac{\partial \delta \varphi^B}{\partial \varphi^A} + \xi^0 \partial_\alpha \frac{\partial L}{\partial \varphi^A,_{\alpha}} - \xi^{\alpha}_{\alpha} \pi_A, \quad (B.1)$$

where $\delta = \delta_0 + \xi^\rho \partial_\rho$.

Applying this general formula to the Poincaré gauge transformations (2.3), we obtain the transformation laws (5.2) for $(\pi_k^\mu, \pi_{ij}^\mu, \pi^{ij}_{\mu\nu})$.

### Appendix C: Dependence of $H_T$ on fields and momenta

Let us consider Lagrangians $L(\varphi^A, \partial \varphi^A)$ that are at most quadratic in the first field derivatives. In this case, the canonical momenta $\pi_A = \partial L/\partial \dot{\varphi}^A$ are functions linear in $\dot{\varphi}^A$. If the canonical Hamiltonian is constructed in the standard way, $H_c = \pi_A \dot{\varphi}^A - L$, the corresponding total Hamiltonian $H_T = H_c + u^k \phi_k$ can be written in the same way:

$$H_T = \pi_A \dot{\varphi}^A - L, \quad (C.1a)$$

where, now, the velocities $\dot{\varphi}^A$ are functions not only of the fields and momenta, but also of the multipliers,

$$\dot{\varphi}^A = \dot{\varphi}^A(\varphi, \pi, u). \quad (C.1b)$$
The velocities $\dot{\varphi}^A$ can depend on momentum derivatives only through the determined multipliers.

Now, using Eq. (C.1), we find how $\mathcal{H}_T$ depends on momentum derivatives:

$$\frac{\partial \mathcal{H}_T}{\partial \pi_{A,\alpha}} = \left( \pi_B - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^B} \right) \frac{\partial \dot{\varphi}^B}{\partial \pi_{A,\alpha}} \approx 0 \ .$$  \hspace{1cm} (C.2)

In a similar way, one can see that $\mathcal{H}_T$ does not depend on higher derivatives of momenta, either.

As for the dependence of $\mathcal{H}_T$ on fields, we can write

$$\frac{\partial \mathcal{H}_T}{\partial \varphi^A} = - \frac{\partial \mathcal{L}}{\partial \varphi^A} + \left( \pi_B - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^B} \right) \frac{\partial \dot{\varphi}^B}{\partial \varphi^A} \approx - \frac{\partial \mathcal{L}}{\partial \varphi^A},$$  \hspace{1cm} (C.3)

and similarly for the field derivatives,

$$\frac{\partial \mathcal{H}_T}{\partial \varphi^{A,\alpha}} \approx - \frac{\partial \mathcal{L}}{\partial \varphi^{A,\alpha}}, \hspace{1cm} \frac{\partial \mathcal{H}_T}{\partial \varphi^{A,\alpha\beta}} \approx - \frac{\partial \mathcal{L}}{\partial \varphi^{A,\alpha\beta}}. \hspace{1cm} (C.4)$$

Notice that the equalities (C.3) and (C.4) are on-shell equalities. Our $\mathcal{H}_T$ in (2.5) is, by construction, of the same type as $\mathcal{H}_T$ in (C.1), and consequently, satisfies the above relations.

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