LOCAL DERIVATIVE ESTIMATES FOR HEAT EQUATIONS ON Riemannian Manifolds

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Abstract. In this short note we present local derivative estimates for heat equations on Riemannian manifolds following the line of W.-X. Shi. As an application we generalize a second derivative estimate of R. Hamilton for heat equations on compact manifolds to noncompact case.

1. Introduction

In [S] W.-X. Shi got local derivative estimates for Hamilton’s Ricci flow which is very important for later developments of geometric evolution equations, see for example Hamilton [H2] and Perelman [P1] [P2].

In this short note we present local derivative estimates of heat equations on Riemannian manifolds (with static metrics) following the line of Shi.

Theorem 1 Let \( \mathcal{M}^n \) be a Riemannian manifold. Suppose \( u \) is a smooth solution to the heat equation on \( B_p(2r) \times [0, T] \) for some \( p \in \mathcal{M}^n \) satisfying \( |u| \leq M \). Then there exist constants \( C_k \) depending only on the dimension and the bounds of curvature and the covariant derivatives (up to order \( k-1 \)) of the curvature on \( B_p(2r) \) such that

\[
|\nabla^k u|^2 \leq C_k M^2 \left( \frac{1}{r^2} + \frac{1}{t} + K^k \right),
\]

hold on \( B_p(r) \times (0, T] \), where \( K \) is the bound of curvature on \( B_p(2r) \).

For a similar but not identical estimate for harmonic map heat flow see Grayson and Hamilton [GH].

As an application we generalize a second derivative estimate of R. Hamilton [H1] for heat equations on compact manifolds to noncompact case.

Theorem 2 Let \( \mathcal{M}^n \) be a complete Riemannian manifold with bounded curvature and covariant derivative (of first order) of Ricci curvature. Suppose \( u \) is a smooth positive solution to the heat equation on \( \mathcal{M}^n \) satisfying \( u \leq M \). Then for \( 0 \leq t \leq 1 \) we have

\[
t \Delta u \leq C u \left[ 1 + \log(M/u) \right],
\]

where \( C \) depends only on the bounds of curvature and covariant derivative (of first order) of Ricci curvature.

In the remaining two sections we prove these two theorems respectively.
2. Proof of Theorem 1

We follow the line of Shi [S] (in particular see the exposition in Cao-Zhu [CZ] which follows Hamilton [H3] in turn). It is a Bernstein-type estimate coupled with a cutoff argument.

First note that for a solution $u$ to the heat equation we have

\[ \frac{\partial}{\partial t} \nabla^k u = \Delta \nabla^k u + \sum_{i=0}^{k-1} \nabla_i Rm * \nabla^{k-i} u, \]

( in fact, when $k = 2$, we have

\[ \frac{\partial}{\partial t} \nabla^2 u = \Delta \nabla^2 u + Rm * \nabla^2 u + \nabla Rc * \nabla u \]

and

\[ \frac{\partial}{\partial t} |\nabla^k u|^2 = \Delta |\nabla^k u|^2 - 2|\nabla^{k+1} u|^2 + \sum_{i=0}^{k-1} \nabla_i Rm * \nabla^{k-i} u \nabla^k u \]

\[ \leq \Delta |\nabla^k u|^2 - 2|\nabla^{k+1} u|^2 + C|\nabla^k u|^2 + C \sum_{i=1}^{k-1} |\nabla^{k-i} u|^2, \]

where and below $C$ denotes various constants depending only on the dimension and the bounds of the curvature and the covariant derivatives of the curvature on $B_p(2r)$, and where * denotes some linear tensor contraction, possible including constants.

Now we prove Theorem 1 by induction. Note that without loss of generality we may assume $r \leq 1/\sqrt{K}$. We first consider the case $k = 1$.

Let $S_1(x, t) = (TM^2 + a^2)|\nabla u|^2$. Then by Cauchy inequality we have

\[ (\frac{\partial}{\partial t} - \Delta) S_1 \leq -|\nabla u|^2 + 16M^2 K |\nabla u|^2 \]

\[ \leq -\frac{1}{2} |\nabla u|^2 + 128M^4 K^2 \]

\[ \leq -\frac{1}{64M^2} S_1 + 128M^4 K^2. \]

Let $F_1 = \frac{S_1}{128M^2}$. Then

\[ \frac{\partial F_1}{\partial t} \leq \Delta F_1 - F_1^2 + K^2. \]

Fix a point $q \in B_p(r)$. As in [CZ] we choose a cutoff function $\varphi$ with support in the ball $B_q(r)$ such that $\varphi(q) = 0, 0 \leq \varphi \leq Ar$ and $|\nabla \varphi| \leq A, |\nabla^2 \varphi| \leq \frac{A}{r}$, $A$ depending only on the dimension of $M$. (For our purpose we may pretend that $\varphi$ be smooth everywhere by Calabi's trick.)

Let $H_1 = \frac{(12 + 4\sqrt{5}) A^2}{\varphi^2} + \frac{1}{t} + K$. Then

\[ \frac{\partial H_1}{\partial t} \geq \Delta H_1 - H_1^2 + K^2. \]

By maximum principle $F_1 \leq H_1$ which implies $|\nabla u|^2 \leq C_1 M^2 (\frac{1}{r^2} + \frac{1}{r} + K)$. Now suppose we have the bounds

\[ |\nabla^i u|^2 \leq C_i M^2 (\frac{1}{r^2} + \frac{1}{r} + K) \]

for $i \leq k$. Then

\[ \frac{\partial}{\partial t} |\nabla^k u|^2 \leq \Delta |\nabla^k u|^2 - 2|\nabla^{k+1} u|^2 + CM^2 (\frac{1}{r^2} + \frac{1}{r} + K) \]

\[ \frac{\partial}{\partial t} |\nabla^{k+1} u|^2 \leq \Delta |\nabla^{k+1} u|^2 - 2|\nabla^{k+2} u|^2 + C|\nabla^{k+1} u|^2 + CM^2 (\frac{1}{r^2} + \frac{1}{r} + K) \]

Let $S_k(x, t) = [B_k M^2 (\frac{1}{r^2} + \frac{1}{r} + K) + |\nabla^k u|^2] \cdot |\nabla^{k+1} u|^2$. Choosing $B_k$ large enough and using Cauchy inequality, we have

\[ (\frac{\partial}{\partial t} - \Delta) S_k \leq [-kB_k M^2 t^{k-1} - 2 |\nabla^{k+1} u|^2 + CM^2 (\frac{1}{r^2} + \frac{1}{r} + K)] \cdot |\nabla^{k+1} u|^2 + 8 |\nabla^{k+1} u|^2 \cdot |\nabla^{k+2} u|^2 + [B_k M^2 (\frac{1}{r^2} + \frac{1}{r} + K) + |\nabla^k u|^2] \cdot [-2 |\nabla^{k+2} u|^2 + C |\nabla^{k+1} u|^2 + CM^2 (\frac{1}{r^2} + \frac{1}{r} + K)] \]

\[ \leq - |\nabla^{k+1} u|^4 + CB_k^2 M^4 (\frac{1}{r^2} + \frac{1}{r} + K) \]

\[ \leq \frac{S_k^2}{(B_k + C_k)^2 M^4 (\frac{1}{r^2} + \frac{1}{r} + K)^2} + CB_k^2 M^4 (\frac{1}{r^2} + \frac{1}{r} + K) \]

Let $v = \frac{1}{r^2} + \frac{1}{r} + K$ and set $F_k = b S_k / v^k$. Then

\[ \frac{\partial F_k}{\partial t} \leq \Delta F_k - \frac{b B_k^2}{b M^4 M^4} + b CB_k^2 M^4 v^k + k F_k v \]

\[ \leq \Delta F_k - \frac{b}{b M^4 M^4} + \frac{b}{b} (2C + k^2) (b + C_k) M^4 v^{k+2}. \]

By choosing $b \leq 2/(2C + k^2)(b + C_k)^2 M^4$, we get
\[ \frac{\partial F_k}{\partial t} \leq \Delta F_k - \frac{1}{v_k} F_k^2 + v^{k+2}. \]

As in [CZ] we introduce
\[ H_k = 5(k+1)(2(k+1) + 1 + \sqrt{n})A^2 \varphi^{-2(k+1)} + Lt^{-(k+1)} + K^{k+1}, \]
where \( L \geq k+2 \). Then we easily check
\[ \frac{\partial H_k}{\partial t} > \Delta H_k - \frac{1}{v_k} F_k^2 + v^{k+2}. \]

By maximum principle we have
\[ F_k \leq H_k, \]
from which one immediately get the desired estimate
\[ |\nabla^{k+1} u|^2 \leq C_{k+1} M^2 \left( \frac{1}{r^2} + \frac{1}{t^2} + K^{k+1} \right). \]

**Corollary** Let \( M^n \) be a complete Riemannian manifold with bounded curvature and covariant derivative (of first order) of Ricci curvature. Suppose \( u \) is a smooth solution to the heat equation on \( M^n \) for \( t \in [0, T] \) satisfying \( u \leq M \). Then for \( t \in (0, T] \) we have
\[ |\nabla^2 u|^2 \leq C^2 M^2 \left( \frac{1}{t^2} + K^2 \right), \]
where \( C^2 \) depends only on the bounds of curvature and covariant derivative (of first order) of Ricci curvature.

### 3. Proof of Theorem 2

The idea is to use Ni-Tam’s generalized maximum principle of noncompact manifold in [NT] (which is originally due to Karp and Li).

Let \( h = \varphi \Delta u + \frac{1}{2} |\nabla u|^2 - u[n + 4\log(M/u)], \) where \( \varphi = (e^{Kt} - 1)/Ke^{Kt}. \) As in [H1], we have
\[ \frac{\partial h}{\partial t} \leq \Delta h \text{ whenever } h \geq 0. \]

It is easy to see that \( \varphi \leq t \) for \( t \geq 0. \) Then using Theorem 1 (actually the Corollary), for any \( p \in M^n \) and \( r > 0 \) we get
\begin{align*}
&\int_0^1 \int_{B_r(p)} e^{-d^2(x,p)}(\varphi \Delta u)^2 dV dt \\
&\leq \int_0^1 \int_{B_r(p)} e^{-d^2(x,p)} nC^2 M^2 \left( \frac{1}{r^2} + K^2 \right) dV dt \\
&\leq nC^2 M^2 (1 + K^2) \int_0^1 \int_{M^n} e^{-d^2(x,p)} dV dt \\
&\leq C < \infty,
\end{align*}
where the constant \( C \) does not depend on \( r \). So we get
\[ \int_0^1 \int_{M^n} e^{-d^2(x,p)}(\varphi \Delta u)^2 dV dt < \infty. \]

Combining with Kotschwar’s estimate in [K] (or one can use the case \( k = 1 \) of our Theorem 1 instead) we get that
\[ \int_0^1 \int_{M^n} e^{-d^2(x,p)} h_+^2 dV dt < \infty, \]
where \( h_+(x, t) := \max\{h(x, t), 0\}. \)

So by the maximum principle in [NT] we get that \( h \leq 0 \), and the desired result follows.

**Remark** A similar argument was used by Kotschwar [K] to generalize the gradient estimate of Hamilton [H1] to noncompact case.

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**Reference**
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