Recent results on small $\mu_B$ QCD from the lattice

A. Pásztor
Department of Physics, Wuppertal University, Gaussstr. 20, D-42119 41 Wuppertal, Germany
E-mail: apasztor@bodri.elte.hu

Abstract. I will briefly review our current understanding of the phase diagram of QCD, from the point of view of recent first principle lattice calculations. Whenever possible, I will also make contact with the experimental efforts of the RHIC Beam Energy Scan Program.

1. Introduction
The equilibrium phase diagram of QCD remains vastly unexplored. At $\mu = 0$ there are some well established results from lattice simulations. E.g. the QCD transition at $\mu = 0$ is a smooth crossover [1]. The equation of state (EoS) near and above the cross-over is also known [2, 3] in 2+1 flavour QCD. Recently, the EoS of 2+1+1 flavour QCD, was also calculated [4]. In this conference contribution I will focus on reviewing some of the recent progress on small finite $\mu_B$ QCD. The large $\mu$ area of the phase diagram still remains unreachable by lattice techniques, due to the complex action problem at finite density.

Probably the most important unresolved issue is about the QCD critical point. It is often conjectured that the crossover line eventually becomes a first order transition line in the $\mu - T$ plane, the two being separated by a critical point. The existence of this critical point is not established however, and if it exists, we do not know where it is. The discovery of the critical point and its location is one of the main challenges in the physics of strong interactions, and one of the main focuses of the RHIC Beam Energy Scan II program, scheduled for 2019 and 2020.

2. Taylor expansion vs. analytical continuation
So far there are only two methods that were used for full QCD (meaning lattice QCD simulations with physical quark masses and also preforming continuum extrapolation). The Taylor expansion method and the analytical continuation method [6, 7]. Both of these start by evaluating fluctuations of conserved charges, i.e. derivatives of the pressure with respect to the chemical potentials:

$$\chi_{u,d,s}^{i,j,k} = \frac{\partial^{i+j+k} (p/T^4)}{\partial \mu_u \partial \mu_d \partial \mu_s} \bigg|_{\mu = 0}.$$

(1)

One can also consider derivatives with respect to $\mu_B/T$, $\mu_I/T$, $\mu_Q/T$ instead. The Taylor expansion method calculates the higher derivatives at $\mu = 0$ directly while the analytical

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1 2+1 here means 2 light quarks with a degenerate mass and a strange quark
2 2+1+1 here means 2 light quark with a degenerate mass, a strange quark and a charm quark
continuation method calculates lower derivatives at different values of $\text{Im}\mu$, and uses fits in $\text{Im}\mu$ to estimate the higher derivatives.

These derivatives are interesting for several reasons. They are sensitive to criticality. They provide stringent tests of the hadron resonance gas (HRG) model at low and resummed perturbation theory at high temperatures\(^3\). They can also be used for extrapolation of observables to small finite $\mu$.

A major difficulty is that remnants of the sign problem are encountered when evaluating the $\chi_{i,j,k}^{a,d,s}$. More concretely, there are some huge cancellations, as say $\chi_{2k}^{T}$ is finite as $V \to \infty$, but is the sum of terms scaling up to $O(V^{k-1})$. The Taylor expansion method therefore penalizes calculating higher derivatives at larger volumes. This already leads us to the main differences between the direct Taylor expansion method and the analytical continuation method.

![Analytical continuation on $N_t = 12$ raw data](image)

**Figure 1.** Illustrating analytical continuation for the observable $\frac{T}{\mu_B} \frac{d(\mu/T^4)}{d(\mu_B/T)}$ for $N_t = 12$ lattice data. The Taylor expansion method only uses, the black $\mu = 0$ point, but with much higher statistics.

The obvious disadvantage of the analytical continuation method is the presence of systematic error, due to the extrapolation to real $\mu$. This is illustrated in Fig. 1. The main advantage is that the analytical continuation method penalizes large volumes much less than the Taylor expansion. This is a consequence of the fact that a lower number of derivatives has to be taken directly.

3. Some recent results

3.1. **Taylor coefficients and EoS at small finite $\mu$**

The QCD pressure can be written as a Taylor expansion around $\mu_B/T = 0$ [8]:

$$
\frac{p(T, \mu_B)}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left( \frac{\mu_B}{T} \right)^{2n}.
$$

The coefficients $c_i(T)$ can be obtained from lattice simulations, either directly at $\mu_B = 0$ or by using analytical continuation techniques from imaginary $\mu_B$. The first results for $c_2$, $c_4$ and $c_6$ were obtained at finite lattice spacing and for heavier-than-physical quark masses [8]. The first continuum extrapolated results for $c_2$ appeared in Ref. [9]; $c_4$ was obtained for the first time in Ref. [10] at finite lattice spacing. The continuum-extrapolated results for $c_6$ were calculated for the first time in [11], and later in [12]. Ref. [13] includes a first determination of $c_8$, at two values of the temperature and for $N_t = 8$. The results from Ref. [11] can be seen in Fig. 2. In our simulations at imaginary baryonic chemical potential, the choice has to be made for

\(^3\) See e.g. [5] for many comparisons.
the strangeness $\mu_S$ and electric charge $\mu_Q$ chemical potential values to be such that we have vanishing net-strangeness and net-electric charge being 0.4 times the net-baryon number, like it is for the lead or gold nuclei used for heavy ion collision experiments.

### 3.2. Curvature of the crossover temperature

Just like with the equation of state, the transition temperature can also be expanded as a Taylor series around $\mu_B = 0$ [16]:

$$\frac{T_c(\mu_B)}{T_c(0)} = 1 - \kappa \left( \frac{\mu_B}{T_c(0)} \right)^2 + \lambda \left( \frac{\mu_B}{T_c(0)} \right)^4 + \ldots .$$  \hspace{1cm} (3)

The Taylor coefficients $\kappa$ and $\lambda$ here can be extracted by either direct Taylor expansion at $\mu = 0$ or analytical continuation from imaginary $\mu$. Here, $\kappa$ is the curvature of the phase diagram at $\mu_B = 0$. The results available in the literature correspond to different choices for $\mu_S$. A value of $\kappa = 0.020(4)$ [15] has been found when $\mu_s = \mu_{u,d}$, and $m_s/m_{u,d} = 20$. Ref. [14] got $\kappa = 0.0135(20)$ at $\mu_S = 0$. In Ref. [16], a value of $\kappa = 0.0149(21)$ was found in the case of strangeness neutrality ($\langle n_s \rangle = 0$). There are no results on the second coefficient $\lambda$ as of now.

### 3.3. Fugacity expansion and hadron chemistry

For simplicity we set $\mu_Q = 0 = \mu_S$. Instead of the Taylor expansion, one can consider the fugacity expansion $p(T, \mu_B)/T^4 = \sum_{k=0}^{\infty} a_k(T) \cosh(k \mu_B/T)$. At $\mu_B = 0$, the pressure is a sum of the $a_k$ coefficients, so these can be interpreted as the partial pressures from the sectors of the Hilbert space with a value for the conserved baryon number. If one considers $\chi_B^\beta$ at an imaginary $\mu_B = i\bar{\mu}_B$, one gets:

$$\chi_B^\beta(T, i\bar{\mu}_B) = i \sum_{k=1}^{\infty} b_k(T) \sin(k \bar{\mu}_B/T),$$  \hspace{1cm} (4)
where \( b_k = k a_k \). If the \( \mu_B \)-dependence of \( \chi_B^R \) is known, then the \( b_k \) can be calculated by a simple Fourier transform. The real \( \mu_B \) fugacity expansion coefficients correspond to Fourier coefficients at imaginary \( \mu_B \). This makes the fugacity expansion more natural to consider, when working with imaginary \( \mu \) lattice data, as compared to the Taylor expansion in \( \mu_B \). If we consider now a nonzero \( \mu_S \) as well, we get a double expansion in both \( e^{\mu_S/T} \) and \( e^{\mu_B/T} \), and we can separate the Hilbert space even further, to get the partial pressure from say strange hadron as opposed to non-strange hadrons. This possibility of decomposing the thermodynamics to partial pressures is especially useful when testing hadronic model near and below \( T_c \), as the contributions from different sectors do not mix. This was pointed out in [17], where the different strangeness sectors were separated, and used to constrain the hadronic spectrum in the HRG model. The strangeness partial pressures obtained in [17] can be seen in Fig. 3. The HRG does not undershoot the full pressure and the strangeless baryonic partial pressure, it does undershoot however every partial pressure with nonzero strangeness, including mesons and baryons.

![Figure 3](image.png)

**Figure 3.** From Ref. [17]. The partial pressures of the different strangeness sectors together with the standard PDG hadron spectrum HRG prediction.

An other application was in [18], where only the baryon number sectors were separated, and used to test repulsive baryonic interactions, introduced as an excluded volume correction. It was found that including repulsive interactions between baryons make hadronic description much better near and slightly above \( T_c \) as opposed to the ideal HRG. A summary of the results can be seen in Fig. 4.

These were just some examples of recent progress in small \( \mu \) physics, a very popular topic in lattice QCD at the moment. Much remains to be done in small \( \mu \) physics still, but to tackle the real challenge, extending out knowledge to higher \( \mu \), fresh ideas are needed.

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Figure 4. The temperature dependence of the first four Fourier coefficients on $N_t = 12$ lattices, and from the excluded volume HRG model with excluded volume $b = 1$ fm$^3$ (solid lines) on a linear (left) and logarithmic (right) scale. The dashed lines show the calculations within the Van der Waals HRG model, with van der Waals parameters $a$ and $b$ fixed by the properties of the nuclear ground state [19]. The arrows are the Stefan-Boltzmann limit. Ideal HRG predicts $b_2 = b_3 = b_4 = \cdots = 0$, while EV-HRG captures the sign structure seen in the lattice data.

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