Approximating Optimal Transport via Low-rank and Sparse Factorization

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November 15, 2021

Abstract

Optimal transport (OT) naturally arises in a wide range of machine learning applications but may often become the computational bottleneck. Recently, one line of works propose to solve OT approximately by searching the transport plan in a low-rank sub-space. However, the optimal transport plan is often not low-rank, which tends to yield large approximation errors. For example, when Monge’s transport map exists, the induced transport plan is full rank. This paper concerns the computation of the OT distance with adequate accuracy and efficiency. A novel approximation for OT is proposed, in which the transport plan can be decomposed into the sum of a low-rank matrix and a sparse one. We theoretically analyze the approximation error. An augmented Lagrangian method is then designed to efficiently calculate the transport plan.

1 Introduction

Optimal transport (OT) defines the distance between two probability measures [Villani, 2009]. It has a wide range of machine learning applications, including generative modeling [Arjovsky et al., 2017], domain adaptation [Courty et al., 2016], and data mining [Xu et al., 2019], to name but a few.

Despite the broad applications, calculating the OT distance easily becomes the computational bottleneck in real-world problems. Originally, the OT problem was solved via linear programming, which involves the formidable computational complexity $O(n^3 \log n)$ where $n$ is the size of discrete measures [Tarjan, 1997]. A popular method, known as Sinkhorn’s method, regularizes the transport plan with its entropy and accelerates the optimization of the transport plan [Cuturi, 2013]. Given a data-dependent cost matrix $C$, each iteration of Sinkhorn’s method takes the form of matrix-vector products $\exp(-C \eta) u$ or $\exp(-C \eta)^T v$ where $\eta$ is the weight of the entropy regularizer and $u, v$ are $n$-dimensional vectors. Such an approach can obtain an $\epsilon$-approximation of the OT distance with complexity $O(n^2 \log \frac{1}{\epsilon})$, which is still computationally expensive when a highly accurate solution is required.

Low-rank kernel factorization. Recently, one line of works speeds up the Sinkhorn’s method by using a rank-$r$ approximation of $\exp(-C \eta)$ [Altschuler et al., 2019; Altschuler and Boix-Adsera, 2020; Scetbon and Cuturi, 2020]. Adopting this approximation, the cost for each iteration of the Sinkhorn’s method can be reduced to $O(nr^2)$. However, to approximate $\exp(-C \eta)$ with sufficient accuracy, $r$ can be large [Altschuler et al., 2019]. When $r^2$ is close to or ever larger than $n$, these methods can hardly yield improved performance over Sinkhorn’s method.

Low-rank transport plan. Another family of works accelerates the calculation by searching the transport plan in a low-rank sub-space [Forrow et al., 2019; Lin et al., 2021; Scetbon et al., 2021]. However, the optimal transport plan may not be low-rank, which leads to poor approximation of these methods. To illustrate this, consider a setting where the target probability measure is obtained by permuting the supports of the source measure. In such a case, the optimal transport plan is a permutation matrix which is full-rank.
In this paper, we propose a novel approximation for the OT distance in which the transport plan is in a new sub-space. The transport plan can be decomposed into the sum of a low-rank matrix and a sparse one. An inexact augmented Lagrangian method is designed to efficiently resolve the resulted optimization problem by solving a series of sub-problems. We handle each sub-problem via a block coordinate descent sub-routine. Our contributions are summarized as follows.

1. We theoretically analyze the error of the proposed approximation.
2. We propose an inexact augmented Lagrangian method to calculate the transport plan.

**Notation.** We use bold lowercase symbols (e.g. \( x \)), bold uppercase letters (e.g. \( X \)), uppercase calligraphic fonts (e.g. \( \mathcal{X} \)), and Greek letters (e.g. \( \alpha \)), to denote vectors, matrices, spaces (sets), and measures, respectively. \( 1^d \in \mathbb{R}^d \) and \( 0^d \in \mathbb{R}^d \) are the all-ones vector and the all-zeros vector respectively, where \( \mathbb{R}^d \) is the \( d \)-dimensional Euclidean space. \( \mathbb{R}^d_+ \) is the subspace of \( \mathbb{R}^d \) and contains non-negative entries. \( x \geq c \) (resp. \( X \geq c \)) means each element of vector \( x \) (resp. matrix \( X \)) is greater than or equal to scalar \( c \). Given a matrix \( X \), we denote by \( \|X\|_F \) its Frobenius norm, by \( |X|_0 \) its number of nonzero entries, by \( \|X\|_1 \) its elementwise \( \ell_1 \) norm (i.e., \( |X|_1 = \sum_{ij} |X_{ij}| \)), and by \( \|X\|_\infty \) its elementwise \( \ell_\infty \) norm (i.e., \( \|X\|_\infty = \max_{ij} |X_{ij}| \)). For two matrices \( A \) and \( B \) that are of the same size, \( (A, B) = \text{trace}(A^\top B) \) is the Frobenius dot-product. \( (a; b) \) denotes the concatenation of vectors \( a \) and \( b \). The vectorization of matrix \( X \) (in the row order) is denoted by \( \text{vec}(X) \). A discrete measure \( \alpha \) can be denoted by \( \alpha = \sum_{i=0}^{m-1} p_i \delta_{x_i} \), where \( \delta_{x_k} \) is the Dirac at position \( x \), i.e., a unit of mass infinitely concentrated at \( x \). With slight abuse of notation, we also use \( p = [p_i] \) to refer to \( \alpha \).

## 2 Preliminaries

### 2.1 Optimal Transport and Sinkhorn Method

The OT distance [Villani, 2009; Cuturi, 2013] between discrete measures \( p \) and \( q \) is defined as

\[
\text{OT}(p, q) = \min_{T \in \Pi(p, q)} \langle C, T \rangle,
\]

where the \((i, j)\)th entry of \( C \) is the distance between the \( i \)th support of \( p \) and the \( j \)th support of \( q \), and the feasible domain of transport plan \( T = [T_{ij}] \) is given by the set \( \Pi(p, q) = \{ T \in \mathbb{R}^{m \times n} | T1^m = p, T^\top 1^n = q \} \).

Cuturi 2013 proposes to solve the following the entropy regularized OT problem,

\[
\min_{T \in \Pi(p, q)} \langle C, T \rangle - \eta H(T),
\]

where \( H(T) \) is the entropy of the transport plan, i.e., \( H(T) = -\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T_{ij} (\log T_{ij} - 1) \). By setting \( \eta = \mathcal{O}\left(\frac{\epsilon}{\log mn}\right) \), the Sinkhorn’s method computes an \( \epsilon \)-approximate solution of the problem (1) in \( \mathcal{O}\left(\frac{\epsilon^2 \log mn}{\epsilon^2} \right) \) operations [Altschuler et al., 2017], which may still be too expensive for large-scale problems especially when a highly accurate solution of (1) is required.

### 2.2 Non-negative factorization of the Transport Plan

Scetbon et al. 2021 force the transport plan to be low-rank by using the notion of the non-negative rank which is formally defined as follows.

**Definition 1** The non-negative rank of matrix \( M \) is the smallest number of non-negative rank-one matrices into which \( M \) can be decomposed additively, i.e.,

\[
\text{rank}_+(M) = \min \{ q | M = \sum_{i=0}^{q-1} R_i, \forall i, \text{rank}(R_i) = 1, R_i \geq 0 \}.
\]
Specifically, they consider problem

\[
\text{LOT}_{r}(p, q) = \min_{T \in \Omega_{r}(p, q)} \langle C, T \rangle,
\]

where \( \Omega_{r}(p, q) = \{ T \in \Pi(p, q) \mid \text{rank}_r(T) \leq r \} \).

From Definition 1, one has \( \text{rank}(T) \leq \text{rank}_r(T) \leq r \) for all \( T \in \Omega_{r}(p, q) \). However, the optimal transport plan is often not low-rank, as we have explained in Introduction.

### 2.3 Low-rank and Sparse Decomposition

Approximating a matrix by the sum of a low-rank matrix and a sparse matrix has a long history [Candès et al., 2011]. Mathematically, such decomposition can be formulated as the following optimization problem

\[
\min_{L, S} \text{rank}(L) + \lambda \|S\|_0, \quad \text{s.t. } L + S = M.
\]

Such non-convex problem is computationally intractable. One common approach is to solve the following surrogate problem which has a convex objective (see, e.g. Wright et al. [2009]; Lin et al. [2011]),

\[
\min_{L, S} \|L\|_* + \lambda \|S\|_1, \quad \text{s.t. } L + S = M,
\]

where the nuclear norm and the \( \ell_1 \) norm induce the sparsity of the singular values of \( L \) and the entries of \( S \) respectively. As we shall see shortly, the low-rank and sparse decomposition yields better approximation for the OT distance (1) than the sole low-rank component.

### 3 Methodology

We first define the approximated distance which decomposes the transport plan into a low-rank matrix and a sparse one. Next, we derive an augmented Lagrangian method for calculating the proposed distance. Finally, we analyze the overall complexity of the proposed method.

#### 3.1 Approximation of OT Distance

Given non-negative integers \( r \ll \min\{m, n\} \) and \( \rho \ll mn \) that control the rank and the sparsity respectively, the proposed approximation of OT distance is

\[
\text{LSOT}_{r, \rho}(p, q) = \min_{T \in \tilde{\Omega}_{r, \rho}(p, q)} \langle C, T \rangle,
\]

where the feasible domain is defined as

\[
\tilde{\Omega}_{r, \rho}(p, q) = \{ T \in \Pi(p, q) \mid T = L + S, L \geq 0, S \geq 0, \text{rank}_r(L) \leq r, \|S\|_0 \leq \rho \}.
\]

We theoretically analyze the approximation error of \( \text{LSOT}_{r, \rho}(p, q) \) in the theorem below.

**Theorem 1** Denote \( T^* \in \arg\min_{T \in \Pi(p, q)} \langle C, T \rangle \). Assume there exist \( L^* \in \mathbb{R}^{m \times n} \) and \( S^* \in \mathbb{R}^{m \times n} \) such that \( T^* = L^* + S^* \), \( \text{rank}_r(L^*) = r^* \), and \( \|S^*\|_0 = \rho^* \). Let \( L^* \) and \( S^* \) be the best approximations of \( L^* \) and \( S^* \) respectively (in terms of the Frobenius norm) satisfying \( \text{rank}_r(L^*) \leq r \) and \( \|S^*\|_0 \leq \rho \). Then,

\[
\text{LSOT}_{r, \rho}(p, q) - \text{OT}(p, q) \leq \frac{U \|C\|_\infty}{\delta} \left( \sqrt{mn(r^* - r)} + (\rho^* - \rho)_+ \right),
\]

where \( U = \max\{\|L^*\|_\infty, \|S^*\|_\infty\} \) and

\[
\delta = \frac{1}{e^2} \min \left\{ \min_{i,j} \{ L^*_{ij} + S^*_{ij} | L^*_{ij} + S^*_{ij} > 0 \}, \min_{i,j} \{ \tilde{L}^*_{ij} + \tilde{S}^*_{ij} | \tilde{L}^*_{ij} + \tilde{S}^*_{ij} > 0 \} \right\}.
\]
The proof is deferred to the appendix. We further have the two following corollaries.

**Corollary 2** \( \text{LSOT}_{r,\rho}(p, q) \) recovers \( \text{OT}(p, q) \) with \( \rho \geq m + n - 1 \), i.e.,

\[
\text{LSOT}_{r,\rho}(p, q) = \text{OT}(p, q), \quad \forall r \geq 0, \text{ and } \rho \geq m + n - 1.
\]

This is the direct result of Theorem 1 and the fact that \( \text{OT} \) distance can be achieved with the transport plan containing up to \( m + n - 1 \) nonzero entries [Brualdi, 2006]. It is generally difficult to determine the non-negative rank of an optimal transport plan. However, \( \text{OT}(p, q) \) can be recovered with \( \rho \geq m + n - 1 \).

**Corollary 3** Denote \( T^* \in \arg\min_{T \in \Pi(p,q)} \langle C, T \rangle \). Assume there exist \( L^* \in \mathbb{R}^{m \times n} \) and \( S^* \in \mathbb{R}^{m \times n} \) such that \( T^* = L^* + S^* \), \( \text{rank}^+ (L^*_p) = r^* \), and \( \|S^*\|_0 = \rho^* \). Let \( L^*_r \) be the best approximations of \( L^* \) (in terms of the Frobenius norm) satisfying \( \text{rank}^+ (L^*_r) \leq r \). Then,

\[
\text{LOT}_r(p,q) - \text{OT}(p,q) \leq \frac{U}{\delta_1} \left( \sqrt{\text{min}(r^* - r, \rho^*)} \right),
\]

where \( U = \max\{\|L^*\|_\infty, \|S^*\|_\infty\} \) and

\[
\delta_1 = \frac{1}{c^2} \min \left\{ \min_{i,j} \{ L^*_{ij} + S^*_i l_{ij} + S^*_j > 0 \}, \min_{i,j} \{ \tilde{L}^*_{ij} l_{ij} > 0 \} \right\}.
\]

Setting \( \rho = 0 \), \( \text{LOT}_r(p,q) \) can obviously be recovered by \( \text{LSOT}_{r,\rho}(p,q) \). As is stated in Theorem 1 and Corollary 3, \( \text{LSOT}_{r,\rho}(p,q) \) approximates \( \text{OT}(p,q) \) better than \( \text{LOT}_r(p,q) \), if \( \rho^* > 0 \) for all \( T^* \).

### 3.2 Optimization

**Surrogate problem.** The \( \|S\|_0 \leq \rho \) constraint in \( \hat{\Omega}_{r,\rho}(p,q) \) makes problem (3) intractable. We thus consider the following surrogate problem

\[
\min_{A,B \in \Omega_{r,\rho}(p,q)} (C, AB^T + S) + \lambda \|S\|_1,
\]

where the feasible domain is given by

\[
\Omega_{r,\rho}(p,q) = \left\{ A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{n \times r}, S \in \mathbb{R}^{m \times n} | 0 \leq A \leq 1, 0 \leq B \leq 1, 0 \leq S \leq 1, (AB^T + S)1 = p, (AB^T + S)^T1 = q \right\}.
\]

Because of the bi-linear terms, (8) has a non-convex objective function and non-convex constraints. Stationary points can be found effectively by an inexact augmented Lagrangian method (ALM).

**Inexact ALM.** ALM is a classical algorithm for constrained optimization [Hestenes, 1969; Powell, 1969]. For solving (8), ALM suggests solving

\[
\min_{A,B,S,y^p,y^q} \mathcal{L}(A,B,S,y^p,y^q,\beta) + \lambda \|S\|_1 + I^A(A) + I^B(B) + I^S(S),
\]

where \( I^A(\cdot), I^B(\cdot) \) and \( I^S(\cdot) \) are indicator functions corresponding to \( 0 \leq A \leq 1, 0 \leq B \leq 1, \) and \( 0 \leq S \leq 1 \) respectively. The augmented Lagrangian function is defined as

\[
\mathcal{L}(A,B,S,y^p,y^q,\beta) = (C, AB^T + S) + \frac{\beta}{2} \left( \| (AB^T + S)1 - p \|^2 + \| (AB^T + S)^T1 - q \|^2 \right) + \langle y^p, (AB^T + S)1 - p \rangle + \langle y^q, (AB^T + S)^T1 - q \rangle.
\]
where \( y^p \in \mathbb{R}^m \) and \( y^q \in \mathbb{R}^n \) are multiplier variables, and \( \beta > 0 \) is the penalty parameter. In the algorithm and the later analysis, we use some additional notation. For ease of notation, variables \( A, B, \) and \( S \) are sometimes referred to as \( x \), where \( x = (\text{vec}(A):\text{vec}(B):\text{vec}(S)) \). Similarly, \( y = (y^p:y^q) \). With slight abuse of notation, we use \( \mathcal{L}(A,B,S,y^p,y^q) \) and \( \mathcal{L}(x,y,\beta) \) interchangeably. Based on \( x \), the equality constraints in \( \Omega_{r,p}(p,q) \) can be rewritten as \( \alpha(x) = 0 \), where \( \alpha: \mathcal{X} \rightarrow \mathbb{R}^{m+n} \) is a vector function with the \( u \)th entry given by

\[
\alpha_u(x) = \begin{cases} 
\sum_{j=0}^{u-1} (\sum_{k=0}^{r-1} A_{u,k} B_{j,k} + S_{u,j}) - p_u, & \text{if } 0 \leq u < m, \\
\sum_{j=0}^{m-1} (\sum_{k=0}^{r-1} A_{u-m,k} B_{j,u-m} + S_{j,u-m}) - q_{u-m}, & \text{otherwise}.
\end{cases}
\]

We further denote

\[
h(x) = \lambda\|S\|_1 + I^A(A) + I^B(B) + I^R(S).
\]

The pseudocode of the proposed inexact ALM is demonstrated in Algorithm 1. The high-level intuition is that we construct a series of strongly-convex and smooth sub-problems, each of which is solved inexactly by the block coordinate descent (BCD) in Algorithm 2. Such an approach is guaranteed by the following proposition. The proof is provided in the appendix.

**Proposition 4** Given \( y \) and \( \beta \), let \( G(x) = \mathcal{L}(x,y,\beta) + L(y,\beta)\|x-\bar{x}\|^2 \) where

\[
L(y,\beta) = \sqrt{2\tau}\|C\|_F + \sqrt{2\tau}(m\sqrt{n} + n\sqrt{m})\|y\| + \beta L_c,
\]

with

\[
B_u = \max_{x \in \mathcal{X}} \max_{\|\nabla \alpha_u(x)\|} \{|\alpha_u(x)|, \|\nabla \alpha_u(x)\|\},
\]

\[
L_c = \sum_{u=0}^{m-1} \sqrt{2m_r B_u} + \sum_{u=m}^{m+n-1} \sqrt{2m_r B_u} + \sum_{u=0}^{m+n-1} B_u^2.
\]

Then, \( G(x) \) is \( 3L(y,\beta) \)-smooth and \( L(y,\beta) \)-strongly convex.

**Algorithm 1** Inexact augmented Lagrangian method

1. **Input**: \( \epsilon, \beta_0 > 0, \sigma > 1, w_0, \) and \( T \).
2. **Output**: \( x_T \).
3. **Initialization**: \( x_0 \in \mathcal{X}, y_0 = 0 \).
4. **for** \( t = 0, 1, \ldots, T-1 \) **do**
5. **Calculate** \( \beta_t = \beta_0 \sigma^t \) and \( L_t = L(y_t,\beta_t) \) as (12).
6. \( x_0,t = x_t \).
7. **for** \( s = 0, 1, \ldots, S-1 \) **do**
8. **Let** \( G_{s,t}() = \mathcal{L}() + L_t\|\cdot - x_{s,t}\|_2^2 \).
9. \( x_{s+1,t} = \text{bcd}(G_{s,t},h,x_{s,t},3L_t,\frac{1}{4}) \).
10. **if** \( 2L_t\|x_{s+1,t} - x_{s,t}\|_2 \leq \frac{\epsilon}{2} \) **then**
11. \( x_{t+1} = x_{s+1,t} \).
12. **Break**.
13. **end if**
14. **end for**
15. \( y_{t+1} = y_t + w_t \alpha(x_t) \) where \( w_t = w_0 \min\{1, \frac{\log^2 2\|\alpha(x_t)\|}{(t+1)\log^2 (t+2)\|\alpha(x_{t+1})\|} \} \)
16. **end for**

**BCD.** Each BCD iteration updates one randomly selected matrix using the partial gradient of \( G() \). Other two matrices are fixed. 
Note that all BCD iterations admit closed-form solutions, since the indicator functions and the \( \ell_1 \) norm regularizer are proximal-friendly.
Algorithm 2 Block coordinate descent method: bcd\((G, h, x_0, L, \delta)\)

1: **Input:** \(x_0 \in \mathcal{X}\), smoothness \(L\), and stationary tolerance \(\delta\).
2: **for** \(\tau = 0, 1, \ldots\) **do**
3:  Uniformly choose \(i_\tau \in \{0, 1, 2\}\).
4:  **if** \(i_\tau = 0\) **then**
5:  \[A_{\tau + 1} = \arg\min_A (\nabla A G(x_\tau), A) + \frac{L}{2} \|A - A_\tau\|_F^2 + h(A, B_\tau, S_\tau), B_{\tau + 1} = B_\tau, S_{\tau + 1} = S_\tau.\]
6:  **else if** \(i_\tau = 1\) **then**
7:  \[B_{\tau + 1} = \arg\min_B (\nabla B G(x_\tau), A) + \frac{L}{2} \|B - B_\tau\|_F^2 + h(A, B, S_\tau), A_{\tau + 1} = A_\tau, S_{\tau + 1} = S_\tau.\]
8:  **else**
9:  \[S_{\tau + 1} = \arg\min_S (\nabla S G(x_\tau), S) + \frac{L}{2} \|S - S_\tau\|_F^2 + h(A_\tau, B_\tau, S), A_{\tau + 1} = A_\tau, B_{\tau + 1} = B_\tau.\]
10: **end if**
11: **if** \(\text{dist} \left( - \nabla G(x_{\tau + 1}), \partial h(x_{\tau + 1}) \right) \leq \delta\) **then**
12:  **Return** \(x_{\tau + 1} = \left( \text{vec}(A_{\tau + 1}); \text{vec}(B_{\tau + 1}); \text{vec}(S_{\tau + 1}) \right)\)
13: **end if**
14: **end for**

3.3 Complexity Analysis

We first bound the number of BCD iterations that is required to reach a stationary point of (10). The computational cost for each BCD iteration is then analyzed. Finally, we obtain the overall complexity of Algorithm 1. For simplicity, we assume \(m \leq n\) without loss of generality in this subsection. Detailed proofs are provided in the appendix.

**Number of BCD iterations.** Following the literature [Sahin et al., 2019; Li et al., 2021], we analyze the complexity for reaching a first-order stationary point which is defined as follows.

**Definition 2** A pair \((x, y)\) is called an \(\epsilon\)-KKT point to (10) [Sahin et al., 2019; Li et al., 2021] if

\[
\sqrt{\sum_{u=0}^{m+n} \alpha_u(x)^2} \leq \epsilon, \tag{14}
\]

and

\[
\text{dist} \left( - \nabla \mathcal{L}(x, y, \beta), \partial h(x) \right) \leq \epsilon, \tag{15}
\]

hold, where the distance function between a vector \(a\) and a set \(B\) is defined as \(\text{dist}(a, B) = \min_{b \in B} \|a - b\|\).

The main convergence result is summarized in the following theorem.

**Proposition 5** In order to produce an \(\epsilon\)-KKT solution of (10) Algorithm 1 updates \(A, B,\) and \(S\) for \(O\left( \frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^2 \right)\) times in expectation.

The number of iterations is the same as the Sinkhorn method in terms of \(\epsilon\) up to logarithm factors.

**Per-iteration complexity.** The cost matrix \(C\) is often low-rank, which can be exploited to accelerate the computation of BCD iterations [Scetbon et al., 2021]. Under suitable assumptions, the per-iteration complexity for each BCD iteration is \(O(nr^2)\), which is stated in the following proposition.

**Proposition 6** When the following assumptions hold,

1. \(\|S_\tau\|_0 \leq \rho\) for all \(\tau\) where \(\rho' = O(nr)\),
2. and \(r \geq \text{rank}(C)\),

the per-iteration complexity for each BCD iteration is \(O(nr^2)\) by exploiting the low-rank structure of \(C\) in evaluating the partial gradients.

The first assumption is mild by choosing a moderately large \(\lambda\).
Conclusion

In this paper, we propose a novel approximation of the OT distance. The optimal transport plan is approximated by the sum of a low-rank matrix and a sparse one. An augmented Lagrangian method is designed to efficiently calculate the transport plan.

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A Omitted Proofs

A.1 Miscellaneous Helpful Lemmas

Lemma 7 (Lemma K of Altschuler et al. [2019]) For any \(a, b > 0\),
\[
|\log a - \log b| \leq \frac{|a - b|}{\min\{a, b\}}.
\]

Lemma 8 (Proposition 1.3 of Bubeck [2015]) Let \(f\) be convex and \(\mathcal{X}\) a closed convex set on which \(f\) is differentiable. Then \(x^* \in \arg\min_{x \in \mathcal{X}} f(x)\), if and only if one has
\[
\langle \nabla f(x^*), x^* - y \rangle \leq 0, \forall y \in \mathcal{X}.
\]

Lemma 9 (Sinkhorn projection) Given \(p \in \Delta^m, q \in \Delta^n\) and \(X \in \mathbb{R}^{m \times n^+}\), the Sinkhorn projection \(\text{Proj}^{\text{KL}}_{\Pi(p, q)}(X)\) of \(X\) onto \(\Pi(p, q)\) defined as
\[
\text{Proj}^{\text{KL}}_{\Pi(p, q)}(X) = \arg\min_{T \in \Pi(p, q)} \text{KL}(T||X),
\] (16)
is the unique matrix in \(\Pi(p, q)\) of the form \(D_1KD_2\) where \(D_1\) and \(D_2\) are diagonal matrices with strictly positive diagonal elements.

Proof:
The strict convexity of KL-divergence and the compactness of \(\Pi(p, q)\) implies that the minimizer exists and is unique. Introducing two dual variables \(f \in \mathbb{R}^m, g \in \mathbb{R}^n\) for each marginal constraint, the Lagrangian of Eq. (16) reads
\[
\mathfrak{L}(T, f, g) = \sum_{ij} T_{ij} \left( \log \frac{T_{ij}}{X_{ij}} - 1 \right) + \langle f, T1 - p \rangle + \langle g, T^T 1 - q \rangle.
\]
First order conditions then yield
\[
\frac{\partial \mathfrak{L}}{\partial T_{ij}} = \log \frac{T_{ij}}{X_{ij}} + f_i + g_j = 0,
\]
which result in the expression
\[
\text{Proj}^{\text{KL}}_{\Pi(p, q)}(X) = \text{diag}(\exp(-f))X\text{diag}(\exp(-g)).
\]

A.2 Missing proofs in Sec. 3.1

We first list some lemmas which are useful to prove Theorem 1.

Lemma 10 Under assumptions of Theorem 1,
\[
\|L^* + S^* - \tilde{L}^* - \tilde{S}^*\|_{\infty} \leq U[(r^* - r)^+\sqrt{mn} + (\rho^* - \rho)^+].
\]

Proof:
When \(r \geq r^*\) (resp. \(\rho \geq \rho^*\)), \(\tilde{L}^*\) (resp. \(\tilde{S}^*\)) can accurately recover \(L^*\) (resp. \(S^*\)). \(\tilde{S}^*\) is the best approximation for \(S^*\) with at most \(\rho\) nonzero entries, which implies
\[
\|S^* - \tilde{S}^*\|_F \leq (\rho^* - \rho)_+ U.
\] (17)
By the definition of the non-negative rank, there exists additive decomposition
\[ L^* = \sum_{i=0}^{r^*-1} R_i, \text{ s.t. rank}(R_i) = 1, R_i \geq 0. \]

When \( r < r^* \),
\[ \|L^* - L^*\|_F \leq \| \sum_{i=r}^{r^*-1} R_i \|_F \leq \sum_{i=r}^{r^*-1} \| R_i \|_F \leq (r^* - r) \| L^* \|_F \leq \sqrt{mn}(r^* - r)U, \]
where the four inequalities are due to the definition of \( L^* \), the definition of the matrix norm, the non-negativity of \( R_i \)'s, and the relation between the Frobenius norm and the infinity norm. We then have
\[ \|L^* - L^*\|_F \leq \sqrt{mn}(r^* - r)U. \] (18)

Combining Eq. (17) and (18), we have
\[ \|L^* + S^* - \tilde{L}^* - \tilde{S}^*\|_\infty \leq \|L^* - L^*\|_\infty + \|S^* - \tilde{S}^*\|_\infty \leq \|L^* - L^*\|_F + \|S^* - \tilde{S}^*\|_F \leq U[(r^* - r), \sqrt{mn} + (\rho^* - \rho)] \]

We now define an auxiliary function which is necessary to prove Theorem 1, i.e.,
\[ \psi(x) = \begin{cases} \log x, & \text{if } x > 0, \\ \log \delta, & \text{otherwise}, \end{cases} \]
where \( \delta \) is defined as Eq. (5). With slight abuse of notation, \( \psi(X) \) is the elementwise operation for matrix \( X \).

**Lemma 11** Let \( \hat{T} = \text{Proj}_{KL}^{(p,q)}(\tilde{L}^* + \tilde{S}^*) \). Under assumptions of Theorem 1,
\[ \|\hat{T} - T^*\|_1 \leq \|\psi(\tilde{L}^* + \tilde{S}^*) - \psi(L^* + S^*)\|_\infty. \] (19)

**Proof:**

The case where \( r \geq r^* \) and \( \rho \geq \rho^* \) is obvious since \( L^* \) and \( S^* \) can be accurately recovered.

Now we consider the case where \( r < r^* \) or \( \rho < \rho^* \) holds, or both hold. For notational simplicity, let \( \tilde{Z} = \tilde{L}^* + \tilde{S}^* \). By Lemma 8 and the form of Sinkhorn projection, \( \sum_{ij} \log \frac{\tilde{T}_{ij}}{\tilde{Z}_{ij}}(T^*_{ij} - \tilde{T}_{ij}) \geq 0 \), which leads to
\[ \sum_{ij} (\psi(\tilde{T}_{ij}) - \psi(\tilde{Z}_{ij}))(T^*_{ij} - \tilde{T}_{ij}) \geq 0. \]

Since \( \sum_{ij} (\psi(T^*_{ij}) - \psi(T^*_{ij}))(T^*_{ij} - \tilde{T}_{ij}) = 0 \), we have
\[ \sum_{ij} (\psi(\tilde{T}_{ij}) - \psi(\tilde{Z}_{ij}) + \psi(T^*_{ij}) - \psi(T^*_{ij}))(\tilde{T}_{ij} - T^*_{ij}) \leq 0, \]
which can be rearranged as
\[ \sum_{ij} (\psi(\tilde{T}_{ij}) - \psi(T^*_{ij}))(\tilde{T}_{ij} - T^*_{ij}) \leq \sum_{ij} (\psi(\tilde{Z}_{ij}) - \psi(T^*_{ij}))(\tilde{Z}_{ij} - T^*_{ij}) \leq \|\psi(\tilde{Z}) - \psi(T^*)\|_\infty \|\hat{T} - T^*\|_1, \]
where we use Hölder’s inequality for the second inequality. To obtain Eq. (19), it suffices to prove that
\[ \|\hat{T} - T^*\|_1^2 \leq \sum_{ij} (\psi(\tilde{T}_{ij}) - \psi(T^*_{ij}))(\tilde{T}_{ij} - T^*_{ij}). \] (20)
ψ(\hat{T}_{ij}) - ψ(T^*_{ij}) has four possible forms

\[ ψ(\hat{T}_{ij}) - ψ(T^*_{ij}) = \begin{cases} 
\log \hat{T}_{ij} - \log T^*_{ij}, & \text{if } \hat{T}_{ij} > 0 \text{ and } T^*_{ij} > 0 \\
\log \hat{T}_{ij} - \log \delta, & \text{if } \hat{T}_{ij} > 0 \text{ and } T^*_{ij} = 0 \\
\log δ - \log T^*_{ij}, & \text{if } \hat{T}_{ij} = 0 \text{ and } T^*_{ij} > 0 \\
\log \delta - \log δ, & \text{otherwise}
\end{cases} \]

which all lead to (ψ(\hat{T}_{ij}) - ψ(T^*_{ij}))(\hat{T}_{ij} - T^*_{ij}) ≥ 0. Let \( y_{ij} = (ψ(\hat{T}_{ij}) - ψ(T^*_{ij}))(\hat{T}_{ij} - T^*_{ij}) \). Denote \( \mathcal{I} = \{(i, j) | y_{ij} > 0\} \). Based on the Cauchy-Schwartz inequality, we can bound the left-hand side of Eq. (20) as follows

\[ ||T - T^*||^2_1 = \left( \sum_{(i, j) \in \mathcal{I}} \sqrt{\frac{|\hat{T}_{ij} - T^*_{ij}|}{y_{ij}}} \right)^2 ≤ \sum_{(i, j) \in \mathcal{I}} y_{ij} \sum_{(i, j) \in \mathcal{I}} \frac{(\hat{T}_{ij} - T^*_{ij})^2}{y_{ij}}. \]

We thus prove \( \sum_{(i, j) \in \mathcal{I}} \frac{(\hat{T}_{ij} - T^*_{ij})^2}{y_{ij}} \leq 1 \), i.e.,

\[ \sum_{(i, j) \in \mathcal{I}} \frac{\hat{T}_{ij} - T^*_{ij}}{ψ(\hat{T}_{ij}) - ψ(T^*_{ij})} ≤ 1. \]  (21)

To do so, we show in the sequel that for all \( (i, j) \in \mathcal{I} \),

\[ \frac{\hat{T}_{ij} - T^*_{ij}}{ψ(\hat{T}_{ij}) - ψ(T^*_{ij})} ≤ \frac{\hat{T}_{ij} + T^*_{ij}}{2}, \]  (22)

which can immediately imply Eq. (21). The left-hand side of the above is positive and thus we can assume without loss of generality \( \hat{T}_{ij} > T^*_{ij} \). We consider separately the cases \( T^*_{ij} > 0 \) and \( T^*_{ij} = 0 \).

(i) \( T^*_{ij} > 0 \). Fix \( \hat{T}_{ij} \) and consider the function

\[ φ(x) = 2(x - T^*_{ij}) - (x + T^*_{ij})(ψ(x) - ψ(T^*_{ij})). \]

We now prove \( φ(x) ≤ 0 \) for \( x ≥ T^*_{ij} \). Clearly \( φ(T^*_{ij}) = 0 \) and

\[ φ(x) = 2(x - T^*_{ij}) - (x + T^*_{ij})(\log x - \log T^*_{ij}). \]

Its derivative is negative,

\[ φ'(x) = 2 - (\log x - \log T^*_{ij}) - (x + T^*_{ij}) \frac{1}{x} = 1 + \log \frac{T^*_{ij}}{x} - \frac{T^*_{ij}}{x} ≤ 0, \]

where in the last inequality we use \( 1 + \log a ≤ a \). We hence have the result.

(ii) \( T^*_{ij} > 0 \). The left-hand side of Eq. (22) can be bounded as follows

\[ \frac{\hat{T}_{ij} - T^*_{ij}}{ψ(\hat{T}_{ij}) - ψ(T^*_{ij})} ≤ \frac{\hat{T}_{ij}}{\log \hat{T}_{ij} - \log δ} ≤ \frac{\hat{T}_{ij}}{2}, \]

where in the last inequality we use the definition of \( δ \).

Combining the two cases above, \( \frac{\hat{T}_{ij} - T^*_{ij}}{ψ(\hat{T}_{ij}) - ψ(T^*_{ij})} ≤ \frac{\hat{T}_{ij} + T^*_{ij}}{2} \) and \( ||T - T^*||^2_1 ≤ \sum_{ij} (ψ(\hat{T}_{ij}) - ψ(T^*_{ij}))(\hat{T}_{ij} - T^*_{ij}) \), which finishes the proof.

\[ \Box \]

Now we prove Theorem 1 which is restated here for convenience.
Theorem 1. Denote $T^* \in \text{argmin}_{T \in \Pi(p, q)}(C, T)$. Assume there exist $L^* \in \mathbb{R}^{n \times n}$ and $S^* \in \mathbb{R}^{m \times n}$ such that $T^* = L^* + S^*$, rank$_r(L^*) = r^*$, and $\|S^*\|_0 = \rho^*$. Let $\hat{L}^*$ and $\hat{S}^*$ be the best approximations of $L^*$ and $S^*$ respectively (in terms of the Frobenius norm) satisfying $\text{rank}_r(\hat{L}^*) \leq r$ and $\|\hat{S}^*\|_0 \leq \rho$. Then,
\[
\text{LSOT}_{r, \rho}(p, q) - \text{OT}(p, q) \leq \frac{U\|C\|_{\infty}}{\delta} (\sqrt{mn(r^* - r)} + (\rho^* - \rho)),
\]
where $U = \max\{\|L^*\|_{\infty}, \|S^*\|_{\infty}\}$ and
\[
\delta = \frac{1}{e^2} \min \{ \min_{i,j} \{ L^*_{ij} + S^*_{ij} | L^*_{ij} + S^*_{ij} > 0 \}, \min_{i,j} \{ \hat{L}^*_{ij} + \hat{S}^*_{ij} | \hat{L}^*_{ij} + \hat{S}^*_{ij} > 0 \} \}.
\]
Proof: Eq. (4) is obvious for $r \geq r^*$ and $\rho \geq \rho^*$.

Now we consider the case where $r < r^*$ or $\rho < \rho^*$ holds, or both hold. Let $\hat{T} = \text{Proj}_{\Pi(p, q)}(\hat{L}^* + \hat{S}^*)$. By the property of non-negative rank and the form of Sinkhorn projection, $\hat{T} \in \hat{\Omega}_{r, \rho}(p, q)$. We thereby have
\[
(C, T) - (C, T') \leq (C, T) - (C, T^*) \leq \|C\|_{\infty} \|T - T^*\|_1 \leq \|C\|_{\infty} \|\text{proj}(L^* + S^*) - \text{proj}(L^* + S^*)\|_{\infty},
\]
where we use Hölder’s inequality and Lemma 11 for the third and the fourth inequalities respectively. Now we bound $\|\text{proj}(L^* + S^*) - \text{proj}(L^* + S^*)\|_{\infty}$ and consider the four following cases.

(i) When $L^*_{ij} + S^*_{ij} > 0$ and $L^*_{ij} + S^*_{ij} > 0$,
\[
\left| \psi(L^*_{ij} + S^*_{ij}) - \psi(L^*_{ij} + S^*_{ij}) \right| = \left| \log(L^*_{ij} + S^*_{ij}) - \log(L^*_{ij} + S^*_{ij}) \right| \leq \frac{L^*_{ij} + S^*_{ij} - L^*_{ij} - S^*_{ij}}{\min\{L^*_{ij} + S^*_{ij}, L^*_{ij} + S^*_{ij}\}} \leq \frac{L^*_{ij} + S^*_{ij} - L^*_{ij} - S^*_{ij}}{\delta},
\]
where we apply Lemma 7 in the first inequality, and use the definition of $\delta$ in the second one.

(ii) When $L^*_{ij} + S^*_{ij} > 0$ and $L^*_{ij} + S^*_{ij} = 0$,
\[
\left| \psi(L^*_{ij} + S^*_{ij}) - \psi(L^*_{ij} + S^*_{ij}) \right| = \left| \log(L^*_{ij} + S^*_{ij}) - \log(\delta) \right| \leq \frac{L^*_{ij} + S^*_{ij} - \delta}{\min\{L^*_{ij} + S^*_{ij}, \delta\}} \leq \frac{L^*_{ij} + S^*_{ij} - L^*_{ij} - S^*_{ij}}{\delta},
\]
where we use Lemma 7 and the fact that $L^*_{ij} + S^*_{ij} = 0$ in the first and the second inequalities respectively.

(iii) When $L^*_{ij} + S^*_{ij} = 0$ and $L^*_{ij} + S^*_{ij} > 0$, we similarly have
\[
\left| \psi(L^*_{ij} + S^*_{ij}) - \psi(L^*_{ij} + S^*_{ij}) \right| \leq \frac{L^*_{ij} + S^*_{ij} - L^*_{ij} - S^*_{ij}}{\delta}.
\]

(iv) When $L^*_{ij} + S^*_{ij} = 0$ and $L^*_{ij} + S^*_{ij} = 0$, we have
\[
\left| \psi(L^*_{ij} + S^*_{ij}) - \psi(L^*_{ij} + S^*_{ij}) \right| = \left| \log(\delta) - \log(\delta) \right| = \frac{L^*_{ij} + S^*_{ij} - L^*_{ij} - S^*_{ij}}{\delta}.
\]
Combining the four cases, we have
\[
\left| \psi(L^* + S^*) - \psi(L^* + S^*) \right|_{\infty} \leq \frac{1}{\delta} \|L^* + S^* - \hat{L}^* - \hat{S}^*\|_{\infty} \leq \frac{U\|C\|_{\infty}}{\delta} \sqrt{mn} (r^* - r) + (\rho^* - \rho),
\]
where we apply Lemma 10 in the second inequality. Substituting Eq. (24) into (23), we have the result.
A.3 Missing Proofs in Sec. 3.2

Proposition 4 is restated here.

**Proposition 4** Given $y$ and $\beta$, let $G(x) = \mathcal{L}(x, y, \beta) + L(y, \beta)\|x - \bar{x}\|^2$ where

$$L(y, \beta) = \sqrt{2r}C\|F + \sqrt{2r(m\sqrt{n} + n\sqrt{m})}\|y\| + \beta L_c,$$

with

$$B_u = \max_{x \in X} \|\alpha_u(x)\|,$$

$$L_c = \sum_{u=0}^{m-1} \sqrt{2nr}B_u + \sum_{u=m}^{m+n-1} \sqrt{2mr}B_u + \sum_{u=0}^{m+n-1} B_u^2.$$

Then, $G(x)$ is $3L(y, \beta)$-smooth and $L(y, \beta)$-strongly convex.

**Proof:**

Recalling the definition $x = (\text{vec}(A); \text{vec}(B); \text{vec}(S))$, it suffices to prove that $\mathcal{L}(x, y, \beta)$ is $L(y, \beta)$-smooth. We proceed by bounding the eigenvalues of the Hessian $\nabla^2 \mathcal{L}(x, y, \beta)$ given by

$$\nabla^2 \mathcal{L}(x, y, \beta) = \nabla^2 (C, AB^T + S) + \nabla^2 \sum_u (\alpha_u(x))^2 + \nabla^2 \sum_y y_u \alpha_u(x).$$

The Hessian of $(C, AB^T + S)$ is given by $H = [H_{iz}]$, where

$$H_{iz} = \begin{cases} C_{ij/lr = j/r_m}, & \text{if } 0 \leq l < mr, mr \leq z < mr + nr, \text{ and } \%r = z\%r \\
C_{iz/lj = l/j_m}, & \text{if } 0 \leq z < mr, mr \leq l < mr + nr, \text{ and } \%r = z\%r \\
0, & \text{otherwise} \end{cases}$$

where // and % is the operation of obtaining the quotient and the remainder of the Euclidean division respectively.

The summands of the second term and the third term are

$$\nabla^2 \beta(\alpha_u(x))^2 = \beta \alpha_u(x)\nabla^2 \alpha_u(x) + \beta \nabla \alpha_u(x)\nabla^\top \alpha_u(x),$$

and

$$\nabla^2 y_u \alpha_u(x) = y_u \nabla^2 \alpha_u(x),$$

respectively both of which involve Hessians $H^{\alpha_u} = [H^{\alpha_u}_{iz}]$, where for $0 \leq u < m$,

$$H^{\alpha_u}_{iz} = \begin{cases} 1, & \text{if } 0 \leq l < mr, mr \leq z < mr + nr, \text{ and } i = l\%r \\
1, & \text{if } 0 \leq z < mr, mr \leq l < mr + nr, \text{ and } i = z\%r \\
0, & \text{otherwise} \end{cases}$$

and for $m \leq u < m + n - 1,$

$$H^{\alpha_u}_{iz} = \begin{cases} 1, & \text{if } 0 \leq l < mr, mr \leq z < mr + nr, \text{ and } j = z/l - m \\
1, & \text{if } 0 \leq z < mr, mr \leq l < mr + nr, \text{ and } j = l/l - m \\
0, & \text{otherwise} \end{cases}$$

Therefore,

$$\|\nabla^2 \mathcal{L}(x, y, \beta)\|_{op} \leq \|\nabla^2 \mathcal{L}(x, y, \beta)\|_{F}$$

$$\leq \|H\|_F + \beta \sum_{u=0}^{m-1} \|\alpha_u(x)\|_F \|H^{\alpha_u}\|_F + \beta \sum_{u=m}^{m+n-1} \|\nabla \alpha_u(x)\|_F + \beta \sum_{u=0}^{m+n-1} y_u \|H^{\alpha_u}\|_F$$

$$\leq \sqrt{2r}C\|F + \beta \sum_{u=0}^{m-1} B_u \sqrt{2nr} + \beta \sum_{u=m}^{m+n-1} B_u \sqrt{2mr} + \beta \sum_{u=0}^{m+n-1} B_u^2 + \sqrt{2r(m\sqrt{n} + n\sqrt{m})}\|y\|,$$

which indicates that $\mathcal{L}(x, y, \beta)$ is $L(y, \beta)$-smooth.
A.4 Missing Proofs in Sec. 3.3

Lemma 12 (Complexity of BCD) Given \( \epsilon > 0 \), within \( \mathcal{O}(\log(1/\epsilon)) \) iterations in expectation, Algorithm 2 outputs a solution \( x \) that satisfies \( \text{dist}(-\nabla G(x), \partial h(x)) \leq \epsilon \).

Proof: For ease of notation, we denote \( F(x) = G(x) + h(x) \). By Proposition 4 and Theorem 7 of Richtárik and Takáč [2014], we have

\[
\mathbb{E}_\tau F(x^\tau) - F(x^*) \leq \left( \frac{8}{9} \right) (F(x_0) - F(x^*)) ,
\]

where \( x^* = \arg\min_x F(x) \). By the 3L(y, \beta)-smoothness of \( F(x) \), we have

\[
F(x^\tau) - F(x^*) \geq \frac{1}{2 \cdot 3L(y, \beta)} \|g(x)\|^2 .
\]

Combining (25) and (26), we have \( T = \mathcal{O}\left( \log \frac{1}{(y, \beta)(F(x_0) - F(x^*))} \right) \).

Proposition 5 In order to produce an \( \epsilon \)-KKT solution of (10) Algorithm 1 updates \( A, B \), and \( S \) for \( \mathcal{O}\left( \frac{1}{\epsilon^2}(\log \frac{1}{\epsilon})^2 \right) \) times in expectation.

Proof: Invoking Theorem 2 of Li et al. [2021], Algorithm 1 terminates with \( T = \mathcal{O}(\log \frac{1}{\epsilon}) \) and \( S = \mathcal{O}(\frac{1}{\epsilon^2}) \). By Lemma 12, Algorithm 2 updates \( A, B \), and \( S \) for \( \mathcal{O}(\log \frac{1}{\epsilon}) \) times in expectation. We hence have the results.

Proposition 6 When the following assumptions hold,

1. \( \|S\|_2 \leq \rho' \) for all \( \tau \) where \( \rho' = \mathcal{O}(nr) \),
2. and \( r \geq \text{rank}(C) \),

the per-iteration complexity for each BCD iteration is \( \mathcal{O}(nr^2) \) by exploiting the low-rank structure of \( C \) in evaluating the partial gradients.

Proof: Denote the operation of clamping each entry of matrix \( X \) to a box \([l, u]\) by \( \text{clamp}(X; l, u) \) and the shrinkage operator by \( \text{shrink}(X, \alpha) \). Then

\[
A_{\tau+1} = \text{clamp}\left( A_\tau - \frac{1}{L} \nabla_A G(x_\tau), 0, 1 \right),
\]

\[
B_{\tau+1} = \text{clamp}\left( B_\tau - \frac{1}{L} \nabla_B G(x_\tau), 0, 1 \right),
\]

\[
S_{\tau+1} = \text{clamp}\left( \text{shrink}\left( S - \frac{1}{L} \nabla_S G(x_\tau), \frac{\lambda}{T} \right), 0, 1 \right),
\]

where

\[
\nabla_A G(x_\tau) = CB_\tau + y^p 1^n 1^T B_\tau + 1^m y^q 1^T B_\tau + \beta \left( A_\tau B_\tau^T 1^n 1^T B_\tau + S_\tau 1^n 1^T B_\tau - p1^n 1^T B_\tau \right)
\]

\[
+ \beta \left( 1^m 1^m 1^T A_\tau B_\tau^T B_\tau + 1^m 1^m 1^T S_\tau B_\tau - 1^m y^q 1^T B_\tau \right) + 2L(A_\tau - A_0)
\]

\[
\nabla_B G(x_\tau) = C^T A_\tau + 1^n y^p 1^T A_\tau + y^q 1^m 1^T A_\tau + \beta \left( 1^n 1^m 1^T B_\tau A_\tau^T A_\tau + 1^n 1^m 1^T S_\tau A_\tau - 1^n p^1 1^T A_\tau \right)
\]

\[
+ \beta \left( B_\tau A_\tau^T 1^m 1^m 1^T A_\tau + S_\tau 1^m 1^m 1^T A_\tau - q1^m 1^T A_\tau \right) + 2L(B_\tau - B_0)
\]

\[
\nabla_S G(x_\tau) = -D + \beta \left( A_\tau B_\tau^T 1^n 1^T S_\tau + S_\tau 1^n 1^T B_\tau \right) + \beta \left( 1^m 1^m 1^T A_\tau B_\tau^T + 1^m 1^m 1^T S_\tau \right) + \rho S_\tau ,
\]

14
and
\[ D = -\left( C + y^P1^\top + 1^m y^q^\top - \beta p_1^1 - \beta q_1^m - \rho S_0 \right). \]

Exploiting the low-rankness of \( C \), the complexity for updating \( A \) and \( B \) is obviously \( O(nr^2) \). Substituting \( \nabla_S G(x_\tau) \) into Eq. (27c), we have
\[ S_{\tau+1} = \text{clamp} \left( \text{shrin} \left( \frac{1}{\beta} S_\tau + \frac{1}{L} D - M, \frac{\lambda}{L} \right), 0, 1 \right), \]
where \( M = \frac{\beta}{L} (A_\tau B_\tau^1 1^\top 1^\top + S_\tau 1^\top 1^\top) + \frac{\beta}{L} (1^m 1^\top A_\tau B_\tau^1 + 1^m 1^\top S_\tau). \) If \( \lambda \) is larger than or equal to the \( nr \)-th largest entry in \( D \), only \( O(\|S_\tau\|_0 + nr) \) entries of \( \frac{1}{\beta} S_\tau + \frac{1}{L} D - M \) will be larger than \( \frac{\lambda}{L} \). Then only \( O(\|S_\tau\|_0 + nr) \) of \( M \) need to be calculated, with complexity \( O(\|S_\tau\|_0 + nr) \).

\[ \square \]