The direct method of functional separation of variables can provide more exact solutions than the compatibility analysis of PDEs based on a single differential constraint

Andrei D. Polyanin\textsuperscript{a,b,c}

\textsuperscript{a} Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia
\textsuperscript{b} Bauman Moscow State Technical University, 5 Second Baumanskaya Street, 105005 Moscow, Russia
\textsuperscript{c} National Research Nuclear University MEPhI, 31 Kashirskoe Shosse, 115409 Moscow, Russia

This note shows that in looking for exact solutions to nonlinear PDEs, the direct method of functional separation of variables can, in certain cases, be more effective than the method of differential constraints based on the compatibility analysis of PDEs with a single constraint (invariant surface condition). This fact is illustrated by examples of nonlinear reaction-diffusion and convection-diffusion equations with variable coefficients, nonlinear Klein–Gordon type equations, and hydrodynamic boundary layer equations. A few new exact solutions are given.

\textbf{Keywords:} method of functional separation of variables, method of differential constraints, nonclassical method of symmetry reduction, direct method of Clarkson and Kruskal, invariant surface condition, exact solutions in implicit form
1 Introduction. The methods concerned

1.1 The direct method for constructing functional separable solutions in implicit form

Let us look at nonlinear PDEs of the form

\[ F(x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \]  \hspace{1cm} (1)

Equation (1) can be analyzed using a direct method of functional separation of variables based on seeking exact solutions in implicit form \[1\]:

\[ \int h(u) \, du = \xi(x)\omega(t) + \eta(x). \] \hspace{1cm} (2)

The functions \( h(u) \), \( \xi(x) \), \( \eta(x) \), and \( \omega(t) \) are to be determined in a subsequent analysis.

The procedure for constructing such solutions is as follows. First, using (2), one calculates the partial derivatives \( u_x, u_t, u_{xx}, \ldots \), which are expressed in terms of the functions \( h, \xi, \eta, \omega \) and their derivatives. Then, these partial derivatives must be substituted into equation (1) followed by eliminating the variable \( t \) with the help of (2). As a result (with a suitable choice of \( \omega \)), one arrives at a bilinear functional-differential equation,

\[ \sum_{j=1}^{N} \Phi_j[x] \Psi_j[u] = 0. \] \hspace{1cm} (3)

Here, \( \Phi_j[x] \equiv \Phi_j(x, \xi, \eta, \xi_x, \eta_x, \ldots) \) and \( \Psi_j[u] \equiv \Psi_j(u, h, h'_u, \ldots) \) are differential forms (in some cases, functional coefficients) that depend, respectively, on \( x \) and \( u \) alone. The following statement is true.

**Proposition** (first formulated by Birkhoff \[2\]). Functional differential equations of the form (3) can have solutions only if the forms \( \Psi_j[u] \) \( (j = 1, \ldots, N) \) are connected by linear relations (see, for example, \[1, 3, 4\]):

\[ \sum_{j=1}^{m_i} k_{ij} \Psi_j[u] = 0, \quad i = 1, \ldots, n, \] \hspace{1cm} (4)
where \( k_{ij} \) are some constants, \( 1 \leq m_i \leq N - 1 \), and \( 1 \leq n \leq N - 1 \). Degenerate cases must also be treated where, in addition to the linear relations, some individual differential forms \( \Psi_j[u] \) vanish.

This proposition is used for the construction of exact solutions to functional differential equations of the form (3) and the corresponding nonlinear PDEs (1). Note that, in the generic case, different linear relations of the form (4) correspond to different solutions of the PDEs under consideration.

1.2 The method of differential constraints

The direct method for constructing functional separable solutions in implicit form based on formula (2) is closely related to the method of differential constraints, which is based on the compatibility theory of PDEs [5].

To show this, we differentiate formula (2) with respect to \( t \) to obtain

\[
 u_t = \xi(x)\bar{\omega}(t)\varphi(u),
\]

where \( \bar{\omega}(t) = \omega'(t) \) and \( \varphi(u) = 1/h(u) \).

Relation (5) can be treated as a first-order differential constraint, which can be used to find exact solutions of equation (1) through a compatibility analysis of the overdetermined pair of equations (1) and (3) with the single unknown \( u \). The differential constraint (5) is equivalent to relation (2); initially, all functions included on the right-hand sides of (2) and (5) are considered arbitrary, and the specific form of these functions is determined in the subsequent analysis.

Differential constraints of the second and higher orders can also be used to construct exact solutions to equation (1); in the general case, any PDE (or ODE, in degenerate cases) that depends on the same variables as the original equation can be treated as a differential constraint. For a description of the method of differential constraints and its relationship with other methods, as well as a number of specific
examples of its application, see, for example, [4–12]. Note that exact solutions can be sought using several differential constraints (see, for example, [4,10]).

The construction of exact solutions by the method of differential constraints is based on a compatibility analysis of PDEs and is carried out in several steps briefly described below.

1. Two PDEs, the original PDE and a differential constraint, are differentiated (sufficiently many times) with respect to \(x\) and \(t\), and then the highest-order derivatives are eliminated from the differential relations obtained and PDEs considered. As a result, one arrives at an equation involving powers of lower-order derivatives, for example, \(u_x\).

2. By equating the coefficients of all degrees of the derivative \(u_x\) with zero in this equation, one obtains compatibility conditions relating the functional coefficients of the PDEs.

3. The compatibility conditions make up a nonlinear system of ODEs for determining the functional coefficients. In this step, it is necessary to find a solution to this system in a closed form.

4. The obtained coefficients are substituted into the differential constraint, which must then be integrated to find a form (or forms) of the unknown function \(u\) (in this step, intermediate solutions are obtained that contain undetermined functions).

5. The final form of the unknown function is determined from the original PDE.

In the last three steps of the method of differential constraints, one has to solve different equations (systems of equations). If no solution can be found in at least one of these steps, the procedure fails and no exact solution to the original equation is obtained.

**Remark 1.** The first-order differential constraint (5) is a special case of an invariant surface condition [13], which characterizes the nonclassical method of
symmetry reduction. In general, an invariant surface condition is a quasilinear first-order PDE of general form. Therefore, the nonclassical method of symmetry reduction can be considered as an important special case of the method of differential constraints; specific examples of its use can be found, for example, in \[3,4,13–19\].

1.3 **Question: which method is more effective?**

Although the differential constraint (5) is equivalent to the functional relation (2), the subsequent procedure for finding exact solutions by the direct method for constructing functional separable solutions in implicit form and that by the method of differential constraints differ significantly. A natural and very important question arises: Do these methods result in the same exact solutions or not?

It will be shown below that the direct method of functional separation of variables based on the implicit representation of solution (2) can provide more exact solutions than the method of differential constraints with the equivalent differential constraint (invariant surface condition) (5).

2 **Non-linear reaction-diffusion equations with variable coefficients**

2.1 **Using the method of differential constraints**

Let us look at nonlinear reaction-diffusion equations with variable coefficients of the form

\[
c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u).
\]

To construct exact solutions to this equation, we use the differential
constraint (invariant surface condition)

\[ u_t = \theta(x, t) \varphi(u), \] (7)

which is more general than constraint (5).

We solve equation (6) for the highest derivative and eliminate \( u_t \) with the help of (7) to obtain

\[ u_{xx} = -\frac{f_u'}{f} u_x^2 - \frac{a'}{a} u_x - \frac{b}{a} f' \varphi + \frac{c \theta \varphi}{a f}. \] (8)

Differentiating (7) twice with respect to \( x \) and taking into account relation (8), we get

\[ u_t = \theta \varphi, \quad u_{tx} = \theta \varphi_u' u_x + \theta_x \varphi, \]
\[ u_{txx} = \theta \varphi_u' u_{xx} + \theta \varphi_{uu} u_x^2 + 2 \theta_x \varphi_u' u_x + \theta_{xx} \varphi \]
\[ = \theta \left( \varphi'' - \frac{f_u'}{f} \varphi_u' \right) u_x^2 + A_1(x, t, u) u_x + A_0(x, t, u). \] (9)

Here \( A_1 \) and \( A_0 \) are independent of \( u_x \) and are expressed in terms of the functions appearing in PDEs (6) and (7).

Differentiating (8) with respect \( t \) and taking into account the first two relations of (9), we find the mixed derivative in a different way:

\[ u_{xxt} = -\theta \left[ \varphi \left( \frac{f_u'}{f} \right)_u' + 2 \frac{f_u'}{f} \varphi_u' \right] u_x^2 + B_1(x, t, u) u_x + B_0(x, t, u). \] (10)

By matching up the third-order mixed derivatives (9) and (10), we get the following relation, quadratic in \( u_x \):

\[ F_2 u_x^2 + F_1 u_x + F_0 = 0, \]
\[ F_2 = \theta \left[ \varphi'' + \varphi_u' \frac{f_u'}{f} + \varphi \left( \frac{f_u'}{f} \right)_u' \right]. \] (11)

The functional coefficients \( F_0 \) and \( F_1 \) depend on \( a, b, c, f, g, \theta, \varphi \) and their derivatives (and are independent of \( u_x \)). By equating the functional coefficients \( F_n \) with zero (the procedure of splitting by the derivative \( u_x \)), one can obtain a
determining system of equations. Next, we only need the first equation of this system (corresponding to \( F_2 = 0 \)), which, after dividing by \( \theta \), takes the form

\[
\varphi''_u + \varphi'_u \frac{f'_u}{f} + \varphi \left( \frac{f'_u}{f} \right)' = 0. \tag{12}
\]

Considering \( f \) to be an arbitrary function and \( \varphi \) to be the unknown, we find the general solution of equation (12):

\[
\varphi = \frac{1}{f} \left( C_1 \int f \, du + C_2 \right), \tag{13}
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants. Thus, the method of differential constraints leads to exact solutions in which the functions \( f \) and \( \varphi \) (involved in the original equation and the differential constraint) are related by (13).

Using the differential constraint (5) is equivalent to representing the solution in the form (2). Since \( \varphi = 1/h \), solution (13) can be rewritten in terms of \( f \) and \( h \) as

\[
h = f \left( C_1 \int f \, du + C_2 \right)^{-1}. \tag{14}
\]

2.2 Using the direct method of functional separation of variables

The study [1] presents a large number of exact solutions to PDEs of the form (6) obtained using the method described in Section 1.1. In particular, it shows that the equation

\[
u_t = \left[ a(x)f(u)u_x \right]_x + \frac{a'_x(x)}{\sqrt{a(x)}} u, \tag{15}
\]

which contains two arbitrary functions \( a(x) > 0 \) and \( f(u) \), admits the exact solution in implicit form

\[
\int \frac{f(u)}{u} \, du = 4t - 2 \int \frac{dx}{\sqrt{a(x)}} + C, \tag{16}
\]

where \( C \) is an arbitrary constant.
Solution (16) is a special case of solutions (2) with $h = f/u$. This solution is different from (14); consequently, it cannot be obtained by the method of differential constraints using relation (5), neither can it be obtained using the more general differential constraint (7).

Solutions of the form (16) are generated by two differential constraints: one of them is (5) and the other (additional) constraint has the form $u_x = p(x)\psi(u)$ (namely, $\sqrt{af}u_x = -2u$). It is important to note that the latter constraint is determined by the functional coefficients of the original equation (6) and cannot be obtained from general a priori considerations.

In addition to solution (16), several other exact solutions of the form (2) were also obtained in [1], which do not satisfy relation (14) and are omitted here; just as above, these solutions cannot be obtained by the method of differential constraints based on a single constraint.

**Remark 2.** It can be shown that solution (16) cannot be obtained by the method of differential constraints using a single constraint of the form $u_t = \varphi(x, t, u)$, which is even more general than (5) and (7).

3 **Non-linear convection-diffusion equations with variable coefficients**

3.1 **Using the method of differential constraints**

Let us look at nonlinear convection-diffusion equations of the form

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u)u_x.$$  \hspace{1cm} (17)

The compatibility analysis of two PDEs, the original equation (17) and differential constraint (7), is performed in the same way as in Section 2.1. As a result, we obtain a relation, quadratic in $u_x$, in which the functional coefficient of $u_x^2$ coincides with $F_2$ from (11).
Therefore, the method of differential constraints based on the single constraint (7) for the convection-diffusion equations (17) also results in relations (13) and (14).

3.2 Using the direct method of functional separation of variables

It can be shown that the nonlinear convection-diffusion equation of special form

\[ u_t = [a(x)f(u)u_x]_x - \frac{1}{2}a'_x(x)f(u)u_x, \]  

where \(a(x)\) and \(f(u)\) are arbitrary functions, admits the pair exact solutions

\[ \int \frac{f(u)}{u} \, du = kt \pm k \int \frac{dx}{\sqrt{a(x)}} + C, \]  

with \(C\) and \(k\) being arbitrary constants.

Solutions (19) are special cases of solutions of the form (2) with \(h = f/u\). These solutions do not satisfy relation (14) and, therefore, cannot be obtained by the method of differential constraints based on the single constraint (5); however, these solutions can be obtained if two differential constraints are used at once.

4 Non-linear Klein–Gordon type equations with variable coefficients

4.1 Using the method of differential constraints

Now let us look at the nonlinear Klein–Gordon type equation with variable coefficients

\[ c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u). \]  

To construct exact solutions to this equation, we also use a more general differential constraint (7) than (5). Differentiating (7) with respect to \(t\) gives

\[ u_t = \theta \varphi \quad \Rightarrow \quad u_{tt} = \theta \varphi'_u u_t + \theta_t \varphi = \theta^2 \varphi \varphi'_u + \theta_t \varphi. \]  

9
We solve equation (20) for $u_{xx}$ and then eliminate $u_{tt}$ with the help of (21) to obtain

$$u_{xx} = -\frac{f'_{u}}{f} u_{x}^{2} - \frac{d_{x}}{a} u_{x} - \frac{b}{a} \frac{g}{f} + \frac{c}{af} (\theta^{2} \varphi_{u} + \theta_{t} \varphi).$$

(22)

Differentiating (7) with respect to $x$ twice and taking into account relation (22), we find $u_{txx}$. Differentiating (22) with respect to $t$ and taking into account the first two relations of (9), we determine the mixed derivative $u_{xxt}$. By matching up the two third-order mixed derivatives, $u_{txx} = u_{xxt}$, we arrive at a relation, quadratic in $u_{x}$, in which the functional coefficient of $u_{x}^{2}$ coincides with $F_{2}$ from (11). Using the same reasoning as in Section 2.1, we obtain the relation (14) between the functions $f$ and $h$ appearing in the equation (20) and differential constraint (7).

4.2 Using the direct method of functional separation of variables

Let us look at the nonlinear Klein–Gordon type equation of special form

$$u_{tt} = [a(x)f(u)u_{x}]_{x} + \frac{x^{2}}{a(x)} g(u),$$

(23)

where $a(x)$ is an arbitrary function; the functions $f(u)$ and $g(u)$ are expressed in terms of the arbitrary function $h = h(u)$ as

$$f(u) = \frac{h'_{u}}{h^{2}}, \quad g(u) = -\frac{1}{h} \left( \frac{h'_{u}}{h^{3}} \right)',$$

(24)

By the method described in Section 1.1, we can construct an implicit exact solution to equation (23) with $f(u)$ and $g(u)$ defined by (24):

$$\int h(u) \, du = t - \int \frac{x \, dx}{a(x)} + C.$$

(25)

It follows from the first relation of (24) and solution (25) that relation (14) is not satisfied, and hence, solution (25) cannot be obtained by the method of differential constraints with the single constrain (5).
5 Axisymmetric boundary layer equations

5.1 Functional separable solutions in explicit form

The system of equations of a laminar unsteady axisymmetric boundary layer on a body of revolution can be reduced through the introduction of a stream function \( w \) (and a suitable new independent variable \( z \)) to a single nonlinear third-order PDE with variable coefficients [20]:

\[
wtz + wzw_{xz} - w_xw_{zz} = \nu r^2(x)w_{zzz} + F(t, x),
\]

where \( r = r(x) \) describes the shape of the body (this function is considered arbitrary here), while \( F(t, x) \) defines the pressure gradient.

Exact solutions to equation (26) can be sought using the method of functional separation of variables in the explicit form [20]

\[
w = f u(\xi) + gz + h, \quad \xi = \varphi z + \psi,
\]

with the functions \( f = f(t, x) \), \( g = g(t, x) \), \( h = h(t, x) \), \( \varphi = \varphi(t, x) \), \( \psi = \psi(t, x) \), and \( u = u(\xi) \) to be determined. Substituting (27) into equation (26) and replacing \( z \) with \( (\xi - \psi)/\varphi \) yields the functional differential equation

\[
\sum_{n=1}^{6} \Phi_n[t, x] \Psi_n[\xi] = \Psi_7[\xi].
\]

Here, \( \Phi_n[t, x] \) are differential forms dependent on the functional coefficients (and their derivatives) involved in (27) and (26), with all \( \Phi_n \) being independent of \( u \). The forms \( \Psi_n = \Psi_n[\xi] \) are expressed as [20]

\[
\begin{align*}
\Psi_1 &= 1, & \Psi_2 &= u'_\xi, & \Psi_3 &= (u'_\xi)^2, & \Psi_4 &= u''_{\xi\xi}, \\
\Psi_5 &= \xi u''_{\xi\xi}, & \Psi_6 &= uu''_{\xi\xi}, & \Psi_7 &= u'''_{\xi\xi\xi}.
\end{align*}
\]

The variables in equation (28) can be separated if we assume that the \( \Phi_n[t, x] \) on the left-hand side of (28) are all proportional to \( r^2 f \varphi^3 \). This leads
to an overdetermined system of PDEs,

$$\Phi_n[t, x] = a_n, \quad n = 1, \ldots, 6 \quad (a_n = \text{const}),$$

(30)

and a nonlinear ODE for $u = u(\xi)$,

$$\sum_{n=1}^{6} a_n \Psi_n = \Psi_7.$$  

(31)

If, for some $a_n$, one succeeds in finding a particular solution to the nonlinear system (30), then the corresponding solution to equation (31) will generate an exact solution to equation (26).

### 5.2 Using multiple differential constraints

It can be shown that the most interesting solutions of the form (27), those involving several arbitrary functions, may be obtained if one uses two or three differential relations that are linear combinations of the forms $\Psi_n$ defined in (29).

Table 1 lists a number of functions $u = u(\xi)$ that generate two or three linear differential constraints among the differential forms (29). The differential constraints shown in the first ten rows were described in [20]; the last four rows show new differential constraints, which generate new exact solutions of the form (27) to equation (26).

It is important that the differential constraints specified in Table 1 are not known in advance. They arise in the course of the analysis and result from the representation of solutions to equation (26) in the form of (27) and while using equation (31).

Similar exact solutions based on several differential connections for other hydrodynamic boundary layer equations are obtained in [21, 22].
Таблица 1. Generating functions $u$ and the corresponding linear relations among $\Psi_n$.

| No. | Generating functions $u$ | Linear constraints between $\Psi_n$ |
|-----|------------------------|-----------------------------------|
| 1   | $u = \xi^2$            | $\Psi_4 = 2\Psi_1$, $\Psi_5 = \Psi_2$, $\Psi_6 = \frac{1}{2}\Psi_3$ |
| 2   | $u = \xi^3$            | $\Psi_5 = 2\Psi_2$, $\Psi_6 = \frac{2}{3}\Psi_3$, $\Psi_7 = 6\Psi_1$ |
| 3   | $u = \xi^4$            | $\Psi_5 = 3\Psi_2$, $\Psi_6 = \frac{3}{4}\Psi_3$ |
| 4   | $u = \xi^{-1}$         | $\Psi_5 = -2\Psi_2$, $\Psi_6 = 2\Psi_3$, $\Psi_7 = -6\Psi_3$ |
| 5   | $u = \xi^n$            | $\Psi_5 = (n-1)\Psi_2$, $\Psi_6 = \frac{n-1}{n}\Psi_3$ ($n \neq -1, 0, 1, 2, 3$) |
| 6   | $u = \exp \xi$         | $\Psi_2 = \Psi_4 = \Psi_7$, $\Psi_6 = \Psi_3$ |
| 7   | $u = \cosh \xi$        | $\Psi_6 = \Psi_1 + \Psi_3$, $\Psi_7 = \Psi_2$ |
| 8   | $u = \sinh \xi$        | $\Psi_6 = \Psi_3 - \Psi_1$, $\Psi_7 = \Psi_2$ |
| 9   | $u = \cos \xi$         | $\Psi_6 = \Psi_3 - \Psi_1$, $\Psi_7 = -\Psi_2$ |
| 10  | $u = \sin \xi$         | $\Psi_6 = \Psi_3 - \Psi_1$, $\Psi_7 = -\Psi_2$ |
| 11  | $u = \tanh \xi$        | $\Psi_6 = -2\Psi_2 + 2\Psi_3$, $\Psi_7 = -2\Psi_2 - 3\Psi_6$ |
| 12  | $u = \coth \xi$        | $\Psi_6 = -2\Psi_2 + 2\Psi_3$, $\Psi_7 = -2\Psi_2 - 3\Psi_6$ |
| 13  | $u = \tan \xi$         | $\Psi_6 = -2\Psi_2 + 2\Psi_3$, $\Psi_7 = 2\Psi_2 + 3\Psi_6$ |
| 14  | $u = \cot \xi$         | $\Psi_6 = 2\Psi_2 + 2\Psi_3$, $\Psi_7 = 2\Psi_2 - 3\Psi_6$ |

6 A note on the direct method of Clarkson and Kruskal

Let us now briefly discuss the direct method of Clarkson and Kruskal [23] (see also [4, 8, 18, 19, 24, 25]), which is based on looking for exact solutions in the form $u = U(x, t, w(z))$ with $z = z(x, t)$. The functions $U(x, t, w)$ and $z(x, t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $w = w(z)$. The requirement that the function $w$ must satisfy a single ODE greatly limits the capabilities of this method and does not allow it to be effectively used to find exact solutions such as presented in this note.

The effectiveness of the direct method of Clarkson and Kruskal will increase significantly if we assume that the function $w$ can satisfy an overdetermined system of several ODEs (see, for example, Section 5.2).
Acknowledgments

The study was supported within the framework of the Russian State Assignment (State Registration Number AAAA-A17-117021310385-6) and partially supported by the Russian Foundation for Basic Research (project No. 18-29-10025).

I would like to express my deep gratitude to Alexei Zhurov and Alexander Aksenov for fruitful discussions.

References

1. A.D. Polyanin, Construction of exact solutions in implicit form for PDEs: New functional separable solutions of non-linear reaction-diffusion equations with variable coefficients, Int. J. Non-Linear Mech, 2019, in press, https://doi.org/10.1016/j.ijnonlinmec.2019.02.005.

2. G. Birkhoff, Hydrodynamics, Princeton University Press, Princeton, 1960.

3. E. Pucci, G. Saccomandi, Evolution equations, invariant surface conditions and functional separation of variables, Physica D 139 (2000) 28–47.

4. A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, 2nd ed., CRC Press, Boca Raton, 2012.

5. N.N. Yanenko, The compatibility theory and methods of integration of systems of nonlinear partial differential equations, In: Proc. All-Union Math. Congress, Nauka, Leningrad, 2 (1964) 613–621.

6. S.V. Meleshko, Differential constraints and one-parameter Lie–Bäcklund groups, Sov. Math. Dokl., 28 (1983) 37–41.

7. V.A. Galaktionov, Quasilinear heat equations with first-order sign-invariants and new explicit solutions, Nonlinear Anal. Theor. Meth. Appl. 23 (1994) 1595–621.
8. P.J. Olver, Direct reduction and differential constraints, Proc. Roy. Soc. London, Ser. A 444 (1994) 509–523.

9. O.V. Kaptsov, Determining equations and differential constraints, Nonlinear Math. Phys. 2(3-4) (1995) 283–291.

10. A.F. Sidorov, V.P. Shapeev, N.N. Yanenko, Method of Differential Constraints and its Applications in Gas Dynamics, Nauka, Novosibirsk, 1984 (in Russian).

11. V.K. Andreev, O.V. Kaptsov, V.V. Pukhnachov, A.A. Rodionov, Applications of Group-Theoretical Methods in Hydrodynamics, Kluwer, Dordrecht, 1998.

12. O.V. Kaptsov, I.V. Verevkin, Differential constraints and exact solutions of nonlinear diffusion equations, J. Phys. A: Math. Gen. 36 (2003) 1401–1414.

13. G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.

14. D. Levi, P. Winternitz, Nonclassical symmetry reduction: Example of the Boussinesq equation, J. Phys. A 22 (1989) 2915–2924.

15. M.C. Nucci, P.A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation, Phys. Lett. A, 164 (1992) 49–56.

16. P.A. Clarkson, Nonclassical symmetry reductions for the Boussinesq equation, Chaos, Solitons & Fractals 5 (1995) 2261–2301.

17. P.J. Olver, E.M. Vorob’ev, Nonclassical and conditional symmetries, In: CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3 (ed. N. H. Ibragimov), CRC Press, Boca Raton, 1996, pp. 291–328.
18. P.A. Clarkson, D.K. Ludlow, T.J. Priestley, The classical, direct and nonclassical methods for symmetry reductions of nonlinear partial differential equations, Methods Appl. Anal. 4(2) (1997) 173–195.

19. G. Saccomandi, A personal overview on the reduction methods for partial differential equations, Note di Matematica 23(2) (2004/2005) 217–248.

20. A.D. Polyanin, A.I. Zhurov, Unsteady axisymmetric boundary-layer equations: Transformations, properties, exact solutions, order reduction and solution method, Int. J. Non-Linear Mech. 74 (2015) 40–50.

21. A.D. Polyanin, A.I. Zhurov, Direct functional separation of variables and new exact solutions to axisymmetric unsteady boundary-layer equations, Commun. Nonlinear Sci. Numer. Simulat., 31 (2016) 11–20.

22. A.D. Polyanin, A.I. Zhurov, One-dimensional reductions and functional separable solutions to unsteady plane and axisymmetric boundary-layer equations for non-Newtonian fluids, Int. J. Non-Linear Mech. 85 (2016) 70–80.

23. P.A. Clarkson, M.D. Kruskal, New similarity reductions of the Boussinesq equation, J. Math. Phys. 30 (1989) 2201–2213.

24. D.K. Ludlow, P.A. Clarkson, A.P. Bassom, Similarity reductions and exact solutions for the two-dimensional incompressible Navier–Stokes equations, Studies Appl. Math. 103 (1999) 183–240.

25. D.K. Ludlow, P.A. Clarkson, A.P. Bassom, New similarity solutions of the unsteady incompressible boundary-layer equations, Quart. J. Mech. and Appl. Math. 53 (2000) 175–206.