ON SEMI-INFINITE COHOMOLOGY OF FINITE DIMENSIONAL ALGEBRAS

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ABSTRACT. We show that semi-infinite cohomology of a finite dimensional graded algebra (satisfying some additional requirements) is a particular case of a general categorical construction. An example of this situation is provided by small quantum groups at a root of unity.

1. INTRODUCTION

Semi-infinite cohomology of associative algebras was studied, in particular, by S. Arkhipov (see [Ar1], [Ar2], [Ar3]). Recall that the definition of semi-infinite cohomology in [Ar1] works in the following set-up. We are given an associative graded algebra $A$, two subalgebras $B, N \subset A$ such that $A = B \otimes N$ as a vector space, satisfying some additional assumptions. In this situation the space of semi-infinite Ext’s, $\text{Ext}^{\infty/2+\bullet}(X, Y)$ is defined for $X, Y$ in the bounded derived category of graded $A$-modules. The definition makes use of explicit complexes.

In this note we show that under some additional assumptions semi-infinite Ext groups $\text{Ext}^{\infty/2+\bullet}(X, Y)$ has a categorical interpretation. More precisely, given a category $\mathcal{A}$ and subcategory $B \subset \mathcal{A}$ one can define for $X, Y \in \mathcal{A}$ the set of morphisms from $X$ to $Y$ “through $B$”; we denote this space by $\text{Hom}_{\mathcal{A}_{B}}(X, Y)$. We then show that if $\mathcal{A}$ is the bounded derived category of $A$-modules, and $B$ is the full triangulated subcategory generated by $B$-projective $A$-modules, then, under certain assumptions one has

$$\text{Ext}^{\infty/2+i}(X, Y) = \text{Hom}_{\mathcal{A}_{B}}(X, Y[i]).$$

Notice that the right hand side of (1) makes sense for a wide class of pairs $(A, B)$ (an associative algebra, and a subalgebra), and $X, Y \in D^b(A - \text{mod})$; in particular we do not need $A, B$ to be graded. Thus one may consider (1) as providing a generalization of the definition of semi-infinite Ext’s to this set up. However, we should warn the reader that under our working assumptions, but not in general, $B$ also equals the full triangulated subcategory generated by $B$-injective modules, or by modules (co)induced from a ”complemental” subalgebra $N \subset A$, so one has at least four different obvious generalizations of the definition of the right-hand side of (1).

In fact, a description of semi-infinite cohomology similar to (1) in a general situation (in particular, in the case of enveloping algebras of infinite-dimensional Lie algebras) requires additional ideas, and is the subject of a forthcoming joint work with Arkhipov and Positselskii.

An example of the situation considered in this paper is provided by a small quantum group at a root of unity [L], or by the restricted enveloping algebra of a simple Lie algebra in positive characteristic. Computation of semi-infinite cohomology in
the former case is due to S. Arkhipov [Ar1] (the answer suggested as a conjecture by B. Feigin). This example was a motivation for the present work. We informally explain the relation of our Theorem 1 to the answer for semi-infinite cohomology of small quantum groups in Remark 5 below (we plan to derive it from Theorem 1 elsewhere).

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2. Categorical preliminaries: morphisms through a functor

Let \( A, B \) be (small) categories, and \( \Phi : B \to A \) be a functor. For \( X, Y \in \text{Ob}(A) \) define the set of "morphisms from \( X \) to \( Y \) through \( \Phi \)" as \( \pi_0 \) of the category of diagrams

\[
(2) \quad X \to \Phi(?) \to Y, \quad ? \in B.
\]

This set will be denoted by \( \text{Hom}_{A \Phi}(X, Y) \). Thus elements of \( \text{Hom}_{A \Phi}(X, Y) \) are diagrams of the form (2), with two diagrams identified if there exists a morphism between them. Composing the two arrows in (2) we get a functorial map

\[
(3) \quad \text{Hom}_{A \Phi}(X, Y) \to \text{Hom}_{A}(X, Y).
\]

If \( A, B \) are additive and \( \Phi \) is an additive functor, then addition of diagrams of the form (2) is defined by

\[
(X \xrightarrow{f} \Phi(Z) \xrightarrow{g} Y) + (X \xrightarrow{f'} \Phi(Z') \xrightarrow{g'} Y) = (X \xrightarrow{f \oplus f'} \Phi(Z \oplus Z') \xrightarrow{g \oplus g'} Y);
\]

it induces an abelian group structure on \( \text{Hom}_{A \Phi}(X, Y) \). Proposition 3 in [M1], VIII.2 shows that for \( Z \in B \) the tautological map

\[
\text{Hom}(X, \Phi(Z)) \otimes_{Z} \text{Hom}(\Phi(Z), Y) \to \text{Hom}_{A \Phi}(X, Y)
\]

is compatible with addition.

We have the composition map

\[
\text{Hom}_{A}(X', X) \times \text{Hom}_{A \Phi}(X, Y) \times \text{Hom}_{A}(Y, Y') \to \text{Hom}_{A \Phi}(X', Y);
\]

in particular, for \( A, B, \Phi \) additive, \( \text{Hom}_{A \Phi}(X, Y) \) is an \( \text{End}(X) - \text{End}(Y) \) bimodule.

Given \( \Phi : A \to B, \Phi' : A' \to B' \) and \( F : A \to A', G : B \to B' \) with \( F \circ \Phi \cong \Phi' \circ G \) we get for \( X, Y \in A \) a map

\[
(4) \quad \text{Hom}_{A \Phi}(X, Y) \to \text{Hom}_{A' \Phi'}(F(X), F(Y)).
\]

If the left adjoint functor \( \Phi^* \) to \( \Phi \) is defined on \( X \), then we have

\[
\text{Hom}_{A \Phi}(X, Y) = \text{Hom}_{A}(\Phi(\Phi^*(X)), Y),
\]

because in this case the above category contracts to the subcategory of diagrams of the form \( X \xrightarrow{\text{can}} \Phi(\Phi^*(X)) \to Y \), where \( \text{can} \) stands for the adjunction morphism.

If the right adjoint functor \( \Phi^! \) is defined on \( Y \), then

\[
\text{Hom}_{A \Phi}(X, Y) = \text{Hom}_{A}(X, \Phi(\Phi^!(Y)))
\]

for similar reasons. In particular, if \( \Phi \) is a full imbedding then (3) is an isomorphism provided either \( X \) or \( Y \) lie in the image of \( \Phi \).

In all examples below \( A \) will be a triangulated category, and \( \Phi : B \to A \) will be an imbedding of a (strictly) full triangulated subcategory. Given \( B \subset A \) we will
tacitly assume $\Phi$ to be the imbedding, and write $\text{Hom}_{A^0}$ ("morphisms through $B^0$") instead of $\text{Hom}_{A^0}$.

**Example 1.** Let $M$ be a Noetherian scheme, and $A = D^b(\text{Coh}_M)$ be the bounded derived category of coherent sheaves on $M$; let $i : B \hookrightarrow A$ be the full subcategory of complexes whose cohomology ia supported on a closed subset $i : N \hookrightarrow M$. Then the functor $i \circ i^! = i \circ i^*$ takes values in a larger derived category of quasi-coherent sheaves (i.e. ind-coherent sheaves), and $i \circ i^! = i \circ i^*$ takes values in the Grothendieck-Serre dual category, the derived category of pro-coherent sheaves (introduced in Deligne's appendix to [H]). Still we have

$$\text{Hom}_{A^0}(X, Y) = \text{Hom}(X, i_*(i^!(Y))) = \text{Hom}(i_*(i^!(X)), Y).$$

In particular, if $X = \mathcal{O}_M$ is the structure sheaf, we get

$$\text{Hom}_{A^0}(\mathcal{O}_M, Y[i]) = H^i_N(Y),$$

where $H^i_N(Y)$ stands for cohomology with support on $N$ (local cohomology) [H].

3. **Recollection of the definition of $\text{Ext}^{\infty/2+\bullet}$**

All algebras below will be associative and unital algebras over a field.

We recall a variant of definition of semi-infinite Ext's (available under certain restrictions on the algebra and subalgebras) suited for our purpose (see e.g. [FS], §2.4, pp 180-183, for this definition in the particular case of small quantum groups; the general case is analogous).

We make the following assumptions. A $\mathbb{Z}$-graded algebra $A$ and graded subalgebras $A^0$, $A^{\leq 0}$, $A^{\geq 0} \subset A$ are fixed and satisfy the following conditions:

1. $A^{\leq 0}$, $A^{\geq 0}$ are graded by, respectively, $\mathbb{Z}^{\leq 0}$, $\mathbb{Z}^{\geq 0}$, and $A^0 = A^{\leq 0} \cap A^{\geq 0}$ is the component of degree 0 in $A^{\geq 0}$ and in $A^{\leq 0}$.

2. The maps $A^{\geq 0} \otimes A^0 A^{\leq 0} \to A$ and $A^{\leq 0} \otimes A^0 A^{\geq 0} \to A$ provided by the multiplication map are isomorphisms.

3. $A$ is finite dimensional; $A^0$ is semisimple, and $A^{\geq 0}$ is self-injective (i.e. the free $A^{\geq 0}$-module is injective).

By a "module" we will mean a finite dimensional graded module, unless stated otherwise. By $A - \text{mod}$ we denote the category of (graded finite dimensional) $A$-modules.

Recall that a bounded below complex of graded modules is called convex if the weights "go down", i.e. for any $n \in \mathbb{Z}$ the sum of weight spaces of degree more than $n$ is finite dimensional. A bounded below complex of graded modules is called concave if the weights "go up" in the similar sense.

**Lemma 1.** i) Any $A$-module admits a right convex resolution by $A$-modules, which are injective as $A^{\geq 0}$-modules. It also admits a right concave resolution by $A$-modules, which are $A^{\leq 0}$-injective.

ii) Any finite complex of $A$-modules is a quasiisomorphic subcomplex of a bounded below convex complex of $A^{\geq 0}$-injective $A$-modules. It is also a quasiisomorphic subcomplex of a bounded below concave complex of $A^{\leq 0}$-injective $A$-modules.

**Proof.** To deduce (ii) from (i) imbed given finite complex $C^\bullet \in \text{Com}(A - \text{mod})$ into a complex of $A$-injective modules $I^\bullet \in \text{Com}^{\geq 0}(A - \text{mod})$ (notice that condition (2) above implies that an $A$-injective module is also $A^{\geq 0}$ and $A^{\leq 0}$ injective), and apply (i) to the module of cocycles $Z^n = I^n/d(I^{n-1})$ for large $n$. 


To check (i) it suffices to find for any $M \in A - \text{mod}$ an imbedding $M \hookrightarrow I$, where $I$ is $A^{\leq 0}$ injective, and if $n$ is such that all graded components $M_i$ for $i < n$ vanish, then $M_n \overset{\sim}{\longrightarrow} I_n$. (This would prove the second part of the statement; the first one is obtained from the first one by renotation.) It suffices to take $I = \text{CoInd}_{A^{\leq 0}}^A(\text{Res}_{A^{\leq 0}}^A(M))$. It is indeed $A^{\leq 0}$-injective, because of the equality
\[(6) \quad \text{Res}_{A^{\leq 0}}^A(\text{CoInd}_{A^{\geq 0}}^A(M)) = \text{CoInd}_{A^{\geq 0}}^A(M),\]
which is a consequence of assumption (2) above. □

We set $D = D^b(A - \text{mod})$.

**Definition 1.** (cf. [FS], §2.4) The assumptions (1–3) are enforced. Let $X, Y \in D$. Let $J^X$ be a convex bounded below complex of $A^{\geq 0}$-injective (= projective) modules quasiisomorphic to $X$, and $J^Y$ be a concave bounded below complex of $A^{\leq 0}$-injective modules quasiisomorphic to $Y$. Then one defines
\[(7) \quad \text{Ext}^{\infty/2+i}(X, Y) = H^i(\text{Hom}^\bullet(J^X \downarrow J^Y \uparrow)).\]

**Remark 1.** Independence of the right-hand side of (7) on the choice of resolutions $J^X, J^Y$ follows from the argument below. Since particular complexes used in [Ar1] to define $\text{Ext}^{\infty/2+i}$ satisfy our assumptions, we see that this definition agrees with the one in loc. cit.

**Remark 2.** Notice that $\text{Hom}$ in the right-hand side of (7) is $\text{Hom}$ in the category of graded modules. As usual, it is often convenient to denote by $\text{Ext}^{\infty/2+i}(X, Y)$ the graded space which in present notations is written down as $\bigoplus_n \text{Ext}^{\infty/2+i}(X, Y(n))$, where $(n)$ refers to shift of grading by $-n$.

**Remark 3.** The next standard Lemma shows that conditions on the resolutions $J^X, J^Y$ used in the (7) can be formulated in terms of the subalgebra $A^{\geq 0}$ alone (or, alternatively, in terms of $A^{\leq 0}$ alone); this conforms with the fact that the left-hand side of (11) in Theorem 1 below depends only on $A^{\geq 0}$. However, existence of a "complemental" subalgebra $A^{\leq 0}$ is used in the construction of a resolution $J^X \downarrow$ with required properties.

**Lemma 2.** An $A$-module is $A^{\leq 0}$-injective iff it is has a filtration with subquotients of the form $\text{CoInd}_{A^{\geq 0}}^A(M), M \in A^{\geq 0} - \text{mod}$.

**Proof.** The "if" direction follows from semisimplicity of $A^0$, and equality (6) above. To show the "only if" part let $M$ be an $A^{\leq 0}$-injective $A$-module. Let $M^-$ be its graded component of minimal degree; then the canonical morphism
\[(8) \quad M \rightarrow \text{CoInd}_{A^{\leq 0}}^A(M^-)\]
is surjective. If $M$ is actually an $A$-module, then the projection $M \rightarrow M^-$ is a surjection of $A^{\geq 0}$-modules, hence yields a morphism
\[(9) \quad M \rightarrow \text{CoInd}_{A^{\geq 0}}^A(M^-).\]

(6) shows that $\text{Res}_{A^{\leq 0}}^A$ sends (8) into (9); in particular (8) is surjective. Thus the top quotient of the required filtration is constructed, and the proof is finished by induction. □
Remark 4. In two special cases \( \mathrm{Ext}^{\infty/2+i}(X,Y) \) coincides with a traditional derived functor. First, suppose that \( \operatorname{Res}^{A}_{A^{\leq 0}}(X) \) has finite injective (equivalently, projective) dimension; then one can use a finite complex \( J^X_\infty \) in \( \overline{\mathbb{C}} \) above. It follows immediately, that in this case we have

\[
\mathrm{Ext}^{\infty/2+i}(X,Y) \cong \mathrm{Hom}(X,Y[i]).
\]

On the other hand, suppose that \( \operatorname{Res}^{A}_{A^{\leq 0}}(Y) \) has finite injective dimension, so that the complex \( J^Y_i \) in \( \overline{\mathbb{C}} \) can be chosen to be finite. To describe semi-infinite \( \mathrm{Ext} \)'s in this case we need another notation. Let \( A^* \) denote the co-regular \( A \)-bimodule; for \( M \in A - \text{mod} \) let \( M* = \mathrm{Hom}_{A}(M,A^*) \) denote the corresponding right \( A \)-module, and we use the same notation for the corresponding functor on the derived categories. Let also \( S : D^b(A - \text{mod}) \rightarrow D^+(A - \text{mod}) \) be given by \( S(Y) = R\mathrm{Hom}_{A}(A^*,Y) \). Notice that \( A^* \) is \( A^{\geq 0} \)-projective by self-injectivity of \( A^{\leq 0} \); thus Lemma \( 2 \) shows that \( \mathrm{Ext}^i_A(A^*,N) = 0 \) for \( i > 0 \) if \( N \) is \( A^{\leq 0} \)-injective. In particular, \( S(Y) \in D^b(A - \text{mod}) \) if \( Y|_{A^{\leq 0}} \) has finite injective dimension. We claim that in this case we have

\[
\mathrm{Ext}^{\infty/2+i}(X,Y) \cong X \otimes_A S(Y).
\]

This isomorphism an immediate consequence of the next Lemma. We also remark that if \( A \) is a Frobenius algebra, then \( S \cong \text{Id} \).

Lemma 3. Let \( M, N \in A - \text{mod} \) be such that \( M \) is \( A^{\geq 0} \)-projective, while \( N \) is \( A^{\leq 0} \)-injective. Then we have

a) \( \mathrm{Ext}^i_A(M,N) = 0; \) \( \mathrm{Ext}^i_A(A^*,N) = (R^iS)(N) = 0, \mathrm{Tor}^A_i(M^*,S(N)) = 0 \) for \( i \neq 0 \).

b) The natural map

\[
(10) \quad M^* \otimes_A S(N) = \mathrm{Hom}_{A}(M,A^*) \otimes_A \mathrm{Hom}_{A}(A^*,N) \rightarrow \mathrm{Hom}_{A}(M,N)
\]

is an isomorphism.

Proof. The first equality in (a) follows from Lemma \( 2 \) and the second one was checked above. Self-injectivity of \( A^{\geq 0} \) shows that \( M^* \) is \( A^{\geq 0} \)-projective, and a variant of Lemma \( 2 \) ensures that it is filtered by modules induced from \( A^{\leq 0} \). Thus it suffices to show that \( S(N) \) is \( A^{\leq 0} \)-projective. This follows from isomorphisms

\[
\mathrm{Hom}_{A}(A^*,\text{CoInd}_{A^{\geq 0}}(N_0)) = \mathrm{Hom}_{A^{\geq 0}}(A^*,N_0) \cong \mathrm{Hom}_{A^{\geq 0}}((A^{\geq 0})^*,N_0) \otimes_{A^{0}} A^{\leq 0}.
\]

Let us now deduce (b) from (a). Notice that (a) implies that both sides of (10) are exact in \( N \) (and also in \( M \)), i.e. send exact sequences \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) with \( N', N'' \) being \( A^{\leq 0} \)-injective into exact sequences. Also (10) is evidently an isomorphism for \( N = A^* \). For any \( A^{\leq 0} \)-injective \( N \) there exists an exact sequence

\[
0 \rightarrow N \rightarrow (A^*)^n \xrightarrow{\phi} (A^*)^m
\]

with image and cokernel of \( \phi \) being \( A^{\leq 0} \)-injective. Thus both sides of (10) turn it into an exact sequence, which shows that (10) is an isomorphism for any \( A^{\leq 0} \)-injective \( N \). \( \square \)
4. Main result

Theorem 1. Let $D_{\infty/2} \subset D$ be the full trinagulated subcategory of $D$ generated by $A^{\geq 0}$-injective (=projective) modules. For $X, Y \in D^b(A - \text{mod})$ we have a natural isomorphism
\[ Hom_{D_{\infty/2}}(X, Y[i]) \cong \text{Ext}^{\infty/2+i}(X, Y). \]

The proof of Theorem 1 is based on the following

Lemma 4. i) Every graded $A^{\geq 0}$-injective $A$-module admits a concave right resolution consisting of $A$-injective modules.

ii) A finite complex of graded $A^{\geq 0}$-injective $A$-modules is quasismorphic to a concave bounded below complex of $A$-injective modules.

Proof. (ii) follows from (i) as in the proof of Lemma 1. (Recall that, according to Hilbert, if a bounded below complex of injectives represents an object $X \in D^b$ which has finite injective dimension, then for large $n$ the module of cocycles is injective.)

To prove (i) it is enough for any $A^{\geq 0}$-injective module $M$ to find an imbedding $M \hookrightarrow I$, where $I$ is $A$-injective, and $M_n \hookrightarrow I_n$ provided $M_i = 0$ for $i < n$. (Notice that cokernel of such an imbedding is $A^{\geq 0}$-injective, because $I$ is $A^{\geq 0}$-injective by condition (2).) We can take $I$ to be $\text{CoInd}_{A^{\geq 0}}^A(\text{Res}_{A^{\geq 0}}^A(M))$. Then $I$ is indeed injective, because $M$ is $A^{\geq 0}$-injective by semi-simplicity of $A^0$, and condition on weights is clearly satisfied.

Proposition 1. a) Let $J_{\prec}$ be a convex bounded below complex of $A$-modules. Let $J_{\prec}^n$ be the $n$-th stupid truncation of $J_{\prec}$ (thus $J_{\prec}^n$ is a quotient complex of $J_{\prec}$).

Let $Z$ be a finite complex of $A^{\geq 0}$-injective $A$-modules. Then we have
\[ Hom_D(X, Z) \cong \varinjlim Hom_D(J_{\prec}^n, Z). \]

In fact, for $n$ large enough we have
\[ Hom_D(X, Z) \cong Hom_D(J_{\prec}^n, Z). \]

Proof. Let $I_{\succ}$ be a concave bounded below complex of $A$-injective modules quasiisomorphic to $Z$ (which exists by Lemma 2(ii)). Then the left-hand side of (12) equals $Hom_{\text{Hot}}(J_{\prec}, I_{\succ})$ where $\text{Hot}$ stands for the homotopy category of complexes of $A$-modules. Conditions on weights of our complexes ensure that there are only finitely many degrees for which the corresponding graded components both in $J_{\prec}$ and $I_{\succ}$ are nonzero; thus any morphism between graded vector spaces $J_{\prec}$, $I_{\succ}$ factors through the finite dimensional sum of corresponding graded components. In particular, $Hom^*(J_{\prec}^n, I_{\succ}) \cong Hom^*(J_{\prec}, I_{\succ})$ for large $n$, and hence
\[ Hom_D(A - \text{mod})(J_{\prec}^n, I_{\succ}) = Hom_{\text{Hot}}(J_{\prec}^n, I_{\succ}) \cong Hom_{\text{Hot}}(J_{\prec}, I_{\succ}) \]
for large $n$.}

Proof of the Theorem. We keep notations of Definition 3. It follows from the Proposition that
\[ Hom_{D_{\infty/2}}(X, Y[i]) = \varinjlim_n Hom_D((J_{\prec}^X)^n, Y[i]). \]
The right-hand side of (11) (defined in (3)) equals $H^i(Hom^*(J_{\prec}^X, J_{\succ}^Y))$. Conditions on weights of $J_{\prec}^X$, $J_{\succ}^Y$ show that for large $n$ we have
\[ Hom^*((J_{\prec}^X)^n, J_{\succ}^Y) \cong Hom^*(J_{\prec}^X, J_{\succ}^Y). \]
Lemma 2 implies that \( \text{Ext}^i_A(M_1, M_2) = 0 \) for \( i > 0 \) if \( M_1 \) is \( A^{\geq 0} \)-projective, and \( M_2 \) is \( A^{\leq 0} \)-injective. Thus

\[
\text{Hom}_D((J_X[n])/Y[i]) = H^i(\text{Hom}^\bullet(J_X[n], J_Y)).
\]

The Theorem is proved. \( \Box \)

Remark 5. This remark concerns with the example provided by a small quantum group. So let \( g \) be a simple Lie algebra over \( \mathbb{C} \), \( q \in \mathbb{C} \) be a root of unity of order \( l \), and let \( A = u_q = u_q(\mathfrak{g}) \) be the corresponding small quantum group \([4]\). Let \( A^{\geq 0} = b_q \subset u_q \) and \( A^{\leq 0} = b_q^- \subset u_q \) be respectively the upper and the lower triangular subalgebras. Then the above conditions (1-3) are satisfied.

Let \( I \) denote the trivial \( u_q \)-module. The cohomology \( \text{Ext}^\bullet(I, I) \), and the semi-infinite cohomology \( \text{Ext}^{\infty/2+\bullet}(I, I) \) were computed respectively in \([3K]\) and \([Ar1]\). Let us recall the results of these computations.

Assume for simplicity that \( l \) is prime to twice the maximal multiplicity of an edge in the Dynkin diagram of \( \mathfrak{g} \). Let \( N \subset \mathfrak{g} \) be the cone of nilpotent elements, and \( n \subset N \) be a maximal nilpotent subalgebra. Then the Theorem of Ginzburg and Kumar asserts that

\[
\text{Ext}^\bullet(I, I) \cong \mathcal{O}(N),
\]

the algebra of regular functions on \( N \). Also, a Theorem of Arkhipov (conjectured by Feigin) asserts that

\[
\text{Ext}^{\infty/2+\bullet}(I, I) \cong H^d_{\mathfrak{n}}(N, \mathcal{O}),
\]

where \( d \) is the dimension of \( n \), and \( H^d_{\mathfrak{n}}(N, \mathcal{O}) \) denotes the cohomology with support on \( \mathfrak{n} \); one also has \( H^i_{\mathfrak{n}}(N, \mathcal{O}) = 0 \) for \( i \neq d \) (here the choice of \( \mathfrak{n} \) is assumed to be compatible with the choice of an upper triangular subalgebra \( b_q \subset u_q \) via isomorphism \([3]\) in a natural sense).

The aim of this remark is to point out a formal similarity between \([4]\) and equality \([3]\) in Example \([1]\) above. Namely, the Ginzburg-Kumar isomorphism \([3]\) yields a functor \( F : D^b(u_q - \text{mod}) \to \text{Coh}(N), F(X) = \text{Ext}^\bullet(I, X) \), such that \( F(I) = \mathcal{O}_X \) is the structure sheaf. It is easy to see that if \( X \in D^b(u_q - \text{mod}) \) has finite projective (equivalently, injective) homological dimension over \( b_q \), then the support of \( F(X) \) lies in \( \mathfrak{n} \) (here by support we mean set-theoretic rather than scheme-theoretic support, so the coherent \( \mathcal{O}_X \) may be annihilated by some power of the ideal of \( \mathfrak{n} \)). Thus if we assume for a moment that the functor \( F \) can be lifted to a triangulated functor \( \tilde{F} : D^b(u_q - \text{mod}) \to D^b(\text{Coh}(N)), \) then \([4]\) and Theorem \([2]\) would yield a morphism from the left-hand side to the right-hand side of \([1]\). Here we say that \( \tilde{F} \) is a lifting of \( F \) if \( F \cong \text{R} \Gamma \circ \tilde{F} \), where \( \text{R} \Gamma \mathcal{F} = \bigoplus_i H^i(\mathcal{F}) \) for \( \mathcal{F} \in D^b(\text{Coh}(N)) \).

It is easy to see that such a functor \( \tilde{F} \) does not exist. A meaningful version of the argument is as follows. Let \( \mathcal{O} \) be the differential graded algebra \( R\text{Hom}_{u_q}(I, I) \) (thus \( \mathcal{O} \) is a well-defined object of the category of differential graded algebras with inverted quasiisomorphisms); the Ginzburg-Kumar theorem \([3]\) shows that the cohomology algebra \( H^\bullet(\mathcal{O}) \cong \mathcal{O}(N) \). Let \( \text{DGmod}(\mathcal{O}) \) be the triangulated category of differential graded modules over \( \mathcal{O} \) with inverted quasiisomorphisms. Let \( D \subset \text{DGmod}(\mathcal{O}) \) be the full subcategory of DG-modules whose cohomology is a finitely generated module over \( H^\bullet(\mathcal{O}) = \mathcal{O}(N) \), and let \( D_{\infty/2} \subset D \) be the full triangulated
subcategory of DG-modules, whose cohomology is a coherent sheaf on $\mathcal{N}$ supported (set-theoretically) on $\mathfrak{n}$.

We have a functor $\tilde{F} : D^b(u_q - \text{mod}) \to D$ given by $\tilde{F} : X \mapsto R\text{Hom}(\mathbb{I}, X)$. It is easy to see that $\tilde{F}$ sends complexes of finite homological dimension over $\mathfrak{b}_q$ to $D_{\infty}/2$; and that $\tilde{F}(\mathbb{I}) = \mathcal{O}$. Thus, by Theorem 1, (3) provides a morphism

$$\text{Ext}^{\infty}_{\mathfrak{b}_q}(\mathbb{I}, \mathbb{I}) \to \text{Hom}_{D_{\infty}/2}^\bullet(\mathcal{O}, \mathcal{O}).$$

One can then show that this morphism is an isomorphism; and also that the DG-algebra $\mathcal{O}$ is formal (quasi-isomorphic to the DG-algebra $H^\bullet(\mathcal{O})$ with trivial differential), which implies that

$$\text{Hom}_{D_{\infty}/2}^\bullet(\mathcal{O}, \mathcal{O}) \cong H^\bullet(\mathcal{N}, \mathcal{O})$$

(notice that the latter isomorphism is not compatible with homological gradings). This yields the isomorphism (14).

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