THE REDUCED WAVE EQUATION IN LAYERED MATERIALS

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1. Introduction

The mathematical theory of wave in layered media is still posing interesting mathematical problems even in the linear, stationary case.

In Jäger-Saitō [9] and [8], we studied the spectrum of the reduced wave operator

\begin{equation}
H_0 = -\mu_0(x)^{-1}\Delta,
\end{equation}

where \( \mu_0(x) \) is a simple function which takes a two positive values \( \mu_{01} \) and \( \mu_{02} \) on \( \Omega_1 \) and \( \Omega_2 \) respectively. Here \( \Omega_\ell, \; \ell = 1, 2, \) are open sets of \( \mathbb{R}^N \) such that

\begin{equation}
\begin{cases}
\Omega_1 \cap \Omega_2 = \emptyset, \\
\overline{\Omega_1} \cup \overline{\Omega_2} = \Omega_1 \cup \overline{\Omega_2} = \mathbb{R}^N,
\end{cases}
\end{equation}

\( \overline{\Omega_\ell} \) being the closure of \( \Omega_\ell \). Under a new condition on the separating surface \( S = \partial \Omega_1 = \partial \Omega_2 \), we have established the limiting absorption principle for \( H_0 \) which implies that \( H_0 \) is absolute continuous. Our condition is satisfied, for example, for the case where \( S \) is a cylinder.

In this work we are going to extend the results in [9] to the multimedia case, the case where \( \mu_0(x) \) can take finitely or infinitely many values (see §2). The limiting absorption principle will be established and, again, the operator \( H_0 \) is absolute continuous. Also we shall consider short-range or long-range perturbation of \( H_0 \), that is, we shall study the operator

\begin{equation}
H = -\mu(x)^{-1}\Delta,
\end{equation}

where

\begin{equation}
\mu(x) = \mu_0(x) + \mu_1(x)
\end{equation}

and \( \mu_1(x) \) is short-range or long-range. In this case we shall prove that the point spectrum, if it exists, is discrete, and the limiting absorption principle holds on any interval which does not contain an eigenvalue.

As for the study of the reduced wave operators with discontinuous coefficients, many works have been done for the stratified media in which the coefficients of the operator are the functions of \( x' \in \mathbb{R}^k \subset \mathbb{R}^N, \; k < N \). Some perturbed operators of the

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above type have been discussed, too. Here we refer Wilcox [16], Ben-Artzi-Dermanjian-Guillot [2], Weder [14, 15], DeBiévre-Pravica [4, 5], Boutet de Monvel-Berthier-Manda [3], and Zhang [17]. In [5] S. DeBiévre and D. W. Pravica proved there is no point spectrum for the stratified propagators without any additional conditions other than sufficient smoothness of the coefficients at infinity.

It seems that there are rather few results for the nonstratified case. Eidus [6] was the first to consider the reduced wave operators $H_0$ with a cone-shape discontinuity. He imposed the following assumptions on the separating surface $S$:

\[ |n^{(1)}_N(x)| \geq c_1 \quad (x \in S), \]

and

\[ |x \cdot n^{(1)}(x)| \leq c_2 \quad (x \in S), \]

where $n^{(\ell)}(x)$, $\ell = 1, 2$, is the unit outward normal of $\Omega_\ell$ at $x$, and $x \cdot n^{(1)}(x)$ is the inner product of $x$ and $n^{(1)}(x)$ in $\mathbb{R}^N$. Note that a cone having its vertex at the origin and the positive $x_N$-axis as its axis satisfies (1.5) and (1.6). Under the above assumptions, Eidus [6] proved the limiting absorption principle for $H_0$, that is, by denoting by $R_0(z)$ the resolvent of $H_0$, the limits

\[ \lim_{\eta \downarrow 0} R_0(\lambda \pm i\eta) = R_0(\lambda) \quad \text{in } B(L_{2,1}(\mathbb{R}^N), L_{2,-1}(\mathbb{R}^N)) \]

exist for $\lambda > 0$, where the weighted $L_2$ space $L_{2,t}(\mathbb{R}^N)$, $t \in \mathbb{R}$, is defined by

\[ L_{2,t}(\mathbb{R}^N) = \{ f : (1 + |x|)^t f(x) \in L_2(\mathbb{R}^N) \} \]

with its inner product and norm

\[ (f, g)_t = \int_{\mathbb{R}^N} f(x)\overline{g(x)}(1 + |x|)^{2t} \, dx, \]

\[ \|f\|_t = (f, f)_t^{1/2}, \]

and $B(X, Y)$ is the Banach space of all bounded linear operators from $X$ into $Y$. Then, Saitô [13] showed that $L_{2,1}(\mathbb{R}^N)$ and $L_{2,-1}(\mathbb{R}^N)$ in (1.7) can be replaced by $L_{2,\delta}(\mathbb{R}^N)$ and $L_{2,-\delta}(\mathbb{R}^N)$ with $\delta > 1/2$, respectively. This means that the limiting absorption principle for $H_0$ holds on the same weighted $L_2$ spaces as are used for the Schrödinger operator (cf. Agmon [1], Ikebe-Saitô [7] and Saitô [11]). Recently Roach-Zhang [10] has shown that $u = R^{\pm}(\lambda)f$, where $\lambda > 0$ and $f \in L_{2,\delta}(\mathbb{R}^N)$ with $\delta > 1/2$, is characterized as a unique solution of the equation

\[ (-\mu_0(x)^{-1} \Delta - \lambda)u = f \]

with the radiation condition

\[ \lim_{R \to \infty} \frac{1}{R} \int_{B_R} |\nabla u + i\sqrt{\lambda\mu(x)}\bar{x}u|^2 \, dx = 0 \quad (\bar{x} = \frac{x}{|x|}), \]
\(B_R\) being the ball with radius \(R\) and center at the origin. The condition (1.11) is a natural extension of the radiation condition for the Schrödinger operators ([7], [11]). [10] also gave another proof of the limiting absorption principle for \(H_0\).

In the recent work [9] and [8], we studied the reduced wave operators \(H_0\) with a cylindrical discontinuity in which the separating surface is assumed to satisfy that

\[
(\mu_2 - \mu_0)(x \cdot n^{(1)}) = (\mu_0 - \mu_2)(x \cdot n^{(2)}) \geq 0 \quad (x \in S).
\]

The condition (1.12) is satisfied if \(\Omega_1\) is an infinite cylindrical domain which contains the origin and \(\mu_0 > \mu_0\). Then it has been shown again that \(H_0\) is absolutely continuous. So far it seems that the absence of the point spectrum can not be obtained without imposing some additional conditions such as (1.5)-(1.6) or (1.12).

In §2 we define the reduced wave operator \(H_0\) with multimedia and we state our assumption on the separating surface \(S\) and the positive function \(\mu_0\) in which \(\mu_0\) can take countably infinite values although the condition is a natural extension of the condition (1.12). §3 is devoted to showing the limiting absorption principle for the unperturbed operator \(H_0\). Here the arguments are quite parallel to the one in [8] or [9], and hence we shall omit some of the proof. In §4 we shall discuss the point spectrum of the perturbed operator (1.3). It will be shown that the point spectrum of \(H\) is discrete. Also some sufficient conditions for the nonexistence of the point spectrum of \(H\) will be given. We shall show in §5 that the limiting absorption principle for \(H\) holds on any closed interval which does not contain the point spectrum.

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2. The operators \(H_0\) and \(H\)

In this section we are going to define a reduced wave operator

\[
H_0 = -\mu_0(x)^{-1}\Delta,
\]

where \(\mu_0(x)\) is a positive, simple function on \(\mathbb{R}^N\) which will be specified below, and its perturbed operator

\[
H = -\mu(x)^{-1}\Delta,
\]

where

\[
\mu(x) = \mu_0(x) + \mu_1(x)
\]

such that \(\mu(x)\) is a positive function on \(\mathbb{R}^N\) and \(\mu_1(x)\) decays to 0 at infinity.
Let us describe the conditions on $\mu_0(x)$. Let $N$ be all positive integers and let $N_-$ be all negative integers. Let $L$ be a subset of integers satisfying one of the following:

\begin{align}
(I) \quad L &= N_- \cup \{0\} \cup N, \\
(II) \quad L &= \{L_-, L_- + 1, \ldots, -1, 0\} \cup N, \\
(III) \quad L &= N_- \cup \{0, 1, \ldots, L_+\}, \\
(IV) \quad L &= \{L_-, L_- + 1, \ldots, -1\} \cup \{0\} \cup \{1, 2, \ldots, L_+\},
\end{align}

where $L_- \in N_- \cup \{0\}$ and $L_+ \in \{0\} \cup N$.

**Assumption 2.1.** Let $N$ be a positive integer such that $N \geq 2$. Let $L$ be as in (2.4). For each $\ell \in L$, let $\Omega_\ell$ be an open set in $\mathbb{R}^N$. Let $\mu_0$ be a positive function on $\mathbb{R}^N$. The family $\{\Omega_\ell\}_{\ell \in L}$ and the function $\mu_0$ are assumed to satisfy the following (i) $\sim$ (iii):

(i) $\{\Omega_\ell\}_{\ell \in L}$ is a disjoint family of open sets of $\mathbb{R}^N$ such that

\begin{align}
\mathbb{R}^N &= \bigcup_{\ell \in L} \overline{\Omega}_\ell,
\end{align}

where $\overline{\Omega}_\ell$ is the closure of $\Omega_\ell$. For any $R > 0$, the open ball $B_R$ with center at the origin and radius $R$ is covered by a union of a finite number of $\overline{\Omega}_\ell$, i.e., for $R > 0$ there is a finite subset $L_R$ of $L$ such that

\begin{align}
\Omega_\ell \cap B_R &= \emptyset \quad (\ell \in L - L_R).
\end{align}

(ii) For each $\ell \in L$, the boundary $\partial \Omega_\ell$ of $\Omega_\ell$ is a disjoint union of two continuous surfaces $S_\ell^(-)$ and $S_\ell^($, i.e.,

\begin{align}
\partial \Omega_\ell &= S_\ell^(-) \cup S_\ell^($,

\begin{align}
S_\ell^(-) \cap S_\ell^($ &= \emptyset
\end{align}

for $\ell \in L$, where $S_\ell^(-)$ and $S_\ell^($ are unions of a finite number of smooth surfaces. Here we assume that $S_{L_-}^(-) = \emptyset$ when $L$ has the smallest number $L_-$ and $S_{L_+}^($ = $\emptyset$ when $L$ has the largest number $L_+$. Further we assume that

\begin{align}
S_\ell^($ = S_{\ell+1}^(-) \quad (\ell \in L),
\end{align}

where we set $S_{\ell+1}^(-) = \emptyset$ if $\ell + 1 \notin L$.

(iii) $\mu_0$ is a simple function which takes the value $\nu_\ell$ on each $\Omega_\ell$, where $\nu_\ell$ is a positive number such that

\begin{align}
0 < m_0 \equiv \inf_{\ell \in L} \nu_\ell \leq \sup_{\ell \in L} \nu_\ell \equiv M_0 < \infty.
\end{align}

Let

\begin{align}
n^{(\ell)}(x) &= (n_1^{(\ell)}(x), n_2^{(\ell)}(x), \ldots, n_N^{(\ell)}(x)) \quad (\ell \in L),
\end{align}
be the unit outward normal of $\Omega_\ell$ at a.e. $x \in \partial \Omega_\ell$. Then we assume that

\[(\nu_\ell - \nu_{\ell+1})(n^{(\ell)}(x) \cdot x) \leq 0 \quad (x \in S^{(+)}_\ell, \ \ell \in L)\]

although $\ell \neq L_+$ if $L$ has the largest number $L_+$, where $n^{(\ell)}(x) \cdot x$ is the usual inner product of $n^{(\ell)}(x)$ and $x$ in $\mathbb{R}^N$.

As for the function $\mu$ we have

**Assumption 2.2.** Let $\mu$ be a measurable function on $\mathbb{R}^N$ satisfying the following (i) and (ii):

(i) We have

\[
0 < \tilde{m}_0 \equiv \inf_{x \in \mathbb{R}^N} \mu(x) \leq \sup_{x \in \mathbb{R}^N} \mu(x) \equiv \tilde{M}_0 < \infty.
\]

(ii) Let $\mu_1 = \mu - \mu_0$. Then either $\mu_1$ is short-range, that is,

\[
|\mu_1(x)| \leq c_1(1 + |x|)^{-1-\epsilon} \quad (x \in \mathbb{R}^N),
\]

or $\mu_1$ is long-range, that is, $\mu_1$ is differentiable such that

\[
\begin{align*}
|\mu_1(x)| & \leq c_1(1 + |x|)^{-\epsilon} \quad (x \in \mathbb{R}^N), \\
|\nabla \mu_1(x)| & \leq c_1(1 + |x|)^{-1-\epsilon} \quad (x \in \mathbb{R}^N),
\end{align*}
\]

with constants $c_1, \epsilon > 0$. Throughout this work we assume that $0 < \epsilon < 1/2$ with no loss of generality.

Let $X_0$ and $X$ be Hilbert spaces given by

\[
\begin{align*}
X_0 &= L_2(\mathbb{R}^N; \mu_0(x)dx), \\
X &= L_2(\mathbb{R}^N; \mu(x)dx).
\end{align*}
\]

The inner product and norm of $X_0$ [or $X$] will be denoted by $(\ , \ )_{X_0}$ and $\| \cdot \|_{X_0}$ [or $(\ , \ )_X$ and $\| \cdot \|_X$], respectively. Then define the operator $H_0$ in $X_0$ by

\[
\begin{align*}
D(H_0) &= H^2(\mathbb{R}^N), \\
H_0u &= -\mu_0(x)^{-1}\Delta u,
\end{align*}
\]

where $D(T)$ is the domain of $T$, $H^2(\mathbb{R}^N)$ is the second order Soblev space on $\mathbb{R}^N$, and $\Delta u$ is defined in the sense of distributions. Similarly the operator $H$ in $X$ is given by

\[
\begin{align*}
D(H) &= H^2(\mathbb{R}^N), \\
Hu &= -\mu(x)^{-1}\Delta u,
\end{align*}
\]

Then it is easy to see that $H_0$ and $H$ are selfadjoint operators in $X_0$ and $X$, respectively.
Now we are going to give some examples of \( \{\Omega_\ell\}_{\ell \in L} \) and \( \mu_0 \) which satisfy Assumption 2.1. In the following examples we take \( N = 3 \) although the \( N \)-dimensional versions of these examples can be easily obtained.

**Example 2.3.** Let \( L = \mathbb{N}_- \cup \{0\} \cup \mathbb{N} \). Let \( \{b_\ell\}_{\ell \in \mathbb{N}_- \cup \mathbb{N}} \) be such that
\[
\begin{align*}
\cdots < b_m < b_{m+1} < \cdots < b_{-1} < 0 < b_1 < \cdots < b_\ell \cdots,
\end{align*}
\]
(2.18)
\[
\begin{align*}
b_\ell &\to \infty \quad (\ell \to \infty), \\
b_m &\to -\infty \quad (m \to -\infty),
\end{align*}
\]
and define \( \{\Omega_\ell\}_{\ell \in L} \) by
\[
\begin{align*}
\Omega_\ell &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : b_\ell < x_3 < b_{\ell+1}\} \quad (\ell \in \mathbb{N}), \\
\Omega_0 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : b_{-1} < x_3 < b_1\}, \\
\Omega_\ell &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : b_{\ell-1} < x_3 < b_\ell\} \quad (\ell \in \mathbb{N}_-),
\end{align*}
\]
(2.19)
Then the separating surfaces \( S_\ell^{(\pm)} \) are given by
\[
\begin{align*}
S_\ell^{(+)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_{\ell+1}\}, \\
S_\ell^{(-)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_\ell\}
\end{align*}
\]
for \( \ell \in \mathbb{N} \),

\[
\begin{align*}
S_0^{(+)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_1\}, \\
S_0^{(-)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_{-1}\},
\end{align*}
\]
(2.21)
and
\[
\begin{align*}
S_\ell^{(+)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_\ell\}, \\
S_\ell^{(-)} &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = b_{\ell-1}\}
\end{align*}
\]
(2.22)
for \( \ell \in \mathbb{N}_- \). Define \( \mu_0 \) by
\[
\mu_0(x) = \nu_\ell \quad (x \in \Omega_\ell)
\]
(2.23)
such that
\[
\begin{align*}
\nu_0 < \nu_1 < \nu_2 < \cdots < \nu_\ell < \cdots,
\end{align*}
\]
(2.24)
with
\[
\begin{align*}
\nu_0 > 0, \\
M_0 \equiv \sup_{\ell \in \mathbb{N} \cup \mathbb{N}_-} \nu_\ell < \infty.
\end{align*}
\]
(2.25)
Since we have
\[
\begin{align*}
  n^{(\ell)}(x) \cdot x &\geq 0 \quad (x \in S^{(+)\ell}, \; \ell \in \{0\} \cup \mathbb{N}), \\
  n^{(\ell)}(x) \cdot x &\leq 0 \quad (x \in S^{(+)\ell}, \; \ell \in \mathbb{N}_-),
\end{align*}
\]
we see that the condition (2.11) is satisfied. Although this is a reduced wave operator in stratified media studied by many authors (see, e.g., [16], [14], [4]), note that \( \Omega_\ell \) can be modified as far as the condition (2.26) holds good.

**Example 2.4.** Let \( L = \{0\} \cup \mathbb{N} \). Let \( \{b_\ell\}_{\ell \in \mathbb{N}} \) be such that
\[
\begin{align*}
  0 < b_1 < b_2 < \cdots < b_\ell < \cdots, \\
  b_\ell \to \infty \quad (\ell \to \infty),
\end{align*}
\]
and define \( \{\Omega_\ell\}_{\ell \in L} \) by
\[
\begin{align*}
  \Omega_0 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < b_1^2\}, \\
  \Omega_\ell &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : b_\ell^2 < x_1^2 + x_2^2 < b_{\ell+1}^2\}, \quad (\ell \in \mathbb{N})
\end{align*}
\]
The separating surfaces \( S^{(\pm)\ell} \) are given by
\[
\begin{align*}
  S^{(+)}_\ell &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = b_{\ell+1}^2\}, \quad (\ell \in L), \\
  S^{(-)}_\ell &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = b_\ell^2\}, \quad (\ell \in \mathbb{N}), \\
  S^{(-)}_0 &= \emptyset,
\end{align*}
\]
Define \( \mu_0 \) by
\[
\mu_0(x) = \nu_\ell \quad (x \in \Omega_\ell)
\]
such that
\[
\nu_0 < \nu_1 < \nu_2 < \cdots < \nu_\ell < \cdots,
\]
with
\[
\begin{align*}
  \nu_0 > 0, \\
  M_0 &\equiv \sup_{\ell \in \mathbb{N}} \nu_\ell < \infty.
\end{align*}
\]
Since
\[
\begin{align*}
  n^{(\ell)}(x) \cdot x &\geq 0 \quad (x \in S^{(+)\ell}_\ell, \; \ell \in L),
\end{align*}
\]
from (2.31) it is seen that the condition (2.11) is satisfied. Again \( \Omega_\ell \) are allowed to be deformed as far as (2.33) holds good.
3. The unperturbed operator $H_0$

In this section we are going to discuss the unperturbed operator $H_0$ given by (2.16). First we shall show the uniqueness theorem for the equation

\[(3.1) \quad (-\mu_0(x)^{-1}\Delta - \lambda)u = f\]

with radiation condition. Then, after showing several a priori estimates of the solution $u$ of the equation (3.1), the limiting absorption principle for $H_0$ will be proved. The arguments in this section are quite parallel to the ones in Jäger-Saitō [9], and hence we shall omit the proof or give a sketch of proof in most of the theorems given in this section.

We shall start with some notations.

**Notation 3.1.** Let $z \in \mathbb{C}$, $x = (x_1, x_2, \ldots, x_N)$, $r = |x|$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N) = x/r$, $\partial_j = \partial/\partial x_j$ and $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_N)$. Then we set

1. $k = k(x) = k(x, z) = [z\mu_0(x)]^{1/2}$, where the branch is taken so that $\text{Im } k(x, z) \geq 0$;
2. $a = a(x) = a(x, z) = \text{Re } k(x, z)$;
3. $b = b(x) = b(x, z) = \text{Im } k(x, z)$;
4. $D u = \nabla u + \{(N - 1)/(2r)\} \tilde{x} u - ik(x) \tilde{x} u$;
5. $D_r u = D u \cdot \tilde{x} = \partial u/\partial r + \{(N - 1)/(2r)\} u - ik(x) u$;

Let $u \in H^2(\mathbb{R}^N)_{\text{loc}}$. Then the restrictions $u|_G$ and $\partial_j u|_G$, $j = 1, 2, \ldots, N$, of $u$ and $\partial_j u = \partial u/\partial x_j$ onto a smooth surface $G$ are defined as the traces of $u$ and $\partial_j u$ on $G$, respectively. Thus $u|_G$ and $\partial_j u|_G$ are considered to belong to $L_2(G)_{\text{loc}}$.

Let $z \in \mathbb{C}$ and let $u \in H^2(\mathbb{R}^N)_{\text{loc}}$. Define $f$ by

\[(3.2) \quad f = -\mu_0(x)^{-1}\Delta u - zu = \mu_0(x)^{-1}(-\Delta u - k^2 u)\]

with $k$ given by (1) of Notation 3.1. Now we are going to show an identity which is an extension of Proposition 3.3 of [9] and will be used throughout this section.

**Proposition 3.2.** Let $u \in H^2(\mathbb{R}^N)_{\text{loc}}$ and let $f$ be given by (3.2). Let $\xi$ be a real-valued, continuous function on $[0, \infty)$ such that $\xi$ has piecewise continuous derivative. Set $\varphi(x) = \alpha(x)\xi(|x|)$, where $\alpha$ is a simple function which is constant on each $\Omega_k$. For $0 < r < R < \infty$, set

\[(3.3) \quad B_{rR} = \{x \in \mathbb{R}^N : r < |x| < R \},\]
Then we have

\[
\int_{B_{r,R}} \left( b\varphi + \frac{1}{2} \frac{\partial \varphi}{\partial r} \right) |Du|^2 \, dx + \sum_{\ell \in L} \int_{\partial \Omega_{\ell} \cap B_{r,R}} \varphi \text{Im}\{ k \frac{\partial u}{\partial n} \} \, dS \\
+ \int_{B_{r,R}} \left( \frac{\varphi}{r} - \frac{\partial \varphi}{\partial r} \right) (|Du|^2 - |Dr u|^2) \, dx \\
+ c_N \int_{B_{r,R}} r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} + b\varphi \right) |u|^2 \, dx
\]

\[(3.4)\]

where \( \Omega_{\ell} \) satisfies (i), (ii) of Assumption 2.1, \( \partial / \partial n \) in the integrand of the surface integral over \( \partial \Omega_{\ell} \cap B_{r,R} \) means the directional derivative in the direction of the outward normal \( n = n^{(\ell)} \) of \( \partial \Omega_{\ell} \), and

\[(3.5)\]

\[c_N = (N - 1)(N - 3)/4.\]

The proof will be omitted since it is essentially the same as the proof of Proposition 3.3 of [9].

**Theorem 3.3.** Assume Assumption 2.1. Let \( u \in H^2(\mathbb{R}^N)_{\text{loc}} \) be a solution of the homogeneous equation

\[(3.6)\]

\[- \mu_0(x)^{-1} \Delta u - \lambda u = 0 \quad (\lambda > 0)\]

on \( \mathbb{R}^N \) such that

\[(3.7)\]

\[\liminf_{R \to \infty} \int_{S_R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) \, dS = 0,\]

for \( N \geq 3 \), or

\[(3.8)\]

\[\liminf_{R \to \infty} R^\alpha \int_{S_R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) \, dS = 0\]

with \( \alpha > 0 \) for \( N = 2 \), where

\[(3.9)\]

\[S_R = \{ x \in \mathbb{R}^N : |x| = R \}.\]

Then \( u \) is identically zero.

**Sketch of Proof.** Theorem 3.3 can be proved by starting with Proposition 3.2 and proceeding as in the proof of Theorem 3.2 of [9] (for \( N \geq 3 \)) or proof of Theorem 7.1 of
[9] (for \( N = 2 \)). Only difference here is that, instead of the last inequality of (3.20) in [9], we have to use

\[
\sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_R} \varphi |k|^2 (\bar{x} \cdot n)|u|^2 dS \leq 0,
\]

where \( n \) in the integrand is the unit outward normal \( n^{(\ell)}(x) \) of \( \Omega_\ell \) at \( x \). In fact, it follows from (i) and (ii) of Assumption 2.1 that

\[
\sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_R} \varphi |k|^2 (\bar{x} \cdot n)|u|^2 dS = \sum_{\ell \in L} \left( \int_{S^{(-)}_\ell \cap B_R} \varphi |k|^2 (\bar{x} \cdot n)|u|^2 dS + \int_{S^{(+)}_\ell \cap B_R} \varphi |k|^2 (\bar{x} \cdot n)|u|^2 dS \right)
\]

\[
= \lambda \sum_{\ell \in L} \int_{S^{(-)}_\ell \cap B_R} \varphi (\nu_\ell - \nu_{\ell + 1}) (\bar{x} \cdot n^{(\ell)})|u|^2 dS,
\]

where we should note that we are dealing with a finite sum because of (i) of Assumption 2.1. Then (3.10) is obtained from (iii) of Assumption 2.1. //

The following corollary guarantees the uniqueness of the inhomogeneous equation

\[
- \mu_0(x)^{-1} \Delta u - \lambda u = f
\]

with one of the conditions

\[
\|D_r^{(\pm)} u\|_{\delta - 1, E_1} < \infty,
\]

where \( \delta > 1/2 \),

\[
D_r^{(\pm)} u = \partial u/\partial r + \{(N - 1)/(2r)\} u \mp ik(x)u,
\]

\[
E_R = \{ x \in \mathbb{R}^N : |x| > R \},
\]

and, for a measurable set \( G \) in \( \mathbb{R}^N \),

\[
\|v\|_{\delta - 1, G}^2 = \int_G (1 + |x|)^{2(\delta - 1)}|v(x)|^2 dx.
\]

**Corollary 3.4.** Let \( \lambda > 0 \) and let \( f \in L^2(\mathbb{R}^N)_{\text{loc}} \). Then the solution \( u \in H^2(\mathbb{R}^N)_{\text{loc}} \) of the equation (3.12) with one of the radiation conditions in (3.13) is unique.

The proof is the same as the proof of Corollary 3.8 of [9].

Let \( L_{2,\delta}(\mathbb{R}^N) \) be the weighted Hilbert space defined by (1.8). Let the resolvent \( (H_0 - z)^{-1} \) of the operator \( H_0 \) be denoted by \( R_0(z) \). Now consider \( u \in X_0 \) defined by

\[
\left\{ \begin{array}{l}
u = R_0(z) f, \\
z = \lambda + i\eta \\
f \in L_{2,\delta}(\mathbb{R}^N).
\end{array} \right.
\]

\( \lambda \geq 0, \eta \neq 0 \),
For $0 < c < d < \infty$ a subset $J_\pm(c, d)$ of $\mathbb{C}$ are defined by

$$
J_+(c, d) = \{ z = \lambda + i\eta : c \leq \lambda \leq d, \ 0 < \eta \leq 1 \},
$$

$$
J_-(c, d) = \{ z = \lambda + i\eta : c \leq \lambda \leq d, \ -1 \leq \eta < 0 \}.
$$

In the next theorem we are going to evaluate the radiation condition terms $\mathcal{D}u$. Here and in the sequel we agree that $C = C(A, B, \cdots)$ in an inequality means a positive constant depending on $A, B, \cdots$. Now we are evaluating the radiation condition term $\mathcal{D}u$.

**Theorem 3.5.** Suppose that Assumption 2.1 holds. Let $1/2 < \delta \leq 1$. Let $u$ be given by (3.17).

(i) Let $N \geq 3$. Then there exists a constant $C = C(\delta, m_0, M_0) > 0$ such that

$$
\|\mathcal{D}u\|_{\delta_1, \ast} \leq C(\|f\|_{\delta} + \|u\|_{-\delta})
$$

where $\mathcal{D}u$ is as in Notation 3.1, $\| \|$ is the norm of $L_{2,t}(\mathbb{R}^N)$, and the constant $C(\delta)$ is independent of $f$ and $z$ satisfying (3.17).

(ii) Let $N = 2$. Let $0 < c < d < \infty$ and let $J_\pm(c, d)$ be as in (3.18). Let $u$ be given by (3.17) with $z \in J_+(c, d) \cup J_-(c, d)$. Then there exists a positive constant $C = C(\delta, c, d, m_0, M_0)$ such that

$$
\|\mathcal{D}u\|_{\delta_1, \ast} \leq C(\|f\|_{\delta} + \|u\|_{-\delta})
$$

where

$$
\|v\|_{L_{t,*}}^2 = \int_{B_1} |x||v(x)|^2 \, dx + \int_{E_1} (1 + |x|)^{2t}|v(x)|^2 \, dx.
$$

Sketch of Proof. We have only to proceed as in the proof of Theorems 4.1 and 7.2 of [9]. Set in (3.4) $\alpha(x) = 1/\sqrt{\rho_0}$,

$$
\xi(r) = \begin{cases} 
\frac{r}{2^{-(2\delta - 1)}(1 + r)^{2\delta - 1}} & (0 \leq r \leq 1), \\
\frac{1}{2^{2\delta}}(1 + r)^{2\delta - 1} & (r \geq 1)
\end{cases}
$$

for $N \geq 3$, and

$$
\xi(r) = \begin{cases} 
\frac{1}{2}r^2 & (r \leq 1/2), \\
\frac{1}{2^{2\delta}}(1 + r)^{2\delta - 1} & (r \geq 1)
\end{cases}
$$

for $N = 2$. Let the second term of the left-hand side of (3.4) be denoted by $I_{L2}$. Then it is easy to see that

$$
I_{L2} = \sum_{\ell \in L} \int_{\partial \Omega \cap B_\ell \cap R} \varphi \text{Im}\{\kappa \frac{\partial u}{\partial n}\} \, dS = 0
$$
(cf. (3.11)). Similarly we see that the second term of the right-hand side of (3.4) is non-positive. All other terms of (3.4) can be evaluated exactly in the same manner as in the proof of Theorems 4.1 and 7.2 of [9], which completes the proof.  

Now that we have established the uniqueness of the solution of the equation (3.12) with the radiation condition (Corollary 3.4) and the estimate of the radiation condition term (Theorem 3.5), we can show the limiting absorption principle for \( H^0_0 \) by proceeding as in \( \S \) 5, \( \S \) 6, and \( \S \) 7 of [9]. Let \( t \in \mathbb{R} \). The weighted Sobolev spaces \( H^j_t(\mathbb{R}^N), j = 1, 2, \) are defined as the completion of \( C_0^\infty(\mathbb{R}^N) \) by the norms

\[
\| u \|_{1,t} = \left[ \int_{\mathbb{R}^N} (1 + r)^{2t} (|\nabla u|^2 + |u(x)|^2) \, dx \right]^{1/2},
\]

and

\[
\| u \|_{2,t} = \left[ \int_{\mathbb{R}^N} (1 + r)^{2t} \sum_{|\gamma| \leq 2} |\partial^\gamma u|^2 \, dx \right]^{1/2},
\]

respectively, where

\[
\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_N), \quad |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_N,
\]

\[
\partial^\gamma u = (\partial_1)^{\gamma_1} \cdots (\partial_N)^{\gamma_N} u \quad (\partial_j = \partial/\partial x_j).
\]

The inner product and norm of \( H^j_t(\mathbb{R}^N) \) will be denoted by \((\ , \)\)\(_{j,t}\) and \(\| \cdot \|_{j,t}\). For an operator \( T \), the operator norm in \( B(H^j_s(\mathbb{R}^N), H^\ell_t(\mathbb{R}^N)) \) will be denoted by \(\| T \|_{(j,s)}^{(\ell,t)}\), where \( j, \ell = 0, 1, 2, \ s, t \in \mathbb{R} \), and we set

\[
H^0_s(\mathbb{R}^N) = L_{2,s}(\mathbb{R}^N).
\]

Let \( D_\pm \subset \mathbb{C} \) be given by

\[
D_+ = \{ z = \lambda + i\eta : \lambda > 0, \eta \geq 0 \}, \quad D_- = \{ z = \lambda + i\eta : \lambda > 0, \eta \leq 0 \}.
\]

Also, for \( 0 < c < d < \infty \), let \( J_\pm(c, d) \) be as in (3.18). The closure \( \overline{J}_\pm(c, d) \) are given by

\[
\overline{J}_+(c, d) = \{ z = \lambda + i\eta : c \leq \lambda \leq d, \ 0 \leq \eta \leq 1 \} \subset D_+,
\]

\[
\overline{J}_-(c, d) = \{ z = \lambda + i\eta : c \leq \lambda \leq d, \ -1 \leq \eta \leq 0 \} \subset D_-.
\]

For \( \lambda > 0 \), let

\[
R_{0\pm}(\lambda) = \lim_{\eta \downarrow 0} R_0(\lambda \pm i\eta),
\]

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and extend the resolvent $R_0(z)$ on $D_\pm$ by

$$R_0(\lambda + i\eta) = \begin{cases} R_0(\lambda + i\eta) & (\lambda > 0, \eta > 0), \\ R_0(\lambda) & (\lambda > 0, \eta = 0) \end{cases}$$

for $z \in D_+$ and

$$R_0(\lambda + i\eta) = \begin{cases} R_0(\lambda + i\eta) & (\lambda > 0, \eta < 0), \\ R_0(\lambda) & (\lambda > 0, \eta = 0) \end{cases}$$

for $z \in D_-$. Then we have

**Theorem 3.6.** Suppose that Assumption 2.1 holds. Let $1/2 < \delta \leq 1$.

(i) Then the limits (3.31) is well-defined in $B(L_{2,\delta}(\mathbb{R}^N), H^2_{\delta-\delta}(\mathbb{R}^N))$, and the extended resolvent $R_0(z)$ is a $B(L_{2,\delta}(\mathbb{R}^N), H^2_{\delta-\delta}(\mathbb{R}^N))$-valued continuous function on each of $D_+$ and $D_-$. 

(ii) For any $z \in D_+$ [ or $D_-$], $R_0(z)$ is a compact operator from $L_{2,\delta}(\mathbb{R}^N)$ into $H^1_{\delta}(\mathbb{R}^N)$.

(iii) The selfadjoint operator $H_0$ is absolutely continuous on the interval $(0,\infty)$. The operator $H_0$ has neither point spectrum nor singular continuous spectrum.

(iv) For $0 < c < d < \infty$ there exists a constant $C = C(c,d,\delta,m_0,M_0) > 0$ such that, for $z \in \mathcal{J}_+(c,d) \cup \mathcal{J}_-(c,d)$,

$$\begin{aligned}
&\left\{ \begin{aligned}
\int_{E_s} (1 + r)^{-2\delta} \left( |\nabla R_0(z)f|^2 + |k|^2 |R_0(z)f|^2 \right) \, dx \\
\leq C^2 (1 + s)^{-2(\delta-1)} \|f\|_{\delta}^2 \\
\|D R_0(z)f\|_{\delta-1} \leq C \|f\|_{\delta}
\end{aligned} \right. \\
&s \geq 1, \quad f \in L_{2,\delta}(\mathbb{R}^N), \\
&f \in L_{2,\delta}(\mathbb{R}^N),
\end{aligned}$$

where, for $\lambda \in D_+ \cap (0,\infty)$ or $D_- \cap (0,\infty)$, $D_0$ should be interpreted as

$$\begin{aligned}
\mathcal{D}^{(\pm)} u &= \nabla u + \{(N - 1)/(2r)\} \bar{x} u - ik(x) \bar{x} u & (\lambda \in D_+ \cap (0,\infty)), \\
\mathcal{D}^{(-)} u &= \nabla u + \{(N - 1)/(2r)\} \bar{x} u + ik(x) \bar{x} u & (\lambda \in D_- \cap (0,\infty)).
\end{aligned}$$

(v) Let $N \geq 3$. Then there exists a constant $C = C(\delta,m_0,M_0) > 0$ such that

$$\begin{aligned}
&\left\{ \begin{aligned}
\int_{E_s} (1 + r)^{-2\delta} |R_0(z)f|^2 \, dx \leq \frac{C^2}{|z|} (1 + s)^{-2(\delta-1)} \|f\|_{\delta}^2 \\
\|R_0(z)|_{(0,\delta)}^0 \leq \frac{C}{\sqrt{|z|}} & (z \in D_+ \cup D_-), \\
\|D R_0(z)f\|_{\delta-1} \leq C \|f\|_{\delta} & (z \in D_+ \cup D_-), \quad f \in L_{2,\delta}.
\end{aligned} \right.
\end{aligned}$$

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Finally we are going to prove a modification of Theorem 3.5, where the range of $\delta$ is slightly wider. This modification will be useful in the next section.

**Proposition 3.7.** Let Assumption 2.1 be satisfied. Let $1/2 < \delta < 3/2$. Let $f \in L_{2,\delta}(\mathbb{R}^N)$ and let $z \in D_+ \cup D_-$. Let $u = R_0(z)f$.

(i) Let $N \geq 3$. Then there exists a constant $C = C(\delta, m_0, M_0) > 0$ such that

$$
\|D_r u\|_{\delta-1} \leq C\|f\|_{\delta} \quad (f \in L_{2,\delta}(\mathbb{R}^N), \ z \in D_+ \cup D_-)
$$

where $D_r u$ is given in Notation 3.1, (6), and for $\lambda \in D_+ \cap (0, \infty)$ [or $D_- \cap (0, \infty)$], $D_r u$ should be interpreted as $D_r^+ \{ \text{or} \ D_r^- \}$.

(ii) Let $N = 2$. Let $0 < c < d < \infty$. Then there exists $C = C(\delta, c, d, m_0, M_0) > 0$ such that

$$
\|D_r u\|_{\delta-1,*} \leq C\|f\|_{\delta} \quad (f \in L_{2,\delta}(\mathbb{R}^N), \ z \in \overline{J_+(c, d)} \cup \overline{J_-(c, d)}),
$$

where $\|D_r u\|_{\delta-1,*}$ is given by (3.21).

**Proof.** In view of the continuity of the extended resolvent of $R_0(z)$, we only have to prove (3.37) and (3.38) for non real $z$. Then we should note that we have $u = R_0(z)f \in H_0^2(\mathbb{R}^N)$ (cf., e.g., [12], Lemma 2.1). As in the proof of Theorem 3.5, we start with (3.4) with $\xi$ given by (3.22) or (3.23) and $\alpha(x) = 1/\sqrt{m_0}$. Let the $j$-th term of the left-hand side be denoted by $I_{Lj}$, where $j = 1, 2, 3, 4$. Then we have $I_{L2} = 0$, and, since

$$
\frac{\varphi}{r} - 2^{-1}\frac{\partial \varphi}{\partial r} \geq 0
$$

and $|Du| \geq |D_r u|$, we have

$$
I_{L1} + I_{L3} \geq \int_{B_{Rr}} \frac{1}{2} \frac{\partial^2 \varphi}{\partial r^2} |Du|^2 \, dx + \int_{B_{Rr}} (\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r})(|Du|^2 - |D_r u|^2) \, dx
$$

$$
= \int_{B_{Rr}} (\frac{\varphi}{r} - \frac{1}{2} \frac{\partial \varphi}{\partial r}) |Du|^2 \, dx - \int_{B_{Rr}} (\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}) |D_r u|^2 \, dx
$$

$$
\geq \int_{B_{Rr}} (\frac{\varphi}{r} - \frac{1}{2} \frac{\partial \varphi}{\partial r}) |D_r u|^2 \, dx - \int_{B_{Rr}} (\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}) |D_r u|^2 \, dx
$$

$$
= \int_{B_{Rr}} \frac{1}{2} \frac{\partial \varphi}{\partial r} |D_r u|^2 \, dx.
$$

As for the fourth term $I_{L4}$, we have $I_{L4} \geq 0$ for $N \geq 3$, and for $N = 2$ we have, as in (7.19) and (7.20) of [9],

$$
-I_{L4} \leq C_1\|u\|^2_{\delta-2} + C_2 \int_{\mathbb{R}^2} |\eta||u|^2 \, dx
$$

$$
\leq C_1\|u\|^2_{\delta-2} + C_3(\|f\|, \|u\|_0)
$$

$$
\leq C_1\|u\|^2_{\delta-2} + \frac{C_3}{2}(\|f\|^2_{\delta} + \|u\|^2_{-\delta})
$$

(3.41)
with $C_1 = C_1(\delta, m_0)$, $C_2 = C_2(c, d)$, and $C_3 = C_3(\delta, c, d, m_0, M_0)$. Here we can evaluate $\|u\|_{\delta - 2}$ as

$$(3.42) \quad \left\{ \begin{array}{ll}
\delta \in (1/2, 1] & \Rightarrow \|u\|_{\delta - 2} \leq \|u\|_{\delta} \leq C'|f|\delta, \\
\delta \in (1, 3/2) & \Rightarrow \|u\|_{\delta - 2} \leq C'''\|f\|_{2 - \delta} \leq C''\|f\|_{\delta},
\end{array} \right.$$

with $C' = C'(\delta, c, d, \mu_0)$ and $C'' = C''(\delta, d, \mu_0)$. We can evaluate the right-hand of (3.4) by proceeding as in the proof of Theorems 4.1 and 7.2 of [9]. Therefore, letting $r \to 0$ and $R \to \infty$, where we have to use the fact that $u = R_0(z)f \in H^2_2(R^N)$, we obtain (3.37) and (3.38), respectively. This completes the proof. //

4. The point spectrum for $H$

Throughout this and the following sections we shall assume that Assumptions 2.1 and 2.2 are satisfied. Let the operator $H$ be as in §2. Let $\sigma_p(H)$ be the set of all eigenvalues of $H$, and let $V_p(H)$ be the set of all eigenvectors of $H$, i.e.,

$$(4.1) \quad V_p(H) = \{u \in H^2(R^N) : u \neq 0, (H - \lambda)u = 0 \text{ with } \lambda \in \sigma_p(H)\}.$$

Now we need to introduce a set $\bar{\sigma}_p(\pm)(H)$ of the extended eigenvalues of $H$ and a set $\bar{V}_p(\pm)(H)$ of the extended eigenvectors of $H$.

**Definition 4.1.** Let $1/2 < \delta < 1/2 + \epsilon$ if $\mu_1$ is short-range, i.e., $\mu_1$ satisfies (2.13), and let $1/2 < \delta < (1 + \epsilon)/2$ if $\mu_1$ is long-range, i.e., $\mu_1$ satisfies (2.14), where $\epsilon$ is given in Assumption 2.2. Denote by $\bar{\sigma}_p(\pm)(H)$ [or $\bar{\sigma}_p^-(H)$] the set of all $\lambda > 0$ such that there exists a function $u$ satisfying

$$(4.2) \quad \begin{array}{ll}
(i) & u \in H^2(R^N)_\text{loc}, \ u \neq 0, \\
(ii) & u \in L_{2, -\delta}(R^N), \\
(iii) & \|D^u\|_{2, -1, E_1} < \infty, \ [\text{or } \|D^u\|_{2, -1, E_1} < \infty], \\
(iv) & -\mu(x)^{-1}\Delta u - \lambda u = 0,
\end{array}$$

where $D^u$ are given by (3.35), and $k = k(x) = [\lambda \mu_0(x)]^{1/2}$ is as in §3. Let $\bar{V}_p(\pm)(H)$ be the set of all $u \in X$ which satisfy (4.2) with $\lambda \in \bar{\sigma}_p(\pm)(H)$. We call $u \in \bar{V}_p(\pm)(H)$ which satisfies (4.2) an extended eigenvector of $H$ associated with the extended eigenvalue $\lambda$.

Since $0 \notin \sigma_p(H)$, we have $V_p(H) \subset \bar{V}_p(\pm)(H)$ and $\sigma_p(H) \subset \bar{\sigma}_p(\pm)(H)$ by definition. In this section we are going to prove that

$$(4.3) \quad \sigma_p(H) = \bar{\sigma}_p^{(+)}(H) = \bar{\sigma}_p^{(-)}(H),$$

and $\sigma_p(H)$ is a discrete set on $(0, \infty)$. 

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Proposition 4.2. Assume that Assumptions 2.1 and 2.2 hold. Let \( u \in H^2(\mathbb{R}^N)_{\text{loc}} \) be a solution of the equation \(- \mu(x)^{-1} \Delta u - \lambda u = 0\). Let \( \varphi(x) = \xi(|x|) \) and let \( \xi(r) \) satisfy the following (a) ~ (c)’:

(a) \( \xi \) is a nonnegative, continuous function on \((0, \infty)\),

(b) \( \xi \) has a piecewise continuous derivative \( \xi' \) such that

\[
\xi'(r) \geq 0,
\]

and

\[
\frac{\xi(r)}{r} - \frac{1}{2} \xi'(r) \geq 0
\]

for almost all \( r > 0 \).

(c) If \( N \geq 3 \),

\[
\xi(r) = O(r) \quad (r \to 0).
\]

(c)’ If \( N = 2 \),

\[
\begin{cases}
\xi(r) = O(r^2) & (r \to 0), \\
\xi'(r) = O(r) & (r \to 0).
\end{cases}
\]

(i) Suppose that \( \mu_1 \) is short-range in the sense of (2.13). Then there exists a constant \( C = C(\lambda, \mu_0) > 0 \) such that, for \( R > 1 \),

\[
\int_{B_R} \frac{\partial \varphi}{\partial r} |u|^2 \, dx \leq 2m_0^{-1} \int_{B_R} \varphi |\mu_1||u||D^{(\pm)}_r u| \, dx
+ C\xi(R) \int_{S_R} |D^{(\pm)}_r u|^2 \, dS
\]

for \( N \geq 3 \), or

\[
\int_{B_R} \frac{\partial \varphi}{\partial r} |u|^2 \, dx \leq 2m_0^{-1} \int_{B_R} \varphi |\mu_1||u||D^{(\pm)}_r u| \, dx
+ (2m_0\lambda)^{-1} \int_{B_R} r^{-2} (\varphi - 2^{1-2} \frac{\partial \varphi}{\partial r}) |u|^2 \, dx
+ C\xi(R) \int_{S_R} |D^{(\pm)}_r u|^2 \, dS
\]

for \( N = 2 \).

(ii) Suppose that \( \mu_1 \) is long-range in the sense of (2.14). Then the relation (4.8) or (4.9) holds again with the first term \( K_{R1} \) of the right-hand side of (4.8) or (4.9) replaced by

\[
K'_{R1} = m_0^{-1} \int_{B_R} \left( |\frac{\partial \varphi}{\partial r} \mu_1| + |\frac{\partial \mu_1}{\partial r} \varphi| \right) |u|^2 \, dx.
\]
Proof. (I) We shall prove (4.8) and (4.9) for $D^{(+)}u$. These formulas for $D^{(-)}u$ can be proved in quite a similar way. For the sake of simplicity of notation we set $D^{(+)}u = D_r u$ and $D^{(-)}u = D u$. Since we have from (iv) of (4.2) $-\Delta u - \lambda \mu_0(x) u = \lambda \mu_1(x) u$, we can apply the formula (3.4) with $f$ and $z$ replaced by $\lambda \mu_0(x) - 1 \mu_1(x) u$ and $\lambda$, respectively, to obtain, for $0 < r < 1 < R < \infty$,

$$
\int_{B_r R} \frac{1}{2} \frac{\partial \varphi}{\partial r} |Du|^2 \, dx + \sum_{\ell \in L} \int_{\partial \Omega \cap B_{r R}} \varphi \text{Im} \{k \frac{\partial u}{\partial n}\} \, dS \\
+ \int_{B_{r R}} \left( \frac{\varphi}{r} - \frac{\partial \varphi}{\partial r} \right) \left( |Du|^2 - |D_r u|^2 \right) \, dx \\
+ c N \int_{B_{r R}} r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx
$$

(4.11)

$$
= \text{Re} \int_{B_{r R}} \lambda \varphi \mu_1(x) u \overline{D_r u} \, dx \\
+ 2^{-1} \sum_{\ell \in L} \int_{\partial \Omega \cap B_{r R}} \varphi k^2 (\overline{\xi} \cdot n) |u|^2 \, dS \\
+ 2^{-1} \int_{S_R} \varphi (2|D_r u|^2 - |Du|^2 - c N r^{-2} |u|^2) \, dS \\
- 2^{-1} \int_{S_r} \varphi (2|D_r u|^2 - |Du|^2 - c N r^{-2} |u|^2) \, dS.
$$

(II) Suppose that $\mu_1$ is short-range. Proceeding as in the proof of [9], Theorems 3.2 or 7.1, we have

$$
\int_{B_{r R}} \frac{1}{2} \frac{\partial \varphi}{\partial r} |Du|^2 \, dx + \sum_{\ell \in L} \int_{\partial \Omega \cap B_{r R}} \varphi \text{Im} \{k \frac{\partial u}{\partial n}\} \, dS \\
\geq \int_{B_{r R}} \frac{1}{2} \frac{\partial \varphi}{\partial r} k^2 |u|^2 \, dx + \int_{B_{r R}} \frac{1}{2} \frac{\partial \varphi}{\partial r} \left( |\nabla u|^2 - |\frac{\partial u}{\partial r}|^2 \right) \, dx \\
- \xi(R) \int_{S_R} \text{Im} \{k \frac{\partial u}{\partial n}\} \, dS,
$$

(4.12)

where we should note that all the integrals in (4.12) is well-defined even in the case of $N = 2$ because of (4.7). Therefore it follows from (4.11) and (4.12) that

$$
\int_{B_{r R}} \frac{1}{2} k^2 \frac{\partial \varphi}{\partial r} |u|^2 \, dx \\
+ \int_{B_{r R}} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) \left( |\nabla u|^2 - |\frac{\partial u}{\partial r}|^2 \right) \, dx \\
+ c N \int_{B_{r R}} r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx
$$
Here we noticed from (c) and (c)' that

\[
\begin{aligned}
\text{(4.14)}
\int_{B_{2\epsilon}} \frac{1}{2} \mu_0(x) \frac{\partial \varphi}{\partial r} |u|^2 \, dx + c_N \int_{B_{2\epsilon}} r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx \\
\leq \lambda \int_{B_{2\epsilon}} \varphi |\mu_1(x)||u| |D_r u| \, dx \\
+ 2^{-1} \xi(R) \int_{S_{2\epsilon}} (2|D_r u|^2 - |D u|^2 - c_N r^{-2}|u|^2) \, dS \\
+ \xi(R) \int_{S_{2\epsilon}} \text{Im}(k \frac{\partial u}{\partial n}) \, dS.
\end{aligned}
\]

It follows from (2.11) that the second term of the right-hand side of (4.13) is nonpositive. Therefore we can drop it from the right-hand side of (4.13). Further, we see from (4.5) in (b) that the second term of the left-hand side of (4.13) is nonnegative, and it can be dropped, too. Then, letting \( r \to 0 \) along an appropriate sequence \( \{r_n\} \), we obtain

\[
\begin{aligned}
\int_{B_{2\epsilon}} \frac{1}{2} \mu_0(x) \frac{\partial \varphi}{\partial r} |u|^2 \, dx + c_N \int_{B_{2\epsilon}} r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) |u|^2 \, dx \\
\leq \lambda \int_{B_{2\epsilon}} \varphi |\mu_1(x)||u| |D_r u| \, dx \\
+ 2^{-1} \xi(R) \int_{S_{2\epsilon}} (2|D_r u|^2 - |D u|^2 - c_N r^{-2}|u|^2) \, dS \\
+ \xi(R) \int_{S_{2\epsilon}} \text{Im}(k \frac{\partial u}{\partial n}) \, dS.
\end{aligned}
\]

Here we noticed from (c) and (c)' that

\[
\begin{aligned}
\lim_{r \to 0} \xi(r) \int_{S_r} (2|D_r u|^2 - |D u|^2 - c_N r^{-2}|u|^2) \, dS = 0, \\
2^{-1} \xi(R) \int_{S_R} (2|D_r u|^2 - |D u|^2 - c_N r^{-2}|u|^2) \, dS \\
+ \xi(R) \int_{S_R} \text{Im}(k \frac{\partial u}{\partial n}) \, dS \\
\leq C \xi(R) \int_{S_R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) \, dS
\end{aligned}
\]

with a constant \( C \). Since \( c_N \geq 0 \) for \( N \geq 3 \) and \( c_N = -1/4 \) for \( N = 2 \), (4.8) and (4.9) follows from (4.14) if we can prove

\[
\int_{S_R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 \right) \, dS \leq C' \int_{S_R} |D_r u|^2 \, dS
\]
with a constant \( C' \).

(III) (Proof of (4.16).) Multiply both sides of the equation \(-\Delta u - \mu(x)\lambda u = 0\) by \( \pi \), integrating over \( B_R \), use partial integration and take the imaginary part to obtain

\[
(4.17) \quad \int_{S_R} \text{Im}\left\{ \frac{\partial u}{\partial r} \pi \right\} dS = 0.
\]

Now we can proceed as in the proof of [9], Theorem 3.7 to obtain (4.16).

(IV) Suppose that \( \mu_1 \) is long-range. We have only to evaluate the first term \( K''_{R1} \) of the right-hand side of (4.11). In fact, we have by partial integration

\[
(4.18) \quad K''_{R1} = \lambda \text{Re} \int_{B_{rR}} \varphi_{\mu_1}(x) u \overline{\sigma_r u} \, dx
\]

\[
= \frac{\lambda}{2} \int_{B_{rR}} \varphi_{\mu_1}(x) \left( \frac{\partial |u|^2}{\partial r} + (N - 1)r^{-1}|u|^2 \right) \, dx
\]

\[
- \frac{\lambda}{2} \int_{S_r} \varphi_{\mu_1} |u|^2 \, dS
\]

Since the third term of the right-hand side of (4.18) converges to 0 as \( r \to 0 \), we see that \( K''_{R1} \) in (4.10) can replace the first term \( K_{R1} \) of the right-hand side of (4.8) or (4.9), which completes the proof. //

**Proposition 4.3.** Assume that Assumptions 2.1 and 2.2 hold. Suppose that \( \mu_1 \) is short-range. Suppose that \( u \in \tilde{V}^+_p(H) \) [or \( u \in \tilde{V}^+_p(H) \)] with an extended eigenvalue \( \lambda \in \tilde{\sigma}_p^+(H) \) [or \( \lambda \in \tilde{\sigma}_p^-(H) \)] such that

\[
|u| \leq \lambda R_0(\lambda)(\mu_0^{-1}\mu_1 u),
\]

where \( C_j^{(N)} = C_j^{(N)}(\delta, \epsilon, m_0, M_0) \) for \( N \geq 3 \) and \( C_j^{(2)} = C_j^{(2)}(\lambda, \delta, \epsilon, m_0, M_0) \).

Proof. (1) We shall prove (4.20) for \( D_r^{(+)} u \), and set \( D_r^{(+)} u = D_r u \). Then \( u \) satisfies the equation \((-\Delta - \lambda\mu_0)u = \lambda\mu_1 u \) and the radiation condition \(||D_r^{(+)} u||_{\delta-1, E_1} < \infty \), i.e.,

\[
(4.22) \quad u = \lambda R_0(\lambda)(\mu_0^{-1}\mu_1 u),
\]
where we set \( R_0(\lambda) = R_0(\lambda) \). Since we have

\[
|\mu_0^{-1} \mu_1 u| \leq c_1 m_0^{-1}(1 + |x|)^{-1-\epsilon-\delta-j\epsilon}[(1 + |x|)^{-\delta+j\epsilon}|u|],
\]

it follows that

\[
\mu_0^{-1} \mu_1 u \in L_{2,1-\delta+(j+1)\epsilon}(\mathbb{R}^N).
\]

Noting that (4.19), \( 1/2 < \delta < 1/2 + \epsilon \) and \( 0 < \epsilon < 1/2 \), we see that

\[
\delta < 1/2 + \epsilon \quad \Rightarrow \quad 1 - \delta + (j+1)\epsilon \geq 1 - \delta + \epsilon > 1/2,
\]

\[
-\delta + j\epsilon \leq 0 \quad \text{and} \quad 0 < \epsilon < 1/2
\]

\[
\Rightarrow \quad 1 - \delta + (j+1)\epsilon \leq 1 + \epsilon < 3/2,
\]

and hence we can apply Proposition 3.7 with \( \delta \) replaced by \( 1 - \delta + (j+1)\epsilon \) to obtain

\[
\|D_r u\|_{-\delta+(j+1)\epsilon} \leq \lambda m_0^{-1} c_1 C_j^\epsilon\|u\|_{-\delta+j\epsilon}
\]

with \( C_j^\epsilon = C_j^\epsilon(\delta, \epsilon, m_0, M_0) \) for \( N \geq 3 \), and

\[
\|D_r u\|_{-\delta+(j+1)\epsilon, \ast} \leq \lambda m_0^{-1} c_1 C_j^\epsilon\|u\|_{-\delta+j\epsilon}
\]

with \( C_j^\epsilon = C_j^\epsilon(\delta, \epsilon, \lambda, m_0, M_0) \) for \( N = 2 \).

(II) Let \( R_0 > 1 \). Set \( \beta = 2(-\delta + (j+1)\epsilon) + 1 \),

\[
\xi(r) = \begin{cases} r & (0 < r \leq 1), \\ 2^{-\beta}(1+r)^\beta & (1 < r \leq R_0), \\ 2^{-\beta}(1+R_0)^\beta & (r > R_0) \end{cases}
\]

if \( N \geq 3 \), and

\[
\xi(r) = \begin{cases} 2^{-1}r^2 & (0 < r \leq 1), \\ 2^{-\beta-1}(1+r)^\beta & (1 < r \leq R_0), \\ 2^{-\beta-1}(1+R_0)^\beta & (r > R_0) \end{cases}
\]

for \( N = 2 \). It follows from (4.25) that \( 0 < \beta < 2 \), and hence \( \xi \) satisfies (a), (b), (c) or (c)' in Proposition 4.2. Therefore we can apply (i) of Proposition 4.2 to obtain

\[
\int_{B_1} |u|^2 \, dx + \int_{B_1 R_0} \beta 2^{-\beta}(1+r)^{\beta-1}|u|^2 \, dx \\
\leq 2m_0^{-1} c_1 \int_{B_1} r(1+r)^{-1-\epsilon}|u||D_r u| \, dx \\
+ 2^{-\beta+1} m_0^{-1} c_1 \int_{B_1 R} (1+r)^{\beta-1-\epsilon}|u||D_r u| \, dx \\
+ C 2^{-\beta}(1+R_0)^\beta \int_{S_R} |D_r u|^2 \, dS
\]
for $R > R_0$ and $N \geq 3$, or

\[
\int_{B_1} r|u|^2 \, dx + \int_{B_{1R}} \beta^2 - \beta - 1 (1 + r)^{\beta - 1} |u|^2 \, dx \\
\leq m_0^{-1} c_1 \int_{B_1} r^2 (1 + r)^{-1-\epsilon} |u| |D_r u| \, dx \\
+ 2^{-\beta} m_0^{-1} c_1 \int_{B_{1R}} (1 + r)^{\beta - 1 - \epsilon} |u| |D_r u| \, dx \\
+ 2^{-\beta + 1} m_0^{-1} \lambda^{-1} \int_{B_{1R}} (1 + r)^{2(-\delta + j\epsilon)} |u|^2 \, dx \\
+ C 2^{-\beta - 1} (1 + R_0)^{\beta} \int_{S_R} |D_r u|^2 \, dS
\]  

(4.31)

for $R > R_0$ and $N = 2$, where we have used the facts that

\[
\begin{cases}
(1 + R_0)^{\beta} \leq (1 + R)^{\beta} & (R \geq R_0), \\
\frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} = 0 & (0 < r < 1),
\end{cases}
\]

and, for $r > 1$,

\[
r^{-2} \left( \frac{\varphi}{r} - 2^{-1} \frac{\partial \varphi}{\partial r} \right) \\
\leq 2^{-\beta - 2} 2^\beta \left( 1 - \frac{\beta}{2} \right) (1 + r)^{\beta - 3} \\
\leq 2^{-\beta + 1} (1 + r)^{2(-\delta + j\epsilon)}.
\]

Since

\[
\beta - 1 - \epsilon = (-\delta + j\epsilon) + (-\delta + (j + 1)\epsilon),
\]

we have

\[
\int_{B_{1R}} (1 + r)^{\beta - 1 - \epsilon} |u| |D_r u| \, dx \leq \|u\|_{-\delta + j\epsilon} \|D_r u\|_{-\delta + (j+1)\epsilon}.
\]

(4.35)

Therefore we have for $N \geq 3$

\[
\|u\|_{-\delta + (j+1)\epsilon, B_{R_0}}^2 \\
\leq C_0 c_1 \|u\|_{-\delta + j\epsilon} \|D_r u\|_{-\delta + (j+1)\epsilon} + \bar{C}(1 + R_0)^{\beta} \int_{S_R} |D_r u|^2 \, dS,
\]

(4.36)

and for $N = 2$

\[
\|u\|_{-\delta + (j+1)\epsilon, B_{R_0}}^2 \\
\leq C_0' c_1 \|u\|_{-\delta + j\epsilon} \|D_r u\|_{-\delta + (j+1)\epsilon, \ast} + \lambda^{-1} \|u\|_{-\delta + j\epsilon}^2 \\
+ \bar{C}'(1 + R_0)^{-\delta + j\epsilon} \int_{S_R} |D_r u|^2 \, dS,
\]

(4.37)
where $C_0$, $C_0'$, $\tilde{C}$, and $\tilde{C}'$ depend on $j, \delta, \epsilon, m_0$, and $M_0$. Note that we have from (iii) of (4.2)

\begin{equation}
\liminf_{R \to \infty} \int_{S_R} |D_r u|^2 \, dS = 0
\end{equation}

Let $R \to \infty$ along an appropriate sequence so that the last terms of the right-hand sides of (4.36) and (4.37) tends to 0. Therefore, noting that $R_0 > 1$ is arbitrary, and using (4.26) and (4.27), too, we obtain (4.21), which completes the proof. //

**Proposition 4.4.** Assume that Assumptions 2.1 and 2.2 hold. Suppose that $\mu_1$ is long-range. Suppose that $u \in \tilde{V}_p^{(+)}(H)$ [ or $u \in \tilde{V}_p^{(-)}(H)$] with an extended eigenvalue $\lambda \in \tilde{\sigma}_p^{(+)}(H)$ [ or $\lambda \in \tilde{\sigma}_p^{(-)}(H)$]. Let $\epsilon' = \epsilon/2$. Suppose that

\begin{equation}
\begin{cases}
u \in L_{2,-\delta+j\epsilon'}(\mathbb{R}^N), \\
-\delta+j\epsilon' \leq 0
\end{cases}
\end{equation}

with a nonnegative integer $j$. Then we have

\begin{equation}
u \in L_{2,-\delta+(j+1)\epsilon'}(\mathbb{R}^N),
\end{equation}

and

\begin{equation}
\|u\|_{-\delta+(j+1)\epsilon'} \leq \sqrt{c_1} C_j^{(N)} \|u\|_{-\delta+j\epsilon'},
\end{equation}

where $C_j^{(N)} = C_j^{(N)}(\delta, \epsilon, m_0, M_0)$ for $N \geq 3$ and $C_j^{(2)} = C_j^{(2)}(\lambda, \delta, \epsilon, m_0, M_0)$.

**Proof.** We shall prove (4.40) and (4.41) in the case that $\lambda \in \tilde{\sigma}_p^{(+)}(H)$. We set $D_r^{(+)} u = D_r u$. Let $R_0 > 1$. Set $\beta = 2(-\delta + (j + 1)\epsilon') + 1$, and let $\xi(r)$ be given by (4.28) (with $\epsilon$ replaced by $\epsilon'$). Here we should note that $0 < \beta < 3/2$, and hence our $\xi(r)$ satisfies the conditions (a), (b) and (c) of Proposition 4.2. Then we have from (ii) of

\begin{equation}
\begin{align*}
\int_{B_1} |u|^2 \, dx + \int_{B_1 R_0} \beta 2^{-\beta} (1 + r)^{\beta - 1} |u|^2 \, dx \\
\leq m_0^{-1} c_1 \int_{B_1} \{(1 + r)^{-\epsilon} + r (1 + r)^{-1-\epsilon}\} |u|^2 \, dx \\
+ m_0^{-1} c_1 (\beta + 1) 2^{-\beta} \int_{B_1 R_0} (1 + r)^{\beta - 1 - \epsilon} |u|^2 \, dx \\
+ m_0^{-1} c_2 2^{-\beta} \int_{B_1 R_0} (1 + r)^{\beta} (1 + r)^{-1-\epsilon} |u|^2 \, dx \\
+ C 2^{-\beta} (1 + R_0)^{\beta} \int_{S_R} |D_r u|^2 \, dS \\
\leq \tilde{C} \int_{B_R} (1 + r)^{2(-\delta+j\epsilon')} |u|^2 \, dx \\
+ C 2^{-\beta} (1 + R_0)^{\beta} \int_{S_R} |D_r u|^2 \, dS,
\end{align*}
\end{equation}
where \( R_0 < R \), \( \bar{C} = \bar{C}(j, \delta, \epsilon, \mu_0) \), and we should note that \( \beta - 1 - \epsilon = 2(\delta + j\epsilon') \).

The inequality (4.41) follows from (4.42). The case that \( N = 2 \) can be treated in quite a similar way, which completes the proof.  

Now we are in a position to show that \( V_p(H) = \tilde{V}_p^{(\pm)}(H) \). Let

\[
\begin{cases}
  j_0 = \min\{j \in \mathbb{N} : -\delta + j\epsilon > 0\}, \\
  \delta_0 = -\delta + j_0\epsilon
\end{cases}
\]

if \( \mu_1 \) is short-range, and let

\[
\begin{cases}
  j_0 = \min\{j \in \mathbb{N} : -\delta + j\epsilon' > 0\}, \\
  \delta_0 = -\delta + j_0\epsilon'
\end{cases}
\]

if \( \mu_1 \) is long-range.

**Theorem 4.5.** Let Assumptions 2.1 and 2.2 be satisfied.

(i) Then we have

\[
\tilde{V}_p^{(\pm)}(H) \subset H^2_{\delta_0}(\mathbb{R}^N),
\]

where \( \delta_0 \) is given by (4.43) or (4.44), and hence

\[
\begin{cases}
  V_p(H) = \tilde{V}_p^{(+)}(H) = \tilde{V}_p^{(-)}(H), \\
  \sigma_p(H) = \tilde{\sigma}_p^{(+)}(H) = \tilde{\sigma}_p^{(-)}(H).
\end{cases}
\]

(ii) Let \( \mu_1 \) be short-range. Let \( u \in V_p(H) \) associated with \( \lambda \in \sigma_p(H) \). Then, for each \( N \geq 2 \), there exists a positive constant \( C^{(N)} \) such that

\[
\begin{cases}
  \|u\|_{\delta_0} \leq C^{(N)}(c_1 \lambda^{j_0}) \|u\|_{-\delta} \quad (N \geq 3), \\
  \|u\|_{\delta_0} \leq C^{(2)}(c_1^{j_0}) \|u\|_{-\delta} \quad (N = 2),
\end{cases}
\]

where \( j_0 \) is given by (4.43), and \( C^{(N)} = C^{(N)}(\delta, \epsilon, \mu_0, \mu_0) \) for \( N \geq 3 \) and \( C^{(2)} = C^{(2)}(\lambda, \delta, \epsilon, \mu_0, \mu_0) \). Further, for \( N \geq 3 \), we have

\[
\sigma_p(H) \subset [c_1^{-2}(C^{(N)})^{-2/j_0}, \infty).
\]

(iii) Let \( \mu_1 \) be long-range. Then, for each \( N \geq 2 \), there exists a positive constant \( C^{(N)} = C^{(N)}(\delta, \epsilon, \mu_0, \mu_0) \) (\( N \geq 3 \), \( = C^{(2)}(\lambda, \delta, \epsilon, \mu_0, \mu_0) \) (\( N = 2 \)) such that

\[
\|u\|_{\delta_0} \leq C^{(N)}(c_1^{j_0/2}) \|u\|_{-\delta},
\]

where \( u \in V_p(H) \), and \( j_0 \) and \( \delta_0 \) are given by (4.44).
Proof. Using Propositions 4.3 and 4.4 repeatedly, we obtain

\[ V_p^{(\pm)}(H) \subset L_{2,\delta_0}(\mathbb{R}^N), \]

and the inequalities (4.47) and (4.49), where \( C^{(N)} = C_0^{(N)} C_1^{(N)} \cdots C_j^{(N)} \) and \( j_0 \) and \( \delta_0 \) are

in (4.43) or (4.44). Let \( u \in V_p^{(\pm)}(H) \) associated with \( \lambda \in \overline{\sigma}_p^{(\pm)}(H) \). Then it follows from

the equation \(-\Delta u - \lambda \mu u = 0\) that \( u, \Delta u \in L_{2,\delta_0}(\mathbb{R}^N)\) which implies that \( u \in H^2_{\delta_0}(\mathbb{R}^N)\).

Thus we have proved (4.45). Let \( N \geq 3 \) and let \( \mu_1 \) is short-range. Since we have from

the first inequality of (4.47)

\[ \|u\|_{-\delta} \leq \|u\|_{\delta_0} \leq C^{(N)}(c_1 \sqrt{\lambda})^j_0 \|u\|_{-\delta}, \]

or

\[ (1 - C^{(N)}(c_1 \sqrt{\lambda})^j_0)\|u\|_{-\delta} \leq 0, \]

whence (4.48) follows. This completes the proof. //

**Theorem 4.6.** Let Assumptions 2.1 and 2.2 be satisfied. Let \( \sigma_p(H) \) be as above.

(i) Then the multiplicity of each \( \lambda \in \sigma_p(H) \) is finite.

(ii) \( \sigma_p(H) \) does not have any accumulation point except \( \lambda = 0 \) and \( \lambda = \infty \). If \( N \geq 3 \), then the only possible accumulation point of \( \sigma_p(H) \) is \( \lambda = \infty \).

Proof. Suppose that \( \sigma_p(H) \) has an accumulation point \( \lambda_0 \in (0, \infty) \). Then there exist infinite sequences \( \{\lambda_n\} \subset \sigma_p(H) \) and \( \{u_n\} \subset V_p(H) \) such that

\[
\begin{align*}
\lambda_n \to \lambda_0 \quad (n \to \infty),
\end{align*}
\]

\[
\begin{align*}
(u_m, u_n)_X = \delta_{mn} \quad (m, n \in \mathbb{N}),
\end{align*}
\]

\[
\begin{align*}
-\mu(x)^{-1} \Delta u_n - \lambda_n u_n = 0 \quad (n \in \mathbb{N}),
\end{align*}
\]

where \( \delta_{mn} \) is Kronecker’s delta. Since

\[ \|\nabla u_n\|_0^2 = \lambda_n (u_n, u_n)_X = \lambda_n \|u_n\|_X^2 \]

\[ = \lambda_n \leq \sup_n \lambda_n < \infty, \]

we can apply the Rellich selection theorem to choose a subsequence \( \{u_{n_m}\} \) which converges in \( L_2(\mathbb{R}^N)_{loc} \) as \( m \to \infty \). Let \( u_0 \in L_2(\mathbb{R}^N)_{loc} \) be the limit function. On the other hand, in view of Theorem 4.5, there exists a positive constant \( C \) such that, for any \( s > 0 \),

\[ \|u_{n_m}\|_{0,E_\epsilon} \leq (1 + s)^{-\delta_0} \|u_{n_m}\|_{\delta_0,E_\epsilon} \leq (1 + s)^{-\delta_0} \|u_{n_m}\|_{\delta_0} \]

\[ \leq C(1 + s)^{-\delta_0} \|u_{n_m}\|_{-\delta} \leq C(1 + s)^{-\delta_0} \|u_{n_m}\|_0, \]

and hence

\[ \|u_{n_m}\|_{0,E_\epsilon} \leq \frac{C}{\sqrt{m_0}} (1 + s)^{-\delta_0} \|u_{n_m}\|_X \leq \frac{C}{\sqrt{m_0}} (1 + s)^{-\delta_0}, \]

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where $\delta_0$ is given by (4.43) or (4.44). Therefore $u_{n,m}$ is small at infinity uniformly for $m \in \mathbb{N}$. Thus it follows that $u_{n,m}$ converges to $u_0$ in $X$ and $\|u_0\|_X = 1$. Noting that $\{u_{n,m}\}$ is an orthonormal system in $X$, we have

$$0 = \lim_{n \to \infty} (u_{n,m}, u_{n,m+1})_X = \|u_0\|_X^2 = 1,$$

which is a contradiction. Therefore $\sigma_p(H)$ is discrete in $(0, \infty)$. If $N \geq 3$, (ii) of Theorem 4.5 implies that $\lambda = 0$ cannot be an accumulation point of $\sigma_p(H)$. This completes the proof. //

Consider the following additional condition on $\mu_1(x)$:

**Assumption 4.7.** (i) The function $\mu_1$ is measurable such that

$$\mu(x) \geq N\mu_1(x) + \lambda_0(|x|\mu_1(x))^2 \quad \text{(a.e. } x \in \mathbb{R}^N).$$

with $\lambda_0 > 0$.

(ii) The function $\mu_1$ is differentiable and $\mu_1$ satisfies

$$\mu(x) + |x|\frac{\partial \mu_1}{\partial |x|} \geq 0 \quad \text{(a.e. } x \in \mathbb{R}^N).$$

The following theorem gives sufficient conditions that the absence of $\sigma_p(H) = 0$ on some interval or whole positive half line (cf. Roach-Zhang [10], the proof of Theorem 3.1).

**Theorem 4.8.** Suppose that $\mu(x) = \mu_0(x) + \mu_1(x)$, $\mu_0$ satisfies Assumptions 2.1, and $\mu$ satisfies (2.12) of Assumptions 2.2. Suppose that (i) or (ii) of Assumption 4.7 hold. Then we have

$$\sigma_p(H) \cap [0, \lambda_0] = \emptyset \quad \text{(if (i) holds),}$$

$$\sigma_p(H) = \emptyset \quad \text{(if (ii) holds).}$$

**Proof.** (I) Let $u \in H^2(\mathbb{R}^N)$ satisfy the homogeneous equation $-\Delta u - \lambda \mu(x)u = 0$ with $\lambda > 0$. We have only to show that $u \equiv 0$. We are going to multiply both sides of the equation by $2r(\partial_r \overline{\mu}) + (N - 1)|\overline{\mu}|$, integrate over $B_R$, $R > 0$, and take the real part.

(II) Using the identity

$$2\text{Re} \left( (\Delta u)r(\partial_r \overline{\mu}) \right) = \text{div} \left\{ 2\text{Re} \{r(\partial_r \overline{\mu})\nabla u\} - |\nabla u|^2x + (N - 2)|\nabla u|^2 \right\}$$

(Roach-Zhang [10], (3.4) with $h(r) \equiv 1$) and the divergence theorem, we have

$$2\text{Re} \int_{B_R} (-\Delta u)r(\partial_r \overline{\mu}) \, dx$$

$$= - \int_{B_R} (N - 2)|\nabla u|^2 \, dx - R \int_{S_R} (2|\partial_r \overline{\mu}|^2 - |\nabla u|^2) \, dS,$$

(4.62)
where \( \partial_v = \partial v / \partial r, \ r = |x| \). Since it is easy to see that

\[
\text{Re} \int_{B_R} (-\Delta u)(N - 1)\overline{\mu} \, dx
\]

(4.63)

\[
= \int_{B_R} (N - 1)|\nabla u|^2 \, dx - (N - 1) \int_{S_R} \text{Re} \, [\partial_r u \overline{\mu}] \, dS,
\]

it follows that

\[
\text{Re} \int_{B_R} (-\Delta u) \{2r(\partial_r \overline{\mu}) + (N - 1)\overline{\mu}\} \, dx
\]

(4.64)

\[
= \int_{B_R} |\nabla u|^2 \, dx - R \int_{S_R} (2|\partial_r \overline{\mu}|^2 - |\nabla u|^2 + \frac{N - 1}{R} \text{Re} \, [\partial_r u \overline{\mu}] \, dS.
\]

(III) Suppose that (ii) of Assumption 4.7 holds. By the use of the integration by parts, we have

\[
2\text{Re} \int_{B_R} (-\lambda \mu u)r(\partial_r \overline{\mu}) \, dx
\]

(4.65)

\[
= -\int_{B_R} (\lambda \mu) r(\partial_r |u|^2) \, dx
= \lambda \int_{B_R} (N \mu + r(\partial_r \mu_1)) |u|^2 \, dx
- \lambda \sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_R} \mu_0 (x \cdot n^{(\ell)}) |u|^2 \, dS - \lambda R \int_{S_R} \mu |u|^2 \, dS,
\]

where we should note that \( \mu_0 \) does not appear in the first term of the right-hand side since it is constant on each \( \Omega_\ell \), and \( \mu_1 \) does not appear in the second term of the right-hand side since it is continuous on \( \mathbb{R}^N \). Also we should note that

\[
- \lambda \sum_{\ell \in L} \int_{\partial \Omega_\ell \cap B_R} \mu_0 (x \cdot n^{(\ell)}) |u|^2 \, dS
\]

(4.66)

\[
= \lambda \sum_{\ell \in L} \int_{S^{(\ell)}_t \cap B_R} (\nu_{t+1} - \nu_t) (x \cdot n^{(\ell)}) |u|^2 \, dS \geq 0
\]

since the integrand is nonnegative by (2.11). Thus,

\[
\text{Re} \int_{B_R} (-\lambda \mu u)[2r(\partial_r \overline{\mu}) + (N - 1)\overline{\mu}] \, dx
\]

(4.67)

\[
= \lambda \int_{B_R} (\mu + r(\partial_r \mu_1)) |u|^2 \, dx
+ \lambda \sum_{\ell \in L} \int_{S^{(\ell)}_t \cap B_R} (\nu_{t+1} - \nu_t) (x \cdot n^{(\ell)}) |u|^2 \, dS - \lambda R \int_{S_R} \mu |u|^2 \, dS
\]

(IV) It follows from (4.64) and (4.67) that

\[
0 = \text{Re} \int_{B_R} (-\Delta u - \lambda \mu u)[2r(\partial_r \overline{\mu}) + (N - 1)\overline{\mu}] \, dx
\]

(4.68)

\[
= \int_{B_R} (|\nabla u|^2 + \lambda (\mu + r(\partial_r \mu_1)) |u|^2) \, dx
+ \lambda \sum_{\ell \in L} \int_{S^{(\ell)}_t \cap B_R} (\nu_{t+1} - \nu_t) (x \cdot n^{(\ell)}) |u|^2 \, dS
+ R \int_{S_R} (|\nabla u|^2 - 2|\partial_r \overline{\mu}|^2 - \frac{N - 1}{R} \text{Re} \, [(\partial_r u \overline{\mu}) - \lambda \mu |u|^2] \, dS.
\]

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Since \(2|\partial_r \varpi|^2 + |\nabla u|^2 + \lambda |u|^2 + (N - 1)|\partial_r u||u| \) is integrable on \(\mathbb{R}^N\), we see that the third term of the right-hand side goes to 0 as \(R \to \infty\) along an appropriate sequence. Therefore it follows from (4.68) that

\[
0 = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda (\mu + r(\partial_r \mu_1))|u|^2) \, dx \\
+ \lambda \sum_{\ell \in L} \int_{\Sigma_{\ell}^+} (\nu_{\ell+1} - \nu_{\ell})(x \cdot n(\ell))|u|^2 \, dS.
\]

Noting that all the integrands in the right-hand side are nonnegative, we have \(\nabla u = 0\) a.e., and hence \(u \equiv 0\) since \(u \in H^2(\mathbb{R}^N)\).

(V) Suppose that (i) of Assumption 4.7 holds. By using partial integration only for the term containing \(\mu_0\), we obtain

\[
\text{Re} \int_{B_R} (-\lambda \mu u)[2r(\partial_r \varpi) + (N - 1)\varpi] \, dx \\
= -\lambda \int_{B_R} [\mu_0 r(\partial_r |u|^2) + (N - 1)\mu_0 |u|^2] \, dx \\
- \lambda \int_{B_R} \mu_1 [2r \text{Re}(u(\partial_r \varpi)) + (N - 1)|u|^2] \, dx \\
= \lambda \int_{B_R} \mu_0 |u|^2 \, dx \\
- \lambda \sum_{\ell \in L} \int_{\partial \Omega \cap B_R} \mu_0 (x \cdot n(\ell))|u|^2 \, dS - \lambda R \int_{\Sigma_R} \mu_0 |u|^2 \, dS,
\]

Let \(h(x)\) be a positive function to be specified later. Since we have

\[
2|r\mu_1 u(\partial_r \varpi)| \leq |r\mu_1|(h|u|^2 + \frac{|\nabla u|^2}{h}),
\]

it follows from (4.70) that

\[
\text{Re} \int_{B_R} (-\lambda \mu u)[2r(\partial_r \varpi) + (N - 1)\varpi] \, dx \\
\geq -\lambda \int_{B_R} \frac{|r\mu_1|}{h} |\nabla u|^2 \, dx \\
+ \lambda \int_{B_R} [\mu_0 - (N - 1)\mu_1 - h|r\mu_1||u|^2 \, dx \\
- \lambda \sum_{\ell \in L} \int_{\partial \Omega \cap B_R} \mu_0 (x \cdot n(\ell))|u|^2 \, dS - \lambda R \int_{S_R} \mu_0 |u|^2 \, dS.
\]

Thus (4.64) and (4.72) are combined to give
\[
0 = \operatorname{Re} \int_{B_R} (-\Delta u - \lambda \mu u)[2r(\partial_r \overline{u}) + (N - 1)\overline{u}] \, dx \\
\geq \int_{B_R} (1 - \frac{\lambda |r\mu_1|}{h})|\nabla u|^2 \, dx \\
+ \lambda \int_{B_R} [\mu - N \mu_1 - h|r\mu_1|]|u|^2 \, dx \\
- \lambda \sum_{\ell \in L} \int_{\partial \Omega \cap B_R} \mu_0(x \cdot n^{(\ell)})|u|^2 \, dS - \lambda R \int_{S_R} \mu_0|u|^2 \, dS \\
- R \int_{S_R} (2|\partial_r \overline{u}|^2 - |\nabla u|^2 + \frac{N - 1}{R} \operatorname{Re} [(\partial_r \overline{u})]) \, dS.
\]

(4.73)

Then, letting \( R \to \infty \) along an appropriate sequence in (4.73), we obtain

\[
0 \geq \int_{\mathbb{R}^N} (1 - \frac{\lambda |r\mu_1|}{h})|\nabla u|^2 \, dx \\
+ \lambda \int_{\mathbb{R}^N} [\mu - N \mu_1 - h|r\mu_1|]|u|^2 \, dx,
\]

where we have used (4.66), too. Let \( \lambda \in [0, \lambda_0) \) be an eigenvalue of \( H \) with its eigenfunction \( u \). Set \( \eta = \frac{\lambda}{\lambda_0} \in (0, 1) \) and

\[
h(x) = \begin{cases} 
1 & \text{(if } \mu_1(x) = 0), \\
\lambda_0 |r\mu_1(x)| & \text{(if } \mu_1(x) \neq 0). 
\end{cases}
\]

(4.75)

Then we have

\[
1 - \frac{\lambda |r\mu_1(x)|}{h(x)} = \begin{cases} 
1 & \text{(if } \mu_1(x) = 0), \\
1 - \eta & \text{(if } \mu_1(x) \neq 0), 
\end{cases}
\]

(4.76)

and hence, by using (4.58)

\[
\mu(x) - N \mu_1(x) - h|r\mu_1(x)| = \begin{cases} 
\mu_0 > 0 & \text{(if } \mu_1(x) = 0), \\
\mu(x) - N \mu_1(x) - \lambda_0(r\mu_1(x))^2 \geq 0 & \text{(if } \mu_1(x) \neq 0). 
\end{cases}
\]

(4.77)

Therefore, we have from (4.74)

\[
0 \geq (1 - \eta) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
\]

(4.78)

i.e., \( \nabla u \equiv 0 \) or \( u \) is identically zero almost everywhere. This completes the proof. //
5. The limiting absorption principle for $H$

Throughout this section we assume that $\delta$ satisfies

$$1/2 < \delta \leq 1/2 + \epsilon/4, \quad \epsilon \text{ as in (2.13) or (2.14)}.$$ \hspace{1cm} (5.1)

Let $u \in X$ be given by

$$u = R(z)f, \begin{cases} &z = \lambda + i\eta, \\
&f \in L^2(\mathbb{R}^N), \end{cases} \quad (\lambda \geq 0, \eta \neq 0), \hspace{1cm} (5.2)$$

where $R(z) = (H - z)^{-1}$. Then $u$ satisfies the inhomogeneous equation $(-\mu^{-1}\Delta - z)u = f$ which is equivalent to

$$(-\mu_0^{-1}\Delta - z)u = g \quad (g = \mu_0^{-1}(\mu f + z \mu_1 u)) \hspace{1cm} (5.3)$$

with $k = \sqrt{z\mu_0}$. Let $\mu_1$ be short-range. Then, since

$$\mu_1u \in L^2_{-\delta}(\mathbb{R}^N), \quad ||\mu_1u||_{-\delta} \leq c_1 ||\mu_1u||_{-\delta}, \hspace{1cm} (5.4)$$

we see that $g \in L^2_{-\delta}(\mathbb{R}^N)$. In the case that $\mu_1$ is long-range, the inequality

$$|\eta||u||^2_{1,0} \leq C(||f||, ||u||_0) \hspace{1cm} (5.5)$$

will be useful, where $C = C(\mu)$, $|| \cdot ||_{1,0}$ is the norm of $H^1(\mathbb{R}^N)$, and $(\cdot, \cdot)_0$ is the inner product of $L_2(\mathbb{R}^N)$. For the proof of (5.5), see, e.g., Eidus [6], [13], Lemma 2.1. Then, by a direct application of Theorem 3.5 to our case, we can evaluate the radiation condition term $Du$.

**Theorem 5.1.** Suppose that Assumptions 2.1 and 2.2 hold. Let $\delta$ be as in (5.1). Let $0 < c < d < \infty$ and let $J_{\pm}(c,d)$ be as in (3.18). Let $u$ be given by (5.2) with $z \in J_{\pm}(c,d) \cup J_{-}(c,d)$. Then there exists a positive constant $C = C(\delta, c,d,m_0,M_0)$ such that

$$||Du||_{-\delta} \leq C(||f|| + ||u||_{-\delta}) \hspace{1cm} (5.6)$$

for $N \geq 3$, and

$$||Du||_{-\delta-1,*} \leq C(||f|| + ||u||_{-\delta}) \hspace{1cm} (5.7)$$

for $N = 2$, where $|| \cdot ||_{t,*}$ is as in (3.21).

**Proof.** We can proceed as in the proof of Theorem 3.5. Since $f$ in the proof of Theorem 3.5 should be replaced by $g = \mu_0^{-1}(\mu f + z \mu_1 u)$ (see (5.3)), our additional task is to prove, for any $e > 0$,

$$I = \text{Re} \int_{B_{3R}} z\varphi \mu_1(x)u\overline{Du}_r u \, dx \leq C(||f||^2 + ||u||^2_{-\delta} + e||Du||^2_{-\delta - 1}) \hspace{1cm} (5.8)$$

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with \( C = C(e, \delta, c, d, m_0, M_0) \), where \( \varphi \) is as in the proof of Theorem 3.5, and \( \|D u\|_{\delta-1} \) in (5.8) should read \( \|D u\|_{\delta-1,*} \) if \( N = 2 \). Suppose that \( \mu_1 \) is short-range. Then (5.8) follows directly from (5.4). Suppose that \( \mu_1 \) is long-range. Then we have from the definition of \( D_r u \ ((6) \) of Notation 3.1) and partial integration

\[
I = \lambda \text{Re} \int_{B_r} \varphi \mu_1(x) u \overline{D_r u} \, dx - \eta \text{Im} \int_{B_r} \varphi \mu_1(x) u \overline{D_r u} \, dx
\]

(5.9)

\[
= -\frac{\lambda}{2} \int_{B_r} \partial_r (\varphi \mu_1)|u|^2 \, dx + \lambda \int_{B_r} b \varphi \mu_1|u|^2 \, dx
\]

\[
- \eta \text{Im} \int_{B_r} \varphi \mu_1(x) u \overline{D_r u} \, dx + I_4(r) + I_5(R)
\]

\[
= I_1 + I_2 + I_3 + I_4(r) + I_5(R),
\]

where the terms \( I_4(r) \) and \( I_5(R) \) tend to zero as \( r \to 0 \) and \( R \to \infty \) along appropriate sequences, respectively. It follows from (5.1) and the definition of \( \varphi \ ((3.22) \) or (3.23)) that \( \varphi \mu_1 \) is bounded on \( \mathbb{R}^N \), and hence \( I_2 \) and \( I_3 \) can be evaluated by using (5.5). On the other hand, since

\[
\partial_r (\varphi \mu_1) = O((1 + r)^{2\delta - 2 - \epsilon}) = O((1 + r)^{-2\delta})
\]

(5.10)

the term \( I_1 \) is evaluated by \( \|u\|^2_{\delta, \delta} \), which completes the proof. \(/ /

As in §4, let \( \sigma_p(H) \) be the set of all eigenvalues of \( H \) which is a discrete set in \( (0, \infty) \) (Theorem 4.6). Let \( \lambda > 0 \) such that \( \lambda \notin \sigma_p(H) \). Let \( u \in H^2(\mathbb{R}^N)_{\text{loc}} \cap L^2_{2-\delta}(\mathbb{R}^N) \) be a solution of the homogeneous equation

\[
-\mu(x)^{-1} \Delta u - \lambda u = 0
\]

with the radiation condition \( \|D u\|_{\delta-1, E_1} < \infty \) or \( \|D u\|_{\delta-1, E_1} < \infty \). Then it follows from Theorem 4.5 that \( u \in L^2_{2, \delta_0}(\mathbb{R}^N) \) where \( \delta_0 \) is given by (4.43) or (4.44). Since \( \lambda \) is supposed not to be an eigenvalue, we have \( u \equiv 0 \). Therefore we can prove the limiting absorption principle for \( \lambda \in (0, \infty) \setminus \sigma_p(H) \) by starting with Theorem 5.1, proceeding as in §5 ~ §7 of [9]. Let \( D_\pm \subset \mathbb{C} \) be given by (3.29). For \( \lambda > 0 \), let

\[
R_\pm(\lambda) = \lim_{\eta \downarrow 0} R(\lambda \pm i\eta),
\]

(5.11)

and extend the resolvent \( R(z) \) on \( D_\pm \) by

\[
R(\lambda + i\eta) = \begin{cases} R(\lambda + i\eta) & (\lambda > 0, \eta > 0), \\ R_+(\lambda) & (\lambda > 0, \eta = 0) \end{cases}
\]

(5.12)

for \( z \in D_+ \) and

\[
R(\lambda + i\eta) = \begin{cases} R(\lambda + i\eta) & (\lambda > 0, \eta < 0), \\ R_-(\lambda) & (\lambda > 0, \eta = 0) \end{cases}
\]

(5.13)

for \( z \in D_- \). Then we have
Theorem 5.2. Suppose that Assumptions 2.1 and 2.2 holds. Let $\delta$ satisfy (5.1).

(i) Then the limits (5.11) is well-defined in $B(L_2(\mathbb{R}^N), H^2_{-\delta}(\mathbb{R}^N))$ for $\lambda \in (0, \infty) \setminus \sigma_p(H)$, and the extended resolvent $R(z)$ is a $B(L_2(\mathbb{R}^N), H^2_{-\delta}(\mathbb{R}^N))$-valued continuous function on each of $D_+ \setminus \sigma_p(H)$ and $D_- \setminus \sigma_p(H)$.

(ii) For any $z \in D_+ \setminus \sigma_p(H)$ [or $D_- \setminus \sigma_p(H)$], $R(z)$ is a compact operator from $L_2(\mathbb{R}^N)$ into $H^1_{-\delta}(\mathbb{R}^N)$.

(iii) The selfadjoint operator $H$ is absolutely continuous on the interval $[c, d]$ such that $0 < c < d < \infty$ and

\begin{equation}
[c, d] \cap \sigma_p(H) = \emptyset.
\end{equation}

The operator $H$ has no singular continuous spectrum.

(iv) For $0 < c < d < \infty$ satisfying (5.14) there exists $C = C(c, d, \delta, m_0, M_0) > 0$ such that, for $z \in \mathcal{J}_+(c, d) \cup \mathcal{J}_-(c, d),

\begin{equation}
\begin{cases}
\int_{E_s} (1 + r)^{-2\delta} (|\nabla R(z)f|^2 + |k|^2 |R(z)f|^2) \, dx \\
\leq C^2 (1 + s)^{-(2\delta - 1)} \|f\|^2_{\delta} \quad (s \geq 1, f \in L_2(\mathbb{R}^N)), \\
\|\mathcal{D}R(z)f\|_{\delta - 1} \leq C \|f\|_{\delta} \quad (f \in L_2(\mathbb{R}^N)),
\end{cases}
\end{equation}

where, for $\lambda \in D_+ \cap (0, \infty)$ [or $D_- \cap (0, \infty)$], $\mathcal{D}u$ should be interpreted as $\mathcal{D}^{(\pm)}$ [or $\mathcal{D}^{(-)}$], and $\mathcal{J}_\pm(c, d)$ are given by (3.30).

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