IDEALS WHOSE FIRST TWO BETTI NUMBERS ARE CLOSE

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Abstract. For an ideal $I$ of a Noetherian local ring $(R, m, k)$ we show that $\beta^R_1(I) - \beta^R_0(I) \geq -1$. It is demonstrated that some residual intersections of an ideal $I$ for which $\beta^R_1(I) - \beta^R_0(I) = -1$ or $0$ are perfect. Some relations between Betti numbers and Bass numbers of the canonical module are studied.

Introduction

The eventual behavior of the Betti sequence $\{\beta^R_i(M)\}$ has been extensively studied, while the behavior of the initial Betti numbers is not that well-known.

In the first section of this paper we are interested in studying the relations between the first two Betti numbers. The motivations for this section are two folds. The first, the inequality in Proposition 1.1 which says that "for an ideal $I$ of a Noetherian local ring $(R, m, k)$, $\beta^R_1(I) - \beta^R_0(I) \geq -1" motivates us to study ideals where the difference of their first two Betti numbers lives in the boarder. The second comes from the roles of perfect ideals of height $2$ for which $\beta_1 - \beta_0 = -1$, and perfect Gorenstein ideals of height $3$ where $\beta_1 - \beta_0 = 0$, in the theory of residual intersection.

As it is known, linkage theory is a basic tool to compare and classify some families of algebraic structures. It involves the direct calculations of the colon ideals $J = a : I$ where $a$ is a complete intersection ideal. Residual intersection arises without some restriction and permits the comparison of objects in very different dimensions. Precisely, we say that $J = a : I$ is an $s$-residual intersection of $I$ if $\text{ht} J \geq s \geq \text{ht} I$ whenever $s \geq \mu(a)$. If in addition $\text{ht}(I + J) \geq s + 1$, $J$ is a geometric $s$-residual intersection of $I$. Linkage preserves the Cohen-Macaulay property (CM property). A famous question in this area is "When is a residual intersection ideal of $I$ CM?" Answering this question involved a lot of attempts started by Artin and Nagata [1]. Huneke [7] showed that in a Cohen-Macaulay local ring any geometric $s$-residual intersection of an ideal $I$ which is strongly Cohen-Macaulay (or at least sliding depth [9]) and satisfies the $G_{s+1}$ condition, is CM. Trying to weaken the conditions imposed on $I$, Huneke and Ulrich [8] proved that if $I$ is evenly linked to an ideal which satisfies $G_{s+1}$ and is strongly Cohen-Macaulay, then any geometric $s$-residual intersection of $I$ is Cohen-Macaulay. Examples of such ideals are perfect ideals of height $2$ and perfect Gorenstein ideals of height $3$. As it already pointed out, perfect ideals of height $2$ has $\beta_1 - \beta_0 = -1$ and perfect Gorenstein ideals of height $3$ has $\beta_1 - \beta_0 = 0$.

2000 Mathematics Subject Classification. 13D07, 13D02.

Key words and phrases. Betti numbers, Residual Intersection.

The research of Borna was in part supported by grant No. 88130035 from IPM.

The research of Hassanzadeh was in part supported by grant No. 88130112 from IPM.
We show that in the presence of some $G_s$ condition, ideals with small $\beta_1 - \beta_0$ admit perfect geometric residual intersections even if they do not satisfy the sliding depth condition. As well, we provide an example of an (non CM) ideal with $\beta_1 - \beta_0 = 0$ which admits a CM 3-residual intersection but does not satisfy the sliding depth condition. In the next propositions in the first section we try to identify some basic properties of ideals with $\beta_1 - \beta_0 = 0, -1$.

Continuing to find more relations among initial Betti numbers, we encountered with the Poincaré series of the canonical module in the second section. In fact we deduced some relations between Bass numbers of an ideal and Betti numbers of a canonical module. Corollary 2.2 which is one of the applications of these relations shows that a partial sum of the Poincaré series of the canonical module $\omega_{R/J}$ is positive or negative if and only if the grade of $J$ is even or odd, respectively.

Setup

Throughout this paper $(R, m, k)$ is a Noetherian local ring of dimension $d$, $I$ is an ideal of $R$ and $M$ is a finitely generated $R$-module. By $\beta_i^R(M)$ (or $\beta_i$ if it does not arise any ambiguity) we mean the $i$th Betti number of $M$, i.e., $\dim_k \operatorname{Tor}_i^R(k, M)$. $P_M(t) = \sum_{i=0}^{\infty} \beta_i^R(M) t^i$ is the Poincaré series of $M$. For an integer $s$, the partial sum of $P_M(t)$ is denoted by $P_M^s(t) = \sum_{i=0}^{s} \beta_i^R(M) t^i$. The $i$th Bass number of $M$, i.e., $\dim_k \operatorname{Ext}_i^R(k, M)$ is denoted by $\mu_i^R(M)$ and $\mu(M)$ is the minimal number of generators of $M$. The analytic spread of $I$, denoted by $\ell(I)$, is the Krull dimension of $R(I) \otimes_R k$, where $R(I)$ is the Rees algebra of $I$. One always has $\operatorname{ht}(I) \leq \ell(I) \leq \min\{\dim R, \mu(I)\}$, cf. [12, 1.89]. Let $I = (i_1, \ldots, i_r)$ and denote by $H_i(I)$ the $i$th homology of the Koszul complex with respect to $i_1, \ldots, i_r$. We say that $I$ satisfies the sliding depth condition (SD) if $\operatorname{depth}(H_i(I)) \geq d - r + i$, $i \geq 0$; see [9], or [8] for an alternate definition. An ideal $I$ satisfies condition $G_{s+1}$ for some integer $s$, if $\mu(I_p) \leq \operatorname{ht} p$ for all prime ideal $I \subset p$ with $\operatorname{ht} p \leq s$. Further, $I$ satisfies $G_\infty$ if $I$ satisfies $G_s$ for all $s$. $M$ is called unmixed if all associated prime ideals of $M$ have a same height. The unmixed part of an ideal $I$, $I^{\text{unm}}$, is the intersection of all primary components of $I$ with height equal to $\operatorname{ht}(I)$. By a perfect module we mean a CM module of finite projective dimension. Finally, we say that $M$ has rank $r$ if $M \otimes_R Q$ is a free $Q$-module of rank $r$ where $Q$ is the total ring of fractions of $R$.

1. Small $\beta_1 - \beta_0$

The following proposition provides one motivation for studying modules with small $\beta_1 - \beta_0$.

**Proposition 1.1.** Let $I$ be an ideal of $R$. Then the followings hold:

(a) $\sum_{j=0}^{2n+1} (-1)^{j+1} \beta_j^R(I) \geq -1$ for all $n \geq 0$; in particular $\beta_0^R(I) - \beta_0^R(I) \geq -1$.

(b) If $R$ is unmixed and equality in (a) holds, then grade($I$) $> 0$.

(c) If grade($I$) $> 0$, then $\sum_{j=0}^{2n} (-1)^{j+1} \beta_j^R(I) \leq -1$ for all $n \geq 0$.

**Proof.** (a) Let $\cdots \to R^{\beta_{i+1}} \to R^{\beta_i} \to I \to 0$ be the minimal free resolution of $I$ and set $Z_i = \ker(R^{\beta_i} \to R^{\beta_{i+1}})$ for all $i \geq 1$. By localizing the exact sequence $0 \to Z_i \to R^{\beta_i} \to R^{\beta_{i+1}} \to \cdots \to I \to 0$ at
the minimal prime \( p \), we get the exact sequence
\[
0 \to (Z_i)_p \to R_p^{\beta_1} \to R_p^{\beta_1-1} \to \cdots \to IR_p \to 0
\]
of \( R_p \)-modules. \( R_p \) is an Artinian local ring, hence each module in this exact sequence is of finite length. We then have
\[
\text{length}_{R_p}(IR_p) + \sum_{j=0}^{i} (-1)^{j+1}\text{length}_{R_p}(R_p^{\beta_j}) + (-1)^{i+2}\text{length}_{R_p}((Z_i)_p) = 0. \tag{\dagger}
\]

This equality in conjunction with \( \text{length}_{R_p}(R_p) \geq \text{length}_{R_p}(IR_p) \) yields
\[
\text{length}_{R_p}(R_p)(1 + \sum_{j=0}^{i} (-1)^{j+1}\beta_j) \geq (-1)^{i+2}\text{length}_{R_p}((Z_i)_p). \tag{\ddagger}
\]

Note that \( \text{length}_{R_p}((Z_i)_p) \geq 0 \); so that for \( i = 2n + 1 \), an odd integer, the desired inequality follows.

(b) Assume that \( R \) is unmixed and that \( \sum_{j=0}^{2n+1} (-1)^{j+1}\beta_j = -1 \) for some integer \( n \geq 0 \). Then it follows from (\ddagger) that \( (Z_{2n+1})_p = 0 \) for all \( p \in \text{Ass}(R) \). Since \( \text{Ass} Z_i \subset \text{Ass} R \) for all \( i, Z_{2n+1} = 0 \). Hence the projective dimension of \( I \) is finite. Thus \( I \) has a rank and \( \text{rank}(I) = \sum_{j=0}^{2n+1} (-1)^{j}\beta_j = 1 \), therefore \( \text{grade}(I) > 0 \).

(c) Since \( \text{grade}(I) > 0 \), there is an \( x \in I \) which is \( R \)-regular. Thus \( \text{length}_{R_p}(R_p) \geq \text{length}_{R_p}(IR_p) \geq \text{length}_{R_p}(xR_p) = \text{length}_{R_p}(R_p) \) for each \( p \in \text{Min}(R) \) (because \( x \notin p \)). This shows that \( \text{length}_{R_p}(IR_p) = \text{length}_{R_p}(R_p) \) for each \( p \in \text{Min}(R) \). employing this fact in (\ddagger), we get \((-1)^i\text{length}_{R_p}((Z_i)_p) = \text{length}_{R_p}(R_p)(1 + \sum_{j=0}^{i} (-1)^{j+1}\beta_j) \). Hence for \( i = 2n \) an even integer, \( \sum_{j=0}^{i} (-1)^{j+1}\beta_j \leq -1 \).

The next theorem is our main theorem on the Cohen-Macaulayness of residual intersections of ideals with small \( \beta_1 - \beta_0 \), here we do not care about the properties of the corresponding Koszul complex.

**Theorem 1.2.** Let \((R, m, k)\) be a Cohen-Macaulay Noetherian local ring, \( I \) an ideal of positive grade \( g \) and with analytic spread \( \ell \), denote \( \beta_i = \beta^R(I) \) for all \( i \). Then

(a) if \( \beta_1 - \beta_0 = 1 \), then any \( s \)-residual intersection of \( I \) is perfect of projective dimension \( s - 1 \);

(b) if \( \beta_1 - \beta_0 = 1 \) and \( R \) is Gorenstein, then \( I^{\text{unm}} \) is perfect;

(c) if \( \beta_1 - \beta_0 = 0 \) and \( I \) satisfies \( G_{\ell+1} \) and \( k \) is an infinite field, then \( I \) admits a CM geometric \( \ell \)-residual intersection of projective dimension \( \ell \).

**Proof.**

(a) Let \( a \subset I \) be an ideal generated by \( s \) elements such that \( J = a : I \) is an \( s \)-residual intersection of \( I \). \( I \) has the presentation \( 0 \to R^{\beta_0 - 1} \to R^{\beta_0} \to I \to 0 \). Let \( \cdots \to R^s \to a \to 0 \) be a free resolution for \( a \). Lifting the inclusion \( a \subset I \) and computing the mapping cone of this lifting map, we have an exact complex
\[
\cdots \to R^{\beta_0 + s - 1} \xrightarrow{\phi} R^{\beta_0} \to I/a \to 0
\]
By Fitting theorem [3, Lemma 20.7], \( J = \text{Ann}(I/a) \subset \sqrt{I_{\beta_0}(\psi)} \). Thus grade \( I_{\beta_0}(\psi) \geq \text{grade}(J) \geq s = (\beta_0 + s - 1) - \beta_0 + 1 \) the largest possible value. Therefore \[3\] Excercise 20.6] implies that \( I_{\beta_0}(\psi) = J \). Now the Eagon-Northcott complex of \( \psi \) provides a free resolution for \( R/I_{\beta_0}(\psi) \) of length \( s \). To see the Cohen-Macaulayness of \( J \) we recall a theorem of Hochster and Eagon which states that if a determinantal ideal attains its maximum height in a Cohen-Macaulay local ring then it is CM; see [4, A2.55].

(b) In the case that \( R \) is Gorenstein, for a maximal regular sequence contained in \( I \) say \( \alpha \), one has \( \alpha : J = \alpha : \alpha : I = \alpha : I^{\text{unm}} = I^{\text{unm}}, \) so that \( I^{\text{unm}} \) is linked to \( J = \alpha : I; \) see [3, 1.1.8 (ii)]. Hence, by theorems in the Linkage theory, the fact that \( J \) is perfect implies that \( I^{\text{unm}} \) is perfect.

(c) This part is based on the following consequence of three results of Eisenbud, Huneke and Ulrich [6, 1.1, 1.2, 3.7]. According to these results, if \( I \) is an ideal of positive grade in a Noetherian local ring with infinite residue field such that \( I \) satisfies \( G_{\ell+1} \), then there exists an ideal \( a \subset I \) generated by \( \ell \) elements which is a minimal reduction of \( I \) such that \( \text{ht}(a : I) \geq \mu(a) + 1 = \ell + 1 \), note that since \( a \) is a reduction of \( I \), \( \text{ht}(I) = \text{ht}(a) \leq \mu(a) < \ell + 1 \), thus \( J = \alpha : I \) is a \( \ell \)-residual intersection of \( I \). The rest of the proof is similar to that of part (a) just note that the beginning of the mentioned mapping cone is of the form

\[
R^{\beta_0 + \mu(a)} \xrightarrow{\psi} R^{\beta_0} \rightarrow I/a \rightarrow 0.
\]

Therefore

\[
\text{ht } I_{\beta_0}(\psi) \geq \text{ht } J \geq \mu(a) + 1 = (\beta_0 + \mu(a) - \beta_0 + 1)(\beta_0 - \beta_0 + 1)
\]

the greatest possible value. The result will now follow as part (a).

The next example shows the benefit of Theorem 1.2 to prove the Cohen-Macaulayness of residual intersections in cases where other approaches can not be applied. For example the sliding depth is a condition which often appears in the proofs of Cohen-Macaulayness of residual intersections; see [11] for instance. In the following we give an example of an (non CM) ideal which admits a CM 3-residual intersection but does not satisfy the sliding depth condition.

**Example 1.3.** Let \( R = \mathbb{Q}[x, y, z] \) and \( I = (x^2, xy, z^2) \). Then \( \text{ht}(I) = 2 \) and the only minimal prime containing \( I \) is \( (x, z) \). One now can see that \( I \) satisfies \( G_{\infty} \). The minimal free resolution of \( I \) is

\[
0 \rightarrow R(-5) \rightarrow R(-3) \oplus R^2(-4) \rightarrow R^3(-2) \rightarrow 0.
\]

That is \( \beta_1^R(I) - \beta_0^R(I) = 0 \), hence \( I \) fulfills the conditions of Theorem 1.2(c). Proposition 1.5 of [12] describes the equations of the Rees algebra \( \mathcal{R}_I \), say \( P \). Using the following procedure in CoCoA, we are able to compute \( P \).

**Define Equations(1)**

\[
S:=\mathbb{Q}[t[1..4],x[1..3],u];
Using S Do
I := Ideal(BringIn(Gens(I))); G := Gens(I); P := Elim(u, Ideal([t[N]-u*G[N] — N In 1..Len(G)])); Minimalize(P); Return P; EndUsing; EndDefine; I := Ideal(x[1]x[1],x[3]x[3],x[1]x[2]); P := Equations(I); P = (-t[3]x[1] + t[1]x[2], -t[2]x[1]^2 + t[1]x[3]^2, t[2]x[1]x[2] - t[3]x[3]^2)

We then obtain that ℓ(I) = 3. Hence Theorem 1.2(c) ensures that there is a geometric 3-residual intersection of I which is CM. On the other hand I does not satisfy SD. To see this notice that the first homology of the Koszul complex of the above generating set of I is ((x^2, z^2) : I = (x, z^2)/I which has depth zero since (x, y, z) = I : xz.

The next lemma will be helpful in the sequel.

Lemma 1.4. Let M be a finitely generated R-module which has a rank. Then for all i ≥ 0,

\[ \beta_R^i(M) - \beta_R^{i+1}(M) \leq \sum_{j=1}^{i} (1)^{i-j} \beta_R^{i-j}(M) + (-1)^i \text{rank}(M). \]

The equality holds if and only if \( \beta_R^{i+2}(M) = 0. \)

Proof. Let i ≥ 0 be an integer. Consider a minimal free resolution \( F_\bullet \) for M and let \( Z_{i-1} \) to be the \( (i-1) \)th syzygy module of this complex. (Take the \( (-1) \)th syzygy of M to be M itself.) By [2, Corollary 1.4.6], \( Z_{i-1} \) has a rank and

\[ \text{rank}(Z_{i-1}) = \sum_{j=1}^{i} (1)^{i-j} \beta_R^{i-j}(M) + (-1)^i \text{rank}(M). \]

Now, the exact sequence 0 → Z_i → R^\beta_i → Z_{i-1} → 0 implies that Z_i, as well, has a rank and rank(Z_{i-1}) + rank(Z_i) = \( \beta_R^i(M) \). Hence using the similar fact that rank(Z_i) = \( \beta_R^{i+1}(M) - \text{rank}(Z_{i+1}) \), one gets that

\[ \beta_R^i(M) - \beta_R^{i+1}(M) = \text{rank}(Z_{i-1}) - \text{rank}(Z_{i+1}) \]

\[ \leq \text{rank}(Z_{i-1}) = \sum_{j=1}^{i} (1)^{j-1} \beta_R^{i-j}(M) + (-1)^i \text{rank}(M). \]

Clearly the equality holds if and only if rank(Z_{i+1}) = 0 which in turn implies that Z_{i+1} = 0. i.e., \( \beta_R^{i+2}(M) = 0. \)

Employing the techniques in the proof of Theorem 1.2 we have the following properties of ideals with small \( \beta_1 - \beta_0 \).

Proposition 1.5. Let I be an ideal of a Gorenstein local ring R with \( \beta_R^1(I) - \beta_R^0(I) = -1 \) and \( \beta_R^1(I) \neq 0. \) Then pd(I) = 1 and moreover either I is a perfect ideal of grade 2 or I is not unmixed and its grade is 1.
Proof. Let $\beta_1 = \beta_1^R(I)$. Since the equality in Lemma 1.4 holds, we have $\beta_2 = 0$, that is $pd I = 1$. Now consider the presentation $0 \to R^{3 \omega - 1} \xrightarrow{\varphi} R^{\omega} \to I \to 0$. By Hilbert-Burch theorem [2, Theorem 1.4.7], $I = aI_{\beta_0 - 1}(\varphi)$ for some non-zero divisor $a$ of $R$, and $I_{\beta_0 - 1}(\varphi)$ is a perfect ideal of grade 2. There are two cases, either $a$ is unit or not. In the former case $I = aI_{\beta_0 - 1}(\varphi)$ is a perfect ideal of grade 2, whereas in the latter one, $\text{grade}(I) = \text{grade}(aI_{\beta_0 - 1}(\varphi)) = \min\{\text{grade}(Ra), \text{grade}(I_{\beta_0 - 1}(\varphi))\} = 1$. Let $\alpha \in I$ be a non-zero divisor. Using the proof of Theorem 1.2(a) one can see that $(\alpha : I)$ is a determinant of a square matrix. Thus it is a principle ideal say $(\gamma)$. By [8, 1.8 (ii)], $(\alpha : \gamma) = I^{unm}$. We note that $\alpha = \gamma \delta$ for some non-zero divisor $\delta$ of $R$, that is $(\delta) = I^{unm}$ which is an ideal of projective dimension zero. In particular, $I \neq I^{unm}$.

Proposition 1.6. Let $I$ be an ideal of a Gorenstein local ring $R$ which satisfies one of the following conditions:

(a) $\beta_1^R(I) - \beta_0^R(I) = -1$ and $\text{grade}(I) = 1$ or
(b) $\beta_1^R(I) - \beta_0^R(I) = 0$ and $\text{grade}(I) = 0$.

Then $\omega_{R/I} \cong R/I^{unm}$.

Proof. For (a), by the same token as Corollary 1.5 for a regular element $\alpha$ in $I$, we have

$$\omega_{R/I} = \alpha : I = (\gamma) \cong R_{(\alpha)} : (\gamma) \cong R_{(\alpha : \gamma)} : (\gamma) = R_{(\alpha : \gamma)} : I^{unm}.$$

For (b) according to the proof of Theorem 1.2(a), one obtains that $(0 : I)$ is principal say $(\gamma)$. Hence $\omega_{R/I} = 0 : I = (\gamma) = R/(0 : \gamma) = R/I^{unm}$.

2. BETTI NUMBERS OF THE CANONICAL MODULE

This section is mostly devoted to study the behavior of the Betti sequence of the canonical module of a Cohen-Macaulay ring $R/J$, where $R$ is a Gorenstein ring, specially those in which $\text{grade}(J)$ is small.

Theorem 2.1. Let $(R, m, k)$ be a CM local ring of dimension $d$ and $J$ an ideal of $R$ of height $g$. Let $\alpha$ be a maximal regular sequence in $J$ and $I = \alpha : J$. Then

(a) $\beta_i^R(\omega_{R/J}) = \beta_{i+1}^R(R/I)$ for all $i \geq g + 1$.
(b) $P_{\omega_{R/J}}^{g - 1}(-1) = (-1)^g \beta_{g+1}^R(R/I) - P_{R/I}^{g - 1}(-1)$.
(c) If $R$ is Gorenstein and $J$ is Cohen-Macaulay, then $\beta_i^R(\omega_{R/J}) = \mu_{d-g}^R(R/J)$ for all $i \geq 0$.
(d) $\beta_0^R(\omega_{R/J}) = \mu_{d-g}^R(R/J) \mu_R^d(R)$.

Proof. (a) Considering the assumptions, we have $I/\alpha = (\alpha : J)/\alpha = \text{Hom}_R(R/J, R/\alpha) = \text{Ext}_R^g(R/J, R) \cong \omega_{R/J}$. Hence we obtain the following exact sequence:

$$0 \to \omega_{R/J} \to R/\alpha \to R/I \to 0$$

The long exact sequence of Tor yields that $\text{Tor}_i^R(R/m, R/I) \cong \text{Tor}_{i+1}^R(R/m, \omega_{R/J})$ for all $i \geq 1$ which finishes the proof of (a).
(b) We consider the beginning terms of the long Tor exact sequence:

\[ 0 \to \text{Tor}^{R}_{g+1}(k, R/I) \to \text{Tor}^{R}_{g}(k, \omega_{R/J}) \to \text{Tor}^{R}_{g}(k, R/\alpha) \to \cdots \to \text{Tor}^{R}_{0}(k, R/I) \to 0. \]

Counting the vector space dimensions, we have

\[ \beta^{R}_{g+1}(R/I) - \sum_{i=0}^{g} (-1)^{i} \beta^{R}_{g-i}(\omega_{R/J}) + \sum_{i=0}^{g} (-1)^{i} \beta^{R}_{g-i}(R/\alpha) - \sum_{i=0}^{g} (-1)^{i} \beta^{R}_{g-i}(R/I) = 0. \]

The fact that \( \sum_{i=0}^{g} (-1)^{i} \beta^{R}_{g-i}(R/\alpha) = \sum_{i=0}^{g} (-1)^{i} \beta^{R}_{g-i} \) implies that

\[ \beta^{R}_{g+1}(R/I) - (-1)^{g}P_{\leq g}^{\leq g}(\omega_{R/J}) - (-1)^{g}P_{\leq g}^{\leq g}(R/I) = 0 \]

which is the assertion of (b).

(c) We construct the following spectral sequence of Foxby. Consider the finite injective resolution of \( R, E^{\bullet} : 0 \to E^{0} \to \cdots \to E^{d} \to 0 \) and the projective resolution of \( k, P^{\bullet} : \cdots \to P_{1} \to P_{0} \to 0 \). The double complex \( P^{\bullet} \otimes_{R} \text{Hom}_{R}(R/J, E^{\bullet}) \) in the second quadrant is pictured in the following:

\[ \cdots \to P_{1} \otimes_{R} \text{Hom}_{R}(R/J, E^{d}) \to P_{1} \otimes_{R} \text{Hom}_{R}(R/J, E^{d-1}) \to P_{1} \otimes_{R} \text{Hom}_{R}(R/J, E^{d}) \to \cdots \]

\[ \cdots \to P_{0} \otimes_{R} \text{Hom}_{R}(R/J, E^{d}) \to P_{0} \otimes_{R} \text{Hom}_{R}(R/J, E^{d-1}) \to P_{0} \otimes_{R} \text{Hom}_{R}(R/J, E^{d}) \to \cdots \]

\[ \cdots \to 0 \to 0 \to 0 \]

The terms of the second horizontal spectral sequence arisen from this double complex is as the following diagram, \( E_{\text{hor}}^{i,j} = \text{Tor}^{R}_{i}(k, \text{Ext}^{d-i}_{R}(R/J, R)) \):

\[ \cdots \to \text{Tor}^{R}_{2}(k, \text{Ext}^{d}_{R}(R/J, R)) \to \cdots \to \text{Tor}^{R}_{2}(k, \text{Ext}^{d-1}_{R}(R/J, R)) \to \text{Tor}^{R}_{2}(k, \text{Ext}^{d}_{R}(R/J, R)) \]

\[ \cdots \to \text{Tor}^{R}_{1}(k, \text{Ext}^{d}_{R}(R/J, R)) \to \cdots \to \text{Tor}^{R}_{1}(k, \text{Ext}^{d-1}_{R}(R/J, R)) \to \text{Tor}^{R}_{1}(k, \text{Ext}^{d}_{R}(R/J, R)) \]

\[ \cdots \to \text{Tor}^{R}_{0}(k, \text{Ext}^{d}_{R}(R/J, R)) \to \cdots \to \text{Tor}^{R}_{0}(k, \text{Ext}^{d-1}_{R}(R/J, R)) \to \text{Tor}^{R}_{0}(k, \text{Ext}^{d}_{R}(R/J, R)) \]
The fact that $R$ is Gorenstein and $R/J$ is Cohen-Macaulay of dimension $d - g$ implies that $\text{Ext}^d_R (R/J, R) = 0$ for all $i \neq d - g$. Thus this spectral sequence has infinite terms $\sim E^{-i,j}_{\text{hor}} = 0$ for $i \neq d - g$ and $\sim E^{-i(d-g)-j,0}_{\text{hor}} = 2 \sim E^{-d-g,j,0}_{\text{hor}}$.

On the other hand the functorial isomorphism $M \otimes_R \text{Hom}_R (N, I) \cong \text{Hom}_R (\text{Hom}_R (M, N), I)$, for a finitely generated $R$-module $M$ and an injective $R$-module $I$ implies the second vertical spectral sequence to be of the following form:

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\text{Ext}^d_R (\text{Ext}^i_R (k, R/J), R) & \text{Ext}^{d-1}_R (\text{Ext}^i_R (k, R/J), R) & \text{Ext}^d_R (\text{Ext}^1_R (k, R/J), R) \\
\text{Ext}^{d-2}_R (\text{Ext}^0_R (k, R/J), R) & \text{Ext}^{d-1}_R (\text{Ext}^0_R (k, R/J), R) & \text{Ext}^d_R (\text{Ext}^0_R (k, R/J), R)
\end{array}
\]

We now notice that $\dim \text{Ext}^i_R (k, R/J) = 0$ for all $i$, in particular $\text{depth} \text{Ext}^i_R (k, R/J) = 0$ for all $i$ and that $\text{Ext}^j_R (\text{Ext}^i_R (k, R/J), R) = 0$ for all $j \neq d$. Thus

\[
\sim E^{-i,j}_{\text{ver}} = \begin{cases} 
0, & i \neq 0; \\
\text{Ext}^d_R (\text{Ext}^j_R (k, R/J), R), & i = 0.
\end{cases}
\]

Now the convergence of these two spectral sequences imply that

\[
\text{Tor}^R_i (k, \omega_{R/J}) \cong \text{Ext}^d_R (\text{Ext}^{d-g+i}_R (k, R/J), R) \text{ for all } i \geq 0.
\]

Accordingly, $\beta_i^R (\omega_{R/J}) = \mu^d_R (\text{Ext}^{d-g+i}_R (R/J), R) = \mu^{d-g+i}_R (R/J)$ for all $i \geq 0$.

(d) To see this part notice that in $\sim E^{2 \sim}_{\text{hor}}, \sim E^{g,0}_{\text{hor}}$ is located in the right-down non-zero corner of the diagram of $\sim E^{2 \sim}_{\text{hor}}$ (recall that $\text{Ext}^{2 \sim} (R/J, R) = 0$ for all $i \neq d - g$); hence $\sim E^{g,0}_{\text{hor}} = 2 \sim E^{g,0}_{\text{hor}}$. Then the convergence of the spectral sequence implies that

\[
\text{Tor}^R_0 (k, \text{Ext}^{d-g}_R (R/J, R)) \cong \text{Tor}^R_0 (k, \omega_{R/J}) \cong \text{Ext}^d_R (\text{Ext}^{d-g}_R (k, R/J), R)
\]

which implies the assertion.

\[
\square
\]

As a corollary of Theorem 2.1 in conjunction with Lemma 1.4 we obtain the following property of the Poincaré series of the canonical module.

**Corollary 2.2.** Suppose that $(R, m)$ is a Cohen-Macaulay local ring and $J$ is an ideal of grade $g \geq 1$. Then

(a) if $g$ is odd, $P_{\omega_{R/J}}^{\leq g} (-1) \leq 0$. In particular, if $g = 1$, $\beta_1^R (\omega_{R/J}) \geq \beta_0^R (\omega_{R/J})$;

(b) if $g$ is even, $P_{\omega_{R/J}}^{\leq g} (-1) \geq 0$. 

Proof. For (a) by Lemma 1.4, $\beta_i^R(R/I) - \beta_i^{R+1}(R/I) \leq \beta_i^R(R/I) - \beta_i^{R+1}(R/I) + \cdots + \beta_i^{R-1}(R/I)$. On the other hand by Theorem 2.1(b), $P^{\leq g}_{\omega_{R/J}}(-1) = \beta^R_{g+1}(R/I) - \beta^R_{g+2}(R/I) - (\beta^R_0(R/I) - \beta^R_1(R/I) + \cdots + \beta^R_{g-1}(R/I))$ which yields the assertion. For (b), $P^{\leq g}_{\omega_{R/J}}(-1) = \beta^R_{g+1}(R/I) + (-\beta^R_0(R/I) + \cdots + \beta^R_{g-1}(R/I)) - \beta^R_g(R/I)$ which is non-negative by Lemma 1.4. □

Acknowledgements. The authors would like to thank Professor M-T. Dibaei for his useful comments which brought improvement in the Proposition 1.1.

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