Neutrino Photon Interaction in a Magnetized Medium II

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In the presence of a thermal medium or an external electro-magnetic field, neutrinos can interact with photon, mediated by the corresponding charged leptons (real or virtual). The effect of a medium or an electromagnetic field is two-fold - to induce an effective $\nu\gamma$ vertex and to modify the dispersion relations of all the particles involved to render the processes kinematically viable. It has already been noted that in a medium neutrinos acquire an effective charge, which in the standard model of electroweak interaction comes from the vector type vertex of weak interaction. On the other hand in a magnetized plasma, the axial vector part also start contributing to the effective charge of a neutrino. This contribution corresponding to the axial vector part in the interaction Lagrangian is denoted as the axial polarisation tensor. In an earlier paper we explicitly calculated the form of the axial polarisation tensor to all odd orders in external magnetic field. In this note we complete that investigation by computing the same, to all even orders in external magnetic field. We further show its gauge invarience properties. Finally we infer upon the zero external momentum limit of this axial polarisation tensor.

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I. INTRODUCTION

Neutrino mediated processes are of great importance in cosmology and astrophysics \cite{1,2}. It is worth mentioning at this stage that various interesting possibilities have been looked into in the context of cosmology e.g., large scale structure formation in the universe, to name one of the few \cite{3}. In this note we would rather consider the astrophysical part of it.

Because of the effective neutrino photon interaction in a medium, it is possible that the neutrinos might dump a fraction of their energy inside the star during stellar evolution. For instance in a type II supernovae collapse\textsuperscript{4} neutrinos produced deep inside the proto-neutron star surge out carrying an effective energy $\sim 10^{52}$ erg/s. It is conjectured that the neutrinos deposit some fraction of its energy during the explosion through different kinds of neutrino electromagnetic interactions, e.g, $\gamma\nu \rightarrow \gamma\nu$, $\nu \rightarrow \nu\gamma$, $\nu\bar{\nu} \rightarrow e^+e^-$, to name a few. It is important to note here, that all these processes are of order $G_F^2$. However the amount of energy dumped by these mechanisms to the mantle of the proto-neutron star seem to be barely sufficient to blow the outer part of the same. However, it is important to note that, inside a star, a nonzero magnetic field would always be present, which in most of the studies performed so far, has not been properly taken into account. Our objective here is to take into account the presence of a strong magnetic field and estimate the corrections coming there off.

It is usually conjectured, taking into account the conservation of surface magnetic field of a proto-neutron star, that during a supernova collapse the magnetic field strength in some regions inside the nascent star can reach upto $B \sim m_e^2c^2$ or more. Here $m_e$ denotes the mass of electron. [Henceforth we would refer to field strengths of this magnitude as critical field strength $B_c$.] This conjecture makes it worthwhile to investigate the role of magnetic field in effective neutrino photon vertex.

Neutrinos do not couple with photons in the tree level in the standard model of particle physics, and this coupling can only take place at a loop level, mediated by the fermions and gauge bosons. This coupling can give birth to off-shell photons only, since for on-shell particles, the processes like $\nu \rightarrow \nu\gamma$ and $\gamma \rightarrow \nu\bar{\nu}$ are restrained kinematically. Only in presence of a medium can all the particles be onshell as there the dispersion relation of the photon changes, giving the much required phase space for the reactions. Intuitively when a neutrino moves inside a thermal medium composed of electrons and positrons, they interact with these background particles. The background electrons and positrons themselves have interac-
tion with the electromagnetic fields, and this fact gives rise to an effective coupling of the neutrinos to the photons. Under these circumstance’s the neutrinos may acquire an “effective electric charge” through which they interact with the ambient plasma.

In this paper we concentrate upon the effective neutrino photon interaction vertex coming from the axial vector part of the interaction. From there we estimate the effective charge of the neutrino inside a magnetised medium. We name the axial contribution in the effective neutrino photon Lagrangian as the axial polarisation tensor. Π_{5μν}, which we call the axial polarisation tensor. Both the vector part of the interaction. From there we estimate the effective charge of the neutrino and the corresponding lepton field respectively. For electron neutrinos,

\[ g_ν = 1 - (1 - 4\sin^2 θ_W)/2, \]
\[ g_A = -1 + 1/2; \]

where the first terms in \( g_ν \) and \( g_A \) are the contributions from the W exchange diagram and the second one from the Z exchange diagram.

With this interaction Lagrangian we can write down the matrix element for the Cherenkov amplitude as,

\[ M = -\frac{G_F}{\sqrt{2}e} Z_ν(γ^μ(1 - γ_5)ν(g_ν G_{μν} + g_A G_5_{μν}), \]

where \( e^ν \) is the photon polarisation tensor, and \( Z \) is the wavefunction renormalisation factor inside a medium. The term \( G_{μν} \) is defined as

\[ i\Pi_{μν} = (-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} Tr [γ_μ iS(p)γ_ν iS(p')] \]

which looks exactly like the photon polarisation tensor, but doesn’t have the same interpretation here. The momentum labels of the propagators can be understood from fig. [1]. Henceforth we would call it the polarisation tensor. \( G_{μν} \) is defined as

\[ i\Pi_{5μν} = (-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} Tr [γ_μ γ_5 iS(p)γ_ν iS(p')] \]

which we call the axial polarisation tensor. Both the polarisation tensor and the axial polarisation tensor are obtained by calculating the Feynman diagram given in fig. [1].

The off-shell electromagnetic vertex function \( Γ_ν \) is defined in such a way, that for on-shell neutrinos, the \( ννγ \) amplitude is given by:

\[ M = -iu(q')Γ_ν u(q)A^ν(k), \]

where, \( k \) is the photon momentum. Here, \( u(q) \) is the neutrino spinor and \( A^ν \) stands for the electromagnetic vector potential. In general \( Γ_ν \) would depend on \( k \) and the characteristics of the medium. With our effective Lagrangian \( Γ_ν \) is given by

\[ Γ_ν = -\frac{1}{\sqrt{2}e} G_F γ^μ(1 - γ_5) (g_ν G_{μν} + g_A G_5_{μν}), \]

The effective charge of the neutrinos is defined in terms of the vertex function by the following relation [3]:

\[ \mathcal{L}_{eff} = -\frac{1}{\sqrt{2}} G_F \bar{γ}^μ(1 - γ_5) ν_5^μ γ_μ (g_ν G_{μν} + g_A G_5_{μν}), \]
\[ e_{\text{eff}} = \frac{1}{2q_0} \tilde{u}(q) \Gamma_0(k_0 = 0, k \to 0) u(q). \] (9)

For massless Weyl spinors this definition can be rendered into the form:

\[ e_{\text{eff}} = \frac{1}{2q_0} \text{Tr} \left[ \Gamma_0(k_0 = 0, k \to 0) (1 + \lambda \gamma^5) \right] \] (10)

where \( \lambda = \pm 1 \) is the helicity of the spinors.

While discussing about \( \Pi^{\mu \nu}_5 \), it should be remembered that for the electromagnetic vertex, we have the current conservation relation,

\[ k^\nu \Pi^{\mu \nu}_5 = 0 \] (11)

which is the gauge invariance condition.

In order to calculate the Cherenkov amplitude or the effective charge of the neutrinos inside a medium, we have to calculate \( \Pi^{\mu \nu}_5 \). The formalism so discussed is a general one and we extend the calculations previously done based upon this formalism to the case where we have a constant background magnetic field in addition to a thermal medium. In doing so we give the full expression of \( \Pi^{\mu \nu}_5 \) in a magnetised medium and explicitly show its gauge invariance. We also comment on the effective charge contribution from the axial polarisation part.

Discussing about the effective charge of the neutrinos in a medium, the way we have done, it should be mentioned that although it is interesting to find it theoretically, it is not the “charge” with which the neutrinos interact that although it is interesting to find it theoretically, it is not the “charge” with which the neutrinos couple with a magnetic field. From the definition of the electromagnetic vertex as given in Eq. (5) and the definition of charge in Eq. (4) it is clear that we are interested to find the coupling of the photon field with \( u T^0 u \) and not \( u \Gamma^a u \). The magnetic interaction will come from the term \( \tilde{u}(q) \Gamma^a u(q) A^a \), but we will see at the end of the calculation that no \( \Gamma^a \) exists in the limit \( k_0 \to 0, \tilde{k} \to 0 \), whereby we cant say of any possible interaction of the neutrinos with the external static uniform magnetic field.

III. GENERAL ANALYSIS

We start this section with a discussion on the possible tensor structure and form factor analysis of \( \Pi^{\mu \nu}_5(k) \), based on the symmetry of the interaction. To begin with we note that, \( \Pi^{\mu \nu}_5(k) \) in vacuum should vanish. This follows from the following arguments. In vacuum the available vectors and tensors at hand are the following,

\[ k_\mu, g_{\mu \nu} \text{ and } \epsilon_{\mu \nu \lambda \sigma}. \] (12)

The two point axial-vector correlation function \( \Pi^{\mu \nu}_5 \) can be expanded in a basis, constructed out of tensors \( g_{\mu \nu}, \epsilon_{\mu \nu \lambda \sigma} \), and vector \( k_\lambda \). Given the parity structure of the theory it is impossible to construct a tensor of rank two using \( g_{\mu \nu} \) and \( k_\mu \). So the only available tensor (with the right parity structure) we have at hand is \( \epsilon_{\mu \nu \lambda \sigma} \). The other vector needed to make it a tensor of rank two is \( k_\lambda \). As we contract \( \epsilon_{\mu \nu \lambda \sigma} \) with \( k_\lambda, k_\sigma \), since \( \epsilon_{\mu \nu \lambda \sigma} \) is completely antisymmetric tensor of rank four, the corresponding term vanishes.

On the other hand, in a medium, we have an additional vector \( u^\mu \), i.e. the velocity of the centre of mass of the medium. Therefore the polarisation tensor can be expanded in terms of form factors along with the new tensors constructed out of \( u^\mu \) and the ones we already had in absence of a medium as,

\[ \Pi_{\mu \nu}(k) = \Pi_T T_{\mu \nu} + \Pi_L L_{\mu \nu}. \] (13)

Here

\[ T_{\mu \nu} = g_{\mu \nu} - L_{\mu \nu} \] (14)
\[ L_{\mu \nu} = \frac{\tilde{u}_\mu u_\nu}{u^2} \] (15)

with

\[ \tilde{g}_{\mu \nu} = g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \] (17)
\[ \tilde{u}_\mu = g_{\mu \rho} u^\rho \] (18)

In the rest frame of the medium the four velocity is given by \( u^\mu = (1, 0, 0, 0) \). It is easy to see that the longitudinal projector \( L_{\mu \nu} \) is not zero in the limit \( k_0 = 0, \tilde{k} \to 0 \) and \( \Pi_L \) is also not zero in the above mentioned limit. This fact is responsible for giving nonzero contribution to the effective charge of neutrino.

As has already been mentioned, that in a medium, we have another extra four vector \( u^\mu \) and hence it is possible to construct the axial polarisation tensor of rank two, out of \( \epsilon_{\mu \nu \sigma \lambda}, u_\mu, k_\mu \), i.e, \( \epsilon_{\mu \nu \sigma \lambda} u^\sigma k_\lambda \), that would verify the Ward identity for the two point function. An explicit calculation of \( \Pi^{\mu \nu}_5(k) \) verifies the tensor structure of it as predicted here. It is worth noting that this contributes to the Cherenkov amplitude, but not to the effective electric charge of the neutrinos since for charge calculation we have to put the index \( \nu = 0 \). In the rest frame only \( u^0 \) exists, that forces the totally antisymmetric tensor to vanish.

In a constant background magnetic field in addition to the ones mentioned in Eq. (12) one has the freedom of having other extra vectors and tensors (to first order in field strength), such as
where $F$ along with the structure of axial polarisation tensor for odd powers in $n$ance with the general parity structure of the theory the presence of possible infrared divergence hence it doesn’t contribute to neutrino effective charge. The first term on the right hand side of Eq.(24) have already been discussed in [3], with the exception that the function $F_1$ now is a even function of external magnetic field $B$: on the other hand the appearence of the second term is new. One can also observe that, in keeping with CP invariance of the theory (i.e background along with the interaction), both the functions $F_1$ and $F_2$ should be odd functions of chemical potential. However this would become clear from Eq.(24) of section 3.

**IV. ONE LOOP CALCULATION OF THE AXIAL POLARISATION TENSOR**

Since we investigate the case with a background magnetic field, without any loss of generality it can be taken to be in the $z$-direction. We denote the magnitude of this field by $B$. Ignoring first the presence of the medium, the electron propagator in such a field can be written down following Schwinger’s approach [11,12,13]:

$$iS_B^{el}(p) = \int_0^\infty ds \, e^{\Phi(p,s)} G(p,s),$$

where $\Phi$ and $G$ are as given below

$$\Phi(p,s) = is \left( p_0^2 - \frac{\tan(eB s)}{eB s} \right) - \epsilon |s|,$$  

$$G(p,s) = \frac{e^{ieB q_z s}}{\cos(eBs)} \left[ \hat{p}_\parallel + \frac{e^{-ieB q_z s}}{\cos(eBs)} \hat{p}_\perp + m \right],$$

where

$$\sigma_z = i\gamma_1 \gamma_2 = -\gamma_0 \gamma_3 \gamma_5,$$

and we have used,

$$e^{ieB q_z s} = \cos(eBs) + is_z \sin(eBs).$$

To make the expressions transparent we specify our convention in the following way,

$$\hat{p}_\parallel = \gamma_0 p^0 + \gamma_3 p^3$$

$$\hat{p}_\perp = \gamma_1 p^1 + \gamma_2 p^2$$

$$p_0^2 = p_0^2 - p_3^2$$

$$p_\parallel^2 = p_\parallel^2 + p_\perp^2.$$
Of course in the range of integration indicated in Eq. (25) \( s \) is never negative and hence \(|s| \) equals \( s \). In the presence of a background medium, the above propagator is now modified to \([4] \):

\[
i S(p) = iS_B^\nu(p) + S_B^\nu(p),
\]

where

\[
S_B^\nu(p) = -\eta_F(p) \left[ iS_B^\nu(p) - iS_B^\nu(p) \right],
\]

and

\[
S_B^\nu(p) = \gamma_0 S_B^\nu(p) \gamma_0,
\]

for a fermion propagator, such that

\[
S_B^\nu(p) = -\eta_F(p) \int_{-\infty}^{\infty} ds \ e^{i(p,s)} G(p,s).
\]

Here \( \eta_F(p) \) contains the distribution function for the fermions and the anti-fermions:

\[
\eta_F(p) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta),
\]

where \( f_F \) denotes the Fermi-Dirac distribution function:

\[
f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u)} + 1},
\]

and \( \Theta \) is the step function given by:

\[
\Theta(x) = 1, \text{ for } x > 0, \\
0, \text{ for } x < 0.
\]

Here the four velocity of the medium is \( u \), in the rest frame it looks like \( u^\mu = (1, 0, 0, 0) \).

### A. The expression for \( \Pi^\nu_\mu \) in thermal medium and in the presence of a background uniform magnetic field

The axial polarisation tensor \( \Pi^\nu_\mu \) is expressed as

\[
i \Pi^\nu_\mu = (-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu \gamma_5 iS(p) \gamma_\nu \gamma iS(p') \right].
\]

Leaving out the vacuum contribution (the contribution devoid of any thermal effects) and the contributions with two thermal factors, we are left with

\[
i \Pi^\nu_\mu(k) = (-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu \gamma_5 S_B^\nu(p) \gamma_\nu iS_B^\nu(p') \right] + \gamma_\mu \gamma_5 S_B^\nu(p) \gamma_\nu S_B^\nu(p>').
\]

The vacuum part has already been done in \([3] \) and the thermal part is related with pure absorption effects in the medium, which we are leaving out for the time being.

Using the form of the fermion propagator in a magnetic field in presence of a thermal medium, as given by expressions \([23] \) and \([13] \) we get

\[
i \Pi^5_\mu(k) = -(-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds \ e^{i(p,s)} \left[ \text{Tr} \left[ \gamma_\mu \gamma_5 G(p,s) \gamma_\nu G(p', s') \right] \eta_F(p) \\
+ \text{Tr} \left[ \gamma_\mu \gamma_5 G(-p', s') \gamma_\nu G(-p, s) \right] \eta_F(-p) \right] \int_{-\infty}^{\infty} ds' \ e^{i(p', s')} R_\mu(p, p', s, s')
\]

where \( R_\mu(p, p', s, s') \) contains the trace part.

### B. \( R_\mu \) to even and odd orders in magnetic field

We calculate \( R_\mu(p, p', s, s') \) to even and odd orders in the external magnetic field and call them \( \Pi^{(e)}_\mu \) and \( \Pi^{(o)}_\mu \). The reason for doing this is that the two contributions have different properties as far as their dependence on medium is concerned, a topic which will be discussed in the concluding section. Calculating the traces we obtain,

\[
\Pi^{(e)}_\mu = 4i\eta_+(p) \left[ \epsilon_{\mu\nu\alpha\beta} p^{\alpha} p^{\beta} (1 + \tan(eBs)\tan(eBs')) \\
+ \epsilon_{\mu\nu\alpha\beta} p^{\alpha} p^{\beta} \sec^2(eBs) + \epsilon_{\mu\nu\alpha\beta} p^{\alpha} p^{\beta} \sec^2(eBs) \right]
\]

and

\[
\Pi^{(o)}_\mu = 4i\eta_+(p) \left[ m^2 \epsilon_{\mu\nu\alpha\beta} (\tan(eBs) + \tan(eBs')) \\
+ \left( (g_{\mu\nu}\tilde{p}_{\alpha}^\mu \tilde{p}_{\beta}^\nu - g_{\mu\nu} \tilde{p}_{\alpha}^\nu \tilde{p}_{\beta}^\mu) + g_{\nu\mu} \tilde{p}_{\alpha}^\nu \tilde{p}_{\beta}^\mu \right) \\
+ \left( (g_{\mu\nu}\tilde{p}_{\alpha}^\mu \tilde{p}_{\beta}^\nu + g_{\nu\mu} \tilde{p}_{\alpha}^\mu \tilde{p}_{\beta}^\nu) \sec^2(eBs) \right) \tan(eBs') \right].
\]

Here

\[
\eta_+(p) = \eta_F(p) + \eta_F(-p) \quad (41)
\]

\[
\eta_-(p) = \eta_F(p) - \eta_F(-p) \quad (42)
\]

which contain the information about the distribution functions. Also it should be noted that, in our convention

\[
a_\mu b^\mu = a_0 b^3 + a_3 b^0.
\]
If we concentrate on the rest frame of the medium, then $p \cdot u = p_0$. Thus the distribution function does not depend on the spatial components of $p$. In this case we can write the expressions of $R_{\mu\nu}^{(c)}$ and $R_{\mu\nu}^{(o)}$ using the relations derived earlier inside the integral sign, as

$$p_\perp^\alpha = \frac{-\tan(eB_\perp)}{\tan(eB_\perp) + \tan(eB_\perp')} k_\perp$$

$$p_\parallel^\beta = \frac{\tan(eB_\parallel)}{\tan(eB_\parallel) + \tan(eB_\parallel')} k_\parallel$$

$$p_\perp^2 = \frac{1}{\tan(eB_\perp) + \tan(eB_\perp')} \left[ -ieB \tan(eB_\perp)^2 k_\perp^2 \right]$$

$$p_\parallel^2 = \frac{1}{\tan(eB_\parallel) + \tan(eB_\parallel')} \left[ -ieB \tan(eB_\parallel)^2 k_\parallel^2 \right]$$

$$m^2 = \left( i \frac{d}{ds} + p_\parallel^2 - \sec^2(eB_\perp) p_\perp^2 \right)$$

and get

$$R_{\mu\nu}^{(c)} = \frac{4i\eta - (p_0)}{\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta (1 + \tan(eB_\parallel) \tan(eB_\parallel'))}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel')$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel)$$

and

$$R_{\mu\nu}^{(o)} = \frac{4i\eta - (p_0)}{\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta (1 + \tan(eB_\parallel) \tan(eB_\parallel'))}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel')$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel)$$

V. GUAGE INVARIANCE

A. Gauge invariance for $\Pi_{\mu\nu}^{(c)}$ to even orders in the external field

The axial polarization tensor even in the external field is given by

$$\Pi_{\mu\nu}^{(c)} = -(-ie)^2 (1 + \tan(eB_\parallel)) \int_0^\infty ds e^{\Phi(p,s)}$$

$$\times \int_0^\infty ds' e^{\Phi(p',s')} R_{\mu\nu}^{(c)}(p,p',s,s')$$

Using Eq.(48) in the rest frame of the medium, we have

$$R_{\mu\nu}^{(c)} = \frac{4i\eta - (p_0)}{\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta (1 + \tan(eB_\parallel) \tan(eB_\parallel'))}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel')$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel)$$

Noting that it is possible to write,

$$q^\alpha p_\alpha = q^\alpha q p_\alpha + q^\alpha p_\alpha$$

Eq.(51) can be written as,

$$R_{\mu\nu}^{(c)} = \frac{4i\eta - (p_0)}{\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta (1 + \tan(eB_\parallel) \tan(eB_\parallel'))}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel')$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta \sec^2(eB_\parallel)$$

Here throughout we have omitted terms such as $\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta$, since by the application of Eq.(8) we have

$$\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta = \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta + \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta$$

$$\frac{\tan(eB_\parallel')}{\tan(eB_\parallel') + \tan(eB_\parallel')} \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta$$

which is zero.

After rearranging the terms appearing in Eq.(52), and by the application of Eqs.(8) and (13), we arrive at the expression

$$R_{\mu\nu}^{(c)} = \frac{4i\eta - (p_0)}{\varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha p_\parallel^\beta (1 + \tan(eB_\parallel) \tan(eB_\parallel'))}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} p_\parallel^\alpha k_\parallel \tan(eB_\parallel') \frac{\tan(eB_\parallel) - \tan(eB_\parallel')}{\tan(eB_\parallel) + \tan(eB_\parallel')}$$

$$+ \varepsilon_{\mu\nu\alpha\beta} k_\parallel^\alpha k_\parallel^\beta$$

Because of the presence of terms like $\varepsilon_{\mu\nu\alpha\beta} k_\parallel^\beta$ and $\varepsilon_{\mu\nu\alpha\beta} k_\parallel^\alpha$ if we contract $R_{\mu\nu}^{(c)}$ by $k_\parallel$, it vanishes.
B. Gauge invariance for $\Pi_{\mu\nu}^{(o)}$ to odd orders in the external field

The axial polarization tensor odd in the external field is given by

$$
\Pi_{\mu\nu}^{(o)} = -(-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int \infty_{-\infty} ds \ e^{\Phi(p,s)}
\times \int \infty_{0} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(o)}(p, p', s, s')
$$

(54)

where $R_{\mu\nu}^{(o)}(p, p', s, s')$ is given by Eq.(49). The general gauge invariance condition in this case

$$
k^{\mu} \Pi_{\mu\nu}^{(o)} = 0
$$

(55)

can always be written down in terms of the following two equations,

$$
k^{\mu} \Pi_{\mu\nu}^{(o)} = 0
$$

(56)

$$
k^{\nu} \Pi_{\mu\nu}^{(o)} = 0
$$

(57)

where $\Pi_{\mu\nu}^{(o)}$ is that part of $\Pi_{\mu\nu}$ where the index $\mu$ can take the values 0 and 3 only. Similarly $\Pi_{\mu\nu}^{(o)}$ stands for the part of $\Pi_{\mu\nu}$ where $\mu$ can take the values 1 and 2 only. $\Pi_{\mu\nu}^{(o)}$ contains $R_{\mu\nu}^{(o)}(p, p', s, s')$ which from Eq.(49) is as follows,

$$
R_{\mu\nu}^{(o)} \equiv 4i\eta_{\nu}(p_{0}) \left[ -\varepsilon_{\mu\nu\rho\lambda} \right] \left( \frac{sec^2(eB\xi) tan^2(eB\xi')}{tan(eB\xi) + tan(eB\xi')} \right) k^\rho
$$

$$
+ \left( k \cdot p \right) \left( tan(eB\xi) + tan(eB\xi') \right)
$$

$$
+ 2\varepsilon_{\mu\rho\lambda} k_{\rho} \left( p'_{\rho} \xi + p'_{\rho} \xi' + p'_{\lambda} \xi + p'_{\lambda} \xi' \right) tan(eB\xi')
$$

$$
+ g_{\mu\nu\lambda} k_{\rho} \left( \frac{sec^2(eB\xi) tan^2(eB\xi')}{tan(eB\xi) + tan(eB\xi')} \right) tan(eB\xi')
$$

$$
+ g_{\mu\nu\lambda} (p \cdot \xi) \left( tan(eB\xi) - tan(eB\xi') \right)
$$

(58)

and $\Pi_{\mu\nu}^{(o)}$ contains $R_{\mu\nu}^{(o)}(p, p', s, s')$ which is

$$
R_{\mu\nu}^{(o)} \equiv 4i\eta_{\nu}(p_{0}) \left[ \left( g_{\mu\nu\lambda} (p \cdot \xi) + g_{\mu\nu\lambda} \xi \xi' \xi' \right) \right] \times \left( tan(eB\xi) - tan(eB\xi') \right)
$$

$$
+ g_{\mu\nu\lambda} k_{\rho} \left( sec^2(eB\xi) tan(eB\xi') \right).
$$

(59)

Eqs.(50),(57) implies one should have the following relations satisfied,

$$
k^{\mu} \int \frac{d^4p}{(2\pi)^4} \int \infty_{-\infty} ds e^{\Phi(p,s)} \int \infty_{0} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(o)} = 0
$$

(60)

and

$$
k^{\nu} \int \frac{d^4p}{(2\pi)^4} \int \infty_{-\infty} ds e^{\Phi(p,s)} \int \infty_{0} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(o)} = 0.
$$

(61)

Out of the two above equations, Eq.(50) can be verified easily since

$$
k^{\mu} R_{\mu\nu}^{(o)} = 0.
$$

(62)

Now we look at Eq.(51). We explicitly consider the case $\mu = 3$ (the $\mu = 0$ case lead to similar result). For $\mu = 3$

$$
k^{\mu} R_{\mu\nu}^{(o)} = -p_{0} \left[ \left( p'_{\parallel}^2 - p_{\parallel}^2 \right) tan(eB\xi) + tan(eB\xi') \right]
$$

$$
+ k_{\parallel}^2 \left( tan(eB\xi) - tan(eB\xi') \right) \left( 4i\eta_{\nu}(p_{0}) \right).
$$

(63)

Apart from the small convergence factors,

$$
\frac{i}{eB} \left( \Phi(p, s) + \Phi(p', s') \right)
$$

$$
= \left( p'_{\parallel}^2 + p_{\parallel}^2 - 2m^2 \right) \xi - \left( p'_{\parallel}^2 - p_{\parallel}^2 \right) \zeta
$$

$$
- p_{\parallel}^2 tan(\zeta - \zeta') - p_{\parallel}^2 tan(\zeta + \zeta'),
$$

(64)

where we have defined the parameters

$$
\xi = \frac{1}{2} eB(s + s'),
$$

$$
\zeta = \frac{1}{2} eB(s - s').
$$

(65)

From the last two equations we can write

$$
ieB \frac{d}{d\xi} e^{\Phi(p,s)} e^{\Phi(p',s')}
$$

$$
e^{\Phi(p,s)} e^{\Phi(p',s')}
$$

$$
\times \left( p'_{\parallel}^2 - p_{\parallel}^2 - p_{\parallel}^2 sec^2(\xi - \zeta) + p_{\parallel}^2 sec^2(\xi + \zeta) \right)
$$

(66)

which implies

$$
p_{\parallel}^2 - p_{\parallel}^2 =
$$

$$
ieB \frac{d}{d\xi} e^{\Phi(p,s)} e^{\Phi(p',s')}
$$

$$
\times \left( p'_{\parallel}^2 sec^2(eB\xi) - p_{\parallel}^2 sec^2(eB\xi') \right).
$$

(67)

The equation above is valid in the sense that both sides of it actually acts upon $e^{\Phi(p,s)}$, where

$$\Phi(p, p', s, s') = \Phi(p, s) + \Phi(p', s').$$

(68)

From Eqs.(53) and (57) we have

$$k^{\mu} R_{3\nu} e^{\Phi} = -4i\eta_{\nu}(p_{0}) \left( p'_{\parallel}^2 sec^2(eB\xi) - p_{\parallel}^2 sec^2(eB\xi') \right)
$$

$$\times \left( tan(eB\xi) + tan(eB\xi') \right)
$$

$$- k_{\parallel}^2 \left( tan(eB\xi) - tan(eB\xi') \right)
$$

$$+ ieBp_{0}(tan(eB\xi) + tan(eB\xi')) d_{\xi} e^{\Phi}.
$$

(69)
Now using the the expressions for $p_\perp^2$ and $p_\parallel^2$ from Eqs.(15) and (16) we can write
\[
k' R_{34} e^{\Phi_0} = 4eB\eta_+(p_0)p_0 \left[(\sec^2(eBS) - \sec^2(eBS')) \right.
\]
\[+ \left(\tan(eBS) + \tan(eBS')\right) \frac{d}{d\xi} e^{\Phi}.\]
(70)

The above equation can also be written as
\[
k' R_{34} e^{\Phi_0} = 4eB\eta_+(p_0)p_0 \frac{d}{d\xi} \left[e^{\Phi}(\tan(eBS) + \tan(eBS'))\right].
\]
(71)

Transforming to $\xi, \zeta$ variables and using the above equation we can write the parametric integrations (integrations over $s$ and $s'$) on the left hand side of Eq.(61) as
\[
\int_{-\infty}^{\infty} ds \int_{0}^{\infty} ds' k' R_{34} e^{\Phi_0}
\]
\[= \frac{8\eta_+(p_0)p_0}{eB} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta (\Theta(\xi - \zeta) \frac{d}{d\xi} F(\xi, \zeta))
\]
(72)

where
\[
F(\xi, \zeta) = e^{\Phi}(\tan(eBS) + \tan(eBS')).
\]

The integration over the $\xi$ and $\zeta$ variables in Eq.(72) can be represented as,
\[
\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\xi} F(\xi, \zeta)
\]
\[= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[\frac{d}{d\xi}(\Theta(\xi - \zeta)F(\xi, \zeta)) - \delta(\xi - \zeta)F(\xi, \zeta)\right]
\]
\[= - \int_{-\infty}^{\infty} d\xi F(\xi, \xi)
\]
(73)

here the second step follows from the first one as the first integrand containing the $\Theta$ function vanishes at both limits of the integration. The remaining integral is now only a function of $\xi$ and is even in $p_0$. But in Eq.(72) we have $\eta_+(p_0)p_0$ sitting, which makes the the integrand odd under $p_0$ integration in the left hand side of Eq.(61), as $\eta_+(p_0)$ is an even function in $p_0$. So the $p_0$ integral as it occurs in the left hand side of equation (57) vanishes as expected, yielding the required result shown in Eq.(73).

VI. EFFECTIVE CHARGE

Now we concentrate on the neutrino effective charge. From the onset it is to be made clear that we are only calculating the axial contribution to the effective charge. We can now write the full expression of the axial polarisation tensor as
\[
i\Pi^{\mu\nu}_5(k) = -(-i\epsilon)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)}
\]
\[\times \int_{0}^{\infty} ds' e^{\Phi(p',s')}(\tan(eBS) + \tan(eBS'))
\]
\[\times \eta_+(p_0) \left[2p_0^2 - (k \cdot p)\right] \varepsilon_{\mu012}
\]
(74)

where $\Pi^{\mu\nu}_5$ and $R^{\mu\nu}$ are given by Eqs.(49) and (38) in the rest frame of the medium.

A. Effective Charge to odd orders in external field

In the limit when the external momentum tends to zero only two terms survive from $\Pi^{\mu\nu}_5(k)$. Denoting $\Pi^{\mu\nu}_5(k_0 = 0, k \to 0) = \Pi^{\mu\nu}_5$, we obtain
\[
\Pi^{\mu0}_5 = \lim_{k_0 = 0 \to k \to 0} 4e^2 \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)}
\]
\[\times \int_{0}^{\infty} ds' e^{\Phi(p',s')}(\tan(eBS) + \tan(eBS'))
\]
\[\times \eta_+(p_0) \left[2p_0^2 - (k \cdot p)\right] \varepsilon_{\mu012}
\]
(75)

the other terms turns out to be zero in this limit. The above equation shows that, except the exponential functions, the integrand is free of the perpendicular components of momenta. This implies we can integrate out the perpendicular component of the loop momentum. Upon performing the gaussian integration over the perpendicular components and taking the limit $k_\perp \to 0$, we obtain,
\[
\Pi^{\mu0}_5 = \lim_{k_0 = 0, k_\perp \to 0} \frac{(-4i\epsilon^3B)}{4\pi} \int \frac{d^2p_\parallel}{(2\pi)^2} \int_{-\infty}^{\infty} ds e^{i\epsilon^2(p_\parallel^2 - m^2) - \epsilon|s|}
\]
\[\times \eta_+(p_0) \left[2p_0^2 - (k \cdot p)\right] \varepsilon_{\mu012}.
\]
(76)

It is worth noting that the $s$ integral gives
\[
\int_{-\infty}^{\infty} ds e^{i\epsilon^2(p_\parallel^2 - m^2)} = 2\pi \delta(p_\parallel^2 - m^2)
\]
(77)

and the $s'$ integral gives
\[
\int_{0}^{\infty} ds' e^{i\epsilon^2(p_\parallel^2 - m^2)} = \frac{i}{(p_\parallel^2 - m^2) + i\epsilon}.
\]
(78)

\*In a forthcoming publication we will comment on the vector contribution to the effective charge of the neutrino.
Using the above results in Eq.(76) and using the delta function constraint, we arrive at,

\[ \Pi_{\mu_0}^5 = \lim_{k_0=0, k \to 0} 2(e^3/2) \int \frac{d^2p||}{(2\pi)^2} \delta(p^2 - m^2) \eta_+(p_0) \]
\[ \times \left[ \frac{2p_0^2}{(k^2 + 2(p,k)_||)} - \frac{1}{2} \right] \varepsilon_{\mu 012}. \] \tag{79} \]

In deriving Eq.(76), pieces proportional to \( k^2 \) in the numerator were neglected. Now if one makes the substitution, \( p^2 \to (p^2 + k^2/2) \) and sets \( k_0 = 0 \) one arrives at,

\[ \Pi_{\mu_0}^5 = -\lim_{k_0=0, k \to 0} 2(e^3/2) \int \frac{dp}{(2\pi)^2} \left( n_+(E'_p) + n_-(E'_p) \right) \]
\[ \times \left[ E_p \left( \frac{1}{2E_p} \right) \right] \varepsilon_{\mu 012}. \] \tag{80} \]

Here \( n_+(E'_p) \) are the functions \( f_{E}(E'_p, -\mu, \beta) \), and \( f_{E}(E'_p, \mu, \beta) \), as given in Eq.(55), which are nothing but the Fermi-Dirac distribution functions of the particles and the antiparticles in the medium. The new term \( E'_p \) is defined as follows,

\[ E'_p = \left( p^2 - k^2/2 \right) + m^2, \]

and it can be expanded for small external momenta in the following way

\[ E'_p \approx p^2 + m^2 - p^2k^2 = E_p^2 - p^2k^2, \]

where \( E_p^2 = p^2 + m^2 \). Noting, that

\[ E'_p = E_p - \frac{p^2k^2}{2E_p} + O(k^4), \] \tag{81} \]

one can use this expansion in Eq.(80), to arrive at:

\[ \Pi_{\mu_0}^5 = -\lim_{k_0=0, k \to 0} 2(e^3/2) \int \frac{dp}{(2\pi)^2} \left( n_+(E'_p) + n_-(E'_p) \right) \]
\[ \times \left[ \frac{E_p}{p^2k^2} + \frac{1}{2E_p} \right] \varepsilon_{\mu 012}. \] \tag{82} \]

The expression for for \( \eta_+(E'_p) = n_+(E'_p) + n_-(E'_p) \) when expanded in powers of the external momentum \( k_3 \) is given by

\[ \eta_+(E'_p) = \left( 1 + \frac{1}{2} \frac{\beta p_3k_3}{E_p} \right) \eta_+(E_p) \] \tag{83} \]

up to first order terms in the external momentum \( k_3 \).

1. Effective Charge For \( \mu \ll m \)

In the limit, when \( \mu \ll m \) one can use the following expansion,

\[ \eta_+(E'_p) = \left[ n_+(E'_p) + n_-(E'_p) \right] \]
\[ = 2 \sum_{n=0}^{\infty} (-1)^n \cosh([n + 1]12/2) e^{-(n+1)12/2} \]
\[ \times \left( 1 + \frac{\beta p_3k_3}{2E_p} + O(k^2) + \ldots \right). \] \tag{84} \]

Now using Eq.(84) in Eq.(82) we get

\[ \Pi_{\mu_0}^5 = -\varepsilon_{\mu 012} \lim_{k_0=0, k \to 0} \frac{(4e^3/2) \sum_{n=0}^{\infty} (-1)^n \cosh([n + 1]12/2) e^{-(n+1)12/2} \cosh([n + 1]12/2) e^{-(n+1)12/2} \}
\[ \times \left( 1 + \frac{\beta p_3k_3}{2E_p} + O(k^2) + \ldots \right). \] \tag{85} \]

The first term vanishes by symmetry of the integral, but the second term is finite and so we get:

\[ \Pi_{\mu_0}^5 = -\beta \varepsilon_{\mu 012} \lim_{k_0=0, k \to 0} \frac{(4e^3/2) \sum_{n=0}^{\infty} (-1)^n \cosh([n + 1]12/2) e^{-(n+1)12/2} \cosh([n + 1]12/2) e^{-(n+1)12/2} \}
\[ \times \int dp \frac{E_p}{p^2k^2} + \frac{1}{2E_p} \right] \varepsilon_{\mu 012}. \] \tag{86} \]

To perform the momentum integration, use of the following limit form turns out to be extremely convenient

\[ e^{-\alpha \sqrt{s}} = \frac{e^{-\alpha \sqrt{s}}}{2\sqrt{s}} \int_0^{\infty} du \frac{e^{-u - e^{-\alpha^{2}/2}u^{3/2}}}{u^{3/2}}. \] \tag{87} \]

Identifying \( \sqrt{s} \) with \( E_p \) and \( [(n + 1)12/2] \) as \( \alpha \) (since the square root opens up), one can easily perform the gaussian \( k_3 \) integration without any difficulty. The result is:

\[ \Pi_{\mu_0}^5 = -\beta \varepsilon_{\mu 012} \frac{(4e^3/2) \sum_{n=0}^{\infty} (-1)^n \cosh([n + 1]12/2) e^{-(n+1)12/2} \cosh([n + 1]12/2) e^{-(n+1)12/2} \}
\[ \times \left( (\beta(n + 1)12/2) \right) \int du \frac{e^{-u - e^{-\alpha^{2}/2}u^{3/2}}}{u^{3/2}}. \] \tag{88} \]

Performing the \( u \) integration the axial part of the effective charge of neutrino in the limit of \( m > \mu \) turns out to be,

\[ e^{-\alpha \sqrt{s}} = \frac{\sqrt{2}g_A m}{} \left( 1 - \lambda \right) \cos(\theta) \]
\[ \times \sum_{n=0}^{\infty} (-1)^n \cosh((n + 1)12/2) K_{-1}(m\beta(n + 1)). \] \tag{89} \]
Here $\theta$ is the angle between the neutrino three momentum and the background magnetic field. The superscript $\nu_\alpha$ on $e'_{\nu_{\alpha}}$ denotes that we are calculating the axial contribution of the effective charge. $K_{-1}(m\beta(n+1))$ is the modified Bessel function (of the second kind) of order one (for this function $K_{-1}(x) = K_1(x)$) which sharply falls off as we move away from the origin in the positive direction. Although as temperature tends to zero Eq. (89) seems to blow up because of the presence of $m\beta$, but $K_{-1}(m\beta(n+1))$ would damp it’s growth as $e^{-m\beta}$, hence the result remains finite.

2. Effective Charge for, $\mu \gg m$

Here we would try to estimate neutrino effective charge when $\mu \gg m$ and $\beta \neq \infty$. We would like to emphasize that the last condition should be strictly followed, i.e., temperature $T \neq 0$. Using Eqs. (82) and (83) we would obtain

$$\Pi_{\mu\nu}^{\mu\nu} = \frac{e^3 B}{2\pi^3} \int \frac{dp}{2\pi} \eta_+(E_p).$$

Neglecting $m$ in the expression in $E_p$ we would obtain,

$$\Pi_{\mu\nu}^{\mu\nu} = \frac{e^3 B}{2\pi^3} \ln[(1 + e^{\beta\mu})(1 + e^{-\beta\mu})].$$

Same can also be written as

$$\Pi_{\mu\nu}^{\mu\nu} = \frac{e^3 B}{\pi^2} \ln \left( \frac{2 \cosh(\beta\mu)}{2} \right).$$

The expression for the effective charge then turns out to be

$$e'_{\nu_{\alpha}} = -\sqrt{2} g_A G_F \frac{e^2 B}{\pi^2} \ln \left( \frac{2 \cosh(\beta\mu)}{2} \right) (1 - \lambda) \cos(\theta)$$

where $\lambda$ is the helicity of the neutrino spinors.

B. Effective Charge At Even Order In The External Field And Coupling With Magnetic Fields

From the part of $\Pi_{\mu\nu}$ which is even in the external fields we see from Eq. (85) that

$$R^{(e)}_{\mu\nu} = 4i\eta_-(p_0) \left[ \varepsilon_{\mu_0\nu_\alpha} p^\alpha k^\beta (1 + \tan(eBs) \tan(eBs')) + \varepsilon_{\mu_0\nu_\alpha} k^\alpha k^\beta \tan(eBs) \tan(eBs') \right] \tan(eBs) - \tan(eBs') \tan(eBs) + \tan(eBs').$$

which shows that $\Pi_{\mu\nu}^{\mu\nu}(k)$ to even orders in the external field will vanish when $k_0 \rightarrow 0, \vec{k} \rightarrow 0$. This implies that there will be no contribution to the effective neutrino charge from the sector which is even in the powers of $B$.

Can the neutrinos which are propagating in a magnetised plasma couple with the classical magnetic field? The situation is a little bit subtle here, as the vertex of the neutrinos with the dynamical photons do get changed here due to the presence of the magnetic field, but this change cannot induce any electromagnetic form factor responsible for coupling of the neutrinos with any magnetic field. In order to find the effective charge of the neutrinos which couples them with time independent magnetic field, one should look for (as given in Eq. (5)), the $\Gamma^i$s, where $i = 1, 2$. A magnetic field in the $z$-direction, is given by a gauge where $A_1, A_2$ are both non zero, or one of them is nonzero. So to calculate the charge which is essential for the neutrino current to couple with a magnetic field, one has to put the index $\nu = 1, 2$ in Eq. (74) and take the limit $k_0 \rightarrow 0, \vec{k} \rightarrow 0$ and see which component of $\Pi_{\mu\nu}^{\mu\nu}(k)$ exists in the prementioned momentum limit. For the odd $B$ part we see from Eq. (83) that

$$R^{(o)}_{\mu\nu} = 4i\eta_+(p_0) \left[ g_{\mu\nu} k_1 \left\{ B^2 (\tan(eBs) - \tan(eBs')) \right\} - k^\alpha \tan^2(eBs') \tan(eBs) \right] + g_{\mu1} (p \cdot \vec{k}) \tan(eBs) - \tan(eBs)) \right]$$

which goes to zero as the photon momentum tends to zero. By the same argument it follows that for $\nu = 2$ there is vanishing contribution. Thus it shows that there is no effective magnetic coupling from $B$ odd part.

For the $B$ even part, as is seen from Eq. (85), that only $R^{(e)}_{12}$ survives, and is given by

$$R^{(e)}_{12} = 4i\eta_-(p_0) \left[ \varepsilon_{1203} (p \cdot \vec{k}) (1 + \tan(eBs) \tan(eBs')) \right],$$

which also perishes in the limit when the external momentum goes to zero. So from this we can say that $\Pi_{\mu\nu}^{\mu\nu}$ has no contribution for any charge of the neutrinos which can couple them with the magnetic field.

VII. CONCLUSION

In our analysis we have calculated the contributions to $\Pi_{\mu\nu}^{\mu\nu}(k)$ to odd and even orders in the external constant magnetic field. The main reason for doing so is the fact that, $\Pi_{\mu\nu}^{\mu\nu}(k)$ and $\Pi_{\mu\nu}^{\mu\nu}(k)$, the axial polarisation tensors to odd and even powers in $eB$, have different dependence on the background matter. Pieces proportional to even
powers in $B$ are proportional to $\eta_-(p_0)$, an odd function of the chemical potential. On the other hand pieces proportional to odd powers in $B$ depend on $\eta_+(p_0)$, and are even in $\mu$ and as a result it survives in the limit $\mu \to 0$. As has already been noted, this is a direct consequence on the charge and parity symmetries of the underlying theory.

In a background magnetic field the field dependence of the form factors, which are usually scalars, can be of the following form:

$$k^\mu F_{\mu\nu}F^{\nu\lambda}k^\lambda \quad \text{and} \quad F_{\mu\nu}F^{\mu\nu}. \quad (97)$$

These forms don’t exhaust all the possibilities, but whatever they are they must contain an even number of $F$’s and $k$’s and hence they will be even functions of $B$.

Of all possible tensorial structures for the axial polarisation tensor in a magnetised plasma, there exists one which satisfies the current conservation condition in the $\nu$ vertex. That is given by,

$$\phi_{\mu\nu} = \epsilon_{\mu\lambda\sigma} F^{\lambda\sigma} u^\alpha \left[ u_\nu - \frac{(k.u)k_\nu}{k^2} \right]. \quad (98)$$

It’s worth noting that the first term in the square bracket in Eq.(98), which is odd in the external field, survives in the zero external momentum limit in the rest frame of the medium. The tensorial structures which are explicitly even in powers of $B$ do have $k$’s also, and so they vanish in the limit when the external momentum goes to zero. We have earlier noted that the form factors which exist in the rest frame of the medium and in the zero momentum limit are even in powers of the external field. This tells us directly that the axial polarisation tensor must be odd in the $\nu$ vertex. That is given by,

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