Shocks and solitary waves in series connected discrete Josephson transmission lines

Eugene Kogan

1Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

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We analytically study shocks and solitary waves in the discrete Josephson transmission line (JTL), constructed from Josephson junctions (JJs) and capacitors. Our approach is based on the quasi-continuum approximation, which we discuss in details. The approximation allows to take into account the intrinsic dispersion in the discrete JTL. Such dispersion, in competition with the nonlinearity of the system, determines the profiles of the waves. We also study the effect of losses in the system. We find that the resistors, shunting the JJs and/or in series with the capacitors, make possible shock waves of more general type, than those existing in the lossless JTLs, and forbid solitary waves. We propose the integral approximation, which generalizes the quasi-continuum approximation.

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I. INTRODUCTION

The concept that in a nonlinear wave propagation system the various parts of the wave travel with different velocities, and that wave fronts (or tails) can sharpen into shock waves, is deeply imbedded in the classical theory of fluid dynamics.1 The methods developed in that field can be profitably used to study signal propagation in nonlinear transmission lines.2–11 In the early studies of shock waves in transmission lines, the origin of the nonlinearity was due to nonlinear capacitance in the circuit.12–14

Interesting and potentially important examples of nonlinear transmission lines are circuits containing Josephson junctions (JJs) - Josephson transmission lines (JTLs).15–19 The unique nonlinear properties of JTLs allow to construct soliton propagators, microwave oscillators, mixers, detectors, parametric amplifiers, and analog amplifiers.17–20

Transmission lines formed by JJs connected in series were studied beginning from 1990s, though much less than transmission lines formed by JJs connected in parallel.20 However, the former began to attract quite a lot of attention recently,21–25 especially in connection with possible JTL traveling wave parametric amplification.26–29

The interest in studies of discrete nonlinear electrical transmission lines, in particular of lossy nonlinear transmission lines, has started some time ago, but it became even more pronounced recently.22–25 These studies should be seen in the general context of waves in strongly nonlinear discrete systems.25–29

In our previous publication,30 we considered shock waves in the continuous JTLs with resistors, studying the influence of those on the shock profile. Now we want to analyse the structure of the shocks and solitary waves in the discrete JTLs, taking into account the intrinsic dispersion of the discrete JTL. Such dispersion is crucial when we study the shocks and solitary waves in the lossless JTLs.

The rest of the paper is constructed as follows. In Section II we formulate quasi-continuum approximation for the equations describing discrete lossless JTL. In Section III for the case of a running wave, the problem is reduced to an effective mechanical problem, describing motion of an effective particle. In Section IV the velocity and the profile of the shock wave, and in Section V - of the solitary wave, are found from the solution of the effective mechanical problem. In Section VI we rigorously justify the quasi-continuum approximation for the shock and solitary waves in certain limiting cases. In Section VII we discuss the effect of the resistors on the shock and solitary waves propagation in the discrete JTL. In Section VIII we check up the quasi-continuum approximation by applying it to the linear discrete transmission line. In Section IX we propose the integral approximation to the discrete equations. Application of the results obtained in the paper and opportunities for their generalization are very briefly discussed in Section X. We conclude in Section XI. In the Appendix we we compare the quasi-continuum approximation with another approximation, developed by P. Rosenau some time ago.

II. THE QUASI-CONTINUUM APPROXIMATION

Consider the model of JTL constructed from identical JJs and capacitors, which is shown on Fig. 1. We take as dynamical variables the phase differences (which we for brevity will call just phases) $\varphi_n$ across the JJs and the charges $q_n$ which have passed through the JJs. The circuit equations are

\begin{align}
\frac{\hbar}{2e} \frac{d\varphi_n}{dt} &= \frac{1}{C} (q_{n+1} - 2q_n + q_{n-1}), \\
\frac{dq_n}{dt} &= I_c \sin \varphi_n,
\end{align}

where $C$ is the capacitor, and $I_c$ is the critical current of the JJ. Differentiating Eq. (1a) with respect to $t$ and substituting $dq_n/dt$ from Eq. (1b), we obtain closed equation

\begin{align}
C \frac{d^2 \varphi_n}{dt^2} &= -\frac{\hbar}{2e} \frac{d^2 \varphi_n}{dt^2} + \frac{1}{2C} (q_{n+1} - 2q_n + q_{n-1}), \\
\frac{dq_n}{dt} &= I_c \sin \varphi_n,
\end{align}
for \( \varphi_n \):

\[
\frac{d^2 \varphi_n}{d\tau^2} = \sin \varphi_{n+1} - 2 \sin \varphi_n + \sin \varphi_{n-1},
\]

(2)

where \( L_J = h/(2eI_c) \), and we have introduced the dimensionless time \( \tau = t/\sqrt{L_JC} \). It is interesting to compare Eq. (2) with a discrete sine-Gordon equation for lattice wave field\(^{15}\)

\[
\frac{d^2 \varphi_n}{d\tau^2} - D(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}) + \sin \varphi = 0,
\]

where \( D \) is some constant, and a sine-lattice discrete double sine-Gordon equation\(^{12}\)

\[
\frac{d^2 u_n}{d\tau^2} - \sin (u_{n+1} - u_n) + \sin (u_n - u_{n-1})
= g (-\sin u_{n-1} + \eta \sin 2u_n),
\]

where \( g \) and \( \eta \) are some constants.

Treating \( \varphi \) in Eq. (2) as a function of the continuous variable \( z \), we can write down Eq. (2) symbolically as

\[
\frac{\partial^2 \varphi}{\partial \tau^2} = 2 \sum_{m=1}^{\Lambda} \frac{\partial^{2m} \sin \varphi}{\partial z^{2m}},
\]

(3)

where \( \Lambda \) is the period of the line. The question how many terms can and should be kept in the sum in the r.h.s. of Eq. (3) is far from being trivial. If we keep only a single term, we obtain the continuum approximation, which attracts by its simplicity. However the phenomena we’ll be talking about are absent in this approximation. Jumping ahead, we state that the width of shocks and solitary waves in this approximation will be equal to zero. So we make the next simplest assumption, by truncating the sum after the first two terms. In such case we obtain a solvable equation for \( \varphi(z,t) \) in the form

\[
\frac{\partial^2 \varphi}{\partial \tau^2} = \frac{\partial^2 \sin \varphi}{\partial z^2} + \frac{\Lambda^2}{12} \frac{\partial^4 \sin \varphi}{\partial z^4},
\]

(4)

thus reducing the original system of ordinary differential equations (1) to a single partial differential equation. We will call such truncation the quasi-continuum approximation. We’ll see later that in certain limiting cases we can rigorously justify the quasi-continuum approximation.

### III. Newtonian Equation

The running wave solution of Eq. (4) is of the form

\[
\varphi(z,t) = \varphi(x),
\]

(5)

where \( x = Ut - z \) (for the sake of definiteness, everywhere in this paper we’ll consider \( V \) to be positive). For the function \( \varphi(x) \) we obtain an ordinary differential equation

\[
\frac{\Lambda^2}{12} \frac{d^4 \sin \varphi}{dx^4} + \frac{d^2 \sin \varphi}{dx^2} - 2 \frac{d^2 \varphi}{dx^2} = 0,
\]

(6)

where \( U \) is the velocity of propagation of small amplitude smooth disturbances of \( \varphi \) on a zero background. Integrating Eq. (6) with respect to \( x \) twice, we obtain

\[
\frac{\Lambda^2}{12} \frac{d^2 \sin \varphi}{dx^2} = -\sin \varphi + U^2 \varphi + F,
\]

(7)

where \( F \) is the constant of integration. The other constant of integration is equal to zero for the problems we are interested in (see below). Note that Eq. (7) can be considered as the balance between the dispersion effects, described by the l.h.s. of the equation, and the nonlinear effects described by the (nonlinear terms of the) r.h.s. of the equation.

Multiplying Eq. (7) by \( d \sin \varphi/dx \) and integrating once again we obtain

\[
\frac{\Lambda^2}{24} \left( \frac{d \sin \varphi}{dx} \right)^2 + \Pi(\sin \varphi) = E,
\]

(8)

where

\[
\Pi(\sin \varphi) = \frac{1}{2} \sin^2 \varphi - U^2 (\varphi \sin \varphi + \cos \varphi) - F \sin \varphi,
\]

(9)

and \( E \) is another constant of integration. Equation (8) can be integrated in quadratures in the general case.

We are interested in the propagation of the waves characterized by the boundary conditions

\[
\lim_{x \to -\infty} \varphi = \varphi_2, \quad \lim_{x \to +\infty} \varphi = \varphi_1
\]

(10)

(this explains presence of only one integration constant in Eq. (7)). We can think about \( x \) as time and about \( \sin \varphi \) as coordinate of the fictitious particle, thus considering (7) as the Newtonian equation. The problem of finding the profile of the wave is reduced to studying the motion of the particle which starts from an equilibrium position, and ends in an equilibrium position.

Using the expertise we acquired in mechanics classes, we come to the conclusion that the initial position corresponds to maxima of the “potential energy” \( \Pi(\sin \varphi) \), and so does the final position. Either these are two different maxima, or the same maximum. In the latter case the particle returns to the initial position after reflection from
a potential wall. (See Figs. 2 (above) and 3 (above).) In the first case the solution describes the shock wave, in the second - the solitary wave.

Introducing the term "shock wave" we actually jumped ahead and took into account that the length scale at which \( \phi \) changes substantially for the solution in question will be shown to be of order of \( \Lambda \), which is a "microscopical" length in our case. (The same will be true for the solitary wave.)

**IV. THE SHOCK WAVE**

In the case of the shock wave, going in Eq. (7) to the limits \( x \to +\infty \) and \( x \to -\infty \), we obtain

\[
U^2 \varphi_1 = \sin \varphi_1 - F , \tag{11a}
\]
\[
U^2 \varphi_2 = \sin \varphi_2 - F . \tag{11b}
\]

Solving \( \text{(11)} \) relative to \( U^2 \) and \( F \) we obtain\[3
\]

\[
U^2 = U_{sh}^2 (\varphi_1, \varphi_2) \equiv \frac{\sin \varphi_2 - \sin \varphi_1}{\varphi_2 - \varphi_1} , \tag{12a}
\]
\[
F = \frac{\varphi_2 \sin \varphi_1 - \varphi_1 \sin \varphi_2}{\varphi_2 - \varphi_1} . \tag{12b}
\]

Hence \( \text{(7)} \) can be written down as

\[
\frac{\Lambda^2 d^2 \sin \varphi}{12 dx^2} = G(\varphi) , \tag{13}
\]

where

\[
G(\varphi) \equiv \frac{(\varphi - \varphi_1)(\varphi_2 - \varphi)}{\varphi_1 - \varphi_2} \left[ U_{sh}^2 (\varphi_1, \varphi) - U_{sh}^2 (\varphi, \varphi_2) \right] . \tag{14}
\]

Adding to \( \text{(12)} \) the equation

\[
E = \Pi (\sin \varphi_1) = \Pi (\sin \varphi_2) \tag{15}
\]

and solving the obtained system we obtain

\[
\varphi_2 = -\varphi_1 , \tag{16a}
\]
\[
F = 0 , \tag{16b}
\]
\[
U^2 = U_{sh}^2 (\varphi_1, -\varphi_1) = \frac{\sin \varphi_1}{\varphi_1} , \tag{16c}
\]
\[
\Pi (\sin \varphi) - E = \frac{1}{2} (\sin \varphi - \sin \varphi_1)^2 - \frac{\sin \varphi_1}{\varphi_1} \left[ (\varphi - \varphi_1) \sin \varphi + \cos \varphi - \cos \varphi_1 \right] . \tag{16d}
\]

Equation \( \text{(16d)} \) and the results of integration of Eq. \( \text{8} \) for this "potential energy" are graphically presented on Fig. 2 (above).

Consider specifically the limiting case \( |\varphi_1| \ll 1 \). Expanding the "potential energy" with respect to \( \varphi \) and \( \varphi_1 \)

and keeping only the lowest order terms we obtain the approximation to Eq. \( \text{8} \) in the form

\[
\Lambda^2 \left( \frac{d\varphi}{dx} \right)^2 = (\varphi_1^2 - \varphi^2)^2 . \tag{17}
\]

The solution of Eq. \( \text{17} \) is

\[
\varphi(x) = |\varphi_1| \tan \frac{\varphi_1 x}{\Lambda} . \tag{18}
\]

Equations \( \text{18} \) coincides with that obtained by Katayama et al. \( \text{[10]} \). So does Eq. \( \text{16c} \), being expanded in series with respect to \( \varphi_1 \) and truncated after the first two terms: \( U_{sh}^2 (\varphi_1 - \varphi_1) = 1 - \frac{\varphi_1^2}{6} \).

**V. THE SOLITARY WAVE**

For a solitary wave \( \varphi_2 = \varphi_1 \), and two equations of \( \text{11} \) become one equation. As an additional parameter we take the amplitude of the solitary wave (maximally different from \( \varphi_1 \) value of \( \varphi \)), which we will designate as \( \varphi_0 \). Adding to \( \text{11} \) the equation

\[
E = \Pi (\sin \varphi_0) = \Pi (\sin \varphi_1) \tag{19}
\]

and solving the obtained system we obtain

\[
U_{sol}^2 (\varphi_0, \varphi_1) = \frac{(\sin \varphi_0 - \sin \varphi_1)^2}{2 [ (\varphi_0 - \varphi_1) \sin \varphi_0 + \cos \varphi_0 - \cos \varphi_1 ]} , \tag{20a}
\]
\[
\Pi (\sin \varphi) - E = \frac{1}{2} (\sin \varphi - \sin \varphi_1)^2 - U_{sol}^2 (\varphi_0, \varphi_1) \left[ (\varphi - \varphi_1) \sin \varphi + \cos \varphi - \cos \varphi_1 \right] . \tag{20b}
\]
Note, that the demand, that the denominator in the r.h.s. of Eq. (20a) should be greater than zero, visualizes the condition \( \cos \varphi_0 > \cos \varphi_1 \). Equation (20b) is graphically presented on Fig. 3 (above).

Considering the limiting case \(|\varphi_1|, |\varphi_0| \ll 1\), expanding Eq. (20b) with respect to all the phases and keeping only the lowest order terms we obtain Eq. (21) in the form

\[
\Lambda^2 \left( \frac{d\varphi}{dx} \right)^2 = (\varphi - \varphi_1)^2 \left( \varphi + (2\varphi_1 + \varphi_0) \right). \tag{21}
\]

Equation (21) can be integrated in elementary functions and we obtain

\[
\sqrt{\frac{(3\varphi_1 + \varphi_0)(\varphi - \varphi_0)}{(\varphi + 2\varphi_1 + \varphi_0)(\varphi_1 - \varphi_0)}} \approx \tanh \sqrt{\frac{(3\varphi_1 + \varphi_0)(\varphi_1 - \varphi_0)}{2\Lambda}}. \tag{22}
\]

Equation (22) is graphically presented on Fig. 3 (below).

- FIG. 3: The "potential energy" (20b) (above) and the solitary wave profile according to Eq. (22) (below). We have chosen \( \varphi_1 = -1.0 \) and \( \varphi_0 = -0.5\varphi_1 \).

In an another limiting case of weak solitary wave (\(|\varphi_1 - \varphi_0| \ll |\varphi_1|\)), Eq. (22) takes the form

\[
\Lambda^2 \left( \frac{d\varphi}{dx} \right)^2 = 4 \tan \varphi_1 \cdot (\varphi - \varphi_1)^2 \left( \varphi + \varphi_0 \right). \tag{23}
\]

The solution of Eq. (23) is

\[
\varphi - \varphi_0 \approx \varphi_1 = \tanh \frac{\sqrt{\tan \varphi_1 \cdot (\varphi_1 - \varphi_0)|x|}}{\Lambda}. \tag{24}
\]

Note that Eq. (24), for \(|\varphi_1| \ll 1\), coincides with Eq. (22), for \( \varphi \approx \varphi_0 \approx \varphi_1 \).

### VI. THE CONTROLLED QUASI-CONTINUUM APPROXIMATION

Let us return to Eq. (2). Looking at Eqs. (18) and (22) we realize with hindsight, that in the description of shock and solitary waves with \(|\varphi_1| \ll 1\), the expansion parameter in the r.h.s. of Eq. (3) is \( \varphi_1 \); thus the quasi-continuum approximation (4) can be rigorously justified. However, strictly speaking, the truncation of the expansion should be performed in accordance with the truncation of the series expansion of the sine function, and Eq. (4), in the consistent approximation should be written as

\[
\frac{1}{\Lambda^2} \frac{\partial^2 \psi}{\partial \tau^2} = \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{6} \frac{\partial^2 \varphi^3}{\partial x^2} + \frac{\Lambda^2}{12} \frac{\partial^4 \varphi}{\partial x^4}. \tag{25}
\]

Note that both (18) and (22) are exact solutions of Eq. (25).

Looking at Eq. (24) we realize that alternatively, the quasi-continuum approximation can be rigorously justified when it is applied to the description of the solitary wave with \(|\varphi_1 - \varphi_0| \ll 1\). The expansion parameter in the r.h.s. of Eq. (3) in this case is \( \sin \varphi_1 \cdot (\varphi_1 - \varphi_0) \). Again, strictly speaking, truncation of the expansion should be performed in accordance with the truncation of the series expansion of the sine function, and Eq. (4) in this case in the consistent approximation should be written as

\[
\frac{1}{\Lambda^2} \frac{\partial^2 \psi}{\partial \tau^2} = \cos \varphi_1 \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2} \sin \varphi_1 \left( \frac{\partial^2 \varphi^2}{\partial x^2} + \cos \varphi_1 \frac{\Lambda^2}{12} \frac{\partial^4 \psi}{\partial x^4} \right). \tag{26}
\]

where \( \psi = \varphi - \varphi_1 \). Note that (24) is an exact solution of Eq. (26).

Here we would like to attract the attention of the reader to the following issue. Common wisdom says that the continuum approximation and the small amplitude approximation are independent - there could be a wave with small amplitude, which allows to expand the sine function, but which varies fast in space (wavelength comparable to lattice spacing), so the continuum limit is not justified. And there could be the opposite situation (large amplitude, long wavelength), in which the sine needs to be retained but the continuum limit is allowed.

However, for the shocks and solitary waves these approximations are not independent. Parametrically, the length scale of the waves is of the order of the lattice spacing \( \Lambda \), so, naively, the continuum (or even the quasi-continuum) limit is not justified. What we have shown above, is that for the waves with small amplitude \(|\varphi_1|\) (\( \tan \varphi_1 (\varphi_1 - \varphi_0) \)), the length scale is \( \Lambda^2/|\varphi_1| \) (\( \Lambda/(\tan \varphi_1 (\varphi_1 - \varphi_0)) \)), thus justifying the quasi-continuum approximation.
VII. THE EFFECT OF THE RESISTORS

Consider JTL with the capacitor and resistor shunting the JJ and another resistor in series with the ground capacitor, shown on Fig. 4. As the result, Eq. (4) changes to

\[ \frac{h}{2e} \frac{d\varphi_n}{dt} = \left( \frac{1}{C} + R \frac{\partial}{\partial t} \right) \left( q_{n+1} - 2q_n + q_{n-1} \right) , \quad (27a) \]

\[ \frac{dq_n}{dt} = I_c \sin \varphi_n + \frac{h}{2eR_J} \frac{d\varphi_n}{dt} + C_J \frac{h}{2e} \frac{d^2\varphi_n}{dt^2} , \quad (27b) \]

where \( R \) is the ohmic resistor in series with the ground capacitor, and \( C_J \) and \( R_J \) are the capacitor and the ohmic resistor shunting the JJ. Equation (27b) takes into account that the current passes through three parallel branches. Equation (4) changes to

\[ \frac{J_L C_J}{\Lambda^2} \frac{\partial^2 \varphi}{\partial t^2} = \left( 1 + RC \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 \varphi}{\partial z^2} + \frac{\Lambda^2}{12} \frac{\partial^4 \varphi}{\partial t^4} \right) \cdot \left( \sin \varphi + \frac{L_J}{R_J} \frac{\partial \varphi}{\partial t} + L_J C_J \frac{\partial^2 \varphi}{\partial t^2} \right) . \quad (28) \]

Opening the parenthesis in Eq. (28) and considering the running wave solutions we obtain

\[ \frac{1}{12U^2} \frac{d^2 \sin \varphi}{d\tau^2} + \left( \frac{C_J}{C} + \frac{R}{R_J} \right) \frac{d^2 \varphi}{d\tau^2} + (\gamma_1 \cos \varphi + \gamma_2) \frac{d\varphi}{d\tau} = -\sin \varphi + \frac{1}{U^2} \varphi + F ; \quad (29) \]

we have introduced the dimensionless "friction coefficients”

\[ \gamma_1 = R/Z_J , \quad \gamma_2 = Z_J/R_J , \quad (30) \]

where \( Z_J = \sqrt{L_J/C} \) is the characteristic impedance of the JTL, and discarded the terms with the derivatives higher than of the forth order.

We impose the boundary conditions (10) and try to understand what part of the analysis of Section IV can be transferred to the present case. The results (11) are determined only by the r.h.s. of Eq. (7), so is (12), following from (11). Since the r.h.s. of Eqs. (7) and (29) are identical, (12) is valid in the present case also. Hence the r.h.s. of Eq. (29) is identical to the r.h.s. of Eq. (13). And comparing the l.h.s. of Eqs. (7) and (29) we understand that in comparison to the lossless case, ohmic resistors make the shocks broader, and so does the capacitor parallel to JJ.

A. The continuum approximation

Looking at Eqs. (29) and (30) we understand, that when \( C_J \) is large enough and/or \( R_J \) is small enough and/or \( R \) is large enough, the first term in the l.h.s. of (29) can be discarded. Actually, the physical meaning and the relevance of the resistor in series with the ground capacitor is not obvious. We included it because we were able to do it for free. The reader who does not like the idea of such resistor may mentally put \( R = 0 \) in all equations of this Section. In simple terms the discrete nature of the JTL becomes not important when the JJ is shunted strong enough. In this case the equation acquires Newtonian form (31)

\[ \left( \frac{C_J}{C} + \frac{R}{R_J} \right) \frac{d^2 \varphi}{d\tau^2} + (\gamma_1 \cos \varphi + \gamma_2) \frac{d\varphi}{d\tau} = G(\varphi) \equiv -\frac{d\Pi(\varphi)}{d\varphi} , \quad (31) \]

where

\[ \Pi(\varphi) = -\cos \varphi \frac{1}{2} \left( \varphi_1 - \varphi_2 \right) \varphi \left( \varphi_1 - \varphi_2 \right) \varphi - \varphi_1 \sin \varphi_2 \varphi_1 - \varphi_2 \sin \varphi_1 \varphi_1 - \varphi_2 \varphi . \quad (32) \]

Using our mechanics expertise, we realize that the resistors, by introducing the effective "friction force", break the "energy" conservation law (3). Absence of the "energy" conservation makes possible the particle motion connecting "potential energy" maximum with a "potential energy" minimum. Hence Eq. (16a) is no longer valid, which allows in lossy JTL the shocks of more general type than in the lossless JTL of Section IV that is with \( \varphi_2 \neq \varphi_1 \). And Eq. (12a) solves the problem of shock velocity in such general case. On the other hand, the solitary waves (in the sense \( \varphi_2 = \varphi_1 \)) are no longer possible for Eq. (31).

The shocks described by Eq. (31) were studied in our previous publication [11]. In simple particular case when \( C_J = 0, R = 0, \) Eq. (31) becomes

\[ \frac{Z_J}{R_J} \frac{d\varphi}{d\tau} = G(\varphi) . \quad (33) \]

If we restore \( U \) in the r.h.s. of Eq. (31), it would be similar to that describing the motion of a fluxon (a shock...
wave of the phase) in biased long JTL. For weak shock $(|\varphi_2 - \varphi_1| << |\varphi_1|)$ we can expand the r.h.s. of Eq. (33) to obtain

$$\frac{Z_J d\psi}{R_J d\tau} = \Delta \varphi \sin \varphi \left( \Delta^2 \varphi - \psi^2 \right), \quad (34)$$

where $\varphi = (\varphi_1 + \varphi_2)/2$, $\Delta \varphi = (\varphi_1 - \varphi_2)/2$, and $\psi = \varphi - \varphi$, with the obvious solution

$$\psi = \Delta \varphi \tanh \alpha \tau, \quad \alpha = \frac{R_J}{Z_J} |\Delta \varphi| \sin \varphi. \quad (35)$$

B. The quasi-continuum approximation

In the case when all three terms in the r.h.s. of Eq. (29) are comparable, it is convenient to go from $\varphi$ to a new variable

$$\bar{\varphi} = \varphi + \frac{1}{12 U^2} \left( \frac{C_J}{C} + \frac{R}{R_J} \right) \sin \varphi. \quad (36)$$

Equation (36) defines $\varphi$ an an implicit function of $\bar{\varphi}$. For our purposes would be enough to know that such function exists. We will not need its explicit form. So in the new variables, Eq. (29) takes Newtonian form

$$\frac{d^2 \bar{\varphi}}{dt^2} + K(\bar{\varphi}) \frac{d\bar{\varphi}}{dt} = G(\varphi) \equiv -\frac{d \Pi(\bar{\varphi})}{d\bar{\varphi}}. \quad (37)$$

We will not use in this Section the expression for the "friction coefficient" $K(\bar{\varphi})$, just the fact, which can be easily checked up, that it is strictly positive. Newtonian form being achieved, the paragraph from the previous Subsection, starting from "Using our mechanics expertise . . . " can be repeated verbatim.

In distinction from the lossless case, Eq. (37) can not be integrated in quadratures (same as Eq. (31)). But one important feature of shocks can be understood without numerical integration, and even without calculating $\Pi(\bar{\varphi})$. We are talking about direction of shock propagation. In fact, looking at the r.h.s. of (13) we realize that the "potential energy" $\Pi(\bar{\varphi})$ is symmetric with respect to interchange of $\varphi_1$ and $\varphi_2$, so is $\Pi_{sh}(\varphi_1, \varphi_2)$. However, the motion of the fictitious particle is possible only if $\bar{\varphi}_2$ is a local maximum of the "potential energy" $\Pi(\bar{\varphi})$, and $\bar{\varphi}_1$ is a local minimum. That maximum/minimum distinction means that

$$\frac{d^2 \Pi(\bar{\varphi})}{d\bar{\varphi}^2} \bigg|_{\bar{\varphi}=\bar{\varphi}_2} < 0, \quad \frac{d^2 \Pi(\bar{\varphi})}{d\bar{\varphi}^2} \bigg|_{\bar{\varphi}=\bar{\varphi}_1} > 0. \quad (38)$$

Taking into account that $d\varphi/d\bar{\varphi} > 0$, we can rewrite the inequalities (38) as

$$\frac{dG(\varphi)}{d\varphi} \bigg|_{\varphi=\varphi_2} < 0, \quad \frac{dG(\varphi)}{d\varphi} \bigg|_{\varphi=\varphi_1} > 0. \quad (39)$$

Taking additionally into account that

$$\mathcal{U}^2_{sh}(\varphi_1, \varphi_1) = \cos \varphi_1 = \pi^2(\varphi_1), \quad (40)$$

where $\pi = u^2 L J / A^2$, and $u(\varphi_1)$ is the velocity of propagation along the lossless JTL of small amplitude smooth disturbances of $\varphi$ on a homogeneous background $\varphi_1$, we can present the inequalities (39) as

$$\pi^2(\varphi_1) > \mathcal{U}^2_{sh}(\varphi_1, \varphi_2) > \pi^2(\varphi_2), \quad (41)$$

thus establishing the connection with the well known in the nonlinear waves theory fact: the shock velocity is lower than the sound velocity in the region behind the shock, but higher than the sound velocity in the region before the shock.

Suppose we have two different asymptotic phases on the ends of the JTL, $\varphi_1$ and $\varphi_2$. The inequality $\pi^2(\varphi_1) > \pi^2(\varphi_2)$ obviously allows only one direction of propagation—from smaller $\cos \varphi$ to larger $\cos \varphi$. In our previous publication (11) same as in the present one) we considered only the solutions which lie completely in the sector $(-\pi/2, \pi/2)$. We have shown that if the phases $\varphi_1, \varphi_2$ have the same sign, the central part of (11) is satisfied automatically. The central part of (11) does bring limitations on the values of positive and negative phases, between which a shock can exist, but not very restrictive.

The shocks with the velocity much less than the velocity of small and smooth perturbations on zero background ($\mathcal{U}^2_{sh} \ll 1$) are of particular interest. Taking $\mathcal{U}_{sh}$ as the parameter, we derive for such shocks the condition

$$\frac{\pi}{2} > \varphi_2 > \varphi_1 > \frac{\pi}{2} - \mathcal{U}^2_{sh}. \quad (42)$$

VIII. THE QUASI-CONTINUUM APPROXIMATION VS. THE EXACT SOLUTION

In this Section we consider the discrete linear transmission line, obtained from that presented on Fig. 1 by substituting linear inductor with the inductance $L$ for the JJ. The line is described by the equation

$$\frac{d^2 q_n(\tau)}{d\tau^2} = q_{n+1}(\tau) - 2q_n(\tau) + q_{n-1}(\tau), \quad (43)$$

where we have introduced the dimensionless time $\tau = t/\sqrt{LC}$. Because the system is linear (but dispersive), it does not allow either shocks or solitary waves, and thus seems to lie outside the scope of the paper. However, we will use the system to check up the (analogue of) Eq. (4) by comparing the exact and the approximate solutions for the propagator.
A. The exact solution

We define the propagator by the initial and the boundary conditions

\[ q_n(0) = \delta_n0, \quad \dot{q}_n(0) = 0, \quad (44a) \]

\[ \lim_{n \to \pm \infty} q_n = 0. \quad (44b) \]

Recalling the recurrence relation satisfied by Bessel functions, we obtain

\[ 2 \frac{dZ_n(\tau)}{d\tau} = Z_{n-1}(\tau) - Z_{n+1}(\tau), \quad (45) \]

where \( Z \) is any Bessel function, and repeating it twice we obtain

\[ 4 \frac{d^2 Z_n(\tau)}{d\tau^2} = Z_{n+2}(\tau) - 2Z_n(\tau) + Z_{n-2}(\tau). \quad (46) \]

Comparing (46) with (44), we obtain plausible solution for half of the problem. This solution for even \( n \) is

\[ q_n(\tau) = J_{2n}(2\tau), \quad (47) \]

where \( J_n \) is the Bessel function of the first kind.

To obtain a rigorous solution (and for the whole problem) we use Laplace transformation

\[ Q_n(s) = \int_0^\infty d\tau e^{-s\tau} q_n(\tau). \quad (48) \]

For \( Q_n(s) \) we obtain the difference equation

\[ Q_{n+1}(s) - (2 + s^2)Q_n(s) + Q_{n-1}(s) = -s\delta_n0. \quad (49) \]

Solving (49) we get

\[ Q_n(s) = \frac{1}{\sqrt{s^2 + 4}} \left( \frac{\sqrt{s^2 + 4} - s}{2} \right)^{2|n|}. \quad (50) \]

Taking into account the inverse Laplace transform correspondence tables, we obtain Eq. (47) for all \( n \).

Closely connected with the problem for the semi-infinite transmission line, which is characterized by the same equation (43) for \( n \geq 1 \) with the initial and the boundary conditions

\[ q_n(0) = \dot{q}_n(0) = 0, \quad (51a) \]

\[ q_0(\tau) = \delta(\tau), \quad \lim_{n \to +\infty} q_n(\tau) = 0. \quad (51b) \]

For brevity we will call such solution the signal.

After Laplace transformation we obtain difference equation

\[ Q_{n+1}(s) - (2 + s^2)Q_n(s) + Q_{n-1}(s) = 0 \quad (52) \]

with the boundary conditions

\[ Q_0(s) = 1, \quad \lim_{n \to +\infty} Q_n(\tau) = 0. \quad (53) \]

Solving (52) we get

\[ Q_n(s) = \left( \frac{\sqrt{s^2 + 4} - s}{2} \right)^{2n}. \quad (54) \]

Taking into account the inverse Laplace transform correspondence tables, we obtain

\[ q_n(\tau) = \frac{2n}{\tau} J_{2n}(2\tau). \quad (55) \]

B. The quasi-continuum approximation

Now let us solve Eq. (43) approximately. We'll consider \( q(z) \) as a function of the continuous variable \( z \), present Eq. (43) as

\[ \frac{\partial^2 q}{\partial \tau^2} = 2 \sum_{m=1}^\infty \frac{1}{(2m)!} \frac{\partial^{2m} q}{\partial z^{2m}}, \quad (56) \]

and truncate the expansion.

In the continuum approximation Eq. (43) takes the form

\[ \frac{\partial^2 q(z, \tau)}{\partial \tau^2} = \frac{\partial^2 q(z, \tau)}{\partial z^2}. \quad (57) \]

The propagator is defined by the initial and the boundary conditions

\[ q(z, 0) = \delta(z), \quad \partial q(z, 0)/\partial \tau = 0, \quad (58a) \]

\[ z \to +\infty, q(z, \tau) = 0. \quad (58b) \]

The solution is obvious

\[ \frac{\partial q(z, \tau)}{\partial \tau} = \frac{1}{2} [\delta(\tau + z) + \delta(\tau - z)]. \quad (59) \]

The result correctly describes the front motion (see below) but completely misses the structure of the exact solution.

In the quasi-continuum approximation we take into account two terms of the expansion (56), which modifies Eq. (57) to

\[ \frac{\partial^2 q(z, \tau)}{\partial \tau^2} = \frac{\partial^2 q(z, \tau)}{\partial z^2} + \frac{1}{12} \frac{\partial^{14} q(z, \tau)}{\partial z^4}. \quad (60) \]

Making Laplace transformation with respect to \( \tau \) and Fourier transformation with respect to \( z \)

\[ Q(k, s) = \int_0^\infty d\tau e^{-s\tau} \int_{-\infty}^{+\infty} dz q(z, \tau) e^{ikz}, \quad (61) \]

we obtain for the propagator equation

\[ \left( s^2 + k^2 - \frac{k^4}{12} \right) Q(k, s) = s. \quad (62) \]
Solving Eq. (62) we get
\[ Q(k, s) = \frac{s}{s^2 + k^2 - k^4/12} . \] (63)

Taking into account the inverse Laplace transform correspondence tables, we obtain
\[ q(k, \tau) = \cos \left( \sqrt{k^2 - k^4/12} \tau \right) . \] (64)

Making inverse Fourier transformation we obtain
\[ q(z, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \cos \left( \sqrt{k^2 - k^4/12} \tau \right) e^{-ikz} . \] (65)

Presenting cosine function as half sum of two exponents we may write down
\[ q(z, \tau) = q^+(z, \tau) + q^-(z, \tau) , \] (66)

where the meaning of \( q^+ \) is obvious.

In the framework of the approximation we are allowed to consider only \( k \ll 1 \). Expanding square root with respect to \( k \) we obtain
\[ q^+(z, \tau) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dk \exp \left[ i(\tau - z)k - i\tau k^3/24 \right] = \tau^{-1/3} \text{Ai} \left[ 2\tau^{-1/3}(z - \tau) \right] , \] (67)

where \( \text{Ai} \) is the Airy function. After some initial time the \( q^+(z, \tau) \) is effectively different from zero only at \( z > 0 \), and \( q^-(z, \tau) \) at \( z < 0 \). Equation (67) describes the signal front at \( z \sim \tau/2 \), exponentially small precursor for \( \tau < 2z \), and oscillations and power law decrease of the signal in the wake for \( \tau > 2z \). The width of the transition region between the two asymptotic forms increases with time as \( \tau^{1/3} \).

Fig. 5 (above) compares Eq. (67) with the exact result (68a) for \( \tau \) from zero up to a couple of \( z \). To compare the results for \( \tau \gg z \), we may use asymptotic forms of Bessel and Airy functions.

We have
\[ J_{2n}(2\tau) \approx \sqrt{\frac{1}{\pi \tau}} (-1)^n \cos \left( 2\tau - \frac{\pi}{4} \right) , \] (68a)
\[ \tau^{-1/3} \text{Ai} \left[ 2\tau^{-1/3}(z - \tau) \right] \sim \sqrt{\frac{1}{\pi \tau}} \cos \left[ A \tau \left( 1 - \frac{z}{\tau} \right)^{3/2} - \frac{\pi}{4} \right] , \] (68b)

where \( A = 2^{5/2}/3 \approx 1.9 \).

Considering the signal problem for the semi-infinite line, we still have Eq. (66), this time for \( z \geq 0 \) and with the initial and the boundary conditions
\[ q(z, 0) = 0 , \quad \partial q(z, 0)/\partial \tau = 0 , \] (69a)
\[ q(0, \tau) = \delta(\tau) , \quad \lim_{z \to -\infty} q(z, \tau) = 0 . \] (69b)

For the Laplace transform we obtain equation
\[ \frac{1}{12} \frac{d^4 Q(z, s)}{dz^4} + \frac{d^2 Q(z, s)}{dz^2} = s^2 Q(z, s) , \] (70)

where
\[ Q(z, s) = \int_{-\infty}^{+\infty} \frac{ds}{2\pi i} e^{ikz} Q(z, s) . \]

Taking into account the inverse Laplace transform correspondence tables, we obtain
\[ q(z, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \exp \left[ s(\tau - z) + zs^3/24 \right] = 2z^{-1/3} \text{Ai} \left[ 2z^{-1/3}(z - \tau) \right] . \] (71)

In the framework of the quasi-continuum approximation we should expand the solution of (72) with respect to \( s \) to obtain
\[ k(s) = -s + s^3/24 . \] (72)

Using Bromwich integral we get
\[ q(z, \tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} Q(z, s) e^{s\tau + k(s)z} ds . \] (73)

Substituting into (73) we finally get
\[ q(z, \tau) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} ds \exp \left[ s(\tau - z) + zs^3/24 \right] = 2z^{-1/3} \text{Ai} \left[ 2z^{-1/3}(z - \tau) \right] . \] (75)
Comparing Eqs. (55) and (76), and looking at Fig. 9, we realize that in the vicinity of the peak of the signal, the agreement between the exact and the approximate results for the semi-infinite line would be as good, as for the infinite line. However for greater $\tau$ the agreement would be worse, because the approximate result decreases with $\tau$ slower than the exact one. That is what we see on Fig. 9 (below).

IX. THE INTEGRAL APPROXIMATION

We must warn the reader that the present Section is mostly of speculative nature. We were not able to advance far on the road we have taken here (if at all). However, some equations obtained in the process look quite amusing to us, and we decided to present them to advance far on the road we have taken here (if at all).

A. The linear transmission line

Let us return to Eq. (43) and try to improve the quasi-continuum approximation by expressing the r.h.s. of the equation in the form alternative to the Taylor expansion (76), thus avoiding the necessity of the truncation of the expansion. Treating $q$ as a function of the continuous variable $z$ (and $\tau$), we approximate Eq. (43) as

$$\frac{d^2 q(z, \tau)}{d\tau^2} = \int_{-\infty}^{+\infty} d\tau' g(z - z') \frac{d^2 q(z', \tau)}{d\tau'^2},$$

(76)

where $g(z)$ is a non-singular function, which is positive, even, has zero moment equal to one

$$\int_{-\infty}^{+\infty} dz g(z) = 1,$$

(77)

and goes to zero fast enough when $z \to \infty$.

Consider again the propagator in the infinite line. Making Laplace transformation with respect to $\tau$ and Fourier transformation with respect to $z$ we obtain, starting from (76), the equation

$$[s^2 + k^2 g(k)] Q(k, s) = s,$$

(78)

where $g(k)$ is the Fourier image of $g(z)$. Proceeding exactly like in the previous Section, we obtain, instead of (65),

$$q(z, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \cos k \sqrt{g(k)} \tau e^{-ikz}.$$

(79)

For (76) to be a reasonable approximation to (43), we should choose $g(z)$ in such a way, that (79) would emulate the exact result (47). We have seen above that choosing $g(k) = 1 + \frac{k^2}{12}$ we obtain quite good agreement between (79) and (47) for $\tau \approx z$. The question is whether we can chose $f(k)$ which will guarantee same good agreement for larger $\tau$? Postponing answering this question until later time, let us try to understand what the approximation similar to (76) gives for the JTL.

B. The Josephson transmission line

Returning to Eq. (3) and treating $\varphi$ as a function of the continuous variable $z$ (which we measure in $\Lambda$), in line what we have tried previously, let us approximate the equation as

$$\frac{\partial^2 \varphi(z, t)}{\partial \tau^2} = \int_{-\infty}^{+\infty} dz' g(z - z') \frac{\partial^2 \varphi(z', t)}{\partial z'^2}. \quad (80)$$

Looking for the running wave (5) solution of (80), we obtain the integro-differential equation (the counterpart of (6)) for the function $\varphi(x)$

$$U^2 \frac{\partial^2 \varphi(x)}{\partial x^2} = \int_{-\infty}^{+\infty} dx' \frac{\partial^2 g(x - x')}{\partial x'^2} \sin \varphi(x') \quad (81)$$

Integrating Eq. (81) with respect to $x$ twice we obtain Hammerstein equation of the second kind

$$\varphi(x) = \int_{-\infty}^{+\infty} dx' g(x - x') \sin \varphi(x') - F. \quad (82)$$

Imposing the boundary conditions (10) and going in Eq. (82) to the limits $x \to +\infty$ and $x \to -\infty$, we recover Eq. (11), and hence, Eq. (12). Substituting $U^2$ and $F$ into Eq. (82) we get the counterpart of Eq. (7)

$$\sin \varphi_2 - \sin \varphi_1 \varphi(x) + \varphi_1 \varphi_2 - \varphi_2 \varphi_1 = \int_{-\infty}^{+\infty} dx' g(x - x') \sin \varphi(x'). \quad (83)$$

Now let us consider Eq. (83) per se, forgetting the properties of $\varphi(x)$ which were postulated to derive it. We realise that if $\varphi(x)$ goes to some limits when $x \to +\infty$ and $x \to -\infty$, each of these limits is either $\varphi_1$, or $\varphi_2$. This is unfortunately all we can say about the equation.

Equation (7) was shown to have the solution only if either $\varphi_2 = \varphi_1$, or $\varphi_2 = -\varphi_1$. We are unable to do the same for Eq. (83), which raises the question whether the relations between the asymptotic phases on both sides of the waves, mentioned above, are the exact ones, or only approximate. This remains unclear for us. However, the relation $\varphi_2 = \varphi_1$ being imposed, Eq. (83) takes the form

$$\frac{\sin \varphi_1 \varphi(x)}{\varphi_1} = \int_{-\infty}^{+\infty} dx' g(x - x') \sin \varphi(x'). \quad (84)$$

This equation is the counterpart of Eq. (7).

The only thing we can prove about the solution of Eq. (84) is that, for any $x$,

$$-\varphi_1 \leq \varphi(x) \leq \varphi_1 \quad (85)$$

(for the sake of definiteness we consider $\varphi_1$ to be positive). In fact, let $\sin \varphi(x)$ reaches maximum value at some point $x_0$. Then

$$\int_{-\infty}^{+\infty} dx' g(x - x') \sin \varphi(x) < \sin \varphi(x_0). \quad (86)$$
Let also \( \varphi(x_0) > \varphi_1 \). Then
\[
\frac{\sin \varphi_1}{\varphi_1} \varphi(x_0) > \sin \varphi(x_0),
\]  
(87)
because \( \sin \varphi / \varphi \) decreases when \( \varphi \) increases for positive \( \varphi \). So we came to a contradiction. Similar for the minimum value of \( \sin \varphi \).

X. DISCUSSION

We hope that the results obtained in the paper are applicable to kinetic inductance based traveling wave parametric amplifiers based on a coplanar waveguide architecture. Onset of shock-waves in such amplifiers is an undesirable phenomenon. Therefore, shock waves in various JTL should be further studied, which was one of motivations of the present work.

Recently, quantum mechanical description of JTL in general and parametric amplification in such lines in particular started to be developed, based on quantisation techniques in terms of discrete mode operators\,[49], continuous mode operators\,[50], a Hamiltonian approach in the Heisenberg and interaction pictures\,[51], the quantum Langevin method\,[52], or on partitions a quantum device into compact lumped or quasi-distributed cells\,[53]. It would be interesting to understand in what way the results of the present paper are changed by quantum mechanics. Particularly interesting looks studying of quantum ripples over a semi-classical shock\,[54] and fate of quantum shock waves at late times\,[55]. Closely connected problem of classical and quantum dispersion-free coherent propagation in waveguides and optical fibers was studied recently in Ref\.[56].

XI. CONCLUSIONS

We analyzed the quasi-continuous approximation for the discrete JTL. The approximation becomes controllable for the case of Josephson phase difference across the JJs being small and for the case of small amplitude disturbances of the phases on a homogeneous background. The approximation is applied to study the shocks and solitary waves which can propagate along the line. The width of such waves turns out to be of the order of the line period. We have found that the resistors shunting the JJs and/or in series with the ground capacitors lead to the existence of the shock waves of more general type than those existing in the lossless JTL. In addition, the resistors lead to broadening of the shocks in comparison to a lossless case.

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Appendix A: The quasi-continuous approximation

It would be interesting to compare the quasi-continuous approximation with the another one, developed extensively some time ago by P. Rosenau\,[57], and called by him the quasi-continuous approximation. Following the pattern of Section VIII, we apply the latter to the calculation of the propagator for the linear transmission line. In the quasi-continuous approximation Eq. (A4) would take the form
\[
\frac{\partial^2 q(z, \tau)}{\partial \tau^2} = \frac{\partial^2 q(z, \tau)}{\partial z^2} + \frac{1}{12} \frac{\partial^4 q(z, \tau)}{\partial z^2 \partial \tau^2},
\]  
(A1)
the initial and the boundary conditions are given by Eq. (55).

Making Laplace transformation with respect to \( \tau \) and Fourier transformation with respect to \( z \) we obtain for the propagator equation
\[
(s^2 + k^2 + \frac{k^2 s^2}{12}) \mathcal{Q}(k, s) = s.
\]  
(A2)
Solving Eq. (78) and taking into account the inverse Laplace transform correspondence tables\,[12], we obtain
\[
q(k, \tau) = \frac{1}{\sqrt{1+k^2/12}} \cos\left(\frac{k \tau}{\sqrt{1+k^2/12}}\right). \]  
(A3)
Expanding with respect to \( k \) we recover the results of Section VIII.

Considering the signal problem for the semi-infinite line, we still have Eq. (A1), this time for \( z \geq 0 \) and with the initial and the boundary conditions (69). For the Laplace transform we obtain equation
\[
\left(1 + \frac{s^2}{12}\right) \frac{d^2 Q(z, s)}{dz^2} = s^2 Q(z, s),
\]  
(A4)
Taking into account the boundary conditions, we can write down the solution of Eq. (A4) as
\[
Q(z, s) = e^{k(s)z},
\]  
(A5)
where \( k(s) = -s/\sqrt{1+s^2/12} \), which for \( s \ll 1 \) coincides with (74).
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