INTRODUCTION TO
THERMAL FIELD THEORY*

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Abstract

Within the next few years experiments at RHIC and the LHC will seek to create in the laboratory a quark-gluon plasma, the phase of matter in which the Universe was initially created. It is believed that the plasma will survive long enough to reach thermal equilibrium. I give an introduction to the formalism of thermal field theory, the combination of statistical mechanics and quantum field theory needed to describe the plasma in thermal equilibrium, in a way that tries to keep close to the physics it describes.

Introduction

Thermal field theory is a combination of quantum field theory and statistical mechanics. This means that it is both difficult and interesting. The reason that we study it is that we want to describe the quark-gluon plasma, the phase that matter is believed to take above some critical temperature $T_c$. Lattice calculations suggest\cite{1} that $T_c$ is about 100 MeV, or $10^{12}$K. In the plasma phase the quarks and gluons are deconfined; they can move rather freely through the whole plasma. This is the phase in which the universe was created at the big bang, and before the end of the century experiments at the new collider RHIC will try to re-create it in the laboratory, by making gold nuclei collide together head-on and dump their kinetic energy into a small volume. Similar experiments, at much higher energy, are planned later for the LHC at CERN.

There is an obvious question: if a plasma is indeed produced, how will we know it? As yet there is no simple answer. There are estimates\cite{2}, necessarily based on very crude non-equilibrium theory, that suggest that the plasma will survive for a time long enough that it reaches thermal equilibrium before it eventually decays back into ordinary matter. So far, it is only equilibrium thermal field theory that is well formulated, and my lectures concentrate on this. For more information, there is a book that was published last year\cite{3} and is already the standard text. As I want my description to stay as close as possible to physics I will develop the theory using operators rather than path integrals, and mostly I will use the so-called real-time formalism.

Because in relativistic theory particles are continually being created and destroyed, it is appropriate to use the grand partition function

$$Z = \sum_i \langle i | e^{-\beta (H - \mu N)} | i \rangle$$  \hspace{1cm} (1)

Here $\beta$ is the inverse temperature, $\beta = 1/k_B T$, and usually we use units in which Boltzmann’s constant $k_B = 1$. The system’s Hamiltonian is $H$ and $N$ is some conserved quantum number, such as baryon

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number, with \( \mu \) the corresponding chemical potential. There may be several conserved quantum numbers, in which case \( \mu N \) is replaced with \( \sum_\alpha \mu_\alpha N_\alpha \).

The states \( |i\rangle \) are a complete orthonormal set of physical states of the system. In scalar field theory all states are physical and so

\[
Z = \text{tr} \ e^{-\beta (H - \mu N)}
\]  

which is invariant under changes in the choice of orthonormal basis of states. In the case of gauge theories there are unphysical states, for example longitudinally-polarised photons or gluons, which must be excluded from the summation in (1). So then

\[
Z = \text{tr} \ \mathbb{P} \ e^{-\beta (H - \mu N)}
\]

where \( \mathbb{P} \) is a projection operator onto physical states. The presence of \( \mathbb{P} \) can make things more complicated, and so to begin with I will consider scalar field theory, where it is not needed.

All the macroscopic properties of the system in thermal equilibrium may be calculated from \( Z \). In particular, for a system that is so large that its surface energy is negligible compared with its volume energy, the equation of state is

\[
P V = T \log Z
\]

Also, the “thermal average” of an observable corresponding to an operator \( Q \) is

\[
\langle Q \rangle = Z^{-1} \text{tr} \ Q e^{-\beta (H - \mu N)}
\]

Notice that, throughout, all operators are familiar zero-temperature ones. The temperature enters only in the exponential, which characterises the particular ensemble of states used to calculate expectation values of the operators.

**Noninteracting scalar bosons**

For many systems of bosons there is no conserved quantum number \( N \); for example, in the case of a heat bath of photons there is no constraint on their total number. Then in the scalar-field-theory case \( Z \) is just \( \text{tr} \ e^{-\beta H} \).

In the absence of interactions, the energies of the separate particles are good quantum numbers. To begin with, quantise the system in a finite volume \( V \), so that the single-boson energies \( \epsilon_r \) are discrete. The states \( |i\rangle \) of the system are labelled by the single-particle occupation numbers \( n_r \), and the eigenvalues of the noninteracting Hamiltonian \( H_o \) are

\[n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3 + \ldots\]

So the noninteracting grand partition function is

\[
Z_0 = \sum_{\{n_r\}} e^{\beta (n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}
\]

\[
= \prod_r \left( \sum_{n_r} e^{-\beta n_r \epsilon_r} \right) = \prod_r \frac{1}{1 - e^{-\beta \epsilon_r}}
\]
and so

$$\log Z_0 = - \sum_r \log (1 - e^{-\beta \epsilon_r})$$  \hspace{1cm} (6b)$$

In the continuum limit

$$\sum_r \to V \int \frac{d^3k}{(2\pi)^3}$$ \hspace{1cm} (7)$$

and so the noninteracting equation of state is

$$P = \frac{T}{V} \log Z_0 = -T \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-\beta k_0})$$ \hspace{1cm} (8)$$

where $k_0 = \sqrt{k^2 + m^2}$. (If the bosons have non zero spin, there is an additional factor $g_s$ corresponding to the spin degeneracy of each single-boson state.)

In the continuum limit we usually work with fields

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} a(k) e^{-ik \cdot x} + h.c.$$ \hspace{1cm} (9a)$$

with

$$[a(k), a^\dagger(k')] = (2\pi)^3 2k_0 \delta^{(3)}(k - k')$$ \hspace{1cm} (10a)$$

In the discrete case, we usually define

$$[a_r, a_s^\dagger] = \delta_{rs}$$ \hspace{1cm} (10b)$$

If we sum this over $r$, the result is 1. But if we apply $V \int d^3k/(2\pi)^3$ to (10a), the result is rather $2k_0 V$. That is (10a) and (10b) have definitions of the operators $a$ differing by a factor $\sqrt{2k_0 V}$. We correct for this by defining the field in the discrete case to be

$$\phi(x) = \sum_r \frac{1}{\sqrt{2\epsilon_r V}} a_r e^{-i\epsilon_r t} e^{ik_r \cdot x} + h.c.$$ \hspace{1cm} (9b)$$

We can now calculate the thermal average

$$\langle T\phi(x)\phi(0) \rangle_0 = Z_0^{-1} \sum_i \langle i | e^{-\beta H_0} T\phi(x)\phi(0) | i \rangle$$ \hspace{1cm} (11)$$

Observe first that

$$T\phi(x)\phi(0) = \langle 0 | T\phi(x)\phi(0) | 0 \rangle + \phi(x)\phi(0) :$$ \hspace{1cm} (12)$$

where $: :$ denotes the usual normal product. The first term contributes to (11) just the usual zero-temperature Feynman propagator. To evaluate the contribution from the second, use the discrete case.
(9b) and so obtain double sums $\sum_{r,s}$ of terms $a_r^\dagger a_s, a_r a_s, a_r^\dagger a_s^\dagger$. When we take the necessary expectation values, only the first survives, and then only for $r = s$. We may replace $a_r^\dagger a_r$ with $n_r$, and use

$$
\left( \frac{1}{1 - e^{-\beta \epsilon}} \right)^{-1} \sum_n n e^{-n\beta \epsilon} = f(\epsilon)
$$

(13a)

where $f$ is the Bose distribution

$$
f(\epsilon) = \frac{1}{e^{\beta \epsilon} - 1}
$$

(13b)

So we find that

$$
\langle : \phi(x) \phi(0) : \rangle_0 = \frac{1}{V} \sum_r \frac{1}{2\epsilon_r} f(\epsilon_r) e^{i\epsilon_r t - i k_r x} + \text{h.c.}
$$

(14)

Going to the continuum limit, we have

$$
\langle T \phi(x) \phi(0) \rangle_0 = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} D_0^I(k)
$$

$$
D_0^I(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi \delta(k^2 - m^2) n(k^0)
$$

(15)

where $n(k^0) = f(|k^0|)$. The second term is the contribution from the heat bath; it contains the $\delta$-function because so far the heat-bath particles do not interact and so they are on shell.

**Perturbation theory**

Suppose now that we introduce an interaction and again calculate $\langle T \phi(x) \phi(0) \rangle$. Now $\phi(x)$ is the interacting Heisenberg-picture field: it is the familiar operator of zero-temperature field theory. We can develop a perturbation theory along the same lines as at zero temperature, by introducing the interaction picture that coincides with the Heisenberg picture at some time $t_0$:

$$
\phi_I(t, x) = \Lambda(t) \phi(t, x) \Lambda^{-1}(t)
$$

$$
\Lambda(t) = e^{i(t-t_0)H_0 I} e^{-i(t-t_0)H}
$$

(16)

where $H_0 I$ is the free-field Hamiltonian in the interaction picture. As usual

$$
\Lambda(t_1) \Lambda^{-1}(t_2) = U(t_1, t_2)
$$

$$
= T \exp \left( -i \int_{t_2}^{t_1} dt' H_I^{\text{INT}}(t') \right)
$$

(17)

We need $U(t_1, t_2)$ for complex $t_1$ and $t_2$, so that we integrate $t$ along some contour running from $t_2$ to $t_1$ in the complex plane and generalise the time ordering $T$ to an ordering $T_c$ along $C$: The operator whose argument is nearest to $t_1$ along the contour comes first.

Now, from the definition of $U$,

$$
e^{-\beta H} = e^{-\beta H_{0I}} U(t_0 - i\beta, t_0)
$$
so that

\[ Z^{-1} \text{tr} e^{-\beta H} \phi(x) \phi(0) = Z^{-1} \text{tr} e^{-\beta H_0} U(t_0 - i\beta, x^0) \phi_I(x) U(x^0, 0) \phi_I(0) U(0, t_0) \]

\[ = Z_0 Z^{-1} \langle U(t_0 - i\beta, x^0) \phi_I(x) U(x^0, 0) \phi_I(0) U(0, t_0) \rangle_0 \]

(18)

where I have used \( U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) \).

Compare with what one has at zero temperature:

\[ \langle 0|\phi(x)\phi(0)|0 \rangle = \langle 0|U(\infty, x^0)\phi_I(x)U(x^0, 0)\phi_I(0)U(0, -\infty)|0 \rangle \]

In (18) we have a non-interacting thermal average instead of a vacuum expectation value, \((t_0 - i\beta)\) instead of \(\infty\), and \(t_0\) instead of \(-\infty\). In fact there is a close similarity between thermal perturbation theory and the usual zero-temperature Feynman perturbation theory. The main difference is that instead of the internal lines in the ordinary Feynman graphs representing vacuum expectation values, in the thermal graphs they are non-interacting thermal averages. In order to derive this, one needs to establish Wick’s theorem. It is a remarkable fact that indeed, for example,

\[ \langle T_c \phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) \rangle_0 = \sum \langle T\phi_I\phi_I \rangle_0 \langle T\phi_I\phi_I \rangle_0 \]

(19)

where the sum is over the possible pairings of the fields. For almost every ensemble other than one in thermal equilibrium there would be correction terms to (19).

One needs to choose a value for \(t_0\). A common choice is \(t_0 = 0\), with the contour \(C\) for the \(t\) integrations running from 0 to \(-i\beta\) along the imaginary axis. This is the imaginary-time formalism. Alternatively, \(t_0 \to -\infty\), which with suitable contour choice gives the real-time formalism.

**Real-time formalism**

One is interested in equilibrium properties of the plasma at finite times. Presumably these are independent of how it reached thermal equilibrium. So, as in familiar scattering theory, we are free to imagine that the interaction slowly switches off as we go into the remote past, and then when we take \(t_0 \to -\infty\) the interaction-picture fields become the usual noninteracting in fields. The corresponding in states are direct products of non-interacting single-particle states. We need to choose how the contour \(C\) runs from \(-\infty\) to \((-\infty - i\beta)\), and the choice that keeps the formalism in the most direct contact with the physics is the so-called Keldysh one: along the real axis from \(-\infty\) to \(\infty\), back to \(-\infty\), then straight down to \((-\infty - i\beta)\).

For most applications (though not all\(^4\)) it turns out that the vertical part of the contour does not contribute. Then we may write the propagator that corresponds to a line of a thermal graph as a \(2 \times 2\) matrix:

\[ D(x_1, x_2) = \begin{bmatrix} \langle T\phi_{in}(x_1)\phi_{in}(x_2) \rangle_0 & \langle \phi_{in}(x_2)\phi_{in}(x_1) \rangle_0 \\ \langle \phi_{in}(x_1)\phi_{in}(x_2) \rangle_0 & \langle T\phi_{in}(x_1)\phi_{in}(x_2) \rangle_0 \end{bmatrix} \]

(20)

When both \(x_1\) and \(x_2\) are on the \(-\infty\) to \(\infty\) part of \(C\), the ordering \(T_c\) is ordinary time ordering \(T\); this corresponds to the element \(D_{11}\) of \(D\). When both are on the \(\infty\) to \(-\infty\) part of \(C\), \(T_c\) is
anti-time-ordering $\bar{T}$; this corresponds to $D_{22}$. The off-diagonal elements correspond to $x_1$ being on one part of $C$ and $x_2$ on the other. There is translation invariance: the elements of $D$ depend only on the difference between $x_1$ and $x_2$.

I have already shown how to calculate $D_{11}$; the result is given in (15). The other elements of $D$ may be calculated in the same way, and its Fourier transform is

$$D(k) = \left[ \frac{i}{\delta^+(k^2 - m^2)} 2\pi \delta^-(k^2 - m^2) \right] + 2\pi \delta(k^2 - m^2) n(k^0) \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$  \hspace{1cm} (21a)

It may also be written in the form

$$M \tilde{D} M$$  \hspace{1cm} (21b)

with

$$\tilde{D} = i \left[ \begin{array}{cc} \frac{1}{k^2 - m^2 + i\epsilon} & 0 \\ 0 & \frac{1}{k^2 - m^2 - i\epsilon} \end{array} \right]$$

$$M = \sqrt{n(k^0)} \left[ \begin{array}{cc} e^{\frac{1}{2} \beta |k^0|} & e^{-\frac{1}{2} \beta k^0} \\ e^{-\frac{1}{2} \beta k^0} & e^{\frac{1}{2} \beta |k^0|} \end{array} \right]$$  \hspace{1cm} (21c)

For the case of a fermion field, there is a rather similar matrix propagator, but with the Fermi-Dirac distribution replacing the Bose distribution.

**Matrix structure**

The elements of the matrix propagator (20) are not independent. For example,

$$D_{21}(x) = \langle \phi_{\text{in}}(x) \phi_{\text{in}}(0) \rangle_0 = Z_0^{-1} \text{tr} \ e^{-\beta H_{0\text{in}}} \phi_{\text{in}}(x) \phi_{\text{in}}(0)$$

$$= Z_0^{-1} \text{tr} \ e^{-\beta H_{0\text{in}}} \phi_{\text{in}}(0) e^{\beta H_{0\text{in}}} \phi_{\text{in}}(x) e^{\beta H_{0\text{in}}}$$

$$= Z_0^{-1} \text{tr} \ e^{-\beta H_{0\text{in}}} \phi(0) \phi(x^0 - i\beta, x)$$

$$= D_{12}(x^0 - i\beta, x)$$  \hspace{1cm} (22a)

Here, I have used a general property of traces, that $\text{tr}(AB) = \text{tr}(BA)$, and the fact that $H_{0\text{in}}$ is the time-translation operator for the noninteracting field $\phi_{\text{in}}$. The Fourier transform of (22a) is

$$D_{12}(k) = e^{\beta k^0} D_{21}(k)$$  \hspace{1cm} (22b)

Also, from their definitions (20), one can see that $D_{11}$ and $D_{22}$ may be expressed in terms of $D_{12}$ and $D_{21}$. It is this, together with (22b), which is responsible for the matrix structure (21).

Define a dressed thermal propagator matrix $D'(x_1, x_2)$ analogous to $D(x_1, x_2)$ in (20), but with the interacting Heisenberg field instead of $\phi_{\text{in}}$. For example, $D'_{12}(x_1, x_2) = \langle \phi(x_2) \phi(x_1) \rangle$. Then, because $H$ is the time-translation operator for $\phi$, we can again derive

$$D'_{12}(k) = e^{\beta k^0} D_{21}(k)$$  \hspace{1cm} (22c)

and so deduce that $D'$ has the structure
Define the thermal self-energy matrix \( \Pi \) by

\[
i \Pi = D^{-1} - D^{-1}
\]  

(24a)

Then \( \Pi \) has the structure

\[
i \Pi = M^{-1} \left( \begin{array}{ccc}
i \Pi'_{k, T} & 0 \\
0 & [i \Pi'(k, T)]^* \end{array} \right) M^{-1}
\]  

(24b)

If we then solve (24a) for \( D' \), we find

\[
D' = M \left( \begin{array}{ccc}
\frac{i}{k^2 - m^2 - \Pi} & 0 \\
0 & \frac{i}{k^2 - m^2 - \Pi} \end{array} \right) M
\]  

(25)

So it is natural to interpret Re \( \hat{\Pi} \) as a temperature-dependent shift to the mass \( m^2 \). \( \hat{\Pi} \) also has an imaginary part, so the propagation of the field through the heat bath decays with time.

In scalar field theory,

\[
\Pi = \bullet + \bullet + \cdots
\]

To calculate the contribution to \( \Pi_{12} \) from the second term, for example, one needs

\[
\begin{array}{cccc}
1 & 2 & 1 & 2 \\
1 & 2 & 1 & 2 \\
1 & 2 & 1 & 2 \\
\end{array}
\]

where each line \( \longrightarrow \) represents \( D_{ij}(k) \) and each vertex 2 is the same as the normal vertex 1, but opposite in sign.

**Imaginary-time formalism**

The real-time formalism stays close to the physics, but has the calculational complication that the propagator is a matrix. In the imaginary-time formalism there is not this complication, though except for a few simple cases there is the need to perform an analytic continuation from imaginary to real time at the end of the calculation. In the imaginary-time formalism the \( t \) integration runs along the imaginary axis, so ordinary time-ordering is replaced with ordering in imaginary time:

\[
\bar{D}(x_1, x_2) = \theta(-\text{Im } t)D_{21}(x_1, x_2) + \theta(\text{Im } t)D_{12}(x_1, x_2)
\]  

(26)

where \( t = x_1^0 - x_2^0 \). Because both \( x_1^0 \) and \( x_2^0 \) are integrated from 0 to \( i\beta \), we need \( \bar{D}(x_1, x_2) \) for values of \( \text{Im } t \) in the range \( -\beta \) to \( +\beta \). In this finite interval it has a Fourier-series expansion.
\[ \bar{D}(t, x) = \frac{i}{\beta} \sum_{n=-\infty}^{\infty} D_n(x) e^{\omega_n t} \]  

(27)

where \( \omega_n = n\pi/\beta \). However, the relation (22a) implies that \( \bar{D}(t, x) = \bar{D}(t + i\beta, x) \), so that only even values of \( n \) contribute to the sum. (In the case of fermions, the anticommutativity of the fields results in a minus sign appearing in the corresponding relation (22a), and so then only odd values of \( n \) contribute.)

If we apply a 3-dimensional Fourier transformation to (27) and invert the Fourier summation over \( n \), we find

\[ D_n(k) = \int_0^{i\beta} dt e^{-w_n t} D_{21}(t, k) \]  

(28)

which turns out to be just the ordinary Feynman propagator with \( k^0 = i\omega_n \). So the Feynman rules are just like the zero-temperature ones, except that the energy-conserving \( \delta \)-function at each vertex is replaced with a Konecker delta which imposes conservation of the discrete energy, and round each loop of a thermal graph

\[ \int \frac{d^4 k}{(2\pi)^4} \rightarrow \frac{i}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \]  

(29)

**Gauge theories**

For gauge theories there is the complication that the grand partition function has to include the projection operator \( \mathbb{P} \) onto physical states: see (3). There are two formalisms for the resulting perturbation theory\(^5\):

**A** Only the two physical degrees of freedom of the gauge field (the transverse polarisations) acquire the additional thermal propagator; the other components of the gauge field, and the ghosts, remain frozen at zero temperature. (This is for the bare propagators; self-energy insertions in the unphysical bare propagators do depend on the temperature.)

**B** All components of the gauge field, and the ghosts, become heated to temperature \( T \).

In the zero-temperature field theory, the ghosts are introduced in order to cancel unwanted contributions from the unphysical components of the gauge field, and the two formalisms lead to the same answers for calculations of physical quantities for that reason. Often, using formalism A makes calculations simpler. It also makes them stay closer to the physics.

**Photon or dilepton emission from a plasma**

As an application, consider the emission of a real or virtual photon of momentum \( q \) from a quark-gluon plasma. This is supposed to be an important diagnostic test of whether a plasma has been created in an experiment and reached thermal equilibrium, and a way to measure its temperature.

Before a photon is emitted, the plasma is described by the density matrix

\[ \rho = Z^{-1} \sum_i |i\text{ in}\rangle\langle i\text{ in}| e^{-\beta H} \]  

(30)

The emission probability is calculated from squared matrix elements of the Heisenberg-picture elec-
tromagnetic current:

\[ W^{\mu\nu}(q) = Z^{-1} \sum f \int d^4x e^{-iq \cdot x} \langle \text{out}|J^\mu(x) \left( \sum_i |\text{in}\rangle \langle \text{in}| e^{-H} \right) J^\nu(0)|\text{out}\rangle \]  

(31a)

where I have introduced also a complete set of out states for the plasma. These satisfy the completeness relation

\[ \sum_f \langle \text{out}|f \rangle \langle f| = 1 \]

and so

\[ W^{\mu\nu}(q) = Z^{-1} \sum_i \int d^4x e^{iq \cdot x} \langle \text{in}|e^{-\beta H}J^\nu(0)J^\mu(x)|\text{in}\rangle \]  

(31b)

If we introduce a matrix \( G^{\mu\nu}(q) \) analagous to \( D' \), but with thermal averages of products of electromagnetic currents instead of fields, \( W^{\mu\nu}(q) \) is just \( G^{\mu\nu}_{12}(-q) \). So we draw thermal graphs where current \(-q\) enters at a 1 vertex and leaves at a 2 vertex, and distribute the labels 1 and 2 in all possible ways on the other vertices. For example,

The emission rate is calculated from a matrix element times its complex conjugate; the vertices labelled 1 correspond to a contribution to the matrix element and those labelled 2 to its complex conjugate. The sets of 1-vertices and of 2-vertices are joined by 12 lines which, according to (21a) are on shell and represent particles in the heat bath. The thermal graphs sum together many physical processes. Consider the first graph, for example. Its right-hand part represents the contributions

\[ + + + \]  

\[ \overleftrightarrow{q} \]

to the amplitude.

In fact, energy-momentum conservation allows only the first one to be non-zero. I have not drawn in the other heat-bath particles, but remember that they are there as spectators. The other part of the thermal graph contains only 11 lines. If I use only the zero-temperature part of \( D_{11} \) in each, I obtain the amplitude

\[ \overleftrightarrow{\cdots} \]

(plus other terms which again vanish for kinematic reasons) and so part of the thermal graph represents the interference between this and \( \overleftrightarrow{\cdots} \). If instead I use the thermal part \( n(k^0) 2\pi\delta(k^2) \) of the 11 gluon propagator, I obtain the amplitudes
In each case, the incoming and outgoing gluon lines must have the same momentum $k$, so that these amplitudes again interfere with $\Rightarrow$, but now with the gluon $k$ being one of the spectator particles in the heat bath:

Similarly, I can identify physical processes that involve the thermal parts of the quark propagators. Even a simple-looking thermal graph corresponds to a large number of physical processes, each of which can be rather complicated. An example is

for which just one of the physical processes is the interference between

A disconnected graph occurs because the thermal graph has an “island” of a pair of 1-vertices entirely surrounded by 2-vertices

**Infrared divergences**

The infrared divergences of zero-temperature become much worse at finite temperature: the Bose distribution diverges at zero energy and causes the usual logarithmic divergences to become power divergences. We know that the infrared divergences must cancel if the theory is to make sense, and in practice they always do, but there is no general theory to show this. To some extent, the situation can be rescued by including thermal self-energy insertions in the propagators, so that they acquire a mass proportional to the temperature. But, in the case of photons or gluons, not all the degrees of freedom have a mass, according to perturbation theory. If the so-called magnetic mass is non zero, it is nonperturbative.
Consider, for example, the effect on the decay rate $\pi^0 \to e^+e^-$ of the microwave background, which is a heat bath consisting only of photons. Let $\Gamma$ be the decay rate in vacuum. The heat bath will change it partly because it gives the electrons an additional temperature-dependent mass $\delta m_e\Gamma/\partial m_e^2$, which is associated with thermal graphs of the form

![Diagram](image)

There is also the thermal graph

![Diagram](image)

One finds that

$$\frac{\Delta \Gamma}{\Gamma} = \frac{\delta m_e^2}{\Gamma} \frac{\partial \Gamma}{\partial m_e^2} + \frac{m_e\alpha_{EM}}{\pi^3 Q} \int d^4k \delta(k^2) n(k^0) \int d^4p_1d^4p_2 \delta^{(+)}(p_1^2 - m_e^2)\delta^{(+)}(p_2^2 - m_e^2)$$

$$\left(\frac{p_1}{p_1 \cdot k} - \frac{p}{p_2 \cdot k}\right)^2 \left\{\delta^{(4)}(p_1 + p_2 - P) - \delta^{(4)}(p_1 + p_2 + k - P)\right\}$$

where $Q^2 = (\frac{4}{3}m_e^2 - m_e^2)$.

In the integral, the first $\delta^{(4)}$-function corresponds to the contribution from the internal vertices in the thermal graphs being both 1 or both 2 and the second to 12 and 21. Each term separately is infrared divergent, like $\int dk/k^2$, but the divergences cancel.

In fact there is more cancellation than just that of the infrared divergences. For $T \ll Q$ one can expand (32) in powers of $T^2/Q^2$. One finds that the first term, of order $\alpha_{EM}T^2/Q^2$, exactly cancels the electron-mass-shift contribution $\delta \Gamma$, so that the net change in the decay rate is of order $\alpha_{EM}T^4$.

The lesson is that, in thermal field theory, it is of importance to calculate all terms; gauge theories at finite temperature are rife with cancellations.

Another example where there is an apparent infrared problem is that of the calculation of the equation of state for the quark-gluon plasma. For instance, in the purely gluonic case thermal graphs of the form

![Diagram](image)

become more and more divergent as more vertical lines are added. To see this, use the imaginary time formalism. The term in the multiple sum over the energies for which all the energies vanish looks just like an integral that would occur in zero-temperature 3-dimensional QCD, and simple power counting at each $k = 0$ reveals the problem. However, it seems that the problem goes away if one sums over all thermal graphs. Give each gluon a mass $m$. Then, apart from possible difficulties with doing this in a gauge theory, in the real-time formalism one can derive the formula

$$P = -\int_0^\infty dm^2 \int \frac{d^4q}{(2\pi)^4} e^{\beta q^0} - 1 \frac{\text{Im} \frac{1}{q^2 - m^2 - \Pi(q,T,m)}}{(2\pi)^4}$$

(33)
where $\Pi$ is the self energy of the gluon defined as in (24b) and $\epsilon(q_0) = \pm 1$ according to whether $q_0$ is positive or negative. As any divergence of $\Pi$ now appears in the denominator, the summation has made it harmless. One has to worry about the denominator possibly vanishing when $q = 0$ and $m = 0$, but this will be rendered harmless by the $q^3$ appearing in $d^4 q = q^3 dq d\Omega$.

**Linear response theory**

Suppose that the thermal equilibrium of a plasma is disturbed by the switching on at $t = 0$ of an external electrostatic potential $A_{\text{ext}}^0(x)$. Then the system’s Hamiltonian acquires an extra term

$$H'(t) = \theta(t) \int d^3 x \, J_0(x) A_{\text{ext}}^0(x)$$

(34)

where $J_0$ is the charge density. This will cause $J_0$ to change. As it is a Heisenberg-picture operator, its equation of motion is

$$\frac{\partial J_0}{\partial t} = i [H + H', J_0]$$

(35a)

where $H$ is the original Hamiltonian. When we take the thermal average of this equation, the contribution from $H$ will disappear because originally there was thermal equilibrium. So the integrated change in $\langle J_0(x) \rangle$ at very large time is

$$\delta \langle J_0(x) \rangle = \int d^4 x' \, G_R(x - x') A_{\text{ext}}^0(x')$$

(35b)

where

$$G_R(x - x') = \theta(t - t') \langle [J_0(x), J_0(x')] \rangle$$

(35c)

Taking the Fourier transform,

$$\delta \langle J_0(k) \rangle = G_R(k) A_{\text{ext}}^0(k)$$

(35d)

We may express the retarded Green’s function $G_R$ in terms of elements of the matrix Green’s function $G^{\mu\nu}$

$$G_R = \frac{1}{2} (G_{11}^{00} - G_{22}^{00} + G_{21}^{00} - G_{12}^{00})$$

(36)

Each of the terms here may be calculated from perturbation theory. However, it is simpler to express $G_R$ in terms of the function $\hat{G}^{00}(k)$ that appears in the diagonal matrix associated with $G^{00}$ (see (23)):

$$G_R = n(k^0) e^{\frac{i}{2} \beta |k^0|} \{ (\hat{G}^{00} + \hat{G}^{00*}) \sinh \frac{1}{2} \beta |k^0| + (\hat{G}^{00} - \hat{G}^{00*}) \sinh \frac{1}{2} \beta k^0 \}$$

(37a)

On the other hand

$$G_{11} = n(k^0) e^{\frac{i}{2} \beta |k^0|} \{ (\hat{G}^{00} + \hat{G}^{00*}) \sinh \frac{1}{2} \beta |k^0| + (\hat{G}^{00} - \hat{G}^{00*}) \cosh \frac{1}{2} \beta k^0 \}$$

So it is sufficient to calculate $G_{11}^{00}$ and change its imaginary part by multiplying it by $\tanh \frac{1}{2} \beta k^0$. 

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References

1. For a recent review, see A Ukawa, [hep-lat/9612011](http://arxiv.org/abs/hep-lat/9612011)
2. K J Eskola and X N Wang, Physical Review D49 (1994) 1284
3. M Le Bellac, *Thermal field theory*, Cambridge University Press (1996)
4. T S Evans and A C Pearson, Physical Review D52 (1995) 4652
5. P V Landshoff and A Rebhan, Nuclear Physics B383 (1992) 607 and B410 (1993) 23
6. N Ashida et al, Physical Review D45 (1992) 2066
   P V Landshoff, Physics Letters B386 (1996) 291
7. M Jacob and P V Landshoff, Physics Letters B281 (1992) 114
8. I T Drummond, R R Horgan, P V Landshoff and A Rebhan, Physics Letters B398 (1997) 326