Key Technique of Almost Exact Simulation for Non-affine Heston Model

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Abstract. In order to sample asset price more accurately under the non-affine Heston model in the situation where the Feller condition was unsatisfied, we proposed the key technique of almost exact simulation for non-affine Heston model, by fusing the approximate analytic solution of conditional variance characteristic function, the second-order Newton’s method and interpolation method. Numerical results on four representative benchmarks show that our newly proposed simulation scheme has both good convergence and accuracy, which are much better than those of the Itô-Taylor schemes. Especially, our scheme performs well in the case where the Feller condition is extremely unsatisfied, while the alternatives don’t work.

1. Introduction
The non-affine Heston model has attracted more and more attention in recent years thanks to its simplicity, flexibility, and good performance in characterizing the sharp peak, thick tail and other characteristics of the distribution of asset return, except that there is no analytical pricing formula for derivatives at this moment. For this reason, Monte Carlo approach is commonly used in practice: Monte Carlo method with antithetic variates is employed to price option[1], Langrené uses the unconditionally stable discretization scheme proposed by Karl and Jäckel [2] to compute the price of a European put option under the IGa stochastic volatility model[3], and so on [4][5]. According to the way of discretising variance process, the existing numerical simulation methods for affine Heston model can be roughly divided into two categories. The first is so-called biased Itô-Taylor scheme, such as Reflection Scheme [6], Absorption Scheme [7], Euler Full Truncated Scheme [8], and Transformed Volatility Scheme [9]. This category is easy to run and fast, but fails when dealing with the case where the Feller condition is not satisfied, since in this category a large number of truncation or reflection formats are inevitably, which results to inestimable discretization biases [2][10][11]. The second category is the so-called exact or nearly-exact scheme, such as Exact Simulation Scheme [7], Quadratic Exponential Scheme [12], Es-Gamma Scheme [13] and Inverse Gaussian Scheme [14]. This category takes full advantage of the analytic density function of the variance process in the affine Heston model. Unfortunately, the density distribution of variance process in non-affine Heston model has no close-form, thus the second category of schemes cannot be directly used. To deal with this problem, we propose the key techniques of almost exact simulation for non-affine Heston model.

In order to sample asset price under the non-affine Heston model more accurately, referring to the treatment of Chacko [15] and Wu [16], we use the perturbation method to deal with the Kolmogorov backward equation based on the characteristic function of the conditional variance, after that, an analytic approximation of the characteristic function is obtained. With the assistance of this function, we sample the asset price. Besides, the second-order Newton’s method is employed to sample the best
variance. Then given the variance, we sample the other key items of the asset price. Finally, four representative benchmarks are tested to validate our scheme.

2. The Non-affine Heston Model

The Non-affine Heston model can be written as

\[
\frac{dlnS_t}{r} = \left( r - \frac{1}{2}V_t \right) dt + \sqrt{V_t} \left( \rho dW^r_t + \sqrt{1 - \rho^2} dW^v_t \right)
\]

\[
dV_t = \kappa(\theta - V_t) dt + \xi \sqrt{V_t} dW^v_t
\]

where \(\ln S_t\) is the logarithm of asset price at time \(t\), \(\sqrt{V_t}\) represents the volatility of \(\ln S_t\). \(r\) denotes the risk-neutral drift, \(\kappa\) denotes the speed of mean reversion, \(\theta\) is the long-term mean of \(V_t\), \(\xi\) is the volatility of volatility. \(W^r_t\) and \(W^v_t\) represent two independent Brownian-motion processes, and \(\rho\) is the correlation between the asset return process and the volatility process.

For \(t_i < t_2\), given the values of \(S_{t_i}\) and \(V_{t_i}\), the asset price at time \(t_2\) can be written as

\[
S_{t_2} = S_{t_i} \exp \left[ rt_2 - rt_i + \rho \int_{t_i}^{t_2} \sqrt{V_t} dW^r_t + \sqrt{1 - \rho^2} \int_{t_i}^{t_2} \sqrt{V_t} dW^v_t - \frac{1}{2} \int_{t_i}^{t_2} V_t dt \right]
\]

3. Key Technique of Almost Exact Simulation Scheme

The algorithm of almost exact simulation scheme can be summarized as follows.

Step 1: Generate \(V_q\) from the approximate cumulative distribution function of \(V^q\) given \(V^q\).

Step 2: Given \(V^q\) and \(V^q\), generate \(\int_{t_i}^{t_q} V_t dt\) by interpolation method.

Step 3: Given \(V^q\) and \(V^q\), generate \(\int_{t_i}^{t_q} \sqrt{V_t} dW^r_t\) by interpolation method.

Step 4: Sample \(\int_{t_i}^{t_q} \sqrt{V_t} dW^v_t\) by generating normal random variable with variance equal to \(\int_{t_i}^{t_q} V_t dt\) and mean equal to 0.

Step 5: Compute \(S_{t_q}\) given \(\int_{t_i}^{t_q} V_t dt\), \(\int_{t_i}^{t_q} \sqrt{V_t} dW^r_t\) and \(\int_{t_i}^{t_q} \sqrt{V_t} dW^v_t\).

The key technique of these steps can be described as follows.

3.1. Sampling Conditional Variance \(\chi_{t_i}^q\) \(V_{t_i}, t_i\)

3.1.1. Characteristic function of \(\chi_{t_i}^q\) \(V_{t_i}, t_i\). Denote \(\Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, \tau)\) as the characteristic function of \(\chi_{t_i}^q\) \(V_{t_i}, t_i\) and it is defined by following

\[
\Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, \tau) = E^Q \left[ \exp(i\omega V_{t_i}) \bigg| V_{t_i}, t_i \right]
\]

where \(\tau = t_2 - t_i\), \(i = \sqrt{-1}\), and \(\omega\) is the frequency. \(E^Q(\cdot)\) represents the conditional expectations under risk neutral.

According to the Feynman-Kac theorem, the characteristic function satisfies the following partial differential equation

\[
\frac{\partial \Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, \tau)}{\partial V_{t_i}} \kappa(\theta - V_{t_i}) + \frac{1}{2} \xi^2 V_{t_i}^2 \frac{\partial^2 \Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, \tau)}{\partial V_{t_i}^2} - \frac{\partial \Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, \tau)}{\partial \tau} = 0
\]

with boundary conditions \(\Phi_{\chi_{t_i}^q}(\omega, V_{t_i}, 0) = \exp(i\omega V_{t_i})\) and \(\Phi_{\chi_{t_i}^q}(0, V_{t_i}, \tau) = 1\).
There is a nonlinear coefficient $V''_\gamma$ in equation (4). This term makes it difficult to obtain the exact analytical solution of equation (4) by common ways. Hence, we apply the perturbation method to approximate its solution[15][16]. Using first-order Taylor expansion at the long-term mean of the variance, we have

$$V''_\gamma \approx \theta' + \gamma \theta'^{-1} (V''_\gamma - \theta) = (1 - \gamma) \theta' + \gamma \theta'^{-1} V''_\gamma .$$

Assume that the characteristic function has a following form like [17]

$$\Phi_{V'_\gamma}(\omega) = \exp[A(\tau) + B(\tau) V'_\gamma].$$

Substituting (5) and (6) into (4), and solving the equation (4), we have

$$B(\tau) = \frac{b}{2a} \frac{1 + \frac{(a \omega)}{(a \omega + b) \exp(b \tau)}}{1 - \frac{(a \omega)}{(a \omega + b) \exp(b \tau)}} \frac{b}{2a}$$

where $a = \frac{1}{2} \xi^2 \theta'^{-1}$ and $b = -\kappa$.

By easy manipulation, we obtain

$$A(\tau) = \frac{1}{4a^2} \left\{ 2a \omega \theta' \xi^2 \beta - 2 \chi \varphi \arctan \left[ \frac{a \omega \exp(b \tau) - a \omega}{b} \right] + a \varphi \ln \left[ b^2 \right] - a \varphi \ln \psi \right\}$$

where $\alpha = b + a \omega \left[ \exp(b \tau) - 1 \right]$, $\beta = (1 - \gamma) \left[ b + a \omega \left[ 1 - \exp(b \tau) \right] \right]$, $\chi = b - i a \omega \left[ \exp(b \tau) - 1 \right]$, $\varphi = 2 a \kappa + b (1 + \gamma) \theta'^{-1} \xi^2$ and $\psi = b^2 + a^2 \omega^2 \left[ \exp(b \tau) - 1 \right]^2$.

3.1.2. Trapezoidal rule. The probability function of $V'_\gamma|V'_\gamma, t_i$ can be computed by Fourier inversion methods [18].

$$F_{V'_\gamma}(v) = \Pr \left[ V'_\gamma | V'_\gamma, t_i \leq v \right] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega v)}{\omega} \Re \left[ \Phi_{V'_\gamma}(\omega, V'_\gamma, \tau) \right] d\omega = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(\omega v)}{\omega} \Re \left[ \Phi_{V'_\gamma}(\omega, V'_\gamma, \tau) \right] d\omega$$

Generally speaking, it is difficult to obtain analytic expression of the infinite integral with integrable function as a complex function. Therefore, the trapezoidal rule is used to approximate the cumulative distribution function, we have

$$F_{V'_\gamma}(v) \approx \frac{h}{\pi} + \frac{2}{\pi} \sum_{j=1}^{N} \sin \left( \frac{j \pi v}{h} \right) \Re \left[ \Phi_{V'_\gamma}(\omega, V'_\gamma, \tau) \right] - e_\gamma(h) - e_\delta(N)$$

$h$ denotes a grid size we set to achieve the desired accuracy, and $N$ represents the last term to be calculated. $e_\gamma(h)$ is the discretization error which can be bounded by a Poisson summation formula as shown in Whitt and Abate [19] and $e_\delta(N)$ is bounded above.

3.1.3. Sampling $V'_\gamma|V'_\gamma, t_i$ with second-order Newton’s method. To simulate $V'_\gamma|V'_\gamma, t_i$, we use the inverse transform method, combining the approximate distribution function. A uniformly distributed random variable $U$ is generated, and then the approximate distribution function is set equal to $U$. Numerical solution is the sample of $V'_\gamma|V'_\gamma, t_i$. Iterative format follows
\[ v_{n+1} = v_n - \frac{f'(v_n)}{f(v_n)} \left[ 1 - \left| \frac{2f(v_n)f'(v_n)}{f(v_n)^2} \right|^{1/2} \right] \]  

where \( f(v) = \frac{h}{\pi}v + 2\sum_{j=1}^{\infty} \frac{\sin(jhv)}{j} \Re[\Phi_{\nu_0}(\omega, v, \tau)] - U \) .

3.2. Sampling \( \int_0^t V \, dt \), \( \int_0^t \sqrt{V} \, dW^V \), \( \int_0^t \sqrt{V} \, dW^V \). Given \( V_0, V_t \).

3.2.1. Sampling \( \int_0^t V \, dt \). Because the characteristic function of \( \int_0^t V \, dt \) is hard to obtain, we use the following interpolation method to replace the accurate method which depends on the real distribution.

\[ \int_0^t V \, dt \approx (\gamma_1 V_0 + \gamma_2 V_t) \tau \]  

where \( \gamma_1 \) and \( \gamma_2 \) are certain constants. In this paper, we set \( \gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{2} \) like literature [12].

3.2.2. Sampling \( \int_0^t \sqrt{V} \, dW^V \). In order to sample \( \int_0^t \sqrt{V} \, dW^V \) given \( V_0 \) and \( V_t \), variance process is divided by \( \frac{\gamma_1}{\gamma_2} \). \( \int_0^t \sqrt{V} \, dW^V \) can be written as

\[ \int_0^t \sqrt{V} \, dW^V = \frac{2}{\xi (3-\gamma)} V_0^{\frac{1-\gamma}{2}} - \frac{2}{\xi (3-\gamma)} V_t^{\frac{1-\gamma}{2}} + \frac{\kappa}{\xi} \int_0^t \sqrt{V} \, (\theta - V) \, dt \]  

Besides, \( \int_0^t \frac{1}{\sqrt{V}} (\theta - V) \, dt \) can be approximated by

\[ \int_0^t \frac{1}{\sqrt{V}} (\theta - V) \, dt \approx \frac{1}{2} \left[ V_0^{\frac{1-\gamma}{2}} (\theta - V_0) + V_t^{\frac{1-\gamma}{2}} (\theta - V_t) \right] \tau \]  

3.2.3. Sampling \( \int_0^t \sqrt{V} \, dW^V \). Because the variance process and the Brownian motion \( W_t \) are independent, \( \int_0^t \sqrt{V} \, dW^V \) is a normal random variable with a mean equal to 0 and a variance equal to \( \int_0^t V \, dt \). Therefore, \( \int_0^t \sqrt{V} \, dW^V \) can be expressed as

\[ \int_0^t \sqrt{V} \, dW^V \approx \sigma(\tau) Z \]  

where \( Z \) represents a standard normal random variable and \( \sigma(\tau) \) is the standard deviation of \( \int_0^t \sqrt{V} \, dW^V \), which is given by

\[ \sigma^2(\tau) = \int_0^t V \, dt \approx \left( \gamma_1 V_0 + \gamma_2 V_t \right) \tau \]  

4. Numerical Results
In this section, we compare the performances of our proposed almost-exact scheme with those of some famous schemes, such as Reflection Scheme [6], Absorption Scheme [7] and Euler Full Truncated Scheme [8] for the non-affine Heston model. The theoretical price of European ATM call option is used as the reference. Four cases are set up in this subsection (see Table 1 for details).
The parameters used in Case 1 and Case 2 stem from Broadie and Kaya [7], and the parameters of Case 3 are from literature [8]. The four cases are used to verify the effectiveness of a simulation scheme in dealing with the situation where the Feller condition is unsatisfied, especially, the parameter setting in Case 3 extremely violates the Feller condition, i.e., \(2\kappa \theta = \xi^2\). In experiments, the non-affine coefficient is set equal to 1.0001, that is, we cautiously make the non-affine Heston model quite close to the affine model. By doing this, a good comparability is guaranteed among numerical schemes and the theoretical ones, since Case 1, Case 2, Case 3 take the European call option price under the affine Heston model as benchmark. Further, to overcome the fact that European call option in the non-affine Heston model has no exact value, we use the quasi-analytical expression [20], developed from the Fourier-Cosine method, as the theoretical value of Case 4. All parameter settings of four cases do not meet the Feller condition.

| Table 1. Parameters. |
|----------------------|
|                     | Case 1     | Case 2     | Case 3     | Case 4     |
| \(\xi\)             | 0.6100     | 1.0000     | 1.0000     | 1.0000     |
| \(\kappa\)           | 6.2100     | 2.0000     | 0.5000     | 2.0000     |
| \(\theta\)           | 0.0190     | 0.0900     | 0.0400     | 0.0900     |
| \(\gamma\)           | 1.0001     | 1.0001     | 1.0001     | 1.5000     |
| \(r\)                | 0.0319     | 0.0500     | 0.0000     | 0.0500     |
| \(\rho\)             | -0.7000    | -0.3000    | -0.9000    | -0.3000    |
| \(V_0\)              | 0.010201   | 0.0900     | 0.0400     | 0.0900     |

In this paper, we take RMS error into account. These errors are reported in Figure1-4 for all schemes mentioned above. Numerical experiments are run on a desktop PC with an AMD Ryzen 5 2.0 Ghz processor and 8G RAM, using Matlab 2016b in Windows 10 environment.

**Figure 1.** RMS error of Case 1. Initial asset price: 100. Maturity: 1 year. Time increment: 1/240. True option price: 6.8061.

**Figure 2.** RMS error of Case 2. Initial asset price: 100. Maturity: 5 years. Time increment: 1/240. True option price: 34.9998.

**Figure 3.** RMS error of Case 3. Initial asset price: 100. Maturity: 10 years. Time increment: 10/240. True option price: 13.0847.
Figure 4. RMS error of Case 4. Initial asset price: 100. Maturity: 5 year. Time increment: 1/240. Quasi-analytical option price: 37.1637.

Figure 1-4 report that as the simulation paths increases, the RMS error obtained by all algorithms decreases. However, the convergence and accuracy of our almost exact scheme are significantly superior to the alternatives. Specifically, figure 3 shows that the Absorption Scheme and Reflection Scheme seem to output completely wrong prices in the case where the Feller condition is extremely unsatisfied, while the numerical results from Euler Full Truncation Scheme deviate a lot from the theoretical value although it is much better than its two formers. Obviously, our almost exact scheme outperforms all others in Case3.

Above results demonstrate that when the Feller condition is not satisfied, using reflection or truncation method to deal with negative variance will bring nonnegligible discretization error, especially when $2\kappa\theta = \bar{\xi}^2$, the huge truncation or reflection errors may drive the simulation results far away from the exact value.

5. Conclusion

In this paper, we propose the key technique of almost exact simulation for non-affine Heston model, by fusing the approximate analytic solution of conditional variance characteristic function, the second-order Newton’s method and interpolation method into a whole scheme. Four European at-the-money call options are taken as benchmarks to test the performance of our almost exact simulation scheme. The numerical results show: (1) our almost exact scheme has both good convergence and accuracy in simulating the non-affine Heston model. (2) our scheme works quite well in the case where the Feller condition is extremely unsatisfied, while the existing schemes may give the wrong option prices.

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