Hyperbolic Extensions and Metrics $\epsilon$-Close to Hyperbolic

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Abstract

We define the Hyperbolic Extension of a Riemannian manifold with a center, and give some properties of it. Our main result says to what extent the hyperbolic extension of $M$ is close to being hyperbolic, if we assume $M$ to be close to hyperbolic.

The results in this paper are used in the problem of smoothing Charney-Davis strict hyperbolizations [2], [4].

Section 0. Introduction.

Recall that hyperbolic $n$-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^k \times \mathbb{H}^{n-k}$ with warped metric $(cosh^2 r) \sigma_{\mathbb{H}^k} + \sigma_{\mathbb{H}^{n-k}}$, where $\sigma_{\mathbb{H}^l}$ denotes the hyperbolic metric of $\mathbb{H}^l$, and $r : \mathbb{H}^{n-k} \to [0, \infty)$ is the distance to a fixed point in $\mathbb{H}^{n-k}$. For instance, in the case $n = 2$, since $\mathbb{H}^1 = \mathbb{R}^1$ we have that $\mathbb{H}^2$ is isometric to $\mathbb{R}^2 = \{(u,v)\}$ with warped metric $cosh^2 v du^2 + dv^2$. In the following paragraph we give a generalization of this construction.

Let $(M^n, h)$ be a complete Riemannian manifold with center $o = o_M \in M$, that is, the exponential map $exp_o : T_o M \to M$ is a diffeomorphism. The warped metric

$$g = (cosh^2 r) \sigma_{\mathbb{H}^k} + h$$

on $\mathbb{H}^k \times M$ is the hyperbolic extension (of dimension $k$) of the metric $h$. Here $r$ is the distance-to-$o$ function on $M$. We write $E_k(M) = (\mathbb{H}^k \times M, g)$, and $g = E_k(h)$. We also say that $E_k(M)$ is the hyperbolic extension (of dimension $k$) of $(M,h)$ (or just of $M$). Hence, for instance, we have $E_k(\mathbb{H}^l) = \mathbb{H}^{k+l}$. For $S \subset M$ we write $E_k(S) = \mathbb{H}^k \times S \subset E_k(M)$.

Also write $\mathbb{H}^k = \mathbb{H}^k \times \{o_M\} \subset E_k(M)$ and we have that any $p \in \mathbb{H}^k$ is a center of $E_k(M)$ (see Section 1). Since $E_k(\mathbb{H}^l) = \mathbb{H}^{k+l}$ one would expect that if $M$ is, in some sense, close to $\mathbb{H}^l$, then $E_k(M)$ would be close to $\mathbb{H}^{k+l}$. As mentioned in the abstract our main result states to what extent the hyperbolic extension of $M$ is close to being hyperbolic, if we assume $M$ to be close to hyperbolic. Our definition of “close to hyperbolic” is a chart-by-chart definition and it is given in the next paragraph (see Section 2 for more details). In this paper we also give some properties of hyperbolic extensions and introduce a set of coordinates well suited to study these objects.

Let $\mathbb{B} \subset \mathbb{R}^{l-1}$ be the unit $(l-1)$-ball, with the flat metric $\sigma_{\mathbb{B}^{l-1}}$. Write $I_\xi = (-1 + \xi, 1 + \xi)$, $\xi > 0$. Our basic models are $T_\xi = \mathbb{B} \times I_\xi$, with hyperbolic metric $\sigma = e^{2t} \sigma_{\mathbb{B}^{l-1}} + dt^2$. The number $\xi$ is called the excess of $T_\xi$. (The reason for introducing $\xi$ will become clear in Theorem A below, see Remark 1). Let $(N^l, g)$ be a Riemannian manifold and $S \subset N$. We say that $S$ is $\epsilon$-close to hyperbolic if there is $\xi > 0$ such that for every $p \in S$ there is an $\epsilon$-close to hyperbolic chart with

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center \( p \), that is, there is a chart \( \phi : \mathbb{T}_\xi \to N, \phi(0,0) = p \), such that \( |\phi^*g - \sigma|_{C^2} < \epsilon \). The number \( \xi \) is called the excess of the charts (which is fixed).

Let \((N^l, g)\) have center \( o \). Using the exponential map \( \exp_o \) we shall sometimes identify \( N \) with \( \mathbb{R}^l \), and \( N - \{ o \} \) with \( S^{l-1} \times \mathbb{R}^+ \). Let \( S \subset N \). We shall say that \( g \) is radially \( \epsilon \)-close to hyperbolic on \( S \) (with respect to \( o \)) if, in addition, the \( \epsilon \)-close to hyperbolic charts \( \phi \) respect the product structure of \( \mathbb{T} \) and \( N - o = S^l \times \mathbb{R}^+ \), that is \( \phi(., t) = (\phi_1(.), t + a) \), for some constant \( a \) depending only on \( \phi \) (see Section 2 for details). Here the “radial” directions are \( -(1 + \xi), 1 + \xi \) and \( \mathbb{R}^+ \) in \( \mathbb{T} \) and \( N - o \), respectively. Sometimes we will just say “\( S \) is radially \( \epsilon \)-close to hyperbolic” when it is clear from the context which metric is being considered.

**Remark 0.1** Our definition of radially \( \epsilon \)-close to hyperbolic metric is well suited to study metrics away from the center, but near the center this definition is not useful. This is because: (1) the need for some space to fit the charts, and (2) the form of our specific fixed model \( \mathbb{T} \). For instance hyperbolic \( n \)-space \( \mathbb{H}^n \) is not radially \( \epsilon \)-close to hyperbolic near a (chosen) center. In fact \( \mathbb{H}^n \) is radially \( \epsilon \)-close to hyperbolic outside the ball \( \mathbb{B}_a(\mathbb{H}^n) \) of radius \( a \), for a constant \( a = a(\epsilon, n, \xi) \) depending only on \( \epsilon, n, \xi \) (see 3.9 in [3]).

The next Theorem states that if \( S \subset M \) is radially \( \epsilon \)-close to hyperbolic then \( \mathcal{E}_k(S) \) is \( \eta \)-close to hyperbolic, where \( \eta \) depends on \( \epsilon \) and the distance \( r \) to \( \mathbb{H}^k \subset \mathcal{E}_k(M) \). We use the notation \( B_a = \mathbb{B}_a(M) \subset M \) for the ball of radius \( a \) centered at the center \( o = o_M \). We assume \( \xi > 0 \).

**Theorem A.** Let \( M^n \) have center \( o \), and \( S \subset M - \{ o \} \). Assume \( S \) is radially \( \epsilon \)-close to hyperbolic, with charts of excess \( \xi \). Then \( \mathcal{E}_k(S - B_a) \) is radially \( \eta \)-close to hyperbolic (with respect to any point in \( \mathbb{H}^k \)), provided

\[
C\left( \epsilon + e^{-a} \right) \leq \eta
\]

where \( C \) is a constant depending on \( k \) and \( \xi \). Moreover, \( \mathcal{E}_k(S - B_a) \) is radially \( \eta \)-close to hyperbolic, with charts of excess \( \xi' \), provided

\[
0 < \xi' < \xi - Le^{-a}
\]

where \( L \) is a constant depending on \( k \) and \( \xi \).

**Remarks.**

1. Note that the excess of the charts decreases. This is the main reason to introduce the excess. In [5] we described another geometric process, warp forcing, which also reduces the excess of the charts.

2. An explicit formula for \( L \) is given at the end of Appendix A. It is implicit in Theorem A that \( r \) cannot be too small, that is, we want \( \xi - Le^{-r} \geq 0 \), hence we want \( r \geq \ln(L) - \ln(\xi) \).

3. We can take \( C = C(k, \xi) = 2(2 + 3\xi + \xi^2)e^{1+\xi}L + C_1(c_{g_k}(k, \xi)), \) with \( C_1 \) as in Corollary 3.3 of [4], and \( c_{g_k} \) is such that \( S^k \) is \( c_{g_k} \)-bounded (see Section 3 of [4]).

We now deal with a natural and useful class of metrics. These are metrics on \( \mathbb{R}^n \) (or on a manifold with center) that are already hyperbolic on the ball \( B_a = B_a(0) \) of radius \( a \) centered at 0, and are radially \( \epsilon \)-close to hyperbolic outside \( B_{a'} \) (here \( a' \) is slightly less than \( a \)). Here is the detailed definition. Let \( M^n \) have center \( o \) and let \( B_a = B_a(o) \) be the ball on \( M \) of radius \( a \).
centered at \( o \). We say that a metric \( h \) on \( M \) is \((B_a, \epsilon)\)-close to hyperbolic, with charts of excess \( \xi \), if

1. On \( B_a - \{o\} = S^{n-1} \times (0, a) \) we have \( h = \sinh^2(t)\sigma_{S^{n-1}} + dt^2 \). Hence \( h \) is hyperbolic on \( B_a \).
2. the metric \( h \) is radially \( \epsilon \)-close to hyperbolic outside \( B_a - 1 - \xi \), with charts of excess \( \xi \).

Remarks.
1. We have dropped the word “radially” to simplify the notation. But it does appear in condition (2), where “radially” refers to the center of \( B_a \).
2. We will always assume \( a > a + 1 \), where \( a \) is as in 0.1. Therefore conditions (1), (2) and remark 0.1 imply a stronger version of (2):

   (2') the metric \( h \) is radially \( \epsilon \)-close to hyperbolic outside \( B_a \), with charts of excess \( \xi \).

This is the reason why we demanded radius \( a - 1 - \xi \) in (2), instead of just \( a \).

Metrics that are \((B_a, \epsilon)\)-close to hyperbolic are very useful, and are key objects in [3]. See also [4], [6]. Our next result answers the following question:

Question. What can we say about the hyperbolic extension of a \((B_a, \epsilon)\)-close to hyperbolic metric?

Theorem B. Let \( M^n \) have center \( o \). Assume \( M \) is \((B_a, \epsilon)\)-close to hyperbolic, with charts of excess \( \xi > 0 \). Then \( E_k(M) \) is \((B_a, C_2\epsilon)\)-close to hyperbolic, with charts of excess \( \xi' \), provided \( a \) is sufficiently large. Explicitly we want

\[
a \geq R = R(\epsilon, k, \xi)
\]

Here \( C_2 = C_2(n, k, \xi) \), and \( \xi' = \xi - e^{-a/2} > 0 \).

Remark. The constant \( R \) is defined as \( R = \ln(\frac{1}{\epsilon}) + \ln(L) + 1 + \xi \). Here \( C_2 = C_2(n, k, \xi) = C'_1 e^{1+\xi} + C \), where \( C'_1 \) is as in 3.8 of [4], and \( C, L \) as in Theorem A.

The results in this paper are used to smooth Charney-Davis strict hyperbolizations [2], [3]. In Section 1 we define hyperbolic extensions and give some properties. In Section 2 we introduce some useful coordinates in hyperbolic extensions. In Section 3 we define (with more detail) \( \epsilon \)-close to hyperbolic metrics. In Section 4 we prove Theorems A and B. There are 2 appendices in which we deal with some technical details.

Section 1. Hyperbolic Extensions.

Let \( M^n \) be a complete Riemannian manifold with center \( o = o_M \in M \), that is, the exponential map \( \exp_o : T_o M \to M \) is a diffeomorphism. In particular \( M \) is diffeomorphic to \( \mathbb{R}^n \). For instance if \( M \) is Hadamard manifold every point is a center point. Denote the metric on \( M \) by \( h \).

In this paper we will use the same symbol “\( o \)” to denote a center of a Riemannian manifold unless it is necessary to specify the manifold, in which case we will write \( o_M \) if \( o \) is a center of \( M \).

Let \( r : M \to [0, \infty) \) be the distance to \( o \). Then

1. we have that \( r(\exp_o v) = h_o(v, v)^{1/2} \), hence \( r \) is continuous and smooth on \( M - \{o\} \). Also \( r^2 \) is smooth on \( M \).
ii. The (images of the) geodesic rays $exp_o(\mathbb{R}^+ v)$ are convex sets in $M$, and the geodesics lines $exp_o(\mathbb{R} v)$ are totally geodesic in $M$. Here $\mathbb{R}^+ = (0, \infty)$.

iii. The function $dr$ is strictly distance decreasing on non-radial vectors. That is, for $v \in TM - T_oM$ we have $|dr(v)| = \mu h(v, v)^{1/2}$ and $|dr(v)| = h(v, v)^{1/2}$ if and only if $v$ is radial, i.e. tangent to a geodesic passing through $o$. (This follows from the Gauss Lemma and the fact that $r \circ (exp_o)^{-1} : T_oM \to \mathbb{R}$ is the euclidean distance to the origin.)

Using the diffeomorphism $exp_o$ onto $M$ and an identification of $T_oM$ with $\mathbb{R}^n$ via some fixed choice of an orthonormal basis in $T_oM$, we can identify $M$ with $\mathbb{R}^n$ and $M - \{o\}$ with $S^{n-1} \times \mathbb{R}^+$. Therefore the metric $h|_{M - \{o\}}$ can be written as $h_r + dr^2$ on $S^{n-1} \times \mathbb{R}^+$. Also we shall call the set of (h-geodesic) rays $t \mapsto (x, t) \in S^{n-1} \times \mathbb{R}^+$ the ray structure of $h$ with respect to $o$.

As mentioned in the Introduction the warped metric $g = (\cosh^2 r) \sigma_{\mu k} + h$ on $\mathbb{H}^k \times M$ is a hyperbolic extension of the metric $h$ on $M$, and sometimes we will also write $g = \mathcal{E}_k(h)$. Note that, even though $r$ is not smooth at $o$, the warping function $\cosh r$ is smooth on $M$ because $\cosh$ is a smooth even function. Since $M$ is complete we have that $\mathcal{E}_k(M)$ is also complete (see [1], p.23).

For instance, if $M = \mathbb{H}^l$ then the hyperbolic extension $\mathcal{E}_k(\mathbb{H}^l)$ is hyperbolic $(k+l)$-space $\mathbb{H}^{k+l} = \mathbb{H}^k \times \mathbb{H}^l$, with metric $(\cosh^2 r) \sigma_{\mu k} + \sigma_{\mu l}$.

For a subset $A \subset \mathbb{H}^k$ we shall write $\mathcal{E}_A(M) = A \times M \subset \mathcal{E}_k(M)$, with the metric $\mathcal{E}_k(h)$ restricted to the set $\mathbb{H}^k \times A$.

We will write $\mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset \mathcal{E}_k(M)$. The hyperbolic extension, away from $\mathbb{H}^k \subset \mathcal{E}_k(M)$ can be described in an alternative way: it is isometric to $(\mathbb{H}^k \times S^{n-1}) \times (0, \infty)$ with metric $(\cosh^2 r) \sigma_{\mu k} + h_r + dr^2$.

Note that $\mathbb{H}^k$ and every $\{y\} \times M$ are convex in $\mathcal{E}_k(M)$ (see [1], p.23). Let $\eta$ be a complete geodesic line in $M$ passing through $o$ and let $\eta^+$ be one of its two geodesic rays (beginning at $o$). Then $\eta$ is a totally geodesic subspace of $M$ and $\eta^+$ is convex (see item (ii) above). Also, let $\gamma$ be a complete geodesic line in $\mathbb{H}^k$.

**Lemma 1.1.** We have that $\gamma \times \eta^+$ is a convex subspace of $\mathcal{E}_k(M)$ and $\gamma \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.

**Proof.** Let $\pi_\gamma : \mathbb{H}^k \to \gamma \subset \mathbb{H}^k$ denote the orthogonal projection, and note that $d\pi_\gamma$ is distance non-increasing, i.e. $\sigma_{\mu k}(v, v) \geq \sigma_{\mu k}(d\pi_\gamma(v), d\pi_\gamma(v))$, for $v \in T\mathbb{H}^k$. Moreover, the equality holds if and only if $v \in T\gamma$.

We assume $\eta : [0, \infty) \to \eta \subset M$ to be parametrized by the arc-length, that is, it is a speed-one geodesic ray. Let $\pi_\eta : M \to \eta^+$ denote the proper map $\pi_\eta(p) = \eta(r(p))$. Note that $\pi_\eta$ is smooth on $M - \{o\}$ and item (iii) above implies that $\pi_\eta$ is strictly distance decreasing on non-radial tangent vectors on $M - \{o\}$.

Let $\alpha : [0, 1] \to \mathbb{H}^k \times M$ and write $\alpha(u) = (a(u), b(u)) \in \mathbb{H}^k \times M$. Assume $b(u) \neq o$ for all $u \in [0, 1]$. Let $\beta = (\pi_\gamma a, \pi_\eta b)$. Hence the length of $\alpha' = (a', b')$ is less or equal the length of $\beta' = (d\pi_\gamma(a'), d\pi_\eta(b'))$. Therefore the length of $\alpha$ is greater than the length of $\beta$, unless $a = \pi_\gamma(a)$ and $b$ is contained in a ray. And, by continuity the same holds without the assumption that $b(u) \neq o$. Therefore $\gamma \times \eta^+$ is convex because $\beta$ is a path in $\gamma \times \eta^+$. 


We prove that $\gamma \times \eta$ is totally geodesic. Let $\eta^- = \eta - \eta^+$ be the “other” geodesic ray of $\eta$. Then $\gamma \times \eta^-$ is also convex. Therefore $\gamma \times \eta - \gamma$ is totally geodesic hence the second fundamental of $\gamma \times \eta$ vanishes there. By continuity this form vanishes on the whole of $\gamma \times \eta$. This proves the lemma.

**Corollary 1.2.** We have that $\mathbb{H}^k \times \eta^+$ and $\gamma \times M$ are convex in $\mathcal{E}_k(M)$. Also $\mathbb{H}^k \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.

**Proof.** For $\mathbb{H}^k \times \eta$ just replace $\gamma$ by $\mathbb{H}^k$ and $\pi_\gamma$ by the identity in the proof of Lemma 1.1. For $\gamma \times M$ replace $\beta$ in the proof of Lemma 1.1 by $\beta = (\pi_\gamma a, b)$. This proves the Corollary.

**Remarks 1.3.**

1. Note that $\mathbb{H}^k \times \eta$ (with metric induced by $\mathcal{E}_k(M)$) is isometric to $\mathbb{H}^k \times \mathbb{R}$ with warped metric $\cosh^2 v s_{\eta^k} + dv^2$, which is just hyperbolic $(k + 1)$-space $\mathbb{H}^{k+1}$. Also $\gamma \times \eta$ is isometric to $\mathbb{R} \times \mathbb{R}$ with warped metric $\cosh^2 v du^2 + dv^2$, which is just hyperbolic 2-space $\mathbb{H}^2$. In particular every point in $\mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset \mathcal{E}_k(M)$ is a center point.

2. It follows from Lemma 1.1 and Remark 1 that the ray structure of $\mathcal{E}_k(h)$ with respect to any center $o_{\eta^k} \in \mathbb{H}^k \subset \mathcal{E}_k(M)$ only depends on the ray structure of $M$ and the center $o_{\eta^k}$.

3. Denote by $\mathbb{B}_r(M)$ the ball of radius $r$ of $M$. Note that if $h$ and $h'$ on $M$ have the same ray structures then the balls $\mathbb{B}_r(M)$ coincide.

4. Recall that $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$. Moreover, for $l \leq k$, we also have $\mathbb{H}^l \subset \mathbb{H}^k \subset \mathcal{E}_k(M)$ is convex. If $h$ and $h'$ on $M$ have the same ray structures then the $r$-neighborhoods (with respect to $h$ and $h'$) of the convex subset $\mathbb{H}^l$ coincide.

**Section 2. Coordinates on $\mathcal{E}_k(M)$.**

Recall that we are identifying $M - \{o\}$, $o = o_M$, with $S^{n-1} \times \mathbb{R}^+$, and sometimes we shall denote a point $v = (u, r) \in S^{n-1} \times \mathbb{R}^+ = M - \{o\}$ by $v = ru$. Fix a center $o = o_{\eta^k} \in \mathbb{H}^k \in \mathcal{E}_k(M)$ and we get a center $o = o_{\mathcal{E}_k(M)} = (o_{\eta^k}, o_M)$ of $\mathcal{E}_k(M)$. Then, for $y \in \mathbb{H}^k - \{o\}$ we can also write $y = tw$, $(w, t) \in S^{k-1} \times \mathbb{R}^+$. Similarly, using the exponential map we can identify $\mathcal{E}_k(M) - \{o\}$ with $S^{k+n-1} \times \mathbb{R}^+$, and for $p \in \mathcal{E}_k(M) - \{o\}$ we can write $p = sx$, $(x, s) \in S^{k+n-1} \times \mathbb{R}^+$.

As before denote the metric on $\mathcal{E}_k(M)$ by $g$ and we can write $g = g_s + ds^2$. Since $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$ we can write $\mathbb{H}^k - \{o\} = S^{k-1} \times \mathbb{R}^+ \subset S^{k+n-1} \times \mathbb{R}^+$ and $S^{k-1} \subset S^{k+n-1}$.

A point $p \in \mathcal{E}_k(M) - o$ has two sets of coordinates: the polar coordinates $(x, s) = (x(p), s(p)) \in S^{k+n-1} \times \mathbb{R}^+$ and the hyperbolic extension coordinates $(y, v) = (y(p), v(p)) \in \mathbb{H}^k \times M$. Write
$M_o = \{o\} \times M$. Therefore we have the following functions:

- the distance to o function: $s : \mathcal{E}_k(M) \to [0, \infty)$, $s(p) = d_{\mathcal{E}_k(M)}(p, o)$
- the direction of p function: $x : \mathcal{E}_k(M) - \{o\} \to \mathbb{S}^{n+k-1}$, $p = s(p)x(p)$
- the distance to $\mathbb{H}^k$ function: $r : \mathcal{E}_k(M) \to [0, \infty)$, $r(p) = d_{\mathcal{E}_k(M)}(p, \mathbb{H}^k)$
- the projection on $\mathbb{H}^k$ function: $y : \mathcal{E}_k(M) \to \mathbb{H}^k$
- the projection on M function: $v : \mathcal{E}_k(M) \to M$
- the projection on $\mathbb{S}^{n-1}$ function: $u : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{S}^{n-1}$, $v(p) = r(p)u(p)$
- the length of y function: $t : \mathcal{E}_k(M) \to [0, \infty)$, $t(w) = d_{\mathbb{H}^k}(y, o)$
- the direction of y function: $w : \mathcal{E}_k(M) - M_o \to \mathbb{S}^{k-1}$, $y(p) = t(p)w(p)$

Note that $r = d_M(v, o)$. Note also that, by 1.1, the functions $w$ and $u$ are constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is $w(sx) = w(x)$ and $u(sx) = u(x)$.

Let $\partial_r$ and $\partial_s$ be the gradient vector fields of $r$ and $s$, respectively. Since the M-fibers $M_y = \{y\} \times M$ are convex the vectors $\partial_r$ are the velocity vectors of the speed one geodesics of the form $a \to (y, a u)$, $u \in \mathbb{S}^{n-1} \subset M$. These geodesics emanate from (and orthogonally to) $\mathbb{H}^k \subset \mathcal{E}_k(M)$. Also the vectors $\partial_s$ are the velocity vectors of the speed one geodesics emanating from $o \in \mathcal{E}_k(M)$. For $p \in \mathcal{E}_k(M)$, denote by $\Delta = \Delta(p)$ the right triangle with vertices $o$, $y = y(p)$, $p$ and sides the geodesic segments $[o, p] \subset \mathcal{E}_k(M)$, $[o, y] \subset \mathbb{H}^k$, $[p, y] \subset \{y\} \times M \subset \mathcal{E}_k(M)$. (These geodesic segments are unique and well defined because: (1) $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$, (2) $(y, o) = o_{\{y\} \times M}$ and $o$ are centers in $\{y\} \times M$ and $\mathbb{H}^k \subset \mathcal{E}_k(M)$, respectively.)

**Lemma 2.1** Let $\eta^+$ (or $\eta$) be a geodesic ray (line) in $M$ through $o$ containing $v = v(p)$ and $\gamma$ a geodesic line in $\mathbb{H}^k$ through $o$ containing $y = y(p)$. Then $\Delta(p) \subset \gamma \times \eta^+ \subset \gamma \times \eta$.

**Proof.** We have that $[o, v] \subset \eta$ and $[o, y] \in \gamma$. By Lemma 1.1 we have $[o, p] \in \gamma \times \eta^+$. This proves the lemma.

Let $\alpha : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{R}$ be the angle from $\partial_r$ to $\partial_x$ (in that order), thus $\cos \alpha = g(\partial_r, \partial_x)$, $\alpha \in [0, \pi]$. Then $\alpha = \alpha(p)$ is the interior angle, at $p = (y, v)$, of the right triangle $\Delta = \Delta(p)$. We call $\beta(p)$ the interior angle of this triangle at $o$, that is $\beta(p) = \beta(x)$ is the spherical distance between $x \in \mathbb{S}^{k+n-1}$ and the totally geodesic sub-sphere $\mathbb{S}^{k-1}$. Alternatively, $\beta$ is the angle between the geodesic segment $[o, p] \subset \mathcal{E}_k(M)$ and the convex submanifold $\mathbb{H}^k$. Therefore $\beta$ is constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is $\beta(sx) = \beta(x)$.

Note that the right geodesic triangle $\Delta(p)$ has sides of length $r = r(p)$, $t = t(p)$ and $s = s(p)$. By Lemma 2.1 and Remark 1.3 we can consider $\Delta$ as contained in hyperbolic 2-space. Hence using hyperbolic trigonometric identities we can find relations between $r$, $t$, $s$, $\alpha$ and $\beta$. For instance, using the hyperbolic law of cosines we get:

\[
(2.2) \quad \cosh(s) = \cosh(r) \cosh(t)
\]
Note that this implies $t \leq s$. Here is an application of this equation.

**Proposition 2.3 (Iterated hyperbolic extensions)** We have that

$$E_l(E_k(M)) = E_{l+k}(M)$$

where we are identifying $\mathbb{H}^{l+k}$ with $\mathbb{H}^l \times \mathbb{H}^k$ with warped metric $(\cosh^2 t) \sigma_{\mathbb{H}^l} + \sigma_{\mathbb{H}^k}$.

**Remarks.**
1. Note that the identification of $\mathbb{H}^{l+k}$ with $\mathbb{H}^l \times \mathbb{H}^k$ (with warp metric) depends on the order of $l$ and $k$, that is, on the order in which the hyperbolic extensions are taken.
2. As before, here the function $t : \mathbb{H}^k \to [0, \infty)$ is the distance in $\mathbb{H}^k$ to the point $o \in \mathbb{H}^k$.

**Proof of Proposition 2.3.** As above let $s : \mathbb{H}^k \times M \to [0, \infty)$ be the distance in $E_k(M)$ to $o$, $r(p) = d_M(v(p), o)$, and $t$ as in the statement of the proposition. Then $E_l(E_k(M))$ is $\mathbb{H}^l \times (\mathbb{H}^k \times M)$ with metric

$$(\cosh^2 s) \sigma_{\mathbb{H}^l} + [(\cosh^2 r) \sigma_{\mathbb{H}^k} + h]$$

On the other hand $E_{l+k}(M)$ is $(\mathbb{H}^l \times \mathbb{H}^k) \times M$ with metric

$$(\cosh^2 r) \left[ (\cosh^2 t) \sigma_{\mathbb{H}^l} + \sigma_{\mathbb{H}^k} \right] + h$$

Hence the Proposition follows from identity (2.2) above. This proves the Proposition.

**Proposition 2.4.** We have the following identity defined outside $\mathbb{H}^k \cup \{o \times M\}$

$$(\sinh^2 s) d\beta^2 + ds^2 = \cosh^2(r) dt^2 + dr^2$$

**Proof.** First a particular case. Take $M = \mathbb{R}$ and $k = 1$, hence $E_k(M) = E_1(\mathbb{R}) = \mathbb{H}^2$. In this case the left-hand side of the identity above is the expression of the metric of $\mathbb{H}^2$ in polar coordinates $(\beta, s)$, and right hand side of the equation is the expression of the same metric in the hyperbolic extension coordinates $(r, t) = (v, y)$. (Here $r$ and $t$ are “signed” distances.) Hence the equation holds in this particular case.

Now, the general case can be reduced to this particular case using Lemma 1.1 and Remark 1 in 1.3. This proves the proposition.

A direct (and longer) proof of the lemma above can be given using hyperbolic trigonometric identities.

**Section 3. $\varepsilon$-close to hyperbolic metrics.**
Let $B = B^{l-1} \subset \mathbb{R}^{l-1}$ be the unit ball, with the flat metric $\sigma_{\mathbb{R}^{l-1}}$. Write $I_\xi = (-1 + \xi, 1 + \xi)$, $\xi > 0$. Our basic models are $T_\xi = B \times I_\xi$, with hyperbolic metric $\sigma = e^{2t} \sigma_{\mathbb{R}^{l-1}} + dt^2$. In what follows we may sometimes suppress the subindex $\xi$, if the context is clear. The number $\xi$ is called the excess of $T_\xi$.

**Remarks.**
1. As mentioned in the Introduction the reason to introduce the excess is that the process of hyperbolic extension decreases the excess of the charts, as shown in the statement of Theorem A.
2. In the applications we may actually need warped metrics with warping functions that are multiples of hyperbolic functions. All these functions are close to the exponential $e^t$ (for $t$ large), so instead of introducing one model for each hyperbolic function we introduced only the exponential model.

Let $|.|$ denote the uniform $C^2$-norm of $\mathbb{R}^n$-valued functions on $\mathbb{T}_\xi = \mathbb{B} \times I_\xi \subset \mathbb{R}^l$. Given a metric $g$ on $\mathbb{T}$, $|g|$ is computed considering $g$ as the $\mathbb{R}^l$-valued function $(x,t) \mapsto (g_{ij}(x,t))$ where, as usual, $g_{ij} = g(e_i,e_j)$, and the $e_i$’s are the canonical vectors in $\mathbb{R}^l$. We will say that a metric $g$ on $\mathbb{T}$ is $\epsilon$-close hyperbolic if $|g - \sigma| < \epsilon$.

A Riemannian manifold $(M^l,g)$ is $\epsilon$-close hyperbolic if there is $\xi > 0$ such that for every $p \in M$ there is an $\epsilon$-close to hyperbolic chart with center $p$, that is, there is a chart $\phi : \mathbb{T}_\xi \to M$, $\phi(0,0) = p$, such that $\phi^*g$ is $\epsilon$-close to hyperbolic. Note that all charts are defined on the same model space $\mathbb{T}_\xi$. The number $\xi$ is called the excess of the charts (which is fixed). More generally, a subset $S \subset M$ is $\epsilon$-close to hyperbolic if every $p \in S$ is the center of an $\epsilon$-close to hyperbolic chart in $M$ with fixed excess $\xi$.

If $N^l$ has center $o$ we say that $S \subset N$ is radially $\epsilon$-close to hyperbolic (with respect to $o$) if, in addition, the charts $\phi$ respect the product structure of $\mathbb{T}$ and $N - o = S^{l-1} \times \mathbb{R}^+$, that is $\phi(.,t) = (\phi_1(.,t+a))$, for some $a$ depending on the $\phi$. Note also that the term “radially” in the definition above refers to the decomposition of the manifold $M - o$ as a product $S^{l-1} \times \mathbb{R}^+$.

Of course a radially $\epsilon$-close to hyperbolic manifold is $\epsilon$-close to hyperbolic.

Remarks.

1. The definition of radially $\epsilon$-close to hyperbolic metrics is well-suited to studying metrics of the form $g + dt^2$ for $t$ large, but for small $t$ this definition is not useful because: (1) we need some space to fit the charts, and (2) the form of our specific fixed model $\mathbb{T}$. An undesired consequence is that punctured hyperbolic space $\mathbb{H}^n - \{o\} = S^{n-1} \times \mathbb{R}^+$ (with warp metric $\sinh^2(t)\sigma_{^{\mathbb{S}^{n-1}}} + dt^2$) is not radially $\epsilon$-close to hyperbolic for $t$ small.

2. Note that a radially $\epsilon$-close hyperbolic chart $\phi : \mathbb{T} = \mathbb{B} \times I_\xi \to M$, $\phi(.,t) = (\phi_1(.,t+a))$, has a natural extension to a smooth chart defined on $\mathbb{B} \times \mathbb{R}^+$ (using the same formula), but this extension may fail to be $\epsilon$-close hyperbolic outside $\mathbb{T}$.

4. The hyperbolic extension of an $\epsilon$-close to hyperbolic metric.

Let $(M^n,h)$ have center $o$ as above and consider the hyperbolic extension $E_k(M)$. As before the metric $E_k(h)$ on $E_k(M)$ is denoted by $g$. The ball of radius $a$ on $M$ centered at $o$ will be denoted by $B_a$. Choose $o \in \mathbb{H}^k \subset E_k(M)$. Recall that $o$ is a center of $E_k(M)$ (see 1.3), hence we can express $E_k(M) - \{o\}$ as $S^{n+k-1} \times \mathbb{R}^+$ with variable metric $g = g_o + ds^2$. The next Theorem is Theorem A in the Introduction.

Theorem 4.1. Let $M$ have center $o$, and $S \subset M^n - \{o\}$. Assume $S$ is radially $\epsilon$-close to hyperbolic, with charts of excess $\xi$. Then $E_k(S - B_a)$ is radially $\eta$-close to hyperbolic, provided

$$C \left( \epsilon + e^{-a} \right) \leq \eta$$

where $C$ is a constant depending on $k$ and $\xi$. Moreover, $E_k(S - B_a)$ is radially $\eta$-close to hyperbolic.
with charts of excess \(\xi'\), provided
\[
0 < \xi' < \xi - Le^{-a}
\]
where \(L\) is a constant depending on \(k\) and \(\xi\).

Remarks.
1. Note that in the statement above \(h\) is radially \(\epsilon\)-close to hyperbolic with respect to the decomposition \(M - \{0\} = S^{n-1} \times \mathbb{R}^+\) and \(g\) is radially \(\eta\)-close to hyperbolic with respect to the decomposition \(E_k(M) - \{0\} = S^{n+k-1} \times \mathbb{R}^+\).
2. An explicit formula for \(L\) is given at the end of Appendix A. It is implicit in Theorem 4.1 that \(r\) cannot be too small, that is, we want \(\xi - Le^{-\tau} \geq 0\), hence we want \(r \geq \ln(L) - \ln(\xi)\).
3. We can take \(C = C(k,\xi) = 2(2 + 3\xi + \xi^2)e^{1+\xi} L + C_4\), where \(C_4 = C_4(k,\xi) = C_1(c_{\xi k}, k, \xi)\), with \(C_1\) as in Corollary 3.3 of [4], and \(c_{\xi k}\) is such that \(S^k\) is \(c_{\xi k}\)-bounded (see Section 3 of [4]).

Before we prove the proposition we shall give a particular type of charts on \(H^{k+1} = E_k(\mathbb{R})\). For this we consider two ways of describing \(H^{k+1}\): using the hyperbolic extension coordinates and the polar coordinates described in Section 2 (for the case \(M = \mathbb{R}\)). Hence a point \(z \in H^{k+1}\) has hyperbolic extension coordinates \((y, r) \in \mathbb{H}^k \times \mathbb{R}\) and polar coordinates \((x, t) \in S^k \times \mathbb{R}^+\). Using the law of sines and the laws of cosines for right triangles in \(H^2\) we can find transformation rules between the coordinates \((x, t)\) and the coordinates \((y, r)\). We are only interested in the explicit expression for \(r = r(x, t)\). In this case we have

\[
(4.2) \quad r(x, t) = \sinh^{-1} \left( \sinh(t) \sin(\beta(x)) \right)
\]

where \(\beta(x)\) is the spherical distance from \(x \in S^k\) to the equator \(S^k \cap H^k \subset H^{k+1}\).

Fix \(\xi \geq 0\). Let \(z_0 = (x_0, t_0) \in S^k \times (2 + \xi, \infty) \subset E_k(\mathbb{R})\) and let \((y_0, r_0)\) be the hyperbolic extension coordinates of \(z_0\). The following definition is a particular version of definition (\(\ast\)) given at the beginning of the proof of Theorem 3.2 in [4]. We define the chart \(\psi = \psi_{z_0} : T_{\xi}^{k+1} \rightarrow S^k \times \mathbb{R}^+ = H^{k+1} - \{0\}\) by

\[
(4.3) \quad \psi(x, t) = \left( \exp_{x_0}(e^{\lambda-t_0} x), t_0 + t \right)
\]

where: (1) we are identifying the euclidean unit ball \(\mathbb{B}^k\) with the unit ball in the tangent space \(T_{x_0}S^k\), (2) \(\exp_{x_0} : T_{x_0}S^k \rightarrow S^k\) is the exponential map, and (3) \(\lambda = \min\{0, t_0 - ln(kc_4)\}\), \(c_4 = \sqrt{k!c^*(S^k)}\). (The \(\lambda\) is a correcting term for \(t\) small, see proof of 3.2 in [4]. Here \(c(S^k)\) is such that \(S^k\) is \(c(S^k)\)-bounded, see Section 3 of [4]). Note that in the formula above the output of the map \(\psi\) is given in polar coordinates.

Lemma 4.4. The chart \(\psi\) is a radially \(\epsilon\)-close to hyperbolic chart, provided

\[
C_4 e^{-t_0} \leq \epsilon
\]

where \(C_4 = C_4(k, \xi)\) is as in Remark 2 after Theorem 4.1.

Proof. This lemma was proven in a more general form in the proof of Theorem 3.2 of [4]: the chart \(\psi\) is a special case of the chart \(\phi\) that appears in equation (\(\ast\)) at the beginning of the proof of 3.2 [4]. To see this note: (1) in the proof of 3.2 [4] it is proven that the chart \(\psi\) in (\(\ast\) of
is \( \eta \)-close to hyperbolic, provided \( \eta \geq C(e^{-t_0} + \epsilon) \), for certain constant \( C \) (different from the \( C \) in 4.1), (2) in our case the map \( \varphi \) in (\( \ast \)) is the exponential map \( \exp_{x_0} \), hence \( A \) in (\( \ast \)) is the identity (the derivative of the exponential is the identity), (3) we can take \( \epsilon = 0 \) in (\( \ast \)) because the family of metrics \( \{ \sigma_{g_k} \} \) is constant hence “zero”-slow (i.e. \( \epsilon \)-slow for every \( \epsilon > 0 \), see Section 3 of \( \mathcal{H} \)), (4) the chart in (\( \ast \)) works to prove Theorem 3.2 \( \mathcal{H} \) (where warping function \( e^t \) is used), and the same chart works to prove Corollary 3.3 \( \mathcal{H} \) (where warping function \( \sinht \) is used), but the constant \( C \) changes to a new constant \( C_1 = C_1(c, k, \xi) \) (see item 3 in Remarks 0.3 \( \mathcal{H} \), Remark 3 after the statement of Theorem 3.2 \( \mathcal{H} \) and Remark 2 after the statement Corollary 3.3 \( \mathcal{H} \)). In our case we can take \( c = c_{g_k} \) such that \( \mathcal{S}^k \) is \( c_{g_k} \)-bounded, therefore our constant becomes \( C_4(k, \xi) = C_1(c_{g_k}, k, \xi) \) as in Remark 2 after 4.1. This proves the lemma.

Denote the hyperbolic extension coordinates of \( \psi \) by \( y = y_{z_0} : \mathbb{T}_\xi \to \mathbb{H}^k \) and \( \tilde{r} = \tilde{r}_{z_0} : \mathbb{T}_\xi \to \mathbb{R} \). That is

\[
\psi(x, t) = (y(x, t), \tilde{r}(x, t)) \in \mathbb{H}^k \times \mathbb{R} = \mathcal{E}_k(\mathbb{R})
\]

Using equation (4.2) we can write

\[
(4.5.)
\tilde{r}(x, t) = \sinh^{-1}\left( \sinh(t_0 + t) \sin(\beta x') \right)
\]

where \( x' = \exp_{x_0}(e^{\lambda-t_0} x) \). Recall that \((y_0, r_0)\) are the hyperbolic extension coordinates of \( z_0 \).

**Lemma 4.6.** We have that

\[
|\tilde{r}(x, t) - (t + r_0)|_{C^2} \leq L e^{-r_0}
\]

where \( L \) is a constant depending on \( k \) and \( \xi \).

The proof of Lemma 4.6 is given in Appendix A. An explicit formula for \( L \) is given at the end of Appendix A.

The next result is the reason why we introduced the variable \( \xi \) in the definition of the models \( \mathbb{T}_\xi \).

**Corollary 4.7.** We have that

\[
\psi(\mathbb{T}_{\xi'}) \subset \mathbb{H}^k \times [r_0 - (1 + \xi), r_0 + (1 + \xi)]
\]

provided \( 0 < \xi' < \xi - Le^{-t_0} \), where \( L \) is as in 4.6.

**Proof.** Write \( \kappa = Le^{-t_0} \). By Lemma 4.6 we have \( (t + r_0) - \kappa \leq \tilde{r}(x, t) \leq (t + r_0) + \kappa \). Hence for \( t \in (-1 - \xi', 1 + \xi') \) we get \( r_0 - (1 + \xi' + \kappa) \leq \tilde{r}(x, t) \leq r_0 + (1 + \xi' + \kappa) \). This together with the choice \( \xi' + \kappa \leq \xi \) implies \( r_0 - (1 + \xi) \leq \tilde{r}(x, t) \leq r_0 + (1 + \xi) \). This proves the corollary.

**Proof of Theorem 4.1.** First some notation. Recall that we are denoting the metric on \( M - \{o\} \) by \( h = h_r + dr^2 \) and the one on \( \mathcal{E}_k(M) \) by \( g \). Also recall \( I_\xi = [-1 - \xi, 1 + \xi] \). For \( u \in \mathbb{S}^{n-1} \) we denote by \( Ru \) the complete geodesic line in \( M \) passing through \( o \) with direction \( u \), i.e. \( Ru = \exp_o(\mathbb{R} u) = \{ p \in M \text{ such that } p = ru \}, r \in \mathbb{R} \). Also write \( R^+ u = \exp_o(\mathbb{R}^+ u) \). Then \( Ru = R(-u) \) but \( R^+ u \cap R^+(-u) = \emptyset \). Hence we get

\[
\mathcal{E}_k(M) - \mathbb{H}^k = \mathbb{H}^k \times (M - \{o\}) = \bigcup_{u \in \mathbb{S}^{n-1}} \left( \mathbb{H}^k \times R^+ u \right)
\]

(1)
Specifically, if \( w = (y, p) \in \mathbb{H}^k \times M = \mathcal{E}_k(M), p \neq o, \) then \( w \in \mathbb{H}^k \times R^+u, \) provided \( p \) has polar coordinates \((u, r), \) for some \( r > 0. \)

Since we are taking \( u \) with length one, we have an obvious identification of \( Ru \) with \( \mathbb{R}, \) given by \( r \mapsto ru \) (this identification does depend on the “sign” of \( u). \) This identification gives a canonical (metric) identification of \( \mathcal{E}_k(Ru) = \mathbb{H}^k \times Ru \) (with metric \( g|_{\mathcal{E}_k(Ru)} \)) with \( \mathcal{E}_k(\mathbb{R}) = \mathbb{H}^k \times \mathbb{R} = \mathbb{H}^{k+1} \) (with the canonical warped metric).

Write \( \mathcal{E}_k(R^+u) = \mathbb{H}^k \times R^+u \subset \mathcal{E}_k(Ru) \) and note that we can canonically identify \( \mathcal{E}_k(R^+u) \) with half hyperbolic \((k+1)\)-space \( \mathbb{H}^{k+1} = \mathbb{H}^k \times \mathbb{R}^+ \subset \mathbb{H}^k \times \mathbb{R} = \mathbb{H}^{k+1}. \)

For \( r > 0 \) and \( y \in \mathbb{H}^k \) denote by \( S_{r,y} \) the set \( \{(y, ru) \in \mathbb{H}^k \times M, u \in S^{n-1}\}. \) Then \( S_{r,y} \) is the geodesic sphere of radius \( r \) of the convex submanifold \( \{y\} \times M \subset \mathcal{E}_k(M). \) Note that every \( \mathcal{E}_k(R^+u) \) intersects every \( S_{r,y} \) orthogonally in the single point \((y, ru). \)

Let \( w_0 = (y_0, p_0) \in \mathbb{H}^k \times M \) and let \((u_0, r_0)\) be the polar coordinates of \( p_0 \in S \subset M. \) Also let \( t_0 \) be the distance in \( \mathcal{E}_k(M) \) from \( o \) to \( w_0. \) Since \( p_0 \in S \) and \( S \) is radially \( \epsilon \)-close to hyperbolic, there is a radially \( \epsilon \)-close to hyperbolic chart \( \phi : T_{\xi}^o \rightarrow M \) with center \( p_0. \) From the definition of a radially \( \epsilon \)-close to hyperbolic chart (see Section 3) we have that for \((x, r) \in T_{\xi}^o = \mathbb{B}^{n-1} \times I_\xi \) we can write

\[
\phi(x, r) = (u(x), r + r_0) = (r + r_0)u(x)
\]

Also write \( z_0 = (y_0, r_0) \in \mathcal{E}_k(\mathbb{R}^+u) \subset \mathcal{E}_k(\mathbb{R}u) = \mathbb{H}^{k+1} \) and let \( \psi = \psi_{z_0} =: \mathbb{B}^k \times \mathbb{R}^+ \rightarrow \mathbb{H}^{k+1}, \) be as defined in (4.3). And, as before, write \( \psi = \psi_{z_0} = (y_0, \bar{r}_0) = (y, \bar{r}), \) where \( (y, \bar{r}) \) are the hyperbolic extension coordinates of \( \psi. \) Note that we are taking the domain of \( \psi \) as \( \mathbb{B}^k \times \mathbb{R}^+ \) and not just \( T_{\xi}^{k+1} = \mathbb{B}^k \times I_\xi \) (see Remark 2 at the end of Section 3). We now define a chart \( \bar{\phi} : \mathbb{B}^k \times \mathbb{B}^{n-1} \times \mathbb{R}^+ \rightarrow \mathcal{E}_k(M) \) by

\[
\bar{\phi}(x_1, x_2, t) = \left(y(x_1, t), \bar{r}(x_1, t)u(x_2)\right) \in \mathbb{H}^k \times M = \mathcal{E}_k(M)
\]

Note that, by Corollary 4.7 we have that

\[
\bar{\phi}(\mathbb{B}^k \times \mathbb{B}^{n-1} \times I_{\xi'}) \subset \mathbb{H}^k \times \phi(T_{\xi})
\]

provided \( \xi' < \xi - Le^\circ. \) By the definition of \( \bar{\phi} \) (see equation (3)) and (2) we have

\[
\bar{\phi}\left(\{x_1\} \times \mathbb{B}^{n-1} \times \{t\}\right) = \phi\left(\mathbb{B}^{n-1} \times \{\bar{r}(x_1, t) - r_0\}\right) \subset S_{y(x_1, t), y(x_1, t)}
\]

and

\[
\bar{\phi}\left(\mathbb{B}^k \times \{x_2\} \times I_{\xi'}\right) \subset \mathcal{E}_k(R^+u(x_2))
\]

Moreover, using (6), the canonical metric identification between \( \mathcal{E}_k(Ru) \) and \( \mathcal{E}_k(\mathbb{R}) = \mathbb{H}^{k+1}, \) and the obvious identification between \( \mathbb{B}^k \times \{x_2\} \times I_{\xi'} \) and \( T_{\xi'} = \mathbb{B}^k \times I_{\xi'}, \) we can say that the chart \( \bar{\phi} \) satisfies

\[
\bar{\phi}|_{\mathbb{B}^k \times \{x_2\} \times I_{\xi'}} = \psi
\]
Also, from (5), (3), (2), and using the obvious identifications of \( \{ x_1 \} \times B^{n-1} \times \{ t \} \) with \( B^{n-1} \times \{ \bar{r}(x_1,t) - r_0 \} \) and \( y(x_1,t) \times M \) with \( M \), we can write
\[
\hat{\phi}|_{\{ x_1 \} \times B^{n-1} \times \{ t \}} = \hat{\phi}|_{B^{n-1} \times \{ \bar{r}(x_1,t) - r_0 \}}
\] (8)

Since, as mentioned above, every \( E_k(R^+u) \) intersects every \( S_{r,h} \) \( g \)-orthogonally in a single point, we have that the \( B^{n-1} \)-fibers \( \{ x_1 \} \times B^{n-1} \times \{ t \} \), and the \( (B^k \times I) \)-fibers \( B^k \times \{ x_2 \} \times I_\xi \), are \( \hat{\phi}^*(g) \)-orthogonal. Also, by (7), \( \hat{\phi}^*(g) \) restricted to a \( (B^k \times I) \)-fiber is canonically hyperbolic, hence, by Lemma 4.4 and the fact that \( r_0 \leq t_0 \), we have

(4.8) The metric \( \hat{\phi}^*(g) \), restricted to a \( B^k \times I \)-fiber, is \( \delta \)-close to hyperbolic, provided \( C_4 e^{-r_0} \leq \delta \)

Therefore \( \hat{\phi} \) has the form \( \hat{\phi}^x g = f_1 + f_2 + dt^2 \) where \( f_1 \) is the restriction of \( \hat{\phi}^x g \) to the \( B^k \)-fibers and \( f_2 \) is the restriction of \( \hat{\phi}^x g \) to the \( B^{n-1} \)-fibers. Also, \( f_1 + dt^2 \) is the restriction of \( \hat{\phi}^x g \) to the \( (B^k \times I) \)-fibers, and \( f_1 + dt^2 \) is a hyperbolic metric. Furthermore, again by (7), we have that \( f_1 + dt^2 \) (hence also \( f_1 \)) is independent of the variable \( x_2 \). This together with (4.8) imply that we only need to consider \( f_2 \).

Claim 4.9. We have that
\[
| f_2(x_1,x_2,t) - e^t \sigma_{r_{n-1}} |_{C^2} \leq 2(2 + 3 \xi + \xi^2)e^{1+\xi L} \left( \epsilon + e^{-r_0} \right)
\]

Proof of Claim 4.9. Let \( a_{ij} \) be the entries of the matrix \( f_2 \). We have to prove that \( | a_{ij}(x_1,x_2,t) - e^t \delta_{ij} |_{C^2} < 2(2 + 3 \xi + \xi^2)e^{1+\xi L} \left( \epsilon + e^{-r_0} \right) \). Let \( b_{ij}(x,r) \) be the entries of the matrix \( \phi^x h_r \). Since, by hypothesis, \( \phi \) is \( \epsilon \)-close to hyperbolic, we have that
\[
| b_{ij}(x,r) - e^t \delta_{ij} |_{C^2} < \epsilon
\]
(9)

On the other hand, equation (8) implies:
\[
a_{ij}(x_1,x_2,t) = b_{ij}(x_2, \bar{r}(x_1,t) - r_0)
\]
(10)

The proof of the claim is obtained by calculating the derivatives of \( a_{ij}(x_1,x_2,t) - e^t \delta_{ij} \) up to order 2 and finding estimates of these derivatives using (9), (10) and Lemma 4.6. This is done in appendix B. (The idea here is that, by (10), \( a_{ij}(x_1,x_2,t) - e^t \delta_{ij} \) is equal to \( b_{ij}(x_2, \bar{r}(x_1,t) - r_0) - e^t \delta_{ij} \), which, by Lemma 4.6 is \( C^2 \)-close to \( b_{ij}(x_2,t) - e^t \delta_{ij} \) which, by (9), is small.)

We now complete the proof of Theorem 4.1. Choose \( \delta = C_4 e^{-r_0} \) in 4.8. Recall that \( f_1 + dt^2 \) is the restriction of \( \hat{\phi}^x g \) to the \( (B^k \times I) \)-fibers. This together with 4.8 and 4.9 imply the following (all norms are \( C^2 \))
\[
| \phi^x g - (e^t \sigma_{r_{n+k-1}} + dt^2) | = | f_1 + f_2 + dt^2 - (e^t \sigma_{r_k} + e^t \sigma_{r_{n-1}} + dt^2) |
\]
\[
\leq | f_1 + dt^2 - (e^t \sigma_{r_k} + dt^2) | + | f_2 - e^t \sigma_{r_{n-1}} |
\]
\[
\leq C_4 e^{-r_0} + 2(2 + 3 \xi + \xi^2)e^{1+\xi L} \left( \epsilon + e^{-r_0} \right)
\]
\[
\leq \left[ 2(2 + 3 \xi + \xi^2)e^{1+\xi L} + C_4 \right] \left( \epsilon + e^{-r_0} \right)
\]
Finally recall that we had chosen \( w_0 \in \mathcal{E}_k(S) \subset \mathcal{E}_k(M) \), with \( w_0 = (y_0, p_0) \in \mathbb{H}^k \times M \) and \( (u_0, r_0) \) the polar coordinates of \( p_0 \in S \subset M \). If \( w_0 \notin \mathcal{E}_k(B_a) \subset \mathcal{E}_k(M) \), then \( r_0 \geq a \), thus \( e^{-r_0} \leq e^{-a} \). This completes the proof of Theorem 4.1.

**Proposition 4.10.** Let \( M^n \) have center \( o \). Assume \( M \) is \((B_a, \epsilon)\)-close to hyperbolic, with charts of excess \( \xi > 0 \). Then \( \mathcal{E}_k(M) \) is \((B_a, \eta)\)-close to hyperbolic, with charts of excess \( \xi' \), provided

\[
C_1' e^{1+\xi} e^{-a} + C \epsilon \leq \eta
\]

where \( 0 < \xi' < \xi - L e^{1+\xi} e^{-a} \). Here \( C_1' = C_1'(n+k, \xi) \), and \( C, L \) are as in 4.1.

**Remark.** Here \( C_1' = C_1'(n+k, \xi) \) is as in 3.8 of [4]. The space \( \mathcal{E}_k(M) \) in the proposition above is radially \( \eta \)-close to hyperbolic with respect to any center \( o \in \mathbb{H}^k \subset \mathcal{E}_k(M) \).

**Proof.** Denote the center of \( \mathcal{E}_k(M) \) by \( o = (a_{ik}, o_M) \). Recall that \( \mathcal{E}_k(\mathbb{H}^n) = \mathbb{H}^{n+k} \). Since \( M \) is (radially) hyperbolic on \( B_a \) we have

\[
\text{the space } \mathcal{E}_k(M) \text{ is (radially) hyperbolic on } \mathbb{H}^k \times \mathbb{B}_a(M) = \mathcal{E}_k(B_a) \quad (a)
\]

Let \( p = (y, v) \in \mathcal{E}_k(M) \). We use the functions (coordinates) in Section 2. In particular \( s = d_{\mathcal{E}_k(M)}(o, p), r = d_{\mathcal{E}_k(M)}(p, \mathbb{H}^k) = d_M(o_M, v) \). Recall that \( s \geq r \) (see 2.2). We will use 3.8 of [4]:

*Corollary 3.8 of [4].* There is \( C_1' = C_1'(n+k, \xi) \) such that hyperbolic \((n+k)\)-space \( \mathbb{H}^{n+k} \) is radially \((C_1' e^{-b})\)-close to hyperbolic outside \( B_a \).

We have two cases.

**First case.** \( r \geq a - 1 - \xi \).

It can be checked that \( C_1'(n+k, \xi) > C(k, \xi) \), hence the hypothesis \( C_1' e^{1+\xi} e^{-a} + C \epsilon \leq \eta \) implies \( C (e^{-(a-1-\xi)} + \epsilon) \leq \eta \). This together with Theorem 4.1 (replace \( a \) by \( a-1-\xi \)) imply that \( \mathcal{E}_k(M) \) is radially \( \eta \)-close to hyperbolic at \( p \) (i.e there is a radially \( \eta \)-close to hyperbolic chart centered at \( p \), with excess \( \xi' < \xi - L e^{1+\xi} e^{-a} \).

**Second case.** \( r < a - 1 - \xi, s \geq a - 1 - \xi \).

The hypothesis \( C_1' e^{1+\xi} e^{-a} + C_3 \epsilon \leq \eta \) implies \( C_1' e^{1+\xi} e^{-a} \leq \eta \). Since \( s \geq a - 1 - \xi \) we can apply Corollary 3.8 of [4] (see above; take \( b = a - 1 - \xi \)) to obtain that \( \mathbb{H}^{n+k} \) is radially \( \eta \)-close to hyperbolic outside the ball \( \mathbb{B}_{a-1-\xi}(\mathbb{H}^{n+k}) \) of radius \( a - 1 - \xi \) on \( \mathbb{H}^{n+k} = \mathcal{E}_k(\mathbb{H}^n) \). By (a) we can identify \( \mathcal{E}_k(B_a) \subset \mathcal{E}_k(M) \) with the corresponding subset of \( \mathbb{H}^{n+k} = \mathcal{E}_k(\mathbb{H}^n) \). Since \( p \in \mathcal{E}_k(B_{a-1-\xi}) \subset \mathcal{E}_k(M) \), we can also consider \( p \in \mathcal{E}_k(B_{a-1-\xi}) \subset \mathcal{E}_k(M) \). Therefore there is a radially \( \eta \)-close to hyperbolic chart \( \phi \) centered at \( p \), with image (a priori) contained in \((n+k)\)-hyperbolic space. Note that, since \( \phi \) is centered at \( p \) and we are assuming \( r < a - 1 - \xi \), by Corollary 4.7 we get that \( \phi(\mathbb{T}_{\xi'}) \subset \{ r < a \} \). Hence, again by (a), the chart \( \phi \) is also a chart for \( \mathcal{E}_k(M) \). This proves the proposition.

The next result is Theorem B in the Introduction.

**Theorem 4.11.** Let \( M^n \) have center \( o \). Assume \( M \) is \((B_a, \epsilon)\)-close to hyperbolic, with charts of excess \( \xi > 0 \). Then \( \mathcal{E}_k(M) \) is \((B_a, C_2 \epsilon)\)-close to hyperbolic, with charts of excess \( \xi' \), provided

\[
a \geq R = R(\epsilon, k, \xi)
\]

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Here $\xi' = \xi - e^{-a/2} > 0$.

Remark. The constant $R$ is defined as $R = \ln(\frac{1}{\xi}) + \ln(L) + 1 + \xi$. Here $C_2 = C_2(n, k, \xi) = C'_1 e^{1+\xi} + C$, where $C'_1$ is as in 4.8, and $C$ as in 4.1.

Proof. We have $a \geq R = \ln(\frac{1}{\xi}) + \ln(L) + 1 + \xi$. Therefore we get: (1) $e^{-a} \leq \epsilon$, and (2) $e^{a/2} \leq Le^{1+\xi}$. Note that (2) implies $\xi - e^{-a/2} \leq \xi - Le^{1+\xi}e^{-a}$. We can now apply 4.8. Note that we are writing $C_2 = C_2(n, k, \xi) = C'_1 e^{1+\xi} + C$. This proves the Theorem.

Appendix A. Proof of Lemma 4.6.

Recall that we are considering $\mathbb{H}^{k+1}$ with two sets of coordinates: the polar coordinates $(x, t)$ and the hyperbolic extension coordinates $(y, r)$. Recall that $x, t, y, r$ are functions defined on $\mathbb{H}^{k+1}$, specifically: $x : \mathbb{H}^{k+1} - \{0\} \to \mathbb{S}^k$, $y : \mathbb{H}^{k+1} \to \mathbb{H}^k$, $r : \mathbb{H}^{k+1} \to \mathbb{R}$, $t : \mathbb{H}^{k+1} \to \mathbb{R}$. Let $\partial_r$ be the gradient vector field of $r$. Then the vectors $\partial_r$ are the velocity vectors of the speed one geodesics emanating orthogonally from $\mathbb{H}^k \subset \mathbb{H}^{k+1}$. Also let $\partial_t$ be the gradient field of $t$ and let $\alpha : \mathbb{H}^{k+1} - \{0\} \to \mathbb{R}$ be the angle between $\partial_t$ and $\partial_r$. Then $\alpha(z)$ is the interior angle, at $z = (y, r)$, of the right triangle with vertices $o, y, z$. We call $\beta(z)$ the interior angle of this triangle at $o$, that is $\beta(z) = \beta(x)$ is the (signed) spherical distance between $x \in \mathbb{S}^k$ and the equator $\mathbb{S}^{k-1} \subset \mathbb{S}^k$, where $(x, t)$ are the polar coordinates of $z$. Note that the triangle mentioned above has sides of length $r = r(z)$, $t = t(z)$ and $\alpha = a(y)$, where we are denoting by $a$ the distance function in $\mathbb{H}^k$ to $o$. Using the hyperbolic law of cosines we get:

\[
\sin \alpha = \frac{\cos \beta}{\cosh r}
\]

(A1)

Therefore

\[
|\sin \alpha| \leq \frac{1}{\cosh r} \quad \text{and} \quad |\cos \alpha| \geq \frac{|\sinh r|}{\cosh r} = |\tanh r|
\]

(A2)

Note that the map $\sin \beta$ is just the height function, i.e. $\sin \beta(x)$ is the (signed) euclidean distance from $x \in \mathbb{S}^k$ to $\mathbb{H}^k$, which is the last coordinate $x_{k+1}$ of $x = (x_1, ..., x_{k+1})$. Therefore the term $\sin \beta(x')$ that appears in the definition of $\bar{r}$ (see 4.5) is the composition

\[
\mathbb{H}^k \xrightarrow{e^{\lambda-t_0}} \mathbb{H}^k \xrightarrow{\exp} \mathbb{S}^k \xrightarrow{\text{proj}} \mathbb{R}
\]

where the first arrow is multiplication by the constant $e^{\lambda-t_0}$, $\exp = \exp_{x_0}$, and the last arrow is the projection take-the-last-coordinate map.

Write $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, ..., k$. Since $\exp_{x_0}$ and proj are smooth and the sphere is compact there is a constant $c$ (independent to $x_0$) with $|\text{proj} \circ \exp|_{C^2} \leq c$.

Remarks.

1. The map $\beta$ is continuous but not smooth at the north pole. On the other hand $\sin \beta$ is smooth.

2. In what follows we will use the fact that we can take $c = 3$. Moreover we can take $|\nabla(\text{proj} \circ \exp)| \leq 3$ (here $|.|$ is Euclidean length). A straightforward calculation (not given here) can show this.

Write $\Lambda = k^{3/2} \sqrt{k!} \mathbf{c}^k/2$, thus $e^{\lambda} \leq \Lambda$ (actually $e^{\lambda} = \Lambda$ for $t_0$ small and $e^{\lambda} = 1$ otherwise). We have then

\[
|\sin \beta(x')|_{C^1} \leq 3\Lambda e^{-t_0} \quad \text{and} \quad |\sin \beta(x')|_{C^2} \leq 3\Lambda^2 e^{-2t_0}
\]

(A3)
Similarly we have $|\nabla \sin (\beta (x'))| \leq 3\Lambda e^{-t_0}$. Write $\bar{t} = t_0 + t$ and note that $\bar{t} > 0$ (recall we are assuming $t_0 > 1 + \xi$). Differentiating equation (4.5) we get

$$\partial_t \bar{r}(x, t) = \frac{\cosh(\bar{t}) \sin(\beta (x'))}{\cosh \bar{r}} = \frac{\cosh(\bar{r}) \cosh(\bar{a}) \sin(\beta (x'))}{\cosh \bar{r}} = \cos \alpha \geq |\tanh \bar{r}|$$

where the second equality is obtained from the first hyperbolic law of cosines and the last from the second hyperbolic law of cosines, and the last inequality comes from (A2). Note also that we get $|\partial_t \bar{r}| \leq 1$. Similarly, using further differentiation, (A2), the two laws of cosines, the law of sines, and a bit of work show

$$|\partial^2_t \bar{r}(x, t)| = \left| (\tanh \bar{r}) \left( \sin^2 \alpha \right) \right| \leq \frac{1}{\cosh^2 \bar{r}}$$

Also, using (A3) we get

$$|\partial_t \bar{r}(x, t)| = \left| \frac{\sinh(t \partial_t \sin(\beta (x'))}{\cosh \bar{r}} \right| \leq 3\Lambda \left( \frac{\sinh(t + t_0)}{e^\alpha} \right) \frac{1}{\cosh \bar{r}} \leq 3\Lambda \left( \frac{1 + \xi}{2} \right) \frac{1}{\cosh \bar{r}}$$

(recall $t \in (-1 + \xi, 1 + \xi)$). A similar argument using $|\nabla \sin (\beta (x'))| \leq 3\Lambda e^{-t_0}$ shows

$$|\nabla \bar{r}(x, t)| \leq 3\Lambda \left( \frac{1 + \xi}{2} \right) \frac{1}{\cosh \bar{r}}$$

Differentiating again, using the two laws of cosines, and (A3) we obtain

$$|\partial_i \partial_j \bar{r}(x, t)| \leq 3\Lambda \left[ \left( \frac{\cosh(t + t_0)}{e^\alpha} \right) + \left( \frac{\sinh(t + t_0)}{e^\alpha} \right) \right] \frac{1}{\cosh \bar{r}} \leq \frac{6\Lambda}{\cosh \bar{r}}$$

provided $t_0 \geq 1 + \xi$. Finally, differentiating and using (A3) we get

$$|\partial_{ij} \bar{r}(x, t)| \leq \Lambda^2 \left( \frac{\sinh(t + t_0)}{e^\alpha} \right) \left[ \frac{3}{e^{2\alpha}} \frac{1}{\cosh \bar{r}} + \frac{\sinh(t + t_0)}{e^\alpha} \frac{9}{\cosh^2 \bar{r}} \right] \leq \frac{8\Lambda^2 e^{2(1+\xi)}}{\cosh \bar{r}}$$

Note that all five terms on the right of the last five equations are less than $\frac{8\Lambda^2 e^{2(1+\xi)}}{\cosh \bar{r}} \leq 16\Lambda^2 e^{2(1+\xi)} e^{-\bar{r}}$.

Now, write $F(x, t) = \bar{r}(x, t) - (t + r_0)$. Since all but one of derivatives of order 1 and 2 of $F$ coincide with the ones of $\bar{r}$ we get that all such derivatives are less than $16\Lambda^2 e^{2(1+\xi)} e^{-\bar{r}}$. The remaining derivative is $\partial_i \bar{F}$. We have

$$|\partial_i \bar{F}| \leq |1 - \tanh \bar{r}| = \frac{e^{-\bar{r}}}{\cosh \bar{r}} \leq \frac{1}{\cosh \bar{r}} \leq 16\Lambda^2 e^{2(1+\xi)} e^{-\bar{r}}$$

It remains to estimate $|\bar{F}|_{C^0}$. For $x \in \mathbb{R}^k$, since $F(0, 0) = 0$, we have $F(x, 0) = \int_0^1 x \cdot \partial_x F(tx, 0) dt$. But $\partial_x F = \langle x, \nabla \bar{r} \rangle$, hence $|F(x, 0)| \leq \frac{2\Lambda e^{1+\xi}}{2 \cosh(\bar{r})}$. Hence

$$|F(x, t)| = |F(x, 0) + \int_0^1 \partial_t F(x, t) dt| \leq \frac{3\Lambda e^{1+\xi}}{2 \cosh(\bar{r})} + \frac{1}{\cosh(\bar{r})} < 4\Lambda e^{1+\xi} e^{-\bar{r}}$$

Therefore

$$|\bar{r}(x, t) - (t + r_0)|_{C^0} \leq 4\Lambda e^{1+\xi} e^{-\bar{r}} \quad (A4)$$
Hence
\[ |\bar{r}(x,t) - (t + r_0)|_{C^2} \leq 16 \Lambda^2 e^{2(1+\xi)} e^{-p} \]  \hspace{1cm} (A5)

To finish the proof we need compare \( \bar{r} \) with \( r_0 \). For this note that, since \( r_0 = \bar{r}(0,0) \), we have
\[ |\bar{r}(x,t) - r_0| \leq |\bar{r}(x,t) - \bar{r}(x,0)| + |\bar{r}(x,0) - \bar{r}(0,0)| \leq \int_0^t |\partial_t \bar{r}(x,t)| \, dt + \int_0^1 |x| |\nabla \bar{r}(tx,0)| \, dt < (1 + \xi) + \frac{3\Lambda e^{1+\xi}}{2\cosh^2} \leq (1 + \xi) + 3\Lambda e^{1+\xi} \]

That is
\[ |\bar{r}(x,t) - r_0|_{C^0} \leq (1 + \xi) + 3\Lambda e^{1+\xi} \]  \hspace{1cm} (A6)

Hence
\[ \bar{r} > r_0 - (1 + \xi + 3\Lambda e^{1+\xi}) \]

This together with (A5) imply
\[ |\bar{r}(x,t) - (t + r_0)|_{C^2} \leq 16 \Lambda^2 e^{1+\xi} e^{(1+\xi)+3\Lambda e^{1+\xi}} e^{-r_0} = Le^{-r_0} \]

with \( L = L(k,\xi) = 16 \Lambda^2 e^{1+\xi} e^{(1+\xi)+3\Lambda e^{1+\xi}} \), where \( \Lambda^2 = k^3 k! c_{2k}^k \). This completes the proof of the lemma.

**Appendix B. Calculations for the proof of Claim 4.9.**

We will use the following abbreviations for the partial derivatives: \( \partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial u_i}, \partial_t = \frac{\partial}{\partial q} \), where \( x_1 = (u_1, \ldots, u_k) \) and \( x_2 = (v_1, \ldots, v_{n-1}) \).

Write \( \kappa = \kappa(r_0) = Le^{-r_0} \). Note that we are assuming \( \kappa \leq \xi \) (see Remark 3 after the statement of 4.1). Also write
\[ \zeta(x_1,t) = \bar{r}(x_1,t) - r_0 \]  \hspace{1cm} (B1)
\[ c = c_{ij}(x_1,x_2,t) = a_{ij}(x_1,x_2,t) - e^t \delta_{ij} = b_{ij}(x_2,\zeta) - e^t \delta_{ij} \]  \hspace{1cm} (B2)

where the last equality follows from (10) in the proof of Claim 4.9 in Section 4. Also write
\[ d = d(x_2,r) = b_{ij}(x_2,r) - e^r \delta_{ij} \]  \hspace{1cm} (B3)

We have to prove that \( |c|_{C^2} < 2(2 + 3\xi + \xi^2) e^{1+\xi} L(e + e^{-r_0}) \). It follows from (B2), (B3) that
\[ c = d(x_2,\zeta) + e^\xi \delta_{ij} - e^t \delta_{ij} \]  \hspace{1cm} (B4)

From Lemma 4.6 and the fact that \( \phi \) is \( \epsilon \)-close to hyperbolic (see (9) in the proof of 4.9 in Section 4) we have
\[ |\zeta - t|_{C^2} \leq \kappa \quad \text{and} \quad |d|_{C^2} < \epsilon \]  \hspace{1cm} (B5)
It follows from (B1), Corollary 4.7, and \( t \in I' \) that
\[
|\zeta|_{C^0} \leq 1 + \xi \tag{B6}
\]
Note that (B5) also implies
\[
|\partial_t \zeta|_{C^0} \leq \kappa \quad |\partial_r \zeta|_{C^0} \leq 1 + \kappa \quad |\partial_r^2 \zeta|_{C^0} \leq \kappa \quad \bar{\partial}_j \zeta = 0 \tag{B7}
\]
From (B3) and (B5) we get that for \( r \in I' \) we have
\[
|\partial_r b_{ij}|_{C^1} \leq \epsilon + |e_t|_{C^2} < \epsilon + \epsilon^{1+\xi} \tag{B8}
\]

The \( C^0 \)-norm of \( c \). Using (B5), (B8) and the Mean Value Theorem we can write
\[
|b_{ij}(x, t) - b_{ij}(x, t)| \leq |\partial_t b_{ij}|_{C^1} |\zeta - t| \leq \kappa \epsilon + \kappa \epsilon^{1+\xi}
\]
And this together with (B5) imply
\[
|c|_{C^0} \leq \kappa \epsilon + \kappa \epsilon^{1+\xi} + |b_{ij}(x, t) - e^t \delta_{ij}|_{C^0} \leq \kappa \epsilon + \kappa \epsilon^{1+\xi} + \epsilon = (1 + \kappa) \epsilon + \epsilon^{1+\xi} \kappa
\]

The \( C^1 \)-norm of \( c \). We have three types of first derivatives. First, from (B4) we have:
\[
\partial_t c = (\partial_t d) (\partial_t \zeta) + (\partial_t \zeta) - 1 e^\zeta I + (e^{\zeta} - e^t) I
\]
This last equation together with (B5), (B6), (B7) imply
\[
|\partial_t c| \leq \epsilon (1 + \kappa) + \kappa e^{1+\xi} + \kappa \epsilon^{1+\xi} = (1 + \kappa) \epsilon + 2 \epsilon^{1+\xi} \kappa
\]
where we are using the Mean Value Theorem, (B5) and (B6) to estimate \( e^\zeta - e^t \). Analogously
\[
|\partial_i c| = |(\partial_i d) (\partial_i \zeta) + (\partial_i \zeta) e^\zeta| \leq \epsilon \kappa + \kappa \epsilon^{1+\xi}
\]
and
\[
|\bar{\partial}_i c| = |\bar{\partial}_i d| < \epsilon
\]

The \( C^2 \)-norm of \( c \). We have six types of first derivatives. As above using (B4), (B5), (B6) and (B7) we can obtain estimates for them. Here are the first three that do not involve the variable
\[ |\partial^2_t c| = \left| (\partial^2_t d)(\partial_t \zeta)^2 + (\partial_t d)(\partial^2_t \zeta) + \left[ (\partial^2_t \zeta) + ((\partial_t \zeta)^2 - 1) \right] e^\xi + (e^\xi - e') \right| \]

\[ \leq \epsilon (1 + \kappa)^2 + \epsilon \kappa + [\kappa + \kappa(\kappa + 2)] e^{1+\xi} + \kappa e^{1+\xi} \]

\[ = (1 + 3\kappa + \kappa^2) \epsilon + e^{1+\xi} (4 + \kappa) \kappa \]

\[ |\partial_t \partial_t c| = \left| (\partial_t \partial_t d)(\partial_t \zeta) + (\partial_t d)(\partial_t \partial_t \zeta) + \left[ (\partial_t \partial_t \zeta) + (\partial_t \zeta)(\partial_t \zeta) \right] e^\xi \right| \]

\[ \leq \epsilon (1 + \kappa) \kappa + \epsilon \kappa + [\kappa + (1 + \kappa)\kappa] e^{1+\xi} \]

\[ = (2\kappa + \kappa^2) \epsilon + e^{1+\xi} (2 + \kappa) \kappa \]

\[ |\partial_j \partial_t c| = \left| (\partial_j \partial_t d)(\partial_j \zeta) + (\partial_j d)(\partial_j \partial_t \zeta) + \left[ (\partial_j \partial_t \zeta) + (\partial_j \zeta)(\partial_t \zeta) \right] e^\xi \right| \]

\[ \leq \epsilon \kappa^2 + \epsilon \kappa + [\kappa + \kappa^2] e^{1+\xi} \]

\[ = (\kappa + \kappa^2) \epsilon + e^{1+\xi} (1 + \kappa) \kappa \]

And the ones involving the \(x_2\) variable:

\[ |\partial_j \partial_j c| = |\partial_j \partial_j d| < \epsilon \]

\[ |\partial_j \partial_j c| = |(\partial_j \partial_j d)(\partial_j \zeta)| < \epsilon \kappa \]

\[ |\partial_j \partial_j c| = |(\partial_j \partial_j d)(\partial_j \zeta)| < \epsilon (1 + \kappa) \]

Note that all the estimates of the derivatives that we have obtained above are less or equal

\[ (1 + 3\kappa + \kappa^2) \epsilon + e^{1+\xi} (4 + 2\kappa) \kappa = (1 + 3\kappa + \kappa^2) \epsilon + e^{1+\xi} (4 + 2\kappa) L e^{-r_0} \]

\[ \leq \left[ 2(2 + 3\xi + \xi^2)e^{1+\xi} L \right](\epsilon + e^{-r_0}) \]

And, since \( \kappa \leq \xi \), we get \( |c|_{C^2} < 2(2 + 3\xi + \xi^2)e^{1+\xi} L(\epsilon + e^{-r_0}) \). This concludes our calculations and the proof of Claim 4.9.

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