The commuting local Hamiltonian on locally-expanding graphs is in NP.

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Abstract

The local Hamiltonian problem is famously complete for the class QMA, the quantum analogue of NP [30]. The complexity of its semi-classical version, in which the terms of the Hamiltonian are required to commute (the CLH problem), has attracted considerable attention recently [4, 18, 27, 38, 28] due to its intriguing nature, as well as in relation to growing interest in the qPCP conjecture [2, 3]. We show here that if the underlying bipartite interaction graph of the CLH instance is a good locally-expanding graph, namely, the expansion of any constant-size set is ε-close to optimal, then approximating its ground energy to within additive factor $O(ε)$ lies in NP. The proof holds for $k$-local Hamiltonians for any constant $k$ and any constant dimensionality of particles $d$. We also show that the approximation problem of CLH on such good local expanders is NP-hard. This implies that too good local expansion of the interaction graph constitutes an obstacle against quantum hardness of the approximation problem, though it retains its classical hardness. The result highlights new difficulties in trying to mimic classical proofs (in particular Dinur’s PCP proof [21]) in an attempt to prove the quantum PCP conjecture. A related result was discovered recently independently by Brandão and Harrow [16], for 2-local general Hamiltonians, bounding the quantum hardness of the approximation problem on good expanders, though no NP-hardness is known in that case.

1 Introduction

Quantum Hamiltonian complexity (QHC) has blossomed into an incredibly active field of research over the past few years (see [34, 3] for an introduction). A key player in QHC is the local Hamiltonian (LH) [30], whose study had led to various unexpected discoveries, ranging from the universality of adiabatic computation [5], to the QMA hardness of one dimensional systems [6], to quantum gadgets [32, 41, 19], to computational complexity lower bounds on notorious open problems in physics [39], to the hardness of translationally invariant systems [25], and more.

In this note we focus on a special case of the LH problem, called the commuting local Hamiltonian problem (CLH), which has attracted considerable attention in the past few years (e.g., [4, 18, 27, 38, 28]). Before we explain our result, let us define the problem more precisely. A $k$-local Hamiltonian is a Hermitian matrix $H = \sum_i H_i$ operating on the Hilbert space of $n$ $d$-dimensional particles (qudits), where each term $H_i$ (sometimes called constraint) acts non-trivially on at most $k$ particles. Kitaev defined in 1999 the local Hamiltonian (LH) problem [30], where one is given a local Hamiltonian on $n$
qudits, and two real numbers $a, b$, with $b - a \geq 1/poly(n)$, and is asked whether the lowest eigenvalue (ground energy) is at most $a$ or at least $b$. Kitaev showed [30] that this problem is complete for the class QMA, the quantum analogue of NP. This provided a quantum counterpart of the celebrated Cook-Levin theorem [35]. To restrict the discussion to commuting local Hamiltonians, we require in addition that every two terms $H_i, H_j$ commute. W.L.O.G we can assume in this case that $H_i$’s are projections. We denote the family of such Hamiltonians acting on $d$ dimensional particles and where each term is $k$-local, by CLH($k, d$); We can now consider the CLH problem (introduced by Bravyi and Vyalyi in 2003 [18]), which is the analogue of the LH problem for the commuting case. Note that since we are concerned with $H_i$’s which are commuting projections, the eigenvalues of $H$ are natural numbers. The problem is now to distinguish between the ground energy being $a = 0$ or at least $b = 1$.

The commutation restriction might seem at first sight to devoid the LH problem of its quantum nature, and place it in NP, since all local terms can be diagonalized simultaneously. Moreover, one usually attributes the interesting features of quantum mechanics to its non-commutative nature, cf. the Heisenberg uncertainty principle. This intuition is importantly wrong. The main source for the interest in CLHs lies in the beautiful and striking fact that the groundspaces of CLHs are capable of possessing intriguing multi-particle entanglement phenomena, such as topological order [31], and quantum error correction [25]; the most well known example is Kitaev’s toric code [31]. Due to this intriguing phenomenon, the study of CLHs has become highly influential in the physics as well as the quantum complexity communities, see e.g. the study of Levin-Wen models [33], quantum double models [31], as well as that of stabilizer error correcting codes [25], which are special cases of CLH’s.

In their 2003 paper, Bravyi and Vyalyi [18] proved a central result regarding the CLH problem: the 2-local case (namely, the case of $k = 2$) lies in NP. The proof uses a clever application of the theory of representations of $C^*$-algebras, with which Bravyi and Vyalyi showed that for such systems, there is always a ground state which can be generated by a constant depth quantum circuit; in particular, the description of the constant depth quantum circuit can be given as a classical witness, from which the energy of the state can be computed efficiently by a classical computer (implying the containment in NP). The CLH question is thus interesting (from the quantum complexity point of view) only with locality $k > 2$. The Bravyi-Vyalyi result was extended later in various special cases [4, 27, 38, 28], but the complexity of the general CLH problem remains open, and as far as we currently know it may lie anywhere between NP and the full power of QMA.\footnote{To be more precise, QMA$_1$ - which is defined like QMA except in case of YES, there exists a state which is accepted with probability exactly 1. Here we will not distinguish between QMA and QMA$_1$ since we are not using those terms technically, but see [1] and [17].}

In this paper, we study the hardness of approximation versions of the CLH problem. Our motivation is two-fold: one, we view this as an important stepping stone towards the clarification of the complexity of the exact CLH problem, which has resisted resolution so far, despite the effort described above. Two, we view it as an important special case of a major conjecture in QHC, namely, the quantum PCP (qPCP) conjecture, to be discussed below.

Our main result states that the approximation problem of CLH lies within NP, whenever the underlying bi-partite interaction graph of the CLH instance is a good enough locally-expanding graph (for definition see below). This means that good local expansion of the underlying interaction graph does not help but in fact disturbs in generating “hard” quantum instances, in sharp contrast to the classical analogue in which the hard instances are excellent locally-expanding graphs (see [21] and Theorem 2 below). The result implies severe constraints on the structure of possible qPCPs for CLHs, as the
output of those cannot be good local expanders. Before we state the result and its implications more formally, we provide some further background and context.

1.1 Relation to the Quantum PCP conjecture

The classical PCP theorem \cite{1, 9} is arguably the most important discovery in classical theoretical computer science over the past quarter century; To state it, recall the Constraint Satisfaction Problem (CSP): Given is a collection of $k$-local constraints on $n$ Boolean or $d$-state variables. The problem is to decide whether the instance is satisfiable or not; this problem is NP complete. The PCP theorem states that there exists an efficient mapping (known as a PCP reduction) which takes satisfiable instances to satisfiable instances and non-satisfiable ones to non-satisfiable ones, such that the NP witness for the new instances can be tested, with constant probability of success, by reading only a constant number of randomly chosen locations in the witness (!). Stated differently, the PCP theorem says that it is NP hard to decide whether a given instance is satisfiable, or at most some constant fraction, say 90%, of its constraints can be satisfied. One may ask \cite{7, 1} does an analogous statement hold in the quantum setting as well? A quantum PCP conjecture \cite{2} can be phrased as follows:

**Conjecture 1 Quantum PCP (qPCP)** There exists $c > 0$, such that it is QMA-hard to decide whether a given $k$-local Hamiltonian $H = \sum_{i=1}^{m} H_i$, with $\|H_i\| \leq 1$ whose terms are all positive semi-definite, is satisfiable (namely, has ground value 0) or its minimal eigenvalue is at least $c \cdot m$.

This conjecture is one of the major open problems in QHC today (see the recent survey \cite{3}). It is important for several reasons. First of all, the question is related to the question of whether quantum entanglement can be made robust at room temperature. Secondly, the question of whether or not qPCPs exist is tightly related to fundamental questions regarding local and global multi-particle entanglement. The resolution of the conjecture, in either direction, would constitute a major advance in QHC and in our understanding of quantum matter. However, despite considerable effort, so far all attempts to extend the classical proofs of the PCP theorem to the quantum settings, or to dispute the conjecture, have encountered severe obstacles \cite{2, 3, 7, 28, 27}. We currently do not even have a good guess whether the conjecture should hold or not.

Just like in the classical case, the qPCP conjecture is equivalent to the following statement: the problem of approximating the ground energy of local Hamiltonians to within a constant fraction is QMA hard. This means that the clarification of the complexity of approximating the local Hamiltonian problem is of major interest. Given the difficulty of this question, it is natural to study hardness of approximation of restricted classes Hamiltonians; we focus here on CLHs, which are a particularly attractive special case, since they are both much easier to analyze than the general case, and in addition, they possess a simplifying structure as introduced in \cite{18}, while still giving rise to extremely complex phenomenon such as topological order, and used as a substrate for almost all known quantum codes.

1.2 Computational Hardness and the geometry of the interaction graph

The geometry of the underlying interaction graph is well known to play a crucial role for the complexity of constraint satisfaction problem. For a start, when a CSP is defined between variables set a one dimensional lattice, with constraints between nearest neighbors only, the problem lies in P (using
dynamic programming), whereas for 2 or more dimensions, the problem is NP-hard. When one is interested in approximations (to within any constant fraction of the number of constraints) the situation changes: approximating CSPs, as well as LHs, to within any constant, lies in P when the interaction graph is a lattice of any constant dimension. To see this, partition the lattice to constant size cubes, and remove the constraints that connect different cubes; we arrive at a collection of disconnected constant-size subproblems, all of which can be optimized together in P. The number of constraints violated by this solution is at most the number of removed constraints, which can be made to be an arbitrarily small fraction. This argument works similarly both in the quantum and classical cases, and implies that to make the approximation problem hard, one has to consider CSPs, CLHs or LHs whose underlying graphs have certain expansion properties.

In light of the above, and motivated by the important role that expanders play in the classical theory of hardness of approximation and PCPs, we turn to the study of approximating CLH systems on expanders. We need our definition of expansion to work for \( k \)-local Hamiltonians, for \( k > 2 \), since 2-local CLHs are already known to be in NP even without approximation [18]. One might consider working with the naturally induced hypergraphs, but there is no consensus on the right definition of expansion for hypergraphs in the literature. Moreover, whereas one is perhaps used to intuitively think of standard graph expansion as the important property for hardness in classical complexity, perhaps due to the strong usage of expansion in Dinur’s PCP proof [21], it is in fact an open question to relate expansion, and in particular small-set expansion, (namely expansion of sets of constant fractional size) to computational hardness [36, 37].

Here we consider, for a given CLH, the bi-partite graph in which variables are on the left side and constraints are on the right, and a variable is connected to the constraints that it appears in. For this bi-partite graph, we consider a very weak notion of expansion, which we call local expansion: we say that the bi-partite graph induced by a \( k \)-local CLH instance is \( \epsilon \)-locally expanding, if for any set \( S \subseteq L \) of at most \( k \) particles, the number of local terms incident on these particles is at least \( D_L |S|(1 - \epsilon) \), where \( D_L \) is the left degree of the graph (here we assume all degrees are equal, see Definition 3 for the general case). Observe that \( D_L |S| \) is the maximal possible number of constraints acting on those particles, so \( \epsilon \) can be viewed a “correction” to this number, or the expansion error. We note that this definition of local expansion cares only about the expansion of constant-size sets, whereas in complexity theory, one is often interested in small-set bi-partite expanders, in which all sets of size up to some constant fraction of \(|L|\) are required to expand [22, 36]; our requirements on bi-partite expanders are thus significantly weaker. It turns out that the notion of local expansion can indeed be related, in the classical world, to computational hardness; as we show in Theorem 2 sufficiently good local expanders are computationally hard for classical computers. As we will see (in Theorem 1) this is not the case in the quantum world.

1.3 Main Result

We can now state our main result, Theorem 1, in which we show that the problem of approximating CLHs on locally-expanding graphs lies inside NP, and thus cannot be QMA hard (under standard complexity assumptions):

\[ \text{Of course, our results regarding CLHs are stronger if they hold for graphs which satisfy weaker requirements} \]

\[ \text{Assuming standard computational complexity assumptions, specifically, NP} \subseteq \text{QMA}_1. \]
Theorem 1 (the approximation of CLH on good locally-expanding bi-partite graphs is in NP) Let $\gamma(\varepsilon) = 2kd\varepsilon$. Let $H = \sum_{i=1}^{m} H_i$ be an instance of CLH$(k,d)$ for constants $k,d > 0$, with bi-partite interaction graph which is $\varepsilon$-locally-expanding, for $\varepsilon < \frac{1}{2}$. Then the $\gamma(\varepsilon)$-approximation problem of CLH$(k,d)$ on such $\varepsilon$ locally-expanding bi-partite graph graphs is in NP. Moreover, for any eigenvalue $\lambda$ of $H$ there exists a state $|\psi\rangle$ such that $|\langle \psi | H | \psi \rangle - \lambda| \leq \gamma(\varepsilon) \cdot m$, such that $|\psi\rangle$ can be generated by a constant depth quantum circuit.

We rule out the possibility that $\gamma(\varepsilon)$ is large enough so that by removing a $\gamma(\varepsilon)$ fraction of the terms in $H$ we can simply trivialize the problem and place it in P (e.g., by disconnecting the graph into small components, as is done in the approximation of CLHs on lattices). We do this by showing that the problem remains NP-hard even with this approximation factor.

Theorem 2 Fix $k \geq 3, d \geq 2$. Let $\varepsilon$ be such that $\gamma(\varepsilon) = 2kd\varepsilon \leq C_0$ where $C_0 < 1/3$ is a universal constant. Then it is NP-hard to approximate CLH$(k,d)$ whose bi-partite interaction graph is $\varepsilon$ locally-expanding, to within a factor $\gamma(\varepsilon) = 2kd\varepsilon$.

$C_0$ in the above statement is some universal constant which appears in the PCP proof of Dinur [21]; we delay its exact definition to Subsection 3.3 but it is not important. Theorem 1 implies that if indeed one could construct a quantum PCP whose output is a CLH, then in contrast to classical PCP (see the proof of Theorem 2), the output cannot have arbitrarily high local expansion.

1.4 Related work

The question of approximating the ground energy of local Hamiltonians was broadly studied in the physics literature for specific Hamiltonians of particular interest in physics. Not much is known regarding the hardness of approximation of the general case. Kempe and Gharibian [23] derived a non-trivial result for a related task: given a $k$-local Hamiltonian on $d$ dimensional qudits, and we are asked to find the state with maximal energy $OPT$. They showed that there always exists a product state with energy at least $OPT/d^{k-1}$, and so the problem of approximating MAX-$k$-local Hamiltonian to within approximation factor $d^{k-1}$ lies in NP (see also a follow up paper by the same authors on approximation versions of problems which are computationally even harder [24]).

The approach of examining commuting Hamiltonians as a precursor for the general case was taken before, e.g. in [2], and by Arad [8] who considered the commuting case as a base for perturbations to derive a partial no-go theorem for quantum PCP. Notably, this approach is the underlying motivation in the recent works of Hastings [27] and Hastings and Freedman [28], which are tightly related to the current paper, as they both attempt to make progress on the qPCP conjecture by studying approximations of CLHs. Freedman and Hastings are searching for Hamiltonians whose low energy states are all highly entangled, and in particular cannot be generated by constant depth quantum circuits (such Hamiltonians are called NLTS for No Low-energy Trivial States). The existence of such Hamiltonians is essential for the qPCP conjecture to hold, for otherwise it would be possible to provide a classical witness for the fact that the energy of the Hamiltonian is low, placing the problem in NP. Freedman and Hastings [28] construct a family of CLHs which is not NLTS but only “one-sided” NLTS (see also [3] for more on this); they leave as an open problem whether true NLTS Hamiltonians exist. Theorem 1 can be applied here to derive an energy upper bound above which such a construction cannot be NLTS, since it shows the existence of trivial states below a certain threshold constant fraction of energy. We note that this upper bound tends to 0 as the local expansion of the graphs
underlying the construction improves, and so one cannot hope to achieve better NLTS constructions by improving the local expansion of the underlying graph.

We finally mention an important recent work discovered independently and around the same time as ours, by Brandão and Harrow [16]. Brandão and Harrow showed that the approximation of 2-local Hamiltonians on graphs whose degree is very large lies in NP (the factor of approximation depends inverse polynomially on the degree). This also implies that when the underlying graph of the 2-local Hamiltonian is an excellent expander (which means that its degree must be very large) the approximation problem (to within a factor which depends inverse polynomially on the degree again) lies in NP.

Both the result of [16] and our main Theorem 1 have the same flavor: too good expanders are an obstruction to quantum hardness, contrary to what one might expect given what we know in the classical case ([21], and Theorem 2). The comparison between the two results, however, is not straight-forward, and in particular none of the results implies the other. First, Theorem 1 holds for any constant locality \( k \geq 2 \), while the result of [16] holds only for \( k = 2 \). One might ask what does Theorem 1 give if we substitute \( k = 2 \); this turns out to be non interesting from the approximation point of view: it is exactly the result [18], which states that the exact 2-local CLH problem lies in NP. On the other hand, it is unclear whether the results of [16] can be extended to \( k \)-local Hamiltonians, for \( k > 2 \). The straight forward attempt to do this, using quantum gadgets [32, 41, 19], that transform \( k \)-local Hamiltonians (\( k > 2 \)) into 2-local Hamiltonians encounters an immediate obstacle: these gadgets (e.g. the best known gadgets so far [19]) significantly change the geometry of the graph, and in particular, may modify its expansion properties considerably. Specifically, they introduce large sets of vertices with no edges between them, with some vertices having very small degree; this implies that the results of [16] will no longer be applicable after such transformations.

1.5 Discussion and Further directions

Our main result implies that surprisingly, good local expansion is an obstruction to quantum hardness of approximation, in the case of commuting local Hamiltonians. It implies that if qPCP were to hold for CLHs, then the output of the qPCP reductions cannot be good local expanders. Our result also directly implies that NLTS Hamiltonians (see discussion in the previous subsection as well as [27, 3]) cannot be constructed on too good bi-partite local-expanders.

The main insight underlying this work is that local Hamiltonian systems made of commuting check terms, are severely limited by a phenomenon called monogamy of entanglement. Essentially, the monogamy of entanglement limits the amount of entanglement that a qudit with \( O(1) \) quantum levels can “handle”. Our study of commuting system thus reveals that monogamy of entanglement is a significant limiting factor, whose influence increases together with expansion. It is intriguing that a similar phenomenon has been identified by completely different methods, by [16] in the context of general (non-commuting) local Hamiltonians of locality \( k = 2 \).

A natural open question is to generalize both our result and that of [16] into one coherent stronger result, which would state roughly that for general, not necessarily commuting, \( k \)-local Hamiltonians (for general \( k \)), the expansion of the underlying interaction graph (where expansion is defined suitably) is an obstruction against quantum hardness. Another problem, is to generalize the results here and those of [16] to richer forms of entanglement. In particular the limitation on quantum hardness need not necessarily come in the form of constant-depth circuit, but rather any form of entanglement which has an efficient description.
We mention another interesting direction: one might hope that Theorem 1 could be extended to show that the approximation of the CLH problem on a general graph is in \( \text{NP} \), by “bridging” between our results on locally-expanding graphs and the easy case of approximation on lattices. We note that a hint about the difficulty of such a result comes from the status of the Unique-Games-Conjecture \([29]\) which is known to be easy on both extreme cases \([13]\) (albeit for standard expander graphs rather than for locally-expanding graphs), though conjectured (or rather, speculated) to be \( \text{NP} \)-hard in general. Still, it may be possible to achieve some weaker approximation for general graphs using “bridging” between the case of lattices and expanders, perhaps using sub-exponential rather than polynomial witnesses, following the works of \([42]\) and \([12]\).

**Organization of paper** In Section 2 we provide some required background. Section 3 proves Theorem 1 as well as Theorem 2.

### 2 Definitions and Basic Lemmas

#### 2.1 Local Hamiltonians

A \((k, d)\)-local Hamiltonian on \(n\) qudits is a Hermitian matrix \(H = \sum_i H_i\) operating on a Hilbert space \(\mathcal{H} = \mathbb{C}^{d^\otimes n}\) of \(d\)-dimensional particles, where we assume for all \(i, \|H_i\| = 1\) and \(H_i\) can be written as: \(H_i = h_i \otimes I_j\), where \(h_i\) is a Hermitian operator acting on at most \(k\) particles. When we say an operator acts on some particle, we mean it acts non-trivially, namely, it cannot be written as an identity on that particle tensor with some operator on the rest.

**Definition 1** The local Hamiltonian (LH) and commuting local Hamiltonian (CLH) problems In the \((k, d)\)-local Hamiltonian problem on \(n d\)-dimensional qudits we are given a \((k, d)\)-local Hamiltonian \(H = \sum_{i=1}^m H_i\) and two constants \(a, b, a - b = \Omega\left(\frac{1}{\text{poly}(n)}\right)\). We are asked to decide whether the lowest eigenvalue of \(H\) is at least \(b\) or at most \(a\), and we are promised that one of the two occurs. In the commuting case, we are guaranteed in addition that \([H_i, H_j] = 0\) for all \(i, j\), and W.L.O.G, we also assume that \(H_i\)'s are projections. \(a\) and \(b\) can be taken to be 0 and 1 in this case, respectively. An instance to this problem is called a CLH\((k, d)\) instance.

#### 2.2 Interaction graphs and their expansion

We define bi-partite expanders, similar to \([40]\), \([20]\), who used them to construct locally-testable classical codes. Note that we require expansion to hold only for sets of constant size \(k\).

**Definition 2** Bi-Partite Interaction Graph For a \((k, d)\)-local Hamiltonian \(H = \{H_i\}_i\) we define the bi-partite interaction graph \(G = (L, R; E)\) as follows: \(L\), the nodes on the left, correspond to the \(n\) particles of \(\mathcal{H} = \mathbb{C}^{d^\otimes n}\) and \(R\) corresponds to the set of local terms \(\{H_i\}_i\). An edge exists between a constraint \(r \in R\) and a particle \(l \in L\) if \(H_r\) acts non-trivially (namely, not as the identity) on \(l\). Note that the right degree is at most \(k\).

**Definition 3** Bi-partite local expansion Given a bi-partite graph \(G(L, R; E)\), we say that a subset \(S \subseteq L\) is \(\varepsilon\)-expanding if \(|\Gamma(S)| \geq (1 - \varepsilon) \sum_{v \in S} D_v\), where \(D_v\) is the degree of \(v\), and \(\Gamma(S)\) is the set of neighbors (in \(R\)) of \(S\). A bi-partite graph \(G = (L, R; E)\) for a \((k, d)\)-local LH is said to be \(\varepsilon\)-locally-expanding, if every subset of particles \(S \subseteq L\) of size \(|S| \leq k\) is \(\varepsilon\)-expanding.
2.3 Commuting terms and $C^\ast$-algebras

We recall the main lemma of [18]:

**Lemma 1** Adapted from [18] Consider two commuting local terms $H_i, H_j$, which intersect on some subset of qudits, whose Hilbert space is $\mathcal{H}_{int}$. Then $\mathcal{H}_{int}$ can be written as a direct sum of subspaces

$$\mathcal{H}_{int} = \bigoplus_\alpha \mathcal{H}^\alpha_{int} = \bigoplus_\alpha \mathcal{H}^\alpha_{int} \otimes \mathcal{H}^\alpha_{int}$$

(1)

where $H_i$ and $H_j$ preserve the $\mathcal{H}^\alpha_{int}$ subspaces, and moreover, when $H_i, H_j$ are restricted to $\mathcal{H}^\alpha_{int}$, then $H_i(H_j)$ acts non trivially only on subsystem $\mathcal{H}^\alpha_{int}$ ($\mathcal{H}^\alpha_{int}$). In particular, in each summand $\alpha$, both $\mathcal{H}^\alpha_{int}, \mathcal{H}^\alpha_{int}$ have dimension strictly less than $\mathcal{H}_{int}$.

Thus, under an isometry on $\mathcal{H}_{int}$ and a restriction to one of the $\mathcal{H}^\alpha_{int}$'s, the terms act on disjoint systems. See Figure 1.

![Figure 1](image)

Figure 1: An example of lemma (1): a pair of 2-local Hamiltonians, are restricted to a subspace $\alpha$ on their intersection. Inside this subspace, they act on separate subsystems.

In this paper we apply a simple corollary of Lemma (1):

**Corollary 1** Let $H$ be a CLH($k,d$) instance, and let $H_0$ be a local term such that for any $H_j \in H \setminus \{H_0\}$ that intersects $H_0$, the intersection is at most on one qudit. Then for each qudit $q$ participating in $H_0$ there exists a direct-sum decomposition of its Hilbert space $\mathcal{H}_q = \bigoplus_{\alpha_q} \mathcal{H}^{\alpha_q}_q$, such that each of these subspaces is preserved by all local terms of $H$, and such that the restriction of all terms in $H$ to any subspace $\mathcal{H}^{\alpha_q}_q$ results in $H_0, H \setminus H_0$ acting on a disjoint subsystems, respectively, $\mathcal{H}^{\alpha_q,0}_q \otimes \mathcal{H}^{\alpha_q,1}_q$. In particular, for any $\alpha_q$, we have that $\dim(\mathcal{H}^{\alpha_q,0}_q), \dim(\mathcal{H}^{\alpha_q,1}_q)$ are strictly less than $\dim(\mathcal{H}_q)$.

**Proof:** For each particle $q$ examined by $H_0$, let the term $H_1$ be the sum of all local terms acting on $q$ except $H_0$. The lemma can now we applied on $q$. Since each term other than $H_0$ intersects $H_0$ at only one location, we can apply the lemma simultaneously for all qudits in $H_0$. The result follows.

Note, that the above corollary does not require that any two local terms acting on a qudit in $H_0$, intersect only on that qudit.

2.4 Notation

d will denote the dimensionality of particles, capital $D$'s will denote degrees in a graph, in particular $D_v$ is the degree of a node $v$. Given $S \subseteq R(L)$ in a bi-partite graph, $\Gamma(S)$ denotes the neighbor set
3 Approximate CLH

3.1 Geometric Preliminaries

Our proof makes use of a very simple property of locally-expanding graphs, namely that most of the neighboring Hamiltonian terms of a set of qudits of size \( k \) only touch this set at one point. We first state a simple fact quantifying this statement (similar observations were used also in \cite{20}) and then explain how it is used in the proof.

**Fact 1** Consider \( S \subseteq L \) in a bi-partite graph \( G(L, R; E) \) and let \( S \) be \( \varepsilon \)-expanding, for \( \varepsilon < \frac{1}{2} \). Then a fraction at most \( 2\varepsilon \) of all vertices of \( \Gamma(S) \) have degree strictly larger than 1 in \( S \).

**Proof:** Put \( D(S) = \sum_{v \in S} D_v \). The average degree of a vertex in \( \Gamma(S) \) w.r.t. \( |S| \) is at most \( \frac{D(S)}{D(S)(1 - \varepsilon)} = \frac{1}{1 - \varepsilon} \). Let \( \alpha_1 \) denote the fraction \( |\Gamma_1(S)|/|\Gamma(S)| \), where \( \Gamma_1(S) \) is the set of neighbors of \( S \) with degree exactly 1 with respect to \( S \). Then

\[
\frac{1}{1 - \varepsilon} \geq \alpha_1 + (1 - \alpha_1)m,
\]

where \( m \) is the average degree of a vertex with at least two neighbors in \( S \). Then by simple algebra

\[
\alpha_1(m) \geq 1 - \frac{1}{m - 1} \cdot \frac{\varepsilon}{1 - \varepsilon},
\]

so \( \alpha_1(m) \) is a monotonously increasing function of \( m \), and since \( m \geq 2 \), then \( \alpha_1 \) is minimized for \( m = 2 \). Hence,

\[
\alpha_1 \geq 1 - \frac{\varepsilon}{1 - \varepsilon}.
\]

and since \( \varepsilon < 1/2 \) we have:

\[
\alpha_1 \geq 1 - \varepsilon(1 + 2\varepsilon) \geq 1 - 2\varepsilon.
\]

3.1.1 Overview of the proof of Theorem

The idea of the proof is that if a constraint \( H_i \) satisfies a simple condition, namely, that it shares at most one particle with any other constraint, then Corollary \( \square \) can be applied (with \( H_i \) playing the role of \( H_0 \)) to “disentangle” the action of \( H_i \) from the rest of the terms (from there we can proceed to other terms iteratively). The problem is that \( H_i \) does not necessarily satisfy this condition. The idea is that because the graph is a good locally-expanding graph, we can remove only a small number of terms to achieve the condition. More formally, we define:

**Definition 4** Isolated constraints and particles Let \( (L, R; E) \) be a bi-partite graph. A constraint \( g \in R \) is called isolated if for any constraint \( v \in R \) except \( g \), we have \( |\Gamma(v) \cap \Gamma(g)| \leq 1 \). We define the isolation penalty of \( g \), to be the minimal number of constraints in \( R \) that we need to remove so that \( g \) is isolated.
We thus can “isolate” $H_i$ by removing all terms that share at least 2 vertices with $H_i$. This is where we use the local-expansion property: by Fact 1, the set of constraints we need to remove constitutes just a small portion of the terms intersecting $H_i$. The remaining terms that intersect $H_i$, intersect it at only one particle. Corollary (1) can now be applied almost directly on $H_i$. The final result is that the term $H_i$ can be separated from the remaining terms and can be viewed (up to the relevant isometries, and after a restriction to one of the subspaces in each qudit) as acting on a separated subsystem.

To approximate the CLH problem in $\text{NP}$, it is not enough to remove for each term, all terms that make it isolated; this will require removing way too many terms. Instead, we use the above idea iteratively, and count how many terms we need to remove more carefully. At each iteration $t$ we choose a local term $v_t$ and “isolate” it (see figure 2). Corollary (1) can then be applied, and $v_t$ can be separated from the rest, after restricting each of its qudits to its relevant subspaces. Some qudits may interact, following this process, with only a single local term of the Hamiltonian, thereby allowing us to remove them. We can now iterate this process, picking another local term; this way we gradually “tear away” local subspaces of particles.

Figure 2: The isolation procedure w.r.t. $v_t$: we remove the bottom local term since it intersects the neighbor-set of $v_t$ at two locations. Then we apply lemma (1), which removes all degrees of freedom of the bottom particle, while reducing the dimension of the next two particles by at least 1.

The analysis of the upper bound on the number of terms that need to be removed altogether is non-trivial, since the number of particles does not necessarily decrease through one iteration; it is only their dimensionality that decreases. We resort to an amortized analysis counting the total number of local dimensions (namely, the sum over the particles of the local dimension of each particle) that are removed in total. To the best of our knowledge this amortized counting of local dimensions is novel.

Note that removing terms here is done not in order to disconnect parts of a graph, but only to isolate terms; loosely speaking, to make the interaction pattern more sparse. This allows a much more economic removal of terms, and enables us to achieve the desired result.
3.2 Approximating CLH on local-expanders - Proof

We are now ready to prove our main theorem:

**Proof:** (Of Theorem \[\text{I}\])
Let $H$ be an instance of CLH($k,d$), and let $(L,R;E)$ be its bi-partite interaction graph. We shall find a state whose energy approximates the ground energy of $H$, (we note that the same procedure can be applied to the approximation of any eigenvalue of $H$). We perform an iterative process. At step $t$, we have a set of remaining terms $R_{\text{rem}}(t)$ acting on the remaining Hilbert space $\mathcal{H}_{\text{rem}}(t)$. We also have a set $R_{\text{bad}}(t)$ which are terms we have collected so far that we want to throw away. We initialize $R_{\text{rem}}(1) = R$, i.e. the set of all local terms of $H$, and $R_{\text{bad}}(1)$ to be the empty set. We also initialize $\mathcal{H}_{\text{rem}}(1) = \mathcal{H}$. Repeat the following:

1. **isolate** Pick an arbitrary local term $v$ in $R_{\text{rem}}(t)$. and isolate it (definition \[\text{I}\]) by removing as few as possible terms from $R_{\text{rem}}(t)$, placing them in $R_{\text{bad}}(t)$.

2. **isometries** Now that $v$ is isolated, the conditions of Corollary \[\text{I}\] hold: there is a tensor product of 1-local unitaries on the qudits of $v$: $W_v = \bigotimes_i W_{i^v}$ such that applying $W_v$ allows to restrict the Hilbert space to some strict subspace containing the zero eigenspace of $R_{\text{rem}}(t)$. Moreover, inside this subspace, the term $v$ acts on a disjoint subsystem than the rest of $R_{\text{rem}}(t)$. We conjugate $v$, and each term of $R_{\text{rem}}(t)$ by $W_v$. We then prune: remove any triviality we encounter, be it removing qudits from Hamiltonian terms or entire terms altogether. After pruning, each local term acts non-trivially on all the qudits that are attached to it.

3. **update** We set $R_{\text{rem}}(t+1)$ as the set of remaining local terms of $R_{\text{rem}}(t)$ after this pruning, and then update $\mathcal{H}_{\text{rem}}(t+1)$ as the support of $R_{\text{rem}}(t+1)$. We then set $R_{\text{good}}(t+1)$ as the union of $R_{\text{good}}(t)$ and the term $v(t)$ restricted to the appropriate subspaces of its qudits, as in corollary \[\text{I}\].

We terminate when there is no longer any intersection between two different terms of $R_{\text{rem}}(t)$. Clearly, this ends after at most polynomially many iterations, since the number of terms in $R_{\text{rem}}(t)$ decreases by at least 1 each iteration. Let the number of iterations be $T$. We claim that a state whose energy is within $|R_{\text{bad}}(T)|$ from the ground energy of the original Hamiltonian, can be recovered from this procedure by finding the ground state of all terms in $R_{\text{good}}(T)$, and applying the inverse of the isometries applied along the way. This is true as long as the subspaces in the direct sum that are chosen at step $t$, contain the groundspace of the current Hamiltonian $R_{\text{rem}}(t)$; the NP prover can provide the indices of the subspaces so that this holds at each step, and the verification procedure is simply to check that those subspaces contain a non-zero kernel.

To see that this implies a constant-depth quantum circuit that generates a groundstate, observe that all isometries on a given qudit $q$, produced along the way as a result of isolating all local terms from $R_{\text{good}}(T) \cup R_{\text{rem}}(T)$ on $q$, commute by definition, and so we can apply them in any order, and in particular, simultaneously. Their application causes the interaction graph to break into linearly many connected components of size at most $k$ each, and so each component can be trivially diagonalized. The composition of these local diagonalizing unitaries, and the tensor-product isometries results in a circuit of depth 2 (just as in \[18\]).

**Bounding the approximation error** We now provide an upper bound on the size of the error $|R_{\text{bad}}(T)|$. For a given $q \in L$, let $D_q$ denote its degree in $G(1)$. Consider the $t$-th “isolation” step,
and denote $v_t$ as the constraint isolated at step $t$, $S_t = \Gamma(v_t)$, and $G(t)$ as the bi-partite interaction graph of $L_{rem}(t)$. We define for each $q \in S_t$, the quantity $p(t, q)$ which is the number of check terms removed to $R_{bad}$ during the $t$-th step, multiplied by the ratio $\frac{D_q}{\sum_{q' \in S_t} D_{q'}}$, where those degrees are taken from $G(1)$. This accounts for the ”relative penalty” shared by $q$, w.r.t. $G(1)$ - note that for any iteration $t$, $\sum_{q \in S_t} p(t, q)$ is exactly the number of check terms removed at the $t$-th iteration. Hence

$$|R_{bad}(T)| = \sum_{t=1}^{T} \sum_{q \in S_t} p(t, q). \quad (2)$$

We claim:

**Fact 2** For any $1 \leq t \leq T$, and any $q \in S_t$, we have $p(t, q) \leq 2\varepsilon D_q$.

**Proof:** We know that in $G(1)$ we had $|\Gamma(S_t)| \geq (1 - \varepsilon) \sum_{q \in S_t} D_q$, and so by Fact 1 in order to isolate $S_t$, it suffices to remove $2\varepsilon \sum_{q \in S_t} D_q$ terms. Since $G(t)$ is derived from $G(1)$ by removing vertices and edges, this bound still holds – the isolation penalty cannot increase by removing elements from the graph. So, at each step, when isolating a term acting on a set of particles $S_t$, we remove at most $2\varepsilon \sum_{q \in S_t} D_q$ constraints. By definition, $p(t, q) \leq 2\varepsilon D_q$. □

On the other hand, let us calculate how many qudit dimensions we remove at such an iteration, out of the total number of qudit dimensions (namely, the sum of dimensions over the particles)\(^4\). For each qudit $q \in S_t$, $q$ is either removed altogether, and thus its local dimension is reduced by $\dim(\mathcal{H}_q)$, or it is restricted to some non-trivial subspace and maybe further divided to two sub-particles, one of which is removed from $\mathcal{H}_{rem}(t)$; in any of these cases, the local dimension decreases by at least 1. Thus, the total number of local dimensions removed by one application of “isolate” is at least $|S_t|$.

We observe by the above that every qudit $q \in L$ participates in at most $d$ iterations. We would like to sum its relative penalties $p(t, q)$ over those iterations. By Fact 2 this sum is at most $d \cdot \max_t p(t, q) \leq d \cdot 2\varepsilon D_q$. We would now like to sum over all $q \in L$; by Equation 2 this will upper bound $|R_{bad}(T)|$. We have

$$|R_{bad}| \leq \sum_{q \in L} 2\varepsilon d D_q = 2\varepsilon d \sum_{q \in L} D_q \leq 2\varepsilon dk|R|.$$ \hspace{1cm} (3)

Therefore, the relative penalty satisfies

$$\frac{|R_{bad}(T)|}{|R|} \leq 2kd\varepsilon,$$

as required. □

### 3.3 NP-hardness of approximation

The proof of Theorem 2 relies on the fact that the output of the PCP reductions in Dinur’s proof [21] can be made into an excellent locally-expanding graph without harming the promise gap, by a small modification of her construction. We provide here only a sketch of the proof, assuming the reader is familiar with [21]. Providing a self contained proof will require essentially repeating Dinur’s PCP proof.

\(^4\) We note that when we reduce the local dimension of a $d$-level qudit by 1, the dimension of the total Hilbert-space is reduced by a factor of $(d-1)/d$; here we are interested however not in the standard dimension of the entire Hilbert space but in the sum of local dimensions over all particles.
Proof: (Of Theorem 2) Let us denote by CSP(3, 2) 3-local CSPs acting on bits. Note that instances of CSP(3, 2) are special cases of CLH(k, d) for k ≥ 3 and d ≥ 2. Suppose that for given parameters k, d we are also given ε, for which γ(ε, k, d) ≤ C0. The theorem will follow for these parameters k, d, ε if we show that it is even NP-hard to approximate CSP(3, 2) whose bipartite interaction graphs are ε-locally-expanding, to within C0.

To do this, recall that Dinur provides in [21] a PCP reduction which maps any 2-local CSP instance on n variables of alphabet Σ0, |Σ0| = O(1), to a CSP instance of poly(n) size, such that the new instance has constant promise gap, say, 1/10. We modify her construction so that the output CSP is 3-local, acting on bits, has local expansion error ε, and the promise gap is larger than C0 where C0 is 1/10 divided by the loss in the promise gap introduced by the alphabet reduction step in Dinur’s construction; this factor is bounded by a universal constant independent of the parameters of the problem [21].

To achieve this we make a small modification of Dinur’s gap amplification procedure. Recall that this procedure consists of iterating three steps: the preprocessing step in which the graph is made into a d-regular expander graph, gap amplification in which the CSP is translated into another CSP, whose new 2-local constraints are defined using conjunctions of constraints in the original CSP along collections of t-walks (walks of length t), thereby significantly amplifying the promise gap of the instance, but increasing the size of the alphabet, ℓ, to be doubly exponential in t; and finally the alphabet reduction in which a 2-CSP instance alphabet Σ, is translated into a 3-local on bits, with some small constant factor loss in the promise gap. We denote 1/10 divided by this loss to be C0. As a final sub-step, the output 3-local CSP is translated into a 2-local CSP on alphabet of size 8. By every such 3-step iteration, the gap is amplified by some constant larger than 1.

We apply Dinur’s procedure, until the output is a (3, 2) - CSP with promise gap at least 1/10. Now we add one additional iteration, in which we make two modifications. First, we choose the parameter t, defined to be the length of walks in the gap amplification step, in such a way that ℓ, the size of the new alphabet, which is doubly exponential in t, satisfies 1/L < xε where L = 2^{2t}, and x is some combinatorial constant to be determined below. Second, we skip the very final step where 3-local constraints are translated into 2-local constraints. Finally, regardless of how the alphabet reduction is defined in previous iterations (there are several versions of Dinur’s proof) apply at the final iteration alphabet reduction using Dinur’s variant [21] of the long code test ([26]). We claim that the resulting CSP is a 3-local CSP whose underlying bi-partite graph has ε local expansion, and whose promise gap is at least C0.

The fact that the promise gap is at least C0 follows from the fact that we have applied Dinur’s iteration to achieve promise gap at least 1/10, and then the alphabet reduction reduces it to at least C0 regardless of the alphabet size (this is how we have defined C0). Let us now show why the local expansion error is at most ε. Let S be some set of 3 bits. The local expansion error of S can be bounded from above by (1 − α1)/2, where α1 = |Γ_1(S)|/|Γ(S)| as in the proof of Fact (I). Hence, it is sufficient to lower bound α1 and show it is at least 1 − 2ε.

We consider in more detail the long-code construction used in the alphabet reduction. This step associates with each 2-local constraint acting on two large alphabet variables (each is of dimension ℓ) with L = 2^{2t} bits, which are supposed to be the long-code encoding of the assignments to those two large alphabet variables. Those bits are called LC bits; for each such L-tuples of bits, the long-code constraints (called LC constraints) are 3-local constraints, one for each triplet among those L bits. In addition, we add for each ℓ-size variable ℓ bits that are responsible for the Consistency of
this variable in all constraints it appears in. These \( \ell \) bits are thus called \( C \)-type; the consistency constraints (\( C \)-constraints), act on one \( C \)-type bit, and two \( LC \)-bits from the long-code encoding of a given constraint. We note that choosing any two bits of the \( C \) constraint determines the third bit.

We first observe, that the only interesting \( S \)'s are those that have at least 2 \( LC \)-type bits in some long-code encoding of the same constraint. This is because in all the other cases, either no constraint can intersect \( S \) in more than one bit (this is the case when \( S \) contains no \( LC \) bits, or it contains 3 \( LC \) bits, all from the long-code encoding of different constraints ) or the number of constraints that can intersect \( S \) in two bits is at most 2 (this is the case if \( S \) contains one \( LC \) bit only, or if \( S \) contains 2 \( LC \) bits, but from different long-code encodings). Since all other constraints intersect \( S \) in exactly one bit, \( \alpha_1 \) is close enough to 1.

So we are left with the case that \( S \) has at least 2 bits in the same long-code encoding. The fraction of \( LC \) constraints that intersect \( S \) in 2 places, out of all possible \( LC \) constraints that intersect it, is at most \( O(1/L) \); The number of \( C \) constraints that intersect \( S \) in 2 places, is at most 3 since every two bits determine the third in a \( C \) constraint. Thus, the fraction of \( C \) constraints that intersect \( S \) in two bits, out of all \( C \) constraints that intersect \( S \), is \( O(1/L) \). Thus, the expansion error in this case is at most \( O(1/L) \), which, by choosing \( x \) correctly in the choice of \( t \) above, is \( < \varepsilon \).

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