NOTES ON PROJECTIVE, CONTACT, AND NULL CURVES

ROBERT L. BRYANT

Abstract. These are notes on some algebraic geometry of complex projective curves, together with an application to studying the contact curves in $\mathbb{P}^3$ and the null curves in the complex quadric $\mathbb{Q}^3 \subset \mathbb{P}^4$, related by the well-known Klein correspondence. Most of this note consists of recounting the classical background. The main application is the explicit classification of rational null curves of low degree in $\mathbb{Q}^3$.

I have recently received a number of requests for these notes, so I am posting them to make them generally available.

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1. Introduction

These are notes containing the details of the proof of my claim \([6]\) that there are no unbranched rational null curves $\gamma : \mathbb{P}^1 \to \mathbb{C}^3$ with simple poles and having total degree 5 or 7. Claims to the contrary that have been made in the literature (c.f., \([10]\)) are in error.

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Along the way, I explain a few other results of interest. Mostly these are consequences of the results in [3]. Some of this material has, in the meantime, been rediscovered by others [2, 3].

For the convenience of the reader, I include some discussion of the algebraic geometry of projective curves. All of this material is classical [5].

2. INVARIANTS OF PROJECTIVE CURVES

Let $V$ be a complex vector space of dimension $n+1 \geq 2$, and let $\mathbb{P}(V)$ be its projectivization. When $V$ is clear from context, I will write $\mathbb{P}^n$ for $\mathbb{P}(V)$.

Let $S$ be a connected Riemann surface and let $f : S \to \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ be a nondegenerate holomorphic curve, i.e., $f(S)$ does not lie in any proper hyperplane $H^{n-1} \subset \mathbb{P}^n$.

When $S$ is compact, the degree of $f$, $\deg(f)$, is the number of points in the pre-image $f^{-1}(H) \subset S$ where $H \subset \mathbb{P}^n$ is any hyperplane that is nowhere tangent to $f$. When $f : S \to \mathbb{P}^n$ is nondegenerate, one knows that $\deg(f) \geq n$.

2.1. Ramification. Given $p \in S$, one can write

\[
(2.1) \quad f = [h_0 \, v^0 + h_1 \, v^1 + \cdots + h_n \, v^n]
\]

for some basis $v^0, \ldots, v^n$ of $V$ where the $h_i$ are meromorphic functions on $S$ that satisfy

\[
(2.2) \quad 0 = \nu_p(h_0) < \nu_p(h_1) < \cdots < \nu_p(h_n),
\]

where $\nu_p(h_i)$ is the order of vanishing of $h_i$ at $p \in S$. The numbers $a_i(p) = \nu_p(h_i) \geq i$ for $0 \leq i \leq n$ depend only on $f$ and $p$, not on the choice of basis $v^i$ and meromorphic functions $h_i$ satisfying (2.1) and (2.2).

For all but a closed, discrete set of points $p \in S$, one will have $a_i(p) = i$ for $0 \leq i \leq n$. It is useful to define, for $i \geq 1$,

\[
(2.3) \quad r_i(p) = a_i(p) - a_{i-1}(p) - 1 \geq 0,
\]

which is known as the $i$-th ramification degree of $f$ at $p$. When $f$ is not clear from context, I will write $r_i(p, f)$.

Since $r_i(p) = 0$ for $0 \leq i \leq n$ for all but a closed, discrete set of points $p \in S$, one can define the $i$-th ramification divisor of $f$ to be the locally finite formal sum

\[
R_i(f) = \sum_{p \in S} r_i(p, f) \cdot p.
\]

When $S$ is compact, this is a finite sum, in which case, $R_i(f)$ is an effective divisor on $S$.

Remark 1 (Branch points). A point $p \in S$ at which $r_1(p, f) > 0$ is said to be a branch point of $f$ of order $r_1(p, f)$. When $R_1(f) = 0$, $f$ is said to be unbranched, which is equivalent to $f$ being an immersion.

2.2. The associated curves. Since $f$ is nondegenerate, there is a well-defined sequence of associated curves, $f_k : S \to \mathbb{P}(\Lambda^k(V))$ for $1 \leq k \leq n$, defined, relative to any local holomorphic coordinate $z : U \to \mathbb{C}$ where $U \subset S$ is an open set, by

\[
f_k = \left[ F \wedge \frac{dF}{dz} \wedge \cdots \wedge \frac{d^{k-1}F}{dz^{k-1}} \right]
\]
where $F : U \to V$ is holomorphic and non-vanishing and $f = \lfloor F \rfloor$ on $U \subset S$. (It is easy to show that $f_k$ is well-defined, independent of the choice of $z$ or $F$.) Of course, $f_1 = f$.

**Remark 2 (Wronskians).** If $h_1, \ldots, h_k$ are meromorphic functions on a connected Riemann surface $S$ and $z : U \to \mathbb{C}$ is a local holomorphic coordinate on $U \subset S$, then the Wronskian differential of $(h_1, \ldots, h_k)$ is the expression

$$W(h_1, \ldots, h_k) = \det \begin{pmatrix} h_1 & h_2 & \cdots & h_k \\ h_1^{(1)} & h_2^{(1)} & \cdots & h_k^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(k-1)} & h_2^{(k-1)} & \cdots & h_k^{(k-1)} \end{pmatrix} dz^{k(k-1)/2},$$

where $h_k^{(j)} = d^j h_k / dz^j$. It is not hard to show that $W(h_1, \ldots, h_k)$ does not depend on the choice of local holomorphic coordinate $z$ and hence is a globally defined (symmetric) differential on $S$.

The Wronskian has two important (and easily proved) properties that will be needed in the rest of these notes.

First, if $\nu_p(h_1) < \nu_p(h_2) < \cdots < \nu_p(h_k)$, then

$$\nu_p(W(h_1, \ldots, h_k)) = \nu_p(h_1) + \cdots + \nu_p(h_k) - \frac{1}{2}k(k-1).$$

Second (and this follows easily from the first fact), $W(h_1, \ldots, h_k)$ vanishes identically if and only if the functions $h_1, \ldots, h_k$ are linearly dependent as functions on $S$.

Note that, when $f : S \to \mathbb{P}^n$ is described as in (2.1), the associated curves can be written in the form

$$(2.4) \quad f_k = \left[ \sum_{0 \leq i_1 < i_2 < \cdots < i_k \leq n} W(h_{i_1}, \ldots, h_{i_k}) v^{i_1} \wedge v^{i_2} \wedge \cdots \wedge v^{i_k} \right].$$

2.3. **The canonical $k$-plane and line bundles.** Since $f_k(p)$ is the projectivization of a nonzero simple $k$-vector for all $p \in S$, it follows that there exists a flag of subspaces

$$\{0\} = E_0(p) \subset E_1(p) \subset \cdots \subset E_n(p) \subset E_{n+1}(p) = V$$

such that $\dim E_i(p) = i$ and $f_i(p) = \mathbb{P}(\Lambda^i(E_i(p)))$ for all $p \in S$ and $i \geq 1$.

It is easy to show that the subset

$$(2.5) \quad E_i = \{(p, v) \in S \times V \mid v \in E_i(p) \}$$

is a holomorphic $i$-plane subbundle of the trivial bundle $E_{n+1} = S \times V$. Since $E_{i-1} \subset E_i$, there are well-defined quotient line bundles over $S$

$$(2.6) \quad L_i = E_i / E_{i-1}$$

for $1 \leq i \leq n+1$. Note that, since $\Lambda^i(E_i) \simeq \Lambda^i (E_{i-1})$ for $1 \leq i \leq n+1$, it follows that

$$L_1 \otimes L_2 \otimes \cdots \otimes L_n \simeq \Lambda^{n+1}(E_{n+1}) = S \times \mathbb{C}.$$

Let $B \subset S \times V \times V \times \cdots \times V$ (with $n+1$ factors of $V$) be the set of $(n+2)$-tuples $(p, v_0, \ldots, v_n)$ that satisfy the conditions $p \in S$, $v_i \in E_{i+1}(p)$ for $0 \leq i \leq n$, and $(v_0, \ldots, v_n)$ is a basis of $V$. This $B$ is a holomorphic submanifold of $S \times V^{n+1}$, the projection $\sigma : B \to S$ onto the first factor is a submersion, and the
$V$-valued functions $e_i : B \to V$ defined by $e_i(p, v_0, \ldots, v_n) = v_i$ for $0 \leq i \leq n$ are holomorphic. Consequently, there are unique holomorphic 1-forms $\omega^j_i$ on $B$ satisfying the structure equations

$$\text{(2.8)} \quad de_i = e_j \omega^j_i,$$

and

$$\text{(2.9)} \quad d\omega^j_i = -\omega^j_i \wedge \omega^k_i.$$

Moreover, since, by construction,

$$\text{(2.10)} \quad f_i \circ \sigma = [e_0 \wedge e_1 \wedge \cdots \wedge e_{i-1}],$$

it follows that $\omega^j_i = 0$ whenever $j > i + 1$ and that $\omega^{i+1}_i$ is $\sigma$-semibasic for $0 \leq i \leq n$.

Now, for $b = (p, v_0, \ldots, v_n) \in B$, one has $v_i \in E_i(p)$ but $v_i \notin E_{i-1}(p)$ for $i \geq 1$. Consequently, there is a unique linear function $e^i(b) : E_i(p) \to \mathbb{C}$ that has $E_{i-1}(p)$ as its kernel and satisfies $e^i(v_i) = 1$. Thus, $e^i(b)$ can be regarded as a nonzero linear function on the line $E_i(p)/E_{i-1}(p) = L_i(p)$, and hence $e^i(b)$ is a nonzero element of the dual line $L_i(p)^*$. In addition, there is an element $[e_i(b)] \in E_i(p)/E_{i-1}(p) = L_i(p)$ that is given by $[e_i(b)] = v_i \mod E_{i-1}(p)$.

With these definitions, it is not difficult to show that there is a well-defined section $\rho_i$ of the line bundle $L_{i+1} \otimes L_i^* \otimes K$ over $S$ (where $K$ is the canonical line bundle of $S$) satisfying

$$\text{(2.11)} \quad \rho_i \circ \sigma = [e_{i+1}] \otimes e^i \otimes \omega^{i+1}_i.$$

Moreover, following the definitions above, one finds that the section $\rho_i$ vanishes to order $r_i(p)$ at $p \in S$.

2.4. The compact case and divisors. Now suppose that $S$ is compact, and fix a nondegenerate $f : S \to \mathbb{P}^n$, which will not be notated in the following discussion.

Then $L_i \simeq \mathcal{O}(-D_i)$ for $1 \leq i \leq n$, where $D_i$ is a divisor on $S$, well-defined up to linear equivalence.

From (2.7), it then follows that

$$\text{(2.12)} \quad D_1 + D_2 + \cdots + D_{n+1} \equiv 0,$$

where `≡' means linear equivalence of divisors.

Moreover, because the zero divisor of the holomorphic section $\rho_i$ of $L_{i+1} \otimes L_i^* \otimes K$ is $R_i$ for $1 \leq i \leq n$, it follows that

$$\text{(2.13)} \quad R_i \equiv -D_{i+1} + D_i + K,$$

where, again, $K$ is the canonical divisor of $S$. In particular, for $\ell > 1$ we have

$$\text{(2.14)} \quad D_\ell \equiv D_1 + (\ell-1)K - R_1 - R_2 - \cdots - R_{\ell-1}.$$

Moreover, using (2.12), one obtains

$$\text{(2.15)} \quad (n+1)D_1 + \binom{n+1}{2}K \equiv nR_1 + (n-1)R_2 + \cdots + R_n.$$

Since $\text{deg} \ D_1 = \text{deg} \ f$, taking degrees of divisors, one has

$$\text{(2.16)} \quad (n+1) \text{deg} \ f + n(n+1)(k-1) = n\ r_1 + (n-1)\ r_2 + \cdots + r_n$$

where $r_i = \text{deg} \ R_i \geq 0$ and $k$ is the genus of $S$. 

Example 1 (Rational normal curves). If $S$ is compact and $f : S \to \mathbb{P}^n$ is nondegenerate and satisfies $r_i = 0$ for all $i$, it follows from (2.16) that $k = 0$ and $\deg f = n$, so that $f(S) \subset \mathbb{P}^n$ is the rational normal curve of degree $n$, i.e., up to projective equivalence,

\begin{equation}
(2.17) \quad f = [1, z, z^2, \ldots, z^n]
\end{equation}

where $z$ is a meromorphic function on $S = \mathbb{P}^1$ with a single, simple pole.

To conclude this subsection, I list a few further useful facts. First,

\begin{equation}
(2.18) \quad \deg f_i = \deg D_1 + \deg D_2 + \cdots + \deg D_i
\end{equation}

for $1 \leq i \leq n$.

Next, the dual curve $f_n : S \to \mathbb{P}^n(\Lambda^*(\Lambda^i(V))) = \mathbb{P}(V^*)$ of $f = f_1$ is nondegenerate, and its ramification divisors are given by

\begin{equation}
R_i(f_n) = R_{n+i}(f_1).
\end{equation}

Moreover, the dual curve of $f_n$ is $f_1$, i.e., $(f_n)_n = f_1 = f$.

Finally, one has the following relation between the first ramification divisor of $f_i$ and the $i$-th ramification divisor of $f$:

\begin{equation}
R_1(f_i) = R_i(f).
\end{equation}

(This follows immediately from (2.11) and the properties of the Wronskian.) However, note that, in general, for $1 < i < n$, the higher ramification divisors of $f_i$ cannot be computed solely in terms of the ramification divisors of $f = f_1$. In fact, the $f_i$ in this range need not even be nondegenerate, as will be seen.

3. Contact curves in $\mathbb{P}^3$

Now let $V$ have dimension 4 and let $\beta \in \Lambda^2(V^*)$ be a nondegenerate 2-form on $V$, i.e., $V$ is a symplectic vector space of dimension 4. (Since any two nondegenerate 2-forms on $V$ are $\text{GL}(V)$-equivalent, the particular choice of $\beta$ is not important.) Let $\text{Sp}(\beta) \subset \text{GL}(V)$ denote the group of linear transformations of $V$ that preserve $\beta$.

The choice of $\beta$ defines a volume form $\Omega = \frac{1}{2} \beta^2 \in \Omega^2(V^*)$ on $V$ and, because of the nondegenerate pairing

\[ \Lambda^2(V) \times \Lambda^2(V^*) \to \mathbb{C}, \]

it also defines a subspace $W = \beta^\perp \subset \Lambda^2(V)$ of dimension 5.

Moreover, by the usual reduction process induced by the $\mathbb{C}^*$-action of scalar multiplication on $V$, the projective space $\mathbb{P}^3 = \mathbb{P}(V)$ inherits a contact structure, i.e., a holomorphic 2-plane field $C \subset T\mathbb{P}^3$ that is nowhere integrable and is invariant under the induced action of $\text{Sp}(\beta)$ on $\mathbb{P}^3$.

A connected holomorphic curve $f : S \to \mathbb{P}^3$ is said to be a contact curve with respect to $\beta$ if $f^*(T_pS) \subset C_{f(p)} \subset T_{f(p)}\mathbb{P}^3$ for all $p \in S$. Equivalently, $f$ is a contact curve if and only if either $f$ is constant or else $f_2(S)$ has image in $\mathbb{P}(W) \subset \mathbb{P}(\Lambda^2(V))$. If $f(S)$ does not lie in a line in $\mathbb{P}^3$, I will say that $f$ is nonlinear.

**Proposition 1.** If $f : S \to \mathbb{P}^3$ is a nonlinear contact curve, then $f$ is nondegenerate. Moreover, $R_1(f) = R_3(f)$, and $f_2 : S \to \mathbb{P}(W) \simeq \mathbb{P}^4$ is nondegenerate, with

\[ R_1(f_2) = R_4(f_2) = R_2(f) \quad \text{and} \quad R_2(f_2) = R_3(f_2) = R_1(f). \]
Proof. If $f$ were degenerate, then $f(S)$ would be linearly full in some $\mathbb{P}^2 \subset \mathbb{P}^3$, and hence it would be expressible on a neighborhood of $p \in S$ in the form

$$f = [v^0 + h_1 v^1 + h_2 v^2],$$

where the $h_i$ are meromorphic functions on $S$ with $\nu_p(h_1) = a_1 > 0$ and $\nu_p(h_2) = a_2 > a_1$, and with $v^0, v^1, v^2$ being linearly independent vectors in $V$. If $z : U \to \mathbb{C}$ is a $p$-centered local holomorphic coordinate on an open $p$-neighborhood $U \subset S$, and we set $dh_i = h'_i dz$, then

$$f_2 = [h'_1 v^0 \wedge v^1 + h'_2 v^0 \wedge v^2 + (h'_1 h'_2 - h'_2 h'_1) v^1 \wedge v^2],$$

where $\nu_p(h'_1) = a_1 - 1 < \nu_p(h'_2) = a_2 - 1 < \nu_p(h_1) h'_2 - h_2 h'_1) = a_2 + a_1 - 1$. It follows that the functions $h'_1, h'_2, h_1 h'_2 - h_2 h'_1$ are linearly independent on $U$, implying that $f_2(S) \subset \mathbb{P}(W)$ is linearly full in the projectivization of the span of $\{v^0, v^1, v^2\}$, and hence that all of the 2-vectors $\{v^0 \wedge v^1, v^0 \wedge v^2, v^1 \wedge v^2\}$ must lie in $W$. However, this implies that $\beta$ vanishes on the 3-plane spanned by $\{v^0, v^1, v^2\}$, which is impossible, since $\beta$ is nondegenerate. Thus, $f$ must be nondegenerate.

Fix $p \in S$ and suppose that $f(p) = [v^0]$. Then one can choose $v^1, v^2, v^3$ in $V$ such that $(v^0, v^1, v^2, v^3)$ is a basis of $V$ for which

$$\beta = \xi_0 \wedge \xi_3 + \xi_1 \wedge \xi_2,$$

where $(\xi_0, \xi_1, \xi_2, \xi_3)$ is the dual basis of $V^*$ corresponding to $(v^0, v^1, v^2, v^3)$.

Write

$$f = [v^0 + h_1 v^1 + h_2 v^2 + h_3 v^3]$$

for some meromorphic functions $h_i$ on $S$ that vanish at $p$ and select a local $p$-centered holomorphic coordinate $z : U \to \mathbb{C}$ on some $p$-neighborhood $U \subset S$. The condition that $f$ be contact with respect to $\beta$ is expressed as the equation

$$h_3' = h_2 h_1' - h_1 h_2'$$

where $dh_i = h'_i dz$ on $U$.

Since $f$ is nondegenerate, $h_2 h_1' - h_1 h_2'$ does not vanish identically. Hence, by making a change of basis in $(v^1, v^2)$, it can be assumed that $0 < \nu_p(h_1) < \nu_p(h_2)$, which, by (3.2) and the fact that $\nu_p(h_3) > 0$, forces

$$\nu_p(h_3) = \nu_p(h_1) + \nu_p(h_2).$$

Set $a_i = \nu_p(h_i)$, so that $a_3 = a_1 + a_2$ and $a_2 > a_1 > 0$.

First, note that this implies that

$$r_3(p, f) = a_3 - a_2 - 1 = a_1 - 1 = r_1(p, f).$$

Since this holds for all $p \in S$, it follows that $R_3(f) = R_1(f)$.

Second, the relation (3.2) implies that

$$f_2 = [h'_1 v^0 \wedge v^1 + h'_2 v^0 \wedge v^1 + h'_3 (v^0 \wedge v^3 - v^1 \wedge v^2) + (h'_1 h'_3 - h'_3 h'_1) v^1 \wedge v^3 + (h'_2 h'_3 - h'_3 h'_2) v^2 \wedge v^3].$$

Now, the sequence of orders of vanishing of these five coefficients of the basis elements of $W$ are five distinct numbers:

$$a_1 - 1 < a_2 - 1 < a_1 + a_2 - 1 < 2a_1 + a_2 - 1 < a_1 + 2a_2 - 1.$$
Hence, $f_2 : S \to \mathbb{P}(W) \simeq \mathbb{P}^4$ is nondegenerate and has the following ramification indices at $p$:

$$r_1(p, f_2) = a_2 - a_1 - 1 = r_2(p, f),$$

$$r_2(p, f_2) = (a_1 + a_2) - a_2 - 1 = r_1(p, f),$$

$$r_3(p, f_2) = (2a_1 + a_2) - (a_1 + a_2) - 1 = r_1(p, f),$$

$$r_4(p, f_2) = (a_1 + 2a_2) - (2a_1 + a_2) - 1 = r_2(p, f).$$

Thus, $R_1(f_2) = R_4(f_2) = R_2(f)$ and $R_2(f_2) = R_3(f_2) = R_1(f)$, as claimed. □

**Remark 3.** Proposition [1] was proved in [6], though it was known classically [1]. It also appears (in slightly different notation) in [2], the authors of which do not appear to have been aware of [6].

Proposition [1] also suggests a slightly more general notion of contact curve, which is described by the following result.

**Proposition 2.** Let $f : S \to \mathbb{P}(\mathbb{C}^4) \simeq \mathbb{P}^3$ be a nondegenerate holomorphic curve for which $f_2 : S \to \mathbb{P}(\Lambda^2(\mathbb{C}^4)) \simeq \mathbb{P}^5$ is degenerate. Then there exists a nondegenerate symplectic form $\beta \in \Lambda^2((\mathbb{C}^4)^*)$, unique up to constant multiples, such that $f$ is a contact curve with respect to $\beta$.

**Proof.** As before, fix a point $p \in S$ and write $f$ in the form

$$f = [v^0 + h_1 v^1 + h_2 v^2 + h_3 v^3]$$

for some meromorphic functions $h_i$ on $S$ that vanish at $p$ and satisfy

$$0 < a_1 = \nu_p(h_1) < a_2 = \nu_p(h_2) < a_3 = \nu_p(h_3).$$

Let $z$ be a meromorphic function on $S$ that has a simple zero at $p$ and write $dh_i = h_i^0 dz$. Then $f_2$ takes the form

$$f_2 = [h_1' v^0 \wedge v^1 + h_2' v^0 \wedge v^2$$

$$+ h_3' v^0 \wedge v^3 + (h_1 h_2' - h_2 h_1') v^1 \wedge v^2$$

$$+ (h_1 h_3' - h_3 h_1') v^1 \wedge v^3 + (h_2 h_3' - h_3 h_2') v^2 \wedge v^3],$$

and the orders of vanishing at $p$ of the meromorphic coefficients of these terms in the order written in (3.5) are

$$a_1 - 1 < a_2 - 1 < a_3 - 1$$

$$a_2 + a_1 - 1 < a_3 + a_1 - 1 < a_3 + a_2 - 1.$$  

If $a_3$ were not equal to $a_2 + a_1$, then these six integers would be distinct, and it would follow that the six coefficient functions were linearly independent as meromorphic functions on $S$. In this case, $f_2$ would be linearly full in $\Lambda^2(\mathbb{C}^4)$, contrary to hypothesis. Thus, we must have $a_3 = a_2 + a_1$, and the inequalities (3.6) become

$$a_1 - 1 < a_2 - 1 < a_2 + a_1 - 1 < a_2 + 2a_1 - 1 < 2a_2 + a_1 - 1.$$

Now, in order for $f_2$ to be degenerate, these six coefficients must satisfy at least one nontrivial linear relation with constant coefficients. Because of the strict inequalities (3.7), this relation cannot involve $h_1'$ or $h_2'$, so it must of the form

$$c_1 h_3' + c_2 (h_1 h_2' - h_2 h_1') + c_3 (h_1 h_3' - h_3 h_1') + c_4 (h_2 h_3' - h_3 h_2') = 0.$$  

Moreover, again because of the strict inequalities (3.7), neither $c_1$ nor $c_2$ can vanish, and, thus, there cannot be two independent linear relations of this kind.
Now, consider the 2-form
\[
\beta = c_1 \xi_0 \wedge \xi_3 + c_2 \xi_1 \wedge \xi_2 + c_3 \xi_1 \wedge \xi_3 + c_4 \xi_2 \wedge \xi_3,
\]
where \((\xi_0, \xi_1, \xi_2, \xi_3)\) is the dual basis in \((\mathbb{C}^4)^*\) to the basis \((v^0, v^1, v^2, v^3)\) of \(\mathbb{C}^4\). Since \(\beta \wedge \beta = 2c_1c_2 \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \neq 0\), the 2-form \(\beta\) is nondegenerate and hence defines a symplectic structure on \(\mathbb{C}^4\). By construction, \(f_2\) lies in the projectivization of \(W \subset \mathbb{A}^2(\mathbb{C}^4)\), the kernel of \(\beta\). Hence, \(f\) is a contact curve in the projectivization of the symplectic space \((\mathbb{C}^4, \beta)\).

Since there is only one linear relation among the meromorphic coefficients appearing in \(f_2\), it follows that \(f_2\) lies linearly fully in \(\mathbb{P}(W) \cong \mathbb{P}^4\), which proves the uniqueness of \(\beta\) up to multiples. \(\square\)

**Example 2** (Rational contact curves of arbitrary degree). Let \(p\) and \(q\) be relatively prime integers satisfying \(0 < p < q\), and consider the curve \(f : \mathbb{P}^1 \to \mathbb{P}^3\), where \(z\) is a meromorphic function on \(\mathbb{P}^1\) possessing a single, simple pole at \(P\) and a single, simple zero at \(Q\), defined by
\[
f = [v^0 + z^p v^1 + z^q v^2 + z^{p+q} v^3],
\]
where \(v^0, v^1, v^2, v^3\) are linearly independent in \(\mathbb{C}^4\). Computation yields
\[
f_2 = [p v^0 \wedge v^1 + q z^{p-q} v^0 \wedge v^2 + z^q ((p+q) v^0 \wedge v^3 + (q-p) v^1 \wedge v^2) + q z^{p+q} v^1 \wedge v^3 + p z^{2q} v^2 \wedge v^3].
\]
Thus, \(f\) is a contact curve for the symplectic structure
\[
\beta = (p-q) \xi_0 \wedge \xi_3 + (p+q) \xi_1 \wedge \xi_2,
\]
and one has \(R_1(f) = (p-1)(P+Q)\), while \(R_2(f) = (q-p-1)(P+Q)\). This example, for \(q = p+1\), appears in [6].

4. **Null curves in \(\mathbb{C}^3\) and \(\mathbb{Q}^3\)**

Endow \(\mathbb{C}^3\) with a nondegenerate (complex) inner product, which will be denoted \(v \cdot w \in \mathbb{C}\) for \(v, w \in \mathbb{C}^3\).

If \(S\) is a connected Riemann surface, then a non-constant meromorphic curve \(\gamma : S \to \mathbb{C}^3\) will be said to be a null curve if the meromorphic symmetric quadratic form \(\nabla \cdot \nabla\gamma\) vanishes identically on \(S\).

In order to treat the poles of meromorphic null curves algebraically, it will be useful to introduce an algebraic compactification of \(\mathbb{C}^3\). The usual compactification that regards \(\mathbb{C}^3\) as an affine open set in \(\mathbb{P}^3\) is not useful in this context, since there is no natural way to extend the notion of ‘null’ to the hyperplane at infinity.

Instead, one embeds \(\mathbb{C}^3\) into \(\mathbb{P}^4\) as a quadric hypersurface \(Q_3\) by identifying \(x \in \mathbb{C}^3\) with the point
\[
[1, x, x \cdot x] = [1, x_1, x_2, x_3, x_1^2 + x_2^2 + x_3^2] \in \mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5).
\]
The resulting image is an affine chart on the projective quadric \(Q_3 \subset \mathbb{P}^4\) defined by the homogeneous equations
\[
(4.1) \quad X_0X_4 - X_1^2 - X_2^2 - X_3^2 = 0.
\]

A meromorphic null curve \(\gamma : S \to \mathbb{C}^3\) completes uniquely to an algebraic curve \(g : S \to Q_3 \subset \mathbb{P}^4\) that is also null, in the sense that the tangent lines to the curve lie in \(Q_3\) as well.
Moreover, \([4,12]\) is the quadratic form associated to an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{C}^5\) with the property that a \(g : S \to \mathbb{Q}^3\) that is the completion of a meromorphic null curve \(\gamma : S \to \mathbb{C}^3\) is of the form \(g = [G]\) where \(G : S \to \mathbb{C}^5\) is meromorphic and satisfies
\[
(4.2) \quad \langle G, G \rangle = \langle G, dG \rangle = \langle dG, dG \rangle = 0.
\]
(In the last equation, \(\langle dG, dG \rangle\) is to be interpreted as a symmetric meromorphic quadratic form.)

**Proposition 3.** If \(g : S \to \mathbb{C}^3\) is a meromorphic null curve with \(d\) simple poles and no other poles, then the completed null curve \(g : S \to \mathbb{Q}^3\) has degree \(d\) (as a map to \(\mathbb{P}^4\)). If, in addition, \(\gamma\) is an immersion away from its poles (i.e., \(\gamma\) is unbranched), then \(g : S \to \mathbb{Q}^3\) is also an immersion.

**Proof.** This follows immediately from local computation. \(\square\)

### 4.1. The Klein correspondence.
I now recall the famous Klein correspondence between nondegenerate contact curves \(f : S \to \mathbb{P}^3\) and nondegenerate null curves \(g : S \to \mathbb{Q}^3 \subset \mathbb{P}^4\).

As before, let \(V\) be a symplectic complex vector space of dimension 4 with symplectic form \(\beta \in \Lambda^2(V^*)\), with \(\Omega = \frac{1}{2}\beta^2 \in \Lambda^4(V^*)\) a volume form on \(V\). Let \(W \subset \Lambda^2(V)\) be the 5-dimensional subspace annihilated by \(\beta\). Then there is a nondegenerate symmetric inner product \(\langle \cdot, \cdot \rangle\) on \(W\) defined by
\[
\langle w_1, w_2 \rangle = \Omega(w_1 \wedge w_2).
\]
The (connected) symplectic group \(\text{Sp}(\beta) \subset \text{GL}(V)\) acts on \(\Lambda^2(V)\) preserving \(W\) and preserving this inner product. Moreover \(g(w) = w\) for all \(w \in W\) if and only if \(g = \pm I_V\), thus defining a double cover \(\text{Sp}(\beta) \to \text{SO}(\langle \cdot, \cdot \rangle)\), which is one of the so-called ‘exceptional isomorphisms’.

Note that \(\langle w, w \rangle = 0\) for a nonzero \(w \in W\) if and only if \(w\) is a decomposable 2-vector, i.e., \(w = v_1 \wedge v_2\) for two linearly independent vectors \(v_1, v_2 \in V\). Such an element \(w\) will be said to be a null vector in \(W\). Define
\[
\mathbb{Q}^3 = \{ [w] \in \mathbb{P}(W) \mid \langle w, w \rangle = 0 \} \subset \mathbb{P}(W).
\]
Then \(\mathbb{Q}^3\) is the null hyperquadric of \(\langle \cdot, \cdot \rangle\). Since \(\langle \cdot, \cdot \rangle\) is nondegenerate, \(\mathbb{Q}^3\) is a smooth hypersurface in \(\mathbb{P}(W) \simeq \mathbb{P}^4\).

If \(f : S \to \mathbb{P}(V)\) is a nondegenerate contact curve, then \(g = f_2\) has image in \(W\) and, moreover, since, by construction, \(g(p)\) is the projectivization of a decomposable 2-vector for all \(p \in S\), it follows that \(g(S) \subset \mathbb{Q}^3\).

In fact, more is true: Writing \(f = [F]\) where \(F : S \to V\) is meromorphic and letting \(z : U \to \mathbb{C}\) be a local holomorphic coordinate on \(U \subset S\) and writing \(dH = H'\,dz\) for any meromorphic \(H\) on \(S\), one obtains \(g = [G]\), where \(G = F \wedge F'\). Since \(G' = F \wedge F''\), it follows that
\[
\begin{align*}
\langle G, G \rangle &= \langle F \wedge F', F' \wedge F' \rangle = \Omega(F \wedge F' \wedge F' \wedge F') = \Omega(0) = 0 \\
\langle G, G' \rangle &= \langle F \wedge F', F \wedge F'' \rangle = \Omega(F \wedge F' \wedge F' \wedge F'') = \Omega(0) = 0 \\
\langle G', G' \rangle &= \langle F \wedge F'', F \wedge F'' \rangle = \Omega(F \wedge F'' \wedge F' \wedge F'') = \Omega(0) = 0
\end{align*}
\]
Hence, \(g : S \to \mathbb{Q}^3\) is a null curve, which, by Proposition \([12]\) is nondegenerate as a curve in \(\mathbb{P}(W)\).

The interesting thing is the converse, which is due to Klein:
Proposition 4. If \( g : S \to \mathbb{Q}^3 \) is a null curve that is nonlinear, i.e., its image is not contained in a linear \( \mathbb{P}^1 \subset \mathbb{P}(W) \), then \( g = f_2 \) for a unique nondegenerate contact curve \( f : S \to \mathbb{P}(V) \).

Proof. The result is local, so write \( g = [G] \) where \( G : U \to W \) is holomorphic and nonvanishing. By hypothesis, \( G \wedge G = 0 \), which, of course, implies that \( G \wedge G' = 0 \). The condition that \( g : S \to \mathbb{Q}^3 \) be null is then equivalent to \( G' \wedge G'' = 0 \). Thus, \( G \) and \( G' \) span a null 2-plane in \( W \). Since \( G \) and \( G' \) are each decomposable, while \( G \wedge G' = 0 \), it follows that they can be written in the form \( G = F \wedge H \) and \( G' = F \wedge K \) for some meromorphic \( F, H, K : U \to W \) that are virtually linearly independent, i.e., \( F \wedge H \wedge K \) vanishes only at isolated points. Since \( G' = F' \wedge H + F \wedge H' = F \wedge K \), it follows that \( F' \wedge H = F \wedge (K - H') \), so that \( F \wedge F' \wedge H \) vanishes identically, implying that \( F, F' \), and \( H \) are linearly dependent and hence span a 2-dimensional vector space.

Now, I claim that \( F \wedge F' \) does not vanish identically, which implies that \( [F \wedge F'] = [F \wedge H] = g \), and hence that \( g = f_2 \) where \( f = [F] \).

Suppose, on the contrary, that \( F \wedge F' \) did vanish identically. In that case \( F = hv^0 \) for some vector \( v^0 \in V \), unique up to multiples, and some meromorphic function \( h \). Thus, we can assume that \( F = v^0 \), and so \( G = v^0 \wedge H \). Consequently, \( G' = v^0 \wedge H' \) and \( G'' = v^0 \wedge H'' \). If \( v^0 \), \( H \), \( H' \), and \( H'' \) were linearly independent on any open set, it would follow that the subspace of \( W \) spanned by \( G, G' \), and \( G'' \), would be of dimension 3 and totally null (since all of the elements are decomposable), which is impossible since \( (,.) \) is nondegenerate and \( W \) has dimension 5. Thus, \( H'' \) is a linear combination of \( v^0 \), \( H \) and \( H' \). Consequently, the 3-plane spanned by \( v^0 \), \( H \), and \( H' \) is constant, implying that the 2-plane in \( W \) spanned by \( G \) and \( G' \) is constant, which implies that \( g(S) \subset \mathbb{Q}^3 \) is a line in \( \mathbb{P}^4 \), contrary to hypothesis.

It is now established that \( g = f_2 \) where \( f = [F] : S \to \mathbb{P}(V) \) is a contact curve, and the uniqueness of \( f \) is clear. Since \( g \) is not constant, \( f(S) \) does not lie in a line in \( \mathbb{P}(V) \) and hence, by Proposition \( \text{II} \), \( g : S \to \mathbb{Q}^3 \) is nondegenerate in \( \mathbb{P}(W) \). \( \square \)

4.2. Ramifications and degrees. Now suppose that \( S \) is a compact (connected) Riemann surface and that \( f : S \to \mathbb{P}(V) \) is a holomorphic contact curve that is not contained in a line and that \( g = f_2 : S \to \mathbb{Q}^3 \subset \mathbb{P}(W) \) is its Klein-corresponding null curve. The Plücker formula \( \text{(2.16)} \), coupled with the fact that \( r_3(f) = r_1(f) \), implies that

\[
4 \deg(f) + 12(k - 1) = 4r_1(f) + 2r_2(f),
\]

where \( k \) is the genus of \( S \). Note that, in consequence, \( r_2(f) \) is always even. Meanwhile, Proposition \( \text{II} \) and \( \text{(2.16)} \) imply

\[
5 \deg(g) + 20(k - 1) = 5r_1(f) + 5r_2(f).
\]

Example 3. The case of most interest in these notes will be when \( g \) is unbranched, i.e., \( r_2(f) = 0 \), and \( S \) has genus \( k = 0 \), in which case, the formulae above reduce to

\[
\deg(g) = \deg(f) + 1 \quad \text{and} \quad r_1(f) = \deg(f) - 3.
\]

These relations will be useful in the sequel.
5. Rational null curves of low degrees

With the above preliminaries out of the way, I can now provide an analysis of the possibilities when $f : \mathbb{P}^1 \to \mathbb{P}^3$ is a rational contact curve of low degree such that $f_2 : \mathbb{P}^1 \to \mathbb{Q}^3$ is unbranched.

A contact curve $f : \mathbb{P}^1 \to \mathbb{P}(V)$ of degree 1 is linear, and a null curve $f : \mathbb{P}^1 \to \mathbb{Q}^3$ is linear. These linear cases will be set aside from now on.

Example 4 (Even degrees). Explicit unbranched null curves $g : \mathbb{P}^1 \to \mathbb{Q}^3$ are provided by Example 2. The curve $g = f_2$ has even degree $2p + 2 \geq 4$ and is unbranched whenever $q = p + 1$.

However describing all the unbranched null curves in $\mathbb{Q}^3$ of any given degree seems to be a harder problem.

5.1. Degree at most 4. The very lowest possible degrees are easy to treat.

Proposition 5. If $f : \mathbb{P}^1 \to \mathbb{P}(V)$ is a nonlinear contact curve of degree at most 3, then $f(\mathbb{P}^1) \subset \mathbb{P}(V)$ is a rational normal curve. All contact rational normal curves are symplectically equivalent.

Proof. By (4.3), if $f : \mathbb{P}^1 \to \mathbb{P}^3$ is a nondegenerate contact curve, then

$$\deg(f) = 3 + r_1(f) + \frac{1}{2}r_2(f).$$

Consequently, $\deg(f) \geq 3$, with equality only when $r_1(f) = r_2(f) = 0$. Since $r_3(f) = r_1(f) = 0$, it follows that $f : \mathbb{P}^1 \to \mathbb{P}^3$ is completely unramified and hence is a rational normal curve of degree 3 (see Example 1).

Conversely, if $\deg(f) = 3$, then, choosing a meromorphic function $z$ on $\mathbb{P}^1$ with exactly one simple pole, write

$$f = [v^0 + zv^1 + z^2 v^2 + z^3 v^3]$$

for some basis $(v^0, v^1, v^2, v^3)$ of $V \simeq \mathbb{C}^4$ with dual basis $(\xi_0, \xi_1, \xi_2, \xi_3)$ of $V^*$. Then

$$g = f_2 = [v^0 \wedge v^1 + 2z v^0 \wedge v^2 + z^2 (3v^0 \wedge v^3 + v^1 \wedge v^2) + 2z^3 v^1 \wedge v^3 + z^4 v^2 \wedge v^3].$$

Thus, $g(\mathbb{P}^1)$ lies linearly fully in the projectivization of the kernel $W \subset \Lambda^2(V)$ of the nondegenerate 2-form $\beta = \xi_0 \wedge \xi_3 - 3 \xi_1 \wedge \xi_2 \in \Lambda^2(V^*)$.

Thus, all contact rational normal curves are symplectically equivalent. 

Corollary 1. If $g : \mathbb{P}^1 \to \mathbb{Q}^3$ is a nonlinear null curve of degree at most 4, then $\deg(g) = 4$ and $g = f_2$ where $f : \mathbb{P}^1 \to \mathbb{P}^3$ is a contact rational normal curve. In particular, there are no nonlinear null curves in $\mathbb{Q}^3$ of degree 2 or 3.

Proof. Write $g = f_2$, where $f : \mathbb{P}^1 \to \mathbb{P}^3$ is a contact curve. Then $\deg(g) = 4 + r_1(f) + r_2(f)$, so $\deg(g) \leq 4$ implies that $\deg(g) = 4$ and $r_1(f) = r_2(f) = 0$. Thus, $f : \mathbb{P}^1 \to \mathbb{P}^3$ is a rational normal curve. 

5.2. Degree 5. To begin, I classify the nonlinear rational contact curves of degree 4.

Proposition 6. Up to symplectic equivalence, there is only one nonlinear contact curve $f : \mathbb{P}^1 \to \mathbb{P}^3$ of degree 4. It satisfies $R_1(f) = 0$ and $R_2(f) = p + q$ where $p, q \in \mathbb{P}^1$ are distinct.
Proof. Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ be a nonlinear contact curve of degree 4. Then by (11), $(r_1(f), r_2(f))$ is either $(1, 0)$ or $(0, 2)$, and $f$ can be written in the form

$$f = [v^0 + zv^1 + z^2v^2 + z^3v^3 + z^4v^4]$$

for five vectors $v^0, \ldots, v^4$ in $\mathbb{C}^4$ that satisfy one linear relation and where $z$ is a meromorphic function on $\mathbb{P}^1$ that has a single pole. Since $f$ has degree 4, neither $v^0$ nor $v^4$ can be zero.

If $r_1(f) = 1$ and $r_2(f) = 0$, then $f$ branches to order 1 at a single point of $\mathbb{P}^1$, which can be taken to be the pole of $z$. This implies that $v^3$ must be a multiple of $v^4$. Hence, by replacing $z$ by $z + c$ for an appropriate constant $c$, it can be assumed that $v^3 = 0$. Thus, the vectors $v^0, v^1, v^2, v^4$ are a basis of $\mathbb{C}^4$. However, when

$$f = [v^0 + zv^1 + z^2v^2 + z^4v^4],$$

one finds that

$$f_2 = [v^0 \wedge v^1 + 2z v^0 \wedge v^2 + z^2 v^1 \wedge v^2 + 4z^3 v^0 \wedge v^4 + 3z^4 v^1 \wedge v^4 + 2z^5 v^2 \wedge v^4],$$

so that $f_2(\mathbb{P}^1)$ is nondegenerate in $\mathbb{P}(\Lambda^2(\mathbb{C}^4)) \simeq \mathbb{P}^5$. Thus, such an $f$ cannot be a contact curve with respect to any symplectic structure on $\mathbb{C}^4$.

If $r_2(f) = 2$, there are two possibilities, either $R_2(f) = 2p$ for some $p \in \mathbb{P}^1$ or else $R_2(f) = p + q$ where $p, q \in \mathbb{P}^1$ are distinct. Moreover, $r_1(f) = 0$, which implies, in particular, that $v^0 \wedge v^1$ and $v^3 \wedge v^4$ are nonzero.

In the first subcase, assume, without loss of generality, that $p$ is the zero of $z$.

Since

$$f_2 = [v^0 \wedge v^1 + 2z v^0 \wedge v^2 + z^2 (3v^0 \wedge v^3 + v^1 \wedge v^2) + \cdots],$$

where the unwritten terms vanish to order 3 or more at $z = 0$, the assumption that $R_1(f_2) = R_2(f) = 2p$ implies that $v^0 \wedge v^2$ and $3 v^0 \wedge v^3 + v^1 \wedge v^2$ are multiples of $v^0 \wedge v^1$. This implies that both $v^2$ and $v^3$ lie in the linear span of $v^0$ and $v^1$, which is impossible, since $v^0, v^1, v^4$ cannot span $\mathbb{C}^4$.

Meanwhile, if $R_2(f) = p + q$, where $p, q \in \mathbb{P}^1$ are distinct, then we can choose $z$ so that $p$ and $q$ are defined by $z = 0$ and $z = \infty$. Since

$$f_2 = [v^0 \wedge v^1 + 2z v^0 \wedge v^2 + \cdots + 2z^5 v^2 \wedge v^4 + z^6 v^3 \wedge v^4],$$

where the unwritten terms vanish to order at least 2 at $z = 0$ and have a pole of at most order 4 at $z = \infty$, it follows from $R_1(f_2) = R_2(f) = p + q$ that $v^0 \wedge v^2$ is a multiple of $v^0 \wedge v^1$ and $v^2 \wedge v^4$ is a multiple of $v^0 \wedge v^4$. In particular, $v^2$ must be both a linear combination of $v^0$ and $v^1$ and a linear combination of $v^3$ and $v^4$. Now, this can only happen if $v^2 = 0$, since $v^0, v^1, v^3, v^4$ must be a basis of $\mathbb{C}^4$. Thus,

$$f = [v^0 + z v^1 + z^3 v^3 + z^4 v^4],$$

which implies

$$f_2 = [v^0 \wedge v^1 + 3z^2 v^0 \wedge v^3 + z^3 (2v^1 \wedge v^3 + 4v^0 \wedge v^4) + 3z^4 v^1 \wedge v^4 + 3z^6 v^3 \wedge v^4].$$

Thus, $f_2$ is linearly full in $\mathbb{P}(W) \simeq \mathbb{P}^4$, where $W \subset \Lambda^2(\mathbb{C}^4)$ is the 5-dimensional subspace annihilated by the symplectic form

$$\beta = \xi_0 \wedge \xi_4 - 2\xi_1 \wedge \xi_3$$

(where $(\xi_0, \xi_1, \xi_3, \xi_4)$ is the basis of $V^* \simeq \mathbb{C}^4$ that is dual to the basis $(v^0, v^1, v^3, v^4)$ of $\mathbb{C}^4$).
Thus, \( f \) is a contact curve with respect to the contact structure on \( \mathbb{P}^3 \) defined by \( \beta \). The uniqueness of \( f \) up to symplectic equivalence is now clear. \( \square \)

**Corollary 2.** There is no nonlinear null curve \( g : \mathbb{P}^1 \to \mathbb{Q}^3 \) of degree 5.

**Proof.** If such a curve \( g \) existed, it would be of the form \( g = f_2 \) where \( f : \mathbb{P}^1 \to \mathbb{P}^3 \) would be a nonlinear contact curve with ramification degrees \( r_1(f) \) and \( r_2(f) \). Now

\[
5 = \deg(g) = 4 + r_1(f) + r_2(f),
\]

which, since \( r_2(f) \) must be even, implies that \( r_1(f) = 1 \) and \( r_2(f) = 0 \). Hence \( \deg(f) = 3 + r_1(f) + \frac{1}{2}r_2(f) = 4 \). However, Proposition 6 shows that the only nonlinear contact curve \( f : \mathbb{P}^1 \to \mathbb{P}^3 \) of degree 4 has \( r_1(f) = 0 \) and \( r_2(f) = 2 \).

Thus, such a \( g \) does not exist. \( \square \)

5.3. **Degree 6.** Now, I will classify the nonlinear rational null curves of degree 6.

**Proposition 7.** Up to projective equivalence, there are only two nonlinear null curves \( g : \mathbb{P}^1 \to \mathbb{Q}^3 \) of degree 6. One of these is unbranched, and the other has two distinct branch points, each of order 1.

**Proof.** Let \( g : \mathbb{P}^1 \to \mathbb{Q}^3 \) be a nonlinear null curve of degree 6 and let \( f : \mathbb{P}^1 \to \mathbb{P}^3 \) be the Klein-corresponding nonlinear contact curve. From the formulae above,

\[
6 = \deg g = 4 + r_1(f) + r_2(f).
\]

Since \( r_2(f) \) is even, there are two possibilities: \( (r_1(f), r_2(f)) \) is either \((0, 2)\) or \((2, 0)\).

First, if \((r_1(f), r_2(f)) = (0, 2)\), then \( \deg f = 3 + 0 + 1 = 4 \), and, by Proposition 6 this \( f \) is unique up to symplectic equivalence. In this case, \( g = f_2 \), since \( R_1(g) = R_2(f) = p + q \) where \( p, q \in \mathbb{P}^1 \) are distinct, \( g \) has two branch points of order 1.

Second, if \((r_1(f), r_2(f)) = (2, 0)\) then \( \deg f = 3 + 2 + 0 = 5 \), and \( R_1(f) = p + q \) where \( p, q \in \mathbb{P}^1 \) may be equal.

In the special case when \( R_1(f) = 2p \), choose a meromorphic function \( z \) on \( \mathbb{P}^1 \) that has a single pole at \( p \), and write

\[
f = \left[ v^0 + z v^1 + z^2 v^2 + z^3 v^3 + z^4 v^4 + z^5 v^5 \right].
\]

where \( v^0, \ldots, v^5 \) span \( \mathbb{C}^4 \) and \( v^0 \) and \( v^5 \) are not zero. The condition \( R_1(f) = 2p \) implies that \( v^3 \) and \( v^4 \) are multiples of \( v^5 \), and so, by replacing \( z \) by \( z + c \) for an appropriate constant, it can be assumes that \( v^4 = 0 \), so that

\[
f = \left[ v^0 + z v^1 + z^2 v^2 + (a z^3 + z^5) v^5 \right].
\]

where \( a \) is a constant. Thus,

\[
g = f_2 = \left[ v^0 \land v^1 + 2z v^0 \land v^2 + z^2 (3a v^0 \land v^5 + v^1 \land v^2) + 2az^3 v^1 \land v^5 + 3z^4 v^1 \land v^5 + 3z^6 v^2 \land v^5 \right]
\]

By inspection, whatever the value of \( a \), the seven coefficients of \( z^k \) in this expression span the entire 6-dimensional space \( \Lambda^2(\mathbb{C}^4) \). Thus, \( f \) is not a contact curve for any symplectic structure on \( \mathbb{C}^4 \).

Supposing, instead, that \( R_1(f) = p + q \), where \( p, q \in \mathbb{P}^1 \) are distinct, let \( z \) be a meromorphic function on \( \mathbb{P}^1 \) with a simple pole at \( p \) and a zero at \( q \). Then \( f \) takes the form

\[
f = \left[ (1 + az) v^0 + z^2 v^2 + z^3 v^3 + (bz^4 + z^5) v^5 \right].
\]
for some constants $a$ and $b$, where $(v^0, v^2, v^3, v^5)$ is a basis of $\mathbb{C}^4$. Thus,
\[
g = f_2 = \left[ 2v^0 \wedge v^2 + z (a v^0 \wedge v^2 + 3v^0 \wedge v^3) + z^2 (2a v^0 \wedge v^3 + 4b v^0 \wedge v^5) + z^3 (5 + 3ab)v^0 \wedge v^5 + v^2 \wedge v^3) + z^4 (4a v^0 \wedge v^5 + 2b v^2 \wedge v^5) + z^5 (3v^2 \wedge v^5 + b v^3 \wedge v^5) + 2z^6 v^3 \wedge v^5 \right]
\]
By inspection, whenever either $a$ or $b$ is nonzero, $g = f_2$ is linearly full in $\Lambda^2(\mathbb{C}^4)$, and, hence, $f$ is not contact for any symplectic structure on $\mathbb{C}^4$.

Meanwhile, if $a = b = 0$, then the formula for $g$ simplifies to
\[
g = \left[ 2v^0 \wedge v^2 + 3z v^0 \wedge v^3 + z^3 (5 v^0 \wedge v^5 + v^2 \wedge v^3) + 3z^5 v^2 \wedge v^5 + 2z^6 v^3 \wedge v^5 \right]
\]
so that $g(\mathbb{P}^1)$ is linearly full in the 5-dimensional subspace $W \subset \Lambda^2(\mathbb{C}^4)$ that is annihilated by the symplectic form
\[
\beta = \xi_0 \wedge \xi_5 - 5 \xi_2 \wedge \xi_3
\]
where $(\xi_0, \xi_2, \xi_3, \xi_5)$ is the basis of $(\mathbb{C}^4)^*$ dual to the basis $(v^0, v^2, v^3, v^5)$ of $\mathbb{C}^4$. Hence, $f$ is contact and $g : \mathbb{P}^1 \to \mathbb{Q}^3 \subset \mathbb{P}(W)$ is an unbranched (since $r_2(f) = 0$) null curve of degree 6.

This argument establishes the uniqueness up to projective equivalence of such an $f : \mathbb{P}^1 \to \mathbb{P}^3$ of degree 5 with $R_1(f) = p + q$ and $R_2(f) = 0$ and hence the uniqueness up to equivalence of an unbranched null curve $g : \mathbb{P}^1 \to \mathbb{Q}^3$ of degree 6. \(\square\)

**Remark 4 (Reducibility of a moduli space).** Note that the two corresponding rational contact curves are
\[
f(\mathbb{P}^1) = [1, z, z^3, z^4],
\]
which has $g = f_2$ branched at $z = 0$ and $z = \infty$, and
\[
f(\mathbb{P}^1) = [1, z^2, z^3, z^5]
\]
which has $g = f_2$ unbranched, though $f$ itself is branched at $z = 0$ and $z = \infty$.

In each case, the projective subgroup $H \subset \text{SL}(4, \mathbb{C})$ that stabilizes $f$ has dimension 1 (and has two components). Consequently, the moduli space of such contact curves for a given symplectic structure $\beta$ is of the form $\text{Sp}(\beta)/H$ and hence has dimension 9.

Thus, the moduli space of nonlinear rational null curves in $\mathbb{Q}^3$ of degree 6 is disconnected. Even when compactified using geometric invariant theory, this moduli space will necessarily be reducible, being the union of two irreducible varieties of dimension 9.

### 5.4. Degree 7

Finally, we treat the unbranched case in degree 7.

**Proposition 8.** There is no unbranched nonlinear null curve $g : \mathbb{P}^1 \to \mathbb{Q}^3$ of degree 7.

**Proof.** Suppose that an unbranched nonlinear null curve $g : \mathbb{P}^1 \to \mathbb{Q}^3$ of degree 7 exists and let $f : \mathbb{P}^1 \to \mathbb{P}^3$ be the Klein-corresponding contact curve. By the formulae (4.5) of Example 3 it follows that $f$ has degree 6 and satisfies $r_1(f) = 3$ and $r_2(f) = 0$. There are three cases to consider, depending on the structure of $R_1(f)$.

First, suppose that $R_1(f) = 3p$ for some $p \in \mathbb{P}^1$. Choose a meromorphic $z$ on $\mathbb{P}^1$ with a simple pole at $p$ (and no other poles). Then $f$ takes the form
\[
f(\mathbb{P}^1) = [v^0 + z v^1 + z^2 v^2 + z^3 v^3 + z^4 v^4 + z^5 v^5 + z^6 v^6]
\]
for some \( v^0, \ldots, v^6 \in \mathbb{C}^4 \) with \( v_0 \) and \( v_6 \) nonzero. Since \( R_1(f) = 3p \), it follows that \( v^3 \), \( v^4 \), and \( v^5 \) are multiples of \( v^6 \). By replacing \( z \) by \( z + c \) for some constant \( c \), I can arrange that \( v^5 = 0 \), so I do that. Then \( f \) takes the form

\[
(5.5) \quad f = [v^0 + z v^1 + z^2 v^2 + (a z^3 + b z^4 + z^6) v^6]
\]

for some constants \( a \) and \( b \), where \((v^0, v^1, v^2, v^6)\) are a basis for \( \mathbb{C}^4 \). Then

\[
(5.6) \quad g = f_2 = [v^0 \wedge v^1 + 2z v^0 \wedge v^2 + z^2 (3a v^0 \wedge v^6 + v^1 \wedge v^2) + z^3 (4b v^0 + 2a v^1) \wedge v^6 + z^4 (3b v^1 - a v^2) \wedge v^6 + z^5 (2b v^2 - 6 v^0) \wedge v^6 + 5z^6 v^1 \wedge v^6 + 4z^7 v^2 \wedge v^6]
\]

By inspection the coefficients of the different powers of \( z \) span \( \Lambda^2(\mathbb{C}^4) \), no matter what the values of \( a \) and \( b \). Hence, this curve is linearly full in \( \mathbb{P}(\Lambda^2(\mathbb{C}^4)) \), and this \( f \) is not contact for any symplectic structure on \( \mathbb{C}^4 \). Second, suppose that \( R_1(f) = \frac{2p + q}{3} \) for some \( p, q \in \mathbb{P}^1 \) that are distinct. Choose a meromorphic \( z \) on \( \mathbb{P}^1 \) with a simple pole at \( p \) and a simple zero at \( q \). Then, because \( R_1(f) = \frac{2p + q}{3} \), it follows that \( f \) can be written in the form

\[
(5.7) \quad f = [(1 + az) v^0 + z^2 v^2 + z^3 v^3 + (b z^4 + c z^5 + z^6) v^6]
\]

for some constants \( a \), \( b \), and \( c \) and vectors \( v^0, v^2, v^3, v^6 \) that form a basis of \( \mathbb{C}^4 \). Then

\[
(5.8) \quad g = f_2 = [2v^0 \wedge v^2 + z v^0 \wedge (a v^2 + 3 v^3) + z^2 v^0 \wedge (4b v^6 + 2a v^3) + z^3 ((3ab+5c) v^0 \wedge v^6 + v^2 \wedge v^3) + z^4 ((4ac+6) v^0 + 2b v^2) \wedge v^6 + z^5 (5a v^0 + 3c v^2 + b v^3) \wedge v^6 + z^6 (2c v^3 + 4 v^2) \wedge v^6 + 3z^7 v^3 \wedge v^6].
\]

Looking at the coefficients of the 0-th, 1-st, 7-th, and 6-th powers of \( z \) in this formula, it follows that \( f_2 \) lies linearly fully in a space \( W \subset \Lambda^2(\mathbb{C}^4) \) that contains \( v^0 \wedge v^2 \), \( v^0 \wedge v^3 \), \( v^3 \wedge v^6 \), and \( v^2 \wedge v^6 \), no matter what the values of \( a \), \( b \), and \( c \). The space \( W \) must also contain the elements in the set

\[
\{a v^0 \wedge v^6, b v^2 \wedge v^6, (4ac+6) v^0 \wedge v^6, v^2 \wedge v^3 + (3ab+5c) v^0 \wedge v^6\}.
\]

No matter what the values of \( a \), \( b \), and \( c \) are, the first three elements will span the multiples of \( v^0 \wedge v^6 \), and this, combined with the fourth element, will force \( W \) to contain \( v^2 \wedge v^3 \) as well. Thus \( W = \Lambda^2(\mathbb{C}^4) \), implying that \( f_2 \) is nondegenerate, which is impossible if \( f \) is to be a contact curve. Thus, this case is also impossible.

Third, and finally, suppose that \( R_1(f) = \frac{p + q + s}{3} \) where \( p, q, s \in \mathbb{P}^1 \) are distinct. Let \( z \) be the meromorphic function on \( \mathbb{P}^1 \) that has a pole at \( p \), a zero at \( q \) and satisfies \( z(s) = 1 \). (This uniquely specifies \( z \).) Then \( f = [F(z)] \) where

\[
(5.9) \quad F(z) = v^0 + z v^1 + z^2 v^2 + z^3 v^3 + z^4 v^4 + z^5 v^5 + z^6 v^6
\]

and where \( v^0, \ldots, v^6 \) span \( \mathbb{C}^4 \) and \( v^0 \) and \( v^6 \) are nonzero. Moreover, because \( p \) and \( q \) are branch points of \( f \), it follows that \( v^0 \wedge v^1 = v^5 \wedge v^6 = 0 \), so that we must actually have

\[
(5.10) \quad F(z) = (1 + az) v^0 + z^2 v^2 + z^3 v^3 + z^4 v^4 + (b z^5 + z^6) v^6,
\]

for some constants \( a \) and \( b \). Moreover, there can be only one linear relation among the 5 vectors \( v^0, v^2, v^3, v^4, v^6 \). Also, because \( f \) must have degree 6, we cannot have \( F(z_0) = 0 \) for any \( z_0 \in \mathbb{C} \), since then \( F(z)/(z - z_0) \) would be a curve of degree 5, forcing \( f \) to have degree at most 5.
Now, it turns out that it greatly simplifies the argument below to make a change of basis so that $F$ is written in the form
\[
F(z) = (1 + az) v^0 + z^2 \left( (v^2 - v^3 - (2a + 3)v^0) 
+ z^3 (2v^3 + (a + 2)v^0 + (b + 2)v^6) 
+ z^4 (v^4 - v^3 - (2b + 3)v^6) \right) 
\]
(5.11)
\[
F'(z) = F(z) \wedge F'(z) = 0 
\]

as can clearly be done. The reason this is useful is that the condition that $f$ have a branch point at $s$, which is where $z = 1$, is equivalent to the condition that $F(1) \wedge F'(1) = 0$, and computation now shows that
\[
0 = F(1) \wedge F'(1) = 2 v^2 \wedge v^4 . 
\]
Hence $v^2$ and $v^4$ must be linearly dependent. Since there can only be one linear relation among the 5 vectors \{ $v^0, v^2, v^3, v^4, v^6$, \}, it follows that $v^2$ and $v^4$ must be multiples, not both zero, of a single vector. Consequently, after a renaming and a choice of two numbers $p$ and $q$, not both zero, we can write $F$ in the form
\[
F(z) = (1 + az) v^0 + z^2 \left( p v^2 - v^3 - (2a + 3)v^0 
+ z^3 (2v^3 + (a + 2)v^0 + (b + 2)v^6) 
+ z^4 (q v^2 - v^3 - (2b + 3)v^6) \right) 
\]
(5.12)
\[
F'(z) = F(z) \wedge F'(z) = 0 
\]

where now $v^0, v^2, v^3, v^6$ are a basis of $\mathbb{C}^4$, and $a$, $b$, $p$, and $q$ are constants, with $p$ and $q$ not both zero.

Now, let
\[
G(z) = \frac{F(z) \wedge F'(z)}{z(z - 1)} = G_0 + G_1 z + G_2 z^2 + \cdots + G_7 z^7 , 
\]
so that $g = f_2 = [G(z)]$. We must determine the conditions on $a$, $b$, $p$, and $q$ in order that $g(\mathbb{P}^3)$ not lie linearly fully in $\mathbb{P}(\Lambda^2(\mathbb{C}^4))$, which is that the eight vectors $G_0, \ldots, G_7$ in $\Lambda^2(\mathbb{C}^4)$ should only span a vector space of dimension at most 5.

Let $B^1 = v^0 \wedge v^2$, $B^2 = v^0 \wedge v^3$, $B^3 = v^0 \wedge v^6$, $B^4 = v^2 \wedge v^3$, $B^5 = v^2 \wedge v^6$, and $B^6 = v^3 \wedge v^6$. Then $B^1, \ldots, B^6$ form a basis of $\Lambda^2(\mathbb{C}^4)$. Thus, there is a 8-by-6 matrix $M_{ab}$ such that $G_a = \sum_{j} M_{aj} B^j$. Calculation now yields that $M$ is
\[
\begin{pmatrix}
-2p & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p(a + 2) & a - 4 & -3(b + 2) & 0 & 0 & 0 & 0 & 0 \\
-(a p + 2 p + 4 q) & -3 a & -2 a b - 4 a + 5 b + 6 & 0 & 0 & 0 & 0 & 0 \\
-q(3 a + 4) & 3 a + 4 & 6 a b + 9 a + 3 b + 12 & -2 p & -p(b + 2) & b + 2 & 0 & 0 \\
q(a + 2) & -(a + 2) & -6 a b - 3 a - 9 b - 12 & -2 q & p(3 b + 4) & -3 b + 4 & 0 & 0 \\
0 & 0 & 2 a b - 5 a + 4 b - 6 & 0 & b q + 4 p + 2 q & 3 b & 3 b & 0 \\
0 & 0 & 3(a + 2) & 0 & q(b + 2) & 4 - b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 q & -2 & 0 
\end{pmatrix}
\]

Now, in order that $g$ not be branched at $z = 0$, we must have $G_0$ and $G_1$ linearly independent, i.e., the first two rows of $M$ must be of rank 2 and inspection shows that this requires that at least one of $p$ and $b + 2$ must be nonzero. Similarly, because $g$ is not branched at $z = \infty$, at least one of $q$ and $a + 2$ must be nonzero.

In order for the rank of $M$ to be at most 5, all of the 6-by-6 minors of $M$ must be zero. By computation, the determinant of the first 6 rows is
\[ -48((3p + q)b + 4p + 2q)^3 \]
while the determinant of the last 6 rows is 

\[-48((p+3q)a + 2p + 4q)^3.\]

Thus, we must have

\[(3p+q)b + 4p + 2q = (p+3q)a + 2p + 4q = 0.\]

Recall that \(p\) and \(q\) cannot simultaneously vanish. It is now apparent that \(3p+q\) cannot be zero either, since the above equations would then imply that \(4p+2q = 0\), forcing \(p = q = 0\), which cannot happen. Similarly, \(p+3q\) cannot be zero. Thus, we can solve for \(a\) and \(b\) in the form

\[a = \frac{-2p+4q}{p+3q} \quad \text{and} \quad b = \frac{-4p+2q}{3p+q}.\]

From these formulae, we can see that, if \(q\) were zero, then \(a\) would be \(-2\), but \(q = a+2 = 0\) is not allowed. Hence \(q\) is non-zero. Similarly \(p\) must be nonzero.

Finally, computing the determinant of the 6-by-6 minor of \(M\) obtained by deleting the third and sixth rows of \(M\) yields

\[
\frac{8640 \ pq \ (p+q)^3}{(p+3q)(3p+q)}.
\]

Consequently, since \(p\) and \(q\) cannot be zero, it must be that \(p+q = 0\), which implies that \(a = -1\) and \(b = -1\). Further, by scaling \(v^2\), we can arrange that \(p = 1\) and \(q = -1\). Thus, the only possibility for \(f = |F|\) is to have

\[
F(z) = (1-z) v^0 + z^2 (v^2 - v^3 - v^6)
+ z^3 (2v^3 + v^0 + v^5)
- z^4 (v^2 + v^3 + v^6) + (z^6 - z^5) v^6,
\]

(5.13)

However, since \(F(1) = 0\), it follows that \(f = |F(z)/(z-1)|\) can only have degree 5 at most. This contradiction shows that the desired \(f\) does not exist.

Hence, there is no unbranched null curve \(g : \mathbb{P}^1 \to \mathbb{Q}^3\) of degree 7, as claimed. \(\square\)

**Remark 5.** There does exist a branched nonlinear null curve \(g : \mathbb{P}^1 \to \mathbb{Q}^3\) of degree 7.

The corresponding contact curve

\[f = [(1-5z^2) v^0 + (z-3z^2) v^1 + (z^4 - 3z^3) v^4 + (z^5 - 5z^3) v^5]\]

for a meromorphic parameter \(z\) on \(\mathbb{P}^1\) and a basis \(v^0, v^1, v^4, v^5\) of \(\mathbb{C}^4\). This \(f\) satisfies \(R_1(f) = s\), where \(z(s) = 1\), and \(R_2(f) = p+q\), where \(p\) is the pole of \(z\) and \(q\) is the zero of \(z\).

It can be shown \([2]\) that, up to projective equivalence, this is the unique contact curve \(f : \mathbb{P}^1 \to \mathbb{P}^3\) with \(r_1(f) = 1\) and \(r_2(f) = 2\).

Since any nonlinear null curve \(g : \mathbb{P}^1 \to \mathbb{Q}^3\) of degree 7 must satisfy \(r_1(f) + r_2(f) = 7 - 4 = 3\) and since \(r_2(f)\) must be even, it follows that such a curve, which must be branched by Proposition\(\[\square\]

Hence all of the nonlinear rational null curves \(g : \mathbb{P}^1 \to \mathbb{Q}^3\) of degree 7 form a single \(\text{Sp}(2, \mathbb{C})\)-orbit of dimension 10.

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