Glitches, torque evolution and the dynamics of young pulsars

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ABSTRACT

The Crab pulsar has suffered in 1975 and 1989 two glitches in which the frequency did not relaxed to the extrapolated pre-glitch value but rather spun up showing long-term changes in the frequency derivative \( \dot{\Omega} \). This particular behaviour has been interpreted as evidence for an evolution of the torque acting upon the star. A variable torque may be related to non-canonical braking indexes, for which some determinations have been possible. We briefly analyse in this work the consistency of postulating a growth in the angle between the magnetic moment and the rotation axis as the cause of such events. We show that this hypothesis leads to the determination of the initial period, initial and present angles, according to the assumed angle growth, for young pulsars whose respective braking indices \( n_{obs} \) and jerk parameters \( m_{obs} \) are known, and some insights on the equation of
state.

**KEY WORDS**: Pulsars: general - Pulsars: individual (PSR B0531+21, PSR B0540-69, PSR B0833-45, PSR B1509-58)
1 INTRODUCTION

The dynamics of compact stars offers a unique opportunity to explore and understand the physics of dense matter. Particularly, timing irregularities have proved to be extremely important (and challenging) for building a coherent picture of those stellar interiors involving hadronic matter at subnuclear and supranuclear densities.

Among the most notorious timing irregularities, sudden discontinuities (glitches) are observed in the pulsar frequency $\Omega$ and spin-down rate $\dot{\Omega}$. Several of such events have been observed from the Crab, Vela and some other pulsars. It is generally agreed that they are the result of a (complex) interplay between the superfluid component(s) and the rest of the star. Theoretical models searched an explanation in terms of starquakes (Baym et al. 1969) or, more recently, vortex motion in the superfluid component (Alpar et al. 1984, Pines & Alpar 1985, Link, Epstein & Baym 1993). While the latter model is flexible enough to accommodate a large body of observations, evidence coming from very timely and detailed observations of two glitches in the Crab pulsar seems to indicate that new physical inputs may be required to explain the data (see below). Since the Crab can be considered as the best studied object, it may be presumed that an analogous behaviour could be present in the young pulsar sample for which an increasing body of data is available. We shall attempt to understand the dynamics of the youngest objects and build a consistent picture of their evolution.

In 1975 and 1989 the spin rate of the Crab pulsar $\Omega$ suddenly increased by amounts $\Delta \Omega/\Omega \sim 10^{-8}$ and after that continued to spin-down at a faster rate (that is, the pulsar continued to spin slower than before) of $\Delta \dot{\Omega}/\dot{\Omega} \sim 10^{-4}$ (Gullahorn et al. 1977, Lohsen 1989).
1981, Lyne, Smith & Pritchard 1992). The same feature is also present in the 1969, 1981 and 1986 events (Lyne & Pritchard 1987, Lyne, Pritchard & Smith 1993). This behaviour has been attributed to a decoupling of some internal shell or a change in the external torque acting on the star (Gullahorn et al. 1977, Demiański & Prószyński 1983). By using a general postglitch relaxation equation of the form
\[
\frac{\partial \omega(r,t)}{\partial t} = f(\omega, r)
\]
for the differential rotation \( \omega \) as a function of the specific external torque \( f(\omega, r) \), Link, Epstein and Baym (1992) have argued that a frequency deficit \textit{can not} be explained by the existing glitch models. In other words, according to their work the postglitch frequency \( \Omega_c(t) \) must be always greater than the extrapolated preglitch frequency \( \Omega_{co}(t) \), contrary to the observations. On the other hand, Alpar & Pines (1993) have discussed the theoretical interpretation of these events in the framework of vortex creep theory. Even though there seems to be enough room for such a behaviour in the latter, several details of the pinning layers are still unclear and complicate the interpretation. Thus, it may be interesting to explore alternatives to bring the theoretical picture closer to the observed phenomenology.

An important aspect of these observations is that, given the much larger amount of data taken from the Crab pulsar and the difficulties of extracting this signature from timing noise, it is entirely possible that other young pulsars also behave similarly. If so, events producing a permanent \( \Delta \dot{\Omega}/\dot{\Omega} \) may be important for the understanding of pulsar dynamics. Models having variable external torques have been considered in the past (see Blandford & Romani 1988 and references therein) and may be helpful for understanding
the data. Our goal in this work is to see to what extent a simple (but consistent) picture of young pulsar dynamics can be built by assuming a specific version of a variable external torque model. Section 2 is dedicated to formulate a minimal dynamical model in which the angle between the magnetic dipole and the rotation axis of the pulsar is allowed to vary according to simple laws. Section 3 presents an application of the model to the Crab, Vela and two other interesting pulsars (PSR B1509-58 and PSR B0540-69). Finally, we present a discussion and conclusions in Section 4. There is also an Appendix with exact solutions for the proposed growth laws.

2 BASIC EQUATIONS

The equation of motion of a rotating pulsar, assumed to have a crust+core normal component (suffix \( c \)) and a superfluid shell (suffix \( s \)) is

\[
I_c \dot{\Omega}_c + I_s \dot{\Omega}_s = \tau_{ext}
\]  

(1)

since the frequency difference \( \Omega_c - \Omega_s = \omega \) is constant on sufficiently long timescales, the rotational equilibrium implies further that

\[
I_{tot} \dot{\Omega}_c = \tau_{ext}.
\]  

(2)

Up to now studies have been based mainly on the use of torque laws of the form \( \tau_{ext} = -K\Omega_c^n \), where \( K \) is a constant (up to its possible decay for very old objects) and the braking index \( n \) is assumed to be 3 as predicted by the magnetic dipole model (see e.g. Manchester & Taylor 1977). The vacuum dipole model expression corresponds
to $K = \frac{2}{3} |M|^2 \sin^2 \alpha$, where $\alpha$ is the angle between $M$ and $\Omega_c$, $|M| = B_o R^3$ and $B_o$, $R$ are the magnetic field and the radius of the star respectively.

From eq. (2) and the form of $\tau_{ext}$ it is clear that the peculiar events of 1975 and 1989 require either a reduction of $I_{tot}$, an increase of the magnitude of the magnetic moment $M$ or an increase of the angle $\alpha$ between $M$ and $\Omega_c$. Although all these hypothesis are in principle allowed by the data, we shall address here the change in the relative orientation of $M$ and $\Omega_c$ axis in those events. The idea is appealing not only because of its simplicity but also because it may be considered as a realisation of Ruderman’s plate tectonic theory (Ruderman 1991) where it finds a natural room. The other two hypothesis also have a theoretical support in the framework of vortex creep theory (Alpar & Pines 1993) and magnetic field generation or surfacing (Blandford, Applegate & Hernquist 1883, Muslimov & Page 1996) respectively. They have been previously considered and we shall not address them here.

At this point it must be noted that a counter-aligning pulsar in which $\alpha$ increases with time is quite remarkable, since the opposite situation is to be expected naively for the rotating/radiating star. Therefore, the presence of ”anomalous” glitch events is per se a signature of the complexity of the internal/magnetospheric dynamics which ultimately determine the behaviour of $\alpha(t)$. Several papers have addressed the question of the internal dynamics and its consequences for $\alpha(t)$ (Michel & Goldwire 1970, Davis & Goldstein 1970, Goldreich 1970, Michel 1973, Lyne & Manchester 1988, Michel 1991). Macy (1974) has given a general treatment by solving simultaneously the system of equations which determine $\alpha(t)$ and $\Omega_c(t) (\equiv \Omega_{pulse})$ adopting a simple vacuum dipole model for the pulsar.
radiation. In order to check the consistency of the increasing angle hypothesis we have
chosen instead to parametrise the growth of \( \alpha \) by simple expressions and solve analytical
models to compare them with the observations.

In first place, we want to know if the angle growth happens solely on glitches, in a
discrete mode, or if there is a inter-glitch continuous contribution. To do this, we can
determine the variation of the angle \( \alpha \) due to persistent shifts \( \Delta \dot{\Omega}/\dot{\Omega} \) and \( \Delta \Omega/\Omega \) which is
easily found from eq.(2) to be

\[
\Delta \alpha = \left( \frac{\Delta \dot{\Omega}}{\dot{\Omega}} - 3 \frac{\Delta \Omega}{\Omega} \right) \tan \alpha/2.
\] (3)

Dividing eq.(3) by \( \Delta t \), here defined as a typical time-scale between glitches, we obtain
a (mean) increase rate which we shall denote as \( \langle \Delta \alpha/\Delta t \rangle \). We shall keep in mind that, since
all we have at disposal is a very short observational span of \( \sim 20 \, yr \) at most, there could
be a discrepancy between \( \langle \Delta \alpha/\Delta t \rangle \) and the continuous modeling in which \( \dot{\alpha} \) is a true local
derivative. In fact, the data from Lyne, Pritchard & Smith (1993) allows us to find for the
Crab pulsar

\[
\langle \Delta \alpha/\Delta t \tan \alpha \rangle \simeq 1.8 \times 10^{-5} \, rad \, yr^{-1}
\] (4)

with \( \Delta t = 4.6 \, yr \). On the other hand, the rate needed to account for the observed braking
index is found from eq.(A14) to be

\[
\frac{\dot{\alpha}}{\tan \alpha} = \frac{n_{obs} - 3 \dot{\Omega}}{2 \Omega} = 9.6 \times 10^{-5} \, rad \, yr^{-1}
\] (5)

about 5 times the rate obtained from glitches. For PSR B0540-69, Vela and PSR B1509-58,
the rates given by eq.(5) are respectively $14.4 \times 10^{-5} \text{ rad yr}^{-1}$, $3.49 \times 10^{-5} \text{ rad yr}^{-1}$ and $2.62 \times 10^{-5} \text{ rad yr}^{-1}$. All these values are within one order of magnitude, strengthening the idea that they could have the same origin. Once we have acknowledged that the main contribution to the angle growth comes of the interglitch slowdown, and because even pulsars (like PSR B1509-58 and PSR B0540-69) which did not display any glitches until now show low braking indexes, we have chosen to describe the angle growth as a continuous function rather than a discrete one. The other hypothesis put forward to explain the low braking index could be invoked here as above, but we will only explore the angle growth scenario.

We have tried four simple cases for the angle growth, namely an exponential, linear, power-law and logarithmic functions (see Appendix). Analytical expressions for $\Omega(t)$ have been obtained in all cases which, in principle, may be acceptable as descriptions of the long-term pulsar’s spindown. However, a closer inspection of these $\Omega(t)$ forms reveals that the linear and power-law solutions are not of general applicability because they critically depend on the specific input parameters (like the present angle $\alpha_p$) to exist and be positive definite. Therefore, we have been led to consider the exponential growth $\alpha(t) = \alpha_o e^{t/t_o}$ and the logarithmic growth $\alpha(t) = \ln \left[ \frac{t}{t_p} \left( e^{\alpha_p} - e^{\alpha_o} \right) + e^{\alpha_o} \right]$ which do not suffer from these misbehaviours (see Appendix).

We now turn to the application of a varying-angle model described by the formulae of the Appendix to specific cases in the next Section.
3 APPLICATION OF THE MODEL

3.1 General Considerations

In order to check the consistency of the varying-angle model we shall apply it to the four pulsars in which $\ddot{\Omega}$ (and hence $n_{\text{obs}}$) has been measured. Starting from the Crab pulsar, in which a rather direct measure of $m_{\text{obs}}$ has been also obtained, we have attempted to unify as far as possible the behaviour of the young pulsar sample. The strategy is as follows: assuming that the braking index and jerk parameter discrepancies are solely due to the growth of $\alpha$, we impose the observed values of $n_{\text{obs}}$ and $m_{\text{obs}}$ (see eqs. (A17) and (A19) in the Appendix) to determine a solution for $\alpha$. Once these values are obtained, eq. (A1), (A2) and (A3) can be used to calculate $P_o$, $\alpha_o$ and the time-scale for maximum torque (that is, the age at which $\alpha$ reaches $\pi/2$) termed $t_m$.

3.2 The Crab Pulsar (PSR B0531+21)

The Crab pulsar has been extensively studied and the basic quantities like $n_{\text{obs}}$ and $m_{\text{obs}}$ determined through measurements of $\Omega$ derivatives. Using the observational data at MJD 40000.0 available from Lyne, Pritchard & Smith (1993), we found for the exponential form (the logarithmic form does not provide a consistent solution for this pulsar)

$$\alpha_p \approx 68^o$$

$$\alpha_o \approx 56^o$$

$$P_o \approx 19\, ms$$ (6)
\[ t_m \simeq 2300 \text{ yr}. \]

We note that the \( t_m \) obtained is larger than the true age of the Crab \( t_p \sim 940 \text{ yr} \), as it should be. It is also worth mentioning that \( \alpha_p \) is more consistent with the new value inferred from optical polarization data \( (\alpha_p \sim 60^\circ, \text{F.G.Smith et al. 1988}) \) than the older one \( (\alpha_p \geq 80^\circ, \text{Kristian et al. 1970}) \) and radio data (Rankin 1990).

As a corollary of the dynamical solution, the structural constant \( C = \frac{2B^2 R^6}{3c^4 I_{\text{tot}}} \) (see Appendix) can be now evaluated to be \( 4.14 \times 10^{-16} \text{ s} \). If the magnetic field is taken to be \( \sim 3.8 \times 10^{12} G \) (Taylor, Manchester and Lyne 1993), a constraint on the product
\[
\left( \frac{R}{10^6 \text{ cm}} \right)^6 \left( \frac{I}{10^{45} \text{ g cm}^2} \right)^{-1} \simeq 1.16 \text{ g cm}^{-4}
\]
is obtained from the torque expression. We thus see that the proposed dynamics offers some insight onto the equation of state, namely, the relationship favours a equation of state slightly softer than the Bethe-Johnson I model (Bethe & Johnson 1974).

3.3 The Vela Pulsar (PSR B0833-45)

A measurement of \( n_{\text{obs}} \) for the Vela pulsar has been recently obtained by Lyne et al. (1996), and it has not been possible determine \( m_{\text{obs}} \) as yet, therefore the procedure used above is not suitable. However, introducing the observational data from the catalog of Taylor, Manchester & Lyne (1993), we find that a logarithmic solution is possible only if \( m_{\text{obs}} > 4.5 \) (and \( \ddot{\Omega} > -8.6 \times 10^{-34} \text{ rad s}^{-4} \)), and an exponential one requires \( m_{\text{obs}} < 3.2 \) (and \( \ddot{\Omega} < -6.1 \times 10^{-34} \text{ rad s}^{-4} \)). An observational determination of the jerk parameter would thus discriminate the angle-growth. Meanwhile, eq.(A15) provides an upper limit for \( m_{\text{obs}} \) of 3.2, calculated \textit{without} considering the term which contains the angle and its
derivatives. Therefore the only acceptable solution is a exponential growth of $\alpha$. For an arbitrary value of $\alpha_p$ we have found the limits

$$\alpha_o < 32^\circ$$

$$P_o > 51 \text{ms}$$

(7)

$$t_m < 1.4 \times 10^5 \text{yr}$$

where the characteristic age has been used as the true age. There are claims (Aschenbach, Egger & Trümper 1995, Lyne et al. 1996) suggesting that Vela is actually older by a factor of 2 or 3. This last possibility would imply in our analysis smaller limits for $\alpha_o$ and $P_o$ for the same $\alpha_p$, and a bigger limit for $t_m$. An older pulsar also implies that $m_{obs}$ becomes a positive number. This is as far as we can go without any further data.

3.4 PSR B0540-69

This pulsar does not have a observational determination of $m_{obs}$ either. As an interesting feature of our models we predict using the data presented in Taylor et al. (1995) (Taylor, Manchester & Lyne 1993) a logarithmic angle growth if $m_{obs} > 7.8 (\bar{\Omega} > -8.4 \times 10^{-31} \text{ rad s}^{-4})$, and an exponential one if $m_{obs} < 7.2 (\bar{\Omega} < -7.7 \times 10^{-31} \text{ rad s}^{-4})$. The upper limit from eq.(A15) is $m_{obs} \simeq 7.3$, so a logarithmic growth is also excluded as in the case of Vela. Letting $\alpha_p$ to be a free parameter, we can calculate for an exponential growth
\[ \alpha_o < 41^o \]

\[ P_o > 23\, ms. \] \hspace{1cm} (8)

\[ t_m < 3.3 \times 10^4\, yr \]

3.5 PSR B1509-58

With the data by Kaspi et al. (1994) and using the characteristic age as \( t_p \simeq 1550 \, yr \), we find that the logarithmic growth is the only one allowed for this pulsar and gives

\[ \alpha_p \simeq 83^o \]

\[ \alpha_o \simeq 61^o \] \hspace{1cm} (9)

\[ P_o \simeq 45\, ms \]

\[ t_m \simeq 2200\, yr. \]

Again, we can calculate the structural constant \( C \simeq 6 \times 10^{-15}\, s \), which leads to

\[ \left( \frac{R}{10^6\, cm} \right)^6 \left( \frac{I}{10^{45}\, g\, cm^2} \right)^{-1} \simeq 1.01\, g\, cm^{-4}, \] assuming \( B_o \simeq 1.55 \times 10^{13}\, G \) (Taylor, Manchester and Lyne 1993). This is approximately the same result obtained for the Crab pulsar,
or about 3% relative difference in the radius of both objects, suggesting that the same equation of state can be applied to them.

There is some dispute about this pulsar’s true age (Thorsett 1992, Blandford & Romani 1988, Van den Bergh & Kramper 1984). If our picture is correct, the maximum admissible value for the pulsar’s age \( t_p \) is \( \sim 1730 \text{ yr} \) but in this case the initial period would have been as low as \( \sim 4 \text{ ms} \), which seems difficult to accept. These figures would support the association of this pulsar with the 185 AD supernovae. On the other hand, if this pulsar is older (the remnant is calculated to be \( 10^4 \text{ yr} \), see Van den Bergh & Kramper and references therein), its angle \( \alpha \) may have reached \( \pi/2 \) in the past, so that our analysis no longer applies to it.

4 CONCLUSIONS

We have studied simple parametric dynamical models of the angle growth which led us to determine the initial features of the four considered pulsars. Even if we have tried to unify as much as possible their dynamical behaviour, some ambiguity related to the intrinsic difficulty of the data analysis remains. As an example, the data analysis of the Crab pulsar made by Lyne, Pritchard & Smith (1993) have not directly calculated the third derivative of frequency \( \dot{\nu} \), but instead proceeded to find it by setting \( m_{\text{obs}} = \dot{\nu} \)

\[ \frac{\nu^2}{\dot{\nu}^3} = n_{\text{obs}}(2n_{\text{obs}} - 1) \].

A inspection of eq.(A15) reveals that if the canonical braking index is replaced by \( n_{\text{obs}} \) (M.P.Allen & J.E.Horvath, in preparation) the angle-dependent term automatically vanishes, but since the variation of \( n_{\text{obs}} \) is small in the Crab, the correction is also small. Strictly speaking, the true \( m_{\text{obs}} \) in the case of a varying angle
would be greater than that value if the "angular" term

\[
\frac{\bar{\Omega}}{\Omega} \left( \frac{\ddot{\alpha}}{\dot{\alpha}} - \frac{2\dot{\alpha}}{\sin(2\alpha)} \right) < n_{\text{obs}} - 1
\]  

(10)

and smaller if the inequality is reversed. If we substitute \( \alpha_p = 68^\circ \) as found in Section 3.2 in eq.(A15), we obtain a correction to \( m_{\text{obs}} \) of + 0.15. It should be noted that this correction in \( \ddot{\Omega} \) implies small corrections in the other derivatives measurements particularly in \( \ddot{\Omega} \). Also, the discrepancy in the braking index, either originated by angle growth, magnetic field increase or inertia moment decrease, necessarily leads to some correction in the estimatives of \( m_{\text{obs}} \) made through the simple assumption above, unless fortuitous cancellations occur.

We consider that an exponential angle growth, with a typical (but non-universal) time-scale \( \leq 10^4 \text{ yr} \) is a good (yet simple) model for the young pulsars dynamics. It accounts for the observed discrepancies in the braking index and the jerk parameter. It also provides limits for the initial parameters of individual pulsars if just \( n_{\text{obs}} \) is known. These preditions can be compared to the observational data to be gathered in a few years.

As a final remark we stress that our results show that the characteristic age may be a rather poor estimative of the true age. The dependence of the torque on \( t \) forces a full calculation of \( t_p \) by inverting the complicated expression of eq.(A1) instead of using the standard form

\[
\tau = -\frac{\Omega_p}{(n-1)\dot{\Omega}_p} \left[ 1 - \left( \frac{\Omega_p}{\Omega_o} \right)^{n-1} \right],
\]

(11)

which nevertheless remains valid (even if \( \dot{\alpha} \neq 0 \)) in \( n_{\text{obs}} = \text{constant} \) (M.P.Allen &
J.E.Horvath, in preparation). We conclude that, even if a exponential growth of the $\alpha$ angle seems to be consistent with several observed aspects of young pulsars behaviour, more observational data (mainly a detection of a change in the average pulse profile at the level $\sim 10^{-3}$) is needed to formulate more complete and complex models of the dynamics. Reliable determinations of $\alpha$ for these pulsars through optical polarization or other methods and $m_{\text{obs}}$ would be decisive to test angle-growing models.

While this paper was being written we have received a preprint by B.Link & R.I.Epstein where some of these ideas were independently exposed and discussed.

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7 APPENDIX

We present in this Appendix the explicit forms $\alpha(t)$ employed for the determination of the dynamical pulsar histories. First let us consider an exponential growth of the angle $\alpha(t) = \alpha_o e^{t/t_\alpha}$. Assuming a vacuum dipole model for $\tau_{ext}$, eq.(2) can be integrated to yield the analytic form for $\Omega$

$$\Omega = \Omega_p \left[ 1 + \frac{t_\alpha}{T \sin^2 \alpha_p} \sum_{n=1}^{\infty} (-1)^n \frac{(2\alpha_p)^{2n} - (2\alpha_o)^{2n}}{(2n)(2n)!} \right]^{-1/2} \quad (A1)$$

where $T = \frac{-\Omega_p}{\Omega_p} = (C \sin^2 \alpha_p \Omega_p^2)^{-1}$ is a slowdown time-scale at the present time, with $C$ being a structural constant depending on the chosen equation of state, $\alpha_p$ and $\alpha_o$ are the present and initial angle respectively, and $t_\alpha = \alpha_p/\dot{\alpha}_p$ is a constant time-scale for the growth of the angle. Now, we can express the time-scale for maximum torque $t_m$, the initial period $P_o$ and $\alpha_o$ in terms of observable quantities yielding

$$t_m = t_\alpha \ln \left( \frac{\pi}{2\alpha_p} \right) + t_p \quad (A2)$$

$$\alpha_o = \alpha_p e^{-t_p/t_\alpha}. \quad (A3)$$

If we assume instead a linear growth law $\alpha(t) = (\frac{\pi}{2} - \alpha_o) \frac{t}{t_m} + \alpha_o$ and integrate again the equation of motion, we obtain the non-linear algebraic equation

$$\Omega = \Omega_p \left[ 1 - \frac{1}{T \dot{\alpha} \sin^2 \alpha_p} \left( \alpha_p - \alpha - \sin(2\alpha_p) + \sin(2\alpha) \right) \right]^{-1/2} \quad (A4)$$

which gives
\[ t_m = \left( \frac{\pi}{2} - \alpha_p \right) \frac{1}{\dot{\alpha}} + t_p \] (A5)

\[ \alpha_o = \alpha_p - \dot{\alpha} \, t_p^\prime. \] (A6)

Integration of a logarithmic law \( \alpha(t) = \ln \left[ \frac{t}{t_p} (e^{\alpha_p} - e^{\alpha_o}) + e^{\alpha_o} \right] \) gives

\[ \Omega = \Omega_p \left\{ 1 - \frac{2 \, t_p}{5 T (e^{\alpha_p} - e^{\alpha_o}) \sin^2 \alpha_p} \times \right. \\
\left. \times \left[ e^{\alpha_p} \left( \sin^2 \alpha_p - \sin(2\alpha_p) + 2 \right) - e^{\alpha} \left( \sin^2 \alpha - \sin(2\alpha) + 2 \right) \right] \right\}^{-1/2} \] (A7)

with the following results

\[ t_m = \frac{e^{\pi/2} - \alpha_p - \ln (1 - t_p \dot{\alpha}_p)}{e^{\alpha_p} \dot{\alpha}_p} \] (A8)

\[ \alpha_o = \alpha_p + \ln (1 - t_p \dot{\alpha}_p). \] (A9)

Finally, integrating a power-law \( \alpha(t) = \left( \frac{\pi}{2} - \alpha_o \right) \left( \frac{t}{t_m} \right)^b + \alpha_o \), where \( b \) is a positive constant \( \neq 1 \), we have

\[ \Omega = \Omega_p \left\{ 1 - \frac{t_p \, t_m}{T \left( \frac{\pi}{2} - \alpha_o \right)^{1/b} \sin^2 \alpha_p} \left[ (\alpha_p - \alpha_o)^{1/b} - (\alpha - \alpha_o)^{1/b} - \left( \frac{1}{b} \right)! \times \right. \\
\left. \times \left[ \cos(2\alpha_p) \sum_{i=0}^{int(\frac{1}{2b}-1)} f_{2i+2}(\alpha_p) - \cos(2\alpha) \sum_{i=0}^{int(\frac{1}{2b}-1)} f_{2i+2}(\alpha) + \right. \right] \right\} \]
\[ + \sin(2\alpha_p) \sum_{i=0}^{\text{int}(\frac{\pi}{4} - \frac{x}{2})} f_{2i+1}(\alpha_p) - \sin(2\alpha) \sum_{i=0}^{\text{int}(\frac{\pi}{4} - \frac{x}{2})} f_{2i+1}(\alpha) \right] \right\}^{-1/2} \] (A10)

where

\[ f_{2i+1}(\alpha) = (-1)^i \frac{(\alpha - \alpha_o)^{1/2} - 2i - 1}{2^{2i+1}(\frac{1}{b} - 2i - 1)!} \]

\[ f_{2i+2}(\alpha) = (-1)^i \frac{(\alpha - \alpha_o)^{1/2} - 2i - 2}{2^{2i+2}(\frac{1}{b} - 2i - 2)!} , \]

and \( \text{int}(x) \) is the greatest integer number lesser than \( x \), for \( b = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \} \). For any other \( b \) we will have

\[ \Omega = \Omega_p \left\{ 1 - \frac{t_p t_m}{T(\frac{\pi}{2} - \alpha_o)^{1/2} \sin^2 \alpha_p} \left[ (\alpha_p - \alpha_o)^{1/2} (\cos^2 \alpha_o - \cos 2\alpha_p) - \right. \right. \]

\[ - (\alpha - \alpha_o)^{1/2} (\cos^2 \alpha_o - \cos 2\alpha) - \left( \frac{1}{b} \right)! \left[ \cos(2\alpha_p) \sum_{i=1}^{\infty} g_{2i}(\alpha_p) - \cos(2\alpha) \sum_{i=1}^{\infty} g_{2i}(\alpha) + \right. \]

\[ \left. + \sin(2\alpha_p) \sum_{i=0}^{\infty} g_{2i+1}(\alpha_p) - \sin(2\alpha) \sum_{i=0}^{\infty} g_{2i+1}(\alpha) \right] \right\}^{-1/2} \] (A11)

where

\[ g_{2i}(\alpha) = (-1)^i \frac{2^{2i}(\alpha - \alpha_o)^{1/2} + 2i}{(\frac{1}{b} + 2i)!} \]

\[ g_{2i+1}(\alpha) = (-1)^i \frac{2^{2i+1}(\alpha - \alpha_o)^{1/2} + 2i + 1}{(\frac{1}{b} + 2i + 1)!} \]
giving

\[ t_m = t_p^{(1 - \frac{1}{b})} \left[ \frac{\pi - \alpha_p}{\alpha_p} \right]^{1/b} \]  \hspace{1cm} (A12)

\[ \alpha_o = \alpha_p - \frac{t_p \dot{\alpha}_p}{b}. \]  \hspace{1cm} (A13)

Note that the expressions eqs. (A1), (A4), (A7), (A10) and (A11) must be now used to define the characteristic age of the pulsar replacing the usual form \( \tau = \frac{1}{(n-1)} |\frac{\Omega}{\dot{\Omega}}| \) (see Section 4).

It also follows that in these angle varying models it can be shown that, even if the power of \( \Omega \) in the torque expression is exactly 3, the observed braking index defined as \( n_{obs} = \frac{\ddot{\Omega}}{\Omega^2} \) picks up an extra term

\[ n_{obs} = 3 - 2 \left| \frac{\Omega}{\dot{\Omega}} \right| \cot \alpha \frac{d\alpha}{dt} \]  \hspace{1cm} (A14)

and the observed jerk parameter defined as \( m_{obs} = \frac{\dddot{\Omega}}{\dot{\Omega}^3} \) is also modified

\[ m_{obs} = 3 \left( 2 n_{obs} - 1 \right) + \left( n_{obs} - 3 \right) \left[ n_{obs} + \frac{\Omega}{\dot{\Omega}} \left( \frac{\ddot{\alpha}}{\dot{\alpha}} - \frac{2\dot{\alpha}}{\sin(2\alpha)} \right) \right]. \]  \hspace{1cm} (A15)

Therefore, using the simple expressions for \( \alpha(t) \) we obtain

\[ \frac{\sin(2\alpha) - 2\alpha}{4 \cos^2 \alpha} = \frac{m_{obs} - n_{obs}(n_{obs} + 3) + 3}{(n_{obs} - 3)^2} \]  \hspace{1cm} (A16)

for the exponential form,

\[ \cos \alpha = \left[ -2 \left( \frac{m_{obs} - n_{obs}(n_{obs} + 3) + 3}{(n_{obs} - 3)^2} \right) \right]^{-1/2} \]  \hspace{1cm} (A17)
for the linear form,

\[
\frac{\sin(2\alpha) + 2}{4 \cos^2 \alpha} = \frac{m_{\text{obs}} - n_{\text{obs}}(n_{\text{obs}} + 3) + 3}{(n_{\text{obs}} - 3)^2} \tag{A18}
\]

for the logarithmic form, and

\[
\cos \alpha = \left[ -2 \left( \frac{m_{\text{obs}} - n_{\text{obs}}(n_{\text{obs}} + 3) + 3}{(n_{\text{obs}} - 3)^2} + \frac{T(b - 1)}{t_p(n_{\text{obs}} - 3)} \right) \right]^{-1/2} \tag{A19}
\]

for the power-law. These forms express the unknown angles in terms of the \( n_{\text{obs}} \) and \( m_{\text{obs}} \).