Solving the liar detection problem using the four-qubit singlet state

Adán Cabello

Departamento de Física Aplicada II, Universidad de Sevilla, 41012 Sevilla, Spain

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A method for solving the Byzantine agreement problem [M. Fitzi, N. Gisin, and U. Maurer, Phys. Rev. Lett. 87, 217901 (2001)] and the liar detection problem [A. Cabello, Phys. Rev. Lett. 89, 100402 (2002)] is introduced. The main advantages of this protocol are that it is simpler and is based on a four-qubit singlet state already prepared in the laboratory.

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I. INTRODUCTION

Some interesting applications of the N-particle N-level singlet states have been recently introduced [1]. Particularly, the three three-level singlet state

\[ |S_3^{(3)}\rangle = \frac{1}{\sqrt{6}}( |012\rangle - |021\rangle - |102\rangle + |120\rangle + |201\rangle - |210\rangle ) \] (1)

(the lower and upper indexes refer, respectively, to the number of constituents and the dimension of the Hilbert space of any of the constituents of the composite system) was shown to be useful for solving the “liar detection problem” [1], which is a simplification of a classically unsolvable problem in quantum computing called the “Byzantine generals’ problem” or the “Byzantine agreement problem” introduced in two seminal papers by Lamport, Pease, and Shostak [2, 3], and inspired by a pseudohistorical scenario [4]. A version of this problem was previously solved by Fitzi, Gisin, and Maurer [5] also using the three three-level singlet state. Ref. [1] ended by remarking that the preparation of the required states would pose a formidable challenge. Meanwhile, Gisin has proposed a method for preparing the three three-level singlet state [5]. To my knowledge, no experimental results have as yet been reported.

On the other hand, the Munich group [9] (see also Ref. [3]) has prepared, using parametric down-converted photons entangled in polarization, a four-qubit entangled state that can be expressed as

\[ |S_4^{(2)}\rangle = \frac{1}{2\sqrt{3}}(|0011\rangle - |0101\rangle - |0110\rangle - |1001\rangle - |1010\rangle + 2|1100\rangle) . \] (2)

This state belongs to a family of states of N (even) qubits that generalizes the familiar two-qubit singlet state

\[ |S_2^{(2)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) . \] (3)

Any member of this family can be expressed as

\[ |S_N^{(2)}\rangle = \frac{1}{N!\sqrt{2^N + 1}} \sum_{\text{permutations}} z! \left( \frac{N}{2} - z \right)! \left( \frac{z}{2} \right)^{N-z} \times |i j \ldots n \rangle , \] (4)

where the sum is extended to all of the states obtained by permuting the state \(|0\ldots01\ldots1\rangle\), which contains the same number of 0’s and 1’s; \(z\) is the number of 0’s in the first \(N/2\) positions (for example, in \(|01\rangle\), \(z = 1\); in \(|1100\rangle\), \(z = 0\); in \(|01101\rangle\), \(z = 2\)). Expression (4) is similar to that introduced in Ref. [1] for the N-particle N-level singlet states (which, according to the notation introduced here, should be denoted as \(|S_N^{(N)}\rangle\)).

The interesting point is that the N-qubit “singlet” \(\mathbb{R}\) states given by Eq. (4) share some properties with the N-particle N-level singlet states introduced in Ref. [1]. Both are nonseparable and N-lateral unitary invariant. None separability means that no local model can mimic the predictions of quantum mechanics for these states \(\mathbb{R}\) and implies that the outcomes of the measurements do not reveal predefined results. N-lateral unitary invariance means that, if we act on any state with the tensor product of \(N\) equal unitary operators, the result will be to reproduce the same state:

\[ U^\otimes N |\psi\rangle = |\psi\rangle , \] (5)

\(U^\otimes N\) being \(U \otimes \ldots \otimes U\), where \(U\) is a unitary operator. This implies that the same correlations between the outcomes of measurements on the \(N\) particles occur for different sets of measurements. Both properties were essential to the solutions to the problems proposed in Refs. [1, 3]. The question is whether the \(|S_N^{(2)}\rangle\) states can be used to perform tasks that, so far, were specific to the \(|S_N^{(N)}\rangle\) states.

The \(|S_N^{(2)}\rangle\) states can be used to distribute cryptographic keys [11, 12], encode quantum information in decoherence-free subspaces [13, 14, 15, 16], perform secret sharing, and teleclone quantum states (telecloning is a process combining quantum teleportation and optimal quantum cloning from 1 input to \(M\) outputs) [12]. In fact, for \(N \geq 4\), the \(|S_N^{(2)}\rangle\) states are the N-lateral unitary invariant version of the telecloning states introduced in

*Electronic address: adan@us.es
In this paper, I shall show that the state $|S_4^{(2)}\rangle$ can also be used to solve the Byzantine generals' and liar detection problems.

The Byzantine generals' problem is a classical problem in distributed computing defined as follows. $N$ generals are connected by secure pairwise classical channels. A commanding general must send an order to his $N-1$ lieutenant generals such that: (a) All loyal lieutenant generals obey the same order; (b) if the commanding general is loyal, then every loyal lieutenant general obeys his order. If the commanding general is loyal, (a) follows from (b). However, the commanding general may be a traitor.

The Byzantine generals' problem is a metaphor for distributed processes, some of which may be faulty. For $N = 3$ generals and 1 traitor the problem has no classical solution, as proved in Refs. 2, 3. However, a version thereof can be solved with the aid of quantum mechanics. The protocol proposed in Ref. 2 solves the Byzantine generals' problem for $N = 3$ in the following sense: if all generals are loyal then, after the protocol, all the generals would obey the commanding general's order. If one general is a traitor then, after the protocol, either the two loyal generals would obey the commanding general's order or abort the process.

As can be easily seen, the Byzantine generals' problem for $N = 3$ generals and 1 traitor is equivalent to the liar detection problem defined as follows: three parties $A$ (lice), $B$ (ob), and $C$ (arol) are connected by secure pairwise classical channels. $A$ sends a message $m$ to $B$ and $C$, and $B$ forwards the message to $C$. If both $A$ and $B$ are honest, then $C$ should receive the same message from $A$ and $B$. However, $A$ could be dishonest and send different messages to $B$ and $C$, $m_{AB} \neq m_{AC}$, or, alternatively, $B$ could be dishonest and send a message that is different from that he has received, $m_{BC} \neq m_{AB}$. For $C$ the problem is to ascertain without a shadow of a doubt who is being dishonest.

In Sec. III I shall introduce a protocol based on the $|S_4^{(2)}\rangle$ state for solving the liar detection problem (and thus also the Byzantine generals' problem) in the following sense: if both $A$ and $B$ are honest then $C$ will receive the same message from $A$ and $B$. If one of $A$ and $B$ is dishonest then, after the protocol, either $C$ ascertain who is being dishonest or $C$ does not accept the message from one of the parties $A$ and $B$.

Before getting into the details of the protocol, it would be useful to give a rough idea of how it works: besides their messages, $A$ and $B$ must send some additional information. For instance, $A$ must also send $B$ some private information correlated with the message. $B$ and $C$ must be able to check the authenticity of the information they have received by using private information. In addition, to convince $C$ that the message $B$ is sending her is actually that $B$ received from $A$, $B$ would also need to send $C$ the information $B$ has received from $A$. In this scenario, being a liar means that she/he did not distribute the appropriate additional information. The advantage derives from the fact that $C$ can detect the origin of the inapprop
(vii) C performs a measurement on her corresponding qubit. For instance, C can measure $M_j$. The outcomes of $B$ and $C$’s measurements must be correlated. For instance, if they have measured $M_j$ on the four qubits, two of them must be “0” and the other two must be “1”; due to unitary invariance, this occurs for any $M_j$. Due to the fact that A’s qubits are entangled with $B$ and $C$’s qubits, the outcomes can be proved [11] to be genuinely unpredictable (if one does not know the results of the other measurements). C checks whether the expected correlations occur.

(viii) For each four-qubit system $j$ of $S_2$, $C$ follows the same steps as in (iv)–(vii), only exchanging the roles of $A$ and $B$.

(ix) Since $C$’s choice of subsets $S_1$ and $S_2$ is known by $A$ and $B$ only after all qubits have been distributed among them, then, if all the outcomes are correctly correlated, $C$ would assume that the remaining $L$ distributed four-qubit systems are in the $|S_4^{(2)}\rangle$ state and use them for the protocol described below. If not, $C$ would conclude that something went wrong and abort any subsequent action.

III. PROTOCOL FOR LIAR DETECTION

Let us suppose that the message $m$ that $A$ sends to $B$ and $C$, and $B$ sends to $C$ is a bit value “0” or “1” and that all three parties agree to use the following protocol.

(I) For each four-qubit system, $C$ asks $A$ and $B$ to perform the same measurement on their individual qubits. After a large number of these measurements, both $B$ and $C$ ($A$) are in possession of a long list of (pairs of) 0’s and 1’s $l_B$, and $l_C$ ($l_A$) such as the following:

| Position | $l_A$ | $l_B$ | $l_C$ |
|----------|-------|-------|-------|
| 1        | 00    | 1     | 1     |
| 2        | 01    | 0     | 1     |
| 3        | 00    | 1     | 1     |
| 4        | 11    | 0     | 0     |
| 5        | 11    | 0     | 0     |
| 6        | 00    | 1     | 1     |
| 7        | 01    | 0     | 1     |
| 8        | 11    | 0     | 0     |
| ...      | ...   | ...   | ...   |

Each of these lists has the following properties.

(a) It is random (i.e., generated by an intrinsically unrepeatable method that gives each possible number the same probability of occurring).

(b) It is correlated to the other two lists. If “00” (“11”) is in position $j$ in $l_A$, then “1” (“0”) is in position $j$ in $l_B$ and $l_C$.

(c) Each party knows only his/her own list.

(II) The message $A$ sends $B$ is denoted as $m_{AB}$. In addition to $m_{AB}$, $A$ must also send $B$ the list $l_{A}^{(m_{AB})}$ of positions in $l_A$ in which $m_{AB}$ appears twice. For instance, if $A$ sends $B$ the message $m_{AB} = 0$, then she must also send $B$ the list $l_{A}^{(m_{AB}=0)} = \{1, 3, 6, \ldots \}$ since in $l_A$, “00” appears in positions 1, 3, 6, . . . . Note that, if the sequences are random and long enough, then the length of $l_{A}^{(m_{AB}=0)}$ must be about one-quarter of the total length $L$ of the list $l_A$.

(III) $B$ would not accept the message if the received list $l_{A}^{(m_{AB})}$ is not compatible with $l_B$ or the length of $l_{A}^{(m_{AB}=0)} \approx L/4$. For instance, if $l_{A}^{(m_{AB}=0)} = \{1, 2, 3, 6, \ldots \}$, then $B$ would not accept the message because “00” is in position 2 in $l_B$, so $A$ cannot have “00” in this position.

Requirements (II) and (III) force $A$ to send $B$ information which is correct but perhaps incomplete. Otherwise, if $A$ sends a list containing $n$ erroneous data, then the probability that $B$ does not accept the message $m_{AB}$ would be at least $(2^n - 1)/2^n$.

(IV) The message $B$ sends $C$ is denoted as $m_{BC}$. $B$ must also send $C$ a list $l_{A}^{(m_{BC})}$, which is supposedly the same $l_{A}^{(m_{AB})}$ that $B$ has received from $A$.

(V) The message $A$ sends $C$ is denoted as $m_{AC}$. $A$ must also send $C$ a list $l_{AC}$, which is supposedly $l_{A}$.

(VI) When $C$ finds that $m_{AC} \neq m_{BC}$, she has three lists $l_C$, $l_{A}^{(m_{BC})}$, and $l_{AC}$, to help her find out whether it is $A$ or $B$ who is being dishonest. $C$ must first check whether $l_{AC}$ is consistent with $l_C$. If not, then $A$ is the liar. If yes, then $C$ must check whether $l_{A}^{(m_{BC})}$ has an appropriate length and is consistent with $l_{AC}$. If not, then $B$ is the liar. At this stage, this is the only possibility.

IV. CONCLUSIONS

The main practical advantage of the introduced protocol over those presented in Refs. [1, 2] is that it requires a simpler quantum state (a four-qubit state instead of a three-qutrit state) which has been already prepared in a laboratory [7]. This would make the new protocol immediately applicable for solving both the Byzantine generals’ and liar detection problems. The main theoretical advantage is that the introduced protocol seems to be fundamental in a greater degree than those in Refs. [1, 2], in the sense that it requires only one party uses trit values, instead of all three parties.

Although both the Byzantine generals’ and liar detection problems assume that the parties are connected by secure pairwise classical channels (secure messengers in Ref. [7]), this is an unrealistic constraint for real applications. Note, however, that the protocol described above also works if the classical channels between $A$ and $C$, and between $B$ and $C$ are not secure but public and unjamable (i.e., which cannot be tampered with). Obviously, such a channel cannot be used to distribute delicate information (like, for instance, whether or not the generals will attack, or the lists $l_{A}^{(m_{BC})}$ and $l_{AC}$), but can be
used, together with an additional quantum channel, to implement a standard quantum key distribution protocol [11, 12] to send this delicate information. The final picture gives us a quantum solution to a problem without classical solution: the liar detection problem without secure classical channels.

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[4] Taken from A. Panconesi (unpublished): “Byzantium, 1453 AD. The city of Constantinople, the last remnants of the hoary Roman Empire, is under siege. Powerful Ottoman battalions are camped around the city on both sides of the Bosphorus, poised to launch the next, perhaps final, attack. Sitting in their respective camps, the generals are meditating. Because of the redoubtable fortifications, no battalion by itself can succeed; the attack must be carried out by several of them together or otherwise they would be thrust back and incur heavy losses that would infuriate the Grand Sultan. Worse, that would jeopardize the prospects of a defeated general to become Vizier. The generals can agree on a common plan of action by communicating thanks to the messenger service of the Ottoman Army, which can deliver messages within an hour, certifying the identity of the sender and preserving the content of the message. Some of the generals, however, are secretly conspiring against the others. Their aim is to confuse their peers so that an insufficient number of generals is deceived into attacking. The resulting defeat will enhance their own status in the eyes of the Grand Sultan. The generals start shuffling messages around, those trying to agree on a time to launch the offensive, the others trying to split their ranks...” To my knowledge, however, no historical account of the fall of Constantinople mentions these treacherous generals.
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[9] The name “singlet” for the $|S^2⟩$ state, the only two-qubit state of total spin zero, was coined, together with the name “triplet” (which denotes three orthogonal states of total spin one). In this paper, I use the name “singlet” for any of the $|S^2⟩$ states because, as Eq. (4) shows, they are a generalization of $|S^2⟩$. However, for $N > 2$, with $N$ even, the dimension of the subspace of total spin zero is $N!/[(N/2)!(N/2 + 1)!]$ (see Ref. [15]).
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