Some functional inequalities under lower Bakry-Émery-Ricci curvature bounds with $\varepsilon$-range

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Abstract

For $n$-dimensional weighted Riemannian manifolds, lower $m$-Bakry-Émery-Ricci curvature bounds with $\varepsilon$-range, introduced by Lu-Minguzzi-Ohta [10], integrate constant lower bounds and certain variable lower bounds in terms of weight functions. In this paper, we prove a Cheng type inequality and a local Sobolev inequality under lower $m$-Bakry-Émery-Ricci curvature bounds with $\varepsilon$-range. These generalize those inequalities under constant curvature bounds for $m \in (n, \infty)$ to $m \in (-\infty, 1] \cup \{\infty\}$.

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1 Introduction

The Ricci curvature plays an important role in geometric analysis. For example, lower bounds of Ricci curvature imply comparison theorems such as the Laplacian comparison theorem and Bishop-Gromov volume comparison theorem. This paper is concerned with the Bakry-Émery-Ricci curvature $\text{Ric}_\psi^m$, which is a generalization of the Ricci curvature for weighted Riemannian manifolds and $m$ is a real parameter called the effective dimension. The condition $\text{Ric}_\psi^m \geq K$ for $K \in \mathbb{R}$ implies many comparison geometric results similar to those for Riemannian manifolds with Ricci curvature bounded from below by $K$ and dimension bounded from above by $m$. Especially the case of $m \geq n$ is now classical and well investigated. Recently, there is a growing interest in the $m$-Bakry-Émery-Ricci curvature in the case of $m \in (-\infty, 1]$. For this range, some Poincaré inequalities [3] (see also [11] for its rigidity), Beckner inequality [2] and the curvature-dimension condition [13] were studied.

It is known that some comparison theorems (such as the Bishop-Gromov volume comparison theorem and the Laplacian comparison theorem) under the constant curvature bound $\text{Ric}_\psi^m \geq Kg$ hold only for $m \in [n, \infty)$ and fail for $m \in (-\infty, 1] \cup \{\infty\}$. Nonetheless, Wylie-Yeroshkin [20] introduced a variable curvature bound

$$\text{Ric}_\psi^m \geq Ke^{-\frac{4}{m-1}\psi}g$$

associated with the weight function $\psi$, and established several comparison theorems. They were then generalized to

$$\text{Ric}_\psi^m \geq Ke^{-\frac{4}{m-1}\psi}g$$

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with $m \in (-\infty, 1)$ by Kuwae-Li [3]. In [10], Lu-Minguzzi-Ohta gave a further generalization of the form

$$\text{Ric}_m^\psi \geq K e^{\frac{(\varepsilon-1)}{m-n} \psi} g$$

for an additional parameter $\varepsilon$ in an appropriate range, depending on $m$, called the $\varepsilon$-range (see also [9] for a preceding work on singularity theorems in Lorentz-Finsler geometry). This is not only a generalization of [20] and [3], but also a unification of both constant and variable curvature bounds by choosing appropriate $\varepsilon$. We refer to [5, 6, 7] for further investigations on the $\varepsilon$-range.

In this paper, we assume lower bounds of the $m$-Bakry-Émery-Ricci curvature with $\varepsilon$-range and study analytic applications on non-compact manifolds. The main contributions of this paper are the following:

- We give an upper bound of the $L_p$-spectrum. In particular, when $p = 2$, this gives an upper bound of the first nonzero eigenvalue of the weighted Laplacian.
- We give an explicit form of a local Sobolev inequality.

An upper bound of the first nonzero eigenvalue of the Laplacian under lower Ricci curvature bounds was first investigated in [1] in 1975 and it is called the Cheng type inequality. Some variants of the Cheng type inequality are known (we refer to [16], for example) under lower bounds of the Bakry-Émery-Ricci curvature in the case of $m \in [n, \infty]$. Our Theorem 6 generalizes them. The local Sobolev inequality is an important tool for the De Giorgi-Nash-Moser theory. Recently in [17], they obtained a Liouville type theorem for the weighted $p$-Laplacian by using a local Sobolev inequality and Moser’s iteration techniques. Our results in Theorem 8 are consistent with the local Sobolev inequality in [17] in the case of constant curvature bounds and the effective dimension $m \in [n, \infty]$.

This paper is organized as follows. In Section 2, we briefly review the Bakry-Émery-Ricci curvature and Cheng type inequalities and local Sobolev inequalities. We show a Cheng type inequality in Section 3 and a local Sobolev inequality in Section 4 under lower bounds of the Bakry-Émery-Ricci curvature with $\varepsilon$-range. In Appendix, we give a variant of Cheng type inequality for deformed metrics under lower bounds of the Bakry-Émery-Ricci curvature with $\varepsilon$-range.

## 2 Preliminaries

### 2.1 $\varepsilon$-range

Let $(M, g, \mu)$ be an $n$-dimensional weighted Riemannian manifold. We assume that $M$ is non-compact in this paper. We set $\mu = e^{-\psi} v_g$ where $v_g$ is the Riemannian volume measure and $\psi$ is a $C^\infty$ function on $M$. For $m \in (-\infty, 1] \cup [n, +\infty]$, the $m$-Bakry-Émery-Ricci curvature is defined as follows:

$$\text{Ric}_m^\psi := \text{Ric}_g + \nabla^2 \psi - \frac{d\psi \otimes d\psi}{m-n},$$

where when $m = +\infty$, the last term is interpreted as the limit 0 and when $m = n$, we only consider a constant function $\psi$, and set $\text{Ric}_n^\psi := \text{Ric}_g$.

In [10], [9], they introduced the notion of $\varepsilon$-range:

$$\varepsilon = 0 \text{ for } m = 1, \quad |\varepsilon| < \frac{m-1}{m-n} \text{ for } m \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } m = n. \tag{1}$$

In this $\varepsilon$-range, for $K \in \mathbb{R}$, they considered the condition

$$\text{Ric}_m^\psi (v) \geq K e^{\frac{(\varepsilon-1)}{m-n} \psi (v)} g(v,v), \quad v \in T_x M.$$  

We also define the associated constant $c$ as

$$c = \frac{1}{m-1} \left(1 - \varepsilon \frac{m-n}{m-1}\right) > 0 \tag{2}$$

for $m \neq 1$ and $c = (n-1)^{-1}$ for $m = 1$. We define the comparison function $s_\kappa$ as

$$s_\kappa (t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{-\kappa}t) & \kappa > 0, \\ t & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & \kappa < 0. \end{cases} \tag{3}$$

We denote $B(x,r) = \{ y \in M \mid d(x,y) < r \}$, $V(x,r) = \mu(B(x,r))$ and $tB = B(x,tr)$ if $B = B(x,r)$.
In this subsection, we explain Cheng type inequalities under lower Bakry-Émery-Ricci curvature bounds.

When $\text{Ric} \leq R$ and $\text{vol} M$ is bounded from below, according to the argument in [10, Theorem 3.6], the condition $\text{Ric}(x) \geq R$ holds for all $x \in M$. Assume that $\text{Ric}(x) \geq R$ holds for all $x \in M$. Theorem 1. (\cite{10}, Theorem 3.11) Let $(M, g, \mu)$ be a complete weighted Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the $\varepsilon$-range \cite{11}, $K \in \mathbb{R}$ and $b \geq a > 0$. Assume that

$$\text{Ric}^m_{\psi}(v) \geq Ke^{\frac{4(e-1)}{n-1}}\psi(g(v, v))$$

holds for all $v \in T_xM \setminus 0$ and $a \leq e^{-\frac{2(e-1)}{n-1}} \psi \leq b$.

Then we have

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{b}{a} \frac{\int_0^{\min\{R/a, \pi/\sqrt{CR}\}} s_{\varepsilon K}(\tau)^{1/c} d\tau}{\int_0^{r/b} s_{\varepsilon K}(\tau)^{1/c} d\tau}$$

for all $x \in M$ and $0 < r < R$, where $R \leq b\pi/\sqrt{CR}$ when $K > 0$ and we set $\pi/\sqrt{CR} := \infty$ for $K \leq 0$.

We briefly review the argument in [10] (where they considered, more generally, Finsler manifolds equipped with measures). Given a unit tangent vector $v \in T_xM$, let $\eta: [0, l) \to \mathbb{R}$ be the geodesic with $\dot{\eta}(0) = v$. We take an orthonormal basis $\{e_i\}_{i=1}^n$ of $T_xM$ with $e_n = v$ and consider the Jacobi fields $E_i(t) := (d\exp_x)_{t_0}(te_i)$, $i = 1, 2, \ldots, n - 1$, along $\eta$. Define the $(n - 1) \times (n - 1)$ matrices $A(t) = (a_{ij}(t))$ by

$$a_{ij}(t) := g(E_i(t), E_j(t)).$$

We define

$$h_0(t) := (\det A(t))^{1/2(n-1)}, \quad h(t) := e^{-c\psi(\eta(t))}(\det A(t))^{c/2}, \quad h_1(\tau) := h(\varphi_{\eta}^{-1}(\tau))$$

for $t \in [0, l)$ and $\tau \in [0, \varphi_{\eta}(l))$, where

$$\varphi_{\eta}(t) := \int_0^t e^{\frac{2(e-1)}{n-1}\psi(\eta(s))} ds.$$

By the definition, we have the following relationship:

$$(e^{-\psi(\eta)}h_0^{n-1})(t) = h(t)^{1/c} = h_1(\varphi_{\eta}(t))^{1/c}. $$

According to the argument in [10] Theorem 3.6, the condition $\text{Ric}^m_{\psi}(v) \geq Ke^{\frac{4(e-1)}{n-1}}\psi(g(v, v))$ implies that

$$(e^{-\psi(\eta)}h_0^{n-1})/s_{\varepsilon K}(\varphi_{\eta})^{1/c} \text{ is non-increasing. (4)}$$

This plays the key role in proving Theorem 1 above.

2.2 Upper bounds of the $L^p$-spectrum

In this subsection, we explain Cheng type inequalities under lower Bakry-Émery-Ricci curvature bounds by constants. We generalize these results to the $\varepsilon$-range in Section 3. For $p > 1$, the $L^p$-spectrum is defined by

$$\lambda_{\mu, p}(M) := \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla \phi|^p d\mu}{\int_M |\phi|^p d\mu}.$$

When $p = 2$, the $L^2$ spectrum is the first nonzero eigenvalue of the weighted Laplacian. Under lower $m$-Bakry-Émery-Ricci curvature bounds with $m \in [n, \infty)$, we have the following theorems.

Theorem 2. (\cite{10}, Theorem 3.2) Let $(M, g, \mu)$ be an $n$-dimensional weighted complete Riemannian manifold. Assume that $\text{Ric}^m_{\psi} \geq -K$ ($K \geq 0$). Then the $L^p$-spectrum satisfies

$$\lambda_{\mu, p}(M) \leq \left( \frac{\sqrt{(m-1)K}}{p} \right)^p.$$

An additional assumption on the weight function leads to the following Cheng type inequality under a lower $\infty$-Bakry-Émery-Ricci curvature bound.
**Theorem 3.** ([16, Theorem 3.3]) Let $(M, g, \mu)$ be an $n$-dimensional complete weighted Riemannian manifold. We fix a point $q \in M$. Assume that $\text{Ric}_m^\infty \geq -K (K \geq 0)$ and $\frac{\text{Ric}_m^\infty}{\text{Vol}} \geq -k (k \geq 0)$ along all minimal geodesic segments from the fixed point $q \in M$, where $r$ is the distance from $q$. Then the $L_p^\mu$-spectrum satisfies

$$\lambda_{\mu,p}(M) \leq \left(\frac{\sqrt{(n-1)K + k}}{p}\right)^p.$$ 

These results are generalizations of the original Cheng type inequality in [1].

### 2.3 Local Sobolev inequality

We have the following local Sobolev inequality under lower bounds of the $m$-Bakry-Émery-Ricci curvature in the case of $m \in (n, \infty)$ and $n \geq 2$. We generalize the following result in Section 4. We refer to [12] for the case of $m = \infty$.

**Theorem 4.** ([14, Lemma 3.2]) Let $(M, g, \mu)$ be an $n$-dimensional weighted complete Riemannian manifold. If $\text{Ric}_m^\infty \geq -(m-1)K$ for some $K \geq 0$ and $m > n \geq 2$, then there exists a constant $C$, depending on $m$, such that for all $B(a, r) \subset M$ we have for $f \in C_0^\infty (B(a, r))$,

$$\left(\int_{B(a, r)} |f|^{\frac{2m}{2m-1}} \, d\mu\right)^{\frac{2m-1}{2m}} \leq e^{C(1+\sqrt{K}r)} \mu(B(a, r))^{-\frac{1}{\nu}} r^2 \int_{B(a, r)} (|\nabla f|^2 + r^{-2} f^2) \, d\mu.$$ 

We will use the next theorem in Subsection 4.2 to prove a local Sobolev inequality under lower Bakry-Émery-Ricci curvature bounds with $\epsilon$-range.

**Theorem 5.** ([14, Theorem 2.2]) Let $e^{-tA}$ be a symmetric submarkovian semigroup acting on the spaces $L^p(M, d\mu)$. Given $\nu > 2$, the following three properties are equivalent.

1. $\|e^{-tA} f\|_\infty \leq C_0 t^{-\nu/2} \|f\|_1$ for $0 < t < t_0$.
2. $\|f\|_{2^{\nu/(\nu-2)}} \leq C_1 \left(\|A^{1/2} f\|_2^2 + t_0^{-1} \|f\|_2^2\right)$.
3. $\|f\|_{2^{\nu+1}/\nu} \leq C_2 \left(\|A^{1/2} f\|_2^2 + t_0^{-1} \|f\|_2^2\right)^{\frac{1}{\nu}} \|f\|_1^{\frac{2}{\nu}}$.

Moreover, 3. implies 1. with $C_0 = (\nu CC_2)^{\nu/2}$ and 1. implies 2. with $C_1 = CC_0^{2/\nu}$, where $C$ is some numerical constant.

### 3 Upper bound of the $L_p^\mu$-spectrum with $\epsilon$-range

**Theorem 6.** Let $(M, g, \mu)$ be an $n$-dimensional weighted complete Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\epsilon \in \mathbb{R}$ in the $\epsilon$-range [1], $K > 0$ and $b \geq a > 0$. Assume that

$$\text{Ric}_m^\infty (v) \geq -Ke^{\frac{4(\epsilon - 1)}{\mu} \psi(x)} g(v, v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \leq e^{-\frac{2(\epsilon - 1)}{\mu} \psi} \leq b.$$ 

Then, for $p > 1$, we have

$$\lambda_{\mu,p}(M) \leq \left(\frac{\sqrt{K}}{c a} \frac{1}{p}\right)^p.$$ 

**Proof.** We apply the argument in [16, Theorem 3.2]. For an arbitrary $\delta > 0$, we set

$$\alpha := -\frac{\sqrt{K} \frac{1}{c a} + \delta}{p}$$

and, for $x \in M$ and $R \geq 2$,

$$\phi(y) := \exp(\alpha r(y)) \phi(y),$$

where $r(y) := \text{dist}(y, M \setminus B(a, r))$ and $\psi(x) := \frac{1}{c a} \text{dist}(x, M \setminus B(a, r))$.
where \( r(y) = d(x, y) \) and \( \varphi \) is a cut off function on \( B(x, R) \) such that \( \varphi = 1 \) on \( B(x, R - 1) \), \( \varphi = 0 \) on \( M \setminus B(x, R) \) and \( |\nabla \varphi| \leq C_3 \), where \( C_3 \) is a constant independent of \( R \). For an arbitrary \( \zeta > 0 \), we have

\[
|\nabla \phi|^p = |a e^{\alpha r} \varphi \nabla r + e^{\alpha r} \nabla \varphi|^p \\
\leq e^{\mu a r} (\alpha \varphi + |\nabla \varphi|)^p \\
\leq e^{\mu a r} \left[ (1 + \zeta)^{p-1} (\alpha \varphi)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} |\nabla \varphi|^p \right].
\]

By the definition of \( \lambda_{\mu, p}(M) \), we find

\[
\lambda_{\mu, p}(M) \leq (1 + \zeta)^{p-1} (-\alpha)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \int_M e^{\mu a r} |\nabla \varphi|^p d\mu \\
= (1 + \zeta)^{p-1} (-\alpha)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \int_{B(x, R)} e^{\mu a r} |\nabla \varphi|^p d\mu \\
\leq (1 + \zeta)^{p-1} (-\alpha)^p + C_3^p \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{e^{\mu (R-1)} \mu(B(x, R))}{\int_{B(x, 1)} e^{\mu a r} d\mu} \\
\leq (1 + \zeta)^{p-1} (-\alpha)^p + C_3^p \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{e^{\mu (R-1)} \mu(B(x, R))}{e^{\mu a} \mu(B(x, 1))}.
\]

(6)

It follows from Theorem[1] that

\[
\mu(B(x, R)) \leq \mu(B(x, 1)) \frac{b}{a} \int_0^{R/a} s^{1/c} K(\tau) d\tau, \quad (7)
\]

To estimate the RHS of (7), we observe

\[
(\sqrt{KR})^{1/c} \int_0^{R/a} s^{1/c} K(\tau) d\tau = \frac{1}{2} \left\{ \exp(\sqrt{K} \tau) - \exp(-\sqrt{K} \tau) \right\} \frac{d\tau}{\sqrt{\frac{c}{a}}}
\]

\[
\leq \int_0^{R/a} \exp \left( \sqrt{\frac{K}{\tau}} \right) d\tau
\]

\[
= \sqrt{\frac{c}{K}} \left\{ \exp \left( \sqrt{\frac{K}{\tau} \frac{R}{a}} \right) - 1 \right\}.
\]

Thus, we have

\[
\mu(B(x, R)) \leq \mu(B(x, 1)) \frac{b}{a} \int_0^{R/a} s^{1/c} K(\tau) d\tau \sqrt{\frac{c}{K}} \exp \left( \sqrt{\frac{K R}{c a}} \right) \frac{1}{(\sqrt{c K})^{1/c}}.
\]

This implies

\[
\frac{e^{\mu (R-1)} \mu(B(x, R))}{e^{\mu a} \mu(B(x, 1))} \leq C_4 \exp \left( \mu a R + \sqrt{\frac{K R}{c a}} \right) = C_4 \exp(-\delta R) \to 0
\]

as \( R \to \infty \), where \( C_4 \) is a constant depending on \( a, b, K, \delta \). Hence, (6) yields

\[
\lambda_{\mu, p}(M) \leq (1 + \zeta)^{p-1} (-\alpha)^p.
\]

Since \( \zeta > 0 \) and \( \delta > 0 \) are arbitrary, this implies the theorem.

When \( m \in [n, \infty) \), \( \varepsilon = 1 \) and \( a = b = 1 \), then \( c = \frac{1}{m-1} \) and it holds

\[
\lambda_{\mu, p}(M) \leq \left( \frac{\sqrt{(m-1)K}}{p} \right)^p,
\]

which recovers Theorem[2].
4 Functional inequalities with $\varepsilon$-range

4.1 Local Poincaré inequality

In this subsection, we prove the following Poincaré inequality.

**Theorem 7.** (Local Poincaré inequality) Let $(M, g, \mu)$ be an $n$-dimensional complete weighted Riemannian manifold and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the $\varepsilon$-range $\Pi$. $K > 0$ and $b \geq a > 0$. Assume that

$$\text{Ric}^m_{\varepsilon}(v) \geq -K e^{\frac{4a(1-\varepsilon)}{n}}(e\varepsilon) g(v, v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \leq e^{\frac{2(1-\varepsilon)}{m}} \leq b.$$  \hspace{1cm} (8)

Then we have

$$\forall f \in C^\infty(M), \int_B |f - f_B|^2 d\mu \leq 2n + 3\left(\frac{2b}{a}\right)^{1/c} \exp\left(\sqrt{\frac{K}{c} \frac{2r}{a}}\right) r^2 \int_{2B} |\nabla f|^2 d\mu$$

for all balls $B \subset M$ of radius $0 < r < \infty$, where

$$f_B := \frac{1}{\mu(B)} \int_B f d\mu.$$  \hspace{1cm} (9)

**Proof.** We apply the argument in [15, Theorem 5.6.6, Lemma 5.6.7]. For any pair of points $(x, y) \in M \times M$, let

$$\gamma_{x,y} : [0, d(x, y)] \to M$$

be a geodesic from $x$ to $y$ parametrized by arclength. We also set

$$l_{x,y}(t) = \gamma_{x,y}(td(x, y))$$

for $t \in [0, 1]$. We have, using Jensen’s inequality,

$$\int_B |f - f_B|^2 d\mu = \int_B \left| \int_B (f(x) - f(y)) \frac{d\mu(y)}{\mu(B)} \right|^2 d\mu(x)$$

$$\leq \frac{1}{\mu(B)} \int_B \int_B |f(l_{x,y}(1)) - f(l_{x,y}(0))|^2 d\mu(x) d\mu(y)$$

$$\leq \frac{1}{\mu(B)} \int_B \int_B \left\{ \int_0^1 \left| \frac{df \circ l_{x,y}(t)}{dt} \right| dt \right\}^2 d\mu(x) d\mu(y)$$

$$\leq \frac{1}{\mu(B)} \int_B \int_B \int_0^1 \left| \frac{df \circ l_{x,y}(t)}{dt} \right|^2 dt d\mu(x) d\mu(y)$$

$$= \frac{2}{\mu(B)} \int_B \int_{1/2}^1 \left| \frac{df \circ l_{x,y}(t)}{dt} \right|^2 dt d\mu(x) d\mu(y).$$

To obtain the last equality we decompose the set

$$\{(x, y, t) : x, y \in B, t \in (0, 1)\}$$

into two pieces,

$$\{(x, y, s) : x, y \in B, t \in (1/2, 1)\}$$

and

$$\{(x, y, s) : x, y \in B, t \in (0, 1/2)\},$$

then use $l_{x,y}(t) = l_{y,x}(1-t)$. Now, suppose that we can bound the Jacobian $J_{x,t}$ of the map

$$\Phi_{x,t} : y \mapsto l_{x,y}(t)$$

from below by

$$\forall x, y \in B, \forall s \in [1/2, 1], \quad J_{x,t}(y) \geq 1/F(r),$$

\hspace{1cm} (10)
where $r$ is the radius of the ball $B$. Then

$$\int_B \int_B \int_{1/2}^1 \left| \frac{df(l_{x,y}(t))}{dt} \right|^2 dt \, d\mu(x) \, d\mu(y) \leq F(r) \int_B \int_B \int_{1/2}^1 \left| \frac{df(l_{x,y}(t))}{dt} \right|^2 J_{x,t}(y) dt \, d\mu(x) \, d\mu(y)$$

$$\leq F(r) \int_B \int_B \int_0^1 |\nabla f(l_{x,y}(t))|^2 d(x,y)^2 J_{x,t}(y) dt \, d\mu(x) \, d\mu(y)$$

$$\leq (2r)^2 F(r) \int_0^1 \int_B \int_B |\nabla f(l_{x,y}(t))|^2 J_{x,t}(y) d\mu(y) \, d\mu(x) \, dt$$

$$= (2r)^2 F(r) \int_0^1 \int_B \left( \int_{\Phi_x(y)} |\nabla f(z)|^2 d\mu(z) \right) d\mu(x) \, dt$$

$$\leq (2r)^2 F(r) \int_0^1 \int_B \left( \int_{2B} |\nabla f(z)|^2 d\mu(z) \right) d\mu(x) \, dt$$

$$\leq (2r)^2 F(r) \mu(B) \int_{2B} |\nabla f(z)|^2 d\mu(z).$$

We finally prove (10). Let $\xi$ be the unit tangent vector at $x$ such that $\partial_s \gamma_{x,y}(s)|_{s=0} = \xi$. Let $I(x, s, \xi)$ be the Jacobian of the map $\exp_x : T_x M \to M$ at $s\xi$ with respect to $\mu$. Then

$$d\mu = I(x, s, \xi) ds \, d\xi,$$

where $d\xi$ is the usual measure on the sphere. Using the notation in Subsection 2.1, we have $I(x, s, \xi) = e^{-\psi(s)} h_n s^{-1}$. According to (11), we find that

$$s \to \frac{I(x, s, \xi)}{s^{-cK(\varphi(s))^{1/c}}},$$

is non-increasing. Under the relationship $l_{x,y}(t) = \gamma_{x,y}(s)$, it follows that

$$J_{x,t}(y) = \left( \frac{s}{d(x,y)} \right)^n \frac{I(x, s, \xi)}{I(x, d(x,y), \xi)} \geq \left( \frac{1}{2} \right)^n \frac{s^{-cK(\varphi(s))^{1/c}}}{s^{-cK(\varphi(d(x,y)))^{1/c}}}$$

for all $s \in (d(x,y)/2, d(x,y))$. Thus, we have, since $s/b \leq \varphi(s) \leq s/a$,

$$J_{x,t}(y) \geq \left( \frac{1}{2} \right)^n \frac{s^{-cK(\varphi(d(x,y)/2))^{1/c}}}{s^{-cK(\varphi(d(x,y)))^{1/c}}}$$

$$\geq \left( \frac{1}{2} \right)^n \left( \frac{\varphi(d(x,y)/2)}{\varphi(d(x,y))} \right)^{1/c} \exp \left( -\sqrt{\frac{K}{c}} \varphi(d(x,y)) \right)$$

$$\geq \left( \frac{1}{2} \right)^n \left( \frac{a}{b} \right)^{1/c} \exp \left( -\sqrt{\frac{K}{c}} \frac{2r}{a} \right).$$

This proves (10) with $F(r) = \left\{ \left( \frac{1}{2} \right)^n \left( \frac{a}{b} \right)^{1/c} \exp \left( -\sqrt{\frac{K}{c}} \frac{2r}{a} \right) \right\}^{-1}$ and the theorem follows. \hfill $\square$

Given that we have the local Poincaré inequality and the volume doubling property (obtained explicitly later in (12)), we can apply [15 Corollary 5.3.5] and we obtain the following inequality.

**Corollary 1.** Under the same assumptions as in Theorem 7, there exist constants $C, P$ such that

$$\forall f \in C^\infty(M), \quad \int_B |f - f_B|^2 \, d\mu \leq P e^{C r^2} \int_B |\nabla f|^2 \, d\mu$$

for all balls $B \subset M$ of radius $r > 0$. 

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4.2 Local Sobolev inequality

It is shown in [14] that the volume doubling property and Poincaré inequality imply a local Sobolev inequality. We follow this line with Theorems [1] and [7].

**Theorem 8.** (Local Sobolev inequality) Let $(M, g, \mu)$ be an $n$-dimensional complete weighted Riemannian manifold with $n \geq 3$ and $m \in (-\infty, 1] \cup [n, +\infty]$, $\varepsilon \in \mathbb{R}$ in the $\varepsilon$-range [1], $K > 0$ and $b \geq a > 0$. Assume that

$$\text{Ric}_\psi^n(v) \geq -K e^{\frac{\varepsilon b - 1}{\varepsilon}} \mu(v, v)$$

holds for all $v \in T_x M \setminus 0$ and

$$a \leq e^{-\frac{2 \varepsilon b}{\varepsilon}} \psi \leq b.$$  \hspace{1cm} (11)

Then there exist constants $D, E$ depending on $c, a, b, n$ such that for all $B(o, r) \subset M$ we have for $f \in C_0^\infty(B(o, r))$,

$$\left(\mu(B(o, r))^{-1} \int_{B(o, r)} |f|^{2(1+\varepsilon)} \mu \right)^{\frac{1}{1+\varepsilon}} \leq E \exp \left( D \left( 1 + \sqrt{\frac{K}{c}} \right) \frac{r}{a} \right) r^2 \mu(B(o, r))^{-1} \int_{B(o, r)} (|\nabla f|^2 + r^{-2} f^2) \mu.$$

We first prove two lemmas. We set

$$f_s(x) = \int \chi_s(x, z) f(z) \mu(z),$$

where $V(x, s) = \mu(B(x, s))$ and $\chi_s(x, z) = \frac{1}{v(x, s)^{-1}} B(x, s)(z)$.

**Lemma 1.** Under the same assumptions as in Theorem [8] there exists a constant $C_5$ such that for all $y \in M$ and all $0 < s < r,$ we have

$$\|f_s\|_2 \leq C_5 r^{-\frac{1}{2}} \left( \frac{r}{a} \right)^{\frac{1}{2} \left( 1 + \frac{\varepsilon}{2} \right)} \|f\|_1,$$

for all $f \in C_0^\infty(B)$, where $B = B(y, r)$ and $V = V(r) = V(y, r)$.

**Proof.** We apply the argument in [14] Lemma 2.3. We use the notations in Subsection 2.1. For $\tau \geq 0$, $0 < s < r$, we set

$$t := \frac{r b}{a s} \tau.$$

Since $\frac{r b}{a s} \geq 1$, we have $\tau \leq t$. Hence, by direct computations, we obtain

$$s^{-c K(t)^{1/c}} \leq s^{-c K(\tau)^{1/c}} \left( \frac{t}{\tau} \right)^{1/c} \exp \left( \sqrt{\frac{K}{c}} \frac{t}{\tau} \right).$$

Integrating both sides in $t$ from 0 to $r/a$, we have

$$\int_0^{r/a} s^{-c K(t)^{1/c}} dt \leq \int_0^{r/a} s^{-c K(\tau)^{1/c}} \left( \frac{t}{\tau} \right)^{1/c} \exp \left( \sqrt{\frac{K}{c}} \frac{t}{\tau} \right) \mu dt$$

$$\leq \left( \frac{r b}{a s} \right)^{1/c} \exp \left( \sqrt{\frac{K}{c}} \frac{r}{a} \right) \int_0^{r/a} s^{-c K(\tau)^{1/c}} dt$$

$$= \left( \frac{r b}{a s} \right)^{1+\frac{\varepsilon}{2}} \exp \left( \sqrt{\frac{K}{c}} \frac{r}{a} \right) \int_0^{s/b} s^{-c K(\tau)^{1/c}} dt.$$
Therefore, we also have the doubling property:

\[
V(2r) \leq V(r) \left( \frac{b}{a} \right)^{2+\frac{1}{s}} 2^{\frac{1}{s}+1} \exp \left( \sqrt{\frac{K}{c}} \frac{2r}{a} \right).
\]  \hspace{1cm} (12)

For \( x, z \in M \) satisfying \( d(z, x) < s \), we have

\[
V(z, s) \leq V(x, 2s) \leq V(x, s) \left( \frac{b}{a} \right)^{2+\frac{1}{s}} 2^{\frac{1}{s}+1} \exp \left( \sqrt{\frac{K}{c}} \frac{2s}{a} \right).
\]

This implies

\[
\chi_s(x, z) \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} 2^{\frac{1}{s}+1} \exp \left( \sqrt{\frac{K}{c}} \frac{2s}{a} \right) \chi_s(z, x).
\]

Thus,

\[
\|f_s\|_1 \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} 2^{\frac{1}{s}+1} \exp \left( \sqrt{\frac{K}{c}} \frac{2s}{a} \right) \|f\|_1.
\]  \hspace{1cm} (13)

We moreover assume \( B \cap B(x, s) \neq \emptyset \). Since

\[
\frac{V(x, 2r + s)}{V(x, s)} \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} \left( \frac{2r + s}{s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{2r + s}{a} \right)
\]

and

\[
\frac{V(x, 4r)}{V(x, 2r + s)} \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} \left( \frac{4r}{2r + s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{4r}{a} \right),
\]

we have

\[
\frac{1}{V(x, s)} \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} \left( \frac{2r + s}{s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{2r + s}{a} \right) \frac{1}{V(x, 2r + s)} \leq \left( \frac{b}{a} \right)^{2(2+\frac{1}{s})} \left( \frac{2r + s}{s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{2r + s}{a} \right) \left( \frac{4r}{2r + s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{4r}{a} \right) \frac{1}{V(x, 4r)} \leq \left( \frac{b}{a} \right)^{2(2+\frac{1}{s})} \left( \frac{4r}{s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{6r + s}{a} \right) \frac{1}{V(y, r)}.
\]

Hence,

\[
\|f_s\|_{\infty} = \left\| \int \chi_s(x, z) f(z) d\mu(z) \right\|_{\infty} \leq \left( \frac{b}{a} \right)^{2(2+\frac{1}{s})} \left( \frac{4r}{s} \right)^{1+\frac{1}{s}} \exp \left( \sqrt{\frac{K}{c}} \frac{6r + s}{a} \right) \frac{\|f\|_1}{V(y, r)}.
\]

Using (13), we have

\[
\|f_s\|_2 = \left( \int f_s^2 d\mu \right)^{\frac{1}{2}} \leq \sqrt{\|f_s\|_{\infty}} \sqrt{\|f\|_1} \leq \left( \frac{b}{a} \right)^{2+\frac{1}{s}} \left( \frac{4r}{s} \right)^{\frac{1}{s}(1+\frac{1}{s})} \exp \left( \sqrt{\frac{K}{c}} \frac{6r + s}{2a} \right) \frac{1}{\sqrt{V(y, r)}} \left( \frac{b}{a} \right)^{(2+\frac{1}{s})} \left( \frac{2r + s}{s} \right)^{\frac{1}{s}(1+\frac{1}{s})} \exp \left( \sqrt{\frac{K}{c}} \frac{4r}{a} \right) \|f\|_1 \leq \left( \frac{b}{a} \right)^{3+\frac{1}{s}} 2^{\frac{1}{s}(1+\frac{1}{s})} \left( \frac{4r}{s} \right)^{\frac{1}{s}(1+\frac{1}{s})} \exp \left( \sqrt{\frac{K}{c}} \frac{6r + 3s}{2a} \right) \frac{1}{\sqrt{V(y, r)}} \|f\|_1.
\]

Setting \( C_5 = \left( \frac{b}{a} \right)^{3+\frac{1}{s}} 2^{\frac{1}{s}(1+\frac{1}{s})} \left( \frac{4r}{3s} \right)^{\frac{1}{s}(1+\frac{1}{s})} \exp \left( \sqrt{\frac{K}{c}} \frac{9r}{3a} \right) \), we get the desired inequality. \( \square \)
Lemma 2. We fix a constant $r > 0$. Under the same assumptions as in Theorem 5, there exists $C_6$ depending only on $c, a, b, r, K, n$ such that

$$\|f - f_s\|_2 \leq C_6 s \|\nabla f\|_2, \quad f \in C_6^\infty(M)$$

for all $0 < s < r$.

Proof. We apply the argument in [13, Lemma 2.4]. Fix $a > 0$, let $\{B_j : j \in J\}$ be a collection of balls of radius $s/2$ such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $M = \bigcup_{i \in J} \overline{B_i}$. For $z \in M$, let $J(z) = \{i \in J : z \in 8B_i\}$ and $N(z) = \#J(z)$. We first estimate $N(z)$ from above. Let $B_2$ be a ball in $\{B_j : j \in J\}$ such that $z \in 2B_2$. For $i \in J(z)$, we have $B_2 \subset 16B_i$. Hence,

$$\mu(B_2) \leq \mu(16B_i) \leq C_7 \mu(B_i),$$

where

$$C_7 := (\frac{b}{a})^{2+\frac{1}{2}} 2^{\frac{7}{2} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{8r}{a}} \right) \geq \left( \frac{b}{a} \right)^{2+\frac{1}{2}} 2^{\frac{7}{2} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{8s}{a}} \right).$$

Therefore, we have

$$\sum_{i \in J(z)} \mu(B_i) \geq N(z) \frac{\mu(B_2)}{C_7^2}.$$  

On the other hand, for $i \in \{j \in J : z \in 8B_j\}$, we have $B_i \subset 16B_2$. Hence,

$$\sum_{i \in J(z)} \mu(B_i) \leq \mu(16B_2) \leq C_7 \mu(B_2).$$

Therefore, we find

$$N(z) \frac{\mu(B_2)}{C_7^2} \leq C_7 \mu(B_2).$$

Letting $N_0 := C_7^2$, we have $N(z) \leq N_0$. We now estimate $\|f - f_s\|_2$. Note that

$$\|f - f_s\|_2 \leq \sum_{i \in J} \left( 2 \int_{2B_i} |f(x) - f_{4B_i}|^2 + |f_{4B_i} - f_s(x)|^2 \, d\mu(x) \right).$$

By the Poincaré inequality (29), we have

$$\int_{4B_i} |f(x) - f_{4B_i}|^2 \, d\mu(x) \leq 2^{n+3} \left( \frac{2h}{a} \right)^{\frac{1}{2}} \exp \left( \sqrt{\frac{K}{c} \frac{4s}{a}} \right) (2s)^2 \int_{8B_i} |\nabla f|^2 \, d\mu \leq C_8 s^2 \int_{8B_i} |\nabla f|^2 \, d\mu,$$

where

$$C_8 = 2^{n+5} \left( \frac{2h}{a} \right)^{\frac{1}{2}} \exp \left( \sqrt{\frac{K}{c} \frac{4s}{a}} \right).$$

Since for any $x \in 2B_i = B(x_i, s),

$$V(x_i, s) \leq V(x, 2s) \leq V(x, s) \left( \frac{b}{a} \right)^{2+\frac{1}{2}} 2^{\frac{7}{2} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{2s}{a}} \right),$$

we have

$$\int_{2B_i} |f_{4B_i} - f_s(x)|^2 \, d\mu(x) \leq \int_{2B_i} \int_{B(x_i, s)} \frac{1}{V(x_i, s)} |f_{4B_i} - f(z)| \, d\mu(z) \, d\mu(x) \leq \int_{2B_i} \int_{B(x_i, s)} \frac{1}{V(x_i, s)} |f_{4B_i} - f(z)| \, d\mu(z) \, d\mu(x) \leq \frac{1}{V(x_i, s)} \left( \frac{b}{a} \right)^{2+\frac{1}{2}} 2^{\frac{7}{2} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{2s}{a}} \right) \int_{2B_i} \int_{4B_i} |f_{4B_i} - f(z)|^2 \, d\mu(z) \, d\mu(x) \leq \left( \frac{b}{a} \right)^{2+\frac{1}{2}} 2^{\frac{7}{2} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{2s}{a}} \right) C_8 s^2 \int_{8B_i} |\nabla f|^2 \, d\mu.$$
Using (14), (15), (16), we have
\[\|f - f_s\|^2 \leq C_9 s^2 \sum_{x \in J} \int_{B_r(x)} |\nabla f|^2 \, d\mu \leq C_9 N_0 s^2 \|\nabla f\|^2,\]
where
\[C_9 = 4 \left( \frac{b}{a} \right)^{2 + \frac{1}{r} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{2r}{a}} \right) C_8 \geq 2 \left( \frac{b}{a} \right)^{2 + \frac{1}{r} + 1} \exp \left( \sqrt{\frac{K}{c} \frac{2r}{a}} \right) C_8 + C_8,\]
Therefore, setting
\[C_6 := \sqrt{N_0 C_9},\]
we have the desired inequality.

**Proof of Theorem 3**
We apply the argument in [14] Theorem 2.1. Fix \(x \in M, r > 0\). For \(0 < s \leq r\) and \(f \in C_0^\infty (B(x, r))\), we have
\[\|f\|_2 \leq \|f - f_s\|_2 + \|f_s\|_2.\]
It follows from Lemmas 1, 2 that
\[\|f\|_2 \leq C_6 s \|\nabla f\|_2 + C_5 V^{-\frac{1}{r}} \left( \frac{r}{s} \right)^{\frac{1}{r}} \|f\|_1,\]
where \(\nu = 1 + \frac{1}{r}\). Hence, we obtain
\[\|f\|_2 \leq 4 C_6 s \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right) + C_5 V^{-\frac{1}{r}} \left( \frac{r}{s} \right)^{\frac{1}{r}} \|f\|_1,\]
(17)
To obtain the minimum of the RHS of (17), we consider its differential with respect to \(s > 0\). At \(s > 0\) which attains the minimum, we have
\[4 C_6 \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right) + C_5 V^{-\frac{1}{r}} \left( \frac{r}{s} \right)^{\frac{1}{r}} \|f\|_1 = 0.\]
Thus,
\[s^{\frac{1}{r} + 1} = \frac{C_{10} V^{-\frac{1}{r} r} \|f\|_1}{\|\nabla f\|_2 + \frac{1}{r} \|f\|_2},\]
(18)
where \(C_{10} = \nu \frac{C_5}{2 C_6}\). Substituting (18) to the RHS of (17), we obtain
\[\|f\|_2 \leq 4 C_6 \left( C_{10} V^{-\frac{1}{r} \frac{r}{s} \|f\|_1} \right) \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right) + C_5 V^{-\frac{1}{r} \frac{r}{s} \|f\|_1} \left( \frac{C_{10} V^{-\frac{1}{r} \frac{r}{s} \|f\|_1}}{\|\nabla f\|_2 + \frac{1}{r} \|f\|_2} \right)^{\frac{1}{r}} \|f\|_1 \]
\[= 4 C_6 C_{10} \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right)^{-\frac{1}{r} \frac{r}{s} + 1} \left( V^{-\frac{1}{r} \frac{r}{s} \|f\|_1} \right)^{\frac{1}{r}} \]
\[+ C_5 C_{10} \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right)^{\frac{1}{r} \frac{1}{r} \frac{r}{s}} \left( V^{-\frac{1}{r} \frac{r}{s} \|f\|_1} \right)^{-\frac{1}{r}} \|f\|_1 \]
\[= \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right)^{\frac{1}{r} \frac{1}{r} \frac{r}{s}} \left( 4 C_6 C_{10} V^{-\frac{1}{r} \frac{r}{s} \|f\|_1} \right) \|f\|_1^{\frac{1}{r}} \]
\[+ C_5 C_{10} V^{-\frac{1}{r} \frac{1}{r} \frac{r}{s} \frac{r}{s}} \|f\|_1^{\frac{1}{r}} \}
\[= 4 C_6 C_{10} + C_5 C_{10} \left( \|\nabla f\|_2 + \frac{1}{r} \|f\|_2 \right)^{\frac{1}{r} \frac{1}{r} \frac{r}{s}} V^{-\frac{1}{r} \frac{1}{r} \frac{r}{s} \frac{r}{s} \|f\|_1^{\frac{1}{r}}}.\]
and also define measures for $K > 0$, we define

$$
\{ 4C_6 C_{10}^{\frac{2}{r}} + C_5 C_{10}^{\frac{2}{r}} \}^{2+\frac{\phi}{r^2}} V^{-\frac{\phi}{r^2}} \left( 2 \left( \| \nabla f \|_2^2 + \| f \|_2^2 \right) \right) \| f \|_1^\phi.
$$

Recalling the expressions of $C_5$ and $C_6$ in the proofs of Lemmas 1, 2, we use constants $E_1$, $E_2$ depending on $c, a, b, n$ such that

$$
C_6 C_{10}^{\frac{2}{r}} = E_1 \exp \left( \frac{\sqrt{K}}{c} \left( \frac{35r}{a} - \frac{61r}{2a} \frac{2}{2 + \nu} \right) \right)
$$

and

$$
C_5 C_{10}^{\frac{2}{r}} = E_2 \exp \left( \frac{\sqrt{K}}{c} \left( \frac{9r}{2a} - \frac{61r}{2a} \frac{2}{2 + \nu} \right) \right).
$$

Thus, there exist constants $D, E_3$ such that

$$
\left\{ 4C_6 C_{10}^{\frac{2}{r}} + C_5 C_{10}^{\frac{2}{r}} \right\}^{2+\frac{\phi}{r^2}} < E_3 \exp \left( D \left( 1 + \sqrt{\frac{K}{c}}\frac{r}{a} \right) \right).
$$

We remark that $D, E_3$ depend only on $c, a, b, n$. Since $c \leq \frac{1}{\pi - 1}$, we have $\nu = 1 + \frac{1}{c} \geq n > 2$ when $n \geq 3$. Hence, we can use Theorem 5 and Theorem 8 follows. \hfill \square

**Remark 1.** At the end of the proof of Theorem 8, we use the fact that $1 + \frac{1}{c} > 2$, it is the only reason why we need the assumption $n \geq 3$. In the case of $n = 2$, we have the local Sobolev inequality when $\varepsilon \neq 0$ under the same curvature bound and 11 in Theorem 8.

**Remark 2.** One of the possible subjects of further research is the gradient estimate of eigenfunctions of the weighted Laplacian with $\varepsilon$-range, which turned out to be difficult. If $\text{Ric}^{\varepsilon}$ is bounded from below with $m > n$, then one way to obtain the gradient estimate is to apply the Li-Yau trick as described in 18, 19 and another way is to use the DeGiorgi-Nash-Moser theory 8 as described in 12. Once we obtained the gradient estimate by the Li-Yau trick, an upper bound of eigenvalues of the weighted Laplacian is obtained as in 18, 19. However, it seems that the Li-Yau trick and Moser’s iteration argument in 17 do not work well in the case where $\text{Ric}^{\varepsilon}$ is bounded from below with $m \leq 1$. The main difficulty stems from the lack of a suitable Bochner formula for analyzing lower Bakry-Émery-Ricci curvature bounds with $\varepsilon$-range. Although 5 Lemma 2.1 obtained the Bochner formula for the distance function with $\varepsilon$-range, a suitable Bochner formula for eigenfunctions of the weighted Laplacian is yet to be known. Finding a suitable Bochner formula for eigenfunctions is our future work.

**Appendix: Upper bound of the $L^p$-spectrum for deformed measures**

Although we considered the Riemannian distance $d$, it is also possible to study comparison theorems associated with a metric deformed by using the weight function (we refer 20, 4, 5, for example). In this appendix, we start from a volume comparison theorem in 5 and prove a variant of Cheng type inequality for the $L^p$-spectrum.

Let $(M, g, \mu = e^{-\psi} v_g)$ be an $n$-dimensional weighted Riemannian manifold, $m \in (-\infty, 1] \cup [n, +\infty)$ and $\varepsilon \in \mathbb{R}$ in the range 1. We fix a point $q \in M$. We define lower semi continuous functions $s_q : M \to \mathbb{R}$ by

$$
s_q(x) := \inf_{\gamma} \int_0^{d(q,x)} e^{-\frac{2(1-\varepsilon)\psi(q(x))}{\varepsilon^2}} d\xi,
$$

where the infimum is taken over all unit speed minimal geodesics $\gamma : [0, d(q,x)] \to M$ from $q$ to $x$. For $r > 0$, we define

$$
B_{\psi,q}(r) := \{ x \in M \mid s_q(x) < r \},
$$

and also define measures

$$
\mu := e^{-\psi} v_g, \quad \nu := e^{\frac{2(1-\varepsilon)\psi}{\varepsilon^2}} \mu.
$$

We set

$$
S_{-K}(r) := \int_0^r s_{-K}^{1/c}(s) ds
$$

for $K > 0$. In 5, they obtained the following theorem.
Theorem 9. ([5] Proposition 4.6, Volume comparison) Let \( (M, g, \mu) \) be an \( n \)-dimensional weighted Riemannian manifold. We assume \( \text{Ric}_\nu^m \geq -Ke^{(c-1)\nu}g \) for \( K > 0 \). Then for all \( r, R > 0 \) with \( r \leq R \) we have
\[
\frac{\nu(B_{\psi, \rho}(R))}{\nu(B_{\psi, \rho}(r))} \leq \frac{S_{-cK}(R)}{S_{-cK}(r)}.
\]

In the following argument, we start from Theorem 9 instead of Theorem 1 to prove a Cheng type inequality of the \( L^p \)-spectrum for the deformed measure \( \nu \).

**Theorem 10.** Let \( (M, g, \mu) \) be a complete weighted Riemannian manifold. We assume that \( s_\eta \) is smooth and there exists a constant \( k > 0 \) such that
\[
|\nabla s_\eta(x)| \leq k
\]
holds for arbitrary \( x \in M \). We also assume
\[
\text{Ric}_\nu^m \geq -K e^{(c-1)\nu}g
\]
for \( K > 0 \). Then we have
\[
\lambda_{\nu, p}(M) \leq \left( \frac{k}{p} \sqrt{\frac{K}{e}} \right)^p.
\]

**Proof.** We apply the argument in Theorem 9.

For \( R \geq 2 \), let \( \eta : \mathbb{R} \to \mathbb{R} \) be a nonnegative smooth function such that \( \eta = 1 \) on \( -(R-1), R-1 \), \( \eta = 0 \) on \( \mathbb{R}\setminus(R,R) \) and \( |\eta'| \leq C_3 \), where \( C_3 \) is a constant independent of \( R \). We set, for an arbitrary \( \delta > 0 \),
\[
\alpha = -\frac{\sqrt{K/e + \delta}}{p}
\]
and
\[
\phi(y) := \exp(\alpha s_\eta(y))\varphi(y),
\]
where \( \varphi(y) := \eta(s_\eta(y)) \). By the assumption of \( s_\eta \), we have
\[
|\nabla \varphi| = |\eta'(s_\eta)||\nabla s_\eta| \leq kC_3.
\]
As in the proof of Theorem 9 we find for an arbitrary \( \zeta > 0 \),
\[
|\nabla \varphi|^p = |ae^{\alpha s_\eta}\varphi \nabla s_\eta + e^{\alpha s_\eta} \nabla \varphi|^p
\]
\[
\leq e^{\alpha s_\eta} (-k\alpha \varphi + |\nabla \varphi|)^p
\]
\[
\leq e^{\alpha s_\eta} \left( (1 + \zeta)^{p-1}(-k\alpha \varphi)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} |\nabla \varphi|^p \right).
\]

By the definition of \( \lambda_{\nu, p}(M) \), we obtain
\[
\lambda_{\nu, p}(M) \leq (1 + \zeta)^{p-1}(-k\alpha \varphi)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{\int_M \exp(\alpha s_\eta)|\nabla \varphi|^p d\nu}{\int_M \exp(\alpha s_\eta) \varphi^p d\nu}
\]
\[
= (1 + \zeta)^{p-1}(-k\alpha \varphi)^p + \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{\int_{B_{\psi, \rho}(R)} \exp(\alpha s_\eta)|\nabla \varphi|^p d\nu}{\int_{B_{\psi, \rho}(R)} \exp(\alpha s_\eta) \varphi^p d\nu}
\]
\[
\leq (1 + \zeta)^{p-1}(-k\alpha \varphi)^p + (kC_3)^p \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{\exp(p\alpha(R-1)) \nu(B_{\psi, \rho}(1))}{\int_{B_{\psi, \rho}(1)} \exp(\alpha s_\eta) d\nu}
\]
\[
\leq (1 + \zeta)^{p-1}(-k\alpha \varphi)^p + (kC_3)^p \left( \frac{1 + \zeta}{\zeta} \right)^{p-1} \frac{\exp(p\alpha(R-1)) \nu(B_{\psi, \rho}(R))}{\exp(\alpha) \nu(B_{\psi, \rho}(1))}.
\]

From Theorem 9 and
\[
(\sqrt{cK})^{1/c} S_{-cK}(R) \leq \int_0^R \left[ \frac{1}{2} \exp(\sqrt{cK}s) - \exp(-\sqrt{cK}s) \right]^{1/c} ds
\]
\[
\leq \sqrt{\frac{c}{K}} \exp \left( \sqrt{\frac{K}{c}} R \right),
\]
we deduce
\[
\frac{\frac{c}{\alpha}(R-1)^\nu(B_{\psi,q}(R))}{\frac{c}{\alpha}(B_{\psi,q}(1))} \leq \frac{1}{c^2} \exp \left( \frac{K}{c} \right) \exp \left( \frac{p\alpha R + \sqrt{K}}{c} \right) \rightarrow 0
\]
as \( R \to \infty \). Letting \( R \to \infty \) in (20), we obtain
\[
\lambda_{\nu,p}(M) \leq (1 + \zeta)^{(p-1)(-k\alpha)^p}.
\] (21)

Since \( \zeta > 0 \) and \( \delta > 0 \) are arbitrary, the theorem follows.

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