IRLS for Sparse Recovery Revisited: Examples of Failure and a Remedy
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Abstract—Compressed sensing is a central topic in signal processing with myriad applications, where the goal is to recover a signal from as few observations as possible. Iterative re-weighting is one of the fundamental tools to achieve this goal. This paper re-examines the iteratively reweighted least squares (IRLS) algorithm for sparse recovery proposed by Daubechies, Devore, Fornasier, and Güntürk in Iteratively reweighted least squares minimization for sparse recovery, Communications on Pure and Applied Mathematics, 63(2010) 1–38. Under the null space property of order $K$, the authors show that their algorithm converges to the unique $k$-sparse solution for $k$ strictly bounded above by a value strictly less than $K$, and this $k$-sparse solution coincides with the unique $\ell_1$ solution. On the other hand, it is known that, for $k$ less than or equal to $K$, the $k$-sparse and $\ell_1$ solutions are unique and coincide. The authors emphasize that their proof method does not apply for $k$ sufficiently close to $K$, and remark that they were unsuccessful in finding an example where the algorithm fails for these values of $k$.

In this note we construct a family of examples where the Daubechies-Devore-Fornasier-Güntürk IRLS algorithm fails for $k = K$, and provide a modification to their algorithm that provably converges to the unique $k$-sparse solution for $k$ less than or equal to $K$ while preserving the local linear rate. The paper includes numerical studies of this family as well as the modified IRLS algorithm, testing their robustness under perturbations and to parameter selection.

I. INTRODUCTION

The fundamental problem in compressed sensing is to recover the sparsest solution $x_*$ to a linear equation of the form $\Phi x = y$ for a given $y$, where $\Phi \in \mathbb{R}^{k \times N}$ is the measurement matrix and $\ell < N$. We denote the set of solutions to the equation $\Phi x = y$ by $\Phi^{-1}(y)$ which is assumed to be non-empty throughout. The problem of obtaining the sparsest solution can be posed as the minimization of the so-called 0-norm, $\|x\|_0$, over $\Phi^{-1}(y)$, where $\|x\|_0$ is the number of non-zero components in the vector $x$. Since the 0-norm problem is NP hard, in practice [7] one replaces this problem with the $\ell_1$ minimization (or basis pursuit) problem

$$\min_{x \in \Phi^{-1}(y)} \|x\|_1.$$  (BP)

The relationship of BP to the 0-norm problem has been intensively studied over the past few years [5], [6], [11], [10]. Compressed sensing has applications to a range of signal processing areas, including image acquisition, sensor networks and image reconstruction [7], [18], [22].

Numerous algorithms have been proposed for solving BP and its various reformulations, which include the basis pursuit denoising (BPDN) problem:

$$\min_{x} \|x\|_1 \text{ s.t. } \|\Phi x - y\|_2 \leq \sigma,$$

the LASSO problem: $\min_{x} \|x\|_1 + \frac{\lambda}{2} \|\Phi x - y\|_2^2$, and the $\ell_1$-regression problem:

$$\min_{x} \|Ax - b\|_1 \quad (\ell_1R)$$

under the correspondences $\text{rge}(A) = \text{Null}(\Phi)$ and $\Phi b = y$ [6] (see Section V for details). Algorithms designed to solve these problems include the iteratively reweighted least squares (IRLS) algorithms [4], [17] which apply to $\ell_1R$, the FISTA algorithm [2], [23] which applies to the LASSO, and the homotopy algorithm [19], the alternating direction method of multipliers (ADMM) [3], [13], and the level-set method described in [1] which all apply to BPDN. However, the focus of this paper is the IRLS algorithm described in [9] which we refer to as the DDFG-IRLS algorithm.

In [9], the authors show that if the matrix $\Phi$ satisfies the the null space property of order $K$ for $0 < \gamma < 1$ (see Section III for details), then the DDFG-IRLS algorithm converges to the unique $k$-sparse solution when $k < K - 2\gamma(1-\gamma)^{-1}$, and this $k$-sparse solution coincides with the unique $\ell_1$ solution, where a vector is $k$-sparse if it has $k$ nonzero components. In addition, the authors also establish the local linear convergence of the DDFG-IRLS algorithm when $0 < \gamma < 1 - 2/(K + 2)$. On the other hand, it is known that for $k \leq K$ the $k$-sparse and $\ell_1$ solutions are unique and coincide [15], [12], [9]. In [9, Remark 5.4], the authors note that their proof method does not apply for $K - 2\gamma(1-\gamma)^{-1} \leq k \leq K$, and state that they were unsuccessful in finding an example where the algorithm fails when $k$ falls in this range. In this note we construct a family of examples where the DDFG-IRLS algorithm fails when $k = K$, and provide a modification to their algorithm that provably converges to the unique $k$-sparse solution for $k \leq K$. In addition, we show that this modification is locally linearly convergent for all $k \leq K$ and $\gamma \in (0, 1)$ which increases the range of $\gamma$ values for which linear convergence is assured.

Iteratively re-weighted least squares algorithms (IRLS) for solving $\ell_p$ minimization problems for $1 \leq p \leq \infty$ have been in the literature for many years beginning with the Ph.D. thesis of Lawson [16]. For $0 < p \leq 1$, IRLS was used to solve sparse reconstruction in [14], and a theory for solving $\ell_p$ minimization problems in general can be found in [20]. We refer the reader to [21] for a survey on IRLS methods applied to robust regression. More recently, cluster point convergence of IRLS smoothing methods for problems of the form $\min f(x) + \lambda\|x\|_p$, where $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, is given in [17]. In addition, an IRLS algorithm has been developed for convex inclusions of the form $A_i x + b_i \in C_i$, $i = 1, \ldots, n$ where the sets $C_i$ are all assumed to be convex [4]. In this case, the authors establish the iteration complexity of their method. However, all of these methods focus on general
linear systems and do not specifically address the problem of compressed sensing where the null space properties play a key role. Daubechies, Devore, Fornasier, and Güntürk [9] focus on the compressed sensing case where \( \|x\|_0 \) is approximated by a smoothing of the norms \( \|x\|_p \) for \( 0 < p \leq 1 \). We follow Daubechies, Devore, Fornasier, and Güntürk in the \( p = 1 \) case and suggest a simple modification to their method for updating the smoothing parameter. This modification allows us to obtain stronger convergence properties.

Our discussion proceeds as follows. In Section 2 we discuss the DDFG-IRLS algorithm and our modification to the smoothing parameter update procedure. In Section 3 we prove the stronger convergence and rate of convergence properties for the modified algorithm. Our proofs closely parallel those given in [9] but contain some simplifications. In Section 4, we construct a family of examples where the DDFG-IRLS algorithm fails but our modifications succeed. These results are illustrated numerically in Section 5 where we also provide a few numerical experiments to illustrate the numerical stability of the modified algorithm. In particular, we show that on randomly chosen problems the two methods have virtually identical performance characteristics.

II. THE MODIFIED IRLS ALGORITHM

Our algorithm is similar to the IRLS algorithm given in [9]. The primary innovation is the manner in which the smoothing parameter \( \epsilon_k \) is updated. In [9], \( \epsilon_k \) is updated by the rule

\[
\epsilon_{k+1} = \min \left\{ \epsilon_k, \frac{r_{K_2}(x^{k+1})}{N} \right\},
\]

where, for \( x = (x_1, \ldots, x_N)^T \in \mathbb{R}^N \),

\[ r_i(x) \]

is the \( i \)th largest element of \( \{ |x_j|, 1 \leq j \leq N \} \).

On the other hand, the algorithm below employs the update rule

\[
\epsilon_{k+1} = \min \left\{ \epsilon_k, \frac{\eta(1 - \gamma)\sigma_{K_2}(x^{k+1})}{N} \right\}, \tag{II.1}
\]

where \( \eta \in (0, 1) \) is chosen and fixed at the beginning of the iteration, the parameters \( \gamma \) and \( K \) come from A2, and

\[ \sigma_j(z) := \sum_{\nu > j} r_{\nu}(z), \quad j = 1, \ldots, N. \tag{II.2} \]

As stated, the algorithm is an iteratively re-weighted least squares algorithm where the weights at each iteration are given by

\[
w_i^{k} := \left((x_i^{k})^2 + \epsilon_i^2\right)^{-1/2} \quad i = 1, \ldots, N. \tag{II.3}
\]

Moreover, given a positive weight vector \( w \in \mathbb{R}^N \), we define the associated inner product by

\[
\langle u, v \rangle_w := \sum_{i=1}^{N} w_i u_i v_i \quad \forall u, v \in \mathbb{R}^N,
\]

and the corresponding weighted 2-norm by \( \|u\|_w := \sqrt{\langle u, u \rangle_w} \). With this notation, our algorithm can be stated as follows.

Algorithm 1: An IRLS algorithm for compressed sensing.

\[
\text{Input : } x^0 \in \mathbb{R}^N \\
\text{Initialize } \epsilon_0 = 1 \text{ and } \eta \in (0, 1)
\]

1. while not converge do
   2. \( w_i^{k} \leftarrow \left((x_i^{k})^2 + \epsilon_i^2\right)^{-1/2} \quad i = 1, \ldots, N \).
   3. \( x^{k+1} \leftarrow \arg \min \left\{ \|x\|_{w^{k}}^2 \mid x \in \Phi^{-1}(y) \right\} \).
   4. \( \epsilon_{k+1} \leftarrow \min \left\{ \epsilon_k, \frac{\eta(1 - \gamma)\sigma_{K}(x^{k+1})}{N} \right\} \).
   5. \( \text{If } \epsilon_{k+1} = 0, \text{ stop.} \)
   6. \( k \leftarrow k + 1. \)
7. end

\text{Output: } x^{k+1}

In general, the null space parameters \( K \) and \( \gamma \) are unknown, however, we show in Section V-B that the performance of both algorithms is robust with respect to their choice. In particular, by taking \( K = N/2 \) and \( \gamma = .9 \), the algorithms DDFG-IRLS and Algorithm 1 perform essentially the same in successfully solving the BP problem.

III. CONVERGENCE

We follow the proof strategy given in [9] for establishing the convergence and rate of convergence of Algorithm 1. Given \( \epsilon > 0 \), consider the smoothed \( \ell_1 \) objective

\[
J(x, \epsilon) := \sum_{i=1}^{n} \sqrt{x_i^2 + \epsilon^2}.
\]

Since \( \epsilon > 0 \), the function \( J(x, \epsilon) \) is strictly convex in \( x \). Hence, the minimizer in \( x \) over any convex set is unique if it exists. For each \( \epsilon \geq 0 \), set

\[
x^\epsilon = \arg \min_{x \in \Phi^{-1}(y)} J(x, \epsilon).
\]

The smoothing function \( J(x, \epsilon) \) is used to measure the progress of the iteratively re-weighted iterates. For this we require that \( \Phi \) satisfies the null space property NSP.

Assumption III.1. [8, Section 3] Null Space Property (NSP)

A matrix \( \Phi \in \mathbb{R}^{t \times N} \) satisfies NSP of order \( K \) for \( \gamma \in (0, 1) \) if and only if

\[
\|z_T\|_1 \leq \gamma \|z_T^*\|_1 \quad \forall z \in \text{Null}(\Phi) \tag{III.1}
\]

and for all index sets \( T \subset \{1, \ldots, N\} \) of cardinality not exceeding \( K \).

Observe that since (III.1) holds for all index sets \( T \subset \{1, \ldots, N\} \) of cardinality \( K \), we must have \( K \leq N/2 \). The null space property is intimately connected to the \( k \)-sparsity of solutions to the basis pursuit problem BP.

Lemma III.2 (NSP + K-sparsity imply uniqueness). [9, Lemma 4.3] Assume A2 holds and \( \Phi^{-1}(y) \) contains a \( K \)-sparse vector \( x^* \). Then \( x^* \) is the unique \( \ell_1 \)-minimizer in \( \Phi^{-1}(y) \) and for all \( v \in \Phi^{-1}(y) \),

\[
\|v - x^*\|_1 \leq \frac{2}{1 - \gamma} \sigma_L(v).
\]
We now show that the null space property guarantees the boundedness of any sequence generated by Algorithm 1.

**Lemma III.3** (Boundness of \( \{x^n\} \)). Let Assumption III.1 hold, and suppose \( \{x^n\} \) is a sequence generated by Algorithm 1. Then the sequence \( \{J(x^n, \epsilon_n)\} \) is non-increasing, \( \|x^n\|_1 \leq J(x^0, \epsilon_0) \), for all \( n \in \mathbb{N} \), and \( \sum_{i=1}^\infty \|x^{n+1} - x^n\|_w^2 < \infty \).

**Proof.** By concavity of the square root function \( \sqrt{b + \frac{1}{2\sqrt{b}}(a-b)} \geq \sqrt{a} \) for \( 0 \leq a, b \), and so
\[
J(x^{n+1}, \epsilon_n) - J(x^n, \epsilon_n) \leq \frac{1}{2} (\|x^{n+1}\|_w^2 - \|x^n\|_w^2) .
\] (III.2)

By completing the square and rearranging terms, we have
\[
\|x^{n+1}\|_w^2 - \|x^n\|_w^2 = -\|x^{n+1} - x^n\|_w^2 + 2\langle x^{n+1}, x^n - x^n \rangle_w .
\] (III.3)

Since \( x^{n+1} = \arg\min_{x \in \Phi^{-1}(y)} \|w\|_w \), we know
\[
\langle x^{n+1}, x^n - x^n \rangle_w = 0 .
\] (III.4)

By combining (III.2), (III.3) and (III.4) and using the fact that \( \{\epsilon_n\} \) is non-increasing, we have
\[
J(x^{n+1}, \epsilon_{n+1}) - J(x^n, \epsilon_n) \leq J(x^{n+1}, \epsilon_n) - J(x^n, \epsilon_n) \leq -\frac{1}{2} \|x^{n+1} - x^n\|_w^2 .
\]

Hence \( \|x^n\|_1 \leq J(x^n, \epsilon_n) \leq J(x^0, \epsilon_0) \). Moreover, by telescoping we know
\[
\sum_{n=1}^\infty \|x^{n+1} - x^n\|_w^2 \leq 2J(x^0, \epsilon_0) < \infty .
\]

Our convergence proof also relies on the following lemma.

**Lemma III.4.** [9, Lemma 4.2] Let Assumption III.1 hold. Then, for any \( z, z' \in \Phi^{-1}(y) \), we have
\[
\|z - z'\|_1 \leq 1 - \frac{\gamma}{1+\gamma} \left( \|z'\|_1 - \|z\|_1 + 2\sigma_K(z) \right) .
\] (III.5)

where \( \sigma_K \) is defined in (II.2).

The main convergence result makes use of the following notation: for \( S \subseteq [N] := \{1, 2, 3, ..., N\} \) and \( x \in \mathbb{R}^N \), define \( x_S \in \mathbb{R}^N \) componentwise by
\[
(x_S)_i = \begin{cases} x_i, & i \in S, \\ 0, & \text{otherwise}. \end{cases}
\]

**Theorem III.5** (Convergence of Algorithm 1). Let Assumption III.1 hold, and let \( y \in \mathbb{R}^m \) and \( x_0 \in \mathbb{R}^N \) be given. If \( \{x_k\} \) is generated by Algorithm 1 initialized at \( x_0 \), then there is an \( \bar{x} \in \mathbb{R}^N \) such that \( x_k \to \bar{x} \). Moreover, the following hold.

1. If \( \epsilon := \lim_{n \to \infty} \epsilon_n = 0 \), then \( \bar{x} \) is a \( K \)-sparse in which case \( \bar{x} \) is the unique \( \ell_1 \)-minimizer.
2. If there exists a \( K \)-sparse \( x^* \in \Phi^{-1}(y) \), then \( \bar{x} = x^* \) is the unique \( \ell_1 \)-minimizer and \( \lim_{n \to \infty} \epsilon_n = 0 \).

**Proof.** Part (1): The proof the part (1) is similar to the proof of [9, Theorem 5.3(i)]. First observe that \( \epsilon \) is well-defined since the sequence \( \{\epsilon_n\}_{n=1}^\infty \) is non-increasing. Moreover, by definition, \( \sigma_K(x) = 0 \) if and only if \( x \) is \( K \)-sparse. Consequently, if for any iteration \( n_0 \) we have \( \epsilon_{n_0+1} = 0 \), then Algorithm 1 terminates at \( x^{n_0} \) with \( x^{n_0} \) \( K \)-sparse, and so part (1) follows from Lemma III.2. Therefore, we assume that the algorithm does not terminate and \( 0 < \epsilon_n \to 0 \). In this case, there must be a subsequence \( N_i \subseteq \mathbb{N} \) such that \( \sigma_K(x^{n_i}) \to 0 \). Since Lemma III.3 tells us that the sequence \( \{x^n\} \) is bounded, there is a further subsequence \( N' \subseteq N_i \) and a point \( \bar{x} \in \Phi^{-1}(y) \) such that \( x^{n_i} \to \bar{x} \) with \( \sigma_K(\bar{x}) = 0 \). Hence, by Lemma III.2, \( \bar{x} \) is the unique \( K \)-sparse \( \ell_1 \)-minimizer.

Next let \( J' \subseteq \mathbb{N} \) be any subsequence. Again, by Lemma III.3, there is a further subsequence \( J'' \subseteq J' \) and a point \( x' \) such that \( x^n \to x' \). Let \( i \in N' \) and \( j \in J'' \) be such that \( i < j \). Then
\[
\|x^i - x^j\|_1 \leq \frac{1 - \gamma}{1+\gamma} \left( \|x^i\|_1 - \|x^j\|_1 + 2\sigma_K(x^i) \right) \quad \text{(by (III.5))}
\]
\[
\leq \frac{1 - \gamma}{1+\gamma} (J(x^i, \epsilon_j) - J(x^i, \epsilon_i) + N\epsilon_i + 2\sigma_K(x^i))
\]
\[
\leq \frac{1 - \gamma}{1+\gamma} (N\epsilon_i + 2\sigma_K(x^i)) .
\] (by Lemma III.3)

Consequently, \( \bar{x} = x' \). Hence the entire sequence \( \{x^n\} \) must converge to \( \bar{x} \) since every subsequence has a further subsequence convergent to \( \bar{x} \).

Part (2): We assume \( \epsilon = \inf_{n} \epsilon_n = \lim_{n \to \infty} \epsilon_n > 0 \) and establish a contradiction. By Lemma III.3, every subsequence \( N_k \subseteq \mathbb{N} \) has a further subsequence \( N'_k \subseteq N_k \) such that \( x^{n_k} \to \bar{x} \) for some \( \bar{x} \in \Phi^{-1}(y) \). For any \( x \in \Phi^{-1}(y) \) and \( i \in N' \), we have
\[
J(x, \epsilon_i) - J(x^i, \epsilon_i) \geq \langle x^i, x - x^i \rangle_w^i .
\] (III.6)
\[
= \langle x^{i+1}, x - x^i \rangle_w^i + \langle x^i, x - x^{i+1} \rangle_w^i
\]
\[
\geq \langle x^{i+1}, x - x^i \rangle_w^i - \|x^i - x^{i+1}\|_w^i \|x - x^i\|_w^i, \quad \text{(III.7)}
\]

where (III.6) follows from the convexity of \( \sqrt{b + \frac{1}{2\sqrt{b}}(a-b)} \) and (III.7) is the Cauchy-Schwarz inequality. Since \( x^{i+1} = \arg\min_{x \in \Phi^{-1}(y)} \|w\|_w \), we have \( \langle x^{i+1}, x - x^i \rangle_w^i = 0 \).

In addition, since \( \epsilon = \inf_{n} \epsilon_n \), we have \( \|x - x^i\|_w^i \leq \epsilon^{-1} \|x - x^i\|_1 \). By combining these two statements with (III.7), we obtain
\[
J(x, \epsilon_i) - J(x^i, \epsilon_i) \geq -\epsilon^{-1} \|x^i - x^{i+1}\|_w^i \|x - x^i\|_1 .
\]

Since, by Lemma III.3, \( \|x^i - x^{i+1}\|_w^i \to 0 \), we find that \( J(x, \epsilon) \geq J(\bar{x}, \epsilon) \). Consequently, \( \bar{x} = x^* \), that is, every subsequence of \( \{x^n\} \) has a further subsequence convergent to \( x^* \) which implies that the entire sequence converges to \( x^* \).

Now set \( T := \{i | x^i \neq 0, 1 \leq i \leq N\} \) so that \( |T| \leq K \), and observe that
\[
\|x^i\|_1 \leq J(x^i, \epsilon) \leq J(x^*, \epsilon) \leq \|x^*\|_1 + N\epsilon .
\] (III.8)
In addition, we have
\[
\|x_{T+1}^*\|_1 = \|x^*\|_1 - \|x_T^*\|_1 \\
\leq \|x^*\|_1 + N\epsilon - (\|x_T^*\|_1 - \|x_T - x_T^*\|_1) \quad \text{(III.9)}
\]
(by (III.8) and \(\Delta\) inequality)
\[
\leq N\epsilon + \|x_T^* - x_T\|_1 \\
(\text{since } \|x^*\|_1 = \|x_T^*\|_1) \\
\leq \gamma \|x_T^*\| + N\epsilon. \quad \text{(III.11)}
\]
(NSP)

Next observe that
\[
N\epsilon = \lim_{n \to \infty} N\epsilon_n \leq \lim_{n \to \infty} \eta(1 - \gamma)\sigma_K(x^n) \\
= \eta(1 - \gamma)\sigma_K(x^e) \leq \eta(1 - \gamma) \|x_T^*\|_1.
\]

Plugging this into (III.12) gives
\[
\|x_T^*\|_1 \leq \gamma \|x_T^*\| + \eta(1 - \gamma) \|x_T^*\|_1. \quad \text{(III.13)}
\]

If \(\|x_T^*\|_1 = 0\), then \(x^e = x^*\) and \(\sigma_K(x^e) = 0\). But then \(\lim_{n \to \infty} \sigma_K(x^n) = \sigma_K(x^e) = 0\) which implies that \(\epsilon_n \to 0\), a contradiction. Therefore, \(\|x_T^*\|_1 > 0\). Dividing (III.13) by \(\|x_T^*\|_1\) gives
\[
1 \leq \gamma + \eta(1 - \gamma) < \gamma + (1 - \gamma) = 1
\]
(a contradiction. Therefore, \(\gamma\) must equal zero which returns us to Part (1) and completes the proof.

We now establish the local linear convergence for Algorithm 1. Recall that a sequence \(\{z^k\} \subset \mathbb{R}^N\) converges locally linearly to \(z^* \in \mathbb{R}^N\) if there are constants \(\kappa \geq 0\) and \(\lambda \in (0, 1)\) and an iteration \(k_0 \in \mathbb{N}\) such that
\[
\|z^k - z^*\| \leq \kappa\lambda^{k - k_0}\|z^{k_0} - z^*\| \quad \forall k \geq k_0.
\]

In [9], the authors refer to linear convergence as exponential convergence.

**Theorem III.6** (The Local Linear Convergence of Algorithm 1). Let Assumption III.1 hold, and suppose that \(\Phi^{-1}(y)\) contains a \(K\)-sparse vector \(x^*\). Set \(T := \{i| x^*_i \neq 0, 1 \leq i \leq N\}\) and choose \(\rho \in (0, 1 - \gamma(1 + \eta(1 - \gamma)))\), where \(\gamma\) is given in A2 and \(\eta \in (0, 1)\) is initialized in Algorithm 1. Then there is a smallest \(n_0 \in \mathbb{N}\) such that
\[
\|x^{n_0} - x^*\|_{w^T} \leq \rho \min_{i \in T} \|x^*_i\|.
\]

Moreover, for all \(n \geq n_0\),
\[
\|x^{n+1} - x^*\|_{w^T} \leq \mu \|x^n - x^*\|_{w^T}, \quad \text{(III.15)}
\]
\[
\|x^n - x^*\|_1 \leq (1 + \gamma)\mu^{n-n_0} \|x^{n_0} - x^*\|_1, \quad \text{(III.16)}
\]
where \(\mu := \gamma(1 + \eta(1 - \gamma))\).

**Proof.** By Theorem III.5, \(x^n \to x^*\) so that for every \(\rho \in (0, 1 - \gamma(1 + \eta(1 - \gamma)))\), there is a smallest \(n_0 \in \mathbb{N}\) such that (III.14) holds. Consequently, \(n_0\) exists.

We follow the proof in [9, Theorem 6.1]. We prove (III.15) by induction. Let \(\tilde{n} \geq n_0\) be such that (III.14) holds with \(n_0\) replaced by \(\tilde{n}\). Since \(x^{\tilde{n}+1} = \arg\min_{x^* \in F^{-1}(y)} \|x^*\|_{w^\tilde{n}}\), the optimality conditions for this problem tell us that
\[
\langle x^{\tilde{n}+1}, x^{\tilde{n}+1} - x^*\rangle_{w^\tilde{n}} = 0.
\]

Consequently,
\[
\|x^{\tilde{n}+1} - x^*\|_{w^\tilde{n}}^2 = - \langle x^*, x^{\tilde{n}+1} - x^*\rangle_{w^\tilde{n}} \\
= - \langle (x^*)_{T^\tilde{n}}, x^{\tilde{n}+1} - x^*\rangle_{w^\tilde{n}} \\
\leq \sum_{i \in T^\tilde{n}} |x^*_i| |x^{\tilde{n}+1}_i - x^*_i| \\
\leq \sum_{i \in T} \sqrt{(x^*_i)^2 + \epsilon_i^2}.
\]

Note, for \(i \in T\), NSP tells us that
\[|x^*_i - x^*_i| \leq \|(x^* - x^*)_T\|_1 \leq \gamma \|(x^* - x^*)_T\| \leq \rho \min_{i \in T} \|x^*_i|,
\]
we have
\[
\frac{|x^*_i|}{\sqrt{(x^*_i)^2 + \epsilon_i^2}} \leq \frac{x^*_i}{|x^*_i|} \leq \frac{|x^*_i|}{|x^*_i| - |x^*_i|} \leq \frac{1}{1 - \rho}.
\]
Hence
\[
\|x^{\tilde{n}+1} - x^*\|_{w^\tilde{n}}^2 \leq \frac{1}{1 - \rho} \|\|(x^* - x^*)_T\|_1
\]
\[
\leq \frac{\gamma}{1 - \rho} \|\|(x^* - x^*)_T\|_1.
\]

Consequently, by Cauchy-Schwartz Inequality,
\[
\|x^{\tilde{n}+1} - x^*\|_{w^\tilde{n}}^2 = \left( \sum_{i \in T} \frac{|x^{\tilde{n}+1}_i - x^*_i|}{\sqrt{(x^*_i)^2 + \epsilon_i^2}} \right)^2 \\
\leq \left( \sum_{i \in T} \frac{|x^{\tilde{n}+1}_i - x^*_i|}{\sqrt{(x^*_i)^2 + \epsilon_i^2}} \right) \left( \sum_{i \in T} \sqrt{(x^*_i)^2 + \epsilon_i^2} \right) \\
\leq \gamma \|\|(x^* - x^*)_T\|_1
\]
\[
\leq \gamma \|\|(x^* - x^*)_T\|_1 + N\epsilon_n\].
\]

Therefore
\[
\|x^{\tilde{n}+1} - x^*\|_{w^\tilde{n}} \leq \frac{\gamma}{1 - \rho} \|\|(x^* - x^*)_T\|_1 + N\epsilon_n\] \\
\leq \frac{\gamma}{1 - \rho} \|\|(x^* - x^*)_T\|_1 + \eta(1 - \gamma)\sigma_K(x^e)\|
\]
\[(\text{Step 4 in Algorithm 1})
\]

Observe \(\sigma_K(x^e) \leq \|\|(x^e)_{T^\tilde{n}}\|_1 = \|\|(x^* - x^*)_T\|_1\|
\]
\[
\|x^{\tilde{n}+1} - x^*\|_{w^\tilde{n}} \leq \frac{\gamma(1 + \eta(1 - \gamma))}{1 - \rho} \|\|(x^* - x^*)_T\|_1 \]
\[
= \mu \|\|(x^* - x^*)_T\|_1. \quad \text{(III.17)}
\]

Since \(n_0\) satisfies (III.14), this shows that (III.15) is satisfied for \(\tilde{n} = n_0\).

Now assume (III.15) holds for \(\{n_0, n_0 + 1, \ldots, n - 1\}\). Then (III.15) tells us that
\[
\|x^n - x^*\|_{w^T} \leq \mu \|\|(x^{n-1} - x^*)_T\|_1 \]
\[
\leq \mu \|\|(x^{n_0} - x^*)_T\|_1 \]
\[
\leq \rho \min_{i \in T} \|x^*_i|,
\]
where the last inequality is by (III.14) and \(\mu < 1\). In particular, we have (III.14) with \(n_0\) replaced by \(n\), and so, by (III.17), (III.15) is satisfied at \(n\) which completes the induction.
Finally, the NSF for \( \Phi \) tells us that
\[
\|x^n - x^*\|_1 \leq (1 + \gamma) \|x^n - x^*\|_T \|_1
\leq (1 + \gamma) \mu^{n-n_0} \|x^{n_0} - x^*\|_T \|_1
\leq (1 + \gamma) \mu^{n-n_0} \|x^{n_0} - x^*\|_1.
\]

IV. Failure of DDFG-IRLS

We construct an example where the DDFG-IRLS algorithm provably fails for \( K = 2\gamma/(1-\gamma) \leq \gamma \leq K \). However, we emphasize that, in general, the failure of this inequality does not imply the failure of the DDFG-IRLS algorithm.

The example is formulated in the context of the \( \ell_1 \) regression problem \( \ell_1 \mathbf{R} \) discussed in the introduction. It is well-known that \( \text{BP} \) is equivalent to this \( \ell_1 \) regression problem under the correspondences \( \text{rge}(A) = \text{Null}(\Phi) \) and \( \Phi b = -y \) [6]. In addition, under these correspondences, the NSF for \( \Phi \) of order \( K \) for \( \gamma \in (0,1) \) is equivalent to the following condition on the matrix \( A \):
\[
\|(Az)_T\|_1 \leq \gamma \|Az\|_T \|_1 \quad \text{for all } z \text{ and } |T| \leq K. \quad (\text{IV.1})
\]

In terms of the DDFG-IRLS algorithm, when the matrix \( A \) has full column rank, then there is a 1-1 correspondence between the iterates of this algorithm and a corresponding IRLS algorithm for solving the \( \ell_1 \mathbf{R} \). If we denote the \( i \)-th row of \( A \) by \( a_i \), for given \( \epsilon_0 \) and \( x^0 \), this correspondence is given by
\[
x^n = Ax^n - b \quad \forall n = 0, 1, \ldots,
\]
where, for \( n = 0, 1, \ldots, \)

**DDFG-IRLS**
\[
x^{n+1} \leftarrow \min_{x \in \mathbb{R}^k} \sum_{i=1}^{N} \frac{x_i^2}{(a_i^T z_i - b_i)^2}
\]
and
\[
\epsilon_{n+1} \leftarrow \min_{z} \sum_{i=1}^{N} \frac{(a_i^T z_i - b_i)^2}{(a_i^T z_i - b_i)^2 + \epsilon_n}.
\]

**\( \ell_1 \mathbf{R} \)-IRLS**
\[
z^{n+1} \leftarrow \min_{z} \sum_{i=1}^{N} \frac{(a_i^T z_i - b_i)^2}{(a_i^T z_i - b_i)^2 + \epsilon_n}
\]
and
\[
\epsilon_{n+1} \leftarrow \min_{z} \sum_{i=1}^{N} \frac{(a_i^T z_i - b_i)^2}{(a_i^T z_i - b_i)^2 + \epsilon_n}.
\]

Therefore, by Lemma III.2, whenever \( \Phi \) satisfies the NSF of order \( K \) for \( \gamma \), or equivalently, \( A \) satisfies (IV.1), if there exists \( z^* \) for which \( Az^* - b \) is \( K \)-sparse, then \( x^* := Az^* - b \) is the unique solution to \( \text{BP} \). If, in addition, \( A \) has full column rank, then \( z^* \) is the unique solution to \( \ell_1 \mathbf{R} \).

We now construct our example. Given \( k \geq 1 \), set \( \tilde{A} := (I_k, \ldots, I_k)^T \in \mathbb{R}^{(2k^2+k) \times k} \) with \( 2k + 1 \) blocks of the identity \( k \times k \) matrix \( I_k \). For any \( a \in \mathbb{R}^k \) and any \( T \subseteq [2k^2 + k] \) with \( |T| = k \), let \( i_0 \in \{ i \mid |z_i| \geq |z_j|, \forall 1 \leq j \leq k \} \). Then
\[
\left\| (\tilde{A}z)_T \right\|_{1|z_{i_0}|} \leq \frac{k}{k+1} \left| z_{i_0} \right| \leq \left\| (\tilde{A}z)_T \right\|_{1}
\]
and
\[
\left\| (\tilde{A}z)_T \right\|_{1|z_{i_0}|} \leq \frac{k}{k+1} \left| z_{i_0} \right| \leq \left\| (\tilde{A}z)_T \right\|_{1}.
\]

Thus, for \( K = k \), \( \tilde{A} \) satisfies (IV.1) with \( \gamma = \frac{k}{k+1} \) and this value for \( \gamma \) is sharp. We now modify \( \tilde{A} \) to obtain an \( A \), whose \( \gamma \) is any element of \((\frac{k}{k+1}, 1)\). To this end, let \( \gamma \in (\frac{k}{k+1}, 1) \) and define \( A_{\gamma} \in \mathbb{R}^{(2k^2+k) \times k} \) so that \( A_{\gamma}(ik + 1, 1) := \frac{k+1}{k} \gamma \)
for all \( 0 \leq i \leq k-1 \), while all other components of \( A_{\gamma} \) coincide with those of \( \tilde{A} \). That is, we only replace the \((1,1)\) entry in each of the first \( k \) identity matrices \( I_k \) of \( \tilde{A} \) by \( \frac{k+1}{k} \gamma \).

By applying the same argument to \( A_{\gamma} \) as above for \( A \), we find that \( A_{\gamma} \) satisfies (IV.1) with \( K = k \) and \( \gamma = \frac{k}{k+1} \), and this \( \gamma \) is also sharp.

Next choose \( z^* \in \mathbb{R}^k \). Given \( \delta \in \mathbb{R} \), set \( b := A_{\gamma} z^* + \delta e \), where \( e := \sum_{j=0}^{k-1} e_{(j+1)} \) with each \( e_{(j+1)} \) the \((j+1)\)th standard unit coordinate vector. Observe that \( x^* := A_{\gamma} z^* - b \) is \( k \)-sparse and \( A_{\gamma} \) has full column rank. Hence, by our previous discussion, Lemma III.2 implies that \( z^* \) is the unique solution to \( \ell_1 \mathbf{R} \) for this choice of \( A \) and \( b \).

Our goal is to show that there is an initialization for the \( \ell_1 \mathbf{R} \)-IRLS algorithm in (IV.2) such that the generated sequence \( \{z^n\} \) satisfies \( z^n \rightarrow z^* \), and hence, the corresponding DDFG-IRLS iterates \( x^n := A_{\gamma} z^n - b \) do not converge to the unique solution \( x^* := A_{\gamma} z^* - b \) to \( \text{BP} \).

**Theorem IV.1.** Let \( z^* \in \mathbb{R}^k, \delta \in (0,k(2k+1)], \text{ and } \gamma \in [\nu,1] \), where
\[
\nu := \sqrt{\frac{1 + 4k^2(2k+1)^2}{1 + 4k^2(2k+1)^2}} = \frac{4k^2(2k+1)^2 + 1}{4k^2(2k+1)^2 + 4}.
\]
Then \( \gamma/(k(2k+1)\sqrt{\xi^2 - 1}) > 1 \) and \( \gamma/(k(2k+1)\sqrt{\xi^2 - 1}) > 1 \).

**Initialization** \( \epsilon_0 := 1 \) and \( z^0 \in \mathbb{R}^k \) componentwise by
\[
z^0 \in \left\{ \begin{array}{l}
z_1^+ + \frac{\delta}{\alpha + \gamma/(k(2k+1)\sqrt{\xi^2 - 1})}, \quad z_1^+ + \frac{\delta}{\alpha + 1} \\
\end{array} \right. \quad \text{and}
\]
and
\[
z_0^0 := z_1^+, \quad i = 2, \ldots, k.
\]
If \( \{z^n\} \) is the sequence generated by the \( \ell_1 \mathbf{R} \)-IRLS algorithm in (IV.2) with this initialization, then \( z^n \rightarrow z^* \).

**Proof.** We first prove the inequalities in (IV.3). The first inequality follows since
\[
\alpha > 1 \implies \nu < \left( \frac{k}{k+1} \right)^2 \quad \implies (k+1)^2 \left( 1 + \frac{1}{4k^2(2k+1)^2} \right) > k^2 \left( 1 + \frac{1}{k^2(2k+1)^2} \right) \quad \implies 2k + 1 + \frac{(k+1)^2}{4k^2(2k+1)^2} > \frac{1}{(2k+1)^2}.
\]
The second inequality in (IV.3) follows directly from the fact that \( \gamma \geq \nu \). The third inequality in (IV.3) follows since
\[
\gamma/(k(2k+1)\sqrt{\xi^2 - 1}) > 1 \implies \xi^2 < 1 + \frac{\gamma^2}{k^2(2k+1)^2} \implies \gamma^2 < 1.
\]
Note that the third inequality in (IV.3) implies that
\[ \delta(\alpha + \gamma / (k(2k+1) + \sqrt{\xi^2 - 1}))^{-1} \leq \delta(\alpha + 1)^{-1} \]
so that \( x_1^n \) is well defined.

We establish the result by showing that \( z_1^n \rightarrow z_1^* \). Observe that
\[
\begin{align*}
b_k z_{k+1} + 1 &= (A_z z)^{k+1} = z_1^* + 1 \\
b_1 &= \alpha z_1^* + \delta = \alpha b_{k+1} + \delta. 
\end{align*}
\]
By the \( l_r \)-R-IRLS algorithm, \( z_{n+1} \) solves the least-squares problem
\[
\min_z \frac{k(z_1^n - b_1)^2}{(b_1 - \alpha z_1^n)^2 + \epsilon^2_n} + \frac{(k + 1)(z_1 - b_{k+1})^2}{(z_1^n - b_{k+1})^2 + \epsilon^2_n} + (2k + 1) \sum_{i=2}^{k} \frac{(z_i - b_i)^2}{(z_i^n - b_i)^2 + \epsilon^2_n}.
\]
Due to the separability of the objective in the variables \( z_i \), \( i = 2, \ldots, k \), we have \( z_i^n = b_i = z_1^* \), \( i = 2, \ldots, k \), for \( n \geq 1 \). The optimality conditions for each subproblem tells us that
\[
z_{1}^{n+1} = \frac{\alpha b_k}{(b_1 - \alpha z_1^n)^2 + \epsilon^2_n} + \frac{(k + 1)b_{k+1}}{(z_1^n - b_{k+1})^2 + \epsilon^2_n} + (2k + 1) \sum_{i=2}^{k} \frac{b_i}{(z_i^n - b_i)^2 + \epsilon^2_n}.
\]
By (IV.6), we have
\[
z_{1}^{n+1} - b_{k+1} = \frac{\alpha b_k}{(b_1 - \alpha z_1^n)^2 + \epsilon^2_n} + \frac{(k + 1)b_{k+1}}{(z_1^n - b_{k+1})^2 + \epsilon^2_n} + (2k + 1) \sum_{i=2}^{k} \frac{b_i}{(z_i^n - b_i)^2 + \epsilon^2_n} \geq 0.
\]
And
\[
b_1 - \alpha z_{1}^{n+1} = \frac{(k + 1)(b_1 - \alpha b_{k+1})}{(z_1^n - b_{k+1})^2 + \epsilon^2_n} + (k + 1) \sum_{i=2}^{k} \frac{b_i}{(z_i^n - b_i)^2 + \epsilon^2_n} \geq 0.
\]
Hence,
\[
z_{1}^{n+1} - b_{k+1} \geq 0, \ b_1 - \alpha z_{1}^{n+1} \geq 0, \text{ and }
\]
\[
s_{n+1} = \gamma \sqrt{(z_1^n - b_{k+1})^2 + \epsilon^2_n} \geq \epsilon_n, \quad \forall n \geq 0,
\]
where \( s_{n+1} := (z_{1}^{n+1} - b_{k+1}) / (b_1 - \alpha z_{1}^{n+1}) \).
If we let \( \epsilon_n := z_{1}^{n} - b_{k+1} \), then \( s_n = \epsilon_n / (\delta - \alpha \epsilon_0) \) by (IV.5). For \( n = 0 \), (IV.4) tells us that
\[
s_0 = \frac{\epsilon_0}{\delta - \alpha \epsilon_0} = \frac{1}{\gamma} \left( \frac{\epsilon_0}{(k(2k+1) + \sqrt{\xi^2 - 1})} \right),
\]
We now show by induction that
\[
s_n > k(2k+1) \sqrt{\xi^2 - 1} \quad \text{and} \quad \epsilon_n = \frac{\epsilon_n}{(k(2k+1))} \quad \forall n \geq 1.
\]
First consider \( n = 1 \). Since \( \epsilon_0 = 1 \), the definition of \( \epsilon_0 \) and \( s_0 \) in conjunction with (IV.5) and (IV.9) tell us that
\[
s_1 = \gamma \sqrt{\frac{\epsilon_0^2 + 1}{(\delta - \alpha \epsilon_0)^2 + 1}} \leq 1.
\]
Observe that
\[
(A_z z^n - b_i) = \begin{cases} 
\alpha z_1^n - b_1, & \text{if } i \in \{jk+1 | j \in \{0, \ldots, k-1\}\}, \\
\alpha z_1^n - b_{k+1}, & \text{if } i \in \{jk+1 | j \in \{k, \ldots, 2k\}\}, \\
0, & \text{otherwise}.
\end{cases}
\]
Hence, since \( (z_1^n - b_{k+1}) / (b_1 - \alpha z_1^n) = s_1 < 1 \), the \((k + 1)\)th largest magnitude of the entries of \( A_z z_1 - b \) is \( |z_1^n - b_{k+1}| \) with \( |z_1^n - b_{k+1}| = |z_1^n - b_{k+1}| \) by (IV.9). Thus \( \epsilon_1 = \min \left\{ \epsilon_0, s_1 / k(2k+1) \right\} \). The given definitions and the inequality \( s_1 < 1 \), yield
\[
z_1^n - b_{k+1} = \epsilon_1 = \frac{\delta s_1}{\alpha s_1 + 1} = \frac{\delta}{\alpha + 1} \leq \frac{\delta}{\alpha + 1} \leq k(2k+1).
\]
Therefore, \( \epsilon_1 = \frac{\epsilon_1}{k(2k+1)} \), since \( \epsilon_0 = 1 \), which proves the second part of (IV.11) for \( n = 1 \). To obtain the first part of (IV.11) for \( n = 1 \), observe that
\[
s_1^2 = k(2k+1) \sqrt{\xi^2 - 1}.
\]
Assume \( s_n > k(2k+1) \sqrt{\xi^2 - 1} \) and \( \epsilon_n = \frac{\epsilon_n}{k(2k+1)} \). Plugging \( \epsilon_n = \frac{\epsilon_n}{k(2k+1)} \) into (IV.9) gives
\[
s_{n+1} = \gamma \sqrt{1 + \frac{(x^n - k(2k+1) \xi^2 + \epsilon_0^2)}{(k(2k+1))(k(2k+1))} + \frac{\epsilon_0^2}{k(2k+1)}} \geq \frac{\epsilon_0^2}{k(2k+1)}.
\]
Since the function \( f(x) := \frac{1}{\sqrt{1+\sqrt{\xi^2(n+1)}}} \) is increasing on \((0, \infty)\), we know
\[
s_{n+1} = \frac{x f(s_n)}{f(k(2k+1) \sqrt{\xi^2 - 1})} = k(2k+1) \sqrt{\xi^2 - 1},
\]
which established the first part of (IV.11) for \( n + 1 \). To establish the second part, observe that (IV.14) and the induction hypothesis gives

\[
s_{n+1} = \xi \frac{s_n}{\sqrt{1 + s_n^2(k^2(2k+1)^2)^{-1}}} \leq \xi \frac{s_n}{\sqrt{1 + \xi^2 - 1}} = s_n,
\]

Thus far, we have shown that \( s_{n+1} \geq k(2k+1)\sqrt{\xi^2 - 1} \) and \( s_{n+1} \leq s_n \). By combining these inequalities with the fact that \( s_n = \frac{\epsilon_n}{\epsilon_n - \alpha s_n} \) for each \( n \geq 1 \), we have \( \epsilon_{n+1} \leq \epsilon_n \) for all \( n \geq 1 \). Therefore, by the induction hypothesis, \( \epsilon_{n+1} = \min\{\epsilon_n, \frac{\epsilon_{n+1}}{\xi(2k+1)}\} = \min\{\epsilon_n, \frac{\epsilon_{n+1}}{\xi(2k+1)}\} = \frac{\epsilon_{n+1}}{\xi(2k+1)} \). This concludes the proof of (IV.11).

Observe that our induction proof also shows that \( \{s_n\} \) is a non-increasing sequence bounded below by \( k(2k+1)\sqrt{\xi^2 - 1} \).

Consequently, \( z_1^n - z_1^* = z_1^n - b_{k+1}^* = \epsilon_n = (\delta s_n)/(1 + \alpha s_n) \)

\[ \rightarrow (\delta s^*)/(1 + \alpha s^*) = \frac{k(2k+1)\sqrt{\xi^2 - 1}}{1 + \alpha k(2k+1)\sqrt{\xi^2 - 1}} > 0. \]

Consequently, \( z_1^n \not\to z_1^* \), and we have arrived at the desired result.

\[ \square \]

V. NUMERICAL EXAMPLES

A. Failure of the DDFG-IRLS Algorithm

We present three numerical experiments illustrating the failure of the DDFG-IRLS algorithm for small perturbations of the example given in Theorem IV.1. Experiment 1 (see Figure 1) simply illustrates the content of Theorem IV.1 for \( k = 5, \gamma = \sqrt{(4k^2(2k+1)^2 + 1)/(4k^2(2k+1)^2 + 4)} = 0.999876, \delta = k(2k+1) = 55 \). The true solution of problem \( \ell_1 \mathbf{R} \), \( z^* \), is sampled from \( N(0, I_k) \). In both algorithms, \( x^0 = A\gamma z^0 - b \) where \( z^0 \) satisfies (IV.4), i.e., \( z_0^0 = z_1^* + (\delta/(\alpha + \gamma/(k(2k+1)\sqrt{\xi^2 - 1})) + \delta/(\alpha + 1))/2 \). For Algorithm 1, \( \eta = 0.9 \).

In experiment 2 (see Figure 2), we examine the sensitivity of the success/failure of the DDFG-IRLS algorithm to the selection of the parameter \( \gamma \) near the critical value \( \gamma_0 := \sqrt{(4k^2(2k+1)^2 + 1)/(4k^2(2k+1)^2 + 4)} \approx 1 - 10^{-3.9} \).

Again, we let \( k = 5 \). To illustrate the effect of the selection of \( \gamma \), we run the DDFG-IRLS algorithm for \( \gamma \in \{1 - 10^{-1}, 1 - 10^{-2}, 1 - 10^{-3}, 1 - 10^{-3.3}, 1 - 10^{-3.6}, 1 - 10^{-4}, 1 - 10^{-5}\} \). Here, 20 instances of the random variable \( N(0, 100\cdot I_5) \) are chosen for the starting point \( z_0 \). All other parameters are the same as those of experiment 1. The iterations are terminated when either \( \|x^n - x^*\|_2 \leq 10^{-3} \) or the number of iterations exceeds \( 10^5 \). In the range \( 1 - 10^{-3.6} \leq \gamma < \gamma_0 \), all the experiments fail to achieve the termination criteria \( \|x^n - x^*\|_2 \leq 10^{-3} \). This illustrates the extremely slow rate of convergence of the DDFG-IRLS algorithm when the critical value \( \gamma_0 \) is approached from below.

In experiment 3 (see Figure 2), we examine the robustness of the success/failure of the DDFG-IRLS algorithm for small perturbations of the example given in Theorem IV.1 obtained by perturbing the matrix \( A_\gamma \). Again, we let \( k = 5 \) and \( \delta = k(2k+1) = 55 \) and use DDFG-IRLS to solve perturbed versions of our basic example with \( A_{\gamma, \sigma} = A_\gamma + \sigma \mathbf{R} \), where \( \mathbf{R} \in \mathbb{R}^{(2k+1)\times k} \) is a random matrix with i.i.d. \( N(0, 1) \).
entries and \(b_\sigma := A_\gamma \sigma z^* + \delta \tilde{e}\), where \(\tilde{e} := \sum_{j=0}^{k-1} e_{(jk+1)}\) with each \(e_{(jk+1)}\) the \((jk+1)\)th standard unit coordinate vector. As in experiment 2, the entries of vector \(z^*\) are realizations of i.i.d. \(N(0, 1)\) random variables. For each \(\sigma \in [10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}]\), construct 50 problems with the entries of \(R\) i.i.d. \(N(0, 1)\). The DDFG-IRLS algorithm is run on all 50 problems with each run of the algorithm initialized at a \(z^0\) with components selected i.i.d. \(N(0, 100)\). The algorithm is terminated when either \(\|x_k - x^*\|_2 < 10^{-3}\) or the number of iterations exceed \(10^5\). The results are presented on the right hand side of Figure 2. Each point with coordinates \((x, y)\) represents the experiment with \(\sigma = 10^{-3}\) terminated after \(x\) iterations. When \(\sigma = 10^{-4}\), the DDFG-IRLS algorithm fails to recover the true \(x^*\) within \(10^5\) steps for all the 50 problems. In other words, the failure of DDFG-IRLS is robust to a small random normal perturbation of matrix \(A_\gamma\) and when it does succeed for slightly large perturbations of \(A_\gamma\) the convergence is still quite slow.

**B. Comparison of DDFG-IRLS and Algorithm 1**

In practice the DDFG-IRLS algorithm and Algorithm 1 have nearly identical performance on randomly generated problems. We illustrate this with two additional numerical experiments.

In experiment 4, the entries of \(\Phi \in \mathbb{R}^{300 \times 500}\) are chosen to be i.i.d. \(N(0, 1)\) with the solution \(x_\gamma \in \mathbb{R}^{500}\) chosen so that the first 100 entries are independent samples from \(N(0, 1)\) and the remaining components are taken to be 0. Set \(y := \Phi x_\gamma\). In practice, the NSP parameters \(K\) and \(\gamma\) are not known even though they appear explicitly in the updating policy for the smoothing parameter \(\epsilon_k\). All that is known is that if the NSP holds, then \(K < N/2\) and \(\gamma \in (0, 1)\). In this regard, it may be that the DDFG-IRLS algorithm has an edge over Algorithm 1 since the performance of Algorithm 1 may be sensitive to the choice of \(\gamma\). Consequently, in this experiment, we examine the robustness of the performance of both algorithms to an ad hoc choice of the NSP parameters \(K\) and \(\gamma\). For each \(K \in \{99, 100, 150, 200, 250, 300\}\) and \(\gamma \in \{0.1, 0.5, 0.9\}\), we run Algorithm 1 one hundred times with a random initialization \(x_0 \sim N(0, 100 \cdot I_5)\) on each run. For each of these values of \(K\), we also run the DDFG-IRLS algorithm for 100 times with same random initializations \(x_0 \sim N(0, 100 \cdot I_5)\). The results are presented in Figure 3. The plot tells us that the success of both algorithms is robust with respect to the choice of \(K\). When \(K\) is strictly smaller than the true number of the nonzero entries in the solution, both algorithms fail regardless of the choice of \(\gamma\). On the other hand, if we take \(K = 250 = N/2\) or \(K = 300\), both algorithms succeed. In addition, the two algorithms have nearly identical performance regardless of the choice of \(K\) when \(\gamma\) is chosen to be 0.9. Overall, a degradation in the performance of Algorithm 1 for the smaller values of \(\gamma\) only occurs when \(K\) is poorly chosen. In practice, we recommend choosing \(K\) be a half of the columns of the measurement matrix \(\Phi\) and set \(\gamma \approx 0.9\). In this case, our experiment indicates that the performance of the two algorithms is essentially identical.
1 in solving problems with randomly generated data. In this experiment we use the fixed parameter setting \((K, \gamma, \eta) = (N/2, 0.9, 0.9)\) with \((N, m) = (500, 300)\). In all of these experiments, the entries of \(\Phi\) are independent samples from \(N(0, 1)\). In all experiments, the first \(k\) entries of \(x^\star\) are i.i.d. sampled from \(N(0, 100)\) with remaining entries set to zero. In Figure 4, \(k = 100\), \(k = 120\) in Figure 5, and \(k = 50\) in Figure 6. The experiment is repeated 50 times for each algorithm. The top panel of Figure 4 shows percentage of problems solved versus the number of iterations, with an iteration max of 12. The bottom panel of Figure 2 shows the average error \((1/50) \sum_{i=1}^{50} \text{error}_i^k\) where \(\text{error}_i^k\) is the value of \(\|x^k - x^\star\|\) in the \(i\)th trial. Figures 5 and 6 show the percentage of problems solved versus the number of iterations for their respective \(k\) values.

Fig. 4. Recovery rate (top panel) and average distance (bottom panel) vs. iterations for \(k = 100\). The bottom figure illustrates the linear rate of convergence of the algorithms.

Fig. 5. Recovery rate vs. iterations for \(k = 50\).

Fig. 6. Recovery rate vs. iterations for \(k = 120\).

The DDFG-IRLS and Algorithm 1 perform essentially the same in these random experiments. The number of iterations required depends on the sparsity of the solution with the iteration count decreasing with the sparsity \(k\). This indicates that these algorithms are most useful when the underlying solution is sparse. Finally, the bottom panel of Figure 4 demonstrates the linear rate of convergence of these methods.

VI. DISCUSSION

In this contribution we provide a concrete example where the DDFG-IRLS fails when \(k = K\), and provide a remedy by changing the updating strategy for the smoothing parameter \(\epsilon_n\). This remedy increases the range of values for both \(k\) and \(\gamma\) for which the algorithm provably converges with a local linear rate to the largest possible intervals \([1, K]\) and \((0, 1)\) for \(k\) and \(\gamma\), respectively. We have also shown through our numerical experiments that on randomly generated problems both algorithms are robust to the choice of \(K\) and \(\gamma\) and that their performance is essentially identical. Therefore, if one is concerned about the possible failure DDFG-IRLS,
then Algorithm 1 should be considered with recommended parameter choices \((K, \eta, \gamma) = (N/2, 0.9, 0.9)\), or equivalently, \(0.05 \leq \eta(1 - \gamma) \leq 0.09\) since knowledge of the product \(\eta(1 - \gamma)\) is all that is required for implementation.

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