CONVERGENCE ANALYSIS OF NUMERICAL SCHEMES FOR NON-LINEAR VARIATIONAL INEQUALITIES, APPLICATION TO THE SEEPAGE PROBLEM

YAHYA ALNASHRI AND JÉRÔME DRONIOU

Abstract. Using the gradient discretisation method (GDM), we provide a complete and unified numerical analysis for non-linear variational inequalities (VIs) based on Leray–Lions operators and subject to non-homogeneous Dirichlet and Signorini boundary conditions. This analysis is proved to be easily extended to the obstacle and Bulkley models, which can be formulated as non-linear VIs. It also enables us to establish convergence results for many conforming and nonconforming numerical schemes included in the GDM, and not previously studied for these models. Our theoretical results are applied to the hybrid mimetic mixed method (HMM), a family of schemes that fit into the GDM. Numerical results are provided for HMM on the seepage model, and demonstrate that, even on distorted meshes, this method provides accurate results.

1. Introduction

Non-linear variational inequalities are related to a wide range of applications. In particular, unconfined seepage models, free boundary problems involving Signorini boundary conditions, can be used to study the construction of earth dams, embankments and hydraulic design. With non-linear variational inequalities, one can also study the Bulkley fluid model, which is applicable to different phenomena and processes, such as blood flow [48], food processing [30] and Bingham fluid flows [43].

We consider here variational inequalities (VIs) related to elliptic equations of the type

\begin{align}
-\text{div } a(x, u, \nabla u) &= f \quad \text{in } \Omega, \\
   u &= g \quad \text{on } \partial \Omega,
\end{align}

where \( \Omega \) is an open bounded connected subset of \( \mathbb{R}^d \), \( d \geq 1 \), with boundary \( \partial \Omega \).

Precise assumptions on data will be stated in the next sections. The purpose of this paper is to provide a complete and unified convergence analysis of numerical schemes for VIs based on (1.1). Our convergence result applies to a wide range of methods, such as finite elements methods (conforming and non-conforming), finite volume methods, mimetic finite difference schemes, etc. To our knowledge, this
result is the first one for non-conforming methods applied to non-linear variational inequalities.

The theory on PDEs of the kind (1.1) has been covered in several works, see [15, 29, 40, 50] and references therein. A number of numerical analyses on these models has also been carried out, starting from the approximation of the p-Laplace equation, with proven rates of convergences, by $P_1$ finite elements in [6]. Subsequent works consider more general Leray–Lions models, possibly transient, and establish either error estimates (under regularity assumptions on the solution to the PDE), or prove the convergence towards a solution with minimal regularity. We refer the reader to [3, 4, 13–15, 22, 33, 44] for a few examples. Several algorithms can be used to compute the solution to the corresponding non-linear numerical schemes, from basic fixed-point iterations (which corresponds to the Kačanov method [40]) to Newton methods, to multigrid techniques [7], to augmented Lagrangian algorithms [38].

The mathematical theory of variational inequalities based on equations of the kind (1.1) is well understood, see e.g. [10, 41, 42, 49]. We note that [49] considers an obstacle problem with measure source terms rather than $W^{1,p}(Ω)$ source terms (the theory for the corresponding PDEs is developed in [8]). [37] studies non-linear quasi-variational inequalities and proposes a semi-smooth Newton iteration to obtain a solution.

The numerical approximation of variational inequalities based on linear operators, including the issues faced with numerical approximations of the convex set described by the obstacle, has been covered in a number of works – see, e.g., [32] and references therein. Some works tackle the question of the numerical approximation of VIs based on non-linear equations such as (1.1). Under strong monotonicity assumptions on the operator, [47] develops a convergence analysis of conforming numerical schemes for non-linear VIs. [31] develops the analysis of conforming finite elements method for VIs involving a non-linear proper function. In [39, 46], $P_1$ finite elements are applied to the obstacle problem for a p-Laplace-like operator, with homogeneous Dirichlet boundary conditions and zero barrier inside the domain; an a priori error estimate is obtained under $W^{2,p}$ regularity on the solution. See also [45] for non-linear parabolic variational inequalities. The Bulkley model also has been approximated by $P_1$ finite elements [11, 12] and the Lagrange methods [11]. In [51], a seepage model is approximated by a finite elements method, but no convergence analysis is carried out. The authors utilise a fixed point method (Kačanov) to treat the non-linearity and compute the solution to the scheme.

All these studies of numerical schemes for non-linear VIs deal with conforming numerical schemes, mostly $P_1$ finite elements. It seems that a lot of work remains to be done, starting from convergences analyses and tests on other kinds of schemes than conforming finite element (FE) schemes (e.g., non-conforming FE, finite volume, mimetic finite differences, etc.). Our work aims at filling this gap. We provide a complete convergence analysis of numerical schemes for variational inequalities based on non-linear Leray–Lions operators, and we present numerical results using the hybrid mimetic mixed method. This method, contrary to FE methods, is applicable on grids with very general cell geometries as encountered in some porous flow applications.

Instead of conducting individual studies for each numerical scheme, we develop a unified convergence analysis that is readily applicable to a several methods. This
is done by adapting the gradient discretisation method (GDM) to non-linear VIs. The GDM is a framework for the analysis of numerical schemes for diffusion PDEs. It covers a variety of methods, such as conforming, non-conforming and mixed finite elements methods (including the non-conforming “Crouzeix–Raviart” method and the Raviart–Thomas method), hybrid mimetic mixed methods (which contain hybrid mimetic finite differences, hybrid finite volumes/SUSHI scheme and mixed finite volumes), nodal mimetic finite differences, and finite volumes methods (such as some multi-points flux approximation and discrete duality finite volume methods). The original GDM identifies a small number of properties required to establish the convergence of numerical schemes for various models based on elliptic and parabolic PDEs: linear and non-linear diffusion, stationary and transient Leray–Lions equations, the Stefan model of melting material, the Richards model of water flow in an unsaturated porous medium, diphasic flows, etc. The GDM is also adapted to various boundary conditions. For more details, we refer the reader to the monograph [20] and to the papers [19, 22, 23, 25, 27, 28].

In this work, we adapt the gradient discretisation method to three non-linear variational inequalities involving Leray–Lions operators. We show that the GDM provides a unified convergence analysis of numerical methods for these models. This analysis yields convergence theorems of numerical schemes for meaningful models of VIs, including the non-linear seepage problems and the Bulkley model. To illustrate our theoretical results, we apply the hybrid mimetic mixed (HMM) method to the seepage problem, and show that – even on distorted meshes – its efficiency is comparable to the $P_1$ finite elements of [51]. One of its additional strengths, however, is that it is applicable on very generic meshes, contrary to the $P_1$ finite element method. As proved in [21], the HMM method contains the hybrid finite volume method of [26], the hybrid mixed mimetic finite difference method of [9] and the mixed finite volume method of [18].

This paper is organised as follows. Section 2 details the non-linear Signorini problem, its approximation by the gradient discretisation method, and the corresponding convergence results. Section 3 shows that the GDM can successfully be adapted to the obstacle problem and the Bulkley fluid model. A short section, Section 4, describes the case where the barriers of the Signorini and obstacle problems are approximated as part of the discretisation process. In Section 5 we show that our result apply to the HMM scheme, and establish its convergence for all three models. Section 6 presents numerical tests that demonstrate the efficiency of the HMM method for solving the seepage model on various meshes, including very distorted ones. An appendix, Section 7, presents an interpolation operator useful for the HMM method (and, more generally, for methods based on cell and face unknowns).
2. Non-linear Signorini problem

2.1. Continuous problem. We first consider the following non-linear Signorini problem:

\[-\text{div} a(x, \bar{u}, \nabla \bar{u}) = f \quad \text{in } \Omega, \quad (2.1)\]
\[\bar{u} = g \quad \text{on } \Gamma_1, \quad (2.2)\]
\[a(x, \bar{u}, \nabla \bar{u}) \cdot n = 0 \quad \text{on } \Gamma_2, \quad (2.3)\]
\[(a - \bar{u})a(x, \bar{u}, \nabla \bar{u}) \cdot n = 0 \quad \text{on } \Gamma_3, \quad (2.4)\]

Here \(n\) denotes the unit outer normal to the boundary \(\partial \Omega\), which is split in three parts \((\Gamma_1, \Gamma_2, \Gamma_3)\). The assumptions on the Leray–Lions operator \(a\) are standard:

\(a : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d\) is a Caratheodory function, \(a\) is measurable, \(a\) is continuous and, for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^d\), \(x \to a(x, s, \xi)\) is measurable and, for some \(p \in (1, \infty)\) and \(p' = \frac{p}{p-1}\),

\[\exists \pi \in L^{p'}(\Omega), \exists \mu > 0 : \quad |a(x, s, \xi)| \leq \pi(x) + \mu |\xi|^{p-1}, \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^d, \quad (2.6)\]
\[\exists a > 0 : \quad a(x, s, \xi) \cdot \xi \geq a |\xi|^p, \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^d, \quad (2.7)\]
\[(a(x, s, \xi) - a(x, s, \chi)) \cdot (\xi - \chi) \geq 0 \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \forall \xi, \chi \in \mathbb{R}^d. \quad (2.8)\]

Assumptions (2.6), (2.7) and (2.8) are respectively called the growth, coercivity and monotonicity conditions. Setting \(a(x, u, \nabla u) = |\nabla u|^{p-2} \nabla u\) in (2.1) gives in particular the \(p\)-Laplacian operator.

Remark 2.1. With \(p = 2\) and \(a(x, u, \nabla u) = \Lambda(x, u - h(x)) \nabla u\), Problem (2.1)–(2.4) covers seepage models, where \(h\) is a fixed function and \(\Lambda\) is defined based on a permeability tensor \(K\) and a regularised Heaviside function. The role of this Heaviside function is to extend the Darcy law to the dry domain. We refer the reader to [51] and references therein for more details.

Assumptions 2.2. The assumptions on the data in Problem (2.1)–(2.4) are the following:

1. the operator \(a\) satisfies (2.5)–(2.8) and the domain \(\Omega\) has a Lipschitz boundary,
2. the parts of the boundary, \(\Gamma_1, \Gamma_2, \text{ and } \Gamma_3\), are assumed to be measurable and pairwise disjoint subsets of \(\partial \Omega\) such that \(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial \Omega\) and the \((d-1)\)-dimensional measure of \(\Gamma_1\) is non zero,
3. the source term \(f\) belongs to \(L^p(\Omega)\), the barrier \(a\) belongs to \(L^p(\partial \Omega)\) and the boundary data \(g\) belongs to \(W^{1-\frac{1}{p}, p'}(\partial \Omega)\),
4. the closed convex set \(\mathcal{K} := \{v \in W^{1,p}(\Omega) : \gamma(v) = g \text{ on } \Gamma_1, \quad \gamma(v) \leq a \text{ on } \Gamma_3\}\) is non-empty.

Based on Assumptions 2.2, Problem (2.1)–(2.4) can be written in the following weak sense:

\[\begin{cases}
\text{Find } \bar{u} \in \mathcal{K} \text{ such that, } \forall v \in \mathcal{K}, \\
\int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla (\bar{u} - v) \, dx \leq \int_{\Omega} f(\bar{u} - v) \, dx.
\end{cases} \quad (2.9)\]
The existence of a solution to Problem (2.9) is ensured by \[42, \text{Theorem 8.2, Chap. 2}\].

2.2. The gradient discretisation method. The GDM consists in replacing, in the weak formulation of the model, the continuous space and operators by discrete ones, obtaining thus a gradient scheme (GS). The discrete elements are gathered in what is called a gradient discretisation (GD). The restrictions put on these discrete elements are rather light, and there are therefore a large choice of possible GDs. It was shown in previous papers (see \[23\] for a review) that, for a number of classical schemes, specific GDs can be chosen such that the corresponding GSs are the considered schemes.

**Definition 2.3.** (Gradient discretisation for Signorini BCs). A gradient discretisation $D$ for Signorini boundary conditions and nonhomogeneous Dirichlet boundary conditions is $D = (X_D, \Pi_D, I_D, T_D, \nabla_D)$, where:

1. the set of discrete unknowns $X_D = X_{D,\Gamma_2,3} \oplus X_{D,\Gamma_1}$ is a direct sum of two finite dimensional spaces on $\mathbb{R}$. The first space corresponds to the interior degrees of freedom and to the boundaries degrees of freedom on $\Gamma_2 \cup \Gamma_3$. The second space corresponds to the boundary degrees of freedom on $\Gamma_1$,

2. the linear mapping $\Pi_D : X_D \to L^p(\Omega)$ reconstructs functions from the degrees of freedom,

3. the linear mapping $I_D, \Gamma_1 : W^{1-\frac{1}{p},p}(\partial \Omega) \to X_{D,\Gamma_1}$ interpolates the traces of functions in $W^{1-\frac{1}{p},p}(\Omega)$ on the degrees of freedom,

4. the linear mapping $T_D : X_D \to L^p(\partial \Omega)$ reconstructs traces from the degrees of freedom,

5. the linear mapping $\nabla_D : X_D \to L^p(\Omega)^d$ reconstructs gradients from the degrees of freedom. It must be such that $\|\nabla_D \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{D,\Gamma_2,3}$.

As already explained, the GS is obtained by taking the weak formulation (2.9) of the model, and replacing the continuous elements (space, function, gradient, trace...) by the discrete elements provided by the chosen GD.

**Definition 2.4.** (Gradient scheme for Signorini problem). Let $D$ be a gradient discretisation in the sense of Definition 2.3. The corresponding gradient scheme for Problem (2.9) is

$$
\begin{aligned}
\text{Find } u \in K_D \text{ such that } \forall v \in K_D,
\int_{\Omega} a(x, \Pi_D u, \nabla_D u) \cdot \nabla_D (u - v) \, dx &\leq \int_{\Omega} f \Pi_D (u - v) \, dx,
\end{aligned}
$$

(2.10)

where $K_D := \{ v \in I_D, \Gamma_1, g + X_{D,\Gamma_2,3} : T_D v \leq a \text{ on } \Gamma_3 \}$.

We presented in [2] three properties called coercivity, GD-consistency and limit-conformity to assess the accuracy of gradient schemes for VIs. These properties were sufficient to establish error estimates and prove the convergence of the GDM for linear differential operator. For non-linear problems, an additional property called compactness is required to ensure the convergence of the GDM. Let us describe these four properties in the context of Signorini boundary conditions.

**Definition 2.5** (Coercivity). If $D$ is a gradient discretisation in the sense of Definition 2.3, set

$$
C_D = \max_{v \in X_{D,\Gamma_2,3} \setminus \{0\}} \left( \frac{\|\Pi_D v\|_{L^p(\Omega)}}{\|\nabla_D v\|_{L^p(\Omega)^d}} + \frac{\|T_D v\|_{L^p(\partial \Omega)}}{\|\nabla_D v\|_{L^p(\Omega)^d}} \right),
$$

(2.11)
A sequence \((\mathcal{D}_m)_{m \in \mathbb{N}}\) of gradient discretisations is coercive if \((C_{\mathcal{D}_m})_{m \in \mathbb{N}}\) remains bounded.

**Definition 2.6** (GD-Consistency). If \(\mathcal{D}\) is a gradient discretisation in the sense of Definition 2.3, define \(S_{\mathcal{D}} : \mathcal{K} \to [0, +\infty)\) by
\[
\forall \varphi \in \mathcal{K}, \quad S_{\mathcal{D}}(\varphi) = \min_{v \in \mathcal{K}_{\mathcal{D}}} \left( \|\Pi_{\mathcal{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|_{L^p(\Omega)^d} \right). \tag{2.12}
\]
A sequence \((\mathcal{D}_m)_{m \in \mathbb{N}}\) of gradient discretisations is GD-consistent (or simply consistent, for short) if \(\lim_{m \to \infty} S_{\mathcal{D}_m}(\varphi) = 0\), for all \(\varphi \in \mathcal{K}\).

**Definition 2.7** (Limit-conformity). If \(\mathcal{D}\) is a gradient discretisation in the sense of Definition 2.3, define \(W_{\mathcal{D}} : C^2(\overline{\Omega})^d \to [0, +\infty)\) by
\[
\forall \psi \in C^2(\overline{\Omega})^d \text{ such that } \psi \cdot \mathbf{n} = 0 \text{ on } \Gamma_2,
\[
W_{\mathcal{D}}(\psi) = \sup_{v \in X_{\mathcal{D}, \mathbf{r}, 2, 3} \setminus \{0\}} \frac{\left| \int_{\Omega} (\nabla_{\mathcal{D}} v \cdot \psi + \Pi_{\mathcal{D}} v \text{div}(\psi)) \, dx - \int_{\Gamma_3} \psi \cdot \mathbf{n} T_{\mathcal{D}} v \, dz \right|}{\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}}. \tag{2.13}
\]
A sequence \((\mathcal{D}_m)_{m \in \mathbb{N}}\) of gradient discretisations is limit-conforming if, for all \(\psi \in C^2(\overline{\Omega})^d\) such that \(\psi \cdot \mathbf{n} = 0 \text{ on } \Gamma_2\), \(\lim_{m \to \infty} W_{\mathcal{D}_m}(\psi) = 0\).

**Definition 2.8** (Compactness). A sequence \((\mathcal{D}_m)_{m \in \mathbb{N}}\) of GDs is compact if, for any sequence \((u_m)_{m \in \mathbb{N}}\) with \(u_m \in \mathcal{K}_{\mathcal{D}_m}\) and such that \((\|\nabla_{\mathcal{D}_m} u_m\|_{L^p(\Omega)^d})_{m \in \mathbb{N}}\) is bounded, the sequence \((\Pi_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}\) is relatively compact in \(L^p(\Omega)\).

### 2.3. Convergence results.

We can now state and prove our main convergence theorem for the gradient discretisation method applied to the non-linear Signorini problem.

**Theorem 2.9** (Convergence of the GDM, non-linear Signorini problem). Under Assumptions 2.2, let \((\mathcal{D}_m)_{m \in \mathbb{N}}\) be a sequence of gradient discretisations in the sense of Definition 2.3, that is coercive, GD-consistent, limit-conforming and compact, and such that \(\mathcal{K}_{\mathcal{D}_m}\) is non-empty for any \(m\). Then, for any \(m \in \mathbb{N}\), the gradient scheme (2.10) has at least one solution \(u_m \in \mathcal{K}_{\mathcal{D}_m}\).

Assume furthermore that
\[
\exists \varphi_g \in W^{1,p}(\Omega) \text{ s.t. } \gamma(\varphi_g) = g \quad \text{and} \quad \lim_{m \to \infty} \min \{ \|\Pi_{\mathcal{D}_m} v - \varphi_g\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}_m} v - \nabla \varphi_g\|_{L^p(\Omega)^d} : v \in \mathcal{K}_{\mathcal{D}_m, \mathbf{r}, 1} \gamma(\varphi_g) \in X_{\mathcal{D}_m, \mathbf{r}, 2, 3} \} = 0. \tag{2.14}
\]
Then, up to a subsequence as \(m \to \infty\), \(\Pi_{\mathcal{D}_m} u_m\) converges strongly in \(L^p(\Omega)\) to a weak solution \(\bar{u}\) of Problem (2.9), and \(\nabla_{\mathcal{D}_m} u_m\) converges weakly in \(L^p(\Omega)^d\) to \(\nabla \bar{u}\).

If moreover \(\mathbf{a}\) is strictly monotonic in the sense
\[
(\mathbf{a}(x, s, \xi) - \mathbf{a}(x, s, \chi)) \cdot (\xi - \chi) > 0 \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \forall \xi, \chi \in \mathbb{R}^d \text{ with } \xi \neq \chi, \tag{2.15}
\]
then \(\nabla_{\mathcal{D}_m} u_m\) converges strongly in \(L^p(\Omega)^d\) to \(\nabla \bar{u}\).

**Remark 2.10.** Assumption (2.14) is obviously always satisfied if \(g = 0\) (take \(\varphi_g = 0\)). For most sequences of gradient discretisations, the convergence stated in (2.14) actually holds for any \(\varphi\) with \(g = \gamma(\varphi)\), and corresponds to the GD-consistency of the method for non-homogeneous Fourier BCs (see [20, Remark 2.58 and Definition 2.49]).
Applying Young’s inequality to this relation shows that
\[ g_D = \arg\min_{v \in K_D} \left( \|\Pi_D v - \tilde{g}\|_{L^p(\Omega)} + \|\nabla_D v - \nabla \tilde{g}\|_{L^p(\Omega)^d} \right). \]

Let \( \langle \cdot, \cdot \rangle \) be the dual product between the finite dimensional space \( X_{D,\Gamma_{2,3}} \) and its dual \( X'_{D,\Gamma_{2,3}} \). Define the operator \( \mathcal{A}_D : X_{D,\Gamma_{2,3}} \rightarrow X'_{D,\Gamma_{2,3}} \) by, for \( \hat{u}, \hat{v} \in X_{D,\Gamma_{2,3}} \),
\[
\langle \mathcal{A}_D(\hat{u}), \hat{v} \rangle = \int_{\Omega} a(x, \Pi_D(\hat{u} + g_D)(x), \nabla_D(\hat{u} + g_D)(x)) : \nabla_D(v + g_D)(x) \, dx.
\]

Applying the same reasoning as in [42], we check that \( \mathcal{A}_D \) is an operator of the calculus of variations (this is extremely easy here, due to the finite dimension of \( X_{D,\Gamma_{2,3}} \)). The existence of a solution to the scheme (2.10) is then a consequence of [42, Theorem 8.2, Chap. 2] since, setting \( \hat{u} = u - g_D \) and \( \hat{v} = v - g_D \), this scheme can be re-written

\[
\text{find } \hat{u} \in K_D - g_D \text{ such that } \forall \hat{v} \in K_D - g_D, \langle \mathcal{A}_D(\hat{u}), \hat{v} - \hat{v} \rangle \leq \ell(\hat{u} - \hat{v}),
\]

where \( \ell \in X'_{D,\Gamma_{2,3}} \) is defined by \( \ell(\hat{w}) = \int_{\Omega} f \Pi_D \hat{w} \, dx \).

**Step 2:** convergence towards the solution to the continuous model.

Let us start by estimating \( \|\nabla_{D_m} u_m\|_{L^p(\Omega)^d} \). In (2.10), set \( u := u_m \), and \( v := v_m \) a generic element in \( K_{D_m} \). By using the Holder’s inequality and due to the coercivity assumption (2.7), it follows that
\[
\|\nabla_{D_m} u_m\|_{L^p(\Omega)^d}^p \leq \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) : \nabla_{D_m} u_m \, dx
\]
\[
\leq \|f\|_{L^p(\Omega)} \|\Pi_{D_m}(u_m - v_m)\|_{L^p(\Omega)}
\]
\[
+ \|a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m)\|_{L^p(\Omega)^d} \|\nabla_{D_m} v_m\|_{L^p(\Omega)^d}.
\]

Since \( u_m - v_m \) is an element in \( X_{D_m,\Gamma_{2,3}} \), applying the coercivity property (see Definition 2.5) gives \( C_p \) not depending on \( m \) such that \( \|\Pi_{D_m}(u_m - v_m)\|_{L^p(\Omega)} \leq C_p \|\nabla_{D_m}(u_m - v_m)\|_{L^p(\Omega)^d} \). Thus, using the growth assumption (2.6),
\[
\|\nabla_{D_m} u_m\|_{L^p(\Omega)^d}^p \leq C_p \|f\|_{L^p(\Omega)} \left( \|\nabla_{D_m} u_m\|_{L^p(\Omega)^d} + \|\nabla_{D_m} v_m\|_{L^p(\Omega)^d} \right)
\]
\[
+ \left( \|\Pi\|_{L^{p'}(\Omega)^d} + \mu \|\nabla_{D_m} u_m\|_{L^p(\Omega)^d}^{p-1} \right) \|\nabla_{D_m} v_m\|_{L^p(\Omega)^d}.
\]

Applying Young’s inequality to this relation shows that
\[
\|\nabla_{D_m} u_m\|_{L^p(\Omega)^d}^p \leq C_1 \left( \|\nabla_{D_m} v_m\|_{L^p(\Omega)^d}^p + \|f\|_{L^{p'}(\Omega)^d}^p + \|\Pi\|_{L^{p'}(\Omega)^d}^p \right)
\]
where \( C_1 \) does not depend on \( m \). Let us now define, for \( \varphi \in K \), an element \( P_{D_m} \varphi \) of \( K_{D_m} \) by
\[
P_{D_m}(\varphi) = \arg\min_{\varphi \in K_{D_m}} (\|\Pi_{D_m} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{D_m} v - \nabla \varphi\|_{L^p(\Omega)^d}).
\]

We have
\[
S_{D_m}(\varphi) = \|\Pi_{D_m} (P_{D_m} \varphi) - \varphi\|_{L^p(\Omega)} + \|\nabla_{D_m} (P_{D_m} \varphi) - \nabla \varphi\|_{L^p(\Omega)^d}.
\]

Set \( v_m := P_{D_m} \varphi \) in (2.16). By the triangle inequality
\[
\|\nabla_{D_m} v_m\|_{L^p(\Omega)^d} \leq S_{D_m}(\varphi) + \|\nabla \varphi\|_{L^p(\Omega)^d},
\]
and the GD-consistency of $D_m$ shows that $\|\nabla D_m u_m\|_{L^p(\Omega)^d}$ is bounded. Used in (2.16), this proves that $\|\nabla D_m u_m\|_{L^p(\Omega)^d}$ remains bounded.

Now, using (2.14), [20, Lemma 2.57] (slightly adjusted to the fact that the limit-conformity involves here functions such that $\psi \cdot n = 0$ on $\Gamma_2$, see (2.13)) asserts the existence of $\bar{u} \in W^{1,p}(\Omega)$ and a subsequence, still denoted by $(D_m)_{m \in \mathbb{N}}$, such that $\gamma \bar{u} = g$ on $\Gamma_1$, $\Pi_{D_m} u_m$ converges weakly to $\bar{u}$ in $L^p(\Omega)$, $\nabla D_m u_m$ converges weakly to $\nabla \bar{u}$ in $L^p(\Omega)^d$ and $T_{D_m} u_m$ converges weakly to $\gamma \bar{u}$ in $L^p(\Gamma_3)$. Since $u_m \in K_{D_m}$, we have $T_{D_m} u_m \leq a$ on $\Gamma_3$, which implies $\gamma \bar{u} \leq a$ on $\Gamma_3$. In other words, $\bar{u}$ belongs to $K$. By the compactness hypothesis, the convergence of $\Pi_{D_m} u_m$ to $\bar{u}$ is actually strong in $L^p(\Omega)$. Up to another subsequence, we can therefore assume that this convergence holds almost everywhere on $\Omega$.

To complete this step, it remains to show that $\bar{u}$ is a solution to (2.9). We use the Minty trick. From assumption (2.6), the sequence $A_{D_m} = a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m)$ is bounded in $L^q(\Omega)^d$ and converges weakly up to a subsequence to some $A$ in $L^q(\Omega)^d$. Owing to the GD-consistency of the gradient discretisations, for all $\varphi \in \mathcal{K}$ we have $\Pi_{D_m} (P_{D_m} \varphi) \to \varphi$ strongly in $L^p(\Omega)$ and $\nabla_{D_m} (P_{D_m} \varphi) \to \nabla \varphi$ strongly in $L^p(\Omega)^d$. Taking $v := P_{D_m} \varphi$ as a test function in the gradient scheme (2.10) and passing to the superior limit gives

$$\limsup_{m \to \infty} \int_\Omega a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \cdot \nabla_{D_m} u_m \, dx$$

$$\leq \limsup_{m \to \infty} \left( \int_\Omega f(\Pi_{D_m} u_m - \Pi_{D_m} P_{D_m} \varphi) \, dx + \int_\Omega A_{D_m} \cdot \nabla_{D_m} P_{D_m} \varphi \, dx \right)$$

$$\leq \int_\Omega f(\bar{u} - \varphi) \, dx + \int_\Omega A \cdot \nabla \varphi \, dx, \quad \text{for all } \varphi \in \mathcal{K}.$$

Choosing $\varphi = \bar{u}$, yields

$$\limsup_{m \to \infty} \int_\Omega a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \cdot \nabla_{D_m} u_m \, dx \leq \int_\Omega A \cdot \nabla \bar{u} \, dx. \quad (2.18)$$

Using the monotonicity assumption (2.8), one writes, for $G \in L^p(\Omega)^d$,

$$\liminf_{m \to \infty} \left[ \int_\Omega a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \cdot \nabla_{D_m} u_m \, dx - \int_\Omega a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \cdot G \, dx \right.$$

$$- \int_\Omega a(x, \Pi_{D_m} u_m, G) \cdot \nabla_{D_m} u_m \, dx + \int_\Omega a(x, \Pi_{D_m} u_m, G) \cdot G \, dx \right]$$

$$= \liminf_{m \to \infty} \int_\Omega \left[ a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) - a(x, \Pi_{D_m} u_m, G) \right] \cdot \left[ \nabla_{D_m} u_m - G \right] \, dx$$

$$\geq 0. \quad (2.19)$$

The a.e. convergence of $\Pi_{D_m} u_m$, the growth property (2.6) and the dominated convergence theorem show that $a(x, \Pi_{D_m} u_m, G) \to a(x, \bar{u}, G)$ strongly in $L^p(\Omega)^d$. Hence, passing to the limit in (2.19),

$$\liminf_{m \to \infty} \int_\Omega a(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \cdot \nabla_{D_m} u_m \, dx - \int_\Omega A \cdot \nabla G \, dx$$

$$- \int_\Omega a(x, \bar{u}, G) \cdot \nabla \bar{u} \, dx + \int_\Omega a(x, \bar{u}, G) \cdot G \, dx \geq 0. \quad (2.20)$$
Combining this inequality with (2.18) yields
\[ \int_{\Omega} A \cdot \nabla \bar{u} \, dx - \int_{\Omega} A \cdot G \, dx - \int_{\Omega} a(x, \bar{u}, G) \cdot \nabla \bar{u} \, dx + \int_{\Omega} a(x, \bar{u}, G) \cdot G \, dx \geq 0. \]

Take \( \phi \in C_c^{\infty}(\Omega)^d \) and \( \alpha > 0 \). Putting \( G = \nabla \bar{u} + \alpha \phi \) and dividing by \( \alpha \), one obtains
\[ - \int_{\Omega} (A - a(x, \bar{u}, \nabla \bar{u} + \alpha \phi)) \cdot \phi \, dx \geq 0, \quad \forall \phi \in C_c^{\infty}(\Omega)^d, \quad \forall \alpha > 0. \]

Letting \( \alpha \to 0 \) and applying the dominated convergence theorem yields
\[ - \int_{\Omega} (A - a(x, \bar{u}, \nabla \bar{u})) \cdot \phi \, dx \geq 0, \quad \forall \phi \in C_c^{\infty}(\Omega)^d. \]

Applied to \( -\phi \) instead of \( \phi \), this leads to
\[ - \int_{\Omega} (A - a(x, \bar{u}, \nabla \bar{u})) \cdot \phi \, dx = 0, \quad \forall \phi \in C_c^{\infty}(\Omega)^d, \]
which implies that
\[ A = a(x, \bar{u}, \nabla \bar{u}) \text{ a.e. on } \Omega. \] (2.21)

Setting \( G = \nabla \bar{u} \) in (2.20), it follows that
\[ \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx \leq \lim_{m \to \infty} \int_{\Omega} a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) \cdot \nabla \mathcal{D}_m u_m \, dx, \] (2.22)
which gives, since \( u_m \) is a solution to the gradient scheme (2.10), for all \( \varphi \in \mathcal{K} \),
\[ \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx \]
\[ \leq \lim_{m \to \infty} \left[ \int_{\Omega} f \Pi_{\mathcal{D}_m} (u_m - P_{\mathcal{D}_m} \varphi) \, dx + \int_{\Omega} a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) \cdot \nabla \mathcal{D}_m (P_{\mathcal{D}_m} \varphi) \, dx \right]. \]

Using (2.21) and the strong convergence of \( \nabla \mathcal{D}_m (P_{\mathcal{D}_m} \varphi) \) to \( \nabla \varphi \) yields
\[ \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx \leq \int_{\Omega} f(\bar{u} - \varphi) \, dx + \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \varphi \, dx. \]

This shows that \( \bar{u} \) is a solution to (2.9).

**Step 3**: strong convergence of the gradients, if \( a \) is strictly monotonic.

Owing to (2.18) and (2.21),
\[ \limsup_{m \to \infty} \int_{\Omega} a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) \cdot \nabla \mathcal{D}_m u_m \, dx \leq \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx. \] (2.23)

Together with (2.22), we conclude that
\[ \lim_{m \to \infty} \int_{\Omega} a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) \cdot \nabla \mathcal{D}_m u_m \, dx = \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx. \] (2.24)

The remaining reasoning to obtain the strong convergence of \( \nabla \mathcal{D}_m u_m \) is exactly like in [22]. For the sake of completeness, we recall it. Equality (2.24) leads to
\[ \lim_{m \to \infty} \int_{\Omega} (a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) - a(x, \bar{u}, \nabla \bar{u})) \cdot (\nabla \mathcal{D}_m u_m - \nabla \bar{u}) \, dx = 0. \]

Making use of the fact that \( (a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) - a(x, \bar{u}, \nabla \bar{u})) \cdot (\nabla \mathcal{D}_m u_m - \nabla \bar{u}) \geq 0 \) a.e. on \( \Omega \), we deduce that
\[ (a(x, \Pi_{\mathcal{D}_m} u_m, \nabla \mathcal{D}_m u_m) - a(x, \bar{u}, \nabla \bar{u})) \cdot (\nabla \mathcal{D}_m u_m - \nabla \bar{u}) \to 0 \text{ in } L^1(\Omega). \]
Up a subsequence, the convergence holds almost everywhere. The strict monotonicity assumption (2.15) and [22, Lemma 3.2] yield \( \nabla \Delta_m u_m \to \nabla \bar{u} \) a.e. as \( m \to \infty \). Furthermore, as a consequence, \( a(x, \Pi_m u_m, \nabla \Delta_m u_m) \cdot \nabla \Delta_m u_m \to a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \) a.e. Since \( a(x, \Pi_m u_m, \nabla \Delta_m u_m) \cdot \nabla \Delta_m u_m \geq 0 \), and taking into account (2.24), [22, Lemma 3.3] gives the strong convergence of \( a(x, \Pi_m u_m, \nabla \Delta_m u_m) \cdot \nabla \Delta_m u_m \to a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \) in \( L^1(\Omega) \) as \( m \to \infty \). As a consequence of this \( L^1 \)-convergence, we obtain the equi-integrability of the sequence of functions \( a(x, \Pi_m u_m, \nabla \Delta_m u_m) \cdot \nabla \Delta_m u_m \). This provides, with (2.7), the equi-integrability of \( (|\nabla \Delta_m u_m|^p)_m \in \mathbb{N} \). The strong convergence of \( \nabla \Delta_m u_m \to \nabla \bar{u} \) in \( L^1(\Omega) \) is then directly implied by the Vitali theorem.

\[ \Box \]

3. Obstacle problem and generalised Bulkley fluid models

3.1. Continuous problems.

3.1.1. Obstacle problem. We are concerned here with other kinds of variational inequalities. The first one is an obstacle model, in which the inequalities are imposed inside the domain \( \Omega \). It is formulated as

\[
(\text{div } a(x, \bar{u}, \nabla \bar{u}) + f)(\psi - \bar{u}) = 0 \quad \text{in } \Omega, \tag{3.1a}
\]

\[-\text{div } a(x, \bar{u}, \nabla \bar{u}) \leq f \quad \text{in } \Omega, \tag{3.1b}
\]

\[ \bar{u} \leq \psi \quad \text{in } \Omega, \tag{3.1c}
\]

\[ \bar{u} = h \quad \text{on } \partial \Omega. \tag{3.1d}
\]

Let us provide the assumptions on the data of this model.

**Assumptions 3.1.**

1. The operator \( a \) and the domain \( \Omega \) satisfy the same properties as in Assumption 2.2,
2. The function \( f \) belongs to \( L^p(\Omega) \), the boundary function \( h \) is in \( W^{1-\frac{1}{2}p,p}(\partial \Omega) \) and the obstacle function \( \psi \) belongs to \( L^p(\Omega) \),
3. The closed convex set \( K := \{ v \in W^{1,p}(\Omega) : v \leq \psi \text{ in } \Omega, \gamma(v) = h \text{ on } \partial \Omega \} \) is non-empty.

The weak formulation of the obstacle problem (3.1) is

\[
\left\{ \begin{array}{l}
\text{Find } \bar{u} \in K \text{ such that, } \forall v \in K, \\
\quad \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla (\bar{u} - v) \, dx \leq \int_{\Omega} f(\bar{u} - v) \, dx.
\end{array} \right. \tag{3.2}
\]

3.1.2. Generalised Bulkley model. The second problem is called the Bulkley model, whose weak formulation is given by

\[
\left\{ \begin{array}{l}
\text{Find } \bar{u} \in W^{1,p}_0(\Omega) \text{ such that, for all } v \in W^{1,p}_0(\Omega), \\
\quad \int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla (\bar{u} - v) \, dx + \int_{\Omega} |\nabla \bar{u}| \, dx - \int_{\Omega} |\nabla v| \, dx \\
\quad \leq \int_{\Omega} f(\bar{u} - v) \, dx.
\end{array} \right. \tag{3.3}
\]

Here the operator \( a \) is assumed to satisfy (2.5)–(2.8) and the domain \( \Omega \) has a Lipschitz boundary. Models considered in the removal of materials from a duct by using fluids [30] are included in (3.3) by setting \( a(x, \bar{u}, \nabla \bar{u}) = |\nabla \bar{u}|^{p-2} \nabla \bar{u} \).

As for the Signorini problem, [42, Theorem 8.2, Chap. 2] yields the existence of a solution to each of the problems (3.2) and (3.3).
3.2. Discrete problems.

3.2.1. Obstacle problem. Let us recall the definition of a gradient discretisation for non-homogeneous Dirichlet boundary conditions [20].

**Definition 3.2** (GD for non-homogeneous Dirichlet boundary conditions). A gradient discretisation \( \mathcal{D} \) for non-homogeneous Dirichlet boundary conditions is defined by \( \mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, I_{\mathcal{D},\partial\Omega}, \nabla_\mathcal{D}) \), where:

1. the set of discrete unknowns \( X_\mathcal{D} = X_{\mathcal{D},0} \oplus X_{\mathcal{D},\partial\Omega} \) is a direct sum of two finite dimensional spaces on \( \mathbb{R} \), representing respectively the interior degrees of freedom and the boundary degrees of freedom,
2. the linear mapping \( \Pi_\mathcal{D} : X_\mathcal{D} \to L^p(\Omega) \) provides the reconstructed function,
3. the linear mapping \( I_{\mathcal{D},\partial\Omega} : W^{1-\frac{1}{p}}(\partial\Omega) \to X_{\mathcal{D},\partial\Omega} \) provides an interpolation operator for the trace of functions in \( W^{1,p}(\Omega) \),
4. the linear mapping \( \nabla_\mathcal{D} : X_\mathcal{D} \to L^p(\Omega)^d \) gives a reconstructed gradient, which must be defined such that \( \|\nabla_\mathcal{D} \cdot \|_{L^p(\Omega)^d} \) is a norm on \( X_{\mathcal{D},0} \).

**Definition 3.3** (GS for the non-linear obstacle problem). Let \( \mathcal{D} \) be a gradient discretisation in the sense of Definition 3.2. The corresponding gradient scheme for (3.3) is given by

\[
\begin{align*}
\text{Find } u & \in K_\mathcal{D} := \{ v \in X_{\mathcal{D},0} + I_{\mathcal{D},\partial\Omega} h : \Pi_\mathcal{D} v \leq \psi \text{ in } \Omega \} \text{ s.t., } \forall v \in K_\mathcal{D}, \\
\int_\Omega a(x, \Pi_\mathcal{D} u, \nabla_\mathcal{D} u) \cdot \nabla_\mathcal{D} (u - v) \, dx & \leq \int_\Omega f \Pi_\mathcal{D} (u - v) \, dx.
\end{align*}
\]  

(3.4)

3.2.2. Generalised Bulkley model.

**Definition 3.4** (GD for homogeneous Dirichlet boundary conditions). A gradient discretisation \( \mathcal{D} \) for homogeneous Dirichlet boundary conditions is defined by \( \mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}) \), where \( X_{\mathcal{D},0} \) is a finite dimensional vector space over \( \mathbb{R} \), taking into account the zero boundary condition, and \( \Pi_\mathcal{D} \) and \( \nabla_\mathcal{D} \) are as in Definition 3.2 but defined on \( X_{\mathcal{D},0} \).

**Definition 3.5** (GS for the Bulkley model). Let \( \mathcal{D} \) be a gradient discretisation in the sense of Definition 3.4. The corresponding gradient scheme for (3.3) is given by

\[
\begin{align*}
\text{Find } u & \in X_{\mathcal{D},0} \text{ such that for all } v \in X_{\mathcal{D},0}, \\
\int_\Omega a(x, \Pi_\mathcal{D} u, \nabla_\mathcal{D} u) \cdot \nabla_\mathcal{D} (u - v) \, dx & + \int_\Omega |\nabla_\mathcal{D} u| \, dx - \int_\Omega |\nabla_\mathcal{D} v| \, dx \\
& \leq \int_\Omega f \Pi_\mathcal{D} (u - v) \, dx.
\end{align*}
\]  

(3.5)

3.2.3. Properties of GDs. Except for the restriction to the convex sets \( \mathcal{K} \) and \( K_\mathcal{D} \) in the GD-consistency, all the properties of GDs required for the convergence analysis of the GDM on the non-linear obstacle and Bulkley models are similar to the corresponding ones for GDs adapted to PDEs [20, 22].

**Definition 3.6** (Coercivity). If \( \mathcal{D} \) is a gradient discretisation in the sense of Definition 3.2 or Definition 3.4, define

\[
C_\mathcal{D} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_\mathcal{D} v\|_{L^p(\Omega)}}{\|\nabla_\mathcal{D} v\|_{L^p(\Omega)^d}}.
\]  

(3.6)

A sequence \((\mathcal{D}_m)_{m \in \mathbb{N}}\) of such gradient discretisations is **coercive** if \((C_{\mathcal{D}_m})_{m \in \mathbb{N}}\) remains bounded.
\textbf{Definition 3.7} (GD-Consistency). If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.2, let $S_{\mathcal{D}} : \mathcal{K} \to [0, +\infty)$ be defined by
\begin{equation}
\forall \varphi \in \mathcal{K}, \quad S_{\mathcal{D}}(\varphi) = \min_{v \in K_D} \left( \|\Pi_{\mathcal{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|_{L^p(\Omega)^d} \right). \tag{3.7}
\end{equation}

If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.4, $S_{\mathcal{D}}$ is defined the same way with $(\mathcal{K}, \mathcal{K}_D)$ replaced by $(W^{1,p}_0(\Omega), X_{\mathcal{D},0})$.

A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of such gradient discretisations is \textit{GD-consistent} if for all $\varphi \in \mathcal{K}$, $\lim_{m \to \infty} S_{\mathcal{D}_m}(\varphi) = 0$.

\textbf{Definition 3.8} (Limit-conformity). If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.2 or Definition 3.4, define $W_{\mathcal{D}} : C^2(\Omega)^d \to [0, +\infty)$ by
\begin{equation}
\forall \psi \in C^2(\Omega)^d, \quad W_{\mathcal{D}}(\psi) = \sup_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}} \left| \int_{\Omega} (\nabla_{\mathcal{D}} v \cdot \psi + \Pi_{\mathcal{D}} \nabla \psi) \, dx \right|. \tag{3.8}
\end{equation}

A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of such gradient discretisations is \textit{limit-conforming} if for all $\psi \in C^2(\Omega)^d$, $\lim_{m \to \infty} W_{\mathcal{D}_m}(\psi) = 0$.

Finally, Definition 2.8 (compactness) remains the same for gradient discretisations in the sense of Definition 3.2 or Definition 3.4, with $K_{\mathcal{D}_m}$ replaced by $X_{\mathcal{D},0,m}$ in the latter case.

### 3.3. Convergence results.

The following two theorems state the convergence properties of the GDM for the non-linear obstacle problem and the Bulkley model. Note that for quasi-linear operators (that is, $a(x, \bar{u}, \nabla \bar{u}) = \Lambda(x, \bar{u}) \nabla \bar{u}$), the convergence of the GDM for the obstacle problem was established in [1].

\textbf{Theorem 3.9} (Convergence of the GDM, non-linear obstacle problem).

Under Assumptions 3.1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 3.2, that is coercive, GD-consistent, limit-conforming and compact, and such that $K_{\mathcal{D}_m}$ is a non-empty set for any $m$.

Then, for any $m \in \mathbb{N}$, the gradient scheme (3.4) has at least one solution $u_m \in K_{\mathcal{D}_m}$, and, up to a subsequence as $m \to \infty$, $\Pi_{\mathcal{D}_m} u_m$ converges strongly in $L^p(\Omega)$ to a weak solution $\bar{u}$ of Problem (3.2) and $\nabla_{\mathcal{D}_m} u_m$ converges weakly in $L^p(\Omega)^d$ to $\nabla \bar{u}$.

If the strict monotonicity (2.15) is assumed, then $\nabla_{\mathcal{D}_m} u_m$ converges strongly in $L^p(\Omega)^d$ to $\nabla \bar{u}$.

\textbf{Theorem 3.10} (Convergence of the GDM, Bulkley model).

Under Assumptions (2.5)–(2.8) and $f \in L^p(\Omega)$, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 3.4, that is coercive, GD-consistent, limit-conforming and compact.

Then, for any $m \in \mathbb{N}$, the gradient scheme (3.5) has at least one solution $u_m \in X_{\mathcal{D},0}$ and, up to a subsequence as $m \to \infty$, $\Pi_{\mathcal{D}_m} u_m$ converges strongly in $L^p(\Omega)$ to a weak solution $\bar{u}$ of Problem (3.3) and $\nabla_{\mathcal{D}_m} u_m$ converges weakly to $\nabla \bar{u}$ in $L^p(\Omega)^d$.

If we also assume that $a$ is strictly monotonic in the sense of (2.15), then $\nabla_{\mathcal{D}_m} u_m$ converges strongly in $L^p(\Omega)^d$ to $\nabla \bar{u}$.

The proof of Theorem 3.9 is extremely similar to the proof of Theorem 2.9. We therefore only provide the proof of Theorem 3.10.
Proof of Theorem 3.10. Let us define the operator \( A_D : X_{D,0} \to X'_{D,0} \) and the functional \( J_D : X_{D,0} \to \mathbb{R}^+ \) as follows:

\[
\langle A_D(u), v \rangle = \int_{\Omega} a(x, \Pi_D u(x), \nabla_D u(x)) \cdot \nabla_D v(x) \, dx \quad \text{and} \quad J_D(u) = \int_{\Omega} |\nabla_D u(x)| \, dx,
\]

where \( \langle \cdot, \cdot \rangle \) is the duality product between \( X'_{D,0} \) and \( X_{D,0} \). Applying the same arguments as in [42], one can easily prove that the operator \( A_D \) is pseudo-monotone and obtain, since \( J_D \geq 0 \),

\[
\frac{\langle A_D(u), u - \phi \rangle + J_D(u)}{\| \nabla_D u \|_{L^p(\Omega)^d}} \to +\infty \quad \text{as} \quad \| \nabla_D u \|_{L^p(\Omega)^d} \to \infty.
\]

A direct application of [42, Theorem 8.5, Chap. 2] then gives the existence of a solution to Problem (3.5). We now show that \( \| \nabla_D u_m \|_{L^p(\Omega)^d} \) is bounded. Choose \( u := u_m \), and \( v := 0 \in X_{D,0} \) in (3.5). Due to the coercivity assumption (2.7) on \( a \), the Hölder inequality and the coercivity of \( (D_m)_{m \in \mathbb{N}} \), one has

\[
g \| \nabla_D u_m \|^p_{L^p(\Omega)^d} \leq \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla_D u_m) \cdot \nabla_D u_m \, dx \\
\leq \int_{\Omega} f \Pi_{D_m} u_m \, dx \\
\leq C_p \| f \|_{L^{p'}(\Omega)} \| \nabla_D u_m \|_{L^p(\Omega)^d}.
\]

This shows that \( \| \nabla_D u_m \|_{L^p(\Omega)^d} \) is bounded. According to [20, Lemma 2.12], there exists \( \bar{u} \in W^{1,p}_0(\Omega) \) and a subsequence, denoted in the same way, such that \( \Pi_{D_m} u_m \) converges weakly to \( \bar{u} \) in \( L^p(\Omega) \) and \( \nabla_D u_m \) converges weakly to \( \nabla \bar{u} \) in \( L^p(\Omega)^d \). In fact, the strong convergence of the sequence \( \Pi_{D_m} u_m \) to \( \bar{u} \) in \( L^p(\Omega) \) is ensured by the compactness property. The growth assumption (2.6) shows that the sequence \( \mathcal{A}_{D_m} = a(x, \Pi_{D_m} u_m, \nabla_D u_m) \) is bounded in \( L^p(\Omega)^d \) and thus, up to a subsequence, that it converges weakly to some \( \mathcal{A} \) in this space.

Define \( P_{D_m} \) as in (2.17) with \( K \) and \( K_{D_m} \) replaced with \( W^{1,p}_0(\Omega) \) and \( X_{D,0} \), respectively. The consistency guarantees that \( \Pi_{D_m} (P_{D_m} \varphi) \to \varphi \) strongly in \( L^p(\Omega) \) and \( \nabla_D (P_{D_m} \varphi) \to \nabla \varphi \) strongly in \( L^p(\Omega)^d \), for all \( \varphi \in W^{1,p}_0 \). Inserting \( v := P_{D_m} \varphi \) into the gradient scheme (3.4), we obtain

\[
\int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla_D u_m) \cdot \nabla_D u_m \, dx \\
\leq \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla_D u_m) \cdot \nabla_D P_{D_m} \varphi \, dx \\
+ \int_{\Omega} f \Pi_{D_m} (u_m - P_{D_m} \varphi) \, dx \\
- \int_{\Omega} \nabla_D u_m \, dx + \int_{\Omega} \nabla_D (P_{D_m} \varphi) \, dx.
\]
All the terms except the last two can be handled as in Theorem 2.9. From the strong convergence of $\nabla D_m P_{D_m} \varphi$, letting $m \to \infty$ in the last term implies

$$\lim_{m \to \infty} \int_{\Omega} |\nabla D_m (P_{D_m} \varphi)| \, dx = \int_{\Omega} |\nabla \varphi| \, dx. \tag{3.10}$$

Estimating $\liminf_{m \to \infty} \int_{\Omega} |\nabla D_m u_m| \, dx$ is rather standard. For any $w \in L^\infty(\Omega)^d$ such that $|w| \leq 1$, write $\int_{\Omega} w \cdot \nabla D_m u_m \, dx \leq \int_{\Omega} |\nabla D_m u_m| \, dx$. The weak convergence in $L^p(\Omega)^d$ of $\nabla D_m u_m$ then yields

$$\int_{\Omega} w \cdot \nabla \bar{u} \, dx = \lim_{m \to \infty} \int_{\Omega} w \cdot \nabla D_m u_m \, dx \leq \liminf_{m \to \infty} \int_{\Omega} |\nabla D_m u_m| \, dx.$$

Taking the supremum over $w$ leads to

$$\int_{\Omega} |\nabla \bar{u}| \, dx \leq \liminf_{m \to \infty} \int_{\Omega} |\nabla D_m u_m| \, dx.$$

From this estimation and (3.10), passing to the superior limit in (3.9) gives

$$\limsup_{m \to \infty} \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla D_m u_m) \cdot \nabla D_m u_m \, dx$$

$$\leq \int_{\Omega} A \cdot \nabla \varphi \, dx + \int_{\Omega} f(\bar{u} - \varphi) \, dx - \int_{\Omega} |\nabla \bar{u}| \, dx + \int_{\Omega} |\nabla \varphi| \, dx. \tag{3.11}$$

Since this inequality holds for any $\varphi \in W^{1,p}(\Omega)$, making $\varphi = \bar{u}$ gives

$$\limsup_{m \to \infty} \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla D_m u_m) \cdot \nabla D_m u_m \, dx \leq \int_{\Omega} A \cdot \nabla \bar{u} \, dx. \tag{3.12}$$

Exactly as Theorem 2.9, it is then shown that $A = a(x, \bar{u}, \nabla \bar{u})$ and

$$\int_{\Omega} a(x, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u} \, dx \leq \liminf_{m \to \infty} \int_{\Omega} a(x, \Pi_{D_m} u_m, \nabla D_m u_m) \cdot \nabla D_m u_m \, dx.$$

Substituting $A$ and using this relation in (3.11) show that $\bar{u}$ is a solution to Problem (3.3). The rest of proof follows the same lines as for Theorem 2.9. \hfill \Box

4. **Approximate barriers**

Let us now discuss the case of approximate barriers. In most numerical methods, as the $P_1$ finite elements for instance, the standard interpolant of a smooth function $v$ is constructed by taking the value of $v$ at interpolation nodes. When $v$ is bounded by the barrier ($a$ for the Signorini problem, $\psi$ for the obstacle problem), this interpolation may not satisfy the barriers conditions at all points on the boundary/in the domain, especially for the case of the non-constant barriers. It is therefore classical to modify these barriers conditions when discretising the model. This modification can often be written in the following way.

Using $a_\mathcal{D} \in L^p(\partial \Omega)$ (for the Signorini problem) or $\psi_\mathcal{D} \in L^p(\Omega)$ (for the obstacle problem), which are respectively approximations of $a$ or $\psi$, we introduce the convex sets

$$K_{\mathcal{D},a_\mathcal{D}} := \{ v \in \mathcal{I}_{\mathcal{D},1} ; g + X_{\mathcal{D},1,2} : \mathcal{T}_\mathcal{D} v \leq a_\mathcal{D} \text{ on } \Gamma_3 \}$$

or

$$K_{\mathcal{D},\psi_\mathcal{D}} := \{ v \in \mathcal{I}_{\mathcal{D},1} ; h + X_{\mathcal{D},0} : \mathcal{P}_\mathcal{D} v \leq \psi_\mathcal{D} \}.$$ 

The schemes (2.10) or (3.4) are then modified by replacing the set $K_{\mathcal{D}}$ by $K_{\mathcal{D},a_\mathcal{D}}$ in the Signorini case, or by $K_{\mathcal{D},\psi_\mathcal{D}}$ in the obstacle case. The convergence results for
this case of approximate barriers are given in the following theorems, whose proofs are identical to that of Theorem 2.9 (see [2, Section 6] for the case of approximate barriers in gradient schemes for linear VIs).

**Theorem 4.1** (Convergence: non-linear Signorini, approximate barrier).
Under the assumptions of Theorem 2.9, let \((D_m)_{m \in \mathbb{N}}\) be a sequence of gradient discretisations in the sense of Definition 2.3, that is coercive, limit-conforming, compact, and GD-consistent (with \(S_D\) defined using \(K_{D,a_D}\) instead of \(K_D\)). Assume that each \(K_{D_m,a_{D_m}}\) is non-empty.

Then, for any \(m \in \mathbb{N}\), there exists at least one solution \(u_m \in K_{D_m,a_{D_m}}\) to the gradient scheme (2.10) in which \(K_{D_m}\) has been replaced with \(K_{D_m,a_{D_m}}\). If moreover \(a_{D_m} \to a\) in \(L^p(\partial \Omega)\) as \(m \to \infty\), then the convergences of \(\Pi_{D_m} u_m\) and \(\nabla_{D_m} u_m\) stated in Theorem 2.9 still holds.

**Theorem 4.2** (Convergence: non-linear obstacle problem, approximate barrier).
Under the assumptions of Theorem 2.9, let \((D_m)_{m \in \mathbb{N}}\) be a sequence of gradient discretisations in the sense of Definition 2.3, that is coercive, limit-conforming, compact, and GD-consistent (with \(S_D\) defined using \(K_{D,\psi_D}\) instead of \(K_D\)). Assume that \(K_{D_m,\psi_{D_m}}\) is non-empty for any \(m\).

Then, for any \(m \in \mathbb{N}\), there exists at least one solution \(u_m \in K_{D_m,\psi_{D_m}}\) to the gradient scheme (3.4) in which \(K_{D_m}\) has been replaced with \(K_{D_m,\psi_{D_m}}\). Furthermore, if \(\psi_{D_m} \to \psi\) in \(L^p(\Omega)\) as \(m \to \infty\), then the convergence of \(\Pi_{D_m} u_m\) and \(\nabla_{D_m} u_m\) given in Theorem 2.9 still holds.

5. **Application to the hybrid mixed mimetic methods**

The gradient discretisation method is used here to design a hybrid mimetic mixed (HMM) scheme for non-linear variational inequalities. It is shown in [21] that the HMM method gathers three different families: the hybrid finite volume method, the (mixed-hybrid) mimetic finite differences methods, and the mixed finite volume methods. The mimetic methods have become efficient tools to discretise heterogeneous anisotropic diffusion problems on generic meshes. In [2] we established the HMM method for the linear Signorini and obstacle problems (i.e., \(a(x,u,\nabla u) = \Lambda(x)\nabla u\)). The only other application, that we are aware of, of mimetic method to variational inequalities only concerns linear variational inequalities and the nodal mimetic finite difference method [5]. The HMM scheme described here for non-linear variational inequalities seems to be the first numerical scheme for these models on generic meshes.

Let us first recall the notion of polytopal mesh [22].

**Definition 5.1** (Polytopal mesh). Let \(\Omega\) be a bounded polytopal open subset of \(\mathbb{R}^d\) \((d \geq 1)\). A polytopal mesh of \(\Omega\) is given by \(\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P})\), where:

1. \(\mathcal{M}\) is a finite family of non empty connected polytopal open disjoint subsets of \(\Omega\) (the cells) such that \(\bar{\Omega} = \bigcup_{K \in \mathcal{M}} \bar{K}\). For any \(K \in \mathcal{M}\), \(|K| > 0\) is the measure of \(K\) and \(h_K\) denotes the diameter of \(K\).
2. \(\mathcal{E}\) is a finite family of disjoint subsets of \(\bar{\Omega}\) (the edges of the mesh in 2D, the faces in 3D), such that any \(\sigma \in \mathcal{E}\) is a non empty open subset of a hyperplane of \(\mathbb{R}^d\) and \(\sigma \subset \bar{\Omega}\). We assume that for all \(K \in \mathcal{M}\) there exists a subset \(\mathcal{E}_K\) of \(\mathcal{E}\) such that \(\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \sigma\). We then set \(\mathcal{M}_\sigma = \{K \in \mathcal{M} : \sigma \in \mathcal{E}_K\}\) and assume that, for all \(\sigma \in \mathcal{E}\), \(\mathcal{M}_\sigma\) has exactly one element and \(\sigma \subset \partial \Omega\), or \(\mathcal{M}_\sigma\) has two elements and \(\sigma \subset \Omega\). \(\mathcal{E}_{\text{int}}\) is the set of all interior faces, i.e.
\( \sigma \in \mathcal{E} \) such that \( \sigma \subset \Omega \), and \( \mathcal{E}_{\text{ext}} \) the set of boundary faces, i.e. \( \sigma \in \mathcal{E} \) such that \( \sigma \subset \partial \Omega \). For \( \sigma \in \mathcal{E} \), the \((d-1)\)-dimensional measure of \( \sigma \) is \( |\sigma| \), the centre of mass of \( \sigma \) is \( \mathcal{I}_\sigma \), and the diameter of \( \sigma \) is \( h_\sigma \).

(3) \( \mathcal{P} = (x_K)_{K \in \mathcal{M}} \) is a family of points of \( \Omega \) indexed by \( \mathcal{M} \) and such that, for all \( K \in \mathcal{M} \), \( x_K \in K \) (\( x_K \) is sometimes called the “centre” of \( K \)). We then assume that all cells \( K \in \mathcal{M} \) are strictly \( x_K \)-star-shaped, meaning that if \( x \notin K \) then the line segment \([x_K, x]\) is included in \( K \).

For a given \( K \in \mathcal{M} \), let \( n_{K, \sigma} \) be the unit vector normal to \( \sigma \) outward to \( K \) and denote by \( d_{K, \sigma} \) the orthogonal distance between \( x_K \) and \( \sigma \in \mathcal{E}_K \). The size of the discretisation is \( h_{\mathcal{M}} = \sup \{ h_K : K \in \mathcal{M} \} \).

5.1. HMM for the Signorini problem. Let \( \mathcal{T} \) be a polytopal mesh that is aligned with the boundaries \((\Gamma_i)_{i=1,2,3}\), that is, for any \( i = 1, 2, 3 \), each boundary edge is either fully included in \( \Gamma_i \) or disjoint from this set. We describe here a gradient discretisation that corresponds, for linear diffusion problems and standard boundary conditions, to the HMM method \([20, 22]\).

Define two discrete spaces as follows:

\[
X_{\mathcal{D}, \Gamma_{1,3}} = \{ v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}, v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\text{ext}} \text{ such that } \sigma \subset \Gamma_1 \} \tag{5.1}
\]

and

\[
X_{\mathcal{D}, \Gamma_1} = \{ v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}, v_K = 0 \text{ for all } K \in \mathcal{M}, v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\text{int}} \text{ and } v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\text{ext}} \text{ such that } \sigma \subset \Gamma_2 \cup \Gamma_3 \} \tag{5.2}
\]

The space

\[
X_{\mathcal{D}} = \{ v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R} \} \tag{5.3}
\]

is the direct sum of these two spaces. The piecewise-constant function reconstruction \( \Pi_{\mathcal{D}} \), the piecewise-constant trace reconstruction \( \Gamma_{\mathcal{D}} \) and the gradient reconstruction \( \nabla_{\mathcal{D}} \) are given by: \( \forall v \in X_{\mathcal{D}} \),

\[
\forall K \in \mathcal{M} : \Pi_{\mathcal{D}} v = v_K \text{ on } K, \\
\forall \sigma \in \mathcal{E}_{\text{ext}} : \Gamma_{\mathcal{D}} v = v_\sigma \text{ on } \sigma, \\
\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K : \nabla_{\mathcal{D}} v = \nabla v + \frac{d}{d_{K, \sigma}} (A_K R_K(v))_{\sigma} n_{K, \sigma} \text{ on } D_{K, \sigma},
\tag{5.4}
\]

where \( D_{K, \sigma} \) is the convex hull of \( \sigma \cup \{x_K\} \) and

- \( \nabla v = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| v_\sigma n_{K, \sigma} \),
- \( R_K(v) = (v_\sigma - v_K - \nabla_K v \cdot (\mathcal{I}_\sigma - x_K))_{\sigma \in \mathcal{E}_K} \in \mathbb{R}^{\mathcal{E}_K} \),
- \( A_K \) is an isomorphism of the vector space \( \text{Im}(R_K) \).

The interpolant \( \mathcal{I}_{\mathcal{D}, \Gamma_1} : W^{1-\frac{1}{p}, p} (\partial \Omega) \to X_{\mathcal{D}, \Gamma_1} \) is defined by

\[
\forall g \in W^{1-\frac{1}{p}, p} (\partial \Omega) : (\mathcal{I}_{\mathcal{D}, \Gamma_1} g)_\sigma = \frac{1}{|\sigma|} \int_{\partial \Omega} g(x) d\sigma(x),
\tag{5.5}
\]

for all \( \sigma \in \mathcal{E}_{\text{ext}} \) such that \( \sigma \subset \Gamma_1 \).

We then have \( \mathcal{K}_{\mathcal{D}} := \{ v \in X_{\mathcal{D}, \Gamma_{1,3}} + \mathcal{I}_{\mathcal{D}, \Gamma_1} g : v_\sigma \leq a \text{ on } \sigma, \forall \sigma \in \mathcal{E}_{\text{ext}} \text{ s.t. } \sigma \subset \Gamma_3 \} \), and the HMM discretisation of Problem (2.9) is the gradient scheme (2.10) corresponding to the gradient discretisation described above.
The convergence of the HMM scheme is a consequence of Theorem 2.9 and of the four properties proved in the following proposition. The condition (2.14) follows from the GD-consistency of HMM for Fourier boundary conditions (see [20, Definition 2.49 and Section 12.2.2]).

**Proposition 5.2.** Let \((\mathcal{D}_m)_{m\in\mathbb{N}}\) be a sequence of HMM GDs given by (5.1)–(5.4), for certain polytopal meshes \((\mathcal{T}_m)_{m\in\mathbb{N}}\). Assume the existence of \(\theta > 0\) such that, for any \(m \in \mathbb{N}\),

\[
\max_{K \in \mathcal{M}_m} \left( \max_{\sigma \in \mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} + \text{Card}(\mathcal{E}_K) \right) + \max_{\sigma \in \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{K,L\}} \left( \frac{d_{K,\sigma}}{d_{L,\sigma}} + \frac{d_{L,\sigma}}{d_{K,\sigma}} \right) \leq \theta
\]  

(5.6)

and, for all \(K \in \mathcal{M}_m\) and \(\mu \in \mathbb{R}^{\mathcal{E}_K}\),

\[
\frac{1}{\theta} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(\mu)}{d_{K,\sigma}} \right|^p \leq \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K R_{K}(\mu))_\sigma}{d_{K,\sigma}} \right|^p 
\]  

\[
\leq \theta \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(\mu)}{d_{K,\sigma}} \right|^p.
\]  

(5.7)

Then the sequence \((\mathcal{D}_m)_{m\in\mathbb{N}}\) is coercive, limit-conforming and compact in the sense of Definitions 2.5, 2.7 and 2.8. If moreover the function \(a\) is piecewise-constant on \(\mathcal{E}_{\text{ext}}\), then the sequence \((\mathcal{D}_m)_{m\in\mathbb{N}}\) is GD-consistent in the sense of Definition 2.6.

**Remark 5.3.** In [31], the convergence of numerical schemes for variational inequalities (with homogenous Dirichlet BC and constant barriers) is established by using the density of \(C^2(\Omega) \cap \mathcal{K}\) in \(\mathcal{K}\). We do not need such a density result in Proposition 5.2, which enables us to treat the case of piecewise-constant barriers.

**Proof.** The coercivity, limit-conformity and compactness follow as in the case of the HMM method for PDEs, see [20, Theorem 12.12, Remark 12.13 and Section 12.2]. To prove the GD-consistency, we make use of the interpolation operator described in the appendix. Let \(\varphi \in \mathcal{K}\) and take \(v_m = P_{\mathcal{D}_m}^a \varphi\), where \(P_{\mathcal{D}_m}^a\) is defined by (7.2) with weights \(\omega_m = (\omega_{m,K})_{K \in \mathcal{M}_m}\) given by Lemma 7.1. Since \(\gamma(\varphi) = g\) on \(\Gamma_1\), the definitions (5.5) and (7.2) of \(\mathcal{I}_{\mathcal{D}_m}^a, \Gamma_1\) and \(P_{\mathcal{D}_m}^a\) show that \(v_m \in \mathcal{I}_{\mathcal{D}_m}^a, \Gamma_1 g + X_{\mathcal{D}_m}^a, \Gamma_{2,3}\). Moreover, if \(\sigma \subset \Gamma_3\) then, since \(a\) is constant on \(\sigma\),

\[
(v_m)_\sigma = \frac{1}{|\sigma|} \int_\sigma \gamma(\varphi)(x) \, ds(x) \leq \frac{1}{|\sigma|} \int_\sigma a(x) \, ds(x) = a|_{\sigma}.
\]  

(5.8)

Hence, \(v_m \in \mathcal{K}_{\mathcal{D}_m}\) and thus

\[
\mathcal{S}_{\mathcal{D}_m}(\varphi) \leq \|\Pi_{\mathcal{D}_m} v_m - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}_m} v_m - \nabla \varphi\|_{L^p(\Omega)}.
\]

Proposition 7.2 shows that the right-hand side of this inequality tends to 0 as \(m \to \infty\), which concludes the proof of the GD-consistency of \((\mathcal{D}_m)_{m\in\mathbb{N}}\). □

**Remark 5.4 (Non-piecewise-constant barrier).** If the barrier \(a\) is not piecewise-constant on \(\mathcal{E}_{\text{ext}}\), in the context of HMM schemes it is natural to consider an approximate barrier as in Section 4. The function \(a_{\mathcal{D}_m}\) is simply defined as the piecewise-constant function such that

\[
\forall \sigma \in \mathcal{E}_{\text{ext}}, \quad (a_{\mathcal{D}_m})_\sigma = \frac{1}{|\sigma|} \int_\sigma a(x) \, ds(x).
\]

The same reasoning as in Step 1 of the proof of Proposition 7.2 shows that, under the assumptions of Proposition 5.2, \(a_{\mathcal{D}_m} \to a\) in \(L^p(\partial \Omega)\) as \(m \to \infty\). Moreover, if
\[ \varphi \in \mathcal{K} \text{ then the inequality in (5.8) shows that } v_m \text{ constructed in the proof above belongs to } \mathcal{K}_{D_m,a_{D_m}}. \text{ Hence, } (D_m)_{m \in \mathbb{N}} \text{ remains GD-consistent if } S_{D_m} \text{ is defined using } \mathcal{K}_{D_m,a_{D_m}} \text{ instead of } \mathcal{K}_{D_m}. \text{ Theorem 4.1 can therefore be applied and establishes the convergence of the HMM method with this approximate barrier.}

**Remark 5.5 (Non-conforming } P_1 \text{ finite elements).** With minor modifications in the proof of Proposition 7.2 regarding the approximation inside the cell, the arguments above can be used to analyse the gradient discretisations corresponding to non-conforming } P_1 \text{ finite elements [20, Chapter 9]. This shows that Theorems 2.9 and 4.1 apply to these non-conforming finite elements.}

5.2. **HMM methods for the obstacle problem and Bulkley model.** We use the notations introduced in Section 5.1. The elements of gradient discretisation \( D \) to consider here are given by

\[
X_{D,0} = \{ v = ((v_K)_{K \in M}, (v_\sigma)_{\sigma \in E}) : v_K \in \mathbb{R}, v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{ext} \},
\]

\[
X_{D,\partial \Omega} = \{ v = ((v_K)_{K \in M}, (v_\sigma)_{\sigma \in E}) : v_\sigma \in \mathbb{R}, v_K = 0 \text{ for all } K \in M, v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{int} \}.
\]

The discrete mappings \( \mathcal{I}_{D,\partial \Omega}, \Pi_D \) and \( \nabla_D \) are as in Section 5.1.

Setting \( \mathcal{K}_D := \{ v \in X_{D,0} + \mathcal{I}_{D,\partial \Omega}h : v_K \leq \psi \text{ on } K, \text{ for all } K \in M \} \), the HMM methods for (3.2) and (3.3) are respectively the gradient schemes (3.4) and (3.5) coming from the above gradient discretisation.

Recall that the coercivity, limit-conformity and compactness for sequences of GDs adapted to the obstacle problem are the same properties as for sequences of GDs for PDEs with non-homogeneous Dirichlet boundary conditions. The proof of these properties follow therefore from [20], under the regularity assumptions (5.6) and (5.7). If the barrier \( \psi \) is constant in each cell, the GD-consistency follows as in Proposition 5.2. Indeed, the weights \( \omega_{m,K} \) being non-negative, for all \( \varphi \in \mathcal{K}, m \in \mathbb{N} \) and \( K \in M \), we have

\[
(P_{D_m}^{\omega_m})|_K = \frac{1}{|K|} \int_K \omega_{m,K}(x) \varphi(x) \, dx 
\]

\[
\leq \frac{1}{|K|} \int_K \omega_{m,K}(x) \psi(x) \, dx = \psi|_K,
\]

which shows that \( P_{D_m}^{\omega_m} \varphi \in \mathcal{K}_{D_m} \).

For the Bulkley model, all the properties of GDs are identical to those for PDEs with homogeneous Dirichlet boundary conditions, and therefore follow (still under the assumptions (5.6) and (5.7)) from [22].

Using these properties, the convergence of the HMM method for each problem is a straightforward consequence of Theorems 3.9 and 3.10.

**Remark 5.6 (Non-piecewise-constant obstacle).** If \( \psi \) is not piecewise-constant on the mesh, we approximate it by the piecewise-constant function \( \psi_D \) defined by

\[
\forall K \in M, \quad (\psi_D)|_K = \frac{1}{|K|} \int_K \omega_K(x) \psi(x) \, dx.
\]

Step 1 in the proof of Proposition 7.2 shows that \( \psi_D \rightarrow \psi \) in \( L^p(\Omega) \) as \( m \rightarrow \infty \). Since \( P_{D_m}^{\omega_m} \varphi \in \mathcal{K}_{D_m,\psi_D,m} \) whenever \( \varphi \in \mathcal{K} \), this establishes the GD-consistency and
Theorem 4.2 then ensures the convergence of the HMM method for the non-linear obstacle problem with approximate barriers.

6. Numerical results

We demonstrate here the efficiency of the HMM method for solving non-linear Signorini problems by considering the meaningful example of the seepage model. Due to the double non-linearity in the model, two iterative algorithms are used in conjunction to compute a numerical solution: fixed point iterations to deal with the non-linear operator, and a monotonicity algorithm for the inequalities coming from the imposed Signorini boundary conditions.

We consider a test case from [51]. The geometry of the domain $\Omega$ representing the dam is illustrated in Fig 6.1. Letting $x = (x, y)$, the model reads

$$- \text{div}(\Lambda(x, \bar{u}) \nabla \bar{u}) = 0 \quad \text{in } \Omega,$$

$$\bar{u} = g \quad \text{on } \Gamma_1,$$

$$\Lambda(x, \bar{u}) \nabla \bar{u} \cdot n = 0 \quad \text{on } \Gamma_2,$$

$$\begin{cases} 
\bar{u} \leq y \\
\Lambda(x, \bar{u}) \nabla \bar{u} \cdot n \leq 0 \\
\Lambda(x, \bar{u}) \nabla u \cdot n(y - \bar{u}) = 0 
\end{cases} \quad \text{on } \Gamma_3,$$

with

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \in [0, 5]\} \cup \{(x, y) \in \mathbb{R}^2 : x + y = 7 \text{ and } y \in [0, 1]\},$$

$$\Gamma_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\},$$

$$\Gamma_3 = \{(x, y) \in \mathbb{R}^2 : y = 5 \text{ and } x \in [0, 2]\} \cup \{(x, y) \in \mathbb{R}^2 : x + y = 7 \text{ and } y \in (1, 5)\}.$$

![Figure 6.1. Geometry for the numerical tests.](image)

The boundary conditions $g$ is defined by $g(0, y) = 5$ for all $y \in [0, 5]$ and $g(x, y) = 1$, for all $x \in (0, 7)$. We set $\Lambda(x, s) = H_\lambda^\varepsilon(s - y)$ with a regularised Heavyside function $H_\lambda^\varepsilon$ given by

$$H_\lambda^\varepsilon(\rho) = \begin{cases} 
1 & \text{if } \rho \geq 0, \\
\frac{1}{\lambda} \varepsilon \rho + 1 & \text{if } -\lambda < \rho < 0, \\
\varepsilon & \text{if } \rho < -\lambda.
\end{cases}$$
Here, both \(\lambda\) and \(\varepsilon\) are taken equal to \(10^{-3}\). As stated above, to obtain the solution to this problem, we first apply simple fixed point iterations (Algorithm 1), the idea of which is to generate a sequence \((u^{(n)})_{n \in \mathbb{N}} \subset K_D\) by solving linear variational inequalities. These linear VIIs are obtained by fixing the non-linearity in the operator to the previous element in the sequence.

**Algorithm 1** Fixed point algorithm

1. Let \(\delta\) be a small number (stopping criteria) and \(u^{(0)} = 0\) \(\triangleright\) For us, \(\delta = 10^{-2}\).
2. for \(n = 1, 2, 3, \ldots\) do
3.   Solve the following linear VI, using Algorithm 2: \(\triangleright u^{(n)}\) is known
\[
\begin{align*}
\text{Find } u^{(n+1)} & \in K_D \text{ such that, for all } v \in K_D, \\
\int_{\Omega} & \Lambda(x, \Pi_D u^{(n)}) \nabla_D u^{(n+1)} \cdot \nabla_D (u^{(n+1)} - v) \, dx \\
& \leq \int_{\Omega} f(x) \Pi_D (u^{(n+1)} - v) \, dx.
\end{align*}
\]
4. if
\[
\|\Pi_D (u^{(n+1)} - u^{(n)})\|_{L^2(\Omega)} + \|
\nabla_D (u^{(n+1)} - u^{(n)})\|_{L^2(\Omega)} &
\leq \delta (\|\Pi_D u^{(n)}\|_{L^2(\Omega)} + \|\nabla_D u^{(n)}\|_{L^2(\Omega)})
\]
then
5. Exit “for” loop
6. end if
7. end for
8. Set \(u = u^{(n+1)}\)

In each iteration \(n\) in Algorithm 1, a linear VI must be solved. To compute its solution, introduce the linear fluxes \(u \mapsto F_{K,\sigma}^u(u)\) (for \(K \in \mathcal{M}\) and \(\sigma \in \mathcal{E}_K\)) defined by: for all \(K \in \mathcal{M}\) and all \(u, v, w \in X_D\),
\[
\sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}^u(u)(v_K - v_\sigma) = \int_K \Lambda(x, w_K) \nabla_D u \cdot \nabla_D v \, dx
\]
Choosing \(w = u^{(n)}\) in this relation, Problem (6.1) can be recast as [2]
\[
\sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}^u(u^{(n+1)}) = m(K)f_K, \quad \forall K \in \mathcal{M}
\]
\[
F_{K,L}^u(u^{(n+1)}) = 0, \quad \forall \sigma \in \mathcal{E}_{\text{int}} \text{ with } \mathcal{M}_{\sigma} = \{K, L\}, \quad u^{(n+1)}_\sigma = g, \quad \forall \sigma \in \mathcal{E}_{\text{ext}} \text{ such that } \sigma \subset \Gamma_1
\]
\[
F_{K,\sigma}^u(u^{(n+1)}) = 0, \quad \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K \text{ such that } \sigma \subset \Gamma_2,
\]
\[
F_{K,\sigma}^u(u^{(n+1)}) (u^{(n+1)} - y_\sigma) = 0, \quad \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K \text{ such that } \sigma \subset \Gamma_3,
\]
\[
-F_{K,\sigma}^u(u^{(n+1)}) \leq 0, \quad \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K \text{ such that } \sigma \subset \Gamma_3,
\]
Here \(y_\sigma\) denotes to \(y\)-coordinate of the centre of mass of edge \(\sigma\). This choice corresponds to the approximate barrier \(n_D\) of \(a(x) = y\) described in Remark 5.4. The monotonicity algorithm given in [36] is used to solve this non-linear system (see
Algorithm 2). It is proved in [34] that the number of iterations of this monotonicity algorithm is bounded by the number of edges in $\Gamma_3$. This algorithm only requires, at each of its steps, to solve a square linear system on unknowns $(w_K)_{K \in \mathcal{M}}, (w_\sigma)_{\sigma \in \mathcal{E}}$.

Algorithm 2: Monotonicity algorithm

1: (Only the first time the algorithm is called):
   Set $A^{(0)} = \{\sigma \in \mathcal{E} : \sigma \subset \Gamma_3\}$, $B = \emptyset$ and $I = \text{Card}(A^{(0)})$ \(\triangleright\) $I$ = theoretical bound on the iterations
2: while $i \leq I$ do
3:   $A^{(i)}$ and $B^{(i)}$ being known, find the solution $w$ to the system (6.2)–(6.5) together with
   $F_{K,\sigma}^{(n)}(w) = 0$, $\forall K \in \mathcal{M}$, $\forall \sigma \in \mathcal{E}_K$ such that $\sigma \in B^{(i)}$, $w_\sigma = \overline{y}_\sigma$, $\forall \sigma \in \mathcal{E}_{\text{ext}}$ such that $\sigma \in A^{(i)}$.
4:   Set $A^{(i+1)} = \{\sigma \in A^{(i)} : -F_{K,\sigma}^{(n)}(w) \leq 0\} \cup \{\sigma \in B^{(i)} : w_\sigma \geq \overline{y}_\sigma\}$
5:   Set $B^{(i+1)} = \{\sigma \in B^{(i)} : \sigma \subset \Gamma_3\}$ \(\triangleright\) $I$ = theoretical bound on the iterations
6:   if $A^{(i+1)} = A^{(i)}$ and $B^{(i+1)} = B^{(i)}$ then
7:     Exit “while” loop
8:   end if
9: end while
10: Set $u^{(n+1)} = w$ \(\triangleright\) Solution to (6.2)–(6.8)
11: (For next call of Algorithm 2) Set $A^{(0)} = A^{(i+1)}$ and $B^{(0)} = B^{(i+1)}$

Figure 6.2. The first mesh type (hexahedral, left) and the second mesh type (Kershaw, right).

Our numerical tests are conducted on two different mesh types given in Fig. 6.2. The first mesh (left) is build on hexagonal cells with 441 cells, maximum size $h_{\mathcal{M}} \approx 0.69$, and a number of edges in $\Gamma_3$ equal to $N = 72$. The second mesh (right) comes from the “Kershaw mesh” in [35]; it has 2601 cells, a maximum size similar to that of the hexagonal mesh, and $N = 92$. For the hexagonal mesh, the fixed point algorithm (Algorithm 1) converges in 5 iterations. For the Kershaw mesh, an oscillating phenomenon occurs: for $n \geq 9$, $u^{(n)} \approx u^{(n-2)}$ but $u^{(n)} \neq u^{(n+1)}$; the Kačanov algorithm does not converge, but essentially alternates between two vectors. To break this oscillation, we use an under-relaxation technique: when it is
found that $|u^{(n)} - u^{(n-2)}|_{\infty} \leq 10^{-2}|u^{(n-2)}|_{\infty}$, where $|\cdot|_{\infty}$ is the maximum norm of vectors in $X_D$, denoting by $\tilde{u}^{(n+1)}$ the solution of (6.1) we actually set $u^{(n+1)} = u^{(n)} + 0.5(\tilde{u}^{(n+1)} - u^{(n)})$; that is, we only progress halfway from $u^{(n)}$ to $\tilde{u}^{(n+1)}$. This tweak enables the fixed-point algorithm to converge in 11 iterations overall, which remains quite low given the distortion of the grid. Note that this oscillation issue is only perceptible on this particular Kershaw mesh; when considering the next two meshes, with sizes $h_M \approx 0.52$ and $h_M \approx 0.42$, in the Kershaw family presented in [35], the fixed-point algorithm converges in 6 and 8 iterations respectively, without the need for under-relaxation.

The maximum number of iterations of the monotonicity algorithm (Algorithm 2) is also very far from the theoretical bound, with 6 for the hexagonal mesh and 5 for the Kershaw mesh. Note that, between two iterations of the fixed-point algorithm, the sets $A$ and $B$ in the monotonicity algorithm are not reset. This means that the final sets obtained at level $n$ of Algorithm 1 are used as initial guesses at level $n+1$ of this algorithm. This considerably reduces the number of iterations of Algorithm 2 and, after the first 2 or 3 iterations of Algorithm 1, the monotonicity algorithm converges in only 1 or 2 iterations.

The monotonicity algorithm offers a way to determine the location of the seepage point. Following the interpretation of the model in [51], the seepage point should split the free boundary $\Gamma_3$ into upper and lower parts in the following way: (1) there is no flow on the upper part (so $F_{K,\sigma} = 0$ for every edge $\sigma$ in this part); (2) the pore pressure vanishes on the lower part (so $\bar{u} = y$ on this part); (3) both conditions are satisfied at the seepage point. The first and second conditions are naturally expressed by Equation (6.9). Since any edge in the set $B$ cannot satisfy the last property (due to the strict inequality $w_\sigma < \bar{y}_\sigma$), the seepage point does not lie on those edges. This point can thus be located at the edge $\sigma$ in the set $A$ whose midpoint has the largest ordinate $y_\sigma$. Considering the mesh size and the fact that the HMM solution is computed at the mid-point of edges, our numerical results locate the seepage point at an ordinate in [3.31, 3.65] for the hexahedral mesh, and in [3.28, 3.63] for the Kershaw mesh. This location is in perfect agreement with the numerical tests in [51].

Testing the scheme on meshes with larger numbers of cells, the seepage location does not change much. We observe that the seepage position moves only by 1% for
a 1681-cell refinement of the hexahedral mesh, and by 2% for a 4624-cell refinement of the Kershaw mesh.

Fig 6.3 shows the streamlines of the Darcy velocity field of the solution. These lines are generated by MATLAB via the streamline function. As expected, the distorted cells at the top of the domain provoke perturbations of the streamlines there. Quite remarkably, though, this grid distortion does not impact the location of the seepage point. Elsewhere, the streamlines are very similar to the ones in [51].

7. Appendix: interpolant for the HMM method

Let \( \mathcal{T} \) be a polytopal mesh of \( \Omega \), and select weights \( \omega = (\omega_K)_{K \in \mathcal{M}} \) with, for all \( K \in \mathcal{M}, \omega_K \in L^\infty(K) \) such that
\[
\frac{1}{|K|} \int_K \omega_K(x) \, dx = 1 \quad \text{and} \quad \frac{1}{|K|} \int_K x \omega_K(x) \, dx = x_K. \tag{7.1}
\]

Recalling the definition (5.3) of the space \( X_D \), the interpolant \( P_D^\omega : W^{1,p}(\Omega) \to X_D \) is then defined by:
\[
\forall \phi \in W^{1,p}(\Omega), \quad P_D^\omega \phi = ((\phi_K^\omega)_{K \in \mathcal{M}}, (\phi_{\sigma})_{\sigma \in \mathcal{E}}) \quad \text{with} \quad \forall K \in \mathcal{M}, \phi_K^\omega = \frac{1}{|K|} \int_K \omega_K(x) \phi(x) \, dx, \tag{7.2}
\]
\[
\forall \sigma \in \mathcal{E}, \phi_{\sigma} = \frac{1}{|\sigma|} \int_{\sigma} \phi(x) \, ds(x).
\]

This interpolant enjoys nice approximation properties for the HMM gradient discretisation. Before stating and proving these, we first establish the existence of weights with suitable properties. Note that [24, Lemma A.7] already gives a construction of such weights (as linear functions), without the positivity property.

**Lemma 7.1** (Existence of weights). Let \( \mathcal{T} \) be a polytopal mesh and let
\[
\rho_\mathcal{T} = \max_{K \in \mathcal{M}} \max_{\sigma \in \mathcal{E}} \min_{K \in \mathcal{M}} \frac{h_K}{d_{K,\sigma}}.
\]
Then there exists weights \( \omega = (\omega_K)_{K \in \mathcal{M}} \) satisfying (7.1) and such that
\[
\forall K \in \mathcal{M}, \forall x \in K, \quad 0 \leq \omega_K(x) \leq \rho_\mathcal{T}. \tag{7.3}
\]

**Proof.** By [20, Lemma B.1], for all \( K \in \mathcal{M} \) the ball \( B_K \) of center \( x_K \) and radius \( \rho_\mathcal{T}^{-1} h_K \) is fully contained in \( K \). Let us define \( \omega = (\omega_K)_{K \in \mathcal{M}} \) by
\[
\forall K \in \mathcal{M}, \forall x \in K, \quad \omega_K(x) = \begin{cases} \frac{|K|}{|B_K|} & \text{if } x \in B_K, \\ 0 & \text{if } x \notin B_K. \end{cases}
\]
Denoting by \( V_1 \) the volume of the unit ball in \( \mathbb{R}^d \), since \( K \) is contained in the ball of center \( x_K \) and radius \( h_K \) we have \( |K| \leq V_1 h_K^d \). On the other hand, \( |B_K| = V_1 (\rho_\mathcal{T}^{-1} h_K)^d \). Hence, \( |K|/|B_K| \leq \rho_\mathcal{T}^d \) and (7.3) is satisfied.

The relations (7.1) are trivial since \( \int_{B_K} 1 \, dx = |B_K| \) and \( \int_{B_K} x \, dx = x_K |B_K| \) (as \( x_K \) is the center of \( B_K \)). \( \square \)

**Proposition 7.2** (Approximation properties of \( P_D^\omega \)). Let \( (\mathcal{T}_m)_{m \in \mathbb{N}} \) be a sequence of polytopal meshes such that \( h_{\mathcal{T}_m} \to 0 \) as \( m \to \infty \) and, for some \( \theta > 0 \), (5.6) holds for all \( m \in \mathbb{N} \). For each \( m \in \mathbb{N} \), take weights \( \omega_m = (\omega_{m,K})_{K \in \mathcal{M}_m} \) given by Lemma 7.1.
For $m \in \mathbb{N}$, let $\mathcal{D}_m = (X_{\mathcal{D}_m}, \Pi_{\mathcal{D}_m}, T_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ be the HMM gradient discretisations defined on $\mathcal{T}_m$ by (5.4), without specific boundary conditions. Then, for all $\varphi \in W^{1,p}(\Omega)$, as $m \to \infty$,

\[
\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) \to \varphi \text{ in } L^p(\Omega),
\]

\[
T_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) \to \gamma(\varphi) \text{ in } L^p(\partial \Omega),
\]

\[
\nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) \to \nabla \varphi \text{ in } L^p(\Omega)^d.
\]

**Proof.** Note that by choice of $\theta$ and by Lemma 7.1, for all $m \in \mathbb{N}$ and $K \in \mathcal{M}_m$, $\|\omega_{m,K}\|_{L^\infty(K)} \leq \theta^d$.

**Step 1:** convergence of the function and trace reconstructions.

Fix $\varepsilon > 0$ and take $\varphi_\varepsilon \in C^\infty_c(\mathbb{R}^d)$ such that $\|\varphi - \varphi_\varepsilon\|_{W^{1,p}(\Omega)} \leq \varepsilon$. A triangle inequality yields

\[
\|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) - \varphi\|_{L^p(\Omega)} \leq \|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m}(\varphi - \varphi_\varepsilon))\|_{L^p(\Omega)} + \|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi_\varepsilon) - \varphi_\varepsilon\|_{L^p(\Omega)} + \|\varphi_\varepsilon - \varphi\|_{L^p(\Omega)}.
\] (7.4)

By Jensen’s inequality, for any $\psi \in W^{1,p}(\Omega)$ and $K \in \mathcal{M}_m$,

\[
|\psi_{|K}^m|^p \leq \frac{1}{|K|} \int_K |\omega_{m,K}(x)|^p |\psi(x)|^p \, dx \leq \frac{\|\omega_{m,K}\|_{L^\infty(K)}}{|K|} \int_K |\psi(x)|^p \, dx.
\]

Multiplying by $|K|$, summing over $K \in \mathcal{M}_m$, and recalling the definition of $\Pi_{\mathcal{D}_m}$ gives, by choice of $\theta$,

\[
\|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \psi)\|_{L^p(\Omega)} \leq \theta^d \|\psi\|_{L^p(\Omega)}.
\] (7.5)

Using this estimate with $\psi = \varphi - \varphi_\varepsilon$ in (7.4) yields

\[
\|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) - \varphi\|_{L^p(\Omega)} \leq (\theta^d + 1)\varepsilon + \|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi_\varepsilon) - \varphi_\varepsilon\|_{L^p(\Omega)}.
\] (7.6)

For all $K \in \mathcal{M}_m$ and $y \in K$, by (7.1) and choice of $\theta$ we have

\[
|\varphi_\varepsilon_{|K}^m - \varphi_\varepsilon(y)| = \frac{1}{|K|} \int_K \omega_{m,K}(x) |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \, dx \leq \theta^d h_K \sup_{\mathbb{R}^d} |\nabla \varphi_\varepsilon|.
\]

Hence, $\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi_\varepsilon) \to \varphi_\varepsilon$ uniformly on $\Omega$ as $m \to \infty$. Taking the superior limit as $m \to \infty$ of (7.6) therefore leads to

\[
\limsup_{m \to \infty} \|\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) - \varphi\|_{L^p(\Omega)} \leq (\theta^d + 1)\varepsilon.
\]

Letting $\varepsilon \to 0$ concludes the proof that $\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi) \to \varphi$ in $L^p(\Omega)$ as $m \to \infty$.

The convergence of the reconstructed traces is identical, since they satisfy an equivalent of the stability estimate (7.5), namely $\|T_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \psi)\|_{L^p(\partial \Omega)} \leq \|\gamma \psi\|_{L^p(\Omega)}$.

**Step 2:** convergence of the gradient reconstructions.

The proof of this convergence follows a similar reasoning, provided that we can establish the two following convergence and stability results:

\[
\nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \varphi_\varepsilon) \to \nabla \varphi_\varepsilon \text{ in } L^p(\Omega)^d \text{ as } m \to \infty,
\] (7.7)

and

\[
\exists C > 0, \forall m \in \mathbb{N}, \forall \psi \in W^{1,p}(\Omega), \|\nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}^{\omega_m} \psi)\|_{L^p(\Omega)^d} \leq C \|\nabla \psi\|_{L^p(\Omega)^d}.
\] (7.8)

In the following, $C$ denotes a generic constant that can change from one line to the other, but depends only on $\Omega$, $p$ and $\theta$. 
We first aim at proving (7.7). Let \( \tilde{P}_m \varphi_\varepsilon \) = \( ((\varphi_\varepsilon(x_K))_{K \in \mathcal{M}}, (\varphi_\varepsilon(\mathbf{r}_\sigma))_{\sigma \in \mathcal{E}}) \in X_{D_m} \). As a consequence of [20, Lemma 12.8 and proof of Proposition 7.36], since \( \varphi_\varepsilon \in C^\infty(\mathbb{R}^d) \),
\[
\nabla_{D_m}(\tilde{P}_m \varphi_\varepsilon) \to \nabla \varphi_\varepsilon \quad \text{in } L^p(\Omega)^d \quad \text{as } m \to \infty.
\] (7.9)

Define the following discrete \( W^{1,p} \)-semi-norm on \( X_{D_m} \):

\[
\forall v \in X_{D_m}, |v|_{T_{m,p}} = \left( \sum_{K \in \mathcal{M}_m} \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} \left| \frac{v_\sigma - v_{K}}{d_{K,\sigma}} \right|^p \right)^{1/p}.
\]

It follows from [22, Lemma 5.3] that

\[
\forall v \in X_{D_m}, \|\nabla_{D_m} v\|_{L^p(\Omega)^d} \leq C|v|_{T_{m,p}}.
\] (7.10)

Since \( \mathbf{r}_\sigma \) is the center of mass of \( \sigma \), a Taylor expansion of order 2 shows that

\[
|\varphi_\varepsilon(\mathbf{r}_\sigma) - (\varphi_\varepsilon)_{\sigma}| = \left| \varphi_\varepsilon(\mathbf{r}_\sigma) - \frac{1}{|\sigma|} \int_\sigma \varphi_\varepsilon(x) \, dx \right| \leq C\|D^2 \varphi_\varepsilon\|_{C_b(\mathbb{R}^d)} h_\sigma^2.
\] (7.11)

Moreover, [24, Lemma A.7] yields

\[
|\varphi_\varepsilon(x_K) - (\varphi_\varepsilon)_{K}| = \left| \varphi_\varepsilon(x_K) - \frac{1}{|K|} \int_\sigma \omega_{m,K}(x) \varphi_\varepsilon(x) \, dx \right| \leq C\|\varphi_\varepsilon\|_{C^2_b(\mathbb{R}^d)} h_K^2.
\] (7.12)

Estimates (7.11) and (7.12), and the properties

\[
\frac{h_\sigma}{d_{K,\sigma}} \leq \frac{h_K}{d_{K,\sigma}} \leq \theta \quad \text{for all } K \in \mathcal{M}_m \text{ and } \sigma \in \mathcal{E}_K
\]

and (see [20, Lemma B.2])

\[
\sum_{K \in \mathcal{M}_m} \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} = \sum_{K \in \mathcal{M}_m} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} = d \sum_{K \in \mathcal{M}_m} |K| = d|\Omega|
\]

show that \( |\tilde{P}_m \varphi_\varepsilon - P_{D_m}^m \varphi_\varepsilon|_{T_{m,p}} \leq C\|\varphi_\varepsilon\|_{C^2_b(\mathbb{R}^d)} h_{M_m} \). Applying then (7.10) to \( v = \tilde{P}_m \varphi_\varepsilon - P_{D_m}^m \varphi_\varepsilon \) gives

\[
\nabla_{D_m}(\tilde{P}_m \varphi_\varepsilon) - \nabla_{D_m}(P_{D_m}^m \varphi_\varepsilon) \to 0 \quad \text{in } L^p(\Omega)^d \quad \text{as } m \to \infty.
\]

Combined with (7.9), this establishes (7.7).

Let us now turn to the stability estimate (7.8). By [20, Proposition 7.15],

\[
|P_{D_m}^1 \psi|_{T_{m,p}} \leq C\|\nabla \psi\|_{L^p(\Omega)^d}
\] (7.13)

where \( P_{D_m}^1 \) is the interpolant (7.2) computed with the constant weights \( \omega_K = 1 \).

Let us estimate \( |P_{D_m}^1 \psi - P_{D_m}^m \psi|_{T_{m,p}} \). For \( K \in \mathcal{M}_m \), since \( \frac{1}{|K|} \int_K \omega_{m,K}(x) \, dx = 1 \), we can write

\[
|P_{D_m}^1 \psi|_K - P_{D_m}^m \psi|_K| K \bigg| = \left| \frac{1}{|K|} \int_K \psi(y) \, dy \right| \bigg| - \left| \frac{1}{|K|} \int_K \omega_{m,K}(x) \psi(x) \, dx \right| \\
= \left| \frac{1}{|K|} \int_K \omega_{m,K}(x) \left( \frac{1}{|K|} \int_K \psi(y) \, dy - \psi(x) \right) \, dx \right| \\
\leq \theta d \left| \int_K \frac{1}{|K|} \int_K \psi(y) \, dy - \psi(x) \right| \, dx.
\]
Use then Jensen’s inequality and [20, Lemma B.7] to write
\[
\left| (P_{Dm}^1 \psi)_K - (P_{Dm}^\sigma \psi)_K \right|^p \leq \frac{\theta^p}{|K|} \int_K \left| \frac{1}{|K|} \int_K \psi(y) \, dy - \psi(x) \right|^p \, dx \\
\leq \frac{Ch^p}{|K|} \int_K |\nabla \psi(x)|^p \, dx.
\] (7.14)

Since \( P_{Dm}^1 \psi \) and \( P_{Dm}^\sigma \psi \) have the same face values, only the difference of their cell values is involved in the computation of \( |P_{Dm}^1 \psi - P_{Dm}^\sigma \psi|_{T_m,p} \). Hence, dividing (7.14) by \( d_{K,\sigma}^p \) for any \( \sigma \in \mathcal{E}_K \), using \( h_K/d_{K,\sigma} \leq \theta \), multiplying by \( |\sigma|d_{K,\sigma} \), summing over \( \sigma \in \mathcal{E}_K \), using \( \sum_{\sigma \in \mathcal{E}_K} |\sigma|d_{K,\sigma} = d|K| \), and summing over \( K \in \mathcal{M}_m \) leads to
\[
|P_{Dm}^1 \psi - P_{Dm}^\sigma \psi|_{T_m,p} \leq C\|\nabla \psi\|_{L^p(\Omega)}^d.
\]

Combined with (7.13), this shows that \( |P_{Dm}^\sigma \psi|_{T_m,p} \leq C\|\nabla \psi\|_{L^p(\Omega)}^d \). The estimate (7.8) then follows from (7.10) applied to \( v = P_{Dm}^\sigma \psi \). □

References

[1] Y. Alnashri and J. Droniou, Gradient schemes for an obstacle problem, in Finite Volumes for Complex Applications VII-Methods and Theoretical Aspects, J. Fuhrmann, M. Ohlberger, and C. Rohde, eds., vol. 77, Springer International Publishing, 2014, pp. 67–75.

[2] Y. Alnashri and J. Droniou, Gradient schemes for the Signorini and the obstacle problems, and application to hybrid mimetic mixed methods, Computers and Mathematics with Applications, (2016), p. 20p. To appear.

[3] B. Andreianov, F. Boyer, and F. Hubert, Besov regularity and new error estimates for finite volume approximations of the p-Laplacian, Numer. Math., 100 (2005), pp. 565–592.

[4] ———, Discrete Duality Finite Volume schemes for Leray–Lions-type elliptic problems on general 2D meshes, Num. Meth. PDEs, 23 (2007), pp. 145–195.

[5] P. Antonietti, L. Beirão da Veiga, N. Bigoni, and M. Verani, Mimetic finite differences for nonlinear and control problems, Mathematical Models and Methods in Applied Sciences, 24 (2014), pp. 1457–1493.

[6] J. W. Barrett and W. Liu, Finite element approximation of the p-Laplacian, mathematics of computation, 61 (1993), pp. 523–537.

[7] R. Bermejo and J.-A. Infante, A multigrid algorithm for the p-Laplacian, SIAM journal on scientific computing, 21 (2000), pp. 1774–1789.

[8] I. Boatto and T. Gallouët, Nonlinear elliptic equations with right hand side measures, Communications in Partial Differential Equations, 17 (1992), pp. 189–258.

[9] F. Brezzi, K. Lipnikov, and V. Simoncini, A family of mimetic finite difference methods on polygonal and polyhedral meshes, Math. Models Methods Appl. Sci., 15 (2005), pp. 1533–1551.

[10] J. Chabrowski, On an obstacle problem for degenerate elliptic operators involving the critical sobolev exponent, Journal of Fixed Point Theory and Applications, 4 (2008), pp. 137–150.

[11] J. C. De los Reyes and S. G. Andrade, Numerical simulation of two-dimensional bingham fluid flow by semismooth neutron methods, Journal of computational and applied mathematics, 235 (2010), pp. 11–32.

[12] J. C. De los Reyes and S. González Andrade, Numerical simulation of two-dimensional Bingham fluid flow by semismooth Newton methods, J. Comput. Appl. Math., 235 (2010), pp. 11–32.

[13] D. A. Di Pietro and J. Droniou, A hybrid high-order method for Leray–Lions elliptic equations on general meshes, Math. Comp., (2016), p. 32p. To appear. http://arxiv.org/abs/1508.01918.

[14] ———, \( W^{s,p} \)-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a hybrid high-order discretisation of Leray–Lions problems, (2016). Submitted.
[15] J. Droniou, Finite volume schemes for fully non-linear elliptic equations in divergence form, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 40 (2006), pp. 1069–1100.

[16] J. Droniou, Finite volume schemes for diffusion equations: introduction to and review of modern methods, Math. Models Methods Appl. Sci., 24 (2014), pp. 1575–1619.

[17] J. Droniou, Introduction to discrete functional analysis techniques for the numerical study of diffusion equations with irregular data, in Proceedings of the 17th Biennial Computational Techniques and Applications Conference (CTAC-2014, Canberra), J. Sharples and J. Bunder, eds., vol. 56 of ANZIAM J., 2015, pp. C101–C127.

[18] J. Droniou and R. Eymard, A mixed finite volume scheme for anisotropic diffusion problems on any grid, Numer. Math., 105 (2006), pp. 35–71.

[19] J. Droniou and R. Eymard, Uniform-in-time convergence of numerical methods for nonlinear degenerate parabolic equations, Numer. Math., 132 (2016), pp. 721–766.

[20] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, and R. Herbin, The gradient discretisation method: A framework for the discretisation and numerical analysis of linear and nonlinear elliptic and parabolic problems, 2016. 425p, https://hal.archives-ouvertes.fr/hal-01382358 (version 3).

[21] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin, A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods, Mathematical Models and Methods in Applied Sciences, 20 (2010), pp. 265–295.

[22] J. Droniou, R. Eymard, and R. Herbin, Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic problems, Mathematical Models and Methods in Applied Sciences, 23 (2013), pp. 2395–2432.

[23] J. Droniou, R. Eymard, and R. Herbin, Gradient schemes: generic tools for the numerical analysis of diffusion equations, M2AN Math. Model. Numer. Anal., 50 (2016), pp. 749–781. Special issue – Polyhedral discretization for PDE.

[24] J. Droniou and N. Nataraj, Improved $L^2$ estimate for gradient schemes and superconvergence of the tpfa finite volume scheme, IMA J. Numer. Anal., (2017), p. 40p. To appear.

[25] R. Eymard, P. Feron, T. Gallouët, R. Herbin, and C. Guichard, Gradient schemes for the Stefan problem, International Journal On Finite Volumes, 10s (2013).

[26] R. Eymard, T. Gallouët, and R. Herbin, Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces, IMA J. Numer. Anal., 30 (2010), pp. 1009–1043.

[27] R. Eymard, C. Guichard, and R. Herbin, Small-stencil 3D schemes for diffusive flows in porous media, ESAIM. Mathematical Modelling and Numerical Analysis, 46 (2012), pp. 265–290. Copyright - EDP Sciences, SMAI, 2011; Last updated - 2013-03-05.

[28] R. Eymard, C. Guichard, R. Herbin, and R. Masson, Gradient schemes for two-phase flow in heterogeneous porous media and Richards equation, ZAMM Z. Angew. Math. Mech., 94 (2014), pp. 560–585.

[29] M. Feistauer and V. Sobotíková, Finite element approximation of nonlinear elliptic problems with discontinuous coefficients, RAIRO-Modélisation mathématique et analyse numérique, 24 (1990), pp. 457–500.

[30] I. A. Frigaard, S. Leimgruber, and O. Scherzer, Variational methods and maximal residual wall layers, Journal of Fluid Mechanics, 483 (2003), pp. 37–65.

[31] R. Glowinski, Numerical methods for non-linear variational problems, Tata Institute of fundamental research, Bombay, 1980.

[32] R. Glowinski, J. Lions, and R. Tremolieres, Numerical analysis of variational inequalities, North-Holland Publishing Company, 8 ed., 1981.

[33] R. Glowinski and J. Rappaz, Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology, ESAIM: Math. Model Numer. Anal. (M2AN), 37 (2003), pp. 175–186.

[34] R. Herbin, A monotonic method for the numerical solution of some free boundary value problems, SIAM journal on numerical analysis, 40 (2002), pp. 2292–2310.

[35] R. Herbin and F. Hubert, Benchmark on discretization schemes for anisotropic diffusion problems on general grids, in Finite volumes for complex applications V, ISTE, London, 2008, pp. 659–692.
[36] R. Herbin and E. Marchand, *Finite volume approximation of a class of variational inequalities*, IMA Journal of Numerical Analysis, 21 (2001), pp. 553–585.

[37] M. Hintermiller and C. N. Rautenberg, *A sequential minimisation technique for elliptic quasi-variational inequalities with gradient constraints*, SIAM Journal on Optimization, 22 (2012), pp. 1224–1257.

[38] R. R. Huilgol and Z. You, *Application of the augmented Lagrangian method to steady pipe flows of Bingham, Casson and Herschel–Bulkley fluids*, Journal of non-newtonian fluid mechanics, 128 (2005), pp. 126–143.

[39] G. Jouvet and E. Bueler, *Steady, shallow ice sheets as obstacle problems: well-posedness and finite element approximation*, SIAM Journal on Applied Mathematics, 72 (2012), pp. 1292–1314.

[40] M. Křížek and P. Goitza, *Finite element approximation of variational problems and applications*, vol. 50, Longman Scientific & Technical, 1990.

[41] V. K. Le and K. Schmitt, *On boundary value problems for degenerate quasilinear elliptic equations and inequalities*, Journal of Differential Equations, 144 (1998), pp. 170 – 218.

[42] J. Lions, *Quelques méthodes de résolution des problemes aux limites non linéaires*, vol. 31, Dunod Paris, 1969.

[43] J.-L. Lions, *Inequalities in mechanics and physics*, Springer, 1976.

[44] W. Liu and N. Yan, *Quasi-norm a priori and a posteriori error estimates for the nonconforming approximation of p-Laplacian*, Numer. Math., 89 (2001), pp. 341–378.

[45] W. B. Liu and J. W. Barrett, *Quasi-norm error bounds for the finite element approximation of some degenerate quasilinear parabolic equations and variational inequalities*, Numerical Functional Analysis and Optimization, 16 (1995), pp. 1309–1321.

[46] Liu, W. B. and Barrett, John W., *Quasi-norm error bounds for the finite element approximation of some degenerate quasilinear elliptic equations and variational inequalities*, ESAIM: M2AN, 28 (1994), pp. 725–744.

[47] J. Oden and N. Kikuchi, *Theory of variational inequalities with applications to problems of flow through porous media*, International Journal of Engineering Science, 18 (1980), pp. 1173 – 1284.

[48] D. Sankar and U. Lee, *Two-fluid Herschel–Bulkley model for blood flow in catheterized arteries*, Journal of Mechanical Science and Technology, 22 (2008), pp. 1008–1018.

[49] C. Scheven, *Gradient potential estimates in non-linear elliptic obstacle problems with measure data*, Journal of Functional Analysis, 262 (2012), pp. 2777–2832.

[50] M. Struwe and J. Jost, *Variational methods, applications to nonlinear partial differential equations and hamiltonian systems*, Jahresbericht der Deutschen Mathematiker Vereinigung, 96 (1994), pp. 24 – 25.

[51] H. Zheng, H. Chao Dai, and D. F. Liu, *A variational inequality formulation for unconfined seepage problems in porous media*, Applied Mathematical Modelling, 33 (2009), pp. 437 – 450.

(Yahya Alnashri) School of Mathematical Sciences, Monash University, Victoria 3800, Australia, and Umm-Alqura University
E-mail address: yanashri@ummalqura.edu

(Jérôme Droniou) School of Mathematical Sciences, Monash University, Victoria 3800, Australia.
E-mail address: jerome.droniou@monash.edu