The causal boundary and its relations with the conformal boundary

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Abstract. Our aim in this note is to present the results (obtained in [2]) which ensure that, under certain regularity conditions, the conformal boundary becomes equal to the causal boundary, not only as a point set, but in a topological and chronological level. In particular, under these conditions the conformal boundary becomes a powerful tool to compute the causal one.

1. Preliminaries
In this section, we will introduce the basic concepts related with the causal and conformal boundaries.

1.1. Causal boundary.
The construction of the causal boundary (called c-boundary for short) will follow from the natural idea (introduced in [3]) of adding ideal points in order to: ensure that any timelike curve have an endpoint; and two timelike curves with the same past (or future) are attached the same ideal point. In order to formalize this idea, the following notions are introduced: A subset \( P \subset M \) is called a past set if it coincides with its past; i.e, \( P = I^{-}[P] := \{ p \in M : p \ll q \text{ for some } q \in P \} \). If \( P \) cannot be expressed as the union of two proper past sets, it will be called indecomposable past set, \( IP \). IPs can be classified in two different sets: \( PIPS \), proper indecomposable past sets, which are IPs defined as the past of a point; \( TIPS \), terminal indecomposable past sets, which are IPs defined as the past of a future-directed inextendible timelike curve. The common past of \( S \subset M \) is defined by \( \downarrow S := I^{-}\{p \in M : p \ll q \forall q \in S\} \). The analogous concepts of future sets, \( IIF, PIF,TIF \) and common future are obtained by interchanging the roles of past and future.

Thanks to the fact that the spacetime is strongly causal (and then, in particular, future and past distinguishing), \( M \) can be identified with the set of \( PIPS \) (\( PIFs \)). Then, the future causal boundary \( \partial M \) of \( M \) will be defined as the set of all \( TIPS \) in \( M \). Finally, the future causal completion \( M \) will be defined as the set of all IPs:

\[
M \equiv PIPS, \quad \partial M \equiv TIPS, \quad \hat{M} \equiv IPs.
\]

Analogously, the concepts of past causal boundary \( \partial M \) and past causal completion \( \hat{M} \) rise naturally as:

\[
M \equiv PIFs, \quad \partial M \equiv TIFs, \quad \hat{M} \equiv IFs.
\]
In order to define a (total) causal boundary, the union of future and past boundaries becomes pathological and make necessary to consider a different approach. After some unsatisfactory approaches, Marolf and Ross introduce in [4] an alternative procedure based on forming pairs with TIPs and TIFs. To formulate the construction, the following relation has to be introduced (see [5] for details): two terminal sets $P, F$ are $S$-related, $P \sim S F$, if $P$ is maximal IP into $\uparrow F$ and $F$ is maximal IF in $\uparrow P$. Then, the $c$-completion and $c$-boundary will be defined as

$$\overline{M} := \{(P, F): P \sim S F, P \text{ an IP}, F \text{ an IF}\} \quad \partial M := \overline{M} \setminus M$$

where $M$ is identified with the pairs $(I^-(p), I^+(p))$ with $p \in M$ and if $P$ (analogously for $F$) is not $S$-related with nobody, the pair $(P, \emptyset)$ (analogously $(\emptyset, F)$) is formed.

The natural way to define a chronological relation to the $c$-completion is the following: $(P, F), (P', F') \in \overline{M}$ are chronologically related, $(P, F) \ll (P', F')$, if $P' \cap F \neq \emptyset$. The topology considered will be the chronological topology (see [1] and [2] for details).

### 1.2. Conformal boundary.

In order to compute the conformal boundary of the spacetime $M$, we will consider a (conformal) envelopment, i.e., an open conformal embedding $i: M \rightarrow M_0$, where $M_0$ is a ("aphysical") strongly causal spacetime. Then, we will define the conformal completion (associated to the envelopment $i$) as the topological adherence of the image of $M$, $\overline{M}_i := \overline{i(M)}$; and the conformal boundary will be defined in the natural way, $\partial_{c} M = \overline{M}_i \setminus i(M)$. The set of points in $\partial_{c} M$ which are endpoints of inextendible timelike curves of $M$ are called the accessible part of the boundary, and it will be denoted by $\partial^a_{c} M$. Without lost of generality, we will suppose that $M \subset M_0$, leaving the sub-index $i$ just in order to distinguish the conformal and causal boundaries (or completions). Finally, the topology on $\overline{M}_i$ will be the induced one from $M_0$ and the chronological and causal relations, denoted by $\ll$ and $\leq$ resp., will be defined in the following way: $p \ll q$ iff there exists some continuous curve $\gamma: [a, b] \rightarrow \overline{M}_i$ with $\gamma(a) = p, \gamma(b) = q$ such that $\gamma_{|[a,b]}$ is future-directed smooth timelike and contained in $M$; and $p \leq q$ iff there exist a continuous curve $\gamma: [a, b] \rightarrow \overline{M}_i$ with $\gamma(a) = p, \gamma(b) = q$ and causal in $M_0$.

In general, any envelopment will not generate a complete conformal completion, i.e., all inextendible timelike curve of $M$ will not have necessarily an endpoint (in the natural sense) in $M_0$; for instance, consider $M$ as the right semiplane of $L^2$ in natural coordinates and, as envelopment, the identity over $\mathbb{L}^2$. In this sense, we will say that an envelopment is chronologically complete if any inextendible timelike curve of $M$ (and so, a curve which generate a TIP or a TIF) have an endpoint in $M_0$.

**Theorem 1.1** An envelopment is chronologically complete if and only if the natural projections of the future and past causal preboundary, $\partial M \rightarrow \partial^F M, \partial M \rightarrow \partial^I M$, which map each TIP $P = I^- [\gamma]$ or TIF $F = I^+ [\gamma]$ in the endpoint of $\gamma$, are well defined on all $\partial M$ and $\partial M$. In this case, the natural projections of the precompletions

$$\hat{\pi} : \hat{M} \rightarrow \overline{M}_i, \quad \hat{\pi} : \hat{M} \rightarrow \overline{M}_i$$

are obviously defined.

### 2. Main result.

In general, we cannot expect a totally general result relating the conformal and the $c$-boundaries or completions (see appendix in [2] for details). In order to obtain relevant results, we have to impose some regular conditions for the envelopment. In this sense, we introduce the concept of regular points:
Definition 2.1 Let $i : M \hookrightarrow M_0$ be an envelopment. We say that a continuous curve $\gamma : [a,b] \to M_i$ such that $\gamma |_{[a,b]}$ is future-directed (resp. past-directed) smooth timelike curve, contained in $M$ and such that $\gamma(b) \in \overline{M}_i$ is future (resp. past) timelike deformable (in its endpoint), if there exists a neighbourhood $U = U_0 \cap \overline{M}_i$ of $\gamma(b)$ (where $U_0$ is an open set of $M_0$) such that $\gamma(a) \ll_i w$ (resp. $w \ll_i \gamma(a)$) for all $w \in U$.

A point $z \in \partial_i^* M$ is regularly accessible if: any TIPs and TIFs associated to $z$ is defined by a timelike deformable curve; and it admits a neighbourhood $V = M_i \cap V_0$ ($V_0$ open in $M_0$) such that for any $x, x' \in M_i$ with $x \ll_i z \ll_i x'$ (or $x \ll_i z \ll_i x'$) then $x \ll_i x'$.

Its boundary is regularly accessible if all the points in $\partial_i^* M$ are regularly accessible.

Then, the following theorem is obtained.

Theorem 2.2 Let $i : M \hookrightarrow M_0$ be a chronologically complete envelopment. If its boundary is regularly accessible then the conformal and c-completion are equivalent, i.e.:

a) The map $\pi : M \to \overline{M}_i^*$ is well defined and bijective.

b) $\pi$ is an homeomorphism and a chronological isomorphism.

where

$$\pi : M \to \overline{M}_i^*, \quad \pi((P, F)) = \begin{cases} \hat{\pi}(P) & \text{if } P \neq \emptyset \\ \hat{\pi}(F) & \text{if } F \neq \emptyset. \end{cases}$$

The following theorem yield sufficient conditions to ensure regular accessibility, which are very easy to check in practice. The condition of strong accessibility ensure that any point $z \in \partial_i^* M$ which is accessible by a future (or past) timelike curve, is accessible by a future (or past) timelike curve which is smoothly extendible to its endpoints with timelike velocity. The other condition, causally tame (see [2] for details), is imposed in order to avoid pathological cases which would not happen in physical examples.

Theorem 2.3 An envelopment $i : M \hookrightarrow M_0$ with $C^1$ boundary, strong accessible and causally tame is regularly accessible. So, its conformal completion and c-completion are equivalent (in the sense of previous theorem).

The following two corollaries are practical applications of previous theorem. The first result allow us to compute the c-boundary of any asymptotically flat spacetime (see [6] for definition).

Corollary 2.4 The c-boundary of the asymptotic part of an asymptotically flat spacetime will be two lightlike cones (for the future and past resp.), which base is topologically a sphere and with vertexes $i^+$ and $i^-$ respectively.

About the second result, the implication to the right is well-known, but we are not aware of a rigorous proof for the implication to the left.

Corollary 2.5 A spacetime which admits a chronologically complete envelopment with $C^1$ boundary $\partial_i M$ is globally hyperbolic if and only if $\partial_i M$ does not have timelike points.

References

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