A Unified Approach to Scalar, Vector, and Tensor Slepian Functions on the Sphere and Their Construction by a Commuting Operator

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Abstract

We present a unified approach for constructing Slepian functions — also known as prolate spheroidal wave functions — on the sphere for arbitrary tensor ranks including scalar, vectorial, and rank 2 tensorial Slepian functions, using spin-weighted spherical harmonics. For the special case of spherical cap regions, we derived commuting operators, allowing for a numerically stable and computationally efficient construction of the spin-weighted spherical-harmonic-based Slepian functions. Linear relationships between the spin-weighted and the classical scalar, vectorial, tensorial, and higher-rank spherical harmonics allow the construction of classical spherical-harmonic-based Slepian functions from their spin-weighted counterparts, effectively rendering the construction of spherical-cap Slepian functions for any tensorial rank a computationally fast and numerically stable task.

1 Introduction

Functions cannot simultaneously have both a spectral and temporal (or spatial) finite support [13, 29]. In scientific or engineering applications, however, it may be desirable to represent signals in a time-limited but spectrally concentrated manner. Slepian, Landau, and Pollak created a suitable orthogonal basis for a range of Euclidean domains [21, 38, 39], see also [16, 35]. Geoscientific or planetary studies typically involve data or models on a sphere, or parts thereof. To reap the benefits of the spatiotemporal analysis previously developed for Euclidean spaces, Albertella, Sansò, and Sneeuw [1], and later Simons, Dahlen, and Wieczorek [35, 37] developed corresponding scalar-valued functions by spatiotemporally optimizing linear combinations of spherical harmonics and named the resulting orthogonal basis “Slepian functions”. These Slepian functions found a wide range of applications in fields such as geodesy and geophysics, gravimetry, geodynamics, cosmology, planetary science, biomedical science, and in computer science (see [20, 31] and the references therein).

As a result of their construction, Slepian functions can be orthonormalized on the sphere, while remaining orthogonal within the target region. The two end member families of Slepian functions include spatially concentrated – spectrally limited bases, and spatially limited – spectrally concentrated bases. Here, we limit our discussions on the former case.

Construction of the spatially concentrated – spectrally limited Slepian functions requires solving a finite-dimensional algebraic eigenvalue problem. Depending on the region of interest and the bandlimit, the underlying matrix can become ill-conditioned, leading to an eigenvalue problem, which is numerically unstable to solve. For some special regions, alternative eigenvalue problems based on commuting operators were discovered for the scalar and the vectorial case on the sphere (see [20, 35, 37]). The alternative problems have the same eigenvectors but are numerically stable.

The construction of Slepian functions for various tensorial ranks typically follows the recipe described in [26], where an abstract Hilbert space setup is used to describe the general “construction manual” of Slepian functions with a particular focus on ill-posed inverse problems but without commuting operators. In this article, we restrict our considerations on the case of a spherical cap region. We show that the scalar, vectorial, and tensorial cases can be considered as particular cases of a generalized setup, which we present here.

For this purpose, we utilize the spin-weighted spherical harmonics of Newman and Penrose [30] (see also [25, 34]). As a consequence the eigenvalue problems for vectorial or higher-ranked Slepian function constructions, which are coupled when using the classical vector or tensor spherical harmonics, decouple. Besides reducing the dimension of the eigenproblem, this decoupling also facilitates the construction of a general commuting operator for polar cap regions. The special case of spin weight 0 leads to the known scalar Slepian functions (see also [33, 37]), while the combination of spin weight 0 with spin weight ±1 yields the known vector Slepian functions (see also [26, 31]).

We demonstrate our method by explicitly constructing tensor (rank 2) Slepian functions using spin weights of 0, ±1, and ±2. We compare the result to a system of Slepian functions constructed from
the basis of the tensor spherical harmonics of Freeden, Gervens, and Schreiner [11], by transforming the latter into the spin-weighted basis system. Tensor Slepian functions have been constructed before using a different ansatz by [8] for the basis of the tensor spherical harmonics of Martinec [23]. Our general ansatz yields, for the first time, a commuting operator for the tensorial case and it opens a way for the consideration of tensors of arbitrary ranks. This paper comprises some of the results of the thesis [34].

2 Preliminaries

Before presenting the spin-weighted spherical harmonics by Newman and Penrose [30], upon which our unified Slepian construction is based, we recall the classical scalar, vector [17, 28], and tensor spherical harmonics [11]. This will allow us to compare the two constructions in Theorem 2.38. And we describe a procedure to construct classical Slepian functions from spin-weighted Slepian functions, should the need arise. As usual, N, Q, R, and C stand for the sets of positive integers, rational numbers, real numbers, and complex numbers, respectively. Correspondingly, \(Q_0^+ := \{x \in Q \mid x \geq 0\} \) etc.

2.1 Basics and Notations

For completeness, we present the definitions for function spaces, norms, unit vectors and tensors, and operators that we use. For further details, see [14, 24].

**Definition 2.1.** Let \(D \subset \mathbb{R}^n\) and \(W \subset \mathbb{C}^m\). Then, \(C^{(k)}(D, W)\) is the space of all functions \(f : D \rightarrow W\), which are at least differentiable to order \(k \in \mathbb{N}_0\) and where the \(k\)-th derivative is continuous. If \(W = \mathbb{C}\), then we denote \(C^{(k)}(D, \mathbb{C}) =: C^{(k)}(D)\) and if \(k = 0\), then we write \(C^{(0)}(D, W) =: C(D, W)\).

Analogously, we define \(c^{(k)}(D, w)\) for all vector functions \(f : D \rightarrow w, w \subset \mathbb{C}^3\), and \(c^{(k)}(D, w)\) for all second-rank tensor functions \(f : D \rightarrow w, w \subset \mathbb{C}^{3 \times 3}\).

**Definition 2.2.** We define the inner product of two vectors \(x, y \in \mathbb{C}^n\) by

\[
\langle x, y \rangle := x \cdot y := \sum_{j=1}^n x_j \overline{y_j}
\]

with the induced norm \(|x| := \sqrt{x \cdot \overline{x}}\).

**Definition 2.3.** The norm for \(F \in C(D), D \subset \mathbb{R}^n\) compact, is given by

\[
\|F\|_{C(D)} := \sup_{x \in D} |F(x)|.
\]

Analogously, the norms on \(C(D, \mathbb{C}^m)\) are defined by using \(|F(x)|\) in the sense of Definition 2.3.

The following definitions contain the notations for the spherical geometry used in the construction of our Slepian functions.

**Definition 2.4.** The unit sphere \(\Omega\) of the three-dimensional Euclidean space \(\mathbb{R}^3\) is represented by

\[
\Omega = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}.
\]

We will use the local orthonormal basis given by

\[
\xi(t, \varphi) = e^r = \begin{pmatrix} \sqrt{1-t^2} \cos \varphi \\ \sqrt{1-t^2} \sin \varphi \end{pmatrix}, \quad e^\varphi = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}, \quad e^t = \begin{pmatrix} -t \cos \varphi \\ -t \sin \varphi \end{pmatrix} / \sqrt{1-t^2}.
\]

Here, \(t \in [-1, 1]\) is the polar distance and \(\varphi \in [0, 2\pi]\) denotes the longitude. The polar distance is related to the latitude \(\theta \in [-\pi/2, \pi/2]\) through the relationship \(t = \sin(\theta)\). Note that \(e^r\) is radially outward, \(e^\varphi\) eastward and \(e^t\) northward. For \(t = -1\), we obtain the South pole and for \(t = 1\) the North pole. Furthermore, we define the unit sphere without the poles

\[
\Omega_0 := \Omega \setminus \{ \xi = \xi(t, \varphi) \mid t = \pm 1 \},
\]

where \(\xi = \xi(t, \varphi)\) is the polar coordinate representation of \(\xi \in \Omega\).
Definition 2.5. We define the tensors

\[ i_{\tan}(\xi) := \varepsilon^\varphi \otimes \varepsilon^\varphi + \varepsilon^t \otimes \varepsilon^t \]
and

\[ j_{\tan}(\xi) := \varepsilon^t \otimes \varepsilon^\varphi - \varepsilon^\varphi \otimes \varepsilon^t \]
for \( \xi = \xi(t, \varphi) \in \Omega \). This definition follows the construction presented by [14].

In the following, we use the shorthand form for differentiation

\[ \partial_x := \frac{\partial}{\partial x} \]
to define some well-known Cartesian and spherical differential operators.

Definition 2.6. The gradient is defined by

\[ \nabla_x := \left( \partial_x i \right)_{i=1,2,3} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \]
and the Laplace operator by

\[ \Delta_x := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \]
for \( x = (x_1, x_2, x_3)^T \in D \subset \mathbb{R}^3 \).

Definition 2.7. The surface gradient

\[ \nabla^{\ast}_x := \varepsilon^\varphi \frac{1}{\sqrt{1 - t^2}} \partial_t + \varepsilon^t \sqrt{1 - t^2} \partial_t \]
and the surface curl gradient

\[ L^{\ast}_x := -\varepsilon^\varphi \sqrt{1 - t^2} \partial_t + \varepsilon^t \frac{1}{\sqrt{1 - t^2}} \partial_t \]
for \( \xi = \xi(t, \varphi) \in \Omega \) are differential operators on the sphere such that \( \nabla^{\ast}_r \xi = \left( \xi \partial_r + \frac{1}{r} \nabla^{\ast}_r \right) \) for \( r \in \mathbb{R}^+ \), \( \xi \in \Omega \) and \( L^{\ast}_r = \xi \wedge \nabla^{\ast}_r \), where \( \wedge \) represents the vector product in \( \mathbb{R}^3 \).

Moreover,

\[ \Delta^{\ast}_x := \partial_t \left( (1 - t^2) \partial_t \right) + \frac{1}{1 - t^2} \partial^2_\varphi \]
is the Beltrami operator such that \( \Delta^{\ast} = \nabla^{\ast} \cdot \nabla^{\ast} = L^{\ast} \cdot L^{\ast} \) and \( \Delta_{\nu} \xi = \frac{\partial^2}{\partial \nu^2} + \frac{2}{r} \frac{\partial}{\partial \nu} + \frac{1}{r^2} \Delta^{\ast}_x \).

This enables us to formulate Green’s second surface identity [24]. Note that all integrals we use are Lebesgue integrals.

Theorem 2.8. Green’s second surface identity is given by

\[ \int_{\Gamma} \left( F(\xi) \Delta^{\ast}_x G(\xi) - G(\xi) \Delta^{\ast}_x F(\xi) \right) \, d\omega(\xi) = \int_{\partial \Omega} \left( F(\xi) \frac{\partial}{\partial \nu(\xi)} G(\xi) - G(\xi) \frac{\partial}{\partial \nu(\xi)} F(\xi) \right) \, d\sigma(\xi), \]
where \( F, G \in C^{(2)}(\overline{\Gamma}), \Gamma \subset \Omega \) with a sufficiently smooth boundary and \( \nu \) is the outward unit normal vector field to \( \partial \Gamma \).

As a special case, Green’s second surface identity over the entire unit sphere leads to

\[ \int_{\Omega} \left( F(\xi) \Delta^{\ast}_x G(\xi) - G(\xi) \Delta^{\ast}_x F(\xi) \right) \, d\omega(\xi) = 0 \]
for \( F, G \in C^{(2)}(\Omega) \).
The proof to Theorem 2.8 can be found in [2, p. 448]. The construction of scalar, vector, and tensor Slepian functions requires norms based on inner products. The following Hilbert spaces will satisfy this requirement.

**Definition 2.9.** For a (Lebesgue) measurable set $D \subset \mathbb{R}^n$, we denote with $L^2(D, \mathbb{C}^m)$ the Hilbert space of (equivalence classes of almost everywhere identical) functions $F : D \to \mathbb{C}^m$ with

$$
\|F\|_2 := \|F\|_{L^2(D, \mathbb{C}^m)} := \left( \int_D |F(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty.
$$

We focus our attention on the cases $m = 1, 3, 3 \times 3$, where $m = 1$ yields scalar, $m = 3$ vectorial, and $m = 3 \times 3$ tensorial function spaces. Consequently, we define $L^2(D, \mathbb{C}^1) =: L^2(D)$, $L^2(D, \mathbb{C}^3) =: l^2(D)$, and $L^2(D, \mathbb{C}^{3 \times 3}) =: l^2(D)$. The inner products are denoted by

$$
\langle F,G \rangle_{L^2(D)} := \int_D F(\xi) \overline{G(\xi)} \, d\omega(\xi)
$$

for $m = 1$, by

$$
\langle f,g \rangle_{l^2(D)} := \int_D f(\xi) \cdot \overline{g(\xi)} \, d\omega(\xi)
$$

for $m = 3$, and by

$$
\langle f,g \rangle_{l^2(D)} := \int_D f(\xi) : \overline{g(\xi)} \, d\omega(\xi)
$$

for $m = 3 \times 3$.

**Theorem 2.10.** The relationships between the Hilbert spaces and the spaces of continuous functions are

$$
\|\cdot\|_{L^2(\Omega)} = L^2(\Omega),
$$

$$
\|\cdot\|_{l^2(\Omega)} = l^2(\Omega),
$$

$$
\|\cdot\|_{l^2(\Omega)} = l^2(\Omega).
$$

The proof to Theorem 2.10 can be found in [44, Theorems 3.2.6 and 3.6.2].

### 2.2 Scalar, Vector, and Tensor Spherical Harmonics

Previously published constructions of scalar, vector, and tensor Slepian functions are based on classical spherical harmonics. Our unified approach for constructing Slepian functions is based on spin-weighted spherical harmonics. This approach will allow us to handle scalar, vectorial, and tensorial functions with the same setup. The reason is that spin-weighted spherical harmonics provide us with an equivalent representation of scalar, vector, and tensor spherical harmonics. Before explaining this in detail, we will first recapitulate a classical definition of the latter functions within this section.

**Definition 2.11.** We denote the (scalar) fully normalized spherical harmonics by

$$
Y_{n,j}(\xi(t,\varphi)) := X_{n,j}(t) e^{ij\varphi} := \begin{cases} (-1)^j \sqrt{\frac{n+1}{4\pi}} \frac{\sqrt{(n-j)!}}{(n+j)!} P_{n,j}(t) e^{ij\varphi}, & j \geq 0, \\ (-1)^j \sqrt{\frac{n+1}{4\pi}} \frac{\sqrt{(n-j)!}}{(n+j)!} P_{n,-j}(t), & j < 0, \end{cases}
$$

with the fully normalized associated Legendre functions given by

$$
X_{n,j}(t) := \begin{cases} (-1)^j \sqrt{\frac{n+1}{4\pi}} \frac{\sqrt{(n-j)!}}{(n+j)!} P_{n,j}(t), & j \geq 0, \\ (-1)^j X_{n,-j}(t), & j < 0, \end{cases}
$$

the associated Legendre functions by

$$
P_{n,j}(t) := (1 - t^2)^{\frac{1}{2}} \left( \frac{d}{dt} \right)^j P_n(t),
$$

where $P_n(t)$ are the Legendre polynomials.
and the Legendre polynomials given by the Rodriguez formula

\[ P_n(t) := \frac{1}{2^n n!} \left( \frac{d}{dt} \right)^n (t^2 - 1)^n, \]

where \( t \in [-1, 1] \), \( \varphi \in [0, 2\pi] \), \( \xi \in \Omega \), and \( n \in \mathbb{N}_0 \), \( j = -n, \ldots, n \).

The set of fully normalized spherical harmonics constructs a basis of \( (L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)}) \), see e.g. [24] for a proof.

The following sets of functions form basis systems for \( (L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)}) \) and \( (L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)}) \), respectively, see e.g. [12] for a proof.

**Definition 2.12.** The vector spherical harmonics by Hill [17] (also called the Morse-Feshbach vector spherical harmonics, see [28]) are defined by

\[
\begin{align*}
y_{n,j}^{(1)}(\xi) &:= \xi Y_{n,j}(\xi), \\
y_{n,j}^{(2)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \nabla_\xi Y_{n,j}(\xi), \\
y_{n,j}^{(3)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} L_\xi^2 Y_{n,j}(\xi)
\end{align*}
\]

for \( \xi \in \Omega \), \( n \in \mathbb{N}_0 \), \( n \geq 0 \), \( j = -n, \ldots, n \), and \( i = 1, 2, 3 \) with

\[
0_i := \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3 \end{cases}
\]

We define the function spaces

\[ \text{harm}_n(\Omega) := \text{span} \left\{ y_{n,j}^{(i)} \mid 1 \leq i \leq 3 \text{ and } j = -n, \ldots, n \right\} \]

and the function spaces

\[ \text{harm}_{p,\ldots,q}(\Omega) := \bigoplus_{n=p}^{q} \text{harm}_n(\Omega) \]

for \( p \leq q \).

Note that the components normal to the unit sphere \( \Omega \) are described by \( y_{n,j}^{(1)} \), while \( y_{n,j}^{(2)} \) and \( y_{n,j}^{(3)} \) describe the tangential components.

**Definition 2.13.** The tensor spherical harmonics by Freeden, Gervens, and Schreiner [11] are defined by

\[
\begin{align*}
y_{n,j}^{(1,1)}(\xi) &:= (\xi \otimes \xi) Y_{n,j}(\xi), \\
y_{n,j}^{(1,2)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} (\xi \otimes \nabla_\xi Y_{n,j}(\xi)), \\
y_{n,j}^{(1,3)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} (\xi \otimes L_\xi^2 Y_{n,j}(\xi)), \\
y_{n,j}^{(2,1)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} (\nabla_\xi Y_{n,j}(\xi) \otimes \xi), \\
y_{n,j}^{(2,2)}(\xi) &:= \frac{1}{\sqrt{2}} i_{\text{tan}}(\xi) Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{2}} Y_{n,j}(\xi) (\varepsilon^x \otimes \varepsilon^x + \varepsilon^\varphi \otimes \varepsilon^\varphi), \\
y_{n,j}^{(2,3)}(\xi) &:= \frac{1}{\sqrt{2n(n+1)(n+1)-2}} \left[ (\nabla_\xi \otimes \nabla_\xi - L_\xi^2 \otimes L_\xi^2) Y_{n,j}(\xi) + 2\nabla_\xi Y_{n,j}(\xi) \otimes \xi \right],
\end{align*}
\]
Definition 2.14. Following the construction of [30], we define the spin-weighted differential operators functions for arbitrary tensor ranks and (ii) construct commuting operators for spherical-cap regions for spherical harmonics is that the former allow us to (i) derive a unified approach to construct Slepian and the vector Slepian case for spherical-cap regions ([16, 20]). The advantage of the spin-weighted functions (see e.g. [8] for the tensorial case). In addition, commuting operators are known for the scalar It is known that scalar, vector, and tensor spherical harmonics can be used to construct spherical Slepian 2.3 Spin-Weighted Spherical Harmonics

left tangential/ right normal, and

\[ y^{(3,1)}_{n,j}(\xi) := \frac{1}{\sqrt{n(n+1)}} (L^{*}_t Y_{n,j}(\xi) \otimes \xi), \]

\[ y^{(3,2)}_{n,j}(\xi) := \frac{1}{\sqrt{2n(n+1)[n(n+1) - 2]}} \left[ (\nabla^{*}_t \otimes L^*_t + L^*_t \otimes \nabla^{*}_t) Y_{n,j}(\xi) + 2L^*_t Y_{n,j}(\xi) \otimes \xi \right], \]

\[ y^{(3,3)}_{n,j}(\xi) := \frac{1}{\sqrt{2}} j_{\text{tan}}(\xi) Y_{n,j}(\xi) \]

\[ = \frac{1}{\sqrt{2}} Y_{n,j}(\xi) \left( e_t \otimes e^t - e^t \otimes e_t \right) \]

for \( \xi \in \Omega, n \in \mathbb{N}_0, n \geq 0, j = -n, \ldots, n, \) and \( i, k = 1, 2, 3 \) with

\[ 0 \leq i, k \leq 3 \] and \( j = -n, \ldots, n \}

As in [13], we define the function spaces

\[ \text{harm}_n(\Omega) := \text{span} \left\{ y^{(i,k)}_{n,j} \mid 1 \leq i, k \leq 3 \right\}, \]

and the function spaces

\[ \text{harm}_{p,n}(\Omega) := \bigoplus_{n=p}^{q} \text{harm}_n(\Omega), \]

for \( p \leq q. \)

With these definitions, \( y^{(1,1)}_{n,j} \) is normal, \( y^{(1,2)}_{n,j}, y^{(1,3)}_{n,j} \) are left normal/right tangential, \( y^{(2,1)}_{n,j}, y^{(2,3)}_{n,j} \) are left tangential/ right normal, and \( y^{(2,2)}_{n,j}, y^{(3,2)}_{n,j}, y^{(3,3)}_{n,j} \) are tangential.

2.3 Spin-Weighted Spherical Harmonics

It is known that scalar, vector, and tensor spherical harmonics can be used to construct spherical Slepian functions (see e.g. [3] for the tensorial case). In addition, commuting operators are known for the scalar and the vector Slepian case for spherical-cap regions ([16, 20]). The advantage of the spin-weighted spherical harmonics by Newman and Penrose [30] (which we introduce in this section) over the classical spherical harmonics is that the former allow us to (i) derive a unified approach to construct Slepian functions for arbitrary tensor ranks and (ii) construct commuting operators for spherical-cap regions for arbitrary tensor ranks (see also [25, 34]).

Definition 2.14. Following the construction of [30], we define the spin-weighted differential operators \( \partial_N : C^{(1)}(\Omega_0) \rightarrow C(\Omega_0) \) and \( \overline{\partial}_N : C^{(1)}(\Omega_0) \rightarrow C(\Omega_0) \) of spin weight \( N \in \mathbb{Q} \) by

\[ \partial_N F(\xi) := \left( \sqrt{1 - \xi^2} \partial_t + \frac{Nt - i\partial_{\varphi}}{\sqrt{1 - \xi^2}} \right) F(\xi), \]

\[ \overline{\partial}_N F(\xi) := \left( \sqrt{1 - \xi^2} \partial_t - \frac{Nt - i\partial_{\varphi}}{\sqrt{1 - \xi^2}} \right) F(\xi), \]

where \( \xi = \xi(t, \varphi) \in \Omega_0 \) and \( F \in C^{(1)}(\Omega_0) \).

Definition 2.15. Symbol \( \overline{\partial}_N^M \) for \( M \in \mathbb{Q}_0^+ \) and \( N \in \mathbb{Q} \) denotes the successive application of spin-weighted operators \( \partial_k \) for spin weights \( k = N, N + 1, \ldots, N + M - 1 \) on a function \( F \in C^{(M)}(\Omega_0) \) such that

\[ \overline{\partial}_N^M F := \partial_{N+M-1} \partial_{N+M-2} \ldots \partial_{N+1} \partial_N F. \]

Mutatis mutandis, we define

\[ \overline{\partial}_N^M F := \overline{\partial}_{N-M+1} \overline{\partial}_{N-M+2} \ldots \overline{\partial}_{N-1} \overline{\partial}_N F. \]

The case \( M = 0 \) denotes the identity operator

\[ \overline{\partial}_N^0 = \text{Id} = \overline{\partial}_N^0. \]
Definition 2.16. The spin-weighted spherical harmonics by Newman and Penrose \([20]\) (see also \([6,15,22,42,43,45,46]\)) are defined for \(n \in \mathbb{N}_0, N \in \mathbb{Q}, n \geq |N|, \) and \(j = -n, \ldots, n\) by

\[
\mathcal{N}Y_{n,j} := \begin{cases} \\
\sqrt{\frac{(n-N)!}{(n+N)!}} \hat{\Omega}_0^N Y_{n,j}, & 0 \leq N \leq n \\
-\sqrt{\frac{(n+N)!}{(n-N)!}} \bar{\Omega}_0^{-N} Y_{n,j}, & -n \leq N \leq 0 \\
0, & n < |N|
\end{cases}
\]

Note that we define the spin-weighted spherical harmonics on \(\Omega_0\), because the operators \(\mathfrak{D}\) and \(\mathfrak{D}^\dagger\) have singularities at the poles.

Lemma 2.17 and Theorem 2.18 provide alternative formulations for the spin-weighted spherical harmonics.

Lemma 2.17. The spin-weighted spherical harmonics fulfill for all \(N \in \mathbb{Z}, n \in \mathbb{N}_0, \) and all \(j = -n, \ldots, n\) the following properties

\[
\mathfrak{D}_N \mathcal{N}Y_{n,j} = \sqrt{n(n+1) - N(N+1)} \mathcal{N}Y_{n+1,j},
\]

and

\[
\bar{\mathfrak{D}}_N \mathcal{N}Y_{n,j} = -\sqrt{n(n+1) - N(N-1)} \mathcal{N}Y_{n-1,j}.
\]

See \([6,15,22,40,42,45,46]\) for a proof.

Theorem 2.18. The spin-weighted spherical harmonics also satisfy \([22,42,43]\)

\[
\mathcal{N}Y_{n,j}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}} e^{ij\varphi} d^n_{j,-N}(\vartheta)
\]

where \(\xi = \xi(t, \varphi) \in \Omega_0, \) \(t = \cos \vartheta, \) \(n \in \mathbb{N}_0, \) \(N \in \mathbb{Z}, n \geq |N|, j = -n, \ldots, n, \) and \(d^n_{j,N}\) is the Wigner D-function.

See \([7,43]\) for a proof.

Furthermore, the spin-weighted spherical harmonics can also be formulated as functions of spin weight. First, we define a function of spin weight \(N\).

Definition 2.19. The coefficients \(d_{i_1,i_2,\ldots,i_{2n}} \in \mathbb{R}\) for \(n \in \mathbb{N}_0\) are called totally symmetric, if they are equal for every permutation of the index \(i_1, i_2, \ldots, i_{2n}\) in \(\mathbb{N}_0^{2n}\).

Definition 2.20. A function \(\mathcal{N}F_n \in L^2(\Omega)\) is called a function of spin weight \(N \in \mathbb{Z}\) and degree \(n \in \mathbb{N}_0, \) if it can be written as \([42]\)

\[
\mathcal{N}F_n = \sum_{i_1, \ldots, i_{2n}=1}^2 d_{i_1i_2\ldots i_{2n}} \hat{\mathfrak{D}}^{i_1} \hat{\mathfrak{D}}^{i_2} \cdots \hat{\mathfrak{D}}^{i_n+N} \hat{\mathfrak{D}}^{i_n+N+1} \cdots \hat{\mathfrak{D}}^{i_{2n}}
\]

where \(|N| \leq n, \) the coefficients \(d_{i_1i_2\ldots i_{2n}} \in \mathbb{R}\) are totally symmetric, and for \(\xi = \xi(t, \varphi) \in \Omega, \) we define

\[
\hat{\mathfrak{D}}^1(\xi) := \hat{\mathfrak{D}}^1 := e^{-i\xi} \sqrt{1+\frac{t}{2}}, \quad \hat{\mathfrak{D}}^2(\xi) := \hat{\mathfrak{D}}^2 := e^{i\xi} \sqrt{1-\frac{t}{2}},
\]

\[
\hat{\mathfrak{D}}^3(\xi) := \hat{\mathfrak{D}}^3 := e^{-i\xi} \sqrt{1-\frac{t}{2}}, \quad \hat{\mathfrak{D}}^4(\xi) := \hat{\mathfrak{D}}^4 := e^{i\xi} \sqrt{1+\frac{t}{2}}.
\]
Lemma 2.21. The spin-weighted spherical harmonics can be represented as functions of spin weight \( N \in \mathbb{Z} \) for \( \xi = \xi(t, \varphi) \in \Omega \) by

\[
NY_{n,j}(\xi) = (-1)^{j} \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{(n-j)!(n+j)!(n-N)!(n+N)!}}{\kappa!(n-j-k)!(n-N-k)!(N-j+k)!} \times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} \left( \alpha_{k}^{j} \right)^{k+N-j} \left( \alpha_{k}^{j} \right)^{-k-N} \left( \alpha_{k}^{j} \right)^{n-k-N} \left( \alpha_{k}^{j} \right)^{n-j+k} \, ,
\]

where \( n \in \mathbb{N}_{0}, n \geq |N|, \) and \( j = -n, \ldots, n. \)

Similar to the classical spherical harmonics, the spin-weighted spherical harmonics also satisfy recursion relations, a Christoffel-Darboux formula, and an addition theorem. These have been proven in [25, 34] for the first time. In the following, we report the results and refer to [25] and [34] for the proofs.

Theorem 2.22. The spin-weighted spherical harmonics satisfy the following recursion relations for \( \xi = \xi(t, \varphi) \in \Omega_{0}: \)

\[
(t^{2} - 1) \partial_{t} NY_{n,j}(\xi) = \left( nt + \frac{Nj}{n} \right) NY_{n,j}(\xi) - (2n+1)\alpha_{n,j}^{N} NY_{n-1,j}(\xi), \quad (3)
\]

\[
= - \left( (n+1)t + \frac{Nj}{n+1} \right) NY_{n,j}(\xi) + (2n+1)\alpha_{n+1,j}^{N} NY_{n+1,j}(\xi), \quad (4)
\]

\[
\left( t + \frac{Nj}{n(n+1)} \right) NY_{n,j}(\xi) = \alpha_{n,j}^{N} NY_{n-1,j}(\xi) + \alpha_{n+1,j}^{N} NY_{n+1,j}(\xi), \quad (5)
\]

where

\[
\alpha_{n,j}^{N} := \frac{\sqrt{(n-N)(n+N)}}{n} c_{n,j} = \frac{\sqrt{(n-N)(n+N)}}{n} \sqrt{\frac{(n-j)(n+j)}{(2n-1)(2n+1)}} \, ,
\]

\( N \in \mathbb{Z}, n \in \mathbb{N}_{0}, n \geq |N+1|, \) and \( j = -n, \ldots, n. \) Furthermore, we denote \( NY_{n,j} := 0 \) for \( n < |j|. \)

Theorem 2.23 (Christoffel-Darboux Formula). For all \( N \in \mathbb{Z}, \) we obtain the Christoffel-Darboux formula for the spin-weighted spherical harmonics

\[
(t_{1} - t_{2}) \sum_{n=-n_{j}}^{L-1} NY_{n,j}(\xi) NY_{n,j}(\eta) = \alpha_{n,j}^{N} \left( NY_{L,j}(\xi) NY_{L-1,j}(\eta) - NY_{L-1,j}(\xi) NY_{L,j}(\eta) \right),
\]

where

\[
n_{j} := \max\{|N|,|j|\},
\]

\( \xi = \xi(t_{1}, \varphi_{1}), \eta = \eta(t_{2}, \varphi_{2}) \) are the polar coordinate representations of \( \xi, \eta \in \Omega_{0}, L > n_{j} \) is the bandlimit, and \( j = -L, \ldots, L. \)

Theorem 2.24. The spin-weighted spherical harmonics are orthonormal with respect to the \( L^{2}(\Omega) \)-inner product,

\[
\int_{\Omega} NY_{n,j}(\xi) NY_{n',j'}(\xi) \, d\omega(\xi) = \delta_{n,n'}\delta_{j,j'},
\]

See [5, 6, 13, 18, 22, 30, 46] for the proof of Theorem 2.24.

Theorem 2.25 (Addition Theorem for Spin-Weighted Spherical Harmonics). The spin-weighted spherical harmonics satisfy the following addition theorem for \( N_{1}, N_{2} \in \mathbb{Z} \) and for \( n \in \mathbb{N}_{0}, n \geq \max\{|N_{1}|,|N_{2}|\}, \)

\[
\sum_{j=-n}^{n} N_{1}Y_{n,j}(\xi_{1}) N_{2}Y_{n,j}(\xi_{2}) = (-1)^{N_{1}} \sqrt{\frac{2n+1}{4\pi}} N_{2}Y_{n,-N_{1}}(\xi) e^{-iN_{2}y},
\]

where \( \xi_{1} = \xi_{1}(t_{1}, \varphi_{1}), \xi_{2} = \xi_{2}(t_{2}, \varphi_{2}), t_{i} = \cos \vartheta_{i}, i = 1, 2, \xi = \xi(t, \alpha) \in \Omega, t = \cos \beta, \) and \( \alpha, \beta, \) and \( \gamma \) are the Euler angles given by...
• for \( \sin(\varphi_1 - \varphi_2) \neq 0 \)

\[
\begin{align*}
\cot \alpha &= \cos \vartheta_1 \cot(\varphi_1 - \varphi_2) - \cot \vartheta_2 \frac{\sin \vartheta_1}{\sin(\varphi_1 - \varphi_2)}, \\
\cos \beta &= \cos \vartheta_1 \cos \vartheta_2 + \sin \vartheta_1 \sin \vartheta_2 \cos(\varphi_1 - \varphi_2), \\
\cot \gamma &= \cos \vartheta_2 \cot(\varphi_1 - \varphi_2) - \cot \vartheta_1 \frac{\sin \vartheta_2}{\sin(\varphi_1 - \varphi_2)}.
\end{align*}
\]

• for \( \sin(\varphi_1 - \varphi_2) = 0 \), so \( \varphi_1 - \varphi_2 = k\pi, k \in \mathbb{Z} \), then

\[
\begin{aligned}
\begin{cases}
\alpha = \pi, \beta = \vartheta_1 - \vartheta_2, \gamma = \pi & \text{, if } k \text{ even, } -\vartheta_1 + \vartheta_2 \in [-\pi, 0) \\
\alpha = 0, \beta = -\vartheta_1 + \vartheta_2, \gamma = 0 & \text{, if } k \text{ even, } -\vartheta_1 + \vartheta_2 \in [0, \pi) \\
\alpha = \pi, \beta = \vartheta_1 + \vartheta_2, \gamma = 0 & \text{, if } k \text{ odd, } \vartheta_1 + \vartheta_2 \in [0, \pi) \\
\alpha = 0, \beta = 2\pi - (\vartheta_1 + \vartheta_2), \gamma = \pi & \text{, if } k \text{ odd, } \vartheta_1 + \vartheta_2 \in [\pi, 2\pi) 
\end{cases}
\end{aligned}
\]

The proof can be found in [25, 34].

**Corollary 2.26.** With \( \xi_1 = \xi_2 = \eta \in \Omega \) and with \( N_1 = N_2 = N \) the addition theorem reduces to

\[
\sum_{j=-n}^{n} N Y_{n,j}(\eta) \overline{N Y_{n,j}(\eta)} = \frac{2n+1}{4\pi}.
\]

**Theorem 2.27.** The spin-weighted spherical harmonics are eigenfunctions of the spin-weighted Beltrami operator

\[
\Delta^\star N := \Delta^\star_{\xi} - \frac{N^2 - 2itN\partial_{\varphi}}{1-t^2},
\]

where the classical Beltrami operator satisfies \( \Delta^\star_{\xi} = \partial_{\xi} \left( (1-t^2)\partial_{\xi} \right) + \frac{1}{1-t^2} \partial^2_{\varphi} \). Hence, for all \( \xi = \xi(t, \varphi) \in \Omega_0 \), all \( N \in \mathbb{Z} \), all \( n \in \mathbb{N}_0 \), \( n \geq |N| \), and all \( j = -n, \ldots, n \)

\[
\Delta^\star N_{Y_{n,j}(\xi)} = -n(n+1)_{N Y_{n,j}(\xi)}.
\]

The spin-weighted spherical harmonics are part of a function space with a series of properties that we will need in the construction of the commuting operator for the spin-weighted Slepian functions.

**Definition 2.28.** We denote by \( X^k(\Gamma) \), \( k \in \mathbb{N}_0 \), the set of all functions \( F \in C^k(\Gamma) \cap L^2(\Gamma) \) which satisfy the following conditions, where \( \xi = \xi(t, \varphi) \in \Gamma \subset \Omega \):

- \( F \) has the form \( H(t)e^{it\varphi} \) for \( j \in \mathbb{Z} \),
- \( F \) is bounded on \( \Gamma \),
- \( \partial_t^j F(\xi) = \theta \left( (1-t^2)^{\frac{j}{2}} \right) \) as \( t \to \pm 1 \) for all \( j = 0, \ldots, k \),
- \( \Delta^\star N F(\xi) = \theta(1) \) for \( N \in \mathbb{Z} \) as \( t \to \pm 1 \).

**Corollary 2.29.** The previous definition is chosen such that

\[
N Y_{n,j} \in X^k(\Omega_0)
\]

with \( \Gamma_{\Omega_0} = \Omega \), for all \( n \in \mathbb{N}_0 \), \( N \in \mathbb{Z} \), \( n \geq |N| \), \( j = -n, \ldots, n \), and all \( k \in \mathbb{N}_0 \).

See [25, 34] for the proof of Corollary 2.29.

**Theorem 2.30** (Green’s Second Surface Identity for the Spin-Weighted Beltrami Operator). Let \( \Gamma \subset \Omega \) with a sufficiently smooth boundary \( \partial \Gamma \). For \( F, G \in X^2(\Gamma) \)

\[
\int_{\Gamma} \left( F(\xi)\Delta^\star N G(\xi) - G(\xi)\Delta^\star N F(\xi) \right) \, d\omega(\xi)
= \int_{\partial \Gamma} \left( F(\xi)\frac{\partial}{\partial \nu(\xi)} G(\xi) - G(\xi)\frac{\partial}{\partial \nu(\xi)} F(\xi) \right) \, d\sigma(\xi)
- \int_{\Gamma} \frac{2iNt}{1-t^2} \partial_{\varphi} \left( F(\xi)\overline{G(\xi)} \right) \, d\omega(\xi),
\]

where \( \overline{G(\xi)} \) denotes the complex conjugate of \( G(\xi) \).
if the integrals exist.

As a special case, Green's second surface identity for the spin-weighted Beltrami operator $\Delta^{*,N}$ over the entire unit sphere yields

$$\int_{\Omega} \left( F(\xi) \Delta^{*,N}_\xi G(\xi) - G(\xi) \Delta^{*,N}_\xi F(\xi) \right) \, d\omega(\xi) = 0$$

and for the region $R$ a polar cap,

$$\int_{R} \left( F(\xi) \Delta^{*,N}_\xi G(\xi) - G(\xi) \Delta^{*,N}_\xi F(\xi) \right) \, d\omega(\xi)$$

$$= \int_{0}^{2\pi} \left[ (1 - t^2) \left( \nabla(\xi) \nabla F(\xi) - F(\xi) \nabla G(\xi) \right) \right]_{t=b} \, dt$$

for $F,G \in X^2(\Omega_0)$, where $\xi = (t, \varphi) \in \Omega$.

See [25, 34] for a proof of Theorem 2.30. It can also be shown that the spin-weighted spherical harmonic functions form a complete orthonormal system for $L^2(\Omega)$, see e.g. [25, 34] for a proof.

**Theorem 2.31.** The set of functions $\{ nY_{n,j} \}_{n \geq |N|, j = -n, \ldots, n}$ forms a complete orthonormal system for $(L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)})$. Consequently, for $F \in L^2(\Omega)$, we obtain

$$\lim_{L \to \infty} \left\| F - \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \langle F, nY_{n,j} \rangle_{L^2(\Omega)} nY_{n,j} \right\|_{L^2(\Omega)} = 0.$$

Hence, for every function $F \in L^2(\Omega)$ and for every $N \in \mathbb{Z}$, there exist unique coefficients $N F_{n,j}$ such that

$$F = \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} N F_{n,j} nY_{n,j}$$

in the sense of $L^2(\Omega)$. Furthermore, for every function $F,G \in L^2(\Omega)$ and every $N \in \mathbb{Z}$, the spin-weighted spherical harmonics satisfy the Parseval identity

$$\langle F, G \rangle_{L^2(\Omega)} = \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} \langle F, nY_{n,j} \rangle_{L^2(\Omega)} \langle G, nY_{n,j} \rangle_{L^2(\Omega)}$$

and consequently,

$$\|F\|^2_{L^2(\Omega)} = \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} \left| \langle F, nY_{n,j} \rangle_{L^2(\Omega)} \right|^2.$$

Analogously to the classical harmonic function spaces (see e.g. [14]), we can now define the function spaces for the spin-weighted spherical harmonics.

**Definition 2.32.** A function $Y_n \in C^2(\Omega_0)$ of degree $n \in \mathbb{N}_0$ is called $(*,N)$-harmonic for $N \in \mathbb{Z}$, if for all $\xi \in \Omega_0$

$$\Delta^{*,N}_\xi Y_n(\xi) = -n(n+1)Y_n(\xi).$$

**Definition 2.33.** With $\text{Harm}^N_n(\Omega_0)$, we denote the set of the $(*,N)$-harmonic functions of degree $n \in \mathbb{N}_0$ and spin weight $N \in \mathbb{Z}$. This means that

$$\text{Harm}^N_n(\Omega_0) : = \left\{ nP_n \mid nP_n \in C^2(\Omega_0) \text{ is a } (*,N) \text{-harmonic function of spin weight } N \right\}. $$
Definition 2.34. Analogously to the (spin-free) notations (cf. Definition 2.12 and [24]), we define the spaces
\[ \text{Harm}^N_{0,...,n}(\Omega_0) := \bigoplus_{i=0}^{n} \text{Harm}^N_i(\Omega_0) \]
and
\[ \text{Harm}^N_{0,...,\infty}(\Omega_0) := \bigcup_{i=0}^{\infty} \text{Harm}^N_i(\Omega_0). \]

Note that multiplying a \((*,N)\)-harmonic function by \(r^n\), \(r \in \mathbb{R}\), \(n \in \mathbb{N}_0\) does not necessarily render it a harmonic function. Similarly, a function of spin weight \(N \neq 0\) is generally not homogeneous. Only for the special case of spin weight zero do we get \(\text{Harm}^N_n(\Omega_0) = \text{Harm}_n(\Omega_0)\), the set of the to the unit sphere restricted harmonic and homogeneous polynomials of degree \(n \in \mathbb{N}_0\).

Corollary 2.35. The spin-weighted spherical harmonics \(N Y_n\) of spin weight \(N \in \mathbb{Z}\) and degree \(n \in \mathbb{N}_0\), \(n \geq |N|\), span the set \(\text{Harm}^N_n(\Omega_0)\). The functions \(\{N Y_{n,j}\}_{j=-n,...,n}\) form an orthonormal system in the space \((\text{Harm}^N_n(\Omega_0), \langle \cdot, \cdot \rangle_{L^2(\Omega)})\).

Remark 2.36. For \(|N| \leq p \leq q \leq \infty\), we define
\[ \text{Harm}^N_{p,...,q}(\Omega_0) := \text{span} \{N Y_{n,j}\}_{n=p,...,q, j=-n,...,n}. \]

Theorem 2.37. For all \(N \in \mathbb{Z}\), all \(n \in \mathbb{N}_0\), \(n \geq |N|\), and all \(j = -n,...,n\), the spin-weighted spherical harmonics \(N Y_{n,j}\) and their linear combinations are the only eigenfunctions of the differential operator \(\Delta^{*,N}\) in \(X^2(\Omega_0)\). Their eigenvalues are \(-n(n+1)\).

See [25, 34] for the proof of Theorem 2.37. The following theorem allows us to relate the spin-weighted spherical harmonics to the classical scalar, vector, and tensor harmonics.

Theorem 2.38. The spin-weighted spherical harmonics multiplied by unit vectors \(\tau_{\pm}\) or unit tensors \(\tau_{\pm} \otimes \xi\), \(\xi \otimes \tau_{\pm}\), or \(\tau_{\pm} \otimes \tau_{\pm}\), for
\[ \tau_{\pm} := -\frac{1}{\sqrt{2}} \left(e^i \pm ie^r\right) \quad \text{and} \quad e^r = \xi \in \Omega \]
are linear combinations of classical scalar, vector, and tensor spherical harmonics in the following way.

- The scalar spherical harmonics are equal to the spin-weighted spherical harmonics of spin weight 0,
  \[ 0 Y_{n,j} = Y_{n,j}. \]

- Spin-weighted spherical harmonics of spin weight \(\pm 1\), multiplied by the unit vectors \(\tau_{\pm}\) are linear combinations of the classical tangential vector spherical harmonics,
  \[ \pm \frac{1}{\sqrt{2}} \left(-y^{(2)}_{n,j}(\xi) \pm iy^{(3)}_{n,j}(\xi)\right) = \pm Y_{n,j}(\xi) \tau_{\pm}. \]

- Spin-weighted spherical harmonics of spin weight \(\pm 1\) multiplied by the unit tensor \(\xi \otimes \tau_{\pm}\) are linear combinations of the left normal/right tangential tensor spherical harmonics,
  \[ \pm \frac{1}{\sqrt{2}} \left(-y^{(1,2)}_{n,j}(\xi) \pm iy^{(1,3)}_{n,j}(\xi)\right) = \pm Y_{n,j}(\xi) (\xi \otimes \tau_{\pm}). \]

Spin-weighted spherical harmonics of spin weight \(\pm 1\) multiplied by the unit tensor \(\tau_{\pm} \otimes \xi\) are linear combinations of the left tangential/right normal tensor spherical harmonics,
\[ \pm \frac{1}{\sqrt{2}} \left(-y^{(2,1)}_{n,j}(\xi) + iy^{(3,1)}_{n,j}(\xi)\right) = \pm Y_{n,j}(\xi) (\tau_{\pm} \otimes \xi). \]

Spin-weighted spherical harmonics of spin weight \(\pm 2\), multiplied by the unit tensors \(\tau_{\pm} \otimes \tau_{\pm}\) are linear combinations of the tangential tensor spherical harmonics,
\[ -\frac{1}{\sqrt{2}} \left(-y^{(2,3)}_{n,j}(\xi) \pm iy^{(3,2)}_{n,j}(\xi)\right) = \pm 2 Y_{n,j}(\xi) (\tau_{\pm} \otimes \tau_{\pm}) \]

The relationship for the scalar spherical harmonics follows directly from the definition of the spin-weighted spherical harmonics. For the remainder of the proof, see [31]. Note that \(\tau_{\pm} \cdot \tau_{\mp} = 1\) and \(\tau_{\pm} \cdot \tau_{\mp} = 0\) for all \(\xi \in \Omega\).
3 Spin-Weighted Slepian Functions

Similar to the case for the classical spherical harmonics presented in [37], we solve the concentration problem on the unit sphere for bandlimited functions of spin weight $N$. Section 3.1 solves the bandlimited optimization problem for general regions by deriving an equivalent finite-dimensional eigenvalue problem whose matrix is Hermitian and positive definite. This eigenvalue problem is equivalent to a homogeneous integral equation of the second kind.

For the special case of polar cap regions and spin weights $N \in \mathbb{Z}$, we find commuting operators (Section 3.2) with corresponding kernel matrices that are tridiagonal and which have simple eigenvalues (Section 3.3). Because the eigenvectors (but not the eigenvalues) of the constructed commuting matrices are equal to the eigenvectors of the original kernel matrix, this reduces the numerical cost of constructing the Slepian functions for polar caps and spin weights $N \in \mathbb{Z}$, and increases the numerical stability. Slepian functions for general spherical caps can be constructed from those for a polar cap of equal opening angle through rotation.

**Definition 3.1.** If it exists, then the bandlimit $L \in \mathbb{N}_0$ of a function $N F \in L^2(\Omega)$ is the smallest non-negative integer such that $N F \in \text{Harm}_{[N],\ldots,L}^N(\Omega)$. If no such number exists, then $L = \infty$.

3.1 Construction for General Regions

As a consequence of Theorem 2.31 and Remark 2.36, every function $N F \in L^2(\Omega)$ of spin weight $N \in \mathbb{Z}$ that is bandlimited by $L$ (or $N F \in \text{Harm}_{[N],\ldots,L}^N(\Omega)$, for short) can be uniquely described by a linear combination of the spin-weighted spherical harmonics with spin weight $N$,

$$N F = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N F_{n,j} N Y_{n,j},$$

where $N F_{n,j} = \langle N F, N Y_{n,j} \rangle_{L^2(\Omega)}$ for the ranges for $n$ and $j$ defined in the sum above. The dimension of $\text{Harm}_{[N],\ldots,L}^N(\Omega)$ is $(L + 1)^2 - N^2$.

To solve the concentration problem in a measurable set $R \subset \Omega$ for bandlimited (by $L$) spin-weighted spherical harmonics with spin weight $N$, we need to find a function $N F \in \text{Harm}_{[N],\ldots,L}^N(\Omega)$ with a maximum fraction of its total energy (in the sense of the $L^2$-norm) within the region $R$, as is formulated in Problem 3.2.

**Problem 3.2** (Spin-Weighted Concentration Problem). For $L \in \mathbb{N}_0$ and a measurable set $R \subset \Omega$, find $N F \in \text{Harm}_{[N],\ldots,L}^N(\Omega)$ such that

$$\lambda = \frac{\int_{R} N F(\xi) \frac{N F(\xi)}{N F(\xi)} d\omega(\xi)}{\int_{\Omega} N F(\xi) \frac{N F(\xi)}{N F(\xi)} d\omega(\xi)}$$

is maximized. With

$$\int_{R} N F(\xi) \frac{N F(\xi)}{N F(\xi)} d\omega(\xi) = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{n'=-n'}^{n'} N F_{n,j} N F_{n',j'} \int_{R} N Y_{n,j}(\xi) N Y_{n',j'}(\xi) d\omega(\xi)$$

and

$$\int_{\Omega} N F(\xi) \frac{N F(\xi)}{N F(\xi)} d\omega(\xi) = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} |N F_{n,j}|^2,$$

we obtain the formulation

$$\lambda = \frac{G^N G^N}{G^N G^N},$$

13
where $G^N := (N F_{00}, \ldots, N F_{L L})^T$ and $K^N := \begin{pmatrix} K^N_{00,00} & \cdots & K^N_{00,LL} \\ \vdots & \ddots & \vdots \\ K^N_{LL,00} & \cdots & K^N_{LL,LL} \end{pmatrix}$.

Problem 3.2 is therefore equivalent to the following problem.

**Problem 3.3** (Matrix Formulation of the Spin-Weighted Concentration Problem). Find the eigenvectors $G^N \in \mathbb{C}^{(L+1)^2-N^2}$ for the eigenvalue problem

$$K^N G^N = \lambda G^N$$

for which the eigenvalue $\lambda \in \mathbb{R}$ is maximized.

**Lemma 3.4.** The kernel matrix $K^N$ is, as a Gramian matrix based on linearly independent functions, Hermitian and positive definite. Its eigenvalues are hence real and positive and its eigenvalues form an orthonormal basis. This also holds true for its complex conjugate $K^N$. From equation (7) follows that $0 \leq \lambda \leq 1$.

**Theorem 3.5.** The matrix elements of the kernel matrix $K^N$ are given by

$$K^N_{n,j,n',j'} = (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \frac{(2n+1)(2n'+1)(2k+1)}{4\pi} \binom{n+k}{j-j'} \int_R X_{k,j-j'}(t) e^{-i(j-j')\omega} \, d\omega(\xi(t,\varphi))$$

for all $N \in \mathbb{Z}$, all $n,n' \in \mathbb{N}_0$, $n,n' \geq |N|$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$, where

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$$

denotes the Wigner 3j-symbol (see [32] [43]).

The proof to Theorem 3.5 is given in [34].

The matrix in Problem 3.3 is Hermitian, hence the errors in the eigenvalues are limited by numerical errors arising from calculating the regional integrals in Theorem 3.5 (see [9]). Numerical experiments show that the matrix $K^N$ has eigenvalues close to 1 as well as close to 0. This supports the intention behind the definition of Slepian functions: we find well-concentrated functions and can distinguish them from badly concentrated functions. On the other hand, this distribution of the eigenvalues implies a very high condition number of $K^N$. It is known that such a discrepancy between large and small eigenvalues has the following effect: perturbations of the matrix components (which are relevant in our case, if the integrals over $R$ are calculated numerically) can cause a large relative error for the small eigenvalues, which is certainly also connected to the error for the associated eigenvectors, see e.g. [33] Section 5.7. Moreover, high bandlimits, that is large matrices in the eigenvalue problem, are often associated to high numerical costs. Previous experiments with known commuting operators showed that the alternative eigenvalue problem is notably faster to solve.

To construct the commuting matrix for the polar cap regions, we first need to derive yet another equivalent problem to Problems 3.2 and 3.3. The following derivation holds for all regions $R$. Upon multiplying (7) by $N Y_{n,j}(\eta)$, $\eta \in \Omega$, and summing over all $n = |N|, \ldots, L$ and $j = -n, \ldots, n$, we obtain

$$\sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \int_R N Y_{n,j}(\xi) N Y_{n',j'}(\xi) \, d\omega(\xi) N F_{n,j} = \lambda \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) N F_{n,j}.$$

By interchanging summation and integration, this leads to

$$\int_R \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\xi) N Y_{n,j}(\eta) \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} N F_{n',j'} N Y_{n',j'}(\xi) \, d\omega(\xi) = \lambda \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N F_{n,j} N Y_{n,j}(\eta).$$

With these considerations, we can reformulate the eigenvalue problem of Problem 3.3.
Problem 3.6. The eigenvalue problem is equivalent to a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel, this means that

\[ \int_R \mathcal{H}_N^N(\xi, \eta) N F(\xi) \, d\omega(\xi) = \lambda N F(\eta), \]

where \( N F \in \text{Harm}_N^{\text{Harm}}(\Omega) \)

We now turn our attention to the special case of spherical cap regions where \( k \) with a finite-rank, symmetric, and Hermitian kernel, this means that

\[ K_{N,j,n',j'}^N = \int_R \sum_{n = |N|}^n \sum_{j = -n}^n \sum_{n' = 0}^{n+n'} \sum_{k = |n-n'|}^{n+n'} \frac{(-1)^{N+j}}{\sqrt{(2n+1)(2n'+1)}} \sum_{k=|n-n'|}^{n+n'} \begin{pmatrix} n & k & n' \\ j & 0 & -j \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \]

for all \( N \in \mathbb{Z} \), all \( n, n' = |N|, \ldots, L \), all \( j = -n, \ldots, n \), and all \( j' = -n', \ldots, n' \).

Theorem 3.7. The kernel matrix for the spherical cap can be calculated by

\[ K_{N,j,n',j'}^N = \int_R \sum_{n = |N|}^n \sum_{j = -n}^n \sum_{n' = 0}^{n+n'} \sum_{k = |n-n'|}^{n+n'} \frac{(-1)^{N+j}}{\sqrt{(2n+1)(2n'+1)}} \sum_{k=|n-n'|}^{n+n'} \begin{pmatrix} n & k & n' \\ j & 0 & -j \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \]

for all \( N \in \mathbb{Z} \), all \( n, n' = |N|, \ldots, L \), all \( j = -n, \ldots, n \), and all \( j' = -n', \ldots, n' \).

The proof for Theorem 3.7 is given in [34].

Next, we define and prove the commuting operator.

Theorem 3.8. For a polar cap with \( b = \cos \theta \leq t \leq t \), the kernel function defined by

\[ \mathcal{H}_N^N(\xi, \eta) := \sum_{n = |N|}^n \sum_{j = -n}^n \sum_{n' = 0}^{n+n'} \sum_{k = |n-n'|}^{n+n'} \frac{(-1)^{N+j}}{\sqrt{(2n+1)(2n'+1)}} \sum_{k=|n-n'|}^{n+n'} \begin{pmatrix} n & k & n' \\ j & 0 & -j \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \]

commutes with the differential operator

\[ \mathcal{F}_N^N := (b - t_1) \Delta_N + (t_1^2 - 1) \partial_{t_1} - L(L + 2)t_1 \]

for all \( N \in \mathbb{Z} \), where \( \Delta_N \) is defined in Theorem 2.27. \( L \) is the bandlimit, and \( \xi = \xi(t_1, \varphi_1), \eta = \eta(t_2, \varphi_2) \in \Omega \). This means that for any function \( u \in X^2(\Omega_0) \), we obtain

\[ \int_R \mathcal{H}_N^N(\xi, \eta) \left[ \mathcal{F}_N^N u(\eta) \right] \, d\omega(\eta) = \int_R \left[ \mathcal{F}_N^N \mathcal{H}_N^N(\xi, \eta) \right] u(\eta) \, d\omega(\eta) = \mathcal{F}_N^N \int_R \mathcal{H}_N^N(\xi, \eta) u(\eta) \, d\omega(\eta). \]

Remark 3.9. Let \( N \in \mathbb{Z} \) and \( \xi, \eta \in \Omega \). To prove Theorem 3.8 we have to show that the following holds:

1. For two functions \( u_1, u_2 \in X^2(\Omega_0) \), the differential operator is self-adjoint, that is

\[ \int_R u_1(\xi) \left[ \mathcal{F}_N^N u_2(\xi) \right] \, d\omega(\xi) = \int_R \left[ \mathcal{F}_N^N u_1(\xi) \right] u_2(\xi) \, d\omega(\xi). \]
2. \( \mathcal{F}_\xi^N \mathcal{H}^N (\xi, \eta) = \mathcal{F}_\eta^N \mathcal{H}^N (\xi, \eta) \).

3. \( \mathcal{F}_\xi^N \mathcal{H}^N (\xi, \eta) = \mathcal{F}_\xi^N \mathcal{H}^N (\xi, \eta) \).

4. We can interchange the integration and the differential operator; this means that for any \( u \in X^2(\Omega_0) \)

\[
\mathcal{F}_\xi^N \int_R \mathcal{H}^N (\xi, \eta) u(\eta) \ d\omega(\eta) = \int_R \mathcal{F}_\xi^N \mathcal{H}^N (\xi, \eta) u(\eta) \ d\omega(\eta).
\]

Theorem 3.8 follows from Remark 3.9 and Corollary 2.29 because

\[
\int_R \mathcal{H}^N (\xi, \eta) [\mathcal{F}_\eta^N u(\eta)] \ d\omega(\eta) \stackrel{!}{=} \int_R [\mathcal{F}_\eta^N \mathcal{H}^N (\xi, \eta)] u(\eta) \ d\omega(\eta)
\]

\[
\stackrel{!}{=} \int_R [\mathcal{F}_\eta^N \mathcal{H}^N (\xi, \eta)] u(\eta) \ d\omega(\eta)
\]

\[
\stackrel{!}{=} \int_R \mathcal{F}_\xi^N \mathcal{H}^N (\xi, \eta) u(\eta) \ d\omega(\eta).
\]

To prove Theorem 3.8, we prove Remark 3.9 in the following.

Proof. Let \( N \in \mathbb{Z} \) be a given spin weight.

1. Let \( \xi = \xi(t, \varphi) \in \Omega \) and \( u_1, u_2 \in X^2(\Omega_0) \). Then, the left-hand side of the first condition in Remark 3.9 can be reformulated to

\[
\int_R u_1(\xi) [\mathcal{F}_\xi^N u_2(\xi)] \ d\omega(\xi) = \int_R (b - t)u_1(\xi)\Delta^*_\xi^N u_2(\xi) \ d\omega(\xi) + \int_R (t^2 - 1) \overline{u_1(\xi)} \partial_t u_2(\xi) \ d\omega(\xi)
\]

\[
- \int_R L(\xi^2 + 2) \overline{u_1(\xi)} u_2(\xi) \ d\omega(\xi).
\]

These integrals exist, because \( u_1, u_2 \in X^2(\Omega_0) \) yields that \( u_1, u_2, (t^2 - 1) \partial_t u_2, \) and \( \Delta^*_\xi^N u_2 \) are all bounded on \( \Omega \). As a consequence, \( \mathcal{F}_\xi^N u_2 \) is also bounded on \( \Omega \). We hence integrate over products of bounded functions, which themselves are bounded. The same holds true for the right-hand side of the first condition from Remark 3.9 with interchanged roles of \( u_1 \) and \( u_2 \). Note that in Theorem 3.8, we use \( \mathcal{H}^N \) instead of \( u_1 \). Corollary 2.29 states that the spin-weighted spherical harmonics are in \( X^2(\Omega_0) \). Therefore, \( \mathcal{H}^N \) consists of products and sums of bounded functions, which are hence bounded themselves.

The first integral on the right-hand side of Equation 8, together with Green’s second surface identity for the spin-weighted Beltrami operator for the spherical cap (Theorem 2.30) leads us to

\[
\int_R (b - t)\overline{u_1(\xi)}\Delta^*_\xi^N u_2(\xi) \ d\omega(\xi)
\]

\[
= \int_R \overline{\Delta^*_\xi^N[(b - t)\overline{u_1(\xi)}]} u_2(\xi) \ d\omega(\xi)
\]

\[
- \int_0^{2\pi} [(1 - t^2) \left( (b - t)\overline{u_1(\xi)} \partial_t u_2(\xi) - u_2(\xi) \partial_t \left( (b - t)\overline{u_1(\xi)} \right) \right)]_{t=b} \ d\varphi.
\]

Because

\[
\overline{\Delta^*_\xi^N} = \Delta^*_\xi - \frac{N^2 + 2iNt\partial_t}{1 - t^2}
\]
(see Theorem 2.27) and \( b - t \) is independent of \( \varphi \), we get, consequently,

\[
\Delta_\xi^2 \left( (b - t)u_1(\xi) \right) \\
= \partial_t \left( (1 - t^2) \partial_t (b - t)\overline{u_1(\xi)} \right) + (b - t) \frac{1}{1 - t^2} \Delta_\xi^2 u_1(\xi) \\
= \partial_t \left( (1 - t^2) \left( -\overline{u_1(\xi)} + (b - t)\partial_t u_1(\xi) \right) \right) + (b - t) \frac{1}{1 - t^2} \Delta_\xi^2 u_1(\xi) \\
= \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) - (1 - t^2) \partial_t u_1(\xi) + (b - t)\partial_t \left( (1 - t^2) \partial_t \overline{u_1(\xi)} \right) \\
+ (b - t) \frac{1}{1 - t^2} \Delta_\xi^2 u_1(\xi) \\
= (b - t)\Delta_\xi^2 \overline{u_1(\xi)} + \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) + (t^2 - 1) \partial_t \overline{u_1(\xi)},
\]

which yields

\[
\Delta_\xi^2 \left( (b - t)\overline{u_1(\xi)} \right) = (b - t)\Delta_\xi^2 \overline{u_1(\xi)} + \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) + (t^2 - 1) \partial_t \overline{u_1(\xi)}.
\]

Furthermore, we obtain

\[
\left[ (1 - t^2) \left( (b - t)\overline{u_1(\xi)} \partial_t u_2(\xi) - u_2(\xi) \partial_t \left( (b - t)\overline{u_1(\xi)} \right) \right) \right]_{t=b} \\
= \left[ (1 - t^2) (b - t)\overline{u_1(\xi)} \partial_t u_2(\xi) \right]_{t=b} - \left[ (1 - t^2) (b - t)u_2(\xi) \partial_t \overline{u_1(\xi)} \right]_{t=b} \\
+ \left[ (1 - t^2) \overline{u_1(\xi)} u_2(\xi) \right]_{t=b} \\
= \left[ (1 - t^2) \overline{u_1(\xi)} u_2(\xi) \right]_{t=b}.
\]

Altogether, the first integral on the right-hand side of Equation (8) can be expressed as

\[
\int_R (b - t)\overline{u_1(\xi)} \Delta_\xi^2 u_2(\xi) \, d\omega(\xi) \\
= \int_R (b - t)u_2(\xi) \Delta_\xi^2 \overline{u_1(\xi)} \, d\omega(\xi) + \int_R u_2(\xi) \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) \, d\omega(\xi) \\
+ \int_R (t^2 - 1) u_2(\xi) \partial_t \overline{u_1(\xi)} \, d\omega(\xi) - \int_0^{2\pi} \left[ (1 - t^2) \overline{u_1(\xi)} u_2(\xi) \right]_{t=b} \, d\varphi,
\]

while integration by parts applied to the second integral on the right-hand side of Equation (8) leads to

\[
\int_R (t^2 - 1) \overline{u_1(\xi)} \partial_t u_2(\xi) \, d\omega(\xi) \\
= \int_0^{2\pi} \int_b^1 (t^2 - 1) \overline{u_1(\xi)} \partial_t u_2(\xi) \, dt \, d\varphi \\
= \int_0^{2\pi} \left[ (t^2 - 1) \overline{u_1(\xi)} u_2(\xi) \right]_{t=1} \, d\varphi - \int_0^{2\pi} \int_b^1 \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) u_2(\xi) \, dt \, d\varphi \\
= \int_0^{2\pi} \left[ (1 - t^2) \overline{u_1(\xi)} u_2(\xi) \right]_{t=b} \, d\varphi - \int_0^{2\pi} \int_b^1 \partial_t \left( (t^2 - 1) \overline{u_1(\xi)} \right) u_2(\xi) \, dt \, d\varphi.
\]
Altogether, we obtain
\[
\int_R \frac{u_1(\xi)}{\mathcal{F}_\xi^N u_2(\xi)} \, d\omega(\xi)
= \int_R (b - t)\Delta^{*N}_\xi u_1(\xi) u_2(\xi) \, d\omega(\xi) + \int_R (t^2 - 1) \left( \partial_t u_1(\xi) \right) u_2(\xi) \, d\omega(\xi)
- \int_R L(L + 2)t u_1(\xi) u_2(\xi) \, d\omega(\xi)
= \int_R \left( \mathcal{F}_\xi^N u_1(\xi) \right) u_2(\xi) \, d\omega(\xi).
\]

2. To prove the second equation in Remark 3.9 for \( \xi \in \Omega \) and \( \eta \in \Omega \), we invoke Theorems 2.27 and 2.22

\[
\left( \mathcal{F}_\xi^N - \mathcal{F}_\eta^N \right) \mathcal{F}^N(\xi, \eta)
= \left( (b - t_1) \Delta^{*N}_\xi + (t_1^2 - 1) \partial_t \right) N Y_{n,j}(\xi) - L(L + 2) N Y_{n,j}(\xi) \left( \begin{array}{c} \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n + 1) - L(L + 2)) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) \\ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (t_1^2 - 1) \partial_t \right) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) \end{array} \right)
\]

\[
= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n + 1) - L(L + 2)) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta)
+ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (t_1^2 - 1) \partial_t \right) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta)
= \left( \begin{array}{c} N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n + 2) - L(L + 2)) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) \\ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (2n + 1) \alpha_{n,j} \left( N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n-1,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n-1,j}(\eta) \right) \end{array} \right).
\]

Note that we used \( \Delta^{*N}_\xi N Y_{n,j}(\xi) = \Delta^{*N}_\xi N Y_{n,j}(\xi) \).

Furthermore, Theorem 2.23 together with the relationship

\[
\sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{k=|n|}^{n-1} b_{n,j,k} = \sum_{k=|N|}^{L-1} \sum_{j=-k}^{k} \sum_{n=|N|}^{L} b_{n,j,k},
\]

yields

\[
\left( \mathcal{F}_\xi^N - \mathcal{F}_\eta^N \right) \mathcal{F}^N(\xi, \eta)
= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n + 2) - L(L + 2)) N Y_{n,j}(\xi) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta)
\]

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\[ + (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} (2n + 1) \sum_{k=n_j}^{n-1} nY_{k,j}(\xi) N Y_{k,j}(\eta) \]
\[ = (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} (n(n + 2) - L(L + 2))nY_{n,j}(\xi) N Y_{n,j}(\eta) \]
\[ + (t_1 - t_2) \sum_{k=n+1}^{L-1} \sum_{|j|-k}^{n} nY_{k,j}(\xi) N Y_{k,j}(\eta) \sum_{n=k+1}^{L} (2n + 1) \]
\[ = (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} nY_{n,j}(\xi) N Y_{n,j}(\eta) \left[ n(n + 2) - L(L + 2) \]
\[ + \sum_{k=n+1}^{L} (2k + 1) \right] \]
\[ = 0, \]

because \( n(n + 2) - L(L + 2) + \sum_{k=n+1}^{L} (2k + 1) = 0 \). This is equivalent to the proposition
\[ \mathcal{F}_\xi^N \mathcal{H}^N(\xi, \eta) = \mathcal{F}_\eta^N \mathcal{H}^N(\xi, \eta). \]

3. To prove the third equality of Remark 3.9, we use that \( \mathcal{H}^N \) is self-adjoint
\[ \mathcal{H}^N(\xi, \eta) = \mathcal{H}^N(\eta, \xi). \]

Using the definition of \( \Delta^{s,N} \) given in Theorem 2.27, we obtain
\[ \mathcal{J}_\xi^N = \partial_t (b - t) (1 - t^2) \partial_t - \left( \frac{N^2(b - t)}{1 - t^2} + L(L + 2)t \right) + \frac{b - t}{1 - t^2} \left( \partial^2_r + 2iNt\partial_r \right). \]

We can now utilize Theorem 2.18 to derive the identity
\[ \mathcal{J}_\xi^N \mathcal{H}^N(\xi, \eta) \]
\[ = \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} nY_{n,j}(\eta) \mathcal{J}_\xi^N \mathcal{H}^N(\xi, \eta) \]
\[ = \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} nY_{n,j}(\eta) \left( \partial_t (b - t) (1 - t^2) \partial_t \mathcal{H}^N(\xi, \eta) \right) \]
\[ - \left( \frac{N^2(b - t)}{1 - t^2} + L(L + 2)t \right) \mathcal{H}^N(\xi, \eta) + \frac{b - t}{1 - t^2} \left( -j^2 - 2jNt \right) \mathcal{H}^N(\xi, \eta). \]

Analogously, the left-hand side of the third condition in Remark 3.9 yields
\[ \mathcal{J}_\xi^N \mathcal{H}^N(\xi, \eta) \]
\[ = \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} nY_{n,j}(\eta) \mathcal{J}_\xi^N \mathcal{H}^N(\xi, \eta) \]
\[ = \sum_{n=|N|}^{L} \sum_{|j|-n}^{n} nY_{n,j}(\eta) \left( \partial_t (b - t) (1 - t^2) \partial_t \mathcal{H}^N(\xi, \eta) \right) \]
\[ - \left( \frac{N^2(b - t)}{1 - t^2} + L(L + 2)t \right) \mathcal{H}^N(\xi, \eta) + \frac{b - t}{1 - t^2} \left( -j^2 - 2jNt \right) \mathcal{H}^N(\xi, \eta). \]

As a consequence, we get
\[ \mathcal{J}_\xi^N \mathcal{H}^N(\xi, \eta). \]
\[
= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} N Y_{n,j}(\eta) \left( \partial_{t} \left( (b - t^2) \partial_{t} N Y_{n,j}(\xi) \right) - \left( \frac{N^2(b - t)}{1 - t^2} + L(L + 2)t \right) N Y_{n,j}(\xi) + \frac{b - t}{1 - t^2} \left( -j^2 - 2jNt \right) N Y_{n,j}(\xi) \right).
\]

Hence, the left- and the right-hand side of the third condition in Remark 3.9 are equal.

4. Since the functions occurring in the fourth condition of Remark 3.9 are sufficiently smooth, it is easy to verify that the conditions for interchanging differentiation and integration are satisfied.

\[\square\]

**Theorem 3.10.** The commuting relation also holds true for an integral over the unit sphere. For \( \xi \in \Omega, \)

\[
\int_{\Omega} \overline{\mathcal{F}^N(\xi, \eta)} \mathcal{F}_\eta^N u(\eta) \ d\omega(\eta) = \int_{\Omega} \left[ \mathcal{F}_\xi^N \overline{\mathcal{F}^N(\xi, \eta)} \right] u(\eta) \ d\omega(\eta) = \mathcal{F}_\xi^N \int_{\Omega} \overline{\mathcal{F}^N(\xi, \eta)} u(\eta) \ d\omega(\eta).
\]

**Proof.** Note that we did not have any constraints on \( R \) which would have excluded the case \( R = \Omega. \) \[\square\]

**Corollary 3.11.** The operator \( \mathcal{F}^N \) is an endomorphism on every \( \text{Harm}_{|N|\ldots L}(\Omega), \) that is \( \mathcal{F}^N Y \in \text{Harm}_{|N|\ldots L}(\Omega) \) for every \( Y \in \text{Harm}_{|N|\ldots L}(\Omega). \)

**Proof.** Clearly, \( \mathcal{F}^N \) from Problem 3.6 is the reproducing kernel of \( \text{Harm}_{|N|\ldots L}(\Omega) \) in the sense that

\[
\int_{\Omega} Y(\eta) \overline{\mathcal{F}^N(\xi, \eta)} \ d\omega(\eta) = Y(\xi)
\]

for all \( \xi \in \Omega \) and all \( Y \in \text{Harm}_{|N|\ldots L}(\Omega). \) More generally, we have automatically

\[
\int_{\Omega} Y(\eta) \overline{\mathcal{F}^N(\xi, \eta)} \ d\omega(\eta) = \mathcal{P}_{\text{Harm}_{|N|\ldots L}(\Omega)} Y(\xi)
\]

for all \( \xi \in \Omega \) and all \( Y \in \text{L}^2(\Omega), \) where \( \mathcal{P}_{\text{Harm}_{|N|\ldots L}(\Omega)} \) is the orthogonal projection onto \( \text{Harm}_{|N|\ldots L}(\Omega). \)

Hence, if now \( Y \in \text{Harm}_{|N|\ldots L}(\Omega) \) is arbitrary, then we obtain together with Theorem 3.10

\[
\mathcal{F}_\xi^N Y(\xi) = \mathcal{F}_\xi^N \int_{\Omega} Y(\eta) \overline{\mathcal{F}^N(\xi, \eta)} \ d\omega(\eta) = \int_{\Omega} \left[ \mathcal{F}_\eta^N Y(\eta) \right] \overline{\mathcal{F}^N(\xi, \eta)} \ d\omega(\eta).
\]

Thus, \( \mathcal{F}^N Y \in \text{Harm}_{|N|\ldots L}(\Omega). \) \[\square\]

### 3.3 Computation of Slepian Functions for Spherical Cap Regions

For polar cap regions, we can use the commuting operator \( \mathcal{F}^N \) of Section 3.2 to construct a commuting Matrix \( I^N, \) which, as we will show, has the same eigenvectors as \( K^N. \) By solving for the eigenvectors of \( I^N \) instead of \( K^N, \) we obtain the same spin-weighted Slepian functions but with an increased numerical stability and at a lower computational cost. See also [37] for the scalar case.

Previously, we showed that \( \mathcal{F}_\xi^N \) and \( \mathcal{F}^N \) commute for all \( N \in \mathbb{Z}. \) Next, we need to show that they have the same eigenfunctions \( \mathcal{G}_\alpha^N, \) meaning that for \( \xi \in \Omega \) and \( N \in \mathbb{Z}, \)

\[
\mathcal{F}_\xi^N \mathcal{G}_\alpha^N(\xi) = \chi_\alpha \mathcal{G}_\alpha^N(\xi),
\]

\[
\int_R \overline{\mathcal{F}^N(\xi, \eta)} \mathcal{G}_\alpha^N(\eta) \ d\omega(\eta) = \lambda_\alpha \mathcal{G}_\alpha^N(\xi),
\]

where \( \lambda_\alpha \) and \( \chi_\alpha \) are not necessarily equal.
Remark 3.12. The matrix problem equivalent to $\mathcal{F}_\xi \mathcal{G}_\alpha^N(\xi) = \chi_\alpha \mathcal{G}_\alpha^N(\xi)$ is $I^N \mathcal{G}_\alpha^N = \chi_\alpha G_\alpha^N$, where

$$I^N_{n,n'} := \int_{\Omega} N Y_{n,j}(\xi) \left( \mathcal{F}_\xi \mathcal{G}_\alpha^N(\xi) \right) d\omega(\xi),$$

$$I^N := \begin{pmatrix}
I^N_{|N|,-|N|,-|N|} & \cdots & I^N_{|N|,-|N|,|N|} \\
\vdots & \ddots & \vdots \\
I^N_{|N|,|N|,-|N|} & \cdots & I^N_{|N|,|N|,|N|}
\end{pmatrix},$$

and $G_\alpha^N = \left( (G_\alpha^N)_{|N|,-|N|}, \ldots, (G_\alpha^N)_{|N|,|N|} \right)^T$, with $(G_{n,j})_\alpha = \langle G_\alpha^N, N Y_{n,j} \rangle_{L^2(\Omega)}$.

Proof. The eigenvalue problem

$$\mathcal{F}_\xi \mathcal{G}_\alpha^N(\xi) = \chi_\alpha \mathcal{G}_\alpha^N(\xi)$$

is equivalent to

$$\sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha \mathcal{F}_\xi N Y_{n',j'}(\xi) = \chi_\alpha \sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha N Y_{n',j'}(\xi).$$

Upon multiplying by $N Y_{n,j}(\xi)$, $n = |N|, \ldots, L$, $j = -n, \ldots, n$, we obtain

$$\sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha N Y_{n,j}(\xi) \left( \mathcal{F}_\xi \mathcal{G}_\alpha^N(\xi) \right) = \chi_\alpha \sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha N Y_{n,j}(\xi) N Y_{n',j'}(\xi).$$

Integration over the unit sphere $\Omega$ and interchanging of sum and integral, together with Theorem 2.24 yields

$$\sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha \int_{\Omega} N Y_{n,j}(\xi) \left( \mathcal{F}_\xi \mathcal{G}_\alpha^N(\xi) \right) d\omega(\xi)$$

$$= \chi_\alpha \sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha \int_{\Omega} N Y_{n,j}(\xi) N Y_{n',j'}(\xi) d\omega(\xi)$$

and hence

$$\sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha I^N_{n,j,n'} = \chi_\alpha (G_{n,j})_\alpha$$

for all $n = |N|, \ldots, L$ and all $j = -n, \ldots, n$.

Vice versa, if (9) holds true, then the linearity of $\mathcal{F}^N$ yields

$$\chi_\alpha \mathcal{G}_\alpha^N(\xi) = \sum_{n=|N|}^L \sum_{j=-n}^{n} \chi_\alpha (G_{n,j})_\alpha N Y_{n,j}(\xi)$$

$$= \sum_{n=|N|}^L \sum_{j=-n}^{n} \sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha I^N_{n,j,n'} N Y_{n,j}(\xi)$$

$$= \sum_{n=|N|}^L \sum_{j=-n}^{n} \sum_{n'=|N|}^L \sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha \int_{\Omega} N Y_{n,j}(\eta) \left( \mathcal{F}_\eta \mathcal{G}_\alpha^N(\eta) \right) d\omega(\eta) N Y_{n,j}(\xi)$$

$$= \sum_{n=|N|}^L \sum_{j=-n}^{n} \int_{\Omega} N Y_{n,j}(\eta) \left[ \mathcal{F}_\eta \mathcal{G}_\alpha^N \right] d\omega(\eta) N Y_{n,j}(\xi).$$
Hence, Theorem 2.31, Remark 2.36, and Corollary 3.11 imply
\[
\chi_\alpha \mathcal{F}_\alpha^N (\xi) = \sum_{n=|N|}^{N} \sum_{j=-n}^{n} \langle \mathcal{F}_n^N \mathcal{F}_\alpha^N, N \alpha_j \rangle_{L^2(\Omega)} N \alpha_j (\xi) = \mathcal{F}_\alpha^N X_\alpha (\xi),
\]
which completes the proof. \(\square\)

**Theorem 3.13.** \(K^N\) and \(I^N\) also commute for all \(N \in \mathbb{Z}\), that is
\[
K^N I^N = I^N K^N.
\]

**Proof.** Let \(N \in \mathbb{Z}\), \(n, l = |N|, \ldots, L\), \(j = -n, \ldots, n\), and \(m = -l, \ldots, l\). Then the left-hand side together with Theorem 3.10, Theorem 2.24, and Corollary 2.29 leads to
\[
(K^N I^N)_{n,j,lm} = \sum_{n'=|N|}^{N} I^N_{n,j,lm} K^N_{n',j',lm}.
\]
The right-hand side in combination with Theorem 3.8 and Corollary 2.29 yields the same result.

Note that we used here again that \(\mathcal{F}^N\) is the reproducing kernel of \(\text{Harm}^N_{|N|,\ldots,L}(\Omega)\) (see also the proof of Corollary 3.11). \(\square\)

Note that because \(I^N, K^N \in \mathbb{R}^{([L+1]^2-N^2) \times ([L+1]^2-N^2)}\) (as a result of \(n = |N|, \ldots, L, j = -n, \ldots, n\)), we obtain \((L + 1)^2 - N^2\) orthogonal eigenvectors \(G^N\), orthogonal eigenfunctions \(G^N\), and eigenvalues \(\lambda^N\), where \(\alpha = 1, \ldots, (L + 1)^2 - N^2\).

**Lemma 3.14.** The components of \(I^N\) have the following form:
\[
\begin{align*}
I_{n,j,n,j}^N &= - \left[ n(n+1)b + N \left( 1 - \frac{L(L+2)+1}{n(n+1)} \right) \right], \\
I_{n,j,n+1,j}^N &= \left( [n(n+1)^2 - 1 - L(L+2)] \alpha_{n+1,j}^N \right) \\
&= [n(n+2) - L(L+2)] \sqrt{(n+1-N)(n+1+N)} \frac{(n+1-j)(n+1+j)}{(2n+1)(2n+3)}, \\
I_{n+1,j,n,j}^N &= [n(n+2) - L(L+2)] \alpha_{n+1,j}^N = I_{n,j,n+1,j}^N, \\
I_{n,j,n',j'}^N &= 0, \quad \text{else}
\end{align*}
\]
for all \(N \in \mathbb{Z}\), all \(n,n' = |N|, \ldots, L\), all \(j = -n, \ldots, n\), and all \(j' = -n', \ldots, n\). Therefore, \(I^N\) is a symmetric tridiagonal matrix.
Theorem 3.16. The Slepian functions and their corresponding coefficient vectors are orthonormal on the unit sphere and orthogonal within the region of interest $R$

$$\sum_{n=|N|}^{L} \sum_{j=-n}^{n} \langle G_{n,j}^{N} \rangle_{\alpha} \langle G_{n,j}^{N} \rangle_{\beta} = \delta_{\alpha,\beta},$$

$$\sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \langle G_{n,j}^{N} \rangle_{\alpha} K_{n,n'}^{N} \langle G_{n',j'}^{N} \rangle_{\beta} = \lambda_{\alpha} \delta_{\alpha,\beta},$$

$$\langle \mathcal{S}_{\alpha}^{N} , \mathcal{S}_{\beta}^{N} \rangle_{L^{2}(R)} = \delta_{\alpha,\beta},$$

$$\langle \mathcal{S}_{\alpha}^{N} , \mathcal{S}_{\beta}^{N} \rangle_{L^{2}(\Omega)} = \lambda_{\alpha} \delta_{\alpha,\beta}$$

for all $\alpha, \beta = 1, \ldots, (L+1)^2 - N^2$.

Theorem 3.17. Each construction of spin-weighted Slepian functions $\{\mathcal{S}_{\alpha}^{N}\}_{\alpha=1,\ldots,(L+1)^2-N^2}$ for any region $R$ forms a complete orthonormal basis system of $\text{Harm}_{|N|}^{N} L(\Omega)$. Therefore, any $\mathcal{F} \in \text{Harm}_{|N|}^{N} L(\Omega)$
can be expressed both in the basis of the spin-weighted spherical harmonics and in the basis of the spin-weighted Slepian functions

\[ N F(\xi) = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \langle N F, N Y_{n,j} \rangle_{L^2(\Omega)} N Y_{n,j}(\xi) = \sum_{\alpha=1}^{(L+1)^2 - N^2} \langle N F, \mathcal{G}^N_{\alpha} \rangle_{L^2(\Omega)} \mathcal{G}^N_{\alpha}(\xi), \]

for \( \xi \in \Omega \).

**Theorem 3.18.** The spin-weighted spherical harmonics for degrees \(|N|\) to \(L\) can be expressed in the basis of the spin-weighted Slepian functions

\[ N Y_{n,j} = \left( \frac{(L+1)^2 - N^2}{2} \right) \sum_{\alpha=1}^{(L+1)^2 - N^2} \langle G_{n,j}^N, \mathcal{G}^N_{\alpha} \rangle \mathcal{G}^N_{\alpha}, \quad (15) \]

where

\[ \sum_{\alpha=1}^{(L+1)^2 - N^2} \langle G_{n,j}^N, \mathcal{G}^N_{\alpha} \rangle \mathcal{G}^N_{\alpha} = \delta_{n,n'} \delta_{j,j'}, \quad (16) \]

**Theorem 3.19.** The spin-weighted Slepian functions also fulfill the following properties

\[ \sum_{\alpha=1}^{(L+1)^2 - N^2} \lambda_{\alpha} \langle G_{n,j}, G_{n',j'} \rangle_{\alpha} = K_{n,j,n',j'}, \quad (17) \]

\[ \sum_{\alpha=1}^{(L+1)^2 - N^2} \lambda_{\alpha} \mathcal{G}^N_{\alpha}(\xi) \mathcal{G}^N_{\alpha}(\eta) = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \langle N Y_{n,j}, K_{n,j,n',j'} N Y_{n',j'} \rangle \mathcal{G}^N_{\alpha}(\xi) \mathcal{G}^N_{\alpha}(\eta), \quad (18) \]

for all \( n,n' = |N|, \ldots, L \), all \( j = -n, \ldots, n \), all \( j' = -n', \ldots, n' \), and all \( \xi, \eta \in \Omega \).

As we mentioned above, eigenvalues of spin-weighted Slepian functions often cluster around \( \lambda^N \approx 1 \) and \( \lambda^N \approx 0 \). The eigenvalue number at which this transition takes place can be predicted by the Shannon number, which we derive for the spin-weighted Slepian functions in the next section.

### 3.5 Shannon Number

If the eigenvalues of a matrix \( K^N \) have a bimodal distribution with clusters at 1 and 0, then the Shannon number

\[ S^N = \sum_{\alpha=1}^{(L+1)^2 - N^2} \lambda_{\alpha} = tr(K^N), \]

predicts the number of eigenvalues close to 1. Therefore, \( S^N \) provides an estimation of the dimension of the space of signals of spin weight \( N \in \mathbb{Z} \) that are both bandlimited by \( L \) and optimally concentrated in \( R \). The basis of this space is given by the eigenfunctions \( \mathcal{G}^N_1, \mathcal{G}^N_2, \ldots, \mathcal{G}^N_{S^N} \).

**Lemma 3.20.** The Shannon number \( S^N \) of spin-weighted Slepian functions and hence the trace of the matrix \( K^N \) only depends on the bandwidth \( L \), the spin weight \( N \), and the area \( A \) of the region \( R \) on the unit sphere

\[ S^N = \left( (L+1)^2 - N^2 \right) \frac{A}{4\pi}. \]
Proof. Corollary 2.26 from the addition theorem yields

\[ S^N = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} K_{n,j}^N \]

\[ = \int_R \sum_{n=|N|}^{L} \sum_{j=-n}^{n} NY_{n,j}(\xi) \cdot NY_{n,j}(\xi) d\omega(\xi) \]

\[ = \frac{1}{4\pi} \sum_{n=|N|}^{L} (2n+1) \int_R d\omega(\xi) \]

\[ = \left((L+1)^2 - N^2\right) \frac{A}{4\pi}. \]

As an obvious consequence, the number of Slepian functions with significant eigenvalues is higher for regions with a large area on the unit sphere, than it is for regions covering a small area. For the special case of a spherical cap \((b = \cos \theta \leq t \leq 1)\), the area satisfies

\[ \frac{A_{\text{cap}}}{4\pi} = \frac{1 - b}{2}. \]

4 Scalar, Vector, and Tensor Slepian Functions

The previously described construction of spin-weighted Slepian functions with the help of Theorem 2.38 allows us to construct scalar, vector, and tensor Slepian functions. In particular for the tensor Slepian functions for spherical cap regions, this approach presents a previously unknown commuting operator approach.

4.1 Scalar Slepian Functions

The scalar Slepian functions have already been well investigated e.g. in \([1, 35, 37]\). From the definition of the spin-weighted spherical harmonics and, consequently, from Theorem 2.38, we know that the spin-weighted spherical harmonics of spin weight zero are the fully normalized spherical harmonics. We therefore obtain the scalar Slepian functions directly from the spin-weighted Slepian functions with spin weight zero.

4.2 Vector Slepian Functions

We revisit the vector Slepian functions presented by Jahn and Bokor \([19]\) and Plattner and Simons \([31]\) by constructing them using the spin-weighted spherical-harmonic approach. A commuting operator using the classical approach was presented in \([20]\). Here we build an alternative vector spherical-harmonic basis using the spin-weighted spherical harmonics.

Definition 4.1. The spin-weighted harmonic-based vector functions with bandlimit \(L\) are

\[ y_{n,j}^1(\xi) := y_{n,j}^{(1)}(\xi) = \xi Y_{n,j}(\xi), \]

\[ y_{n,j}^2(\xi) := \frac{1}{\sqrt{2}} \left(-y_{n,j}^{(2)}(\xi) + iy_{n,j}^{(3)}(\xi)\right) = \tau_+ Y_{n,j}(\xi), \]

\[ y_{n,j}^3(\xi) := \frac{1}{\sqrt{2}} \left(-y_{n,j}^{(2)}(\xi) - iy_{n,j}^{(3)}(\xi)\right) = \tau_- Y_{n,j}(\xi), \]

for \(\xi \in \Omega\) and \(n = 0, \ldots, L\), where \(-n \leq j \leq n\), with

\[ 0_i = \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3 \end{cases}. \]
Remark 4.2. The functions in Definition 4.1 form an orthonormal basis of $\text{harm}_{0,L}(\Omega)$. Moreover,
\[ y_{n,j}^i(\xi) \cdot y_{m,l}^k(\xi) = 0, \quad \text{if } i \neq k, \text{ for all } \xi \in \Omega \text{ and all } n, j, m, l \quad (\text{pointwise orthogonality}), \]
where $\cdot$ denotes the standard inner product.

Pointwise orthogonality follows from the pointwise orthonormality of $\xi, \tau_+$, and $\tau_-$. The spin-harmonic-based vector functions with maximum degree $L$ form a basis of $\text{harm}_{0,L}(\Omega)$ because of their non-degenerate linear relationship to the functions $y_{n,j}^{(i)}$ for $i \in \{1, 2, 3\}, n = 0, \ldots, L$, and $-n \leq j \leq n$, which themselves form a basis of $\text{harm}_{0,L}(\Omega)$, see Definition 2.12.

We can therefore represent any vector function $g \in \text{harm}_{0,L}(\Omega)$ as a linear combination of the spin-weighted harmonic-based vector functions
\[ g(\xi) = \sum_{i=1}^{3} \sum_{n=0}^{L} \sum_{j=-n}^{n} g_{n,j}^i y_{n,j}^i(\xi) \]
for all $\xi \in \Omega$ with coefficients
\[ g_{n,j}^i = \int_{\Omega} g(\xi) \cdot \overline{y_{n,j}^i(\xi)} \, d\omega(\xi) \]
for all $(i, n, j) \in J_L$. Here, we used the set of indices
\[ J_L := \{(i, n, j) | i = 1, 2, 3; n = 0, \ldots, L; j = -n, \ldots, n\}. \]

The spatial concentration problem for bandlimited vector functions, independent of the selected basis, is
\[ \lambda = \frac{\int_{\Omega} g(\xi) \cdot \overline{g(\xi)} \, d\omega(\xi)}{\int_{\Omega} g(\xi) \cdot d\omega(\xi)} = \max. \quad (19) \]

The classical vector spherical harmonic functions $y_{n,j}^{(i)}$ for $(i, n, j) \in J_L$ lead to a blockdiagonal matrix, where the normal component is decoupled from the tangential component (see [19, 31]). As we show in the following, the pointwise orthogonality of the three types of spin-weighted harmonic-based functions $y_{n,j}^1, y_{n,j}^2, y_{n,j}^3$ leads to a blockdiagonal matrix with three blocks. One for the radial component and two for the tangential component. Moreover, due to the decoupling, we can solve the concentration problem for each of the spin weights individually, allowing us to take full advantage of the derivations in previous sections. In particular, the commuting operator solution for spherical caps for spin-weights $0, \pm 1$ translates directly into vector Slepian functions.

**Problem 4.3.** Concentration problem (19) expressed in the spin-weighted harmonic-based vector functions from Definition 4.1 yields the eigenvalue problem
\[ kg = \lambda g, \]
where
\[ k := \begin{pmatrix} k^1 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^3 \end{pmatrix} := \begin{pmatrix} K^0 & 0 & 0 \\ 0 & K^+ & 0 \\ 0 & 0 & K^- \end{pmatrix} \in \mathbb{R}^{3(L+1)^2-2 \times 3(L+1)^2-2} \]
with the matrices $k^i$ defined by their components
\[ k_{n,j,n',j'}^{i} := \int_{\mathbb{R}} \overline{y_{n,j}^i(\xi)} \cdot y_{n',j'}^i(\xi) \, d\omega(\xi). \]
Hence, the vector problem reduces to three spin-weighted problems for spin weights $0, \pm 1$, and $-1$, which we solved in the previous chapter. To find the eigenvectors $g_{\alpha}$ of Problem 4.3, we simply pad the eigenvectors of the spin-weighted problems with zeros. Again, we sort the eigenvectors by decreasing eigenvalues. Note that this sorting implies that the types of vectors are not sorted any more. We represent this with a mapping $\tilde{\alpha} : \{1, \ldots, 3(L+1)^2-2\} \rightarrow \{1, \ldots, (L+1)^2\} \times \{1, 2, 3\}, \alpha \mapsto (\tilde{\alpha}_1, \tilde{\alpha}_2)$ which associates
the number $\alpha$ of a vectorial Slepian function to a vector type (i.e., here, a matrix block) $\tilde{\alpha}_2(\alpha) \in \{1, 2, 3\}$ and the number $\tilde{\alpha}_1(\alpha)$ of a spin-weighted scalar Slepian function. Correspondingly, we obtain the vector Slepian functions $g_\alpha$ from the spin-weighted Slepian functions $\mathcal{G}_N^{\tilde{\alpha}_1(\alpha)}$ described in Equation (10) by

$$g_\alpha(\xi) = \begin{cases} \xi \mathcal{G}_N^{\tilde{\alpha}_1(\alpha)}(\xi) & \text{if } \tilde{\alpha}_2(\alpha) = 1, \\ \tau_+ \mathcal{G}_N^{\tilde{\alpha}_1(\alpha)}(\xi) & \text{if } \tilde{\alpha}_2(\alpha) = 2, \\ \tau_- \mathcal{G}_N^{1-\tilde{\alpha}_1(\alpha)}(\xi) & \text{if } \tilde{\alpha}_2(\alpha) = 3. \end{cases}$$

for $\alpha = 1, \ldots, 3(L + 1)^2 - 2$.

We obtain the same Shannon numbers as [31],

$$S_{\text{vector}} = \left[3(L + 1)^2 - 2\right] \frac{A}{4\pi}.$$ 

For the special case of a spherical cap with $b = \cos \theta \leq t \leq 1$, this yields

$$S_{\text{vector}} = \left[3(L + 1)^2 - 2\right] \frac{1 - b}{2}.$$ 

### 4.3 Tensor Slepian Functions

Tensor Slepian functions on the sphere have been investigated by Eshagh [8] with a choice of basis for which, to date, no commuting operator is known. Here, we follow the recipe used for the spin-weighted harmonic-based vector functions to construct tensor Slepian functions for which we derived a commuting operator for polar cap regions in Section 3.2. As for the vector case, the first step involves defining a basis of spin-weighted harmonic-based tensor functions.

**Definition 4.4.** The spin-weighted harmonic-based tensor functions with bandlimit $L$ are

$$y_{n,j}^1(\xi) = y_{n,j}^{(1,1)}(\xi) = (\xi \otimes Y_{n,j})(\xi),$$

$$y_{n,j}^2(\xi) = \frac{1}{\sqrt{2}} \left( -y_{n,j}^{(1,2)}(\xi) + iy_{n,j}^{(1,3)}(\xi) \right) = (\xi \otimes \tau_+) + 1 Y_{n,j}(\xi),$$

$$y_{n,j}^3(\xi) = -\frac{1}{\sqrt{2}} \left( -y_{n,j}^{(1,2)}(\xi) - iy_{n,j}^{(1,3)}(\xi) \right) = (\xi \otimes \tau_-) - 1 Y_{n,j}(\xi),$$

$$y_{n,j}^4(\xi) = \frac{1}{\sqrt{2}} \left( -y_{n,j}^{(2,1)}(\xi) + iy_{n,j}^{(3,1)}(\xi) \right) = (\tau_+ \otimes \xi) + 1 Y_{n,j}(\xi),$$

$$y_{n,j}^5(\xi) = -\frac{1}{\sqrt{2}} \left( -y_{n,j}^{(2,1)}(\xi) - iy_{n,j}^{(3,1)}(\xi) \right) = (\tau_- \otimes \xi) - 1 Y_{n,j}(\xi),$$

$$y_{n,j}^6(\xi) = y_{n,j}^{(2,2)}(\xi) = \frac{1}{\sqrt{2}} h_{\text{tan}} Y_{n,j}(\xi),$$

$$y_{n,j}^7(\xi) = y_{n,j}^{(3,3)}(\xi) = \frac{1}{\sqrt{2}} h_{\text{tan}} Y_{n,j}(\xi),$$

$$y_{n,j}^8(\xi) = -\frac{1}{\sqrt{2}} \left( -y_{n,j}^{(2,3)}(\xi) + iy_{n,j}^{(3,2)}(\xi) \right) = (\tau_+ \otimes \tau_+) + 2 Y_{n,j}(\xi),$$

$$y_{n,j}^9(\xi) = -\frac{1}{\sqrt{2}} \left( -y_{n,j}^{(2,3)}(\xi) - iy_{n,j}^{(3,2)}(\xi) \right) = (\tau_- \otimes \tau_-) - 2 Y_{n,j}(\xi)$$

for $\xi \in \Omega$ and $n = 0, \ldots, L$, where $-n \leq j \leq n$, with

$$0_i := \begin{cases} 0, & i = 1, 6, 7 \\ 1, & i = 2, 3, 4, 5 \\ 2, & i = 8, 9 \end{cases}$$

**Remark 4.5.** The functions in Definition 4.4 form an orthonormal basis of $\text{harm}_{0 \ldots L}(\Omega)$. Moreover, $y_{n,j}^i(\xi) : y_{m,l}^i(\xi) = 0$, if $i \neq k$ for all $\xi \in \Omega$ and all $n, j, m, l$ (pointwise orthogonality of different types) where $\cdot$ denotes the tensor inner product.
Remark 4.5 follows using the same arguments as for Remark 4.2.

We can therefore represent any tensor function \( g \in \text{harm}_{0...L}(\Omega) \) as a linear combination of the spin-weighted harmonic-based tensor functions

\[
g(\xi) = \sum_{i=1}^{9} \sum_{n=0}^{L} \sum_{j=-n}^{n} g_{n,j}^i y_{n,j}^i(\xi)
\]

for all \( \xi \in \Omega \) with coefficients

\[
g_{n,j}^i = \int_{\Omega} g(\xi) : y_{n,j}^i(\xi) \, d\omega(\xi)
\]

for all \((i, n, j) \in J_L\), where we define the set of indices by

\[
J_L := \{(i, n, j) \mid i = 1, \ldots, 9; n = 0, \ldots, L; j = -n, \ldots, n\}
\]

We can formulate the concentration problem independently of the basis

\[
\lambda = \frac{\int_{\Omega} g(\xi) : g(\xi) \, d\omega(\xi)}{\int_{\Omega} g(\xi) : g(\xi) \, d\omega(\xi)} = \max.
\]  

Similarly to the vector case, choosing the spin-weighted harmonic-based tensor basis from Definition 4.4 leads to a decoupling of the eigenvalue problem as a result of the pointwise orthogonality of the different types of spin-weighted harmonic-based tensor functions.

**Problem 4.6.** Concentration problem (20) expressed in the spin-weighted harmonic-based tensor functions from Definition 4.4 yields the eigenvalue problem

\[
k g = \lambda g,
\]

where

\[
k := \begin{pmatrix}
k^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & k^4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k^5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k^6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k^7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k^8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k^9
\end{pmatrix}
\]

\[
i = \begin{pmatrix}
K^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & K^+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K^+1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K^0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K^+2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & K^{-2}
\end{pmatrix}
\in \mathbb{R}^{[9(L+1)^2-12] \times [9(L+1)^2-12]}
\]

with the matrices \( k^i \) given by their components

\[
k_{n,j,n',j'}^i := \int_{\mathbb{R}} y_{n,j}^i(\xi) : y_{n',j'}^i(\xi) \, d\omega(\xi).
\]

The tensor problem reduces to nine spin-weighted problems corresponding to spin weights 0, +1, −1, +2, and −2, which we solved in Section 3. To find the eigenvectors \( g_\alpha \) of Problem 4.6 we simply pad
We obtain the Shannon number

\[ S_{\text{tensor}} = \left[ 9(L + 1)^2 - 12 \right] \frac{A}{4\pi} \]

for general domains, and

\[ S_{\text{tensor}} = \left[ 9(L + 1)^2 - 12 \right] \frac{1 - b}{2}, \]

for the spherical cap with \( b = \cos \theta \leq t \leq 1 \).

5 Conclusions

Scalar and vector Slepian functions on the sphere have proven to be a useful tool in a variety of studies. The construction of Slepian functions can be rendered numerically stable and computationally efficient through an aptly designed commuting operator. In this article, we contributed to the understanding of Slepian functions in two ways: (i) we presented a unified approach for constructing Slepian functions for arbitrary rank tensors and (ii) we designed commuting operators for polar caps for arbitrary rank tensor Slepian functions. For the tensor Slepian functions, no such commuting operator had been known.

5.1 Summary of the construction

Designing the spin-weighted Slepian functions required us to solve the Slepian concentration problem for the spin-weighted spherical harmonics of general spin weight \( N \). We reformulated the concentration problem as a spin-weighted eigenvalue problem \( K^N G^N = \lambda G^N \). Furthermore, we derived a spin-weighted Shannon number allowing for the estimation of the number of eigenvalues close to 1 and hence the number of Slepian functions that are well concentrated within the region of interest.

For the spin-weighted kernel function \( K^N \) for polar cap regions we derived a commuting operator \( F^N \) leading to a tridiagonal matrix \( I^N \) which commutes with the kernel matrix \( K^N \) and which has simple eigenvalues. As a result, the eigenvectors of \( I^N \) (which are numerically stable and computationally inexpensive to compute) are equal to the eigenvectors of \( K^N \) (which are the coefficients for the spin-weighted spherical-harmonic-based Slepian functions).

We used the linear relationships between the spin-weighted spherical harmonics and the scalar spherical harmonics, the vector spherical harmonics by Hill [17], and the tensor spherical harmonics by Freeden, Gervens, and Schreiner [11] to design the corresponding Slepian functions on the sphere. For the tensor Slepian functions, this work presents the first construction of a commuting operator.
5.2 Outlook

We presented a framework for the construction of tensor Slepian functions for the spherical cap as well as for arbitrary measurable domains but not implemented it. Development of software to construct tensor Slepian functions for arbitrary domains hence remains an open problem. If such a software were to be designed to solve the concentration problem for general spin-weighted spherical-harmonics, then this would allow the construction of tensor Slepian functions for arbitrary ranks and arbitrary regions. Moreover, the construction of a commuting operator for the polar double cap and belt for the vector and tensor Slepian functions is unknown at present. While the Shannon number does provide an estimation for the number of well-concentrated Slepian functions, it typically overestimates that number. Hence a better constraint on the number of well-concentrated Slepian functions would provide a valuable contribution.

As another potential avenue for future research, tensor Slepian functions could form part of a dictionary-based method for tensor-valued inverse problems in analogy to [9, 27, 40].

To date, tensor Slepian functions have not been used to invert for potential field models from second-derivative data such as, for example, for gravity potential from the satellite mission GOCE, or to invert for cosmic microwave background polarization. As is shown in [32], when inverting for potential field models on the planet’s surface from satellite data, the spatially concentrated spectrally limited Slepian functions are not well suited, as they are typically poorly conditioned under downward continuation (as a result of each function including a wide range of spherical-harmonic degrees). The approach of solving a related, continuation-cognizant problem as in [32] could be translated to the tensor Slepian case.

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