Black Holes, Space-Filling Chains and Random Walks

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Abstract

Many approaches to a semiclassical description of gravity lead to an integer black hole entropy. In four dimensions this implies that the Schwarzschild radius obeys a formula which describes the distance covered by a Brownian random walk. For the higher-dimensional Schwarzschild-Tangherlini black hole, its radius relates similarly to a fractional Brownian walk. We propose a possible microscopic explanation for these random walk structures based on microscopic chains which fill the interior of the black hole.

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1 Introduction

The full non-perturbative formulation of quantum gravity continues to pose a formidable challenge. Though loop quantum gravity had some success in dealing with 4d non-supersymmetric black holes, we believe, following string-theory, that to incorporate the electroweak and strong gauge interactions next to gravity we have to leave four dimensions and start with ten or eleven dimensions in the first place. The appearance of four non-compact dimensions must then be explained by some dynamical partial decompactification process enlarging four of the ten resp. eleven dimensions during the cosmic evolution of the very early universe. The success of this idea hinges very much on our ability to better understand M-theory at a non-perturbative level. Progress in this direction has been made by the proposal of M-theory as a matrix-theory [1].

Also on the frontier of deriving predictions from M/string-theory in order to address phenomenology and test the theory, decisive developments took place over the last few years. Various moduli have been stabilized by means of fluxes, non-perturbative effects or a combination of both (see e.g. [2]-[11]). To really see whether M/string-theory is a phenomenologically viable theory, it seems most promising to consider heterotic M-theory [12],[13] and compactify it down to four dimensions [14]-[18]. Though the effective supergravity description allows a phenomenologically realistic stabilization of most moduli, one realizes that ultimately the structure of the complete theory requires the knowledge of the full non-perturbative quantum M-theory. This problem was already pointed out in the original work [12],[13].

Given therefore the urgent fundamental and phenomenological need to understand the microscopic formulation of M/string-theory, one of the best guidance principles could be the Bekenstein-Hawking (BH) entropy [19]-[25], which every candidate quantum gravity theory should successfully explain microscopically. In this respect string-theory was remarkably successful in explaining the entropy of certain supersymmetric extremal and near-extremal black holes correctly - without having to adjust an arbitrary proportionality constant appropriately [26]-[28]. What is however still problematic in conventional string-theory is the counting of the microscopic degrees of freedom directly in the non-perturbative regime where the weakly coupled string technology is no longer applicable. This, however, is required when the black hole is non-supersymmetric and for instance of Schwarzschild or Kerr type.

By starting from a 10 or 11-dimensional spacetime whose event horizon and com-
plete internal compactification space are wrapped by two Euclidean electric-magnetic dual brane pairs \((E_1, M_1) + (E_2, M_2)\), progress towards explaining the BH-entropy directly in the non-perturbative regime was made recently in \([29-32]\). Namely by interpreting the inverse tension of a Euclidean brane as a smallest volume unit on the brane which is a direct consequence of the ‘worldvolume uncertainty principle’ \([49]\), the wrapped Euclidean branes can be thought of as being composed out of a finite number of smallest cells (units of smallest worldvolume). On this lattice-like structure chain excitations were proposed \([29,32]\) whose microcanonical ensemble entropy correctly reproduces both the BH-entropy and its known leading logarithmic correction. For some other earlier interesting ideas on how to understand the entropy of non-supersymmetric black holes from string-theory see \([33-41]\).

2 Black Holes and Brownian Random Walks

One of the results of the proposal made in \([29,32]\) has been that the BH-entropy equals an integer

\[
S_{BH} = N \in \mathbb{N} ,
\]

which represents the total number of cells on the dual brane pair lattice. This relation arose not only in four \([29]\) but also in higher dimensions \([32]\). Moreover, it appeared in many other approaches to black hole quantization, as well \([42-44]\). Let us study its consequences for the \(d\)-dimensional hyperspherically symmetric Schwarzschild-Tangherlini black hole with topology \(\mathbb{R} \times S^{d-2}\). These black holes are described by a metric \([45]\)

\[
ds^2 = -\left(1 - \left(\frac{r_S}{r}\right)^{d-3}\right) dt^2 + \left(1 - \left(\frac{r_S}{r}\right)^{d-3}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2
\]

with \(r\) the radial coordinate and \(d\Omega_{d-2}^2\) the line-element on the unit \(d-2\) sphere. We will consider values \(d \geq 4\) (in \(d = 3\) a vanishing Ricci-tensor implies a flat spacetime and conventional black holes are absent).

Given the mass \(M_d\) of the black hole, its Schwarzschild radius \(r_S\) is defined as

\[
M_d = \frac{(d-2)A_{d-2}}{16\pi G_d} r_S^{d-3} , \quad A_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}
\]

with \(A_{d-2}\) the ‘area’ of the unit \(d-2\) sphere. The BH-entropy for these black holes is
given by the standard expression

\[ S_{BH} = \frac{\text{vol}(\mathcal{H}_{d-2})}{4G_d} \]  

(4)

where \( G_d \) is the \( d \)-dimensional Newton’s Constant and \( \mathcal{H}_{d-2} \) the black hole surface of topology \( S^{d-2} \) with ‘area’

\[ \text{vol}(\mathcal{H}_{d-2}) = r_S^{d-2}A_{d-2} \cdot \]  

(5)

A direct consequence of the integer-valued BH-entropy \( (1) \) in conjunction with \( (4) \) and \( (5) \) is that the Schwarzschild radius \( r_S \) becomes a discrete function of the integer \( N \)

\[ r_S(N) = l_d \left( \frac{4}{A_{d-2}} \right)^{\frac{1}{d-2}} N^{\frac{1}{d-2}} , \]  

(6)

where the \( d \)-dimensional Planck-length \( l_d \) is given by

\[ l_d^{d-2} = G_d . \]  

(7)

In four dimensions, for \( d = 4 \), this relation becomes

\[ r_S(N) = \frac{l_4}{\sqrt{\pi}} \sqrt{N} , \]  

(8)

which is precisely the relation for the root-mean-square (rms) distance covered by a Brownian random walk of average step-size \( l_4/\sqrt{\pi} \). In general, a fractional Brownian random walk of step-size \( l_{\text{walk}} \) and dimension \( d_{\text{walk}} \) is characterized by the relation (see e.g. [46])

\[ r_{\text{walk}} = l_{\text{walk}} N^{\frac{1}{d_{\text{walk}}}} \]  

(9)

expressing the rms-distance covered, \( r_{\text{walk}} \), as a function of the number of steps \( N \). Hence, we recognize that as a consequence of the integer-valued BH-entropy the Schwarzschild radius \( r_S \) of a \( d \)-dimensional black hole is given by a fractional Brownian random walk relation with step-size

\[ l_{\text{walk}} = l_d \left( \frac{4}{A_{d-2}} \right)^{\frac{1}{d-2}} \]  

(10)

and integer dimension

\[ d_{w} = d - 2 \geq 2 . \]  

(11)
Though a fractional Brownian random walk is mathematically also defined for fractional dimensions \( d_w \notin \mathbb{N} \) not being integers, this possibility is not realized by black holes where all \( d_w \) happen to be integers.

As an aside, let us remark that a fractional Brownian walk can also be regarded as a diffusion process with ‘time’ \( \tau = N \Delta \tau \) and ‘time-increment’ \( \Delta \tau \) (during \( \Delta \tau \) the diffusion spreads out over the step-length \( l_{walk} \)). The diffusion coefficient \( D \) is defined via the diffusion equation

\[
\tau_w^2 = 2D \tau.
\]

Hence, with (12) the diffusion coefficient becomes

\[
D = \left( \frac{l_{walk}}{\sqrt{2} \Delta \tau^{1/d_{walk}}} \right)^2 \frac{1}{\tau(d_{walk}-2)/d_{walk}} = \frac{C(d_{walk})}{\tau(d_{walk}-2)/d_{walk}},
\]

with some constant \( C(d_{walk}) \) depending on \( d_{walk} \). One learns that the higher-dimensional \( d > 4 \) case with \( d_{walk} > 2 \) corresponds to an anomalous diffusion coefficient, i.e. one with non-trivial ‘time’ \( \tau \) dependence, while the 4-dimensional \( d = 4 \) case with \( d_{walk} = 2 \) is distinguished by a truly constant \( \tau \) independent diffusion coefficient. Thus for the higher-dimensional case \( D \) decreases with ‘time’ \( \tau \) and it becomes increasingly difficult for the diffusion process to spread out as ‘time’ \( \tau \) progresses. Hence, in this case the diffusion process is called ‘subdiffusion’ [47]. Such processes are known to occur as well in disordered or poorly connected media in which the subdiffusion process can be studied e.g. by the ‘ant in a labyrinth’ model of de Gennes describing a particle constrained to diffuse through a percolation cluster [48]. It would be interesting to understand better this distinction of four dimensions which we will, however, not attempt in this work.

An interesting and important property of a fractional Brownian random walk of dimension \( d_{walk} \) is its capability to fill the volume of a finite space of dimension \( d_{walk} \) completely in the large \( N \) limit [46], [47]. This generalizes the well-known result of a Brownian random walk which covers a 2-dimensional space in the large \( N \) limit. The pressing question which arises is whether and how these random walk behaviors occurring for black holes can be understood directly from the microscopic physics leading to the integer-valuedness of the BH-entropy [1] in the first place. We will attempt in the next section a simple explanation within the chain-approach developed in [29]-[32]. This approach has its origin in non-perturbative M/string-theory and therefore requires us to start in ten resp. eleven dimensions. But before going into details here, let us mention yet another consequence of (11). Namely, by using the random walk results for the Schwarzschild radius and plugging
it into the mass formula (3), the $d$-dimensional black hole’s mass $M_d$ becomes a discrete function of $N$ as well

$$M_d(N) = \frac{(d-2)}{4\pi} \left( \frac{A_{d-2}}{4} \right)^{\frac{1}{(d-2)}} \times \frac{N^{\frac{d-3}{d-2}}}{l_d}. \quad (14)$$

This expression becomes more suggestive when written in terms of the Schwarzschild radius as

$$M_d(N) = \frac{(d-2)}{4\pi} \times \frac{N}{r_S(N)}, \quad (15)$$

in which form it resembles the $N$th Kaluza-Klein excitation of a field compactified on a circle of radius $4\pi r_S/(d-2)$. It might therefore be an interesting hint at the microscopic quantum gravitational Hamiltonian.

### 3 Space-Filling Chains

The approach of [29]-[32] to understand the BH-entropy in terms of microscopic chain-states was proposed to address the problem of black hole state counting directly in the non-perturbative regime where the black hole lives. Though firmly rooted in M/string-theory, this microscopic derivation of the BH-entropy uses a direct counting mechanism which does not rely on supersymmetry and is therefore also applicable to non-supersymmetric spacetimes such as Schwarzschild black holes or their higher-dimensional cousins, the Schwarzschild-Tangherlini black holes. Let us be a bit more specific about this approach.

Starting with 10-dimensional type II string-theory (alternatively one could start from 11-dimensional M-theory [29,32]), a $d$-dimensional spacetime $\mathcal{M}^{1,d-1}$ with $d-2$ dimensional horizon $\mathcal{H}^{d-2}$ (the intersection of the spacetime’s future event horizon with a partial Cauchy surface) is embedded into ten dimensions through the direct product $\mathcal{M}^{1,d-1} \times \mathcal{M}^{10-d}$, with $\mathcal{M}^{10-d}$ being compact. This describes a compactification from ten to $d$ dimensions. For the uncharged, non-dilatonic Schwarzschild-Tangherlini black holes one would consider a doublet of Euclidean $(D3, D3) + (\overline{D3}, \overline{D3})$ brane pairs [30,32], each wrapped around $\mathcal{H}^{d-2} \times \mathcal{M}^{10-d}$ where $\mathcal{H}^{d-2} \simeq S^{d-2}$. Here, the first entry in each brane pair is always orthogonal to the second. By applying the ‘brane worldvolume uncertainty principle’ [19] which splits the brane worldvolume into a discrete number of smallest cells, whose size is given by the inverse of the brane’s tension [29], one concludes that the joined worldvolume of the $(D3, D3) + (\overline{D3}, \overline{D3})$ doublet consists out of a finite number of $N$ cells.
The presence of the branes allows for a simple rewriting of the $d$-dimensional BH-entropy associated with the spacetime $M^{d-1}$, implying the entropy’s integer-valuedness \cite{1}.

With $N$ the number of cells on the $(D3, D3) + (\overline{D3}, \overline{D3})$ worldvolume, the natural excitations to look at (in the non-perturbative regime where the string coupling constant $g_s$ is of $O(1)$ and the finite cell sizes, being proportional to $g_s^2$, become relevant) are links starting and ending on any of these cells. Indeed chains composed out of $N$ such links (closed chain case) or $N - 1$ links (open chain case) possess a microcanonical ensemble entropy which correctly reproduces the $d$-dimensional BH-entropy and its logarithmic correction \cite{29,32}. It is these chains which exhibit already an inherent random structure which will be of interest to us now. Actually such discrete random-walk structures are also suggested by string-theory in the weakly coupled regime, namely at high temperatures near the Hagedorn temperature. Here, it was known for a long time (see e.g. \cite{50,51}) that the string in this regime is best described by a random walk. Moreover, composites of open strings which could generate chain structures in string theory appeared in \cite{52}.

Our aim will now be to show that the fractional Brownian random walk relation \cite{6} can arise from such a chain when it is assumed to fill the interior of the black hole, bounded by the horizon $\mathcal{H}^{d-2} \simeq S^{d-2}$, i.e. the $d-1$ dimensional hyperball $D^{d-1}$. To this end we will need to determine the volume covered by the chain.
3.1 D=10 Type II String-Theory Case

For the purpose of explaining the $d$-dimensional fractional random walk relation (6), it will turn out to be sufficient to assume that the size of the internal compactification manifold, $\mathcal{M}^{10-d}$, is much smaller than the Schwarzschild radius $r_S$ of the external $d$-dimensional black hole in $\mathcal{M}^{1,d-1}$. This means that on the internal $\mathcal{M}^{10-d}$ all links will end on just a very few cells while there is much more freedom for the links in $\mathcal{M}^{1,d-1}$ to end on many different cells on the much larger horizon sphere $S^{d-2}$. Hence, it will be enough to consider only the freedom of the links to vary over the external horizon sphere. As, by assumption, each link has the freedom to start and end on any of the cells covering $S^{d-2}$ with equal probability, most links will then stretch right through the interior of the hyperball $D^{d-1}$ which is bounded by the horizon sphere $S^{d-2}$ (see fig.1).

The average length of a single link $\langle L_{\text{link}} \rangle$ is determined by keeping one end of the link fixed and letting the other vary over all $N$ cells

$$\langle L_{\text{link}} \rangle = \frac{1}{N} \sum_{i=1}^{N} L_{\text{link},i} ,$$

where $L_{\text{link},i}$ is the value $L_{\text{link}}$ assumes when the link stretches between the fixed cell and the $i$th cell. In view of the assumed smallness of the internal compactification radius, the length $L_{\text{link},i}$ reduces to the length the link stretches inside $D^{d-1}$. The cell volume, being given by [32]

$$V_{\text{cell}} = 8l_d^{-2} \text{vol}(\mathcal{M}^{10-d})$$

implies that the cell size lies between the $d$-dimensional Planck-length and the compactification radius. Hence, it should be considerably smaller than $r_S$ as well, and we are entitled to approximate the average over the discrete cells by an integral average over the hypersphere $S^{d-2}$

$$\langle L_{\text{link}} \rangle = \frac{1}{A_{d-2}} \int_{S^{d-2}} L_{\text{link}} d\Omega_{d-2}$$

$$= \frac{1}{A_{d-2}} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta_1 d\theta_1 \int_{0}^{\pi} \sin^2 \theta_2 d\theta_2 \ldots \int_{0}^{\pi} \sin^{d-3} \theta_{d-3} d\theta_{d-3} L_{\text{link}} .$$

Here $(\theta_{d-3}, \ldots, \theta_2, \theta_1, \phi)$ are spherical coordinates parameterizing $S^{d-2}$. The result which will be derived in the appendix is

$$\langle L_{\text{link}} \rangle = 2r_S \frac{B(d-\frac{3}{2}, d-2)}{B(d-1, d-\frac{3}{2})} = \frac{2^{d-2} \Gamma^2(d-\frac{1}{2})}{\sqrt{\pi} \Gamma(d-\frac{3}{2})} r_S ,$$

where $B(a, b)$ is the beta function and $\Gamma(a)$ is the gamma function.
Table 1: The dependence of the average link length $\langle L_{\text{link}} \rangle$ in units of the Schwarzschild radius $r_S$ on the dimension $d$ of the external spacetime.

| $d$ | $\langle L_{\text{link}} \rangle / r_S$ |
|-----|-------------------------------------|
| 4   | 1.333                               |
| 5   | 1.358                               |
| 6   | 1.371                               |
| 7   | 1.380                               |
| 8   | 1.385                               |
| 9   | 1.389                               |
| 10  | 1.392                               |

where $B(x, y)$ denotes the Euler Beta-function. It leads to the numerical values displayed in table 1. The value for $\langle L_{\text{link}} \rangle$ increases slowly but monotonically with $d$. By using Stirling’s formula for the asymptotic behavior of the Gamma-function one can show that $\langle L_{\text{link}} \rangle$ converges for large $d$ towards

$$\langle L_{\text{link}} \rangle \to \sqrt{2} r_S , \quad d \gg 1 . \quad (21)$$

This behavior is shown in fig.2. The asymptotic value $\sqrt{2} r_S$ corresponds to a link which stretches e.g. from the north pole to the equator. In critical M/string-theory (for the M-theory case see next subsection 3.2), of course, only the values $d \leq 11$ will be relevant.

Since the size of the 8-dimensional cross-section of each link is always given by $V_{\text{cell}}$, it follows that the average volume occupied by a link is

$$V_{\text{link}} = \langle L_{\text{link}} \rangle V_{\text{cell}} . \quad (22)$$

The average volume of the chain, $V_{\text{chain}}$, is then given by multiplication with $N$, the number of links which the chain contains (as we will work in the large $N$ limit, there is no need to distinguish between ‘closed’ chains with $N$ links and ‘open’ chains with $N - 1$ links).

$$\langle L_{\text{link}} \rangle / (\sqrt{2} r_S)$$

Figure 2: The convergence of the average link length $\langle L_{\text{link}} \rangle$ towards $\sqrt{2} r_S$ for large values of the dimension $d$.  

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links, see [32] – both will occupy the same volume)

\[ V_{\text{chain}} = NV_{\text{link}} = N \langle L_{\text{link}} \rangle V_{\text{cell}}. \]  

The next step will consist in considering chains at large \( N \), the regime where one connects to black holes of macroscopic size. Because of the random nature of the chains – each link can start and end on any of the \( N \) cells with the same probability – such chains tend to cover at sufficiently large \( N \) the full finite space \( D^{d-1} \times \mathcal{M}^{10-d} \) through which they stretch. Notice that the filling of the \( D^{d-1} \times \mathcal{M}^{10-d} \) space by the chains does not occur due to a fractal behavior of the chain, as the links already possess the same dimensionality, nine, as the space \( D^{d-1} \times \mathcal{M}^{10-d} \). For such large \( N \) space-filling chains one is led to the following volume equality

\[ 1 = \frac{\text{Volume of Chain}}{\text{Volume of } D^{d-1} \times \mathcal{M}^{10-d}} = \frac{N \langle L_{\text{link}} \rangle V_{\text{cell}}}{A_{d-2} \left( \frac{r_{d-1}}{d-1} \right) \text{vol}(\mathcal{M}^{10-d})} \]  

By plugging in the expression (17) for \( V_{\text{cell}} \), the internal volume drops out of this equation. Employing furthermore the result (20) for \( \langle L_{\text{link}} \rangle \), we obtain

\[ \frac{\text{Volume of Chain}}{\text{Volume of } D^{d-1} \times \mathcal{M}^{10-d}} = \left( \frac{l_d}{r_S} \right)^{d-2} N f(d) \]  

where the function

\[ f(d) = (d-1) \frac{2^d \Gamma^3 \left( \frac{d-1}{2} \right)}{\pi^{d/2} \Gamma \left( d - \frac{3}{2} \right)} \]  

depends only on the dimension \( d \). With this relation the equality (24) leads to the following relation for the radius \( r_S(N) \) of the ball \( D^{d-1} \)

\[ r_S(N) = l_d f(d) \left( \frac{1}{\pi^{1/2}} N^{1/(d-2)} \right). \]  

Since the hypersurface \( S^{d-2} \) has been identified with the horizon boundary \( \mathcal{H}^{d-2} \) of the Schwarzschild-Tangherlini black hole, we see that the chains at large \( N \) which become \( D^{d-1} \times \mathcal{M}^{10-d} \) space-filling, give indeed the correct fractional Brownian random walk relation \( r_S(N) \propto l_d N^{1/(d-2)} \) in each dimension \( d \). Though the step-sizes of the walks (6) and (27) do not agree exactly, their difference is not large and even vanishes when \( d \) becomes large. To see this quantitatively, let us look at the ratio of both step-sizes given by

\[ g(d) = \left( \frac{f(d)}{4/A_{d-2}} \right)^{\frac{1}{\pi^{1/2}}} \]  

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Figure 3: The convergence of the ratio of the step-sizes, $g(d)$, towards 1 for large values of the dimension $d$.

It approaches unity for large values of $d$ (see fig 3):

$$g(d) \simeq \sqrt{2^{1-d}} \to 1, \quad d \gg 1 \quad (29)$$

as one can see by using Stirling’s approximation for the Gamma-function at large argument. For the physical values of $d$ ranging from 4 to 10 in type II string-theory we collect the numerical values of $g(d)$ in table 2.

Let us comment on the possible origin and resolution of this small discrepancy between the step-sizes at finite $d$. One observes that $g(d)$ is always larger than one. This means that the chain volume as given by (23) is slightly bigger than required in order to give the correct step-size (10). Notice that it is only the $d$-dependent numerical factor contained in (23) which is slightly too large – the dependence on all the other parameters $N, r_S, l_d, \text{vol}(M^{10-d})$ is the correct one. The fact that both step-sizes become equal in the large $d$ limit already hints at a possible resolution. For this, let us note that when a chain crosses in a random way a finite space so often that eventually it fills this space completely, self-intersections of the chain become inevitable. Notice further that the chains are not infinitely thin but have a non-zero 8-dimensional transverse cross-section given by $V_{cell}$. The chains can therefore intersect also in higher dimensions $d > 4$ in contrast

| $d$ | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $g(d)$ | 2.83 | 2.22 | 1.92 | 1.75 | 1.64 | 1.56 | 1.50 |

Table 2: The ratio of the step-sizes, $g(d)$, as a function of the dimension $d$. 
to infinitely thin 1-dimensional mathematical chains which cannot intersect in dimensions bigger than two (or if behaving as Brownian walks with fractional dimension 2 such mathematical chains wouldn’t intersect in dimensions bigger than four). Moreover, when self-intersections occur, one would expect that the probability for them would decrease the larger the dimension \( d \) becomes. This simply because for higher \( d \) there is more ‘room’ for two randomly placed links to avoid each other. Therefore, by including also the effect of chain self-intersections, one would expect to reduce the \( d \)-dependent factor in the chain volume while maintaining the large \( d \) equality (29). This would then hopefully reconcile both step-sizes with each other. We will, however, not pursue this effect further here.

### 3.2 D=11 M-Theory Case

Let us finally address the 11-dimensional M-theory case which proceeds in an analogous manner. Here, the neutral Schwarzschild-Tangherlini black holes can only be associated with the charge-neutral combination of Euclidean \((M2, M5) + (M2, M5)\) brane pairs wrapping \(H^{d-2} \times \mathcal{M}^{11-d}\), with \(\mathcal{M}^{11-d}\) the compact internal space. The different, now \(11 - d\) dimensional, internal compactification space will result in a different cell volume

\[
V_{cell} = 8 l_d^{d-2} \text{vol}(\mathcal{M}^{11-d}) .
\]  

(30)

Apart from this, the analysis in the D=11 case will be exactly the same as in D=10. In particular, under the same assumption that \( r_S \) is much larger than the size of the compact \(\mathcal{M}^{11-d}\), the average (20) of the link-length over the hypersphere \(S^{d-2} \simeq H^{d-2}\) will be unaffected.

The chains will now be \(D^{d-1} \times \mathcal{M}^{11-d}\) space filling at large \(N\) which leads to the equality

\[
1 = \frac{\text{Volume of Chain}}{\text{Volume of } D^{d-1} \times \mathcal{M}^{11-d}} = \frac{N \langle L_{\text{link}} \rangle V_{cell}}{A_{d-2} (\frac{r_s^{d-1}}{d-1}) \text{vol}(\mathcal{M}^{11-d})} .
\]  

(31)

Since the different internal volume \(\text{vol}(\mathcal{M}^{11-d})\) drops out by virtue of (30), this ratio coincides exactly with the one for the D=10 case, given in (25). Consequently one obtains also for the D=11 M-theory case the same fractional Brownian walk relation (27) as in the D=10 analysis and thus a potential microscopic explanation of the semiclassical random walk relation (6).
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A The Average Link Length $\langle L_{\text{link}} \rangle$

We will provide here the calculation of the average link-length $\langle L_{\text{link}} \rangle$, deriving the result \(^{(20)}\). The definition of $\langle L_{\text{link}} \rangle$ was given in \(^{(19)}\). Let us first introduce standard spherical coordinates

\[
x_{d-1} = r_s \cos \theta_{d-3}
\]
\[
x_{d-2} = r_s \sin \theta_{d-3} \cos \theta_{d-4}
\]
\[
\vdots
\]
\[
x_2 = r_s \sin \theta_{d-3} \ldots \sin \theta_1 \cos \phi
\]
\[
x_1 = r_s \sin \theta_{d-3} \ldots \sin \theta_1 \sin \phi
\]

which parameterize the hypersphere $S^{d-2}$ defined by the relation

\[
x_1^2 + \ldots x_{d-1}^2 = r_S^2 .
\]

With the Cartesian coordinate origin at the center of the sphere, the north pole $NP$ (see fig.4) is described by

\[
\vec{r}_{NP} = (x_1 = 0, \ldots, x_{d-2} = 0, x_{d-1} = r_S).
\]

Without loss of generality, we can fix for the determination of the average link length the starting point of all links at the north pole while the end-point of the links, described by the vector $\vec{L}_{\text{link}}$ (see fig.1) varies over $S^{d-2}$. In terms of the difference vector (see fig.1)

\[
\Delta \vec{L}_{\text{link}} = (x_1, \ldots, x_{d-1})
\]

$\vec{L}_{\text{link}}$ becomes

\[
\vec{L}_{\text{link}} = \Delta \vec{L}_{\text{link}} - \vec{r}_{NP} = (x_1, \ldots, x_{d-2}, x_{d-1} - r_S),
\]

where the Cartesian coordinates obey \(^{(33)}\). $L_{\text{link}}$, the norm of the vector $\vec{L}_{\text{link}}$, can then be expressed in spherical coordinates as

\[
L_{\text{link}} = 2r_S \sin \left(\frac{\theta_{d-3}}{2}\right).
\]
Figure 4: In the regime where the cell size is much smaller than the radius \( r_S \) of the hypersphere \( S^{d-2} \), the links become approximately thin lines and their average length over the hypersphere can be determined by integration. The picture defines the vectors used in the calculation.

The integration \((38)\) can now be performed trivially over \( \phi, \theta_1, \ldots, \theta_{d-4} \) and contributes the ‘area’ \( A_{d-3} \) of the unit \( d-3 \) sphere

\[
\langle L_{\text{link}} \rangle = 2r_S \frac{A_{d-3}}{A_{d-2}} I_{d-3}. \tag{38}
\]

where

\[
I_{d-3} = \int_0^\pi d\theta_{d-3} \sin^{d-3} \theta_{d-3} \sin \left( \frac{\theta_{d-3}}{2} \right). \tag{39}
\]

It remains to determine this integral which can be done by switching to the integration variable \( x = \theta_{d-3}/2 \)

\[
I_{d-3} = 2^{d-2} \int_0^{\pi/2} dx \sin^{d-2} x \cos^{d-3} x. \tag{40}
\]

This integral is equal to (see e.g. \[53\])

\[
I_{d-3} = 2^{d-3} \frac{\Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-2}{2} \right)}{\Gamma \left( d - \frac{3}{2} \right)} \tag{41}
\]

which can be brought into the simpler form

\[
I_{d-3} = \sqrt{\pi} \frac{\Gamma (d-2)}{\Gamma (d - \frac{3}{2})} \tag{42}
\]

by using Legendre’s duplication formula for the product of two Gamma-functions.
Writing the ‘areas’ $A_{d-2}$ and $A_{d-3}$ explicitly out using (3) one arrives at

$$\langle L_{\text{link}} \rangle = 2r_S \frac{\Gamma\left(\frac{d-1}{2}\right) \Gamma(d-2)}{\Gamma\left(\frac{d-1}{2}-1\right) \Gamma\left(d-\frac{3}{2}\right)} = 2r_S \frac{B\left(\frac{d-1}{2}, d-2\right)}{B\left(\frac{d-1}{2}-1, d-\frac{3}{2}\right)}$$

(43)

where the Beta-function allows a quite compact expression. However, more useful for us will be an expression which is obtained by employing once more Legendre’s duplication formula in the form

$$\Gamma\left(\frac{d}{2}-1\right) = \frac{\sqrt{\pi} \Gamma(d-2)}{2^{d-3} \Gamma\left(\frac{d-1}{2}\right)}$$

(44)

to finally obtain the result (20) quoted in the main text

$$\langle L_{\text{link}} \rangle = r_S \frac{2^{d-2} \Gamma^2\left(\frac{d-1}{2}\right)}{\sqrt{\pi} \Gamma\left(d-\frac{3}{2}\right)}.$$ 

(45)

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