QUALITATIVE ANALYSIS OF A FOURTH ORDER DIFFERENCE EQUATION

H. S. Alayachi¹,†, M. S. M. Noorani¹ and E. M. Elsayed²,³

Abstract In this paper, we will investigate some qualitative behavior of solutions of the following fourth order difference equation

\[ x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, \ldots \]

where the initial conditions \( x_{-3}, x_{-2}, x_{-1}, \) and \( x_0 \) are arbitrary real numbers and the values \( a, b, c, \) and \( d \) are defined as positive real numbers.

Keywords Stability, periodicity, global attractor, boundedness, difference equations.

MSC(2010) 39A10.

1. Introduction

Our main objective in this paper is to obtain the qualitative behavior of the solutions of the following recursive equation:

\[ x_{n+1} = ax_n + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, \ldots \]

where the initial conditions \( x_{-3}, x_{-2}, x_{-1}, \) and \( x_0 \) are arbitrary nonzero real numbers and \( a, b, c, \) and \( d \) are positive constants.

In recent years, the theory of difference equations has been studied by a large number of researchers due to the importance of this field in modeling a large number of real-life problems. Difference equations are used in modeling some natural phenomena that appear in biology, physics, economy, engineering, etc. Difference equations become apparent in the study of discretization methods for differential equations. Some results in the theory of difference equations have been obtained in the corresponding results of differential equations as more or less natural discrete analogues. Some recent studies of the dynamics of difference equations are given as follows. Agarwal and Elsayed [3] studied the periodicity character and global stability and provided a solution form for several special cases of the recursive sequence

\[ x_{n+1} = ax_n + \frac{bx_n x_3}{cx_{n-2} + dx_{n-3}}. \]

† the corresponding author. Email address: HSSHAREEF@taibahu.edu.sa (H. S. Alayachi), msn@ukm.edu.my (M. S. M. Noorani), emmelsayed@yahoo.com (E. M. Elsayed)

¹ School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi, Selangor, Malaysia
² Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
³ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
Cinar [7] investigated the solution of the difference equation
\[ x_{n+1} = \frac{ax_{n-1}}{1 + bx_nx_{n-1}}. \]

Ibrahim [25] presented some relevant results of the difference equation
\[ x_{n+1} = \frac{x_nx_{n-2}}{x_{n-1}(a + bx_nx_{n-2})}. \]

Elsayed [16] analyzed the global stability and examined the periodic solution of the following difference equation:
\[ x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-1}}. \]

Elabbasy et al. [9] investigated the global stability and periodicity character and gave the solution of the special case of the difference equation
\[ x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_n}. \]

Additionally, Yağlınkaya [39] addressed the difference equation
\[ x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}. \]

Yang et al. [40] examined the global and local stability of the equilibrium points of the following recursive equation:
\[ x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1}x_{n-2}}. \]

Other results of the qualitative behavior of difference equations can be obtained in refs. [1]- [42].

2. Some Basic Properties and Definitions

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let \( I \) be some interval of real numbers, and the function \( f \) have continuous partial derivatives on \( I^{k+1} \), where \( I^{k+1} = I \times I \times \cdots \times I \) (\( k + 1 \) times). Then, for initial conditions \( x_{-k}, x_{-k+1}, ..., x_0 \in I \), the difference equation
\[ x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, \ldots \quad (2.1) \]
has a unique solution \( \{x_n\}_{n=-k}^{\infty} \).

A point \( \overline{x} \in I \) is called an equilibrium point of Eq.(2.1) if
\[ \overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}). \]
That is, \( x_n = \overline{x} \) for \( n \geq 0 \) is a solution of Eq.(2.1), or equivalently, \( \overline{x} \) is a fixed point of \( f \).
Definition 2.1 (Stability).

(i) The equilibrium point $\bar{x}$ of Eq. (2.1) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_k - \bar{x}| + |x_{k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point $\bar{x}$ of Eq. (2.1) is locally asymptotically stable if $\bar{x}$ is a locally stable solution of Eq. (2.1) and there exists $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_k - \bar{x}| + |x_{k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Eq. (2.1) is a global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Eq. (2.1) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (2.1).

(v) The equilibrium point $\bar{x}$ of Eq. (2.1) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Eq. (2.1) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \quad (2.2)$$

Now, assume that the characteristic equation associated with Eq. (2.2) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \ldots + p_k = 0, \quad (2.3)$$

where $p_i = \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}}$.

Theorem A ( [30]). Assume that $p_i \in R$, $i = 1, 2, \ldots$ and $k \in \{0, 1, 2, \ldots\}$. Then,

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + \ldots + p_k y_n = 0, \quad n = 0, 1, \ldots$$

Next, we introduce a fundamental theorem to prove the global attractor of the fixed points.

Theorem B ( [30]). Let $g : [a, b]^{k+1} \to [a, b]$ be a continuous function, where $k$ is a positive integer and $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \quad (2.4)$$
Suppose that $g$ satisfies the following conditions.

1. For each integer $i$ with $1 \leq i \leq k+1$, the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in $z_i$ for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

2. If $m, M$ is a solution of the system
   
   \[ \begin{align*}
   m &= g(m_1, m_2, ..., m_{k+1}), \\
   M &= g(M_1, M_2, ..., M_{k+1}),
   \end{align*} \]

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

\[ \begin{align*}
   m_i &= \begin{cases} m, & \text{if } g \text{ is non-decreasing in } z_i, \\
   M, & \text{if } g \text{ is non-increasing in } z_i, \end{cases} \\
   M_i &= \begin{cases} M, & \text{if } g \text{ is non-decreasing in } z_i, \\
   m, & \text{if } g \text{ is non-increasing in } z_i. \end{cases}
   \end{align*} \]

Then, there exists exactly one equilibrium point $\bar{x}$ of Eq. (2.4), and every solution of Eq. (2.4) converges to $\bar{x}$.

3. Local Stability of the Equilibrium Point of Eq.(1.1)

This section studies the local stability character of the equilibrium point of Eq.(1.1). Eq.(1.1) has an equilibrium point given by

\[ \bar{x} = a\bar{x} + \frac{b\bar{x}}{c\bar{x} - d\bar{x}}. \]

If $(c - d)(1 - a) > 0$, then the only positive equilibrium point of Eq.(1.1) is given by

\[ \bar{x} = \frac{b}{(c - d)(1 - a)}. \]

Let $f : (0, \infty)^2 \rightarrow (0, \infty)$ be a continuous function defined by

\[ f(u, v) = au + \frac{du}{cu - dv}. \]  

(3.1)

Therefore, it follows that

\[ \begin{align*}
   \frac{\partial f(u, v)}{\partial u} &= a - \frac{bdv}{(cu - dv)^2}, \\
   \frac{\partial f(u, v)}{\partial v} &= \frac{bdu}{(cu - dv)^2}.
   \end{align*} \]

Then, we see that

\[ \begin{align*}
   \frac{\partial f(\bar{x}, \bar{x})}{\partial u} &= a - \frac{d(1 - a)}{(c - d)} = p_0, \\
   \frac{\partial f(\bar{x}, \bar{x})}{\partial v} &= \frac{d(1 - a)}{(c - d)} = p_1.
   \end{align*} \]

Then, the linearized equation of Eq.(1.1) about $\bar{x}$ is

\[ y_{n+1} - p_0y_{n-1} - p_1y_{n-3} = 0. \]
Theorem 3.1. Assume that
\[ |ac - d| + d|1 - a| < |c - d|. \]
Then, the positive fixed point of Eq.(1.1) is locally asymptotically stable.

Proof. It follows by Theorem A that Eq.(1.1) is asymptotically stable if
\[ |p_1| + |p_0| < 1. \]
That is,
\[ \left| a - \frac{d(1 - a)}{c - d} \right| + \left| \frac{d(1 - a)}{(c - d)} \right| < 1, \]
then,
\[ |a(c - d) - d(1 - a)| + |d(1 - a)| < |c - d|. \]
Thus
\[ |ac - d| + d|1 - a| < |c - d|. \]
According to Theorem A, the fixed point of Eq.(1.1) is asymptotically stable. Hence, the proof is complete.

4. Global Attractivity of the Equilibrium Point of Eq.(1.1)

In this section, we investigate the global attractivity character of the solutions of Eq.(1.1).

Theorem 4.1. The fixed point \( \bar{x} \) of Eq.(1.1) is a global attractor if \( ac > d \).

Proof. Let \( \alpha \) and \( \beta \) be real numbers and assume that \( g : [\alpha, \beta]^2 \rightarrow [\alpha, \beta] \) is a function defined by Eq.(3.1). Then,
\[
\frac{\partial g(u, v)}{\partial u} = a - \frac{bdv}{(cu - dv)^2},
\]
\[
\frac{\partial g(u, v)}{\partial v} = \frac{bdu}{(cu - dv)^2}.
\]

Case (1) If \( a - \frac{bdv}{(cu - dv)^2} > 0 \), then we can easily see that the function \( g(u, v) \) is increasing in \( u, v \). Suppose that \( (m, M) \) is a solution of the system
\[ m = g(m, m) \text{ and } M = g(M, M). \]
Then from, Eq.(1.1), we see that
\[ m = am + \frac{bm}{cm - dm}, \]
\[ M = aM + \frac{bM}{cM - dM}. \]
This result gives
\[ (M - m) = a(M - m), \ a \neq 1. \]
Thus,

\[ M = m. \]

It follows by Theorem B that \( \pi \) is a global attractor of Eq.(1.1). Therefore, the proof is complete.

**Case (2)** If \( a - \frac{bdv}{cu - dv} < 0 \), let \( \alpha \) and \( \beta \) be a real numbers and assume that \( g : [\alpha, \beta]^2 \to [\alpha, \beta] \) is a function defined by

\[
g(u, v) = au + \frac{bu}{cu - dv}.
\]

Then, we can easily see that the function \( g(u, v) \) is decreasing in \( u \) and increasing in \( v \). Suppose that \((m, M)\) is a solution of the system

\[
M = g(m, M) \text{ and } m = g(M, m).
\]

Then, from Eq.(1), we see that

\[
m = aM + \frac{bM}{cM - dm},
\]

\[
M = am + \frac{bm}{cm - dM},
\]

\[
cMm - acM^2 - dm^2 + adMm = bM,
\]

\[
cMm - acn^2 - dM^2 + adMm = bm,
\]

then

\[
(M^2 - m^2)(d - ac) = b(M - m), \quad ac > d.
\]

Thus,

\[ M = m. \]

It follows by Theorem B that \( \pi \) is a global attractor of Eq.(1.1). Hence, the proof is complete.

\[ \square \]

5. Existence of Periodic Solutions

In this section, we study the existence of periodic solutions of Eq.(1.1). The following theorem states the necessary and sufficient condition that this equation does not have periodic solutions of prime period two.

**Theorem 5.1.** Eq.(1.1) has no positive solutions of prime period two.

**Proof.** First suppose that there exist prime period two solutions

\[
..., p, q, p, q, ...
\]

of Eq.(1.1). Then,

\[
p = ap + \frac{bp}{cp - dp},
\]

\[
q = aq + \frac{bq}{cq - dq}.
\]
Then,

\[ p(1-a) = \frac{b}{c-d}, \]
\[ q(1-a) = \frac{b}{c-d}. \]

This result contradicts the fact that \( p \neq q \). Hence, this completes the proof. \( \square \)

6. Special case of Eq.(1.1)

In this section, we study the following special case of Eq.(1.1)

\[ x_{n+1} = x_{n-1} + \frac{x_{n-1}}{x_{n-1} - x_{n-3}}, \quad (6.1) \]

where the initial conditions \( x_{-3}, x_{-2}, x_{-1} \) and \( x_0 \) are arbitrary real numbers with \( x_{-3} \neq x_{-1} \) and \( x_{-2} \neq x_0 \).

**Theorem 6.1.** Let \( \{x_n\}_{n=-3}^{\infty} \) be the solution of Eq.(6.1) satisfying \( x_{-3} = t, x_{-2} = s, x_{-1} = k, x_0 = h \). Then, for \( n = 0, 1, 2, \ldots \)

\[ x_{4n-3} = \frac{[nk^2 + k(n^2 - t(2n - 1)) + (n-1)t(t-n)]}{k-t}, \]
\[ x_{4n-2} = \frac{[nh^2 + h(n^2 - s(2n - 1)) + (n-1)s(s-n)]}{h-s}, \]
\[ x_{4n-1} = \frac{[(n+1)k^2 + k(n(n + 1) - t(2n + 1)) + nt(t-n)]}{k-t}, \]
\[ x_{4n} = \frac{[(n+1)h^2 + h(n(n + 1) - s(2n + 1)) + ns(s-n)]}{h-s}. \]

**Proof.** For \( n = 0 \), the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n-1 \). That is,

\[ x_{4n-7} = \frac{[(n-1)k^2 + ((n-1)^2 - t(2n - 3)) + (n-2)t(t-(n-1))]}{k-t}, \]
\[ x_{4n-6} = \frac{[(n-1)h^2 + ((n-1)^2 - s(2n - 3)) + (n-2)s(s-(n-1))]}{h-s}, \]
\[ x_{4n-5} = \frac{[nk^2 + k(n(n - 1) - t(2n - 1)) + (n-1)t(t-(n-1))]}{k-t}, \]
\[ x_{4n-4} = \frac{[nh^2 + h(n(n - 1) - s(2n - 1)) + (n-1)s(s-(n-1))]}{k-t}. \]

Now, it follows from Eq.(6.1) that

\[ x_{4n-3} = x_{4n-5} + \frac{x_{4n-5}}{x_{4n-5} - x_{4n-7}} \]
\[ = \frac{nk^2 + k(n(n - 1) - t(2n - 1)) + (n-1)t(t-(n-1))}{k-t}. \]
Additionally, according to Eq. (6.1),

\[
x_{4n-1} = x_{4n-3} + \frac{x_{4n-3}}{x_{4n-3} - x_{4n-5}}
\]

\[
= \frac{\left(nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right)}{k - t} + \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t}
\]

\[
= \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t}
\]

\[
+ \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t}
\]

\[
= \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t}
\]

\[
+ \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{nk}
\]

\[
= \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t}
\]

\[
= \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{k - t} \left(\frac{1}{k - t} + \frac{1}{nk}\right)
\]

\[
= \frac{\left[nk^2 + k(n^2 - t(2n - 1)) + (n - 1)t(t - n)\right]}{nk(k - t)}
\]

\[
= \frac{(n + 1)k^2 + k(n(n + 1) - t(2n + 1) + nt(t - n))}{k - t}
\]

Also the other relations can be proofed similarly.
7. Numerical examples

We now present some numerical examples to confirm the theoretical work.

Example 7.1. We plot the stability of our equation under the values $a = 0.2$, $b = 1$, $c = 5$, $d = 0.5$, $x_{-3} = 5$, $x_{-2} = 0.3$, $x_{-1} = 2$ and $x_0 = -1$. See Figure 1.

Example 7.2. This example shows the global stability of our equation under the values $a = 0.5$, $b = 2$, $c = 6$, $d = 0.1$, $x_{-3} = 6$, $x_{-2} = -5$, $x_{-1} = 4$ and $x_0 = -3$. See Figure 2.

Example 7.3. An unstable solution of Eq. (6.1) is shown in Figure 3 under the values $a = 0.2$, $b = 1$, $c = 5$, $d = 0.5$, $x_{-3} = 5$, $x_{-2} = 0.3$, $x_{-1} = 2$ and $x_0 = -1$.

References

[1] M. A. E. Abdelrahman and O. Moaaz, On the new class of the nonlinear rational difference equations, Electronic Journal of Mathematical Analysis and Applications, 2018, 6(1), 117–125.
[2] R. Abo-Zeid and C. Cinar, *Global Behavior of The Difference Equation* $x_{n+1} = Ax_n - 1 / (B - Cx_n x_{n-2})$, Boletim da Sociedade Paranaense de Matemática, 2013, 31(1), 43–49.

[3] R. P. Agarwal and E. M. Elsayed, *Periodicity and stability of solutions of higher order rational difference equation*, Advanced Studies in Contemporary Mathematics, 2008, 17(2), 181–201.

[4] M. Aloqeili, *Dynamics of a Rational Difference Equation*, Applied Mathematics and Computation, 2006, 176(2), 768–774.

[5] F. Belhannache, N. Touafek and R. Abo-zeid, *On a Higher-Order Rational Difference Equation*, Journal of Applied Mathematics & Informatics, 2016, 34(5–6), 369–382.

[6] F. Belhannache, *Asymptotic stability of a higher order rational difference equation*, Electronic Journal of Mathematical Analysis and Applications, 2019, 7(2), 1–8.

[7] C. Cinar, *On The Positive Solutions of The Difference Equation $x_{n+1} = ax_n - 1 / (1 + bx_n x_{n-1})$*, Applied Mathematics and Computation, 2004, 156, 587–590.

[8] C. Cinar, T. Mansour and Y Yalçinkaya, *On the difference equation of higher order*, Utilitas Mathematica, 2013, 92, 161–166.

[9] E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, *On The Difference Equation $x_{n+1} = ax_n - bx_n / (cx_n - dx_{n-1})$*, Advances in Difference Equations, 2006, 2006, 1–10.

[10] E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, *On The Difference Equation $x_{n+1} = (ax_n^2 + bx_n - 1 x_{n-k}) / (cx_n^2 + dx_n - 1 x_{n-k})$*, Sarajevo Journal of Mathematics, 2008, 4(17), 1–10.

[11] E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, *On The Difference Equation $x_{n+1} = (ax_{n-1} + bx_{n-k}) / (Ax_{n-1} + Bx_{n-k})$*, Acta Mathematica Vietnamica, 2008, 33(1), 85–94.

[12] E. M. Elabbasy and E. M. Elsayed, *Global Attractivity and Periodic Nature of a Difference Equation*, World Applied Sciences Journal, 2011, 12(1), 39–47.

[13] M. M. El-Dessoky and M. El-Moneam, *On The Higher Order Difference Equation $x_{n+1} = Ax_n + Bx_{n-1} + Cx_{n-k} + (\gamma x_{n-k}) / (Dx_{n-s} + Ex_{n-l})$*, Journal of Computational Analysis and Applications, 2018, 25(2), 342–354.

[14] H. El-Metwally and E. M. Elsayed, *Solution and Behavior of a Third Rational Difference Equation*, Utilitas Mathematica, 2012, 88, 27–42.

[15] E. M. Elsayed, *Behavior and Expression of The Solutions of Some Rational Difference Equations*, Journal of Computational Analysis and Applications, 2013, 15(1), 73–81.

[16] E. M. Elsayed, *Dynamics of a Recursive Sequence of Higher Order*, Communications on Applied Nonlinear Analysis, 2009, 16(2), 37–50.

[17] E. M. Elsayed, A. Alghamdi, *Dynamics and Global Stability of Higher Order Nonlinear Difference Equation*, Journal of Computational Analysis and Applications, 2016, 21(3), 493–503.
18. E. M. Elsayed and F. Alzahrani, *Periodicity and solutions of some rational difference equations systems*, Journal of Applied Analysis and Computation, 2019, 9(6), 2358–2380.

19. E. M. Elsayed, F. Alzahrani, I. Abbas and N. H. Alotaibi, *Dynamical Behavior and Solution of Nonlinear Difference Equation Via Fibonacci Sequence*, Journal of Applied Analysis and Computation, 2020, 10(1), 282–296.

20. E. M. Elsayed, F. Alzahrani and H. S. Alayachi, *Formulas and Properties of some Class of Nonlinear Difference Equation*, Journal of Computational Analysis and Applications, 2018, 4(1), 141–155.

21. E. M. Elsayed and M. Alzubaidi, *The form of the solutions of system of rational difference equation*, Journal of Mathematical Sciences and Modelling, 2018, 1(3), 181–191.

22. E. M. Elsayed, S. R. Mahmoud and A. T. Ali, *Expression and Dynamics of The Solutions of Some Rational Recursive Sequences*, Iranian Journal of Science & Technology, 2014, 38(A3), 295–303.

23. M. Gumus, *Global Dynamics of Solutions of A New Class of Rational Difference Equations*, Konuralp Journal of Mathematics, 2019, 7(2), 380–387.

24. T. F. Ibrahim, *Generalized partial ToDD’s difference equation in n-dimensional space*, Journal of Computational Analysis and Applications, 2019, 26(5), 910–926.

25. T. F. Ibrahim, *On The Third Order Rational Difference Equation $x_{n+1} = (x_n x_{n-2})/(x_{n-1}(a + bx_n x_{n-2}))$*, International Journal of Contemporary Mathematical Sciences, 2009, 4(27), 1321–1334.

26. R. Karatas, *Global Behavior of a Higher Order Difference Equation*, International Journal of Contemporary Mathematical Sciences, 2017, 12(3), 133–138.

27. A. Khaliq, F. Alzahrani and E. M. Elsayed, *Global Attractivity of a Rational Difference Equation of Order Ten*, Journal of Nonlinear Sciences and Applications, 2016, 9, 4465–4477.

28. A. Khaliq and E. Elsayed, *The Dynamics and Solution of Some Difference Equations*, Journal of Nonlinear Sciences and Applications, 2016, 9(3), 1052–1063.

29. Y. Kostrov, *On a Second-Order Rational Difference Equation with a Quadratic Term*, International Journal of Difference Equations, 2016, 11(2), 179–202.

30. M. R. S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall / CRC Press, 2001.

31. I. Okumus and Y. Soykan, *On the Solutions of Four Second-Order Nonlinear Difference Equations*, Universal Journal of Mathematics and Applications, 2019, 2(3), 116–125.

32. M. Saleh and M. Aloqeili, *On The Difference Equation $y_{n+1} = A + y_n/y_{n-k}$*, Applied Mathematics and Computation, 2006, 176(1), 359–363.

33. S. Sadiq and M. Kalim, *Global attractivity of a rational difference equation of order twenty*, International Journal of Advanced and Applied Sciences, 2018, 5(2), 1–7.
[34] D. Simsek, C. Cinar and I. Yalcinkaya, *On The Recursive Sequence* \( x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}} \), International Journal of Contemporary Mathematical Sciences, 2006, 1(10), 475–480.

[35] D. Şimşek and P. Esengul, *Solutions of the Rational Difference Equations*, MANAS Journal of Engineering, 2018, 6(2), 177–192.

[36] Y. Su and W. Li, *Global Asymptotic Stability of a Second-Order Nonlinear Difference Equation*, Applied Mathematics and Computation, 2005, 168, 981–989.

[37] D. T. Töllu and İ. Yalcınkaya, *Global behavior of a three-dimensional system of difference equations of order three*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 2019, 68(1), 1–16.

[38] D. T. Töllu, Y. Yazlık, N. Taskara, *Behavior of Positive Solutions of a Difference Equation*, Journal of Applied Mathematics & Informatics, 2017, 35(3–4), 217–230.

[39] İ. Yalcınkaya, *On the difference equation* \( x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k} \), Discrete Dynamics in Nature and Society, 2008, Vol. 2008, Article ID 805460, 8 pages, doi: 10.1155/2008/805460.

[40] X. Yang, W. Su, B. Chen, G. M. Megson and D. J. Evans, *On The Recursive Sequence* \( x_{n+1} = (ax_{n-1} + bx_{n-2})/(c + dx_{n-1}x_{n-2}) \), Applied Mathematics and Computation, 2005, 162, 1485–1497.

[41] Y. Yazlık and M. Kara, *On a solvable system of difference equations of higher-order with period two coefficients*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 2019, 68(2), 1675–1693.

[42] E. Zayed and M. El-Moneam, *On The Rational Recursive Sequence* \( x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k})/(C dx_n + Ddx_{n-k}) \), Acta Applicandae Mathematicae, 2010, 111(3), 287–301.