Generalized quantum teleportation of shared quantum secret with quantum walks

Hengji Li\textsuperscript{a,b}, Jian Li\textsuperscript{b,c,*}, Xiubo Chen\textsuperscript{c,d}

\textsuperscript{a}Information Security Center, State Key Laboratory Networking and Switching Technology, Beijing University of Posts Telecommunications, Beijing 100876, China
\textsuperscript{b}School of Artificial Intelligence, Beijing University of Posts Telecommunications, Beijing 100876, China
\textsuperscript{c}Center for Quantum Information Research, ZaoZhuang University, ZaoZhuang Shandong 277160, China
\textsuperscript{d}GuiZhou University, Guizhou Provincial Key Laboratory of Public Big Data, Guizhou Guiyang, 550025, China

Abstract

Very recently, Lee \textit{et al.} proposed the first secure quantum teleportation protocol, where quantum information shared by an arbitrary number of senders can be transferred to another arbitrary number of receivers. Here, by introducing quantum walks, a novel secure \((n,m)\) quantum teleportation of shared quantum secret between \(n\) senders and \(m\) receivers is presented. Firstly, two kinds of \((n,2)\) teleportation schemes are proposed by \(n\)-walker quantum walks on the line, the first walker of which is driven by three coins, respectively, based on two kinds of coin operators: the homogeneous coins and the position-dependent coins. Secondly, by increasing the amount of the coins of the first walker to \(m+1\), the previous \((n,2)\) scheme can be generalized to \((n,m)\) teleportation scheme. Then, we give the proof of the information security of our proposed scheme, in which neither any single nor subparties of senders and receivers can fully access the secret quantum information. Moreover, the projective measurements are needed, instead of the joint Bell measurements that are necessary in Lee \textit{et al.}'s protocol. Our work can also be extended further to QWs on the cycle. This work provides an additional relevant instance of the richness of quantum walks for quantum information processing tasks and thus opens the wider application purpose of quantum walks.

1. Introduction

Quantum teleportation, first proposed by Bennett \textit{et al.}\cite{1}, allows us to transfer an unknown quantum state from the sender to the receiver without transferring the quantum particle. Nowadays, it is a fundamental building block of formal quantum information theory\cite{2} and quantum technology, such as quantum computation\cite{3}, quantum network\cite{4}, and quantum repeaters\cite{5}.

The original protocol transfers quantum information from one sender to one receiver, however, to realize the versatile quantum networks, it has been extended to the protocol that split quantum information from one sender to multiple receivers\cite{6,14}, in which no single receiver can fully access the information unless collaborated by all other receivers. Recently, Lee \textit{et al.} proposed the first teleportation protocol of shared quantum secret state among multiple senders and receivers, in which a share quantum information is directly transferred to others among multiple parties without concentrating the information in the location of single or subparties\cite{11}. Following Lee \textit{et al.}'s work, by introducing quantum walks, we devote to presenting a novel kind of secure quantum teleportation scheme of shared quantum secret.

Quantum walks(QWs), the quantum mechanical analogs of classical random walks is a relatively hot research topic\cite{12,13}. Two models of QWs have been suggested: discrete QWs (also call coined QWs)\cite{12} and continuous QWs\cite{13}. In coined QWs, an additional quantum system: a coin is introduced, which might produce plenty of interesting properties. Here, we focus on coined QWs. Different classes of coined QWs have been explored in the past decades, such as QWs with many coins\cite{14,26}, multiple walkers\cite{27,39}, and nonhomogeneous coin operators\cite{40,51}. Very recently, coined QWs have found several applications ranging from quantum computation\cite{37,52,53} to quantum information\cite{15,26,36,38,39,48,50,54}. For example, via utilizing the model of QWs with many coins, some secure quantum communication protocols are proposed\cite{18,20} and via utilizing the model of QWs with nonhomogeneous coin operators, generalized quantum measurement are realized\cite{42,50}.

*Corresponding author

\textit{Email address:} lijian@bupt.edu.cn (Jian Li)

Preprint submitted to XXXXX

December 8, 2020
As the coherent action of the step and coin operators could lead to entanglement between the position and the coin state, different kinds of modes of QWs would produce various of entanglement resources depending on homogeneous and nonhomogeneous coin operators, which might open exciting new possibilities for the realization of quantum information protocols. In this manuscript, via using the model of multi-walker quantum walks with multiple coins, a novel \((n,m)\) quantum teleportation protocol of shared quantum secret state between \(n\) senders and \(m\) receivers is presented for the first time. Based on two different coin operators: the homogeneous coins and the position-dependent coins, two corresponding novel \((n,2)\) teleportation schemes are presented with \(n\)-walker QWs, the first walker of which is driven by three coins. And then by increasing the amount of the coins of the first walker to \(n+1\), the previous \((n,2)\) scheme can be generalized to the \((n,m)\) teleportation scheme.

In our proposed scheme, neither any single nor subparties of senders and receivers can fully access the secret quantum information and it allows us to relay quantum information over a network in an efficient and distributed manner without requiring fully trusted central or intermediate nodes. In addition, the prior entangled state is not necessarily prepared, as the shift operator can generate entanglement between the position and the coin state. Moreover, the projective measurements are needed, instead of multiple times of standard distributed Bell-state measurements that are necessary in Lee et al.’s protocol \cite{11}. Our results provide an additional relevant instance of the richness of QWs for quantum information processing tasks, notwithstanding their simplicity. This work thus opens the wider application purpose of quantum walks and shows a new route to the realization of secure distributed quantum communications in quantum network.

The paper is structured as follows. First the preliminaries are provided about the knowledge of QWs. In Sec. 3 two novel kinds of \((n,2)\) and \((n,m)\) schemes of shared quantum secret state are presented, respectively, based on two kinds of coin operators: the homogeneous coins and the position-dependent coins. Then, we prove the information security of our scheme and discuss our main results. Finally, Sec. 5 contains our conclusions.

2. preliminaries

2.1. QWs on the line

QWs on the line \cite{55} takes place in the product space \(H = \mathcal{H}_p \otimes \mathcal{H}_c\), with \(\mathcal{H}_p\) as the position Hilbert space \(|x\rangle \rangle x \in \mathbb{Z}\), and \(\mathcal{H}_c\) as the two-dimensional auxiliary “coin” space \(|0\rangle, |1\rangle\). One step of the walk is defined by \(U = S(I \otimes \mathbb{C})\), where \(S\) is the conditional shift operator, \(\mathbb{C}\) is any unitary operator, and \(I\) is the identity operator. The shift operator is defined as

\[
E = S \otimes |0\rangle \langle 0| + S^\dagger \otimes |1\rangle \langle 1|,
\]

with \(S = \sum_x |x+1\rangle \langle x|\) and \(S^\dagger = \sum_x |x-1\rangle \langle x|\). Consequently, after \(t\) steps of the walk the initial state \(|\psi_0\rangle\) will become \(|\psi_t\rangle = U^t |\psi_0\rangle\).

2.2. QWs with multiple coins

Brun et al. presented the model of QWs with many coins, which can be utilized as one possible route to classical behavior of coined QWs \cite{14}. In this model, one could replace the coin with a new quantum coin for each flip and after a time \(t\), one would have accumulated \(t\) coins, all of which are entangled with the position of the particle. At each step, the particle moves in the direction dictated by the coin that is active at that step, with the other coins remaining inert until it is their turn once again. Therefore, the unitary transformation that results from flipping the \(m\)-th coin is

\[
E_m = S_m \otimes |0\rangle \langle 0| + S_m^\dagger \otimes |1\rangle \langle 1|,
\]

with \(S_m = S\). As the introduction of the new coins might produce some novel properties, the model has been studied widely \cite{13,17} and been used as a tool to design some quantum communication protocols \cite{18,21,36} and quantum Parrondo’s game \cite{22,23}.

2.3. QWs with multiple walkers

With involving more walkers, QWs with multiple walkers are extensively researched \cite{27,39}, which unlock additional possibilities which are worth investigating, for instance, the two walkers can be also entangled while the initial coin states are entangled \cite{27}. Multi-walker QWs could be capable of universal quantum computation \cite{37} and can be used as a tool in some fields such as graph isomorphism testing \cite{31}, constructing quantum Hash scheme \cite{33}, and quantum network coding \cite{39}. QWs with \(n\) walkers takes place in the Hilbert space \(H = \bigotimes_{i=1}^n \mathcal{H}_i\), with \(\mathcal{H}_i = \mathcal{H}_p \otimes \mathcal{H}_c\) for walker \(i\) and thus each step of QWs will be given by \(U_{i,\ldots,n} = \bigotimes_{i=1}^n U_i\), where \(U_i\) is the unitary operator on the \(i\)-th walker.
2.4. QWs with nonhomogeneous coin operators

In the early models, researchers devoted on the study on homogeneous coin operators, one of most studied which is Hadamard (H) coin. Later, QWs with various of nonhomogeneous coin operators are studied widely, such as site-dependent coins \(C_{n} \), time-dependent coins \(C_{n} \) as well as site- and time-dependent coins \(C_{n} \), which make it possible to obtain the desired distribution. In the homogeneous models, the coin operator is independent of the time and position, however, in the nonhomogeneous models, it will change and the new unitary operator can be denoted as \(C(p)I \) with \(p \) and \(t \) respectively representing the position of the walker and the steps of the evolution. The ability to operate with different coins and the ease of addressing individual position states opens exciting new possibilities for the realization of quantum information protocols, such as perfect state transfer and generalized quantum measurement.

3. Scheme of teleportation between multiple senders and receivers

Here, we succeed in teleporting shared quantum secret from the senders, i.e., a group of \(n \) parties to the receivers, i.e., another group of \(m \) parties, by utilizing multi-walker QWs with multiple coins.

Firstly, two novel \((n, 2)\) teleportation schemes are presented with \(n\)-walker QWs, the first walker of which is driven by three coins, based on two different coin operators: the homogeneous coins and the position-dependent coins. Secondly, the previous \((n, 2)\) scheme can be generalized to the \((n, m)\) teleportation scheme, by increasing the amount of the coins of the first walker to \(m + 1\). The receivers can reconstruct and share the secret by the appropriate joint work of local operations, and no participants can access the full secret quantum information during the whole process. Suppose that a quantum secret in joint work of local operations, and no participants can access the full secret quantum information during the whole process. Therefore, the overall initial tensor state will be

\[
|\Psi\rangle = \otimes_{i=1}^{n} |0\rangle_{p_{i}} \otimes (\alpha \otimes_{i=1}^{n} |0\rangle_{C_{i,1}} + \beta \otimes_{i=1}^{n} |1\rangle_{C_{i,1}}) \otimes |0\rangle_{c_{1,2}} \otimes |0\rangle_{c_{1,3}}.
\]

For simplifying the expression, we will omit the subscripts in the following statement. In the case that we take \(C_{i,1} = I \) as the coin operators in the first step without loss of generality, the initial state thus will evolve into

\[
|\Psi'\rangle = \alpha |1\rangle \otimes_{n} |0\rangle \otimes_{n} |0\rangle + \beta |1\rangle \otimes_{n} |1\rangle \otimes_{n} |0\rangle |0\rangle,
\]
There are many types of coin operators such as nonhomogeneous and site-dependent coin operators [40–43], which will generate various of different entanglement resources. In order to accomplish the task, two kinds of coin operators, the homogenous coins and the position-dependent coins, are utilized to produce the entanglement state in the second step.

**Case 1: homogenous coins.** Taking $C_{c_{1},2} = C_{n,3} = H$ as the coin operators (the choice of $H$ is for producing the maximum entanglement state), the state (6) will evolve into

$$|\Psi_n^{\prime}\rangle = \alpha |3\rangle \otimes |0\rangle \otimes |0\rangle + \alpha |1\rangle \otimes |0\rangle \otimes |1\rangle + \beta |1\rangle \otimes |0\rangle \otimes |0\rangle (\text{7})$$

which is not normalized. For brevity, we will ignore the normalization of quantum states in the following description. Furthermore, for understanding the quantum state (7) more intuitively, by adjusting the order of the terms we can rewrite the final state (7) as

$$|\Psi_n^{\prime}\rangle = \beta |1\rangle \otimes |0\rangle \otimes |0\rangle + \beta |0\rangle \otimes |1\rangle \otimes |1\rangle + \beta |0\rangle \otimes |1\rangle \otimes |0\rangle + \beta |1\rangle \otimes |0\rangle \otimes |1\rangle (\text{8})$$

where the two terms enclosed in parentheses are the ultima quantum state obtained by the receivers. Then, each sender $s_i$ performs two single-particle measurements on $s_{p_k}$ and $s_{c_{i,n}}$ with the corresponding basis

$$\{ \Lambda_0 = \{ |3\rangle, |1\rangle, |\bar{1}\rangle, |\bar{3}\rangle \}, s_{p_k}, \}$$

$$\{ \Theta_0 = \{ |\bar{1}\rangle, |\bar{3}\rangle \}, s_{p_k}, k = 2, \cdots, n, \}$$

$$\Delta_{c_{i,n}} = \{ |\bar{1}\rangle, |+\rangle \}, s_{c_{i,n}}, i = 1, 2, \cdots, n, \}$$

with

$$|3\rangle = (|3\rangle + |1\rangle)/\sqrt{2}, |\bar{1}\rangle = (|3\rangle - |1\rangle)/\sqrt{2}, |\bar{3}\rangle = (|3\rangle - |1\rangle)/\sqrt{2}, |+\rangle = (|1\rangle + |0\rangle)/\sqrt{2}, \text{ and } |0\rangle = (|1\rangle - |0\rangle)/\sqrt{2}, \text{ which are for } s_{p_k}$$. Then mark the results 00, 01, 10 and 11 corresponding to $|3\rangle$, $|\bar{1}\rangle$, $|\bar{3}\rangle$ and $|\bar{0}\rangle$ for $s_{p_k}$. And mark the results 1 and 0 corresponding to $|\bar{1}\rangle$(|--$\rangle$) and $|\bar{3}\rangle$(|+$\rangle$) for $s_{p_k}(s_{c_{i,n}})$. Therefore, conditioned on the results, the final state obtained by the receivers can be written as

$$|\Phi_n^\prime\rangle = \left\{ \begin{array}{l}
|\Phi_n^\prime_{c_{i,n}}\rangle : (1\alpha|00\rangle + \alpha|11\rangle + (1\alpha\beta|01\rangle + \beta|10\rangle), p_1 = 0s, (9a) \\
|\Phi_n^\prime_{c_{i,n}}\rangle : (\alpha|00\rangle + \alpha|11\rangle + (1\alpha\beta|01\rangle + (1\alpha\beta|10\rangle), p_1 = 1t. (9b)
\end{array} \right.$$
obviously that \( |0\rangle \sigma_x \sigma_z |\phi\rangle + |1\rangle |\phi\rangle = \sigma_y |\phi\rangle |0\rangle + |\phi\rangle |1\rangle \). Therefore, any of the receivers \( \{ r_1, r_2 \} \) performs the unitary operator \( \sigma_x \sigma_z \), and the other one performs the unitary operator \( \sigma_y \sigma_z \), to obtain the final quantum state at receivers’ locations
\[
|\Phi'_{h,r}\rangle = |0\rangle \sigma_x \sigma_z |\phi\rangle + |1\rangle |\phi\rangle ,
\]
(12)
Similarly, in the case that the result is 1t, it can be deduced that the receivers \( \{ r_1, r_2 \} \) can also perform the same unitary operators as the result is 0s, however, it will yield
\[
|\Phi'_{h,r}\rangle = |0\rangle |\phi\rangle + |1\rangle \sigma_x \sigma_z |\phi\rangle ,
\]
(13)
To sum up, after the QWs and the local Pauli operators, eventually the initial shared quantum state \( |\Phi'_s\rangle \) becomes
\[
|\Phi'_s\rangle = \begin{cases} p_1 = 0s : |\Phi'_{h,r}\rangle^2, \\ p_1 = 1t : |\Phi'_{h,r}\rangle^2. \end{cases}
\]
(14)
which could make us feel confused owing to the difference with \( |\Phi'_2\rangle \) (The explanation will be shown in Section 3.2).

Case 2: position-dependent coins. Considering the fact that the coin unitary operator can be selected arbitrarily and it would generate various of entangled states \( |\Phi'_2\rangle \). So whether it is possible by choosing the proper coin operator to obtain some special quantum state that is \( |\Phi'_2\rangle \) plus local unitary operation. To do it, we have resorted to a position-dependent coin operator.

Taking \( C_{c,1,2} = C_{c,1,3} = I \), \( C_{c,1,2} = -I \), \( C_{c,1,3} = -2 \) as the coin operators \( (C_{c,1,4}(x) \) is the coin operation on the k-th coin whose action depends on the corresponding position \( x \)), thus after the second step \( U^{(2)} \), the state evolves into
\[
|\Psi_n\rangle = \alpha |3\rangle |0\rangle \otimes |1\rangle |0\rangle \otimes |0\rangle \otimes |0\rangle + \beta |3\rangle |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle ,
\]
(15)
Compared with the basis on \( s_{p_1} \) in Case 2, the measurement basis turns to be \( \Lambda_p = \{ |3\rangle, |\bar{3}\rangle \} \), with \( |\bar{3}\rangle = (|3\rangle + |3\rangle)/\sqrt{2} \); \( |\bar{3}\rangle = (|3\rangle - |3\rangle)/\sqrt{2} \), however, the measurement basis on \( s_{p_2} \) and \( s_{p_1} \) denoted as \( \Theta_p \) and \( \Delta_p \) respectively remains unchanged, that is \( \Theta_p = \Theta_h \) and \( \Delta_p = \Delta_h \). Mark the measurements results \( 0 \) and \( 1 \) corresponding to \( |3\rangle \) and \( |\bar{3}\rangle \) for \( s_{p_1} \). Therefore, the state owned by the receivers will be
\[
|\Phi_p\rangle^2 = \alpha |00\rangle + (-1)^{\omega_p} \beta |11\rangle.
\]
(16)
with \( \omega_p = \sum_{i=1}^n (p_i + c_i) \) mod 2. Then \( r_1 \) or \( r_2 \) can reconstruct \( |\Phi_p\rangle^2 \) by the local Pauli operator \( \sigma_z \otimes \sigma_z \) at the location of \( r_1 \) or \( r_2 \).

3.2. Scheme of teleportation between n senders and m receivers

In Section 3.1 the teleportation scheme between \( n \) senders and 2 receivers is explicitly demonstrated by utilizing \( n \)-walker QWs, the first walker of which is driven by three coins. The next question is whether it is possible to implement the teleportation scheme between \( n \) senders and \( m \) receivers with QWs? Here, we generalize the previous \((n, 2)\) scheme to the \((n, m)\) scheme by increasing the amount of the coins of the first walker to \( m + 1 \). The scheme is presented as follows.

The initial setting of the senders is analogously to the setting in the \((n, 2)\) scheme. The difference takes place in the setting of the receivers. By adjusting the number of the coins of the first walker to \( m + 1 \), every receiver \( r_j (j = 1, 2, \cdots, m) \) separately has one particle \( r_{c_1+j} \) corresponding to the \( j+1 \) coin of the first walker. Thus, the joint Hilbert space of the composite system is \( \mathcal{H}_{1,2,\cdots,n} = \otimes_{i=1}^n \mathcal{H}_{p_i} \otimes_{i=1}^n \mathcal{H}_{c_i,1} \otimes_{j=1}^m \mathcal{H}_{c_1,j+1} \) and we can derive the corresponding unitary operators of the two steps as follows

(i)The unitary operator of the first step is \( U^{(1)} = \mathcal{U}_m^i \mathcal{C}_m^i \) with
\[
C_m^i = \otimes_{i=1}^n (I_{p_i} \otimes C_{c_i,1}) \otimes_{j=1}^m I_{c_1,j+1}.
\]
\[
E_m^i = \otimes_{i=1}^n (S_{p_i} \otimes |0\rangle_{c_1,1} \langle 0| + S_{p_i}^i \otimes |1\rangle_{c_1,1} \langle 0|) \otimes_{j=1}^m I_{c_1,j+1},
\]
(17)
(ii)The unitary operator of the second step is \( U^{(2)} = \mathcal{U}_j^m \mathcal{E}_j^m \) with
\[
C_j^m = \otimes_{i=1}^n (I_{p_i} \otimes I_{c_1,1}) \otimes_{k=1}^{j-1} I_{c_1,k+1} \otimes C_{c_1,j+1} \otimes I_{c_1,k+1},
\]
\[
E_j^m = S_{p_1} \otimes I_{c_1,1} \otimes_{i=1}^n (I_{p_i} \otimes I_{c_1,1}) \otimes_{k=1}^m I_{c_1,k+1} |0\rangle_{c_1,j+1} \langle 0| \otimes_{k=j+1}^m I_{c_1,k+1} + S_{p_1}^i \otimes I_{c_1,1} \otimes_{i=1}^n (I_{p_i} \otimes I_{c_1,1}) \otimes_{k=1}^m I_{c_1,k+1} |1\rangle_{c_1,j+1} \langle 1| \otimes_{k=j+1}^m I_{c_1,k+1}.
\]
(18)
Also, the overall initial state can be written while assuming the coin state in \( r_{c_1,j+1} \) is \( |0\rangle \)
\[
|\Psi_m\rangle = \otimes_{i=1}^n |0\rangle \otimes (\alpha \otimes_{i=1}^n |0\rangle_{c_1,1} + \beta \otimes_{i=1}^n |1\rangle_{c_1,1}) \otimes_{j=1}^m |0\rangle_{c_1,j+1}.
\]
(19)
And in the first step, we keep taking $C_{c_1,1} = I$ as the coin operators, and the initial state \( |\Psi\rangle_m \) will become
\[
|\Psi\rangle_m = \alpha |1\rangle^\otimes n |0\rangle^\otimes n |0\rangle^\otimes m + \beta |-1\rangle^\otimes n |1\rangle^\otimes n |0\rangle^\otimes m .
\]
(20)
In order to accomplish the task, in analogy to the \((n, 2)\) scheme, two kinds of coin operators that are the homogeneous coins and the position-dependent coins, are utilized to produce the entanglement state in the second step.

**Case 1: homogeneous coins.** We take $C_{c_1,j+1} = H$ as the coin operators, thus after the second step $U^{(2)}$, the state evolves into
\[
|\Psi''\rangle_m = \sum_{k=0}^{m'} (\alpha |m_k\rangle |1\rangle^\otimes(n-1) |0\rangle^\otimes n + \beta |m_{k+1}\rangle |-1\rangle^\otimes(n-1) |1\rangle^\otimes n) P[|1\rangle^\otimes k |0\rangle^\otimes (m-k)]
\]
\[
= \sum_{k=0}^{m'} \alpha |m_{2k}\rangle |1\rangle^\otimes(n-1) |0\rangle^\otimes n P[|1\rangle^\otimes 2k |0\rangle^\otimes (m-2k-1)] |0\rangle + \sum_{k=0}^{m'} \beta |m_{2k+1}\rangle |-1\rangle^\otimes(n-1) |1\rangle^\otimes n P[|1\rangle^\otimes 2k |0\rangle^\otimes (m-2k-1)] |1\rangle
\]
\[
+ \sum_{k=0}^{m'-1} \alpha |m_{2k+1}\rangle |1\rangle^\otimes(n-1) |0\rangle^\otimes n P[|1\rangle^\otimes 2k+1 |0\rangle^\otimes (m-2k-2)] |0\rangle + \sum_{k=1}^{m'} \beta |m_{2k}\rangle |-1\rangle^\otimes(n-1) |1\rangle^\otimes n P[|1\rangle^\otimes 2k-1 |0\rangle^\otimes (m-2k-2)] |1\rangle
\]
\[
+ \sum_{k=0}^{m'} \alpha |m_{2k+1}\rangle |1\rangle^\otimes(n-1) |0\rangle^\otimes n P[|1\rangle^\otimes 2k+1 |0\rangle^\otimes (m-2k-1)] |1\rangle + \sum_{k=1}^{m'} \beta |m_{2k}\rangle |-1\rangle^\otimes(n-1) |1\rangle^\otimes n P[|1\rangle^\otimes 2k |0\rangle^\otimes (m-2k-1)] |0\rangle,
\]
(21)
where $m_k = m + 1 - 2k$, $m' = \left[ \frac{m}{2} \right]$, and $P[\cdot]$ denotes the sum of all possible permutation, for example, taking $m = 3$ and $k = 2, P[|110\rangle |011\rangle] \equiv [101] + [101] + [110]$ (See the appendix. A for the calculation process of this formula (21)). Then, each sender $s_i$ performs two single-particle measurements on $s_{p_i}$ and $s_{c_{i,t}}$ with the corresponding basis
\[
A^n_h = \{ |\tilde{m}_i\rangle, |\tilde{m}'_i\rangle | \tilde{m}_i = m+1-4s, \tilde{m}'_i = m-1-4t, s, t \in Z, 0 \leq s \leq m'' = m' + 1, 0 \leq t \leq m'\},
\]
\[
\Theta^n_h = \Theta_h ; \; \Delta^n_h = \Delta_h,
\]
(22)
with
\[
|\tilde{m}_i\rangle = \frac{1}{\sqrt{m''+1}} \sum_{l=0}^{m''} e^{-\frac{2\pi i ls}{m''+1}} |\tilde{m}_i\rangle,
\]
\[
|\tilde{m}'_i\rangle = \frac{1}{\sqrt{m'+1}} \sum_{l=0}^{m'} e^{-\frac{2\pi i lt}{m'+1}} |\tilde{m}_i\rangle,
\]
(23a)
(23b)
Mark the measurements results 0s and 1t corresponding $|\tilde{m}_i\rangle$ and $|\tilde{m}'_i\rangle$ for $s_{p_1}$. Therefore, conditioned on the results, the final state obtained by the receivers can be written as
\[
|\Phi_h\rangle_{\pi} = \left\{ \begin{array}{l}
|\Phi_{h,s}\rangle_{\pi} := |1\rangle^\otimes (m'-1) \sigma_z^{\text{wh,m}} |\phi\rangle + |1\rangle^\otimes (m'-1) \sigma_z^{\text{evn,m}} R_z(\theta_s) |\phi\rangle, \theta_s = 2\pi s/(m'' + 1), p_1 = 0, s, \\
|\Phi_{h,t,\pi} := |1\rangle^\otimes (m'-1) \sigma_z^{\text{wh,m}} |\phi\rangle + |1\rangle^\otimes (m'-1) \sigma_z^{\text{evn,m}} R_z(\theta_t) |\phi\rangle, \theta_t = 2\pi t/(m' + 1), p_1 = 1, t.
\end{array} \right.
\]
(24a)
(24b)
where $\omega_h = \omega$ and $R_z(\theta)$ is the rotation operator along $z$ axis.

The analysis is similar with the \((n, 2)\) scheme. In Lee et al.’s protocol\(^{[1]}\), the final state at the receivers’ locations is $|\Phi\rangle_{\pi}^m$ plus logical Pauli operations and the receivers can perform the local Pauli operations to retrieve $|\Phi\rangle_{\pi}$. The shared secret state $|\Phi\rangle_{\pi}^m$ can be rewritten as
\[
|\Phi_{s,\pi}^m = |\rangle^\otimes (m'-1) \sigma_z |\phi\rangle + |\rangle^\otimes (m'-1) \sigma_z |\phi\rangle,
\]
(25)
with
\[
|\rangle^\otimes (m'-1) \sigma_z = \sum_{j=\text{even(odd)}}^{m'-1} P[|\rangle^\otimes j |\rangle^\otimes (m-1-j)],
\]
(26)
e.g. $P[|\rangle|\rangle|\rangle] = |+ - + - + | - + + + + | + + + + + | - - + + +$, while taking $m = 4$ and $j = 2$. 

6
Next, taking the outcome \( p_1 = 0 \) as an example to illustrate it. Then \( r_m \) performs \( \sigma_z^{\omega_{i,m}} \sigma_x \) yielding
\[
|1\rangle_{\text{odd}}^{\otimes m} (\phi) + (-1)^{\omega_{i,m}} |1\rangle_{\text{even}}^{\otimes m} \sigma_x R_z(\theta_j) \phi),
\]
and then all of the other receivers \( \{r_1, r_2, \ldots, r_{m-1}\} \) take the same Pauli operator \( \sigma_z^{\omega_{i,m}} \), which will make the expression become
\[
|\Phi_{h,s,r}^{m}\rangle = |1\rangle_{\text{odd}}^{\otimes m} (\phi) + |1\rangle_{\text{even}}^{\otimes m} \sigma_x R_z(\theta_s) \phi).
\]

Note that \( r_m \) and all of the other receivers \( \{r_1, r_2, \ldots, r_{m-1}\} \) could respectively also perform \( R_z(\theta_s) \sigma_z^{\omega_{i,m}} \) and the Pauli operator \( \sigma_z^{\omega_{i,m}} \), which will arrive at \( |1\rangle_{\text{odd}}^{\otimes m} R_z(-\theta_s) \sigma_x {\phi} + |1\rangle_{\text{even}}^{\otimes m} {\phi} \). However, the non-Pauli operator \( R_z(\theta_s) \) is introduced, which might not be a good choice. Consequently, we take the former choice only with the Pauli operator.

Furthermore, considering the fact that
\[
|a\rangle \sigma_x \sigma_z^{\omega_{i,m}} \phi) + |(a+1) \mod 2\rangle \sigma_z^{\omega_{i,m}} R_z(\theta_s) \phi) = \sigma_x \sigma_z^{\omega_{i,m}} |a\rangle \phi + \sigma_z^{\omega_{i,m}} R_z(\theta_s) \phi) |(a+1) \mod 2\rangle,
\]
it can be concluded that any receivers \( r_j \) and the other \( m-1 \) receivers can also obtain \( |\Phi_{h,s,r}^{m}\rangle \) by performing the operators \( \sigma_z^{\omega_{i,m}} \sigma_x \) and \( \sigma_z^{\omega_{i,m}} \), respectively ((See the appendix. B for the explanation). Similarly, while the result on \( s_p \), is 0s, the receivers can do the same as the result is 1t, however, which will yield
\[
|\Phi_{h,t,r}^{m}\rangle = |1\rangle_{\text{odd}}^{\otimes m} |\phi\rangle + |1\rangle_{\text{even}}^{\otimes m} \sigma_x R_z(\theta_t) |\phi\rangle,
\]

To sum up, in our \((n, m)\) scheme, after the QWs and the local Pauli operators, the initial shared quantum state \(|\Phi_s^m\rangle\) become
\[
|\Phi_s^m\rangle = \begin{cases} 
  p_1 = 0:s_1: |\Phi_{h,s,r}^{m}\rangle, \\
  p_1 = 1t: |\Phi_{h,t,r}^{m}\rangle.
\end{cases}
\]

which can also be regarded as a new multiparty entanglement state to split the quantum secret state \(|\phi\rangle\) among \( m \) parties, compared with the quantum state \(|\Phi_s^m\rangle\). In addition, the state \(|\Phi_s^m\rangle\) is symmetric meaning that any receiver can reconstruct the quantum secret state with the help of the other receivers. Taking \( m = 2 \), the \((n, m)\) scheme will degenerate into the previous \((n, 2)\) scheme in Section 3.1.

**Case 2: position-dependent coins.** Here, by choosing the proper position-dependent coin operators, we can obtain some special quantum state that is local Pauli equivalent with \(|\Phi_s^m\rangle\). Taking \( C_{c_1,j+1}(j) = I, C_{c_1,j+1}(-j) = \sigma_x (j = 1, 2, \ldots, m) \) as the coin operators, the state will be after the QWs
\[
|\Psi_{p,m}^n\rangle = \alpha |m+1\rangle \otimes (n-1) |0\rangle \otimes n |0\rangle \otimes m + \beta |-(m+1)\rangle \otimes (n-1) |1\rangle \otimes n |1\rangle \otimes m,
\]
The measurement basis on \( s_p \) will be \( \Lambda_{p}^m = \{|-\tilde{m}_p\rangle, |\tilde{m}_p\rangle\} \) with \( |\tilde{m}_p\rangle = (|m_p\rangle + |m_p\rangle)/\sqrt{2} \). The measurement basis on the other \( m-1 \) walkers satisfy the condition
\[
\Theta_{m}^p = \Theta_{h}^m \text{ and } \Delta_{m}^p = \Delta_{h}^m.
\]
Mark the measurements results 0 and 1 corresponding \( |\tilde{m}_p\rangle \) and \(|-\tilde{m}_p\rangle \) for \( s_p \).

Therefore, the state owned by the receivers will be the owners by the receivers will be
\[
|\Phi_{p,m}^n\rangle = \alpha |0\rangle \otimes m + (-1)^{\omega_{p,m}} \beta |1\rangle \otimes m.
\]
with \( \omega_{p,m} = \omega_p \). Then any receiver \( r_j \) can reconstruct \(|\Phi_{p,m}^n\rangle\) by the Pauli operator \( \sigma_z^{\omega_{p,m}} \) at his location.

4. Discussion

4.1. Information security

The proposed protocol is aimed at achieving the goal that no single or subparts of the senders or receivers is allowed to access the secret during the whole process due to that none of the participants is fully trusted here. The \( n \) senders initially share a quantum secret \(|\Phi_s^m\rangle = \alpha \otimes_{i=1}^n |0\rangle_{s_i} + \beta \otimes_{i=1}^n |1\rangle_{s_i}\). In our protocol, as soon as any subgroup of the senders, say \( \{s_{i_1}, s_{j_2}, \ldots, s_{i_p}\} \leq p \leq n \), perform the measurements, all the remaining \( n + m - p \) participants, \( \{s_{j_1+1}, \ldots, s_{j_p}, r_1, \ldots, r_m\} \) are entangled such that none of their sub-parties can fully access the shared secret. As presented above, we have proposed two kinds of \((n, m)\) teleportation protocols based on the different choices of the coin operator: position-dependent coins and homogeneous coins, which are respectively denoted as **Protocol 1** and **Protocol 2** for brevity. Next, we will give the proof of the information security clearly and take \((n, 2)\) scheme as an example to illustrate it in detail.

**Protocol 1:** To begin with, we give the reduced state of the remaining participants after the arbitrary \( p \) senders \( \{s_{j_1}, s_{j_2}, \ldots, s_{j_p}\} \) carry out the measurements, which will bring out two kinds of results:
case 1: \( s_1 \notin \{ s_{j_1}, s_{j_2}, \ldots, s_{j_p} \} \). The corresponding reduced quantum state will be
\[
|\Psi'_{h_1/m}^{n-p} \rangle = \sum_{k=0}^{m} (\alpha |m_k\rangle [1]^{\otimes(n-p-1)} |0\rangle^{\otimes(n-p)} \pm \beta |m_{k+1}\rangle [-1]^{\otimes(n-p-1)} |1\rangle^{\otimes(n-p)} P_1 [1]^{\otimes k} |0\rangle^{\otimes(m-k)} ,
\]
(34)
By tracing out any sub-parties of \( \{ s_{j_{p+1}} = s_1, s_{j_{p+2}}, \ldots, s_{j_m}, r_1, \ldots, r_m \} \), it will result in decoherence with the amplitude information only, so that the secret is not accessible to any single or sub-parties among the senders and receivers.

Next, we begin to minutely analyze the \((n, 2)\) scheme. Taking \( m = 2 \), the reduced state of the remaining parties \( \{ s_1, s_{j_{p+2}}, \ldots, s_{j_m}, r_1, r_2 \} \) is
\[
|\Psi'_{h_2/m}^{n-p} \rangle = \left\{ \begin{array}{ll}
|\tilde{\alpha}_1\rangle : \sum_{k=0}^{m'} \alpha e^{2\pi ik s / (m'+1)} [1]^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k)} |0\rangle^{\otimes(m-2k)} \\
\pm \sum_{k=0}^{m''} \beta e^{2\pi ik (s+1) / (m'+1)} [-1]^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k+1)} |0\rangle^{\otimes(m-2k-1)}, \\
\end{array} \right.
(35)
\]
which can be derived that only the amplitude information is accessible by tracing out any sub-parties. For example, tracing out the sub-group \( \{ s_1, s_{j_{p+2}}, \ldots, s_{j_m} \} \) will yield a mixed quantum state \( \mathbb{I}_4 + |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| \), with the amplitude information only. Similarly, tracing out any sub-parties will yield the same result.

case 2: \( s_1 \in \{ s_{j_1}, s_{j_2}, \ldots, s_{j_p} \} \). As taken will result in two corresponding quantum states below
\[
|\Psi''_{h_2/m}^{n-p} \rangle = \left\{ \begin{array}{ll}
|\tilde{\alpha}_1\rangle : \sum_{k=0}^{m'} \alpha e^{2\pi ik s / (m'+1)} [1]^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k)} |0\rangle^{\otimes(m-2k)} \\
\pm \sum_{k=0}^{m''} \beta e^{2\pi ik (s+1) / (m'+1)} [-1]^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k+1)} |0\rangle^{\otimes(m-2k-1)} \\
\end{array} \right.
(36a)
\]
It can also be proved that only the amplitude information is accessible while tracing out any sub-parties of \( \{ s_{j_{p+1}}, s_{j_{p+2}}, \ldots, s_{j_m}, r_1, \ldots, r_m \} \), which means that the secret is not accessible to any single or sub-parties among the senders and receivers.

Next, we take \((n, 2)\) scheme to explain it in detail. Here, without loss of generality, we take the case \( |\tilde{\alpha}_s\rangle \) as the measurement basis to show it. In this case, \( m' = 1, m'' = 0, m'' = 1, s = \{ 0, 1 \} \), therefore, the reduced state at the remain parties will become
\[
|\Psi''_{h_2/m}^{n-p} \rangle = \left\{ \begin{array}{ll}
\alpha |1\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} \pm \alpha (-1)^s |1\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k)} |0\rangle^{\otimes(m-2k)} \\
\pm \sum_{k=0}^{m''} \beta e^{2\pi ik (s+1) / (m'+1)} [-1]^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} P_1 [1]^{\otimes(2k+1)} |0\rangle^{\otimes(m-2k-1)} \\
\end{array} \right.
(37)
\]
for example, by tracing out the sub-group \( \{ s_{j_{p+1}}, s_{j_{p+2}}, \ldots, s_{j_m} \} \), it will yield a mixed quantum state
\[
|\alpha|^2 |00\rangle \langle 00| + |11\rangle \langle 11| + (-1)^s |00\rangle \langle 11| + |11\rangle \langle 00| \\
+ \beta^2 |01\rangle \langle 01| + |10\rangle \langle 10| + (-1)^s |01\rangle \langle 10| + |10\rangle \langle 01|).
(38)
\]
with the amplitude information only. Similarly, the same holds for any subparties of senders and receivers for information security.

Protocol 2: The accessible information in Protocol 1 has been analyzed in detail. Here, we will show the result in Protocol 2 directly. Firstly, according to the expression \( (39) \), we give the following reduced quantum state of all the remaining \( n+m-p \) participants \( \{ s_{j_{p+1}}, \ldots, s_{j_m}, r_1, \ldots, r_m \} \)
\[
|\Psi''_{h_1/m}^{n-p} \rangle = \alpha [m+1]^{\otimes(n-p-1)} |0\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} \pm \beta |m\rangle^{\otimes(n-p-1)} |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes(m)} ,
(39a)
\]
\[
|\Psi''_{h_2/m}^{n-p} \rangle = \alpha [1]^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} \pm \beta |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes(m)} ,
(39b)
\]
where \( |\Psi''_{h_1/m}^{n-p} \rangle \) and \( |\Psi''_{h_2/m}^{n-p} \rangle \) are similar with \( |\Psi'_{h_1/m}^{n-p} \rangle \) and \( |\Psi'_{h_2/m}^{n-p} \rangle \), respectively. Specially, taking \( m = 2 \), the corresponding reduced quantum state
\[
|\Psi''_{h_1/2}^{n-p} \rangle = \alpha [3]^{\otimes(n-p-1)} |0\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} \pm \beta |3\rangle^{\otimes(n-p-1)} |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes2},
(40a)
\]
\[
|\Psi''_{h_2/2}^{n-p} \rangle = \alpha [1]^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} |0\rangle^{\otimes(n-p)} \pm \beta |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes(n-p)} |1\rangle^{\otimes2}.
(40b)
\]
By tracing out any sub-parties of \( \{s_{p+1}, s_{j_p+2}, \ldots, s_{j_n}, r_1, \ldots, r_m\} \), it will also result in decoherence with the amplitude information only, so that the secret is not accessible to any sub-parties of senders and receivers.

Above all, we have proved that any subparties cannot fully access the quantum secret during the teleportation procedures in the proposed \((n, m)\) schemes Protocol 1 and Protocol 2.

### 4.2. The analysis of our scheme

Multipartite entangled states are a fundamental resource for a wide range of quantum information processing tasks and we accomplished the transfer of the multipartite entangled states from \(n\) to \(m\) receivers. The \((n, 2)\) teleportation scheme was firstly clearly presented via the model of \(n\)-walker QWs, the first walker of which is driven by three coins and then it can be generalized to the \((n, m)\) scheme by increasing the amount of the coins of the first walker to \(m + 1\). Specially, \(n\) senders \(s_i(1 \leq i \leq n)\) hold \(2n\) particles, the corresponding position and the first coin state of \(n\) walkers, and the shared secret quantum state \(|\Phi^m\rangle\) is encoded in the coin states. Similarly, each receiver \(r_i(1 \leq i \leq m)\) has one particle \(c_{e_1,i+1}\) corresponding to the \(i + 1\) coin of the first walker. We note that our work is not limited to QWs on the line and it can be extended further to QWs on the cycle [56].

Neither any single nor subparties of the participants can fully access the secret quantum information in our proposed scheme, which has been proved in Sec. [11]. It implies that the participants can relay quantum information over a network without requiring fully trusted central or intermediate nodes, which might be useful to establish a long-distance quantum communication via distributed nodes, none of which necessarily relays the full quantum information. Compared with Lee et al.’s protocol [11], single-particle measurements are needed, instead of joint measurements. In particular, \(2n\) times of projective measurements are needed in our scheme and the measurement basis are \(\{\Lambda^m_p, \Theta^m_p, \Delta^m_p\}\) and \(\{\Lambda_p^r, \Theta^r_p, \Delta^r_p\}\), respectively corresponding Protocol 1 and Protocol 2. However, \(n\) times of standard Bell-state measurements are needed, in which a distributed Bell-state measurement is introduced that can be jointly performed by separated parties [11].

In Protocol 1, the initial shared quantum state \(|\Phi^m_r\rangle\) become \(|\Phi^m_{h_i}\rangle\) instead of \(|\Phi^m_r\rangle\), which might seem that the teleportation task of the quantum state \(|\Phi^m_r\rangle\) failed. Now let us turn our attention to the original motivation for the scheme, which is aimed at transferring the shared quantum secret information \(|\rho\rangle\) by altering the shared quantum secret information \(|\rho\rangle\) to \(|\Phi^m_{h_i}\rangle\) instead of \(|\Phi^m_r\rangle\). However, it would not take place while introducing position-dependent coins in Protocol 2. By choosing the proper coin operators depending on the position, the receivers can obtain the reduced state \(|\Phi^m_{p_j}\rangle = |\Phi^m_r\rangle + \text{logical Pauli operators, meaning that the receivers can carry out the local Pauli operators to retrieve } |\Phi^m_r\rangle\).

During the recovery process, conditioned on all the measurement results, the receivers can carry out the local Pauli operators to retrieve the initial state quantum state, specially (i) in Protocol 1, any one of the receivers \(r_j\) needs to perform the unitary operator \(\sigma^z_i \sigma_x\), and all the other ones need to perform the unitary operator \(\sigma^z_i \sigma_m\); (ii) in Protocol 2, only any one of the receivers need to take the Pauli operator \(\sigma^z_i \sigma_m\) at his location. In addition, both \(|\Phi^m_{p_j}\rangle\) and \(|\Phi^m_{r_j}\rangle\) are symmetric meaning that any receiver can reconstruct the quantum secret state with the help of the other receivers. Moreover, as the logical outcome is irrespective of the order of performed measurements, it is possible to separate all the measurements spatially or temporally with the help of classical communications to share their results among the nodes.

Additionally, as shown in Lee et al.’s protocol [11], the GHZ-entanglement state \(|\Phi^{(n+m)}\rangle = |0\rangle^{\otimes(n+m)} + |1\rangle^{\otimes(n+m)}\) is utilized as the network channel to fulfill the teleportation scheme. Due to QWs can contain entanglement by the conditional shift operator between the walkers and the coins [14, 27], the entangled state is not necessarily prepared in advance in our schemes. Furthermore, conditioned on the various choices of the coin unitary operator, different kinds of entanglement sources can be generated to meet our requirements. In Protocol 1, the model of the homogeneous coins is used and we take \(\mathbb{H}\) as the coin operator for producing the maximal entanglement state, which is the unbiased coin flipping operator and can change \(|0\rangle\) to \(|0\rangle + |1\rangle\) with the same amplitude, and therefore meets the requirement. In Protocol 2, for obtaining the specific quantum state, we introduce position-dependent coins and show an approach to solving it, that is \(C_{e_1,j+1}(j) = I, C_{e_1,j+1}(-j) = \sigma_x(j = 1, 2, \ldots, m)\).

### 5. Conclusion

In this manuscript, by using the model of multi-walker QWs with multiple coins and two different coin operators, the homogeneous coin operators and the position-dependent coin operators, two kinds of corresponding \((n, m)\) teleportation schemes of shared quantum secret state among \(n\) senders and \(m\) receivers, were proposed. This work thus not only opens a route to the realization of secure distributed quantum communications and computations in quantum networks, but also provides an additional relevant instance of the richness
of QWs for quantum information processing tasks and thus opens the wider application purpose of quantum walks.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos.U1636106, 61671087, 61170272), the BUPT Excellent Ph.D. Students Foundation (No.CX2020310), Natural Science Foundation of Beijing Municipality (No.4182006), Technological Special Project of Guizhou Province (Grant No. 20183001), and the Foundation of Guizhou Provincial Key Laboratory of Public Big Data (Grant No.2018BDKFJJ016, 2018BDKFJJ018) and the Fundamental Research Funds for the Central Universities (No.2019XD-A02).

Appendix

Appendix A. The derivation process of the formula (22)

Firstly, according to the parity of $k$, the quantum state $|\Phi''\rangle$ can be written as $|\Phi''\rangle = |\Phi''\rangle_{even} + |\Phi''\rangle_{odd}$, defined by

$$|\Phi''\rangle_{even} = \sum_{k=0}^{m'} \alpha |m_{2k}\rangle |1\rangle^{(n-1)} |0\rangle^\otimes n P[|1\rangle^{\otimes 2k} |0\rangle^{\otimes (m-2k)}] + \sum_{k=0}^{m''} \beta |m_{2k+2}\rangle |1\rangle^{(n-1)} |1\rangle^\otimes n P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-1)}]$$

(A.1a)

$$|\Phi''\rangle_{odd} = \sum_{k=0}^{m'} \alpha |m_{2k+1}\rangle |1\rangle^{(n-1)} |0\rangle^\otimes n P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-1)}] + \sum_{k=0}^{m''} \beta |m_{2k+1}\rangle |1\rangle^{(n-1)} |1\rangle^\otimes n P[|1\rangle^{\otimes (2k)} |0\rangle^{\otimes (m-2k)}]$$

(A.1b)

where $m' = \frac{m}{2}$, $m'' = \frac{m-1}{2}$. Next, by utilizing the following equations:

$$P[|1\rangle^{\otimes 2k} |0\rangle^{\otimes (m-2k)}] = P[|1\rangle^{\otimes 2k} |0\rangle^{\otimes (m-2k-1)}]P[|1\rangle^{\otimes (m-2k)} |1\rangle]$$

(A.2a)

$$P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-1)}] = P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-2)}]P[|1\rangle^{\otimes (m-2k-1)} |1\rangle]$$

(A.2b)

the state $|\Phi''\rangle$ can be rewritten as

$$|\Phi''\rangle = \sum_{k=0}^{m''} \alpha |m_{2k}\rangle |1\rangle^{(n-1)} |0\rangle^\otimes n P[|1\rangle^{\otimes 2k} |0\rangle^{\otimes (m-2k)}] + \sum_{k=1}^{m'} \alpha |m_{2k}\rangle |1\rangle^{(n-1)} |0\rangle^\otimes n P[|1\rangle^{\otimes (2k-1)} |0\rangle^{\otimes (m-2k)}]P[|1\rangle^{\otimes (m-2k)} |1\rangle] + \sum_{k=0}^{m''} \beta |m_{2k+2}\rangle |1\rangle^{(n-1)} |1\rangle^\otimes n P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-1)}]P[|1\rangle^{\otimes (m-2k-1)} |1\rangle]$$

(A.3)

Then, by adjusting the order of the terms, it will become

$$|\Phi''\rangle = \sum_{k=1}^{m''} \alpha |m_{2k}\rangle |1\rangle^{(n-1)} |0\rangle^\otimes n P[|1\rangle^{\otimes 2k} |0\rangle^{\otimes (m-2k)}] + \sum_{k=0}^{m'} \beta |m_{2k+2}\rangle |1\rangle^{(n-1)} |1\rangle^\otimes n P[|1\rangle^{\otimes (2k+1)} |0\rangle^{\otimes (m-2k-2)}]\langle 0|^{\otimes (m-2k)} |1\rangle$$

(A.4)
Appendix B.

To begin with, we give the following equation
\[
|0\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |1\rangle \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle = |0\rangle |\alpha| 1 + (-1)^{\omega, m} \beta |0\rangle | + |1\rangle |\alpha| 0 + (-1)^{\omega, m} e^{i \theta_s} |\beta| 1\rangle |
\]
\[
= |\alpha| 1 + (-1)^{\omega, m} \beta |0\rangle | + |\alpha| 0 + (-1)^{\omega, m} e^{i \theta_s} |\beta| 1\rangle |
\]
\[
= \sigma_x \sigma_z^{\omega, m} |\phi\rangle | + \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |1\rangle, \tag{B.1}
\]

Similarly, it can be easily deduced that
\[
|1\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |0\rangle \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle = \sigma_x \sigma_z^{\omega, m} |\phi\rangle | + \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |0\rangle, \tag{B.2}
\]

Next, we will prove that the quantum states of \( r_{m-1} \) and \( r_m \) are exchangeable, that is, the composite state keeps constant after carrying out the Swap gate between \( r_{m-1} \) and \( r_m \). Recall \( |\Phi_{h,s}\rangle|_r \) in (24m) and by using the (B.1) and (B.2), it can be rewritten as
\[
|\Phi_{h,s}\rangle|_r^m = |1\rangle^{\text{even}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |1\rangle^{\text{odd}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle
\]
\[
= |1\rangle^{\text{odd}} |0\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |1\rangle^{\text{even}} |1\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |1\rangle^{\text{even}} |0\rangle \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle + |1\rangle^{\text{even}} |1\rangle \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle
\]
\[
= |1\rangle^{\text{odd}} |0\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |1\rangle^{\text{odd}} |1\rangle \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle + |1\rangle^{\text{even}} |1\rangle \sigma_x \sigma_z^{\omega, m} |\phi\rangle + |0\rangle^{\text{even}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle
\]
\[
= |1\rangle^{\text{odd}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle | + |1\rangle^{\text{even}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |1\rangle + |1\rangle^{\text{even}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle + |1\rangle^{\text{even}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |0\rangle
\]
\[
= |1\rangle^{\text{odd}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle | + |1\rangle^{\text{even}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |1\rangle + |1\rangle^{\text{odd}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle + |1\rangle^{\text{even}} \sigma_z^{\omega, m} R_z(\theta_s) |\phi\rangle |0\rangle
\] \tag{B.3}

which says that the quantum states of \( r_{m-1} \) and \( r_m \) are exchangeable. It can be obviously concluded that any two receivers of \( \{r_1, \cdots, r_{m-1}\} \) are also exchangeable owing to the symmetry of \( |1\rangle^{\text{even}} \) and \( |1\rangle^{\text{odd}} \). Thus, the quantum states of \( r_j \) and \( r_m \) are exchangeable and then it will give
\[
|\Phi_{h,s}\rangle|_r^m = |1\rangle^{\text{even}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle^{\text{odd}} + |1\rangle^{\text{odd}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle^{\text{even}} + |1\rangle^{\text{odd}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle^{\text{od}} + |1\rangle^{\text{even}} \sigma_x \sigma_z^{\omega, m} |\phi\rangle |1\rangle^{\text{od}} \tag{B.4}
\]

Therefore, any one of the receivers and the other \( m-1 \) receivers can perform respectively the Pauli operator \( \sigma_z^{\omega, m} = \sigma_z \) and \( \sigma_x^{\omega, m} = \sigma_x \), which will arrive at
\[
|1\rangle^{\text{odd}} \sigma_x |\phi\rangle |1\rangle^{\text{even}} + |1\rangle^{\text{even}} \sigma_x |\phi\rangle |1\rangle^{\text{odd}} + |1\rangle^{\text{odd}} \sigma_z R_z(\theta_s) |\phi\rangle |1\rangle^{\text{even}} + |1\rangle^{\text{even}} \sigma_z R_z(\theta_s) |\phi\rangle |1\rangle^{\text{odd}} \tag{B.5}
\]

which is equal to \( |\Phi_{h,s}\rangle|_r^m \), which can be contained by the same way above based on the following equation
\[
|a\rangle |\phi\rangle + |(a+1) \text{mod} 2\rangle \sigma_x R_z(\theta_s) |\phi\rangle = |\phi\rangle |a\rangle + \sigma_x R_z(\theta_s) |\phi\rangle |(a+1) \text{mod} 2\rangle, a = \{0, 1\}. \tag{B.6}
\]

6. References

References

[1] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters, Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels, Physical Review Letters 70(13), 1895 (1993)
[2] M.A. Nielsen, I. Chuang. Quantum computation and quantum information (2002)
[3] R. Raussendorf, H.J. Briegel, A one-way quantum computer, Physical Review Letters 86(22), 5188 (2001)
[4] T.A. Brun, H.A. Carteret, A. Ambainis, Quantum random walks, Physical Review A 66(2), 020302 (2002)
[5] T.A. Brun, H.A. Carteret, A. Ambainis, Quantum walks driven by many coins, Physical Review A 67(5), 052317 (2003)
