Riesz-type inequalities and maximum flux exchange flow

I McGillivray
School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
United Kingdom
e maiemg@bristol.ac.uk
t + 44 (0)117 3311663
f + 44 (0)117 9287999

Abstract

Let \( D \) stand for the open unit disc in \( \mathbb{R}^d \) (\( d \geq 1 \)) and \((D, \mathcal{B}, m)\) for the usual Lebesgue measure space on \( D \). Let \( \mathcal{H} \) stand for the real Hilbert space \( L^2(D, m) \) with standard inner product \((\cdot, \cdot)\). The letter \( G \) signifies the Green operator for the (non-negative) Dirichlet Laplacian \(-\Delta\) in \( \mathcal{H} \) and \( \psi \) the torsion function \( G\chi_D \). We pose the following problem. Determine the optimisers for the shape optimisation problem

\[
\alpha_t := \sup \left\{ (G\chi_A, \chi_A) : A \subseteq D \text{ is open and } (\psi, \chi_A) \leq t \right\}
\]

where the parameter \( t \) lies in the range \( 0 < t < (\psi, 1) \). We answer this question in the one-dimensional case \( d = 1 \). We apply this to a problem connected to maximum flux exchange flow in a vertical duct. We also show existence of optimisers for a relaxed version of the above variational problem and derive some symmetry properties of the solutions.

Key words: shape optimisation

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1 Introduction

Let \( \Omega \) stand for a bounded open set in \( \mathbb{R}^d \) (\( d \geq 1 \)) and \((\Omega, \mathcal{B}, m)\) for the usual Lebesgue measure space on \( \Omega \). Let \( \mathcal{H} \) stand for the real Hilbert space \( L^2(\Omega, m) \) with standard inner product \((\cdot, \cdot)\). The letter \( G \) signifies the Green operator for the (non-negative) Dirichlet Laplacian \(-\Delta\) in \( \mathcal{H} \) and \( \psi \) the torsion function \( G\chi_\Omega \). We pose the following problem. Determine the optimisers for the shape optimisation problem

\[
\alpha_t := \sup \left\{ (G\chi_A, \chi_A) : A \subseteq \Omega \text{ is open and } (\psi, \chi_A) \leq t \right\}
\]

where the parameter \( t \) lies in the range \( 0 < t < (\psi, 1) \). We show that optimisers exist for a relaxed version of this problem and derive certain symmetry properties of the solutions when \( \Omega \) is replaced by the open unit ball \( D \). We obtain the explicit form of the optimisers in the one-dimensional case \( d = 1 \) for the open interval \( D = (-1, 1) \).

Define

\[
V_t := \left\{ f \in \mathcal{H} : 0 \leq f \leq 1 \text{ m.a.e. on } \Omega \text{ and } (f, \psi) \leq t \right\}
\]

for \( t \) in the range \( 0 < t < (\psi, 1) \) and consider the relaxed variational problem

\[
\beta_t := \sup \left\{ J(f) : f \in V_t \right\},
\]

where \( J(f) = (f, Gf) \). The first main result runs as follows.
Theorem 1.1. For each \( t \) in the range \( 0 < t < (\psi, 1) \), there exists \( f \in V_t \) such that \( \beta_t = J(f) \).

In case \( \Omega \) is replaced by the open unit ball \( D \) centred at the origin, we can say more about the symmetry properties of optimisers. In fact,

Theorem 1.2. Let \( f \in V_t \) such that \( \beta_t = J(f) \). Then \( f \) possesses circular cap symmetry.

We now turn to the one-dimensional case \( d = 1 \) so that \( D = (-1, 1) \) and the torsion function \( \psi \) is given explicitly by \( \psi(x) = (1/2) (1 - x^2) \) for \( x \in D \). Noting that \( (\psi, 1) = 2/3 \), define \( \varphi : D \to (0, 2/3) \) by

\[
\varphi(x) := \chi(x, 1) \chi(1-x, 1),
\]

and specify \( \xi_t \in (-1, 1) \) uniquely via the relation

\[
\varphi(\xi_t) = t \tag{1.3}
\]

for each \( t \in (0, 2/3) \). Set \( A_t := (\xi_t, 1) \). Then

Theorem 1.3. For any open subset \( A \) in \( D \) satisfying \( (\psi, \chi_A) \leq t \) it holds that

\[
(G \chi_A, \chi_A) \leq (G \chi_{A_t}, \chi_{A_t}),
\]

and equality occurs precisely when either \( A = A_t \) or \( A = -A_t \).

The inequality is somewhat reminiscent of the Riesz rearrangement inequality: this justifies the epithet in the title. This problem has a probabilistic interpretation in so far as the function \( G \chi_A \) is the expected occupation time in \( A \) spent by absorbing Brownian motion in \( D \) (associated to the Laplacian \( \Delta \)). The \( d \geq 2 \) case has not yet been resolved. It is tempting to speculate that a hyperbolic cap optimises (1.1) in this case. Numerical evidence does not seem to bear this out, however [5].

One reason why this problem is intriguing is because of its connection to maximum flux exchange flow in a vertical duct, a model of lava flow in a volcanic vent (see [4]). In the two-dimensional case \( d = 2 \), we imagine a configuration of two immiscible fluids in \( D \times \mathbb{R} \) with different physical characteristics in a state of steady flow. The densities of the fluids are labelled \( \rho, \rho' \) and we take \( \rho > \rho' \). Each fluid has unit viscosity. With respect to cylindrical coordinates \( (x, z) \in D \times \mathbb{R} \), gravity acts in the direction \((0, -1)\) according to the model. The pressure \( p \) depends only upon \( z \) and has constant gradient \( \partial p/\partial z = -G \). Suppose that the fluid with density \( \rho \) occupies a region in \( D \times \mathbb{R} \) with cross-section \( A \subseteq D \). Restricting the problem to \( D \), the velocity \( u \) of the components of the fluid may be described (informally) using the Navier-Stokes equation via

\[
0 = \Delta u + G - \rho g \quad \text{on} \quad A;
\]
\[
0 = \Delta u + G - \rho' g \quad \text{on} \quad D \setminus A.
\]

Non-slip (Dirichlet) boundary conditions are imposed on the boundary of \( D \). It is also assumed that \( u \) and its gradient are continuous on the interface between the two regions \( A \) and \( D \setminus A \) (continuity of velocity and stress).

The parameter \( G \) lies in the interval \((\rho' g, \rho g)\). This allows the possibility of a bi-directional flow. Upon rescaling (and relabelling the velocities) we obtain the system

\[
0 = \Delta u - \lambda - 1 \quad \text{on} \quad A;
\]
\[
0 = \Delta u - \lambda + 1 \quad \text{on} \quad D \setminus A; \tag{1.4}
\]

where

\[
\lambda := \frac{(\rho' + \rho)g - 2G}{(\rho - \rho')g} \in (-1, 1)
\]
is a proxy for the pressure gradient. Two problems arise. One is to maximise the flux \( Q := (\chi_{D\setminus A}, u) \) amongst all regions \( A \) which satisfy the flux balance condition \((u, 1) = 0 \) with constant \( \lambda \); the other in which we optimize also over \( \lambda \). In detail, we seek optimisers for the problems
\[
\gamma := \sup \left\{ (\chi_{D\setminus A}, u) : (u, 1) = 0, A \subseteq D \text{ open} \right\},
\]
\[
\gamma_\lambda := \sup \left\{ (\chi_{D\setminus A}, u) : (u, 1) = 0, A \subseteq D \text{ open} \right\},
\]
where in the latter \( \lambda \) is fixed in the interval \((-1, 1)\). It turns out that problem (1.1) is closely related to the two problems above. Note too that these problems have obvious analogues for the case \( d = 1 \).

We come to our last main result. Note that the \( d = 2 \) analogue is discussed as a marginal case in [4].

**Theorem 1.4.** In case \( d = 1 \),

(i) for each \( \lambda \in (-1, 1) \), the problem (1.6) is optimised precisely when either \( A = A_1 - \lambda \) or \( A = -A_1 - \lambda \);

(ii) the problem (1.5) is optimised precisely when either \( A = (0, 1) \) or \( A = (-1, 0) \) and has optimal value \( 1/12 \).

We give a brief sketch of the organisation of the paper. In Section 2, we obtain existence of optimisers for the relaxed problem (1.2) and derive some symmetry properties when \( \Omega \) is replaced by the ball \( D \). Sections 3 to 8 deal with the proof of Theorem 1.3. Section 9 contains an application to maximum flux exchange flow (Theorem 1.4).

## 2 Existence of optimisers and symmetry in a general relaxed setting

For the sake of clarity, we first of all remark that the (non-negative) Dirichlet Laplacian \((D(-\Delta), -\Delta)\) is associated with the Dirichlet form \((\mathcal{F}, \mathcal{E})\) in \( H \) with form domain \( \mathcal{F} := W^{1,2}_0(\Omega) \) and
\[
\mathcal{E}(u, v) = (Du, Dv) \quad (u, v \in \mathcal{F}).
\]

We begin with the proof of Theorem 1.4.

**Proof of Theorem 1.4** Let \((f_n)_{n \in \mathbb{N}}\) be a maximising sequence for \( \beta \). Now, \( V_t \) is weakly sequentially compact in \( \mathcal{H} \). This follows by appeal to [3] Theorem 10.2.9 due to the fact that \( V_t \) is bounded, closed and convex in the reflexive Banach space \( \mathcal{H} \). So we may assume that \((f_n)\) converges weakly in \( \mathcal{H} \) to some \( f \in V_t \) as \( n \to \infty \) after choosing a subsequence if necessary.

Put \( u_n := Gf_n \). Then for each \( n \),
\[
\|u_n\|_{W^{1,2}_0(\Omega)}^2 \leq \sqrt{2 \delta m(\Omega)} \|u_n\|_{W^{1,2}_0(\Omega)}.
\]

Additionally,
\[
\|u_n\|_{W^{1,2}_0(\Omega)}^2 = \delta(u_n, u_n) + (u_n, u_n) = (f_n, Gf_n) + (Gf_n, Gf_n) \leq (1, \psi) + (\psi, \psi).
\]

In short, the sequence \((u_n)\) is bounded in \( W^{1,2}_0(\Omega) \). In case \( d \geq 2 \) by the Rellich-Kondrachov compactness theorem ([3] 5.7 for example), we may assume that \((u_n)\) converges in \( L^1(\Omega, m) \) to some element \( u \) after extracting a subsequence if necessary. In case \( d = 1 \), we use Morrey’s inequality (see [3] 5.6.2, for example) and the Arzela-Ascoli compactness criterion to extract a uniformly convergent subsequence. The details are described in the proof of Theorem 1.4.

For each \( n \in \mathbb{N} \),
\[
(u_n, \varphi) = (Gf_n, \varphi) = (f_n, G\varphi) \quad \text{for all} \ \varphi \in \mathcal{H},
\]
which yields
\[(u, \varphi) = (f, G\varphi) = (Gf, \varphi)\] for all \(\varphi \in \mathcal{H}\)
upon taking limits. Therefore, \(u = Gf\) m.a.e. on \(\Omega\). Moreover,
\[J(f) - J(f_n) = (u, f) - (u_n, f_n) = (u, f - f_n) + (f_n, u - u_n)\]
and the right-hand side converges to zero as \(n \to \infty\) in virtue of the weak respectively strong \(L^1(\Omega, m)\) (or uniform in the case \(d = 1\)) convergence of the sequences \((f_n)\) respectively \((u_n)\). As \(\beta_t = \lim_{n \to \infty} J(f_n)\) it follows that \(\beta_t = J(f)\).

In the remainder of this section, we replace \(\Omega\) by the open unit ball \(D\) in \(\mathbb{R}^d\) centred at the origin. We first discuss the operation of polarisation for integrable functions on \(D\) (see [2] and references therein). For \(\nu \in S^{d-1}\) the closed half-space \(H = H_\nu\) is defined by
\[H_\nu := \{x \in \mathbb{R}^d : x \cdot \nu \geq 0\}\]
with an associated reflection
\[\tau_H : \mathbb{R}^d \to \mathbb{R}^d; x \mapsto x - 2(x \cdot \nu)\nu\]
Refer to the collection of all these closed half-spaces by \(\mathcal{H}\). The polarisation \(f_H\) of \(f \in L^1_c(D, m)\) with respect to \(H \in \mathcal{H}\) is defined as follows. Choose an \(m\)-version of \(f\), which we again denote by \(f\). Set
\[f_H(x) := \begin{cases} f(x) \wedge f(\tau_H x) & \text{for } x \in D \cap H, \\ f(x) \vee f(\tau_H x) & \text{for } x \in D \setminus H. \end{cases}\]
Its \(m\)-equivalence class is the polarisation of \(f\). The definition is well-posed.

The Green kernel \(G(x, y)\) is given by
\[G(x, y) = \Phi(y - x) - \Phi(|x|(|y - x^*|))\] for \((x, y) \in D \times D \setminus d\),
where \(\Phi\) is the fundamental solution of Laplace’s equation in \(\mathbb{R}^d\), \(d\) stands for the diagonal in \(D \times D\) and the decoration \(*\) refers to inversion in the unit sphere. We note the inequality
\[G(x, y) > G(x, \tau_H y)\] for any \(x, y \in D \cap \text{int } H\),
(2.1)
which follows from the strong maximum principle.

**Theorem 2.1.** Let \(f \in L^1_c(D, m)\) and \(H \in \mathcal{H}\). Then \(J(f) \leq J(f_H)\) with equality if and only if either \(f = f_H\) or \(f \circ \tau_H = f_H\) m.a.e. on \(D\).

**Proof.** We work with an \(m\)-version of \(f\), again denoted \(f\). Define
\[A^+ := \{x \in D \cap H : f(x) < f(\tau_H x)\}\]
and similarly \(B^+\) but with the strict inequality replaced by the sign \(>\). Put \(A^- := \tau_H A^+\) and \(A := A^+ \cup A^-\). Set \(S := D \setminus A\). In this notation,
\[f_H = \chi_A f \circ \tau_H + \chi_S f.\]
As a consequence,
\[J(f_H) = J(\chi_A f \circ \tau_H) + 2(\chi_A f \circ \tau_H, G\chi_S f) + J(\chi_S f) = J(\chi_A f) + 2(\chi_A f \circ \tau_H, G\chi_S f) + J(\chi_S f)\]
Lemma 2.1. A geometric argument establishes the following lemma.

Before proving Theorem 1.2, we prepare a number of lemmas. We first discuss a useful two-point inequality. We introduce the notation

\[ f \] 

Equip \( Q \) for each \( m \) and a similar identity holds for \( J(f) \) but without composition with reflection. We may then write that

\[
J(f_H) - J(f) = 2 \int_{A^+} \int_{B^+} (f(\tau_H x) - f(x))(g(x, y) - g(\tau_H x, y))(f(y) - f(\tau_H y)) m(dy) m(dx).
\]

It is clear from this representation with the help of (2.1) that \( J(f) \leq J(f_H) \).

In the case of equality, it holds that either \( m(A^+) = 0 \) or \( m(B^+) = 0 \). In the former case, \( f = f_H \) while in the latter, \( f \circ \tau_H = f_H \) m.a.e. on \( D \).

The spherical cap symmetrisation (see [7], [8], [9] for example) of \( A \in \mathcal{B} \) with respect to the direction \( \omega \in S^{d-1} \) is the set \( A^\ast \in \mathcal{B} \) specified uniquely by the conditions

\[
A^\ast \cap \{0\} = A \cap \{0\},
A^\ast \cap \partial B(0, r) = B(r\omega, \rho) \cap \partial B(0, r) \quad \text{for some } \rho \geq 0,
\]

for each \( r \in (0, 1) \). Here, \( \sigma_r \) stands for the surface area measure on \( \partial B(0, r) \). The spherical cap symmetrisation of \( f \in L^1_+(D, m) \) (denoted \( f^\ast \) for brevity) is defined as follows. Choose an \( m \)-version of \( f \), which we again denote by \( f \). Let \( f^\ast \) be the unique function such that

\[
\{f^\ast > t\} = \{f > t\}^\ast \quad \text{for each } t \in \mathbb{R}.
\]

Its \( m \)-equivalence class is the polarisation of \( f \). The definition is again well-posed. We also write \( f^\ast \) as \( C_\omega f \).

Before proving Theorem 1.2 we prepare a number of lemmas. We first discuss a useful two-point inequality. We introduce the notation

\[
Q := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0\},
R := \{(x_1, x_2) \in Q : 0 \leq x_2 < x_1\},
S := \{(x_1, x_2) \in Q : 0 < x_1 \leq x_2\}.
\]

Equip \( Q \) with the \( \ell^1 \)-norm \( \|x\|_1 := |x_1| + |x_2| \) where \( x = (x_1, x_2) \in Q \). Define a mapping \( \varphi : Q \to Q \) via

\[
(x_1, x_2) \mapsto (x_1 \lor x_2, x_1 \land x_2).
\]

A geometric argument establishes the following lemma.

Lemma 2.1. For any \( x, y \in Q \),

\[
\|\varphi x - \varphi y\|_1 \leq \|x - y\|_1
\]

with strict inequality if and only if \( x \in R \) and \( y \in S \) or \( x \in \overline{S} \) and \( y \in R \) or the same with the rôles of \( x \) and \( y \) interchanged.

For \( \omega \in S^{d-1} \) introduce the collection of closed half-spaces

\[
\mathcal{H}_\omega := \{x \in \mathbb{R}^d : x \cdot \nu \geq 0\}.
\]

Lemma 2.2. Let \( f \in L^1_+(D, m) \) and \( \omega \in S^{d-1} \). For any \( H \in \mathcal{H}_\omega \),

\[
\|f_H - C_\omega f\|_{L^1(D, m)} \leq \|f - C_\omega f\|_{L^1(D, m)} \leq \|f - C_\omega f\|_{L^1(D, m)}
\]

with strict inequality if

\[
m(\{f \circ \tau_H > f\}) > 0.
\]
For each \( x \), this means that Lemma 2.1, Lemma 2.3.

Lemma 3.9.

The next lemma is a spherical cap symmetrisation counterpart to \([2]\) Lemma 6.3, and extends \([7]\) Lemma 3.9.

**Lemma 2.3.** Let \( f \in L^1(\mathbb{D}, m) \) and \( \omega \in S^{d-1} \) and assume that \( f \neq C_\omega f \). Then there exists \( H \in \mathcal{H}_\omega \) such that

\[
\|f_H - C_\omega f\|_{L^1(\mathbb{D}, m)} < \|f - C_\omega f\|_{L^1(\mathbb{D}, m)}.
\]

**Proof.** For shortness, write \( f^* \) for \( C_\omega f \). As \( f \neq f^* \) there exists \( t > 0 \) such that

\[
m\{\{f > t\} \setminus \{f^* > t\}\} > 0.
\]

It follows that the sets \( A := \{f \leq t < f^*\} \) and \( B := \{f^* \leq t < f\} \) are disjoint and have identical positive \( m \)-measure.

We claim that there exists \( H \in \mathcal{H}_\omega \) such that \( m(A \cap \tau_H B) > 0 \). Taking this as read, on \( A \cap \tau_H B \) we have that \( f^* > t \geq f^* \circ \tau_H \) so that \( A \cap \tau_H B \subseteq H \). Also, \( f \leq t < f^* \circ \tau_H \) there. In short, \( A \cap \tau_H B \subseteq \{f \circ \tau_H > f\} \cap H \). So \( m(\{f \circ \tau_H > f\}) > 0 \) and there is strict inequality in (2.2) by Lemma 2.2.

To prove the claim, we assume for a contradiction that \( m(A \cap \tau_H B) = 0 \) for all \( H \in \mathcal{H}_\omega \). Let \( F \) be a countable dense subset in \( S^{d-1} \cap \mathcal{H}_\omega \). Then

\[
m(A \cap \bigcup_{\nu \in F} \tau_H, B) = 0.
\]

Therefore, for all \( r \in (0, 1) \), it holds that

\[
\sigma_r(A_r \cap \tau_{H_r} B_r) = 0 \quad \text{for every } \nu \in F,
\]

except on a \( \lambda \)-null set \( N \). Here, \( \lambda \) stands for Lebesgue measure on the Borel sets in \( \mathbb{R} \), and \( A_r := A \cap \partial B(0, r) \) for the section of \( A \) (likewise for \( B_r \)). Let \( \nu \in S^{d-1} \cap \mathcal{H}_\omega \) with corresponding reflection \( \tau = \tau_{H_r} \). Select a sequence \( \nu_j \) in \( F \) which converges to \( \nu \) in \( S^{d-1} \). Write \( \tau_j \) for the reflection associated to closed half-space \( H_{\nu_j} \). For \( r \in (0, 1) \setminus N \),

\[
|\sigma_r(A_r \cap \tau B_r) - \sigma_r(A_r \cap \tau_{H_r} B_r)| \leq \|\chi_B - \chi_B \circ \tau \circ \tau_{H_r}\|_{L^1(\partial B(0, r), \sigma_r)},
\]

and this latter converges to zero as \( j \to \infty \). This is due to the fact that the special orthogonal group \( SO(d) \) acts continuously on \( L^1(S^{d-1}, \sigma) \). We derive therefore that

\[
\sigma_r(A_r \cap \tau_{H_r} B_r) = 0 \quad \text{for every } \nu \in S^{d-1} \cap \mathcal{H}_\omega \quad (2.4)
\]

for all \( r \in (0, 1) \setminus N \).

To conclude the argument, choose \( r \in (0, 1) \setminus N \) such that \( \sigma_r(A_r) = \sigma_r(B_r) > 0 \). Use Lebesgue’s density theorem to select a density point \( x \) for \( A_r \) lying in \( A_r \), and choose \( y \) in \( B_r \) similarly. Then \( f^*(x) > t \geq f^*(y) \). So there exists \( \nu \in S^{d-1} \cap \mathcal{H}_\omega \) such that with \( \tau = \tau_{H_r} \) we have that \( \tau y = x \).

But this means that

\[
\lim_{\varepsilon \downarrow 0} \frac{\sigma_r(A_r \cap \tau B_r \cap B(x, \varepsilon))}{\sigma_r(\partial B(0, r) \cap B(x, \varepsilon))} = 1,
\]

for each \( x \), this means that Lemma 2.1, Lemma 2.3.
so that, in fact, $\sigma_r(A_r \cap \tau B_r) > 0$, contradicting (2.4). \hfill \Box

**Proof of Theorem 1.2** Assume for a contradiction that $f \neq C_\omega f$ for each $\omega \in S^{d-1}$. Then there exists $\omega \in S^{d-1}$ such that

$$\delta := \inf_{\nu \in S^{d-1}} \|f - C_\nu f\|_{L^1(D,m)} = \|f - C_\omega f\|_{L^1(D,m)} > 0.$$ 

By Lemma 2.3 there exists $H \in H_\omega$ such that

$$\|f_H - C_\omega f\|_{L^1(D,m)} < \|f - C_\omega f\|_{L^1(D,m)}.$$ 

It is plain that $f \neq f_H$. But also $f \circ \tau_H \neq f_H$, for otherwise,

$$\|f - C_\sigma f\|_{L^1(D,m)} = \|f_H - C_\omega f\|_{L^1(D,m)} < \|f - C_\omega f\|_{L^1(D,m)},$$

contradicting optimality of $\omega$. It follows by Theorem 2.1 that $J(f) < J(f_H)$ and this contradicts the optimality of $f$ in the expression for $\beta_t$. \hfill \Box

### 3 Preliminaries for the one-dimensional problem

In the remainder of the article we work in the one-dimensional setting where $D = (-1, 1)$. In this context, the corresponding Green operator $G$ has kernel given by

$$G(x,y) = \begin{cases} \frac{1}{2} (1-y)(1+x) & \text{for } x \leq y, \\ \frac{1}{2} (1+y)(1-x) & \text{for } x > y, \end{cases} \tag{3.1}$$

for $x, y \in D$. We record the useful inequality

$$|G(x,y) - G(x,x)| \leq |y-x| \text{ for all } x, y \in D, \tag{3.2}$$

for future use. As noted above, the torsion function $\psi := G \chi_D$ is given explicitly by $\psi(x) = (1/2)(1-x^2)$ for $x \in D$, and

$$\psi = 2/3. \tag{3.3}$$

The Green kernel may be bounded in terms of $\psi$; that is,

$$G(x,y) \leq \psi(x) \text{ for all } y \in D, \tag{3.4}$$

with fixed $x \in D$.

For $t \in (0, 2/3)$ introduce the shape space

$$U_t := \left\{ f = \chi_A : A \subseteq D \text{ is open and } (f, \psi) \leq t \right\}.$$ 

We may then write

$$\alpha_t = \sup \left\{ J(f) : f \in U_t \right\}. \tag{3.5}$$

For each $t \in (0, 2/3)$ and $m \in \mathbb{N}$ define $U_t^{(m)}$ to be the collection of all functions of the form $f = \chi_A$ where $A$ is a union of at most $m$ disjoint open intervals in $D$ with the additional requirement that $(f, \psi) \leq t$. We occasionally refer to the condition

$$\text{int } A = A. \tag{3.6}$$

We also introduce the variational problem

$$\alpha_t^{(m)} := \sup \left\{ J(f) : f \in U_t^{(m)} \right\}. \tag{3.7}$$

We now derive the crucial property that (3.7) attains its optimum.
Theorem 3.1. For each \( t \in (0, 2/3) \) and \( m \in \mathbb{N} \) there exists \( f \in U_t^{(m)} \) with \( (f, \psi) = t \) such that \( \alpha_t^{(m)} = J(f) \).

Proof. Let \((f_n)_{n \in \mathbb{N}}\) be a maximising sequence for \( \alpha_t^{(m)} \). Each \( f_n \) may be written in the form 
\[
 f_n = \sum_{j=1}^{n_k} \chi_{A_j}, \quad \text{for some } 1 \leq k_n \leq m \text{ where } A_j = (a_{nj}, b_{nj}) \text{ and }
\]
\[
 -1 \leq a_{n1} < b_{n1} \leq a_{n2} < b_{n2} \leq \cdots \leq a_{nk_n} < b_{nk_n} \leq 1.
\]

After selecting a subsequence if necessary we may suppose that \( k_n \) takes a fixed value \( k \) for some \( k \) between 1 and \( m \). On appeal to the Bolzano-Weierstrass theorem, we may assume (perhaps after discarding a subsequence) that \( a_{nj} \to a_j \) and \( b_{nj} \to b_j \) as \( n \to \infty \) where
\[
 -1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_k \leq b_k \leq 1.
\]

Let \( f := \sum_{j=1}^{k} \chi_{A_j} \) where \( A_j = (a_j, b_j) \). By the dominated convergence theorem, \((f_n)\) converges weakly to \( f \) in \( \mathcal{H} \).

Put \( u_n := Gf_n \) as before. Then the sequence \((u_n)\) is bounded in \( W_0^{1,2}(D) \) as in the proof of Theorem 3.1. By Morrey’s inequality (see [3] 5.6.2, for example),
\[
 \|u_n\|_{C^{0,1/4}(\overline{D})} \leq c \quad \text{for all } n \in \mathbb{N}
\]
for some finite constant \( c \); in particular,
\[
 |u_n(x) - u_n(y)| \leq c |x - y|^{1/2} \quad \text{for any } x, y \in \overline{D}
\]
and any \( n \in \mathbb{N} \). Thus, \((u_n)\) forms a bounded and equicontinuous sequence in \( C(\overline{D}) \). By the Arzela-Ascoli compactness criterion, we may assume that \((u_n)\) converges uniformly to some \( u \in C(\overline{D}) \) as \( n \to \infty \) after extracting a subsequence if necessary. Now continue the argument as in the proof of Theorem 3.1 to conclude that \( \alpha_t^{(m)} = J(f) \).

We now show that \((f, \psi) = t \). First note that \((f, \psi) \leq t \); this flows from the fact that \( f \) is a weak limit of elements in \( U_t^{(m)} \). Suppose for a contradiction that \((f, \psi) < t \). As \((f, \psi) < 2/3 \), in \([3,8]\) there must exist \( j = 0, \ldots, k \) such that \( b_j < a_{j+1} \) with the understanding that \( b_0 := -1 \) and \( a_{k+1} := 1 \). By choosing \( B \) to be a suitable (semi-)open interval in \([b_j, a_{j+1}]\) we can arrange that the function \( f_1 := f + \chi_B \) satisfies the requirement \((f_1, \psi) \leq t \) as well as \( J(f) < J(f_1) \). This contradicts the optimality of \( f \).

We now revisit the operation of polarisation in the one-dimensional setting. We use the letter \( P \) to signify the polarisation operator with respect to the closed half-space \([0, \infty)\). Thus, for \( f \in U_t \), the polarisation is defined by
\[
 Pf(x) := \begin{cases} 
 f(x) \vee f(-x) & \text{if } 0 \leq x < 1, \\
 f(x) \wedge f(-x) & \text{if } -1 < x < 0. 
\end{cases}
\]

Alternatively, suppose that \( f = \chi_A \) where \( A \) is an open subset of \( D \). Then \( Pf = \chi_{PA} \) where \( PA \) denotes the polarisation of the set \( A \); in other words,
\[
 PA = A \cap \tau A \bigcup (A \cup \tau A) \cap (0, 1)
\]
where \( \tau : D \to D \) stands for the reflection \( x \mapsto -x \). We shall sometimes refer to the symmetric resp. non-symmetric parts of \( PA \); that is,
\[
 A_1 := A \cap \tau A; \\
 A_2 := (A \cup \tau A) \cap (0, 1) \setminus A \cap \tau A.
\]

Lemma 3.1. Let \( f \in U_t \) for some \( t \in (0, 2/3) \). Then the following statements are equivalent:
(i) $f \in PU_1$;
(ii) $f = 1$ on $S := \{x \in (0, 1) : f(-x) = 1\}$.

**Proof.** Let $f \in PU_1$ so that $f = Pg$ for some $g \in U_t$. Let $x \in (0, 1)$ with $f(-x) = 1$. Then $1 = f(-x) = Pg(-x) = g(x) \land g(-x)$. So $g(x) = 1$ and $f(x) = Pg(x) = g(x) \lor g(-x) = 1$. On the other hand, suppose that $f = 1$ on $S$. For $x \in S$,

$$PF(x) = 1 \lor f(-x) = 1 = f(x) \quad \text{while} \quad PF(-x) = 1 \land f(-x) = f(-x),$$

and for $x \in (0, 1) \setminus S$,

$$PF(-x) = f(x) \lor 0 = f(x) \quad \text{while} \quad PF(-x) = f(x) \land 0 = f(-x).$$

In other words, $f = Pf$.

It is sometimes useful to polarise with respect to the closed half-space $(-\infty, 0]$. To distinguish between these two polarisations we use the notations $P_+, P_-$. In particular,

$$P_+ f(x) := \begin{cases} f(x) \land f(-x) & \text{if} \quad 0 < x < 1, \\ f(x) \lor f(-x) & \text{if} \quad -1 < x \leq 0. \end{cases}$$

(3.12)

**Lemma 3.2.** Let $f = \chi_A \in P_+ U_t^{(m)}$ for some $m \in \mathbb{N}$ and $t \in (0, 2/3)$ where $A$ satisfies condition (3.7). Put $g := \chi_B$ where $B := D \setminus \overline{A}$. Then $g$ is an $m$-version of $1 - f$ and $g \in P_- U_{3/2-t}$.

**Proof.** We may suppose that $A = \bigcup_{j=1}^k A_j$ for some $1 \leq k \leq m$ and $A_j = (a_j, b_j)$ with $-1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq 1$.

We use the criterion in Lemma 3.1. Let $x \in (-1, 0)$ such that $g(-x) = 1$. We first note that $x$ cannot be a boundary point (that is, $x \not\in \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$). For if it is, then either $-x$ is a boundary point or $-x \in A$. This is due to the fact that $f$ is polarised to the right. In either case, we obtain the contradiction that $g(x) = 0$. We want to show that $g(x) = 1$ so suppose on the contrary that $g(x) = 0$. Then for $y = -x \in (0, 1)$, it holds that $f(-y) = 1$, but $f(y) = 0$. This counters the fact that $f \in P_+ U_t$ by the criterion.

4 A (non-)optimality criterion

In this section we develop a (non-)optimality criterion for configurations $f$ in $U_t^{(m)}$. Given $f \in U_t$ define $u := Gf$. It is known that $D(-\Delta) = W^{1,2}_0(D) \cap W^{1,2}(D)$. Thus, $u \in W^{2,2}(D)$ and by a Sobolev inequality (see [3], 5.6.3 for example), $u$ belongs to the Hölder space $C^{1,1/2}(\overline{D})$. Define

$$h = h_f := \frac{u}{\psi},$$

Then $h \in C(D)$ and by l’Hôpital’s rule,

$$h(-1) = \lim_{x \to -1} \frac{u'(x)}{-x} = u'(-1),$$

and similarly $h(1) = -u'(1)$ at the right-hand end-point. In short, $h \in C(\overline{D})$.

**Lemma 4.1.** Suppose that $f = \chi_A$ for some open subset $A$ in $D$. Let $u \in C(D)$. Given $a \in \overline{A} \cap D$, put $A_\eta := [a - \eta, a + \eta]$ for $\eta > 0$ small. Then

(i) $\lim_{\eta \to 0} \frac{\int_{\partial A_\eta} \left(\frac{\chi_{A_\eta}(u)}{\chi_{A_\eta}(\psi)} - h(a)\right)}{\int_{A_\eta} \psi} = h(a)$;
(ii) \( \lim_{\eta \to 0} \frac{(f \chi_{A_\eta}, G(f \chi_{A_\eta}))}{(f \chi_{A_\eta}, \psi)} = 0. \)

Proof. (i) Notice that \( A \cap (a - \eta, a + \eta) \neq \emptyset \) for each \( \eta > 0 \). Consequently, \( f \chi_{A_\eta}, 1 = m(A \cap A_\eta) > 0 \) for each \( \eta > 0 \) (small) and likewise for \( (f \chi_{A_\eta}, \psi) \). Write

\[
\frac{(f \chi_{A_\eta}, u)}{(f \chi_{A_\eta}, \psi)} = \frac{u(a)(f \chi_{A_\eta}, 1) + (f \chi_{A_\eta}, u - u(a))}{\psi(a)(f \chi_{A_\eta}, 1) + (f \chi_{A_\eta}, \psi - \psi(a))} = h(a) + \frac{\psi(a)\zeta_1 - u(a)\zeta_2}{\psi(a)(\psi(a) + \zeta_2)}
\]

where

\[
\zeta_1 := \frac{(f \chi_{A_\eta}, u - u(a))}{(f \chi_{A_\eta}, 1)}
\]

\[
\zeta_2 := \frac{(f \chi_{A_\eta}, \psi - \psi(a))}{(f \chi_{A_\eta}, 1)}
\]

Both these last vanish in the limit \( \eta \to 0 \) and this leads to the identity.

(ii) From the estimate \( 3.4 \), for \( \eta > 0 \) small,

\[
\psi^{-1} G[f \chi_{A_\eta}] \leq \psi^{-1} G[f \chi_{A_\eta}] \leq 2 \eta,
\]

and this establishes the limit. \( \square \)

With this preparation in hand we arrive at the crucial (non-)optimality condition.

**Theorem 4.1.** Let \( t \in (0, 2/3) \), \( m \in \mathbb{N} \) and \( f = \chi_A \in U_t^{(m)} \). Assume that \( A \) satisfies condition \( \text{(3.7)} \). Suppose that \( a, b \in D \) with \( a \neq b \) such that

(i) \( h(a) < h(b) \);

(ii) \( a \in \partial A \);

(iii) \( b \in \partial A \).

Then there exists \( f_1 \in U_t^{(m)} \) such that \( J(f) < J(f_1) \).

Proof. Write \( f \) in the form \( f = \sum_{j=1}^k \chi_{A_j} \) for some \( 1 \leq k \leq m \) where \( A_j = (a_j, b_j) \) and

\(-1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq 1.\)

Given \( \eta > 0 \) put \( A_\eta := [a - \eta, a + \eta] \) and \( B_\eta := (b - \eta, b + \eta) \). Set \( g := 1 - f \). The functions

\[
\eta \mapsto (\chi_{A_\eta}, f, \psi) \quad \text{and} \quad \eta \mapsto (\chi_{B_\eta}, g, \psi)
\]

are strictly increasing at least for \( \eta > 0 \) small. For \( \varepsilon > 0 \) sufficiently small, there exist unique \( \eta > 0 \) and \( \zeta > 0 \) depending upon \( \varepsilon \) such that

\( \varepsilon = (\chi_{A_\eta}, f, \psi) = (\chi_{B_\eta}, g, \psi). \)

Define

\( f_\varepsilon := f - f \chi_{A_\eta} + g \chi_{B_\eta}. \)

Then \( f_\varepsilon \in U_t^{(m)} \) for \( \varepsilon > 0 \) small. Now

\[
J(f_\varepsilon) - J(f) = (f_\varepsilon, G f_\varepsilon) - (f, G f)
\]

\[
= (f_\varepsilon - f, G [f_\varepsilon + f])
\]

\[
= (-f \chi_{A_\eta}, g \chi_{B_\eta}, G [2 f - f \chi_{A_\eta} + g \chi_{B_\eta}])
\]

\[
= 2 (g \chi_{B_\eta}, u) - 2 (f \chi_{A_\eta}, u) + (g \chi_{B_\eta} - f \chi_{A_\eta}, G [g \chi_{B_\eta} - f \chi_{A_\eta}]) \quad (4.2)
\]

where \( u = G f \) as usual. Thus, by Lemma 4.1 (with the help of the Cauchy-Schwarz inequality to deal with the cross-terms),

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \left\{ J(f_\varepsilon) - J(f) \right\} = 2 (h(b) - h(a)) > 0.
\]

In particular, there exists \( \varepsilon > 0 \) (small) such that \( f_1 := f_\varepsilon \in U_t^{(m)} \) satisfies \( J(f_1) > J(f) \). \( \square \)
5 More on the (non-)optimality condition

In this section we verify condition (i) in Theorem 4.1 for some particular configurations $\mathbf{f}$ in $U_t$.

**Lemma 5.1.** Let $\mathbf{f} \in U_t$ for some $t \in (0, 2/3)$ and set $u := G\mathbf{f}$. Then

(i) $u'(-1) + u'(1) = -\int_D x f \, dm$;

(ii) $u'(-1) - u'(1) = \int_D f \, dm$.

**Proof.** (i) Using $\psi' = -x$ and the integration-by-parts formula,

\[
\int_D x f \, dm = \int_D \psi' \, u'' \, dm
= \psi'(1) u'(1) - \psi'(-1) u'(-1) - \int_D \psi'' \, u' \, dm
= -u'(1) - u'(-1).
\]

(ii) This follows from $\int_D u'' \, dm = -\int_D f \, dm = u'(1) - u'(-1)$. \qed

Assuming that $f \not\equiv 0$, define $a, b \in \mathbb{R}$ by

\[
a := \inf \left\{ x > -1 : (\chi_{(-1, x)}, f) > 0 \right\},
1 - b := \inf \left\{ x > 0 : (\chi_{[1 - x, 1)}, f) > 0 \right\};
\]

so that $a \in [-1, 1)$ and $b \in (-1, 1]$.

**Lemma 5.2.** For each $\mathbf{f} \in U_t$ with $\mathbf{f} \not\equiv 0$,

(i) $h(y) = \frac{1}{1 - y} \int_D (1 - x) f \, dm$ for each $y \in [-1, a]$;

(ii) $h(y) = \frac{1}{1 + y} \int_D (1 + x) f \, dm$ for each $y \in [b, 1]$.

**Proof.** (i) Suppose that $a = -1$. By (4.1) and Lemma 5.1,

\[
h(-1) = u'(-1) = (1/2) \int_D (1 - x) f \, dm.
\]

Now suppose that $a \in (-1, 1)$. Using integration-by-parts,

\[
u(y) = \int_{(-1, y]} u' \, dm
= -\int_{(-1, y]} u' \psi'' \, dm
= -u'(y) \psi'(y) + u'(-1) \psi'(-1) + \int_{(-1, y]} u'' \psi' \, dm
= -u'(y) \psi'(y) + u'(-1)
\]

as $u'' = -f = 0$ m-a.e. on $(-1, a]$. For the same reason,

\[
u'(y) - u'(-1) = \int_{(-1, y]} u'' \, dm = 0.
\]

Therefore,

\[
u(y) = (1 - \psi'(y)) u'(-1) = \frac{1 + y}{2} \int_D (1 - x) f \, dm
\]

from which the statement is clear. Part (ii) follows in a similar fashion. \qed
Proposition 5.1. Let \( f \in PU_t \) for some \( t \in (0, 2/3) \). With \( a, b \) as in (5.7) assume that

(i) \(-1 < a < 0 < -a < b < 1\);
(ii) \( f = 1 \) \(m\text{-a.e.} \) on \((-a, b)\).

Then \( h(a) < h(b) \).

Proof. From Lemma 5.2

\[
 h(b) - h(a) = -\frac{a + b}{(1 - a)(1 + b)} \int_D f \, dm + \frac{2 - a + b}{(1 - a)(1 + b)} \int_D x f \, dm.
\]

Put \( S := \{x \in (0, 1) : f(-x) = 1\} \) as in Lemma 3.1. Then

\[
 \int_D x f \, dm = \int_S x \{f(x) - f(-x)\} \, m(dx) + \int_{(0,1)\setminus S} x f \, dm
\]

\[
 = \int_{(0,1)\setminus S} x f \, dm \geq \int_{(-a,b)} x f \, dm = (1/2)(b^2 - a^2)
\]

making use of (ii). Thus,

\[
(2 - a + b) \int_D x f \, dm \geq (1/2) (2 - a + b)(b^2 - a^2) > b^2 - a^2 \geq (a + b) \int_D f \, dm
\]

and hence \( h(b) - h(a) > 0 \).

In the next two sections, we show non-optimality of polarised configurations in three broad cases.

6 Two non-symmetric cases

Let \( t \in (0, 1/3] \) and imagine a configuration polarised to the right that charges the left-hand interval \((-1, 0)\) but which is not symmetric under reflection in the origin. We show this is non-optimal.

Lemma 6.1. Let \( m \in \mathbb{N} \) and \( t \in (0, 1/3] \). Suppose that \( f \in U_t^{(m)} \) satisfies the properties

(i) \( f = Pf \);
(ii) \( f, \chi_{(-1,0)} > 0 \).
(iii) \( f, \chi_{(0,1)} > (f, \chi_{(-1,0)}) \).

Then there exists \( f_1 \in U_t^{(m)} \) with the property that \( f_1 = Pf_1 \) such that \( J(f) < J(f_1) \).

Proof. We may assume that \( f = \chi_A \) where \( A \) satisfies condition 3.12. We may then write \( f \) in the form described at the beginning of the proof of Theorem 4.1. By (ii), \( a_1 < 0 \); and by (i), \( b_k \geq -a_1 \).

Case (a): \(-1 < a_1 \) and \( b_k < 1 \). Then, in fact, \(-1 < a_1 < 0 < -a_1 \leq b_k < 1 \). Put

\[
k_1 := \min \{ j = 1, \ldots, k : b_j \geq -a_1 \}.
\]

Suppose first of all that \(-a_1 = b_{k_1} \). Decompose \( A \) into its symmetric and non-symmetric parts \( A_1 \) and \( A_2 \) as in 3.11. By (iii), \( A_2 \neq \emptyset \). Write \( f_1 := \chi_{A_1} \) and \( f_2 := \chi_{A_2} \). By symmetry, \( h_{f_1}(a_1) = h_{f_2}(-a_1) \). Further, \( h_{f_1}(a_1) < h_{f_2}(-a_1) \), this being a consequence of 4.1. Therefore, as \( h = h_{f_1} + h_{f_2} \), we obtain \( h(a_1) < h(-a_1) \). The conclusion follows with an application of Theorem 4.1.
If \(-a_i \neq b_i\) then \(-a_i < b_i\), and \(f = 1\) on \((-a_i, b_i)\). If \(k_i = k\) then \(h(a_1) < h(b_k)\) by Proposition 5.1. On the other hand, if \(k_1 < k\) define

\[
f_1 := \sum_{j=1}^{k_1} \chi_{A_j} \quad \text{and} \quad f_2 := \sum_{j=k_1+1}^{k} \chi_{A_j},
\]

By Proposition 5.1, \(h_{f_1}(a_1) < h_{f_2}(b_k)\). It can be seen from the representation in Lemma 5.2 that \(h_{f_2}\) is increasing on \([-1, a_{k_1+1}]\). In sum, then, \(h(a_1) < h(b_{k_1})\). Now apply Theorem 4.1 once more.

**Case (b):** \(-1 < a_1\) and \(b_k = 1\). In this situation, \(-1 < a_1 < 0 < -a_1 < b_k = 1\). Define \(k_1\) as before. The case \(k_1 < k\) may be dealt with in a similar way to case (a) above. So assume that \(k_1 = k\). As \(f\) is polarised to the right, the interval \((-b_1) \cup (0, -a_1)\) must sit inside \(A\) and so it must hold that \(a_k < -a_1\). In case \(t \in (0, 1/3)\), it must also hold that \(0 < a_k\). The situation \(t = 1/3\) and \(a_k = 0\) forces \((f, \chi_{(-1, 0)}) = 0\) contradicting (ii). In either case, therefore, \(0 < a_k < -a_1\) and \(k \geq 2\).

Consider the function \(g := \chi_B\) where \(B := D \setminus \mathbb{T}\). By Lemma 6.2 \(g \in P_{U_{3/2\cdot t}}\). Thus,

\[-1 < a_1 < -a_k < 0 < a_k < b_k = 1,\]

and \(g = 1\) just to the right of \(-a_k\) as \(g\) is polarised to the left. This situation corresponds to the one described at the start of the consideration of this case but for \(g\) instead of \(f\). Use the fact that \(h_g = 1 - h_f\).

**Case (c):** \(a_1 = -1\). Then \(b_k = 1\) and \(a_k \leq -b_1\) as \(f = Pf\). Apply the arguments in case (a) to the function \(g\).

We now take \(t \in (0, 1/3)\) and imagine a configuration that lies entirely in the right-hand interval \((0, 1)\) but that has not yet been pushed rightwards to the maximum extent. We again show non-optimality.

**Lemma 6.2.** Let \(m \in \mathbb{N}\) and \(t \in (0, 1/3)\). Suppose that \(f \in U_t^{(m)}\) satisfies the properties

- (i) \((f, \chi_{(-1, 0)}) = 0;\)
- (ii) \((f, \chi_{(0, \xi)}) > 0.\)

Then there exists \(f_1 \in U_t^{(m)}\) with the property that \(f_1 = Pf_1\) such that \(J(f) < J(f_1)\).

**Proof.** Again take \(f = \chi_A\) where \(A\) satisfies condition 3.6 and suppose \(f\) takes the form described at the beginning of the proof of Theorem 4.1. By (i), \(a_1 \geq 0\) and by (ii), \(a_1 < \xi\). Therefore \(a_1 < b_1 < 1\); for otherwise, if \(b_1 = 1\) then

\[(f, \psi) > (\chi_{(a_1, 1)}, \psi) > (\chi_{(\xi, 1)}, \psi) = t.\]

Again borrowing the notation of Theorem 4.1 put \(f_1 := \chi_{A_1}\) and \(f_2 := \sum_{j=2}^{k} \chi_{A_j}\). By Lemma 5.2

\[h_{f_1}(a_1) = \frac{1}{1 - a_1} \int_{(a_1, b_1)} (1 - x) \, dm = \frac{b_1 - a_1}{1 - a_1} \left(1 - (1/2)(a_1 + b_1)\right)\]

and

\[h_{f_1}(b_1) = \frac{1}{1 + b_1} \int_{(a_1, b_1)} (1 + x) \, dm = \frac{b_1 - a_1}{1 + b_1} \left(1 + (1/2)(a_1 + b_1)\right).\]

A little algebra yields \(h_{f_2}(b_1) > h_{f_1}(a_1)\). Lemma 5.2 also indicates that \(h_{f_2}\) is monotone increasing on \([-1, a_2]\). Therefore, \(h(b_1) > h(a_1)\). The conclusion now follows with the help of Theorem 4.1. 

qed
7 The symmetric case

In the last of the three cases, we consider a configuration that is symmetric under reflection in the origin.

**Proposition 7.1.** Let \( m \in \mathbb{N} \) and \( t \in (0, 2/3) \). Suppose that \( f \in U_t^{(m)} \) satisfies the properties

(i) \( f = Pf \);
(ii) \( \langle f, \chi_{(-1, 0)} \rangle > 0 \);
(iii) \( \langle f, \chi_{(0, 1)} \rangle = \langle f, \chi_{(-1, 0)} \rangle \).

Then there exists \( f_1 \in U_t^{(m)} \) with the property that \( f_1 = Pf_1 \) such that \( J(f) < J(f_1) \).

Before embarking on the proof of Proposition 7.1, we require a number of supplementary results.

**Lemma 7.1.** Suppose that \( f = \chi_A \in U_t^{(m)} \) for some \( m \in \mathbb{N} \) and \( t \in (0, 2/3) \). Assume that \( A \) satisfies condition (3.16). Suppose that \( a \in \partial A \cap D \). Then

\[
\lim_{\eta \downarrow 0} \frac{(f_{\chi_A}, G[f_{\chi_A}])}{(f_{\chi_A}, \psi)^2} = \psi(a)^{-1},
\]

where \( A_\eta = [a - \eta, a + \eta] \) as before.

**Proof.** Write

\[
G[f_{\chi_A}](x) = \psi(a) (f_{\chi_A}, 1) + \{ \psi(x) - \psi(a) \} (f_{\chi_A}, 1) + r(x)
\]

where \( r(x) := (G(x, \cdot) - \psi(x), f_{\chi_A}) \) for \( x \in D \). Since \( \psi(x) = G(x, x) \), the estimate (5.22) gives

\[
| (f_{\chi_A}, r) | \leq 2\eta (f_{\chi_A}, 1)^2.
\]

Forming the inner product we obtain

\[
(f_{\chi_A}, G[f_{\chi_A}]) = \psi(a) (f_{\chi_A}, 1)^2 + (f_{\chi_A}, \psi - \psi(a)) (f_{\chi_A}, 1) + (f_{\chi_A}, r).
\]

It is clear from this that

\[
\lim_{\eta \downarrow 0} \frac{(f_{\chi_A}, G[f_{\chi_A}])}{(f_{\chi_A}, 1)^2} = \psi(a).
\]

A short step leads to the assertion.

**Lemma 7.2.** Suppose that \( f = \chi_A \in U_t^{(m)} \) for some \( m \in \mathbb{N} \) and \( t \in (0, 2/3) \) and that \( A \) satisfies condition (3.16). Suppose that \( a, b \in D \) with \( a \neq b \) such that both \( a \in \partial A \) and \( b \in \partial A \). Put \( g := 1 - f \). Given \( \varepsilon > 0 \) sufficiently small there exist unique \( \eta > 0 \) and \( \zeta > 0 \) depending upon \( \varepsilon \) such that \( \varepsilon = (f_{\chi_A}, \psi) = (g_{\chi_B}, \psi). \) Then

\[
\lim_{\varepsilon \downarrow 0} \frac{(f_{\chi_A}, G[g_{\chi_B}])}{\varepsilon^2} = \frac{G(a, b)}{\psi(a) \psi(b)}.
\]

**Proof.** Write

\[
G[g_{\chi_B}](x) = (G(x, \cdot), g_{\chi_B}) = G(a, b) (g_{\chi_B}, 1) + r(x)
\]

where \( r(x) := (G(x, \cdot) - G(a, b), g_{\chi_B}) \). For \( x \in A_\eta \) and \( y \in B_\zeta \),

\[
| G(x, y) - G(a, b) | \leq \eta + \zeta
\]
by (4.2). Consequently,

\[ |(f_{XA}, r)| \leq (\eta + \zeta) (f_{XA}, 1) (g_{XB}, 1). \]

Now,

\[ (f_{XA}, G[g_{XB}]) = G(a, b) (f_{XA}, 1) (g_{XB}, 1) + (f_{XA}, r), \]

from which we derive

\[ \lim_{\varepsilon \downarrow 0} \frac{(f_{XA}, G[g_{XB}])}{(f_{XA}, 1) (g_{XB}, 1)} = G(a, b), \]

and the conclusion follows straightforwardly.

**Lemma 7.3.** Let \( b \in (0, 1) \) and \( a := -b \). Let \( \eta > 0 \) small and define \( \zeta = \zeta(\eta) \) via the relation

\[ (\chi_{[\eta, \eta + \eta]}, \psi) = (\chi_{[b, b + \zeta]}, \psi). \]

Then \( \zeta \) depends smoothly upon \( \eta \) in a neighbourhood of \( \eta = 0 \) and

\[ \zeta = \eta + \frac{2b}{1 - b^2} \eta^2 + O(\eta^3) \]

in the limit \( \eta \downarrow 0 \).

**Proof.** A short computation gives that

\[ (\chi_{[\eta, \eta + \eta]}, \psi) = (1/2) \left\{ (1 - a^2) \eta - a \eta^2 - (1/3) \eta^3 \right\}. \]

Define smooth functions \( f, g : \mathbb{R} \to \mathbb{R} \) by

\[ f(\eta) := (1 - b^2) \eta + b \eta^2 - (1/3) \eta^3 \text{ and } g(\zeta) := (1 - b^2) \zeta - b \zeta^2 - (1/3) \zeta^3. \]

Now \( f'(0) = g'(0) = 1 - b^2 > 0 \). In particular, \( f \) is strictly increasing in a neighbourhood of \( \eta = 0 \) and \( g \) possesses a local smooth inverse \( h \) in the neighbourhood of \( \zeta = 0 \) by the inverse function theorem. Note that \( \zeta \) is characterised by the relation \( g(\zeta) = f(\eta) \) for \( \eta > 0 \) small. Thus \( \zeta = (h \circ f)(\eta) \) and depends smoothly upon \( \eta \) for \( \eta > 0 \) small. Implicit differentiation yields

\[ \zeta'(0) = 1 \text{ and } \zeta''(0) = \frac{2b}{1 - b^2} \text{ Taylor’s theorem with remainder then yields the expansion.} \]

**Proof of Proposition 7.1.** We may suppose that \( f = \chi_A \) where \( A \) satisfies condition (3.3). Define \( a \) as in (5.1). Assume in the first instance that \( a \in (-1, 0) \). Put \( b := -a \). Conditions (i)-(iii) entail that \( f \) is even. In particular, \( a, b \in \partial A \). As in Lemma 7.3 we write

\[ \varepsilon = (f_{XA}, \psi) = (g_{XB}, \psi) \]

for \( \varepsilon > 0 \) small. We aim to show that \( J(f_{\varepsilon}) - J(f) > 0 \) at least for \( \varepsilon > 0 \) small as in Theorem 4.1 and we shall borrow notation without comment from its proof. We first claim that (see (4.2))

\[ \lim_{\varepsilon \downarrow 0} \frac{J(f_{\varepsilon}) - J(f)}{\varepsilon^2} = \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} \left\{ 2 (g_{XB}, u) - 2 (f_{XA}, u) + (g_{XB} - f_{XA}) \right\}
\]

\[ = \frac{2}{\psi(b)^2} \{ b \psi(b) + u'(b) + b(1 - b) \}. \]

As \( u \in C^{1,1/2}(\overline{D}) \) we have that

\[ (f_{XA}, u) = u(a) \eta + u'(a) (1/2) \eta^2 + O(\eta^{5/2}). \]
Also, with the help of Lemma 7.3
\[
(g \chi_{B_{t}}, u) = u(b) \zeta + u'(b) (1/2) \zeta^2 + O(\zeta^{5/2})
\]
\[
= u(b) \left\{ \eta + \frac{2 b}{1 - b^2} \eta^2 \right\} + u'(b) (1/2) \eta^2 + O(\eta^{5/2})
\]
\[
= u(b) \eta + \left\{ \frac{2 b}{1 - b^2} u(b) + \frac{1}{2} u'(b) \right\} \eta^2 + O(\eta^{5/2}).
\]
Now \( u'(a) = -u'(b) \) because \( u \) is even so
\[
\lim_{\varepsilon \to 0} \frac{(g \chi_{B_{t}}, u) - (f \chi_{A_{t}}, u)}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{(g \chi_{B_{t}}, u) - (f \chi_{A_{t}}, u)}{\eta^2} = \frac{1}{\psi(b)^2} \{ b h(b) + u'(b) \}.
\]
On the other hand, from Lemmas 7.1 and 7.2
\[
\lim_{\varepsilon \to 0} \frac{(g \chi_{B_{t}} - f \chi_{A_{t}}) G[ (g \chi_{B_{t}} - f \chi_{A_{t}}) ]}{\varepsilon^2} = \frac{2}{\psi(a)} - \frac{2 G(a, b)}{\psi(a) \psi(b)} = \frac{2 b (1 - b)}{\psi(b)^2}
\]
as \( \psi(b) - G(a, b) = b (1 - b) \). The combination of these identities establishes the claim 7.1.

We now show that the expression in 7.1 is positive. From Lemma 5.2 and the even property of \( f \),
\[
h(b) = \frac{1}{1 + b} \int_{D} f \ dm
\]
and from Lemma 5.1
\[
u'(b) = u'(1) = -\frac{1}{2} \int_{D} f \ dm.
\]
So we may write
\[
b h(b) + u'(b) + b (1 - b) = \left\{ \frac{b}{1 + b} - \frac{1}{2} \right\} \int_{D} f \ dm + b (1 - b)
\]
\[
\geq \left\{ \frac{b}{1 + b} - \frac{1}{2} \right\} 2 b + b (1 - b)
\]
\[
= b^2 (1 - b) \frac{1}{1 + b} > 0.
\]
The conclusion now follows by Theorem 4.1.

The case \( a = -1 \) may be dealt with by applying the above argument (with appropriate modifications) to \( g := 1 - f \). \( \Box \)

8  The main result

We are now in a position to prove the main result Theorem 1.3.

**Theorem 8.1.** Let \( t \in (0, 1/3] \) and \( m \in \mathbb{N} \) and put \( A_{t} := (\xi_{t}, 1) \) with \( \xi_{t} \) as in (1.3). Then \( \alpha_{t}^{(m)} = J(f) \) where \( f = \chi_{A_{t}} \).

**Proof.** By Theorem 5.1 there exists \( f \in U_{t}^{(m)} \) such that \( \beta_{t}^{(m)} = J(f) \). Now \( U_{t}^{(m)} \) is closed under polarisation. So \( P \) \( f \in U_{t}^{(m)} \) and \( J(f) \leq J(Pf) \) by Theorem 4.1. We may assume therefore that \( f = Pf \).

Assume that \( f, \chi_{(-1, 0)} > 0 \). Since \( f = Pf \) it must be the case that \( f, \chi_{(0, 1)} \geq (f, \chi_{(-1, 0)}) \) (in consequence of Lemma 5.1). By Lemma 5.1 and Proposition 7.1 there exists \( f_{1} \in U_{t}^{(m)} \) with
the conclusion that \( f_1 = Pf_1 \) and \( J(f) < J(f_1) \). This contradicts the optimality of \( f \). We conclude that \( (f, \chi_{(-1, 0)}) = 0 \).

If \( t = 1/3 \) this compels \( f = \chi_{(0, 1)} \) bearing in mind that \( f \in U^{(m)}_{1/3} \) and \( f = Pf \). So let us now take \( t \in (0, 1/3) \). Suppose that \( (f, \chi_{(0, t)}) > 0 \). Then the requirements of Lemma \([6.2]\) are satisfied and hence there exists \( f_1 \in U^{(m)}_t \) with the property that \( f_1 = Pf_1 \) such that \( J(f) < J(f_1) \). Again this contradicts optimality. Hence, \( (f, \chi_{(0, t)}) = 0 \) in fact, \( (f, \chi_{(-1, t)}) = 0 \). As \( f \in U^{(m)}_t \) we draw the conclusion that \( f = \chi_{A_t} \).

**Corollary 8.1.** Let \( t \in (0, 1/3) \). Then \( \alpha_t = J(f) \) where \( f = \chi_{A_t} \).

**Proof.** Let \( f \in U_t \). By Lindelöf’s theorem, we may write \( f = \chi_A \) where \( A = \bigcup_{k=1}^{\infty} A_k \) is a countable union of disjoint open intervals \( A_k \) in \( D \). Put \( f_n := \sum_{k=1}^{\infty} \chi_{A_k} \). By the monotone convergence theorem, \( J(f) = \lim_{n \to \infty} J(f_n) \). Note that \( f_n \in U^{(m)}_t \) and \( J(f_n) \leq J(\chi_{A_t}) \) on taking limits.

**Proof of Theorem 1.3** We only need to deal with the case \( t \in (1/3, 2/3) \) in view of Corollary 8.1. Let \( f \in V_t \) for such a \( t \). Then \( g := 1 - f \in V_{2/3 - t} \) and

\[
J(f) = 2(t - 1/3) + J(g) = 2(t - 1/3) + J(\chi_{A_{2/3-t}}) = J(1 - \chi_{A_{2/3-t}}) = J(\chi_{A_t}).
\]

This clinches the result in the final case.

**9 Application: maximum flux exchange flow**

In this section we prove Theorem 1.4.

**Proposition 9.1.** It holds that

\[
(i) \quad \gamma = 2a - (1/3)(1 - \lambda)^2 \quad \text{for} \quad \lambda \in (-1, 1); \\
(ii) \quad \gamma = \sup_{\lambda \in (-1, 1)} \gamma \lambda.
\]

**Proof.** (i). Fix \( \lambda \in (-1, 1) \). Let \( A \) be an open subset in \( D \). Suppose that \( u \) satisfies \((1.4)\) along with the flux-balance condition \((u, 1) = 0\). Put

\[
f = f_{A, \lambda} := \begin{cases} 
-(\lambda + 1) & \text{on} \ A, \\
-(\lambda - 1) & \text{on} \ D \setminus A,
\end{cases}
\]

Then \( u = Gf \). From the flux-balance condition and symmetry of the Green operator,

\[
0 = (1, Gf) = (\psi, f) = -(\lambda + 1)(\psi, \chi_A) - (\lambda - 1)(\psi, \chi_{D \setminus A});
\]

so that \( \lambda = (\psi, 1)^{-1}(\psi, \chi_{D \setminus A} - \chi_A) \) and \( (\psi, \chi_A) = \frac{1 - \lambda}{2}(\psi, 1) \). Moreover,

\[
(\chi_{D \setminus A}, u) = (G \chi_{D \setminus A}, f) = -(\lambda + 1)(G \chi_{D \setminus A}, \chi_A) - (\lambda - 1)(G \chi_{D \setminus A}, \chi_{D \setminus A})
\]

\[
= 2(\psi, 1)^{-1}\left\{ - (\psi, \chi_{D \setminus A})(G \chi_{D \setminus A}, \chi_A) + (\psi, \chi_A)(G \chi_{D \setminus A}, \chi_{D \setminus A}) \right\}
\]

\[
= 2(\psi, 1)^{-1}\left\{ (\psi, 1)(G \chi_A, \chi_A) - (\psi, \chi_A)^2 \right\}
\]

\[
= 2J(\chi_A) - (1/2)(1 - \lambda)^2(\psi, 1),
\]

upon writing \( \chi_{D \setminus A} = \chi_D - \chi_A \) on each occurrence in the penultimate line. Now simplify using \((3.3)\). These considerations lead to the reformulation \((i)\). The statement in \((ii)\) then follows immediately.
Proof of Theorem 1.4  Part (i) follows from Theorem 1.3. For \( t \in (0, 2/3) \),
\[
\alpha_t = 2(t - 1/3) + \alpha_{2/3-t},
\]
as can be seen from the proof of Theorem 1.3. Therefore, for \( \lambda \in (-1, 1) \),
\[
\gamma_{-\lambda} = 2 \alpha_{1+\lambda} - (1/3)(1 + \lambda)^2 = 2 \alpha_{1-\lambda} - (1/3)(1 - \lambda)^2 = \gamma_{\lambda}.
\]
We now show that \( \gamma_{\lambda} < \gamma_0 \) for each \( \lambda \in (0, 1) \).
Let \( \xi \in (-1, 1) \) and \( u := G f \) where \( f := \chi_{(\xi, 1)} \). Then
\[
u(x) = \begin{cases}
\frac{1}{8}(\xi - 1)^2(x + 1) & \text{if } -1 < x \leq \xi, \\
-\frac{1}{2}x^2 + \frac{1}{4}(\xi + 1)^2x + \frac{1}{2} - \frac{1}{4}(\xi + 1)^2 & \text{if } \xi \leq x < 1.
\end{cases}
\]
A computation leads to
\[
J(f) = -\frac{1}{8} \xi^4 + \frac{1}{6} \xi^3 + \frac{1}{4} \xi^2 - \frac{1}{2} \xi + \frac{5}{24}.
\]
Also (see (1.3)),
\[
\frac{1 - \lambda}{3} = (\psi, f) = \varphi(\xi) = \frac{1}{6} \left( 2 - 3 \xi + \xi^3 \right).
\]
Therefore,
\[
\gamma_{\lambda} = 2 \alpha_{1+\lambda} - (1/3)(1 - \lambda)^2 \\
= 2 J(f) - 3 \varphi(\xi)^2 \\
= \frac{11}{12} \xi^2 + \frac{1}{4} \xi^4 - \frac{1}{12} \xi^6 \\
=: h(\xi).
\]
Now \( h(0) = \frac{1}{12} \) and \( h(1) = 0 \) and \( h'(\xi) = -(1/2) \xi (1 - \xi^2)^2 < 0 \) for \( \xi \in (0, 1) \). This shows that \( \gamma_{\lambda} < \gamma_0 \) for each \( \lambda \in (0, 1) \) as desired. The result follows from this and (i) of the Theorem. \( \square \)

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