AN SDP APPROACH FOR SOLVING QUADRATIC FRACTIONAL PROGRAMMING PROBLEMS

VAN-BONG NGUYEN†, RUEY-LIN SHEU‡§, AND YONG XIA¶

Abstract. This paper considers a fractional programming problem (P) which minimizes a ratio of quadratic functions subject to a two-sided quadratic constraint. As is well-known, the fractional objective function can be replaced by a parametric family of quadratic functions, which makes (P) highly related to, but more difficult than a single quadratic programming problem subject to a similar constraint set. The task is to find the optimal parameter $\lambda^*$ and then look for the optimal solution if $\lambda^*$ is attained. Contrasted with the classical Dinkelbach method that iterates over the parameter, we propose a suitable constraint qualification under which a new version of the S-lemma with an equality can be proved so as to compute $\lambda^*$ directly via an exact SDP relaxation. When the constraint set of (P) is degenerated to become an one-sided inequality, the same SDP approach can be applied to solve (P) without any condition. We observe that the difference between a two-sided problem and an one-sided problem lies in the fact that the S-lemma with an equality does not have a natural Slater point to hold, which makes the former essentially more difficult than the latter. This work does not, either, assume the existence of a positive-definite linear combination of the quadratic terms (also known as the dual Slater condition, or a positive-definite matrix pencil), our result thus provides a novel extension to the so-called “hard case” of the generalized trust region subproblem subject to the upper and the lower level set of a quadratic function.

Key words. Quadratic fractional programming; Dinkelbach algorithm; Nonconvex quadratic programming; S-lemma; Semidefinite relaxation; Slater point; positive-definite matrix pencil; generalized trust region subproblem

AMS subject classifications. 90C09, 90C10, 90C20

1. Introduction. In this paper we study a single ratio quadratic fractional programming problem taking the following format:

$$\min_{\lambda^*} \left\{ \frac{f_1(x)}{f_2(x)} : x \in X \right\}$$

This research was supported by Taiwan National Science Council under grant 102-2115-M-006-010, by National Center for Theoretical Sciences (South), by National Natural Science Foundation of China under grants 11001006 and 91130019/A011702, and by the fund of State Key Laboratory of Software Development Environment under grant SKLSDE-2013ZX-13

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where $X = \{ x \in \mathbb{R}^n : u \leq g(x) \leq v \}$; $f_i(x) = x^T A_i x + 2 b_i^T x + c_i$, $i = 1, 2$; $g(x) = x^T B x + 2 d^T x + \alpha$. The matrices $A_1, A_2, B$ are assumed to be symmetric and can be indefinite, $u \in \mathbb{R} \cup \{-\infty\}$, $v \in \mathbb{R} \cup \{+\infty\}$ such that $X \neq \emptyset$. To avoid the denominator becoming 0, we call (P) well-defined if $f_2(x) > 0$ for all $x \in X$. In this paper, we only consider a well-defined (P), but characterize conditions under which (P) can be well-defined in the last section. Denote $x^*$ to be the optimal solution of (P) if it is attained, and $\lambda^*$ the infimum of the problem, which could be $-\infty$ when (P) is unbounded below. By setting $f_2(x) = 1$, problem (P) is reduced to the “interval bounded generalized trust region subproblem (I-GTRS)” [18] which is essentially a quadratic programming problem with two quadratic constraints (QP2QC). Problem (I-GTRS) was studied in [19] by Stern and Wolkowicz for a homogenous $g(x)$; in [25] by Ye and Zhang under a primal and dual Slater condition; and in [18] by Pong and Wolkowicz for a necessary and sufficient optimality condition with an algorithm solving the “regular case” (to be explained later). Due to the fractional structure in the objective, (P) is in general more difficult than (I-GTRS).

As is well-known, the fractional objective function can be replaced by a parametric family of quadratic functions. Dinkelbach [12] in 1967 proposed a family of subproblems parameterized by $\lambda$ :

$$\text{(P)}_{\lambda} \quad f(\lambda) = \inf \{ f_1(x) - \lambda f_2(x) : x \in X \}$$

(1.2)

and developed an iterative algorithm on $\lambda$ to find a value $\lambda_0$ such that $f(\lambda_0) = 0$. When $X$ is compact, it was shown that $\lambda_0 = \lambda^*$. Moreover, (P) and (P)$_{\lambda_0}$ share the same optimal solution set [12, 14, 26]. Applying the Dinkelbach method to solve (P) amounts to solving globally a sequence of (I-GTRS)’s. Each (I-GTRS) (P)$_{\lambda}$ could be unbounded below or unattainable. Otherwise, under the primal Slater condition:

**Assumption A**

$$\inf_{x \in \mathbb{R}^n} g(x) < u \leq v \leq \sup_{x \in \mathbb{R}^n} g(x),$$

a global optimal solution $x(\lambda)$ to (P)$_{\lambda}$ can be characterized with a Lagrange multiplier $\mu(\lambda)$ such that the first order condition $$(A_1 - \lambda A_2 - \mu(\lambda) B) x(\lambda) = -b_1 + \lambda b_2 + \mu(\lambda) d;$$ the second order condition $A_1 - \lambda A_2 - \mu(\lambda) B \succeq 0$; together with the complementarity become necessary and sufficient [18]. The real task is to find algorithmically the pair of saddle point $(x(\lambda), \mu(\lambda))$ for each $\lambda$, suppose they exist, from the set of optimality conditions. So far, existing methods such as SDP with a rank one decomposition procedure [25] or a matrix pencil secular function approach [18] must rely on the existence of a positive definite matrix pencil $A_1 - \lambda A_2 - \mu B > 0$ for some $\mu \in \mathbb{R}$. This is also known to be the dual Slater condition [25], the stability condition [16], or the “regular (ease)” case [19, 18]. We notice that, while the primal Slater
condition is quite natural and easy to satisfy, the dual Slater condition is very strict. A sufficient condition for the dual Slater condition is that at least one of the matrices $A_1 - \lambda A_2$ and $B$ is positive definite. A necessary condition is that $A_1 - \lambda A_2$ and $B$ can be simultaneously diagonalizable via congruence (SDC). Namely, there exists a nonsingular matrix $C$ (depending on $\lambda$) such that both matrices $C^T(A_1 - \lambda A_2)C$ and $C^TBC$ are diagonal. Therefore, assuming the dual Slater condition for each $(P)_\lambda$ is impractical. Nevertheless, there were some papers which solve quadratically constrained quadratic fractional problem using the iterative method. For example, Beck et al. [3] considered a special case of $(P)$ with $g(x) = x^T B x$, $B \succ 0$, $u \geq 0$, $v > 0$. Zhang and Hayashi [26] studied a CDT-type quadratic fractional problem subject to two quadratic constraints, one of which is a ball, by an iterative generalized Newton method for finding $f(\lambda_0) = 0$.

On the other hand, $\lambda^*$ could be directly computed via an exact semi-definite reformulation (SDR), rather than iteratively. In particular, Beck and Teboulle [5] considered an one-sided homogeneous constrained quadratic problem below:

$$(RQ) \inf \left\{ \frac{f_1(x)}{f_2(x)} : \|Lx\|^2 \leq \rho \right\}$$  \hspace{1cm} (1.3)

where $L \in \mathbb{R}^{r \times n}$ is a full row rank matrix and $\rho > 0$. Under some technical conditions, Problem $(RQ)$ was shown to possess a “hidden convexity” that it admits an exact SDR. Therefore, the optimal value $\lambda^*$ can be evaluated in a polynomial time. The result was later strengthened in [23] by Xia that the $(RQ)$ problem indeed admits an exact SDR without any condition. Moreover, it is attained if and only if the associated SDR (3.10) has a unique solution. Unfortunately, problems beyond $(RQ)$ are more complicate. An exact SDR is in general not available for $(P)$ even when $\|Lx\|^2 \leq \rho$ is relaxed to become a convex nonhomogeneous constraint $g(x) = x^T B x + 2d^T x + \alpha$. See Example 3.1 in Sect. 3 for an explanation.

Later, Beck and Teboulle proposed a framework that minimizes the ratio of two quadratic functions over $m$ quadratic inequalities [6]:

$$(QCRQ) \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} : g_i(x) = x^T B_i x + 2d_i^T x + \alpha_i \leq 0, i = 1, 2, \ldots, m \right\}.$$  \hspace{1cm} (1.4)

It covers quadratically constrained quadratic programming (QPQC) as a special case. It is known that (QPQC) is NP-hard and there is no surprise that an even more generic (QCRQ) can be studied only under very restrictive situations. Based on the homogenization technique, $(QCRQ)$ can be made homogeneous by substituting $x = \frac{y}{t}$, $t \neq 0$:

$$\inf \left\{ \frac{f_1^H(y,t)}{f_2^H(y,t)} : g_i^H(y,t) \leq 0, i = 1, 2, \ldots, m, t \neq 0 \right\},$$  \hspace{1cm} (1.5)
where \(f^H_1(y, t), f^H_2(y, t), g^H_i(y, t), i = 1, 2, ..., m\) are homogeneous versions of \(f_1(x), f_2(x), g_i(x), i = 1, 2, ..., m\), respectively. Notice that the homogenization yields Problem (1.5) which is valid only for \(t \neq 0\), but the non-triviality occurs normally in the case \(t = 0\) when homogenizing a quadratic system. Beck and Teboulle further relaxed \(t \neq 0\) to be \((y, t) \neq (0, 0)\) and considered a slightly different “mutated” problem.

\[
\inf_{(y, t) \neq (0, 0)} \left\{ \frac{f^H_1(y, t)}{f^H_2(y, t)} : g^H_i(y, t) \leq 0, i = 1, 2, ..., m \right\}, \tag{1.6}
\]

By imposing \(f^H_2(y, t) = 1\), (1.6) was proven to be equivalent to the following non-fractional problem:

\[
v(H) = \min_{(H)} f^H_1(z, s) \quad \text{s.t.} \quad f^H_2(z, s) = 1, \quad g^H_i(z, s) \leq 0 \tag{1.7}
\]

where \(v(\cdot)\) denotes the optimal value of the problem \((\cdot)\). Restricting \(s = 0\) in (1.7), a related problem \((H_0)\) is used as a reference to be compared with \((H)\):

\[
v(H_0) = \min_{(H_0)} f^H_1(z, 0) \quad \text{s.t.} \quad f^H_2(z, 0) = 1, \quad g^H_i(z, 0) \leq 0. \tag{1.8}
\]

Then, (QCRQ) was shown to have a tight semi-definite relaxation under the following three conditions:

\[
\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \succ 0; \tag{1.9}
\]

\[
v(H) < v(H_0); \tag{1.10}
\]

The semi-definite relaxation admits a rank-one optimal solution. \(\tag{1.11}\)

As we shall see later, the three assumptions (1.9)-(1.11) put (QCRQ) in a very rigid class. In Sect. 2.3 we provide two examples of (P), Examples 2.2 and 2.3 which violate at least (1.9) and (1.10) but can be solved by our method. The drawback of the direct method for finding \(\lambda^*\) “once for all” lies on the fact that there are not too many special cases of (P) that possess a hidden convexity. More sophisticated analysis is often necessary.

Our idea to compute \(\lambda^*\) relies on a new S-Lemma. See Sect. 2.2 Theorem 2.4. When the optimal solution \(x^*\) is an interior point of \(X = \{u \leq g(x) \leq v\}\), the case is somehow simple and we show that \(\lambda^*\) can be computed by an SDP. See Sect. 2.4 Theorem 2.3. Otherwise, \(x^*\) resides on one of the two boundaries satisfying \(g(x^*) = u\).
or \( g(x^*) = v \). In either case, to find \( \lambda^* \), one faces a parametric family of one equality-constrained quadratic programming problems (1.12). By a coordinate change, we need to only consider \( h(x) = x^T B x + 2d^Tx = 0 \). Then, we can compute \( \lambda^* \) also by an SDP provided the family (1.12) can be converted to the other one (1.13):

\[
\lambda^* = \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} \left| h(x) = 0 \right. \right\}
\]

\[
= \sup \left\{ \lambda : \left\{ x \in \mathbb{R}^n | \lambda > \frac{f_1(x)}{f_2(x)}, h(x) = 0 \right\} = \emptyset \right\}
\]

\[
= \sup \left\{ \lambda : \left\{ x \in \mathbb{R}^n | f_1(x) - \lambda f_2(x) < 0, h(x) = 0 \right\} = \emptyset \right\}
\]

\[
= \sup \left\{ \lambda : f_1(x) - \lambda f_2(x) + \mu h(x) \geq 0, \forall x \in \mathbb{R}^n, \mu \in \mathbb{R} \right\}
\]

\[
(1.12)
\]

\[
= \sup_{\lambda, \mu \in \mathbb{R}} \left\{ \lambda : \begin{pmatrix} A_1 - \lambda A_2 + \mu B & b_1 - \lambda b_2 + \mu d \\ c_1 - \lambda c_2 \end{pmatrix} \succeq 0 \right\}
\]

\[
(1.13)
\]

The equivalence of (1.12) and (1.13) is indeed a very strong statement since it requires the S-lemma of the equality version to hold for every parameter \( \lambda \). Moreover, since \( h(x) = 0 \) cannot have any Slater point, this variant of S-Lemma is more difficult to obtain than the inequality version with \( h(x) \leq 0 \). We show that, under the following constraint qualification for equality constraint:

**Assumption B** There exists \( \zeta \in X = \{ x \in \mathbb{R}^n : h(x) = 0 \} \) such that

\[
x^T B x = 0 \Rightarrow (B \zeta + d)^T x = 0,
\]

(1.12) and (1.13) can be made equivalence. In addition, Assumption B has an important feature that it relates merely to \( g(x) \) (or \( h(x) \)), not to the parametric family of functions \( f_1(x) - \lambda f_2(x) \). In contrast, the extended Finsler’s theorem [4], Thm A.2 can not apply as it requires a condition (in our format and notations)

\[
A_1 - \lambda A_2 \succeq \eta B, \text{ for some } \eta \in \mathbb{R}
\]

in which \( \eta \) might vary for different \( \lambda \)’s. We provide an example, Sect. 2.2 Remark (3.4), which can be solved by our extended S-lemma, while there is no \( \eta \) satisfying (1.15) right at the optimal value \( \lambda^* \). Assumption B is also more general than a condition imposed in [17] Prop. 3.1, where \( h(x) \) was assumed to be strictly convex or strictly concave. Compared with the dual Slater condition, Assumption B is easier to obtain. For example, if \( A_1 - \lambda A_2 = 0 \) and \( B \) is positive semidefinite, singular and \( d \) is in the range of \( B \), then Assumption B can be satisfied while the dual Slater condition is obviously violated. Consequently, some hard cases of (I-GTRS) that can not be solved due to lack of a positive definite matrix pencil can now be done under Assumption B.
The paper is organized as follows. In Section 2, we study Problem (P) under Assumption A. The first step of our algorithm tries to determine whether the optimal solution $x^*$ could lie in the interior of $X$, followed by checks on both boundaries otherwise. For each inspection, we use an SDP to compute a potential $\lambda^*$ and then verify whether $f(\lambda^*) = 0$ by solving a (constrained) quadratic programming problem $(P)_{\lambda^*}$. We show that (P) can be solved in polynomial time under the constraint qualification Assumption B, which is independent of the usual primal and dual Slater conditions. In Section 3, the one-sided (P) for which Assumption A is violated is treated. Our result is that the one-sided (P) can be completely solved in polynomial time without any condition. An interesting comparison between the two-sided original (P) and the one-sided case is elaborated in Remark 3.1. The (RQ) problem (1.3) as a special case of the one-sided (P) can now be resolved without any technical conditions. In Section 4, we characterize conditions for the ultimate assumption of (P) that the denominator function $f_2(x) > 0$ on $X$ such that (P) is well-defined. It turns out the well-definedness property can be related to simultaneous diagonalization via congruence. The final section concludes the paper.

2. Quadratic Fractional programming problem with two-sided quadratic inequality constraint. In this section, we first characterize conditions under which (P) is bounded from below and under which (P) can be attained. Then, we show how to compute $\lambda^*$ using a semi-definite programming approach. Some difficult cases of (P) are resolved with the help of a new version of S-Lemma under Assumption B, which is more powerful than the primal/dual Slater condition; a similar result in [17] Prop. 3.1; and the extended Finsler’s theorem [11, Thm A.2]. Examples are given to illustrate all the ideas.

2.1. Boundedness, attainment, and unconstrained cases. In fractional programming, it is often assumed that the feasible set $X$ is compact. In general, a well-defined (P) is not necessarily bounded from below and can not be always attained. The following two lemmas, generalizing some basic results in fractional programming, characterize completely the boundedness and the attainment properties of (P) without the compactness assumption. We omit the proof as the original compactness assumption was only used to guarantee that the optimal value of (P) is attained and each iteration of the Dinkelbach method is defined. The reader can refer to Dinkelbach’s original proof [12] or a more general discussion on a multi-ratios case. See for example [9][10][7][1][8].

Lemma 2.1 (The boundedness problem). Suppose that (P) is well defined. It is bounded below if and only if there exists a $\bar{\lambda} \in \mathbb{R}$ such that $f(\bar{\lambda}) \geq 0$. Furthermore, if
\( \lambda^* > -\infty, \) then

\[
\lambda^* = \max_{f(\lambda) \geq 0} \lambda.
\]

The following Example 2.1 shows that it is possible for a bounded (P) to have \( f(\lambda^*) > 0, \) in which case (P) is unattainable. That is, the optimal value \( \lambda^* \) can not be attained.

**Example 2.1.** It is easy to check

\[
\lambda^* = \inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + 1}{x_2^2 + 1} : g(x) = x_1^2 + 2x_3 - 1 \leq 0 \right\} = 0
\]

by letting \( x_1 = 0 \) and \( x_2 \) go to infinity. Solving its parametric problem

\[
f(\lambda) = \inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + 1 - \lambda(x_2^2 + 1)}{x_2^2} : g(x) = x_1^2 + 2x_3 - 1 \leq 0 \right\}
\]

we observe that \( f(\lambda^*) = f(0) = 1 > 0. \)

**Lemma 2.2 (The attainment problem).** Suppose that (P) is well defined. Then, \( \lambda^* = v(P) \) is attained at \( x^* \in X \) if and only if \( \lambda^* \) is a root of \( f(\lambda) = 0 \) and \( x^* \) is an optimal solution to \( (P)_{\lambda^*}. \)

**Remark 2.1.** In fact, Lemmas 2.1 and 2.2 hold for any well-defined fractional programming problem where the ratio of functions are not necessary to be quadratics, and the constraint set \( X \) can be arbitrary.

**Remark 2.2.** Due to Lemma 2.2, we can freely exchange and mention the two types of problems: either (P) or \( (P)_{\lambda^*} \) with \( f(\lambda^*) = 0. \)

In the following until the end of the section, we assume that problem (P) is always attained and satisfies Assumption A. All other cases not satisfying this assumption can be treated separately. When \( u \leq \inf_{x \in \mathbb{R}^n} g(x), \) the constraint \( u \leq g(x) \leq v \) is reduced to \( g(x) \leq v. \) Similarly, if \( \sup_{x \in \mathbb{R}^n} g(x) \leq v, \) then \( u \leq g(x) \leq v \) becomes just \( u \leq g(x). \) The two cases where \( g(x) \leq v \) or \( u \leq g(x) \) will be studied in next section.

If \( u \leq \inf g(x) \leq \sup g(x) \leq v, \) problem (P) is an unconstrained quadratic fractional programming problem. According to Lemma 2.1, the optimal value can be computed directly by

\[
\lambda^* = \max_{f(\lambda) \geq 0} \lambda
\]

\[
= \max \{ \lambda \in \mathbb{R} : \inf_{x \in \mathbb{R}^n} f_1(x) - \lambda f_2(x) \geq 0 \}
\]

\[
= \max \{ \lambda \in \mathbb{R} : f_1(x) - \lambda f_2(x) \geq 0, \ \forall x \in \mathbb{R}^n \}
\]

\[
= \max \left\{ \lambda \in \mathbb{R} : \left( A_1 - \lambda A_2 \right) b_1 - \lambda b_2, \ b_1^T - \lambda b_2^T \right\} \succeq 0, c_1 - \lambda c_2 \geq 0 \right\}.
\]
It indicates that an unconstrained (P), if not unbounded below, must be equivalent to the convex unconstrained problem: \( \inf_{x \in \mathbb{R}^n} f_1(x) - \lambda^* f_2(x) \). If (P) is attained, from Lemma 2.2, the optimal solution \( x^* \) can be also found by solving \((P)_{\lambda^*}\).

Now we elaborate how to solve (P) under Assumption A. First notice that the optimal solution \( x^* \) of \((P)_{\lambda^*}\) will be either an interior point of \( X = \{ u \leq g(x) \leq v \} \), or resides on one of the two boundaries satisfying \( g(x^*) = u \) or \( g(x^*) = v \). By the first order and the second order necessary conditions, \( x^* \) is an interior point only when \( A_1 - \lambda^* A_2 \succeq 0 \) and also satisfies \((A_1 - \lambda^* A_2)x^* = -(b_1 - \lambda^* b_2)\). Therefore, problem (P) can be analyzed by the following three (possibly overlapped) cases.

**Case 1.** \( f(\lambda^*) = 0, \ A_1 - \lambda^* A_2 \succeq 0 \) and

\[
\begin{aligned}
x^* \in S &= \{ x \in \mathbb{R}^n | (A_1 - \lambda^* A_2)x = -(b_1 - \lambda^* b_2) \} \\
\text{satisfying } u &\leq g(x^*) \leq v.
\end{aligned}
\]

**Case 2.** \( f(\lambda^*) = 0 \), and \( x^* \) solves

\[
\begin{aligned}
\inf_{x \in \mathbb{R}^n} f_1(x) - \lambda^* f_2(x) \\
\text{s.t. } g(x) &= u
\end{aligned}
\] (2.2)

**Case 3.** \( f(\lambda^*) = 0 \), and \( x^* \) solves

\[
\begin{aligned}
\inf_{x \in \mathbb{R}^n} f_1(x) - \lambda^* f_2(x) \\
\text{s.t. } g(x) &= v
\end{aligned}
\] (2.3)

Theorem 2.3 below shows that Case 1 can be directly solved as if an unconstrained (P), while Case 2 and Case 3 must do, namely to find \( \lambda^* \) and \( x^* \), with a new version of S-Lemma. See Theorem 2.4 below.

**THEOREM 2.3.** If \((\lambda^*, x^*)\) happens to satisfy Case 1, then \( \lambda^* \) can be computed by

\[
\lambda^* = \sup \left\{ \lambda \in \mathbb{R} : \begin{pmatrix} A_1 - \lambda A_2 & b_1 - \lambda b_2 \\ b_1^T - \lambda b_2^T & c_1 - \lambda c_2 \end{pmatrix} \succeq 0 \right\}. \quad (2.4)
\]

**Proof.** Since Case 1 is assumed, \( f_1(x) - \lambda^* f_2(x) \) is convex. By (2.1), \( x^* \) is also a global minimizer of the unconstrained quadratic problem

\[
f(\lambda^*) = \inf_{x \in \mathbb{R}^n} f_1(x) - \lambda^* f_2(x) = 0.
\]

Then, we have \( f_1(x) - \lambda^* f_2(x) \geq 0, \forall x \in \mathbb{R}^n \), which is equivalent to

\[
\begin{pmatrix} A_1 - \lambda^* A_2 & b_1 - \lambda^* b_2 \\ b_1^T - \lambda^* b_2^T & c_1 - \lambda^* c_2 \end{pmatrix} \succeq 0.
\]
To show that $\lambda^*$ is the largest one satisfying the matrix inequality (2.4), we suppose that there exists $\bar{\lambda} > \lambda^*$ also satisfying that matrix inequality:

$$
\begin{pmatrix}
A_1 - \bar{\lambda}A_2 & b_1 - \bar{\lambda}b_2 \\
\bar{b}_1^T - \bar{\lambda}\bar{b}_2^T & c_1 - \bar{\lambda}c_2
\end{pmatrix} \succeq 0.
$$

Then

$$f_1(x) - \bar{\lambda} f_2(x) \geq 0, \forall x \in \mathbb{R}^n.$$

Equivalently,

$$\frac{f_1(x)}{f_2(x)} \geq \bar{\lambda} > \lambda^*, \forall x \in \mathbb{R}^n,$$

which indicates that $f(\lambda^*) > 0$, a contradiction.

To apply Theorem 2.3, we first solve the SDP problem (2.4) to get a candidate value $\lambda^*$. If $f(\lambda^*) = 0$, it has satisfied the first criterion in Case 1. Since $A_1 - \lambda^* A_2 \succeq 0$, $(P)_{\lambda^*}$ is convex. Any unconstrained optimizer $x^* \in S$ satisfying $u \leq g(x^*) \leq v$ must also solve $(P)_{\lambda^*}$. Then, the value $\lambda^*$ computed by (2.4) is the optimal value of $(P)$ with the optimal solution $x^*$. Otherwise, if either $f(\lambda^*) > 0$ or there is no such $x^* \in S$ satisfying $u \leq g(x^*) \leq v$, Case 1 does not happen and we have to look for Case 2 and Case 3. That is,

$$\lambda^* = \inf \left\{ \frac{f_1(x)}{f_2(x)} \mid x \in X \right\} = \min \left\{ \lambda_1 \equiv \inf \left\{ \frac{f_1(x)}{f_2(x)} \mid g(x) = u \right\}; \lambda_2 \equiv \inf \left\{ \frac{f_1(x)}{f_2(x)} \mid g(x) = v \right\} \right\}. \quad (2.5)$$

It is possible that at least one of $\lambda_1$ and $\lambda_2$ is negative infinity, then $(P)$ is unbounded below. Another possibility is that, in (2.5), we have $\lambda_1 = \lambda_2$. Then, we need to check additionally which one, $f(\lambda_1) = \inf_{g(x) = u} \{ f_1(x) - \lambda_1 f_2(x) \} = 0$ or $f(\lambda_2) = \inf_{g(x) = v} \{ f_1(x) - \lambda_2 f_2(x) \} = 0$, holds in order to determine $\lambda^*$. If neither $f(\lambda_1) = 0$ nor $f(\lambda_2) = 0$, $(P)$ is unattainable according to Lemma 2.2. In the following subsection, we focus on solving the quadratic fractional programming problem subject to one quadratic equality constraint.

### 2.2. An extended S-Lemma with equality

Since Case 2 and Case 3 have the same pattern, we only discuss Case 2 in which $x^*$ satisfies $g(x^*) = u$. Without loss of generality, we define $h(x) = g(x) - u$ and assume that $h(0) = 0$. Otherwise, by replacing $x$ with $x + x'$ for some nonzero vector $x' \in \{ x \in \mathbb{R}^n : g(x) = u \}$, the change of coordinate makes $\tilde{h}(x) = h(x + x')$ satisfy $\tilde{h}(0) = h(x') = 0$. Then, the problem casted in the new coordinate system

$$\inf_{x \in \mathbb{R}^n} \left[ f_1(x) - \lambda^* \tilde{f}_2(x) = f_1(x + x') - \lambda^* \tilde{f}_2(x + x') \right]$$

s.t. $h(x) = h(x + x') = 0 \quad (2.6)$
is equivalent to (2.2) in the sense that if \( x^\ast \) is an optimal solution of (2.6), then \( x^\ast + x' \) is optimal to (2.2). Conversely, if \( x^\ast \) is optimal to (2.2), \( x^\ast - x' \) is optimal to (2.6). Therefore, we only have to deal with

\[
f(\lambda^\ast) = \inf_{x \in \mathbb{R}^n} f_1(x) - \lambda^\ast f_2(x) \quad \text{s.t.} \quad h(x) = 0\]

(2.7)

where \( h(x) = x^T B x + 2d^T x \) and \( \lambda^\ast \) is the optimal value of the following problem:

\[
\lambda^\ast = \inf_{x \in \mathbb{R}^n} \frac{f_1(x)}{f_2(x)} \quad \text{s.t.} \quad h(x) = 0.
\]

(2.8)

Theorem 2.4 below is an extended version of S-Lemma which, under Assumption B, converts the fractional programming problem (2.8) to an equivalent SDP problem (2.21). Assumption B plays the role of constraint qualification, which used to be the primal Slater condition when an inequality system \( f_1(x) - \lambda f_2(x) < 0, h(x) \leq 0 \) is otherwise considered. Naturally, \( h(x) = 0 \) does not possess any Slater point so that another type of constraint qualification like Assumption B is needed.

**Theorem 2.4 (Extended S-Lemma with equality).** Under Assumptions A and B, the following two statements are equivalent for each given \( \lambda \).

(i) The system

\[
f_1(x) - \lambda f_2(x) < 0, h(x) = 0
\]

(2.9)

is unsolvable.

(ii) There exists \( \mu \in \mathbb{R} \) such that

\[
f_1(x) - \lambda f_2(x) + \mu h(x) \geq 0, \forall x \in \mathbb{R}^n.
\]

Proof. Notice that statement (ii) trivially implies statement (i) without any condition. We only prove for the converse under Assumptions A and B. The proof will be presented in two cases: either \( B\zeta = 0 \) or \( B\zeta \neq 0 \).

(a) \( B\zeta = 0 \). Then Assumption B becomes

\[
x^T B x = 0 \Rightarrow d^T x = 0.
\]

(2.10)

We first rewrite system (2.9) as

\[
x^T (A_1 - \lambda A_2) x + 2(b_1^T - \lambda b_2^T) x + c_1 - \lambda c_2 < 0; \quad \text{(2.11a)}
\]

\[
x^T B x + 2d^T x \leq 0; \quad \text{(2.11b)}
\]

\[
x^T (-B) x - 2d^T x \leq 0, \quad \text{(2.11c)}
\]
which can be made homogeneous by introducing a new variable \( t \in \mathbb{R} \) as follows:

\[
x^T(A_1 - \lambda A_2)x + 2(b_1^T - \lambda b_2^T)xt + (c_1 - \lambda c_2)t^2 < 0; \quad (2.12a)
\]
\[
x^T Bx + 2d^T xt \leq 0; \quad (2.12b)
\]
\[
x^T(-B)x - 2d^T xt \leq 0. \quad (2.12c)
\]

We want to assert that if (2.11) is unsolvable, then (2.12) is unsolvable too. Suppose in contrary that (2.12) has a solution \((\bar{x}, \bar{t})\).

If \( \bar{t} \neq 0 \), by dividing both sides of (2.12a)-(2.12c) by \( \bar{t}^2 \), we see that \( \bar{x} \bar{t} \) is a solution to (2.11), which is a contradiction.

If \( \bar{t} = 0 \), system (2.12) becomes

\[
\bar{x}^T(A_1 - \lambda A_2)\bar{x} < 0; \quad (2.13a)
\]
\[
\bar{x}^T B\bar{x} \leq 0; \quad (2.13b)
\]
\[
\bar{x}^T(-B)\bar{x} \leq 0. \quad (2.13c)
\]

Inequalities (2.13b) and (2.13c) together imply that \( \bar{x}^T B\bar{x} = 0 \). According to (2.10), \( d^T \bar{x} = 0 \) and thus (2.11b) and (2.11c) are satisfied by \( \bar{x} \). Moreover, we observe that \( \beta \bar{x} \) is a solution to (2.13) for any \( \beta > 0 \). By (2.13a), \( \bar{x}^T(A_1 - \lambda A_2)\bar{x} < 0 \), we can then choose \( \beta \) large enough such that \( \beta \bar{x} \) satisfies (2.11a), and also (2.11b)-(2.11c).

Therefore, if the system (2.11) does not have a solution, the homogeneous system (2.12) must be also unsolvable.

The system (2.12) can be put into the quadratic form as follows:

\[
y^T P y < 0
\]
\[
y^T Q y \leq 0
\]
\[
y^T (-Q)y \leq 0
\]

with

\[
P = \begin{pmatrix}
A_1 - \lambda A_2 & b_1 - \lambda b_2 \\
b_1^T - \lambda b_2^T & c_1 - \lambda c_2
\end{pmatrix},
\]
\[
Q = \begin{pmatrix}
B & d \\
d^T & 0
\end{pmatrix}
\]

and

\[
y = \begin{pmatrix}
x \\
t
\end{pmatrix} \in \mathbb{R}^{n+1}.
\]

It is unsolvable if and only if

\[
\{ (y^T P y, y^T Q y) : y \in \mathbb{R}^{n+1} \} \cap \{ (a, b) \in \mathbb{R}^2 : a < 0, b = 0 \} = \emptyset.
\]

Notice that \( C \) is convex (see, e.g. [11]), \( 0 \in C \), and \( D \) is also convex. If we express

\[
D = \left\{ z = \begin{pmatrix}
a \\
b
\end{pmatrix} \in \mathbb{R}^2 : e_1^T z < 0, e_2^T z \leq 0, e_3^T z \leq 0 \right\}
\]

with \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \), by the separation theorem [2], there exist \( \eta_1, \eta_2 \in \mathbb{R}, \eta_1^2 + \eta_2^2 \neq 0 \), such that

\[
\eta_1 (y^T P y) + \eta_2 (y^T Q y) \geq 0, \forall y \in \mathbb{R}^{n+1} \quad (2.14)
\]
and

$$\left( \eta_1, \eta_2 \right)^T z \leq 0, \; \forall z \in D. \quad (2.15)$$

Applying Farkas’ Lemma \cite{2} to (2.15), there exists \((\xi_1, \xi_2, \xi_3) \geq 0\) such that

$$\left( \eta_1, \eta_2 \right)^T = \xi_1 e_1^T + \xi_2 e_2^T + \xi_3 e_3^T.$$  \hspace{1cm} \text{(2.15)}

Namely,

$$\eta_1 = \xi_1 \geq 0, \quad \eta_2 = \xi_2 - \xi_3.$$ \quad \text{(2.16)}

Substituting (2.16) into (2.14), we have

$$\xi_1 (y^T P y) + (\xi_2 - \xi_3)(y^T Q y) \geq 0, \quad \forall y \in \mathbb{R}^{n+1}$$ \quad \text{(2.17)}

and \(\xi_1 \geq 0.\)

If \(\xi_1 = 0\), we must have \(\xi_2 - \xi_3 \neq 0\) since \(\eta_1, \eta_2\) can not be both zero. Then \(2.17\) implies that either \(y^T Q y \geq 0\) or \(y^T Q y \leq 0\), \(\forall y \in \mathbb{R}^{n+1}\). That is, \(Q\) can not be indefinite. By Assumption A, we have

$$\inf_{x \in \mathbb{R}^n} h(x) < 0 < \sup_{x \in \mathbb{R}^n} h(x)$$

and there exist \(x', x'' \in \mathbb{R}^n\) such that \(h(x') < 0, h(x'') > 0\). Let \(y' = (x', 1)\) and \(y'' = (x'', 1)\) then \((y')^T Q(y') < 0\) and \((y'')^T Q(y'') > 0\). This contradiction leads to \(\xi_1 > 0\). Dividing \(\xi_1 > 0\) throughout \(2.17\) and letting \(\mu = \frac{\xi_2 - \xi_3}{\xi_1}\), we obtain

$$y^T P y + \mu(y^T Q y) \geq 0, \quad \forall y \in \mathbb{R}^{n+1}.$$ \quad \text{(2.17)}

In particular, for \(y = \begin{pmatrix} x \\ 1 \end{pmatrix}, x \in \mathbb{R}^n\), there is

$$f_1(x) - \lambda f_2(x) + \mu h(x) \geq 0,$$

which shows the validity of statement (ii).

\(\text{(b)} \) \(B\zeta \neq 0\). In this case, letting \(z = x - \zeta\), we have

$$f_1(x) - \lambda f_2(x) = f_1(z + \zeta) - \lambda f_2(z + \zeta) = z^T A z + 2b^T z + c,$$ \quad \text{(2.18)}

where

$$A = A_1 - \lambda A_2, b = A\zeta + b_1 - \lambda b_2 \text{ and } c = \zeta^T A\zeta + 2b^T \zeta + c_1 - \lambda c_2.$$ \hspace{1cm} \text{(2.18)}

Also,

$$h(x) = h(z + \zeta) = \zeta^T B\zeta + 2d^T \zeta + z^T Bz + 2(B\zeta + d)^T z = z^T Bz + 2(d')^T z,$$ \quad \text{(2.19)}

which is positive semidefinite.
where $\zeta^T B\zeta + 2d^T \zeta = 0$ since $\zeta \in X$ and $d' = B\zeta + d$. Obviously, System (2.19) is unsolvable if and only if
\[
z^T Ax + 2b^T z + c < 0, z^T Bz + 2(d')^T z = 0
\]
does not have a solution. Moreover, Assumption B says that, if $z^T Bz = 0$, there must be $(d')^T z = 0$ (where $d' = B\zeta + d$). Therefore, under Assumption B, if System (2.20) is unsolvable, we can apply the proof for case (a) to get $\mu'$ such that
\[
z^T Ax + 2b^T z + c + \mu'(z^T Bz + 2(d')^T z) \geq 0, \forall z \in \mathbb{R}^n,
\]
which is equivalent to
\[
f_1(x) - \lambda f_2(x) + \mu' h(x) \geq 0, \forall x \in \mathbb{R}^n
\]
by (2.18) and (2.19). This completes the proof of Theorem 2.4.

Applying Theorem 2.4, we can compute the optimal value $\lambda^*$ of (2.8) by the SDP problem (2.21) below. The proof of Theorem 2.5 was already sketched in Sect. 1 (1.12)-(1.14) and thus will not be repeated here.

**Theorem 2.5.** Under Assumptions A and B, the optimal value of problem (2.8)
\[
\lambda^* = \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} \mid h(x) = x^T Bx + 2d^T x = 0 \right\}
\]
can be computed by
\[
\lambda^* = \sup_{\lambda, \mu \in \mathbb{R}} \left\{ \lambda : \begin{pmatrix} A_1 - \lambda A_2 + \mu B & b_1 - \lambda b_2 + \mu d \\ b_1^T - \lambda b_2^T + \mu d^T & c_1 - \lambda c_2 \end{pmatrix} \succeq 0 \right\}.
\]

Now we can compute potential values for both $\lambda_1$ and $\lambda_2$ in (2.5) by the SDP problem (2.21), but yet to check $f(\lambda_1) = 0$ or $f(\lambda_2) = 0$ (and also to find the optimal solution). It requires to solve the quadratic fractional problem with an equality quadratic constraint of type (2.7). Moré ([16], Thm 3.2) has shown that, under a constraint qualification (similar to our Assumption A) and assuming that $A_1 - \lambda^* A_2 \neq 0$, every optimal solution $x^*$ of (2.7) admits a Lagrange multiplier $\mu^*$ with no duality gap. However, we do not know in advance whether $x^*$ exists. Moreover, computing the saddle point $(x^*, \lambda^*)$ algorithmically requires the existence of a positive definite matrix pencil $A_1 - \lambda^* A_2 + \mu B > 0$ for some $\mu \in \mathbb{R}$. Failing to have a positive definite matrix pencil leads to a difficult unstable (2.7) that a small perturbation could make (2.7) become unbounded below. Our new version of S-lemma hence provides an alternative way to deal with (2.7). We show that, under Assumption B, (2.7) admits the strong duality so that the rank one decomposition [20, 17] can be applied to get $x^*$. 
We first notice that Assumption A implies \( \{ x \in \mathbb{R}^n : g(x) = v \} \neq \emptyset \), or equivalently, problem (2.7) is feasible.

**Theorem 2.6.** Under Assumptions A and B, the strong duality holds for problem (2.7).

**Proof.** Based on the extended S-Lemma Theorem 2.4, we can evaluate \( f(\lambda^*) \) of (2.7) by an SDP as follows:

\[
f(\lambda^*) = \inf_{x \in \mathbb{R}^n} \{ f_1(x) - \lambda^* f_2(x) | h(x) = 0 \}
= \sup \{ \nu \in \mathbb{R} | \{ x \in \mathbb{R}^n : f_1(x) - \lambda^* f_2(x) < \nu, h(x) = 0 \} = \emptyset \}
= \sup \{ \nu \in \mathbb{R} | f_1(x) - \lambda^* f_2(x) - \nu + \mu h(x) \geq 0, \forall x \in \mathbb{R}^n \}
= \sup_{\nu, \mu \in \mathbb{R}} \left\{ \nu \in \mathbb{R} \left| \begin{pmatrix} A_1 - \lambda^* A_2 & b_1 - \lambda^* b_2 + \mu d \\ b_1^T - \lambda^* b_2^T + \mu d^T & c_1 - \lambda^* c_2 - \nu \end{pmatrix} \geq 0 \right\} \right. \tag{2.22}
\]

Notice that (2.22) is the SDP reformulation for the Lagrange dual problem of (2.7), e.g., see [22]. It means that the strong duality holds for Problem (2.7). \( \square \)

Due to the strong duality of Theorem 2.6, the conic dual problem of (2.22) is attained, then one of the values \( \lambda \) (SDR) of (2.22) can be obtained by applying the matrix rank-one decomposition procedure [20, 17] to an optimal solution \( Z \) of (SDR). Hence Assumption B is trivially true.

**Remark 2.3.** We comment on the applicability of Assumption B.

(3.1) If \( B \succ 0 \) or \( B \prec 0 \) as assumed in [17] Prop. 3.1, then \( x^T B x = 0 \) if and only if \( x = 0 \). Hence Assumption B is trivially true.

(3.2) If \( B \succeq 0 \) or \( B \preceq 0 \) but not definite, then Assumption B is equivalent to the fact that \( d \) is in the range space of \( B \). Indeed, since \( B \succeq 0 \) (the case \( B \preceq 0 \) is similarly considered), \( x^T B x = 0 \) if and only if \( Bx = 0 \). Suppose that \( d \) is in the range space of \( B \) such that \( d = Bz \) for some \( z \). Then, for all \( \zeta \in X \), there is \( (B\zeta + d)^T x = (B\zeta + Bz)^T x = \zeta^T B x + z^T B x = 0 \) if \( x \) satisfies \( Bx = 0 \). That is, Assumption B holds. Conversely, if Assumption B holds, then \( (B\zeta + d)^T x = 0 \) for all \( x \) such that \( Bx = 0 \). We have \( (B\zeta + d)^T x = (B\zeta)^T x + d^T x = \zeta^T B x + d^T x = 0 \), where \( \zeta^T B x = 0 \) since \( Bx = 0 \). It implies
that $d^T x = 0$ for all $x$ such that $B x = 0$. That is, $d$ is in the orthogonal complementary space of the null space of $B$, then $d$ must be in the range space of $B$.

(3.3) Assumption B also covers cases in which $B$ is indefinite. For example, let $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $d^T = (1, 1)$. Then Assumption B holds if and only if there exists $\zeta \in \{ x : x_1^2 - x_2^2 + 2x_1 + 2x_2 = 0 \}$ such that $B\zeta + d = 0$. It happens that $\zeta = (-1, 1)^T$ is the only possibility.

(3.4) The extended Finsler’s theorem [4, Thm A.2] is also a version of S-Lemma with equality, but it can not be applied to compute $\lambda^*$ because the condition (1.15) is very difficult to satisfy for all $\lambda$. For example, $\lambda^* = \inf_{x \in \mathbb{R}^3} \{ f_1(x) - \lambda^* f_2(x) \} = \inf_{x \in \mathbb{R}^3} \{ x_1^2 + x_3^2 + 2x_3 + 1 \} = 0$, which is attained at $\hat{x} = (0, 0, -1)^T$. Since $\hat{x}$ does not satisfy the constraint $0 \leq x_3^2 + 2x_3 \leq 3$, Case 1 does not hold. We go to the next step.

2.3. Examples. In this subsession, two examples are used to demonstrate the entire procedure of our ideas to solve (P). From the examples, we can also observe that, although (QCRQ) studied by Beck and Teboulle [6] is the most generic framework for quadratic fractional programming problems, their approach fails to solve both examples since the conditions (1.9)-(1.11) are too restrictive to be satisfied.

Example 2.2. Solve

$$\inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + x_3^2 + 2x_3}{x_2^2 + 1} : 0 \leq x_3^2 + 2x_3 \leq 3 \right\}.$$  \hspace{1cm} (2.23)

We first notice that $f_2(x) = x_2^2 + 1 > 0$ and thus (2.23) is well-defined. Moreover, Assumption A holds. To solve (2.23), we check Case 1 first.

Step 1. Solve the SDP problem (2.24) to get a candidate $\lambda^* = -1$. Then, for this $\lambda^*$,

$$f(\lambda^*) = \inf_{x \in \mathbb{R}^3} \{ f_1(x) - \lambda^* f_2(x) \} = \inf_{x \in \mathbb{R}^3} \{ x_1^2 + x_3^2 + 2x_3 + x_2^2 + 1 \} = 0,$$

which is attained at $\hat{x} = (0, 0, -1)^T$. Since $\hat{x}$ does not satisfy the constraint $0 \leq x_3^2 + 2x_3 \leq 3$, Case 1 does not hold. We go to the next step.

Step 2. At this step we need to check Assumption B and it is indeed satisfied. Solve the SDP problem (2.21) for two cases: $h(x) = g(x) - u$ and $h(x) = g(x) - v$.

- $h(x) = g(x) - u = x_3^2 + 2x_3$. Since $h(0) = 0$ we do not need to make any change of coordinate. An immediately result from solving (2.21) gives $\lambda_1 = 0$. 

• $h(x) = g(x) - v = x_3^2 + 2x_3 - 3$. Since $h(0) \neq 0$, we select

$$x' = (0, 0, 1)^T \in \{x \in \mathbb{R}^3 : g(x) - v = x_3^2 + 2x_3 - 3 = 0\}$$

and make a coordinate change by replacing $x$ with $x + x'$ so that

$$f_1(x + x') = x_1^2 + (x_3 + 1)^2 + 2(x_3 + 1) = x_1^2 + x_3^2 + 4x_3 + 3; \quad f_2(x + x') = x_2^2 + 1;$$

and $h(x + x') = (x_3 + 1)^2 + 2(x_3 + 1) - 3 = x_3^2 + 4x_3$. Solving the SDP (2.21) we also get $\lambda_2 = 0$.

Since $\lambda_1 = \lambda_2 = 0$, we have to compute $f(\lambda_1)$ and $f(\lambda_2)$ to see which one is 0. It turns out that

$$0 = f(\lambda_1) = \inf_{x \in \mathbb{R}^3} x_1^2 + x_3^2 + 2x_3 \quad \text{s.t.} \quad x_3^2 + 2x_3 = 0, \quad (2.24)$$

whereas

$$3 = f(\lambda_2) = \inf_{x \in \mathbb{R}^3} x_1^2 + x_3^2 + 4x_3 + 3 \quad \text{s.t.} \quad x_3^2 + 4x_3 = 0. \quad (2.25)$$

Therefore, $\lambda^* = \lambda_1 = 0$ and the optimal solution set for (2.24) is

$$X^* = \{(0, a, 0)^T, (0, b, -2)^T | a, b \in \mathbb{R}\} \subset \mathbb{R}^3.$$ 

Since there is no change of coordinate in computing $\lambda_1$, the set $X^*$ is also the optimal solution set for (2.23).

**Remark 2.4.** In Example 2.2, since $A_2$ is a singular positive semi-definite matrix, Condition (1.9) is violated. After homogenization, the related problems $(H)$ (1.7) and $(H_0)$ (1.8) are formulated as

$$(H) \quad \min z_1^2 + z_3^2 + 2z_3s \quad \text{s.t.} \quad z_2^2 + s^2 = 1 \quad 0 \leq z_3^2 + 2z_3s \leq 3s^2;$$

and

$$(H_0) \quad \min z_1^2 + z_2^2 \quad \text{s.t.} \quad z_2^2 = 1 \quad 0 \leq z_3^2 \leq 0.$$ 

It is easy to see $v(H) = v(H_0)$. Then condition (1.11) is also violated. In other words, Beck and Teboulle’s algorithm proposed in [6] cannot be used to solve Example 2.2.

**Example 2.3.** Let $X = \{(x_1, x_2)^T | 0 \leq x_2^2 + 2x_1 \leq 2\}$ and solve

$$\inf_{x \in \mathbb{R}^2} \left\{ \frac{2x_2}{x_2^2 + 1} : x \in X \right\}. \quad (2.26)$$
Again, (2.26) is well-defined and Assumption A is satisfied. However, since \( d = (1, 0)^T \) is not in the range space of \( B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \), Assumption B is violated. Fortunately, we will see that (2.26) meets Case 1, which does not need Assumption B. To justify, we solve the SDP problem (2.4) to get \( \lambda^* = -1 \) and find that \( A_1 - \lambda^* A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \).

Moreover, since

\[
\begin{align*}
f(\lambda^*) &= f(-1) = \inf_{x \in \mathbb{R^n}} \{ f_1(x) - \lambda^* f_2(x) = x_2^2 + 2x_2 + 1 \} = 0.
\end{align*}
\]

The stationary points of \( f_1(x) - \lambda^* f_2(x) = x_2^2 + 2x_2 + 1 \) is defined as follows.

\[
S = \{ x = (x_1, x_2)^T \in \mathbb{R}^2 : 2x_2 + 2 = 0 \} = \{(x_1, -1)^T \} \subset \mathbb{R}^2.
\]

Now the intersection \( S \cap X = \{(x_1, -1)^T : -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \} \) is the optimal solution set of (2.26) and \( \lambda^* = -1 \) is the optimal value.

**Remark 2.5.** Since Example 2.2 and Example 2.3 share the same \( f_2(x) \), Condition (1.9) is again violated. Similarly, for Example 2.3 we have:

\[
(H) \quad \min 2z_2 s \\
\text{s.t.} \quad z_2^2 + s^2 = 1 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \leq z_2^2 + 2z_1 s \leq 2s^2
\]

and \((H_0)\):

\[
(H_0) \quad \min 0 \\
\text{s.t.} \quad z_2^2 = 1 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \leq z_2^2 \leq 0.
\]

Since Problem \((H_0)\) is infeasible, Condition (1.10) can not be true. Beck and Teboulle’s algorithm fails to solve Example 2.3.

### 3. Quadratic Fractional Programming Problem with One Inequality

#### Quadratic Constraint (QF1QC)

As analyzed in Sect. 2.1 when Assumption A is violated, Problem \((P)\) becomes either an unconstrained problem or having one-sided constraint \( g(x) \leq v \) or \( u \leq g(x) \). The unconstrained quadratic fractional programming problem is, in fact, equivalent to the convex unconstrained quadratic problem as studied in Sect. 2.1. In this section we study Problem \((P)\) with an one-sided quadratic constraint taking the following form:

\[
(QF1QC) \quad \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} : g(x) \leq 0 \right\} \tag{3.1}
\]

where \( f_1(x), f_2(x) \) and \( g(x) \) are quadratic functions as defined at the beginning of the paper. The parametric problem \((P)_\lambda\) is now reduced to a quadratic programming
problem having one quadratic inequality constraint (QP1QC):
\[
(QF1QC)_\lambda \quad f(\lambda) = \inf_{x \in \mathbb{R}^n} \left\{ f_1(x) - \lambda f_2(x) \right\}
\text{ s.t. } g(x) \leq 0. \tag{3.2}
\]

Assume in this section that problem (QF1QC) satisfies the Slater condition, i.e., there exists \( \bar{x} \in \mathbb{R}^n \) such that \( g(\bar{x}) < 0 \). Otherwise, the problem (QF1QC) is either infeasible or reduced to an unconstrained fractional programming problem, the latter of which has been discussed in Section 2.

**Lemma 3.1.** If Problem (QF1QC) has no Slater point, it is either infeasible or equivalent to an unconstrained quadratic fractional programming problem.

**Proof.** The Slater condition is violated only when \( g(x) \geq 0, \forall x \in \mathbb{R}^n \). This implies that \( B \succeq 0 \), i.e., \( g(x) \) is convex, and \( d \in \mathcal{R}(B) \), where \( \mathcal{R}(B) \) is the range space of \( B \). That is, the affine space
\[
\{ x \in \mathbb{R}^n : Bx + d = 0 \} \neq \emptyset.
\]
Then \( Bx + d = 0 \iff x = -B^+d + Wz \), where \( B^+ \) is the Moore-Penrose generalized inverse of \( B \) and \( W \) is a matrix whose columns form a basis for the null space of \( B \) if \( B \) is singular; and \( W = 0 \) if \( B \) is nonsingular. Since \( g(x) \) is convex, \( x = -B^+d + Wz \) is the global minimizer of \( g(x) \) with the minimum value \(-d^TB^+d + \alpha \). If \(-d^TB^+d + \alpha > 0, g(x) \geq -d^TB^+d + \alpha > 0 \) implies that (QF1QC) is infeasible. If \(-d^TB^+d + \alpha = 0 \), then \( g(x) \geq 0 \). In this case, the feasible domain \( X = \{ x | g(x) \leq 0 \} \) is reduced to \( X = \{ x | g(x) = 0 \} \). That is
\[
\{ x \in \mathbb{R}^n : g(x) \leq 0 \} = \begin{cases} \{ -B^+d + Wz, z \in \mathbb{R}^m \}, & \text{if } d^TB^+d = \alpha \\ \emptyset, & \text{if } d^TB^+d < \alpha \end{cases}
\]
where \( m \) is the dimension of the null space of \( B \). In the case that (QF1QC) is feasible, it can be expressed in term of \( z \in \mathbb{R}^m \) and becomes the following unconstrained fractional programming problem:
\[
\lambda^* = \inf_{z \in \mathbb{R}^m} \left\{ \frac{f_1(z)}{f_2(z)} : g(z) \leq 0 \right\} = \inf_{z \in \mathbb{R}^m} \frac{\bar{f}_1(z)}{\bar{f}_2(z)}, \tag{3.3}
\]
where \( \bar{f}_i(z) = f_i(-B^+d+Wz) = z^TQ_iz - 2q_i^Tz + \gamma_i, Q_i = W^TA_iW, q_i^T = (d^TB^+A_i - b_i^T)W, \gamma_i = d^TB^+A_iB^+d - 2b_i^TB^+d + c_i, i = 1, 2 \). \( \Box \)

**Theorem 3.2.** For any well-defined problem (QF1QC) satisfying the Slater condition, its optimal value \( \lambda^* \) can be determined by solving the following semi-definite programming problem
\[
\lambda^* = \sup_{\lambda \in \mathbb{R}, \mu \geq 0} \left\{ \lambda : \begin{pmatrix} A_1 - \lambda A_2 + \mu B & b_1 - \lambda b_2 + \mu d \\ b_1^T - \lambda b_2 + \mu d^T & c_1 - \lambda c_2 + \mu \alpha \end{pmatrix} \succeq 0 \right\}. \tag{3.4}
\]
Proof. We have
\[
\lambda^* = \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} : g(x) \leq 0 \right\} \\
= \sup \left\{ \lambda : \{ x \in \mathbb{R}^n | \lambda > f_1(x) \}, g(x) \leq 0 \} = \emptyset \right\} \\
= \sup \{ \lambda : \{ x \in \mathbb{R}^n | f_1(x) - \lambda f_2(x) < 0, g(x) \leq 0 \} = \emptyset \} \\
= \sup \{ \lambda : f_1(x) - \lambda f_2(x) + \mu g(x) \geq 0, \forall x \in \mathbb{R}^n, \mu \geq 0 \} \\
= \sup \left\{ \lambda : \left( \begin{array}{c} A_1 b_1 \\ b_1^T c_1 \end{array} \right) - \lambda \left( \begin{array}{c} A_2 b_2 \\ b_2^T c_2 \end{array} \right) \right\} \\
= \sup_{\lambda \in \mathbb{R}, \mu \geq 0} \left\{ \lambda : \left( \begin{array}{c} A_1 - \lambda A_2 + \mu B \\ b_1^T - \lambda b_2 + \mu d \\ c_1 - \lambda c_2 + \mu \alpha \end{array} \right) \right\} \\
\tag{3.5a}
\end{equation}
\[
\lambda^* = \inf_{x \in \mathbb{R}^n} \left\{ \frac{f_1(x)}{f_2(x)} : g(x) \leq 0 \right\} \\
= \sup \left\{ \lambda : \{ x \in \mathbb{R}^n | \lambda > f_1(x) \}, g(x) \leq 0 \} = \emptyset \right\} \\
= \sup \{ \lambda : f_1(x) - \lambda f_2(x) + \mu g(x) \geq 0, \forall x \in \mathbb{R}^n, \mu \geq 0 \} \\
= \sup \left\{ \lambda : \left( \begin{array}{c} A_1 b_1 \\ b_1^T c_1 \end{array} \right) - \lambda \left( \begin{array}{c} A_2 b_2 \\ b_2^T c_2 \end{array} \right) \right\} \\
= \sup_{\lambda \in \mathbb{R}, \mu \geq 0} \left\{ \lambda : \left( \begin{array}{c} A_1 - \lambda A_2 + \mu B \\ b_1^T - \lambda b_2 + \mu d \\ c_1 - \lambda c_2 + \mu \alpha \end{array} \right) \right\} \\
\tag{3.5b}
\end{equation}
where the equivalence of (3.5a) and (3.5b) is due to a standard S-lemma under the Slater condition \[17, 24\].

To know whether (QF1QC) is attained and to find \(x^*\) that solves (QF1QC), we need to check whether \(\lambda^*\) satisfies \(f(\lambda^*) = 0\) and to solve \((QF1QC)_{\lambda^*}\). We have
\[
f(\lambda^*) = \sup \{ \nu \in \mathbb{R} : \{ x \in \mathbb{R}^n | f_1(x) - \lambda f_2(x) < \nu, g(x) \leq 0 \} = \emptyset \} \cdot \tag{3.6}
\]
Since the Slater condition is assumed, we can apply S-lemma to (3.6) and obtain
\[
f(\lambda^*) = \sup \{ \nu \in \mathbb{R} : f_1(x) - \lambda^* f_2(x) - \nu + \eta g(x) \geq 0, \forall x \in \mathbb{R}^n, \eta \geq 0 \},
\]
which is equivalent to a convex SDP formulation:
\[
f(\lambda^*) = \sup \left\{ \nu \in \mathbb{R} : \left( \begin{array}{c} A_1 - \lambda^* A_2 + \eta B \\ b_1^T - \lambda^* b_2 + \eta d \\ c_1 - \lambda^* c_2 + \eta \alpha - \nu \end{array} \right) \right\} \geq 0, \eta \geq 0 \}. \tag{3.7}
\]
We notice that (3.7) is the Lagrange dual problem of \((QF1QC)_{\lambda^*}\). \[22\]. It means that, the strong duality holds for \((QF1QC)_{\lambda^*}\). Therefore, \((QF1QC)_{\lambda^*}\) has the following tight SDP relaxation:
\[
\inf \quad M(f_1 - \lambda^* f_2) \bullet Z \\
\text{s.t.} \quad M(g) \bullet Z \leq 0 \tag{3.8} \\\nZ \geq 0, I_{nn} \bullet Z = 1,
\]
where \(M(f_1 - \lambda^* f_2), M(g), I_{nn}\) and \(Z\) are similarly defined as in Sect. \[2\]. Then an optimal solution \(x^*\) of \((QF1QC)_{\lambda^*}\), if exists, can be obtained from an optimal solution of (3.8) followed by the matrix rank-one decomposition procedure. See \[20, 17\].

Remark 3.1. In Sect. \[2\] our analysis showed that the difficulty of the two-sided \((P)\) lies mainly on the equality constrained problem \[22\], which can only be solved
under the constraint qualification Assumption B. Interestingly, we also showed that
the one-sided (P), namely (QF1QC), can be solved completely without any condition.
This leads to a conclusion that the equality constrained version is more difficult than
its counterpart with an inequality constraint. They are not identical, even though for
each $\lambda$ the two-sided (P)$_\lambda$ (1.2); the equality version of (P)$_\lambda$ (2.2); and the inequality
version of (P)$_\lambda$ (3.2) all possess a set of (similar in format, but the difficulty in
solving them might differ) necessary and sufficient conditions that guarantee a strong
duality, respectively in ([18], Thm 2.3); ([16], Thm 3.2); and ([16], Thm 3.3). We
have some reasons for it. Geometrically, even for a convex $g(x)$, $g(x) \leq 0$ leads to a
convex set whereas $g(x) = 0$ not. Technically, the S-lemma is crucial in both cases.
For the inequality version $g(x) \leq 0$, the proof of the S-lemma must rely on the Slater
point. Fortunately, when $g(x) \leq 0$ fails the Slater condition, it leads to a fact that
$g(x)$ must be convex as shown in Lemma 3.1. On the other hand, the equality version
$g(x) = 0$ can not have a Slater point. It must rely on a more sophisticated constraint
qualification like Assumption B. Failing that constraint qualification does not conclude
any convexity of $g(x)$.

In the remaining part of this section, we shall discuss the (RQ) problem (1.3) as
a special case of (QF1QC), where $g(x)$ in (RQ) is convex ($B \succeq 0$); no linear term
($d = 0$) and $\alpha = -\rho < 0$. Suppose the rank of matrix $L$ is $r$ such that $0 < r \leq n$.
Due to the special property of (RQ), Lemma 2.2 and Theorem 3.2 can be combined
to have a stronger version as Theorem 3.4 below. To prove it, we quote and use a
result from [15], which states: Consider a quadratic problem

$$\min_{x \in \mathbb{R}^n} \left\{ g_0(x) : g_i(x) \leq 0, i = 1, 2, \ldots, m \right\}$$

(3.9)

where $g_i(x) = x^T Q_i x + 2q_i^T x + c_i$, $i = 0, 1, 2, \ldots, m$ are quadratic functions. Then,

**Lemma 3.3.** ([15], Theorem 2) Suppose that $Q_1 \succeq 0$ and $Q_i = 0$ for $i = 2, \ldots, m$.
Then if the objective function $g_0(x)$ is bounded from below over the feasible set $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$ then problem (3.3) attains its minimum.

**Theorem 3.4.** (The attainment of the (RQ) problem) For any well-defined (RQ),
the following three statements are equivalent:

(i) $\lambda^* = v(RQ)$ is attained.

(ii) The following semi-definite programming problem (D) has a unique solution $(\lambda^*, \eta^*)$ :

$$\max_{\lambda \in \mathbb{R}, \eta \geq 0} \left\{ \lambda : \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} - \lambda \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} + \eta \begin{pmatrix} L^T L & 0 \\ 0 & -\rho \end{pmatrix} \succeq 0 \right\}. \quad (3.10)$$

(iii) $f(\lambda^*) = 0$.

**Proof.** The equivalence of (i) and (ii) was indeed proved in [23] Theorem 3.3. It
suffices to show the equivalence between (i) and (iii), which strengthens the general
result Lemma 2.2 that \( \lambda^* = v(RQ) \) is attained if and only if \( f(\lambda^*) = 0 \) and

\[
(RQ)_{\lambda^*} : f(\lambda^*) = \inf_{x \in \mathbb{R}^n} \{ f_1(x) - \lambda^* f_2(x) : ||Lx||^2 \leq \rho \}
\]

is attained. However, if \( f(\lambda^*) = 0 \), (RQ)_{\lambda^*} is bounded below. Since \( ||Lx||^2 \leq \rho \) is convex, Lemma 3.3 assures that a bounded (RQ)_{\lambda^*} must be attained. In other words, \( f(\lambda^*) = 0 \) implies the attainment of problem (RQ)_{\lambda^*}.

**Remark 3.2.** The equivalence of (i) and (ii) in Theorem 3.4 only holds for (RQ) problem. It can not be extended to (QF1QC) in general as the following Example 3.1 shows.

**Example 3.1.** Consider Example 2.1 again as follows:

\[
\inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + 1}{x_2^2 + 1} : g(x) = x_1^2 + 2x_3 - 1 \leq 0 \right\}.
\]

It has been verified in Example 2.1 that \( \lambda^* = 0 \) and \( f(\lambda^*) = f(0) = 1 > 0 \), so the problem is unattainable. However, the SDP problem (3.10):

\[
\max \left\{ \lambda : \begin{pmatrix} 1 + \eta & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & \eta & 1 - \lambda - \eta \end{pmatrix} \succeq 0, \; \eta \geq 0 \right\}
\]

has a unique solution \((\lambda^*, \eta^*) = (0, 0)\).

Some similar results of the attainment of the (RQ) problem were also discussed in \[5\] under stricter conditions. For comparison, we quote the conditions and the results from \[5\].

**Assumption C (5)** There exists \( \eta \geq 0 \) such that

\[
\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} + \eta \begin{pmatrix} L^T L & 0 \\ 0 & -\rho \end{pmatrix} \succ 0
\]

**Assumption D (5)** Either \((r = n)\) or \((r < n \text{ and } \lambda_{\min}(M_1, M_2) < \lambda_{\min}(F^T A_1 F, F^T A_2 F))\)

where

\[
M_1 = \begin{pmatrix} F^T A_1 F & F^T b_1 \\ b_1^T F & c_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} F^T A_2 F & F^T b_2 \\ b_2^T F & c_2 \end{pmatrix}
\]

with \( F \in \mathbb{R}^{n \times (n-r)} \) a matrix whose columns form an orthonormal basis for the null space of \( L \).

**Theorem 3.5.** (5) If Assumptions C and D are satisfied, the minimum of (RQ) is attained and \( v(RQ) \leq \lambda_{\min}(M_1, M_2) \).
Theorem 3.6. (12) Let \( n \geq 2 \) and suppose that Assumptions C and D are satisfied. Then

\[ v(D) = \lambda^*, \]

where \((D)\) is the semi-definite problem \((3.10)\).

It was proved by Example 3.5 in [23] that Assumption D is not a necessary condition for the attainment of (RQ). Therefore, the necessary and sufficient statements (i) and (ii) in Theorem 3.4 strictly generalize Theorem 3.5. Since Example 3.1 further shows that the equivalence of (i) and (ii) in Theorem 3.4 does not hold for (QF1QC), our Lemma 2.2 thus improves Theorem 3.5 sharply. Secondly, our Theorem 3.2 shows that the conclusion of Theorem 3.6 is indeed true for a more general (QF1QC) problem without any condition.

4. On well-definedness of (QF1QC). In this section we characterize the well-definedness property for the problem (QF1QC) (i.e. \( f_2(x) > 0 \) on \( X = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \} \)). To this end, we assume the primal Slater condition and that the two matrices \( A_2 \) and \( B \) are simultaneously diagonalizable via congruence (SDC). Then, there exists a nonsingular matrix \( C \) such that both matrices \( C^T A_2 C \) and \( C^T B C \) are diagonal. It has been argued in [13] that, under the (SDC) condition, the following quadratic problem

\[ \inf_{x \in \mathbb{R}^n} \{ f_2(x) : g(x) \leq 0 \} \quad (4.1) \]

would be either unbounded below, or attained, or eventually reduced to an unconstrained quadratic problem, but can never be unattainable.

However, if \((4.1)\) is indeed an unconstrained problem, it is attainable if and only if \( A_2 \succeq 0 \) and the vector \( b_2 \) lies in the range space of \( A_2 \); or it must be unbounded below. In other words, under the SDC condition, \((4.1)\) can never be unattainable if bounded from below. It leads to a similar, but slightly more general result than Lemma 3.3:

Lemma 4.1. If \( A_2 \) and \( B \) are SDC, the quadratic problem \((4.1)\) is either attained or unbounded below.

Remark 4.1. The question as to “Simultaneous diagonalization via congruence of a finite collection of symmetric matrices” was proposed to be the twelfth open problem in [21]. Even for just two matrices, the complexity to check whether or not they are indeed SDC remains unanswered.

The well-definedness of (QF1QC) can be checked computationally as follows.

Theorem 4.2. Suppose that \( A_2 \) and \( B \) are SDC. The following three statements are equivalent under the primal Slater condition:
(a) Problem \((QF1QC)\) is well-defined. That is, \(f_2(x) > 0\) on \(X = \{x \in \mathbb{R}^n | g(x) \leq 0\}\).

(b) There exist \(\delta > 0, \eta \geq 0\) such that
\[
\begin{pmatrix}
A_2 & b_2 \\
b_2^T & c_2 - \delta
\end{pmatrix} + \eta \begin{pmatrix}
B & d \\
d^T & \alpha
\end{pmatrix} \succeq 0.
\] (4.2)

(c) There exists \(\delta > 0\) such that
\[
\inf_{x \in \mathbb{R}^n} \{f_2(x) : g(x) \leq 0\} \geq \delta > 0.
\]

Proof. We observe that (c) trivially implies (a). It remains to show that (a) implies (b) and (b) implies (c).

(a) \(\Rightarrow\) (b). Since \((QF1QC)\) is well-defined, \(f_2(x)\) is bounded from below by 0 over \(\{x \in \mathbb{R}^n : g(x) \leq 0\}\) and thus Problem (4.1) attains its minimum, say at \(x^*\), by Lemma 4.1. Let \(\delta = f_2(x^*) > 0\). The following system
\[
\begin{align*}
f_2(x) &< \delta \\
g(x) &\leq 0
\end{align*}
\]
is hence unsolvable. By S-Lemma, there exists \(\eta \geq 0\) such that
\[
f_2(x) - \delta + \eta g(x) \geq 0, \forall x \in \mathbb{R}^n,
\]
which is exactly (4.2).

(b) \(\Rightarrow\) (c). Suppose that there exist \(\delta > 0, \eta \geq 0\) such that the matrix inequality (4.2) holds. Since \(g(x) \leq 0\) and \(\eta \geq 0\), we have
\[
\inf_{x \in \mathbb{R}^n} \{f_2(x) : g(x) \leq 0\} \geq \inf_{x \in \mathbb{R}^n} \{f_2(x) + \eta g(x) : g(x) \leq 0\}.
\]
Moreover, the matrix inequality (4.2) is equivalent to
\[
f_2(x) - \delta + \eta g(x) \geq 0, \forall x \in \mathbb{R}^n
\]
so that
\[
\inf_{x \in \mathbb{R}^n} \{f_2(x) + \eta g(x) : g(x) \leq 0\} \geq \delta > 0.
\]

Thus
\[
\inf_{x \in \mathbb{R}^n} \{f_2(x) : g(x) \leq 0\} \geq \delta > 0
\]
which completes the proof.

Remark 4.2. The assumption that \(A_2\) and \(B\) are SDC in Theorem 4.2 cannot be relaxed as the following example explains.
Example 4.1. \( \inf_{x \in \mathbb{R}^2} \left\{ \frac{x_1^2 + x_2^2 + 5}{x_1^2} : 1 - 2x_1x_2 \leq 0 \right\} \). We note here that the matrices
\[
A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]
are not SDC. However, the problem is well-defined since vector \((0, x_2)^T\) is not in the feasible set. We can also verify that the matrix
\[
\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 - \delta \end{pmatrix} + \eta \begin{pmatrix} B & d \\ d^T & \alpha \end{pmatrix} = \begin{pmatrix} 1 - \eta & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & \eta - \delta \end{pmatrix}
\]
is not positive semi-definite for any \( \eta \geq 0, \delta > 0 \). The statement (b) in Theorem 4.2 fails.

Finally, it was proved in [5] that Assumption C implies the well-definedness of Problem (RQ). The following example indicates that (4.2) in Theorem 4.2 is more general than Assumption C.

Example 4.2. \( \inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + x_2^2 + x_3}{x_1^2 + 1} : x_1^2 + x_2^2 \leq 1 \right\} \) (4.3)

In can be observed that (4.3) is an (RQ) problem which satisfies the SDC condition. It is well-defined so that condition (4.2) is satisfied. However, there is no \( \eta \geq 0 \) satisfying the matrix inequality in Assumption C.

5. Conclusion and Further Research. In this paper, we study a quadratic fractional programming problem (P) over the intersection of an upper and a lower level set of a quadratic function \( g(x) \). In contrast to the traditional Dinkelbach iterative method, we solve (P) by establishing the equivalence between the parametric form \( (P)_{\lambda^*} \) and the related SDP formulations. Therefore, computational efficiency for (P) is greatly improved over the tedious and slow convergence of the repeated iterations.

The problem (P) is posed intensionally over the two-sided constraint set in order to also shed some light on the old existing quadratic programming with more than one quadratic constraint. However, our study shows that the major difficulty of (P) lies in solving a quadratic fractional minimization problem subject to a quadratic equality constraint. The future research will be naturally to obtain a stronger version of the extended S-Lemma and study its geometric insights.

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