INERTIA OF HEAT IN ADVECTIVE ACCRETION DISKS AROUND KERR BLACK HOLES

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ABSTRACT

In the innermost region of an advective accretion disk orbiting a black hole of high spin, the inertia of heat stored in the accreting gas is comparable to that of the gas' rest mass itself. Accounting for this effect, we derive additional terms in the disk structure equations and show that the heat inertia plays a significant role in the global energy conservation and dynamics of accretion in relativistic advective disks.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — relativity

1. INTRODUCTION

In advective accretion disks (see Narayan 1997 for a review), the released gravitational energy is not radiated away as in the standard thin disk model (Shakura 1972; Shakura & Sunyaev 1973). Instead, it is stored in the form of internal heat, being eventually advected into the black hole. Optically thin advective disks have been applied to X-ray transients, low-luminosity active galactic nuclei, and the Galactic center (Narayan 1997). Accretion can also proceed in the advection-dominated regime in optically thick super-Eddington accretion disks (Abramowicz et al. 1988), and the observational significance of these disk models for quasars has recently been discussed (e.g., Szuszkiewicz, Malkan, & Abramowicz 1996).

Most of the models of the advective disk have been constructed under assumptions similar to those used in the standard model: that the flow is stationary and axially symmetric and that a one-dimensional approximation, with “integrated” vertical structure, is physically adequate. At the same time, the equations for such a slim disk differ considerably from the standard ones. In particular, the slim radial Euler equation accounts for a deviation of gas motion from the Keplerian case, which becomes significant in an advection-dominated disk. A Newtonian version of the slim disk equations was derived by Paczynski & Bisnovatyi-Kogan (1981) some time ago and has been used by many authors. The relativistic version of these equations was derived more recently by Lasota (1994; see also Abramowicz et al. 1996; Abramowicz, Lanza, & Percival 1997; Peitz & Appl 1997). Lasota's equations have been integrated numerically to obtain relativistic models of a hot, low-luminosity disk (Abramowicz et al. 1996; Jaroszyński & Kjurkchiev 1997; Peitz & Appl 1997; Igumenshchev, Abramowicz, & Novikov 1997).

Accreting black holes are expected to acquire a large spin, and then relativistic effects become especially important. A special feature of an accretion disk around a rapidly rotating black hole is that the viscously dissipated power becomes comparable to $Mc^2$, $M$ being the accretion rate. As a result, the inertial mass associated with heat stored in the advective disk is approximately equal to the rest mass of the accreting gas. Lasota's equations, being fully relativistic in all other respects, neglect the inertia of internal heat, and Peitz & Appl (1997) and Jaroszyński & Kjurkchiev (1997) have partly included it in their calculations. In this paper, we show that accounting for the heat inertia is necessary to make the disk model consistent with the global energy conservation law, and we derive the disk structure equations including this effect. In particular, an additional term appears in the radial Euler equation. We dub it “heat deceleration,” as it describes the back-reaction of energy release on the radial velocity of accreting matter.

2. NOTATION, ASSUMPTIONS, AND METHOD

Our approach and notation are close to those in Abramowicz et al. (1996), except that we do not neglect the contribution of internal energy and pressure to the inertial mass of the flow. We denote by $r_p = 2GM/c^2$ the gravitational radius of the black hole with mass $M$, and by $a = J/Mc$ the Kerr parameter connected to the black hole angular momentum $J$. We use the Boyer-Lindquist coordinates $x^a = (t, r, \theta, \phi)$ and the signature $(-++++)$. The metric tensor $g_{ij}$ is given, e.g., by Misner, Thorne, & Wheeler (1973). The four-velocity of the accreting gas has components $u^\mu = (u^t, u^r, u^\theta, u^\phi)$ in these coordinates. The disk is assumed to lie at the equatorial plane of the Kerr geometry. In the slim approximation, $u^\theta$ is neglected and all the metric and connection coefficients are evaluated at the equatorial plane.

The gas Lorentz factor measured in the frame of local observers with zero angular momentum is connected to $u^t$ by $\gamma = u^t(-g_{tt})^{-1/2}$. The angular velocity of the gas rotation is defined as $\Omega = u^\phi/u^t$. It is not equal in general to the Keplerian angular velocity, 

$$\Omega_k^+ = \pm \frac{c}{r(2r/r_g)^{1/2} + a}$$

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and are the shear tensor components, expressed by the equations

\[ a_r = \frac{d}{dr} (u_r u') + \frac{1}{2} \frac{\partial g_{\phi \phi}}{\partial r} g^{\gamma \gamma} (\Omega^r - \Omega \Delta) (\Omega - \Omega \gamma) \]

(Abramowicz et al. 1996). The dynamical equations are derived from the conservation laws \( \partial_i T^i_k = 0 \), where \( \partial_i \) is the covariant derivative in the Kerr metric and \( T^i_k \) is the stress-energy tensor of the viscous gas flow,

\[ T^i_k = (\rho + p) u^i u^k + p \delta^i_k - 2 \eta \sigma^i_k + c^{-1} (u^i q_k + u_k q^i) \]

(Misner et al. 1973), where \( \rho \) and \( p \) are respectively the total energy density and pressure in the comoving frame, \( \sigma^i_k \) is the shear tensor, and \( q^i \) is the energy flux vector, assumed to be directed vertically in the disk. The dynamic viscosity \( \eta \) is related to the kinematic viscosity \( \nu \) by \( \eta = \nu (\rho + p)/c^2 \) (note that this differs from the previously used relation, which includes only the rest mass of the gas).

The final equations are written for the vertically integrated thermodynamic quantities: surface rest-mass density \( \Sigma \), surface energy density \( U \), and the vertically integrated pressure \( P \). \( F^- \) denotes the surface rate of viscous heating, and \( F^+ \) denotes the radiation flux radiated from both faces of the disk. All the thermodynamic quantities and both fluxes \( (F^-, F^+) \) are measured in the comoving frame. The dimensionless specific enthalpy is defined as

\[ \mu = \frac{U + P}{\Sigma c^2} = 1 + \frac{\Pi + P}{\Sigma c^2} . \]

Neglecting the internal heat contribution to the inertia of the flow is the same as assuming that \( \mu = 1 \). In this paper we do not make this assumption, but retain in all equations \( \mu > 1 \). Consistently, we keep the viscous term in the radial Euler equation (see §4).

We will skip all computational details, because our derivation follows the same standard lines explained, e.g., by Page & Thorne (1974) and used recently by Lasota (1994) and all the other authors cited above. In the next sections we give only the final results.

3. CONSERVATION OF ENERGY AND ANGULAR MOMENTUM

The conservation of energy and angular momentum are expressed by the equations

\[ \frac{d}{dr} \left[ \mu \left( \frac{\dot{M} u_i}{2\pi} + 2 \Sigma \sigma^i_r \right) \right] = \frac{F^-}{c^2} r u_r , \]  
\[ \frac{d}{dr} \left[ \mu \left( \frac{\dot{M} u_\phi}{2\pi} + 2 \Sigma \sigma^\phi_r \right) \right] = \frac{F^-}{c^2} r u_\phi . \]

(1)  
(2)

Here \( \dot{M} \) is the accretion rate, related to \( u' \) and \( \Sigma \) by the baryon conservation law,

\[ 2\pi c u' \Sigma = - \dot{M} , \]  
(3)

and \( \sigma^i_r \) and \( \sigma^\phi_r \) are the shear tensor components,

\[ \sigma^i_r = \frac{1}{2} g^{ij} g_{\phi \phi} \sqrt{-g} \frac{d}{dr} \left( \frac{\partial g_{\phi \phi}}{\partial r} \right) , \quad \sigma^\phi_r = - \Omega \sigma^\phi_r . \]

These equations differ from those with neglected inertia of heat (see Abramowicz et al. 1996) only by the presence of the factor \( \mu > 1 \). They also differ from the equations in Peitz & Appel (1997) and Jarozyński & Kurpiowski (1997), who take into account the contribution of heat to the specific orbital energy and angular momentum of the flow but neglect the similar term in the relation between the dynamic and kinematic viscosity.

In the standard thin disk \( \mu \approx 1 \) with high accuracy, and the rotation is very close to Keplerian. Then equations (1) and (2) become the same as those derived by Page & Thorne (1974). For the advective disk, however, the factor \( \mu \) should not be neglected. This can easily be seen from the global energy conservation law. Integrating equation (1) from the transonic inner edge of the disk, \( r_{\text{in}} \), to infinity, and neglecting the viscous stress at \( r_{\text{in}} \), we obtain

\[ L = - \frac{2\pi}{c} \int_{r_{\text{in}}}^\infty u_r F^- r \, dr = \dot{M} c^2 \left( 1 + \frac{\mu_{\text{in}} u_{\text{in}}}{c} \right) . \]  
(4)

It follows that the radiative efficiency of the disk equals

\[ \epsilon = 1 - \mu_{\text{in}} (1 - \frac{b_{\text{in}} c^2}{e^2}) , \]  
(5)

where \( b_{\text{in}} = c^2 + u_{\text{in}}^2 \) is the specific binding energy at the inner edge. In advective disks, most of the dissipated binding energy is stored as internal heat and the radiative losses are negligible, \( \epsilon \to 0 \). From this we conclude that \( \mu_{\text{in}} \to (1 - b_{\text{in}} c^2)^{-1} \). In the particular case of an extremely rotating black hole, assuming, e.g., that \( b_{\text{in}} \) is close to the corresponding standard value \( 1 - 1/\sqrt{3} \approx 0.42 \), one has \( \mu_{\text{in}} = \sqrt{3} \). Note that from the usual assumption \( \mu = 1 \) it follows that \( b_{\text{in}} = 0 \) for any advective disk. In fact, the position and binding energy of the inner edge can be found only by integrating the disk structure equations, and it is the difference \( \mu - 1 \) that adjusts to keep \( \epsilon \approx 0 \). In this respect the heat inertia is essential in the case of a nonrotating black hole, as well as that of an extremely rotating black hole.

It is instructive to compare the angular momentum equation (eq. [2]) with its commonly used standard version (Novikov & Thorne 1973; Page & Thorne 1974). While \( \mu = 1 \) can be safely assumed in the standard disk, the radiative losses of angular momentum represented by the right-hand side of equation (2) become important when an accreting black hole is rapidly rotating (see, e.g., Lamb 1997). Contrary to this situation, in an advection-dominated disk the radiative losses are always small, but the deviation of \( \mu \) from unity becomes significant. In a sense, the angular momentum that would be radiated away by the standard disk remains in place, being carried by the increased inertia of internal energy.

4. RADIAL MOTION AND VISCOUS HEATING

To clarify the role of heat inertia in the equation of radial motion, we derive this equation in two ways. First, we vertically integrate the \( r \)-component of the equation \( \partial_i T^i_k = 0 \), to obtain

\[ u' u' \left( \frac{d}{dr} (\Pi + P) - \zeta \frac{\Pi + P}{\Sigma} \frac{d\Sigma}{dr} \right) + a_r (U + P) \]
\[ + \frac{dP}{dr} + u_r F^- = 0 , \]  
(6)

where \( \zeta \approx 1 \) is a numerical factor accounting for the inhomogeneity of the disk in the vertical direction. In this equa-
tion, we have taken into account the fact that the divergence of the vertical energy flux $q^i$ becomes equal to $F^{-}$ after the vertical integration.

Second, we project the equation $\nabla_i T^i_k = 0$ onto the hypersurface orthogonal to $u^i$ to obtain the relativistic Euler equation (cf. Lightman et al. 1979, problem 5.31). Then the flux term vanishes, but the viscous term arises in the radial equation, and we obtain after the vertical integration

$$a_r(U + P) + \frac{dP}{dr} (1 + u'u_r) + \frac{F^+}{c} u_r = 0,$$  \hspace{1cm} (7)

where

$$F^+ = 2\Sigma \mu \sigma^2 c^2, \quad \sigma^2 = \frac{1}{2} g^r g_{\phi\phi} - g^{\phi\phi}, \quad \frac{d\Omega}{dr}.$$  \hspace{1cm} (8)

Keeping the $F^\pm$ terms in equations (6) and (7) allows one to obtain the first law of thermodynamics for a vertically integrated accretion disk as a consequence of these two equations:

$$F^+ - F^- = c\sqrt{u} \left( \frac{d\Omega}{dr} - \frac{\Xi}{P \frac{d\Sigma}{dr}} \right).$$  \hspace{1cm} (9)

Finally, substituting $a_r$ into equation (7), we obtain the equation of radial motion with $u'u_r < 1$,

$$\frac{1}{2} \frac{d}{dr} (u'^2 u) = \frac{1}{2} \frac{\partial}{\partial r} (g^{rr} (\Omega - \Omega_0^r))^2 (\Omega - \Omega_0^r)$$

$$- \frac{1}{c^2 \Sigma \mu} \frac{dP}{dr} (F^+ u') - \frac{F^+ u}{c^2 \Sigma \mu}.$$  \hspace{1cm} (10)

This equation differs from the radial equation in Abramowicz et al. (1996) by the presence of the factor $\mu$ and by the additional term proportional to $F^+$. Without this term, equation (10) would have a simple meaning: radial acceleration is a combined result of the deviation from Keplerian rotation and the radial pressure gradient. The viscous term $\propto F^+$ represents the heat deceleration effect, which can be interpreted in the following way: The mass of the accreting gas measured in its local rest frame increases because of the stored heat. Thus, from the local observer's point of view, the dissipation of binding energy is an external source of mass-energy, and the matter inflow proceeds like motion of a body with changing mass. In this case, in addition to acting forces there is a contribution to acceleration due to the change of mass. The mass of the accreting gas increases, and this tends to decelerate the matter inflow. Note that the "deceleration by heating" described by the additional term is due to viscous heating only. Adiabatic heating does not contribute to this term; for example, in the case of inviscid radial accretion, there would be no additional deceleration term even though the gas would be heated in the adiabatic contraction.\(^5\)

The heat deceleration is a purely relativistic effect, as it accounts for the mass-energy relation. It is proportional to the ratio of the released power to $M c^2$ and can be essential near a rapidly rotating black hole. There it becomes of the same order as the left-hand side of equation (10), and both terms should be taken into account even when they are much smaller than the other two: neglect of these terms would correspond to the hydrostatic approximation ($u^r = 0$), and the dynamics of accretion would be lost in the model.

Note also that the deceleration term is independent of the radiative losses, $F^-$, and remains the same when $F^+ \sim F^+$. In this case, the bulk of the dissipated energy is radiated away. However, this energy first appears as heat decelerating the radial inflow. Only then it is radiated away. The net momentum taken away by the radiation flux vanishes in the rest frame of the accreting gas and does not contribute to the velocity change.

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\(^5\) We are grateful to the referee for this observation.

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