Bell inequalities and entanglement

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We discuss general Bell inequalities for bipartite and multipartite systems, emphasizing the connection with convex geometry on the mathematical side, and the communication aspects on the physical side. Known results on families of generalized Bell inequalities are summarized. We investigate maximal violations of Bell inequalities as well as states not violating (certain) Bell inequalities. Finally, we discuss the relation between Bell inequality violations and entanglement properties currently discussed in quantum information theory.

Keywords: Bell inequalities, entanglement, local hidden variable theories

1. Introduction and historical survey

Quantum mechanics was born in a remarkably short period around the year 1926, when the long period of guessing turned into the successful building of the theory, starting a golden age of remarkable discoveries. In a similar way we can put a date to the beginning of a particular branch of quantum theory, the theory of entanglement. It is the year 1935, when both, the paper of Einstein, Podolsky and Rosen ("EPR"\textsuperscript{1}), and (motivated by the EPR paper) Schrödinger’s article\textsuperscript{2} in which he coined the term “verschränkter Zustand” (entangled state).

Einstein and Schrödinger had both made crucial contributions to the development of quantum theory, yet they were both expressing a deep dissatisfaction with the “present situation of quantum theory” (Schrödinger’s title). And both articles were dismissed by some of the younger generation as the grumblings of old men who were just not able to follow the new lines of thought. Bohr’s reply\textsuperscript{3} to the EPR paper, although little more than a refusal to accept the problem, was hailed as a conclusive rebuttal, and almost everybody went back to business. In hindsight, however, one must admit that Einstein was struggling with the deepest departure from classical physics contained in quantum physics: not the discrete “jumps” and other such conspicuous features, but entanglement. And he was

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remarkably lucid, even though we may not share his conclusions.

There are perhaps two reasons in the EPR paper itself which led to its rather delayed impact on the physics community. One was that the concern of “completeness”, seemed like something to worry about in the distant future, for a community buzzing with successful applications of the theory. The other was that the example seemed a bit contrived: the state discussed is a highly singular object, and even now there are papers trying to make mathematical sense out of it. As a result the perfect correlation between results at two distant locations, which was a crucial part of the argument was not even rigorously true. This defect was overcome in Bohm’s version of the argument\(^4\), using spins (“qubits”). The purely meta-theoretical appeal was changed dramatically by John Bell\(^5\) in 1964, by the observation that the EPR dilemma could be formulated in the form of assumptions, which naturally led to a falsifiable prediction. It is hardly possible to underrate the importance of this discovery, which made it possible to rule out not just a particular scientific theory, but the very way scientific theories had been formulated for centuries.

The history of experimental verifications of the violations of Bell’s inequalities, predicted by quantum theory, is essentially the story of building efficient sources for entangled systems. A breakthrough, bringing the first reliable violations of the inequalities, was Alain Aspect’s atomic cascade\(^6\), which used the then relatively new technology of optical pumping. The search of good sources of entangled systems has become much more intensive with the advent of quantum information theory, in which entanglement is a key resource. The emphasis has thus shifted from demonstrating entanglement to using it, and experimental violations of Bell’s inequalities are often merely a first check whether the source is working properly. New sources (e.g., using parametric down conversion) now admit Bell experiments in the student lab. Even the infamous “detection loophole” (related to the fact that only a small fraction of the produced pairs are really detected) is rapidly being closed now\(^7\).

On the theoretical side, “violation of Bell’s inequalities” had become synonymous with “non-classical correlation”, i.e., entanglement. One of the first papers in which finer distinctions were made was the construction of states with the property (Ref.\(^8\), see Sec.4.2. below) that they satisfy all the usual assumptions leading to the Bell inequalities, but can still not be generated by a purely classical mechanism (are not “separable” in modern terminology). This example pointed out a gap between the obviously entangled states (violating a Bell inequality) and the obviously non-entangled ones, which are merely classical correlated (separable). In 1995 Popescu\(^9\) (and later\(^10\)) narrowed this gap considerably by showing that after local operations and classical communication one could “distill” entanglement, leading once again to violations, even from states not violating any Bell inequality initially. Similar examples were then constructed by Gisin\(^11\) even for the case of two qubits. To summarize this phase: it became clear that violations of Bell inequalities, while still a good indicator for the presence of non-classical correlations by no means capture all kinds of “entanglement”.

The natural conjecture in this situation was that “violation of some Bell inequality after suitable distillation” might be synonymous with entanglement, i.e., distillation should be possible for every non-separable state. But in 1998 the Horodecki family\(^12\) constructed
counterexamples, the so-called bound entangled states (see Sec.4.3.). Due to a certain property (the positivity of the partial transpose) these states turned out to satisfy any of the known Bell inequalities\textsuperscript{13,14}. Up to now, for the bipartite case, it is neither clear whether the violation of a Bell inequality already implies distillability nor do we know whether there is any Bell inequality, which is violated by a state having positive partial transpose. For multipartite systems, however, the structure of the state space with respect to entanglement properties is much richer and Dür\textsuperscript{15} recently showed that there exist indeed undistillable multipartite states violating a Bell inequality.

The purpose of this article is to give a theoretical review of the derivation of Bell inequalities from classical assumptions, discuss their quantum violations and to illuminate relations to entanglement properties and quantum information theory in general. Moreover, we emphasize the connection with convex geometry in the appendix.

As nowadays new papers concerning Bell inequalities or closely related topics are posted on the Los Alamos e-print archive almost every day this will by no means be an exhaustive discussion. We will for instance disregard related topics like the Kochen-Specker theorem, nonlocal hidden variable theories, experimental implementations, and the “Bell theorem without inequalities\textsuperscript{16}”. However, these restrictions will enable us to give an otherwise rather self-contained review of Bell inequalities and entanglement. Other review like articles and extensive discussions emphasizing different topics can be found in Ref.\textsuperscript{17,18,19,20}.

2. Derivations of the Inequalities

There are many derivations of Bell inequalities in the literature. This may at first be a bit surprising for such a simple mathematical statement. However, the hard work in such a derivation is almost never mathematical but conceptual: if we want to draw far-reaching conclusions ruling out whole classes of theories, or ways of formulating natural laws, we have to analyze theories on a very general and abstract level in order to even state the assumptions of “Bell's Theorem”. Naturally, there are many ways to say what the really essential assumptions are, depending on philosophical taste and scientific background.

However, in all derivations two types of elements can be identified

| locality       | classicality                  |
|----------------|-------------------------------|
| no-signalling  | hidden variables              |
| non-contextuality | classical logic               |
|                | joint distributions            |
|                | counterfactual definiteness   |
|                | “realism”                     |

Since Bell’s inequalities are found to be violated in Nature\textsuperscript{7}, one of these two assumptions needs to be dropped. Quantum mechanics (in statistical interpretation) chooses locality, whereas hidden variable theories drop locality in order to retain a description by classical parameters. In either case, however, fundamental features of the pre-quantum way of describing the world are lost.

2.1. Basic notation
Bell type inequalities always refer to correlations between two or more “parties” or sites. It is helpful to imagine that the experiments at the sites are conducted by physicists, traditionally named Alice and Bob in the bipartite (two party) case. Each of the parties gets a particle (or “subsystem”) from a common source, and makes a measurement on her/his subsystem. The basic object of the theory are the joint probabilities obtained in this way. We will denote a typical measured probability by

\[ P(a_2, b_1 \mid A_2, B_1), \]

where after the vertical bar we write the devices used, in this case device \( A_2 \) by Alice and \( B_1 \) by Bob, and before the bar we denote the particular outcomes: \( a_2 \) a possible outcome of the device \( A_2 \) and \( b_1 \) an outcome of \( B_1 \). For simplicity, we will assume throughout that only finitely many outcomes are possible for each measurement. The collection of all these numbers are the basic raw data, we might call the correlation table.

Of course, these data have to satisfy some constraints which follow already from the probability interpretation: \( P(a_2, b_1 \mid A_2, B_1) \geq 0 \), and all the probabilities in a particular setup \((A, B)\) have to add up to 1:

\[ \sum_{a, b} P(a, b \mid A, B) = 1. \]

An interesting role is played by the marginals, which we denote by

\[ P(a \mid A, B) = \sum_b P(a, b \mid A, B). \]

These are the probabilities measured by Alice in a given setup \((A, B)\). For general correlation tables such marginals might depend on the whole setup and, in particular, on the device \( B \) chosen by Bob. For example, the device \( B \) might be a transmitter with a particular input fed into it, and \( A \) might be a receiver. Then this dependence on \( B \) would be precisely what is required for Alice to ‘get the message’. Note, however, that this usually requires some signal-carrying physical system to go from Bob to Alice, contrary to the basic description of the correlation setup (“all parties get particles from a common source”). What we expect in a general correlation experiment (without communication between the parties) is the following no-signalling condition:

\[ \sum_b P(a, b \mid A, B) \equiv P(a \mid A) \quad \text{is independent of } B \]

and similarly for all other sites, in this case \( A \) instead of \( B \).

Before coming to the conditions leading to Bell inequalities we have to clear up two common misunderstandings concerning hidden variables and nonlocal effects. These two subsections can be skipped; the formal development continues in Sec.2.4.

2.2. Hidden variables exist
“Hidden variables” have a bad name in the physics community. Yet the question whether we can understand the observed quantum randomness as arising from our ignorance of some underlying classical variables (this is what the term means) is a fundamental question, which must be addressed seriously if we want to understand quantum mechanics at all.

There was an early argument by von Neumann proving the non-existence of such variables. But von Neumann’s proof was making heavy use of the quantum mechanical structure, so it was really convincing only for those who had already accepted the conclusion. The unfortunate consequence was perhaps that some people began to think that constructing a hidden variable theory, and thereby contradicting the great von Neumann, was somehow a non-trivial achievement in itself. What we want to show here is that, quite to the contrary, it is trivial to construct such a theory. Moreover, this will allow us later to point out more precisely the price to be paid in all such theories, namely some kind of non-locality.

The simplest classical structure from which all measured results can be obtained was already mentioned: it is the collection of correlation data itself. This is saying little more than that gathering statistical data is an activity completely within the domain of classical probability. Thus in this “theory”, which is remarkable only for having explanatory power exactly equal to zero, the hidden variables are the data to be measured, and they are hidden in the same way that the future is. The hidden variable thus contains a complete description of the experimental setup, i.e., of the devices (A, B, C, ...) chosen by all the parties. This feature marks so-called contextual or, more simply, “non-local” hidden variable theories.

Contextuality is, of course, not always as blatant, especially in hidden variable theories focusing on dynamical laws (such as the Bohm/Nelson theory and its generalizations), where the “setups” are not apparent, but enter via a description of the measuring devices inside the theory. By far the most wide-spread hidden variable theory is the “individual state” interpretation of quantum mechanics, according to which some wave function is somehow attached to each individual system, and constitutes a “catalog of all expectations” to be measured on the system. Technically, this is indeed nothing but the description of a hidden variable theory, although such statements can also be found by the Copenhagen Masters, who are not usually associated with hidden variable views.

2.3. The Ping Pong Ball Test

This shows that the temptation is very great to use a language, which is too naively classical. It is especially great when one has to explain the quantum world to a general public. This is the only excuse for the amount of confusing explanations one can find. Here is a simple guideline for spotting many of the misleading ones.

Take any explanations of Bell inequalities or quantum non-locality, and substitute ping

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*We could also say that there is a separate probability space to be chosen for each experimental setup, although we can equivalently put them all in a single probability space declaring different setups to be independent.*
Bell inequalities and entanglement

Pong balls for every quantum particle in the account. Then, if what the author is selling as paradoxical still remains true, he/she isn’t telling you anything about quantum mechanics after all.

Surprisingly many texts fail this test and lead to “paradoxes” like: imagine a box containing a ping pong ball, which can be separated in two parts, without looking at the ball. The two parts are then shipped far apart from each other and after opening one box we then know “instantaneously” whether the ball is at the distant location or not. This is true, but hardly paradoxical, and certainly utterly useless for sending a message.

2.4. Local hidden variable theories and the CHSH inequality

To formalize the idea of a local hidden variable theory, let us explicitly introduce a hidden variable \( \lambda \) which takes values in a space \( \Lambda \). We assume that the systems sent to Alice and Bob (and maybe the others) are described by \( \lambda \) in all details necessary to compute their response to any measurement, or at least to determine the probabilities of all such responses. Thus for any measuring device \( A \) of Alice, and any possible outcome \( a \) of this device we get a response probability function \( \lambda \mapsto \chi_A(a, \lambda) \). The source of the correlation experiment is characterized by the probabilities with which the different \( \lambda \) occur, i.e., by a probability measure \( M \) on \( \Lambda \). With these data we can thus compute all the correlations

\[
P(a, b \mid A, B) = \int M(d\lambda) \chi_A(a, \lambda)\chi_B(b, \lambda) .
\]

We will say that a correlation table allows a local classical model, if it can be represented in this form.

But should this not be always the case? After all, we have only written down, what any probabilist would write anyhow: there is a random variable \( \chi_A \) for each device, and we are looking at the joint distribution as we should. But a comparison with the trivial contextual theories shows immediately where the additional locality assumption is: in principle the response probabilities \( \chi_A(a, \lambda) \) might also depend on the devices \( B, \ldots \) chosen at other sites, and by excluding this dependence we have increased the demands on our model. This is precisely the modification leading to Bell’s inequalities.

The standard example of a Bell inequality is the \textit{Clauser-Horne-Shimony-Holt} (CHSH) inequality, which refers to correlation experiments with two \( \pm 1 \) valued observables on two sites. From the response probabilities \( \chi_A(a, \lambda) \) \((a = \pm 1)\) we form the mean value of the random variable \( a \) (given the hidden variable \( \lambda \)):

\[
\hat{a}(\lambda) = \chi_A(+1, \lambda) - \chi_A(-1, \lambda) ,
\]

and from these the correlation function

\[
B(\lambda) = \frac{1}{2} [\hat{a}_1(\lambda)(\hat{b}_1(\lambda) + \hat{b}_2(\lambda)) + \hat{a}_2(\lambda)(\hat{b}_1(\lambda) - \hat{b}_2(\lambda))] .
\]

\(^{b}\text{A subtle way of failing the ping pong ball test is to supplement the description by the statement that the properties of quantum particles (in contrast to those of ping pong balls or socks) remain “objectively undecided” until the measurement is made. Of course, this merely shifts the burden of explanation to that rather cryptic phrase.}\)
We claim that $B$ satisfies the pointwise inequality $|B(\lambda)| \leq 1$: indeed, the extreme values are attained, when each $\hat{a}_i(\lambda), \hat{b}_i(\lambda)$ is extremal, i.e., $\pm 1$. But then $2B(\lambda)$ is an even integer, and since $2B(\lambda) = 4$ requires $\hat{a}_1(\lambda) = \hat{b}_1(\lambda) = \hat{b}_2(\lambda) = -\hat{b}_2(\lambda)$, a contradiction, we must have $2B(\lambda) \leq 2$.

Since $B$ is pointwise bounded, its expectation, the so-called Bell correlation,

$$\beta = \int M(d\lambda)B(\lambda)$$

is also bounded by unity. But $\beta$ can be expressed directly in terms of the measured correlation table: Introducing the expectation values $E(A,B) = \sum_{a,b=\pm 1} a b \operatorname{P}(a,b \mid A,B)$, the inequality $|\beta| \leq 1$ becomes the CHSH inequality:

$$\frac{1}{2} |E(A_1,B_1) + E(A_1,B_2) + E(A_2,B_1) - E(A_2,B_2)| \leq 1.$$  

(9)

Of course, the existence of local models underlying this inequality is very reminiscent of the no-signalling condition. In fact, the locality of the classical model is precisely the condition that the no-signalling property persists even when we know the value of $\lambda$, or the source has been upgraded to produce only one $\lambda$. Conversely, given a local classical model, the no-signalling condition for the experimental correlation data is merely “locality property on average”.

2.5. Deterministic models and classical configurations

An important motivation for the search for hidden variables was to restore the determinism of classical physics or, using Einstein’s famous metaphor, to allow God to quit gambling. We can easily formulate determinism as a requirement for a local classical model: the knowledge of $\lambda$ should not only allow us to predict the probabilities of outcomes, but the outcomes themselves with certainty. Thus we call a local hidden variable model deterministic, if the response functions take only the values 0 and 1. It turns out, however, that this seemingly much stronger constraint on the model does not lead to sharper conditions on the correlation data.

The reason is that we can upgrade any non-deterministic model to a deterministic one. To do this we only need to incorporate the randomness in the measuring device into the hidden variable. Mathematically, we replace the hidden variable $\lambda$ by $\tilde{\lambda} = (\lambda, \xi_A, \xi_B)$, where $\xi_A$ and $\xi_B$ are uniformly distributed random variables on $[0,1]$, which are independent of each other and of $\lambda$. We then set

$$\tilde{\chi}_A(a, \tilde{\lambda}) \equiv \tilde{\chi}_A(a, (\lambda, \xi_A, \xi_B)) = \begin{cases} 1 & \xi_A \leq \chi_A(a, \lambda) \\ 0 & \text{otherwise} \end{cases}$$

(10)

and similarly for $B$. Obviously, this model is deterministic, and it is straightforward to check that it produces the same correlation table as (5).

For a fixed value of the hidden variable $\lambda$ the response function $\chi_A(a, \lambda)$ in a deterministic model takes on the value one for one outcome $a$ and vanishes for all the others. If we now consider an $n$-partite system, where each of the parties has the choice of $m$
$v$-valued observables to be measured, any of the $nm$ observables thus divides the hidden variable space $\Lambda$ into $v$ pieces (which do not necessarily have to be connected). In this way $\Lambda$ is build up of $v^{nm}$ regions $\Lambda_c$, such that every region is characterized by a single classical configuration $c$, i.e., an assignment of one of the $v$ outcomes to each of the $nm$ observables (for the CHSH case for instance with $(n,m,v) = (2,2,2)$ there are 16 classical configurations). This enables us to rewrite Eq.(5) in form of a sum (analogous for more than two sites)

$$P(a,b \mid A,B) = \sum_c \int_{\Lambda_c} M(d\lambda) \chi_A(a,\lambda)\chi_B(b,\lambda) = \sum_c p_c \chi_A(a,c)\chi_B(b,c),$$

where $p_c = \int_{\Lambda_c} M(d\lambda)$ is the probability corresponding to the classical configuration $c$. Locality is here expressed in the fact that the assignment of a value to an observable at one site does not depend on the observables chosen at other sites.

In Sec.3 and in the appendix we will see that classical configurations play a crucial role in the construction of Bell inequalities as the extreme points of the classically accessible region.

2.6. Bell’s Telephone

We will call a Bell telephone any device, which enables Alice to send messages to Bob using only the correlations in the particles the two have obtained from a common source. In other words, this is precisely the device declared impossible by the no-signalling or locality conditions. In this section we will give an alternative proof of the CHSH inequalities, which emphasizes this communication aspect: we assume a rather weak “classicality” condition, and show that Bell’s telephone will work, whenever the CHSH inequality is violated. In fact, we show that the quality of the transmission is directly related to the Bell correlation.

We will discuss this in the framework of correlation experiments used in the previous sections. The framework contains no space-time aspects so we cannot say that the communication would be “superluminal” (for that, see Sec.5.7.). However, since we are free to move the partners arbitrarily apart, and the effect has nothing to do with distance, we can make the communication superluminal if we want.

If we accept the experimental evidence of violations of the inequalities, and also uphold the causality principle, which forbids Bell’s telephone, then something must be wrong with the classicality condition, which is the basic assumption of our proof. What we assume is that Bob has a joint measurement device, which simultaneously replaces the two devices he uses in the experiment for the Bell correlation. In this way, the violations of Bell’s inequalities become a direct experimental verification of a well known feature of quantum mechanics, namely that there are observables which do not admit a joint measurement. It also implies the impossibility of other tasks such as cloning (copying) and teleportation (transmission of quantum information on a classical channel).

Let us state the basic assumption in the given framework: Bob has a joint measuring
device $B_1 & B_2$ for his two observables, which produces pairs of outcomes $(b_1, b_2)$ with probabilities $p_i(a_i, b_1, b_2) = P(a_i, (b_1, b_2) | A_i, B_1 & B_2)$. The defining characteristic of such a device is that the statistics of the outcomes is the same as for the single devices $B_1$ and $B_2$, i.e. for $i = 1, 2$:

\begin{align*}
\sum_{b_1} p_i(a_i, b_1, b_2) &= P(a_i, b_2 | A_i, B_2) \quad \text{and} \quad (12) \\
\sum_{b_2} p_i(a_i, b_1, b_2) &= P(a_i, b_1 | A_i, B_1).
\end{align*}

Having this kind of device Bob may guess what apparatus Alice has chosen by simply interpreting a coincidence of his outcomes $b_1 = b_2$ as “$A_1$” and suspecting “$A_2$” whenever they differ. If the probability $p_{ok}$ for Bob to be right is better than chance ($p_{ok} > \frac{1}{2}$), then the two can clearly construct a Bell telephone. This, however, immediately takes place as soon as the CHSH inequality is violated since we can estimate

\begin{align*}
p_{ok} &= \frac{1}{2} \sum_{a_1, b_1, b_2} \left| \frac{b_1 + b_2}{2} \right| |a_1|p_1(a_1, b_1, b_2) \\
&\quad + \frac{1}{2} \sum_{a_2, b_1, b_2} \left| \frac{b_1 - b_2}{2} \right| |a_2|p_2(a_2, b_1, b_2) \\
&\quad \geq \frac{1}{4} \sum_{a_1, b_1, b_2} (b_1 + b_2)a_1p_1(a_1, b_1, b_2) \\
&\quad + \frac{1}{4} \sum_{a_2, b_1, b_2} (b_1 - b_2)a_2p_2(a_2, b_1, b_2) \\
&= \beta.
\end{align*}

Hence the experimental fact that nature allows $\beta > 1$ together with the no-signalling assumption rules out joint measuring devices. But this also forbids the existence of universal cloning and classical teleportation since these could be used to construct a joint measuring device\textsuperscript{37}.

3. All the Bell inequalities

So far we have only discussed the CHSH inequality as one specific example of a Bell inequality. However, there is an infinite hierarchy of such Bell type inequalities, which can basically be classified by specifying the type of correlation experiments they deal with. The essential assumption leading to any Bell inequality is the existence of a local realistic model, which describes the outcomes of a certain class of correlation measurements. The modus operandi for the derivation of a class of Bell inequalities would therefore be the following: We first fix the type of correlation measurements we want to deal with – say we consider $n$-partite systems, where each of the parties has the choice of $m$ $v$-valued observables to be measured.\textsuperscript{5} Then we consider the space spanned by the entire set of the raw experimental

\textsuperscript{5} Of course we are free to require further restrictions, e.g. we might just want to look at a subset of all
data, i.e., the \((mv)^n\) probabilities, and ask for the inequalities, which bound the region that is accessible within the framework of a local realistic model. Whatever this underlying model looks like, if only it is “classical”, i.e. a local hidden variable model, the accessible region will be contained in a convex polytope, whose extreme points are the classical configurations (for the connection to convex geometry see the appendix). The classical region is thus bounded by a finite albeit huge number of linear inequalities. These are the natural generalizations of the original inequality John Bell published in 1964\(^5\).

Hence we are faced with a whole hierarchy of inequalities. The task of finding a minimal set of these inequalities, which is complete in the sense that they are satisfied if and only if the correlations considered allow a local classical model, is however closely related to some known hard problems in computational complexity\(^{28,29}\). So it is not surprising that complete solutions only exist either in cases, where additional symmetries can be exploited\(^{14,30}\), or for small values of \((n, m, v)\) where we can utilize today’s computing power for a brute force approach. An extensive numerical search for the cases \((n, m, v) = (3, 2, 2)\) and \((2, 3, 2)\) was performed by Pitowsky and Svozil\(^{31}\). Unfortunately, however, the result of such a search is typically a list of the coefficients of thousands of inequalities\(^{32}\), from which generalizable insights cannot easily be extracted.

Various aspects of the hierarchy of Bell inequalities have been investigated. Garg and Mermin\(^{19}\) for instance have resumed the idea of Bell and discussed systems, with maximal (anti-) correlation, and \(v > 2\). Gisin\(^{33}\) investigated setups with more than two dichotomic observables per site (however for arbitrary states). Another closely related subject of interest, which we will however not discuss, is to expose the non-local (or non-classical) character of nature without making use of Bell-type inequalities (cf. Ref.\(^{34,35}\)).

In the following we will again briefly discuss the CHSH\(^{26}\) inequalities as the first complete set of inequalities for the case \((n, m, v) = (2, 2, 2)\). Then we recapitulate the complete characterization for the multipartite case \((n, 2, 2)\)\(^{14,30}\), where additional symmetries enable us to give a rather exhaustive discussion.

### 3.1. The CHSH case as a complete set of inequalities

The CHSH inequalities are by far the best studied case of Bell inequalities. In this case, the characterization is complete in the following sense, first established by Arthur Fine\(^{25}\).

The following conditions on a correlation table for two parties with two dichotomic observables each \(((n, m, v) = (2, 2, 2))\) are equivalent:

1. The correlation table allows a local realistic model in the sense of Eq. (5).

2. The CHSH inequalities hold, i.e., Eq. (9) holds, also when any observables at one site or outcomes of a any observable are interchanged.

3. There exists a joint probability distribution for the outcomes of the four observables, which returns the measured correlation probabilities as marginals.

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possible correlations or restrict to a special class of observables or systems we want to investigate.
We have shown above that 1 and 3 are equivalent, and imply 2. Hence the non-trivial “completeness” part is to show that 2 implies the others, which can be done by analyzing the polytope discussed above.

Item 3 points to another interesting generalization of the CHSH inequalities. Given a set of probability distributions \( P_1, \ldots, P_k \), there always exists joint distribution that returns them as marginals, namely the product distribution \( \prod_{i=1}^{k} P_i \). However, if we fix in addition the joint distributions \( P_{ij} \) for a certain subset of pairs \((i, j)\), best visualized as a graph, a joint distribution with these marginals in general no longer exists. Bell inequalities thus appear as the obstructions to extending partial joint probability distributions.

3.2. All multipartite correlation Bell inequalities for two dichotomic observables per site



\( n \)-particle generalizations of the CHSH inequality were first proposed by Mermin\(^{37}\), and further developed by Ardehali\(^{38}\), Belinskii and Klyshko\(^{40}\) and others\(^{39,41}\). In these works the emphasis was to find just one inequality for every \( n \). In this section we give a complete set, first constructed in Refs.\(^{14,30}\).

The data under consideration are the \( 2^n \) full correlation functions of an arbitrary \( n \)-partite system, with two dichotomic observables per site. Each of the \( 2^n \) different experimental setups is labeled by the choice of observables at each site. We will parameterize these choices by binary variables \( s_k \in \{0, 1\} \) so that \( s_k \) indicates the choice of the \( \pm 1 \) valued observable \( A_k(s_k) \) at site \( k \). A “full correlation function” is then the expectation of a product

\[
\xi(s) = E\left( \prod_{k} A_k(s_k) \right),
\]

where the bit string \( s = (s_1, \ldots, s_n) \) labels the respective experimental setup. Hence \( \xi(s) \) can be considered as a component of a vector \( \xi \) in the \( 2^n \) dimensional space spanned by the experimental data, and any Bell inequality is therefore of the form

\[
\sum_s \beta(s) \xi(s) \leq 1,
\]

where we have normalized the coefficients \( \beta(s) \) such that the maximal classical value is 1 (i.e., for the CHSH case \( \beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \)). The linear combination in Eq. (16) may also be computed under the expectation value, so that this inequality can be stated as an upper bound to the expectation of

\[
\mathcal{B} = \sum_s \beta(s) \prod_{k=1}^{n} A_k(s_k).
\]

We will call such expressions Bell polynomials. They can be used directly in the quantum case as Bell operators, where all variables \( A_k(s_k) \) are substituted by operators with \( -1 \leq \)
$A_k(s_k) \leq 1$, acting in the Hilbert space of the $k$-th site, and the product is taken as the tensor product.

For the construction of a complete set of inequalities it suffices to consider the extremal cases, i.e., classical configurations, where each of the observables takes on one of its two values with certainty. The restriction to full correlation functions, i.e., disregarding correlations of less than $n$ sites, then enables us to exploit the invariance of $\xi$ under swapping the values of both observables on two sites. It is basically this symmetry, which leads to the fact that any of the $2^n$ binary vectors $f \in \{-1, 1\}^{2^n}$ with components $f(r)$, $r \in \{0,1\}^n$ corresponds to one Bell inequality via

$$\beta(s) = 2^{-n} \sum_r f(r)(-1)^{\langle r,s \rangle}, \quad (18)$$

where $\langle r,s \rangle = \sum_{i=1}^n r_is_i$ denotes the inner product. Moreover, Eq. (18) indeed provides a set, which is complete in the sense that the considered correlations allow a local classical model if and only if all these inequalities are satisfied.

Surprisingly, these $2^{2^n}$ linear inequalities are equivalent to a single non-linear inequality, namely

$$\sum_r |\hat{\xi}(r)| \leq 1 \quad \text{with} \quad \hat{\xi}(r) = 2^{-n} \sum_s \xi(s)(-1)^{\langle r,s \rangle}. \quad (19)$$

This is the characterizing inequality of a hyper-octahedron in $2^n$ dimensions. Hence the classical accessible region in this case has surprisingly high symmetry, which is unfortunately not a symmetry inherent to the underlying problem. One should thus not expect to find an analogous structure for other cases of $(n,m,v)$.

4. Quantum states with no violation

The violation of Bell’s inequality was the first mathematically sharp criterion for entanglement. In this section we describe cases in which this criterion fails to detect any entanglement, even though in some of these cases the quantum state may be “entangled” according to now current terminology.

4.1. Separable states

Generally, entanglement is defined in terms of its negation, and a quantum state is said to be unentangled, separable or classically correlated iff it can be written as a convex combination of product states.

Let $\rho$ be a density matrix corresponding to a composite quantum system described on the Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. Then by definition a separable state can be written as

$$\rho = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)}, \quad (20)$$

$^6$The possibility of replacing the set of linear inequalities by a single nonlinear one was apparently first recognized by Cirelson for the CHSH case in Ref. 63.
where the positive weights \( p_j \) sum up to one and \( \rho^{(i)} \) describes a state on \( \mathcal{H}^{(i)} \). The terminology “classical correlated” is justified due to the fact, that the preparation leading to the correlations can be assumed to be classical in the following sense. Suppose we have two independent preparing devices, one for each subsystem, which prepare a certain state \( \rho_j \) depending on some classical input \( j \). Then in order to obtain a state of the form (20) we have just to add a random generator, which produces numbers \( j \) with probability \( p_j \). Combining these three devices thus leads to such a “separable” state and the expectation value of two observables \( A^{(1)}, A^{(2)} \) is then given by

\[
\text{tr}(\rho A^{(1)} \otimes A^{(2)}) = \sum_j p_j \text{tr}(\rho_j^{(1)} A^{(1)}) \text{tr}(\rho_j^{(2)} A^{(2))}).
\]

The correlation thus just depend on the random generator, which however can be chosen to be a purely classical device\(^1\). Moreover, it is obvious from Eq. (21) that any separable state admits a description within the framework of a local classical model (as described in Sec.2.4.) and therefore satisfies any Bell inequality. The following subsection is concerned with the fact that the converse however is not true.

4.2. \( U \otimes U \) invariant states

They key idea in Ref.\(^8\) in order to circumvent at least to some extent both difficulties, the construction of a local classical model and the proof of non-separability, is to make extensive use of symmetries. The states considered are those commuting with all unitaries of the form \( U \otimes U \) and can be written as

\[
\rho(p) = (1 - p) \frac{P_+}{r_+} + p \frac{P_-}{r_-}, \quad 0 \leq p \leq 1,
\]

where \( P_+ (P_-) \) is the projector onto the symmetric (antisymmetric) subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \) and \( r_+ = \text{tr} P_+ = \frac{d^2 + d}{2} \) are the respective dimensions. It has been shown that states of the form (22) are separable iff \( p \leq \frac{1}{2} \) independent of the dimension of the system.

Now, consider a von Neumann measurement is performed on each of the two subsystems \( (i = 1, 2) \):

\[
A^{(i)} = \sum_{\mu} \alpha_{\mu}^{(i)} Q_{\mu}^{(i)} \quad \text{with} \quad \sum_{\mu} Q_{\mu}^{(i)} = 1,
\]

where the \( Q_{\mu}^{(i)} \) are one-dimensional orthogonal projectors. A description within a local classical model then would require that there exist a measure \( M \) on a probability space \( \Lambda \ni \lambda \) and response functions \( \chi^{(i)}(\mu, \lambda) \geq 0 \) (with \( \sum_{\mu} \chi^{(i)}(\mu, \lambda) = 1 \)) for any observable, such that

\[
\text{tr}(\rho Q_{\mu}^{(1)} \otimes Q_{\nu}^{(2)}) = \int M(d\lambda) \chi^{(1)}(\mu, \lambda) \chi^{(2)}(\nu, \lambda).
\]

For \( \Lambda \) being the unit sphere \( \{ \lambda \in \mathbb{C}^d | |\lambda| = 1 \} \) and the choice

\[
\chi^{(1)}(\mu, \lambda) = \langle \lambda, Q_{\mu}^{(1)} \lambda \rangle,
\]

\(^1\)We note that classical correlation does not mean that the state has actually been prepared in the manner described, but only that its statistical properties can be reproduced by a classical mechanism.
Bell inequalities and entanglement

\[ \chi^{(2)}(\nu, \lambda) = \begin{cases} 
1, & \langle \lambda, Q^{(1)}_{\nu} \lambda \rangle < \langle \lambda, Q^{(1)}_{\mu} \lambda \rangle \ \forall \mu \neq \nu \\
0, & \text{else} \end{cases} \]

it can be shown that Eq.(24) indeed holds for

\[ p = 1 - \frac{d + 1}{2d^2}, \tag{25} \]

which is in any nontrivial case larger than one half and thus corresponds to an entangled state. For increasing dimension Eq.(25) even approaches 1, which is (within the family of \( U \otimes U \) invariant states) as far removed from the classically correlated state \( p = \frac{1}{2} \) as possible.

4.3. PPT states

Another class of states for which it has been shown\(^{13,14}\) that none of the inequalities discussed in Sec.3. are violated is the class of states having a positive partial transpose (PPT states). Peres\(^{42}\) even conjectured that these states in general admit a description within a local classical model. Note that without additional assumptions the converse, however, is not true since the state (25) discussed in the previous subsection admits such a local classical description although it is no PPT state.

The partial transpose \( A^{T_1} \) of an operator \( A \) on \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) is defined in terms of its matrix elements with respect to a given basis by \( \langle k\ell|A^{T_1}|mn \rangle = \langle m\ell|A|kn \rangle \), and \( \rho \) is said to be a PPT state if \( \rho^{T_1} \geq 0 \).

The key idea for showing that PPT states satisfy any inequality coming from Eq.(18) is to utilize the positivity of the partial transpose by applying the variance inequality

\[ \text{tr}(\rho B)^2 = \text{tr}(\rho^{T_1} B^{T_1})^2 \leq \text{tr}(\rho^{T_1} B^{T_1})^2, \tag{26} \]

where \( B \) is the Bell operator, whose expectation value has to be bounded by one within a local classical model. Additionally averaging over all partial transposes with respect to any partition of the multipartite system into two subsystems then shows that in fact \( \text{tr}(\rho B) \leq 1 \).

Though the positivity of the partial transpose is known to be one of the most efficient separability criteria\(^{43}\) it is in general not a sufficient one. Hence there exist states, which are not classically correlated although they satisfy the PPT condition and therefore admit a (however possibly restricted) local classical description\(^{12}\). These states are often referred to as PPT bound entangled states since they have the additional property that their entanglement cannot be recovered by entanglement distillation\(^{44}\).

5. Quantum violations: The CHSH case

5.1. The See-Saw iteration

The derivation of the maximal quantum violation of a Bell inequality for an arbitrary state is a high dimensional variational problem for which we are not aware of any explicit
solution except for the case of the CHSH inequality for two qubit systems. Therefore we begin with providing a simple iterative algorithm, the See-Saw iteration, which turned out to be an efficient method for maximizing an affine functional with respect to hermitian operators with $-1 \leq A \leq 1$ corresponding to the expectation of a Bell operator with dichotomic observables. Since generalization to the multipartite case is straightforward, we content ourselves with bipartite systems where the functional to be maximized is of the form

$$\text{tr}(B \rho) = \sum_{i,j} \text{tr}(\beta_{ij} A_i \otimes B_j \rho),$$

(27)

where it suffices to consider unitary observables $A = A^* = A^{-1}$ (and the same for $B$), as these are extremal in the convex set of Hermitian operators with $-1 \leq A \leq 1$. The idea is now to maximize this functional with respect to observables on one site while keeping the other ones fixed, and then to iterate this procedure. Therefore, we rewrite Eq.(27) by taking the partial trace over that site of the system, which we keep fixed, i.e.:

$$\text{tr}(B \rho) = \sum_i \text{tr}_A \left( X_i A_i \right) = \sum_j \text{tr}_B \left( Y_j B_j \right) \quad \text{with}$$

(28)

$$X_i = \text{tr}_B \left( \sum_j \beta_{ij} (1 \otimes B_j) \rho \right)$$

(29)

$$Y_j = \text{tr}_A \left( \sum_i \beta_{ij} (A_i \otimes 1) \rho \right)$$

(30)

The maximization in Eq.(28), however, can be made explicit just by taking $A_i = \text{sign}(X_i)$ resp. $B_j = \text{sign}(Y_j)$. Of course, the See-Saw iteration is faced with the usual problem of most numerical optimization methods: it cannot guarantee the convergence on absolute extrema. Nevertheless it turned out to be a useful tool in the search for Bell violations (e.g. in Ref.14), which converges already after a few iterations and is in general very stable with respect to variations of the initial values.

5.2. Cirelson’s inequality

The best upper bound for the violation of the CHSH inequality, first derived by Cirelson45, is obtained by squaring the Bell operator and utilizing the variance inequality46, which already appeared in the previous section in Eq.(26). Taking again into account that it suffices to consider unitary observables of the form $A = A^* = A^{-1}$ we get

$$B^2 = 1 - \frac{1}{4} [A_1, A_2] \otimes [B_1, B_2].$$

(31)

Since the commutators are bounded by two as $\| [A, B] \| \leq 2 \| A \| \| B \|$ this leads to

$$\text{tr}(\rho B) \leq \sqrt{2},$$

(32)

which is usually referred to as Cirelson’s inequality.

5.3. Operators for maximal violation
The bound $\|A, B\| \leq 2\|A\|\|B\|$ used in the previous section is clearly saturated when $A$ and $B$ are Pauli matrices. It is therefore no surprise that experiments coming close to the maximal violation $\beta \approx \sqrt{2}$ are possible with qubit systems, and indeed this is precisely the idealized description of Aspect’s experiments. In this subsection we argue (following Ref.47) that the qubit example is even the only possibility to get the maximal violation in any dimension.

To give a more precise, but simple statement, suppose that both the restricted density operators are faithful (have no zero eigenvalues), and suppose $A_1, A_2, B_1, B_2$ are operators giving $\beta = \sqrt{2}$. Then $A_1, A_2$ and $A_3 = iA_2A_1$ satisfy all the algebraic relations of the Pauli matrices: $A_k^2 = 1$, $A_1A_2 = iA_3$ and cyclic permutations thereof.

The basic idea of the proof is to write the inequalities in terms of the two operators

$$A = \frac{1}{2}(A_1 + iA_2)$$

$$B = \frac{1}{2\sqrt{2}}[(B_1 + B_2) + i(B_1 - B_2)]$$

which allow a simple representation of the Bell operator as $B = \sqrt{2}(A^*B + AB^*)$. One readily checks that, on the other hand, $A^*A + AA^* \leq 1$ and $B^*B + BB^* \leq 1$. The core of the proof is the decomposition

$$1 - B/\sqrt{2} = (A - B)^*(A - B) + (A - B)(A - B)^* + (1 - A_1^2) + (1 - A_2^2) + (1 - B_1^2) + (1 - B_2^2).$$

Clearly, the right hand side is a positive operator, which provides an alternative proof of Cirelson’s inequality (32). States with maximal violation are those for which this equation, and hence every single term on the right hand side has expectation zero. In particular, for any vector $\Phi$ in the support of the density operator we get $(A - B)\Phi = (A^* - B^*)\Phi = (1 - A_1^2)\Phi = \cdots = 0$. Hence $(A^2 + A^*A)\Phi = (1/2)(A_2^2 - A_2^2)\Phi = 0$ and $(B^2 + B^*B)\Phi = (i/2)(B_2^2 - B_2^2)\Phi = 0$. Combining this with $A^2\Phi = B^2\Phi$ and $A^*A\Phi = B^*B\Phi$ we get $A^2\Phi = 0$ and similarly $(A^*A + AA^*)\Phi = \Phi$ for every vector $\Phi$ in the support of the density operator. Since the reduced density operator for Alice has full support, this implies the identities $A^*A + AA^* = 1$ and $A^2 = 0$, so that $A_1 = A + A^*$, $A_2 = i(A^* - A)$, and $A_3 = A^*A - AA^*$ are a realization of the Pauli matrices.

5.4. Qubits: Structure of the Bell operator

It is obvious from Eq.(31) that as soon as the observables on only one of the two subsystems commute, the inequality is satisfied. We may therefore disregard the case $A = 1$ and for the case of two qubits restrict to observables of the form $A_k(s_k) = \vec{a}_k(s_k)\vec{\sigma}$, where $\vec{\sigma}$ is the vector of Pauli matrices and $\vec{a}_k(s_k)$ is a normalized vector in $\mathbb{R}^3$. Furthermore, we can use the homomorphism between $SU(2)$ and $SO(3)$ and do a local unitary transformation such that the vectors belonging to the four observables all lie in the $x - y$ plane, i.e.:

$$A_k = \sigma_1 \sin \alpha_k + \sigma_2 \cos \alpha_k$$

(36)
With this choice of observables we get
\[ B^2 = 1 - \sin(\alpha_1 - \alpha'_1)\sin(\alpha_2 - \alpha'_2)\sigma_3 \otimes \sigma_3, \] (37)
such that the four Bell states \(|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)\) and \(|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\) turn out to be eigenstates of the CHSH operator. Moreover, \(B\) has symmetric spectrum since
\[ (1 \otimes \sigma_3)B = -B(1 \otimes \sigma_3) \]
so that we end up with a Bell operator which is up to local unitary transformations always of the form
\[ B = \lambda_1 (P_{\Phi^+} - P_{\Phi^-}) + \lambda_2 (P_{\Psi^+} - P_{\Psi^-}), \] (38)
where \(P\) denotes the projector onto the respective Bell states and the eigenvalues have to satisfy \(\lambda_1^2 + \lambda_2^2 = 2\), which follows from \(\text{tr}B^2 = 4\).

In fact, the spectral decomposition in Eq.(38) exhibits a property of the Bell operator, which is typical for any multipartite inequality for two dichotomic observables per site (see Sec.6.2.).

5.5. Qubits: Maximal violation for arbitrary states

Let us now consider an arbitrary quantum state \(\rho\) of two qubits and let \(R_{ij} = \text{tr}(\rho \sigma_i \otimes \sigma_j)\). Following Ref.\(^{50}\) the maximal violation of the CHSH inequality is then given by
\[ \beta(\rho) = \max_{\vec{a}_1, \vec{a}_1', \vec{a}_2, \vec{a}_2'} \frac{1}{2} \left( \vec{a}_1 \cdot R(\vec{a}_2 + \vec{a}_2') + \vec{a}_1' \cdot R(\vec{a}_2 - \vec{a}_2') \right) \]
\[ = \max_{\vec{a}_2, \vec{a}_2'} \frac{1}{2} \left( ||R(\vec{a}_2 + \vec{a}_2')|| + ||R(\vec{a}_2 - \vec{a}_2')|| \right) \]
\[ = \max_{\phi, \vec{c}, \vec{c}'} \cos \phi ||R\vec{c}|| + \sin \phi ||R\vec{c}'|| \]
\[ = \max_{\vec{c}, \vec{c}'} \sqrt{||R\vec{c}||^2 + ||R\vec{c}'||^2}, \] (39)
where the maxima are always taken over all unit vectors. Evaluating the last maximum we obtain
\[ \beta(\rho) = \sqrt{\nu + \nu'}, \] (40)
where \(\nu, \nu'\) are the two largest eigenvalues of the matrix \(R^TR\). For pure two qubit states, which can always be written in their Schmidt form as \(|\Psi\rangle = \cos \varphi |00\rangle + \sin \varphi |11\rangle\) this can further be evaluated to
\[ \beta(\Psi) = \sqrt{1 + \sin^2(2\varphi)}, \] (41)
which means that as soon as a pure two qubit state is entangled it violates the CHSH inequality.\(^{51}\)

5.6. Continuous variable systems

\(^{9}\)This result was generalized to higher dimensional bipartite systems by Gisin and Peres.\(^{52}\)
There is recent effort in order to adopt the CHSH inequality to the continuous variable case. One possibility in order to derive dichotomic observables in this case is to utilize the apparent analogy between the parity operator in Fock space and the “spin-measurement” in $\mathbb{C}^2$ associated to the Pauli $\sigma_3$ operator. As admissible observables in the continuous variable case we may then either use the set of coherently displaced parity operators or explicitly construct a direct analogue to the three Pauli operators by establishing the isomorphism $\ell^2(\mathbb{N}) \simeq \mathbb{C} \otimes \ell^2(\mathbb{N})$ via collecting parities. Let

\begin{align*}
  s_z &= \sum_{n=0}^{\infty} \left( |2n\rangle \langle 2n| - |2n+1\rangle \langle 2n+1| \right), \\
  s_+ &= \sum_{n=0}^{\infty} |2n\rangle \langle 2n+1|, \quad s_- = s_+^*,
\end{align*}

be the parity and parity flip operators in Fock state representation, then $(s_x, s_y, s_z)$ with $s_x \pm is_y = 2s_z$ is again a representation of the Pauli matrices. So we can in principle make use of all the above results and Eq.(39,40) lead to a lower bound on the maximal violation of the CHSH inequality for continuous variable states.

A special kind of such states are Gaussian states, i.e., states having a Gaussian Wigner distribution, which play a crucial role in quantum optics, since coherent, squeezed and thermal states are all Gaussians. If we only consider observables given by the field quadratures or simple functions thereof, then the positive Wigner function itself provides a “hidden variable distribution”. Hence no violation of a Bell inequality can occur. Observables obtained from Eq.(42,43), however, are not of that kind. For the pure Gaussian two mode state

\[ |\psi(r)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \otimes |n\rangle, \quad c_n = \frac{\tanh^n(r)}{\cosh(r)}, \]

which is characterized by the squeezing parameter $r$, the maximal violation with respect to these observables is

\[ \beta(r) = \sqrt{1 + \tanh^2(2r)} \]

analogous to Eq.(41).

### 5.7. Quantum Field Theory

It is not surprising that Quantum Field Theory should contain violations of Bell inequalities. After all, this theory is supposed to describe Aspects experiment, too. There are three special reasons, though to at this theory specifically. The first is that here one can take the locality assumption of the general derivations literally in the sense of Einstein causality. Thus Alice and Bob are assigned space time regions in which they can perform their experiments, and these regions are chosen to be spacelike separated, so that relativistic causality forbids any signaling between the two. Since we are looking at a quantum

\[ ^5 \text{For a different approach see for instance Ref.}^{54}, \text{where the chosen observables distinguish between photon counts and no photon counts.} \]
field theory, it is clear that quantum features are also contained in the description from the outset. The best adapted framework for this discussion is “algebraic quantum field theory” also called “Local Quantum Physics”: here the concepts of quantum structure and relativistic localization are taken as the axiomatic starting point.

The second feature making quantum field theory interesting is that we have here a distinguished state, the vacuum. As it is well known that there are always vacuum fluctuations, it is natural to ask whether these fluctuations are classical or not. More concretely: if Alice and Bob are in spacelike separated laboratories, can they get a violation of Bell’s inequalities from vacuum fluctuations alone? It turns out that they can, although the effect is extremely small for large spatial separation: in a massive theory, it decreases exponentially with the separation of the regions on the scale of the Compton wavelength. On the other hand, one gets maximal violation if the regions are very close.

The third reason to look a quantum field theory emerges from these studies: it turns out that not only the vacuum produces maximal violations at short range, but any state, which is not extremely singular (e.g., requires only finite total energy). This is a new possibility arising only in theories with infinitely many degrees of freedom, and quantum field theory is an ideal testing ground for ideas about this phenomenon.

We remark that it is still an open problem to show that even at arbitrarily large distance a (necessarily exponentially small) violation of the CHSH inequality can be detected. What is known so far is only that the positive partial transpose property always fails, and hence the vacuum is not classical at any distance.

6. Quantum Violations: Beyond CHSH

6.1. Bipartite systems with more than two observables

For the case of more than two dichotomic observables per site only little is known. In particular there is yet no explicit characterization of the extremal inequalities, although constructing some inequalities, e.g., by chaining CHSH inequalities, is not difficult. However, Cirelson recognized that the quantum correlation functions, which are in general rather cumbersome objects, can be reexpressed in terms of finite dimensional vectors in Euclidean space.

If we have two sets of observables $\{A_1(s)\}$ and $\{A_2(t)\}$ (again hermitian with $-1 \leq A \leq 1$) where $s \in \{1, \ldots, p\}$ and $t \in \{1, \ldots, q\}$, then for any state $\rho$ there exist sets of real unit vectors $\{x_s\}$ and $\{y_t\}$ in the Euclidean space of dimension $q + p$ such that

$$\text{tr}(\rho A_1(s) \otimes A_2(t)) = \langle x_s, y_t \rangle \quad \forall s, t.$$  \hfill (46)

For the case of two observables on one site and an arbitrary number on the other Cirelson showed that the maximal quantum violation is $\sqrt{2}$ and thus already obtained for the CHSH inequality ($p = q = 2$). For an increasing number of observables on both sites, however, he obtained the Grothendieck constant ($\approx 1.782$), known from the geometry of Banach spaces, as a limit for the maximal violation.

\footnote{Technically speaking: any state which is locally normal with respect to the vacuum}
6.2. The multipartite case – the role of GHZ states and Mermin’s inequality

Like the four Bell states are of particular importance for the CHSH operator, the generalized GHZ states, which are up to local unitaries of the form

\[ |\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}((|00\ldots0\rangle + |11\ldots1\rangle)), \]  

(47)

play a special role for all multipartite inequalities with two dichotomic observables per site. In fact, they provide a basis of eigenstates for all the Bell operators with extremal observables and lead thus to maximal violations.

If we again consider observables of the form in Eq. (36), which are obtained after applying local unitaries, and let \( \Omega \in \{0, 1\}^n \) and its complement \( \Omega^C \) with \( \Omega^C_k = 1 - \Omega_k \) characterize vectors on the \( n \)-fold tensor product, then

\[ \bigotimes_{k=1}^n A_k(s_k)|\Omega\rangle = f_{\Omega}(s)|\overline{\Omega}\rangle, \quad \text{with} \]

\[ f_{\Omega}(s) = \exp \left[ i \sum_{k=1}^n \alpha_k(s_k)(-1)^{\Omega_k} \right]. \]  

(49)

Therefore the set of \( 2^n \) GHZ-like basis vectors \( |\Psi_{\Omega}\rangle = \frac{1}{\sqrt{2}}(e^{i\theta_{\Omega}}|\Omega\rangle + |\overline{\Omega}\rangle) \) satisfies the eigenvalue equations \( B|\Psi_{\Omega}\rangle = \lambda_{\Omega}|\Psi_{\Omega}\rangle \) if \( \theta_{\Omega} \) is chosen such that

\[ \lambda_{\Omega} = e^{i\theta_{\Omega}} \sum_s \beta(s)f_{\Omega}(s) \]  

(50)

is real. Hence any Bell operator for multipartite systems with two dichotomic observables per site indeed admits spectral decomposition into GHZ states, and it is an immediate corollary thereof that GHZ states lead to maximal violations. It was shown in Ref. 14 that the computation of the latter can be reduced to a variational problem with just one free variable per site. Moreover, even any extreme point of the convex body of the quantum mechanically attainable correlation functions is found in the generalized GHZ states.

It is crucial for the derivation of all the above results that we have no more than two dichotomic observables, since three or more vectors do in general not lie in a plane and we would therefore not be able to restrict to observables of the form in Eq. (36).

If we now fix the state to be of the GHZ form (47) and ask for the inequality within the set (18) leading to the maximal violation, we obtain Mermin’s inequality. For an \( n \)-partite system the Bell operator corresponding to Mermin’s inequality is defined recursively starting with \( B_1 = A_1 \) by

\[ B_n = \frac{B_{n-1}}{2} \otimes (A_n + A'_n) + \frac{B'_{n-1}}{2} \otimes (A_n - A'_n), \]  

(51)

\(^1\)In fact, Mermin\(^37\) derived the inequality corresponding to Eq. (51) for odd \( n \), whereas Ardehali\(^38\) in turn obtained the one for even \( n \). It was then Klyshko and Belinskii\(^40\), who recognized that Eq. (51) covers both types of inequalities (see also Ref.\(^14\)). However, since the basic idea is going back to David Mermin it seemed justifiable to us to refer to the inequality as “Mermin’s inequality.”
Permitting arbitrary unrelated choice for the \( n - 1 \) partite Bell operators \( B_{n-1} \) and \( B'_{n-1} \) it was shown in Ref.\(^{14}\) that any inequality of the set (18) can be obtained from Eq.(51), i.e., by nesting CHSH inequalities. Squaring the Bell operator as in Eq.(31) leads to
\[
\text{tr}(\rho B_n) \leq 2^{n-1},
\]
which is however only saturated for Mermin’s inequality\(^{14}\). Hence the maximal violations grow exponentially with the number of subsystems. However, one has to keep in mind that the joint efficiency of \( n \) independent detectors would in turn decline exponentially in \( n \).

Finally we should mention that for the case of multipartite systems with an infinite set of settings, i.e., more than two dichotomic observables per site, Zukowski\(^{65}\) derived an inequality for which the GHZ state leads to an exponentially growing violation of \( \frac{1}{2} \left( \frac{\pi}{2} \right)^n \).

### 7. Relations to Quantum Information Theory

One of the essential innovations of quantum information theory is to think of entanglement as a resource for quantum information processing purposes. This new point of view led to a dramatic increase of knowledge about the structure of the state space with respect to entanglement properties. Whereas in the late eighties there was hardly any difference between entangled states and states violating some Bell inequality, we have a much more subtle discrimination nowadays.

Figure 1 summarizes the previously discussed relations between various degrees of classicalness, i.e., negations of entanglement properties. The implication “PPT \( \Rightarrow \) Bell inequalities” thereby means, that this holds for all inequalities of the form in Eq.(18). In other words, there exists a local hidden variable model if we only consider full correlation functions of two dichotomic observables per site. Since no counterexample could be found for other cases so far one might follow Asher Peres\(^{42}\) and conjecture that positivity of the partial transpose generally implies the existence of a local hidden variable model.

Other open problems are associated with the distillability of entangled states, i.e., the possibility of extracting maximally entangled states by means of classical communication and local operations on several copies of the input state. It is for instance still an open question whether positivity of the partial transpose is necessary for undistillability\(^{66}\). Moreover, it is yet not clear whether the violation of a bipartite Bell inequality already implies distillability. For multipartite systems, however, the structure of the state space with respect to entanglement properties is much richer and Dür\(^{15}\) recently showed that there exist indeed undistillable multipartite states\(^7\) violating Mermin’s inequality.

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\(^{6}\)A different way of deriving Mermin’s inequality is obtained by identifying the expectation values of observables \( A \) and \( A' \) of a single site with a square in the complex plane. After a suitable linear transformation (a \( \pi/4 \) rotation and a dilation) we can take it as the square \( S \) with corners \( \pm 1 \) and \( \pm i \). The pair of expectation values of \( A \) and \( A' \) is thus replaced by the single complex number \( \text{tr}(\rho a) \), where
\[
a = \frac{1}{2} \left( (A + A') + i(A' - A) \right) = e^{-i\pi/4} \frac{(A + iA')}{\sqrt{2}}.
\]
The basic idea behind this transformation is that products of complex numbers lying in \( S \) again lie in \( S \). Since this also holds for convex combinations and pure states within a local classical model are always of a product form, the statement that the product of such complex expectations lies in \( S \) indeed corresponds to a Bell inequality. In fact, this is essentially Mermin’s inequality (see Ref.\(^{13}\)).

\(^{7}\)The example in Ref.\(^{15}\) makes use of at least eight parties. However, the same techniques can be applied
Of course Fig. 1 is far from being complete. Relations between Bell inequalities and usefulness for teleportation\textsuperscript{67}, quantum key distribution and quantum secret sharing\textsuperscript{68,69} have also been studied. Scarani and Gisin\textsuperscript{69} for instance have recently argued that in a secret sharing protocol the authorized partners have a higher mutual information than the unauthorized ones, iff they could violate Mermin’s inequality. Such a connection has already been suggested by the coincidence that for two-partner key distribution a secret key can be established using one-way privacy amplification iff the two partners can violate the CHSH inequality\textsuperscript{70,71}.

In this way Bell inequalities seem to appear in a new context, playing the role of witnesses for the usefulness of a state for certain quantum communication purposes.

However, the resource point of view of entanglement also requires a quantitative description, which tells us how much entanglement is present in a given quantum state. So why not take the maximal violation of a Bell inequality as a measure of entanglement? Although, this seems to be quite reasonable at first, the works of Popescu\textsuperscript{9} and Gisin\textsuperscript{11} have shown that the maximal violation is in fact not capable of measuring entanglement. By definition entanglement is that part of correlations between several subsystems, which is not “classical”. Therefore a measure of entanglement has to be able to distinguish this non-classical part from classical correlation, which can increase under local operations and classical communication (LOCC). However, Ref.\textsuperscript{9,11} have shown in an impressive way, that the maximal violation of a Bell inequality does not behave monotonously under LOCC operations. Therefore Bell violations merely give a hint for the strength of entanglement but they do not fulfill the usual requirements for entanglement measures.

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\textsuperscript{9} for 4-partite systems.
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References

1. A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
2. E. Schrödinger, Naturwissenschaften 23, 807-812; 823-828; 844-849 (1935).
3. N. Bohr, Nature 136, 65 (1935); Phys. Rev. 48, 696 (1935).
4. D. Bohm, Quantum Theory (Prentice-Hall, Englewood Cliffs, New York, 1951).
5. J.S. Bell, Physics 1, 195 (1964).
6. A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. 47, 460 (1981).
7. M. A. Rowe et al., Nature 409, 791 (2001).
8. R.F. Werner, Phys. Rev. A 40, 4277 (1989).
9. S. Popescu, Phys. Rev. Lett. 74, 2619 (1995).
10. C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W.K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
11. N. Gisin, Phys. Lett. A 210, 151 (1996).
12. M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
13. R.F. Werner and M.M. Wolf, Phys. Rev. A 61, 062102 (2000).
14. R.F. Werner and M.M. Wolf, quant-ph/0102024 (2001).
15. W. Dürr, quant-ph/0107050 (2001).
16. A. Cabello, Phys. Rev. Lett. 86, 1911 (2001).
17. J.F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978).
18. D. Home and F. Selleri, Riv. Nuovo Cimento 14, 1 (1991).
19. A. Garg and N.D. Mermin, Found. Phys. 14, 1 (1984).
20. N.D. Mermin, Rev. Mod. Phys. 65, 803 (1993).
21. J.v. Neumann, Mathematische Grundlagen der Quantenmechanik, (Springer, Berlin, 1932).
22. D. Bohm, Phys. Rev. 85, 166 (1952); 85, 180 (1952).
23. E. Nelson, Dynamical Theories of Brownian Motion, (Princeton University Press 1967).
24. R.F. Werner, Phys. Rev D 34, 463 (1986).
25. A. Fine, Phys. Rev. Lett. 48, 291 (1982); A. Fine, J. Math, Phys. 23, 1306 (1982).
26. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
27. R.F. Werner in Quantum Information – an introduction to basic theoretical concepts and experiments (Springer tracts in modern physics, Berlin, 2001).
28. I. Pitowsky, Quantum Probability – Quantum Logic (Springer, Berlin, 1989).
29. I. Pitowsky, Mathematical Programming 50, 395 (1991).
30. M. Zukowski and C. Brukner, quant-ph/0102039 (2001).
31. I. Pitowsky and K. Svozil, quant-ph/0011060 (2000).
32. A list of the coefficients belonging to the 53856 inequalities for the case $(n, m, v) = (3, 2, 2)$ can be found on http://tph.tuwien.ac.at/~svozil/publ/ghzbigs.html.
33. N. Gisin, Phys. Lett. A 260, 1 (1999).
34. N.D. Mermin, Phys. Rev. Lett. 65, 3373 (1990).
35. D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett. 85, 4418 (2000).
36. A. Garg and N.D. Mermin, Phys. Rev. Lett. 49, 1220 (1982).
37. N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
38. M. Ardehali, Phys. Rev. A 46, 5375 (1992).
39. S. M. Roy and V. Singh, Phys. Rev. Lett. 67, 2761 (1991).
24 BELL INEQUALITIES AND ENTANGLEMENT

40. A. V. Belinskii and D. N. Klyshko, Sov. Phys. Usp. 36, 653 (1993).
41. N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998).
42. A. Peres, Found. Phys. 29, 589 (1999).
43. A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
44. C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
45. B. S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
46. L. J. Landau, Phys. Lett. A 120, 54 (1987).
47. S.J. Summers and R.F. Werner, Commun. Math. Phys. 110, 247 (1987).
48. S.J. Summers and R.F. Werner, Ann. Inst. Henri Poincaré 49, 215 (1988).
49. V. Scarani and N. Gisin, quant-ph/0103068 (2001).
50. R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).
51. N. Gisin, Phys. Lett. A 154, 201 (1991).
52. N. Gisin and A. Peres, Phys. Lett. A 162, 15 (1992).
53. K. Banaszek and K. Wódkiewicz, Phys. Rev. A 58, 4345 (1998).
54. K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. 82, 2009 (1999).
55. H. Jeong, J. Lee, M.S. Kim, Phys. Rev. A 61, 052101 (2000).
56. Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, quant-ph/0103051 (2001).
57. R. Haag and D. Kastler, J. Math. Phys. 5, 848 (1964).
58. R. Haag, Local Quantum Physics, Springer (1992).
59. S.J. Summers and R.F. Werner, Phys. Lett. A 110, 257 (1985).
60. R. Verch and R.F. Werner, in preparation.
61. S.L. Braunstein and C.M. Caves, Ann. Phys. 202, 22 (1990).
62. B.S. Cirel’son, Lett. Math. Phys., 4, 93 (1980).
63. B.S. Tsirel’son, J. Sov. Math., 36, 557 (1987).
64. D.M. Greenberger, M. Horne, and A. Zeilinger in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, M. Kafatos, ed., Kluwer, Dordrecht (1989).
65. M. Zukowski, Phys. Lett. A 177, 290 (1993).
66. D.P. DiVincenzo, P.W. Shor, J.A. Smolin, B.M. Terhal, and A.V. Thapliyal, Phys. Rev. A 61, 062312 (2000); W. Dür, J.J. Cirac, M. Lewenstein, D. Bruss, Phys. Rev. A 61, 062313 (2000); T. Eggeling, K.G.H. Vollbrecht, R.F. Werner, and M.M. Wolf, quant-ph/0104095 (2001).
67. R. Horodecki, M. Horodecki, and P. Horodecki, quant-ph/9606027 (1996).
68. V. Scarani and N. Gisin, quant-ph/0104016 (2001).
69. V. Scarani and N. Gisin, quant-ph/0104016 (2001).
70. B. Huttner and N. Gisin, Phys. Lett. A 228, 13 (1997).
71. C. Fuchs, N. Gisin, R.B. Griffiths, C.-S. Niu, and A. Peres, Phys. Rev. A 56, 1163 (1997).
72. P.M. Gruber and J.M. Wills (editors), Handbook of convex geometry, North-Holland (1993).
73. H.H. Schaefer, Topological Vector Spaces (Springer, Berlin, 1980).
74. R. Webster, Convexity (Oxford University Press, 1994).
75. P. McMullen, J. Combin. Theory Ser. B 10, 187 (1971).
76. I. Bárány and A. Pór, 0-1 polytopes with many facets, manuscript, Rényi Institute of Mathematics, Hungarian Academy of Sciences; \url{http://www.renyi.hu/~barany/} (2000).
77. C.K. Yap, Fundamental problems in algorithmic algebra (Oxford University Press, 2000); see also \url{http://www.ifor.math.ethz.ch/fukuda/polyfaq/polyfaq.html} for a brief review.
78. B. Chazelle, Discrete Comput. Geom. 10, 377 (1993).
79. D. Avis and K. Fukuda, Discrete Compute. Geom. 8, 295 (1992).
80. F. Barahona, J. Phys. A 15, 3241 (1982).
Appendix A

8. Bell inequalities and convex geometry

In this appendix we return to the construction of Bell inequalities, which we started to discuss in Sec.3., and emphasize that finding all the Bell inequalities is a special instance of a standard problem in convex geometry known as the convex hull problem. This point of view provides a rather intuitive geometrical interpretation of Bell inequalities as well as making contact to one of the oldest fields in mathematics.

Let us again consider an $n$-partite system, where each party has the choice of $m^v$-valued observables to be measured. Note that we choose such a symmetric setting just in order to circumvent cumbersome notation – the basic idea, however, is obviously independent of the type of considered correlations.

Hence, we have $m^n$ different experimental setups and each of them may lead to $v^n$ different outcomes, such that the raw experimental data are made up of $(mv)^n$ probabilities. These numbers form a vector $\xi$ lying in a space of dimension $(mv)^n$ (minus a few for normalization constraints), to which we will refer to as the correlation space.

Now, we ask for the region $\Xi$ in this correlation space, which is accessible within any local classical model. The crucial characteristic of these models is that any vector $\xi$ is generated by specifying probabilities for each classical configuration, i.e., for every assignment of one of the $v$ values to each of the $nm$ observables. Locality is thereby expressed in the fact that the assignment of a value to an observable at one site does not depend on the choice of observables at the other sites.

Since every classical configuration $c$ is also represented by a vector $\epsilon_c$ of probabilities, the classical accessible region is just the convex hull of at most $v^{(nm)}$ explicitly known extreme points:

$$\Xi = \text{conv}\{\epsilon_c\}. \quad (A.1)$$

Since the numbers of configurations is finite $\Xi$ is a convex polytope.

8.1. Representations of convex polytopes and the hull problem

Every polytope has two representations. It can either be expressed in terms of a finite number of extreme points ($V$ representation) or as the intersection of halfspaces ($H$ representation), i.e., as a set of solutions to a system of linear inequalities – which in our case are the Bell inequalities. The set of linear inequalities corresponding to all halfspaces containing $\Xi$ is represented by the set

$$\Xi^* = \{\beta : \forall c : \langle \beta, \epsilon_c \rangle \leq 1\}, \quad (A.2)$$

called the polar of $\Xi$, which is in turn a convex polytope of the same dimension. The duality between a convex set and its polar is a generalization of the duality between regular platonic solids, under which the octahedron and the cube as well as the dodecahedron and the icosahedron are polars of each other.
It is obvious from the convexity of $\Xi$ that for each $\beta \in \Xi^*$ the inequality $\langle \beta, \xi \rangle \leq 1$ is necessary for $\xi \in \Xi$. Moreover, the Bipolar theorem\cite{bipolar} says that the collection of all these inequalities is also sufficient and since $\Xi^*$ is also convex it suffices to look at the extreme points of $\Xi^*$. We are therefore left with the following problem known as the convex hull or face enumeration problem: given the extreme points of a polytope find the extreme points of its polar. If there are initially more points than extreme points, then the convex hull problem is said to be degenerate.

8.2. The number of facets

The total number of Bell inequalities for a given class of correlations is not known except for the case described in Sec.3. However, convex geometry provides a number of general results and upper bounds, which we will just briefly discuss. The most well-known result in the combinatorial theory of polytopes is probably the Euler-Poincaré\cite{euler} relation stating that the numbers $\{f_j\}$ of faces of dimension $j$ are related via

$$\sum_{j=0}^{D-1} (-1)^j f_j = 1 - (-1)^D,$$

where $D$ is the dimension of the polytope. Moreover, McMullen's\cite{mcmullen} upper bound theorem implies that a polytope with $N$ vertices has at most $f_{D-1} \sim O(N^{[D/2]})$ facets, which is in our case the number of Bell inequalities. This bound is also tight as exhibited by cyclic polytopes. However, the class of polytopes occurring in the construction of Bell inequalities is of a rather special type often referred to as 0-1 polytopes since the components of any extreme point $\epsilon_c$ are either 0 or 1. The question whether the number of facets of such polytopes is bounded by an exponential in $D$ was just recently given a negative answer to by Bárány and Pór\cite{barany}, who showed that there exists a positive constant $c$ such that

$$f_{D-1} > \left( \frac{cD}{\log D} \right)^{D/4},$$

(A.3)

for some 0-1 polytopes. Hence the growth can in general be superexponential.

8.3. Complexity of convex hull algorithms

There are several ways of measuring the complexity of a convex hull algorithm\cite{complexity}. Basically, there are two points in which different approaches differ from each other: the elementary operations (bit operations vs. elementary arithmetic operations) and the role of the output (whether it is a parameter of the measure or not).

One way would be just to count the number of elementary arithmetic operations and to assume that the storage of an integer number takes a unit space (which is essentially the unit cost RAM model as opposed to the Turing model). Fixing the dimension $D$ of the polytope and then looking at the worst-case running time as a function of $N$ an optimal algorithm is known\cite{algorithm} that runs in time $O(N^{[D/2]})$, as already suggested by McMullen’s upper bound theorem.

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\cite{barany}See also the problem page http://www.imaph.tu-bs.de/qi/problems/1.html.
\cite{barany}A subset of the polytope $P$ is called a face if it is the intersection of the polytope with one of its supporting hyperplanes, i.e., a plane $h$ such that one of the closed halfspaces of $h$ contains $P$. The faces of dimension 0, 1, $D-1$ are called vertices (extreme points), edges and facets. Every face is again a convex polytope.
For the general nondegenerate convex hull problem there are algorithms with run times which are polynomially bounded by $N, D$ and $f_{D-1}$ (e.g.\textsuperscript{79}). For the degenerate case however no such polynomial algorithm is known.

For a more detailed discussion of the complexity of finding a complete set of Bell inequalities we would like to refer to the work of Pitowsky\textsuperscript{28,29}, who also discusses the relation between the convex hull problem and the notorious $NP = P$ resp. $NP = coNP$ questions. It is shown, that deciding membership in a “correlation polytope” is an NP-complete problem, whereas deciding facets is probably not even in NP. Moreover, in Ref.\textsuperscript{29} the relation to the minimum energy problem for Ising spin systems is discussed, which in turn was shown to be NP-hard by Barahona\textsuperscript{80}.