Conformally invariant cosmology based on Riemann–Cartan spacetime

Yuri V Shtanov\textsuperscript{a,b} and Sergei A Yushchenko\textsuperscript{a}

\textsuperscript{a}Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine
\textsuperscript{b}Department of Physics, Brown University, Providence, RI 02912, USA

E-mail: shtanov@bitp.kiev.ua, yushchenko@bitp.kiev.ua

\textbf{Abstract.} Conformally invariant GUT-like model including gravity based on Riemann–Cartan space-time $U_4$ is considered. Cosmological scenario that follows from the model is discussed and standard quantum gravitational formalism in the Arnowitt–Deser–Misner form is developed. General formalism is then illustrated on Bianchi-IX minisuperspace cosmological model. Wave functions of the universe in the de Sitter minisuperspace model with Vilenkin and Hartle–Hawking boundary conditions are considered and corresponding probability distributions for the scalar field values are calculated.

PACS numbers: 04.60-m, 98.80.Qc

\textit{Preprint} BROWN-HET-937
1. Introduction

In unifying gravity with other interactions concepts of symmetries, undoubtedly, play an important role. Among various symmetries that have been put forward at different times the symmetry with respect to local conformal transformations remains to be of special significance and keeps attracting attention of many researches. First, because theories with such a symmetry typically exhibit better quantum behaviour, and the issue of renormalizability might lead to conformal invariance. Second, because such kind of symmetry might help to resolve the cosmological constant problem (the usual $\Lambda$-term breaks the conformal symmetry of the action and therefore is excluded simply by a demand of such a symmetry).

The original idea of local conformal invariance belongs to Weyl [1] who introduced a compensating gauge vector field for the transformations under consideration. The corresponding geometric structure is usually called Weylian. It was afterwards developed in various directions [2–10].

Invariance with respect to local conformal transformations can be implemented into theory of gravity in different manners [11]. One of the rather simple ways is to base a theory on Riemann–Cartan spacetime which naturally arises within the framework of the Poincaré gauge theory of gravity. Some work has already been done along these lines. Conformally invariant dynamics of spinor fields on a background Riemann–Cartan space were studied in [12]. In the works [13] a simple example of the gravitational part of a theory of such a kind has been considered. It incorporated a scalar field into the term $\varphi^2 R$ in the Lagrangian. The aim of the present paper is to generalize these proposals and to consider a realistic theory which possesses Weyl invariance. In doing this we take into account that the vector trace $Q_\mu$ of the torsion tensor transforms under the action of the local conformal group similarly to the Weyl vector. Making use of this property we construct a generic GUT-like model, based on the Riemann–Cartan geometric structure, which contains a scalar field multiplet with its kinetic term in the Lagrangian and which is invariant with respect to local conformal transformations. The key difference of the theory considered here from the theory of Weyl is that in the last one a special vector field is introduced as a compensating gauge field for the local conformal transformations whereas in our theory this role is played by the torsion trace vector. We therefore avoid to introduce new specific entities to make our theory conformally invariant.

The role of the scalar fields in the theory considered here, on the one hand, is the same as in ordinary GUT models. They give masses to gauge vector bosons and to fermions through the symmetry breaking mechanism of Higgs, making it possible to preserve the (ordinary) gauge invariance of the theory. On the other hand, they allow one to use the term of type $\varphi^2 R$ in the Lagrangian and thus to generate the gravitational coupling constant. In such a way the theory is extended to include gravity and, at the same time, remains to be locally conformally invariant. In this paper we will consider some of the basic features of such a theory, in particular, those related to inflationary and quantum cosmology.
We will see below that the cosmological constant problem is not actually solved in our theory. However, the value of $\Lambda$ in the effective $\Lambda$-term is now not an independent parameter, but is a function of some other parameters of the model. It is therefore at least restricted and further investigation might discover the possibility of having it equal to zero without fine tuning the constants of the theory.

The paper is organized as follows. In the next section we present a locally conformally invariant version of a generic GUT-like model coupled to gravity. In Section 3 a cosmological inflationary scenario based on our theory is discussed. In Section 4 we develop a standard quantum cosmological formalism in the Arnowitt–Deser–Misner (ADM) form for the theory considered. General formalism will then be illustrated in Section 5 on a minisuperspace cosmological model of Bianchi-IX type. In the Appendix we provide the necessary geometric background including the description of the Riemann–Cartan geometric structure and local conformal transformations.

2. Conformally invariant theory

Our theory will be based on Riemann–Cartan space $U_4$ (for more detailed description and basic notations see the Appendix). Riemann–Cartan structure implies the presence of the affine connection form $\omega^a{}_b$ and the metric tensor $g$ which is covariantly constant:

$$\nabla^\omega g = 0,$$

where the symbol $\nabla^\omega$ denotes the covariant derivative specified by the affine connection $\omega^a{}_b$. The affine connection is supposed to satisfy only the metricity condition (1) hence torsion tensor is not assumed to vanish.

We are going to consider a rather natural conformally invariant generalization of GUT-like model with the Lagrangian of the form

$$L = \frac{1}{2} m^2 (\varphi) R(\omega) + i \frac{1}{2} \left( \bar{\psi} \gamma^\mu D_\mu \psi - (\bar{D}_\mu \psi) \gamma^\mu \psi \right) - \{ f \varphi \bar{\psi} \psi \}$$

$$- \frac{1}{2} |D \varphi|^2 - V(\varphi) - \frac{1}{4e^2} Tr F_{\mu\nu} F^{\mu\nu}. \tag{2}$$

It is constructed from a multiplet $\psi$ of spinor fields, a multiplet $\varphi$ of scalar (Higgs) fields (multiplet indices of $\varphi$ and $\psi$ will often be omitted), gauge connection field $A = A_\mu dx^\mu$ of some gauge group $G$, metric $g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} e^a \otimes e^b$ which can be presented by the orthonormal tetrad components $e^a_\mu$ or $e^a_\mu$, and affine connection field $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu = \omega^a{}_{bc} e^c$. The metric signature is taken to be $(-, +, +, +)$. Greek indices are raised and lowered by the metric components $g_{\mu\nu}$ and $g^{\mu\nu}$ in the coordinate base, and Latin indices - by the metric components $\eta_{ab}$ and $\eta^{ab}$ in the orthonormal tetrad base. $R(\omega)$ is the Riemann–Cartan curvature scalar. "Long" derivatives $D_\mu$ which enter the Lagrangian (2) are defined by the equations

$$D_\mu \psi = (\partial_\mu + \omega_\mu - Q_\mu + A_\mu) \psi, \tag{3}$$
Conformally invariant cosmology based on Riemann–Cartan spacetime

\[ D_\mu \varphi = (\partial_\mu - \frac{2}{3}Q_\mu + A_\mu)\varphi, \]  
(4)

where \( Q_\mu \) is the vector trace of the torsion tensor: \( Q_\mu = Q^a_{\mu a} \), and

\[ \omega = -\frac{1}{8} \omega^{ab} [\gamma_a, \gamma_b] \]  
(5)

is the spin connection form. \( \gamma^a \) are the usual constant Dirac matrices. The quantities \( F_{\mu \nu} \) are the components of the curvature two-form of the gauge connection form \( A = A_\mu dx^\mu \):

\[ iF = dA + A \wedge A. \]  
(6)

The symbol \( \{ f \varphi \bar{\psi} \bar{\psi} \} \) denotes the sum of various possible Yukawa couplings between the spinor and the scalar Higgs multiplets with coupling constants \( \{ f \} \). The values \( m^2(\varphi) \) and \( V(\varphi) \) are assumed to be analytic in \( \varphi \), so in order to preserve conformal invariance of the action (about the conformal transformations see below) they must represent respectively a quadratic and a quartic forms of \( \varphi \). We assume \( m^2(\varphi) \) to be positive definite. Thus for the multiplet \( \{ \varphi^A, A = 1, \ldots, k \} \) of real scalar fields we put

\[ m^2(\varphi) = \xi_{AB} \varphi^A \varphi^B, \]  
(7)

where \( m^2(\varphi) > 0 \) for \( \varphi \neq 0 \), and

\[ V(\varphi) = \lambda_{ABCD} \varphi^A \varphi^B \varphi^C \varphi^D, \]  
(8)

where \( \xi_{AB} \) and \( \lambda_{ABCD} \) are real dimensionless constants symmetric in their indices. The functions \( m^2(\varphi) \), \( V(\varphi) \) and \( \{ f \varphi \bar{\psi} \bar{\psi} \} \) are assumed also to be \( G \)-invariant. Finally, the trace in (2) is taken in the representation space of the group \( G \).

The action of the theory is written as

\[ S = \int_M L \sqrt{-g} d^4x + \int_{\partial M} m^2(\varphi) K(\omega) \sqrt{h} d^3x, \]  
(9)

where \( K(\omega) \) is the scalar extrinsic curvature (with respect to the connection \( \omega^a_{\ b} \)) of the boundary \( \partial M \) of the integration region \( M \), \( h_{ij} \) \((i, j = 1, 2, 3)\) are the induced metric components on this boundary, and \( h = \det(h_{ij}) \).

The action (9) with the Lagrangian (2) was constructed so as to differ as little as possible from the general relativity action. The only basic difference between them is the presence of the function \( m^2(\varphi) \) instead of a constant \((8\pi G_N)^{-1}\), and the presence of torsion in the scalar curvature \( R \) and in the metric sector of the theory.

The action (9) is invariant with respect to the group of local conformal transformations

\[ g(x) \rightarrow g'(x) = \exp (2\sigma(x)) g(x), \]  
(10)

\[ \varphi(x) \rightarrow \varphi'(x) = \exp (-\sigma(x)) \varphi(x), \]  
(11)

\[ \psi(x) \rightarrow \psi'(x) = \exp \left( -\frac{3}{2} \sigma(x) \right) \psi(x), \]  
(12)
Conformally invariant cosmology based on Riemann–Cartan spacetime

\[ \bar{\psi}(x) \rightarrow \tilde{\psi}(x) = \exp \left( -\frac{3}{2} \sigma(x) \right) \bar{\psi}(x), \]

\[ A(x) \rightarrow A'(x) = A(x), \]

\[ \omega^a_{\ b}(x) \rightarrow \omega'^a_{\ b}(x) = \omega^a_{\ b}(x), \]

where \( \sigma(x) \) is an arbitrary real function. Such an invariance is provided in particular by using the “long” derivatives \( D_\mu \) as defined in (3) and (4). They involve torsion trace vector \( Q_\mu \) which under the action of the local conformal group transforms similarly to the Weyl gauge vector field (see the Appendix):

\[ Q_\mu(x) \rightarrow Q'_\mu(x) = Q_\mu(x) - \frac{3}{2} \partial_\mu \sigma(x). \]

Due to this property “long” derivatives (3) and (4) transform under the local conformal group just like the corresponding (spinor and scalar) fields themselves. Note that the torsion trace \( Q_\mu \) in the derivative (3) of a spinor in fact drops out of the spinor kinetic term of the Lagrangian (2) due to Hermitian form of the latter. Conformal invariance of the action (9) is also provided by the specific shape (7) and (8) of the functions \( m^2(\varphi) \) and \( V(\varphi) \) respectively.

Both terms of the action (9) are invariant with respect to the local conformal transformations written just above. The second, boundary, term does not affect the equations of motion. We have added it to the action in order to recover, in the natural gauge \( m^2(\varphi) = \text{const} \), the usual Gibbons-Hawking boundary term [14] of the Hilbert-Einstein action for gravity, as we will see a bit later.

The key difference of the theory considered here from the theory of Weyl is that in the last one a special vector field is introduced as a compensating gauge field for the local conformal transformations whereas in our theory this role is played by the torsion trace vector. We therefore use only the geometric structure already at our disposal and avoid to introduce new entities to make our theory conformally invariant.

One can think of the action (9) with the Lagrangian (2) as of the first lowest terms in the expansion in field derivatives of some generic conformally invariant action based on Riemann–Cartan geometry.

The equations of motion are obtained by varying the action (9) over the independent variables \( \varphi, \psi, \bar{\psi}, A_\mu, \omega^a_{\ b\mu}, \) and \( e^a_{\ \mu} \). Varying over the affine connection components \( \omega^a_{\ b\mu} \) we obtain the following equations for the torsion tensor

\[ m^2(\varphi) \left( Q^\mu_{\ \nu\sigma} + \frac{2}{3} \delta^\mu_{\ [\nu} Q_{\sigma]} \right) = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \bar{\psi} \gamma^\tau \gamma^5 \gamma^\rho \psi, \]

\[ Q_\mu = \frac{3}{4} \partial_\mu \ln M^2(\varphi), \]

where \( \epsilon_{\mu\nu\rho\sigma} \) is the antisymmetric tensor with the components \( \epsilon_{0123} = -\sqrt{-g}, \gamma^\mu = e^\mu_{\ a} \gamma^a, \gamma^5 = \frac{i}{4} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \) and

\[ M^2(\varphi) = m^2(\varphi) + \frac{1}{6} \varphi^2, \quad \varphi^2 = \sum_{A=1}^k \varphi^A \varphi^A. \]
Purely algebraic equations (17), (18) for the torsion tensor components reflect the fact that torsion in our theory is non-propagating.

At this point we can substitute the expressions for the torsion from (17) and (18) back into the action (9). We will then obtain the following torsion-free effective action

\[ S_{\text{eff}} = \int_M L_{\text{eff}} \sqrt{-g} \, d^4x + \int_{\partial M} m^2(\varphi) K(\Gamma) \sqrt{h} \, d^3x, \tag{20} \]

the Lagrangian of which is

\[ L_{\text{eff}} = \frac{1}{2} m^2(\varphi) R(\Gamma) + \frac{i}{2} \left( \overline{\psi} \gamma^\mu D_\mu \psi - (D_\mu \psi) \gamma^\mu \psi \right) - \{ f \varphi \overline{\psi} \psi \] \[ - \frac{3}{16m^2(\varphi)} \left( \overline{\psi} \gamma^\mu \gamma^5 \psi \right) \left( \overline{\psi} \gamma_\mu \gamma^5 \psi \right) - \frac{1}{2} |D\varphi|^2 + 3 (\nabla M(\varphi))^2 - V(\varphi) - \frac{1}{4e^2} Tr F_{\mu \nu} F^{\mu \nu}, \tag{21} \]

and in particular contains the term that describes four-fermionic axial current \( \times \) current interaction. Here \( R(\Gamma) \) is the usual Riemannian curvature scalar, and \( K(\Gamma) \) is the extrinsic curvature of the boundary \( \partial M \) with respect to the Riemannian connection form \( \Gamma^a_b \). The "long" derivatives \( D_\mu \psi \) and \( D_\mu \varphi \) in (21) are defined as follows

\[ D_\mu \psi = (\partial_\mu + \Gamma_\mu + A_\mu) \psi, \tag{22} \]
\[ D_\mu \varphi = (\partial_\mu + A_\mu) \varphi, \tag{23} \]

where

\[ \Gamma_\mu = -\frac{1}{8} \Gamma^{ab}_\mu [\gamma^a, \gamma^b] \tag{24} \]

is the spin connection constructed from the Riemannian (torsion-free) affine connection \( \Gamma^a_b \). The symbol \( \nabla \) will denote the usual covariant derivative with respect to the affine connection \( \Gamma^a_b \).

The action (20) remains to be locally conformally invariant, although its two terms are not invariant separately, contrary to the expression (9) for the former action. Note that in the gauge \( m^2(\varphi) = \text{const} \) the second, boundary, term in (20) reproduces the Gibbons-Hawking boundary term [14] of the Hilbert-Einstein theory.

Equations of motion for the scalar multiplet fields \( \varphi \) that stem from the action (20) can be put in the following form (multiplet indices are omitted, so, for example, the equality \( \varphi = 0 \) denotes that all the multiplet components \( \varphi^A \) are zero, and \( \varphi \neq 0 \) means that some of the components are nonzero)

\[ D^\dagger_\mu D^\mu \varphi - \frac{\nabla^2 M(\varphi)}{M(\varphi)} \varphi - \frac{\partial V(\varphi)}{\partial \varphi} - \{ f \varphi \overline{\psi} \psi \] \[ - \frac{\xi(\varphi)}{m^2(\varphi)} \left( \varphi D^\dagger_\mu D^\mu \varphi - \frac{\nabla^2 M(\varphi)}{M(\varphi)} \varphi^2 - 4V(\varphi) - \{ f \varphi \overline{\psi} \psi \} \right) = 0, \tag{25} \]

where \( D^\dagger_\mu \) is Hermitian conjugate of \( D_\mu \), \( \{ f \varphi \overline{\psi} \psi \} \) stands for the derivative \( \partial \{ f \varphi \overline{\psi} \psi \} / \partial \varphi \), and

\[ \xi(\varphi) = \frac{1}{2} \frac{\partial m^2(\varphi)}{\partial \varphi} \tag{26} \]
Conformally invariant cosmology based on Riemann–Cartan spacetime

or, writing explicitly the multiplet indices

$$\xi_A(\varphi) = \frac{1}{2} \frac{\partial m^2(\varphi)}{\partial \varphi^A} = \xi_{AB} \varphi^B.$$  \hspace{1cm} (27)

Note that the left-hand-side of Eq. (25) if multiplied by $\varphi$ with the summation over the scalar multiplet index becomes identically zero.

For classical vacuum configurations we set $A = 0, \psi = \bar{\psi} = 0, \varphi = \text{const}$, and one gets from (25) the following equation

$$\frac{\partial V(\varphi)}{\partial \varphi} - \frac{\xi(\varphi)}{m^2(\varphi)} 4V(\varphi) = 0,$$  \hspace{1cm} (28)

which is simply the extremum condition of the function $V(\varphi)$ on the hypersurface given by the equation $m^2(\varphi) = \text{const}$ in the space of $\varphi$. Due to the positivity property of the form $m^2(\varphi)$ this hypersurface is compact so Eq. (28) has non-trivial solutions. Among them there are those which minimize the potential $V(\varphi)$ on the hypersurface considered. The values $\varphi_0$ of this solution will determine in a usual way the fermionic masses through the Yukawa coupling terms in the Lagrangian and the masses of the gauge vector bosons through the gauge interactions.

Variation of the action (20) over the fermionic fields yields the following equations of motion

$$i \left( \gamma^\mu (\partial_\mu + A_\mu) \psi + \frac{1}{2\sqrt{-g}} \partial_\mu (\sqrt{-g} \gamma^\mu) \psi \right) + \frac{1}{4} \epsilon^{abcd}_a \Gamma^b_{\mu c} \gamma^d \gamma^5 \psi$$

$$- \frac{3}{8m^2(\varphi)} (\bar{\psi} \gamma^5 \psi) \gamma^5 \psi - \{ \varphi \psi \} = 0.$$  \hspace{1cm} (29)

Variation of the action (20) over the metric with the equations of motion (29) taken into account yields

$$m^2(\varphi) \left( R_{\mu\nu}(\Gamma) - \frac{1}{2} g_{\mu\nu} R(\Gamma) \right) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2)m^2(\varphi) + D^{\dagger}_{(\mu} \varphi D_{\nu)} \varphi$$

$$- 6 \partial_\mu M(\varphi) \partial_\nu M(\varphi) - \frac{i}{2} (\bar{\psi} \gamma(\mu D_\nu) \psi - (D_\mu \psi) \gamma^\nu \psi) + \frac{1}{e^2} \text{Tr} F_{\mu\sigma} F^{\sigma}_\nu$$

$$- \frac{1}{2} g_{\mu\nu} \left( |D \varphi|^2 - 6(\nabla M(\varphi))^2 + 2V(\varphi) \right)$$

$$- \frac{3}{8m^2(\varphi)} (\bar{\psi} \gamma^5 \psi) (\bar{\psi} \gamma^5 \psi) + \frac{1}{2e^2} \text{Tr} F_{\sigma \tau} F^{\sigma \tau} \right).$$  \hspace{1cm} (30)

The equations obtained are invariant with respect to local conformal transformations considered above. All the observables of the theory are regarded to be invariant as well. This allows one to fix the conformal gauge freedom by imposing some appropriate condition on the solutions. Two of such conditions are especially convenient as can be
seen from the Lagrangian (21) or from the equations of motion written just above. The first one is the gauge already mentioned which is defined by the equation

$$m^2(\varphi) = \text{const}. \quad (31)$$

The second is the gauge-fixing condition

$$M^2(\varphi) = \text{const}. \quad (32)$$

The gauge (31) is the most convenient one for the cosmological interpretation of the theory as in this gauge the gravitational coupling (or, equivalently, the Planck mass) is explicitly constant.

3. Cosmological scenario

A viable cosmological scenario now can hardly be built without inflationary stage. So we start with the question of whether and under what conditions does our model allow for inflation. Doing this we will have in mind mostly the chaotic inflation scenario which is more natural and for which the analysis is more simple as compared to the new inflation (we will study more thoroughly both these scenarios, as they appear in our model, in our subsequent papers, for their good review see [15]). As is usual for such an analysis, we will assume that during inflation scalar field contribution dominates in the energy-momentum tensor, so the rest of the matter fields will be put to zero. Considering the standard Friedmann–Robertson–Walker cosmology we obtain the equations for the dynamics of the universe filled by a homogeneous scalar field multiplet \( \varphi(t) \). It is convenient to write these equations in the gauge (31). In this gauge we can apply the standard analysis of the plausible conditions for inflation. The dynamics equations in the gauge (31) are written as follows

$$H^2 + \frac{\kappa}{a^2} = \frac{1}{3m^2} \left( \frac{1}{2} \dot{\varphi}^2 - \frac{(\varphi\ddot{\varphi})^2}{12M^2} + V \right), \quad (33)$$

$$\dot{H} - \frac{\kappa}{a^2} = -\frac{1}{m^2} \left( \frac{1}{2} \dot{\varphi}^2 - \frac{(\varphi\ddot{\varphi})^2}{12M^2} \right), \quad (34)$$

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\dot{M} + 3H\dot{M}}{M} \varphi + \frac{\partial V}{\partial \varphi} - \frac{\xi(\varphi)}{m^2} \left( \varphi\ddot{\varphi} + 3H\varphi\dot{\varphi} - \frac{\dot{M} + 3H\dot{M}}{M} \varphi^2 + 4V \right) = 0, \quad (35)$$

where \( H \equiv \dot{a}/a \) is the Hubble parameter, \( \kappa = 0, \pm 1 \) determines the spatial curvature of the universe, and the quantity \( \xi(\varphi) \) was defined in (26). Dots denote the derivatives with respect to the cosmological time \( t \). It is easy to see that

$$\frac{1}{2} \dot{\varphi}^2 - \frac{(\varphi\ddot{\varphi})^2}{12M^2} \geq \frac{1}{2} \left( \dot{\varphi}^2 - \frac{(\varphi\ddot{\varphi})^2}{\varphi^2} \right) \geq 0, \quad (36)$$
hence the right-hand-side of Eq. (33) is not negative and the right-hand-side of Eq. (34) is not positive.

The condition for inflation \( |\dot{H}| \ll H^2 \) implies then

\[
\varphi^2 - \frac{(\varphi \dot{\varphi})^2}{6M^2} \ll V(\varphi).
\]

During inflation soon it becomes possible to neglect the second derivatives of the scalar fields (the latter begin to roll slowly down the scalar field potential on the hypersurface \( m^2(\varphi) = \text{const} \)). From Eqs. (35) and (33) we then have a very rough estimate

\[
\dot{\varphi}^2 \sim \frac{V(\varphi)}{\varphi^2} m^2.
\]

Taking into account the definition (7) we obtain from (37) the following both inflation and slow-rolling condition

\[
m^2(\varphi) \ll \varphi^2,
\]

or

\[
\xi_{AB} \ll 1.
\]

This estimate is not difficult to understand. The value \( m^2(\varphi) \) determines the Planck mass \( M_P \) through

\[
m^2(\varphi) = \frac{M_P^2}{8\pi}.
\]

In terms of the Planck mass the condition (39) reads

\[
M_P^2 \ll \varphi^2,
\]

and looks quite familiar to those who deal with chaotic inflationary cosmology (see [15]).

The estimate (39) is sufficient for inflation to take place, but not necessary. In fact, it is the condition for the chaotic type inflation. It is clear that the condition (37) can be fulfilled in the plateau regions of the potential \( V(\varphi) \) on the hypersurface \( m^2(\varphi) = \text{const} \) without the condition (39). In this case we would have inflationary dynamics similar to new inflation. Concrete realization of both these possibilities will be a subject of our future studies.

Besides (39) another condition which is necessary for the sufficient amount of inflation to take place is the following

\[
\lambda_{ABCD} \ll 1.
\]

It stems from (41) and from the requirement \( V(\varphi) \lesssim (m^2(\varphi))^2 \) which allows one to consider space-time as classical.

Neglecting terms with the second time derivative of the scalar fields \( \varphi \) in Eq. (35) and using the condition (39) we are able to write down the following approximate equation for \( \varphi \)

\[
3H \left( \varphi - \frac{\varphi \dot{\varphi}}{\varphi^2} \right) + \frac{\partial V(\varphi)}{\partial \varphi} - \frac{\xi(\varphi)}{m^2(\varphi)} 4V(\varphi) = 0.
\]
Note that if multiplied by $\varphi$ with the summation over the scalar multiplet index this equation gives identically zero, as is also the case with the precise equation of motion for $\varphi$.

The slow-rolling regime of the scalar field dynamics terminates as the scalar field values approach close to a (may be local) minimum of the potential $V(\varphi)$ on the hypersurface of constant $m^2(\varphi)$. After that the scalar fields start oscillating around their stationary point. Due to various couplings between fields these oscillations give birth to particles of various kinds. This process heats the universe in the standard manner.

The value $V_0$ of the scalar field potential in its local minimum (on the hypersurface $m^2(\varphi) = \text{const}$) generates the present-day $\Lambda$-term whose value can be estimated as $\Lambda \sim (\lambda/\xi^2)M_P^4$ (here $\lambda$ has the order of magnitude of the $\lambda$’s in (8), and $\xi$ has the order of magnitude of the $\xi$’s in (7)). In general this $\Lambda$-term is nonzero. For this not to be the case some fine tuning of the potential parameters $\lambda_{ABCD}$ seems to be necessary. If we assume the scalar field potential $V(\varphi)$ to be non-negative then to have $V_0 = 0$ in the minimum it is necessary and sufficient that such $\varphi_0 \neq 0$ exists for which $V(\varphi_0) = 0$. An interesting suggestion is that this property somehow can be provided by other symmetries of the theory (besides the local conformal one).

4. Conformally invariant quantum cosmology

In this section our aim will be to see how the standard formalism of quantum cosmology (for reviews see [16–19]) is applied to the conformally invariant theory which we study in this paper. For simplicity we will restrict ourselves only to the scalar-gravitational sector of the full theory (see Eqs. (20) and (21) for the action), which is described by the Lagrangian

$$L = \frac{1}{2} m^2(\varphi) R(\Gamma) - \frac{1}{2} (\nabla \varphi)^2 + 3 (\nabla M(\varphi))^2 - V(\varphi),$$

and by the action

$$S = \int_M L \sqrt{-g} d^4x + \int_{\partial M} m^2(\varphi) \mathcal{K}(\Gamma) \sqrt{h} d^3x,$$

in which the boundary term was explained in Section 2 after Eq. (21).

The metric $g$ in the ADM form (see [16] for the best description) is written as

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dt \otimes dx^i + dx^i \otimes dt) + h_{ij} dx^i \otimes dx^j.$$

Latin indices from the middle of the alphabet run from 1 to 3. The quantities $N$ and $N_i$ are respectively lapse function and shift vector. Substituting the metric in this form into the action (45) we obtain for a spatially closed universe

$$S = \int_M L d^3x dt,$$
where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2} N \sqrt{h} m^{2}(\varphi) \left( \mathcal{R} + \mathcal{K}_{ij} \mathcal{K}^{ij} - \mathcal{K}^{2} \right)$$

$$+ N \sqrt{h} \left( \frac{2K}{N} \xi_{A}(\varphi) \dot{\varphi}^{A} - (m^{2}(\varphi))^{i}_{i} - \frac{2KN_{i}}{N} \xi_{A}(\varphi) \partial_{i} \varphi^{A} - V(\varphi) \right)$$

$$+ N \sqrt{h} T_{AB}(\varphi) \left( \frac{1}{2N^{2}} \dot{\varphi}^{A} \dot{\varphi}^{B} - \frac{N_{i}}{N^{2}} \xi_{A}(\varphi) \partial_{i} \varphi^{B} - \frac{1}{2} \left( h^{ij} - \frac{N_{i}N_{j}}{N^{2}} \right) \partial_{i} \varphi^{A} \partial_{j} \varphi^{B} \right),$$

and

$$\mathcal{K}_{ij} = \frac{1}{2N} \left( 2N_{(i;j)} - \dot{h}_{ij} \right)$$

is the extrinsic curvature of the hypersurface $t = \text{const}$ in the ADM form. Vertical lines denote covariant derivatives with respect to the metric $h_{ij}$, with the aid of which also small Latin indices are raised and lowered, $\mathcal{R}$ is the three-curvature scalar built of the metric $h_{ij}$, and

$$T_{AB}(\varphi) = \delta_{AB} - \frac{6}{M^{2}(\varphi)} \left( \xi_{A}(\varphi) + \frac{1}{6} \varphi^{A} \right) \left( \xi_{B}(\varphi) + \frac{1}{6} \varphi^{B} \right)$$

is a non-degenerate symmetric matrix of signature $(-, +, +, ..., +)$.

The theory described by the Lagrangian (48) is degenerate: due to coordinate reparametrization invariance the time derivatives (velocities) of the values $N$ and $N_{i}$ do not enter the Lagrangian at all; due to local conformal invariance the matrix of the second derivatives of $\mathcal{L}$ with respect to the velocities $\dot{h}_{ij}$ and $\dot{\varphi}^{A}$ has one zero eigenvalue. So some linear combination of the velocities $\dot{h}_{ij}$ and $\dot{\varphi}^{A}$ will not be expressed through the corresponding generalized momenta. As a consequence new constraint will appear.

Let us denote the generalized momenta for the variables $h_{ij}$ and $\varphi^{A}$ by $p_{ij}$ and $p_{A}$ respectively. Then given the Lagrangian (48) we can express the velocities $\dot{\varphi}^{A}$ through the corresponding momenta

$$\dot{\varphi}^{A} = N^{i} \partial_{i} \varphi^{A} + \frac{N}{\sqrt{h}} T^{AB}(\varphi) \left( p_{B} - 2\sqrt{h} \mathcal{K}^{B}(\varphi) \right),$$

where $T^{AB}(\varphi)$ is the matrix inverse of $T_{AB}(\varphi)$. Trying then to express the velocities $\dot{h}_{ij}$, we find that this is possible only for the traceless part of the tensor $\dot{h}_{ij}$. We have

$$\tilde{\mathcal{K}}^{ij} = -\frac{2}{m^{2}(\varphi) \sqrt{h}} \tilde{p}^{ij},$$

where $\tilde{\mathcal{K}}^{ij}$ and $\tilde{p}^{ij}$ are the traceless parts of $\mathcal{K}^{ij}$ and $p^{ij}$, respectively, the values $\mathcal{K}^{ij}$ being given by (49). The trace $p = p^{ij} h_{ij}$ is involved into the constraint

$$\mathcal{F} \equiv \frac{1}{3} \left( p^{ij} h_{ij} - \frac{1}{2} p_{A} \varphi^{A} \right) = 0.$$
Conformally invariant cosmology based on Riemann–Cartan spacetime

The action written in the Hamiltonian form is

\[ S = \int_M \left( p_A \dot{\varphi}^A + p^{ij} \ddot{\tilde{h}}_{ij} - v \mathcal{F} - N \mathcal{H} - N^i \mathcal{H}_i \right) d^3x dt. \]  

(54)

Variables \( v, N, N^i \) are Lagrange multipliers. Variations with respect to them give constraint equations. The expression for \( \mathcal{F} \) is written in (53). Expressions for \( \mathcal{H} \) and \( \mathcal{H}_i \) are

\[ \mathcal{H} = \frac{2}{\sqrt{\hbar m^2(\varphi)}} \tilde{p}^{ij} \tilde{p}_{ij} + \frac{1}{2\sqrt{\hbar}} T^{AB}(\varphi) p_A p_B - \frac{1}{2} m^2(\varphi) \sqrt{\hbar} \mathcal{R} \]

\[ + \sqrt{\hbar} \left( m^2(\varphi) \right)^{|i} i + \frac{1}{2} \sqrt{\hbar} h^{ij} T_{AB}(\varphi) \partial_i \varphi^A \partial_j \varphi^B + \sqrt{\hbar} V(\varphi), \]

(55)

\[ \mathcal{H}_i = -\frac{1}{3} \left( \varphi^A p_A \right)_i - 2 \tilde{p}_{ij} + \partial_i \varphi^A p_A. \]

(56)

Note that the trace of \( p^{ij} \) does not enter the constraints \( \mathcal{H} \) and \( \mathcal{H}_i \). To simplify the equations of the theory it is convenient to make a canonical transformation to new canonical variables related to the old ones through the following equations

\[ \tilde{h}_{ij} = h^{-1/3} h_{ij}, \quad \tilde{\pi}^{ij} = h^{1/3} \tilde{p}^{ij}, \]

(57)

\[ h = \text{det}(h_{ij}), \quad \pi_h = \frac{1}{3h} \left( p^{ij} h_{ij} - \frac{1}{2} p_A \varphi^A \right) = \frac{1}{h} \mathcal{F}, \]

(58)

\[ \chi^A = h^{1/6} \varphi^A, \quad \pi_A = h^{-1/6} p_A. \]

(59)

The meaning of the first line (57) in the above relations is the following: \( \tilde{h}_{ij} \) is a function of some five parameters (not specified here) which determine this matrix with unitary trace, \( \tilde{\pi}^{ij} \) is a function of the corresponding five generalized momenta. Denoting also \( h^{-1/6} N = \tilde{N}, h^{1/6} \mathcal{H} = \tilde{\mathcal{H}} \), we will have \( N \mathcal{H} = \tilde{N} \tilde{\mathcal{H}} \), and

\[ S = \int_M \left( \pi_A \dot{\chi}^A + \tilde{\pi}^{ij} \dot{\tilde{h}}_{ij} - \left( v - \frac{\dot{h}}{h} \right) \mathcal{F} - \tilde{N} \tilde{\mathcal{H}} - N^i \mathcal{H}_i \right) d^3x dt, \]

(60)

where in terms of new variables

\[ \mathcal{F} = h \pi_h, \]

(61)

\[ \tilde{\mathcal{H}} = \frac{2}{m^2(\chi)} \tilde{\pi}^{ij} \tilde{\pi}_{ij} + \frac{1}{2} T^{AB}(\chi) \pi_A \pi_B - \frac{1}{2} m^2(\chi) \tilde{\mathcal{R}} \]

\[ + \left( m^2(\chi) \right)^{|i} i + \frac{1}{2} h^{ij} T_{AB}(\chi) \partial_i \chi^A \partial_j \chi^B + V(\chi), \]

(62)

\[ \mathcal{H}_i = -\frac{1}{3} \left( \chi^A \pi_A \right)_i - 2 \tilde{\pi}_{ij} + \partial_i \chi^A \pi_A. \]

(63)

Dots in front of small Latin indices denote covariant derivatives with respect to the metric \( \dot{\tilde{h}}_{ij} \), \( \tilde{\mathcal{R}} \) is the curvature scalar built of this metric. All small Latin indices in (62), (63) and below are also lowered and raised by the metric \( \tilde{h}_{ij} \) and its inverse \( \tilde{h}^{ij} \).
From the expression (60) for the action we see that the variables $h$, $\pi_h$ are non-dynamical. Their role becomes manifest if we write down a formal quantum path integral of the theory described by the action (60). For the wave function $\Psi$ of the universe we will have

$$\Psi = \int e^{iS} [d\tilde{N}] [dN_i] [dv] [d\pi_h] [dh] [d\tilde{\pi}^{ij}] [d\tilde{h}_{ij}] [d\pi_A] [d\chi^A] , (64)$$

In this path integral it is convenient to shift the integration variable $v \to v + \dot{h}$ (see the expression (60) for the action). Then the integrals over the (shifted) Lagrange multiplier $v$ and over the canonical variables $\pi_h$ and $h$ result in the overall factor

$$\int e^{-i\int vF d^3x dt} [dv] [d\pi_h] [dh] = \int \delta[F] [d\pi_h] [dh] = \int [d\sigma] , (65)$$

where by $\sigma$ we denoted the integration variable $\ln h$. In deriving (65) the expression (61) for the constraint $F$ has been taken into account. The last path integral over $\sigma$ is just the integral over the gauge conformal group. We see that this integral has been factorized automatically.

The quantum operator constraint $\hat{\mathcal{H}}$ imposed on the wave function of the universe with the operator arrangement as written in Eq. (61) implies that the wave function does not depend on the variable $h$. Note that the remaining variables $\tilde{h}_{ij}$, $\chi^A$ are conformally invariant. Thus the additional constraint implies that the wave function depends only on conformally invariant combinations of the initial variables, that is, it is conformally invariant. The constraint $\tilde{\mathcal{H}}_i$ as usual means invariance of the wave function with respect to three-metric and matter field variations induced by coordinate diffeomorphisms. All the dynamical content of the theory considered is expressed by the analogue of the Wheeler–DeWitt equation

$$\tilde{\mathcal{H}} \Psi = 0 . (66)$$

It might seem that by elimination of the conformally non-invariant variable $h$ from the equations of the theory we succeeded in building a consistent conformally invariant quantum gravity. This would be the case if the theory defined by the constraint equation (66) were well-defined. But the latter condition is not true in our theory as well as in the similar statement is not true in the Einstein theory of gravity. The reason is the infinity of the number of the degrees of freedom together with high non-linearity of the theory. By all this we are led to the necessity of regularizations. Nevertheless it is interesting to note that the theory developed here is no "worse" in this respect than the usual Einstein theory of gravity. Ascribing one or the other meaning to Eq. (66) (for example, restricting it to minisuperspace) we obtain a well-defined theory. Only in our case we are to respect the possible remnants of both the coordinate reparametrization invariance and local conformal invariance. In the following section we will consider quantum theory of minisuperspace based on our model, which is invariant with respect to spatially homogeneous conformal transformations.

The basic features of the constraint $\tilde{\mathcal{H}}$ resemble those of the well-known Wheeler–DeWitt constraint in the Einstein theory of gravity. The quadratic form in momenta
in (62) is non-degenerate and has eigenvalue signs \((-,-,\ldots,+)\) and the potential part of \(\widetilde{\mathcal{H}}\) is not bounded from below. In our theory however negative sign comes from the matter (not metric) kinetic term in the expression (62) for \(\widetilde{\mathcal{H}}\). Pseudo-Euclidean superspace metric signature enables us to have a timelike variable as in standard quantum cosmology. Such variable can be taken to be proportional to \(m(\chi)\).

To introduce time explicitly it is convenient once again to proceed to new variables

\[
\mu = \sqrt{6} m(\chi), \quad \vartheta^r = \vartheta^r(\chi), \quad r = 1, \ldots, k-1, \tag{67}
\]

with corresponding conjugate momenta \(\pi_\mu, \pi_\vartheta^r, r = 1, \ldots, k - 1\). If coordinates \(\vartheta^r \) are chosen so that \(\vartheta^r = \text{const}\) are rays in the \(\chi\)-space which begin at the origin \(\chi = 0\), then the quadratic forms in \(\pi_\mu\) and \(\pi_\vartheta^r\) decouple in \(\widetilde{\mathcal{H}}\) and the matter part of the constraint \(\widetilde{\mathcal{H}}\) takes the following form:

\[
\widehat{\mathcal{H}}_{\text{matter}} = -\frac{1}{2} \pi_\mu^2 - \mu_i \mu^i + \frac{1}{6} (\mu^2)^i - i + \frac{1}{2\mu^2} E^{rs}(\vartheta) \pi_\vartheta^r \pi_\vartheta^s
\]

\[
+ \frac{1}{6} \left(\frac{\mu}{m(n)}\right)^2 \left(n^A n_B + \frac{1}{M^2(n)} m(n)_i m(n)^i\right) + \frac{1}{2} \mu^4 W(\vartheta), \tag{68}
\]

where \(E^{rs}(\vartheta), r, s = 1, \ldots, k-1\), is the symmetric matrix of some positive-definite form, \(n^A = n^A(\vartheta), A = 1, \ldots, k\), is a unit vector in the \(\chi\)-space as a function of \(\vartheta\), and

\[
W(\vartheta) = \frac{1}{18 m^4(n(\vartheta))} V(n(\vartheta)) \tag{69}
\]

is the (rescaled) scalar field potential on the hypersurface \(m^2(\chi) = \text{const}\).

5. Conformally invariant quantum minisuperspace cosmology

In the previous section we were able to see how the well-developed ADM formalism of quantum cosmology is applied to the conformally invariant theory described above. For simplicity we restricted ourselves only to the scalar - gravitational sector of the full theory, which is described by the Lagrangian (44) with the action (45). In the previous section a theory of full superspace was considered. For many purposes however, especially when one deals with quantum cosmology, it is desirable to perform a more simple analysis of a minisuperspace model. This will be our main task in what follows.

For definiteness we start from homogeneous but not necessarily isotropic closed cosmology known as Bianchi-IX. The metric for this model can be written as (see, e.g., [19])

\[
ds^2 = \frac{1}{2\pi^2} \left(-N^2(t) dt \otimes dt + e^{2\alpha(t)} \left(e^{2\beta(t)}\right)_{ij} \sigma^i \otimes \sigma^j\right), \tag{70}
\]

where \(N(t)\) is the lapse function, \(\{\sigma^i\}\) is a homogeneous basis of one-forms on a unit spatial three-sphere, \(e^{\alpha(t)}\) is the scale factor of the space parametrized by the function \(\alpha(t)\) while the symmetric traceless matrix \(\beta_{ij}(t)\) parametrizes the anisotropy. The shift vector is absent due to spatial homogeneity of the metric. The overall factor out front
Conformally invariant cosmology based on Riemann–Cartan spacetime

is a convenient scaling. The traceless matrix $\beta_{ij}$ may be chosen to be diagonal. It is convenient to write it in the form

$$\beta_{ij} = \frac{1}{\sqrt{6}} \text{diag} \left( \beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+ \right).$$  \hspace{1cm} (71)

For the metric (70) and for a homogeneous scalar field multiplet $\varphi(t)$ the action (45), after taking integral over the space, is written as

$$S = \int L \, dt,$$  \hspace{1cm} (72)

with the Lagrangian

$$L = \frac{e^{3\alpha}}{2N} m^2(\varphi) \left( \dot{\beta}_+^2 + \dot{\beta}_-^2 - 6\dot{\alpha}^2 + T_{AB}(\varphi)\dot{\varphi}^A\dot{\varphi}^B - 12\dot{\alpha}\xi_A(\varphi)\dot{\varphi}^A \right) + Ne^\alpha m^2(\varphi) R_{IX}(\beta) - Ne^{3\alpha} V(\varphi),$$  \hspace{1cm} (73)

in which the potential $V(\varphi)$ has been rescaled by $2\pi^2$ for convenience, and

$$R_{IX}(\beta) = \text{Tr} \left( 2e^{-2\beta} - e^{4\beta} \right)$$  \hspace{1cm} (74)

is half of the spatial three-curvature scalar of the unitary ($\alpha = 0$) Bianchi-IX space. The non-degenerate matrix $T_{AB}(\varphi)$ was defined in (50) and the value $\xi_A(\varphi)$ - in (27).

The theory described by the Lagrangian (73) is degenerate: due to time reparametrization invariance the time derivative (velocity) of the lapse function $N$ does not enter the Lagrangian at all; due to the remnant of the conformal invariance the matrix of the second derivatives of $L$ with respect to the velocities $\dot{\alpha}$ and $\dot{\varphi}$ has one zero eigenvalue. So, as we already know, some linear combination of the velocities $\dot{\alpha}$ and $\dot{\varphi}$ will not be expressed through the corresponding generalized momenta. As a consequence new constraint will appear.

Let us denote the generalized momenta for the variables $\varphi^A$, $\alpha$, $\beta_+$, and $\beta_-$ by $p_A$, $p$, $p_+$, and $p_-$ respectively. Then given the Lagrangian (73) we can express the velocities $\dot{\varphi}^A$ and $\dot{\beta}_\pm$ through the corresponding momenta

$$\dot{\varphi}^A = T^{AB}(\varphi) \left( Ne^{-3\alpha} p_B + 6\dot{\alpha}\xi_B(\varphi) \right), \hspace{1cm} (75)$$

$$\dot{\beta}_\pm = \frac{N}{m^2(\varphi)} e^{-3\alpha} p_\pm,$$  \hspace{1cm} (76)

where $T^{AB}(\varphi)$ is the matrix inverse of $T_{AB}(\varphi)$. Trying then to express the velocity $\dot{\alpha}$ through the momenta we find that this is not possible. The value $p$ turns to be involved in the constraint

$$\mathcal{F}(\varphi^A, p_A, \alpha, p) \equiv p - \varphi^A p_A = 0.$$  \hspace{1cm} (77)

The action written in the Hamiltonian form is

$$S = \int \left( p_A \dot{\varphi}^A + p_+ \dot{\beta}_+ + p_- \dot{\beta}_- + p\dot{\alpha} - v\mathcal{F} - N\mathcal{H} \right) dt.$$  \hspace{1cm} (78)
The variables $v$ and $N$, are the Lagrange multipliers. Variations with respect to them give constraint equations. The expression for $\mathcal{F}$ is written above in (77). The expression for $\mathcal{H}$ is

$$\mathcal{H} = e^{-3\alpha} \left( \frac{1}{2m^2(\varphi)} (p^2_+ + p^2_-) + \frac{1}{2} T^{AB}(\varphi)p_A p_B \right) - e^\alpha m^2(\varphi) R_{IX}(\beta) + e^{3\alpha} V(\varphi).$$  

(79)

Note that the canonical variable $p$ does not enter the constraint $\mathcal{H}$. To simplify the equations of the theory it is convenient to make a canonical transformation from $\varphi_A$, $p_A$, and $p$ to new canonical variables $\chi_A$, $\pi_A$, and $\pi_\alpha (\alpha, \beta_\pm$, and $p_\pm$ being untouched) related to the old ones through the following equations (compare with (57)-(59))

$$\chi_A = e^\alpha \varphi_A, \quad \pi_A = e^{-\alpha} p_A, \quad \pi_\alpha = \mathcal{F}(\varphi_A, p_A, \alpha, p) = p - \varphi^A p_A.$$  

(80)

Denoting also $e^{-\alpha} N = \tilde{N}$, $e^\alpha \mathcal{H} = \tilde{\mathcal{H}}$, we will have $N \mathcal{H} = \tilde{N} \tilde{\mathcal{H}}$, and

$$S = \int \left( \pi_A \chi^A + p_+ \dot{\beta}_+ + p_- \dot{\beta}_- - (v - \dot{\alpha}) \pi_\alpha - \tilde{N} \tilde{\mathcal{H}} \right) dt,$$

(81)

where in terms of new variables

$$\tilde{\mathcal{H}} = \frac{1}{2m^2(\chi)} (p^2_+ + p^2_-) + \frac{1}{2} T^{AB}(\chi) \pi_A \pi_B - m^2(\chi) R_{IX}(\beta) + V(\chi).$$

(82)

From the expression for the action (81) we see that the variables $\alpha$ and $\pi_\alpha$ are non-dynamical. Their role becomes manifest if we write down the formal quantum path integral of the theory described by the action (81). For the wave function $\Psi(\alpha, \beta, \chi^A)$ we have

$$\Psi = \int e^{iS} [d\tilde{N}] [dv] [d\pi_\alpha] [d\alpha] [dp_\pm] [d\beta_\pm] [d\pi_A] [d\chi^A],$$

(83)

After shifting the integration variable $v \rightarrow v + \dot{\alpha}$ (see the expression (81) for the action) the integrals over the (shifted) Lagrange multiplier $v$ and over the canonical variables $\pi_\alpha$ and $\alpha$ lead to an overall factor

$$\int e^{-i \int v \pi_\alpha dt} [dv] [d\pi_\alpha] [d\alpha] = \int \delta[\pi_\alpha] [d\pi_\alpha] [d\alpha] = \int [d\alpha].$$

(84)

The last path integral over $\alpha(t)$ is just the (infinite) integral over the remnant of the local conformal group. It is the analogue of the factor (65) of the theory of full superspace. Again we see that this integral factorizes automatically.

Quantum operator constraint $\hat{\pi}_\alpha$ imposed on the wave function of the universe implies that the wave function $\Psi(\alpha, \beta, \chi^A)$ does not depend on the variable $\alpha$. Note that the remaining variables $\beta_\pm$, $\chi$ are conformally invariant. Thus the additional constraint implies that the wave function depends only on conformally invariant combinations of initial variables, that is, that it is conformally invariant. All the dynamical content of the theory is then expressed by the analogue of the Wheeler-DeWitt equation for minisuperspace (see [15–19] for a review of the standard minisuperspace quantum cosmology)

$$\tilde{\mathcal{H}} = 0.$$  

(85)
In imposing constraints on the wave function some particular operator ordering is to be chosen. Different choices as usual correspond to different quantum versions of the principal classical theory. We will not discuss this topic here.

The basic features of the constraint $\tilde{H}$ were described in general in the previous section. They resemble those of the well-known Wheeler-DeWitt constraint in the Einstein theory of gravity (see [15–19]). The quadratic form in momenta in (82) is non-degenerate and has eigenvalue signs $(-, +, \ldots, +)$. Negative sign comes from the matter kinetic term in the expression (82) for $\tilde{H}$ and enables us to have a timelike variable as in standard minisuperspace quantum cosmology.

To introduce time explicitly it is convenient to proceed to new variables (67) with corresponding conjugate momenta $\pi_\mu, \pi_\vartheta^r, r = 1, \ldots, k - 1$. If the coordinates $\vartheta^r$ are chosen as in the previous section (see the text following Eq. (67)) then the quadratic forms in $\pi_\mu$ and $\pi_\vartheta^r$ decouple in the expression for $\tilde{H}$ which then takes the following shape

$$
\tilde{H} = \frac{1}{2\mu^2}(p_\mu^2 + p_-^2) - \frac{1}{2}\pi^2_\mu + \frac{1}{2\mu^2}E^{rs}(\vartheta)\pi^\vartheta_s\pi^\vartheta_r - \frac{1}{6}\mu^2 R_{IX}(\beta) + \frac{1}{2}\mu^4 W(\vartheta), \tag{86}
$$

where $E^{rs}(\vartheta), r, s = 1, \ldots, k - 1$, is the matrix of the same positive-definite form, as in (68), and $W(\vartheta)$ is given by (69).

From the description given at the end of the previous section it is clear that $\{\vartheta\}$ can be regarded as just (arbitrary) coordinates on the unit $(k - 1)$-sphere in the $\chi$-space. This in particular has the following consequence. If $\mu$ is a good time variable (the wave function of the universe has WKB form in $\mu$) then the matter field probability is distributed over a manifold with the topology of $(k - 1)$-sphere. This manifold is compact hence the potential $W(\vartheta)$ given by (69) is bounded on it and the probability distribution may be well defined on it everywhere. To illustrate this idea let us consider the case when the wave function has a WKB form in the time variable $\mu$ and, for simplicity, let us ignore the variables $\beta_{\pm}$ i.e. put $\beta_{\pm} = 0$ (this means that we turn to the de Sitter minisuperspace model). Let us also assume that the action of the third term in (86) on the wave function is negligible. The effective constraint operator is then

$$
\mathcal{H}_{\text{eff}} = -\frac{1}{2}\left(\pi^2_\mu + \mu^2 - \mu^4 W(\vartheta)\right), \tag{87}
$$

and the wave function is written in the WKB form as

$$
\Psi(\mu, \vartheta) = A(\vartheta) \exp(I(\mu, \vartheta)), \tag{88}
$$

where $A(\vartheta)$ is the normalization factor. Hence $\vartheta$ (in fact $W(\vartheta)$) play the role of parameters in the equation

$$
\hat{H}_{\text{eff}} \Psi = 0. \tag{89}
$$

In the leading order of the WKB approximation the solution for $I(\mu, \vartheta)$ is

$$
I(\mu, \vartheta) = \begin{cases}
\pm \frac{1}{3W(\vartheta)} (1 - \mu^2 W(\vartheta))^{3/2}, & \mu^2 W(\vartheta) \leq 1; \\
\pm \frac{i}{3W(\vartheta)} (\mu^2 W(\vartheta) - 1)^{3/2}, & \mu^2 W(\vartheta) \geq 1.
\end{cases} \tag{90}
$$
In order to be able to neglect the action of the term \( \frac{1}{2}\mu^2E^{rs}(\vartheta)\pi^e_r \pi^e_s \) on the wave function the normalization factor \( A(\vartheta) \) is to be taken

\[
A(\vartheta) = \exp \left( \pm \frac{1}{3W(\vartheta)} \right).
\] (91)

The WKB solution (88) is then valid for not very large values of \( \mu \). The signs in (90) and (91) are determined by the boundary conditions. Typically the wave function is a linear combination of the exponents of (90) with different signs.

Let us consider, as an example, two classical choices of the boundary conditions and the corresponding wave functions. The tunneling boundary condition of Vilenkin [20–22] demands that there be only an outgoing wave at the singular boundary \( \mu \to \infty \) of the minisuperspace. The corresponding wave function in the leading order of the WKB approximation is given by

\[
\Psi_T(\mu, \vartheta) = \begin{cases} 
\exp \left( -\frac{1}{3W(\vartheta)} \left[ 1 - (1 - \mu^2W(\vartheta))^{3/2} \right] \right), & \mu^2W(\vartheta) \leq 1; \\
\exp \left( -\frac{1}{3W(\vartheta)} \left[ 1 + i (\mu^2W(\vartheta) - 1)^{3/2} \right] \right), & \mu^2W(\vartheta) \geq 1.
\end{cases}
\] (92)

The boundary condition of Hartle and Hawking [23–26] is formulated in terms of Euclidean path integral representation of the wave function similar to (83). The proposal is that the wave function is given by the path integral over compact configurations without the second boundary. Application of this proposal to the model considered here can be performed just along the same lines as it is done for the minisuperspace model based on the Einstein theory of gravity [27]. We obtain the following result:

\[
\Psi_H(\mu, \vartheta) = \begin{cases} 
\exp \left( \frac{1}{3W(\vartheta)} \left[ 1 - (1 - \mu^2W(\vartheta))^{3/2} \right] \right), & \mu^2W(\vartheta) \leq 1; \\
\exp \left( \frac{1}{3W(\vartheta)} \cos \left[ \frac{1}{3W(\vartheta)} (\mu^2W(\vartheta) - 1)^{3/2} - \frac{\pi}{4} \right] \right), & \mu^2W(\vartheta) \geq 1.
\end{cases}
\] (93)

According to the widespread interpretation of the wave function [28] the probability density is to be defined through the probability flux vector on any of the hypersurfaces in the minisuperspace which is crossed by the probability flux lines from the same side everywhere. In our case such a hypersurface is most conveniently chosen as \( \mu = \text{const} \). The probability density is then defined in the space of variables \( \{\vartheta\} \) and is given by

\[
dP(\vartheta) = J^{(\mu)}(\vartheta)d\Sigma_{(\mu)}(\vartheta),
\] (94)

where

\[
J^{(\mu)} = \text{Im} \left( \frac{\partial}{\partial \mu} \Psi^* \right)
\] (95)

is the probability flux vector component in the direction of \( \mu \), and \( d\Sigma_{(\mu)}(\vartheta) \) is the surface volume element of the surface \( \mu = \text{const} \). This volume element is easily calculable to be

\[
d\Sigma_{(\mu)} = \frac{\mu^{k-1}}{m^k(n(\vartheta))} \sqrt{\frac{m^2(n(\vartheta)) + 6\xi^2(n(\vartheta))}{m^2(n(\vartheta)) + \frac{1}{6}}} dS^{k-1}(\vartheta),
\] (96)
where $\xi^2 = \sum_A \xi_A \xi_A$, $k$ is the dimension of the $\chi$-space, and $dS^{k-1}(\vartheta)$ is the surface volume element of the unit sphere $S^{k-1}$ in the $\chi$-space. Calculating the fluxes $J_T^{(\mu)}$ and $J_H^{(\mu)}$ correspondingly for the tunneling wave function of Vilenkin (92) and for the expanding-universe component of the wave function of Hartle and Hawking (93) we obtain the expressions for the probability densities in the corresponding cases ($\mu^2 W > 1$)

$$dP_T(\vartheta) = C_T \sqrt{\mu^2 W(\vartheta) - 1} \exp\left(-\frac{2}{3W(\vartheta)}\right) \left(\frac{\mu}{m(n(\vartheta))}\right)^k \times \sqrt{\frac{m^2(n(\vartheta)) + 6\xi^2(n(\vartheta))}{m^2(n(\vartheta)) + \frac{1}{6}}} dS^{k-1}(\vartheta),$$

$$dP_H(\vartheta) = C_H \sqrt{\mu^2 W(\vartheta) - 1} \exp\left(\frac{2}{3W(\vartheta)}\right) \left(\frac{\mu}{m(n(\vartheta))}\right)^k \times \sqrt{\frac{m^2(n(\vartheta)) + 6\xi^2(n(\vartheta))}{m^2(n(\vartheta)) + \frac{1}{6}}} dS^{k-1}(\vartheta),$$

where $C_T$ and $C_H$ are corresponding normalization constants. The expressions obtained differ only in the signs in the powers of the leading exponent factors. Thus for the tunneling boundary conditions the probability to have large potential $W(\vartheta)$ at the onset of classical universe evolution is exponentially large whereas for the boundary conditions of Hartle and Hawking such probability is exponentially suppressed. This means high probability of inflation in the first case and low in the second one.

To conclude this section we wish to stress once again that the probabilities (97) and (98) are distributed on a compact manifold with the topology of $(k-1)$-sphere in the space of the fields $\{\chi\}$. On this manifold the field potential $W(\vartheta)$ is bounded, and if it is also smooth enough then the expressions like (97), (98) may be applicable everywhere on this manifold. This situation is much different from what we have in usual cases when probability distributions are defined typically on spaces like $R^n$ with the scalar field potential unbounded.

6. Discussion

In this paper we presented a generalization of a GUT-like model which incorporated gravity and which was invariant under the group of local conformal transformations. The model was based on Riemann–Cartan geometry and the vector trace of torsion played the role of gauge vector potential for the conformal group. We have considered inflationary universe dynamics based on our model and found that inflationary stage is allowable provided the couplings satisfy some natural constraints. We also developed standard quantum gravitational formalism for the scalar-gravitational sector of the model. We have seen that conformally non-invariant dynamical variables can be eliminated from the equations of the theory so that the wave function of the universe turns to be independent of them. Although our treatment was rather formal, nevertheless, it may indicate that
it is possible to construct and operate with conformally invariant quantum theory of gravity. We also considered a simple minisuperspace formulation of the theory under discussion. It has been illustrated that invariance with respect to local conformal transformations (to be precise, their spatially homogeneous remnant) can be consistently implemented into a quantum theory of minisuperspace by an appropriate operator arrangement choice. Then this symmetry becomes manifest as conformal invariance of the wave function, i.e. its dependence only on conformally invariant variables.

An interesting issue which remained unresolved concerns the possibility of having zero cosmological constant at the present cosmological epoch. In our model cosmological constant, although in certain sense constrained, is not automatically zero. Whether or not it can be made equal to zero (exactly or approximately) without unnatural fine tuning of the parameters is an open question. Other topics to be elaborated in frames of the theory considered here are: the origin of primordial energy density fluctuations, their magnitude and spectrum, and the universe reheating after inflation. We hope to turn to these topics in future.

Acknowledgments

One of us (Yu.Sh.) would like to express gratitude to Brown University for hospitality, and especially to thank Robert Brandenberger for invitation to visit Brown University.

Appendix

Our theory is based on Riemann–Cartan space $U_4$ (for a good detailed description see e.g. [29]). This space naturally arises within the framework of the Poincaré gauge theory of gravity. Riemann-Cartan structure in $U_4$ implies the presence of the affine connection form and the metric tensor which is covariantly constant. Let $\{e_a(x)\}$ be an arbitrary field of bases in the tangent space of $U_4$ at each point $x$, and $\{\epsilon^a(x)\}$ - the field of their dual bases, that is

$$\langle e^a(x), e^b(x) \rangle = \delta^a_b. \quad (A1)$$

Latin indices run from 0 to 3. The affine connection form $\omega^a_b(x)$ referred to the basis $\{e^a(x)\}$ defines covariant derivative (denoted $\nabla^a$) of tensors in $U_4$.

The metric tensor $g$ can be developed as follows

$$g = g_{ab}e^a \otimes e^b = g_{\mu\nu}dx^\mu \otimes dx^\nu, \quad (A2)$$

where $g_{ab}(x)$ and $g_{\mu\nu}(x)$ are symmetric components. In Riemann–Cartan space $U_4$ the affine connection form $\omega^a_b$ satisfies the metricity condition

$$\nabla^a g = 0. \quad (A3)$$

The metric components $g_{ab}$ ($g_{\mu\nu}$), and their inverse (in the sense of matrices) $g^{ab}$ ($g^{\mu\nu}$) allow one to raise and lower Latin (Greek) indices. The metricity condition (A3)
Conformally invariant cosmology based on Riemann–Cartan spacetime

written in terms of the components $g^{ab}$ of the metric reads

$$\omega^{(ab)} \equiv \frac{1}{2}(\omega^{ab} + \omega^{ba}) = \frac{1}{2}dg^{ab}. \quad (A4)$$

Torsion tensor components $Q^a_{bc} = Q^a_{[bc]} \equiv \frac{1}{2}(Q^a_{bc} - Q^a_{cb})$ are defined by the following relations

$$de^a + \omega^a_b \wedge e^b = -Q^a_{bc}e^b \wedge e^c. \quad (A5)$$

From Eq. (A3) it follows that

$$\omega^a_{bc} = \Gamma^a_{bc} + Q^a_{bc} + Q^a_{ab}, \quad (A6)$$

where $\Gamma^a_{bc}$ are the components of the Riemannian connection form (which is constructed in the usual way from the metric tensor) written in the basis $\{e_a\}, \{e^a\}$.

In order to describe spinors one has to choose an orthonormal vector basis (tetrad) $\{e_a\}$, in which

$$g^{ab} = g(e_a, e_b) = \eta^{ab} = \text{diag}(-1, 1, 1, 1). \quad (A7)$$

In this basis the metricity condition (A4) becomes

$$\omega^{(ab)} = 0, \text{ or } \omega^{ab} = \omega^{[ab]}. \quad (A8)$$

Henceforth all the formulas will refer to an arbitrary orthonormal basis.

The spin connection form which determines parallel transport and covariant derivative of four-spinors is defined by

$$\omega = -\frac{1}{8}\omega^{ab}[\gamma_a, \gamma_b], \quad (A9)$$

where $\gamma^a$ are the usual constant Dirac $\gamma$-matrices, which satisfy

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}. \quad (A10)$$

The covariant derivative of a spinor is then defined as follows

$$\nabla^\omega \psi = d\psi + \omega\psi. \quad (A11)$$

It transforms like spinor under the action of local Lorentz group $L_6$.

For any group $G$ of intrinsic gauge transformations by $A = A_\mu dx^\mu$ we will denote its connection form. Then both locally Lorentz and $G$-invariant derivative of any field multiplet $f(x)$ will be

$$Df = \nabla^\omega f + Af, \quad (A12)$$

where $A$ implies the matrix of the corresponding representation of the algebra of the group $G$.

Let us consider now the group of local conformal transformations whose action on the metric $g$ is

$$g(x) \to g'(x) = \exp(2\sigma(x))g(x), \quad (A13)$$
where $\sigma(x)$ is an arbitrary smooth function. Eq. (A13) means that the metric $g$ has conformal weight two:

$$w(g) = 2.$$  

(A14)

From (A1) and (A7) we immediately obtain conformal weights of the tetrad basis vectors and their dual one-forms

$$w(e_a) = -1, \quad w(e^a) = 1.$$  

(A15)

For the affine and gauge connection forms we are to set

$$w(\omega^a_{\ b}) = 0, \quad w(A) = 0,$$

(A16)

and conformal weights of scalar and spinor fields are, as usual,

$$w(\varphi) = -1, \quad w(\psi) = -\frac{3}{2}.$$  

(A17)

Given the affine connection form $\omega^a_{\ b}$ we can construct its Riemann–Cartan curvature two-form

$$R^a_{\ b\ c\ d}(\omega) = d\omega^a_{\ b\ c} + \omega^a_{\ c\ e} \wedge \omega^e_{\ b\ d},$$  

(A18)

which is conformally invariant and whose components relative to the coordinate basis \( \{dx^\mu\} \)

$$R^a_{\ b\ \mu\ \nu}(\omega) = \partial_\mu \omega^a_{\ b\ \nu} - \partial_\nu \omega^a_{\ b\ \mu} + \omega^a_{\ c\ \mu} \omega^c_{\ b\ \nu} - \omega^a_{\ c\ \nu} \omega^c_{\ b\ \mu},$$  

(A19)

constitute those of curvature tensor. From this last one constructs the scalar curvature

$$R(\omega) = R^a_{\ b\ \mu\ \nu}(\omega) e^a_{\ e\ b\ e},$$  

(A20)

whose conformal weight is

$$w(R(\omega)) = -2.$$  

(A21)

From Eq. (A5) there follows the transformation law for the torsion tensor components

$$Q^a_{\ bc}(x) \to Q^a_{\ bc}(x) = \exp(-\sigma(x)) \left( Q^a_{\ bc}(x) + \delta^a_{\ [b} \right. < d\sigma(x), e_{c\ ]}(x) > ) .$$  

(A22)

As it can be shown these transformations affect only the vector trace part of the torsion tensor

$$Q_a = Q^b_{\ ab} = Q_\mu e^\mu_a,$$  

(A23)

so that

$$Q_\mu(x) \to Q'_\mu(x) = Q_\mu(x) - \frac{3}{2} \partial_\mu \sigma(x).$$  

(A24)

† A field $f(x)$ will be said to have conformal weight $w(f)$ if under the action of local conformal group it transforms as

$$f(x) \to f'(x) = \exp(w(f)\sigma(x)) f(x).$$
Conformally invariant cosmology based on Riemann–Cartan spacetime

Locally Lorentz and $G$-invariant derivative $Df$ defined by the expression (A12) does not preserve conformal properties of the fields. Using transformation law (A24) of the torsion trace vector and taking into account (A16) and (A17) we can construct a new derivative

$$\mathcal{D}f = Df + \frac{2}{3}w(f)Qf$$

with the desired property

$$w(\mathcal{D}f) = w(f)
,$$

where $Q$ is the one-form of the torsion trace

$$Q = Q_a e^a = Q_\mu dx^\mu.$$

The derivative $\mathcal{D}$ is used in our paper in constructing locally conformally invariant actions.

References

[1] Weyl H. (1918) Sitzung. d. Preuss. Akad. d. Wiss. 465; Weyl H. (1920) Raum. Zeit. Matterie. (Berlin: Springer).
[2] Dirac P.A.M. (1973) Proc. Roy. Soc. London A333, 403.
[3] Utiyama R. (1973) Prog. Theor. Phys. 50, 2080.
[4] Freund P.G.O. (1974) Ann. Phys. (N.Y.) 84, 440.
[5] Utiyama R. (1975) Prog. Theor. Phys. 53, 565.
[6] Hayashi K., Kasuya M. and Shirafugi T. (1977) Prog. Theor. Phys. 57, 431.
[7] Hayashi K. and Kugo T. (1979) Prog. Theor. Phys. 61, 334.
[8] Cheng H. (1988) Phys. Rev. Lett. 61, 2182.
[9] Wheeler J.T. (1990) Phys. Rev. D41, 431.
[10] Hochberg D. and Plunien G. (1991) Phys. Rev. D43, 3358.
[11] Zhytnikov V.V. (1993) Int. J. Mod. Phys. A8, 5141.
[12] Nieh H.T. and Yan M.L. (1982) Ann. Phys. (N.Y.) 138, 237.
[13] Obukhov Yu.N. (1982) Phys. Lett. A90, 13; Dereli T. and Tucker R.W. (1982) Phys. Lett. B110, 206.
[14] Gibbons G.W. and Hawking S.W. (1977) Phys. Rev. D15, 2752.
[15] Linde A.D. (1990) Particle physics and inflationary cosmology. (Chur, Switzerland: Harwood Academic Publishers).
[16] Misner C.N., Thorn K.S. and Wheeler J.A. (1973) Gravitation. (San Francisco: Freeman).
[17] Alvarez E. (1989) Rev. Mod. Phys. 61, 561.
[18] Halliwell J.J. (1990) Introductory lectures on quantum cosmology, in: Proceedings of the Jerusalem Winter School on Quantum Cosmology and Baby Universes (Ed. by T.Piran).
[19] Duncan M.J. (1990) Quantum geometrodynamics. Lectures presented at TASI summer school; UMN-TH-916/90.
[20] Vilenkin A. (1984) Phys. Rev. D30, 509.
[21] Vilenkin A. (1986) Phys. Rev. D33, 3560.
[22] Vilenkin A. (1988) Phys. Rev. D37, 888.
[23] Hawking S. (1982) in Astrophysical cosmology: Proceedings of the study week on cosmology and fundamental physics, Eds. H.A.Bruck, G.V.Coyne and M.S.Longair (Vatican: Pontificae Academiae Scientarium Scripta Varia).
[24] Hawking S. (1983) in Relativity, Groups and Topology II, Eds. B.S.DeWitt and R.Stora (Amsterdam: North Holland Physics Publishing).
[25] Hartle J.B. and Hawking S.W. (1983) Phys. Rev. D28, 2960.
[26] Hawking S. (1984) Nucl. Phys. B239, 257.
[27] Halliwell J.J. and Louko J. (1989) Phys. Rev. D39, 2206.
[28] Vilenkin A. (1989) Phys. Rev. D39, 1116.
[29] Hehl F.W., Von der Heyde P., Kerlick G.D. and Nester J.M. (1976) Rev. Mod. Phys. 48, 393.