Chaos Synchronization of a Class of Fractional-order Economic Systems with Time-delay

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Abstract: In this paper, the chaos synchronization problem of a class of economic systems with time delays was studied on Lyapunov stability theory and fractional-order calculus theory. Two methods of fractional-order active sliding mode control and single controller sliding mode control are used, and the differences between the two methods are compared. Numerical simulations verify the feasibility of the two methods. Results show that active sliding mode control scheme is highly resistant to interference.

1. Introduction

Time delays is common to nature. In practical systems, factors such as mechanical friction can cause time delay, such as economics, biology, chemistry, physics, and engineering and so on, fractional-order chaotic systems are closer to real life and dynamic, learning behavior is more complicated. In the actual physical system, changing the order for the fractional order can enrich the form of the current chaotic system. Therefore, the study of fractional-order time-delay chaotic systems has important theoretical and practical value. For one thing, many scholars have found that fractional-order time-delay systems have more dynamic behaviors. But at present, the research on the dynamic behavior of fractional-order chaotic systems with time-delay is still in its infancy, and many places need to be improved. For another thing, the synchronous control of fractional-order chaotic systems has great application values in the field of secure communication, however, there is less research on the synchronization of fractional-order chaotic systems with time-delay in the field of secure communication. Therefore, it is of great value to study the synchronization of chaotic systems with fractional delays.

In the fractional-order nonlinear systems having a time delay of. Score Lyapunov direct method is one of the fractional-order delay nonlinear systems most commonly used method. The sliding mode control is a nonlinear control technique featuring remarkable properties of accuracy, robustness and easy tuning and implementation with a very large application fields. So in this view, we choose a class of fractional-order financial chaotic systems with time-delay and apply two methods of controlling synchronization. The synchronization between the drive system and the response system is realized, and numerical simulation is performed to verify the effectiveness of the two methods.
2. Prerequisite knowledge

2.1. Consider a class of fractional-order chaotic systems with time-delay and be used as a drive system \(^6\).

\[ D^q x = Ax(t - \tau) + F(x, x(t - \tau)) \]  \hspace{1cm} (2.1.1)

Where \( x(t) \in \mathbb{R}^n \) is a n-dimensional states vector of the drive system. \( A \in \mathbb{R}^{n \times n}, \quad q \in (0,1) \) is the order of the fractional differential equation. \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the linear and nonlinear parts of the drive system with time delay.

Let the response system is:

\[ D^q y = Ay(t - \tau) + F(y, y(t - \tau)) + u(t) \]  \hspace{1cm} (2.1.2)

Where \( y(t) \in \mathbb{R}^n \) is an n-dimensional state vector of the response system, \( u(t) \) is a controller to be designed.

The error system has been defined as \( e = y - x \), The error of a fractional-order system with time delays can be described as

\[ D^q e = Ae(t - \tau) + F(y, y(t - \tau)) - F(x, x(t - \tau)) + u(t) \]

\[ = Ae(t - \tau) + G(x, y) + u(t) \]  \hspace{1cm} (2.1.3)

Where \( G(x, y) = F(y, y(t - \tau)) - F(x, x(t - \tau)) \).

Definition 1. 1. Fractional defined for drive and response systems, if the controller \( u(t) \) satisfy the following condition is called chaotic synchronization

\[ \lim_{t \to \infty} \|e(t)\| = \lim_{t \to \infty} \|y(t) - x(t)\| = 0 \]  \hspace{1cm} (2.1.4)

2.2. Chaotic Phenomena Model of Fractional Delay Economic System

The drive system is described by:

\[
\begin{align*}
D^q_1 x_1 &= x_2(t - \tau) - ax_1 + bx_2 x_3 \\
D^q_2 x_2 &= cx_2 - x_1(t - \tau)x_3 + x_3 \\
D^q_3 x_3 &= dx_1 x_2 - mx_3(t - \tau)
\end{align*}
\]  \hspace{1cm} (2.2.1)

For \( q = 0.9, \tau = 0.2 \) and \( x(0) = (-2.5,1) \), the chaotic attractor of system (2.2.1) maps has been given in the literature [7].

Response system can be given:

\[
\begin{align*}
D^q_1 y_1 &= y(t - \tau) - ay_1 + by_2 y_3 + u_1 \\
D^q_2 y_2 &= cy_2 - y_1(t - \tau)y_3 + y_3 + u_2 \\
D^q_3 y_3 &= dy_1 y_2 - my_3(t - \tau) + u_3
\end{align*}
\]  \hspace{1cm} (2.2.2)

3. Two control methods of financial systems with time-delay

3.1. Active sliding mode control for time-delay financial systems

Active sliding mode control is divided into two parts, Designed as follows.

3.1.1. Active controller designed

The input vector is designed as:

\[ u(t) = H(t) - G(x, y) \]  \hspace{1cm} (3.1.1)

Bring it into (2.1.3), the error system has been given

\[ D^q Ae(t - \tau) + H(t) \]  \hspace{1cm} (3.1.2)

Sliding mode control law is designed as: \( H(t) = Kw(t), \quad K = [k_1, k_2, \ldots, k_n] \) is a constant gain vector, \( w(t) \in \mathbb{R} \) is the control input and satisfies. Here \( s = s(e) \) is the dynamic switching plane that satisfies the needs. Dynamic switching plane to meet the needs, So we can get:

\[ D^q e = Ae(t - \tau) + Kw(t) \]  \hspace{1cm} (3.1.3)
3.1.2. Sliding surface designed

Sliding surface is designed as:

\[ s(t) = CD^q e(t) \]  \hspace{1cm} (3.1.4)

Where \( C = [C_1, C_2, \ldots, C_n] \) is a constant gain vector. \( D^q \) is a Caputo operator of order \( q - 1 \) order. Sliding mode control law is:

\[ w(t) = w_{eo} + w_r \]  \hspace{1cm} (3.1.5)

Where \( w_{eo} \) is to keep the state track of the switching surface, \( w_r \) is switching control. The controlled system must meet the following conditions:

\[ s(e) = 0, \dot{s}(e) = 0 \]  \hspace{1cm} (3.1.6)

By (3.1.4) and (3.1.6):

\[ \dot{s}(e(t)) = CD^q e(t) = C(\Delta e(t - \tau) + Kw(t)) = 0 \]  \hspace{1cm} (3.1.7)

Where the existence of \((CK)^{-1}\) is a necessary condition.

3.1.3. Synovial controller designed

The following synovial controller is designed as:

\[ \dot{s}(t) = -\mu s - psgn(s) \]  \hspace{1cm} (3.1.8)

Where \( sgn(\cdot) \) represents a symbolic function, \( \mu > 0, p > 0 \) is the gain constant. The conversion control by (3.1.6) and (3.1.8) is:

\[ w_e(t) = -(CK)^{-1}CAe(t - \tau) - (CK)^{-1}[\mu s + psgn(s)] \]

Theorem 1[7]. If and only if the following conditions are met, sliding surface (3.1.4) is progressively stable.

\[ s(t) \cdot \dot{s}(t) < 0 \]

Prove: The Lyapunov function is considered as:

\[ V(t) = \frac{1}{2} s^2(t) \]

Then \( \dot{V} = s \cdot \dot{s} = s \cdot (-\mu s - psgn(s)) = -\mu s^2 - ps^2 < 0 \) is certified.

Theorem 2. If the control input \( w(t) \) in the error system (2.1.3) satisfies the following formula

\[ w(t) = -(CK)^{-1}CAe(t - \tau) - (CK)^{-1}[\mu s + psgn(s)] \]

Then the zero solution to the error system (2.1.3) is progressively stable.

Prove: The Lyapunov function is considered as:

\[ V(t) = \frac{1}{2} \|s^2(t)\| \]

Then \( \dot{V} = s^T \cdot \dot{s} = s^T \cdot CD^q e(t) = s^T (-\mu s - psgn(s)) \leq -\mu_{min} \|s\|^2 - p_{min} \|s\| < 0 \]

Where \( \mu_{min} = \min\{\mu_i\}, p_{min} = \min\{p_i\}, i = 1, \ldots, n. \) Certificate.

When the error system has been reached the sliding surface

\[ D^q e = Ae(t - \tau) + Kw_{ea}(t) = A^*e(t - \tau) \]  \hspace{1cm} (3.1.9)

Where \( A^* = A - K(CK)^{-1}CA \).

Theorem 3. The necessary and sufficient condition for the drive system and the response system error system to achieve synchronization after the sliding surface is (3.1.9) satisfied

\[ \|arg(A^*)\| > \frac{\pi}{2}, det[s^q I - A^*e^{-s\tau}] = 0 \] does not have a pure virtual root.

Lemma 1[7]. If \( q \in (0,1), \lambda s \) is all of the eigenvalues of \( A \), and satisfy \( |arg(\lambda)| > \frac{\pi}{2}, \tau_{ij} > 0, i,j = 1, \ldots, n, \) \( det(\Delta(s)) = 0 \) is characteristic equation and does not have a pure virtual root.

Then the zero solution to the system (2.1.1) is Lyapunov’s global progressive stability.

Prove: We know from Lemma 1, Theorem 3 is clearly established.

3.2. Single sliding mode control for time-delay financial systems

In order to make (2.1.4) gradually stable to 0. The design method of the controller has the following form. It is designed as a first sliding surface [8]:

\[ s = D^{q-1}e_1(t) + \int_0^t ae_1(\tau)d\tau \]  \hspace{1cm} (3.2.1)
Guided by it

\[ \dot{s} = D^q e_1(t) + ae_1(t) \]  

(3.2.2)

In the sliding surface, combine sliding surface and sliding mode motion rules, obtain the equivalent control law

\[ u_{e\phi} = -b(y_2y_3 - x_2x_3) + ae_1 \]  

(3.2.3)

Study to design discontinuous switching rate

\[ w_e = ksgn(s) \]  

(3.2.4)

Where \( k \) is constant less than 0, combine (3.1.6) and (3.2.2) to get a complete sliding mode controller:

\[ u = -b(y_2y_3 - x_2x_3) + ae_1 + ksgn(s) \]  

(3.2.5)

Prove: Constructing Lyapunov functions \( V = \frac{s^2}{2} \), Guided by it

\[ \dot{V} = s \cdot \dot{s} = s \cdot [D^q e_1(t) + ae_1(t)] = s \cdot [ksgn(s)] = k|s| \leq 0 \]  

(3.2.6)

Certified.

4. Numerical Simulation

4.1. Numerical Simulation of Active Sliding Mode Control Method

The drive system can be shown in the form of equation (2.2.1), where

\[
A = \begin{pmatrix}
-3 & 1 & 0 \\
0 & 4.7 & 1 \\
0 & 0 & -9 \\
\end{pmatrix}, 
F = \begin{pmatrix}
2.7(y_2y_3 - x_2x_3) \\
-3y_1(t - \tau)y_3 + x_1(t - \tau)x_3 \\
2y_1y_2 - x_1x_2 \\
\end{pmatrix}
\]

According to the Theorem 2, synchronization between system (2.1.2) can be realized with \( K = (-1,2,-0.1)^T \) and \( C = [1,4,6] \), when the error system reaches the sliding surface, Error system is:

\[
\begin{align*}
D^q e_1 &= -22.2e_1 + 127.72e_2(t - \tau) - 320e_3 \\
D^q e_2 &= 38.4e_1 - 248.74e_2 + 641e_3 \\
D^q e_3 &= -1.92e_1 + 12.672e_2 - 41e_3(t - \tau)
\end{align*}
\]

It can be known from numerical simulation, active sliding mode control and synchronization methods are effective. The simulation results are shown in figure 1.

4.2. Numerical Simulation of a single sliding mode controller

The error system is shown as

\[
\begin{align*}
D^q e_1 &= e_2(t - \tau) - ae_1 + b(y_2y_3 - x_2x_3) + u \\
D^q e_2 &= ce_2 - e_1(t - \tau) + e_3 \\
D^q e_3 &= d(y_1y_2 - x_1x_2) - me_3(t - \tau)
\end{align*}
\]

The initial conditions for synchronization are set to \([x_1, x_2, x_3, y_1, y_2, y_3] = [-2.5, 1, 1, 2, 3], h = 0.01, \tau = 2, k = -1\). The simulation results are shown in figure 2. Visible from the figure, Sliding mode motion can reach and stay on the sliding surface, synchronization error gradually converges to 0 with time evolution, simulation results verify the effectiveness of the control method.

![Figure 1](sim1.png) Error synchronization under active sliding mode control

![Figure 2](sim2.png) Single controller error synchronization
5. Conclusion
Based on these two methods, different controllers are designed for the financial system. The main advantage of the design of a single synovial controller are its practicality, low cost and poor control effect on multiple disturbances. The active sliding mode controls sets the controller of the overall system, and the cost is high, but it can control a variety of uncertain interference factors. Therefore, the two control methods have their own advantages and disadvantages. According to the development of the times, the active sliding mode control method is selected to predict and protect the financial situation as a whole.

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