Entanglement, particularly multipartite entanglement state, is one of the most fundamental and puzzling aspect in quantum mechanics. However, entanglement is such a fragile resource that it may be easily degraded during its interaction with the environmental noise. To this point, entanglement distillation (always in a probabilistic way) has been proposed to increase the entanglement in the noise-disturbed entangled state[1, 2]. Restricted by the famous No-Go theorem in continuous variable (CV) entanglement distillation[3–5], lots of efforts have been devoted to the non-Gaussian operations. As an example, photon subtraction (PS) operation[6], proposed by Opatrný et al in 2000, is principally simple and can be readily implemented with beamsplitter and photon detectors. Very recently, about 10 years after Opatrný et al’s pioneering work, an experiment which faithfully implements the PS-based two mode entanglement distillation has been reported [7]. One of the challenge in this experiment is the extremely low successful probability, which is mainly due to the rather high transmittance beamsplitter used in PS operation—For one thing, the beamsplitter must own a relatively high transmittance to guarantee the successful distillation decreases exponentially with the number of parties $N$. However, here, we shall propose an entanglement distillation scheme whose success probability scales as a constant with $N$. Our protocol employs several local squeezers, but it requires only a single PS operation. Using the logarithmic negativity as a measure of entanglement, we find that both the success probability and the distilled entanglement can be improved at the same time. Moreover, an $N$-mode transfer theorem (transferring states from phase space to Hilbert space) is presented.

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In this paper, we study entanglement distillation of multipartite continuous-variable Gaussian entangled states. Following Opatrný et al.’s photon subtraction (PS) scheme, the probability of successful distillation decreases exponentially with the number of parties $N$. However, here, we shall propose an entanglement distillation scheme whose success probability scales as a constant with $N$. Our protocol employs several local squeezers, but it requires only a single PS operation. Using the logarithmic negativity as a measure of entanglement, we find that both the success probability and the distilled entanglement can be improved at the same time. Moreover, an $N$-mode transfer theorem (transferring states from phase space to Hilbert space) is presented.

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the unbiased states, namely \( \{\psi_i\}_{i=1}^{N} \). To be simple, we will mainly focus on \( \{\psi_i\}_{i=1}^{N} \) and its Fourier transform. A standard normalized Wigner function is defined as \( \psi(\chi) = \int \exp[-i\chi \cdot \vec{r}] f(\vec{r}) d\vec{r} \), where \( f(\vec{r}) \) is the probability density. In the following, we also use the Wigner function, the variance matrix \( \bar{\chi} = \chi^T \chi \) and \( \bar{\chi} \cdot \bar{\chi} \) in the distilling of a family of genuinely \( N \)-partite symmetric Gaussian entangled state. The covariance matrix is given by \( \bar{\chi} = \frac{1}{2} \left( \begin{array}{cc} \psi & \psi^T \\ \psi^T & \psi \end{array} \right) \), where \( \psi \) denotes the covariance matrix. In this paper, we are mainly interested in the distilling of a family of genuinely \( N \)-partite symmetric Gaussian entangled state. The covariance matrix is given by

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**Multi-partite entanglement states.** Multiparticle CV Gaussian entanglement plays a prominent role in future quantum network and quantum communication protocol. In this paper, we are mainly interested in the distilling of a family of genuinely \( N \)-partite symmetric Gaussian entangled state. The covariance matrix is given by

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in which \( \bar{\chi} = \frac{1}{2} \left( \begin{array}{cc} \psi & \psi^T \\ \psi^T & \psi \end{array} \right) \).

**Distillation with local squeezing and one-time PS.** Let’s now derive the state evolution of our one-time PS distillation protocol. As shown in Fig.1(b), \( N \) local squeezing symplectic transformations \( S(s_i), i=1,\ldots,N \) are applied before PS. This corresponds in phase space to a transformation of covariance matrix \( \bar{\chi} = \frac{1}{2} \left( \begin{array}{cc} \psi & \psi^T \\ \psi^T & \psi \end{array} \right) \). Finally, a successful distillation is heralded if the detector register non-vacuum results. According to AppendixA, one can find the distilled state is a linear combination of two gaussian states

\[
\bar{\chi}_\text{dis} = \frac{\delta}{\delta - 1} \bar{\chi}_1 - \frac{1}{\delta - 1} \bar{\chi}_2,
\]

where \( \delta = \sqrt{\text{det}(\Gamma_2 + I_2/2)} \) and \( \rho(\Gamma) \) being a normalized \( N \)-partite gaussian state with covariance matrix \( \Gamma \). The \( \Gamma_1 \) and \( \Gamma_2 \) are defined by partitioning of matrix.

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strategy(r) to the rather low initial squeezing (stronger squeezing. In Fig.(c), we increase photon subtracted. Our method can be applied for even mode, which certainly decrease the probability of being success probability (Fig.3) is about neg and success probability in Fig.2(d) and Fig.3. The optimal squeezing scales linearly with the initial parameter r. s\_opt scales linearly with the initial parameter r: s\_opt \approx 1.4r. (d) The corresponding entanglement with s\_opt. In the numerical simulation, we truncate the photon number of each mode at D = 7, namely, we consider only the contribution of |0\>, |1\>, |2\>, \cdots |6\>. By choosing D = 7, the error during our state transfer can be controlled with in 5 x 10^{-5}. In Fig.2, we evaluate the entanglement after and before distillation as the example. For a fixed initial squeezing r = 0.05, as shown in Fig.2(a), we plot the entanglement as a function of s\_1 = s\_2 = s\_3 = s\_opt. Through our simulation, we assume the transmittance of PS beamsplitter is T = 0.90 which is available in recent experiments[7]. The probability of success is briefly shown in Fig.2(b). It should be noted that the success probability which is about O(10^{-5}). This is mainly due to the rather low initial squeezing (r = 0.05), which results extremely low photon number in each transmission mode, which certainly decrease the probability of being photon subtracted. Our method can be applied for even stronger squeezing. In Fig.(c), we increase r and find the optimal squeezing s\_opt which may maximize the log-neg of output entanglement state. The numerical results support the linearity reliance of s\_opt upon the increasing r. Also, we plot the corresponding optimized log-neg and success probability in Fig.2(d) and Fig.3. The success probability (Fig.3) is about O(10^{-3}) which is an pronounced improvement compared with the 3-time PS strategy(O(10^{-9})).

Discussions and Prospectives. We presented here a photon-subtraction based entanglement distillation for arbitrary N-partite continuous variable entanglement state. As an example, now in this paper, only the three-partite symmetric Gaussian state is involved. This method is applicable for arbitrary N-partite CV state. Indeed, even for N = 2, this improvement in both log-neg and success probability also applies. As an auxiliary result, we also derive the transfer theorem for N-partite Gaussian state from Phase space to Hilbert space. We can envisage that this theorem could find more application in the entanglement evaluation tasks, such as entanglement swapping and entanglement distribution.

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Appendix A: Wigner function description of Photon Subtraction

In ideal cases, a perfect photon subtraction is described with the annihilation operation: a = \sum_{n=1}^{\infty} \sqrt{n}(n - 1)|n\rangle in Hilbert space. However, this is not an unitary operation and cannot be implemented deterministically. A convenient way is to use beamsplitter and photon detectors[6]. For case, we consider a single partite state \rho_i as input, the PS operation \hat{\Pi}^\rho_i(see Fig.1) can be represented with a completely-positive map from \rho_i to nor-
normalized output state $\tilde{\rho}_i = \mathcal{E}(\rho_i)/\text{Tr}[\mathcal{E}(\rho_i)]$, with

$$\mathcal{E}(\rho_i) = \text{Tr}_B \left[ U_{AB} \left( \rho_i \otimes \ket{0}_{B_i}\bra{0} \right) U_{AB}^\dagger \left( \hat{I}_A \otimes \hat{\Pi}_{i}^{\text{on}} \right) \right] \tag{A1}$$

where $\hat{\Pi}_{i}^{\text{on}} = \sum_{n=1}^{\infty} \ket{n}_{B_i} \bra{n}$ denotes the positive operators projecting to non-vacuum subspace and $U_{AB}$ is $\exp[\text{arccos}(\sqrt{T})](a_i^\dagger a_{A_i} - a_{A_i} a_i^\dagger)$ denotes the Beam-splitting operation between $A_i$ and $B_i$ modes.

In our calculation, indeed, it is convenient to use the wigner function to describe the PS process above (Eq. (A1)). In fact, the operator $\hat{\Pi}_{i}^{\text{on}} = I - \ket{0}_{B_i}\bra{0}$ is a difference of two operation whose wigner function are both Gaussian[16], i.e.,

$$W(\hat{r}_i, \hat{\Pi}_{i}^{\text{on}}) = \frac{1}{2\pi} \left( 1 - 2 \exp[\hat{r}_i J_2 \hat{r}_i^\dagger] \right)^2 \tag{A2}$$

In case that the input state $\rho_i$ is Gaussian, the wigner-function of distilled entanglement can be easily formulated with the linear combination of a series of Gaussian function, each of which can be conveniently expressed with the covariance marices.

Appendix B: Quantum State from phase-space to Hilbert Space

In this section, we give the detailed techniques we use in the processing of multi-partite Gaussian quantum state from phase-space to Hilbert Space. For a $N$-partite CV state, the density matrix follows

$$\rho = \int \frac{d\vec{\mu}}{\pi^N} \text{Tr} \left[ \rho D(\vec{\mu}) \right] D(-\vec{\mu}), \vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{C}^N$$

with $D(\vec{\mu}) = \exp \left[ \vec{\mu}^T \vec{a} - \vec{\mu}^T \vec{a}^\dagger \right] = \exp \left[ (\vec{\mu}, \vec{\mu}^T)(\vec{a}^\dagger, -\vec{a})^T \right]$ being the $N$ mode displacement operator.

The matrix entries of $\rho$ can be conveniently obtained by observing the equation[9]:

$$\langle k_1, k_2, \ldots, k_N | D(-\vec{\mu}) | m_1, m_2, \ldots, m_N \rangle = \frac{\prod_{i=1}^{N} \partial_{k_i} \prod_{i=1}^{N} \partial_{m_i}}{\sqrt{\prod_{i=1}^{N} k_i! \prod_{i=1}^{N} m_i!}} \exp \left[ \vec{t}^\dagger \vec{\mu} + \vec{\mu}^\dagger \vec{t} \right] \bigg|_{\vec{t} = \vec{\mu} = 0}$$

where $\vec{t} = (t_1, t_2, \ldots, t_N)$, $\vec{t} = (t_1', t_2', \ldots, t_N')$ is the $N$-dimensional real vector.

By noticing the fact that

$$\langle \vec{a}^\dagger, -\vec{a} \rangle = iL_N(\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_N, \hat{p}_N)^T \tag{B1}$$

one obtains that (after integrating the $\vec{\mu}$)

$$\langle k_1, k_2, \ldots, k_N | \rho | m_1, m_2, \ldots, m_N \rangle = \frac{\prod_{i=1}^{N} \partial_{k_i} \prod_{i=1}^{N} \partial_{m_i}}{\sqrt{\prod_{i=1}^{N} k_i! \prod_{i=1}^{N} m_i!}} \frac{F = \exp \left[ \frac{1}{2} \left( \vec{t}^\dagger, \vec{t} \right) \left( \vec{t}^\dagger, \vec{t} \right)^T \right] / \sqrt{\det(\Gamma + I_{2N}/2)}} {R = \sigma_x \otimes I_N + (\sigma_z \otimes I_N) L_N^\dagger (\Gamma + I_{2N}/2)^{-1} L_N^\dagger (\sigma_z \otimes I_N)} \tag{B2}$$

in which

$$F = \exp \left[ \frac{1}{2} \left( \vec{t}^\dagger, \vec{t} \right) \left( \vec{t}^\dagger, \vec{t} \right)^T \right] / \sqrt{\det(\Gamma + I_{2N}/2)} \tag{B3}$$

$$R = \sigma_x \otimes I_N + (\sigma_z \otimes I_N) L_N^\dagger (\Gamma + I_{2N}/2)^{-1} L_N^\dagger (\sigma_z \otimes I_N) \tag{B4}$$

and $\sigma_x, \sigma_z$ are the Pauli matrices. Then, one can check that the state $\rho$ (Eq. (B2)) is now automatically normalized, i.e., $\text{Tr}[\rho] = 1$.

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