On a Spector ultrapower of the Solovay model*

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Abstract

We prove that a Spector–like ultrapower extension \( N \) of a countable Solovay model \( M \) (where all sets of reals are Lebesgue measurable) is equal to the set of all sets constructible from reals in a generic extension \( M[\alpha] \) where \( \alpha \) is a random real over \( M \). The proof involves an almost everywhere uniformization theorem in the Solovay model.

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Introduction

Let \( \mathcal{U} \) be an ultrafilter in a transitive model \( \mathcal{M} \) of \( \text{ZF} \). Assume that an ultrapower of \( \mathcal{M} \) via \( \mathcal{U} \) is to be defined. The first problem we meet is that \( \mathcal{U} \) may not be an ultrafilter in the universe because not all subsets of the index set belong to \( \mathcal{M} \).

We can, of course, extend \( \mathcal{U} \) to a true ultrafilter, say \( \mathcal{U}' \), but this may cause additional trouble. Indeed, if \( \mathcal{U} \) is a special ultrafilter in \( \mathcal{M} \) certain properties of which were expected to be exploit, then most probably these properties do not transfer to \( \mathcal{U}' \); assume for instance that \( \mathcal{U} \) is countably complete in \( \mathcal{M} \) and \( \mathcal{M} \) itself is countable. Therefore, it is better to keep \( \mathcal{U} \) rather than any of its extensions in the universe, as the ultrafilter.

If \( \mathcal{M} \) models \( \text{ZFC} \), the problem can be solved by taking the inner ultrapower. In other words, we consider only those functions \( f : I \rightarrow \mathcal{M} \) (where \( I \in \mathcal{M} \) is the carrier of \( \mathcal{U} \)) which belong to \( \mathcal{M} \) rather than all functions \( f \in \mathcal{M}^I \), to define the ultrapower. This version, however, depends on the axiom of choice in \( \mathcal{M} \); otherwise the proofs of the basic facts about ultrapowers (e. g. Los’ theorem) will not work.

The “choiceless” case can be handled by a sophisticated construction of Spectr [1991], which is based on ideas from both forcing and the ultrapower technique. As presented in Kanovei and van Lambalgen [1994], this construction proceeds as follows. One has to add to the family of functions \( \mathcal{F}_0 = \mathcal{M}^I \cap \mathcal{M} \) a number of new functions \( f \in \mathcal{M}^I \), \( f \notin \mathcal{M} \), which are intended to be choice functions whenever we need such in the ultrapower construction.

In this paper, we consider a very interesting choiceless case: \( \mathcal{M} \) is a Solovay model of \( \text{ZF} \) plus the principle of dependent choice, in which all sets of reals are Lebesgue measurable, and the ultrafilter \( \mathcal{L} \) on the set \( I \) of Vitali degrees of reals in \( \mathcal{M} \), generated by sets of positive measure.
1 On a.e. uniformization in the Solovay model

In this section, we recall the uniformization properties in a Solovay model. Thus let $\mathcal{M}$ be a countable transitive Solovay model for Dependent Choices plus “all sets are Lebesgue measurable”, as it is defined in SOLOVAY [1970], – the ground model. The following known properties of such a model will be of particular interest below.

**Property 1** [True in $\mathcal{M}$]
$V = \mathbb{L}(\text{reals})$; in particular every set is real-ordinal-definable. □

To state the second property, we need to introduce some notation.

Let $N = \omega^\omega$ denote the Baire space, the elements of which will be referred to as real numbers or reals.

Let $P$ be a set of pairs such that $\text{dom } P \subseteq N$ (for instance, $P \subseteq N^2$). We say that a function $f$ defined on $N$ uniformizes $P$ a.e. (almost everywhere) iff the set

$$\{ \alpha \in \text{dom } P : \langle \alpha, f(\alpha) \rangle \notin P \}$$

has null measure. For example if the projection $\text{dom } P$ is a set of null measure in $N$ then any $f$ uniformizes a.e. $P$, but this case is not interesting. The interesting case is the case when $\text{dom } P$ is a set of full measure, and then $f$ a.e. uniformizes $P$ iff for almost all $\alpha$, $\langle \alpha, f(\alpha) \rangle \in P$. □

**Property 2** [True in $\mathcal{M}$]
Any set $P \in \mathcal{M}$, $P \subseteq N^2$, can be uniformized a.e. by a Borel function. (This implies the Lebesgue measurability of all sets of reals, which is known to be true in $\mathcal{M}$ independently.) □

This property can be expanded (with the loss of the condition that $f$ is Borel) on the sets $P$ which do not necessarily satisfy $\text{dom } P \subseteq N$.

**Theorem 3** In $\mathcal{M}$, any set $P$ with $\text{dom } P \subseteq \mathcal{M}$ admits an a.e. uniformisation.

**Proof** Let $P$ be an arbitrary set of pairs such that $\text{dom } P \subseteq N$ in $\mathcal{M}$. Property [1] implies the existence of a function $D : (\text{Ord } \cap \mathcal{M}) \times (N \cap \mathcal{M})$ onto $\mathcal{M}$ which is $\varepsilon$-definable in $\mathcal{M}$.

We argue in $\mathcal{M}$. Let, for $\alpha \in N$, $\xi(\alpha)$ denote the least ordinal $\xi$ such that

$$\exists \gamma \in N \ [ \langle \alpha, D(\xi, \gamma) \rangle \in P ] .$$

(It follows from the choice of $D$ that $\xi(\alpha)$ is well defined for all $\alpha \in N$. ) It remains to apply Property [2] to the set $P' = \{ \langle \alpha, \gamma \rangle \in N^2 : \langle \alpha, D(\xi(\alpha), \gamma) \rangle \in P \}$.
The functions to get the Spector ultrapower

We use a certain ultrafilter over the set of Vitali degrees of reals in $\mathcal{M}$, the initial Solovay model, to define the ultrapower.

Let, for $\alpha, \alpha' \in \mathbb{N}$, $\alpha \text{ vit} \alpha'$ if and only if $\exists m \forall k \geq m (\alpha(k) = \alpha'(k))$, (the Vitali equivalence).

- For $\alpha \in \mathbb{N}$, we set $\alpha = \{\alpha' : \alpha' \text{ vit} \alpha\}$, the Vitali degree of $\alpha$.

- $\mathbb{N} = \{\alpha : \alpha \in \mathbb{N}\}$; $i, j$ denote elements of $\mathbb{N}$.

As a rule, we shall use underlined characters $\underline{f}, \underline{F}, \ldots$ to denote functions defined on $\mathbb{N}$, while functions defined on $\mathbb{N}$ itself will be denoted in the usual manner.

Define, in $\mathcal{M}$, an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ by: $X \subseteq \mathbb{N}$ belongs to $\mathcal{U}$ iff the set $X = \{\alpha \in \mathbb{N} : \alpha \in X\}$ has full Lebesgue measure. It is known (see e.g. van Lambalgen [1992], Theorem 2.3) that the measurability hypothesis implies that $\mathcal{U}$ is $\kappa$-complete in $\mathcal{M}$ for all cardinals $\kappa$ in $\mathcal{M}$.

One cannot hope to define a good $\mathcal{U}$-ultrapower of $\mathcal{M}$ using only functions from $\mathcal{F}_0 = \{f \in \mathcal{M} : \text{dom } f = \mathbb{N}\}$ as the base for the ultrapower. Indeed consider the identity function $i \in \mathcal{M}$ defined by $i(i) = i$ for all $i \in \mathbb{N}$. Then $i(i)$ is nonempty for all $i \in \mathbb{N}$ in $\mathcal{M}$, therefore to keep the usual properties of ultrapowers we need a function $\underline{f} \in \mathcal{F}_0$ such that $\underline{f}(i) \in i$ for almost all $i \in \mathbb{N}$, but Vitali showed that such a choice function yields a nonmeasurable set.

Thus at least we have to add to $\mathcal{F}_0$ a new function $\underline{f}$, not an element of $\mathcal{M}$, which satisfies $\underline{f}(i) \in i$ for almost all $i \in \mathbb{N}$. Actually it seems likely that we have to add a lot of new functions, to handle similar situations, including those functions the existence of which is somehow implied by the already added functions. A general way how to do this, extracted from the exposition in Spector [1991], was presented in Kanovei and van Lambalgen [1994]. However in the case of the Solovay model the a.e. uniformization theorem (Theorem 3) allows to add essentially a single new function, corresponding to the $i$-case considered above.

The generic choice function for the identity

Here we introduce a function $\tau$ defined on $\mathbb{N} \cap \mathcal{M}$ and satisfying $\tau(i) \in i$ for all $i \in \mathbb{N} \cap \mathcal{M}$. $\tau$ will be generic over $\mathcal{M}$ for a suitable notion of forcing.

The notion of forcing is introduced as follows. In $\mathcal{M}$, let $\mathbb{P}$ be the set of all functions $p$ defined on $\mathbb{N}$ and satisfying $p(i) \subseteq i$ and $p(i) \neq \emptyset$ for all $i$. (For example $i \in \mathbb{P}$.) We order $\mathbb{P}$ so that $p$ is stronger than $q$ iff $p(i) \subseteq q(i)$ for all $i$. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathcal{M}$, $G$ defines a function $\tau$ by

$$\tau(i) = \text{the single element of } \bigcap_{p \in G} p(i)$$

1Or, equivalently, the collection of all sets $X \subseteq \mathbb{N}$ which have a nonempty intersection with every Vitali degree. Perhaps this forcing is of separate interest.
for all $i \in \mathbb{N} \cap M$. Functions $r$ defined this way will be called \(P\)-generic over $M$. Let us fix such a function $r$ for the remainder of this paper.

**The set of functions used to define the ultrapower**

We let $F$ be the set of all superpositions $f \circ r$ where $r$ is the generic function fixed above while $f \in M$ is an arbitrary function defined on $N \cap M$. Notice that in particular any function $f \in M$ defined on $N \cap M$ is in $F$: take $f(\alpha) = f(r(\alpha))$.

To see that $F$ can be used successfully as the base of an ultrapower of $M$, we have to check three fundamental conditions formulated in Kanovei and van Lambalgen [1994].

**Proposition 4** [Measurability] Assume that $E \in M$ and $f_1, ..., f_n \in F$. Then the set $\{i \in N \cap M : E(f_1(i), ..., f_n(i))\}$ belongs to $M$.

**Proof** By the definition of $F$, it suffices to prove that $\{i : r(i) \in E\} \in M$ for any set $E \in M$, $E \subseteq N$. By the genericity of $r$, it remains then to prove the following in $M$: for any $p \in \mathbb{P}$ and any set $E \subseteq N$, there exists a stronger condition $q$ such that, for any $i$, either $q(i) \subseteq E$ or $q(i) \cap E = \emptyset$. But this is obvious. $\square$

**Corollary 5** Assume that $V \in M$, $V \subseteq N$ is a set of null measure in $M$. Then, for $L$-almost all $i$, we have $r(i) \notin V$.

**Proof** By the proposition, the set $I = \{i : r(i) \in V\}$ belongs to $M$. Suppose that, on the contrary, $I \in L$. Then $A = \{\alpha : r(\alpha) \in I\}$ is a set of full measure. On the other hand, since $r(i) \in i$, we have $A \subseteq \bigcup_{\beta \in V} \beta$, where the right-hand side is a set of full measure because $V$ is such a set, contradiction. $\square$

**Proposition 6** [Choice] Let $f_1, ..., f_n \in F$ and $W \in M$. There exists a function $f_1 \in F$ such that, for $L$-almost all $i \in N \cap M$, it is true in $M$ that

$$\exists x W(f_1(i), ..., f_n(i), x) \rightarrow W(f_1(i), ..., f_n(i), f(i)).$$

**Proof** This can be reduced to the following: given $W \in M$, there exists a function $f_1 \in F$ such that, for $L$-almost all $i \in N \cap M$,

$$\exists x W(r(i), x) \rightarrow W(r(i), f(i))$$

in $M$.

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$^2$To make things clear, $f \circ r(i) = f(r(i))$ for all $i$.  

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We argue in $\mathcal{M}$. Choose $p \in \mathbb{P}$, and let $p'(i) = \{ \beta \in p(i) : \exists x W(\beta, x) \}$, and $X = \{ i : p'(i) \neq \emptyset \}$. If $X \notin \mathcal{L}$, then an arbitrary $f$ defined on $\mathbb{N}$ will satisfy $(*),$ therefore it is assumed that $X \in \mathcal{L}$. Let

$$q(i) = \begin{cases} p'(i) & \text{iff } i \in X \\ p(i) & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$; then $q \in \mathbb{P}$ is stronger than $p$. Therefore, since $r$ is generic, one may assume that $r(i) \in q(i)$ for all $i$.

Furthermore, DC in the Solovay model $\mathcal{M}$ implies that for every $i \in X$ the following is true: there exists a function $\phi$ defined on $q(i)$ and such that $W(\beta, \phi(\beta))$ for every $\beta \in q(i)$. Theorem 3 provides a function $\Phi$ such that for almost all $\alpha$ the following is true: the value $\Phi(\alpha, \beta)$ is defined and satisfies $W(\beta, \Phi(\alpha, \beta))$ for all $\beta \in q(\alpha)$. Then, by Corollary 3, we have

$$\text{for all } \beta \in q(r(i)), \ W(\beta, \Phi(r(i), \beta))$$

for almost all $i$. However, $r(i) = i$ for all $i$. Applying the assumption that $r(i) \in q(i)$ for all $i$, we obtain $W(r(i), \Phi(r(i), r(i)))$ for almost all $i$. Finally the function $f(i) = \Phi(r(i), r(i))$ is in $\mathcal{F}$ by definition. \hfill $\square$

**Proposition 7**  [Regularity] For any $f \in \mathcal{F}$ there exists an ordinal $\xi \in \mathcal{M}$ such that for $\mathcal{L}$-almost all $i$, if $f(i)$ is an ordinal then $f(i) = \xi$.

**Proof** To prove this statement, assume that $f = f \circ r$ where $f \in \mathcal{M}$ is a function defined on $\mathbb{N}$ in $\mathcal{M}$.

We argue in $\mathcal{M}$. Consider an arbitrary $p \in \mathbb{P}$. We define a stronger condition $p'$ as follows. Let $i \in \mathbb{N}$. If there does not exist $\beta \in p(i)$ such that $f(\beta)$ is an ordinal, we put $p'(i) = p(i)$ and $\xi(i) = 0$. Otherwise, let $\xi(i) = \xi$ be the least ordinal such that $f(\beta) = \xi$ for some $\beta \in p(i)$. We set $p'(i) = \{ \beta \in p(i) : f(\beta) = \xi(i) \}$.

Notice that $\xi(i)$ is an ordinal for all $i \in \mathbb{N}$. Therefore, since the ultrafilter $\mathcal{L}$ is $\kappa$-complete in $\mathcal{M}$ for all $\kappa$, there exists a single ordinal $\xi \in \mathcal{M}$ such that $\xi(i) = \xi$ for almost all $i$.

By genericity, we may assume that actually $r(i) \in p'(i)$ for all $i \in \mathbb{N} \cap \mathcal{M}$. Then $\xi$ is as required. \hfill $\square$

**The ultrapower**

Let $\mathcal{M} = \text{Ult}_\mathcal{L} \mathcal{F}$ be the ultrapower. Thus we define:

- $f \approx g$ iff $\{ i : f(i) = g(i) \} \in \mathcal{L}$ for $f, g \in \mathcal{F}$;
- $[f] = \{ g : g \approx f \}$ (the $\mathcal{L}$-degree of $f$);
• \([f] \in^* [g] \text{ iff } \{i : f(i) \in g(i)\} \in \mathcal{L}\);

• \(\mathfrak{N} = \{[f] : f \in \mathcal{F}\}\), equipped with the above defined membership \(\in^*\).

**Theorem 8** \(\mathfrak{N}\) is an elementary extension of \(\mathfrak{M}\) via the embedding which associates
\(x^* = [\mathfrak{N} \times \{x\}]\) with any \(x \in \mathfrak{M}\). Moreover \(\mathfrak{N}\) is wellfounded and the ordinals in \(\mathfrak{M}\) are isomorphic to the \(\mathfrak{M}\)-ordinals via the mentioned embedding.

**Proof** See Kanovei and van Lambalgen [1994].

**Comment.** Propositions \([\text{ }]\) and \([\text{ }]\) are used to prove the Loś theorem and the property of elementary embedding. Proposition \([\text{ }]\) is used to prove the wellfoundedness part of the theorem.

## 3 The nature of the ultrapower

Theorem 8 allows to collapse \(\mathfrak{N}\) down to a transitive model \(\hat{\mathfrak{N}}\); actually \(\hat{\mathfrak{N}} = \{\hat{X} : X \in \mathfrak{N}\}\) where
\(\hat{X} = \{\hat{Y} : Y \in \mathfrak{N} \text{ and } Y^* \in X\}\).

The content of this section will be to investigate the relations between \(\mathfrak{M}\), the initial model, and \(\hat{\mathfrak{N}}\), the (transitive form of its) Spector ultrapower. In particular it is interesting how the superposition of the “asterisk” and “hat” transforms embeds \(\mathfrak{M}\) into \(\hat{\mathfrak{N}}\).

**Lemma 9** \(x \mapsto \hat{x}^*\) is an elementary embedding \(\mathfrak{M}\) into \(\hat{\mathfrak{N}}\), equal to identity on ordinals and sets of ordinals (in particular on reals).

**Proof** Follows from what is said above.

Thus \(\hat{\mathfrak{N}}\) contains all reals in \(\mathfrak{M}\). We now show that \(\hat{\mathfrak{N}}\) also contains some new reals. We recall that \(r \in \mathcal{F}\) is a function satisfying \(r(i) \in i\) for all \(i \in \mathfrak{N} \cap \mathfrak{M}\).

Let \(a = [\hat{r}]\). Notice that by Loś \([r]\) is a real in \(\mathfrak{N}\), therefore \(a\) is a real in \(\hat{\mathfrak{N}}\).

**Lemma 10** \(a\) is random over \(\mathfrak{M}\).

**Proof** Let \(B \subseteq \mathfrak{N}\) be a Borel set of null measure coded in \(\mathfrak{M}\); we prove that \(a \notin B\). Being of measure 0 is an absolute notion for Borel sets, therefore \(B \cap \mathfrak{M}\) is a null set in \(\mathfrak{M}\) as well. Corollary \([\text{ }]\) implies that for \(\mathcal{L}\)-almost all \(i\), we have \(r(i) \notin B\). By Loś, \(\neg ([r] \in^* B^*)\) in \(\mathfrak{N}\). Then \(a \notin B^*\) in \(\hat{\mathfrak{N}}\). However, by the absoluteness of the Borel coding, \(B^* = B \cap \mathfrak{N}\), as required.

Thus \(\hat{\mathfrak{N}}\) contains a new real number \(a\). It so happens that this \(a\) generates all reals in \(\hat{\mathfrak{N}}\).
Lemma 11  The reals of \( \hat{\mathcal{N}} \) are exactly the reals of \( \mathcal{M}[a] \).

Proof  It follows from the known properties of random extensions that every real in \( \mathcal{M}[a] \) can be obtained as \( F(a) \) where \( F \) is a Borel function coded in \( \mathcal{M} \). Since \( a \) and all reals in \( \mathcal{M} \) belong to \( \hat{\mathcal{N}} \), we have the inclusion \( \supseteq \) in the lemma.

To prove the opposite inclusion let \( \beta \in \hat{\mathcal{N}} \cap N \). Then by definition \( \beta = \hat{[F]} \), where \( F \in \mathcal{F} \). In turn \( F = f \circ \tau \), where \( f \in \mathcal{M} \) is a function defined on \( N \cap \mathcal{M} \). We may assume that in \( \mathcal{M} \) \( f \) maps reals into reals. Then, first, by Property \( \overline{2} \), \( f \) is a.e. equal in \( \mathcal{M} \) to a Borel function \( g = B_{\gamma} \) where \( \gamma \in N \cap \mathcal{M} \) and \( B_{\gamma} \) denotes, in the usual manner, the Borel subset (of \( N^2 \) in this case) coded by \( \gamma \). Corollary \( \overline{3} \) shows that we have \( F(i) = B_{\gamma} (\tau(i)) \) for \( \mathcal{L} \)-almost all \( i \). In other words, \( F(i) = B_{\gamma^*(i)} (\tau(i)) \) for \( \mathcal{L} \)-almost all \( i \). By Loś, this implies \( [F] = B_{\gamma^*} ([\tau]) \) in \( \mathcal{N} \), therefore \( \beta = B_{\gamma} (a) \) in \( \hat{\mathcal{N}} \). By the absoluteness of Borel coding, we have \( \beta \in L[\gamma, a] \), therefore \( \beta \in \mathcal{M}[a] \).

We finally can state and prove the principal result.

Theorem 12  \( \hat{\mathcal{N}} \subseteq \mathcal{M}[a] \) and \( \hat{\mathcal{N}} \) coincides with \( L^{\mathcal{M}[a]}(\text{reals}) \), the smallest subclass of \( \mathcal{M}[a] \) containing all ordinals and all reals of \( \mathcal{M}[a] \) and satisfying all the axioms of ZF.

Proof  Very elementary. Since \( \mathcal{V} = L(\text{reals}) \) is true in \( \mathcal{M} \), the initial Solovay model, this must be true in \( \hat{\mathcal{N}} \) as well. The previous lemma completes the proof.

Corollary 13  The set \( N \cap \mathcal{M} \) of all “old” reals does not belong to \( \hat{\mathcal{N}} \).

Proof  The set in question is known to be non–measurable in the random extension \( \mathcal{M}[a] \); thus it would be non–measurable in \( \hat{\mathcal{N}} \) as well. However \( \hat{\mathcal{N}} \) is an elementary extension of \( \mathcal{M} \), hence it is true in \( \hat{\mathcal{N}} \) that all sets are measurable.

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