Research Article

A Novel of Ideals and Fuzzy Ideals of Gamma-Semigroups

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In this paper, we define new types of ideals, fuzzy ideals, almost ideals, and fuzzy almost ideals of \(\Gamma\)-semigroups by using the elements of \(\Gamma\). We investigate properties of them.

1. Introduction and Preliminaries

Ideal theory in semigroups, like all other algebraic structures, plays an important role in studying them. Bi-ideals in semigroups were introduced by Good and Hughes [1] in 1952. Steinfeld [2] gave a notion and studied quasi-ideals in semigroups in 1956. In 1965, Zadeh introduced the concept of fuzzy subsets in [3]. Since then, fuzzy subsets are now applied in groups in 1956. In 1965, Zadeh introduced the concept of fuzzy subsets. Bi-ideals in semigroups were introduced by Good and Hughes [1] in 1952.

Almost ideals (A-ideals) in semigroups were studied by Grošek [4] and Satko [5–7] in 1980-1981. Next year, Bogdanović [8] studied bi-ideals in semigroups. Sen [9] introduced almost bi-ideals in semigroups by using concepts of almost ideals and bi-ideals in semigroups. Sen [9] introduced \(\Gamma\)-semigroups in 2007. Moreover, Iampan [10] gave remarkable notes on bi-\(\Gamma\)-ideals in \(\Gamma\)-semigroups in 2009. Recently, Lamban trope [11] studied quasi-ideals, and fuzzy almost ideals in semigroups in 2018. So, all of these give the inspiration to study about new types of ideals and fuzzy ideals in \(\Gamma\)-semigroups in this paper.

First, we recall the definition of \(\Gamma\)-semigroups which was defined by Sen and Saha [15].

**Definition 1** (see [15]). Let \(S\) and \(\Gamma\) be nonempty sets. Then, \(S\) is called a \(\Gamma\)-semigroup if there exists a mapping \(S \times \Gamma \times S \rightarrow S\) written as \((a, \gamma, b) \mapsto ab\) satisfying the axiom \((a\alpha b)\beta c = a\alpha (b\beta c)\) for all \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\).

**Remark 1**

1. In case \(|\Gamma| = 1\), the definition of \(\Gamma\)-semigroup is a semigroup.
2. Every semigroup \((S, \cdot)\) can be considered to be a \(\Gamma\)-semigroup where \(\Gamma := \{\}\).
3. If \(S\) is a \(\Gamma\)-semigroup, then for each \(\alpha \in \Gamma\), \((S, \alpha)\) is a semigroup.

Let \(S\) be a \(\Gamma\)-semigroup. For nonempty subsets \(A, B\) of \(S\), let

\[
\Gamma_{AB} = \{ab \mid a \in A, b \in B, \alpha \in \Gamma\}. \tag{1}
\]

If \(x \in S\) and \(\alpha \in \Gamma\), we let \(\Gamma_x := \Gamma\{x\}\), \(x\Gamma_A := \{x\}\Gamma_A\), and \(\Gamma_{A\alpha} := A\{\alpha\}\).

**Definition 2.** Let \(S\) be a \(\Gamma\)-semigroup. A nonempty subset \(A\) of \(S\) is called

1. A sub-\(\Gamma\)-semigroup of \(S\) if \(\Gamma\alpha A \subseteq A\)
2. A left ideal of \(S\) if \(\Gamma\alpha A \subseteq A\)
3. A right ideal of \(S\) if \(\Gamma\alpha S \subseteq A\)
4. An ideal of \(S\) if it is both a left ideal and a right ideal of \(S\)
A quasi-ideal of $S$ if $S^2A \cap A^2 \subseteq A$

(6) A bi-ideal of $S$ if $A$ is a sub-\(\Gamma\)-semigroup of $S$ and $A^2 S A \subseteq A$

Recently, Wattanatripop and Changphas [13] defined the concepts of left almost ideals and right almost ideals of \(\Gamma\)-semigroups. A \(\Gamma\)-semigroup containing no proper left (respectively, right) almost ideals was characterized.

Now, we recall the definitions and some notations of fuzzy subsets. A fuzzy subset of a set $S$ is a function from $S$ into the closed interval $[0, 1]$. For any two fuzzy subsets $f$ and $g$ of a set $S$,

1. $f \cap g$ is a fuzzy subset of $S$ defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$, for all $x \in S$. (2)

2. $f \cup g$ is a fuzzy subset of $S$ defined by $(f \cup g)(x) = \max\{f(x), g(x)\}$, for all $x \in S$. (3)

3. $f \circ g$ is a fuzzy subset of $S$ defined by $$(f \circ g)(x) = \begin{cases} \sup\{\min\{f(a), g(b)\}\}, & \text{if } x \in S^2, \\ 0, & \text{otherwise.} \end{cases}$$ (4)

4. $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

For a fuzzy subset $f$ of a set $S$, the support of $f$ is defined by $$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}.$$ (5)

The characteristic mapping of a subset $A$ of $S$ is a fuzzy subset of $S$ defined by $$C_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$ (6)

The definition of fuzzy points was given by Pu and Liu [16]. For $x \in S$ and $t \in (0, 1]$, a fuzzy point $x_t$ of a set $S$ is a fuzzy subset of $S$ defined by $$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$ (7)

Some basic concepts of fuzzy semigroup theory can be seen in [17].

For a \(\Gamma\)-semigroup $S$, let $\mathcal{F}(S)$ be the set of all fuzzy subsets of $S$. For each $a \in \Gamma$, define a binary operation $\ast_a$ on $\mathcal{F}(S)$ by $$\ast_a^a (x) = \begin{cases} \sup \{\min\{f(a), g(b)\}\}, & \text{if } x \in S^2, \\ 0, & \text{otherwise.} \end{cases}$$ (8)

Let $\Gamma^* := \{\ast_a \mid a \in \Gamma\}$. Then, $(\mathcal{F}(S), \Gamma^*)$ is a $\Gamma$-semigroup.

In 2017, Wattanatripop and Changphas [13] defined the concepts of left A-ideals and right A-ideals (almost left ideals and almost right ideals) of a $\Gamma$-semigroup as follows.

In 1981, Bogdanovic [8] gave the definition of almost bi-ideals of semigroups as follows.

A nonempty subset $B$ of a semigroup $S$ is called an almost bi-ideal of $S$ if $$\{sB \cap B \cap s \neq \emptyset \mid s \in S \}.$$ (9)

In 2018, Wattanatripop et al. [14, 18] introduced the notions of almost quasi-ideals and right almost quasi-ideals as follows.

A nonempty subset $Q$ of a semigroup $S$ is called an almost quasi-ideal of $S$ if $$\{sQ \cap Q \cap s \neq \emptyset \mid s \in S \}.$$ (10)

Let $f$ be a fuzzy subset of a semigroup $S$ such that $f \neq 0$. Then, $f$ is called

1. A fuzzy almost bi-ideal of $S$ if for all $s \in S$, $$f \circ C_s \circ f \cap f \neq 0.$$ (12)

2. A fuzzy almost left ideal of $S$ if for all $s \in S$, $$C_s \circ f \cap f \neq 0.$$ (13)

3. A fuzzy almost right ideal of $S$ if for all $s \in S$, $$f \circ C_s \cap f \neq 0.$$ (14)

4. A fuzzy almost quasi-ideal of $S$ if for all $s \in S$, $$(C_s \circ f \cap f \circ C_s) \cap f \neq 0.$$ (15)

In 1981, Kuroki [4] introduced the notion of fuzzy ideals of semigroups as follows: A fuzzy subset $f$ of a semigroup $S$ is called:

1. A fuzzy left ideal of $S$ if $f(ab) \geq f(b)$ for all $a, b \in S$.

2. A fuzzy right ideal of $S$ if $f(ab) \geq f(a)$ for all $a, b \in S$.

3. A fuzzy ideal of $S$ if it is both a fuzzy left ideal and a fuzzy right ideal of $S$.

The aim of this paper is to define new types of ideals and fuzzy ideals of a $\Gamma$-semigroup $S$ by using elements in $\Gamma$. In Section 2, we consider new types of ideals of $S$. In Section 3, we study new types of fuzzy ideals of $S$. In Section 4, we consider new types of almost ideals of $S$. In Section 5, we study new types of fuzzy almost ideals of $S$.

2. New Types of Ideals

In this section, we will focus on $(\alpha, \beta)$-ideals, $(\alpha, \beta)$-quasi-ideals, and $(\alpha, \beta)$-bi-ideals of $\Gamma$-semigroups for $\alpha, \beta \in \Gamma$. 

2.1. \((α, β)-\text{Ideals.}\) First, we will define \((α, β)\)-ideals of Γ-semigroups as follows.

**Definition 3.** Let \(S\) be a \(Γ\)-semigroup, \(A\) be a nonempty subset of \(S\), and \(α, β ∈ Γ\). Then, \(A\) is called

1. A left \(α\)-ideal of \(S\) if \(SaA ⊆ A\).
2. A right \(β\)-ideal of \(S\) if \(AβS ⊆ A\).
3. An \((α, β)\)-ideal of \(S\) if it is both a left \(α\)-ideal and a right \(β\)-ideal of \(S\).
4. An \(α\)-ideal of \(S\) if it is an \((α, α)\)-ideal of \(S\).

**Remark 2**

1. Every left ideal of a \(Γ\)-semigroup \(S\) is a left \(α\)-ideal of \(S\) for all \(α ∈ Γ\).
2. Every right ideal of a \(Γ\)-semigroup \(S\) is a right \(β\)-ideal of \(S\) for all \(β ∈ Γ\).
3. Every ideal of a \(Γ\)-semigroup \(S\) is an \((α, β)\)-ideal of \(S\) for all \(α, β ∈ Γ\).

However, the converse of Example 1 is not generally true. We can see in the following example.

**Example 1.** Let \(S = Γ = \mathbb{N}\) and \((a, y, b) → a + y + b\) for all \(a, b ∈ S\) and \(y ∈ Γ\). Then, \(S\) is a \(Γ\)-semigroup. Let \(A = \{1\} ∪ \{6, 7, 8, 9, \ldots\}\). It is easy to show that \(A\) is a left \(4\)-ideal but not a left ideal of \(S\).

**Theorem 1.** The following statements are true:

1. If \(L\) is a left \(α\)-ideal of a \(Γ\)-semigroup \(S\), then \(L\) is a left ideal of a semigroup \((S, α)\).
2. If \(R\) is a right \(β\)-ideal of a \(Γ\)-semigroup \(S\), then \(R\) is a right ideal of a semigroup \((S, β)\).
3. If \(I\) is an \(α\)-ideal of a \(Γ\)-semigroup \(S\), then \(I\) is an ideal of a semigroup \((S, α)\).

For a nonempty subset \(A\) of a \(Γ\)-semigroup \(S\), let \((A)_{(α)}\), \((A)_{(β)}\), and \((A)_{(α, β)}\) be the left \(α\)-ideal, the right \(β\)-ideal, and the \((α, β)\)-ideal of \(S\) generated by \(A\), respectively.

**Theorem 2.** Let \(A\) be a nonempty subset of a \(Γ\)-semigroup \(S\). Then,

1. \((A)_{(α)} = A ∪ SaA\).
2. \((A)_{(β)} = A ∪ AβS\).
3. \((A)_{(α, β)} = A ∪ SaA ∪ AβS ∪ SaAβS\).

**Proof**

1. Let \(L\) be a nonempty subset of a \(Γ\)-semigroup \(S\). Let \(L = A ∪ SaA\). Clearly, \(A ⊆ L\). Since \(S\) is a \(Γ\)-semigroup, \(SaL = Sa(A ∪ SaA) = SaA ∪ SaA = SaA ⊆ L\). Therefore, \(L\) is a left \(α\)-ideal of \(S\). Next, let \(C\) be any left \(α\)-ideal of \(S\) containing \(A\). Since \(C\) is a left \(α\)-ideal of \(S\) and \(A ⊆ C\), \(SaC\). Therefore, \(L = A ∪ SaA ⊆ C\). Hence, \(L\) is the smallest left \(α\)-ideal of \(S\) containing \(A\). Therefore, \((A)_{(α)} = L = A ∪ SaA\), as required.

The proofs of (2) and (3) are similar to the proof of (1).

**Theorem 3.** Let \(L\) be a left \(α\)-ideal and \(R\) a right \(β\)-ideal of a \(Γ\)-semigroup \(S\). Then, \(LyR\) is an \((α, β)\)-ideal of \(S\) for all \(γ ∈ Γ\).

**Proof.** Let \(L\) and \(R\) be a left \(α\)-ideal and a right \(β\)-ideal of \(S\), respectively, and let \(γ ∈ Γ\). Clear that \(LyR ≠ ∅\). We have \(Sa(LyR) = (SaL)γR ⊆ LyR\) and \((LyR)βS = Ly(RβS) ⊆ LyR\). Therefore, \(LyR\) is an \((α, β)\)-ideal of \(S\).

**Theorem 4.** Let \(L_1\) and \(L_2\) be two left \(α\)-ideals of a \(Γ\)-semigroup \(S\). The following statements are true:

1. \(L_1 ∪ L_2\) is a left \(α\)-ideal of \(S\).
2. \(L_1 ∩ L_2\) is a left \(α\)-ideal of \(S\), where \(L_1 ∩ L_2 ≠ ∅\).

**Proof**

1. Let \(L_1\) and \(L_2\) be two left \(α\)-ideals of \(S\). Clear that \(L_1 ∪ L_2 ≠ ∅\). Then, \(Sa(L_1 ∪ L_2) ⊆ SaL_1 ∪ SaL_2 ⊆ L_1 ∪ L_2\). Hence, \(L_1 ∪ L_2\) is a left \(α\)-ideal of \(S\).
2. Since \(L_1 ∩ L_2 ≠ ∅\), we have \(Sa(L_1 ∩ L_2) ⊆ SaL_1 ∩ SaL_2 ⊆ L_1 ∩ L_2\). Hence, \(L_1 ∩ L_2\) is a left \(α\)-ideal of \(S\).

**Theorem 5.** Let \(R_1\) and \(R_2\) be two right \(β\)-ideals of a \(Γ\)-semigroup \(S\). Then,

1. \(R_1 ∪ R_2\) is a right \(β\)-ideal of \(S\).
2. \(R_1 ∩ R_2\) is a right \(β\)-ideal of \(S\), where \(R_1 ∩ R_2 ≠ ∅\).

**Proof.** It is similar to Theorem 4.

**Theorem 6.** Let \(I_1\) and \(I_2\) be two \((α, β)\)-ideals of a \(Γ\)-semigroup \(S\). Then,

1. \(I_1 ∪ I_2\) is an \((α, β)\)-ideal of \(S\).
2. \(I_1 ∩ I_2\) is an \((α, β)\)-ideal of \(S\), where \(I_1 ∩ I_2 ≠ ∅\).

**Proof.** It follows by Theorems 4 and 5.

2.2. \((α, β)\)-Quasi-Ideals. We define \((α, β)\)-quasi-ideals of \(Γ\)-semigroups as follows.

**Definition 4.** Let \(S\) be a \(Γ\)-semigroup. A nonempty subset \(Q\) of \(S\) is called

1. An \((α, β)\)-quasi-ideal of \(S\) if \(SaQ ∩ QβS ⊆ Q\).
2. An \(α\)-quasi-ideal of \(S\) if it is an \((α, α)\)-quasi-ideal of \(S\).

**Theorem 7.** Let \(S\) be a \(Γ\)-semigroup and \(Q_i\) an \((α, β)\)-quasi-ideal of \(S\) for all \(i ∈ I\). If \(∩_{i∈I}Q_i ≠ ∅\), then \(∩_{i∈I}Q_i\) is an \((α, β)\)-quasi-ideal of \(S\).

**Proof.** Let \(S\) be a \(Γ\)-semigroup and \(Q_i\) an \((α, β)\)-quasi-ideal of \(S\) for all \(i ∈ I\). Then, \((Sa ∩_{i∈I}Q_i) ∩ (∩_{i∈I}Q_iβS) ⊆ SaQ_i ∩ \)
$Q_i\beta\subseteq Q_i$ for all $i \in I$, so $(S \cap \cap_{i \in I} Q_i) \cap (\cap_{i \in I} Q_i) \beta S \subseteq \cap_{i \in I} Q_i$. Therefore, $\cap_{i \in \mathbb{N}} Q_i$ is an $(\alpha, \beta)$-quasi-ideal of $S$.

Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$. Let $(A)_{[(a, \beta)]}$ be the $(\alpha, \beta)$-quasi-ideal of $S$ generated by $A$.

**Theorem 8.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$. Then, $$(A)_{[(a, \beta)]} = A \cup (SA \cap A \beta S). \quad (16)$$

**Proof.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$. Let $Q = A \cup (SA \cap A \beta S)$. Clearly, $A \subseteq Q$. We have $SA \cap Q \beta S = SA \cap (A \cup (SA \cap A \beta S)) \beta S \subseteq Q$. Therefore, $Q$ is an $(\alpha, \beta)$-quasi-ideal of $S$.

Let $C$ be any $(\alpha, \beta)$-quasi-ideal of $S$ containing $A$. Since $C$ is an $(\alpha, \beta)$-quasi-ideal of $S$ and $A \subseteq C$, $SA \cap A \beta S \subseteq C$. Therefore, $Q = A \cup (SA \cap A \beta S) \subseteq C$.

Hence, $Q$ is the smallest $(\alpha, \beta)$-quasi-ideal of $S$ containing $A$. Therefore, $(A)_{[(a, \beta)]} = Q = A \cup (SA \cap A \beta S)$, as required.

**Theorem 9.** Let $S$ be a $\Gamma$-semigroup. Let $L$ and $R$ be a left $\alpha$-ideal and a right $\beta$-ideal of $S$, respectively. If $L \cap R \neq \emptyset$, then $L \cap R$ is an $(\alpha, \beta)$-quasi-ideal of $S$.

**Proof.** Let $L$ and $R$ be a left $\alpha$-ideal and a right $\beta$-ideal of a $\Gamma$-semigroup $S$, respectively. Then, $SA \cap (L \cap R) \beta S \subseteq SA \cap A \beta S \subseteq L \cap R$, and also, $R \beta S \subseteq R$. Hence, $L \cap R$ is an $(\alpha, \beta)$-quasi-ideal of $S$.

**Corollary 1.** Let $S$ be a $\Gamma$-semigroup. Let $L$ and $R$ be a left $\alpha$-ideal and a right $\beta$-ideal of $S$, respectively. Then, $L \cap R$ is an $\alpha$-quasi-ideal of $S$.

**Proof.** We have $RA \subseteq L \cap R$, which implies $L \cap R \neq \emptyset$. By Theorem 9, $L \cap R$ is an $\alpha$-quasi-ideal of $S$.

**Theorem 10.** Every $(\alpha, \beta)$-quasi-ideal $Q$ of a $\Gamma$-semigroup $S$ is the intersection of a left $\alpha$-ideal and a right $\beta$-ideal of $S$.

**Proof.** Let $Q$ be an $(\alpha, \beta)$-quasi-ideal of a $\Gamma$-semigroup $S$. Let $L = Q \cup SaQ$ and $R = \cup Q \beta S$. Then, $SA \cup (Q \cap SaQ) = SA \cup (Q \cap SaQ) \subseteq S \subseteq L$, and also, $R \beta S \subseteq R$. We have that $L \cap R = (Q \cup SaQ) \cap (Q \cup Q \beta S) = Q \cup (Q \cap Q \beta S) = Q$. Therefore, $Q = L \cap R$.

**Definition 5.** A $\Gamma$-semigroup $S$ is called $(\alpha, \beta)$-quasi-simple if $S$ does not contain proper $(\alpha, \beta)$-quasi-ideals.

A $\Gamma$-semigroup $S$ is called $\alpha$-quasi-simple if $S$ is $(\alpha, \alpha)$-quasi-simple.

**Theorem 11.** Let $S$ be a $\Gamma$-semigroup. Then, $S$ is $(\alpha, \alpha)$-quasi-simple if and only if $S \cap saS = S$ for all $s \in S$.

**Proof.** Assume that $S$ is $\alpha$-quasi-simple. Let $s \in S$; we claim that $S \cap saS$ is an $\alpha$-quasi-ideal of $S$. We have $sas \in S \cap saS$; this implies $S \cap saS \neq \emptyset$. Moreover, $S \cap (S \cap saS) = (S \cap saS) \subseteq S \cap saS$; $S \cap saS \subseteq S \cap saS$; $S \cap saS \subseteq S \cap saS$; $S \cap (S \cap saS) \subseteq S \cap saS$; $S \cap saS \subseteq S \cap saS$; $S \cap saS \subseteq S \cap saS$. Therefore, $S \cap saS$ is an $\alpha$-quasi-ideal of $S$. Since $S$ is $\alpha$-quasi-simple, we have $S = S \cap saS$.

Conversely, assume that $S \cap saS = S$ for all $s \in S$. Let $Q$ be an $\alpha$-quasi-ideal of $S$ and $q \in Q$. By assumption, $S = S \cap qaS$. Since $Q$ is an $\alpha$-quasi-ideal of $S$, $S \cap qaS \subseteq S \cap Q \cap Q \subseteq Q$. Therefore, $Q = S$. Hence, $Q$ is an $\alpha$-quasi-simple.

**Definition 6.** An $(\alpha, \beta)$-quasi-ideal $Q$ of a $\Gamma$-semigroup $S$ is called minimal if for all $(\alpha, \beta)$-quasi-ideal $C$ of $S$, if $C \subseteq Q$, then $C = Q$.

**Theorem 12.** Let $S$ be a $\Gamma$-semigroup and $Q$ an $(\alpha, \beta)$-quasi-ideal of $S$. If $Q$ is $(\alpha, \beta)$-quasi-simple, then $Q$ is a minimal $(\alpha, \beta)$-quasi-ideal of $S$.

**Proof.** Assume that $S$ is a $\Gamma$-semigroup and $Q$ an $(\alpha, \beta)$-quasi-ideal of $S$. Let $C$ be a $(\alpha, \beta)$-quasi-ideal of $S$ such that $C \subseteq Q$. Then, $Q \cap C \subseteq Q \cap C \beta \subseteq C$. Therefore, $C$ is an $(\alpha, \beta)$-quasi-ideal of $Q$. Since $Q$ is $(\alpha, \beta)$-quasi-simple, $C = Q$. Hence, $Q$ is a minimal $(\alpha, \beta)$-quasi-ideal of $S$.

2.3. $(\alpha, \beta)$-Bi-Ideals. We will define $(\alpha, \beta)$-bi-ideals of $\Gamma$-semigroups as follows.

**Definition 7.** Let $S$ be a $\Gamma$-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset $B$ of $S$ is called

(1) An $(\alpha, \beta)$-bi-ideal of $S$ if $BaS \beta B \subseteq B$.

(2) An $\alpha$-$\beta$-bi-ideal of $S$ if it is an $(\alpha, \alpha)$-bi-ideal of $S$.

**Theorem 13.** Every $(\alpha, \beta)$-quasi-ideal of a $\Gamma$-semigroup $S$ is a $(\beta, \alpha)$-bi-ideal of $S$.

**Proof.** Let $Q$ be an $(\alpha, \beta)$-quasi-ideal of $S$. Then,

$$Q \cap saQ \subseteq Q \beta S \cap saQ \subseteq Q. \quad (17)$$

Hence, $Q$ is an $(\beta, \alpha)$-bi-ideal of $S$.

**Theorem 14.** Let $S$ be a $\Gamma$-semigroup and $B_i$ an $(\alpha, \beta)$-bi-ideal of $S$ for all $i \in I$. If $\cap_{i \in I} B_i \neq \emptyset$, then $\cap_{i \in I} B_i$ is an $(\alpha, \beta)$-bi-ideal of $S$.

**Proof.** Let $S$ be a $\Gamma$-semigroup and $B_i$ an $(\alpha, \beta)$-bi-ideal of $S$ for all $i \in I$. Then, $\cap_{i \in I} B_i \subseteq \cap_{i \in I} B_i \subseteq B_i \subseteq B_i \subseteq B_i \subseteq B_i$. Therefore, $\cap_{i \in I} B_i$ is an $(\alpha, \beta)$-bi-ideal of $S$.

Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$, and let $(A)_{[(a, \beta)]}$ denote the $(\alpha, \beta)$-bi-ideal of $S$ generated by $A$.

**Theorem 15.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$, and $A, \beta \in \Gamma$. Then,

$$(A)_{[(a, \beta)]} = A \cup (A \alpha S \beta A). \quad (18)$$
Let $S$ be a nonempty subset of a $\Gamma$-semigroup $S$. Let $B = A \cup (AaS\beta A)$. Clearly, $A \subseteq B$. We have that $\Gamma(\alpha, \beta)$ is an $(\alpha, \beta)$-ideal of $S$, and $A \subseteq C$, $AaS\beta A \subseteq C$. Therefore, $B = A \cup (AaS\beta A) \subseteq C$. Hence, $B$ is the smallest $(\alpha, \beta)$-ideal of $S$ containing $A$. Therefore, $(A)_{b(\alpha, \beta)} = B = A \cup (AaS\beta A)$.

**Theorem 16.** Let $S$ be a $\Gamma$-semigroup, $A$ a nonempty subset of $S$, and $B$ an $(\alpha, \beta)$-ideal of $S$. The following statements are true:

1. $BaA$ is an $(\alpha, \beta)$-ideal of $S$.
2. $A\beta B$ is an $(\alpha, \beta)$-ideal of $S$.

**Proof.** We have that

\[
(BaA)_{aS\beta}(BaA) = Ba(AaS\beta)(BaA) \subseteq (BaS\beta B)aA \subseteq BaA.
\]

Then, $BaA$ is an $(\alpha, \beta)$-ideal of $S$. Similarly, $A\beta B$ is an $(\alpha, \beta)$-ideal of $S$.

**Theorem 17.** Let $S$ be a $\Gamma$-semigroup. Assume that $B_1, B_2,$ and $B_3$ are $(\alpha, \beta)$-ideals of $S$. Then, $B_1aB_2B_3$ is an $(\alpha, \beta)$-ideal of $S$.

**Proof.** Since $B_1, B_2,$ and $B_3$ are $(\alpha, \beta)$-ideals of $S$, we have

\[
(B_1aB_2B_3)a(B_1aB_2B_3) \subseteq B_1aB_2B_3,
\]

\[
(B_1aB_2B_3)B_2(B_1aB_2B_3) \subseteq B_1aB_2B_3.
\]

Then,

\[
(B_1aB_2B_3)aS\beta(B_1aB_2B_3)
\]

\[
= (B_1aB_2aB_3B_2)B_2B_3
\]

\[
\subseteq (B_1aS\beta B_2aB_3B_2B_3)
\]

\[
\subseteq B_1B_2B_3.
\]

Therefore, $B_1aB_2B_3$ is an $(\alpha, \beta)$-ideal of $S$.

**Definition 9.** An $(\alpha, \beta)$-ideal $B$ of a $\Gamma$-semigroup $S$ is called minimal if for all $(\alpha, \beta)$-ideal $C$ of $S$, if $C \subseteq B$, then $C = B$.

**Theorem 19.** Let $S$ be a $\Gamma$-semigroup and $B$ an $(\alpha, \beta)$-ideal of $S$. If $B$ is $(\alpha, \beta)$-simple, then $B$ is a minimal $(\alpha, \beta)$-ideal of $S$.

**Proof.** Assume that $S$ is a $\Gamma$-semigroup and $B$ an $(\alpha, \beta)$-ideal of $S$. Let $B$ be $(\alpha, \beta)$-simple. Let $C$ be an $(\alpha, \beta)$-ideal of $S$ such that $C \subseteq B$. Then, $C \subseteq B \subseteq C$. Therefore, $C$ is an $(\alpha, \beta)$-ideal of $B$. Since $B$ is $(\alpha, \beta)$-simple, $C = B$. Then, $B$ is a minimal $(\alpha, \beta)$-ideal of $S$.

**3. New Types of Fuzzy Ideals**

3.1. **Fuzzy $(\alpha, \beta)$-Ideals.** We will define fuzzy $(\alpha, \beta)$-ideals of $\Gamma$-semigroups as follows.

**Definition 10.** Let $\alpha, \beta \in \Gamma$ and $f$ be a fuzzy subset of a $\Gamma$-semigroup $S$. Then, $f$ is called

1. A fuzzy left $\alpha$-ideal of $S$ if $f(xay) \geq f(y)$ for all $x, y \in S$.
2. A fuzzy right $\beta$-ideal of $S$ if $f(xy) \geq f(x)$ for all $x, y \in S$.
3. A fuzzy $(\alpha, \beta)$-ideal of $S$ if it is both a fuzzy left $\alpha$-ideal and a fuzzy right $\beta$-ideal of $S$.

The following theorems show the relationship between $(\alpha, \beta)$-ideals and fuzzy $(\alpha, \beta)$-ideals.

**Theorem 20.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $S$. Then, the following statements are true:

1. $A$ is a left $\alpha$-ideal of $S$ if and only if $C_A$ is a fuzzy left $\alpha$-ideal of $S$.
2. $A$ is a right $\beta$-ideal of $S$ if and only if $C_A$ is a fuzzy right $\beta$-ideal of $S$.
3. $A$ is an $(\alpha, \beta)$-ideal of $S$ if and only if $C_A$ is a fuzzy $(\alpha, \beta)$-ideal of $S$.

**Proof.**

1. Suppose that $A$ is a left $\alpha$-ideal of $S$. Let $x, y \in S$. If $y \in A$, then $xay \in A$. Thus, $C_A(xay) = 1$ so that $C_A(xay) \geq C_A(y)$.

If $y \notin A$, then $C_A(y) = 0 \leq C_A(xay)$.

Therefore, $C_A$ is a fuzzy left $\alpha$-ideal of $S$.

Conversely, assume that $C_A$ is a fuzzy left $\alpha$-ideal of $S$. Let $x \in S$ and $y \in A$. Then, $C_A(y) = 1$. Since $C_A(xay) \geq C_A(y)$, we have $xay \in A$. Hence, $A$ is a left $\alpha$-ideal of $S$.

2. is similar to (1).

3. follows by (1) and (2).

**Theorem 21.** Let $f$ be a fuzzy subset of a $\Gamma$-semigroup $S$. Then, the following properties hold:

1. $f$ is a fuzzy left $\alpha$-ideal of $S$ if and only if $f \star s \subseteq f$.
2. $f$ is a fuzzy right $\beta$-ideal of $S$ if and only if $f \ast s \subseteq f$. 
(3) \( f \) is a fuzzy \((a, \beta)\)-ideal of \( S \) if and only if \( S \circ_a f \subseteq f \) and \( f \circ_\beta S \subseteq f \).

**Proof**

(1) Assume that \( f \) is a fuzzy left \( \alpha \)-ideal of a \( \Gamma \)-semigroup \( S \). Let \( x \in S \) and \( \alpha \in \Gamma \).

If \( x \notin S_\alpha a \), then \( (S \circ_a f)(x) = 0 \). So, \( (S \circ_a f)(x) \leq f(x) \).

If \( x \in S_\alpha a \), then there exist \( y, z \in S \) such that \( x = yaz \).

Since \( f \) is a fuzzy left \( \alpha \)-ideal of \( S \), we have

\[
(S \circ_a f)(x) = \sup_{x = yaz} \min \{S(y), f(z)\}
\]

\[
\leq \min \{S(y), f(z)\}
\]

\[
= f(z)
\]

\[
\leq f(yaz)
\]

(22)

\[
= f(x).
\]

We conclude that \( S \circ_a f \subseteq f \).

Conversely, assume that \( S \circ_a f \subseteq f \). Let \( x, y, z \in S \) be such that \( x = yaz \). Then,

\[
f(yaz) = f(x)
\]

\[
\geq (S \circ_a f)(x)
\]

\[
= \sup_{x = yaz} \min \{S(y), f(z)\}
\]

\[
\geq \min \{S(y), f(z)\}
\]

\[
= \min \{1, f(z)\}
\]

\[
= f(z)
\]

Hence, \( f \) is a fuzzy left \( \alpha \)-ideal of \( S \).

(2) and (3) can be seen in a similar fashion.

**Theorem 22.** Let \( f \) and \( g \) be fuzzy left \( \alpha \)-ideals of a \( \Gamma \)-semigroup \( S \). Then,

(1) \( f \cap g \) is a fuzzy left \( \alpha \)-ideal of \( S \).

(2) \( f \cup g \) is a fuzzy left \( \alpha \)-ideal of \( S \).

**Proof.** It is similar to Theorem 22.

**Theorem 24.** Let \( f \) and \( g \) be fuzzy \((a, \beta)\)-ideals of a \( \Gamma \)-semigroup \( S \). Then,

(1) \( f \cap g \) is a fuzzy \((a, \beta)\)-ideal of \( S \).

(2) \( f \cup g \) is a fuzzy \((a, \beta)\)-ideal of \( S \).

**Proof.** It follows by Theorems 22 and 23.

**Theorem 25.** Let \( f \) be a nonzero fuzzy subset of a \( \Gamma \)-semigroup \( S \) and \( f_t = \{ x \in S | f(x) \geq t \} \). The following statements are true:

(1) \( f \) is a fuzzy left \( \alpha \)-ideal of \( S \) if and only if for all \( t \in (0, 1] \), if \( f_t \) is a nonempty set, then \( f_t \) is a left \( \alpha \)-ideal of \( S \).

(2) \( f \) is a fuzzy right \( \beta \)-ideal of \( S \) if and only if for all \( t \in (0, 1] \), if \( f_t \) is a nonempty set, then \( f_t \) is a right \( \beta \)-ideal of \( S \).

(3) \( f \) is a fuzzy \((a, \beta)\)-ideal of \( S \) if and only if for all \( t \in (0, 1] \), if \( f_t \) is a nonempty set, then \( f_t \) is an \((a, \beta)\)-ideal of \( S \).

(2) is similar to (1).

(3) follows by (1) and (2).

3.2. Fuzzy \((a, \beta)\)-Quasi-Ideals. We will define fuzzy \((a, \beta)\)-quasi-ideals of \( \Gamma \)-semigroups as follows.

**Definition 11.** Let \( a, \beta \in \Gamma \) and \( f \) be a fuzzy subset of a \( \Gamma \)-semigroup \( S \). Then, \( f \) is called a fuzzy \((a, \beta)\)-quasi-ideal of \( S \) if \( (S \circ_a f) \cap (f \circ_\beta S) \subseteq f \).

A fuzzy subset \( f \) of \( S \) is called a fuzzy \( a \)-quasi-ideal of \( S \) if \( f \) is a fuzzy \((a, \beta)\)-quasi-ideal of \( S \).

**Theorem 26.** Let \( S \) be a \( \Gamma \)-semigroup. Let \( f \) and \( g \) be a fuzzy left \( \alpha \)-ideal and a fuzzy right \( \beta \)-ideal of \( S \), respectively. Then, \( f \cap g \) is a fuzzy \( \alpha \)-quasi-ideal of \( S \).

**Proof.** Let \( f \) and \( g \) be a fuzzy left \( \alpha \)-ideal and a fuzzy right \( \beta \)-ideal of a \( \Gamma \)-semigroup \( S \), respectively. We have \( g \circ_\beta f \subseteq f \cap g \); this implies \( f \cap g \neq \emptyset \). Then, \( S \circ_a (f \cap g) \cap (f \cap g) \circ_\beta S \subseteq f \cap g \). Hence, \( f \cap g \) is a fuzzy \( \alpha \)-quasi-ideal of \( S \).
Theorem 27. Every fuzzy \((\alpha, \beta)\)-quasi-ideal of a \(\Gamma\)-semigroup \(S\) is the intersection of a fuzzy left \(\alpha\)-ideal and a fuzzy right \(\beta\)-ideal of \(S\).

Proof. Let \(f\) be a fuzzy \((\alpha, \beta)\)-quasi-ideal of a \(\Gamma\)-semigroup \(S\). Let \(g = f \cup (S \circ \alpha)\) and \(h = f \cup (f \circ \beta)\). Then, \(S \circ \alpha g = S \circ \alpha (f \cup (S \circ \alpha)) = (S \circ \alpha f) \cup (S \circ \alpha (S \circ \alpha f)) \subseteq S \circ \alpha f \subseteq g\), and also, \(h \circ \beta S \subseteq h\). Thus, \(g\) and \(h\) are a fuzzy left \(\alpha\)-ideal and a fuzzy right \(\beta\)-ideal of \(S\), respectively. We claim that \(f = g \cap h\). That is, \(f \subseteq (f \cup (S \circ \alpha)) \cap (f \cup (f \circ \beta)) \subseteq g \cap h\), and conversely, \(g \cap h = (f \cup (S \circ \alpha f)) \cap (f \cap (f \circ \beta)) \subseteq f \cup (S \circ \alpha f) \cap (f \circ \beta) \subseteq f\). Therefore, \(f = g \cap h\).

Theorem 28. Let \(Q\) be a nonempty subset of a \(\Gamma\)-semigroup \(S\). Then, \(Q\) is an \((\alpha, \beta)\)-quasi-ideal of \(S\) if and only if \(C_Q\) is a fuzzy \((\alpha, \beta)\)-quasi-ideal of \(S\).

Proof. Assume that \(Q\) is an \((\alpha, \beta)\)-quasi-ideal of a \(\Gamma\)-semigroup \(S\). If \(y \notin (S \circ \alpha Q) \cap (Q \circ \alpha)\), then \((S \circ \alpha Q) \cap (Q \circ \alpha) = 0 \subseteq C_Q(y)\). Let \(y \in (S \circ \alpha Q) \cap (Q \circ \alpha)\). Then, \(y \in Q\). So \(C_Q(y) = 1\). Hence, \((S \circ \alpha Q) \cap (Q \circ \alpha) \subseteq C_Q\). Therefore, \(C_Q\) is a fuzzy \((\alpha, \beta)\)-quasi-ideal of \(S\).

Conversely, assume that \(C_Q\) is a fuzzy \((\alpha, \beta)\)-quasi-ideal of \(S\). Then, \((S \circ \alpha C_Q) \cap (Q \circ \alpha) \subseteq C_Q\). Let \(x \in (S \circ \alpha Q) \cap (Q \circ \alpha)\). Then, \([(S \circ \alpha C_Q) \cap (Q \circ \alpha)](x) = 1\), and this implies that \(C_Q(x) = 1\). So, \(S \circ \alpha Q \cap (Q \circ \alpha) \subseteq C_Q\). Consequently, \(Q\) is an \((\alpha, \beta)\)-quasi-ideal of \(S\).

3.3 Fuzzy \((\alpha, \beta)\)-Bi-Ideals

Definition 12. Let \(\alpha, \beta \in \Gamma\) and \(f\) be a fuzzy subset of a \(\Gamma\)-semigroup \(S\). Then, \(f\) is called a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\) if \(f \circ \alpha S \circ \beta f \subseteq f\).

Theorem 29. Let \(S\) be a \(\Gamma\)-semigroup, \(g\) a fuzzy subset of \(S\), and \(f\) a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\). Then, the following statements are true:

(1) \(f \circ \alpha g\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).
(2) \(g \circ \beta f\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

Proof
(1) Let \(x_i\) be a fuzzy point of \(S\). Then,

\[
(f \circ \alpha g) \circ \alpha S \circ \beta (f \circ \alpha g) = f \circ \alpha (g \circ \alpha x_i) \circ \beta (f \circ \alpha g) \subseteq \left( f \circ \alpha x_i \circ \beta f \right) \circ \alpha g \subseteq f \circ \alpha g.
\]

Hence, \(f \circ \alpha g\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).
(2) is similar to (1).

Theorem 30. Let \(S\) be a \(\Gamma\)-semigroup. If \(f_1, f_2\) and \(f_3\) are fuzzy \((\alpha, \beta)\)-bi-ideals of \(S\), then \(f_1 \circ \alpha f_2 \circ \beta f_3\) is a fuzzy bi-(\(\alpha, \beta\))-ideal of \(S\).

Proof. Let \(x_i\) be a fuzzy point of \(S\). Then,

\[
(f_1 \circ \alpha f_2 \circ \beta f_3) \circ \alpha x_i \circ \beta (f_1 \circ \alpha f_2 \circ \beta f_3) \subseteq f_1 \circ \alpha f_2 \circ \beta f_3 \circ \beta f_3 \subseteq f_1 \circ \alpha f_2 \circ \beta f_3.
\]

Hence, \(f_1 \circ \alpha f_2 \circ \beta f_3\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

Theorem 31. Let \(S\) be a \(\Gamma\)-semigroup and \(B\) a nonempty subset of \(S\). Then, \(B\) is an \((\alpha, \beta)\)-bi-ideal of \(S\) if and only if \(C_B\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

Proof. Assume that \(B\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\). Then, \(B \circ \beta B \subseteq B\). If \(z \notin B \circ \beta B\), we have \(C_B \circ \alpha \beta \subseteq C_B\). Let \(z \in B \circ \beta B\). Then, \(B \circ \beta B \subseteq B\). Thus, \(B \circ \beta B \subseteq B\). Hence, \(B\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

Conversely, assume that \(B\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\) and \(C_B \circ \alpha \beta \subseteq C_B\). Let \(z \in B \circ \beta B\). Then, \(B \circ \beta B \subseteq B\). Thus, \(B \circ \beta B \subseteq B\). This implies that \(B \circ \beta B \subseteq B\). Thus, \(B\) is an \((\alpha, \beta)\)-bi-ideal of \(S\).

Theorem 32. Let \(f\) and \(g\) be two fuzzy \((\alpha, \beta)\)-bi-ideals of a \(\Gamma\)-semigroup \(S\). If \(f \cap g \neq \emptyset\), then \(f \cap g\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

Proof. Let \(f\) and \(g\) be fuzzy \((\alpha, \beta)\)-bi-ideals of a \(\Gamma\)-semigroup \(S\). Then, \((f \cap g) \circ \alpha S \circ \beta (f \cap g) \subseteq f \circ \alpha S \circ \beta f \cap g \circ \alpha S \circ \beta f \subseteq f \cap g\). Hence, \(f \cap g\) is a fuzzy \((\alpha, \beta)\)-bi-ideal of \(S\).

4. New Types of Almost Ideals

4.1. Almost \((\alpha, \beta)\)-Ideals

Definition 13. Let \(S\) be a \(\Gamma\)-semigroup and \(L, R\), and \(I\) be nonempty subsets of \(S\). Let \(\alpha, \beta \in \Gamma\). Then,

(1) \(L\) is called an almost left \(\alpha\)-ideal of \(S\) if \(S \circ \alpha L \cap L \neq \emptyset\).
(2) \(R\) is called an almost right \(\beta\)-ideal of \(S\) if \(R \circ \beta S \cap R \neq \emptyset\).
(3) \(I\) is called an almost \((\alpha, \beta)\)-ideal of \(S\) if \(I\) is both an almost left \(\alpha\)-ideal and an almost right \(\beta\)-ideal of \(S\).

Theorem 33. Let \(S\) be a \(\Gamma\)-semigroup. If \(L\) is a left \(\alpha\)-ideal of \(S\), then \(L\) is an almost left \(\alpha\)-ideal of \(S\).

Proof. Let \(L\) be a left \(\alpha\)-ideal of \(S\). Then, \(S \circ \alpha L \subseteq L\) so that \(S \circ \alpha L \cap L \neq \emptyset\). Thus, \(L\) is an almost left \(\alpha\)-ideal of \(S\).

Theorem 34. Let \(S\) be a \(\Gamma\)-semigroup. If \(R\) is a right \(\beta\)-ideal of \(S\), then \(R\) is an almost right \(\beta\)-ideal of \(S\).

Proof. It is similar to Theorem 33.

Theorem 35. Let \(S\) be a \(\Gamma\)-semigroup. If \(I\) is an \((\alpha, \beta)\)-ideal of \(S\), then \(I\) is an almost \((\alpha, \beta)\)-ideal of \(S\).

Proof. It follows by Theorems 33 and 34.
Theorem 36. Let S be a Γ-semigroup. If L is an almost left α-ideal of S and L ⊆ H ⊆ S, then H is an almost left α-ideal of S.

Proof. Let L be an almost left α-ideal of S and L ⊆ H ⊆ S. Since S \alpha L \cap L \neq \emptyset, and S \alpha L \cap L \subseteq S \alpha H \cap H, we have S \alpha H \cap H \neq \emptyset. Therefore, H is an almost left α-ideal of S.

Theorem 37. Let S be a Γ-semigroup. If R is an almost right β-ideal of S and R ⊆ H ⊆ S, then H is an almost right β-ideal of S.

Proof. It is similar to Theorem 36.

Theorem 38. Let S be a Γ-semigroup. If I is an almost (α, β)-ideal of S and I ⊆ H ⊆ S, then H is an almost (α, β)-ideal of S.

Proof. It follows by Theorems 36 and 37.

Corollary 2. Let S be a Γ-semigroup. If L1 and L2 are almost left α-ideals of S, then L1 ∪ L2 is an almost left α-ideal of S.

Corollary 3. Let S be a Γ-semigroup. If R1 and R2 are almost right β-ideals of S, then R1 ∪ R2 is an almost right β-ideal of S.

Corollary 4. Let S be a Γ-semigroup. If I1 and I2 are almost (α, β)-ideals of S, then I1 ∪ I2 is an almost (α, β)-ideal of S.

Example 2. Consider a Γ-semigroup S = {a, b, c, d, e} with Γ = {α, β} and

|   | a   | b   | c   | d   | e   |
|---|-----|-----|-----|-----|-----|
| a | a   | b   | c   | d   | e   |
| b | b   | c   | d   | e   | b   |
| c | c   | d   | e   | b   | a   |
| d | d   | e   | b   | a   | c   |
| e | e   | a   | c   | d   | b   |

We have {a, b, d} and {a, c, d} are almost left α-ideals of S. However, {a, b, d} ∩ {a, c, d} = {a, d} is not an almost left α-ideal of S.

Remark 3. The intersection of two almost left α-ideals of a Γ-semigroup S need not be an almost left α-ideal of S.

Remark 4. The intersection of two almost right β-ideals of a Γ-semigroup S need not be an almost right β-ideal of S.

Remark 5. The intersection of two almost (α, β)-ideals of a Γ-semigroup S need not be an almost (α, β)-ideal of S.

Definition 14. A Γ-semigroup S is called

1. Almost left α-simple if S does not contain proper almost left α-ideals.
2. Almost right β-simple if S does not contain proper almost right β-ideals.
3. Almost (α, β)-simple if S does not contain proper almost (α, β)-ideals.

Theorem 39. Let S be a Γ-semigroup. Then, S is almost left α-simple if and only if for each a ∈ S there exists s ∈ S such that sa(S\−\{a\}) = \{a\}.

Proof. Assume that S is almost left α-simple. Then, S has no proper almost left α-ideals. Let a ∈ S. Then, S\−\{a\} is not an almost left α-ideal of S. Thus, there exists s ∈ S such that sa(S\−\{a\}) ∩ (S\−\{a\}) = \emptyset; this implies that sa(S\−\{a\}) = \{a\}.

Conversely, for each a ∈ S, there exists s ∈ S such that sa(S\−\{a\}) = \{a\}. Assume that L is a proper almost left α-ideal of S, and let a ∈ S\−L. Since L ⊆ S\−\{a\}, we have S\−\{a\} is an almost left α-ideal of S by Theorem 36. Thus, sa(S\−\{a\}) ∩ (S\−\{a\}) ≠ \emptyset; we get a contradiction. Hence, S has no proper left almost α-ideals. Therefore, S is almost left α-simple.

Theorem 40. Let S be a Γ-semigroup. Then, S is almost right β-simple if and only if for each a ∈ S, there exists s ∈ S such that (S\−\{a\})\beta s = \{a\}.

Proof. It is similar to Theorem 39.

4.2. Almost (α, β)-Quasi-Ideals

Definition 15. A nonempty subset Q of a Γ-semigroup S is called an almost (α, β)-quasi-ideal of S if saQ ∩ Qβs ∩ Q ≠ \emptyset for all s ∈ S.

Proposition 1. Every (α, β)-quasi-ideal of a Γ-semigroup S is either saQ ∩ Qβs ∩ Q = \emptyset for some s ∈ S or an almost (α, β)-quasi-ideal of S.

Proof. Assume that Q is an (α, β)-quasi-ideal of a Γ-semigroup S. Assume that saQ ∩ Qβs ∩ Q ≠ \emptyset for all s ∈ S. Let s ∈ S. Then, saQ ∩ Qβs ⊆ SαQ ∩ Qβs ⊆ Q. That is, (saQ ∩ Qβs) ∩ Q ≠ \emptyset. Hence, Q is an almost (α, β)-quasi-ideal of S.

Theorem 41. Every almost (α, β)-quasi-ideal of a Γ-semigroup S is an almost left α-ideal of S.

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S. Let s ∈ S. Then, \emptyset ≠ (saQ ∩ Qβs) ∩ Q \subseteq saQ ∩ Q. Hence, Q is an almost left α-ideal of S.

Similarly, every almost (α, β)-quasi-ideal of a Γ-semigroup S is an almost right β-ideal of S, but the converse is not true.

Theorem 42. If Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S and Q⊆ H⊆ S, then H is an almost (α, β)-quasi-ideal of S.
Proof. Assume that $Q$ is an almost $(a, \beta)$-quasi-ideal of a $\Gamma$-semigroup $S$ and $Q \subseteq H \subseteq S$. Let $s \in S$. Then, $\emptyset \neq (saQ \cap Q\beta s) \cap Q_e (saH \cap H\beta s) \cap H$. Therefore, $H$ is an almost $(a, \beta)$-quasi-ideal of $S$.

**Corollary 5.** The union of two almost $(a, \beta)$-quasi-ideals of a $\Gamma$-semigroup $S$ is an almost $(a, \beta)$-quasi-ideal of $S$.

**Example 3.** Consider a $\Gamma$-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{a, \beta\}$ and

$$
\begin{array}{cccccc}
a & a & a & a & a & a \\
b & b & c & d & e & e \\
c & c & d & e & a & a \\
d & d & a & b & c & e \\
e & e & a & b & c & d \\
\end{array}
$$

We have $\{b, c, e\}$ and $\{b, d, e\}$ are almost $(a, \beta)$-quasi-ideals of $S$.

**Remark 6.** The intersection of two almost $(a, \beta)$-quasi-ideals of a $\Gamma$-semigroup $S$ need not be an almost $(a, \beta)$-quasi-ideal of $S$.

**Definition 16.** A $\Gamma$-semigroup $S$ is called almost $(a, \beta)$-quasi-simple if $S$ does not contain proper almost $(a, \beta)$-quasi-ideals.

**Theorem 43.** A $\Gamma$-semigroup $S$ is almost $(a, \beta)$-quasi-simple if and only if for any $a \in S$, there exists $s_a$ such that $s_a a (S \{a\}) \cap (S \{a\}) \beta s_a \subseteq \{a\}$.

**Proof.** Assume that a $\Gamma$-semigroup $S$ is almost $(a, \beta)$-quasi-simple and $S \{a\}$ is not an almost $(a, \beta)$-quasi-ideal of $S$. Then, there exists $s_a \in S$ such that $s_a a (S \{a\}) \cap (S \{a\}) \beta s_a \nsubseteq \{a\}$. Therefore, $s_a a (S \{a\}) \cap (S \{a\}) \beta s_a \subseteq \{a\}$.

Conversely, assume that, for any $a \in S$, there exists $s_a$ such that $s_a \beta (S \{a\}) \nsubseteq \{a\}$. Then, $s_a \beta (S \{a\}) \nsubseteq \{a\}$. Hence, $S \{a\}$ is not an almost $(a, \beta)$-quasi-ideal of $S$. Let $A$ be a proper almost $(a, \beta)$-quasi-ideal of $S$. Then, $A \subseteq S \{a\} \nsubseteq \{a\}$ for some $a \in S$; this is a contradiction. Therefore, $S$ has no proper almost $(a, \beta)$-quasi-ideals. Hence, $S$ is almost $(a, \beta)$-quasi-simple.

**4.3. Almost $(a, \beta)$-Bi-Ideals**

**Definition 17.** A nonempty subset $B$ of a $\Gamma$-semigroup $S$ is called an almost $(a, \beta)$-bi-ideal of $S$ if $BasB \cap B \nsubseteq \emptyset$ for all $s \in S$.

**Theorem 44.** If $B$ is an almost $(a, \beta)$-bi-ideal of a $\Gamma$-semigroup $S$ and $B \subseteq C \subseteq S$, then $C$ is an almost $(a, \beta)$-bi-ideal of $S$.

Proof. Let $B$ be an almost $(a, \beta)$-bi-ideal of a $\Gamma$-semigroup $S$ and $B \subseteq C \subseteq S$. Since $BasB \cap B \nsubseteq \emptyset$ for all $s \in S$ and $BasB \cap B \subseteq Bas\beta C \cap B \subseteq Cas\beta C \cap B$, we have $Cas\beta C \nsubseteq \emptyset$. Therefore, $C$ is an almost $(a, \beta)$-bi-ideal of $S$.

**Corollary 6.** The union of two almost $(a, \beta)$-bi-ideals of a $\Gamma$-semigroup $S$ is also an almost $(a, \beta)$-bi-ideal of $S$.

**Example 4.** Consider a $\Gamma$-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{a, \beta\}$ and

$$
\begin{array}{cccccc}
a & a & a & a & a & a \\
b & b & c & d & e & e \\
c & c & d & e & a & a \\
d & d & a & b & c & e \\
e & e & a & b & c & d \\
\end{array}
$$

We have $\{b, c, e\}$ and $\{b, d, e\}$ are almost $(a, \beta)$-bi-ideals of $S$.

**Remark 7.** The intersection of two almost $(a, \beta)$-bi-ideals of a $\Gamma$-semigroup $S$ need not be an almost $(a, \beta)$-bi-ideal of $S$.

**Theorem 45.** A $\Gamma$-semigroup $S$ has a proper almost $(a, \beta)$-bi-ideal if and only if there exists an element $a \in S$ such that $(S \{a\})aS (S \{a\}) \cap (S \{a\}) \nsubseteq \emptyset$ for all $a \in S$.

**Proof.** Assume that a $\Gamma$-semigroup $S$ contains a proper almost $(a, \beta)$-bi-ideal $B$ and $a \notin B$. Then, $B \subseteq S \{a\} \subseteq S$. By Theorem 44, $S \{a\}$ is a proper almost $(a, \beta)$-bi-ideal of $S$, that is, $(S \{a\})aS (S \{a\}) \cap (S \{a\}) \nsubseteq \emptyset$ for all $a \in S$.

Conversely, let $a \in S$ such that $(S \{a\})aS (S \{a\}) \cap (S \{a\}) \nsubseteq \emptyset$ for all $a \in S$. Since $S \{a\} \subseteq S$, we have $S \{a\}$ is a proper almost $(a, \beta)$-bi-ideal of $S$.

**Definition 18.** A $\Gamma$-semigroup $S$ is called almost $(a, \beta)$-bi-simple if $S$ does not contain proper almost $(a, \beta)$-bi-ideals.

**Theorem 46.** A $\Gamma$-semigroup $S$ is almost $(a, \beta)$-bi-simple if and only if for all $a \in S$, there exists $s \in S$ such that $(S \{a\})aS (S \{a\}) \cap (S \{a\}) \nsubseteq \emptyset$.

**Proof.** Assume that a $\Gamma$-semigroup $S$ is almost $(a, \beta)$-bi-simple. Then, $S$ has no proper almost $(a, \beta)$-bi-ideals, and let $a \in S$. Then, $S \{a\}$ is not an almost $(a, \beta)$-bi-ideal of $S$. Thus, there exists $s \in S$ such that $(S \{a\})aS (S \{a\}) \cap (S \{a\}) \nsubseteq \emptyset$. This implies that $(S \{a\})aS (S \{a\}) = \{a\}$.

Conversely, suppose that $S$ has a proper almost $(a, \beta)$-bi-ideal $B$, that is, $B \subseteq S$. Since $B \subseteq S \{a\} \subseteq S$, we have $S \{a\}$ is an almost $(a, \beta)$-bi-ideal of $S$ by Theorem 44; this is a contradiction. Hence, $S$ has no proper almost $(a, \beta)$-bi-ideals. Therefore, $S$ is almost $(a, \beta)$-bi-simple.
5. New Types of Fuzzy Almost Ideals

5.1. Fuzzy Almost \((a,\beta)\)-Ideals

Definition 19. Let \(a, \beta \in \Gamma\). Let \(f\) be a fuzzy subset of a \(\Gamma\)-semigroup \(S\). Then, \(f\) is called

(1) A fuzzy almost left \(a\)-ideal of \(S\) if \((x_{i} \circ f) \cap f \neq \emptyset\) for all fuzzy point \(x_{i}\) of \(S\).

(2) A fuzzy almost right \(\beta\)-ideal of \(S\) if \((f \circ x_{i}) \cap f \neq \emptyset\) for all fuzzy point \(x_{i}\) of \(S\).

(3) A fuzzy almost \((a,\beta)\)-ideal of \(S\) if it is both a fuzzy almost left \(a\)-ideal and a fuzzy almost right \(\beta\)-ideal of \(S\).

Theorem 47. Let \(f\) be a fuzzy almost left \(a\)-ideal of a \(\Gamma\)-semigroup \(S\) and \(g\) be a fuzzy subset of \(S\) such that \(f \subseteq g\). Then, \(g\) is a fuzzy almost left \(a\)-ideal of \(S\).

Proof. Assume that \(f\) is a fuzzy almost left \(a\)-ideal of a \(\Gamma\)-semigroup \(S\) and \(g\) is a fuzzy subset of \(S\) such that \(f \subseteq g\). Then, for each fuzzy point \(x_{i}\), \((x_{i} \circ f) \cap f \neq \emptyset\). We have \((x_{i} \circ f) \cap f \subseteq (x_{i} \circ g) \cap g\); this implies \((x_{i} \circ g) \cap g \neq \emptyset\). Therefore, \(g\) is a fuzzy almost left \(a\)-ideal of \(S\).

Theorem 48. Let \(f\) be a fuzzy almost right \(\beta\)-ideal of a \(\Gamma\)-semigroup \(S\) and \(g\) be a fuzzy subset of \(S\) such that \(f \subseteq g\). Then, \(g\) is a fuzzy almost right \(\beta\)-ideal of \(S\).

Proof. It is similar to Theorem 47.

Theorem 49. Let \(f\) be a fuzzy almost \((a,\beta)\)-ideal of a \(\Gamma\)-semigroup \(S\) and \(g\) be a fuzzy subset of \(S\) such that \(f \subseteq g\). Then, \(g\) is a fuzzy almost \((a,\beta)\)-ideal of \(S\).

Proof. It follows by Theorem 47 and Theorem 48.

Corollary 7. The union of two fuzzy almost left \(a\)-ideals of a \(\Gamma\)-semigroup \(S\) is a fuzzy almost left \(a\)-ideal of \(S\).

Corollary 8. The union of two fuzzy almost right \(\beta\)-ideals of a \(\Gamma\)-semigroup \(S\) is a fuzzy almost right \(\beta\)-ideal of \(S\).

Corollary 9. The union of two fuzzy almost \((a,\beta)\)-ideals of a \(\Gamma\)-semigroup \(S\) is a fuzzy almost \((a,\beta)\)-ideal of \(S\).

Theorem 50. Let \(A\) be a nonempty subset of a \(\Gamma\)-semigroup \(S\).

(1) \(A\) is an almost left \(a\)-ideal of \(S\) if and only if \(C_{A}\) is a fuzzy almost left \(a\)-ideal of \(S\).

(2) \(A\) is an almost right \(\beta\)-ideal of \(S\) if and only if \(C_{A}\) is a fuzzy almost right \(\beta\)-ideal of \(S\).

(3) \(A\) is an almost \((a,\beta)\)-ideal of \(S\) if and only if \(C_{A}\) is a fuzzy almost \((a,\beta)\)-ideal of \(S\).

Proof

(1) Assume that \(A\) is an almost left \(a\)-ideal of a \(\Gamma\)-semigroup \(S\). Then, \(\forall x \in S\), \(\exists y \in S\) such that \(\forall x \in A\), \(\exists y \in A\) and \(\forall x \in A\). So, \((x_{i} \circ C_{A}) (y) = 1\) and \((C_{A} (y) = 1\). Hence, \((x_{i} \circ C_{A}) (C_{A}) \neq 0\). Therefore, \(C_{A}\) is a fuzzy almost left \(a\)-ideal of \(S\).

Conversely, assume that \(C_{A}\) is a fuzzy almost left \(a\)-ideal of \(S\). Let \(x \in S\). Then, \((x_{i} \circ C_{A}) (C_{A}) \neq 0\). Then, there exists \(a \in S\) such that \([x_{i} \circ C_{A}] (a) \neq 0\). Hence, \(a \in xA\). So, \(xA \cap A \neq \emptyset\). Consequently, \(A\) is an almost left \(a\)-ideal of \(S\).

The proofs of (2) and (3) are similar to the proof of (1).

Theorem 51. Let \(f\) be a fuzzy subset of a \(\alpha\)-semigroup \(S\).

(1) \(f\) is a fuzzy almost left \(a\)-ideal of \(S\) if and only if \(\text{supp}(f)\) is an almost left \(a\)-ideal of \(S\).

(2) \(f\) is a fuzzy almost right \(\beta\)-ideal of \(S\) if and only if \(\text{supp}(f)\) is an almost right \(\beta\)-ideal of \(S\).

(3) \(f\) is a fuzzy almost \((a,\beta)\)-ideal of \(S\) if and only if \(\text{supp}(f)\) is an almost \((a,\beta)\)-ideal of \(S\).

Proof

(1) Assume that \(f\) is a fuzzy almost left \(a\)-ideal of a \(\Gamma\)-semigroup \(S\). Let \(x \in S\). Then, \((x_{i} \circ f) \cap f \neq \emptyset\). Hence, there exists \(a \in S\) such that \((x_{i} \circ f) \cap f \neq \emptyset\). So, there exists \(a \in S\) such that \((x_{i} \circ f) \cap f \neq \emptyset\). That is, \(a \in \text{supp}(f)\). Thus, \((x_{i} \circ C_{\text{supp}(f)} (a)) \neq 0\) and \((C_{\text{supp}(f)} (a)) \neq 0\). Therefore, \((C_{\text{supp}(f)} (a)) \neq 0\). Hence, \(C_{\text{supp}(f)} (a) \neq 0\). This means \((x_{i} \circ f) \cap f \neq \emptyset\). Therefore, \(f\) is a fuzzy almost left \(a\)-ideal of \(S\).

Conversely, assume that \(\text{supp}(f)\) is an almost left \(a\)-ideal of \(S\). By Theorem 50, \(\text{supp}(f)\) is an almost left \(a\)-ideal of \(S\).

The proofs of (2) and (3) are similar to the proof of (1).

Definition 20. A fuzzy almost left \(a\)-ideal \(f\) of a \(\Gamma\)-semigroup \(S\) is minimal if for all fuzzy almost left \(a\)-ideal \(g\) of \(S\) such that \(g \subseteq f\), we obtain \(\text{supp}(g) = \text{supp}(f)\).

Theorem 52. Let \(A\) be a nonempty subset of a \(\Gamma\)-semigroup \(S\).

(1) \(A\) is a minimal almost left \(a\)-ideal of \(S\) if and only if \(C_{A}\) is a minimal fuzzy almost left \(a\)-ideal of \(S\).

(2) \(A\) is a minimal almost right \(\beta\)-ideal of \(S\) if and only if \(C_{A}\) is a minimal fuzzy almost right \(\beta\)-ideal of \(S\).

(3) \(A\) is a minimal almost \((a,\beta)\)-ideal of \(S\) if and only if \(C_{A}\) is a minimal fuzzy almost \((a,\beta)\)-ideal of \(S\).

Proof

(1) Assume that \(A\) is a minimal almost left \(a\)-ideal of a \(\Gamma\)-semigroup \(S\). By Theorem 50 (1), \(C_{A}\) is a fuzzy almost left \(a\)-ideal of \(S\). Let \(g\) be a fuzzy almost left \(a\)-ideal of \(S\) such that
Corollary 10. Let A be a sub-Γ-semigroup of a Γ-semigroup S. Then,

1. A is almost left α-simple if and only if for each fuzzy almost left α-ideal f of S, supp(f) = A.
2. A is almost right β-simple if and only if for each fuzzy almost right β-ideal f of S, supp(f) = A.
3. A is almost (α, β)-simple if and only if for each fuzzy almost (α, β)-ideal of S, supp(f) = A.

5.2. Fuzzy Almost (α, β)-Quasi-Ideals

Definition 21. Let α, β ∈ Γ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called a fuzzy almost (α, β)-quasi-ideal of S if \((f ∗ αx_i) \cap (x_i ∗ βf))\( ≠ 0\) for all fuzzy points \(x_i\) of S.

Theorem 53. Let f be a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S and g a fuzzy subset of S such that f ⊆ g. Then, g is a fuzzy almost (α, β)-quasi-ideal of S.

Proof. Assume that f is a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S and g is a fuzzy subset of S such that f ⊆ g. Then, for all fuzzy point \(x_i\) of S, \(\((f ∗ αx_i) \cap (x_i ∗ βf))\( ≠ 0\)\). We have that \(\((f ∗ αx_i) \cap (x_i ∗ βf))\( ∩ f ≠ 0\)\) and \(\((f ∗ αx_i) \cap (x_i ∗ βf))\( ∩ g ≠ 0\)\). This implies that \(\((f ∗ αx_i) \cap (x_i ∗ βf))\( ∩ g ≠ 0\)\). Therefore, g is a fuzzy almost (α, β)-quasi-ideal of S.

Corollary 11. Let f and g be fuzzy almost (α, β)-quasi-ideals of a Γ-semigroup S. Then, f ∪ g is a fuzzy almost (α, β)-quasi-ideal of S.

Proof. Since f ⊆ f ∪ g, by Theorem 53, f ∪ g is a fuzzy almost (α, β)-quasi-ideal of S.

Example 5. Consider the Γ-semigroup \(Z_5\) where \(Γ = \{0, 1, 2, 3, 4\}\) and \(πYB = π + γ + B\), where \(π, B ∈ Z_5\) and \(γ ∈ Γ\). Let f: \(Z_5 \rightarrow [0, 1]\) defined by

\[
\begin{align*}
f(0) &= 0, \\
f(1) &= 0.6, \\
f(2) &= 0, \\
f(3) &= 0.4, \\
f(4) &= 0.4,
\end{align*}
\]

and g: \(Z_5 \rightarrow [0, 1]\) defined by

\[
\begin{align*}
g(0) &= 0, \\
g(1) &= 0.3, \\
g(2) &= 0.6, \\
g(3) &= 0, \\
g(4) &= 0.8.
\end{align*}
\]

We have f and g are fuzzy almost \((0, 0)\)-quasi-ideals of \(Z_5\), but f ∩ g is not a fuzzy almost \((0, 0)\)-quasi-ideal of \(Z_5\).

Theorem 54. Let Q be a nonempty subset of a Γ-semigroup S. Then, Q is an almost (α, β)-quasi-ideal of S if and only if CQ is a fuzzy almost (α, β)-quasi-ideal of S.

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S, and let \(x_1\) be a fuzzy point of S. Then, \(\((Qαx) \cap (xβQ)\) ∩ Q ≠ 0\). Thus, there exists \(y ∈ (Qαx) \cap (xβQ)\) and \(y ∈ Q\). So, \(\((CQ ∘ αx_1) ∩ (x_1 ∘ βCQ)\) ∩ Q ≠ 0\) and \(CQ(γ) = 1\). Hence, \(\((CQ ∘ αx_1) ∩ (x_1 ∘ βCQ)\) ∩ CQ ≠ 0\). Therefore, CQ is a fuzzy almost (α, β)-quasi-ideal of S.

Conversely, assume that CQ is a fuzzy almost (α, β)-quasi-ideal of S. Let \(s ∈ S\). Then, \(\((CQ ∘ αs_1) \cap (s_1 ∘ βCQ)\) ∩ CQ ≠ 0\). Hence, there exists \(x ∈ S\) such that \(\((CQ ∘ αs_1) \cap (s_1 ∘ βCQ)\) ∩ CQ ≠ 0\). Consequently, Q is an almost (α, β)-quasi-ideal of S.

Theorem 55. Let f be a fuzzy subset of a Γ-semigroup S. Then, f is a fuzzy almost (α, β)-quasi-ideal of S if and only if supp(f) is an almost (α, β)-quasi-ideal of S.

Proof. Assume that f is a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S. Let \(s ∈ S\) and \(t ∈ (0, 1)\). Then, \(\((f ∗ αs_1) \cap (s_1 ∘ βf)\) ∩ f ≠ 0\). Hence, there exists \(x ∈ S\) such that

\[
\left(\left(f ∗ αs_1\right) \cap (s_1 ∘ βf)\right) ∩ f ≠ 0.
\]

So, there exist \(y_1, y_2 ∈ S\) such that \(x = y_1 sαy_2, f(x) ≠ 0\), \(f(y_1) ≠ 0\), and \(f(y_2) ≠ 0\). That is, \(x, y_1, y_2 ∈ supp(f)\). Thus, \(\((Csupp(f) ∘ αs_1) \cap (s_1 ∘ βCsupp(f))\) ∩ f ≠ 0\) and \(Csupp(f)(x) ≠ 0\). Therefore, \(\((Csupp(f) ∘ αs_1) \cap (s_1 ∘ βCsupp(f))\) ∩ f ≠ 0\). Hence, \(Csupp(f)\) is fuzzy almost (α, β)-quasi-ideal of S. By Theorem 54, supp(f) is an almost (α, β)-quasi-ideal of S.

Conversely, assume that supp(f) is an almost (α, β)-quasi-ideal of S. By Theorem 54, Csupp(f) is a fuzzy almost (α, β)-quasi-ideal of S. Then, for each fuzzy point \(s_1\) of S, we have \(\((Csupp(f) ∘ αs_1) \cap (s_1 ∘ βCsupp(f))\) ∩ Csupp(f) ≠ 0\). Then, there exists \(x ∈ S\) such that

\[
\left(Csupp(f) ∘ αs_1\right) \cap (s_1 ∘ βCsupp(f)) ∩ Csupp(f) ≠ 0.
\]

Hence, \(\((Csupp(f) ∘ αs_1) \cap (s_1 ∘ βCsupp(f))\) ∩ f ≠ 0\) and \(Csupp(f)(x) ≠ 0\). Then, there exist \(y_1, y_2 ∈ S\) such that \(x = y_1 sαy_2, f(x) ≠ 0\), \(f(y_1) ≠ 0\), and \(f(y_2) ≠ 0\). This means that \(\((f ∗ αs_1) \cap (s_1 ∘ βf)\) ∩ f ≠ 0\). Therefore, f is a fuzzy almost (α, β)-quasi-ideal of S.

Next, we define minimal fuzzy almost (α, β)-quasi-ideals in Γ-semigroups and give some relationship between
minimal almost \((α, β)\)-quasi-ideals and minimal fuzzy almost \((α, β)\)-quasi-ideals of \(Γ\)-semigroups.

**Definition 22.** A fuzzy almost \((α, β)\)-quasi-ideal \(f\) of a \(Γ\)-semigroup is called minimal if for each fuzzy almost \((α, β)\)-quasi-ideal \(g\) of \(S\) such that \(g ⊆ f\), we have \(\text{supp}(g) = \text{supp}(f)\).

**Theorem 56.** Let \(Q\) be a nonempty subset of a \(Γ\)-semigroup \(S\). Then, \(Q\) is a minimal almost \((α, β)\)-quasi-ideal of \(S\) if and only if \(C_Q\) is a minimal fuzzy almost \((α, β)\)-quasi-ideal of \(S\).

**Proof.** Assume that \(Q\) is a minimal almost \((α, β)\)-quasi-ideal of a \(Γ\)-semigroup \(S\). By Theorem 54, \(C_Q\) is a fuzzy almost \((α, β)\)-quasi-ideal of \(S\). Let \(g\) be a fuzzy almost \((α, β)\)-quasi-ideal of \(S\) such that \(g ⊆ C_Q\). Then, \(\text{supp}(g) ⊆ \text{supp}(C_Q)\). Since \(g ⊆ C_Q\), we have \(\text{supp}(g) = \text{supp}(C_Q)\) is a fuzzy almost \((α, β)\)-quasi-ideal of \(S\). By Theorem 54, \(\text{supp}(g)\) is an almost \((α, β)\)-quasi-ideal of \(S\). Since \(Q\) is minimal, \(\text{supp}(g) = Q = \text{supp}(C_Q)\). Therefore, \(C_Q\) is minimal.

Conversely, assume that \(C_Q\) is a minimal fuzzy almost \((α, β)\)-quasi-ideal of \(S\). Let \(Q'\) be an almost \((α, β)\)-quasi-ideal of \(S\) such that \(Q' ⊆ C_Q\). By Theorem 54, \(C_Q\) is a fuzzy almost \((α, β)\)-quasi-ideal of \(S\) such that \(C_Q' ⊆ C_Q\). Since \(C_Q\) is minimal, \(Q' = \text{supp}(C_Q') = \text{supp}(C_Q) = Q\). Therefore, \(Q\) is minimal.

**Corollary 12.** Let \(Q\) be a sub \(Γ\)-semigroup of a \(Γ\)-semigroup \(S\). Then, \(Q\) is almost \((α, β)\)-quasi-simple if and only if for all fuzzy almost \((α, β)\)-quasi-ideal \(f\) of \(S\), \(\text{supp}(f) = Q\).

### 5.3. Fuzzy Almost \((α, β)\)-Bi-Ideals

**Definition 23.** Let \(α, β ∈ Γ\) and \(f\) be a fuzzy subset of a \(Γ\)-semigroup \(S\). Then, \(f\) is called a fuzzy almost \((α, β)\)-bi-ideal of \(S\) if \((f ◦ αx_1 ◦ βf) ∩ f ≠ ∅\) for all fuzzy point \(x_1\) of \(S\).

**Theorem 57.** If \(f\) is a fuzzy almost \((α, β)\)-bi-ideal of a \(Γ\)-semigroup \(S\) and \(g\) is a fuzzy subset of \(S\) such that \(f ⊆ g\). Then, \(g\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

**Proof.** Assume that \(f\) is a fuzzy almost \((α, β)\)-bi-ideal of a \(Γ\)-semigroup \(S\) and \(g\) is a fuzzy subset of \(S\) such that \(f ⊆ g\). Then, for each fuzzy point \(x_1\), \((f ◦ αx_1 ◦ βf) ∩ f ≠ ∅\). We have \((f ◦ αx_1 ◦ βf) ∩ f ⊆ (g ◦ αx_1 ◦ βg) ∩ g\); this implies \((g ◦ αx_1 ◦ βg) ∩ g ≠ ∅\). Therefore, \(g\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

**Corollary 13.** Let \(f\) and \(g\) be fuzzy almost \((α, β)\)-bi-ideals of a \(Γ\)-semigroup \(S\). Then, \(f \cup g\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

**Example 6.** Consider the \(Γ\)-semigroup \(Z_5\) where \(Γ = \{0, 1\}\) and \(\overline{α}B = α + γ + β\), where \(α, β, γ ∈ Z_5\) and \(γ ∈ Γ\). Let \(f: Z_5 → [0, 1]\) defined by

\[
\begin{align*}
f(0) &= 0, \\
f(1) &= 0.7, \\
f(2) &= 0, \\
f(3) &= 0.6, \\
f(4) &= 0.4, \\
\end{align*}
\]

and \(g: Z_5 → [0, 1]\) defined by

\[
\begin{align*}
g(0) &= 0, \\
g(1) &= 0.1, \\
g(2) &= 0.6, \\
g(3) &= 0, \\
g(4) &= 0.8. \\
\end{align*}
\]

We have \(f\) and \(g\) are fuzzy almost \((0, 1)\)-bi-ideals of \(Z_5\).

**Remark 8.** The intersection of two fuzzy almost \((α, β)\)-bi-ideals of a \(Γ\)-semigroup \(S\) need not be a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

**Theorem 58.** Let \(B\) be a nonempty subset of a \(Γ\)-semigroup \(S\). Then, \(B\) is an almost \((α, β)\)-bi-ideal of \(S\) if and only if \(C_B\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

**Proof.** Assume that \(B\) is an almost \((α, β)\)-bi-ideal of a \(Γ\)-semigroup \(S\). Then, \(BαβB ∩ B ≠ ∅\) for all \(x ∈ S\). Thus, there exists \(y ∈ BαβB\) and \(y ∈ B\). So, \((C_B ◦ αx ◦ βC_B)(y) = 1\) and \(C_B(y) = 1\). Hence, \((C_B ◦ αx ◦ βC_B) ∩ C_B ≠ ∅\). Therefore, \(C_B\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\).

Conversely, assume that \(C_B\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\). Let \(s ∈ S\). Then, \((C_B ◦ αx ◦ βC_B) ∩ C_B ≠ ∅\). Thus, there exists \(x ∈ S\) such that \((C_B ◦ αx ◦ βC_B)(x) ≠ ∅\). Hence, \(x ∈ BαβB ∩ B ≠ ∅\). Consequently, \(B\) is an almost \((α, β)\)-bi-ideal of \(S\).

**Theorem 59.** Let \(f\) be a fuzzy subset of a \(Γ\)-semigroup \(S\). Then, \(f\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\) if and only if \(\text{supp}(f)\) is an almost \((α, β)\)-bi-ideal of \(S\).

**Proof.** Assume that \(f\) is a fuzzy almost \((α, β)\)-bi-ideal of a \(Γ\)-semigroup \(S\). Let \(s ∈ S\), then \((f ◦ αx ◦ βf)(x) ≠ ∅\). Hence, there exists \(x ∈ S\) such that \((f ◦ αx ◦ βf)(x) ≠ ∅\). So, there exist \(y_1, y_2 ∈ S\) such that \(x = y_1αβy_2\) with \(f(y_1) ≠ ∅\) and \(f(y_2) ≠ ∅\). That is, \(x, y_1, y_2 ∈ \text{supp}(f)\). Thus, \((C_{\text{supp}(f)} ◦ αx ◦ βC_{\text{supp}(f)})(x) ≠ ∅\) and \(C_{\text{supp}(f)}(x) ≠ ∅\). Therefore, \((C_{\text{supp}(f)} ◦ αx ◦ βC_{\text{supp}(f)})(x) ≠ ∅\). Hence, \(C_{\text{supp}(f)}\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\). By Theorem 58, \(\text{supp}(f)\) is an almost \((α, β)\)-bi-ideal of \(S\).

Conversely, assume that \(\text{supp}(f)\) is an almost \((α, β)\)-bi-ideal of \(S\). By Theorem 58, \(C_{\text{supp}(f)}\) is a fuzzy almost \((α, β)\)-bi-ideal of \(S\). Then, \((C_{\text{supp}(f)} ◦ αx ◦ βC_{\text{supp}(f)})(x) ≠ ∅\) for all \(s ∈ S\). Then, there exists \(x ∈ S\) such that \(x ∈ S\).
Let \( \alpha \) there exist \( \beta \) such that Theorem 60. Therefore, \( f \) is a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

We define minimal fuzzy almost \( (\alpha, \beta) \)-bi-ideals in \( \Gamma \)-semigroups and give some relationship between minimal almost \( (\alpha, \beta) \)-bi-ideals and minimal fuzzy almost \( (\alpha, \beta) \)-bi-ideals of \( \Gamma \)-semigroups.

Definition 24. A fuzzy almost \( (\alpha, \beta) \)-bi-ideal \( f \) of a \( \Gamma \)-semigroup \( S \) is called minimal if for all fuzzy almost \( (\alpha, \beta) \)-bi-ideal \( g \) of \( S \) such that \( g \subseteq f \), we have \( \supp(g) = \supp(f) \).

Theorem 60. Let \( B \) be a nonempty subset of a \( \Gamma \)-semigroup \( S \). Then, \( B \) is a minimal almost \( (\alpha, \beta) \)-bi-ideal of \( S \) if and only if \( C_B \) is a minimal fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

Proof. Assume that \( B \) is a minimal almost \( (\alpha, \beta) \)-bi-ideal of a \( \Gamma \)-semigroup \( S \). By Theorem 58, \( C_B \) is a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( g \) be a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \) such that \( g \subseteq C_B \). Then, \( \supp(g) \subseteq \supp(C_B) = B \). Since \( g \subseteq \supp(g) \), and by Theorem 57, we have \( \supp(g) \) is a fuzzy almost bi-ideal of \( S \). By Theorem 58, \( \supp(g) \) is an almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Since \( B \) is minimal, \( \supp(g) = B = \supp(C_B) \). Therefore, \( C_B \) is minimal.

Conversely, assume that \( C_B \) is a minimal fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( B' \) be an almost \( (\alpha, \beta) \)-bi-ideal of \( S \) such that \( B' \subseteq B \). Then, \( C_{B'} \) is a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \) such that \( C_{B'} \subseteq C_B \). Hence, \( B' = \supp(C_{B'}) = \supp(C_B) = B \). Therefore, \( B \) is minimal.

Corollary 14. Let \( B \) be a sub-\( \Gamma \)-semigroup of a \( \Gamma \)-semigroup \( S \). Then, \( B \) is almost \( (\alpha, \beta) \)-bi-simple if and only if for all fuzzy almost \( (\alpha, \beta) \)-bi-ideal \( f \) of \( S \), \( \supp(f) = B \).

Next, we give the relationship between \( \alpha \)-prime almost \( (\alpha, \beta) \)-bi-ideals and \( \alpha \)-prime fuzzy almost \( (\alpha, \beta) \)-bi-ideals.

Definition 25. Let \( S \) be a \( \Gamma \)-semigroup and \( \gamma \in \Gamma \).

(1) An almost \( (\alpha, \beta) \)-bi-ideal \( A \) of \( S \) is called \( \gamma \)-prime if for all \( x, y \in S \), \( xy \in A \) implies \( x \in A \) or \( y \in A \).

(2) A fuzzy almost \( (\alpha, \beta) \)-bi-ideal \( f \) of \( S \) is called \( \gamma \)-prime if for all \( x, y \in S \), \( f(xy) \leq \max\{f(x), f(y)\} \).

Theorem 61. Let \( A \) be a nonempty subset of a \( \Gamma \)-semigroup \( S \). Then, \( A \) is a \( \gamma \)-prime almost \( (\alpha, \beta) \)-bi-ideal of \( S \) if and only if \( C_A \) is a \( \gamma \)-prime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

Proof. Assume that \( A \) is a \( \gamma \)-prime almost \( (\alpha, \beta) \)-bi-ideal of \( S \). By Theorem 58, \( C_A \) is a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( x, y \in S \). We consider two cases:

Case 1: \( xy \in A \). So, \( x \in A \) or \( y \in A \). Then, \( \max\{C_A(x), C_A(y)\} = 1 \geq C_A(xy) \).

Case 2: \( xy \notin A \). Then, \( C_A(xy) = 0 \leq \max\{C_A(x), C_A(y)\} \).

Thus, \( C_A \) is a \( \gamma \)-prime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Conversely, assume that \( C_A \) is a \( \gamma \)-prime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). By Theorem 58, \( A \) is an almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( x, y \in S \) be such that \( xy \in A \). Then, \( C_A(xy) = 1 \). By assumption, \( C_A(xy) \leq \max\{C_A(x), C_A(y)\} \). Therefore, \( \max\{C_A(x), C_A(y)\} = 1 \). Hence, \( x \in A \) or \( y \in A \). Thus, \( A \) is a \( \gamma \)-prime almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

In this section, we give the relationship between \( \gamma \)-semiprime almost \( (\alpha, \beta) \)-bi-ideals and \( \gamma \)-semiprime fuzzy almost \( (\alpha, \beta) \)-bi-ideals.

Definition 26. Let \( S \) be a \( \Gamma \)-semigroup and \( \alpha \in \Gamma \).

(1) An almost \( (\alpha, \beta) \)-bi-ideal \( A \) of \( S \) is called a \( \gamma \)-semiprime if for all \( x \in S \), \( xy \in A \) implies \( x \in A \).

(2) A fuzzy almost \( (\alpha, \beta) \)-bi-ideal \( f \) of \( S \) is called a \( \gamma \)-semiprime if for all \( x \in S \), \( f(xy) \leq f(x) \).

Theorem 62. Let \( A \) be a nonempty subset of \( S \). Then, \( A \) is a \( \gamma \)-semiprime almost \( (\alpha, \beta) \)-bi-ideal of \( S \) if and only if \( C_A \) is a \( \gamma \)-semiprime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

Proof. Assume that \( A \) is a \( \gamma \)-semiprime almost \( (\alpha, \beta) \)-bi-ideal of \( S \). By Theorem 58, \( C_A \) is a fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( x \in S \). We consider two cases:

Case 1: \( xy \in A \). Then, \( x \in A \). So, \( C_A(x) = 1 \). Hence, \( C_A(xy) \leq \max\{C_A(x), C_A(y)\} \).

Case 2: \( xy \notin A \). Then, \( C_A(xy) = 0 \leq C_A(x) \).

Thus, \( C_A \) is a \( \gamma \)-semiprime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

Conversely, assume that \( C_A \) is a \( \gamma \)-semiprime fuzzy almost \( (\alpha, \beta) \)-bi-ideal of \( S \). By Theorem 58, \( A \) is an almost \( (\alpha, \beta) \)-bi-ideal of \( S \). Let \( x \in S \) be such that \( xy \in A \). Then, \( C_A(xy) = 1 \). By assumption, \( C_A(xy) \leq \max\{C_A(x), C_A(y)\} \). Therefore, \( \max\{C_A(x), C_A(y)\} = 1 \). Hence, \( x \in A \) or \( y \in A \). Thus, \( A \) is a \( \gamma \)-semiprime almost \( (\alpha, \beta) \)-bi-ideal of \( S \).

6. Discussion and Conclusion

In this paper, we define new types of ideals and fuzzy ideals by using elements in \( \Gamma \). We show interesting properties of these ideals and fuzzy ideals. Moreover, we show the relationships between these ideals and their fuzzifications.

Data Availability

No data were used to support this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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