Integrated perturbation theory for cosmological tensor fields. I. Basic formulation

Takahiko Matsubara1,2,*

1Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Oho 1-1, Tsukuba 305-0801, Japan
2The Graduate Institute for Advanced Studies, SOKENDAI, Tsukuba 305-0801, Japan
(Dated: September 23, 2024)

In order to extract maximal information about cosmology from the large-scale structure of the Universe, one needs to use every bit of signal that can be observed. Beyond the spatial distributions of astronomical objects, the spatial correlations of tensor fields, such as galaxy spins and shapes, are ones of promising sources that can be accessed in the era of large surveys in the near future. The perturbation theory is a powerful tool to analytically describe the behaviors and evolutions of correlation statistics on large scales for a given cosmology. In this paper, we formulate a nonlinear perturbation theory of tensor fields in general, based on the formulation of integrated perturbation theory for the scalar-valued bias, generalizing it to include the tensor-valued bias. To take advantage of rotational symmetry, the formalism is constructed on the basis of the irreducible decomposition of tensors, identifying physical variables which are invariant under the rotation of the coordinates system.

I. INTRODUCTION

The large-scale structure of the Universe is an inexhaustible fountain of cosmological information and has been playing important roles in cosmology. The various patterns in the spatial distribution of galaxies have emerged from initial density fluctuations of the Universe, whose origin is believed to be generated by quantum fluctuations in the inflationary phase of the very early Universe [1]. Considerable information on the evolution of the Universe is also contained in the large-scale structure. For example, the spatial clustering of galaxies is a promising probe to reveal the nature of dark energy [2] and the initial condition in the early Universe such as the primordial spectrum and non-Gaussianity etc. [3]. While the large-scale structure is used as a probe of the density field in the relatively late Universe, the distribution of observable objects, such as galaxies, quasars, and 21 cm lines, etc., are biased tracers of the underlying matter distribution of the Universe. The difference between distributions of galaxies and matter is referred to as “galaxy bias” [4]. The phenomena of biasing are not restricted to the galaxies, but are unavoidable when we use astronomical objects as tracers of underlying matter distribution.

The number density of galaxies is the primary target to probe the properties of large-scale structure. Theoretical predictions for the spatial distribution of galaxies is investigated by various methods, including numerical simulations, analytic modeling, and the perturbation theory. On one hand, the dynamical evolutions of the mass density fields on relatively large scales are analytically described by the perturbation theory [5–8], which is valid as long as the density fluctuations are small and are in the quasilinear regime. On the other hand, the dynamical structure formation on relatively small scales, including the formation of galaxies and other astronomical objects, is not analytically tractable because of the full nonlinearity of the problem. One should resort to numerical simulations and nonlinear modeling to describe and understand the small-scale structure formation. For example, the halo model [9–11] statistically predicts spatial distributions of halos depending on the initial conditions, using simple assumptions based on the Press-Schechter theory [12] and its extensions [13, 14] of the nonlinear structure formation.

The biasing of tracers affects not only spatial distributions on small scales, but also those on large scales. However, the biasing effects are not so complicated on large scales as those on small scales. For example, the density contrast δs of biased tracers on sufficiently large scales, where the dynamics are described by linear theory, is proportional to the linear density contrast δL, and we have a simple relation δs = bδL, where a proportional constant b is called the linear bias parameter [15, 16]. While nonlinear dynamics complicate the situation, the nonlinear perturbation theory to describe the spatial clustering of biased tracers has been also developed [4, 8]. Since the perturbation theory cannot describe the nonlinear structure formation which takes place on small scales, it is inevitable to introduce some unknown parameters or functions whose values strongly depend on the small-scale nonlinear physics which cannot be predicted from the first principle. These nuisance parameters are called bias parameters or bias functions in the nonlinear perturbation theory. When a model of bias is defined in some way, one can calculate the gravitational evolution of the spatial distribution of biased tracers by applying the nonlinear perturbation theory.

There are several different ways of characterizing the bias model in the nonlinear perturbation theory [4]. Among others, the integrated perturbation theory (iPT) [17–20], which is based on the Lagrangian perturbation theory [21, 22] and orthogonal decomposition of the bias relation [23], provides a method to generically include any model of bias formulated in Lagrangian space, irrespective of whether the bias is local or nonlocal. In this formalism, the concept of the renormalized bias functions [17] is introduced. Dynamical evolutions on large scales which are described by perturbation theory are separated from complicated nonlinear processes of structure formation on small scales. The latter complications are all encoded in the renormalized bias functions in the formalism of iPT. The renormalized bias functions are given for any models of bias, whether they are local or nonlocal in general. For example, the halo bias is one of the typical models of Lagrangian bias, and the renormalized bias functions are uniquely deter-
The number density of biased objects is not the only quantity we can observe. For example, the position and shape of galaxies are simultaneously observed in imaging surveys of galaxies, which are essential in observations of weak lensing fields to probe the nature of dark matter and dark energy [24]. Before the lensing effects, the galaxy shapes are more or less correlated to the mass density field through, e.g., gravitational tidal fields and other environmental structures in the underlying density field. The shapes of galaxies, i.e., sizes and intrinsic alignments [25–33] are expected to be promising probes of cosmological information [34] in the near future when unprecedentedly large imaging surveys are going to take place. For example, the shape statistics of galaxies offer a new probe of particular features in primordial non-Gaussianity generated during the inflation in the presence of higher-spin fields [35], whose features are dubbed “cosmological collider physics.”

Motivated by recent progress in observational techniques of imaging surveys, analytical modelings of galaxy shape statistics by the nonlinear perturbation theory have been actively considered in recent years [36–41]. Among others, the authors of Ref. [39] developed a formalism based on spherical tensors to describe the galaxy shapes, with which rotation and parity symmetries in the shape statistics are respected. They use an approach called the effective field theory of the large-scale structure (EFTofLSS) with the Eulerian perturbation theory, and galaxy biasing is modeled by introducing a set of semilocal terms that are generally allowed by conceivable symmetries of the problem and the coefficients of these terms are free parameters in the theory.

The galaxy shape statistics are usually characterized by the second-order moments of galaxy images in the two-dimensional sky, which correspond to the projection of the three-dimensional second-order moments of galaxy shapes to the two-dimensional sky. In principle, the galaxy images contain information about other moments higher than the second order. The higher-order moments in galaxy images turn out to be possible probes for the higher-spin fields in the context of cosmological collider physics through angle-dependent primordial non-Gaussianity. In particular, Ref. [42] showed that the shape statistics of the order-$s$ moment turn out to be a probe for spin-$s$ fields.

In this paper, inspired by the above developments, we consider a generalization of the iPT formalism to predict correlations of tensor fields of rank-$l$ in general with the perturbation theory. In order to have rotational symmetry apparent, we decompose the tensor fields into irreducible tensors on the basis of spherical tensors, just similarly in the pioneering work of Refs. [39, 40]. However, unlike the previous work, we do not fix the coordinates system to the directions of wave vectors of Fourier modes, so as to make the theory fully covariant with respect to the rotational symmetry in three-dimensional space. It turns out that the generalizing iPT formalism on the basis of irreducible tensors results in an elegant way of describing the perturbation theory only with rotationally invariant quantities in the presence of tensor-valued bias. Various methods developed for the theory of angular momentum in quantum mechanics, such as the $3n_j$-symbols and their orthogonality relations, sum rules, the Wigner-Eckart theorem, and so on, are effectively utilized in the course of calculations for predicting statistical quantities of tensor fields.

This paper defines a basic formalism of the tensor generalization of the methods of iPT. Several applications of the fully general formalism to relatively simple examples of calculating statistics are also illustrated, i.e., the lowest-order predictions of the power spectrum of tensor fields in real space and in redshift space, lowest-order effects of primordial non-Gaussianity in the power spectrum, and tree-level predictions of the bispectrum of tensor fields in real space. The main purpose of this paper is to show the fundamental formulation of tensor-valued iPT, and to give useful methods and formulas for future applications to calculate statistics of concrete tensor fields, such as galaxy spins, shapes, etc. Further developments for calculating loop corrections in the higher-order perturbation theory are described in a separate paper in the series, Paper II [43], which follows this paper. Since the three-dimensional tensor fields themselves are not directly observable, instead, the projected two-dimensional tensors on the sky are observable. The projection effects for the tensor components are studied in another separate paper in the series, Paper III [44]. In Papers I–III, the distant-observer approximation is assumed, and this approximation is mostly good as long as the galaxies are located at greater distances from the observer than the scales of clustering we are interested in. Beyond the distant-observer approximation, full-sky or wide-angle effects are studied in yet another separate paper in the series, Paper IV [45].

This paper is organized as follows. In Sec. II, various notations regarding the spherical basis are introduced and defined, and their fundamental properties are derived and summarized. In Sec. III, the iPT formalism originally introduced for scalar-valued fields is generalized to those for tensor-valued fields. In Sec. IV, rotationally invariant components of renormalized bias functions and higher-order propagators are identified. Additional symmetries in them are also described. Explicit forms of several lower-order propagators are calculated and presented. In Sec. V, the calculation of the power spectrum, correlation function, and bispectrum of tensor fields by our formalism is illustrated and presented in several cases that are not relatively complicated. In Sec. VI, we define a class of bias models, which we call semilocal models, in which the Lagrangian bias of the tensor field is determined only by derivatives of the gravitational potential at the same position in Lagrangian space. The relations between the renormalized bias functions in iPT and bias renormalizations in conventional perturbation theory are clarified within the scheme of semilocal models of bias. Conclusions are given in Sec. VII. In Appendix A, explicit expressions of higher-order spherical tensor basis are derived and given. Formulas related to spherical harmonics as well as to Wigner’s $3j$, $6j$, and $9j$-symbols, which are both repeatedly used throughout Papers I–IV, are summarized, respectively, in Appendixes B and C.
II. DECOMPOSITION OF TENSORS BY SPHERICAL BASIS

In this paper, the irreducible representations of the tensor field in spherical basis are extensively used, for which non-standard conventions of the rotation group are employed for reasons of notational ease. We define our conventions below in this section.

For a given set of orthonormal basis vectors \( \hat{e}_i (i = 1, 2, 3) \) in a flat, three-dimensional space, the spherical basis \((e_0, e_s)\) is defined by \([46, 47]\)

\[
e_0 = \hat{e}_3, \quad e_s = \pm \hat{e}_1 \pm i \hat{e}_2 / \sqrt{2}.
\]  

(1)

It is convenient to introduce the dual basis with an upper index by taking complex conjugates of the spherical basis:

\[
e^0 \equiv e_0^* = \hat{e}_3, \quad e^s \equiv e_s^* = \mp \hat{e}_1 \mp i \hat{e}_2 / \sqrt{2}.
\]  

(2)

The orthogonality relations among these bases are given by

\[
e_m \cdot e_{m'} = g^{(1)}_{mm'}, \quad e^m \cdot e^{m'} = \delta^{m'}_m, \quad e^m \cdot e^{m'} = g^{pm'}_{mm'},
\]  

(3)

where \(m, m'\) are azimuthal indices of the spherical basis which represent one of \((-0,+)\). The spherical metric tensors are defined by

\[
g^{(1)}_{mm'} = g^{ mm'} = (-1)^m \delta_{m,-m'},
\]  

(4)

where \(\delta^{m'}_m = \delta_{mm'}\) is the Kronecker’s delta symbol which is unity for \(m = m'\) and zero for \(m \neq m'\). The label “(1)” in \(g^{(1)}_{mm'}\) and \(g^{(1)}_{mm'}\) suggests that the matrix composed by these tensors has dimensions \((2l + 1) \times (2l + 1)\) with \(l = 1\), and they are special cases of \(g^{(i)}_{mm'}\) or \(g^{(i)}_{mm'}\) defined in Eqs. (27) or (28) below. The matrices \(g^{(i)}_{mm'}\) and \(g^{(i)}_{mm'}\) are inverse matrices of each other,

\[
g^{(1)}_{mm'} g^{(1)}_{mm'} = \delta^{mm'},
\]  

(5)

where the repeated index \(m''\) is assumed to be summed over, omitting summation symbols as commonly adopted in the convention of tensor analysis (Einstein summation convention). Throughout this paper, the Einstein convention of summation for azimuthal indices is always assumed, unless otherwise stated. They act as metric tensors of the spherical basis,

\[
e^m = g^{mm'} e_{m'}, \quad e_m = g^{(1)}_{mm} e^{m'}.
\]  

(6)

Any vector can be represented by \( V \propto V^m e_m = V_s e^s\) up to normalization constant, and thus \(e_s\) is considered as the basis of covariant spherical vectors, and \(e^s\) is considered as the basis of contravariant spherical vectors.

One of the advantages of the spherical basis over the Cartesian basis is that the Cartesian tensors are reduced into irreducible tensors on the spherical basis under a rotation of coordinates system \([48]\). We define the spherical basis of traceless, symmetric tensors \([39, 49] as

\[
\mathcal{Y}^{(0)} = 1, \quad \mathcal{Y}^{(m)} = e^m_i
\]  

(7)

for tensors of rank-0 and rank-1, and

\[
\mathcal{Y}^{(0)}_{ij} = \sqrt{3/2} \left( e^0_i e^0_j - 1/3 \delta_{ij} \right),
\]  

(8)

\[
\mathcal{Y}^{(1)}_{ij} = \sqrt{2} e^0_i (e^0_j),
\]  

(9)

\[
\mathcal{Y}^{(2)}_{ij} = e^0_i e^0 j
\]  

(10)

tensors of rank-2, where \(e^0_i = [e^0_i]\) is the Cartesian components of the spherical basis, and round brackets in the indices of the right-hand side (rhs) of Eq. (9) indicate symmetrization with respect to the indices inside the brackets, e.g., \(x_i y_j = (x_i y_j + x_j y_i)/2\). These bases satisfy orthogonality relations,

\[
\mathcal{Y}^{(0)} \mathcal{Y}^{(0)} = 1, \quad \mathcal{Y}^{(m)} \mathcal{Y}^{(m')} = g^{mm'}_{(1)}, \quad \mathcal{Y}^{(m)} \mathcal{Y}^{(m')}_{ij} = g^{mm'}_{(2)} i j,
\]  

(11)

where, in the last equation, \(g^{mm'}_{(2)} = (-1)^m \delta_{m,-m'}\) is the \(5 \times 5\) spherical metric, and the indices \(m, m'\) run over the integers \(-2, -1, 0, +1, +2\) in this case. The Einstein summation convention is also applied for the Cartesian indices in the above, so that \(i\) and \(j\) are summed over on the left-hand side (lhs) of the above equations. Taking complex conjugates of the spherical basis virtually lowers the azimuthal indices,

\[
\mathcal{Y}^{(0)*} = \mathcal{Y}^{(0)}, \quad \mathcal{Y}^{(m)*} = g^{mm'}(1) \mathcal{Y}^{(m')}_{i j}, \quad \mathcal{Y}^{(m)*} = g^{mm'}(2) i j.
\]  

(12)

For a unit vector \(n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) whose direction is given by spherical coordinates \((\theta, \phi)\), the spherical tensors are related to the spherical harmonics \(Y_{lm}(\theta, \phi)\) as

\[
Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \mathcal{Y}^{(0)*}, \quad Y_{1m}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \mathcal{Y}^{(m)*}_i n_i,
\]  

\[
Y_{2m}(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \mathcal{Y}^{(m)*}_i n_i n_j.
\]  

(13)

where the normalization convention of the spherical harmonics is given by a standard one,

\[
Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} \frac{(-1)^m}{(l - m)!} P^m_l(\cos \theta) \mathcal{Y}^{(m)*}_i e^{i m}_{i j},
\]  

(14)

and

\[
P^m_l(x) = \frac{(-1)^m}{2^l l!} \left(1 - x^2\right)^{l/2} \frac{d^{l+m}}{dx^{l+m}} \left(1 - x^2\right)^l.
\]  

(15)

1. The normalization of the spherical tensors in this paper is somehow different from those in Refs. [39, 49]. Our normalization is chosen so that Eq. (11) below should hold. In addition, in the previous literature, the axis \(e^0\) of the coordinates system is specifically chosen to be parallel to the separation vector \(r\) of the correlation function [49], or, to the wave vector \(k\) of Fourier modes [39], while our axis can be arbitrarily chosen, so that fully rotational symmetry is explicitly kept in our formalism, not choosing a special coordinates system.
are associated Legendre polynomials that specifically include the so-called Condon-Shortley phase factor of $(-1)^m$. Similarly, spherical bases for higher-rank symmetric tensors are uniquely defined by imposing the similar properties of Eq. (13),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)!!}{4\pi l!}} Y_{l123...i_l123...i_l}^{(m)} n_{i_1} n_{i_2} \cdots n_{i_l}. \quad (16)$$

General procedures to construct spherical bases of arbitrary rank are given and explicit forms of the spherical bases of the rank-0 to 4 are derived in Appendix A.

The standard normalization of spherical harmonics turns out to be so not convenient to represent various equations in this work. Instead, the spherical harmonics with Racah’s normalization, defined by

$$C_{lm}(\theta, \phi) \equiv \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) = \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad (17)$$

turn out to simplify various equations. One of the reasons is that the last normalization is closely related to the standard normalization of Legendre polynomials, as we have $C_0(\theta, \phi) = P_0(\cos \theta)$, where $P_l(\cos \theta)$ are the ordinary Legendre polynomials. With the spherical harmonics with Racah’s normalization, a general relation of Eq. (16) is given by

$$Y_{l123...i_l123...i_l}^{(m)} n_{i_1} n_{i_2} \cdots n_{i_l} = A_l C_{lm}^*(\theta, \phi), \quad (18)$$

where we define a factor,

$$A_l \equiv \sqrt{\frac{l!}{(2l-1)!!}}, \quad (19)$$

which repeatedly appears in later formulas. The spherical bases are irreducible representations of the rotation group SO(3) in the same way as the spherical harmonics are. We consider a passive rotation of the Cartesian basis,

$$\hat{e}_i \rightarrow \hat{e}'_i = \hat{e}_j R_{ji}, \quad (20)$$

where $R_{ji}$ are components of a real orthogonal matrix which satisfies $R^T R = I$, and $I$ is the $3 \times 3$ unit matrix. In terms of the Euler angles $(\alpha, \beta, \gamma)$ with the $y$ convention, the rotation matrix $R$ is explicitly given by

$$R(\alpha, \beta, \gamma) = \begin{pmatrix}
   c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\
   s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\
   -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta
\end{pmatrix}, \quad (21)$$

where $c_\alpha = \cos \alpha$, $s_\alpha = \sin \alpha$, $c_\beta = \cos \beta$, and so forth. Cartesian components of a unit vector $\mathbf{n}$, where $|\mathbf{n}| = 1$, transform as

$$n_i \rightarrow n'_i = n_j R_{ji}, \quad (22)$$

because we have $n_i \hat{e}_i = n'_i \hat{e}'_i$. Denoting the spherical coordinates of $n_i$ and $n'_i$ as $(\theta, \phi)$ and $(\theta', \phi')$, respectively, the spherical harmonics transform as

$$C_{lm}(\theta', \phi') = C_{lm}(\theta, \phi) D^m_{lm}(R), \quad (23)$$

where $D^m_{lm}(R) = D^m_{lm}(\alpha, \beta, \gamma)$ is the Wigner’s rotation matrix of the passive rotation $R$ with the Euler angles $(\alpha, \beta, \gamma)$. Therefore, combining the properties of Eqs. (18), (22) and (23), we find a useful identity for the spherical basis,

$$R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_l j_l} Y_{l123...i_l123...i_l}^{(m)} = D_{lm}^{(m)}(R^{-1}) Y_{l123...i_l123...i_l}^{(m)}, \quad (24)$$

where Wigner’s matrix of the inverse rotation is given by

$$D_{lm}^{(m)}(R^{-1}) = D_{lm}^{(m)}(-\gamma, -\beta, -\alpha) = D_{lm}^{(m)*}(R). \quad (25)$$

The orthogonality relations of Eq. (11) are generalized for higher-rank tensors. In fact, using Eqs. (24) and (25) with $R^T R = I$, one can show that they are given by

$$Y_{l123...i_l123...i_l}^{(m)} Y_{l123...i_l123...i_l}^{(m)*} = \delta_{l123...i_l123...i_l}^{(m)m}, \quad (26)$$

where $(2l+1) \times (2l+1)$ matrix $\delta_{l123...i_l123...i_l}^{(m)m}$ is defined just as Eq. (4) but the indices $m, m'$ run over integers from $-l$ to $+l$. The inverse matrix is similarly denoted by $\delta_{l123...i_l123...i_l}^{(m)m'}$, whose elements are the same as $\delta_{l123...i_l123...i_l}^{(m)m}$. Thus, similar relations as in Eqs. (4) and (5) hold for $g_{l123...i_l123...i_l}^{(m)m}$ and $g_{l123...i_l123...i_l}^{(m)m'}$:

$$g_{l123...i_l123...i_l}^{(m)m} = (-1)^m \delta_{l123...i_l123...i_l}^{(m)m'}, \quad (27)$$

The relations to the $1jm$-symbol introduced by Wigner [50] are given by

$$(-1)^{l} g_{l123...i_l123...i_l}^{(m)m} = \begin{pmatrix} m \\ m' \end{pmatrix} = (-1)^{l} g_{l123...i_l123...i_l}^{(m)m'}. \quad (28)$$

Taking the complex conjugate of the basis virtually lowers the azimuthal index as in Eq. (12), i.e., we have

$$Y_{l123...i_l123...i_l}^{(m)m'} = g_{l123...i_l123...i_l}^{(m)m'} Y_{l123...i_l123...i_l}^{(m)m'}, \quad Y_{l123...i_l123...i_l}^{(m)m} = g_{l123...i_l123...i_l}^{(m)m'} Y_{l123...i_l123...i_l}^{(m)m'}, \quad (29)$$

The same properties also apply to spherical harmonics:

$$C_{lm}(\theta, \phi) = g_{l123...i_l123...i_l}^{(m)m'} C_{lm}(\theta, \phi), \quad C_{lm}(\theta, \phi) = g_{l123...i_l123...i_l}^{(m)m'} C_{lm}(\theta, \phi). \quad (30)$$

Consequently, the orthogonality relations of Eq. (26) are also equivalent to

$$Y_{l123...i_l123...i_l}^{(m)m'} Y_{l123...i_l123...i_l}^{(m)m'} = \delta_{l123...i_l123...i_l}^{(m)m'}, \quad Y_{l123...i_l123...i_l}^{(m)m} Y_{l123...i_l123...i_l}^{(m)m'} = \delta_{l123...i_l123...i_l}^{(m)m'}. \quad (31)$$

We decompose a scalar function $S$ and a vector function $V_i$ into spherical components as

$$S = S_{00} Y_0^0, \quad V_i = V_{l1} Y_i^{(m)}. \quad (32)$$

Because of the orthogonality relations of Eq. (11), the inverted relations of the above are given by

$$S_{00} = S Y_0^0, \quad V_{l1} = V_i Y_i^{(m)}. \quad (33)$$
Since the spherical basis is a traceless tensor, a symmetric rank-2 tensor function \( T_{ij} \), which is not necessarily traceless, is decomposed into spherical components of the trace part and traceless part as

\[
T_{ij} = \frac{1}{3} \delta_{ij} T_{00} \mathcal{Y}^{(0)} + \sqrt{\frac{2}{3}} T_{2m} \mathcal{Y}^{(m)}, \tag{34}
\]

where

\[
T_{00} = T_{ii} \mathcal{Y}^{(0)*}, \quad T_{2m} = \sqrt{\frac{3}{2}} T_{ij} \mathcal{Y}^{(m)*}. \tag{35}
\]

The prefactor \( \sqrt{2/3} \) in front of the second term on the rhs of Eq. (34) is again our normalization convention. The general rules of our normalization convention are shortly explained at the end of this section.

Higher-rank tensors can be similarly decomposed into spherical tensors. We first note that any symmetric tensor of rank-\( j \) can be decomposed into [51]

\[
T_{i_1 i_2 \ldots i_{j-1}} = T^{(l)}_{i_1 i_2 \ldots i_{j-1}} + \frac{l(l-1)}{2(2l-1)} \delta_{i_1 i_2} T^{(l-2)}_{i_3 i_4} + \cdots , \tag{36}
\]

where \( T^{(l)}_{i_1 i_2 \ldots i_{j-1}} \) is a symmetric traceless tensor of rank-\( l \), \( T^{(l-2)}_{i_3 i_4} \) is traceless part of \( T^{(l)}_{i_1 i_2 \ldots i_{j-1}} \), and so forth (one should note that the Einstein summation convention is applied also to Cartesian indices, and thus \( j = 1, 2, 3 \) is summed over in the last tensor). The traceless part of \( T_{i_1 \ldots i_l} \) is generally given by a formula [51, 52]

\[
T^{(l)}_{i_1 \ldots i_l} = \frac{l!}{(2l-1)!! \sum_{k=0}^{[l/2]} (-1)^k (2l-2k-1)!!} \times \delta_{i_1 i_2} \cdots \delta_{i_{2k} i_{2k+1}} T^{(l, k)}_{i_{2k+1} \ldots i_l}, \tag{37}
\]

where \([l/2]\) is the Gauss symbol, i.e., \([l/2] = l/2\) if \( l \) is even and \([l/2] = (l-1)/2\) if \( l \) is odd, and

\[
T^{(l, k)}_{i_{2k+1} \ldots i_l} = \delta_{i_{1}i_{3}} \delta_{i_2i_4} \cdots \delta_{i_{2k-2}i_{2k}} T_{i_{1}i_{2k+1} \ldots i_{l}} = T_{j_1 j_2 j_3 \ldots j_{2k+1} i_{l}} \tag{38}
\]

is the trace of the original tensor taken \( k \) times. Tensors of rank-0 and rank-1 correspond to a scalar and a vector, respectively, and are already traceless as they do not have any trace component from the first place. The traceless part of the tensor components forms an irreducible representation of the rotation group SO(3). Decomposition of higher-rank tensors can be similarly obtained according to the procedure described in Ref. [51]. Recursively using Eq. (37), one obtains explicit expressions of Eq. (36). Up to the fourth rank, they are given by

\[
T_{ij} = T^{(2)}_{ij} + \frac{1}{3} \delta_{ij} T^{(0)}, \tag{39}
\]

\[
T_{ijk} = T^{(3)}_{ijk} + \frac{3}{5} \delta_{ij} T^{(1)}_{k}, \tag{40}
\]

\[
T_{ijkl} = T^{(4)}_{ijkl} + \frac{6}{7} \delta_{ij} T^{(2)}_{kl} + \frac{1}{5} \delta_{ij} \delta_{kl} T^{(0)}. \tag{41}
\]

The traceless part of a tensor is decomposed by spherical bases as

\[
T^{(l)}_{i_1 i_2 \ldots i_l} = A_l T_{lm} \mathcal{Y}^{(m)}_{i_1 i_2 \ldots i_l}, \tag{42}
\]

\[
T_{lm} = \frac{1}{A_l} T^{(l)}_{i_1 i_2 \ldots i_l} \mathcal{Y}^{(m)*}_{i_1 i_2 \ldots i_l}. \tag{43}
\]

The prefactor \( A_l \) on the rhs of Eq. (42) is our normalization convention, where \( A_l \) is given by Eq. (19). The merits of this particular choice of normalization for the irreducible tensor \( T_{lm} \) in general appear in Sec. VI below. However, the equations presented in Secs. III–V of this paper are not affected by the choice of normalization \( A_l \), because this normalization factor simultaneously appears on both sides and cancels to each other in most of the equations.

The traceless part of a traced tensor is similarly decomposed, e.g., as

\[
T^{(l-2k)}_{i_1 i_2 \ldots i_{l-2k}} = A_{l-2k} T_{i_1 i_2 \ldots i_{l-2k}} \mathcal{Y}^{(m)}_{i_1 i_2 \ldots i_{l-2k}}, \tag{44}
\]

\[
T_{i_1 i_2 m} = \frac{1}{A_{l-2k}} T^{(l)}_{i_1 i_2 i_3 \ldots i_{l-2k}} \mathcal{Y}^{(m)*}_{i_1 i_2 i_3 \ldots i_{l-2k}}, \tag{45}
\]

Under the passive rotation of the basis, Eq. (20), Cartesian tensors of rank-\( l \) transform as

\[
T^{(l)}_{i_1 i_2 \ldots i_l} \rightarrow T^{(l)}_{i_1 i_2 \ldots i_l} = T^{(l)}_{j_1 j_2 \ldots j_l R_{j_1 i_1} \cdots R_{j_l i_l}}, \tag{46}
\]

and the corresponding transformation of spherical tensors is given by

\[
T_{lm} \rightarrow T'_{lm} = T_{lm} D^{m'}_{l(m')}(R), \tag{47}
\]

as derived from the transformation of the spherical base, Eq. (24). The contravariant components of the spherical tensors are defined by

\[
T^{m}_{i_1 i_2 \ldots i_l} = g^{m m'} T'_{i_1 i_2 \ldots i_l}, \tag{48}
\]

and transform under the rotation as

\[
T^{m}_{i_1 i_2 \ldots i_l} \rightarrow T'^{m}_{i_1 i_2 \ldots i_l} = D^{m'}_{l(m')}(R^{-1}) T^{m}_{i_1 i_2 \ldots i_l}. \tag{49}
\]

III. THE INTEGRATED PERTURBATION THEORY OF TENSOR FIELDS

A. Formulating the iPT of tensor fields

The integrated perturbation theory (iPT) is a systematic method of perturbation theory to describe the quasinonlinear evolution of the large-scale structure. This method is based on the Lagrangian scheme of the biasing and nonlinear structure formation. The formalism of this method is explicitly developed to describe the scalar fields, such as the number density fields of galaxies or other biased objects in the Universe [17, 19]. In this section, the formalism is generalized to be able to describe quasinonlinear evolutions of objects which generally have tensor values, such as the angular momentum of galaxies (vector), the second moment of intrinsic alignment.
of galaxy shapes (rank-2 tensor), or higher moments of intrinsic alignment (rank-3 or higher tensors), etc.

The fundamental formalism of iPT to calculate spatial correlations of biased objects is described in Ref. [17]. Basically, we can follow the formalism with a generalization of assigning the tensor values to the objects. We denote the observable objects $X$, where $X$ is the class of objects selected in a given cosmological survey, such as a certain type of galaxies. We consider each object of a $X$ has a tensor value $F_{i_1i_2...}$ in general, and define the tensor field by

$$F_{X_{i_1i_2...}}(x) = \frac{1}{\bar{n}_X} \sum_{a \in X} F^a_{i_1i_2...} \delta^3_D (x - x_a),$$  \hspace{1cm} (50)$$

where $x_a$ is the location of a particular object $a$, $\bar{n}_X$ is the mean number density of objects $X$, and $\delta^3_D(x)$ is the three-dimensional Dirac’s delta function. It is crucial to note that the above definition of the tensor field is weighted by the number density of the objects $X$. In the above, although the field really depends on the time $F_{X_{i_1i_2...}}(x,t)$, the argument of time $t$ is omitted in the notation for simplicity and the same suppression is employed for other functions in this paper.

In general, any type of tensor can be decomposed into a sum of symmetric and isotropic tensors [51], where the latter are of lower rank than the parent tensor. Bearing that in mind, below we consider the case that tensor field $F^a_{i_1i_2...}$ is a symmetric tensor, which can be one of a decomposed symmetric tensor even if the parent tensor is not symmetric. We decompose the symmetric tensor of each object $F_{i_1i_2...}$ into irreducible tensors $F^a_{i_1i_2...}$ according to the scheme described in the previous section. Accordingly, corresponding components of Eq. (50) are given by

$$F_{X_{i_1i_2...}}(x) = \frac{1}{\bar{n}_X} \sum_{a \in X} F^a_{i_1i_2...} \delta^3_D (x - x_a),$$  \hspace{1cm} (51)$$

As this tensor field is a number density-weighted quantity by construction, we also consider the unweighted field $G_{X_{i_1i_2...}}(x)$ defined by

$$F_{X_{i_1i_2...}}(x) = [1 + \delta_X(x)] G_{X_{i_1i_2...}}(x),$$  \hspace{1cm} (52)$$

where $\delta_X(x)$ is the number density contrast of the objects $X$, defined by

$$1 + \delta_X(x) = \frac{1}{\bar{n}_X} \sum_{a \in X} \delta^3_D (x - x_a).$$  \hspace{1cm} (53)$$

Substituting Eq. (53) into Eq. (52) and comparing with Eq. (51), the unweighted field satisfies

$$G_{X_{i_1i_2...}}(x_a) = F^a_{X_{i_1i_2...}}$$  \hspace{1cm} (54)$$

at each position of object $a$, as naturally expected. The function $G_{X_{i_1i_2...}}(x)$ can arbitrarily be interpolated where $x \neq x_a$ for all $a \in X$, because the field $F_{X_{i_1i_2...}}(x)$ is not supported at those points.

The iPT utilizes the Lagrangian perturbation theory (LPT) for the dynamical nonlinear evolutions (see Ref. [53] for notations of LPT employed in this paper). The Eulerian coordinates $x$ and the Lagrangian coordinates $q$ of a mass element are related by

$$x = q + \Psi(q),$$  \hspace{1cm} (55)$$

where $\Psi(q)$ is the displacement field. The number density of the objects in Lagrangian space $n^L_X(q)$ is given by the Eulerian counterpart by a continuity relation,

$$n_X(x) d^3 x = n^L_X(q) d^3 q,$$  \hspace{1cm} (56)$$

i.e., the Lagrangian number density is defined so that the Eulerian positions of objects are displaced back into the Lagrangian positions, and the number density in Lagrangian space also depends on the time of evaluating $n_X(x)$. The mean number density of objects $\bar{n}_X$ is the same in Eulerian and Lagrangian spaces by construction, thus we have

$$1 + \delta_X(x) = [1 + \delta^L_X(q)] / j(q),$$

where the Eulerian coordinate $x$ on the lhs is a function of the Lagrangian coordinates $q$ according to Eq. (55), and $j(q) = (\partial \bar{n}_X / \partial \bar{n}_X)$ is the Jacobian of the mapping. The density contrasts between these spaces are therefore related by

$$1 + \delta_X(x) = \int d^3 q [1 + \delta^L_X(q)] \delta^3_D (x - q - \Psi(q)).$$  \hspace{1cm} (57)$$

As in the Eulerian space, the tensor field in Lagrangian space is also naturally defined with the weight of the number density of objects. The unweighted tensor field $G^L_{X_{i_1i_2...}}(q)$ in Lagrangian space is simply defined by taking the same value displaced back from the Eulerian position of Eq. (52),

$$G^L_{X_{i_1i_2...}}(q) \equiv G_{X_{i_1i_2...}}(q),$$  \hspace{1cm} (58)$$

where $x$ on the rhs is just given by Eq. (55) and the number density-weighted tensor field in Eulerian space, $F^L_{i_1i_2...}$, is defined by

$$F^L_{X_{i_1i_2...}}(q) = [1 + \delta^L_X(q)] G^L_{X_{i_1i_2...}}(q).$$  \hspace{1cm} (59)$$

As obviously derived from Eqs. (57) and (58), the number density-weighted tensor fields in Eulerian space and Lagrangian space is related by

$$F_{X_{i_1i_2...}}(x) = \int d^3 q F^L_{X_{i_1i_2...}}(q) \delta^3_D (x - q - \Psi(q)).$$  \hspace{1cm} (60)$$

In cosmology, the properties of observable quantities are determined by the initial condition of density fluctuations in the Universe, and the growing mode of linear density contrast, $\delta_L$, is a representative of initial conditions. Therefore, the tensor field $F_{X_{i_1i_2...}}$ is considered as a functional of the linear density field $\delta_L$. The ensemble average of the $n$th functional derivatives of the evolved field is called multipoint propagators [23, 54, 55], which are defined in equivalently various manners. Here we explain a comprehensive definition according to Ref. [23], starting from a propagator in configuration space:

$$\Gamma^{(n)}_{X_{i_1i_2...}}(x, x_1, \ldots, x_n) \equiv (-i)^n \left( \frac{\partial^n F_{X_{i_1i_2...}}(x)}{\partial \delta_L(x_1) \cdots \partial \delta_L(x_n)} \right).$$  \hspace{1cm} (61)$$
where $\delta_l(x)$ is the linear density contrast, $\delta/\delta \delta_l(x)$ is the functional derivative with respect to the linear density contrast, and $\langle \cdots \rangle$ represents the ensemble average of the field configurations. The phase factor $(-i)^l$ in front of Eq. (61) is our convention to define the propagators of the tensor field, which we find convenient. Because of the translational invariance of the Universe in a statistical sense, the rhs should be invariant under the homogeneous translation, $x \rightarrow x + x_0$ and $x \rightarrow x + x_0$ for $i = 1, \ldots, n$ with any fixed displacement $x_0$, and thus the arguments on the lhs of Eq. (61) should be given as those in the expression.

We apply the Fourier transform of the linear density contrast and the tensor field,

$$\tilde{\delta}_l(k) = \int d^3x e^{-ik \cdot x} \delta_l(x),$$

$$F_{Xlm}(k) = \int d^3x e^{-ik \cdot x} F_{Xlm}(x).$$

and the functional derivative in Fourier space is given by

$$\frac{\delta}{\delta \tilde{\delta}_l(k)} = \frac{1}{(2\pi)^3} \int d^3x e^{ik \cdot x} \frac{\delta}{\delta \delta_l(x)}.$$  \hspace{1cm} (64)

Using the above equations, the Fourier counterpart of the rhs of Eq. (61) is calculated, and we have

$$\left(\frac{\delta^l_F Xlm(k)}{\delta \tilde{\delta}_l(k_1) \cdots \delta \tilde{\delta}_l(k_n)}\right) = (-i)^l (2\pi)^{3n} \delta_3^l(k_1 + \cdots + k_n - k) \Gamma_{Xlm}^{(n)}(k_1, \ldots, k_n).$$

$$\Gamma_{Xlm}^{(n)}(k_1, \ldots, k_n) = \int d^3x_1 \cdots d^3x_n e^{i(k_1 \cdot x_1 + \cdots + k_n \cdot x_n)} \times \Gamma_{Xlm}^{(n)}(x_1, \ldots, x_n)$$ \hspace{1cm} (66)

is the Fourier transform of the propagator. The appearance of the delta function on the rhs of Eq. (65) is the consequence of the statistical homogeneity of the Universe. Integrating Eq. (65), we have

$$\Gamma_{Xlm}^{(n)}(k_1, \ldots, k_n) = (-i)^l (2\pi)^{3n} \int \frac{d^3k}{(2\pi)^3} \left(\frac{\delta^l_F Xlm(k)}{\delta \tilde{\delta}_l(k_1) \cdots \delta \tilde{\delta}_l(k_n)}\right).$$  \hspace{1cm} (67)

The Fourier transform of Eq. (60) is given by

$$F_{Xlm}(k) = \int d^3q e^{-ik \cdot q} \psi(q) P_{Xlm}^{(l)}(q).$$

In the iPT, the displacement field $\Psi(q)$ is expanded according to the LPT,

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{i}{n!} \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} \frac{\delta^l_F Xlm(k)}{\delta \tilde{\delta}_l(k_1) \cdots \delta \tilde{\delta}_l(k_n)} q^k \times L_n(k_1, \ldots, k_n) \tilde{\delta}_l(k_1) \cdots \tilde{\delta}_l(k_n).$$

The perturbation kernels $L_n$ are recursively obtained both for the model of the Einstein-de Sitter Universe [56, 57] and for general models [53]. The propagators of Eq. (67) are systematically evaluated by applying diagrammatic rules in the iPT. The detailed derivation of the diagrammatic rules for the number density field $\delta_l$ is given in Ref. [17]. Exactly the same derivation applies to the tensor field, simply replacing $1 + \delta x \rightarrow F_{Xlm}$ and $1 + \delta x \rightarrow F_{Xlm}^T$ in that reference. In this method, the renormalized bias functions play important roles. In the present case of the tensor field, the renormalized bias functions $c^{(n)}_{Xlm}(k_1, \ldots, k_n)$ in Fourier space are defined by

$$\left(\frac{\delta^l_F Xlm(k)}{\delta \tilde{\delta}_l(k_1) \cdots \delta \tilde{\delta}_l(k_n)}\right) = i^l (2\pi)^{3-3n} \delta_3^l(k_1 + \cdots + k_n - k) c^{(n)}_{Xlm}(k_1, \ldots, k_n),$$

$$c^{(n)}_{Xlm}(k_1, \ldots, k_n) = (-i)^l (2\pi)^{3n} \int \frac{d^3k}{(2\pi)^3} \left(\frac{\delta^l_F Xlm(k)}{\delta \tilde{\delta}_l(k_1) \cdots \delta \tilde{\delta}_l(k_n)}\right).$$

As can be seen by comparing the above definition with Eqs. (65) and (67), the renormalized bias functions are the counterparts of the propagator for the biasing in Lagrangian space.

The diagrammatic rules of iPT are summarized in the Appendix of Ref. [19]. The same rules apply, simply replacing $1 + \delta x \rightarrow F_{Xlm}$, $\Gamma^{(n)} \rightarrow \Gamma^{(n)}_{Xlm}$, etc. For the terms without $c^{(n)}_{Xlm}$ in that reference, an additional factor $c^{(0)}_{Xlm}$ should be multiplied, which is unity in the original formulation for scalar fields. The propagators are given in a form,

$$\Gamma_{Xlm}^{(n)}(k_1, \ldots, k_n) = \Pi(k) \tilde{\Gamma}_{Xlm}^{(n)}(k_1, \ldots, k_n),$$

where

$$\Pi(k) = \left(\begin{array}{c} \psi \cdot \psi \end{array}\right) = \exp \left[ \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} \langle [k \cdot \Psi]^n \rangle_c \right]$$

is the vertex resummation factor in terms of the displacement field, and $\langle \cdots \rangle_c$ denotes the $n$-point connected part of the random variables. The propagator $\tilde{\Gamma}_{Xlm}^{(n)}$ without the resummation factor $\Pi(k)$ is called reduced propagators in the following.

Below, we give explicit forms of lower-order propagators, which can be used to estimate power spectra, bispectra, etc. The vertex resummation factor in the one-loop approximation of iPT is given by

$$\Pi(k) = \exp \left\{ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ k \cdot L_1(p) \right]^2 P_L(p) \right\},$$

where $P_L(k)$ is the linear power spectrum defined by

$$\langle \tilde{\delta}_l(k) \tilde{\delta}_l(k) \rangle = (2\pi)^3 \delta_D^l(k + k') P_L(k).$$

In the above equation, the delta function on the rhs appears due to the statistical homogeneity of space in the initial density field, and the linear power spectrum is a function of only the absolute magnitude of wave vector $k = |k|$ due to the statistical isotropy. As noted above, reduced propagators of
tensor fields are given by straightforward generalizations of the corresponding propagators for the scalar fields derived in Ref. [19].

The reduced propagator of the first order, up to the one-loop approximation, is given by

$$\Gamma^{(1)}_{Xlm}(k) = c^{(1)}_{Xlm}(k) + [k \cdot L_1(k)] c^{(0)}_{Xlm}(k)$$

$$+ \int \frac{d^3p}{(2\pi)^3} P_1(p) \left( [k \cdot L_1(-p)] c^{(1)}_{Xlm}(k, p) + [k \cdot L_2(k, -p)] c^{(0)}_{Xlm}(p) \right)$$

$$+ \frac{1}{2} [k \cdot L_3(k, p, -p)] c^{(0)}_{Xlm}(p)$$

$$+ [k \cdot L_1(p)] [k \cdot L_2(k, -p)] c^{(0)}_{Xlm}(p), \quad (76)$$

The first line of the above expression corresponds to the lowest order. The reduced propagator of the second order in the tree-level approximation is given by

$$\Gamma^{(2)}_{Xlm}(k_1, k_2) = c^{(2)}_{Xlm}(k_1, k_2)$$

$$+ [k_{12} \cdot L_1(k_1)] c^{(1)}_{Xlm}(k_2) + [k_{12} \cdot L_1(k_2)] c^{(1)}_{Xlm}(k_1)$$

$$+ [k_{12} \cdot L_2(k_1)] [k_{12} \cdot L_1(k_2)] + k_{12} \cdot L_2(k_1, k_2) c^{(0)}_{Xlm}, \quad (77)$$

where $k_{12} = k_1 + k_2$, and the reduced propagator of the third order, again in the tree-level approximation, is given by

$$\Gamma^{(3)}_{Xlm}(k_1, k_2, k_3) = c^{(3)}_{Xlm}(k_1, k_2, k_3)$$

$$+ [k_{123} \cdot L_1(k_1)] c^{(2)}_{Xlm}(k_2, k_3) + \text{cyc.}$$

$$+ [k_{123} \cdot L_1(k_1)] [k_{123} \cdot L_2(k_2)] + [k_{123} \cdot L_2(k_1, k_2)]$$

$$\times c^{(1)}_{Xlm}(k_3) + \text{cyc.}$$

$$+ [k_{123} \cdot L_1(k_1)] [k_{123} \cdot L_1(k_2)] [k_{123} \cdot L_1(k_3)]$$

$$+ [k_{123} \cdot L_1(k_1)] [k_{123} \cdot L_2(k_2, k_3)] + \text{cyc.}$$

$$+ k_{123} \cdot L_3(k_1, k_2, k_3) c^{(0)}_{Xlm}, \quad (78)$$

where $k_{123} = k_1 + k_2 + k_3$, and + cyc. represents two terms which are added with cyclic permutations of each previous term.

In real space, the kernels of LPT in the standard theory of gravity (in the Newtonian limit) are given by [58, 59]

$$L_1(k) = \frac{k}{k^2}, \quad (79)$$

$$L_2(k_1, k_2) = \frac{3}{7} k_{12} \left[ 1 - \left( \frac{k_1 \cdot k_2}{k_{12}} \right)^2 \right], \quad (80)$$

$$L_3(k_1, k_2, k_3) = \frac{1}{3} \left[ L_3(k_1, k_2, k_3) + \text{cyc.} \right], \quad (81)$$

$$\tilde{L}_3(k_1, k_2, k_3)$$

$$= \frac{3}{7} k_{123} \left[ \frac{5}{7} \left[ 1 - \left( \frac{k_1 \cdot k_2}{k_{12}} \right)^2 \right] \left[ 1 - \left( \frac{k_2 \cdot k_3}{k_{23}} \right)^2 \right] - 1 \right] \left[ 1 - 3 \left( \frac{k_1 \cdot k_2}{k_{12}} \right)^2 + 2 \left( \frac{k_1 \cdot k_2}{k_{12}} \right) \left( \frac{k_2 \cdot k_3}{k_{23}} \right) \left( \frac{k_3 \cdot k_1}{k_{31}} \right) \right]$$

$$+ \frac{3}{7} k_{123} \times \left( \frac{1}{k_{12}} \right)^2 \left( \frac{1}{k_{23}} \right)^2 \left[ 1 - \left( \frac{k_2 \cdot k_3}{k_{23}} \right)^2 \right]. \quad (82)$$

While the transverse part of the last line of Eq. (82) does not contribute to evaluating the power spectra up to one-loop order in perturbation theory [19], this part does contribute in more general occasions, such as two-loop corrections of power spectrum, one-loop corrections of bispectrum, etc.\footnote{These properties can be shown, e.g., by the diagrammatic method of iPT [17].}

In the above, weak dependencies on the time $t$ in the kernels are neglected [8, 53]. Taking into account the weak dependencies is also possible [53, 60–62], while the expressions are more complicated. In Ref. [53], for example, complete expressions of the displacement kernels of LPT up to the seventh order are explicitly given, together with a general way of recursively deriving the kernels including weak dependencies on the time in general cosmology and subleading growing modes. This method is also generalized to obtain kernels of LPT in modified theories of gravity [63].

One of the benefits of the Lagrangian picture is that redshift-space distortions are relatively easy to incorporate into the theory. The displacement vector in redshift space $\Psi$ is obtained from a linear mapping of that in real space by $\Psi = \Psi + H^{-1}(\hat{z} \cdot \Psi) \hat{z}$, where $\Psi = \partial \Psi / \partial t$ is the time derivative of the displacement field, $\hat{z}$ is a unit vector directed to the line of sight, $H = \dot{a}/a$ is the time-dependent Hubble parameter and $a(t)$ is the scale factor. A displacement kernel in redshift space $L^*_n$ is simply related to the kernel in real space in the same order by a linear mapping [64]

$$L_n \rightarrow L^*_n = L_n + nf(\hat{z} \cdot L_n) \hat{z}, \quad (83)$$

where $f = d \ln D / d \ln a = \dot{D} / HD$ is the linear growth rate, $D(t)$ is the linear growth factor. Throughout this paper, the distant-observer approximation is assumed in redshift space and the unit vector $\hat{z}$ denotes the line-of-sight direction. The expressions of Eqs. (74)–(78) apply as well in redshift space when the displacement kernels $L_n$ are replaced by those in redshift space $L^*_n$. 

B. Symmetries

1. Complex conjugate

We assume the Cartesian tensor field of Eq. (50) is a real tensor:

$$F^{e}_{X_{li_{i_2}}}(x) = F_{X_{li_{i_2}}}(x),$$  

(84)

as it should be for physically observable quantities. For the spherical decomposition of the traceless part, $F_{Xlm}(x)$, Eqs. (29), (43), (48) and the above equation indicate

$$F^{e}_{Xlm}(x) = g^{mn}_{(l)} F_{Xlm}(x) = F_{m}^{(m)}(x),$$  

(85)

and thus the Fourier transform of Eq. (63) satisfies

$$\tilde{F}^{e}_{Xlm}(k) = g^{mn}_{(l)} \tilde{F}_{Xlm}(-k) = \tilde{F}_{m}^{(m)}(-k).$$  

(86)

That is, the complex conjugate of the irreducible tensor field in Fourier space raises the azimuthal index, and inverts the sign of the wave vector in the argument. The linear density contrast $\delta_{i}(x)$ is also a real field, and its Fourier transform satisfies $\tilde{\delta}_{i}(k) = \delta_{i}(-k)$. Therefore, the renormalized bias functions satisfy

$$c^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) = (-1)^{l} g^{mn}_{(l)} c^{(n)}_{Xlm}(-k_{1}, \ldots, -k_{n})$$  

$$= (-1)^{l} F_{m}^{(m)}(-k_{1}, \ldots, -k_{n}).$$  

(87)

Similarly, the tensor propagator of Eq. (67) satisfies

$$\tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) = (-1)^{l} g^{mn}_{(l)} \tilde{\Gamma}^{(n)}_{Xlm}(-k_{1}, \ldots, -k_{n})$$  

$$= (-1)^{l} \tilde{\Gamma}_{m}^{(m)}(-k_{1}, \ldots, -k_{n}).$$  

(88)

In redshift space, the lines of sight also rotate, $z \rightarrow z' = R^{-1}z$, although simultaneous rotations of wave vectors and lines of sight do not change the scalar products in the propagators, as in Eqs. (74)–(83). Thus we have

$$\tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}; \tilde{z}) \equiv \tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}; \tilde{z})$$  

$$= \tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}; \tilde{z}) D^{mn}_{(l)(m)}(R).$$  

(95)

2. Rotation

Under the passive rotation of Eq. (20), vector components $V_{i}$ generally transform as

$$V_{i} \rightarrow V'_{i} = (R^{-1})_{ij} V_{j} = V_{j} R_{ij}.$$  

(90)

We denote the rotation of the position and wave vector components as, e.g., $x \rightarrow x' = R^{-1}x$ and $k \rightarrow k' = R^{-1}k$, respectively. This notation does not mean the physical rotations of vectors and does mean the rotation of components, $x_{i} \rightarrow x'_{i} = x_{i} R_{ij}$ and $k_{i} \rightarrow k'_{i} = k_{i} R_{ij}$, respectively, as a result of the passive rotation of Cartesian basis, Eq. (20). The Cartesian tensor field of Eq. (50) transforms as

$$F^{e}_{X_{li_{i_2}}}(x) \rightarrow F^{e}_{X_{li_{i_2}}}(x') = F_{X_{lj_{j_2}}}(x) R_{j_{j_1}} R_{j_{j_2}} \cdots,$$  

(91)

and correspondingly the irreducible tensor of rank-1 transforms as

$$F_{Xlm}(x) \rightarrow F'_{Xlm}(x') = F_{Xlm}(x) D^{mn}_{(l)(m)}(R).$$  

(92)

The corresponding transformation in Fourier space, Eq. (63), is given by

$$\tilde{F}_{Xlm}(k) \rightarrow \tilde{F}'_{Xlm}(k') = \tilde{F}_{Xlm}(k) D^{mn}_{(l)(m)}(R),$$  

(93)

because the volume element $d^{3}x$ and the inner product $k \cdot x$ are invariant under the rotation.

The same transformations apply to the tensor fields in Lagrangian space, $\tilde{F}^{e}_{Xlm}(q)$. Therefore, the renormalized bias functions, Eq. (71), transform as

$$c^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) \rightarrow c^{(n)}_{Xlm}(k'_{1}, \ldots, k'_{n})$$  

$$= c^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) D^{mn}_{(l)(m)}(R).$$  

(94)

Similarly, the propagators of Eq. (67) transform as

$$\tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) \rightarrow \tilde{\Gamma}^{(n)}_{Xlm}(k'_{1}, \ldots, k'_{n})$$  

$$= \tilde{\Gamma}^{(n)}_{Xlm}(k_{1}, \ldots, k_{n}) D^{mn}_{(l)(m)}(R).$$  

(95)

3. Parity

Next, we consider the property of parity symmetry. Keeping the left-handed coordinates system, we consider an active parity transformation of the physical system, instead of flipping the axes of the coordinates system. With the active parity transformation, the field values at a position $x$ are mapped into those at $-x$, and the functional form of the tensor field is transformed as

$$F_{X_{li_{i_2}}}(x) \rightarrow F'_{X_{li_{i_2}}}(x) = (-1)^{p_{X}} F_{X_{li_{i_2}}}(-x),$$  

(97)

where $p_{X} = 0$ for ordinary tensors and $p_{X} = 1$ for pseudotensors. The angular momentum is a typical example of a pseudotensor of rank-1. The corresponding transformation for the irreducible tensor is given by

$$F_{Xlm}(x) \rightarrow F'_{Xlm}(x) = (-1)^{p_{X}} F_{Xlm}(-x)$$  

(98)

in configuration space, and

$$\tilde{F}_{Xlm}(k) \rightarrow \tilde{F}'_{Xlm}(k) = (-1)^{p_{X}} \tilde{F}_{Xlm}(-k)$$  

(99)
in Fourier space. The same applies to tensor fields in Lagrangian space. Accordingly, the renormalized bias functions transform as

\[
\tilde{c}^{(n)}_{Xlm}(k_1, \ldots, k_n) \rightarrow \tilde{c}^{(n)}_{Xlm}(k_1, \ldots, k_n) = (-1)^{p+1} \tilde{c}^{(n)}_{Xlm}(-k_1, \ldots, -k_n),
\]

for the linear density field transforms as \(\tilde{\delta}_L(k) \rightarrow \tilde{\delta}_L(-k)\). Similarly, the propagators transform as

\[
\tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n) \rightarrow \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n) = (-1)^{p+1} \tilde{\Gamma}^{(n)}_{Xlm}(-k_1, \ldots, -k_n).
\]

In redshift space, we have

\[
\tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n; \hat{z}) \rightarrow \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n; \hat{z}) = (-1)^{p+1} \tilde{\Gamma}^{(n)}_{Xlm}(-k_1, \ldots, -k_n; \hat{z}).
\]

4. Interchange of arguments

An obvious symmetry of the renormalized bias functions and propagators is that they are invariant under permutation of the wave vectors in the argument. For the renormalized bias function, we have

\[
c^{(n)}_{Xlm}(k_1, \ldots, k_{\sigma(n)}, \ldots, k_n) = c^{(n)}_{Xlm}(k_1, \ldots, k_n),
\]

where \(\sigma \in S_n\) is a permutation of the symmetric group \(S_n\) of order \(n\). Any permutation can be realized by a series of interchange operations of adjacent arguments:

\[
c^{(n)}_{Xlm}(k_1, \ldots, k_i, k_{i+1}, \ldots, k_n) = c^{(n)}_{Xlm}(k_1, \ldots, k_{i+1}, k_i, \ldots, k_n),
\]

with arbitrary \(i = 1, \ldots, n - 1\). Similarly, we have

\[
\tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_i, k_{i+1}, \ldots, k_n) = \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_{i+1}, k_i, \ldots, k_n),
\]

for the propagators in real space, and

\[
\tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_i, k_{i+1}, \ldots, k_n; \hat{z}) = \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_{i+1}, k_i, \ldots, k_n; \hat{z}),
\]

for the propagators in redshift space.

IV. ROTATIONALLY INVARIANT FUNCTIONS

Throughout this paper, we assume that the Universe is statistically isotropic. In redshift space, the statistical quantities such as the power spectrum and correlation functions are apparently anisotropic and the line of sight is a special direction. However, when we rotate the whole system including the direction to the line of sight, \(\hat{z}\), rotations of the coordinates system should not alter the physical degrees of freedom of the statistics. In order to explicitly represent the property in our formalism, we introduce invariant functions for the renormalized bias functions and propagators so that the invariant functions contain only physical degrees of freedom which do not depend on a particular coordinates system of arbitrary choice.

A. Renormalized bias functions of tensor fields

In the following of this paper, we mostly work in Fourier space. For notational simplicity, we omit the tildes above variables in Fourier space, and denote \(\delta_L(k), F^L_{Xlm}(k), \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n)\) etc. instead of \(\tilde{\delta}_L(k), \tilde{F}^L_{Xlm}(k), \tilde{\Gamma}^{(n)}_{Xlm}(k_1, \ldots, k_n)\), respectively, as long as they are not confusing.

One of the essential parts of iPT is the introduction of the renormalized bias function defined by Eq. (71). They characterize complicated fully nonlinear processes in the formation of astronomical objects, such as various types of galaxies and clusters, etc. If these complicated processes are described by some kind of models and the function \(F^L_{Xlm}\) in Lagrangian space is analytically given in terms of the linear density field \(\delta_L\), then we can analytically calculate the renormalized bias function \(c^{(n)}_{Xlm}\) according to the model. In the case of the number density of halos, the Press-Schechter model and its extensions are applied to the concrete calculations along this line [18–20]. When this kind of models is not known, the renormalized bias functions have infinite degrees of freedom which is difficult to calculate from the fundamental level. However, the rotational symmetry considered in the previous subsection places some constraints on the functional form of the renormalized bias functions as we show below.

We consider the renormalized bias functions given in the form of Eq. (71). In the simplified notation without tildes,

\[
c^{(n)}_{Xlm}(k_1, \ldots, k_n) = (-i)^n (2\pi)^2 \int \frac{d^3 \delta \hat{k}}{(2\pi)^3} \left\{ \frac{\delta^4 F^L_{Xlm}(\hat{k})}{\delta \delta_L(\hat{k})} \right\}.
\]

The angular dependencies of the wave vectors in the arguments can be decomposed in a series expansion with spherical harmonics with Racah’s normalization, Eq. (17), as

\[
c^{(n)}_{Xlm}(k_1, \ldots, k_n) = \sum_{l_1, \ldots, l_n} c^{(n)}_{Xlm, l_1, \ldots, l_n} (k_1, \ldots, k_n) \times C_{l_1, m_1}(\hat{k}_1) \cdots C_{l_n, m_n}(\hat{k}_n),
\]

where \(k_i = |k_i|\) are absolute values of the wave vectors, and \(\hat{k}_i = (\theta_i, \phi_i)\) represents the spherical coordinates for the direction of wave vectors \(k_i\). The repeated indices \(m_1, m_1, \ldots, m_n\) in Eq. (108) are summed over according to the Einstein summation convention as in the previous section. Because of the orthonormality relation for the spherical harmonics, Eq. (B6), the expansion of Eq. (108) can be inverted and we have

\[
c^{(n)}_{Xlm, l_1, \ldots, l_n} (k_1, \ldots, k_n) = \sum_{l_1, \ldots, l_n} c^{(n)}_{Xlm, l_1, \ldots, l_n} (k_1, \ldots, k_n) \times C_{l_1, m_1}(\hat{k}_1) \cdots C_{l_n, m_n}(\hat{k}_n),
\]

where \(d^2 \hat{k}_i = \sin \theta_i d\theta_i d\phi_i\) represents the angular integration of the wave vector \(k_i\). Under the passive rotation of Eq. (20), the function of Eq. (109) covariantly transforms as

\[
c^{(n)}_{Xlm, l_1, \ldots, l_n} (k_1, \ldots, k_n) \rightarrow D^{(n)}_{l_1, m_1} (R) \cdots D^{(n)}_{l_n, m_n} (R),
\]

\[
\times c^{(n)}_{Xlm, l_1, \ldots, l_n} (k_1, \ldots, k_n).
\]
as obviously indicated by the lower positions of the azimuthal indices \( m, m_1, \ldots, m_\ell \).

For the statistically isotropic Universe, the functions of Eq. (109) are rotationally invariant as these functions are given by ensemble averages of the field variables as seen by Eqs. (107) and (109), and they should not depend on the choice of coordinates system \( \mathbf{e} \), to describe wave vectors \( k_1, \ldots, k_\ell \) of the renormalized bias functions \( c^{(1)}_{\ell m}(k_1, \ldots, k_\ell) \). Therefore, they are invariant under the transformation of Eq. (110), and we have

\[
c^{(n)}_{\ell m_1 m_2 \ldots m_\ell} (k_1, \ldots, k_\ell) = c^{(n)}_{\ell m_1 m_2 \ldots m_\ell} (k_1, \ldots, k_\ell)
\]

\[
\times \frac{1}{8\pi^2} \int [dR] D^{m_\ell}_{(\ell) m_\ell} (R) D^{m_1}_{(1) m_1} (\mathbf{R}) \cdots D^{m_3}_{(3) m_3} (\mathbf{R}),
\]

where

\[
\int [dR] \cdots = \int_0^{2\pi} d\alpha \int_0^\pi d\beta \sin \beta \int_0^{2\pi} dy \cdots
\]

represents the integrals over Euler angles (\( \alpha \beta \gamma \)) of the rotation \( R \).

A product of two Wigner’s rotation matrices of the same rotation reduces to a single matrix through vector-coupling coefficients as [46]

\[
D^{m_\ell}_{(\ell) m_\ell} (R) D^{m_1}_{(1) m_1} (\mathbf{R}) = \sum_{l,m,m'} (2l+1) \left( \begin{array}{ccc} l_1 & l_2 & 1 \\ m_1 & m_2 & m' \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & 1 \\ m_1 & m_2 & l \end{array} \right) D^{m'}_{l m l} (\mathbf{R}),
\]

using Wigner’s 3-j-symbol. The complex conjugate of the rotation matrix is given by \( D^{m_\ell}_{m_\ell (\ell)} (R) = g_{l(1)} g_{m_2 m_2'(1)} D^{m_\ell}_{m_\ell (\ell)} (R) \) using our metric tensor for spherical basis.

In the following, we use a simplified notation for the 3-j-symbols,

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} \equiv \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right),
\]

which is nonzero only when \( m_1 + m_2 + m_3 = 0 \). We consider that the azimuthal indices can be raised or lowered by spherical metric, e.g.,

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} = g_{l(1)} g_{m_2 m_3(1)} (l_1 l_2 l_3)_{m_1 m_2 m_3}
\]

\[
= (-1)^{m_2+m_3} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & -m_2 & -m_3 \end{array} \right),
\]

and so forth. The properties and formulas for Wigner’s 3-j-symbol with our notation, which are repeatedly used in this paper below and in subsequent papers in the series, are summarized in Appendix C. With our notation of the 3-j-symbol, Eq. (113) is equivalent to an expression,

\[
D^{m_\ell}_{(\ell) m_\ell} (R) D^{m_1}_{(1) m_1} (\mathbf{R}) = (-1)^{l_1+l_2} \sum_{l=0}^\infty (-1)^l (2l+1)
\]

\[
\times (l_1 l_2 l_3)_{m_1 m_2 m_3} (l_1 l_2 l_3)_{m_1 m_2 m_3} D^{m'}_{l m l} (\mathbf{R}),
\]

in which the rotational covariance is explicit. We use a property of Eq. (C2) for the 3-j-symbol to derive the above equation.

Consecutive applications of Eq. (116) to Eq. (111), the dependence on the rotation \( R \) in the integrand on the rhs can be represented by a linear combination of products of less than or equal to three rotation matrices and the coefficients of the expression are given by products of 3-j-symbols. The averages of products of rotation matrices of three or fewer are given by

\[
\frac{1}{8\pi^2} \int [dR] D^{m_\ell}_{(\ell) m_\ell} (R) = \delta_{\ell 0} \delta_{m_\ell 0} \delta_{m_\ell 0},
\]

\[
\frac{1}{8\pi^2} \int [dR] D^{m_\ell}_{(\ell) m_\ell} (R) D^{m_1}_{(1) m_1} (\mathbf{R}) = \frac{\delta_{l_1 l_2}}{2l_1 + 1} g^{(1)}_{m_1 m_2},
\]

\[
\frac{1}{8\pi^2} \int [dR] D^{m_\ell}_{(\ell) m_\ell} (R) D^{m_1}_{(1) m_1} (\mathbf{R}) D^{m_2}_{(2) m_2} (\mathbf{R}) = (-1)^{l_1+l_2+l_3} (l_1 l_2 l_3)_{m_1 m_2 m_3},
\]

Equations (117) and (118) are also derived from Eq. (119) noting \( D^{0}_{00} (R) = 1 \) and Eqs. (C8). Also, Eqs. (118) and (119) are derived from further applications of Eq. (116) to reduce the number of products and finally using Eq. (117).

As a result of the above procedure, averaging over the rotation \( R \), the dependence of the indices of \( m \)’s in Eq. (111) are all represented by spherical metrics and 3-j-symbols. First, we specifically derive the results for lower orders of \( n \). For \( n = 0 \), using Eqs. (111) and (117), we derive

\[
\epsilon^{(0)}_{Xlm} = \epsilon^{(0)}_{X00} \delta_{l 0} \delta_{m 0}.
\]

This result is trivial because the corresponding function of Eq. (107) for \( n = 0 \) does not depend on the direction of the wave vector. For notational simplicity, we simply denote \( \epsilon^{(0)}_{X} \equiv \epsilon^{(0)}_{X00} \), and we have

\[
\epsilon^{(0)}_{Xlm} = \delta_{l 0} \delta_{m 0} \epsilon^{(0)}_{X}.
\]

For a scalar function, \( F_X \), the coefficient \( \epsilon^{(0)}_{X} \) just corresponds to the mean value, \( \epsilon^{(0)}_{X} = \langle F_X \rangle = (\langle F_X \rangle)^{\gamma(0)} = \langle F_X \rangle \), as seen from Eqs. (7) and (32). Because of the assumption that the field \( F_X \) is a real field, the coefficient \( \epsilon^{(0)}_{X} \) has a real value:

\[
\epsilon^{(0)}_{Xlm} = \epsilon^{(0)}_{Xlm}.
\]

The parity transformation is given by

\[
\epsilon^{(0)}_{Xlm} \rightarrow (-1)^{l_1+l_2+l_3} \epsilon^{(0)}_{Xlm}.
\]

If the Universe is statistically invariant under the parity, we should have \( p_X = 0 \), because the mean value of a scalar field \( \langle F_X \rangle \) vanishes if the field is a pseudoscalar in the Universe with parity symmetry.

For \( n = 1 \), using Eqs. (111) and (118), we derive

\[
\epsilon^{(1)}_{Xlm} (k) = \frac{\delta_{l 1}}{2l_1 + 1} g^{(1)}_{lm} \epsilon^{(1)}_{Xlm} (k),
\]
and therefore both indices $m, m_1$ appear only in the metric $g^{(2)}_{lm}$. Defining a rotationally invariant function,

$$c_X^{(1)}(k) \equiv \frac{(-1)^{l}}{\sqrt{2l+1}} g^{mm_1} c_{Xlm,m_1}(k),$$  \hspace{1cm} (125)

one finds that Eq. (124) reduces to a simple form,

$$c_{Xlm,m_1}^{(1)}(k) = \frac{(-1)^{l}}{\sqrt{2l+1}} \delta_{m,m_1} c^{(1)}(k).$$  \hspace{1cm} (126)

Substituting this expression into Eq. (108) with $n = 1$, we derive

$$c_{Xlm}^{(1)}(k) = \frac{(-1)^l}{\sqrt{2l+1}} c_X^{(1)}(k) C_{lm}(\hat{k}).$$  \hspace{1cm} (127)

The invariant coefficient $c_X^{(1)}(k)$ can also be read off from an expression of the renormalized bias function $c_{Xlm}^{(1)}(k)$ in which the angular dependence on $k$ is expanded by the spherical harmonics $C_{lm}(\hat{k})$. One can also explicitly invert the above equation by using the orthonormality relation of spherical harmonics.

Combining the properties of complex conjugate for the spherical harmonics, Eq. (B3), and for the renormalized bias function, Eq. (87) with $n = 1$, we have

$$c^{(1)*}_X(k) = c^{(1)}_X(k),$$  \hspace{1cm} (128)

and thus the reduced function is a real function. Combining the properties of parity for the spherical harmonics, Eq. (B2), and for the renormalized bias function, Eq. (100) with $n = 1$, we have

$$c^{(1)}_X(k) \overset{\text{F}}{\to} (-1)^{p_X} c^{(1)}_X(k).$$  \hspace{1cm} (129)

If the Universe is statistically invariant under the parity, the function is nonzero only when $p_X = 0$. This means that pseudotensors do not have the first-order renormalized bias function in the presence of parity symmetry. This property is partly because we consider the case that only scalar perturbations $\delta_L$ are responsible for the properties of tensor fields. For a typical example that is a manifestation of the last conclusion, angular momenta of galaxies are not generated by the first-order effect in linear perturbation theory [65].

For $n = 2$, using Eqs. (111) and (119), we derive

$$c_{Xlm,m_1,m_2}^{(2)}(k_1, k_2) = (l_1 l_2)_m^{m_1 m_2} c_{Xlm_1,l_2}(k_1, k_2),$$  \hspace{1cm} (130)

where we define a rotationally invariant function,

$$c_{Xlm_1,l_2}^{(2)}(k_1, k_2) \equiv (-1)^{l_1+1+l_2} (l_1 l_2)_m^{m_1 m_2} c_{Xlm,m_1,m_2}^{(2)}(k_1, k_2).$$  \hspace{1cm} (131)

The above Eq. (130) essentially corresponds to the Wigner-Eckart theorem of representation theory and quantum mechanics [48], and the dependences on azimuthal indices $m, m_1, m_2$ appear only through $3j$-symbols. Substituting Eq. (130) into Eq. (108) with $n = 2$, we derive

$$c_{Xlm}^{(2)}(k_1, k_2) = \sum_{l_1,l_2} c_{Xlm_1,l_2}^{(2)}(k_1, k_2) (l_1 l_2)_m^{m_1 m_2} \times C_{lm_1}(\hat{k}_1) C_{lm_2}(\hat{k}_2).$$  \hspace{1cm} (132)

The last expression contains the bipolar spherical harmonics [47] which is defined by Eq. (B9) in Appendix B with our notation. Rather than using the conventional notation for the bipolar spherical harmonics, $\{Y_{l_1}(\hat{k}_1) \otimes Y_{l_2}(\hat{k}_2)\}_{lm}$, we introduce a notation with convenient normalization of the bipolar spherical harmonics,

$$X_{lm}^{(2)}(\hat{k}_1, \hat{k}_2) \equiv \frac{(-1)^{l} 4\pi}{\sqrt{\ell_1! \ell_2! \ell_3!}} \{Y_{l_1}(\hat{k}_1) \otimes Y_{l_2}(\hat{k}_2)\}_{lm},$$  \hspace{1cm} (133)

where we employ a notation,

$$\{L\} \equiv 2L + 1.$$  \hspace{1cm} (134)

for a non-negative integer $L$ throughout the paper henceforth, as this type of factor repeatedly appears later. Please do not confuse with the curly brackets of the bipolar spherical harmonics, in which the argument is not a non-negative integer. The bipolar spherical harmonics satisfy the orthonormality relation, Eq. (B14), and product formula, Eq. (B16), as shown in Appendix B. With the above notation, Eq. (132) is concisely given by

$$c_{Xlm}^{(2)}(k_1, k_2) = \sum_{l_1,l_2} c_{Xlm_1,l_2}^{(2)}(k_1, k_2) X_{lm}^{(2)}(\hat{k}_1, \hat{k}_2).$$  \hspace{1cm} (135)

Combining the properties of complex conjugate for the bipolar spherical harmonics, Eq. (B13), and for the renormalized bias function, Eq. (87) with $n = 2$, we have

$$c_{Xlm_1,l_2}^{(2)}(k_1, k_2) \overset{\text{F}}{\to} (-1)^{p_X + l_1 + l_2} c_{Xlm_1,l_2}^{(2)}(k_1, k_2).$$  \hspace{1cm} (136)

and thus the reduced function is a real function. Combining the properties of parity for the bipolar spherical harmonics, Eq. (B12), and for the renormalized bias function, Eq. (100) with $n = 2$, we have

$$c_{Xlm_1,l_2}^{(2)}(k_1, k_2) \overset{\text{F}}{\to} (-1)^{p_X + l_1 + l_2} c_{Xlm_1,l_2}^{(2)}(k_1, k_2).$$  \hspace{1cm} (137)

If the Universe is statistically invariant under the parity, the function is nonzero only when $p_X + l_1 + l_2 = 0$. The interchange symmetry of Eq. (104) in this case is given by $c_{Xlm_1,l_2}^{(2)}(k_1, k_2) = c_{Xlm_1,l_2}^{(2)}(k_2, k_1)$. Because of the symmetry of $3j$-symbols of Eq. (C4), the bipolar spherical harmonics satisfy

$$X_{lm}^{(2)}(\hat{k}_1, \hat{k}_2) = (-1)^{l_1+l_2} X_{lm}^{(2)}(\hat{k}_2, \hat{k}_1),$$  \hspace{1cm} (138)

and therefore the invariant function satisfies

$$c_{Xlm_1,l_2}^{(2)}(k_1, k_2) = (-1)^{l_1+l_2} c_{Xlm_1,l_2}^{(2)}(k_2, k_1).$$  \hspace{1cm} (139)

For $n = 3$, using Eqs. (111), (116) and (119), we derive

$$c_{Xlm,m_1,m_2,m_3}^{(3)}(k_1, k_2, k_3) = \sum_{l_1,l_2,l_3} (-1)^{l_1} \sqrt{[l]} (l_1 l_2)_m^{m_1 m_2} \times (L l_2 l_3)_M^{m_1 m_2 m_3} c_{Xlm_1,l_2,l_3}^{(3)}(k_1, k_2, k_3),$$  \hspace{1cm} (140)

where we define a rotationally invariant function,

$$c_{Xlm_1,l_2,l_3}^{(3)}(k_1, k_2, k_3) \equiv (-1)^{l_1+1+l_2+1+l_3+1} c_{Xlm_1,l_2,l_3}^{(3)}(k_1, k_2, k_3).$$  \hspace{1cm} (141)
Substituting Eq. (140) into Eq. (108) with \( n = 3 \), we derive
\[
\begin{align*}
  c^{(3)}_{\ell m}(k_1, k_2, k_3) &= \sum_{l_1, l_2, l_3} (-1)^l \sqrt{[L]}
  \times c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3) (L l_1)_{m_1} M L_2(l_2)_{m_2} M L_3(l_3)_{m_3}
  \times C_{l_1 m_1}(\hat k_1) C_{l_2 m_2}(\hat k_2) C_{l_3 m_3}(\hat k_3).
\end{align*}
\]
(142)

The last expression contains the triproperal spherical harmonics [47]. Similar to Eq. (133) of bipolar spherical harmonics, we introduce a notation for the triproperal spherical harmonics,
\[
X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_2, \hat k_3) = (-1)^l (4\pi)^{3/2}
\times \{Y_{l_1} (\hat k_1) \otimes \{Y_{l_2} (\hat k_2) \otimes Y_{l_3} (\hat k_3)\}\}_{L}\text{l m}.
\]
(143)

The explicit form of the above harmonics is defined by Eq. (B17), and their orthonormality relation, Eq. (B21) and product formula, Eq. (B22) are given in Appendix B with our notations for convenience. Thereby Eq. (142) is concisely represented as
\[
X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_2, \hat k_3) = \sum_{l_1, l_2, l_3} c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3) X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_2, \hat k_3).
\]
(144)

Combining the properties of complex conjugate for the triproperal spherical harmonics, Eq. (B20), and for the renormalized bias function, Eq. (87) with \( n = 3 \), we have
\[
c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3) = c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3),
\]
(145)
and thus the reduced function is a real function. Combining the properties of parity for the triproperal spherical harmonics, Eq. (B19), and for the renormalized bias function, Eq. (100) with \( n = 3 \), we have
\[
c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3) = (-1)^{l_1 + l_2 i_s + i_t} c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3).
\]
(146)

If the Universe is statistically invariant under the parity, the function is nonzero only when \( p_X + l_1 + l_2 + l_3 = \) even. The interchange symmetries of Eq. (104) in terms of invariant functions in this case are derived from properties of interchange arguments in triproperal spherical harmonics. For an interchange of the last two arguments, \( 2 \leftrightarrow 3 \), we have
\[
X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_2, \hat k_3) = (-1)^{l_1 + l_2 i_s + l_3 i_t} X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_3, \hat k_2),
\]
(147)
just similarly in the case of Eq. (138). For an interchange of the first two arguments, \( 1 \leftrightarrow 2 \), however, a recoupling of the 3j-symbols appears, and we have
\[
X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_1, \hat k_2, \hat k_3) = (-1)^{l_1 l_2} \sum_{L' \ell_L} \{L_L \} \begin{pmatrix} 1 & l & L \\ l_1 & l_2 & L \end{pmatrix} \times X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3} (\hat k_2, \hat k_1, \hat k_3),
\]
(148)
where the factor in front of the triproperal spherical harmonics on the rhs is a 6j-symbol defined by Eq. (C15) or equivalently by Eq. (C16). Useful properties of 6j-symbols are summarized in Eqs. (C17), (C18) and (C24) of Appendix C. The above relation of Eq. (148) is derived by applying a sum rule of the 3j-symbols, Eq. (C24), to the definition of the triproperal spherical harmonics, Eq. (B17). Correspondingly, the interchange symmetries of invariant coefficients of Eq. (144) are given by
\[
c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3) = (-1)^{l_1 l_2 + l_2 l_3} c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_1, k_2, k_3),
\]
(149)
and
\[
c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_2, k_1, k_3) = (-1)^{l_1 i_s + l_2 i_t} \sum_{L' \ell_L} \{L_L \} \begin{pmatrix} 1 & l & L \\ l_1 & l_2 & L \end{pmatrix} \times c^{(3)LL}_{\ell_1\ell_2\ell_3}(k_2, k_1, k_3).
\]
(150)

Combining Eqs. (149) and (150), all the other symmetries concerning the permutation of \( (1, 2, 3) \) in the subscripts of the arguments are straightforwardly obtained. A similar kind of considerations of the interchange symmetry in a formulation of angular trispectrum of the cosmic microwave background is found in Ref. [66] to enforce permutation symmetry in the angular trispectrum.

In the same way, one derives the results for general orders \( n > 3 \). The expansion coefficient of order \( n \) is given by
\[
c^{(n)}_{\ell_1 l_1 - l_n} X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}(k_1, \ldots, k_n) = \sum_{k_{2n-1}, \ldots, k_1} (-1)^{l_1 + l_2} \sqrt{[L_2] \cdots [L_{n-1}]} (L_1 L_2 \ell_1 M_{m_1 M_{l_1}})
\times (L_2 L_3 \ell_2 M_{M_{l_2 M_{m_2}}}) \cdots (L_{n-2} L_{n-1} \ell_{n-1} M_{M_{m_{n-2} M_{l_{n-1}}}})
\times (L_{n-1} L_n \ell_n M_{M_{m_n M_{l_n}}}) c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(k_1, \ldots, k_n),
\]
(151)
where we define
\[
c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(k_1, \ldots, k_n) \equiv (-1)^{l_1 l_2 + l_2 l_3} \times
\times (-1)^{l_1 l_2 + l_2 l_3} \times \cdots \times \sqrt{[L_2] \cdots [L_{n-1}]} \times (L_1 L_2 \ell_1 M_{m_1 M_{l_1}})
\times (L_2 L_3 \ell_2 M_{M_{l_2 M_{m_2}}}) \cdots (L_{n-2} L_{n-1} \ell_{n-1} M_{M_{m_{n-2} M_{l_{n-1}}}})
\times (L_{n-1} L_n \ell_n M_{M_{m_n M_{l_n}}}) c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(k_1, \ldots, k_n).
\]
(152)
The azimuthal indices of the decomposed renormalized bias functions, Eq. (109) are all represented by combinations of 3j-symbols, and the physical contents of the renormalized bias functions are given by the invariant functions, \( c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(\ldots) \) are represented by the combinations of 3j-symbols. Because of the symmetry of 3j-symbols, we have \( m + m_1 + \cdots + m_n = 0 \), which is a manifestation of the axial symmetry of the rotation around the third axis \( \hat k_3 \).

The renormalized bias function of Eq. (108) is given by
\[
c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(k_1, \ldots, k_n) = \sum_{l_1, \ldots, l_{n-1}} (-1)^{l_1 l_2 + l_2 l_3} \sqrt{[L_2] \cdots [L_{n-1}]} \times c^{(n)}_{X^{(i_l+i_s+i_t)}_{\ell_1 \ell_2 \ell_3}}(k_1, \ldots, k_n)
\times (L_2 L_3 \ell_2 M_{M_{l_2 M_{m_2}}}) \cdots (L_{n-2} L_{n-1} \ell_{n-1} M_{M_{m_{n-2} M_{l_{n-1}}}})
\times (L_{n-1} L_n \ell_n M_{M_{m_n M_{l_n}}}) C^{(n)}_{\ell_1 m_1}(\hat k_1) \cdots C^{(n)}_{\ell_n m_n}(\hat k_n).
\]
(153)
The last expression contains the polypolar spherical harmonics defined by Eq. (B23), and their orthonormality relation, Eq. (B27) and product formula, Eq. (B28) are given in Appendix B with our notations for convenience. Similar to Eqs. (133) and (143) of bipolar and triplor spherical harmonics, we introduce a notation for the polypolar spherical harmonics,

\[ X_{L_1-\ldots-L_n-\lambda m}(\hat{k}_1, \ldots, \hat{k}_n) \equiv \frac{(-1)^l (4\pi)^{n/2}}{\sqrt{l!}!} \left\{ Y_{L_1}(\hat{k}_1) \otimes \left[ \cdots \otimes Y_{L_n}(\hat{k}_n) \right] \right\} , \tag{154} \]

and Eq. (153) is concisely represented as

\[ c_{Xlm}^{(n)}(k_1, \ldots, k_n) = \sum_{l_1,\ldots,l_n, l_1' \ldots l_n'} c_{Xl_1-\ldots-l_n}^{(n)L_1-\ldots-L_n-\lambda m}(k_1, \ldots, k_n) \times X_{L_1-\ldots-L_n-\lambda m}^{l_1-\ldots-l_n}(\hat{k}_1, \ldots, \hat{k}_n). \tag{155} \]

Combining the properties of complex conjugate for the polypolar spherical harmonics, Eq. (B26), and for the renormalized bias function, Eq. (87), we have

\[ c_{Xlm}^{(n)}(k_1, \ldots, k_n) = c_{Xlm}^{(n)}(k_1, \ldots, k_n), \tag{156} \]

and thus the reduced function is a real function. Combining the properties of parity for the polypolar spherical harmonics, Eq. (B25), and for the renormalized bias function, Eq. (100), we have

\[ c_{Xlm}^{(n)}(k_1, \ldots, k_3) \equiv (-1)^{p_k+\ell_1+\cdots+\ell_n} c_{Xlm}^{(n)}(k_1, \ldots, k_3). \tag{157} \]

If the Universe is statistically invariant under the parity, the function is nonzero only when \( p_k + \ell_1 + \cdots + \ell_n = \text{even} \).

The interchange symmetries of Eq. (104) in terms of invariant function in this case are also derived following a similar way in the case of \( n = 3 \). For the interchange of the last two arguments, we have

\[ c_{Xlm}^{(n)}(k_1, \ldots, k_{n-2}, k_n, k_{n-1}) = (-1)^{p_k+\ell_1+\cdots+\ell_n} c_{Xlm}^{(n)}(k_1, \ldots, k_n), \tag{158} \]

and for the interchange of the other adjacent arguments, \( i \leftrightarrow i+1 \), we have

\[ c_{Xlm}^{(n)}(k_1, \ldots, k_i, k_{i+1}, k_i, k_{i+2}, \ldots, k_n) = (-1)^{p_k+\ell_1+\cdots+\ell_n} \sum_{L_1' \cdots L_n'} \left\{ \begin{array}{ccc} L_1 & L_i & L_{i+1} \\ i_1 & i_{i+2} & L' \end{array} \right\} \times c_{Xlm}^{(n)}(k_1, \ldots, k_n), \tag{159} \]

where \( i = 1, \ldots, n-2 \). Combining Eqs. (158) and (159), all the other symmetries concerning the permutation of \( 1, \ldots, n \) in the subscripts of the arguments are straightforwardly obtained.

### B. Propagators of tensor fields

The renormalized bias functions describe the bias mechanisms in the physical space, and thus there is not any corresponding concept of redshift space. However, the apparent clustering in redshift space introduces anisotropy in the propagators in Eulerian space. In the following, we separately consider the propagators in real space and redshift space.

#### 1. Real space

In real space, where the fully rotational symmetry is satisfied even in Eulerian space, the reduced propagators \( \hat{\Gamma}^{(n)}_{Xlm} \) are similarly decomposed into rotationally invariant coefficients as in the case of renormalized bias functions explained in the previous subsection. The derivation of the invariant propagators is mostly parallel to the latter case. In real space, formally replacing the renormalized bias functions \( c_{Xlm}^{(n)} \) with normalized propagators \( \hat{\Gamma}^{(n)}_{Xlm} \) in Sec. IV A is enough to derive the necessary equations of invariant propagators. We summarize some important equations explicitly below for later use. The derivations are exactly the same as those of renormalized bias functions.

The propagator of the first order, \( n = 1 \), is given by

\[ \hat{\Gamma}_{Xlm}^{(1)}(k) = \frac{(-1)^{1}}{\sqrt{2l+1}} \hat{\Gamma}_{Xlm}^{(1)}(\hat{k}), \tag{160} \]

with the invariant propagator \( \hat{\Gamma}_{Xlm}^{(1)}(k) \) which is a real function. If the Universe is statistically invariant under the parity, the first-order propagator is nonzero only when \( p_k = 0 \) and vanishes for pseudotensors. The parity transformation of the invariant function is given by

\[ \hat{\Gamma}_{Xlm}^{(1)}(k) \rightarrow (-1)^{p_k} \hat{\Gamma}_{Xlm}^{(1)}(k). \tag{161} \]

The propagator of second order, \( n = 2 \), is given by

\[ \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2) = \sum_{l_1, l_2} \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2) \hat{\Gamma}_{l_1l_2}^{(2)}(\hat{k}_1, \hat{k}_2), \tag{162} \]

with the invariant propagator \( \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2) \) which is a real function.

The parity transformation of the invariant function is given by

\[ \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2) \rightarrow (-1)^{p_k+p_l} \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2). \tag{163} \]

If the Universe is statistically invariant under the parity, the function is nonzero only when \( p_k + l_1 + l_2 = \text{even} \). The interchange symmetry for the second-order propagator is given by

\[ \hat{\Gamma}_{Xlm}^{(2)}(k_2, k_1) = \frac{(-1)^{p_k+p_l}}{\sqrt{2l_1+1}} \hat{\Gamma}_{Xlm}^{(2)}(k_1, k_2). \tag{164} \]

The propagator of third order, \( n = 3 \), is given by

\[ \hat{\Gamma}_{Xlm}^{(3)}(k_1, k_2, k_3) = \sum_{l_1, l_2, l_3} \hat{\Gamma}_{Xlm}^{(3)}(k_1, k_2, k_3) \times \hat{\Gamma}_{l_1l_2l_3}^{(3)}(\hat{k}_1, \hat{k}_2, \hat{k}_3), \tag{165} \]
with the invariant propagator \( \hat{\Gamma}^{(3)LL}_{Xl_3/l_1/l_2}(k_1, k_2, k_3) \) which is a real function. The parity transformation of the invariant function is given by

\[
\hat{\Gamma}^{(3)LL}_{Xl_1/l_2/l_3}(k_1, k_2, k_3) \xrightarrow{\mathcal{P}} (-1)^{p_X + l_1 + l_2 + l_3} \hat{\Gamma}^{(3)LL}_{Xl_1/l_2/l_3}(k_1, k_2, k_3). \tag{166}
\]

If the Universe is statistically invariant under the parity, the function is nonzero only when \( p_X + l_1 + l_2 + l_3 \) is even. The interchange symmetries for the third-order propagator are given by

\[
\hat{\Gamma}^{(3)LL}_{Xl_1/l_2/l_3}(k_1, k_2, k_3) = (-1)^{l_1 + l_2 + l_3} \hat{\Gamma}^{(3)LL}_{Xl_2/l_3/l_1}(k_1, k_2, k_3), \tag{167}
\]

and

\[
\hat{\Gamma}^{(3)LL}_{Xl_1/l_2/l_3}(k_2, k_1, k_3) = (-1)^{l_1 + l_2 + l_3} \sum_{L} \left( \begin{array}{ccc} l_1 & l & L \\ l_2 & l_3 & L' \end{array} \right) \times \hat{\Gamma}^{(3)LL}_{Xl_1/l_2/l_3}(k_1, k_2, k_3). \tag{168}
\]

The propagators of order \( n > 3 \) are given by

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n) = \sum_{l_1, l_2, \ldots, l_n} \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n) \times \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n), \tag{169}
\]

with the invariant propagators \( \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n) \) which are real functions. The parity transformation is given by

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_3) \xrightarrow{\mathcal{P}} (-1)^{p_X + l_1 + l_2 + \cdots + l_3} \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_3). \tag{170}
\]

The interchange symmetries for the higher-order propagators are given by

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_{n-2}, k_n, k_{n-1}) = (-1)^{l_1 + l_2 + \cdots + l_n} \hat{\Gamma}^{(n-2)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n), \tag{171}
\]

and

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \ldots, k_n) = (-1)^{l_1 + l_2 + \cdots + l_{i-1}} \times \hat{\Gamma}^{(n-2)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n). \tag{172}
\]

Symmetric properties of all the other permutations are obtained by combining the above two cases.

2. Redshift space

In redshift space, the propagators also depend on the direction of the line of sight, \( \hat{z} \). In this paper, we approximately consider the direction of the line of sight to be fixed in the observed space. This approximation is usually called the plane-parallel, or distant-observer approximation, and is known as a good approximation for most practical purposes. In order to keep the rotational covariance in the theory, the line of sight, \( \hat{z} \), is considered to be arbitrarily directed in the coordinates system. In literature, it is a common practice to fix the direction of the line of sight in the third axis \( \hat{e}_3 \) in the distant-observer approximation. In this paper, we do not fix the direction, and we have \( \hat{z} \neq \hat{e}_3 \) in general.

In redshift space, the propagators depend on the direction of the line of sight \( \hat{z} \) as well as the wave vectors. We decompose the directional dependencies on directions of both the line of sight \( \hat{z} \) and wave vectors \( \hat{k}_1, \ldots, \hat{k}_n \) by spherical harmonics in a similar manner of Eq. (108) in the case of renormalized bias functions. However, it turns out to be often useful not to expand all the angular dependencies, and keeping the angular dependencies unexpanded for the sum of the wave vectors, \( \hat{k} = \hat{k}_1 + \cdots + \hat{k}_n \). The sum of wave vectors corresponds to “the external wave vector” [17], which is not integrated over in the nonlinear spectra calculated by the perturbation theory. Therefore, the amplitude \( k = |\hat{k}| \) and directional cosine with respect to the line of sight \( \mu = \hat{z} \cdot \hat{k} \) can be fixed in the calculation of propagators in redshift space. Some parts of the directional dependencies of the propagators on the line of sight \( \hat{z} \) and the wave vectors \( \hat{k}_1, \hat{k}_2, \ldots \), which is included in the parameters \( k \) and \( \mu \) are optionally not need to be expanded in the spherical harmonics. While it is sometimes required to expand every angular dependency into spherical harmonics (e.g., Paper IV), the calculations are more complicated than keeping the dependencies unexpanded in \( k \) and \( \mu \). In the following, we consider the general case that the propagators conveniently have an explicit dependence on \( k \) and \( \mu \).

The expansion is given by

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n, \hat{z}, k, \mu) = \sum_{l_1, l_2, \ldots, l_n} \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n) \times C^*_{l_1} \hat{C}_{l_1} \times \cdots \times C^*_{l_n} \hat{C}_{l_n}. \tag{173}
\]

As the parameters \( k \) and \( \mu \) are functions of \( \hat{z} \) and \( k_1, \ldots, k_n \), the above expansion is not unique in general. Nevertheless, if which parts of the dependencies on the parameters \( k \) and \( \mu \) are consistently fixed in the expression on the lhs of Eq. (173) throughout, the expansion is unique, at least artificially. Using the expansion of Eq. (173), deriving the invariant propagators in redshift space is quite similar to the previous cases of renormalized bias functions and propagators in real space. Comparing Eq. (173) with Eq. (109), the similarity is apparent, while there are additional dependencies on the direction to the line of sight, \( \hat{z} \), and also possible dependencies on \( k \) and \( \mu \). Bearing these differences in mind, we apply the similar derivation in Sec. IV A to the present case below.

The transformation under the passive rotation of Eq. (20) is given by

\[
\hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n, k, \mu) \xrightarrow{\mathcal{R}} \hat{\Gamma}^{(n)LL}_{Xl_1/l_2/l_3}(k_1, \ldots, k_n, k, \mu) D^m_{\ell_1 m_1} (R) \times D^m_{\ell_2 m_2} (R) D^m_{\ell_3 m_3} (R) \cdots D^m_{\ell_n m_n} (R). \tag{174}
\]
When we rotate the direction of the line of sight together under the rotation of the coordinate system in the Universe which satisfies the cosmological principle. Thus we have

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k_1, \ldots, k_n; k, \mu)) = \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k_1, \ldots, k_n; k, \mu)) \left( \frac{1}{8\pi^2} \int [dR] D_{(1)mm}^m \times D_{(1)mm}^{m_0} \cdots D_{(1)mm}^{m_n} \right). \quad (175)
$$

Following the same procedure below Eq. (111) in the case of renormalized bias functions, one can represent the dependence of propagators in redshift space by spherical metric, 3j-symbols, and invariant variables. The main difference is that an angular dependence on the line of sight and dependencies on $k$ and $\mu$ also appear in the expansion.

For the propagator of first order, $n = 1$, we have $k = k_1$ and one can omit the dependence on $k_1$. Thus we have

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) = (I \z_1^{(1)})_{mm,m_1} ^{(1)} \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)), \quad (176)
$$

with

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) \equiv (-1)^{s \z_1^{(1)} + l} \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)). \quad (177)
$$

The propagator is given by

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) = \sum_{l,m} \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) X_{lm}^{(1)}(\hat{z}, \hat{k}). \quad (178)
$$

Combining the properties of complex conjugate for the bipolar spherical harmonics, Eq. (B13), and for the propagator in redshift space, Eq. (89) with $n = 1$, we have

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) = (-1)^{\z_1^{(1)} + l} \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)). \quad (179)
$$

Combining the properties of parity for the bipolar spherical harmonics, Eq. (B12), and for the propagator in redshift space, Eq. (102) with $n = 1$, we have

$$
\tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)) \rightarrow (-1)^{p \z_1^{(1)} + l + l_1} \tilde{\Gamma}(x_l^{(1)}; x_l^{(1)}(k, \mu)). \quad (180)
$$

If the Universe is statistically invariant under the parity, the function is nonzero only when $p \z_1^{(1)} + l + l_1 = 0$.

For the propagator of second order, $n = 2$, we have

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) = \sum_{L,M} (-1)^{L} \sqrt{(L)(l \cdot L)}_{mm,M} \times (L \cdot L)_M^{M \cdot M M^{MM}_M} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)), \quad (181)
$$

with

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) \equiv (-1)^{s \z_1^{(2)} + l_1 + l_2 + L} \sqrt{(L)(l \cdot L)}_{mm,M} \times (L \cdot L)_M^{M \cdot M M^{MM}_M} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)). \quad (182)
$$

The propagator is given by

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; \hat{z}, \hat{k}, \mu)) = \sum_{l_1, l_2, l_3} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) X_{Llm}^{(2)}(\hat{z}, \hat{k}_1, \hat{k}_2). \quad (183)
$$

Combining the properties of complex conjugate for the bipolar spherical harmonics, Eq. (B20), and for the propagator in redshift space, Eq. (89) with $n = 2$, we have

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) = (-1)^{L} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)). \quad (184)
$$

Combining the properties of parity for the bipolar spherical harmonics, Eq. (B19), and for the propagator in redshift space, Eq. (102) with $n = 2$, we have

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) \rightarrow (-1)^{p \z_1^{(2)} + l_1 + l_2} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)). \quad (185)
$$

If the Universe is statistically invariant under the parity, the function is nonzero only when $p \z_1^{(2)} + l_1 + l_2 = 0$.

The interchange symmetry for the second-order propagator is given by

$$
\tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)) = (-1)^{L} \tilde{\Gamma}(x_l^{(2)}; x_l^{(2)}(k_1, k_2; k, \mu)). \quad (186)
$$

For the propagator of higher orders, $n \geq 3$, we have

$$
\tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) = \sum_{L_1, L_2, \ldots, L_n} (-1)^{L_1 + \ldots + L_n} \sqrt{(L_1) \cdots (L_n)} \times (L \cdot L)_{MM}^{M \cdot M M^{MM}_M} \tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) \quad (187)
$$

with

$$
\tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) \equiv (-1)^{s \z_1^{(n)} + l_1 + \ldots + l_n} \sqrt{(L_1) \cdots (L_n)} \times (L \cdot L)_{MM}^{M \cdot M M^{MM}_M} \tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)). \quad (188)
$$

The propagator is given by

$$
\tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; \hat{z}, \hat{k}, \mu)) = \sum_{L_1, L_2, \ldots, L_n} \tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) \times X_{L_{n-1}lm}^{(n)}(\hat{z}, \hat{k}_1, \ldots, \hat{k}_n). \quad (189)
$$

Combining the properties of complex conjugate for the poly-polar spherical harmonics, Eq. (B26), and for the propagator in redshift space, Eq. (89), we have

$$
\tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) = (-1)^{L} \tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)). \quad (190)
$$

Combining the properties of parity for the poly-polar spherical harmonics, Eq. (B25), and for the propagator in redshift space, Eq. (102), we have

$$
\tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)) \rightarrow (-1)^{p \z_1^{(n)} + l_1 + \ldots + l_n} \tilde{\Gamma}(x_l^{(n)}; x_l^{(n)}(k_1, \ldots, k_n; k, \mu)). \quad (191)
When the Universe is statistically invariant under the parity, the function is nonzero only when $p x + p y + p z = 0$, as discussed in Sec. IV A and also the lhs consistently vanishes as discussed in Sec. IV B.

2. Tree-level first-order propagator in redshift space with the complete expansion

In redshift space, the tree-level approximation of the first-order propagator is given by

$$\Gamma_{Xlm}^{(1)}(z, \hat{z}) = c_{Xlm}^{(1)}(k) + \left[ k \cdot L_i^{(1)}(k; \hat{z}) \right] c_{Xlm}^{(0)}$$

(197)

where the displacement kernel is replaced by the one in redshift space as in Eq. (83), and the relevant factor in the above equation is given by

$$k \cdot L_i^{(1)}(k; \hat{z}) = 1 + f(\hat{z} \cdot \hat{k})^2 = 1 + \frac{2f}{3} P_2(\hat{z} \cdot \hat{k})$$

(198)

where $P_2(x) = (3x^2 - 1)/2$ is the second Legendre polynomial, $P_l(x)$ with $l = 2$.

First, we explain step-by-step calculations to obtain the invariant propagator in the case that the dependence of $\mu$ is completely expanded into spherical harmonics. The following calculations can be regarded as a prototype for more involved calculations in later sections and in a subsequent Paper II. The Legendre polynomials $P_l(z \cdot \hat{k})$ can be decomposed into a superposition of products of spherical harmonics by the addition theorem of Eq. (B4), and represented by a special case of bipolar spherical harmonics as

$$P_l(z \cdot \hat{k}) = (-1)^l \sqrt{2l + 1} X_{00}^l(z, \hat{k})$$

(199)

where the last function on the rhs is a special case of normalized bipolar spherical harmonics given by Eq. (133).

Combining Eqs. (197)--(199) above, we derive

$$\Gamma_{Xlm}^{(1)}(z, \hat{z}) = \frac{(-1)^l}{\sqrt{2l + 1}} c_{Xl}^{(1)}(k) C_{lm}(\hat{k}) + \delta_{00} \delta_{m0} X_{00}^l(z, \hat{k})$$

(200)

Each term of the above equation can be represented by special cases of bipolar spherical harmonics, because we have

$$\delta_{00} \delta_{m0} = \delta_{00} X_{00}^0(z, \hat{k})$$

(201)

$$C_{lm}(\hat{k}) = (-1)^l \sqrt{2l + 1} X_{lm}^l(z, \hat{k})$$

(202)

as straightforwardly shown from the definitions. One can also rewrite the bipolar spherical harmonics appearing in Eq. (200) as

$$\delta_{00} \delta_{m0} X_{00}^l(z, \hat{k}) = \delta_{00} X_{00}^l(z, \hat{k})$$

(203)

Substituting the above identities into Eq. (200), one can readily read off the coefficient of the bipolar spherical harmonics $X_{lm}^l(z, \hat{k})$ of Eq. (178) as

$$\Gamma_{Xll}^{(1)}(z, \hat{z}) = \delta_{l0} \delta_{l0} c_{Xl}^{(1)}(k) + \delta_{00} c_{Xl}^{(0)} \left[ 1 + \frac{f}{3} \delta_{l0} \delta_{l0} + \frac{2\sqrt{3}f}{3} \delta_{l2} \delta_{l2} \right]$$

(204)
Putting \( f = 0, l_1 = 0 \) and \( l_1 = l \) in the above equation consistently recovers the result in real space of Eq. (196). When the Universe is statistically invariant under the parity, the rhs of Eq. (204) only nonzero when \( p_X = 0 \) as discussed in Sec. IV A, and apparently the rhs is nonzero only for \( l + l + l = 1 \). Thus, the lhs is nonzero only when \( p_X = 0 \) and \( l + l + l + l = 1 \), which is consistent with the discussion in Sec. IV B. The same result of Eq. (204) can also be derived more straightforwardly by using an orthonormality relation of the bipolar spherical harmonics given in Eq. (B14). According to this orthonormality relation, the Eq. (178) is inverted as

\[
\hat{\Gamma}_{Xl}^{(1)} \left( k, \ell \right) = (-1)^{l_1 + l_3} \ell l_1 l_3 \int \frac{d^2 \hat{k}}{4 \pi} \frac{d^2 \hat{\ell}}{4 \pi} \hat{X}_{\ell 0}^{(1)} \left( \hat{k}, \hat{\ell} \right) \hat{X}_{l 0}^{(1)} \left( \hat{\ell}, \hat{\ell} \right).
\]

(205)

Substituting Eq. (200) into the above equation, using Eqs. (201)–(203) with the orthonormal relation of Eq. (B14), we derive the same result of Eq. (204) as well.

3. Tree-level first-order propagator in redshift space with a partial expansion

The above result of Eq. (204) for the first-order invariant propagator in redshift space is obtained by completely expanding the angular dependencies in the propagator. As explained in Sec. IV B, the complete expansion is required in some situations. In some other simple situations, angular dependencies that are included in the external wave vector can optionally be kept unexpanded to make the calculations simpler. In the case of the tree-level first-order propagator, the argument of the propagator in Eq. (197) is itself the external wave vector.

Combining Eqs. (197) and (198), we trivially have

\[
\hat{\Gamma}_{Xlm}^{(1)} \left( k, \ell \right) = c_{Xlm}^{(1)} \left( k, \ell \right) + \left( 1 + f \mu^2 \right) c_{Xlm}^{(0)}
\]

(206)

Substituting Eqs. (121), (127), (201), and (202) into the above equation, the invariant propagator can be read off from Eq. (178), resulting in

\[
\hat{\Gamma}_{Xlm}^{(1)} \left( k, \ell \right) = \delta_{l,0} \delta_{l,0} \left[ c_{Xlm}^{(1)} \left( k, \ell \right) + \delta_{l,0} (1 + f \mu^2) c_{Xlm}^{(0)} \right].
\]

(207)

The same result is also derived by substituting Eq. (206) into Eq. (205), formally regarding as if the variable \( \mu \) does not explicitly depend on \( \hat{k} \) and \( \hat{\ell} \).

4. Tree-level second-order propagator in real space

For another example of calculating the invariant propagators, we consider the second-order propagator \( \Phi_{Xlm}^{(2)} \) in real space, Eq. (162). The calculation in this case is quite similar to the derivation of Eq. (204). With the tree-level approximation, the second-order propagator is given by Eq. (77). Applying a similar procedure as in the above derivation using Eqs. (199) and (201)–(203), the rhs of Eq. (77) is represented in a form of the rhs of Eq. (162), and we can read off the invariant coefficient from the resulting expression.

In the course of calculation, one needs to represent a product of spherical harmonics of the same argument through the formula given by Eq. (B8). The coefficient of the spherical harmonics in this formula is known as the Gaunt integral. Similarly to the simplified notation for the 3-j-symbols we introduced above, it is useful to use a simplified symbol for the Gaunt integral \( \left[ l_1 l_2 l_3 \right]_{lm' m'' m''} \) defined by Eq. (C9) in our notation. We consider the azimuthal indices in the Gaunt integral can be raised and lowered by the spherical metric, such as

\[
\left[ l_1 l_2 l_3 \right]_{lm' m'' m''} = \delta_{m' m''} \delta_{m'' m},
\]

(208)

and so forth. The Gaunt integral of Eq. (C9) is nonzero only when \( l_1 + l_2 + l_3 \) is an even number. With this notation, Eq. (B8) is simply represented by

\[
\hat{C}_{l m'} \left( \hat{k} \right) \hat{C}_{l m} \left( \hat{k} \right) = \sum_l \left( 2l + 1 \right) \left[ l_1 l_2 l_3 \right]_{lm' m''} c_{l m} \left( \hat{k} \right)
\]

(209)

Properties of the Gaunt integral, Eqs. (C9)–(C14) are useful in analytical deductions of invariant propagators.

Following the procedures above to derive the invariant propagator of first order, and additionally using the formula of Eq. (209), the invariant propagator of the second order can be straightforwardly derived. The result in real space is given by

\[
\hat{\Phi}_{Xlm}^{(2)} \left( k_1, k_2 \right) = c_{Xlm}^{(2)} \left( k_1, k_2 \right)
\]

\[
+ c_{Xlm}^{(1)} \left( k_1 \right) \left[ \delta_{l,0} \delta_{l,0} + \frac{k_1 \left( -1 \right)}{k_2} \left( \left| l_1 \right| \left( 1 0 0 \right) \delta_{l,1} \right) \right.
\]

\[
+ c_{Xlm}^{(1)} \left( k_2 \right) \left[ \delta_{l,0} \delta_{l,0} + \frac{k_2 \left( -1 \right)}{k_1} \left( \left| l_2 \right| \left( 1 0 0 \right) \delta_{l,1} \right) \right.
\]

\[
+ \delta_{l,0} \left( \left| X \right| \left( 21 \right) \left( k_1 \right) \left( k_2 \right) \delta_{l,1} \delta_{l,1}
\]

\[
+ \frac{8 \sqrt{5}}{21} \delta_{l,2} \delta_{l,2} \right].
\]

(210)

If the Universe is statistically invariant under the parity, the terms on the rhs of Eq. (210) except the first term is nonzero only for normal tensors, \( p_X = 0 \), as discussed in Sec. IV A, and apparently nonzero only for \( l + l + l = 1 \). The first term on the rhs is nonzero only for \( p_X + l + l + l = 1 \). The lhs is nonzero only for \( p_X + l + l + l = 1 \), and therefore the properties are consistent with each other. The corresponding result in redshift space is explicitly given in Paper II [43].

Other propagators to arbitrary orders, both in real space and redshift space, can be calculated in a similar way as illustrated above, although the calculations are more or less demanding in the case of calculation including higher-order perturbations.

V. THE SPECTRA OF TENSOR FIELDS IN PERTURBATION THEORY

In this section, we consider how one can calculate the power spectrum and higher-order spectra such as the bispectrum and so forth, for tensor fields in general.
A. The power spectrum

1. The power spectrum in real space

The power spectrum $P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k)$ of irreducible components of tensor fields $F_{\text{X}_{1}\text{X}_{2}}(k)$ can be defined by

$$
\langle F_{\text{X}_{1}m_{1}}(k_{1})F_{\text{X}_{2}m_{2}}(k_{2}) \rangle_{c} = (2\pi)^{3} \delta_{3}(k_{1} + k_{2}) P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k_{1}),
$$

(211)

where $\langle \cdots \rangle_{c}$ indicates the two-point cumulant, or the connected part, and the appearance of the delta function is due to translational symmetry. We consider generally the cross power spectra between two fields, $X_{1}$ and $X_{2}$ with irreducible components $l_{1}$ and $l_{2}$, respectively. The autopower spectrum is straightforwardly obtained by putting $X_{1} = X_{2} = X$.

The power spectrum $P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k)$ defined above depends on the coordinates system, as it has azimuthal indices and depends on the direction of wave vector $k$ in the arguments. One can also construct the reduced power spectrum which is invariant under the rotation, in a similar manner to introducing the invariant propagators in Sec. IV B.

First, we consider the power spectrum in real space. The directional dependence on the wave vector can be expanded by spherical harmonics as

$$
P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k) = \sum_{l} P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k) C^{l}_{\text{lm}}(\hat{k}).
$$

(212)

Because of the rotational transformations of Eqs. (93) and (23), and the rotational invariance of the delta function in the definition of the power spectrum, Eq. (211), the rotational transformation of the expansion coefficient is given by

$$
P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k) \xrightarrow{\text{rotation}} P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k) D^{m}_{l_{1}m_{1}}(R) D^{m_{2}}_{l_{2}m_{2}}(R).
$$

(213)

When the Universe is statistically isotropic, the functional form of the power spectrum should not depend on the choice of coordinates system. Following the same procedure as in Sec. IV A, the power spectrum in a statistically isotropic Universe is shown to have a form,

$$
P^{l_{1}l_{2}}_{X_{1}X_{2}m_{1}m_{2}}(k) = \sum_{l_{1} l_{2} m_{1} m_{2} m_{12}} C^{l_{1}l_{2}m}_{\text{lm}}(\hat{k}) P^{l_{1}l_{2}m_{1}m_{2}}_{X_{1}X_{2}}(k),
$$

(214)

where

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k) = \delta^{l_{1}l_{2}}(-1)^{l_{1}}(l_{1} l_{2})^{m_{1}m_{2}m} P^{l_{1}l_{2}m_{1}m_{2}m_{12}}_{X_{1}X_{2}m_{12}}(k)
$$

(215)

is the invariant power spectrum. The physical degrees of freedom of the power spectrum are represented by the last invariant spectrum, $P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k)$. There are a finite number of independent functions of power spectra which depend only on the modulus of the wave vector $k = |k|$ of Fourier modes, due to the triangle inequality of the $3j$-symbol, $|l_{1} - l_{2}| \leq l \leq l_{1} + l_{2}$ for a fixed set of integers $(l_{1}, l_{2})$. For example, when $l_{1} = l_{2} = 2$, only $l = 0, 1, \ldots, 4$ are allowed and there are five independent components in the power spectrum.$^3$

2. Symmetries in real space

According to Eqs. (86) and (211), the complex conjugate of the power spectrum is given by

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}m_{1}m_{2}}(k) = P^{l_{1}l_{2}m}_{X_{1}X_{2}}(-k).
$$

(216)

Using the property of the $3j$-symbol, Eq. (C2), the symmetry of the complex conjugate is shown to be given by

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k) = P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k),
$$

(217)

and thus the invariant power spectrum is a real function. Additionally, the parity transformation of the power spectrum is given by

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k) \xrightarrow{\text{parity}} (-1)^{p_{X_{1}} + p_{X_{2}} + l_{1} + l_{2}} P^{l_{1}l_{2}m}_{X_{1}X_{2}}(-k),
$$

(218)

because of Eqs. (99) and (211). Accordingly, the parity transformation of the invariant power spectrum is given by

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k) \xrightarrow{\text{parity}} (-1)^{p_{X_{1}} + p_{X_{2}} + l_{1} + l_{2}} P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k).
$$

(219)

If the Universe is statistically invariant under the parity, the power spectrum is invariant under the parity transformation, we thus have

$$
p_{X_{1}} + p_{X_{2}} + l_{1} + l_{2} + l = \text{even}.
$$

(220)

Together with the triangle inequality, $|l_{1} - l_{2}| \leq l \leq l_{1} + l_{2}$, mentioned above, the numbers of independent components in the invariant power spectrum $P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k)$ are further reduced in the presence of parity symmetry. For example, when $l_{1} = l_{2} = 2$ and $p_{X_{1}} = p_{X_{2}} = 0$, only $l = 0, 2, 4$ are allowed and there are three independent components in the power spectrum.$^4$

The power spectrum in real space has an interchange symmetry,

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}m_{1}m_{2}}(k) = P^{l_{1}l_{2}m}_{X_{1}X_{2}}(-k),
$$

(221)

and the corresponding symmetry for the invariant spectrum is given by

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k) \xrightarrow{\text{parity}} (-1)^{l_{1} + l_{2}} P^{l_{1}l_{2}m}_{X_{1}X_{2}}(k).
$$

(222)

3. The power spectrum in redshift space

Next, we consider the power spectrum in redshift space. The directional dependence of the wave vector and the line of sight can be expanded by spherical harmonics as

$$
P^{l_{1}l_{2}m}_{X_{1}X_{2}m_{1}m_{2}}(k; \hat{z}) = \sum_{l_{1} l_{2} \mu \nu} P^{l_{1}l_{2}m}_{X_{1}X_{2}m_{1}m_{2}}(k, \mu) C^{l_{1}l_{2}}_{\text{lm}}(\hat{z}) C^{\mu}_{\text{ln}}(\hat{k}).
$$

(223)

$^3$ In Ref. [39], the power spectrum $P^{l_{1}l_{2}}(k)$ of intrinsic alignment is defined in

$^4$ In Ref. [39], their power spectrum of intrinsic alignment satisfies $P^{00}(k) = P^{22}(k)$ in the presence of parity symmetry, and thus only three components, $q = 0, 1, 2$, are independent. Again, the degree of freedom of theirs is consistent with ours.
Corresponding to the discussion about the propagator in Sec. IV B 2, some part of the dependence on directional cosine \( \mu = \hat{z} \cdot \hat{k} \) can be kept unexpanded, and we generally include the argument \( \mu \) in the coefficients in the above expansion. When the dependence on \( \mu \) is completely expanded into the spherical harmonics, the argument \( \mu \) is absent, and the expansion of Eq. (223) is unique. Otherwise, unless which dependence on \( \mu \) is specified and fixed, the expansion is not unique.

Following similar considerations of rotational symmetry in the rest of this paper, e.g., in deriving Eq. (178), the power spectrum in the statistically isotropic universe is shown to have a form,

\[
P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}; \hat{z}) = \delta^{l+z} \sum_{L} (-1)^{L} \sqrt{L} \sum_{m_{1},m_{2}}^{L} \left( \frac{l}{m_{1}} \frac{l_{z}}{m_{2}} \right) \frac{L}{(l_{z} + l_1 + l_2 + l_{m_{2}} - l_{m_{1}})} \times \frac{C_{lm_{1},m_{2}}^{m_{m_{2}}}(\hat{z}, \mathbf{k}) P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu)}{L}, \tag{224}
\]

where

\[
P^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, \mu) = \delta^{l+z} (-1)^{l+z} \sqrt{L} \right. \left. \times \sum_{l_{1},l_{2}} L \left( m_{12}^{l_{1}l_{2}} M \right) M_{m_{m_{1}}M_{m_{2}}}, \tag{225}
\]

is the invariant power spectrum in redshift space.

Practically, it should be sometimes convenient to choose a coordinate system in which the direction to the line of sight is chosen to be the \( z \) axis, \( \hat{z} = (0, 0, 1) \). In this case, we have \( X_{1}^{l_{1}l_{2}}(\hat{z}, \mathbf{k}) = \langle L_{1}, l_{1}\rangle_{M_{m_{1}M_{m_{2}}}}^{m_{m_{2}}}(\hat{z}, \mathbf{k}) \), and Eq. (224) reduces to

\[
P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}; \hat{z}) = \delta^{l+z} \sum_{L} (-1)^{L} \sqrt{L} \sum_{m_{1},m_{2}}^{L} \left( \frac{l}{m_{1}} \frac{l_{z}}{m_{2}} \right) \frac{L}{(l_{z} + l_1 + l_2 + l_{m_{2}} - l_{m_{1}})} \times \sum_{l_{1},l_{2}} \left( \frac{l}{m_{12}} \frac{l_{z}}{m_{2}} \right) C_{lm_{1},m_{2}}^{m_{m_{2}}}(\hat{z}, \mathbf{k}) P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu), \tag{226}
\]

where \( m_{12} \equiv m_{1} + m_{2} \) and \( \mu = k_{c}/k \) in this coordinates system. In a simple case that the invariant spectrum does not explicitly depend on the direction cosine \( \mu \), the above equation can be inverted by using orthogonality relations of the spherical harmonics and 3-j symbols, Eqs. (B6) and (C5). The result is given by

\[
P^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, \mu) = \delta^{l+z} (-1)^{l} \left( \frac{l}{m_{12}} \frac{l_{z}}{m_{2}} \right) \frac{L}{(l_{z} + l_1 + l_2 + l_{m_{2}} - l_{m_{1}})} \right. \left. \times \sum_{l_{1},l_{2}} \left( \frac{l}{m_{12}} \frac{l_{z}}{m_{2}} \right) C_{lm_{1},m_{2}}^{m_{m_{2}}}(\hat{z}, \mathbf{k}) P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu), \tag{227}
\]

As the dependence of azimuthal angle \( \phi_{3} \) of the wave vector \( \mathbf{k} \) in the power spectrum of Eq. (226) is given by \( e^{im_{3}\phi_{3}} \) in the spherical harmonics of Eq. (17), the above equation further reduces to

\[
P^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, \mu) = \delta^{l+z} (-1)^{l} \left( \frac{l}{m_{12}} \frac{l_{z}}{m_{2}} \right) \frac{L}{(l_{z} + l_1 + l_2 + l_{m_{2}} - l_{m_{1}})} \right. \left. \times \sum_{l_{1},l_{2}} \left( \frac{l}{m_{12}} \frac{l_{z}}{m_{2}} \right) C_{lm_{1},m_{2}}^{m_{m_{2}}}(\hat{z}, \mathbf{k}) P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu), \tag{228}
\]

where \( P^{(l)}_{m}(\mu) \) are associate Legendre polynomials, and \( p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu) \) is a simplified notation for the power spectrum \( p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}) \) with a constraint \( \phi_{k} = 0 \) (i.e., \( \mathbf{k} \) is on the \( x-z \) plane).

4. Symmetries in redshift space

As we work in the distant-observer approximation, it is reasonable to assume that the power spectrum is unchanged when the direction of the line of sight is flipped, \( \hat{z} \rightarrow -\hat{z} \), keeping the physical system unchanged:

\[
P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}; -\hat{z}) = p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}; \hat{z}). \tag{229}
\]

On one hand, when the directional dependence of \( \mu = \hat{z} \cdot \hat{k} \) is completely expanded in the expansion of Eq. (224), the invariant spectrum satisfies \( p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}) = (-1)^{l} p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}) \), and thus the integers \( l \) in Eq. (224) should be even numbers to have nonzero values of the power spectrum. On the other hand, when the dependence of \( \mu \) is included in the invariant spectrum as \( p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu) \), the integers \( l \) are not necessarily even numbers, and the invariant spectrum instead satisfies

\[
p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, -\mu) = (-1)^{l} p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, -\mu). \tag{230}
\]

According to Eqs. (86) and (211) and the symmetry of the 3-j symbol, Eq. (C2), the complex conjugate of the invariant function is shown to be given by

\[
p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, -\mu) = (-1)^{l} p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, -\mu). \tag{231}
\]

Combining the above equation with the flip symmetry of Eq. (230), we derive

\[
p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \mu) = p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, -\mu), \tag{232}
\]

i.e., the invariant spectra are real functions. This conclusion holds regardless of whether the dependence of \( \mu \) is present or not in the invariant spectrum.

The parity transformation of the power spectrum is given by

\[
P^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}, \hat{z}) \overset{\mathcal{P}}{\rightarrow} (-1)^{p_{X_{1}}+p_{X_{2}}+l_{z}} p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(-\mathbf{k}, -\hat{z}), \tag{233}
\]

because of Eqs. (99) and (211). Accordingly, the transformation of the invariant power spectrum is given by

\[
p^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, \mu) \overset{\mathcal{P}}{\rightarrow} (-1)^{p_{X_{1}}+p_{X_{2}}+l_{z}} p^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, -\mu). \tag{234}
\]

When the Universe is statistically invariant under the parity, the power spectrum in redshift space is invariant under the parity transformation, we thus have

\[
p_{X_{1}} + p_{X_{2}} + l_{z} + l_{1} + l_{2} + l_{m_{12}} \equiv \text{even} \tag{235}
\]

The power spectrum in redshift space has an interchange symmetry,

\[
p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(-\mathbf{k}; \hat{z}) = p^{(l,z)}_{X_{1}X_{2}m_{m_{2}}}(\mathbf{k}; -\hat{z}), \tag{236}
\]

and the corresponding symmetry for the invariant spectrum is given by

\[
p^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, -\mu) = (-1)^{l} p^{(l,z)}_{X_{1}X_{2}}(\mathbf{k}, \mu). \tag{237}
\]
We first consider the power spectrum with the lowest-order approximation, or the linear power spectrum, which is the simplest application of our formalism. The formal expression for the power spectrum of scalar fields in the propagator formalism with Gaussian initial conditions [17, 23, 54] is straightforwardly generalized to the case of tensor field. In the lowest order, we simply have

\[ P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(k) = i^{l_1 + l_2} \Pi^2(k) \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) (-k) P_L(k), \]  

(238)

where the parity symmetry for the vertex resummation factor, \( \Pi(-k) = \Pi(k) \) is taken into account.

The vertex resummation factor \( \Pi(k) \) is given by Eq. (74). In the lowest order of the perturbation theory, this factor can be approximated by \( \Pi(k) = 1 \). This factor strongly dampens the power spectrum on small scales, where the linear theory does not apply. However, sometimes it is useful to keep this exponent even in the linear theory, especially in redshift space where the peculiar velocities damp the power on small scales. When the resummation factor is retained, the linear approximation for this factor is derived from Eqs. (74), (79), and (83), and we have [17, 19, 64]

\[ \Pi(k) = \exp \left( -\frac{k^2}{12\pi^2} \int dp P_L(p) \right), \]  

(239)

in real space, and

\[ \Pi(k, \mu) = \exp \left( -\frac{k^2}{12\pi^2} \left[ 1 + f(f + 2)\mu^2 \right] \int dp P_L(p) \right), \]  

(240)

in redshift space, where \( \mu \equiv \hat{z} \cdot \hat{k} \) is the direction cosine between the wave vector and the line of sight, as usual. Substituting \( f = 0 \) in the expression in redshift space reduces to the expression in real space as a matter of course.

In real space, the first-order propagator in terms of the invariant propagator is given by Eq. (160). Thus Eq. (238) reduces to

\[ P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(k) = i^{l_1 + l_2} \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) (-k) \Pi^2(k) P_L(k) \]

\[ \times \sum_i [l_1 l_2 l_{m_1 m_2} c_{m}(\hat{k})]. \]  

(241)

The invariant power spectrum is easily read off from the above expression and Eq. (214), and we have

\[ P^{(l_1 l_2)}_{X_1 X_2}(k) = (-1)^{l_1 + l_2} \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) (-k) \Pi^2(k) P_L(k) \]

\[ \times \left\{ \frac{(-1)^{l_1} [l_1 l_2 l]}{\sqrt{[l_1][l_2]^2}} \right\} \left\{ \frac{(-1)^{l_2} [l_1 l_2 l]}{\sqrt{[l_1][l_2]^2}} \right\} \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) \Pi^2(k) P_L(k). \]  

(242)

Further substituting the lowest-order approximation of the first-order propagator, Eq. (196), the above equation is explicitly given by

\[ P^{(l_1 l_2)}_{X_1 X_2}(k) = \Pi^2(k) P_L(k) \]

\[ \times \left\{ \frac{(-1)^{l_1} [l_1 l_2 l]}{\sqrt{[l_1][l_2]^2}} \right\} \left\{ \frac{(-1)^{l_2} [l_1 l_2 l]}{\sqrt{[l_1][l_2]^2}} \right\} \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) \Pi^2(k) P_L(k). \]  

(243)

This is one of the generic predictions of our theory.

As a consistency check, we consider the scalar case, \( l_1 = l_2 = 0 \) of autopower spectrum, \( X_1 = X_2 = X \). In this case, Eq. (243) simply reduces to

\[ P^{00}_{XX}(k) = [b_X(k)]^2 \Pi^2(k) P_L(k), \]  

(244)

where

\[ b_X(k) \equiv c^{(1)}_{X0}(k) + c^{(0)}_{X0}(k), \]  

(245)

corresponds to the linear bias factor. Equation (244) is consistent with the result of the usual linear power spectrum of biased density field.

The Kronecker’s symbols in Eq. (243) indicate that the resulting expressions are quite different between scalar fields \( (l_1 = 0 \text{ or } l_2 = 0 \) and nonscalar fields \( l_1 \geq 1 \text{ or } l_2 \geq 1 \). When the fields \( X_1 \) and \( X_2 \) are both nonscalar fields with \( l_1 \neq 0 \) and \( l_2 \neq 0 \), Eq. (243) simply reduces to

\[ P^{l_1 l_2}_{X_1 X_2}(k) = \delta_{l_1 l_2} \bar{\Pi}^{(1)}(X_1 l_1) \bar{\Pi}^{(1)}(X_2 l_2) \Pi^2(k) P_L(k), \]  

(246)

When \( X_1 \) is a nonscalar field and \( X_2 \) is a scalar field with \( l_1 \neq 0 \) and \( l_2 = 0 \), Eq. (243) reduces to

\[ P^{l_1 0}_{X_1 X_2}(k) = \delta_{l_1 l_2} \bar{\Pi}^{(1)}(X_1 l_1) b_{X_2}(k) \Pi^2(k) P_L(k), \]  

(247)

where \( b_{X}(k) \) is given by Eq. (245). The above equation survives only when \( l = l_1 \). When \( X_1 \) and \( X_2 \) are both scalar fields with \( l_1 = l_2 = 0 \), Eq. (243) reduces to

\[ P^{00}_{X_1 X_2}(k) = \Pi^2(k) P_L(k), \]  

(248)

which survives only when \( l = 0 \). The last equation coincides with Eq. (244) when \( X_1 = X_2 = X \) and \( l = 0 \).

6. The linear power spectrum in redshift space

In redshift space, the lowest-order propagator is given by Eq. (178), which can be substituted into Eq. (238). As a result, there appears a product of bipolar spherical harmonics, which can be represented by a superposition of a single bipolar spherical harmonics using a formula with 9-j-symbols [47], as given in Eq. (B16) of Appendix B in our notation. The definition of the 9-j-symbol is given by Eq. (C19), and useful
properties are summarized in Eqs. (C20)–(C23). After some calculations along the line described above, we have

\[
P_{X_{i}X_{j}m_{i}m_{j}}^{l_{i}l_{j}}(k, \hat{z}) = \Pi^{l_{i}l_{j}l_{k}}(k)P_{l_{k}}(k) \sum_{l_{1},l_{2},l_{3},l_{4}}(-1)^{l_{1}+l_{2}}X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) \times \hat{X}_{l_{k}}^{(l_{1}l_{2}l_{3}l_{4})(k, \mu)}(k, \mu) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \hat{z}, \hat{k}). \quad (249)
\]

Substitution of Eq. (204) into the above gives an explicit result. The invariant power spectrum is easily read off from the expression of Eq. (224), including the case that the dependence of the directional cosine \(\mu\) in invariant propagators is present. We derive

\[
P_{X_{i}X_{j}}^{l_{i}l_{j}l_{k}}(k, \mu) = \Pi^{l_{i}l_{j}l_{k}}(k, \mu)P_{l_{k}}(k) \left(-1\right)^{l_{i}+l_{j}}\hat{X}_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu). \quad (250)
\]

This is one of the generic predictions of our theory.

As a consistency check, one can recover the result of the power spectrum in real space, Eqs. (241) and (242). In fact, substituting \(l_{3} = l_{2} = 0\) into Eqs. (249) and (250), using identities \(\hat{X}_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) = \delta_{l_{1},l_{2}}\hat{X}_{X_{i}X_{j}}^{l_{1}l_{2}}(k)\) when \(f = 0\), \(000000 = \delta_{l_{0}}\), and a special case of 9-j-symbols [47],

\[
\left\{\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) = \delta_{l_{1},l_{2}}\delta_{l_{2},l_{3}}\delta_{l_{3},l_{4}} \sqrt{\frac{l_{1}l_{2}l_{3}l_{4}}{l_{3}l_{4}l_{5}l_{6}}}, \quad (251)
\]

the corresponding equations in real space are recovered.

As another consistency check, one can also see if Eq. (249) reproduces the well-known result for the scalar case, i.e., the Kaiser’s formula [67]. In the case that the directional dependence is completely expanded, substituting Eq. (204) into Eq. (249), putting \(l_{1} = l_{2} = m_{1} = m_{2} = 0\), \(X_{1} = X_{2} = X\), and using Eqs. (C8) and (251), we derive

\[
P_{X_{i}X_{j}000}^{l_{i}l_{j}0}(k, \hat{z}) = \Pi^{l_{i}l_{j}0}(k, \mu)\left[b_{X}(k)\right]^{2}P_{l_{k}}(k) \times \left[1 + \frac{2}{3}\beta + \frac{1}{3}\beta^{2}\right]P_{0}(\mu) + \frac{4}{3}\beta + \frac{7}{8}\beta^{2}\right]P_{2}(\mu) + \frac{8}{35}\beta^{3}P_{3}(\mu), \quad (252)
\]

where \(\beta \equiv c_{X}^{(0)}f/b_{X}(k)\) corresponds to the redshift-space distortion parameter. When the scalar field is given by density field of biased objects, this expression is equivalent to a known result of the linear power spectrum of a biased (scalar) field in redshift space [68].

In the lowest-order approximation of the first-order propagator in redshift space, Eq. (204), either \(l_{i}\) or \(l_{j}\) is zero in each term. Therefore, at least two of the indices \(l_{1}, l_{2}, l_{3}, l_{4}\) are zero in the 9-j-symbol of Eq. (250). When two of the indices in the 9-j-symbol are zero, there are formulas [47],

\[
\left\{\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) = \delta_{l_{1},l_{2}}\delta_{l_{2},l_{3}}\delta_{l_{3},l_{4}} \sqrt{\frac{l_{1}l_{2}l_{3}l_{4}}{l_{3}l_{4}l_{5}l_{6}}}, \quad (253)
\]

\[
\left\{\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) = (-1)^{l_{1}+l_{2}+l_{3}-l_{4}}\delta_{l_{1},l_{2}}\delta_{l_{2},l_{3}}\delta_{l_{3},l_{4}} \sqrt{\frac{l_{1}l_{2}l_{3}l_{4}}{l_{3}l_{4}l_{5}l_{6}}}, \quad (254)
\]

A 9-j-symbol, in which two of the indices are zero, can always be represented in a form of the lhs of the above equations by means of the symmetric properties of the 9-j-symbol, Eqs. (C20)–(C22). Thus, with the lowest-order approximation of the first-order propagator given by Eq. (204), the summation of Eq. (250) is explicitly expanded to give

\[
P_{X_{i}X_{j}}^{l_{i}l_{j}l_{k}}(k, \mu) = \Pi^{l_{i}l_{j}l_{k}}(k, \mu)P_{l_{k}}(k) \times \left\{\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) \times \left(\frac{1}{\sqrt{l_{1}l_{2}l_{3}l_{4}}} \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) \times \left(\frac{1}{\sqrt{l_{1}l_{2}l_{3}l_{4}}} \right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times \left(\begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array}\right) \times X_{X_{i}X_{j}}^{l_{1}l_{2}l_{3}l_{4}}(k, \mu) \times \left(\frac{1}{\sqrt{l_{1}l_{2}l_{3}l_{4}}} \right)
\]

where an identity

\[
\left\{\begin{array}{ccc} 2 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right) = \frac{1}{5}\delta_{l_{0}} + \frac{2}{35}\delta_{l_{2}} + \frac{2}{35}\delta_{l_{4}} \quad (256)
\]

is used in deriving the above result. Equation (255) is one of the generic predictions of our theory.

The above Eq. (255) is nonzero only when \(l_{z} = 0, 2, 4\), and the redshift-space distortions in linear theory contain
monopole ($l = 0$), quadrupole ($l = 2$) and hexadecapole ($l = 4$) components, as is well known in the scalar perturbation theory, Eq. (252), while higher-order corrections contain higher-order multipoles for the redshift-space distortions.

In linear theory besides the resummation factor $\Pi(k, \mu)$, higher-rank tensors are not affected by the redshift-space distortions as the first-order propagator of Eq. (204) does not neither. Thus, if neither $l_1$ nor $l_2$ is zero, Eq. (255) is simply given by

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

which survives only when $l_0 = 0$ and $l = L$. The above equation without the resummation factor exactly coincides with the result in real space, Eq. (246). Namely, redshift-space distortions on higher-rank tensors are nonlinear effects. In fact, nonlinear loop corrections in the power spectrum do introduce the redshift-space distortion effects, as explicitly shown in Paper II [43].

As in the case of real space, Kronecker’s symbols in Eq. (255) indicate that the resulting expression is different between scalar fields ($l_1 = 0$ or $l_2 = 0$) and nonscalar fields ($l_1 \geq 1$ or $l_2 \geq 1$). When the fields $X_1$ and $X_2$ are both nonscalar fields with $l_1 \neq 0$ and $l_2 = 0$, the expression reduces to Eq. (257) above. When $X_1$ is a nonscalar field and $X_2$ is a scalar field with $l_1 \neq 0$ and $l_2 = 0$, Eq. (255) reduces to

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

spherical harmonics, and the propagator is given by Eq. (204). We now consider the other case that the dependence of the first-order normalized propagator in redshift space is completely expanded into

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

we substitute Eq. (207) into Eq. (250). Because of the simple form of Eq. (207), the summations over $l_1$, $l_2$, $l_1'$ and $l_2'$ are trivially evaluated, resulting in,

$$P_{X_1X_2}^{(l_1,l_2)}(k, \mu) = \delta_{l_0, 0} \delta_{l_1, l_2} \Pi(k_0, k) P_{l_0}(k)$$

which survives only when $l_0 = 0$ and $l = L$. The above equation without the resummation factor exactly coincides with the result in real space, Eq. (246). Namely, redshift-space distortions on higher-rank tensors are nonlinear effects. In fact, nonlinear loop corrections in the power spectrum do introduce the redshift-space distortion effects, as explicitly shown in Paper II [43].

As in the case of real space, Kronecker’s symbols in Eq. (255) indicate that the resulting expression is different between scalar fields ($l_1 = 0$ or $l_2 = 0$) and nonscalar fields ($l_1 \geq 1$ or $l_2 \geq 1$). When the fields $X_1$ and $X_2$ are both nonscalar fields with $l_1 \neq 0$ and $l_2 = 0$, the expression reduces to Eq. (257) above. When $X_1$ is a nonscalar field and $X_2$ is a scalar field with $l_1 \neq 0$ and $l_2 = 0$, Eq. (255) reduces to

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

spherical harmonics, and the propagator is given by Eq. (204). We now consider the other case that the dependence of the first-order normalized propagator in redshift space is completely expanded into

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

As a consistency check, one immediately sees that this equation reduces to Eq. (243) when $f = 0$.

When the fields $X_1$ and $X_2$ are both nonscalar fields with $l_1 \neq 0$ and $l_2 \neq 0$, Eq. (256) simply reduces to

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

which reduces only when $l_1 = 0$ and $L = l$. This equation is exactly the same as Eq. (257). When $X_1$ is a nonscalar field and $X_2$ is a scalar field with $l_1 \neq 0$ and $l_2 = 0$, Eq. (256) reduces to

$$P_{l_1, l_2, l_3}^{(d)}(k) = \delta_{l_0, 0} \delta_{l_1, l_2} \delta_{l_3, l_0} \Pi(k_0, k) P_{l_0}(k)$$

which survives only when $l_0 = 0$ and $L = l$. The last equation corresponds to the Kaiser’s formula for the density power spectrum in redshift space [67].

**B. The correlation function**

While the power spectrum is defined in Fourier space, the counterpart in configuration space is the correlation function, $\tilde{c}_{X_1X_2lm}(\mathbf{x})$, which is defined by

$$\tilde{c}_{X_1X_2lm}(\mathbf{x}) = \tilde{c}_{X_1X_2lm}(\mathbf{x}_1 - \mathbf{x}_2),$$

where the tensor field $F_{Xlm}(\mathbf{x})$ on the lhs now corresponds to the one in the configuration space. On the rhs, the correlation function is a function of the relative vector between the two positions, $\mathbf{x}_1 - \mathbf{x}_2$, because of the statistical homogeneity of
the Universe. It is standard that the correlation function and the power spectrum are related by a three-dimensional Fourier transform as

$$
\xi^{(l_z)}_{X_iX_{m_1m_2}}(x) = \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x} P^{(l_z)}_{X_iX_{m_1m_2}}(k),
$$

(265)

First, we consider the correlation function in real space. In exactly the same way of deriving Eq. (214), the rotational symmetry requires that the correlation function should have a form,

$$
\xi^{(l_z)}_{X_iX_{m_1m_2}}(x) = \hat{r}^{l_z} \int l \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x),
$$

(266)

where the last factor $\xi^{(l_z)}_{X_iX_2}(x)$ corresponds to the invariant correlation function. Compared with the definition of the invariant power spectrum of Eq. (214), we should note that the phase factor $\hat{r}$ is additionally present. One can show that the invariant correlation function of $\xi^{(l_z)}_{X_iX_2}(x)$ defined above with the phase factor is a real function.

Substituting Eq. (214) into Eq. (265), and using a formula of Eq. (B7) in Appendix B, we have

$$
\xi^{(l_z)}_{X_iX_2}(x) = \int \frac{k^2 dk}{2\pi^2} j_l(kx) P^{(l_z)}_{X_iX_2}(k),
$$

(267)

i.e., the invariant correlation function in real space is given by a Hankel transform of the invariant power spectrum. Using the completeness relation of the spherical Bessel functions, the inverse relation of the above transform is given by

$$
P^{(l_z)}_{X_iX_2}(k) = 4\pi \int x^2 dx j_l(kx) \xi^{(l_z)}_{X_iX_2}(x).
$$

(268)

The correlation function in redshift space is also represented by the invariant spectrum in redshift space. The relation between the correlation function and the power spectrum is just given by Eq. (265) as well in redshift space, provided that both explicitly depend on the direction of the line of sight, $\hat{z}$,

$$
\xi^{(l_z)}_{X_iX_{m_1m_2}}(x; \hat{z}) = \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x} P^{(l_z)}_{X_iX_{m_1m_2}}(k; \hat{z}).
$$

(269)

Because of the rotational symmetry, we have

$$
\xi^{(l_z)}_{X_iX_{m_1m_2}}(x; \hat{z}) = \hat{\theta}^{l_z+1} \int L \, \tilde{L} \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x),
$$

(270)

just as in the case of the power spectrum of Eq. (224), and the last factor $\xi^{(l_z)}_{X_iX_2}(x)$ corresponds to the invariant correlation function. When the invariant power spectrum in redshift space $P^{(l_z)}_{X_iX_2}(k)$ does not contain any dependence on the directional cosine of the wave vector, $\mu = \cos \theta_k$, the invariant correlation function $\xi^{(l_z)}_{X_iX_2}(x)$ is straightforwardly related to the invariant spectrum via the Hankel transform just in the similar way in the case of the real space above,

$$
\xi^{(l_z)}_{X_iX_2}(x) = \int \frac{k^2 dk}{2\pi^2} j_l(kx) P^{(l_z)}_{X_iX_2}(k),
$$

(271)

and

$$
P^{(l_z)}_{X_iX_2}(k) = 4\pi \int r^2 dr j_l(kr) \xi^{(l_z)}_{X_iX_2}(x).
$$

(272)

As in Eqs. (226)–(228) for the power spectrum, we consider expressions of the correlation function in a coordinates system in which the line of sight is directed to the $z$ axis, $\hat{z} = (0, 0, 1)$. In this case, Eq. (270) reduces to

$$
\xi^{(l_z)}_{X_iX_{m_1m_2}}(x) = \hat{\theta}^{l_z+1} \int L \, \tilde{L} \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x),
$$

(273)

where $m_1 \equiv m_2$. Similar to the case of the power spectrum, Eq. (227), the above equation can be inverted to give

$$
\xi^{(l_z)}_{X_iX_2}(x) = \hat{\theta}^{l_z+1} \int L \, \tilde{L} \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x),
$$

(274)

and also

$$
\xi^{(l_z)}_{X_iX_2}(x) = \hat{\theta}^{l_z+1} \int L \, \tilde{L} \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x),
$$

(275)

where $\xi^{(l_z)}_{X_iX_2}(x, \mu_x)$ is a simplified notation for the correlation function $\xi^{(l_z)}_{X_iX_{m_1m_2}}(x)$ with a constraint $\phi_x = 0$ (i.e., $x$ is on the $x$-$z$ plane), and $\cos \theta_k = \mu_x$.

However, when the invariant spectrum in redshift space $P^{(l_z)}_{X_iX_2}(k, \mu)$ does depend on the directional cosine, the above equations no longer hold. In this case, the invariant correlation function can still be represented by the invariant power spectrum. In order to derive the relation, we first note that Eq. (270) can be inverted by applying orthogonality relations of the $3j$-symbols, Eq. (C5), and those of the bipolar spherical harmonics, Eq. (B14). As a result, we have

$$
\xi^{(l_z)}_{X_iX_2}(x) = \hat{\theta}^{l_z+1} \int L \, \tilde{L} \, (l_z \, l)_{m_1m_2} \, m \, C_m(k) \xi^{(l_z)}_{X_iX_2}(x).
$$

(276)

We substitute the expression of the power spectrum, Eq. (224), into Eq. (269), and then the result is substituted into Eq. (276). The resulting equation is calculated by using orthogonality relations of the $3j$-symbols, Eq. (C5), the Rayleigh expansion formula, Eq. (B5), sum rules of spherical harmonics and $6j$-symbols, Eqs. (B4) and (C24), and the formulas of Eqs. (B8)
and (B9). After these manipulations, the expression of the invariant correlation function in redshift space finally reduces to
\[ \xi^{L_1L_2L_3}_{X_1X_2}(x) = (-1)^{l_1} \sum_{L} \left\{ \left( \begin{array}{ccc} l_1 & 0 & L_1 \\ 0 & l_2 & L_2 \\ 0 & 0 & L_3 \end{array} \right) - \left( \begin{array}{ccc} l_1 & l_2 & L_1 + L_2 \\ 0 & 0 & L_3 \end{array} \right) \right\} \times \int \frac{d\mu}{2} P_{L}^{\mu}(\mu) \int \frac{k^2 dk}{2\pi^2} j_l(kx) P_{X_1X_2}^{L_1L_2L_3}(k,\mu), \] (277)
where \( P_{L}^{\mu}(\mu) \) is the Legendre polynomial of order \( L' \). When the invariant spectrum does not depend on \( \mu \), it is straightforwardly shown that the above equation reduces to the previous result of Eq. (271), noting a special case of the 6-j symbol [47],
\[ \left\{ \begin{array}{ccc} l_1 & l_2 & 0 \\ 0 & i & L \end{array} \right\} = (-1)^{l_1+l_2+L} \frac{\delta_{L_1}^{l_1} \delta_{l_2}^{l_2}}{\sqrt{|l|! |0|!}} \] (278)

C. Scale-dependent bias in the power spectrum with non-Gaussian initial conditions

In the presence of bias, the local-type non-Gaussianity is known to produce the scale-dependent bias in the power spectrum on very large scales [3]. The same effect appears even in the shape correlations [69, 70], which is considered to be a second-rank \((l = 2)\) tensor field in our context. Higher moments of the shape correlations are also investigated [42].

In our formalism, these previous findings are elegantly reproduced as shown below. The derivation of scale-dependent bias from the primordial non-Gaussianity in the iPT formalism of scalar fields is already given in Ref. [18]. We can simply generalize the last method to the case of higher-rank tensor fields. We consider the clustering in real space below, while the generalization to that in redshift space is straightforward.

The primordial bispectrum \( B_L(k_1, k_2, k_3) \) is defined by three-point correlations of linear density contrast in Fourier space,
\[ \langle \delta_L(k_1) \delta_L(k_2) \delta_L(k_3) \rangle_c = (2\pi)^3 \delta_D^3(k_1 + k_2 + k_3) B_L(k_1, k_2, k_3), \] (279)
where \( \langle \cdots \rangle_c \) denotes the three-point cumulant, or the connected part. Applying a straightforward generalization of the corresponding result of iPT [18] to tensor fields, the lowest-order contribution of the primordial bispectrum to the power spectrum in the formalism of iPT is given by
\[ \Gamma^{NG(l_1)}_{X_1X_2;m_1m_2}(k) = \frac{\delta_k^{l_1}}{2} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) \times \int \frac{d^3 p}{(2\pi)^3} \Gamma^{(2)}_{X_1X_2;m_1m_2}(p, k-p, -p) B_L(k, -p, p-k) \times \left( \begin{array}{c} (X_1l_1m_1) \leftrightarrow (X_2l_2m_2) \end{array} \right), \] (280)
where we implicitly assume the parity symmetry of the Universe. The linear power spectrum with the Gaussian initial condition, Eq. (238), is added to the above in order to obtain the total power spectrum up to the linear order of the initial power spectrum and bispectrum. However, the Gaussian contribution does not generate scale-dependent bias in the limit of large scales, \( k \to 0 \), and thus the non-Gaussian contribution of Eq. (280) dominates in that limit.

The prefactor of the integral in Eq. (280) is given by Eq. (195):
\[ \Gamma^{NG(l_1)}_{X_1X_2;m_1}(k) = \frac{(-1)^{l_1}}{\sqrt{2l_1+1}} \left[ \xi^{(1)}_{X_1X_2}(k) + \delta_{l_1c} \xi_{X_1X_2}^{(0)} \right] C_{l_1m_1}(k), \] (281)
where we consider the lowest-order approximation and the summation factor \( \Pi(k) \) is just replaced by unity. In order to evaluate the integral in Eq. (280) with the spherical basis, one can insert a unity \( \int d^3 p \delta_k^{l_1}(p + p' - k) = 1 \) in the integral, and reexpress the delta function by a Fourier integral, and primordial bispectrum can be also expanded into spherical harmonics by means of the plane-wave expansion of Eq. (B5). In this way, the angular integrations are analytically performed, leaving only the radial integrals over \( p \) and \( p' \).

We are interested in the scale-dependent bias from the primordial non-Gaussianity, and instead of deriving a general expression according to the above procedure, we directly consider a limiting case of \( k \to 0 \) in Eq. (280). In this case, the integral is approximately given by
\[ \int \frac{d^3 p}{(2\pi)^3} \Gamma^{(2)}_{X_1X_2;m_1m_2}(p, -p) B_L(k, -p, p). \] (282)
The second-order propagator in the integrand is given by Eq. (77) with \( k_{12} = 0, k_1 = p \) and \( k_2 = -p \), and thus we have
\[ \Gamma^{(2)}_{X_1X_2;m_1m_2}(p, -p) = c_{X_1X_2}^{(2)}_{m_1m_2}(p, -p), \] (283)
where Eq. (135) is substituted. According to Eqs. (B9), (209) and (C5), we have
\[ X_{l_1m_1}^{l_2m_2}(\hat{p}, \hat{p}) = \left( \begin{array}{ccc} l_2 & l_1 & l_1 \\ 0 & 0 & 0 \end{array} \right) C_{l_1m_1}(\hat{p}). \] (284)

We decompose the primordial bispectrum with the spherical harmonics as
\[ B_L(k_1, k_2, k_3) = \sum_{l_1l_2l_3} \hat{B}^{(l_1l_2l_3)}_{l_1l_2l_3}(k_1, k_2, k_3) (l_1 l_2 l_3)^{m_1m_2m_3} \times C_{l_1m_1}(\hat{k}_1) C_{l_2m_2}(\hat{k}_2) C_{l_3m_3}(\hat{k}_3). \] (285)
One can always decompose the primordial bispectrum in the above form, which is shown in a similar way of deriving Eq. (142) when \( l = m = 0 \). The appearance of the \( 3-j \) symbols is due to the rotational symmetry. The inverse relation of the above is given by
\[ \hat{B}^{(l_1l_2l_3)}_{l_1l_2l_3}(k_1, k_2, k_3) = (-1)^{l_1+l_2+l_3} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) \times \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{4\pi^3} B_L(k_1, k_2, k_3) \times C_{l_1m_1}(\hat{k}_1) C_{l_2m_2}(\hat{k}_2) C_{l_3m_3}(\hat{k}_3). \] (286)
The particular bispectrum in the integrand of Eq. (282) is given by

\[
B_h(k, -p, p) = \sum_{l} C_{lm}(\hat{k}+\hat{p})C_{lm}(\hat{p}) \sum_{l',l''}(1)^l \times \left( \begin{array}{ccc} \tilde{1} & \tilde{1} & 0 \\ 0 & 0 & 0 \end{array} \right) B_{l'lt''}^{ll'}(k, p, p). \quad (287)
\]

Using the above equations together for the integral of Eq. (282), the angular integration over \( \hat{p} \) can be analytically performed by using the orthonormality relation of the spherical harmonics, Eq. (B6). Comparing the resulting equation with Eq. (214), the corresponding invariant power spectrum is derived, and the result is given by

\[
P_{X_1X_2}^{NG, l, l', l''}(k) = \left( \frac{-1}{2} \right)^l \frac{|l|}{\sqrt{\Omega}} \left| \begin{array}{ccc} l_1 & l_2 & l \\ l' & l'' & l \end{array} \right| \left[ c_{X_1l}^{(1)}(k) + \delta_{l_0} c_{X_1l}^{(0)} \right] \times \sum_{l_1'} \left(\begin{array}{ccc}
-l_1 + l' & l_1' & l_1 \\
0 & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
l_2 & l_2' & l_2 \\
0 & 0 & 0
\end{array}\right)
\times \int \frac{p^2 dp}{2\pi^2} c_{XXl'}^{(2)}(p, p) B_{l'lt''}^{ll'}(k, p, p)
\]

\[+ \{(X_1l_1) \leftrightarrow (X_2l_2)\}. \quad (288)
\]

This is one of the generic predictions of our theory.

The scale-dependent bias from the primordial non-Gaussianity is conveniently derived from the cross power spectrum between the linear density field and biased objects. In particular, we define the lowest-order scale-dependent bias factor by

\[
\Delta b_{Xlm}(k) = \frac{P_{Xlm}^{NG, (0)}}{P_{lm}(k)}, \quad (289)
\]

where \( X_1 = \delta \) indicates that the biased field \( F_{X_1l_1m}(k) \) is replaced by the mass density field \( \delta(k) \) with \( l_1 = m_1 = 0 \), and thus \( c_{X_1l}^{(0)} = 1 \) and \( c_{X_1l}^{(0)} = 0 \) for \( n \geq 1 \). Because of the rotational symmetry, and as explicitly seen from Eq. (214), the directional dependence on the wave vector on the rhs of Eq. (289) should be proportional to \( C_{lm}(\hat{k}) \), so naturally we define an invariant bias factor \( \Delta b_{Xlm}(k) \) by

\[
\Delta b_{Xlm}(k) = \frac{(-1)^l}{\sqrt{l!}} \Delta b_{Xl}(k)C_{lm}(\hat{k}). \quad (290)
\]

Substituting Eq. (214) into Eq. (289), the above definition is equivalent to

\[
\Delta b_{Xlm}(k) = \frac{P_{Xlm}^{NG, (0)}}{P_{lm}(k)} \times \frac{1}{\sqrt{l!}} \Delta b_{Xl}(k)C_{lm}(\hat{k}). \quad (291)
\]

Substituting \( l_1 = 0, l_2 = l, c_{X_1l}^{(0)} = 1, \) and \( c_{X_1l}^{(1)} = c_{X_2l}^{(2)} = 0 \) into Eq. (288), the invariant scale-dependent bias of Eq. (291) reduces to

\[
\Delta b_{Xlm}(k) = \frac{1}{2P_{lm}(k)} \sum_{l_1'} \left(\begin{array}{ccc}
-l_1 + l' & l_1' & l_1 \\
0 & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
l_2 & l_2' & l_2 \\
0 & 0 & 0
\end{array}\right)
\times \int \frac{p^2 dp}{2\pi^2} c_{XXl'}^{(2)}(p, p) B_{l'lt''}^{ll'}(k, p, p). \quad (292)
\]

This is one of the generic predictions of our theory.

The above formula is applicable for arbitrary models of primordial non-Gaussianity with a given bispectrum at the lowest order. We consider a particular model of non-Gaussianity, which is a generalization of the so-called local-type model [69, 71]:

\[
B_h(k_1, k_2, k_3) = \frac{2M(k_3)}{M(k_1)M(k_2)} \times \int \frac{f_{NL}^{(0)} P_l(k_1)P_l(k_2) + \text{cyc.}}{3(1 + z_s)D(z_s)H_0^2\Omega_m}, \quad (293)
\]

The function \( M(k) \) in the prefactor is given by

\[
M(k) = \frac{2}{3} \frac{D(z)}{(1 + z_s)D(z_s)} \frac{H_0^2\Omega_m}{k^2T(k)}, \quad (294)
\]

where \( D(z) \) is the linear growth factor with an arbitrary normalization, \( z_s \) is an arbitrary redshift at the matter-dominated epoch, \( T(k) \) is the transfer function, \( H_0 \) is the Hubble constant, and \( \Omega_m \) is the density parameter of total matter. The factor \( (1 + z_s)D(z_s) \) does not depend on the choice of \( z_s \) as long as \( z_s \) is deep in the matter-dominated epoch. Some authors choose a normalization of the linear growth factor by \( (1 + z_s)D(z_s) = 1 \) in which case the expression of Eq. (294) has the simplest form. Substituting Eq. (293) into Eq. (286), we have

\[
P_{l'lt''}^{NG, (2)} (k_1, k_2, k_3) = \frac{2M(k_3)}{M(k_1)M(k_2)} \frac{P_l(k_1)P_l(k_2)}{P_l(k)} \times \delta_{l_1l_2} \delta_{0l_0} (-1)^l \frac{1}{\sqrt{l!}} f_{NL}^{(0)} + \text{cyc.} \quad (295)
\]

We need a special form of the bispectrum to further evaluate Eqs. (288) and (292). With the above model of Eq. (293), we have

\[
P_{l'lt''}^{NG, (0)} (k, p, p) = \frac{2M(k)}{M(k)} \frac{P_l(p)}{P_l(k)} \delta_{l_0} \delta_{0l_0} \frac{1}{\sqrt{l!}} f_{NL}^{(0)} \times (-1)^l \frac{1}{\sqrt{l!}} f_{NL}^{(0)} \quad (296)
\]

where only two terms in which \( M(k) \) appeared in the denominator are retained because the other term is negligible in the limit of \( k \to 0 \).

Substituting Eq. (296) into Eq. (288), and using Eq. (C8), we have

\[
P_{X_1X_2}^{NG, l, l', l''}(k) = \left[1 + (-1)^l \frac{f_{NL}^{(0)} P_l(k)}{M(k)} \frac{1}{\sqrt{l!}} \left| \begin{array}{ccc} l_1 & l_2 & l \\ l' & l'' & l \end{array} \right| \right]
\times \left[ c_{X_1l}^{(1)}(k) + \delta_{l_0} c_{X_1l}^{(0)} \right] \sum_{l_1'} \left(\begin{array}{ccc}
-l_1 + l' & l_1' & l_1 \\
0 & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
l_2 & l_2' & l_2 \\
0 & 0 & 0
\end{array}\right)
\times \int \frac{p^2 dp}{2\pi^2} c_{XXl'}^{(2)}(p, p) P_l(p) \times \{(X_1l_1) \leftrightarrow (X_2l_2)\}. \quad (297)
\]

Substituting \( l_1 = 0, l_2 = l, c_{X_1l}^{(0)} = 1, \) and \( c_{X_1l}^{(1)} = c_{X_2l}^{(2)} = 0 \) into
the above equation, and using Eq. (291), we derive
\[
\Delta b_X(k) = \left[ 1 + (-1)^l \right] \frac{f_{NL}^{(l)}}{M(k)} \frac{1}{\sqrt{l(l+1)}} \sum_{l', l''} (-1)^{l''} \left[ \begin{array}{ccc} l'' & l & l' \\ 0 & 0 & 0 \end{array} \right] \int \frac{p^2 dp}{2\pi^2} \epsilon_{l'l''l'}(p, p) P_L(p). \tag{298}
\]

Apparently, the rhs survives only in the case of \( l = \) even. This proves that the scale-dependent bias of the rank-1 tensor field is sensitive to the multipole moment \( l \) of the Legendre polynomials \( P_l(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \) in the model of Eq. (293) for the primordial non-Gaussianity. That is previously shown in Ref. [42] by scalar perturbations are derived from the general result above.

As a consistency check, we can see if the known results of scalar perturbations are derived from the general result above. In the scalar case of \( l = 0 \), Eq. (298) reduces to
\[
\Delta b_{X0}(k) = \frac{2 f_{NL}^{(0)}}{M(k)} \sum_{l} \frac{1}{\sqrt{l(l+1)}} \int \frac{k^2 dk}{2\pi^2} \epsilon_{l,l,l}^{(0)}(p, p) P_L(p), \tag{299}
\]
where \( f_{NL}^{(l)} \equiv f_{NL}^{(0)}(l) \) is the original parameter of local-type non-Gaussianity. One can confirm that the last equation is consistent with the previously known results for the scale-dependent bias in the halo models. In fact, the second-order renormalized bias function of a simple halo model is given by [18]
\[
c_h^{(2)}(k_1, k_2) = \frac{\delta_0 \delta_0}{\sigma^2} W(k_1 R) W(k_2 R), \tag{300}\]
where
\[
\sigma^2 = \int \frac{k^2 dk}{2\pi^2} W^2(k R) P_L(k), \tag{301}\]
and \( b_1 \) is a Lagrangian bias parameter of linear order. Equation (300) is a result for the simplest case with a high-peaks (or high-mass) limit of halos in the Press-Schechter mass function (see Ref. [18] for results of other extended halo models). The Lagrangian bias parameter \( b_1 \) is related to the Eulerian bias parameter \( b_1 \) by \( b_1 = b_1^L = b_1 - 1 \). Comparing the above function with Eq. (132) or Eq. (135), and noting \( c_{h i 0}^{(2)}(k_1, k_2) = c_h^{(2)}(k_1, k_2) \), the invariant coefficient in this case is given by
\[
c_{h11i1}^{(2)}(k_1, k_2) = \delta_{i0} \delta_{i0} \frac{\delta \delta b_1^L}{\sigma^2} W(k_1 R) W(k_2 R). \tag{302}\]

Substituting the above expression into Eq. (299), we have
\[
\Delta b_{00}(k) = \frac{2 f_{NL} \delta \delta b_1^L}{M(k)}. \tag{303}\]
Because of the normalization of the spherical basis for the scalar component, Eqs. (7) and (32), the scale-dependent bias of the halo number density is given by \( \Delta b_h(k) = \Delta b_{00}(k) Y^{(0)} = \Delta b_{00} \), and thus we have
\[
\Delta b_h(k) = \frac{2(b_1 - 1) f_{NL} \delta \delta c}{M(k)}. \tag{304}\]
This exactly reproduces a well-known result of the scale-dependent bias in the simplest halo model, which was first derived from a quite different method [3].

D. The bispectrum

1. The invariant bispectrum

For yet another example of applications of our formalism, we consider the bispectrum of the tensor field. For an illustrative purpose, we only consider the bispectrum in real space below, while generalizing the results to those in redshift space is straightforward with similar methods employed for the power spectrum in Sec. V A.

The bispectrum \( B_{X1X2X3,m,m,m}(k_1, k_2, k_3) \) of irreducible components of tensor field \( F_{Xlm}(k) \) can be defined by
\[
\langle F_{X1l_1m_1}(k_1) F_{X2l_2m_2}(k_2) F_{X3l_3m_3}(k_3) \rangle_c = (2\pi)^3 \delta^3_D(k_1 + k_2 + k_3) B_{X1X2X3,m,m,m}(k_1, k_2, k_3), \tag{305}\]
as a simple generalization of the definition of the power spectrum of Eq. (211). We consider generally the cross bispectrum among three fields, \( X_1 \), \( X_2 \) and \( X_3 \) with irreducible components \( l_1, l_2 \) and \( l_3 \), respectively. The auto-bispectrum is straightforwardly obtained by putting \( X_1 = X_2 = X_3 \).

The bispectrum \( B_{X1X2X3,m,m,m}(k_1, k_2, k_3) \) defined above depends on the coordinates system. Generalizing the method of constructing the invariant power spectrum, one can also construct the invariant bispectrum as explicitly shown below. Because of the presence of the delta function on the rhs of Eq. (305), the third argument \( k_3 \) in the bispectrum can be replaced by \( -k_1 - k_2 \), and regarded as a function of only \( k_1 \) and \( k_2 \). However, the resulting expression loses apparent symmetry of permutating the subscripts \( (1, 2, 3) \) of \( k_i, l_i \) and \( m_i \). As is usually the case in the analytic predictions in perturbation theory, we can decompose the symmetric bispectrum into a sum of asymmetric components as
\[
B_{X1X2X3,m,m,m}(k_1, k_2, k_3) = B_{X1X2X3,m,m,m}(k_1, k_2) + \text{cyc}. \tag{306}\]
Although the decomposition into asymmetric components is not unique, it is always possible to specify the originally symmetric bispectrum by giving the asymmetric bispectra, and the predictions of perturbation theory are usually given in the above form.

The directional dependence of the asymmetric bispectrum on the wave vectors can be expanded by spherical harmonics as
\[
B_{X1X2X3,m,m,m}(k_1, k_2) = \sum_{l_1, l_2} B_{X1X2X3,m,m,m}^{l_1, l_2}(k_1, k_2) \times C_{l_1 m_1}^*(\hat{k}_1) C_{l_2 m_2}^*(\hat{k}_2). \tag{307}\]
Because of the rotational transformations of Eqs. (93) and (23), and rotational invariance of the delta function in the definition of the bispectrum, Eq. (305), the rotational transforma-
tion of the expansion coefficients is given by

\[ B_{X_1X_2X_3m_1m_2m_3m_4}(k_1, k_2) \]

\[ \to X_1X_2X_3m_1m_2m_3m_4(k_1, k_2)D_{l_1m_1}^{m_1}(R) \]

\[ \times D_{l_2m_2}^{m_2}(R)D_{l_3m_3}^{m_3}(R)D_{l_4m_4}^{m_4}(R). \]  

(308)

For the statistically isotropic Universe, the functional form of the bispectrum is shown to be given by

\[ \tilde{B}_{X_1X_2X_3m_1m_2m_3m_4}(k_1, k_2) = \frac{p_{X_1} + p_{X_2} + p_{X_3} + l_1 + l_2 + l_3 + l_4 + l_5 + l_6}{p_{X_1} + p_{X_2} + p_{X_3} + l_1 + l_2 + l_3 + l_4 + l_5 + l_6} \]  

(314)

2. The lowest-order bispectrum

We consider the bispectrum of the lowest-order approximation or the tree-level bispectrum. The formal expression of the lowest-order bispectrum in terms of propagators is given by

\[ B^{(l_1l_2l_3)}_{X_1X_2X_3m_1m_2m_3m_4}(k_1, k_2) = i^{l_1+l_2+l_3} \Gamma^{(1)}_{X_1X_2X_3m_1m_2m_3m_4}(k_1, k_2) \]

\[ \times \Gamma^{(2)}_{X_1X_2X_3m_1m_2m_3m_4}(k_1, k_2). \]  

(315)

We do not use the higher-order resummation factor and substitute \( \Pi(k) = 1 \). Substituting Eqs. (160) and (162), and applying Eq. (209), the above equation is represented in terms of invariant functions as

\[ B^{(l_1l_2l_3)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2) = \frac{i^{l_1+l_2+l_3}}{\sqrt{|l_1|^2|l_1|^2}} \Gamma^{(1)}_{X_1X_2X_3}(k_1) \Gamma^{(1)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2) \]

\[ \times \Gamma^{(2)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2). \]  

(316)

Substituting Eqs. (195) and (210), the above expression is given by invariant forms of renormalized bias functions, \( \Gamma^{(1)}_{X_1X_2X_3}(k_1, k_2) \) and \( \Gamma^{(2)}_{X_1X_2X_3}(k_1, k_2) \).

The sum of the products of three \( 3j \)-symbols appearing in the above expression is represented by a \( 9j \)-symbol using the formulas of Eqs. (C9) and (C25) in Appendix C. Comparing the resulting equation with Eq. (309), the invariant bispectrum is derived, and the result is given by

\[ B^{(l_1l_2l_3l_4l_5l_6)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2) = (\cdots) \Gamma^{(1)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2) \]

\[ \times \Gamma^{(2)}_{X_1X_2X_3m_1m_2m_3m_4m_5m_6m_7}(k_1, k_2). \]  

(317)

This is one of the generic predictions of our theory.

As a consistency check, let us see if the bispectrum of the scalarly biased field in the lowest-order perturbation theory can be correctly reproduced. Considering the case of \( l_1 = l_2 = l_3 = m_1 = m_2 = m_3 = 0 \) and \( X_1 = X_2 = X_3 \equiv X \) in Eq. (316), and using Eqs. (C8), (C14) and (B4), we straightforwardly derive

\[ B^{(000)}_{XXX000}(k_1, k_2, k_3) = P_L(k_1)P_L(k_2) \sum_{l_1} \frac{(1)_l}{(1)_l} P_l(\hat{k}_1 \cdot \hat{k}_2) \]

\[ \times \Gamma^{(1)}_{X_0X_0X_0}(k_1, k_2, k_3) \Gamma^{(2)}_{X_0X_0X_0}(k_1, k_2, k_3). \]  

(318)

The same result is also derived from Eqs. (309) and (317) with Eq. (251). We substitute Eqs. (196) and (210) in the above
equation, and straightforward calculations derive the scalar bispectrum as
\[
B_X(k_1, k_2, k_3) = B_{XXX}^{(000)}(k_1, k_2, k_3)
\]
\[
= b_1(k_1) b_1(k_2) P_L(k_1) P_L(k_2) \left\{ b_2^L(k_1, k_2) + b_1(k_1) + b_1(k_2) \right\}
\]
\[
- \frac{4}{7} \left[ \frac{k_2}{k_1} b_1(k_1) + \frac{k_1}{k_2} b_1(k_2) \right] \frac{k_1 \cdot k_2}{k_1 k_2} + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right\}
\]
+ cyc., \quad (319)

where \( b_1(k) = c_X^{(0)} + c_X^{(1)}(k) \) corresponds to the Eulerian bias factor and is the same as \( b_X(k) \) defined in Eq. (245), and
\[
b_2^L(k_1, k_2) \equiv c_X^{(2)}(k_1, k_2) = \sum_l \frac{(-1)^l}{\sqrt{l!}} P_l(\hat{k}_1 \cdot \hat{k}_2) c_X^{(2)}(k_1, k_2)
\]
(320) corresponds to the Lagrangian (nonlocal) bias function of the second order [17]. The more commonly known form of the bispectrum in the lowest-order perturbation theory is represented by the Eulerian bias parameters \( b_1, b_2, b_3 \) [72–75]. It is natural to take a normalization \( \langle F_X^L \rangle = 1 \) [cf., Eq. (50)] and we have \( c_X^{(0)} = 1 \) in this scalar case. On one hand, in the special case of local bias, in which \( b_1 \) and \( b_2^L \) are constants and do not depend on wave vectors, Eq. (319) reduces to
\[
B_X(k_1, k_2, k_3) = b_1^2 P_L(k_1) P_L(k_2)
\]
\[
\times \left\{ b_2^L + 2 b_1 - \frac{4}{7} b_1 \left( \frac{k_1}{k_2} \right) \frac{k_1 \cdot k_2}{k_1 k_2} + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right\}
\]
+ cyc., \quad (321)

On the other hand, the prediction of scalar bispectrum in the standard (Eulerian) perturbation theory is given by [75],
\[
B_X(k_1, k_2, k_3) = b_1^2 P_L(k_1) P_L(k_2)
\]
\[
\times \left\{ b_2 + 2 b_3 \left[ \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 - \frac{1}{3} \right] + b_1 F_2(k_1, k_2) \right\}
\]
+ cyc., \quad (322)

where
\[
F_2(k_1, k_2) = \frac{10}{7} + \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) \frac{k_1 \cdot k_2}{k_1 k_2} + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2
\]
(323) is the second-order kernel$^5$ of the Eulerian perturbation theory [8]. Comparing the coefficients of Eqs. (321) and (322), we find the relations between Lagrangian local bias parameters and Eulerian local bias parameters as
\[
b_1 = 1 + b_1^L, \quad b_2 = b_2^L - \frac{8}{21} b_1^L, \quad b_3 = -\frac{2}{7} b_1^L.
\]
(324)

The above relations exactly agree with those which are found in the literature [74, 75]. It is also possible to include the nonlocal Lagrangian bias \( b_2^L \) in \( b_2^L(k_1, k_2) \), and derive consistent results with those in the literature [74, 75].

VI. SEMILOCAL MODELS OF BIAS FOR TENSOR FIELDS

So far the formulation is general enough, and we have not assumed any model of bias for the tensor fields so far. The bias model is naturally specified in Lagrangian space through the renormalized bias functions \( c_X^{(0)} \) defined by Eq. (71), which is evaluated once a model of bias \( F_X^{Xlm}(q) \) is analytically given as a functional of the linear density field \( \delta_L(q) \). In this section, we consider a category of bias models which are commonly adopted in models of cosmological structure formation. We call this category “semilocal models” of bias.

A. The semilocal models of bias

1. Defining the semilocal models

The concept of the semilocal models is somehow similar to the EFTofLSS approach in perturbation theory of biased tracers [4, 76], where the biased field is given by a finite set of semilocal operators made of spatial and temporal derivatives of the gravitational potential in the perturbative expansions order by order. However, while the EFTofLSS approach is based on phenomenologically perturbative expansions of the biased field, our iPT approach does not assume the underlying operators are perturbative, because we essentially use orthogonal expansions instead of Taylor expansions to include the fully nonlinear biasing into the cosmological perturbation theory through the renormalized bias functions.

This category of semilocal models of bias is defined so that the field \( F_X^{Xlm}(q) \) is given by functions (instead of functionals) of the spatial derivatives of the gravitational potential at the same position \( q \), smoothed with (generally, multiple numbers of) window functions \( W^{(a)}(k) \):
\[
\chi_{Xlm}^{(a)}(q) = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_a} \psi^{(a)}(q),
\]
(325)

where
\[
\psi^{(a)}(q) = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot q} \delta_L(k) k^{L_a} W^{(a)}(k)
\]
(326) is the smoothed linear density field with an isotropic window function \( k^{L_a} W^{(a)}(k) \). The label "a" distinguishes different kinds of the linear tensor field with a particular rank \( L_a \), i.e., the label \( a \) uniquely specifies the rank \( L_a \) and the form of the window function \( W^{(a)}(k) \). The window function usually contains a parameter of smoothing radius \( R_a \), which can take different values for each window function \( W^{(a)} \). While typical window functions include the Gaussian window, \( W^{(a)}(k) = \exp(-k^2 R_a^2 / 2) \), and the top-hat window, \( W^{(a)}(k) = 3 j_1(k R_a) / (k R_a) \), we do not assume any specific form

$^5$ Our choice of the normalizations for kernel functions \( F_a \) is different from those in many previous references, where our \( F_a \) corresponds to \( n! F_a \) in the latter, minimizing the occurrence of \( n! \) in resulting expressions [17].
of the window function here. When only a single linear tensor field of a fixed window function is considered, one can omit the label $a$ in the above. We include the possibility of using multiple numbers of fields and window functions in the general formulation below.

For a concrete example of biasing using multiple numbers of fields and window functions, see Ref. [20]. For example, the second derivatives of the (normalized and smoothed) linear gravitational potential $\varphi$ correspond to $\varphi = \Delta^{-1} \delta R = -\psi(a)$ with a rank of $L_a = 2$, and $\chi_{i,j}^{(a)} = -\partial_i \partial_j \Delta^{-1} \delta R$ where $\delta R$ is a smoothed linear density field with a smoothing window function $W^{(a)}(k) = W(kR)$, and $\Delta^{-1}$ is the inverse operator of the Laplacian $\Delta = \partial_i \partial_i$.

The linear density contrast $\delta_l$ is a real function in configuration space, and thus we have $\delta^*_l(k) = \delta_l(-k)$ as shown from Eq. (62). If the derivative field $\chi_{i,j}^{(a)}(q)$ and thus $\psi(a)(q)$ are real functions, the window $W^{(a)}(k)$ is a real function.

In the semilocal models of bias, the field value $F^{L}_{Xlm}(q)$ at a particular position is given by a function of the field values $X_{lm}^{(a)}(q)$ at the same position in Lagrangian space. This function is common among all the positions. Therefore, without loss of generality, one can consider the particular position at the origin, $q = 0$, to describe the relation in the semilocal model of bias, and thus the field $F^{L}_{Xlm}$ is a function of

$$
X_{i_1\cdots i_n}^{(a)}(q) = i^n \int \frac{d^3k}{(2\pi)^3} \hat{k}_{i_1} \cdots \hat{k}_{i_n} \delta_l(k) W^{(a)}(k),
$$

(327)

where $\hat{k}_i = k_i/k$. The tensor field $F^{L}_{Xlm}(q)$ can depend on multiple local tensors of various ranks, $X^{(a)}_{lm}$, $X^{(a)}_{lj}$, $X^{(a)}_{ij}$, $X^{(a)}_{ijk}$, etc.

There is a caveat in the terminology here. In conventional perturbation theory, the bias relations involving the nondiagonal part of second-order derivatives of the gravitational potential, $\partial_i \partial_j \varphi$, are often referred to as a “nonlocal bias.” In the terminology of this paper and of iPT in general, this type of bias also falls into the category of “semilocal bias.”

2. Irreducible decomposition of semilocal variables

A local tensor can be decomposed by an irreducible spherical basis according to the procedure explained in Sec. II. The tensor of Eq. (327) is decomposed into traceless tensor of rank $L_a$, $L_a - 2$, . . . , and each component of traceless tensor is decomposed by a spherical basis, as in the Eqs. (36)–(43). The decomposition is given by

$$
X_{i_1\cdots i_l}^{(a)} = X_{i_1\cdots i_l}^{(al)} + \frac{l(l-1)}{2(l-1)} \delta_{i_1,i_l} X_{i_2\cdots i_{l-2}}^{(a,l-2)} + \cdots,
$$

(328)

where we simply denote $L_a = l$ in the above, and $X_{i_1\cdots i_l}^{(a,l-2)}$ is the traceless part of the first trace $X_{i_1\cdots i_l}^{(al)}$, and so forth.

The decomposed traceless tensors are further decomposed by spherical basis. Similarly to Eqs. (42) and (44), we define

$$
X_{i_1\cdots i_l}^{(al)} = A_l X_{lm}^{(al)} Y_{i_1\cdots i_l}^{(m)},
$$

(329)

$$
X_{i_1\cdots i_l}^{(a,l-2)} = A_{l-2} X_{l-2,m}^{(a,l-2)} Y_{i_1\cdots i_l}^{(m)},
$$

(330)

and so forth. In our simplified notation with $l = L_a$, the rank of the linear tensor field $a$ is not obvious if we write, e.g., $X_{i_1\cdots i_l}^{(a)}$, the first trace of the original tensor, and thus we instead employ a notations $X_{lm}^{(al)}$, $X_{l-2,m}^{(a,l-2)}$, etc. to remind the rank of the original field $a$ is $l$ in the lower-rank trace parts of the tensor. Namely,

$$
X_{lm}^{(al)} = X_{l,m}^{(a)} |_{L_a = l}, \quad X_{l-2,m}^{(a,l-2)} = X_{l-2,m}^{(a)} |_{L_a = l},
$$

(331)

and so forth.

The relations of Eqs. (329) and (330) are inverted as well as Eqs. (43) and (45), and further using Eqs. (18) and (43), the spherical tensors are given by

$$
X_{lm}^{(al)} = \frac{1}{A_l} X_{l,m}^{(a)} Y_{lm}^{(m)},
$$

(332)

$$
X_{l-2,m}^{(a,l-2)} = \frac{1}{A_{l-2}} X_{l-2,m}^{(a,l-2)} Y_{l-2,m}^{(m)}.
$$

(333)

and so forth, where $A_l$ is given by Eq. (19), and $C_{lm}(\hat{k})$ is the spherical harmonics with Racah’s normalization. In general, we have

$$
X_{l-2,p,m}^{(a,l-2)} = \frac{1}{A_{l-2}} X_{l-2,p,m}^{(a,l-2)} Y_{l-2,p,m}^{(m)},
$$

(334)

where $p = 0, 1, \ldots, [l/2]$. Changing the labels of ranks by $l \to L$ and $l - 2p \to l$, Eq. (334) is equivalently represented by

$$
X_{lm}^{(a,l)} = i^l \int \frac{d^3k}{(2\pi)^3} \delta_l(k) C_{l,m}(\hat{k}) W^{(a)}(k),
$$

(335)

where $l = L, L - 2, \ldots, (0 or 1)$ for a linear tensor field $a$ of rank $L$. The smallest value of $l$ in the last equation is 0 if $L$ is even, and 1 if $L$ is odd. When the rank $L_a$ of the original tensor $X_{i_1\cdots i_l}^{(a)}$ is obvious, one can employ a simplified notation $X_{lm}^{(a)}$ instead of $X_{lm}^{(al)}$, where $l = L, L - 2, \ldots$. In Eq. (335), $L - l$ is an even number, and the complex conjugate of the variable is given by

$$
X_{lm}^{(a)*} = \psi^{(a)}(k) X_{lm}^{(a)}.
$$

(336)

where a simplified notation $X_{lm}^{(a)}$ is adopted to represent the original variable, $X_{lm}^{(al)}$.

3. Simple examples

For a simple example, we consider the second derivatives of the normalized potential $\varphi = \Delta^{-1} \delta R$, smoothed with a window
function $W(kR)$. In this case, we have
\[
\chi_{ij}^{(\psi)} = -\partial_i \partial_j \varphi = -\int \frac{d^3k}{(2\pi)^3} \hat{k}_i \hat{k}_j \rho_L(k) W(kR). \tag{337}
\]
The general variables in Eqs. (326) and (327) correspond to $\vartheta^{(a)} = -\varphi$ and $W^{(a)}(k) = W(kR)$. The second-rank tensor is decomposed into irreducible components as
\[
\chi_{ij}^{(\psi)} = -\left( \partial_i \partial_j - \frac{\delta_{ij}}{3} \right) \varphi - \frac{\delta_{ij}}{3} \varphi \partial_R + \frac{2}{3} \chi_{ij}^{(\phi)} W_{(m)}^{(0)}, \tag{338}
\]
where, corresponding to Eqs. (332) and (333), we have
\[
\chi_{00}^{(\psi[2])} = -\int \frac{d^3k}{(2\pi)^3} \delta_L(k) \rho_L(k) W(kR), \tag{339}
\]
The last variable is apparently given by
\[
\chi_{00}^{(\psi[2])} = -\int \frac{d^3k}{(2\pi)^3} \delta_L(k) \rho_L(k) W(kR) \equiv -\delta_R, \tag{340}
\]
where $\delta_R$ is the smoothed density field with a smoothing function $W(kR)$.

Similarly, the second derivatives of the smoothed density field are given by
\[
\chi_{ij}^{(\delta)} = \partial_i \partial_j \delta_R = -\int \frac{d^3k}{(2\pi)^3} k^2 \hat{k}_i \hat{k}_j \rho_L(k) W(kR). \tag{341}
\]
The general variables in Eqs. (326) and (327) correspond to $\vartheta^{(a)} = \delta_R$ and $W^{(a)}(k) = k^2 W(kR)$. The second-rank tensor is decomposed into irreducible components as
\[
\chi_{ij}^{(\delta)} = \partial_i \partial_j \delta_R = \left( \partial_i \partial_j - \frac{\delta_{ij}}{3} \right) \delta_R + \frac{\delta_{ij}}{3} \partial_R \delta_R + \frac{2}{3} \chi_{ij}^{(\phi)} W_{(m)}^{(0)}, \tag{342}
\]
where
\[
\chi_{2m}^{(\psi[2])} = -\int \frac{d^3k}{(2\pi)^3} \delta_L(k) C_{2m}(k) W(kR), \tag{343}
\]
and the last variable is apparently given by
\[
\chi_{00}^{(\psi[2])} = -\int \frac{d^3k}{(2\pi)^3} \delta_L(k) W(kR) = \Delta \delta_R. \tag{344}
\]

B. Renormalized bias parameters

1. Renormalized bias functions in the semilocal models

In the semilocal models of bias, the tensor field $F_{Xlm}^L$ is generally considered as a function of various irreducible tensors
\[
F_{Xlm}^L = F_{Xlm}^L \left( \chi_{mn}^{(a)} \right) \tag{345}
\]
at every position in (Lagrangian) configuration space. In this case, the functional derivative in the definition of renormalized bias functions, Eq. (71), can be replaced by (with a simplified notation $\delta_L \to \delta_L$ in Fourier space)
\[
(2\pi)^3 \frac{\delta}{\delta \delta_L(k)} \to \sum_{a} \int d^2k \ W^{(a)}(k) \sum_{l=1}^{L_a} \ C_{lm}(k) \frac{\partial}{\partial \chi_{lm}^{(a)}} \tag{346}
\]
where the integer $L_a$ is the original rank of the linear tensor $\chi_{1-1}^{(a)}$. The summation over $m$ in Eq. (348) is implicitly assumed according to the Einstein summation convention, just as in the rest of this paper. Therefore, the renormalized bias functions of Eq. (71) are given by
\[
\psi_X^{(n)}(k_1, \ldots, k_n) = (-i)^l \sum_{a_1 \ldots a_n} \int \delta^{l_1 + \ldots + l_n} \times W^{(a_1)}(k_1) \ldots W^{(a_n)}(k_n) \ C_{a_1 a_2} \ldots \ C_{a_{m_1}, a_{m_2}} \times \left\{ \frac{\partial^n F_{Xlm}}{\partial \chi_{lm}^{(a_1)} \cdots \partial \chi_{lm}^{(a_n)}} \right\}, \tag{347}
\]
where integers $l, (i = 1, \ldots, n)$ run over $l = L_{a_1}, L_{a_2} = 2, \ldots, 0$ or 1. Comparing the above equation with Eq. (108), we have
\[
\psi_X^{(n)}(k_1, \ldots, k_n) = (-i)^l \sum_{a_1 \ldots a_n} \int \delta^{l_1 + \ldots + l_n} \times W^{(a_1)}(k_1) \ldots W^{(a_n)}(k_n) \ g_{m_1}^{(a_1)} \ldots \ g_{m_n}^{(a_n)} \times \left\{ \frac{\partial^n F_{Xlm}}{\partial \chi_{lm}^{(a_1)} \cdots \partial \chi_{lm}^{(a_n)}} \right\}. \tag{348}
\]
The renormalized bias functions in the form of Eq. (349) are evaluated for a given model of the biased tensor field, once the underlying statistics of the fields $\chi_{lm}^{(a)}$ are specified. These fields just linearly depend on the linear density contrast $\delta_L$ through Eqs. (332)–(335) and therefore the statistics are straightforwardly given by those of linear density field.

Using Eqs. (75) and (335), and the orthonormality relation of the spherical harmonics, Eq. (B6), the covariance of the fields (at the same position, the same applies hereafter) can be straightforwardly calculated and is given by
\[
\langle \chi_{lm}^{(a)}(k) \chi_{lm}^{(b)}(k) \rangle = \delta_{l}^{b} \gamma_{l}^{ab}(k)^{m_m}, \tag{349}
\]
where
\[
\gamma_{l}^{ab} = \frac{2l+1}{2l+1} \int \frac{k^2dk}{2\pi^2} \ W^{(a)}(k) W^{(b)}(k) P_l(k). \tag{350}
\]
One can regard the set of parameters $\gamma_{l}^{ab}$ as matrix elements of a matrix $\gamma_{l}$, whose components are given by $[\gamma_{l}]_{ab} = \gamma_{l}^{ab}$. We denote the matrix elements of the inverse matrix as
\[
\gamma_{l}^{ab} = [\gamma_{l}^{-1}]_{ab}. \tag{351}
\]
When one considers a set of variables $\chi_{lm}^{(a)}$ as a vector with the set of indices $(a, l, m)$, the inverse of the covariance matrix of Eq. (351) is given by $\delta_{l}^{b} \gamma_{l}^{ab}(k)^{m_m}$ with our notations.
In Eq. (351), the integer of rank-\(l\) should satisfy a condition that integers \(L_a - l\) and \(L_b - l\) are non-negative even numbers, where \(L_a\) and \(L_b\) are original ranks of tensors. Therefore, the integer \(L_a + L_b\) is also an even number, and \(l \leq \min(L_a, L_b)\). The matrix \(\gamma_{ab}\) composed from matrix elements of Eq. (352) is a real symmetric matrix when the constraint above is satisfied, and so is the inverse matrix, i.e.,
\[
\gamma_{ab} = \gamma_{ba}, \quad \gamma_{ab} = \gamma_{ba},
\]
for physically meaningful set of indices in Eq. (351). When the initial density field \(\delta_{l}\) is a Gaussian random field, the two-point covariance of Eq. (351) contains all the information for the statistical distribution of the variables. In this case, the distribution function is given by a multivariate Gaussian distribution function,
\[
P_G\left(\{\chi_{lm}^{(a)}\}\right) = \frac{1}{(2\pi)^{N/2} \det \gamma(l)} \exp\left[-\frac{1}{2} \sum_{l} \gamma_{ab} g(l) m_{lm}^{(a)} \chi_{lm}^{(a)} m_{lm}^{(b)}\right],
\]
where the Einstein summation convention is applied also to the indices \(a, b\) as well as azimuthal indices \(m, m'\) and thus the summation over these indices is implicitly assumed in the exponent. While the variables \(\chi_{lm}^{(a)}\) are complex numbers, the exponent of Eq. (355) is a real number, when the original derivative field \(\chi_{lm}^{(a)}\) is real. This property is readily shown by noting that each term in the summation is represented by \(\sum m_{lm}^{(a)} \gamma_{ab} m_{lm}^{(b)}\), and the matrix elements \(\gamma_{ab}\) make a real symmetric matrix (real Hermitian matrix) due to Eq. (354), and any quadractic form of a real symmetric matrix is a real number.

2. Defining the renormalized bias parameters

When the initial density fluctuations are Gaussian, the expectation value in Eq. (349) is calculated by multivariate Gaussian integrals with the distribution function of Eq. (355), whose evaluation is analytically possible in many cases when the semilocal bias function of Eq. (347) is given by an analytic function of a finite number of variables. When there is a small non-Gaussianity in the initial condition, and the linear density field is not exactly a random Gaussian field, one can evaluate non-Gaussian corrections to the Gaussian distribution function of Eq. (355). The procedure of deriving the non-Gaussian corrections is found in Refs. [23, 77], which is straightforward to apply in this case. However, for the illustrative examples below of this paper, we do not need to include the non-Gaussian corrections in the evaluations of renormalized bias functions.

The expectation value on the rhs of Eqs. (349) and (350) is represented by rotationally invariant variables, just in the case of the renormalized bias function \(c_{X_{l_1 L_1}}^{(a_1, \ldots, a_n)}(k_1, \ldots, k_n)\), while the arguments of \(k\) are not present here. Similarly to Eqs. (126), (130), (140), and (151), we have
\[
\left(\frac{\partial^2 F_{X_m}}{\partial x_{l_1 m_1} \partial x_{l_2 m_2}}\right) = \left(-i\frac{e^{i L_1}}{2\pi l^1} \right) \delta_{ij} \delta_{m_1} \delta_{m_2} b_{X}^{(a_1 a_2)}(k_1),
\]
and
\[
\left(\frac{\partial^2 F_{X_m}}{\partial x_{l_1 m_1} \partial x_{l_2 m_2} \partial x_{l_3 m_3}}\right) = \left(-i\frac{e^{i L_1 + L_2}}{2\pi l^1 l^2} \right) m_1 m_2 m_3 b_{X}^{(a_1 a_2 a_3)}(k_1, k_2),
\]
and
\[
\left(\frac{\partial^2 F_{X_m}}{\partial x_{l_1 m_1} \partial x_{l_2 m_2} \partial x_{l_3 m_3} \partial x_{l_4 m_4}}\right) = \left(-i\frac{e^{i L_1 + L_2 + L_3}}{2\pi l^1 l^2 l^3} \right) m_1 m_2 m_3 m_4 b_{X}^{(a_1 a_2 a_3 a_4)}(k_1, k_2, k_3),
\]
and
\[
c_{X_{l_{n-1} L_{n-1}}}^{(a_1, \ldots, a_n)}(k_1, \ldots, k_n) = \sum_{a_{n-1}} b_{X_{l_{n-1} L_{n-1}}}^{(a_1 a_2 a_3 \ldots a_n)}(k_1, \ldots, k_n) W^{(a_n)}(k_n).
\]
Thus, the renormalized bias functions are given by superpositions of products of window functions with constant parameters which are determined by a given semilocal model of bias. We naturally call these parameters as the “renormalized bias parameters.” The scale dependencies in the invariant renormalized functions are all contained in the window functions which are fixed by the construction of the semilocal model with a finite number of scale-independent parameters. These properties simplify the calculations of loop corrections in our applications. As the invariant renormalized functions and the window functions \(W^{(a)}(k)\) are both real, the renormalized bias parameters are also real parameters.

Corresponding to the interchange symmetries of the renormalized bias functions with orders greater than 2, Eqs. (139), (140) and (151) are satisfied.
for the third-order bias, and
\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} b^{(3, a_1 a_2 a_3)}_{Xlkl} \]
(364)
for the second-order bias,
\[ b^{(2, a_1 a_2)}_{Xlkl} = (-1)^{l_1 + l_2} b^{(2, a_1 a_2)}_{Xlkl} \]
(365)
\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} \sum_{L'} (2L' + 1) \left\{ \begin{array}{ccc} l_1 & l & L \\ l_1 & l_1 & L' \end{array} \right\} b^{(3, a_1 a_2 a_3)}_{Xlkl} \]
(366)
for the third-order bias, and
\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} b^{(3, a_1 a_2 a_3)}_{Xlkl} \]
(367)
\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} \sum_{L'} (2L' + 1) \left\{ \begin{array}{ccc} l_1 & l & L \\ l_1 & l_1 & L' \end{array} \right\} b^{(2, a_1 a_2 a_3)}_{Xlkl} \]
(368)
for higher-order bias with \( n > 3 \), where \( i = 1, \ldots, n - 2 \).

3. An example: The scale-dependent bias in tensor fields

As a specific example, there is a loop integral in the scale-dependent bias of Eq. (298). Applying an expression Eq. (361) of semilocal models, the integral is given by

\[ \int \frac{p^2 dp}{2\pi^2} T^{(2)}(p, p) P_L(p) = \sum_{a_1, a_2} b^{(2, a_1 a_2)}_{Xlkl} \sigma^{(a_1 a_2)}, \]
(369)
where

\[ \sigma^{(a_1 a_2)} = \int \frac{k^2 dk}{2\pi^2} P_L(k) W^{(a_1)}(k) W^{(a_2)}(k). \]
(370)
Thus Eq. (298) reduces to

\[ \Delta b_{Xl}(k) = \frac{2/j_{NL}(k)}{M(k)} \frac{1}{\sqrt{|l|}} \sum_{l_1, l_2} (-1)^{l_1 + l_2} \left\{ \begin{array}{ccc} l_1 & l_2 & l \\ 0 & 0 & 0 \end{array} \right\} \sum_{a_1, a_2} b^{(2, a_1 a_2)}_{Xlkl} \sigma^{(a_1 a_2)}, \]
(371)
for \( l \) even and \( \Delta b_{Xl}(k) = 0 \) for \( l \) odd.

For a simple case of scalar bias with the high-mass limit of the halo model, comparison of Eq. (302) with Eq. (361) shows that only a term with \( b^{(2, 0)}_{Xlkl} = \sigma^2 \delta \delta^2_{NL} \) survives, and one can readily see the result of Eq. (303) is reproduced as a consistency check.

If we consider a simple model that the tensor bias is a local function of only the second-order derivatives of the gravitational potential \( \partial \phi \), the window function is given by \( W^{(a)}(k) = W(kR) \) as shown in Eq. (371). We can omit the label \( (\phi) \) in this case, because the linear tensor field consists of only a single tensor. Thus Eq. (370) simply reduces to

\[ \sigma^2 = \int \frac{k^2 dk}{2\pi^2} P_L(k) W^2(kR), \]
(372)
which is the variance of the smoothed linear density field. The ranks \( l_1 \) and \( l_2 \) take only values of 0 and 2 in the summation of Eq. (371) in this case, and due to the \( 3j \)-symbol in Eq. (371), only the cases of \( l = 0, 2, 4 \) are nonzero. Substituting concrete numbers of the \( 3j \)-symbols, Eq. (371) is explicitly expanded in a finite number of terms and the result is given by

\[ \Delta b_{Xl}(k) = \frac{2\pi^2}{M(k)} \left[ \delta_{0,0} b^{(2, 0)}_{Xlkl} + \frac{1}{\sqrt{5}} b^{(2, 0)}_{Xlkl} \right] \]
\[ + \frac{2}{3} \delta_{0,2} b^{(2, 0)}_{Xlkl} - \frac{1}{\sqrt{14}} b^{(2, 0)}_{Xlkl} \]
\[ + \frac{1}{3} \frac{2}{\sqrt{35}} \delta_{0,2} b^{(2, 0)}_{Xlkl} \]
(373)

where we use an interchange symmetry \( b^{(2)}_{2,20} = b^{(2)}_{2,02} \) derived from Eq. (364). Therefore, if the tensor bias is modeled as a local function of only a second-rank linear tensor field, the scale-dependent bias of the tensor field is nonzero only when the rank of the biased tensor is 0, 2, or 4. Because of the rotational symmetry, only a small number of bias parameters, \( b^{(2)}_{0,00}, b^{(2)}_{0,22}, b^{(2)}_{2,02}, b^{(2)}_{2,22}, \) and \( b^{(2)}_{4,22} \) appear in this particular semilocal model.

The above examples probably do not sufficiently exhibit the practical merits of scale-dependent models of bias. Their advantages are more apparent in the calculation of loop corrections in the higher-order perturbation theory. The present formalism with the full use of the spherical basis is quite compatible with the FFT-PT framework [78, 79] or the FAST-PT framework [80, 81], which dramatically reduces the dimensionality of multidimensional integrals in the calculation of loop corrections in the higher-order perturbation theory. Technical details of calculating the loop corrections in the present formalism will be considered in a separate paper of the series, Paper II [43].

C. Relations to the conventional approach of bias renormalization

On one hand, the renormalized bias functions in the IPT are fully renormalized from the first place [17], and we do not need to renormalize the bias parameters order by order in perturbation theory. On the other hand, the bias parameters in conventional approaches in the literature of perturbation theory should be renormalized order by order. These conventional approaches consider only semilocal models of bias in the sense of our terminology above. While our approach of IPT can include generally nonlocal bias characterized by a hierarchy of renormalized bias functions, our approach with semilocal bias models described above is related to and consistent with other approaches.

In order to illustrate the relation, let us consider the simplest example, the local-in-matter density (LIMD) model of bias in Lagrangian space [4, 73]. For simplicity, we ignore gravitational evolutions and assume that the initial density fluctuations are Gaussian. In this toy model, the density contrast of

(149), (150), (158), and (159), the same symmetries for the constant coefficients of bias are given by

\[ b^{(2, a_1 a_2)}_{Xlkl} = (-1)^{l_1 + l_2} b^{(2, a_1 a_2)}_{Xlkl} \]
(364)
for the second-order bias,

\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} b^{(3, a_1 a_2 a_3)}_{Xlkl} \]
(365)
\[ b^{(3, a_1 a_2 a_3)}_{Xlkl} = (-1)^{l_1 + l_2} \sum_{L'} (2L' + 1) \left\{ \begin{array}{ccc} l_1 & l & L \\ l_1 & l_1 & L' \end{array} \right\} b^{(3, a_1 a_2 a_3)}_{Xlkl} \]
(366)
biased objects \( X \) in Lagrangian space is given by
\[
\delta_X(q) = a_1 \delta_R(q) + \frac{a_2}{2!} \left[ \delta_R(q) \right]^2 - \sigma^2
+ \frac{a_3}{3!} \left[ \delta_R(q) \right]^3 + \frac{a_4}{4!} \left[ \delta_R(q) \right]^4 - 3\sigma^2 + \cdots, \tag{374}
\]
where \( \delta_R(q) \) is the smoothed linear density field with smoothing radius \( R \), \( \sigma^2 \equiv \left\langle \delta_R^2 \right\rangle \) is the variance, and \( a_n \) are bare bias parameters in the LIMD model. The correlation function of the biased field is straightforwardly calculated as
\[
\xi_X(q) = (a_1^2 + a_1 a_3 \sigma^2 + \cdots) \xi_0(q)
+ \frac{1}{2} \left( a_2^2 + a_2 a_4 \sigma^2 + \cdots \right) \left[ \xi_0(q) \right]^2 + \cdots, \tag{375}
\]
where \( \xi_0(q_1 - q_2) = \left( \langle \delta_R(q_1) \delta_R(q_2) \rangle \right) \) is the correlation function of smoothed initial density field.

In the conventional procedure of bias renormalization, the resulting coefficients on the rhs of the above equation correspond to the renormalized bias parameters, as defined by
\[
b_1^l = a_1 + \frac{a_2 \sigma^2}{2} + \cdots, \quad b_2^l = a_2 + \frac{a_4 \sigma^2}{2} + \cdots, \tag{376}
\]
so that Eq. (375) can be represented by a simple form,
\[
\xi_X(q) = (b_1^l)^2 \xi_0(q) + \frac{1}{2} (b_2^l)^2 \left[ \xi_0(q) \right]^2 + \cdots. \tag{377}
\]
On large scales where the correlation function \( \xi_0 \) is small enough, the above form is properly a perturbative series of expansion, and the renormalized parameters \( b_1^l, b_2^l, \ldots \) correspond to the quantities that can be determined by observations.

While the bias parameters should be renormalized order by order in the conventional procedure explained above, the renormalized bias functions in iPT are renormalized from the beginning. The renormalized bias functions in iPT exactly correspond to renormalized bias parameters to all orders in the semilocal models of bias in the conventional procedure. One can show this property model by model. For an illustrative example, we consider the LIMD model of Eq. (374). In our notation, the density field corresponds to a scalar field \( F_{X_0}^L(q) \) with \( l = m = 0 \), so that \( F_{X_0}^L(q) = 1 + \delta_X(q) \), as understood by, e.g., comparing Eqs. (57) and (60). The Fourier transform of Eq. (374) is given by
\[
F_{X_0}^L(k) = \sum_{n=1}^{\infty} \frac{a_n}{n!} \int_0^\infty \frac{d^3 k_1}{(2\pi)^3} \cdots \frac{d^3 k_n}{(2\pi)^3} \delta_D(k_1 + \cdots + k_n - k)
\times \delta_1(k_1) \cdots \delta_1(k_n) W(k_1 R) \cdots W(k_n R), \tag{378}
\]
where we exclude the zero mode and assume \( k \neq 0 \). Substituting the above equation into the definition of the renormalized bias functions, Eqs. (70) or (71) with \( l = m = 0 \), and using a standard formula of Gaussian moments, \( \langle \delta_R^p \rangle = \sigma^p (p - 1)! \) where \( p \) is a non-negative integer, we straightforwardly derive
\[
c_{X_0}^{(n)}(k_1, \ldots, k_n) = W(k_1 R) \cdots W(k_n R) \sum_{m=0}^{\infty} \frac{a_{2m+n} \sigma^{2m}}{(2m-1)!!}. \tag{379}
\]
Using this bias functions, the prediction of iPT for the correlation function in Lagrangian space (neglecting the gravitational evolution) is given by \[17\]
\[
\xi_X(q) = \sum_{n=1}^{\infty} \frac{b_n^l}{n!} \left[ \xi_0(q) \right]^n, \tag{380}
\]
where
\[
b_n^l = \sum_{m=0}^{\infty} \frac{a_{2m+n} \sigma^{2m}}{(2m-1)!!}, \tag{381}
\]
The last parameters are consistent with the renormalized bias parameters in the conventional theory, Eq. (376). We note that the renormalized bias parameters are directly derived to full orders with infinite terms at once, and we do not need to derive them order by order in iPT. In the notations of semilocal bias in this section, \( c_{X_0}^{(n)} = \delta \delta_{aal} \delta_R \) with \( L_a = 0 \) and \( W_{X_0}^{(n)}(k) = W(k R) \) in Eq. (332), and the renormalized bias parameters in our formalism, generally derived from Eqs. (359) and (363), simply correspond to
\[
b_n^{X_0} = b_1^l, \quad b_2^{X_0 \gamma} = b_2^l, \quad b_{n \gamma \gamma}^{X_0 \gamma \gamma} = b_n^l \tag{382}
\]
in the LIMD model for scalar density fields.

The above example shows that the renormalized bias functions in semilocal models exactly correspond to the renormalized bias parameters in conventional theory of bias renormalization. With the iPT, the bias parameters are renormalized from the first place of introducing renormalized bias functions, and we do not need order-by-order renormalizations of parameters as in the case of conventional perturbation theory. The main reason is that the bias relations between the biased field and mass density field are not expanded into perturbative series in iPT, and are treated nonperturbatively throughout the calculation. While the dynamics of gravitational evolution for the mass density field are treated perturbatively, the bias relations are not. On large scales where amplitude of the correlation is small, e.g., \( \xi_0(q) \ll 1 \) in Eq. (381), the expansion scheme is well defined. In the above, the LIMD model is employed just for an illustration, and one can extend the arguments to more general cases with more complicated models of bias and arbitrary ranks of tensors.

VII. SUMMARY AND CONCLUSIONS

In this paper, the formalism iPT is generalized to calculate statistics of generally tensor-valued objects, using the nonlinear perturbation theory. Higher-rank tensors are conveniently decomposed into spherical tensor-valued objects which are irreducible representations of the three-dimensional rotation group \( SO(3) \), and mathematical techniques developed in the theory of angular momentum in quantum mechanics are effectively applied.

The fundamental formalism of iPT can be recycled without essential modification, and the only difference is that the biased field is colored by tensor values. Because the original formalism of iPT is supposed to be applied to scalar fields,
the rotational symmetry in this original version of the theory is relatively trivial. When the theory is generalized to include tensor-valued fields, the statistical properties of biasing are largely constrained by symmetries. In particular, the renormalized bias functions, which are ones of the important ingredients in the iPT, can be represented by a set of invariant functions under rotations of the coordinate system. Explicit constructions of these invariants are derived and presented in this paper. These invariants can also be seen as coefficients of angular expansions by polypolar spherical harmonics.

The iPT offers a systematic way of calculating the propagators of nonlinear perturbation theory, both in real space and in redshift space. The methodology of deriving propagators of tensor-valued fields is generally explained in detail and several examples of lower-order propagators are explicitly derived in this paper. For an illustrative purpose, these examples of propagators are applied to simple cases of predicting the correlation statistics, including the linear power spectra and correlation functions, the lowest-order power spectrum with primordial non-Gaussianity, the bispectrum in the tree-level approximation, for generally tensor fields. So as to check consistencies in the examples, we confirm that each derived formula reproduces the known result for scalar fields, just substituting \( l = m = 0 \) in the general formulas. In Fig. 1, an example of the procedure to derive analytic predictions of iPT for power spectra of tensor fields in this paper is illustrated with relevant equations in this paper. The correlation functions and bispectra are also derived in similar procedures.

In the last section, the concept of semilocal models of bias is introduced. The formalism of iPT does not have to assume any model of bias, and all the uncertainties of bias are included in the renormalized bias functions, which are determined by a nonlocal functional of the biased field in terms of the linear density field, and have infinite degrees of freedom in general. The semilocal models of bias are defined so that the bias functional should be a function of a finite number of variables derived from the linear density field in Lagrangian space. As a result, the biasing is modeled by a finite number of parameters in this category of models. Almost all the existing models of bias introduced in the literature fall into this category, including the halo model, the peak theory of bias, excursion set peaks, EFTofLSS approach, and so forth. In this paper, only the formal definition of the semilocal models of bias is described, and rotationally invariant parameters in the models are identified. Their applications to individual models of concrete targets will be addressed in future work.

In this paper, the power spectrum and bispectrum are only evaluated in a lowest-order, or tree-level approximation. Some further techniques are required in calculating higher-order loop corrections, and they are described in a subsequent paper of the series, Paper II [43]. In realistic observations, we can only observe projected components of tensors onto the two-dimensional sky. Predictions of the statistics of three-dimensional tensors given in this paper can be transformed to those of projected tensors. Explicit formulations of this transformation are given in another subsequent paper of the series, Paper III [44]. While we assume the distant-observer approximation in these first three papers of the series, Paper IV [45] in the series gives a full-sky formulation without the last approximation.

We hope that the present formalism offers a way to carve out the future of applying the cosmological perturbation theory to the analysis of observations in the era of precision cosmology.

**ACKNOWLEDGMENTS**

I thank Y. Urakawa, K. Kogai, K. Akitsu, and A. Taruya for useful discussions. This work was supported by JSPS KAKENHI Grants No. JP19K03835 and No. 21H03403.
The spherical tensors $\mathcal{Y}_{l_{i_1}i_i}^{(m)}$ are the basis to expand symmetric tensors in the Cartesian coordinates system to the irreducible tensors on the spherical basis. In this Appendix, the expressions of the spherical tensor basis for ranks $l = 0, 1, 2, 3, 4$ are explicitly given in terms of the spherical bases $e^i$ and $e^0$, whose components are denoted by $e^m_i = [e^m_i]$, with $m = 0, \pm$. The Cartesian indices of the spherical tensor basis are totally symmetric, $Y_{t_{i_1}i_i} = \mathcal{Y}_{l_{i_1}i_i}^{(m)}$, where the round brackets indicate the symmetrization with respect to the indices inside the brackets, i.e.,

$$A_{i_{i_1}i_i} = \frac{1}{n!} \sum_{\sigma \in S_n} A_{i_{\sigma(i_1)}i_{\sigma(i_i)}}$$  \hspace{1cm} (A1)

and $S_n$ is the symmetric group of order $n$. The following results are straightforwardly derived from the explicit form of spherical harmonics $Y_{lm}(\theta, \phi)$, which uniquely determine the form of spherical basis tensors $\mathcal{Y}_{l_{i_1}i_i}^{(m)}$ to satisfy Eq. (A1), i.e.,

$$Y_{lm}(\theta, \phi) = \frac{(2l+1)!!}{4\pi l!} \mathcal{Y}_{l_{i_1}i_i}^{(m)} Y_{(i_1i_i)i_{i_1i_i}} Y_{i_{i_1}i_i}.$$  \hspace{1cm} (A2)

The procedure to derive the results are given as follows: First, we represent the complex conjugate of the spherical harmonics $Y_{lm}(\theta, \phi)$ in terms of the components of $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (n_1, n_2, n_3)$, and the resulting expressions only contain factors of $n_1 \mp in_2$ and $n_3$, because of the structure of spherical harmonics. Second, we substitute $n_1 \mp in_2 \to \mp \sqrt{2} e^m n_1$ and $n_3 \to e^0 n_3$ [cf., Eq. (2)] into the expression. Comparing the results with Eq. (A2), the explicit forms of spherical basis tensors are uniquely and obviously determined. We give the explicit expressions up to $l \leq 4$ below for the reader’s convenience.

**Spherical basis tensors of rank-0 and rank-1:**

$$\mathcal{Y}_{ij}^{(0)} = 1, \quad \mathcal{Y}_{ij}^{(m)} = e^m_i.$$  \hspace{1cm} (A3)

**Spherical basis tensors of rank-2:**

$$\mathcal{Y}_{ij}^{(0)} = \sqrt{\frac{3}{2}} \left[ e^0_i e^0_j - \frac{1}{3} \delta_{ij} \right], \quad \mathcal{Y}_{ij}^{(\pm 1)} = \sqrt{2} e^0_i e^{\pm}_j, \quad \mathcal{Y}_{ij}^{(\pm 2)} = e^i_+ e^{\pm}_j.$$  \hspace{1cm} (A4)

**Spherical tensor basis of rank-3:**

$$\mathcal{Y}_{ijk}^{(0)} = \sqrt{\frac{5}{2}} \left[ e^0_i e^0_j e^0_k - \frac{3}{2} \delta_{ij} e^0_k \right], \quad \mathcal{Y}_{ijk}^{(\pm 1)} = \sqrt{\frac{15}{2}} \left[ e^0_i e^0_j e^{\pm}_k \right], \quad \mathcal{Y}_{ijk}^{(\pm 3)} = e^i_+ e^{\pm}_j e^{\pm}_k.$$  \hspace{1cm} (A5)

**Spherical tensor basis of rank-4:**

$$\mathcal{Y}_{ijkl}^{(0)} = \frac{1}{2} \sqrt{\frac{35}{2}} \left[ e^0_i e^0_j e^0_k e^0_l - \frac{6}{7} \delta_{ij} e^0_k e^0_l + \frac{3}{35} \delta_{ij} \delta_{kl} \right], \quad \mathcal{Y}_{ijkl}^{(\pm 1)} = \sqrt{7} \left[ e^0_i e^0_j e^{\pm}_k e^{\pm}_l \right], \quad \mathcal{Y}_{ijkl}^{(\pm 3)} = 2 e^0_i e^0_j e^{\pm}_k e^{\pm}_l.$$  \hspace{1cm} (A7)

**Appendix B: Formulas for spherical harmonics and polypolar spherical harmonics**

In this Appendix, various formulas regarding the spherical harmonics are summarized, which are repeatedly used in the main text. As we mainly use Racah’s normalization of the spherical harmonics in this paper, the corresponding formulas look different from those found in standard textbooks. Accordingly bipolar, tripolar, and generally polypolar spherical harmonics are also defined in convenient normalizations used in this paper. Most of the formulas in this Appendix are derived from those found in standard literature [46, 47].

While the spherical harmonics are frequently represented by $Y_{l}^{m}(\theta, \phi)$ in the literature, this function transforms as a covariant tensor with respect to the coordinate rotation in the spherical basis introduced in Sec. II, thus we instead prefer the notation $Y_{lm}(\theta, \phi)$ with the lower position of the azimuthal index $m$. For simplicity, the spherical harmonics are frequently represented
by \( Y_{lm}(n) = Y_{lm}(\theta, \phi) \), where the angular coordinates are represented by a unit vector \( n \) whose spherical coordinates are \((\theta, \phi)\). We also use the simplified notation that the integration over the angular coordinates \( \int \sin \theta \, d\theta \, d\phi \cdots \) is represented by \( \int d^2 n \cdots \).

The standard normalization of the spherical harmonics \( Y_{lm}(n) \) is given by Eqs. (14) and (15). However, as described above, we intensively use the Racah's normalization of the harmonics, defined by Eq. (17), i.e.,

\[
C_{lm}(n) \equiv \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(n) = \sqrt{(l-m)! \over (l+m)!} P_l^m(\cos \theta) e^{im\phi}, \tag{B1}
\]

where \( P_l^m(\chi) \) is the associated Legendre polynomial.

The parity symmetry of the spherical harmonics is given by

\[
C_{lm}(-n) = (-1)^l C_{lm}(n). \tag{B2}
\]

The complex conjugate is given by

\[
C_{lm}^*(n) = g_{lm}^{nm} C_{lm}(n) = (-1)^m C_{l,-m}(n), \tag{B3}
\]

where \( g_{lm}^{nm} = (-1)^m \delta_{m,-m'} \) is the metric tensor in the spherical basis and the summation over \( m' \) is implicitly assumed when the same azimuthal index appears as a pair of lower and upper indices, just as in the Einstein summation convention. An addition theorem is given by

\[
g_{lm}^{nm} C_{lm}(n_1) C_{lm'}(n_2) = P_l(n_1 \cdot n_2), \tag{B4}
\]

where \( P_l(x) \) is the Legendre polynomial of order \( l \). The Rayleigh expansion or plane-wave expansion is given by

\[
e^{i \mathbf{k} \cdot \mathbf{x}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\pm i)^l (2l+1) j_l(kx) g_{lm}^{nm} C_{lm}(\hat{k}) C_{lm'}(\hat{x}), \tag{B5}
\]

where \( k = |\mathbf{k}|, \hat{k} = \mathbf{k}/k, x = |\mathbf{x}|, \) and \( \hat{x} = \mathbf{x}/x \). The orthonormality relation is given by

\[
\int \frac{d^2 n}{4\pi} C_{lm}(n) C_{lm'}(n) = \frac{\delta_{l,l'} \delta_{m,m'}}{2l+1} g_{l}^{lm} \tag{B6}
\]

where \( g_{lm}^{nm} = (-1)^m \delta_{m,-m'} \) is the inverse of \( g_{lm}^{nm} \), whose matrix elements are the same as \( g_{lm}^{nm} \). Combining Eqs. (B5) and (B6), we have

\[
\int \frac{d^2 \mathbf{k}}{4\pi} e^{i \mathbf{k} \cdot \mathbf{x}} C_{lm}(\mathbf{k}) = (\pm i)^l j_l(kx) C_{lm}(\hat{x}). \tag{B7}
\]

A product of two spherical harmonics with the same direction is given by

\[
C_{l_1 m_1}(n_1) C_{l_2 m_2}(n_2) = \sum_{l,m} (2l_1 + 1) \sum_{m} (l_1 l_2) \begin{pmatrix} l_1 & l_2 & l \cr 0 & 0 & 0 \cr m_1 & m_2 & m \end{pmatrix} C_{l m}(n_1) C_{l m}(n_2) \tag{B8}
\]

where the last two factors but one are Wigner's 3-j-symbols.

The bipolar spherical harmonics in our notation are defined by

\[
X_{lm}^{l_1 l_2}(n_1, n_2) = (l_1 l_2)_{m}^{m} C_{l_1 m_1}(n_1) C_{l_2 m_2}(n_2), \tag{B9}
\]

where

\[
(l_1 l_2)_{m}^{m} = g_{l_1 (l_2)}^{m_1 m} g_{l_2 (l_2)}^{m_2 m} \begin{pmatrix} l_1 & l_2 & l \cr m_1 & m_2 & m \end{pmatrix} = (-1)^{m_1+m_2} \begin{pmatrix} l & l \cr m & -m \end{pmatrix} \tag{B10}
\]

is a 3-j-symbol with two azimuthal indices raised by the metric tensor of the spherical basis. The relation to the bipolar spherical harmonics introduced in standard textbooks [47], \((Y_{l_1}(n_1) \otimes Y_{l_2}(n_2))_{lm}\), is given by

\[
(Y_{l_1}(n_1) \otimes Y_{l_2}(n_2))_{lm} = \frac{(-1)^l}{4\pi} \sqrt{(2l_1+1)(2l_2+1)} X_{lm}^{l_1 l_2}(n_1, n_2). \tag{B11}
\]
The orthonormality relation for the bipolar spherical harmonics is given by
\[ X_{lm}^{l_1 l_2}(-n_1, -n_2) = (-1)^{l_1 + l_2} X_{lm}^{l_1 l_2}(n_1, n_2). \] (B12)

The complex conjugate of the bipolar spherical harmonics is given by
\[ X_{lm}^{l_1 l_2*}(n_1, n_2) = (-1)^{l_1 + l_2} g_{mn} \cdot X_{lm}^{l_1 l_2}(n_1, n_2). \] (B13)

The orthonormality relation for the bipolar spherical harmonics is given by
\[ \int \frac{d^2 n_1}{4\pi} \frac{d^2 n_2}{4\pi} X_{lm}^{l_1 l_2}(n_1, n_2) \overline{X}_{l'm'}^{l_1 l_2}(n_1, n_2) = \begin{cases} 1, & \text{if } |l_1 - l_2| \leq l \leq l_1 + l_2, \\ 0, & \text{otherwise,} \end{cases} \] (B14)
where
\[ \delta_{l_1 l_2} = \begin{cases} 1, & \text{if } |l_1 - l_2| \leq l \leq l_1 + l_2, \\ 0, & \text{otherwise,} \end{cases} \] (B15)
i.e., \( \delta_{l_1 l_2} \) is unity when the set of three numbers \((l_1, l_2, l_3)\) satisfies the triangular condition, and is zero otherwise. A product of two bipolar spherical harmonics with the same set of directions is given by
\[ X_{lm}^{l_1 l_2}(n_1, n_2) X_{l'm'}^{l_1 l_2}(n_1, n_2) = \sum_{l''} (-1)^{l''} (2l'' + 1) (l l' l'')_{mn} n'' \times \sum_{l''} (-1)^{l''} (2l'' + 1) (2l'' + 1) (l l' l'')_{mn} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_1 & l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) X_{l'm'}^{l_1 l_2}(n_1, n_2). \] (B16)
where the last factor but one is Wigner’s \( j \)-symbol.

The tripolar spherical harmonics are defined by
\[ X_{L;lm}^{l_1 l_2 l_3}(n_1, n_2, n_3) = (-1)^{l_L} \sqrt{2L + 1} (l_L l_L)_{m_0} m_M m_{m_0} C_{l_L m_0} C_{ml_2 m_2} C_{l_m m_1} (n_1, n_2, n_3). \] (B17)
The relation to the tripolar spherical harmonics introduced in standard textbooks [47] is given by
\[ \left\{ Y_{l_1}(n_1) \otimes Y_{l_2}(n_2) \otimes Y_{l_3}(n_3) \right\}_{L;lm} = \frac{(-1)^{l_L}}{(4\pi)^{3/2}} \sqrt{2L + 1} (2l_1 + 1)(2l_2 + 1)(2l_3 + 1) X_{L;lm}^{l_1 l_2 l_3}(n_1, n_2, n_3). \] (B18)
The parity symmetry of the tripolar spherical harmonics is given by
\[ X_{lm}^{l_1 l_2 l_3}(-n_1, -n_2, -n_3) = (-1)^{l_1 + l_2 + l_3} X_{lm}^{l_1 l_2 l_3}(n_1, n_2, n_3). \] (B19)
The complex conjugate of the bipolar spherical harmonics is given by
\[ X_{lm}^{l_1 l_2 l_3*}(n_1, n_2, n_3) = (-1)^{l_1 + l_2 + l_3} g_{mn} \cdot X_{lm}^{l_1 l_2 l_3}(n_1, n_2, n_3). \] (B20)
The orthonormality relation for the tripolar spherical harmonics is given by
\[ \int \frac{d^2 n_1}{4\pi} \frac{d^2 n_2}{4\pi} \frac{d^2 n_3}{4\pi} X_{L;lm}^{l_1 l_2 l_3}(n_1, n_2, n_3) \overline{X}_{L';m'}^{l_1 l_2 l_3}(n_1, n_2, n_3) = \begin{cases} 1, & \text{if } |l_1 - l_2 - l_3| \leq l \leq l_1 + l_2 + l_3, \\ 0, & \text{otherwise,} \end{cases} \] (B21)
\[ \text{where } l = l_1 + l_2 + l_3. \]
A product of two tripolar spherical harmonics with the same set of directions is given by
\[ X_{L;lm}^{l_1 l_2 l_3}(n_1, n_2, n_3) X_{L';m'}^{l_1 l_2 l_3}(n_1, n_2, n_3) = \sqrt{(2L + 1)(2L' + 1)} \sum_{l''} (-1)^{l''} (2l'' + 1) (l l' l'')_{mn} n'' \times \sum_{l''} (-1)^{l''} (2l'' + 1) (2l'' + 1) (l l' l'')_{mn} n'' \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_1 & l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) X_{L';m'}^{l_1 l_2 l_3}(n_1, n_2, n_3). \] (B22)
While they are not found in standard textbooks, we generalize the bipolar and tripolar spherical harmonics to higher-order counterparts, and define poly-polar spherical harmonics of arbitrary order by

\[
X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) = (-1)^{\frac{1}{2}+l_1+l_2+\cdots+l_m} \sqrt{(2l_1+1)(2l_2+1)\cdots(2l_n+1)} (l_1 l_2 \cdots l_m m_1 m_2 \cdots m_n)
\]

They satisfy recursive relations,

\[
X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) = (-1)^{\frac{1}{2}+l_1+l_2+\cdots+l_m} \sqrt{(2l_1+1)(l_1 l_2 \cdots l_m m_1 m_2 \cdots m_n) C_{l_1 m_1}(n_1) C_{l_2 m_2}(n_2) \cdots C_{l_m m_n}(n_n)}.
\]

The parity symmetry of the poly-polar spherical harmonics is given by

\[
X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) = (-1)^{l_1+l_2+\cdots+l_m} X_{l_2 l_1 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n).
\]

The complex conjugate of the bipolar spherical harmonics is given by

\[
X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) = (-1)^{l_1+l_2+\cdots+l_m} g_{mm'} X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n).
\]

The orthonormality relation for the poly-polar spherical harmonics in general is straightforwardly derived from the orthonormality of the 3-j-symbols, and the result is given by

\[
\int \frac{d^3 n_1}{4\pi} \cdots \frac{d^3 n_n}{4\pi} X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) X_{l_1' l_2' \cdots l_m'}^{l_1' l_2' \cdots l_m'}(n_1, n_2, \ldots, n_n) = \sum_{m'} \sum_{l_1', \ldots, l_m'} \frac{(l_1' l_2' \cdots l_m')_{mm'}}{(2l_1' + 1)(2l_2' + 1)\cdots(2l_m' + 1)}
\]

A product of poly-polar harmonics with the same set of directions is given by

\[
X_{l_1 l_2 \cdots l_m}^{l_1 l_2 \cdots l_m}(n_1, n_2, \ldots, n_n) X_{l_1' l_2' \cdots l_m'}^{l_1' l_2' \cdots l_m'}(n_1, n_2, \ldots, n_n) = \sqrt{(2l_1+1)(2l_2+1)\cdots(2l_n+1)\cdots(2l_1'+1)(2l_2'+1)\cdots(2l_n'+1)}
\]

The last result can be derived by applying the sum rules of Eqs. (C25) and (C26) in Appendix C.

**Appendix C: The 3-j, 6-j and 9-j-symbols**

Wigner’s 3-j, 6-j and 9-j-symbols are the coefficients in adding and recoupling of angular momenta in quantum mechanics. These coefficients are frequently used in the main text of this paper, because they are useful as well in our formulation, regarding rotational symmetry of the Universe in a statistical sense. We summarize some of the formulas in this Appendix for the readers’ convenience. All the formulas below can be found in, or derived from, Refs. [46, 47]. In the following, we assume \( l, l_1, l_2, \ldots \) are non-negative integers (excluding the possibility of half-integers) as well as azimuthal indices, and we assume \((-1)^l = (-1)^{l_1}, (-1)^m = (-1)^{m_1}\).

In this paper, we employ a simplified notation for the 3-j-symbol,

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} \equiv \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\]

where the 3-j-symbol is nonzero only when \( m_1 + m_2 + m_3 = 0 \) in the above, and azimuthal indices \( m_1 \)'s are raised or lowered by spherical metric tensors, \( g_{m'm'}^{(l)} = (-1)^m g_{mm'}^{(l)} = g_{mm'}^{(l)} \), as exemplified in Eq. (B10). In particular, the symmetries of the 3-j-symbol indicates

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} = (-1)^{l_1+l_2+l_3} (l_1 l_2 l_3)_{m_1 m_2 m_3} ,
\]

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} = (-1)^{l_1+l_2+l_3} (l_1 l_2 l_3)_{m_1 m_2 m_3} ,
\]
since \( m_1 + m_2 + m_3 = 0 \). Even permutations of the 3\( j \)-symbols have the same value,

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} = (l_2 l_3 l_1)_{m_1 m_2 m_3} = (l_3 l_1 l_2)_{m_1 m_2 m_3},
\]

while odd permutations are given by

\[
(l_2 l_1 l_3)_{m_1 m_2 m_3} = (l_1 l_3 l_2)_{m_1 m_2 m_3} = (l_3 l_2 l_1)_{m_1 m_2 m_3} = (-1)^{l_1+l_2+l_3} (l_1 l_2 l_3)_{m_1 m_2 m_3}.
\]

The above equalities hold even when some of the azimuthal indices are raised on both sides of the equations. The orthogonality relations of the 3\( j \)-symbol are represented by

\[
(l_1 l_2)_{m_1 m_2} (l' l_2)_{m'_1 m'_2} = \frac{(-1)^{l_1+l_2+l'_1+l'_2}}{2l+1} \delta_{m m'} \delta_{l l'},
\]

\[
\sum_l (-1)^l (2l+1) (l_1 l_2)_{m_1 m_2} (l_1 l_2)^{m_1 m_2}_{m'_1 m'_2} = (-1)^{l_1+l_2} g_{m_1 m'_1} g_{m_2 m'_2},
\]

where \( \delta_{l_1 l_2 l_3} = 1 \) only when \((l_1, l_2, l_3)\) satisfies the triangle inequality, and is zero otherwise. From Eq. (C5), we have a derived formula,

\[
(l_1 l_2 l_3)_{m_1 m_2 m_3} (l_1 l_2 l_3)^{m_1 m_2 m_3} = (-1)^{l_1+l_2+l_3} \delta_{l_1 l_2 l_3},
\]

When one or two of the \( l \)'s in the 3\( j \)-symbols are zero, we have

\[
(100)_{m m 0} = \delta_{m m} \delta_{m 0}, \quad (l' 0)_{m m 0} = \frac{(-1)^{l'}}{\sqrt{2l+1}} g_m^{(l)}.
\]

In this paper, the Gaunt integral with our notation is denoted by

\[
[l_1 l_2 l_3]_{m_1 m_2 m_3} \equiv \int \frac{d^3 n}{4\pi} C_{l_1 m_1}(\hat{n}) C_{l_2 m_2}(\hat{n}) C_{l_3 m_3}(\hat{n}) = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix},
\]

and azimuthal indices are raised and lowered by the spherical metric \( g^{m m'}_{(l)} \), \( g^{(l)}_{m m'} \) as in the case of 3\( j \)-symbol. A 3\( j \)-symbol with vanishing \( m \)'s are nonzero only when the sum of \( l \)'s are even, i.e.,

\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad \text{when } l_1 + l_2 + l_3 = \text{odd}.
\]

Correspondingly, the Gaunt integrals are nonzero only when \( l_1 + l_2 + l_3 = \) even in the above equation. The covariant version of the Gaunt integral has the same components as the contravariant one:

\[
[l_1 l_2 l_3]_{m_1 m_2 m_3} = [l_1 l_2 l_3]^ {m_1 m_2 m_3},
\]

because of a symmetry of the 3\( j \)-symbols, Eq. (C2). The Gaunt integrals are totally symmetric under the permutation,

\[
[l_1 l_2 l_3]_{m_1 m_2 m_3} = [l_2 l_3 l_1]_{m_1 m_2 m_3} = [l_3 l_1 l_2]_{m_1 m_2 m_3} = [l_2 l_1 l_3]_{m_1 m_2 m_3} = [l_1 l_3 l_2]_{m_1 m_2 m_3} = [l_3 l_2 l_1]_{m_1 m_2 m_3}.
\]

An orthogonality relation of the 3\( j \)-symbols, Eq. (C5), indicates the orthogonality relation of the Gaunt integral:

\[
[l_1 l_2]_{m_1 m_2}, [l' l_2]_{m'_1 m'_2} = \frac{1}{2l+1} \begin{pmatrix} l_1 & l_2 \\ 0 & 0 \end{pmatrix}^2 \delta_{(l)} g_{m m'}^{(l)}.
\]

When one or two of the \( l \)'s in the Gaunt integral are zero, we have

\[
[100]_{m 0 0} = \delta_{m m} \delta_{m 0}, \quad [l' 0]_{m m 0} = \frac{\delta_{l'}}{2l+1} g_{m m}^{(l')}.
\]

The Wigner's 6\( j \)-symbol is associated with the recoupling of three angular momenta, and defined by

\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{pmatrix} = (-1)^{l_1+l_2+l_4+l_5+l_6} (l_1 l_2 l_3)^{m_1 m_2 m_3} (l_1 l_5 l_6)^{m_4 m_5 m_6} (l_4 l_2 l_6)^{m_4 m_5 m_6} (l_4 l_5 l_6)^{m_4 m_5 m_6}.
\]
The $6j$-symbols are invariant against any permutation of the columns:

$$\begin{align*}
\{l_1 & l_2 l_3 \} = \{l_1 l_3 l_2 \} = \{l_1 l_2 l_3 \} = \{l_2 l_1 l_3 \} = \{l_2 l_3 l_1 \} = \{l_3 l_1 l_2 \} = \{l_3 l_2 l_1 \} ,
\end{align*}$$

(C17)

and also invariant against the interchange of the upper and lower arguments in each of any two columns,

$$\begin{align*}
\{l_1 & l_2 l_3 \} = \{l_1 l_5 l_6 \} = \{l_4 l_2 l_3 \} = \{l_4 l_5 l_6 \} = \{l_7 l_8 l_9 \} ,
\end{align*}$$

(C18)

The Wigner's $9j$-symbol is associated with the recoupling of four angular momenta, and defined by

$$\begin{align*}
\{l_1 & l_2 l_3 l_4 \} = (l_1 l_2 l_3)^{m_1 m_2 m_3} (l_4 l_5 l_6)^{m_4 m_5 m_6} (l_7 l_8 l_9)^{m_7 m_8 m_9} ,
\end{align*}$$

(C19)

The $9j$-symbols are invariant under transposition,

$$\begin{align*}
\{l_1 & l_2 l_3 \\
 l_4 & l_5 l_6 \\
 l_7 & l_8 l_9 \} = \{l_1 l_4 l_7 \\
 l_2 l_5 l_8 \\
 l_3 l_6 l_9 \} ,
\end{align*}$$

(C20)

and also invariant against cyclic (even) permutations of the columns or lows:

$$\begin{align*}
\{l_1 & l_2 l_3 \\
 l_4 & l_5 l_6 \\
 l_7 & l_8 l_9 \} = \{l_2 l_3 l_1 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} = \{l_3 l_1 l_2 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} = \{l_1 l_3 l_2 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} ,
\end{align*}$$

(C21)

They change the sign by noncyclic (odd) permutations of the columns or lows, if the sum of all the arguments is odd:

$$\begin{align*}
(-1)^R \{l_1 & l_2 l_3 \\
 l_4 & l_5 l_6 \\
 l_7 & l_8 l_9 \} = \{l_2 l_3 l_1 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} = \{l_3 l_1 l_2 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} = \{l_1 l_3 l_2 \\
 l_4 l_5 l_6 \\
 l_7 l_8 l_9 \} ,
\end{align*}$$

(C22)

where

$$R = l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 + l_8 + l_9 .$$

(C23)

Many formulas among $3j$, $6j$, and $9j$-symbols, including orthogonality relations and sum rules, can be found in the standard literature mentioned above. In particular, we use the following sum rules in this paper for products of $3j$-symbols:

$$\begin{align*}
(l_1 l_2 L)^{M}_{m_1 m_2} (L l_3 l_4)^{M'}_{m_3 m_4} &= (-1)^{l_1 + l_2} \sum_{L'} (2L' + 1) (l_1 l_3 L')^{M'}_{m_1 m_3} (l_2 l_4 L')^{M'}_{m_2 m_4} ,
\end{align*}$$

(C24)

and

$$\begin{align*}
(L l_1 l_2)^{m_1 m_2}_{M_1} (L' l_1 l_2)^{m_1' m_2'}_{M_1'} (l_1 l_2 l_1')^{m_1'' m_2''}_{m_1'} (l_2 l_1 l_2')^{m_1'' m_2''}_{m_2'} &= \sum_{L_1 L_1'} (2L_1' + 1) (2L_1'' + 1) (L L' L'')^{M''}_{M'} (L'' l_1 l_2)^{m_1'' m_2''}_{M'} (l_1 l_1 l_2')^{m_1'' m_2''}_{m_1'} (l_2 l_1 l_2')^{m_1'' m_2''}_{m_2'} ,
\end{align*}$$

(C25)

and

$$\begin{align*}
(L l_1 l_2)^{m_1 m_2}_{M} (L' l_1 l_2)^{m_1' m_2'}_{M'} (l_1 l_1 l_1')^{m_1'' m_2''}_{m_1'} (l_2 l_1 l_1')^{m_1'' m_2''}_{m_2'} &= \sum_{L_1''} (2L_1'' + 1) (L L' L'')^{M''}_{M'} (L'' l_1 l_2)^{m_1'' m_2''}_{M'} (l_1 l_1 l_2')^{m_1'' m_2''}_{m_1'} (l_2 l_1 l_2')^{m_1'' m_2''}_{m_2'} ,
\end{align*}$$

(C26)

[1] D. H. Lyth and A. R. Liddle, *The primordial density perturbation: Cosmology, inflation and the origin of structure* (Cambridge University Press, Cambridge 2009)
[81] X. Fang, J. A. Blazek, J. E. McEwen and C. M. Hirata, JCAP 02, 030 (2017)