Plane waves in a general Robertson-Walker background

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We present an exact solution for the plane wave mode functions of a massless, minimally coupled scalar propagating in an arbitrary homogeneous, isotropic and spatially flat geometry. Our solution encompasses all previous solvable special cases such as de Sitter and power law expansion. Moreover, it can generate the mode functions for gravitons. We discuss some of the many applications that are now possible.

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1. Introduction: The universe is homogeneous and isotropic on the largest observable scales [1]. Recent precision measurements of anisotropies in the cosmic microwave background strongly support the idea that it is also spatially flat [2, 3, 4]. This is in any case the theoretical prejudice of inflationary cosmology [5]. The invariant element of such a universe can be simply expressed in co-moving coordinates,

\[ ds^2 = -dt^2 + a^2(t)d\mathbf{x} \cdot d\mathbf{x}. \]  

(1)

Although the scale factor \( a(t) \) is not measurable, its derivatives can be formed to give observables,

\[ H(t) \equiv \dot{a}^{-1}, \quad q(t) \equiv -1 - \dot{H}a^{-2}. \]  

(2)

The Hubble parameter \( H(t) \) is the cosmological expansion rate while the deceleration parameter \( q(t) \) provides a measure of the rate at which this expansion is slowing down \( (q > -1) \) or speeding up \( (q < -1) \).

It has long been realized that an expanding universe can result in particle creation [6]. A particularly interesting and likely candidate for this effect is the massless, minimally coupled scalar,

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - g g^{\mu \nu} \varphi \cdot \nabla - g. \]  

(3)

Because these particles are massless without classical conformal invariance, there can be an appreciable amplitude for virtual pairs to appear with wave lengths comparable to \( H^{-1} \). When this happens the pairs are pulled apart by the expansion of spacetime and there is physical particle creation [7]. When a scalar potential is added to drive inflation, essentially this mechanism [8] is the leading candidate for explaining the primordial cosmological perturbations which seeded structure formation and whose imprint is observed in the anisotropies of the cosmic microwave background [9].

An even more direct application derives from the observation [10] that linearized gravitons obey the same equation of motion in the background [11]. This leads to a fundamental tensor component in the primordial spectrum of cosmological perturbations [11, 12, 13]. There is also the possibility that quantum gravitational back-reaction slows inflation [14]. Although standard inflationary cosmology is realized via a scalar field, it is probably not natural to have interacting scalars which are much lighter than the Hubble parameter during inflation. In that respect the graviton can achieve all the advantages of inflation in a far more economical and natural way [14].

It is therefore frustrating that the scalar mode functions are only known for certain choices of \( a(t) \) [15]. The fact that typical inflationary backgrounds are not of this type means that either perturbative or numerical techniques must be used to predict the spectrum of cosmological perturbations [11, 12, 13]. The absence of general mode solutions is most noticeable when back-reaction becomes important. By working in de Sitter background one can see that the slowing effect eventually becomes strong [14], but it is not possible to reliably follow where the physics is trying to lead.

The purpose of this paper is to solve the scalar mode equation for any scale factor \( a(t) \). Section 2 gives the equation and defines what we mean by the “mode function”. In Section 3 we present a solution which is valid for any period during which the deceleration parameter \( q \) is constant. In Section 4 these solutions are extended to cover general evolution by representing it as a sequence of ever finer steps between regions of constant \( q(t) \). The transfer matrix which makes this possible is expressed as the path-ordered product of the exponential of a line integral. In Section 5 we improve the constant deceleration solution by explicitly extracting the determinant of this matrix for general \( a(t) \). We also identify and normalize the positive frequency solution. Expansions for the ultraviolet and infrared regimes are derived in Sections 6 and 7, respectively. Our results are collected in Section 8 and expressed in terms of standard parameters. We also discuss applications, including an improved result for the gravitational wave contribution to the power spectrum of
anisotropies in the cosmic microwave background.

2. The scalar mode equation: We take advantage of spatial translation invariance to Fourier transform in the $(D-1)$-dimensional spatial coordinates,

\[ \tilde{\varphi}(t, \vec{k}) \equiv \int a^{D-1} x e^{-i\vec{k} \cdot x} \varphi(t, x) . \quad (4) \]

The equation of motion of \( \tilde{\varphi}(t, \vec{k}) \) is,

\[ \ddot{\tilde{\varphi}} + (D-1)H \dot{\tilde{\varphi}} + k^2 a^{-2} \tilde{\varphi} = 0 . \quad (5) \]

The middle term, \((D-1)H \dot{\tilde{\varphi}}\), is responsible for the phenomenon of “Hubble friction” in which the expansion of spacetime retards the scalar’s evolution.

Our work was motivated by a recent paper of Finelli, Marozzi, Vaccar and Venturi [16] who studied the case of a minimally coupled scalar with arbitrary mass. We follow them in the standard step of extracting a factor of \(a^{-D-1} \) to eliminate the Hubble friction,

\[ \tilde{\varphi}(t, \vec{k}) = a^{-\frac{D+1}{2}} \left\{ u(t, k) \alpha(\vec{k}) + u^*(t, k) \alpha^*(\vec{k}) \right\} . \quad (6) \]

The quantities \( \alpha(\vec{k}) \) and \( \alpha^*(\vec{k}) \) are time independent quantum operators. What remains is the \( C \)-number mode function \( u(t, k) \) and its conjugate. It obeys,

\[ \ddot{u}(t, k) + H^2 \left[ (\frac{D+1}{2})^2 - (\frac{D-1}{2})^2 \right] u(t, k) = 0 . \quad (7) \]

To fix the normalization, suppose the creation and annihilation operators commute canonically,

\[ \left[ \alpha(\vec{k}), \alpha^*(\vec{k'}) \right] = (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k'}) . \quad (8) \]

Taking the time derivative of (8) gives,

\[ \left[ \tilde{\varphi}(t, \vec{k}), \tilde{\varphi}(t, \vec{k'}) \right] = \left( \frac{2\pi}{a(t)} \right)^{D-1} \delta^{D-1}(\vec{k} + \vec{k'}) \times \left\{ u(t, k) \dot{u}^*(t, k) - u^*(t, k) \dot{u}(t, k) \right\} . \quad (9) \]

Since the momentum canonically conjugate to \( \varphi(t, x) \) is \( aD^{-1}(t) \dot{\varphi}(t, x) \), we must also have,

\[ \left[ \tilde{\varphi}(t, \vec{k}), \tilde{\varphi}(t, \vec{k'}) \right] = i \left( \frac{2\pi}{a(t)} \right)^D \delta^{D-1}(\vec{k} + \vec{k'}) . \quad (10) \]

Comparing (9) with (10) determines the Wronskian of the mode function,

\[ u(t, k) \dot{u}^*(t, k) - u^*(t, k) \dot{u}(t, k) = i . \quad (11) \]

3. The solution for constant deceleration: Just as in [10], we organize the dependence upon time and wave number into two dimensionless parameters,

\[ x(t, k) \equiv \frac{k}{aH} , \quad y(t) \equiv -\frac{\dot{H}}{H^2} . \quad (12) \]

The meaning of \( x(t, k) \) is the physical wave number in Hubble units. Super-horizon modes have \( x < 1 \) whereas sub-horizon modes have \( x > 1 \). The quantity \( y(t) \) is related to the deceleration parameter, \( q(t) = -1 + y(t) \). The Weak Energy Condition implies \( y(t) \geq 0 \). Slow roll inflation is characterized by \( y(t) \ll 1 \).

Note that \( y(t) \) is constant whenever \( a(t) \) is a power of any linear function of \( t \). For example, a radiation dominated universe has \( y = 2 \); the result for a matter dominated universe is \( y = \frac{1}{2} \), and curvature domination gives \( y = 1 \). Of course \( \phi \) is trivially reducible to Bessel’s equation whenever the scale factor obeys a power law, but the order of the necessary Bessel functions varies with the power. This is why it has been so difficult to find a general solution. In [10], however, it was recognized that the general solution should involve a Bessel function whose order depends upon \( y(t) \). They worked only to first order in \( y \) but it is straightforward to extend their result to all orders. The required generalization is,

\[ \mathcal{H}(x, y) \equiv x^\mu \mathcal{H}^{(1)}_{\nu}(\lambda x) , \quad (13) \]

where the parameters are the following functions of \( y \),

\[ \mu \equiv \frac{-\frac{D+1}{2} + y}{1 - y} , \quad \nu \equiv \frac{D-1 - \frac{D+1}{2}}{1 - y} , \quad \lambda \equiv \frac{1}{1 - y} . \quad (14) \]

Although the parameter \( x \) is always positive, \( \lambda \) passes from \( +\infty \) to \( -\infty \) as \( y \) evolves from \( 1^- \) to \( 1^+ \), which signals the end of inflation. This does not lead to wild oscillations in \( \mathcal{H}(x, y) \) because \( \nu = \frac{1}{2} + \frac{y}{2} - \frac{D-1}{2} \) makes the same transition, and the factor of \( 1/\Gamma(n + \nu + 1) \) in the series expansion of the Hankel function compensates the growth of \( \lambda \). However, the possibility of \( \lambda \) becoming negative does raise the issue of how we define \( (\lambda x)^\nu \).

Throughout this paper we shall understand negative \( \lambda \) to mean \( e^{i\pi|\lambda|} \), so that \((-|\lambda| x)^\nu \equiv e^{i\pi \nu(|\lambda| x)^\nu} \).

To see that (13) solves (10) for constant \( y \), first compute the (general) time derivatives of \( x(t, k) \) and \( y(t) \),

\[ \dot{y} = H \left[ 2\frac{\dot{H}^2}{H^4} - \frac{\ddot{H}}{H^3} \right] , \quad (15) \]

\[ \dot{x} = -k \left[ 1 + \frac{\dot{H}}{H^2} \right] = -Hx(1 - y) , \quad (16) \]

\[ \ddot{x} = Hk \left[ 1 + \frac{\dot{H}}{H^2} + 2\frac{\dot{H}^2}{H^4} - \frac{\ddot{H}}{H^3} \right] \]

\[ = H^2x \left( 1 - y + \frac{\dot{y}}{H} \right) . \quad (17) \]

Now use Bessel’s equation to derive an identity for the second \( x \) derivative of \( \mathcal{H}(x, y) \),

\[ \mathcal{H}_{xx} = \left( \frac{2\mu - 1}{x} \right) \mathcal{H}_x - \lambda^2 \mathcal{H} + \left( \frac{\nu^2 - \mu^2}{x^2} \right) \mathcal{H} , \quad (18) \]

\[ = -\frac{\mathcal{H}_x}{x(1 - y)} - \left( \frac{x^2 + \frac{D-1}{2} y - \left( \frac{D+1}{2} \right)^2}{x^2(1 - y)^2} \right) \mathcal{H} . \quad (19) \]
When $\dot{y} = 0$ we have,
\begin{align}
\dot{\mathcal{H}}_{y=0} &= \dot{x} \mathcal{H}_x , \\
\dot{\mathcal{H}}_{y=0} &= \dot{x} \mathcal{H}_x + \dot{x}^2 \mathcal{H}_{xx} ,
\end{align}
(20)
(21)

\begin{align}
H^2 x(1-y) \mathcal{H}_x + H^2 x^2 (1-y)^2 \mathcal{H}_{xx} ,
= -H^2 \left[ x^2 + \frac{2}{x} y - \left( \frac{2}{x} \right)^2 \right] \mathcal{H} .
\end{align}
(22)
(23)

Suppose further that the scale factor is continuous at the transition.
Since the differential equation has no factors of its solution must be continuous at the transition,
$w(t) = w(t) \quad \forall t < t_1,$
(24)
$u_2(t, k) = c_2 \mathcal{H}(x, y) + d_2 \mathcal{H}^*(x, y) \quad \forall t > t_1.$
(25)

Suppose further that the scale factor is continuous at the transition.
Since the differential equation has no factors of $\delta(t - t_1)$ its solution must also be continuous at $t_1,$
$c_2 \mathcal{H}_{12} + d_2 \mathcal{H}_{12} = c_1 \mathcal{H}_{11} + d_1 \mathcal{H}_{11}^* ,
(26)

where we have compressed the notation by making the definition, $\mathcal{H}_{ij} \equiv \mathcal{H}(x(t_i), y_i).$
Suppose finally that the Hubble constant is continuous. Since the differential equation has no factors of $\delta(t - t_1)$ the first derivative of its solution must be continuous at the transition,
\begin{align}
(1-y_2) 
\begin{bmatrix}
c_2 \mathcal{H}_{12,x} + d_2 \mathcal{H}_{12,x}^*
\end{bmatrix}
= (1-y_1) 
\begin{bmatrix}
c_1 \mathcal{H}_{11} + d_1 \mathcal{H}_{11}^*
\end{bmatrix} .
\end{align}
(27)

We represent the linear transformation between the “before” and “after” combination coefficients as a $2 \times 2$ matrix $T(t_1),$
\begin{align}
\begin{bmatrix}
c_2 \\
d_2
\end{bmatrix}
= T(t_1) 
\begin{bmatrix}
c_1 \\
d_1
\end{bmatrix} .
\end{align}
(28)

Solving for its components is a straightforward exercise in linear algebra,
\begin{align}
T^{11}(t_1) &= \left( T^{22}(t_1) \right)^* \\
&= \frac{1}{W_2} \begin{bmatrix}
\mathcal{H}_{11} \mathcal{H}_{12,x} - \left( \frac{1-y_1}{1-y_2} \right) \mathcal{H}_{11,x} \mathcal{H}_{12}
\end{bmatrix} ,
(29)

T^{21}(t_1) &= \left( T^{12}(t_1) \right)^* \\
&= \frac{1}{W_2} \begin{bmatrix}
-\mathcal{H}_{11} \mathcal{H}_{12,x} + \left( \frac{1-y_1}{1-y_2} \right) \mathcal{H}_{11,x} \mathcal{H}_{12}
\end{bmatrix} .
(30)

Here $W_2 \equiv W(x(t_1), y_2)$ and $W(x, y)$ is the Wronskian of our constant $y$ solution $[13],$
\begin{align}
W(x, y) \equiv \mathcal{H}^*_x - \mathcal{H}_x \mathcal{H}^* = -\frac{4i}{\pi} x^{2\mu-1} \text{sgn}(\lambda) .
\end{align}
(31)

Now consider a sequence of such transitions at times $t_1 < t_2 < ...$ The solution for $t > t_{n-1}$ can be expressed as the inner product, through a product of transformation matrices $T(t_k),$ between the row vector $(\mathcal{H}, \mathcal{H}^*)$ and the column vector composed of the original combination coefficients,
\begin{align}
\begin{bmatrix}
u_n(t, k) \\
\end{bmatrix} = \left( \mathcal{H}(x, y_n) \right) \prod_{k=1}^{n-1} T(t_{n-k}) \begin{bmatrix}
c_1 \\
d_1
\end{bmatrix} .
(32)
\end{align}

We can approach continuous evolution in $y(t)$ by representing any fixed interval as a series of more and more transitions of ever smaller magnitude. In the infinitesimal limit the product of the $T$ matrices exponentiates.
To demonstrate exponentiation consider the transition at $t = t_1$ for $y_1 = y(t_1) - \frac{1}{2} \Delta y$ and $y_2 = y(t_1) + \frac{1}{2} \Delta y.$ The Wronskian and the constant $y$ solutions can all be expanded in powers of $\Delta y,$
\begin{align}
\left( \frac{1-y_1}{1-y_2} \right) &= 1 + \frac{\Delta y}{1-y} + \ldots ,
(33)

\frac{1}{W_2} &= \frac{1}{W} - \frac{W_y \Delta y}{W^2} + \ldots ,
(34)

\mathcal{H}_{11} &= \mathcal{H} - \mathcal{H}_y \frac{\Delta y}{2} + \ldots ,
(35)

\mathcal{H}_{12} &= \mathcal{H} + \mathcal{H}_y \frac{\Delta y}{2} + \ldots .
(36)
\end{align}

Expanding the coefficients of $T(t_1)$ to first order gives,
\begin{align}
T^{11}(t_1) &= 1 + \zeta \left(x(t_1), y(t_1) \right) \Delta y + \ldots ,
(37)

T^{21}(t_1) &= 0 + \xi \left(x(t_1), y(t_1) \right) \Delta y + \ldots ,
(38)
\end{align}

where we define,
\begin{align}
\zeta(x, y) &= \frac{1}{W} \left[ -\mathcal{H}_y \mathcal{H}_{x,x} + \mathcal{H}_{x,y} \mathcal{H}^* - \mathcal{H}_x \mathcal{H}^* \right] ,
(39)

\xi(x, y) &= \frac{1}{W} \left[ \mathcal{H}_y \mathcal{H}_x - \mathcal{H}_{x,y} \mathcal{H} + \mathcal{H}_x \mathcal{H}^* \right] .
(40)
\end{align}

Taking the infinitesimal limit gives the path-ordered exponential of the line integral of the following matrix:
\begin{align}
A(t, k) \equiv \dot{y}(t) \left( \zeta(x, y) \quad \xi \left(x, y \right) \right) .
(41)
\end{align}

We define the transfer matrix as,
\begin{align}
M(t, t_i, k) &= P \left\{ \exp \left[ \int_{t_i}^{t} dt' A(t', k) \right] \right\} ,
(42)

&= \sum_{n=0}^{\infty} \int_{t_i}^{t} dt_1 \int_{t_1}^{t} dt_2 \ldots \int_{t_{n-1}}^{t} dt_n \\
&\times A(t_1, k) \cdots A(t_n, k) .
(43)
\end{align}
Here $t_i$ is the initial time. The full solution assumes the form,

\[ u(t, k) = (\mathcal{H}(x, y), \mathcal{H}^*(x, y)) \mathbf{M}(t, t_i, k) \begin{pmatrix} c \\ d \end{pmatrix}. \quad (44) \]

Explicitly verifying that \( \mathbf{H} \) obeys \( \mathbf{H} = \mathbf{H}^* \) is facilitated by two identities implied by the Wronskian,

\[ \zeta \mathcal{H} + \xi \mathcal{H}^* = -\mathcal{H}_{xy}, \quad (45) \]

\[ \zeta \mathcal{H}_x + \xi \mathcal{H}^*_x = -\mathcal{H}_{xy} + \frac{\mathcal{H}_x}{1-y}. \quad (46) \]

Another very useful fact is that the time derivative of the transfer matrix is proportional to it,

\[ \mathbf{M}(t, t_i, k) = \mathbf{A}(t, k) \mathbf{M}(t, t_i, k) \quad (47) \]

Finally, we recall that time derivatives of $\mathcal{H}(x(t, k), y(t))$ follow from the chain rule,

\[ \dot{\mathcal{H}}(x, y) = \dot{x} \mathcal{H}_x + \dot{y} \mathcal{H}_{xy}. \quad (48) \]

With these facts in hand it is straightforward to compute the first derivative,

\[ \dot{u}(t, k) = \left[ (\dot{\mathcal{H}}, \mathcal{H}^*) + (\mathcal{H}, \dot{\mathcal{H}}^*) \right] \mathbf{M} \begin{pmatrix} c \\ d \end{pmatrix}, \quad (49) \]

\[ = \dot{x} \mathcal{H}_x + \dot{\mathcal{H}}^*_x \mathbf{M} \begin{pmatrix} c \\ d \end{pmatrix}. \quad (50) \]

The second derivative requires only a little more effort,

\[ \ddot{u}(t, k) = \left[ \ddot{x} \mathcal{H}_x, \mathcal{H}^*_x \right] + \dot{x} \left( \mathcal{H}_x, \dot{\mathcal{H}}^*_x \right) \]

\[ + \dot{\mathcal{H}}^*_x \mathbf{M} \begin{pmatrix} c \\ d \end{pmatrix}, \quad (51) \]

\[ = \left( \ddot{x} \mathcal{H}_x + \dot{x}^2 \mathcal{H}_{xx} + \dot{x} \mathcal{H}_x \frac{x}{1-y}, \text{ c.c.} \right) \mathbf{M} \begin{pmatrix} c \\ d \end{pmatrix}, \quad (52) \]

\[ = -H^2 \left( \dddot{x}^2 + \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial x^2} \right) u(t, k). \quad (53) \]

In the last step we have substituted expressions \( \text{(16)-(17)} \) for $\dddot{x}$ and $\dot{x}$, and we have exploited relation \( \text{(18)} \) for $\mathcal{H}_{xx}$.

5. **Evaluating the transfer matrix:** During most periods of interesting cosmologies one has $|\dot{y}/H| \ll 1$. Since $\mathbf{M}$ vanishes with $\dot{y}$ it is reasonable to expect that the mode functions derive most of their time dependence from the constant deceleration solutions, $\mathcal{H}(x, y)$ and $\mathcal{H}^*(x, y)$. This is one reason our solution is effective in spite of the cumbersome definition \( \text{(39)} \) of the transfer matrix. The role of $\mathbf{M}(t, t_i, k)$ is to slowly mix positive and negative frequency solutions as cosmological particle production progresses. It must be a nonlocal summation over the past.

The nonlocal character of $\mathbf{M}(t, t_i, k)$ does not preclude learning a great deal about it. We begin by recalling that $\mathcal{H}(x, y) \equiv x^\mu H^{(1)}_\nu(\lambda x)$ with $\mu = \frac{1}{2} - \frac{1}{2} \nu$ and $\nu = \frac{1}{2} + (\frac{D}{2} - 1)\lambda$. Now apply the chain rule to show,

\[ \mathcal{H}_x = \mu x^{\mu-1} H^{(1)}_\nu(\lambda x) + \lambda x^\mu H^{(1)}_\nu(\lambda x), \quad (54) \]

\[ \mathcal{H}_{xy} = \mu x^{\nu+1} H^{(1)}_\nu(\lambda x) \]

\[ + \lambda x^{\nu+1} H^{(1)}_\nu(\lambda x) + \nu' x^\mu \partial_x H^{(1)}_\nu(\lambda x). \quad (55) \]

Here a prime means differentiation with respect to the function’s argument. Upon substitution in the definition \( \text{(39)} \) for $\xi(x, y)$ and making some tedious rearrangements we find,

\[ \xi = \frac{1}{2} \lambda^2 \ln(x) + \frac{1}{2} \lambda \]

\[ - \frac{i\pi}{4} |\lambda| \left\{ \frac{1}{2} E_\nu(\lambda x) + F_\nu(\lambda x) + (\nu - \frac{1}{2}) G_\nu(\lambda x) \right\}, \quad (56) \]

where $E_\nu(z), F_\nu(z)$ and $G_\nu(z)$ are the following real products of Hankel functions,

\[ E_\nu(z) \equiv z \left| H^{(1)}_\nu(z) \right|^2, \quad (57) \]

\[ F_\nu(z) \equiv z^2 \left[ \left| \partial_z H^{(1)}_\nu(z) \right|^2 + \left( 1 - \frac{\nu^2}{z^2} \right) \left| H^{(1)}_\nu(z) \right|^2 \right], \quad (58) \]

\[ G_\nu(z) \equiv z \left[ \partial_z H^{(1)}_\nu(z) \partial_z \left( H^{(1)}_\nu(z) \right)^* - \partial_z \partial_z H^{(1)}_\nu(z) \left( H^{(1)}_\nu(z) \right)^* \right]. \quad (59) \]

It might be thought that these three functions require separate expansions but the last two actually follow from $E_\nu(z)$ through the identities,

\[ F_\nu(z) = 2E_\nu(z), \quad G_\nu(z) = \frac{2\nu}{z^2} E_\nu(z). \quad (60) \]

The same procedure can be used to bring $\xi(x, y)$ to a closely related form,

\[ \xi = \frac{i\pi}{4} |\lambda| \left\{ \frac{1}{2} E_\nu(\lambda x) + F_\nu(\lambda x) + (\nu - \frac{1}{2}) G_\nu(\lambda x) \right\}. \quad (61) \]

In this case each function is the product of the same two Hankel functions,

\[ E_\nu(z) \equiv z \left( H^{(1)}_\nu(z) \right), \quad (62) \]

\[ F_\nu(z) \equiv z^2 \left[ \left( \partial_z H^{(1)}_\nu(z) \right)^2 + \left( 1 - \frac{\nu^2}{z^2} \right) \left( H^{(1)}_\nu(z) \right)^2 \right], \quad (63) \]

\[ G_\nu(z) \equiv z \left[ \partial_z H^{(1)}_\nu(z) \partial_z H^{(1)}_\nu(z) - \partial_z \partial_z H^{(1)}_\nu(z) H^{(1)}_\nu(z) \right]. \quad (64) \]
These functions are not purely real, unlike those for $\zeta$. However, they still bear the same relation to one another,\footnote{The subscript “i” in this and subsequent expressions refers to quantities evaluated at the initial time $t = t_i$, for example, $a_i = a(t_i)$. It is worth emphasizing that extracting the determinant of the transfer matrix already gives an improvement on the previous constant $y$ solutions.}

$$\mathcal{F}_\nu'(z) = 2\mathcal{E}_\nu(z), \quad \mathcal{G}_\nu'(z) = -\frac{2\nu}{z^2}\mathcal{E}_\nu(z).$$ \hfill (65)

It is straightforward to obtain explicit series expansions for $\zeta(x, y)$ and $\xi(x, y)$. One begins by expressing $\mathcal{E}_\nu(z)$ and $\mathcal{E}_\nu(z)$ in terms of Bessel functions,

$$E_\nu(z) = \frac{1}{\sin^2(\nu\pi)} \left\{ zJ^2_{\nu}(z) - 2\cos(\nu\pi)J_{\nu}(z)J_{\nu}(z) + zJ^2_{\nu}(z) \right\},$$ \hfill (66)

$$\mathcal{E}_\nu(z) = \frac{1}{\sin^2(\nu\pi)} \left\{ -zJ^2_{\nu}(z) + 2e^{-i\nu\pi}J_{\nu}(z)J_{\nu}(z) - e^{-2i\nu\pi}zJ^2_{\nu}(z) \right\}.$$ \hfill (67)

Now exploit relations (60) and (65) to show,

$$\zeta = \frac{1}{2} \lambda^2 \ln(x) + \frac{1}{2} \lambda - \left[ \frac{i\pi}{2} \frac{1}{\sin^2\nu\pi} - 2 \right] \left\{ b_\nu(\lambda|x) - 2 + 4 \sin^2(\nu\pi) \theta(-\lambda) + \cos(\nu\pi) c_\nu(\lambda|x) + d_\nu(\lambda|x) \right\},$$ \hfill (68)

$$\xi = \left[ \frac{i\pi}{2} \frac{1}{\sin^2\nu\pi} - 2 \right] \left\{ b_\nu(\lambda|x) - 2 + 4 \sin^2(\nu\pi) \theta(-\lambda) + \cos(\nu\pi) c_\nu(\lambda|x) + d_\nu(\lambda|x) \right\},$$ \hfill (69)

where the various coefficient functions are,

$$b_\nu(z) = \frac{1}{2 \sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n - \nu - \frac{1}{2}) z^{2n-2}\Gamma(n-\nu-1)}{\Gamma(n) \Gamma(n + \nu + 1) \Gamma(n - 2\nu + 1)},$$ \hfill (70)

$$c_\nu(z) = -\frac{4}{\pi} \sin(\nu\pi) \left\{ \psi(\nu) - 1 - \ln(4z) \right\} - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n - \nu - \frac{1}{2}) z^{2n} n^{-1}}{\Gamma(n) \Gamma(n + \nu + 1) \Gamma(n - 2\nu + 1)},$$ \hfill (71)

$$d_\nu(z) = \frac{1}{2 \sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n + \nu - \frac{1}{2}) z^{2n+2}\Gamma(n + \nu + 1)}{\Gamma(n + 2\nu) \Gamma(n + \nu + 1) \Gamma(n - 2\nu + 1)}. $$ \hfill (72)

Recall that $\psi(z) \equiv \psi'(z)/\Gamma(z)$.

It is illuminating to express the matrix $\mathbf{A}(t, k)$ \footnote{From the time derivative $H' = \mathbf{A}(t, k)$ and our normalization we see that the mean solution is,}
in terms of the real and imaginary parts of $\zeta$ and $\xi$,

$$\mathbf{A}(t, k) = \gamma \text{Re}(\mathbf{I}) + \gamma \left\{ \text{Re}(\zeta) \sigma^1 + \text{Im}(\zeta) \sigma^2 + i \text{Im}(\zeta) \sigma^3 \right\},$$ \hfill (73)

$$= \frac{1}{2} \frac{d}{dt} \ln(\lambda x^\lambda a) \mathbf{I} + \delta \mathbf{A}(t, k).$$ \hfill (74)

The unit matrix $\mathbf{I}$ commutes with all other matrices so we can factor this out of path-ordered product. Since its coefficient is a total derivative we can also give an explicit form for the contribution it makes to $\mathbf{M}(t, t_i, k)$,

$$\mathbf{M}(t, t_i, k) = \sqrt{\frac{\lambda x^\lambda a}{\lambda x^\lambda a}} \mathbf{M}(t, t_i, k).$$ \hfill (75)
Now substitute (81) and integrate termwise to obtain an asymptotic expansion for \( \zeta(x,y) \),
\[
\zeta(x,y) = -i\lambda^2 x + i\tilde{\zeta}(\nu - \xi) - 2i\nu + \frac{1}{2}
\]
\[
- \frac{i(\nu - \xi)}{2\pi} \int_0^{\infty} \frac{d\tau}{\nu_0 + \frac{n+1}{2} \Gamma(n+1) \Gamma(\nu-n+\frac{1}{2})} (2n+1)
\]
\[
(85)
\]
It is simplest to express the asymptotic expansion for \( \xi(x,y) \) in terms of the function \( h_\nu(z) \) defined in (82),
\[
\xi(x,y) = \frac{i}{2} \lambda e^{2i(\lambda x - \nu z - \xi)} \left\{ i\nu h_\nu(z) - (\nu^2 - \xi) \frac{h_\nu^2(z)}{z} + \frac{\partial_x h_\nu(z) h_\nu^\prime(z) - \partial_x h_\nu^\prime(z) h_\nu(z)}{i} \right\}
\]
\[
(86)
\]
From the expansion of \( h_\nu(z) \) we see that \( \xi(x,y) \) is negligible in the ultraviolet,
\[
\xi(x,y) = \frac{i}{2} \lambda e^{2i(\lambda x - \nu z - \xi)} \left\{ i\nu h_\nu(z) - (\nu^2 - \xi) \frac{h_\nu^2(z)}{z} + \frac{\partial_x h_\nu(z) h_\nu^\prime(z) - \partial_x h_\nu^\prime(z) h_\nu(z)}{i} \right\}
\]
\[
(87)
\]
Ignoring \( \xi(x,y) \) defines the ultraviolet regime,
\[
\tilde{A}_{uv}(t,k) \equiv \tilde{y} \zeta(x,y) \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right)
\]
\[
(88)
\]
In the notation of (89) the ultraviolet regime is characterized by,
\[
\rho_{uv}(t,k) = 0 ,
\]
\[
\tau_{uv}(t,k) = \int_{t_{i}}^{t} dt' \tilde{y} \zeta(x,y) .
\]
\[
(89)
\]
Since \( \rho_{uv} = 0 \) the value of \( \omega_{uv} \) is irrelevant.
In the extreme ultraviolet limit we need only keep the first two terms from the expansion of \( \zeta(x,y) \),
\[
\tau_{uv}(t,k) \rightarrow \int_{t_{i}}^{t} dt' \tilde{y} \left[ -\lambda x + \frac{\tilde{\zeta}(\nu - \xi)}{\nu} \right],
\]
\[
(90)
\]
The integral proportional to \(-\lambda x \) on the last line is just the conformal time interval \( \eta - \eta_i \). Substituting as well the asymptotic expansion of the Hankel function in \( H(x,y) \) gives the following leading ultraviolet result,
\[
u_{uv}(t,k) \rightarrow \frac{1}{2\sqrt{2}} e^{-ik(\eta - \eta_i) + i\lambda},
\]
\[
(91)
\]
where \( \chi(t,k) \equiv \lambda(t) x(t,k) - [\nu(t) + \frac{1}{2}] \tilde{\zeta} \). Recovering the correct correspondence limit is an important check on the consistency of the formalism.

7. The infrared regime: For phenomenology it is the infrared regime \( (x \ll 1) \) that is more useful. In this case it is best to decompose the Hankel functions into Bessel functions of positive and negative order,
\[
H(x,y) = icsc(\nu \pi) x^\mu \left[ e^{2i\nu \pi} J_\nu(\lambda x) - J_{-\nu}(\lambda x) \right].
\]
\[
(92)
\]
The conjugate solution takes slightly different forms for \( \lambda \) positive (inflation) and negative (subluminal expansion),
\[
H^*(x,y) = icsc(\nu \pi) x^\mu \left[ e^{2i\nu \pi} J_\nu(\lambda x) + J_{-\nu}(\lambda x) \right].
\]
\[
(93)
\]
We seek a matrix \( O(t) \) that effects the change of basis from \( H \) and \( H^* \) to solutions based upon \( J_\nu \) and \( J_{-\nu} \),
\[
\frac{1}{\sqrt{2}} \left( H(x,y), H^*(x,y) \right) \equiv O(t)
\]
\[
(94)
\]
For positive \( \lambda \) (inflation) the required matrix and its inverse are,
\[
O(t) = \frac{1}{\sqrt{2}} \left( 1 - i cot(\nu \pi), 1 + i cot(\nu \pi) \right),
\]
\[
(95)
\]
\[
O^{-1}(t) = \frac{1}{\sqrt{2}} \left( 1 + i cot(\nu \pi), 1 - i cot(\nu \pi) \right).\]
\[
(96)
\]
For negative \( \lambda \) (subluminal expansion) we have,
\[
O(t) = \frac{1}{\sqrt{2}} \left( 1 + i cot(\nu \pi), 1 - i cot(\nu \pi) \right),
\]
\[
(97)
\]
\[
O^{-1}(t) = \frac{1}{\sqrt{2}} \left( 1 - i cot(\nu \pi), 1 + i cot(\nu \pi) \right).\]
\[
(98)
\]
In the new basis this matrix is,
\[
M(t_2,t_1,k) = O^{-1}(t_2) \tilde{M}(t_2,t_1,k) O(t_1,k),
\]
\[
(99)
\]
The new exponent matrix is related to the old as follows,
\[
A(t,k) = \tilde{O}^{-1}(t) O(t) + O^{-1}(t) \tilde{A}(t,k) O(t).
\]
\[
(100)
\]
This is just an exercise in matrix multiplication. For either sign of \( \lambda \) the result is,
\[
A = \frac{\pi}{4} \nu \left( \frac{\csc(\nu \pi) e_{\nu}(\lambda x)}{2\nu \csc(\nu \pi) b_{\nu}(\lambda x) - \csc(\nu \pi) e_{\nu}(\lambda x)} \right),
\]
\[
(101)
\]
For small \( x \) the coefficients of \( A \) go like,
\[
\beta \sim x^{2-2\nu}, \gamma \sim \ln(x), \delta \sim x^{2\nu}.
\]
\[
(102)
\]
where the various components are,

\[ -\text{icsc}(\nu \pi)J_\nu , J_\nu \}
\]

The small \( x \) behavior of \( \gamma J_\nu \) is \( \sim \ln(x)x^\nu \), whereas \( \beta J_\nu \) is \( \sim x^{2+\nu} \). Now consider two factors of \( \mathcal{A} \) for any times during which \( x \ll 1 \),

\[ \mathcal{A}(t_2,k)\mathcal{A}(t_1,k) = \left( \gamma_\nu \gamma_{n-1} \cdots \gamma_1 \right) \left( -i[\gamma_n \cdots \gamma_2 \delta_1 \cdots] \right) . \]

(108)

Since \( \beta \delta \sim x^2 \), these terms are negligible with respect to \( \gamma^2 \sim \ln^2(x) \).

To see this, first consider the basis functions times a single factor of \( \mathcal{A} \),

\[ -\text{icsc}(\nu \pi)\gamma J_\nu - i\beta J_\nu , -\text{cs}(\nu \pi)\delta J_\nu - \gamma J_\nu \}
\]

(107)

The small \( x \) behavior of \( \gamma J_\nu \) is \( \sim \ln(x)x^\nu \), whereas \( \beta J_\nu \) is \( \sim x^{2+\nu} \). Now consider two factors of \( \mathcal{A} \) for any times during which \( x \ll 1 \),

\[ A(t_2,k)\mathcal{A}(t_1,k) = \left( \gamma_\nu \gamma_{n-1} \cdots \gamma_1 \right) \left( -i[\gamma_n \cdots \gamma_2 \delta_1 \cdots] \right) . \]

(109)

In this expression the off diagonal elements are,

\[ -i[\gamma_n \cdots \gamma_2 \delta_1 \cdots] \equiv -i[\gamma_n \cdots \gamma_2 \delta_1 \cdots] \]

(108)

\[ -i[\beta_n \gamma_{n-1} \cdots \gamma_1 \cdots] \equiv -i[\beta_n \gamma_{n-1} \cdots \gamma_1 \cdots] \]

(111)

\[ -i[\gamma_n \delta_n \gamma_{n-2} \cdots \gamma_1 \cdots] \equiv -i[\gamma_n \delta_n \gamma_{n-2} \cdots \gamma_1 \cdots] \]

(111)

This form can be exponentiated,

\[ \mathcal{M}(t_2,t_1,k) \rightarrow \left( \begin{array}{ccc}
\Gamma(t_2,t_1,k) & -i\Delta(t_2,t_1,k) \\
-iB(t_2,t_1,k) & \Gamma^{-1}(t_2,t_1,k)
\end{array} \right) , \]

(122)

where the various components are,

\[ \Gamma(t_2,t_1,k) = \exp \left[ \int_{t_1}^{t_2} dt \gamma(t,k) \right] , \]

(113)

\[ B(t_2,t_1,k) = \int_{t_1}^{t_2} dt \Gamma^{-1}(t_2,t,k) \beta(t,k) \Gamma(t,t_1,k) , \]

(114)

\[ \Delta(t_2,t_1,k) = \int_{t_1}^{t_2} dt \Gamma(t_2,t,k) \delta(t,k) \Gamma^{-1}(t_2,t_1,k) . \]

(115)

Because (122) is only valid up to factors of order \( x^2 \) it is superfluous to evaluate the components using anything but the leading small \( x \) forms for \( \beta, \gamma \) and \( \delta \),

\[ \beta(t,k) = \frac{1}{2\pi} \frac{\nu}{\nu - \frac{1}{2}} \frac{\Gamma^2(\nu - 1)}{2} + O(x^{4-2\nu}) , \]

(116)

\[ \gamma(t,k) = \nu \left[ \ln \left( \frac{1}{2} \lambda x \right) + 1 - \psi(\nu) \right] + O(x^2) , \]

(117)

\[ \delta(t,k) = \left[ \frac{\pi}{2\nu} \frac{\nu}{\nu - \frac{1}{2}} \frac{\Gamma^2(\nu)}{2} \right] + O(x^{2\nu+2}) . \]

(118)

The leading term in (117) is a total derivative,

\[ \gamma(t,k) = \frac{d}{dt} \ln \left[ a^{\nu-1}(\frac{1}{2} \lambda x)^{\nu-\frac{1}{2}} \right] + O(x^2) , \]

(119)

so we can obtain explicit expressions for \( \Gamma, B \) and \( \Delta \), up to correction factors of order \( x^2 \),

\[ \Gamma(t_2,t_1,k) \rightarrow \frac{a^{\nu-1}(\frac{1}{2} \lambda x_2)^{\nu-\frac{1}{2}}}{a_1^{\nu-1}(\frac{1}{2} \lambda x_1)^{\nu-\frac{1}{2}}} \Gamma(\nu) , \]

(120)

\[ B(t_2,t_1,k) \rightarrow \frac{a^{\nu-1}(\frac{1}{2} \lambda x_2)^{\nu-\frac{1}{2}}}{a_1^{\nu-1}(\frac{1}{2} \lambda x_1)^{\nu-\frac{1}{2}}} \Gamma(\nu) , \]

(121)

which can win out over the other terms and make \( \Gamma(t_2,t_1,k) \rightarrow \Gamma(\nu) \) dominate \( \Gamma(t_2,t_1,k) \rightarrow \Gamma(\nu) \). After inflation the parameter \( \nu(t) \) changes sign and \( -iB(t_2,t_1,k) \) is insignificant compared with \( \Gamma(t_2,t_1,k) \). (The only exception is when \( \nu_2 \) passes through a negative integer, at which point \( \Gamma(\nu_2) \) diverges and the infrared basis becomes degenerate.)
It is straightforward to partially integrate once, giving
\[ u^-(t, t_1, k) = \sqrt{\frac{\pi a}{k}} \left( -\csc(\nu \pi) J_\nu(\lambda x), \right), \]
\[ J_\nu(\lambda x) \mathcal{M}(t, t_1, k) \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad (126) \]
\[ u^+(t, t_1, k) = \sqrt{\frac{\pi a}{k}} \left( -\csc(\nu \pi) J_\nu(\lambda x), \right), \]
\[ J_\nu(\lambda x) \mathcal{M}(t, t_1, k) \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad (127) \]

It is straightforward to verify that these solutions obey
\[ u^-(t, t_1, k) u^+(t, t_1, k) - u^-(t, t_1, k) u^+(t, t_1, k) = i. \quad (128) \]

If \( x \ll 1 \) between \( t_1 \) and \( t \), we need only keep the \( n = 0 \) terms of the Bessel functions (126). We can also employ the infrared simplifications (128) of the transfer matrix. Even though \( B(t, t_1, k) \) can be larger than \( \Gamma(t, t_1, k) \), its contribution to \( u^-(t, t_1, k) \) is still insignificant because it multiplies the \( J_\nu \).

\[ u^-(t, t_1, k) \rightarrow -i \sqrt{\frac{\pi a}{k}} \left[ -\csc(\nu \pi) J_\nu(\lambda x) \Gamma(t, t_1, k) \right. \]
\[ \left. + J_\nu(\lambda x) B(t, t_1, k) \right], \quad (129) \]
\[ -i \frac{d \nu}{\sqrt{2\pi \kappa}} \frac{\Gamma(\nu_1)}{\kappa^{\nu_1 - 1} (\frac{\lambda}{a} \lambda x)^{\nu_1 - \frac{1}{2}}}. \quad (130) \]

Most of our limiting form (128) consists of constants; its time dependence derives entirely from the factor of \( a^{-\frac{\nu}{2}} \).

This corresponds to a constant scalar field, and we see from (3) that \( \tilde{\varphi}(t, k) = \text{constant} \) is indeed a solution for the infrared limit of \( \tilde{k} = 0 \).

Even in the infrared limit both terms contribute to \( u^+(t, t_1, k) \),
\[ u^+(t, t_1, k) \rightarrow \sqrt{\frac{\pi a}{k}} \left[ -\csc(\nu \pi) J_\nu(\lambda x) \Delta(t, t_1, k) \right. \]
\[ \left. + J_\nu(\lambda x) \Gamma^{-1}(t, t_1, k) \right], \quad (131) \]
\[ \rightarrow \sqrt{\pi a} \kappa^{-\nu} \frac{a^{\nu-1} (\frac{\lambda}{a} \lambda x)^{\nu-\frac{1}{2}}}{\Gamma(\nu_1)} \]
\[ \times \left[ -\frac{1}{2D-4} \int_{t_1}^t dt' \frac{\dot{H}}{\nu^2 H a^{D-1}} + \frac{1}{2 D a^{D-1}} \right]. \quad (132) \]

It is straightforward to partially integrate once,
\[ \int_{t_1}^t dt' \frac{\dot{H}}{\nu^2 H a^{D-1}} = -\frac{1}{2 \nu H a^{D-1}} + \left( \frac{D}{d} \right) \frac{D - 1 - y}{\nu a^{D-1}}. \quad (133) \]

From the identities,
\[ \frac{D - 1 - y}{\nu} = 2 - 2y = 2 - \frac{d}{dt} \left( \frac{2}{H} \right), \quad (134) \]
\[ \frac{1}{\nu} + 2 = (D - 2) \frac{\lambda}{\nu}, \quad (135) \]

we see that a second partial integration brings the integral to the form,
\[ \frac{1}{2D-4} \int_{t_1}^t dt' \frac{\dot{H}}{\nu^2 H a^{D-1}} = -\frac{\lambda}{2 \nu H a^{D-1}} + \frac{D - 1 - y}{2 \nu a^{D-1}}. \quad (136) \]
Substitution in (132) gives the infrared limit for the small mode function,
\[ u^+(t, t_1, k) \rightarrow \sqrt{\pi a} \kappa^{\nu} \frac{a^{\nu-1} (\frac{\lambda}{a} \lambda x)^{\nu-\frac{1}{2}}}{\Gamma(\nu_1)} \]
\[ \times \left[ -\frac{1}{2 \nu_1 H a^{D-1}} - \int_{t_1}^t dt' \frac{\dot{H}}{a^{D-1}} \right]. \quad (137) \]

Note that the infrared limits (138) of our solutions retain their canonical normalization (128).

The two independent solutions of the scalar field equation (3) in the infrared limit of \( \tilde{k} = 0 \) are,
\[ \tilde{\varphi}(t, \tilde{0}) \sim 1 \quad \tilde{\varphi}(t, \tilde{0}) \sim \int_{t_1}^t \frac{dt' \dot{H}}{a^{D-1}}. \quad (138) \]

We have seen that \( u^-(t, t_1, k) \) becomes proportional to the constant solution, but the second solution in (138) also contains a constant term from its dependence upon the lower limit of integration. Since the integrand falls as the universe expands, care must be taken to isolate the small, time-dependent term from the potentially much larger constant. This is something our formalism does automatically. For if \( y(t) \) is constant for \( t \geq t_1 \) we can solve for the Hubble parameter and the scale factor,
\[ H(t) \bigg|_{y=y_1} = \frac{H_1}{1 + y_1 H_1 (t - t_1)}, \quad (139) \]
\[ a(t) \bigg|_{y=y_1} = a_1 \left[ 1 + y_1 H_1 (t - t_1) \right]^{\frac{\nu}{2}}. \quad (140) \]

In this case the integral can be explicitly evaluated,
\[ \int_{t_1}^t \frac{dt' \dot{H}}{a^{D-1}} \bigg|_{y=y_1} = -\frac{\lambda_1}{2 \nu_1 H a^{D-1}} + \frac{\lambda_1}{2 \nu_1 H_1 a^{D-1}}. \quad (141) \]

Hence the constant on the second line of our infrared form (138) for \( u^+(t, t_1, k) \) cancels the large constant from the lower limit, up to very small terms coming from the weak time dependence of \( y(t) \). So our second solution is indeed dominated by the small, time-dependent, lower limit term.

It remains only to express the original mode solution (3) in the infrared basis,
\[ u(t, k) = \frac{1}{\sqrt{2}} \left( u^-(t, t_1, k), u^+(t, t_1, k) \right) \times \mathcal{M}(t_1, t_1, k) \left( \begin{array}{c} 1 \\ 1 + i \cot(\nu_1 \pi) \end{array} \right). \quad (142) \]

It is reasonable for \( t_1 \) to be the time of first horizon crossing, that is, \( x(t_1, k) = 1 \). Even though \( x(t, k) \) is just
becoming small at this time, the expansion is so rapid during inflation that we have $x \ll 1$ almost immediately afterwards. Because the transfer matrix up to this time is of order one, only $u^-(t, t_1, k)$ contributes significantly,

$$u_{tt}(t, k) \rightarrow \frac{-ia\nu}{\sqrt{2k}} \frac{\Gamma(1-\nu)J_{-\nu}(\lambda x)}{(\frac{1}{2}x\nu)^{\nu}} \times (\frac{H_1}{k})^{\frac{D-1}{2}} C_1(k)C_t(k). \quad (143)$$

We have expressed $u_{tt}$ with the standard normalization times two correction factors. The correction that depends upon the system at first horizon crossing is,

$$C_t(k) \equiv \frac{\gamma}{\sin(\nu\pi)} \frac{\Gamma^2(\nu)}{(\frac{1}{2}\lambda_1)^{\nu-\frac{1}{2}}} . \quad (144)$$

The other correction factor depends upon evolution up to this point,

$$C_i(k) \equiv \mathcal{M}^{11}(t_1, t_1, k) + \frac{ie^{-\nu\pi}}{\sin(\nu\pi)} \mathcal{M}^{12}(t_1, t_1, k). \quad (145)$$

Further corrections to (143) are down by a factor $x^2(t', k)$ for some $t'$ between first and second horizon crossings. (Second horizon crossing is the time $t_2$, after inflation, at which $x(t_2, k) = 1$.) Before second horizon crossing we need retain only the leading term in the series expansion of the Bessel function (125), but the form (143) should remain valid to arbitrarily late times provided that $y(t)$ does not experience violent evolution at late times. This is a very safe assumption given what we know about late time cosmology.

8. Summary and discussion: This has been a long and technical series of derivations, and it was not clear at the beginning what would be the most economical form in which to express the final answer. Now that we are done, it is well to summarize the important results in a simple way. It is also desirable to replace the parameter $y(t) = -\dot{H}/H^2$, and the quantities $\lambda$ and $\mu$ which are derived from it, with the familiar deceleration parameter, $q(t) = -1 + y(t)$. The only exception we make is the parameter $\nu = 1 - \frac{D-1}{2} q$, which appears in too many indices to be conveniently expunged.

Our mode solution is specified in terms of parameters constructed from the scale factor and its derivatives,

$$x(t, k) \equiv \frac{k}{aH} , \quad q(t) \equiv -1 \frac{\dot{H}}{H^2} \quad \text{and} \quad \nu(t) \equiv 1 - \frac{D-1}{2} \frac{q}{q} . \quad (146)$$

The solution to equation (17) which is positive frequency at the initial time $t = t_1$ takes the form of a row vector of Bessel functions multiplied into a transfer matrix,

$$u(t, k) \equiv \sqrt{\frac{\pi a}{2k}} \left( -\frac{x}{2q} \right)^{\frac{1}{2}} \left( -\nu \pi J_{-\nu}(\nu \pi) , J_{-\nu}(\nu \pi) \right) \times \mathcal{M}(t, t_1, k) \left( 1 + i\cot[\nu(t_1)\pi] \right) . \quad (147)$$

The transfer matrix $\mathcal{M}(t, t_1, k)$ is the time-ordered product of the exponential of a line integral,

$$\mathcal{M}(t, t_1, k) \equiv P \exp \left\{ \int_{t_1}^{t} dt' \mathcal{A}(t', k) \right\} , \quad (148)$$

$$= \sum_{n=0}^{\infty} \int_{t_1}^{t} dt_1 \int_{t_1}^{t_2} dt_2 \cdots \int_{t_n}^{t_{n-1}} \mathcal{A}(t_1, k) \cdots \mathcal{A}(t_n, k) . \quad (149)$$

The exponent matrix $\mathcal{A}(t, k)$ vanishes whenever $q(t)$ is constant. It has the form,

$$\mathcal{A}(t, k) = \frac{\nu \pi}{4} \left( \csc(\nu\pi) c_{\nu}(\nu \pi) - 2i d_{\nu}(\nu \pi) \right) , \quad (150)$$

where the various coefficient functions are,

$$b_{\nu}(z) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n-\nu+\frac{1}{2} z^{2n-2\nu(n-\nu)^{-1}}}{\Gamma(n) \Gamma(n+1) \Gamma(n-2\nu+1)} . \quad (151)$$

$$c_{\nu}(z) = -\frac{4i}{\sin(\nu\pi)} \left( \sin(\nu\pi) - 1 - \sinh(z) \right)$$

$$- \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n-\nu+\frac{1}{2})}{\Gamma(n+1) \Gamma(n+2\nu+1)} z^{2n-2\nu(n-\nu)^{-1}} . \quad (152)$$

$$d_{\nu}(z) = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\nu+\frac{1}{2}) z^{2n+2\nu(n+\nu)^{-1}}}{\Gamma(n+2\nu+1) \Gamma(n+1) \Gamma(n+1)} . \quad (153)$$

In the ultraviolet ($x(t, k) \gg 1$) our general solution (147) approaches,

$$u_{uv}(t, k) \rightarrow \sqrt{\frac{a(t)}{2k}} e^{ik\eta(t_1)+ix_1} , \quad (154)$$

were $\chi(t, k) \equiv \lambda(t) x(t, k) - [\nu(t) + \frac{1}{2} \frac{q}{q}]$. This is just the flat space solution with a trivial redshift factor. Corrections to are of order $1/x$ times (154).

It is common to assume all modes are originally subhorizon ($x(t_1, k) > 1$). During inflation $x(t, k)$ falls, typically exponentially. This carries more and more modes into the infrared regime of $x(t, k) \ll 1$. First horizon crossing is the time, $t_1(k)$, at which $x(t_1, k) = 1$. After this time our general solution (147) takes the form,

$$u_{tt}(t, k) \rightarrow \frac{-ia\nu}{\sqrt{2k}} \frac{\Gamma(1-\nu)J_{-\nu}(-\frac{q}{2})}{\frac{1}{2}x\nu} \times (\frac{H_1}{k})^{\frac{D-1}{2}} C_1(k)C_t(k) . \quad (155)$$

Corrections are of order $x^2(t', k)$ times this result, where $t'$ is some time for which $x(t', k) \ll 1$.

The factor $C_1(k)$ depends upon the state of the system at first horizon crossing,

$$C_1(k) \equiv \frac{\gamma}{\sin(\nu\pi)} \frac{\Gamma^2(\nu)}{(\frac{1}{2}\lambda_1)^{\nu-\frac{1}{2}}} . \quad (156)$$
The factor \( C_i(k) \) depends upon evolution up to this time,
\[
C_i(k) \equiv \mathcal{M}^{11}(t_1, t_i, k) + \frac{i e^{-i\nu_i \pi}}{\sin(\nu_i \pi)} \mathcal{M}^{12}(t_1, t_i, k). \tag{157}
\]
Since both \( y(t) \) and \( \dot{y}(t) \) are small during inflation, we can obtain an excellent approximation for \( C_i(k) \) by simply expanding the transfer matrix,\[
\mathcal{M}^{11}(t_1, t_i, k) = 1 + \int_{t_i}^{t_1} dt \gamma(t, k) + \int_{t_i}^{t_1} dt \int dt' \left[ \gamma(t, k)\gamma(t', k) - \delta(t, k)\beta(t', k) \right] + \ldots \tag{158}
\]
\[
\mathcal{M}^{12}(t_1, t_i, k) = 0 - i \int_{t_i}^{t_1} dt \delta(t, k) - i \int_{t_i}^{t_1} dt \int dt' \left[ \gamma(t, k)\delta(t', k) - \delta(t, k)\gamma(t', k) \right] + \ldots \tag{159}
\]
For completeness we remind the reader of the relations,
\[
\beta(t, k) = \frac{\pi \nu}{2 \sin^2(\nu \pi)} b_{\nu}(\frac{\nu}{k}), \tag{160}
\]
\[
\gamma(t, k) = \frac{\pi \nu}{4 \sin(\nu \pi)} c_{\nu}(-\frac{\nu}{k}), \tag{161}
\]
\[
\delta(t, k) = \frac{\pi \nu}{2} d_{\nu}(\frac{\nu}{k}). \tag{162}
\]

The formalism we have just summarized has many applications. One of these is to derive an improved estimate for the graviton contribution to the power spectrum of anisotropies in the cosmic microwave background. We shall present the derivation elsewhere \[18\] but the result is,
\[
\mathcal{P}_h(k) = GH^2_\nu(k) ||C_i(k)||^2 C^2_i(k). \tag{163}
\]

The standard result is \( GH^2_\nu \), so the factors \( C_i(k) \) and \( C_1(k) \) represent improvements. Because the literature abounds with different conventions for this quantity we correspond \( \mathcal{P}_h(k) \) below to the symbol \( \delta_\nu(k) \) used by Mukhanov, Feldman and Brandenberger \[11\], to the variable \( \mathcal{P}_g(k) \) used by Liddle and Lyth \[12\], and to the quantity \( A^2_T(k) \) used by Lidsey et al. \[13\].
\[
\mathcal{P}_h(k) = \frac{9\pi}{4} ||\delta_\nu(k)||^2 = \frac{\pi}{16} \mathcal{P}_g(k) = \frac{25\pi}{4} A^2_T(k). \tag{164}
\]

The most unambiguous definition of \( \mathcal{P}_h(k) \) is given by the (purely gravitational wave contribution to the) correlation function between temperature fluctuations observed from directions \( \hat{e}_1 \) and \( \hat{e}_2 \),
\[
\left< \left| \frac{\Delta T_R(\hat{e}_1)}{T_R} \frac{\Delta T_R(\hat{e}_2)}{T_R} \right|^2 \right> = \int_0^{\infty} dk P_h(k) \int \frac{d^2 \tilde{k}}{4\pi} \Theta(\hat{e}_1, \tilde{k}) \times \Theta^*(\tilde{e}_2, \tilde{k}) \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 \left[ \Pi_{im} \Pi_{jn} - \frac{1}{2} \delta_{ij} \Pi_{mn} \right]. \tag{165}
\]
In this formula \( \Pi_{ij} \equiv \delta_{ij} - \hat{k}_i \hat{k}_j \) is the transverse projection matrix coming from the sum over graviton polarization tensors. The angular factor is,
\[
\Theta(\hat{e}, \tilde{e}) = \frac{e^{2i\theta_{x_0}}}{\sqrt{4\pi}} \left\{ 2 - 3\theta^2 - \frac{3}{2} \Theta(\theta^2) \left[ \ln \left( \frac{1 + \theta}{1 - \theta} \right) + i\pi \right] \right\}, \tag{166}
\]
where \( x_0 \equiv x(t_0, k) \) is the physical wave number in current Hubble units and we define \( \theta \equiv \tilde{k} \cdot \hat{e} \).

There are many other applications. One particularly interesting question we now have the technology to answer is the response to multiple periods of inflation. Whereas the weak energy condition implies that the Hubble parameter can only decrease or stay constant, the evolution of \( q(t) \) need not consist of monotonic growth from its inflationary value of nearly \( -1 \). Indeed, after inflation we know that \( q(t) \) decreased from the era of radiation domination \( (q = +1) \) to the era of matter domination \( (q = +\frac{1}{3}) \), and considerable evidence exists that it has now dropped below zero \[4\ \[15\ \[20\]. There is no reason it could not have experienced such oscillations at very early times. The standard formalism for analyzing density perturbations breaks down in this case, but our techniques should be applicable to any number of horizon crossings.

Note that our formalism depends only upon the scale factor \( a(t) \), not on the matter theory which supports it. Results derived in this way are therefore independent of the details of particular inflationary models. For example, it does not matter how many scalars, if any, participate in driving inflation.

It should be possible to compute the one loop effective action of the massless, minimally coupled scalar as a functional of general \( a(t) \). Of course the ultraviolet divergence is universal and was derived long ago \[15\], but the ultraviolet finite, infrared contributions have never been computed and are, in many ways more interesting. Our infrared expression \[14\] \[14\] for the mode function should be of great use there.

Another application is computing the ultraviolet finite part of the coincidence limit of the propagator. Since linearized gravitons have the same mode functions, this would also apply for quantum gravity. Proceeding in this way to compute the infrared part of the non-coincident propagator, it should be possible to derive the quantum gravitationally induced stress tensor in the presence of a class of backgrounds guaranteed to include any solution which preserves homogeneity and isotropy. This is an essential step in checking whether quantum gravitational back-reaction can quench \( \Lambda \)-driven inflation \[14\].

Finally, a simple modification of the formalism developed here can be used to derive the mode functions for general perturbations in gravity plus a single, minimally coupled scalar \[21\]. These mode functions can be used to improve the scalar power spectrum as was done for its tensor counterpart \[13\].
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