HOMOGENEOUS EINSTEIN METRICS ON $G_2/T$

ANDREAS ARVANITOEYORGOS, IOANNIS CHRYSIKOS, AND YUSUKE SAKANE

(Communicated by Lei Ni)

Abstract. We construct the Einstein equation for an invariant Riemannian metric on the exceptional full flag manifold $M = G_2/T$. By computing a Gröbner basis for a system of polynomials on six variables we prove that this manifold admits exactly two non-Kähler invariant Einstein metrics. Thus $G_2/T$ turns out to be the first known example of an exceptional full flag manifold which admits a non-Kähler and not normal homogeneous Einstein metric.

Introduction

A Riemannian manifold $(M, g)$ is called Einstein if the metric $g$ has constant Ricci curvature, that is, $\text{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$, where $\text{Ric}_g$ is the Ricci tensor corresponding to $g$. The question whether $M$ carries an Einstein metric, and if so, how many, is a fundamental one in Riemannian geometry. A number of interesting results in geometry have been motivated and inspired by this hard problem. The Einstein equation is a nonlinear second order PDE, and a good understanding of its solutions in the general case seems far from being attained. It becomes more manageable in the homogeneous setting. Most known examples of compact simply connected Einstein manifolds are homogeneous. In the homogeneous case the Einstein equation reduces to a system of algebraic equations for which we are looking for positive solutions. For some cases such solutions have been obtained explicitly. We refer to [14] and the references therein for more details on compact homogeneous Einstein manifolds. The low-dimensional cases were also examined in [5].

Let $K$ be a compact, connected and semisimple Lie group. A full flag manifold is a compact homogeneous space of the form $K/T$, where $T$ is a maximal torus in $K$. It is known that $K/T$ admits a unique (up to isometry) $K$-invariant Kähler-Einstein metric (cf. [13]).

Non-Kähler homogeneous Einstein metrics on full flag manifolds corresponding to classical Lie groups have been studied by several authors (cf. [2], [16], [10]). Although various existence results of homogeneous Einstein metrics on these spaces have been obtained, the classification of such metrics is a demanding task which remains widely open. In the present paper we study the classification problem of homogeneous Einstein metrics on the full flag manifold $G_2/T$. The isotropy representation of this space decomposes into six inequivalent irreducible submodules.
There are three (nonisomorphic) flag manifolds corresponding to the exceptional Lie group $G_2$, since there are exactly three different ways to paint black the simple roots in the Dynkin diagram of $g_2$, as shown in Figure 1 (for the classification of generalized flag manifolds in terms of painted Dynkin diagrams, we refer to [1] or [6]).

If we paint black one simple root in the Dynkin diagram of $g_2$, then we obtain a flag manifold of the form $G_2/U(2)$ with two or three isotropy summands, depending on the height of this simple root. Recall that for $g_2$ we can choose a set of simple roots by $\Pi_M = \{\alpha_1, \alpha_2\}$ with $(\alpha_1, \alpha_1) = 3(\alpha_2, \alpha_2)$, so the highest root has the form $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2$ (see Section 5). Thus, the flag manifold $G_2(\alpha_2)$ in Figure 1 has two isotropy summands, and $U(2)$ is represented by the short root of $g_2$. For this space, all $G_2$-invariant Einstein metrics have been obtained explicitly in [3]. The second flag manifold $G_2(\alpha_1)$ in Figure 1 has three isotropy summands, and the isotropy group $U(2)$ is represented by the long root of $g_2$. For this space, the $G_2$-invariant Einstein metrics were studied in [12], [2].

The full flag manifold $G_2/T$, where $T = U(1) \times U(1)$ is a maximal torus in $G_2$, is obtained by painting black both simple roots in the Dynkin diagram of $g_2$. According to [19] a full flag manifold $K/T$ is a normal homogeneous Einstein manifold if and only if all roots of $K$ have the same length, and in this case the normal metric of $K/T$ is never Kähler. Therefore, if $K$ is an exceptional Lie group, then $K/T$ is a normal homogeneous Einstein manifold if and only if $K \in \{E_6, E_7, E_8\}$, so $G_2/T$ is not normal. Our main result is the following:

**Theorem A.** The full flag manifold $G_2/T$ admits exactly three $G_2$-invariant Einstein metrics (up to isometry). There is a unique Kähler-Einstein metric given (up to a scalar) by $g = (3,1,4,5,6,9)$, and the other two are non-Kähler. The approximate values of these invariant metrics are given in Theorem [41].

As a consequence of Theorem A, $G_2/T$ is the first known example of an exceptional full flag manifold which admits a non-Kähler and not normal homogeneous Einstein metric. Also, the present work on $G_2/T$ is the first attempt towards the classification of homogeneous Einstein metrics on generalized flag manifolds with six isotropy summands.

The paper is organised as follows: In Section 2 we recall the Lie-theoretic description of a full flag manifold $K/T$ of a compact and connected semisimple Lie group $K$, and we study its isotropy representation. Next, following [16], we describe the structure constants of $K/T$ relative to the associated isotropy decomposition, and we give the expression of the Ricci tensor of a $K$-invariant metric on $K/T$. In Section 3 we consider the exceptional full flag manifold $G_2/T$ and we give its Lie-theoretic description. Then we construct the Einstein equation for a $G_2$-invariant Riemannian metric. In the last section, we give the corresponding polynomial system, and by computing a Gröbner basis for this system, we prove Theorem A and obtain the full classification of homogeneous Einstein metrics on $G_2/T$. 

The full flag manifold $G_2/T$, where $T = U(1) \times U(1)$ is a maximal torus in $G_2$, is obtained by painting black both simple roots in the Dynkin diagram of $g_2$. According to [19] a full flag manifold $K/T$ is a normal homogeneous Einstein manifold if and only if all roots of $K$ have the same length, and in this case the normal metric of $K/T$ is never Kähler. Therefore, if $K$ is an exceptional Lie group, then $K/T$ is a normal homogeneous Einstein manifold if and only if $K \in \{E_6, E_7, E_8\}$, so $G_2/T$ is not normal. Our main result is the following:

**Theorem A.** The full flag manifold $G_2/T$ admits exactly three $G_2$-invariant Einstein metrics (up to isometry). There is a unique Kähler-Einstein metric given (up to a scalar) by $g = (3,1,4,5,6,9)$, and the other two are non-Kähler. The approximate values of these invariant metrics are given in Theorem [41].

As a consequence of Theorem A, $G_2/T$ is the first known example of an exceptional full flag manifold which admits a non-Kähler and not normal homogeneous Einstein metric. Also, the present work on $G_2/T$ is the first attempt towards the classification of homogeneous Einstein metrics on generalized flag manifolds with six isotropy summands.

The paper is organised as follows: In Section 2 we recall the Lie-theoretic description of a full flag manifold $K/T$ of a compact and connected semisimple Lie group $K$, and we study its isotropy representation. Next, following [16], we describe the structure constants of $K/T$ relative to the associated isotropy decomposition, and we give the expression of the Ricci tensor of a $K$-invariant metric on $K/T$. In Section 3 we consider the exceptional full flag manifold $G_2/T$ and we give its Lie-theoretic description. Then we construct the Einstein equation for a $G_2$-invariant Riemannian metric. In the last section, we give the corresponding polynomial system, and by computing a Gröbner basis for this system, we prove Theorem A and obtain the full classification of homogeneous Einstein metrics on $G_2/T$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
2. FULL FLAG MANIFOLDS

Let \( K/T \) be a full flag manifold where \( T \) is a maximal torus of a compact semisimple Lie group \( K \). We will give a characterization of \( K/T \) in terms of root system theory, and we will describe some topics of the associated Kähler geometry. Then, we study the isotropy representation of \( K/T \) and we give the expression of the Ricci tensor for a \( K \)-invariant metric on \( K/T \).

2.1. Lie-theoretic description of \( K/T \). Assume that \( \dim_{\mathbb{R}} T = \text{rank} \, G = \ell \). We denote by \( \mathfrak{t}, \mathfrak{t} \) the Lie algebras of \( K \) and \( T \) respectively, and by \( \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}, \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t} \), the corresponding complexifications. Let \( \mathfrak{t}^* \) and \( \mathfrak{t}_{\mathbb{C}}^* \) be the dual spaces of \( \mathfrak{t} \) and \( \mathfrak{t}_{\mathbb{C}} \), respectively. The subalgebra \( \mathfrak{t}_{\mathbb{C}} \) is a Cartan subalgebra of the complex semisimple Lie algebra \( \mathfrak{t}_{\mathbb{C}} \), and thus we obtain the root space decomposition \( \mathfrak{t}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\lambda \in \Lambda} \mathfrak{t}_{\mathbb{C}}^\lambda \), where \( R \) is the root system of \( \mathfrak{t}_{\mathbb{C}} \) relative to \( \mathfrak{t}_{\mathbb{C}} \) and \( \mathfrak{t}_{\mathbb{C}}^\lambda \) are the corresponding root spaces. Recall that by \( \mathbb{C} \)-linearity, a root \( \alpha \in R \) is completely determined by its restriction to either \( \mathfrak{t} \) or \( i\mathfrak{t} \). Since the Killing form \( B \) of \( \mathfrak{t}_{\mathbb{C}} \) is nondegenerate, for any \( \lambda \in \mathfrak{t}_{\mathbb{C}}^* \) we define \( H_{\lambda} \in i\mathfrak{t} \) by the equation \( B(H_{\lambda}, H) = \lambda(H) \) for all \( H \in \mathfrak{t}_{\mathbb{C}} \). Let \( \mathfrak{t}^* \) denote the real linear subspace of \( \mathfrak{t}_{\mathbb{C}}^* \) which consists of all \( \lambda \in \mathfrak{t}_{\mathbb{C}}^* \) such that the restriction \( \lambda|_t \) has values in \( i\mathbb{R} \). Note that the restriction map \( \lambda \mapsto \lambda|_i \) defines an isomorphism from \( \mathfrak{t}^* \) onto the real linear dual space \( \mathfrak{t}^* \), which allows us to identify these spaces. Then, it is well known that \( R \) spans \( \mathfrak{t}^* \) and that \( R \) is a finite subset of \( \mathfrak{t}^* \). Thus, if \( \alpha \in R \), then \( \alpha \in \mathfrak{t}^* \).

Let \( ( \ , \ ) \) be the bilinear form on \( \mathfrak{t}_{\mathbb{C}}^* \) induced by the Killing form, that is, \( (\lambda, \mu) = B(H_{\lambda}, H_{\mu}) \), for any \( \lambda, \mu \in \mathfrak{t}_{\mathbb{C}}^* \). Then, since \( B \) is negative definite on \( \mathfrak{t} \) and positive definite on \( i\mathfrak{t} \), the restriction of \( ( \ , \ ) \) on \( \mathfrak{t}^* \) is a positive definite inner product. The weight lattice of \( \mathfrak{t}_{\mathbb{C}} \) with respect to \( \mathfrak{t}_{\mathbb{C}} \) is given by \( \Lambda = \{ \lambda \in \mathfrak{t}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha \in R \} \). Let \( \Pi = \{ \alpha_1, \ldots, \alpha_\ell \} \) be a simple root system of \( R \), and let \( R^+ \) be the set of all positive roots with respect to \( \Pi \). Consider the fundamental weights corresponding to \( \Pi \), that is, \( \Lambda_1, \ldots, \Lambda_\ell \in \Lambda \) such that

\[
(2.1) \quad \frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (1 \leq i, j \leq \ell).
\]

Then \( \{\Lambda_1, \ldots, \Lambda_\ell\} \) forms a \( \mathbb{Z} \)-basis for the weight lattice \( \Lambda \), and since \( \mathfrak{t} \cong i\mathfrak{t}^* \), it is \( \mathfrak{t} = \sum_{i=1}^\ell \mathbb{R}\Lambda_i \). In the weight lattice \( \Lambda \) there is a distinguished subset \( \Lambda^+ \) given by \( \Lambda^+ = \{ \lambda \in \Lambda \mid (\lambda, \alpha) > 0 \text{ for any } \alpha \in R^+ \} \). One can see that \( \Lambda^+ = \Lambda \cap C(\Pi) \), where \( C(\Pi) \) is the fundamental Weyl chamber corresponding to \( \Pi \), given by \( C(\Pi) = \{ \lambda \in i\mathfrak{t}^* \mid (\lambda, \alpha_i) > 0 \text{ for all } \alpha_i \in \Pi \} \). The elements of \( \Lambda^+ \) are usually called dominant weights relative to \( R^+ \), and any dominant weight can be expressed as a linear combination of the fundamental weights with nonnegative coefficients. For example, set \( \delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{t}^* \). Then, \( \delta = \sum_{i=1}^\ell \Lambda_i \) and thus \( \delta \in \Lambda^+ \) (cf. [S, p. 168]).

We now define the complex Lie subalgebras of \( \mathfrak{t}_{\mathbb{C}} \) by \( \mathfrak{n} = \sum_{\alpha \in R^+} \mathfrak{t}_{\mathbb{C}}^\alpha \) and \( \mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n} \). One can easily show that \( \mathfrak{n} \) is a nilpotent ideal of \( \mathfrak{t}_{\mathbb{C}} \) and that \( \mathfrak{b} \) is a maximal solvable Lie subalgebra of \( \mathfrak{t}_{\mathbb{C}} \); i.e. it is a Borel subalgebra. Let \( K_C \) denote the complex simply connected semisimple Lie group whose Lie algebra is \( \mathfrak{t}_{\mathbb{C}} \). Then, the connected subgroup \( B \subset K_C \) with Lie algebra \( \mathfrak{b} \) is a Borel subgroup of \( K_C \) and \( K_C/B \cong K/T \) as \( C^\infty \)-manifolds.
Since $K_C$ is a complex Lie group and $B$ a closed complex subgroup, the quotient $K_C/B$ admits a $K$-invariant complex structure. Furthermore, the $K$-invariant complex structures on $K_C/B = K/T$ are in 1-1 correspondence with different choices of positive roots for $\mathfrak{t}_C$ (cf. [13]). Since the Weyl group $W(R)$ of the root system of $\mathfrak{t}_C$ acts transitively on the sets of systems of positive roots, all these complex structures are equivalent. Moreover, the following holds:

**Theorem 2.1** ([7], [17]). There is a 1-1 correspondence between $K$-invariant Kähler metrics on $K_C/B$ and dominant weights in $\Lambda^+$. In particular, the $K$-invariant Kähler metric on $K_C/B$ corresponding to $2\delta$ is a Kähler–Einstein metric.

According to [7] p. 504 a full flag manifold admits a unique (up to equivalence) invariant complex structure, hence a unique (up to scale) Kähler–Einstein metric (cf. also [13]). This Kähler–Einstein metric will be computed in Section 2.

### 2.2. The isotropy representation of $K/T$.

We will now examine the isotropy representation of a full flag manifold $K_C/B = K/T$. Consider the reductive decomposition $\mathfrak{t} = \mathfrak{t} \oplus \mathfrak{m}$ of $\mathfrak{t}$ with respect to the negative of the Killing form $Q = -B(\ ,\ )$, that is, $\mathfrak{m} = \mathfrak{t}^\perp$ and $\text{Ad}(T)\mathfrak{m} \subset \mathfrak{m}$. As usual, we identify $\mathfrak{m} = T_o(K/T)$ (where $o = eT$ is the identity coset of $K/T$).

Choose a Weyl basis $\{H_{\alpha_1}, \ldots, H_{\alpha_\ell}\} \cup \{E_\alpha \in \mathfrak{t}_C^a | \alpha \in R\}$ with $B(E_\alpha, E_{-\alpha}) = -1$, $[E_\alpha, E_{-\alpha}] = -H_\alpha$. Recall for later use that the root vectors satisfy $[E_\alpha, E_\beta] = N_{\alpha, \beta}E_{\alpha + \beta}$ if $\alpha, \beta, \alpha + \beta \in R$ and $[E_\alpha, E_\beta] = 0$ otherwise. The numbers $N_{\alpha, \beta} \in \mathbb{R}$ satisfy $N_{\alpha, \beta} = -N_{\beta, \alpha}$, $N_{\alpha, \beta} = N_{-\alpha, -\beta} \in \mathbb{R}$ if $\alpha, \beta, \alpha + \beta \in R$, and $N_{\alpha, \beta} = 0$ if $\alpha, \beta \in R$, $\alpha + \beta \notin R$. They can also be chosen so that $N_{\alpha, -\beta} = N_{-\alpha, \beta}$. Then the real subalgebra $\mathfrak{t}$ is given by

$$\mathfrak{t} = \sum_{j=1}^\ell \mathbb{R}iH_{\alpha_j} \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) = \mathfrak{t} \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

where $A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = i(E_\alpha - E_{-\alpha})$ ($\alpha \in R^+$).

Since $\mathfrak{t} = \text{span}_\mathbb{R}\{iH_{\alpha_j} | 1 \leq j \leq \ell\}$, the reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ implies that

$$\mathfrak{m} = T_o(K/T) = \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

Set $\mathfrak{m}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha$ for any $\alpha \in R^+$. The linear space $\mathfrak{m}_\alpha$ is an irreducible $\text{Ad}(T)$-module which does not depend on the choice of an ordering in $R$. Furthermore, since the roots of $\mathfrak{t}_C$ with respect to $\mathfrak{t}_C$ are distinct and the root spaces are one-dimensional, it is obvious that $\mathfrak{m}_\alpha \nsubseteq \mathfrak{m}_\beta$ as $\text{Ad}(T)$-representations, for any two roots $\alpha, \beta \in R^+$. Thus, by using (2.3) we obtain the following:

**Proposition 2.2.** Let $M = K/T$ be a full flag manifold of a compact simple Lie group $K$. Then the isotropy representation of $M$ decomposes into a direct sum of 2-dimensional pairwise inequivalent irreducible $T$-submodules $\mathfrak{m}_\alpha$ as follows:

$$\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha.$$ 

The number of these submodules is equal to the cardinality $|R^+|$. 


2.3. The Ricci tensor for a $K$-invariant metric on $K/T$. Since $K/T$ is a reductive homogeneous space, there is a natural 1-1 correspondence between $K$-invariant symmetric covariant 2-tensors on $K/T$ and $\text{Ad}(T)$-invariant symmetric bilinear forms on $\mathfrak{m}$. For example, in this correspondence a $K$-invariant Riemannian metric $g$ on $K/T$ corresponds to an $\text{Ad}(T)$-invariant inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{m}$. In particular, since $\mathfrak{m}$ admits the decomposition (2.4) and the $\text{Ad}(T)$-submodules are mutually inequivalent, the space of $K$-invariant Riemannian metrics on $K/T$ is given by

$$
\left\{ g = \langle \cdot , \cdot \rangle = \sum_{\alpha \in R^+} x_\alpha \cdot Q|_{\mathfrak{m}_\alpha} \mid x_\alpha \in \mathbb{R}^+ \right\}.
$$

The $K$-invariant Kähler-Einstein metric on $K_C/B = K/T$ corresponding to $2\delta = 2\sum_{i=1}^\ell \Lambda_i$ is given by

$$
g_{2\delta} = \sum_{\alpha \in R^+} 2(\Lambda_1 + \cdots + \Lambda_\ell, \alpha) \cdot Q|_{\mathfrak{m}_\alpha}.
$$

Similarly, the Ricci tensor $\text{Ric}_g$ of a $K$-invariant metric $g$ on $K/T$, as a $K$-invariant covariant 2-tensor, will be described by an $\text{Ad}(T)$-invariant symmetric bilinear form on $\mathfrak{m}$ given by

$$
\text{Ric}_g = \sum_{\alpha \in R^+} r_\alpha x_\alpha \cdot Q|_{\mathfrak{m}_\alpha},
$$

where $r_\alpha (\alpha \in R^+)$ are the components of the Ricci tensor on each submodule $\mathfrak{m}_\alpha$. Since $\mathfrak{m}_\alpha \not\subset \mathfrak{m}_\beta$ for any $\alpha, \beta, \in R^+$, it is $\text{Ric}_g(\mathfrak{m}_\alpha, \mathfrak{m}_\beta) = 0$ (cf. [20]).

There is a useful description of the components $r_\alpha$ associated to the isotropy decomposition (2.4). Let $K/L$ be a compact homogeneous space of a compact semisimple Lie group $K$ whose isotropy representation $\mathfrak{m}$ decomposes into $s$ pairwise inequivalent irreducible $\text{Ad}(L)$-submodules $\mathfrak{m}_i$, as $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$. Following [20] and [15] we choose a $Q$-orthonormal basis $\{ e_p \}$ adapted to $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$. Let $A^r_{pq} = Q([e_p, e_q], e_r)$ so that $[e_p, e_q]_\mathfrak{m} = \sum_{\gamma} A^r_{pq} e_r$, and set

$$
\begin{bmatrix} k \\ i \\ j \end{bmatrix} = \sum (A^r_{pq})^2 = \sum (Q([e_p, e_q], e_r))^2,
$$

where the sum is taken over all indices $p, q, r$ with $e_p \in \mathfrak{m}_i$, $e_q \in \mathfrak{m}_j$, and $e_r \in \mathfrak{m}_k$. The triples $\begin{bmatrix} k \\ i \\ j \end{bmatrix}$ are called the structure constants of $K/L$ with respect to the decomposition $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$ and are symmetric to all three indices.

For the case of a full flag manifold $K/T$ we study its structure constants with respect to the $Q$-orthonormal decomposition $\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha$, where $Q = -B(\cdot, \cdot)$ and $\mathfrak{m}_\alpha = \mathbb{R} A_\alpha + \mathbb{R} B_\alpha$. Note that we can rewrite the previous splitting as $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$, where $|R^+|$. Since $B(A_\alpha, E_{-\beta}) = -\delta_{\alpha,\beta}$ one can verify that the vectors $A_\alpha$ and $B_\alpha$ are such that $B(A_\alpha, A_\beta) = B(B_\alpha, B_\beta) = -2\delta_{\alpha,\beta}$ and $B(A_\alpha, B_\beta) = 0$. Therefore, the set $\{ X_\alpha = A_\alpha/\sqrt{2}, Y_\alpha = B_\alpha/\sqrt{2} \mid \alpha \in R^+ \}$ is a $Q$-orthonormal basis of each $\mathfrak{m}_\alpha$.

We use the notation $\begin{bmatrix} \gamma \\ \alpha \beta \end{bmatrix}$ for $\alpha, \beta, \gamma \in R$ instead of $\begin{bmatrix} k \\ i \\ j \end{bmatrix}$ for submodules $\mathfrak{m}_\alpha = \mathfrak{m}_{-\alpha}, \mathfrak{m}_\beta = \mathfrak{m}_{-\beta}$, and $\mathfrak{m}_\gamma = \mathfrak{m}_{-\gamma}$. Recall that if $\alpha, \beta, \alpha + \beta \in R$, then $[e^\alpha, e^\alpha + \beta] = e^\alpha + \beta$ and $B(e^\alpha, e^\beta) = 0$ (cf. [11] p. 168]). Since $\begin{bmatrix} \gamma \\ \alpha \beta \end{bmatrix} \neq 0$ if and only if $Q([\mathfrak{m}_\alpha, \mathfrak{m}_\beta], \mathfrak{m}_\gamma) \neq 0$, we can easily conclude that $\begin{bmatrix} \gamma \\ \alpha \beta \end{bmatrix} \neq 0$ if and only if the roots $\alpha, \beta, \gamma$ satisfy one of
the relations \( \alpha + \beta - \gamma = 0, \alpha - \beta + \gamma = 0, -\alpha + \beta + \gamma = 0 \). Note that if \( [\gamma_{\alpha}] \neq 0 \) then we can rewrite \( [\gamma_{\alpha}] \) as \( [\gamma_{\alpha'}] \) with \( \alpha', \beta', \gamma' \in R^+ \) by rearranging roots and changing the sign of roots.

By using the above notation, it can be shown (16 p. 75) that for each \( \alpha \in R^+ \), the Ricci component \( r_\alpha \) corresponding to the isotropy summand \( m_\alpha \) is given by

\[
(2.8) \quad r_\alpha = \frac{1}{2x_\alpha} + \frac{1}{8} \sum_{\beta, \gamma \in R^+} \frac{x_\alpha}{x_\beta x_\gamma} [\beta\gamma] - \frac{1}{4} \sum_{\beta, \gamma \in R^+} \frac{x_\gamma}{x_\alpha x_\beta} [\alpha\beta].
\]

Hence, a \( K \)-invariant metric \( (2.5) \) on \( K/T \) is an Einstein metric with Einstein constant \( k \) if and only if it is a positive real solution of the system \( \{r_\alpha = k | \alpha \in R^+\} \).

**Proposition 2.3.** For a full flag manifold \( K/T \) the triples \( [\alpha + \beta] \) are given by

\[
(2.9) \quad [\alpha + \beta] = 2N_{\alpha, \beta}^2.
\]

**Proof.** By definition \( (2.7) \) we see that

\[
\begin{align*}
[\alpha + \beta] = & \quad (B([X_\alpha, X_\beta], X_{\alpha + \beta}))^2 + (B([X_\alpha, X_\beta], Y_{\alpha + \beta}))^2 \\
& + (B([Y_\alpha, X_\beta], X_{\alpha + \beta}))^2 + (B([Y_\alpha, X_\beta], Y_{\alpha + \beta}))^2 \\
& + (B([X_\alpha, Y_\beta], X_{\alpha + \beta}))^2 + (B([X_\alpha, Y_\beta], Y_{\alpha + \beta}))^2 \\
& + (B([Y_\alpha, Y_\beta], X_{\alpha + \beta}))^2 + (B([Y_\alpha, Y_\beta], Y_{\alpha + \beta}))^2.
\end{align*}
\]

Since \( B(B_{\alpha + \beta}, A_{\alpha + \beta}) = 0 \) and \( B(A_{\alpha + \beta}, A_{\alpha + \beta}) = B(B_{\alpha + \beta}, B_{\alpha + \beta}) = -2 \), a straightforward computation using the properties of the root vectors and the numbers \( N_{\alpha, \beta} \) gives that

\[
\begin{align*}
B([X_\alpha, X_\beta], X_{\alpha + \beta}) = & \quad 1/(2\sqrt{2})B(N_{\alpha, \beta}A_{\alpha + \beta} + N_{\alpha, -\beta}A_{\alpha - \beta}, A_{\alpha + \beta}) = -N_{\alpha, \beta}/\sqrt{2}, \\
B([Y_\alpha, X_\beta], Y_{\alpha + \beta}) = & \quad B([X_\alpha, Y_\beta], Y_{\alpha + \beta}) = -B([Y_\alpha, Y_\beta], X_{\alpha + \beta}) = -N_{\alpha, \beta}/\sqrt{2}, \\
B([X_\alpha, X_\beta], Y_{\alpha + \beta}) = & \quad B([Y_\alpha, X_\beta], X_{\alpha + \beta}) \\
& = B([X_\alpha, Y_\beta], X_{\alpha + \beta}) = B([Y_\alpha, Y_\beta], Y_{\alpha + \beta}) = 0,
\end{align*}
\]

and the result follows.

**Remark 2.4.** Two roots \( \alpha, \beta \in R \) have the same length with respect to the Killing form \( B \) if and only if there is an element \( w \) of the Weyl group \( W(R) \) of the root system \( R \) such that \( \beta = w(\alpha) \) (see for example 18 p. 242). Thus, because of the invariance of the Killing form under \( W(R) \), it is obvious that for any element \( w \in W(R) \) we have that \( [\gamma_{w(\alpha)}] = [\gamma_{w(\beta)}] \).

### 3. The Full Flag Manifold \( G_2/T \)

We now study the geometry of the full flag manifold \( G_2/T \), where \( T \) is a maximal torus of \( G_2 \). We start by describing its isotropy representation.

**3.1. The decomposition of the isotropy representation of \( G_2/T \).** The root system of the exceptional complex simple Lie algebra \( g_2 \) can be chosen to be \( R = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (\alpha_1 + 2\alpha_2), \pm (\alpha_1 + 3\alpha_2), \pm (2\alpha_1 + 3\alpha_2) \} \). We fix a system of simple roots to be \( \Phi = \{ \alpha_1, \alpha_2 \} \). With respect to \( \Phi \) the positive roots are given by

\[
(3.1) \quad R^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2 \}.
\]
The highest root is \( \tilde{\alpha} = 2\alpha_1 + 3\alpha_2 \) (see Figure 2). Also, it is \( ||\alpha_1|| = \sqrt{3}||\alpha_2|| \), and the roots of \( g_2 \) make successive angles of \( \pi/6 \). The Weyl group is generated by rotations of \( \mathbb{R}^2 \) about the origin through an angle \( \pi/6 \) and reflections about the vertical axis.

![Figure 2. The root system of \( g_2 \)](https://example.com/root_system.png)

The full flag manifold \( G_2/T \) is obtained by painting black both simple roots in the Dynkin diagram of \( g_2 \). Proposition 2.2 implies that the isotropy representation \( m \) of \( G_2/T \) decomposes into six inequivalent irreducible ad(\( k \))-submodules, i.e.
\[
\begin{align*}
m_1 &= m_{\alpha_1}, \\
m_2 &= m_{\alpha_2}, \\
m_3 &= m_{\alpha_1 + \alpha_2}, \\
m_4 &= m_{\alpha_1 + 2\alpha_2}, \\
m_5 &= m_{\alpha_1 + 3\alpha_2}, \\
m_6 &= m_{2\alpha_1 + 3\alpha_2}.
\end{align*}
\]

### 3.2. Kähler-Einstein metrics

It is well known ([7, p. 504]) that a full flag manifold \( K/T \) admits \( |W(K)|/2 \) invariant complex structures (here \( W(K) \) is the Weyl group of \( K \)), which are all equivalent under an automorphism of \( K \). We now compute the unique Kähler–Einstein metric which is compatible with the natural invariant ordering \( J_{\text{nat}} \), that is, the complex structure corresponding to the natural invariant ordering \( R^+ \) given by (3.1). From (2.5) a \( G_2 \)-invariant Riemannian metric on \( G_2/T \) is given by
\[
g = x_1 \cdot Q|_{m_1} + \cdots + x_6 \cdot Q|_{m_6},
\]
where we have set \( x_1 = x_{\alpha_1}, x_2 = x_{\alpha_2}, x_3 = x_{\alpha_1 + \alpha_2}, x_4 = x_{\alpha_1 + 2\alpha_2}, x_5 = x_{\alpha_1 + 3\alpha_2}, x_6 = x_{2\alpha_1 + 3\alpha_2} \), and the \( m_k \) \((k = 1, \ldots, 6)\) are given by (3.2). We will denote such metrics by \( g = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+ \).

**Theorem 3.1.** The full flag manifold \( G_2/T \) admits six invariant Kähler–Einstein metrics which are isometric to each other. The Kähler–Einstein metric \( g_{2\delta} \) which is compatible with the natural invariant ordering \( J_{\text{nat}} \) is given (up to a scale) by \( g_{2\delta} = (3, 1, 4, 5, 6, 9) \).

**Proof.** According to the notation of Section 2.1, the weight \( \delta \) for \( G_2/T \) is given by \( \delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \sum_{i=1}^{2} \Lambda_i \), where \( \Lambda_1 \) and \( \Lambda_2 \) are the fundamental weights corresponding to the simple roots \( \alpha_1 \) and \( \alpha_2 \), respectively. In Figure 2 one can easily distinguish the long roots
\[
L_1 = \alpha_1, \quad L_2 = \alpha_1 + 3\alpha_2, \quad L_3 = 2\alpha_1 + 3\alpha_2
\]
from the short roots
\[ S_1 = \alpha_2, \quad S_2 = \alpha_1 + \alpha_2, \quad S_3 = \alpha_1 + 2\alpha_2. \]

Then \( \| L_i \| = \sqrt{3}\| S_j \| \), where \( 1 \leq i, j \leq 3 \) and \( i, j \) are independent. We set \( (L_i, L_j) = 3 \) and \( (S_i, S_j) = 1 \), for any \( 1 \leq i \leq 3 \). We denote the Kähler-Einstein metric \( g_{25} \) by

\[ g_{\alpha_1} \cdot Q|_{m_1} + g_{\alpha_2} \cdot Q|_{m_2} + g_{\alpha_1 + \alpha_2} \cdot Q|_{m_3} + g_{\alpha_1 + 2\alpha_2} \cdot Q|_{m_4} + g_{\alpha_1 + 3\alpha_2} \cdot Q|_{m_5} + g_{2\alpha_1 + 3\alpha_2} \cdot Q|_{m_6}, \]

which is compatible with the natural invariant complex structure \( J_{\text{nat}} \) defined by the ordering \( R^+ \). By using (2.1) and applying relation (2.6) we obtain the following values for the components \( g_{\alpha} = (2\delta, \alpha) \), \( \alpha \in R^+ \) of this metric:

\[ g_{\alpha_1} = 2(\Lambda_1, \alpha_1) = (\alpha_1, \alpha_1) = 3, \quad g_{\alpha_2} = 2(\Lambda_2, \alpha_2) = (\alpha_2, \alpha_2) = 1, \]
\[ g_{\alpha_1 + \alpha_2} = 2(\Lambda_1, \alpha_1 + \alpha_2) = 4, \quad g_{\alpha_1 + 2\alpha_2} = 2(\Lambda_1, \alpha_1) + 4(\Lambda_2, \alpha_2) = 5, \]
\[ g_{\alpha_1 + 3\alpha_2} = 2(\Lambda_1 + 6(\Lambda_2, \alpha_2) = 6, \quad g_{2\alpha_1 + 3\alpha_2} = 4(\Lambda_1, \alpha_1) + 6(\Lambda_2, \alpha_2) = 9. \]

\[ \square \]

### 3.3. Homogeneous Einstein metrics.

We now proceed to the calculation of the Ricci tensor \( \text{Ric}_g \) corresponding to a \( G_2 \)-invariant metric \( \langle \mathcal{G} \rangle \) on \( G_2/T \). Following the notation of Section 2.2, the tensor \( \text{Ric}_g \) as a \( G_2 \)-invariant symmetric covariant 2-tensor on \( G_2/T \) is given by \( \text{Ric}_g = r_1 x_1 \cdot Q|_{m_1} + \cdots + r_6 x_6 \cdot Q|_{m_6} \), where for simplicity we have set \( r_1 = r_{\alpha_1}, r_2 = r_{\alpha_2}, r_3 = r_{\alpha_1 + \alpha_2}, r_4 = r_{\alpha_1 + 2\alpha_2}, r_5 = r_{\alpha_1 + 3\alpha_2} \) and \( r_6 = r_{2\alpha_1 + 3\alpha_2} \). In order to apply (2.8) we first need to find the nonzero structure constants \( \langle k \rangle \) of \( G_2/T \). By using (3.1) and (3.2) it follows that these are

\[ c^3_{12} = \left[ \begin{array}{c} \alpha_1 + \alpha_2 \\ \alpha_1 \alpha_2 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 12 \end{array} \right], \quad c^5_{24} = \left[ \begin{array}{c} \alpha_1 + 2\alpha_2 \\ \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} 4 \\ 23 \end{array} \right], \quad c^6_{24} = \left[ \begin{array}{c} \alpha_1 + 3\alpha_2 \\ \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} 5 \\ 24 \end{array} \right], \]

\[ c^6_{15} = \left[ \begin{array}{c} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 \alpha_1 + 3\alpha_2 \end{array} \right] = \left[ \begin{array}{c} 6 \\ 15 \end{array} \right], \quad c^6_{34} = \left[ \begin{array}{c} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} 6 \\ 34 \end{array} \right]. \]

By Remark 2.3 and the remarks on the notation for \( \langle \gamma \rangle \), we obtain the following:

**Lemma 3.2.** The triples \( c^3_{12}, c^5_{24} \) and \( c^6_{34} \) are equal.

**Proof.** The Weyl group \( W(R) \) is generated by the reflections \( \{ s_{\alpha_1}, s_{\alpha_2} \} \), and we have that \( s_{\alpha_1}(\alpha_1) = -\alpha_1, s_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2, s_{\alpha_2}(\alpha_2) = -\alpha_2, s_{\alpha_2}(\alpha_1) = \alpha_1 + 3\alpha_2 \). Now we see that \( s_{\alpha_2}(\alpha_1 + \alpha_2) = \alpha_1 + 2\alpha_2 \) and hence we have that

\[ c^5_{24} = \left[ \begin{array}{c} \alpha_1 + 3\alpha_2 \\ \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} s_{\alpha_2}(\alpha_1) \\ -s_{\alpha_2}(\alpha_2) \end{array} \right] = \left[ \begin{array}{c} \alpha_1 \\ -\alpha_2 \alpha_2 \alpha_1 + \alpha_2 \end{array} \right] = \left[ \begin{array}{c} \alpha_1 \\ -\alpha_2 \alpha_1 + \alpha_2 \end{array} \right] = \left[ \begin{array}{c} \alpha_1 \\ \alpha_1 \alpha_2 \end{array} \right] = \left[ \begin{array}{c} 5 \\ 24 \end{array} \right]. \]

We also see that \( s_{\alpha_1}(\alpha_1 + 3\alpha_2) = 2\alpha_1 + 3\alpha_2 \) and \( s_{\alpha_1}(\alpha_1 + 2\alpha_2) = \alpha_1 + 2\alpha_2 \), and hence we have that

\[ c^6_{34} = \left[ \begin{array}{c} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} s_{\alpha_1}(\alpha_1 + 3\alpha_2) \\ s_{\alpha_1}(\alpha_1 + 2\alpha_2) \end{array} \right] = \left[ \begin{array}{c} \alpha_1 + 3\alpha_2 \\ \alpha_2 \alpha_1 + 2\alpha_2 \end{array} \right] = \left[ \begin{array}{c} 5 \\ 24 \end{array} \right]. \]
For the calculation of the above triples we use Proposition 2.3 and the fact that
\begin{equation}
N^2_{\alpha,\beta} = N_{\alpha,\beta}N_{-\alpha,-\beta} = \frac{q(p+1)}{2}Q(\alpha,\alpha),
\end{equation}
where $p,q$ are the largest nonnegative integers such that $\beta + k\alpha \in R$, with $-p \leq k \leq q$.

We first proceed to the calculation of $c^3_{12} = [\alpha_1 + \alpha_2]$. By using the relation $(\alpha_1,\alpha_1) = 3(\alpha_2,\alpha_2)$ and equation (3.4) we obtain that $N^2_{\alpha_1,\alpha_2} = 3Q(\alpha_2,\alpha_2)/2$, so Proposition 2.3 implies that $c^3_{12} = 3Q(\alpha_2,\alpha_2)$. The normalizing value $Q(\alpha_2,\alpha_2)$ is given by $Q(\alpha_2,\alpha_2) = 1/12$ (cf. [8]); thus $c^3_{12} = 1/4$. Similarly, we obtain that
\begin{equation}
c^4_{23} = \left[\frac{\alpha_1 + 2\alpha_2}{\alpha_2 + \alpha_2}\right] = 2N^2_{\alpha_2,\alpha_1 + \alpha_2} = \frac{1}{3},
\end{equation}
\begin{equation}
c^6_{15} = \left[\frac{2\alpha_1 + 3\alpha_2}{\alpha_1 + \alpha_2}\right] = 2N^2_{\alpha_1,\alpha_1 + 3\alpha_2} = \frac{1}{4}.
\end{equation}

The above computations combined with Lemma 3.2 give the following:

**Proposition 3.3.** The nonzero triples $[k_{ij}]$ of the full flag manifold $G_2/T$ are given by
\[
\begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix} = \begin{bmatrix} 6 \\ 34 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix} = \frac{1}{4} \quad \text{and} \quad \begin{bmatrix} 4 \\ 23 \end{bmatrix} = \frac{1}{3}.
\]

Therefore, we obtain the following proposition for the Ricci tensor from (2.8):

**Proposition 3.4.** The components $r_i$ ($i = 1, \ldots, 6$) of the Ricci tensor associated to the $G$-invariant Riemannian metric $g$ given by (3.3) are the following:
\[
\begin{align*}
    r_1 &= \frac{1}{2x_1} + \frac{1}{16} \left( \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right) + \frac{1}{16} \left( \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5} \right), \\
    r_2 &= \frac{1}{2x_2} + \frac{1}{16} \left( \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right) + \frac{1}{12} \left( \frac{x_2}{x_3x_4} - \frac{x_3}{x_2x_4} - \frac{x_4}{x_2x_3} \right) \\
        &\quad + \frac{1}{16} \left( \frac{x_2}{x_4x_5} - \frac{x_4}{x_2x_5} - \frac{x_5}{x_2x_4} \right), \\
    r_3 &= \frac{1}{2x_3} + \frac{1}{16} \left( \frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} \right) + \frac{1}{12} \left( \frac{x_3}{x_2x_4} - \frac{x_2}{x_3x_4} - \frac{x_4}{x_2x_3} \right) \\
        &\quad + \frac{1}{16} \left( \frac{x_3}{x_4x_6} - \frac{x_4}{x_3x_6} - \frac{x_6}{x_3x_4} \right), \\
    r_4 &= \frac{1}{2x_4} + \frac{1}{12} \left( \frac{x_4}{x_2x_3} - \frac{x_2}{x_3x_4} - \frac{x_3}{x_2x_4} \right) + \frac{1}{16} \left( \frac{x_4}{x_2x_5} - \frac{x_2}{x_4x_5} - \frac{x_5}{x_2x_4} \right) \\
        &\quad + \frac{1}{16} \left( \frac{x_4}{x_3x_6} - \frac{x_3}{x_4x_6} - \frac{x_6}{x_3x_4} \right), \\
    r_5 &= \frac{1}{2x_5} + \frac{1}{16} \left( \frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5} \right) + \frac{1}{16} \left( \frac{x_5}{x_2x_4} - \frac{x_2}{x_4x_5} - \frac{x_4}{x_2x_5} \right), \\
    r_6 &= \frac{1}{2x_6} + \frac{1}{16} \left( \frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} \right) + \frac{1}{16} \left( \frac{x_6}{x_3x_4} - \frac{x_3}{x_4x_6} - \frac{x_4}{x_3x_6} \right).
\end{align*}
\]

A $G_2$-invariant Riemannian metric on the full flag manifold $G_2/T$ is Einstein if and only if there is a positive constant $k$ such that
\begin{equation}
r_1 = k, \quad r_2 = k, \quad r_3 = k, \quad r_4 = k, \quad r_5 = k, \quad r_6 = k,
\end{equation}
where $r_i$ ($i = 1, \ldots, 6$) are given in Proposition 3.4.
4. Proof of Theorem A

Note that the action of the Weyl group of \( g_2 \) on its root system (cf. Figure 2) induces an action on the components of the \( G_2 \)-invariant metric \( (3.3) \). In particular, if \( (x_1, x_2, x_3, x_4, x_5, x_6) = (a_1, a_2, a_3, a_4, a_5, x_6) \) is a solution for the system of equations \((3.5)\), then

\[
(x_1, x_2, x_3, x_4, x_5, x_6) = (a_5, a_2, a_4, a_3, a_1, a_6)
\]

is also a solution of the system \((3.5)\). In fact, if \( w \) is a reflection about \( 2\alpha_1 + 3\alpha_2 \) in the root diagram of \( g_2 \), then \( w(\alpha_1) = \alpha_1 + 3\alpha_2, w(\alpha_1 + \alpha_2) = \alpha_1 + 2\alpha_2 \). This induces an interchange of \( x_1 \) with \( x_5 \) and \( x_3 \) with \( x_4 \), and keeps \( x_2, x_6 \) fixed. Similarly we see that

\[
(x_1, x_2, x_3, x_4, x_5, x_6) = (a_6, a_3, a_4, a_2, a_1, a_5),
\]

\[
(x_1, x_2, x_3, x_4, x_5, x_6) = (a_1, a_3, a_2, a_4, a_6, a_5),
\]

\[
(x_1, x_2, x_3, x_4, x_5, x_6) = (a_5, a_4, a_2, a_3, a_6, a_1),
\]

\[
(x_1, x_2, x_3, x_4, x_5, x_6) = (a_6, a_4, a_3, a_2, a_5, a_1)
\]

are also solutions of system \((3.5)\). These metrics are all isometric to each other.

The above analysis using the Weyl group suggests splitting the study of solutions for the system \((3.5)\) into two cases: Case A \( (x_1 - x_5)(x_1 - x_6)(x_5 - x_6) = 0 \) and Case B \( (x_1 - x_5)(x_1 - x_6)(x_5 - x_6) \neq 0 \).

Note that the system of equations \((3.5)\) is equivalent to the equations

\[
\begin{align*}
    f_1 &= -3x_2^2x_3x_6 - 6x_2^2x_4x_5x_6 - 4x_2^2x_5x_6 - 3x_2x_3x_4x_5^2 \\
        &+ 24x_2x_3x_4x_5x_6 - 3x_2x_3x_4x_6^2 + 3x_2x_3x_4 + 4x_3^2x_5x_6 + 3x_3x_4^2x_6 \\
    f_2 &= 3x_2^2x_3x_6 + 6x_2^2x_4x_5x_6 + 8x_2^2x_5x_6 - 3x_2x_3x_5^2 + 3x_2x_4^2x_5 \\
        &- 24x_2x_4x_5x_6 + 3x_2x_5^2x_6 - 6x_2^2x_5x_6 - 8x_3^2x_5x_6 - 3x_3x_4^2x_6 \\
        &+ 24x_3x_4x_5x_6 - 3x_3x_5^2x_6 = 0,
    f_3 &= 3x_2^2x_3x_6 - 3x_2^2x_4x_5x_6 + 6x_2^2x_3x_5x_6 - 24x_2x_3x_5x_6 - 6x_2x_4^2x_5 \\
        &+ 24x_2x_4x_5x_6 + 3x_2^2x_4x_5x_6 + 8x_3^2x_5x_6 - 3x_3x_4^2x_6 + 3x_3x_5^2x_6 \\
        &- 8x_4^2x_5x_6 - 3x_4x_5x_6 = 0,
    f_4 &= -4x_2^2x_5x_6 - 3x_2x_3^2x_5 - 3x_2x_3x_4x_5^2 + 3x_2x_3x_4^2x_6 \\
        &- 24x_2x_3x_4x_6 + 3x_2x_3x_4 + 24x_2x_3x_5x_6 + 3x_2x_4^2x_5 - 3x_2x_5x_6^2 \\
        &- 4x_3^2x_5x_6 + 6x_3x_4^2x_6 - 6x_3x_5^2x_6 + 4x_4^2x_5x_6 = 0,
    f_5 &= -2x_2^2x_3x_6 + x_2x_3x_5 + 2x_2x_3x_4x_5^2 - 8x_2x_3x_5x_6 - 2x_2x_3x_4x_6^2 \\
        &+ 8x_2x_3x_4x_6 + x_2x_4^2x_5 - x_2x_5x_6^2 - 3x_3x_4^2x_6 + x_3x_5^2x_6 = 0
\end{align*}
\]

for solutions with \( x_2x_3x_4x_5x_6 \neq 0 \).

To find nonzero solutions of equations \((4.2)\), we compute a Gröbner basis (see for example \(9\)) by using algebraic manipulations in a computer system.

Case A. We may assume that \( x_1 = x_5 = 1 \). If \( x_6 = 1 \), we consider a polynomial ring \( R_1 = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6] \) and an ideal \( I_1 \) generated by

\[
\{f_1, f_2, f_3, f_4, f_5, x_5 - 1, x_6 - 1, yx_2x_3x_4x_5x_6 - 1\}.
\]
We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on $R_1$. Then a Gröbner basis for the ideal $I_1$ is given by
\[ \{ x_6 - 1, x_5 - 1, 15x_4^2 - 20x_4 + 9, x_3 - x_4, x_2 - x_3, -2600 + 3975x_4 + 729y \}. \]

Now the equation $15x_4^2 - 20x_4 + 9 = 0$ has no real solutions. Thus there are no Einstein metrics for this case.

If $x_6 \neq 1$, we consider an ideal $I_2$ generated by
\[ \{ f_1, f_2, f_3, f_4, f_5, x_5 - 1, (x_6 - 1)y x_2 x_3 x_4 x_5 x_6 - 1 \}. \]

We take a lexicographic order $>$ with $y > x_6 > x_5 > x_2 > x_3 > x_4$ for a monomial ordering on $R_1$. Then we see that $x_3 - x_4$ is an element of a Gröbner basis for the ideal $I_2$. Thus we obtain the following expression for the Ricci components in this case:
\[
\begin{align*}
    r_1 &= r_5 = \frac{1}{2} + \frac{1}{16} \left( \frac{x_2 x_3 - x_2 - x_3}{x_2 x_3} \right) - \frac{x_6}{16}, \\
    r_2 &= \frac{1}{2x_2} + \frac{1}{12} \left( \frac{x_2 x_3^2 - 2}{x_2^2} \right) + \frac{1}{8} \left( \frac{x_2 - x_3 - x_2}{x_2 x_3} \right), \\
    r_3 &= r_4 = \frac{1}{2x_3} + \frac{1}{16} \left( \frac{x_2 x_3 - x_2 - x_3}{x_2 x_3} \right) - \frac{x_2}{12x_3^2} - \frac{x_6}{16x_3^2}, \\
    r_6 &= \frac{1}{2x_6} + \frac{1}{16} \left( x_6 - \frac{2}{x_6} \right) + \frac{1}{16} \left( \frac{x_6 - 2}{x_6} \right). 
\end{align*}
\]

Now the system of equations (4.3) is equivalent to the equations
\[
(4.3) \quad r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_6. 
\]

Moreover, we see that the system of equations (4.3) is equivalent to the equations
\[
(4.4) \quad \begin{cases} 
    h_1 = -9x_2^2 x_3 - 4x_4^2 - 3x_2 x_3^2 x_6 + 24x_2 x_3^2 + 3x_3^3 - 16x_3^2 + 9x_3 = 0, \\
    h_2 = 9x_2^2 x_3 + 8x_2^2 - 24x_2 x_3 + 3x_2 x_6 - 9x_3^3 + 16x_3^2 - 3x_3 = 0, \\
    h_3 = -3x_2^2 x_3 x_6 - 4x_2^2 x_6 - 3x_2 x_3 x_6^2 - 12x_2 x_3^2 + 24x_2 x_3 x_6, \\
    &-6x_2 x_6^2 + 3x_3^3 x_6 - 3x_3 x_6 = 0, 
\end{cases}
\]

for solutions with $x_2 x_3 x_6 \neq 0$.

We consider a polynomial ring $R_2 = \mathbb{Q}[y, x_2, x_3, x_6]$ and an ideal $I_3$ generated by
\[ \{ h_1, h_2, h_3, y x_2 x_3 x_6 - 1 \}. \]

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_6$ for a monomial ordering on $R_2$. Then a Gröbner basis for the ideal $I_3$ contains the following polynomials $p_1, p_2, p_3$:
\[
\begin{align*}
p_1 &= 28431 x_6^{14} - 589032 x_6^{13} + 5435343 x_6^{12} - 29379024 x_6^{11} + 100757208 x_6^{10} \\
&\quad - 224163176 x_6^9 + 336260186 x_6^8 - 371473808 x_6^7 + 339968604 x_6^6 - 262478048 x_6^5 \\
&\quad + 152856152 x_6^4 - 69550016 x_6^3 + 35706576 x_6^2 - 17407872 x_6 + 3888000, 
\end{align*}
\]
By solving the equation \( p_1 = 0 \) numerically, we obtain exactly two real solutions which are approximately given by \( x_6 \approx 0.74403477990 \) and \( x_6 \approx 1.78960022 \). We substitute these values for \( x_6 \) into equations \( p_2 = 0 \) and \( p_3 = 0 \), and we obtain two positive solutions approximately given by \( x_2 \approx 0.21737 \), \( x_3 \approx 1.02343 \) and \( x_2 \approx 0.27624 \), \( x_3 \approx 1.03473 \). Moreover, we obtain the value for \( k \) from (3.5). Thus we have the following:
Theorem 4.1. The full flag manifold $G_2/T$ admits two non-Kähler $G_2$-invariant Einstein metrics. These metrics are given approximately as follows:

$$x_1 = 1, \quad x_2 \approx 0.2762, \quad x_3 = x_4 \approx 1.0347, \quad x_5 = 1, \quad x_6 \approx 1.7896, \quad k \approx 0.3560,$$

$$x_1 = 1, \quad x_2 \approx 0.2173, \quad x_3 = x_4 \approx 1.0234, \quad x_5 = 1, \quad x_6 \approx 0.7440, \quad k \approx 0.4269.$$

Case B. We consider a polynomial ring $R_1 = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6]$ and an ideal $I_4$ generated by

$$\{f_1, f_2, f_3, f_4, f_5, (x_5 - 1)(x_6 - 1) y x_2 x_3 x_4 x_5 x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on $R_1$. Then a Gröbner basis for the ideal $I_4$ contains a polynomial of the form

$$(x_6 - 3)(x_6 - 2)(2x_6 - 3)(2x_6 - 1)(3x_6 - 2)(3x_6 - 1)q_1(x_6),$$

where $q_1(x_6)$ is an explicitly given polynomial of degree 84 with integer coefficients.

For the case when $(x_6 - 3)(x_6 - 2)(2x_6 - 3)(2x_6 - 1)(3x_6 - 2)(3x_6 - 1) = 0$, we first consider an ideal $I_5$ generated by

$$\{f_1, f_2, f_3, f_4, f_5, (x_6 - 3) y x_2 x_3 x_4 x_5 x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on $R_1$. Then a Gröbner basis for the ideal $I_5$ is given by

$$\{x_6 - 3, x_5 - 2, 3x_4 - 5, 3x_3 - 4, 3x_2 - 1, 40y - 9\}.$$

We also compute the Gröbner basis for other cases and we obtain the following sets of solutions for equations \((4.2)\):

$$(x_6 = 3, x_5 = 2, x_4 = \frac{5}{3}, x_3 = \frac{4}{3}, x_2 = \frac{1}{3}), \quad (x_6 = 2, x_5 = 3, x_4 = \frac{5}{3}, x_3 = \frac{1}{3}, x_2 = \frac{4}{3}),$$

$$(x_6 = \frac{2}{3}, x_5 = \frac{1}{3}, x_4 = \frac{4}{3}, x_3 = \frac{5}{3}, x_2 = \frac{1}{3}), \quad (x_6 = \frac{3}{2}, x_5 = \frac{3}{2}, x_4 = \frac{1}{3}, x_3 = \frac{4}{3}, x_2 = \frac{5}{3}),$$

$$(x_6 = \frac{2}{3}, x_5 = \frac{1}{3}, x_4 = \frac{1}{3}, x_3 = \frac{5}{3}, x_2 = \frac{4}{3}), \quad (x_6 = \frac{1}{3}, x_5 = \frac{2}{3}, x_4 = \frac{1}{3}, x_3 = \frac{4}{3}, x_2 = \frac{5}{3}).$$

Note that these are the six Kähler-Einstein metrics of Theorem 3.1.

Now, by solving the equation $q_1(x_6) = 0$ numerically, we obtain 14 positive solutions, which are approximately given by

$$x_6 \approx 0.1101296649906623, \quad x_6 \approx 0.1276467609933986, \quad x_6 \approx 0.1654266507070432,$$

$$x_6 \approx 0.2010643285289733, \quad x_6 \approx 0.306532888396123, \quad x_6 \approx 0.5181203151843693, $$

$$x_6 \approx 0.5477334830916693, \quad x_6 \approx 1.825705440455314, \quad x_6 \approx 1.930053639460474,$$

$$x_6 \approx 3.262293320377869, \quad x_6 \approx 4.973532636625297, \quad x_6 \approx 6.044975194298747,$$

$$x_6 \approx 7.834119661302769, \quad x_6 \approx 9.080205592968872.$$

To get the solutions of equations \((4.2)\) for the variables $x_2, x_3, x_4, x_5$ corresponding to the solution $x_6$, we compute a Gröbner basis under the condition $(x_6 - 3)(x_6 - 2)(2x_6 - 3)(2x_6 - 1)(3x_6 - 2)(3x_6 - 1) \neq 0$; that is, we consider an ideal $I_6$ generated by \(\{f_1, f_2, f_3, f_4, f_5, (x_6 - 3)(x_6 - 2)(2x_6 - 3)(2x_6 - 1)(3x_6 - 2)(3x_6 - 1) y x_2 x_3 x_4 x_5 x_6 - 1\}$ and take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on $R_1$. Then $q_1$ is contained in this Gröbner basis, and by examining the other elements of the obtained Gröbner basis, we see that the other variables $x_2, x_3, x_4, x_5$ can be expressed by polynomials of $x_6$ with degree 83. Let $x_2 = q_2(x_6)$, $x_3 = q_3(x_6)$, $x_4 = q_4(x_6)$, and $x_5 = q_5(x_6)$ be these polynomials. Now we substitute these 14 values for $x_6$ into the expressions of the
polynomials $q_2, q_3, q_4, q_5$ of $x_6$ with degree 83. Then we get the following solutions which are approximately given by

$$
x_6 \approx 0.11013, \ x_5 \approx 0.547733, \ x_4 \approx 1.61358, \ x_3 \approx 0.399131, \ x_2 \approx -0.277481,
$$

$$
x_6 \approx 0.127647, \ x_5 \approx -0.775539, \ x_4 \approx 0.202709, \ x_3 \approx 1.7601, \ x_2 \approx -0.203265,
$$

$$
x_6 \approx 0.165427, \ x_5 \approx -0.021892, \ x_4 \approx 0.308989, \ x_3 \approx 0.00455279, \ x_2 \approx 0.5435,
$$

$$
x_6 \approx 0.201064, \ x_5 \approx 1.82571, \ x_4 \approx 0.728695, \ x_3 \approx 2.94591, \ x_2 \approx -0.506599,
$$

$$
x_6 \approx 0.306533, \ x_5 \approx -1.52438, \ x_4 \approx 0.207857, \ x_3 \approx 1.64949, \ x_2 \approx 5.33389,
$$

$$
x_6 \approx 0.51812, \ x_5 \approx -0.100239, \ x_4 \approx -0.120371, \ x_3 \approx -2.58645, \ x_2 \approx -0.539579,
$$

$$
x_6 \approx 0.547733, \ x_5 \approx 0.11013, \ x_4 \approx 1.61358, \ x_3 \approx -0.277481, \ x_2 \approx 0.399131,
$$

$$
x_6 \approx 1.82571, \ x_5 \approx 0.201064, \ x_4 \approx 0.728695, \ x_3 \approx -0.506599, \ x_2 \approx 2.94591,
$$

$$
x_6 \approx 1.93005, \ x_5 \approx -0.193467, \ x_4 \approx -1.04142, \ x_3 \approx -4.99198, \ x_2 \approx -0.232323,
$$

$$
x_6 \approx 3.26229, \ x_5 \approx -4.97297, \ x_4 \approx 17.4007, \ x_3 \approx 5.38113, \ x_2 \approx 0.678092,
$$

$$
x_6 \approx 4.97353, \ x_5 \approx 9.08021, \ x_4 \approx -2.51959, \ x_3 \approx 14.6516, \ x_2 \approx 3.62419,
$$

$$
x_6 \approx 6.04498, \ x_5 \approx -0.132336, \ x_4 \approx 3.28544, \ x_3 \approx 0.0275215, \ x_2 \approx 1.86783,
$$

$$
x_6 \approx 7.83412, \ x_5 \approx -6.07566, \ x_4 \approx -1.5924, \ x_3 \approx 13.7889, \ x_2 \approx 1.58805,
$$

$$
x_6 \approx 9.08021, \ x_5 \approx 4.97353, \ x_4 \approx -2.51959, \ x_3 \approx 3.62419, \ x_2 \approx 14.6516.
$$

We see that at least one of the $x_i$’s in these solutions is negative. Thus there are no invariant Einstein metrics for these cases, and this completes the proof of Theorem A.

**References**

1. D. V. Alekseevsky and A. M. Perelomov: *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl. **20** (1986) 171–182. MR868557 (88c:53049)

2. A. Arvanitoyeorgos: *New invariant Einstein metrics on generalized flag manifolds*, Trans. Amer. Math. Soc. **337** (1993) 981–995. MR1097162 (93b:53043)

3. A. Arvanitoyeorgos and I. Chrysikos: *Invariant Einstein metrics on generalized flag manifolds with two isotropy summands*, J. Aust. Math. Soc. **90** (2011) 237–251. MR2821781

4. A. Arvanitoyeorgos and I. Chrysikos: *Invariant Einstein metrics on generalized flag manifolds with four isotropy summands*, Ann. Glob. Anal. Geom. **37** (2010) 185-219. MR2578265

5. C. Böhm and M. Kerr: *Low-dimensional homogeneous Einstein manifolds*, Trans. Amer. Math. Soc. **358** (2006) 1455–1468. MR2186982 (2006g:53056)

6. M. Borderman, M. Forger and H. Römer: *Homogeneous Kähler manifolds: paving the way towards new supersymmetric sigma models*, Comm. Math. Phys. **102** (1986) 605–617. MR824094 (87c:53096)

7. A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958) 458–538. MR0102800 (21:1586)

8. N. Bourbaki: *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968. MR0240238 (39:1590)

9. D. Cox, J. Little and D. O’Shea: *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992. MR1189133 (93j:13031)

10. E. C. F. Dos Santos and C. J. C. Negreiros: *Einstein metrics on flag manifolds*, Revista Della, União Matemática Argentina, **47** (2006) 77–84. MR2301378 (2008b:53061)

11. S. Helgason: *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978. MR0514561 (80k:53081)

12. M. Kimura: *Homogeneous Einstein metrics on certain Kähler C-spaces*, Adv. Stud. Pure Math. **18-I**, Academic Press, Boston, MA, 1990, 303–320. MR1145261 (93b:53039)

13. J. L. Koszul: *Sur la forme hermitienne canonique des espaces homogènes complexes*, Can. J. Math. **7** (1955) 562–576. MR0077879 (17:1109a)
14. Yu. G. Nikonorov, E. D. Rodionov and V. V. Slavskii: *Geometry of homogeneous Riemannian manifolds*, J. Math. Sciences 146 (2007) 6313–6390. MR2568572 (2011a:53083)
15. J-S. Park and Y. Sakane: *Invariant Einstein metrics on certain homogeneous spaces*, Tokyo J. Math. 20 (1997) 51–61. MR1451858 (98c:53058)
16. Y. Sakane: *Homogeneous Einstein metrics on flag manifolds*, Lobachevskii J. Math. 4 (1999) 71–87. MR1743146 (2000m:53069)
17. M. Takeuchi: *Homogeneous Kähler submanifolds in complex projective spaces*, Japan. J.Math. 4 (1978) 171–219. MR528871 (80i:32059)
18. P. Tauvel and R. W. T. Yu: *Lie Algebras and Algebraic Groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. MR2146652 (2006c:17001)
19. M. Wang and W. Ziller: *On normal homogeneous Einstein manifolds*, Ann. Scient. Éc. Norm. Sup. 18 (1985) 563–633. MR839687 (87k:53113)
20. M. Wang and W. Ziller: *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math. 84 (1986) 177–194. MR830044 (87e:53081)

Department of Mathematics, University of Patras, GR-26500 Rion, Greece
E-mail address: arvanito@math.upatras.gr

Department of Mathematics and Statistics, Masaryk University, Kotlarska 2, 611 37 Brno, Czech Republic
E-mail address: xrysikos@master.math.upatras.gr

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: sakane@math.sci.osaka-u.ac.jp