STRONGLY SELF-ABSORBING $C^\ast$-ALGEBRAS ARE $\mathbb{Z}$-STABLE

WILHELM WINTER

Abstract. We prove the title. This characterizes the Jiang–Su algebra $\mathbb{Z}$ as the uniquely determined initial object in the category of strongly self-absorbing $C^\ast$-algebras.

0. Introduction

A separable unital $C^\ast$-algebra $\mathcal{D} \neq \mathbb{C}$ is called strongly self-absorbing, if there is an isomorphism $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor embedding, cf. [12]. The interest in such algebras largely arises from Elliott’s program to classify nuclear $C^\ast$-algebras by $K$-theoretic invariants. In fact, examples suggest that classification will only be possible up to $\mathcal{D}$-stability (i.e., up to tensoring with $\mathcal{D}$) for a strongly self-absorbing $\mathcal{D}$, cf. [11], [4], [15]. While the known strongly self-absorbing examples are quite well understood, and are entirely classified, it remains an open problem whether these are the only ones. From a more general perspective, the question is in how far abstract properties allow for comparison with concrete examples. For nuclear $C^\ast$-algebras, this question prominently manifests itself as the UCT problem (i.e., is every nuclear $C^\ast$-algebra $KK$-equivalent to a commutative one); a positive answer even in the special setting of strongly self-absorbing $C^\ast$-algebras would be highly satisfactory, and likely shed light on the general case.

In this note we shall be concerned with a closely related interpretation of the aforementioned question: we will show that any strongly self-absorbing $C^\ast$-algebra $\mathcal{D}$ admits a unital embedding of a specific example, the Jiang–Su algebra $\mathbb{Z}$ (we refer to [8] and to [10] for an introduction and various characterizations of $\mathbb{Z}$). It then follows immediately that $\mathcal{D}$ is in fact $\mathbb{Z}$-stable. The result answers some problems left open in [12] and in [1]; in particular it implies that strongly self-absorbing $C^\ast$-algebras are always $K_1$-injective. Moreover, it shows that the Jiang–Su algebra is an initial object in the category of strongly self-absorbing $C^\ast$-algebras (with unital $*$-homomorphisms); there can only be one such initial object, whence $\mathbb{Z}$ is characterized this way. It is interesting to note that the Cuntz algebra $\mathcal{O}_2$ is the uniquely determined final object in this category, and that $\mathcal{O}_\infty$ can be characterized as the initial object in the category of infinite strongly self-absorbing $C^\ast$-algebras.

The proof of our main result builds on ideas from [10] and from [1], where the problem was settled in the case where $\mathcal{D}$ contains a nontrivial projection.

Date: May 5, 2009.
2000 Mathematics Subject Classification. 46L35, 46L85.
Key words and phrases. strongly self-absorbing $C^\ast$-algebra, Jiang–Su algebra.
Supported by: EPSRC First Grant EP/G014019/1.
1. Small elementary tensors

In this section, we generalize a technical result from [1] to a setting that does not require the existence of projections, see Lemma 1.4 below. We refer to [9] for a brief account of the Cuntz semigroup.

1.1 Proposition: Let $A$ be a unital $C^*$-algebra, $0 \leq g \leq 1_A$.

Then, for any $0 \neq n \in \mathbb{N}$, we have

$$1_A \otimes n - g \otimes n \geq (1_A - g) \otimes g \otimes \ldots \otimes g + g \otimes (1_A - g) \otimes g \otimes \ldots \otimes g \ldots + g \otimes \ldots \otimes g \otimes (1_A - g).$$

Proof: The statement is trivial for $n = 1$. Suppose now we have shown the assertion for some $0 \neq n \in \mathbb{N}$. We obtain

$$1_{A \otimes (n+1)} - g^{\otimes (n+1)} = 1_{A \otimes n} \otimes g - g^{\otimes n} \otimes g + 1_{A \otimes n} \otimes (1_A - g)$$

$$= (1_{A \otimes n} - g^{\otimes n}) \otimes g + 1_{A \otimes n} \otimes (1_A - g)$$

$$\geq ((1_A - g) \otimes g \otimes \ldots \otimes g) \otimes g + (g \otimes (1_A - g) \otimes g \otimes \ldots \otimes g) \otimes g \ldots + (g \otimes \ldots \otimes g \otimes (1_A - g)) \otimes g + g^{\otimes n} \otimes (1_A - g),$$

where for the inequality we have used our induction hypothesis as well as the fact that $1_{A \otimes n} \otimes (1_A - g) \geq g^{\otimes n} \otimes (1_A - g)$. Therefore, the statement also holds for $n + 1$.

1.2 Proposition: Let $D$ be strongly self-absorbing, $0 \leq d \leq 1_D$.

Then, for any $0 \neq k \in \mathbb{N}$,

$$[1_D \otimes k - d^{\otimes k}] \leq k \cdot [(1_D - d) \otimes 1_{D \otimes (k-1)}] \text{ in } W(D^{\otimes k}).$$

Proof: The assertion holds trivially for $k = 1$. Suppose now it has been verified for some $k \in \mathbb{N}$. Then,

$$[1_D - d^{\otimes (k+1)}] = [1_{D^{\otimes k}} \otimes (1_D - d) + 1_{D^{\otimes k}} \otimes d - d^{\otimes k} \otimes d]$$

$$\leq [1_{D^{\otimes k}} \otimes (1_D - d)] + [(1_{D^{\otimes k}} - d^{\otimes k}) \otimes 1_D]$$

$$\leq [(1_D - d) \otimes 1_{D^{\otimes k}}] + k \cdot [(1_D - d) \otimes 1_{D^{\otimes (k-1)}} \otimes 1_D]$$

$$= (k + 1) \cdot [(1_D - d) \otimes 1_{D^{\otimes k}}]$$

(using that $D$ is strongly self-absorbing as well as our induction hypothesis for the second inequality), so the assertion also holds for $k + 1$.

1.3 The following is only a mild generalization of [1] Lemma 1.3].

Lemma: Let $D$ be strongly self-absorbing and let $0 \leq f \leq g \leq 1_D$ be positive elements of $D$ satisfying $1_D - g \neq 0$ and $fg = f$.
Then, there is $0 \neq n \in \mathbb{N}$ such that
\[ [f^\otimes n] \leq [1_D^\otimes n - g^\otimes n] \] in $W(D^\otimes n)$.

**Proof:** Since $D$ is simple and $1_D - g \neq 0$, there is $n \in \mathbb{N}$ such that
\[ [f] \leq n \cdot [1_D - g]. \]

Then,
\[
[f^\otimes n] \leq n \cdot [(1_D - g) \otimes f \otimes \ldots \otimes f] \\
= [(1_D - g) \otimes f \otimes \ldots \otimes f] + \ldots + [f \otimes \ldots \otimes f \otimes (1_D - g)] \\
= [(1_D - g) \otimes f \otimes \ldots \otimes f + \ldots + f \otimes \ldots \otimes f \otimes (1_D - g)] \\
\leq [(1_D - g) \otimes g \otimes \ldots \otimes g + \ldots + g \otimes \ldots \otimes g \otimes (1_D - g)] \\
\leq [1_D^\otimes n - g^\otimes n],
\]

where for the first equality we have used that $D$ is strongly self-absorbing, for the second equality we have used that the terms are pairwise orthogonal by our assumptions on $f$ and $g$, and the last inequality follows from Proposition 1.1.

**1.4** The following is a version of [1, Lemma 2.4] for positive elements rather than projections.

**Lemma:** Let $D$ be strongly self-absorbing and let $0 \leq f \leq g \leq 1_D$ be positive elements satisfying $1_D - g \neq 0$ and $fg = f$; let $0 \neq d \in D_+$.

Then, there is $0 \neq m \in \mathbb{N}$ such that
\[ [f^\otimes m] \leq [d \otimes 1_D^\otimes (m-1)] \] in $W(D^\otimes m)$.

**Proof:** By Lemma 1.3 there is $0 \neq n \in \mathbb{N}$ such that
\[ [f^\otimes n] \leq [1_D^\otimes n - g^\otimes n]; \]
since $f^\otimes n \perp 1_D^\otimes n - g^\otimes n$, this implies
\[ 2 \cdot [f^\otimes n] \leq [1_D^\otimes n]. \]

By an easy induction argument we then have
\[ 2^k \cdot [f^\otimes nk] \leq [1_D^\otimes nk] \]
for any $k \in \mathbb{N}$.

By simplicity of $D$ and since $d$ is nonzero, there is $\tilde{k} \in \mathbb{N}$ such that
\[ [f] \leq 2^{\tilde{k}} \cdot [d]. \]

Set
\[ m := n\tilde{k} + 1, \]
then
\[
[f^\otimes m] \leq 2^{\tilde{k}} \cdot [d \otimes f^\otimes (m-1)] \\
= 2^{\tilde{k}} \cdot [d \otimes f^\otimes \tilde{k}] \\
\leq [d \otimes 1_D^\otimes \tilde{k}] \\
= [d \otimes 1_D^\otimes (m-1)].
\]
2. LARGE ORDER ZERO MAPS

Below we establish the existence of nontrivial order zero maps from matrix algebras into strongly self-absorbing $C^*$-algebras, and we show certain systems of such maps give rise to order zero maps with small complements. We refer to [16] and [17] for an introduction to order zero maps.

2.1 Proposition: Let $\mathcal{D}$ be strongly self-absorbing and $0 \neq d \in \mathcal{D}_+$. Then, for any $0 \neq k \in \mathbb{N}$, there is a nonzero c.p.c. order zero map

$$\psi : M_k \to dDd.$$ 

Proof: Let us first prove the assertion in the case where $d = 1_D$ and $k = 2$. Since $\mathcal{D}$ is infinite dimensional, there are orthogonal positive normalized elements $e, f \in \mathcal{D}$. Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ is strongly self-absorbing, there is a sequence of unitaries $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D} \otimes \mathcal{D}$ such that

$$u_n(e \otimes f)u_n^* \xrightarrow{n \to \infty} f \otimes e;$$

since $e \otimes f \perp f \otimes e$ this implies that there is a c.p.c. order zero map

$$\bar{\sigma} : M_2 \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$$

given by

$$\bar{\sigma}(e_{11}) = e \otimes f, \bar{\sigma}(e_{22}) = f \otimes e, \bar{\sigma}(e_{21}) = \pi((u_n(e \otimes f))_{n \in \mathbb{N}}),$$

(cf. [16]), where $\pi : \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$ denotes the quotient map.

Since order zero maps with finite dimensional domains are semiprojective (cf. [16]), $\bar{\sigma}$ has a c.p.c. order zero lift $M_2 \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$, which in turn implies that there is a nonzero c.p.c. order zero map

$$\tilde{\sigma} : M_2 \to D \otimes D \cong D.$$ 

Next, if $k = 2^r$ for some $r \in \mathbb{N}$, then

$$M_{2^r} \cong (M_2)^{\otimes r} \xrightarrow{\tilde{\sigma} \otimes r} D^{\otimes r} \cong D$$

is a nonzero c.p.c. order zero map; for an arbitrary $k \in \mathbb{N}$, we may take $r$ large enough and restrict $\tilde{\sigma} \otimes r$ to $M_k \subset M_{2^r}$ to obtain a nonzero c.p.c. order zero map

$$\sigma : M_k \to D.$$ 

This settles the proposition for arbitrary $k$ and for $d = 1_D$. Now if $d$ is an arbitrary nonzero positive element (which we may clearly assume to be normalized), we can define a c.p.c. map

$$\tilde{\psi} : M_k \to \prod_{\mathbb{N}} \frac{dDd}{\bigoplus_{\mathbb{N}} dDd} \subset \prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}$$

by setting

$$\tilde{\psi}(x) := \pi((d\sigma_n(x)d)_{n \in \mathbb{N}}) \text{ for } x \in M_k,$$

where again $\pi : \prod_{\mathbb{N}} \frac{dDd}{\bigoplus_{\mathbb{N}} dDd} \to \prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}$ denotes the quotient map and $\sigma_n : M_k \to \mathcal{D}$ is a sequence of c.p.c. maps lifting the c.p.c. order zero map

$$\mu \sigma : M_k \to (\prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}) \cap \mathcal{D}'$$

with

$$\mu : \mathcal{D} \to (\prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}) \cap \mathcal{D}'$$
being a unital $\ast$-homomorphism as in [12, Theorem 2.2]. It is straightforward to check that $\bar{\psi}$ is nonzero and has order zero. Again by semiprojectivity of order zero maps, this implies the existence of a nonzero c.p.c. order zero map

$$\bar{\psi} : M_k \to dD_d.$$

2.2 Propostition: Let $B$ be a unital $C^*$-algebra and $\varrho : M_2 \to B$ a unital $\ast$-homomorphism. Define

$$E := \{ f \in C([0,1], B \otimes M_2) \mid f(0) \in B \otimes 1_{M_2}, \ f(1) \in 1_B \otimes M_2 \}.$$

Then, there is a unital $\ast$-homomorphism

$$\tilde{\varrho} : M_2 \to E.$$

Proof: This follows from simply connecting the two embeddings $\varrho \otimes 1_{M_2}$ and $1_{M_2} \otimes \text{id}_{M_2}$ of $M_2$ into $\varrho(M_2) \otimes M_2 \cong M_2 \otimes M_2$ along the unit interval.

2.3 Lemma: Let $m \in \mathbb{N}$ and $A$ a unital $C^*$-algebra. Let

$$\varphi_1, \ldots, \varphi_m : M_2 \to A$$

be c.p.c. order zero maps such that

$$\sum_{i=1}^m \varphi_i(1_{M_2}) \leq 1_A$$

and

$$[\varphi_i(M_2), \varphi_j(M_2)] = 0 \text{ if } i \neq j.$$

Then, there is a c.p.c. order zero map

$$\bar{\varphi} : M_2 \to C^*(\varphi_i(M_2)) \mid i = 1, \ldots, m \subset A$$

such that

$$\bar{\varphi}(1_{M_2}) = \sum_{i=1}^m \varphi_i(1_{M_2}).$$

Moreover, if $d \in A^+$ satisfies $\varphi_m(e_{11})d = d$, we may assume that $\bar{\varphi}(e_{11})d = d$.

Proof: In the following, we write $C_i$, $i = 1, \ldots, m$, for various copies of the $C^*$-algebra $C_0((0,1], M_2)$; these come equipped with c.p.c. order zero maps $\varrho_i : M_2 \to C_i$ given by

$$\varrho_i(x)(t) = t \cdot x \text{ for } t \in (0,1] \text{ and } x \in M_2.$$

By [14, Proposition 3.2(a)], the c.p.c. order zero maps $\varphi_i : M_2 \to A$ induce unique $\ast$-homomorphisms $C_i \to A$ via $\varrho_i(x) \mapsto \varphi_i(x)$, for $x \in M_2$.

We now define a universal $C^*$-algebra

$$B := C^*(C_i, 1 \mid \sum_{i=1}^m \varrho_i(1_{M_2}) \leq 1, \ [C_i, C_j] = 0 \text{ if } i \neq j \in \{1, \ldots, m\}).$$

Then, $B$ is generated by the $\varrho_i(x)$, $i \in \{1, \ldots, m\}$ and $x \in M_2$; the assignment

$$\varrho_i(x) \mapsto \varphi_i(x)$$

for $i \in \{1, \ldots, m\}$ and $x \in M_2$

induces a unital $\ast$-homomorphism

$$\pi : B \to C^*(\varphi_i(M_2), 1_A \mid i \in \{1, \ldots, m\}) \subset A$$
satisfying
\[ \sum_{l=1}^{m} \pi_l(1_{M_2}) = \sum_{l=1}^{m} \varphi_l(1_{M_2}). \]

Now if we find a c.p.c. order zero map
\[ \bar{\varphi} : M_2 \to B \]
satisfying
\[ \bar{\varphi}(1_{M_2}) = \sum_{l=1}^{m} \varphi_l(1_{M_2}), \]
then
\[ \bar{\varphi} := \pi \bar{\varphi} \]
will have the desired properties, proving the first assertion of the lemma. We proceed to construct \( \bar{\varphi} \).

For \( k = 1, \ldots, m \), let
\[ J_k := J_1(1 - \sum_{l=k}^{m} \varphi_l(1_{M_2})) \subset B \]
denote the ideal generated by \( 1 - \sum_{l=k}^{m} \varphi_l(1_{M_2}) \) in \( B \); let
\[ B_k := B/J_k \]
denote the quotient. We clearly have
\[ J_1 \subset J_2 \subset \ldots \subset J_m \]
and surjections
\[ B \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_m} B_m. \]

Observe that
\[ \pi_m \circ \ldots \circ \pi_1 \circ \varphi_m : M_2 \to B_m \]
is a unital surjective c.p. order zero map, hence a \(*\)-homomorphism by [14, Proposition 3.2(b)]; therefore,
\[ B_m \cong M_2. \]

For \( k = 1, \ldots, m - 1 \), set
\[ E_k := \{ f \in C([0,1], B_{k+1} \otimes M_2) \mid f(0) \in B_{k+1} \otimes 1_{M_2}, f(1) \in 1_{B_{k+1} \otimes M_2} \}; \]
and surjections
\[ B \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_{m}} B_m \]
one easily checks that the maps
\[ \sigma_k : B_k \to E_k \]
induced by
\[ \pi_k \ldots \pi_1 \varphi_i(x) \mapsto \begin{cases} (t \mapsto (1 - t) \cdot \pi_{k+1} \ldots \pi_1 \varphi_i(x) \otimes 1_{M_2}) & \text{for } i = k + 1, \ldots, m \text{ and } x \in M_2 \\ (t \mapsto t \cdot 1_{B_{k+1} \otimes x}) & \text{for } i = k \text{ and } x \in M_2 \end{cases} \]
are well-defined \(*\)-isomorphisms. Similarly, the map
\[ \sigma_0 : B \to E_0 := \{ f \in C([0,1], B_1) \mid f(1) \in C \cdot 1_{B_1} \} \]
induced by
\[ \varphi_i(x) \mapsto (t \mapsto (1 - t) \cdot \pi_1 \varphi_i(x)) \text{ for } i = 1, \ldots, m \text{ and } x \in M_2, \]
\[ 1_B \mapsto 1_{E_0} \]
is a well-defined *-isomorphism; note that
\[ \sigma_0(\sum_{i=1}^m \theta_i(1_{M_2})) = (t \mapsto (1 - t) \cdot 1_{B_1}). \]

By (1) together with Proposition 2.2 and an easy induction argument, the unital *-homomorphism
\[ \pi_m \cdots \pi_1 \theta_m : M_2 \to B_m \]
pulls back to a unital *-homomorphism
\[ \tilde{\theta} : M_2 \to B_1; \]
This in turn induces a c.p.c. order zero map
\[ \tilde{\phi} : M_2 \to E_0 \]
by
\[ \tilde{\phi}(x) := (t \mapsto (1 - t) \cdot \tilde{\theta}(x)); \]
note that this map satisfies
\[ \tilde{\phi}(1_{M_2}) = (t \mapsto (1 - t) \cdot 1_{B_1}). \]

We now define a *-homomorphism
\[ \tilde{\rho} := \sigma_0^{-1} \circ \tilde{\theta} : M_2 \to B; \]
note that \( \tilde{\rho}(1_{M_2}) = \sum_{i=1}^m \theta_i(1_{M_2}) \), whence \( \tilde{\rho} \) is as desired.

For the second assertion of the lemma, note that \( \tilde{\rho} \) and \( \theta_m \) agree modulo \( J_m \). Therefore, \( \varphi = \pi \tilde{\rho} \) and \( \varphi_m = \pi \theta_m \) agree up to \( \pi(J_m) \). However, one checks that \( \pi(J_m) \perp d \), whence \( (\varphi(x) - \varphi_m(x))d = 0 \) for all \( x \in M_2 \). This implies \( \varphi(e_{11})d = \varphi_m(e_{11})d = d \).

2.4 **Proposition:** Let \( \mathcal{D} \) be strongly self-absorbing, \( 0 \neq m \in \mathbb{N} \) and
\[ \varphi_0 : M_2 \to \mathcal{D} \]
a c.p.c. order zero map.

Then, there are c.p.c. order zero maps
\[ \varphi_1, \ldots, \varphi_m : M_2 \to \mathcal{D} \otimes \mathcal{D} \]
such that
\begin{enumerate}
    \item \( \varphi_1 = \varphi_0 \otimes 1_{\mathcal{D} \otimes (m-1)} \)
    \item \( \varphi_i(M_2), \varphi_j(M_2) = 0 \) if \( i \neq j \)
    \item \( 1_{\mathcal{D} \otimes m} - \sum_{i=1}^m \varphi_i(1_{M_2}) = (1_{\mathcal{D}} - \varphi_0(1_{M_2})) \otimes \mathcal{D} \)
\end{enumerate}

**Proof:** For \( k \in \{1, \ldots, m\} \), define
\[ \varphi_k := (1_{\mathcal{D}} - \varphi_0(1_{M_2})) \otimes (k-1) \otimes \varphi_0 \otimes 1_{\mathcal{D} \otimes (m-k)}, \]
then the \( \varphi_k \) obviously satisfy (2.4(i) and (ii)).

A simple induction argument shows that, for \( k = 1, \ldots, m \),
\[ 1_{\mathcal{D} \otimes m} - \sum_{i=1}^k \varphi_i(1_{M_2}) = (1_{\mathcal{D}} - \varphi_0(1_{M_2})) \otimes k \otimes 1_{\mathcal{D} \otimes (m-k)}, \]
which is (2.4(iii)) when we take \( k = m \).
3. \(\mathcal{Z}\)-stability

We now assemble the techniques of the preceding sections and a result from [10] to prove our main result; we also derive some consequences.

3.1 Theorem: Any strongly self-absorbing \(C^*\)-algebra \(\mathcal{D}\) absorbs the Jiang–Su algebra \(\mathcal{Z}\) tensorially.

Proof: Let \(k \in \mathbb{N}\). By Proposition 2.1 there is a nonzero c.p.c. order zero map \(\phi : M_2 \rightarrow \mathcal{D}\). Using functional calculus for order zero maps (cf. [17]), we may assume that there is

\[
0 \leq d \leq \phi(e_{11})
\]

such that

\[
d \neq 0 \text{ and } \phi(e_{11})d = d.
\]

Note that

\[
(1_D - \phi(1_{M_2}))(1_D - d) = 1_D - \phi(1_{M_2}).
\]

By Proposition 2.1 there is a nonzero c.p.c. order zero map

\[
\psi : M_k \rightarrow d_Dd;
\]

note that

\[
\phi(e_{11})\psi(x) = \psi(x) \text{ for } j = 1, \ldots, k \text{ and } x \in M_k.
\]

Apply Lemma 1.4 (with \(D^\otimes k, \psi(e_{11})^\otimes k, (1_D - \phi(1_{M_2})) \otimes 1_{D^\otimes (k-1)} \) and \((1_D - d) \otimes 1_{D^\otimes (k-1)}\) in place of \(D, d, f, g\), respectively) to obtain \(0 \neq m \in \mathbb{N}\) such that

\[
(1_D - \phi(1_{M_2})) \otimes 1_{D^\otimes (k-1)}^\otimes m \leq \psi(e_{11})^\otimes k \otimes 1_{D^\otimes (m-1)}^\otimes m,
\]

in \(W((D \otimes k)^\otimes m)\). From Proposition 2.4 (with \(D^\otimes k\) in place of \(D\) and \(\phi_0 := \phi \otimes 1_{D^\otimes (k-1)}\)) we obtain c.p.c. order zero maps

\[
\varphi_1, \ldots, \varphi_m : M_2 \rightarrow (D^\otimes k)^\otimes m
\]
satisfying (2.4(i), (ii) and (iii)). By relabeling the \(\varphi_i\) we may assume that actually \(\varphi_m = \varphi_0 \otimes 1_{(D^\otimes k)^\otimes (m-1)}\) in (2.4(i)).

From Lemma 2.3, we obtain a c.p.c. order zero map

\[
\bar{\varphi} : M_2 \rightarrow C^*(\varphi_i(M_2) \mid i = 1, \ldots, m) \subset (D^\otimes k)^\otimes m
\]
such that

\[
\bar{\varphi}(1_{M_2}) = \sum_{i=1}^m \varphi_i(1_{M_2}).
\]

By the second assertion of Lemma 2.3 and since

\[
\varphi_m(e_{11})(\psi(1_{M_k}) \otimes 1_{D^\otimes (k-1)}) = (\varphi(e_{11}) \otimes 1_{D^\otimes (km-1)})(\psi(1_{M_k}) \otimes 1_{D^\otimes (km-1)}),
\]

we may furthermore assume that

\[
\bar{\varphi}(e_{11})(\psi(1_{M_k}) \otimes 1_{D^\otimes (km-1)}) = \psi(1_{M_k}) \otimes 1_{D^\otimes (km-1)},
\]

which in turn yields

\[
\psi(1_{M_k}) \otimes 1_{D^\otimes (km-1)} \leq \bar{\varphi}(e_{11})
\]
since $\psi$ is contractive. Note that we have

$$\left[1_{(D \otimes k) \otimes m} - \tilde{\varphi}(1_{M_2})\right] = \left[1_{D \otimes k} - \varphi(1_{M_2})\right] \otimes 1_{(D \otimes (k-1)) \otimes m} \leq [\psi(e_{11}) \otimes 1_{(D \otimes (m-1)) \otimes k}]$$

in $W((D \otimes k) \otimes m)$. Define a c.p.c. order zero map

$$\Phi : M_{2k} \cong (M_2) \otimes k \to (D \otimes k) \otimes \alpha = \mathcal{D} \otimes kmk$$

by

$$\Phi := \tilde{\varphi} \otimes k.$$  

We have

$$\left[1_{((D \otimes k) \otimes m) \otimes k} - \Phi(1_{(M_2) \otimes k})\right] \leq \left[k \cdot \left[1_{(D \otimes k) \otimes m} - \tilde{\varphi}(1_{M_2})\right] \otimes 1_{((D \otimes k) \otimes m) \otimes (k-1)}\right] \leq \left[k \cdot [\psi(e_{11}) \otimes 1_{(D \otimes (m-1)) \otimes k}] \otimes 1_{((D \otimes k) \otimes m) \otimes (k-1)}\right] \leq [\tilde{\varphi}(1_{M_k}) \otimes 1_{(D \otimes (m-1)) \otimes k}] \leq [\Phi(e_{11})]$$

in $W((D \otimes k) \otimes m)$. From [10, Proposition 5.1] we now see that there is a unital $\ast$-homomorphism

$$\theta : Z^{2k,2k+1} \to D \otimes kmk \cong D.$$  

Since $k$ was arbitrary, by [13, Proposition 2.2] this implies that $D$ is $Z$-stable.

3.2 COROLLARY: The Jiang–Su algebra is the uniquely determined (up to isomorphism) initial object in the category of strongly self-absorbing $C^*$-algebras (with unital $\ast$-homomorphisms).

PROOF: By Theorem 3.1, the Jiang–Su algebra does embed unitally into any strongly self-absorbing $C^*$-algebra, so it is an initial object. If $D$ is another initial object, then $Z$ and $D$ embed unitally into one another, whence they are isomorphic by [12, Proposition 5.12].

Sometimes an object in a category is called initial only if there is a unique morphism to any other object; this remains true in our setting if one takes approximate unitary equivalence classes of unital $\ast$-homomorphisms as morphisms.

3.3 REMARK: By [9], $Z$-stable $C^*$-algebras are $K_1$-injective, whence $K_1$-injectivity is unnecessary in the hypotheses of the main results of [12, 2, 3, 5, 6] and 7.

REFERENCES

[1] Marius Dădălăt and Mikael Rørdam, Strongly self-absorbing $C^*$-algebras which contain a nontrivial projection, arXiv preprint math.OA/0902.3886, 2009.

[2] Marius Dădălăt and Wilhelm Winter, On the $KK$-theory of strongly self-absorbing $C^*$-algebras, arXiv preprint math.OA/0704058, to appear in Math. Scand., 2007.

[3] Marius Dădălăt and Wilhelm Winter, Trivialization of $C(X)$-algebras with strongly self-absorbing fibres, Bull. Soc. Math. France 136 (2008), no. 4, 575–606. MR MR2443037
[4] George A. Elliott and Andrew S. Toms, Regularity properties in the classification program for separable amenable C*-algebras, Bull. Amer. Math. Soc. (N.S.) 45 (2008), no. 2, 229–245. MR MR2383304
[5] Ilan Hirshberg, Mikael Rørdam, and Wilhelm Winter, C(X)-algebras, stability and strongly self-absorbing C*-algebras, Math. Ann. 339 (2007), no. 3, 695–732. MR MR2336064 (2008j:46040)
[6] Ilan Hirshberg and Wilhelm Winter, Rokhlin actions and self-absorbing C*-algebras, Pacific J. Math. 233 (2007), no. 1, 125–143. MR MR2366371
[7] ______, Permutations of strongly self-absorbing C*-algebras, Internat. J. Math. 19 (2008), no. 9, 1137–1145. MR MR2458564
[8] Xinhui Jiang and Hongbing Su, On a simple unital projectionless C*-algebra, Amer. J. Math. 121 (1999), no. 2, 359–413. MR MR1680321 (2000a:46104)
[9] Mikael Rørdam, The stable and the real rank of Z-absorbing C*-algebras, Internat. J. Math. 15 (2004), no. 10, 1065–1084. MR MR2106263 (2005k:46164)
[10] Mikael Rørdam and Wilhelm Winter, The Jiang–Su algebra revisited, arxiv preprint math.OA/0801.2259; to appear in J. Reine Angew. Math., 2008.
[11] Andrew S. Toms, On the classification problem for nuclear C*-algebras, Ann. of Math. (2) 167 (2008), no. 3, 1029–1044. MR MR2415391
[12] Andrew S. Toms and Wilhelm Winter, Strongly self-absorbing C*-algebras, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029 (electronic). MR MR2302521 (2008c:46086)
[13] ______, Z-stable ASH algebras, Canad. J. Math. 60 (2008), no. 3, 703–720. MR MR2414961
[14] Wilhelm Winter, Covering dimension for nuclear C*-algebras, J. Funct. Anal. 199 (2003), no. 2, 535–556. MR MR1971906 (2004c:46134)
[15] ______, Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, arXiv preprint math.OA/0708.0283v3, with an appendix by H. Lin, 2007.
[16] ______, Covering dimension for nuclear C*-algebras II, Trans. Amer. Math. Soc. 361 (2009), no. 8, 4143–4167.
[17] Wilhelm Winter and Joachim Zacharias, Completely positive maps of order zero, arXiv preprint math.OA/0903.3290v1, 2009.