A characterisation of translation ovals in finite even order planes

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Abstract

In this article we consider a set $C$ of points in $PG(4, q)$, $q$ even, satisfying certain combinatorial properties with respect to the planes of $PG(4, q)$. We show that there is a regular spread in the hyperplane at infinity, such that in the corresponding Bruck-Bose plane $PG(2, q^2)$, the points corresponding to $C$ form a translation hyperoval, and conversely.

1 Introduction

In this article we first consider a non-degenerate conic in $PG(2, q^2)$, $q$ even. We look at the corresponding point set in the Bruck-Bose representation in $PG(4, q)$, and study its combinatorial properties (details of the Bruck-Bose representation are given in Section 2). Some properties of this set were investigated in [4]. In this article we are interested in combinatorial properties relating to planes of $PG(4, q)$. We consider a set of points in $PG(4, q)$ satisfying certain of these combinatorial properties and find that the points correspond to a translation oval in the Bruck-Bose plane $PG(2, q^2)$.

In [3], the case when $q$ is odd is considered, and we show that given a set of points in $PG(4, q)$ satisfying the following combinatorial properties, we can reconstruct the conic in $PG(2, q^2)$. We use the following terminology in $PG(4, q)$: if the hyperplane at infinity is denoted $\Sigma_\infty$, then we call the points of $PG(4, q) \setminus \Sigma_\infty$ affine points.

Theorem 1.1 Let $\Sigma_\infty$ be the hyperplane at infinity in $PG(4, q)$, $q \geq 7$, $q$ odd. Let $C$ be a set of $q^2$ affine points, called $C$-points, and suppose there exists a set of planes called $C$-planes satisfying the following properties:

1. Each $C$-plane meets $C$ in a $q$-arc.
2. Any two distinct C-points lie in a unique C-plane.

3. The affine points of PG(4, q) are of three types: points of C; points on no C-plane; and points on exactly two C-planes.

4. If a plane meets C in more than four points, it is a C-plane.

Then there exists a unique spread S in \(\Sigma_\infty\) so that in the Bruck-Bose translation plane \(\mathcal{P}(S)\), the C-points form a \(q^2\)-arc of \(\mathcal{P}(S)\). Moreover, the spread S is regular, and so \(\mathcal{P}(S) \cong \text{PG}(2, q^2)\), and the \(q^2\)-arc can be completed to a conic of \(\text{PG}(2, q^2)\).

The case when \(q\) is even is more complex. The combinatorial properties only allow us to reconstruct a translation oval in \(\text{PG}(2, q^2)\). The main result of this article is the following theorem.

**Theorem 1.2** Consider \(\text{PG}(4, q)\), \(q\) even, \(q > 2\), with the hyperplane at infinity denoted by \(\Sigma_\infty\). Let C be a set of \(q^2\) affine points, called C-points and consider a set of planes called C-planes which satisfies the following:

(A1) Each C-plane meets C in a \(q\)-arc.

(A2) Any two distinct C-points lie in a unique C-plane.

(A3) The affine points that are not in C lie on exactly one C-plane.

(A4) Every plane which meets C in at least three points either meets C in exactly four points or is a C-plane.

Then there exists a regular spread S in \(\Sigma_\infty\) such that in the Bruck-Bose plane \(\mathcal{P}(S) \cong \text{PG}(2, q^2)\), the C-points, together with two extra points on \(\ell_\infty\), form a translation hyperoval of \(\text{PG}(2, q^2)\).

We begin in Section 2 with the necessary background material on the Bruck-Bose representation. In Section 3 we investigate combinatorial properties of conics and translation ovals in \(\text{PG}(2, q^2)\), \(q\) even, and show that they satisfy properties (A1-4) of Theorem 1.2. The rest of the article is devoted to proving Theorem 1.2. In Section 5 we discuss further aspects of the problem, and avenues for further work.
2 The Bruck-Bose representation

We use the Bruck-Bose representation of \( \text{PG}(2, q^2) \) in \( \text{PG}(4, q) \) introduced in \([1, 4, 6]\). See \([2]\) for more details of this representation. Let \( \Sigma_\infty \) be the hyperplane at infinity of \( \text{PG}(4, q) \) and let \( \mathcal{S} \) be a spread of \( \Sigma_\infty \). Call the points of \( \text{PG}(4, q) \setminus \Sigma_\infty \) affine points and the points in \( \Sigma_\infty \) infinite points. The lines and planes of \( \text{PG}(4, q) \) that are not contained in \( \Sigma_\infty \) are called affine lines and affine planes respectively.

Consider the incidence structure whose points are the affine points of \( \text{PG}(4, q) \), whose lines are the affine planes of \( \text{PG}(4, q) \) which contain a line of the spread \( \mathcal{S} \), and where incidence is inclusion. This incidence structure is an affine plane, and can be completed to a translation plane denoted \( \mathcal{P}(\mathcal{S}) \) by adjoining the line at infinity \( \ell_\infty \) whose points are the elements of the spread \( \mathcal{S} \). The translation plane \( \mathcal{P}(\mathcal{S}) \) is the Desarguesian plane \( \text{PG}(2, q^2) \) if and only if the spread \( \mathcal{S} \) is regular. Note that the affine planes of \( \text{PG}(4, q) \) that do not contain a line of \( \mathcal{S} \) correspond to Baer subplanes of \( \text{PG}(2, q^2) \) secant to \( \ell_\infty \).

We introduce the coordinate notation that we will use in the Bruck-Bose representation of \( \text{PG}(2, q^2) \) in \( \text{PG}(4, q) \) (so \( \mathcal{S} \) is a regular spread). See \([2]\) for more details of this coordinatisation. In \( \text{PG}(2, q^2) \), points have coordinates \((x, y, z) \in \text{GF}(q^2) \) and the line at infinity has equation \( z = 0 \). In \( \text{PG}(4, q) \), points have coordinates \((x_0, \ldots, x_4), x_i \in \text{GF}(q) \), and we let the hyperplane at infinity \( \Sigma_\infty \) have equation \( x_4 = 0 \). Let \( \tau \) be a primitive element in \( \text{GF}(q^2) \) with primitive polynomial \( x^2 - t_1 x - t_0 \) over \( \text{GF}(q) \). Let \( \alpha, \beta \in \text{GF}(q^2) \), then we can uniquely write \( \alpha = a_0 + a_1 \tau, \beta = b_0 + b_1 \tau \) for \( a_i, b_i \in \text{GF}(q) \). The Bruck-Bose map takes a point \( P = (\alpha, \beta, 1) \in \text{PG}(2, q^2) \setminus \ell_\infty \) to the point \( P = (a_0, a_1, b_0, b_1, 1) \in \text{PG}(4, q) \setminus \Sigma_\infty \). In \( \text{PG}(2, q^2) \), \( \ell_\infty \) has points \( \{P_\infty = (0, 1, 0)\} \cup \{P_\delta = (1, \delta, 0) : \delta \in \text{GF}(q^2)\} \).

If \( \delta = d_0 + d_1 \tau \), for \( d_0, d_1 \in \text{GF}(q) \), then in \( \text{PG}(4, q) \), these points correspond to lines \( p_\infty, p_\delta \) of the regular spread \( \mathcal{S} \) where

\[
\begin{align*}
p_\infty &= \{(0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\}, \\
p_\delta &= \{(1, 0, d_0, d_1, 0), (0, 1, t_0 d_1, d_0 + t_1 d_1, 0)\}.
\end{align*}
\]

3 Properties of conics and translation ovals

Let \( \overline{\mathcal{C}} \) be a non-degenerate conic in \( \text{PG}(2, q^2) \), \( q \) even, that meets \( \ell_\infty \) in a point \( P_\infty \), so \( \overline{\mathcal{C}} \) has nucleus \( N \subseteq \ell_\infty \). Let \( \mathcal{C} = \overline{\mathcal{C}} \setminus \{P_\infty\} \). We use the term \( \mathcal{C} \)-points for (affine) points of \( \text{PG}(2, q^2) \) in \( \mathcal{C} \). If \( \alpha \) is a Baer subplane secant to \( \ell_\infty \), then \( \alpha \) meets \( \overline{\mathcal{C}} \) in either a subconic, or in at most four points; so \( \alpha \) meets \( \mathcal{C} \) in either a \( q \)-arc or in at most four points. Baer subplanes secant to \( \ell_\infty \) that meet \( \mathcal{C} \) in a \( q \)-arc are called \( \mathcal{C} \)-planes. The following properties about \( \mathcal{C} \)-planes and \( \mathcal{C} \)-points is straightforward to prove, and is also a special case of Theorem 3.2.
Lemma 3.1 Let \( \mathcal{C} = \mathcal{C} \cup \{ P_\infty \} \) be a conic in \( \text{PG}(2,q^2) \), \( q \) even, that meets \( \ell_\infty \) in a point \( P_\infty \). Define \( \mathcal{C} \)-points to be points of \( \mathcal{C} \), and \( \mathcal{C} \)-planes to be Baer subplanes secant to \( \ell_\infty \) that meet \( \mathcal{C} \) in a \( q \)-arc. Then the following hold.

1. Two distinct \( \mathcal{C} \)-points lie in a unique \( \mathcal{C} \)-plane.
2. Every affine point not in \( \mathcal{C} \) lies in a unique \( \mathcal{C} \)-plane.
3. If a Baer subplane secant to \( \ell_\infty \) contains at least three \( \mathcal{C} \)-points, then it either contains exactly four \( \mathcal{C} \)-points, or is a \( \mathcal{C} \)-plane.

Our aim in this article is to work in the Bruck-Bose representation in \( \text{PG}(4,q) \), and consider a set of \( \mathcal{C} \)-points and \( \mathcal{C} \)-planes that satisfy the properties given in Lemma 3.1, and see if we can reconstruct the conic. In fact, it turns out that these geometric properties do not uniquely reconstruct the conic, rather, they determine a translation oval. So we first show that a translation oval satisfies the required geometric properties.

Recall that a translation oval in \( \text{PG}(2,2^h) \) is projectively equivalent to \( \mathcal{O}(2^n) = \{(t^{2^n}, t, 1) : t \in \text{GF}(q) \cup \{ \infty \}\} \) where \( (n, h) = 1 \). Note that \( \mathcal{O}(2^n) \) has nucleus \( N = (0, 1, 0) \), and \( \mathcal{O} \cup \{ N \} \) is a translation hyperoval. Further, if \( n = 1 \), then \( \mathcal{O}(2) \) is a non-degenerate conic.

Theorem 3.2 Let \( \mathcal{O}(2^n) = \{(t^{2^n}, t, 1) : t \in \text{GF}(q) \cup \{ \infty \}\} \) be a translation oval in \( \text{PG}(2,q^2) \), \( q = 2^h \), \( (n, h) = 1 \), where \( \ell_\infty \) has equation \( z = 0 \). Define \( \mathcal{C} \)-points to be points of \( \mathcal{O} = \mathcal{O}(2^n) \setminus \{(1,0,0)\} \). Define \( \mathcal{C} \)-planes to be Baer subplanes secant to \( \ell_\infty \) that contain \( q \) points of \( \mathcal{O} \). Then the following hold.

1. Each \( \mathcal{C} \)-plane meets \( \mathcal{O} \) in a \( q \)-arc.
2. Any two distinct \( \mathcal{C} \)-points lie in a unique \( \mathcal{C} \)-plane.
3. Every affine point is either a \( \mathcal{C} \)-point, or lies in exactly one \( \mathcal{C} \)-plane.
4. Every Baer subplane secant to \( \ell_\infty \) which meets \( \mathcal{O} \) in at least three points, either meets \( \mathcal{O} \) in exactly four points, or is a \( \mathcal{C} \)-plane.

Proof Note that part 1 holds trivially. We begin the proof by calculating the group \( G \) of homographies of \( \text{PG}(2,q^2) \) fixing \( \mathcal{O} \) and \( \ell_\infty \). Let \( \sigma \) be a homography that fixes \( \mathcal{O} \) and \( \ell_\infty \), so \( \sigma \) has matrix

\[
M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad a, \ldots, i \in \text{GF}(q^2),
\]

4
and \( \sigma(x) = Mx \), where \( x \) is the column vector representing the coordinates of a point. As \( \sigma \) fixes \( \ell_\infty \) and \( \mathcal{O} \), it fixes \( P_\infty = (1,0,0) \) and the nucleus \( N = (0,1,0) \). Hence \( b = h = d = g = 0 \). An affine point \((x,y,z)\) of \( \text{PG}(2,q^2) \) belongs to \( \mathcal{O} \) if and only if \( xz = y^{2n} \), so the image of the point \((t^{2^k}, t, 1)\) is on \( \mathcal{O} \) if its co-ordinates satisfy \( xz = y^{2n} \). This gives

\[
M = \begin{pmatrix}
e^{2^k} & 0 & f^{2^k} \\
0 & e & f \\
0 & 0 & 1
\end{pmatrix}, \quad e, f \in \text{GF}(q^2).
\]

(1)

The determinant of \( M \) is \( 1 \cdot e \cdot e^{2^n} \) so we require \( e \neq 0 \), but there are no restrictions on \( f \). Thus there are \( q^2(q^2 - 1) \) matrices \( M \), so \( |G| = q^2(q^2 - 1) \).

Let \( \alpha \) be the Baer subplane \( \text{PG}(2,q) \), then the subgroup of \( G \) fixing \( \alpha \) has matrices with form given in (1), but with \( e, f \in \text{GF}(q) \), hence this subgroup has size \( q(q - 1) \). Thus by the orbit-stabilizer theorem, the orbit of \( \alpha \) is of size \( q(q + 1) \). Now \( \alpha \) meets \( \mathcal{O} \) in a \( q \)-arc, so is a \( \mathcal{C} \)-plane, so there are at least \( q(q + 1) \) \( \mathcal{C} \)-planes. Now we look at the subgroup \( H \) of \( G \) that stabilizes the \( \mathcal{C} \)-point \( P = (0,0,1) \). The group \( H \) consists of homographies with matrices with form given in (1), with \( f = 0 \), so \( |H| = q^2 - 1 \). Hence by the orbit-stabilizer theorem, \( P \) has orbit in \( G \) of size \( q^2 \), that is, \( G \) is transitive on the \( q^2 \) \( \mathcal{C} \)-points. Next consider the subgroup \( I \) of \( H \) fixing the \( \mathcal{C} \)-point \( Q = (1,1,1) \). The group \( I \) consists of homographies with matrices with form given in (1), with \( f = 0 \) and \( e = 1 \), so \( |I| = 1 \). Hence by the orbit-stabilizer theorem, \( G \) is 2-transitive on the \( \mathcal{C} \)-points.

We now show that part 2 holds. Let \( A, B \) be any two \( \mathcal{C} \)-points. Any \( \mathcal{C} \)-plane containing \( A \) and \( B \) is a Baer subplane that contains the line \( \ell_\infty \) and a \( q \)-arc of \( \mathcal{O} \), and so contains \( N \) and \( P_\infty \), the unique completion of the \( q \)-arc to a \( (q + 2) \)-arc. The four points \( A, B, P_\infty, N \) form a quadrangle, and so lie in a unique Baer subplane \( \alpha \). Hence there is at most one \( \mathcal{C} \)-plane through the two \( \mathcal{C} \)-points \( A, B \). Now count in two ways the triples \((A, B, \beta)\), where \( A \) and \( B \) are distinct \( \mathcal{C} \)-points in the \( \mathcal{C} \)-plane \( \beta \). Firstly we have \( q^2(q^2 - 1) \) pairs \((A, B)\) with at most one \( \mathcal{C} \)-plane containing them. Thus the number of triples is at most \( q^2(q^2 - 1) \) with equality if and only if every pair is on a unique \( \mathcal{C} \)-plane. Alternatively, we calculated in the first paragraph that there are at least \( q(q + 1) \) \( \mathcal{C} \)-planes \( \beta \). Each of these \( \mathcal{C} \)-planes contains \( q \) \( \mathcal{C} \)-points, so there are at least \( q(q + 1) \times q(q - 1) = q^2(q^2 - 1) \) triples. Hence there are exactly \( q^2(q^2 - 1) \) triples, exactly \( q(q + 1) \) \( \mathcal{C} \)-planes, and any two distinct points lie on exactly one \( \mathcal{C} \)-plane. Hence part 2 holds.

For part 3, consider the affine point \( P = (0,1,1) \), it is on the line joining two points \((1,0,0)\) and \((1,1,1)\) of the oval \( \mathcal{O} \), and and so is not a \( \mathcal{C} \)-point. The image of \( P \) under the homography \( \sigma \) with matrix \( M \) given in (1) is \( P' = MP = (f^{2^k}, e + f, 1) \). This is equal to \( P \) if and only if \( f = 0 \) and \( e = 1 \). Hence the only element of \( G \) fixing \( P \) is the identity, so by the orbit-stabilizer theorem, \( G \) is transitive on the \( q^4 - q^2 \) affine non-\( \mathcal{C} \)-points of \( \text{PG}(2,q^2) \). As each of the \( q^2 + q \) \( \mathcal{C} \)-planes contains \( q^2 - q \) affine non-\( \mathcal{C} \)-points, it follows that every affine non-\( \mathcal{C} \)-point is on exactly one \( \mathcal{C} \)-plane and part 3 is proved.
For part 4, let \( \alpha \) be a Baer subplane secant to \( \ell_\infty \) that contains three \( C \)-points \( A, B, C \). As the group \( G \) is 2-transitive on the \( C \)-points, without loss of generality, we may assume the points are \( A = (0, 0, 1) \), \( B = (1, 1, 1) \) and \( C = (t^{2n}, t, 0) \), where \( t \in GF(q^2) \setminus \{0, 1\} \). So \( \alpha \) contains the three points \( X = AB \cap \ell_\infty = (1, 1, 0) \), \( Y = AC \cap \ell_\infty = (t^{2n}, t, 0) \) and \( Z = BC \cap \ell_\infty = (t^{2n} + 1, t + 1, 0) \) on \( \ell_\infty \). As \( \alpha \) is a subplane, it follows that the point \( D = BY \cap XC = ((t + 1)^2n, t + 1, 1) \) is in \( \alpha \). Note that \( D \) also lies on the line \( ZA \). However, \( D \) is a \( C \)-point which is not equal to \( A, B \) or \( C \), as \( t \neq 0, 1 \). Thus if \( \alpha \) contains at least three \( C \)-points, then \( \alpha \) contains at least four \( C \)-points.

We want to show that either \( \alpha \) contains exactly four \( C \)-points, or \( \alpha \) is a \( C \)-plane. To do this we find the coordinates of the remaining points of \( \alpha \). The homography \( \rho \) with matrix

\[
K = \begin{pmatrix} t^{2n} + 1 & t^{2n} & 0 \\ t + 1 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

maps \((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\) to the points \( Z, Y, A, B \) respectively. Hence \( \rho \) maps the Baer subplane \( PG(2, q) \) to \( \alpha \). By part 2, there is a unique \( C \)-plane through \( A, B \), namely \( PG(2, q) \). Note that \( \alpha = PG(2, q) \) if and only if \( t \in GF(q) \) (since \( C \in PG(2, q) \) if and only if \( t \in GF(q) \)). Suppose \( t \notin GF(q) \), so \( \alpha \) is not a \( C \)-plane. We need to show that \( \alpha \) contains no further \( C \)-points. Note that \( \rho \) maps points of \( \ell_\infty \) to points of \( \ell_\infty \), so the affine points of \( \alpha \) are the image under \( \rho \) of affine points of \( PG(2, q) \). An affine point \( P = (x, y, 1), x, y \in GF(q) \) of \( PG(2, q) \) maps under \( \rho \) to the affine point \( P' = KP = (x(t^{2n} + 1) + yt^{2n}, x(t + 1) + yt, 1) \) of \( \alpha \). So the affine points of \( \alpha \) are the points with coordinates of form \( P' \) for \( x, y \in GF(q) \). Now \( P' \) is a \( C \)-point if and only if \( x(t^{2n} + 1) + yt^{2n} = (x(t + 1) + yt)^2n \). Rearranging gives \( t^{2n}(x + x^{2n} + y + y^{2n}) + (x + x^{2n}) = 0 \). As \( t \notin GF(q) \), we have \( t^{2n} \notin GF(q) \) as \( (n, h) = 1 \), and so it follows that \( x + x^{2n} + y + y^{2n} = 0 \) and \( x + x^{2n} = 0 \). The second equation implies that \( x \in GF(2) \), as \( (n, h) = 1 \). Hence \( x = 0 \) or 1, and then the first equation implies that \( y = 0 \) or 1. Thus the only points of \( \alpha \) which are \( C \)-points are \( \rho(0, 0, 1) = K(0, 0, 1) = A \), \( \rho(0, 1, 1) = C \), \( \rho(1, 0, 1) = D \) and \( \rho(1, 1, 1) = B \). Hence if \( \alpha \) is not a \( C \)-plane, then it contains exactly four \( C \)-points. That is, if a plane contains more than four \( C \)-points, it is a \( C \)-plane, completing the proof of part 4. \( \Box \)

We briefly discuss the interpretation of these properties in the Bruck-Bose representation of \( PG(2, q^2) \) in \( PG(4, q) \). Recall that Baer subplanes of \( PG(2, q^2) \) secant to \( \ell_\infty \) correspond exactly to affine planes of \( PG(4, q) \) that do not contain a line of the spread. So when we consider \( C \)-planes in \( PG(4, q) \), we are interested in affine planes of \( PG(4, q) \). If \( \pi \) is a Baer subplane of \( PG(2, q^2) \) secant to \( \ell_\infty \), and \( K \) is a \( k \)-arc of \( \pi \), then in \( PG(4, q) \), \( K \) is a \( k \)-arc in the affine plane corresponding to \( \pi \). Moreover, if \( K \) is a conic in the Baer subplane \( \pi \) of \( PG(2, q^2) \), then \( K \) corresponds to a conic in \( PG(4, q) \) (see [10]).
4 Proof of Theorem 1.2

For the remainder of this article, we will assume that \( q \) is even, and suppose that in \( \text{PG}(4,q) \) we have a set of \( q^2 \) (affine) \( C \)-points, and a set of \( C \)-planes that satisfy the conditions (A1-4) of Theorem 1.2.

The proof of Theorem 1.2 proceeds as follows. In Section 4.1 we investigate \( C \)-planes, and show that the \( C \)-planes each meet \( \Sigma_\infty \) in one of \( q+1 \) \( C \)-lines which are pairwise skew. In Section 4.2 we study the \( C \)-lines and show that there are two disjoint lines \( t_N, t_\infty \) in \( \Sigma_\infty \) that meet every \( C \)-line. In Section 4.3 we use the Klein correspondence to coordinatising the \( C \)-lines. The remaining proof of Theorem 1.2 in Section 4.4 begins by coordinatising the \( C \)-points. Then we construct a regular spread \( S \) containing the two special lines \( t_N, t_\infty \), so that in the Bruck-Bose plane \( \mathcal{P}(S) \simeq \text{PG}(2,q^2) \), the points corresponding to \( C \) and \( t_N \) and \( t_\infty \) form a translation hyperoval.

4.1 Properties of \( C \)-planes

In this section we begin by showing that the \( C \)-points and \( C \)-planes form an affine plane, and then investigate the parallel classes of this affine plane.

**Lemma 4.1** Consider the incidence structure \( A \) with points the \( C \)-points, and lines the \( C \)-planes, and natural incidence. Then \( A \) is an affine plane of order \( q \).

**Proof** By assumptions (A1) and (A2), \( A \) is a \( 2-(q^2,q,1) \) design, and hence is an affine plane of order \( q \). \( \square \)

**Lemma 4.2** No three \( C \)-points are collinear.

**Proof** Suppose three \( C \)-points \( A, B, C \) are collinear. By (A2) there is a unique \( C \)-plane containing \( A, B \), this \( C \)-plane contains the line \( AB \), and so contains the point \( C \), which contradicts (A1). Hence is it not possible for three \( C \)-points to be collinear. \( \square \)

Consider two distinct \( C \)-planes in \( \text{PG}(4,q) \); they either meet in a line \( \ell \) or in a point \( P \). So there are five possibilities: (a) they meet in an infinite line; (b) they meet in an affine line; (c) they meet in an infinite point; (d) they meet in an affine point \( P \) that is a \( C \)-point; and (e) they meet in an affine point \( P \) that is not a \( C \)-point. Note that as \( q > 2 \), by (A3) cases (b) and (e) cannot occur. We now show that case (c) cannot occur either.

**Lemma 4.3** Two \( C \)-planes cannot meet exactly in a point of \( \Sigma_\infty \).
Proof  Note that if two C-planes $\alpha$ and $\beta$ are in the same parallel class of the affine plane $A$, then they have no affine points in common, and so must meet in a line or a point of $\Sigma_\infty$ (that is, case (a) or (c)). If $\alpha$ and $\beta$ are in different parallel classes of $A$, then as case (b) and (e) cannot occur, they must meet in a point of $\Sigma_\infty$.

Suppose two C-planes $\alpha$ and $\beta$ meet exactly in a point $X \in \Sigma_\infty$, so $\alpha$ and $\beta$ are in the same parallel class of $A$. Let $A$ be any C-point of $\alpha$ and let $B, C$ be two C-points of $\beta$ such that the line $BC$ does not contain $X$ (this is possible as $q > 2$). Consider the 3-space $\Sigma = \langle A, B, C, X \rangle$, note that $\beta \subset \Sigma$. Consider the plane $\pi = \langle A, B, C \rangle$, it contains three C-points and hence by (A4) contains a fourth C-point $D$. Now $D$ does not belong to $\alpha$ as if so, the two lines $AD$ and $BC$ on the plane $\pi$ must meet, and must do so at $\alpha \cap \beta = X$, a contradiction as $BC$ does not contain $X$. Further, $D$ does not belong to $\beta$ as $\pi$ meets $\beta$ in a line which already contains two C-points (and no three C-points are collinear by Lemma 4.2). Now consider the two C-points $A$ and $D$, by (A2) there is a unique C-plane $\gamma$ containing $A,D$. As $\gamma$ meets $\alpha$ in the C-point $A$, and $\alpha$ and $\beta$ are in the same parallel class, $\gamma$ meets $\beta$ in a C-point $E$, say. Now $A,D \in \pi \subset \Sigma$ and $E \in \beta \subset \Sigma$ so $\gamma \subset \Sigma$. So the 3-space $\Sigma$ contains two C-planes $\beta$ and $\gamma$. However, $\gamma, \beta$ are in different parallel classes of $A$, and so meet exactly in the C-point $E$. Hence $\gamma, \beta$ cannot lie in a 3-space. So no two C-planes meet exactly at a point in $\Sigma_\infty$. \qed

Hence, C-planes can meet in one of two ways, and so in the affine plane $A$ defined in Lemma 4.1, we have the following properties about parallel classes.

**Lemma 4.4** 1. Two C-planes lie in the same parallel class of $A$ if and only if their intersection is a line in $\Sigma_\infty$.

2. Two C-planes lie in distinct parallel classes of $A$ if and only if their intersection is exactly a point of $C$.

As a direct consequence, the parallel classes of $A$ allow us to define a set of lines in $\Sigma_\infty$ that lie on C-planes.

**Corollary 4.5** Each parallel class of C-planes meets $\Sigma_\infty$ in a common line, called a C-line. There are $q + 1$ distinct C-lines, and they are pairwise skew.

### 4.2 Properties of C-lines

In this section we investigate properties of the C-lines in $\Sigma_\infty$ defined in Corollary 4.5. In particular, we show that we can construct two lines $t_N, t_\infty$ that meet every C-line.

Let $\alpha$ be a C-plane, then by (A1) the set $Q = \alpha \cap C$ is a $q$-arc. As $q > 2$, $Q$ is contained in a unique oval of $\alpha$, and hence a unique hyperoval $Q^+$, see [11]. The two points of $Q^+ \setminus Q$
are called $\infty$-points (we use this name as we will show in Corollary 4.8 that these two points lie in the hyperplane at infinity $\Sigma_\infty$). The line of the $C$-plane $\alpha$ joining the two $\infty$-points is called the $\infty$-line of $\alpha$ (again, we will show that this line is in $\Sigma_\infty$). Note that the $\infty$-line contains no $C$-points, otherwise this would contradict $Q^+$ being a hyperoval. We also need notation for the $q - 1$ points of the $\infty$-line of $\alpha$ that are not $\infty$-points; these points are called $\ast$-points. It is straightforward to verify the following lemma.

**Lemma 4.6** Let $\alpha$ be a $C$-plane, and let $Q = \alpha \cap C$. Let $X$ be of a point of $\alpha$, with $X \notin Q$. Then

1. If $X$ is a $\infty$-point, then the lines of $\alpha$ through $X$ are $q$ 1-secants and one 0-secant of $Q$.

2. If $X$ is a $\ast$-point, then the lines of $\alpha$ through $X$ are the $\infty$-line, and $\frac{q}{2}$ 2-secants, and $\frac{q}{2}$ 0-secants of $Q$.

3. If $X$ is not on the $\infty$-line, then the lines of $\alpha$ through $X$ are $(\frac{q}{2} - 1)$ 2-secants, two 1-secants (through the $\infty$-points) and $\frac{q}{2}$ 0-secants of $Q$.

We now investigate the $\infty$-points and will show they all lie in $\Sigma_\infty$.

**Lemma 4.7** If two $C$-planes meet in a line, then their $\infty$-points coincide, and lie in $\Sigma_\infty$.

**Proof** Let $\alpha, \beta$ be two $C$-planes that meet in a line $t$. By Lemma 4.3, $t$ is in $\Sigma_\infty$. Suppose that $\alpha$ and $\beta$ do not share $\infty$-points. We construct a 1-secant $\ell$ of $C$ in $\alpha$ and a 2-secant $m$ in $\beta$ as follows. Firstly, if at least one plane, $\alpha$ say, has $\infty$-points in $\Sigma_\infty$, then let $X$ be an $\infty$-point of $\alpha$ which is not an $\infty$-point of $\beta$, and let $\ell$ be a 1-secant of $C \cap \alpha$ through $X$. Secondly, if neither $\alpha$ nor $\beta$ has an $\infty$-point in $\Sigma_\infty$, let $X$ be any point of $\alpha \cap \beta$ not on the $\infty$-lines of $\alpha$ or $\beta$. By Lemma 4.6, we can join $X$ to an $\infty$-point of $\alpha$ to form a 1-secant $\ell$. In both cases, as $q > 2$, again by Lemma 4.6, there exists a 2-secant $m$ in $\beta$ through $X$. Let $A$ be the $C$-point on $\ell$ and let $B$ and $C$ be the $C$-points on $m$.

Consider the plane $\pi = \langle A, B, C \rangle$; it meets $C$ in three points, so by assumption (A4) contains a further $C$-point $D$. Note that $\pi \cap \beta = m$ is a 2-secant of $C$, and $\pi \cap \alpha = \ell$ is a 1-secant, so $D$ does not belong to $\alpha$ or $\beta$. By (A2), there is a $C$-plane $\gamma$ containing $A$ and $D$. As $\gamma$ meets $\alpha$ in a $C$-point, and $\alpha$ and $\beta$ belong to the same parallel class, $\gamma$ also meets $\beta$ in a $C$-point $E$. Consider the 3-space $\Sigma = \langle \pi, E \rangle$. Now $A, B, E$ are three non-collinear points in $\Sigma \cap \gamma$, so $\gamma \subset \Sigma$. Similarly, $B, C, E$ are three non-collinear points in $\Sigma \cap \beta$, so $\beta \subset \Sigma$. But $\beta, \gamma$ have a common $C$-point $E$, so by Lemma 4.4 they meet in exactly $E$ and so cannot lie in a common 3-space, a contradiction. Thus $\alpha$ and $\beta$ must have the same $\infty$-points, and so their $\infty$-points lie in $\Sigma_\infty$, and hence their $\infty$-lines lie in $\Sigma_\infty$. □

By Corollary 4.3 $C$-planes in the same parallel class meet in a line of $\Sigma_\infty$, hence it follows that all the $\infty$-points lie in $\Sigma_\infty$, and the $\infty$-lines are $C$-lines.
Corollary 4.8 All the $\infty$-points and $\infty$-lines lie in $\Sigma_{\infty}$.

We now investigate lines of $\Sigma_{\infty}$ that meet two $C$-lines.

Lemma 4.9 Any line which meets two $C$-lines in $\star$-points meets exactly one other $C$-line, and meets it in a $\star$-point.

Proof Let $\ell$, $m$ be $C$-lines (so by Corollary 4.8 they are $\infty$-lines). Let $X \in \ell$, $Y \in m$ be $\star$-points. By Lemma 4.6 there is a line through $X$ containing two $C$-points $A, B$. By Corollary 4.5 $m$ defines a parallel class, and so $\langle m, A \rangle$ is a $C$-plane in this parallel class. So by Lemma 4.6, the line $AY$ in the $C$-plane $\langle m, A \rangle$ contains another $C$-point, $C$ say. By (A2), $B, C$ lie in a $C$-plane $\gamma$, further $\gamma$ does not belong to the parallel class defined by either $\ell$ or $m$ as $Z = BC \cap \Sigma_{\infty}$ is not in $\ell$ or $m$. So $\gamma$ meets $\Sigma_{\infty}$ in a $C$-line $n$ which by Corollary 4.5 is disjoint from $\ell$ and $m$. Consider the plane $\pi = \langle A, B, C \rangle$. We have $X, Y \in \pi$, so the point $Z$ lies on the line $XY$, that is $Z = XY \cap BC \in n$. By Corollary 4.8 $n$ is the $\infty$-line of $\gamma$, so as $Z$ lies on a 2-secant $BC$ of $C$, by Lemma 4.6 $Z$ is a $\star$-point. By Lemma 4.6 the line $AZ$ contains a further $C$-point, $D$ say, so the plane $\pi$ contains four $C$-points, $A, B, C, D$. Note that $\pi \cap \Sigma_{\infty} = XY$ is not a $C$-line, as $C$-lines are disjoint, so $\pi$ is not a $C$-plane.

So we have shown that $XY = \pi \cap \Sigma_{\infty}$ meets another $C$-line $n$ in a $\star$-point $Z$. Suppose that $XY$ meets a further $C$-line $t$ in a point $W$. Note that $X, Y \in \pi$, so $W \in \pi$. As $B, C, W, A \in \pi$, let $E = BC \cap WA$ and note that $E$ is distinct from $A, B, C, D$. If $E$ is a $C$-point, then $\pi$ contains five $C$-points, $A, B, C, D, E$, contradicting (A4) as $\pi$ is not a $C$-plane. If $E$ is not a $C$-point, then $E$ is on two $C$-planes, namely $\langle A, t \rangle$ and $\langle B, n \rangle$, this contradicts (A3). Hence $XY$ cannot meet four $C$-lines. That is, $XY$ meets exactly one other $C$-line, and it meets it in a $\star$-point. □

By Corollary 4.8 the $\infty$-lines are $C$-lines, and the $\infty$-points lie on the $C$-lines. By Lemma 4.7 $C$-planes in the same parallel class have the same $\infty$-points. By Corollary 4.5 there are $q + 1$ distinct, pairwise disjoint $C$-lines. Hence there are $2(q + 1)$ distinct $\infty$-points. We now show that the $2(q + 1)$ $\infty$-points form two disjoint lines in $\Sigma_{\infty}$.

Lemma 4.10 The $2(q + 1)$ $\infty$-points form two disjoint lines of $\Sigma_{\infty}$. We label these lines $t_{N}$, $t_{\infty}$.

Proof In this proof we are interested in the lines of $\Sigma_{\infty}$ that meet three $C$-lines in $\star$-points. These lines exist by Lemma 4.9, and in this proof, we call these lines $\star$-lines. Let $A, B$ be $\star$-points on distinct $C$-lines. By Lemma 4.9 $AB$ is a $\star$-line, and contains exactly one further $\star$-point, $C$ say. We will count in two ways the number $x$ of ordered triples
(A, B, C) where A, B, C are *-points on a common *-line. Firstly, there are \((q + 1)(q - 1)\) choices for \(A\); then \(q(q-1)\) choices for \(B\), yielding a unique \(C\). So \(x = (q+1)(q-1)q(q-1)\).

Secondly, we count the number of *-lines first. Let \(r_0\) be a *-line, so \(r_0\) meets three \(C\)-lines \(l, m, n\). These three \(C\)-lines determine a unique regulus \(\mathcal{R}\), with opposite regulus \(\mathcal{R}' = \{r_0, \ldots, r_q\}\). Each line of \(\mathcal{R}'\) meets the three \(C\)-lines \(l, m, n\), so by Lemma 1.9 they meet no further \(C\)-line. Further, if any of the lines \(r_i\) contains two *-points, then it contains three *-points, and is a *-line. The maximum number of *-lines in \(\mathcal{R}'\) is \(q - 1\), which occurs when the \(\infty\)-points of \(l, m, n\) all lie on two lines of \(\mathcal{R}'\). So the total number of *-lines is at most \(\left(\frac{q+1}{3}\right) \times (q-1)\). The number of ordered triples of *-points on a *-line is \(3!\). Hence this count gives \(x \leq \left(\frac{q+1}{3}\right) \times (q-1)! = (q+1)q(q-1)^2\). Thus \(x = (q+1)q(q-1)^2\), and for each set of three \(C\)-lines, the \(\infty\)-points lie on two common lines. Hence all the \(\infty\)-points lie on two lines of \(\Sigma_\infty\).

\[\square\]

### 4.3 Coordinatising the \(C\)-lines

In Sections 4.1 and 4.2, we showed that given a set of \(C\)-points and \(C\)-planes in \(PG(4,q)\) satisfying (A1-4), we can construct from them \(q+1\) \(C\)-lines, and two special lines \(t_N\) and \(t_\infty\) in \(\Sigma_\infty\). We now show that without loss of generality, we can coordinatise these special lines of \(\Sigma_\infty\) as outlined in Theorem 4.11.

To determine this coordinatisation, we use the Klein correspondence between lines of \(\Sigma_\infty \cong PG(3,q)\) and points of the Klein quadric \(\mathcal{H}_5\) in \(PG(5,q)\). The line \(PQ\) of \(PG(3,q)\) with \(P = (p_0, p_1, p_2, p_3), Q = (q_0, q_1, q_2, q_3)\) maps to the point \((\ell_01, \ell_02, \ell_03, \ell_12, \ell_23)\) of \(\mathcal{H}_5\) in \(PG(5,q)\) where \(\ell_{ij} = p_iiq_j - p_jiq_i\). The Klein quadric \(\mathcal{H}_5\) has equation \(x_0x_5 + x_1x_4 + x_2x_3 = 0\). For background on the Klein correspondence, see \([8]\).

**Theorem 4.11** In \(PG(4,q)\) we can without loss of generality coordinatise the lines \(t_N\), \(t_\infty\) and the \(C\)-lines \(m_t\), \(t \in GF(q) \cup \{\infty\}\) as follows:

\[
\begin{align*}
t_N &= \langle (0, 0, 1, 0, 0), (0, 0, 0, 1, 0) \rangle, \\
t_\infty &= \langle (1, 0, 0, 0, 0), (0, 1, 0, 0, 0) \rangle, \\
m_t &= \langle (t^{2^n}, 1, 0, 0, 0), (0, 0, t, 1, 0) \rangle, \quad t \in GF(q) \cup \{\infty\}.
\end{align*}
\]

**Proof** In this proof, we use the Klein correspondence between the lines of \(\Sigma_\infty \cong PG(3,q)\) and the points of the Klein quadric \(\mathcal{H}_5\) in \(PG(5,q)\). In \(PG(4,q)\), a point \((x_0, x_1, x_2, x_3, x_4)\) in \(\Sigma_\infty\) has last coordinate \(x_4 = 0\), so we write the coordinates of points in \(\Sigma_\infty\) as \((x_0, x_1, x_2, x_3)\). Let \(A = (1, 0, 0, 0), B = (0, 1, 0, 0), C = (0, 0, 1, 0), D = (0, 0, 0, 1)\) be points of \(\Sigma_\infty\). Without loss of generality we can let \(t_\infty = AB\) and \(t_N = CD\). Consider the lines \(\ell_1 = AC, \ell_2 = AD, \ell_3 = BC, \ell_4 = BD\). In \(\Sigma_\infty\), the \(C\)-lines and \(\ell_1, \ldots, \ell_4\) all meet \(t_N\) and \(t_\infty\), so they lie in the hyperbolic congruence consisting of the \((q+1)^2\) transversals of \(t_N, t_\infty\). This hyperbolic congruence maps under the Klein correspondence to a hyperbolic
quadric $\mathcal{H}_3$ contained in a 3-space $\Sigma$ in $\text{PG}(5, q)$ (see [8]). Hence in $\text{PG}(5, q)$, the $\mathcal{C}$-lines correspond to a set $\mathcal{K}$ of $q + 1$ points that lie on $\mathcal{H}_3$. By Corollary 4.4, the $\mathcal{C}$-lines are pairwise skew, hence in $\text{PG}(5, q)$ the points of $\mathcal{K}$ are such that no two lie on a generator line of $\mathcal{H}_3$. Hence there is exactly one point of $\mathcal{K}$ on each generator of $\mathcal{H}_3$.

Now consider the result of Lemma 4.9. Let $m_1, m_2$ be two $\mathcal{C}$-lines of $\Sigma_\infty$, and let $\ell$ be a line that meets $m_1, m_2$ in $\ast$-points. Then $\ell$ meets exactly one more $\mathcal{C}$-line, $m_3$ say. In $\text{PG}(5, q)$, $\ell$ corresponds to a point $L$ not in $\Sigma$ (the 3-space containing $\mathcal{H}_3$), and the line $m_i$ corresponds to a point $M_i$ in $\mathcal{K}$. Now Lemma 4.9 says that if $LM_1, LM_2$ are lines of the Klein quadric $\mathcal{H}_5$, then there is exactly one more line $LM_3$ of $\mathcal{H}_5$ through $L$ that contains a point $M_3$ of $\mathcal{K}$. Now $\langle L, M_1, M_2, M_3 \rangle$ is a 3-space as $M_1, M_2, M_3$ are not on a generator of $\mathcal{H}_3$. So $\langle L, M_1, M_2, M_3 \rangle$ meets $\mathcal{H}_5$ in a conic cone, and so meets $\Sigma$ in a conic. Conversely, a conic of $\mathcal{H}_3$ will lie on such a conic cone of $\mathcal{H}_5$. Hence any conic of $\mathcal{H}_3$ contains at most three points of $\mathcal{K}$. Hence each plane of $\Sigma$ meets $\mathcal{K}$ in at most three points. Thus $\mathcal{K}$ is a $(q + 1)$-arc of $\Sigma$.

The images of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ of $\Sigma_\infty$ under the Klein correspondence are $L_1 = (0, 1, 0, 0, 0, 0), L_2 = (0, 0, 1, 0, 0, 0), L_3 = (0, 0, 0, 1, 0, 0), L_4 = (0, 0, 0, 0, 1, 0)$. These four points lie in the 3-space $\Sigma$ determined by the equations $x_0 = 0, x_5 = 0$. If we use $(x_1, x_2, x_3, x_4)$ for the coordinates of $\Sigma$, then $\Sigma$ meets $\mathcal{H}_5$ in the hyperbolic quadric of equation $\mathcal{H}_3 : x_1x_4 + x_2x_3 = 0$.

In [8] Section 21.3], a line $\ell$ through a point $P$ of $\mathcal{K}$ is called a special unisecant of $\mathcal{K}$ if every plane through $\ell$ contains at most one other point of $\mathcal{K}$. Further, it is shown that there are exactly two special unisecants through each point $P$ of $\mathcal{K}$, and the special unisecants are the generators of a quadric $\mathcal{H}(\mathcal{K})$. We note that since there is exactly one point of $\mathcal{K}$ on each generator $g$ of $\mathcal{H}_3$, then the $q + 1$ planes through $g$ defined by $g$ and one of $q + 1$ lines in the opposite regulus of $\mathcal{H}_3$ contains at most one point of $\mathcal{K}$. This argument shows that $\mathcal{H}(\mathcal{K}) = \mathcal{H}_3$.

By [8] Theorem 21.3.15, the $(q+1)$-arc $\mathcal{K}$ is projectively equivalent to $\mathcal{C}(2^n) = \{(t^{2^n+1}, t^{2^n}, t, 1) : t \in \text{GF}(q) \cup \{\infty\}\}$ for some $n$ with $(n, h) = 1$. Further, $\mathcal{C}(2^n)$ determines through it special unisecants the hyperbolic quadric $\mathcal{H}(\mathcal{C}(2^n))$ which can be shown to equal $\mathcal{H}_3$ (with equation $x_1x_4 + x_2x_3 = 0$). Thus the homography $\sigma$ mapping $\mathcal{K}$ to $\mathcal{C}(2^n)$ fixes $\mathcal{H}_3$. The homographies of $\text{PGO}_+(6, q)$ that fix the Klein quadric are in 1-1 correspondence with the homographies of $\text{PGL}(4, q)$ that map lines of $\Sigma_\infty$ to lines of $\Sigma_\infty$. Hence $\sigma$ corresponds to a homography of $\Sigma_\infty$ that fixes the hyperbolic congruence consisting of all transversals of $t_N, t_\infty$. So without loss of generality, we can coordinatise the $\mathcal{C}$-lines to be the lines that correspond to points of $\mathcal{C}(2^n)$. Straightforward calculations show that the line $m_t = \langle (t^{2^n}, 1, 0, 0), (0, 0, t, 1) \rangle$ for $t \in \text{GF}(q) \cup \{\infty\}$ corresponds under the Klein map to the point $M_t = (0, t^{2^n+1}, t^{2^n}, t, 1, 0)$ which lies in $\mathcal{C}(2^n)$. Hence we can without loss of generality coordinatise the $\mathcal{C}$-lines and $t_N, t_\infty$ as stated in the theorem. ∎
4.4 Proof of the Main Theorem

In this section we complete the proof of Theorem 1.2.

Proof of Theorem 1.2 Suppose we have a set of \( \mathcal{C} \)-points and \( \mathcal{C} \)-planes in \( \text{PG}(4,q) \) that satisfy (A1-4). Then we have shown that in the hyperplane at infinity \( \Sigma_\infty \), there are two special lines \( t_N, t_\infty \) and \( q + 1 \mathcal{C} \)-lines \( m_t, t \in \text{GF}(q) \cup \{\infty\} \). In Theorem 4.11 we coordinatised these lines. We now coordinatis the \( \mathcal{C} \)-points.

Without loss of generality we can choose \( A = (0,0,0,0,1) \) to be a \( \mathcal{C} \)-point. The plane spanned by \( A \) and the \( \mathcal{C} \)-line \( m_\infty \) is a \( \mathcal{C} \)-plane, that is

\[
\alpha_{(A,\infty)} = \langle A, m_\infty \rangle = \langle (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0) \rangle
\]

is a \( \mathcal{C} \)-plane. Every line through \( A \) in this plane either meets \( t_N \) or \( t_\infty \), or contains exactly one further \( \mathcal{C} \)-point. Points in \( \text{PG}(4,q) \) have coordinates \( (x_0, x_1, x_2, x_3, x_4) \). Points in this plane \( \alpha = \alpha_{(A,\infty)} \) have \( x_1 = x_3 = 0 \), so we use the coordinates \( (x_0, x_2, x_4) \) to represent points of \( \alpha \). With this notation, \( A = (0,0,1) \), and consider the points \( R_N = \alpha \cap t_N = (0,1,0) \), \( R_\infty = \alpha \cap t_\infty = (1,0,0) \). Consider the point \( Q = (1,1,0) \) on the line \( R_NR_\infty \).

The line \( AQ \) consists of points \( A, Q \) and \( (s, s, 1) \), \( s \in \text{GF}(q) \setminus \{0\} \). As \( Q \neq R_N, R_\infty \), there exists some \( s \in \text{GF}(q) \setminus \{0\} \) such that \( B = (s, s, 1) \) is a \( \mathcal{C} \)-point. Hence in PG \( (4,q) \), we can take \( B = (s, 0, 0, 1) \) to be a \( \mathcal{C} \)-point, where \( s \) is some nonzero element of \( \text{GF}(q) \).

We can use \( A \) and \( B \) to calculate the coordinates of the remaining \( \mathcal{C} \)-points and \( \mathcal{C} \)-planes. Firstly, the plane spanned by \( A \) and any \( \mathcal{C} \)-line \( m_t \) is a \( \mathcal{C} \)-plane, similarly for \( B \). This gives us the \( \mathcal{C} \)-planes

\[
\alpha_{(A,t)} = \langle A, m_t \rangle = \langle (0,0,0,0,1), (t^{2^n}, 1, 0, 0, 0), (0,0,1,0) \rangle, \quad t \in \text{GF}(q) \cup \{\infty\},
\]

\[
\alpha_{(B,u)} = \langle B, m_u \rangle = \langle (s,0,0,1), (u^{2^n}, 1, 0, 0, 0), (0,0,u,1) \rangle, \quad u \in \text{GF}(q) \cup \{\infty\}.
\]

Note that

\[
\alpha_{(A,\infty)} = \langle (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0) \rangle,
\]

\[
\alpha_{(B,\infty)} = \langle (s,0,0,1), (u^{2^n}, 1, 0, 0, 0), (0,0,u,1) \rangle.
\]

By (A2), \( A, B \) lie on a unique \( \mathcal{C} \)-plane, which is \( \alpha_{(A,\infty)} = \alpha_{(B,\infty)} \). So this gives coordinates of \( (2q + 1) \mathcal{C} \)-planes. By Lemma 4.11 two distinct \( \mathcal{C} \)-planes meet in a unique \( \mathcal{C} \)-point. The coordinates of the \( q(q - 1) \mathcal{C} \)-points \( P_{(A,t),(B,u)} = \alpha_{(A,t)} \cap \alpha_{(B,u)} \) for \( t, u \in \text{GF}(q), \quad t \neq u \) are

\[
P_{(A,t),(B,u)} = \left( \frac{st^{2^n}}{(t+u)^{2^n}}, \frac{s}{(t+u)^{2^n}}, \frac{st}{t+u}, \frac{s}{u}, 1 \right).
\]

The remaining \( q - 1 \mathcal{C} \)-points are in \( \alpha_{(A,\infty)} \), we calculate their coordinates next. As \( B \in \alpha_{(A,\infty)} \), we first need to calculate some \( \mathcal{C} \)-planes that meet \( \alpha_{(A,\infty)} \). Let \( u \in \text{GF}(q) \setminus \{0\} \) and consider the \( \mathcal{C} \)-point

\[
P_{(A,0),(B,u)} = \left( 0, \frac{s}{u^{2^n}}, 0, \frac{s}{u}, 1 \right)
\]
which lies in the \( \mathcal{C} \)-plane \( \alpha_{(A,0)} \). Consider the \( q \) \( \mathcal{C} \)-planes \( \alpha_{(P,t)} = \langle P_{(A,0),(B,u),m_t} \rangle \) for \( t \in \text{GF}(q) \setminus \{0\} \). These \( q \) \( \mathcal{C} \)-planes each meet \( \alpha_{(A,\infty)} \) in a \( \mathcal{C} \)-point. Hence the \( q-1 \) \( \mathcal{C} \)-points in \( \alpha_{(A,\infty)} \) distinct from \( A \) are

\[
\alpha_{(A,\infty)} \cap \alpha_{(P,t)} = \left( \frac{st^{2n}}{u^{2n}}, 0, \frac{st}{u}, 0, 1 \right),
\]

\( t \in \text{GF}(q) \setminus \{0\} \). Note that we only need one parameter here, let \( \psi = t/u \), then the \( q-1 \) \( \mathcal{C} \)-points are

\[
Q_\psi = (s\psi^{2n}, 0, s\psi, 0, 1),
\]

\( \psi \in \text{GF}(q) \setminus \{0\} \). Hence we have calculated the coordinates of the \( q^2 \) \( \mathcal{C} \)-points.

We show that we can choose a regular spread \( \mathcal{S} \) such that in the corresponding plane \( \mathcal{P}(\mathcal{S}) \cong \text{PG}(2,q^2) \) (under the Bruck-Bose map) the \( \mathcal{C} \)-points form a translation oval.

Let \( \tau \) be a primitive element of \( \text{GF}(q^2) \), and let \( n < h \) be a positive integer such that \( (n,h) = 1 \). Then we can uniquely write \( \tau^{2n} \) as \( \tau^{2n} = a_0 + a_1 \tau \) for some \( a_0, a_1 \in \text{GF}(q) \). Note that as \( (n,h) = 1 \), \( \text{GF}(2^n) \cap \text{GF}(2^h) = \text{GF}(2) \), hence \( a_1 \neq 0 \). Now consider the homography \( \sigma \) of \( \text{PG}(4,q) \) with matrix

\[
M = \begin{pmatrix}
1 & a_0 & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The \( \mathcal{C} \)-points \( P_{(A,t),(B,u),Q_\psi} \) are mapped under \( \sigma \) to

\[
P_{(t,u)} = \sigma(P_{(A,t),(B,u)}) = \left( \frac{s(t^{2n} + a_0)}{(t + u)^{2n}}, \frac{a_1}{(t + u)^{2n}}, \frac{st}{t + u}, \frac{s}{t + u}, 1 \right),
\]

\[
Q_\psi = \sigma(Q_\psi) = (s\psi^{2n}, 0, s\psi, 0, 1).
\]

The \( \mathcal{C} \)-lines \( m_\infty, m_t, t \in \text{GF}(q) \) are mapped under \( \sigma \) to

\[
m_\infty' = \langle (1,0,0,0,0), (0,0,1,0,0) \rangle = m_\infty,
\]

\[
m_t' = \langle (t^{2n} + a_0, a_1, 0, 0, 0), (0,0,t,1,0) \rangle.
\]

Note that the special lines \( t_N, t_\infty \) are fixed by the homography \( \sigma \).

So without loss of generality, we can map the \( \mathcal{C} \)-points, \( \mathcal{C} \)-lines and \( \mathcal{C} \)-planes so that the special lines are \( t_N = \langle (0,0,1,0,0), (0,0,1,0,0) \rangle, t_\infty = \langle (1,0,0,0,0), (0,1,0,0,0) \rangle \); the \( \mathcal{C} \)-lines are \( m_t' = \langle (t^{2n} + a_0, a_1, 0, 0, 0), (0,0,t,1,0) \rangle, t \in \text{GF}(q) \cup \{\infty\} \); and the \( \mathcal{C} \)-points are \( P_{(t,u)}, Q_\psi, t, u, \psi \in \text{GF}(q), t \neq u, \psi \neq 0 \). We now look at this in the Bruck-Bose plane \( \text{PG}(2,q^2) \) using the regular spread \( \mathcal{S} \) coordinatised in Section 2. Note that the primitive element \( \tau \) used to extend \( \text{GF}(q) \) to \( \text{GF}(q^2) \) in this coordinatisation is the same.
as the $\tau$ used to determine $a_0, a_1$ in the matrix $M$ (so $\tau^{2^n} = a_0 + a_1\tau$). Also note that $t_N, t_\infty$ are two lines of this regular spread, since using the notation of Section 2 we have $t_N = p_\infty$ and $t_\infty = p_0$. Using this regular spread $S$, we look in the related Bruck-Bose plane $P(S) \cong PG(2, q^2)$ and show that the points of $C$ correspond to a translation oval in $PG(2, q^2)$.

Under the Bruck-Bose map, in $PG(2, q^2)$, the point $P_{t,u}$ corresponds to the point

$$P_{t,u} = \left( \frac{s(t^{2^n} + a_0)}{(t + u)^{2^n}} + \frac{s a_1}{(t + u)^{2^n}}\tau, \frac{st}{t + u} + \frac{s}{t + u}\tau, 1 \right).$$

Let

$$\theta = \theta(t, u) = \frac{t}{t + u} + \frac{1}{t + u}\tau,$$

then

$$\{P_{t,u} : t, u \in GF(q), t \neq u\} = \{(s\theta^{2^n}, s\theta, 1) : \theta \in GF(q^2) \setminus GF(q)\}.$$ 

The point $Q_\psi, \psi \in GF(q) \setminus \{0\}$ of $PG(4, q)$ corresponds in $PG(2, q^2)$ to the point $Q_\psi = (s\psi^{2^n}, s\psi, 1)$. The spread lines $t_\infty$ and $t_N$ correspond in $PG(2, q^2)$ to the points $P_\infty = (1, 0, 0)$ and $N = (0, 1, 0)$ respectively on the line at infinity $\ell_\infty$. Consider the set of points $K$ in $PG(2, q^2)$ consisting of the images of $P_{t,u}, Q_{t,u}, P_\infty, N$, so

$$K = \{(s\theta^{2^n}, s\theta, 1) : \theta \in GF(q^2)\} \cup \{(1, 0, 0)\} \cup \{(0, 1, 0)\}.$$ 

Then $K$ is the image of the translation oval $O(2^n)$ with nucleus $N(0, 1, 0)$ under the projectivity with matrix

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Hence if we have a set of $C$-points and $C$-planes satisfying (A1-4), then they arise from a translation oval in $PG(2, q^2)$. This completes the proof of Theorem 1.2. □

5 Discussion

In the case when $q$ is odd, (see [3]) the assumptions of Theorem 1.1 allow us to construct a unique spread $S$, such that in the Bruck-Bose plane $P(S) \cong PG(2, q^2)$ the $C$-points, together with $P_\infty$, form a conic. In the case when $q$ is even, the assumptions of Theorem 1.2 do not lead to a unique spread. We have shown that there exists a regular spread $S$ in $\Sigma_\infty$ such that in the Bruck-Bose plane $P(S) \cong PG(2, q^2)$, the $C$-points together with the point at infinity $P_\infty$ form a translation oval. It is possible that there are other spreads $S'$ for which the $C$-points in the Bruck-Bose plane $P(S')$ form an arc. If there is such a spread $S'$, we show it must contain $t_N$ and $t_\infty$, and that the remaining lines of $S'$ each meet exactly one $C$-line.
Lemma 5.1 Let $S'$ be any spread of $\Sigma_\infty$ containing $t_N$ and $t_\infty$. Then in the Bruck-Bose plane $\mathcal{P}(S')$, the set $\mathcal{C} \cup \{P_\infty, N\}$ is a hyperoval if and only if each spread line (other than $t_N$ and $t_\infty$) meets exactly one $\mathcal{C}$-line.

Proof Let $S'$ be a spread of $\Sigma_\infty$ containing $t_N$ and $t_\infty$ such that each spread line (other than $t_N$ and $t_\infty$) meets exactly one $\mathcal{C}$-line. We show that $\mathcal{C} \cup \{P_\infty, N\}$ is a hyperoval in the Bruck-Bose plane $\mathcal{P}(S')$. If an affine plane $\pi$ of $\text{PG}(4, q)$ meets $\mathcal{C}$ in at least three points, $A, B, C$, then we use a very similar argument to the proof of Lemma 4.9 to find a fourth point $\mathcal{C}$-point $D$ such that the diagonal points of the quadrangle $ABCD$ are $\star$-points. Hence a plane that meets $\mathcal{C}$ in more than two points is either a $\mathcal{C}$-plane or meets three $\mathcal{C}$-lines. Thus every plane through a line of $S'$ (other than $t_N$ and $t_\infty$) contains at most two $\mathcal{C}$-points. Note that the affine planes through $t_N$ and $t_\infty$ all contain exactly one $\mathcal{C}$-point by Lemma 4.6, as these lines contain only $\infty$-points. Hence in the Bruck-Bose plane $\mathcal{P}(S')$, every line meets $\mathcal{C} \cup \{N, P_\infty\}$ in at most two points, and so it is a hyperoval.

Conversely, consider a spread $S'$ containing $t_N$ and $t_\infty$. If in the Bruck-Bose plane $\mathcal{P}(S')$, the set $\mathcal{C} \cup \{P_\infty, N\}$ is a hyperoval, then in $\text{PG}(4, q)$, every affine plane through a spread line contains at most two $\mathcal{C}$-points. So if a spread line (other than $t_N$ and $t_\infty$) meets two $\mathcal{C}$-lines, then by Lemma 4.9 and its proof, we can construct a plane through the spread line with four $\mathcal{C}$-points, contradicting $\mathcal{C} \cup \{N, P_\infty\}$ being a hyperoval. Thus each spread line (other than $t_N$ and $t_\infty$) meets exactly one $\mathcal{C}$-line. □

Note that we can create a spread satisfying Lemma 5.1 by considering derivation. See [2] for details on derivation and its representation in the Bruck-Bose correspondence. Let $\mathcal{C}$ be a conic of $\text{PG}(2, q^2)$, $q = 2^h$, meeting $\ell_\infty$ in a point $P_\infty$ with nucleus $N \in \ell_\infty$. Let $h$ be even, and let $\mathcal{D}$ be a derivation set (that is, a Baer subline of $\ell_\infty$) not containing $N$ or $P_\infty$, and such that $P_\infty$ and $N$ are conjugate points with respect to $\mathcal{D}$. In [7] it was shown that upon deriving with respect to $\mathcal{D}$, the points of $\mathcal{C}$ correspond to a translation oval in the derived plane $\mathcal{H}(q^2)$ (the Hall plane). In $\text{PG}(4, q)$, the derivation set $\mathcal{D}$ corresponds to a regulus $\mathcal{R}$ of the regular spread $\mathcal{S}$. Let $\mathcal{R}'$ be the opposite regulus of $\mathcal{R}$. Then the spread $S' = S \backslash \mathcal{R} \cup \mathcal{R}'$ corresponds (via the Bruck-Bose map) to the derived plane $\mathcal{P}(S') = \mathcal{H}(q^2)$. By [7], the $\mathcal{C}$-points with $P_\infty$ form a translation oval in this plane. We conject that the only spread $S'$ in $\Sigma_\infty$ for which the $\mathcal{C}$-points form a $q^2$-arc in the Bruck-Bose plane $\mathcal{P}(S')$ are those obtained from the regular spread $\mathcal{S}$ by reversing one or more disjoint reguli not containing $t_N$ or $t_\infty$.

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