Nonlinearity
Macroscopic non-uniqueness and limits of Hamiltonian dynamics

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Abstract
We construct explicit examples of spontaneous energy generation and non-uniqueness for the compressible Euler system, with and without pressure, by taking limits of Hamiltonian dynamics as the number of molecules increases to infinity. The examples come from rescalings of well-posed, deterministic systems of molecules that either collide elastically or interact via singular pair potentials.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Non-uniqueness for weak solutions of hydrodynamic equations is well known. Examples include the construction of Scheffer [Sch] and Shnirelman [Sh] of non-trivial weak solutions of (incompressible, two-dimensional) Euler equations with compact time and space support, and the work by De Lellis and Székelyhidi [dLS] showing that non-uniqueness (of the incompressible and compressible Euler equations in dimension greater or equal to two) persists even under ‘admissibility’ conditions. [D] is a standard reference on the non-uniqueness of weak solutions of hyperbolic conservation laws in general.

In an attempt to investigate the origin of this behavior, we adopt here the point of view that hydrodynamic equations are the result of averaging microscopic evolution equations (see [M], p 81, and [B], Part I, section 20) to construct explicit examples of spontaneous macroscopic energy generation and non-uniqueness for the compressible Euler system, with and without
pressure, as limits of Hamiltonian dynamics. Our examples are rescaled limits of well-posed, deterministic systems of molecules that either collide elastically or interact via rescaled, singular pair interaction potentials, at the limit of infinitely many molecules, see Morrey’s work [Mor]. For each moment \( t \) and finite \( N \), the positions and velocities of the molecules define the probability measure \( M_t^{(N)}(dx, dv) := \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k, v_k)}(dx, dv) \). In all examples here, the sequence \( M_t^{(N)} \) converges weakly as \( N \to \infty \). Denoting by \( M_t \), this weak limit measure, for each \( t \) and \( x \) the macroscopic density is given by the first marginal of \( M_t \), and the macroscopic velocity by the barycentric projection of \( M_t \) at \( x \) with respect to this marginal.

The first part of this article, consisting of sections 3 and 4, is centered on an example showing spontaneous generation of macroscopic velocity. The microscopic systems start with groups of motionless molecules and a single molecule, macroscopically undetectable, initially at a sufficiently large distance from the group, moving towards the group. Macroscopically, the limit of these flows describes a line segment in \( \mathbb{R}^2 \), staying at rest for \( t \in (-\infty, 0] \), which splits into two equal parts moving away from each other with velocities \( \pm v \) as soon as \( t \) becomes positive. The macroscopic velocity and the macroscopic density from \( M_t \) turn out to be a weak solution of the two-dimensional (2D) pressureless Euler for all \( t \) in \( \mathbb{R} \). This solution is macroscopically as ‘inadmissible’ as those of Scheffer and Shnirelman in that kinetic energy is spontaneously created at \( t_0 \). (Microscopically, total energy is, of course, conserved.) As Hamiltonian flows are time reversible, in section 4 the flows \( M_t^{(N)} \) are reversed to produce a solution to the 2D pressureless Euler that does decrease energy. When compared to an elementary transverse flow, this provides an example of non-uniqueness for the pressureless Euler under the admissibility condition of non-increasing energy.

In the second part of this article, section 5 provides an interpretation, via a microscopic derivation, of the well known non-uniqueness of the Cauchy problem for the one-dimensional (1D) Euler system. We show how the moment equations derived from the transport equation

\[
\frac{\partial}{\partial t} M_t + v \frac{\partial}{\partial x} M_t = 0
\]  

(1.1)
can result in the 1D Euler system. The main point here is that two flows of probability measures solving the same transport equation, even if their moments coincide at \( t = 0 \), in general will not have identical moments for all later times. Indeed, we construct two limit measures \( M_t \) and \( \tilde{M}_t \) both solving (1.1) and resulting in the 1D Euler system. At \( t = 0 \), both \( M_t \) and \( \tilde{M}_t \) give the same macroscopic density, velocity, and pressure. Macroscopically, the solutions produced by \( M_t \) and \( \tilde{M}_t \) can be pictured as a segment of two and three, respectively, layers on top of each other moving freely, see figures 11 and 13. The solutions in section 5 are surrounded by vacuum (zero density).

2. Preliminaries and notation

2.1. Measure theory

Recall that a sequence of finite measures \( M_n(dx) \) converges weakly to a finite measure \( M(dx) \) if for any \( f(x) \) continuous and bounded \( \int f(x)M_n(dx) \to \int f(x)M(dx), n \to \infty \). We then write \( M_n \Rightarrow M \).

For \( f : X \to Y \) measurable and \( M \) a probability measure on \( X \) the push-forward measure \( f_#\mu \) of \( f \) on \( Y \) (the distribution measure of the random variable \( f \)) is \( (f_#\mu)(B) = M(f^{-1}(B)) \). We often write \( f_#\mu \) for this push-forward.

If \( M \) is on \( \mathbb{R}^{2d} \) its first marginal will be \( \pi_1 \) for \( \pi_1 : \mathbb{R}^{2d} \to \mathbb{R}^d, \pi_1(x, v) = x \).

\[ \int M_n(dv)\mu(dx) \] is a shorthand for the measure \( f \to \int (f f(x, v)M_n(dv)) \mu(dx) \).
The disintegration of $M(dx, dv)$ with respect to its first marginal $\mu(dx)$ is the unique, up to a $\mu$-measure 0, family $M_\delta(dx)$ such that $M(dx, dv) = \int M_\delta(dx) \mu(dx)$. The barycentric projection of this disintegration is $\nabla(x) = \int \nabla M_\delta(dx) \mu(dx)$ for $x$ in the support of $\mu$, $\nabla = 0$ otherwise. For details see [AGS], section 5.3, or [DJX], section 3.1.

2.2. Finite systems

A system of $N$ molecules in $\mathbb{R}^d$ will be described by the positions and velocities of the molecules, $(x_k(t), u_k(t)), 1 \leq k \leq N$, evolving via Hamiltonian dynamics with pairwise interaction $\Phi_\sigma(r)$ of finite range $\sigma$:

$$\frac{d}{dt}x_k(t) = u_k(t),$$
$$\frac{d}{dt}u_k(t) = -\frac{1}{N} \sum_{j \neq k}^N \Phi'_\sigma(|x_k(t) - x_j(t)|) \frac{x_k(t) - x_j(t)}{|x_k(t) - x_j(t)|}. \quad (2.1)$$

Following Morrey [Mor], we shall take $\Phi_\sigma(r) = \Phi \left( \frac{r}{\sigma} \right)$ for some $\Phi : (0, \infty) \to [0, \infty)$ satisfying:

$$\lim_{r \to 0} \Phi(r) = +\infty, \quad \Phi' \leq 0, \quad \Phi'' \geq 0, \quad \Phi(r) \neq 0 \iff 0 < r < 1. \quad (2.2)$$

For each $N$, suppose that a system $(x_k^{(N)}(t), u_k^{(N)}(t)), k = 1, \ldots, N$ evolves according to (2.1). Of central importance will be the corresponding $t$-family of probability measures on $\mathbb{R}^{2d}$:

$$M_t^{(N)}(dx, dv) := \frac{1}{N} \sum_{k=1}^N \delta(x_k^{(N)}(t), u_k^{(N)}(t)) (dx, dv), \quad t \geq 0, \text{ or } t \in \mathbb{R}. \quad (2.3)$$

When $M_t^{(N)}$ converges weakly to some $M_t$, it is crucial to note that the empirical measure formed by neglecting a single molecule converges weakly to the same $M_t$. In fact, neglecting $o(N)$ number of molecules has the same effect. In this sense, any single molecule is macroscopically invisible. The construction in section 3 relies heavily on this observation.

3. Spontaneous macroscopic velocity generation from Hamiltonian dynamics

This section presents an example of a microscopic Hamiltonian flow with macroscopic limit, as $N \to \infty$, that shows spontaneous velocity generation. The microscopic systems start with groups of motionless molecules and a single molecule, initially at a sufficiently large distance from the group, moving towards the group with large velocity. For $t < 0$, as $N \to \infty$, the moving molecule is invisible and the macroscopic system is motionless. However, as the moving molecule starts interacting with the group at $t = 0$, its energy is transferred to the rest of the system in such a way that all other molecules acquire speed 1 to create macroscopic velocity for $t > 0$.

There are similarities here with Lanford [L], pp 50–53, although Lanford works with an infinite system of hard balls that always remains discrete, rather than the limit of finite Hamiltonian systems with interaction, and he does not obtain hydrodynamic equations.

Throughout this section we use $Q_t$ for the segment

$$\{(x, y) : 0 \leq x \leq 1, y = t\} \subset \mathbb{R}^2 \quad (3.1)$$

and $\Delta_t(dx)$ for the normalized 1-dimensional Lebesgue measure on $Q_t$. 

4456
Theorem 3.1. For each $N \in \mathbb{N}$, there exists $\sigma_N > 0$ and $(x_k^{(N)}(t), u_k^{(N)}(t))$, $k = 1, \ldots, N$, solution of the Hamiltonian system (2.1) with interaction $\Phi_{\sigma_N}$ for all $t \in \mathbb{R}$, such that for all $t \in \mathbb{R}$, the sequence of empirical measures

$$M_t^{(N)}(dx, dv) := \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k^{(N)}(t), u_k^{(N)}(t))}(dx, dv)$$

(3.2)

converges weakly, as $N \to \infty$, to

$$M_t(dx, dv) = \begin{cases} 
\Delta_0(dx) \otimes \delta_{(0,0)}(dv) & t \leq 0 \\
\frac{1}{2} \Delta_+ (dx) \otimes \delta_{(0,1)}(dv) + \frac{1}{2} \Delta_- (dx) \otimes \delta_{(0,-1)}(dv) & t > 0.
\end{cases}$$

(3.3)

The proof of this theorem occupies the rest of this section. For the moment, note that the first marginal (macroscopic density) of $M_t(dx, dv)$ in (3.3) is

$$\mu_t(dx) = \begin{cases} 
\Delta_0(dx) & t \leq 0 \\
\frac{1}{2} \Delta_+ (dx) + \frac{1}{2} \Delta_- (dx) & t > 0.
\end{cases}$$

(3.4)

If we disintegrate

$$M_t(dx, dv) = \int M_{t,x}(dv) \mu_t(dx)$$

(3.5)

then, for $\chi_Q$ the characteristic function of the set $Q$,

$$M_{t,x}(dv) = \begin{cases} 
\chi_Q(x) \delta_{(0,0)}(dv) & t \leq 0 \\
\chi_Q(x) \delta_{(0,1)}(dv) + \chi_{Q_-}(x) \delta_{(0,-1)}(dv) & t > 0.
\end{cases}$$

(3.6)

Notice that in (3.6) we only needed to specify $M_{t,x}(dv)$ for $x$ in the support of $\mu_t(dx)$. The macroscopic velocity is the barycentric projection of this disintegration:

$$u(t, x) := \int_{\mathbb{R}^2} v M_{t,x}(dv) = \begin{cases} 
(0,0) & t \leq 0 \\
\chi_Q(x) \cdot (0,1) + \chi_{Q_-}(x) \cdot (0,-1) & t > 0.
\end{cases}$$

(3.7)

The macroscopic density (3.4) and velocity (3.7) show clearly a macroscopic velocity generation (see figure 1): before $t = 0$, the macroscopic system stays at rest, while, starting at $t = 0$, two equal mass fronts split and move away from each other with velocity $\pm 1$. The sudden increase of macroscopic kinetic energy, of course, comes from interaction with an invisible molecule as we will see in the proof of theorem 3.1 (sections 3.1–3.3). In section 3.4 we examine the macroscopic hydrodynamic equation solved by the density (3.4) and velocity (3.7).
3.1. Interaction with one particle at rest

Start with two identical molecules $P, Q$ interacting with potential $\Phi_\sigma$ as in (2.1). Denote the positions and velocities of $P, Q$ as $x_P = (x_P, y_P), x_Q = (x_Q, y_Q), v_P,$ and $v_Q.$ Consulting figure 2, let $D$ be the disc with center $(x_0, y_0)$ and radius $r > 0$ and assume that at $t = 0$

1. $(x_P, y_P) \in D$ and $x_Q = x_0 + d$ with $d > r + \sigma,$ i.e. $P$ is inside $D$ and $Q$ is on the vertical line $x = x_0 + d.$
2. $v_P = v(\cos \phi, \sin \phi)$ with $-\frac{\pi}{2} < \phi < \frac{\pi}{2},$ $v > 0$ and $v_Q = (0, 0),$ i.e. $P$ moves with speed $v$ and $Q$ is at rest.

We say that there is interaction between $P$ and $Q$ whenever their distance is smaller than $\sigma.$ Since $Q$ is at rest at $t = 0,$ there will be no interaction between $P$ and $Q$ as long as $P$ is inside $D.$ The following lemma on the interaction between $P$ and $Q$ is the building block of the rest of this section.

**Lemma 3.2.** Let $P, Q$ be as above:

1. For any $\theta$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ there exists $y_Q$ such that $P$ and $Q$ will eventually interact (i.e. $P$ and $Q$ will interact at some time $t > 0$), and after interaction $P$ and $Q$ will move in directions perpendicular to each other with constant velocities $v'_P = v \cos \theta (\cos \phi', \sin \phi')$ and $v'_Q = v \sin \theta (\sin \phi', -\cos \phi'),$ respectively, where $\phi' = \phi + \theta.$
2. If interaction takes place then $y_Q$ satisfies

   $|y_Q - (y_0 + d \tan \phi)| < \frac{r + \sigma}{\cos \phi}.$

3. Whenever $P$ and $Q$ interact, they are both inside the disc with center $(x_0 + d, y_0 + d \tan \phi)$ and radius $\frac{r + \sigma}{\cos \phi} + 5\sigma.$

**Figure 2.** The initial disc $D$ and the segment $I$ for $\phi < 0.$
Proof.

(1) Consulting figure 3 (which is similar to [LL]’s figure 17, p 47), for θ the deflection angle from $v_P$ to $v'_P$, conservation of momentum and energy gives the formulas of $v'_P$ and $v'_Q$.

That any θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ is attained by some $y_Q$ follows from corollary A.3 in the appendix and the formulas in [LL], section 13 that show how to transform from motion in a central field to a system of two molecules.

(2) Let $S$ be the strip between the two lines tangent to $D$ and parallel to $v_P$, $S_\sigma$ the set of all points with distance smaller than $\sigma$ from $S$, and $I$ the interval of intersection of $S_\sigma$ with the line $x = x_0 + d$, see figure 2. Then if $Q$ has second coordinate anywhere out of $I$, $P$ ignores it and continues with unaltered velocity $v_P$. Elementary geometry shows that $I$ has midpoint $y_0 + d \tan \phi$ and half-length $\frac{r + \sigma}{\cos \phi}$, consult figure 2.

(3) By lemma A.1 in the appendix, when $P$ and $Q$ interact, their interaction time is less than $\frac{4\sigma}{v}$ and by conservation of energy ($\Phi$ is positive) the speed of $Q$ will never be more than $v$ during interaction. Therefore, during interaction $Q$ travels less than $4\sigma$, i.e. it stays in the disc centered at $(x_0 + d, y_0 + d \tan \phi)$ with radius $\frac{r + \sigma}{\cos \phi} + 4\sigma$. As the distance between $P$ and $Q$ is always less than $\sigma$ during interaction, $P$ is always inside the circle centered at $(x_0 + d, y_0 + d \tan \phi)$ with radius $\frac{r + \sigma}{\cos \phi} + 5\sigma$. □

3.2. A system of molecules on the plane

We describe now a system consisting of $N + 1$ molecules $P, Q_k, k = 1, \ldots, N$ where $P$ interacts (only once) with each $Q_k$ (in the order of increasing $k$) and interactions are independent ($P$ does not interact with $Q_j, j \neq k$, when interacting with $Q_k$, and there is no interaction between the $Q_k$’s). In addition, the moment before interacting with $Q_k$ the speed of $P$ will be greater than 1 and the speed of $Q_k$ after interaction will be 1.

We use $\theta_k$ for the deflection angle of $P$ due to the interaction with $Q_k$. Assume that before interacting with $Q_k$, $P$ moves along the x-axis. Then $\phi_k = \sum_{j=1}^{k-1} \theta_j$ will be the angle from the x-axis to the direction of the velocity of $P$ right after its interaction with $Q_k$. The angle from the x-axis to the direction of the velocity of $Q_k$ after its interaction with $P$ will be denoted by $\hat{\phi}_k$. By figure 3, $\hat{\phi}_k = (-1)^{k+1} \frac{\pi}{2} + \phi_k$. 

Figure 3. The two possible deflection triangles for given $|v'_Q|$. 

Nonlinearity 30 (2017) 4454

S Dostoglou and J Xue
Lemma 3.3. For $N \in \mathbb{N}$ fixed and $k = 1, 2, \ldots, N$, let
\[
\theta_k = (-1)^k \arcsin \frac{1}{\sqrt{N + 2 - 2k}}, \quad \phi_k = \sum_{j=1}^{k} \theta_j.
\] (3.8)

Then
(1) $\phi_k < 0$, when $k$ is odd and $\phi_k > 0$, when $k$ is even,
(2) $|\phi_k| < |\phi_{k+2}|$.
(3) $|\phi_1| = |\theta_1| \leq \frac{\pi}{4}$ and $|\phi_k| < |\theta_k| \leq \frac{\pi}{4}$, for $k > 1$.

Proof of Lemma 3.3. For (1), observe that the $\theta_k$s start negative, increase strictly in absolute value and alternate sign. Therefore for $k$ odd and $k > 1$
\[
\phi_k = \theta_1 + (\theta_2 + \theta_3) + \ldots + (\theta_{k-1} + \theta_k) < \theta_1 < 0.
\] (3.9)

whereas for $k$ even
\[
\phi_k = (\theta_1 + \theta_2) + \ldots + (\theta_{k-1} + \theta_k) > 0.
\] (3.10)

For (2), notice that $\theta_{k+1} + \theta_{k+2}$ always has the same sign as $\phi_k$, hence
\[
|\phi_{k+2}| = |\phi_k| + |\theta_{k+1} + \theta_{k+2}| > |\phi_k|.
\] (3.11)

(3) For $1 \leq k \leq N$
For odd $k$ and $k > 1$

$$|\phi_k| = -\phi_k = -\phi_{k-1} - \theta_k < -\theta_k \leq \frac{\pi}{4}.$$  

(3.13)

whereas for even $k$

$$|\phi_k| = \phi_k = \phi_{k-1} + \theta_k < \theta_k \leq \frac{\pi}{4}.$$  

(3.14)

Lemma 3.3 shows that the even $\phi_k$ s are positive, increasing, and never more than $\pi/4$ (and therefore the even $\hat{\phi}_k$ s are negative, increasing, and never more than $-\pi/4$), whereas the odd $\phi_k$ s are negative, decreasing, and never less than $-\pi/4$ (and therefore the odd $\hat{\phi}_k$ s are positive, decreasing, and never less than $\pi/4$). Figure 4 summarizes the behavior of $\phi_k$ and $\hat{\phi}_k$.

In the description of the interaction of $P$ and $Q_k$, for $\phi_j$ as in (3.8) (assuming $\phi_0 = 0$), the point

$$\left( x_k, y_k \right) = \left( \frac{k}{N}, 1 - \frac{1}{N} \sum_{j=0}^{k-1} \tan \phi_j \right)$$  

(3.15)

will play the same role as $(x_0 + d, y_0 + d \tan \phi)$ in lemma 3.2. The segments and half-lines

$$P_k := \{(x, y) : x_k \leq x \leq x_{k+1}, y = (x - x_k) \tan \phi_k + y_k \}, \quad k = 1, \ldots, N-1,$$

$$P_N := \{(x, y) : x_N \leq x, y = (x - x_N) \tan \phi_N + y_N \},$$

$$Q_k := \{(x, y) : x \geq x_k, y = (x - x_k) \tan \hat{\phi}_k + y_k \}, \quad k = 1, \ldots, N,$$  

(3.16)

will be useful in describing the trajectories of $P$ and of each $Q_k$, respectively, see figure 5.

Define the distance between any two of these sets as

$$d(A, B) := \inf \{||a - b|| : a \in A, b \in B\}.$$  

(3.17)

Lemma 3.4. Let $m, n = 1, 2, \ldots, N$. For $Q_m, P_n$ as above,

$$d(Q_m, Q_n) > \frac{1}{N}, \quad m \neq n,$$

$$d(Q_m, P_n) > \frac{1}{N}, \quad m < n.$$  

(3.18)

Proof of Lemma 3.4. Recalling (3.16), we use here ‘right half plane of $Q_n$’ to mean the half-plane to the right of the $y$-axis defined by: $Q_n$ is the positive $y$-axis when $\hat{\phi}_n$ is positive; $Q_n$ is the negative $y$-axis when $\hat{\phi}_n$ is negative.

Observe first that for any fixed $n$, the point $(x_n, y_n)$ is always in the right half plane of $Q_n$ for all $m < n$: this holds by the relation of the angles $\phi_i$ to the angles $\hat{\phi}_j$, see figures 4 and 5.

To get the first estimate in (3.18), it suffices to consider $n > m$. If $n - m$ is even, then the
angle of $Q_n$ (i.e. $\hat{\phi}_n$) is of smaller absolute value than the angle of $Q_m$ (i.e. $\hat{\phi}_m$). If $n - m$ is odd, then the angles of $Q_m$ and $Q_n$ differ by more than $\pi/2$. In either case the point on $Q_n$ closest to $Q_m$ is $(x_n, y_n)$.

Similarly, the angle of $P_n$ (i.e. $\phi_n$), is always of absolute value smaller than the angle of any $Q_m$ (i.e. $\hat{\phi}_m$). Therefore the point on $P_n$ closest to $Q_m$, for $m < n$, is $(x_n, y_n)$.

Now it suffices to notice that the distance from $(x_n, y_n)$ to each $Q_m$ is greater or equal to $|P_m|$ which is clearly bigger than $\frac{1}{N}$ (consult figures 4 and 5). □

We are now ready to establish the evolution of $P, Q_1, \ldots, Q_N$.

**Proposition 3.5.** For each $N \in \mathbb{N}$ and $\Sigma_N < \frac{1}{\sqrt{2N(N+1)}}$, consider the system $P, Q_1, \ldots, Q_N$ with interaction $\Phi_{\Sigma_N}$. Then there exist $y_{Q_k}$’s, $k = 1, \ldots, N$, such that the system evolves as follows: for $t \leq 0$,

$$P(t) = t v_P, \quad v_P(t) = \left(\sqrt{N+1}, 0\right),$$

$$Q_k(t) = \left(\frac{k}{N}, y_{Q_k}\right), \quad v_{Q_k}(t) = (0, 0), \quad k = 1, \ldots, N,$$

and for $t > 0$,

1. there exist times $0 < t'_1 < t'_2 < t'_3 < \ldots < t'_N$ such that for any $1 \leq k \leq N$,
   - $P$ starts to interact with $Q_k$ at $t = t'_k$ and completes this interaction at $t = t''_k$;
2. for any $1 \leq k \leq N$, the molecule $Q_k$ does not interact with any other molecule for $t < t'_k$ or $t > t''_k$ and its velocity is given by

$$v_{Q_k}(t) = \begin{cases} 
(0, 0) & t < t'_k \\
(\sin|\phi_k|, (-1)^{k+1}\cos \phi_k) & t > t''_k,
\end{cases}$$

for $\phi_k$ as in (3.8);
(3) the velocity of $P$ satisfies

\[ v_P(t) = \begin{cases} 
(\sqrt{N+1}, 0) & t \leq t'_0 \\
\sqrt{N+1-k} (\cos \phi_k, \sin \phi_k) & t'_0 < t \leq t'_{k+1}, k = 1, 2, \ldots, N-1 \\
(\cos \phi_N, \sin \phi_N) & t \geq t''_N,
\end{cases} \tag{3.21} \]

for $\phi_k$ as in (3.8);

(4) during the time interval $[t'_k, t''_k]$ for $1 \leq k \leq N$ the molecules $P$ and $Q_k$ are in the disc of center $(x_k, y_k)$ as in (3.15) and radius given recursively by

\[ r_k = \frac{r_{k-1} + \sigma_N}{\cos \phi_{k-1}} + 5\sigma_N, \quad r_0 = 0. \tag{3.22} \]

In particular,

\[ r_k < 2\sqrt{2}(N + 3)^{3/2}\sigma_N. \tag{3.23} \]

**Proof of proposition 3.5.** For all $t \leq 0$ and any choice of $y_{Q_k}$, $k = 1, \ldots, N$, take $v_P(t) = (\sqrt{N+1}, 0)$, $P(t) = tv_P$, $Q_k(t) = (\frac{1}{r_k}, y_{Q_k})$, and $v_{Q_k}(t) = (0, 0)$. For all $\sigma_N < \frac{1}{8}$, it is clear that $P$, $Q_1, \ldots, Q_N$ solve the Hamiltonian system for $t \leq 0$ (as there is no interaction). We now specify $y_{Q_k}$ for the evolution for $t > 0$.

Applying lemma 3.2 for $x_0 = y_0 = 0$, $r = 0$, $\phi = 0$, $\nu = \sqrt{N+1}$, $d = 1/N$ and $\theta = \theta_1 = \phi_1 = -\arcsin(1/\sqrt{N+1})$ there is $y_{Q_1}$ such that $P$ will interact with $Q_1$ and after interaction

\[ v_P = \sqrt{N} (\cos \phi_1, \sin \phi_1), \quad v_{Q_1} = (-\sin \phi_1, \cos \phi_1). \tag{3.24} \]

In this way, the position of $Q_1$, depending on $\sigma_N$, is now determined. The whole interaction, according to lemma 3.2, takes place in the disc of radius $r_1 = 6\sigma_N$ and center $(1/N, 0)$. Let $[t'_1, t''_1]$ be the time interval of this interaction. Preparing for the next interaction, make a new choice of $\sigma_N$ so that $r_1 = 6\sigma_N < 1/N$, and note that everything in this first step still holds for the new choice of $\sigma_N$.

For induction, fix $k \in \mathbb{N}$ and assume that $r_1, \ldots, r_k$ satisfy (3.22), and therefore $r_j < 1/N$, $j = 1, \ldots, k$, for all $\sigma_N$ small enough. Further assume that $y_{Q_1}, \ldots, y_{Q_k}, t'_j, t''_j, \nu_j$, $v_P(t), v_{Q_k}(t)$, for $t \leq t''_j$, have all been determined and satisfy (3.21) and (3.20).

Apply lemma 3.2 for $(x_0, y_0) = (x_k, y_k)$, for $(x_k, y_k)$ as in (3.15), $r = r_k$, $\phi = \phi_k$, $\nu = \sqrt{N+1-k}$, $d = 1/N$ and $\theta = \theta_k$ as in (3.8), to find that $r_{k+1}$ is determined by formula (3.22) and to determine $y_{Q_{k+1}}$, the times $t'_k, t''_k$, and the velocities $v_P(t), v_{Q_{k+1}}(t)$ for $t \in [t'_k, t''_k]$ that will satisfy (3.21) and (3.20). Therefore $Q_{k+1}$ is always in the $r_{k+1}$-neighborhood of $Q_{k+1}$, as defined in (3.16). Choose $\sigma_N$ so that $r_{k+1}$ is smaller than $1/N$. Using lemma 3.4, $Q_{k+1}$ does not interact with $Q_1, \ldots, Q_k$ during the interval $(-\infty, t''_{k+1}]$ (consult figure 6).

For (3.23), rewrite first (3.22) as

\[
\begin{align*}
    r_k &= \sec \phi_{k-1} r_{k-1} + \sec \phi_{k-1} \sigma_N + 5\sigma_N \\
    &= \prod_{j=0}^{k-1} \sec \phi_j r_0 + \sum_{j=0}^{k-1} \sum_{m=j}^{k-1} \sec \phi_m \sigma_N + \left( \sum_{j=1}^{k-1} \sum_{m=j}^{k-1} \sec \phi_m + 1 \right) 5\sigma_N \tag{3.25}
\end{align*}
\]
and, using $r_0 = 0$ and $|\phi_j| \leq |\theta_j|$ (lemma 3.3), estimate this by
\[
\leq \sum_{j=0}^{k-1} \prod_{m=j}^{k-1} \sec \theta_m \sigma_N + \left( \sum_{j=1}^{k-1} \prod_{m=j}^{k-1} \sec \theta_m + 1 \right) 5 \sigma_N, \tag{3.26}
\]
and then, increasing $k$ to $N$ and using (3.8), estimate the same by
\[
\leq \sum_{j=0}^{N-1} \sqrt{N+2-j} \sigma_N + \left( \sum_{j=1}^{N-1} \sqrt{N+2-j} + 1 \right) 5 \sigma_N
\leq 6 \sum_{j=0}^{N-1} \sqrt{N+2-j} \sigma_N \leq 3 \sqrt{2} \sum_{j=3}^{N+2} \sqrt{\sigma_N}
\leq 3 \sqrt{2} \sigma_N \int_3^{N+3} \sqrt{x} \, dx < 2 \sqrt{2} (N + 3)^{3/2} \sigma_N. \tag{3.27}
\]
In particular, $\sigma_N < \frac{1}{2 \sqrt{2} N (N+3)^{3/2}}$ implies $r_k < 1/N$ for all $k$.

\[\square\]

\textbf{Remark 3.6.} Notice that, for each $N$, proposition 3.5 provides examples of the general theory of Galperin and Vaserstein, [G] and [V], according to which, for finite range interactions, molecules evolve by eventually separating into independent clusters. Each cluster here consists of a single molecule.

### 3.3. The limit system as $N \to \infty$

In the notation of proposition 3.5, let $T'_N := \sum_{j=1}^{N} (t''_j - t'_j)$, the time during which $P$ interacts with some $Q_k$. Then $T''_N = t''_N - T'_N$ is the time during $[0, t'_N]$ when $P$ is not interacting at all.

**Proposition 3.7.** $t''_N \to 0$, as $N \to \infty$.

**Proof.** According to (3.21), the speed of $P$ at $t'_k$ is $\sqrt{N+2-k}$. Then, by lemma A.1,
\[
T'_N \leq \sum_{k=1}^{N} \frac{4 \sigma_N}{\sqrt{N+2-k}} = 4 \sigma_N \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \tag{3.28}
\]
After the interaction of $P$ with $Q_k$ is complete, $P$ moves with speed $\sqrt{N+1-k}$, forming angle $\phi_k$ with the x-axis. The distance $d_k$ that $P$ will travel until its interaction with $Q_{k+1}$ begins, satisfies
\[
d_k \leq \frac{1}{N \cos \phi_k} \leq \frac{1}{N \cos \theta_k}, \tag{3.29}
\]
see figure 2. Recalling that $|\theta_k| \leq \frac{\pi}{4}$ from lemma 3.3 gives
\[
T''_N \leq \sum_{k=0}^{N-1} \frac{1}{N \cos \theta_k} \frac{1}{\sqrt{N+1-k}} \leq \frac{\sqrt{2}}{N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{k}} \frac{1}{\sqrt{N+1-k}} \leq \frac{\sqrt{2}}{N} \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \tag{3.30}
\]
This and (3.28) imply
\[ t''_N < \left( 4\sigma_N + \frac{\sqrt{3}}{N} \right) \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \] (3.31)

As \( \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}} < 2\sqrt{N+1} \), and for \( \sigma_N \) as in proposition 3.5, we conclude that \( t''_N \to 0 \) as \( N \to \infty \). \( \Box \)

**Proposition 3.8.** \( \max_{0 \leq t \leq N} y_{Q_k} \to 0 \) as \( N \to \infty \).

**Proof.** Noting that \( y_{Q_k} \) is the second coordinate of \( Q_k \) before \( t = t'_k \), whereas \( y_k \) is the second coordinate of the center of the \( k \)-interaction disc, it follows from the definition of \( r_k \) and (3.23) that
\[
|y_{Q_k}| < |y_k| + 2\sqrt{2(N+3)^{3/2}} \sigma_N.
\] (3.32)

For the second term on the right use \( \sigma_N \) as in proposition 3.5 and estimate the first term as
\[
|y_k| = \frac{1}{N} \left| \sum_{m=0}^{k-1} \tan \phi_m \right| \leq \frac{1}{N} \left| \sum_{m=0}^{k-1} \tan |\phi_m| \right|
\leq \frac{1}{N} \sum_{m=0}^{k-1} \tan |\phi_m| = \frac{1}{N} \sum_{m=0}^{k-1} \tan \left( \arcsin \frac{1}{\sqrt{N+2-m}} \right)
\leq \sum_{m=0}^{k-1} \frac{1}{N\sqrt{N+1-m}} < \sum_{m=0}^{N-1} \frac{1}{N\sqrt{N+1-m}}
\leq \frac{1}{N} \sum_{m=2}^{N+1} \frac{1}{\sqrt{m}} \to 0,
\] as \( N \to \infty \). \( \Box \)

For each fixed \( N \), writing \( v = (v_x, v_y) \) and following (2.3), set for \( t \in \mathbb{R} \)
\[
M^{(N+1)}_t(dx, dy, dv_x, dv_y)
\leq \frac{1}{N+1} \left( \delta_{(P(t),V(t))}(dx, dy, dv_x, dv_y) + \sum_{k=1}^{N} \delta_{(Q_k(t),\nu_{Q_k}(t))}(dx, dy, dv_x, dv_y) \right).
\] (3.34)

The crucial observation in the following proposition is that, due to the factor \( 1/N \), no single molecule shows as \( N \to \infty \), but its interaction with many other molecules, if their number is of order \( N \), shows macroscopically.

**Proposition 3.9.** As \( N \to \infty \), and for \( \sigma_N \) as in proposition 3.5: for \( t \leq 0 \),
\[
M^{(N+1)}_t(dx, dy, dv_x, dv_y) \Rightarrow \chi_{[0,1]}(x)dx \otimes \delta_0(dy) \otimes \delta_{(0,0)}(dv_x, dv_y),
\] (3.35)

and for \( t > 0 \),
\[
M^{(N+1)}_t(dx, dy, dv_x, dv_y)
\Rightarrow \chi_{[0,1]}(x)dx \otimes \left( \frac{1}{2} \delta_t(dy) \otimes \delta_{(0,1)}(dv_x, dv_y) + \frac{1}{2} \delta_{-t}(dy) \otimes \delta_{(0,-1)}(dv_x, dv_y) \right).
\] (3.36)
**Proof.** It suffices to check the statement on the integrals of bounded Lipschitz functions, see [AGS], p. 109. For this, for \( f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) bounded and Lipschitz
\[
\int_{\mathbb{R}^3} f(x, y, v_x, v_y) M_t^{(N+1)}(dx, dy, dv_x, dv_y)
= \frac{1}{N+1} f(P(t), v_P(t)) + \frac{1}{N+1} \sum_{k=1}^N f(Q_k(t), v_{Q_k}(t)).
\] (3.37)

Since \( f \) is bounded, the first term vanishes as \( N \to \infty \). The rest of the proof examines the convergence of the second term.

Fix any \( t \leq 0 \). Recalling (3.19),
\[
\frac{1}{N+1} \sum_{k=1}^N f(Q_k(t), v_{Q_k}(t)) = \frac{1}{N+1} \sum_{k=1}^N f\left(\frac{k}{N}, y_{Q_k}, 0, 0\right).
\] (3.38)

For \( L_f \) be the Lipschitz constant of \( f \), and using proposition 3.8,
\[
\left| \frac{1}{N+1} \sum_{k=1}^N f\left(\frac{k}{N}, y_{Q_k}, 0, 0\right) - \frac{1}{N+1} \sum_{k=1}^N f\left(\frac{k}{N}, 0, 0, 0\right) \right| 
\leq \frac{N}{N+1} L_f \max_{1 \leq k \leq N} |y_{Q_k}| \to 0.
\] (3.39)

By the definition of the Riemann integral,
\[
\frac{1}{N+1} \sum_{k=1}^N f\left(\frac{k}{N}, 0, 0, 0\right) \to \int_0^1 f(x, 0, 0, 0) \, dx.
\] (3.40)

Therefore
\[
\int_{\mathbb{R}^3} f(x, y, v_x, v_y) M_t^{(N+1)}(dx, dy, dv_x, dv_y) \to \int_0^1 f(x, 0, 0, 0) \, dx.
\] (3.41)

This is exactly (3.35). Now fix \( t > 0 \). By proposition 3.7 there exists \( N_1 \) such that for all \( N > N_1 \), \( t'' < t \), i.e. for each time we can choose \( N \) large enough so that all interactions have already happened and all molecules are moving at time \( t \), and are moving with their terminal velocities. We consider such \( N \)s only. According to proposition 3.5, and since now \( t \geq t'' \),
\[
x_{Q_k}(t) = x_{Q_k}(t'') + (t - t'') v_{Q_k}(t''),
y_{Q_k}(t) = y_{Q_k}(t'') + (t - t'') v_{Q_k}(t'').
\] (3.42)

For \( \alpha_N = \lfloor N - \sqrt{N} \rfloor \), the integer part of \( N - \sqrt{N} \), and by (3.20), for any \( 1 \leq k \leq \alpha_N \)
\[
|v_{Q_k}(t'')| = |\sin \phi_k| \leq |\sin \theta_k| \leq \frac{1}{\sqrt{N + 2 - \alpha_N}},

|v_{Q_k}(t'') - (-1)^{k+1}| = |\cos \phi_k - 1| \leq |\sin \phi_k| \leq \frac{1}{\sqrt{N + 2 - \alpha_N}},

\left| x_{Q_k}(t'') - \frac{k}{N} \right| \leq r_k, \quad |y_{Q_k}(t'')| < |y_k| + r_k.
\] (3.43)
Therefore for $1 \leq k \leq \alpha N$, by (3.23), propositions 3.7 and 3.8,

$$\left| x_{Q_k}(t) - \frac{k}{N} \right| \leq \left| x_{Q_k}(t') - \frac{k}{N} \right| + (t - t') |v_{x,Q_k}(t')|$$

$$< r_k + \frac{t}{\sqrt{N + 2 - \alpha N}} \to 0,$$

$$\left| y_{Q_k}(t) - (-1)^{k+1}t \right| \leq \left| y_{Q_k}(t') \right| + t |v_{y,Q_k}(t') - (-1)^{k+1}| + t'' |v_{y,Q_k}(t'')|$$

$$< |y_k| + r_k + \frac{t}{\sqrt{N + 2 - \alpha N}} + t'' \to 0. \quad (3.44)$$

Since $f$ is Lipschitz, (3.43) and (3.44) imply that

$$\left| \frac{1}{N+1} \sum_{k=1}^{\alpha N} f(Q_k(t), v_{Q_k}(t)) \right| \leq \frac{c(k)}{N+1} \to 0,$$

for $C_f = \max |f|$,}

$$\left| \frac{1}{N+1} \sum_{k=\alpha N+1}^{N} f(Q_k(t), v_{Q_k}(t)) \right| \leq \frac{C_f}{N+1} \to 0,$$

$$\left| \frac{1}{N+1} \sum_{k=\alpha N+1}^{N} f \left( \frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1} \right) \right| \leq \frac{C_f}{N+1} \to 0.$$  \quad (3.46)

Therefore,

$$\left| \frac{1}{N+1} \sum_{k=1}^{N} f(Q_k(t), v_{Q_k}(t)) - \frac{1}{N+1} \sum_{k=1}^{N} f \left( \frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1} \right) \right| \to 0. \quad (3.47)$$

By the definition of the Riemann integral,

$$\frac{1}{N+1} \sum_{k=1}^{N} f \left( \frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1} \right) \to \int_0^1 \frac{1}{2} \left( f(x, t, 0, 1) + f(x, -t, 0, -1) \right) dx \quad (3.48)$$

which implies (3.36).

With

$$x_k^{(N+1)}(t) = Q_k(t), \quad u_k^{(N+1)}(t) = v_{Q_k}(t), \quad k = 1, \ldots, N,$$

$$x_{N+1}^{(N+1)}(t) = P(t), \quad u_{N+1}^{(N+1)}(t) = v_P(t), \quad (3.49)$$

Theorem 3.1 follows immediately from propositions 3.5 and 3.9.

### 3.4. Macroscopic equations

We now examine the hydrodynamic equations for $Mt(dx, dv)$ as in theorem 3.1. It is easy to check that for any $\phi(t, \mathbf{x}) \in C_c^\infty (\mathbb{R} \times \mathbb{R}^2)$

$$4467$$
Using disintegration (3.5), for $\mu_t(dx)$ and $u(t,x)$ as in (3.4) and (3.7), we rewrite (3.50) as

$$
\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t,x) \mu_t(dx) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_x \phi(t,x) \cdot u(t,x) \mu_t(dx) dt = 0,
$$

(3.51)

Notice that at each $t,x$ the $M_{t,x}(dv)$ is singular, therefore

$$
\int_{\mathbb{R}^2} v \otimes v M_{t,x}(dv) = u \otimes u.
$$

(3.52)

Then (3.51) becomes

$$
\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t,x) \mu_t(dx) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_x \phi(t,x) \cdot u \mu_t(dx) dt = 0.
$$

(3.53)

In other words $(\mu_t(dx), u(t,x))$, $t \in \mathbb{R}$ solves weakly two dimensional Euler system without pressure:

$$
\partial_t \mu_t + \text{div}(u \mu_t) = 0,
$$

$$
\partial_t (u \mu_t) + \text{div}(u \otimes u \mu_t) = 0.
$$

(3.54)

For the naturalness of measure solutions in the pressureless Euler system, see [ERS], p 354.

**Remark 3.10.** The trivial solution $\tilde{\mu}_t(dx) = \Delta_0(dx)$, $\tilde{u} = (0,0)$ also solves (3.53) for all $t$, and coincides with $(\mu_t, u)$ for $t \leq 0$. Note that $(\mu_t, u)$ is not ‘energy admissible’ since the kinetic energy of $(\mu_t, u)$ increases in time:

$$
\int |u|^2 \mu_t(dx) = \int |v|^2 M_t(dx, dv) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases}
$$

(3.55)

A solution to (3.53) with decreasing energy can be obtained by reversing the direction of time, as in the next section. The value of the construction in this section lies in the microscopic, Hamiltonian interpretation of spontaneous velocity generation in weak solutions of hydrodynamic equations as in [Sch, Sh].

**4. Time reversal and macroscopic non-uniqueness**

**4.1. Reverse flow with decreasing energy**

We now reverse time in the construction of the previous section to establish macroscopic non-uniqueness in the class of energy decreasing solutions. It is standard that for $(x_k^{(N)}(t), u_k^{(N)}(t))$
a Hamiltonian flow as in theorem 3.1 the reverse flow \( \left( x_k^{(N)}(-t), -u_k^{(N)}(-t) \right) \) also solves the Hamiltonian system (2.1). Roughly speaking, for each \( N \) the reverse system consists of \( N \) molecules moving with speed 1 for \( t < 0 \). At \( t = 0 \), through interaction, one of the \( N \) molecules gathers all the energy from the rest \( N-1 \) molecules and leaves the rest of the group. Therefore for \( t > 0 \), macroscopically the system is motionless. If we still use \( \left( x_k^{(N)}(t), u_k^{(N)}(t) \right) \) for the reverse flow then the measure \( M_t^{(N)} \) converges weakly to

\[
M_t(dx, dv) = \begin{cases}
\frac{1}{2} \Delta_t(dx) \otimes \delta_{(0,1)}(dv) & t < 0 \\
\Delta_0(dx) \otimes \delta_{(0,0)}(dv) & t \geq 0,
\end{cases}
\]

with

\[
\mu_t(dx) = \begin{cases}
\frac{1}{2} \Delta_t(dx) + \frac{1}{2} \Delta_{-t}(dx) & t < 0 \\
\Delta_0(dx) & t \geq 0,
\end{cases}
\]

\[
u(t,x) = \begin{cases}
\chi_{Q_1}(x) \cdot (0,1) + \chi_{Q_3}(x) \cdot (0,-1) & t < 0 \\
0 & t \geq 0,
\end{cases}
\]

and decreasing energy:

\[
\int_{\mathbb{R}^2} |u|^2 \mu_t(dx) = \int_{\mathbb{R}^2} |v|^2 M_t(dx, dv) = \begin{cases}
1 & t < 0 \\
0 & t \geq 0,
\end{cases}
\]

see [BN], definition 2.1.

**Remark 4.1.** This describes two fronts approaching each other up until \( t = 0 \), when they merge and stay at rest, see figure 7. In the context of the pressureless Euler system this is a ‘sticky’ macroscopic solution, see [BN]. Rather than using particle systems with adhesion dynamics, here we obtain the solution as the limit of Hamiltonian dynamics with repulsive force. We also provide an explanation for the loss of energy: all the energy is transferred to a macroscopically invisible part of the system.

### 4.2. Transverse flow

It is known that merely requiring decreasing energy does not guarantee uniqueness of measure solutions to the system (3.54), see [BN]. This persists when comparing the flow of the previous section with the limit of a trivial Hamiltonian flow: for this we take the \( N \)-system to
consist of molecules that stay far enough from each other so that they never interact. We obtain a solution to the system (3.53) that coincides with (4.2) for all \( t < 0 \). But at \( t = 0 \), the moment the two fronts meet, instead of merging and staying at rest, they go through each other. More precisely, for each \( N = 2n \in \mathbb{N}, j = 1, 2, \ldots, N \), let

\[
\tilde{\xi}^{(N)}_j = \left( \frac{j}{N}, 0 \right), \quad \tilde{\mu}^{(N)}_j = \begin{cases} (0, 1) & \text{if } j \text{ odd} \\ (0, -1) & \text{if } j \text{ even}. \end{cases}
\] (4.4)

For \( t \in \mathbb{R} \) the orbits

\[
\tilde{\xi}^{(N)}_j(t) = \tilde{\xi}^{(N)}_j + t\tilde{\mu}^{(N)}_j
\] (4.5)

satisfy the Hamiltonian system (2.1) provided that the interaction range is sufficiently short, for example, \( \sigma < 1/N \). (Notice that \( \sigma_N \) in theorem 3.1, and therefore in section 4 satisfies \( \sigma_N < 1/N \).) Recalling definition (2.3), set

\[
\tilde{M}^{(N)}_t(dx, dv) = \frac{1}{N} \sum_{j=1}^{N} \delta(\tilde{\xi}^{(N)}_j(t), \tilde{\mu}^{(N)}_j(t)) (dx, dv).
\] (4.6)

By the definition of Riemann integral, for any continuous bounded \( f(x, v) \) we have

\[
\lim_{N \to \infty} \int_{\mathbb{R}^2} f(x, v)\tilde{M}^{(N)}_t(dx, dv) = \frac{1}{2} \int_0^1 f(x, t, 0, 1) dx + \frac{1}{2} \int_0^1 f(x, -t, 0, -1) dx.
\] (4.7)

Therefore

\[
\tilde{M}^{(N)}_t(dx, dv) \Rightarrow \tilde{M}_t(dx, dv) := \frac{1}{2} \Delta_t(dx) \otimes \delta_{(0,1)} (dv) + \frac{1}{2} \Delta_{-t}(dx) \otimes \delta_{(0,-1)} (dv).
\] (4.8)

The macroscopic density and velocity are

\[
\tilde{\mu}_t(dx) = \frac{1}{2} \Delta_t(dx) + \frac{1}{2} \Delta_{-t}(dx),
\]

\[
\tilde{u}(t, x) = \chi_{Q^+}(x) \cdot (0, 1) + \chi_{Q^-}(x) \cdot (0, -1), \quad t \in \mathbb{R},
\] (4.9)

see figure 8.

It is easily checked that (3.50) and (3.51) hold, and that for all \( t \neq 0 \)

\[
\int_{\mathbb{R}^2} v \otimes v \tilde{M}_t(dx) = \tilde{u} \otimes \tilde{u}.
\] (4.10)

Therefore \((\tilde{\mu}_t(dx), \tilde{u}(t, x))\) also solves weakly the pressureless Euler system for \( t \in \mathbb{R} \). Since

\[
\int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_t(dx) = 1 \text{ except for } t = 0,
\]

we can alter \( \tilde{u} \) at time \( t = 0 \) so that
\[ \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_0(\mathrm{d}x) = 1, \quad (4.11) \]

still solving equation (3.53). If we still use \( \tilde{\mu}_t(\mathrm{d}x), \tilde{u}(t, x) \) for the modified solution, we then have constant macroscopic kinetic energy in time:

\[ \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_t(\mathrm{d}x) = 1, \quad t \in \mathbb{R}. \quad (4.12) \]

Clearly for all \( t < 0, (\tilde{\mu}_t(\mathrm{d}x), \tilde{u}(t, x)) \), modified or not, coincides with \((\mu_t(\mathrm{d}x), u(t, x))\). Macroscopically, the same two fronts are approaching each other and, unless we know their microscopic origin, we are not be able to tell what will happen for \( t > 0 \).

**Remark 4.2.** Notice here the total macroscopic energy of the limit system is conserved in time:

\[ \int_{\mathbb{R}^4} |v|^2 \tilde{M}_t(\mathrm{d}x, \mathrm{d}v) = 1, \quad t \in \mathbb{R}, \quad (4.13) \]

and the macroscopic kinetic energy \( \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_t(\mathrm{d}x) \) is only part of the total energy in general:

\[ \int_{\mathbb{R}^4} |v|^2 \tilde{M}_t(\mathrm{d}x, \mathrm{d}v) = \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_t(\mathrm{d}x) + \int_{\mathbb{R}^2} |v - \tilde{u}|^2 \tilde{M}_t(\mathrm{d}x, \mathrm{d}v). \quad (4.14) \]

Let \( h(t) = \int_{\mathbb{R}^2} |v - \tilde{u}|^2 \tilde{M}_t(\mathrm{d}x, \mathrm{d}v) \). Then

\[ \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_t(\mathrm{d}x) + h(t) = 1, \quad t \in \mathbb{R}. \quad (4.15) \]

Notice that \( h(t) = 0 \) when \( t \neq 0 \) and \( h(0) = 1 \). Therefore for \( t < 0 \), all the energy of the system (4.8) is macroscopic kinetic energy which becomes \( h(0) \), the fluctuation energy, at \( t = 0 \). For \( t > 0 \) all the energy is again macroscopic kinetic energy.

By (4.3), for the reverse flow in section 4.1, the total energy \( \int_{\mathbb{R}^4} |v|^2 \tilde{M}_t(\mathrm{d}x, \mathrm{d}v) \) is decreasing in time. Trivially, the corresponding fluctuation energy \( h(t) = 0 \) for all \( t \in \mathbb{R} \).

**Remark 4.3.** It is possible that from a statistical mechanics point of view the non-uniqueness described here can be avoided by excluding a set of flows \( M_t \) negligible with respect to some probability measure. Notwithstanding this, our aim here is to understand specific non-uniqueness examples.

**Figure 8.** Macroscopic flow of (4.9).

\[ \int_{\mathbb{R}^2} |\tilde{u}|^2 \tilde{\mu}_0(\mathrm{d}x) = 1, \]
5. Non-uniqueness from moments of measures satisfying identical transport equations

Section 4 has shown non-uniqueness by comparing moments of the two limit flows $M_t(dx, dv)$ of (4.1) and $\tilde{M}_t(dx, dv)$ of (4.8). Note that $M_t$ satisfies weakly the transport equation

$$\partial_t M_t + v \cdot \nabla_x M_t = 0, \quad (5.1)$$

while $\tilde{M}_t$ satisfies the same with a nonzero kick at $t = 0$:

$$\partial_t \tilde{M}_t + v \cdot \nabla_x \tilde{M}_t = \left(\tilde{M}_0^+ - \tilde{M}_0^-\right) \otimes \delta_0(dt), \quad t \in \mathbb{R}, \quad (5.2)$$

for $\tilde{M}_0^\pm = \lim_{t \to 0^\pm} \tilde{M}_t$. In this section we present two examples where two different measures solve the same transport equation (5.1), give identical macroscopic density and velocity at $t = 0$, but the macroscopic density and velocity evolve differently to provide a non-uniqueness result for the Cauchy problem of the compressible Euler system in space dimension one.

5.1. Finite systems with velocity exchange

For systems in space dimension 1, we use identical molecules that move freely until they collide. The arguments in this section also hold for systems (2.1) of (finite range, at least) interactions, rescaled as in (2.2). In fact, there exist $\sigma_N$s such that, for space dimension 1, the limit of elastic collisions coincides with the limit of rescaled interactions, see [X]. However, such $\sigma_N$s might be too small for the rescaled interaction model to be physically better than elastic collisions. For simplicity then, we shall use elastic collisions. The complications of finite range interactions were evident in section 3.

In the elastic collision model collisions are instantaneous. Momentum and energy are conserved. Here it will be enough to consider only two kinds of collisions, both compatible with finite range interaction dynamics.

- (1) Binary collisions with incoming velocities $v_1, v_2$ and outgoing velocities $v'_1, v'_2$ satisfying

$$\begin{align*}
v_1 + v_2 &= v'_1 + v'_2 \\
v_1^2 + v_2^2 &= (v'_1)^2 + (v'_2)^2
\end{align*} \Rightarrow v_1 = v'_2, \quad v_2 = v'_1, \quad (5.3)$$

i.e. the molecules exchange velocities (as they are not allowed to go through each other).

- (2) Triple collisions, consisting of two molecules exactly as in item (1) and a third molecule in between that stays motionless.

As Zemlyakov shows in his delightful article [Z], several important questions for such systems can be answered using the graphs of the molecule positions as functions of time. Following this, the two types of collision we consider are shown in figure 9.

Consider a 1-dimensional point system $(x^{(N)}_k(t), u^{(N)}_k(t)), k = 1, \ldots, N$ obeying elastic collision dynamics. Fix any $T \in (0, \infty)$. For all $t \in [0, T]$ assume that all collisions are binary or triple as above.

**Proposition 5.1.** Let $S_t(x, v) = (x + vt, v)$. For all $t \in [0, T]$ the empirical measures

$$M_t(dx, dv) = \frac{1}{N} \sum_{k=1}^N \delta_{(x^{(N)}_k(t), u^{(N)}_k(t))}(dx, dv) \quad (5.4)$$

S Dostoglou and J Xue
satisfy
\[ M_t^{(N)}(dx, dv) = S_t M_0^{(N)}(dx, dv). \] (5.5)

Proof. Merely notice that for each \( t \)
\[ \frac{1}{N} \sum_{k=1}^{N} \delta \left( x_k^{(N)}(t), u_k^{(N)}(t) \right) = \frac{1}{N} \sum_{k=1}^{N} \delta \left( x_k^{(N)}(0) + u_k^{(N)}(0), u_k^{(N)}(0) \right) \] (5.6)
since there is a bijection, if multiplicities are taken into account:
\[ \left\{ (x_k^{(N)}(t), u_k^{(N)}(t)) \right\} \leftrightarrow \left\{ (x_k^{(N)}(0) + u_k^{(N)}(0), u_k^{(N)}(0)) \right\}. \] (5.7)
Indeed, the exchange of velocities between the moving molecules of a collision establishes a bijection between the orbits before and after that collision. Iterating this finitely many times brings us back to the initial orbits given by \( (x_k^{(N)}(0) + u_k^{(N)}(0), u_k^{(N)}(0)) \). □

The following lemma will be used repeatedly.

**Lemma 5.2.** Suppose that
\[ M_t^{(N)}(dx, dv) = S_t M_0^{(N)}(dx, dv), \quad M_0^{(N)}(dx, dv) \Rightarrow M_0(dx, dv). \] (5.8)

Then \( M_t^{(N)}(dx, dv) \Rightarrow S_t M_0(dx, dv) \).

Proof. Use the definitions of weak convergence and push forward under \( S_t \). □

As it is standard that \( M_t(dx, dv) = S_t M_0(dx, dv) \) solves weakly the free transport equation
\[ \partial_t M_t + v \partial_x M_t = 0 \] (5.9)
we shall refer to it as the a free transport flow.

**5.2. Euler system from free transport flow**

We find here conditions that imply that averages with respect to free transport flow satisfy the compressible Euler system in dimension 1. The next two sections provide examples satisfying such conditions.
Lemma 5.3. Suppose that $M_t(dx, dv) = S_t M_0(dx, dv)$. Then for all $\phi(t, x) \in C^1_c((0, T) \times \mathbb{R})$ and $g(v)$ such that $vg(v) \in L^1(M_0)$, we have

$$
\int_0^T \int_{\mathbb{R}^2} [\partial_t \phi(t, x) g(v) + \partial_x \phi(t, x) v g(v)] M_t(dx, dv) dt + \int_{\mathbb{R}^2} \phi(0, x) g(v) M_0(dx, dv) = 0.
$$

(5.10)

**Proof.** This is a straightforward calculation using the definition of the push forward under $S_t$ and the assumption that $\phi$ is compactly supported.

Disintegrating $M_t(dx, dv)$ of lemma 5.3 as

$$
M_t(dx, dv) = \int M_{t,x}(dv) \mu_t(dx),
$$

(5.11)

and for

$$
g(v)(t, x) = \int g(v) M_{t,x}(dv),
$$

(5.12)

(5.10) becomes

$$
\int_0^T \int_{\mathbb{R}} \left[ \partial_t \phi(t, x) g(v)(t, x) + \partial_x \phi(t, x) v g(v)(t, x) \right] \mu_t(dx) dt + \int_{\mathbb{R}} \phi(0, x) g(v)(0, x) \mu_0(dx) = 0.
$$

(5.13)

To apply lemma 5.3 for $g(v) = 1, v$, and $\frac{1}{2} v^2$, assume $v^3 \in L^1(M_0)$. Noting that

$$
u(t, x) = \tau(t, x) = \int v M_{t,x}(dv),
$$

(5.14)

and using the notation

$$
\xi^2(t, x) = \int (v - u(t, x))^2 M_{t,x}(dv), \quad \xi^3(t, x) = \int (v - u(t, x))^3 M_{t,x}(dv),
$$

(5.15)

it follows that

$$
\overline{\xi^3}(t, x) = u^3(t, x) + \xi^3(t, x), \quad \overline{\xi^3}(t, x) = u^3(t, x) + 3u(t, x)\xi^2(t, x) + \xi^3(t, x).
$$

(5.16)

Then (5.13) for $g(v) = 1, v$, and $\frac{1}{2} v^2$ gives

$$
\int_0^T \int_{\mathbb{R}} (\partial_t \phi + \partial_x \phi u) \mu_t(dx) dt + \int_{\mathbb{R}} \phi(0, x) \mu_0(dx) = 0,
$$

$$
\int_0^T \int_{\mathbb{R}} (\partial_t \phi u + \partial_x \phi (u^2 + \xi^2)) \mu_t(dx) dt + \int_{\mathbb{R}} \phi(0, x) u \mu_0(dx) = 0,
$$

$$
\int_0^T \int_{\mathbb{R}} \left\{ \partial_t \phi \left( \frac{1}{2} u^2 + \frac{1}{2} \xi^2 \right) + \partial_x \phi \left( \frac{1}{2} u^2 + \frac{3}{2} \xi^2 \right) u + \frac{3 \xi^3}{2} \right\} \mu_t(dx) dt + \int_{\mathbb{R}} \phi(0, x) \left( \frac{1}{2} u^2(0, x) + \frac{1}{2} \xi^2(0, x) \right) \mu_0(dx) = 0.
$$

(5.17)
Moreover, if \( \mu_r(\mathrm{dx}) = \rho(t,x)\mathrm{dx} \), \( \xi^0(t,x) = 0 \) and for \( e(t,x) = \frac{\xi^0(t,x)}{2} \), \( p = 2\rho e \), (5.17) shows that \( \rho, u, e \) solve weakly the Cauchy problem

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x ((\rho u^2) + \partial_t \rho) &= 0, \\
\partial_t \left( \frac{\rho u^2}{2} + pe \right) + \partial_x \left( \rho u \left( \frac{\rho u}{2} + e \right) + pu \right) &= 0, \\
p &= 2\rho e, \quad \rho|_{t=0} = \rho(0,x), \quad u|_{t=0} = u(0,x), \quad e|_{t=0} = e(0,x),
\end{align*}
\]  

(5.18)

the one dimensional Euler system, see [CF], p 7. In summary, we have shown:

**Proposition 5.4.** For \( M_i(\mathrm{dx},\mathrm{dv}) = S_i M_0(\mathrm{dx},\mathrm{dv}) \), suppose that \( v^3 \in L^1 (M_0) \), \( \mu_r(\mathrm{dx}) = \rho(t,x)\mathrm{dx} \), and \( \xi^0(t,x) = 0 \). Then \( \rho(t,x), u(t,x), e(t,x) \) as defined above is a weak solution to the 1D Euler system (5.18).

The definition of initial conditions for weak solutions here is compatible with the one in [dp], p 2 and [VF], section VII.10. Two examples satisfying the conditions of this proposition now follow.

### 5.3. Two-layer system

For \( N \) fixed, consider \( N = 2n \) point molecules \( x_1, x_2, \ldots, x_N \) on the real line, with

\[
x_k(0) = \frac{k}{N}, \quad u_k(0) = \begin{cases} 1 & \text{for } k \text{ odd} \\ -1 & \text{for } k \text{ even.} \end{cases}
\]  

(5.19)

Let the system evolve as in section 5.1. After the first \( n \) simultaneous collisions take place the molecules with labels 1 and \( N \) move with velocities 1 and \(-1\), respectively, without ever interacting with any other molecule again. The remaining molecules now form a replica of the initial system, reduced by two molecules.

As in [Z], the graphs of the positions as functions of time show the evolution of the system, figure 10.

For

\[
M_{i}^{(N)} = \frac{1}{N} \sum_{k=1}^{N} \delta(x_k(t), u_k(t)),
\]  

(5.20)

according to proposition 5.1,

\[
M_{i}^{(N)} = S_i M_{0}^{(N)}.
\]  

(5.21)

On the other hand, it is easy to check that as \( N \to \infty \),

\[
M_{0}^{(N)}(\mathrm{dx},\mathrm{dv}) \Rightarrow M_{0}(\mathrm{dx},\mathrm{dv}) = \chi_{[0,1]}(x)\mathrm{dx} \otimes \left( \frac{1}{2} \delta_{-1}(\mathrm{dv}) + \frac{1}{2} \delta_{1}(\mathrm{dv}) \right),
\]  

(5.22)

therefore, by lemma 5.2,

\[
M_{i}^{(N)} \Rightarrow M_{i} = S_i M_0, \quad N \to \infty.
\]  

(5.23)

It is straightforward to calculate that

\[
M_{i}(\mathrm{dx},\mathrm{dv}) = \frac{1}{2} \chi_{[\frac{1}{2},1]}(x)\mathrm{dx} \otimes \delta_{1}(\mathrm{dv}) + \frac{1}{2} \chi_{[-\frac{1}{2},-1]}(x)\mathrm{dx} \otimes \delta_{-1}(\mathrm{dv}).
\]  

(5.24)
Mt describes two layers, each of total mass $1/2$, initially overlapping on the interval $[0, 1]$, moving with velocities $\pm 1$ for $t \geq 0$, see figure 11.

The macroscopic density, velocity and energy density given by $Mt$ are

\[
\rho(t, x) = \frac{1}{2} \chi_{[t, 1+t]}(x) + \frac{1}{2} \chi_{[-t,1-t]}(x), \\
u(t, x) = \chi_{[t,1+t]}(x) - \chi_{[-t,1-t]}(x), \\
e(t, x) = \frac{1}{2} \chi_{[t,1+t]}(x) \cdot \chi_{[-t,1-t]}(x).
\]
(5.25)

Notice that \( \int_{\mathbb{R}} |v|^3 M_0(x, dv) < \infty \) and

\[
\xi^T = \int_{\mathbb{R}} (v - u(t, x))^3 M_{t,x}(dv) = 0.
\]
(5.26)

Therefore, by proposition 5.4, \((\rho, u, e)\) is a solution to the Euler system

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p &= 0, \\
\partial_t \left( \frac{\rho u^2}{2} + e \right) + \partial_x \left( \rho u \left( \frac{u^2}{2} + e \right) + pu \right) &= 0, \\
p &= 2\rho e,
\end{align*}
\]
(5.27)

5.4. Three-layer system

Consider now for each $N = 3n$ a second system, consisting of $N$ molecules $x_1, x_2, \ldots, x_N$ on the real line with
\[ x_k(0) = \frac{k}{N}, \quad k = 1, \ldots, N. \]

\[ u_k(0) = \begin{cases} 
\sqrt{6}/2 & \text{for } k = 3m - 2 \\
0 & \text{for } k = 3m - 1 \\
-\sqrt{6}/2 & \text{for } k = 3m \end{cases} \quad \text{for } m = 1, \ldots, n, \quad (5.28) \]

also evolving under elastic collisions as in section 5.1.

The evolution of the system initialized by (5.28) is shown in figure 12. Again, if for the current system

\[ \tilde{M}_t(N) = \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k(t), u_k(t))}, \quad (5.29) \]

by proposition 5.1,

\[ \tilde{M}_t(N) = S_t \tilde{M}_0(N). \quad (5.30) \]

On the other hand, as \( N \to \infty \),

\[ \tilde{M}_0^{(N)}(dx, dv) \Rightarrow \tilde{M}_0(dx, dv) \]

\[ = \chi_{[0,1]}(x)dx \otimes \left( \frac{1}{3} \delta_{-\sqrt{6}/2}(dv) + \frac{1}{3} \delta_0(dv) + \frac{1}{3} \delta_{\sqrt{6}/2}(dv) \right), \quad (5.31) \]

therefore

\[ \tilde{M}_t^{(N)} \Rightarrow \tilde{M}_t = S_t \tilde{M}_0, \quad N \to \infty. \quad (5.32) \]

It is again a straightforward calculation that
Figure 12. Microscopic evolution of section 5.4.

Figure 13. Macroscopic evolution of section 5.4.
\[ \tilde{M}_t(dx, dv) = \frac{1}{3} \chi \left[ -\frac{\sigma}{2}, 0 \right] (x) dx \otimes \delta_{\tilde{\sigma}}(dv) + \frac{1}{3} \chi \left[ 0, \frac{\sigma}{2} \right] (x) dx \otimes \delta_{\tilde{\sigma}}(dv) + \frac{1}{3} \chi \left[ \frac{\sigma}{2}, 0 \right] (x) dx \otimes \delta_{\tilde{\sigma}}(dv). \] (5.33)

\( \tilde{M}_t \) describes three layers, each of total mass \( \frac{1}{3} \), initially overlapping on the interval \([0, 1]\). Two of them move with velocities \( \pm \sqrt{6}/2 \) for \( t > 0 \), while the third stays at rest, see figure 13. The macroscopic density, velocity and energy density given by \( \tilde{M}_t \) are

\[
\begin{align*}
\tilde{\rho}(t, x) &= \frac{1}{2} \chi \left[ -\frac{\sigma}{2}, 0 \right] (x) + \frac{1}{2} \chi \left[ 0, \frac{\sigma}{2} \right] (x) + \frac{1}{3} \chi \left[ \frac{\sigma}{2}, 0 \right] (x) \\
\tilde{u}(t, x) &= -\frac{\tilde{\rho}(t, x)}{\rho(t, x)} \\
\tilde{e}(t, x) &= \frac{1}{2} \chi \left[ -\frac{\sigma}{2}, 0 \right] (t) \chi \left[ 0, \frac{\sigma}{2} \right] (x) + \frac{1}{3} \chi \left[ \frac{\sigma}{2}, 0 \right] (t) \chi \left[ 0, \frac{\sigma}{2} \right] (x) - \frac{1}{3} (\tilde{\rho}(t, x) \tilde{\sigma}(t, x))
\end{align*}
\] (5.34)

When \( \tilde{\rho}(t, x) = 0 \), take \( \tilde{u}(t, x), \tilde{e}(t, x) = 0 \). Notice that

\[ \int_{\mathbb{R}} (v - \tilde{u}(t, x))^2 \tilde{M}_t(dx, dv) = 0. \] (5.35)

By proposition 5.4, \((\tilde{\rho}, \tilde{u}, \tilde{e})\) is also a solution to the Cauchy problem (5.27), clearly distinct from the solution \((\rho, u, e)\).

**Remark 5.5.** It is well known that weak solutions to systems like (5.27) are not unique, see [D]. This section provides a microscopic interpretation of such macroscopic non-uniqueness, showing that such phenomena are quite natural from a Hamiltonian point of view.

**Appendix. Motion in a central field**

We establish some facts for the motion in dimension 2 of a single particle in an external field of potential energy \( \Phi \) of finite range \( \sigma \):

\[ x''(t) = -\Phi'(|x|) \frac{x}{|x|}. \] (A.1)

To accommodate (2.2), assume that \( \Phi : (0, \infty) \to [0, \infty) \) satisfies

\[ \lim_{r \to 0} \Phi(r) = +\infty, \quad \Phi'(0) \leq 0, \quad \Phi'' \geq 0, \quad \Phi(r) \neq 0 \quad \Leftrightarrow \quad 0 < r < \sigma. \] (A.2)

Consulting figure A1, let \( O \) be the center of the potential \( \Phi \). A molecule \( m \) enters the range of \( \Phi \) at \( A \) with velocity \( v \) and leaves at \( B \). For \( D \) the middle of \( AB \), the path of \( m \) in the range of \( \Phi \) is symmetric about \( OD \), by the reversibility of the equations of motion. Decompose \( v(t) \) into \( v_1(t) \) and \( v_2(t) \) along \( AB \) and \( OD \), respectively, and let \( E \) be the intersection of \( OD \) and the trajectory of \( m \). When \( m \) crosses \( OD \) it has moved \( d \) on the direction of \( OD \). If \( \theta \) is the angle between \( v \) and \( AB \) and \( C \) is the point on \( OD \) with \( AC \) of direction \( v \), then

\[ d = DE < CD = AC \cdot \sin \theta < AO \cdot \sin \theta = \sigma \sin \theta. \] (A.3)

Let \( T \) be the time it takes \( m \) to travel from \( A \) to \( B \).
Lemma A.1. For \( \sigma \) the range of \( \Phi \), \( T \) and \( v \) as above satisfy \( T < \frac{4v}{v} \).

Proof. From (A.1),

\[
\nu_2'' = -\Phi''(|x|) \frac{x \cdot x'}{|x|^2} \nu_2 - \Phi'(|x|) \frac{x'}{|x|} + \Phi'(|x|) \frac{x \cdot x'}{|x|^3} \nu_2. \tag{A.4}
\]

For \( x_2 < 0 \) and as \( \frac{d|x|}{dt} < 0 \) for \( t \in (0, T/2) \), and as \( \Phi \) is convex, the first term of this is negative and, if \( x_1 \) is also negative, the sum of the remaining two terms is also negative provided that

\[
-x_2' |x|^2 + x_2 (x \cdot x') > 0 \iff -x_2' x_1 + x_2 x_1' < 0,
\]  

since \( \Phi \) is decreasing. Now note that \(-x_2' x_1 + x_2 x_1'\) stays constant in time and the inequality is satisfied at \( t = 0 \). Therefore \( \nu_2 \) is concave and by (A.3)

\[
\frac{\nu_2(0)}{2} - \frac{T}{2} < d < \sigma \sin \theta, \tag{A.6}
\]

which, along with \( \nu_2(0) = v \sin \theta \), concludes the proof. \( \square \)

Still in figure A1, let \( \angle ACD = \phi \). Denoting the distance of \( O \) from \( AC \) (the impact parameter) by \( \alpha \), by [LL], p 49

\[
\phi(\alpha) = \int_{r_{\min}}^{\infty} \frac{\alpha}{r^2 \sqrt{1 - \frac{\alpha^2}{r^2} - \frac{x(r)}{r}}} dr, \tag{A.7}
\]

1 Note here that [LL]'s analysis of motion in a central field in their section 14 is valid for any central field, including the ones with finite range.
where $E = \frac{1}{2} mv^2$ and $r_{\text{min}}$ is a zero of the radicand:

$$1 - \frac{\alpha^2}{r_{\text{min}}^2} - \frac{\Phi(r_{\text{min}})}{E} = 0.$$  \hfill (A.8)

**Lemma A.2.** For interaction potential as in (2.2), $r_{\text{min}} = r_{\text{min}}(\alpha)$ is increasing and $\phi(\alpha)$ is continuous on $[0, \infty)$.

**Proof.** For fixed $\alpha$ and $E$, the function

$$r \mapsto \frac{\alpha^2}{r^2} + \frac{\Phi(r)}{E}$$ \hfill (A.9)

is strictly decreasing from $+\infty$ to 0 for $r > 0$ and the pre-image $r_{\text{min}}$ of 1 satisfies (A.8), or

$$\alpha = \left(1 - \frac{\Phi(r_{\text{min}})}{E}\right)^{1/2} r_{\text{min}}$$ \hfill (A.10)

showing that $\alpha = \alpha(r_{\text{min}})$, and therefore $r_{\text{min}} = r_{\text{min}}(\alpha)$, is increasing.

To show that $\phi$ is continuous, change the variable in (A.7) via $r = r_{\text{min}} y$:

$$\phi(\alpha) = \int_1^{\infty} \frac{1}{y^2 \sqrt{\frac{r_{\text{min}}}{\alpha^2}} \left(1 - \frac{\Phi(r_{\text{min}} y)}{E}\right) - \frac{1}{y^2}} dy$$ \hfill (A.11)

By (A.10)

$$= \int_1^{\infty} \frac{1}{y^2 \sqrt{\frac{E - \Phi(r_{\text{min}} y)}{E - \Phi(r_{\text{min}})}} - \frac{1}{y^2}} dy.$$

From (A.8) we have

$$E > \Phi(r_{\text{min}})$$ \hfill (A.12)

and since $\Phi(r)$ is decreasing,

$$\Phi(r_{\text{min}}) \geq \Phi(r_{\text{min}} y), \quad y \geq 1,$$ \hfill (A.13)

therefore

$$\frac{1}{y^2 \sqrt{\frac{E - \Phi(r_{\text{min}} y)}{E - \Phi(r_{\text{min}})}} - \frac{1}{y^2}} \leq \frac{1}{y^2 \sqrt{1 - \frac{1}{y^2}}}$$ \hfill (A.14)

with

$$\int_1^{\infty} \frac{1}{y^2 \sqrt{1 - \frac{1}{y^2}}} dy = \frac{\pi}{2}.$$ \hfill (A.15)

In other words, the integrand of $\phi$ is dominated by an integrable function. This, and the continuity of $r_{\text{min}}$ in $\alpha$, show that $\phi$ is continuous in $\alpha$. \qed

**Corollary A.3.** For any $0 \leq \phi_0 \leq \pi/2$, there exists $0 \leq \alpha_0 \leq \sigma$ such that $\phi(\alpha_0) = \phi_0$.

**Proof.** Just use continuity and that $\phi(0) = 0$ (‘head-on collision’), $\phi(\sigma) = \frac{\pi}{2}$ (no interaction). \qed
As is well known, motion in a central field also describes a system of two bodies interacting with each other via \( \Phi \), a function of their distance, in a coordinate system with its origin at the center of mass of the system. The formulas for this transformation are in [LL], section 13.

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