GENERIC $\mathfrak{gl}_2$-FOAMS, WEB AND ARC ALGEBRAS

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Abstract. We define parameter dependent $\mathfrak{gl}_2$-foams and their associated web and arc algebras and verify that they specialize to several known $\mathfrak{sl}_2$ or $\mathfrak{gl}_2$ constructions related to higher link and tangle invariants. Moreover, we show that all these specializations are equivalent, and we deduce several applications, e.g. for the associated link and tangle invariants, and their functoriality.

Contents

1. Introduction 1
2. A family of singular TQFTs, foams and web algebras 5
   2.1. Webs and prefoams 5
   2.2. Generic singular TQFTs 7
   2.3. Foam 2-categories 13
   2.4. Known specializations 16
   2.5. Web algebras 18
   2.6. Web bimodules and foam 2-categories 20
3. A family of arc algebras 22
   3.1. Combinatorics of arc diagrams 22
   3.2. The linear structure of the arc algebras 24
   3.3. The algebra structure 25
   3.4. Arc bimodules 29
4. Isomorphisms, equivalences and their consequences 31
   4.1. Web and arc algebras 31
   4.2. Arc algebras: isomorphisms 33
   4.3. Arc bimodules: bimodule homomorphisms 35
   4.4. Arc bimodules: co-structure 37
   4.5. Consequences 40
5. Applications 41
   5.1. Connection to category $\mathcal{O}$ 41
   5.2. Connection with link and tangle invariants 44
6. Main proofs 52
Index of notation 63
References 64

1. Introduction

Let $P = \{\alpha, \tau_0^{\pm1}, \tau_0^{\pm1}, \tau_0^{\pm1}, \omega^{\pm1}, \omega^{\pm1}\}$ be a set of generic parameters. In this paper we introduce a $P$-version of a two-dimensional singular topological quantum field theories (TQFTs) which we use to define a 5-parameter foam 2-category $\mathfrak{F}[P]$ which unifies several topological variants of Khovanov homology. By specialization of this
5-parameter version one can for instance obtain the following foam 2-categories studied in the context of higher link and tangle invariants:

KBN: Khovanov/Bar-Natan’s cobordisms [23], [2] can be obtained by specializing $P = \{\alpha, \tau, \omega, \omega^{-1}\}$ to $\{0, 1, 1, 1, 1\}$.

(Sp) Ca: Capraru’s foams [10] by specializing $P$ to $\{0, 1, 1, i, -i\}$.

CMW: Clark–Morrison–Walker’s disoriented cobordisms [14] by specializing $P$ to $\{0, 1, 1, i, -i\}$.

Bl: Blanchet’s foams [3] by specializing $P$ to $\{0, 1, -1, 1, -1\}$.

We also study the web algebra $\mathcal{W}[P]$ corresponding to $\mathfrak{g}[P]$. The web algebra comes with a 2-category of certain bimodules which in fact gives a (fully) faithful 2-representation of $\mathfrak{g}[P]$. (Similarly for any specialization of $P$.)

For $Q = \{\alpha, \varepsilon, \omega, \omega^{-1}\}$, obtained by specializing $P$ via $\tau_\mu = 1$, $\tau_\nu = \varepsilon \omega^2$, $\omega = \omega$ and $\omega_- = \varepsilon \omega$, with $\varepsilon = \pm 1$, we define an algebraic model simultaneously for $\mathfrak{g}[Q]$ and $\mathcal{W}[Q]$, that is, an arc algebra $\mathfrak{A}[Q]$ encoding algebraically/combinatorially the topological data coming from $\mathfrak{g}[Q]$ and $\mathcal{W}[Q]$. We call these signed 2-parameter versions, and they still specialize to the examples from (Sp). Further, we call the $\varepsilon = 1$ specializations the $sl_2$ specializations and the $\varepsilon = -1$ specializations the $gl_2$ specializations, since they correspond to the web algebras describing the tensor categories of finite-dimensional representations of the respective complex Lie algebra, cf. Section 1.3.

Our main result is, surprisingly, that any two specializations of $\varepsilon, \omega$ to values in some ring $R$, with $\varepsilon = \pm 1$, are isomorphic/equivalent. That is, if we denote by $\ast$ any such specialization, then (we only extend scalars to get an isomorphism of $Q$-algebras):

**Theorem.** Let $Q = \mathbb{Z}[\alpha, \varepsilon, \omega, \omega^{-1}], \varepsilon = \pm 1$. There are graded algebra isomorphisms

$$\Psi : \mathfrak{A}[Q] \xrightarrow{\cong} \mathfrak{A}[\ast] = \mathfrak{A}_R[\ast] \otimes_{\mathbb{Z}} Q.$$ (Similarly for the corresponding web algebras.)

Additionally one can also specialize $\alpha$. But in contrast to the other parameters involved this sometimes has to be done on both sides of the isomorphisms/equivalences. We will call this simultaneous specialization for short. From this we obtain:

**Theorem.** The isomorphisms from above induce isomorphisms of graded, $Q$-linear 2-categories of certain graded bimodules

$$\Psi : \mathfrak{A}[Q] \text{-biMod}_{gr}^P \xrightarrow{\cong} \mathfrak{A}[\ast] \text{-biMod}_{gr}^P,$$

giving on the topological side equivalences of graded, $Q$-linear 2-categories

$$\mathfrak{A}[Q] \text{-biMod}_{gr}^P \cong \mathfrak{g}[Q] \cong \mathfrak{g}[\ast] \cong \mathfrak{A}[\ast] \text{-biMod}_{gr}^P.$$ (Similarly for any further simultaneous specialization of $\alpha$.)

An almost direct consequence of the above results is:

**Corollary.** The KBN, Ca, CMW and Bl foam 2-categories, obtained via specialization as in (Sp), are all equivalent when one works over $\mathbb{Z}[i]$. ❑

We like to stress that our main results are proven by explicit isomorphisms/equivalences, which we construct. However, the main ingredients for the proofs of all these statements are non-trivial, and some of them are also rather lengthy. We have moved these proofs into an extra section, see Section 6.
1.1. Applications. Two consequences of our main theorem are:

- We show in Section 5.2 that the higher tangle invariants constructed from the various 2-categories are the same, i.e. they get identified by the above equivalence, and not just the associated link homologies. Note hereby:
  - In case of links, the algebras acting are trivial and a weaker result is sufficient to show that the homologies agree (see e.g. [14, Theorem 4.1]).
  - Using our explicit isomorphisms and the functoriality of the Ca, CMW and Bl specializations, see [10, Theorem 3.5], [14, Theorem 1.1] and [3, Theorem 5.1], one can redefine the original KBN complex (defined via the famous cube construction) to make it functorial, without changing its simple framework—changing the bimodule structure instead. To be precise, in the setup we discuss this works directly for braid cobordisms, but a bit more care should establish the same for link cobordisms, cf. Section 5.1. (Another approach to functoriality is given in [46]. We do not know how these results relate to ours.)

- We discuss in Section 5.1 how one can obtain a “singular TQFT model” for the graded BGG parabolic category \(\mathcal{O}\) for a two-block parabolic in type \(A\). (This is the category used in the Lie theoretical construction of Khovanov homology, see e.g. [40] or [42].) By our main theorem it follows that all of the specializations from \(\text{Sp}\) describe the same Lie theoretical instances.

1.2. Outlook. There are also several potential applications.

- Arc algebras of type \(A\) are extensively studied, see e.g. [24], [5], [6], [7], [8], [9], [17]. However, outside of type \(A\) they are not well-understood at present, see e.g. [15] and [16] for a type \(D\) version. For these the appearing scalars are quite delicate and twisting their multiplication structure cases to work with them a lot. This was already successfully applied in [19] following a strategy similar to ours (using a version of type \(D\) foams), and one could hope for further results in this direction. For other arc algebras, as e.g. a \(\mathfrak{gl}_1\)-variation developed in [39], and an odd version [33], analog questions arise.

- Similarly, web algebras of type \(A\) are studied beyond the rank one case, see e.g. [32], [34], [35], [43], [31] and [44]. However, for higher ranks basically nothing is known at present. Both, algebraic models of these or twisting the associated singular TQFTs of [36], might lead to interesting results. Moreover, such a description is still missing for example for symmetric type \(A\)-webs in the sense of [37] or [45] and our construction might be helpful to find these.

- The \(\mathfrak{gl}_2\) specializations tend to give functorial higher link and tangle invariants, as e.g. the Ca, CMW and Bl specializations. In contrast, the \(\mathfrak{sl}_2\) versions usually do not, as e.g. the KBN specialization, cf. [21]. We believe it can be proven without too much effort, following [18], that this is the case for all such specializations. The analog question in higher ranks is probably also true, but technically much more involved.

1.3. \(\mathfrak{sl}_2\) versus \(\mathfrak{gl}_2\). The original cobordism 2-category of Khovanov–Bar-Natan and the associated web algebra “categorify” the Temperley–Lieb category, see [24, Proposition 23]. Similarly, for the corresponding arc algebra, see [40, Section 6] (combined with [7, Theorem 1.2]). The Temperley–Lieb category gives a presentation of the category of quantum \(\mathfrak{sl}_2\)-intertwiners, as neatly explained in [28]. In contrast, \(\mathfrak{sl}_2\) and its associated web and arc algebra categorify \(\mathfrak{gl}_2\)-webs (webs for short) which give a
presentation of the category of quantum \( gl_2 \)-intertwiners. (These webs come from a Howe duality of \( U_q(gl_M) \) and \( U_q(gl_2) \), see \cite{12} or more specifically \cite{45, Remark 1.1}.)

Here certain “phantom edges” correspond to the determinant representation \( \bigwedge^2 q C_2 q \) of quantum \( gl_2 \). For cases where \( \bigwedge^2 q C_2 q \) is not trivially categorified—e.g. for \( \varepsilon = -1 \)—we can say that such a categorification encodes \( gl_2 \) instead of \( sl_2 \).

1.4. Abstract reasons for the existence of our main isomorphisms. The idea that our theorems from above should hold grew out of the following.

Using arguments as in \cite[Section 5.3, Proposition 5.18]{32} one can show that \( \mathfrak{W}_C[KBN] \) is Morita equivalent to a certain KL–R algebra of level 2 (using \( \mathbb{C} \) as a ground field). Hereby, the two main ingredients in the corresponding proof in \cite{32} were a categorification of an instance of \( q \)-Howe duality as well as Rouquier’s universality theorem \cite[Proposition 5.6 and Corollary 5.7]{38} (“such categorifications are unique”). Now, it was shown in \cite[Theorem 9.2]{7} that another instance of \( q \)-Howe duality can be categorified using \( \mathfrak{A}_C[KBN] \). Moreover, one can deduce from \cite[Propositions 3.5 and 3.3]{29}—in the light of Proposition 2.45—the same for \( \mathfrak{W}_{\mathbb{C}}[CMW] \) and \( \mathfrak{W}_{\mathbb{C}}[Bl] \). Since \( \mathfrak{A}_C[KBN] \) is constructed from \( \mathfrak{W}_{\mathbb{C}}[KBN] \), “uniqueness of categorification in type \( A \)” should yield our theorems.

However, we should stress that there is still work to be done for this abstract approach since it is only known for \( \mathfrak{A}_C[KBN] \) that it categorifies the corresponding highest weight module of the “Howe dual” quantum group, and one would need to check the same for the \( \mathfrak{Ca}, CMW, Bl \) setups or the signed 2-parameter version.

In contrast to these abstract reasons, our work is completely explicit. This has many advantages. For example one can not deduce just from the abstract existence of such isomorphisms any of our applications with respect to higher link and tangle invariants in Section 5.2 since such isomorphisms could be “uncontrollable”—e.g. on the bimodules used to define these invariants. Moreover, outside of type \( A \) the first incarnations of foams exist \cite{19} and can potentially be parameter twisted as well, but it is less clear how to apply uniqueness of categorification arguments there.

1.5. Conventions used throughout.

**Convention 1.1.** By a ring \( R \) we always understand a commutative, unital ring without zero divisors. By an algebra we always mean an \( R \)-algebra \( A \)—over which ring will be clear from the context. We do not assume that such \( A \)’s are (locally) unital, associative or free over \( R \) and it will be a non-trivial fact that all \( A \)’s which we consider are actually (locally) unital, associative and free. (To be precise, they are direct sums of unital, associative, free algebras of finite rank.) Given two algebras \( A \) and \( B \), then an \( A \)-\( B \)-bimodule is a free \( R \)-module \( M \) with a left action of \( A \) and a right action of \( B \) in the usual compatible sense. If \( A = B \), then we also write \( A \)-bimodule for short. We denote the category of \( A \)-bimodules which are free over \( R \) by \( A\text{-biMod} \). Moreover, we call an \( A \)-\( B \)-bimodule \( M \) biprojective, if it is projective as a left \( A \)-module as well as a right \( B \)-module.

**Convention 1.2.** Graded should be read as \( \mathbb{Z} \)-graded. By a graded algebra we mean an algebra \( A \) which decomposes into graded pieces \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) such that \( A_i A_j \subset A_{i+j} \) for all \( i, j \in \mathbb{Z} \). Given two graded algebras \( A \) and \( B \), we study (and only consider) graded \( A \)-\( B \)-bimodules, i.e. \( A \)-\( B \)-bimodules \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) such that \( A_i M_j B_k \subset M_{i+j+k} \) for all \( i, j, k \in \mathbb{Z} \). We also set \( M\{s\}_i = M_{i-s} \) for \( s \in \mathbb{Z} \). (Thus, positive integers shift up.)
If $A$ is a graded algebra and $M$ is a graded $A$-bimodule, then $\mathcal{M}$ obtained from $M$ by forgetting the grading is in $A\text{-biMod}$. Given such $A$-bimodules $\mathcal{M}, \mathcal{N}$, then
\begin{equation}
\text{Hom}_{A\text{-biMod}}(\mathcal{M}, \mathcal{N}) = \bigoplus_{s \in \mathbb{Z}} \text{Hom}_0(\mathcal{M}, \mathcal{N}_s).
\end{equation}
Here $\text{Hom}_0$ means all degree-preserving $A$-homomorphisms $\phi$, i.e. $\phi(M_i) \subset N_i$.

**Convention 1.3.** We consider three diagrammatic calculi in this paper: webs, foams and arc diagrams. Our reading convention for all of these is from bottom to top and left to right. Furthermore, diagrammatic left respectively right actions will be given by acting on the bottom respectively on the top. Moreover, we often only illustrate local pieces. The corresponding diagram is meant to be the identity or arbitrary outside of the displayed part—which one will be clear from the context.

**Remark 1.4.** We use colors in this paper. It is only necessary to distinguish colors for webs and foams. For the readers with a grayscale version: we illustrate colored web edges using dashed lines, while colored foam facets appear shaded.

**Acknowledgements:** We like to thank Jonathan Comes, Jonathan Grant, Martina Lanini, David Rose, Pedro Vaz, Paul Wedrich and Arik Wilbert for helpful comments and discussions. Special thanks to Jonathan Grant, a referee, Pedro Vaz and Paul Wedrich for comments on a draft of this paper, and to the bars of Cologne for helping to write down one of the main isomorphisms of this paper.

# 2. A family of singular TQFTs, foams and web algebras

In this section we introduce a 5-parameter version of singular TQFTs. We use these to define the 5-parameter foam 2-category $\mathcal{F}[P]$ and its web algebra $\mathcal{W}[P]$.

## 2.1. Webs and prefoams

We start by recalling the definition of webs and of prefoams, where we closely follow [17, Section 2].

**Definition 2.1.** A (topological) web is a labeled (with either 1 or 2), oriented, piecewise linear, one-dimensional CW complex such that each internal point of it has a local neighborhood of the following form.

\begin{equation}
\begin{aligned}
\begin{diagram}
\node{(1)} \arrow{s} \arrow{e} \node{(2)} \\
\node{(3)} \arrow{w} \node{(4)}
\end{diagram}
\end{aligned}
\end{equation}

(The outer circle is for illustration only and omitted later on.) As in the two rightmost illustration in (2.1), called split and merge, we indicate the labels by coloring. By convention, the empty web $\emptyset$ and circle components are also webs.

A web is called closed, if its boundary is the empty set.

Edges come in two different types, namely as ordinary edges which are labeled 1, and phantom edges which are labeled 2, cf. (2.1). We draw phantom edges dashed (and colored); one should think of them as “non-existing”. If we talk for instance about circles in webs, then we will always just ignore all phantom edges.

Next, by a surface we mean a marked, orientable, compact surface with finitely many—possibly none—boundary and connected components. Additionally, by a trivalent surface we understand the same as in [26, Section 3.1], i.e. certain embedded, marked, singular cobordisms whose boundaries are closed webs.

Precisely, fix the following data denoted by $S$, which will be the main ingredient to define an associated closed prefoam.
(SI) A surface $S$ with connected components divided into two sets $S_0^1, \ldots, S_p^1$ and $S_0^2, \ldots, S_p^2$, called ordinary and phantom surfaces, respectively.

(SII) The boundary components of $S$ are partitioned into triples $(C_i^o, C_j^o, C_k^p)$ containing precisely one boundary component $C_k^p$ of a phantom surface.

(SIII) The three circles $C_i^o, C_j^o$, and $C_k^p$ in each triple are identified via diffeomorphisms

$\varphi_{ij} : C_i^o \rightarrow C_j^o$ and $\varphi_{jk} : C_j^o \rightarrow C_k^p$.

(SIV) A finite (possible empty) set of markers per connected components $S_0^1, \ldots, S_p^1$ and $S_0^2, \ldots, S_p^2$, that move freely around its connected component.

**Definition 2.2.** Let $S$ be as above. The closed, singular trivalent surface $f_c = f_c^S$ attached to $S$ is the CW-complex obtained as the quotient of $S$ by the identifications $\varphi_{ij}$ and $\varphi_{jk}$. We call all such $f_c$’s closed prefoams (following [26]) and their markers dots. A triple $(C_i^o, C_j^o, C_k^p)$ becomes one circle in $f_c$ which we call a singular seam, while the interior of the connected components $S_1^1, \ldots, S_p^1$ and $S_1^2, \ldots, S_p^2$ are facets of $f_c$, called ordinary facets and phantom facets. We only consider prefoams which can be properly embedded into $\mathbb{R}^2 \times [-1, 1]$ such that the three annuli glued to a singular seam are consistently oriented—which induces an orientation on the singular seam, compare to (2.2)—and we fix the embedding as well. We consider closed prefoams modulo isotopies in $\mathbb{R}^2 \times [-1, 1]$.

We color phantom facets in what follows. An example of our construction from Definition 2.2 is illustrated in [17, Example 2.3].

We furthermore need prefoams which are not necessarily closed. Following [26, Section 3.3], we consider the $xR$-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ and say that $\mathbb{R}^2$ intersects a closed prefoam $f_c$ generically, if $\mathbb{R}^2 \cap f_c$ is a web.

**Definition 2.3.** A (not necessarily closed) prefoam $f$ is defined as the intersection of $\mathbb{R}^2 \times [-1, 1]$ with some closed foam $f_c$ such that $\mathbb{R}^2 \times \{ \pm 1 \}$ intersects $f_c$ generically. We consider such prefoams modulo isotopies in $\mathbb{R}^2 \times [-1, 1]$ which fix the horizontal boundary. We see such a prefoam $f$ as a singular cobordism between $(\mathbb{R}^2 \times \{-1\}) \cap f_c$ (bottom, source) and $(\mathbb{R}^2 \times \{+1\}) \cap f_c$ (top, target) embedded in $\mathbb{R}^2 \times [-1, 1]$, cf. (2.2). Moreover, there is an evident composition $g \circ f$ via gluing and rescaling (if the top boundary of $f$ and the bottom boundary of $g$ coincide). Similarly, we construct prefoams properly embedded in $\mathbb{R} \times [-1, 1] \times [-1, 1]$ with vertical boundary components. These vertical boundary components should be the boundary of the webs at the bottom/top times $[-1, 1]$. We finally take everything modulo isotopies that preserve the vertical boundary as well as the horizontal boundary.

We call prefoam parts ordinary, if they do not contain singular seams or phantom facets, and we call prefoam parts phantom, if they contain phantom facets only.

By definition, all generic slices of prefoams are webs, and the only vertical boundary components of prefoams $f$ come from the boundary points of webs times $[-1, 1]$.

**Example 2.4.** By construction, each interior point of a prefoam has a local neighborhood of one of the four forms

\begin{equation}
(2.2)
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{image1} & , & \includegraphics[width=0.1\textwidth]{image2} \\
\end{array}
\end{equation}

The facet on the left is an ordinary facet. Whereas the second facet is a phantom facet and the reader might think of it as “non-existing”, similar to a phantom edge. In
We consider Convention 2.8. When we write formulations as “similarly for any specialization of (2.3) let Definition 2.7. We use the following two sets of parameters (which we always view as being ordered as indicated) and associated rings.

\[ P = \{ \alpha, \tau^\pm_0, \tau^\pm_1, \omega^\pm_+, \omega^\pm_- \}, \quad P = \mathbb{Z}[\alpha, \tau^\pm_0, \tau^\pm_1, \omega^\pm_+, \omega^\pm_-], \quad \deg_P(\alpha) = 4. \]

\[ Q = \{ \alpha, \varepsilon, \omega^\pm_1 \}, \quad \varepsilon = \pm 1, \quad Q = \mathbb{Z}[\alpha, \omega^\pm_1], \quad \deg_Q(\alpha) = 4. \]

We consider P and Q to be graded with the indicated degrees.

Remark 2.5. Prefoams are considered modulo boundary preserving isotopies that do preserve the condition that each generic slice is a web. These isotopies form a finite list: isotopies coming from the two cobordism theories associated to the two different types of facets as explained below (see for example [27, Section 1.4]), and isotopies coming from isotopies of the singular seams seen as tangles in \( \mathbb{R}^2 \times [-1,1] \).

2.2. Generic singular TQFTs. We now define P, Q and specializations.

Definition 2.6. We use the following two sets of parameters (which we always view as being ordered as indicated) and associated rings.

\[ \mathbb{P} = \{ \alpha, \tau^\pm_0, \tau^\pm_1, \omega^\pm_+, \omega^\pm_- \}, \quad \mathbb{P} = \mathbb{Z}[\alpha, \tau^\pm_0, \tau^\pm_1, \omega^\pm_+, \omega^\pm_-], \quad \deg_{\mathbb{P}}(\alpha) = 4. \]

\[ \mathbb{Q} = \{ \alpha, \varepsilon, \omega^\pm_1 \}, \quad \varepsilon = \pm 1, \quad \mathbb{Q} = \mathbb{Z}[\alpha, \omega^\pm_1], \quad \deg_{\mathbb{Q}}(\alpha) = 4. \]

We consider P and Q to be graded with the indicated degrees.

We will need Q from Section 3 onwards. Moreover, we denote any mathematical object X with underlying ring P, respectively Q, by \( X[\mathbb{P}] \), respectively by \( X[\mathbb{Q}] \).

Definition 2.7. Let \( p : \mathbb{P} \to \mathbb{R} \) be a ring homomorphism to some ring \( \mathbb{R} \). We denote by \( X_{\mathbb{R}}[p(\alpha), p(\tau_0), p(\tau_1), p(\omega_+), p(\omega_-)] \) the corresponding mathematical object obtained from \( X[\mathbb{P}] \) by specialization via \( p \). Similarly, given a ring homomorphism \( q : \mathbb{Q} \to \mathbb{R} \), we denote by \( X_{\mathbb{R}}[q(\alpha), q(\varepsilon), q(\omega)] \) the corresponding specialization via \( q \).

Abusing notation, we will always use \( p \), respectively \( q \), as a symbol for any specialization of \( \mathbb{P} \), respectively \( \mathbb{Q} \), and sometimes even omit to write \( p(\cdot) \) or \( q(\cdot) \). For example, \( X_{\mathbb{Z}}[0,1,1,1,1] \) will denote the specialization of \( X[\mathbb{P}] \) via \( p(\alpha) = 0 \in \mathbb{Z} \) and all other parameters to 1 \( \in \mathbb{Z} \), and \( X_{\mathbb{Z}}[0,1,1,1] \) will denote the specialization of \( X[\mathbb{Q}] \) via \( q(\alpha) = 0 \in \mathbb{Z} \) and \( q(\varepsilon) = q(\omega) = 1 \in \mathbb{Z} \).

The following degree preserving specialization is very important for us:

\[
(2.3) \quad p : \mathbb{P} \to \mathbb{Q}, \quad p(P) = \{ \alpha, 1, \varepsilon \omega^x, \omega, \varepsilon \omega \}.
\]

Convention 2.8. When we write formulations as “similarly for any specialization of \( \mathbb{P} \)” after some statement, then this is to be understood that the statement holds for any specialization ignoring the grading. (Some specializations do not preserve the grading, but only preserve the filtration obtained from the grading. We will not elaborate on these filtered versions in this paper, but everything works analogously).

To work with the 5-parameter foam 2-category it will be enough (for our purposes) to consider its image under certain monoidal functors from the category of prefoams to the category of free \( P \)-modules called singular TQFTs. To understand our construction, recall that equivalence classes of TQFTs for surfaces are in 1 : 1 correspondence with
isomorphism classes of associative, commutative Frobenius algebras which are free $P$-modules of finite rank. The reader unfamiliar with this might consult Kock’s book [27], which is our main source for these kind of TQFTs. (In fact, Kock works over an arbitrary field, but his arguments work over $P$ as well.) Given such a Frobenius algebra $F$ corresponding to a TQFT $Z_F$, then the association is as follows. To a disjoint union of $m$ objects one associates the $m$-fold tensor product $F^\otimes m$. (If not mentioned otherwise, $\otimes = \otimes_P$.) To a cobordism $\Sigma$ with distinguished incoming and outgoing boundary components consisting of $m$ and $m'$ circles, one assigns a $P$-linear map from $F^\otimes m$ to $F^\otimes m'$. Hereby the usual cup, cap and pants cobordisms correspond to the unit, counit, multiplication and comultiplication maps given by the Frobenius structure. Then the TQFT assigns to a surface $\Sigma$ a $P$-linear map $Z_F(\Sigma) : F^\otimes m \to F^\otimes m'$, which is obtained by decomposing $\Sigma$ into basic cup, cap and pants cobordisms.

To get a singular TQFT we glue two such Frobenius algebras. The Frobenius algebras/TQFT’s we use are as follows.

$$
\begin{align*}
F_o &= P[X]/(X^2 - \alpha), \\
\Delta_o(1) &= 1 \otimes X + X \otimes 1, \\
\tau_o(1) &= 0, \\
\tau_o(X) &= \tau_o, \\
F_p &= P, \\
\Delta_p(1) &= \tau_p^{-1}, \\
\tau_p(1) &= \tau_p.
\end{align*}
$$

(2.4)

These should have the evident units and multiplications, and the indicated counits/traces and comultiplications. The associated TQFTs are denoted by $Z_{F_o}$ and $Z_{F_p}$.

The following construction is inspired from [3] and [17], but generalizes both. We want to construct a 5-parameter, singular TQFT $\mathcal{T}[P]$ on the monoidal category $\tilde{F}$ whose objects are closed webs as in Definition 2.1, whose morphisms are prefoams, with composition being gluing and monoidal product the disjoint union. We define for $a, b, c, d \in P$ gluing maps

$$
\begin{align*}
gl_{F_o} : F_o \otimes F_o \to F_o, \\
(a + bX) \otimes (c + dX) \mapsto (a + \omega \tau_o^{-1} bX)(c + \omega^{-1} \tau_o^{-1} dX), \\
gl_{F_p} : F_p \to F_p, \\
1 \mapsto 1.
\end{align*}
$$

(2.5)

These will play a crucial role for defining the singular TQFT on prefoam because they give us the following way to evaluate closed prefoams.

**Definition 2.9.** Given a closed prefoam $f_c$, let $\tilde{f}_c = f_o \cup f_p$ be the prefoam obtained by cutting $f_c$ along the singular seams, of which we assume to have $m$ in total. Here $f_o$ is the surface which in $f_c$ is associated to the ordinary parts and $f_p$ is the surface which in $f_c$ is associated to phantom parts. Note that the boundary of $f_o$ splits into $\sigma_i^+$ and $\sigma_i^-$ for each $i \in \{1, \ldots, m\}$, determined as follows: use the right-hand rule with the index finger pointing in the direction of the singular seam and the middle finger pointing in direction of the attached phantom facet, then the thumb points in direction of the direction of the path-component whose boundary is $\sigma_i^+$. In contrast, $f_p$ has only boundary components $\sigma_i$ for each $i \in \{1, \ldots, m\}$. Now

$$
\begin{align*}
Z_{F_o}(f_o) &\in \bigotimes_{i=1}^m (Z_{F_o}(\sigma_i^+) \otimes Z_{F_o}(\sigma_i^-)) \cong (F_o \otimes F_o)^\otimes m, \\
Z_{F_p}(f_p) &\in \bigotimes_{i=1}^m Z_{F_p}(\sigma_i) \cong F_p^\otimes m.
\end{align*}
$$

(2.6)

Let $\tau_{F_o} : F_o \to P$ be as in (2.4), and let $\gl_{F_o}, \gl_{F_p}$ be as in (2.5). Then we set $T[P](f_c) = \tau_{F_o}(\gl_{F_o}(Z_{F_o}(f_o))) \otimes \gl_{F_p}(Z_{F_p}(f_p))) \in P^\otimes m \cong P$. This procedure, called evaluation, assigns to any prefoam $f_c$ a value $T[P](f_c) \in P$. □

For an example we refer to the proof of Lemma 2.15 below.
Theorem 2.10. The construction from Definition 2.9 with (2.6) can be extended to a monoidal functor $T[P] : \mathcal{F} \rightarrow P\text{-Mod}_{\text{free}}$.

Proof. This follows by using the universal construction from [4]: First assign to a closed web the free $P$-module of all prefoams bounding it, and to each prefoam the associated $P$-linear map given by composition. Next, consider the $P$-bilinear form induced by the evaluation from Definition 2.9 and taking the quotient by its radical to get the functor $T[P]$. Finally, the relations (2.8) to (2.19) for sure suffice to show that the assignment yields free $P$-modules of finite rank, cf. Lemma 2.21.

We call $T[P]$ the 5-parameter, singular TQFT. Similarly, we call all such monoidal functors singular TQFTs, e.g. for any specialization of $P$.

Note that $\mathcal{F}$ has two important subcategories, i.e. those prefoams with only ordinary parts and those with only phantom parts. We associate the Frobenius algebra $F_o$ to the ordinary parts and the Frobenius algebra $F_p$ to the phantom parts of a prefoam $f$ in the sense that $Z_{F_o}$ can be seen as a monoidal functor on the subcategory with only ordinary parts and $Z_{F_p}$ as a monoidal functor on the subcategory with only phantom parts, both coming from $T[P]$ via restriction.

Example 2.11. In our context, dots correspond to multiplication by $\tau_o^{-1} X$ or $\tau_p^{-1}$:

\[
\begin{align*}
\ast & \mapsto \tau_o^{-1} X : F_o \rightarrow F_o, \\
\ast & \mapsto \tau_p^{-1} : F_p \rightarrow F_p.
\end{align*}
\]

Moreover, if we view a $P$-linear map $\phi : P \rightarrow F_i^\otimes m$ as $\phi(1) \in F_i^\otimes m$, then

\[
\begin{align*}
\ast & \mapsto 1 \in F_o, \\
\ast & \mapsto X \in F_o, \\
\ast & \mapsto 1 \in F_p.
\end{align*}
\]

These are $\iota_o$, $(X) \circ \iota_o$ and $\iota_p$ as maps. The $\iota$'s are called units. The counits/traces $\text{tr}_i$ are obtained by flipping the pictures (and scaling by $\tau_o$ or $\tau_p$).

The $P$-linear maps associated to non-closed prefoams can be determined by closing them in all possible ways using (2.7) and its dual.

Specializations 2.12. Using the specializations $p : P \rightarrow \mathbb{Z}$ given by $p(\alpha) = 0$ respectively $p(\alpha) = 1$ (all other parameters are send to 1), we obtain

\[
\begin{align*}
F & \cong \mathbb{Z}[X]/(X^2) \quad \text{respectively} \quad F_{\text{Lee}} \cong \mathbb{Z}[X]/(X^2 - 1),
\end{align*}
\]

with the latter studied by Lee in her deformation of Khovanov’s complex, see [30]. Specializing via $p(\alpha) = 0$, $p(\tau_o) = 1$, $p(\tau_p) = -1$, $p(\omega+) = 1$ and $p(\omega-) = -1$ we obtain the singular TQFT studied in [3] as well as in [17, Section 2.2].

Next and throughout, we say for short that a relation $f = g$—where $f,g$ are formal $P$-linear combinations of prefoams—lies in the kernel of a (singular) TQFT $\mathcal{T}$, if $\mathcal{T}(f) = \mathcal{T}(g)$ as $P$-linear maps.

Remark 2.13. Later we are often using the specialization of $P$ to $Q$ from (2.3). For convenience, we also indicate in small print, with brackets and in gray the values of the relations in the kernel of (singular) TQFTs under the specialization to $Q$. ■
Lemma 2.14. The ordinary and phantom sphere relations and the dot removing relations as displayed here

\[(2.8) \quad \begin{array}{c}
\begin{array}{c}
\text{dot removing relations}
\end{array}
\end{array} = 0,
\begin{array}{c}
\begin{array}{c}
\text{sphere relations}
\end{array}
\end{array} = \tau_o \quad \text{(1)},
\begin{array}{c}
\begin{array}{c}
\text{dot removing relations}
\end{array}
\end{array} = \tau_p \quad \text{(2.8)},
\begin{array}{c}
\begin{array}{c}
\text{ordinary and phantom neck cutting relations}
\end{array}
\end{array} = \tau_o - 1 \quad \text{(2.9)}\]

as well as the ordinary and phantom neck cutting relations

\[(2.9) \quad \begin{array}{c}
\begin{array}{c}
\text{neck cutting relations}
\end{array}
\end{array} = \tau^{-1} o \quad \text{(1)},
\begin{array}{c}
\begin{array}{c}
\text{neck cutting relations}
\end{array}
\end{array} = \tau^{-1} p \quad \text{(2.10)}\]

are in the kernel of \(Z_{F_o}\) (ordinary) respectively of \(Z_{F_p}\) (phantom).

Proof. A direct computation. For example, the traces from (2.4) immediately give the sphere relations from (2.8). The remaining local relations can be shown by closing the local pictures in all possible ways, e.g.

\[(2.9) \quad \begin{array}{c}
\begin{array}{c}
\text{local relations}
\end{array}
\end{array} = \alpha \quad \text{(2.8)} = 0,\]

(We have indicated one possible closure.)

The neck cutting relations (2.10) give a topological interpretation of dots as a shorthand notation for handles, see also [2, Equation (4)].

By construction, the relations from (2.8), (2.9) and (2.10) are in the kernel of \(T[P]\) as well. The following lemmas give some additional relations in its kernel.

Lemma 2.15. The sphere (or theta foam) relations, i.e.

\[(2.11) \quad \begin{array}{c}
\begin{array}{c}
\text{sphere relations}
\end{array}
\end{array} = \begin{cases}
\omega^+ & \text{if } a = 1, b = 0,
\omega^- & \text{if } a = 0, b = 1,
0 & \text{otherwise},
\end{cases}\]

(with \(a, b \in \mathbb{Z}_{\geq 0}\) dots), are in the kernel of \(T[P]\).

Proof. We prove the case \(a = 0, b = 1\). The others are similar and omitted for brevity.

Decompose the sphere \(f_o\) into (t=thumb, i=index finger, m=middle finger)

Now, because of the assignment in (2.7), we have \(Z_{F_o}(f_o) = 1 \otimes X\) and \(Z_{F_p}(f_p) = 1\). Thus, \(gl_{F_o}(Z_{F_o}(f_o)) = \omega^+ \tau^{-1} X\) and \(gl_{F_p}(Z_{F_p}(f_p)) = 1\), both considered in \(F_o\). Applying the trace \(tr_o\) to \((\omega^- \tau^{-1} X) \otimes 1\) gives \(\omega^-\) as in (2.11).
Remark 2.16. In our story closed foam evaluations, by repeated neck cutting, boil down to (2.8) and (2.11). We can thus say that $P$ gives the most generic twisting (which preserve the grading from Definition 2.18) of the foam evaluation envisioned in [26], [3]. Further, one could also make our approach here $GL_2$-equivariant as in [25] by introducing another parameter for (2.9). This $GL_2$-equivariant story works completely similar to what we do in this paper. But introducing such a parameter makes the notion more cumbersome, so we decided not to do it. $lacksquare$

Lemma 2.17. The bubble removals (where we have a sphere in a phantom plane, with the top dots on the front facets and the bottom dots on the back facets)

\[
\tau_p \omega^{-1} \cdot = \tau_p \omega^{-1} \cdot + (\epsilon \omega) \cdot = 0 = \tau_p \omega^{-1} \cdot \]

are in the kernel of $T[P]$. The (singular) neck cutting relation

\[
(2.13) \quad = \tau_p \omega^{-1} \cdot + \tau_p \omega^{-1} \cdot
\]

(2.13)

(2.13) (with top dot on the front facet and bottom dot on the back facet) is also in the kernel of $T[P]$. Furthermore, the ordinary-to-phantom neck cutting relation (in the leftmost picture the upper closed circle is an ordinary facet) and the squeezing relation

\[
= \tau_p^{-1} \omega^2 \cdot \quad (2.14) \quad = \tau_p \omega^{-2} \cdot
\]

\[
= \tau_p^{-1} \omega^2 \cdot \quad (2.14) \quad = \tau_p \omega^{-2} \cdot
\]

(2.14) (2.15)

(2.14) (2.15)

the dot migration relations

\[
= \omega_+ \omega_-^{-1} \cdot \quad , \quad = \omega_+ \omega_-^{-1} \cdot
\]

\[
= \omega_+ \omega_-^{-1} \cdot \quad , \quad = \omega_+ \omega_-^{-1} \cdot
\]

(2.16)

(2.16)

as well as the closed seam removal relation

\[
= \omega_- \cdot + \omega_+ \cdot
\]

\[
= \omega_- \cdot + \omega_+ \cdot
\]

(2.17)

(2.17)
and the phantom cup removal, the phantom squeeze relation

\[
\begin{align*}
\tau_{P\omega}^{-1}\omega^2 & \quad \Rightarrow \quad \tau_{\epsilon}\omega_+^2 \quad \text{(2.18)} \\
\tau_{P\omega}^{-2}\omega & \quad \Rightarrow \quad \tau_{\epsilon}\omega_+^2 \quad \text{(2.19)}
\end{align*}
\]

and their counterparts—having the phantom facets in the back, and \(\omega_+\) instead of \(\omega_-\)—are also in the kernel of \(T[P]\).

**Proof.** We only prove (2.17). The other relations are verified similarly. First note that we have to consider all possible ways to close the prefoams on the left-hand and on the right-hand side of the equations. For (2.17) we consider the closure

A direct computation, using the relations (2.11) on the left-hand side and (2.8) on the right-hand side, shows that they agree for this closure, i.e. both give \(\omega_-\). All other closures work in the same way—there are only finitely many to consider, cf. Remark 2.16—which shows that (2.17) is in the kernel of \(T[P]\). \(\square\)

The leftmost situation in (2.13) is called a cylinder - as are all local parts of a prefoam \(f\) which are cylinders after removing the phantom facets. Note that the squeezing relation (2.15) enables us to use the neck cutting (2.13) on more general cylinders with possibly internal phantom facets.

If we define a grading on \(F\) by setting \(\deg_F(1) = -1\) and \(\deg_F(x) = 1\), then the TQFT \(Z_F\) respects the grading, where the degree of a cobordism \(\Sigma\) is given by \(\deg(\Sigma) = -\chi(\Sigma) + 2 \cdot \#\text{dots}\). Here \(\chi(\Sigma)\) is the topological Euler characteristic of \(\Sigma\), that is, the number of vertices minus the number of edges plus the number of faces of \(\Sigma\) seen as a CW complex, and \(\#\text{dots}\) is the number of dots. Additionally, we can see \(Z_F\) as being trivially graded. Motivated by this we define the following.

**Definition 2.18.** Given a prefoam \(f\), let \(\hat{f}\) be the CW complex obtained from it by removing the phantom edges and phantom facets. We define a degree of \(f\) via

\[
\deg(f) = -\chi(\hat{f}) + 2 \cdot \#\text{dots} + \frac{1}{2} \#\text{vbound},
\]

where \(\#\text{vbound}\) is the total number of vertical boundary components. \(\blacksquare\)

**Example 2.19.** If \(\hat{f} = \emptyset\), then \(\chi(\hat{f}) = 0\). Moreover, recalling that \(P\) is graded, we can see prefoams now as forming a graded, free \(P\)-module. For example,

\[
\begin{align*}
\alpha & \quad \text{deg} = 6 \\
\beta & \quad \text{deg} = 1 \\
\gamma & \quad \text{deg} = -1 \\
\delta & \quad \text{deg} = -1
\end{align*}
\]

The second prefoam on the right is called a saddle (as well as its horizontal mirror). The prefoams on the left are called cup foam respectively cap foam. \(\blacksquare\)
Having the singular TQFT $\mathcal{T}[P]$, we define the following category for which the relations identified above directly make sense.

**Definition 2.20.** Let $\mathcal{F}_P^{\text{ker}}$ be the monoidal category obtained from $\mathcal{F}$ via $P$-linear extensions and by taking the quotient by the relations in the kernel of $\mathcal{T}[P]$. ■

By Remark 2.16, $\mathcal{F}_P^{\text{ker}}$ is a graded category, and, by the above relations, there are isomorphisms as follows. (We leave it to the reader to check these isomorphisms.)

(2.20) $\begin{array}{c}
\rotatebox{90}{$\cong$} \\
\text{By (2.8),(2.10)}
\end{array}$

Hereby the notation $\{\pm 1\}$ means that the corresponding isomorphisms are of the indicated degree. Moreover, the horizontal mirrors of the isomorphisms from (2.20) hold as well. Taking all of these together we get:

**Lemma 2.21.** $\text{End}_{\mathcal{F}_P^{\text{ker}}}(\emptyset) \cong P$, and all hom spaces of $\mathcal{F}_P^{\text{ker}}$ are free $P$-modules of finite rank. Any specialization is of the same rank as for the 5-parameter version. □

**Proof.** The first statement is clear by the closed foam evaluation. The second statement follows from the first by observing that the isomorphisms in (2.20) suffice to evaluate closed webs, i.e.

$$\text{Hom}_{\mathcal{F}_P^{\text{ker}}}(\emptyset, w_0) \cong \bigoplus_{\text{finite}} \text{Hom}_{\mathcal{F}_P^{\text{ker}}}(\emptyset, w_1) \cong \ldots \cong \bigoplus_{\text{finite}} \text{Hom}_{\mathcal{F}_P^{\text{ker}}}(\emptyset, w_r) \cong \bigoplus_{\text{finite}} P,$$

where each $w_i$ is obtained from its predecessor by one of the isomorphisms in (2.20), and $w_r = \emptyset$. The claim about general hom spaces is then evident by construction. ■

2.3. **Foam 2-categories.** We like to study the following 2-category which we call the 5-parameter foam 2-category.

To this end, we denote by $\mathfrak{b}$ the set of all vectors $\vec{k} = (k_i)_{i \in \mathbb{Z}} \in \{0, 1, 2\}^\mathbb{Z}$ with $k_i = 0$ for $|i| \gg 0$. Abusing notation, we also sometimes write $\vec{k} = (k_a, \ldots, k_b)$ for some fixed part of $\vec{k}$ (with $a < b \in \mathbb{Z}$) where it is to be understood that all non-displayed entries are zero. By convention, the empty vector $\emptyset \in \mathfrak{b}$ is the unique vector containing only zeros. We consider $\vec{k} \in \mathfrak{b}$ as a set of discrete labeled points in $\mathbb{R} \times \{\pm 1\}$ (or in $\mathbb{R} \times \{0\}$) by putting the symbols $k_i$ at position $(i, \pm 1)$ (or $(i, 0)$). If not stated otherwise, then the first non-zero entry of such $\vec{k}$’s is assumed to be $k_i$ for $i = 0$.

Moreover, we say a web is **upwards oriented** if it is embedded in $\mathbb{R} \times \{\pm 1\}$ (with $\mathbb{R}$ being the horizontal direction) such that for each generic horizontal slice its orientations point upwards through this slice. For such webs we can identify its bottom and top boundary with $\vec{k} \in \mathfrak{b}$ in the evident way.

Recall from e.g., [32, Lemma 3.7] that we can regard open prefoams as morphism in $\mathcal{F}$, and hence, in $\mathcal{F}_P^{\text{ker}}$, via clapping, with the example to keep in mind:

(2.21) $\begin{array}{c}
\text{By } (2.18),(2.19)
\end{array}$
Given two webs $u, v$, let $\text{clap}(u), \text{clap}(v)$ be the webs obtained via clapping, e.g.:

Hereby we abuse notation, since there are several ways of clapping. However, one easily sees using the clapping of foams exemplified in (2.21) that the choice of how to clap does not affect the statements we are going to make, and we use them interchangeable, see for example Lemma 2.24 below. Moreover, from now on we use upwards oriented webs if not stated otherwise. By clapping, this is for convenience only.

**Definition 2.22.** Let $\mathfrak{F}[P]$ be the $P$-linear 2-category given by:

- The objects are all $\vec{k} \in b1$ (which includes $\emptyset = (\ldots, 0, 0, \ldots)$).
- The 1-morphisms spaces $\text{Hom}_{\mathfrak{F}[P]}(\vec{k}, \vec{l})$ consists of all upwards oriented webs whose bottom boundary is $\vec{k}$ and whose top boundary is $\vec{l}$ (which includes $\emptyset \in \text{End}_{\mathfrak{F}[P]}(\emptyset)$). We have $\text{Hom}_{\mathfrak{F}[P]}(\vec{k}, \vec{l}) = \emptyset$ iff $k_u + \cdots + k_b = 0 \neq l_{0'} + \cdots + l_{0'y}$.
- The 2-morphisms spaces $2\text{Hom}_{\mathfrak{F}[P]}(u, v)$ are finite, formal $P$-linear combinations of prefoams with bottom boundary $u$ and top boundary $v$.
- We regard the elements of $2\text{Hom}_{\mathfrak{F}[P]}(u, v)$ via clapping as morphisms in $\tilde{\mathcal{F}}^\text{ker}_P$, which induces relations as above.
- Composition of webs $v \circ u = uv$ is stacking $v$ on top of $u$, vertical composition $g \circ f$ of prefoams is stacking $g$ on top of $f$, horizontal composition $g \circ f$ is putting $g$ to the right of $f$, whenever those operations make sense. Hereby we additionally scale the results accordingly.

With the grading from Definition 2.18, the relations are homogeneous, cf. Remark 2.16, which endows $2\text{Hom}_{\mathfrak{F}[P]}(u, v)$ with the structure of a graded $P$-module whose grading is additive under composition. Hence, $\mathfrak{F}[P]$ is a graded, $P$-linear 2-category. 

We call the 2-morphisms in $\mathfrak{F}[P]$ foams, and we adapt all notions we had for prefoams to the setting of foams. Note now that, if one fixes a ring $R$ and a specialization $p: P \to R$, then there exists an induced specialization 2-functor and an induced specialized 2-category $\text{Sp}_{p}: \mathfrak{F}[P] \to \tilde{\mathcal{F}}^\text{ker}_P[p(\alpha), p(\tau_{\omega}), p(\tau_{\omega}^+), p(\omega_+)]$. We keep on calling the 2-morphisms in such specializations foams.

**Example 2.23.** If we see $R$ as trivially graded, then any specialization of $\mathfrak{F}[P]$ with $p(\alpha) = 0$ respects the grading because the relation on the left in (2.9) will be a homogeneous relation, while the others are clearly homogeneous. Thus, in this case, specializations of $\mathfrak{F}[P]$ with $p(\alpha) = 0$ are graded, $R$-linear 2-categories.

We denote by $^\ast$ the involution which flips webs upside down and reverses their orientations. Let $\ell \in \mathbb{Z}_{\geq 0}$ and let $\omega_\ell = (1, \ldots, 1, 0, \ldots, 0)$ with $\ell$ numbers equal 1, and let $1_{2\omega_\ell}$ denote the identity web on $2\omega_\ell$.

Given $\vec{k} \in b1^\ast$ with $\sum_{i \in \mathbb{Z}} k_i = 2\ell$, define the shift $d(\vec{k}) = \ell - \sum_{i \in \mathbb{Z}} k_i(k_i - 1)$. For example, for $\vec{k} \in b1^\ast$ with $k_i \neq 2$ we have $d(\vec{k}) = \ell$. By construction we get a lemma which we will use silently throughout:

**Lemma 2.24.** Given two webs $u, v$. Then

$$2\text{Hom}_{\mathfrak{F}[P]}(u, v) \cong 2\text{Hom}_{\mathfrak{F}[P]}(\text{clap}(u), \text{clap}(v))$$

$$\cong 2\text{Hom}_{\mathfrak{F}[P]}(1_{2\omega_\ell}, \text{clap}(u)\text{clap}(v)^\ast \{d(\vec{k})\}) \cong \text{Hom}_{\tilde{\mathcal{F}}^\text{ker}_P}(\emptyset, \text{clap}(u)\text{clap}(v)^\ast \{d(\vec{k})\})$$
as graded, free $P$-modules.

The following easy, yet important, lemma implies that 2-hom spaces of $\mathfrak{g}[P]$ are free $P$-modules of finite rank (as we show below). Moreover, it also justifies to think of foams between webs which have only phantom edges as being “closed”.

**Lemma 2.25.** $\text{2End}_{\mathfrak{g}[P]}(1_{2\omega}) \cong P$, and all 2-hom spaces of $\mathfrak{g}[P]$ are free $P$-modules of finite rank. Any specialization is of the same rank as for the 5-parameter version. □

**Proof.** Immediately from Lemma 2.21 and Lemma 2.24. □

Let $\text{CUP}(\vec{k}) = \text{Hom}_{\mathfrak{g}[P]}(2\omega\ell, \vec{k})$, with its elements called cup webs.

**Remark 2.26.** Fix cup webs $u, v \in \text{CUP}(\vec{k})$ and consider $\text{2Hom}_{\mathfrak{g}[P]}(1_{2\omega\ell}, uv^*)$. We recall a certain basis $u B \circ (\vec{k}) v$ for this 2-hom space which we call the cup foam basis. Its formal definition is a bit involved, and we refer the reader to [17, Definition 4.12] which can be adapted to our setup without any problems.

Its informal description—which suffices for our paper—is easy: Note that the isomorphisms from (2.20) induce isomorphisms—including a right-handed version of the second isomorphism—given by foams

\[
\begin{pmatrix}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{pmatrix}
\xrightarrow{\tau}
\begin{pmatrix}
\begin{array}{c}
\{+1\} \\
\{-1\}
\end{array}
\end{pmatrix},
\]

which we fix as above. Now, fix a basis point $\uparrow$ for each circle in $uv^*$ to be on the segment of the circles in question which contains the point with the biggest $x$ coordinate and the lowest $y$ coordinate (“the rightmost bottom segment”), recalling that we see $uv^*$ as being embedded in the $xy$-plane. Moreover, fix an evaluation of $uv^*$, i.e. a sequence of isomorphisms as in (2.22) which recursively reduce $uv^*$ to $1_{2\omega\ell}$. This evaluation has to be chosen such that the segments of $uv^*$ containing the basis points are only removed using the leftmost isomorphism in (2.22). Having fixed such an evaluation, we can define the cup foam basis inductively by reading the corresponding isomorphisms—which we have fixed as in (2.22)—from $1_{2\omega\ell}$ to $uv^*$. □

The reader might think about cup foam basis elements as “cups” in the most naive sense of the word.

**Example 2.27.** The cup foam basis is best explained in an example. Let $\uparrow$ denote the fixed facet—in general, where there will be several such facets—on which all dots appearing on foams in $u B^{\circ}(\vec{k}) v$ will be placed. Next, the cup foam basis can be read off from the webs in question, if we consider a movie of generic slices as e.g.

Hereby each arrow represents half of the corresponding foams in (2.13) or (2.15). In this case, we have chosen a particular evaluation of the web. (In fact, there was no
choice in this case, since we want to keep the basis points till the very end.) The corresponding cup foam basis elements—there are four of them—can be read off by following the arrows from right to left.

**Lemma 2.28.** Let \( u, v \in \text{CUP}(\tilde{k}) \). The set \( _uB^p(\tilde{k})_v \) is a homogeneous, \( P \)-linear basis of the space \( 2\text{Hom}_P(1_{2\omega}, u^*v^*) \). (Similarly for any specialization of \( P \).) □

**Proof.** Almost word-by-word as in [17, Lemma 4.13] and left to the reader. □

2.4. **Known specializations.** Several 2-categories which appear in the literature (i.e. the examples from (Sp), but keeping \( \alpha \) generic) are specializations of \( \mathfrak{F}[P] \).

**Definition 2.29.** We define three 2-categories, denoted by \( \mathfrak{F}_{\alpha}[\text{KBN}] \), \( \mathfrak{F}_{\alpha}[\text{Ca}] \) and \( \mathfrak{F}_{\alpha}[\text{CMW}] \), as in Definition 2.22 except for the following differences.

\( \triangleright \) In all three cases: as objects one allows only \( \tilde{k} \in \mathcal{B} \) without entries 2.

\( \triangleright \) As 1-morphisms one has “webs” generated by the elements in (2.24) (we have already indicated the assignment for the 2-functors we define below)

\[
\begin{align*}
\text{(2.23)} & \quad \begin{array}{ccc}
\includegraphics[width=2cm]{web1.png} & \quad \begin{array}{ccc}
\includegraphics[width=2cm]{web2.png} & \quad \begin{array}{ccc}
\includegraphics[width=2cm]{web3.png} & \quad \begin{array}{ccc}
\includegraphics[width=2cm]{web4.png}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

rathet than webs in the sense of Definition 2.1.

\( \triangleright \) On the 2-morphisms level they are defined in the proof of Proposition 2.31.

The 2-category \( \mathfrak{F}_{\alpha}[\text{Bl}] \) is defined as the specialization of the 2-category \( \mathfrak{F}[P] \) via \( p(\alpha) = 0, p(\tau_o) = 1, p(\tau_p) = -1, p(\omega_+) = 1 \) and \( p(\omega_-) = -1 \), with values in \( \mathbb{Z}[\alpha] \).

We consider all of them as graded, \( R \)-linear 2-categories (for \( R \) being either \( \mathbb{Z}[\alpha] \), in cases KBN and Bl, or \( \mathbb{Z}[\alpha, i] \), in cases Ca and CMW, with \( \deg_R(\alpha) = 4 \)). □

**Remark 2.30.** The 2-category \( \mathfrak{F}_{\alpha}[\text{KBN}] \) coincides with the 2-category studied in [23, Section 2.3] and also in [2, Section 11.2]. The 2-categories \( \mathfrak{F}_{\alpha}[\text{Ca}], \mathfrak{F}_{\alpha}[\text{CMW}] \) and \( \mathfrak{F}_{\alpha}[\text{Bl}] \) are only 2-subcategories of the 2-categories considered in [10, Section 2], in [14, Section 2.2] and in [3, Section 1] respectively, since we only allow upwards oriented webs and also only allow disorientation lines coming from singular seams. These 2-subcategories however suffice for the construction of the Khovanov complex and the corresponding higher link and tangle invariants. □

**Proposition 2.31.** There are specializations of the parameters \( P \) and equivalences of graded, \( R \)-linear 2-categories (which are in fact isomorphisms)

\[
\begin{align*}
\mathfrak{F}_{\alpha}[\mathbb{Z}[\alpha, 1, 1, 1, 1]] & \xrightarrow{\cong} \mathfrak{F}_{\alpha}[\text{KBN}], & \mathfrak{F}_{\alpha}[\mathbb{Z}[\alpha, 1, 1, i, -i]] & \xrightarrow{\cong} \mathfrak{F}_{\alpha}[\text{Ca}], & \mathfrak{F}_{\alpha}[\mathbb{Z}[\alpha, 1, 1, i, -i]] & \xrightarrow{\cong} \mathfrak{F}_{\alpha}[\text{CMW}], & \mathfrak{F}_{\alpha}[\mathbb{Z}[\alpha, 1, -1, 1, -1]] & \xrightarrow{\cong} \mathfrak{F}_{\alpha}[\text{Bl}],
\end{align*}
\]

extending (2.23), (2.24). Here \( R = \mathbb{Z}[\alpha] \) in the first and fourth case and \( R = \mathbb{Z}[\alpha, i] \) in the other two cases. (Similarly for any further specialization of \( \alpha \).) □

From now on we will identify the various 2-categories and their specializations.

**Proof.** We need to define the 2-categories in question on the level of 2-morphisms and then the 2-functors which provide the equivalences. The 2-morphisms of \( \mathfrak{F}_{\alpha}[\text{KBN}] \) are \( \mathbb{Z}[\alpha] \)-linear combinations (modulo the relations below) of prefoams with only
ordinary parts. The 2-morphisms of $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{Ca}]$ are $\mathbb{Z}[\alpha, i]$-linear combinations of the topological CW complexes obtained from prefoams by removing the phantom edges and phantom facets, but keeping the singular seams (modulo the relations below). Moreover, the 2-morphisms of $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{CMW}]$ are $\mathbb{Z}[\alpha, i]$-linear combinations of these with extra disorientation tags and lines (modulo the relations below). For example

Here we assume that such disorientation lines all come from singular seams as explained below, see also Remark 2.30. The relations for $\hat{\mathcal{Z}}_{[\alpha]}[\text{KBN}]$, $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{Ca}]$ and $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{CMW}]$ which are imposed upon the 2-morphisms are the ordinary sphere, dot removing and neck cutting relations from (2.8), (2.9) and (2.10).

For $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{Ca}]$ we we remove the phantom facets and additionally impose the relations from Lemma 2.15 and Lemma 2.17 for the specialization $p(\tau_o) = 1, p(\tau_p) = 1, p(\omega_+) = i$ and $p(\omega_-) = -i$; for $\hat{\mathcal{Z}}_{[\alpha, i]}[\text{CMW}]$ we additionally impose the disorientation removals and all local relations they induce by closing in all possible ways:

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{disorientation1} \\
\end{array} = i \cdot \begin{array}{c}
\includegraphics[width=1cm]{disorientation2} \\
\end{array}, \\
\begin{array}{c}
\includegraphics[width=1cm]{disorientation3}
\end{array} = -i \cdot \begin{array}{c}
\includegraphics[width=1cm]{disorientation4}
\end{array}
\end{align*}$$

The grading in all cases is, as in Definition 2.18, induced by the topological Euler characteristic. In particular, disorientation lines do not change the degree.

Thus, we have 2-functors

$$\begin{align*}
\Gamma_{\text{KBN}} : \hat{\mathcal{Z}}_{[\alpha]}[\alpha, 1, 1, 1] &\to \hat{\mathcal{Z}}_{[\alpha]}[\text{KBN}], \\
\Gamma_{\text{Ca}} : \hat{\mathcal{Z}}_{[\alpha, i]}[\alpha, 1, i, -i] &\to \hat{\mathcal{Z}}_{[\alpha, i]}[\text{Ca}]
\end{align*}$$

given on objects by replacing every entry 2 in a given $\vec{k}$ by a 0, on 1-morphisms by (2.24) and on 2-morphisms by removing the phantom edges and phantom facets for $\Gamma_{\text{Ca}}$ and additionally by removing the singular seams for $\Gamma_{\text{KBN}}$. For example

That these 2-functors are well-defined, grading preserving $\mathbb{Z}[\alpha]$-linear (respectively $\mathbb{Z}[\alpha, i]$-linear) follows directly by comparing the resulting specialized relations from (2.8), (2.9) and (2.10), and from Lemma 2.15 and Lemma 2.17. Clearly, $\Gamma_{\text{KBN}}$ and $\Gamma_{\text{Ca}}$ are essential surjective on objects and 1-morphisms and full on 2-morphisms. That they are faithful on 2-morphisms is evident (one can check this on the cup basis from Remark 2.26 which one easily writes down, mutatis mutandis, for $\Gamma_{\text{KBN}}$ and $\Gamma_{\text{Ca}}$ as well), which shows that they are equivalences as claimed.

We define a 2-functor $\Gamma_{\text{CMW}} : \hat{\mathcal{Z}}_{[\alpha]}[\alpha, 1, i, -i] \to \hat{\mathcal{Z}}_{[\alpha]}[\text{CMW}]$ on objects and on 1-morphisms analogously to the two 2-functors from above, but using the third assignment in (2.24). On 2-morphisms it is defined by removing all phantom edges and phantom facets and replacing singular seams by disorientation lines, where the
orientation of the seam induces the direction of the disorientation line:

with disorientation lines pointing into the paper. One can check directly that the relations from $\delta_{Z[\alpha,i]}[\alpha,1,1,-i,-i]$ hold in the image of $\Gamma_{CMW}$ which shows that $\Gamma_{CMW}$ is well-defined. For example, the disorientation removals (2.25) on level of foams are (with $p(\tau_o) = 1$, $p(\tau_p) = 1$, $p(\omega_+) = i$ and $p(\omega_-) = -i$):

These imply that the relation [14, Figure 3] holds in the image of $\Gamma_{CMW}$. As above, it follows also that $\Gamma_{CMW}$ is a grading preserving, $\mathbb{Z}[\alpha,i]$-linear 2-functor which is essential surjective on objects and 1-morphisms, as well as fully faithful on 2-morphisms. This shows that $\Gamma_{CMW}$ gives the claimed equivalence. Last, $\delta_{Z[\alpha,i][Bl]}$ is defined precisely as in Definition 2.22, but with the choice of parameters $p(\alpha) = 0$, $p(\tau_o) = 1$, $p(\tau_p) = -1$, $p(\omega_+) = 1$ and $p(\omega_-) = -1$. Thus, the statement for this case follows directly from the definition (which formally uses Lemma 2.28 again). Moreover, the cases with specialized $\alpha$ work similarly which finishes the proof. ■

2.5. Web algebras. We define the following algebraic version $\mathcal{W}[P]$ of $\mathfrak{F}[P]$. As we will see later in Proposition 4.34, when passing to $Q$, the 2-category $\mathcal{F}[P]$ will be equivalent to a certain $\mathcal{W}[P]$-bimodule 2-category as defined in Definition 2.44.

Definition 2.32. Denote by $b^{\circ} \subset b1$ the set of all $\bar{k} \in b1$ which have an even number of entries 1. We call elements of $b^{\circ}$ balanced. ■

Definition 2.33. Let $\bar{k} \in b^{\circ}$, $u,v \in \text{CUP}(\bar{k})$. We denote by $u(\mathcal{W}[P]_{\bar{k}})_{v}$ the space $\mathcal{2}\text{Hom}_{\mathfrak{F}}(1_{2\omega}, uv^{\ast})\{d(\bar{k})\}$. The 5-parameter web algebra $\mathcal{W}[P]_{\bar{k}}$ and the (full) 5-parameter web algebra $\mathcal{W}[P]$ are the $P$-modules

$$\mathcal{W}[P]_{\bar{k}} = \bigoplus_{u,v \in \text{CUP}(\bar{k})} u(\mathcal{W}[P]_{\bar{k}})_{v}, \quad \mathcal{W}[P] = \bigoplus_{\bar{k} \in b^{\circ}} \mathcal{W}[P]_{\bar{k}}.$$ We consider these as graded $P$-modules by using the degree from Definition 2.18. Moreover, we endow them with a multiplication

$$\text{Mult}: \mathcal{W}[P]_{\bar{k}} \otimes \mathcal{W}[P]_{\bar{k}} \rightarrow \mathcal{W}[P]_{\bar{k}}, \quad f \otimes g \mapsto \text{Mult}(f,g) = fg$$

using multiplication foams as follows. To multiply $f \in u(\mathcal{W}[P]_{\bar{k}})$, with $g \in z(\mathcal{W}[P]_{\bar{k}})$ to obtain $fg$ stack the diagram $\tilde{v}w^{\ast}$ on top of $wv^{\ast}$ and obtain $wv^{\ast}\tilde{v}w^{\ast}$. Then $fg = 0$ if $v \neq \tilde{v}$. Otherwise, pick any cup-cap pair as in (2.27) and perform a so-called surgery inductively: Start with $v_0 = v$ and with sad $o$ being the identity foam on $v^{\ast}v$. Then replace in each step $v_r$ by $v_{r+1}$ and set $\text{sur}_{r+1} = \text{saddle foam} \circ \text{sur}_r$:

$$v_r \quad \text{saddle foam} \quad v_{r+1}$$
where the saddle foam is locally of the following form (and the identity elsewhere)

\[(2.28)\]

Now, start with a foam \(f \in 2\text{Hom}_\mathfrak{F}(1_{2\omega}, uv^*vw^*)\) and stack on top of it the foam \(\text{id}_u\text{sur}_l\text{id}_w^*\) for \(l\) being the last step in the surgery process. This gives inductively rise to a multiplication foam. (After the last step we collapse the webs and foams as is easiest explained in an example, see Example 2.35.) Compare also to [32, Definition 3.3]. ■

One nice feature of web algebras is that the following lemma is clear, since the web algebras are defined topologically via singular TQFTs.

**Lemma 2.34.** The map \(\text{Mult} : \mathcal{W}_P[\ell] \otimes \mathcal{W}_P[\ell] \rightarrow \mathcal{W}_P[\ell]\) given above is degree preserving and independent of the order in which the surgeries are performed. This turns \(\mathcal{W}_P[\ell]\) into a graded, associative, unital algebra, which is a free \(P\)-module. (Similarly for the locally unital algebra \(\mathcal{W}_P[\ell]\).) □

**Proof.** That they are free \(P\)-modules follows from Lemma 2.28. Everything else follows by identifying the multiplication in the general web algebras with composition in \(\mathfrak{F}[\ell]\) via clapping, see e.g. [17, Lemma 2.26] or [32, Lemma 3.7]. ■

**Example 2.35.** An easy multiplication example for \(u = v = w \in \text{CUP}((1,1))\) is

\[
\begin{array}{c}
\text{mult. foam} \\
\text{underneath}
\end{array}
\]

where the reader should think about any foam \(f : 1_{2\omega} \rightarrow uv^*vw^*\) sitting underneath (as illustrated in one case above). The rightmost step above is the collapsing step and usually omitted from illustrations. The saddle is of degree 1 and thus, taking the shift \(d((1,1)) = 1\) into account, the multiplication foam is of degree zero. ■

**Remark 2.36.** Everything from this and the next section goes through for \(Q\) or any other specialization as well. In particular, we have a graded \(Q\)-linear 2-category \(\mathfrak{F}[Q]\), called the signed 2-parameter foam 2-category, and graded algebras \(\mathcal{W}_Q[\ell]\) and \(\mathcal{W}_Q[\ell]\) called signed 2-parameter web algebras. These still include the examples from (Sp). ■

**Specializations 2.37.** By Proposition 2.31, the specialization \(\mathcal{W}_{Z[\alpha]}[\alpha, 1, 1, 1, 1]\) is graded isomorphic to the algebra \(\mathcal{W}_{Z[\alpha]}[\text{KBN}]\). Similarly, \(\mathcal{W}_{Z[\alpha]}[\alpha, 1, -1, 1, -1]\) is graded isomorphic to \(\mathcal{W}_{Z[\alpha]}[\text{Bl}]\). Moreover, we can view \(\mathcal{W}_{Z[\alpha]}[\alpha, 1, 1, i, -i]\) as describing the setups of \(\text{Ca}\) or \(\text{CMW}\), see also Specializations 2.47. ■
2.6. Web bimodules and foam 2-categories. We still consider only \( \vec{k}, \vec{l} \in b1^\circ \).

**Definition 2.38.** Given any \( u \in \Hom_\mathfrak{A}[\vec{k}, \vec{l}] \) (with boundaries \( \vec{k} \) and \( \vec{l} \) summing up to \( 2\ell \)), we consider the \( \mathfrak{W}[P] \)-bimodule

\[
\mathcal{W}[P](u) = \bigoplus_{v \in \CUP(\vec{k})} \Hom_\mathfrak{A}[\vec{k}](1_{2\omega}, vv^*)
\]

with left (bottom) and right (top) action of \( \mathfrak{W}[P] \) as in **Definition 2.33**. We call all such \( \mathfrak{W}[P] \)-bimodules \( \mathcal{W}[P](u) \) web bimodules.

**Definition 2.39.** Given \( u \in \Hom_\mathfrak{A}[\vec{k}, \vec{l}] \), define a cup foam basis \( \mathfrak{B}^\circ(u) \) of \( \mathcal{W}[P](u) \) as in **Remark 2.26** by considering all webs \( vv^* \) for \( v \in \CUP(\vec{k}), w \in \CUP(\vec{l}) \).

**Lemma 2.40.** Let \( u \in \Hom_\mathfrak{A}[\vec{k}, \vec{l}] \). The set \( \mathfrak{B}^\circ(u) \) is a homogeneous, \( P \)-linear basis of the web bimodule \( \mathcal{W}[P](u) \). (Similarly for any specialization of \( P \).)

**Proof.** Analogous to **Lemma 2.28** and thus, omitted. See also [17, Lemma 4.14] for the proof with specialized parameters (which still works almost word-by-word).

The following is now evident.

**Corollary 2.41.** All web bimodules are free \( P \)-modules with finite-dimensional subspaces for all pairs \( v, w \). Any specialization of these subspaces is of the same rank as for the 5-parameter version.

**Lemma 2.42.** Let \( u \in \Hom_\mathfrak{A}[\vec{k}, \vec{l}] \) be a web. Then the left (bottom) action of \( \mathfrak{W}[P]_\vec{k} \) as well as the right (top) action of \( \mathfrak{W}[P]_\vec{l} \) on \( \mathcal{W}[P](u) \) are well-defined and commute. Hence, \( \mathcal{W}[P](u) \) is a \( \mathfrak{W}[P]_\vec{k} \)-\( \mathfrak{W}[P]_\vec{l} \)-bimodule and thus, a \( \mathfrak{W}[P] \)-bimodule. (Similarly for any specialization of \( P \).)

**Proof.** Let \( u \in \Hom_\mathfrak{A}[\vec{k}, \vec{l}] \). The left (bottom) action of \( \mathfrak{W}[P]_\vec{k} \) and the right (top) action of \( \mathfrak{W}[P]_\vec{l} \) on \( \mathcal{W}[P](u) \) commute since they are “far apart”. Hence, \( \mathcal{W}[P](u) \) is a \( \mathfrak{W}[P]_\vec{k} \)-\( \mathfrak{W}[P]_\vec{l} \)-bimodule (and thus, a \( \mathfrak{W}[P] \)-bimodule). The same works word-by-word for any specialization of \( P \) which shows the statement.

**Proposition 2.43.** All \( \mathcal{W}[P](u) \) are graded biprojective, \( \mathfrak{W}[P] \)-bimodules which are free \( P \)-modules with finite-dimensional subspaces for all pairs \( v, w \). (Similarly for any specialization of \( P \).)

**Proof.** Clearly, they are graded. By **Lemma 2.42** and **Corollary 2.41** it remains to show that they are biprojective. This follows, since they are direct summands of some \( \mathfrak{W}[P]_\vec{k} \) (of \( \mathfrak{W}[P]_\vec{l} \)) as left (right) modules and for suitable \( \vec{k} \in b1^\circ \) (or \( \vec{l} \in b1^\circ \)). Again, the arguments are parameter independent which shows the statement.

**Definition 2.44.** Let \( \mathfrak{W}[P] \)-\( \text{biMod}^{P}_{gr} \) be the following 2-category:

- Objects are the various \( \vec{k} \in b1^\circ \).
- 1-morphisms are finite direct sums and tensor products (taken over the algebra \( \mathfrak{W}[P] \)) of the \( \mathfrak{W}[P] \)-bimodules \( \mathcal{W}[P](u) \).
- 2-morphisms are \( \mathfrak{W}[P] \)-bimodule homomorphisms.
- The composition of web bimodules is the tensor product \( \cdot \otimes_{\mathfrak{W}[P]} \cdot \). The vertical composition of \( \mathfrak{W}[P] \)-bimodule homomorphisms is the usual composition. The horizontal composition is given by tensoring (over \( \mathfrak{W}[P] \)).

We consider \( \mathfrak{W}[P] \)-\( \text{biMod}^{P}_{gr} \) as a graded 2-category by turning the 2-homs into graded \( P \)-modules (in the sense of (1.2)) via **Definition 2.18**.
As usual, we also consider specializations of $\mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}}$, e.g. any specialization $p(\alpha) = 0$ yields a graded 2-category.

This 2-category provides a faithful 2-representation of the 2-category $\mathcal{S}[\mathcal{P}]$ we are interested in as follows. Recall that the additive closure $\oplus(C)$ of a 2-category $\mathcal{C}$ has the same objects as $\mathcal{C}$, but one allows finite formal direct sums of 1-morphisms from $\mathcal{C}$ and matrices between these these as 2-morphisms. The reader unfamiliar with this construction is referred to [2, Definition 3.2] for a thorough treatment.

**Proposition 2.45.** There is an embedding of graded, $P$-linear 2-categories

$$\Upsilon : \oplus(\mathcal{S}[\mathcal{P}]) \hookrightarrow \mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}},$$

which is bijective on objects and essential surjective on 1-morphisms. (Similarly for any specialization of $P$.)

**Remark 2.46.** The web bimodules $\mathcal{W}[\mathcal{P}](u)$ are infinite-dimensional in a “stupid” way since they are defined by taking all possible closures with webs. This can be easily avoided by restricting to closures with so-called basis webs as defined in (4.6). In particular, there is a 2-category $\mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}}$ consisting of based web bimodules, which is closely related to $\mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}}$. In fact, we will see in Proposition 4.34 that $\Upsilon$ gives rise to an equivalence onto $\mathcal{W}[\mathcal{Q}]-\text{biMod}^p_{\text{gr}}$. This is non-trivial and relies on the isomorphisms from Section 4, since there could potentially be plenty of uncontrollable $\mathcal{W}[\mathcal{P}]-bimodule$ homomorphisms.

**Proof.** Define $\Upsilon : \oplus(\mathcal{S}[\mathcal{P}]) \to \mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}}$ via (and then extend additively):

- On objects $\vec{k}$ we set $\Upsilon(\vec{k}) = \vec{k}$.
- On 1-morphisms $u \in \text{Hom}_{\mathcal{S}[\mathcal{P}]}(\vec{k}, \vec{l})$ we set $\Upsilon(u) = \mathcal{W}[\mathcal{P}](u)$.
- On 2-morphisms $f \in \text{2Hom}_{\mathcal{S}[\mathcal{P}]}(u, v)$ we set $\Upsilon(f) : \mathcal{W}[\mathcal{P}](u) \to \mathcal{W}[\mathcal{P}](v)$ given by stacking $f$ on top of the elements of $\mathcal{W}[\mathcal{P}](u)$.

Note that $\Upsilon(f)$ is a $\mathcal{W}[\mathcal{P}]-bimodule$ homomorphism. This can be seen topologically: $\mathcal{W}[\mathcal{P}]$ acts on elements of web bimodules “horizontally”, while $f$ is stacked “vertically”. (The meticulous reader can copy the arguments from [24, Section 2.7].) Thus, $\Upsilon$ extends to a $P$-linear 2-functor. Since $\Upsilon$ is clearly bijective on objects, it remains to show that $\Upsilon$ is essential surjective on 1-morphisms and faithful.

- **Essential surjective on 1-morphisms.** Each 1-morphism of $\mathcal{W}[\mathcal{P}]-\text{biMod}^p_{\text{gr}}$ is by definition of the form $\mathcal{W}[\mathcal{P}](u)$, a finite direct sum or a tensor product (over $\mathcal{W}[\mathcal{P}]$) of these. Note that $\mathcal{W}[\mathcal{P}](u) \otimes_{\mathcal{W}[\mathcal{P}]} \mathcal{W}[\mathcal{P}](v)$ is isomorphic to $\mathcal{W}[\mathcal{P}](uv)$. This follows as in [24, Theorem 1]. We note hereby that Khovanov’s arguments are parameter free. Thus, $\Upsilon$ is essential surjective on 1-morphisms.

- **Faithful.** By clapping we have a cup foam basis for $2\text{Hom}_{\mathcal{S}[\mathcal{P}]}(u, v)$ using these identifications. By construction, these are sent via $\Upsilon$ to linearly independent $\mathcal{W}[\mathcal{P}]-$homomorphisms. This shows faithfulness of $\Upsilon$, since passing to the additive closure does not change the arguments from above.

Clearly, $\Upsilon$ is degree preserving. Note also that the arguments from above are independent of the precise form of the parameters from $P$. Hence, the same holds word-by-word for any specialization. The statement follows.
Specializations 2.47. By Proposition 2.31 and Proposition 2.45 we get embeddings of graded, $R$-linear 2-categories (for $R = \mathbb{Z}[\alpha]$ in case one and four or $R = \mathbb{Z}[\alpha, i]$ else)
\[
\begin{align*}
\oplus(\mathcal{R}[\text{KBN}]) & \hookrightarrow \mathcal{W}[\text{KBN}]-\text{biMod}_{gr}^p, & \oplus(\mathcal{R}[\text{Ca}]) & \hookrightarrow \mathcal{W}[\text{Ca}]-\text{biMod}_{gr}^p, \\
\oplus(\mathcal{R}[\text{CMW}]) & \hookrightarrow \mathcal{W}[\text{CMW}]-\text{biMod}_{gr}^p, & \oplus(\mathcal{R}[\text{Bl}]) & \hookrightarrow \mathcal{W}[\text{Bl}]-\text{biMod}_{gr}^p.
\end{align*}
\]
We will see later in Specializations 4.35 that these are actually equivalences.

3. A family of arc algebras

Now we define a $\mathbb{Q}$-version of Khovanov’s arc algebra [24]. (Recall that the parameters from $\mathbb{Q}$ are specializations of those from $\mathbb{P}$, see (2.3).) Combinatorially this will follow [17], but the multiplication will be more involved, incorporating $\mathbb{Q}$.

3.1. Combinatorics of arc diagrams. In this section we summarize the combinatorics of arc diagrams. The examples to keep in mind while reading the formal definitions are (oriented) cups, caps and rays, and weights and block sequences:

\[
\begin{align*}
\text{(3.1)} & \quad \begin{array}{c}
\bigwedge \lambda = \lor \\
\times \circ \lambda = \lhd \\
\bigvee \lambda = \land
\end{array}, & \begin{array}{c}
\bigcirc \times \lambda = \lor \\
\bigcirc \circ \lambda = \lhd \\
\bigcirc \bigvee \lambda = \land
\end{array}, & \begin{array}{c}
\bigwedge \lambda = \land \\
\bigvee \lambda = \lor
\end{array}, & \begin{array}{c}
\bigcirc \times \text{seq}(\lambda) = \lor \times \\
\bigcirc \circ \text{seq}(\lambda) = \lhd \times \\
\bigcirc \bigvee \text{seq}(\lambda) = \land \times
\end{array}.
\end{align*}
\]

Definition 3.1. A (diagrammatical) weight is a sequence $\lambda = (\lambda_i)_{i \in \mathbb{Z}}$ with entries $\lambda_i \in \{\lhd, \lor, \land, \star\}$, such that $\lambda_i = \lor$ for $|i| \gg 0$. Two weights $\lambda$ and $\mu$ are said to be equivalent if one can obtain $\mu$ from $\lambda$ by permuting some symbols $\land$ and $\lor$. The equivalence classes of weights are called blocks, whose set will be denoted by $\mathfrak{b}l$.

If we display such weights or blocks, then the first entry which is not $\lor$ is assumed to be at $i = 0$ (if not stated otherwise).

As in [17, Definition 3.2] a block can be determined by giving a block sequence and demanding a certain number of symbols $\land$ and $\lor$ to appear in its weights.

Definition 3.2. Let $\Lambda \in \mathfrak{b}l$ be a block. To $\Lambda$ we associate its (well-defined) block sequence $\text{seq}(\Lambda) = (\text{seq}(\Lambda_i))_{i \in \mathbb{Z}}$ by taking any $\lambda \in \Lambda$ and replacing the symbols $\land, \lor$ by $\star$. Moreover, we define $\text{up}(\Lambda)$, respectively $\text{down}(\Lambda)$, to be the total number of $\land$’s, respectively $\lor$’s, in $\Lambda$ where we count $\times$ as both, $\land$ and $\lor$.

The definitions presented in this section will only make use of balanced blocks, i.e. blocks $\Lambda$ with $\text{up}(\Lambda) = \text{down}(\Lambda)$, see [17, Definition 3.3], and we denote by $\mathfrak{b}l^b \subset \mathfrak{b}l$ the set of balanced blocks. For example, the third weight in (3.1) does not belong to a balanced block, but all others therein do.

As in [17, Section 3.1], a cup diagram $c$ is a finite collection of non-intersecting arcs inside $\mathbb{R} \times [-1, 0]$ such that each arc intersects the boundary exactly in its endpoints, and either connecting two distinct points $(i, 0)$ and $(j, 0)$ with $i, j \in \mathbb{Z}$ (called a cup), or connecting one point $(i, 0)$ with $i \in \mathbb{Z}$ with a point on the lower boundary of $\mathbb{R} \times [-1, 0]$ (called a ray). Furthermore, each point in the boundary is the endpoint of at most one arc. Two cup diagrams are equal if the arcs contained in them connect the same points. One can reflect a cup diagram $c$ along the axis $\mathbb{R} \times \{0\}$, denote this operation by $\ast$, to obtain a cap diagram $c^\ast$ (defined inside $\mathbb{R} \times [0, 1]$). Clearly, $(c^\ast)^\ast = c$.

A cup diagram $c$ (and similarly a cap diagram $d^\ast$) is compatible with a block $\Lambda \in \mathfrak{b}l$ if $\{ (i, 0) | \text{seq}(\Lambda_i) = \star \} = (\mathbb{R} \times \{0\}) \cap c$.

We will view a weight $\lambda$ as labeling integral points, called vertices, of the horizontal line $\mathbb{R} \times \{0\} \subset \mathbb{R} \times [-1, 0]$ (or $\mathbb{R} \times \{0\} \subset \mathbb{R} \times [0, 1]$ for caps) by putting the symbol $\lambda_i$ at position $(i, 0)$. Together with a cup diagram $c$ this forms a new diagram $c\lambda$. 


Definition 3.3. We say that $c\lambda$ is oriented if:

(OrI) An arc in $c$ only contains vertices labeled $\land$ or $\lor$, and every vertex labeled $\land$ or $\lor$ is contained in an arc.

(OrII) The two vertices of a cup are labeled by exactly one $\land$ and one $\lor$.

(OrIII) For $i < j$ with $\lambda_i = \lor$, $\lambda_j = \land$ at most one, $\lambda_i$ or $\lambda_j$, is contained in a ray. ■

Remark 3.4. We will restrict to arc diagrams consisting of cups and caps only for the main part of our paper. To get the generalized arc algebra—which is the quasi-hereditary cover of the arc algebra we are going to define below—one needs to include the rays as well. However, as we will explain in Section 5.1, the generalized arc algebra can be recovered from the arc algebra as a subquotient. Consequently, restricting to cups and caps, is for convenience only. Note hereby that the condition (OrIII) is needed for this subquotient construction to work. ■

Similarly, a cap diagram $d^\ast$ together with a weight $\lambda$ forms a diagram $\lambda d^\ast$, which is called oriented if $d\lambda$ is oriented. A cup, respectively a cap, in such diagrams is called anticlockwise, if its rightmost vertex is labeled $\land$ and clockwise otherwise.

Putting a cap diagram $d^\ast$ on top of a cup diagram $c$ such that they are connected to the line $\mathbb{R} \times \{0\}$ at the same points creates a circle diagram, denoted by $cd^\ast$. All connected component of this diagram that do not touch the boundary of $\mathbb{R} \times [-1,1]$ are called circles, all others are called lines. Together with $\lambda \in \Lambda$ such that $c\lambda$ and $\lambda d^\ast$ are oriented it forms an oriented circle diagram $c\lambda d^\ast$.

Definition 3.5. We define the degree of an oriented cup diagram $c\lambda$, of an oriented cap diagram $\lambda d^\ast$ and of an oriented circle diagram $c\lambda d^\ast$ as follows.

$$\deg(c\lambda) = \#\{\text{clockwise cups in } c\lambda\}, \quad \deg(\lambda d^\ast) = \#\{\text{clockwise caps in } \lambda d^\ast\},$$

(3.2)

$$\deg(c\lambda d^\ast) = \deg(c\lambda) + \deg(\lambda d^\ast).$$

Example 3.6. The cup diagrams which we mostly use are all similar to the ones displayed in (4.5). In this case the block $\Lambda$ has sequence $\ast \ast \ast \ast \ast$. The weight $\lambda$ given by $\lor \land \lor \land$ gives rise to an orientation for both diagrams. With these orientations the degree of the left cup diagram (4.5) would be 1 and of the right it would be 0. For more examples see (3.1) or [17, Example 3.6]. ■

Finally, we associate to each $\lambda \in \Lambda$ a unique cup diagram, denoted by $\lambda$, via:

(I) Connect neighboring pairs $\lor \land$ with a cup, ignoring symbols of the type $\circ$ and $\times$ as well as symbols already connected. Repeat this process until there are no more $\lor$'s to the left of any $\land$.

(II) Put a ray under any remaining symbols $\lor$ or $\land$.

It is an easy observation that $\lambda$ always exists for a fixed $\lambda$. Furthermore, $\lambda$ is the (unique) orientation of $\lambda$, such that $\lambda \lambda$ has minimal degree. Each cup diagram $c$ is of the form $\lambda$ for $\lambda \in \Lambda$, a block compatible with $c$.

Similarly we can define $\lambda = \lambda^\ast$, and, as before, in an oriented circle diagram $\lambda \circ \lambda$ a circle $C$ is said to be oriented anticlockwise if the rightmost vertex contained in the circle is $\land$ and clockwise otherwise.

Note that, by [17, Lemma 3.9], the contribution to the degree of the arcs contained in a given circle $C$ inside an oriented circle diagram is equal to

$$\deg(C) = (\text{number of cups in } C) \pm 1,$$

with $+1$, if the circle $C$ is oriented clockwise and $-1$ otherwise.

The following statistics will come up in the coefficients of the multiplication.
For $i \in \mathbb{Z}$ and a block $\Lambda$ define the position of $i$ as
\[
p_\Lambda(i) = \# \{ j \mid j < i, \text{seq}(\Lambda)_j = \star \} + 2 \cdot \# \{ j \mid j < i, \text{seq}(\Lambda)_j = \times \}.
\]
For a cup or cap $\gamma$ in a diagram connecting vertices $(i, 0)$ and $(j, 0)$ we define its distance $d_\Lambda(\gamma)$ and saddle width $s_\Lambda(\gamma)$ by
\[
d_\Lambda(\gamma) = |p_\Lambda(i) - p_\Lambda(j)| \quad \text{and} \quad s_\Lambda(\gamma) = \frac{1}{2} (d_\Lambda(\gamma) + 1).
\]
For a ray $\gamma$ set $d_\Lambda(\gamma) = 0$. For a collection or concatenation $\gamma_1 \cdots \gamma_r$ of distinct arcs (e.g. a circle or sequence of arcs connecting two vertices) set
\[
d_\Lambda(\gamma_1 \cdots \gamma_r) = \sum_{1 \leq k \leq r} d_\Lambda(\gamma_k).
\]
\[\text{Example 3.8.} \] Given $\Lambda \in \mathfrak{b}^1\varnothing$ with sequence $\star \star$ and the circle as in (3.13). Then $p_\Lambda(0) = 0$ and $p_\Lambda(1) = 1$. Moreover, if $\gamma$ is either the cup or cap of the circle, then $d_\Lambda(\gamma) = 1$, while the saddle (where the surgery is performed) in the multiplication has $s_\Lambda(\gamma) = 1$. Changing to $\star \times \star$ will leave the circle as it is diagrammatically. But now $p_\Lambda(0) = 0$, $p_\Lambda(1) = 1$, $p_\Lambda(2) = 3$, $d_\Lambda(\gamma) = 3$ and $s_\Lambda(\gamma) = 2$. \[\square\]

3.2. The linear structure of the arc algebras. Fix a block $\Lambda \in \mathfrak{b}^1$, and consider the set $\mathbb{B}(\Lambda) = \{ \Delta^{\mathfrak{p}} \mid \Delta^{\mathfrak{q}} \text{ is oriented and } \lambda, \mu, \nu \in \Lambda \}$. We call this set basis set of oriented circle diagrams. This set is subdivided into smaller sets of the form $\lambda \mathbb{B}(\Lambda)\mu$, which are those diagrams in $\mathbb{B}(\Lambda)$ which have $\lambda$ as cup part and $\mathfrak{p}$ as cap part.

From now on, we restrict to circle diagrams that only contain cups and caps. Formally this is done as follows: for a block $\Lambda \in \mathfrak{b}1$ denote by $\Lambda^\varnothing$ the set of weights $\lambda$ such that $\Lambda$ only contains cups. Note that $\Lambda^\varnothing \neq \emptyset$ iff $\Lambda$ is balanced. Define
\[
\mathbb{B}^\varnothing(\Lambda) = \{ \Delta^{\mathfrak{q}} \mid \Delta^{\mathfrak{p}} \text{ is oriented and } \lambda, \mu \in \Lambda^\varnothing, \nu \in \Lambda \} = \bigcup_{\lambda, \mu \in \Lambda^\varnothing} \mathbb{B}(\Lambda)\mu.
\]
We equip the elements of $\mathbb{B}(\Lambda)$ and of $\mathbb{B}^\varnothing(\Lambda)$ with the degree from Definition 3.5.

\[\text{Example 3.9.} \] Collapsing of the middle of each diagram in Section 3.3.4 gives typical elements from $\mathbb{B}^\varnothing(\Lambda)$. \[\square\]

For any ring $R$ let $\langle \cdot \rangle_R$ be the $R$-linear span. We define graded, free $Q$-modules via
\[
\mathfrak{A}[Q]_\lambda = \langle \mathbb{B}^\varnothing(\Lambda) \rangle_Q = \bigoplus_{\Delta^{\mathfrak{q}} \in \mathbb{B}^\varnothing(\Lambda)} Q(\Delta^{\mathfrak{p}}), \quad \mathfrak{A}[Q] = \bigoplus_{\lambda \in \mathfrak{b}^1} \mathfrak{A}[Q]_\lambda,
\]
which we call signed 2-parameter arc algebra for $\Lambda \in \mathfrak{b}^1\varnothing$ respectively (full) signed 2-parameter arc algebra. As usual, we also have their specializations.

Denote by $\lambda(\mathfrak{A}[Q]_\lambda)_\mu$ the $Q$-linear span of the basis vectors inside $\lambda \mathbb{B}^\varnothing(\Lambda)\mu$.

\[\text{Proposition 3.10.} \] The map $\text{mult} : \mathfrak{A}[Q]_\lambda \otimes \mathfrak{A}[Q]_\lambda \to \mathfrak{A}[Q]_\lambda$ given in Section 3.3 below endows $\mathfrak{A}[Q]_\lambda$ with the structure of a graded, unital algebra with pairwise orthogonal, primitive idempotents $\lambda \mathbb{1} = \lambda \mathbb{1} \lambda$ for $\lambda \in \Lambda$ and unit $\mathbb{1} = \sum_{\lambda \in \Lambda} \lambda \mathbb{1}$. (Similarly for the locally unital algebra $\mathfrak{A}[Q]$ and any specialization of $Q$. ) \[\square\]

\[\text{Proof.} \] As in [17, Proposition 3.12] where we leave it to the reader to incorporate the parameters (which can be done without problems). \[\square\]

\[\text{Remark 3.11.} \] Note that so far we do not know whether $\mathfrak{A}[Q]_\lambda$ is associative. It will follow from the identification of $\mathfrak{A}[Q]_\lambda$ with $\mathfrak{W}[Q]_\lambda$ that $\text{mult}$ is independent of the chosen order in which the surgeries are performed and that $\mathfrak{A}[Q]_\lambda$ is associative, see Corollary 4.9. (Similarly for any specialization of $Q$.) \[\square\]
3.3. The algebra structure. We define \( \text{mult} \) in two steps: we first recall the maps used in each step (without any parameters), compare to [17, Section 3.3], and afterward go into details about how we modify these maps incorporating \( Q \). The reader who wants to see examples may jump to Section 3.3.4.

For \( \lambda, \mu, \mu', \eta \in \Lambda \) we define a map

\[
\text{mult}_{\lambda, \mu, \mu', \eta} : \lambda(\mathfrak{A}[Q]_\Lambda)_{\mu} \otimes \mu'(\mathfrak{A}[Q]_\Lambda)_\eta \to \lambda(\mathfrak{A}[Q]_\Lambda)_\eta
\]

as follows. If \( \mu \neq \mu' \), then we declare the map to be identically zero. Thus, assume that \( \mu = \mu' \), and stack the diagram, without orientations, \( \mu\eta \) on top of the diagram \( \mu \), creating a diagram \( D_0 = \lambda \mu \eta \). Given such a diagram \( D_r \), starting with \( r = 0 \), we construct below a new diagram \( D_{r+1} \) by choosing a certain symmetric pair of a cup and a cap in the middle section. If \( l \) is the number of cups in \( \mu \), then this can be done a total number of \( l \) times. We call this procedure a surgery at the corresponding cup-cap pair. For each such step we define below a map \( \text{mult}_{D_r, D_{r+1}} \). Observing that the space of orientations of the final diagram \( D_r \) is equal to the space of orientations of \( \lambda \eta \), we define

\[
\text{mult}_{\lambda, \mu, \mu', \eta} = \text{mult}_{D_{l-1}, D_l} \circ \ldots \circ \text{mult}_{D_0, D_1}.
\]

Then \( \text{mult} \) is defined as the direct sum of all of these. In order to make \( \text{mult} \) a priori well-defined, we always pick the leftmost available cup-cap pair. We stress that it will be a non-trivial fact that one could actually pick any pair.

3.3.1. The surgery procedure. To obtain \( D_{r+1} \) from \( D_r = \lambda d^* d \eta \) — for some cup diagram \( d^* \) — choose the symmetric cup-cap pair with the leftmost endpoint in \( d^* d \) that can be connected without crossing any arcs (this means that the cup and cap are not nested inside any other arcs). Cut open the cup and the cap and stitch the loose ends together to form a pair of vertical line segments, call this diagram \( D_{r+1} \):

3.3.2. The map without parameters. The multiplication without parameters will closely resemble the one from [5]. One of the key differences is that we incorporate the parameter \( \alpha \) which changes some cases. The map \( \text{mult}_{D_{r+1}, D_{r}} \) without any additional coefficients only depends on how the components change when going from \( D_r \) to \( D_{r+1} \).

We have illustrated the above very much in the spirit of Section 2.5.

Merge. If two circles, say \( C_i \) and \( C_j \), are merged into a circle \( C \) proceed as follows.

\( \triangleright \) If \( C_i \) and \( C_j \) are oriented anticlockwise, then orient \( C \) anticlockwise.

\( \triangleright \) If exactly one of \( C_i \) and \( C_j \) is oriented clockwise, then orient \( C \) clockwise.

\( \triangleright \) If \( C_i \) and \( C_j \) are oriented clockwise, then orient \( C \) anticlockwise.

Split. If one circle \( C \) splits into two circles, say \( C_i \) and \( C_j \), proceed as follows.

\( \triangleright \) If \( C \) is oriented anticlockwise, then take the sum of two copies of the diagram \( D_{r+1} \). In one copy orient \( C_i \) clockwise and \( C_j \) anticlockwise, in the other vice versa.
If $C$ is oriented clockwise, then take the sum of two copies of the diagram $D_{r+1}$. In one copy orient both $C_i$ and $C_j$ clockwise, in the other orient $C_i$ and $C_j$ anticlockwise.

3.3.3. The map with parameters. In general, the formulas below include signs (recall that $\epsilon = \pm 1$) as well as coefficients coming from the parameters $\alpha$ and $\omega$.

The signs can be divided into the dot moving signs, the topological sign and the saddle sign. The latter two are topological in nature and quite involved. As we will see in Section 4.1, these signs encode how to rewrite foams in terms of basis foams.

These signs are as follows (explained for each case in detail below).

Dot moving signs: $\epsilon d_\Lambda(\gamma^\text{dot}_k)$ and $\epsilon d_\Lambda(\gamma^\text{ndot}_k)$.

Topological sign: $\epsilon^\frac{1}{4} d_\Lambda(C_{in})^{-2}$. Saddle sign: $\epsilon^s\Lambda(\gamma)$.

The dot moving signs can appear in any situation, the topological sign will appear for nested merges and splits, and the saddle sign for nested merges and non-nested splits. Each case can pick up some extra factors $\alpha$, $\epsilon$ or $\omega$ as we are going to describe below.

We note that one can always produce examples such that two of the three signs from (3.6) are trivial (that is, their exponents are $0 \mod 2$), but one is not (its exponent is $1 \mod 2$). Hence, all of them are needed for the general formula.

We distinguish whether the two circles, that are merged together or appear after a split, are nested inside each other or not. Fix for each circle

$$t(C) = \text{(a choice of)} \text{ a rightmost point in the circle } C.$$ 

Let $\gamma$ denote the cup in the cup-cap pair we use to perform the surgery procedure in this step connecting vertices $i < j$.

Non-nested Merge. The non-nested circles $C_i$ and $C_j$—containing vertices $i$ respectively $j$—are merged into $C$. The cases from above are modified as follows.

- **Both circles oriented anticlockwise.** As in Section 3.3.2 (no extra coefficients).
- **One circle oriented clockwise, one oriented anticlockwise.** Let $C_k$ (for $k = i$ or $k = j$) be the clockwise oriented circle and let $\gamma^\text{dot}_k$ be a sequence of arcs in $C$ connecting $t(C_k)$ and $t(C)$. (Neither $t(C_k), t(C)$ nor $\gamma^\text{dot}_k$ are unique, but possible choices differ in distance by 2, making the sign well-defined, see also [16, Lemma 5.7].) Proceed as in Section 3.3.2 and multiply by the dot moving sign

$$\epsilon d_\Lambda(\gamma^\text{dot}_k).$$

- **Both circles oriented clockwise.** Let $\gamma^\text{dot}_k$ be a sequence of arcs in $C$ connecting $t(C_k)$ and $t(C)$ (for both $k = i, j$). Proceed as in Section 3.3.2 and multiply by

$$\alpha \cdot \epsilon d_\Lambda(\gamma^\text{dot}_k) \cdot \epsilon d_\Lambda(\gamma^{\text{ndot}_k}).$$

Nested Merge. The nested circles $C_i$ and $C_j$ (with notation as before) are merged into $C$. Denote by $C_{in}$ the inner of the two original circles. Then:

- **Both circles oriented anticlockwise.** Proceed as above, but multiply by

$$\epsilon \cdot \epsilon^\frac{1}{4} d_\Lambda(C_{in})^{-2} \cdot \epsilon^s\Lambda(\gamma),$$

where $s\Lambda(\gamma)$ is the saddle width of the cup where the surgery is performed.

- **One circle oriented clockwise, one oriented anticlockwise.** Again perform the surgery procedure as described in Section 3.3.2 and multiply by

$$\epsilon \cdot \epsilon d_\Lambda(\gamma^\text{dot}_k) \cdot \epsilon^\frac{1}{4} d_\Lambda(C_{in})^{-2} \cdot \epsilon^s\Lambda(\gamma).$$
where $\gamma^\text{dot}_i$ (for $k = i$ or $k = j$) is defined as in (3.8), and $s_A(\gamma)$ as in (3.10).

- Both circles oriented clockwise. Again perform the surgery procedure as described in Section 3.3.2 and multiply by

$$\alpha \cdot \varepsilon \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)} \cdot \varepsilon^{s_A(\gamma)},$$

where $\gamma^\text{dot}_i$ and $\gamma^\text{dot}_j$ are defined as in (3.9), and $s_A(\gamma)$ as in (3.10).

Non-nested Split. The circle $C$ splits into the non-nested circles $C_i$ and $C_j$—containing vertices $i$ respectively $j$.

- $C$ oriented anticlockwise. Use the map as in Section 3.3.2, but the copy where $C_i$ is oriented clockwise is multiplied with

$$\omega \cdot \varepsilon^{d_A(\gamma^\text{ndot}_i)} \cdot \varepsilon^{s_A(\gamma)},$$

while the one where $C_j$ is oriented clockwise is multiplied with

$$\varepsilon \cdot \omega \cdot \varepsilon^{d_A(\gamma^\text{ndot}_j)} \cdot \varepsilon^{s_A(\gamma)}.$$

Here $\gamma^\text{ndot}_i$ and $\gamma^\text{ndot}_j$ are sequences of arcs connecting $(i, 0)$ and $t(C_i)$ inside $C_i$ respectively $(j, 0)$ and $t(C_j)$ in $C_j$, and $s_A(\gamma)$ being again as in (3.10).

- $C$ oriented clockwise. Multiply the copy with both circles oriented clockwise by

$$\omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)},$$

and the one with both circles oriented anticlockwise by

$$\varepsilon \cdot \omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)}.$$

Nested Split. We use here the same notations as in the non-nested split case, and we denote by $C_{in}$ and $C_{out}$ the inner and outer of the two circles $C_i$ and $C_j$.

- $C$ oriented anticlockwise. We use the map as defined in Section 3.3.2, but the copy where $C_{in}$ is oriented clockwise is multiplied with

$$\omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)},$$

while the copy where $C_{out}$ is oriented clockwise is multiplied with

$$\varepsilon \cdot \omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)}.$$

- $C$ oriented clockwise. Multiply the copy with both circles oriented clockwise by

$$\omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)},$$

and the one with both circles oriented anticlockwise by

$$\alpha \cdot \varepsilon \cdot \omega \cdot \varepsilon^{\frac{1}{l}(d_A(C_{in}) - 2)}.$$

3.3.4. Examples for the surgery procedure. We give now examples for some of the shapes that can occur during the surgery procedure and determine the coefficients. In all examples assume that outside of the shown strip all entries are $\circ$. 
Example 3.12. In a simple, non-nested merge we have no coefficients at all:

\[
\begin{array}{c}
\xymatrix{
&&&
\}
\end{array}
\]

The rightmost step above, called \textit{collapsing}, is always performed at the end of a multiplication procedure and is omitted in what follows.

Further, consider a merge of two anticlockwise, nested circles:

\[
\begin{array}{c}
\xymatrix{
&&&
\}
\end{array}
\]

Example 3.13. In both examples given here a non-nested merge is performed, followed by a split into two non-nested respectively nested circles. First, the H-shape:

\[
\begin{array}{c}
\xymatrix{
&&&
\}
\end{array}
\]

Here we have \( s_{\Lambda} (\gamma) = 1 \), but \( \frac{1}{4} (d_{\Lambda}(C_{in}) - 2) = 0 \) for the left multiplication step and \( \frac{1}{4} (d_{\Lambda}(C_{in}) - 2) = 1 \) for the right multiplication step.

\[
\begin{array}{c}
\xymatrix{
&&&
\}
\end{array}
\]

Remark 3.14. The \( \mathcal{C} \) shape cannot appear as long as we impose the choice of the order of cup-cap pairs from left to right in the surgery procedure.

Specializations 3.15. If we specialize \( q(\alpha) = 0, q(\varepsilon) = 1 \) and \( q(\omega) = 1 \), then we obtain the multiplication rules of the algebra from [5]. Specializing \( q(\alpha) = 0, q(\varepsilon) = -1 \) and \( q(\omega) = 1 \) gives the multiplication rule for the algebra from [17].
3.4. **Arc bimodules.** Very similar to [6, Section 3] and [17, Section 3.4], we define graded $\mathfrak{A}[\mathbb{Q}]$-bimodules by introducing additional diagrams moving from one block $\Lambda$ to another block $\Gamma$. That is, fix two blocks $\Lambda, \Gamma \in \frak{B}\Lambda$ such that $\text{seq}(\Lambda)$ and $\text{seq}(\Gamma)$ coincide except at positions $i$ and $i+1$. Following [6], a $(\Lambda, \Gamma)$-admissible matching (of type $\pm \alpha_i$) is a diagram $t$ inside $\mathbb{R} \times [0,1]$ consisting of vertical lines connecting $(k,0)$ with $(k,1)$ if we have that $\text{seq}(\Lambda)_k = \text{seq}(\Gamma)_k = \star$ and, depending on the sign of $\alpha_i$, an arc at positions $i$ and $i+1$ of the form

\[
\alpha_i : \begin{array}{ccc}
\circ & \times & \circ \\
i & i & \circ \\
\end{array} \quad \text{ and } \quad -\alpha_i : \begin{array}{ccc}
\circ & \times & \circ \\
i & i & \circ \\
\end{array}
\]

where we view $\text{seq}(\Lambda)$ as decorating the integral points of $\mathbb{R} \times \{0\}$ and $\text{seq}(\Gamma)$ as decorating the integral points of $\mathbb{R} \times \{1\}$. Again, the first two moves in each row are called rays, the third ones cups and the last ones caps. Note that for the first arc in each row it holds $d_\Lambda(\gamma) = 0$, while for the second it holds $d_\Lambda(\gamma) = 2$.

For $t$ a $(\Lambda, \Gamma)$-admissible matching, $\lambda \in \Lambda$, and $\mu \in \Gamma$ we say that $\lambda \mu$ is oriented if its cups, respectively caps, connect one $\wedge$ and one $\vee$ in $\mu$, respectively in $\lambda$, and rays connect the same symbols in $\lambda$ and $\mu$. For a sequence of blocks $\vec{\Lambda} = (\Lambda_0, \ldots, \Lambda_r)$ a $\vec{\Lambda}$-admissible composite matching is a sequence of diagrams $\vec{t} = (t_1, \ldots, t_r)$ such that $t_k$ is a $(\Lambda_{k-1}, \Lambda_k)$-admissible matching of some type. We view the sequence of matchings as being stacked on top of each other. A sequence of weights $\lambda_i \in \Lambda_i$ such that $\lambda_{k-1} t_k \lambda_k$ is oriented for all $k$ is an orientation of the $\vec{\Lambda}$-admissible composite matching $\vec{t}$. For short, we tend to drop the word admissible, since the only matchings we consider are admissible.

We stress that $\vec{\Lambda}$-composite matching can contain lines and “floating” circles.

**Example 3.16.** The following is a $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5)$-composite matching (we assume that outside of the indicated areas all symbols are equal to $\circ$).

![Diagram]

The types of the matchings are $-\alpha_0, \alpha_2, \alpha_1, \alpha_0, \alpha_1$ (read from bottom to top).

We now want to consider bimodules between arc algebras for different blocks, or said differently, bimodules for the algebra $\mathfrak{A}[\mathbb{Q}]$.

To a $\vec{\Lambda}$-composite matching $\vec{t}$ we again associate a set of diagrams from which to create a graded, free $Q$-module (with degree as in Definition 3.5)

\[
(3.18) \quad \mathcal{B}^0(\vec{\Lambda}, \vec{t}) = \left\{ \Lambda(\vec{t}, \vec{t}) \mid \lambda \in \Lambda_0, \mu \in \Lambda_r, \vec{t} = (\nu_0, \ldots, \nu_r) \text{ with } \nu_i \in \Lambda_i, \begin{array}{c}
\Lambda_0 \text{ oriented, } \nu_0 \text{ oriented, } \\
\nu_i \text{ oriented for all } 1 \leq i \leq r.
\end{array} \right\}
\]

As before we obtain the set $\mathcal{B}^0(\vec{\Lambda}, \vec{t})$ by allowing $\lambda \in \Lambda_0$ and $\mu \in \Lambda_r$ in (3.18).

**Example 3.17.** Let $\Lambda$ be the block with block sequence $\star \circ \times$, and $\Gamma$ the block with sequence $\star \star \star \star$ (both with $\circ$ everywhere else). Assume both blocks are...
balanced. Then an example for a \((\Lambda, \Gamma)\)-matching of type \(a_2\) is the third diagram in the first row of (3.17) denoted here by \(t_1\). Taking this as our composite matching we obtain a graded, free \(Q\)-module of rank 6 with basis consisting of

These are of degrees 0, 2, 4, 1 and 3 (read from left to right). □

**Definition 3.18.** Let \(\tilde{t}\) be a \(\tilde{\Lambda}\)-composite matching for \(\tilde{\Lambda} = (\Lambda_0, \ldots, \Lambda_r)\). Set

\[
A[\tilde{Q}](\tilde{\Lambda}, \tilde{t}) = \left\langle B^\sigma(\tilde{\Lambda}, \tilde{t}) \right\rangle_Q^- \{-(\text{up}(\Lambda_k) + \text{down}(\Lambda_k))\}, \ k \in \{0, \cdots, r\}
\]

as a graded, free \(Q\)-module, using \(\text{up}(\Lambda_k)\) and \(\text{down}(\Lambda_k)\) from Definition 3.2. (Neither \(\text{up}(\Lambda_k)\) nor \(\text{down}(\Lambda_k)\) depend on \(k\).) We call all such \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\) are bimodules.

The left (bottom) action of a basis element \(\lambda \mu \bar{\pi} \in A[\tilde{Q}]_{\Lambda_0}\) on a basis element of the form \(\mu'(\tilde{t}, \bar{\nu} \bar{\eta})\) is given similar as for the algebra itself. As before we obtain zero, if \(\mu \neq \mu'\), and otherwise we perform the same surgeries as before. The only difference is that local moves from \(\times \star\) to \(\star \times\) and vice versa contribute length 2 to \(\lambda_1(C)\) if they are contained in the circle \(C\) while those between \(\times \circ\) and \(\circ \times\) do not. The right (top) action is defined in complete analogy.

It is not clear that the above actions are well-defined and commute and we need the translation between \(\mathcal{W}[Q]\) and \(\mathfrak{A}[Q]\) from Section 4.1 to prove it.

**Proposition 3.19.** Let \(\tilde{t}\) be a \(\tilde{\Lambda}\)-composite matching with \(\tilde{\Lambda} = (\Lambda_0, \ldots, \Lambda_r)\). Then the left action of \(\mathfrak{A}[Q]|_{\Lambda_0}\) as well as the right action of \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\) are well-defined and commute. Hence, \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\) is an \(\mathfrak{A}[Q]|_{\Lambda_0}\)-\(\mathfrak{A}[Q]|_{\Lambda_r}\)-bimodule and thus, a \(\mathfrak{A}[Q]\)-bimodule. (Similarly for any specialization of \(Q\).)

**Proof.** We identify \(\mathfrak{A}[Q]|_{\Lambda}\) with \(\mathcal{W}[Q]|_{\bar{\Xi}}\) via Theorem 4.7. Then we identify \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\) with \(\mathcal{W}[Q](w(\tilde{\Lambda}, \tilde{t}))\) via Lemma 4.5. The latter isomorphism intertwines the actions of the two algebras on the bimodules by construction and hence, proves the claim. □

**Proposition 3.20.** All \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\) are graded biprojective, \(\mathfrak{A}[Q]\)-bimodules which are free \(Q\)-modules of finite rank. (Similarly for any specialization of \(Q\).)

**Proof.** We show that they are projective as left \(\mathfrak{A}[Q]\)-modules and omit the similar proof for the right action. Denote by \(\lambda_1 \mathfrak{A}[Q] \lambda_1\) the idempotent obtained from \(\Delta(\tilde{t}, \bar{\rho} \bar{\sigma})\) via downwards reduction (see [17, Section 3.4]). Then \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t}) \cong \bigoplus_{\lambda \in \Lambda} \mathfrak{A}[Q] \cdot \lambda_1 \mathfrak{A}[Q] \lambda_1\) and hence, it is projective. The other statements are clear and the claim follows. □

This proposition motivates the definition of the following 2-category.

**Definition 3.21.** Let \(\mathfrak{A}[Q]\)-\(\text{biMod}\)\(_{\mathbf{gr}}^\circ\) be the following 2-category:

- **Objects** are the various \(\Lambda \in \mathfrak{b}_1\).
- **1-morphisms** are finite direct sums and tensor products (taken over the algebra \(\mathfrak{A}[Q]\)) of the \(\mathfrak{A}[Q]\)-bimodules \(A[\tilde{Q}](\tilde{\Lambda}, \tilde{t})\).
- **2-morphisms** are \(\mathfrak{A}[Q]\)-bimodule homomorphisms.
- The composition of arc bimodules is the tensor product \(\otimes_{\mathfrak{A}[Q]}\). The vertical composition of \(\mathfrak{A}[Q]\)-bimodule homomorphisms is the usual composition. The horizontal composition is given by tensoring (over \(\mathfrak{A}[Q]\)).
We consider $\mathfrak{A}[Q]$-$\text{biMod}^p_{gr}$ as a graded 2-category by turning the 2-hom-spaces into graded $Q$-modules (in the sense of (1.2)) via Definition 3.5.

As usual, we also consider specializations of $\mathfrak{A}[Q]$-$\text{biMod}^p_{gr}$.

4. ISOMORPHISMS, EQUIVALENCES AND THEIR CONSEQUENCES

This section has two main goals. First, we will construct an isomorphism of graded algebras $\Phi: \mathfrak{W}[Q]^p \overset{\cong}{\longrightarrow} \mathfrak{A}[Q]$, where $\mathfrak{W}[Q]^p$ is a certain subalgebra of $\mathfrak{W}[Q]$ defined in (4.7). This isomorphism works for any specialization of $\Lambda$ as well and provides an algebraic model of $\mathfrak{W}[Q]$. Form this we obtain (with $w(\cdot)$ as in (4.4)):

**Theorem 4.1.** There is an equivalence of graded, $Q$-linear 2-categories

$$\Phi: \mathfrak{W}[Q]_{\text{biMod}}^p \overset{\cong}{\longrightarrow} \mathfrak{A}[Q]_{\text{biMod}}^p$$

induced by $\Phi$ under which the web bimodules $\mathcal{W}[Q](w(\Lambda, \bar{1}))$ and the arc bimodules $\mathcal{A}[q](\Lambda, \bar{1})$ are identified. (Similarly for any specialization of $Q$.)

Second, let $R[\alpha]$ be a ring with a grading so that all $r \in R$ are of degree 0 and $\alpha$ is of degree 4. Let $q: Q \rightarrow R[\alpha]$ be any ring homomorphism with $q(\alpha) = \alpha$. Set

$$\mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] = \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] \otimes_{\mathbb{Z}} Q.$$ 

(WWe need these scalar extensions for technical reasons, e.g. to make statements as “isomorphisms of $Q$-algebras”. We omit the subscript for these.) We show in Section 4.2, where we explicitly construct the isomorphism from (4.2), the following.

**Theorem 4.2.** There is an isomorphism

$$\Psi: \mathfrak{A}[Q] \overset{\cong}{\longrightarrow} \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]$$

of graded $Q$-algebras. (Similarly for any further simultaneous specialization of $\alpha$.)

From this we obtain:

**Theorem 4.3.** Let $R[\alpha], q$ and $\mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]$ be as above. There is an equivalence (which is, in fact, even an isomorphism) of graded, $Q$-linear 2-categories

$$\Psi: \mathfrak{A}[Q]_{\text{biMod}}^p \overset{\cong}{\longrightarrow} \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]_{\text{biMod}}^p$$

induced by $\Psi$ under which $\mathcal{A}[q](\Lambda, \bar{1})$ and $\mathcal{A}[\alpha, q(\varepsilon), q(\omega)](\Lambda, \bar{1})$ are identified. (Similarly for any further simultaneous specialization of $\alpha$.)

Taking Proposition 2.31, the equivalences (4.1) and Theorem 4.3 together (and working over $\mathbb{Z}[\alpha, \bar{i}]$), we obtain that $\mathfrak{A}[\alpha, \bar{i}]$ [KBN], $\mathfrak{A}[\alpha, \bar{i}]$ [Ca], $\mathfrak{A}[\alpha, \bar{i}]$ [CMW] and $\mathfrak{A}[\alpha, \bar{i}]$ [BI] are all equivalent, see Corollary 4.37.

4.1. **Web and arc algebras.** We start by constructing a graded algebra isomorphism $\Phi: \mathfrak{W}[Q]^p \rightarrow \mathfrak{A}[Q]$. For this purpose, recall that there is a bijection

$$b1^\circ \rightarrow b1^\circ, \quad \bar{k} \mapsto \Lambda, \quad \text{given by } 0 \mapsto o, \quad 1 \mapsto \ast, \quad 2 \mapsto \times.$$ 

Here $o, \ast, \times$ are entries of $\text{seq}(\Lambda)$ and $\Lambda$ is determined demanding that $\Lambda$ is balanced. We identify, using (4.3), such $\bar{k}$’s and $\Lambda$’s in what follows. Moreover, recall that for $\Lambda \in b1^\circ$ and $\lambda \in \Lambda$, there is a unique web $w(\lambda)$ associated to the cup diagram $\Lambda$, see [17, Lemma 4.8]. That is, there is a map

$$w(\cdot): \Lambda \rightarrow \text{CUP}(\bar{k}), \quad \lambda \mapsto w(\lambda)$$

isomorphisms of $Q$ (which is, in fact, even an isomorphism) of graded, $Q$ (We need these scalar extensions for technical reasons, e.g. to make statements as (4.7). This isomorphism works for any specialization of $Q$ as well and provides an algebraic model of $\mathfrak{W}[Q]$. Form this we obtain (with $w(\cdot)$ as in (4.4)):

**Theorem 4.1.** There is an equivalence of graded, $Q$-linear 2-categories

$$\Phi: \mathfrak{W}[Q]_{\text{biMod}}^p \overset{\cong}{\longrightarrow} \mathfrak{A}[Q]_{\text{biMod}}^p$$

induced by $\Phi$ under which the web bimodules $\mathcal{W}[Q](w(\Lambda, \bar{1}))$ and the arc bimodules $\mathcal{A}[q](\Lambda, \bar{1})$ are identified. (Similarly for any specialization of $Q$.)

Second, let $R[\alpha]$ be a ring with a grading so that all $r \in R$ are of degree 0 and $\alpha$ is of degree 4. Let $q: Q \rightarrow R[\alpha]$ be any ring homomorphism with $q(\alpha) = \alpha$. Set

$$\mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] = \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] \otimes_{\mathbb{Z}} Q.$$ 

(WWe need these scalar extensions for technical reasons, e.g. to make statements as “isomorphisms of $Q$-algebras”. We omit the subscript for these.) We show in Section 4.2, where we explicitly construct the isomorphism from (4.2), the following.

**Theorem 4.2.** There is an isomorphism

$$\Psi: \mathfrak{A}[Q] \overset{\cong}{\longrightarrow} \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]$$

of graded $Q$-algebras. (Similarly for any further simultaneous specialization of $\alpha$.)

From this we obtain:

**Theorem 4.3.** Let $R[\alpha], q$ and $\mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]$ be as above. There is an equivalence (which is, in fact, even an isomorphism) of graded, $Q$-linear 2-categories

$$\Psi: \mathfrak{A}[Q]_{\text{biMod}}^p \overset{\cong}{\longrightarrow} \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)]_{\text{biMod}}^p$$

induced by $\Psi$ under which $\mathcal{A}[q](\Lambda, \bar{1})$ and $\mathcal{A}[\alpha, q(\varepsilon), q(\omega)](\Lambda, \bar{1})$ are identified. (Similarly for any further simultaneous specialization of $\alpha$.)

Taking Proposition 2.31, the equivalences (4.1) and Theorem 4.3 together (and working over $\mathbb{Z}[\alpha, \bar{i}]$), we obtain that $\mathfrak{A}[\alpha, \bar{i}]$ [KBN], $\mathfrak{A}[\alpha, \bar{i}]$ [Ca], $\mathfrak{A}[\alpha, \bar{i}]$ [CMW] and $\mathfrak{A}[\alpha, \bar{i}]$ [BI] are all equivalent, see Corollary 4.37.

4.1. **Web and arc algebras.** We start by constructing a graded algebra isomorphism $\Phi: \mathfrak{W}[Q]^p \rightarrow \mathfrak{A}[Q]$. For this purpose, recall that there is a bijection

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Here $o, \ast, \times$ are entries of $\text{seq}(\Lambda)$ and $\Lambda$ is determined demanding that $\Lambda$ is balanced. We identify, using (4.3), such $\bar{k}$’s and $\Lambda$’s in what follows. Moreover, recall that for $\Lambda \in b1^\circ$ and $\lambda \in \Lambda$, there is a unique web $w(\lambda)$ associated to the cup diagram $\Lambda$, see [17, Lemma 4.8]. That is, there is a map

$$w(\cdot): \Lambda \rightarrow \text{CUP}(\bar{k}), \quad \lambda \mapsto w(\lambda)$$
constructed from the cup diagram $\Lambda$. Similarly, for each $\tilde{\Lambda}$-composite matching $\vec{t}$ there is a unique associated web $w(\tilde{\Lambda}, \vec{t})$ (given by an analogous map). The images of these maps are called basis webs. All the reader needs to know about these basis webs is summarized in Example 4.4 below. Details can be found in [17, Section 4.1].

**Example 4.4.** Given a web $u$, then we can associate to it an arc diagram $a(u)$ via

\[ \begin{align*}
\vdash & \mapsto \mid, \quad \vdash \mapsto \emptyset, \quad \Upsilon & \mapsto \cup, \quad \Upsilon & \mapsto \cap
\end{align*} \]

We do not consider any relations on the set of webs. Hence, isotopic webs are not equal and there are plenty of webs giving the same arc diagram, but there is a preferred choice of a preimage which defines a split of the map $u \mapsto a(u)$ and gives the map $w(\cdot)$. An example is

\[ (4.5) \]

How this choice of preimage can be made precise is not important in what follows. That is, we only need the fact that there is a preferred choice. The only thing we additionally note is that this association is parameter independent. □

Moreover, as indicated in Example 4.4, the set of basis webs

\[ \text{Cup}(\vec{k}) = \{ u \in \text{CUP}(\vec{k}) \mid u = w(\lambda) \text{ for some } \lambda \in \Lambda \} \]

is always a strict and finite subset of $\text{CUP}(\vec{k})$. Now, given $\lambda, \mu \in \Lambda$, let us denote

\[ (4.6) \]

\[ (4.7) \]

Clearly, $\mathfrak{W}[P]_{\vec{k}}$ is a graded subalgebra of $\mathfrak{W}[P]$ with based web bimodules $\mathfrak{W}[P]^\bullet(u)$ given as in (2.29), but using Cup($\vec{k}$) instead of CUP($\vec{k}$). We can view these as $\mathfrak{W}[P]$-bimodules as well, and we thus, get a 2-category $\mathfrak{W}[P]_{\vec{k}}$-$\text{biMod}$ consisting of based $\mathfrak{W}[P]$- or $\mathfrak{W}[P]_{\vec{k}}$-bimodules. (Similarly for any specialization of $P$.)

Let us switch back to $Q$. Recalling the cup foam bases from Remark 2.26 and Definition 2.39 and the bases from (3.4) and (3.18), we have the following lemmas.

**Lemma 4.5.** Let $u, v \in \text{Cup}(\vec{k})$ be webs such that $u = w(\lambda)$ and $v = w(\mu)$. There is an isomorphism of graded, free $P$-modules

\[ (4.8) \]

\[ (4.9) \]

which sends $u^\circ(\vec{k})_v$ to $\chi(\mathfrak{A}[Q]_{\vec{t}})_{\lambda}$ by identifying the basis cup foams without dots with anticlockwise circles and the basis cup foams with dots with clockwise circles.

Let $u \in \text{Hom}_{\mathfrak{A}[Q]}(\vec{k}, \vec{t})$ be a web such that $u = w(\tilde{\Lambda}, \vec{t})$. There is an isomorphism of graded, free $Q$-modules

\[ (4.10) \]

which sends $\mathbb{B}(u)$ to $\mathbb{B}(\tilde{\Lambda}, \vec{t})$ by identifying the basis cup foams without dots with anticlockwise circles and the basis cup foams with dots with clockwise circles.

(For both statements: similarly for any specialization of $Q$.) □
Proof. The arguments used in [17, Lemmas 4.15 and 4.16] as well as the construction of the two bases in question are parameter independent. Thus, we can adapt [17, Lemmas 4.15 and 4.16] without difficulties and the claim follows.

Lemma 4.6. For any $\lambda, \mu \in \Lambda$ and $u = w(\lambda), v = w(\mu)$: the isomorphisms $\Phi_{\lambda\mu}^k$ from (4.8) extend to isomorphisms of graded, free $Q$-modules

$$\Phi_{\lambda\mu}^k: \mathfrak{M}[Q]_k \to \mathfrak{A}(\Lambda), \quad \Phi: \mathfrak{M}[Q]_k \to \mathfrak{A}(\Lambda).$$

(Similarly for any specialization of $Q$.)

Proof. Clear by Lemma 4.5.

Theorem 4.7. The maps from (4.10) are isomorphisms of graded algebras. (Similarly for any specialization of $Q$.)

The non-trivial and lengthy proof of Theorem 4.7 is given in Section 6.

Remark 4.8. We use the specialization of the parameters $\mathbb{P}$ to $\mathbb{Q}$ from (2.3) to not having to worry about the difference between the “directions” in which we squeeze, migrate dots or perform ordinary-to-phantom neck cutting. Being more careful with the performed steps in the topological rewriting process leads to an analogue of Theorem 4.7 for $\mathbb{P}$ as well. Since this would require the introduction of some involved (but straightforward) notions for the diagram combinatorics keeping track of directions, we have decided, for brevity and clearness, to only do the $\mathbb{Q}$ case here—which includes our list of examples from (Sp) anyway. Moreover, we could also relay the condition of webs being upwards oriented. This makes the involved scalars more cumbersome, and we decided not to pursue this direction further.

Corollary 4.9. The multiplication rule from Section 3.3 is independent of the order in which the surgeries are performed. This turns $\mathfrak{A}(\mathbb{Q})$ into a graded, associative, unital algebra. (Similarly for the locally unital algebra $\mathfrak{A}(\mathbb{Q})$ and for any specialization of $\mathbb{Q}$.)

Proof. The claimed properties are clear for the web algebras $\mathfrak{W}[\mathbb{Q}]_k$ and $\mathfrak{W}[\mathbb{Q}]_k$, see Lemma 2.34. Thus, using Theorem 4.7 provides the claim.

We are now ready to prove our first main result, i.e. the equivalence from (4.1).

Proof of Theorem 4.1. The algebras $\mathfrak{M}[\mathbb{Q}]_k$ and $\mathfrak{M}[\mathbb{Q}]_k$ are graded Morita equivalent (this can be seen as in [17, Proof of Theorem 4.1]) and the statement follows from Theorem 4.7: the identification of the bimodules as graded, free $Q$-modules is clear by Lemma 4.5, while the actions agree by Theorem 4.7 and the construction of the actions. Everything in these arguments is independent of the parameters and thus, the theorem follows.

4.2. Arc algebras: isomorphisms. In this section we show that the signed 2-parameter arc algebra $\mathfrak{A}[\mathbb{Q}]$ and the (scalar extended) KBN specialization

$$\mathfrak{A}[\text{KBN}] = \mathfrak{A}_{\mathbb{Z}[\alpha]}[\text{KBN}] \otimes \mathbb{Z}[\omega^{\pm 1}]$$

are isomorphic as graded algebras. Here, as usual, $\alpha$ is of degree 4. As we explain, this gives rise to the isomorphisms from (4.2), which enables us to prove Theorem 4.3.

Both, $\mathfrak{A}[\text{KBN}]_\Lambda$ and $\mathfrak{A}[\mathbb{Q}]_\Lambda$, are isomorphic as graded, free $\mathbb{Q}$-modules to $\langle \mathfrak{B}(\Lambda) \rangle_{\mathbb{Q}}$, with $\mathfrak{B}(\Lambda)$ being as in (3.4). By definition, the multiplication differs only by the appearing coefficients in the result. Hence, we will give the isomorphism from $\mathfrak{A}[\text{KBN}]_\Lambda$
to $\mathbb{Z}[q]_A$ by defining a coefficient for each of the diagrams appearing in the multiplication and show that the maps intertwine the two multiplication rules.

**Definition 4.10.** We call any diagram appearing in an intermediate step of the multiplication procedure from Section 3.3 a stacked diagram. We denote such diagrams throughout this section by $D$ (possibly with decorations and indices), and choices of orientations of it by $D^o$ (possibly with decorations and indices). ■

**Definition 4.11.** We define sets of arcs inside a circle $C$ in a fixed diagram $D$:

- $\mathring{\cup}(C)$ denotes all cups in $C$ such that the exterior of $C$ is above the cup.
- $\mathring{\cap}(C)$ denotes all caps in $C$ such that the exterior of $C$ is below the cap.
- $\mathring{\cup}(C)$ denotes all cups in $C$ such that the interior of $C$ is above the cup.
- $\mathring{\cap}(C)$ denotes all caps in $C$ such that the interior of $C$ is below the cap.

The exterior and interior is meant here with respect to the circle $C$ only (ignoring all possible other components of $D$).

**Example 4.12.** The outer circle $C_{\text{out}}$ in the third diagram in (3.14) has $\mathring{\cup}(C_{\text{out}}) = 1$, $\mathring{\cap}(C_{\text{out}}) = 0$, $\mathring{\cup}(C_{\text{out}}) = 1$ and $\mathring{\cap}(C_{\text{out}}) = 2$. The inner circle $C_{\text{in}}$ in the same stacked diagram has exactly the same numbers. The circle $C$ in the rightmost diagram in (3.14) has $\mathring{\cup}(C) = 1$, $\mathring{\cap}(C) = 1$, $\mathring{\cup}(C) = 2$ and $\mathring{\cap}(C) = 2$. Moreover, $$(\mathring{\cup}(C_{\text{out}}) \cup \mathring{\cap}(C_{\text{out}}) \cup \mathring{\cup}(C_{\text{in}}) \cup \mathring{\cap}(C_{\text{in}})) \setminus \text{surg} = \mathring{\cup}(C) \cup \mathring{\cap}(C).$$

Here “surg” means the set containing the cup-cap involved in the surgery.

We denote by $\mathfrak{B}(D)$ the set of all possible orientations of a given $D$.

**Definition 4.13.** For a fixed $D$, we define its $Q$-linear coefficient map via:

$$\text{coeff}_D : \langle \mathfrak{B}(D) \rangle_Q \rightarrow \langle \mathfrak{B}(D) \rangle_Q, \quad D^o \mapsto (\prod_{\text{circles}} \text{coeff}_\varepsilon(C, D^o) \cdot \text{coeff}_\omega(C, D^o)) D^o.$$

Here the product runs over all circles in $D$, and the involved terms (i.e. for each such circle $C$) are defined as follows.

- If $C$ is oriented anticlockwise when looking at the orientation $D^o$, then set $\text{coeff}_\varepsilon(C, D^o) = \prod_{\gamma \in \mathring{\cup}(C)} \varepsilon^{s_\lambda(\gamma) + 1} p_\lambda(\gamma) \cdot \prod_{\gamma \in \mathring{\cap}(C)} \varepsilon^{s_\lambda(\gamma) (p_\lambda(\gamma) + 1)}$,

$$\text{coeff}_\omega(C, D^o) = \prod_{\gamma \in \mathring{\cup}(C)} \omega^{s_\lambda(\gamma)} \cdot \prod_{\gamma \in \mathring{\cap}(C)} \omega^{-s_\lambda(\gamma) - 1},$$

where, as usual, the $\gamma$’s denote the corresponding cups and caps, $p_\lambda(\gamma)$ denotes the position of their leftmost points and $s_\lambda(\gamma)$ is the saddle width as in Definition 3.7.

- If $C$ is oriented clockwise when looking at the orientation $D^o$, then we use the same coefficient and additionally multiply by $\varepsilon^{t(C)}$. (Recalling $t(C)$ from (3.7)—the reader might think of $\varepsilon^{t(C)}$ as keeping track of “dot moving” again.) ■

Since $\varepsilon = \pm 1$, its powers matter only mod 2.

**Example 4.14.** The circle $C_{\text{out}}$ in the third diagram $D_3$ in (3.14) has only one cup $\gamma$ “pushing inwards” with $s_\lambda(\gamma) = 1$ and $p_\lambda(\gamma) = 3$. Thus, if $D_{\varepsilon}^o$ denotes the orientation from (3.14), then $\text{coeff}(C_{\text{out}}, D_{\varepsilon}^o) = \varepsilon^4 \omega^{-1} = \omega^{-1}$. Similarly one obtains $\text{coeff}(C_{\text{in}}, D_{\varepsilon}^o) = \varepsilon^4 \omega^{-1} = \omega^{-1}$. Moreover, the circle $C$ in the rightmost diagram $D_4$ in (3.14) has one cup $\gamma$ and one cap $\gamma'$ with $s_\lambda(\gamma) = 2$, $p_\lambda(\gamma) = 1$, $s_\lambda(\gamma') = 1$ and $p_\lambda(\gamma') = 1$ “pushing inwards”. Thus, $\text{coeff}(C, D_{\varepsilon}^o) = \varepsilon^2 \omega^{-2} \cdot \varepsilon^2 = \varepsilon \omega^{-2}$. ■
We will usually write \( \text{coeff}(C^{\text{anti}}) = \text{coeff}(C, D^{\text{or}}) \) etc. to denote the coefficient for the circle \( C \) when the orientation is chosen such that \( C \) is oriented anticlockwise, and similarly \( \text{coeff}(C^{\text{cl}}) = \text{coeff}(C, D^{\text{or}}) \) when it is chosen such that \( C \) is oriented clockwise. For example, we have by definition

\[
\text{coeff}(C^{\text{cl}}) = \text{coeff}(C^{\text{anti}}) \cdot \varepsilon^{\text{(C)}},
\]

**Definition 4.13** restricts to a homogeneous, \( Q \)-linear map

\[
\text{coeff}_{\lambda,\mu} : \langle \lambda \mathbb{B}^\circ(\Lambda) \rangle_Q \rightarrow \langle \lambda \mathbb{B}^\circ(\Lambda) \rangle_Q,
\]

for \( \lambda, \mu \in \Lambda \). By summing all of these up we obtain a homogeneous, \( Q \)-linear map

\[
\text{coeff}_{\lambda} : \mathfrak{A}[\text{KBN}]_\Lambda \rightarrow \mathfrak{A}[q]_\Lambda
\]

by using \( \mathfrak{A}[\text{KBN}]_\Lambda \cong \langle \mathbb{B}^\circ(\Lambda) \rangle_Q \cong \mathfrak{A}[q]_\Lambda \), as graded, free \( Q \)-modules.

In fact, the \( Q \)-linear map from (4.12) is actually an isomorphism of graded algebras:

**Proposition 4.15.** The maps from (4.12) are isomorphisms of graded, \( Q \)-algebras for all \( \Lambda \in \mathfrak{b}^\circ \). These can be extended to an isomorphism of graded \( Q \)-algebras

\[
\text{coeff} : \mathfrak{A}[\text{KBN}] \overset{\cong}{\rightarrow} \mathfrak{A}[q].
\]

Again, the proof of Proposition 4.15 is rather lengthy and is given in Section 6. The main point hereby, as we explain in detail in the proof, is to show that

\[
\text{coeff}_{\lambda}(D_l^{\text{or}}) \cdot \text{coeff}(q) = \text{coeff}_{\lambda}(\tilde{D}_l^{\text{or}})).
\]

Here \( \text{coeff}_{\lambda}(D_l^{\text{or}}) \) and \( \text{coeff}_{\lambda}(\tilde{D}_l^{\text{or}}) \) denote the coefficients of the stacked diagrams before and after the \( l \)th step in the multiplication procedure, and \( \text{coeff}(q) \) denotes the coefficients (for \( \mathfrak{A}[q] \)) coming from this step. We give an example—which serves as a road map—illustrating the reasoning.

**Example 4.16.** In Example 4.14 we have already calculated \( \text{coeff}(C_{\text{cont}}) = \omega^{-1} \), \( \text{coeff}(C_{\text{in}}) = \omega^{-1} \) and \( \text{coeff}(C) = \varepsilon \omega^{-2} \) for the three circles appearing in the diagram on the right-hand side of (3.14). Moreover, \( \text{coeff}(q) = \varepsilon \). Thus, (4.13) holds.

Given the setup as in the beginning of this section, we define the map \( \Psi \) from (4.2) as follows. Let \( \text{coeff}_{\alpha,q(\varepsilon),q(\omega)} : \mathfrak{A}[\text{KBN}] \rightarrow \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] \) be the homogeneous, \( Q \)-linear map obtained in the same way as \( \text{coeff} : \mathfrak{A}[\text{KBN}] \rightarrow \mathfrak{A}[q] \), but using the specialized parameters \( q(\varepsilon) \) and \( q(\omega) \). Then, by Proposition 4.15, set

\[
\Psi : \mathfrak{A}[q] \rightarrow \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)], \quad \Psi = \text{coeff}_{\alpha,q(\varepsilon),q(\omega)} \circ (\text{coeff})^{-1}.
\]

We are now ready to prove Theorem 4.2, assuming Proposition 4.15.

**Proof of Theorem 4.2.** The proof of Proposition 4.15 only uses that \( \varepsilon = \pm 1 \) and that \( \omega \) is invertible. Thus, the same arguments work for any \( q(\varepsilon) \) and \( q(\omega) \) providing a homogeneous isomorphism \( \text{coeff}_{\alpha,q(\varepsilon),q(\omega)} \) between \( \mathfrak{A}[\text{KBN}] \) and \( \mathfrak{A}[\alpha, q(\varepsilon), q(\omega)] \). (Similarly for any further simultaneous specialization of \( \alpha \).)

**4.3. Arc bimodules: bimodule homomorphisms.** In the last section we have identified \( \mathfrak{A}[\text{KBN}] \) with \( \mathfrak{A}[q] \) using the coefficient map. Thus, there is also an identification of their bimodules. The aim of this section is to make this explicit. For the identification of the bimodules \( \mathfrak{A}[\text{KBN}](\tilde{\Lambda}, \tilde{t}) \) and \( \mathfrak{A}[q](\tilde{\Lambda}, \tilde{t}) \) for a fixed admissible matching \((\tilde{\Lambda}, \tilde{t})\) we need to introduce some additional notations and slightly modify the coefficient map. But otherwise the identification works as for the algebras.
Definition 4.17. As in Definition 4.10, we call any diagram appearing in the intermediate step of the multiplication procedure from Section 3.4 a stacked diagram (using similar notations). Furthermore, fixing a circle $C$ in such a stacked diagram, we define subsets of arcs containing the arcs in the basic moves of the second type for $\alpha_i$ and $-\alpha_i$, i.e., local moves from $\star \times$ to $\star \times$, or from $\times \star$ to $\star \times$, see (3.17). We divide these depending on the exterior or interior of $C$:

- $\mathcal{X}(C)$ denotes the arcs in local moves from $\times \star$ to $\star \times$, where the exterior of the circle is to the lower left of the arc.
- $\mathcal{Y}(C)$ denotes the arcs in local moves from $\star \times$ to $\star \times$, where the exterior of the circle is to the lower right of the arc.
- $\mathcal{X}(C)$ denotes the arcs in local moves from $\star \times$ to $\times \star$, where the interior of the circle is to the lower left of the arc.
- $\mathcal{Y}(C)$ denotes the arcs in local moves from $\times \star$ to $\star \times$, where the interior of the circle is to the lower right of the arc.

Again, the exterior and interior is meant here with respect to the circle $C$ only. 

Definition 4.18. For a fixed $D$, we define its $Q$-linear coefficient map via:

$$
\text{coeff}_D : \mathcal{B}(D)_Q \rightarrow \mathcal{B}(D)_Q, \\
D^\omega \mapsto \prod_{\text{circles}} \text{coeff}_x(C, D^\omega) \cdot \text{coeff}_\omega(C, D^\omega) D^\omega.
$$

Here the product runs over all circles in $D$, and the involved terms (i.e. for each such circle $C$) are defined as follows.

\begin{equation}
\text{coeff}_x(C, D^\omega) = \prod_{\gamma \in \mathcal{U}(C)} e^{(s_\lambda(\gamma)+1)p_{\lambda}(\gamma)} \cdot \prod_{\gamma \in \mathcal{A}(C)} e^{s_\lambda(\gamma)(p_{\lambda}(\gamma)+1)} \cdot \prod_{\gamma \in \mathcal{X}(C)} e^{p_{\lambda}(\gamma)} \cdot \prod_{\gamma \in \mathcal{Y}(C)} e^{p_{\lambda}(\gamma)+1},
\end{equation}

$$
\text{coeff}_\omega(C, D^\omega) = \prod_{\gamma \in \mathcal{U}(C)} \omega^{-s_\lambda(\gamma)} \prod_{\gamma \in \mathcal{A}(C)} \omega^{s_\lambda(\gamma)-1} \cdot \omega^{\mathcal{X}(C) \cup \mathcal{Y}(C)},
$$

where we use the same notations as in Definition 4.13.

\begin{itemize}
    \item If $C$ is oriented anticlockwise when looking at the orientation $D^\omega$, then set
    $$
    \text{coeff}_x(C, D^\omega) = \prod_{\gamma \in \mathcal{U}(C)} e^{(s_\lambda(\gamma)+1)p_{\lambda}(\gamma)} \cdot \prod_{\gamma \in \mathcal{A}(C)} e^{s_\lambda(\gamma)(p_{\lambda}(\gamma)+1)} \cdot \prod_{\gamma \in \mathcal{X}(C)} e^{p_{\lambda}(\gamma)} \cdot \prod_{\gamma \in \mathcal{Y}(C)} e^{p_{\lambda}(\gamma)+1},
    $$
    $$
    \text{coeff}_\omega(C, D^\omega) = \prod_{\gamma \in \mathcal{U}(C)} \omega^{-s_\lambda(\gamma)} \prod_{\gamma \in \mathcal{A}(C)} \omega^{s_\lambda(\gamma)-1} \cdot \omega^{\mathcal{X}(C) \cup \mathcal{Y}(C)},
    $$
\end{itemize}

where we use the same notations as in Definition 4.13.

\begin{itemize}
    \item If $C$ is oriented clockwise when looking at the orientation $D^\omega$, then we use the same coefficient and additionally multiply by $e^{t(C)}$.
\end{itemize}

Similar to (4.12), we use these maps to define a homogeneous, $Q$-linear map

\begin{equation}
\text{coeff}_{\tilde{\lambda}, \tilde{t}} : \mathcal{A}[KBN]((\tilde{\Lambda}, \tilde{t}) \rightarrow \mathcal{A}[q](\tilde{\Lambda}, \tilde{t}),
\end{equation}

Proposition 4.19. The map

$$
\text{coeff}_{\tilde{\lambda}, \tilde{t}} : \mathcal{A}[KBN]((\tilde{\Lambda}, \tilde{t}) \rightarrow \mathcal{A}[q](\tilde{\Lambda}, \tilde{t})
$$

is an isomorphism of graded, free $Q$-modules that intertwines the actions of $\mathcal{A}[KBN]$ and $\mathcal{A}[q]$, i.e., for any $x \in \mathcal{A}[KBN]$ and any $m \in \mathcal{A}[KBN]((\tilde{\Lambda}, \tilde{t})$ it holds that $\text{coeff}_{\tilde{\lambda}, \tilde{t}}(x \cdot m) = \text{coeff}(x) \cdot \text{coeff}_{\tilde{\lambda}, \tilde{t}}(m)$. (Similarly for the right action.)

Again, the proof of this proposition appears in Section 6.
be the homogeneous, $Q$-linear map obtained in the same way as $\text{coeff}_{\vec{\Lambda}, \vec{t}}$ from (4.16), but using the specialized parameters $q(\varepsilon)$ and $q(\omega)$ instead of $\varepsilon$ and $\omega$. Then set
\begin{equation}
(4.17) \quad \text{coeff}_\psi = \text{coeff}_{\vec{\Lambda}, \vec{t}} \circ (\text{coeff}_{\vec{\Lambda}, \vec{t}})^{-1} : \mathcal{A}[Q](\vec{\Lambda}, \vec{t}) \to \mathcal{A}[\alpha, q(\varepsilon), q(\omega)](\vec{\Lambda}, \vec{t}).
\end{equation}

**Corollary 4.20.** The map $\text{coeff}_\psi$ is an isomorphism of graded, free $Q$-modules that intertwines the actions of $\mathcal{A}[Q]$ and $\mathcal{A}[\alpha, q(\varepsilon), q(\omega)]$. □

**Proof.** As in the proof Theorem 4.2, but using Proposition 4.19. ■

**Proof of Theorem 4.3.** This follows from Theorem 4.2 and Corollary 4.20. ■

### 4.4. Arc bimodules: co-structure

The aim of this section is to describe a co-structure topologically on web bimodules $\mathcal{W}[Q](v)$ and algebraically on arc bimodules $\mathcal{A}[Q](\vec{\Lambda}, \vec{t})$. Then we match these structures—which again comes with sophisticated scalars—for different specializations of $Q$ using an isomorphism similar, but not equal, to the coefficient map from (4.16).

We start on the side of $\mathcal{W}[Q]$. (The whole definition works of course more general for $P$.) We, as usually, only consider balanced $\vec{k}, \vec{l} \in \mathfrak{b}1^\omega$.

**Definition 4.21.** Let $v \in \text{Hom}_{\mathfrak{g}[Q]}(\vec{k}, \vec{l})$. Recalling that we consider in $\mathfrak{g}[Q]$ webs without relations, we can pick any pair of neighboring vertical usual edges, ignoring possible phantom edges, and perform a reverse surgery on $\mathcal{W}[Q](v)$:

Here the saddle foam is locally of the form as in (2.28), but read from top to bottom:

One ends up with a new web $v' \in \text{Hom}_{\mathfrak{g}[Q]}(\vec{k}, \vec{l})$. This should be read as follows: start with $f \in 2\text{Hom}_{\mathfrak{g}[P]}(1_{2\omega}, uv^*vw)$ and stack on top of it a foam which is the identity at the bottom ($u$ part) and top ($w$ part) of the web, and the saddle in between. Repeat this for all $u \in \text{CUP}(\vec{k}), w \in \text{CUP}(\vec{l})$. ■

Note that we make a certain choice where to perform the reverse surgery. But fixing $v'$ determines this choice. Thus, we can write $\text{rMult}_{v'}$ etc. without ambiguity.

**Lemma 4.22.** The procedure from Definition 4.21 defines a $\mathfrak{g}[Q]$-bimodule homomorphism $\text{rMult}_{v'} : \mathcal{W}[Q](v) \to \mathcal{W}[Q](v')$. □

**Proof.** Clear by construction. ■

We define the same on the side of $\mathcal{A}[Q]$. As usual, all blocks are balanced.

**Definition 4.23.** Let $\vec{t}$ be a $\vec{\Lambda}$-composite matching. Recalling that we construct these using the basic moves from (3.17), we can pick any pair of neighboring vertical arcs
(ignoring possible symbols ◦ or ⋆ in between) and perform a reverse surgery on $A[\mathbb{Q}](\tilde{\Lambda}, \tilde{\mu})$ giving us a new composite matching $\tilde{\mu}'$ for $\tilde{\Lambda}'$:

\[
\begin{array}{c}
\begin{array}{c}
\mu = e \\
\Lambda = e \\
\end{array}
\end{array}
\quad \rightarrow 
\begin{array}{c}
\begin{array}{c}
\mu = e \\
\Lambda = e \\
\end{array}
\end{array}
\]

Define a $Q$-linear map $\text{rmult} : A[\mathbb{Q}](\tilde{\Lambda}, \tilde{\mu}) \rightarrow A[\mathbb{Q}](\tilde{\Lambda}', \tilde{\mu}')$ precisely as in Section 3.3 to be the identity on circles not involved in the reversed surgery, and with the following differences for involved circles:

**Non-nested Merge.** The non-nested circles $C_i$ and $C_j$ are merged into $C$. We use the same conventions and spread the same scalars as in Section 3.3.3.

**Nested Merge.** The nested circles $C_i$ and $C_j$ are merged into $C$. We use the same conventions and scalars as in Section 3.3.3, but additionally multiply with $\varepsilon$.

**Non-nested Split.** The circle $C$ splits into the non-nested circles $C_{\text{bot}}$ and $C_{\text{top}}$ (being at the bottom or top of the picture). We use the same conventions and spread almost the same scalars as in Section 3.3.3, but in case $C$ is oriented anticlockwise, we use (for bottom, respectively top, circle oriented clockwise)

\[
\omega \cdot e^{d_\Lambda(\gamma_{\text{bot}}^\text{ndot})} \cdot e^{s_\Lambda(\gamma)} \quad \text{respectively} \quad e \cdot \omega \cdot e^{d_\Lambda(\gamma_{\text{top}}^\text{ndot})} \cdot e^{s_\Lambda(\gamma)}.
\]

Here $\gamma_{\text{bot}}^\text{ndot}$ respectively $\gamma_{\text{top}}^\text{ndot}$ are to be understood similar to (3.11) and (3.12). In case $C$ is oriented clockwise, we use

\[
\omega \cdot e^{d_\Lambda(\gamma_{\text{bot}}^\text{ndot})} \cdot e^{d_\Lambda(\gamma_{\text{top}}^\text{ndot})} \cdot e^{s_\Lambda(\gamma)} \quad \text{respectively} \quad \alpha \cdot e \cdot \omega \cdot e^{d_\Lambda(\gamma_{\text{bot}}^\text{ndot})} \cdot e^{d_\Lambda(\gamma_{\text{top}}^\text{ndot})} \cdot e^{s_\Lambda(\gamma)}
\]

for both circles oriented clockwise respectively anticlockwise.

**Nested Split.** The circle $C$ splits into the non-nested circles $C_{\text{in}}$ and $C_{\text{out}}$. We use the same conventions and spread the same scalars as in Section 3.3.3.

**Remark 4.24.** We note that the web algebra is in fact a (symmetric) Frobenius algebra. (This can be seen by copying [24, Proposition 30] or [32, Theorem 3.9].) The same holds for the arc algebra. (This can be seen by copying [5, Theorem 6.3].) Thus, both come with co-multiplications. The reverse surgeries from above can be used to give rise to these co-multiplications. We skip the details, since we do not use this co-multiplication in this paper. We only point out that our results of this section match the various co-multiplications on web or arc algebras for different parameters (similar, but “co”, as in Section 4.2), but not the Frobenius structures since the isomorphisms in the present section are different from the ones in Section 4.2.

**Example 4.25.** An illustration of the reverse multiplication is
**Lemma 4.26.** The procedure from Definition 4.23 defines an $\mathfrak{A}[Q]$-bimodule homomorphism $\text{rmult}: \mathcal{A}[Q](\vec{\Lambda}, \vec{t}) \to \mathcal{A}[Q](\vec{\Lambda}', \vec{t}')$. □

**Proof.** This follows from Proposition 4.27 below. ■

As before, we have the following, recalling the equivalence $\Phi$ from Theorem 4.1 induced by the isomorphism $\Phi$ from (4.10) under which the web bimodules $W[Q](w(\vec{\Lambda}, \vec{t}))$ and the arc bimodules $\mathfrak{A}[Q](\vec{\Lambda}, \vec{t})$ are identified.

**Proposition 4.27.** Fixing the $Q$-linear isomorphisms $\Phi$ from (4.9), it holds

$$\Phi(\vec{\Lambda}, \vec{t}) \circ \text{rmult}^{\vec{\Lambda}}_{\vec{t}} = \text{rmult}^{\vec{\Lambda}'}_{\vec{t}'} \circ \Phi(\vec{\Lambda}, \vec{t}).$$

(Similarly for any specialization of $Q$.) □

**Proof.** Very similar to the proof of Theorem 4.7. Indeed, we can use the same argumentation as given there (noting that the shifting basic moves as in the first two columns of (3.17) can be incorporated without difficulties), but we turn the corresponding pictures by $\pi_2$ (which gives the slight changes for the scalars in the algebraic setting). We skip the calculations for brevity. ■

We now aim to match the bimodule maps for different specializations of $Q$ as in Section 4.2 and Section 4.3. For this purpose, we define a coefficient map which is again slightly modified. In particular, we use the same notations as in Definition 4.17.

**Definition 4.28.** For fixed $D$, we define its reverse coefficient map $\text{coeff}$ as

$$\text{coeff}_{\vec{\Lambda}, \vec{t}}(C, D_{\text{gr}}) = \prod_{\gamma \in \mathcal{U}(C)} \epsilon^{s}_\Lambda(\gamma)(p_{\Lambda}(\gamma)+1) \cdot \prod_{\gamma \in \mathcal{N}(C)} \epsilon^{p}_{\Lambda}(\gamma) \cdot \prod_{\gamma \in \mathcal{J}(C)} \epsilon^{p}_{\Lambda}(\gamma).$$

Instead of (4.15), and a further factor of $\epsilon^{t}(C)$ for the clockwise circle. ■

For the following proposition we use the evident notation to distinguish the reverse multiplication maps from Definition 4.21 for different choices of specializations.

**Proposition 4.29.** The homogeneous, $Q$-linear map (defined as in (4.16), but using $\text{coeff}$ instead of $\text{coeff}$) $\text{coeff}^{\vec{\Lambda}, \vec{t}}: \mathcal{A}[[KBN]](\vec{\Lambda}, \vec{t}) \to \mathcal{A}[Q](\vec{\Lambda}, \vec{t})$ is an isomorphism of graded, free $Q$-modules such that the following commutes:

$$\begin{array}{ccc}
\mathcal{A}[[KBN]](\vec{\Lambda}, \vec{t}) & \xrightarrow{\text{coeff}^{\vec{\Lambda}, \vec{t}}[KBN]} & \mathcal{A}[Q](\vec{\Lambda}, \vec{t}) \\
\text{coeff}^{\vec{\Lambda}, \vec{t}} & \xrightarrow{\text{coeff}^{\vec{\Lambda}', \vec{t}'}[KBN]} & \mathcal{A}[Q](\vec{\Lambda}', \vec{t}').
\end{array}$$

(4.18)

**Proof.** We omit the details of this proof. It can be proven similar (but “co”) to the proof that $\text{coeff}^{\vec{\Lambda}, \vec{t}}: \mathcal{A}[[KBN]](\vec{\Lambda}, \vec{t}) \to \mathcal{A}[Q](\vec{\Lambda}, \vec{t})$ from Proposition 4.19 intertwines the actions of $\mathfrak{A}[KBN]$ and $\mathfrak{A}[Q]$ (again checking the cases (i)-(iv)) as in the proof of Theorem 4.7 with the following differences: the non-nested cases work analogously, while in the nested cases one needs to successively apply Lemma 6.6 as in the presented nested merge case in the proof of Proposition 4.19. ■
Similar to (4.17), but using Proposition 4.29 and the corresponding maps, we define 
\[ \text{coeff}_\Psi : A[Q](\vec{\Lambda}, \vec{t}) \to A[KB\mathbb{N}](\vec{\Lambda}, \vec{t}) \to A[\alpha, q(\varepsilon), q(\omega)](\vec{\Lambda}, \vec{t}). \]

The following is now clear because the proof of Proposition 4.29 does not use the specific form of the parameters in question.

**Corollary 4.30.** The map \( \text{coeff}_\Psi \) is an isomorphism of graded, free \( Q \)-modules such that the corresponding diagram in (4.18) commutes. (Similarly for any further simultaneous specialization of \( \alpha \).)

**Example 4.31.** Denote the diagrams in Example 4.25 from left to right by \( D_1, D_2 \) and \( D_3 \). Then \( \text{coeff}_{D_1}(D^\alpha_{or}) = \varepsilon \cdot \omega \cdot D^\alpha_{or} \), \( \text{coeff}_{D_2}(D^\alpha_{or}) = 1 \cdot D^\alpha_{or} \) and \( \text{coeff}_{D_3}(D^\alpha_{or}) = \varepsilon \cdot D^\alpha_{or} \). Thus, we see that (4.18) commutes in this example.

**Remark 4.32.** By Lemma 6.7 (which we state later), \( \text{coeff}_{D}(D^\alpha_{or}) \) can be expressed in terms of \( \text{coeff}_{D}(D^\alpha_{or}) \) times a constant that can either be determined by counting cups or by counting caps as well as shifts. Hence, it is evident that for \( x \in A[KB\mathbb{N}] \) and any \( m \in A[KB\mathbb{N}](\vec{\Lambda}, \vec{t}) \) it holds that \( \text{coeff}_{\vec{\Lambda}, \vec{t}}(x \cdot m) = \text{coeff}(x) \cdot \text{coeff}_{\vec{\Lambda}, \vec{t}}(m) \). (Similarly for the right action.) Thus, \( \text{coeff} \) is a graded, free \( Q \)-modules isomorphism intertwining the two actions.

**Lemma 4.33.** The compositions 
\( \text{coeff}_{\Psi}^{-1} \circ \text{coeff}_{\Psi}, \text{coeff}_{\Psi}^{-1} \circ \text{coeff}_{\Psi} : A[Q](\vec{\Lambda}, \vec{t}) \to A[Q](\vec{\Lambda}, \vec{t}) \)

are \( A[Q] \)-bimodule maps. (Similarly for any specialization of \( Q \).)

**Proof.** This follows from Remark 4.32 and Proposition 4.19 (and, as before, that our arguments do not use the specific form of the parameters in question).

**4.5. Consequences.** Using Theorem 4.7, Theorem 4.2 and Theorem 4.3, we have the following. Note that \( \Upsilon \) from Proposition 2.45 gives rise to a based version \( \Upsilon^b \).

**Proposition 4.34.** There is an equivalence of graded, \( Q \)-linear 2-categories
\[ \Upsilon^b : \mathfrak{F}[Q] \rightarrow \mathfrak{W}[Q]^{\ast} \text{-biMod}_{gr}, \]
which is bijective on objects. Similarly for \( q : Q \rightarrow R \) such that either:

- (gen) \( q(\alpha) = \alpha \) is generic or \( q(\alpha) = 0 \).
- (sesi) \( q(\alpha) \) is invertible, \( \sqrt{q(\alpha)} \in R \) and \( \frac{1}{2} \in R \).

The proof is given in Section 6. (The main step is to calculate the ranks of hom-spaces between bimodules. We only note here that the algebras in question are semisimple under the circumstances of (sesi).)

**Specializations 4.35.** The embeddings from Specializations 2.47 are, by Proposition 2.45, actually equivalences.

**Theorem 4.36.** Let \( R[\alpha] \) and \( q \) be as in Theorem 4.3. Then there are equivalences of graded, \( Q \)-linear 2-categories
\[ \mathfrak{F}[Q] \cong \mathfrak{F}[\alpha, q(\varepsilon), q(\omega)]. \]
(Similarly for any simultaneous specialization of \( \alpha \) satisfying the conditions (gen) or (sesi) from Proposition 4.34.)
Proof. This is just assembling all the pieces. First we use Proposition 4.34 to see that both sides are equivalent to the corresponding module categories of (specialized) web algebras. Then we use Theorem 4.1 to translate it to the corresponding arc algebras. Finally, using Theorem 4.3 provides the statement.

If one works over \( \mathbb{Z}[\alpha] \), then Theorem 4.36 shows that the 2-categories coming from the KBN and Bl setups are equivalent. Having a square root of \(-1\) gives a stronger result, i.e. the following is a direct consequence of Proposition 2.31 and Theorem 4.36.

Corollary 4.37. There are equivalences of graded, \( \mathbb{Z}[\alpha, i] \)-linear 2-categories

\[
\mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{KBN}] \cong \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{Ca}] \cong \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{CMW}] \cong \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{Bl}].
\]

(Similarly, by using \( \mathbb{Z}[\sqrt{q(\alpha)}] \) for any simultaneous specialization of \( \alpha \) satisfying the condition (sesi) from Proposition 4.34.)

Example 4.38. The equivalences from Corollary 4.37 are obtained by using the translation from web to arc algebras. In particular, these equivalences are given by evaluating foams on the cup foam basis.

Let us compare for instance the two endomorphism \( f, g \) of the singular neck cut \((2.13)\). In this case the cup foam basis is, for all 2-categories from \((4.19)\),

\[
u = \begin{array}{c}
\includegraphics[scale=0.5]{cup foam basis}
\end{array}, \quad \mathbb{B}^\circ(u) = \left\{ x = \begin{array}{c}
\includegraphics[scale=0.5]{cup foam basis}
\end{array}, y = \begin{array}{c}
\includegraphics[scale=0.5]{cup foam basis}
\end{array}, f = \begin{array}{c}
\includegraphics[scale=0.5]{cup foam basis}
\end{array}, g = \begin{array}{c}
\includegraphics[scale=0.5]{cup foam basis}
\end{array}\right\},
\]

In general we need to match the two bases via the coefficient map from Definition 4.18. But in this case we have the following identification (given on the cup foam basis):

\[
f(x) = 0 = g(y), \quad f(y) = \begin{cases} x, & \text{for KBN, Bl}, \\ i \cdot x, & \text{for Ca, CMW}, \end{cases} \quad g(x) = \begin{cases} -y, & \text{for KBN, Bl}, \\ -i \cdot y, & \text{for Ca, CMW} \end{cases}.
\]

Thus, the equivalence from \( \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{KBN}] \) to \( \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{Ca}], \mathfrak{F}_{\mathbb{Z}[\alpha, i]}[\text{CMW}] \) respectively rescales \( f, g \leftrightarrow i \cdot f, i \cdot g \) respectively \( f, g \leftrightarrow f, g \). Similarly for more complicated situations, where also the cup foam basis might be already changed. ■

5. Applications

Now we discuss some applications of our isomorphisms and equivalences.

5.1. Connection to category \( \mathcal{O} \). To obtain a connection to parabolic category \( \mathcal{O}^\bullet \) for some maximal parabolic of the complex general linear Lie algebra, we first need to define the generalized signed 1-parameter arc algebra \( \mathbf{C}[0, e, \omega]_\Lambda \). This algebra might be seen as the quasi-hereditary cover of the signed 1-parameter arc algebra, as it was shown in the KBN case in [5, Corollary 5.4].

Let us give details. To this end, we work over \( \mathbb{Z}[\omega^{\pm 1}] \) or \( \mathbb{C} \) and specialize \( \alpha = 0 \) throughout this section. Denote by \( \text{KBN}_0 \) the further specialization of the KBN setup with \( q(\alpha) = 0 \). (Similarly for the \( \text{Ca}, \text{CMW} \) and \( \text{Bl} \) specializations.) The corresponding algebras are called signed 1-parameter (web or arc) algebras.

Our construction here follows closely [5, Section 4] and [16, Section 6]. In particular, we fix a not necessarily balanced block \( \Lambda \in \mathfrak{b} \mathfrak{l} \). Further, fix \( n \in \mathbb{Z}_{\geq 2} \) and take two integers \( p, q \in \mathbb{Z}_{\geq 0} \) such that \( p + q = n \). Consider \( \mathfrak{g} \mathfrak{l}_n \) with fixed Cartan and Borel subalgebras \( \mathfrak{h} \subset \mathfrak{b} \), and fix the standard parabolic subalgebra \( \mathfrak{p} = \mathfrak{p}_n^P \) with respect to \( \mathfrak{b} \) such that its Levi factor is isomorphic to \( \mathfrak{g} \mathfrak{l}_p \times \mathfrak{g} \mathfrak{l}_q \).
Denote by $\mathcal{O}^{p,q}$ the corresponding parabolic category $\mathcal{O}$, i.e., the full subcategory of the category $\mathcal{O}$ for $\mathfrak{gl}_n$ consisting of modules which are locally finite for the action of $p$. (The reader unfamiliar with this construction is referred to e.g. [20, Chapter 9].) We can associate to $\Lambda$ with $\text{up}(\Lambda) = p$ and $\text{down}(\Lambda) = q$ a block of $\mathcal{O}^{p,q}$ which we denote by $\mathcal{O}_\Lambda^{p,q}$. (This works as in [7, (1.3) and (1.4)].) Our aim is to match our signed 1-parameter arc algebra with the projective-injective modules in $\mathcal{O}_\Lambda^{p,q}$, see Remark 5.10, and furthermore construct a generalized signed 1-parameter arc algebra describing these parabolic blocks, see Theorem 5.8.

The idea for the construction is to embed the set of elements of the basis $\mathcal{B}(\Lambda)$ into a set $\mathcal{B}(\mathfrak{hl}_m(\Lambda))$ for some balanced block $\mathfrak{hl}_m(\Lambda) \in \mathfrak{bl}^p$ called the $m$-hull of $\Lambda$. This will enable us to define the generalized signed 1-parameter arc algebra $\mathfrak{A}[0, \varepsilon, \omega]_{\mathfrak{hl}_m(\Lambda)}$ as a subquotient of the signed 1-parameter arc algebra $\mathfrak{A}[0, \varepsilon, \omega]_{\Lambda}$.

We start by introducing the $m$-closure of a weight and corresponding to this the $m$-hull of a block. Morally one puts “enough symbols $\vee$ respectively $\wedge$ to the left respectively right of $\Lambda$ such that one can close any diagram bounding $\Lambda$”.

**Definition 5.1.** Fix $\lambda \in \Lambda$. Let $m \gg 0$ such that $\lambda_i = \text{seq}(\Lambda)_i = \varnothing$ for $|i| > m$. The $m$-closure $\text{cl}_m(\lambda)$ of $\lambda$ is defined as the sequence

$$\text{cl}_m(\lambda)_i = \begin{cases} \lambda_i, & \text{for } |i| \leq m, \\ \vee, & \text{for } -m - \text{up}(\Lambda) \leq i < -m, \\ \wedge, & \text{for } m < i \leq m + \text{down}(\Lambda), \\ \varnothing, & \text{otherwise}. \end{cases}$$

The $m$-hull $\mathfrak{hl}_m(\Lambda)$ of $\Lambda$ is the equivalence class (modulo permutations of $\vee$ and $\wedge$) of $\text{cl}_m(\lambda)$. In addition, we have the subset $\text{cl}_m(\lambda) = \{ \text{cl}_m(\lambda) | \lambda \in \Lambda \}$ inside $\mathfrak{hl}_m(\Lambda)$.

Recalling that $\lambda \cdot \mu \varnothing = \mu \cdot \varnothing$, we fix the idempotent

$$\mathbb{I}_{\text{cl}_m(\lambda)} = \sum_{\nu \in \text{cl}_m(\lambda)} \mathbb{I}_\nu \in \mathfrak{A}[0, \varepsilon, \omega]_{\mathfrak{hl}_m(\Lambda)}.$$

**Lemma 5.2.** Fix an $m$-hull $\mathfrak{hl}_m(\Lambda)$. Then the graded, free $\mathbb{Z}[\omega^{\pm 1}]$-module

$$\mathbb{I}(\Lambda, m) = \langle \lambda' \mathfrak{hl}_m(\Lambda) \lambda' | \lambda', \mu' \in \text{cl}_m(\lambda), \nu' \notin \mathfrak{hl}_m(\Lambda) \setminus \text{cl}_m(\lambda) \rangle_{\mathbb{Z}[\omega^{\pm 1}]}$$

is an ideal in $\mathfrak{A}[0, \varepsilon, \omega]_{\text{cl}_m(\lambda)} = \mathbb{I}_{\text{cl}_m(\lambda)} \cdot \mathfrak{A}[0, \varepsilon, \omega]_{\mathfrak{hl}_m(\Lambda)} \cdot \mathbb{I}_{\text{cl}_m(\lambda)}$.

**Proof.** The proof follows the same arguments as in [16, Section 5.3] (in fact, it is easier since one does not need to consider the dotted cups from [16]), since the exact coefficient do not matter in the argument given therein.

**Remark 5.3.** The set $\mathbb{I}(\Lambda, m)$ is not an ideal if $\alpha$ is not specialized to 0. This is evident from the arguments in [16, Section 5.3].

**Remark 5.4.** Using our isomorphism from Theorem 4.7: in terms of the web algebra the ideal above is given by cup foam basis elements which have a dot on a component touching the boundary in a point with $|i| > m$.

Note that the generalized arc algebras for different hulls of $\Lambda$ are isomorphic.

**Lemma 5.5.** Fix $m' \geq m \gg 0$ such that $\text{seq}(\Lambda)_i = \varnothing$ for $|i| > m$. Then there is an isomorphism of graded $\mathbb{Z}[\omega^{\pm 1}]$-algebras

$$\mathfrak{A}[0, \varepsilon, \omega]_{\mathfrak{hl}_m(\Lambda)} \cong \mathfrak{A}[0, \varepsilon, \omega]_{\mathfrak{hl}_{m'}(\Lambda)}.$$

This isomorphism identifies the subalgebras $\mathfrak{A}[0, \varepsilon, \omega]_{\text{cl}_m(\lambda)}$ and $\mathfrak{A}[0, \varepsilon, \omega]_{\text{cl}_{m'}(\lambda)}$ as well as the ideals $\mathbb{I}(\Lambda, m)$ and $\mathbb{I}(\Lambda, m')$.
Proof. The claim follows immediately since the only difference between the two algebras is the number of symbols $\circ$ between the symbols from the block and the newly added ones, and these do not interfere at all with the multiplication rules.

We then define the generalized version via the indicated quotient construction.

**Definition 5.6.** The generalized signed 1-parameter arc algebra is defined as

$$\mathcal{C}[0, \varepsilon, \omega]_{\Lambda} = \mathfrak{A}[0, \varepsilon, \omega]_{\text{clm}(\Lambda)} / I(\Lambda, m).$$

Up to isomorphisms (induced from Lemma 5.5), this is independent of $m \gg 0$. Moreover, everything above works for specializations of $\varepsilon$ and $\omega$ as well.

**Remark 5.7.** By Theorem 4.7 we have indeed no problems to define the same notions as in Definition 5.6 on the side of the web algebras. The result for the $KBN_0$ specialization of this will be exactly as in [13], see also Remark 5.4.

We are now ready to give the representation theoretical meaning of $\mathcal{C}_\Lambda[0, \varepsilon, \omega]_{\Lambda}$ in case the ground ring is $\mathbb{C}$. By [7, Theorem 1.1] there is an equivalence of categories (5.1)

$$O_{\Lambda}^{p,q} \cong \mathcal{C}[KBN_{0}]_{\Lambda}-\text{Mod}^{fd},$$

sending a minimal projective generator to $\mathcal{C}[KBN_{0}]_{\Lambda}$. Here $\text{Mod}^{fd}$ denotes categories of finite-dimensional modules. Since $\mathcal{C}C[KBN_{0}]_{\Lambda}$ is clearly graded, this allows to define the block $O_{\Lambda}^{p,q}$ of graded category $\mathcal{O}$ as the category of graded, finite-dimensional modules of $\mathcal{C}C[KBN_{0}]_{\Lambda}$.

Thus, using our results from Section 4, we obtain an alternative algebraic description as well as a “singular TQFT model” of category $\mathcal{O}$:

**Theorem 5.8.** For any specialization $q : q \rightarrow \mathbb{C}$ with $q(\alpha) = 0$ it holds that

$$O_{\Lambda}^{p,q} \cong \mathcal{C}[0, q(\varepsilon), q(\omega)]-\text{Mod}^{fd}_{\Lambda} \cong \mathcal{C}[KBN_{0}]_{\Lambda}-\text{Mod}^{fd}_{\Lambda} \cong \mathcal{C}[\text{Car}]_{\Lambda}-\text{Mod}^{fd}_{\Lambda},$$

(Similarly for the corresponding web algebras.)

**Proof.** This follows directly from (5.1) and Proposition 4.15. The claim about the web algebras follows then from Theorem 4.1.

**Example 5.9.** Take $\Lambda$ balanced with block sequence $\star \star$. For $m = 2$ we obtain for $\lambda \in \Lambda$ with sequence $\wedge \vee$ the 2-closure with sequence $\vee \wedge \vee$, while for $\mu \in \Lambda$ with sequence $\vee \wedge$ we have the 2-closure with sequence $\vee \wedge \wedge$. Thus, since $I(\Lambda, 2)$ “consists of cup foam basis elements without a dot touching the two outer points”, we have (where we marked the components touching the outer points)

as a $\mathbb{Z}[\omega^{\pm 1}]$-basis of $\mathcal{C}[0, q(\varepsilon), q(\omega)]$ (denoted from left to right by $I_\lambda, x_\lambda^\mu, x_\mu^\lambda, I_\mu, x_\mu^\mu$).

This is the path algebra of the quiver (with $x_\mu^\mu \circ x_\lambda^\mu = 0$ and $x_\lambda^\mu \circ x_\lambda^\mu = q(\omega)x_\mu^\mu$)

Working over $\mathbb{C}$, this is the quiver of the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_2$, see for example [41, Section 5.1.1] for an explicit calculation of this quiver.
Remark 5.10. Let $\pi(O_{\Lambda}^{p,q})$ be the category of finite-dimensional modules for the endomorphism algebra of all indecomposable projective-injectives in $O_{\Lambda}^{p,q}$. We have $\pi(O_{\Lambda}^{p,q}) \cong \mathbb{A}[0,q(\varepsilon), q(\omega)]-\text{Mod}^{\text{id}}$.

This can be seen as in the proof of Theorem 5.8. That is, one first uses the known equivalence to the $KBN_0$ setup, see [7, Lemma 4.3], and then the equivalence to any other specialization in $\mathbb{C}$ with $q(\alpha) = 0$ from Theorem 4.2.

5.2. Connection with link and tangle invariants. Given a tangle diagram $T$, we will now construct a chain complex $\mathbb{s} \mapsto \mathbb{C}^T : \mathbb{s} \mapsto \mathbb{C}^T$ with values in $\mathbb{W}[\text{biMod}]$. We show in Proposition 5.23 that its homotopy type is an invariant of the corresponding tangle, up to isotopy. Hence, $\mathbb{s} \mapsto \mathbb{C}^T$ extends to a complex for tangles (and thus, for links) called the $5$-parameter complex. We see this as a generalization of Khovanov’s original construction [24]. Indeed, the $5$-parameter complex specializes to the original $KBN$ complex $\mathbb{s} \mapsto \mathbb{KBN}$, as well as to the versions $\mathbb{s} \mapsto \mathbb{Ca}$, $\mathbb{s} \mapsto \mathbb{CMW}$ and $\mathbb{s} \mapsto \mathbb{Bl}$. Using our various isomorphisms, we can show in Theorem 5.25 that all of these give the same tangle invariant.

5.2.1. Tangles and tangled webs. Akin to upwards oriented webs as in Section 2.3, we define tangles (algebraically).

Definition 5.11. An (oriented) tangle diagram is an oriented, four-valent graph, whose vertices are labeled by crossings, which can be obtained by gluing (whenever this makes sense) or juxtaposition of finitely many of the following pieces:

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\bigcirc \\
\bigcirc
\end{array}
\end{array}
\]

The third generator is called a positive crossing and the fourth a negative crossing. We assume that these are embedded in $\mathbb{R} \times [-1,1]$ in the same ways as webs. In particular, we can associate to each such tangle diagram a (bottom) source $\vec{s}$ and a (top) target $\vec{t}$—both being elements of $\{0, +, -\}^2$—in the evident way using the conventions indicated in (5.2). We only consider tangle diagrams with an even number of bottom and top boundary points, called even tangle diagrams.

Remark 5.12. The restriction to even tangle diagrams comes from the fact that we work with $\mathbb{W}[\cdot]$ and $\mathbb{A}[\cdot]$. One could treat arbitrary tangles by using the generalized algebras from Section 5.1—restricting the parameters as therein—i.e.: it is clear by our results that one can follow [13, Section 5] or [42, Section 5] to define parameter dependent complexes of bimodules for these generalized algebras giving rise to an invariant of arbitrary tangles. We have decided for brevity to only do the $\mathbb{W}[\cdot]$ and $\mathbb{A}[\cdot]$ versions here, since we treat these in detail in this paper.

We study the following category, cf. [22, Theorem XII.2.2].

Definition 5.13. The category of tangles $1\text{-Tan}$ consists of:

\begin{itemize}
\item Objects are sequences $\vec{s}, \vec{t} \in \{0, +, -\}^2$ with only a finite and even number of non-zero entries (which includes $\emptyset = (\ldots, 0, 0, 0, \ldots)$).
\item 1-Morphisms from $\vec{s}$ to $\vec{t}$ are all tangle diagrams with source $\vec{s}$ and target $\vec{t}$.
\item The relations are the usual tangle Reidemeister moves, which can be found in [22, Section XII.2, Figures 2.2 to 2.9].
\end{itemize}
Composition of tangles is given via the evident gluing.

Remark 5.14. The tangle Reidemeister moves can be roughly described as:

(iR) “Isotopies”, i.e. zig-zag moves and relations to twist crossings.

(uR) The usual Reidemeister moves $R_1$, $R_2$, and $R_3$, with the latter two seen as braid moves, i.e. pointing only upwards.

(mR) Some mixed (oriented) $R_2$ moves $mR_2$, cf. (5.11).

Note that $1\cdot\text{Tan}$ gives a generators and relations description of the topological category of tangles (as explained e.g. in [22, Section XII.2]). We denote by $\vec{s}$ $\vec{t}$ $1$-morphisms in $\text{Hom}_{1\cdot\text{Tan}}(\vec{s}, \vec{t})$ and by $\vec{s}$ a choice of a diagram representing $\vec{s}$ $\vec{t}$.

Definition 5.15. Define (upwards oriented) tangled webs, i.e. (upwards oriented) webs as in Section 2.3, but additionally allow local generators of the form

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
\end{array}
\]

called positive crossing, negative crossing and phantom crossings. Clearly, tangle webs have the same boundary sequences as webs, e.g. $\vec{k}, \vec{l} \in \mathfrak{b}^{1^2}$. We write $\text{Hom}_{tw}(\vec{k}, \vec{l})$ for the set of tangle webs with bottom sequences $\vec{k}$ and top sequence $\vec{l}$.

We now give a (straightforward) translation of a tangle diagram $\vec{s}$ $\vec{t}$ into a tangled web. We assume for simplicity that we fix an $\ell \in \mathbb{Z}_{\geq 0}$ which is $\ell \gg 0$ (“big enough”). Moreover, we assume that $\vec{s}$ has $s \in 2\mathbb{Z}_{\geq 0}$ non-zero elements etc. in what follows.

Definition 5.16. Given a tangle diagram $\vec{s}$ $\vec{t}$, we define a map $w(\cdot)$: $\vec{s}$ $\vec{t}$ $\mapsto w(\vec{s}$ $\vec{t}$) to tangled webs in $\text{Hom}_{tw}(\vec{k}, \vec{l})$ with $\vec{k} = \omega_{\ell+x} + \omega_s$ and $\vec{l} = \omega_{\ell+t} + \omega_t$ locally as

\[
\begin{array}{cccc}
\uparrow & \sim & \downarrow & \\
\uparrow & \sim & \downarrow & \\
\uparrow & \sim & \downarrow & \\
\uparrow & \sim & \downarrow & \\
\uparrow & \sim & \downarrow & \\
\end{array}
\]

(5.4)

(By using the phantom crossings from (5.3), one can always rearrange everything such that one can start in $\omega_{\ell+x} + \omega_s$ and end in $\omega_{\ell+t} + \omega_t$. This association is far from being unique, but what we are going to do will not depend on the choice of the map $w(\cdot)$ and the concerned reader can pick any such choice.)

Example 5.17. One example of this association is the following.

Here we have to pull the phantom edge to the right, because we demand that we start and end in a sequences all of whose entries equal to 2 are placed on the left.

5.2.2. The 5-parameter complex: definition and invariance. Fix an additive 2-category $\mathcal{X}$. A chain complex $(C_i, d_i)_{i \in \mathbb{Z}}$ with values in $\mathcal{X}$ is a chain complex whose chain groups $C_i$ are the 1-morphisms from $\mathcal{X}$ and whose differentials $d_i$ are the 2-morphisms of $\mathcal{X}$ such that $d_i \circ d_i = 0$ for all $i \in \mathbb{Z}$. Such a complex $(C_i, d_i)_{i \in \mathbb{Z}}$ is called bounded, if $C_i = 0$ for $|i| \gg 0$. Denote by $1\cdot\text{CC}(\mathcal{X})$ the category of bounded complexes with values.
in \( X \). These can be related via 2-morphism, i.e. chain maps with entries from \( X \). We consider these in the graded setup, allowing only 2-morphisms of degree 0.

To construct \([\tau_{\mathcal{T}}]^p\) for an oriented tangle diagram \( \tau_{\mathcal{T}} \), we first define \([u_t]^p\) for a tangle web \( u_t \in \text{Hom}_{\mathcal{CW}}(\vec{k}, \vec{l}) \). To this end, we define the following basic complexes, where we denote by \( \{ \cdot \} \) the shift of the chain groups in their internal degree.

**Definition 5.18.** The basic complexes are

\[
\begin{align*}
[\vec{x}]^p &= \begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {+1};
\node (B) at (1,0) {+2};
\node (C) at (0,-1) {-1};
\node (D) at (1,-1) {-2};
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}
\end{array}, \\
[\vec{x}]^p &= \begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {+1};
\node (B) at (1,0) {+2};
\node (C) at (0,-1) {-1};
\node (D) at (1,-1) {-2};
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}
\end{array}, \\
[\vec{x}]^p &= \begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {+1};
\node (B) at (1,0) {+2};
\node (C) at (0,-1) {-1};
\node (D) at (1,-1) {-2};
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}
\end{array}, \\
[\vec{x}]^p &= \begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {+1};
\node (B) at (1,0) {+2};
\node (C) at (0,-1) {-1};
\node (D) at (1,-1) {-2};
\draw (A) -- (B);
\draw (C) -- (D);
\end{tikzpicture}
\end{array}
\end{align*}
\]  

with the underlined terms sitting in homological degree zero. We see these as objects in \( 1-\text{CC}(\mathbb{W}[P]^p)-\text{biMod}_{\text{bi}}^p \), i.e. the chain groups are (shifts of) based web bimodules and the differentials are \( \mathbb{W}[P] \)-bimodule homomorphisms.

Using these, we can associate to any tangled web \( u_t \in \text{Hom}_{\mathcal{CW}}(\vec{k}, \vec{l}) \) an object \([u_t]^p\) via the “tensor products” of the basic complexes. (Similarly for any specialization of \( P \).) We skip the details of this “tensor product procedure” and refer the reader to [24, Section 3.4], which we can copy, incorporating \( P \), without problems.

**Remark 5.19.** The whole definition of \([u_t]^p\) can be also made using Khovanov’s famous cube construction (as in [23, Section 3] or [2, Section 2.3]). Here the basic complexes from (5.5) and (5.6) give rise to 0/1-resolutions, while the basic complexes from (5.7) have only one resolution (the one given in (5.7)).

**Definition 5.20.** Let \( \tau_{\mathcal{T}} \) be a tangle diagram. Set \([\tau_{\mathcal{T}}]^p = [w(\tau_{\mathcal{T}})]^p\), which is an object of \( 1-\text{CC}(\mathbb{W}[P]^p)-\text{biMod}_{\text{bi}}^p \). (Similarly for any specialization of \( P \).)

**Specializations 5.21.** Working with the based version, we have as chain complexes:

\[
\begin{align*}
[\tau_{\mathcal{T}}]^p\alpha_{1,1,1,1}^e &\cong [\tau_{\mathcal{T}}]^p\text{KBN}, \\
[\tau_{\mathcal{T}}]^p\alpha_{1,1,1,-1}^e &\cong [\tau_{\mathcal{T}}]^p\text{Ca}, \\
[\tau_{\mathcal{T}}]^p\alpha_{1,1,1,1}^e &\cong [\tau_{\mathcal{T}}]^p\text{CMW},
\end{align*}
\]

Hereby we denote specializations in our usual 5-term notation.

**Remark 5.22.** Note that all the chain groups are isomorphic to the one in the KBN setup from [24], but the differential are crucially different from the KBN differentials—as can be seen in small examples—and e.g. not just a scalar times the KBN differentials. Similarly for other specialization.

Denote by \( 1-\text{HCC}(\mathbb{W}[P]^p)-\text{biMod}_{\text{bi}}^p \) the same category as the one defined before **Definition 5.18**, but modulo chain homotopy, where we only use \( \mathbb{W}[P] \)-bimodule homomorphisms for all maps in question.

**Proposition 5.23.** The association \([.]^p\) from **Definition 5.20** extends to a functor

\[
[.]^p : 1-\text{Tan} \to 1-\text{HCC}(\mathbb{W}[P]^p)-\text{biMod}_{\text{bi}}^p.
\]
Thus, $[\cdot]^p$ is an invariant of tangles. (Similarly for any specialization of $P$.)

**Proof.** Note that composition of tangles corresponds, by construction, to tensoring of bimodule complexes. (The careful reader might want to copy [24, Proposition 13] to see this.) Thus, there are two things to show: we have to show that $[\cdot]^p$ does not depend on our choice of the map $w(\cdot)$ from Definition 5.16, and we have to show invariance under the tangle Reidemeister moves from Definition 5.13.

> **Independence of choice.** Given some $\mathbf{sT}\, t$. Assume that we have two different choices $w_1(\mathbf{sT}\, t)$ and $w_2(\mathbf{sT}\, t)$. To show independence we have to prove that

$$[w_1(\mathbf{sT}\, t)]^p = [w_2(\mathbf{sT}\, t)]^p \text{ as complexes in } 1\text{-HCC}(\mathfrak{W}[P]^p\text{-biMod}^p).$$

Now, two different choices $w_1(\mathbf{sT}\, t)$ and $w_2(\mathbf{sT}\, t)$ can only differ by the following local moves: The **ordinary-ordinary-phantom** $R3$ moves (similarly for a negative crossing)

(5.8)

The **ordinary-phantom** $R2$ moves and the **pure-phantom** $R2$ move

(5.9)

Third, the **ordinary-phantom-phantom** $R3$ moves and the **pure-phantom** $R3$ move

(5.10)

Thus, it suffices to show that the chain complex stays the same—up to chain homotopy—if the two choices $w_1(\mathbf{sT}\, t)$ and $w_2(\mathbf{sT}\, t)$ differ by one of these moves. For this purpose, we have by [17, Lemmas 4.4 and 4.5] (which work for $P$ as well) that the chain groups are isomorphic—the slogan is “isotopic webs give isomorphic web bimodules”. These isomorphisms are given by the evident foams, which are clearly $\mathfrak{W}[P]$-bimodule homomorphisms. Thus, it remains to show that these commute with the differentials in the associated complexes. This is clear for a difference as in (5.9) or (5.10). For a difference as in (5.8) one easily checks that these isomorphisms commute with the differentials. Indeed, instead of checking that these work out locally, we could also perform the necessary arrangements globally, i.e. outside of the illustrated picture. We thus, avoid the crossings and the saddle foams will commute with these, since all non-trivial changes are “far apart”.

> **Tangle Reidemeister moves.** Denote by $\mathbf{sT}\, t$ and $\mathbf{sT}\, t$ two tangle diagrams that differ by one of the moves $(i\mathbf{R})-(m\mathbf{R})$. Again, if we show that

$$[\mathbf{sT}\, t]^p = [\mathbf{sT}\, t]^p \text{ as complexes in } 1\text{-HCC}(\mathfrak{W}[P]^p\text{-biMod}^p),$$

then we are done. The main point is invariance under a move from $(u\mathbf{R})$. Indeed, invariance under a move from $(i\mathbf{R})$ follows as above, because e.g. we can by the above assume that a zig-zag move looks locally as in Example 5.17 and then use the same arguments as before (“isotopic webs give isomorphic web bimodules”). Hence, it
remains to show invariance under the moves from ($uR$) and ($mR$), i.e. we have to check the following, together with variations of these:

\begin{align}
(5.11) \quad \begin{array}{c}
\xymatrix{ R_1 \ar[r]^a & \bullet \\ R_2 \ar[r]^b & \circ \\ R_3 \ar[r]^c & \nabla \\ mR_2 \ar[u]^d } \\
\end{array}
\end{align}

(i) **$R_1$ move, right curl with a positive or negative crossing.** For the positive crossing we take the same cobordisms as in [3, Section 3.3]—adding phantom edges/facets similar to (5.4)—but set $f_0 = \tau_p \omega_+^2 f_{BI}$, $g_0 = -\omega - g_{BI}^L$, $g_0^p = \omega + g_{BI}^R$ and $D_0 = \tau_p \omega_+^2 D_{BI}$. (We use Blanchet’s notation, where $g_{BI}^L$ and $g_{BI}^R$ mean the left and right summand of Blanchet’s $g$ from [3, Fig. 13] with the dot in the back for $g_{BI}^R$). Similarly for the negative crossing with exchanged roles of $f$ and $g$ (beware the slight typos in [3, Section 3.3].)

(ii) **$R_2$ move, left curl with a positive or negative crossing.** Similarly as for the right curl, but exchanging the roles of $\omega_+$ and $\omega_-$. 

(iii) **$R_3$ move, both versions.** We use the same cobordisms—plus phantom edges and facets—and coefficients as in [3, Sections 3.4 and 3.5].

(iv) **$R_4$ move, both versions.** This can be showed using the usual (abstract) Gauss elimination argument, as pioneered in [1]. To be precise, one uses the $\mathcal{P}$-analogs of [17, Lemma 4.3]—the “circle removal”—and then twice the Gauss elimination from [1, Lemma 3.2]. One obtains that the two complexes for both sides of the $R_3$ move have isomorphic chain groups. These can then be matched directly. We leave the details to the reader, where we note that all appearing coefficients are trivial, because the “complicated” maps in the Gauss elimination are at extremal parts of the complexes.

(v) **$mR_2$ move, version in (5.11).** Due to our setup using upwards oriented webs, this move is quite different from the one in e.g. [3]. Thus, we—ignoring shifts—just give the chain maps and homotopies explicitly:
The second illustration gives the chain maps $f_1, f_2, g_1, g_2$ and the third the non-trivial homotopies $h_1, h_2$—all of which are the composites of the displayed foams—as well as all their scalars. We leave it to the reader to verify that $f_1, f_2, g_1, g_2$ commute with the differentials $d_1, d_2, d_3, d_4$, that $f_1, f_2$ are mutually inverses, $g_1 \circ g_2 = 0$ and $h_1 \circ d_2 - d_4 \circ h_2 = g_2 \circ g_1$.

(vi) mR2 move, other versions. As above but keeping in mind that some scalars depend on directions, e.g. the ones turning up for the squeeze (2.15).

Everything above works for any specialization of $\mathcal{P}$ and the claim follows.

Thus, we can write $\mathcal{J}_{T_i}^\mathcal{P}$ etc. without ambiguity.

5.2.3. The signed 2-parameter complex: comparison. Our results of Section 4 almost immediately imply that the link and tangle homologies from above “are the same”. Let us make this precise—using our results from Section 4.

Denote by $\mathcal{J}_{T_i}^{KBN}$ the functor obtained via specializing $q(\alpha) = \alpha$, $q(\varepsilon) = 1$ and $q(\omega) = 1$ (and scalar extension). Moreover, by abusing notation, we denote by

$$\Psi : \mathcal{M}^{[KBN]} \text{-biMod}_{gr}^\mathcal{P} \cong \mathcal{M}^{[Q]} \text{-biMod}_{gr}^\mathcal{P}$$

the equivalence obtained by combining Theorem 4.1 and Theorem 4.3.

Proposition 5.24. The following diagram commutes.

$$\begin{align*}
\begin{array}{ccc}
1-\text{HCC}([\mathcal{J}_{T_i}^{KBN}] \text{-biMod}_{gr}^\mathcal{P}) & \cong & 1-\text{HCC}([\mathcal{J}_{T_i}^{Q}] \text{-biMod}_{gr}^\mathcal{P}) \\
\mathcal{J}_{T_i}^{KBN} & \mapsto & \Psi \\
1-\text{Tan} & \mapsto & \mathcal{J}_{T_i}^{Q}
\end{array}
\end{align*}$$

(Similarly for any further simultaneous specialization of $\alpha$.)

Proof. Let $\mathcal{J}_{T_i}$ be any tangle. The Khovanov cubes associated to $\Psi([\mathcal{J}_{T_i}^{KBN}])$ and $[\mathcal{J}_{T_i}^{Q}]$—mentioned in Remark 5.19—are the same combinatorially, i.e. all vertices and all edges are at the same positions. Moreover, by Theorem 4.1 and Theorem 4.3, the web bimodules associated to vertices are isomorphic. Recall that edges of a Khovanov cube have an associated “multiplication foam” $f : W(u) \to W(v)$ (as in (5.6)) or a “reversed multiplication foam” $g : W(v) \to W(u)$ (as in (5.5)). Clearly, the “type” of such a foam associated to an edges is the same for the two complexes under consideration. To be precise, for each edge in the Khovanov cubes we have

$$
\begin{align*}
f : W[\mathcal{KBN}](u) & \to W[\mathcal{KBN}](v) \mapsto \text{coeff}_v \circ f \circ \text{coeff}_u^{-1} : W[\mathcal{Q}](u) \to W[\mathcal{Q}](v), \\
g : W[\mathcal{KBN}](v) & \to W[\mathcal{KBN}](u) \mapsto \text{coeff}_u \circ g \circ \text{coeff}_v^{-1} : W[\mathcal{Q}](v) \to W[\mathcal{Q}](u),
\end{align*}
$$
where we use the evident notation from Section 4, but for the web algebras side instead of the arc algebra used therein. It remains to analyze these differentials, i.e. we have to compare $\Psi(f)$ to $f_Q$ and $\Psi(g)$ to $g_Q$ (with the latter being the differentials for $[\mathbb{T}]^\mathbb{Q}$).

With the work already done this is not hard. Indeed, it follows from Proposition 4.19 that $\Psi(f) = f_Q$, while it follows from Proposition 4.29 that $g_Q = \text{coeff}_{v} \circ \Psi(g) \circ \text{coeff}_{u}^{-1}$.

Thus, the differentials of two complexes $\Psi([\mathbb{T}]^\mathbb{KBN})$ and $[\mathbb{T}]^\mathbb{Q}$ differ only by “units”, and we can use the usual unit sprinkling (see [14, Lemma 4.5]) to get a chain isomorphism between them. Hence, it remains to verify that the maps used in this chain isomorphism are actually entrywise $W[\mathbb{Q}]$-bimodule homomorphisms. This is true by the above and Lemma 4.33. The statement follows.

Let $R[\alpha]$ and $q$ be as at the beginning of Section 4. Moreover, denote by $[\cdot]^\mathbb{Q}$ the functor obtained from $[\cdot]^\mathbb{K}$ via the specialization $q: Q \to R[\alpha]$ (and scalar extension).

Abusing notation, we keep on writing $\Psi$ for the equivalence.

Theorem 5.25. The following diagram commutes.

$$
\begin{array}{ccc}
1-\text{HCC}(\mathbb{W}[\mathbb{Q}]-\text{biMod}_{\mathfrak{gr}}^p) & \xrightarrow{[\cdot]^\mathbb{Q}} & 1-\mathbb{T}an \\
\downarrow \Psi & & \downarrow \\
1-\text{HCC}(\mathbb{W}[\mathbb{Q}]-\text{biMod}_{\mathfrak{gr}}^p).
\end{array}
$$

(Similarly for any further simultaneous specialization of $\alpha$.)

Proof. Exactly as in the proof of Proposition 5.24, since we have not used the specific form of the parameters in question.

These result are stronger than just saying that the corresponding chain complexes are homotopy equivalent since we match the bimodules structures as well.

Let us write $\approx$ for short if two homologies obtained via specialization of $[\cdot]^\mathbb{Q}$ can be matched as in (5.12). In this case, we say that they give the same invariant.\hfill $\blacksquare$

Specializations 5.26. Set $R = \mathbb{Z}[\alpha]$ and specialize $q(\alpha) = \alpha$, $q(\epsilon) = 1$ and $q(\omega) = 1$ respectively $q(\alpha) = \alpha$, $q(\epsilon) = -1$ and $q(\omega) = 1$. Then

$$
[\mathbb{T}]_{\mathbb{Z}[\alpha]}^{\mathbb{KBN}} \approx [\mathbb{T}]_{\mathbb{Z}[\alpha]}^{\mathbb{Bl}}.
$$

(Similarly for e.g. $q(\alpha) = 0$.) This shows that Khovanov’s original link homology and Blanchet’s version of it give the same invariant—even for tangles.\hfill $\blacksquare$

Remark 5.27. The result of Section 5.1 give a way to relate our link and tangle invariants constructed here to the link and tangle invariants $[\cdot]_\mathcal{C}^\mathbb{O}$ constructed from category $\mathcal{O}$. We refer the reader to [42, Section 5.10] for details.\hfill $\blacksquare$

Working over $R = \mathbb{Z}[\alpha, i]$ or $R = \mathbb{C}$ gives a stronger result whose proof is now evident by using Theorem 5.25 and Remark 5.27.

Corollary 5.28. We have (with the last $\approx$ only for $R = \mathbb{C}$ and $q(\alpha) = 0$)

$$
[\mathbb{T}]_{R}^{\mathbb{KBN}} \approx [\mathbb{T}]_{R}^{\mathbb{Ca}} \approx [\mathbb{T}]_{R}^{\mathbb{CMW}} \approx [\mathbb{T}]_{R}^{\mathbb{Bl}} \approx [\mathbb{T}]_{\alpha = 0}^{\mathbb{O}}.
$$

(Similarly for any further simultaneous specialization of $\alpha$.)\hfill $\blacksquare$

This was known for links, but, to the best of our knowledge, not for tangles.
5.2.4. The signed 2-parameter complex: functoriality. In fact, our results are even stronger: It follows from Section 4 that all of them can be arranged such that they give functorial invariants of (upwards oriented) braids. Moreover, using the arc algebra setup, calculations of these functorial invariants can be made explicit.

In order to give some detail, let us denote by $\mathcal{2-Tan}$ the 2-category of tangles. This is the 2-category whose underlying 1-category is $\mathcal{1-Tan}$ and whose 2-morphisms are certain cobordisms called 2-tangles. There is a generators and relations description of $\mathcal{2-Tan}$ in the spirit of the one for $\mathcal{1-Tan}$ as well, with relations given by the movie moves. There is a certain full 2-subcategory $\mathcal{2-Tan}^\uparrow$ whose underlying 1-morphisms are (upwards oriented) braids. We do not recall the details here and refer the reader to [2, Section 8] or [11, Chapter 1], and to [11, Section 3.4] for the braid version.

Hence, in the light of Proposition 5.23, it makes sense to ask, if there is a 2-functor

$$[\ ]^P : \mathcal{2-Tan}^\uparrow \to \mathbf{2-HCC}(\mathcal{W}[P]-\text{biMod}_{\text{gr}}^P)$$

Here $\mathbf{2-HCC}(\mathcal{W}[P]-\text{biMod}_{\text{gr}}^P)$ means that we identify homotopic 2-morphisms.

Again, we specialize to $\mathcal{Q}$. Let $\mathcal{2-Tan}^\uparrow$ be the 2-subcategory consisting of only upwards oriented braids, which we do not consider up to isotopy; see [3, Remark 5.2] for the reason for this. In this case we have $\mathbf{2-HCC}(\mathcal{Q}-\text{Mod}_{\text{free}})$ as a target 2-category. Now, Caprau, Clark–Morrison–Walker and Blanchet showed that their construction of Khovanov homology extends to 2-functors (for $R = \mathbb{Z}[i]$ or $R = \mathbb{Z}[1/2]$ and $q(\alpha) \in \{0, 1\}$)

$$(5.13) \quad [\ ]^\text{Ca}, [\ ]^\text{CMW}, [\ ]^\text{Bl}_{\mathbb{Z}[1/2]} : \mathcal{2-Tan}^\uparrow \to \mathbf{2-HCC}(R-\text{Mod}_{\text{free}}),$$

see—using additionally the ideas from [18] to include tangles—[10, Theorem 3.5], [14, Theorem 1.1] and [3, Theorem 5.1].

Thus, the above gives a way to fix functoriality of Khovanov homology without changing the framework of KBN. Namely, use any of the functorial invariants from (5.13) and “pull it over”. To be more precise, one uses the coefficient maps (from the KBN setup to any of $\text{Ca}$, $\text{CMW}$ or $\text{Bl}$) from Definition 4.18 on the chain groups (bimodules) to get a different, scalar adjusted, cup foam basis. Then one can rearrange the differentials (web bimodule homomorphisms) as in Example 4.38. The resulting complex is functorial.

Remark 5.29. We only get the braid version, since we associate the “wrong” chain homotopies to e.g. the $mR2$ move:

$$\text{Resolution} \xleftarrow{\text{mR2 in } 1-\text{Tan}} \quad \text{mR2} \xrightarrow{\text{Resolution}}$$

Here the left-hand side illustrates what we associated as a particular resolution to the $\text{mR2}$ move, while the right-hand side illustrates what e.g. Blanchet [3] associates to this move. This makes a difference for the functoriality, but not on the level of the link and tangle invariants.

However, copying what we have done in this paper in the more flexible setup of all webs instead of upwards oriented would give a 2-functor

$$[\ ]^P : \mathcal{2-Tan} \to \mathbf{2-HCC}(\mathcal{W}[P]-\text{biMod}_{\text{gr}}^P)$$

whose specializations could again be compared as we did throughout the paper. ■
6. Main proofs

In this final section we give the more technical proofs of our main statements (together with some technical lemmas needed to proof these statements).

6.1. Proof of Theorem 4.7. We will use the notation from Section 2 and Section 4.1.

Proof. Our proof here follows [17, proof of Theorem 4.18]. That is, we show that each step in the multiplication procedure from Definition 2.33 locally agrees with the one from Section 3.3. Here we use Lemma 4.6, i.e. it suffices to show that they agree on the cup foam basis on the side of $\mathbb{M}[Q]$. Note that the setup of $\mathbb{M}[Q]$ is more flexible and thus, harder to work with. That is, throughout the whole proof, we first check the multiplication steps for $\mathbb{M}[Q]$ where some rewriting has to be done, and then for $\mathbb{M}[Q]$ where we can read off the multiplication directly using the rules from Section 3.3.

Now, there are four topologically different situations to check:

(i) Non-nested merge. Two non-nested circles are replaced by one circle.
(ii) Nested merge. Two nested circles are replaced by one circle.
(iii) Non-nested split. One circle is replaced by two non-nested circles.
(iv) Nested split. One circle is replaced by two nested circles.

As in [17, proof of Theorem 4.18], we will go through the following cases:

(A) Basic shape. The involved components are as small as possible with the minimal number of phantom edges.
(B) Minimal saddle. While the components themselves are allowed to be of any shape, the involved saddle has only a single phantom facet.
(C) General case. Both, the shape as well as the saddle, are arbitrary.

Our proof here is in principle the same as [17, proof of Theorem 4.18], but harder and more delicate, because the appearing factors are more involved. Thus, for brevity, we only do here the basic shapes in detail and sketch the remaining ones.

We start with (A), where—including a horizontal flip of the (ii) case—we have:

The two rightmost cases are called $H$-shape and $C$-shape. (Recall that, by our convention, the $C$-shape does not occur.) Here we have displayed both, the web and its corresponding arc diagram.

To understand the following calculations recall that we use the specialization from (2.3). What is of paramount importance about it is that we do not have to worry about the “direction” in which we apply squeezing (2.15), dot migrations (2.16) or ordinary-to-phantom neck cutting (2.14), since all of them will just contribute an $\varepsilon$.

Non-nested merge - basic shape. This case works almost exactly in the same way as in [17, proof of Theorem 4.18]. That is, multiplication of basis cup foams yields topologically basis cup foams again, except in case where we start with two dotted
basis cup foams. But in this case we can use (2.9) to create a basis cup foam without dots and a factor $\alpha$. The same happens for $\mathcal{A}[Q]$, see (3.13).

\textbf{Nested merge - basic shape.} In this case something has to be done on the side of $\mathcal{W}[P]$. In fact, this is the most complicated case and we go through all details and will be shorter in the other cases afterward. The multiplication step is given by

\textbf{Non-nested split - basic shape.} The multiplication step is
We have again illustrated the dots which are created while topologically rearranging the resulting foam. Assuming that a basis cup foam without dots is sitting underneath the leftmost picture, we see that we almost get a basis cup foam after stacking the saddle on top: we get two cup foams sitting underneath the left and right circle which touch each other in the middle in a closed singular seam, and a corresponding phantom facet. Thus, by using the singular seam removal (2.17)—creating dots as illustrated above—and dot migration (2.16), we get two basis cup foams, one with a dot on the rightmost facets of the left circle and one with a dot on the rightmost facet of the right circle. The singular seam removal gives a factor $\varepsilon\omega$ for the first and a factor $\omega$ for the second. Additionally, the second gets a factor $\varepsilon$ coming from the dot migration.

Recalling $\varepsilon = \pm 1$, this matches the side of $A[Q]$ which was computed in (3.15). On the other hand, if a basis cup foam with a dot on the rightmost facet is sitting underneath the leftmost picture, we can move the dot topologically aside, proceed as above and create, after using the singular seam removal (2.17) and dot migration (2.16), two basis cup foams. Remembering that we started with a dot, we see that these two are now a basis cup foam with one dot on the rightmost facets of the two circles and a foam that is topological a basis cup foam, but with two dots on the rightmost facet of the right circles. Thus, using (2.9), we get the same result as for $A[Q]$, see (3.15).

$\triangleright$ Nested split - basic shape. The multiplication foam is now (indicating again the cylinder we want to cut and the dots we created via cutting)

Again we can apply neck cutting. This time to the internal cylinder in the second foam between the middle web and the rightmost web connecting the two nested circles that we can cut using (2.13). First assume that the original basis cup foam sitting underneath has no dots. After neck cutting we get a sum of two basis cup foams, so nothing needs to be done topologically. One has a factor $\omega$ and a dot sitting on the rightmost facet of the nested circle, the other has a factor $\varepsilon\omega$ and a dot sitting on the next to leftmost facets of the outer circle, as illustrated above. Moving this dot across two phantom facets to the rightmost facets picks up, by dot migration (2.16), a factor $\varepsilon^2 = 1$—recalling that the dot is sitting underneath the place where we applied neck cutting and hence, is on a foam with a generic slice as in the leftmost picture above. Thus, we end with the same as for $A[Q]$, see (3.16). Similarly, starting with a basis cup foam sitting underneath having a dot on the rightmost facet, we can move the dot topologically aside and proceed as before. As above in the non-nested split case, we get a sum of two basis cup foams, one with one dot on each rightmost facet, and one with two dots on the rightmost facet of the outer circle. Hence, using (2.9) again, we get the same result as in (3.16).

The remaining cases (B) and (C) from above can be proven by copying the arguments from [17, Proof of Theorem 4.18]. In particular, non-interfering foam parts can be topologically moved away and do not matter in the rewriting process. The only thing that changes is that the dot moving signs, the topological sign and the saddle
sign from (3.6) are now powers of $\varepsilon$ instead of powers of $-1$.

**General shape, but minimal saddle.** The dot moving signs are precisely the same on both sides (recalling that moving across phantom facets always gives $\varepsilon$). Furthermore, we can always move existing dots topologically aside and we do not have to worry about them until the very end where we possibly apply (2.8) to remove two of them. In particular, if we understand the undotted case, then the dotted follows. So let us consider only basis cup foams without dots. In case of the non-nested merge, the resulting foams are topologically basis cup foams and we are done. In case of the nested merge we have to topologically manipulate the result until it is a basis cup foam again. This can be done as in [17, Proof of Theorem 4.18] with the difference that the formula [17, (43)] gives

$$\varepsilon^{1/4(dA(C)m-2)} \cdot \varepsilon^{-1} \quad \text{instead of} \quad (-1)^{1/4(dA(C)m-2)} \cdot (-1)^{1/4}.$$  

This matches the side of $Z[Q]$. For the case of the non-nested split we can proceed as above and we get the same factors which matches the case of $Z[Q]$. Last, for the nested split we copy the argument in [17, Proof of Theorem 4.18], but picking up

$$\varepsilon^{1/4(dA(C)m-2)} \quad \text{instead of} \quad (-1)^{1/4(dA(C)m-2)}.$$  

Again, this is as in case $Z[Q]$. ▶

**General shape.** The non-nested merge works as above, i.e. this case does not depend on the “size” of the saddle. Incorporating a general saddle in the cases of a nested merge is as in [17, Proof of Theorem 4.18] but with

$$\varepsilon^{sA(\gamma)} \quad \text{instead of} \quad (-1)^{sA(\gamma)}.$$  

The non-nested split case can be done as above for the basic shape, but the two cups foams touch each other now locally as (we have illustrated the case $sA(\gamma) = 2$)

Using (2.17) once followed by $sA(\gamma) - 1$ applications of (2.14) (as well as $2(sA(\gamma) - 1)$ applications of (2.16) which do not contribute because $\varepsilon = \pm 1$) gives the above, where again $sA(\gamma) = 2$—the left $\varepsilon$ has an exponent $sA(\gamma)$ in general.

The case of a nested split does not depend on the saddle and can be done as above in case of the minimal saddle. In all cases, we get the same on the side of $Z[Q]$ which finishes the arguments for the general cases. ▶

The case of specialized $Q$ works analogously. Thus, the claim follows. ■

Next, we prove Proposition 4.15. We will use the notation from Section 4.2.

**Proof.** First note that the maps from (4.12) are clearly homogeneous and $Q$-linear. Moreover, it suffices to show the isomorphism for some fixed, but arbitrary, $\Lambda \in b\mathbb{L}$. Thus, we fix $\Lambda \in b\mathbb{L}$ in what follows.

The main idea of the proof is to show that the maps coeff$_D$ successively intertwine the two multiplication rules for $Z[KBN]_\Lambda$ and $Z[Q]_\Lambda$. Consequently, we compare two
intermediate multiplications steps in the following fashion:

\[
\begin{align*}
D_t &\xrightarrow{\text{mult}^\text{KBN}_{D_1,D_{t+1}}} D_{t+1} \\
\text{coeff}_{D_t} &\Downarrow \text{coeff}_{D_{t+1}}
\end{align*}
\]

with the notation as in Section 3.3. The goal is to show that each such diagram, i.e. for each \(D_t\) and \(D_{t+1}\), commutes. Since the multiplication in \(\mathfrak{A}[\text{KBN}]_\Lambda\) always trivial coefficients—up to a factor \(\alpha\)—and

\[
\text{mult}^\text{KBN}_{D_t,D_{t+1}}(\mathcal{D}_t) = \text{coeff}(\mathcal{Q}) \cdot \bar{\mathcal{D}}_{t+1} + \text{REST},
\]

where \(\text{coeff}(\mathcal{Q})\) is the coefficient coming from \(\mathfrak{A}[\mathcal{Q}]_\Lambda\), this amounts to prove

(6.1) \hspace{1cm} \text{coeff}_{D_t}(\mathcal{D}_t) \cdot \text{coeff}(\mathcal{Q}) = \text{coeff}_{D_{t+1}}(\bar{\mathcal{D}}_{t+1})

holds up to a factor \(\alpha\) which always appears on neither side or on both sides of (6.1).

To this end, we need to check the same cases (i)-(iv) as in the proof of Theorem 4.7.

In contrast to the situation of Theorem 4.7, we additionally need to distinguish the cases with different orientations of the circles in question. Next, all circles not involved in the surgery from \(D_t\) to \(D_{t+1}\) remain unchanged, and we ignore them in the following.

\begin{itemize}
  \item [\textbf{Non-nested merge.}] Assume that circles \(C_i\) and \(C_j\) are merged into a circle \(C\). In this case we have (as one easily sees)

  \begin{equation}
  \mathcal{U}(C_i) \cup \mathcal{A}(C_i) \cup \mathcal{U}(C_j) \cup \mathcal{A}(C_j) = \mathcal{U}(C) \cup \mathcal{A}(C).
  \end{equation}

  For an example see (3.13). Now let us look at possible orientations.

  \begin{itemize}
  \item [\textbf{Both, \(C_i\) and \(C_j\), are oriented anticlockwise.}] By (6.2), we directly obtain

  \begin{equation}
  \text{coeff}(C_i^{\text{anti}}) \cdot \text{coeff}(C_j^{\text{anti}}) = \text{coeff}(C^{\text{anti}}).
  \end{equation}

  Since \(\text{coeff}(\mathcal{Q}) = 1\) in this case, we see that (6.1) holds.

  \item [\textbf{One circle is oriented anticlockwise, the other clockwise.}] If \(C_i\) is oriented clockwise, then the left-hand side of (6.3) picks up the coefficient \(\varepsilon^{\text{d}_{\text{i}}(\gamma_i^{\text{anti}})} = \varepsilon^{\text{t}(C) - \text{t}(C_i)}\) from the multiplication rule for \(\text{mult}^\text{KBN}_{D_t,D_{t+1}}\). We again obtain (6.1):

  \begin{equation}
  \text{coeff}(C_i^{\text{cl}}) \cdot \text{coeff}(C_j^{\text{anti}}) \cdot \varepsilon^{\text{t}(C) - \text{t}(C_i)} = \text{coeff}(C_i^{\text{anti}}) \cdot \varepsilon^{\text{t}(C_i)} \cdot \text{coeff}(C_j^{\text{anti}}) \cdot \varepsilon^{\text{t}(C) - \text{t}(C_j)}
  \end{equation}

  \begin{equation}
  = \text{coeff}(C^{\text{anti}}) \cdot \varepsilon^{\text{t}(C)}
  \end{equation}

  The case of \(C_j\) being clockwise and \(C_i\) being anticlockwise instead is similar.

  \item [\textbf{Both, \(C_i\) and \(C_j\), are oriented clockwise.}] In this case we have

  \begin{equation}
  \text{coeff}(C_i^{\text{cl}}) \cdot \text{coeff}(C_j^{\text{cl}}) \cdot \varepsilon^{\text{t}(C) - \text{t}(C_i)} \cdot \varepsilon^{\text{t}(C) - \text{t}(C_j)}
  \end{equation}

  \begin{equation}
  \overset{(4.11)}{=} \text{coeff}(C_i^{\text{anti}}) \cdot \text{coeff}(C_j^{\text{anti}}) \cdot \varepsilon^{\text{t}(C)} \overset{(6.3)}{=} \text{coeff}(C^{\text{anti}}) \cdot \varepsilon^{\text{t}(C)}
  \end{equation}

  which again give (6.1), because the multiplication rule for \(\text{mult}^\text{KBN}_{D_t,D_{t+1}}\) picks up the coefficient \(\text{coeff}(\mathcal{Q}) = \alpha \varepsilon^{\text{d}_{\text{i}}(\gamma_i^{\text{anti}})} \varepsilon^{\text{d}_{\text{j}}(\gamma_j^{\text{anti}})}\).
  \end{itemize}
\end{itemize}
Nested merge. In this case two nested circles, \(C_{\text{out}}\) and \(C_{\text{in}}\), are merged into one circle \(C\). In the nested situation—also for the nested split below—the notion of exterior and interior swaps for the nested circle \(C_{\text{in}}\). Moreover, in case of the nested merge, the cup-cap pair involved in the surgery is of the form \(\cup \cdot \cap\) or of the form \(\cap \cdot \cup\) and hence, is “lost” after the surgery. That is, we have altogether
\[
(\cup(C_{\text{out}}) \cup \cap(C_{\text{out}}) \cup (C_{\text{in}}) \cup \cap(C_{\text{in}})) \setminus \text{surg} = \cup(C) \cup \cap(C).
\]
Here “surg” is the set containing the cup-cap of the surgery, see e.g. (3.14).

Both, \(C_{\text{out}}\) and \(C_{\text{in}}\), are oriented anticlockwise. First note that we get the coefficient
\[
\text{coeff}(Q) = \varepsilon \cdot t_{\text{ex}}(d_{\text{ex}}(C_{\text{in}})^{-2}) \cdot \varepsilon^{s_{\Lambda}(\gamma)}\text{ from } \text{mult}^2_{D_1,D_2+1}.
\]
We get (6.1):
\[
(6.4) \quad \text{coeff}(\cap_{\text{out}}) \cdot \text{coeff}(\cap_{\text{in}}) \cdot \varepsilon \cdot t_{\text{ex}}(d_{\text{ex}}(C_{\text{in}})^{-2}) \cdot \varepsilon^{s_{\Lambda}(\gamma)}
\]
\[
= \text{coeff}(\cap_{\text{out}}) \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cap(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1}
\]
\[
\quad \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cap(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \varepsilon^{s_{\Lambda}(\gamma)}
\]
\[
\overset{(1)}{=} \text{coeff}(\cup_{\text{out}}) \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cap(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1}
\]
\[
\quad \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cap(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1}
\]
\[
\overset{(1)}{=} \text{coeff}(\cup_{\text{out}}) \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cap(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1} \cdot \prod_{\gamma^* \in \cup(C_{\text{in}})} \varepsilon^{s_{\Lambda}(\gamma^*)+1}
\]
\[
\overset{(4)}{=} \text{coeff}(\cap_{\text{out}}) \cdot \text{coeff}(\cap_{\text{in}}) = \text{coeff}(\cap_{\text{out}}).
\]
Here (1) follows from Lemma 6.2 and Lemma 6.4 (since \(\varepsilon = \pm 1\)), and (II) from Lemma 6.3. Moreover, note that \(e^{p_{\Lambda}(\gamma)+s_{\Lambda}(\gamma)}\) is the inverse of the coefficient coming from the cup-cap pair in the surgery (counting them both).

\(C_{\text{out}}\) is oriented clockwise and \(C_{\text{in}}\) anticlockwise. In this case both sides are just multiplied with \(e^{t(C)} = e^{t(C_{\text{in}})}\). Hence, the calculation from (6.5) gives
\[
\text{coeff}(\cap_{\text{out}}) \cdot \text{coeff}(\cap_{\text{in}}) \cdot \varepsilon \cdot t_{\text{ex}}(d_{\text{ex}}(C_{\text{in}})^{-2}) \cdot \varepsilon^{s_{\Lambda}(\gamma)} = \text{coeff}(\cap_{\text{out}}).
\]
Thus, we again obtain (6.1), since \(\text{mult}^2_{D_1,D_2+1}\) does not give extra factors additionally to the ones from above.

\(C_{\text{in}}\) is oriented clockwise and \(C_{\text{out}}\) anticlockwise. In this case the coefficient of \(C\) is multiplied with \(e^{t(C)}\), while the one for \(C_{\text{in}}\) is multiplied with \(e^{t(C_{\text{in}})}\). But in addition the multiplication also introduces a dot moving. Hence, by (6.5),
\[
(6.6) \quad \text{coeff}(\cap_{\text{out}}) \cdot \text{coeff}(\cap_{\text{in}}) \cdot \varepsilon \cdot t_{\text{ex}}(C_{\text{in}})^{-2} \cdot \varepsilon^{s_{\Lambda}(\gamma)} = \text{coeff}(\cap_{\text{out}}),
\]
which again gives (6.1), since \(\text{mult}^2_{D_1,D_2+1}\) gives, additionally to the factors from above, the extra coefficient \(e^{d_{\text{ex}}(\gamma_{\text{out}})} = e^{t(C_{\text{in}})}\).

Both, \(C_{\text{in}}\) and \(C_{\text{out}}\), are oriented clockwise. In this case we obtain two dot moving signs, but the one for \(C_{\text{out}}\) is, as before, equal to 1. Thus, we obtain the same as in (6.6), but multiplied on both sides with \(\alpha \cdot e^{t(C)}\) which shows (6.1).

Non-nested split. In this case a circle \(C\) is split into two non-nested circles \(C_i\) and \(C_j\) (containing the vertexes at positions \(i\) or \(j\)). We clearly have
\[
(6.7) \quad \cup(C) \cup \cap(C) = \cup(C_i) \cup \cap(C_i) \cup \cup(C_j) \cup \cap(C_j) \cup \text{surg},
\]
where “surg” is as above. For an example see (3.15).

$C$ is oriented anticlockwise. By (6.7), we get

$$\text{coeff}(C_{\text{anti}}) = \text{coeff}(C_{\text{anti}}^{(1)}) \cdot \text{coeff}(C_{\text{anti}}^{(2)}) \cdot \varepsilon^{\beta_\Lambda(\gamma)} \cdot \varepsilon^{-1},$$

since, as above, $\varepsilon^{\beta_\Lambda(\gamma)} \cdot \varepsilon^{-1}$ is the coefficient coming from the cup-cap pair in the surgery (recalling that $\varepsilon = \pm 1$). Now, we have $\text{coeff}(Q) = \omega \varepsilon^{d_\Lambda(\gamma')} \varepsilon^{s_\Lambda(\gamma)}$. By Lemma 6.2 and $\varepsilon = \pm 1$ we have $\varepsilon^{d_\Lambda(\gamma')} \varepsilon^{s_\Lambda(\gamma)}$. This in turn gives

$$\text{coeff}(C_{\text{anti}}) \cdot \varepsilon \cdot \varepsilon^{d_\Lambda(\gamma')} \cdot \varepsilon^{s_\Lambda(\gamma)},$$

(6.8)

$$= \text{coeff}(C_{\text{anti}}^{(1)}) \cdot \text{coeff}(C_{\text{anti}}^{(2)}) \cdot \varepsilon \cdot \varepsilon^{t(C) - p_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{\beta_\Lambda(\gamma)} \cdot \varepsilon^{-1}$$

$$= \text{coeff}(C_{\text{anti}}^{(1)}) \cdot \varepsilon^{(C)} \cdot \text{coeff}(C_{\text{anti}}^{(2)}) \cdot \varepsilon^{\beta_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{-1}.$$

The term with $C_j$ is oriented clockwise instead is dealt with completely analogously using the fact that $\varepsilon^{p_\Lambda(\gamma)} = \varepsilon^{p_\Lambda(\gamma) + 1}$ (by definition). We obtain (6.1).

$C$ is oriented clockwise. We first compare the coefficients for the term where both, $C_i$ and $C_j$, are oriented clockwise (thus, $\text{coeff}(Q) = \omega \varepsilon^{d_\Lambda(\gamma')} \varepsilon^{d_\Lambda(\gamma') \cdot \varepsilon^{s_\Lambda(\gamma)}}$) and obtain by rewriting the dot moving signs similar as above (using $\varepsilon = \pm 1$)

$$\text{coeff}(C_{\text{anti}}) \cdot \varepsilon \cdot \varepsilon^{d_\Lambda(\gamma')} \cdot \varepsilon^{d_\Lambda(\gamma')} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{-1},$$

(6.8)

$$\text{coeff}(C_{\text{anti}}^{(1)}) \cdot \varepsilon^{(C)} \cdot \text{coeff}(C_{\text{anti}}^{(2)}) \cdot \varepsilon^{\beta_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{-1} \text{coeff}(C_{\text{anti}}^{(1)}) \cdot \text{coeff}(C_{\text{anti}}^{(2)}).$$

Hence, we have (6.1). For the term where both $C_i$ and $C_j$ are oriented anticlockwise (where we have $\text{coeff}(Q) = \alpha \varepsilon \omega \varepsilon^{d_\Lambda(\gamma')} \varepsilon^{d_\Lambda(\gamma')} \varepsilon^{s_\Lambda(\gamma)}$) we obtain

$$\text{coeff}(C_{\text{anti}}) \cdot \alpha \cdot \varepsilon \cdot \omega \cdot \varepsilon^{d_\Lambda(\gamma')} \cdot \varepsilon^{d_\Lambda(\gamma')} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{-1},$$

(6.8)

$$\text{coeff}(C_{\text{anti}}^{(1)}) \cdot \alpha \cdot \varepsilon \cdot \omega \cdot \varepsilon^{p_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \alpha \cdot \varepsilon \cdot \omega \cdot \varepsilon^{p_\Lambda(\gamma)} \cdot \varepsilon^{s_\Lambda(\gamma)} \cdot \varepsilon^{-1} \text{coeff}(C_{\text{anti}}^{(1)}) \cdot \text{coeff}(C_{\text{anti}}^{(2)}).$$

where we again use the crucial fact that $\varepsilon = \pm 1$. Thus, we obtain (6.1).

\[\triangleright\text{Nested split.}\] In this case one circle $C$ is split into two nested circles $C_\text{out}$ and $C_\text{in}$. The steps are very similar to the case of the nested merge before, with the main difference that, instead of (6.4), we have

(6.9) \[\bigcup(C) \cup C(C) = \bigcup(C_{\text{out}}) \cup C(C_{\text{out}}) \cup \bigcup(C_{\text{in}}) \cup C(C_{\text{in}}).\]

For an example see (3.16). By (6.9), we obtain (with (III) similar as in (6.5))

$$\text{coeff}(C_{\text{anti}})$$

$$= \text{coeff}(C_{\text{out}}) \cdot \prod_{\gamma' \in C(C_{\text{out}})} \varepsilon^{\sigma_\Lambda(\gamma') \cdot p_\Lambda(\gamma')} \cdot \prod_{\gamma'' \in C(C_{\text{in}})} \varepsilon^{(\sigma_\Lambda(\gamma') + 1)} p_\Lambda(\gamma')$$

$$= \text{coeff}(C_{\text{out}}) \cdot \prod_{\gamma' \in C(C_{\text{out}})} \omega^{\sigma_\Lambda(\gamma') - 1} \cdot \prod_{\gamma'' \in C(C_{\text{in}})} \omega^{\sigma_\Lambda(\gamma')}$$

(III) \[\text{coeff}(C_{\text{out}}) \cdot \text{coeff}(C_{\text{in}}) \cdot \varepsilon^{-1} \cdot \varepsilon^{d_\Lambda(\gamma)} - 2 \cdot \varepsilon^{t(C)} ,\]
\( C \) is oriented anticlockwise. We have to multiply the coefficient \( \text{coeff}(C^{\text{anti}}) \) by
\[
\text{coeff}(Q) = \omega \cdot e^{i/4(d_\Lambda(C^{\text{in}})-2)}
\]
and compare it to the coefficient of \( \text{coeff}(C^{\text{anti}}) \text{coeff}(C^{\text{cl}}) \) respectively to the coefficient of \( \text{coeff}(C^{\text{out}}) \text{coeff}(C^{\text{in}}) \). In both cases (6.1) follows then by Lemma 6.4.

\( C \) is oriented clockwise. This is done in an analogous way. Since we have \( \text{coeff}(C^{\text{cl}}) = \text{coeff}(C^{\text{anti}}) e^{i/4(C_{\text{out}})} \), this fits for both appearing terms.  

Taking everything together proves the theorem.

\[
\text{Proof.}
\]

6.2. Some rather dull lemmas needed for the proof of Proposition 4.15. We fix \( \Lambda \in \mathcal{L}_2 \).

**Lemma 6.1.** For a circle \( C \in D \) it holds that
\[
(6.10) \quad \#(\mathcal{U}(C)) + 1 = \#(\mathcal{C}(C)) \quad \text{and} \quad \#(\mathcal{C}(C)) + 1 = \#(\mathcal{U}(C)).
\]

**Proof.** This is clear for a circle containing only a single cup and cap. Any other circle can be constructed from such a small circle by successively adding “zig-zags”:
\[
\begin{align*}
\text{in} & \quad \text{ex} \quad \text{ex} \quad \text{in} \quad \text{ex} \quad \text{ex} \quad \text{in} \quad \text{ex} \quad \text{ex} \quad \text{in} \quad \text{ex} \quad \text{ex} \quad \text{in}
\end{align*}
\]
This increases both sides of the equalities from (6.10) by 0 or 1. The claim follows.  

**Lemma 6.2.** Let \( C \) be any circle in a stacked diagram \( D \).

1. If \( \gamma \in \mathcal{U}(C) \), then \( p_\Lambda(\gamma) \equiv t(C) \mod 2 \).
2. If \( \gamma \in \mathcal{C}(C) \), then \( p_\Lambda(\gamma) \equiv t(C) \mod 2 \).
3. If \( \gamma \in \mathcal{C}(C) \), then \( p_\Lambda(\gamma) \equiv t(C) + 1 \mod 2 \).
4. If \( \gamma \in \mathcal{U}(C) \), then \( p_\Lambda(\gamma) \equiv t(C) + 1 \mod 2 \).

**Proof.** All four statements are clear for a circle \( C' \) with a single cup and cap. The circle \( C \) is obtained by adding successively “zig-zags” to \( C' \). Adding such a zig-zag somewhere gives the following (we have illustrated where to read off \( p_\Lambda(\gamma) \) and \( t )
\[
\begin{align*}
(a) : p_\Lambda(\gamma) & \equiv \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{in} \quad \text{ex} \quad \text{ex} \quad \text{in} \quad \text{ex} \quad \text{ex} \quad \text{in} \\
(b) : p_\Lambda(\gamma) & \equiv \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \\
(c) & \equiv \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \\
(d) & \equiv \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex} \quad \text{ex}
\end{align*}
\]
Observe that \( t \) might not be the rightmost point \( t(C) \) on the circle \( C \). But since clearly \( t \equiv t(C) \mod 2 \), these do not change the congruences and we are done.  

**Lemma 6.3.** We have
\[
- \sum_{\gamma \in \mathcal{U}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{C}(C)} (s_\Lambda(\gamma) - 1) = 1 - \sum_{\gamma \in \mathcal{U}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{C}(C)} (s_\Lambda(\gamma) - 1)
\]
for any circle \( C \) in a stacked diagram \( D \).

**Proof.** By comparison of definitions, we get
\[
\begin{align*}
\sum_{\gamma \in \mathcal{U}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{C}(C)} (s_\Lambda(\gamma) - 1) &= 1/4 (d_\Lambda(C) - 2), \\
\sum_{\gamma \in \mathcal{C}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{U}(C)} (s_\Lambda(\gamma) - 1) &= 1/4 (d_\Lambda(C) - 2).
\end{align*}
\]
Now we apply Lemma 6.1.  

Lemma 6.4. Let $C \in D$. Then
\[
\sum_{\gamma \in \mathcal{U}(C_{in})} (s\lambda(\gamma) + 1)p\lambda(\gamma) + \sum_{\gamma \in \mathcal{A}(C_{in})} s\lambda(\gamma)(p\lambda(\gamma) + 1) + t(C) + \frac{1}{4}(d\lambda(C) - 2) = \sum_{\gamma \in \mathcal{U}(C_{in})} (s\lambda(\gamma) + 1)p\lambda(\gamma) + \sum_{\gamma \in \mathcal{A}(C_{in})} s\lambda(\gamma)(p\lambda(\gamma) + 1) \mod 2. \]
\]
Proof. Via a direct calculation: one starts with the first line and rewrites all $p\lambda(\gamma)$ in terms of $t(C)$ using Lemma 6.2. Then we use the same equalities as in the proof of Lemma 6.3. Finally one has to use Lemma 6.1 to arrive at the second line.

6.3. Proof of Proposition 4.19. We use the notation from Section 4.3.

Proof. The proof is done in complete analogy to the proof of Proposition 4.15. We show that in each step the coefficient maps defined above for stacked diagrams intertwine the multiplication steps, i.e., in each step Equality (6.1) holds true. Since the coefficient map is only modified slightly, it is clear that all arguments for the non-merged merge and non-nested split are valid in the exact same way as before. For the nested cases the swap of exterior and interior of the inner circle $C_{in}$ is more involved. We illustrate this by giving the proof for the nested merge situation.

\[\text{△ Nested merge.} \] Two nested circles $C_{out}$ and $C_{in}$ are merged into one circle $C$. As in the proof of Proposition 4.15, the notion of exterior and interior swaps for the nested circle $C_{in}$. Overall the situation is similar in the sense that the cup-cap pair involved in the surgery is of the form $\mathcal{U}(\mathcal{C})$ or of the form $\mathcal{U}(\mathcal{A})$. That is, we have

\[(6.11) \quad (\mathcal{U}(C_{out}) \cup \mathcal{A}(C_{out}) \cup \mathcal{U}(C_{in}) \cup \mathcal{A}(C_{in})) \setminus \text{surg} = \mathcal{U}(C) \cup \mathcal{A}(C), \]

where “surg" is the set containing the cup and cap of the surgery, and we have

\[(6.12) \quad \mathcal{X}(C_{out}) \cup \mathcal{X}(C_{out}) \cup \mathcal{X}(C_{in}) \cup \mathcal{X}(C_{in}) = \mathcal{X}(C) \cup \mathcal{X}(C). \]

Both, $C_{out}$ and $C_{in}$, are oriented anticlockwise. Similarly as before, we get (6.1):

\[
\begin{align*}
\text{coeff}(C_{out}) \cdot \text{coeff}(C_{in}) & = \text{coeff}(C_{out}) \cdot \prod_{\gamma' \in \mathcal{U}(C_{in})} e^{s\lambda(\gamma') + 1}p\lambda(\gamma') \cdot \prod_{\gamma' \in \mathcal{A}(C_{in})} e^{s\lambda(\gamma')(p\lambda(\gamma') + 1)} \\
& = \prod_{\gamma' \in \mathcal{X}(C_{in})} e^{p\lambda(\gamma')} \cdot \prod_{\gamma' \in \mathcal{X}(C_{in})} e^{p\lambda(\gamma')} + 1 \cdot e \cdot e^{1/4(d\lambda(C_{in}) - 2)} \cdot e^{s\lambda(\gamma)} \\
& \overset{(\text{I})}{=} \text{coeff}(C_{out}) \cdot \prod_{\gamma' \in \mathcal{U}(C_{in})} e^{s\lambda(\gamma') + 1}p\lambda(\gamma') \cdot \prod_{\gamma' \in \mathcal{A}(C_{in})} e^{s\lambda(\gamma')(p\lambda(\gamma') + 1)} \\
& \quad \cdot \prod_{\gamma' \in \mathcal{X}(C_{in})} e^{p\lambda(\gamma')} \cdot \prod_{\gamma' \in \mathcal{X}(C_{in})} e^{p\lambda(\gamma')} + 1 \cdot e \cdot e^{p\lambda(\gamma) + s\lambda(\gamma)} \\
\end{align*}
\]

\[(6.11) + (6.12) \]

\[
\begin{align*}
\text{coeff}(C_{out}) \cdot \text{coeff}(C_{in}) & = \text{coeff}(C_{out}) \cdot \prod_{\gamma' \in \mathcal{U}(C_{in})} e^{s\lambda(\gamma')} \cdot \prod_{\gamma' \in \mathcal{A}(C_{in})} e^{s\lambda(\gamma') - 1} \cdot \omega \#(\mathcal{X}(C) \cup \mathcal{X}(C)) \\
& \overset{(\text{II})}{=} \text{coeff}(C_{out}) \cdot \prod_{\gamma' \in \mathcal{U}(C_{in})} e^{s\lambda(\gamma')} \cdot \prod_{\gamma' \in \mathcal{A}(C_{in})} e^{s\lambda(\gamma') - 1} \cdot \omega \#(\mathcal{X}(C) \cup \mathcal{X}(C)) \cdot \omega \\
& \quad = \text{coeff}(C_{out}) \cdot \text{coeff}(C_{in}). 
\end{align*}
\]

Here (I) follows from Lemma 6.4 and Lemma 6.5, while (II) follow from Lemma 6.6. The arguments for the other orientations are as for Proposition 4.15.

The claim of the proposition follows than analogously to Proposition 4.15.
6.4. Some rather dull lemmas needed for the proof of Proposition 4.19. Again, fix $\Lambda \in b1^\circ$.

**Lemma 6.5.** Let $C$ be any circle in a stacked diagram $D$.

(a) If $\gamma \in \nabla(C)$, then $p_\Lambda(\gamma) \equiv t(C) + 1 \mod 2$.
(b) If $\gamma \in \omega(C)$, then $p_\Lambda(\gamma) \equiv t(C) \mod 2$.
(c) If $\gamma \in \varpi(C)$, then $p_\Lambda(\gamma) \equiv t(C) + 1 \mod 2$.
(d) If $\gamma \in \mu(C)$, then $p_\Lambda(\gamma) \equiv t(C) \mod 2$.

**Proof.** Recall that the symbol $\times$ counts as being of length 2. Hence, moving across parts in (a)-(d) preserves the parity. Thus, the claim follows as in Lemma 6.2. □

**Lemma 6.6.** Let $C$ be any circle in a stacked diagram $D$. Then

$$-\sum_{\gamma \in \cup(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in \mathcal{R}(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in \mathcal{Z}(C)} 1 + \sum_{\gamma \in \mathcal{X}(C)} 1$$

$$= -\sum_{\gamma \in \cup(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{R}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{Z}(C)} 1 + \sum_{\gamma \in \mathcal{X}(C)} 1,$$

and both equal $\frac{1}{4}(d_\Lambda(C) - 2)$. □

**Proof.** This follows immediately by interpreting $\frac{1}{4}(d_\Lambda(C) - 2)$ as the number of internal phantom edges of the circle as done in [17, Lemma 4.10]. □

6.5. Two other dull, yet important, lemmas.

**Lemma 6.7.** For a stacked diagram $D^{or}$ and a circle $C$ in it, we have

$$\text{coeff}_\varepsilon(C, \gamma) = \text{coeff}_\varepsilon(C, D^{or}) \cdot \varepsilon(C),$$

$$\text{coeff}_\omega(C, D^{or}) = \text{coeff}_\omega(C, D^{or}) \cdot \chi(C),$$

$$\chi(C) = \prod_{\gamma \in \cup(C) \cup \mathcal{R}(C)} e^{p_\Lambda(\gamma) + s_\Lambda(\gamma)},$$

$$= \prod_{\gamma \in \mathcal{R}(C) \cup \mathcal{Z}(C)} e^{p_\Lambda(\gamma) + s_\Lambda(\gamma)} \cdot e^{\#(\mathcal{X}(C) \cup \mathcal{Z}(C) \cup \mathcal{Z}(C) \cup \mathcal{Z}(C))},$$

$$\chi(C) = \omega^\#(\mathcal{R}(C) \cup \mathcal{Z}(C)) = \omega^\#(\mathcal{R}(C) \cup \mathcal{Z}(C)).$$

**Proof.** For $\text{coeff}_\varepsilon(C, D^{or})$, after rewriting all positions with respect to the rightmost point by using Lemma 6.2 and Lemma 6.5, the left side, by Lemma 6.6, is (all congruences below are modulo 2)

$$\sum_{\gamma \in \cup(C)} s_\Lambda(\gamma) t(C) + \sum_{\gamma \in \mathcal{R}(C)} (s_\Lambda(\gamma) - 1)(t(C) + 1)$$

$$+ \sum_{\gamma \in \mathcal{Z}(C)} (t(C) + 1) + \sum_{\gamma \in \mathcal{X}(C)} (t(C) + 1)$$

$$= \left(\sum_{\gamma \in \cup(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in \mathcal{R}(C)} (s_\Lambda(\gamma) - 1) + \#(\mathcal{X}(C) \cup \mathcal{Z}(C))\right) t(C)$$

$$+ \#(\mathcal{R}(C)) t(C) + \sum_{\gamma \in \mathcal{Z}(C)} (s_\Lambda(\gamma) - 1) + \#(\mathcal{X}(C) \cup \mathcal{Z}(C))$$

$$= \left(\sum_{\gamma \in \cup(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{R}(C)} s_\Lambda(\gamma) + \#(\mathcal{X}(C) \cup \mathcal{Z}(C))\right) t(C)$$

$$+ \#(\mathcal{R}(C)) t(C) + \sum_{\gamma \in \mathcal{Z}(C)} (s_\Lambda(\gamma) - 1) + \#(\mathcal{X}(C) \cup \mathcal{Z}(C)).$$

Collecting all terms that belong to $\text{coeff}_\varepsilon(C, D^{or})$ we are left with

$$\#(\mathcal{X}(C) \cup \mathcal{Z}(C) \cup \mathcal{Z}(C) \cup \mathcal{Z}(C))$$

$$+ \#(\mathcal{R}(C)) t(C) + \sum_{\gamma \in \mathcal{R}(C)} s_\Lambda(\gamma) + \#(\mathcal{R}(C)) t(C) + \sum_{\gamma \in \mathcal{Z}(C)} (s_\Lambda(\gamma) - 1)$$

$$\equiv \sum_{\gamma \in \mathcal{R}(C)} (p_\Lambda(\gamma) + s_\Lambda(\gamma)) + \sum_{\gamma \in \mathcal{Z}(C)} (p_\Lambda(\gamma) + s_\Lambda(\gamma)).$$
That this can be rewritten with respect to cups instead is just an application of Lemma 6.6 to both sums in the first line above.

Next, the (easier) $\omega$-term: $\text{coeff}_\omega(C, D^{\text{or}})$ can be rewritten as

$$-\sum_{\gamma \in \mathcal{U}(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in \mathcal{F}(C)} s_\Lambda(\gamma) + \#(\mathcal{X}(C) \cup \mathcal{Y}(C))$$

Let us write

$$6.6. \quad \text{Lemma 6.6}$$

$$\equiv -\sum_{\gamma \in \mathcal{U}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \mathcal{F}(C)} (s_\Lambda(\gamma) - 1) + \#(\mathcal{X}(C) \cup \mathcal{Y}(C))$$

$$+ \#(\mathcal{A}(C) \cup \mathcal{A}(C)).$$

The first three summands are the powers in $\text{coeff}_\omega(C, D^{\text{or}})$, while the last term is the power in $\chi_\omega(C)$. That $\chi_\omega(C)$ can be written in the two ways is evident. 

6.6. Proof of Proposition 4.34. We will use the notation from Section 4.5.

Proof. Let us write $\mathcal{W}[s]_a$ respectively $\mathcal{W}[s]_b$ etc., if we are in the generic situation or in case (gen) respectively in case (sesi). We also write $\mathcal{W}[s]_*$ etc. if we mean both cases (for simplicity of notation we extend scalars to $Q$).

By Proposition 2.45, it suffices to show, that $\text{Hom}_{\mathcal{W}[s]_a}(\mathcal{W}[s]_b(u), \mathcal{W}[s]_d(v))$ is a free $Q$-module of finite rank, and then calculate its rank. The first task follows directly from Corollary 2.41. The main difficulty is to “control” the number of $\mathcal{W}[s]_*$-bimodule homomorphisms. We do so by analyzing the decomposition structure.

To this end, recall from Section 4.1 that, given a web $u$, then we can associate to it a $K$-composite matching $a(u)$ by erasing orientations and phantom edges. Here choose any presentation of the associated $K$-composite matching $a(u)$ in terms of the basic moves from (3.17). From this we obtain an $\mathfrak{A}[s]$-bimodule $\mathcal{A}(s)(a(u))$ associated to $\mathcal{W}[s]_b(u)$. (The careful reader might want to check that different choices in terms of basic moves give isomorphic $\mathfrak{A}[s]$-bimodules.)

Case (gen). The main ingredient in order to control the number of $\mathcal{W}[s]_b$-bimodule homomorphisms is to first use the results from Section 4.1 to identify $\mathcal{W}[s]_b$ and its web bimodules with $\mathfrak{A}[a]$ and its arc bimodules. Then further identify $\mathfrak{A}[a]$ with $\mathfrak{A}[\mathcal{KBN}]$ and their arc bimodules by using the results from Section 4.2 and Section 4.3. Hence, we can use statements obtained in [6] and [7] as we explain below. Hereby we note that these results work, mutatis mutandis, in the generic case as well.

Next, we have $2\text{Hom}_{\mathfrak{A}[a]}(\mathcal{W}[s](u, v) \cong 2\text{Hom}_{\mathfrak{A}[a]}(\mathcal{W}[s](u, v) \cup \mathcal{W}[s](v) \cup \mathcal{W}[s](u) \cup \mathcal{W}[s](v))$ as graded, free $Q$-modules, cf. the proof of Proposition 2.45. Thus, using the cup foam basis and the translation to the side of $\mathfrak{A}[a]$ from Lemma 4.5, the (graded) rank of $2\text{Hom}_{\mathfrak{A}[a]}(\mathcal{W}[s](u, v))$ is precisely given by all orientations of the composite matching for $a(\text{clap}(u)\text{clap}(v)\text{clap}(v)^*\text{clap}(u)^*)$ and their degrees. Thus, we have to show the same on the side of $\mathfrak{A}[\mathcal{KBN}]$:

$$\text{rank}_Q(\text{Hom}_{\mathfrak{A}[\mathcal{KBN}]}(\mathcal{A}[\mathcal{KBN}](a(u), \mathcal{A}[\mathcal{KBN}](a(v)\{s\}))) \nabla \text{need to show # orientations of } a(\text{clap}(u)\text{clap}(v)^*) \text{ of degree } s.$$ 

Assume first that neither $u$ nor $v$ have internal circles. Then $a(u)$ and $a(v)$ fit into the framework from [6, Section 4], i.e. [6, Theorems 3.6 and 4.14] show that $\mathcal{A}[\mathcal{KBN}](a(u))$ is indecomposable iff $a(u)$ does not contain internal circles.

It follows now from [7, Theorem 3.5] that (6.13) holds in case $\alpha = 0$ and $R = \mathbb{C}$. Scrutiny of the arguments used in [6] and [7] shows that these work under the circumstances of case (a) as well. Next, if $C$ is any circle in $u$, and thus, in $a(u)$, then

$$\mathcal{A}[\mathcal{KBN}](a(u)) \cong \mathcal{A}[\mathcal{KBN}](a(u)-C)\{+1\} \oplus \mathcal{A}[\mathcal{KBN}](a(u)-C)\{-1\},$$
which follows as in [17, Example 3.22]. (Similarly for v.) By (3.3), the right-hand side of (6.13) behaves in the same way, i.e. for \( w = a(\text{clap}(u-C)\text{clap}(v)^*) \) we have

\[
\begin{align*}
\# \{ \text{orientations of } a(\text{clap}(u)\text{clap}(v)^*) \text{ of degree } s \} \\
= \# \{ \text{orientations of } w \text{ of degree } s+1 \} + \# \{ \text{orientations of } w \text{ of degree } s-1 \}
\end{align*}
\]

(Similarly for v.) The claim follows in case (gen).

\( \blacktriangleright \) Case (sesi). As before, it suffices to study the case where \( u \) and \( v \) do not have internal circles. In this case \( 2\text{End}_\mathcal{G}(u) \) has a basis which locally looks like

\[
\begin{array}{ccc}
\bl & * & \br
\end{array}
\]

This can be shown by using the cup foam basis. Now, because of (2.8), (2.9) and (2.10), we do not need to worry about the phantom parts of any foam \( f \in 2\text{End}_\mathcal{G}(u) \), and we ignore these in what follows. A direct calculation shows that

\[
e_+ = \frac{1}{2} \left( \begin{array}{c}
\bl \\
\br
\end{array} + \sqrt{q} \alpha^{-1} \cdot \begin{array}{c}
* \\
\cdot
\end{array} \right) \quad \text{and} \quad e_- = \frac{1}{2} \left( \begin{array}{c}
\bl \\
\br
\end{array} - \sqrt{q} \alpha^{-1} \cdot \begin{array}{c}
* \\
\cdot
\end{array} \right)
\]

are idempotents satisfying \( e_+e_- = 0 = e_-e_+ \) and \( 1 = e_+ + e_- \). If \( u \) has \( c \) connected components—ignoring phantom edges, but counting both adjacent usual edges—then

\[
E = \{ \vec{e} = (e_1, \ldots, e_{2^c}) \mid e_i = e_{\pm i}, i = 1, \ldots, 2^c \} \setminus \{0\}
\]

(some \( \vec{e} \)'s might be zero, see below) gives a complete set of pairwise orthogonal idempotents in \( 2\text{End}_\mathcal{G}(u) \). Here the idempotents \( \vec{e} \) are obtained by separating the idempotents \( e_+ \) and \( e_- \) locally around a trivalent vertex as follows:

\[
\begin{aligned}
e = 1: \quad & \bl \bl \bl \quad \text{and} \quad \bl \bl \bl \\
e = -1: \quad & \bl \bl \bl \quad \text{and} \quad \bl \bl \bl
\end{aligned}
\]

These are idempotents as one easily checks, while the other possible combinations give zero. This shows that, with \( w = \text{clap}(u)\text{clap}(v)^* \) being the clapped web,

\[
(6.14) \quad \text{rank}_Q(2\text{Hom}_\mathcal{G}(u, v)) = \# \{ \text{non-zero “colorings” of } w \text{ with the } e_{\pm i} \text{’s} \}.
\]

Using \( E \): As in [32, Proposition 3.13], one can show that \( \mathfrak{M}[b]|_k \) is semisimple for all \( k \in \mathfrak{B}^2 \). A web bimodule \( W[b](u) \) for \( u \) having \( c \) connected components decomposes into pairwise non-isomorphic copies of \( Q \), i.e. \( W[b](u) \cong \bigoplus_{\vec{e} \in E} \mathcal{C}W[b](u)\vec{e} \). The claim follows, since—for \( w = \text{clap}(u)\text{clap}(v)^* \)—we get that the \( Q \)-rank of \( \text{Hom}_{\mathfrak{M}[b]}(W[b](u), W[b](v)) \) equals the number of non-zero “colorings” of \( w \) with the \( e_{\pm i} \)'s. By (6.14), \( \text{rank}_Q(2\text{Hom}_\mathcal{G}(u, v)) = \text{rank}_Q(\text{Hom}_{\mathfrak{M}[b]}(W[b](u), W[b](v))) \).

Altogether, this shows the claim.

\[
\begin{array}{|c|}
\hline
\text{KBN} & \text{KBN specializations} \\
\hline
\text{Ca} & \text{Ca specializations} \\
\hline
\text{CMW} & \text{CMW specializations} \\
\hline
\text{BI} & \text{BI specializations} \\
\hline
P & \text{The set } \{ \alpha, \tau_{\pm 1}, \tau_{\pm 1}^\pm, \omega_{\pm 1}, \omega_{\pm 1}^\pm \} \\
Q & \text{The set } \{ \alpha, \varepsilon_{\pm 1}, \varepsilon_{\pm 1}^\pm \} \\
\alpha & \text{The “two-dots” parameter} \\
\varepsilon & \text{The “sA vs. gI” parameter} \\
\omega & \text{The gluing parameter} \\
\bullet & \text{The ring } \mathbb{Z}[\alpha, \tau_{\pm 1}, \tau_{\pm 1}^\pm, \omega_{\pm 1}, \omega_{\pm 1}^\pm] \\
Q & \text{The ring } \mathbb{Z}[\alpha, \omega_{\pm 1}] \\
p, q & \text{Specialization maps} \\
\mathcal{P} & \text{The foam 2-category over } P \\
\mathfrak{M}[b] & \text{Various cup foam bases} \\
\mathcal{P}[+] & \text{Various specializations of } \mathfrak{P} \\
\mathcal{B} & \text{The web algebra over } P \\
\mathfrak{M}[b][+] & \text{Various specializations of } \mathfrak{B} \\
\mathfrak{P} & \text{The foam 2-category over } Q \\
\hline
\end{array}
\]
\[ \mathfrak{w}[Q] \] The web algebra over \( Q \)
\[ \mathcal{W}(u) \] The web bimodules
\[ \mathfrak{w}[\mathfrak{A}] \] -biMod\(_P^Q\) Web bimodule 2-category
\( \oplus(\cdot) \) Additive closure
\( p_\Lambda(\cdot) \) The position on arc diagrams
\( s_\Lambda(\gamma) \) The saddle width
d\( (\cdot) \) The distance of e.g. an arc
\( \mathfrak{B}^\circ(\cdot) \) Various arc diagram bases
\( \mathfrak{A}[\cdot] \) Various specializations of \( \mathfrak{A}[Q] \)
\( e^{A\Lambda} s^{(A\Lambda)^{\text{dot}}} \) The dot moving sign
\( e^{A\Lambda} s^{(A\Lambda)^{\text{dot}}} \) The topological sign
\( t(C) \) A rightmost point on a circle
\( \Lambda(A,t) \) The arc bimodules
\( \mathfrak{A}[\cdot] \) -biMod\(_P^Q\) Arc bimodule 2-category
\( \Phi \) etc. Various \( \mathfrak{w}[\cdot] \xrightarrow{\sim} \mathfrak{A}[\cdot] \)
\( R(\cdot) \) A ring with specialized \( \varepsilon \) and \( \omega \)
\( \Psi \) etc. Various \( \mathfrak{A}[Q] \xrightarrow{\sim} \mathfrak{A}[\cdot] \)
\( \mathfrak{w}(\cdot) \) An associated web
\( \mathfrak{w}[\cdot] \) Based versions
\( \mathfrak{w}(C) \) Cups “pushing inwards”
\( \mathfrak{w}(C) \) Caps “pushing inwards”
\( \mathfrak{w}(C) \) Caps “pushing outwards”
\( \mathfrak{w}(C) \) Caps “pushing outwards”
coeff (\cdot) Various coefficient maps
\#(\cdot) Various number of elements
\( \mathfrak{X}(C) \) Left shift of \( x \) - left exterior
\( \mathfrak{X}(C) \) Right shift of \( x \) - right exterior
\( \mathfrak{X}(C) \) Left shift of \( x \) - left interior
\( \mathfrak{X}(C) \) Right shift of \( x \) - right interior
coeff (\cdot) Various reverse coefficient maps
\( \mathfrak{C}(\cdot) \) Cover(s) of arc algebra(s)
\( \mathcal{O}^{p,q} \) Two block parabolic category \( \mathcal{O} \)
[\( \cdot \) ] Various higher tangle invariants
1-CC, 1-HCC Chain complex categories
2-CC, 2-HCC Their 2-versions
1-Tan, 2-Tan Tangle (2-)categories

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