EXACT SOLUTION OF D=1 KAZAKOV-MIGDAL INDUCED GAUGE THEORY

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Abstract

We give the exact solution of the Kazakov-Migdal induced gauge model in the case of a D=1 compactified lattice with a generic number S of sites and for any value of N. Due to the peculiar features of the model, the partition function that we obtain also describes the vortex-free sector of the D=1 compactified bosonic string, and it coincides in the continuum limit with the one obtained by Boulatov and Kazakov in this context.

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1. Introduction

Recently, a new interesting approach to gauge theories has been proposed [1] and solved in the large N limit [1, 2] by Kazakov and Migdal. The hope is to be able to describe within this approach, the asymptotically free fixed point of QCD in 4 dimensions. The Kazakov-Migdal model seem to be a promising tool in this direction, but several question must be understood in order to reach this goal. Let us mention two of them: first, one should make sure that there is no phase transition at any finite value of $N$; for that reason it would be important to have some example where the exact explicit dependence on $N$ is known. Second, one would like to understand the nature of the transition point which appears in the model at a critical value $m_c$ of the mass parameter. Indeed, looking at the induced gauge theory, due to the fact that matter fields are in the adjoint representation of the $U(N)$ group (see below), one finds a super-confining behaviour [3], and the hope is to reach an ordinary confining phase through the above mentioned phase transition. Hence it would be interesting to study models which are simple enough to be exactly solvable, but still having all the desired non trivial properties.

In this letter we will give the exact solution for any value of $N$ in the case of a $d=1$ compactified lattice made of $S$ matter fields, for any value of $S$. This model fulfills the above requirements: it has a non trivial phase transition, and the solution can be obtained by using simple combinatoric properties of the permutation group, the key trick being the reduction of the permutation group to its cyclic representations. Moreover, as one would expect from the definition of the model (see below), our solution also describes the vortex-free sector of the $d=1$ compactified bosonic string [4]. In this context it is rather interesting to notice that the analytic continuation in the mass parameter from the strong to the weak coupling phase provides the correct prescription to obtain the physical properties of the upside-down oscillators from the standard matrix oscillators [5].

This letter is organized as follows: after a brief introduction on the Kazakov-Migdal model (sect.2), we give in sect.3 the exact solution of the model. The solution is discussed in sect. 4 which includes also some concluding remark.

2. The Kazakov-Migdal model

The starting point of Kazakov-Migdal suggestion is to induce the Yang-Mills interaction using massive scalar fields in the adjoint representation of $U(N)$. The action they propose is defined on a generic $d$-dimensional lattice and has the following form:

$$S = \sum_x N \text{Tr}[m^2 \phi^2(x) - \sum_\mu \phi(x)U(x, x + \mu)\phi(x + \mu)U^\dagger(x, x + \mu)]$$ (1)

where $\phi(x)$ is an Hermitian $N \times N$ matrix defined on the sites $x$ of the lattice and $U(x, x + \mu)$ is a Unitary $N \times N$ matrix, defined on the links $(x, x + \mu)$, and plays
the role, as in the usual lattice discretization of Yang-Mills theories, of the gauge field. Integrating over the scalar field $\phi$ one can induce an effective action for the gauge field,

$$\int DUD\Phi exp(-S) \sim \int DU exp(-S_{ind}[U])$$

with:

$$S_{ind}[U] = - \frac{1}{2} \sum_{[\Gamma]} \frac{|\text{Tr} U[Y]|^2}{l[\Gamma]m^2[\Gamma]} ,$$

where $l[\Gamma]$ is the length of the loop $\Gamma$, $U[\Gamma]$ is the ordered product of link matrices along $\Gamma$ and the summation is over all closed loops.

In a similar way, integrating over the gauge fields one can induce an effective interaction for the scalar fields, which turns out to be deeply related to the matrix approach to 2D Quantum Gravity and noncritical strings.

Integration can be achieved by using the well known formula first discovered by Harish-Chandra [7], rediscovered in the context of matrix models by Itzykson and Zuber [6] and fully exploited by Mehta [8]:

$$I(\phi(x),\phi(y)) = \int DU \exp\left( N\text{tr}\phi(x)U\phi(y)U^\dagger \right) \propto \det_{ij} \exp(N\lambda_i(x)\lambda_j(y)) \Delta(\lambda(x))\Delta(\lambda(y))$$

where $\lambda_i(x)$ are the eigenvalues of the matrix $\phi(x)$

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$$

is the Vandermonde determinant, and $(x,y)$ are nearest neighbour links of the lattice.

The main difference between this approach and the usual one lies in the fact that now, since the angular variables $U(x, x+\mu)$ are themselves degrees of freedom and their self-interaction term is explicitly absent, we can integrate them out safely, while the same is not possible in the usual description of 2d quantum gravity coupled with matter defined on lattices with closed loops. It is exactly this kind of obstruction which doesn’t allow, in the context of the matrix approach, a description of $d > 1$ bosonic strings in terms of the eigenvalues only, and which manifests itself in the case of the $d=1$ compactified bosonic string as a vortex contribution. This means that in this last example the Kazakov-Migdal model should exactly correspond to the singlet, vortex free, solution of the compactified bosonic string.

A further important feature of the Kazakov-Migdal model is the presence of a phase transition, which should occur at a finite, non-zero value $m_c$ of the mass parameter, between a strong coupling regime ($m > m_c$) and a weak coupling phase ($m < m_c$). This transition was discussed in [1, 2] in the context of the induced matrix model (after integration on the gauge fields) and was conjectured in [3] to be related (in the context of the induced gauge theory, after integration on the matter fields) to the change from ordinary to local confinement. We will show below
that this same transition in the $d=1$ case separates the upside-down oscillator phase from the standard matrix oscillator description of the $d=1$ bosonic string.

### 3. Exact solution for a $d=1$ compactified lattice

The partition function of the Kazakov-Migdal model defined on a 1d lattice with $S$ sites (labelled by $\alpha$) compactified on a circle is:

$$ Z = \sum_{\alpha=1}^{S} \int d\phi(\alpha) dU(\alpha, \alpha + 1) e^{-N \text{Tr}[m^2\phi(\alpha)^2 - \phi(\alpha)U(\alpha, \alpha + 1)\phi(\alpha + 1)U(\alpha, \alpha + 1)^\dagger]} . \quad (6) $$

The model (6) is reduced, by integrating over the unitary matrices on each link, to

$$ Z = \int \prod_{\alpha, i} d\lambda_i(\alpha) e^{-m^2N \sum_{\alpha, i} \lambda_i(\alpha)^2} \frac{1}{N^S N(N-1)/2} \prod_{\alpha} \det \left( e^{N\lambda_i(\alpha)\lambda_j(\alpha + 1)} \right) \quad (7) $$

where $\lambda_i^{(\alpha)}$ is the $i^{th}$ eigenvalue of the matrix $\phi(\alpha)$.

It is easy to see that this expression can be rewritten as follows:

$$ Z = \sum_{P_\alpha} (-1)^{P_1 + \cdots + P_S} \int \prod_{\alpha, i} d\lambda_i(\alpha) e^{-\frac{N}{2} \sum_{i=1}^{N} \sum_{\alpha=1}^{S} [a\lambda_i^{(\alpha)} - b(P_\alpha \lambda^{(\alpha + 1)})_i]^2} \frac{1}{N^S N(N-1)/2} \quad (8) $$

where the $P_\alpha$’s are $S$ independent permutations of $N$ objects and we have defined $2m^2 = a^2 + b^2$ with $ab = 1$. Here $a$ and $b$ are restricted to be real, hence $m > 1$. The region with $m < 1$, where the integral (7) is not defined, can be reached by analytic continuation as described later. Let us introduce the new variables of integration

$$ \hat{\lambda}_i^{(1)} = \lambda_i^{(1)} $$
$$ \hat{\lambda}_i^{(2)} = (P_1 \lambda^{(2)})_i $$
$$ \hat{\lambda}_i^{(3)} = (P_1 P_2 \lambda^{(3)})_i $$
$$ \vdots $$
$$ \hat{\lambda}_i^{(S)} = (P_1 P_2 \cdots P_{S-1} \lambda^{(S)})_i $$

and define $P = P_1 \cdots P_S$

Then we have
Z = \frac{1}{N^{SN(N-1)/2}}(N!)^{S-1}\sum_{P}(-1)^{P}\int \prod_{\alpha,i} d\hat{\lambda}_{i}^{(\alpha)} \times \\
\exp -\frac{N}{2} \sum_{i=1}^{N}\{[a\hat{\lambda}_{i}^{(1)} - b\hat{\lambda}_{i}^{(2)}]^2 + [a\hat{\lambda}_{i}^{(2)} - b\hat{\lambda}_{i}^{(3)}]^2 + \cdots + [a\hat{\lambda}_{i}^{(s)} - b(P\hat{\lambda}_{i}^{(1)})]^2\} \\
(9)

It is natural now to change variables in the integral and define

\zeta^{(\alpha)} = a\hat{\lambda}^{(\alpha)} - b\hat{\lambda}^{(\alpha+1)} \\
(10)

where \hat{\lambda}^{(S+1)} = (P\hat{\lambda}_{i}^{(1)})_{i}. Now we have to calculate the Jacobian. This is a simple task if we take a cyclic permutation, say of order r, namely the permutation: 1 → 2 → 3 → \cdots → r → 1 and if we order the variables in the following way:

\zeta_{1}^{(1)}, \zeta_{1}^{(2)}, \cdots, \zeta_{1}^{(s)}, \zeta_{2}^{(1)}, \zeta_{2}^{(2)}, \cdots, \zeta_{2}^{(s)}, \cdots, \zeta_{r}^{(1)}, \zeta_{r}^{(2)}, \cdots, \zeta_{r}^{(s)},

then the Jacobian is given by:

\left| \frac{\partial \zeta}{\partial \lambda} \right| = 
\begin{vmatrix} 
 a & -b \\
 a & -b \\
 a & -b \\
 \end{vmatrix} = a^{rS} - b^{rS}, \\
(11)

where we have chosen a > b due to the absolute value in the Jacobian. Hence:

\begin{align*}
a^2 &= m^2 + \sqrt{m^4 - 1} \\
b^2 &= m^2 - \sqrt{m^4 - 1}
\end{align*}

As each permutation P can be decomposed into products of cycles, each integral at the r.h.s. of (11) decomposes into the product of integrals corresponding to the cycles in the decomposition of P.

Any permutation is characterized by a set of numbers \{r_{1}, \cdots, r_{j}, \cdots, r_{N}\}, \ r_{j} being the number of times that the cycle of order j appears in the decomposition of P. Obviously one has the condition

\sum_{j=1}^{N} j r_{j} = N \\
(12)

Among the N! permutations there are N! \prod_{j=1}^{N} \left(\frac{1}{r_{j}}\right)^{r_{j}!} which are characterized by the same set of numbers \{r_{i}\}.
By putting everything together, we find for the partition function the following expression:

\[
Z = \frac{(N!)^S}{N^{SN(N-1)/2}} \sum_{r_1, \ldots, r_N} \delta(\sum j r_j - N)(-1)^{(j-1)r_j} \prod_{j=1}^{N} \left( \frac{F_{j,S,N}}{r_j!} \right) \frac{1}{r_j!}
\]

\[
= \frac{(N!)^S}{N^{SN(N-1)/2}} \int_0^{2\pi} d\theta \frac{2\pi}{2\pi} \exp -iN\theta - \sum_{r=1}^{\infty} \frac{(-1)^{(r)}}{r} F_{r,S,N} \exp i\theta r
\]

where \( F_{r,S,N} \) is the integral corresponding to a cycle of order \( r \):

\[
F_{r,S,N} = \int \prod_{i=1}^{r} d\lambda_i^{(a)} \sum_{\alpha=1}^{S} \left( a\lambda_i^{(a)} - b\lambda_i^{(a+1)} \right)^2
\]

(14)

where as before \( \lambda_i^{(S+1)} = \lambda_i^{(1)} \).

But the r.h.s. of eq.(14) is a gaussian integral in the \( \zeta_i^{(a)} \) variables, hence:

\[
F_{r,S,N} = \frac{1}{a^{rS} - b^{rS}} \int \prod_{i=1}^{r} d\zeta_i^{(a)} \sum_{\alpha=1}^{S} \zeta_i^{(a)^2} = \left( \frac{\pi}{N} \right)^{rS} \frac{1}{a^{rS} - b^{rS}}
\]

(15)

By inserting (15) into the expression (13) for \( Z \) we obtain

\[
Z = \frac{(N!)^S}{N^{SN(N-1)/2}} \int_0^{2\pi} d\theta \frac{2\pi}{2\pi} \exp -iN\theta \prod_{K=0}^{\infty} \left( 1 + e^{i\theta} \left( \frac{\pi}{Na^2} \right)^{S/2} a^{-2SK} \right)
\]

(16)

In this infinite product we recognize the grand-canonical partition function of a set of fermions where \( A \equiv e^{i\theta} \left( \frac{\pi}{Na^2} \right)^{S/2} \) plays the role of the fugacity. Then the only effect of the integral over \( \theta \) is to select in the product \( \prod_{K=0}^{\infty}(1 + AX^{SK}) \) the coefficient of the \( A^N \) term. Such coefficient is

\[
\sum_{K_1<K_2<\ldots<K_N} X^{SK_1} X^{SK_2} \ldots X^{SK_N} = \frac{1}{X^{NS}} \prod_{K=1}^{N} \frac{X^{SK}}{1 - X^{SK}}
\]

(17)

So we find the final expression for the partition function

\[
Z = \frac{1}{2\pi} \frac{(N!)^S}{N^{SN(N-1)/2}} \left( \frac{\pi}{N} \right)^{NS/2} \prod_{K=1}^{N} \frac{(a^{-2S})^{N^2/2}}{1 - (a^{-2S})^K}
\]

(18)
4. Discussion of the results and concluding remarks

Apart from some irrelevant factors the partition function \( Z^{(N)}(q) \) is of the form
\[
Z^{(N)}(q) = \frac{q^{N^2/2}}{(1-q)(1-q^2) \cdots (1-q^N)}
\]  
with \( q = a^{-2S} \).

The same partition function has been obtained in a completely different fashion by Boulatov and Kazakov in ref. [5] as the one describing the singlet (vortex free) part of the partition function for a 1d string. This is not surprising as the singlet is obtained by integrating over the residual angular variables which play therefore the role of gauge variables. The crucial difference is that the theory considered in [5] has a continuous compactified target space. As a consequence the argument \( q = a^{-2S} \) in (19) is identified in [5] with \( q = e^{-\beta \omega} \) where \( \beta \) is the length of the string and \( \omega \) is the frequency of the oscillators. A rescaling of time changes both \( \beta \) and \( \omega \) in such a way to keep this product constant. The most convenient way to formulate such scaling in our case is to define
\[
a^2 = e^\varphi
\]
so that
\[
m^2 = \cosh \varphi
\]
The partition function (18) is then left invariant by the following rescaling:
\[
S \rightarrow S', \quad \varphi \rightarrow \frac{S}{S'} \varphi
\]
which for the mass of the scalar implies
\[
m^2 \equiv \cosh \varphi \rightarrow m'^2 \equiv \cosh \left( \frac{S}{S'} \varphi \right)
\]
With an infinite rescaling \( (S' \rightarrow \infty) \) one should recover the continuum theory of ref. [5]. In fact it can be easily checked that the action in (5) becomes in such limit:
\[
S_{\text{cont}} = N \text{Tr} \int_0^\beta dt \left[ \frac{1}{2} (D\hat{\varphi})^2 + \frac{1}{2} \omega^2 \hat{\varphi}^2 \right]
\]
where \( t = \frac{\alpha}{S'} \beta \), \( \hat{\varphi} = \sqrt{\frac{\beta}{S'}} \varphi \) and
\[
\beta^2 \omega^2 = 2S'^2 (m'^2 - 1) = 2S'^2 [\cosh \left( \frac{S}{S'} \varphi \right) - 1] = \varphi^2 + O \left( \frac{1}{S'^2} \right)
\]
It is apparent from (25) that the continuum limit always correspond to the critical point \( m^2 = 1 \), although it leads to two completely different phases according
to whether the point \( m^2 = 1 \) is reached from above or from below, the former corresponding to a real and the latter to an imaginary frequency \( \omega \).

The fact that, except for a rescaling of \( a \), the same partition function is obtained irrespective of the value of \( S \) is quite remarkable and tells us that the whole information about the partition function is already contained in the simplest case \( S = 1 \) where the lattice is reduced to just one site and one link. Such drastic reducibility in the number of degrees of freedom seems to denote that all relevant quantities, such as for instance the vacuum density of the eigenvalues of the scalar field, are space independent.

The calculation leading to eq. (18) has been done in the regime \( m > 1 \). In such regime the quadratic potential is stable, \( a \) and \( b \) are real and all integrals are well defined. In the weak coupling regime \( m < 1 \) the quadratic potential is unstable and such instability manifests itself in the divergence of the integrals over the eigenvalues. It is easy to see for instance that for \( a \) and \( b \) on the unit circle the integral in (14) is divergent.

The natural way out is to define the partition function for \( m < 1 \) as the analytic continuation from \( m > 1 \). In terms of the variable \( q = a^{-2S} \) it means an analytic continuation from the real axis with \( q < 1 \) to the unit circle.

As discussed above, in the continuum theory we have \( m \to m_c = 1 \) and two different phases originate corresponding to real (resp. imaginary) frequency oscillators if \( m_c \) is approached from above (resp. below). It was argued in ref. [5] that the physical properties of the oscillators with imaginary frequencies - the so called upside-down oscillators - can be obtained from the ones of ordinary oscillators by the replacement \( \omega \to i\omega \). It was shown this to be consistent with the introduction of an SU(N) invariant cutoff at large \( \lambda \)’s.

In the discrete theory on the other hand it is possible to analytically continue from the strong to the weak coupling regime and then perform the continuum limit. In this way one recovers the correct prescription for the upside-down oscillators, namely that their properties are obtained via the substitution \( \omega \to i\omega \).

Finally we want to remark that it is very simple in this theory to integrate over the matrix fields \( \phi \) and obtain the partition function as a function of the gauge fields only. By taking advantage of the fact that the partition function is independent from \( S \) we can do the calculation for \( S = 1 \). Gauge invariance can be used to choose the unitary matrix \( U \) to be diagonal and the integral over each matrix element of the matrix field \( \phi \) is gaussian. The result is then:

\[
Z(\beta) = \int_0^{2\pi} \prod_{k=1}^{N} \frac{d\theta_k}{2\pi} |e^{i\theta}|^2 \left[ \frac{1}{m^2 - 1} \right]^{N/2} \prod_{i<j} \frac{1}{2[m^2 - \cos(\theta_i - \theta_j)]} \prod_{k,m=1}^{N} \left[ \frac{q^{1/2}}{1 - qe^{i(\theta_k - \theta_m)}} \right]^{q^{1/2}}
\]

where, for \( S = 1 \), \( q = 1/a^2 \) and the \( \theta \)’s are the invariant angles of the \( U(N) \) matrix. In order to obtain the expression for arbitrary \( S \) it is sufficient to replace \( q \) with \( a^{-2S} \).
and interpret the $\theta_i$’s as the invariant angles of the product of the unitary matrices over the plaquette. It should be noticed that eq. (26) is much more suitable than eq. (3) to study the weak and strong coupling behaviour of the induced gauge theory since it can be analytically continued from the strong to the weak coupling regime.

Eq. (26) was also derived, in the continuum limit, in ref. \cite{5} (see eq. (4.34)) by performing the gaussian integral over $\hat{\phi}(t)$ and it was used as an intermediate step in deriving eq. (13).

Note added

After completing this paper we have become aware of a paper by S. Dalley entitled "The Weingarten model à la Polyakov" and published on Mod. Phys. Lett. A7 (1992) 1651. Induced gauge theories on a lattice are considered there in the context of a complex matrix model; in particular the case $D = 1$, that has the same physical content as the corresponding Kazakov-Migdal model, is studied in detail.

Acknowledgments

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