Multiple multi-orbit pairing algebras in shell model and interacting boson models

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Abstract

In nuclei with valence nucleons are say identical nucleons and say these nucleons occupy several-j orbits, then it is possible to consider pair creation operator $S_+$ to be a sum of the single-j shell pair creation operators $S_+(j)$ with arbitrary phases, $S_+ = \sum_j \alpha_j S_+(j); \alpha_j = \pm 1$. In this situation, it is possible to define multi-orbit or generalized seniority that corresponds to the quasi-spin $SU(2)$ algebra generated by $S_+, S_- = (S_+)^\dagger$ and $S_0 = (\hat{n} - \Omega)/2$ operators; $\hat{n}$ is number operator and $\Omega = \lfloor \sum_j (2j + 1) \rfloor / 2$. There are now multiple pairing quasi-spin $SU(2)$ algebras, one for each choice of $\alpha_j$’s. Clearly, with $r$ number of $j$ shells there will be $2^{r-1}$ quasi-spin $SU(2)$ algebras. Also, the $\alpha_j$’s and the generators of the corresponding generalized seniority generating sympletic algebras $Sp(2\Omega)$ in $U(2\Omega) \supset Sp(2\Omega)$ have one-to-one correspondence. Using these, derived is the condition that a general one-body operator of angular momentum rank $k$ to be a quasi-spin scalar or a vector vis-a-vis the $\alpha_j$’s. These then will give special seniority selection rules for electromagnetic transitions. A particular choice for $\alpha_j$’s as advocated by Arvieu and Moszkowski (AM), based on SDI interaction, when applied to these conditions will give the selection rules discussed in detail in the past by Talmi. We found, using the correlation coefficient defined in the spectral distribution method of French, that the $\alpha_j$ choice of AM gives pairing Hamiltonians having maximum correlation with well known effective interactions. The various results derived for identical fermion systems are shown to extend to identical boson systems with the bosons occupying several-\ell orbits as for example in sd, sp, sdg and sdpf IBM’s. The quasi-spin algebra here is $SU(1,1)$ and the generalized seniority quantum number is generated by $SO(2\Omega)$ in $U(2\Omega) \supset SO(2\Omega)$. The different pairing $SO(2\Omega)$ algebras in the interacting boson models along with the tensorial nature of $E2$ and $E1$ operators in these models with respect to the corresponding $SU(1,1)$ are presented. These different $SO(2\Omega)$ algebras will be important in the study of quantum phase transitions and order-chaos transitions in nuclei.

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I. INTRODUCTION

Pairing force and the related quasi-spin or seniority quantum number continue to play an important role in shell model in particular and nuclear structure in general [1, 2]. There are several single-$j$ shell nuclei that are known to carry seniority quantum number as a good or useful quantum number [1, 3, 4]. Even when single shell seniority is a broken symmetry, seniority quantum number provides a basis for constructing shell model Hamiltonian matrices [5]. Pairing symmetry with nucleons occupying several $j$-orbits is more complex and less well understood from the point of view of its goodness or usefulness in nuclei. Restricting to nuclei with valence nucleons are identical nucleons (protons or neutrons), and say these nucleons occupy several-$j$ orbits, then it is possible to consider pair creation operator $S_+$ to be a sum of the single-$j$ shell pair creation operators $S_+(j)$ with arbitrary phases, $S_+ = \sum_j \alpha_j S_+(j); \alpha_j = \pm 1$. In this situation, it is possible to define multi-orbit or generalized seniority that corresponds to the quasi-spin $SU(2)$ algebra. However, with $r$ number of $j$ shells there will be $2^{r-1}$ quasi-spin $SU(2)$ algebras. Also, the $\alpha_j$‘s and the generators of the corresponding generalized seniority generating sympletic algebras $Sp(2\Omega)$ in $U(2\Omega) \supset Sp(2\Omega)$ have one-to-one correspondence. In this paper we will examine in detail these multiple pairing $SU(2)$ algebras and also the corresponding multiple pairing algebras for interacting boson systems. The usefulness or goodness of these multiple pairing algebras is not well known except a special situation was studied long time back by Arvieu and Moszkowski (AM) [6] in the context of surface delta interaction. In addition, pair states with $\alpha_j$ being free parameters (need not be $+1$ or $-1$) are used in generating low-lying states with good generalized seniority [1] and they are also employed in the so called broken pair model [7]. On the other hand these are also used in providing a microscopic basis for the interacting boson model [8]. Going beyond all these, there are also attempts to solve and apply more general pairing Hamiltonian’s by Pan Feng et al [9] and also a pair shell model is being studied by Zhao et al [10]. Now we will give a preview.

Section II gives in some detail the algebraic structure of the multiple multi-orbit pairing quasi-spin $SU(2)$ and the complimentary $Sp(N)$ algebras in $j - j$ coupling shell model for identical nucleons. Using the multiple algebras, in Section III derived are the selection rules for electromagnetic transitions with multi-orbit seniority. In section IV, correlation between realistic effective interactions and pairing operator with a given set of phases ($\alpha_j$) is studied
and shown that the choice advocated by AM gives maximum correlation. Section V gives
details of the algebraic structure of the multiple multi-orbit pairing quasi-spin SU(1,1) and
the complimentary SO(N) algebras in interacting boson models with identical bosons such
as sd, sp, sdg and sdpf IBM’s. Here, again derived are the selection rules for electromagnetic
transition operators as a function of the given set of phases in the generalized boson pair
operator. In Section VI presented are the results for the particle number dependence of the
matrix elements of one-body operators that are quasi-spin scalar or vector for both fermion
and boson systems. In Section VII presented are some applications of the multi-orbit pairing
algebras in shell model and interacting boson models. Finally, Section VIII gives conclusions
and future outlook.

II. MULTIPLE MULTI-ORBIT PAIRING QUASI-SPIN SU(2) AND THE COM-
PLIMENTARY Sp(N) ALGEBRAS IN j − j COUPLING SHELL MODEL

A. Multiple multi-orbit pairing quasi-spin SU(2) algebras

Let us say there are \( m \) number of identical fermions (protons or neutrons) in \( j \) orbits \( j_1, \)
\( j_2, \ldots, j_r \). Now, it is possible to define a generalized pair creation operator \( S_+ \) as

\[
S_+ = \sum_j \alpha_j S_+(j) ; \quad S_+(j) = \sum_{m>0} (-1)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger = \frac{\sqrt{2j+1}}{2} \left(a_{j}^\dagger a_{j}^\dagger\right)^0 . \tag{1}
\]

Here, \( \alpha_j \) are free parameters and assumed to be real. The \( m \) used for number of particles
should not be confused with the \( m \) in \( jm \). Given the \( S_+ \) operator, the corresponding pair
annihilation operator \( S_- \) is

\[
S_- = (S_+)^\dagger = \sum_j \alpha_j S_-(j) ; \quad S_-(j) = (S_+(j))^\dagger = -\frac{\sqrt{2j+1}}{2} \left(\tilde{a}_{j}\tilde{a}_{j}\right)^0 . \tag{2}
\]

Note that \( a_{jm} = (-1)^{j-m}\tilde{a}_{j-m} \). The operators \( S_+ \), \( S_- \) and \( S_0 \), with \( \hat{n} = \sum_{jm} a_{jm}^\dagger a_{jm} \) the
number operator,

\[
S_0 = \hat{n} - \Omega \quad ; \quad \Omega = \sum_j \Omega_j , \quad \Omega_j = (2j+1)/2 . \tag{3}
\]
form the generalized quasi-spin SU(2) algebra [hereafter called SU_Q(2)] only if

\[
\alpha_j^2 = 1 \text{ for all } j . \tag{4}
\]
With Eq. (4) we have,

\[
[S_0, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_0.
\] (5)

Thus, in the multi-orbit situation for each

\[
\{\alpha_j_1, \alpha_j_2, \ldots, \alpha_j_r\}
\]

with \(\alpha_j_i = \pm 1\) there is a \(SU_Q(2)\) algebra defined by the operators in Eqs. (1), (2) and (3). For example, say we have three \(j\) orbits \(j_1, j_2\) and \(j_3\). Then, without loss of generality we can choose \(\alpha_1 = +1\) and then \((\alpha_2, \alpha_3)\) can take values \((+1, +1), (+1, -1), (-1, +1), (-1, -1)\) giving four pairing \(SU_Q(2)\) algebras. Similarly, with four \(j\) orbits, there will be eight \(SU_Q(2)\) algebras and in general for \(r\) number of \(j\) orbits there will \(2^{r-1}\) number of \(SU_Q(2)\) algebras. The consequences of having these multiple pairing \(SU_Q(2)\) algebras will be investigated in the following.

Though well known, for later use and for completeness, some of the results of the \(SU_Q(2)\) algebra are that the \(S^2 = S_+S_--S_0+S_0^2\) operator and the \(S_0\) operator in Eq. (3) define the quasi-spin \(s\) and its \(z\)-component \(m_s\) with \(S^2 |sm_s\rangle = s(s+1) |sm_s\rangle\) and \(S_0 |sm_s\rangle = m_s |sm_s\rangle\).

Also, from Eq. (3) we have \(m_s = (m - \Omega)/2\); the \(m\) here is number of particles. Moreover, it is possible to introduce the so called seniority quantum number \(v\) such that \(s = (\Omega - v)/2\) giving,

\[
\begin{align*}
  s &= (\Omega - v)/2, \\
  m_s &= (m - \Omega)/2, \\
  v &= m, m-2, \ldots, 0 \text{ or } 1 \text{ for } m \leq \Omega \\
  &= (2\Omega - m), (2\Omega - m) - 2, \ldots, 0 \text{ or } 1 \text{ for } m \geq \Omega.
\end{align*}
\] (6)

Note that the total number of single particle states is \(N = 2\Omega\) and therefore for \(m > \Omega\) one has fermion holes rather than particles. The following results will provide a meaning to the seniority quantum number “\(v\)”,

\[
\begin{align*}
  \langle S_+S_- \rangle^{sm_s} &= \langle S_+S_- \rangle^{mv} = \langle mv | S_+S_- | mv \rangle \\
  &= \frac{1}{4}(m - v)(2\Omega - m - v + 2), \\
  |m, v, \beta\rangle &= \sqrt{\frac{(\Omega - v - p)!}{(\Omega - v)!p!}} (S_+) \frac{m - v}{2} |v, v, \beta\rangle; \quad p = \frac{(m - v)}{2}.
\end{align*}
\] (7)

With these, it is clear that for a given \(v\) and \(m\) there are \((m - v)/2\) zero coupled pairs. Thus, \(v\) gives the number of particles that are not coupled to angular momentum zero. In Eq. (8), \(\beta\) is an extra label that is required to specify a \((j_1, j_2, \ldots, j_r)^m\) state completely.
Before going further, an important result (to be used later) that follows from Eqs. (1) and (2) is,

\[ 4S_+S_- = 4 \sum_j S_+(j)S_-(j) + \sum_{j_1 > j_2} \alpha_{j_1}\alpha_{j_2} \]

\[ \times \sum_k \sqrt{2k+1} \left\{ \left[ (a_{j_1}^\dagger a_{j_2})^k (a_{j_1}^\dagger a_{j_2})^k \right]^0 + \left[ (a_{j_2}^\dagger a_{j_1})^k (a_{j_2}^\dagger a_{j_1})^k \right]^0 \right\} . \]  

(9)

**B. Multiple multi-orbit complimentary pairing \(Sp(N)\) algebras**

In the \((j_1, j_2, \ldots, j_r)^m\) space, often it is more convenient to start with the \(U(N)\) algebra generated by the one-body operators \(u_q^k(j_1, j_2)\),

\[ u_q^k(j_1, j_2) = (a_{j_1}^\dagger a_{j_2})^k_q . \]  

(10)

Total number of generators is obviously \(N^2\) and \(N = 2\Omega\). All \(m\) fermion states will be antisymmetric and therefore belong uniquely to the irreducible representation (irrep) \(\{1^m\}\) of \(U(N)\). The quadratic Casimir invariant of \(U(N)\) is easily given by

\[ C_2(U(N)) = \sum_{j_1, j_2} (-1)^{j_1-j_2} \sum_k u^k(j_1, j_2) \cdot u^k(j_2, j_1) , \]  

(11)

with eigenvalues

\[ \langle C_2(U(N)) \rangle^m = m(N + 1 - m) ; \quad N = 2\Omega . \]  

(12)

Eq. (12) can be proved by writing the one and two-body parts of \(C_2(U(N))\) and then showing that the one-body part is \(2\Omega \tilde{n}\) and the two-body part will have two-particle matrix elements diagonal with all of them having value \(-2\).

More importantly, \(U(N) \supset Sp(N)\) and the \(Sp(N)\) algebra is generated by the \(N(N+1)/2\) number of generators \(u_k^k(j, j)\) with \(k=\text{odd}\) only and \(V_q^k(j_1, j_2)\), \(j_1 > j_2\) with

\[ V_q^k(j_1, j_2) = [\mathcal{N}(j_1, j_2, k)]^{1/2} \left[ (a_{j_1}^\dagger a_{j_2})^k_q + X(j_1, j_2, k) (a_{j_2}^\dagger a_{j_1})^k_q \right] , \quad \{ X(j_1, j_2, k) \}^2 = 1 . \]  

(13)

The quadratic Casimir invariant of \(Sp(N)\) is given by,

\[ C_2(Sp(N)) = 2 \sum_j \sum_{k=\text{odd}} u^k(j, j) \cdot u^k(j, j) + \sum_{j_1 > j_2; k} V^k(j_1, j_2) \cdot V^k(j_1, j_2) . \]  

(14)

The \(Sp(N)\) algebra will be complimentary to the quasi-spin \(SU(2)\) algebra defined for a given set of \(\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\}\) provided

\[ \mathcal{N}(j_1, j_2, k) = (-1)^{k+1} \alpha_{j_1}\alpha_{j_2} , \quad X(j_1, j_2, k) = (-1)^{j_1+j_2+k}\alpha_{j_1}\alpha_{j_2} . \]  

(15)
Using Eqs. (11) and (13)-(15) along with Eq. (9) it is easy to derive the following important relation,

\[ C_2(U(N)) - C_2(Sp(N)) = 4S_+S_- - \hat{n}. \]  

(16)

Now, Eqs. (16), (12) and (7) will give

\[ \langle C_2(Sp(N)) \rangle^{m,v} = v(2\Omega + 2 - v) \]  

(17)

and this proves that the seniority quantum number \( v \) corresponds to the \( Sp(N) \) irrep \( \langle 1^v \rangle \).

In summary, given the \( SU_Q(2) \) algebra generated by \{ \( S_+, S_-, S_0 \) \} operators for a given set of \( \{ \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \} \) with \( \alpha_{j_i} = +1 \) or \(-1\), there is a complimentary \( \leftrightarrow \) \( Sp(N) \) subalgebra of \( U(N) \) generated by

\[ Sp(N) : u^k(j,j) = \left( a_{j_1} a_{j_2} \right)_q^k \text{ with } k = \text{ odd}, \]

\[ V_q^k(j_1,j_2) = \left( (-1)^{k+1} \alpha_{j_1} \alpha_{j_2} \right)^{1/2} \left[ \left( a_{j_1} a_{j_2} \right)_q^k + (-1)^{j_1+j_2+k} \alpha_{j_1} \alpha_{j_2} \left( a_{j_1} a_{j_2} \right)_q^k \right] \text{ with } j_1 > j_2. \]  

(18)

As the \( Sp(N) \) generators are one-body operators and that \( Sp(N) \leftrightarrow SU_Q(2) \), there will be special selection rules for electro-magnetic transition operators connecting \( m \) fermion states with good seniority. Though these are well known for a special choice of \( \alpha \)'s [1], their relation to the multiple \( SU(2) \) algebras or equivalently to the \( \{ \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \} \) set is, to our best of knowledge, is not discussed before. We will turn to this now.

III. SELECTION RULES FOR ELECTRO-MAGNETIC TRANSITIONS WITH MULTI-ORBIT SENIORITY

Electro-magnetic operators are essentially one-body operators (two and higher-body terms are usually not considered). In order to derive selection rules and matrix elements for allowed transitions, let us first consider the commutator of \( S_+ \) with \( \left( a_{j_1} a_{j_2} \right)_q^k \). Firstly we have easily,

\[ \left[ S_+(j) , \left( a_{j_1}^\dagger a_{j_2} \right)_q^k \right] = -\delta_{j_1,j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger \right)_q^k. \]  

(19)
Therefore, 

\[
\begin{align*}
S_+ , \quad & \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^k_q + X \left( a_{j_2}^\dagger \tilde{a}_{j_1} \right)^k_q \\
= & -\alpha_{j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger \right)^k_q \{ 1 - X \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+k} \} \\
= & 0 \text{ if } X = \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+k} \\
\neq & 0 \text{ if } X = -\alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+k} .
\end{align*}
\]

(20)

Note that the commutator is zero implies that the operator is a scalar \( T_0^0 \) with respect to \( SU_Q(2) \) and otherwise it will be a quasi-spin vector \( T_0^1 \). In either situation the \( S_z \) component of \( T \) is zero as a one-body operator can not change particle number. Thus, for \( j_1 \neq j_2 \) we have

\[
U_q^{k}(j_1, j_2) = N_u \left\{ \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^k_q + \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+k} \left( a_{j_2}^\dagger \tilde{a}_{j_1} \right)^k_q \right\} \rightarrow T_0^0 ,
\]

\[
W_q^{k}(j_1, j_2) = N_w \left\{ \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^k_q - \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+k} \left( a_{j_2}^\dagger \tilde{a}_{j_1} \right)^k_q \right\} \rightarrow T_0^1 .
\]

(21)

Here \( N_u \) and \( N_w \) are some constants. Similarly, for \( j_1 = j_2 \) we have

\[
\begin{align*}
\left( a_{j_1}^\dagger \tilde{a}_{j_1} \right)^k_q \text{ with } k \text{ odd } & \rightarrow T_0^0 , \\
\left( a_{j_1}^\dagger \tilde{a}_{j_1} \right)^k_q \text{ with } k \text{ even } & \rightarrow T_0^1 \text{ except for } k = 0 .
\end{align*}
\]

(22)

The results in Eqs. (21) are easy to understand as \( U_q^{k} \) in Eq. (21) is to within a factor same as \( V_q^{k} \) of Eq. (18) and therefore a generator of \( Sp(N) \). Hence it can not change the \( v \) quantum number of a \( m \)-particle state. Also, as \( Sp(N) \leftrightarrow SU_Q(2) \), clearly \( U_q^{k} \) will be a \( SU_Q(2) \) scalar. Similarly turning to Eq. (22), as \( \left( a_{j_1}^\dagger \tilde{a}_{j_1} \right)^k_q \) with \( k \) odd are generators of \( Sp(N) \) and hence they are also \( SU_Q(2) \) scalars.

General form of electric and magnetic multipole operators \( T^{EL} \) and \( T^{ML} \) respectively with \( L = 1, 2, 3, \ldots \) is, with \( X = E \) or \( M \),

\[
T_q^{XL} = \sum_{j_1, j_2} \epsilon_{j_1, j_2}^{XL} \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^L_q
\]

\[
= \sum_j \epsilon_{j_1, j_2}^{XL} \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^L_q + \sum_{j_1 > j_2} \epsilon_{j_1, j_2}^{XL} \left[ \left( a_{j_1}^\dagger \tilde{a}_{j_2} \right)^L_q + \frac{\epsilon_{j_1, j_2}^{XL}}{\epsilon_{j_2, j_1}^{XL}} \left( a_{j_2}^\dagger \tilde{a}_{j_1} \right)^L_q \right] .
\]

(23)

Therefore, \( \epsilon_{j_2, j_1}^{XL} / \epsilon_{j_1, j_2}^{XL} \) along with Eqs. (21)and (22) will determine the selection rules. Then,

\[
\begin{align*}
\frac{\epsilon_{j_2, j_1}^{XL}}{\epsilon_{j_1, j_2}^{XL}} = & \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+L} \rightarrow T_0^0 \text{ w.r.t. } SU_Q(2) , \\
\frac{\epsilon_{j_2, j_1}^{XL}}{\epsilon_{j_1, j_2}^{XL}} = & -\alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+L} \rightarrow T_0^1 \text{ w.r.t. } SU_Q(2) .
\end{align*}
\]

(24)
Thus, the $SU_Q(2)$ tensorial nature of $T^{XL}$ depend on the $\alpha_i$ choice. For $T^0_0$ we have $v \rightarrow v$ and for $T^1_0$ we have $v \rightarrow v, v \pm 2$ transitions. It is well known [1, 6] that for $T^{EL}$ and $T^{ML}$ operators,

$$\frac{\epsilon^{EL}_{j_2,j_1}}{\epsilon^{EL}_{j_1,j_2}} = (-1)^{\ell_1+\ell_2+j_1+j_2+L}, \quad \frac{\epsilon^{ML}_{j_2,j_1}}{\epsilon^{ML}_{j_1,j_2}} = (-1)^{\ell_1+\ell_2+j_1+j_2+L}. \quad (25)$$

In Eq. (25) $\ell_i$ is the orbital angular momentum of the $j_i$ orbit. Therefore, combining results in Eqs. (21)-(25) together with parity selection rule will give seniority selection rules, in the multi-orbit situation, for electro-magnetic transition operators when the observed states carry seniority quantum number as a good quantum number. The selection rules with the choice $\alpha_{j_i} = (-1)^{\ell_i}$ for all $i$ are as follows.

1. $T^{EL}$ with $L$ even will be $T^1_0$ w.r.t. $SU_Q(2)$.

2. $T^{EL}$ with $L$ odd will be $T^1_0$ w.r.t. $SU_Q(2)$. However, if all $j$ orbits have same parity, then $T^{EL}$ with $L$ odd will not exist. Therefore here, for the transitions to occur, we need minimum two orbits of different parity.

3. $T^{ML}$ with $L$ odd will be $T^0_0$ w.r.t. $SU_Q(2)$.

4. $T^{ML}$ with $L$ even will be $T^0_0$ w.r.t. $SU_Q(2)$. However, if all $j$ orbits have same parity, then $T^{ML}$ with $L$ even will not exist. Therefore here, for the transitions to occur, we need minimum two orbits of different parity.

5. For $T^0_0$ only $v \rightarrow v$ transitions are allowed while for $T^1_0$ both $v \rightarrow v$ and $v \rightarrow v \pm 2$ transition are allowed. For both $m$ is not changed.

The above rules were given already by AM [6] and described by Talmi [1]. As stated by Arvieu and Moszkowski, they have introduced the choice $\alpha_i = (-1)^{\ell_i}$ “for convenience” and then found that it will make surface delta interaction a $SU_Q(2)$ scalar. It is important to note that for $SU_Q(2)$ generated by $\alpha_i \neq (-1)^{\ell_i}$, the above rules (1)-(4) will be violated and then Eq. (24) has to be applied. This is a new result not reported before, to our knowledge, in the literature. A similar result applies to interacting boson models as presented ahead in Section V. Before going further, within shell model context it is necessary to conform that a realistic pairing operator do respect the condition $\alpha_i = (-1)^{\ell_i}$. In order to test this, we will use correlation coefficient between operators as defined in French’s spectral distribution method [11].
IV. CORRELATION BETWEEN OPERATORS AND PHASE CHOICE IN THE PAIRING OPERATOR

Given an operator $O$ acting in $m$ particle spaces and assumed to be real, its $m$-particle trace is $\langle\langle O \rangle\rangle^m = \sum_{\alpha} \langle m, \alpha \mid O \mid m, \alpha \rangle$ where $|m, \alpha\rangle$ are $m$-particle states. Similarly, the $m$-particle average is $\langle O \rangle^m = [d(m)]^{-1} \langle\langle O \rangle\rangle^m$ where $d(m)$ is $m$-particle space dimension. In $m$ particle spaces it is possible to define, using the spectral distribution method of French [11, 12], a geometry [12, 13] with norm (or size or length) of an operator $O$ given by $||O||_m = \sqrt{\langle\tilde{O}\tilde{O} \rangle^m}$; $\tilde{O}$ is the traceless part of $O$. With this, given two operators $O_1$ and $O_2$, the correlation coefficient

$$
\zeta(O_1, O_2) = \frac{\langle\tilde{O}_1\tilde{O}_2 \rangle^m}{||O_1||_m ||O_2||_m},
$$

(26)
gives the cosine of the angle between the two operators. Thus, $O_1$ and $O_2$ are same within a normalization constant if $\zeta = 1$ and they are orthogonal to each other if $\zeta = 0$ [11, 13]. Most recent application of norms and correlation coefficients defined above to understand the structure of effective interactions is due to Draayer et al [14, 15].

Clearly, in a given shell model space, given a realistic effective interaction Hamiltonian $H$, the $\zeta$ in Eq. (26) can be used as a measure for its closeness to the pairing Hamiltonian $H_P = S_+S_-$ with $S_+$ defined by Eq. (1) for a given set of $\alpha_j$’s. Evaluating $\zeta(H, H_P)$ for all possible $\alpha_j$ sets, it is possible to identify the $\alpha_j$ set that gives maximum correlation of $H_p$ with $H$. Following this, $\zeta(H, H_P)$ is evaluated for effective interactions in $(0f_{7/2}, 0f_{5/2}, 1p_{3/2}, 1p_{1/2}), (0f_{5/2}, 1p_{3/2}, 1p_{1/2}, 0g_{9/2})$ and $(0g_{7/2}, 1d_{5/2}, 1d_{3/2}, 2s_{1/2}, 0h_{11/2})$ spaces using GXPF1 [16], JUN45 [17] and jj55-SVD [18] interactions respectively. As we are considering only identical particle systems and also as we are interested in studying the correlation of $H$’s with $H_P$’s, only the $T = 1$ part of the interactions is considered (dropped are the $T = 0$ two-body matrix elements and also the single particle energies). With this $\zeta(H, H_P)$ are calculated in the three spaces for different values of the particle number $m$ and for all possible choices of $\alpha_j$’s defining $S_+$ and hence $H_P$. Results are given in Table I. It is clearly seen that the choice $\alpha_j = (-1)^{\ell_i}$ gives the largest value for $\zeta$ and hence it should be the most preferred choice. This is a significant result justifying the choice made by AM [6], although the magnitude of $\zeta$ is not more than 0.3. Thus, realistic $H$ are far, on a global $m$-particle space scale, from the simple pairing Hamiltonian. However, it is likely that the generalized
pairing quasi-spin or sympletic symmetry may be an effective symmetry for low-lying state and some special high-spin states [4]. Evidence for this will be discussed in Section VII.

Before turning to interacting boson systems, it is useful to add that in principle the spectral distribution method can be used to study the mixing of seniority quantum number in the eigenstates generated by a given Hamiltonian by using the so called partial variances [3, 11]. The \( v_i \rightarrow v_f \) partial variances, with \( v_i \neq v_f \), are defined by

\[
\sigma^2(m, v_i \rightarrow m, v_f) = \left[ d(m, v_i) \right]^{-1} \sum_{\alpha, \beta} |\langle m, v_f, \beta | H | m, v_i, \alpha \rangle|^2 .
\] (27)

In Eq. (27), \( d(m, v) \) is the dimension of the \((m, v)\) space. It is important to note that the partial variances can be evaluated without constructing the \( H \) matrices but by using the propagation equations. These are available both for fermion and boson systems; see [19, 20]. However, propagation equations for the more realistic \( \sigma^2(m, v_i, J \rightarrow m, v_f, J) \) partial variances are not yet available.

V. MULTIPLE MULTI-ORBIT PAIRING QUASI-SPIN \( SU(1,1) \) AND THE COMPLEMENTARY \( SO(N) \) ALGEBRAS IN INTERACTING BOSON MODELS

Going beyond the shell model, also within the interacting boson models, i.e. for example in \( sd, sp, sdg \) and \( spdf \) IBM’s, again it is possible to have multiple pairing symmetry algebras as we have several \( \ell \) orbits in these models with bosons [21–24]. Here, as it is well known, the pairing algebra is \( SU_Q(1,1) \) instead of \( SU_Q(2) \) [25]. Let us consider IBM with identical bosons carrying angular momentum \( \ell_1, \ell_2, \ldots, \ell_r \) and the parity of an \( \ell_i \) orbit is \((-1)^{\ell_i}\).

Now, again it is possible to define a generalized boson pair creation operator \( S^B_+ \) as

\[
S^B_+ = \sum_\ell \beta_\ell S^B_+(\ell) ; S^B_+(\ell) = \frac{1}{2} \sum_m (-1)^m b_{\ell m}^\dagger b_{\ell-m}^\dagger = \frac{\sqrt{2\ell+1}}{2} (-1)^{\ell} \left( b_{\ell}^\dagger b_{\ell}^\dagger \right)^0 = \frac{1}{2} b_{\ell}^\dagger \cdot b_{\ell}^\dagger .
\] (28)

Here, \( \beta_\ell \) are free parameters and assumed to be real. Given the \( S^B_+ \) operator, the corresponding pair annihilation operator \( S^B_- \) is

\[
S^B_- = (S^B_+)^\dagger = \sum_\ell \beta_\ell S^B_- (\ell) ; S^B_- (\ell) = (S^B_+(\ell))^\dagger = (-1)^{\ell} \frac{\sqrt{2\ell+1}}{2} \left( b_{\ell} b_{\ell}^\dagger \right)^0 = \frac{1}{2} b_{\ell} \cdot b_{\ell}^\dagger .
\] (29)

Note that \( b_{\ell m} = (-1)^{\ell-m} b_{\ell-m}^\dagger \). The operators \( S^B_+ \), \( S^B_- \) and \( S^B_0 \), with \( \hat{n}^B = \sum_\ell \hat{a}_\ell^\dagger \hat{a}_\ell \) the number operator,

\[
S^B_0 = \frac{\hat{n}^B + \Omega^B}{2} ; \Omega^B = \sum_\ell \Omega^B_\ell , \quad \Omega^B_\ell = (2\ell + 1)/2 .
\] (30)
TABLE I. Correlation coefficient ζ between a realistic interaction \((H)\) and the pairing Hamiltonian \(H_p\) for various particle numbers \((m)\) in three different spectroscopic spaces. The single particle (sp) orbits for these three spaces are given in column #1. The range of \(m\) values used for each sp space is given in column #3. The phases \(α_j\) for each orbit in the generalized pair creation operator are given in column #4 (the order is same as the sp orbits listed in column #1). The variation in \(ζ\) with particle number \(m\) is given in column #5. Results for the phase choices that give \(|ζ| < 0.1\) for all \(m\) values are not shown in the table. See text for other details.

| sp orbits          | interaction | \(m\)    | \(α_j\)         | \(ζ(H,H_p)\) |
|--------------------|-------------|----------|-----------------|---------------|
| \(^0g_{7/2}, ^1d_{5/2}, ^1d_{3/2}, ^2s_{1/2}, ^0h_{11/2}\) | jj55-SVD    | 2 \(-30\) | \((+,+,+,+,-)\) | 0.33-0.11     |
|                    |             |          | \((+,+,+,-,-)\) | 0.26-0.09     |
|                    |             |          | \((+,+,-,+,-)\) | 0.17-0.06     |
|                    |             |          | \((+,+,-,-,-)\) | 0.13-0.04     |
|                    |             |          | \((+,-,+,+,-)\) | 0.11-0.04     |
| \(^0f_{5/2}, ^1p_{3/2}, ^1p_{1/2}, ^0g_{9/2}\)       | jun45       | 2 \(-20\) | \((+,+,+,+)\)   | 0.42-0.21     |
|                    |             |          | \((+,+,+,-)\)   | 0.27-0.13     |
|                    |             |          | \((+,+,-,+,-)\) | 0.15-0.07     |
|                    |             |          | \((+,-,+,-,-)\) | 0.12-0.06     |
| \(^0f_{7/2}, ^1p_{3/2}, ^0f_{5/2}, ^1p_{1/2}\)       | gxpf1       | 2 \(-18\) | \((+,+,+,+)\)   | 0.36-0.33     |
|                    |             |          | \((+,+,+,-)\)   | 0.22-0.20     |
|                    |             |          | \((+,+,-,+,-)\) | 0.13-0.12     |
|                    |             |          | \((+,-,+,-,-)\) | 0.13-0.11     |
|                    |             |          | \((+,-,-,-,-)\) | 0.11-0.10     |

form the generalized quasi-spin SU(1,1) algebra [hereafter called \(SU^B_Q(1,1)\)] only if

\[
β_ℓ^2 = 1 \text{ for all } ℓ . \tag{31}
\]

With Eq. (31) we have,

\[
[S^B_0 S^B_±] = ±S^B_± , \quad [S^B_+ S^B_-] = -2S^B_0 . \tag{32}
\]

Thus, in the multi-orbit situation for each

\[
\{β_ℓ_1, β_ℓ_2, \ldots, β_ℓ_r\}
\]
with $\beta_{\ell_1} = \pm 1$ there is a $SU_Q^U(1,1)$ algebra defined by the operators in Eqs. (28), (29) and (30). In general for $r$ number of $\ell$ orbits there will $2^{r-1}$ number of $SU_Q^U(1,1)$ algebras. Let us mention that $(S^B)^2 = (S_0^B)^2 - S_+^B S_-^B$ and $S_0^B = (\hat{n}^B + \Omega^B)/2$ provide the quasi-spin $s$ and the $s_z$ quantum number $m_s$ giving the basis $|s, m_s\rangle$ \cite{21,22},

\[(S^B)^2 |s, m_s, \gamma\rangle = s(s-1) |s, m_s, \gamma\rangle, \quad S_0 |s, m_s, \gamma\rangle = m_s |s, m_s, \gamma\rangle; \]

\[m_s = s, s+1, s+2, \ldots \]

\[\Rightarrow \]

\[s = (\Omega^B + \omega^B)/2, \quad m_s = (\Omega^B + N^B)/2, \quad \omega^B = N^B, N^B - 2, \ldots, 0 \text{ or } 1, \]

\[S_+^B S_-^B |s, m_s, \gamma\rangle = S_+^B S_-^B |N^B, \omega^B, \gamma\rangle = \frac{1}{4} (N^B - \omega^B)(\omega^B + N^B + 2\Omega^B - 2) |N^B, \omega^B, \gamma\rangle. \tag{33} \]

Here, $N^B$ is number of bosons. Just as for fermions, corresponding to each $SU_Q^U(1,1)$ there will be, in the $(\ell_1, \ell_2, \ldots, \ell_r)^{N^B}$ space, a $SO(N)$ in $U(N)$ with $N = 2\Omega^B = \sum_{\ell}(2\ell + 1)$. The $U(N)$ algebra is generated by the $N^2$ number of operators

\[u^k_q(\ell_1, \ell_2) = \left(\beta_{\ell_1}^{\dagger} \tilde{b}_{\ell_2}\right)_q^k. \]

As all the $N^B$ boson states will be symmetric, they belong uniquely to the irrep $\{N^B\}$ of $U(N)$. The quadratic Casimir invariant of $U(N)$ is easily given by

\[C_2(U(N)) = \sum_{\ell_1, \ell_2} (-1)^{\ell_1+\ell_2} \sum_k u^k(\ell_1, \ell_2) \cdot u^k(\ell_2, \ell_1), \tag{34} \]

with eigenvalues

\[\langle C_2(U(N)) \rangle^{N^B} = N^B(N^B + N - 1). \tag{35} \]

More importantly, $U(N) \supset SO(N)$ and the $N(N-1)/2$ generators of $SO(N)$ are \cite{23},

\[SO(N) : u^k_q(\ell, \ell) \text{ with } k \text{ odd}, \]

\[V^k_q(\ell_1, \ell_2) = \{(1)^{\ell_1+\ell_2} Y(\ell_1, \ell_2, k)\}^{1/2} \left[\left(b_{\ell_1}^{\dagger} \tilde{b}_{\ell_2}\right)_q^k + Y(\ell_1, \ell_2, k) \left(b_{\ell_2}^{\dagger} \tilde{b}_{\ell_1}\right)_q^k \right]; \tag{36} \]

\[Y(\ell_1, \ell_2, k) = (-1)^{k+1} \beta_{\ell_1} \beta_{\ell_2}. \]
Just as for fermion systems, the $SO(N)$ is complimentary to the quasi-spin $SU_Q^B(1,1)$ and this follows from the relations (proved in [23]),

$$4S^B_+S^B_-=C_2(U(N))-\hat{n}^B-C_2(SO(N)),$$

$$C_2(SO(N)) = \sum_\ell C_2(SO(N_\ell)) + \sum_{\ell_i<\ell_j} \sum_k V^k(\ell_i,\ell_j) \cdot V^k(\ell_i,\ell_j);$$

(37)

$$C_2(SO(N_\ell)) = 2 \sum_{k=\text{odd}} u^k(\ell,\ell) \cdot u^k(\ell,\ell);$$

$$\Longrightarrow \langle C_2(SO(N)) \rangle_{^N_B^\omega^B} = \omega^B(\omega^B+N-2).$$

In the last step we have used Eqs. (33) and (35). Thus, the irreps of $SO(N)$ are labeled by the symmetric irreps $[\omega^B]$ with

$$\omega^B = N^B, N^B-2, \ldots, 0 \text{ or } 1.$$  

(38)

A. Seniority selection rules for one-body transition operators

Given a general one-body operator

$$T^k_q = \sum_{\ell_1,\ell_2} \epsilon^k_{\ell_1,\ell_2} \left( b_{\ell_1}^\dagger \tilde{b}_{\ell_2} \right)_q^k$$

$$= \sum_{\ell} \epsilon^k_{\ell,\ell} \left( b_{\ell_1}^\dagger \tilde{b}_{\ell_2} \right)_q^k + \sum_{\ell_1<\ell_2} \epsilon^k_{\ell_1,\ell_2} \left[ \left( b_{\ell_1}^\dagger \tilde{b}_{\ell_2} \right)_q^k + \epsilon^k_{\ell_2,\ell_1} \left( b_{\ell_2}^\dagger \tilde{b}_{\ell_1} \right)_q^k \right],$$

(39)

as $SO(N) \leftrightarrow SU_B(1,1)$, it should be clear from the generators in Eq. (36) that the diagonal $(b_{\ell_1}^\dagger \tilde{b}_{\ell_2})_q^k$ part will be $SU_Q^B(1,1)$ scalar $T^0_0$ for $k$ odd and vector $T^1_0$ for $k$ even (except for $k = 0$). Similarly, the off diagonal

$$\left[ \left( b_{\ell_1}^\dagger \tilde{b}_{\ell_2} \right)_q^k + \epsilon^k_{\ell_2,\ell_1} \left( b_{\ell_2}^\dagger \tilde{b}_{\ell_1} \right)_q^k \right]$$

part will be

$$\frac{\epsilon^k_{\ell_2,\ell_1}}{\epsilon^k_{\ell_1,\ell_2}} = (-1)^{k+1} \beta_{\ell_1,\ell_2} \rightarrow T^0_0, \quad \frac{\epsilon^k_{\ell_2,\ell_1}}{\epsilon^k_{\ell_1,\ell_2}} = (-1)^k \beta_{\ell_1,\ell_2} \rightarrow T^1_0.$$  

(40)

Thus, the selection rules for the boson systems are similar to those for the fermion systems. Results in Eqs. (36) and (40) together with a condition for the seniority tensorial structure will allow us to write proper forms for the EM operators in boson systems. Let us say that $S^B_+$ is given by

$$S^B_+ = \sum_\ell \frac{\beta_\ell^\dagger}{2} \cdot \beta_\ell; \quad \beta_i = +1 \text{ or } -1.$$  

(41)
If we impose the condition that the $T^{E,L=\text{even}}$ and $T^{M,L=\text{odd}}$ operators are $T_0^1$ and $T_0^0$ w.r.t. $SU_Q^B(1,1)$, just as the fermion operators are w.r.t. $SU_Q(2)$ (see Section III), then

$$T^L = \sum_{\ell} \epsilon^L_{\ell\ell} (b^\dagger_{\ell} \tilde{b}_\ell)_q + \sum_{\ell_1 > \ell_2} \epsilon^L_{\ell_1\ell_2} \left[ \left( b^\dagger_{\ell_1} \tilde{b}_{\ell_2} \right)_q + \beta_{\ell_1} \beta_{\ell_2} \left( b^\dagger_{\ell_2} \tilde{b}_{\ell_1} \right)_q \right] ;$$

$$\Rightarrow \quad T^{EL} \rightarrow T_0^1, \quad T^{ML} \rightarrow T_0^0. \quad (42)$$

Note that for $\ell_1 \neq \ell_2$, parity selection rule implies that $(-1)^{\ell_1 + \ell_2}$ must be +1. Similarly, the parity changing $T^{E,L=\text{odd}}$ and $T^{M,L=\text{even}}$ operators are,

$$T^L = \sum_{\ell_1 > \ell_2} \epsilon^L_{\ell_1\ell_2} \left[ \left( b^\dagger_{\ell_1} \tilde{b}_{\ell_2} \right)_q - \beta_{\ell_1} \beta_{\ell_2} \left( b^\dagger_{\ell_2} \tilde{b}_{\ell_1} \right)_q \right] ;$$

$$\Rightarrow \quad T^{EL} \rightarrow T_0^1, \quad T^{ML} \rightarrow T_0^0. \quad (43)$$

Note that for $\ell_1 \neq \ell_2$, parity selection rule implies that $(-1)^{\ell_1 + \ell_2}$ must be −1 and therefore here we need orbits of different parity as in $sp$ and $sdpf$ IBM’s. On the other hand, if we impose the condition that $T^{EL}$ is $T_0^0$ w.r.t. $SU_Q^B(1,1)$, then

$$T^{E,L=\text{even}} = \sum_{\ell_1 > \ell_2} \epsilon^L_{\ell_1\ell_2} \left[ \left( b^\dagger_{\ell_1} \tilde{b}_{\ell_2} \right)_q - \beta_{\ell_1} \beta_{\ell_2} \left( b^\dagger_{\ell_2} \tilde{b}_{\ell_1} \right)_q \right] ; \quad (-1)^{\ell_1 + \ell_2} = +1 ,$$

$$T^{E,L=\text{odd}} = \sum_{\ell_1 > \ell_2} \epsilon^L_{\ell_1\ell_2} \left[ \left( b^\dagger_{\ell_1} \tilde{b}_{\ell_2} \right)_q + \beta_{\ell_1} \beta_{\ell_2} \left( b^\dagger_{\ell_2} \tilde{b}_{\ell_1} \right)_q \right] ; \quad (-1)^{\ell_1 + \ell_2} = -1 \quad (44)$$

$$\Rightarrow \quad T^{EL} \rightarrow T_0^0 .$$

Similarly, $T^{ML}$ can be chosen to be $T_0^1$ w.r.t. $SU_Q^B(1,1)$. Examples for $sd$, $sp$, $sdg$ and $sdpf$ systems are discussed in Section VII.

VI. NUMBER DEPENDENCE OF MANY PARTICLE MATRIX ELEMENTS OF $T_0^0$ AND $T_0^1$ OPERATORS

Applying Wigner-Ekart theorem for the many particle matrix elements in good seniority states, number dependence of the matrix elements of $T_0^0$ and $T_0^1$ operators is easily determined. For fermions one uses the $SU_Q(2)$ Wigner coefficients [26] and for bosons $SU_Q^B(1,1)$ Wigner coefficients [25]. Results for fermions systems are given for example in [1]. For
completeness we will give these here and also those for boson systems. For fermions, using $SU_Q(2)$ algebra, we have

$$
\langle m, v, \alpha \mid T^0_v \mid m, v', \beta \rangle = \delta_{v, v'} \langle v, v, \alpha \mid T^0_v \mid v, v, \beta \rangle,
$$

$$
\langle m, v, \alpha \mid T^1_v \mid m, v, \beta \rangle = \frac{\Omega - m}{\Omega - v} \langle v, v, \alpha \mid T^1_v \mid v, v, \beta \rangle,
$$

$$
\langle m, v, \alpha \mid T^1_v \mid m, v - 2, \beta \rangle = \sqrt{\frac{(2\Omega - m - v + 2)(m - v + 2)}{4(\Omega - v + 1)}} \langle v, v, \alpha \mid T^1_0 \mid v, v - 2, \beta \rangle.
$$

Similarly, for bosons, using $SU_Q^B(1, 1)$ algebra (see [25]), we have

$$
\langle N^B, \omega^B, \alpha \mid T^0_v \mid N^B, \omega^B, \beta \rangle = \langle \omega^B, \omega^B, \alpha \mid T^0_v \mid \omega^B, \omega^B, \beta \rangle,
$$

$$
\langle N^B, \omega^B, \alpha \mid T^1_v \mid N^B, \omega^B, \beta \rangle = \frac{\Omega^B + N^B}{\Omega^B + \omega^B} \langle \omega^B, \omega^B, \alpha \mid T^1_v \mid \omega^B, \omega^B, \beta \rangle,
$$

$$
\langle N^B, \omega^B, \alpha \mid T^1_v \mid N^B, \omega^B - 2, \beta \rangle = \sqrt{\frac{(2\Omega^B + N^B + \omega^B - 2)(N^B - \omega^B + 2)}{4(\Omega^B + \omega^B - 1)}} \langle \omega^B, \omega^B, \alpha \mid T^0_v \mid \omega^B, \omega^B - 2, \beta \rangle.
$$

Note the well-established $\Omega \rightarrow -\Omega$ symmetry between the fermion and boson system formulas in Eqs. (45) and (46); see also [22, 23]. Also, $T^1_0$ generates both $v(\omega^B) \rightarrow v(\omega^B)$ and $v(\omega^B) \rightarrow v(\omega^B) \pm 2$ transitions while $T^0_v$ only $v(\omega^B) \rightarrow v(\omega^B)$ transitions for fermion (boson) systems. The later matrix elements are independent of number of particles.

In order to understand the variation of $B(EL)$ [similarly $B(ML)$] for fermion systems, two numerical example are shown in Fig. 1. Firstly, considered an electric multipole (of multipolarity $L$) transition between two states with same $v$ value. Then, the $B(EL) \propto \left[\Omega - m)/(\Omega - v)\right]^2$ as seen from the second equation in Eq. (45). Note that, with $\alpha_j = (-1)^j$, the $T^{EL}$ operators are $T^1_0$ w.r.t. $SU_Q(2)$. Assuming $v = 2$, variation of $B(EL)$ with particle number $m$ is shown for three different values of $\Omega$ and $m$ varying from 2 to 2$\Omega - 2$. It is clearly seen that $B(EL)$ decrease up to mid-shell and then again increases, i.e. $B(EL)$ vs $m$ is an inverted parabola. This behavior is seen for example in Sn isotopes [27] as discussed further in Section VII.A. Going further, assuming the ground $0^+$ and first excited $2^+$ states of a nucleus belong to $v = 0$ and $v = 2$ respectively, $B(E2; 2^+ \rightarrow 0^+)$ variation with particle number is calculated using the third equation in Eq. (45) giving $B(E2) \propto (2\Omega - m - v + 2)(m - v + 2)/4(\Omega - v + 1)$. The variation of $B(E2)$ is that it will increase up to mid-shell and then decrease; i.e. the $B(E2; 2^+ \rightarrow 0^+)$ vs $m$ is a parabola. The behavior seen in Fig. 1b is used to explain the variation in $B(E2)$’s in Sn isotopes [28] as discussed further in Section VII.A.
FIG. 1. (a) Variation of $B(EL)$ with particle number $m$ for three different values of $\Omega$ and seniority $v = 2$ for $v \rightarrow v$ transitions. (b) Variation of $B(E2; v = 2, 2^+ \rightarrow v = 0, 0^+)$ with particle number. Results are for fermion systems and they are obtained by applying the last two equations in Eq. (45). The $B(EL)$ and $B(E2)$ values are scaled such that the maximum value is 100 and they are not in any units.

In Fig. 2, variation of $B(EL)$ for boson systems with boson number assuming $\omega^B = 2$ for both $\omega^B \rightarrow \omega^B$ and $\omega^B \rightarrow \omega^B - 2$ transitions are shown by employing the last two equations in Eq. (46). The $B(EL)$ values increase with $N^B$ and this variation is quite different from the variation seen in Fig. 1 for fermion systems. The increase in $B(E2)$'s with increase in $N^B$ is seen in Ru isotopes [29].

VII. APPLICATIONS

A. Shell model applications

First examples for the goodness of generalized seniority in nuclei are Sn isotopes. Note that for Sn isotopes the valence nucleons are neutrons with $Z=50$ a magic number. From Eq. (7) it is easy to see that the spacing between the first $2^+$ state (it will have $v = 2$) and the ground state $0^+$ (it will have $v = 0$) will be independent of $m$, i.e. the spacing should be same for all Sn isotopes and this is well verified by experimental data [1]. Going beyond
FIG. 2. (a) Variation of $B(EL)$ with particle number $N^B$ for $\Omega = 16$ and seniority $\omega^B = 2$ for $\omega^B \to \omega^B$ transitions. (b) Variation of $B(E2; \omega^B = 2, 2^+ \to \omega^B = 0, 0^+)$ with particle number $N^B$. Results are for boson systems and they are obtained by applying the last two equations in Eq. (46). The $B(EL)$ and $B(E2)$ values are scaled such that the maximum value is 100 and they are not in any units.

In addition to $B(E2; 2^+ \to 0^+)$ data, there is now good data available for $B(E2)$’s and $B(E1)$’s for some high-spin isomer states in even Sn isotopes. These are: $B(E2; 10^+ \to 8^+)$ data for $^{116}\text{Sn}$ to $^{130}\text{Sn}$ and $B(E2; 15^- \to 13^-)$ for $^{120}\text{Sn}$ to $^{128}\text{Sn}$ and $B(E1; 13^- \to 12^+)$ in $^{120}\text{Sn}$ to $^{126}\text{Sn}$. The states $10^+$ and $8^+$ are interpreted to be $v = 2$ states while $15^-, 13^-$ and $12^+$ are $v = 4$ states. Therefore, all these transitions are $v \to v$ transitions and the their variation with $m$ will be as shown in Fig. 1a. This is well verified by data [27] by assuming that the active sp orbits are $^0h_{11/2}$, $^1d_{3/2}$ and $^2s_{1/2}$ with $\Omega = 9$ (see also Fig. 1a with $\Omega = 9$). The results with $\Omega = 8$ and $\Omega = 7$ obtained by dropping $^2s_{1/2}$ and $^1d_{3/2}$ orbits respectively,
are not in good accord with data.

In summary, both the $B(E2; 2^+ \rightarrow 0^+)$ data and the $B(E2)$ and $B(E1)$ data for high-spin isomer states are explained by assuming goodness of generalized seniority with the choice $\beta_j = (-1)^{\ell_j}$ but with effective $\Omega$ values. Although the sp orbits (and hence $\Omega$ values) used are different for the low-lying levels and the high-spin isomer states, the good agreements between data and effective generalized seniority description on one hand and the correlation coefficients presented in Section IV on the other show that for Sn isotopes generalized seniority is possibly an ‘emergent symmetry’.

In addition to even Sn isotopes, $B(E2)$ data for $10^+$ isomers in $N = 82$ isotones ($A=148$-$162$), $12^+$ isomers in Pb isotopes ($A=176$-$198$) and also high-spin isomers in odd-A Sn isotopes, $N=82$ isotones and Pb isotopes are analyzed, though the data is sparse, using the results in Fig. 1 and Eq. (45) [30].

B. Interacting boson model applications

Turning to the interacting boson models, let us first consider the $SO(6)$ limit of $sd$IBM. Then, we have $U(6) \supset SO(6)$ and the complimentary $SU(1,1)$ algebra corresponds to the $sd$ pair $S_+ = s^\dagger s^\dagger \pm d^\dagger \cdot d^\dagger$. Arima and Iachello [31] used the choice $S_+ = s^\dagger s^\dagger - d^\dagger \cdot d^\dagger$. The corresponding $SU(1,1)$ we denote as $SU^-(1,1)$. Similarly, the $SU(1,1)$ with $S_+ = s^\dagger s^\dagger + d^\dagger \cdot d^\dagger$ is denoted by $SU^+(1,1)$. Corresponding to the two $SU(1,1)$ algebras, there will be two $SO(6)$ algebras as pointed out first in [32]. Their significance is seen in quantum chaos studies [33, 34]. For illustration, let us consider the tensorial structure of the $E2$ operator. Following the discussion in Section V, the $E2$ transition operator will be $T_0^0$ w.r.t. $SU^-(1,1)$ if we choose $T^{E2} = \alpha (s^\dagger \tilde{d} + d^\dagger \tilde{s})^2_\mu$ where $\alpha$ is a constant. This is the choice made in [31] and this operator will not change the seniority quantum number (called $\sigma$ in [31]) defining the irreps of $SO(6)$ that is complimentary to $SU^-(1,1)$. However, if we demand that the $T^{E2}$ operator should be $T_0^1$ w.r.t. $SU^-(1,1)$, then we have $T^{E2} = \alpha_1 (d^\dagger \tilde{d})^2_\mu + i\alpha_2 (s^\dagger \tilde{d} - d^\dagger \tilde{s})^2_\mu$. This operator will have both $\sigma \rightarrow \sigma$ and $\sigma \rightarrow \sigma \pm 2$ transitions. On the other hand, $T^{E2} = \alpha_1 (d^\dagger \tilde{d})^2_\mu + \alpha_2 (s^\dagger \tilde{d} + d^\dagger \tilde{s})^2_\mu$ will be a mixture of $T_0^0$ and $T_0^1$ operators.

In the second example we will consider the $sp$ boson model, also called vibron model with applications to diatomic molecules [35] and two-body clusters in nuclei [36]. Just as in $sd$IBM, here we have $U(4) \supset SO(4)$ and there will be two $SO(4)$ algebras with
FIG. 3. Energy spectra for 50 bosons and \((\omega_{sd}^B, \omega_g^B) = (0, 0)\) in \(sdg\)IBM with the Hamiltonian
\[
H_{sdg} = \frac{[1 - \xi]}{N^B} \hat{n}_g + \frac{[\xi/(N^B)^2]}{4(S_{sd}^+ + xS_{sd}^0)(S_{sd}^- + xS_{sd}^0) - N^B(N^B + 13)}
\]
where \(S_{sd}^+\) is the \(S^+\) operator for the \(sd\) boson system and \(S_+^g\) for the \(g\) boson system. In each panel, energy spectra are shown as a function of the parameter \(\xi\) taking values from 0 to 1. Results are shown in the figures for \(x = 1, 0.8, 0.5, 0.2, 0, -0.2, -0.5, -0.8\) and \(-1\). Note that \(x = 1\) and \(-1\) correspond to the two \(SU(1, 1)\) algebras in the model. In the figures, energies are not in any units.

\[S_+ = s^\dagger s^\dagger + \beta p^\dagger \cdot p^\dagger; \ \beta = \pm 1.\]

The general form of the \(E1\) operator (to lowest order) in this model is
\[T^{E1} = \epsilon_{sp} \left( s^\dagger \tilde{p} \mp p^\dagger \tilde{s} \right)^1.\]

With \(SU^+(1, 1)\) defined by \(S_+ = s^\dagger s^\dagger + p^\dagger \cdot p^\dagger\), from Eq. (43) we see that \(T^{E1} = i \epsilon \left( s^\dagger \tilde{p} - p^\dagger \tilde{s} \right)^1\) will be \(T_0^1\) w.r.t. \(SU^+(1, 1)\). Similarly, with \(SU^-(1, 1)\) defined by \(S_+ = s^\dagger s^\dagger - p^\dagger \cdot p^\dagger\), from Eq. (42) we see that \(T^{E1} = \epsilon_{sp} \left( s^\dagger \tilde{p} + p^\dagger \tilde{s} \right)^1\) will be \(T_0^1\) w.r.t. \(SU^-(1, 1)\). If the definitions of \(T^{E1}\) are interchanged, then they will be \(T_0^1\) w.r.t. the corresponding \(SU(1, 1)\) algebras. These results are described and applied in [35, 37, 38].

In the third example we will consider the \(sdg\) interacting boson model [39] and there is new interest in this model in the context of quantum phase transitions (QPT) [40]. With \(s, \]
$d$ and $g$ bosons, the generalized pair operator here is $S_+ = s^\dagger s^\dagger \pm d^\dagger \cdot d^\dagger \pm g^\dagger \cdot g^\dagger$ giving four $SU^{+,\pm,\pm}(1,1)$ algebras and the corresponding $SO^{+,\pm,\pm}(15)$ algebras in $U(15) \supset SO(15)$; the superscripts in $SU^{+,\pm,\pm}(1,1)$ and similarly in $SO(15)$ are the signs of the $s$, $d$ and $g$ pair operators in $S_+$. In QPT studies, VanIsacker et al have chosen [40] the operators $(s^\dagger \tilde{d} + d^\dagger \tilde{s})^2_\mu$ and $(s^\dagger \tilde{g} + g^\dagger \tilde{\tilde{s}})^2_\mu$ to be $SO(15)$ scalars. Then, from Eq. (40) it is seen that the $SO(15)$ will correspond to the $SU^{+,\pm,\mp}(1,1)$ algebra with $H_p = S_+ S_-$ where $S_+ = s^\dagger s^\dagger - d^\dagger \cdot d^\dagger - g^\dagger \cdot g^\dagger$. Note that here the $sd$-part is same as the one used by Arima and Iachello (see the $sd$IBM discussion above). In another recent study, the $E2$ operator in $sdg$IBM was chosen to be [41]

$$T^{E2} = \alpha_1 (d^\dagger \tilde{d})^2_\mu + \alpha_2 (g^\dagger \tilde{g})^2_\mu + \alpha_3 (s^\dagger \tilde{d} + d^\dagger \tilde{s})^2_\mu + \alpha_4 (d^\dagger \tilde{g} + g^\dagger \tilde{\tilde{s}})^2_\mu.$$  

With respect to the $SU^{+,\pm,\pm}(1,1)$ above, this operator will be a mixture of $T^0_0$ and $T^1_0$ operators. However, w.r.t. $SU^{+,+,+}(1,1)$, it will be a pure $T^1_0$ operator. It should be clear that with different choices of $SU(1,1)$ algebras (there are four of them), the QPT results for transition to rotational $SU(3)$ limit in $sdg$IBM will be different. It is important to investigate this going beyond the results presented in [40].

In the final example, let us consider the $sdpf$ model [24] applied recently with good success in describing $E1$ strength distributions in Nd, Sm, Gd and Dy isotopes [42] and also spectroscopic properties (spectra, and $E2$ and $E1$ strengths) of even-even $^{98-110}$Ru isotopes [29]. Note that the parities of the $p$ and $f$ orbit are negative. In $sdpf$IBM, following the results in Section V, there will be eight generalized pairs $S_+$ and the algebra complimentary to the $SU(1,1)$ is $SO(16)$ in $U(16) \supset SO(16)$. Keeping the $SO(6)$ pair structure, as chosen by Arima and Iachello, of $sd$IBM intact we will have four $S_+$ pairs, $S_+ = s^\dagger s^\dagger - d^\dagger \cdot d^\dagger \pm p^\dagger \cdot p^\dagger \pm f^\dagger \cdot f^\dagger$ giving $SU^{+,\pm,\pm}(1,1)$ and correspondingly four $SO^{+,\pm,\pm}(16)$ algebras. For each of the four choices, one can write down the $T^{E2}$ and $T^{E1}$ operators that transform as $T^0_0$ or $T^1_0$ w.r.t. $SU(1,1)$. In [24], $SU^{+,\pm,\mp}(1,1)$ is employed. Then, the $E2$ and $E1$ operators employed in [24, 29, 42] will be mixture of $T^0_0$ and $T^1_0$ w.r.t. to $SU^{+,\pm,\mp}(1,1)$. For example, the $E1$ operator used is,

$$T^{E1} = \alpha_{sp} \left(s^\dagger \tilde{p} + p^\dagger \tilde{s}\right)^1_\mu + \alpha_{pd} \left(p^\dagger \tilde{d} + d^\dagger \tilde{p}\right)^1_\mu + \alpha_{df} \left(d^\dagger \tilde{f} + f^\dagger \tilde{d}\right)^1_\mu.$$  

(47)

The first term in the operator will be $T^1_0$ and the remaining two terms will be $T^0_0$ w.r.t. $SU^{+,\pm,\mp}(1,1)$. However if we use $\alpha_{sp} \left(s^\dagger \tilde{p} - p^\dagger \tilde{s}\right)^1_\mu$ in the above, then the whole operator
will be $T_0^0$. It will be interesting to employ the $H_p = S_+S_-$ with $S_+$ given above (there will be four choices) in the analysis made in [29] and confront the data.

VIII. CONCLUSIONS

In this article an attempt is made to bring focus, bringing all known and new results to one place, to multiple multi-orbit pairing algebras in $j - j$ coupling shell model for identical nucleons and similarly, for identical boson systems described by multi-orbit interacting boson models such as $sd$, $sp$, $sdg$ and $sdpf$ IBM’s. The relationship between quasi-spin tensorial nature of one-body transition operators and the phase choices in the multi-orbit pair creation operator is derived for both identical fermion (described by shell model) and boson (described by interacting boson model) systems. These results are presented in Sections II and III for fermion systems and V for boson systems. As pointed out in these sections, some of the results here are known before for some special situations. In Section IV, results for the correlation coefficient between the pairing operator with different choices for phases in the generalized pair creation operator and realistic effective interactions are presented. It is found that the choice advocated by AM [6] gives maximum correlation though its absolute value is no more than 0.3. Particle number variation in electromagnetic transition strengths is discussed in Section VI.

Applications of multiple pairing algebras are briefly discussed in Section VII. As discussed in Section VII.A, drawing from the recent analysis by Maheswari and Jain [27, 28], shell model generalized seniority with phase choice advocated by AM appear to describe $B(E2)$ and $B(E1)$ data in Sn isotopes both for low-lying states and high-spin isomeric states. Though deviations from the results obtained using AM choice is a signature for multiple multi-orbit pairing algebras, direct experimental evidence for the multiple pairing algebras is not yet available.

Turning to interacting boson model description of collective states, imposing specific tensorial structure, with respect to pairing $SU(1,1)$ algebras, is possible as discussed with various examples in Section VII.B. It will be interesting to derive results for B(E2)’s (say in $sdg$ and $sdpf$ IBM’s) and B(E1)’s (in $sdpf$ IBM) with fixed tensorial structure for the transition operator but with wavefunctions that correspond to different $SU(1,1)$ algebras. Such an exercise was carried out before for $sd$IBM [32]. Also, with recent interest in $sdg$
[40] and sdpf [29, 42] IBM’s, it will be interesting to study quantum phase transitions and order-chaos transitions in these models, in a systematic way, employing Hamiltonians that interpolate the different pairing algebras in these models. Such studies for the simpler sd and sp IBM’s are available; see for example [33, 34, 43]. Construction of the Hamiltonian matrix for the interpolating Hamiltonians is straightforward as described briefly in Appendix-A. As an example, results for the spectra for a sdgIBM system are shown in Fig. 3.

Finally, going beyond multiple pairing algebras for identical fermion or boson systems, there are also multiple pairing algebras for fermions and bosons carrying internal degrees of freedom such as isospin and spin. Though these are identified [22, 23, 44], they are not studied in any detail till now. Similarly, there are multiple rotational SU(3) algebras both in shell model and IBM’s as discussed with some specific examples in [3, 31, 43, 45]. A systematic study of these multiple extended pairing algebras with internal degrees of freedom and multiple SU(3) algebras will be the topics of future papers.

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APPENDIX A

Let us consider identical bosons in \( r \) number of \( \ell \) orbits. Now, given the general pairing Hamiltonian

\[
H^G_p = \sum_{\ell} \epsilon_\ell \hat{n}^B_\ell + S^B_+ S^B_-;
\]

\[
S^B_+ = \sum_{\ell} x_\ell S^B_+(\ell), \quad S^B_-(\ell) = \frac{1}{2} b^\dagger_\ell \cdot b^\dagger_\ell,
\]

it will interpolate \( U(\mathcal{N}) \supset SO(\mathcal{N}) \supset \sum_\ell [U(\mathcal{N}_\ell) \oplus \text{pairing algebra}] \) for arbitrary values of \( x_\ell \)'s and \( \epsilon_\ell \)'s. Matrix representation for the Hamiltonian \( H^G_p \) is easy to construct by choosing the basis

\[
\Phi = \left| N^B_{\ell_1}, \omega^B_{\ell_1}, \alpha_{\ell_1}; N^B_{\ell_2}, \omega^B_{\ell_2}, \alpha_{\ell_2}; \ldots N^B_{\ell_r}, \omega^B_{\ell_r}, \alpha_{\ell_r} \right\rangle
\]

where \( \alpha_{\ell} \) are additional labels required for complete specification of the basis states (they play no role in the present discussion) and total number of bosons \( N^B = \sum_{\ell} N^B_{\ell} \). The first
term (one-body term) in $H_p^G$ is diagonal in the $\Phi$ basis giving simply $\sum_\ell \epsilon_\ell N_\ell^B$. The second term can be written as

$$S_+^B S_-^B = \left[ \sum_\ell (x_\ell)^2 S_+^B(\ell) S_-^B(\ell) \right] + \left[ \sum_{\ell_i \neq \ell_j} x_{\ell_i} x_{\ell_j} S_+^B(\ell_i) S_-^B(\ell_j) \right]. \tag{A-3}$$

In the basis $\Phi$, the first term is diagonal and its matrix elements follow directly from Eq. (33) and it is the second term that mixes the basis states $\Phi$'s. The mixing matrix elements follow from,

$$S_+^B(\ell_i) S_-^B(\ell_j) \langle N_{\ell_i}^B, \omega_{\ell_i}^B, \alpha_{\ell_i}; N_{\ell_j}^B, \omega_{\ell_j}^B, \alpha_{\ell_j} \rangle = \frac{1}{4} \sqrt{(N_{\ell_i}^B - \omega_{\ell_i}^B) (2\Omega_{\ell_i}^B + N_{\ell_i}^B + \omega_{\ell_i}^B - 2)} \left( N_{\ell_i}^B - \omega_{\ell_i}^B + 2 \right) \left( 2\Omega_{\ell_i}^B + N_{\ell_i}^B + \omega_{\ell_i}^B \right) \tag{A-4}$$

It is important to note that the action of $H_p^G$ on the basis states $\Phi$ will not change the $\omega_\ell^B$ quantum numbers. For boson numbers not large, it is easy to apply Eqs. (A-3) and (A-4) and construct the $H_p^G$ matrices. It is easy to extend the above formulation to fermion systems and also for the situation where two or more orbits are combined to a larger orbit. The later, for example for $sdg$IBM gives $U(15) \supset SO(15) \supset SO_{sd}(6) \oplus SO_g(9)$ and $U(15) \supset [U(6) \supset SO(6)] \oplus [U(9) \supset SO(9)]$ interpolation [similarly with $SO_{dg}(14)$ and $SO_{sg}(10)$ algebras]. Finally, it is also possible to use the exact solution for the generalized pairing Hamiltonians as given in [46] which is more useful for large boson numbers.

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