Finitely Many Dirac-Delta Interactions on Riemannian Manifolds

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Abstract

This work is intended as an attempt to study the non-perturbative renormalization of bound state problem of finitely many Dirac-delta interactions on Riemannian manifolds, S², H², and H³. We formulate the problem in terms of a finite dimensional matrix, called the characteristic matrix Φ. The bound state energies can be found from the characteristic equation Φ(−ν²)A = 0. The characteristic matrix can be found after a regularization and renormalization by using a sharp cut-off in the eigenvalue spectrum of the Laplacian, as it is done in the flat space, or using the heat kernel method. These two approaches are equivalent in the case of compact manifolds. The heat kernel method has a general advantage to find lower bounds on the spectrum even for compact manifolds as shown in the case of S². The heat kernels for H², and H³ are known explicitly, thus we can calculate the characteristic matrix Φ. Using the result, we give lower bound estimates of the discrete spectrum.

1 Introduction

It is well-known that the exactly solvable Dirac-delta interactions on the plane and 3-dimensional Euclidean space in quantum mechanics give rise to some unphysical results for physical observables, i.e., bound state energy and scattering cross section are infinite and the problem is said to be ultraviolet divergent. Nevertheless, there is a systematic way to dispense with these infinities by means of a so-called regularization and renormalization, which is first introduced in quantum field theory for the same reason. This problem constitutes an analytical example of regularization and renormalization in quantum mechanics so that it helps us to understand and deal with it in a more elementary context rather than field theory and it has been studied in the literature from several point of views [11] - [13]. Moreover, a single point interaction in two dimensional flat space is an instructive example of dimensional transmutation in non-relativistic quantum mechanics [3, 14, 15, 16]. That is, the original Hamiltonian does not contain any intrinsic energy scale due to the dimensionless coupling constant in natural units. Nevertheless,
a new parameter $\mu^2$, which species the bound state energy, must be introduced after the renormalization procedure which then fixes the energy scale of the system. (A detailed discussion of dimensional transmutation in nonrelativistic quantum mechanics is given in a relatively recent article [16]).

In this study, we consider a bound state problem in which a non-relativistic particle living in a Riemannian manifold (in particular $S^2$, $H^2$, and $H^3$) interacts with finitely many Dirac-delta interactions. Similar to the corresponding bound state problem on $\mathbb{R}^2$ and $\mathbb{R}^3$, we encounter divergences in this case as well. The main purpose of this paper is to show how to non-perturbatively regularize and renormalize the problem by means of heat kernel (even in the case where we do not have an explicit expression for it). After the renormalization, we estimate a lower bound for the ground state energy for each particular Riemannian manifold. This problem on two dimensional Riemannian manifolds, such as $S^2$ and $H^2$ also displays a kind of dimensional transmutation [16], where new energy scales different from the intrinsic energy scales of the system appear after the renormalization. We will briefly discuss it in sections 3.1 and 4.2.

Many body version of this problem on $\mathbb{R}^2$ and $\mathbb{R}^3$ is known as the formal non-relativistic limit of the $\lambda \phi^4$ scalar field theory in (2+1) and (3+1) dimensions. All these are extensively discussed in [17]. Our primary motivation here is coming from the question how the renormalization method for the singular interactions in quantum mechanics would be performed on Riemannian manifolds, hoping that this may help us to understand the problem in the realm of quantum field theory. However, we shall postpone the discussion of the many body extension of it for future work and study first the one-particle Schrödinger problem.

The paper is organized as follows. In section 2 we first define the bound state problem on compact and connected Riemannian manifolds and reformulate the problem in terms of a finite dimensional matrix $\Phi$, which we will call as the characteristic matrix [17]. Then, we emphasize the relation of the characteristic matrix with the corresponding spectral functions, resolvent and heat kernel. This allows us to reformulate the renormalization in terms of heat kernel. After that we continue to the discussion in the following sections by working out concrete examples. In section 3 we consider the delta interaction problem on $S^2$ as an example for compact and connected manifolds. Considering the properties of the operator $\Phi$ and using some properties and upper bound estimates of the heat kernel, Geršgorin theorem allows us to estimate a lower bound for the ground state energy of the system. In section 4 we apply the similar methodology, developed in the section of heat kernel method for $S^2$, to the non-compact manifolds, such as $H^2$ and $H^3$ and show that the methods developed for compact manifolds work for some particular non-compact manifolds as well. Therefore, we renormalize the problem on hyperbolic spaces and give estimates on the ground state energy of each system.

2 Renormalization of Finitely Many Dirac-Delta Interactions on Compact and Connected Riemannian Manifolds $(M, g)$

The canonical quantization on non-trivial manifolds is known to have some ambiguities in quantum mechanics. For the path integral approach to the quantum system, the ambiguity in the canonical formalism is replaced by the undetermined parameter $\lambda$ and it can take various possible values [18]. We remove this term for simplicity in all our examples, in which the
curvature term is constant and it corresponds to an overall shift in energy levels so that we can safely set \( \lambda \) to be zero.

Now, we consider a non-relativistic point particle living on a Riemannian manifold \( M \) interacting with a finite number of delta interactions located on the manifold and study bound states of the problem. We first investigate the delta interactions on a compact and connected Riemannian manifold \((M, g)\) without boundary, of dimension \( D = 2, 3 \) with the Riemannian metric \( g \). The kinetic energy operator on Riemannian manifold \((M, g)\) is just the Laplace-Beltrami operator or simply Laplacian, which is defined, in local coordinates \( x \equiv (x^1, ..., x^D) \) for a neighborhood in the manifold, as follows:

\[
\triangle_g = -\frac{1}{\sqrt{\det g}} \sum_{\alpha, \beta = 1}^D \frac{\partial}{\partial x^\alpha} \left( g^{\alpha\beta} \sqrt{\det g} \frac{\partial}{\partial x^\beta} \right),
\]

where \( g_{\alpha\beta} \) is the metric tensor and \( g \equiv (g_{\alpha\beta}) \). We shall usually denote the Laplacian as \( \triangle_g \) to specify which metric structure on Riemannian manifold it is associated with.

The spectral theorem \([19, 20]\) states that the eigenvalue problem \( \triangle_g \phi_l = \lambda_l \phi_l \) on a compact and connected Riemannian manifold \((M, g)\) has a complete orthonormal system of \( C^\infty \) eigenfunctions \( \phi_0, \phi_1, \ldots \) in \( L^2(M) \) and the spectrum \( \text{Spec} (\triangle_g) \equiv \text{Spec} (M, g) = \{ \lambda_l \} = \{ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \} \), with \( \lambda_l \) tending to infinity as \( l \to \infty \). As a corollary of this theorem, the Laplacian on \((M, g)\) provides us with all the tools of Fourier analysis, so that we can expand any “sufficiently good” function \( \psi(x) \) on \( M \) in terms of the complete orthonormal eigenfunctions \( \phi_l(x) \)

\[
\psi(x) = \sum_{l \geq 0} C_l \phi_l(x),
\]

with the normalization

\[
\int_M \phi_l(x) \phi_l^*(x) \sqrt{\det g} \, dx^1 \wedge ... \wedge dx^D = \delta_{ll'},
\]

where \( C_l \)'s are expansion coefficients. Note that extra labels in the eigenfunction expansion must be taken into account if the problem admits degeneracy. Delta functions on \( M \) can also assumed to be represented by these eigenfunctions

\[
\delta^D(x - a_i) = \sum_{l \geq 0} \phi_l(x) \phi_l^*(a_i),
\]

with \( a_i \in M \) and \( \delta^D(x - a_i) \) being the \( D \)-dimensional normalized delta function at point \( a_i \),

\[
\int_M \delta^D(x - a_i) \sqrt{\det g} \, dx^1 \wedge ... \wedge dx^D = 1.
\]

A typical Hamiltonian operator in quantum theory consists of a kinetic term, the Laplacian \( \triangle_g \) with the factor \( \hbar^2/2m \), and a potential function of position, attractive delta interactions in our problem. The time-independent Schrödinger equation on \( M \) for the bound states of a particle under the influence of \( N \) attractive delta interactions reads

\[
\left[ \frac{\hbar^2}{2m} \triangle_g - \sum_{i=1}^N g_i \delta^D(x - a_i) \right] \psi(x) = -\nu^2 \psi(x),
\]
where \( g_i \in \mathbb{R}^+ \) is the strength of the delta interaction at \( a_i \) and \(-\nu^2\) is the bound state energy of the system. If we substitute (2) and (3) into the Schrödinger equation, it yields

\[
\sum_{l \geq 0} \left[ \frac{\hbar^2}{2m} \lambda_l C_l - \sum_{i=1}^{N} A_i g_i \phi_l^*(a_i) + \nu^2 C_l \right] \phi_l(x) = 0,
\]

where \( A_i \equiv \psi(a_i) \) for simplicity of notation. The fact that \( \phi_l \)'s form a complete orthonormal system allows us to write \( C_l \) in terms of them:

\[
C_l = \frac{1}{\frac{\hbar^2}{2m} \lambda_l + \nu^2} \sum_{i=1}^{N} A_i g_i \phi_l^*(a_i).
\]

Substituting (5) into the definition of \( A_i \)

\[
A_i = \sum_{j=1}^{N} A_j g_j \sum_{l \geq 0} \frac{\phi_l(a_i) \phi_l^*(a_j)}{\frac{\hbar^2}{2m} \lambda_l + \nu^2},
\]

and grouping the \( A_i \) terms we find

\[
\left[ g_i^{-1} - \sum_{l \geq 0} \frac{|\phi_l(a_i)|^2}{\frac{\hbar^2}{2m} \lambda_l + \nu^2} \right] A_i - \sum_{j=1}^{N} \left[ \frac{g_j}{g_i} \sum_{l \geq 0} \frac{\phi_l(a_i) \phi_l^*(a_j)}{\frac{\hbar^2}{2m} \lambda_l + \nu^2} \right] A_j = 0 .
\]

The observation that the preceding equation is linear in \( A_i \) permits us to write it naturally as a matrix equation

\[
\Phi(-\nu^2) A = 0 ,
\]

where \( \Phi(-\nu^2) \) is called the characteristic matrix and defined as

\[
\Phi_{ij}(-\nu^2) = \begin{cases} 
 g_i^{-1} - \sum_{l \geq 0} \frac{|\phi_l(a_i)|^2}{\frac{\hbar^2}{2m} \lambda_l + \nu^2} & \text{if } i = j \\
 -\frac{g_j}{g_i} \sum_{l \geq 0} \frac{\phi_l(a_i) \phi_l^*(a_j)}{\frac{\hbar^2}{2m} \lambda_l + \nu^2} & \text{if } i \neq j .
\end{cases}
\]

As we shall see below that the resolvent is intimately related to it and this allows us to state that the equation \( \det \Phi(-\nu^2) = 0 \) gives the bound state energies of our problem. In other words, this equation is considered to be the determining equation of the ground state energy. Unfortunately, this nontrivial eigenvalue problem can not be solved analytically, that is, we can not obtain an exact expression for the bound state energy for arbitrary \( N \) since the characteristic matrix depends nonlinearly on the bound state energy. Indeed, the problem is even worse than that, because we have not a finite expressions in the matrix elements of \( \Phi_{ij}(-\nu^2) \). Fortunately, there exist a way to redefine the problem so that the physical observables yield finite values with the help of regularization and renormalization. Before introducing this procedure for our problem, it would be good to review first the problem in flat spaces. The infinite sums in the characteristic matrix on \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) is then replaced by integrals. The idea in that case is to take Fourier transform of the wave function

\[
\psi(x) = \int \tilde{\psi}(k) e^{ik.x} \frac{d^Dk}{(2\pi)^D} ,
\]
and substitute into the Schrödinger equation. Then we find that the diagonal part of the characteristic matrix is

\[
\frac{1}{g_i} - \frac{1}{(2\pi)^D} \int \frac{d^D k}{k^2 + \nu^2},
\]

where \( D = 2, 3 \). This integral does not converge as it stands. The well-known method to remove the divergence is to put a cut-off \( \Lambda \) to the integral’s upper limit and consider the equation as a determining equation of bound state energy for a given coupling constant \( g \). If this regularization is performed, we realize that as the cut-off goes to infinity, ground state energy becomes divergent. In order to get a physically acceptable result, one assumes that the coupling constant depends on this cut-off and performs the limit \( \Lambda \to \infty \) in such a way that bound state energy remains finite. These infinities should be removed properly since all the physical observables are measured experimentally as finite quantities. The cut-off dependence of the coupling constant is chosen as

\[
\frac{1}{g_i(\Lambda)} = \frac{1}{(2\pi)^D} \int_{|k|<\Lambda} \frac{d^D k}{k^2 + \mu^2_i}.
\]

The determination of this coupling constant is called renormalization. Now, we follow the same idea to remove the divergence from our problem. By using Weyl’s asymptotic formula \[21\], one expects that the diagonal term \[ \sum_{l \geq 0} \frac{|\phi_l(a_i)|^2}{2m^2 \lambda_l + \nu^2} \] in the above matrix does not converge and this will be explicitly seen for a particular manifold \( S^2 \). For a general compact manifold, we introduce cut-off to the upper bound of the infinite sum and choose the coupling constant as

\[
g_{i}^{-1}(\Lambda) = \sum_{l=0}^{\Lambda} \frac{|\phi_l(a_i)|^2}{2m^2 \lambda_l + \mu^2_i},
\]

where \(-\mu^2_i\) is the measured binding energy to a single delta interaction. Then, we take the limit \( \Lambda \to \infty \)

\[
\lim_{\Lambda \to \infty} \sum_{l=0}^{\Lambda} \frac{|\phi_l(a_i)|^2}{2m^2 \lambda_l + \mu^2_i} - \sum_{l=0}^{\Lambda} \frac{|\phi_l(a_i)|^2}{2m^2 \lambda_l + \nu^2},
\]

and this should give us a finite result in two and three dimensions. Hence, the divergence has been removed and bound state energy becomes finite. A rigorous proof of this is not trivial, so we will stay at a heuristic level and study special cases only.

As we will show in the next subsection, the heat kernel is intimately related to the characteristic matrix \( \Phi \) and this relation helps us to see easily which part of the matrix is divergent or convergent and then how to renormalize the problem non-perturbatively. Furthermore, heat kernel is especially very helpful to remove the divergences for our problem on non-compact manifolds, as we shall discuss in section \[4\]. We will see that the above method can easily be extended to find the renormalized resolvent of the singular Hamiltonian.

### 2.1 The Relation of Matrix \( \Phi \) with Heat Kernel and Resolvent

The resolvent (or Green’s function) and heat kernel play very essential role in establishing the connection between spectral properties of the operator and corresponding geometrical notions. Up to now, we have been dealing with a matrix \( \Phi \), and do not refer to resolvent and heat kernel. In order to see the relation between the matrix \( \Phi \) and heat kernel we consider the separable
Hamiltonians $H = H_0 - \sum_{i=1}^N g_i |f_i\rangle \langle f_i|$\], where $|f_i\rangle$ is a particular Dirac ket. We work out the resolvent formula of $H$ in terms of $H_0$ and assume that the two Dirac kets $|\psi\rangle$ and $|\chi\rangle$ are related in such a way that the equality $(H - z)|\psi\rangle = |\chi\rangle$ is satisfied. Then, we have

$$\left[ H_0 - z - \sum_{j=1}^N g_j |f_j\rangle \langle f_j| \right] |\psi\rangle = |\chi\rangle, \tag{11}$$

assuming complex number $z \not\in \text{Spec}(H_0)$. Acting the operator $(H_0 - z)^{-1}$ on both sides and projecting it onto $\langle f_i\rangle$, we obtain

$$\sum_{j=1}^N \Phi_{ij}(z) |f_j\rangle \langle f_j| = g_i^{-1} \langle f_i| (H_0 - z)^{-1} |\chi\rangle, \tag{12}$$

where we define a matrix $\Phi_{ij}(z)$ as

$$\Phi_{ij}(z) = \begin{cases} g_i^{-1} - \langle f_i| (H_0 - z)^{-1} |f_i\rangle & \text{if } i = j \\ -g_j g_i^{-1} \langle f_i| (H_0 - z)^{-1} |f_j\rangle & \text{if } i \neq j. \end{cases} \tag{12}$$

After a little algebra, it is evident that

$$(H - z)^{-1} = (H_0 - z)^{-1} + (H_0 - z)^{-1} \sum_{i,j=1}^N |f_i\rangle \Phi_{ij}(z)^{-1} \langle f_j| (H_0 - z)^{-1}, \tag{13}$$

as long as $\Phi_{ij}(z)^{-1}$ exists. Such formulae were extensively discussed in problems associated with self-adjoint extensions of operators, notably by Krein and his school, and also for such singular interactions in flat spaces \[12, 22\]. Therefore, our problem can also be considered as a kind of self-adjoint extension of the free Hamiltonian. It is defined through regulating (or controlling) the behavior of the wave function in the vicinity of these interaction points.

If we take the matrix element of (13) by projecting on to the Dirac kets $\langle x|\rangle$ and $|y\rangle$, we have found the resolvent kernel $R(x, y|z) \equiv \langle x|(H - z)^{-1}|y\rangle$ corresponding to (11)

$$R(x, y|z) = R_0(x, y|z) + \int dx' dy' R_0(x, x'|z) \left[ \sum_{i,j=1}^N f_i(x') \Phi_{ij}(z)^{-1} f_j(y') \right] R_0(y', y|z)$$

$$= R_0(x, y|z) + \sum_{i,j=1}^N \left[ \int dx' R_0(x, x'|z) f_i(x') \right] \Phi_{ij}(z)^{-1} \left[ \int dy' R_0(y', y|z) f_j(y') \right].$$

By choosing the functions $f_i(x)$’s as bump functions centered at $x = a_i$ such that the sequences of the functions admit the limit $f_i(x) \to \delta^D(x - a_i)$ (in the appropriate topology), it turns out that

$$R(x, y|z) = R_0(x, y|z) + \sum_{i,j=1}^N R_0(x, a_i|z) \Phi_{ij}(z)^{-1} R_0(a_j, y|z). \tag{14}$$

\[1\]There is no confusion in notation because we will see that this matrix $\Phi$ is exactly the same matrix considered in the previous sections.
The important point to note here is the relation between the resolvent operator, defined on an infinite dimensional space and the characteristic matrix, defined on a finite dimensional space. This allows us to find the bound state spectrum of the separable Hamiltonian operator in a compact manifold. Hence, the matrix $\Phi$ is written in terms of the heat kernel in the following way,

$$\Phi_{ij}(z) = \begin{cases} 
    g_i^{-1} - \frac{1}{\hbar} \int_0^\infty e^{\frac{\hbar}{2m} t} K_t(a, a) dt & \text{if } i = j \\
    -\frac{g_i}{g_j} \frac{1}{\hbar} \int_0^\infty e^{\frac{\hbar}{2m} t} K_t(a, a) dt & \text{if } i \neq j.
\end{cases}$$  

(17)

The matrix $\Phi_{ij}$ is exactly the same matrix mentioned in the previous sections. This can be shown easily from the spectral theorem \cite{19} for compact manifolds

$$K_t(a, a) = \sum_{l \geq 0} e^{-\frac{\hbar^2}{4m} \lambda_l^2 t} \phi_l(a) \phi_l^*(a),$$

(18)

which converges uniformly on $M \times M$ for each $t > 0$:

$$\langle f_i | (H_0 - z)^{-1} | f_j \rangle \to \int_0^\infty e^{\frac{\hbar}{2m} \lambda_l^2 t} K_t(a, a) dt = \sum_{l \geq 0} \phi_l(a) \phi_l^*(a) \int_0^\infty e^{-\frac{\hbar^2}{4m} \lambda_l^2 t} \frac{dt}{\hbar}$$

$$= \sum_{l \geq 0} \frac{\phi_l(a) \phi_l^*(a)}{\frac{\hbar^2}{2m} \lambda_l - z},$$

where summation and integral are interchanged since summation converges uniformly. This is the same result for $z = -\nu^2$ that we already obtained for non-diagonal part of the characteristic matrix in the section 2 One can understand how the non-diagonal part of it in \eqref{17} is convergent by using the smooth behaviour of the heat kernel and the integral $\int_0^\infty e^{\frac{\hbar}{2m} t} K_t(a, a) dt$ is convergent for $a_i \neq a_j$. However, the asymptotic behaviour of the heat kernel as $t \to 0^+$ for every point $x$ on a compact manifold $M$ \cite{19} is given by

$$K_t(x, x) \sim \left(4\pi \frac{\hbar t}{2m} \right)^{-D/2} \sum_{k=0}^\infty u_k(x, x) \left(\frac{\hbar t}{2m}\right)^k,$$ 

(19)
where $D$ is the dimension of the manifold and the $u_k(x, x)$ are functions given in terms of the curvature tensor of $M$ and its covariant derivatives at the point $x$. This result shows that diagonal part of the heat kernel as $t \to 0^+$ for $D = 2, 3$ leads to a divergence since $u_0(x, x) = 1$ (there is no infinities for $D = 1$ as it can be easily realized). In other words, the sum in the diagonal term in $\Phi$ is divergent while the sum in the non-diagonal term is convergent. However, we have already shown that bound state energies are related to the characteristic matrix, i.e., $\det \Phi(z) = 0$ contains information about bound states. If some of the elements of the characteristic matrix have infinities, it is impossible to get sensible bound state energies for our problem. Before establishing the renormalization of our problem with the help of heat kernel, we must indicate why this problem occurs. Although the delta interactions may approximately describe a system in which a particle interacting with a point-like centers when its de Broglie wavelength is large compared to the typical range of a potential, we have not encountered in nature this type of contact interaction. This means that the substituting the Dirac-Delta interactions into the Hamiltonian for $D = 2, 3$ directly is not a proper way. Therefore, we must modify our problem such that it has a finite range and then consider the zero range limit. In our renormalization method with heat kernel, short range is replaced with the short time as we will see.

We introduce a small constant $\epsilon$, in the lower limit of the integral. We then take the limit as the cut-off $\epsilon$ goes to zero in such a way that the experimentally measured ground state energy remains finite. This requires that some quantities in the problem, e.g. coupling constant, should have a cut-off dependence in a definite way. For our problem, we naturally choose

$$g_i^{-1}(\epsilon, \mu_i) = \frac{1}{\hbar} \int_\epsilon^{\infty} e^{-\frac{\nu^2 t}{\hbar}} K_t(a_i, a_i) \, dt. \quad (20)$$

After performing the limit $\epsilon \to 0$, we have the renormalized characteristic matrix

$$\Phi_{ij}(z) = \begin{cases} \frac{1}{\hbar} \int_0^\infty K_t(a_i, a_i) \left[ e^{-\frac{\nu^2 t}{\hbar}} - e^{\frac{\nu^2 t}{\hbar}} \right] dt & \text{if } i = j \\ -\frac{1}{\hbar} \int_0^\infty e^{\frac{\nu^2 t}{\hbar}} K_t(a_i, a_j) dt & \text{if } i \neq j \end{cases}, \quad (21)$$

where $\Re(z) < 0$ and $\Phi_{ij}(z)$ can be analytically continued to its largest set in the entire complex plane. One can naturally ask whether the renormalization performed with heat kernel is compatible with the one introduced in section 2. The answer is affirmative and one can easily show that the cut-off $\Lambda$ for the infinite sum introduced in section 2 corresponds to the cut-off $\epsilon$ for the lower bound of integral in the heat kernel method. This can be realized easily by using the spectral theorem in the diagonal part of equation (21) and taking $z = -\nu^2$:

$$g_i^{-1}(\epsilon, \mu_i) = \frac{1}{\hbar} \int_\epsilon^{\infty} e^{-\frac{\nu^2 t}{\hbar}} \sum_{l \geq 0} e^{-\frac{\nu^2}{2\hbar^2} \lambda_l} [\phi_l(a_i) \phi_l^*(a_i)] dt$$

$$= g_i^{-1}(\epsilon, \mu_i) - \frac{1}{\hbar} \sum_{l \geq 0} \phi_l(a_i) \phi_l^*(a_i) \int_\epsilon^{\infty} e^{-\frac{\nu^2 t}{\hbar}} e^{-\frac{\nu^2}{2\hbar^2} \lambda_l} [\phi_l(a_i)] dt, \quad (22)$$

where we have used the uniform convergence of the sum. Now, in order to remove the divergence, we can naturally choose the coupling constant as

$$g_i^{-1}(\epsilon, \mu_i) = \frac{1}{\hbar} \sum_{l \geq 0} \phi_l(a_i) \phi_l^*(a_i) \int_\epsilon^{\infty} e^{-\frac{\nu^2 t}{\hbar}} e^{-\frac{\nu^2}{2\hbar^2} \lambda_l} [\phi_l(a_i)] dt. \quad (23)$$
Then, we have

\[
\lim_{\varepsilon \to 0} \left\{ \frac{1}{\hbar} \sum_{l \geq 0} \phi_l(a_i) \phi^*_l(a_i) \left[ \int_{\epsilon}^{\infty} e^{-\frac{\nu^2 t}{\hbar}} e^{-\frac{\hbar^2}{2m} \lambda_l} dt - \int_{\epsilon}^{\infty} e^{-\frac{\nu^2 t}{\hbar}} e^{-\frac{\hbar^2}{2m} \lambda_l} dt \right] \right\} = \frac{1}{\hbar} \sum_{l \geq 0} \frac{[\mu^2 - \mu^2_l]}{\left[ \frac{\hbar^2}{2m} \lambda_l + \mu^2 \right]} \phi_l(a_i) \phi^*_l(a_i),
\]

which is the same result we would have obtained by the eigenfunction expansion by introducing a cut-off \( \Lambda \) (equation (10)). After finding the renormalized characteristic matrix, the resolvent which is the same result we would have obtained by the eigenfunction expansion by introducing normalization constant can be found easily

\[
R(x, y|z) = R_0(x, y|z) + \sum_{i,j=1}^{N} R_0(x, a_i|z) \Phi_{ij}(z)^{-1} R_0(a_j, y|z),
\]

where

\[
R_0(x, y|z) = \frac{1}{\hbar} \int_{0}^{\infty} e^{\frac{\nu^2}{\hbar} t} K_i(x, y) dt.
\]

Once we have given the resolvent of an operator, all the information about the operator is contained in it. Nevertheless, it is instructive to check that the wave functions can also be obtained and they are normalizable. We write the normalized wave function with a cut-off \( \Lambda \) contained in it. Nevertheless, it is instructive to check that the wave functions can also be obtained and they are normalizable. We write the normalized wave function with a cut-off \( \Lambda \) and then take the limit \( \Lambda \to \infty \), this way we will not get a vanishing wave function. So the normalization constant can be found easily

\[
|C(\Lambda)|^2 = \sum_{i,j=1}^{N} g_i(\Lambda) g_j(\Lambda) A^*_i(\Lambda) A_j(\Lambda) \int d^D x \sqrt{g} \sum_{l,l'=0}^{\Lambda} \phi_l(a_i) \phi^*_l(x) \phi^*_l(a_j) \phi_l(x) \frac{\hbar^2}{2m} \lambda_l + \mu^2 \frac{\hbar^2}{2m} \lambda_{l'} + \mu^2 \]

\[
= \sum_{i,j=1}^{N} g_i(\Lambda) g_j(\Lambda) A^*_i(\Lambda) A_j(\Lambda) \sum_{l=0}^{\Lambda} \frac{\phi_l(a_i) \phi^*_l(a_j)}{\left( \frac{\hbar^2}{2m} \lambda_l + \mu^2 \right)^2}.
\]

One expects from the Weyl asymptotic formula that the wave function is not normalizable if we are on a space of dimension bigger than three. Moreover, we can see that the summation over the eigenmodes is exactly the derivative of \( \Phi(-\nu^2) \) with respect to \( \nu \), hence we get:

\[
|C(\Lambda)|^2 = \frac{1}{2\nu} \sum_{i,j=1}^{N} g_j(\Lambda)^2 A^*_j(\Lambda) \frac{\partial \Phi_{ij}(\Lambda, -\nu^2)}{\partial \nu} A_j(\Lambda).
\]

Performing the limit \( \Lambda \to \infty \), the properly normalized wave function of \( n^{th} \) state becomes

\[
\psi_n(x) = \sqrt{2\nu_n} \left[ \sum_{r,s=1}^{N} A^*_r(\nu_n) \frac{\partial \Phi_{rs}(-\nu^2)}{\partial \nu} \bigg|_{\nu = \nu_n} A_s(\nu_n) \right]^{-\frac{1}{2}} \sum_{l \geq 0} \sum_{i=1}^{N} A_i(\nu_n) \phi^*_i(a) \phi_l(x) \frac{\hbar^2}{2m} \lambda_l + \nu^2_n,
\]

where \( \nu_n \) is the \( n^{th} \) root of the energy equation \( \det \Phi(-\nu^2) = 0 \). This can further be simplified
to an expression in terms of the heat kernel

\[
\psi_n(x) = \sqrt{2\nu_n} \left[ \sum_{r,s=1}^{N} A_r^\ast(\nu_n) \frac{\partial \Phi_{rs}(-\nu^2)}{\partial \nu} \bigg|_{\nu=\nu_n} A_s(\nu_n) \right]^{-\frac{1}{2}} \times \sum_{l \geq 0} \sum_{i=1}^{N} A_i(\nu_n) \phi_i^\ast(a_i) \phi_l(x) \int_0^\infty e^{-\frac{t}{\hbar} \left( \frac{\nu_n^2}{2m} + \nu_n^2 \right)} \frac{dt}{\hbar} \right]_{\nu=\nu_n} A_s(\nu_n) \right]^{-\frac{1}{2}} \int_0^\infty e^{-\frac{t}{\hbar} \nu_n^2} \sum_{i=1}^{N} A_i(\nu_n) K_t(a_i, x) \frac{dt}{\hbar},
\]

in which one can easily see that \(\psi_n(x)\) is finite.

3 Finitely Many Dirac-Delta Interactions on \(S^2\)

Since the simplest and one of the most familiar compact manifolds is the sphere \(S^2\), we shall work out the problem of point interactions on a sphere as a concrete example. Suppose that point interactions are located at the points given by the local coordinates \((\theta_i, \phi_i)\) on a sphere of radius \(R\). Then, the Schrödinger equation for the bound states of a particle living on the sphere under the influence of \(N\) attractive delta interactions becomes

\[
\left[ \frac{\hbar^2}{2m} \Delta_{S^2} - \sum_{i=1}^{N} g_i \delta^2(\theta - \theta_i, \phi - \phi_i) \right] \psi = -\nu^2 \psi, \tag{30}
\]

where \(\Delta_{S^2}\) is Laplacian on the sphere in spherical coordinates

\[
\Delta_{S^2} = -\frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \tag{31}
\]

and \(\delta^2(\theta - \theta_i, \phi - \phi_i) = \frac{\delta(\theta - \theta_i) \delta(\phi - \phi_i)}{R^2 \sin^2 \theta} \) is the two-dimensional delta function on the sphere centered at \((\theta_i, \phi_i)\). It is well known that spherical harmonics \(Y_l^m\) are eigenfunctions of the Laplacian \(\Delta_{S^2}\) with the eigenvalues \(l(l + 1)/R^2\) and form a complete orthonormal basis on \(S^2\). In order to be consistent with the standard normalization of spherical harmonics, we choose \(\phi_{lm} = \frac{Y_l^m}{\sqrt{R}}\). From the following identity

\[
\sum_{m=-l}^{l} Y_l^m(\theta_i, \phi_i) Y_l^{m^\ast}(\theta_j, \phi_j) = \frac{2l + 1}{4\pi} P_l \left( \cos \theta_i \cos \theta_j + \cos(\phi_i - \phi_j) \sin \theta_i \sin \theta_j \right) = \frac{2l + 1}{4\pi} P_l \left( 1 - \frac{d_{ij}^2}{2} \right),
\]

\(d_{ij}^2 = 1 - \cos \theta_i \cos \theta_j - \cos(\phi_i - \phi_j) \sin \theta_i \sin \theta_j\).
where \( d_{ij} = \frac{d_{ij}}{R} = |\hat{r}_i - \hat{r}_j| \in [0,2] \) being rescaled distance between point centers with radius of the sphere \( R \), the matrix \( \Phi_{ij}(-\nu^2) \) in (11) becomes

\[
\Phi_{ij}(-\nu^2) = \begin{cases} 
  g_i^{-1} - \frac{1}{4\pi R^2} \sum_{l \geq 0} \frac{2l + 1}{2mR^2} l(l+1) + \nu^2, & i = j \\
  -\frac{g_j}{g_i^4\pi R^2} \sum_{l \geq 0} \frac{2l + 1}{2mR^2} l(l+1) + \nu^2 P_l \left(1 - \frac{d_{ij}^2}{2}\right), & i \neq j.
\end{cases}
\]

(32)

It follows easily from the Cauchy-MacLaurin integral test that the infinite sum

\[
\frac{1}{4\pi R^2} \sum_{l \geq 0} \frac{2l + 1}{2mR^2} l(l+1) + \nu^2
\]

is divergent. To get a sensible results for our problem, we must modify our original problem as outlined in section 2. Therefore, considering our problem in the light of this method, we first define the coupling constant \( g_i \) as a function of the parameter \( \Lambda \) (cut-off). Then, by choosing \( g_i^{-1}(\Lambda)'s naturally

\[
g_i^{-1}(\Lambda) = \frac{1}{4\pi R^2} \sum_{l = 0}^{\Lambda} \frac{2l + 1}{2mR^2} l(l+1) + \mu_i^2,
\]

where \( \mu_i \) is experimentally measured value of bound state energy for the single delta interaction and taking the limit \( \Lambda \to \infty \) of the difference, we have obtained

\[
\lim_{\Lambda \to \infty} \left[ \frac{1}{4\pi R^2} \sum_{l = 0}^{\Lambda} \frac{2l + 1}{2mR^2} l(l+1) + \mu_i^2 - \frac{1}{4\pi R^2} \sum_{l = 0}^{\Lambda} \frac{2l + 1}{2mR^2} l(l+1) + \nu^2 \right]
\]

\[
\xrightarrow[\Lambda \to \infty]{\frac{1}{4\pi R^2 \mu_R^2}} \left[ \phi \left(\frac{\mu_i}{\mu_R}\right) - \phi \left(\frac{\nu}{\mu_R}\right) \right],
\]

where \( \mu_R^2 \equiv \frac{\hbar^2}{2mR^2} \). The function \( \phi \) here is defined as

\[
\phi(x) \equiv \frac{1}{x^2} - H_{\frac{1}{2}} - \sqrt{\frac{1}{4} - x^2} - H_{\frac{1}{2} + \sqrt{\frac{1}{4} - x^2}}, \quad x \in \mathbb{R}^+,
\]

where \( H \)'s are the harmonic numbers, commonly defined on integers as \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) and can be extended by analytical continuation to its largest domain in the entire complex plane as \( H_z = \psi(z + 1) + \gamma \), where \( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) being the digamma function and \( \gamma \) being the Euler-Mascheroni constant. The digamma function has several useful integral representations [23], some of which are

\[
\psi(z) = \int_{0}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right) dt, \quad (33)
\]

\[
\psi(z) = \log z + \int_{0}^{\infty} \left( \frac{1}{1 - e^{-t}} + \frac{1}{t} - 1 \right) e^{-zt} dt, \quad (34)
\]

where \( \Re(z) > 0 \) and these can be useful for the estimates of its upper and lower bounds. Due to the Schwarz reflection principle of harmonic numbers (\( H_z = H_{\bar{z}} \)), the function \( \phi(x) \) is real.
valued ($\phi \in \mathbb{R}$) for all $x \in \mathbb{R}^+$. It is also easy to check $\lim_{\Lambda \to \infty} \frac{g_j(\Lambda)}{g_j(\Lambda)} \to 1$ in the non-diagonal part of (32), simply because of their same form of the divergence. Then, the renormalized matrix $\Phi(-\nu^2)$ for bound states can be eventually written as

$$
\Phi_{ij}(-\nu^2) = \frac{1}{4\pi R^2 \mu_R^2} \begin{cases}
\phi \left( \frac{\mu_i}{\mu_R} \right) - \phi \left( \frac{\nu}{\mu_R} \right) & i = j \\
- \sum_{l \geq 0} \frac{2l + 1}{l(l+1) + \mu_R^2} P_l \left( 1 - \frac{d_{ij}^2}{2} \right) & i \neq j.
\end{cases}
$$

By the analytical continuation of the characteristic matrix to its largest domain in the entire complex plane, we have

$$
\Phi_{ij}(z) = \frac{1}{4\pi R^2 \mu_R^2} \begin{cases}
\phi \left( \frac{\mu_i}{\mu_R} \right) - \phi \left( \frac{\sqrt{-z}}{\mu_R} \right) & i = j \\
- \sum_{l \geq 0} \frac{2l + 1}{l(l+1) - \frac{\mu_R^2}{\mu_R^2}} P_l \left( 1 - \frac{d_{ij}^2}{2} \right) & i \neq j,
\end{cases}
$$

from which we can write the resolvent equation (14). Hence, we have obtained a well-defined formulation of our problem, that is, the infinities have been removed. Moreover, we see that the problem realizes a generalized dimensional transmutation. In this case, the coupling constants $g_i$ have the same dimension as $\frac{\hbar^2}{2m}$ by dimensional analysis. In contrast to the flat case, we have one more parameter $R$ coming from the geometry of the space. Thus, we expect that the system must have an intrinsic energy scale $\frac{\hbar^2}{2mR^2}$ as well as $\frac{\hbar^2}{md_{ij}^2}$ terms. However, after the renormalization, we obtain a set of new dimensional parameters $\mu_R^2$. Hence, the first set of scales we expect by naive dimensional analysis at the beginning is not sufficient. Instead, a specific combination of all these parameters together determine the scale of our problem. This means that delta potentials on a sphere is an example of a kind of dimensional transmutation. However, there is a slight difference, especially in the case of single delta attractor: in the flat case there is no combination of dimensional parameters to come up with an energy scale, whereas in the case of a sphere we have a geometric length scale $R$ which already defines an energy scale $\frac{\hbar^2}{2mR^2}$. The dimensional transmutation is most striking in such cases where there is no intrinsic energy scale.

In order to estimate the non-diagonal part of the matrix $\Phi$ for sphere $S^2$, we follow a different strategy, using the heat kernel.

### 3.1 Lower Bound of $E_{gr}$ by Heat Kernel Method for $S^2$

Heat kernel $K_t(x,y)$ is the unique fundamental solution to the heat equation $\frac{\hbar^2}{2m} \Delta_g \phi = -\hbar \phi_t$. It has the symmetry ($K_t(x,y) = K_t(y,x)$) and semi-group property [19, 20]. As well as being a useful computational tool in establishing the existence and some of the properties of the spectrum of the Laplacian of the eigenfunctions on a Riemannian manifolds, it is very helpful to understand the nature of the divergences for our purposes, as we have shown in the previous section.

By means of the relation (21) and explicit form of the heat kernel, one can calculate the matrix $\Phi_{ij}$. However, there are some situations in which one can not calculate the heat kernel
explicitly, e.g. we do not have an explicit expression of the heat kernel for two dimensional sphere. In this case, one can still find some bound estimates on matrix $\Phi_{ij}$ without having explicit form of the heat kernel, instead some properties of it. In order to analyze this for $S^2$, we will use some estimates on the heat kernel, based on a work by Li and Yau [24]. Let us recall the corollary of the theorem (3.1) in [24]:

Let $M$ be a complete manifold without boundary. If the Ricci curvature of $M$ is bounded from below by $-K$, for some constant $K \geq 0$, then for $1 < \alpha < 2$ and $0 < \varepsilon < 1$, the heat kernel satisfies

$$K_t(x, y) \leq \frac{C(\varepsilon)^\alpha}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} e^{C_7 \varepsilon^{(\alpha-1)} K_t - \frac{d(x,y)^2}{(4+\varepsilon)t}},$$

where $V(x, r) = \mu(B(x, r))$, $B(x, r)$ is the geodesic ball of radius $r$ centered at $x \in M$ and $d(x, y)$ is the geodesic distance between two points $x$ and $y$ on the manifold. The constant $C_7$ depends only on the dimension of the manifold $D$, while $C(\varepsilon)$ depends on $\varepsilon$ with $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. When $K = 0$, the above estimate, after letting $\alpha \to 1$, can be written as

$$K_t(x, y) \leq \frac{C(\varepsilon)}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} e^{-\frac{d(x,y)^2}{(4+\varepsilon)t}}. \quad (37)$$

Since $S^2$ satisfies the conditions above as a particular case, this corollary can be applied to it as well. On the other hand, we have a different purpose from the original corollary of the theorem for the estimates on the upper bound of the heat kernel, in which the sharp estimate for the heat kernel is found. Instead, we are trying to find a best lower bound of the ground state energy of the system. Therefore, we shall modify the original corollary in [24]. Using this theorem with relaxed condition $0 < \varepsilon < 1$, we have found the upper bound estimate for heat kernel of sphere $S^2$ in our problem:

$$K_t(a_i, a_j) \leq \frac{C'(\delta)}{\sqrt{V(x, \frac{ht}{2m})V(y, \frac{ht}{2m})}} e^{-\frac{2m^2 d^2_{ij}}{D(\delta)t}}, \quad (38)$$

where

$$C'(\delta) \equiv (1 + \delta)^2 \exp \left[ \frac{1}{4\delta (1 + \delta) (1 + 2\delta)} + \frac{1}{2\delta (2 + \delta)} + \frac{1}{4\delta} \right], \quad (39)$$

and

$$D(\delta) \equiv 4(1 + 2\delta)(1 + \delta)^2, \quad (40)$$

$\delta$ is merely required to be positive. When we want to find a lower bound for the energy, the numerical values of the coefficients $C'(\delta)$ and $D(\delta)$ will be determined explicitly. It is easy to see that $V \left( x, \frac{ht}{2m} \right) = V \left( y, \frac{ht}{2m} \right) = 2\pi R^2 \left( 1 - \cos \sqrt{\frac{ht}{2mR^2}} \right)$ as long as $0 \leq t \leq \frac{2m^2 \pi^2 R^2}{h}$. For $t \geq \frac{2m^2 \pi^2 R^2}{h}$, we have $V \left( x, \frac{ht}{2m} \right) = V \left( y, \frac{ht}{2m} \right) = 4\pi R^2$. According to our corollary and positive definiteness of heat kernel, the following integral has an upper bound:

$$\frac{1}{h} \int_0^\infty e^{-\frac{x^2}{2h}} K_t(a_i, a_j) dt \leq \frac{C'(\delta)}{h} \int_0^{\frac{2m^2 \pi^2 R^2}{h}} e^{-\frac{2m^2 d^2_{ij}}{D(\delta)t} - \frac{x^2}{2h}} dt + \frac{C'(\delta)}{4\pi R^2 h} \int_{\frac{2m^2 \pi^2 R^2}{h}}^\infty e^{-\frac{2m^2 d^2_{ij}}{D(\delta)t}} \frac{e^{\nu^2 t}}{t} dt,$$
where we have taken $z$ as $-\nu^2$. With the help of the identity $1 - \cos \sqrt{\frac{\hbar}{2mR^2}} = 2 \sin^2 \sqrt{\frac{\hbar}{8mR^2}}$
and the inequality $\frac{1}{\sin \theta} \leq \frac{\pi}{\theta}$ for $0 \leq \theta \leq \pi/2$, we obtain

$$\frac{1}{\hbar} \int_0^\infty e^{-\frac{\pi^2}{2\hbar} K_t(a_i, a_j)} dt \leq \frac{m \pi C'(\delta)}{2\hbar^2} \int_0^\infty e^{-\frac{2md^2_{ij}}{mR^2} \frac{\nu^2 t}{2}} dt + \frac{C'(\delta)}{4\pi R^2 \hbar} \int_0^\infty e^{-\frac{2md^2_{ij}}{mR^2} \frac{\nu^2 t}{2}} dt .$$

Evaluating these integrals, we find

$$| - \Phi_{i\neq j}(-\nu^2) | \equiv |K_{ij}| = \frac{1}{\hbar} \int_0^\infty e^{-\frac{\pi^2}{2\hbar} K_t(a_i, a_j)} dt \leq \frac{C'}{m \pi} [\frac{m \pi}{\hbar^2} K_0(\alpha_{ij} \nu) + \frac{\alpha_{ij}}{4\pi} \frac{K_1(\alpha_{ij} \nu)}{\nu R^2}] ,$$

where

$$\alpha_{ij} \equiv \sqrt{\frac{8m d^2_{ij}}{D(\delta) \hbar^2}} ,$$

and $K_0(x), K_1(x)$ are modified Bessel functions. This shows us that the infinite series in the non-diagonal part of the characteristic matrix is finite and bounded from above according to (41). In order to find a lower bound for the diagonal part, denoted by $\mathcal{D}$, of the matrix $\Phi$ for sphere $S^2$, we first recall how the diagonal part of the matrix $\Phi$ appears in (36):

$$\mathcal{D}_i = \frac{1}{4\pi R^2} \lim_{\Lambda \to \infty} \left[ \sum_{l=0}^\Lambda \frac{2l+1}{2m^2 l(l+1)+\mu_i^2} - \sum_{l=0}^\Lambda \frac{2l+1}{2m^2 l(l+1)+\nu^2} \right] \geq 0 .$$

Instead of calculating explicitly this limit as we have done in section 3, we estimate a lower bound of it by means of integrals replaced by the sums as follows

$$\mathcal{D}_i \geq \frac{1}{4\pi R^2} \lim_{\Lambda \to \infty} \left[ \int_0^{\Lambda+1} \frac{2t+1}{2m^2 t(t+1)+\mu_i^2} dt - \int_0^\Lambda \frac{2t+1}{2m^2 t(t+1)+\nu^2} dt - \frac{1}{\nu^2} \right] .$$

After taking the limit we find

$$\mathcal{D}_i \geq \left[ \frac{m}{\pi \hbar^2} \log(\nu/\mu_i) - \frac{1}{4\pi R^2 \nu^2} \right] ,$$

and using the estimate for logarithmic functions in [25]

$$\log x > \frac{x - 1}{x} \quad \text{for } x > 0, x \neq 1 ,$$

we obtain

$$\mathcal{D}_i \geq \left[ \frac{m}{\pi \hbar^2} \log(\nu/\mu_i) - \frac{1}{4\pi R^2 \nu^2} \right] > \left[ \frac{m}{\pi \hbar^2} \cdot \frac{m \mu_i}{\pi \hbar^2 \nu} - \frac{1}{4\pi R^2 \nu^2} \right] > 0 .$$

For positive definiteness, we have assumed $\nu$ is sufficiently large, which is not a particularly restrictive condition. In fact, one can try to find sharper estimates by means of the integral representations of digamma functions (33) and (34) without this assumption. However, the estimated functions in this case are too complicated to suggest a bound for ground state energy.
A well-known theorem in matrix analysis, called Geršgorin Theorem [26] states that all the eigenvalues $\lambda_i$ of the renormalized matrix $\Phi$ are located in the union of $N$ discs

$$
\bigcup_{i=1}^{N} \{|\lambda_i - \Phi_{ii}| \leq R'_i(\Phi)\} \equiv G(\Phi),
$$

where $R'_i(\Phi) \equiv \sum_{i \neq j=1}^{N} |\Phi_{ij}|$ and $1 \leq i \leq N$. If we want $\lambda = 0$ not to be an eigenvalue, then none of the discs should contain $\lambda = 0$. Then, we should impose

$$
| - D_i(\nu) | > \sum_{i \neq j}^{N} |K_{ij}(\nu)|,
$$

for all $i$. This is possible for a critical value $\nu > \nu^*$ since the left hand side is an increasing function of $\nu$ and the right hand side is a decreasing function of it for a given $d$ and $N$. In fact, this inequality obviously provide a lower bound for the bound state energy by just plotting the functions on both sides in spite of how complicated the form of functions are. However, we shall try to find an explicit expression for the lower bound of the ground state energy depending on the number of delta interactions. In order to achieve this, we choose $\nu$ such that;

$$
| - D_i(\nu) | > \left[ \frac{m}{\pi \hbar^2} - \frac{m \mu_i}{\pi \hbar^2 \nu} - \frac{1}{4\pi R'^2 \nu^2} \right] (N - 1)C'(\delta) \left[ m \pi \hbar^2 K_0(\alpha \nu) + \frac{\alpha}{4\pi} K_1(\alpha \nu) \right] > \sum_{i \neq j}^{N} |K_{ij}(\nu)|,
$$

where we have used the monotonic behavior of the functions in $D_{ij}$ and $K_{ij}$ and defined $\mu \equiv \max_i \mu_i$ and $\alpha \equiv \min_{i \neq j} \alpha_{ij}$ or $d \equiv \min_{i \neq j} d_{ij}$. From the integral representations of the Bessel functions for $z \in \mathbb{R}^+$ [27]

$$
K_0(z) = \int_{0}^{\infty} e^{-z \cosh t} dt,
$$

$$
K_1(z) = z \int_{0}^{\infty} e^{-z \cosh t} \sinh^2 t dt,
$$

and using the inequalities $\frac{\alpha}{2} < \cosh t, \sinh^2 t < \frac{\alpha}{4}$ for all $t \in \mathbb{R}^+$, we can find the upper bounds for the functions $K_0$ and $K_1$

$$
K_0(\alpha \nu) < \frac{2 e^{-\frac{\alpha \nu}{2}}}{\alpha \nu},
$$

$$
K_1(\alpha \nu) < e^{-\frac{\alpha \nu}{2}} \left( \frac{1}{\alpha \nu} + \frac{1}{2} \right),
$$

where $\alpha \nu \in \mathbb{R}^+$. Considering the estimated bounds for Bessel functions, it is easy to see that

$$
\frac{m}{\pi \hbar^2} (N - 1)C'(\delta) \left[ \frac{2\pi^2 e^{-\frac{\alpha \nu}{2}}}{\alpha \nu} + \frac{e^{-\frac{\alpha \nu}{2}} \mu^2}{2 \nu^2} + \frac{e^{-\frac{\alpha \nu}{2}} \mu^2 \alpha}{4 \nu} \right] > (N - 1)C'(\delta) \left[ \frac{m \pi}{\hbar^2} K_0(\alpha \nu) + \frac{\alpha}{4\pi} \frac{K_1(\alpha \nu)}{\nu R'^2} \right].
$$

15
Using the argument \( \nu > \sqrt{2} \mu_R \) in equation (44) and last inequality, we impose the following inequality with the help of Geršgorin Theorem:

\[
\nu > \mu + \frac{\mu_R}{2\sqrt{2}} + (N-1)C'(\delta)e^{-\frac{\mu}{\alpha}} \left[ \frac{2\pi^2}{\alpha} + \frac{\mu_R}{2\sqrt{2}} + \frac{\mu_R^2}{4} \right]. 
\tag{49}
\]

Let us make the following reasonable assumptions and take these for granted for the present (we will later show that they indeed satisfy these conditions by finding the extremum of ground state energy with respect to the parameter \( \delta \))

\[
\frac{\mu_R^2}{4} < \frac{2\pi^2}{\alpha}, \tag{50}
\]
\[
\frac{\mu_R}{2\sqrt{2}} < \frac{2\pi^2}{\alpha}, \tag{51}
\]
\[
\frac{1}{\sqrt{D(\delta)}} > \frac{1}{5}, \tag{52}
\]

so that the inequality becomes

\[
\nu > \mu + \frac{\mu_R}{2\sqrt{2}} + 3\pi^2 \mu_d (N-1) C'(\delta) \sqrt{D(\delta)} e^{-\frac{\mu}{\alpha}}, \tag{53}
\]

from which we conclude that there exists a critical value \( \nu > \nu^* \) for a given \( N \) such that \( \lambda \neq 0 \) and then, the ground state energy cannot be less than \( -\nu^{*2} \):

\[
E_{gr} \geq -\nu^{*2} = - \left\{ \mu + \frac{\mu_R}{2\sqrt{2}} + 5\mu_d W \left[ \frac{3\pi^2}{5} C'(\delta) \sqrt{D(\delta)} (N-1) e^{-\frac{\mu + \mu_R}{\alpha \mu_d}} \right] \right\}^2, \tag{54}
\]

where \( W \) is the Lambert W-function, also called Omega function or product-log function \[28\].

Now, we choose \( \delta \) in such a way that the energy bound take its minimum value. This is accomplished if \( \delta \) is chosen approximately 0.508, which is independent of the parameters in the problem. This independence can be easily realized from the form of inequality (53). By substituting the values of \( C'(\delta) \) and \( D(\delta) \), we estimate a lower bound for the ground state energy:

\[
E_{gr} \geq -\nu^{*2} = - \left\{ \mu + \frac{\mu_R}{2\sqrt{2}} + 5\mu_d W \left[ 28\pi^2 (N-1) e^{-\frac{\mu + \mu_R}{\alpha \mu_d}} \right] \right\}^2. \tag{55}
\]

By using this value of \( \delta \) and the fact that \( d < 2\pi R \), the consistency of the assumption we made can be shown easily. Finally, we shall consider the large \( N \) behavior of the ground state energy. The asymptotic expansion of product-log function \( W \) \[28\] for large \( z \) is given as

\[
W(z) \sim \log z - \log \log z. \tag{56}
\]

Hence, this leads to

\[
E_{gr} \sim -\mu^2 \left[ \log (N) - \log \log (N) \right]^2. \tag{57}
\]

The method we have introduced for the two dimensional sphere \( S^2 \) can also be applied to a general compact manifold. The main idea is based on finding an upper and lower bound for the characteristic matrix or heat kernel (based on the work by Li and Yau). Then, Geršgorin theorem allows us to estimate a lower bound for the ground state energy.
4 Finitely Many Dirac-Delta Interactions on Hyperbolic Spaces

The hyperbolic space $\mathbb{H}^n$ is defined as maximally symmetric and simply connected complete $n$-dimensional Riemannian manifold with a constant negative sectional curvature $-1/R$, which is also in some sense considered to be the negative curvature analog of the sphere $S^n$. We shall deal with the delta interactions on the hyperbolic spaces $\mathbb{H}^3$ and $\mathbb{H}^2$ in the following sections. The method developed in the previous sections for $S^2$ will be useful as well for the hyperbolic spaces. The heat kernel on hyperbolic spaces [29], written in terms of dimensionless quantities:

$$K_t(x, y) = \frac{1}{(4\pi t)^{3/2}} \frac{d}{\sinh d} e^{-\frac{d^2}{4t}} \quad \text{on } \mathbb{H}^3,$$

$$K_t(x, y) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-\frac{d^2}{4t}} \int_0^\infty \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh d)^{1/2}} ds \quad \text{on } \mathbb{H}^2,$$

where $d \equiv \text{dist}(x, y)$, geodesic distance between two points $x$ and $y$ on $\mathbb{H}^n$.

Although spectral theorem and asymptotic expansion of heat kernel discussed in the previous sections may not be valid for general non-compact manifolds, we shall demonstrate that for the specific examples in non-compact manifolds, such as $\mathbb{H}^2$ and $\mathbb{H}^3$, our viewpoint still works. It would be desirable to show the equivalence between the eigenfunction expansion and the heat kernel method for the regularization in non-compact manifolds rigourously. Nevertheless, we have not been able to do this. The main idea is similar in spirit to the renormalization procedure introduced for the compact manifolds.

4.1 Finitely Many Dirac-Delta Interactions on Hyperbolic Space $\mathbb{H}^3$

In the hyperbolic space $\mathbb{H}^3 = \{x \in \mathbb{R}^3 | x_3 > 0\}$, the geodesic distance $d$ is defined as

$$\cosh \frac{d(x, y)}{R} = 1 + \frac{|x - y|^2}{2 x_3 y_3},$$

where $R$ is the scaling parameter. The Schrödinger equation for the bound states of a particle living on $\mathbb{H}^3$ under the influence of $N$ attractive delta interactions is

$$\left[ \frac{\hbar^2}{2m} \Delta_{\mathbb{H}^3} - \sum_{i=1}^N g_i \delta^3(\chi - \chi_i, \theta - \theta_i, \phi - \phi_i) \right] \psi = -\nu^2 \psi,$$

where Laplacian $\Delta_{\mathbb{H}^3}$ in polar coordinates $(\chi, \theta, \phi)$

$$\Delta_{\mathbb{H}^2} = -\frac{1}{R^2} \frac{\partial^2}{\partial \psi^2} - \frac{2 \coth \psi}{R^2} \frac{\partial}{\partial \psi} + \frac{1}{R \sinh^2 \psi} \Delta_{\mathbb{S}^2}. $$

We have an explicit formula [29] for the heat kernel of the three dimensional hyperbolic plane $\mathbb{H}^3$ written by using physical constants

$$K_t(x, y) = \frac{1}{R^3} \frac{d(x, y)}{4\pi \left[ \frac{\hbar}{2m R^2} \right]^3/2} \sinh \frac{d(x, y)}{R} \exp \left( -\frac{\hbar t}{2m R^2} - \frac{md(x, y)^2}{2ht} \right).$$
such that as $R \to \infty$, we can obtain the heat kernel on $\mathbb{R}^3$. Hence we have the free resolvent kernel as

$$
\langle a_i | (H_0 - z)^{-1} | a_j \rangle = \frac{1}{\hbar R^3} \int_0^\infty \frac{d_{ij}}{R} \exp \left( \frac{zt}{\hbar} - \frac{ht}{2mR^2} - \frac{m^2 t^2}{2mR^2} \right) dt
$$

$$
= \left[ \frac{1}{4\pi R^3} \frac{d_{ij}}{\sinh \frac{d_{ij}}{R}} \exp \left( -\frac{\mu R}{\mu_{d_{ij}}} \sqrt{1 - \frac{z}{\mu_R^2}} \right) \right] \frac{\mu_{d_{ij}}}{\mu_R^3},
$$

where $d_{ij} \equiv d(a_i, a_j)$, $\mu_R^2 \equiv \frac{\hbar^2}{2mR^2}$, $\mu_{d_{ij}}^2 \equiv \frac{\hbar^2}{2mR_{d_{ij}}}$. It follows easily that this term gives infinity when $i = j$, that is, the diagonal term in the characteristic matrix is divergent. Then, we can now proceed the regularization and renormalization schemes analogously for the hyperbolic spaces. However, the divergence in hyperbolic space $\mathbb{H}^3$ is due to fact that the lower bound of integral (62) is zero. Hence we regularize the divergent term by introducing a lower cut-off $\epsilon$, as we have shown in section 2.1, we expect this should in some way related to the ultraviolet regularization. We next define the coupling constant as a function of this cut-off:

$$
\Phi_{ii}(z) = \lim_{\epsilon \to 0^+} \left[ g_i^{-1}(\epsilon) - \frac{1}{(4\pi)^{3/2} \mu_i^2 R^3} \int_\epsilon^\infty u^{-3/2} e^{- \left[ 1 - \frac{\mu_i}{\mu_R} \right] u} \right],
$$

where the integration variable $u \equiv \frac{\mu_i^2}{2mR^2} t$ is introduced for simplicity. The natural choice for $g_i^{-1}(\epsilon)$ is simply

$$
g_i^{-1}(\epsilon) = \frac{1}{(4\pi)^{3/2} \mu_i^2 R^3} \int_\epsilon^\infty u^{-3/2} e^{- \left[ 1 + \frac{\mu_i^2}{\mu_R^2} \right] u} du,
$$

where $\mu_i$ is an experimentally measured bound state energy for the single delta interaction and it helps us to keep track of the strength of point interactions. In $\epsilon \to 0^+$ limit, we have found the explicit renormalized characteristic matrix for $\mathbb{H}^3$

$$
\Phi_{ij}(z) = \frac{1}{4\pi} \frac{1}{\mu_i^2 R^3} \left\{ \begin{array}{ll}
\sqrt{1 - \frac{z}{\mu_i^2}} - \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} & \text{if } i = j \\
-\frac{\mu_{d_{ij}}}{\mu_i} \frac{d_{ij}}{\sinh \frac{d_{ij}}{R}} \exp \left( -\frac{\mu_R}{\mu_{d_{ij}}} \sqrt{1 - \frac{z}{\mu_R^2}} \right) & \text{if } i \neq j.
\end{array} \right.
$$

Then, we have the resolvent equation (14) with the free resolvent kernel $R_0(x, y|z)$ for $\mathbb{H}^3$ given by

$$
R_0(x, y|z) = \frac{1}{4\pi} \frac{1}{\mu_i^2 R^3} \frac{\mu_{d(x,y)}}{\mu_R} \frac{d(x,y)}{R} \sinh \frac{d(x,y)}{R} \exp \left( -\frac{\mu_R}{\mu_{d(x,y)}} \sqrt{1 - \frac{z}{\mu_R^2}} \right),
$$

from which we can get all information about the system. Using the Geršgorin Theorem (15) for this matrix, and following the same ideas introduced for $S^2$ we obtain

$$
\left[ \sqrt{1 + \frac{\nu^2}{\mu_R^2}} - \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right] > (N - 1) \frac{\mu_{d}}{\mu_R} \frac{d}{R} \sinh \frac{d}{R} \exp \left( -\frac{\mu_R}{\mu_{d}} \sqrt{1 + \frac{\nu^2}{\mu_R^2}} \right).
$$

where we have taken $z = -\nu^2$ and chosen $d \equiv \min_{i \neq j} d_{ij}$, and $\mu \equiv \max_i \mu_i$. It turns out that this inequality indicates that there exist a critical $\nu > \nu^*$ for a given $d$ and $N$ for which this
inequality is satisfied and zero is not an eigenvalue. Therefore, the ground state energy cannot be less than $-\nu^2$:

$$E_{gr} \geq -\nu^2 = -\mu^2 - 2\mu_d \sqrt{\mu_R^2 + \mu^2} \left[ e^{\frac{-\mu_R}{\mu_d} \sqrt{1 + \nu^2 \mu^2} \frac{d}{R} (N-1)} \sinh \frac{d}{R} \right]$$

For the large $N$ behavior of the ground state energy, the estimate becomes

$$E_{gr} \sim -2\mu_d \sqrt{\mu^2_R + \mu^2} \left[ \log N - \log \log N \right] - \mu_d^2 \left[ \log N - \log \log N \right]^2.$$

Now let us consider the two center case on the hyperbolic plane $\mathbb{H}^3$ and assume again that their strengths (or bound state energies of each center) are the same. In this way, determining equation (det $\Phi = 0$) becomes

$$\sqrt{1 + \frac{\nu^2}{\mu_R^2}} - \sqrt{1 + \frac{\mu^2}{\mu_R^2}} = \pm \frac{\mu_d}{\mu_R} \frac{d}{R} \exp \left( -\frac{\mu_R}{\mu_d} \sqrt{1 + \nu^2 \mu^2} \right).$$

If we expand it for small $d$ we have

$$\sqrt{1 + \frac{\nu^2}{\mu_R^2}} - \sqrt{1 + \frac{\mu^2}{\mu_R^2}} = \pm \frac{\mu_d}{\mu_R} \left[ 1 - \sqrt{1 + \frac{\mu^2}{\mu_R^2} + \frac{\nu^2}{\mu_R^2}} \right],$$

from which we can conclude

$$E_{gr} = -\nu^2 \sim \frac{3}{4} \mu^2_R - \frac{\mu^2}{4} - \frac{\mu_d^2 + \mu^2}{2} \sqrt{1 + \frac{\mu^2}{\mu_R^2}},$$

Similarly, for large values of $d$, the right hand side of the energy equation for two dirac delta interactions vanishes, so that we obtain the ground state energy $E_{gr} = -\nu^2 = -\mu^2$.

### 4.2 Finitely Many Dirac-Delta Interactions on Hyperbolic Plane $\mathbb{H}^2$

The geodesic distance on the hyperbolic plane $\mathbb{H}^2$ is defined by

$$\cosh \frac{d(x,y)}{R} = 1 + \frac{|x - y|^2}{2 x_2 y_2},$$

where $R$ is a scale distance. Then, the Schrödinger equation for the bound states of a particle living on $\mathbb{H}^2$ in the presence of $N$ attractive delta interactions is

$$\left[ \frac{\hbar^2}{2m} \Delta_{\mathbb{H}^2} - \sum_{i=1}^N g_i \delta^2(\theta - \theta_i, \phi - \phi_i) \right] \psi = -\nu^2 \psi,$$
where the Laplacian $\Delta_{H^2}$ in polar coordinates $(\theta, \phi)$ is given by
\[
\Delta_{H^2} = -\frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} - 2 \coth \theta \frac{\partial}{\partial \theta} - \frac{1}{R^2 \sinh^2 \theta} \frac{\partial^2}{\partial \phi^2}.
\] (67)

The heat kernel for $H^2$ with the proper physical parameters is
\[
K_t(x, y) = \frac{\sqrt{2}}{(4\pi \sqrt{\frac{\hbar^2}{2mR^2}}) t^{3/2}} e^{-\frac{h}{2mR^2} \frac{1}{t}} \int_0^\infty \frac{r e^{-\frac{2}{R} \frac{\mu^2}{\hbar^2} \frac{1}{r^2}}}{\sqrt{\cosh r - \cosh \frac{d_{ij} R}{R}}} dr.
\] (68)

One can check that this goes to the heat kernel on $\mathbb{R}^2$ as $R \to \infty$. Then, the free resolvent kernel is immediately obtained
\[
\langle a_i| (H_0 - z)^{-1} |a_j \rangle = \frac{1}{hR^2} \int_0^\infty e^{\pm \frac{\sqrt{2}}{2R^2 t^{3/2}}} e^{-\frac{h}{2mR^2} \frac{1}{t}} \left[ \int_0^\infty \frac{r e^{-\frac{2}{R} \frac{\mu^2}{\hbar^2} \frac{1}{r^2}}}{\sqrt{\cosh r - \cosh \frac{d_{ij} R}{R}}} dr \right] dt
\]}

We see that the diagonal term, which corresponds to $d_{ij} = 0$ is divergent, as expected. Therefore we again repeat the similar regularization and renormalization procedure as we have done for $H^3$. After introducing a cut-off to the lower limit of the integral
\[
\Phi_{ii}(z) = \lim_{\epsilon \to 0^+} \left[ g_i^{-1}(\epsilon) - \frac{1}{4\pi \mu_R^2 R^2} \int_\epsilon^\infty e^{-\frac{1}{2} \sqrt{1 - \frac{4\epsilon^2}{\nu_R^2}}} \sinh u du \right],
\]
and by the natural choice for $g_i^{-1}(\epsilon)$
\[
g_i^{-1}(\epsilon) = \frac{\sqrt{2}}{4\pi \mu_R^2 R^2} \int_\frac{\epsilon}{R}^\infty e^{-u R} \sinh u du ,
\]
we have obtained the renormalized characteristic matrix for $H^2$ in the $\epsilon \to 0$ limit,
\[
\Phi_{ij}(z) = \begin{cases} \frac{\sqrt{2}}{4\pi R^2} \frac{1}{\mu_R^2} \left[ \psi \left( \frac{1}{2} + \sqrt{1 - \frac{z}{\mu_R^2}} \right) - \psi \left( \frac{1}{2} + \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right) \right] & \text{if } i = j \\ -\frac{1}{4\pi R^2} e^{-\frac{1}{2} \sqrt{1 - \frac{4\epsilon^2}{\nu_R^2}}} \sinh u du & \text{if } i \neq j, \end{cases}
\] (69)
where $\psi$ is the digamma function. Then, we have the resolvent equation (14) in which the free resolvent kernel $R_0(x, y|z)$ for $\mathbb{H}^2$ is given by

$$R_0(x, y|z) = \frac{1}{4\pi R^2} \frac{1}{\mu_R^2} \int_{\frac{d(x,y)}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1 - \frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh \frac{d(x,y)}{R}}} \, dr. \quad (70)$$

The integral on the right hand side is in fact one of the integral representation of the Legendre polynomials of second type \[27\]

$$\sqrt{2}Q_\lambda \left( \cosh \frac{d(x, y)}{R} \right) = \int_{\frac{d(x,y)}{R}}^{\infty} \frac{e^{-(\lambda + \frac{1}{2})r}}{\sqrt{\cosh r - \cosh \frac{d(x,y)}{R}}} \, dr, \quad (71)$$

which are defined for $\Re(\lambda) > -1$ and in our case $\Re(\lambda) = \Re\left( \frac{1}{2} \sqrt{1 - \frac{4z}{\mu_R^2}} - \frac{1}{2} \right) > -1$. Therefore, the free resolvent in terms of $Q_\lambda$

$$R_0(x, y|z) = \frac{1}{4\pi R^2} \frac{1}{\mu_R^2} \sqrt{2}Q_\frac{1}{2} \sqrt{1 - \frac{4z}{\mu_R^2}} \frac{1}{2} \left( \cosh \frac{d(x,y)}{R} \right). \quad (72)$$

Geršgorin theorem allows us to estimate the lower bound for the bound state energy as done for $S^2$ and $\mathbb{H}^3$. In order not to have zero as an eigenvalue, we must have

$$\sqrt{2} \left[ \psi \left( 1 + \sqrt{\frac{1}{4} + \frac{\nu}{\mu_R^2}} \right) - \psi \left( 1 + \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right) \right] > \sum_{i \neq j} \int_{\frac{d_{ij}}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1 + \frac{4\nu^2}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh \frac{d_{ij}}{R}}} \, dr, \quad (73)$$

for all $i$ and we have taken $z = \nu^2$ and $\nu > \max_i \mu_i$. It is easy to see this inequality is satisfied for some values of $\nu$ because the left hand side is an increasing function, whereas the right hand side is a decreasing function of $\nu$. However, it is not so easy to give an explicit estimate for $\nu$ by this inequality so we will estimate the functions on both sides. The inequality for the digamma functions \[30\]

$$\psi(x) > \log x - \frac{1}{x}, \quad x > 0, \quad (74)$$

which can be obtained from the integral representation \[34\], and $1 + \sqrt{\frac{1}{4} + x^2} \geq x$ for all $x > 0$ helps us that we can find the following inequality by assuming $\nu$ is sufficiently large

$$\left[ \psi \left( 1 + \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right) - \psi \left( 1 + \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right) \right] > \left[ \log \frac{\nu}{\mu_R} - \frac{\mu_R}{\nu} - \psi \left( 1 + \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} \right) \right]. \quad (75)$$

Since the right hand side of equation (73) is $(N - 1)\sqrt{2}Q_\lambda \left( \cosh \frac{d_{ij}}{R} \right)$ we can find an upper bound for this function, using another integral representation of the second type Legendre polynomials \[27\]:

$$Q_\lambda \left( \cosh \frac{d_{ij}}{R} \right) = \frac{1}{\Gamma(\lambda + 1)} \int_{0}^{\infty} e^{-t\cosh \frac{d_{ij}}{R}} K_0 \left( t \sinh \frac{d_{ij}}{R} \right) t^\lambda \, dt, \quad (76)$$
where \( \Im \left( \frac{d_{ij}}{R} \right) = 0 \) and \( \lambda = \frac{1}{2} \sqrt{1 + \frac{4\nu^2}{\mu^2} - \frac{1}{2}} \). Using the estimate for the function \( K_0 \) given in equation (18), we obtain

\[
\sqrt{2}Q_\lambda \left( \cosh \frac{d_{ij}}{R} \right) < \frac{2\sqrt{2}}{\Gamma(\lambda + 1) \sinh \frac{d_{ij}}{R}} \int_0^\infty e^{-t \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right) t^{\lambda - 1}} \, dt
\]

and the right hand side is just the Gamma function, then the estimate becomes

\[
\sqrt{2}Q_\lambda \left( \cosh \frac{d_{ij}}{R} \right) < \frac{2\sqrt{2}\Gamma(\lambda)}{\Gamma(\lambda + 1) \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right)^\lambda \sinh \frac{d_{ij}}{R}}.
\]

Using identity \( \Gamma(\lambda + 1) = \lambda \Gamma(\lambda) \) and the assumption \( \nu/\mu_R > 1 \) and \( \sqrt{1 + \frac{4\nu^2}{\mu^2}} > \frac{2\nu}{\mu_R} \), we get

\[
\sqrt{2}Q_\lambda \left( \cosh \frac{d_{ij}}{R} \right) < \frac{4\sqrt{2}}{\left( \frac{2\nu}{\mu_R} - 1 \right) \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right)^\lambda} \sinh \frac{d_{ij}}{R}.
\]

Also, by choosing \( d \equiv \min_{i \neq j} d_{ij} \) and \( \mu \equiv \max_i \mu_i \), we easily find

\[
\left[ \log \frac{\nu}{\mu_R} - \frac{\mu_R}{\nu} - \psi \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu^2}{\mu_R^2}} \right) \right] > \left[ \log \frac{\nu}{\mu_R} - \frac{\mu_R}{\nu} - \psi \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu^2}{\mu_R^2}} \right) \right],
\]

and

\[
\frac{e^{-\frac{1}{2}} \log \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right)}{\sinh \frac{d_{ij}}{R}} < \frac{e^{-\frac{1}{2}} \log \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right)}{\sinh \frac{d_{ij}}{R}}.
\]

Therefore, we impose

\[
\left[ \log \frac{\nu}{\mu_R} - \frac{\mu_R}{\nu} - \psi \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu^2}{\mu_R^2}} \right) \right] > \frac{4(N - 1)}{\nu} e^{-\frac{1}{2}} \log \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right).
\]

It is immediately seen that there exists a critical value \( \nu > \nu^* \) for a given \( d \) and \( N \) for which this inequality is satisfied and zero is not an eigenvalue. Last inequality can be written as

\[
e^{-\frac{1}{2}} \log \left( \cosh \frac{d_{ij}}{R} + \frac{1}{2} \sinh \frac{d_{ij}}{R} \right) \left( \frac{\nu}{\mu_R} \log \left( \frac{\nu}{\mu_R} e^{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu^2}{\mu_R^2}}} \right) - 1 \right) > \frac{4(N - 1)}{\sinh \frac{d_{ij}}{R}}.
\]

If \( \frac{\nu}{\mu_R} > \psi \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu^2}{\mu_R^2}} \right) + 1 \) (independent of \( N \)), then we have the lower bound of the ground state energy

\[
E_{gr} \geq -\nu^2 = -\mu_R^2 \left[ A + W \left( \frac{4(N - 1)A e^{-A/2}}{\sinh \frac{d_{ij}}{R}} \right) \right]^{-2}.
\]
where we define $A \equiv \log \left( \cosh \frac{d}{R} + \frac{1}{2} \sinh \frac{d}{R} \right)$ for simplicity of notation. For large values of $N$ as long as the ratio $\frac{\mu}{\mu_R}$ and $\frac{d}{R}$ is finite, the behavior of the bound state energy is given by

$$E_{gr} \sim -\mu_R^2 \left[ \log N - \log \log N \log \left( \cosh \frac{d}{R} + \frac{1}{2} \sinh \frac{d}{R} \right) \right]^2.$$  \hspace{1cm} (82)

This problem again is an example of a certain kind of dimensional transmutation in non-relativistic quantum mechanics. By dimensional analysis, the hamiltonian of the system contains intrinsic energy scales $\frac{\hbar^2}{md_{ij}}$ and $\frac{\hbar^2}{2mR^2}$. However, after the renormalization, we obtain new parameters $\mu_i^2$ with energy dimensions. Hence, the number of parameters we expect for the energy at the beginning has changed after the renormalization. As it happens in the $S^2$ case, the delta potentials on $\mathbb{H}^2$ is an example of a generalized dimensional transmutation.

5 Conclusion

In this work, we studied a particle moving under the influence of $N$ attractive Dirac delta interactions on some special Riemannian manifolds. We renormalized the problem and find a finite dimensional matrix $\Phi$, called the characteristic matrix, by means of which a well defined expression for the resolvent can be written. All the information about the bound states can be obtained from the characteristic matrix. The renormalization can be done by means of the heat kernel and this is equivalent to the sharp cut-off method for the eigenvalues of the Laplacian, in the case of compact manifolds. We have studied the problem on particular compact and non-compact manifolds, $S^2$, $\mathbb{H}^2$, and $\mathbb{H}^3$ and we give explicit lower bound estimates on the bound state energies for each problem. Although we are concerned with particular manifolds, the basic idea for the renormalization can be applied also to general manifolds.

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