Completeness of the Lattice-Boltzmann IKT approach for classical incompressible fluids\textsuperscript{§}

E. Fonda\textsuperscript{a}, M. Tessarotto\textsuperscript{a,b}, P. Nicolini\textsuperscript{a,b} and M. Ellero\textsuperscript{c}

\textsuperscript{a}Department of Mathematics and Informatics, University of Trieste, Italy
\textsuperscript{b}Consortium for Magneto-fluid-dynamics, University of Trieste, Italy
\textsuperscript{c}Technical University of Munich, Germany

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Abstract

Despite the abundant literature on the subject appeared in the last few years, the lattice Boltzmann method (LBM) is probably the one for which a complete understanding is not yet available. As an example, an unsolved theoretical issue is related to the construction of a discrete kinetic theory which yields \textit{exactly} the fluid equations, i.e., is non-asymptotic (here denoted as \textit{LB inverse kinetic theory}). The purpose of this paper aims at investigating discrete inverse kinetic theories (IKT) for incompressible fluids. We intend to show that the discrete IKT can be defined in such a way to satisfy, in particular, the requirement of \textit{completeness}, i.e., \textit{all} fluid fields are expressed as moments of the kinetic distribution function and \textit{all} hydrodynamic equations can be identified with suitable moment equations of an appropriate inverse kinetic equation IKE.

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I. INTRODUCTION

Basic issues concerning the foundations classical hydrodynamics still remain unanswered. A remarkable aspect is related the construction of inverse kinetic theories (IKT) for hydrodynamic equations in which the fluid fields are identified with suitable moments of an appropriate kinetic probability distribution. The topic has been the subject of theoretical investigations both regarding the incompressible Navier-Stokes (NS) equations (INSE) and the quantum hydrodynamic equations associated to the Schrödinger equation. The importance of the IKT-approach for classical hydrodynamics goes beyond the academic interest. In fact, INSE represent a mixture of hyperbolic and elliptic pde’s, which are extremely hard to study both analytically and numerically. As such, their investigation represents a challenge both for mathematical analysis and for computational fluid dynamics. The discovery of IKT provides, however, a new starting point for the theoretical and numerical investigation of INSE. In fact, an inverse kinetic theory yields, by definition, an exact solver for the fluid equations: all the fluid fields, including the fluid pressure \( p(r,t) \), are uniquely prescribed in terms of suitable momenta of the kinetic distribution function, solution of the kinetic equation. In the case of INSE this permits, in principle, to determine the evolution of the fluid fields without solving explicitly the Navier-Stokes equation, nor the Poisson equations for the fluid pressure. Previous IKT approaches have been based on continuous phase-space models. However, the interesting question arises whether similar concepts can be adopted also to the development of discrete inverse kinetic theories based on the lattice Boltzmann (LB) theory. The goal of this investigation is to propose a novel LB theory for INSE, based on the development of an IKT with discrete velocities, here denoted as lattice Boltzmann inverse kinetic theory (LB-IKT). In this paper we intend to analyze the theoretical foundations and basic properties of the new approach useful to display its relationship with previous CFD and lattice Boltzmann methods (LBM) for incompressible isothermal fluids. In particular, we wish to prove that it delivers an inverse kinetic theory, i.e., that it realizes an exact Navier-Stokes and Poisson solver. The motivations of this work are related to some of the basic features of customary LB theory representing, at the same time, assets and weaknesses. One of the main reasons of the popularity of the LB approach lays in its simplicity and in the fact that it provides an approximate Poisson solver, i.e., it permits to advance in time the fluid fields without explicitly solving numerically the Pois-
son equation for the fluid pressure. However customary LB approaches can yield, at most, only asymptotic approximations for the fluid fields. This is because of two different reasons. The first one is the difficulty in the precise definition of the kinetic boundary conditions in customary LBM’s, since sufficiently close to the boundary the form of the distribution function prescribed by the boundary conditions is not generally consistent with hydrodynamic equations. The second reason is that the kinetic description adopted implies either the introduction of weak compressibility [7, 8, 9, 10, 11, 12] or temperature [13] effects of the fluid or some sort of state equation for the fluid pressure [14]. These assumptions, although physically plausible, appear unacceptable from the mathematical viewpoint since they represent a breaking of the exact fluid equations. A fundamental issue is, therefore, related to the construction of more accurate, or higher-order, LBM’s, applicable for arbitrary values of the relevant physical (and asymptotic) parameters. However, the route which should permit to determine them is still uncertain, since the very existence of an underlying exact (and non-asymptotic) discrete kinetic theory, analogous to the continuous inverse kinetic theory [2, 3], is not yet known. According to some authors [15, 17, 18] this should be linked to the discretization of the Boltzmann equation, or to the possible introduction of weakly compressible and thermal flow models. However, the first approach is not only extremely hard to implement [19], since it is based on the adoption of higher-order Gauss-Hermite quadratures (linked to the discretization of the Boltzmann equation), but its truncations yield at most asymptotic theories. Other approaches, which are based on ‘ad hoc’ modifications of the fluid equations (for example, introducing compressibility and/or temperature effects [20]), by definition cannot provide exact Navier-Stokes solvers. The aim of this work is the development of an inverse kinetic theory for the incompressible Navier-Stokes equations (INSE) which, besides realizing an exact Navier-Stokes (and Poisson) solver, overcomes some of the limitations of previous LBM’s. Unlike Refs. [2, 3], where a continuous IKT was considered, here we construct a discrete theory based on the LB velocity-space discretization. In such a type of approach, the kinetic description is realized by a finite number of discrete distribution functions \( f_i(\mathbf{r}, t) \), for \( i = 0, k \), each associated to a prescribed discrete constant velocity \( \mathbf{a}_i \) and defined everywhere in the existence domain of the fluid fields (the open set \( \Omega \times I \)). The configuration space \( \Omega \) is a bounded subset of the Euclidean space \( \mathbb{R}^3 \) and the time interval \( I \) is a subset of \( \mathbb{R} \). The kinetic theory is obtained as in [2, 3] by introducing an inverse kinetic equation (LB-IKE) which advances in time the distribution function and
by properly defining a correspondence principle, relating a set of velocity momenta with the relevant fluid fields.

II. 2 - LB INVERSE KINETIC THEORY (LB-IKT)

There are several important motivations for seeking an exact solver based on LBM. The lack of a theory of this type represents in fact a weak point of LB theory. Besides being a still unsolved theoretical issue, the problem is relevant in order to determine the exact relationship between the LBM’s and traditional CFD schemes based on the direct discretization of the Navier–Stokes equations. Following ideas recently developed [2, 3, 4, 6], we show that such a theory can be formulated by means of an inverse kinetic theory (IKT) with discrete velocities. By definition such an IKT should yield exactly the complete set of fluid equations and which, contrary to customary kinetic approaches in CFD (in particular LB methods), should not depend on asymptotic parameters. This implies that the inverse kinetic theory must also satisfy an exact closure condition. As a further condition, we require that the fluid equations are fulfilled independently of the initial conditions for the kinetic distribution function (to be properly set) and should hold for arbitrary fluid fields. The latter requirement is necessary since we must expect that the validity of the inverse kinetic theory should not be limited to a subset of possible fluid motions nor depend on special assumptions, like a prescribed range of Reynolds numbers. In principle a phase-space theory, yielding an inverse kinetic theory, may be conveniently set in terms of a quasi-probability, denoted as kinetic distribution function, \( f(x, t) \). A particular case of interest (investigated in Refs. [2, 3]) refers to the case in which \( f(x, t) \) can actually be identified with a phase-space probability density. In the sequel we address both cases, showing that, to a certain extent, in both cases the formulation of a generic IKT can actually be treated in a similar fashion. This requires the introduction of an appropriate set of constitutive assumptions (or axioms). These concern in particular the definitions of the kinetic equation - denoted as inverse kinetic equation (IKE) - which advances in time \( f(x, t) \) and of the velocity momenta to be identified with the relevant fluid fields (correspondence principle). However, further assumptions, such as those involving the regularity conditions for \( f(x, t) \) and the prescription of its initial and boundary conditions must clearly be added. The concept [of IKT] can be easily extended to the case in which the kinetic distribution function takes on only discrete
values in velocity space. In the sequel we consider for definiteness the case of the so-called *LB discretization*, whereby - for each \((r, t) \in \Omega \times I\) - the kinetic distribution function is discrete, and in particular admits a finite set of discrete values \(f_i(r, t) \in \mathbb{R}\), for \(i = 0, k\), each one corresponding to a prescribed constant discrete velocity \(a_i \in \mathbb{R}^3\) for \(i = 0, k\). Let us now introduce the constitutive assumptions (*axioms*) set for the construction of a LB-IKT for INSE, whose form is suggested by the analogous continuous inverse kinetic theory \[2, 3\].

The axioms, define the ”generic” form of the discrete kinetic equation, its functional setting, the momenta of the kinetic distribution function and their initial and boundary conditions, are the following ones:

1) **Axiom I - LB–IKE and functional setting.** Let us require that the extended fluid fields \(\{V, p_1\}\) are strong solutions of INSE, with suitable initial and boundary conditions and that the pseudo pressure \(p_o(t)\) is an arbitrary, suitably smooth, real function. In particular we impose that the fluid fields and the volume force belong to the minimal functional setting:

\[
\begin{align*}
\phi & \in C^{(2,1)}(\Omega \times I), \\
V & \in C^{(3,1)}(\Omega \times I), \\
f_1 & \in C^{(1,0)}(\Omega \times I).
\end{align*}
\]

We assume that in the set \(\Omega \times I\) the following equation

\[
L_{D(i)} f_i = \Omega_i(f_i) + S_i
\]

[*LB inverse kinetic equation (LB-IKE)*] is satisfied identically by the discrete kinetic distributions \(f_i(r, t)\) for \(i = 0, k\). Here \(\Omega_i(f_i)\) and \(L_{D(i)}\) are respectively the BGK and the differential streaming and operators, while \(S_i\) is a source term to be defined. We require that KB-IKE is defined in the set \(\Omega \times I\), so that \(\Omega_i(f_i)\) and \(S_i\) are at least that \(C^{(1)}(\Omega \times I)\) and continuous in \(\overline{\Omega \times I}\). Moreover \(\Omega_i(f_i)\), to be identified as usual with the BGK operator, is considered for generality and will be useful for comparisons with customary LB approaches. We remark that the choice of the equilibrium kinetic distribution \(f_i^{eq}\) in the BGK operator remains completely arbitrary. We assume furthermore that in terms of \(f_i\) the fluid fields \(\{V, p_1\}\) are determined by means of functionals of the form \(M_{X_j} [f_i] = \sum_{i=0,8} X_j f_i\) (denoted as *discrete velocity momenta*). For \(X = X_1, X_2\) (with \(X_1 = c^2, X_2 = \frac{2}{\rho_o} a_i\)) these are related to the fluid fields by means of the equations (*correspondence principle*) defined by the
equations \( p_1(r, t) - \Phi(r) = c^2 \sum_{i=0,8} f_i = c^2 \sum_{i=0,8} f_{i}^{eq}, \) and \( V(r, t) = \frac{3}{\rho_o} \sum_{i=1,8} a_i f_i = \frac{3}{\rho_o} \sum_{i=1,8} a_i f_{i}^{eq}, \)
where \( c = \min \{|a_i|, i = 1,8\} \) is the test particle velocity and, without loss of generality, \( f_{i}^{eq} \)
can be identified with a polynomial expression, with the kinetic pressure \( p_1 \) replacing the fluid pressure \( p \) adopted previously \([16]\). These equations are assumed to hold identically in the set \( \Omega \times I \) and by assumption, \( f_i \) and \( f_{i}^{eq} \) belong to the same functional class of real functions defined so that the extended fluid fields belong to the minimal functional setting \([1]\). Moreover, without loss of generality, we consider the D2Q9 LB discretization.

2) **Axiom II - Kinetic initial and boundary conditions.** The discrete kinetic distribution function satisfies, for \( i = 0, k \) and for all \( r \) belonging to the closure \( \overline{\Omega} \), the initial conditions \( f_i(r, t_o) = f_{o\alpha}(r, t_o) \), where \( f_{o\alpha}(r, t_o) \) (for \( i = 0, k \)) is a initial distribution function defined in such a way to satisfy in the same set the initial conditions for the fluid fields \( p_{1\alpha}(r) \equiv P_o(t_o) + p_o(r) - \Phi(r) = c^2 \sum_{i=0,8} f_{o\alpha}(r) \) and \( V_o(r) = \frac{3}{\rho_o} \sum_{i=1,8} a_i f_{o\alpha}(r) \) To define the analogous kinetic boundary conditions on \( \delta \Omega \), let us assume that \( \delta \Omega \) is a smooth, possibly moving, surface. Let us introduce the velocity of the point of the boundary determined by the position vector \( r_w \in \delta \Omega \), defined by \( V_w(r_w(t), t) = \frac{d}{dt} r_w(t) \) and denote by \( n(r_w, t) \) the outward normal unit vector, orthogonal to the boundary \( \delta \Omega \) at the point \( r_w \). Let us denote by \( f_i^{(+)}(r_w, t) \) and \( f_i^{(-)}(r_w, t) \) the kinetic distributions which carry the discrete velocities \( a_i \) for which there results respectively \( (a_i - V_w) \cdot n(r_w, t) > 0 \) (outgoing-velocity distributions) and \( (a_i - V_w) \cdot n(r_w, t) \leq 0 \) (incoming-velocity distributions) and which are identically zero otherwise. We assume for definiteness that both sets, for which \( |a_i| > 0 \), are non empty (which requires that the parameter \( c \) be suitably defined so that \( c > |V_w| \)). The boundary conditions are obtained by suitably prescribing the incoming kinetic distribution \( f_i^{(-)}(r_w, t) \), i.e., imposing (for all \( (r_w, t) \in \delta \Omega \times I \) \( f_i^{(-)}(r_w, t) = f_{o\alpha}^{(-)}(r_w, t) \). Here \( f_{o\alpha}^{(-)}(r_w, t) \) are suitable functions, to be assumed non-vanishing and defined only for incoming discrete velocities for which \( (a_i - V_w) \cdot n(r_w, t) \leq 0 \).

3) **Axiom III - Moment equations.** If \( f_i(r, t) \), for \( i = 0, k \), are arbitrary solutions of LB-IKE \([\text{Eq.}(2)]\) which satisfy Axioms I and II validity of Axioms I and II, we assume that the moment equations of the same LB-IKE, evaluated in terms of the moment operators \( M_{X_j} [\cdot] = \sum_{i=0,8} X_j f_i \), with \( j = 1,2 \), coincide identically with INSE, namely that there results identically [for all \( (r, t) \in \Omega \times I \) \( M_{X_1} [L_i f_i - \Omega_i(f_i) - S_i] = \nabla \cdot V = 0 \) and \( M_{X_2} [L_i f_i - \Omega_i(f_i) - S_i] = N \nabla = 0 \).
4) *Axiom IV - Source term.* The source term is required to depend on a finite number of momenta of the distribution function. It is assumed that these include, at most, the extended fluid fields \( \{V, p_1\} \) and the kinetic tensor pressure \( \Pi = 3 \sum_{i=0}^{8} f_i a_i a_i - \rho_0 V V \). Furthermore, we also normally require (except for the LB-IKT described in Appendix B) that \( S_i(\mathbf{r}, t) \) results independent of \( f_i^{eq}(\mathbf{r}, t), f_{\alpha}(\mathbf{r}) \) and \( f_{wi}(\mathbf{r}_w, t) \) (for \( i = 0, k \)).

Although, the implications will made clear in the following sections, it is manifest that these axioms do not specify uniquely the form (and functional class) of the equilibrium kinetic distribution function \( f_i^{eq}(\mathbf{r}, t) \), nor of the initial and boundary kinetic distribution functions suitably defined. Thus, both \( f_i^{eq}(\mathbf{r}, t), f_{\alpha}(\mathbf{r}, t_o) \) and the related distribution they still remain in principle *completely arbitrary.* Nevertheless, by construction, the initial and (Dirichlet) boundary conditions for the fluid fields are satisfied identically. In the sequel we show that these axioms define a (non-empty) family of parameter-dependent LB-IKT’s, depending on two constant free parameters \( \nu_c, c > 0 \) and one arbitrary real function \( P_o(t) \). The examples considered are reported respectively in the following Sec. 5,6 and in the Appendix B.

**III. 3 - A POSSIBLE REALIZATION: THE INTEGRAL LB-IKT**

We now show that, for arbitrary choices of the distributions \( f_i(\mathbf{r}, t) \) and \( f_i^{eq}(\mathbf{r}, t) \) which fulfill axioms I-IV, an explicit (and non-unique) realization of the LB-IKT can actually be obtained. We prove, in particular, that a possible realization of the discrete inverse kinetic theory, to be denoted as *integral LB-IKT*, is provided by the source term \[ S_i = \]

\[
S_i = \sum_{j=0}^{8} \frac{w_i}{c^2} \left[ \frac{\partial p_1}{\partial t} - a_i \cdot (f_1 - \mu \nabla^2 V - \nabla \cdot \Pi + \nabla p) \right] \equiv \tilde{S}_i,
\]

where \( \sum_{j=0}^{8} \frac{w_i}{c^2} \) is denoted as first pressure term. Then the following theorem hols.

**Theorem - Integral LB-IKT**

In validity of axioms I-IV the following statements hold. For an arbitrary particular solution \( f_i \) and for arbitrary extended fluid fields: A) if \( f_i \) is a solution of LB-IKE \([\text{Eq.}(2)]\) the moment equations coincide identically with INSE in the set \( \Omega \times I \); B) the initial
conditions and the (Dirichlet) boundary conditions for the fluid fields are satisfied identically; C) in validity of axiom IV the source term $\tilde{S}_i$ is non-uniquely defined by Eq. (3).

Proof

A) We notice that by definition there results identically

$$\sum_{i=0}^{8} \tilde{S}_i = \frac{1}{c^2} \frac{\partial p_1}{\partial t}$$

(4)

$$\sum_{i=0}^{8} a_i \tilde{S}_i =$$

(5)

$$= -\frac{1}{3} [f - \mu \nabla^2 V - \nabla \cdot \Pi + \nabla p]$$

On the other hand, by construction (Axiom I) $f_i (i = 1, k)$ is defined so that there results identically $\sum_{i=0}^{8} \Omega_i = 0$ and $\sum_{i=0}^{8} a_i \Omega_i = 0$. Hence the momenta $M_{X_1}, M_{X_2}$ of LB-IKE deliver respectively

$$\nabla \cdot \sum_{i=1,8} a_i f_i = 0$$

(6)

$$3 \frac{\partial}{\partial t} \sum_{i=1,8} a_i f_i + \rho_0 V \cdot \nabla V + \nabla p_1 + f - \mu \nabla^2 V = 0$$

(7)

where the fluid fields $V, p_1$ are defined by appropriate moments [16]. Hence Eqs. (6) and (7) coincide respectively with the isochoricity and Navier-Stokes equations. As a consequence, $f_i$ is a particular solution of LB-IKE iff the fluid fields $\{V, p_1\}$ are strong solutions of INSE.

B) Initial and boundary conditions for the fluid fields are satisfied identically by construction thanks to Axiom II.

C) However, even prescribing $\nu_c, c > 0$ and the real function $P_o(t)$, the functional form of the equation cannot be unique The non uniqueness of the functional form of the source term $\tilde{S}_i (r, t)$ is assumed to be independent of $f_i^{eq} (r, t)$ [and hence of Eq. (2)] is obvious. In fact, let us assume that $\tilde{S}_i$ is a particular solution for the source term which satisfies the previous axioms I-IV. Then, it is always possible to add to $\tilde{S}_i$ arbitrary terms of the form $\tilde{S}_i + \delta S_i$, with $\delta S_i \neq 0$ which depends only on the momenta indicated above, and gives vanishing contributions to the first two moment equations, namely $M_{X_j} [\delta S_i] = \sum_{i=0,8} X_j \delta S_i = 0$, with $j = 1, 2$. To prove the non-uniqueness of the source term $S_i$, it is sufficient to notice that, for example, any term of the form $\delta S_i = \left( \frac{3 \nu^2}{2 c^2} - 1 \right) F(r, t)$, with $F(r, t)$ an arbitrary real
function (to be assumed, thanks to Axiom IV, a linear function of the fluid velocity), gives vanishing contributions to the momenta $M_{X1}, M_{X2}$. Hence $\tilde{S}_i$ is non-unique.

The implications of the theorem are straightforward. First, manifestly, it holds also in the case in which the BGK operator vanishes identically. This occurs letting $\nu_c = 0$ in the whole domain $\Omega \times I$. Hence the inverse kinetic equation holds independently of the specific definition of $f_{eq}^i(r,t)$.

An interesting feature of the present approach lies in the choice of the boundary condition adopted for $f_i(r,t)$, which is different from that usually adopted in LBM’s [see for example [9] for a review on the subject]. In particular, the choice adopted is the simplest permitting to fulfill the Dirichlet boundary conditions [imposed on the fluid fields]. This is obtained prescribing the functional form of $f_i(r,t)$ on the boundary of the fluid domain ($\partial \Omega$), which is identified with a function $f_{oi}(r,t)$.

Second, the functional class of $f_i(r,t)$, $f_{eq}^i(r,t)$ and of $f_{oi}(r,t)$ remains essentially arbitrary. Thus, in particular, the initial and boundary conditions, specified by the same function $f_{oi}(r,t)$, can be suitably defined. As further basic consequence, $f_{eq}^i(r,t)$ and $f_i(r,t)$ need not necessarily be Galilei-invariant (in particular they may not be invariant with respect to velocity translations), although the fluid equations must be necessarily fully Galilei-covariant. As a consequence it is always possible to select $f_{eq}^i(r,t)$ and $f_{oi}(r,t)$ based on convenience and mathematical simplicity. Thus, besides distributions which are Galilei invariant and satisfy a principle of maximum entropy (see for example [22, 23]), it is always possible to identify them [i.e., $f_{eq}^i(r,t), f_{oi}(r,t)$] with a non-Galilean invariant polynomial distribution. We mention that the non-uniqueness of the source term $\tilde{S}_i$ can be exploited also by imposing that $f_{eq}^i(r,t)$ results a particular solution of the inverse kinetic equation Eq.(2) and there results also $f_{oi}(r,t) = f_{eq}^i(r,t)$.

IV. 4 - CONCLUSIONS

In this paper we have presented the theoretical foundations of a new phase-space model for incompressible isothermal fluids, based on a generalization of customary lattice Boltzmann approaches. We have shown that many of the limitations of traditional (asymptotic) LBM’s can be overcome. As a main result, we have proven that the LB-IKT can be developed in such a way that it furnishes exact Navier-Stokes and Poisson solvers, i.e., it is - in a proper
sense - an inverse kinetic theory for INSE. The theory exhibits several features, in particular we have proven that the integral LB-IKT (see Sec.3):

1. determines uniquely the fluid pressure \( p(r, t) \) via the discrete kinetic distribution function without solving explicitly (i.e., numerically) the Poisson equation for the fluid pressure. Although analogous to traditional LBM’s, this is interesting since it is achieved without introducing compressibility and/or thermal effects. In particular the present theory does not rely on a state equation for the fluid pressure.

2. is complete, namely all fluid fields are expressed as momenta of the distribution function and all hydrodynamic equations are identified with suitable moment equations of the LB inverse kinetic equation.

3. allows arbitrary initial and boundary conditions for the fluid fields.

4. is self-consistent: the kinetic theory holds for arbitrary, suitably smooth initial conditions for the kinetic distribution function. In other words, the initial kinetic distribution function must remain arbitrary even if a suitable set of its momenta are prescribed at the initial time.

5. the associated the kinetic and equilibrium distribution functions can always be chosen to belong to the class of non-Galilei-invariant distributions. In particular the equilibrium kinetic distribution can always be identified with a polynomial of second degree in the velocity.

6. is non-asymptotic, i.e., unlike traditional LBM’s it does not depend on any small parameter, in particular it holds for finite Mach numbers.

The main result of the paper is represented by the construction of an explicit realization of the LB-IKT for the incompressible Navier-Stokes equations. The construction of a discrete inverse kinetic theory of this type for the incompressible Navier-Stokes equations represents an exciting development for the phase-space description of fluid dynamics, providing a new starting point for theoretical and numerical investigations based on LB theory. In our view, the route to more accurate, higher-order LBM’s, here pointed out, will be important in order to achieve substantial improvements in the efficiency of LBM’s in the near future.
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**Notice**

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[1] M. Ellero and M. Tessarotto, Bull. Am Phys. Soc. 45 (9), 40 (2000).

[2] M. Tessarotto and M. Ellero, Proc. 24th RGD, Bari, Italy (July 2004), Ed. M. Capitelli, AIP Conf. Proc. 762, 108 (2005).

[3] M. Ellero and M. Tessarotto, Physica A 355, 233 (2005).

[4] M. Tessarotto and M. Ellero, Physica A 373, 142 (2007); arXiv: physics/0602140 Proc. 25th RGD (International Symposium on Rarefied gas Dynamics, St. Petersburg, Russia, July 21-28, 2006), Ed. M.S. Ivanov and A.K. Rebrov (Novosibirsk Publ. House of the Siberian Branch of the Russian Academy of Sciences), p.1001; arXiv:physics/0611113 (2007).

[5] M. Tessarotto, M. Ellero, N. Aslan, M. Mond and P. Nicolini, “An exact pressure evolution equation for the incompressible Navier-Stokes equations Inversed”, arXiv:physics/0612072 (2006).

[6] M. Tessarotto, M. Ellero and P. Nicolini, Phys.Rev. A 75, 012105 (2007); arXiv:quantum-ph/060691.

[7] G.R. McNamara and G. Zanetti, Phys. Rev. Lett. 61, 2332 (1988).

[8] F. Higuera, S. Succi and R. Benzi, Europhys. Lett. 9.345 (1989).

[9] S. Succi, The Lattice Boltzmann Equation for Fluid Dynamics and Beyond (Numerical Mathematics and Scientific Computation), Oxford Science Publications (2001).

[10] R. Benzi et al., Phys. Rep. 222, 145 (1992).

[11] S. Chen, H. Chen, D. O. Martinez, and W. H. Matthaeus, Phys.Rev. Lett. 67, 3776 (1991).

[12] H. Chen, S. Chen, and W. Matthaeus, Phys. Rev. A 45, R5339 (1992).

[13] S. Ansumali and I. V. Karlin, Phys. Rev. E 65, 056312 (2002).

[14] Y. Shi, T. S. Zhao and Z. L. Guo, Phys. Rev.E 73, 026704 2006.

[15] X. Shan and X. He, Phys. Rev. Lett. 80, 65 (1998).

[16] X. He and L.S. Luo, Phys. Rev. E 55, R6333 (1997).

[17] S. Ansumali, I.V. Karlin, and H. C. Öttinger, Europhys.Lett. 63, 798 (2003).

[18] S.S. Chikatamarla and I.V. Karlin, Phys. Rev.Lett. 97, 190601 (2006).

[19] A. Bardow, I.V. Karlin, and A. A. Gusev, Europhys. Lett. 75, 434 (2006).

[20] S. Ansumali and I.V. Karlin, Phys. Rev. Lett. 95, 260605 (2005).

[21] Enrico Fonda, Massimo Tessarotto and Marco Ellero, Lattice Boltzmann inverse kinetic ap-
proach for the incompressible Navier-Stokes equations, arXiv:0704.0339 (2007).

[22] I.V. Karlin and S. Succi, Phys. Rev. E 58, R4053 (1998).

[23] I.V. Karlin, A. Ferrante, and H. C. Öttinger, Europhys. Lett. 47, 182 (1999).