Nonlinear perturbations of Reissner–Nordström black holes

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We develop a nonlinear perturbation theory of Reissner–Nordström black holes. We show that, at each perturbation level, Einstein–Maxwell equations can be reduced to four inhomogeneous wave equations, two for polar and two for axial sector. Gravitational part of these equations is similar to Regge–Wheeler and Zerilli equations with source and additional coupling to the electromagnetic sector. We construct solutions to the inhomogeneous part of wave equations in terms of sources for Einstein–Maxwell equations. We discuss \( \ell = 0 \) and \( \ell = 1 \) cases separately.

I. INTRODUCTION

Perturbative methods play an important role in General Relativity. They find application to stability analysis, gravitational radiation, cosmology, rotating stars, accretion disc, self-force, etc. Sometimes linear analysis give sufficient insight into physical phenomena, but sometimes going beyond linear order can change qualitatively linear predictions (e.g. Bizó–Rostworowski conjecture of instability of anti-de Sitter spacetime [1]). In the following paper, we study nonlinear perturbations of spherically symmetric solutions to Maxwell–Einstein equations. Linear perturbation theory of Schwarzschild solution was formulated by Regge and Wheeler [2] and Zerilli [3] and then generalised to Reissner–Nordström black hole by Zerilli [4] (see also [5], [6], [7], [8]). Linear perturbations of Reissner–Nordström have also been recently discussed in context of strong cosmic censorship conjecture (see [9], [10]). Taking into account higher–order perturbation terms makes the computations significantly more difficult: equations at each order beyond linear include all the previous–order terms. This issue has been treated by some authors - e.g. second order perturbations of Schwarzschild were studied by Tomita and Tajima [11], Garat and Price [12], Gleiser et al. [13], Nakano and Ioka [14], Brizuela et al. [15]. Recently, Rostworowski [16] provided a robust framework to deal with nonlinear (in principle of any order) gravitational perturbations of spherically symmetric spacetimes. Present article is an extension of [16] to both gravitational and electromagnetic nonlinear perturbations of Reissner–Nordström black holes.

Our approach is based on assumptions similar to those from [16]. We rewrite them explicitly here, since there are some differences:

1. At each perturbation level, there are four master scalar variables, two in polar and two in axial sector. In each sector, they fulfill a system of two linearly–coupled inhomogeneous (homogeneous at the linear order) wave equations with potentials.

2. At each perturbation level, Regge–Wheeler variables and electromagnetic tensor components are linear combinations of master scalar variables from the suitable sector and their derivatives up to the second order. At the nonlinear orders, one needs to include additional functions to fulfil Maxwell–Einstein equations.

3. At the linear level, relations from the previous point can be inverted to express master scalars as combinations of RW variables and electromagnetic tensor components. At the nonlinear level, we take the same expressions for the master scalar functions.

In our considerations, we restrict ourselves to axially–symmetric perturbations only (going beyond axial symmetry is a straightforward procedure, that conceptually adds little to this paper). During calculations, we stick to the Regge–Wheeler (RW) gauge. For practical implementations, after finding a solution in the RW gauge, one should move to an asymptotically flat gauge to ensure regularity of higher order source functions (see Brizuela et al. [15]).

The paper is organised as follows: in section II we briefly introduce Reissner–Nordström metric and in section III we discuss general form of perturbation expansion of Einstein–Maxwell equations. In sections IV and V we remind polar expansion in axial symmetry, discuss gauge choice and present source identities. The main result of this paper, namely providing inhomogeneous wave equations for Einstein–Maxwell equations of any perturbation order, is contained in VII.

II. BACKGROUND METRIC

Reissner–Nordström solution describes a static, spherically symmetric black hole with an electric charge. In static coordinates \((t \in (-\infty, \infty), r \in (r_+, \infty), \theta \in (0, \pi), \phi \in [0, 2\pi))\) it’s metric is given by (we use \(G = c = 4\pi\epsilon_0 = 1\)):

\[
\tilde{g} = -Adt^2 + \frac{1}{A}dr^2 + r^2d\Omega^2,
\]

(1)

where \(A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \ r_+ = M + \sqrt{M^2 - Q^2}, \) and \(M \) and \(Q\) are mass and charge of a black hole, respectively (we assume \(|Q| < M\)). Together with an electromagnetic tensor \(\tilde{F}\) with only nonzero terms \(\tilde{F}_{tr} = -\tilde{F}_{rt} = \frac{Q}{r}\), metric
solves Einstein–Maxwell equations:

\[ \bar{R}_{\mu\nu} = 8\pi \bar{T}_{\mu\nu}, \]
\[ \bar{\nabla}_\mu \bar{F}^{\mu\nu} = 0, \]
\[ \bar{F}_{(\mu\nu,\lambda)} = 0, \]

where \( \bar{\nabla} \) and \( \bar{R}_{\mu\nu} \) are, respectively, covariant derivative and Ricci tensor w.r.t. metric \( \bar{g} \) and comma denotes partial derivative. \( \bar{T}_{\mu\nu} \) is given by

\[ \bar{T}_{\mu\nu} = \frac{1}{4\pi} \left( \bar{F}_{\mu\alpha} \bar{F}_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} \bar{F}_{\alpha\beta} \bar{F}^{\alpha\beta} \right) \tag{5} \]

### III. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS OF EINSTEIN–MAXWELL SYSTEMS

Let’s assume that metric \( \bar{g} \) and electromagnetic tensor \( \bar{F} \) solve Einstein–Maxwell equations \( \text{(2)-(4)} \). Now we seek for new solutions \( g \) and \( F \) that we expand around \( \bar{g} \) and \( \bar{F} \) w.r.t. to the perturbation parameter \( \epsilon \):

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i>0} \epsilon^i h_{\mu\nu}^{(i)}, \]
\[ F_{\mu\nu} = \bar{F}_{\mu\nu} + \sum_{i>0} \epsilon^i f_{\mu\nu}^{(i)}, \]

We plug \( \text{(2)}, \text{(4)} \) into Einstein–Maxwell equations, to obtain their perturbative form of order \( i \):

\[ \Delta_L^{\mu\nu} h_{\mu\nu} - 8\pi^{(i)} t_{\mu\nu} = {^i}S^G_{\mu\nu}, \]
\[ \nabla^\mu f_{\mu\nu} - (i)\Theta_{\nu} = {^i}S^M_{\nu}, \]
\[ {^i}f_{(\mu\nu,\lambda)} = 0, \] \tag{10}

where

\[ \Delta_L^{\mu\nu} = \frac{1}{2} \left( -\bar{\nabla}_\mu \bar{\nabla}_\nu (i) h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu (i) h_{\alpha\beta} - 2\bar{R}_{\mu\nu\rho\sigma} (i) h_{\rho\sigma} + \bar{\nabla}_\mu \bar{\nabla}_\nu (i) h_{\nu\alpha} + \bar{\nabla}_\nu \bar{\nabla}_\mu (i) h_{\mu\alpha} \right), \]
\[ (i)h_{\mu\nu} = 2 (i) f_{\alpha(\mu} \bar{F}_{\nu)\alpha} - \frac{1}{2} (i) f_{\alpha\beta} \bar{F}_{\alpha\beta} g_{\mu\nu} + \left( \frac{1}{2} \bar{F}_{\alpha\sigma} \bar{F}_{\beta}^{\sigma} \bar{g}_{\mu\nu} - \bar{F}_{\mu\alpha} \bar{F}_{\nu\beta} \right) (i) h_{\alpha\beta} - \frac{1}{4} \bar{F}^2 (i) h_{\mu\nu} - (i) h_{\alpha(\mu} \bar{T}_{\nu)\alpha} \right), \]
\[ (i)\Theta_{\nu} = \bar{g}^{\alpha\beta} \left( \bar{F}_{\sigma\nu} (i) \bar{\Gamma}_{\alpha\beta}^{\sigma} + \bar{F}_{\beta\sigma} (i) \bar{\Gamma}_{\alpha\beta}^{\sigma} \right), \]
\[ (i)\bar{\Gamma}_{\alpha\beta} = \frac{1}{2} \bar{g}^{\rho\delta} \left( -\bar{\nabla}_\delta (i) h_{\alpha\beta} + \bar{\nabla}_\alpha (i) h_{\beta\delta} + \bar{\nabla}_\beta (i) h_{\delta\alpha} \right). \] \tag{14}

Tensor sources \( {^i}S^G_{\mu\nu} \) and vector sources \( {^i}S^M_{\nu} \) are expressed by \( (\ell<\ell) h_{\mu\nu} \) and \( (\ell<\ell) f_{\mu\nu} \), therefore perturbative Einstein equations should be solved by order (see Appendix A for the construction of sources). For \( i = 1 \) both sources vanish.

### IV. POLAR EXPANSION

In a spherically symmetric background, in 3+1 dimensions, vector and tensor components split into two sectors that transform differently under rotations: polar and axial (for the details see e.g. [2], [3], [18], [19]). Symmetric tensors have 7 polar and 3 axial components, and antisymmetric tensors have 3 polar and 3 axial components. Below we list expansions of all the components of both symmetric and antisymmetric tensors and of vectors in axial symmetry (\( P_\ell \) denotes \( \ell \)-th Legendre polynomial).
Antisymmetric tensor, polar sector:

\[ A_{tr}(t, r, \theta) = \sum_{0 \leq \ell} A_{\ell tr}(t, r) P_{\ell}(\cos \theta), \quad (21) \]

\[ A_{a\theta}(t, r, \theta) = \sum_{1 \leq \ell} A_{\ell a\theta}(t, r) \partial_\theta P_{\ell}(\cos \theta), \quad a = t, r. \quad (22) \]

Antisymmetric tensor, axial sector:

\[ A_{a\phi}(t, r, \theta) = \sum_{0 \leq \ell} A_{\ell a\phi}(t, r) \sin \theta \partial_\theta P_{\ell}(\cos \theta), \quad a = t, r, \quad (23) \]

\[ A_{\theta\phi}(t, r, \theta) = \sum_{0 \leq \ell} A_{\ell \theta\phi}(t, r) \sin \theta P_{\ell}(\cos \theta). \quad (24) \]

Vector, polar sector:

\[ V_a(t, r, \theta) = \sum_{0 \leq \ell} V_{\ell a}(t, r) P_{\ell}(\cos \theta), \quad a = t, r, \quad (25) \]

\[ V_\theta(t, r, \theta) = \sum_{1 \leq \ell} V_{\ell \theta}(t, r) \partial_\theta P_{\ell}(\cos \theta). \quad (26) \]

Vector, axial sector:

\[ V_\phi(t, r, \theta) = \sum_{1 \leq \ell} V_{\ell \phi}(t, r) \sin \theta \partial_\theta P_{\ell}(\cos \theta). \quad (27) \]

Since the background is spherically symmetric, differential operators acting on \( h_{\mu\nu} \) and \( f_{\mu\nu} \) do not mix axial and polar sectors, therefore Einstein–Maxwell equations split into two sectors as well: there are 7 Einstein and 3 Maxwell equations in polar sector, and 3 Einstein and 1 Maxwell equations in axial sector. In our paper we consider separately \( \ell \geq 2 \), \( \ell = 1 \) and \( \ell = 0 \).

**V. GAUGE CHOICE**

Under a gauge transformation induced by a vector \( X^\mu \), tensors transform as \( t_{\mu\nu} \to t_{\mu\nu} + \mathcal{L} X t_{\mu\nu} \) (see Appendix \[ \text{B} \] for the explicit form of transformations). Throughout the paper we use Regge–Wheeler gauge \[ \text{[2]} \], namely we set \((i) h_{\ell tr\ell}, (i) h_{\ell \theta\ell} \) and \((i) h_{\ell r\ell} \) to zero in polar sector, and \((i) h_{\ell \theta\phi} = 0 \) in axial sector. When the background quantities \( \tilde{g} \) and \( \tilde{F} \) fulfill Einstein equations, left hand sides of perturbation equations (39–40) of order \( i \) do not feel gauge transformations of order \( i \), but source functions \((i) S^G_{\mu\nu} \) and \((i) S^\mu_{\ell \nu} \) depend on the gauge transformations of order \( j < i \) explicitly, so such a formulation is not fully gauge invariant. This, however, is not a problem, since equations are solved order by order and for the practical implementations one goes to the asymptotically flat gauge before moving to the next order anyway.

**VI. SOURCES FOR EINSTEIN–MAXWELL EQUATIONS**

Sources \((i) S^G_{\ell \mu\nu} \) and \((i) S^M_{\ell \nu} \) are built of \((j) h_{\ell \mu\nu} \) and \((j) f_{\ell \mu\nu} \) with \( j < i \). These sources are not independent, but fulfill five identities:

\[ \nabla^\mu (i) S^G_{\mu\nu} - \frac{1}{2} \nabla^\nu (i) S^G_{\mu\mu} - 2 \bar{F}^\mu_{\nu} (i) S^M_{\mu\nu} = 0, \quad (28) \]

\[ \nabla^\mu (i) S^M_{\mu\nu} = 0, \quad (29) \]

which come from Bianchi identity and contracted Jacobi identity for tensor \( F_{\mu\nu} \). One can check that they hold using (39–40) directly. Explicit form of identities (28), (29) for polar–expanded sources in polar sector reads (we introduce \( \tau = \sqrt{(\ell - 1)(\ell + 2)} \)):

\[
\left( A' + \frac{2A}{r} \right) (i) S^G_{\ell \mu\nu} + \frac{2Q}{r^2} (i) S^M_{\ell \mu} + A \partial_\tau (i) S^G_{\ell \mu\nu} - \frac{1}{2A} \partial_\ell (i) S^G_{\ell \mu\nu} - \frac{1}{2} A \partial_\ell (i) S^G_{\ell \mu\nu} - \frac{\ell (\ell + 1)}{r^2} (i) S^G_{\ell \theta\ell} - \frac{1}{r^2} \partial_\ell (i) S^G_{\ell \ell} + 0,
\]

\[
\left( A' + \frac{2A}{r} \right) (i) S^G_{\ell \mu\nu} + \frac{2Q}{r^2} (i) S^M_{\ell \mu} + \frac{1}{2} A \partial_\tau (i) S^G_{\ell \mu\nu} + \frac{1}{2} A \partial_\ell (i) S^G_{\ell \mu\nu} - \frac{\ell (\ell + 1)}{r^2} (i) S^G_{\ell \theta\ell} - \frac{1}{r^2} (i) S^G_{\ell \ell} + 0,
\]

\[
\left( A' + \frac{2A}{r} \right) (i) S^G_{\ell \mu\nu} + \frac{1}{2A} (i) S^G_{\ell \mu\nu} - \frac{1}{2} A (i) S^G_{\ell \mu\nu} + A \partial_\tau (i) S^M_{\ell \mu\nu} - \frac{1}{2} A \partial_\ell (i) S^M_{\ell \mu\nu} - \frac{\tau^2}{r^2} (i) S^G_{\ell \ell} + 0,
\]

\[
\left( A' + \frac{2A}{r} \right) (i) S^M_{\ell \mu\nu} + A \partial_\tau (i) S^M_{\ell \mu\nu} - \frac{\ell (\ell + 1)}{r^2} (i) S^M_{\ell \mu\nu} = 0,
\]

and in axial sector:

\[
\left( A' + \frac{2A}{r} \right) (i) S^G_{\ell \mu\nu} + A \partial_\tau (i) S^G_{\ell \mu\nu} - \frac{\partial_\ell (i) S^G_{\ell \mu\nu}}{A} - \frac{\tau^2 (i) S^G_{\ell \mu\nu}}{r^2} = 0.
\]
VII. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS

Now we polar–expand equations \(38\)–\(40\):

\[(i) E_{\ell \mu \nu} = \Delta_L (i) h_{\ell \mu \nu} - 8\pi (i) t_{\ell \mu \nu} = (i) S^G_{\ell \mu \nu}, \]

\[(i) J_{\ell \nu} = \nabla u (i) f_{\ell \mu \nu} - (i) \Theta_{\ell \nu} = (i) S^M_{\ell \nu}, \]

\[(i) f_{\ell (\mu \nu, \alpha)} = 0. \]  

(A) Polar sector, \(\ell \geq 2\)

Firstly, from \(41\) we have:

\[(i) f_{\ell tr} = \partial_r (i) f_{\ell t \theta} - \partial_\ell (i) f_{\ell r \theta}, \]

and from \((i) E_{\ell -}\):

\[
\frac{1}{4} \left( A (i) h_{\ell uu} - A (i) h_{\ell rr} \right) - (i) S_{\ell -} = 0. \]

We use relations \(38\) and \(39\) to eliminate \((i) f_{\ell tr}\) and \((i) h_{\ell uu}\) from equations \(35\) and \(37\). Then we are left with 5 variables: \((i) h_{\ell uu}, (i) h_{\ell tr}, (i) h_{\ell +}, (i) f_{\ell t \theta}\) and \((i) f_{\ell r \theta}\).

Remaining equations can be fulfilled by introducing two master scalar variables \((i) \Phi^p_{\ell}\) and \((i) \Psi^p_{\ell}\) which solve a system of two coupled inhomogeneous (homogeneous at the linear order) wave equations \(20\):

\[
r (\square + V^P_{G \ell}) (i) \Phi^p_{\ell} = (i) \Phi^p_{\ell}, \]

\[
r (\square + V^P_{M \ell}) (i) \Psi^p_{\ell} = (i) \Phi^p_{\ell}. \]

Following the idea of \(17\), we express leftover variables by linear combinations of master scalar functions, their partial derivatives up to the second order (to solve homogeneous part of Einstein–Maxwell equations) and additional source functions (to solve inhomogeneous part of equations). These combinations and potentials \(V^P_{G \ell}, V^P_{M \ell}, V^P_{MG \ell}\) are defined uniquely:

\[
V^P_{G \ell} = \tau^2 V^P_{G \ell} = \frac{\tau^2}{r^2} \left( -\tau^2 A + 2 A (2 r^2 A + (r + 1)^2) - \ell^2 (\ell + 1)^2 \right) + \frac{8 Q^2 \tau^2 A}{r^4 (r A' - 2 A + \ell (\ell + 1))}, \]

\[
V^P_{M \ell} = \frac{r}{r^2} \left( 2 r^2 A + r^2 A' + (r + 1)^2 \right) + \frac{4 Q^2}{r^6 (r A' - 2 A + \ell (\ell + 1))} + \frac{8 r^2 A}{r^4 (r A' - 2 A + \ell (\ell + 1))}, \]

\[
V^P_{MG \ell} = \tau^2 V^P_{MG \ell} = 2 r Q A \left( 2 A (2 r^2 A + (r + 1)^2) - \ell^2 (\ell + 1)^2 \right) + \frac{8 Q^2 \tau^2 A}{r^4 (r A' - 2 A + \ell (\ell + 1))}, \]

\[
(i) h_{\ell uu} = - r \partial_r (i) \Phi^p_{\ell} + \frac{r A'}{2 A} \left( \frac{\tau^2}{r^2} A + (r + 1)^2 \right) - \ell^2 (\ell + 1)^2 + \frac{2 r Q \partial_r}{r (r A' - 2 A + \ell (\ell + 1))} (i) \Phi^p_{\ell} + (i) \alpha_{\ell}, \]

\[
(i) h_{\ell rr} = - r \partial_r (i) \Psi^p_{\ell} + \frac{r A'}{2 A} \left( \frac{\tau^2}{r^2} A + (r + 1)^2 \right) - \ell^2 (\ell + 1)^2 + \frac{2 r Q \partial_r}{r (r A' - 2 A + \ell (\ell + 1))} (i) \Phi^p_{\ell} + (i) \beta_{\ell}, \]

\[
(i) f_{\ell t \theta} = \frac{A}{4} \partial_r (i) \Phi^p_{\ell} + \frac{Q A}{2 r} \partial_r (i) \Phi^p_{\ell} + \frac{Q A}{2 r} (i) \Phi^p_{\ell} + (i) \lambda_{\ell}, \]

\[
(i) f_{\ell r \theta} = \frac{\tau}{4 A} \partial_r (i) \Psi^p_{\ell} - \frac{Q}{2 r A} \partial_r (i) \Phi^p_{\ell} + (i) \kappa_{\ell}. \]

At the linear level \((i) \alpha_{\ell} = (i) \beta_{\ell} = (i) \gamma_{\ell} = (i) \lambda_{\ell} = (i) \kappa_{\ell} = 0\) and relations \(35\)–\(37\) can be inverted to express \((i) \Phi^p_{\ell}\) and \((i) \Psi^p_{\ell}\) as functions of \((i) h_{\ell uu}\) and \((i) f_{\ell uu}\). At higher orders, we treat linear level expressions for \((i) \Phi^p_{\ell}\) and \((i) \Psi^p_{\ell}\) as definitions of \((i) \Phi^p_{\ell}\) and \((i) \Psi^p_{\ell}\):
Having these definitions, we may express left hand side of \([40], [41]\) as combinations of \((i)E_{\mu
u}, (i)J_{\nu},\) and their derivatives. Finding these combinations, we use \([35]\) and \([36]\) to build sources for wave equations:

\[
(i)\Phi_{\ell} = \frac{4rA (r \partial_{r}(i)h_{\ell+} - A(i)h_{\ell r r})}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))} - \frac{2r(i)h_{\ell+}}{\ell(\ell+1)},
\]

\[
(i)\Psi_{\ell} = \frac{4r^2 (\partial_{r}(i)f_{\ell+} - \partial_{t}(i)f_{\ell+})}{\ell(\ell+1)} + \frac{8QA (r \partial_{r}(i)h_{\ell+} - A(i)h_{\ell r r})}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))}.
\]

\[(i)\tilde{S}_{G_{\ell}}^{P} = \frac{4A^2 \left( 2r^2 r^2 + 4Q^2 \right) (i)S_{G_{\ell}}^{rr} - \frac{4(i)S_{G_{\ell}}^{G_{\ell} tt} (2r^3 A' - 4r^2 A + (\ell(\ell+1)) r - 4Q^2)}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))} + \frac{8A \partial_{r}(i)S_{G_{\ell}}^{r r}}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))} + \frac{8A(i)S_{G_{\ell}}^{G_{\ell} r r}}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))} - \frac{4V_{G_{\ell}}^P(i)S_{G_{\ell}}^{G_{\ell} +}}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))},
\]

\[
\frac{(i)\tilde{S}_{M_{\ell}}^{P}}{4} = \frac{(i)\tilde{S}_{M_{\ell}}^{P}}{4}
\]

\[
\frac{r^2 \partial_{r}(i)S_{M_{\ell}}^{P}}{4} - \frac{r^2 \partial_{t}(i)S_{M_{\ell}}^{r r}}{4 - (i)S_{M_{\ell}}^{r r}} \left( 2r - \frac{8Q^2}{r (r A' - 2A + \ell(\ell+1))} \right) + \frac{8Q \left( \frac{r^2 (r A' - 2A + 2(\ell(\ell+1))) + Q^2}{4} \right)}{r^2 (r A' - 2A + \ell(\ell+1))} + \frac{r A^2 (i)S_{M_{\ell}}^{G_{\ell} rr}}{A} - \frac{2Q A(i)S_{G_{\ell}}^{G_{\ell} +}}{A} - \frac{2Q A(i)S_{G_{\ell}}^{G_{\ell} -}}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))}.
\]

\[(i)\alpha_{\ell} = \frac{-2r^2 \left( r^2 A^2 (i)S_{G_{\ell}}^{rr} + r^2 (i)S_{G_{\ell}}^{G_{\ell} tt} + 2A^2 (i)S_{G_{\ell}}^{G_{\ell} +} \right)}{\ell(\ell+1)r^2 (r A' - 2A + \ell(\ell+1))} + \frac{16Q^2 A(i)S_{G_{\ell}}^{G_{\ell} -}}{\ell(\ell+1)(r A' - 2A + \ell(\ell+1))},
\]

\[(i)\beta_{\ell} = r \left( \frac{2r(i)S_{G_{\ell}}^{G_{\ell} tr}}{\ell(\ell+1)} + \frac{\partial_{t}(i)\alpha_{\ell}}{A} \right),
\]

\[(i)\gamma_{\ell} = \frac{r \partial_{t}(i)\alpha_{\ell} + (i)\alpha_{\ell}}{A} - \frac{(i)\alpha_{\ell} (r A' + \ell(\ell+1))}{A^2},
\]

\[(i)\kappa_{\ell} = \frac{r (i)S_{G_{\ell}}^{G_{\ell} r r}}{\ell(\ell+1) + \frac{2Q A(i)S_{G_{\ell}}^{G_{\ell} -}}{A(\ell+1)}},
\]

\[(i)\lambda_{\ell} = \frac{r^2 (i)S_{M_{\ell}}^{G_{\ell} r r}}{\ell(\ell+1)} + \frac{2QA(i)S_{G_{\ell}}^{G_{\ell} +}}{\ell(\ell+1)}.
\]
B. Polar sector, \( \ell = 1 \)

For \( \ell = 1 \), there is no \( S_\ell \) coefficient in a symmetric tensor decomposition, therefore we don’t have \( \hat{h}_\ell \) metric coefficient and we loose algebraic Einstein equation (39). However, since one of the gauge conditions was \( \hat{h}_\ell = 0 \), we gain additional gauge freedom, which we can use to keep algebraic relation (39). That means, the only obstacle is that for \( \ell = 1 \) coefficient \( \tau = 0 \) and singular terms appear in the source for wave equation (41) and in the definition (51). We can deal with it introducing \( \hat{h}_\ell \), which, together with \( \hat{\Phi}_\ell \), fulfils a set of wave equations:

\[
\begin{align*}
 r(-\square + \tau^2 \hat{V}_G^\ell) \frac{(i)\hat{\Phi}_\ell^P}{r} + \hat{V}_{MG}^\ell \frac{(i)\hat{\Phi}_\ell^P}{r} &= (i)\hat{S}_G^\ell, \\
 r(-\square + V_M^\ell) \frac{(i)\hat{\Phi}_\ell^P}{r} + \tau^2 \hat{V}_{MG}^\ell \frac{(i)\hat{\Phi}_\ell^P}{r} &= (i)\hat{S}_M^\ell,
\end{align*}
\]

where \( \hat{V}_G^\ell \), \( \hat{V}_{MG}^\ell \) and \( (i)\hat{S}_M^\ell \) are defined in (42), (44) and (45). For \( \ell = 1 \) the system is simpler - there is no coupling to gravitational master scalar in (67). Now scalar sources for both equations are regular for \( \ell = 1 \). Metric and electromagnetic tensor perturbations are then given by:

\[
\begin{align*}
(i)h_{1\,tr} &= -r \partial_r (i)\Phi_i^P + \frac{r A'}{2A} \partial_r (i)\Phi_i^P + \frac{2Q\partial_r}{r (rA' - 2A + 2)} \frac{(i)\hat{\Phi}_\ell^P}{r} + (i)\alpha_I, \\
(i)h_{1\,rr} &= -r \partial_r (i)\Phi_i^P + \frac{r A'}{2A} \partial_r (i)\Phi_i^P + \frac{2Q\partial_r}{r (rA' - 2A + 2)} \frac{(i)\hat{\Phi}_\ell^P}{r} + (i)\beta_I, \\
(i)h_1 &= -A \partial_r (i)\Phi_i^P + \frac{A - 1}{r} \frac{(i)\hat{\Phi}_\ell^P}{r} - \frac{2QA}{r^2 (rA' - 2A + 2)} \frac{(i)\hat{\Phi}_\ell^P}{r} + (i)\gamma_I, \\
(i)f_{1\,t\theta} &= \frac{A}{4} \partial_r (i)\Phi_i^P - \frac{QA}{2r} \partial_r (i)\Phi_i^P + \frac{QA}{2r^2} (i)\Phi_i^P + (i)\lambda_I, \\
(i)f_{1\,r\theta} &= \frac{Q}{4A} \partial_r (i)\Phi_i^P - \frac{Q}{2rA} \partial_r (i)\Phi_i^P + (i)\kappa_I.
\end{align*}
\]

Since there is no \( (i)S_\ell^G \) - source term, \( (i)\alpha_I, (i)\beta_I, (i)\gamma_I, (i)\lambda_I, (i)\kappa_I \) for \( \ell = 1 \) are given by:

\[
\begin{align*}
(i)\alpha_I &= -\frac{r^2 A^2 (i)\Phi_i^P + r A' (i)\Phi_i^P + 2A (i)\Phi_i^P}{(rA' - 2A + 2)}, \\
(i)\beta_I &= \frac{r^2 A^2 (i)\Phi_i^P + r \partial_r (i)\Phi_i^P}{A}, \\
(i)\gamma_I &= \frac{r \partial_r (i)\Phi_i^P + (i)\alpha_I (rA' + 2)}{2A^2}, \\
(i)\lambda_I &= \frac{r^2}{2} (i)\Phi_i^P, \\
(i)\kappa_I &= \frac{r^2}{2} (i)\Phi_i^P.
\end{align*}
\]

Although direct implementation of previous results provides a general solution to \( \ell = 1 \) equations, it can be misleading: it looks like there are two dynamical variables, whereas there should be only one \( \hat{h}_\ell \) (for Schwarzschild case \( \ell = 1 \) gravitational modes are pure gauge (21)). However, by the following gauge transformation, one can get rid of \( (i)\Phi_i^P \) from (82)-(86):

\[
\begin{align*}
(i)\xi_{1\,t} &= -\partial_t (i)\Phi_i^P, \\
(i)\xi_{1\,r} &= \frac{r (i)\Phi_i^P}{r} - \partial_r (i)\Phi_i^P, \\
(i)\xi_{1\,\theta} &= -\frac{r}{2} (i)\Phi_i^P.
\end{align*}
\]
pendent equations) we have respectively:

\begin{align}
(i) h_{1tt} &= -\frac{2A^2Q}{r (r A' - 2A + 2)} \partial_r (i) \tilde{\Psi}_1 - \frac{r A}{2} V_{MG}^P (i) \tilde{\Psi}_1^P + A^2(i) \beta_1 + r A (i) \tilde{S}_G^\ell, \\
(i) h_{1tr} &= -\frac{2Q \partial_r}{r (r A' - 2A + 2)} (i) \tilde{\Psi}_1^P + (i) \alpha_1, \\
(i) h_{1rr} &= -\frac{2Q}{r (r A' - 2A + 2)} \partial_r (i) \tilde{\Psi}_1^P + \frac{r}{2A} V_{MG}^P (i) \tilde{\Psi}_1^P + (i) \beta_1, \\
(i) h_{1+} &= -\frac{2QA}{r (r A' - 2A + 2)} (i) \tilde{\Psi}_1^P + r^2 (i) \gamma_1, \\
(i) f_{1\theta} &= \frac{A}{4} \partial_r (i) \tilde{\Psi}_1^P + (i) \lambda_1, \\
(i) f_{1\varphi} &= \frac{1}{4A} \partial_t (i) \tilde{\Psi}_1^P + (i) \kappa_1.
\end{align}

(81) - (86)

The cost of performing this transformation is the loss of algebraic relation (39). From our results one can also move to a gauge used by some authors (6, 9) in which \((i) h_{1+} = 0\).

C. Polar sector, \(\ell = 0\)

In this case we follow Rostworowski [17]. Using gauge freedom we set \((i) h_{0+} = 0\) and \((i) h_{0tr} = 0\) and leftover nonzero variables are \((i) h_{0tt}, (i) h_{0rr}\) and \((i) f_{0tr}\). From \((i) E_\theta 01, (i) E_{\varphi 00} + A^2(i) E_{\varphi 11}\) and \((i) J_{\theta 1}\) (the only independent equations) we have respectively:

\begin{align}
\frac{A}{r} \partial_t (i) h_{0rr} &= (i) S_{0r}^G, \\
\frac{A}{r} \partial_r \left(A (i) h_{0rr} - \frac{(i) h_{0tt}}{A}\right) &= (i) S_{0r}^G + A (i) S_{rr}^G, \\
\partial_t \left[(i) f_{0tr} + Q \frac{(i) h_{0tt}}{A} - A (i) h_{0rr}\right] &= -A (i) S_{r}^M.
\end{align}

(87) - (89)

These equations can be therefore integrated directly, starting from (81).

D. Axial sector, \(\ell \geq 2\)

Firstly, we use (37) to obtain:

\begin{align}
(i) f_{\ell \theta} &= -\frac{\partial_t f_{\ell \varphi}}{\ell (\ell + 1)}, \\
(i) f_{\ell \varphi} &= -\frac{\partial_r f_{\ell \varphi}}{\ell (\ell + 1)}.
\end{align}

(90) - (91)

We are left with three variables \((i) h_{\ell \theta}, (i) h_{\ell \varphi}\) and \((i) f_{\ell \varphi}\). In the same manner as before, we can fulfill equations (35) - (39) by introducing two master scalar variables \((i) \Phi_\ell^A\) and \((i) \Psi_\ell^A\), which solve a system of two coupled wave equations:

\begin{align}
r (-\Box + V_{G\ell}^A) \frac{(i) \Phi_\ell^A}{r} + V_{MG\ell}^A (i) \Psi_\ell^A &= (i) \tilde{S}_G^\ell, \\
r (-\Box + V_{M\ell}^A) \frac{(i) \Psi_\ell^A}{r} + V_{MG\ell}^A (i) \Phi_\ell^A &= (i) \tilde{S}_M^\ell.
\end{align}

(92) - (93)

Following the procedure described in the previous section, we find three potentials and express \(h_{\ell \theta}, h_{\ell \varphi}\) and \(f_{\ell \varphi}\) by master scalars and their derivatives:

\begin{align}
V_{G\ell}^A &= r^2 (A - 3r A') + (r^2 + 1) r^2 - Q^2, \\
V_{M\ell}^A &= -A r^3 + \ell (\ell + 1) r^2 + 4Q^2, \\
V_{MG\ell}^A &= -\frac{2Q}{r^3}, \\
(i) h_{\ell \theta} &= A \partial_t \left[ (i) \Phi_\ell^A + (i) \sigma_\ell \right], \\
(i) h_{\ell \varphi} &= A \partial_r \left[ (i) \Phi_\ell^A + (i) \chi_\ell \right], \\
(i) f_{\ell \varphi} &= \frac{1}{2} \ell (\ell + 1) r (i) \Phi_\ell^A + (i) \delta_\ell.
\end{align}

(94) - (99)

Now we invert above relations for linear order and treat the following expressions as definitions of \((i) \Phi_\ell^A\) and \((i) \Psi_\ell^A\) at the nonlinear order:

\begin{align}
(i) \Phi_\ell^A &= \frac{r}{\ell (\ell + 1)} \left[ (i) \partial_t h_{\ell \theta} - (i) \partial_r h_{\ell \varphi} - 2 (i) h_{\ell \varphi}\right] + \\
&\quad + \frac{4Q (i) f_{\ell \varphi}}{r^2}, \\
(i) \Psi_\ell^A &= \frac{2 (i) f_{\ell \varphi}}{\tau \ell (\ell + 1)}.
\end{align}

(100) - (101)

Finally, we find inhomogeneous functions \((i) \sigma_\ell, (i) \chi_\ell, (i) \delta_\ell\),
gauge freedom to set (i)\(\delta t = 0\),

and scalar sources (i)\(\tilde{S}_G^A\), (i)\(\tilde{S}_M^A\):}

\[
(i)\tilde{S}_G^A = \frac{2r}{\tau^2} (\partial_t (i)S_{t\phi}^G - \partial_t (i)S_{r\phi}^G),
\]

\[
(i)\tilde{S}_M^A = \frac{2r}{\tau} (i)S_{t\phi}^M.
\]

E. Axial sector, \(\ell = 1\)

Since (i)\(h_{t\phi}\) does not appear for \(\ell = 1\), we can use gauge freedom to set (i)\(h_{1, r\phi} = 0\). From (37) we have:

\[
(i)f_{1, t\phi} = -\frac{\partial_t (i)f_{1, \phi}}{2},
\]

\[
(i)f_{1, r\phi} = -\frac{\partial_t (i)f_{1, \phi}}{2}.
\]

Remaining equations contain (i)\(h_{1, t\phi}\) and (i)\(f_{1, \theta\phi}\) only. From (i)\(E_{1, r\phi} = (i)S_{1, r\phi}\) we find:

\[
-\frac{r^2}{2A} \partial_r \left( \frac{(i)h_{1, t\phi}}{r^2} \right) - \frac{Q(i)f_{1, \phi}}{Ar^2} + \eta(r) = \int^t (i)S_{r\phi}^G dt',
\]

where \(\eta(r)\) is some function of \(r\). It is not arbitrary - from (i)\(E_{1, t\phi} = (i)S_{1, t\phi}\) and source identity (33) we find \(\eta = \frac{C_1}{r^2}\), \(C_1\) being an arbitrary constant.

Let’s introduce (i)\(\Psi_i^A\) such that (i)\(\Psi_i^A = (i)\Psi_i^A + \frac{4\ell Q}{\ell Q(r^2A^2 + 2A - 1)}\). From (39) we find that (i)\(\Psi_i^A\) fulfills an inhomogeneous (homogeneous at the linear level) wave equation:

\[
r(-\Box + V_{Mi}\frac{(i)\Psi_i^A}{r}) = (i)\tilde{S}_M^A,
\]

where:

\[
V_{Mi}^A = \frac{4Q^2 - r^3A' + 2r^2}{r^4},
\]

\[
(i)\tilde{S}_M^A = \frac{2(i)S_{t\phi}^M}{\tau} - \frac{4AQ}{\tau^2} \int^t (i)S_{r\phi}^G dt'.
\]

We note that at the linear level setting (i)\(\Psi_i^A = 0\) corresponds to the linearised Kerr–Newman metric.

VIII. SUMMARY

Nonlinear perturbation theory of Reissner–Nordström solution was not present in the literature so far and present article fills this gap. Basing on a systematic approach to gravitational perturbations by Rostworowski [17], we have shown that one can fulfill perturbative Einstein–Maxwell equations at any perturbation order by solving two inhomogeneous master wave equations at each sector (cases \(\ell = 0, 1\) needed special treatment). This makes treatment of higher-order perturbations of Reissner–Nordström clear and would be especially useful for the the numerical purposes. To summarise, a complete order by order algorithm of solving Einstein–Maxwell equations within our formalism would be:

1. Solve wave equations (40), (41), (92), (93) and calculate RW variables and electromagnetic tensor components according to (55), (60), (105), (106).

2. Move to asymptotically flat gauge and calculate sources to Einstein–Maxwell equations (Appendix A).

3. Construct sources to wave equations (equations (55), (60), (105), (106)) and move to the next order.

Applications of presented calculations possibly include nonlinear studies on strong censorship conjecture and on astrophysical systems, where electromagnetic filed is taken into account.

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Appendix A: Sources for Einstein–Maxwell equations

Let’s fix index \(i\) and assume that we already know the solution to Einstein–Maxwell equations (55), (60) up to \(i\)-th order:

\[
\tilde{g}_{\mu\nu} = \sum_{j=1}^{i} \sum_{\ell} (j)\hat{h}_{\ell\mu\nu},
\]

\[
\tilde{F}_{\mu\nu} = \sum_{j=1}^{i} \sum_{\ell} (j)\hat{f}_{\ell\mu\nu}.
\]

Using this solution we can calculate Einstein tensor \(G_{\mu\nu}(\tilde{g})\) and energy–momentum tensor \(T_{\mu\nu}(\tilde{g}, \tilde{F})\). Although these tensors fulfill Einstein–Maxwell equations up to order \(i\), they contribute to the \(i+1\) (and higher) perturbation equations. Finally, tensor and vector sources
of order $i+1$ are given by:

$$(i+1)S_{G}^{\mu\nu} = [i + 1] \left( -G_{\mu\nu}(\tilde{g}) + 8\pi T_{\mu\nu}(\tilde{g}, \tilde{F}) \right),$$

$$(i+1)S_{E}^{\nu} = [i + 1] \left( -\nabla^{\mu}(\tilde{g}\omega_{\nu})\tilde{F}_{\mu\nu} \right),$$

where $[k] (...)$ denotes the k-th order expansion in $\epsilon$ of a given quantity.

Although in most cases expressions for the sources $(i+1)S_{G}^{\mu\nu}$ and $(i+1)S_{E}^{\nu}$ are complicated, their construction is a purely algebraic task and can be easily performed using computer algebra.

**Appendix B: Gauge transformations**

Under a gauge transformation $x^\mu \rightarrow x^\mu + X^\mu$, tensors transform as $t_{\mu\nu} \rightarrow t_{\mu\nu} + L_{\alpha} t_{\alpha\mu\nu}$. For $X^\mu = (i)\zeta^\mu \epsilon^i$,

perturbation functions of order $i$ transform in the following way:

$$(i)h_{t\nu} \rightarrow (i)h_{t\nu} + L(\zeta)\tilde{g}_{t\nu},$$

$$(i)f_{t\nu} \rightarrow (i)f_{t\nu} + L(\zeta)\tilde{F}_{t\nu}.$$  

Explicit form of these transformations in polar sector is the following:

$$(i)h_{\ell\ell} \rightarrow (i)h_{\ell\ell} + 2\partial_{\ell}(i)\zeta_{\ell} - A\partial^{(i)}(\zeta_{\ell}),$$

$$(i)h_{t\ell} \rightarrow (i)h_{t\ell} + \partial_{t}(i)\zeta_{\ell} + \partial_{\ell}(i)\zeta_{t} - \frac{A}{r}(i)\zeta_{t},$$

$$(i)h_{t\theta} \rightarrow (i)h_{t\theta} + \partial_{t}(i)\zeta_{\theta} + (i)\zeta_{t},$$

$$(i)h_{trr} \rightarrow (i)h_{trr} + 2\partial_{r}(i)\zeta_{\theta} + A\partial_{t}(i)\zeta_{\theta}.$$  

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