Super-replication with transaction costs under model uncertainty for continuous processes

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Abstract
We formulate a superhedging theorem in the presence of transaction costs and model uncertainty. Asset prices are assumed continuous and uncertainty is modeled in a parametric setting. Our proof relies on a new topological framework in which no Krein–Smulian type theorem is available.

1 INTRODUCTION

Robust super-replication has a rich literature, starting from the seminal work of Hobson (1998) about lookback options. In that paper, bounds for option prices and the connection with the Skorokhod embedding are established. The approach was employed further by Brown et al. (2001) for barrier options, Cox and Obłój (2011) for double no-touch options, Hobson and Neuberger (2012) for forward start options, Carr and Lee (2010) for variance options, among others. We refer to the survey of Hobson (2011) and references therein.

The use of optimal transport in robust hedging was developed in Galichon et al. (2014), Tan and Touzi (2013), Beiglböck et al. (2013), Dolinsky and Soner (2014b). These works provided a dual formulation, which transforms the superhedging problem into a martingale optimal transportation problem. In a continuous time setting, Dolinsky and Soner (2014b) proved duality and the existence of a family of simple, piecewise constant super-replication strategies that...
asymptotically achieve the minimal super-replication cost. In the quasi-sure setting, Denis and
Martini (2006) established a theoretical framework to construct stochastic integral and duality
results, while Nutz (2014) and Bouchard and Nutz (2015) obtained optimal super-replication
strategies in discrete time. Various frameworks are investigated in Acciaio et al. (2016), Davis et al.
(2014), Cheridito et al. (2017), Hou and Oblój (2018), Burzoni et al. (2016), and Bartl et al. (2019).

Under transaction costs, we refer to the discrete-time studies (Bouchard et al., 2019; Dolinsky & Soner,
2014a). As pointed out by Bouchard et al. (2019), it is not easy to come up with a proof of the superhedging
duality using the local method, developed in Bouchard and Nutz (2015), which uses one-period argument and then measurable selection techniques to paste the periods

In Bouchard et al. (2019), the authors introduced a fictitious market without transaction cost using a randomization technique, then the results of Bouchard and Nutz (2015) were applied.

In continuous-time markets with transaction costs, however, the only results we know of are those
of Dolinsky and Soner (2017). They establish, under technical conditions, that the pathwise super-
hedging price is the same as the superhedging price for a continuous price process satisfying the
conditional full support property.

The study Biagini et al. (2017) is close in spirit to our paper, hence, a brief comparison is
in place. In Biagini et al. (2017), arbitrage in the quasi-sure sense was analyzed in frictionless
markets with continuous price processes, using the “no arbitrage of the first kind” assumption
NA₁(P), where P denotes the set of possible prior probabilities. A superhedging theorem is provided
under this hypothesis. A crucial step in their arguments shows that NA₁(P) is equivalent to having
NA₁(P) for all individual P ∈ P. In the present paper, somewhat analogously, we assume a
“no free lunch with vanishing risk” condition (stronger than no arbitrage of the first kind) for
each individual model and deduce the superhedging theorem (with transaction costs) under this
assumption. Note that we operate with “martingale” price systems while Biagini et al. (2017) uses
“supermartingale deflators.”

In the present article, we prove a fairly general superhedging theorem. We take a parametriza-
tion approach, different from the pathwise or quasi-sure settings, first used in Chau and Rásonyi
(2019), Rásonyi and Meireles-Rodrigues (2021), see also Chau (2020). As far as we know, the
present paper is the first to apply to a wide range of continuous-time markets under transaction
costs and model uncertainty. We now list the main features of the present paper.

First, we use the new topological framework of Chau (2020) for studying hedging under model
uncertainty, this time in a continuous-time setting. The working space is the product topological
vector space L∞ = ⨉θ∈Θ(L∞, w⁺), where we denote the product topology by w⁺. Here Θ is the
set of parameters describing uncertainty, and w⁺ is the weak star topology on L∞, the space of a.s.
equivalence classes of bounded random variables. Unlike the local method of Bouchard and Nutz
(2015), this approach offers a global method, which is suitable for handling transaction costs in
continuous time.

Second, for the separation arguments to work, we need the w⁺-closedness of the set of hedge-
able payoffs C in the product space L∞. This is typically shown relying on the Krein–Smulian
theorem. Usual versions of that result are stated for Fréchet spaces, see for example Schachermayer
(1994). Our product space L∞ has the predual ⨁θ∈Θ L₁, which is, in general, not Fréchet,
as Θ is typically uncountable. Therefore, we are not able to apply the Krein–Smulian theorem
directly. To remedy this, we prove a convex compactness property for the set of strategies in a
suitable topology. Such a property fails in frictionless markets, it is a particularity of markets with
transaction costs. Next, we apply Krein–Smulian for finite direct sums of Fréchet spaces and the
w⁺-closedness of C will be ensured by convex compactness.
Third, we apply the Hahn–Banach theorem in $\mathbb{L}^\infty$ to obtain what will be called consistent price systems in the robust sense. These systems are, in fact, infinite dimensional vectors with finitely many nonzero components such that discounting price processes by them has the same effect as the usual consistent price systems.

The paper is organized as follows. In Section 2, we introduce the market model, consistent price systems in the robust sense, and the main result. Proofs are given in Section 3. Section 4 recalls important facts about topological vector spaces and establishes the necessary results about convex compactness. Extensions of our results to the multi-asset, conic framework of Kabanov and Safarian (2009) seem straightforward but they would involve technical complications so we are staying in a one-asset framework for now.

It seems less obvious to include stock prices with jumps where trading strategies are not necessarily right continuous, see Guasoni et al. (2012). This would necessitate finding a topological space in which the set of nice trading strategies is convexly compact. It would also be interesting to see if the present framework can be adapted to continuous-time frictionless markets, see Chau (2020) about what happens in the discrete-time case.

2 | THE MODEL

Let $T > 0$ be the time horizon, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a filtered probability space, where the filtration is assumed to be right-continuous, and $\mathcal{F}_0$ coincides with the $P$-completion of the trivial sigma-algebra. For simplicity, we also assume $\mathcal{F} = \mathcal{F}_T$. Consider a financial market model with one risky asset and one risk-free asset (cash) whose price is assumed to be constant one.

Let $\Theta$ be an (non-empty) arbitrary index set, which is interpreted as the parametrization of uncertainty. Consider a family $(S_{\theta}^0)_{\theta \in \Theta}$ of adapted, positive processes with continuous trajectories, which represent the possible evolutions of the price of the risky asset in consideration.

The risky asset is traded under proportional transaction cost $\lambda \in (0, 1)$, where the possible ask prices and bid prices are $(1 + \lambda)S_t^0$, $(1 - \lambda)S_t^0$, respectively. Let $H^\uparrow_t$, $H^\downarrow_t$, $t \in [0, T]$ be predictable processes with nondecreasing left-continuous trajectories. $H^\uparrow_t$ denotes the cumulative amount of transfers from the riskless asset to the risky one up to time $t$, and $H^\downarrow_t$ represents the cumulative transfers in the opposite direction. The set of all such $(H^\uparrow, H^\downarrow)$ is denoted by $\mathcal{V}$. A useful metric structure can be defined on $\mathcal{V}$, see Section 4.3, which will be key for later developments.

The portfolio position in the risky asset at time $t$ equals

$$H_t := H_0 + H^\uparrow_t - H^\downarrow_t, \ t \in [0, T].$$

Here $H_0$ is a ($\mathcal{F}_0$-measurable, hence deterministic) initial transfer. We also set $H_{0^-} := 0$.

For any real number $x \in \mathbb{R}$, we denote

$$x^+ := \max\{0, x\}, \ x^- := \max\{0, -x\}.$$ 

For a pair of initial capital and initial transfer $(z, H_0) \in \mathbb{R}^2$ and a strategy $H \in \mathcal{V}$, the corresponding liquidation value
for parameter \( \theta \in \Theta \) at time \( t \in [0, T] \) is defined by

\[
W_t^\vartheta(S^\vartheta, H, \lambda) := z - H_0^+ S^\vartheta_0 (1 + \lambda) + H_0^- S^\vartheta_0 (1 - \lambda) - \int_0^t (1 + \lambda) S^\vartheta_u dH_u^+ \\
+ \int_0^t (1 - \lambda) S^\vartheta_u dH_u^- + H_t^+ (1 - \lambda) S^\vartheta_t - H_t^- (1 + \lambda) S^\vartheta_t.
\] (1)

**Definition 2.1.** A finite variation process \((H_0, H) \in \mathbb{R} \times \mathbf{V}\) is called an \( x \)-admissible strategy for the model \( \vartheta \) if

\[
W_0^\vartheta(t)(S^\vartheta, H, \lambda) \geq -x, \text{ a.s. for all } t \in [0, T].
\] (2)

[Correction added on 30 June 2022, after first online publication: ‘X’in Equation (2) has been removed as it was introduced in error during production.]

Denote by \( \mathcal{A}^\vartheta_x(\lambda) \) the set of \( x \)-admissible strategies for the model \( \vartheta \), set \( \mathcal{A}^\vartheta(\lambda) = \bigcup_{x>0} \mathcal{A}^\vartheta_x(\lambda) \). The set of admissible strategies for the robust model is defined by \( \mathcal{A}(\lambda) := \bigcap_{\vartheta \in \Theta} \mathcal{A}^\vartheta(\lambda) \).

In the following, we will work with different product spaces. Let \( \mathbf{L}_0^t = \prod_{\vartheta \in \Theta} (\mathbf{L}_0(L^0_{F_t, P}), d_0) \) be the product space with the corresponding product topology when each \( \mathbf{L}_0(L^0_{F_t, P}) \) is equipped with the topology of convergence in measure given by the metric \( d_0 \). Similarly, \( \mathbf{L}_\infty^t = \prod_{\vartheta \in \Theta} (\mathbf{L}_\infty(L^0_{F_t, P}), \| \cdot \|_\infty) \) is the product space with the corresponding product topology when each \( \mathbf{L}_\infty(L^0_{F_t, P}) \) is equipped with the sup norm. Furthermore, if each \( \mathbf{L}_\infty(L^0_{F_t, P}) \) is equipped with the weak star topology \( w^* = \sigma (\mathbf{L}_\infty, \mathbf{L}_1) \), then we define by \( \mathbf{w}^* \) the product topology of \( \prod_{\vartheta \in \Theta} (\mathbf{L}_\infty(L^0_{F_t, P}), w^*) \), see Section 4.1 for more details. Other product spaces are defined similarly. For notational convenience, we use bold notations for vectors in these product spaces. For \( t \in [0, T] \), we denote by \( S_t := (S^\vartheta_t)_{\vartheta \in \Theta} \) a vector in \( \mathbf{L}_0^0 \). For \( H \in \mathcal{A}(\lambda) \), we denote by

\[
\mathbf{W}^0_t(H) = (W^0_t(S^\vartheta, H, \lambda))_{\vartheta \in \Theta},
\]

the vector consists of attainable payoffs from the strategy \( H \) under model uncertainty. Set \( \mathbf{L}_0^0 := \mathbf{L}_0^t, \mathbf{L}_\infty^0 := \mathbf{L}_\infty^t \). \( \mathbf{L}_0^0 \) is the non-negative orthant of \( \mathbf{L}_0^0 \), that is, the subset of \( \mathbf{L}_0^0 \) consisting elements \( f = (f^\vartheta)_{\vartheta \in \Theta} \) such that \( P[f^\vartheta < 0] = 0 \) for all \( \vartheta \in \Theta \). Define \( \mathbf{K}_0 - \mathbf{L}_0^+ = \{ \mathbf{W}^0_t(H) - \mathbf{h}, H \in \mathcal{A}(\lambda), \mathbf{h} \in \mathbf{L}_0^+ \} \)

\[
\mathbf{K}_0 = \{ \mathbf{W}^0_t(H) : H \in \mathcal{A}(\lambda) \}, \quad \mathbf{C} = (\mathbf{K}_0 - \mathbf{L}_0^+) \cap \mathbf{L}_\infty^0.
\]

For each \( \vartheta \in \Theta \), the classical no free lunch with vanishing risk condition for \( S^\vartheta \), denoted by \( NFLVR(\lambda, \vartheta) \), is recalled from Guasoni et al. (2012).

**Definition 2.2.** The price process \( S^\vartheta \) together with the transaction cost \( \lambda \) satisfy the \( NFLVR(\lambda, \vartheta) \) condition if for any sequence \( H_n, n \in \mathbb{N} \) such that \( H_n \in \mathcal{A}^\vartheta_{1/n} \), and \( \mathbf{W}^0_t(S^\vartheta, H_n, \lambda) \) converges a.s. to some limit \( W \in [0, \infty] \) a.s. then \( W = 0 \), a.s.

The following conditions will be imposed throughout the paper:

**Assumption 2.3.**

(i) There is a sequence of stopping times \( T_n, n \in \mathbb{N} \) increasing almost surely to infinity such that \( \sup_{\vartheta \in \Theta} \sup_{t \in [0, T]} S^\vartheta_{t \wedge T_n} \leq K_n \) a.s. for some constants \( K_n, n \in \mathbb{N} \).
(ii) For all $\theta \in \Theta$, the stock price $S^\theta$ satisfies $NFLVR(\lambda, \theta)$ for all $0 < \lambda < 1$.

Condition (i) is technical, requiring uniform boundedness of the prices up to a sequence of stopping times. Condition (ii) assumes that none of the possible price processes admits (model-dependent) arbitrage. It holds obviously that $C \cap L^\infty_\kappa = \{0\}$.

**Remark 2.4.** To develop an intuition about the current, parametric framework of uncertainty we sketch a rather general example. Let $Y$ be some stochastic process with values in $\mathbb{R}^m$ for some $m$ which generates the filtration. We imagine that $Y$ describes economics factors (dividend yields, interest rates, unemployment rates, statements of firms), which are known (at least we have a reliable statistical model for them). The asset price $S$ is assumed to be a nonlinear functional of these factors, that is, $S^\theta_t = F(\theta, Y_t), \theta \in \Theta$ for some parameter set $\Theta$ and a function $F: \Theta \times \mathbb{R}^m \to \mathbb{R}_+$. Here the parameter $\theta$ is unknown. Note that if $Y$ is locally bounded and $F$ is regular enough, then a localizing sequence $T_n$ for $Y$ satisfies (i) in Assumption 2.3.

**Example 2.5.** Consider a simple volatility uncertain model where $dS^\theta_t = \sigma^\theta S^\theta_t dW_t$, where $\sigma^\theta \in [\sigma_1, \sigma_2]$. The continuous process $S^*_t = \sup_\theta S^\theta_t$ is locally bounded with the sequence of stopping times $T_n = \inf\{t \geq 0 : S^*_t \geq n\}$. It is clear that Assumption 2.3 holds true.

**Example 2.6.** Let the filtration be generated by a Brownian motion $W_t$. Consider the possible price processes

$$S^\theta_t = F\left(\int_0^t \mu^\theta(s) \, ds + \int_0^t K^\theta(t, s) dW_s\right), \quad t \in [0, T]$$

where $\theta$ runs in an index set $\Theta$, $K^\theta(t, s)$ are kernels of a suitable regularity, $\mu^\theta$ are suitable processes and $F: \mathbb{R} \to (0, \infty)$ is a function (e.g., exponential). This is a family of models which possibly fail the semimartingale property but satisfy $NFLVR(\lambda, \theta)$ for every $0 < \lambda < 1$ under mild conditions, see for example, Guasoni et al. (2010) and Gasbarra et al. (2011).

**Definition 2.7.** A consistent price system in the robust sense for $S$ is a pair $(Z, M)$ where

(i) $0 \leq Z_t \in \bigoplus_{\theta \in \Theta} L^1$ and $E[\sum_{\theta \in \Theta} Z^\theta_t] = 1$.

(ii) $M$ is a local martingale such that

$$\sum_{\theta \in \Theta} (1 - \lambda) Z^\theta_t S^\theta_t \leq M_t \leq \sum_{\theta \in \Theta} (1 + \lambda) Z^\theta_t S^\theta_t, \text{ a.s.,}$$

where $Z^\theta_t := E[Z^\theta_t | F_t], t \in [0, T]$ for all $\theta \in \Theta$.

If $(Z, M)$ is a consistent price system in the robust sense for $S$, Definition 2.7 (i) implies that $E[Z^\theta_T] > 0$ for some $\theta$. The set of such pairs is denoted by $C(\bar{\theta}, \lambda)$. Let

$$Z(\lambda) := \bigcup_{\theta \in \Theta} C(\bar{\theta}, \lambda).$$
By definition of direct sum, $Z_\theta^T = 0$ for all but finitely many $\theta$, therefore, the sums in Equation (3) are well-defined. The quantity $Z_T$ may be interpreted as a local martingale density. In the setting without uncertainty, it is usually required that $Z_\theta^T > 0$, $a.s.$ Here, our density $Z_\theta^T$ has to be positive only on some $A \in \mathcal{F}_T$ with $P[A] > 0$. The new densities might look strange at the first glance, however, they are comparable to “absolutely continuous local martingale measures”, introduced in Delbaen and Schachermayer (1995a) and Delbaen and Schachermayer (1995b). The main reason for using this notion is that, $\Theta$ being possibly uncountable, no exhaustion argument (as in the Kreps–Yan separation theorem) can be applied. It will become clear in the proof of Theorem 2.9 that the new densities work better than the densities with $Z_\theta^T > 0$, $a.s.$.

When $\Theta$ is a singleton, it is possible to use the exhaustion argument and we may assume $Z_\theta^T > 0$, $a.s.$ so this definition reduces to the usual definition of consistent price systems, see Guasoni et al. (2010, 2012).

**Remark 2.8.** Let $D \subset \Theta$ be finite and let $Z_i^\theta = 0$ for all the coordinates $i \notin D$. If, for each $\theta \in D$,

$$
(1 - \lambda)Z_i^\theta S_i^\theta \leq M_i^\theta \leq (1 + \lambda)Z_i^\theta S_i^\theta, \text{ a.s.}
$$

for suitable martingales $M^\theta$ (that is, the processes $(Z^\theta, M^\theta)$ determine consistent price systems for the individual models $S^\theta, \theta \in D$) then, defining $M_i := \sum_{\theta \in D} M_i^\theta$ defines a martingale satisfying Equation (3). This shows how “usual” consistent price systems provide objects in $C(\bar{\theta}, \lambda)$. The important point is that there are many other elements in $C(\bar{\theta}, \lambda)$. This is explained in Chau (2020) where frictionless toy examples are presented.

We state the main result of the paper.

**Theorem 2.9.** Let Assumption 2.3 be in force. Let $G = (G_\theta)_{\theta \in \Theta} \in L^0$ be bounded or let $G \geq 0$. There exists $(z, H) \in \mathbb{R} \times A(\lambda)$ such that for any $\theta \in \Theta$

$$
W^T_z(S^\theta, H, \lambda) \geq G^\theta, \text{ a.s.}
$$

if and only if

$$
z \geq \sup_{(Z, M) \in Z(\lambda)} E \left[ \sum_{\theta \in \Theta} Z_\theta^T G^\theta \right].
$$

The proof of Theorem 2.9 is given in Section 3 below. We now make some comparisons to the previous results. In Dolinsky and Soner (2014a), the authors studied a similar superhedging problem with transaction costs in a discrete time setting using the pathwise approach. Dual objects are “approximate martingale laws” compatible with the prices of given European options. They established a duality result with additional assumptions that the option payoff is upper semicontinuous and at most quadratic growth. Without transaction cost, Dolinsky and Soner (2014b) proved a duality in continuous time, and again (Lipschitz) continuity is imposed on the option payoffs. It is noticed that although there are no transaction costs in that paper, trading strategies are limited to the family of finite variation processes, for the purpose of defining wealth processes pathwise. Dolinsky and Soner (2017) considered the hedging problem closest to ours. Their main result is a superhedging duality for Lipschitz continuous option payoffs. Using the notion of prediction set
to define possible paths, Hou and Obłój (2018) obtained similar results for duality. In our paper, Assumption 2.3 (ii) is compatible with the existence of consistent price systems appeared in the mentioned studies. Nevertheless, we made no continuity assumptions on option payoffs.

3 | PROOF

3.1 | The $w^*$-closedness of $C$

The following result is classical.

**Lemma 3.1.** For any $x > 0$, and any $\vartheta \in \Theta$, the set $\{\|H\|_T : H \in A^\vartheta_1(\lambda)\}$ is bounded in probability, that is,

$$\lim_{n \to \infty} \sup_{H \in A^\vartheta_1(\lambda)} P[\|H\|_T > n] = 0.$$ 

**Proof.** We can assume that $x = 1$ and adopt the argument in Guasoni et al. (2012). By Assumption 2.3 (ii), for each $\vartheta \in \Theta$, $0 < \hat{\lambda} < \lambda$, the process $S^\vartheta$ satisfies the classical condition $NFLVR(\hat{\lambda}, \vartheta)$. Let us denote

$$S^\vartheta = (\lambda - \hat{\lambda}) \inf_{t \in [0,T]} S^\vartheta_t > 0, \text{ a.s.}$$

Assume that there exist $\alpha > 0$ and a sequence $H^n \in A^\vartheta_1(\lambda), n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$P\left[\|H^n\|_T > n\right] \geq \alpha.$$

By Definition 2.1, we have that

$$1 + W_1^0(S^\vartheta, H^n, \hat{\lambda})$$

$$\geq W_1^0(S^\vartheta, H^n, \hat{\lambda}) - W_1^0(S^\vartheta, H^n, \lambda)$$

$$= \int_0^T \left((1 + \lambda)S^\vartheta_u - (1 + \hat{\lambda})S^\vartheta_u\right) dH_u^n,^+$$

$$+ \int_0^T \left((1 - \hat{\lambda})S^\vartheta_u - (1 - \lambda)S^\vartheta_u\right) dH_u^n,^\downarrow$$

$$+ (H^n)^+ \left((1 - \hat{\lambda})S^\vartheta - (1 - \lambda)S^\vartheta\right) + (H^n)^- \left((1 + \lambda)S^\vartheta - (1 + \hat{\lambda})S^\vartheta\right)$$

$$\geq \|H^n\|_t.$$ 

It is clear that $H^n = H^n/n \in A^\vartheta_1(\lambda)$ and then

$$1/n + W_1^0(S^\vartheta, H^n, \hat{\lambda}) \geq \|H^n\|_T, \text{ a.s.}$$ (5)
and \( P[S^\beta \| R^n \| T > 1] \geq \alpha \) for all \( n \in \mathbb{N} \). By Lemma 9.8.1 of Delbaen and Schachermayer (2006), there exist convex combinations \( f^n \in \text{conv}\{S^\beta \| R^n \| T, S^\beta \| R^{n+1} \| T, \ldots\} \) such that \( f^n \to f, \text{a.s.} \) and

\[
P[f > 0] > 0. \tag{6}
\]

Using the same weights as in the construction of \( f^n \), we obtain a sequence of strategies \( \hat{H}^n \) such that

\[
1/n + W^0_T(S^\delta, \hat{H}^n, \hat{\lambda}) \geq f^n, \text{a.s.}
\]

Therefore, the random variable \( f \) is a \( FLVR(\hat{\lambda}, \theta) \) for \( S^\delta \), which is a contradiction. \( \square \)

**Remark 3.2.** If \( H \) is a convex combination of \( H^1, H^2 \), it may happen that there are overlapping regions where \( H^1 \uparrow H^1 \neq 0 \), for example, when \( H^2 \uparrow H^1 \downarrow \neq 0 \). However, we can always remove these redundant transactions from \( H \) and the obtained strategy generates a payoff, which is at least as that from \( H \). Without further notice, from now on we interpret \( H \) as the strategy after removing redundancy.

Let \( D \) be a non-empty finite subset of \( \Theta \). We say that a sequence \( f_n, n \in \mathbb{N} \) in \( \prod_{\theta \in D} L^0 \) Fatou-converges to \( f \) if for each \( \theta \in D \), \( f^n_\theta \) converges to \( f^\theta \) a.s. and \( f^n_\theta \geq -x^\theta, \text{a.s.} \) for some \( x^\theta > 0 \). Define \( B^\infty_{x^\theta} = \{ y \in L^\infty(F_T, P) : \| y \|_{L^\infty} \leq x^\theta \} \) and

\[
C(D) = \left\{ (W^0_T(S^\delta, H, \lambda) - h^\theta)_{\theta \in D} : H \in \bigcap_{\theta \in D} A^\theta(\lambda), h^\theta \in L^0_+, \theta \in D \right\} \bigcap \prod_{\theta \in D} L^\infty(F_T, P).
\]

**Proposition 3.3.** The set \( C(D) \) is Fatou-closed, that is \( C(D) \cap \prod_{\theta \in D} B^\infty_{x^\theta} \) is closed in the product space \( \prod_{\theta \in D}(L^0(F_T, P), d_0) \), for each \( (x^\theta)_{\theta \in D} \in \mathbb{R}^D_+ \).

**Proof.** Let \( f_n, n \in \mathbb{N} \) be a sequence in \( C(D) \) such that for every \( \theta \in D \), \( f^n_\theta \to f^\theta \) in probability and \( f^n_\theta \geq -x^\theta \). We need to find \( H \in \bigcap_{\theta \in D} A^\theta(\lambda) \) such that for every \( \theta \in D \), \( W^0_T(S^\delta, H, \lambda) \geq f^\theta, \text{a.s.} \).

By taking a subsequence, we may assume that \( f^n_\theta \to f^\theta, \text{a.s.} \) for all \( \theta \in D \). By definition,

\[
f^n_\theta \leq W^0_T(S^\delta, H^n, \lambda), \forall \theta \in D,
\]

for some \( H^n \in \bigcap_{\theta \in D} A^\theta(\lambda) \). Using Theorem 1 of Schachermayer (2014), for every \( \theta \in D \) we get that \( H^n \in A^\theta(\lambda) \) for all \( n \in \mathbb{N} \). Fix \( \theta_0 \in D \) arbitrarily. Lemma 3.1 implies that the set \( \{ H \| T \in A^\theta_{x^\theta}\(\lambda) \} \) is bounded in probability. Using Lemma B.4 of Guasoni et al. (2012), there are convex combinations \( \tilde{H}^{n,1} \in \text{conv}(H^{n,1}, H^{n+1,1}, \ldots) \) and a finite variation process \( H^1 \) such that \( \tilde{H}^{n,1} \to H^1 \) pointwise [Correction added on 5th July 2022, after first online publication: ‘s’ has been removed in this version. Similarly, we find another convex combinations, still denoted by \( \tilde{H}^{n,1} \), such that \( \tilde{H}^{n,1} \to H^1 \) pointwise.
Next we prove that $H \in \bigcap_{\theta \in D} A^\theta(\lambda)$. First, using Lemma 4.3 of Guasoni (2002), we obtain that $\hat{H}_n,\uparrow \rightarrow H^\uparrow$ and $\hat{H}_n,\downarrow \rightarrow H^\downarrow$ weakly. It follows that for every $\theta \in \Theta$, $t \in [0, T]$,

$$W^0_t(S^\theta, \hat{H}^n, \lambda) \rightarrow W^0_t(S^\theta, H, \lambda), a.s..$$

Therefore, $W^0_t(S^\theta, H, \lambda) \geq f^\theta$, a.s. for all $\theta \in D$. Second, for every $\theta \in D$, the triangle inequality and Theorem 1 of Schachermayer (2014) again yield that

$$W^0_t(S^\theta, \hat{H}^n, \lambda) = -\int_0^t (1 + \lambda) S^\theta_u d\hat{H}^n_u + \int_0^t (1 - \lambda) S^\theta_u d\hat{H}^n_u$$

$$\geq -\chi^\theta, \text{ a.s. for } 0 \leq t \leq T.$$ 

In other words, $H \in A^\theta_x(\lambda)$ for all $\theta \in D$. 

It is worth mentioning that this proposition relies on the admissibility result of Schachermayer (2014), which is stated for one-dimensional settings. Extensions to multidimensional ones seem to be difficult.

**Remark 3.4.** Fatou-closedness is studied in the quasi-sure approach by Maggis (2018). In their topological setting, Fatou-closedness, denoted by $(FC)$, does not imply “weak star” closedness, denoted by $(WC)$. An additional condition, namely $P$-sensitivity, is required. And it is proved that for a convex and monotone set, $(WC) = (FC) + P$-sensitivity. When the set of priors $P$ is dominated, $P$-sensitivity is always satisfied, but this is not the case when $P$ is non-dominated. In comparison, our Proposition 3.3 resembles the dominated case (although the laws of $S^\theta$ are not necessarily dominated), where the number of uncertain models is finite. Nevertheless, our “weak star” closedness, that is $w^*$-closedness, is proved in Proposition 3.5 below by using a different technique. Fatou-closedness is also discussed in the pathwise approach by Cheridito et al. (2021) as a given property rather than a proved one.

Recall

$$C = \{ (W^0_t(H, \lambda) - h) : H \in A(\lambda), h \in L^0_+ \} \bigcap \prod_{\theta \in \Theta} L^\infty(F, P).$$

Now, we are able to prove

**Proposition 3.5.** The convex cone $C$ is $w^*$-closed.

**Proof.** Let $f_\alpha, \alpha \in I$ be a net in $C$, that is, $f_\alpha \leq W^0_t(H_\alpha), H_\alpha \in A(\lambda)$, such that $f_\alpha \rightarrow f$ in the $w^*$ topology for some $f \in L^\infty$. We need to prove that $f \in C$, that is there exists $H \in A(\lambda)$ such that for all $\theta \in \Theta$, $W^0_t(S^\theta, H, \lambda) \geq f^\theta$, a.s.. For each $\theta \in \Theta$, we define

$$H^\theta = \{ H \in V : H \in A^\theta(\lambda) \text{ and } W^0_t(S^\theta, H, \lambda) \geq f^\theta, a.s. \}.$$
The set $H^\theta$ is clearly convex for each $\theta \in \Theta$. In addition, using Lemma 4.3 of Guasoni (2002), we can prove that $H^\theta$ is closed in $V$. If we can prove

$$\bigcap_{\theta \in \Theta} H^\theta \neq \emptyset,$$

then the proof is complete. First, we will prove that

$$H^D = \bigcap_{\theta \in D} H^\theta \neq \emptyset,$$

where $D$ is an arbitrary finite subset of $\Theta$.

Proposition 3.3 implies that the set $C(D)$ is Fatou-closed. Therefore, using Proposition 4.1, the set $C(D)$ is closed in the $w^*$ topology of $\prod_{\theta \in D} L^\infty(F_T, P)$. Since $f^\theta_\delta \to f^\theta$ in the $w^*$ topology for each $\theta \in D$, and we obtain that $(f^\theta_\delta)_{\delta \in D} \in \mathcal{C}(D)$, and thus, Equation (8) holds true.

Fix $\theta_0 \in \Theta$ arbitrarily. Since $x^\theta_0 = \|f^\theta_0\| < \infty$, Lemma 3.1 shows that the set $\{\|H\|_T, H \in \mathcal{A}_{x^\theta_0}^\theta(\lambda)\}$ is bounded in $L^0(F_T, P)$, and thus convexly compact, by Proposition 4.5. Since

$$\bigcap_{\theta \in \Theta} H^\theta = \bigcap_{D \in \text{Fin}(\Theta)} H^D \cup \{\theta_0\},$$

we conclude that Equation (7) holds. The proof is complete. □

**Corollary 3.6.** For every $0 < \lambda < 1$, $\theta \in \Theta$, the set $C(\theta, \lambda)$ is non-empty. Furthermore, for any $A \in \mathcal{F}_T$ with $P[A] > 0$, there exists $Z_T \in C(\theta, \lambda)$ with $E[Z_T 1_A] > 0$.

**Proof.** Let us fix $\tilde{\theta} \in \Theta$ arbitrarily. By Proposition 3.5, the convex set $C$ is $w^*$-closed. The compact set $1_A \mathbf{1}_{\tilde{\theta}}$ and the closed convex set $C$ are disjoint. Applying the Hahn–Banach theorem, there exists $Q \in (\prod_{\theta \in \Theta}(L^\infty(F_T, P), w^*))^*$ such that

$$\sup_{f \in C} Q(f) \leq \alpha < \beta \leq Q(1_A \mathbf{1}_{\tilde{\theta}}).$$

By Equation (14), we identify $Q = (Z_{T,\delta})_{\delta \in \Theta} \in \bigoplus_{\delta \in \Theta} L^1(F_T, P)$. Since $0 \in C$, it follows that $\alpha \geq 0$. Since $C$ is a cone, we must have

$$Q(f) \leq 0, \forall f \in C,$$

and as a consequence, $Z_{T,\delta}^\delta \geq 0, \forall \theta \in \Theta$. Note that $E[Z_{T,\delta}^\delta 1_A] > 0$.

For any stopping times $\sigma \leq \tau \leq T_\eta$ and $B \in \mathcal{F}_\sigma$, the strategy $H = \pm 1_B 1_{[\sigma, \tau]}$ belongs to $\mathcal{A}(\lambda)$. Therefore, from Equation (9), we obtain

$$Q(((1 - \lambda)S_\tau - (1 + \lambda)S_\sigma)1_B) \leq 0, \quad Q(((1 - \lambda)S_\sigma - (1 + \lambda)S_\tau)1_B) \leq 0. \quad (10)$$

Define

$$X_\tau = \sum_{\theta \in \Theta} Z_{T,\theta}^\theta (1 - \lambda)S_\tau^\theta, \quad Y_\sigma = \sum_{\theta \in \Theta} Z_{\sigma,\theta}^\theta (1 + \lambda)S_\sigma^\theta.$$
where $Z_i^\bar{\theta} = E[Z_i^\bar{\theta} | F_t]$. We compute that
\[
E[(X_t - Y_t)1_B]
= E \left[ \left( \sum_{\theta \in \Theta} Z_i^\bar{\theta} (1 - \lambda) S_t^\bar{\theta} - \sum_{\theta \in \Theta} Z_i^\bar{\theta} (1 + \lambda) S_t^\bar{\theta} \right) 1_B \right]
= E \left[ \left( \sum_{\theta \in \Theta} E[Z_i^\bar{\theta} | F_t] (1 - \lambda) S_t^\bar{\theta} \right) 1_B \right] - E \left[ \left( \sum_{\theta \in \Theta} E[Z_i^\bar{\theta} | F_\sigma] (1 + \lambda) S_t^\bar{\theta} \right) 1_B \right]
\leq 0,
\]
by the tower law of conditional expectation and Equation (10). Similarly, we obtain
\[
E[(X_\sigma - Y_\sigma)1_B] \leq 0.
\]
Using Lemma 3.8, there is a martingale $M^n$ such that on $[0, T_n]$, 
\[
(1 - \lambda) \sum_{\theta \in \Theta} Z_i^\bar{\theta} S_t^\bar{\theta} \leq M^n_t \leq (1 + \lambda) \sum_{\theta \in \Theta} Z_i^\bar{\theta} S_t^\bar{\theta}, \text{a.s.}
\]
However, from the proof of Lemma 6.2 of Guasoni et al. (2012), it can be checked that $M^{n+1}$ and $M^n$ coincide on $[0, T_n]$. Therefore, the local martingale $M$ obtained by pasting the processes $M^n, n \in \mathbb{N}$ together satisfies
\[
(1 - \lambda) \sum_{\theta \in \Theta} Z_i^\bar{\theta} S_t^\bar{\theta} \leq M_t \leq (1 + \lambda) \sum_{\theta \in \Theta} Z_i^\bar{\theta} S_t^\bar{\theta}, \text{a.s., } t \in [0, T].
\]
In other words, we obtain an element in $C(\bar{\theta}, \lambda)$ with the required properties. \hfill \Box

Remark 3.7. One may ask: if Assumption 2.3 (ii) is relaxed but a robust no free lunch condition is imposed, is the set $C$ (or an appropriate enlargement of it) $\mathbf{w}^*$-closed? This would result in a robust version of FTAP but we do not know yet how to handle this case. We refer to Dolinsky and Soner (2014a); Bayraktar and Zhang (2016) for the results in discrete-time settings.

We recall Lemma 6.3 of Guasoni et al. (2012).

**Lemma 3.8.** Let $(X_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$ be two càdlàg bounded processes. The following conditions are equivalent:

(i) There exists a càdlàg martingale $(M_t)_{t \in [0,T]}$ such that
\[
X \leq M \leq Y, \quad \text{a.s.}
\]
(ii) For all stopping times $\sigma, \tau$ such that $0 \leq \sigma \leq \tau \leq T$, a.s., we have

$$E[X_\tau | F_\sigma] \leq Y_\sigma, \quad \text{and} \quad E[Y_\tau | F_\sigma] \geq X_\sigma, \quad \text{a.s.}$$

3.2 Proof of Theorem 2.9

Let $z$ and $H \in A(\lambda)$ satisfy

$$z + W^0_T(S^\vartheta, H, \lambda) \geq G^\vartheta, \quad \text{a.s., } \vartheta \in \Theta.$$

For any $(Z, M) \in \mathcal{Z}(\lambda)$, we have

$$z + E \left[ \sum_{\vartheta \in \Theta} Z^\vartheta_i W^0_i(S^\vartheta, H, \lambda) \right] \geq E \left[ \sum_{\vartheta \in \Theta} Z^\vartheta_i G^\vartheta \right]. \quad (11)$$

Suppose that $Z^\vartheta_i$ has positive values only if $\vartheta \in D$ for some $D \in Fin(\Theta)$. Denote by $H^\vartheta_1, H^\vartheta_\downarrow$ the continuous parts of $H^\vartheta_1, H^\vartheta_\downarrow$, respectively. We define the process $I^\vartheta_i := \int_0^t (1 + \lambda) S^\vartheta_u dH^\vartheta_u$. Using integration by parts, we obtain that

$$d (Z^\vartheta_i I^\vartheta_i) = I^\vartheta_i - dZ^\vartheta_i + Z^\vartheta_i(1 + \lambda) S^\vartheta_i dH^\vartheta_1 + d[I^\vartheta_i, I^\vartheta_i].$$

Noting that $S^\vartheta$ is continuous and $I^\vartheta$ is of finite variation. Similarly, we define $J^\vartheta_i := \int_0^t (1 - \lambda) S^\vartheta_u dH^\vartheta_\downarrow$ and compute

$$d (Z^\vartheta_i J^\vartheta_i) = J^\vartheta_i - dZ^\vartheta_i + Z^\vartheta_i(1 - \lambda) S^\vartheta_i dH^\vartheta_\downarrow + Z^\vartheta_i(1 - \lambda) S^\vartheta_i \Delta H^\vartheta_1. \quad (13)$$

Therefore, Equations (12) and (13) and the property of $M$ yield

$$\sum_{\vartheta \in D} Z^\vartheta_i W^0_i(S^\vartheta, H, \lambda) \leq \sum_{\vartheta \in D} \int_0^t (J^\vartheta_u - I^\vartheta_u) dZ^\vartheta_u$$

$$- \sum_{u \leq s \leq t} M_u \Delta H_u - \int_0^t M_u dH^c_u - \sum_{u \leq s \leq t} M_u \Delta H_u + H_t M_t - H_0 M_0$$

$$= \sum_{\vartheta \in D} \int_0^t (J^\vartheta_u - I^\vartheta_u) dZ^\vartheta_u + \int_0^t H_u dM_u.$$
The RHS of the above inequality is a local martingale and thus a supermartingale as $W_T^0(S^\vartheta, H, \lambda)$ is uniformly bounded from below. From Equation (11), we have

$$E \left[ \sum_{\vartheta \in D} Z_{iT}^\vartheta G^\vartheta \right] \leq z + E \left[ \sum_{\vartheta \in D} Z_{iT}^\vartheta W_T^0(S^\vartheta, H, \lambda) \right] \leq z.$$ 

Therefore, $z \geq \sup_{(Z, M) \in Z(\lambda)} E \left[ \sum_{\vartheta \in \Theta} Z_{i}^\vartheta G^\vartheta \right]$. 

Next, we prove the reverse inequality for the case $G^\vartheta, \vartheta \in \Theta$ are bounded. Let $z \in \mathbb{R}$ be such that there is no strategy $H \in \mathcal{A}(\lambda)$ satisfying

$$W_T^z(S^\vartheta, H, \lambda) \geq G^\vartheta, \text{ a.s., } \forall \vartheta \in \Theta.$$ 

In other words, $(G^\vartheta)_{\vartheta \in \Theta} - z \notin \mathcal{C}$. Applying the Hahn–Banach theorem, there exists $Q = (Z_{i}^\vartheta)_{\vartheta \in \Theta} \in \bigoplus_{\vartheta \in \Theta} L^1(F_T, P)$ such that

$$\sup_{f \in \mathcal{C}} Q(f) \leq \alpha < \beta \leq Q((G^\vartheta)_{\vartheta \in \Theta} - z).$$ 

Since $\mathcal{C}$ is a cone containing $-L_\infty^\lambda$, it is necessarily that

$$\sup_{f \in \mathcal{C}} Q(f) = 0, \quad Q((G^\vartheta)_{\vartheta \in \Theta} - z) > 0.$$ 

We also deduce that $Z_{i}^\vartheta \geq 0, \text{ a.s., } \vartheta \in \Theta$ and it is possible to normalize $Q$ such that $Q(1) = 1$. A similar argument as in the proof of Corollary 3.6 gives the martingale $M$ associated to $Z$. This means $(Z, M) \in Z(\lambda)$ and that

$$z < Q((G^\vartheta)_{\vartheta \in \Theta}) \leq \sup_{(Z, M) \in Z(\lambda)} E \left[ \sum_{\vartheta \in \Theta} Z_{i}^\vartheta G^\vartheta \right].$$ 

Finally we investigate the case $G^\vartheta \geq 0, \vartheta \in \Theta$. Let $z \in \mathbb{R}$ be the number such that $z \geq \sup_{(Z, M) \in Z(\lambda)} E \left[ \sum_{\vartheta \in \Theta} Z_{i}^\vartheta G^\vartheta \right]$. Then, for all $n \in \mathbb{N}$, we have

$$z \geq \sup_{(Z, M) \in Z(\lambda)} E \left[ \sum_{\vartheta \in \Theta} Z_{i}^\vartheta G^\vartheta \wedge n \right].$$ 

The result for bounded $G$ implies for each $n \in \mathbb{N}$, there exists $H^n \in \mathcal{A}(\lambda)$ such that for any $\vartheta \in \Theta$

$$z + W_T^0(S^\vartheta, H^n, \lambda) \geq G^\vartheta \wedge n, \text{ a.s.}$$ 

For each $\vartheta \in \Theta$, Theorem 1 of Schachermayer (2014) yields $H^n \in \mathcal{A}^\vartheta_2(\lambda)$ for all $n \in \mathbb{N}$. Now, we repeat the argument in Proposition 3.3. For a fixed $\vartheta_0 \in \Theta$, the set $\{||H||_T, H \in \mathcal{A}_{\vartheta_0}(\lambda)\}$ is bounded in probability by Lemma 3.3. Using Lemma B.4 of Guasoni et al. (2012), there are convex combinations $H^{i, \lambda}, H^{n, \lambda}$ and finite variation processes $H^i, H^n$ such that $H^{i, \lambda} \rightarrow H^i, H^{n, \lambda} \rightarrow H^n$ pointwise. It can be checked, for example from the proof of Lemma 3.4 of Guasoni (2002), that the choice of $\vartheta$ is not important here and the limiting processes do not depend on $\vartheta$. From Lemma 4.3 of Guasoni...
we obtain $\tilde{H}^n \uparrow \rightarrow H^\uparrow$ and $\tilde{H}^n \downarrow \rightarrow H^\downarrow$ weakly. It follows that for every $\vartheta \in \Theta, t \in [0, T]$,

$$W_0^0(S^\Theta, \tilde{H}^n, \lambda) \rightarrow W_0^0(S^\Theta, H, \lambda), a.s.$$ 

Since $G$ is bounded from below, $H \in \mathcal{A}(\lambda)$ and $z + W_0^0(S^\Theta, H, \lambda) \geq G^\Theta, a.s.$ for any $\vartheta \in \Theta$. The proof is complete.

4 | AUXILIARY RESULTS

4.1 | Direct sums and product spaces

A bilinear pairing is a triple $(X, Y, \langle \cdot, \cdot \rangle)$ where $X, Y$ are vector spaces over $\mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is a bilinear map from $X \times Y$ to $\mathbb{R}$. Let $(E, u)$ be a topological vector space. Let $E^* = (E, u)^*$ be its dual, that is, the set of all continuous linear maps from $E$ to $\mathbb{R}$. Then there is a natural bilinear pairing $(E, E^*, \langle \cdot, \cdot \rangle)$. We denote by $\sigma(E, E^*)$ the usual weak topology on $E$ and by $\sigma(E^*, E)$ the weak-star topology on $E^*$.

Let $I$ be a non-empty set and, for each $i \in I$, let $(X_i, \tau_i)$ be a locally convex topological spaces. The topological direct sum of the family $(X_i, \tau_i)$, denoted by $\bigoplus_{i \in I} (X_i, \tau_i)$, is the locally convex space defined as follows. The vector space $\bigoplus_{i \in I} X_i$ is the set of tuples $(x_i)_{i \in I}$ with $x_i \in X_i$ such that $x_i = 0$ for all but finitely many $i$. It is equipped with the inductive topology with respect to the canonical embeddings

$$e_i : (X_i, \tau_i) \rightarrow X$$

$$x_i \mapsto x = (x_i),$$

where $x^i = x_i$ and $x^j = 0$ whenever $j \neq i$, that is, the strongest locally convex topology on $\bigoplus_{i \in I} X_i$ such that all these embeddings are continuous.

The product space of the family $(X_i, \tau_i)$, denoted by $\prod_{i \in I} (X_i, \tau_i)$, consists of the product set $\prod_{i \in I} X_i$ and a topology $\tau$ having as its basis the family

$$\left\{ \prod_{i \in I} O_i : O_i \in \tau_i \text{ and } O_i = X_i \text{ for all but a finite number of } i \right\}.$$ 

The topology $\tau$ is called the product topology, which is the coarsest topology for which all the projections are continuous. Note that the product space defined in this way is also a topological vector space, see Theorem 5.2 of Charalambos and Border (2006). Since each $(X_i, \tau_i)$ is locally convex, $\prod_{i \in I} (X_i, \tau_i)$ is locally convex, too, see Proposition 2.1.3 of Bogachev et al. (2017). If $I$ is uncountable, the product space is not normable.

For any index set $I$, it holds that

$$\left( \bigoplus_{i \in I} (X_i, \tau_i) \right)^* = \prod_{i \in I} X_i^*, \quad \left( \prod_{i \in I} (X_i, \tau_i) \right)^* = \bigoplus_{i \in I} X_i^*, \quad (14)$$
We will be using the pairing 

$$\langle f, g \rangle = \sum_{i \in I} \langle f^i, g^i \rangle_i,$$

$$\forall f \in \bigoplus_{i \in I} X_i, g \in \prod_{i \in I} X_i^*,$$

where $$\langle \cdot, \cdot \rangle_i$$ is the natural pairing for $$X_i, X_i^*$$. From Corollary 1, page 138 of Schaefer (1971), it holds that

$$\sigma\left(\prod_{i \in I} X_i^*, \bigoplus_{i \in I} X_i\right) = \prod_{i \in I} \sigma(X_i^*, X_i).$$

(15)

Let $$D$$ be a finite index set. In what follows, we will be interested in the duality between

$$E := \bigoplus_{\gamma \in D} (L^1(F_T, P), \| \cdot \|_1), \quad E^* = \prod_{\gamma \in D} L^\infty(F_T, P).$$

(16)

We define $$B^\infty_r = \{ f \in L^\infty(F_T, P) : \| f \|_\infty \leq r \}$$, the closed ball of radius $$r \geq 0$$ in $$L^\infty(F_T, P)$$. The following result is analogous to Proposition 5.2.4 of Delbaen and Schachermayer (2006).

**Proposition 4.1.** Let $$C \subset E^*$$ be a convex set, where $$E^*$$ is defined in Equation (16). The set $$C$$ is closed in the $$w^*$$ topology if and only if $$C \cap \prod_{\gamma \in D} B^\infty_r$$ is closed in $$\prod_{\gamma \in D} L^0(F_T, P)$$ for each $$r \geq 0$$.

**Proof.** We follow the proof of Proposition 4.4 in Kabanov and Last (2002). ($\Rightarrow$) Let $$f_n, n \in \mathbb{N}$$ be a sequence in $$C \cap \prod_{\gamma \in D} B^\infty_r$$ such that $$f_n \to f$$ in $$\prod_{\gamma \in D} L^0(F_T, P)$$. We have to show that $$f \in C \cap \prod_{\gamma \in D} B^\infty_r$$. For each $$g \in L^1(F_T, P)$$ and $$\gamma \in D$$, the dominated convergence theorem implies that

$$\lim_{n \to \infty} E[g f^\gamma_n] = E[g f^\gamma].$$

Therefore, we have that $$f \in C \cap \prod_{\gamma \in D} B^\infty_r$$.

($\Leftarrow$) Assume that $$C \cap \prod_{\gamma \in D} B^\infty_r$$ is closed in $$\prod_{\gamma \in D} L^0(F_T, P)$$. It is easy to check that $$C \cap \prod_{\gamma \in D} B^\infty_r$$ is closed in the Hilbert space $$\prod_{\gamma \in D} (L^2, \| \cdot \|_2)$$, hence also in its weak topology $$\sigma(\prod_{\gamma \in D} L^2, \prod_{\gamma \in D} L^2)$$, which is the same as $$\sigma(\prod_{\gamma \in D} L^\infty, \prod_{\gamma \in D} L^2)$$-closedness. Since $$L^2 \subset L^1$$, the set $$C \cap \prod_{\gamma \in D} B^\infty_r$$ is also $$\sigma(\prod_{\gamma \in D} L^\infty, \prod_{\gamma \in D} L^1)$$-closed so, by the Krein–Smulian theorem, $$C$$ is closed in the weak-star topology. \( \square \)

### 4.2 Convex compactness in $$L^0_+$$

Let $$L^\#$$ denote the set of $$[0, \infty]$$-valued random variables, equipped with the topology of convergence in probability. A set $$A \subset L^0_+$$ is bounded if $$\sup_{X \in A} P(X \geq n) \to 0, n \to \infty$$. Now consider the topological product $$L := (L^0_+)^\mathbb{N}$$. We call a subset $$C \subset L$$ c-bounded, if $$\pi_k(C)$$ is bounded in $$L^0_+$$ for all coordinate mappings $$\pi_k, k \in \mathbb{N}$$.

For any set $$A$$, we denote by Fin($$A$$) the family of all non-empty finite subsets of $$A$$. This is a directed set with respect to the partial order induced by inclusion. We reproduce Definition 2.1 of Žitković (2010).

**Definition 4.2.** A convex subset $$C$$ of some topological vector space is convexly compact, if for any non-empty set $$A$$ and any family $$F_a, a \in A$$ of closed and convex subsets of $$C$$, one has $$\bigcap_{a \in A} F_a \neq \emptyset$$.
whenever

$$\forall B \in \text{Fin}(A), \ \cap_{a \in B} F_a \neq \emptyset.$$  

It was established, independently in both Pratelli (2005) and Žitković (2010), that every closed and bounded convex subset of $L_0^+$ is convexly compact. In this section, we will show the following.

**Proposition 4.3.** Any c-bounded, convex, and closed subset $C \subset L$ is convexly compact.

For an element $f \in L$, we write $f^k := \pi_k(f), k \in \mathbb{N}$. Let $I$ be a directed set and $f_i, i \in I$ a net in $L$. For each $i \in I$, let $\Gamma_i$ denote the set of (finite) convex combinations of the elements $\{f_j : j \geq i\}$. The next lemma goes back to Lemma A1.1 of Delbaen and Schachermayer (1994). Its proof closely follows that of Lemma 2.1 in Pratelli (2005), see also Theorem 3.1 in Žitković (2010).

**Lemma 4.4.** There exist $g_i \in \Gamma_i, i \in I$ such that the nets $g_i^k, i \in I$ converge to $g^k$ in probability, for each $k \in \mathbb{N}$, where $g^k \in L^\$.

**Proof.** Set $u(x) := 1 - e^{-x}, x \in [0, \infty]$ and note that, for given $\alpha > 0$, there is $\beta > 0$ such that

$$u\left(\frac{x+y}{2}\right) \geq \frac{u(x) + u(y)}{2} + \beta, \text{ when } |x - y| \geq \alpha \text{ and } \min(x, y) \leq 1/\alpha. \quad (17)$$

Set, for $i \in I$,

$$s_i = \sup\left\{ \sum_{k=0}^{\infty} 2^{-k} E u(g^k) : g \in \Gamma_i \right\}.$$ 

As $s_i, i \in I$ is a nonincreasing net of numbers, it converges to $s_\infty := \inf_{i \in I} s_i$. Choose a nondecreasing $i_m, m \in \mathbb{N}$ such that $s_\infty = \lim_{m \to \infty} s_{i_m}$ and, for each $m$,

$$|s_\infty - s_{i_m}| \leq \frac{1}{m + 1}, \quad (18)$$

and let $g_{i_m}$ be such that $\sum_{k=0}^{\infty} 2^{-k} E u(g_{i_m}^k) \geq s_{i_m} - 1/(m + 1)$.

For elements $p \in I \setminus \{i_m, m \in \mathbb{N}\}$, there is $l = l(p)$ such that $s_{i_{l+1}} \leq s_p \leq s_{i_l}$ and choose $g_p \in \Gamma_p$ such that $\sum_{k=0}^{\infty} 2^{-k} E u(g_p^k) \geq s_{i_{l+1}} - 1/(l + 1)$.

In order to prove that $g_i^k, i \in I$ is a Cauchy net in $L^\$ for each $k$, we need to establish that, for each $k$ and each $\alpha, \varepsilon > 0$, there is $i(\varepsilon)$ such that, for $p, q \geq i(\varepsilon)$,

$$P\left( |g_p^k - g_q^k| \geq \alpha, \min(g_p^k, g_q^k) \leq 1/\alpha \right) \leq \varepsilon.$$ 

Let $p, q \geq i_m$. Notice that $(g_p + g_q)/2 \in \Gamma_{i_m}$ so

$$\sum_{k=0}^{\infty} 2^{-k} E u\left(\frac{g_p^k + g_q^k}{2}\right) \leq s_{i_m}.$$
However, by construction, \( l(p) \geq m \) so

\[
\sum_{k=0}^{\infty} 2^{-k} E_u(g^k_p) \geq s_{l(p)+1} - \frac{1}{l(p) + 1} \geq s_m - \frac{1}{m + 1} - (s_m - s_{l(p)+1}),
\]

and the latter is \( \geq s_m - \frac{2}{m+1} \), by Equation (18). A similar estimate holds for \( \sum_{k=0}^{\infty} 2^{-k} E_u(g^k_q) \), hence, we conclude, from Equation (17), that

\[
\beta \sum_{k=0}^{\infty} 2^{-k} P(|g^k_p - g^k_q| \geq \alpha, \min(g^k_p, g^k_q) \leq 1/\alpha) \leq s_m - \frac{s_m}{2} + \frac{1}{m + 1} - s_m - \frac{1}{m + 1} \leq \frac{2}{m + 1}.
\]

This entails, for each \( k \in \mathbb{N} \),

\[
P(|g^k_p - g^k_q| \geq \alpha, \min(g^k_p, g^k_q) \leq 1/\alpha) \leq \frac{2^{k+1}}{\beta(m + 1)},
\]

which can be made arbitrarily small if \( m \) is large enough. The statement is proved.

**Proof of Proposition 4.3.** Let \( A \) be an arbitrary index set and let \( C_a, a \in A \) be closed, convex subsets of \( C \). Assume that for each \( a \in \text{Fin}(A) \) with \( a = \{a_1, ..., a_K\} \), we have \( C_{a_1} \cap ... \cap C_{a_K} \neq \emptyset \). Let us pick an element \( c(a) \) from this intersection. Apply Lemma 4.4 to the net \( c(a), a \in \text{Fin}(A) \) to obtain convex combinations \( g(a) \in C_{a_1} \cap ... \cap C_{a_K} \) such that the net \( g^k(a), a \in \text{Fin}(A) \) converges to some \( g^k \in L^\delta \) in probability, for each \( k \in \mathbb{N} \).

As \( C \) is c-bounded, \( g^k(a), a \in A \) are bounded in \( L^0_+ \), so, actually, \( g^k \in L^0_+ \). It follows that \( g \in L \) and, by the definition of topological products, the net \( g(a) \), \( a \in \text{Fin}(A) \) converges to \( g \). However, for each fixed \( a \in A \), \( g(a) \in C_a \) for each \( a \in \text{Fin}(A) \) containing \( a \), hence, by the closedness of \( C_a \), we also have \( g \in C_a \). It follows that \( \cap_{a \in A} C_a \neq \emptyset \) since \( g \) is in this intersection.

### 4.3 Convex compactness for finite variation processes

Let \( \mathcal{V} \) denote the family of nondecreasing, left-continuous functions on \([0, T] \), which are 0 at 0. Let \( r_k, k \in \mathbb{N} \) be an enumeration of \((\mathbb{Q} \cap [0, T]) \cup \{T\} \) with \( r_0 = T \). For \( f, g \in \mathcal{V} \), define

\[
\rho(f, g) := \sum_{k=0}^{\infty} 2^{-k} |f(r_k) - g(r_k)|.
\]

The series converges since \( |f(r_k) - g(r_k)| \leq f(T) + g(T) \) for each \( k \), and it defines a metric. The corresponding Borel-field is denoted by \( \mathfrak{G} \).

Let \( \mathcal{W} \) denote the set of pairs \( H = (H^\uparrow, H^\downarrow) \) where \( H^\uparrow, H^\downarrow, t \in [0, T] \) are predictable processes such that \( H^\uparrow(\omega), H^\downarrow(\omega) \in \mathcal{V} \) for each \( \omega \in \Omega \). Considered as mappings \( H^\uparrow, H^\downarrow : (\Omega, F) \to (\mathcal{V}, \mathfrak{G}) \), they are measurable, by the definition of the metric \( \rho \). We identify elements of \( \mathcal{W} \) when they
coincide outside a $P$-null set. We equip $V$ with the topology coming from the metric

$$\varphi(H, G) := E[\rho(H^\uparrow, G^\uparrow) \land 1] + E[\rho(H^\downarrow, G^\downarrow) \land 1].$$

Although this metric was not defined, a related convergence structure was introduced already in Chau and Rásonyi (2019).

Similarly to Proposition 4.3, we obtain the following convex compactness result for subsets of $V$.

**Proposition 4.5.** Let $C$ be a convex and closed subset of $V$. If

$$\{H^\uparrow_T + H^\downarrow_T, H \in C\},$$

is bounded in $L^0_+$ then $C$ is convexly compact.

**Proof.** Let $A$ be an arbitrary index set and let $C_a$, $a \in A$ be closed, convex subsets of $C$. Assume that for any $a \in \text{Fin}(A)$ with $a = \{a_1, \ldots, a_K\}$ we have $C_{a_1} \cap \ldots C_{a_K} \neq \emptyset$. We prove that $\cap_{a \in A} C_a \neq \emptyset$. We identify $L$ with $(L^0_+)^{\mathbb{Q} \cap [0,T]} \cup \{T\}$.

Denote

$$D = \left\{ (H^\uparrow_q, H^\downarrow_q)_{q \in (\mathbb{Q} \cap [0,T]) \cup \{T\}, H \in C} \right\} \subset L,$$

and similarly

$$D_a = \left\{ (H^\uparrow_q, H^\downarrow_q)_{q \in (\mathbb{Q} \cap [0,T]) \cup \{T\}, H \in C_a} \right\} \subset D.$$

Clearly, $D_a, D$ are convex and closed in $L$. By hypothesis, the set $D$ is $c$-bounded. Therefore, Proposition 4.3 implies there exists $(\tilde{H}^\uparrow_q, \tilde{H}^\downarrow_q)_{q \in (\mathbb{Q} \cap [0,T]) \cup \{T\}}$ in $\cap_{a \in A} D_a \neq \emptyset$. Define, for $t \in [0, T] \setminus \mathbb{Q}$,

$$H^\uparrow_t = \lim_{q \uparrow t, q \in \mathbb{Q}} H^\uparrow_q, \quad H^\downarrow_t = \lim_{q \uparrow t, q \in \mathbb{Q}} H^\downarrow_q.$$

Then $(\tilde{H}^\uparrow, \tilde{H}^\downarrow) \in \cap_{a \in A} C_a$ as required. □

**Corollary 4.6.** Let $C$ be a convex and closed subset of $V$. If $\{H^\uparrow_T + H^\downarrow_T, H \in C\}$ is bounded in probability then, for each $a > 0$, $[-a, a] \times C$ is convexly compact (as a subset of $\mathbb{R} \times V$).

**Proof.** Follows by obvious modifications of the arguments of Section 4.2 and Proposition 4.5. □

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