Glauber dynamics in a zero magnetic field and eigenvalue spacing statistics

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Abstract

We discuss the eigenvalue spacing statistics of the Glauber matrix for various models of statistical mechanics (a one dimensional Ising model, a two dimensional Ising model, a one dimensional model with a disordered ground state, and a SK model with and without a ferromagnetic bias). The dynamics of the one dimensional Ising model are integrable, and the eigenvalue spacing statistics are non-universal. In the other cases, the eigenvalue statistics in the high temperature regime are intermediate between Poisson and G.O.E (with $P(0)$ of the order of 0.5). In the intermediate temperature regime, the statistics are G.O.E.. In the low temperature regime, the statistics have a peak at $s = 0$. In the low temperature regime, and for disordered systems, the eigenvalues condense around integers, due to the fact that the local field on any spin never vanishes. This property is still valid for the Ising model on the Cayley tree, even if it is not disordered. We also study the spacing between the two largest eigenvalues as a function of temperature. This quantity seems to be sensitive to the existence of a broken symmetry phase.
Nous discutons les statistiques d’écart entre valeurs propres de la matrice de Glauber pour différents modèles de mécanique statistique (modèle d’Ising unidimensionnel, modèle d’Ising bidimensionnel, modèle unidimensionnel avec un état fondamental désordonné, modèle SK avec ou sans biais ferromagnétique). La dynamique du modèle d’Ising unidimensionnel est intégrable, et la statistique d’écart entre valeurs propres est non universelle. Dans les autres cas, la statistique de valeurs propres dans le régime haute température est intermédiaire entre la loi de Poisson et la loi G.O.E. (avec \( P(0) \) de l’ordre de 0.5). Dans le régime intermédiaire, les statistiques sont G.O.E.. Dans le régime de basses températures, les statistiques présentent un pic pour \( s = 0 \). À basse température et pour des systèmes désordonnés, les valeurs propres condensent autour des entiers, à cause du fait que le champ local sur aucun spin ne s’annule jamais. Cette propriété est encore vraie pour le modèle d’Ising sur l’arbre de Cayley bien qu’il ne soit pas désordonné. Nous étudions également l’écart entre les deux plus grandes valeurs propres en fonction de la température. Cette quantité semble être sensible à l’existence d’une phase avec brisure de symétrie.
1 Introduction

The ideas of level spacing statistics emerged for the first time in the context of nuclear physics [1][2][3], where Wigner proposed computing statistical quantities from consideration of deterministic spectra. Later, these ideas were applied to quantum systems whose classical analogs are chaotic [4][5]. The idea of level spacing statistics is to calculate the difference between two consecutive levels, and to study the probability of occurrence $P(s)$ of a level spacing $s$. The different generic behaviors of $P(s)$ are classified according to random matrix theory [4][6]. A generic case is the integrable spectrum. Each level is labeled by a set of quantum numbers, the energy levels are decorrelated and the statistics are Poissonian: $P(s) = \exp(-s)$. If the number of conserved quantities is too small, it is not possible to find a set of quantum numbers for each level, and the levels are correlated, that is, there exists level repulsion. The repulsion is linear and the level spacing statistics have the Gaussian Orthogonal Ensemble (G.O.E.) shape:

$$P(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}.$$  (1)

If time reversal invariance is broken, and if the system is chaotic, the level spacing statistics has a Gaussian Unitary Ensemble (G.U.E.) shape:

$$P(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi} s^2},$$  (2)

where the repulsion is quadratic. The ideas of quantum chaos have been applied to various fields of condensed matter physics, such as disordered systems [7]. Another field of application is strongly correlated electron systems [8][9], where the hope is to extract more information from finite size systems. In the present paper we wish to analyze the dynamics of classical spin systems using eigenvalue spacing statistics. We use here the term ‘eigenvalue’ rather than ‘level’ since there are no energy levels as in quantum mechanics. We consider the $2^N \times 2^N$ Glauber matrix, with $N$ the number of Ising spins, and diagonalize it for small clusters. The only symmetries are the lattice symmetries, that we treat using group theory, and the global $Z_2$ symmetry. We can thus only study the dynamics for a small number of sites, typically on the order of 10 sites. The eigenstates are not physical, except for the Boltzmann distribution which corresponds to the upper eigenvalue $\lambda = 0$. The other eigenvectors are not probability distributions, since the sum of their components is zero, so that their interpretation is not obvious.

We have studied two different quantities. The first quantity is the distance $\Delta_N(\beta)$ between the two largest eigenvalues (one of them being zero for all $\beta$) for $N$ sites. In the infinite temperature limit, $\Delta_N(0) = 1$ (see below for a proof of this fact). Notice that in what follows one unit of time corresponds to a single spin flip step, whereas a Monte Carlo Step would
correspond to $N$ single spin flip steps. As a consequence, with the normalized time units the spectrum lies in the interval $[-1, 1]$, and the normalized distance between the two largest eigenvalues is $\tilde{\Delta}_N(\beta) = \Delta_N(\beta)/N$. In particular, $\tilde{\Delta}_N(0) = 1/N$ in the infinite temperature limit. One would expect the function $\Delta_N(\beta)$ to decrease as the inverse temperature increases since the relaxation times are expected to be larger for smaller temperatures. In the presence of a broken symmetry (for instance in the case of the two dimensional Ising model), one expects that, in the thermodynamic limit, $\Delta_\infty(\beta) = 0$ if $\beta > \beta_c$ since the broken symmetry states becomes degenerate with the Boltzmann distribution in the range of temperatures $\beta > \beta_c$. By contrast, in the absence of a broken symmetry (for instance for a one dimensional Ising model or for a model with a disordered ground state), the quantity $\Delta_\infty(\beta)$ should be finite, except in the limit $\beta = +\infty$. However, our finite size study is far from the thermodynamic limit since we could only diagonalize the Glauber matrix for about ten sites. We were not able to distinguish between the different conjectured behaviors in the thermodynamic limit. Nonetheless, we study finite size effects and show that $\Delta_N(\beta)$ decreases as the number of sites $N$ increases. For some models (one dimensional Ising model and the frustrated one dimensional Ising model), we find that $\Delta_N(\beta)$ is close to an exponential. In the case of the two dimensional Ising model, $\Delta_N(\beta)$ is clearly not exponential.

Another quantity of interest is the eigenvalue spacing statistics of the full spectrum. We first consider the one dimensional Ising model. In this case, the dynamics are shown to be integrable. The corresponding eigenvalue spacing statistics are found to be non-universal, with a peak at $s = 0$ which increases as the temperature decreases. In the case of the two dimensional Ising model with nearest neighbor coupling, the statistics are intermediate between Poisson and G.O.E. for very small $\beta$, with $P(0) \simeq 0.5$. As $\beta$ increases, the statistics evolves towards a G.O.E. shape ($\beta \sim 1$). At low temperatures no eigenvalue repulsion exists, and the statistics exhibit a peak at $s = 0$. The weight of the peak increases with $\beta$. Next, we consider a frustrated one dimensional model with an extensive entropy at low temperatures. The evolution of the eigenvalue spacing statistics is similar to the case of the two dimensional Ising model. In the case of the Sherrington-Kirkpatrick (SK) model, we also have the following evolution of the eigenvalues spacing statistics: no repulsion at very low temperatures, repulsion for intermediate temperature ($\beta \sim 1$), and no repulsion at low temperatures. An important property of disordered models is that their eigenvalues condense around integers at low temperatures. This is due to the fact that, except for disorder realizations with zero probability measure, the local field on any site is never zero. However, such a behavior also exists for non random systems, for instance for the nearest neighbor Ising model on the Cayley tree.
The Glauber matrix

Glauber dynamics \([11]\) are a single spin flip dynamics with a continuous time. If \(p(\{\sigma\}, t)\) is the probability to find the spin system in the configuration \(\{\sigma\}\) at time \(t\), the master equation for the single spin flip dynamics is

\[
\frac{d}{dt}p(\{\sigma\}, t) = -\left( \sum_{i=1}^{N} w_i(\{\sigma\}) \right) p(\{\sigma\}, t) + \sum_{i=1}^{N} w_i(\{\sigma_1, ..., -\sigma_i, ..., \sigma_N\}) p(\{\sigma_1, ..., -\sigma_i, ..., \sigma_N\}, t).
\]

(3)

The single spin flip transition probabilities are defined as the probabilities that the spin \(\sigma_i\) flips from \(\sigma_i\) to \(-\sigma_i\) while the other spins remain fixed. Since the Boltzmann distribution is a fixed point of the dynamics (3), the transition probabilities have the form

\[
w_i(\{\sigma\}) = \frac{1}{2} \left( 1 - \sigma_i \tanh (\beta J \sum_{j \in V(i)} \sigma_j) \right),
\]

(4)

where \(V(i)\) is the set of neighbors of the site \(i\). Denoting the \(2^N\) vector of the \(p(\{\sigma\}, t)\) as \(p(t)\), equation (3) can be written as

\[
\frac{d}{dt}p(t) = G p(t),
\]

(5)

where the matrix \(G\) is the Glauber matrix. Since the Boltzmann distribution is a steady state of the dynamics, its corresponding eigenvalue is zero regardless of temperature. The matrix \(G\) is not symmetric. It can however be related to a symmetric matrix \(M\). To do so, we notice that the Glauber matrix satisfies the detailed balance, that is \(G_{\alpha,\beta} p(0)_{\beta} = G_{\beta,\alpha} p(0)_{\alpha}\), where \(p(0)\) is the Boltzmann distribution. As a consequence,

\[
\left(p_{\alpha}(0)^{-1/2} G_{\alpha,\beta} \left(p_{\beta}(0)^{1/2}\right) \right) = \left(p_{\beta}(0)^{-1/2} G_{\beta,\alpha} \left(p_{\alpha}(0)^{1/2}\right) \right).
\]

(6)

We call \(M\) the matrix defined by

\[
M_{\alpha,\beta} = \left(p_{\alpha}(0)^{-1/2} G_{\alpha,\beta} \left(p_{\beta}(0)^{1/2}\right) \right).
\]

(7)

Then, \(M\) is symmetric. If \(p\) is a right eigenvector of the Glauber matrix, then

\[
\sum_{\beta} G_{\alpha,\beta} p_{\beta} = \lambda p_{\alpha}
\]

(8)

is equivalent to

\[
\sum_{\beta} M_{\alpha,\beta} \left(p_{\beta}(0)^{-1/2}\right) p_{\beta} = \lambda \left(p_{\alpha}(0)^{-1/2}\right) p_{\alpha},
\]

(9)

so that \(\left(p_{\alpha}(0)^{-1/2}\right) p_{\alpha}\) is an eigenvector of \(M\). We conclude that \(G\) is diagonalizable, and that all its eigenvalues are real.
The spectrum in the infinite temperature limit can be understood as follows. If we call
\[ |\psi\rangle = \sum |\sigma_1\rangle \otimes ... \otimes |\sigma_N\rangle, \] 
(10)
then the dynamics reads
\[ \frac{d}{dt} |\psi\rangle = -\frac{N}{2} |\psi\rangle + \frac{1}{2} \sum_{i=1}^{N} \sigma_i^x |\psi\rangle, \] 
(11)
so that the eigenvalues of the Glauber matrix at infinite temperature are of the form
\[ \lambda = -\frac{N}{2} + \frac{1}{2} \sum_{i=1}^{N} \mu_i, \] 
(12)
where \(\mu_i = \pm 1\). The spectrum in the infinite temperature limit is thus made of eigenvalues at integer values between \(-N\) and 0, with a degeneracy given by the binomial coefficients.

Another property of \(G\) is that for bipartite lattices, such as the square lattice or the Cayley tree, the spectrum of \(G\) is symmetric: if \(\lambda\) belongs to the spectrum then \(-N - \lambda\) is an eigenvalue also. The proof is as follows. Let \(X\{\sigma\}\) be an eigenvector of \(M\), with an eigenvalue \(\lambda\):
\[ \lambda X\{\sigma\} = -\sum_{i=1}^{N} \frac{1}{2} (1 - \sigma_i \tanh(\beta J h_i)) X\{\sigma\} + \sum_{i=1}^{N} \frac{1}{2 \cosh(\beta J h_i)} X\{\sigma_1, ..., -\sigma_i, ..., \sigma_N\}, \] 
(13)
where \(h_i\) is defined by
\[ h_i = \sum_{j \in V(i)} \sigma_j. \] 
(14)
Let \(Y\{\sigma\}\) be defined as
\[ Y\{\sigma\} = (-1)^{\nu(\sigma)} X\{\tilde{\sigma}\}, \] 
(15)
where \(\nu(\sigma)\) is the number of up spins in the configuration \{\sigma\}. \{\tilde{\sigma}\} is deduced from \{\sigma\} by flipping the spins of one of the two sublattices. Then,
\[ (MY)\{\sigma\} = -\sum_{i=1}^{N} \frac{1}{2} (1 - \sigma_i \tanh(\beta H h_i)) (-1)^{\nu(\sigma)} X\{\tilde{\sigma}\} \] 
(16)
\[ + \sum_{i=1}^{N} (-1)^{\nu(\sigma_1, ..., -\sigma_i, ..., \sigma_N)} \frac{1}{2 \cosh(\beta J h_i)} X\{\tilde{\sigma}_1, ..., -\tilde{\sigma}_i, ..., \tilde{\sigma}_M\} \]
\[ = (-1)^{\nu(\sigma)} \left[ -\sum_{i=1}^{N} \frac{1}{2} (1 + \tilde{\sigma}_i \tanh(\beta J h_i)) X\{\tilde{\sigma}\} \right. \]
(17)
\[ - \sum_{i=1}^{N} \frac{1}{2 \cosh(\beta J h_i)} X\{\tilde{\sigma}_1, ..., -\tilde{\sigma}_i, ..., \tilde{\sigma}_N\} \]
\[ = -(N + \lambda)(-1)^{\nu(\sigma)} X\{\tilde{\sigma}\} = -(N + \lambda)Y\{\sigma\}. \] 
(18)
Given an eigenvector $X$ for the eigenvalue $\lambda$, we have built an eigenvector $Y$ for the eigenvalue $-N - \lambda$.

The difference between (3) and the Schrödinger equation is that quantum mechanics preserves the scalar product, leading to Hermitian Hamiltonians. Moreover, the space of physical states is a Hilbert space, and each state of the Hilbert state is physical. In the case of the Glauber matrix, no vector space is present in the sense that the sum of two probability distributions is not a probability distribution. However, some quantities are conserved by the dynamics. It is easy to show that the eigenvectors of $G$ for the non-zero eigenvalues have the property that

$$\sum_{\{\sigma\}} p(\{\sigma\}) = 0.$$  \hspace{1cm} (19)

This is a simple consequence of the fact that the Glauber matrix preserves the quantity

$$\sum_{\{\sigma\}} p(\{\sigma\}).$$  \hspace{1cm} (20)

3 One dimensional Ising model

3.1 Integrability of the dynamics

In the case of the one dimensional model, the Glauber dynamics is integrable. To show this, we follow Glauber and write evolution equations for the correlation functions. We call $R_{i_1,\ldots,i_n}(t)$ the $n$-point correlation function, with $i_\alpha \neq i_\beta$ if $\alpha \neq \beta$, that is

$$R_{i_1,\ldots,i_n}(t) = \langle \prod_{\alpha} \sigma_{i_\alpha}(t) \rangle.$$  \hspace{1cm} (21)

Then following Glauber, we write the evolution equation of $R_{i_1,\ldots,i_n}(t)$ under the form

$$\frac{d}{dt} R_{i_1,\ldots,i_n}(t) = -2 \langle \sigma_{i_1}(t) \ldots \sigma_{i_n}(t) (w_{i_1}(\{\sigma\}) + \ldots + w_{i_n}(\{\sigma\})) \rangle,$$  \hspace{1cm} (22)

where the transition probabilities are given by (4). We notice that in the one dimensional case, each spin has two neighbors, so that $w_i(\{\sigma\})$ can be written as

$$w_i(\{\sigma\}) = \frac{1}{2} \left( 1 - \frac{\gamma}{2} \sigma_i (\sigma_{i+1} + \sigma_{i-1}) \right),$$  \hspace{1cm} (23)

with $\gamma = \tanh 2\beta J$. Here, we take periodic boundary conditions, but the case of open boundary conditions is similar. Inserting (23) into (22) we get

$$\frac{d}{dt} R_{i_1,\ldots,i_n}(t) = \gamma \sum_{i=\pm 1} \sum_{\alpha=1}^{n} \langle \sigma_{i_\alpha+\epsilon} \prod_{\beta \neq \alpha} \sigma_{i_\beta} \rangle.$$  \hspace{1cm} (24)
The terms with $\epsilon = 1$ collect the right neighbors, and $\epsilon = -1$ corresponds to the left neighbors. The correlation function in (24) leads to a (n-2)-point correlator if $\exists \beta, i_{\alpha + \epsilon} = i_\beta$ or to an n-point correlator if not. The expression (24) can be brought under the form

$$\frac{d}{dt} R^{(n)}_{i_1,\ldots,i_n}(t) = -nR^{(n)}_{i_1,\ldots,i_n}(t) + \frac{\gamma}{2} \sum_{\epsilon = \pm 1} \sum_{\alpha=1}^{n} \left( \sum_{\beta=1}^{n} \delta_{i_\beta, i_{\alpha + \epsilon}} R^{(n-2)}_{i_1,\ldots,i_{\alpha - 1},i_\alpha + 1,\ldots,i_{\beta - 1},i_\beta + 1,\ldots,i_n}(t) \right)$$

$$+ \left( 1 - \sum_{\beta=1}^{n} \delta_{i_\beta, i_{\alpha + \epsilon}} \right) R^{(n)}_{i_1,\ldots,i_{\alpha - 1},i_\alpha + \epsilon,i_{\alpha + 1},\ldots,i_n}(t).$$

If none of the sites $i_1,\ldots,i_n$ are neighbors, the term containing $R^{(n-2)}$ vanishes. However, if at least two sites in the set $i_1,\ldots,i_n$ are neighbours, we have to take into account a term containing $R^{(n-2)}$ in the evolution of $R^{(n)}$. It is clear that (25) is nothing but a rewriting of (3) in the case where all the sites have only two neighbors. The number of distinct correlation functions is

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N,$$

which is equal to the number of spin configurations. The system (25) is integrable. Glauber gives the explicit solution for $R^{(1)}(t)$. The equation giving $dR^{(2)}(t)/dt$ contains only linear combinations of $R^{(2)}$. In order to solve for the three points correlation functions, we inject Glauber’s solution into the evolution equation for $R^{(3)}$, diagonalize the associated matrix and get a first order differential equation, which is explicitly integrable and yields the second order correlation functions. The entire hierarchy can be solved by this method since $dR^{(k)}(t)/dt$ does not contain $R^{(p)}$ with $p > k$.

### 3.2 Eigenvalue spacing statistics

In order to calculate the eigenvalue spacing statistics, we need to take all the symmetries of the lattice into account. Here, the symmetries are so obvious that we do not require a group theory treatment. We work with an open Ising chain. This graph is invariant under the reflection and the identity operators. We denote the basis of our ‘Hilbert’ space as $\{\sigma\}$. We use quotes since there is no vector space structure on the probability distributions. However, to diagonalize the Glauber matrix, we can use an analogy to quantum mechanics. If $R$ is the reflection operator, we form the combinations

$$|\{\sigma\}\rangle_\epsilon = \frac{1}{\sqrt{2}} (|\{\sigma\}\rangle + \epsilon R|\{\sigma\}\rangle).$$

This operation leads to states with a well defined behavior under the reflection. The resulting state is either symmetric ($\epsilon = 1$) or antisymmetric ($\epsilon = -1$). The antisymmetric state may be
zero if \(|\{\sigma\}\)> is invariant under the reflection. The dimension of the antisymmetric sector is
\[
\frac{1}{2} \left( 2^N - 2^{\left\lfloor \frac{N+1}{2} \right\rfloor} \right),
\]
and the dimension of the symmetric sector is
\[
\frac{1}{2} \left( 2^N + 2^{\left\lfloor \frac{N+1}{2} \right\rfloor} \right),
\]
where \([\ ]\) denotes the integer part. Finally, we take into account the global \(Z_2\) symmetry of the Glauber matrix. [The matrix elements of the Glauber matrix are invariant under the transformation \(\{\sigma_i\} \rightarrow \{-\sigma_i\}\). Taking into account all the symmetries, we diagonalize the Glauber matrix in the reflection symmetric and the antisymmetric sectors, and in the \(Z_2\) symmetric and antisymmetric sectors. The evolution of the eigenvalues of one sector as a function of the inverse temperature is plotted in figure 1 (a). No avoided crossings are present, which is what is expected for an integrable system. The difference between the two largest eigenvalues is plotted on figure 2 as a function of the inverse temperature. Eventhough some deviations are visible, the behavior of \(\Delta_N(\beta)\) is close to an exponential decay. Finite size effects are visible: \(\Delta_N(\beta)\) decreases as the number of sites \(N\) increases.

The eigenvalue spacing statistics are found to be non universal. For instance, \(P(s)\) is plotted on figure 3 for \(\beta = 1\). The height of the peak at \(s = 0\) decreases as the temperature decreases. Eventhough the dynamics is integrable, the eigenvalue spacing statistics are not of the Poisson form (1). Such a behavior for the spectrum of integrable systems has already been found in the context of integrable quantum fluids (see [9]).

4 Bidimensional Ising model

4.1 Dynamics

In this case, the dynamics are no longer integrable. The evolution of the correlation function is not a linear equation, as it was in the case of the Ising chain. This is essentially due to the fact that, with four neighbors, one has to introduce a cubic term in \(w_i(\{\sigma\})\) given by equation (23):
\[
\tanh (\beta J(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)) = \alpha(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) + \alpha'(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^3,
\]
with
\[
\alpha = \frac{1}{12} (8 \tanh 2\beta J - \tanh 4\beta J),
\]
\[
\alpha' = \frac{1}{48} (\tanh 4\beta J - 2 \tanh 2\beta J).
\]
In the one dimensional case, we could integrate the dynamics because $dR^{(n)}/dt$ was only a function of $R^{(k)}$ with $k \leq n$. In the two dimensional case $dR^{(n)}/dt$ is also a function of $R^{(k)}$ with $k > n$, so that the hierarchy is no longer integrable by this method. It is not because one does not know how to solve the dynamics that these dynamics are not integrable. The analysis of the spectral statistics of the Glauber matrix may be useful to determine whether or not there exists some conserved quantities in the Glauber dynamics of the the Ising model.

4.2 Use of group theory

We use group theory to find the symmetries of the clusters for which we shall diagonalize the Glauber matrix. Notice that we are restricted to small sizes since the size of the ‘Hilbert’ space is equal to $2^N$. In practice, and to have reasonable execution times, we are restricted to $N \leq 13$. The first step is to determine the symmetry group of the lattice, that is to enumerate all the permutations that leave the lattice invariant. To do so, we do not test all the $N!$ possible permutations since the computation time may be huge. Instead, we use the following procedure. We first label the lattice sites and give the list of bonds. We determine all the possible images $\sigma(1)$ of the site 1, that is all the $N - 1$ sites. Then, for each of the possible images of the site 1, we determine the images $\sigma(2)$ of site 2 which respect the lattice symmetry: if there is a bond between 1 and 2, there must be a bond between $\sigma(1)$ and $\sigma(2)$. If there is no bond between 1 and 2, there must be no bond between $\sigma(1)$ and $\sigma(2)$. At this point, we have a list of potential permutations beginning with $\sigma(1)$ and $\sigma(2)$. Then, we determine all the possible images of site 3 which leave the lattice invariant. We thus get a tree structure, but, during the construction, some branches shall stop. At the end of the process, that is when $\sigma(N)$ has been determined, we get all the permutations which leave the graph invariant. The second step is to determine the classes and the table of characters of the group. We use the program in [12], which automatically determines the classes and the table of characters. In a third step, we have to determine the size of the blocs corresponding to the irreducible representations, and how many times a given irreducible representation appears. The dimension of the blocs corresponding to the irreducible representation $(j)$ is equal to

$$\text{Dim}^{(j)} = \frac{1}{h} \sum_{g \in G} \chi(\tilde{g}) \chi^{(j)}(\tilde{g}).$$

(33)

$\tilde{g}$ is the representation of the group element $g$ in the ‘Hilbert’ space, $h$ is the cardinal of the group $G$, $\chi(\tilde{g})$ is the trace of $\tilde{g}$ and $\chi^{(j)}(g)$ is read from the table of characters at the intersection of the line corresponding to the representation $(j)$ and the column of the class of $\tilde{g}$. The fourth step is to implement a Gram-Schmidt procedure to determine the basis of one
block corresponding to the representation \((j)\). We first use the projector
\[
P^{(j)} = \sum_{g \in G} \chi^{(j)}(\bar{g})\bar{g}.
\] (34)

A basis element of the Hilbert space is coded as a binary number of size \(N\). Zero corresponds to a down spin, and 1 codes an up spin. In order to label the basis vectors, we use the decimal representation of the binary number of size \(N\). We denote the corresponding vector \(|\psi_k\rangle\). The procedure consists of scanning all the states \(|\psi_k\rangle\) and to determine \(k_0\) such as \(P^{(j)}|\psi_k\rangle = 0\) if \(k < k_0\) and \(P^{(j)}|\psi_{k_0}\rangle \neq 0\). The state \(P^{(j)}|\psi_{k_0}\rangle\) is the first vector of the basis that we are looking for. Once we have found the first vector of the basis, we continue to scan all the states \(|\psi_k\rangle\), but we project them with
\[
P^{(j)}_0 = \sum_{g \in G} \langle \psi_{k_0} | \bar{g} | \psi_{k_0} \rangle \bar{g}.
\] (35)

If \(P^{(j)}_0|\psi_k\rangle = 0\), we forget about \(|\psi_k\rangle\) and project \(|\psi_{k+1}\rangle\). If \(P^{(j)}_0|\psi_k\rangle \neq 0\), we try to incorporate \(P^{(j)}_0|\psi_k\rangle\) into the basis using a modified Gram-Schmidt procedure \[13\]. If \(P^{(j)}_0|\psi_k\rangle\) is a linear combination of the basis vectors, then we discard it and project \(|\psi_{k+1}\rangle\). If it is not, we incorporate it into the basis, after having orthogonalized it, and we make the projection test for \(|\psi_{k+1}\rangle\). At the end of the procedure, the dimension of the basis must be equal to \((33)\). We note that it is not possible to store all the components of the orthonormal basis because of limited storage capacity. In order to save memory, we stored only the non zero components.

The fifth step is to take into account the global \(Z_2\) symmetry. The sixth and last step is to diagonalize the Glauber matrix using the basis that has been determined at the fourth step. The size of the matrices to be diagonalized are small enough, so that we can use the Jacobi method.

### 4.3 Results

We work with a 3x4 lattice with periodic boundary conditions. The number of representations is equal to 15, and the maximal block dimension is 335. The spectrum in a given sector of symmetry of the 3x4 square lattice is pictured in Figure 1 (b) as a function of the inverse temperature. In the limit \(\beta \rightarrow 0\), we recover degeneracies for integer eigenvalues (see relation \((12)\)). As the inverse temperature increases, the degeneracies present for \(\beta = 0\) are lifted, but the eigenvalues from two different degeneracies are not free to cross, due to eigenvalue repulsion.

We studied the evolution of \(\Delta_N(\beta)\) as a function of \(\beta\) for the 3x4 lattice and for the 3x3 lattice. The result is plotted in figure 4. Finite size effects are visible; \(\Delta_N(\beta)\) decreases if \(N\) increases, and the evolution of \(\Delta_N(\beta)\) is clearly non exponential. In the thermodynamic
limit, $\Delta_\infty(\beta) = 0$ if $\beta > \beta_c$ and $\Delta_\infty(\beta) > 0$ if $\beta < \beta_c$, which means that $\ln \Delta_\infty(\beta) \to -\infty$ if $\beta \to \beta_c^-$. Our finite size study is consistent with such a behavior. However, it would be useful to analyze larger samples, which will be done in the near future.

We now discuss the shape of the eigenvalue spacing statistics $P(s)$. If the inverse temperature is very small, the degeneracies of the $\beta = 0$ case are lifted and the once degenerate eigenvalues spread out linearly. The corresponding statistics are plotted in Figure 5 for $\beta = 0.01$. In this case, we find $P(0) \simeq 0.5$ and the statistics are close to the Poisson law for large $s$. If $\beta$ increases, one reaches the eigenvalue repulsion regime (see figure 1 (b)). Linear eigenvalue repulsion is visible on figure 5 since, for $\beta = 1$, $P(s) \sim s$ for small $s$. However, for $s$ of order unity, large deviations to the G.O.E. law occur. For large $\beta$, the statistics are not universal (see Figure 6 for $\beta = 10$), with a peak at $s = 0$. The weight of this peak increases as $\beta$ increases.

5 Frustrated one dimensional model

5.1 The model

We consider the one dimensional antiferromagnetic Ising model with antiferromagnetic next-nearest-neighbor interactions. This model can be seen as a succession of triangles, as pictured in Figure 7 and can be solved via a transfer matrix formalism, with the sites gathered as shown on figure 7. The transfer matrix has the form

$$ T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, $$

(36)

with

$$ A = \begin{pmatrix} e^{-4\beta J} & e^{-2\beta J} \\ e^{2\beta J} & 1 \end{pmatrix}, \ B = \begin{pmatrix} e^{2\beta J} & 1 \\ 1 & e^{2\beta J} \end{pmatrix}. $$

(37)

In (38), the states are ordered in the form $| \uparrow, \uparrow \rangle, | \uparrow, \downarrow \rangle, | \downarrow, \downarrow \rangle, | \downarrow, \uparrow \rangle$. Because of the form (36) of the transfer matrix, if $(\psi, \varphi)$ is an eigenvector of $T$, then $(\psi + \varphi, \psi + \varphi)$ and $(\psi - \varphi, \psi - \varphi)$ are eigenvectors of $T$ for the same eigenvalue, so that the eigenvalues of $T$ are the eigenvalues of $A + B$ and $A - B$, and the initial 4x4 problem is reduced to two 2x2 problems, due to the time reversal invariance. The partition function is simply $Z_N = Tr T^{N/2}$ for a $N$ site chain ($N$ is assumed to be even). The zero temperature entropy is found to be extensive, of the form $S(0)/N = \ln 2/2$, whereas in the corresponding ferromagnetic problem, the entropy is finite at zero temperatures. This one dimensional antiferromagnetic model has thus the same properties as the triangular antiferromagnet [14], namely the number of zero temperature ground states is proportional to $\exp \alpha N$, with $\alpha$ a constant.
5.2 Results

We work with an open chain version, so that the only symmetries are the inversion and the global $Z_2$ symmetry. We have already explained how to treat these symmetries (section 3.2).

The evolution of $\Delta_N(\beta)$ as a function of the inverse temperature $\beta$ is plotted in Figure 8 for different values of $N$. We observe finite size effects: $\Delta_N(\beta)$ decreases with $N$. The variations of $\Delta_N(\beta)$ as a function of $\beta$ are linear in a semi-log plot, so that $\Delta_N(\beta)$ decreases exponentially with $\beta$.

The evolution of the eigenvalues of the Glauber matrix as a function of the inverse temperature is plotted in figure 1 (c). Level repulsion is visible for $0.2 < \beta < 1$.

The eigenvalue spacing statistics in the high temperature regime is plotted in Figure 9 ($\beta = 0.01$). In this range of temperatures, the statistics are intermediate between Poisson and G.O.E. $P(s)$ has a maximum, but it occurs for smaller separations than in the G.O.E. case. If the inverse temperature increases, the maximum occurs for larger separations, and the eigenvalue spacing statistics is close to the G.O.E. law for $\beta = 1$ (see Figure 9). For larger temperature ($\beta = 3$ on Figure 10), $P(0)$ is not zero (for instance, $P(0) \simeq 0.4$ for $\beta = 3$) and $P(s)$ is close to the Poisson law for large spacings.

6 SK model

6.1 The model

This model was proposed in 1975 as an ‘exactly solvable’ spin glass model [15]. For general reviews on the problems of spin glasses, we refer the reader to references [16] [17] [18]. The SK model is defined by the disordered, infinite range interaction Hamiltonian

$$H = \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j,$$

with quenched random interactions $J_{ij}$ with a gaussian distribution

$$P(J_{ij}) = \left(\frac{N}{2\pi J^2}\right)^{1/2} \exp\left(-\frac{(NJ_{ij} - \delta)^2}{2J^2}\right).$$

In this section, we study the symmetric case $\delta = 0$. The case $\delta > 1$ is the object of section 6. We refer the reader to the reviews previously quoted for the solution of this model. For $T < J$, the model is a spin glass, and a paramagnet for $T > J$. In the glass phase, the model has a large number of thermodynamic phases, with no symmetry connecting ground states. The different ground states are separated by large barriers, proportional to the system size. An other feature of glassiness is the presence of aging, associated with slow relaxation processes.
6.2 Eigenvalue spacing statistics

In the case of the SK model, we do not need to look for the lattice symmetries since, except for some special cases of zero probability, the random infinite range interactions break all the lattice symmetries. The only symmetry to be taken into account is the global $Z_2$ symmetry.

The evolution of the eigenvalues as a function of the inverse temperature is plotted in Figure 11 (a) in a given symmetry sector. We observe that, in the limit $\beta \to +\infty$, the eigenvalues condense around integers. This property is due to the fact that, in this limit, the symmetric representation $M$, of $G$, is diagonal since the local field never vanishes (except for some disorder configurations of zero probability measure), and that the diagonal coefficients are integers. This cluster property in the zero temperature limit is an effect of disorder, and is not related to the existence of glassiness. However, as we shall see in section 8, we find a similar behavior for a glassy ferromagnetic system with no disorder!

The evolution of $\Delta N(\beta)$ as a function of $\beta$ is plotted in Figure 12. The variations of $\Delta N(\beta)$ as a function of $\beta$ are approximately linear in a semi-log plot, so that $\Delta N(\beta)$ decays exponentially as a function of $\beta$.

The eigenvalue spacing statistics $\overline{P(s)}$ in the high temperature regime are intermediate between the Poisson law and the G.O.E. law (see Figure 13 for $\beta = 0.01$). In this regime, $\overline{P(0)} \simeq 0.5$ and $\overline{P(s)}$ is close to the Poisson law for large separations. If the inverse temperature $\beta$ increases, the maximum of $P(s)$ occurs for larger separations (see Figure 13 for $\beta = 0.1$) and finally $\overline{P(s)}$ reaches the G.O.E. law (see figure 14 for $\beta = 1$). In the very low temperature limit, the statistics are close to the Poisson law, with a peak for $s = 0$ (see Figure 14 for $\beta = 10$). The height of this peak increases as $\beta$ increases.

6.3 Zero temperature density of zero eigenvalues

At zero temperature, the eigenvalues of the Glauber matrix of the SK model are integers, and the Glauber matrix is diagonal in the natural basis, excepted for some disorder configurations of zero probability measure. As a matter of fact, some eigenvalues are zero. The states corresponding to the zero temperature zero eigenvalue are the metastable states of the SK model, and their degeneracy is expected to increase exponentially with the system size, as expected from the number of solutions to the TAP equations [16]. It is thus interesting to calculate the density of zero eigenvalues in the zero temperature limit as a function of the system size, even if our approach is restricted only to small system sizes. First, we analytically treat the case of the 3 site SK model, as a warm up exercise.
6.3.1 Warmup: the 3 site SK model

We call $\sigma_i$ the spin variable on site $i$ ($i = 0, 1, 2$), and $J (K, L)$ the random bond between $\sigma_0$ and $\sigma_1$ ($\sigma_0$ and $\sigma_2$, $\sigma_1$ and $\sigma_2$). Since the local field never vanishes, the Glauber matrix is diagonal, and the diagonal coefficients are

\[
\Delta_1 = \Delta(+, +, +) = \Delta(-, -, -) = -\frac{3}{2} + \frac{1}{2} \epsilon(J + K) + \frac{1}{2} \epsilon(J + L) + \frac{1}{2} \epsilon(K + L) \quad (40)
\]

\[
\Delta_2 = \Delta(+, +, -) = \Delta(-, -, +) = -\frac{3}{2} + \frac{1}{2} \epsilon(J - K) + \frac{1}{2} \epsilon(J - L) - \frac{1}{2} \epsilon(K + L) \quad (41)
\]

\[
\Delta_3 = \Delta(+, -, +) = \Delta(-, +, -) = -\frac{3}{2} + \frac{1}{2} \epsilon(K - J) - \frac{1}{2} \epsilon(J + L) + \frac{1}{2} \epsilon(K - L) \quad (42)
\]

\[
\Delta_4 = \Delta(-, -, +) = \Delta(+, -, -) = -\frac{3}{2} - \frac{1}{2} \epsilon(J + K) - \frac{1}{2} \epsilon(J - L) - \frac{1}{2} \epsilon(K - L), \quad (43)
\]

where $\epsilon(x)$ denotes the sign of $x$. The gaussian bond distribution is replaced by

\[
P_{\delta'}(J) = \theta\left(\frac{1}{\sqrt{3}} + \delta' - J\right)\theta\left(J + \frac{1}{\sqrt{3}} - \delta'\right) \quad (44)
\]

in order to analytically perform the forthcoming integrations. The variable $\delta'$ is equal to $\delta/3$. After lengthy but straightforward calculations, we get the probability $P_i(\delta')$ that $\Delta_i$ ($i = 0, 1, 2, 3$) takes the value zero:

- if $\delta' > 1/\sqrt{3}$, $P_0(\delta') = 1$ and $P_1(\delta') = 0$.

- if $0 \leq \delta' \leq 1/\sqrt{3}$,

\[
P_0(\delta') = \frac{1}{4} + \frac{3\sqrt{3}}{4} \delta' + \frac{9}{4} \delta^2 - \frac{9\sqrt{3}}{4} \delta^3 \quad (45)
\]

\[
P_1(\delta') = \frac{1}{4} - \frac{\sqrt{3}}{4} \delta' - \frac{3}{4} \delta^2 + \frac{3\sqrt{3}}{4} \delta^3. \quad (46)
\]

- if $-1/\sqrt{3} \leq \delta' \leq 0$,

\[
P_0(\delta') = \frac{3\sqrt{3}}{4} \left(\frac{1}{\sqrt{3}} + \delta'\right)^3 \quad (47)
\]

\[
P_1(\delta') = \frac{1}{4} - \frac{\sqrt{3}}{4} \delta' - \frac{3}{4} \delta^2 - \frac{\sqrt{3}}{4} \delta^3. \quad (48)
\]

- finally, if $\delta' \leq -1/\sqrt{3}$, $P_0(\delta') = 0$ and $P_1(\delta') = 1/3$.

Moreover, $P_1 = P_2 = P_3$. We observe that $P_0 + 3P_1 = 1$ regardless of the value of the ferromagnetic bias $\delta'$. In the ferromagnetic region, only one state (ferromagnetic) has a zero
eigenvalue. In the frustrated region, three ‘disordered’ configurations share the zero eigenvalue, while the probability of the ferromagnetic state is vanishing. For this three site toy model, the probability of finding a zero eigenvalue is a constant as a function of the ferromagnetic bias. This example shows that two mechanisms are responsible for the appearance of zero eigenvalues: (frustration + disorder) or broken symmetry. Notice that both frustration and disorder are important here since without disorder, the Glauber matrix would not be diagonal in the zero temperature limit.

6.4 N site SK model

We studied the total probability $P_N(\delta)$ of finding a zero eigenvalue at zero temperature. This quantity is plotted in Figure 15 in the frustrated region ($\delta = -2$), in the zero bias case ($\delta = 0$) and in the ferromagnetic region ($\delta = 2$). We observe that in both cases $P_N(\delta)$ increases with $N$, but more slowly in the ferromagnetic case than in the zero bias case. This suggests that the (frustration + disorder) mechanism to generate zero eigenvalues of the Glauber matrix at zero temperature is more efficient than the broken symmetry mechanism, in the limit of large $N$. Moreover, we see that the variations of $P_N(\delta)$ as a function of $N$ are compatible with an exponential growth. However, we are restricted to small systems, so that it would be interesting to do the same study for larger systems. This shall be done in the near future.

Another quantity of interest is the complete distribution function $P(\Delta_N(\beta))$. This quantity is plotted in Figure 16 in the low temperature and high temperature regime. We observe that, in the low temperature regime, the standard deviation of $\Delta_N(\beta)$ is of the same order as $\Delta_N(\beta)$, whereas in the high temperature regime, the standard deviation is small compared to the mean value.

7 SK model with a ferromagnetic bias

We now consider the SK model with a ferromagnetic bias $\delta > 1$. In this regime, the SK model possesses a paramagnetic/ferromagnetic transition as a function of temperature. The spectrum of the Glauber matrix is plotted on Figure 11 (b) as a function of the inverse temperature $\beta$. Level repulsion is visible, and the eigenvalues are attracted by integers in the limit of large $\beta$, which is consistent with the fact that the Hamiltonian is disordered. We plotted on Figure 12 $\Delta_N(\beta)$ for different values of the bias $\delta$. We observe that $\Delta_N(\beta)$ decreases as the ferromagnetic bias increases. This is due to the fact that the paramagnetic/ferromagnetic transition temperature of the biased SK model increases with the bias $\delta$ [15], so that the inverse transition temperature decreases as $\delta$ increases, leading to a decrease of $\Delta_N(\beta)$ as $\delta$ increases. Moreover, the variations of $\Delta_N(\beta)$ become non-exponential if the ferromagnetic bias
increases (see the case \( \delta = 3 \) of Figure 12), which is consistent with what has been observed in the case of the two dimensional Ising model. The eigenvalue spacing statistics are plotted in Figure 17 and Figure 18. Level repulsion appears even for small values of \( \beta \).

8 Ferromagnetic Cayley tree

Trees were introduced in statistical physics as early as the 1930’s in order to implement mean field theories [19] [20]. In this case, only the central spin was considered and the border of the tree (which represents a finite fraction of the spins) was sent to infinity. This limit is the Bethe-Peierls limit. However, it is possible to develop statistical physics on a Cayley tree, with the border included. This was done in the seventies, and a continuous transition was discovered [21] [22]. The dynamics was studied only recently [10]. A cross-over (not a transition like in the SK model) to a glassy regime was then found below the temperature scale \( T_g \sim J/ \ln n \), with \( n \) the number of generations. The temperature scale \( T_g \) decreases in the limit \( n \to +\infty \), but very slowly, so that the glassy regime exists even in the macroscopic regime. The zero temperature barriers scale like \( J \ln n \), with \( n \) the number of generations. If the coordination of the bulk sites is equal to 3, the local field of the bulk sites cannot vanish. This property is reminiscent on the behavior of disordered systems. The analysis of the spectrum of the Glauber matrix at low temperatures reveals that the clustering property also exists for the Cayley tree, as shown in Figure 19, where we plotted the eigenvalues of the Glauber matrix on the Cayley tree at low temperature. This indicates a striking analogy between the ferromagnetic Ising model on the Cayley tree and disordered systems.

9 Conclusion

Since the contents of this paper were already summarized in the introduction, we end here with some final remarks and open questions. The one dimensional case is very special due to the existence of an underlying integrable dynamics. In the other cases, we have the following scenario for the evolution of the eigenvalue spacing statistics as a function of \( \beta \): there exists an intermediate regime \((\beta \sim 1)\) where the statistics are dominated by the non integrability of the dynamics, and are of the G.O.E. type. In the low temperature limit and in the high temperature limit, no eigenvalue repulsion was found. Another important feature of the spectrum of the Glauber matrix is that for disordered systems, the eigenvalues are close to integers in the zero temperature limit. This is a consequence of the fact that the local field never vanishes on any site. This property is not restricted to disordered systems, since we also observe eigenvalue clustering at low temperatures for the nearest neighbor Ising model on the ferromagnetic Cayley tree. Notice that the clustering property is not related to glassiness since disordered
systems without glassiness may exist (the biased SK model with a sufficient ferromagnetic bias) and ferromagnets exist which are known to be glassy at low temperatures and which are not expected to have this property (regular fractals, percolation clusters, hyperbolic lattices).

We also studied the separation between the two largest eigenvalues. This quantity exhibits strong finite size effects. In the case of the one dimensional Ising model, the frustrated one dimensional model, and the unbiased SK model, this quantity decays approximately exponentially with the inverse temperature. However, in the case of the two dimensional Ising model and in the case of a biased SK model, the decay is faster than exponential. This suggests that the separation between the two largest eigenvalues is sensitive to the existence of a second order transition. In the thermodynamic limit, $\Delta_N(\beta)$ is expected to diverge at the transition temperature in a semi-log plot.

We also studied the density of zero eigenvalues in the SK model. At zero temperature, this quantity increases with the number of sites. We found variations that are compatible with an exponential growth. However, it would be interesting to investigate larger systems. As far as the highest non-zero eigenvalue probability distribution is considered, we found that at low temperature, the standard deviation of this distribution is of the order of the average value of the distribution, whereas in the high temperature regime, the standard deviation is small compared to the average value. This result is compatible with the fact that glassiness is due to an accumulation of eigenvalues in the vicinity of zero.

In the present study, we only treated the case of small systems. A generalization to larger systems is under way, in which it would be especially interesting to study the separation between the two largest eigenvalues, as well as the probability of occurrence of zero eigenvalues at zero (and non zero) temperatures. Another important point is that it would be interesting to have analytic expressions for $\Delta_N(\beta)$. This question is also under study. We shall also address in the near future the case of a Glauber dynamics with an external magnetic field.

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Figure captions

Figure 1:
Evolution of the eigenvalues of the Glauber matrix as a function of the inverse temperature for: (a) an 8 site Ising chain in the reflection antiperiodic sector and the Z2 antiperiodic sector. No avoided crossings are present, as expected for an integrable system; (b) the 3x4 Ising model on a square lattice, in the representation number 3 and the antisymmetric $Z_2$ sector. The evolution of 32 eigenvalues is plotted on this figure; (c) the frustrated one dimensional model with nearest neighbor interactions, for 8 sites in the antisymmetric reflection sector, and the antisymmetric $Z_2$ sector.

Figure 2:
Evolution of the parameter $\Delta_N(\beta)$ as a function of the inverse temperature $\beta$, for different sizes (open Ising chain).

Figure 3:
Eigenvalue spacing statistics of the one dimensional Ising model with nearest neighbors couplings. The inverse temperature is $\beta = 1$. The statistics are not universal.

Figure 4:
Evolution of $\Delta_N(\beta)$ as a function of the inverse temperature $\beta$ for the 3x4 lattice ($N = 12$) and the 3x3 lattice ($N = 9$). Finite size effects are visible, and the variations of $\Delta_N(\beta)$ are not exponential.

Figure 5:
Eigenvalue spacing statistics of the 3x4 square lattice for $\beta = 0.01$ and $\beta = 1$. The Poisson and G.O.E. distribution are plotted in dashed lines. In the regime where initially degenerate eigenvalues spread linearly ($\beta = 0.01$), $P(0) \simeq 0.5$ and the statistics are close to the Poisson law for large $s$. In the eigenvalue repulsion regime ($\beta = 1$), the statistics exhibit eigenvalue repulsion ($P(0) = 0$), but the shape of $P(s)$ is distinctly different from the G.O.E. shape.

Figure 6:
Eigenvalue spacing statistics of the 3x4 square lattice for $\beta = 10$. The Poisson distribution is plotted in dashed lines.
The one dimensional Ising model with next-nearest-neighbor coupling, and its representation as a succession of triangles. The dashed lines represent how the sites are gathered in the transfer matrix formalism.

Evolution of $\Delta_N(\beta)$ as a function of $\beta$ for $N = 6, 7, 8, 9, 10$ in the case of the disordered one dimensional model of Figure 7. If $N > N'$, $\Delta_N(\beta) < \Delta_{N'}(\beta)$; $\Delta_N(\beta)$ decreases exponentially with the inverse temperature $\beta$.

Eigenvalue spacing statistics of the frustrated one dimensional model with next nearest neighbor interactions, for an inverse temperature $\beta = 0.01$ and $\beta = 1$.

Eigenvalue spacing statistics of the frustrated one dimensional model with next nearest neighbor interactions, for an inverse temperature $\beta = 3$.

Evolution of the eigenvalues as a function of the inverse temperature $\beta$ (a) in the $Z_2$ antisymmetric sector of a 7 site SK model, and for a given disorder configuration; (b) in the $Z_2$ antisymmetric sector of the 7 site SK model with a ferromagnetic bias, and for a given disorder configuration. The ferromagnetic bias is $\delta = 1.5$.

Evolution of $\Delta_N(\beta)$ as a function of $\beta$ for $N = 9$ in the case of the SK model, for different values of the ferromagnetic bias $\delta$. $\Delta_N(\beta)$ has been averaged over 200 realizations of the disorder. We plotted $\Delta_N(\beta)$ for $\delta = 0, 1, 1.5, 2, 3$. $\Delta_N(\beta)$ decreases if the ferromagnetic bias $\delta$ increases, which is consistent with the fact that the transition temperature increases as a function of $\delta$.

Eigenvalue spacing statistics of the SK model for an inverse temperature $\beta = .01$ and $\beta = 0.1$. The G.O.E. shape and the Poisson shape are plotted in dotted lines.

Eigenvalue spacing statistics of the SK model for an inverse temperature $\beta = 1$ and $\beta = 10$. The G.O.E. shape and the Poisson shape are plotted in dotted lines.

Probability to find a zero eigenvalue at zero temperature, as a function of the number of sites, for $\delta = -2, \delta = 0$ and $\delta = 2$. This is a semi-log plot.
Figure 16:
Distribution of the largest non-zero eigenvalue $-\Delta_N(\beta)$ for a 9 site SK model, and for $\beta = 0.2$ and $\beta = 1.7$. At low temperatures, the standard deviation is of the same order of magnitude as the mean value, whereas this is not the case at high temperatures.

Figure 17:
Eigenvalue spacing statistics of the biased SK model for an inverse temperature $\beta = 0.01$ and $\beta = 0.1$. The bias is $\delta = 1.5$. The G.O.E. shape and the Poisson shape are plotted in dotted lines.

Figure 18:
Eigenvalue spacing statistics of the biased SK model for an inverse temperature $\beta = 1$. The bias is $\delta = 1.5$. The G.O.E. shape and the Poisson shape are plotted in dotted lines.
Figure 19:
Eigenvalues of the Glauber matrix on the Cayley tree with 2 generations and coordination 3 (10 sites). The inverse temperature is $\beta = 2$. 