Nearest-Neighbor Sample Compression: Efficiency, Consistency, Infinite Dimensions

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Abstract
We examine the Bayes-consistency of a recently proposed 1-nearest-neighbor-based multiclass learning algorithm. This algorithm is derived from sample compression bounds and enjoys the statistical advantages of tight, fully empirical generalization bounds, as well as the algorithmic advantages of a faster runtime and memory savings. We prove that this algorithm is strongly Bayes-consistent in metric spaces with finite doubling dimension — the first consistency result for an efficient nearest-neighbor sample compression scheme. Rather surprisingly, we discover that this algorithm continues to be Bayes-consistent even in a certain infinite-dimensional setting, in which the basic measure-theoretic conditions on which classic consistency proofs hinge are violated. This is all the more surprising, since it is known that $k$-NN is not Bayes-consistent in this setting. We pose several challenging open problems for future research.

1 Introduction
This paper deals with Nearest-Neighbor (NN) learning algorithms in metric spaces. Initiated by Fix and Hodges in 1951 [16], this seemingly naive learning paradigm remains competitive against more sophisticated methods [8, 46] and, in its celebrated $k$-NN version, has been placed on a solid theoretical foundation [11, 44, 13, 47].

Although the classic 1-NN is well known to be inconsistent in general, in recent years a series of papers has presented variations on the theme of a regularized 1-NN classifier, as an alternative to the Bayes-consistent $k$-NN. Gottlieb et al. [18] showed that approximate nearest neighbor search can act as a regularizer, actually improving generalization performance rather than just injecting noise. In a follow-up work, [27] showed that applying Structural Risk Minimization to (essentially) the margin-regularized data-dependent bound in [18] yields a strongly Bayes-consistent 1-NN classifier. A further development has seen margin-based regularization analyzed through the lens of sample compression: a near-optimal nearest neighbor condensing algorithm was presented [20] and later extended to cover semimetric spaces [21]; an activated version also appeared [26]. As detailed in [27], margin-regularized 1-NN methods enjoy a number of statistical and computational advantages over the traditional $k$-NN classifier. Salient among these are explicit data-dependent generalization bounds, and considerable runtime and memory savings. Sample compression affords additional advantages, in the form of tighter generalization bounds and increased efficiency in time and space.

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In this work we study the Bayes-consistency of a compression-based 1-NN multiclass learning algorithm, in both finite-dimensional and infinite-dimensional metric spaces. The algorithm is essentially the passive component of the active learner proposed by Kontorovich, Sabato, and Urner in [26], and we refer to it in the sequel as KSU; for completeness, we present it here in full (Alg. 1). We show that in finite-dimensional metric spaces, KSU is both computationally efficient and Bayes-consistent. This is the first compression-based multiclass 1-NN algorithm proven to possess both of these properties. We further exhibit a surprising phenomenon in infinite-dimensional spaces, where we construct a distribution for which KSU is Bayes-consistent while \( k \)-NN is not.

**Main results.** Our main contributions consist of analyzing the performance of KSU in finite and infinite dimensional settings, and comparing it to the classical \( k \)-NN learner. Our key findings are summarized below.

- In Theorem 2, we show that KSU is computationally efficient and strongly Bayes-consistent in metric spaces with a finite doubling dimension. This is the first (strong or otherwise) Bayes-consistency result for an efficient sample compression scheme for a multiclass (or even binary) 1-NN algorithm. This result should be contrasted with the one in [27], where margin-based regularization was employed, but not compression; the proof techniques from [27] do not carry over to the compression-based scheme. Instead, novel arguments are required, as we discuss below. The new sample compression technique provides a Bayes-consistency proof for multiple (even countably many) labels; this is contrasted with the multiclass 1-NN algorithm in [28], which is not compression-based, and requires solving a minimum vertex cover problem, thereby imposing a \( 2 \)-approximation factor whenever there are more than two labels.

- In Theorem 4, we make the surprising discovery that KSU continues to be Bayes-consistent in a certain infinite-dimensional setting, even though this setting violates the basic measure-theoretic conditions on which classic consistency proofs hinge, including Theorem 2. This is all the more surprising, since it is known that \( k \)-NN is not Bayes-consistent for this construction [9]. We are currently unaware of any separable metric probability space on which KSU fails to be Bayes-consistent; this is posed as an intriguing open problem.

Our results indicate that in finite dimensions, an efficient, compression-based, Bayes-consistent multiclass 1-NN algorithm exists, and hence can be offered as an alternative to \( k \)-NN, which is well known to be Bayes-consistent in finite dimensions [12, 41]. In contrast, in infinite dimensions, our results show that the condition characterizing the Bayes-consistency of \( k \)-NN does not extend to all NN algorithms. It is an open problem to characterize the necessary and sufficient conditions for the existence of a Bayes-consistent NN-based algorithm in infinite dimensions.

**Related work.** Following the pioneering work of [11] on nearest-neighbor classification, it was shown by [13, 47, 14] that the \( k \)-NN classifier is strongly Bayes consistent in \( \mathbb{R}^d \). These results made extensive use of the Euclidean structure of \( \mathbb{R}^d \), but in [41] a weak Bayes-consistency result was shown for metric spaces with a bounded diameter and a bounded doubling dimension, and additional distributional smoothness assumptions. More recently, some of the classic results on \( k \)-NN risk decay rates were refined by [10] in an

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1 An efficient sample compression algorithm was given in [20] for the binary case, but no Bayes-consistency guarantee is known for it.

2 Cérou and Guyader [9] gave a simple example of a nonseparable metric on which all known nearest-neighbor methods, including \( k \)-NN and KSU, obviously fail.
analysis that captures the interplay between the metric and the sampling distribution. The worst-case rates have an exponential dependence on the dimension (i.e., the so-called \textit{curse of dimensionality}), and Pestov \cite{pestov1, pestov2} examines this phenomenon closely under various distributional and structural assumptions.

Consistency of NN-type algorithms in more general (and in particular infinite-dimensional) metric spaces was discussed in \cite{pestov1, pestov2, pestov3, pestov4, pestov5}. In \cite{pestov1, pestov2}, characterizations of Bayes-consistency were given in terms of Besicovitch-type conditions (see Eq. (3)). In \cite{pestov1}, a generalized “moving window” classification rule is used and additional regularity conditions on the regression function are imposed. The \textit{filtering} technique (i.e., taking the first $d$ coordinates in some basis representation) was shown to be universally consistent in \cite{pestov5}. However, that algorithm suffers from the cost of cross-validating over both the dimension $d$ and number of neighbors $k$. Also, the technique is only applicable in Hilbert spaces (as opposed to more general metric spaces) and provides only asymptotic consistency, without finite-sample bounds such as those provided by KSU. The insight of \cite{pestov5} is extended to the more general Banach spaces in \cite{pestov6} under various regularity assumptions.

None of the aforementioned generalization results for NN-based techniques are in the form of fully empirical, explicitly computable sample-dependent error bounds. Rather, they are stated in terms of the unknown Bayes-optimal rate, and some involve additional parameters quantifying the well-behavedness of the unknown distribution (see \cite{koltchinskii2} for a detailed discussion). As such, these guarantees do not enable a practitioner to compute a numerical generalization error estimate for a given training sample, much less allow for a data-dependent selection of $k$, which must be tuned via cross-validation. The asymptotic expansions in \cite{koltchinskii2, koltchinskii3, koltchinskii4, koltchinskii5} likewise do not provide a computable finite-sample bound. The quest for such bounds was a key motivation behind the series of works \cite{koltchinskii2, koltchinskii18, koltchinskii20}, of which KSU \cite{koltchinskii26} is the latest development.

The work of Devroye et al. \cite{devroye2} Theorem 21.2] has implications for 1-NN classifiers in $\mathbb{R}^d$ that are defined based on data-dependent majority-vote partitions of the space. It is shown that under some conditions, a fixed mapping from each sample size to a data-dependent partition rule induces a strongly Bayes-consistent algorithm. This result requires the partition rule to have a bounded VC dimension, and since this rule must be fixed in advance, the algorithm is not fully adaptive. Theorem 19.3 ibid. proves weak consistency for an inefficient compression-based algorithm, which selects among all the possible compression sets of a certain size, and maintains a certain rate of compression relative to the sample size. The generalizing power of sample compression was independently discovered by \cite{devroye2}, and later elaborated upon by \cite{devroye1}. In the context of NN classification, \cite{devroye2} lists various condensing heuristics (which have no known performance guarantees) and leaves open the algorithmic question of how to minimize the empirical risk over all subsets of a given size.

The first compression-based 1-NN algorithm with provable optimality guarantees was given in \cite{devroye1}; it was based on constructing $\gamma$-nets in spaces with a finite doubling dimension. The compression size of this construction was shown to be nearly unimprovable by an efficient algorithm unless $P=NP$. With $\gamma$-nets as its algorithmic engine, KSU inherits this near-optimality. The compression-based 1-NN paradigm was later extended to semimetrics in \cite{devroye1}, where it was shown to survive violations of the triangle inequality, while the hierarchy-based search methods that have become standard for metric spaces (such as \cite{devroye4, devroye18} and related approaches) all break down.

It was shown in \cite{koltchinskii2} that a margin-regularized 1-NN learner (essentially, the one proposed in \cite{koltchinskii18}, which, unlike \cite{devroye1}, did not involve sample compression) becomes strongly Bayes-consistent when the margin is chosen optimally in an explicitly prescribed sample-dependent fashion. The margin-based technique developed in \cite{koltchinskii18} for the binary case was extended to multiclass in \cite{koltchinskii28}. Since the algorithm relied on computing a minimum vertex cover, it was not possible to make it both computationally efficient and Bayes-consistent when the number of labels exceeds two. An additional improvement over \cite{koltchinskii28} is that the generalization bounds presented there had an explicit (logarithmic) dependence on the number of labels, while our compression
scheme extends seamlessly to countable label spaces.

**Paper outline.** After fixing the notation and setup in Sec. 2 in Sec. 2, we present KSU, the compression-based 1-NN algorithm we analyze in this work. Sec. 4 discusses our main contributions regarding KSU, together with some open problems. High-level proof sketches are given in Sec. 5 for the finite-dimensional case, and Sec. 6 for the infinite-dimensional case. Full detailed proofs are found in the appendices.

## 2 Setting and Notation

Our instance space is the metric space $(\mathcal{X}, \rho)$, where $\mathcal{X}$ is the instance domain and $\rho$ is the metric. (See Appendix A for relevant background on metric measure spaces.) We consider a countable label space $\mathcal{Y}$. The unknown sampling distribution is a probability measure $\mu$ over $\mathcal{X} \times \mathcal{Y}$, with marginal $\mu$ over $\mathcal{X}$. Denote by $(X, Y)$ a pair drawn according to $\mu$. The generalization error of a classifier $f : \mathcal{X} \to \mathcal{Y}$ is given by $\text{err}_\mu(f) := \mathbb{P}_\mu(Y \neq f(X))$, and its empirical error with respect to a labeled set $S' \subseteq \mathcal{X} \times \mathcal{Y}$ is given by $\hat{\text{err}}(f, S') := \frac{1}{|S'|} \sum_{(x, y) \in S'} 1[y \neq f(x)]$. The optimal Bayes risk of $\mu$ is $R^*_\mu := \inf \text{err}_\mu(f)$, where the infimum is taken over all measurable classifiers $f : \mathcal{X} \to \mathcal{Y}$. We say that $\mu$ is realizable when $R^*_\mu = 0$. We omit the overline in $\mu$ in the sequel when there is no ambiguity.

For a finite labeled set $S \subseteq \mathcal{X} \times \mathcal{Y}$ and any $x \in \mathcal{X}$, let $X_m(x, S)$ be the nearest neighbor of $x$ with respect to $S$ and let $Y_m(x, S)$ be the nearest neighbor label of $x$ with respect to $S$:

$$
(X_m(x, S), Y_m(x, S)) := \arg\min_{(x', y') \in S} \rho(x, x'),
$$

where ties are broken arbitrarily. The 1-NN classifier induced by $S$ is denoted by $h_S(x) := Y_m(x, S)$. The set of points in $S$, denoted by $X = \{X_1, \ldots, X_{|S|}\} \subseteq \mathcal{X}$, induces a Voronoi partition of $\mathcal{X}$, $\mathcal{V}(X) := \{V_i(X), \ldots, V_{|S|}(X)\}$, where each Voronoi cell is $V_i(X) := \{x \in \mathcal{X} : \arg\min_{j \in \{1, \ldots, |S|\}} \rho(x, X_j) = i\}$. By definition, $\forall x \in V_i(X), h_S(x) = Y_i$.

A 1-NN algorithm is a mapping from an i.i.d. labeled sample $S_n \sim \bar{\mu}^n$ to a labeled set $S'_n \subseteq \mathcal{X} \times \mathcal{Y}$, yielding the 1-NN classifier $h_{S'_n}$. While the classic 1-NN algorithm sets $S'_n := S_n$, in this work we study a compression-based algorithm which sets $S'_n$ adaptively, as discussed further below.

A 1-NN algorithm is strongly Bayes-consistent on $\mu$ if $\text{err}(h_{S'_n})$ converges to $R^*$ almost surely, that is $\mathbb{P}[\lim_{n \to \infty} \text{err}(h_{S'_n}) = R^*] = 1$. An algorithm is weakly Bayes-consistent on $\mu$ if $\text{err}(h_{S'_n})$ converges to $R^*$ in expectation, $\lim_{n \to \infty} \mathbb{E}[\text{err}(h_{S'_n})] = R^*$. Obviously, the former implies the latter. We say that an algorithm is Bayes-consistent on a metric space if it is Bayes-consistent on all distributions in the metric space.

A convenient property that is used when studying the Bayes-consistency of algorithms in metric spaces is the **doubling dimension**. Denote the open ball of radius $r$ around $x$ by $B_r(x) := \{x' \in \mathcal{X} : \rho(x, x') < r\}$ and let $B_r(x)$ denote the corresponding closed ball. The doubling dimension of a metric space $(\mathcal{X}, \rho)$ is defined as follows. Let $n$ be the smallest number such that every ball in $\mathcal{X}$ can be covered by $n$ balls of half its radius, where all balls are centered at points of $\mathcal{X}$. Formally,

$$
n := \min\{n \in \mathbb{N} : \forall x \in \mathcal{X}, r > 0, \exists x_1, \ldots, x_n \in \mathcal{X} \text{ s.t. } B_r(x) \subseteq \bigcup_{i=1}^n B_{r/2}(x_i)\}.
$$

Then the doubling dimension of $(\mathcal{X}, \rho)$ is defined by $\text{ddim}(\mathcal{X}, \rho) := \log_2 n$.

For an integer $n$, let $[n] := \{1, \ldots, n\}$. Denote the set of all index vectors of length $d$ by $I_{n,d} := [n]^d$. Given a labeled set $S_n = (X_i, Y_i)_{i \in [n]}$ and any $i = \{i_1, \ldots, i_d\} \in I_{n,d}$, denote the sub-sample of $S_n$
Algorithm 1

where ties are broken arbitrarily. This procedure creates a labeled set $S_i := \{(X_{i1}, Y_{i1}), \ldots, (X_{i, n_i}, Y_{i, n_i})\}$. Similarly, for a vector $Y' = \{Y'_1, \ldots, Y'_{d'}\} \in \mathcal{Y}^{d'}$, denote by $S_n(i, Y') := \{(X_{i, n_i}, Y'_{i, n_i}), \ldots, (X_{i, d}, Y'_{i, d})\}$, namely the sub-sample of $S_n$ as determined by $i$ where the labels are replaced with $Y'$. Lastly, for $i, j \in I_{n,d}$, we denote $S_n(i; j) := \{(X_{i1}, Y_{j1}), \ldots, (X_{i, d}, Y_{j, d})\}$.

3 1-NN majority-based compression

In this work we consider the 1-NN majority-based compression algorithm proposed in [26], which we refer to as KSU. This algorithm is based on constructing $\gamma$-nets at different scales; for $\gamma > 0$ and $A \subseteq \mathcal{X}$, a set $X \subseteq A$ is said to be a $\gamma$-net of $A$ if $\forall a \in A, \exists x \in X : \rho(a, x) \leq \gamma$ and for all $x \neq x' \in X, \rho(x, x') > \gamma$.

The algorithm (see Alg. 1) operates as follows. Given an input sample $S_n$, whose set of points is denoted $X_n = \{X_{i1}, \ldots, X_{n}\}$, KSU considers all possible scales $\gamma > 0$. For each such scale it constructs a $\gamma$-net of $X_n$. Denote this $\gamma$-net by $X(\gamma) := \{X_{i1}, \ldots, X_{i, m}\}$, where $m = m(\gamma)$ denotes its size and $i \equiv i(\gamma) := \{i_1, \ldots, i_m\} \in I_{n,m}$ denotes the indices selected from $S_n$ for this $\gamma$-net. For every such $\gamma$-net, the algorithm attaches the labels $V := (V_1, \ldots, V_m)$, which are the empirical majority-vote labels in the respective Voronoi cells in the partition $\mathcal{V}(X(\gamma)) = \{V_1, \ldots, V_m\}$. Formally, for $i \in [m], V_i = \arg\max_{y \in \mathcal{Y}}|\{j \in [n] | X_j \in V_i, Y_j = y\}|, \quad (1)

where ties are broken arbitrarily. This procedure creates a labeled set $S_n'(\gamma) := S_n(i(\gamma), Y'(\gamma))$ for every relevant $\gamma \in \{\rho(X_i, X_j) | i, j \in [n] \ \backslash \ \{0\}$ The algorithm then selects a single $\gamma^*$, denoted $\gamma^* := \gamma_n^*$, and outputs $h_{\delta, n}(\gamma)$. The scale $\gamma^*$ is selected so as to minimize a generalization error bound, which upper bounds $err(S_n'(\gamma))$ with high probability. This error bound, denoted $Q$ in the algorithm, can be derived using a compression-based analysis, as described below.

Algorithm 1 KSU: 1-NN compression-based algorithm

Require: Sample $S_n = (X_i, Y_i)_{i \in [n]}$, confidence $\delta$
Ensure: A 1-NN classifier

1. Let $\Gamma := \{\rho(X_i, X_j) | i, j \in [n] \ \backslash \ \{0\}$
2. for $\gamma \in \Gamma$ do
3. Let $X(\gamma)$ be a $\gamma$-net of $\{X_1, \ldots, X_n\}$
4. Let $m(\gamma) := \#X(\gamma)$
5. For each $i \in [m(\gamma)]$, let $Y_i'$ be the majority label in $V_i(X(\gamma))$ as defined in Eq. (1)
6. Set $S_n'(\gamma) := (X(\gamma), Y'(\gamma))$
7. end for
8. Set $\alpha(\gamma) := c\delta T(h_{\delta, n}(\gamma), S_n)$
9. Find $\gamma_n^* \arg\min_{\gamma} Q(n, \alpha(\gamma), 2m(\gamma), \delta)$, where $Q$ is, e.g., as in Eq. (2)
10. Set $S_n' := S_n'(\gamma_n^*)$
11. return $h_{\delta, n}$

We say that a mapping $S_n \rightarrow S_n'$ is a compression scheme if there is a function $C : \bigcup_{n=0}^{\infty}(\mathcal{X} \times \mathcal{Y})^m \rightarrow 2^{\mathcal{X} \times \mathcal{Y}^m}$, from sub-samples to subsets of $\mathcal{X} \times \mathcal{Y}$, such that for every $S_n$ there exists an $m$ and a sequence $i \in I_{n,m}$ such that $S_n' = C(S_n(i))$. Given a compression scheme $S_n \rightarrow S_n'$ and a matching function $C$,

\[\text{Algorithm 1 KSU: 1-NN compression-based algorithm}\]

\[\text{Require:} \quad S_n = (X_i, Y_i)_{i \in [n]}, \text{confidence } \delta\]
\[\text{Ensure: A 1-NN classifier}\]

\[1. \quad \text{Let } \Gamma := \{\rho(X_i, X_j) \mid i, j \in [n] \ \backslash \ \{0\}\}
2. \quad \text{for } \gamma \in \Gamma \text{ do}\]
3. \quad \text{Let } X(\gamma) \text{ be a } \gamma\text{-net of } \{X_1, \ldots, X_n\}
4. \quad \text{Let } m(\gamma) := \#X(\gamma)
5. \quad \text{For each } i \in [m(\gamma)], \text{ let } Y_i' \text{ be the majority label in } V_i(X(\gamma)) \text{ as defined in Eq. (1)}
6. \quad \text{Set } S_n'(\gamma) := (X(\gamma), Y'(\gamma))
7. \quad \text{end for}\]
8. \quad \text{Set } \alpha(\gamma) := c\delta T(h_{\delta, n}(\gamma), S_n)
9. \quad \text{Find } \gamma_n^* \arg\min_{\gamma} Q(n, \alpha(\gamma), 2m(\gamma), \delta), \text{where } Q \text{ is, e.g., as in Eq. (2)}
10. \quad \text{Set } S_n' := S_n'(\gamma_n^*)
11. \quad \text{return } h_{\delta, n}\]

\[\text{We say that a mapping } S_n \rightarrow S_n' \text{ is a compression scheme if there is a function } C : \bigcup_{n=0}^{\infty}(\mathcal{X} \times \mathcal{Y})^m \rightarrow 2^{\mathcal{X} \times \mathcal{Y}^m}, \text{ from sub-samples to subsets of } \mathcal{X} \times \mathcal{Y}, \text{ such that for every } S_n \text{ there exists an } m \text{ and a sequence } i \in I_{n,m} \text{ such that } S_n' = C(S_n(i)). \text{ Given a compression scheme } S_n \rightarrow S_n' \text{ and a matching function } C,\]

\[\text{For technical reasons, having to do with the construction in Sec. 6 we depart slightly from the standard definition of a } \gamma\text{-net } X \subseteq A. \text{ The classic definition requires that (i) } \forall a \in A, \exists x \in X : \rho(a, x) < \gamma \text{ and (ii) } \forall x \neq x' \in X : \rho(x, x') \geq \gamma. \text{ In our definition, the relations } < \text{ and } \geq \text{ in (i) and (ii) are replaced by } \leq \text{ and } >.\]
we say that a specific $S'_n$ is an $(\alpha, m)$-compression of a given $S_n$ if $S'_n = \mathcal{C}(S_n(i))$ for some $i \in I_{n, m}$ and \( \bar{\epsilon}(h_{|S'_n|}, S_n) \leq \alpha \). The generalization power of compression was recognized by [17] and [22]. Specifically, it was shown in [21, Theorem 8] that if the mapping $S_n \rightarrow S'_n$ is a compression scheme, then with probability at least $1 - \delta$, for any $S'_n$ which is an $(\alpha, m)$-compression of $S_n$, we have (omitting the constants, explicitly provided therein, which do not affect our analysis)

$$\text{err}(h_{S'_n}) \leq \frac{n}{n - m} - \alpha + O\left(\frac{m \log(n) + \log(1/\delta)}{n - m}\right) + O\left(\sqrt{\frac{\log(n)}{n - m} \log(1/\delta)}\right). \tag{2}$$

Defining $Q(n, \alpha, m, \delta)$ as the RHS of Eq. (2) provides KSU with a compression bound. The following proposition shows that KSU is a compression scheme, which enables us to use Eq. (2) with the appropriate substitution.

**Proposition 1.** The mapping $S_n \rightarrow S'_n$, defined by Alg. 1, is a compression scheme whose output $S'_n$ is a $(\bar{\epsilon}(h_{|S'_n|}), 2|S'_n|)$-compression of $S_n$.

**Proof.** Define the function $\mathcal{C}$ by $\mathcal{C}((\bar{X}_i, \bar{Y}_i)_{i \in [2m]}) = (\bar{X}_i, \bar{Y}_i + m)_{i \in [m]}$, and observe that for all $S_n$, we have $S'_n = \mathcal{C}(S_n(\cdot, \gamma); \cdot, \gamma))$, where $\gamma = \{i_1, \ldots, i_m\} \in I_{\overline{\gamma}}$ is some index vector such that $S'_n(\cdot, \gamma) = Y_{i_1}$ for every $i \in [m(\gamma)]$. Since $Y_{i_1}$ is an empirical majority vote, clearly such a $\gamma$ exists. Under this scheme, the output $S'_n$ of this algorithm is a $(\bar{\epsilon}(h_{|S'_n|}), 2|S'_n|)$-compression.

KSU is efficient, for any countable $\mathcal{Y}$. Indeed, Alg. 1 has a naive runtime complexity of $O(n^4)$, since $O(n^2)$ values of $\gamma$ are considered and a $\gamma$-net is constructed for each one in time $O(n^2)$ (see [20, Algorithm 1]). Improved runtimes can be obtained, e.g., using the methods in [29, 18]. In this work we focus on the Bayes-consistency of KSU, rather than optimize its computational complexity. Our Bayes-consistency results below hold for KSU, whenever the generalization bound $Q(n, \alpha, m, \delta_n)$ satisfies the following properties:

**Property 1** For any integer $n$ and $\delta \in (0, 1)$, with probability $1 - \delta$ over the i.i.d. random sample $S_n \sim \bar{\mu}^n$, for all $\alpha \in [0, 1]$ and $m \in [n]$: If $S'_n$ is an $(\alpha, m)$-compression of $S_n$, then err($h_{S'_n}$) $\leq Q(n, \alpha, m, \delta)$.  

**Property 2** $Q$ is monotonically increasing in $\alpha$ and in $m$.

**Property 3** There is a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n \in (0, 1)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and for all $m$,

$$\lim_{n \to \infty} \sup_{\alpha \in [0, 1]} (Q(n, \alpha, m, \delta_n) - \alpha) = 0.$$  

The compression bound in Eq. (2) clearly satisfies these properties. Note that Property 3 is satisfied by Eq. (2) using any convergent series $\sum_{n=1}^{\infty} \delta_n < \infty$ such that $\delta_n = e^{-o(n)}$, in particular, the decay of $\delta_n$ cannot be too rapid.  

\[\text{In [25], the analysis was based on compression with side information, and does not extend to infinite } \mathcal{Y}.\]
4 Main results

In this section we describe our main results. The proofs appear in subsequent sections. First, we show that KSU is Bayes-consistent if the instance space has a finite doubling dimension. This contrasts with classical 1-NN, which is only Bayes-consistent if the distribution is realizable.

**Theorem 2.** Let \((X, \rho)\) be a metric space with a finite doubling-dimension. Let \(Q\) be a generalization bound that satisfies Properties 1-3, and let \(\delta_n\) be as stipulated by Property 3 for \(Q\). If the input confidence \(\delta\) for input size \(n\) is set to \(\delta_n\), then the 1-NN classifier \(h_{S_n^*}(\gamma_n)\) calculated by KSU is strongly Bayes consistent on \((X, \rho)\):
\[
P(\lim_{n \to \infty} \text{err}(h_{S_n^*}) = R^*) = 1.
\]

The proof, provided in Sec. 5, closely follows the line of reasoning in [27], where the strong Bayes-consistency of an adaptive margin-regularized 1-NN algorithm was proved, but with several crucial differences. In particular, the generalization bounds used by KSU are purely compression-based, as opposed to the Rademacher-based generalization bounds used in [27]. The former can be much tighter in practice and guarantee Bayes-consistency of KSU even for countably many labels. This however requires novel technical arguments, which are discussed in detail in Appendix B.1. Moreover, since the compression-based bounds do not explicitly depend on \(\text{ddim}\), they can be used even when \(\text{ddim}\) is infinite, as we do in Theorem 4 below. To underscore the subtle nature of Bayes-consistency, we note that the proof technique given here does not carry to an earlier algorithm, suggested in [20, Theorem 4], which also uses \(\gamma\)-nets. It is an open question whether the latter is Bayes-consistent.

Next, we study Bayes-consistency of KSU in infinite dimensions (i.e., with \(\text{ddim} = \infty\)) — in particular, in a setting where \(k\)-NN was shown by [9] not to be Bayes-consistent. Indeed, a straightforward application of [9] Lemma A.1 yields the following result.

**Theorem 3** (Cérou and Guyader [9]). There exists an infinite dimensional separable metric space \((X, \rho)\) and a realizable distribution \(\bar{\mu}\) over \(X \times \{0, 1\}\) such that no \(k\)-NN learner satisfying \(k_n/n \to 0\) when \(n \to \infty\) is Bayes-consistent under \(\bar{\mu}\). In particular, this holds for any space and realizable distribution \(\bar{\mu}\) that satisfy the following condition: The set \(C\) of points labeled 1 by \(\bar{\mu}\) satisfies
\[
\mu(C) > 0 \quad \text{and} \quad \forall x \in C, \quad \lim_{r \to 0} \frac{\mu(C \cap \bar{B}_r(x))}{\mu(B_r(x))} = 0. \tag{3}
\]

Since \(\mu(C) > 0\), Eq. (3) constitutes a violation of the Besicovitch covering property. In doubling spaces, the Besicovitch covering theorem precludes such a violation [15]. In contrast, as [35][36] show, in infinite-dimensional spaces this violation can in fact occur. Moreover, this is not an isolated pathology, as this property is shared by Gaussian Hilbert spaces [45].

At first sight, Eq. (3) might appear to thwart any 1-NN algorithm applied to such a distribution. However, the following result shows that this is not the case: KSU is Bayes-consistent on a distribution with this property.

**Theorem 4.** There is a metric space equipped with a realizable distribution for which KSU is weakly Bayes-consistent, while any \(k\)-NN classifier necessarily is not.

The proof relies on a classic construction of Preiss [35] which satisfies Eq. (3). We show that the structure of the construction, combined with the packing and covering properties of \(\gamma\)-nets, imply that the majority-vote classifier induced by any \(\gamma\)-net with a sufficiently small \(\gamma\) approaches the Bayes error.

We conclude the main results by posing intriguing open problems.
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KSU

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In this section we give a high-level proof of Theorem 2, showing that

Bayes-consistency of

KSU

is not? Does there exist

Open problem 1.

Cérou and Guyader [9] distill a certain Besicovitch condition which is necessary and sufficient for k-NN to be Bayes-consistent in a metric space. Our Theorem [4] shows that the Besicovitch condition is not necessary for KSU to be Bayes-consistent. Is it sufficient? What is a necessary condition?

5 Bayes-consistency of KSU in finite dimensions

In this section we give a high-level proof of Theorem[2], showing that KSU is strongly Bayes-consistent in finite-dimensional metric spaces. A fully detailed proof is given in Appendix B.

Recall the optimal empirical error \( \alpha_n^* \equiv \alpha(\gamma^*_n) \) and the optimal compression size \( m_n^* \equiv m(\gamma^*_n) \) as computed by KSU. As shown in Proposition[1] the sub-sample \( S'_n(\gamma^*_n) \) is an \( (\alpha_n^*, 2m_n^*) \)-compression of \( S_n \). Abbreviate the compression-based generalization bound used in KSU by

\[
Q_n(\alpha, m) := Q(n, \alpha, 2m, \delta_n).
\]

To show Bayes-consistency, we start by a standard decomposition of the excess error over the optimal Bayes into two terms:

\[
err(h_{S'_n(\gamma^*_n)}) - R^* = (err(h_{S'_n(\gamma^*_n)}) - Q_n(\alpha_n^*, m_n^*)) + (Q_n(\alpha_n^*, m_n^*) - R^*) =: T_I(n) + T_{II}(n),
\]

and show that each term decays to zero with probability one. For the first term, Property 1 for \( Q \), together with the Borel-Cantelli lemma, readily imply \( \limsup_{n \to \infty} T_I(n) \leq 0 \) with probability one. The main challenge is showing that \( \limsup_{n \to \infty} T_{II}(n) \leq 0 \) with probability one. We do so in several stages:

1. Loosely speaking, we first show (Lemma[8]) that the Bayes error \( R^* \) can be well approximated using 1-NN classifiers defined by the true (as opposed to empirical) majority-vote labels over fine partitions of \( \mathcal{X} \). In particular, this holds for any partition induced by a \( \gamma \)-net of \( \mathcal{X} \) with a sufficiently small \( \gamma > 0 \). This approximation guarantee relies on the fact that in finite-dimensional spaces, the class of continuous functions with compact support is dense in \( L_1(\mu) \) (Lemma[7]).

2. Fix \( \tilde{\gamma} > 0 \) sufficiently small such that any true majority-vote classifier induced by a \( \tilde{\gamma} \)-net has a true error close to \( R^* \), as guaranteed by stage 1. Since for bounded subsets of finite-dimensional spaces the size of any \( \gamma \)-net is finite, the empirical error of any majority-vote \( \gamma \)-net almost surely converges to its true majority-vote error as the sample size \( n \to \infty \). Let \( n(\tilde{\gamma}) \) sufficiently large such that \( Q_n(\tilde{\gamma})(\alpha(\tilde{\gamma}), m(\tilde{\gamma})) \) as computed by KSU for a sample of size \( n(\tilde{\gamma}) \) is a reliable estimate for the true error of \( h_{S'_n(\tilde{\gamma})} \).

3. Let \( \gamma \) and \( n(\gamma) \) be as in stage 2. Given a sample of size \( n = n(\gamma) \), recall that KSU selects an optimal \( \gamma^* \) such that \( Q_n(\alpha(\gamma), m(\gamma)) \) is minimized over all \( \gamma > 0 \). For margins \( \gamma \ll \tilde{\gamma} \), which are prone to over-fitting, \( Q_n(\alpha(\gamma), m(\gamma)) \) is not a reliable estimate for \( h_{S'_n(\gamma^*)} \) since compression may not yet taken place for samples of size \( n \). Nevertheless, these margins are discarded by KSU due to the penalty term in \( Q \). On the other hand, for \( \gamma \)-nets with margin \( \gamma \gg \tilde{\gamma} \), which are prone to under-fitting, the true error is well estimated by \( Q_n(\alpha(\gamma), m(\gamma)) \). It follows that KSU selects \( \gamma_n^* \approx \tilde{\gamma} \) and \( Q_n(\alpha^*_n, m^*_n) \approx R^* \), implying \( \limsup_{n \to \infty} T_{II}(n) \leq 0 \) with probability one.
As one can see, the assumption that $\mathcal{X}$ is finite-dimensional plays a major role in the proof. A simple argument shows that the family of continuous functions with compact support is no longer dense in $L_1$ in infinite-dimensional spaces. In addition, $\gamma$-nets of bounded subsets in infinite dimensional spaces need no longer be finite.

6 On Bayes-consistency of NN algorithms in infinite dimensions

In this section we study the Bayes-consistency properties of 1-NN algorithms on a classic infinite-dimensional construction of Preiss [35], which we describe below in detail. This construction was first introduced as a concrete example showing that in infinite-dimensional spaces the Besicovich covering theorem [15] can be strongly violated, as manifested in Eq. (3).

Example 1 (Preiss’s construction). The construction (see Figure 7) defines an infinite-dimensional metric space $(\mathcal{X}, \rho)$ and a realizable measure $\mu$ over $\mathcal{X} \times \mathcal{Y}$ with the binary label set $\mathcal{Y} = \{0, 1\}$. It relies on two sequences: a sequence of natural numbers $\{N_k\}_{k \in \mathbb{N}}$ and a sequence of positive numbers $\{a_k\}_{k \in \mathbb{N}}$. The two sequences should satisfy the following:

$$\sum_{k=1}^{\infty} a_k N_1 \ldots N_k = 1; \quad \lim_{k \to \infty} a_k N_1 \ldots N_{k+1} = \infty; \quad \text{and} \quad \lim_{k \to \infty} N_k = \infty. \quad (4)$$

These properties are satisfied, for instance, by setting $N_k := k!$ and $a_k := 2^{-k} / \prod_{i \in \mathbb{N}} N_i$. Let $Z_0$ be the set of all finite sequences $(z_1, \ldots, z_k)_{k \in \mathbb{N}}$ of natural numbers such that $z_i \leq N_i$, and let $Z_{\infty}$ be the set of all infinite sequences $(z_1, z_2, \ldots)$ of natural numbers such that $z_i \leq N_i$.

Define the example space $\mathcal{X} := Z_0 \cup Z_{\infty}$ and denote $\gamma_k := 2^{-k}$, where $\gamma_{\infty} := 0$. The metric $\rho$ over $\mathcal{X}$ is defined as follows: for $x, y \in \mathcal{X}$, denote by $x \wedge y$ their longest common prefix. Then,

$$\rho(x, y) = (\gamma_{|x \wedge y|} - \gamma_{|x|}) + (\gamma_{|x \wedge y|} - \gamma_{|y|}).$$

It can be shown (see [35]) that $\rho(x, y)$ is a metric; in fact, it embeds isometrically into the square norm metric of a Hilbert space.

To define $\mu$, the marginal measure over $\mathcal{X}$, let $\nu_{\infty}$ be the uniform product distribution measure over $Z_{\infty}$, that is: for all $i \in \mathbb{N}$, each $z_i$ in the sequence $z = (z_1, z_2, \ldots) \in Z_{\infty}$ is independently drawn from a uniform distribution over $[N_i]$. Let $\nu_0$ be an atomic measure on $Z_0$ such that for all $z \in Z_0$, $\nu_0(z) = a_{|z|}$. Clearly, the first condition in Eq. (4) implies $\nu_0(Z_0) = 1$. Define the marginal probability measure $\mu$ over $\mathcal{X}$ by

$$\forall A \subseteq Z_0 \cup Z_{\infty}, \quad \mu(A) := \alpha \nu_{\infty}(A) + (1 - \alpha) \nu_0(A).$$

In words, an infinite sequence is drawn with probability $\alpha$ (and all such sequences are equally likely), or else a finite sequence is drawn (and all finite sequences of the same length are equally likely). Define the realizable distribution $\bar{\mu}$ over $\mathcal{X} \times \mathcal{Y}$ by setting the marginal over $\mathcal{X}$ to $\mu$, and by setting the label of $z \in Z_{\infty}$ to be 1 with probability 1 and the label of $z \in Z_0$ to be 0 with probability 1.

As shown in [35], this construction satisfies Eq. (3) with $C = Z_{\infty}$ and $\mu(C) = \alpha > 0$. It follows from Theorem 3 that no $k$-NN algorithm is Bayes-consistent on it. In contrast, the following theorem shows that KSU is weakly Bayes-consistent on this distribution. Theorem 4 immediately follows from this result.

Theorem 5. Assume $(\mathcal{X}, \rho), \mathcal{Y}$ and $\bar{\mu}$ as in Example 7. KSU is weakly Bayes-consistent on $\bar{\mu}$. 


The proof, provided in Appendix C, first characterizes the Voronoi cells for which the true majority-vote yields a significant error for the cell (Lemma 13). In finite-dimensional spaces, the total measure of all such “bad” cells can be made arbitrarily close to zero by taking \( \gamma \) to be sufficiently small, as shown in Lemma 8 of Theorem 2. However, it is not immediately clear whether this can be achieved for the infinite dimensional construction above.

Indeed, we expect such bad cells, due to the unintuitive property that for any \( x \in C \), we have \( \mu(B_\gamma(x) \cap C)/\mu(B_\gamma(x)) \to 0 \) when \( \gamma \to 0 \), and yet \( \mu(C) > 0 \). Thus, if for example a significant portion of the set \( C \) (whose label is 1) is covered by Voronoi cells of the form \( V = B_\gamma(x) \) with \( x \in C \), then for all sufficiently small \( \gamma \), each one of these cells will have a true majority-vote 0. Thus a significant portion of \( C \) would be misclassified. However, we show that by the structure of the construction, combined with the packing and covering properties of \( \gamma \)-nets, we have that in *any* \( \gamma \)-net, the total measure of all these “bad” cells goes to 0 when \( \gamma \to 0 \), thus yielding a consistent classifier.

**Remark 1.** In a previous version of the manuscript the following erroneous claim was made (Theorems 5 and 7 in previous version): “For the example constructed in Theorem 5 there exists a sequence of [countable] Voronoi partitions with a vanishing diameter, such that the induced true majority-vote classifiers are not Bayes consistent”. Unfortunately, the proof contains technical errors. In fact, such sequences do not exist, as recently established by [24, Lemma 4.3].

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### A Background on metric measure spaces

Here we provide some general relevant background on metric measure spaces. Our metric space \((\mathcal{X}, \rho)\) is doubling, but in this section finite diameter is not assumed. We recall some standard definitions. A topological space is *Hausdorff* if every two distinct points have disjoint neighborhoods. It is a standard (and obvious) fact that all metric spaces are Hausdorff.
A metric space $X$ is complete if every Cauchy sequence converges to a point in $X$. Every metric space may be completed by (essentially) adjoining to it the limits of all of its Cauchy sequences [38, Exercise 3.24]; moreover, the completion is unique up to isometry [32, Section 43, Exercise 10]. We implicitly assume throughout the paper that $X$ is complete. Closed subsets of complete metric spaces are also complete metric spaces under the inherited metric.

A topological space $X$ is locally compact if every point $x \in X$ has a compact neighborhood. It is a standard and easy fact that complete doubling spaces are locally compact. Indeed, consider any $x \in X$ and the open $r$-ball about $x$, $B_r(x) := \{ y \in X : \rho(x, y) < r \}$. We must show that $\operatorname{cl}(B_r(x))$ — the closure of $B_r(x)$ — is compact. To this end, it suffices to show that $\operatorname{cl}(B_r(x))$ is totally bounded (that is, has a finite $\varepsilon$-covering number for each $\varepsilon > 0$), since in complete metric spaces, a set is compact iff it is closed and totally bounded [32, Theorem 45.1]. Total boundedness follows immediately from the doubling property. The latter posits a constant $k$ and some $x_1, \ldots, x_k \in X$ such that $B_r(x) \subseteq \bigcup_{i=1}^k B_{r/2}(x_i)$. Then certainly $\operatorname{cl}(B_r(x)) \subseteq \bigcup_{i=1}^k B_{r/3}(x_i)$. We now apply the doubling property recursively to each of the $B_{2r/3}(x_i)$, until the radius of the covering balls becomes smaller than $\varepsilon$.

We now recall some standard facts from measure theory. Any topology on $X$ (and in particular, the one induced by the metric $\rho$), induces the Borel $\sigma$-algebra $\mathcal{B}$. A Borel probability measure is a function $\mu : \mathcal{B} \to [0, 1]$ that is countably additive and normalized by $\mu(X) = 1$. The latter is complete if for all $A \subseteq B \in \mathcal{B}$ for which $\mu(B) = 0$, we also have $\mu(A) = 0$. Any Borel $\sigma$-algebra may be completed by defining the measure of any subset of a measure-zero set to be zero [39, Theorem 1.36]. We implicitly assume throughout the paper that $(X, \mathcal{A}, \mu)$ is a complete measure space, where $\mathcal{A}$ contains all of the Borel sets.

The measure $\mu$ is said to be outer regular if it can be approximated from above by open sets: For every $E \in \mathcal{A}$, we have

$$\mu(E) = \inf \{ \mu(V) : E \subseteq V, V \text{ open} \}.$$ 

A corresponding inner regularity corresponds to approximability from below by compact sets: For every $E \in \mathcal{A}$,

$$\mu(E) = \sup \{ \mu(K) : K \subseteq E, K \text{ compact} \}.$$ 

The measure $\mu$ is regular if it is both inner and outer regular. Any probability measure defined on the Borel $\sigma$-algebra of a metric space is regular [25, Lemma 1.19]. (Dropping the “metric” or “probability” assumptions opens the door to various exotic pathologies [7, Chapter 7], [39, Exercise 2.17].)

Finally, we have the following technical result, adapted from [39, Theorem 3.14] to our setting:

**Theorem 6.** Let $X$ be a complete doubling metric space equipped with a complete probability measure $\mu$, such that all Borel sets are $\mu$-measurable. Then $C_c(X)$ (the collection of continuous functions with compact support) is dense in $L_1(\mu)$.

## B Bayes-consistency proof of KSU in finite dimensions

In this section we prove Theorem 2 in detail. Let $(X, \rho)$ be a metric space with doubling-dimension $\ddim < \infty$. Given a sample $S_n \sim \hat{\mu}_n$, we abbreviate the optimal empirical error $\alpha_n^* = \alpha(\gamma_n^*)$ and the optimal compression size $m_n^* = m(\gamma_n^*)$ as computed by KSU. As shown in Sec. 3 the labeled set $S_n^*(\gamma_n^*)$ computed by KSU is an $(\alpha_n^*, 2m_n^*)$-compression of the sample $S_n$. For brevity we denote $Q_n(\alpha, m) := Q(n, \alpha, 2m, \delta_n)$. 

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To prove Theorem 2, we decompose the excess error over the Bayes into two terms:

$$\text{err}(h_{S_n(\gamma_2^+)}(x)) - R^* = (\text{err}(h_{S_n(\gamma_2^+)}(x)) - Q_n(\alpha_n^*, m_n^*)) + (Q_n(\alpha_n^*, m_n^*) - R^*)$$

$$= T_I(n) + T_{II}(n),$$

and show that each term decays to zero with probability one.

For the first term, $T_I(n)$, from Property 1 of generalization bound $Q$, we have that for any $n > 0$,

$$P_{S_n} [\text{err}(h_{S_n(\gamma_2^+)}(x)) - Q_n(\alpha_n^*, m_n^*) > 0] \leq \delta_n.$$

Since $\sum \delta_n < \infty$, the Borel-Cantelli lemma implies $\limsup_{n \to \infty} T_I(n) \leq 0$ with probability one.

The main challenge is to prove that $\limsup_{n \to \infty} T_{II}(n) \leq 0$ with probability one. We begin by showing that the Bayes error $R^*$ can be approached using classifiers defined by the true majority-vote labeling over fine partitions of $X$. Formally, let $V = \{V_1, \ldots\}$ be a finite partition of $X$, and define the function $I_V : X \to \mathcal{Y}$ such that $I_V(x)$ is the unique $V \in \mathcal{V}$ for which $x \in V$. For any measurable set $\emptyset \neq E \subseteq X$ define the true majority-vote label $y^*(E)$ by

$$y^*(E) = \arg\max_{y \in \mathcal{Y}} P_{\mu}(Y = y \mid X \in E),$$

where ties are broken lexicographically. To ensure that $y^*$ is always well-defined, when $E = \emptyset$ we arbitrarily define it to be the lexicographically first $y \in \mathcal{Y}$. Given $\mathcal{V}$ and a measurable set $M \subseteq X$, consider the true majority-vote classifier $h^*_{V,M} : X \to \mathcal{Y}$ given by

$$h^*_{V,M}(x) = y^*(I_V(x) \cap M).$$

Note that if $x \notin M$, this classifier attaches a label to $x$ based on the true majority-vote in a set that does not contain $x$. To bound the error of $h^*_{V,M}$ for any conditional distribution of labels, we use the fact that on doubling metric spaces, continuous functions are dense in $L_1(\mu)$.

**Lemma 7.** For every probability measure $\mu$ on a doubling metric space $X$, the set of continuous functions $f : X \to \mathbb{R}$ with compact support is dense in $L_1(\mu) = \{f : \int_X |f| d\mu(x) < \infty\}$. Namely, for any $\varepsilon > 0$ and $f \in L_1(\mu)$ there is a continuous function $g \in L_1(\mu)$ with compact support such that $\int_X |f - g| d\mu(x) < \varepsilon$.

**Proof.** This is stated as Theorem 4 in Appendix A.

We have the following uniform approximation bound for the error of classifiers in the form of (7), essentially extending the approximation analysis done in the proof of [14, Theorem 21.2] for the special case $|\mathcal{Y}| = 2$ and $X = \mathbb{R}^d$ to the more general multi-class problem in doubling metric spaces.

**Lemma 8.** Let $\bar{\mu}$ be a probability measure on $X \times \mathcal{Y}$ where $X$ is a doubling metric space. For any $\nu > 0$, there exists a diameter $\beta = \beta(\nu) > 0$ such that for any finite measurable partition $\mathcal{V} = \{V_1, \ldots\}$ of $X$ and any measurable set $M \subseteq X$ satisfying

(i) $\mu(X \setminus M) \leq \nu$

(ii) $\text{diam}(\mathcal{V} \cap M) \leq \beta$,

the true majority-vote classifier $h^*_\mathcal{V,M}$ defined in (7) satisfies $\text{err}(h^*_\mathcal{V,M}) \leq R^* + 5\nu$.
Proof. Let \( \eta_y : \mathcal{X} \to [0, 1] \) be the conditional probability function for label \( y \in \mathcal{Y} \),
\[
\eta_y(x) = \mathbb{P}_{\mu}(Y = y | X = x).
\]
Define \( \tilde{\eta}_y : \mathcal{X} \to [0, 1] \) as \( \eta_y \)'s conditional expectation function with respect to \( (\mathcal{Y}, M) \),
\[
\tilde{\eta}_y(x) = \mathbb{P}_{\mu}(Y = y | X \in I_{\mathcal{Y}}(x) \cap M) = \frac{\int_{I_{\mathcal{Y}}(x) \cap M} \eta_y(z) d\mu(z)}{\mu(I_{\mathcal{Y}}(x) \cap M)}.
\]
(And, say, \( \tilde{\eta}_y(x) = 1[y = 1] \) for \( x \notin M \).) Note that \( (\tilde{\eta}_y)_{y \in \mathcal{Y}} \) are piecewise constant on the cells of the restricted partition \( \mathcal{Y} \cap M \). By definition, the Bayes classifier \( h^* \) and the true majority-vote classifier \( h^*_{V,M} \) satisfy
\[
h^*(x) = \operatorname{argmax}_{y \in \mathcal{Y}} \eta_y(x),
\]
\[
h^*_{V,M}(x) = \operatorname{argmax}_{y \in \mathcal{Y}} \tilde{\eta}_y(x).
\]
It follows that
\[
\mathbb{P}_{\mu}(h^*_{V,M}(X) \neq Y | X = x) - \mathbb{P}_{\mu}(h^*(X) \neq Y | X = x) = \eta_{h^*(x)}(x) - \eta_{h^*_{V,M}(x)}(x) = \max_{y \in \mathcal{Y}} \eta_y(x) - \max_{y \in \mathcal{Y}} \tilde{\eta}_y(x)
\]
\[
\leq \max_{y \in \mathcal{Y}} |\eta_y(x) - \tilde{\eta}_y(x)|.
\]
By condition \((i)\) in the theorem statement, \( \mu(\mathcal{X} \setminus M) \leq \nu \). Thus,
\[
err(h^*_{V,M}) - R^* = \mathbb{P}_{\mu}(h^*_{V,M}(X) \neq Y) - \mathbb{P}_{\mu}(h^*(X) \neq Y)
\]
\[
\leq \mu(\mathcal{X} \setminus M) + \int_{M} \max_{y \in \mathcal{Y}} |\eta_y(x) - \tilde{\eta}_y(x)| d\mu(x)
\]
\[
\leq \nu + \sum_{y \in \mathcal{Y}} \int_{M} |\eta_y(x) - \tilde{\eta}_y(x)| d\mu(x).
\]
Let \( \mathcal{Y}' \subseteq \mathcal{Y} \) be a finite set of labels such that \( \mathbb{P}_{\mu}[Y \in \mathcal{Y}'] \geq 1 - \nu \). Then
\[
err(h^*_{V,M}) - R^* \leq 2\nu + \sum_{y \in \mathcal{Y}'} \int_{M} |\eta_y(x) - \tilde{\eta}_y(x)| d\mu(x).
\]  (8)
To bound the integrals in (8), we approximate \((\eta_y)_{y \in \mathcal{Y}'}\) with functions from the dense set of continuous functions, by applying Lemma[7] Since \( \eta_y \in L_1(\mu) \) for all \( y \in \mathcal{Y}' \) and \( |\mathcal{Y}'| < \infty \), Lemma[7] implies that there are \(|\mathcal{Y}'|\) continuous functions \((r_y)_{y \in \mathcal{Y}'}\) with compact support such that
\[
\max_{y \in \mathcal{Y}'} \int_{\mathcal{X}} |\eta_y(x) - r_y(x)| d\mu(x) \leq \nu / |\mathcal{Y}'|.
\]  (9)
Similarly to \((\tilde{\eta}_y)_{y \in \mathcal{Y}'}\), define the piecewise constant functions \((\tilde{r}_y)_{y \in \mathcal{Y}'}\) by
\[
\tilde{r}_y(x) = \mathbb{E}_{\mu}[r_y(X) | X \in I_{\mathcal{Y}}(x) \cap M] = \frac{\int_{I_{\mathcal{Y}}(x) \cap M} r_y(z) d\mu(z)}{\mu(I_{\mathcal{Y}}(x) \cap M)}.
\]
We bound each integrand in (8) by

$$|\eta_y(x) - \tilde{\eta}_y(x)| \leq |\eta_y(x) - r_y(x)| + |r_y(x) - \tilde{r}_y(x)| + |\tilde{r}_y(x) - \eta_y(x)|.$$  \hspace{1cm} (10)$$

The integral of the first term in (10) is smaller than $\nu/|\mathcal{Y}'|$ by the definition of $r_y$ in (9). For the integral of the third term in (10),

$$\int_M |\tilde{r}_y(x) - \eta_y(x)| \, d\mu(x) = \sum_{V \in \mathcal{V}} \int_{V \cap M} \left| r_y(x) - \eta_y(x) \right| \, d\mu(x)$$

$$= \sum_{V \in \mathcal{V}} \int_{V \cap M} (r_y(x) - \eta_y(x)) \, d\mu(x)$$

$$\leq \int_M |r_y(x) - \eta_y(x)| \, d\mu(x) \leq \nu/|\mathcal{Y}'|.$$  

Finally, for the integral of the second term in (10),

$$\int_M |r_y(x) - \tilde{r}_y(x)| \, d\mu(x) = \sum_{V \in \mathcal{V}: \mu(V \cap M) \neq 0} \int_{V \cap M} \left| r_y(x) - \frac{\mathbb{E}_x[r_y(x)1[X \in V \cap M]]}{\mu(V \cap M)} \right| \, d\mu(x)$$

$$= \sum_{V \in \mathcal{V}: \mu(V \cap M) \neq 0} \frac{1}{\mu(V \cap M)} \int_{V \cap M} |r_y(x)\mu(V \cap M) - \mathbb{E}_x[r_y(x)1[X \in V \cap M]]| \, d\mu(x)$$

$$= \sum_{V \in \mathcal{V}: \mu(V \cap M) \neq 0} \frac{1}{\mu(V \cap M)} \int_{V \cap M} \left| r_y(x) \int_{V \cap M} d\mu(z) - \int_{V \cap M} r_y(z) d\mu(z) \right| \, d\mu(x)$$

$$= \sum_{V \in \mathcal{V}: \mu(V \cap M) \neq 0} \frac{1}{\mu(V \cap M)} \int_{V \cap M} \int_{V \cap M} (r_y(x) - r_y(z)) \, d\mu(x) \, d\mu(z)$$

$$\leq \sum_{V \in \mathcal{V}: \mu(V \cap M) \neq 0} \frac{1}{\mu(V \cap M)} \int_{V \cap M} \int_{V \cap M} |r_y(x) - r_y(z)| \, d\mu(x) \, d\mu(z).$$

Since $|\mathcal{Y}'| < \infty$ and any continuous function with compact support is uniformly continuous on all of $\mathcal{X}$, the collection $\{r_y: y \in \mathcal{Y}'\}$ is equicontinuous. Namely, there exists a diameter $\beta = \beta(\nu) > 0$ such that for any $A \subseteq \mathcal{X}$ with $\text{diam}(A) \leq \beta$, $\max_{y \in \mathcal{Y}'} |r_y(x) - r_y(z)| \leq \nu/|\mathcal{Y}'|$ for every $x, z \in A$ (note that $\beta(\nu)$ does not depend on $(\mathcal{V}, M)$). By condition (ii) in the theorem statement, $\text{diam}(V \cap M) \leq \beta$ for all $V \in \mathcal{V}$. Hence,

$$\frac{1}{\mu(V \cap M)} \int_{V \cap M} \int_{V \cap M} |r_y(x) - r_y(z)| \, d\mu(x) \, d\mu(z) \leq \nu/|\mathcal{V}| \mu(V \cap M).$$

Summing over all cells $V \in \mathcal{V}$ with $\mu(V \cap M) \neq 0$, the integral of the second term in (10) satisfies

$$\int_M |r_y(x) - \tilde{r}_y(x)| \, d\mu(x) \leq \nu/|\mathcal{Y}'|. $$
Putting the bounds for the three terms together,

$$
\sum_{y \in Y} \int_M |\eta_y(x) - \tilde{\eta}_y(x)| d\mu(x) \leq \sum_{y \in Y} \frac{3\nu}{|Y'|} = 3\nu.
$$

Applying this bound to (8), we conclude \(\text{err}(h^*_V, M) - R^* \leq 5\nu\).

Next, we prepare to use Lemma 8 to show that the generalization bound \(Q_n(\alpha^*_n, m^*_n)\) also approaches the Bayes error \(R^*\), thus proving \(\lim sup_{n \to \infty} T_{II}(n) \leq 0\) with probability one. Given \(\varepsilon > 0\), fix

$$
\gamma = \gamma(\varepsilon) = \beta(\varepsilon/10)/4,
$$

where \(\beta\) is as guaranteed by Lemma 8. Let \(S'_n(\gamma) = (X(\gamma), Y')\) be the labeled set calculated by Alg. 1 for this value of \(\gamma\). Let \(\alpha(\gamma)\) and \(m(\gamma)\) as defined in Alg. 1. We show that there exist \(N = N(\varepsilon) > 0\), a universal constant \(c > 0\), and a function \(C(\gamma, \varepsilon) > 0\) that does not depend on \(n\), such that \(\forall n \geq N\),

$$
\mathbb{P}(Q_n(\alpha^*_n, m^*_n) > R^* + \varepsilon) \leq C(\gamma, \varepsilon)e^{-cn\varepsilon^2}.
$$

Let \(S_n(\alpha(\gamma), m(\gamma))\) be the labeled set calculated by Alg. 1. We show that there exist \(N = N(\varepsilon) > 0\), a universal constant \(c > 0\), and a function \(C(\gamma, \varepsilon) > 0\) that does not depend on \(n\), such that \(\forall n \geq N\),

$$
\mathbb{P}(Q_n(\alpha^*_n, m^*_n) > R^* + \varepsilon) \leq C(\gamma, \varepsilon)e^{-cn\varepsilon^2}.
$$

By the Borel-Cantelli lemma, this implies that with probability one,

$$
\lim sup_{n \to \infty} T_{II}(n) = \lim sup_{n \to \infty} (Q_n(\alpha^*_n, m^*_n) - R^*) \leq 0.
$$

Since \(\forall n, T_I(n) + T_{II}(n) \geq 0\), this implies \(\lim_{n \to \infty} T_{II}(n) = 0\) with probability one, thus completing the proof of Theorem 2.

We now proceed to prove (12). For \(A \subseteq \mathcal{X}\), denote \(UB_\gamma(A) := \cup_{x \in A} B_\gamma(x)\) and consider the random variable

$$
L_\gamma(S_n) := \mu(\mathcal{X} \setminus UB_\gamma(S_n)),
$$

also known as the \(\gamma\)-missing mass of \(S_n\). We bound

$$
\mathbb{P}(Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon) \\
\leq \mathbb{P}(Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon \land L_\gamma(S_n) \leq \frac{\varepsilon}{10}) + \mathbb{P}(L_\gamma(S_n) > \frac{\varepsilon}{10}].
$$

To bound the second term in (14), we apply the following theorem, due to [2,3], bounding the mean and deviation of the \(\gamma\)-missing mass \(L_\gamma(S_n)\). Denote

$$
\mathcal{N}_\gamma := \left[\frac{\text{diam}(\mathcal{X})}{\gamma}\right]^{d_{\text{dim}}} < \infty.
$$
Theorem 9 \((2, 3)\). Let \((\mathcal{X}, \rho)\) be a doubling metric space and let \(S_n \sim \mu^n\). Then

\[
\mathbb{E}_{S_n}[L_\gamma(S_n)] \leq \frac{\mathcal{N}_\gamma/2}{en}
\]

and for any \(\xi > 0\),

\[
\mathbb{P}_{S_n}\left(L_\gamma(S_n) > \mathbb{E}_{S_n}[L_\gamma(S_n)] + \xi\right) \leq \exp\left(-n\xi^2\right).
\]

Taking \(n\) sufficiently large so that \(\frac{\mathcal{N}_\gamma/2}{en} \leq \varepsilon/20\) and applying Theorem 9 with \(\xi = \varepsilon/20\), we have

\[
\mathbb{P}_{S_n}[L_\gamma(S_n) > \varepsilon/10] \leq e^{-\frac{\varepsilon^2}{16en}}.
\]

We are left to bound the first term in Eq. (14). To this end, we use the fact that any \(\gamma\)-net \(X(\gamma)\) of \(\mathcal{X}\) must satisfy \(13\)

\[
|X(\gamma)| \leq \mathcal{N}_\gamma,
\]

so the compression size \(m(\gamma)\) computed by KSU while using the margin \(\gamma\) satisfies \(\mathbb{P}[m(\gamma) \leq \mathcal{N}_\gamma] = 1\). Hence, the first term in Eq. (14) is bounded by

\[
\mathbb{P}_{S_n}[Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon \wedge L_\gamma(S_n) \leq \frac{\varepsilon}{10} \wedge m(\gamma) = d]
\]

\[
\leq \sum_{d=1}^{\mathcal{N}_\gamma} \mathbb{P}_{S_n}[Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon \wedge L_\gamma(S_n) \leq \frac{\varepsilon}{10} \wedge m(\gamma) = d].
\]

Thus, it suffices to bound each term in the right-hand sum separately. We do so in the following lemma.

Lemma 10. Fix \(\varepsilon > 0\) and let \(\gamma = \gamma(\varepsilon) = \beta(\varepsilon/10)/4\) as in Eq. (17). Under the same conditions as Theorem 2 there exists an \(n_0\) such that for all \(n \geq n_0\), and for all \(d \in [\mathcal{N}_\gamma]\),

\[
p_d := \mathbb{P}_{S_n}[Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon \wedge L_\gamma(S_n) \leq \frac{\varepsilon}{10} \wedge m(\gamma) = d] \leq e^{-ne^2/32}.
\]

Applying Lemma 10 and summing over all \(1 \leq d \leq \mathcal{N}_\gamma\), we have that the first term in (14) satisfies

\[
\mathbb{P}_{S_n}[Q_n(\alpha(\gamma), m(\gamma)) > R^* + \varepsilon \wedge L_\gamma(S_n) \leq \frac{\varepsilon}{10}] \leq \sum_{d=1}^{\mathcal{N}_\gamma} p_d \leq \mathcal{N}_\gamma \cdot e^{-ne^2/32} = C_1(\gamma, \varepsilon)e^{-ne^2/32}.
\]

Plugging Eq. (19) and Eq. (17) into Eq. (14), we get that (12) holds, which completes the proof of Theorem 2. The proof of Lemma 10 follows.

Proof of Lemma 10. Let \(i = i(\gamma) \in I_{n,d}\) be the set of indices of samples from \(S_n = (X_i, Y_i)_{i \in [n]}\) selected by the algorithm for the construction of \(S'_n(\gamma)\), so

\[
X \equiv X(i) = \{X_{i_1}, \ldots, X_{i_d}\}.
\]

By construction, the algorithm guarantees that \(X\) is a \(\gamma\)-net of \(S_n\). This \(\gamma\)-net induces the Voronoi partition \(\mathcal{V}(X) = \{V_1, \ldots, V_d\}\) of \(\mathcal{X}\), where

\[
V_j = \{x \in \mathcal{X} : X_{ma}(x) = X_{i_j}\}, \quad j \in [d].
\]
Let $Y^*(i) \in \mathcal{Y}^d$ be the true majority-vote labels with respect to the restricted partition $\mathcal{V}(X) \cap \text{UB}_{2\gamma}(X)$,

$$(Y^*)_j = y^*(V_j \cap \text{UB}_{2\gamma}(X)), \quad j \in [d].$$

We pair $X(i)$ with the labels $Y^*(i)$ to obtain the labeled set

$$S_n(i, \ast) := S_n(i, Y^*(i)) = (X, Y^*(i)) \in (\mathcal{X} \times \mathcal{Y})^d.$$  \hfill (22)

Note that $S_n(i, \ast)$ is completely determined by $X$ and does not depend on the rest of $S_n$.

The induced 1-NN classifier $h_{S_n(i, \ast)}(x)$ can be written as $h_{S_n(i, \ast)}^\ast(x) = y^*(I_{\mathcal{Y}'}(x) \cap M)$ with $\mathcal{V} = \mathcal{V}(X)$ and $M = \text{UB}_{2\gamma}(X)$ (see Eq. (7) for the definition of $h_{S_n(i, \ast)}^\ast(x)$). We now show that

$$L_\gamma(S_n) \leq \frac{\varepsilon}{10} \implies \text{err}(h_{S_n(i, \ast)}) \leq R^* + \varepsilon/2,$$  \hfill (23)

by showing that under the assumption $L_\gamma(S_n) \leq \frac{\varepsilon}{10}$ the conditions of Lemma 8 hold for $\mathcal{V}, M$ as defined above. For this purpose, we bound the diameter of the partition $\mathcal{V}(X) \cap \text{UB}_{2\gamma}(X)$ and the measure of the missing mass $L_\gamma(X)$ under the assumption.

To bound the diameter of the partition $\mathcal{V}(X) \cap \text{UB}_{2\gamma}(X)$, let $x \in V_j \cap \text{UB}_{2\gamma}(X)$. Then $\rho(x, x_j) = \min_i \rho(x, x_i)$ and, since $x \in \text{UB}_{2\gamma}(X)$, $\min_i \rho(x, x_i) \leq 2\gamma$. Therefore

$$\text{diam}(\mathcal{V} \cap M) = \text{diam}(\mathcal{V}(X) \cap \text{UB}_{2\gamma}(X)) \leq 4\gamma.$$
Thus, it suffices to bound each term in the right-hand sum in (25) separately. Define
\[
  r_{d,n} = \sup_{\alpha \in (0,1)} (Q_n(\alpha, d) - \alpha).
\]
From Property 3 of $Q$, we have that $\lim_{n \to \infty} r_{d,n} = 0$. We have
\[
  Q_n(\tilde{\text{err}}(h_{S_n(i,*)}, S_n), d) \leq \tilde{\text{err}}(h_{S_n(i,*)}, S_n) + r_{d,n}.
\]

Let $i' = \{1, \ldots, n\} \setminus i$ and note that
\[
  \tilde{\text{err}}(h_{S_n(i,*)}, S_n) \leq \frac{n - d}{n} \cdot \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) + \frac{d}{n}.
\]
Combining the two inequalities above, we get
\[
  Q_n(\tilde{\text{err}}(h_{S_n(i,*)}, S_n), d) \leq \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) + \frac{d}{n} + r_{d,n}.
\]
Taking $n$ sufficiently large so that for all $d \leq N_g$,
\[
  \frac{d}{n} + r_{d,n} \leq \frac{\varepsilon}{4},
\]
we have
\[
  Q_n(\tilde{\text{err}}(h_{S_n(i,*)}, S_n), d) \leq \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) + \frac{\varepsilon}{4}.
\]

Therefore, for such an $n$,
\[
  \left\{ Q_n(\tilde{\text{err}}(h_{S_n(i,*)}, S_n), d) > \text{err}(h_{S_n(i,*)}) + \frac{\varepsilon}{2} \right\}
  \Rightarrow \left\{ \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) > \text{err}(h_{S_n(i,*)}) + \frac{\varepsilon}{4} \right\}.
\]
Thus, each term in (25) is bounded above by
\[
  \mathbb{P}_{S_n} \left[ \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) > \text{err}(h_{S_n(i,*)}) + \frac{\varepsilon}{4} \right]
  = \mathbb{E}_{S_n(i)} \left[ \mathbb{P}_{S_n(i')} \left| S_n(i) \right. \left[ \tilde{\text{err}}(h_{S_n(i,*)}, S_n(i')) > \text{err}(h_{S_n(i,*)}) + \frac{\varepsilon}{4} \right] \right].
\]

Since $\mathbb{P}_{S_n(i')} | S_n(i)$ is a product distribution, by Hoeffding’s inequality, we have that (26) is bounded above by $e^{-2(n-d)(\frac{\varepsilon}{4})^2}$. Therefore, from Eq. (25)
\[
  p_d \leq \sum_{i \in I_{n,d}} e^{-2(n-d)(\frac{\varepsilon}{4})^2} \leq |I_{n,d}| e^{-2(n-d)(\frac{\varepsilon}{4})^2} \leq (n)^d e^{-2(n-d)(\frac{\varepsilon}{4})^2} = e^{d\log(n) - 2(n-d)(\frac{\varepsilon}{4})^2}.
\]

Selecting $n$ large enough so that $d \log(n) \leq (n - d)(\frac{\varepsilon}{4})^2$ and $d \leq n/4$, we get the statement of the lemma. \qed
**B.1 Comparison to previous strong Bayes-consistency results**

The strong Bayes-consistency result in Theorem 2 can be seen as an extension of the general-purpose strong Bayes-consistency result in [14, Theorem 21.2] for data-dependent partitioning rules, so as to overcome two additional entangled technical challenges: (i) KSU is adaptive; and (ii) it relies on compression-based generalization bounds. Indeed, since a 1-NN algorithm is a partition of $X$ into majority-vote Voronoi cells, [14, Theorem 21.2] could be applied to show the strong Bayes-consistency of a non-adaptive version of KSU. In our setting, being non-adaptive means the scale $\gamma_n$ used for a sample of size $n$ must be fixed in advance, as opposed to the optimized scale $\gamma^*$ chosen by KSU.

To enable adaptivity, one typically turns to the framework of Structural Risk Minimization [42]. Such an approach was taken by [18], where an adaptive margin-regularized 1-NN algorithm (termed here GKK) is analyzed based on the hierarchy of the class of $1/\gamma$-Lipschitz functions for $\gamma > 0$. These classes come with corresponding generalization bounds that can be minimized to obtain an optimal scale constant $L^*_n = 2/\gamma^*_n$. To minimize the bound, GKK computes, for any choice of $\gamma > 0$, a minimum vertex cover for removing from the sample the smallest number of points so as to ensure that it is $\gamma$-separated (meaning: no two points with conflicting labels are $\gamma$-close). It was shown in [27] that this approach leads to a strongly Bayes-consistent 1-NN classifier.

Despite being adaptive, GKK has some major limitations. In particular, it is not computationally efficient for the multiclass problem [28], the generalization bounds explicitly depend on the dimension of the space (see however [19]), as well as the number of labels, and it suffers from the time and memory inefficiency of storing a large sub-sample.

To mitigate the last two limitations, one might consider the binary classification algorithm implicit in [20] (termed here GKN). GKN minimizes the same empirical error as GKK, but the remaining portion of the sample $S$ is further compressed to a $\gamma$-net of size roughly $(\text{diam}(S)/\gamma)^{\text{dim}}$. GKN then seeks the $\gamma$ which minimizes a compression-based generalization bound, qualitatively similar to Eq. (2). We conjecture that GKN is Bayes-consistent, but were unable to prove it, despite numerous attempts. Due to its reliance on the minimum vertex cover, GKN is also inefficient for the multiclass problem.

KSU overcomes all of these limitations by minimizing a compression-based generalization bound, which amounts to minimizing the empirical error via an efficient majority-vote rule. In particular, it allows the efficient minimization of the bounds that hold for infinite dimensional spaces and countable number of labels.

**C Bayes-consistency proof of KSU on Preiss’s construction**

In this section we prove Theorem 4. We break the proof into three parts. After deriving the necessary properties of Preiss’s construction in Sec. C.1 we prove in Sec. C.2 that if KSU used the true majority-vote labels, it would be Bayes consistent. We then prove in Sec. C.3 that the same holds for empirical majority labels.

**C.1 Preliminaries**

In this subsection, we characterize all possible forms a Voronoi cell can take in partitions of $X$ that are induced by $\gamma$-nets. As we show below, due to the structure of the construction and the packing and covering properties of $\gamma$-nets, each Voronoi cell can be one among only 4 specific types as stated in Lemma 13. These types will be used in subsequent sections to characterize the error incurred by any such majority-vote partition of $X$. 


Balls and subtrees. For any $z \in Z_0 \cup Z_\infty$ of length $|z| = l \in \mathbb{N} \cup \{\infty\}$, we denote by $z_{1:k} = (z_1, \ldots, z_k)$ its prefix of length $k \leq l$ and write $z_{1:k} \preceq z$ for short. By convention, $z_{1:0} = \emptyset$. For any $l \geq 1$ and $z_{1:l} \in Z_0$, we define the subtree rooted at $z_{1:l}$ by

$$T(z_{1:l}) = \{z' \in Z_0 \cup Z_\infty : z_{1:l} \preceq z'\}.$$  

For a subtree $T(z_{1:l})$, we define $\overline{T}(z_{1:l})$ as $T(z_{1:l})$ augmented with its ancestor, $z_{1:l-1}$.

$$\overline{T}(z_{1:l}) = T(z_{1:l}) \cup \{z_{1:l-1}\}.$$  

We will make extensive use of the following properties (see Figure 1).

Lemma 11. Let $z \in Z_\infty = C$. For any $l \geq 1$,

(i) $\forall y, w \in \overline{T}(z_{1:l})$, $\rho(y, w) \leq 2\gamma_l = \gamma_{l-1}$.

(ii) $\forall y \notin \overline{T}(z_{1:l})$ and $w \in \overline{T}(z_{1:l})$, $\rho(y, w) \geq \gamma_l + (\gamma_l - \gamma_{|z_{1:l}|})$.

In particular, taking $w = z$,

(iii) $\forall y \notin \overline{T}(z_{1:l})$, $\rho(y, z) \geq \gamma_{l-1} + \gamma_l > \gamma_{l-1}$.

Following from (i) and (iii), the closed ball about any $z \in C$ satisfies $\overline{B}_{\gamma_{l-1}}(z) = \overline{T}(z_{1:l})$.

(iv) For any $l < \infty$ and any $y \in Z_0$ such that $|y| = l$, $\overline{B}_{\gamma_l}(y) = \overline{T}(y)$.

Proof. (i) Let $w \in T(z_{1:l})$ and note that $|w| \geq l$ and $z_{1:l} \preceq w$. For the case $y = z_{1:l-1}$, $\rho(y, w) = \gamma_{l-1} - \gamma_{|w|} \leq \gamma_{l-1}$. For the other cases $y \in T(z_{1:l})$, note that $\gamma_{|w \wedge y|} \geq \gamma_l$ and thus

$$\rho(y, w) = 2\gamma_{|w \wedge y|} - \gamma_{|w|} - \gamma_{|y|} \leq 2\gamma_{|w \wedge y|} - 2\gamma_{|w|} \leq 2\gamma_l = \gamma_{l-1}.$$  

To show (ii) note that for $y \notin \overline{T}(z_{1:l})$ and $w \in \overline{T}(z_{1:l})$, we have $|w \wedge y| \leq l - 1$. There are two possible cases, $y \in T(z_{1:l-1})$ and $y \notin T(z_{1:l-1})$. For the first case, since $y \notin \overline{T}(z_{1:l})$ we have $y \in T(z_{1:l-1}) \setminus \overline{T}(z_{1:l})$ and thus $|w \wedge y| = l - 1$ and $|y| \geq l$. Hence,

$$\rho(y, w) = 2\gamma_{l-1} - \gamma_{|w|} - \gamma_{|y|} \geq 2\gamma_{l-1} - \gamma_{|w|} = \gamma_l + (\gamma_l - \gamma_{|w|}).$$  

For the second case, $y \notin T(z_{1:l-1})$, we have $|w \wedge y| < l - 1$ and by definition $|y| \geq |w \wedge y|$, thus

$$\rho(y, w) \geq 2\gamma_{|w \wedge y|} - \gamma_{|w|} - \gamma_{|w \wedge y|} \geq \gamma_l - \gamma_{|w|} > \gamma_l + (\gamma_l - \gamma_{|w|}).$$  

Part (iii) readily follows from (ii). To show (iv) note that since $|y| = l$, for any $w \in T(y)$ we have $\rho(y, w) \leq \gamma_l$. Similarly, $\rho(y_{1:l-1}, y) = \gamma_l$. Thus, $\overline{T}(y) \subseteq \overline{B}_{\gamma_l}(y)$. In addition, for any $w \notin \overline{T}(y)$, part (ii) implies $\rho(w, y) \geq \gamma_l + (\gamma_l - \gamma_l) > \gamma_l$, so $\overline{B}_{\gamma_l}(y) \subseteq \overline{T}(y)$.

Bad Voronoi cells. The following two lemmas show that the tree structure of the construction, combined with the packing and covering properties of $\gamma$-nets, imply that Voronoi cells with $V \cap C \neq \emptyset$ have a special form. Call a Voronoi cell $V$ pure iff $V \cap C = \emptyset$, and impure otherwise.

Lemma 12. Let $\gamma_k = 2^{-k}$ and $X_k$ be a $\gamma_k$-net with induced Voronoi cells $V_k$. Let $V \in V_k$ be an impure cell, $x \in V$ be its anchor, and $z \in V \cap C$. Then,
Applying the fact that Lemma 11, we prove the above parts by their order. Proof. We prove the above parts by their order.

Part (I). Since $V$ is a Voronoi cell of a $\gamma_k$-net, we must have $\rho(z, \bar{x}) \leq \gamma_k$. Hence, $\bar{x} \in \overline{B}_{\gamma_k}(z)$. By Lemma $\Xi$, $\overline{B}_{\gamma_k}(z) = \overline{T}(z_{1:k+1})$. In addition, by the $\gamma_k$-net condition, any other anchor $\bar{q} \in X_k$ must satisfy $\rho(\bar{x}, \bar{q}) > \gamma_k$. However, if $\bar{q} \in \overline{T}(z_{1:k+1})$, part (i) of Lemma $\Xi$ implies $\rho(\bar{x}, \bar{q}) \leq \gamma_k$. Hence there is no other anchor besides $\bar{x}$ in $\overline{T}(z_{1:k+1})$.

Part (II). To show $T(z_{1:k+1}) \subseteq V$ note that by part (I) there is no other anchor in $\overline{T}(z_{1:k+1})$ besides $\bar{x}$. However, this implies that any other anchor $\bar{q} \neq \bar{x}$ is too far to be the anchor of any $y \in T(z_{1:k+1})$, implying $y \in V$. To see this, note that since $\bar{q} \notin \overline{T}(z_{1:k+1})$, we have $|\bar{q} \land y| \leq k$. Similarly, $|\bar{x} \land y| \geq k$. Consider the case $|\bar{x} \land y| = k$. Then it must be that $\bar{x} = z_{1:k}$ and $\bar{x} < y$. In addition, since $\bar{q} \notin \overline{T}(z_{1:k+1})$ we have $\rho(\bar{x}, \bar{q}) > 0$. So by the definition of $\rho$,

$$\rho(y, \bar{q}) = \rho(y, \bar{x}) + \rho(\bar{x} \land \bar{q}) + \rho(\bar{x} \land \bar{q}, \bar{q}) = \rho(y, \bar{x}) + \rho(\bar{x}, \bar{q}) > \rho(y, \bar{x}).$$

Hence, $\bar{x}$ is too far from $y$ to be its anchor as claimed. Next, consider the case $|\bar{x} \land y| > k$. By definition, both $y, \bar{x} \in T(\bar{x} \land y)$, and

$$\rho(y, \bar{x}) \leq \rho(y, \bar{x} \land y) + \rho(\bar{x} \land y, \bar{x}) \leq \rho(y, \bar{x} \land y) + \gamma_{|\bar{x} \land y|} \leq \rho(y, \bar{x} \land y) + \gamma_{k+1}.$$  \(\text{(27)}\)

On the other hand, since $\bar{q} \notin \overline{T}(z_{1:k+1})$ and $z_{1:k} < \bar{x} \land y < y$,

$$\rho(y, \bar{q}) = \rho(y, \bar{x} \land y) + \rho(\bar{x} \land y, \bar{q}).$$

By part (ii) of Lemma $\Xi$ (with $w = \bar{x} \land y$ and $l = k + 1$),

$$\rho(\bar{x} \land y, \bar{q}) \geq \gamma_{k+1} + (\gamma_k - \gamma_{|\bar{x} \land y|}).$$

Applying the fact that $|\bar{x} \land y| > k$, we thus get

$$\rho(y, \bar{q}) \geq \rho(y, \bar{x} \land y) + \gamma_{k+1} + (\gamma_k - \gamma_{k+1}) = \rho(y, \bar{x} \land y) + \gamma_k.$$ \(\text{(28)}\)

Hence, by (27) and (28), $\rho(y, \bar{q}) - \rho(y, \bar{x}) \geq \gamma_k - \gamma_{k+1} = \gamma_{k+1} > 0$. Thus $\bar{q}$ is too far from $y$ to be its anchor in this case as well. It follows that $T(z_{1:k+1}) \subseteq V$ as claimed.

To show $V \subseteq \overline{T}(z_{1:k})$, note that by part (I), $\bar{x} \in \overline{T}(z_{1:k+1})$ and thus $|\bar{x}| \geq k$. For any $y \notin \overline{T}(z_{1:k})$, by part (ii) of Lemma $\Xi$ (with $w = \bar{x}$ and $l = k$),

$$\rho(y, \bar{x}) \geq \gamma_k + (\gamma_{k-1} - \gamma_{|\bar{x}|}) \geq \gamma_{k-1} > \gamma_k.$$  \(\text{(29)}\)

However, by the $\gamma_k$-net condition, for any $y \in V$ we must have $\rho(y, \bar{x}) \leq \gamma_k$, thus $y \notin V$.  

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Proof of Lemma 13. By part (II) of Lemma 12, \( T(z_{1:k+1}) \subseteq V \). When equality holds, \( V = T(z_{1:k+1}) \), which is of type \( I_a \). By part (IV), when \( V \setminus T(z_{1:k+1}) \neq \emptyset \), \( V \) must contain \( z_{1:k} \) and thus \( T(z_{1:k+1}) \subseteq V \). When equality holds, \( V = T(z_{1:k+1}) \) and we get type \( I_a \). When \( V \setminus T(z_{1:k+1}) \neq \emptyset \), part (III) implies \( T(z_{1:k}) \subseteq V \). When equality holds, \( V = T(z_{1:k}) \), which is of type \( I_b \). By part (II), we have \( V \subseteq T(z_{1:k}) \), so we are left with \( V = T(z_{1:k}) \) which is of type \( I_a \). Hence (A) follows.

To show (B), consider first cells of type \( I_a \). For these we have
\[
\mu(T(z_{1:k}) \cap C) = \Pr[y \in Z_\infty : z_{1:k} < y] = \alpha/(N_1 \ldots N_k).
\]
Similarly,
\[
\mu(T(z_{1:k})) = \frac{\alpha/(N_1 \ldots N_k) + \Pr[y \in Z_0 : z_{1:k} \prec y]}{\Pr[y \in Z_0 : z_{1:k} \prec y]}
\]
\[
= \frac{\alpha/(N_1 \ldots N_k) + (1 - \alpha)(a_k + \sum_{j=1}^\infty a_{k+j}N_{k+1} \ldots N_{k+j})}{1 + \frac{1 - \alpha}{\alpha} \sum_{j=0}^\infty a_{k+j}N_{k+1} \ldots N_{k+j}}.
\]

Thus,
\[
\frac{\mu(T(z_{1:k}) \cap C)}{\mu(T(z_{1:k}))} = \frac{1 + \frac{1 - \alpha}{\alpha} N_1 \ldots N_k(a_k + \sum_{j=1}^\infty a_{k+j}N_{k+1} \ldots N_{k+j})}{1 + \frac{1 - \alpha}{\alpha} \sum_{j=0}^\infty a_{k+j}N_{k+1} \ldots N_{k+j}}.
\]

By the first condition in Eq. (4), \(\sum_{k=1}^\infty a_k N_1 \ldots N_k = 1\). Thus, we have \(\sum_{j=0}^\infty a_{k+j}N_{k+1} \ldots N_{k+j} = \sum_{j=k}^\infty a_j N_1 \ldots N_j \to 0\) when \(k \to \infty\). It follows that
\[
\frac{\mu(T(z_{1:k}) \cap C)}{\mu(T(z_{1:k}))} \overset{k \to \infty}{\to} 1.
\]

Hence, for sufficiently large \(k\), cells of type \(I_a\) have true-majority-vote 1. An identical argument shows that cells of type \(I_b\) have true-majority-vote 1 as well.

For cells of type \(I_{a^1}\),
\[
\mu(T(z_{1:k}) \cap C) = \mu(T(z_{1:k+1}) \cap C) = \Pr[y \in Z_\infty : z_{1:k} \prec y] = \alpha/(N_1 \ldots N_k),
\]
the same as for type \(I_a\). However, since \(z_{1:k} \in T(z_{1:k+1})\), we have \(\mu(T(z_{1:k+1})) \geq (1 - \alpha)a_k\). Thus, by the second condition in Eq. (4),
\[
\frac{\mu(T(z_{1:k}) \cap C)}{\mu(T(z_{1:k}))} \leq \frac{\alpha}{(1 - \alpha)a_k N_1 \ldots N_{k+1}} \overset{k \to \infty}{\to} 0.
\]

Thus, for sufficiently large \(k\), cells of type \(I_{a^1}\) have true-majority-vote 0. Same argument shows that cells of type \(I_{b^1}\) have true-majority-vote 0 as well.

\[\square\]

### C.2 Consistency with true majority vote labels

Consider a \(\gamma\)-net \(X = V(\gamma)\) of \(X\). Recall that \(V(X) = \{V_1, \ldots\}\) is the Voronoi partition induced by \(X\) and \(I_V : X \to \mathcal{V}\) such that \(I_V(x)\) is the unique \(V \in \mathcal{V}\) for which \(x \in V\). Define the true majority-vote classifier \(h_X : X \to \mathcal{Y}\) given by
\[
h_X(x) = y^*(I_V(x)),
\]
where \(y^*(A)\) is the true majority vote label of \(A \subseteq X\) as given in Eq. (6).

**Theorem 14.** Assume \((X, \rho), \mathcal{Y}\) and \(\bar{\mu}\) as in Example [7]. For \(k \in \mathbb{N}\), let \(X_k\) be some \(\gamma_k\)-net over \(X\). The sequence \((h_{X_k})_{k \in \mathbb{N}}\) of true majority-vote classifiers is Bayes consistent on \(\bar{\mu}\):
\[
\lim_{k \to \infty} \text{err}(h_{X_k}) = 0.
\]
Proof of Theorem 14. To bound the error of $h_{X_k}$, first note that $h_{X_k}$ may incur an error only from impure Voronoi cells. Indeed, any pure $V$ contains only points from $X \setminus C$, which all have label 0 which is in agreement with the true majority-vote of $V$ as required.

As for impure cells, Lemma 13 implies that these can be only of types $I_a, I_b, II_a, II_b$, and for all sufficiently large $k \in \mathbb{N}$, all cells of type $I_a, I_b$ have true majority-vote 1, while all those of type $II_a, II_b$ have true majority-vote 0.

Due to their true majority-vote of 1, each cell $V$ of type $I_a$ incurs an error

$$\mu(V \setminus C) = \mu(T(z_{1:k}) \setminus C) = \Pr[y \in Z_0 : z_{1:k} \prec y] = a_k + \sum_{j=1}^{\infty} a_{k+j}N_{k+1} \ldots N_{k+j}.$$ 

There are at most $N_1 \ldots N_k$ cells of type $I_a$. Denoting by $V_{I_a}^k \subseteq V_k$ the set of all cells of type $I_a$, we have

$$\mu(V_{I_a}^k \setminus C) \leq N_1 \ldots N_k (a_k + \sum_{j=1}^{\infty} a_{k+j}N_{k+1} \ldots N_{k+j}) = \sum_{j=k}^{\infty} a_j N_1 \ldots N_j \xrightarrow{k \to \infty} 0.$$ 

Similarly,

$$\mu(V_{I_b}^k \setminus C) \leq \sum_{j=k+1}^{\infty} a_j N_1 \ldots N_j \xrightarrow{k \to \infty} 0.$$ 

As for cells of type $II_a$, due to their true majority-vote of 0, each such cell incurs an error

$$\mu(V \cap C) = \mu(T(z_{1:k}) \cap C) = \mu(T(z_{1:k}) \cap C) = \Pr[y \in Z_\infty : z_{1:k} \prec y] = \alpha / (N_1 \ldots N_k).$$

Since for cells of type $II_a$ we have $z_{1:k-1} \in V$, there are at most $N_1 \ldots N_k$ such cells in total, thus

$$\mu(V_{II_a}^k \cap C) \leq \frac{\alpha N_1 \ldots N_{k-1}}{N_1 \ldots N_k} = \frac{\alpha}{N_k} \xrightarrow{k \to \infty} 0,$$

where the limit above follows from the last property of the construction in Eq. (4). Similarly, the total error incurred by cells of type $II_b$ is

$$\mu(V_{II_b}^k \cap C) \leq \frac{\alpha N_1 \ldots N_k}{N_1 \ldots N_{k+1}} = \frac{\alpha}{N_{k+1}} \xrightarrow{k \to \infty} 0.$$ 

Hence, we conclude that

$$\text{err}(h_{X_k}) = \mu(V_{I_a}^k \setminus C) + \mu(V_{I_b}^k \setminus C) + \mu(V_{II_a}^k \cap C) + \mu(V_{II_b}^k \cap C) \xrightarrow{k \to \infty} 0.$$ 

C.3 Consistency with empirical majority vote labels

The following theorem proves the consistency of KSU on the Preiss construction. Theorem 4 immediately follows from this result.

Theorem 15. Assume $(X, \rho), Y, \tilde{\mu}$ as in Example 1. KSU is weakly-Bayes-consistent on $\tilde{\mu}$. 

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The following lemmas allow us to reason about empirical $\gamma_k$-nets.

**Lemma 16.** For $z \in Z_0$, denote $P_z := \{ y \in Z_\infty : z < y \}$. Denote $A_k = \{ z \in Z_0 : |z| \leq k + 1 \}$ and let $B \subseteq X$ such that

$$A_k \subseteq B \quad \text{and} \quad \forall z \in A_k, B \cap P_z \neq \emptyset. \quad (33)$$

Then any $\gamma_k$-net of $B$ is also a $\gamma_k$-net of $X$.

**Proof.** For any $z \notin B$, we have $|z| > k + 1$ and there is some $y \in P_{z,k+1} \cap B$. Let $X_k$ be a $\gamma_k$-net of $B$. Following the same steps as in the proof of part (II) in Lemma 12, the Voronoi cell $V$ that includes $y$ has $T(z_{1:k+1}) \subseteq V$. Hence $z \in V$. It follows that $X$ is covered by $X_k$ and thus $X_k$ is a $\gamma_k$-net of $X$. □

**Lemma 17.** For any $\gamma_k > 0$, any $\gamma_k$-net of $X$ in Example 1 is of finite size.

**Proof.** Let $A_k, P_z$ be defined as in Lemma 16. First note that $X = X_{k-1} \cup \{ \bigcup_{z \in A_k} T(z) \}$ and $|A_k| < \infty$. For any $z \in Z_0$ with $|z| = k + 1$, there exists in the $\gamma_k$-net a Voronoi cell $V \supseteq T(z)$. This follows by noting that for any $y \in P_z$, the Voronoi cell $I_V(y)$ has $I_V(y) \cap C \neq \emptyset$, thus by part (II) of Lemma 12, $T(z) \subseteq I_V(y)$. Since there are $N_1 \ldots N_{k+1} < \infty$ number of subtrees $T(z)$ with $|z| = k + 1$, and also $|A_k| < \infty$, there is a finite number of Voronoi cells in the partition of $X$.

**Proof of Theorem 5.** Let $X_n$ be the set of points in the sample $S_n$. Recall that for any $\gamma > 0$, $S_n(\gamma)$ is a $\gamma$-net of $X_n$ labeled by the empirical majority vote in $S_n$. Let $X_n(\gamma_k)$ be the set of points in $S_n(\gamma_k)$. Let $h_{S_n}^*$ be the hypothesis returned by KSU when run with $\delta_n$ that satisfies Property 3 for $Q$, and recall that $S_n^* = S_n(\gamma_n^*)$, where $\gamma_n^*$ is the scale selected by KSU for $S_n$.

We show that for any $\epsilon, \delta \in (0,1)$ there exists an $n_0$ such that for any $n > n_0$,

$$\mathbb{P}[\text{err}(h_{S_n}^*) \leq \epsilon] \geq 1 - \delta.$$ 

This implies $\lim_{n \to \infty} \mathbb{P}[\text{err}(h_{S_n}^*) \leq \epsilon] = 1$, as required for weak consistency.

For each $k$, let $X_k$ be some $\gamma_k$-net over $X$. From Theorem 14, we have $\lim_{k \to \infty} \text{err}(h_{X_k}) = 0$. Let $k$ such that for all $\gamma_k$-nets $X$ of $X$, $\text{err}(h_X) \leq \epsilon$. Such a $k$ exists, since there is a finite number of possible impure cells in a $\gamma_k$-net over $X$, and the partitioning of pure cells does not affect the error of $h_X$.

Let $A_k, P_z$ be defined as in Lemma 16. From Example 1 it can be verified that $|A_k| < \infty, \mu(A_k) > 0$ and $\min z \in A_k \mu(P_z) > 0$. Thus there exists an integer $n'$ such that for any $n > n'$, with a probability at least $1 - \delta$. The condition Eq. (33) in Lemma 16 holds for $B = X_n$. Thus in this case, $X_n(\gamma_k)$ is a $g_k$-net of $X$. Therefore all the impure Voronoi cells $V$ induced by $X_n(\gamma_k)$, are of one of the types listed in Lemma 15.

Denote this set of Voronoi cells by $V_k$.

Let $\beta = \min_{y \in V_k} [\mu(V \cap C) - \mu(V \setminus C)]$. From part (B) of Lemma 13 we have that for sufficiently large $k, \beta > 0$. Assume w.l.o.g that the selected $k$ satisfies this.

Invoking Property 3 of $Q$, let $n$ be large enough such that

$$\sup_{\alpha \in [0,1], m \leq M_k} Q(n, \alpha, m, \delta_n) - \alpha \leq \epsilon, \quad (34)$$

where $M_k < \infty$ is the maximal size of a $\gamma_k$-net on $X$ (such an $M_k$ exists by Lemma 17) and such that with a probability at least $1 - \delta$, the following conditions all hold:

1. $X_n$ satisfies Eq. (33) with $X_n = B$.
2. The empirical majority vote of $S_n$ in each $V \in V_k$ is equal to the true majority vote in $V$.

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3. The following holds:

\[
\sum_{V \in \mathcal{V}_k} \left| \frac{|V \cap C \cap X_n|}{|X_n|} - \mu(V \cap C) \right| + \left| \frac{|V \setminus C \cap X_n|}{|X_n|} - \mu(V \setminus C) \right| \leq \epsilon.
\]

4. \(\text{err}(h_{S_n'}) \leq Q(n, \hat{\text{err}}(h_{S_n}, S_n), 2|S_n'|, \delta_n).\)

Conditions 2, 3 can be satisfied for large enough \(n\) because \(\beta > 0\) and \(\mathcal{V}_k\) is finite. Condition 4 can be satisfied because of Property 1 of \(Q\), by setting \(n\) such that \(\delta_n \ll \delta\).

Assume that the conditions above hold. Then by condition 1 all impure Voronoi cells in \(X_n(\gamma_k)\) are in \(\mathcal{V}_k\). For pure cells, clearly both the true and the empirical majority votes are zero. For cells in \(\mathcal{V}_k\), by condition 2 the true majority vote and the empirical majority vote are equal. Therefore, \(h_{S_n(\gamma_k)} = h_{X_n(\gamma_k)}\).

Moreover, from condition 3 \(\hat{\text{err}}(h_{S_n(\gamma_k)}, S_n) \leq \text{err}(h_{S_n(\gamma_k)}) + \epsilon\). Therefore,

\[
\hat{\text{err}}(h_{S_n(\gamma_k)}, S_n) \leq \text{err}(h_{X_n(\gamma_k)}) + \epsilon. \tag{35}
\]

Now, KSU selects \(\gamma_n^* = \arg\min_\gamma Q(n, \hat{\text{err}}(h_{S_n(\gamma)}, S_n), 2|S_n(\gamma)|, \delta_n)\), and sets \(S_n' = S_n(\gamma_n^*)\). Thus we have

\[
\text{err}(h_{S_n'}) \leq Q(n, \hat{\text{err}}(h_{S_n(\gamma_k)}, S_n), 2|S_n(\gamma_n^*)|, \delta_n)
\leq Q(n, \hat{\text{err}}(h_{S_n(\gamma_k)}, S_n), 2|S_n(\gamma_k)|, \delta_n)
\leq \hat{\text{err}}(h_{S_n(\gamma_k)}, S_n) + \epsilon.
\]

The first inequality follows from condition 4. In the last inequality we used Eq. (34). Combining this with Eq. (35) gives that with a probability at least \(1 - \delta\),

\[
\text{err}(h_{S_n'}) \leq \text{err}(h_{X_n(\gamma_k)}) + 2\epsilon.
\]

Moreover, since Eq. (33) holds, \(X_n(\gamma_k)\) is a \(\gamma_k\)-net of \(\lambda\). Therefore \(\text{err}(h_{X_n(\gamma_k)}) \leq \epsilon\). It follows that with a probability at least \(1 - \delta\), for all large enough \(n\),

\[
\mathbb{P}[	ext{err}(h_{S_n'}) \leq 3\epsilon] \geq 1 - \delta.
\]

Substituting \(3\epsilon\) by \(\epsilon\), this proves weak consistency of KSU on Example 1.

\[\square\]

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