Adaptive FE–BE Coupling for Strongly Nonlinear Transmission Problems with Friction II

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Abstract

This article discusses the well-posedness and error analysis of the coupling of finite and boundary elements for transmission or contact problems in nonlinear elasticity. It concerns “pseudoplastic”, $p$–Laplacian-type Hencky materials with an unbounded stress–strain relation, as they arise in the modelling of ice sheets, non-Newtonian fluids or porous media. For $1 < p < 2$ the bilinear form of the boundary element method fails to be continuous in natural function spaces associated to the nonlinear operator. We propose a functional analytic framework for the numerical analysis and obtain a priori and a posteriori error estimates for Galerkin approximations to the resulting boundary/domain variational inequality. The a posteriori estimate complements recent estimates obtained for mixed finite element formulations of friction problems in linear elasticity.

1 Introduction

Let $n = 2$ or $3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We consider transmission and frictional contact problems between a nonlinear, uniformly $W^{1,p}(\Omega)$–monotone operator in $\Omega$ and the homogeneous Lamé equation in the exterior domain. Adaptive finite element / boundary element procedures provide an efficient and extensively investigated tool for the numerical solution when the nonlinear operator is uniformly elliptic [12]. Their analysis, however, does not apply to the above “pseudoplastic” material laws arising in the modelling of ice sheets, non-Newtonian fluids or porous media [1,7], because for $p < 2$ the bilinear form of the boundary element method fails to be continuous on natural function spaces related to the nonlinear operator. This article provides a functional analytic framework to study the wellposedness and an error analysis of FE / BE coupling procedures in this situation.

Formulation of Problem: We consider the following contact problem for $(u, u_c) \in (W^{1,p}(\Omega))^n \times (W^{1,2}_{loc}(\Omega_c))^n$, where $p \in (1, \infty)$ and $\partial \Omega = \Gamma_s \cup \Gamma_t$ is

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We have denoted the strains by well as Carreau-type laws that the data belong to the following spaces: We assume \( \Gamma \)

and a radiation condition \( u(x) = o(1) \), \( \text{grad} \ u(x) = \mathcal{O}(|x|^{-1}) \) resp. \( u(x) = \mathcal{O}(|x|^{-2}) \) is satisfied for \( n = 2, 3 \) as \( |x| \to \infty \). On \( \Gamma_s \) contact conditions corresponding to Tresca friction are imposed. If \( \nu \) denotes the unit outer normal to \( \partial \Omega \), the conditions are given in terms of the normal and tangential components of \( u \), \( u_n = \nu \cdot u \) and \( u_t = u - u_n \nu \), and of the stress, \( \sigma_n(u) = -\nu A'(\varepsilon(u))\nu \) and \( \sigma_t(u) = -A'(\varepsilon(u))\nu - \sigma_n(u)\nu \):

\[
\begin{align*}
\sigma_n(u) &\leq 0, \quad u_{0,n} + u_{c,n} - u_n \leq 0, \quad \sigma_n(u)(u_{0,n} + u_{c,n} - u_n) = 0, \\
|\sigma_t(u)| &\leq \mathcal{F}, \quad \sigma_t(u)(u_{0,t} + u_{c,t} - u_t) + \mathcal{F}|u_{0,t} + u_{c,t} - u_t| = 0.
\end{align*}
\]

We have denoted the strains by \( \varepsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \) and the natural conormal derivative \( 2\mu \partial_{\nu} + \lambda \text{div} + \mu \nu \times \text{curl} \) at the boundary by \( T^\ast \). The exterior problem is strongly elliptic provided \( \mu > 0, \lambda > -\mu \). The function \( A': L^p(\Omega) \otimes \mathbb{R}^{n \times n} \to L^p(\Omega) \otimes \mathbb{R}^{n \times n} \) is assumed to be a bounded, continuous and uniformly monotone operator, so that in particular for \( p \in (1, 2) \):

\[
\begin{align*}
\langle A'(x) - A'(y), x - y \rangle &\gtrsim (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2}\|x - y\|_{L^p(\Omega)}^2, \\
\langle A'(x) - A'(y), z \rangle &\lesssim \|x - y\|_{L^p(\Omega)}^{p-1}\|z\|_{L^p(\Omega)}.
\end{align*}
\]

When \( p \in [2, \infty) \), we require

\[
\begin{align*}
\langle A'(x) - A'(y), x - y \rangle &\gtrsim \|x - y\|_{L^p(\Omega)}^p, \\
\langle A'(x) - A'(y), z \rangle &\lesssim (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2}\|x - y\|_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.
\end{align*}
\]

We assume \( \Gamma_t \neq \emptyset \), the compatibility condition \( \int_{\Omega} f + \langle t_0, 1 \rangle = 0 \) for \( n = 2 \) and that the data belong to the following spaces:

\[
f \in (L^p(\Omega))^n, \quad u_0 \in (W^{\frac{1}{2}, 2}(\partial\Omega))^n, \quad t_0 \in (W^{-\frac{1}{2}, 2}(\partial\Omega))^n, \quad 0 \leq \mathcal{F} \in L^\infty(\Gamma_s).
\]

In Theorem 3.2 we will show that Problem \( \text{(I)} \) admits a unique weak solution \( (u_1, u_2) \in W^{1,p}(\Omega)^n \times W^{-1,2}(\mathcal{O}^c)^n \). Examples include, in particular, \( p \)-Laplacian materials with \( A'(x) = |x|^{p-2}x \) as well as Carreau-type laws \( A'(x) = (|x|^{1-\delta}(1 + |x|^2)^\delta)^{\frac{p-2}{2}}x \) with \( \delta \in [0, 1] \).

For the symmetric coupling of finite and boundary elements, the Poincaré–Steklo operator \( S \) of the Lamé equation on \( \Omega^c \) is used to reduce Problem \( \text{(I)} \) to a variational inequality in the Banach space

\[
X_p = \{(u, v) \in (W^{1,p}(\Omega))^n \times (W^{-1,2}(\mathcal{O}^c))^n : u|_{\partial\Omega} + v \in W^{\frac{1}{2}, 2}(\partial\Omega)^n \}.
\]
where \( r = \min\{p, 2\} \).

**Main Results:** This article complements the analysis of [6], which concerned a scalar \( p \)-Laplacian-type problem with frictional contact in the simpler case of “dilatant” material laws with \( 2 \leq p < \infty \). In [6] numerical approximations of the variational inequality could be studied in \( \bar{X}^p = (W^{1,p}(\Omega))^n \times (\bar{W}^\frac{1}{2},2(\Gamma_s))^n \), as \( \bar{X}^p = X^p \) for \( p \geq 2 \), with an emphasis on the transmission problem. Numerical examples confirmed the theoretical estimates.

Here we show that the space \( X^p \) provides the proper setting for the numerical analysis for all \( p \in (1, \infty) \), and we focus on the more intricate wellposedness and a sharp error analysis of the friction problem when \( p \in (1, 2) \): While the a posteriori estimate in [6] was aimed at the pure transmission problem, Theorem 6.1 gives a sharp a posteriori estimate for the error of Galerkin approximations to the variational inequality. It complements recent results for mixed finite element formulations of friction problems [9, 10, 11] and is new even in the elliptic case.

The existence of a unique \( X^p \)-solution is shown in Theorem 3.2, and Theorem 4.1 gives an a priori estimate for Galerkin approximations. Finally, in Section 6 we sketch the analysis when the discretization of the Poincaré–Steklov operator is included. As an example of the added difficulty when \( p \in (1, 2) \), the variational inequality no longer splits into an equality on \( \Omega \) and an inequality on \( \partial \Omega \), unless the artificial regularity assumption \( u|_{\partial \Omega} \in W^\frac{1}{2},2(\partial \Omega)^2 \) is imposed.

The results in this article are stated for \( p \in (1, \infty) \), but we refer to [6] for most of the arguments when \( p \geq 2 \). Conversely, an appendix adapts the new a posteriori estimate for the frictional term to the setting considered there.

The mathematical differences between \( p < 2 \) and \( p \geq 2 \) are not artificial. They reflect the different physical behavior: While pseudoplastic materials like ice or molasses (\( p < 2 \)) get stiffer and stiffer under a smaller stress, possibly infinitely so, the opposite happens in the dilatant case like a thick emulsion of sand and water (\( p > 2 \)).

### 2 Preliminaries

Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega \). Set \( p' = \frac{p}{p-1} \) whenever \( p \in (1, \infty) \). We will also denote \( r = \min\{p, 2\} \) and \( q = \max\{p, 2\} \).

Before analyzing a variational formulation of (16), we recall some properties of \( L^p \)-Sobolev spaces on \( \Omega \):

**Remark 2.1.**

- a) \( (W^{s,p}(\partial \Omega))' = W^{-s,p'}(\partial \Omega) \) and \( W^{s,2}(\partial \Omega) = H^s(\partial \Omega) \).
- b) \( W^{s,2}(\Omega) \hookrightarrow W^{s,p}(\Omega) \) and \( \|u\|_{W^{s,p}(\Omega)} \leq |\Omega|^{1-\frac{s}{2}}\|u\|_{W^{s,2}(\Omega)} \) for \( 1 < p \leq 2 \).
- c) If \( \partial \Omega \) is smooth, pseudodifferential operators of order \( m \) with \( \mathbb{C}^{k\times k} \)-valued symbol in the Hörmander class \( S^m_{0,0}(\partial \Omega) \) map \( (W^{s,p}(\partial \Omega))^{k} \) continuously to \( (W^{s-m,p}(\partial \Omega))^{k} \). For Lipschitz \( \partial \Omega \), at least the first–order Steklov–Poincaré
operator $S$ of the Lamé operator on $\Omega^c$ is continuous between $(W^{1,2}(\partial \Omega))^n$ and $(W^{-\frac{1}{2},2}(\partial \Omega))^n$.  

d) Points a) to c) imply that the quadratic form $\langle Su, u \rangle$ associated to $S$ is well-defined on $(W^{1,2,p}(\partial \Omega))^n$ if $p \geq 2$. $S$ being elliptic, the form is unbounded for $p < 2$ even if $\partial \Omega$ is smooth.

The fundamental solution for the Lamé operator in $\mathbb{R}^2$,  
\[ G(x,y) = \frac{x + 3\mu}{4\pi \mu(\lambda + 2\mu)} \left\{ \log(|x-y|^{-1}) \text{Id} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x-y)(x-y)^T}{|x-y|^2} \right\} , \]

resp. $\mathbb{R}^3$ 
\[ G(x,y) = \frac{x + 3\mu}{4\pi \mu(\lambda + 2\mu)} \left\{ \frac{1}{|x-y|} \text{Id} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x-y)(x-y)^T}{|x-y|^3} \right\} , \]

allows to define layer potentials on $\partial \Omega$ associated to the exterior problem in the usual way:

\[ \mathcal{V}\phi(x) = \int_{\partial \Omega} \phi(x') G(x,x') \, dx', \]
\[ \mathcal{K}\phi(x) = \int_{\partial \Omega} \phi(x') \partial_{x'} G(x,x') \, dx', \]
\[ \mathcal{K}'\phi(x) = \int_{\partial \Omega} \phi(x') \partial_{x} G(x,x') \, dx', \]
\[ \mathcal{W}\phi(x) = \partial_{x} \int_{\partial \Omega} \phi(x') \partial_{x'} G(x,x') \, dx'. \]

They extend from $C^\infty(\partial \Omega)^n$ to a bounded map \( \begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}' \end{pmatrix} \) on the Sobolev space $W^{\frac{1}{2},2}(\partial \Omega)^n \times W^{-\frac{1}{2},2}(\partial \Omega)^n$. If (for $n = 2$) the capacity of $\partial \Omega$ is less than 1, which can always be achieved by scaling, $\mathcal{V}$ and $\mathcal{W}$ considered as operators on $W^{-\frac{1}{2},2}(\partial \Omega)^n$ are selfadjoint, $\mathcal{V}$ is positive and $\mathcal{W}$ non-negative. The Steklov-Poincaré operator for the exterior Lamé problem is given as  
\[ S = \mathcal{W} + (1-\mathcal{K}')\mathcal{V}^{-1}(1-\mathcal{K}) : W^{\frac{1}{2},2}(\partial \Omega)^n \subset W^{-\frac{1}{2},2}(\partial \Omega)^n \rightarrow W^{-\frac{1}{2},2}(\partial \Omega)^n \]

and defines a positive and selfadjoint operator with the main property  
\[ T^* u_2|_{\partial \Omega} = -S(u_2|_{\partial \Omega}) \]

for solutions $u_2$ of the Lamé equation on $\Omega^c$ satisfying the decay condition at $\infty$. $S$ therefore gives rise to a coercive and symmetric bilinear form $\langle Su, u \rangle$ on $W^{\frac{1}{2},2}(\partial \Omega)^n$.

Existence of a unique solution to (1) will be shown using Korn’s inequality and coercivity:

**Proposition 2.2.** ([1], Proposition 2) Assume $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\Gamma \subset \partial \Omega$ has positive $(n-1)$–dimensional measure. Then there is a $C > 0$ such that  
\[ \|u\|_{1,p} \leq C(\|\varepsilon(u)\|_p + \|u|_\Gamma\|_{L^1(\Gamma)}) \quad \text{for all } u \in (W^{1,2}(\Omega))^n. \]
3 Analysis of the boundary integral formulation

For $r = \min\{p, 2\}$, we consider the space

$$X^p = \{(u, v) \in (W^{1,p}(\Omega))^n \times (\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s))^n : u|_{\partial\Omega} + v \in W^{\frac{1}{2},2}(\partial\Omega)^n\}$$

equipped with the norm

$$\|u, v\|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$ 

Note that $X^p = (W^{1,p}(\Omega))^n \times (\tilde{W}^{\frac{1}{2},2}(\Gamma_s))^n$ when $p \geq 2$, so that we recover a vector–valued variant of the Banach spaces considered in [6].

**Lemma 3.1.** $(X^p, \| \cdot \|_{X^p})$ is a Banach space, and

$$|u, v|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}$$
defines an equivalent norm on $X^p$.

**Proof.** It is readily verified that $\| \cdot \|_{X^p}$ defines a norm on $X^p$. To show completeness, let $(u_j, v_j) \in X$ be a Cauchy sequence. Then $(u_j, v_j)$ converges to a limit $(u, v)$ in the Banach space $W^{1,p}(\Omega)^n \times \tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)^n$. Also $u_j|_{\partial\Omega} + v_j$ converges to a limit $w$ in $W^{\frac{1}{2},2}(\partial\Omega)^n$. However, the continuity of the trace operator assures that $u_j|_{\partial\Omega} \to u|_{\partial\Omega}$ in $W^{1-\frac{1}{p},p}(\partial\Omega)^n$. Therefore in $W^{1-\frac{1}{p},p}(\partial\Omega)^n$, hence also in $W^{1-\frac{1}{p},p}(\partial\Omega)^n$, $u_j|_{\partial\Omega} + v_j$ converges both to $u|_{\partial\Omega} + v$ and to $w$. This means that $u|_{\partial\Omega} + v = w \in W^{\frac{1}{2},2}(\partial\Omega)^n$, or $(u, v) \in X^p$.

To see the equivalence of norms, note that $|u, v|_{X^p} \leq \|u, v\|_{X^p}$. On the other hand, the continuous inclusion of $W^{\frac{1}{2},2}(\partial\Omega)$ into $W^{1-\frac{1}{p},p}(\partial\Omega)$, of $W^{1-\frac{1}{p},p}(\partial\Omega)$ into $W^{1-\frac{1}{p},p}(\partial\Omega)$, and the continuity of the trace operator from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega)$ imply

$$\|u, v\|_{X^p} \leq \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}$$
$$+ \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}$$
$$+ \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} = |u, v|_{X^p}.$$ 

The assertion follows. \[\square\]

We consider a variational formulation of the contact problem in terms of the functional

$$J(u, v) = \langle A(\varepsilon(u)), \varepsilon(u) \rangle + \frac{1}{2} \langle S(u|_{\partial\Omega} + v, u|_{\partial\Omega} + v \rangle - L(u, v)$$
Lemma 3.3. The operator associated to $X^p$. Here $A$ is derived from $A'$ by an explicit formula, $v = u_0 + u_c - u$,

$$j(v) = \int_{\Gamma_a} F |v_t|,$$

and

$$L(u, v) = \int_{\Omega} f u + \langle t_0 + Su_0, u|_{\partial\Omega} + v \rangle.$$

This paper investigates the numerical approximation of the following nonsmooth variational problem over the closed convex subset

$$K = \{(u, v) : v_n \leq 0, \langle S1, u|_{\partial\Omega} + v - u_0 \rangle = 0\}$$

of $X^p$:

Find $(\hat{u}, \hat{v}) \in K$ such that

$$J(\hat{u}, \hat{v}) + j(\hat{v}) = \min_{(u, v) \in K} J(u, v) + j(v).$$

(4)

Note that $j$ is Lipschitz, but not differentiable.

As in [6] one observes that Problem (4) is equivalent to the contact problem (3). The existence of a unique solution to the latter is therefore a consequence of the following theorem.

**Theorem 3.2.** There exists a unique minimizer $(\hat{u}, \hat{v}) \in K$ of $J + j$ over $K$.

The crucial ingredient in the proof is a monotonicity estimate:

**Lemma 3.3.** The operator associated to $J$ is strongly monotone on $X^p$.

Let $r = \min\{p, 2\}$, $q = \max\{p, 2\}$ and $C > 0$. Then for every $(u_1, v_1), (u_2, v_2) \in X^p$ with $\|u_1, v_1\|_{X^p(\Omega)}, \|u_2, v_2\|_{X^p(\Omega)} < C$, there holds

$$\|u_2 - u_1, v_2 - v_1\|_{X^p} \leq C \langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1) \rangle + \langle S((u_2 - u_1)|_{\partial\Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial\Omega} + v_2 - v_1 \rangle \leq C \|u_2 - u_1, v_2 - v_1\|_{X^p}^r.$$

**Proof.** The upper bound is a consequence of the estimates (2), (3) for the nonlinear operator and the boundedness of $S$ from $W^{1,2}(\partial\Omega)^n$ to $W^{-1,2}(\partial\Omega)^n$. For $p \geq 2$, we refer to [6], Lemma 3, for the proof of an analogous lower estimate. When $p < 2$ the monotony of $A'$ resp. coercivity of $S$ imply for any $\delta \in (0, 1)$

$$\langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1) \rangle$$

$$+ \langle S((u_2 - u_1)|_{\partial\Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial\Omega} + v_2 - v_1 \rangle \geq \|\varepsilon(u_2 - u_1)||_{L^p(\Omega)}^p + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|^2_{W^{1,2}(\partial\Omega)}$$

$$\geq \|\varepsilon(u_2 - u_1)||_{L^p(\Omega)}^p + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|^2_{W^{1,2}(\Gamma_1)} + \|u_2 - u_1\|^2_{W^{1,2}(\Gamma)}$$

$$+ \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|^2_{W^{1,2}(\partial\Omega)}$$

$$\geq \|\varepsilon(u_2 - u_1)||_{L^p(\Omega)}^p + \delta \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|^2_{W^{1,\frac{p}{2}}(\Gamma_1)} + \|u_2 - u_1\|^2_{W^{1,\frac{p}{2}}(\Gamma_1)}$$

$$+ \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|^2_{W^{1,\frac{p}{2}}(\partial\Omega)}.$$
In the last inequality we use the continuous inclusion \( W^{\frac{1}{2},2}(\Gamma_s) \subset W^{1-\frac{1}{p},p}(\Gamma_s) \).

Korn’s inequality, Proposition 2.2 implies

\[
\|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}^2 + \|u_2 - u_1\|_{W^{\frac{1}{2},2}(\Gamma_s)}^2 \geq \|u_2 - u_1\|_{W^{1,p}(\Omega)}^2.
\] (6)

Further note from the triangle inequality, the convexity of \( x \mapsto x^2 \) as well as the continuity of the trace map from \( W^{1,p}(\Omega) \) to \( W^{1-\frac{1}{p},p}(\Gamma_s) \):

\[
\|v_2 - v_1\|_{W^{1-\frac{1}{p},p}(\Gamma_s)}^2 \leq \left( \|u_2 - u_1\|_{\Gamma_s} + 2\|v_2 - v_1\|_{W^{1-\frac{1}{p},p}(\Gamma_s)} + 2\|u_2 - u_1\|_{W^{1-\frac{1}{p},p}(\Gamma_s)} \right)^2
\]
\[
\leq 2\|u_2 - u_1\|_{\Gamma_s} + 2\|v_2 - v_1\|_{W^{1-\frac{1}{p},p}(\Gamma_s)} + 2\|u_2 - u_1\|_{W^{1,p}(\Omega)}^2.
\] (7)

The asserted estimate follows from (5), (6) and (7), after choosing \( \delta > 0 \) sufficiently small.

Strong monotony on all of \( X^p \) is shown similarly, but for large \( \|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)} \) the exponent 2 in the lower bound has to be replaced by \( p \).

Proof (of Theorem 3.2). By Lemma 3.3 the operator associated to \( J \) is bounded and strongly monotone. Existence and uniqueness for the perturbation \( J + j \) of \( J \) follow e.g. by applying the perturbation result [13], Proposition 32.36.

4 Discretization and a priori error analysis

Let \( \{\mathcal{T}_h\}_{h \in I} \) a regular triangulation of \( \Omega \) into disjoint open regular triangles \( (n = 2) \) resp. tetrahedra \( (n = 3) \) \( T \), so that \( \overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \). Each element has at most one edge resp. face on \( \partial \Omega \), and the closures of any two of them share at most a single vertex, edge or face. Let \( \rho_T \) denote the diameter of \( T \in \mathcal{T}_h \) and \( \rho_T \) the diameter of the largest inscribed ball. We assume that \( 1 \leq \max_{T \in \mathcal{T}_h} \frac{\rho_T}{\rho_T} \leq R \) independent of \( h \) and that \( h = \max_{T \in \mathcal{T}_h} h_T \). \( \mathcal{E}_h \) is going to be the set of all edges of the triangles / faces of the tetrahedra in \( \mathcal{T}_h \). Associated to \( \mathcal{T}_h \) is the space \( W^{1,p}_h(\Omega) \subset W^{1,p}(\Omega) \) of functions whose restrictions to any \( T \in \mathcal{T}_h \) are linear.

The boundary \( \partial \Omega \) is triangulated by \( \{l \in \mathcal{E}_h : l \subset \partial \Omega\} \). For \( r = \min\{p,2\} \), \( W^{1-\frac{1}{r},r}_h(\partial \Omega) \) denotes the corresponding space of continuous, piecewise linear functions, and \( \widetilde{W}^{1-\frac{1}{r},r}_h(\Gamma_s) \) the subspace of those supported on \( \Gamma_s \). Finally, \( W^{\frac{1}{2},2}_h(\partial \Omega) \subset W^{\frac{1}{2},2}(\partial \Omega) \) is the space of piecewise constant functions, and \( X^p_h = W^{1,p}_h(\Omega) \times \widetilde{W}^{\frac{1}{2},2}_h(\Gamma_s) \subset X^p \).

We denote by \( i_h : W^{1,p}_h(\Omega) \hookrightarrow W^{1,p}(\Omega) \), \( j_h : \widetilde{W}^{\frac{1}{2},2}_h(\Gamma_s) \hookrightarrow \widetilde{W}^{\frac{1}{2},2}(\Gamma_s) \) and \( k_h : W^{\frac{1}{2},2}_h(\partial \Omega) \hookrightarrow W^{\frac{1}{2},2}(\partial \Omega) \) the canonical inclusion maps.

The discrete problem involves the discretized functional

\[
J_h(u_h,v_h) = (A(\varepsilon(u_h)),\varepsilon(u_h)) + \frac{1}{2}(S(\rho_h\varepsilon(u_h) + v_h), u_h\varepsilon(\rho_h + v_h) - L_h(u_h,v_h)
\]
on $X^p_h$. Here

$$S_h = \frac{1}{2}(W + (I - K')k_h(k_h V k_h)^{-1}kt_h(I - K))$$

and

$$L_h(u_h, v_h) = \int_{\Omega} f u_h + \langle t_0 + S_h u_0, u_h|_{\partial \Omega} + v_h \rangle .$$

There exists $h_0 > 0$ such that the approximate Steklov–Poincaré operator $S_h$ is coercive uniformly in $h < h_0$, i.e. $\langle S_h u_h, u_h \rangle \geq \alpha_S \|u_h\|^2_{W^{1,2}(\partial \Omega)}$ with $\alpha_S$ independent of $h$. Therefore, as in the previous section the discrete minimization problem

$$J(\hat{u}_h, \hat{v}_h) + j(\hat{v}_h) = \min_{(u_h, v_h) \in K \cap X^p_h} J(u_h, v_h) + j(v_h) . \quad (8)$$

is associated to a perturbation of a strongly monotone operator on $X^p_h$ and admits a unique minimizer.

Our Galerkin method for the numerical approximation relies on an equivalent reformulation of the continuous and discretized minimization problems (4), (8) as variational inequalities:

Find $(\hat{u}, \hat{v}) \in K$ such that

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u - \hat{u}) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (u - \hat{u})|_{\partial \Omega} + v - \hat{v} \rangle$$

$$+ j(v) - j(\hat{v}) \geq L(u - \hat{u}, v - \hat{v}) \quad (9)$$

for all $(u, v) \in K$.

The discretized variant reads as follows:

Find $(\hat{u}_h, \hat{v}_h) \in K \cap X^p_h$ such that

$$\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S_h(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial \Omega} + v_h - \hat{v}_h \rangle$$

$$+ j(v_h) - j(\hat{v}_h) \geq L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \quad (10)$$

for all $(u_h, v_h) \in K \cap X^p_h$.

**Theorem 4.1.** a) The following a priori estimate holds with $q = \max\{p, 2\}$:

$$\|u - \hat{u}_h, \hat{v} - \hat{v}_h\|_{X^p} \leq \inf_{(u_h, v_h) \in K \cap X^p_h} \{ \|\varepsilon(\hat{u}_h)\|_{L^p(\Omega)} + \|(u_h - \hat{u}_h)|_{\partial \Omega} + \hat{v} - v_h\|_{W^{1,2}(\partial \Omega)}$$

$$+ \|\hat{v} - v_h\|_{L^1(\Gamma_\delta)} \} + \text{dist}_{W^{-\frac{1}{2},2}(\partial \Omega)}(V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2},2}(\partial \Omega))^2 .$$

b) If $\hat{v} \in \overline{W^{\frac{1}{2},2}(\Gamma_s)}$, e.g. for $p \geq 2$ or $\Gamma_s = \emptyset$, the estimate can be improved to

$$\|u - \hat{u}_h, \hat{v} - \hat{v}_h\|_{X^p} \leq \inf_{(u_h, v_h) \in K \cap X^p_h} \{ \|\varepsilon(\hat{u}_h)\|_{L^p(\Omega)} + \|(u_h - \hat{u}_h)|_{\partial \Omega} + \hat{v} - v_h\|_{W^{1,2}(\partial \Omega)}$$

$$+ \|\hat{v} - v_h\|_{L^1(\Gamma_\delta)} \} + \text{dist}_{W^{-\frac{1}{2},2}(\partial \Omega)}(V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2},2}(\partial \Omega))^2 .$$

Here $\beta = \frac{2}{p - p}$ for $p < 2$ resp. $\beta = p' = \frac{p}{p - 1}$ for $p \geq 2$. 

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Proof. Adding the continuous and discrete variational inequalities, we see that

\[ 0 \leq \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}) - \varepsilon(\tilde{u}) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial \Omega} + \hat{v}_h - \hat{v} \rangle + j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v}) \]

Hence,

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}) - \varepsilon(\tilde{u}) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h), (\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \rangle \\
+ \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\tilde{u}_h) \rangle + \langle S(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (\hat{u}_h - \tilde{u}_h)|_{\partial \Omega} + \hat{v}_h - \tilde{v}_h \rangle \\
+ j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v}) \\
+ \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\tilde{u}_h) \rangle + \langle S(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (\hat{u}_h - \tilde{u}_h)|_{\partial \Omega} + v_h - \tilde{v}_h \rangle \\
+ j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, v_h - \hat{v}) \\
\]

Let \( p < 2 \). To bound \( \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}) - \varepsilon(\hat{u}_h) \rangle \), we use the estimate \( \| \| \) and Young’s inequality for any \( \delta > 0 \):

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\hat{u}) \rangle \lesssim \| \varepsilon(\hat{u}_h) \|_{L^p(\Omega)}^{p-1} \| \varepsilon(\hat{u}) \|_{L^p(\Omega)} \\
\lesssim \delta^n \| \varepsilon(\hat{u}_h) \|_{L^p(\Omega)}^2 + \delta^{-n} \| \varepsilon(\hat{u}) \|_{L^p(\Omega)}^2 .
\]

On the other hand, for \( p \geq 2 \) the upper bound \( \| \| \) yields

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\hat{u}) \rangle \lesssim \| \varepsilon(\hat{u}_h) \|_{L^p(\Omega)} \| \varepsilon(\hat{u}) \|_{L^p(\Omega)} \\
\lesssim \delta^p \| \varepsilon(\hat{u}_h) \|_{L^p(\Omega)}^{p} + \delta^{-p} \| \varepsilon(\hat{u}) \|_{L^p(\Omega)}^{p} .
\]

As for the second term, we use the boundedness of \( S \) from \( W^{\frac{1}{2}, 2}(\partial \Omega)^n \) to \( W^{\frac{1}{2} - \frac{1}{2}, 2}(\partial \Omega)^n \) to estimate

\[
\langle S((\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h), (\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \rangle \\
\lesssim \| (\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \|_{W^{\frac{1}{2}, 2}(\partial \Omega)}^2 + \delta^{-1} \| (\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \|_{W^{\frac{1}{2}, 2}(\partial \Omega)}^2 .
\]

Without further assumptions on \( \hat{v} \), we estimate the second line using Cauchy–Schwarz by a multiple of

\[
\| \varepsilon(\hat{u}_h - \hat{u}) \|_{L^p(\Omega)} + \| (u_h - \hat{u})|_{\partial \Omega} + v_h - \hat{v} \|_{W^{\frac{1}{2}, 2}(\partial \Omega)} .
\]
For part b), where $\hat{v} \in \overline{W}^{\frac{1}{2}, 2}(\Gamma_s)$, one may use the variational inequality for an improved estimate: Substituting $(u, v) = (u_h, \hat{v})$ and $(u, v) = (2\hat{u} - u_h, \hat{v})$ into the variational inequality on $X^p$, we obtain

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h) - \varepsilon(\hat{u}) \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u})|_{\partial\Omega} \rangle = L(u_h - \hat{u}, 0).$$

With this, the second line reduces to $\langle S(\hat{u}|_{\partial\Omega} + \hat{v}), v_h - \hat{v} \rangle + L(0, \hat{v} - v_h)$, i.e. to $-\langle t_0 - S(\hat{u}|_{\partial\Omega} + \hat{v} - u_0), v_h - \hat{v} \rangle = -\langle A'(\varepsilon(\hat{u})) \cdot \nu, v_h - \hat{v} \rangle \leq \|F\|_{L^\infty(\Gamma_s)} \|v_{n,h} - \hat{v}_n\|_{L^1(\Gamma_s)}$.

For the third line,

$$j(v_h) - j(\hat{v}) = \int_{\Gamma_s} F(|v_h| - |\hat{v}|) \leq \int_{\Gamma_s} F(|v_h - \hat{v}|) \leq \|F\|_{L^\infty(\Gamma_s)} \|v_{\nu,h} - \hat{v}_{\nu}|_{L^1(\Gamma_s)}.$$ 

Finally, the last line simplifies as follows:

$$-\langle L_h - L |(u_h - \hat{u}_h, v_h - \hat{v}_h) + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle 
\lesssim \delta^{-1} \| (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) \|_{W^{-\frac{1}{2}, 2}(\partial\Omega)} + \delta \|(u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \|_{W^{-\frac{1}{2}, 2}(\partial\Omega)} 
\leq \delta^{-1} \| (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) \|_{W^{-\frac{1}{2}, 2}(\partial\Omega)} 
+ \delta \|(u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \|_{W^{-\frac{1}{2}, 2}(\partial\Omega)}.$$ 

The term involving $S_h - S$ is known to be bounded by $\text{dist}_{W^{-\frac{1}{2}, 2}(\partial\Omega)} (V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2}, 2}(\partial\Omega))^2$.

To sum up, for general $\hat{v}$ we obtain for $\alpha = \frac{p}{p-1}, \beta = \frac{2}{3-p}$ (p < 2) resp. $\alpha = p, \beta = p'$ (p ≥ 2)

$$\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(\hat{u}_h) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle 
\lesssim \delta^\alpha \| \varepsilon(\hat{u} - \hat{u}_h) \|_{L^p(\Omega)} + \delta \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} + \delta^{-\beta} \| \varepsilon(\hat{u} - \hat{u}_h) \|_{L^p(\Omega)} 
+ \| \varepsilon(u_h - \hat{u}_h) \|_{L^p(\Omega)} + \| (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} 
+ \delta^{-1} \| (u_h - \hat{u}_h)|_{\partial\Omega} + \hat{v} - v_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} + \| v_h - \hat{v} \|_{L^p(\Gamma_s)} + \| (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} 
+ \delta^{-1} \text{dist}_{W^{\frac{1}{2}, 2}(\partial\Omega)} (V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2}, 2}(\partial\Omega))^2.$$ 

The lowest exponents dominate.

When $\hat{v} \in \overline{W}^{\frac{1}{2}, 2}(\Gamma_s)^n$, the estimates yield:

$$\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(\hat{u}_h) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle 
\lesssim \delta^\alpha \| \varepsilon(\hat{u} - \hat{u}_h) \|_{L^p(\Omega)} + \delta \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} + \delta^{-\beta} \| \varepsilon(\hat{u} - \hat{u}_h) \|_{L^p(\Omega)} 
+ \delta^{-1} \| (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} + \| v_h - \hat{v} \|_{L^p(\Gamma_s)} + \| (u_h - \hat{u})|_{\partial\Omega} + v_h - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} 
+ \delta^{-1} \text{dist}_{W^{\frac{1}{2}, 2}(\partial\Omega)} (V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2}, 2}(\partial\Omega))^2.$$
Note that as in Lemma 3.3 the monotony of $A'$ and coercivity of $S$ allow to bound the left hand side from below by

$$\|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)}^p + \|\varepsilon(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W_{1,2}^2(\partial\Omega)}.$$  

Choosing $\delta > 0$ sufficiently small, the claimed estimates follow.  

**Remark 4.2.** a) Theorem 4.1 proves convergence of the proposed FE–BE coupling procedure for quasi–uniform grid refinements. However, generic weak solutions to the contact problem (11) only belong to $X^p$ and not to any higher-order Sobolev space. Therefore the convergence can be arbitrarily slow as the grid size $h$ tends to 0.

b) Like for the $p$–Laplacian operators in [6], under additional assumptions on $A'$ slightly sharper estimates can be obtained with respect to certain quasinorms on $X^p$.

5 An a posteriori estimate

If we consider the variational inequality (10) for $v_h = \hat{v}_h$ and with $u_h \mapsto u_h$ resp. $u_h \mapsto 2\hat{u}_h - u_h$, Problem (11) splits into an interior equation and an inequality on the boundary: For all $(u_h, v_h) \in K \cap X^p_h$:

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), u_h|_{\partial\Omega} \rangle = \int_{\Omega} f u_h + \langle t_0 + S_h u_0, u_h \rangle = L_h(u_h, 0),$$

$$\langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), v_h - \hat{v}_h \rangle + j(v_h) - j(\hat{v}_h) \geq \langle t_0 + S u_0, v_h - \hat{v}_h \rangle = L_h(0, v_h - \hat{v}_h).$$

(11)

For the continuous inequality, we only get a weaker assertion because $u|_{\partial\Omega} + v$ needs to be in $W_{1,2}^2(\partial\Omega)$. Choosing $u = \hat{u} + \hat{u}_h - u_h$, $v = \hat{v} + \hat{v}_h - v_h$ for any $(u_h, v_h) \in X^p_h$ with $v_h \leq \hat{v} + \hat{v}_h$ transforms (9) into the estimate

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle$$

$$\leq \langle \hat{v} + \hat{v}_h - v_h \rangle - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h).$$

(12)

In combination with the coercivity estimates, we may start to derive an a posteriori estimate:

$$\|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)} + \|\varepsilon(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W_{1,2}^2(\partial\Omega)}$$

$$\lesssim \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h) \rangle + \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle$$

$$+ \langle S((\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h \rangle$$

$$+ \langle S((\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle$$

We consider the second and fourth term on the right hand side,

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle$$

$$+ \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}),(u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle.$$
Applying the equality in (11) to

\[
\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{|{\partial \Omega}} + v_h - \hat{v}_h
\]

and inequality (12) to

\[
\langle A'\varepsilon(\hat{u}), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (u_h - \hat{u}_h) \rangle_{|{\partial \Omega}} + v_h - \hat{v}_h
\]

we estimate their sum by

\[
-L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h)
\]

\[
-\langle S_h(u_h|_{\partial \Omega} + \hat{v}_h), v_h - \hat{v}_h \rangle + \langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{|{\partial \Omega}} + v_h - \hat{v}_h.
\]

For

\[
\langle A'\varepsilon(\hat{u}), \varepsilon(u_h - u_h) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (u_h - u_h) \rangle_{|{\partial \Omega}} + \hat{v} - \hat{v}_h
\]

we use the variational inequality (9) with \((u, v) = (u_h, v_h)\) to conclude

\[
\|\varepsilon(u - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\varepsilon(u - \hat{u}_h)\|_{|{\partial \Omega}} + \hat{v} - \hat{v}_h
\]

\[
\lesssim L(u - u_h, \hat{v} - v_h) + j(v_h) - \langle A'\varepsilon(\hat{u}_h), \varepsilon(u - u_h) \rangle
\]

\[
-\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u - u_h) \rangle - \langle S_h(u_h|_{\partial \Omega} + \hat{v}_h), (u_h - u_h) \rangle_{|{\partial \Omega}} + \hat{v} - v_h
\]

\[
-\langle S_h(u_h|_{\partial \Omega} + \hat{v}_h), v_h - \hat{v}_h \rangle - \langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{|{\partial \Omega}} + \langle t_0 + S u_0, v_h - \hat{v}_h \rangle
\]

\[
-\langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{|{\partial \Omega}} + \hat{v} - \hat{v}_h.
\]

The first term is estimated as usual for \(u_h = \hat{u}_h + \Pi_h(\hat{u} - \hat{u}_h)\) using the Hölder inequality and the properties of a Clement interpolation operator \(\Pi_h\) (see e.g. [2]):

\[
\int_{\Omega} f(\hat{u} - u_h) \lesssim \|\hat{u} - u_h\|_{W^{1,p}(\Omega)} \left( \sum_{T \subset \Omega} h_T^{p'} \|f\|_{L^{p'}(T)}^{p'} \right)^{1/p'}
\]

\((p' = \frac{p}{p-1})\)

Similarly, integrating by parts we obtain

\[\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u - u_h) \rangle = \sum_{E \subset \Omega} \int_E [A'(\varepsilon(\hat{u}_h))\nu](u - u_h) |_{\partial \Omega} + \langle A'(\varepsilon(\hat{u}_h))\nu, (u - u_h) |_{\partial \Omega} \rangle_{|{\partial \Omega}} \]

with

\[
\sum_{E \subset \Omega} \int_E [A'(\varepsilon(\hat{u}_h))\nu](\hat{u} - u_h) |_{\partial \Omega} \lesssim \|\hat{u} - u_h\|_{W^{1,p}(\Omega)} \left( \sum_{E \subset \Omega} h_E \|A'(\varepsilon(\hat{u}_h))\nu\|_{L^{p'}(E)}^{p'} \right)^{1/p'}.
\]
It remains to consider the boundary contributions. To do so, recall the strong formulation of the contact conditions in terms of $\sigma_n(u)$ and $\sigma_t(u)$ on $\Gamma_s$,

$$\sigma_n(u) \leq 0, \quad v_n \leq 0, \quad \sigma_n(u)v_n = 0,$$

$$|\sigma_t(u)| \leq F, \quad \sigma_t(u)v_t + F|v_t| = 0.$$

Then, substituting $v_h = \hat{v}_h$, we obtain

$$j(\hat{v} + \hat{v}_h - v_h) = j(\hat{v}) = \int_{\Gamma_s} F|\hat{v}_t| = -\langle \sigma_t(\hat{u}), \hat{v}_t \rangle = -\langle \sigma(\hat{u}), \hat{v} \rangle.$$

Also,

$$\hat{v}(\hat{v}_h) - \langle A'(\varepsilon(\hat{u}_h))\nu, \hat{v}_h \rangle \leq \int_{\Gamma_s} (F|\hat{v}_h,t| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}) + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h} +) .$$

Together, the terms

$$j(\hat{v} + \hat{v}_h - v_h) + j(v_h) - 2j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|\partial\Omega \rangle \partial\Omega$$

$$= -j(\hat{v}) + j(\hat{v}_h) - \langle A'(\varepsilon(\hat{u}_h))\nu, \hat{v}_h \rangle \partial\Omega - \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|\partial\Omega + \hat{v} - \hat{v}_h - \hat{v} \rangle \partial\Omega$$

are hence dominated by

$$\langle \sigma(\hat{u}), \hat{v} \rangle + \int_{\Gamma_s} (F|\hat{v}_h,t| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}) + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h}) +$$

$$- \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|\partial\Omega + \hat{v} - \hat{v}_h - \hat{v} \rangle \partial\Omega$$

$$= \int_{\Gamma_s} (F|\hat{v}_h,t| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}) + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h}) +$$

$$- \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|\partial\Omega + \hat{v} - \hat{v}_h \rangle \partial\Omega + \langle \sigma(\hat{u}) - \sigma(\hat{u}_h), \hat{v} \rangle .$$

We split the $\sigma$–term into tangential and normal parts

$$\langle \sigma(\hat{u}) - \sigma(\hat{u}_h), \hat{v} \rangle = \langle \sigma_n(\hat{u}) - \sigma_n(\hat{u}_h), \hat{v}_n \rangle + \langle \sigma_t(\hat{u}) - \sigma_t(\hat{u}_h), \hat{v}_t \rangle ,$$

and estimate the normal part as follows ($r' = \frac{r}{r - 1}$):

$$\langle \sigma_n(\hat{u}) - \sigma_n(\hat{u}_h), \hat{v}_n \rangle \leq -\langle \sigma_n(\hat{u}_h), \hat{v}_n \rangle \leq \|\sigma_n(\hat{u}_h)\|_{W^{1,\frac{1}{4},r'}(\Gamma_s)} .$$

For the tangential contribution, involving the Tresca friction, we find it convenient to write $\sigma_t(\hat{u}) = -\zeta F$ with $|\zeta| \leq 1$ and $|v_t| = \zeta v_t$. Then

$$\langle \sigma_t(\hat{u}) - \sigma_t(\hat{u}_h), \hat{v}_t \rangle = -\zeta \langle F, \hat{v}_t \rangle - \langle \sigma_t(\hat{u}_h), \hat{v}_t \rangle = -\langle F, \hat{v}_t \rangle - \langle \sigma_t(\hat{u}_h), \hat{v}_t \rangle$$

$$\leq \langle \sigma_t(\hat{u}_h) - \sigma_t(\hat{u}_h), \hat{v}_t \rangle \leq \|\sigma_t(\hat{u}_h) - \sigma_t(\hat{u}_h)\|_{H^{-1,\frac{1}{4},r'}(\Gamma_s)} .$$
We conclude

\[
\| \varepsilon (\hat{u} - \hat{u}_h) \|^q_{L^p(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \hat{v}_h \|^q_{W^{1/2,2}(\partial \Omega)} \\
\lesssim \| \varepsilon (\hat{u} - \hat{u}_h) \|^q_{L^p(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - \hat{v}_h \|^q_{W^{1/2,2}(\partial \Omega)} \\
\lesssim \| \hat{u} - \hat{u}_h \|_{W^{1,p}(\Omega)} \left( \sum_{T \subset \Omega} h_T^p \| f \|_{L^{p'}(T)}^p \right)^{1/p'} \\
+ \| \hat{u} - \hat{u}_h \|_{W^{1,p}(\Omega)} \left( \sum_{E \subset \Omega} h_E \| [A'(\varepsilon (\hat{u}_h)) \nu] \|_{L^{p'}(E)}^p \right)^{1/p'} \\
+ \int_{\Gamma_s} \{ F | \hat{v}_{h,t} | + \sigma_t (\hat{u}_h) \hat{v}_{h,n} \} + \int_{\Gamma_s} (\sigma_n (\hat{u}_h) \hat{v}_{n,h}) + \\
+ \| t_0 + S_h (u_0 - \hat{u}_h)|_{\partial \Omega} + \hat{v}_h ) - A'(\varepsilon (\hat{u}_h)) \nu \|^q_{W^{1/2,2'}(\partial \Omega)} \\
+ \| \sigma_n (\hat{u}_h) + \| \hat{v}_{h,t} | + \sigma_t (\hat{u}_h) \hat{v}_{n,h} \} + \int_{\Gamma_s} (\sigma_n (\hat{u}_h) \hat{v}_{n,h}) + \\
+ \| S_h - S)(\hat{u}_h)|_{\partial \Omega} + \hat{v}_h + \hat{v}_h \|^2_{W^{1/2,2}(\Gamma_s)}.
\]

Summing up:

**Theorem 5.1.** Let \( r = \min \{ p, 2 \} \) and \( q = \max \{ p, 2 \} \). The following a posteriori estimate holds:

\[
\| \hat{u} - \hat{u}_h \|_{\tilde{X}^p} \\
\lesssim \left( \sum_{T \subset \Omega} h_T^p \| f \|_{L^{p'}(T)}^p \right)^{1/p'} \\
+ \int_{\Gamma_s} \{ F | \hat{v}_{h,t} | + \sigma_t (\hat{u}_h) \hat{v}_{h,n} \} + \int_{\Gamma_s} (\sigma_n (\hat{u}_h) \hat{v}_{n,h}) + \\
+ \| S_h - S)(\hat{u}_h)|_{\partial \Omega} + \hat{v}_h + \hat{v}_h \|^2_{W^{1/2,2}(\Gamma_s)}.
\]

**Remark 5.2.** Adapting the interpolation operator \( \Pi_h \) to include \( \hat{v} - \hat{v}_h \) on \( \Gamma_s \), it might be possible to improve the term \( \| t_0 + S_h (u_0 - \hat{u}_h)|_{\partial \Omega} + \hat{v}_h ) - A'(\varepsilon (\hat{u}_h)) \nu \|^q_{W^{1/2,2'}(\Gamma_s)} \) to \( \| t_0 + S_h (u_0 - \hat{u}_h)|_{\partial \Omega} + \hat{v}_h ) - A'(\varepsilon (\hat{u}_h)) \nu \|^2_{W^{1/2,2}(\Gamma_s)} \).

### 6 Formulation in terms of layer potentials

In practice, one would like to estimate the numerical error without a priori information about \( S - S_h \). This is achieved by formulating the problem directly in terms of the layer potentials \( \mathcal{V}, \mathcal{W}, \mathcal{K}, \mathcal{K}' \) rather than \( S = \mathcal{W} + (1 - \mathcal{K}') \mathcal{V}^{-1} (1 - \mathcal{K}) \). The arguments are a notationally more involved variant of those in Section 5 and we only outline them.
We consider the space

\[ Y^p = X^p \times W^{-\frac{1}{2},2}(\partial\Omega)^n, \]

equipped with the norm

\[ \|u, v, \phi\|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,\frac{1}{p}}(\Gamma_s)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}. \]

From Lemma 3.1 we conclude that \((Y^p, \| \cdot \|_{Y^p})\) is a Banach space and

\[ \|u, v, \phi\|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)} \]

an equivalent norm on \(Y^p\). We consider the discretization in finite dimensional subspaces \(Y_h^p = X_h^p \times W_h^{-\frac{1}{2},2}(\partial\Omega)^n\) of \(Y^p\).

In order to show coercivity, we use a theoretical stabilization as in [5]; Let \(r_1, \ldots, r_D\) a basis of the space of rigid body motions, and consider their orthogonal projections \(\xi_1, \ldots, \xi_D\) onto \(L^2(\partial\Omega)\). The arguments in [5], Lemma 4 and Proposition 5, show that \(|u, v, \phi|_{Y^p}\) is equivalent to the norm

\[ |u, v, \phi|_{Y^p,s} = \|\varepsilon(u)\|_{L^p(\Omega)}^2 + \langle W(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v \rangle + \langle \phi, \nabla \phi \rangle + \sum_{j=1}^D |\langle \xi_j, (1 - K)(u|_{\partial\Omega} + v) + \nabla \phi \rangle|^2. \]

On \(Y^p\), we have the following equivalent formulation of the contact problem:

\[ B(\tilde{u}, \tilde{v}, \tilde{\phi}; u - \tilde{u}, v - \tilde{v}, \phi - \tilde{\phi}) + j(v) - j(\tilde{v}) \geq \Lambda(u - \tilde{u}, v - \tilde{v}, \phi - \tilde{\phi}) \]

with

\[ B(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) = \langle A'(\varepsilon(u)), \varepsilon(\tilde{u}) \rangle + \langle W(u|_{\partial\Omega} + v), (K - 1)\phi, \tilde{u}|_{\partial\Omega} + \tilde{v} \rangle + \langle \phi, \nabla \phi + (1 - K)(u|_{\partial\Omega} + v) \rangle, \]

\[ \Lambda(u, v, \phi) = \langle t_0 + \nabla u_0, u|_{\partial\Omega} + v \rangle + \int_\Omega f + \langle \phi, (1 - K)u_0 \rangle. \]

The discretized problem is obtained by restricting to \(Y_h^p\), and we denote its solution by \((\tilde{u}_h, \tilde{v}_h, \tilde{\phi}_h)\). We also consider a stabilized problem that for all \((u_h, v_h, \phi_h) \in K' \cap Y_h^p\)

\[ \tilde{B}(\tilde{u}_s, \tilde{v}_s, \tilde{\phi}_s; u_h - \tilde{u}_s, v_h - \tilde{v}_s, \phi_h - \tilde{\phi}_s) + j(v_h) - j(\tilde{v}_s) \geq \tilde{\Lambda}(u_h - \tilde{u}_s, v_h - \tilde{v}_s, \phi_h - \tilde{\phi}_s), \]

where

\[ \tilde{B}(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) = B(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) + \sum_{j=1}^D \langle \xi_j, \nabla \phi + (1 - K)(u|_{\partial\Omega} + v) \rangle \langle \xi_j, \nabla \phi + (1 - K)(\tilde{u}|_{\partial\Omega} + \tilde{v}) \rangle \]

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respectively

\[ \tilde{\Lambda}(u, v, \phi) = \Lambda(u, v, \phi) + \sum_{j=1}^{D} \langle \xi_j, (1 - K)u_0 \rangle \langle \xi_j, \nabla \phi + (1 - K)(u|_{\partial \Omega} + v) \rangle. \]

Because the variational inequality (16) is an equality in \( \phi \), as in [3], Proposition 3, the solution to the stabilized and nonstabilized problems coincide, \((\hat{u}_h, \hat{v}_h, \hat{\phi}_h) = (\check{u}_{s.h}, \check{v}_{s.h}, \check{\phi}_{s.h})\). However, the stabilized variational inequality is coercive in the stabilized norm (13):

\[
\| \varepsilon(\hat{u} - \check{u}_h) \|^2_{L^p(\Omega)} + \langle \mathcal{W}((\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h), (\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h \rangle \\
+ \langle \mathcal{W}(\hat{\phi} - \check{\phi}_h), \hat{\phi} - \check{\phi}_h \rangle + \sum_{j=1}^{D} \langle \xi_j, (1 - K)((\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h) + \mathcal{W}(\hat{\phi} - \check{\phi}_h)) \rangle^2
\leq \langle A'(\varepsilon(\hat{u})), A'(\varepsilon(\check{u}_h)), \varepsilon(\hat{u} - \check{u}_h) \rangle \\
+ \langle \mathcal{W}((\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h) + (K' - 1)(\hat{\phi} - \check{\phi}_h), (\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h \rangle \\
+ ((1 - K)((\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h) + \mathcal{W}(\hat{\phi} - \check{\phi}_h), \hat{\phi} - \check{\phi}_h) \\
+ \sum_{j=1}^{D} \langle \xi_j, (1 - K)((\hat{u} - \check{u}_h)|_{\partial \Omega} + \hat{v} - \check{v}_h) + \mathcal{W}(\hat{\phi} - \check{\phi}_h) \rangle^2
\]

Proceeding as in Section 5, we obtain:

**Theorem 6.1.** Let \( r = \min\{p, 2\} \) and \( q = \max\{p, 2\} \). The following a posteriori estimate holds:

\[
\| \hat{u} - \check{u}_h, \hat{v} - \check{v}_h, \hat{\phi} - \check{\phi}_h \|^q_{\tilde{L}^p(\Omega)} \\
\leq \left( \sum_{T \subseteq \Omega} h_T^{q'} \| f \|^p_{L^p(T)} \right)^{q'/p'} + \left( \sum_{E \subseteq \Omega} h_E \| [A'(\varepsilon(\check{u}_h)) \nu] \|^p_{L^p'(E)} \right)^{q'/p'} \\
+ \| t_0 - \mathcal{W}(\check{u}_h|_{\partial \Omega} + \check{v}_h - u_0) - (K' - 1)\hat{\phi}_h - A'(\varepsilon(\hat{u}_h)) \nu \|^q_{W^{1, q'}(\partial \Omega)} \\
+ \| \mathcal{W}(\hat{\phi}_h + (1 - K)(\check{u}_h|_{\partial \Omega} + \check{v}_h - u_0)) \|^2_{W^{1, 2}(\partial \Omega)} \\
+ \int_{\Gamma_s} \{ \mathcal{F} \hat{u}_h, \sigma_t(\hat{u}_h) \hat{v}_h, \nu \} + (\sigma_t(\hat{u}_h) \check{v}_{n.h}) + \int_{\Gamma_s} (\sigma_t(\hat{u}_h) \check{v}_{n.h}) + \| \sigma_t(\hat{u}_h) \|^2_{W^{1, q'}(\Gamma_s)} \\
+ \| \sigma_t(\check{u}_h) \|^2_{W^{1, q'}(\Gamma_s)} + \| \mathcal{F} + \sigma_t(\hat{u}_h) - \check{u}_h \|^2_{W^{1, q'}(\Gamma_s)}.
\]

**7 Appendix – An improved error estimate for the scalar \( p \)-Laplacian**

Consider the following scalar transmission problem for \( p \geq 2 \):

\[
- \text{div} A'(\nabla u) = f \quad \text{in} \ \Omega, \\
- \Delta u_c = 0 \quad \text{in} \ \Omega^c, \\
A'(u) \nu - \partial_{\nu} u_c = t_0 \quad \text{on} \ \partial \Omega, \\
u - u_c = u_0 \quad \text{on} \ \Gamma_t, \quad (16)
\]
On $\Gamma_s$, contact conditions corresponding to Tresca friction are imposed in terms of the stress $\sigma(u) = -A'(\nabla u)\nu$,

$$|\sigma(u)| \leq g, \quad \sigma(u)(u_0 + u_c - u) + g|u_0 + u_c - u| = 0.$$ 

A radiation condition holds for $|x| \to \infty$:

$$u(x) = a + o(1),$$

and for simplicity of notation we assume $a = 0$. Here $A': L^p(\Omega)^2 \to L^{p'}(\Omega)^2$ is assumed to be a bounded, continuous and uniformly monotone operator, so that in particular

$$\langle A'(x) - A'(y), x - y \rangle \geq \|x - y\|_{L^p(\Omega)}^p,$$

$$\langle A'(x) - A'(y), z \rangle \leq (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2})\|x - y\|_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.$$ 

The data belong to the following spaces:

$$f \in L^{p'}(\Omega), \quad u_0 \in W^{\frac{1}{2},2}(\partial \Omega), \quad t_0 \in W^{-\frac{1}{2},2}(\partial \Omega), \quad 0 \leq g \in L^\infty(\Gamma_s), \quad a \in \mathbb{R}.$$ 

In addition, $\int_{\Omega} f + t_0 = 0$. We are looking for weak solutions $(u, u_c) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c)$.

The above contact problem is equivalent to the following variational inequality in the space

$$X^p = W^{1,p}(\Omega) \times \tilde{W}^{\frac{1}{2},2}(\Gamma_s), \quad \tilde{W}^{\frac{1}{2},2}(\Gamma_s) = \{ u \in W^{\frac{1}{2},2}(\partial \Omega) : \text{supp} u \subset \Gamma_s \} :$$

Find $(\hat{u}, \hat{v}) \in X^p$ such that for all $(u, v) \in X^p$,

$$\langle A'(\nabla \hat{u}), \nabla u \rangle + \langle S(\hat{u} + \hat{v}), u|_{\partial \Omega} \rangle = \int_{\Omega} fu + \langle t_0 + Su_0, u|_{\partial \Omega} \rangle = L(u, 0),$$

$$\langle S(\hat{u} + \hat{v}), v - \hat{v} \rangle + j(v) - j(\hat{v}) \geq \langle t_0 + Su_0, v - \hat{v} \rangle = L(0, v - \hat{v}).$$

We obtain a variant of Galerkin orthogonality in the interior:

$$\langle A'(\nabla \hat{u}) - A'(\nabla \tilde{u}_h), \nabla u_h \rangle + \langle S((\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h), u_h|_{\partial \Omega} \rangle$$

$$+ \langle (S - S_h)(\tilde{u}_h|_{\partial \Omega} + \tilde{v}_h - u_0), u_h|_{\partial \Omega} \rangle = 0.$$ 

As in [6], Theorem 2, the monotony of $A'$ and coercivity of $S$ imply

$$\|\hat{u} - \tilde{u}_h, \hat{v} - \tilde{v}_h\|_p \lesssim |\hat{u} - \tilde{u}_h|_{1,s,p}^2 + \|(\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h\|_{W^{1/2,2}(\partial \Omega)}^2$$

$$\lesssim \langle A'(\nabla \hat{u}) - A'(\nabla \tilde{u}_h), \nabla (\hat{u} - \tilde{u}_h) \rangle$$

$$+ \langle S((\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h), (\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \rangle.$$ 

Using the variational equality in $\Omega$, the right hand side becomes

$$\langle A'(\nabla \hat{u}) - A'(\nabla \tilde{u}_h), \nabla (\hat{u} - \tilde{u}_h) \rangle$$

$$+ \langle S((\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h), (\hat{u} - \tilde{u}_h)|_{\partial \Omega} + \hat{v} - \tilde{v}_h \rangle$$

$$= L(\hat{u} - \tilde{u}_h, 0) + \langle S(\tilde{u}_h|_{\partial \Omega} + \tilde{v}_h), \hat{v} - \tilde{v}_h \rangle$$

$$- \langle A'(\nabla \tilde{u}_h), \nabla (\hat{u} - \tilde{u}_h) \rangle - \langle S(\tilde{u}_h|_{\partial \Omega} + \tilde{v}_h), (\hat{u} - \tilde{u}_h)|_{\partial \Omega} \rangle$$

$$- \langle S(\tilde{u}_h|_{\partial \Omega} + \tilde{v}_h), \hat{v} - \tilde{v}_h \rangle.$$
Let \( u_h \in W^{1,p}_h(\Omega) \) arbitrary and \((e, \tilde{e}) = (\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h)\), \( e_h = \hat{u} - u_h \), whence \( e - e_h = \hat{u} - u_h \). With the help of Galerkin orthogonality in \( \Omega \), the right hand side turns into

\[
L(e - e_h, 0) = \langle S(\bar{u}|_{\partial \Omega} + \hat{\nu}), \tilde{e} \rangle - \langle A' (\nabla \hat{u}_h), \nabla (e - e_h) \rangle - \langle S(\bar{u}_h|_{\partial \Omega} + \hat{\nu}_h), e - e_h \rangle \\
+ \langle S(\bar{u}_h|_{\partial \Omega} + \hat{\nu}_h), \tilde{e} \rangle + \langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0), e_h \rangle .
\]

Recall that \( L(e - e_h, 0) = \int_{\Omega} f(e - e_h) + \langle t_0 + S u_0, (e - e_h)|_{\partial \Omega} \rangle \). In [6] it was shown for a suitable interpolant \( e_h = \pi e \) and any \( \varepsilon > 0 \),

\[
\int_{\Omega} f(e - e_h) \lesssim \varepsilon |e|_{(1, \bar{u}, p)}^2 + C(\varepsilon) \eta_f^2 + \varepsilon \eta_{gr}^2
\]

where

\[
\eta_{gr}^2 = \sum_{K \in T_h} \int_K G_{p, \delta} (\nabla \hat{u}_h, \nabla \hat{u}_h - G_h \hat{u}_h) ,
\]

\[
\eta_f^2 = \sum_{K \in T_h} \int_K G_{p', 1} (|\nabla \hat{u}_h|^{p-1}, h_K (f - f_K)) ,
\]

involve the gradient recovery resp. the approximation error of \( f \). Integrating by parts in the term \(-\langle A'(\nabla \hat{u}_h), \nabla (e - e_h) \rangle\) yields two terms,

\[
- \sum_{l \subseteq \partial \Omega} \int_{l} \nu \cdot A'(\nabla \hat{u}_h) (e - e_h)
\]

and

\[
- \sum_{l \subseteq \partial \Omega} \int_{l} A_l (e - e_h) \lesssim \eta_{gr}^2 + \varepsilon |e|_{(1, \bar{u}, p)}^2 + \eta_{gr}^2 .
\]

Altogether we conclude

\[
\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|_{X_p}^2 \lesssim |\hat{u} - \hat{u}_h|_{(1, \bar{u}, p)}^2 + |(\hat{u} - \hat{u}_h)|_{(1, \bar{u}, p)} + \hat{v} - \hat{v}_h|_{W^{1,2}_{\Omega}(\partial \Omega)} \\
\lesssim 2\varepsilon |e|_{(1, \bar{u}, p)}^2 + C(\varepsilon) \eta_f^2 + (1 + 2\varepsilon) \eta_{gr}^2 \\
+ \langle \nu \cdot A'(\nabla \hat{u}_h) + S(\hat{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0) - t_0, \pi e - e \rangle \\
+ \langle S((\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{\nu} - \hat{v}_h), \hat{v} - \hat{v}_h \rangle \\
+ \langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0), \pi e \rangle .
\]

We write the second–to–last term as \( \langle \sigma(\hat{u}) - \sigma(\hat{u}_h), \hat{v} - \hat{v}_h \rangle \) and the friction
conditions as \( \sigma(\dot{u}) = -\zeta g, |\ddot{v}| = \zeta \dot{v} \) for some \( |\zeta| \leq 1 \). Then

\[
\langle \sigma(\dot{u}) - \sigma(\dot{u}_h), \dot{v} - \dot{v}_h \rangle \\
= -\langle \zeta g, \dot{v} \rangle - \langle \sigma(\dot{u}_h), \dot{v} \rangle + \langle \zeta g, \dot{v}_h \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle \\
= -\langle g, |\dot{v}| \rangle - \langle \sigma(\dot{u}_h), \dot{v} \rangle + \langle \zeta g, \dot{v}_h \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle \\
\leq \langle \langle |\sigma(\dot{u}_h)| - g \rangle_+, |\dot{v}| \rangle + \langle \zeta g, \dot{v}_h \rangle - \langle \langle |\sigma(\dot{u}_h)|, |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle \\
\leq \langle \langle |\sigma(\dot{u}_h)| - g \rangle_+, |\dot{v} - \dot{v}_h| + |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle \\
\lesssim \| \langle |\sigma(\dot{u}_h)| - g \rangle_+ \|_{\dot{W}^{-\frac{1}{2},2}(\Gamma_s)} \| \dot{v} - \dot{v}_h \|_{\dot{W}^{\frac{1}{2},2}(\Gamma_s)} \\
+ \langle \langle |\sigma(\dot{u}_h)| - g \rangle_+ + g - \langle \sigma(\dot{u}_h), |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle \\
= \| \langle |\sigma(\dot{u}_h)| - g \rangle_+ \|_{\dot{W}^{-\frac{1}{2},2}(\Gamma_s)} \| \dot{v} - \dot{v}_h \|_{\dot{W}^{\frac{1}{2},2}(\Gamma_s)} \\
+ \int_{\Gamma_s} \langle |\sigma(\dot{u}_h)| - g \rangle_+ \| \dot{v}_h \| + 2 \int_{\Gamma_s} \langle \sigma(\dot{u}_h) \dot{v}_h \rangle_+ .
\]

This proves the following a posteriori estimate:

**Theorem 7.1.** Let \( f \in L^p(\Omega) \) and denote by \( (e, \ddot{v}) \) the error between the Galerkin solution \((\dot{u}_h, \dot{v}_h) \in X_h^p \) and the true solution \((\dot{u}, \dot{v}) \in X^p \). Then

\[
\| \ddot{u} - \dot{u}_h, \dot{v} - \dot{v}_h \|_{X^p} \lesssim \eta_{gr}^2 + \eta_f^2 + \eta_S^2 + \eta_\delta^2 + \eta_g^2,
\]

where

\[
\eta_{gr}^2 = \sum_{K \subseteq \bar{T}_h} \int_K G_{p,\delta}(\nabla \dot{u}_h, \nabla \ddot{u}_h - G_h \dot{u}_h),
\]

\[
\eta_f^2 = \sum_{K \subseteq \bar{T}_h} \int_K G_{p',1}(\nabla \dot{u}_h)^{p-1}, h_K(f - f_K),
\]

\[
\eta_S^2 = \text{dist}_{W^{-\frac{1}{2},2}(\partial \Omega)} \left( V^{-1}(1 - K)(\dot{u} + \dot{v} - u_0), W^{-\frac{1}{2},2}_h(\partial \Omega) \right)^2
\]

\[
\eta_\delta^2 = \| \nu \cdot A'(\nabla \dot{u}_h) + S(\dot{u}_h \nabla \dot{v}_h + \dot{v}_h - u_0) - t_0 \|_{W^{-1, p_0'}(\Omega)}^p
\]

\[
\eta_g^2 = \| \langle |\sigma(\dot{u}_h)| - g \rangle_+ \|_{\dot{W}^{-\frac{1}{2},2}(\Gamma_s)} \| \dot{v}_h \| + \int_{\Gamma_s} \langle |\sigma(\dot{u}_h)| - g \rangle_+ \| \dot{v}_h \| + \int_{\Gamma_s} \langle \sigma(\dot{u}_h) \dot{v}_h \rangle_+ .
\]

**Remark 7.2.** As \( p \geq 2 \), we are here able to split both the discretized and the continuous variational inequality into an equation in \( \Omega \) and an inequality on \( \partial \Omega \). This explains the slightly different form of the frictional terms compared to Theorems 5.1 and 6.1.

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