Complexity of the Einstein-Born-Infeld-Massive Black holes

S. H. Hendi$^{1,2}$ and B. Bahrami-Asl$^{1,\dagger}$

$^1$ Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
$^2$ Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha, Iran

Motivated by interesting correspondence between computational complexity in a CFT and the action evaluated on a WDW patch in the bulk, we study the complexity of the Einstein-massive black holes in the presence of BI nonlinear electrodynamic. The upper limit of Llyod bound according to the WDW patch is investigated and it is proved that Llyod bound is held.

I. INTRODUCTION

Thermodynamical systems, obeying the laws of classical or quantum mechanics, are characterized by a set of intensive and extensive quantities. Among these quantities, the main two characteristics of thermodynamical systems are entropy and temperature. It is known that in the equilibrium state the former has the greatest value and the later is a constant. However, one can use the first law of thermodynamics to calculate the absolute temperature for non-equilibrium situation. In order to understand the conception of entropy, one has to focus on the concept of information which is related to the measure of uncertainty in physical systems. Considering an identical concept for the entropy and information is a confusion and in order to prevent such misunderstanding, the complexity statement is used to discuss characteristics of physical thermodynamical systems. The value of the complexity of a system can change due to some influences of environment or strictly speaking, transmission of information between the environment and system.

According to the Bekenstein suggested \cite{1}, black holes have entropy, and therefore, laws of thermodynamics can be valid for them. Regarding black hole entropy, it is natural to think about the existence of information inside the event horizon as well as information paradox after Hawking radiation. In addition, it is interesting to look for an approach for distinguishing black holes with different information. The black hole information is related to its entropy and consequently complicatedness or complexity. As a result, measuring the complexity of black holes may help us to resolve the information paradox and quantum nature of black hole as well.

The formal definition of computational complexity in the context of quantum mechanics is related to the minimum number of quantum gates in quantum circuit which is required to prepare the boundary state from a simple state \cite{2}. In other words, complexity is the minimal difficulty of taking the system from a simple reference state to a particular state of interest. In the context of black hole physics, the complexity is the boundary state complicatedness of a geometric property of the black hole interior. It may suggest that the structure of black hole interior is a geometric representation of a quantum circuit or is matched with the geometry of tensor network, and therefore, one can use the holography principle to estimate the complexity \cite{3–6}.

In order to find a suitable relation for the complexity based on AdS/CFT correspondence, we consider a situation of sending a signal through Einstein-Rosen bridge (ERB). We need an appropriate duality which connects the quantity of theoretical information with a geometric concept for calculating the complicatedness. It is known that the computational complexity of the boundary state is proportional to the volume of ERB or in general case is proportional to the volume of black hole interior \cite{2}

$$C \sim \frac{V}{Gl},$$

where $G$ is the Newton’s constant and $l$ is a length scale which is related to the AdS radius for large black holes and for small ones it is proportional to the Schwarzschild radius \cite{3, 4}. Multiplying Eq. (1) by $l$, one can propose a new perspective of the complexity

$$C \sim \frac{W}{Gl^2},$$

where $W = lV$ has the units of space-time volume and $l^2$ is proportional to the cosmological constant of the AdS

* email address: hendi@shirazu.ac.ir
† email address: banafsheh.bahrami@shirazu.ac.ir
Equation (2) inspires new conjecture that connects the complexity with the gravitational dynamics \[ C \sim \frac{A}{\pi \hbar} \] (3)

where \( A \) is the action which is calculated by integrating of the bulk Lagrangian over \( W \) with an appropriate boundary term. Equation (3) indicates complexity-action conjecture which suggests that the complexity is proportional to the action. This equation induces a deep connection between quantum information and gravitational dynamics, and strictly speaking, it explains a connection between tensor networks and geometry which means that the geometry is defined by the smallest tensor network preparing the state [6].

Another interesting aspect of the complexity is its time evolution. There is an interesting conjecture that bounds variation of the complexity which is inspired by Llyod with the following form [8]

\[ \frac{\partial}{\partial t} C (e^{-iHt}|\psi>) \leq \frac{2E_\psi}{\pi \hbar} |\psi>, \] (4)

where \( E_\psi \) is the average energy of state \( \psi \) which is related to the ground state. For charged rotating black holes, Eq. (4) reduces to [8]

\[ \frac{\partial}{\partial t} C \leq \frac{2}{\pi \hbar} \left( (M - \mu Q - \Omega J) - (M - \mu Q - \Omega J)_{ground \ state} \right), \] (5)

where \( M, \mu, Q, \Omega \) and \( J \) are, respectively, mass, chemical potential, conserved charge, angular velocity and angular momentum of the black hole.

For most cases, one may propose that the quantum complexity of boundary state is equal to the classical action of spacetime in the maximally extended black hole defined with respect to two choices of time, on each boundary; the enclosed area is called the Wheeler-De Witt (WDW) patch [4,5].

As we mentioned, there are two complexity conjectures; the complexity-volume duality and the complexity-action duality. The complexity-volume conjecture states that the complexity of black holes is dual to the volume of the black hole interior while the complexity-action conjecture provides a relation between the complexity of black holes and the action of the associated WDW patch. In this paper, we are going to investigate the complexity of black holes in Einstein-massive gravity in the presence of nonlinear electrodynamics. Although Einstein theory is one of the best theory with some correct predictions, there are some mismatches that motivate one to generalize it. As an example we refer the reader to the non-renormalizable properties of general relativity which is arisen from the fact that this theory is consistent with interaction of massless spin-2 gravitons. As a result, it is logical to modifying general relativity to the case of massive gravity with massive spin-2 particles. Massive gravity has some advantages with respect to Einstein theory. Among them, one may refer to explanation of accelerated expansion of the universe without including dark energy and also renormalizable property which helps us to understand the conceptions of quantum gravity [9–17]. Different aspects of massive gravity have been investigated in literature. AdS massive gravity is investigated in [18] while charged massive gravity is studied in [19]. In addition, there are some interesting papers in the context of massive gravity in the presence of nonlinear electrodynamics [20–41]. The main motivation of considering the nonlinear electrodynamics is overcoming on the main problem of the Maxwell theory which is the infinite self-energy of the point-like charges. In this regard, Born and Infeld introduced a nonlinear electrodynamics which is known as Born-Infeld (BI) theory [21]. One of the interesting properties of the BI electrodynamics is that its effective action arises in an open superstring theory and D-brains with nonsingular self-energy of the point-like charges [22,23] (we refer the reader to see [24] for reviewing aspects of BI theory in the context of string theory). Recently many papers have published with the subject of complexity, which are investigated the complexity of black holes in the presence of dilaton field, Maxwell field and nonlinear electrodynamics with Einstein or modified gravity theories [27–31]. It seems that this subject will be one of the hot topics for equipping the classical theories of gravitation with some quantum characteristics.

In this paper, we interested in studying complexity of the AdS black holes in the massive gravity with BI electrodynamics. First, we introduce the suitable action of the Einstein-BI-massive black hole and its metric according to the symmetry of the spacetime. Next, in the context of CA duality, the complexity of holographic state dual to the Einstein-BI-massive black hole in the AdS space is obtained and then growth of the complexity is calculated and discussed.
II. EINSTEIN-BORN-INFELD-MASSIVE GRAVITY

Our starting point is the dRGT action for ghost-free massive gravity with a nonlinear electrodynamics and negative cosmological constant

\[ S = -\frac{1}{16\pi} \int d^4x\sqrt{-g} \left[ R - 2\Lambda + L(f) + m^2 \sum_{i=1}^n c_i U_i(g,f) \right] \]  

in which \( R \) is the scalar curvature of dynamical metric \( g_{\mu\nu} \), \( \Lambda = -\frac{(d-1)(d-2)}{2d^2} \) is the negative cosmological constant and \( L(f) \) describes the Lagrangian of a nonlinear model of electrodynamics which is called BI theory

\[ L(f) = 4b^2 \left( 1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}} \right), \]

where \( b \) is the Born-Infeld parameter and for \( b \rightarrow \infty \) the Born-Infeld theory reduces to Maxwell theory with \( L(f) = -f \), where \( f \) is the Maxwell invariant \( f = F_{\mu\nu}F^{\mu\nu} \) with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) as the Faraday tensor with the gauge potential \( A_\mu \). The last term of Eq. (7) has a potential term role containing no derivatives of the dynamical metric but it depends explicitly on a non-dynamical symmetric reference metric \( f_{\mu\nu} \). In addition, \( c_i \)'s are some constants and \( U_i \)'s denote the symmetric polynomials of the eigenvalues of \( d \times d \) matrix \( \kappa^{\mu\nu}_\nu = \sqrt{g^{\mu\nu}} f_{\alpha\nu} \) which can be written as

\[ U_1 = [\kappa], \]
\[ U_2 = [\kappa]^2 - [\kappa^2], \]
\[ U_3 = [\kappa]^3 - 3[\kappa][\kappa^2] + 2[\kappa^3], \]
\[ U_4 = [\kappa]^4 - 6[\kappa^2][\kappa^2] + 8[\kappa^3][\kappa] + 3[\kappa^2]^2 - 6[\kappa^4], \]
\[ ... \]

It is worth mentioning that \( U_n \)'s have no contribution in the field equations for \( n \geq d \). Since higher order terms of \( U_n \)'s (\( 4 < n < d \)) have no significant effect on the geometrical behavior of the solutions, we restrict the solutions up to \( U_4 \). Variation of the action with respect to the dynamical metric and also gauge potential leads to the following field equations

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} L(f) - \frac{2F_{\rho\lambda}F^{\rho\lambda}}{\sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}} + m^2 \chi_{\mu\nu} = 0, \]  

\[ \partial_\mu \left( \frac{\sqrt{-g} F^{\mu\nu}}{\sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}} \right) = 0, \]

where \( G_{\mu\nu} \) is the Einstein tensor and \( \chi_{\mu\nu} \) is related to the massive term with the following explicit form

\[ \chi_{\mu\nu} = -\frac{c_1}{2}(U_1g_{\mu\nu} - \kappa_{\mu\nu}) - \frac{c_2}{2}(U_2g_{\mu\nu} - 2U_1\kappa_{\mu\nu} + 2\kappa^2_{\mu\nu}) - \frac{c_3}{2}(U_3g_{\mu\nu} - 3U_2\kappa_{\mu\nu} + 6U_1\kappa^2_{\mu\nu} - 6\kappa^3_{\mu\nu}) - \frac{c_4}{2}(U_4g_{\mu\nu} - 4U_3\kappa_{\mu\nu} + 12U_2\kappa^2_{\mu\nu} - 24U_1\kappa^3_{\mu\nu} + 24\kappa^4_{\mu\nu}) + ... \]

Now, we obtain static nonlinearly charged black holes in context of massive gravity with adS asymptotes. For this purpose, we adopt a static metric of \( d \)-dimensional spacetime in the following form

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^idx^j, \quad i, j = 1, 2, 3, ..., d - 2, \]

where \( h_{ij}dx^idx^j \) is a \((d - 2)\)-dimensional line element for the Euclidian space with constant curvature \((d - 2)(d - 3)k\) and volume \( V_{d-2} \). We should note that the constant \( k \) indicates the boundary of \( t = constant \) and \( r = constant \), and it can be negative, zero and positive which indicates hyperbolic, flat and elliptic hypersurface, respectively. In addition, we consider the following ansatz for the non-dynamic metric

\[ f_{\mu\nu} = \text{diag}(0, 0, c^2h_{ij}), \]
where \( c \) is a positive constant. Using the mentioned ansatz for \( f_{\mu\nu} \), one can find that \( U_i \)’s are simplified as

\[
U_i = \left( \frac{c}{r} \right)^i \prod_{j=2}^{d-i} (d-j).
\]  

The gauge potential that supports dynamical metric, from Eq. (10), follows as

\[
h(r) = -\sqrt{\frac{d-2}{d-3}} \frac{q}{r^{d-3}} _2F_1\left( \begin{bmatrix} 1 & d-3 \cr \frac{3(d-7)}{2(d-2)} \end{bmatrix} ; 1 \right) \left( \frac{1}{2}, \frac{d-3}{2(d-2)} \right) \frac{r^3}{\Gamma} \right),
\]

where \( \Gamma = \frac{(d-2)(d-3)q^2}{b_2^2r^2(d-2)^2} \) and \( q \) is integration constant which is related to the electric charge. It is straightforward to show that the nonzero component of electromagnetic field tensor is \( F_{tr} = \frac{\sqrt{(d-2)(d-3)}}{r(d-2)} \). In addition, regarding the nonzero components of the gravitational field equation (Eq. (19)), simultaneously, the metric function is obtained

\[
f(r) = k - \frac{m_0}{r^{d-3}} + \left( \frac{4b^2 - 2\Lambda}{(d-1)(d-2)} \right) r^2 + \frac{4(d-2)q^2}{(d-1)r^{2(d-3)}} _2F_1\left( \begin{bmatrix} 1 & d-3 \cr \frac{3(d-7)}{2(d-2)} \end{bmatrix} ; 1 \right) \left( \frac{1}{2}, \frac{d-3}{2(d-2)} \right) \frac{r^3}{\Gamma} \right) - \frac{4b^2r^2}{(d-1)(d-2)} \sqrt{1+\Gamma} + \frac{m_0}{d-2} \left( \sum_{i=1}^{n} c^i c_i r^{2-i} \prod_{j=2}^{d-i} (d-j) \right),
\]

in which \( m_0 \) is an integration constant which is related to the total mass of the black hole. Calculations confirm that there is a curvature singularity at the origin which is covered by an event horizon, and therefore, the solutions can be interpreted as black holes [34]. It is also notable that by considering different values for the parameters the roots of metric function have different behaviors (see [34] for more details). In order to study the conformal behavior of the solutions, one can use the conformal compactification method to plot conformal (Penrose) diagrams (see Fig. 1 and also [34]).

### III. ACTION OF THE WDW PATCH

According to the CA conjecture, the complexity of a boundary state is proportional to the classical action of a region of spacetime called the WDW patch which is the region in the bulk enclosed between rays. The mentioned patch in the adS-Einstein-Born-Infeld-massive black hole spacetime with considering the case with two horizons (outer one is extreme) is evolved in time from \( t_1 \) to \( t_2 \).

We focus on the rate change of action as a function of time rather than its absolute value. The different parts that may contribute to the action growth are bulk region \( V_1 \) and \( V_2 \), and null-null surface joints \( A, B, C \) and \( D \) (for more details see [42])

\[
\partial S = \int_{V_1} dr dt d^{d-2}x_1 \sqrt{-g} \mathcal{L} - \int_{V_2} dr dt d^{d-2}x_2 \sqrt{-g} \mathcal{L} + \frac{1}{8\pi G} \int_B a_B \sqrt{\gamma} d^{d-2}x_i + \frac{1}{8\pi G} \int_A a_A \sqrt{\gamma} d^{d-2}x_i + \frac{1}{8\pi G} \int_D a_D \sqrt{\gamma} d^{d-2}x_i - \frac{1}{8\pi G} \int_C a_C \sqrt{\gamma} d^{d-2}x_i,
\]

where \( \gamma \) is the determinant of induced metric and the integrant \( a \) has the following form

\[
a = \ln(-\frac{1}{2} N N),
\]

in which \( N \) is the future-directed null normal to the left-moving null surface and \( \bar{N} \) denotes the future-directed null normal to the right-moving null surface. For calculating the contribution of volume, we can write

\[
S_{V_1} - S_{V_2} = \frac{V_{d-2}}{16\pi} \delta t \left[ 4\Lambda v^{d-1} - m^2 X + BI \right]^{t_2}_{t_1},
\]

where \( BI \) and \( X \) are

\[
BI = -\frac{8b^2 v^{d-1}}{(d-1)(d-2)} \left[ 1 + _2F_1\left( \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \cr \frac{3}{2d-4} \end{bmatrix} ; 1, \frac{1}{2d-4} \right), \frac{d-3}{2d-4} \frac{q^2}{b^2} \right]^{t_2}_{t_1},
\]

(19)
FIG. 1: The three cases for behaviors of the metric function and their Penrose diagrams are plotted. Three horizons (continuous line of metric function and related Penrose diagram in the right-down panel), two horizons which inner one is extreme (dashed line of metric function and related Penrose diagram in the left-up panel) and two horizons which outer one is extreme (bold line of metric function and related Penrose diagram in the right-down panel).

\[
X = \frac{r^{(d-5)}}{(d-1)(d-2)} (-48c_4r^4 + 2c_1r^4 - 4c_2r^4 + 12c_3r^4 + 48c_4r^d + 2c_2c_2r^2d^2 - 6c_2c_2r^2d^2 + 6c_2c_2r^2d^2 + 2c_3c_3r^2d^2 + 12c_3c_3r^3d^2 + 22c_3c_3r^3d^2 - 3c_2c_3r^3d^2 + 9c_3c_3r^2d^2 - 6c_2c_2r^2d^2 - 18c_3c_3r^3d^2 - 4c_3c_3r^3d^2 + 32c_3c_3rd^2 - 92c_3c_3rd^2 + 112c_3c_3rd^2 + 12c_3c_3rd^2 + 36c_2c_2r^2d^2 + 24c_3c_2r^2d^2 - 24c_3c_3r^3d^2 + 72c_3c_3r^3d^2 - c_1r^4d - 4c_2r^4d + 12c_3c_3r^3d^2 - 6c_3c_3r^3d^2 + 2c_3c_3d^3 - 20c_3c_3d^3 + 70c_3c_3d^2 - 100c_3c_3d^2 - 48c_3c_3r - 48c_3c_3r^3 + 24c_3c_3r^4d - 2cc_1r^3 - 12c_3c_3r + 4c_2c_2r^2). \tag{20}
\]

Before calculating the contributions for joints, we transform \( N \) and \( \bar{N} \) under an affine parametrization

\[
N_\alpha = -b_1 \partial_\alpha (t - r^*), \quad \bar{N} = b_2 \partial_\alpha (t + r^*),
\]

in which \( b_1 \) and \( b_2 \) are two arbitrary positive constants and \( r^* \) is defined as

\[
r^* = \int \frac{dr}{f(r)}. \tag{21}
\]

Finally the contributions for joints are calculated as

\[
S_B - S_A = \frac{V_{d-2} \delta t}{16\pi} \left[ r^{d-2} \left( \frac{d - 2}{r} f(r) \ln \left[ \frac{f(r)}{b_1 b_2} \right] + f'(r) \right) \right]_{r_A}, \tag{22}
\]

\[
S_D - S_C = \frac{V_{d-2} \delta t}{16\pi} \left[ r^{d-2} \left( \frac{d - 2}{r} f(r) \ln \left[ \frac{f(r)}{b_1 b_2} \right] + f'(r) \right) \right]_{r_C}. \tag{23}
\]
where \( f'(r) = \frac{df(r)}{dr} \). Now, by combining the volume and joint contributions, we obtain

\[
\frac{dS}{dt} = -\frac{V_d-2}{16\pi} \left[ 4\Lambda r^{d-1} - m^2 X + BI \right]_{r_\infty}^{r_-} + \frac{V_d-2}{16\pi} \left[ r^{d-2} \left( \frac{d-2}{r} f(r) \ln \left( \frac{f(r)}{b_1 b_2} \right) + f'(r) \right) \right]_{r_\infty}^{r_C} .
\] (24)

At the late time \( r_A \) and \( r_C \) approach, respectively, to \( r_- \) and \( r_+ \) and \( f(r) \) goes to zero, and therefore, Eq. (24) becomes

\[
\frac{dS}{dt} = \frac{V_d-2}{16\pi} \left[ 4\Lambda r^{d-1} - m^2 X + BI \right]_{r_\infty}^{r_+} + \frac{V_d-2}{16\pi} \left[ r^{d-2} \left( f'(r) \right) \right]_{r_\infty}^{r_+} .
\] (25)

Regarding the above relation and Eq. (33) the rate of complexity (the left hand side of the Eq. (33) ) will be calculated. Here, we should determine the right hand side of Eq. (33). According to the presence of nonlinear electrodynamics and massive term in the action, it is notable that the equation for upper bound of the rate of complexity must be modified by additional terms as

\[
\frac{\partial C}{\partial t} \leq \frac{2}{\pi \hbar} \left[ (M - \mu Q - Bb - \sum_{i=1}^{c} C_i c_i)_{r_+} - (M - \mu Q - Bb - \sum_{i=1}^{c} C_i c_i)_{r_-} \right].
\] (26)

Now, we regard the electric charge, massive charge and the nonlinearity parameter as extensive parameters and then we obtain their intensive conjugates. The chemical potential is intensive parameter which conjugates to the electric charge, and is obtained from Hamiltonian approach with the following explicit form

\[
\mu = -\frac{\sqrt{(d-2)(d-3)q}}{(d-3)d-3} 2F(1, \frac{d-3}{2(d-2)}, \frac{3(d-\frac{7}{3})}{2(d-2)}, \frac{(d-2)(d-3)q^2}{b^2 r^{2d-4}}).
\] (27)

Next, we should obtain conjugate quantities of the massive parameters. For this purpose, the total mass should be obtained from Hamiltonian approach with the following explicit form

\[
M = \frac{(d-2) V_{d-2}}{2k^2} m_0,
\] (28)

where \( m_0 \) can be replaced from the fact that the metric function vanishes at the horizon. Thus, we can write

\[
M = \frac{(d-2) V_{d-2}}{2k^2} r_+^{d-3} \left[ k - \frac{4b^2 r_+^2}{(d-1)(d-2)} \sqrt{1 + \frac{q^2(d-2)(d-3)}{b^2 r_+^{2d-4}}} + \frac{r_+^2}{2(d-2)(d-3)} \right] + \frac{4b^2}{(d-1)(d-2)} + \frac{4(d-2)q^2}{(d-1)r_+^{2d-6}} 2F\left(1, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, \frac{q^2(d-2)(d-3)}{b^2 r_+^{2d-4}} \right) + m^2 \left( \frac{c_1 r_+ + c_2}{r_+} + \frac{(d-3)(d-4)c_3 c_4}{r_+^2} \right).
\] (29)

In order to calculate the conjugate quantities, we can use the first law of thermodynamics. Differentiating of \( M \) with respect to the massive parameters gives their conjugates, as

\[
C_1 = \frac{dM}{dc_1} = \frac{V_{d-2}}{2} m^2 c_1 r_+^{d-2},
\] (30)

\[
C_3 = \frac{dM}{dc_3} = \frac{(d-2)(d-3)m^2 k V_{d-2}}{2} r_+^{d-4} c_3^3,
\] (31)

\[
C_4 = \frac{dM}{dc_4} = \frac{(d-2)(d-3)(d-4)m^2 k V_{d-2}}{2} r_+^{d-5} c_4^4.
\] (32)

As one can see, we did not calculate the conjugate of \( c_2 \). It is due to the fact that this term has no contribution to the Smarr relation, and consequently, has no contribution to Eq. (28) (see 39 for more details). Regardless of \( c_2 \), in general, there are \( d \) massive parameters, and therefore, \( d \) conjugate quantities. The \( n^{th} \) conjugate quantity related to \( c_n \) is introduced in the following form

\[
C_n = \frac{m^2 k V_{d-2}}{2} c_n r_+^{d-n-1} \prod_{i=2}^{n} (d-i).
\] (33)
Eventually intensive parameter which is conjugate to the nonlinearity parameter, $b$, is

$$B = \frac{dM}{db} = \frac{(d-2)kV_{d-2}r_+^{d-1}}{2} \left[ \frac{8b}{(d-1)(d-2)} \left( 1 + \sqrt{1 + \frac{(d-2)(d-3)q^2}{b^2r_+^{2d-4}}} \right) + \frac{4q^2(d-3)r_+^{4-2d}}{b(d-1)^2 + 1 + \frac{(d-2)(d-3)q^2}{b^2r_+^{2d-4}}} + \frac{4(d-2)^2(d-3)^2q^2}{(3d-7)(d-1)b^6r_+^{12-4d}} \times \right. \\
\left. \frac{2F_1 \left( \frac{3}{2}, \frac{3d-7}{2d-4}, \frac{5d-11}{2d-4}, -\frac{q^2(d-2)(d-3)}{b^2r_+^{2d-4}} \right) }{ } \right]$$

(34)

At this situation, we calculate both the right and left hand sides of Eq. (26), and therefore, we can numerically check the validity of the inequality (26). We can plot a figure for both LHS and RHS of Eq. (26) versus $c_1, c_3, m$ and $b$. According to these plots, we find that the inequality (26) is always held because the diagram for the RHS is constant and the diagram for LHS is negative and with increasing of each parameter decreases therefore $RHS - LHS$ has always positive value.

IV. SHOCKWAVE AND DISCONTINUITY

In this section, we study CA duality in the presence of a bulk shockwave which is dual to the insertion of a perturbation in the past of the thermofield double state. The motivation of adding perturbation is to investigate entanglement between two states before and after adding perturbation. Suppose that two typical systems $A$ and $B$ (which are small subsystems in left and right state) at time $t = 0$ are highly entangled. Then we consider the effect of injecting a small amount of energy $E$ into the left system, by throwing a few quanta towards the horizon at time $t_w$. One expects that the CFTs dual to black holes have sensitive dependence on the initial conditions, and this small perturbation should touch off chaotic behavior in the left theory and then should have less entanglement between $A$ and $B$. The added complexity by this perturbation can be understood in terms of the minimal quantum circuit needed to apply the Heisenberg operator $W(t_w) = e^{iHt_w}W e^{-iHt_w}$ to the thermofield double state. According to Ref. [43, 44], we expect a partial cancelation of the forward and backward time evolutions generating $W(t_w)$, and therefore, the total additional complexity for large $t_w$ is proportional to $2(t_w - t_*)$ in which $t_*$ is the scrambling time. For small values of $t_w$ the geometry will not be substantially affected by the perturbation. The Schwarzschild time evolution acts near the horizon as a boost.

For understanding the effect of shockwave, it is helpful to change the coordinates $t$ and $r$ to $u$ and $v$ with the following definition

$$u = t + r^*, \quad v = t - r^*,$$

in which $r^*$ is defined in Eq. (21). With applying these change of coordinates, the line element changes to

$$ds^2 = -f(r)dudv + r^2h_{ij}dx^idx^j, \quad i, j = 1, 2, 3, ..., d - 2.$$

(35)

It is convenient to carry out the shockwave calculations in the Kruskal-Szekeres coordinates. These coordinates can be defined throughout the eternal black hole spacetime as

$$U = -e^{-\frac{4\pi}{\beta}u}, V = e^{\frac{4\pi}{\beta}v}(right\ exterior\ region)$$

$$U = e^{-\frac{4\pi}{\beta}u}, V = e^{\frac{4\pi}{\beta}v}(black\ hole\ region)$$

$$U = e^{-\frac{4\pi}{\beta}v}, V = e^{\frac{4\pi}{\beta}u}(left\ exterior\ region)$$

$$U = -e^{-\frac{4\pi}{\beta}v}, V = -e^{\frac{4\pi}{\beta}u}(white\ hole\ region)$$

where $\beta$ is the inverse of temperature with the following explicit form

$$\beta = \frac{4\pi}{\partial_r f(r)}|_{r=r_+}$$

(36)

The null shell is injected from the left boundary at time $t_w \rightarrow \infty$ with infinitesimal energy $\delta \epsilon$. In addition, the stress energy distribution is highly compressed in $u$ direction but stretched in $v$ direction, and we can replaced it by a stress tensor that localized at $u = 0$ horizon

$$T_{uu} = \frac{\delta \epsilon}{\beta} e^{-\frac{4\pi}{\beta}u} \delta(u).$$

(37)
Since the shockwave makes a discontinuity in the metric at \( u = 0 \), it leads to a finite shift in \( v \)

\[
\delta v = h \sim e^{\frac{2\pi r_{h} - r_{0}}{\beta}}.
\] (38)

Here, we are going to calculate the complexity in the geometry which is perturbed by a spherically symmetric null shell falling into the black hole. The null shell sets of a shockwave whose physical manifestation is a null shift along the shockwave. The metric is discontinuous along the \( EG \), and so, if we select two points with the same \( r \) but different \( t \), we should not expect the action of both points to add up to zero. We calculate the effect of discontinuity by comparing the two null surfaces \( u = \epsilon \) and \( u = -\epsilon \) which approach to \( EG \) when \( \epsilon \rightarrow 0 \). The contribution of the discontinuity is

\[
S_{\text{discontinuity}} = S_{E'G'} + S_{E''G''} + S_{E'} + S_{G'} + S_{E''} + S_{G''}.
\] (39)

The lines \( E'G' \) and \( E''G'' \) are null, and therefore, the first two terms in Eq. (39) becomes zero. It is notable that the metric function \( f(r) \) is positive and negative for outside and inside the horizon, respectively and therefore, the actions of joints are

\[
S_{E'} = \frac{\Omega^{d-2}}{8\pi G} r^{d-2} \ln[f(r)],
\]

\[
S_{G'} = -\frac{\Omega^{d-2}}{8\pi G} r^{d-2} \ln[-f(r)],
\]

\[
S_{G''} = \frac{\Omega^{d-2}}{8\pi G} r^{d-2} \ln[f(r)],
\]

\[
S_{E''} = -\frac{\Omega^{d-2}}{8\pi G} r^{d-2} \ln[-f(r)].
\]

Since the radius \( r \rightarrow r_{h} \) for \( \epsilon \rightarrow 0 \), we expand the contributions of the joints \( E', G' \) and \( H'' \) around \( r_{h} \). The expansion of \( r^{d-2} \ln[-f(r)] \) (for \( r < r_{h} \)) is

\[
r^{d-2} \ln[-f(r)]|_{r \to r_{h}} = (r_{h}^{d-2} \ln[-(r - r_{h})H])_{r < r_{h}},
\]

where \( H \) is a constant which is defined as

\[
H = 2k - \frac{8b^{2}r_{h}}{(d-1)} \sqrt{1 + \frac{q^{2}(d-2)(d-3)}{b^{2}r_{h}^{2(d-2)}} + 2r_{h}(d-2) \left( \frac{1}{l^{2}} + \frac{4b^{2}}{(d-1)(d-2)} \right) + 2m^{2}(d-3) \frac{1}{r_{h}} \left( \frac{cc_{1}r_{h}}{d-2} + \frac{c_{2}c^{2}}{r_{h}} + \frac{(d-3)c^{3}c_{3}}{r_{h}} + \frac{(d-3)(d-4)c^{4}c_{4}}{r_{h}} \right) + 4r_{h}(d-2)(d-3)q^{2}r_{h}^{2(d-2)} (d-1) \left( \frac{1}{b^{2}r_{h}^{2(d-2)}} \right) + \frac{4}{3} (d-2)^{3}(d-3)^{2}q^{4}} + 2F_{1} \left( \left[ \frac{3}{2} - 2(d-2) + 1 \right], \left[ \frac{3(d - \frac{2}{3})}{2(d-2)} + 1 \right], \frac{(d-3)q^{2}(d-2)}{b^{2}r_{h}^{2(d-2)}} \right) + \frac{m^{2}}{d-2} \left( \frac{cc_{1}}{r_{h}^{2}} - \frac{2(d-3)(d-4)c^{4}c_{4}}{r_{h}^{4}} \right).}
\]

On the other hand, by using the definition of Kruskal-Szekeres coordinates, we can write

\[
UV = e^{-\frac{4\beta}{\pi} r^{*}}; \text{ for inside the horizon,}
\]

\[
UV = -e^{-\frac{4\beta}{\pi} r^{*}}; \text{ for outside the horizon.}
\] (40)

In addition, we can use the series expansion of \( r^{*} \) for \( r \rightarrow r_{h} \) to obtain

\[
r^{*}|_{r \to r_{h}} = \frac{1}{H} \ln((r - r_{h})H)|_{r > r_{h}}.
\]
Considering Eq. (40), one finds that \( r^* \) can be written as 
\[ r^*(E') = -\frac{\beta}{4\pi} \ln(\epsilon U_0^{-1}), \]
\[ r^*(G') = -\frac{\beta}{4\pi} \ln(\epsilon V_0 + \epsilon h), \]
\[ r^*(E'') = -\frac{\beta}{4\pi} \ln(\epsilon U_0^{-1} + \epsilon h), \]
\[ r^*(G'') = -\frac{\beta}{4\pi} \ln(\epsilon V_0), \]
where \( U_0 \) and \( V_0 \) are two arbitrary positive constants. Now, we are in a position to replaced our results in the action of discontinuity, yielding

\[ S_{E'} = \frac{\beta H \Omega^{d-2}}{32\pi^2 G} \epsilon h^{d-2} \ln(\epsilon U_0^{-1}), \]
\[ S_{G'} = \frac{\beta H \Omega^{d-2}}{32\pi^2 G} \epsilon h^{d-2} \ln(\epsilon V_0 + \epsilon h), \]
\[ S_{E''} = \frac{\beta H \Omega^{d-2}}{32\pi^2 G} \epsilon h^{d-2} \ln(\epsilon U_0^{-1} + \epsilon h), \]
\[ S_{G''} = \frac{\beta H \Omega^{d-2}}{32\pi^2 G} \epsilon h^{d-2} \ln(\epsilon V_0), \]
where, in final form, we can obtain

\[ S_{\text{discontinuity}} = \frac{\beta H \Omega^{d-2}}{32\pi^2 G} \epsilon h^{d-2} \ln((1 + V_0^{-1}h)(1 + U_0 h)). \] (41)

It is obvious that \( h, U_0 \) and \( V_0 \) are positive, and therefore, \( S_{\text{discontinuity}} \) is positive too and its value depends only on \( h \). For vanishing \( h \), one finds that \( S_{\text{discontinuity}} \) vanishes too (since the effect of shockwave is vanished) and for nonzero \( h \), \( S_{\text{discontinuity}} \) is an increasing function of \( h \). In addition, it is interesting to note that for \( d > 2 \), increasing the horizon radius (or decreasing the temperature) leads to increasing \( S_{\text{discontinuity}} \). For describing physical meaning of \( S_{\text{discontinuity}} \) one should note that at first complexity is calculated for the Einstein-Rosen (ER) bridge. In this way a signal is sent to ER bridge and difficulties which faced to exiting of the signal (definition of the complexity) is created which adds more difficulty (complexity) in exiting of the signal.

V. CONCLUSION

In this paper, we have examined the holographic complexity for the black holes in Einstein-massive gravity in the presence of BI electrodynamics with CA proposal. For this purpose, we have calculated the complexity based on the action including of bulk term and joints contributions. We have also obtained intensive parameters based on the extended first law of thermodynamics and their conjugate extensive quantities.

In addition, we have calculated the both side of Lloyd equation, separately, and found numerically that such inequality is always held. Finally, we have regarded a geometrical perturbation in the context of bulk shockwave and calculated the contribution of the possible discontinuity. We have shown that the discontinuity is affected by the horizon radius (and also temperature), significantly.

According to the result of the rate of complexity for the case Einstein-Born-Infeld-Massive black holes, we can compute some limited cases as the subclass of our model: Einstein-Maxwell-Massive black holes, Einstein-Born-Infeld black holes and Einstein-Maxwell black holes which are well-known as the Reissner-Nordström black holes. The results are summarized as follows: for the case: Einstein-Maxwell-Massive black hole, we should regard \( b \rightarrow \infty \) case with the following result

\[ \frac{dC}{dt} = -\frac{V_d-2}{16\pi^2 h} \left[ 4\Lambda r^{d-1} - m^2 X + \frac{4}{d-2} \frac{g^2 r^{3-d}}{r^+} \right]_{r^+}^{r_-} + \frac{V_d-2}{16\pi^2 h} \lim_{b \rightarrow \infty} \left[ r^{d-2} f' (r) \right]_{r^+}^{r_-} \]
\[ \leq \frac{2}{\pi h} \left[ (M - \mu Q - \sum_{i=1} C_i c_i)_{r^+} - (M - \mu Q - \sum_{i=1} C_i c_i)_{r_-} \right]. \] (42)
For the case of the Einstein-BI black holes we can write:

\[
\frac{dC}{dt} = -\frac{V_{d-2}}{16\pi^2 \hbar} \left[ 4\Lambda r^{d-1} + BI \right]_{r_+} + \frac{V_{d-2}}{16\pi^2 \hbar} \left[ r^{d-2} f'(r) \right]_{r_-} \bigg|_{m=0}
\]

\[
\leq \frac{2}{\pi \hbar} [(M - \mu Q - bB)_{r_+} - (M - \mu Q - bB)_{r_-}].
\]

For the case of Reissner-Nordström black holes the following result is obtained

\[
\frac{dC}{dt} = -\frac{V_{d-2}}{16\pi^2 \hbar} \left[ 4\Lambda r^{d-1} + \frac{4}{d-2} q^2 r^3 - d \right]_{r_+} + \frac{V_{d-2}}{16\pi^2 \hbar} \lim_{b \to \infty} \left[ r^{d-2} f'(r) \right]_{r_-} \bigg|_{m=0}
\]

\[
\leq \frac{2}{\pi \hbar} [(M - \mu Q)_{r_+} - (M - \mu Q)_{r_-}].
\]

It is notable that these results are in agreement with the previous works reported in Refs. [27, 28].

It is interesting to compute the complexity of different black holes with more than two horizons in the presence of different gauge fields and investigate their perturbation with the bulk of shockwave, and then comparing their \( S_{\text{discontinuity}} \) with each other in order to find a possible relation between \( S_{\text{discontinuity}} \) and the gauge fields.

\[\text{References}\]

[1] J. D. Bekenstein, "Black holes and entropy", Phys. Rev. D 7, 2333 (1973).
[2] J. Aspnes, "Notes on Computational Complexity Theory CPSC", 468/568: Spring 2017. (2017).
[3] A. Brown, D. A. Roberts, L. Susskind, B. Swingle, Y. Zhao, "Complexity, action, and black holes", Phys. Rev. D 93, 086006 (2016).
[4] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, Y. Zhao, "Quantum aspects of massive gravity", Phys. Rev. Lett. 116, 191301 (2016).
[5] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, Y. Zhao, "Holographic complexity equals bulk action?", Phys. Rev. Lett. 116, 191301 (2016).
[6] M. A. Nielsen, M. R. Dowling, M. Gu, A. C. Doherty, "Quantum computation as geometry", Science 311, 1133 (2006).
[7] A. R. Brown, L. Susskind, Y. Zhao, "Quantum complexity and negative curvature", Phys. Rev. D 95, 045010 (2017).
[8] S. Lloyd, "Quantum algorithms for physical simulation", Nature 408, 1040 (2000).
[9] C. De Rham, G. Gabadadze, "Generalization of the Fierz-Pauli action", Phys. Rev. D 82, 044020 (2010).
[10] M. Fierz, "Dirac-Born-Infeld action from Dirichlet 3c3-model", Phys. Rev. D 24, 2767 (1981).
[11] D. G. Boulware, S. Deser, "Can gravitation have a finite range?", Phys. Rev. D 6, 3368 (1972).
[12] S. F. Hassan, R. A. Rosen, "Resolving the ghost problem in nonlinear massive gravity", Phys. Rev. Lett. 108, 041101 (2012).
[13] S. F. Hassan, R. A. Rosen, A. Schmidt-May, "Ghost-free massive gravity with a general reference metric", JHEP 02, 026 (2012).
[14] M. Park, "Quantum aspects of massive gravity", Class. Quant. Grav. 28, 105012 (2011).
[15] C. De Rham, G. Gabadadze, "Generalization of the Fierz-Pauli action", Phys. Rev. D 82, 044020 (2010).
[16] C. De Rham, G. Gabadadze, A. J. Tolley, "Resummation of massive gravity", Phys. Rev. Lett. 106, 231101 (2011).
[17] K. Hinterbichler, "Theoretical aspects of massive gravity", Rev. Mod. Phys. 84, 671 (2012).
[18] D. Vegh, "Holography without translational symmetry", arXiv:1301.0537.
[19] S. F. Hassan, R. A. Rosen, "On non-linear actions for massive gravity", JHEP 07, 009 (2011).
[20] R. G. Cai, Y. P. Hu, Q. Y. Pan, Y. L. Zhang, "Thermodynamics of black holes in massive gravity", Phys. Rev. D 91, 024032 (2015).
[21] M. Born, L. Infeld, "Foundations of the new field theory", Proc. Roy. Soc. Lond. A. 144, 425 (1934).
[22] E. S. Fradkin, A. A. Tseytlin, "Non-linear electrodynamics from quantized strings", Phys. Lett. B 163, 123 (1985).
[23] D. L. Wilzshier, "Black holes in string-generated gravity models", Phys. Rev. D 38, 2445 (1988).
[24] R. G. Leigh, "Dirac-Born-Infeld action from Diraclet 3c3-model", Mod. Phys. Lett. A 4, 2767 (1989).
[25] G. W. Gibbons, C. A. R. Herdeiro, "The Melvin universe in Born-Infeld theory and other theories of nonlinear electrodynamics", Class. Quant. Grav. 18, 1677 (2001).
[26] G. W. Gibbons, "Aspects of Born-Infeld theory and string/M theory", Rev. Mex. Fis. 49S1, 19 (2003).
[27] R. G. Cai, M. Sasaki, S. J. Zhang, "Action growth of charged black holes with a single horizon", Phys. Rev. D 95, 124002 (2017).
[28] B. Swingle, "Holographic Complexity of Einstein-Maxwell-Dilaton Gravity", arXiv:1712.09826.
[29] W. D. Guo, Z. W. Wei, Y. Y. Li, Y. X. Liu, "Complexity growth rates for AdS black holes in massive gravity and f (R) gravity", Eur. Phys. J. C 77, 904 (2017).
[30] S. H. Hendi, S. Panahiyan, B. Eslam Panah, M. Momennia, "Phase transition of charged black holes in massive gravity through new methods", Ann. Phys. 528, 819 (2016).
[33] S. H. Hendi, S. Panahiyan, B. Eslam Panah, "Charged black hole solutions in Gauss-Bonnet-massive gravity" JHEP 01, 129 (2016).

[34] S. H. Hendi, B. Eslam Panah, S. Panahiyan, "Einstein-Born-Infeld-massive gravity: adS-black hole solutions and their thermodynamical properties", JHEP 11, 157 (2015).

[35] S. H. Hendi, B. Eslam Panah, S. Panahiyan, "Topological charged black holes in massive gravity’s rainbow and their thermodynamical analysis through various approaches", Phys. Lett. B 769, 191 (2017).

[36] S. H. Hendi, B. Eslam Panah, S. Panahiyan, "Massive charged BTZ black holes in asymptotically (a) dS spacetimes" JHEP 05, 029 (2016).

[37] S. H. Hendi, GQ. Li, JX. Mo, S. Panahiyan, B. Eslam Panah, "New perspective for black hole thermodynamics in Gauss–Bonnet–Born–Infeld massive gravity", Eur. Phys. J. C 76, 571 (2016).

[38] S. H. Hendi, N. Riazi, S. Panahiyan, "Holographical aspects of dyonic black holes: massive gravity generalization", Ann. Phys. 530, 1700211 (2018).

[39] S. H. Hendi, S. Panahiyan, S. Upadhyay, B. Eslam Panah, "Charged BTZ black holes in the context of massive gravity’s rainbow", Phys. Rev. D 95, 084036 (2017).

[40] S. H. Hendi, B. Eslam Panah, S. Panahiyan, M. Momennia, "Three dimensional magnetic solutions in massive gravity with (non) linear field", Phys. Lett. B 775, 251 (2017).

[41] S. H. Hendi, B. Eslam Panah, S. Panahiyan, "Black Hole Solutions in Gauss-Bonnet-Massive Gravity in the Presence of Power-Maxwell Field", Fortschr. Phys. 66, 1800005 (2018).

[42] L. Lehner, R. C. Myers, E. Poisson, R. D. Sorkin, "Gravitational action with null boundaries", Phys. Rev. D 94, 084046 (2016).

[43] D. Stanford, L. Susskind, "Complexity and shock wave geometries", Phys. Rev. D 90, 126007 (2014).

[44] L. Susskind, "The Stretched Horizon and Black Hole Complementarity", Phys. Rev. D 48, 3743 (1993).

[45] S. H. Shenker, D. Stanford, "Black holes and the butterfly effect", JHEP 03, 067 (2014).