An extension of a cubic 2-connected plane graph $G$ to a hamiltonian plane graph contained in $G^2$

Jan Florek

Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, Poland

Abstract

Let $G$ be a simple cubic 2-connected plane graph. For every 2-factor $X$ of $G$ having $n$-components there exists a simple hamiltonian plane graph $J \subset G^2$ such that $|E(J)| = |E(G)| + 2n - 2$ and $\Delta(J) \leq 5$.

1. Introduction

We use [1] as a reference for undefined terms. Let $F$ be a simple connected graph. $V(F)$ is the vertex set and $E(F)$ is the edge set of $F$. A spanning subgraph of $F$ is a subgraph whose vertex set is the entire vertex set of $F$. A spanning cycle (path) is called Hamilton cycle (Hamilton path) and a spanning $k$-regular subgraph is called $k$-factor. A graph is hamiltonian if it admits a Hamilton cycle. A graph is hamiltonian-connected if for every pair $u, v$ of distinct vertices of $F$, there exists a Hamilton $u - v$ path.

Given a positive integer $k$, we denote by $F^k$ the graph on $V(F)$ in which two vertices are adjacent if and only if they have distance at most $k$ in $F$. The graph $F^2$ and $F^3$ are also referred to as the square and cube, respectively, of $F$. Karaganis [4] and Sekanina [7] proved that the cube of a connected graph is hamiltonian-connected, and Fleischner [3] discovered that the square of a 2-connected graph is hamiltonian (see also Rihá [6]). The strengthened result (employing Fleischner’s work) that the square of such a graph is hamiltonian-connected was proved by Chartrand, Kappor, and Nash-Williams [2].
Let $G$ be the family of all simple cubic 2-connected plane graphs. We prove the following theorem.

**Theorem 1.1.** Let $G \in \mathcal{G}$. For every 2-factor $X$ of $G$ having $n$-components there exists a simple plane graph $J$, $G \subseteq J \subset G^2$, having a Hamilton cycle omitting all edges of $E(G) \setminus E(X)$, $|E(J)| = |E(G)| + 2n - 2$ and $\Delta(J) \leq 5$.

Notice that Petersen [5] proved that every simple bridgeless cubic graph has a 1-factor (see Distel [1] Corollary 2.2.2). It follows that that each graph of $G$ has a 2-factor.

2. Main result

Let $G \in \mathcal{G}$. Each face $f \in F(G)$ is bounded by a cycle $\partial(f)$ called a facial cycle of this face. A cyclic sequence $f_1f_2\ldots f_kf_1$ (a sequence $f_1f_2\ldots f_k$) of different faces in $G$ is called a cyclic sequence of faces (a sequence of faces from $f_1$ to $f_k$, respectively) if any two successive faces $f_i, f_{i+1}$ are adjacent. We say that an edge belongs to a cyclic sequence of faces (to a sequence of faces) if it is a common edge of two successive faces of this sequence. Then, we also say that this cyclic sequence of faces (sequence of faces) contains this edge. Notice that $B \subseteq E(G)$ is a bond of $G$ if and only if it is the set of all edges belonging to a some cyclic sequence of faces.

**Proof of Theorem 1.1** Let $X$ be a 2-factor of $G$ which has $n$ components. Without loss of generality we can assume that $n > 1$. We will define a face 2-colouring $a : F(G) \to \{\alpha, \beta\}$. Fix a face $f$ of $F(G)$. For every $g \in F(G)$, $g \neq f$, there exists a sequence of faces from $f$ to $g$ containing only edges of $E(X)$, because $G$ is cubic and $X$ is a 2-factor of $G$. We set $a(f) = \alpha$ and we colour faces of this sequence with $\alpha$ and $\beta$ alternately. The colouring of $g$ is independent on the choice of the sequence of faces. Indeed, if $D$ is a cyclic sequence of faces and $B \subseteq E(X)$ is the set of all edges belonging to $D$, then $|B|$ is even, because $X$ is a 2-factor of $G$. It follows that

1. any two adjacent faces in $G$ are incident with the same edge belonging to $E(G) \setminus E(X)$ if and only if they are coloured identically by $a$.

Further, every facial cycle of $G$ has an orientation assigned by $a$. Namely, we can assume that a facial cycle of an inner (the outer) face has the clockwise-orientation (counter clockwise-orientation, respectively) if and only if this face is coloured $\alpha$. 

2
We can enumerate, by induction, all components of $X$ as $C_1, C_2, \ldots, C_n$ in such a way that there exists an edge $b_i \in E(G) \setminus E(X)$ connecting a subgraph $C_1 \cup \ldots \cup C_i$ with $C_{i+1}$, for every $1 \leq i < n$. Certainly any two edges of $M = \{b_1, \ldots, b_n\}$ are different and non-adjacent, because $G$ is cubic and $X$ is a 2-factor. Since each $b_i \in M$ is connecting two different components of $X$, we obtain:

(2) each facial 3-cycle of $G$ does not contain any edge of $M$,
(3) each facial 4-cycle of $G$ contains at most one edge of $M$.

Suppose that $K$ is the family of all faces in $G$ each of which is incident with an edge of $M$. Let $f$ be any face of $K$ and suppose that $\partial(f) = c_1c_2\ldots c_nc_1$ is the facial cycle of $f$ with the orientation assigned by $a$. Let $c_i, c_{i+1}, c_{i+2}, c_{i+3}$ (where $p$ depends on the face $f$) be all successive edges of $\partial(f)$ belonging to $M$. By condition (2), vertices $c_i$ and $c_{i+2}$ are not adjacent in $G$, for every $j = 1, \ldots, p$. Since edges of $M$ are not adjacent, we can draw new edges

$$e_1(f) = c_{i_1}c_{i_1+2}, e_2(f) = c_{i_2}c_{i_2+2}, \ldots, e_p(f) = c_{i_p}c_{i_p+2}$$

in such a way that they are not crossing and their interiors are contained in $f$. By adding to $G$ all edges of $\bigcup_{f \in K}\{e_1(f), \ldots, e_p(f)\}$ we obtain a plane graph $J$ such that $G \subseteq J \subset G^2$ and $|E(J)| = |E(G)| + 2n - 2$. Since $G$ is simple, by conditions (2) -- (3), $J$ is simple too.

Let $e = ab$ be any edge of $M$ and suppose that $e$ is incident with faces $f_1, f_2 \in F(G)$. Since $e \in E(G) \setminus E(X)$, by condition (1), faces $f_1$ and $f_2$ are coloured the same by $a$. Assume that $a, b, c$ and $b, a, d$ are successive vertices of $\partial(f_1)$ and $\partial(f_2)$, respectively. If faces $f_1, f_2$ are coloured $\alpha$ (or $\beta$), then a 4-cycle $D_e = abca$ is called a diamond of type $\alpha$ (diamond of type $\beta$, respectively). Let $D^1_{e_1}$ (or $D^2_{e_2}$) denote a set of two edges of $D_e$ belonging to $E(G)$ (to $E(J) \setminus E(G)$, respectively). Then, $D^1_{e_1} = \{ad, bc\}$ and $D^2_{e_1} = \{db, ca\}$ (see Fig 1 and Fig 2). Since edges of $M$ are not adjacent, from condition (3) it follows that

(4) any two different diamonds have no common edge.

Since $G$ is cubic each vertex of $G$ belongs to at most two diamonds. Hence, $\Delta(J) \leq 5$.

Define, by induction, a subgraph of $J$: $H_1 = C_1$ and

$$H_{i+1} = (C_1 \cup \ldots \cup C_{i+1} - (D^1_{e_1} \cup \ldots \cup D^1_{e_1})) + (D^2_{e_1} \cup \ldots \cup D^2_{e_1}),$$

for $1 \leq i < n$. 

3
Figure 1: A subgraph of $G$ (without edges $bd, ac, cf, eg$) and a subgraph of a 2-factor $X$ of $G$ (in bold). Diamonds $D(ab) = adbca$ and $D(ce) = egefc$ are of type $\alpha$.

Figure 2: A subgraph of $G$ (without edges $bd, ac, cg, ef$) and a subgraph of a 2-factor $X$ of $G$ (in bold). Diamonds $D(ab) = adbca$ is of type $\alpha$ and $D(ce) = egefc$ is of type $\beta$. 
Notice that $H_i$ is omitting all edges of $E(G) \setminus E(X)$, because $C_j$ is contained in $X$, for $1 \leq j \leq n$, and $D_{b_j}^2$ is contained in $E(J) \setminus E(G)$, for $1 \leq j < n$. Further, $H_i$ is disjoint with $C_{i+1}$ because $C_{i+1}$ is disjoint with $C_j$, for $j < i$.

Assume that $H_i$ is a cycle of $J$. We prove that $H_{i+1}$ is also a cycle of $J$. Since $b_i$ is an edge connecting $C_1 \cup \ldots \cup C_i$ with $C_{i+1}$, by condition (4), any edge of $D_{b_i}^1$ is not an edge of $D_{b_j}^1$, for $j < i$. Hence, one edge of $D_{b_i}^1$ is an edge of $H_{i+1}$ and another one is an edge of $C_{i+1}$. Further, by condition (4), any edge of $D_{b_i}^2$ is not an edge of $D_{b_j}^2$, for $j < i$. Hence, any edge of $D_{b_i}^2$ is not an edge of $H_i \cup C_{i+1}$. Therefore, $H_{i+1}$ is a cycle of $J$. Thus, $H_n$ is a Hamilton cycle of $J$, because $V(H_n) = V(C_1) \cup \ldots \cup V(C_n) = V(G) = V(J)$. ■

References

[1] R. Diestel, Graph Theory, Springer-Verlag, (2005).

[2] G. Chartrand, A. Kappor and C.St.J.A. Nash-Williams, The square of a block is hamiltonian-connected. J. Combin. Theory 16B (1974) 290-2.

[3] H. Fleischner, The square of every two-connected graph is hamiltonian. J. Combin. Theory 16B (1974) 399-404.

[4] J.J. Karaganis, On the cube of a graph. Canad. Math. Bull. 11 (1968) 295-6.

[5] J. Petersen, Die Theorie der regulären Graphen. Acta Math. 15 (1891) 193–220.

[6] S. Říha, A new proof of the theorem by Fleischner, J. Combin. Theory 52B (1991) 117-123.

[7] M. Sekanina, On an ordering of the set of vertices of a connected graph. Publ. Fac. Sci. Univ. Brno, 412 (1960) 137–42.