HEAT KERNEL COEFFICIENTS ON KÄHLER MANIFOLDS

KEFENG LIU AND HAO XU

Abstract. Polterovich proved a remarkable closed formula for heat kernel coefficients of the Laplace operator on compact Riemannian manifolds involving powers of Laplacians acting on the distance function. In the case of Kähler manifolds, we prove a combinatorial formula simplifying powers of the complex Laplacian and use it to derive a graph theoretic formula for heat kernel coefficients based on Polterovich’s formula.

1. Introduction

The Laplace operator $\Delta$ on a Riemannian manifold $(M, g)$ of dimension $d$ is given by

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \partial_i (g^{ij} \sqrt{\det g} \partial_j).$$

The heat kernel is a smooth function $H(x, y, t) \in C^\infty(M \times M \times \mathbb{R}^+)$ that solves the heat equation $\frac{\partial H}{\partial t} + \Delta_x H = 0$ and satisfies $H(x, y, t) = H(y, x, t)$ and

$$\lim_{t \to 0} \int_M H(x, y, t) f(y) dV = f(x)$$

for any smooth function $f$ of compact support.

For example, the heat kernel of $\mathbb{R}^d$ is

$$H(x, y, t) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t}.$$  

If $M$ is compact, there is a unique heat kernel $H(x, y, t)$ on $M$ with the asymptotic expansion as $t \to 0^+$:

$$H(x, x, t) = (4\pi t)^{-d/2} \left( a_0(x) + a_1(x) t + a_2(x) t^2 + \cdots \right).$$

It was first proved by Minakshisundaram-Pleijel [12]. Here the coefficients $a_j(x)$ are curvature invariants, i.e., invariant polynomials of jets of metrics.

For compact $M$, there exists a complete orthonormal basis $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of $L^2(M)$, consisting of eigenfunctions of $\Delta$, with corresponding eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ arranged in increasing order $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, we have

$$H(x, y, t) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

with uniformly convergence for any fixed $t$. Therefore by (1), we get

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_M H(x, x, t) = (4\pi t)^{-d/2} \left( \text{Vol}(M) + t \int_M \frac{1}{6} \rho dV + \cdots \right).$$

The heat kernel method was extensively used in index theory, moduli space, spectral geometry and quantum gravity (see e.g., [4, 6, 8, 9, 10, 11, 16, 25, 26]).
Various recursive mechanisms of computing \(a_k(x)\) were developed, and explicit formulas were known for \(k \leq 5\) (see e.g., [2, 6, 17, 18]). Quite surprisingly, Polterovich [14] proved a remarkable closed formula for all heat kernel coefficients using a generalization of the Agmon–Kannai expansion [1]. Polterovich’s formula was applied to obtain new formulas for KdV hierarchy [13, 14] and heat invariants of spheres [15].

**Theorem 1.1** (Polterovich [14, 15]). Let \(w \geq 3n\). Then the heat kernel coefficients \(a_n(x)\) are equal to

\[
a_n(x) = (-1)^n \sum_{j=0}^{w} \binom{w + \frac{d}{2}}{j} \frac{1}{4^j j! (j + n)!} \Delta^{j+n}(\text{dist}(y, x)^2)|_{y=x},
\]

where \(\text{dist}(y, x)\) is the distance function.

As noted in [14], it is very difficult to convert powers of the Laplacian and the distance function to curvature tensors and their covariant derivatives.

By Weyl’s work on the invariants of the orthogonal group, any curvature invariants on a Riemannian manifold can be formed from Riemannian curvature tensor by covariant differentiations, multiplications and contractions. In particular, the formal expression of \(a_n\) are universal curvature polynomials independent of the dimension \(d\). But this is not clear a priori from (2).

**Remark 1.2.** It was proved by Weingart [19] that Polterovich’s formula holds for \(w \geq n\). As noted by Weingart [21], Polterovich’s formula was originally proved under Riemannian normal coordinates, which are quite different with Kählerian case. In fact, Riemannian normal coordinates are never complex analytic unless the manifold is flat, which was first noticed by Bochner (cf. [3, p. 22]). In normal coordinates around \(x = 0\) of Kähler manifold, the squared distance function has the form

\[
\text{dist}(0, z)^2 = 2|z|^2 + O(|z|^4).
\]

Following the argument of Polterovich in the proof of [14] Thm. 2.1.3], by Morse Lemma, there exists new holomorphic coordinates \(z'\) such that

\[
\text{dist}(0, z')^2 = 2|z'|^2
\]

and the Kähler metric remains Euclidean at the point 0, i.e., \(g_{\bar{z}z}(0) = \delta_{ij}\). The situation is similar to Polterovich’s generalization of his formula in [14] Thm. 2.1.3]. This observation is needed in the proof of Theorem 2.16.

In this paper, we show that in the case of Kähler manifolds, Polterovich’s formula produces an explicit graph theoretic formula for \(a_n\). More precisely, if we use graphs to represent Weyl invariants in the sense that each vertex represents a partial derivative of the Kähler metric and each edge represents the contraction of index pairs, then the heat kernel coefficients satisfy

\[
a_n = \sum_{\text{stable} G: w(G) = n} \frac{z(G) G}{|\text{Aut}(G)|},
\]

where \(G\) runs over stable digraphs of weight \(n\) (cf. Definition 2.11) and \(z(G)\) is a graph invariant given by

\[
z(G) = \frac{(-1)^{|V(G)|}}{|\text{Aut}(G)|} \sum_{C \in \mathcal{E}(G)} \frac{(-1)^{m(C)} \varphi(\Gamma_C)}{(m(C) + w(G))!}
\]
Here \( \mathcal{C}(G) \) is the set of all \( 2^{|E(G)|} \) possible ways to cut edges of \( G \) and \( w(G) = |E(G)| - |V(G)| \) is the weight of \( G \). Given an edge-cutting \( C \in \mathcal{C}(G) \), \( m(C) \) is the number of edges being cut and \( \Gamma_C \) is a pointed graph obtained by connecting all loose ends to a new vertex \( \bullet \) (cf. Definition 2.12). Finally \( \varphi(\Gamma_C) \) is the number of stable reductions of \( \Gamma_C \) (see Definition 2.7).

The graph invariant \( z(G) \) has the property that if \( G \) is a disjoint union of connected subgraphs \( G = \bigcup_{i=1}^{k} G_i \), then we have

\[
(5) \quad z(G) = \prod_{j=1}^{k} z(G_j) / |\text{Sym}(G_1, \ldots, G_k)|,
\]

where \( \text{Sym}(G_1, \ldots, G_k) \) is the permutation group of the connected subgraphs. It means that we only need to know \( z(G) \) for connected graphs.

In contrast to the graph theoretic formula for Bergman kernel coefficients proved in [22], the invariant \( z(G) \) here may be nonzero for weakly connected graphs.

Note that Weingart [20] proved a graph theoretic formula of heat kernel coefficients of a twisted Laplacian involving only covariant derivatives.

Our formula will be proved in Theorem 2.16 of the next section and we will apply it to compute \( a_k, k \leq 3 \) in the last section.

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### 2. A graph theoretic formula of heat kernel coefficients

We introduce some notations and concepts of graph theory.

**Definition 2.1.** In this paper, a graph always means a multi-digraph \( G = (V, E) \), which is defined to be a finite directed graph allowing to have multi-edges and loops. Here \( V \) and \( E \) are the set of vertices and edges respectively. The weight \( w(G) \) of \( G \) is defined to be \( |E| - |V| \). The adjacency matrix \( A = A(G) \) of a digraph \( G \) with \( n \) vertices is a square matrix of order \( n \) whose entry \( A_{ij} \) is the number of directed edges from vertex \( i \) to vertex \( j \). The outdegree \( \deg^+(v) \) and indgree \( \deg^-(v) \) of a vertex \( v \) are defined to be the number of outward and inward edges at \( v \) respectively.

A vertex \( v \) of \( G \) is called **stable** if \( \deg^-(v) \geq 2, \deg^+(v) \geq 2 \). We call \( G \) **stable** if each vertex \( v \) is stable.

A vertex \( v \) of \( G \) is called **semistable** if \( \deg^{-}(v) \geq 1, \deg^{+}(v) \geq 1 \) and \( \deg^{-}(v) + \deg^{+}(v) \geq 3 \). We call \( G \) **semistable** if each vertex \( v \) is semistable.

A digraph \( G \) is **strongly connected** if there is a directed path from each vertex in \( G \) to every other vertex.

**Definition 2.2.** A **pointed graph** is a multi-digraph \( \Gamma = (V \cup \{\bullet\}, E) \) with a distinguished vertex denoted by \( \bullet \). Let \( \Gamma_- \) be a subgraph of \( \Gamma \) obtained by removing the distinguished vertex \( \bullet \). We call \( \Gamma \) **semistable** (stable) if each ordinary vertex \( v \in V(\Gamma_-) \) is semistable (stable). The weight \( w(\Gamma) \) of \( \Gamma = (V \cup \{\bullet\}, E) \) is defined to be \( |E| - |V| \). Denote by \( \text{Aut}(\Gamma) \) the set of all automorphisms of \( \Gamma \) fixing the distinguished vertex \( \bullet \).
Definition 2.3. A directed edge $uv$ of a semistable pointed graph $\Gamma$ is called \textit{contractible} if $u \neq v$ and at least one of the following two conditions holds: (i) $u \in V(\Gamma_-)$ and $\deg^+(u) = 1$; (ii) $v \in V(\Gamma_-)$ and $\deg^-(v) = 1$.

A semistable pointed graph $\Gamma$ is called \textit{stabilizable} if after contractions of a finite number of contractible edges of $\Gamma$, the resulting graph becomes stable, which is called the \textit{stabilization graph} of $\Gamma$ and denoted by $\Gamma^s$.

The following lemma was proved in [23, Lem. 4.5].

Lemma 2.4. Let $\Gamma$ be a semistable graph.

(i) If $\Gamma$ is strongly connected, then it is stabilizable.

(ii) If $\Gamma$ is stabilizable semistable graph and its stabilization graph $\Gamma^s$ is strongly connected, then $\Gamma$ is also strongly connected.

Definition 2.5. Given a strongly connected pointed graph $\Gamma$, an edge $e$ is called \textit{redundant} if its removal from $\Gamma$ produces a strongly connected subgraph. Otherwise, it is called \textit{essential}. We define its admissible contractions to be.

Lemma 2.6. Every strongly connected semistable pointed graph with at least two vertices has a redundant edge.

Proof. Let $T^+(\bullet)$ and $T^-(\bullet)$ be directed spanning trees of $\Gamma$ rooted at $\bullet$ with all edges directed away from and towards $\bullet$ respectively. The existence of $T^+(\bullet)$ and $T^-(\bullet)$ is guaranteed by the strongly connectedness of $\Gamma$. Their union $T^+(\bullet) \cup T^-(\bullet)$ is a strongly connected spanning subgraph of $\Gamma$ and contains all essential edges of $\Gamma$. This implies that $\Gamma$ contains at most $2|V(\Gamma)| - 2$ essential edges.

If $\Gamma$ is stable and $|V(\Gamma)| \geq 2$, then $2|E(\Gamma)| \geq 4(|V(\Gamma)| - 1) + 2$, i.e., $|E(\Gamma)| \geq 2|V(\Gamma)| - 2$. Thus $\Gamma$ must contain at least one redundant edge.

If $\Gamma$ is merely semistable, consider its stabilization graph $\Gamma^s$, which must contain a redundant edge $e$. It is not difficult to see that $e$ is also redundant in $\Gamma$. \qed

Definition 2.7. For any pointed graph $\Gamma$, we introduce a graph invariant $\varphi(\Gamma)$ as follows:

(i) if $\Gamma$ is not strongly connected, then $\varphi(\Gamma) = 0$,

(ii) if $\Gamma$ has only one vertex with $l$ loops, then $\varphi(\Gamma) = l!$,

(iii) if there is an ordinary vertex $v \in V(\Gamma_-)$ that satisfies $\deg^+(v) = \deg^-(v) = 1$, denote by $\Gamma/\{v\}$ the graph obtain by smoothing out $v$ in $\Gamma$ (i.e., removing $v$ and connecting its two neighboring vertices), then $\varphi(\Gamma) = \varphi(\Gamma/\{v\})$,

(iv) if $\Gamma$ is a strongly connected pointed graph, then

\[
\varphi(\Gamma) = \sum_{e \in E(\Gamma)} \varphi(\Gamma - \{e\}),
\]

where $\Gamma - \{e\}$ denotes deleting an edge $e$ from $\Gamma$ while keeping the endpoints.

Remark 2.8. Define a \textit{strong reduction} of a strongly connected pointed graph $\Gamma$ to be a procedure: at each step removes a redundant edge and smooths out ordinary vertices $v \in V(\Gamma_-)$ with $\deg^+(v) = \deg^-(v) = 1$ until a single vertex $\bullet$ is reached. Then $\varphi(\Gamma)$ counts the number of all strong reductions of $\Gamma$. It is not difficult to see that a strong reduction of $\Gamma$ removes exactly $w(\Gamma)$ edges, since smoothing out a vertex reduces the number of edges by one. By Lemma 2.6, it is not difficult to see that $\varphi(\Gamma) > 0$ when $\Gamma$ is strongly connected.
Proof. First we introduce the operation of attaching a directed arc \( uv \). We have two ways of attaching the endpoints of \( \Gamma \), (i) attach them to any vertices of \( \Gamma \) by creating a new vertex at the attaching edge.

Recall that at each point \( x \) on a Kähler manifold, there exists a normal coordinate system such that at \( x \) the Kähler metric satisfies

\[
g_{ij}(x) = \delta_{ij}, \quad g_{ijkl\ldots k_r}(x) = g_{ijkl\ldots k_r}(x) = 0
\]

for all \( r \leq N \in \mathbb{N} \), where \( N \) can be chosen arbitrary large.

The curvature tensor is given by

\[
R_{ijkl} = -g_{ijkl} + g^{m\bar{p}}g_{m\bar{j}\bar{i}k}.
\]

The covariant derivative of a covariant tensor field \( T_{\beta_1\ldots \beta_p} \) is defined by

\[
T_{\beta_1\ldots \beta_p/\gamma} = \partial_\gamma T_{\beta_1\ldots \beta_p} - \sum_{i=1}^{p} \Gamma_{\gamma \beta_i}^\delta T_{\beta_1\ldots \beta_i-1\delta \beta_i+1\ldots \beta_p},
\]

The above two identities can be used to convert partial derivatives of metrics in (7,8) to covariant derivatives of curvature tensors and vice versa.

Thanks to the Kähler condition \( \partial_\gamma g_{ij} = \partial_j g_{i\bar{k}} \) and \( \partial_k g_{ij} = \partial_k g_{j\bar{i}} \), we can canonically associate a polynomial in the variables \( \{g_{ij} \} |\alpha| \geq 1 \) to a stable graph \( G \), such that each vertex represents a partial derivative of \( g_{ij} \) and each edge represents the contraction of a pair of barred and unbarred indices. Similarly a pointed graph \( \Gamma \) represents a differential operator.

**Proposition 2.9.** Let \( \square \) be the complex Laplacian defined by \( \square T = T_{ijj} \) for any covariant tensor \( T \). Then

\[
\square^k = \sum_{\Gamma:\, w(\Gamma) = k} \text{strong stable} (-1)^{|V(\Gamma)|-1} \frac{\varphi(\Gamma)}{\text{Aut}(\Gamma)} \Gamma,
\]

where \( \Gamma \) runs over all strongly connected stable pointed graphs of weight \( k \).

**Proof.** First we introduce the operation of attaching a directed arc \( uv \) to a graph \( \Gamma \). We have two ways of attaching the endpoints \( u, v \): (i) attach them to any vertices of \( \Gamma \), (ii) attach them to any edges of \( \Gamma \) by creating a new vertex at the attaching edge.

Given \( k \geq 0 \), denote by \( S_k \) the multiset of graphs obtained by attaching \( k \) edges consecutively to \( \bullet \) in all possible ways. Obviously all graphs in \( S_k \) are strongly connected. By (7), (8) and an argument similar to that of [24 §4], \( \square^k \) is equal to
bars which are arranged as follows:

- to a new vertex
- \(\partial g \partial g\)

Given a pointed graph \(\Gamma\) of weight \(k\), it is not difficult to see that there is a natural map from the set of all strong reductions of \(\Gamma\) (cf. Remark 2.8) to \(S_k\) by reversing the reduction procedure. Two strong reductions map have the same image in \(S_k\) if and only if they differ by an automorphism of \(\Gamma\), which implies that \(\Gamma\) occurs in \(S_k\) exactly \(\frac{\#G}{\|\text{Aut}(\Gamma)\|}\) times. So we conclude the proof. \(\square\)

**Lemma 2.10.** Let \(\Gamma\) be a pointed graph that satisfies \(\deg^+(\bullet) = \deg^-(\bullet)\) and has \(l\) loops at \(\bullet\). Denote \(m = \deg^+(\bullet) - l\). As a differential operator, \(\Gamma\) satisfies

\[
\Gamma \left[ \frac{|z|^{2(l+m)}}{(l+m)!} \right]_{z=0} = \frac{(l + m + d - 1)!}{(m + d - 1)!} \sum_{P \in \mathcal{P}(\Gamma)} G_P,
\]

where \(|z| = \sqrt{|z_1| + \cdots + |z_d|}\).

**Proof.** By a straightforward computation, it is reduced to prove the following combinatorial identity: for \(m_1 + \cdots + m_d = m\) with \(m_i \geq 0\),

\[
\sum_{a_1 + \cdots + a_d = 1} \prod_{j=1}^{d} \binom{m_j + a_j}{m_j} = \binom{l + m + d - 1}{m + d - 1}.
\]

The right-hand side has the following enumerative interpretation: in a sequence of \(l + m + d - 1\) symbols, we choose \(m + d - 1\) objects and make \(d - 1\) of them to be bars which are arranged as follows:

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
m_1 & m_2 & \cdots & m_d
\end{array}
\]

It is not difficult to see that all these arrangements are in one-to-one correspondence with that of the left-hand side. \(\square\)

**Definition 2.11.** If a pointed graph \(\Gamma\) satisfies \(\deg^+(\bullet) = \deg^-(\bullet)\) and has \(l\) loops at \(\bullet\), we define \(m(\Gamma) = \deg^+(\bullet) - l\). When we remove \(\bullet\) and all loops at \(\bullet\) from \(\Gamma\), then there are \(m(\Gamma)\) inward and \(m(\Gamma)\) outward loose ends; an inward loose end may be paired with an outward loose end to form an edge. Denote by \(\mathcal{P}(\Gamma)\) the set of all \(m(\Gamma)!\) possible complete pairings of these loose ends. Given a pairing \(P \in \mathcal{P}(\Gamma)\), denote by \(G_P\) the graph generated from the pairing \(P\). Note that two different complete pairings may produce isomorphic graphs.

**Definition 2.12.** Given a graph \(G\), we can arbitrary cut \(m\) edges of \(G\) and get \(2m\) loose ends, then we can construct a pointed graph by connecting all these loose ends to a new vertex \(\bullet\). Denote by \(\mathcal{C}(G)\) the set of all \(2|E(G)|\) possible edge-cuttings of \(G\). Given an edge-cutting \(C \in \mathcal{C}(G)\), denote by \(\mathcal{m}(C)\) the number of edges being cut and \(\Gamma_C\) the corresponding pointed graph obtained by connecting all loose ends to a new vertex \(\bullet\).

**Lemma 2.13.** Let \(\Gamma\) be a pointed graph that has no loops at the distinguished vertex \(\bullet\). Then

\[
\sum_{P \in \mathcal{P}(\Gamma)} G_P = \sum_{G} \frac{\#\{C \in \mathcal{C}(G) \mid \Gamma_C \cong \Gamma\}}{|\text{Aut}(G)|} G.
\]
Proof. The group $\text{Aut}(\Gamma)$ has a natural action on the set $\mathcal{P}(\Gamma)$ by permuting the loose ends (and thus the pairings) resulting from removing $\bullet$. It is not difficult to see that the set of orbits corresponds to isomorphism classes of graphs generated from the pairings, i.e., $\{G_P \mid P \in \mathcal{P}(\Gamma)\}$. Given a pairing $P \in \mathcal{P}(\Gamma)$ and $G = G_P$, denote by $E_P$ the set of edges in $G$ resulting from the pairing. Then the isotropy group at $P$ of the above action is $\text{Aut}(G)'$, which is the subgroup of $\text{Aut}(G)$ that leaves $E_P$ invariant. Note that $\text{Aut}(G)'$ is determined up to conjugacy in $\text{Aut}(G)$ for any $P \in \mathcal{P}(\Gamma)$ satisfying $G_P \cong G$. Note also that the set $\{C \in \mathcal{C}(G) \mid \Gamma_C \cong \Gamma\}$ is in one-to-one correspondence with the coset of $\text{Aut}(G)'$ in $\text{Aut}(G)$. Therefore

$$\sum_{P \in \mathcal{P}(\Gamma)} G_P = \sum_G \frac{|\text{Aut}(\Gamma)|}{|\text{Aut}(G)|} G = |\text{Aut}(\Gamma)| \sum_G \frac{\#\{C \in \mathcal{C}(G) \mid \Gamma_C \cong \Gamma\}}{|\text{Aut}(G)|} G,$$

as claimed. \hfill \square

Remark 2.14. (10) can be equivalently written as

$$(11) \quad \frac{\#\{P \in \mathcal{P}(\Gamma) \mid G_P \cong G\}}{|\text{Aut}(\Gamma)|} = \frac{\#\{C \in \mathcal{C}(G) \mid \Gamma_C \cong \Gamma\}}{|\text{Aut}(G)|}$$

for any graph $G$ and pointed graph $\Gamma$.

Before we prove our main result, we need a combinatorial lemma.

Lemma 2.15. Let $w \geq m \geq 1$ and $d \geq 0$. Then

$$(12) \quad \sum_{j=m}^{w} (-1)^j \binom{w+d}{j} \binom{j+d-1}{m+d-1} = (-1)^m.$$

Proof. Denote the left-hand side by $f(m, w)$. Obviously $f(m, m) = (-1)^m$. Using

$$\frac{(w+d)}{j+d} = \frac{(w+d)}{j+d} + \frac{(w+d)}{j+d-1},$$

it is not difficult to get

$$f(m, w+1) = f(m, w) + \frac{(w+d)!}{(m+d-1)!(w+1-m)!} \sum_{j=m}^{w+1} (-1)^j \binom{w+1-m}{w+1-j}.$$

Therefore (12) follows by induction. \hfill \square

Theorem 2.16. On a Kähler manifold $M$ of dimension $d$, the heat kernel coefficients of the (real) Laplacian is given by

$$(13) \quad a_n = \sum_{G : w(G)=n} ^{\text{stable}} \frac{(-1)^{|V(G)|} 2^n}{|\text{Aut}(G)|} \sum_{C \in \mathcal{C}(G)} \frac{(-1)^{m(C)} \varphi(\Gamma_C)}{(m(C) + n)!} G,$$

where $G$ runs over all stable graphs of weight $n$ and $C$ runs over all edge-cuttings of $G$ (cf. Definition 2.12).

Proof. By Remark 1.2, we can choose holomorphic coordinates $z = (z_1, \ldots, z_d)$ on $M$ centered at 0 such that the Kahler metric is locally Euclidean at 0 and $\text{dist}(0, z) = \sqrt{2} |z|$. The real Laplacian $\Delta$ and complex Laplacian $\Box$ are related by
\[ \Box = -2\Delta. \] By applying Proposition 2.9 and Lemma 2.10 to Polterovich’s formula (2). Note that we take \( w \geq 2n \) below.

\[
a_n = (-1)^n \sum_{j=0}^{w} \binom{w+d}{j+d} \frac{1}{4j!(j+n)!} (\Box)^j + n (z^2) \bigg|_{z=0} \\
= 2^n \sum_{j=0}^{w} \binom{w+d}{j+d} \frac{(-1)^j}{(j+n)!} \left[ |z|^{2j} \right]_{j=0} \\
= 2^n \sum_{j=0}^{w} \binom{w+d}{j+d} \frac{(-1)^j}{(j+n)!} \text{st. stab. loops at } \gamma \in (\Gamma) \\
\times \sum_{\Gamma: \ w(\Gamma) = j + n} \sum_{\phi, \deg^+ (\bullet) = \deg^- (\bullet) = j} \frac{(j + d - 1)!}{(m(\Gamma) + d - 1)!} \frac{(z^2)^{\phi}}{\Aut(\Gamma)} \sum_{P \in \mathcal{P}(\Gamma)} G_P.
\]

If \( \Gamma \) has \( l \) loops at \( \bullet \) and \( w(\Gamma) = j + n \), denote by \( \Gamma' \) the graph obtained from \( \Gamma \) by removing all loops at \( \bullet \). It is not difficult to see (cf. Definition 2.7) that \( \phi(\Gamma) = \phi(\Gamma') \) and \( \Aut(\Gamma) = \phi(\Gamma') \). Together with Lemma 2.13 and Lemma 2.15, we get

\[
a_n = 2^n \sum_{j=0}^{w} \binom{w+d}{j+d} (-1)^j \\
\times \sum_{\Gamma: \ w(\Gamma) = n} \sum_{m=1}^{w} \sum_{j=0}^{w} (-1)^j \left( \binom{w+d}{j+d} \right) \frac{j + d - 1}{m(\Gamma) + d - 1} \frac{(z^2)^{\phi}}{\Aut(\Gamma)} \sum_{P \in \mathcal{P}(\Gamma)} G_P \\
= \sum_{\text{st. stab. } \Gamma: \ w(\Gamma) = n} \sum_{m=1}^{w} \frac{(-1)^m}{m+1} \left( \frac{(z^2)^{\phi}}{\Aut(\Gamma)} \right) G \\
= \sum_{\text{st. stab. } \Gamma: \ w(\Gamma) = n} \frac{(-1)^m}{m+1} \left( \frac{(z^2)^{\phi}}{\Aut(\Gamma)} \right) G.
\]

The second to last equation used the crucial fact that \( w \geq 2n \geq |E(G)| \). \( \Box \)

Denote by \( z(G) \) the coefficient of \( G \) in (13),

\[
z(G) = \frac{(-1)^{|V(G)|} 2^{w(G)}}{|\Aut(G)|} \sum_{C \in \mathcal{C}(G)} (-1)^{m(C)} \phi(\Gamma_C). \]

Proposition 2.17. If $G$ is a disjoint union of connected subgraphs $G = \bigcup_{i=1}^{k} G_i$, then we have

$$z(G) = \prod_{j=1}^{k} z(G_j)/|\text{Sym}(G_1, \ldots, G_k)|,$$

where $\text{Sym}(G_1, \ldots, G_k)$ is the permutation group of the connected subgraphs.

Proof. For any two graphs $G_1, G_2$, we have $\mathcal{C}(G_1 \cup G_2) = \mathcal{C}(G_1) \times \mathcal{C}(G_2)$. Let

$$\tilde{z}(G) = \sum_{C \in \mathcal{C}(G)} (-1)^{|C|} \varphi(\Gamma_C)/(|C| + w(G))^!.$$

If $C_1 \in \mathcal{C}(G_1)$ and $C_2 \in \mathcal{C}(G_1)$, then

$$\varphi(\Gamma_{C_1 \times C_2}) = \left(\frac{m(C_1) + m(C_2) + w(G_1) + w(G_2)}{m(C_1) + w(G_1)}\right).$$

Now it is easy to see that $\tilde{z}(G_1 \cup G_2) = \tilde{z}(G_1) \tilde{z}(G_2)$, which implies (14). \qed

3. Computations of $a_k$ for $k \leq 3$

On a Kähler manifold, we denote by $\mathcal{R}_{ijkl}, \mathcal{Ric}_{ij}, \mathcal{P}$ the Riemannian curvature tensor, Ricci curvature tensor and scalar curvature respectively, in the sense of Riemannian geometry, and denote by $R_{ijkl}, Ric_{ij}, \rho$ the corresponding Kählerian curvature tensors. A proof of the following lemma is in the appendix.

Lemma 3.1. On a Kähler manifold, we have the identities $|\mathcal{R}|^2 = 4|R|^2$, $|\mathcal{Ric}|^2 = 2|Ric|^2$ and $\mathcal{P} = 2\rho$.

Example 3.2. We represent a digraph as a weighted graph. The weight of a directed edge is the number of multi-edges. The number attached to a vertex denotes the number of its self-loops. A vertex without loops is denoted by a small circle.

There is only one stable graph with weight 1. By (13),

$$a_1 = -\frac{1}{3} [\{2\}] = -\frac{1}{3} g_{i\bar{i}j} = \frac{1}{3} R_{i\bar{i}j} = \frac{1}{3} \rho,$$

where $(i, \bar{i}), (j, \bar{j})$ are paired indices to be contracted.

There are four stable graphs with weight 2. By (13),

$$a_2 = -\frac{2}{15} [\{3\}] + \frac{1}{18} [\{2\} | \{2\}] + \frac{23}{90} \left[ \begin{array}{c} 1 \circ \circ \\ 1 \circ \circ \\ \end{array} \right] + \frac{7}{45} \left[ \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \\ \end{array} \right].$$

$$= -\frac{2}{15} g_{i\bar{i}j\bar{k}} + \frac{1}{18} g_{i\bar{i}j} g_{k\bar{k}l} + \frac{23}{90} g_{i\bar{i}k} g_{j\bar{j}l} + \frac{7}{45} g_{i\bar{i}j} g_{j\bar{j}k}

= -\frac{2}{15} (-\Box \rho + |R|^2 + 2|Ric|^2) + \frac{1}{18} \rho^2 + \frac{23}{90} |Ric|^2 + \frac{7}{45} |R|^2

= \frac{2}{15} \Box \rho + \frac{1}{18} \rho^2 - \frac{1}{90} |Ric|^2 + \frac{1}{45} |R|^2.
By Lemma 3.1 and the relation of real and complex Laplacians $\Delta = -2\Box$, we get the well-known $a_1, a_2$ in Riemannian tensors.

$$a_1 = \frac{1}{6}P, \quad a_2 = -\frac{1}{30}\Delta P + \frac{1}{72}P^2 - \frac{1}{180}|\text{Ric}|^2 + \frac{1}{180}|\mathbb{R}|^2.$$ 

**Example 3.3.** Now we compute $a_3$. There are fifteen stable graphs of weight 3.

$$\tau_1 = [2 \mid 2 \mid 2], \quad \tau_2 = \begin{bmatrix} 1 & 1 & 1 \mid 2 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 0 & 2 & 0 \mid 2 \end{bmatrix},$$

$$\tau_4 = \begin{bmatrix} 1 & 1 & 1 \mid 1 \end{bmatrix}, \quad \tau_5 = \begin{bmatrix} 1 & 1 & 1 \mid 1 \end{bmatrix}, \quad \tau_6 = \begin{bmatrix} 1 & 1 & 1 \mid 1 \end{bmatrix},$$

$$\tau_7 = \begin{bmatrix} 1 & 1 & 1 \mid 1 \end{bmatrix}, \quad \tau_8 = [2 \mid 3], \quad \tau_9 = \begin{bmatrix} 1 & 1 & 2 \mid 1 \end{bmatrix},$$

$$\tau_{10} = \begin{bmatrix} 2 & 2 & 0 \mid 0 \end{bmatrix}, \quad \tau_{11} = \begin{bmatrix} 2 & 1 & 2 \mid 1 \end{bmatrix}, \quad \tau_{12} = \begin{bmatrix} 1 & 1 & 2 \mid 1 \end{bmatrix},$$

$$\tau_{13} = \begin{bmatrix} 0 & 3 & 0 \mid 2 \end{bmatrix}, \quad \tau_{14} = [4], \quad \tau_{15} = \begin{bmatrix} 2 & 0 & 2 \mid 0 \end{bmatrix}. $$

By (13), we have

$$a_3 = z_1 \tau_1 + z_2 \tau_2 + \cdots + z_{15} \tau_{15},$$

where the coefficients are given by

$$z_1 = \frac{1}{162}, \quad z_2 = \frac{23}{270}, \quad z_3 = \frac{7}{135}, \quad z_4 = \frac{17}{135}, \quad z_5 = \frac{332}{945},$$

$$z_6 = -\frac{307}{2835}, \quad z_7 = \frac{274}{405}, \quad z_8 = \frac{2}{45}, \quad z_9 = \frac{64}{315}, \quad z_{10} = \frac{26}{105},$$

$$z_{11} = \frac{17}{630}, \quad z_{12} = \frac{89}{315}, \quad z_{13} = \frac{1}{10}, \quad z_{14} = \frac{1}{35}, \quad z_{15} = \frac{206}{2835}.$$ 

We will express $a_3$ as a linear combination of the following independent invariants as used in [5 p. 23],

$$\sigma_1 = \rho^3, \quad \sigma_2 = \rho R_{ij} R_{ji}, \quad \sigma_3 = \rho R_{ijkl} R_{ijkl},$$

$$\sigma_4 = R_{ijkl} R_{ijk}, \quad \sigma_5 = R_{ijkl} R_{klmn}, \quad \sigma_6 = R_{ijkl} R_{jkl},$$

$$\sigma_7 = R_{ijkl} R_{ijk} R_{klm}, \quad \sigma_8 = \rho \rho_i \rho_j \rho_k,$$

$$\sigma_9 = \rho R_{ijk}, \quad \sigma_{11} = \rho R_{ijkl} R_{klm}, \quad \sigma_{12} = R_{ijkl} R_{jkl},$$

$$\sigma_{13} = R_{ijkl} R_{jkl}, \quad \sigma_{14} = \Box^2 \rho, \quad \sigma_{15} = R_{ijkl} R_{jkl} R_{klm}.$$ 

We have computed in [22 Appx. A] that

$$\tau_1 = -\sigma_1, \quad 1 \leq i \leq 7,$$

$$\tau_8 = -2\sigma_2 - \sigma_3 + \sigma_8, \quad \tau_9 = -\sigma_4 - \sigma_5 - \sigma_6 + \sigma_9, \quad \tau_{10} = -2\sigma_5 + \sigma_{10} - \sigma_{15},$$

$$\tau_{11} = \sigma_{11}, \quad \tau_{12} = \sigma_{12}, \quad \tau_{13} = \sigma_{13},$$
\[ \tau_{14} = -3\sigma_4 - 12\sigma_5 - 3\sigma_6 + 6\sigma_7 + 7\sigma_9 + 8\sigma_{10} + 10\sigma_{13} - \sigma_{14} - 6\sigma_{15}, \]

\[ \tau_{15} = -\sigma_{15}. \]

Substitute these into (16), we get

\[ (17) \quad a_3 = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_{15}\sigma_{15}, \]

where the coefficients are given by

\[
\begin{align*}
  c_1 & = \frac{1}{162}, & c_2 & = -\frac{1}{270}, & c_3 & = \frac{1}{135}, & c_4 & = \frac{8}{945}, & c_5 & = -\frac{4}{945}, \\
  c_6 & = -\frac{26}{2835}, & c_7 & = \frac{32}{2835}, & c_8 & = \frac{2}{45}, & c_9 & = \frac{1}{315}, & c_{10} & = \frac{2}{105}, \\
  c_{11} & = \frac{17}{630}, & c_{12} & = -\frac{1}{315}, & c_{13} & = \frac{1}{70}, & c_{14} & = \frac{1}{35}, & c_{15} & = -\frac{2}{567}.
\end{align*}
\]

With a little more work, our result can be combined with some well-known heat kernel coefficients or invariants on special manifolds (e.g., spheres) to get Gilkey’s formula [6] of \(a_3\) on general Riemannian manifolds.

**Appendix A. Comparison of Riemannian and Kähler curvature tensors**

We give a proof of Lemma 3.1. Denote by \(J\) the complex structure. If \(e_1, \ldots, e_{2d}\) is an orthonormal basis such that \(Je_i = e_{d+i}\) for \(i = 1, \ldots, d\), then \(y_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i), 1 \leq i \leq d\) is a unitary basis. By using \(R(u, v, w, y) = R(Ju, Jv, w, y) = R(u, v, Jw, Jy)\), it is not difficult to get

\[ (18) \quad R(y_i, \bar{y}_j, y_k, \bar{y}_l) = R(e_i, e_j, e_k, e_l) + R(e_i, Je_j, Je_k, e_l) \]

\[ - \sqrt{-1}R(Je_i, e_j, e_k, e_l) - \sqrt{-1}R(e_i, e_j, Je_k, e_l). \]

By definition, we have

\[
\begin{align*}
|R|^2 &= \sum_{i,j,k,l=1}^{d} R(y_i, \bar{y}_j, y_k, \bar{y}_l) R(y_j, \bar{y}_i, y_l, \bar{y}_k) \\
&= \sum_{i,j,k,l=1}^{d} R(y_i, \bar{y}_j, y_k, \bar{y}_l) R(y_i, \bar{y}_j, y_k, \bar{y}_l) \\
&= \sum_{i,j,k,l=1}^{d} [R(e_i, e_j, e_k, e_l) + R(e_i, Je_j, Je_k, e_l)]^2 \\
&\quad + \sum_{i,j,k,l=1}^{d} [R(Je_i, e_j, e_k, e_l) + R(e_i, e_j, Je_k, e_l)]^2 \\
&= \sum_{i,j,k,l=1}^{d} [R(e_i, e_j, e_k, e_l)]^2 + R(e_i, Je_j, Je_k, e_l)]^2 \\
&\quad + R(Je_i, e_j, e_k, e_l)^2 + R(e_i, e_j, Je_k, e_l)^2. 
\end{align*}
\]
The last equality follows from

\[(19) \quad \sum_{i,j,k,l=1}^{d} R(e_i, e_j, e_k, e_l)R(e_i, Je_j, Je_k, e_l) = 0,\]

\[(20) \quad \sum_{i,j,k,l=1}^{d} R(Je_i, e_j, e_k, e_l)R(e_i, e_j, Je_k, e_l) = 0.\]

We prove the first identity, the proof of second identity is similar.

\[\sum_{i,j,k,l=1}^{d} R(e_i, e_j, e_k, e_l)R(e_i, Je_j, Je_k, e_l)\]

\[= \sum_{i,j,k,l=1}^{d} R(e_i, e_j, e_k, e_l)R(e_i, Je_j, Je_k, e_l)\]

\[= - \sum_{i,j,k,l=1}^{d} R(e_i, e_j, e_k, e_l)R(e_i, Je_j, e_k, e_l)\]

\[= - \sum_{i,j,k,l=1}^{d} R(e_i, e_j, e_k, e_l)R(e_i, Je_j, Je_k, e_l),\]

which implies (19). On the other hand,

\[|\mathcal{R}|^2 = \sum_{i,j,k,l=1}^{d} [R(e_i, e_j, e_k, e_l)^2 + R(Je_i, Je_j, e_k, e_l)^2 + R(e_i, e_j, Je_k, e_l)^2 + R(Je_i, Je_j, e_k, e_l)^2]\]

\[+ \sum_{i,j,k,l=1}^{d} [R(e_i, Je_j, e_k, e_l)^2 + R(e_i, Je_j, Je_k, e_l)^2 + R(Je_i, e_j, e_k, e_l)^2 + R(Je_i, e_j, Je_k, e_l)^2]\]

\[+ \sum_{i,j,k,l=1}^{d} [R(Je_i, e_j, e_k, e_l)^2 + R(Je_i, e_j, Je_k, e_l)^2 + R(e_i, Je_j, e_k, e_l)^2 + R(e_i, Je_j, Je_k, e_l)^2]\]

\[+ \sum_{i,j,k,l=1}^{d} [R(e_i, e_j, Je_k, e_l)^2 + R(e_i, e_j, Je_k, e_l)^2 + R(Je_i, Je_j, e_k, e_l)^2 + R(Je_i, Je_j, Je_k, e_l)^2]\]

\[= 4 \sum_{i,j,k,l=1}^{d} [R(e_i, e_j, e_k, e_l)^2 + R(e_i, e_j, Je_k, e_l)^2 + R(Je_i, e_j, e_k, e_l)^2 + R(Je_i, e_j, Je_k, e_l)^2]\]

\[= 4|\mathcal{R}|^2.\]

Next we look at the Ricci curvature. By applying a unitary transformation if necessary, we may assume that both $\text{Ric}(y_i, \bar{y}_j)_{1 \leq i,j \leq d}$ and $\text{Ric}(e_i, e_j)_{1 \leq i,j \leq 2d}$ are
diagonal matrices. From \cite{18}, we have

\[\text{Ric}(y_i, \bar{y}_i) = \sum_{k=1}^{d} R(y_i, \bar{y}_i, y_k, \bar{y}_k)\]

\[= -\sum_{k=1}^{d} R(e_i, Je_i, e_k, Je_k)\]

\[= \sum_{k=1}^{d} R(Je_i, e_k, Je_k) + \sum_{k=1}^{d} R(e_i, e_k, Je_k)\]

\[= \sum_{k=1}^{d} R(e_i, Je_k, Je_k) + \sum_{k=1}^{d} R(e_i, e_k, e_i)\]

\[= \text{Ric}(e_i, e_i)\]

where we used the first Bianchi identity in the third equality. Similarly we can prove \(\text{Ric}(y_i, \bar{y}_i) = \text{Ric}(Je_i, Je_i)\). Therefore \(|\text{Ric}|^2 = 2|\text{Ric}|^2\) and \(\mathcal{P} = 2\rho\).

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Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China; Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095-1555, USA

*E-mail address*: liu@math.ucla.edu, liu@cms.zju.edu.cn

Department of Mathematics, University of Pittsburgh, 301 Thackery Hall, Pittsburgh, PA 15260, USA

*E-mail address*: mathxuhao@gmail.com