Unimodal singularities and boundary divisors in the KSBA moduli of a class of Horikawa surfaces

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Abstract
Smooth minimal surfaces of general type with $K^2 = 1$, $p_g = 2$, and $q = 0$ constitute a fundamental example in the geography of algebraic surfaces, and the 28-dimensional moduli space $\mathcal{M}$ of their canonical models admits a modular compactification $\overline{\mathcal{M}}$ via the minimal model program. We describe eight new irreducible boundary divisors in such compactification parameterizing reducible stable surfaces. Additionally, we study the relation with the GIT compactification of $\mathcal{M}$ and the Hodge theory of the degenerate surfaces that the eight divisors parameterize.

KEYWORDS
Hodge theory, Horikawa surface, moduli space, stable pair compactification, unimodal singularity

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1 | INTRODUCTION

The interplay between geometric compactifications and Hodge theory is one of the driving forces in moduli theory. Well-studied cases include abelian varieties [2], K3 surfaces [4–8], algebraic curves [13], cubic surfaces [28], and cubic fourfolds [47]. Recently, there has been a great focus on generalizing this interplay in the case of surfaces of general type—the so-called nonclassical cases [32, 40]. A particular well-posed case for such generalization is given by algebraic surfaces...
TABLE 1 Geometric features of the stable surfaces $S_\Sigma$.

| Sing. $\Sigma$ | $E_{12}$ | $E_{13}$ | $E_{14}$ | $Z_{11}$ | $Z_{12}$ | $Z_{13}$ | $W_{12}$ | $W_{13}$ |
|---------------|---------|---------|---------|---------|---------|---------|---------|---------|
| No. $\tilde{Y}_\Sigma$ in [38, section 13.3] | 88 | 70 | 53 | 71 | 51 | 35 | 41 | 30 |
| $K^2_{\tilde{Z}_\Sigma}$ | 19 | 4 | 2 | 5 | 2 | 7 | 3 | 1 |

See Section 3.5

of general type with geometric genus $p_g = 2$, irregularity $q = 0$, and $K^2 = 1$. These were described by Enriques [25] and later studied by Horikawa [37]. In this work, we refer to these surfaces, which are sometimes called I-surfaces, simply as Horikawa surfaces. By the work of Gieseker [30], the canonical models of such surfaces form a 28-dimensional quasi-projective coarse moduli space $\mathcal{M}$.

In this paper, we consider the compactification perspective offered by the minimal model program. By the work of Kollár, Shepherd-Barron, and Alexeev [1, 43, 45], there exists a projective and modular compactification parameterizing stable surfaces with $K^2 = 1$ and $\chi(\mathcal{O}) = 3$, which contains $\mathcal{M}$. This is also known as the KSBA compactification, and in this specific case has several irreducible components by [27]. In this work, we focus on the main irreducible component $\overline{\mathcal{M}}$ of this compactification, which parameterizes smoothable surfaces. Our first contribution toward the understanding of $\overline{\mathcal{M}}$ is the following theorem that combines Theorems 3.15 and 4.2.

**Theorem 1.1.** Let $\Sigma$ be any of the following eight non–log canonical isolated unimodular surfaces singularities:

$$E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13},$$

which are classified by [10] and appear as the unique singularity of the degenerations of Horikawa surfaces described in Definition 3.7. Then, there exists an irreducible boundary divisor $D_\Sigma \subseteq \overline{\mathcal{M}}$ generically parameterizing stable surfaces $S_\Sigma$, which are the gluing of two surfaces $\tilde{Y}_\Sigma$ and $\tilde{Z}_\Sigma$ along a $\mathbb{P}^1$. The surfaces $\tilde{Y}_\Sigma$ and $\tilde{Z}_\Sigma$ satisfy $h^2(\mathcal{O}) = 1$, $h^1(\mathcal{O}) = 0$, and have only finite cyclic quotient singularities. Moreover, $\tilde{Y}_\Sigma$ is an ADE K3 hypersurface within a weighted projective space, see Table 1.

The stable limits of surfaces in Theorem 1.1 are constructed in Section 3 via explicit stable replacement. In Section 4, we prove that these degenerate stable surfaces describe divisors in the moduli space. In fact, we prove a bit more. Recall that a smooth Horikawa surface has $h^{1,1} = 29$. For each of the singularity types listed in Theorem 1.1, the subscript is equal to the Milnor number $\mu_\Sigma$ of the singularity (e.g., $E_{12}$ has $\mu_{E_{12}} = 12$). The moduli count for the surfaces $\tilde{Y}_\Sigma$ is $\mu_\Sigma - 2$, whereas the moduli count for the surfaces $\tilde{Z}_\Sigma$ is $29 - \mu_\Sigma$. The fact that $D_\Sigma$ is a divisor corresponds to the fact that we can deform $\tilde{Y}_\Sigma$ and $\tilde{Z}_\Sigma$ independently. In Corollaries 3.26 and 3.30, we show that $\chi_{\text{top}}(\tilde{Y}_\Sigma) = \mu_\Sigma + 3$ and $\chi_{\text{top}}(\tilde{Z}_\Sigma) = 36 - \mu_\Sigma$. Therefore, $\chi_{\text{top}}(\tilde{Y}_\Sigma \cup \tilde{Z}_\Sigma) = (36 - \mu_\Sigma) + (\mu_\Sigma + 3) - \chi_{\text{top}}(\mathbb{P}^1) = 37$.

**Remark 1.2.** Our work is complementary to the work of Coughlan, Franciosi, Pardini, Rana, and Rollenske. In [26], the authors described the locus $\mathcal{M}^{\text{Gor}} \subseteq \overline{\mathcal{M}}$ parameterizing Gorenstein stable surfaces. The dimension of the boundary of $\mathcal{M}^{\text{Gor}}$ is 20 (not pure), and it parameterizes surfaces with at worst elliptic singularities of degree 1 and 2 (also known as $E_8$ and $E_7$, respectively). In [27], the authors found two boundary divisors $D_1, D_2 \subseteq \overline{\mathcal{M}}$ generically parameterizing stable surfaces with precisely one isolated singularity of type $\frac{1}{4}(1, 1)$ and $\frac{1}{18}(1, 5)$ (see also [34, Example 1.3.1]). In [17], the authors identified a third boundary divisor $D_3$ generically parameterizing stable surfaces with a $\frac{1}{25}(1, 14)$ singularity and of cuspidal type. The divisor $D_3$ is precisely the intersection of $\overline{\mathcal{M}}$ with another irreducible component of the moduli space of stable surfaces with $K^2 = 1$ and $\chi(\mathcal{O}) = 3$ (the stable surfaces generically parameterized by such a component were first constructed in [54]). We remark that the surfaces generically parameterized by $D_1, D_2, D_3$ are irreducible, while the ones generically parameterized by $D_2$ have two irreducible components.

The boundary divisors $D_2$ are not necessarily the only strata in the KSBA compactification $\overline{\mathcal{M}}$ associated to the singularities $\Sigma$. As discussed above, a full description of this compactification is currently an effort lead by multiple groups, for example, [17, 18, 26, 27].

For our second result, we observe that there is a geometric invariant theory (GIT) compactification $\overline{\mathcal{M}}^\text{git}$ of the moduli space $\mathcal{M}$, see [61]. Therefore, it is natural to compare it with $\overline{\mathcal{M}}$. We show that $\overline{\mathcal{M}}^\text{git}$ essentially forgets the information contained in $\tilde{Y}_\Sigma$. More precisely, in Section 5.2, we prove the following.
Theorem 1.3. The birational map $\overline{M} \to \overline{M}^{gb}$ given by the identity on the interior $\overline{M}$ of the compactifications extends to a dense open subset of the boundary divisors $D_{\Sigma}$. If $f$ denotes such an extension, then the relative dimension of $f |_{D_{\Sigma}}$ equals $\mu_{\Sigma} - 2$.

Hodge theoretically, the degenerations generically parameterized by the boundary divisors $D_{\Sigma}$ constructed in this paper are the analogs of curves of compact type. More precisely, the period map has finite monodromy about the generic point of each $D_{\Sigma}$. Accordingly, the period map extends holomorphically across one-dimensional arcs meeting $D_{\Sigma}$ transversely at a generic point. More generally, in Section 6, we prove the following results, which are applicable in our setting.

Theorem 1.4 (Theorem 6.1). Let $\pi : S \to \Delta$ be a one-parameter degeneration of complex projective surfaces, which is smooth over $\Delta^* = \Delta \setminus \{0\}$ such that

(a) if $t \neq 0$, then $S_t = \pi^{-1}(t)$ has geometric genus 2;
(b) the central fiber $S_0 = \pi^{-1}(0)$ is the union of two irreducible components $\tilde{Y}$ and $\tilde{Z}$, each of which has $h^2(\mathcal{O}) = 1$ and at worst rational singularities.

Then, the local system $V_{\mathbb{Q}} = R^2\pi_*(\mathbb{Q})$ over $\Delta^*$ has finite monodromy.

To determine the limit mixed Hodge structure of Theorem 1.4, we recall that given a semistable degeneration $X' \to \Delta$, the Clemens–Schmid sequence relates the mixed Hodge structure of the central fiber $X_0$ with the limit mixed Hodge structure of the period map, which is constructed on a generic fiber $X_\eta$. We also recall that one of the basic Hodge theoretic birational invariants of a complex projective surface $X$ is its transcendental lattice $T(\mathbb{Q})$. Let $T[\mathbb{H}_2(\mathbb{Q})]$ denote the underlying rational Hodge structure of $T(\mathbb{Q})$.

Theorem 1.5 (Corollary 6.11). Let $S \to \Delta$ be as in Theorem 1.4, and assume that $H^2(\tilde{Y}, \mathbb{Q})$ and $H^2(\tilde{Z}, \mathbb{Q})$ are pure of weight 2. Let $\tilde{S} \to \tilde{\Delta}$ be a semistable degeneration obtained from $S \to \Delta$ via a composition of covers and birational modifications of the central fiber, so that its central fiber $\tilde{S}_0$ is reduced and simple normal crossing. Also denote by $H^2_{\text{lim}}(\tilde{S}_\eta, \mathbb{Q})$ the limit mixed Hodge structure on a generic fiber $\tilde{S}_\eta$ of $\tilde{S} \to \tilde{\Delta}$. Then,

$$T[H^2_{\text{lim}}(\tilde{S}_\eta, \mathbb{Q})] \cong T[H^2(\tilde{Z}, \mathbb{Q})] \oplus T[H^2(\tilde{Y}, \mathbb{Q})],$$

where $T[A]$ is the transcendental part of a $\mathbb{Q}$-Hodge structure $A$ of weight 2 with $F^3A = 0$ (cf. Definition (6.6)).

Remark 1.6. In our setting (see Theorem 1.1), $H^2(\tilde{Z}_\Sigma)$ carries a pure Hodge structure of weight 2 since $\tilde{Z}_\Sigma$ is a V-manifold as it has only finite quotient singularities [53]. Likewise, the Hodge structure on $H^2(\tilde{Y}_\Sigma)$ is pure because $\tilde{Y}_\Sigma$ is an ADE K3 surface.

In the case of singularities of type $Z_{11}, Z_{12}, Z_{13}, W_{12}$, and $W_{13}$, the associated surface $\tilde{Z}_\Sigma$ appearing in Theorem 1.1 is birational to a K3 surface, which can be presented as the double cover of $\mathbb{P}^2$ branched along a sextic curve $C_\Sigma$. The geometry of the curves $C_\Sigma$ is described in Proposition 7.1. The singular locus of a generic curve $C_\Sigma$ is $\emptyset, \{A_1\}, \{A_2\}, \emptyset, \{A_1\}$ if $\Sigma = Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}$, respectively.

2 | PRELIMINARIES

2.1 | Horikawa surfaces and their moduli space

Recall from the introduction that we call Horikawa surface a minimal smooth projective surface $S$ of general type satisfying

$$h^2(\mathcal{O}_S) = 2, h^1(\mathcal{O}_S) = 0, \text{ and } K^2_S = 1.$$

In the literature, these specific surfaces are also called I-surfaces. For such a surface $S$, the divisor $2K_S$ induces a degree 2 morphism $S \to \mathbb{P}(1,1,2)$ with branch curve given by the vanishing of a weighted degree 10 polynomial $F_{10}(x,y,z)$.
Therefore, if \([x : y : z : w]\) are the coordinates in \(\mathbb{P}(1, 1, 2, 5)\), then \(S\) is isomorphic to the weighted degree 10 hypersurface in \(\mathbb{P}(1, 1, 2, 5)\) given by \(w^2 = F_{10}(x, y, z)\).

Following the approach in [61], we review the construction of the moduli space of Horikawa surfaces and its GIT compactification.

**Definition 2.1.** Let \(F_{10}(x, y, z)\) be a weighted degree 10 polynomial such that the coefficient of \(z^5\) is nonzero. Up to rescaling the coefficients of \(F_{10}(x, y, z)\) by an element of \(\mathbb{C}^*\), we can assume that such a coefficient is 1. We can then expand \(F_{10}(x, y, z)\) as follows:

\[
F_{10}(x, y, z) = z^5 + q_2(x, y)z^4 + q_4(x, y)z^3 + q_6(x, y)z^2 + q_8(x, y)z + q_{10}(x, y),
\]

where \(q_d(x, y)\) are homogeneous polynomials of degree \(d \in \{2, 4, 6, 8, 10\}\). After applying the following invertible change of coordinates (also known as a Tschirnhaus transformation):

\[
x = \tilde{x}, \ y = \tilde{y}, \ z = \tilde{z} - \frac{1}{5} q_2(\tilde{x}, \tilde{y}),
\]

we obtain

\[
G_{10}(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{z}^5 + g_4(\tilde{x}, \tilde{y})\tilde{z}^3 + g_6(\tilde{x}, \tilde{y})\tilde{z}^2 + g_8(\tilde{x}, \tilde{y})\tilde{z} + g_{10}(\tilde{x}, \tilde{y}),
\]

for some new homogeneous polynomials \(g_d(\tilde{x}, \tilde{y})\) of degree \(d \in \{4, 6, 8, 10\}\). We call the above degree 10 polynomial and the corresponding hypersurface \(V(w^2 - G_{10}(\tilde{x}, \tilde{y}, \tilde{z})) \subseteq \mathbb{P}(1, 1, 2, 5)\) in normal form.

**Definition 2.2.** The vector space of coefficients for a polynomial in normal form

\[
F_{10}(x, y, z) = z^5 + q_4(x, y)z^3 + q_6(x, y)z^2 + q_8(x, y)z + q_{10}(x, y)
\]

is the vector space

\[
\mathcal{V}_{10} = \bigoplus_{k=2}^{5} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2k)) \cong \mathbb{C}^{32}.
\]

Let \(U\) be the open subset of points in the affine space \(\text{Spec}(\text{Sym}(\mathcal{V}_{10}'))) \cong \mathbb{A}^{32}\) parameterizing smooth Horikawa surfaces in normal form. There is a natural \(\text{GL}_2\)-action on \(U\) given by a linear change of coordinate in \(x\) and \(y\). By [61, Lemma 4 and Lemma 7], we have that the points of the 28-dimensional quotient \(M = U/\text{GL}_2\) are in bijection with the isomorphism classes of Horikawa surfaces considered. We refer to \(M\) as the coarse moduli space of Horikawa surfaces. This coincides with the Gieseker moduli space of canonical surfaces \(S\) of general type with \(K_S^2 = 1, p_g(S) = 2,\) and \(q(S) = 0\) [30]. The family of Horikawa surfaces is defined over \(U\) by the relative equation

\[
H' = V(w^2 - z^5 - g_4(x, y)z^3 - g_6(x, y)z^2 - g_8(x, y)z - g_{10}(x, y)) \subseteq U \times \mathbb{P}(1, 1, 2, 5),
\]

with proper flat morphism \(H' \rightarrow U\) given by the restriction of the projection onto the first factor. This family descends to the geometric quotient by \(\text{GL}_2\) giving the family of Horikawa surfaces \(H \rightarrow M\) with pairwise nonisomorphic fibers.

**Definition 2.3.** As discussed in [61, Proposition 9], we have a natural projective GIT compactification of \(M\) given by

\[
\overline{M}^{\text{git}} := \mathbb{A}^{32} \bigsqcup \mathbb{A}^{32}/\mathbb{C}^* \bigr/ \mathbb{A}^{32}/\mathbb{C}^* \bigsqcup \mathbb{P}(4^5, 6^7, 8^9, 10^{11}) \bigsqcup \text{SL}_2,
\]

where the GIT quotients are with respect to appropriate linearizations (see [61]) and \(\mathbb{P}(4^5, 6^7, 8^9, 10^{11})\) denotes a weighted projective space with \(n^m\) representing \(n, \ldots, n\) repeated \(m\)-times.

### 2.2 Stable pair compactification

By the work of Kollár, Shepherd-Barron, and Alexeev [1, 45], we can construct a projective compactification \(\overline{M} \subseteq \overline{M}\), which is a coarse moduli space parameterizing stable surfaces. Let us review the main definitions of interest.
TABLE 2  Local models of the eight isolated non–log canonical singularities of modality 1 that can be attained as degeneration in \( \mathbb{P}(1, 1, 2, 5) \) of Horikawa surfaces.

|   |   |   |   |
|---|---|---|---|
| \( E_{12} \) & \( z^3 + y^7 + ay^3z \) & \( Z_{11} \) & \( yz^3 + y^5 + ay^4z \) & \( W_{12} \) & \( z^4 + y^3 + ay^3z^2 \) |
| \( E_{13} \) & \( z^3 + y^5z + ay \) & \( Z_{12} \) & \( yz^3 + y^4z + ay^3z^2 \) & \( W_{13} \) & \( z^4 + y^4z + ay^6 \) |
| \( E_{14} \) & \( z^3 + y^8 + ay^6z \) & \( Z_{13} \) & \( yz^3 + y^6 + ay^5z \) & \( W_{14} \) & \( z^4 + y^6z + ay^3z^2 \) |

**Definition 2.4.** Let \( X \) be a variety and \( \sum b_i B_i \) a \( \mathbb{Q} \)-divisor on \( X \) with \( 0 < b_i \leq 1 \) and \( B_i \) prime divisors. Then, the pair \((X, B)\) is called stable provided it has semilog canonical singularities [42, Definition–Lemma 5.10] and \( K_X + B \) is ample. If \( B = 0 \), then \( X \) is called a stable variety.

Stable pairs with divisor \( B = 0 \) can be used to construct a geometric, functorial, and projective compactification of \( M \). Here, we follow [3, Definitions 1.4.2, 1.4.3].

**Definition 2.5.** Let \( d, N \) be positive integers and \( C \) a positive rational number. For any reduced complex scheme \( S \), define \( \overline{V}(S) = \overline{V}_{d,N,C}(S) \) to be the set of proper flat families \( \mathcal{X} \to S \) with the following properties:

1. Every geometric fiber \( X_s \) is a stable variety, \( \dim(X_s) = d \), and \( K_{X_s}^d = C \).
2. There exists an invertible sheaf \( L \) on \( \mathcal{X} \) such that for every geometric fiber \( X_s \), \( L|_{X_s} \cong \mathcal{O}_{X_s}(N(K_{X_s})) \).

The above stack \( \overline{V} \) is called the Viehweg moduli stack. In the notation of [43, section 8], this moduli functor is \( SP(0, d, C) \).

**Definition 2.6.** Consider the Viehweg moduli stack \( \overline{V} \) for \( d = 2, C = 1 \), and \( N \) large enough (see Definition 4.1). Let \( \overline{V} \) be the corresponding projective coarse moduli space. The family of Horikawa surfaces \( H \to M \) in Definition 2.2 induces a morphism \( h : M \to \overline{V} \), which is injective on \( \mathbb{C} \)-points. We denote by \( \overline{M} \) the normalization of the closure of the image of \( h \) in \( \overline{V} \), and we will refer to \( \overline{M} \) as the KSBA compactification of the moduli space of Horikawa surfaces.

**Remark 2.7.** Equivalently, we could have defined \( \overline{M} \) using the Kollár moduli stack [3, Definition 1.4.2]. We chose to work with \( \overline{V} \) instead because it simplifies the discussion when constructing the boundary divisors \( D_\Sigma \) in Definition 4.1.

### 2.3  Hypersurface singularities

The GIT compactification \( M^{\text{git}} \) given in Definition 2.3 parameterizes classes of degenerations of degree 10 hypersurfaces in \( \mathbb{P}(1, 1, 2, 5) \) that appear as double covers of \( \mathbb{P}(1, 1, 2) \). Therefore, we can determine the surface singularities by considering the singularities of the branch curve \( V(F_{10}(x, y, z)) \subseteq \mathbb{P}(1, 1, 2) \). If the curve is away from the singularity of \( \mathbb{P}(1, 1, 2) \), then the singularities are locally isomorphic to a plane curve singularity. We now describe the plane curve singularities of interest for the current work.

Given a singularity \( T \), we can associate two invariants: the Milnor number \( \mu(T) \) and the modality \( m(T) \) of the singularity (also called modulus). These invariants measure the complexity of the singularity and provide a criterion to classify them. From the moduli theory perspective, the modality \( m(T) \) is particularly interesting because it is equal to the dimension of stratum in the base space of the versal deformation where \( \mu \) is constant, minus 1 (see [10, section 4]). Another relevant fact is that both \( \mu(T) \) and \( m(T) \) are upper semicontinuous, see [33, Chapter I, section 2]. The hypersurface singularities with \( m(T) = 0 \) are precisely the ADE singularities. Therefore, the next case of interest are the singularities with \( m(T) = 1 \). By the classification of log canonical two-dimensional hypersurface singularities [49, table 1] in combination with [10, section 1.1], we obtain that there are eight plane curve singularities of modality 1 that are not log canonical. These are given in Table 2. Here, the parameter \( a \) is generic, and as it varies describes nonisomorphic plane curves with the given isolated singularities.

We remark that we label the variables in a different way than in [10, section 1.1]. Our choice is to be compatible with the discussion that follows. Furthermore, the above notation denotes germs of singularities up to stable equivalence, that is, up to terms of the form \( x_1^2 \). Therefore, for instance, we use the notation \( E_{12} \) for both the plane...
curve singularity and for the associated surface singularity \( w^2 = z^3 + y^7 + ay^5z \). Summarizing, we have the following lemma.

**Lemma 2.8.** Let \( \mathcal{X} \rightarrow \Delta \) be a one-dimensional family of smooth Horikawa surface degenerating a double cover of \( \mathbb{P}(1, 1, 2) \) with a unique isolated singularity of modality 1. Then, the singularity is one of the following:

\[ E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}. \]

We conclude with the following diagram in Figure 1, where the arrow \( A \leftarrow B \) means that the germ of an \( A \) singularity degenerates to the germ of a \( B \) singularity.

**Remark 2.9.** Within the context of the main theorems in the introduction (e.g., Theorem 1.1), we highlight that the adjacency diagrams in Figure 1 do not mean that the divisors \( D_\Sigma \) are contained in each other or that they intersect generically in \( \mathcal{M} \). Additionally, in terms of deformation theory, we note that the germs of \( \tilde{E}_7 \) and \( \tilde{E}_8 \) degenerate to \( Z_{11}, Z_{12}, Z_{13} \) and \( E_{12}, E_{13}, E_{14} \), respectively. The equations of \( \tilde{E}_7 \) and \( \tilde{E}_8 \) are \( x^4 + y^4 + ax^2y^2 = 0 \) with \( a^2 \neq 4 \) and \( x^3 + y^6 + ax^2y^2 = 0 \) with \( 4a^3 + 27 \neq 0 \), respectively. However, after stable replacement, these describe codimension one strata of the boundary of \( \mathcal{M} \). In general, in a compact moduli space of stable varieties, it is an open problem to understand the reciprocal relations among the boundary strata corresponding to the singularities in an adjacency diagram of germ of singularities.

### 2.4 | Weighted blow ups

A key step in our work is a partial resolution of isolated singularities via weighted blow ups at a point, which, we briefly describe here. See [46, section 6.38] for a reference. Let \( (a_1, \ldots, a_n) \) be a sequence of relatively prime positive integers. We have a natural map

\[
\mathbb{A}^n \rightarrow \mathbb{P}(a_1, \ldots, a_n)
\]

\[
(x_1, \ldots, x_n) \mapsto (x_1^{a_1}, \ldots, x_n^{a_n}).
\]

The weighted blow up of \( \mathbb{A}^n \) with local coordinates \( (x_1, \ldots, x_n) \) and weights \( (a_1, \ldots, a_n) \) at the origin is the closure of the graph of the above rational map. Its exceptional divisor is isomorphic to \( \mathbb{P}(a_1, \ldots, a_n) \), and the associated ideal is the integral closure of the ideal \( (x_1^{N/a_1}, \ldots, x_n^{N/a_n}) \) for sufficiently divisible \( N \). In particular, if \( a_1 = \cdots = a_n = 1 \), then we recover the simple blow up of \( \mathbb{A}^n \) at the origin.

### 3 | STABLE REPLACEMENTS OF GENERIC ONE-PARAMETER DEGENERATIONS OF HORIKAWA SURFACES

In this section, we construct the **KSBA stable replacements** for one-parameter degenerations of Horikawa surfaces over DVRs whose central fiber has a singularity of type \( \Sigma \). We begin by first analyzing a special class of degenerations defined by a \( \mathbb{C}^* \)-action. The general case of a DVR is handled in Section 3.10.

Let \( \mathbb{C}[x, y, z, w] \) denote the homogeneous coordinate ring of \( \mathbb{P}(1, 1, 2, 5) \), where \( x \) and \( y \) have degree 1, \( z \) has degree 2, and \( w \) has degree 5. Let \( \Sigma \) be one of the eight exceptional families of isolated unimodular surface singularities in Lemma 2.8.


Let
\[ S(f) = \{ ([x : y : z : w], t) \in \mathbb{P}(1,1,2,5) \times \mathbb{C} \mid w^2 - f(x,y,z) = 0 \}, \]

where \( f(x,y,z) \) is a homogeneous polynomial of weighted degree 10 with coefficients in \( \mathbb{C}[t] \). Let \( \pi : S \to \mathbb{C} \) denote the projection onto the second factor with fibers \( S_t = \pi^{-1}(t) \) and \( \Delta \) be a neighborhood of 0 ∈ \( \mathbb{C} \) such that, after restriction to \( \Delta \), \( S \) becomes a one-parameter family of smooth Horikawa surfaces for \( t \neq 0 \) and for \( t = 0 \) the fiber \( S_0 \) has exactly one isolated singularity of type \( \Sigma \). After a sequence of birational modifications of the central fiber of \( \pi : S \to \Delta \) and possibly base changes, we obtain a new family \( S' \to \Delta' \) whose central fiber \( S'_0 \) now has semi–log canonical singularities and it has ample canonical class, that is, \( S'_0 \) is a stable surface. The surface \( S'_0 \) is called the stable replacement of the central fiber of \( S \to \Delta \). The isomorphism class of \( S'_0 \) corresponds to the limit point in the KSBA compactification \( \overline{M} \) of the arc \( \Delta^\circ := \Delta \setminus \{0\} \to M \) induced by the family \( S^\circ := S \setminus S_0 \to \Delta^\circ \). More precisely, for each singularity \( \Sigma \), we describe some explicit families \( \mathbb{V}_10 \to \Delta \) such that the corresponding isomorphism classes of stable surfaces \( S'_0 \) generically describe a divisor in \( \overline{M} \). The discussion is organized as follows. First, in Section 3.1 we define such families. Then, in Section 3.2, we describe the stable replacements \( S'_0 \). Afterward, the remaining subsections contain the proofs of the claims in Section 3.2. We show that all the isomorphism classes of \( S'_0 \) give rise to boundary divisors in \( \overline{M} \) later in Section 4.

### 3.1 Definition of the families

Let \( \Sigma \) be one of the eight singularity types and let \((p, q), d\) as in Table 3.

The meaning of these constants is the following: If we assign weight \((p, q)\) to \((y, z)\), the lower degree part of the local singularity for \( \Sigma \) becomes homogeneous of degree \( d \), and this information will be used to compute the stable replacement of the central fiber \( S_0 \).

**Definition 3.1.** Let \( \mathbb{V}_{10} \) be the complex vector space spanned by the monomials \( x^a y^b z^c \) satisfying \( a + b + 2c = 10 \). If \( x^a y^b z^c \) is a monomial in \( \mathbb{V}_{10} \), then we define its weight with respect to \( \Sigma \) as \( \text{wt}_\Sigma(x^a y^b z^c) = pb + qc - d \).

It will be useful to keep track of which monomials have positive, negative, or zero weight across the eight singularity types.

**Proposition 3.2.** Consider the degree 10 monomials in \( \mathbb{P}(1,1,2) \). Identify the monomial \( x^a y^b z^c \) with the triple \((a, b, c)\). Then, the following 15 monomials have positive weight across the eight singularity types:

\[
(0,0,5), (1,1,4), (2,2,3), (0,2,4), (3,3,2), (1,3,3), (2,4,2), (0,4,3),
(1,5,2), (2,6,1), (0,6,2), (1,7,1), (0,8,1), (1,9,0), (0,10,0).
\]

The following other 12 monomials always have negative weight across the eight singularity types:

\[
(10,0,0), (8,0,1), (6,0,2), (9,1,0), (7,1,1), (5,1,2),
(8,2,0), (6,2,1), (4,2,2), (7,3,0), (5,3,1), (6,4,0).
\]

The remaining nine monomials can be of positive, negative, or zero weight as the singularity type changes:
In particular, for each singularity type, there exist precisely two monomials of weight 0 as listed below.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Sing.} & E_{12} & E_{13} & E_{14} & Z_{11} & Z_{12} & Z_{13} & W_{12} & W_{13} \\
\hline
m_1(x, y, z) & x^3y^7 & x^3y^2z & x^4y^8 & x^4y^3z & x^4y^6 & x^4y^5 & & \\
m_2(x, y, z) & x^4z^3 & x^4z^3 & x^4z^3 & x^3yz^3 & x^3yz^3 & x^3yz^3 & x^2z^4 & x^2z^4 \\
\hline
\end{array}
\]

Proof. This is an immediate computer-assisted check.

Definition 3.3. We let \( U_\Sigma \) denote the codimension 1 subspaces of \( V_{10} \) consisting of elements for which \( m_1 \) and \( m_2 \) have the same coefficient. Then, the weight function gives a direct sum decomposition \( U_\Sigma = U_{\Sigma,+} \oplus U_{\Sigma,0} \oplus U_{\Sigma,-} \) where:

\[
U_{\Sigma,+} = \text{Span}_C \{ x^a y^b z^c \in V_{10} \mid \text{wt}_\Sigma(x^a y^b z^c) > 0 \},
\]

\[
U_{\Sigma,0} = \text{Span}_C \{ m_1 + m_2 \},
\]

\[
U_{\Sigma,-} = \text{Span}_C \{ x^a y^b z^c \in V_{10} \mid \text{wt}_\Sigma(x^a y^b z^c) < 0 \}.
\]

We let \( \pi_+ \), \( \pi_0 \), and \( \pi_- \) denote the projections from \( U_\Sigma \) to \( U_{\Sigma,+} \), \( U_{\Sigma,0} \), and \( U_{\Sigma,-} \) respectively. If \( W \) is a subspace of \( U \), then we define \( W_{\text{reg}} = W \setminus \pi_0^{-1}(0) \) and \( P(W)_{\text{reg}} = \{ [w] \in P(W) \mid w \in W_{\text{reg}} \} \). We have that \( P(W)_{\text{reg}} \) is an affine patch of the projective space \( P(W) \).

Lemma 3.4. \( P(U_\Sigma)_{\text{reg}} \cong P(U_{\Sigma,+} \oplus U_{\Sigma,0})_{\text{reg}} \times P(U_{\Sigma,0} \oplus U_{\Sigma,-})_{\text{reg}} \).

Proof. Consider the following morphisms:

\[
\varphi : P(U_\Sigma)_{\text{reg}} \to P(U_{\Sigma,+} \oplus U_{\Sigma,0})_{\text{reg}} \times P(U_{\Sigma,0} \oplus U_{\Sigma,-})_{\text{reg}}
\]

\[
[\alpha] \mapsto ([\pi_+(\alpha) + \pi_0(\alpha)], [\pi_0(\alpha)]),
\]

\[
\psi : P(U_{\Sigma,+} \oplus U_{\Sigma,0})_{\text{reg}} \times P(U_{\Sigma,0} \oplus U_{\Sigma,-})_{\text{reg}} \to P(U_\Sigma)_{\text{reg}}
\]

\[
([\alpha], [\beta]) \mapsto [\alpha' + \beta' - m_1 - m_2],
\]

where \( \alpha', \beta' \) are lifts of \( [\alpha], [\beta] \) to \( U_{\Sigma,+} \oplus U_{\Sigma,0} \) and \( U_{\Sigma,0} \oplus U_{\Sigma,-} \), respectively such that \( \pi_0(\alpha') = \pi_0(\beta') = m_1 + m_2 \). It can be checked directly that \( \varphi \) and \( \psi \) are inverse of each other.

Definition 3.5. We now define a \( C^* \)-action on \( V_{10} \) by describing how an element \( t \in C^* \) acts on a given monomial \( x^a y^b z^c \in V \) of weight \( \omega \). We define

\[
t \star x^a y^b z^c = \begin{cases} 
  t^{-\omega} x^a y^b z^c & \text{if } \omega \leq 0 \\
  x^a y^b z^c & \text{if } \omega > 0.
\end{cases}
\]
In particular, \( t \star (m_1 + m_2) = m_1 + m_2 \), and hence this \( \mathbb{C}^* \)-action descends to an action on \( U_\Sigma \).

**Remark 3.6.** If \( v \in \mathbb{V}_{10} \), then \( t \star v \) converges to \( \pi_+(v) + \pi_0(v) \) as \( t \to 0 \), and hence \( t \star v \in \mathbb{C}[t] \otimes \mathbb{V}_{10} \). Moreover, if \( u \in (U_\Sigma)_{\text{reg}} \), then \( \lim_{t \to 0} t \star u \neq 0 \).

**Definition 3.7.** If \( u \in (U_\Sigma)_{\text{reg}} \), we define the associated family of surfaces to be

\[
S = S(t \star u) = \{(x : y : z : w), t) \in \mathbb{P}(1,1,2,5) \times \mathbb{C} \mid w^2 - t \star u = 0\} \subseteq \mathbb{P}(1,1,2,5) \times \mathbb{C}.
\]

The corresponding family of curves is

\[
B = B(t \star u) = \{(x : y : z), t) \in \mathbb{P}(1,1,2) \times \mathbb{C} \mid t \star u = 0\} \subseteq \mathbb{P}(1,1,2) \times \mathbb{C}
\]

with projection \( \pi : B \to \mathbb{C} \) and fiber \( B_t = \pi^{-1}(t) \subseteq \mathbb{P}(1,1,2) \).

**Remark 3.8.** The family \( B \) depends only on \([u] \in \mathbb{P}(U_\Sigma)_{\text{reg}}\). The limit branch curve

\[
B_0 = \lim_{t \to 0} B_t = V\left(\lim_{t \to 0} t \star u\right) = V(\pi_+(u) + \pi_0(u)) \subseteq \mathbb{P}(1,1,2)
\]

is obtained by composing \( V(\cdot) \) with the morphism \( P(U_\Sigma)_{\text{reg}} \to P(U_\Sigma, + \oplus U_\Sigma, 0)_{\text{reg}} \) of Lemma 3.4. The central fiber \( S_0 \) is the double cover of \( \mathbb{P}(1,1,2) \) with branch curve \( B_0 \).

**Definition 3.9.** Let \( u \in U_\Sigma \) and consider the one-parameter family \( S(t \star u) \to \Delta \) in Definition 3.7. The central fiber \( S_0 \subseteq S(t \star u) \) is given by \( V(w^2 - (\pi_0 + \pi_+)(u)) \subseteq \mathbb{P}(1,1,2,5) \). We say that \( u \) is \( \Sigma \)-generic provided the following hold:

1. The central fiber \( S_0 \) is singular only at \([1 : 0 : 0 : 0]\), where it has a singularity of type \( \Sigma \).
2. The other fibers of \( S(t \star u) \to \Delta \) are smooth Horikawa surfaces in \( \mathbb{P}(1,1,2,5) \).

Such conditions are verified for a generic choice of the coefficients of the polynomial \( u \). This can be observed at the level of the branch curve (3.2), for which we need to show it has only one singularity of type \( \Sigma \) at \([1 : 0 : 0 : 0]\). The idea is that for a generic \( u \), the curve \( V((\pi_0 + \pi_+)(u)) \) has a singularity of type \( \Sigma \) at \([1 : 0 : 0 : 0]\), and one can choose a specific \( u \) for which the corresponding curve only has a singularity of type \( \Sigma \) at \([1 : 0 : 0 : 0]\). By the upper semicontinuity of Milnor numbers [33, Theorem 2.6], the generic \( u \) has the claimed property. We illustrate an analogous argument in the proof of Theorem 7.1.

**Remark 3.10.** There exists \( r > 0 \) such that \( S_t \) is smooth for \( 0 < |t| < r \), that is, \( u \) is \( \Sigma \)-generic in the sense of the Definition 3.9. In particular, in order to be \( \Sigma \)-generic, the coefficient of \( z^5 \) must be nonzero, otherwise \( S_0 \) will also pass through the singular point \([0 : 1 : 0 : 0]\) \( \in \mathbb{P}(1,1,2,5) \).

Finally, the following construction will be used in Section 3.3.

**Definition 3.11.** Let \( \theta : \mathbb{C}[t, x, y, z] \to \mathbb{C}[t, \alpha, \beta] \) be the ring homomorphism defined by the rules \( \theta(t) = t, \theta(x) = 1, \theta(y) = \alpha, \theta(z) = \beta \). Let \( u \in P(U_\Sigma)_{\text{reg}} \). Then, \( \theta((\pi_0 + \pi_+)(t \star u)) \) is a homogeneous polynomial of degree \( d \) after assigning \( t \) degree 1, \( x \) degree \( p \), and \( y \) degree \( q \). In this way, by composing \( V(\cdot) \) with the morphism \( P(U_\Sigma)_{\text{reg}} \to P(U_\Sigma, 0 \oplus U_\Sigma, -)_{\text{reg}} \) produces a curve

\[
T_0(u) = V(\theta((\pi_0 + \pi_+)(t \star u)))
\]

of degree \( d \) in \( \mathbb{P}(1, p, q) \).

**Example 3.12.** Let us choose \( \Sigma = W_{12} \), so that \((p, q) = (4, 5)\) and \( d = 20 \). Let us construct an example of the family described together with the \( \mathbb{C}^* \)-action. The weight function is given by \( w_{t\Sigma}(x^a y^b z^c) = 4b + 5c - 20 \). So, we can consider

\[
u = z^5 + x^5 y^5 + x^2 z^4 + x^{10},
\]
where the weights of the monomials are 5, 0, 0, −20, respectively. We have that
\[ t \ast u = z^5 + x^3y^5 + x^2z^4 + t^{20}x^{10}. \]
Therefore, in the central fiber for \( t = 0 \), the limiting curve is given by the vanishing of \( z^5 + x^5y^5 + x^2z^4 = 0 \). Additionally, we have that
\[ \theta((\pi_0 + \pi_-)(t \ast u))([t : \alpha : \beta]) = \alpha^5 + \beta^4 + t^{20}, \]
which is homogeneous of degree 20 in \( \mathbb{P}(1, 4, 5) \) with coordinate \([t : \alpha : \beta]\).

### 3.2 Stable replacement of the central fiber of the families

Fix one of the eight singularity types \( \Sigma \) and let \( S = S(t \ast u) \rightarrow \Delta \) be one of the families in Definition 3.7 with \( u \Sigma \)-generic. Then, \( S \rightarrow \Delta \) comes equipped with a fiberwise \( \mathbb{Z}_2 \)-action. The quotient by this action \( S \rightarrow \mathfrak{X} = \mathbb{P}(1, 1, 2) \times \Delta \) has branch divisor \( B \subseteq \mathfrak{X} \), which fiberwisely gives a curve of weighted degree 10.

Let \( S' \rightarrow \Delta' \) denote the stable replacement of \( S \rightarrow \Delta \). As this is obtained after a combination of birational modifications of the central fiber and possibly after base changes branched at the origin of \( \Delta \), then also \( S' \rightarrow \Delta' \) comes equipped with a fiberwise \( \mathbb{Z}_2 \)-action away from the central fiber. It is a standard argument that this action extends to the whole \( S' \) (see, for instance, [30, Lemma 3.14]). Let \( S' \rightarrow \mathfrak{X}' \) be the quotient by this action and let \( \mathfrak{Y}' \subseteq \mathfrak{X}' \) be the branch locus. Then, \( (\mathfrak{X}', \frac{1}{2}B') \rightarrow \Delta' \) is also a family of stable pairs by the work of Alexeev–Pardini [9]. In particular, the central fiber \( S_0' \subseteq S' \) is an appropriate double cover of \( X_0' \subseteq X' \). From this discussion, it follows that the first goal is to compute the stable replacement of the central fiber of \( (\mathfrak{X}', \frac{1}{2}B') \rightarrow \Delta \).

**Definition 3.13.** Let \( \mathfrak{X}' \rightarrow \mathfrak{X} \) be the weighted blow up of the central fiber \( X_0 \subseteq \mathfrak{X} \) at the point \( \xi = [1 : 0 : 0] \) with weights \((p, q)\) according to the singularity type \( \Sigma \) (see Table 3). Denote by \( Y \subseteq \mathfrak{X}' \) the exceptional divisor of the blow up, which is isomorphic to \( \mathbb{P}(1, p, q) \). Let \( Z \) be the strict transform of the central fiber \( X_0 \subseteq \mathfrak{X} \), and let \( B' \subseteq \mathfrak{X}' \) the strict transform of \( B \) with central fiber \( B_0' \). Let \( E \) be the exceptional divisor of \( Z \rightarrow X_0 \) and \( G \subseteq \mathbb{P}(1, p, q) \) the curve \( V(t) \), where we denote by \([t : \alpha : \beta]\) the coordinate of \( \mathbb{P}(1, p, q) \).

The first step is then to prove the following theorem.

**Theorem 3.14.** The central fiber \((X_0', \frac{1}{2}B_0') \) of \((\mathfrak{X}', \frac{1}{2}B') \rightarrow \Delta \) is a stable pair. The pair \((X_0', \frac{1}{2}B_0') \) is obtained by gluing
\[ (Y, G + \frac{1}{2}B'|_Y), (Z, E + \frac{1}{2}B'|_Z), \]
where \( Y \cong \mathbb{P}(1, p, q) \) and \( Z = \text{Bl}(\xi) \mathbb{P}(1, 1, 2) \) are glued along \( G \cong \mathbb{P}^1 \cong E \).

**Proof.** We first observe that \( K_{X_0'} + \frac{1}{2}B_0' \) is Cartier: \( B_0' \) is the restriction to \( X_0' \) of \( B' \), which is Cartier, and \( K_{X_0'} \) is also Cartier by applying the adjunction formula on the central fiber of the family \( X_0' \subseteq \mathfrak{X}' \) (see [16, Proposition 16.4]). Then, proving that \((X_0', \frac{1}{2}B_0') \) is semi-log canonical boils down to show that \((Y, G + \frac{1}{2}B'|_Y) \) and \((Z, E + \frac{1}{2}B'|_Z) \) are log canonical. This is proved in Propositions 3.21 and 3.22. Finally, we check the ampleness of \( K_Y + G + \frac{1}{2}B'|_Y \) and \( K_Z + E + \frac{1}{2}B'|_Z \) in Propositions 3.24 and 3.25, which imply that \( K_{X_0'} + \frac{1}{2}B_0' \) is ample. So, \((X_0', \frac{1}{2}B_0') \) is a stable pair. \( \square \)

**Theorem 3.15.** The stable replacement \( S_0' \) of the one-parameter family \( S = S(t \ast u) \rightarrow \Delta \) with \( u \Sigma \)-generic is the gluing of a K3 surface \( \tilde{Y} \) with singularities of type \( A_\mu \) with a surface \( \tilde{Z} \) satisfying \( h^1(O_Z) = 0, h^2(O_Z) = 1, \) and \( K_Z^2 \) as given in Table 1. The gluing locus \( \tilde{Y} \cap \tilde{Z} \) is isomorphic to \( \mathbb{P}^1 \). The components \( \tilde{Y} \) and \( \tilde{Z} \) have finite cyclic quotient singularities, and these are contained in the gluing locus. Moreover, the topological Euler characteristics of \( \tilde{Y} \) and \( \tilde{Z} \) are \( \mu_\Sigma + 3 \) and \( 36 - \mu_\Sigma \), respectively, where \( \mu_\Sigma \) is the Milnor number of the singularity \( \Sigma \).
By the discussion at the beginning of Section 3.2, the stable replacement $S'_0$ is obtained by taking an appropriate double cover of $X'_0$. In Section 3.5 and Section 3.7, we describe the double covers $\tilde{Y} \to Y$ and $\tilde{Z} \to Z$. These two surfaces are glued along $\tilde{G} \to G$ and $\tilde{E} \to E$, which are isomorphic to $\mathbb{P}^1$ as we prove in Section 3.6. We prove that $\tilde{Y}$ and $\tilde{Z}$ have finite quotient singularities, which lie along the gluing locus, in Proposition 3.31. The claimed values of $h^1(\mathcal{O}_{\tilde{Z}})$, $h^2(\mathcal{O}_{\tilde{Z}})$, $K_{\tilde{Z}}^2$, and of the topological Euler characteristic of $\tilde{Z}$ are computed in Proposition 3.27 and Corollary 3.30. The statement about the topological Euler characteristic of $\tilde{Y}$ is proved in Corollary 3.26.

**Remark 3.16.** Consider $S = S(t \star u) \to \Delta$ as in Definition 3.7 with $u \Sigma$-generic. One may want to describe the stable replacement directly on $S$ and not passing through $(\mathcal{U}, 1/2)$ as we described so far. This is done as follows. In the cases where $d$ is even, it is sufficient to take the weighted blowup of $S$ at $(t, y, z, w) = (0, 0, 0, 0)$ with weights $(1, p, q, d/2)$ in the affine patch $x \neq 0$. If $d$ is odd, then we first have to perform the base change $\tilde{\Delta} \to \Delta$ such that $s \mapsto t^2$ obtaining a new family $\tilde{S} \to \tilde{\Delta}$, and then blow up $\tilde{S}$ at $(t, y, z, w) = (0, 0, 0, 0)$ with weights $(1, 2p, 2q, d)$ in the affine patch $x \neq 0$.

Before we move on with the proofs of the above claims, we recall some preliminaries about weighted projective planes. A reference for the following well-known facts is [36, section 5.1]. Let $a, b$ be two positive coprime integers. Consider the weighted projective plane $\mathbb{P}(1, a, b)$ with coordinates $[x : y : z]$. Let $D_x = V(x)$, $D_y = V(y)$, $D_z = V(z)$. Then, we have the following linear equivalences:

$$abD_x \sim bD_y \sim aD_z.$$ 

Moreover, $abD_x$ generates the Picard group of $\mathbb{P}(1, a, b)$, and the intersection numbers among $D_x, D_y, D_z$ are given by

|       | $D_x$   | $D_y$   | $D_z$   |
|-------|---------|---------|---------|
| $D_x$ | $1/ab$  | $1/b$   | $1/a$   |
| $D_y$ | $1/b$   | $a/b$   | $1$     |
| $D_z$ | $1/a$   | $1$     | $b/a$   |

**Remark 3.17.** Let $C \subseteq \mathbb{P}(1, 1, 2)$ be an irreducible curve of weighted degree 10. As $\mathbb{P}(1, 1, 2)$ is $\mathbb{Q}$-factorial and since $2D_x$ generates $\text{Pic}(\mathbb{P}(1, 1, 2))$, there exists a rational constant $c$ such that $C = cD_x$. Intersecting both sides with $D_z$, we obtain that $c = 10$.

### 3.3 Proof of semi-log canonicity

Let us prove that the pair $\left( X'_0, \frac{1}{2}B'_0 \right)$ has semi-log canonical singularities. We already explained that $K_{X'_0} + \frac{1}{2}B'_0$ is $\mathbb{Q}$-Cartier, so we focus on showing that $\left( Y, G + \frac{1}{2}B'_Y \right)$ and $\left( Z, E + \frac{1}{2}B'_Z \right)$ are log canonical.

Let us first focus on the former pair. We have that $Y \cong \mathbb{P}(1, p, q)$ has coordinates $[t : \alpha : \beta]$ and $G = V(t)$. So, along $G$, $Y$ has a $\frac{1}{p}(1, q)$ singularity at $[0 : 1 : 0]$ and a $\frac{1}{q}(1, p)$ singularity at $[0 : 0 : 1]$. We now describe the curve $B'_Y$. For this, we start with the equation of $B \subseteq \mathbb{X}$. By (3.1), this is $V(t \star u)$, where $u \in U_{\text{reg}}$. The inhomogeneous form of $t \star u$ in the affine patch $x = 1$ is just $\theta(t \star u)$ (see Definition 3.11). By construction, $\theta(t \star u) \in \mathbb{C}[t, \alpha, \beta]$ is a sum of homogeneous polynomials of degree $\geq d$. Moreover, the degree $d$ part of $\theta(t \star u)$ is just $\theta((\pi_0 + \pi_\infty)(t \star u))$. Therefore, using (3.3),

$$B'_Y|_Y = V(\theta((\pi_0 + \pi_\infty)(t \star u))) = \Gamma_0(u).$$

**Lemma 3.18.** Let $W_\Sigma$ denote the vector space of homogeneous polynomials of degree $d$ in $\mathbb{P}(1, p, q)$. Then, the morphism

$$\mathbb{C}m_1 \oplus \mathbb{C}m_2 \oplus U_{\Sigma -} \to W_\Sigma$$

$$u \mapsto \theta(t \star u)$$

is an isomorphism. Moreover, for generic $\omega \in W_\Sigma$, there exists $\sigma \in \text{Aut}(\mathbb{P}(1, p, q))$ such that $\sigma(\omega) \in \theta(U_{\Sigma, 0} \oplus U_{\Sigma, -})$. 

**TABLE 4** Picture of the gluing of \( (Z, E + \frac{1}{2} F|_Z) \) and \( (Y, G + \frac{1}{2} F|_Y) \) along \( G \cong E \) depending on the singularity \( \Sigma \). If the divisor passes through a singular point, we report the singularity type with respect to \( Z \) and \( Y \), respectively. The top boundary point in \( Y \) is \([0:1:0]\) and the bottom one is \([0:0:1]\).

![Table Image](image)

**Proof.** In the following table, we report for each singularity type the number of monomials of degree \( d \) in \( \mathbb{P}(1, p, q) \).

| Sing.   | \( E_{12} \) | \( E_{13} \) | \( E_{14} \) | \( Z_{11} \) | \( Z_{12} \) | \( Z_{13} \) | \( W_{12} \) | \( W_{13} \) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Number of monomials of deg \( d \) in \( \mathbb{P}(1, p, q) \) | 17 | 18 | 19 | 15 | 16 | 17 | 16 | 17 |

This matches the number of degree 10 monomials \( x^a y^b z^c \) with \( \text{wt}_{\Sigma}(x^a y^b z^c) \leq 0 \), which were listed in Proposition 3.2. So, to show the map in the statement is an isomorphism, it suffices to show it is surjective. An arbitrary degree \( d \) monomial in \( W_\Sigma \) is in the form \( t^a \alpha^b \beta^c \) with \( a + pb + qc = d \). This is the image of \( x^{10-b-2c} y^b z^c \), which we need to show has nonpositive weight. But this is true as \( \text{wt}_{\Sigma}(x^{10-b-2c} y^b z^c) = pb + qc - d = -a \leq 0 \). For the statement about the existence of \( \sigma \in \text{Aut}(\mathbb{P}(1, p, q)) \), one simply checks case by case constructing an automorphism \( \sigma \), which makes the nonzero coefficients of \( \theta(m_1) \) and \( \theta(m_2) \) equal, which is the required condition to be in \( \theta(U_{\Sigma,0} \oplus U_{\Sigma,-}) \).

In the next lemma, we discuss the intersection \( B'|_Y \cap G \) depending on the singularity \( \Sigma \).

**Lemma 3.19.** The points in which \( B'|_Y \) intersects with \( G \) are summarized in the table below.

| Sing. \( B'|_Y \cap G \) | \( E_{12} \) | \( E_{13} \) | \( E_{14} \) | \( Z_{11} \) |
|-----------------|-------------|-------------|-------------|-------------|
| \([0:1:-1]\) | \([0:1:0]\), | \([0:1:-1]\) | \([0:0:1]\), |
| \([0:-1:1]\) | \([0:1:0]\) | \([0:1:-1]\) | \([0:0:1]\) |

| Sing. \( B'|_Y \cap G \) | \( Z_{12} \) | \( Z_{13} \) | \( W_{12} \) | \( W_{13} \) |
|-----------------|-------------|-------------|-------------|-------------|
| \([0:1:0]\), | \([0:0:1]\), | \([0:1:0]\), | \([0:1:-1]\) |
| \([0:0:1]\), | \([0:-1:1]\) | \([0:1:0]\) | \([0:0:1]\) |

**Proof.** The intersection of \( B'|_Y \) with \( V(t) \) is given by \( V(\theta(m_1 + m_2)) \cap V(t) \). Working case by case, we see that \( \theta(m_1 + m_2) = \alpha^a \beta^b (\alpha^a + \beta^b) \). The only solution to \( \alpha^a + \beta^b = 0 \) and \( t = 0 \) in \( \mathbb{P}(1, p, q) \) is either \([0:1:-1]\) or \([0:-1:1]\) according to the table above. By inspection \((a,b) = (0,0)\) if \( \Sigma = E_{12}, E_{14}, W_{12} \). For \( \Sigma = E_{13}, W_{13} \), we have \((a,b) = (0,1)\). For \( \Sigma = Z_{11}, Z_{13} \), we have \((a,b) = (1,0)\). For \( Z_{12} \), we have \((a,b) = (1,1)\).

**Remark 3.20.** Part of the information of Lemma 3.19 can be found in Table 4, where in each cell the triangle on the right represents \( (Y, G + \frac{1}{2} B'|_Y) \).

**Proposition 3.21.** The pair \( (Y, G + \frac{1}{2} B'|_Y) \) is log canonical.
Proof. By Lemma 3.18 and by the genericity assumption on the curve \( T_0(u) \), we have that the curve \( B'_1|_Y \) is smooth away from \( G \), which contains the singular points of \( P(1, p, q) \). So, we only have to check the log canonicity in a neighborhood of \( G \). As illustrated in Lemma 3.19, \( G \) and \( B'_1|_Y \) intersect transversely at a smooth point of \( Y \), or at a singular toric fixed point of the toric variety \( Y \). So, we have that \( \left( Y, G + \frac{1}{2} B'_1|_Y \right) \) is log canonical by combining [42, Theorem 3.32] with [19, Proposition 11.4.24 (a)]. □

We now focus on the other pair.

**Proposition 3.22.** The pair \( (Z, E + \frac{1}{2} B'_1|_Z) \) is log canonical.

Proof. By the \( \Sigma \)-genericity assumption, the curve \( B'_1|_Z \) is smooth away from \( E \). So, we only have to check log canonicity of the pair in a neighborhood of the exceptional divisor \( E \). The blow up \( Z = \text{Bl}(p, q) \mathbb{P}(1, 1, 2) \) may be singular along \( E \) at the torus fixed points \( t_1, t_2 \). These singularities are toric, and dictated by the weights \( p \) and \( q \) (see the next Remark 3.23).

As we discussed in the previous section, depending on the singularity \( \Sigma \), the curve \( B'_1|_Z \) may pass through \( t_1 \) or \( t_2 \). So we argue again by cases. These are summarized in Table 4. We can conclude that \( \left( Z, E + \frac{1}{2} B'_1|_Z \right) \) is log canonical again by combining [42, Theorem 3.32] with [19, Proposition 11.4.24 (a)]. □

Remark 3.23. Let us compute the singularities of \( Z \) along \( E \). From a toric perspective, the cone in \( \mathbb{R}^2 \) corresponding to \( \mathbb{P}(1, 1, 2) \) is \( C(\mathbb{P}) = \langle (1, 0), (0, 1) \rangle \), where \( \mathbb{R}_{\geq 0} \mathbb{P}(1, 0) \) corresponds to a torus fixed fiber of \( \mathbb{P}(1, 1, 2) \) and hence \( \mathbb{R}_{\geq 0} \mathbb{P}(0, 1) \) to a torus fixed section. By [36, Remark after Proposition 4.4], we know that \( C(\mathbb{P}) \) is subdivided by the ray \( \mathbb{R}_{\geq 0}(q, p) \) by the weighted blow up (i.e., the blow up of the ideal \( (y^q, z^p) \)). Therefore, let \( t_1, t_2 \) be the torus fixed points of \( Z \) along \( E \) with associated cones \( \langle (1, 0), (q, p) \rangle \) and \( \langle (0, 1), (q, p) \rangle \), respectively. By [19, Proposition 10.1.2], these give rise to cyclic quotient singularities of type \( \frac{1}{p}(1, -q) \) and \( \frac{1}{q}(1, -p) \), respectively.

3.4 Proof of ampleness

We now show \( K_{X'} + \frac{1}{2} B'_1 \) is ample. This boils down to showing that \( K_Y + G + \frac{1}{2} B'_1|_Y \) and \( K_Z + E + \frac{1}{2} B'_1|_Z \) are ample.

**Proposition 3.24.** \( K_Y + G + \frac{1}{2} B'_1|_Y \) is ample.

Proof. Since \( G \) generates \( \text{Pic}(Y) \otimes \mathbb{Q} \), there exists a rational constant \( c \) such that \( B'_1|_Y \sim_{\mathbb{Q}} cG \). Additionally, we have that \( D_\alpha + D_\beta \sim_{\mathbb{Q}} (p + q)G \). Therefore,

\[
K_Y + G + \frac{1}{2} B'_1|_Y \sim_{\mathbb{Q}} -G - D_\alpha - D_\beta + G + \frac{c}{2}G \sim_{\mathbb{Q}} \left( \frac{c}{2} - p - q \right)G.
\]

So, \( K_Y + G + \frac{1}{2} B'_1|_Y \) is ample provided \( \frac{c}{2} - p - q > 0 \). To compute the constant \( c \), we intersect both sides of \( B'_1|_Y \sim_{\mathbb{Q}} cG \) with \( G \) to obtain \( B'_1|_Y \cdot G = cG^2 = \frac{c}{pq} \). Hence,

\[
c = pq(B'_1|_Y \cdot G).
\]

To compute the intersection \( B'_1|_Y \cdot G \), we can use the calculations carried out in the proof of Lemma 3.19 (these are visually summarized in Table 4). For instance, for \( \Sigma = E_{12} \), \( B'_1|_Y \cdot G = 1 \), hence \( c = 21 \). Repeating this for each singularity, we obtain the following table:

| Sing. | \( E_{12} \) | \( E_{13} \) | \( E_{14} \) | \( Z_{11} \) | \( Z_{12} \) | \( Z_{13} \) | \( W_{12} \) | \( W_{13} \) |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| \( c \) | 21 | 15 | 24 | 15 | 11 | 18 | 20 | 16 |
| \( \frac{c}{2} - p - q \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | 1 | \( \frac{1}{2} \) | \( \frac{1}{2} \) | 1 | 1 | 1 |
The inequality $\frac{c}{2} - p - q > 0$ is then verified by the above table (see Table 3 for the values of $p$ and $q$). In particular, $K_Y + G + \frac{1}{2} B'_Y |_Y$ is ample for each singularity type.

**Proposition 3.25.** $K_Z + E + \frac{1}{2} B'_Z |_Z$ is ample.

**Proof.** We have that $B_0 = 10D_x$ by Remark 3.17. Recall that

$$Z = B_{t}^{(p,q)} X \rightarrow \mathbb{P}(1,1,2),$$

so, in the affine patch $\{x \neq 0\}$, we assign weight $p$ to $y$ and $q$ to $z$. For a divisor $D$ in $\mathbb{P}(1,1,2)$, let $\hat{D}$ denote its strict transform. As $Z$ is a toric surface, we have that the divisor $K_Z + E + \frac{1}{2} B'_Z$ is ample if and only if it intersects each boundary curve $\hat{D}_x, \hat{D}_y, \hat{D}_z, E$ positively. In what follows, we compute these four intersection numbers.

We have that $B_0 \sim 10D_x, D_y \sim D_x, D_z \sim 2D_x$, and $\xi = [1:0:0] \notin D_x$. Then, if $\sigma: Z \rightarrow X$ denotes the weighted blow up at $\xi$, the following equalities hold:

$$10\hat{D}_x = \sigma^*(B_0) = B'_Z |_Z + dE \Rightarrow \hat{B}'_Z |_Z = 10\hat{D}_x - dE,$$

$$\hat{D}_x = \sigma^*D_y = \hat{D}_y + pE \Rightarrow \hat{D}_y = \hat{D}_x - pE,$$

$$2\hat{D}_x = \sigma^*D_z = 2\hat{D}_x - qE \Rightarrow \hat{D}_z = 2\hat{D}_x - qE.$$

Combining these equalities with $K_Z = -\hat{D}_x - \hat{D}_y - \hat{D}_z - E$, we can rewrite

$$K_Z + E + \frac{1}{2} B'_Z = \hat{D}_x + \left(p + q - \frac{d}{2}\right)E.$$

We can then compute the following intersection numbers:

$$\hat{D}_x \cdot \left(K_Z + E + \frac{1}{2} B'_Z |_Z\right) = \hat{D}_x \cdot \left(\hat{D}_x + \left(p + q - \frac{d}{2}\right)E\right) = \frac{1}{2},$$

$$\hat{D}_y \cdot \left(K_Z + E + \frac{1}{2} B'_Z |_Z\right) = (\hat{D}_x - pE) \cdot \left(\hat{D}_x + \left(p + q - \frac{d}{2}\right)E\right)$$

$$= \frac{1}{2} + \left(p + q - \frac{d}{2}\right) \frac{1}{q} = \frac{2p + 3q - d}{2q},$$

$$\hat{D}_z \cdot \left(K_Z + E + \frac{1}{2} B'_Z |_Z\right) = (2\hat{D}_x - qE) \cdot \left(\hat{D}_x + \left(p + q - \frac{d}{2}\right)E\right)$$

$$= 1 + \left(p + q - \frac{d}{2}\right) \frac{1}{p} = \frac{4p + 2q - d}{2p},$$

$$E \cdot \left(K_Z + E + \frac{1}{2} B'_Z |_Z\right) = E \cdot \left(\hat{D}_x + \left(p + q - \frac{d}{2}\right)E\right) = \frac{-2p - 2q + d}{2pq},$$

where in the last equality we used $E^2 = -\frac{1}{pq}$ (see [11, Theorem 4.3 (3)]). The intersection with $D_x$ is $\frac{1}{2}$, independently of the singularity type. The intersections with $D_y, D_z, E$ are also positive, as shown in the table below as the singularity type varies.
The double cover $\tilde{Y} \to Y$

If $d$ is even, we can form the double cover of $\mathbb{P}(1, p, q)$ with branch curve $B'|_Y = T_0(u)$ as the hypersurface of degree $d$ in $\mathbb{P}(1, p, q, d/2)$ given by $w^2 = \partial((\pi_0 + \pi_-)(t \star u))$. If $d$ is odd, we use the isomorphism $\mathbb{P}(1, p, q) \cong \mathbb{P}(1, 2p, 2q)$ to construct the double cover as a hypersurface in $\mathbb{P}(1, 2p, 2q, d)$. This amounts to replacing $t$ by $t^2$ and doubling the degrees of $\alpha$ and $\beta$. More geometrically, when $d$ is odd, one is constructing the double cover of $\mathbb{P}(1, p, q)$ branched along $T_0(u) \cup V(t)$. We summarize this information in the table below.

| Sing. | $E_{12}$ | $E_{13}$ | $E_{14}$ | $Z_{11}$ |
|-------|----------|----------|----------|----------|
| $D = \tilde{D}_1, \tilde{D}_2, E$ | $\frac{1}{3} \frac{1}{3} \frac{1}{12}$ | $\frac{1}{10} \frac{2}{10} \frac{1}{15}$ | $\frac{1}{10} \frac{2}{10} \frac{1}{20}$ | $\frac{1}{4} \frac{2}{3} \frac{1}{12}$ |
| $D = \tilde{D}_1, \tilde{D}_2, E$ | $\frac{3}{7} \frac{1}{7} \frac{1}{12}$ | $\frac{3}{7} \frac{1}{7} \frac{1}{20}$ | $\frac{3}{7} \frac{1}{7} \frac{1}{28}$ | $\frac{3}{7} \frac{1}{7} \frac{1}{34}$ |

3.6 The gluing curve $\tilde{Y} \cap \tilde{Z}$

To describe the curve along which the stable surfaces $\tilde{Y}$ and $\tilde{Z}$ are glued, we view $\tilde{G} := \tilde{Y} \cap \tilde{Z} \subseteq \tilde{Y}$ as the double cover of $G \subseteq Y$. The advantage is that for $\tilde{G}$ we have an explicit equation

$$w^2 = \partial(\pi_0(u)) \quad (3.4)$$

in $\mathbb{P}(p, q, d/2)$ or $\mathbb{P}(2p, 2q, d)$, depending whether $d$ is even or odd, respectively. We will prove that $\tilde{G}$ is isomorphic to $\mathbb{P}^1$.

If the singularity type is $E_{12}, E_{13}, Z_{11}$, or $Z_{12}$ (which correspond to $d$ odd), then the isomorphism $\mathbb{P}(2p, 2q, d) \cong \mathbb{P}(p, q, d)$ induces an isomorphism of the curve (3.4) with $w = \partial(\pi_0(u))$. This proves that $G$ is in the branch locus of the cover $\tilde{Y} \to Y$, as each point on it has only one preimage. In particular, $\tilde{G} \cong \mathbb{P}^1$.

We now analyze the case of $E_{14}, Z_{13}, W_{12}, W_{13}$ (which correspond to $d$ even). In these cases, the restriction to $\tilde{G}$ of the projection $\mathbb{P}(p, q, d/2) \to \mathbb{P}(p, q)$ such that $[\alpha : \beta : w] \to [\alpha : \beta]$ gives a 2 : 1 morphism branched at two distinct points (this can be checked inspecting the four cases). In conclusion, $\tilde{G}$ is isomorphic to $\mathbb{P}^1$ also if $d$ is even. We illustrate...
this strategy with one of the cases, since the other ones are analogous. For $E_{14}$ and under the isomorphisms $\mathbb{P}(3,8,12) \cong \mathbb{P}(3,2,3) \cong \mathbb{P}(1,2,1)$, the curve $\tilde{G}$ becomes identified with

$$\tilde{G} = \{ w^2 = \alpha^8 + \beta^3 \} \cong \{ w^2 = \alpha^2 + \beta^3 \} \cong \{ w^2 = \alpha^2 + \beta \} = C.$$ 

The restriction to $C$ of the projection $\mathbb{P}(1,2,1) \rightarrow \mathbb{P}(1,2)$ such that $[\alpha : \beta : w] \mapsto [\alpha : \beta]$ is 2 : 1 and it is branched at the points $[0 : 1]$ and $[1 : -1]$. So $\tilde{G} \cong C \cong \mathbb{P}^1$.

### 3.7 The double cover $\tilde{Z} \rightarrow Z$

We now study the geometry of the double cover $\tilde{Z} \rightarrow Z = \text{Bl}^{(p,q)}_{\xi} \mathbb{P}(1,1,2)$.

**Proposition 3.27.** The double cover $\tilde{Z} \rightarrow Z$ satisfies $h^1(\mathcal{O}_{\tilde{Z}}) = 0$, $h^2(\mathcal{O}_{\tilde{Z}}) = 1$. Moreover, $K^2_{\tilde{Z}}$ is given as follows:

| Sing. | $E_{12}$ | $E_{13}$ | $E_{14}$ | $Z_{11}$ | $Z_{12}$ | $Z_{13}$ | $W_{12}$ | $W_{13}$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| $K^2_{\tilde{Z}}$ | $\frac{19}{21}$ | $\frac{4}{5}$ | $\frac{2}{3}$ | $\frac{2}{6}$ | $\frac{2}{3}$ | $\frac{2}{15}$ | $\frac{1}{5}$ | $\frac{1}{3}$ |

**Proof.** Recall from Section 3.6 that $\bar{V} \cap \bar{Z}$ is part of the ramification divisor if and only if $d$ is odd. Therefore, we distinguish two cases.

If $d$ is even, the branch divisor of $\tilde{Z} \rightarrow Z$ equals $B'|_{\bar{Z}}$. Using the expressions for $B'|_{\bar{Z}}, \hat{D}_y, \hat{D}_z$ computed in the proof of Proposition 3.25, we obtain that

$$K_{\tilde{Z}} = \pi^* \left( K_{\bar{Z}} + \frac{1}{2} B'|_{\bar{Z}} \right)$$

$$= \pi^* \left( -\hat{D}_x - \hat{D}_y - \hat{D}_z - E + 5\hat{D}_x - \frac{d}{2}E \right)$$

$$= \pi^* \left( \hat{D}_x + \left( p + q - 1 - \frac{d}{2} \right)E \right)$$

$$\Rightarrow K^2_{\tilde{Z}} = 2 \left( \hat{D}_x + \left( p + q - 1 - \frac{d}{2} \right)E \right)^2 = 1 - \frac{1}{2pq}(2p + 2q - 2 - d)^2,$$

from which we obtain the claimed values of $K^2_{\tilde{Z}}$. To compute the cohomology of $\mathcal{O}_{\tilde{Z}}$, we use $\pi_* \mathcal{O}_{\tilde{Z}} \cong \mathcal{O}_{\tilde{Z}} \oplus \mathcal{O}_Z(\frac{d}{2}E - 5\hat{D}_x)$. We have $h^1(\mathcal{O}_{\tilde{Z}}) = h^2(\mathcal{O}_{\tilde{Z}}) = 0$ because $Z$ is a rational surface with rational singularities. Hence, we obtain

$$h^1(\mathcal{O}_{\tilde{Z}}) = h^1 \left( \frac{d}{2}E - 5\hat{D}_x \right) = 0, \quad h^2(\mathcal{O}_{\tilde{Z}}) = h^2 \left( \frac{d}{2}E - 5\hat{D}_x \right) = 1,$$

where we used [19, Proposition 9.1.6]. Alternatively, one can use the following Macaulay2 code:

```macaulay2
i1: loadPackage "NormalToricVarieties";
i2: rayList = 1,0,q,p,0,1,-1,-2;
i3: coneList = 0,1,1,2,2,3,3,0;
i4: WX = normalToricVariety(rayList,coneList);
i5: D = toricDivisor(0,d/2,0,-5,WX);
i6: SD = OO D;
i7: HH^1(WX,SD), HH^2(WX,SD)
```

If $d$ is odd, the branch divisor of $\tilde{Z} \rightarrow Z$ equals $B'|_{\bar{Z}} + E$ instead. Hence,

$$K_{\tilde{Z}} = \pi^* \left( K_{\bar{Z}} + \frac{1}{2}(B'|_{\bar{Z}} + E) \right)$$
\[
\pi^* \left( -\tilde{D}_x - \tilde{D}_y - \tilde{D}_z - E + \frac{1-d}{2} E \right) = \pi^* \left( \tilde{D}_x + \left( p + q - \frac{1+d}{2} \right) E \right)
\]

\[
\Rightarrow K^2_{\tilde{Z}} = 2 \left( \tilde{D}_x + \left( p + q - \frac{1+d}{2} \right) E \right)^2 = 1 - \frac{1}{2pq}(2p + 2q - 1 - d)^2,
\]

from which we obtain the remaining values of \(K^2_{\tilde{Z}}\) in the table. For the cohomology of \(\mathcal{O}_{\tilde{Z}}\), we use \(\pi_* \mathcal{O}_{\tilde{Z}} \cong \mathcal{O}_Z \oplus \mathcal{O}_Z \left( \frac{d-1}{2} E - 5\tilde{D}_x \right)\). As \(h^1(\mathcal{O}_{\tilde{Z}}) = h^2(\mathcal{O}_{\tilde{Z}}) = 0\), we obtain

\[
h^1(\mathcal{O}_{\tilde{Z}}) = h^1 \left( \frac{d-1}{2} E - 5\tilde{D}_x \right) = 0, \quad h^2(\mathcal{O}_{\tilde{Z}}) = h^2 \left( \frac{d-1}{2} E - 5\tilde{D}_x \right) = 1,
\]

by [19, Proposition 9.1.6], or by the same Macaulay2 code as above with \(d\) replaced by \(d-1\).

Next, we compute the topological Euler characteristic of \(\tilde{Z}\) across the eight singularity types. Preliminarily, we find the Euler characteristic of the singular curve \(B_0 \subseteq \mathbb{P}(1, 1, 2)\) in (3.2). To do this, we start by recalling the following geometric genus formula for plane curves (see [15]): Let \(D \subseteq \mathbb{P}^2\) be a smooth curve of degree \(d\) and \(C \subseteq \mathbb{P}^2\) be an integral curve of degree \(d\) with normalization \(\pi : \hat{C} \rightarrow C\). Then,

\[
g(\hat{C}) = g(D) - \sum_{p \in \text{sing}(C)} \delta_p,
\]

where

1. \(g(E)\) is the genus of the curve \(E\);
2. \(2\delta_p = \mu_p + |\pi^{-1}(p)| - 1\);
3. \(\mu_p\) is the Milnor number of \(C\) at \(p\).

Let \(P_\omega = \mathbb{P}(w_0, w_1, w_2)\) where \(\gcd(w_i, w_j) = 1\) for \(i \neq j\). Assume that \(P_\omega\) contains a smooth curve of \(D\) degree \(d\). Then, by [15, Theorem 5.6], the geometric genus formula (3.5) holds for any integral curve \(C \subseteq P_\omega\) of degree \(d\), which does not pass through a singular point of \(P_\omega\). Moreover, by [15, Corollary 5.4], in this case

\[
g(D) = \frac{d(d - w_0 - w_1 - w_2)}{2w_0w_1w_2} + 1,
\]

which simplifies to the genus-degree formula for curves in \(\mathbb{P}^2\), but need not be an integer if there are no smooth curves in \(P_\omega\).

**Lemma 3.28.** The topological Euler characteristic of the curve \(B_0\) with singularity \(\Sigma\) at \(\xi = [1 : 0 : 0]\) is given by \(\mu_\Sigma - 30\), where \(\mu_\Sigma\) is the Milnor number of \(B_0\) at \(\xi\).

**Proof.** Consider the short exact sequence (see [24, section 5])

\[
0 \rightarrow H^0(\{\xi\}) \rightarrow H^0(\pi^{-1}(\xi)) \rightarrow H^1(B_0) \rightarrow H^1(\hat{B}_0) \rightarrow 0.
\]

This implies that

\[
\text{rk } H^1(B_0) = \text{rk } H^1(\hat{B}_0) + \text{rk } H^0(\pi^{-1}(\xi)) - \text{rk } H^0(\{\xi\})
\]

\[
= 2g(\hat{B}_0) + |\pi^{-1}(\xi)| - 1.
\]

By (3.5), we have \(g(\hat{B}_0) = 16 - \delta_\xi\). Here, we chose as \(D\) the Fermat curve of degree 10 in \(P(1, 1, 2)\), which is smooth of genus \(g(D) = 16\), and we used that the generic curve \(B_0\) is singular only at the point \(\xi = [1 : 0 : 0]\), which is not an orbifold point.
of $\mathbb{P}(1,1,2)$. By substituting this in (3.7) together with the equality \(2\delta_p = \mu_p + |\pi^{-1}(p)| - 1\), we obtain

\[
\text{rk } H^1(B_0) = 32 - \mu_\Sigma.
\]

Therefore, $\chi_{\text{top}}(B_0) = \mu_\Sigma - 30$. \hfill \Box

Remark 3.29. The sequence (3.6) is an exact sequence of a mixed Hodge structure in which every term except $H^1(B_0)$ is pure of weight equal to the cohomological degree. Consequently, $\text{Gr}^W H^1(B_0) \cong H^1(B_0)$, whereas $W_0 H^1(B_0)$ has rank $|\pi^{-1}(\xi)| - 1$. Moreover, the normalization $\pi : \hat{B}_0 \to B_0$ in this case is given by the strict transform of $B_0$ relative to the weighted blow up $Z \to B_0^{(p,q)} \mathbb{P}(1,1,2)$, and hence $|\pi^{-1}(\xi)|$ is just the number of times the red curve intersects the exceptional divisor in Table 4.

Corollary 3.30. The topological Euler characteristic of the surface $\tilde{Z}$ is $36 - \mu_\Sigma$, where $\mu_\Sigma$ is the Milnor number of the singularity $\Sigma$.

Proof. The surface $S_0$ is the double cover of $\mathbb{P}(1,1,2)$ branched along $B_0 \cup \{\xi\}$, where $\xi = [0 : 0 : 1]$ is the singular point of $\mathbb{P}(1,1,2)$. Therefore,

\[
\chi_{\text{top}}(S_0) = 2\chi_{\text{top}}(\mathbb{P}(1,1,2) \setminus (B_0 \cup \{\xi\})) + \chi_{\text{top}}(B_0) + 1
\]

\[
= 2\chi_{\text{top}}(\mathbb{P}(1,1,2)) - \chi_{\text{top}}(B_0) - 1 = 6 - (\mu_\Sigma - 30) - 1 = 35 - \mu_\Sigma,
\]

where we used Lemma 3.28 for $\chi_{\text{top}}(B_0)$. As $\tilde{Z}$ is a weighted blow up with exceptional divisor $E \cong \mathbb{P}^1$ of $S_0$ at a single point, using again the additivity of the topological Euler characteristic, we obtain $\chi_{\text{top}}(\tilde{Z}) = \chi_{\text{top}}(S_0) + 1 = 36 - \mu_\Sigma$. \hfill \Box

3.8 | The singularities of $\tilde{Y}$ and $\tilde{Z}$

We conclude the proof of Theorem 3.15 with the following proposition.

Proposition 3.31. Across the eight singularity types, the surfaces $\tilde{Y}, \tilde{Z}$ only have finite cyclic quotient singularities along $\tilde{Y} \cap \tilde{Z}$.

Proof. Consider the curves $G \subseteq \tilde{Y}$ and $E \subseteq \tilde{Z}$, which are double covers of $G \subseteq Y$ and $E \subseteq Z$, respectively. From the work we carried out so far, we know that the pairs $(\tilde{Y}, G)$ and $(\tilde{Z}, E)$ are log canonical. The singularities of $\tilde{Y}$ and $\tilde{Z}$ only occur along the gluing curves $G$ and $E$. As the pairs $(\tilde{Y}, G)$ and $(\tilde{Z}, E)$ are log canonical, we can apply [39, Lemma 5.5] to conclude that the isolated singularities of $\tilde{Y}$ and $\tilde{Z}$ along the respective gluing loci are log terminal singularities. Furthermore, by [45, section 4], we know these singularities are cyclic quotient ones. \hfill \Box

3.9 | Summary of the construction of the stable surface $\tilde{Y} \cup \tilde{Z}$

The goal of this subsection is to summarize the construction of the limit surfaces $\tilde{Y} \cup \tilde{Z}$ described so far. Along the way, we use the case of $W_{12}$ as a guiding example. Let $[x : y : z]$ be the coordinate of $\mathbb{P}(1,1,2)$. Let $V_{10}$ denote the complex vector space of degree 10 polynomials in $x, y, z$. Let $\mathbb{M}$ be the basis of $V_{10}$ consisting of the possible degree 10 monomials. For each singularity type $\Sigma$ considered above, let $(p, q, d)$ be as in Table 3. For $W_{12}$, we have $(p, q) = (4, 5)$ and $d = 20$. Consider the weight function $w_{\Sigma}(x^iy^jz^k) = pb + qc - d$. Let $m_1, m_2 \in \mathbb{M}$ be the only two monomials of weight 0. For $W_{12}$, these are $x^4y^5, x^5z^4$. Let $U_{\Sigma}$ denote the subspace of $V_{10}$ consisting of elements such that $m_1$ and $m_2$ have the same coefficient. Given a $\Sigma$-generic $u \in U_{\Sigma}$ in the sense of Definition 3.9, decompose it as $u = \pi_{-}(u) + \pi_0(u) + \pi_{+}(u)$, see Definition 3.3. This notation is nothing more than the monomials of negative, zero, and positive degree with respect to the weight function.

We consider the one-parameter family $S(t \ast u) \to \Delta$ (see Definitions 3.5 and 3.7), of which we want to compute the stable replacement of the central fiber, which is given by $\tilde{Y}(u) \cup \tilde{Z}(u)$. 
(1) If $d$ is even, consider $\mathbb{P}(1, p, q, d/2)$ with coordinate $\langle t : \alpha : \beta : w \rangle$. Let $\tilde{Y}(u)$ be the hypersurface of degree $d$ in $\mathbb{P}(1, p, q, d/2)$ given by the polynomial equation

$$w^2 = \vartheta((\pi_- + \pi_0)(t \star u))(\langle t : \alpha : \beta \rangle),$$

where $\vartheta$ was introduced in Definition 3.11. For an example in the case of $W_{12}$, see Example 3.12. If $d$ is odd consider instead $\mathbb{P}(1, 2p, 2q, d)$ again with coordinate $\langle t : \alpha : \beta : w \rangle$. Let $\tilde{Y}(u)$ be the hypersurface of degree $2d$ in $\mathbb{P}(1, 2p, 2q, 2d)$ given by

$$w^2 = \vartheta((\pi_- + \pi_0)(t^2 \star u))(\langle t : \alpha : \beta \rangle).$$

In either case, let $\tilde{G} \subseteq \tilde{Y}(u)$ be the curve $V(t) \cap \tilde{Y}(u)$, which is isomorphic to $\mathbb{P}^1$ as shown in Section 3.6.

(2) The surface $\tilde{Z}(u)$ is the double cover of $\text{Bl}(p, q)$ by $\mathbb{P}(1, 1, 2)$, where $\xi = [1 : 0 : 0]$, with branch curve $V((\pi_0 + \pi_+)(u))$ if $d$ is even, or $V((\pi_0 + \pi_+)(u))$ union the exceptional divisor of the blow up $E \subseteq \text{Bl}(p, q)$ if $d$ is odd. Let $\tilde{E} \subseteq \tilde{Z}$ be the preimage of $E$.

The surfaces $\tilde{Y}(u)$ and $\tilde{Z}(u)$ are glued along the curves $\tilde{G} \cong \mathbb{P}^1 \cong \tilde{E}$.

3.10 One-parameter degenerations over a DVR

Thus far, we have computed the stable replacement of the central fiber of the families $S(t \star u) \to \Delta$ described in Definition 3.7. Such stable replacement can be understood as in Remark 3.16. Although this is not going to be used later in the paper, we point out that this actually extends to other more general families.

Suppose that $R$ is a DVR with residue field $\mathbb{C}$ and $\mathcal{D} = \text{Spec}(R)$ with uniformizing parameter $s$. In this section, we explain how to modify our previous work to determine the KSBA stable replacement for one-parameter degenerations of Horikawa surfaces of type $\Sigma$ over $\mathcal{D}$. For simplicity of exposition, we focus on the case where $\Delta \subseteq \mathbb{C}$ is a disk and $R = \mathcal{O}_p$ is the ring of germs of holomorphic functions at $p \in \Delta$.

Given $g \in \mathcal{O}_p$, let $k = \text{ord}(g)$ denote the order of vanishing of $g$ at $p$ and let

$$\tau(g) = g^{(k)}(0)s^k/k!$$

be the truncation of $g$ to its lowest order term. Analogously, the truncation of $f \in \mathbb{V}_{10} \otimes \mathcal{O}_p$ is defined component by component relative to the basis $\mathcal{M}$ of degree 10 monomials in $\mathbb{P}(1, 1, 2)$, that is,

$$f = \sum_{m \in \mathcal{M}} f_m(s)m \Rightarrow \tau(f) = \sum_{m \in \mathcal{M}} \tau(f_m)m.$$

Definition 3.32. Given a singularity type $\Sigma$, we say that an element $f \in U_\Sigma \otimes \mathcal{O}_p$ is $\Sigma$-generic if there exists $u \in (U_\Sigma)_{\text{reg}}$ which is $\Sigma$-generic and satisfying $\tau(f) = s \star u$.

If $f \in U_\Sigma \otimes \mathcal{O}_p$ is $\Sigma$-generic, then

$$S(f) := \{(x : y : z : w), q) \in \mathbb{P}(1, 1, 2, 5) \times \Delta | w^2 - f(s(q)) = 0\}$$

is a one-parameter degeneration of smooth Horikawa surfaces. It turns out that the stable replacement of the central fiber of $S(f)$ is computed in the same way as for the degenerations in Definition 3.7. Before proving this, we first introduce the following notation. We denote by $S(f)$ the base change of $S(f)$ with respect to $s \mapsto s^2$.

Proposition 3.33. Let $f$ be $\Sigma$-generic. Consider the one-parameter families $S(f)$ and $S(\tau(f))$. As constructed in Remark 3.16, consider the following modified families:

(1) Assume $d$ is even. Let $S(f)'$ and $S(\tau(f))'$ be, respectively, the weighted blow up of $S(f)$ and $S(\tau(f))$ with respect to the ideal $(s^d, y^q, z^p, w^2)$. 
(2) Assume \(d\) is odd. Let \(S(f)\) and \(S(\tau(f))\) be, respectively, the weighted blow up of \(\overline{S}(f)\) and \(\overline{S}(\tau(f))\) with respect to the ideal \((s^{2d}, y^{2q}, z^{2p}, w^2)\).

Then, the central fibers of \(S(f)\) and \(S(\tau(f))\) are isomorphic. In particular, \(S(f) \to \Delta\) provides the stable replacement of the central fiber of \(S(f) \to \Delta\).

Proof. Let \(\overline{Z}_\tau \cup \overline{Y}_\tau\) and \(\overline{Z} \cup \overline{Y}\) be the central fibers of \(S(\tau(f))\) and \(S(f)\), respectively, where \(\overline{Y}_\tau, \overline{Y}\) denote the exceptional divisors. We already know that \(\overline{Z}_\tau \cup \overline{Y}_\tau\) is a stable surface by the discussion in Section 3.2 (see in particular Remark 3.16).

As \(V(\lim_{s \to \tau(f)} f) = V(\lim_{s \to 0} f) =: C\), then we have that \(\overline{Z}_\tau\) and \(\overline{Z}\) are isomorphic because they are both the double cover of \(B_{\{\theta\}} \mathbb{P}(1, 1, 2)\) with branch divisor the strict transform of the curve \(C\) (union the exceptional divisor if \(d\) is odd).

Let \(\phi: \mathbb{C}[s][x, y, z] \to \mathbb{C}[t][x, y, z]\) denote the map obtained by setting \(\phi(s) = t, \phi(x) = 1, \phi(y) = u,\) and \(\phi(z) = v\) (\(\phi\) is the analog of \(\delta\) in Definition 3.11). We distinguish two cases. If \(d\) is even, then \(Y_{\tau}\) is given by \(V((\pi_0 + \pi_-)(\phi(\tau(f)))) \subseteq \mathbb{P}(1, p, q, d/2)\). On the other hand, \(\overline{Y}_\tau\) is given by the vanishing of the lower degree part of \((\pi_0 + \pi_-)(\phi(\tau(f)))\), which is precisely \((\pi_0 + \pi_-)(\phi(f))\)), so \(\overline{Y}_\tau\) and \(\overline{Y}\) coincide. If \(d\) is odd, then the argument is analogous to the previous one, with the difference that \(\overline{Y}_\tau\) and \(\overline{Y}\) are hypersurfaces in the weighted projective space \(\mathbb{P}(1, 2p, 2q, d)\).

\[\square\]

4 \| DIMENSION COUNT OF BOUNDARY STRATA

We now use the families in Definition 3.7 to define eight closed and irreducible subsets of the boundary of \(\overline{M}\), one for each singularity type \(\Sigma\). The starting point is the construction of a family of degenerate stable surfaces over \(\mathbb{P}(U_{\Sigma})_{\text{reg}}\).

**Definition 4.1.** In a slight abuse of notation with our conventions so far, we let \(\Delta = \text{Spec}(\mathbb{C}[[t]])\). Let

\[\mathfrak{G} := \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta \times \mathbb{P}(1, 1, 2) \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\]

be the usual projection map. Let \(u \in \mathbb{P}(U_{\Sigma})_{\text{reg}}\). Define \(\mathfrak{D} \subseteq \mathfrak{G}\) to be the closed subset given by \(V(t \star u)\). Therefore, the fiber of \((\mathfrak{G}, 1/2\mathfrak{D}) \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) over \((u, t) \in \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) is \((\mathbb{P}(1, 1, 2), 1/2\mathfrak{D}_{(u, t)})\), where the curve has a singularity of type \(\Sigma\) at the point \([1:0:0]\) if \(t = 0\) as in Definition 3.7. The divisor \(\mathfrak{D}\) is \(\mathbb{Q}\)-Cartier.

If \(\mathfrak{G}\) is the scheme associated to the ideal \((t^d, y^q, z^p)\), then define \(\mathfrak{G}' := \text{Bl}_{\mathfrak{D}}\mathfrak{G}\), and let \(\mathfrak{D}' \subseteq \mathfrak{G}'\) be the strict transform of \(\mathfrak{D}\). Then, the fiber of \((\mathfrak{G}', 1/2\mathfrak{D}') \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) over \((u, 0) \in \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) is the gluing of \((Y, G + 1/2B'|_Y)\) and \((Z, E + 1/2B'|_Z)\) as in Section 3.2. We have that \(K_{\mathfrak{G}'}\) and \(\mathfrak{D}'\) are both \(\mathbb{Q}\)-Cartier.

In particular, for \(N\) large enough and divisible by \(p\) and \(q\) across the eight singularity types, we have that \(N (K_{\mathfrak{G}'} + 1/2\mathfrak{D}')\) is Cartier and it restricts to the fibers \(\mathfrak{F} \subseteq \mathfrak{G}'\) giving the Cartier divisor \(N (K_{\mathfrak{F}} + 1/2D)\). Then \((\mathfrak{G}', 1/2\mathfrak{D}') \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) is a family of KSBA stable pairs as in Section 3.2. Both \((\mathfrak{G}', 1/2\mathfrak{D}') \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) and \(\mathfrak{D}' \to P(U)_{\text{reg}} \times \Delta\) are flat as they are dominant morphisms from integral schemes to normal schemes with reduced fibres of constant dimension [35, Lemma 10.12]. In particular, \((\mathfrak{G}', 1/2\mathfrak{D}') \to \mathbb{P}(U_{\Sigma})_{\text{reg}} \times \Delta\) is a well-defined family of KSBA stable pairs for the Viehweg’s moduli stack with \(N\) as above (see Definition 2.5). In particular, it induces a morphism \(f_{\Sigma}: \mathbb{P}(U)_{\text{reg}} \times \Delta \to \overline{M}\) to the KSBA compactification of the moduli space of Horikawa surfaces. Then, we define the boundary stratum \(D_{\Sigma} \subseteq \overline{M}\) as the Zariski closure with the reduced scheme structure of the image \(f_{\Sigma}(\mathbb{P}(U)_{\text{reg}} \times \{0\})\).

For the rest of this section, the goal is to prove the following result.

**Theorem 4.2.** For each singularity type \(\Sigma\), \(D_{\Sigma}\) is a divisor in \(\overline{M}\).

Let \(\Sigma\) be one of the eight singularity types and denote by \(\mu_{\Sigma}\) its Milnor number. The proof of Theorem 4.2, which we are about to discuss, boils down to checking the following for each singularity \(\Sigma\):

1. The dimension of the space of \(\overline{Y}\) will be shown to be \(\mu_{\Sigma} - 2\);
2. The dimension of the space of \(\overline{Z}\) is \(29 - \mu_{\Sigma}\), where \(29\) is the rank of \(h^{1,1}\) of a smooth Horikawa surface;
(3) The deformations of $\tilde{Y}$ and $\tilde{Z}$ are independent.

We first need some preliminaries.

**Lemma 4.3** [36, Proposition 5.1]. Let $a$ and $b$ be positive integers such that $1 < a < b$, $\gcd(a, b) = 1$. Then,

$$\dim(\text{Aut}(\mathbb{P}(1, a, b))) = 4 + \left\lfloor \frac{b}{a} \right\rfloor, \quad \dim(\text{Aut}(\mathbb{P}(1, 1, a))) = 5 + a.$$  

In particular, $\dim(\text{Aut}(\mathbb{P}(1, 1, 2))) = 7$ and the dimensions of $\text{Aut}(\mathbb{P}(1, p, q))$ across the eight singularity types are given by the table below.

| Sing. | $E_{12}$ | $E_{13}$ | $E_{14}$ | $Z_{11}$ | $Z_{12}$ | $Z_{13}$ | $W_{12}$ | $W_{13}$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| dim(\text{Aut}(\mathbb{P}(1, p, q))) | 6       | 6       | 6       | 5       | 5       | 5       | 5       | 5       |

**Lemma 4.4.** Let $\Gamma_{\Sigma}$ be the subgroup of the automorphism group of $\mathbb{P}(1, 1, 2)$ that preserves $\mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0})_{\text{reg}}$. Then, the dimension of $\Gamma_{\Sigma}$ is as follows:

| Sing. | $E_{12}$ | $E_{13}$ | $E_{14}$ | $Z_{11}$ | $Z_{12}$ | $Z_{13}$ | $W_{12}$ | $W_{13}$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| dim($\Gamma_{\Sigma}$) | 2       | 2       | 2       | 3       | 3       | 3       | 3       | 3       |

**Proof.** In the current proof, we denote $d$ in Table 3 by $\deg$ instead. Let us start by describing the automorphisms in $\Gamma_{\Sigma}$. A generic automorphisms of $\mathbb{P}(1, 1, 2)$ has the following form:

$$\varphi : [x : y : z] \mapsto [a' x + b' y : c' x + d' y : e' z + f' x^2 + g'xy + h'y^2],$$

where $a', b', c', d', e', f', g', h'$ are generic constants. Since $\varphi$ is injective, we must have that $e' \neq 0$. Moreover $a' \neq 0$ because we consider automorphisms sending $[1 : 0 : 0]$ to itself. So, after rescaling we can normalize $e'$ to $1$ obtaining

$$\varphi : [x : y : z] \mapsto [\tilde{x} : \tilde{y} : \tilde{z}] = [ax + by : cx + dy : z + ex^2 + fxy + gy^2]$$

with $a \neq 0$. To preserve $\mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0})_{\text{reg}}$, first we must have that for each monomial $x^iy^jz^k$ such that $i + j + 2k = 10$ and $\text{wt}_\Sigma(x^iy^jz^k) \geq 0$, the monomials appearing in

$$\tilde{x}^i\tilde{y}^j\tilde{z}^k = (ax + by)(cx + dy)(z + ex^2 + fxy + gy^2)^k = \sum_{\ell,m,n} c_{\ell,m,n} x^\ell y^m z^n$$

also have nonnegative weight. With this we can explicitly describe $\varphi$: For each $x^iy^jz^k$, let $C_{i,j,k}$ be the set of coefficients $c_{\ell,m,n}$ such that $\text{wt}_\Sigma(x^\ell y^m z^n) < 0$. Define $I_{\Sigma}$ to be the ideal generated by the sets $C_{i,j,k}$ for all $i, j, k$. The construction of $I_{\Sigma}$ and a primary decomposition for it can be automatized with a computer using the following SageMath code [60]: (Here, we use $p = 3$, $q = 4$, $\deg = 15$ as an example, which correspond to $\Sigma = Z_{11}$. These values can be changed according to $\Sigma$.)

```python
R.<a,b,c,d,e,f,g>=PolynomialRing(QQ)
P=R.fraction_field()
S.<x,y,z>=PolynomialRing(P)
p=3; q=4; deg=15;
Coeff=[]
for i in range(0,10+1):
    for j in range(0,10+1):
        for k in range(0,5+1):
            if i+j+2*k==10 and 0*i+p*j+q*k==deg:
                Coeff.append([i,j,k])
```

\[ M = x^i y^j z^k \]
NewM=M.substitute(x=a*x+b*y,y=c*x+d*y,z=z+e*x^2+f*x*y+g*y^2);
for v in NewM.exponents():
    if v[0]*0+v[1]*p+v[2]*q<deg:
        Cijk=NewM.coefficient(x^v[0]*y^v[1]*z^v[2])
Coeff.append(Cijk)
I=Ideal(Coeff)
P.<a,b,c,d,e,f,g>=PolynomialRing(QQ)
J=I.change_ring(P)
print(((J.radical())))

If \( I_\Sigma \) denotes the radical of \( I_\Sigma \), we obtain
\[ J_{E_{12}} = J_{E_{13}} = J_{E_{14}} = (c, e, f, a g), J_{Z_{11}} = J_{Z_{12}} = J_{Z_{13}} = J_{W_{12}} = J_{W_{13}} = (c, e, f), \]
where recall \( a \neq 0 \). On the other hand, to be an element of \( \Gamma_\Sigma \), we must also have that the weight zero monomials of the transformed polynomial have equal coefficients. This imposes one condition on the coefficients of \( f \). To understand this, it is enough to prove this for the associated Lie algebra. More precisely, the action of \( \Gamma_\Sigma \) on \( \mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0}) \) reg gives a representation of the corresponding Lie algebra \( \gamma_\Sigma \) on the tangent space \( T_u(\mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0}) \) reg) for any \( u \in \mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0}) \) reg. So, the Lie algebra \( \gamma_\Sigma \) acts linearly on \( T_u(\mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0}) \) reg), showing that we only need to impose one linear condition of the fact that these two coefficients are equal.

These considerations together give the dimension count for \( \Gamma_\Sigma \) in the statement.

Proof of Theorem 4.2. As discussed in Section 3.2, it will be equivalent to count the dimension of the space of isomorphism classes of stable pairs \( (X'_0, 1_{B'_0}) \), which recall is the gluing of two pairs: \( (Y, G + \frac{1}{2} B'_1|_Y) \) and \( (Z, E + \frac{1}{2} B'_1|_Z) \).

The dimension of the space of pairs \( (Y, G + \frac{1}{2} B'_1|_Y) \) is equal to the dimension of the projectivized vector space \( V_{p,q,d} \) of degree \( d \) curves in \( \mathbb{P}(1,p,q) \) with equal nonzero coefficient for \( \theta(m_1) \) and \( \theta(m_2) \) (see Lemma 3.18) minus the dimension of the subgroup \( G_\Sigma \leq \text{Aut}(\mathbb{P}(1,p,q)) \), which preserves the equality of these two coefficients, and hence has codimension 1 in \( \text{Aut}(\mathbb{P}(1,p,q)) \) (for the dimension of the latter see Lemma 4.3). Therefore, we obtain the following:

| Sing. | \( E_{12} \) | \( E_{13} \) | \( E_{14} \) | \( Z_{11} \) | \( Z_{12} \) | \( Z_{13} \) | \( W_{12} \) | \( W_{13} \) |
|-------|----------------|----------------|----------------|-------------|-------------|-------------|-------------|-------------|
| dim(\( \mathbb{P}(V_{p,q,d}) \)) - dim(\( G_\Sigma \)) | 10 | 11 | 12 | 9 | 10 | 11 | 10 | 11 |

The dimension of the space of pairs \( (Z, E + \frac{1}{2} B'_1|_Z) \) is equal to the dimension of the projectivized vector space of coefficients \( U_{\Sigma, +} \oplus U_{\Sigma, 0} \) (for this dimension, we refer to the tables in Proposition 3.2) minus the dimension of the group \( \Gamma_\Sigma \), which was computed in Lemma 4.4. Therefore, we obtain the following:

| Sing. | \( E_{12} \) | \( E_{13} \) | \( E_{14} \) | \( Z_{11} \) | \( Z_{12} \) | \( Z_{13} \) | \( W_{12} \) | \( W_{13} \) |
|-------|----------------|----------------|----------------|-------------|-------------|-------------|-------------|-------------|
| dim \( \mathbb{P}(U_{\Sigma, +} \oplus U_{\Sigma, 0}) \) - dim(\( \Gamma_\Sigma \)) | 17 | 16 | 15 | 18 | 17 | 16 | 17 | 16 |

Finally, let us discuss the gluing of the two pairs along \( G \) and \( E \). Let \( g_p,g_q \in G \) (resp. \( e_p,e_q \in E \)) be the torus fixed points with singularities of type \( \frac{1}{p}(1,q), \frac{1}{q}(1,p) \) (resp. \( \frac{1}{p}(1,-q), \frac{1}{q}(1,-p) \)). Denote by \( g_b \in G \) (resp. \( e_b \in E \)) the point in \( B'_1|_Y \cap G \) (resp. \( B'_1|_Z \cap E \)) different from \( g_p,g_q \) (resp. \( e_p,e_q \)) (see Lemma 3.19). Then, the pointed curves \( (G;g_p,g_q,g_b) \) and \( (E;e_p,e_q,e_b) \) are identified via the unique isomorphism such that \( g_p \mapsto e_p, g_q \mapsto e_q, g_b \mapsto e_b \). Having that the coefficients of the monomials \( m_1,m_2 \) are equal, implies that we fix the points \( g_b \) and \( e_b \). In particular, there is no moduli associated with the gluing.

Adding up the two contributions for each \( \Sigma \), we obtain 27 as the dimension of the KSBA boundary stratum \( D_\Sigma \).
5 | RELATION WITH THE GIT COMPACTIFICATION

5.1 | GIT stability of the eight singularity types

Recall from Section 2.1 that the parameter space of Horikawa surfaces is the 32-dimensional vector space

\[ W = H^0(P^1, \mathcal{O}_{P^1}(4)) \oplus H^0(P^1, \mathcal{O}_{P^1}(6)) \oplus H^0(P^1, \mathcal{O}_{P^1}(8)) \oplus H^0(P^1, \mathcal{O}_{P^1}(10)). \]

We have a natural action \( GL_2 \bowtie W \), which is induced by linear change of coordinates in \( x \) and \( y \). It turns out that the isomorphism classes of Horikawa surfaces coincide with the orbits of this \( GL_2 \)-action [61, Lemma 7]. Therefore, we obtain the quotient described in Definition 2.3:

\[ \overline{M}^{\text{GIT}} = P\left( \bigoplus_{k=2}^{5} H^0(P^1, \mathcal{O}_{P^1}(2k)) \right) // \langle 1 \rangle SL_2. \]

**Theorem 5.1.** Let \( X \) be a double cover of \( \mathbb{P}(1,1,2) \) with branch curve of degree 10. If \( X \) has isolated log canonical singularities or isolated singularities of type \( E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13} \), then \( X \) is GIT stable.

**Proof.** First, we characterize the GIT nonstable points. It will be convenient to denote a point in \( P(W) \) by \([q(x,y)]\), where

\[ q(x,y) = (q_4(x,y), q_6(x,y), q_8(x,y), q_{10}(x,y)) \]

is identified with the vector in \( W \) of coefficients of \( q_4(x,y), ..., q_{10}(x,y) \). The point \([q(x,y)]\) is stable if and only if, for every one-parameter subgroup \( \lambda(t) \) of \( SL_2 \), the limit

\[ \lim_{t \to 0} \lambda(t) \cdot q(x,y) \]

does not exist (see [52, Theorem 7.4]). Therefore, a point is not stable if and only if there exists a one-parameter subgroup \( \lambda(t) \) such that each of the following limits exist:

\[ \lim_{t \to 0} \lambda(t) \cdot q_4(x,y), \quad \lim_{t \to 0} \lambda(t) \cdot q_6(x,y), \quad \lim_{t \to 0} \lambda(t) \cdot q_8(x,y), \quad \lim_{t \to 0} \lambda(t) \cdot q_{10}(x,y). \]

Up to a change of coordinates, we can suppose that \( \lambda(t) = \text{diag}(t^a, t^{-a}) \) with \( a > 0 \). The existence of the limits implies that \( q_4(x,y), ..., q_{10}(x,y) \) can be written as follows:

\[ q_4(x,y) = x^2 h_2(x,y), \quad q_6(x,y) = x^3 h_3(x,y), \quad q_8(x,y) = x^4 h_4(x,y), \quad q_{10}(x,y) = x^5 h_5(x,y), \]

where the polynomials \( h_i(x,y) \) are homogeneous of degree \( i \). Suppose that \( h_i(x,y) \) are generic and define

\[ [h(x,y)] := [x^2 h_2(x,y) : x^3 h_3(x,y) : x^4 h_4(x,y) : x^5 h_5(x,y)]. \]

Then, a point \([q(x,y)] \in P(W)\) is not stable if and only if it is in the closure of the \( SL_2 \)-orbit of \([h(x,y)]\), because the nonstable locus is closed in \( P(W) \).

Let \( X_h \) be the hypersurface in \( P(1,1,2,5) \) given by

\[ w^2 = z^5 + x^2 h_2(x,y)z^3 + x^3 h_3(x,y)z^2 + x^4 h_4(x,y)z + x^5 h_5(x,y). \]

The above surface has a singularity at \([0:1:0:0]\). So, if we consider the affine patch associated to \( y \neq 0 \), then the affine equation of our singularity

\[ w^2 = z^5 + x^2 h_2(x,1)z^3 + x^3 h_3(x,1)z^2 + x^4 h_4(x,1)z + x^5 h_5(x,1), \]
can be written as
\[ w^2 = p_5(x, z) + r(x, z), \quad \text{deg } (r(x, z)) > 5, \]
where \( p_5(x, z) \) is a homogeneous polynomial of degree 5. Therefore, by definition, we have that \( X_h \) has either a \( N_{16} \) singularity [10, p. 13] or a degeneration of it. By using Arnold’s work in [10], we know that the Milnor number and the modality of the \( N_{16} \) singularity are 16 and 3, respectively. As the Milnor number \( \mu \) and the modality \( m \) are upper semicontinuous invariants, see [33, section I.2.1], any nonstable surface with isolated singularities must have a singularity that satisfies \( \mu \geq 16 \) and \( m \geq 3 \). On the other hand, the classification in [10] and [49] implies that the isolated log canonical singularities and the eight singularity types in the statement satisfy the inequalities \( \mu \leq 14 \) and \( m \leq 1 \). Therefore, they are stable. \( \square \)

### 5.2 Extending the morphism from KSBA to GIT

**Theorem 5.2.** The rational map \( \mathbf{M} \to \mathbf{M}^{\text{git}} \) extends to a dense open subset of each of the eight boundary divisors \( D_\Sigma \subseteq \mathbf{M} \) in Definition 4.1.

To prove this, we need a preliminary lemma, which is a slight generalization of [8, Lemma 3.18] (see also [29, Theorem 7.3]).

**Lemma 5.3.** Let \( X \) and \( Y \) be proper varieties with \( X \) normal. Let \( \varphi : X \to Y \) be a rational map, which is regular on an open dense subset \( U \subseteq X \). Let \( (C, 0) \) be a regular curve and \( f : C \to X \) a morphism whose image meets \( U \). Let \( g_f : C \to Y \) be the unique extension of \( \varphi \circ f \), which exists by the properness of \( Y \).

Let \( V \subseteq X \) be another dense open subset containing \( U \). Assume that for all \( f \) with the same \( f(0) \in V \), there are only finitely many possibilities for \( g_f(0) \). Then, \( \varphi \) can be extended uniquely to a regular morphism \( V \to Y \).

**Proof.** Following the proof of [8, Lemma 3.18], let \( Z \subseteq X \times Y \) be the closure of the graph of \( U \to Y \). By hypothesis, the proper birational morphism \( Z \to X \) is finite on \( V \), so the base change \( Z_V := Z \times_X V \to V \) is also proper, finite, and birational. As \( V \) is normal, \( Z_V \to V \) is an isomorphism by the Zariski Main Theorem, hence we obtain the claimed extension \( V \to Y \) by composing \( V \to Z_V \) with the restriction to \( Z_V \) of \( Z \to Y \). \( \square \)

**Proof of Theorem 5.2.** Fix one of the eight singularity types \( \Sigma \). A dense open subset \( D_\Sigma \subseteq D_\Sigma \) parameterizes stable pairs \( (X_0, \frac{1}{2}B_0') \) given by the gluing of
\[
\left( Y, G + \frac{1}{2}B'|_Y \right) \quad \text{and} \quad \left( Z, E + \frac{1}{2}B'|_Z \right)
\]
as discussed in Section 3.2. We want to show that the birational map \( \mathbf{M} \to \mathbf{M}^{\text{git}} \), which is an isomorphism on \( \mathbf{M} \), extends to a birational morphism on \( \mathbf{M} \sqcup \bigsqcup_\Sigma D_\Sigma \), which is open in \( \mathbf{M} \).

To prove this, let \( x \in D_\Sigma \) and consider an arbitrary \( f : (C, 0) \to \mathbf{M} \), where \( C \) is a regular curve, \( 0 \in C \), \( f(0) = x \), and \( f(C) \cap \mathbf{M} \neq \emptyset \). Let \( g_f : C \to \mathbf{M}^{\text{git}} \) be the unique extension. Then, there is only one possibility for \( g_f(0) \), which parameterizes the following GIT stable orbit. The point \( x \) parameterizes a pair \( (X_0', \frac{1}{2}B_0') \) given by the gluing of \( \left( Y, G + \frac{1}{2}B'|_Y \right) \) and \( \left( Z, E + \frac{1}{2}B'|_Z \right) \). Recall that \( Z \) is the weighted blow up of \( \mathbb{P}(1, 1, 2) \) at the point \( \xi = [1 : 0 : 0] \), and under this blow up the curve \( B'|_Z \) is mapped to a curve in \( \mathbb{P}(1, 1, 2) \) with an equation given by
\[
(\pi_+(u) + \pi_0(u))(x, y, z) = 0,
\]
where \( u \in \mathbb{P}(U_\Sigma)_{\text{reg}} \) (see Section 3.1). This has a unique singularity of type \( \Sigma \) at \( [1 : 0 : 0] \), which we know is GIT stable by Theorem 5.1. In other words, \( g_f(0) \) can be uniquely reconstructed from \( \left( Z, E + \frac{1}{2}B'|_Z \right) \), which only depends on the point \( x \) and not from the choice of \( f : (C, 0) \to (\mathbf{M}, x) \). Since \( \mathbf{M} \) is normal (see Definition 2.6), we are done by Lemma 5.3. \( \square \)
Next, we study the behavior of the Hodge structure associated with our stable surfaces. Let $f : \mathcal{X} \rightarrow \Delta$ be a semistable degeneration with central fiber $X_0 = f^{-1}(0)$. Let $X_\eta = f^{-1}(\eta)$ be a generic fiber of $f$ and $H^k_{\text{lim}}(X_\eta, \mathbb{Q})$ denote the $\mathbb{Q}$-limit mixed Hodge structure of $R^k f_*(\mathbb{Q})$, that is, the underlying $\mathbb{Q}$-vector space is $H^k(X_\eta, \mathbb{Q})$, but the Hodge and weight filtrations arise from the asymptotic behavior of the period map. See [51, 53] for an introduction.

**Theorem 6.1.** Let $\pi : S \rightarrow \Delta$ be a one-parameter degeneration of complex projective surfaces, which is smooth over $\Delta^* = \Delta \setminus \{0\}$ such that

(a) if $t \neq 0$, then $S_t = \pi^{-1}(t)$ has geometric genus 2;
(b) the central fiber $S_0 = \pi^{-1}(0)$ is the union of two irreducible components $\bar{Y}$ and $\bar{Z}$, each of which has $h^2(\mathcal{O}) = 1$ and at worst rational singularities.

Then, the local system $\mathcal{Y}_\mathbb{Q} = R^2 \pi_*(\mathbb{Q})$ over $\Delta^*$ has finite monodromy.

**Proof.** Let us consider a semistable degeneration

$$
\begin{array}{ccc}
\hat{S} & \longrightarrow & S \\
\downarrow & & \downarrow \\
\hat{\Delta} & \longrightarrow & \Delta
\end{array}
$$

where $\hat{\Delta} \rightarrow \Delta$ is a morphism of the form $t \mapsto t^n$ for some $n \geq 1$ and the central fiber $\hat{S}_0$ is reduced and simple normal crossing. In particular, in order to prove that $\mathcal{Y}_\mathbb{Q}$ has finite local monodromy, it is sufficient to prove that the corresponding local system attached to $\hat{S} \rightarrow \hat{\Delta}$ has trivial local monodromy operator $T$.

Let

$$
\hat{S}_0 = \bigcup_{i=1}^n \hat{S}_{0i}
$$

be the decomposition into irreducible components of the central fiber of $\hat{S} \rightarrow \hat{\Delta}$. Given $\eta \in \hat{\Delta} \setminus \{0\}$, by [51, p. 118], we have that

$$
p_g(\hat{S}_\eta) \geq \sum_{i=1}^n p_g(\hat{S}_{0i}),
$$

and equality holds if and only if $N = \log(T) = 0$. So let us prove that equality holds.

By the semistable reduction process, we have that the surfaces $\bar{Y}$ and $\bar{Z}$ are birational to $\hat{S}_{0j}$ and $\hat{S}_{0k}$ for some distinct $j, k \in \{1, \ldots, n\}$. Since $\bar{Y}$ and $\bar{Z}$ have rational singularities, we can conclude that $p_g(\hat{S}_{0j}) = h^2(\mathcal{O}_{\bar{Y}})$ and $p_g(\hat{S}_{0k}) = h^2(\mathcal{O}_{\bar{Z}})$, which are both equal to 1 by hypothesis. Thus,

$$
2 = p_g(\hat{S}_\eta) \geq \sum_{i=1}^n p_g(\hat{S}_{0i}) \geq h^2(\mathcal{O}_{\bar{Y}}) + h^2(\mathcal{O}_{\bar{Z}}) = 2,
$$

and hence equality holds in (6.2). \qed

In particular, this theorem applies to the one-parameter stable degeneration of Horikawa surfaces whose central fiber $S_0$ is in the form $\bar{Y} \cup \bar{Z}$ as described in Section 3.2. In this case,

1. the generic surface is a smooth Horikawa surface which has $p_g = 2$;
2. the surface $\bar{Y}$ is an ADE K3 surface by Proposition 3.27;
3. the surface $\bar{Z}$ has only finite cyclic quotient singularities by Proposition 3.31 (hence rational singularities by [44, Proposition 5.15]) and $h^2(\mathcal{O}_{\bar{Z}}) = 1$ by Proposition 3.27.
Looking ahead to Theorem 6.10, we note that the mixed Hodge structures on $H^2(\tilde{Y}, \mathbb{Q})$ and $H^2(\tilde{Z}, \mathbb{Q})$ are pure of weight 2. This is a well-known result in the case of ADE K3 surfaces. On the other hand, since $\tilde{Z}$ has only finite cyclic quotient singularities, it is a Kähler V-manifold, and hence $H^2(\tilde{Z}, \mathbb{Q})$ admits a pure Hodge structure of weight 2.

To continue, we recall the following result of Griffiths.

**Theorem 6.2** [56, section 4.11]. Let $\varphi : \Delta^* \to \Gamma \setminus \mathbb{D}$ be the period map of a variation of pure Hodge structure over the punctured disk $\Delta^* = \Delta \setminus \{0\}$. If the local monodromy operator $T$ of $\varphi$ has finite order, then $\varphi$ extends holomorphically to the disk $\Delta$.

**Corollary 6.3.** The period maps defined by the families $S$ and $\tilde{S}$ of Theorem 6.1 extend holomorphically to the full disk. The limit mixed Hodge structure of the semistable degeneration $\tilde{S}$ is pure.

**Proof.** The proof of Theorem 6.1 shows that both $S$ and $\tilde{S}$ have finite local monodromy. Hence, apply Theorem 6.2. In the case of $\tilde{S}$, the local monodromy $T = e^N = id$ and hence the limit mixed Hodge structure is pure by Theorem 6.16 of [56].

Returning to the first paragraph of this section, let $f : \mathcal{X} \to \Delta$ be a semistable degeneration with central fiber $X_0$ and $X_\eta$ be a generic fiber of $\mathcal{X}$. Let $H^k(X_0, \mathbb{Q}) \to H^k(X_\eta, \mathbb{Q})$ denote the composite map

$$H^k(X_0, \mathbb{Q}) \xrightarrow{\cong} H^k(\mathcal{X}, \mathbb{Q}) \to H^k(X_\eta, \mathbb{Q})$$

defined via the inclusion $X_\eta \hookrightarrow \mathcal{X}$ and the retraction $\mathcal{X} \to X_0$.

**Theorem 6.4** Clemens–Schmid Sequence [20, 51, 53]. Let $f : \mathcal{X} \to \Delta$ be a semistable degeneration and $d = \dim_{\mathbb{C}}(\mathcal{X})$. Then,

$$\cdots \to H_{2d-k}(X_0, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}) \to H^k(\mathcal{X}, \mathbb{Q}) \xrightarrow{\cong} H^k(X_\eta, \mathbb{Q}) \to H_{2d-k-2}(X_0) \to \cdots$$

is an exact sequence of mixed Hodge structures (after appropriate Tate twists), where $T = e^N$ denotes the local monodromy of $R^k f_*(\mathbb{Q})$.

We now specialize the previous theorem to the case where $d = 3, k = 2$, and $N = 0$ on $H^2_{\lim}(X_\eta, \mathbb{Q})$. Let $\mathbb{Q}(\ell)$ denote the pure $\mathbb{Q}$-Hodge structure of type $(-\ell, -\ell)$ of rank 1 with $\mathbb{Q}$-structure $(2\pi i)^\ell \mathbb{Q} \subseteq \mathbb{C}$. Then,

$$0 \to H^0_{\lim}(X_\eta, \mathbb{Q}) \to H_4(X_0, \mathbb{Q}) \to H^2(X_0, \mathbb{Q}) \to H^2_{\lim}(X_\eta, \mathbb{Q}) \to 0$$

is an exact sequence. The local system $R^0 f_*(\mathbb{Q})$ over $\Delta^*$ is the constant variation of Hodge structure $\mathbb{Q}(0)$, and hence the previous sequence becomes

$$0 \to \mathbb{Q}(0) \to H_4(X_0, \mathbb{Q}) \to H^2(X_0, \mathbb{Q}) \to H^2_{\lim}(X_\eta, \mathbb{Q}) \to 0.$$

Adding the correct Tate twists [51, p. 108], the sequence becomes:

$$0 \to \mathbb{Q}(0) \xrightarrow{(-2, -2)} H_4(X_0, \mathbb{Q}) \xrightarrow{(3,3)} H^2(X_0, \mathbb{Q}) \xrightarrow{(0,0)} H^2_{\lim}(X_\eta, \mathbb{Q}) \to 0. \quad (6.3)$$

To simplify the previous sequence, we note that since we are considering a degeneration of surfaces, it follows that (see [51, p. 117])

$$F^{-1} \text{Gr}_W^H H_4(X_0, \mathbb{C}) = 0,$$

and hence $\text{Gr}_W^H H_4(X_0, \mathbb{Q})$ is pure of type $(-2, -2)$, that is,

$$\text{Gr}_W^H H_4(X_0, \mathbb{Q}) \cong \mathbb{Q}(2)^{\oplus(r+1)} \quad (6.4)$$
for some integer \( r \geq 0 \). Therefore, combining (6.3) with (6.4), we obtain an exact sequence of pure Hodge structures of weight 2

\[
0 \to \mathbb{Q}(-1)^{\oplus r} \to \text{Gr}_2^W H^2(X_0, \mathbb{Q}) \to H^2_{\text{lim}}(X_\eta, \mathbb{Q}) \to 0. \tag{6.5}
\]

To show that this sequence splits, we recall the following.

**Theorem 6.5** [21, 22]. *The mixed Hodge structure on the rational cohomology of a complex algebraic variety is graded-polarizable.*

Accordingly, after selecting a choice of polarization of \( \text{Gr}_2^W H^2(X_0, \mathbb{Q}) \), we obtain a direct sum decomposition

\[
\text{Gr}_2^W H^2(X_0, \mathbb{Q}) \cong \mathbb{Q}(-1)^{\oplus r} \oplus (\mathbb{Q}(-1)^{\oplus r})^\perp, \quad (\mathbb{Q}(-1)^{\oplus r})^\perp \cong H^2_{\text{lim}}(X_\eta, \mathbb{Q}). \tag{6.6}
\]

**Definition 6.6.** Let \( A \) be a \( \mathbb{Q} \)-Hodge structure of weight 2 with \( F^3A = 0 \). Then, the **transcendental part of** \( A \), denoted by \( T[A] \), is the smallest \( \mathbb{Q} \)-sub-Hodge structure of \( A \) such that \( F^2A \subseteq T[A] \mathbb{C} \).

**Lemma 6.7.** Assume that \( A \) and \( B \) are pure \( \mathbb{Q} \)-Hodge structures of weight 2 such that \( F^3A = F^3B = 0 \). Then, \( T[A \oplus B] = T[A] \oplus T[B] \).

**Proof.** The Hodge filtration is an exact functor from the category of mixed Hodge structures to the category of \( \mathbb{C} \)-vector spaces. In particular, \( F^2(A \oplus B) = F^2A \oplus F^2B \). By definition, \( T[A] \mathbb{C} \supseteq F^2A \) and \( T[B] \mathbb{C} \supseteq F^2B \) and hence \( (T[A] \oplus T[B]) \mathbb{C} \supseteq F^2A \oplus F^2B = F^2(A \oplus B) \). Therefore, \( T[A] \oplus T[B] \supseteq T[A \oplus B] \). On the other hand, \( T[A \oplus B] \cap (A \oplus 0) \) is a sub-Hodge structure of \( A \oplus 0 \) containing \( F^2A \). So \( T[A \oplus B] \cap (A \oplus 0) \supseteq F^2A \oplus 0 \), which, obviously implies that \( T[A \oplus B] \supseteq T[A] \oplus 0 \). Symmetrically, \( T[A \oplus B] \supseteq 0 \oplus T[B] \), and hence \( T[A \oplus B] \supseteq T[A] \oplus T[B] \). \( \Box \)

By applying Lemma 6.7 to (6.6) we obtain the following result.

**Corollary 6.8.** In the setting of Equation (6.6), \( T[\text{Gr}_2^W H^2(X_0, \mathbb{Q})] \cong T[H^2_{\text{lim}}(X_\eta, \mathbb{Q})] \).

As a prelude to the next result, we recall that if \( S = S_1 \cup S_2 \) is the union of nonsingular projective surfaces intersecting transversely, then there exists a Mayer–Vietoris sequence

\[
\ldots \to H^1(S_1 \cap S_2) \to H^2(S) \to H^2(S_1) \oplus H^2(S_2) \to H^2(S_1 \cap S_2) \to \ldots
\]

With the exception of \( H^2(S) \), all of the terms in this sequence carry pure Hodge structures of weight equal to the cohomological degree. Moreover, all of these maps are morphisms of mixed Hodge structure. Therefore,

\[
\text{Gr}_2^W H^2(S) \cong \ker(H^2(S_1 \cup S_2) \to H^2(S_1 \cap S_2)),
\]

since \( H^2(S_1 \cup S_2) = H^2(S_1) \oplus H^2(S_2) \). Equivalently, after extending the definition of the Néron–Severi group additively across disjoint unions, the previous equation becomes

\[
\text{Gr}_2^W H^2(S, \mathbb{Q}) \cong \ker(T[H^2(S_1 \cup S_2)] \oplus \text{NS}_\mathbb{Q}(S_1 \cup S_2) \to H^2(S_1 \cap S_2, \mathbb{Q})).
\]

In particular, since elements of \( T[H^2(S_1, \mathbb{Q})] \) vanish upon pullback to \( H^2(S_1 \cap S_2, \mathbb{Q}) \), it follows that

\[
T[\text{Gr}_2^W H^2(S, \mathbb{Q})] \cong T[H^2(S_1 \cup S_2, \mathbb{Q})].
\]

More generally, we have the following.

**Lemma 6.9.** Let \( S \) be a projective surface, which has only simple normal crossing singularities. Let \( S = \cup_1 S_i \) denote the decomposition of \( S \) into irreducible components. Then,

\[
T[\text{Gr}_2^W H^2(S, \mathbb{Q})] \cong \bigoplus_i T[H^2(S_i, \mathbb{Q})].
\]
Proof. Let
\[ \Sigma_0 = \bigsqcup_i \delta_i, \quad \Sigma_1 = \bigsqcup_{i < j} \delta_i \cap \delta_j. \]

Then, via the theory of semisimplicial varieties (see [14, section 11]),
\[ \text{Gr}_W^2 H^2(\delta, \mathbb{Q}) \cong \ker(H^2(\Sigma_0, \mathbb{Q}) \xrightarrow{\delta^*} H^2(\Sigma_1, \mathbb{Q})), \]
where the map \( \delta^* \) is constructed from an alternating sum of pullbacks along the inclusion maps \( \delta_i \cap \delta_j \hookrightarrow \delta_i \).

In analogy with our previous discussion of the case where \( \delta \) had only two irreducible components,
\[ H^2(\Sigma_0, \mathbb{Q}) = T[H^2(\Sigma_0, \mathbb{Q})] \oplus \text{NS}_\mathbb{Q}(\Sigma_0). \]
Likewise, the pullback of an element of \( T[H^2(\delta_i)] \) along the inclusion map \( \delta_i \cap \delta_j \hookrightarrow \delta_i \) is zero. Therefore, \( T[H^2(\Sigma_0, \mathbb{Q})] \subset \ker(\delta^*) \) and hence
\[ \ker(H^2(\Sigma_0, \mathbb{Q}) \xrightarrow{\delta^*} H^2(\Sigma_1, \mathbb{Q})) = T[H^2(\Sigma_0, \mathbb{Q})] \oplus \ker(\text{NS}(\Sigma_0)_\mathbb{Q} \xrightarrow{\delta^*} H^2(\Sigma_1, \mathbb{Q})). \]
Thus,
\[ T[\text{Gr}_W^2 H^2(\delta, \mathbb{Q})] = T[H^2(\Sigma_0, \mathbb{Q})] \cong \bigoplus_i T[H^2(\delta_i)]. \]

\[ \square \]

Theorem 6.10. Let \( f : \mathcal{X} \to \Delta \) be a semistable degeneration of projective surfaces with trivial local monodromy (as in the paragraph above (6.5)). Let \( X_0 = \bigcup_j D_j \) be the decomposition of the central fiber \( X_0 \) into irreducible components. Then,
\[ T[H^2_{\text{lim}}(X_\eta, \mathbb{Q})] \cong T[\text{Gr}_W^2 H^2(X_0, \mathbb{Q})] \cong \bigoplus_j T[H^2(D_j, \mathbb{Q})]. \]

Proof. The first isomorphism is Corollary 6.8. The second isomorphism is Lemma 6.9.

\[ \square \]

Corollary 6.11. Let \( \pi : S \to \Delta \) be as in Theorem 6.1 and \( \tilde{S} \to \tilde{\Delta} \) be the corresponding semistable degeneration (6.1). Then,
\[ T[H^2_{\text{lim}}(\tilde{S}_\eta, \mathbb{Q})] \cong T[H^2(\tilde{Z}, \mathbb{Q})] \oplus T[H^2(\tilde{Y}, \mathbb{Q})]. \]

Proof. By Theorem 6.10, the left-hand side is equal to the sum of the transcendental parts of the irreducible components of the central fiber \( \tilde{S}_0 \) of \( \tilde{S} \to \tilde{\Delta} \). If \( D \) is an irreducible component of \( \tilde{S}_0 \) with geometric genus zero, then \( T[H^2(D, \mathbb{Q})] = 0 \). By the proof of Theorem 6.1, this is true for every irreducible component of \( \tilde{S}_0 \) except for the two corresponding to \( \tilde{Z} \) and \( \tilde{Y} \). Since the transcendental part of \( H^2 \) of a surface is a birational invariant, the result follows.

Remark 6.12. In [41], the authors consider various generalizations of the Clemens–Schmid sequence using the decomposition theorem. Of particular relevance to the class of degenerations considered in this paper is Corollary 9.9(i), which asserts the following: Let \( f : \mathcal{X} \to \Delta \) be a flat projective family with \( \mathcal{X} - X_0 \) smooth. Assume that \( X_0 \) is reduced with semi-log canonical singularities and \( \mathcal{X} \) is normal and \( \mathbb{Q} \)-Gorenstein. Then, \( \text{Gr}_F^0 H^k(X_0) \cong \text{Gr}_F^0 H^k_{\text{lim}}(X_i) \) for all \( k \). One consequence of this result is the equality of the Hodge–Deligne numbers \( h^k(X_0)^{p,q} = h^k_{\text{lim}}(X_i)^{p,q} \) for \( pq = 0 \).

7 | BIRATIONAL TYPE OF LIMIT SURFACES

In this section, we show that the minimal model of a generic surface \( S_0 \) in the sense of Definition 3.9 of type \( Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13} \) is a K3 surface. In fact, our method constructs the minimal model as the minimal resolution of a double sextic. Our techniques do not apply to the \( E_{12}, E_{12}, E_{14} \) cases because the method does not produce such plane curve.
**Proposition 7.1.** Let $\Sigma \in \{\Sigma_{11}, \Sigma_{12}, \Sigma_{13}, \Sigma_{12}, \Sigma_{13}\}$ and let $S_0$ be the central fiber of the one-parameter degeneration $S = S(t \star u) \to \Delta$ as in Definition 3.7 with $u \in (U_\Sigma)_{\text{reg}}$ $\Sigma$-generic. In particular, $S_0$ is the double cover of $\mathbb{P}(1, 1, 2)$ with branch curve $B_0$ in (3.2). Then, $S_0$ is birational to a $K3$ surface with ADE singularities, which is the double cover of $\mathbb{P}^2$ branched along a plane sextic $V(H_\Sigma)$. Furthermore, a plane sextic $C$ is projectively equivalent to $V(H_\Sigma)$ if and only if

(i) $\Sigma = \Sigma_{11}$, $C$ is smooth, and there exists a line $L$ such that $L \cap C$ is the union of three points of multiplicities three, two, and one;

(ii) $\Sigma = \Sigma_{12}$, $C$ is smooth, and there exists a line $L$ such that $L \cap C$ is the union of two points of multiplicities two and four;

(iii) $\Sigma = \Sigma_{13}$, $C$ has an $A_1$ singularity at a point $p$, and there exists a line $L$ such that $L \cap C$ is the union of $p$ with multiplicity four and another double point;

(iv) $\Sigma = \Sigma_{12}$, $C$ has an $A_1$ singularity at a point $p$, and there exists a line $L$ such that $L \cap C$ is the union of $p$ with multiplicity three, and other two points with multiplicities two and one;

(v) $\Sigma = \Sigma_{13}$, $C$ has an $A_2$ singularity at a point $p$, and there exists a line $L$ such that $L \cap C$ is the union of $p$ with multiplicity three, and other two points with multiplicities two and one.

Moreover, let $L(\Sigma) \subseteq \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(6)))$ be the locus parameterizing plane sextics that are projectively equivalent to $V(H_\Sigma)$. Then, the following holds:

| Sing. | $Z_{11}$ | $Z_{12}$ | $Z_{13}$ | $W_{12}$ | $W_{13}$ |
|-------|---------|---------|---------|---------|---------|
| dim$(L(\Sigma)) - \dim(\text{Aut}(\mathbb{P}^2))$ | 18 | 17 | 16 | 17 | 16 |

**Proof.** Consider the rational map

$$\mathbb{P}(1, 1, 2, 5) \dasharrow \mathbb{P}(2, 2, 2, 6) \cong \mathbb{P}(1, 1, 1, 3)$$

$$[x : y : z : w] \mapsto [xy : y^2 : z : yw] = [x_0 : x_1 : x_2 : x_3].$$

This is birational as it restricts to an isomorphism of the smooth affine charts where $y \neq 0$ and $x_1 \neq 0$. Notice that this induces a birational map $\mathbb{P}(1, 1, 2) \dasharrow \mathbb{P}^2$, which at the level of the monomials $x^a y^b z^c \in U_{\Sigma, \geq 0} := U_{\Sigma, +} \cup U_{\Sigma, 0}$ is given by

$$\mu(x^a y^b z^c) = \mu((xy)^d y^2 (b-a)/2 z^e) = x_0^a x_1^{(b-a)/2} x_2^e.$$

Indeed, a direct inspection using Proposition 3.2 reveals that $(b-a)/2 + 1$ is always a nonnegative integer for the $Z$ and $W$ series of singularities. Let $P_\Sigma(x, y, z)$ be the polynomial of degree 10 given by a linear combination of monomials in $U_{\Sigma, \geq 0}$ with general coefficients $a_{ijk}$. The vanishing of $P_\Sigma(x, y, z)$ defines the curve $B_0$ in the statement of the proposition. The transformed polynomial

$$H_\Sigma(x_0, x_1, x_2) := \mu(P_\Sigma(x, y, z)) \cdot x_1 = \sum_{x_1 y^2 z^e \in U_{\Sigma, \geq 0}} a_{ijk} \mu(x_1 y^j z^k) \cdot x_1$$

defines a plane sextic curve (this is also verified by inspection using Proposition 3.2), and the surface $S_0$ is birational to the double cover of $\mathbb{P}^2$ branched along $V(H_\Sigma)$. (As remarked at the beginning of Section 7, this construction does not give a plane sextic for the $E$ series because the monomial $x^4 z^3$ has weight zero and is transformed to $x_0^4 x_1^{-1} x_2^3$.) To describe $S_0$, we study the singularities of this plane sextic as follows. We choose a specialization $F_\Sigma$ of the polynomial $H_\Sigma$. If the curve $V(F_\Sigma)$ is smooth, then the general $V(H_\Sigma)$ is also smooth. Or, if the particular curve $V(F_\Sigma)$ has exactly one isolated $A_n$ singularity at $p$ and the general $V(H_\Sigma)$ also has one $A_n$ singularity at $p$, then the singularity of $V(H_\Sigma)$ is unique as well. The reason is that any other singularity of $V(H_\Sigma)$ would degenerate to $p$ along with the $A_n$ one. Therefore, the Milnor number of the singularity at $p$ of the degeneration has to be strictly larger than $n$. This is in contradiction with the characterization of $V(F_\Sigma)$, by considering the sum of the Milnor numbers of the general curve $V(H_\Sigma)$ and the fact that the Milnor number is upper semicontinuous, see [33, Theorem 2.6].

The special polynomials $F_\Sigma$ are provided below, together with the Macaulay2 code [31] that computes their singularities (we use the packages [48, 58]).
needsPackage("Resultants")

needsPackage("NumericalAlgebraicGeometry")

CC\[x_0,x_1,x_2\]

\[FZ11 = x_1^* (x_0^5 + x_1^5 + x_2^5) + x_2^3 * x_0^3\]

\[FZ12 = x_1^* (x_2^* (x_0^4 + x_1^4 + x_2^4) + x_1^* (x_0^4 + x_1^4)) + x_0^2 * x_2^3 * (x_0 + x_2)\]

\[FZ13 = x_1^* (x_2^2 * (x_0^4 + x_1^4)) + x_2^* x_0^3 * x_0^4 * x_0^3\]

\[FW12 = x_1^* (x_0^5 + x_1^5 + x_2^5) + x_0^2 * x_2^3 * x_0^3\]

\[FW13 = x_1^* (x_1^5 + x_2^4 * x_2^4) + x_2^4 * x_2^4\]

for \(i\) in \([FZ11, FZ12, FZ13, FW12, FW13]\) do \(\)

\[\text{if discriminant } i \neq 0 \text{ then}\]

\[\text{print("Non-zero discriminant", discriminant } i )\]

else \(\)

\[Q = \text{sub}(i, x_1 = 1) ;\]

\[R = \text{sub}(i, x_2 = 1) ;\]

\[\text{solsQ = solveSystem diff(x_0, Q), diff(x_2, Q), Q ;}\]

\[\text{solsR = solveSystem diff(x_0, R), diff(x_1, R), R ;}\]

\[\text{print("complex solutions", solsQ, solsR ) ;}\]

The chosen special polynomials \(F_z\) we listed satisfy the condition that the two monomials \(m_1, m_2\) of weight zero (see Proposition 3.2) have equal nonzero coefficient. We have that \(V(\Sigma)\) is smooth for \(\Sigma = Z_{11}, W_{12}\) and it has exactly one singular point if \(\Sigma = Z_{12}, Z_{13}, W_{13}\). More specifically, we prove that \(V(H_{Z_{12}}), V(H_{Z_{13}}), V(H_{W_{13}})\) have an \(A_1, A_2, A_1\) singularity at \(p\), respectively. This will be done on the way as we prove the claimed geometric characterization of the plane sextics.

We now prove the geometric characterization of the plane sextic \(V(H_5)\) given in the statement. In what follows, for a positive integer \(d\), \(p_d\) and \(q_d\) denote homogeneous polynomials of degree \(d\).

(1) If \(\Sigma = Z_{11}\), then, after enumerating the monomials in \(U_{Z_{11}, \geq 0}\), one can observe that

\[H_{Z_{11}} = x_1 p_5(x_0, x_1, x_2) + x_0^2 x_2^3 p_1(x_0, x_2),\]  

(7.1)

where \(p_5(x_0, x_1, x_2) = c_0 x_0^5 + x_1 q_4(x_0, x_1, x_2), p_1(x_0, x_2) = c_0 x_0 + c_2 x_2\) with \(c_0 \neq 0\). The transformations \(\mu(x^3 y^5) \cdot x_1 = x_0^3 x_1^3\) and \(\mu(x^3 y^5) \cdot x_2 = x_0^2 x_2^2\) of the two weight zero monomials have equal nonzero coefficients.

First, observe that \(H_{Z_{11}}\) in (7.1) satisfies the claimed condition with respect to the line \(L = V(x_1)\). For the converse, suppose that \(C\) and \(L\) are as in part (i). Let us write the homogeneous polynomial of degree 6 describing \(C\) as

\[x_1 q_5(x_0, x_1, x_2) + q_6(x_0, x_2).\]

Up to projectivity, we can suppose that \(L = V(x_1)\), and that the intersection points \(C \cap L\) with multiplicities two and three are at \([0:0:1]\) and \([1:0:0]\), respectively. In this case, \(q_6(x_0, x_2) = x_0 x_2^3 q_1(x_0, x_2)\), which shows \(C\) is described by the vanishing of an equation in the form (7.1).

(2) If \(\Sigma = W_{12}\), then using the monomials in \(U_{W_{12}, \geq 0}\), we obtain that

\[H_{W_{12}} = x_1 p_5(x_0, x_1, x_2) + x_0^2 x_2^3 q_2(x_0, x_2),\]

where \(p_5(x_0, x_1, x_2) = c_0 x_0^5 + x_1 p_4(x_0, x_1, x_2) + x_2 q_4(x_0, x_1, x_2)\) with \(c_0 \neq 0\). The transformations \(\mu(x^5 y^5) \cdot x_1 = x_0^5 x_1\) and \(\mu(x^5 y^5) \cdot x_2 = x_0^3 x_2^3\) of the two weight zero monomials have equal nonzero coefficients given by \(c_0\). The claimed characterization for the plane sextic \(C\) is analogous to the one in the \(Z_{11}\) case.

(3) If \(\Sigma = W_{13}\), then from the monomials in \(U_{W_{13}, \geq 0}\), we obtain that

\[H_{W_{13}} = x_1 (x_1 p_4(x_0, x_1, x_2) + x_2 q_4(x_0, x_2)) + c_0 x_0^2 x_2^3,\]

(7.2)

where \(q_4(x_0, x_2) = c_0 x_0^4 + x_2 q_3(x_0, x_2)\) with \(c_0 \neq 0\). The transformations \(\mu(x^4 y^4 z) \cdot x_1 = x_0^4 x_1 x_2\) and \(\mu(x^2 z^4) \cdot x_1 = x_0^2 x_2^4\) of the two weight zero monomials have equal nonzero coefficients given by \(c_0\). The line \(L = V(x_1)\) has the claimed intersections with the plane sextic defined by (7.2). To conclude, we have to show the point of multiplicity 4 is an
A\textsubscript{1} singularity at [1 : 0 : 0]. But this is clear after restricting to the affine patch \(x_0 \neq 0\) and considering the lowest degree part.

In the other direction, by using the action of \(\text{SL}_3\) we can suppose that the line \(L = V(x_1)\), that the intersection points \(C \cap L\) are [1 : 0 : 0] and [0 : 0 : 1] with multiplicities four and two, respectively, so that the singular point \(p \in C\) is supported at [1 : 0 : 0]. Any degree 6 polynomial can be written as

\[
x_1 p_5(x_0, x_1, x_2) + p_6(x_0, x_2).
\]

The condition that \(C \cap L\) has multiplicity two and four at the mentioned points implies \(p_6(x_0, x_2) = a x_0^2 x_2^4\) for some nonzero constant \(a\). Observe that

\[
x_1 p_5(x_0, x_1, x_2) + a x_0^2 x_2^2
\]

\[
= x_1 (x_1 p_4(x_0, x_1, x_2) + x_2 q_4(x_0, x_2) + b x_0^3) + a x_0^2 x_2^2.
\]

After restricting to the affine patch \(x_0 \neq 0\) and considering the lower degree terms, we find that

\[
x_1 (x_1 p_4(1, x_1, x_2) + x_2 q_4(1, x_2) + b x_0^3) + a x_0^2 x_2^2 + \text{higher order terms}.
\]

From this local description, we conclude that \(C\) is singular if and only if \(b = 0\). Furthermore, the singularity is a double point with nondegenerated quadratic terms. Therefore, it is an \(A\textsubscript{1}\) singularity.

(4) If \(\Sigma = Z_{12}\), then using the monomials in \(U_{Z_{12}} \geq 0\), we obtain that

\[
H_{Z_{12}} = x_1 (x_2 p_3(x_0, x_1, x_2) + x_1 x_2 p_3(x_0, x_1) + x_1 p_4(x_0, x_1)) + x_0^2 x_2^2 p_1(x_0, x_1),
\]

where we have \(p_1(x_0, x_2) = c_0 x_0 + c_2 x_2\) and \(p_4(x_0, x_1, x_2) = c_0 x_0^4 + x_1 p_3(x_0, x_1, x_2) + x_2 q_3(x_0, x_1, x_2)\) with \(c_0 \neq 0\). The transformations \(\mu(x^4 y^2 z) \cdot x_1 = x_0^4 x_1 x_2\) and \(\mu(x^4 y z^2) \cdot x_1 = x_0^3 x_1^2\) of the two weight zero monomials have equal nonzero coefficients given by \(c_0\). The characterization of the plane sextic \(V(H_{Z_{12}})\) is analogous to the one for \(W_{13}\).

(5) If \(\Sigma = Z_{13}\), then using the monomials in \(U_{Z_{13}} \geq 0\), we obtain that

\[
H_{Z_{13}} = x_1 (x_2^2 p_3(x_0, x_1, x_2) + x_1 x_2 p_3(x_0, x_1) + x_1 p_4(x_0, x_1)) + x_0^2 x_2^2 p_1(x_0, x_1),
\]

where \(p_1(x_0, x_1) = c_0 x_0 + c_1 x_1\) and \(p_4(x_0, x_1) = c_0 x_0^4 + x_1 p_3(x_0, x_1)\) with \(c_0 \neq 0\). The transformations \(\mu(x^4 y^6) \cdot x_1 = x_0^4 x_1^4\) and \(\mu(x^4 y z^3) \cdot x_1 = x_0^4 x_1\) of the two weight zero monomials have equal nonzero coefficients given by \(c_0\). The line \(L = V(x_1)\) intersects \(V(H_{Z_{13}})\) as claimed in the statement. Let us explain why we have an \(A_2\) singularity at [1 : 0 : 0].

By restricting the curve to the affine patch \(x_0 \neq 0\), we find that the lowest degree monomials in degree 2 and 3 are \(x_1^2, x_1 x_2, x_2, x_1, x_2^3, x_1^2 x_2, x_2^2, x_1^3\). By [12, Lemma 1], we recognize this is an \(A_2\) singularity.

Next, we recover the above shape of the equation from the claimed incidence relation between a plane sextic \(C\) and a line \(L\). If \(C\) is a general curve of degree 6, then it can be written as

\[
x_1 p_5(x_0, x_1, x_2) + p_6(x_0, x_2).
\]

Up to projective transformation, we can assume that \(L = V(x_1)\) and that the two points of tangency are [1 : 0 : 0] and [0 : 0 : 1], so that the equation of \(C\) is

\[
x_1 p_5(x_0, x_1, x_2) + x_0^2 x_2^3 p_1(x_0, x_2),
\]

which we can rewrite as

\[
x_1 (x_2^2 p_3(x_0, x_1, x_2) + a x_2 x_0^4 + x_2 x_1 p_3(x_0, x_1) + x_1 p_4(x_0, x_1) + b x_0^5) + x_0^2 x_2^3 p_1(x_0, x_2).
\]

The hypothesis that \(C\) is singular at [1 : 0 : 0] implies that \(b = 0\). By restricting to the affine patch \(x_0 \neq 0\), we obtain

\[
x_1 (x_2^2 p_3(1, x_1, x_2) + a x_2 + x_2 x_1 p_3(1, x_1) + x_1 p_4(1, x_1)) + x_2^2 p_1(1, x_2).
\]
whose lower degree part is a linear combination of the monomials \( \{ ax_1x_2, x_2^3 \} \). If \( a \neq 0 \), then we would have an \( A_1 \) singularity at \([1 : 0 : 0]\), which we know it is not. Hence, we obtain \( a = 0 \), and we can recover the equation

\[
x_1 \left( x_2^2 p_3(x_0, x_1, x_2) + x_2 x_1 p_3(x_0, x_1) + x_1 p_4(x_0, x_1) \right) + x_0^2 x_3^2 p_1(x_0, x_2),
\]

which has an \( A_2 \) singularity at \([1 : 0 : 0]\) again by [12, Lemma 1].

Finally, we discuss the dimension count for the plane sextics \( V(H_2) \). For \( \Sigma = Z_{11} \), the form of the polynomial \( H_{Z_{11}} \) is given by (7.1), which has 23 distinct monomials. We do not yet impose the equality of the coefficients of \( x_1x_5^3 \) and \( x_3x_0x_3^2 \) because this can be achieved using the \( SL_3 \)-action, which we will account for at the end. We have that \( \dim(L(\Sigma)) - \dim(\text{Aut}(\mathbb{P}^2)) \) is then obtained by considering the number of monomials of \( H_{Z_{11}} \) in (7.1) up to projective scaling of the coefficients, the choice of the line \( L \) (which is \( V(x_1) \) in (7.1)), the choice of the two points in \( L \), and the \( SL_3 \)-action. These considerations yield

\[
\dim(L(Z_{11})) - \dim(\text{Aut}(\mathbb{P}^2)) = (23 - 1 + 2 + 1 + 1) - 8 = 18.
\]

Alternatively, fixed the plane sextic, the line \( L \) is determined up to a finite choice, we choose two points on this line, and then subtract the dimension of the subgroup of automorphisms of \( \mathbb{P}^2 \) that preserve the two points. An analogous argument gives the dimension of \( L(\Sigma) \) for \( \Sigma = Z_{12}, Z_{13}, W_{12}, W_{13} \).

Remark 7.2. Let \( M \) be a complex projective surface. Then, by [57, Lemma 3.1], the transcendental lattice of \( M \) is a birational invariant of \( M \). This implies that the transcendental part \( T[H^2(\hat{S}_0, \mathbb{Q})] \) considered in Section 6 depends only on the birational type the component surfaces \( \bar{Z} \) and \( \bar{Y} \).

Let \( S_0 \) be the central fiber of \( S \to \Delta \) as in the statement of Proposition 7.1. We note that \( S_0 \) is a simply connected variety [23, (B21) Corollary]. As an application of the fact that \( S_0 \) is birational to a K3 surface, we show that \( \bar{Z} \) is also simply connected.

Corollary 7.3. If \( \Sigma \in \{ Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13} \} \), then the corresponding \( \bar{Z} \), as defined in Section 3.7, is simply connected and \( h^{1,1}(\bar{Z}) = 32 - \mu_\Sigma \).

Proof. For a fixed \( \Sigma \) as in our hypothesis, Proposition 7.1, the surface \( \bar{Z} \) is birational to a K3 surface \( P \) with canonical singularities. Therefore, we have a smooth surface \( W \) such that \( \bar{Z} \leftarrow W \to P \). The surfaces \( P \) and \( \bar{Z} \) have log terminal singularities (\( P \) has ADE singularities and for \( \bar{Z} \) see the discussion in the proof of Proposition 3.31), and \( W \) is smooth. Then, their fundamental groups are isomorphic by [59, Theorem 1.1]. Simply connectedness implies \( H^1(\bar{Z}) = 0 \), and since our surfaces have cyclic quotient singularities, Serre duality applies and \( H^3(\bar{Z}) = 0 \) by [53, Corollary 2.48]. By Corollary 3.30, we have \( \chi_{\text{top}}(\bar{Z}) = 36 - \mu_\Sigma \) and \( p_g = 1 \) by Proposition 3.27. Therefore, we obtain \( h^{1,1}(\bar{Z}) = 32 - \mu_\Sigma \).}

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