A Neural-Network based Approach for Nash Equilibrium Seeking in Mixed-order Multi-player Games

Maojiao Ye and Jizhao Yin

Abstract—Noticing that agents with different dynamics may work together, this paper considers Nash equilibrium computation for a class of games in which first-order integrator-type players and second-order integrator-type players interact in a distributed network. To deal with this situation, we first exploit a centralized method for full information games. In the considered scenario, the players can employ its own gradient information, though it may rely on all players’ actions. Based on the proposed centralized algorithm, we further develop a distributed counterpart. Different from the centralized one, the players are assumed to have limited access to the other players’ actions. In addition, noticing that unmodeled dynamics and disturbances are inevitable for practical engineering systems, the paper further considers games in which the players’ dynamics are suffering from unmodeled dynamics and time-varying disturbances. In this situation, an adaptive neural network is utilized to approximate the unmodeled dynamics and disturbances, based on which a centralized Nash equilibrium seeking algorithm and a distributed Nash equilibrium seeking algorithm are established successively. Appropriate Lyapunov functions are constructed to investigate the effectiveness of the proposed methods analytically. It is shown that if the considered mixed-order game is free of unmodeled dynamics and disturbances, the proposed method would drive the players’ actions to the Nash equilibrium exponentially. Moreover, if unmodeled dynamics and disturbances are considered, the players’ actions would converge to arbitrarily small neighborhood of the Nash equilibrium. Lastly, the theoretical results are numerically verified by simulation examples.

Index Terms—Nash equilibrium seeking; mixed-order dynamics; games; distributed network.

I. INTRODUCTION

Game theory is a powerful tool for analyzing decision-making processes in which multiple rational decision-makers interact with each other. For example, optimal charging of plug-in electric vehicles [1], economic dispatch [2], coordination control in mobile sensor networks [3], formation control [4], energy consumption control in smart grids [8][9], to mention just a few, are representatives that fall into the game theoretic framework. Inspired by the broad applications of game theoretic approaches, many researchers devoted themselves to the development of Nash equilibrium seeking strategies and quite a few Nash equilibrium seeking strategies have been reported in the existing literature. For example, games with first-order dynamic players were investigated in [5]-[7] and [27]-[29]. Nash equilibrium seeking protocols for second-order dynamic players and linear time-invariant dynamic players were studied in [10][11] and [12], respectively. Nevertheless, only a few works reported results on Nash equilibrium seeking for heterogeneous multi-player games. In [13], distributed Nash equilibrium seeking algorithms were developed for heterogeneous Euler-Lagrange systems. The work in [14] considered distributed Nash equilibrium computation for games with first-order continuous-time players and discrete-time players. However, due to the different computation capabilities of distinct computing units, various hardware environments and diversities of the agents’ dynamics, multi-agent systems show remarkable and versatile heterogeneities. Inspired by the above observations, heterogeneous multi-agent systems have been widely explored. For example, linear heterogeneous multi-agent systems with distinct constant matrices in the agents’ dynamics were investigated for formation control, output regulation problems and distributed optimal coordination problems in [16]-[18] and [24], respectively. Nonlinear heterogeneous multi-agent systems, in which both the state dynamics and dimensions can be different were addressed in [19]. Heterogeneous multi-agent systems consisting of first-order continuous-time agents and first-order discrete-time agents were studied in [20]. Besides, second-order heterogeneous multi-agent systems, in which the agents’ inertias and control gains were time-varying, were investigated in [21]. In particular, as velocity-actuated vehicles and acceleration-actuated vehicles might work together, heterogeneous multi-agent systems composed of first-order agents and second-order agents occupy an important position [22].

Consensus of heterogeneous multi-agent systems composed of first-order agents and second-order agents without utilizing velocity measurements was investigated in [23]. Average consensus tracking for sensor networks in which velocity-actuated sensors and force-actuated sensors exist simultaneously was investigated in [22]. Two stationary consensus algorithms were designed for discrete-time heterogeneous multi-agent systems composed of first-order agents and second-order agents in [15] with bounded communication delays considered. However, Nash equilibrium computation for mixed-order multi-player games consisting of first-order players and second-order players hasn’t been addressed yet though it is a problem of great importance. Inspired by the above observation, this paper tries to accommodate Nash equilibrium computation for games with mixed-order integrator-type dynamics. Moreover, noticing that in many practical situations, e.g., physical hydraulic systems [32], air hybrid vehicles [33] and marine
surface vessels [34], external disturbances and unmodeled dynamics are inevitable due to complex working environment of engineering actuators and limited knowledge about the explicit system model, this paper further addresses Nash equilibrium computation for mixed-order games in which the players’ dynamics are suffering from unmodeled dynamics and disturbances. Noticing that radial basis function neural network (RBFNN) has been shown to be capable of approximating unknown continuous functions over a compact set (see, e.g., [31][35]-[38]). This paper takes the benefits of the RBFNN to establish robust Nash equilibrium seeking strategies for the considered mixed-order multi-player games. Compared with the existing works, the main contributions of the paper are summarized as follows.

1) Nash equilibrium seeking for mixed-order multi-player games, in which first-order integrator-type players and second-order integrator-type players coexist is investigated in this paper. To the best of the authors’ knowledge, mixed-order multi-player games have rarely been investigated by the existing works. The exploration of this paper would broaden the applicable fields of game theoretic approaches for distributed games with mixed-order players.

2) Games with ideal mixed-order players are investigated, followed by the case in which the players’ dynamics are suffering from unmodeled dynamics and external disturbances. For both situations, a centralized algorithm and a distributed algorithm are proposed. In particular, the unmodeled dynamics are accommodated by neural networks. Compared with the RISE-based method in [26], the conditions on the unmodeled dynamics and disturbances are relaxed to some extent.

3) The convergence results of the proposed algorithms are analytically investigated by utilizing Lyapunov stability analysis. It is theoretically proven that the players’ actions would be driven to the Nash equilibrium if there is no unmodeled dynamics and disturbances. Moreover, with unmodeled dynamics and disturbances, the players’ actions can be driven to arbitrarily small neighborhood of the Nash equilibrium.

The rest of this paper is organized as follows. Notations and preliminaries are given in Section II. The problem formulation is given in Section III and the main results are presented in Sections IV-V. For games with mixed-order players, a centralized algorithm and a distributed algorithm will be given successively. Moreover, the cases in which the game is subject to unmodeled dynamics and disturbances will be considered following the ideal disturbance-free situation. The numerical examples are presented in Section VI and the conclusions are given in Section VII.

II. NOTATIONS AND PRELIMINARIES

Notations: In this paper, we use $\mathbb{R}$ to denote the set of real numbers. The notation max{\(l_i\)} (min{\(l_i\)}) defines the maximum (minimum) value of \(l_i\) for \(i \in \{1, 2, \ldots, N\}\). \(A = [a_{ij}]\) defines a matrix whose entry on the \(ij\)th row and \(j\)th column is \(a_{ij}\). For a symmetric matrix \(Q \in \mathbb{R}^{N \times N}\), \(\lambda_{\min}(Q)\) and \(\lambda_{\max}(Q)\) are the minimum and maximum eigenvalues of \(Q\), respectively. Moreover, \(\text{diag}\{a_{ij}\}\) for \(i, j \in \{1, 2, \ldots, N\}\) denotes a diagonal matrix whose diagonal elements are \(a_{11}, a_{12}, \ldots, a_{1N}, a_{21}, \ldots, a_{NN}\) and \(\otimes\) is the Kronecker product.

Graph theory: For a graph defined as \(G = (\mathcal{N}, \mathcal{E})\), where \(\mathcal{N} = \{1, 2, \ldots, N\}\) is the set of vertices and \(\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}\) is the set of edges. The network is undirected if for every \((i, j) \in \mathcal{E}\), we have \((j, i) \in \mathcal{E}\). In addition, the undirected graph is connected if there is a path between any pair of distinct vertices. The adjacency matrix associated with graph \(G\) is defined as \(A = [a_{ij}]\), where \(a_{ij} = 1\) if node \((j, i) \in \mathcal{E}\), else, \(a_{ij} = 0\) \((a_{ii} = 0)\). Furthermore, the Laplacian matrix associated with \(G\) is defined as \(L = D - A\) where \(D\) is a diagonal matrix whose \(i\)th diagonal entry is equal to the out degree of node \(i\), represented by \(\sum_{j=1}^{N} a_{ij}\) [5].

Radial basis function neural networks: A continuous function \(l(z) : \mathbb{R}^{q} \rightarrow \mathbb{R}^{N}\) can be approximated on a compact set \(z \in \Omega_{z} \subset \mathbb{R}^{N}\) by

\[
l_{NN}(z) = W^{T}S(z),
\]

where \(W \in \mathbb{R}^{q \times N}\) is an adjustable weight matrix, \(q\) is the number of neuron, and \(S(z) = [s_{1}(z), s_{2}(z), \ldots, s_{q}(z)]^{T}\) is the activation function given by

\[
s_{i}(z) = \exp \left( -\frac{(z - \mu_{i})^{T}(z - \mu_{i})}{\rho_{i}^{2}} \right), i = 1, 2, \ldots, q, \tag{2}\]

where \(\mu_{i} = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iN}]^{T}\) is the center of the receptive field, and \(\rho_{i}\) denotes the width of the Gaussian function [31].

Lemma 1: [31] For any arbitrary small positive constant \(\varepsilon\) and \(z \in \Omega_{z}\), there exists a weight matrix \(W^{*} \in \mathbb{R}^{q \times N}\) such that

\[
l(z) = W^{*T}S(z) + \varepsilon, \tag{3}\]

where \(\varepsilon\) is the approximation error that satisfies \(|\varepsilon| \leq \bar{\varepsilon}\).

Lemma 2: [31] Let \(V(t) \geq 0\) be a continuous function defined for all \(t \geq 0\). Suppose that there are positive constants \(a, b\) such that

\[
\dot{V}(t) \leq -aV(t) + b, \tag{4}\]

then

\[
V(t) \leq V(0)e^{-at} + \frac{b}{a}(1 - e^{-at}). \tag{5}\]

Lemma 3: [30] For any \(\varepsilon > 0\) and \(\eta \in \mathbb{R}\),

\[
0 \leq |\eta| - \eta \tanh \left( \frac{\eta}{\varepsilon} \right) \leq K\varepsilon, \tag{6}\]

where \(K\) is a constant that satisfies \(K = e^{-K+1}\).

III. PROBLEM FORMULATION

Consider a game with \(N\) players whose dynamics are governed by

\[
\dot{x}_{f} = u_{f} + \mathcal{Y}(g_{f}(x_{f}) + d_{f}(t)), \quad f \in \nu_{f}
\]

\[
\dot{x}_{s} = u_{s} + \mathcal{T}(g_{s}(x_{s}) + d_{s}(t)), \quad s \in \nu_{s},
\]

where \(x_{f}, x_{s} \in \mathbb{R}, u_{f}, u_{s} \in \mathbb{R}\) are the actions and the control inputs of players \(f\) and \(s\), respectively. Furthermore, \(\nu_{f} = \ldots\)
\{1, 2, \cdots, n\} \text{ and } \nu_s = \{n+1, n+2, \cdots, N\} \quad (N > n). \text{ It’s worth mentioning that following the above definitions, we get that } \mathcal{N} = \nu_f \bigcup \nu_s. \text{ Note that } g_f(x), g_s(x) \text{ and } d_f(t), d_s(t) \text{ are the unmodeled terms and external disturbances whose explicit expressions are unknown. In addition, } \Upsilon \text{ is a variable that is equal to } 0 \text{ or } 1 \text{ and these two cases will be investigated successively in the rest. This paper aims to design control laws to seek the Nash equilibrium } x^* = (x^*_f, x^*_s) \text{ on which }

\begin{equation}
F_t(x^*_f, x^*_s) \leq F_t(x_i, x^*_s),
\end{equation}

for \( x_i \in \mathbb{R}, i \in \mathcal{N} \), and \( x^*_s = [x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]^T \). In addition, \( F_t(x) \), where \( x = [x_1, x_2, \cdots, x_N]^T \), is the cost function of player \( i \).

By Assumption 4, there exists a positive constant \( l_i \), such that \( \|\nabla F_i(x) - \nabla F_i(z)\| \leq l_i \|x - z\| \) for \( x, z \in \mathbb{R}_+^N, i \in \mathcal{N} \). Moreover, \( \|\bar{P}(x) - \bar{P}(z)\| \leq \bar{l}_1 \|x - z\|, \)

\begin{equation}
\|\bar{P}(x) - \bar{P}(z)\| \leq \bar{l}_1 \|x - z\|,
\end{equation}

where \( \bar{P}(x) = [\nabla F_1(x), \nabla F_2(x), \cdots, \nabla F_N(x)]^T \), and \( \bar{l}_1 = \sqrt{N} \max\{l_i\} \).

Theorem 1: Suppose that Assumptions \([13][4]\) are satisfied and

\begin{equation}
k_2 > \frac{(k^2 \bar{l}_1 + 1)^2}{4k_1 m} + k_1 \bar{h}.
\end{equation}

Then, the Nash equilibrium is globally exponentially stable under \([13]\).

Proof: Let \( v_s = \bar{v}_s - k_1 \bar{P}_s(x) \).

Then, by \([13]\) and \([16]\), we get that

\begin{equation}
\dot{v}_s = -k_2 v_s + k_1 \bar{P}_s(x) = -k_2 \bar{v}_s + k_1 H(x) \bar{x}.
\end{equation}

Define the Lyapunov candidate function as

\begin{equation}
V = \frac{1}{2} \|x - x^*\|^2 + \frac{1}{2} \bar{v}_s^T \bar{v}_s.
\end{equation}
Then,
\[
\dot{V} = (x - x^*)(T \dot{x} + \dot{\bar{v}}_s^T \dot{\bar{v}}_s,
\]
\[
= (x_f - x^f)^T \dot{x}_f + (x_s - x_s^*)^T \dot{x}_s
+ \dot{\bar{v}}_s^T (k_2 \bar{v}_s + k_1 H(x) \dot{x})
= - (x - x^*)^T k_1 \bar{P}_f(x)
+ (x_s - x_s^*)^T (\bar{v}_s - k_1 \bar{P}_s(x))
= -k_2\|\bar{v}_s\|^2 + k_1 \bar{v}^T S H(x) \dot{x}
= -k_1(x - x^*)^T \bar{P}(x) + (x_s - x_s^*)^T \bar{v}_s
= -k_2\|\bar{v}_s\|^2 + k_1 \bar{v}^T S H(x) \dot{x} + k_1 \bar{v}^T S H(x) \dot{\phi}_2,
\]
where \( \dot{\phi}_1 = [-k_1(\bar{P}(x))^T \bar{v}_s]^T = 0_n \) and \( \dot{\phi}_2 = [-k_1(\bar{P}(x))^T \bar{v}_s]^T \).

By Assumption 1, \( \|\bar{P}(x)\| = \|\bar{P}(x) - \bar{P}(x^*)\| \leq l_1 \|x - x^*\| \). Hence, \( k_2^2\bar{v}^T S H(x) \bar{P} \leq k_2^2 l_1 h \|x - x^*\|^2 \|\bar{v}_s\| \) by Assumptions 1 and 4. Therefore,
\[
\dot{V} \leq -k_1 m \|x - x^*\|^2 - (k_2 - k_1 h) \|\bar{v}_s\|^2
+ (k_2^2 l_1 h + 1) \|x - x^*\|^2 \|\bar{v}_s\|,
\]
by further utilizing Assumption 3.

Define \( A = \begin{bmatrix} k_1 m & -k_2 l_1 + 1 \\ -k_2 l_1 + 1 & k_2 - k_1 h \end{bmatrix} \) and let \( k_2 > (k_2^2 l_1 m + 1)^2 / k_2 l_1 m + 1 \). Then, \( A \) is symmetric positive definite and
\[
\dot{V} \leq -\lambda_{\min}(A) \|E\|^2,
\]
where \( \lambda_{\min}(A) > 0 \) and \( E = ((x - x^*), \bar{v}_s)^T \).

Recalling the definition of the Lyapunov candidate function, it is clear that
\[
V = \frac{1}{2} \|E\|^2,
\]
Hence,
\[
\dot{V} \leq -2\lambda_{\min}(A) V,
\]
which further indicates that
\[
\|E(t)\| \leq e^{-\lambda_{\min}(A) t} \|E(0)\|.
\]

Define \( E_r(t) = [(x - x^*), \bar{v}_s]^T \). Then,
\[
\|E_r(t)\| \leq \|E(t)\| + k_1 \|\bar{P}_s(x)\| \leq (1 + l_1 k_1) \|E(t)\| \leq (1 + l_1 k_1) e^{-\lambda_{\min}(A) t} \|E(0)\| \leq (1 + l_1 k_1) e^{-\lambda_{\min}(A) t} \|E(0)\| + k_1 \|\bar{P}_s(0)\| \leq (1 + l_1 k_1) e^{-\lambda_{\min}(A) t} \|E_r(0)\|,
\]
thus arriving at the conclusion.

In Section IV.A the players are supposed to have access to their own gradient values. However, the players’ gradient values rely on all the players’ actions, which can hardly be obtained in many practical situations. Hence, in the following, a distributed seeking algorithm will be established.

B. A distributed algorithm for mixed-order multi-player games

To establish a distributed seeking algorithm, the control inputs are designed as
\[
u_f = -k_3 \nabla_F F(y_f), \quad f \in \nu_f
\]
\[
u_s = -k_2 \bar{u}_s - k_1 \bar{v}_s \nabla_s F_s(y_s), \quad s \in \nu_s,
\]
where \( k_1, k_2 \) are positive parameters, \( y_i = [y_{i1}, y_{i2}, \ldots, y_{iN}]^T \) stands for player \( i \)'s local estimates on \( x \) and \( y_{ij} \) is player \( i \)'s estimate on \( x_j \). In addition, \( \nabla_i F_i(x_i) = \nabla_i F_i(x)|_{x=x_i} \). Moreover, motivated by [5], \( y_{ij} \) is generated by
\[
y_{ij} = -k_3 \left( \sum_{k=1}^N a_{ik} (y_{ij} - y_{kj}) + a_{ij} (y_{ij} - x_j) \right),
\]
for all \( i, j \in \mathbb{N} \), where \( k_3 \) is a positive control gain.

Recalling that the players’ dynamics are governed by (11), it can be obtained that
\[
\dot{x}_f = -k_1 \nabla F F(y_f), \quad f \in \nu_f
\]
\[
\dot{x}_s = \bar{u}_s
\]
\[
\dot{v}_s = -k_2 \bar{u}_s - k_1 \bar{v}_s \nabla_s F_s(y_s), \quad s \in \nu_s
\]
\[
\dot{y}_{ij} = -k_3 \left( \sum_{k=1}^N a_{ik} (y_{ij} - y_{kj}) + a_{ij} (y_{ij} - x_j) \right),
\]
in where in the last equation, \( i, j \in \mathbb{N} \).

Writing (28) in its concatenated-vector form yields
\[
\dot{x}_f = -k_1 \bar{P}_f(y_f)
\]
\[
\dot{x}_s = \bar{u}_s
\]
\[
\dot{v}_s = -k_2 \bar{u}_s - k_1 \bar{v}_s \nabla_s F_s(y_s)
\]
\[
\dot{y} = -k_3 (C \otimes I_{N \times N} + A) (y - I_N \otimes x),
\]
where \( y = [y_{11}^T, y_{12}^T, \ldots, y_{1N}^T, \ldots, y_{N1}^T, \ldots, y_{NN}^T]^T \), \( \dot{y}_f = [y_{11}^T, y_{12}^T, y_{1N}^T]^T \), \( \dot{y}_s = [y_{N1}^T, y_{N2}^T, \ldots, y_{NN}^T]^T \), \( \dot{y} = [y_{N1}^T, y_{N2}^T, \ldots, y_{NN}^T]^T \), and \( \bar{P}(\bar{y}) = [\nabla_1 F_1(y_1), \nabla_2 F_2(y_2), \ldots, \nabla_N F_N(y_N)]^T \) and \( \bar{P}(\bar{y}) = [\nabla_1 F_1(y_1), \nabla_2 F_2(y_2), \ldots, \nabla_N F_N(y_N)]^T \).

Moreover, for notational convenience, let \( \bar{P}(y) = [\bar{P}(y_f)^T, \bar{P}(y_s)^T]^T \) and \( H_1(\bar{y}_s) \in \mathbb{R}^{(N - n) \times N^2} \) whose jth row is \( [\bar{A}_{N(1-j)}, \bar{A}_{N(2-j)}, \ldots, \bar{A}_{N(N-j)}] \), \( \bar{A}_{N(1-j)}, \bar{A}_{N(2-j)}, \ldots, \bar{A}_{N(N-j)} \).

Then, the following result can be obtained.

**Theorem 2.** Suppose that Assumptions [14] are satisfied and
\[
k_1 > \frac{1}{2m} + \frac{(\max_{t \in [0, T]} ||I||_1 + 2 \sqrt{N} ||P|| ||I||_1^2) \sigma_1}{2m}
\]
\[
k_2 > 2k_1 ||P|| ||I||_1 + \frac{k_2^2 k_3 b}{2 \sigma_2}
\]
in which \( b = \sup_{y \in [0, T]} \|H_1(\bar{y}_s)||L \otimes I_{N \times N} + A\| \) and \( \sigma_1, \sigma_2 \) are positive constants that can be arbitrarily chosen. Then, the Nash equilibrium is globally exponentially stable under (28).
Proof: Let
\[ \nu_s = \bar{v}_s - k_1 \bar{P}_s(\bar{y}_s). \] (31)
Then, by (29), we have
\[ \bar{v}_s = -k_2 \bar{v}_s. \] (32)
Hence,
\[ \dot{x}_s = v_s - k_1 \bar{P}_s(\bar{y}_s) \]
\[ \dot{\bar{v}}_s = v_s + k_1 H_1(\bar{y}_s) \dot{\bar{y}}_s \]
\[ = -k_2 \bar{v}_s + k_1 H_1(\bar{y}_s) \dot{\bar{y}}_s. \] (33)
Define the Lyapunov candidate function as
\[ V = \frac{1}{2} (x - x^*)^T (x - x^*) + \frac{1}{2} \bar{v}_s^T \bar{v}_s \]
\[ + (y - 1_N \otimes x)^T P(y - 1_N \otimes x). \] (34)
Then, it can be easily concluded that \( d_1 \|E\|^2 \leq V \leq d_2 \|E\|^2 \), where \( d_1 = \min\{\frac{1}{2}, \lambda_{\min}(P)\} \), \( d_2 = \max\{\frac{1}{2}, \lambda_{\max}(P)\} \) and \( E = [(x - x^*)^T, \bar{v}_s^T, (y - 1_N \otimes x)^T]. \)

Moreover,
\[ \dot{V} = (x - x^*)^T \dot{x} + \bar{v}_s^T \dot{\bar{v}}_s \]
\[ + (y - 1_N \otimes x)^T P(y - 1_N \otimes x) \]
\[ + (y - 1_N \otimes x)^T P(y - 1_N \otimes x), \] (35)
in which
\[ (x - x^*)^T \dot{x} = -(x_f - x_f)^T k_1 \bar{P}_f(\bar{y}_f) \]
\[ + (x_s - x_s)^T (\tilde{v}_s - k_1 \bar{P}_s(\bar{y}_s)) \]
\[ = -k_1 (x - x^*)^T P(y) + (x_s - x_s)^T \bar{v}_s \]
\[ = -k_1 (x - x^*)^T P(y) + k_1 (x - x^*)^T (P(x) - \bar{P}(y)) \]
\[ + (x_s - x_s)^T \bar{v}_s. \] (36)
By Assumption 3 - (x - x^*)^T P(x) \leq -m \|x - x^*\|^2.
Moreover, by Assumption 1, we get that \( \|P(x) - \bar{P}(y)\| \leq \max_{i \in I} \{l_i\} \|y - 1_N \otimes x\|. \) Hence,
\[ (x - x^*)^T \dot{x} \leq -k_1 m \|x - x^*\|^2 + \|x - x^*\| \|\bar{v}_s\| \]
\[ + k_1 \max_{i \in I} \{l_i\} \|x - x^*\| \|y - 1_N \otimes x\|. \] (37)
Moreover,
\[ \bar{v}_s^T \dot{\bar{v}}_s = \bar{v}_s^T (-k_2 \bar{v}_s + k_1 H_1(\bar{y}_s) \dot{\bar{y}}_s). \] (38)
Recalling the definition of \( \dot{\bar{y}}_s \), we get that
\[ \bar{v}_s^T \dot{\bar{v}}_s \leq -k_2 \|\bar{v}_s\|^2 + k_1 k_3 b \|v_s\| \|y - 1_N \otimes x\|, \] (39)
where \( b = \sup_{\bar{y}_s \in \mathbb{R}^{N(N-n)}} \|H_1(\bar{y}_s)\| \|\mathcal{C} \otimes 1_{N \times N} + A\|. \)

Furthermore,
\[ (\dot{y} - 1_N \otimes \dot{x})^T P(y - 1_N \otimes x) \]
\[ + (y - 1_N \otimes x)^T P(y - 1_N \otimes x) \]
\[ = -k_3 (\mathcal{C} \otimes 1_{N \times N} + A)(y - 1_N \otimes x) \]
\[ - 1_N \otimes \bar{P}_3 \|y - 1_N \otimes x\|^2 \]
\[ + (y - 1_N \otimes x)^T P(k_2 \bar{P}_3(\bar{y})) \]
\[ + (y - 1_N \otimes x)^T P((y - 1_N \otimes x)^T). \] (40)
in which \( \bar{P}_3 = \left[-k_1 \bar{P}_f(\bar{y}) \right]^T \left[ \bar{P}_s(\bar{y}_s) \right]^T \).
By Assumption 1 it can be obtained that
\[ 2(y - 1_N \otimes x)^T P(1_N \otimes \bar{P}_3) \]
\[ \leq 2k_1 \|P\| |l_1| \|y - 1_N \otimes x\|^2 \]
\[ + 2k_1 \sqrt{N} \|P\| |l_1| \|y - 1_N \otimes x\| \|x - x^*\| \]
\[ + 2 \sqrt{N} \|P\| \|y - 1_N \otimes x\| \|\bar{v}_s\|. \] (41)
Therefore,
\[ (\dot{y} - 1_N \otimes \dot{x})^T P(y - 1_N \otimes x) \]
\[ + (y - 1_N \otimes x)^T P(y - 1_N \otimes x) \]
\[ \leq -k_3 \lambda_{\min}(Q) \|y - 1_N \otimes x\|^2 \]
\[ + 2k_1 \|P\| |l_1| \|y - 1_N \otimes x\|^2 \]
\[ + 2k_1 \sqrt{N} \|P\| |l_1| \|y - 1_N \otimes x\| \|x - x^*\| \]
\[ + 2 \sqrt{N} \|P\| \|y - 1_N \otimes x\| \|\bar{v}_s\|. \]

Hence,
\[ \dot{V} \leq -k_1 \|x - x^*\|^2 - k_2 \|v_s\|^2 + \|v_s\| \|x - x^*\| \]
\[ - (k_3 \lambda_{\min}(Q) - 2k_1 \|P\| |l_1|) \|y - 1_N \otimes x\|^2 \]
\[ + k_1 (\max_{i \in I} \{l_i\} + 2 \sqrt{N} \|P\| |l_1|) \|y - 1_N \otimes x\| \|x - x^*\| \]
\[ + (2 \sqrt{N} \|P\| + k_1 k_3 b) \|v_s\| \|y - 1_N \otimes x\| \]
\[ \leq -\Psi_1 \|x - x^*\|^2 \]
\[ - (k_2 - \frac{1}{2} - \sqrt{N} \|P\| - k_3 \lambda_{\min}(Q)) \|ar{v}_s\|^2 \]
\[ - \Psi_2 \|y - 1_N \otimes x\|^2, \] (42)
where \( \sigma_1 \) and \( \sigma_2 \) are positive constants that can be arbitrarily chosen and \( \Psi_1 = k_1 m - \frac{1}{2} - (\max_{i \in I} \{l_i\} + 2 \sqrt{N} \|P\| |l_1|) \sigma_1 \iota \), \( \Psi_2 = k_3 \lambda_{\min}(Q) - 2k_1 \|P\| |l_1| - \frac{k_2 k_3 b}{2 \sigma_2} \|P\| - \sigma_2 b \sigma_2 \).
Therefore,
\[ \dot{V} \leq -K \|E\|^2, \] (43)
where \( K = \min\{\Psi_1, k_2 - \frac{1}{2} - \sqrt{N} \|P\| - \frac{k_3 \lambda_{\min}(Q)}{2 \sigma_2} \} \).

Choosing \( k_1 \) according to 30a follows by choosing \( k_3 \) and \( k_2 \) according to and , successively for fixed \( \sigma_1, \sigma_2 \).

Hence,
\[ \|E(t)\| \leq \sqrt{d_2} e^{-\frac{k_1 m}{d_1}} \|E(0)\|, \] (44)
by using the Comparison Lemma [25].

Furthermore, define $E_r(t) = [(x - x^*)^T, \nu_s^T, (y - 1_N \otimes x)^T]^T$. Then,

$$
||E_r(t)|| \leq ||E(t)|| + k_1||P_s(\tilde{y}_s)||
$$

$$
\leq (1 + k_1\sqrt{N - n \max_{j \in \nu_s}(|\tilde{l}_j|)})||E(t)||
$$

$$
\leq (1 + k_1\sqrt{N - n \max_{j \in \nu_s}(|\tilde{l}_j|)})\sqrt{\frac{d_2}{d_1}}e^{-\frac{\delta k_1^2}{2}t}||E(0)||
$$

$$
\leq (1 + k_1\sqrt{N - n \max_{j \in \nu_s}(|\tilde{l}_j|)})\sqrt{\frac{d_2}{d_1}}e^{-\frac{\delta k_1^2}{2}t}||E_r(0)||
$$

$$
\leq (1 + k_1\sqrt{N - n \max_{j \in \nu_s}(|\tilde{l}_j|)})^2\sqrt{\frac{d_2}{d_1}}e^{-\frac{\delta k_1^2}{2}t}||E_r(0)||,
$$

thus arriving at the conclusion. \hfill \square

In this section, we consider that the mixed-order game is free of unmodeled dynamics and external disturbances. However, due to limited knowledge about explicit system model and complicated working environments of actuators and sensors, unmodeled dynamics and disturbances are inevitable in practice. Hence, in the following section, we consider mixed-order games with unmodeled dynamics and disturbances.

V. MIXED-ORDER MULTI-PLAYER GAMES WITH UNMODELED DYNAMICS AND THE EXTERNAL DISTURBANCE

In this section, we consider that the players’ dynamics are given by

$$
\dot{x}_f = u_f + g_f(x) + d_f(t), \quad f \in \nu_f
$$

$$
\dot{x}_s = u_s + g_s(x) + d_s(t), \quad s \in \nu_s.
$$

In the following, a centralized seeking method and a distributed seeking method will be presented, successively.

A. A centralized algorithm for mixed-order multi-player games

In this section, we consider that the players’ gradient values are accessible. Moreover, a RBFNN is adopted to deal with the unmodeled dynamics and external disturbances based on the following condition.

Assumption 5: For each $i \in \mathfrak{N}$, $g_i(x)$ is globally Lipschitz and $d_i(t)$ is bounded.

Remark 4: Note that in [26], it is required that the unmodeled dynamics $g_i(x)$ is sufficiently smooth with its first two partial derivatives being bounded given that $x$ is bounded. Similarly, the disturbance $d_i(t)$ was supposed to be sufficiently smooth with its first two time derivatives being bounded in [26]. From Assumption 5 we see that these conditions are relaxed to some extent in this paper.

Based on the RBFNN, the control inputs are designed as

$$
u_f = -k_4(x_f - z_f) - \hat{W}_f^T S_f(\tilde{x}) - \phi_f
$$

$$
\dot{z}_f = -k_1\nabla F_f(\tilde{x}), \quad f \in \nu_f
$$

$$
\dot{u}_s = -k_2\nu_s - k_1 k_2 F_s(\tilde{x}) - \hat{W}_s^T S_s(\tilde{x}) - \phi_s, \quad s \in \nu_s
$$

where $k_1, k_2, k_4$ are positive control gains, $z_f$ is an auxiliary variable generated by player $f$ and $\tilde{x} = [z_1, z_2, \cdots, z_n, x_{n+1}, x_{n+2}, \cdots, x_N]^T$. Moreover, $\hat{W}_i \in \mathbb{R}^{q \times 1}$, in which $q$ is the number of neurons, defines the weight matrix of the RBFNN. Motivated by [31], we update the weight matrices $\hat{W}_f$ and $\hat{W}_s$ by

$$
\dot{\hat{W}}_f = \begin{cases}
\beta S_f(\bar{x})(x_f - z_f), & \text{if } Tr(\hat{W}_f^T \hat{W}_f) < W_{\max} \\
\text{or } Tr(\hat{W}_s^T \hat{W}_s) = W_{\max} \text{ and } (x_f - z_f)\hat{W}_f^T S_f(\bar{x}) < 0 \\
\beta S_f(\bar{x})(x_f - z_f) - \beta \frac{(x_f - z_f)\hat{W}_f^T S_f(\bar{x})}{Tr(\hat{W}_f^T \hat{W}_f)}, & \text{if } Tr(\hat{W}_f^T \hat{W}_f) = W_{\max} \text{ and } (x_f - z_f)\hat{W}_f^T S_f(\bar{x}) \geq 0
\end{cases}
$$

$$
\dot{\hat{W}}_s = \begin{cases}
\beta S_s(\bar{x})\bar{v}_s, & \text{if } Tr(\hat{W}_s^T \hat{W}_s) < W_{\max} \\
\text{or } Tr(\hat{W}_s^T \hat{W}_s) = W_{\max} \text{ and } \bar{v}_s \hat{W}_s^T S_s(\bar{x}) < 0 \\
\beta S_s(\bar{x})\bar{v}_s - \beta \frac{(x_f - z_f)\hat{W}_f^T S_f(\bar{x})}{Tr(\hat{W}_f^T \hat{W}_f)} \hat{W}_s, & \text{if } Tr(\hat{W}_s^T \hat{W}_s) = W_{\max} \text{ and } \bar{v}_s \hat{W}_s^T S_s(\bar{x}) \geq 0
\end{cases}
$$

where $\beta, W_{\max}$ are positive constants, $\bar{v}_s = v_s + k_1 \nabla F_s(\tilde{x})$ and $\text{Tr}(\hat{W}_i^T(0)\hat{W}_i(0)) \leq W_{\max}$. Moreover, $\hat{W}_s \in \mathbb{R}^{q \times 1}$.

Remark 5: Note that if $\text{Tr}(\hat{W}_i^T(0)\hat{W}_i(0)) \leq W_{\max}$, then, $\text{Tr}(\hat{W}_f^T(t)\hat{W}_f(t)) \leq W_{\max}$ and $||\hat{W}_f||_F = ||\hat{W}_s - W_s^*||_F \leq 2\sqrt{W_{\max}}$.

Furthermore,

$$
\phi_f = \delta \tanh\left(\frac{K\beta(x_f - z_f)}{\epsilon}\right), \quad \phi_s = \delta \tanh\left(\frac{K\beta\bar{v}_s}{\epsilon}\right),
$$

in which $\epsilon > 0$ is a constant, and $\delta$ is a constant that satisfies $|\delta| \geq ||\epsilon|| + ||d(t)||$, where $\epsilon = [\epsilon_1, \epsilon_2, \cdots, \epsilon_N]^T$ and $d(t) = [d_1(t), d_2(t), \cdots, d_N(t)]^T$.

Recalling the players’ dynamics in (46), the centralized Nash equilibrium seeking strategy is given as

$$
\dot{x}_f = -k_4(x_f - z_f) - \hat{W}_f^T S_f(\tilde{x}) - \phi_f + g_f(x) + d_f(t)
$$

$$
\dot{z}_f = -k_1\nabla F_f(\tilde{x}), \quad f \in \nu_f
$$

$$
\dot{x}_s = u_s
$$

$$
\dot{v}_s = -k_2 v_s - k_1 k_2 F_s(\tilde{x}) - \hat{W}_s^T S_s(\tilde{x}) - \phi_s + g_s(x) + d_s(t), \quad s \in \nu_s.
$$

Writing (50) in its concatenated-vector form gives

$$
\dot{x}_f = -k_4(x_f - z_f) - \hat{W}_f^T S_f(\tilde{x}) - \phi_f + g_f(x) + d_f(t)
$$

$$
\dot{z}_f = -k_1\nabla F_f(\tilde{x})
$$

$$
\dot{x}_s = u_s
$$

$$
\dot{v}_s = -k_2 v_s - k_1 k_2 F_s(\tilde{x}) - \hat{W}_s^T S_s(\tilde{x}) - \phi_s + g_s(x) + d_s(t),
$$

where $z_f = [z_1, z_2, \cdots, z_n]^T$, $\hat{W}_f S_f(x) = ([W_1^T S_1(x))]_f, [W_2^T S_2(x))]_f, \cdots, [W_N^T S_N(x))]_f$, $\hat{W}_s S_s(x) = ([W_1^T S_1(\tilde{x}_s))]_s, [W_2^T S_2(\tilde{x}_s))]_s, \cdots, [W_N^T S_N(\tilde{x}_s))]_s$, $\phi_f = [\phi_1^T, \cdots, \phi_n^T]^T$, $\phi_s = [\phi_{n+1}^T, \cdots, \phi_{N+n-1}^T]^T$, $g_f(x) = [g_1^T(x), \cdots, g_n^T(x)]^T$, $g_s(x) = [g_{n+1}^T(x), \cdots, g_N^T(x)]^T$.\]
Define the Lyapunov candidate function as 

$$ V(t) = \frac{1}{2} \left[ (x(t) - x^*)^T (x(t) - x^*) + \sum_{i \in \mathcal{E}} \frac{1}{2} v_i^T \nabla_i f_i(x(t)) \right]^2 $$

The following lemma is given to support the stability analysis.

**Lemma 4:** Suppose that Assumptions 1, 3-5 are satisfied and

$$ k_2 > \left( \frac{k_2^2 h_1 + 1}{4 k_2^2 L} \right)^2 + k_1 h, k_4 > \Phi_2^2 = \frac{k_1 m}{k_2^2 L h_1 + 1}, \quad \Phi_2 = k_1 l_1 + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}} + \sqrt{\frac{2}{N - \max_{i \in \mathcal{E}} \{ \eta_i \}}} $$

where $A_1 = \left( \frac{k_1 m}{k_2^2 L h_1 + 1}, -k_1 L + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}} + \sqrt{\frac{2}{N - \max_{i \in \mathcal{E}} \{ \eta_i \}}} \right)$ and $\eta_i$ is the Lipschitz constant of $g_i(x)$. Then, $v_i(t)$, $x(t)$, and $z_i(t)$ generated by (50) are bounded given that their initial values are bounded.

**Proof:** Let

$$ v_s(t) = \frac{d}{dt} \mathcal{P}(\tilde{x}) $$

Then,

$$ \dot{x} = \dot{v}_s - k_1 \mathcal{P}(\tilde{x}) $$

Define the Lyapunov candidate function as

$$ V = \frac{1}{2} (x - x^*)^T (x - x^*) + \frac{1}{2} v_i^T \nabla_i f_i(x(t)) $$

Then,

$$ (x - x^*)^T \dot{x} = (x - x^*) \left[ \nabla_i f_i(x(t)) - \dot{v}_s + \frac{1}{2} (x - x^*) + \frac{1}{2} (x - x^*)^T (x - x^*) \right] $$

Moreover,

$$ \dot{v}_s = \nabla_i f_i(x(t)) - \dot{v}_s $$

in which $a_s$ is a positive constant that satisfies $|d_i(t)| < d_i, |g_i(x^*)| < g$ and $\eta_i$ is the Lipschitz constant of $g_i(x)$. Moreover, the last inequality is obtained by utilizing $\| \mathcal{P}(\tilde{x}) \| = \| \mathcal{P}(\tilde{x}) - \mathcal{P}(x^*) \| \leq \mathcal{L}_1 \| \tilde{x} - x^* \|$ based on Assumption 1.

Furthermore,

$$ (x_f - z_f)^T (x_f - z_f) $$

$$ = (x_f - z_f)^T (-k_4 (x_f - z_f) + k_1 \mathcal{P}(\tilde{x})) $$

$$ + (x_f - z_f)^T (-\nabla_i f_i(x_f^*) - \mathcal{P}(\tilde{x}) - \mathcal{P}(x_f^*)) $$

$$ + (x_f - z_f)^T (-\mathcal{P}(\tilde{x}) + \mathcal{P}(x_f^*)) $$

in which $a_f = \sqrt{\frac{2}{N - \max_{i \in \mathcal{E}} \{ \eta_i \}}}$. Hence,

$$ \dot{V} \leq -k_1 m \| x - x^* \|^2 + \| x - x^* \| \| v_s \| $$

Define $A_1 = \left( \frac{k_1 m}{k_2^2 L h_1 + 1}, -k_1 l_1 + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}} + \sqrt{\frac{2}{N - \max_{i \in \mathcal{E}} \{ \eta_i \}}} \right)$, and let $k_4 > \frac{\Phi_2^2}{4 k_2^2 L h_1 + 1} + k_1 h$. Then, $A_1$ is symmetric positive definite and

$$ \dot{V} \leq -k_2 m \| x - x^* \|^2 + \| x - x^* \| \| v_s \| $$

in which $\lambda_{\min}(A_1) > 0$, $E_0 = (\tilde{x} - x^*)^T, \tilde{v}_s^T$.

Define $A_2 = \left( \frac{\lambda_{\min}(A_1)}{k_1 l_1 + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}}}, -\frac{a_f}{\theta} \right)$, where $\Phi_2 = k_1 l_1 + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}} + \sqrt{\frac{2}{N - \max_{i \in \mathcal{E}} \{ \eta_i \}}}$, and let $k_4 > \frac{\Phi_2^2}{4 k_2^2 L h_1 + 1} + \sqrt{\max_{i \in \mathcal{E}} \{ \eta_i \}}$. Then, $A_2$ is symmetric positive matrix. Hence

$$ \dot{V} \leq -\lambda_{\min}(A_2) \| E \|^2 + (a_f + a_s) \| E \| $$

$$ = -\lambda_{\min}(A_2) - \theta \| E \|^2 $$

in which $\lambda_{\min}(A_2) > 0$, $E = (\tilde{x} - x^*)^T, \tilde{v}_s^T, (x_f - z_f)^T$ and $0 < \theta < \lambda_{\min}(A_2)$.

Hence,

$$ \dot{V} \leq -\lambda_{\min}(A_2) \| E \|^2 $$

for $\forall \| E \| = \| a_f + a_s \|$. Therefore, according to Theorem 4.18 in [25], it can be obtained that $\| E \|$ is bounded given that the states are initialized to be bounded. Recalling the definition of $\tilde{v}_s$, the conclusion can be obtained.
By Lemma 4, the trajectories generated by the proposed method belong to a compact set given that they are initialized to be bounded. Hence, by Lemma 1, it is clear that \( g_i(x) \) can be approximated as \( g_i(x) = W_i^* x S_i(x) + \epsilon_i \), where \( W_i^* \in \mathbb{R}^{s \times 1} \) is the optimal weight matrix and \( \epsilon_i \in \mathbb{R} \) is the approximation error. Therefore, the centralized Nash equilibrium seeking strategy can be written as

\[
\begin{align*}
\dot{x}_f &= -k_4(\bar{x}_f - z_f) - \hat{W}_f^T S_f(\bar{x}) \\
&\quad + d_f(t) \quad \text{if} \, f \in \nu_f \\
\dot{z}_f &= -k_1 \nabla f(\bar{x}_f), \quad f \in \nu_f \\
\dot{x}_s &= v_s \\
\dot{v}_s &= -k_2 v_s - k_1 k_2 \nabla s F_s(\bar{x}) - \hat{W}_s^T S_s(\bar{x}) \\
&\quad + d_s(t) + \epsilon_s - \phi_s + g_s(x) - g_s(\bar{x}) \quad s \in \nu_s.
\end{align*}
\]

Moreover, the concatenated-vector form of (63) is

\[
\begin{align*}
\dot{x}_f &= -k_4(\bar{x}_f - z_f) - \hat{W}_f^T S_f(\bar{x}) \\
&\quad + (d_f(t) + \epsilon_f - \phi_f) + g_f(x) - g_f(\bar{x}) \\
\dot{z}_f &= -k_1 \bar{P}_f(\bar{x}) \\
\dot{x}_s &= v_s \\
\dot{v}_s &= -k_2 v_s - k_1 k_2 \bar{P}_s(\bar{x}) - \hat{W}_s^T S_s(\bar{x}) \\
&\quad + (d_s(t) + \epsilon_s - \phi_s) + g_s(x) - g_s(\bar{x}).
\end{align*}
\]

The following theorem establishes the stability of (63).

**Theorem 3:** Suppose that Assumptions [13][3] are satisfied.

Then, for any pair of positive constants \( \Lambda \) and \( \Xi \), there exists a positive constant \( k_1^* (\Lambda, \Xi) \) such that for each fixed \( k_1 > k_1^* \), there exist positive constants \( k_2^*, k_3^*, \beta^* \) such that for each \( k_2 > k_2^*, k_3 > k_3^*, \beta > \beta^* \), there exists a positive constant \( T \) such that

\[
\|x(t) - x^*\| + \|v_s(t)\| \leq \Xi, \forall t > T,
\]

given that \( \|\tilde{x}(0) - x^*\|^T v_s(0), (x_f(0) - z_f(0))v_f^T \| + \sum_{i=1}^{N} T \text{Tr}(W_i^T W_i^*) \leq \Lambda \) and \( \text{Tr}(W_i^T W_i^*) \leq W_{\text{max}} \).

**Proof:** Define the Lyapunov candidate function as

\[
V = V_1 + V_2 + V_3 + V_4,
\]

in which

\[
\begin{align*}
V_1 &= \frac{1}{2} (\bar{x} - x^*)^T (\bar{x} - x^*) \quad V_2 = \frac{1}{2} \tilde{v}_s^T \tilde{v}_s \\
V_3 &= \frac{1}{2} (x_f - z_f)^T (x_f - z_f) \quad V_4 = \frac{1}{2 \beta} \sum_{i=1}^{N} T \text{Tr}(W_i^T W_i^*).
\end{align*}
\]

Then, by (66),

\[
\dot{V}_1 \leq -k_1 m \|\bar{x} - x^*\|^2 + \|\bar{x} - x^*\| \|\tilde{v}_s\|.
\]

and

\[
\dot{V}_2 = \tilde{v}_s^T \tilde{v}_s = \tilde{v}_s^T (-k_2 \tilde{v}_s + k_1 H(\bar{x}) \hat{x} + g_s(x) - g_s(\bar{x})) \\
&\quad - \tilde{v}_s^T \tilde{W}_s^T S_s(\bar{x}) + \tilde{v}_s^T (d_s(t) - \phi_s + \epsilon_s) \\
&\quad = -k_2 \|\tilde{v}_s\|^2 - k_1 \| \tilde{W}_s^T H(\bar{x}) \hat{P}_s(\bar{x}) \\
&\quad + k_2 \tilde{v}_s^T H(\bar{x})(d_s(t) - \phi_s + \epsilon_s + g_s(x) - g_s(\bar{x})) \\
&\quad - \tilde{v}_s^T \tilde{W}_s^T S_s(\bar{x}) + \tilde{v}_s^T (d_s(t) - \phi_s + \epsilon_s) \\
&\quad \leq - (k_2 - k_1 h) \|\tilde{v}_s\|^2 + (N - n) \epsilon \\
&\quad + k_2 \| h \| \|\bar{x} - x^*\| \|\tilde{v}_s\| - \tilde{v}_s^T \tilde{W}_s^T S_s(\bar{x}) \\
&\quad + \sqrt{N - n} \max_{i \in \nu_s} \eta_i \|v_i\| \|\bar{x} - z_f\|,
\]

in which we utilize that \( \tilde{v}_s^T (d_s(t) + \epsilon_f - \phi_f) \leq (N - n) \epsilon \), which can be easily proved by Lemma 3.

Moreover,

\[
\dot{V}_3 = (x_f - z_f)^T (x_f - z_f) \\
&\quad = (x_f - z_f)^T (-k_4 (x_f - z_f) + k_1 \bar{P}_f(\bar{x})) \\
&\quad - (x_f - z_f)^T (\tilde{W}_f^T S_f(\bar{x}) + g_s(x) - g_s(\bar{x})) \\
&\quad + (x_f - z_f)^T (-\phi_f + \epsilon_f + d_f(t)) \\
&\quad \leq -k_4 \|x_f - z_f\|^2 + n \epsilon + k_1 l_1 \|x_f - z_f\| \|\bar{x} - x^*\| \\
&\quad - (x_f - z_f)^T (\tilde{W}_f^T S_f(\bar{x}) + \sqrt{n} \max_{i \in \nu_f} \eta_i \|v_i\| \|\bar{x} - z_f\|, 
\]

in which \( (x_f - z_f)^T (-\phi_f + \epsilon_f + d_f(t)) \leq n \epsilon \) can be easily proved by Lemma 3.

Furthermore,

\[
\begin{align*}
\dot{V}_4 &= \sum_{i=1}^{N} \frac{\text{Tr}(\tilde{W}_s^T W_i^*)}{\beta} \\
&\quad = \sum_{f=1}^{n} \frac{\text{Tr}(\tilde{W}_s^T W_f^*)}{\beta} + \sum_{s=n+1}^{N} \frac{\text{Tr}(\tilde{W}_s^T W_s^*)}{\beta}.
\end{align*}
\]

Hence,

\[
\begin{align*}
\dot{V} &\leq -k_1 m \|\bar{x} - x^*\|^2 - (k_2 - k_1 h) \|\tilde{v}_s\|^2 \\
&\quad - k_4 \|x_f - z_f\|^2 + N \epsilon \\
&\quad + (k_1 + k_2 h) \|\bar{x} - x^*\| \|\tilde{v}_s\| \\
&\quad + k_1 l_1 \|x_f - z_f\| \|\bar{x} - x^*\| \\
&\quad + \sqrt{n} \max_{i \in \nu_f} \eta_i \|v_i\| \|\bar{x} - z_f\| \\
&\quad + \sqrt{n} \max_{i \in \nu_s} \eta_i \|v_i\| \|\bar{x} - z_f\| \\
&\quad \leq -k_1 m \|\bar{x} - x^*\|^2 - (k_2 - k_1 h) \|\tilde{v}_s\|^2
\end{align*}
\]
\[ -k_4 \| \mathbf{x}_f - \mathbf{z}_f \|^2 + N \epsilon + (k_2^2 h l_1 + 1) \| \bar{x} - x^* \| \| \bar{v}_s \| \\
+ k_1 l_1 \| \mathbf{x}_f - \mathbf{z}_f \| \| \bar{x} - x^* \|^2 - \sum_{i=1}^{N} \text{Tr} \left( \hat{W}_i^T \hat{W}_i \right) \\
+ \sqrt{N - n} \max_{s \in \varnothing_s} (\eta_i) \| \bar{v}_s \| \| \mathbf{x}_f - \mathbf{z}_f \| \\
+ \sqrt{N} \max_{s \in \varnothing_s} (\eta_i) \| \mathbf{x}_f - \mathbf{z}_f \|^2 + 4N W_{\text{max}} , \]

in which \( \sum_{i=1}^{N} \text{Tr} \left( \hat{W}_i^T \left( \frac{W}{k} - S_\beta(x)(x_f - z_f) \right) \right) \leq 0 \)

and \( \sum_{s=1}^{N} \text{Tr} \left( \hat{W}_i^T \left( \frac{W}{k} - S_\beta(x) \bar{v}_s \right) \right) \leq 0 \), which can be obtained by following the proof of Theorem 2 in [31].

Hence,

\[ \dot{V} \leq - \left( k_1 m - \frac{1}{2 \rho_1} \right) \| \bar{x} - x^* \|^2 - \Psi_1 \| \bar{v}_s \|^2 + 2N \epsilon \\
- \left( k_1 m - \frac{1}{2 \rho_2} \right) \| \mathbf{x}_f - \mathbf{z}_f \| + \frac{k_1}{2} \| \mathbf{x}_f - \mathbf{z}_f \|^2 \\
+ \sqrt{N - n} \max_{i \in \varnothing_s} (\eta_i) \| \mathbf{x}_f - \mathbf{z}_f \|^2 \]

(73)

where \( \rho_1 \) is a positive constant that can be arbitrarily chosen and \( \Psi_1 = k_2 - k_3 - \frac{1}{2} \rho_2 (k_2^2 h l_1 + 1) + \sqrt{N - n} \max_{i \in \varnothing_s} (\eta_i) \).

In addition,

\[ \dot{V} \leq - \Psi_2 \| \mathbf{x} - x^* \|^2 - \Psi_1 \| \bar{v}_s \|^2 - \Psi_3 \| \mathbf{x}_f - \mathbf{z}_f \|^2 \\
- \sum_{i=1}^{N} \text{Tr} \left( \hat{W}_i^T \hat{W}_i \right) + 4N W_{\text{max}} + N \epsilon , \]

(74)

where \( \rho_2 \) is a positive constant that can be arbitrarily chosen, \( \Psi_2 = k_1 m - \frac{1}{2 \rho_2} - \frac{1}{2} \rho_2 k_2^2 \) and \( \Psi_3 = k_4 - \frac{1}{2} \rho_2 k_2^2 - \sqrt{N - n} \max_{i \in \varnothing_s} (\eta_i) \).

Therefore,

\[ \dot{V} \leq -K V + \Delta , \]

(75)

where \( K = 2 \min \{ \Psi_1, \Psi_2, \Psi_3, \beta \} \) and \( \Delta = N \epsilon + 4N W_{\text{max}} \).

Hence,

\[ V \leq V(0) e^{-K t} + \frac{\Delta}{K} (1 - e^{-K t}) , \]

(76)

by Lemma 2.

Recalling the definition of the Lyapunov candidate function, it can be obtained that

\[ \| \bar{x}(t) - x^* \|^2 + \| \bar{v}_s(t) \|^2 + \| \mathbf{x}_f(t) - \mathbf{z}_f(t) \|^2 \]

\[ \leq 2V(0) e^{-K t} + 2 \frac{\Delta}{K} . \]

(77)

In addition, with fixed \( \rho_1 \) and \( \rho_2 \), we can choose \( k_3 \) to be sufficiently large such that \( \Psi_2 > 0 \) and is sufficiently large. Moreover, with fixed \( k_1, \rho_1 \) and \( \rho_2 \), we can choose \( k_2, k_4 \) and \( \beta \) to be sufficiently large such that \( K \) is sufficiently large. Therefore, we arrive at the conclusion.

In this section, we consider mixed-order games in which the players’ dynamics are subject to unmodeled dynamics and external disturbance by designing a centralized method. In the following, we propose a distributed counterpart for the considered problem.

B. A distributed algorithm for mixed-order multi-player games

To achieve disturbance rejection for the mixed-order games in (46), the distributed Nash equilibrium seeking strategy is designed as

\[ \dot{x}_f = -k_4 (x_f - z_f) - \hat{W}_f^T S_\beta(y_f) \]

\[ - \phi_f + g_f (\mathbf{x}) + d_f (t) \]

\[ \dot{z}_f = -k_1 \dot{f}_j f_j (y_f), \]

\[ f \in \varnothing_f \]

\[ \dot{x}_s = \bar{v}_s \]

\[ \bar{v}_s = -k_2 v_s - k_3 k_2 \nabla f_j (y_s) - \hat{W}_s^T S_\beta(y_s) \]

\[ - \phi_s + g_s (\mathbf{x}) + d_s (t), s \in \varnothing_s \]

\[ \dot{\bar{y}}_{ij} = -k_3 \left( \sum_{k=1}^{N} \hat{a}_k (y_{ij} - y_{kj}) + \hat{a}_j (y_{ij} - \bar{x}_j) \right) , \]

\[ i, j \in \varnothing, \]

where \( k_1, k_2, k_3, k_4 \) are positive constants, \( \bar{x}_j = z_j \) for \( j \in \varnothing_f \) and \( \bar{x}_j = x_j \) for \( j \in \varnothing_s \). In addition, the weight matrices \( \hat{W}_f \) and \( \hat{W}_s \) are updated according to

\[ \dot{\hat{W}}_f = \left\{ \begin{array}{ll}
\beta S_\beta (y_f) (x_f - z_f), & \text{if } \text{Tr}(\hat{W}_f^T \hat{W}_f) < W_{\text{max}} \\
\text{or } \text{Tr}(\hat{W}_f^T \hat{W}_f) = W_{\text{max}} \text{ and } (x_f - z_f) \hat{W}_f^T S_\beta (y_f) < 0 \\
\beta S_\beta (y_f) (x_f - z_f) - \beta (x_f - z_f) \hat{W}_f^T S_\beta (y_f) \hat{W}_f, & \text{if } \text{Tr}(\hat{W}_f^T \hat{W}_f) = W_{\text{max}} \text{ and } (x_f - z_f) \hat{W}_f^T S_\beta (y_f) \geq 0
\end{array} \right. \]

(78)

and

\[ \dot{\hat{W}}_s = \left\{ \begin{array}{ll}
\beta S_\beta (y_s) \bar{v}_s, & \text{if } \text{Tr}(\hat{W}_s^T \hat{W}_s) < W_{\text{max}} \\
\text{or } \text{Tr}(\hat{W}_s^T \hat{W}_s) = W_{\text{max}} \text{ and } \bar{v}_s \hat{W}_s^T S_\beta (y_s) < 0 \\
\beta S_\beta (y_s) \bar{v}_s - \beta \hat{v}_s \hat{W}_s^T S_\beta (y_s) \hat{W}_s, & \text{if } \text{Tr}(\hat{W}_s^T \hat{W}_s) = W_{\text{max}} \text{ and } \bar{v}_s \hat{W}_s^T S_\beta (y_s) \geq 0
\end{array} \right. \]

(79)

Moreover, the concatenated-vector form of (78) is

\[ \dot{x}_f = -k_4 (x_f - z_f) - \hat{W}_f^T S_\beta (y_f) \]

\[ - \phi_f + g_f (\mathbf{x}) + d_f (t) \]

\[ \dot{z}_f = -k_1 \hat{P}_j (y_f) \]

\[ \mathbf{x}_s = \mathbf{v}_s \]

\[ \mathbf{v}_s = -k_2 \mathbf{v}_s - k_3 k_2 \hat{P}_j (y_s) - \hat{W}_s^T S_\beta (y_s) \]

\[ - \phi_s + g_s (\mathbf{x}) + d_s (t) \]

\[ \dot{\mathbf{y}} = -k_3 (L \otimes I_{N \times N} + \mathcal{A}) (y - 1_N \otimes \bar{x}) \]

The following lemma is given to support the upcoming stability analysis.

Lemma 5: Suppose that Assumptions [1][5] are satisfied. Then, there exists a positive constant \( k_1^* \) such that for each \( k_1 > k_1^* \), there exist positive constants \( k_3^*, k_4^* \) such that for \( k_3 > k_3^* \), \( k_4 > k_4^* \), there exists a positive constant \( k_2^* \) such that for \( k_2 > k_2^* \), \( \mathbf{x}(t), \mathbf{z}_f(t), \mathbf{v}_s(t) \), and \( y(t) \) generated by the proposed method in (46) stay bounded given that their initial values are bounded.

Proof: Let

\[ \mathbf{v}_s = \mathbf{v}_s - k_1 \hat{P}_j (y_s) . \]

(81)
Then,
\[
\dot{x}_s = \dot{v}_s - k_1 \bar{P}_s(y_s)
\]
\[
\dot{v}_s = \dot{v}_s + k_1 H_1(y_s) \dot{y}_s = -k_2 \dot{v}_s + k_1 H_1(y_s) \dot{y}_s
\]
\[
- \dot{W}^T_s S_s(y_s) - \phi_s + g_s(x) + d_s(t).
\]

Define the Lyapunov candidate function as
\[
V = V_1 + V_2 + V_3 + V_4,
\]
where
\[
V_1 = \frac{1}{2}(\dot{x} - x^*)^T(\dot{x} - x^*),
\]
\[
V_2 = \frac{1}{2}V^T_s \dot{v}_s,
\]
\[
V_3 = \frac{1}{2}(x_f - z_f)^T(x_f - z_f),
\]
\[
V_4 = (y - 1_N \otimes \bar{x})^T P(y - 1_N \otimes \bar{x}).
\]

Then,
\[
\dot{V}_1 = (\dot{x} - x^*)^T \ddot{x} = (\ddot{x} - x^*)^T \bar{A}_T^s, x^* - x^*)^T[\bar{A}_T^s, x^* - x^*)^T]
\]
\[
- k_1 (\dot{x} - x^*)^T P(y) + (\dot{x} - x^*)^T \bar{A}_T^s, x^* - x^*)^T
\]
\[
+ k_1 (\dot{x} - x^*)^T (P(\bar{x}) - P(x^*))
\]
\[
+ (\dot{x} - x^*)^T [\bar{A}_T^s, x^* - x^*)^T - k_1 \bar{H}_1(y_s) \dot{y}_s
\]
\[
\leq -k_1 m \| \bar{x} - x^* \|^2 + a_s \| \bar{x} - x^* \| \| \bar{v}_s \|
\]
\[
+ k_1 \max_{i \in \mathbb{N}} \{ \bar{i}_i \} \| x - x^* \| \| y - 1_N \otimes \bar{x} \|.
\]

and
\[
\dot{V}_2 = \bar{v}_s^T (-k_2 \bar{v}_s + k_1 H_1(y_s) \dot{y}_s)
\]
\[
+ \bar{v}_s^T (-\dot{W}^T_s S_s(y_s) + d_s(t) - \phi_s + g_s(x))
\]
\[
= -k_2 \| \bar{v}_s \|^2 + k_1 \bar{v}_s^T H_1(y_s) \dot{y}_s
\]
\[
+ \bar{v}_s^T (-\dot{W}^T_s S_s(y_s) + d_s(t) - \phi_s + g_s(x))
\]
\[
\leq -k_2 \| \bar{v}_s \|^2 + a_s \| \bar{v}_s \| \| \bar{v}_s \|
\]
\[
+ k_1 \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x - x^* \| \| \bar{v}_s \|.
\]

in which \( b = \sup_{y_s \in \mathbb{R}^n} \| H_1(y_s) \| \| L \otimes I_{N \times N} + A \| \) and \( a_s \) is defined in the proof of Lemma 4.

Moreover,
\[
\dot{V}_3 = (x_f - z_f)^T (x_f - z_f)
\]
\[
= (x_f - z_f)^T (-k_4 \bar{v}_s + k_4 \bar{P}_f(y_f))
\]
\[
+ (x_f - z_f)^T (-\dot{W}^T_s S_s(y_f) - \phi_f + g_f(x) + d_f(t))
\]
\[
\leq -k_4 \| x_f - z_f \|^2 + a_f \| x_f - z_f \|
\]
\[
+ k_1 \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x_f - z_f \| \| y - 1_N \otimes \bar{x} \|
\]
\[
+ (x_f - z_f)^T (g_f(x) - g_f(x^*))
\]

Furthermore,
\[
\dot{V}_4 = (y - 1_N \otimes \bar{x})^T P(y - 1_N \otimes \bar{x})
\]
\[
+ (y - 1_N \otimes \bar{x})^T P(y - 1_N \otimes \bar{x})
\]
\[
= -k_3 (y - 1_N \otimes \bar{x})^T (L \otimes I_{N \times N} + A) P(y - 1_N \otimes \bar{x})
\]
\[
- k_3 (y - 1_N \otimes \bar{x})^T P(y - 1_N \otimes \bar{x})
\]
\[
- 2(y - 1_N \otimes \bar{x})^T P(1_N \otimes \bar{x})
\]
\[
\leq -k_3 \lambda_{\min}(Q) \| y - 1_N \otimes \bar{x} \|^2
\]
\[
+ 2k_1 (y - 1_N \otimes \bar{x})^T P(1_N \otimes \bar{x})
\]
\[
- 2(y - 1_N \otimes \bar{x})^T P(1_N \otimes \bar{x})
\]
\[
\leq -k_3 \lambda_{\min}(Q) \| y - 1_N \otimes \bar{x} \|^2
\]
\[
+ 2k_1 \sqrt{N} \| P \| \| y - 1_N \otimes \bar{x} \| \| \bar{x} - x^* \|
\]
\[
+ 2\sqrt{N} \| P \| \| y - 1_N \otimes \bar{x} \| \| \bar{v}_s \|.
\]

Hence,
\[
\dot{V} \leq -k_1 m \| \bar{x} - x^* \|^2 - k_4 \| x_f - z_f \|^2 - k_2 \| \bar{v}_s \|^2
\]
\[
- (k_3 \lambda_{\min}(Q) - 2k_1 l_1 \| P \|) \| y - 1_N \otimes \bar{x} \|^2
\]
\[
+ (1 + \sqrt{N} - n \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x - x^* \| (\| \bar{v}_s \| + a_s \| \bar{v}_s \|)
\]
\[
+ 2k_1 l_1 \sqrt{N} \| P \| \| y - 1_N \otimes \bar{x} \| \| \bar{x} - x^* \|
\]
\[
+ (k_1 \lambda_{\min}(Q) - 2k_1 l_1 \| P \| \| y - 1_N \otimes \bar{x} \| \| \bar{x} - x^* \|
\]
\[
+ (k_1 \lambda_{\min}(Q) - 2k_1 l_1 \| P \| \| y - 1_N \otimes \bar{x} \| \| \bar{x} - x^* \|
\]
\[
+ \sqrt{N} \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x_f - z_f \| (\| \bar{v}_s \| + a_s \| \bar{v}_s \|).
\]

Therefore,
\[
\dot{V} \leq -\bar{\Psi}_1 \| \bar{x} - x^* \|^2 - \bar{\Psi}_2 \| x_f - z_f \|^2 - \bar{\Psi}_3 \| \bar{v}_s \|^2
\]
\[
- \bar{\Psi}_4 \| y - 1_N \otimes \bar{x} \|^2 + a_f \| x_f - z_f \| + a_s \| \bar{v}_s \|.
\]

where \( \rho_{i, i} \in \{1, 2, 3, 4\} \) are positive constants that can be arbitrarily chosen, \( \bar{\Psi}_1 = k_1 m - (1 + \sqrt{N} - n \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x - x^* \| \| \bar{v}_s \| + a_s \| \bar{v}_s \|) \)
\[
\bar{\Psi}_2 = k_4 - \sqrt{N} \max_{i \in \mathbb{N}} \{| \bar{i}_i \} \| x - x^* \| \| \bar{v}_s \| + a_s \| \bar{v}_s \|
\]
\[
\bar{\Psi}_3 \| x_f - z_f \| + 2k_1 l_1 \| P \| \| \bar{x} - x^* \|
\]
\[
\bar{\Psi}_4 = k_3 \lambda_{\min}(Q) - 2k_1 l_1 \| P \| \| \bar{x} - x^* \|
\]

Hence, by choosing \( k_1 \) to be sufficiently large, \( \bar{\Psi}_1 > 0 \).

Then, by fixed \( k_1 \), we can choose \( k_3 \) and \( k_4 \) to be sufficiently large such that \( \bar{\Psi}_3 > 0 \) and \( \bar{\Psi}_4 > 0 \). Then, by fixing \( k_1, k_3, k_4 \), we can choose \( k_2 \) to be sufficiently large such that \( \bar{\Psi}_3 > 0 \). By such a tuning rule, there exists a positive constant \( \bar{\theta} \) such that
\[
\dot{V} \leq -\bar{\theta} V + a_f \| x_f - z_f \| + a_s \| \bar{v}_s \|.
\]
from which the conclusion can be easily concluded.
By Lemma 5, we can conclude that $g_i(y_i) = W_i^T S_i(y_i) + \varepsilon_i$ by Lemma 1. Hence, we can obtain that

$$
\dot{x}_f = -k_4(x_f - z_f) - W_f^T S_f(y_f) \\
+ g_f(x) - g_f(y_f) + d_f(t) + \varepsilon_f - \phi_f
$$

$$
\dot{z}_f = -k_1 \nabla f_f(y_f), f \in \nu_f
$$

$$
\dot{v}_s = -k_2 v_s - k_1 k_2 \nabla S_s(y_s) - \tilde{W}_s^T S_s(y_s) \\
+ g_s(x) - g_s(y_f) + d_s(t) + \varepsilon_s - \phi_s, s \in \nu_s
$$

$$
\dot{y}_{ij} = -k_3 \sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - \bar{x}_j)
$$

Moreover, the concatenated-vector form of (92) is

$$
\dot{x}_f = -k_4(x_f - z_f) - W_f^T S_f(y_f) \\
+ g_f(x) - g_f(y_f) + (d_f(t) + \varepsilon_f - \phi_f)
$$

$$
\dot{z}_f = -k_1 \tilde{P}_f(y_f)
$$

$$
\dot{v}_s = -k_2 v_s - k_1 k_2 \tilde{P}_s(y_s) - \tilde{W}_s^T S_s(y_s) \\
+ g_s(x) - g_s(y_f) + (d_s(t) + \varepsilon_s - \phi_s)
$$

$$
\dot{y} = -k_3 (L \otimes 1_{N \times N} + A) (y - 1_N \otimes \bar{x})
$$

**Theorem 4** Suppose that Assumptions 1, 3, 4, and 5 are satisfied. Then, for any pair of positive constants $A$ and $\Xi$, there exists a positive constant $\beta^*$ and $k_1^*$ such that for $\beta > \beta^*$ and $k_1 > k_1^*$ such that there exist positive constants $k_2$ and $k_4$ such that for $k_2 > k_2^*$, there exists a positive constant $k_3^*$ such that for $k_3 > k_3^*$, there exists a positive constant $T$ such that

$$
\|x(t) - x^*\| + \|v(t)\| \leq \Xi, \forall t > T,
$$

given that $\|[(\tilde{x}(0) - x^*)^T, v_s(0)^T, (y(0) - 1_N \otimes \tilde{x}(0))^T, (x_f(0) - z_f(0))^T] + \sum_{i=1}^N Tr(W_i(0)^T \tilde{W}_i(0)) \leq A$.

**Proof** Define the Lyapunov candidate function as

$$
V = V_1 + V_2 + V_3 + V_4 + V_5,
$$

where

$$
V_1 = \frac{1}{2}(x - x^*)^T (x - x^*)
$$

$$
V_2 = \frac{1}{2} \tilde{v}_s^T \tilde{v}_s
$$

$$
V_3 = \frac{1}{2}(x_f - z_f)^T (x_f - z_f)
$$

$$
V_4 = (y - 1_N \otimes \bar{x})^T P(y - 1_N \otimes \bar{x})
$$

$$
V_5 = \frac{1}{2} \sum_{i=1}^N Tr(W_i^T \tilde{W}_i)
$$

Then,

$$
\dot{V}_1 \leq -k_1 m \|x - x^*\|^2 + \|x - x^*\| \|\tilde{v}_s\| \\
+ k_1 \max_{i \in \mathbb{N}} \{\tilde{l}_i\} \|x - x^*\| \|y - 1_N \otimes \bar{x}\|,
$$

and

$$
\dot{V}_3 = (x_f - z_f)^T (x_f - z_f) \\
\leq -k_4 \|x_f - z_f\|^2 + n \varepsilon_f \\
+ k_1 \max_{i \in \mathbb{N}} \{\tilde{l}_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\|
$$

$$
\dot{V}_4 \leq -k_3 \lambda_{min}(Q) \|y - 1_N \otimes \bar{x}\|^2 \\
+ 2k_1 \sum_{i=1}^N \|\tilde{P}\| \|y - 1_N \otimes \bar{x}\| \|\tilde{x} - x^*\| \\
+ 2 \sqrt{N} \|\tilde{P}\| \|y - 1_N \otimes \bar{x}\| \|\tilde{x} - x^*\|
$$

Furthermore,

$$
\dot{V}_5 = \sum_{i=1}^N \frac{Tr(W_i^T \tilde{W}_i)}{\beta} \\
\leq \sum_{f=1}^n \frac{Tr(W_f^T \tilde{W}_f)}{\beta} + \sum_{s=n+1}^N \frac{Tr(\tilde{W}_s^T \tilde{W}_s)}{\beta}
$$
Hence,
\[
\dot{V} \leq -k_1 m \|\bar{x} - \bar{x}^*\|^2 - k_4 \|x_f - z_f\|^2 - k_2 \|\bar{v}_s\|^2 \\
- (k_3 \lambda_{\min}(Q) - 2k_1 \|P\| \|y - 1_N \otimes \bar{x}\|^2 \\
+ \|\bar{x} - \bar{x}^*\||\bar{v}_s\| + N \epsilon + (2k_1 \sqrt{N} \|P\| \\
+ k_1 \max\{\bar{l}_i\}) \|y - 1_N \otimes \bar{x}\| \|\bar{x} - x^*\| \\
+ (2 \sqrt{N} \|P\| + k_1k_3b) \|y - 1_N \otimes \bar{x}\| \|\bar{v}_s\| \\
+ k_1 \max\{\bar{l}_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\| \\
+ k_1 \max\{\bar{l}_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\|
\]
\[+ \sum_{f=1}^{N} Tr \left( \dot{W}_f^T \left( \frac{\dot{W}_f}{\beta} - S_f(\bar{y}_f)(x_f - z_f) \right) \right) \\
+ \sum_{s=n+1}^{N} Tr \left( \dot{W}_s^T \left( \frac{\dot{W}_s}{\beta} - S_s(\bar{y}_s)\bar{v}_s \right) \right) \\
+ (x_f - z_f)^T (g_f(x) - g_f(\bar{y}_f)) + \bar{v}_s^T (g_s(x) - g_s(\bar{y}_s)) \\
\leq -k_1 m \|\bar{x} - \bar{x}^*\|^2 - k_4 \|x_f - z_f\|^2 - k_2 \|\bar{v}_s\|^2 \\
- (k_3 \lambda_{\min}(Q) - 2k_1 \|P\| \|y - 1_N \otimes \bar{x}\|^2 \\
+ (2k_1 \sqrt{N} \|P\| + k_1 \max\{\bar{l}_i\}) \|y - 1_N \otimes \bar{x}\| \|\bar{x} - x^*\| \\
+ (2 \sqrt{N} \|P\| + k_1k_3b) \|y - 1_N \otimes \bar{x}\| \|\bar{v}_s\| \\
+ k_1 \max\{\bar{l}_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\| \\
+ \sum_{i=1}^{N} Tr(\dot{W}_i^T \dot{W}_i) + N \epsilon + 4NW_{\text{max}} \\
+ (x_f - z_f)^T (g_f(x) - g_f(\bar{y}_f)) + \bar{v}_s^T (g_s(x) - g_s(\bar{y}_s)).
\]

Noticing that
\[
\|\bar{x} - x^*\| \|\bar{v}_s\| \leq \frac{1}{2 \rho_1} \|\bar{x} - x^*\|^2 + \frac{\rho_1}{2} \|\bar{v}_s\|^2,
\]
(103)
where \(\rho_1\) is a positive constant that can be arbitrarily chosen and
\[
\frac{(2k_1 \sqrt{N} \|P\| + k_1 \max\{\bar{l}_i\}) \|y - 1_N \otimes \bar{x}\| \|\bar{x} - x^*\|}{2 \rho_2} \\
\leq \left( \frac{2 \sqrt{N} \|P\| + \max_{i \in \mathcal{R}} \{\bar{l}_i\} \|y - 1_N \otimes \bar{x}\|}{\rho_3} \right)^2 \|\bar{x} - x^*\|^2 \\
+ \frac{\rho_2 k_4^2}{2} \|y - 1_N \otimes \bar{x}\|^2,
\]
(104)
where \(\rho_2\) is a positive constant that can be arbitrarily chosen. In addition,
\[
\frac{(2 \sqrt{N} \|P\| + k_1k_3b) \|y - 1_N \otimes \bar{x}\| \|\bar{v}_s\|}{\rho_3} \\
\leq \left( \frac{\sqrt{N} \|P\| + k_3^2b}{\rho_3} \right) \|\bar{v}_s\|^2 \\
+ \left( \frac{\sqrt{N} \|P\| \rho_3 + \frac{k_3^2b \rho_3}{2} \|y - 1_N \otimes \bar{x}\|}{\rho_3} \right) \|\bar{v}_s\|^2,
\]
(105)
where \(\rho_3\) is a positive constant that can be arbitrarily chosen.

Furthermore,
\[
k_1 l_1 \|x_f - z_f\| \|\bar{x} - x^*\| \leq \frac{l_1}{2 \rho_4} \|\bar{x} - x^*\|^2 + \frac{\rho_4 l_1 k_4^2}{2} \|x_f - z_f\|^2
\]
and
\[
k_1 l_1 \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\| \\
\leq \frac{l_1}{2 \rho_5} \|y - 1_N \otimes \bar{x}\|^2 + \frac{\rho_5 l_1 k_4^2}{2} \|x_f - z_f\|^2,
\]
(106)
where \(\rho_4, \rho_5\) are positive constants that can be arbitrarily chosen.

Hence,
\[
\dot{V} \leq -\bar{\Phi}_1 \|\bar{x} - x^*\|^2 - (k_4 - \frac{\rho_4 l_1 k_4^2}{2} - \frac{\sqrt{N} \|P\|}{\rho_3}) \|\bar{v}_s\|^2 \\
- (k_3 \lambda_{\min}(Q) - 2k_1 \|P\| \|y - 1_N \otimes \bar{x}\|^2 \\
+ (2k_1 \sqrt{N} \|P\| + k_1 \max\{\bar{l}_i\}) \|y - 1_N \otimes \bar{x}\| \|\bar{x} - x^*\| \\
+ (2 \sqrt{N} \|P\| + k_1k_3b) \|y - 1_N \otimes \bar{x}\| \|\bar{v}_s\| \\
+ k_1 \max\{\bar{l}_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\| \\
+ \sum_{i=1}^{N} Tr(\dot{W}_i^T \dot{W}_i) + N \epsilon + 4NW_{\text{max}} \\
+ (x_f - z_f)^T (g_f(x) - g_f(\bar{y}_f)) + \bar{v}_s^T (g_s(x) - g_s(\bar{y}_s)),
\]
(102)
where \(\bar{\Phi}_1 = k_1 m - \frac{1}{2 \rho_1} - \frac{(2l_1 \sqrt{N} \|P\| + \max_{i \in \mathcal{R}} \{\bar{l}_i\})^2}{2 \rho_2} - \frac{l_1}{2 \rho_4} \)
By further noticing that
\[
(x_f - z_f)^T (g_f(x) - g_f(\bar{y}_f)) \\
\leq \sqrt{n} \max_{i \in \mathcal{R}} \{\eta_i\} \|x_f - z_f\|^2 + \max_{i \in \mathcal{R}} \{\eta_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\|,
\]
(109)
and similarly,
\[
\bar{v}_s^T (g_s(x) - g_s(\bar{y}_s)) \\
\leq \sqrt{n} \max_{i \in \mathcal{R}} \{\eta_i\} \|\bar{v}_s\| \|x_f - z_f\| + \max_{i \in \mathcal{R}} \{\eta_i\} \|x_f - z_f\| \|y - 1_N \otimes \bar{x}\|,
\]
(110)
where \(K = 2 \min\{\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3, \bar{\Phi}_4, \beta\} / \max\{\frac{1}{2}, \lambda_{\max}(P)\}\).

Hence, by Lemma 2 we can obtain that
\[
V(t) \leq V(0)e^{-Kt} + N \epsilon + 4NW_{\text{max}},
\]
(112)
where \(K\) can be arbitrarily large by the following tuning rule: choose \(k_1\) to be sufficiently large such that \(\bar{\Phi}_1\) is sufficiently large. Then, for fixed \(k_1\), choose \(k_3, k_4\) such that \(\bar{\Phi}_3\) and
\( \Phi_4 \) are sufficiently large. Then, for fixed \( k_3 \), choose \( k_2 \) to be sufficiently large such that for fixed \( k_3 \), choose \( k_2 \) to be sufficiently large such that \( \Phi_2 \) is sufficiently large. If this is the case, \( K \) is sufficiently large with sufficiently large \( \beta \), indicating that \( V(t) \) can converge to an arbitrarily small neighborhood of zero by such a tuning rule. Recalling the definitions of the Lyapunov candidate function and \( \bar{v}_s \), the conclusion can be obtained.

\[ \text{Fig. 1: The trajectories of players’ positions generated by (13).} \]

\[ \text{Fig. 2: The trajectories of } v_s(t) \text{ generated by (13).} \]

\[ \text{Fig. 3: The communication graph among the players.} \]

\[ \text{Fig. 4: The trajectories of players’ positions generated by (28).} \]

VI. SIMULATION STUDIES

In this section, we consider the connectivity control of a network of 5 sensors considered in [26] in which the objective function of sensor \( i \) is given as

\[ F_i(x) = h_i(x_i) + l_i(x), \quad (113) \]

where \( x_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2 \) and

\[ h_i(x_i) = x_i^T m_{ii} x_i + x_i^T m_i + i, \quad (114) \]

in which \( m_{ii} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \), \( m_i = [i, i]^T \). Moreover, \( l_1(x) = ||x_1 - x_2||^2 \), \( l_2(x) = ||x_2 - x_3||^2 \), \( l_3(x) = ||x_3 - x_2||^2 \), \( l_4(x) = ||x_4 - x_2||^2 + ||x_4 - x_5||^2 \) and \( l_5(x) = ||x_5 - x_1||^2 \).

It was calculated in [26] that on the Nash equilibrium of the game, \( x_i^* = -\frac{1}{k} \) for \( i \in \{1, 2, 3, 4, 5\} \), \( j \in \{1, 2\} \). In the following simulations, we suppose that sensors 1-3 are first-order integrator-type sensors and sensors 4-5 are second-order integrator-type sensors.

A. Nash equilibrium seeking for mixed-order integrator-type games

In this section, we consider that the players’ dynamics are given by

\[ \dot{x}_f = u_f, \quad f \in \nu_f \]
\[ \dot{x}_s = u_s, \quad s \in \nu_s. \quad (115) \]

In the subsequent simulations, the proposed methods in [13] and [28] will be numerically verified one by one.

1) Centralized Nash equilibrium seeking: Let \( x(0) = [-8, 5, -2, -4, -5, 7, 1, -8, -1, 9]^T, v_s(0) = [0, 0, 0, 0]^T \). Then, the simulation results generated by the proposed method in [13] are shown in Figs. 1-2 in which Fig. 1 shows the evolutions of the players’ positions and Fig. 2 illustrates \( v_s(t) \). From the simulation results, we see that the players’ actions would converge to the Nash equilibrium.

2) Distributed Nash equilibrium seeking: In this section, we suppose that \( x(0) = [-5, 3, -4, -6, 1, 8, 0, -8, -1, 10]^T \), and \( v_s(0) = [-10, -10, 20, 20]^T \). Moreover, \( y(0) \) is initialized at zero. With the communication graph given in Fig. 3 the simulation results generated by [28] are shown in Figs. 4-5 in which Fig. 4 shows the evolutions of the players’ positions. In addition, Fig. 5 plots the trajectories of \( v_s(t) \). The simulation results demonstrate that the players’ actions would converge to the Nash equilibrium, thus verifying Theorem 2.
Furthermore, the variances are all set as $5, -2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5$, respectively.

B. Mixed-order games with unmodel dynamics and disturbances

In the section, we consider that the players’ actions are given by
\[
\begin{align*}
\dot{x}_f &= u_f + g_f(x) + d_f(t), \quad f \in \nu_f \\
\dot{x}_s &= u_s + g_s(x) + d_s(t), \quad s \in \nu_s.
\end{align*}
\]
(116)

In addition, $g_f(x) + d_f(t)$ in the players’ dynamics are $x_{21} + \sin(t), x_{22} + \sin(t), x_{21}^2 + x_{31} + 2\sin(2t), x_{22} + 2\sin(2t), 3x_{31} + 3\sin(3t), 3x_{32} + 3\sin(3t), 4x_{41} + 4\sin(4t), 4x_{42} + 4\sin(4t), 5x_{51} + 5\sin(5t), 5x_{52} + 5\sin(5t)$, respectively.

In the simulation, the number of neurons of the RBFNN is chosen as 11 and the centers of RBFFN activation function are $-2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5$, respectively. Furthermore, the variances are all set as $5\sqrt{2}$. By setting $W_{\max} = 500$, $\beta = 100$, $\delta = 10$, $\epsilon = 0.01$ and $W_i(0)$ as a zero matrix, the centralized algorithm in (50) and the distributed algorithm in (78) will be simulated, successively.

1) Centralized Nash equilibrium seeking: With $x(0) = [-8, 5, -2, -4, -5, 7, 1, -8, -1, 9]^T, v_1(0) = [0, 0, 0, 0]^T$, the simulation results produced by (50) are given in Figs. 6 and Fig. 7 illustrates the evolutions of the players’ positions and $v_s(t)$ generated by the proposed method. From the simulation results, it can be concluded that the players’ actions would be driven to the Nash equilibrium based on the proposed methods.

2) Distributed Nash equilibrium seeking: With $x(0) = [-5, 8, -2, -4, -6, 1, 8, 0, -8, -1, 10]^T, v_2(0) = [0, 0, 0, 0]^T$, the simulation results produced by (78) are given in Figs. 8 and by utilizing the communication graph in Fig. 3, Figs. 9 illustrate the evolutions of the players’ positions and $v_s(t)$, from which we see that the players’ actions would converge to the Nash equilibrium.

VII. Conclusions

This paper considers Nash equilibrium seeking for mixed-order multi-player games consisting of first-order integrator-type players and second-order integrator-type players. A centralized algorithm is designed based on the gradient search, followed by a distributed seeking strategy. Considering that unmodeled dynamics and external disturbances are inevitable in many practical situations, we further address mixed-order games in which the players are subject to unmodeled dynamics and time-varying disturbances. In this situation, a centralized algorithm and a distributed algorithm are proposed one by one. The convergence of the proposed seeking strategies are investigated analytically based on Lyapunov stability analysis.

REFERENCES

[1] L. Deori, K. Margellos, M. Prandini, “Price of anarchy in electric vehicle charging control games: When Nash equilibria achieve social welfare,” Automatica, vol. 96, pp. 150-158, 2018.
[2] A. Cherukuri, J. Cortês, “Decentralized Nash equilibrium learning by strategic generators for economic dispatch,” American Control Conference, pp. 1082-1087, 2016.
[3] M. Stankovic, K. Johansson, D. Stipanovic, “Distributed seeking of Nash equilibria with applications to mobile sensor networks,” IEEE Transactions on Automatic Control, vol. 57, no. 4, pp. 904-919, 2011.
[4] W. Lin, C. Li, Z. Qu, M. Samaan, “Distributed formation control with open-loop Nash strategy,” Automatica, vol. 106, pp. 266-273, 2019.
[5] M. Ye, G. Hu, “Distributed Nash equilibrium seeking under switching communication topologies,” IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4811-4818, 2017.
[6] M. Ye, G. Hu, “Distributed Nash equilibrium seeking in multiagent games under switching communication topologies,” IEEE Transactions on Cybernetics, vol. 48, no. 11, pp. 3208 - 3217, 2017.
Fig. 8: The evolutions of players’ positions generated by (78).

Fig. 9: $v_s(t)$ generated by (78).

[7] C. Persis, S. Grammatico, “Distributed averaging integral Nash equilibrium seeking on networks,” Automatica, published online, DOI:10.1016/j.automatica.2019.108548.

[8] M. Ye, G. Hu, “Game design and analysis for price-based demand response: an aggregate game approach,” IEEE Transactions on Cybernetics, vol. 4, no. 3, pp. 720–730, 2017.

[9] H. Hua, Y. Qin, C. Hao, and J. Cao, “Optimal energy management strategies for energy Internet via deep reinforcement learning approach,” Applied Energy, vol. 239, pp. 598-609, 2019.

[10] A. Ibrahim, T. Hayakawa, “Nash equilibrium seeking with second-order dynamic agents,” IEEE Conference on Decision and Control, pp. 2514-2518, 2018.

[11] M. Ye, “Distributed Nash equilibrium seeking for games in systems with bounded control inputs,” arXiv preprint [arXiv:1901.09333], 2019.

[12] A. Ibrahim, T. Hayakawa, “Nash equilibrium seeking with linear time-invariant dynamic agents,” American Control Conference, pp. 1202-1207, 2019.

[13] Z. Deng, S. Liang, “Distributed algorithms for aggregative games of multiple heterogeneous Euler-Lagrange systems,” Automatica, vol. 99, pp. 246-252, 2019.

[14] M. Ye, “Nash equilibrium seeking for games in hybrid systems,” International Conference on Control, Automation, Robotics and Vision, pp. 140-145, 2018.

[15] C. Liu, F. Liu, “Stationary consensus of heterogeneous multi-agent systems with bounded communication delays,” Automatica, vol. 47, no. 9, pp. 2130-2133, 2011.

[16] W. Jiang, G. Wen, Z. Peng et al, “Fully distributed formation-containment control of heterogeneous multilateral multi-agent systems,” IEEE Transactions on Automatic Control, vol. 64, no. 9, pp. 3889-3896, 2018.

[17] Y. Hua, X. Dong, Q. Li, and Z. Ren, “Distributed time-varying formation robust tracking for general linear multiagent systems with parameter uncertainties and external disturbances,” IEEE Transactions on Cybernetics, vol. 48, no. 7, pp. 1959-1969, 2017.

[18] F. Yaghmaie, F. Lewis, R. Su, “Output regulation of linear heterogeneous multi-agent systems via output and state feedback,” Automatica, vol. 67, pp. 157-164, 2016.

[19] A. Bidram, F. Lewis, A. Davoudi, S. Ge, “Adaptive and distributed control of nonlinear and heterogeneous multi-agent systems,” IEEE Conference on Decision and Control, pp. 6238-6243, 2013.

[20] J. Ma, M. Ye, Y. Zheng, Y. Zhu, “Consensus analysis of hybrid multi-agent systems: A game-theoretic approach,” International Journal of Robust and Nonlinear Control, vol. 29, no. 6, pp. 1840-1853, 2019.

[21] J. Mei, W. Ren, J. Chen, “Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph,” IEEE Transactions on Automatic Control, vol. 61, no. 8, pp. 2019-2034, 2015.

[22] M. Zheng, C. Liu, F. Liu, “Average-consensus tracking of sensor network via distributed coordination control of heterogeneous multi-agent systems,” IEEE Control Systems Letters, vol. 3, no. 1, pp. 132-137, 2018.

[23] Y. Zheng, L. Wang, “Consensus of heterogeneous multi-agent systems without velocity measurements,” International Journal of Control, vol. 85, no. 7, pp. 906-914, 2012.

[24] Z. Li, Z. Wu, Z. Li, Z. Ding, “Distributed optimal coordination for heterogeneous linear multi-agent systems with event-triggered mechanisms,” IEEE Transactions on Automatic Control, published online, DOI:10.1109/TAC.2019.2937500.

[25] H. Khalil, Nonlinear Systems, Upper Saddle River, NJ: Prentice Hall, 2002.

[26] M. Ye, “A RISE-based distributed robust Nash equilibrium seeking strategy for networked games,” IEEE Conference on Decision and Control, 2019, pp. 4047-4052, 2019.

[27] M. Ye, G. Hu, F. Lewis, “Nash equilibrium seeking for N-coalition noncooperative games,” Automatica, vol. 95, pp. 266-272, 2018.

[28] M. Ye, G. Hu, F. Lewis, L. Xie, “A unified strategy for solution seeking in graphical N-coalition noncooperative games,” IEEE Transactions on Automatic Control, vol. 64, no. 11, pp. 4645-4652, 2019.

[29] M. Ye, G. Hu and S. Xu, “An extremum seeking-based approach for Nash equilibrium seeking in N-cluster noncooperative games,” Automatica, vol. 114, 108815, 2020.

[30] M. Polycarpou, “Stable adaptive neural control scheme for nonlinear systems,” IEEE Transactions on Automatic Control, vol. 41, no. 3, pp. 447-451, 1996.

[31] Z. Hou, L. Cheng, M. Tan, “Decentralized robust adaptive control for the multiagent system consensus problem using neural networks,” IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), vol. 39, no. 3, pp. 636-647, 2009.

[32] X. Yao, W. Deng, Z. Jiao, “RISE-based adaptive control of hydraulic systems with asymptotic tracking,” IEEE Transactions on Automation Science and Engineering, vol. 14, no. 3, pp. 1524-1531, 2017.

[33] A. Fazeli, M. Zeinali, A. Khajepour, “Application of adaptive sliding mode control for regenerative braking torque control,” IEEE/ASME Transactions On Mechatronics, vol. 17, no. 4, pp. 745-755, 2012.

[34] S. He, S. Dai, F. Luo, “Asymptotic trajectory tracking control with guaranteed transient behavior for MSV With uncertain dynamics and external disturbances,” IEEE Transactions on Industrial Electronics, vol. 66, no. 5, pp. 3712-3720, 2019.

[35] P. Poggio, F. Giroi, “Networks for approximation and learning,” Proceedings of the IEEE, vol. 78, no. 9, pp. 1481-1497, 1990.

[36] L. Cheng, Z. Hou, M. Tan, Y. Lin, W. Zhang, “Neural-network-based adaptive leader-following control for multiagent systems with uncertainties,” IEEE Transactions on Neural Networks, vol. 21, no. 8, pp. 1351-1358, 2010.

[37] C. Chen, G. Wen, Y. Liu, F. Wang, “Adaptive consensus control for a class of nonlinear multiagent time-delay systems using neural networks,” IEEE Transactions on Neural Networks and Learning Systems, vol. 25, no. 6, pp. 1217-1226, 2014.

[38] G. Wen, C. Chen, Y. Liu, Z. Liu, “Neural-network-based adaptive leader-following consensus control for second-order non-linear multi-agent systems,” IET Control Theory and Applications, vol. 9, no. 13, pp. 1927-1934, 2015.