Light deflection by Damour-Solodukhin wormholes and Gauss-Bonnet theorem

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In this paper, using the idea of Gibbons and Werner, we apply the Gauss-Bonnet theorem to the optical metric of the non-rotating and rotating Damour-Solodukhin wormholes spacetimes to study the weak gravitational lensing by these objects. Furthermore, we study the strong gravitational lensing by the non-rotating Damour-Solodukhin wormholes using the Bozza’s method to see the differences between the weak lensing and the strong lensing. We demonstrate the relation between the strong deflection angle and quasinormal modes of the Damour-Solodukhin wormholes. Interestingly it is found that the wormhole parameter \( \lambda \) affects the deflection of light in strong and weak limits compared to the previous studies of gravitational lensing by Schwarzschild black holes. Hence, the results provide a unique tool to shed light on the possible existence of wormholes.

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I. INTRODUCTION

Einstein-Rosen (ER) bridge is a consequence of Einstein’s theory of relativity similarly to black holes. The ER equation glues to distant points of spacetime. This was firstly introduced by Einstein and Rosen in 1935 and then ER are refereed to as a wormhole [1]. On the other hand, Morris and Thorne in 1988 showed that constructing traversable wormhole solution is also possible, however it costs to necessity of exotic matter [2, 3]. Afterwards, many physicists are inspired by the Morris-Thorne paper, study wormholes in different aspects [4–40].

To detect the wormholes, a possible method is the application of optical gravitational lensing. The gravitational lensing by wormholes was studied widely in the literature of astrophysics as well as theoretical physics [41–59]. In this article, we use new approach, which is found by Gibbons and Werner, to calculate weak gravitational lensing [60–62]. The Gibbons and Werner method (GWM) use the Gauss-Bonnet theorem (GBT) to calculate gravitational lensing that shows its global properties. The gravitational lensing effect, either in the weak gravitational field or in the strong gravitational field, it always requires the null geodesic equations. GWM demonstrates that when the GBT is used within the optical metric, the deflection angle \( \hat{\alpha} \) can be calculated by [62]

\[
\hat{\alpha} = - \int_0^{\infty} \int_{\sigma_0}^{\infty} K \, d\sigma,
\]

where \( d\sigma \) is an areal element, \( K \) is the Gaussian curvature of the optical metric. There have been several recent works on the gravitational lensing by blackholes/wormholes using the GBT [29, 34–36, 63–74].

Here, the main aim of the paper is to show that GBT is valid for the calculating weak gravitational lensing by Damour-Solodukhin wormholes (DSW) which are the static Schwarzschild-like wormholes solution recently found by Damour and Solodukhin in [4] and then rotating Kerr-like case is found by Bueno et al. in [5]. Moreover, we try to show the deflecting angle how much is shifted according to the parameter \( \lambda \) from the Schwarzschild/Kerr black holes. Moreover recently, it is shown that there is a relation between the strong gravitational lensing and the quasinormal modes (QNMs) in the context of black holes [75]. For this purpose, we study the strong gravitational lensing by DSW using the method of Bozza [76] who showed that the logarithmic divergence of the deflection angles at photon sphere exits and we show the relation of deflection angle with QNMs in strong regime.

The organization of the paper is as follows. In section 2 we briefly summarize the DSW, then we present the calculations of the weak gravitational lensing using the GBT and we calculate the strong gravitational lensing by the DSW. In section 3, we briefly give information about the rotating DSW and calculate the deflection angle using the GBT. We conclude the paper in section 4.
II. DAMOUR-SOLODUKHIN WORMHOLE

In this section, we consider the static Schwarzschild-like wormhole solution, namely DSW [4] with metric:

\[ ds^2 = -(1 - 2M/r + \lambda^2)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2d\Omega^2. \]  

(2.1)

Note that this metric reduces to the Schwarzschild black hole at \( \lambda = 0 \). For non-zero values of the parameter \( \lambda^2 \), the Einstein tensor of 2.1 has a zero \( G_{tt} \), and need some matter to become toy model wormhole. Because of \( \lambda \) not corresponding to the time of an asymptotic observer, we can redefine the metric in 2.1 by using \( t \rightarrow t/\sqrt{1 + \lambda^2} \) and \( M \rightarrow M(1 + \lambda^2) \):

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h(r)d\Omega^2, \]  

(2.2)

where

\[ f(r) = 1 - \frac{2M}{r}, \quad g(r) = 1 - \frac{2M(1 + \lambda^2)}{r}, \]  

(2.3)

and \( h(r) = r^2 \).

In the next subsection, we will study the weak gravitational lensing using the Gauss-Bonnet theorem, and obtain the deflection angle in weak limits.

A. Weak gravitational lensing by Damour-Solodukhin wormhole using the Gauss-Bonnet Theorem

To calculate the deflection angle by DSW using the GBT [62], we use the equatorial plane \( \theta = \pi/2, d\theta = 0 \) without losing generality, due to the spherical symmetry, and the 2.2 spacetime reduces to orbital plane of light rays:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\phi^2. \]  

(2.4)

Using the two constants of motion in an affine parameter (\( \lambda \)):

\[ E = f(r)\frac{dt}{d\lambda}, \quad L = r^2\frac{d\phi}{d\lambda}, \]  

(2.5)

where \( E \) and \( L \) are the energy and the angular momentum, respectively. Then one can derive the another constant namely, impact parameter \( b \) is defined as follows:

\[ b = f(r)\frac{\phi}{d\phi/da}. \]  

(2.6)

and the following relation is obtained \( \frac{d\phi}{da} = \frac{bf(r)}{r^2} \).

To define the optical metric \( g_{ij} \) which is also known as the optical reference geometry \( \mathcal{M}^{opt} \), we use the fact that each light ray satisfies the equation for null geodesics \( ds^2 = 0 \), and the the optical metric \( g_{ij} \) is written as follows:

\[ dt^2 \equiv g_{rr}dr^2 + g_{\phi\phi}d\phi^2 = \frac{1}{f(r)g(r)}dr^2 + \frac{r^2}{g(r)}d\phi^2. \]  

(2.7)

Afterwards, we use the slice of the constant time \( t \) of the Eq. (2.4), and we obtain a spatial part of spacetime in two-dimension curved subspace \( \mathcal{M}^{sub} \) as follows:

\[ dt^2 \equiv g_{rr}dr^2 + g_{\phi\phi}d\phi^2 = \frac{dr^2}{g(r)} + r^2d\phi^2. \]  

(2.8)

After using the conformal transformation with conformal factor \( \omega^2(x) \) between Eqs (2.7) and (2.8):

\[ g_{\mu\nu} = \omega^2(x)g_{\mu\nu}, \]  

(2.9)

It is noted that the conformal factor \( \omega^2(x) = \frac{1}{\sqrt{g(r)}} \) and it does not change the condition of null geodesics.

It should be noticed that on the optical reference geometry \( \mathcal{M}^{opt} \), \( t \) plays the role of an arc length parameter because

\[ \int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} \sqrt{g_{rr}(kr)^2 + g_{\phi\phi}(k\phi)^2} dt \]  

(2.10)

where the unit tangent vector \( k^i \) of light ray paths on \( \mathcal{M}^{opt} \) as \( k^i = \frac{dx^i}{dt} \) with the unit vector condition \( 1 = g_{ij}k^i k^j \). Hence the GBT can be used on the optical reference geometry \( \mathcal{M}^{opt} \) as follows:

\[ \int_{D_R} K dS + \oint_{\partial D_R} k d\Gamma + \sum_i \theta_i = 2\pi\chi(D_R), \]  

(2.11)

where \( D_R \) is a non-singular domain outside the light ray, within boundary \( \partial D_R = \gamma_R \cup C_R \), \( \kappa \) stands for the geodesic curvature, \( K \) is used for the Gaussian curvature of optical metric, \( \theta_i \) is the exterior jump angles at the \( i \)th vertex, and \( \chi(D_R) = 1 \) is the Euler characteristic number.

The geodesic curvature \( \kappa \) can be calculated with the following equation for the unit speed condition \( \bar{g} = 1 \):

\[ \kappa = \bar{g} \left( \nabla \kappa, \kappa \right). \]  

(2.12)

When \( R \) goes to infinity \( R \rightarrow \infty \), the summation of the jump angles \( \sum_i \theta_i \) are calculated as \( \pi \) for the the source \( S \), and observer \( O \). Then the GBT is written as follows:

\[ \int_{D_R} K dS + \oint_{\partial D_R} k d\Gamma \bigg|_{\kappa \rightarrow \infty} = \int_{D_\infty} K dS + \int_0^{\pi + \kappa} d\phi = \pi, \]  

(2.13)

where the \( K \) is the Gaussian curvature (gives information about how surface is curved) and the \( \kappa \) is defined.
as follows:

\[ K = \frac{-1}{\sqrt{g_{rr}g_{\phi\phi}}} \left( \frac{\partial}{\partial r} \left( \frac{1}{\sqrt{g_{rr}}} \frac{\partial \sqrt{g_{rr}}}{\partial r} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sqrt{g_{\phi\phi}}} \frac{\partial \sqrt{g_{\phi\phi}}}{\partial \phi} \right) \right). \]  

(2.14)

Then the Gaussian curvature for the optical metric of DSW in 2.7 is calculated:

\[ K = \frac{(6 \lambda^2 + 6) M^3 + (7 \lambda^2 - 7) r M^2 + r^2 (\lambda^2 + 2) M}{(r + 2 M)^4}. \]  

(2.15)

The Gaussian curvature in 2.15 reduces to in this form up to leading orders:

\[ K = 6 \frac{M^3}{(r + 2 M)^4} - 7 \frac{M^2}{(r + 2 M)^3} + \frac{M}{(r + 2 M)^2} + \frac{M (6 M^2 - 7 M r + r^2) \lambda^2}{(r + 2 M)^4} + O(\lambda^3). \]  

(2.16)

Afterwards, we calculate the geodesic curvature \( \kappa \) which shows how far the curve \( C_R \) deviates from the geodesic, using the following equation:

\[ \kappa = \frac{1}{2 \sqrt{g_{rr}g_{\phi\phi}}} \left( \frac{\partial g_{\phi\phi}}{\partial r} \frac{\partial \phi}{\partial t} - \frac{\partial g_{rr}}{\partial \phi} \frac{\partial r}{\partial t} \right). \]  

(2.17)

Note that if the trajectory of light ray \( \gamma \) is geodesic, the geodesic curvature is zero \( \kappa(\gamma) = 0 \) so that we can choose \( C_R := r(\phi) = R = \text{const.} \) At \( R \) goes to \( \infty \), the geodesic curvature \( \kappa \) reduces to

\[ \lim_{R \to \infty} \kappa(C_R) \to \frac{1}{R}. \]  

(2.18)

Additionally, at \( R \) goes to \( \infty \), optical metric also goes to:

\[ \lim_{R \to \infty} dt \to R \, d\phi. \]  

(2.19)

We can define the trajectory of \( \gamma \) (the light ray) where it is \( r = b/\sin \phi \), and the deflection angle by the DSW can be calculated using the GBT as follows:

\[ \hat{\kappa} = -\int \int_0^{\pi} K \, dr \, d\phi, \]  

(2.20)

where \( d\sigma = \sqrt{\det[g_{\mu\nu}]} \, dr \, d\phi \) is an areal element and the \( \kappa(C_R) \, dt = d\phi \) is used.

Using the Gaussian curvature \( K \) Eq. (2.16) into the above integral, the deflection angle by DSW within the leading order terms (weak lensing) is calculated as follows:

\[ \hat{\kappa} \simeq \frac{4 M}{b} + \frac{2 M \lambda^2}{b}. \]  

(2.21)

The deflection angle by DSW is increased with the ratio of the parameter \( \lambda \) as seen in the Eq. 2.21 with compared to Schwarzschild black hole [62].

B. Strong deflection limit of Damour-Solodukhin wormhole and its relation with QNMs

Using the Bozza’s procedure [76], we study the strong gravitational lensing (SGL) of the DSW in the case of photons passing very close to the photon sphere, with radius \( r_m \). We use the assumption (\( \theta = \pi/2 \)), with the light ray’s trajectory

\[ r^2 = g(r) \left( \frac{E^2}{f(r)} - \frac{L^2}{h(r)} \right) = 0, \]  

(2.22)

where “dot” is for the derivative respect to an affine parameter. Note that if the conserved energy is \( E \equiv f(r) \lambda > 0 \) whereas the angular momentum defined as \( L \equiv r^2(r) \phi \) in Eq. 2.5. From the null circular orbit, we can find by the largest positive solution of the equation,

\[ \frac{h(r)'}{h(r)} = f(r)'). \]  

(2.23)

Using the Eq. (2.23), it is found that \( r_m = 3M \) and \( r_m > r_{\text{throat}} \) is satisfied. Moreover, light ray is deflected from the closest approach distance of the photon \( r_c \) smaller than \( r \) so that this condition should be considered where the light ray is supposed to come from infinity and deflect by DSW. Using the conservation of the angular momentum, the closest approach distance \( r_c \) which is related to the impact parameter is calculated as

\[ u = \sqrt{\frac{h(r_c)}{f(r_c)}} = \sqrt{r_c^2 \left( 1 - \frac{2 M}{r_c} \right)^{-1}}. \]  

(2.24)

After we use the definition of the critical impact parameter \( u_{cr} \) at strong deflection limit \( r_c \to r_m \) or \( u \to u_m \) as

\[ u_m(r_m) \equiv \lim_{r_c \to r_m} \sqrt{\frac{h(r_c)}{f(r_c)}}, \]  

(2.25)

we obtain:

\[ u_m = 3 \sqrt{3M}. \]  

(2.26)

Then we calculate the exact deflection angle \( \alpha \) for the DSW as follows:

\[ \alpha(r_c) = I(r_c) - \pi \]  

(2.27)

where \( I(r_c) \) is calculated by

\[ I = 2 \int_{r_c}^{r_m} \frac{dr}{\sqrt{g(r)h(r)} \sqrt{\frac{h(r)f(r_c)}{h(r_c)f(r)}} - 1}. \]  

(2.28)

Note that when \( u \) decreases, the bending angle \( \alpha \) increases that the light rays encircle the DSW completely till \( 2\pi \). At \( r_c = r_m \) due to \( u = u_m \) the photon will
be trapped inside an orbit. As it diverges in the SGL, 
\( u \rightarrow u_m \) or \( r_c \rightarrow r_m \), we rewrite the deflection angle in 
this form which is used for ultrastatic spacetimes:

\[
\alpha(u) = -a \log \left( \frac{r_c}{r_m} - 1 \right) + b + O([r_c - r_m)] \log(r_c - r_m),
\]

where \( a \) and \( b \) are SGL constants. The impact parameter 
\( u \) is also related to angular separation of the image from 
the lens: \( \theta = \frac{u}{\Delta u_m} \), where \( \Delta \) is the distance between 
the lens \( D_{OL} \) and the observer. Then the strong field limit of 
the deflection angle can be calculated as follows:

\[
\alpha(u) = -\hat{a} \log \left( \frac{\theta D_{OL}}{u_m} - 1 \right) + \hat{b} + O([u - u_m)] \log(u - u_m)).
\]

\[
\hat{a} = \sqrt{\frac{2f(r_m)}{g(r_m)[h''(r_m)f(r_m) - f(r_m)h''(r_m)]}}
\]

\[
\hat{b} = \hat{a} \log \left[ \frac{2}{r_m} \left( \frac{h''(r_m)}{h(r_m)} - \frac{f''(r_m)}{f(r_m)} \right) \right] + I(r_m) - \pi
\]

where expression for \( I(r_m) \) can be found in [76] (only 
solve numerically). Note that prime denotes the deriva-
tive with respect to the radial coordinate \( r \). Moreover, 
all the information about the SGL is encoded into \( \hat{a} \) and 
\( \hat{b} \).

Then we find the relation with the QNM of BIBHs 
[75]:

\[
\omega_{\text{QNM}} = \Omega_p I - i(n + 1/2) |\Lambda|,
\]

where

\[
\Lambda = c \sqrt{\frac{g(r_m)[f(r_m)h''(r_m) - f''(r_m)h(r_m)]}{2h(r_m)}}
\]

\[
\Lambda = \frac{c\sqrt{3 - 6\lambda^2}}{9M}
\]

with speed of light \( c \). The parameter \( \Lambda \) which appears 
in the imaginary part is the Lyapunov exponent which 
determines the instability timescale of the orbit. Then 
the simple relation can be written as follows:

\[
\Lambda = \frac{c}{u_m D_{OL}}.
\]

The other important relation is the coordinate angular 
velocity with the impact parameter of the lens

\[
\Omega_m = c \sqrt{f(r_m)g(r_m)} = \frac{c}{r_m}.
\]

(2.37)

After using the equations (2.36) and (2.37), it is easily 
observed that

\[
\pi = \frac{\Omega_p}{\Lambda}.
\]

(2.38)

III. ROTATING DAMOUR-SOLODUKHIN 
WORMHOLE

In this section, we have briefly describe rotating 
Damour-Solodukhin wormhole spacetime. The Kerr-
like wormhole metric is constructed using the method 
of Damour and Solodukhin [5]:

\[
ds^2 = -\left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar\sin^2 \theta}{\Sigma} dtd\phi + \frac{\Sigma}{\Delta} dr^2
\]

\[
+ \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2Ma^2r^2\sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2,
\]

(3.1)

where \( \Sigma \equiv r^2 + a^2 \cos^2 \theta \), \( \Delta \equiv r^2 - 2(M + \lambda^2)r + a^2 \).

Note that the \( M \) is the mass and the \( aM \) stands for the 
angular momentum. For the case of \( \lambda^2 = 0 \), we 
can recover the Kerr metric, on the other hand, for non-
vanning \( \lambda^2 \), the structure of the spacetime is totally 
changed. We have calculated the positive root of \( \Delta \): 
\( r_+ = (1 + \lambda^2)M + \sqrt{M^2(1 + \lambda^2)^2 - a^2} \) that gives special 
surface, but not the surface of the event horizon, and 
the throat of the wormhole is located at \( r = r_+ \). For the non 
vanishing values of the \( \lambda \neq 0 \), the Kerr-like wormhole 
is constructed and its QNMs are recently studied in [5].

Now, we will study the weak gravitational lensing by 
rotating DSW using the Gauss-Bonnet theorem, and ob-
tain the deflection angle in weak limits.

A. Deflection angle of rotating Damour-Solodukhin 
Wormhole

Here, we rewrite the rotating DSW by using the equa-
torial plane \( \theta = \pi/2 \), to study the deflection angle:

\[
ds^2 = -f(r)dt^2 + \frac{\Sigma}{\Delta} dr^2 - 2a (1 - f(r)) d\phi dt
\]

\[
+ \left[ \Sigma - a^2 (f(r) - 2) \right] d\phi^2,
\]

(3.3)

with

\[
f(r) = 1 - \frac{2Mr}{\Sigma},
\]

(3.4)

\[
\Sigma = r^2,
\]

(3.5)

\[
\Delta = r^2 - 2M(1 + \lambda^2)r + a^2.
\]

(3.6)
Using the method of Werner [60] that used the GBT for stationary spacetimes by transforming into the Finsler-Randers type metric of the general form:

\[ F(x, v) = \sqrt{\bar{g}_{ij}(x)v^i v^j + \eta_i(x)v^i}. \]  \hspace{1cm} (3.7)

Here \( \bar{g}_{ij} \) is the Riemannian metric, with the condition \( (\bar{g}^i_j \eta_i \eta_j < 1) \) and \( \eta_i \) is the one-form. For stationary spacetimes optical geometry from the null geodesics \( (ds^2 = 0) \) are calculated by using Finsler geometry \( F \) so that one can use \( dt = F(x, dx) \), and the Fermat’s principle is found as follows:

\[ \delta \int_{\gamma_F} dt = \delta \int_{\gamma_F} F(x, \dot{x}) dt = 0, \]  \hspace{1cm} (3.8)

where the rotating DSW-Randers optical metric is obtained as:

\[ F = \left[ \frac{a^2(1 - f(r))^2}{f(r)^2} + \frac{\Sigma - a^2(f(r) - 2)}{f} \right] \left( \frac{d\phi}{dt} \right)^2 + \frac{\Sigma (\frac{dr}{dt})^2}{f}, \]  \hspace{1cm} (3.9)

Note that the Randers metrics are among the simplest Finsler metrics. Then we use the Nazim’s method to the Finsler metrics to construct Riemannian manifold when the Hessian

\[ g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 F^2(x, v)}{\partial v^i \partial v^j}, \]  \hspace{1cm} (3.10)

where \( x \in \mathcal{M}, \ v \in T_x \mathcal{M} \), that \((\mathcal{M}, \bar{g})\) osculates the rotating DSW-Randers manifold \((\mathcal{M}_r, \bar{F})\). After the vector field \( \bar{v} \) tangent to the geodesic \( \gamma_{\bar{F}} \) is chosen as \( \bar{v}(\gamma_{\bar{F}}) = \dot{\bar{x}} \), the Hessian reduces to

\[ \bar{g}_{ij}(x) = g_{ij}(x, \bar{v}(x)). \]  \hspace{1cm} (3.11)

It is noted that the geodesic of the Randers manifold \( \gamma_{\bar{F}} \) is also a geodesic \( \gamma_{\bar{g}} \) of \((\mathcal{M}, \bar{g}) \) \((\gamma_{\bar{F}} = \gamma_{\bar{g}})\):

\[ \ddot{x}^i + \Gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k = \ddot{x}^i + \Gamma^i_{jk}(x) \dot{x}^j \dot{x}^k = 0. \]  \hspace{1cm} (3.12)

Choosing the nonsingular region \( S_R \subset M \) that is bounded by \( \gamma_{\bar{F}} \) light ray and \( \gamma_R \) within \( R \) (radial distance). At the asymptotically flat limits, the equation for the light rays with the impact parameter \( b \) is chosen as follows:

\[ r(\phi) = \frac{b}{\sin \phi}. \]  \hspace{1cm} (3.13)

The corresponding vector fields according to the equation of the light ray are:

\[ \sigma^r = \frac{dr}{dt} = -\cos \phi, \ \sigma^\phi = \frac{d\phi}{dt} = \frac{\sin^2 \phi}{b}. \]  \hspace{1cm} (3.14)

Here we use the straight line approximation, where the deflection angle is calculated in leading order terms so that the vector field is ruled with the \( r \) light ray. Then, we derive the non-zero components of the metric using the Eqs. (3.10), and (3.14):

\[ g_{rr} = -\frac{2r (\sin(\phi))^6 a m}{b^3} \left( \frac{(\sin(\phi))^4 r^2 + (\cos(\phi))^2}{b^2} \right)^{-3/2} + \frac{4 \lambda m r^2}{r} + \frac{4 m r}{r} + 1, \]  \hspace{1cm} (3.15)

\[ g_{r\phi} = \frac{2 (\cos(\phi))^3 a m}{r} \left( \frac{(\sin(\phi))^4 r^2 + (\cos(\phi))^2}{b^2} \right)^{-3/2}, \]  \hspace{1cm} (3.16)

\[ g_{\phi\phi} = -\frac{4 r^3 (\sin(\phi))^6 a m}{b^3} \left( \frac{(\sin(\phi))^4 r^2 + (\cos(\phi))^2}{b^2} \right)^{-3/2} - \frac{6 m r (\cos(\phi))^2 (\sin(\phi))^2}{b} \left( \frac{(\sin(\phi))^4 r^2 + (\cos(\phi))^2}{b^2} \right)^{-3/2} + 2 \lambda^2 m r + 2 m r + r^2. \]  \hspace{1cm} (3.17)
The determinant of the above metric is calculated as follows:

\[
\det g = -6 \left( \lambda^2 m + m + r/6 \right) \sqrt{\left( \cos(\phi) \right)^4 r^2 + \left( b^2 - 2 r^2 \right) \left( \cos(\phi) \right)^2 + r^2 + m a \left( \cos(\phi) - 1 \right) \left( \cos(\phi) + 1 \right)} r 
\]

(3.18)

Then we calculate the Gaussian optical curvature using the following equation:

\[
\mathcal{K} = \frac{1}{\sqrt{\det g}} \left[ \frac{\partial}{\partial \phi} \left( \frac{\sqrt{\det g} \Gamma_r^\phi}{g_{rr}} \right) - \frac{\partial}{\partial r} \left( \frac{\sqrt{\det g} \Gamma_r^r}{g_{rr}} \right) \right] ,
\]

(3.19)

and \( \mathcal{K} \) is calculated as follows:

\[
\mathcal{K} = \left( 6 \frac{\lambda^4 m^3}{r^4} + 12 \frac{\lambda^2 m^3}{r^4} - 2 \frac{\lambda^2 m^2}{r^3} + 6 \frac{m^3}{r^4} - 2 \frac{m^2}{r^3} \right) B
\]

(3.20)

with the complicated function \( B \):

\[
B = -6 \frac{ma \left( \sin(\phi) \right)^2 \left( \sin(\phi) \right)^4 r^2 + \left( \cos(\phi) \right)^2 b^2}{r^4 b^7} \left( \frac{4 m}{r^4} \left( -r/4 + m \left( \lambda^2 + 1 \right) \right) \right) \left( \sin(\phi) \right)^{12} - \left( -r/3 + m \left( \lambda^2 + 1 \right) \right) \left( \sin(\phi) \right)^{10}
\]

\[
+ 21/2 \left( \frac{5 m}{21} + 3/7 \right) r + m \left( \lambda^2 + 1 \right) \left( m - \frac{9}{7} \right) \left( \cos(\phi) \right)^2 b^2 r^4 \left( \sin(\phi) \right)^8
\]

\[
+ 12 \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^2 b^2 r^3 \left( \sin(\phi) \right)^7
\]

\[
+ 6 \left( \frac{-5/2}{r^2} \left( \cos(\phi) \right)^2 + b^2 \right) \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^2 b^2 r^2 \left( \sin(\phi) \right)^6
\]

\[
+ 24 \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^4 b^3 r^3 \left( \sin(\phi) \right)^5
\]

\[
+ \left( -11/2 + m \right) r + \frac{33 m \left( \lambda^2 + 1 \right)}{2} \left( \cos(\phi) \right)^4 b^4 r^2 \left( \sin(\phi) \right)^4
\]

\[
3 \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^4 b^5 r \left( \sin(\phi) \right)^3
\]

\[
+ 15 \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^6 b^4 r^2 \left( \sin(\phi) \right)^2
\]

\[
- 6 \left( \lambda^2 m + m - r/3 \right) \left( \cos(\phi) \right)^6 b^5 r \sin(\phi) + \left( \cos(\phi) \right)^6 b^6 m^2 \left( \lambda^2 + 1 \right)
\]

(3.21)

Then using the GBT equation defined in Eq. 2.20, we calculate the total deflection angle of the rotating DSW:
The total deflection angle is found

\[ \alpha = \frac{4m}{b} + \frac{2m\lambda^2}{b} \pm \frac{4am}{b^2} + O(Q^2, m), \]

(3.22)

where the positive sign stands for a retrograde light rays and negative sign is for the prograde light rays [60]. Note that for the \( \lambda = 0 \), the deflecting angle of the Kerr black hole is recovered.

IV. CONCLUSIONS

In summary, the observation of wormholes by studying the gravitational lensing is one of the most effective way to testify them in the universe.

For this purpose, first, we have explicitly calculated the deflection angle of the light by DSW in the weak-field limit using the method developed by Gibbons and Werner.

Second, we have studied the strong gravitational lensing using the method of Bozza, and show the relation of the strong deflection angle with its QNMs. Last, we have briefly described the Damour-Solodukhin wormhole in a rotating Kerr-like black hole spacetime and studied the weak gravitational lensing using the GBT.

The significant of this result is that the deflection of a light ray is calculated by outside of the lensing region which means that the effect of the gravitational lensing is a global effect such that there are more than one light ray converging between the source and observer. Hence, we are able to find accurately deflecting angle in weak-field limits. We finally conclude that the deflection angle by the DSW is increased with the DS wormhole parameter of \( \lambda \).

With regards to future work, it would be interesting to see whether this approach could also be extended to the other compact objects and also see whether there is an effect of dark matter on the deflection angle. A careful studies on the gravitational lensing and also gravitational waves with QNMs may shed some light on possible signature of the existence of the wormholes.

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[1] Albert Einstein and N. Rosen. The Particle Problem in the General Theory of Relativity. Phys. Rev., 48:73–77, 1935.
[2] M. S. Morris and K. S. Thorne. Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity. Am. J. Phys., 56:395–412, 1988.
[3] M. S. Morris, K. S. Thorne, and U. Yurtsever. Wormholes, Time Machines, and the Weak Energy Condition. Phys. Rev. Lett., 61:1446–1449, 1988.
[4] Thibault Damour and Sergey N. Solodukhin. Wormholes as black hole foils. Phys. Rev., D76:024016, 2007.
[5] Pablo Bueno, Pablo A. Cano, Frederik Goelen, Thomas Hertog, and Bert Vercnocke. Echoes of Kerr-like wormholes. Phys. Rev., D97(2):024040, 2018.
[6] Matt Visser. Traversable wormholes: Some simple examples. Phys. Rev., D39:3182–3184, 1989.
[7] Francisco S. N. Lobo. Stability of phantom wormholes. Phys. Rev., D71:124022, 2005.
[8] Jose`P. S. Lemos, Francisco S. N. Lobo, and Sergio Quinte de Oliveira. Morris-Thorne wormholes with a cosmological constant. Phys. Rev., D68:064004, 2003.
[9] Juan Maldacena and Leonard Susskind. Cool horizons for entangled black holes. Fortsch. Phys., 61:781–811, 2013.
[10] S. W. Hawking. Wormholes in Space-Time. Phys. Rev., D37:904–910, 1988. [363(1988)].
[11] Sergey V. Sushkov. Wormholes supported by a phantom energy. Phys. Rev., D71:043520, 2005.
[12] K. A. Bronnikov and Sung-Won Kim. Possible wormholes in a brane world. Phys. Rev., D67:064027, 2003.
[13] Valeri P. Frolov and Igor D. Novikov. Physical Effects in Wormholes and Time Machine. Phys. Rev., D42:1057–1065, 1990.
[14] M. S. R. Delgaty and Robert B. Mann. Traversable wormholes in (2+1)-dimensions and (3+1)-dimensions with a cosmological constant. Int. J. Mod. Phys., D4:31–46, 1995.
[15] G. P. Perry and Robert B. Mann. Traversable wormholes in (2+1)-dimensions. Gen. Rel. Grav., 24:305–321, 1992.
[16] John G. Cramer, Robert L. Forward, Michael S. Morris, Matt Visser, Gregory Benford, and Geoffrey A. Landis. Natural wormholes as gravitational lenses. Phys. Rev., D51:3117–3120, 1995.
[17] Julio Oliva, David Tempo, and Ricardo Troncoso. Three-dimensional black holes, gravitational solitons, kinks and wormholes for BHT massive gravity. JHEP, 07:011, 2009.
[18] Juan Martin Maldacena and Liat Maoz. Wormholes in AdS. JHEP, 02:053, 2004.
[19] Gerard Clement. A Class of Wormhole Solutions to Higher Dimensional General Relativity. Gen. Rel. Grav., 16:131, 1984.
[20] Gerard Clement. Wormhole cosmic strings. Phys. Rev., D51:6803–6809, 1995.
[21] Gerard Clement. Flat wormholes from straight cosmic strings. J. Math. Phys., 38:5807–5819, 1997.
[22] E. I. Guendelman. Wormholes and the construction of compactified phases. Gen. Rel. Grav., 23:1415–1419, 1991.
[23] Eduardo I Guendelman, Alexander Kaganovich, Emil Nissimov, and Svetlana Pacheva. Einstein-Rosen ‘Bridge’ Needs Lightlike Brane Source. Phys. Lett., B681:457–462, 2009.
[24] Eduardo I. Guendelman, Alexander Kaganovich, Emil Nissimov, and Svetlana Pacheva. Spherically Symmetric and Rotating Wormholes Produced by Lightlike Branes.
Naoki Tsukamoto. Strong deflection limit analysis and gravitational lensing of light in the Kerr spacetime is a wormhole. *Phys. Rev.*, D95(6):064035, 2017.

M. Sharif and Sehrish Iftikhar. Strong gravitational lensing in non-commutative wormholes. *Astrophys. Space Sci.*, 357(1):85, 2015.

Rajibul Shaikh and Sayan Kar. Gravitational lensing by scalar-tensor wormholes and the energy conditions. *Phys. Rev.*, D96(4):044037, 2017.

S. N. Sajidi and N. Riazi. Gravitational Lensing by Polytrropic Wormholes. 2016.

Regina Lukmanova, Aliya Kulbakova, Ramil Izmailov, and Alexander A. Potapov. Gravitational Microlensing by Ellis Wormhole: Second Order Effects. *Int. J. Theor. Phys.*, 55(11):4723–4730, 2016.

Takao Kitamura. Gravitational lensing in an exotic spacetime. PhD thesis, Hiroasaki U., 2016-03-03.

Hideki Asada. Gravitational lensing by exotic objects. *Mod. Phys. Lett.*, A32(34):1730031, 2017.

Izzet Sakalli, Ali Ogün, and Seyyedeh Fatemeh Mirekhtiai. Gravitational Lensing Effect on the Hawking Radiation of Dyonic Black Holes. *Int. J. Geom. Meth. Mod. Phys.*, 11(08):1450074, 2014.

Kamal K. Nandi, Ramil N. Izmailov, Almir A. Yanbekov, and Azat A. Shyakhmetov. Ring-down gravitational waves and lensing observables: How far can a wormhole mimic those of a black hole? *Phys. Rev.*, D95(10):104011, 2017.

Peter K. F. Kuhfittig. Gravitational lensing of wormholes in noncommutative geometry. 2015.

Chul-Moon Yoo, Tomohiro Harada, and Naoki Tsukamoto. Wave Effect in Gravitational Lensing by the Ellis Wormhole. *Phys. Rev.*, D87:084045, 2013.

Naoki Tsukamoto, Tomohiro Harada, and Kohji Yajima. Can we distinguish between black holes and wormholes by their Einstein ring images? *Phys. Rev.*, D86:104062, 2012.

Amrita Bhattacharya and Alexander A. Potapov. Bending of light in Ellis wormhole geometry. *Mod. Phys. Lett.*, A25:2399–2409, 2010.

Kamal Kanti Nandi, Yuan-Zhong Zhang, and Alexander A. Potapov. Gravitational Microlensing by Ellis wormholes. *Phys. Rev.*, D96(4):044037, 2017.

M. C. Werner. Gravitational lensing in the Kerr-Randers optical geometry. *Gen. Rel. Grav.*, 44:3047–3057, 2012.

G. W. Gibbons, C. M. Warnick, and M. C. Werner. Light-bending in Schwarzschild-de-Sitter: Projective geometry of the optical metric. *Class. Quant. Grav.*, 25:245009, 2008.

G. W. Gibbons and M. C. Werner. Applications of the Gauss-Bonnet theorem to gravitational lensing. *Class. Quant. Grav.*, 25:235009, 2008.

Kemkit Jusufi, Izzet Sakalli, and Ali Ogün. Gravitational lensing of wormholes in an exotic spacetime. *Phys. Rev.*, D96(2):024020, 2016.

Peter K. F. Kuhfittig. Gravitational lensing of wormholes by noncommutative spacetime. 2015.

Chul-Moon Yoo, Tomohiro Harada, and Naoki Tsukamoto. Wave Effect in Gravitational Lensing by the Ellis Wormhole. *Phys. Rev.*, D87:084045, 2013.

Naoki Tsukamoto, Tomohiro Harada, and Kohji Yajima. Can we distinguish between black holes and wormholes by their Einstein ring images? *Phys. Rev.*, D86:104062, 2012.

Amrita Bhattacharya and Alexander A. Potapov. Bending of light in Ellis wormhole geometry. *Mod. Phys. Lett.*, A25:2399–2409, 2010.

Kamal Kanti Nandi, Yuan-Zhong Zhang, and Alexander A. Potapov. Gravitational Microlensing by Ellis wormholes. *Phys. Rev.*, D96(4):044037, 2017.

M. C. Werner. Gravitational lensing in the Kerr-Randers optical geometry. *Gen. Rel. Grav.*, 44:3047–3057, 2012.

G. W. Gibbons, C. M. Warnick, and M. C. Werner. Light-bending in Schwarzschild-de-Sitter: Projective geometry of the optical metric. *Class. Quant. Grav.*, 25:245009, 2008.

G. W. Gibbons and M. C. Werner. Applications of the Gauss-Bonnet theorem to gravitational lensing. *Class. Quant. Grav.*, 25:235009, 2008.

Kemkit Jusufi, Izzet Sakalli, and Ali Ogün. Gravitational lensing of wormholes in an exotic spacetime. *Phys. Rev.*, D96(2):024020, 2016.

Peter K. F. Kuhfittig. Gravitational lensing of wormholes by noncommutative spacetime. 2015.

Chul-Moon Yoo, Tomohiro Harada, and Naoki Tsukamoto. Wave Effect in Gravitational Lensing by the Ellis Wormhole. *Phys. Rev.*, D87:084045, 2013.

Naoki Tsukamoto, Tomohiro Harada, and Kohji Yajima. Can we distinguish between black holes and wormholes by their Einstein ring images? *Phys. Rev.*, D86:104062, 2012.

Amrita Bhattacharya and Alexander A. Potapov. Bending of light in Ellis wormhole geometry. *Mod. Phys. Lett.*, A25:2399–2409, 2010.

Kamal Kanti Nandi, Yuan-Zhong Zhang, and Alexander A. Potapov. Gravitational Microlensing by Ellis wormholes. *Phys. Rev.*, D96(4):044037, 2017.

M. C. Werner. Gravitational lensing in the Kerr-Randers optical geometry. *Gen. Rel. Grav.*, 44:3047–3057, 2012.

G. W. Gibbons, C. M. Warnick, and M. C. Werner. Light-bending in Schwarzschild-de-Sitter: Projective geometry of the optical metric. *Class. Quant. Grav.*, 25:245009, 2008.

G. W. Gibbons and M. C. Werner. Applications of the Gauss-Bonnet theorem to gravitational lensing. *Class. Quant. Grav.*, 25:235009, 2008.

Kemkit Jusufi, Izzet Sakalli, and Ali Ogün. Gravitational lensing of wormholes in an exotic spacetime. *Phys. Rev.*, D96(2):024020, 2016.

Peter K. F. Kuhfittig. Gravitational lensing of wormholes by noncommutative spacetime. 2015.

Chul-Moon Yoo, Tomohiro Harada, and Naoki Tsukamoto. Wave Effect in Gravitational Lensing by the Ellis Wormhole. *Phys. Rev.*, D87:084045, 2013.

Naoki Tsukamoto, Tomohiro Harada, and Kohji Yajima. Can we distinguish between black holes and wormholes by their Einstein ring images? *Phys. Rev.*, D86:104062, 2012.

Amrita Bhattacharya and Alexander A. Potapov. Bending of light in Ellis wormhole geometry. *Mod. Phys. Lett.*, A25:2399–2409, 2010.

Kamal Kanti Nandi, Yuan-Zhong Zhang, and Alexander A. Potapov. Gravitational Microlensing by Ellis wormholes. *Phys. Rev.*, D96(4):044037, 2017.

M. C. Werner. Gravitational lensing in the Kerr-Randers optical geometry. *Gen. Rel. Grav.*, 44:3047–3057, 2012.

G. W. Gibbons, C. M. Warnick, and M. C. Werner. Light-bending in Schwarzschild-de-Sitter: Projective geometry of the optical metric. *Class. Quant. Grav.*, 25:245009, 2008.
[67] I. Sakalli and A. Ovgun. Hawking Radiation and Deflection of Light from Rindler Modified Schwarzschild Black Hole. *EPL*, 118(6):60006, 2017.

[68] Kimet Jusufi, Farook Rahaman, and Ayan Banerjee. Semi-classical gravitational effects on the gravitational lensing in the spacetime of topological defects. *Annals Phys.*, 389:219–233, 2018.

[69] Prieslei Goulart. Phantom wormholes in Einstein-Maxwell-dilaton theory. *Class. Quant. Grav.*, 35(2):025012, 2018.

[70] Kimet Jusufi. Deflection angle of light by wormholes using the Gauss-Bonnet theorem. *Int. J. Geom. Meth. Mod. Phys.*, 14(12):1750179, 2017.

[71] Kimet Jusufi, Nayan Sarkar, Farook Rahaman, Ayan Banerjee, and Sudan Hansraj. Deflection of light by black holes and massless wormholes in massive gravity. *Eur. Phys. J.*, C78(4):349, 2018.

[72] Toshiaki Ono, Asahi Ishihara, and Hideki Asada. Gravitomagnetic bending angle of light with finite-distance corrections in stationary axisymmetric spacetimes. *Phys. Rev.*, D96(10):104037, 2017.

[73] Koki Nakajima and Hideki Asada. Deflection angle of light in an Ellis wormhole geometry. *Phys. Rev.*, D85:107501, 2012.

[74] Gabriel Crisnejo and Emanuel Gallo. Light deflection in a plasma medium and gravitational lensing of massive particles using the Gauss-Bonnet theorem. A unified treatment. 2018.

[75] Ivan Zh. Stefanov, Stoytcho S. Yazadjiev, and Galin G. Gyulchev. Connection between Black-Hole Quasinormal Modes and Lensing in the Strong Deflection Limit. *Phys. Rev. Lett.*, 104:251103, 2010.

[76] V. Bozza. Gravitational lensing in the strong field limit. *Phys. Rev.*, D66:103001, 2002.