Retractions and Gorenstein Homological Properties

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Plan

1. Retractions of Algebras
2. Gorenstein Homological Properties
3. Nakayama Algebras
Localizable modules

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- $A$-mod = the category of finite generated left $A$-modules
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- an equivalence $S^\perp \simeq A$-mod/add $S$
Left retractions of algebras

- there is an algebra homomorphism $\iota : A \to L(A)$ such that the functor $i = \text{Hom}_{L(A)}(L(A), -)$ induces an equivalence $L(A)\text{-mod} \simeq S^\perp$. 
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Remark: $L(A)$ is Morita equivalent to $eAe$ for an idempotent $e$
A recollement

£ : A → L(A) is a left localization [Silver 1967], and thus a homological epimorphism [Geigle-Lenzing 1991]: an embedding of derived categories induced by

i : L(A)-mod → A-mod

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- \( \iota: A \to L(A) \) induces a recollement

\[ D^b(L(A)\text{-mod}) \leftrightarrow D^b(A\text{-mod}) \leftrightarrow D^b(\Delta\text{-mod}) \]

where \( \Delta = \text{End}_A(S)^{\text{op}} \); compare [C.-Krause 2011].
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Singularity categories

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- a **singular equivalence** means a triangle equivalence between singularity categories

- the left retraction $\iota: A \to L(A)$ induces a singular equivalence

$$\mathcal{D}_{\text{sg}}(L(A)) \simeq \mathcal{D}_{\text{sg}}(A)$$
An $A$-module $G$ is Gorenstein projective (or MCM) provided that $\text{Ext}^i_A(G, A) = 0 = \text{Ext}^i_{A^{\text{op}}}(G^*, A)$ for $i \geq 1$ and $G$ is reflexive, where $G^* = \text{Hom}_A(G, A)$; [Auslander-Bridger 1969/Enochs-Jenda 1995]
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- $A\text{-proj} \subseteq A\text{-Gproj} \subseteq A\text{-mod}$

- $A\text{-Gproj}$ is Frobenius, and thus $A\text{-Gproj}$ is triangulated

- there exists a canonical triangle embedding

  $$F_A : A\text{-Gproj} \rightarrow \text{D}_{sg}(A);$$

  [Buchweitz 1987/Keller-Vossieck 1987/Happel 1991 ....]
Gorenstein algebras

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- Gorensteinness $\iff$ finite resolutions by Gorenstein projectives
  $\iff$ the functor $F_A : A\text{-}Gproj \rightarrow D_{sg}(A)$ is dense, and thus a triangle equivalence
Gorenstein homological properties: a trichotomy

- CM-free algebras: Gorenstein projective = projective, or equivalently, $A$-Gproj is trivial.
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A trivial trichotomy from Gorenstein homological algebra:
(1) Gorenstein algebras
(2) non-Gorenstein CM-free algebras
(3) non-Gorenstein algebras, that are not CM-free
The results
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Theorem

Let $S, L(A)$ be as above. Then $A$ is Gorenstein if and only if $L(A)$ is Gorenstein and $\text{proj.dim } L(A) \otimes_A E(S) < \infty$. 
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**Proposition**

The CM-freeness of $L(A)$ implies the CM-freeness of $A$. 

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**Proposition**

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The proof: the functor $i_\lambda: A\text{-mod} \to L(A)\text{-mod}$ sends Gorenstein projectives to Gorenstein projectives.
Notation

- A a connected Nakayama algebra: $Q(A)$ is a line or an oriented cycle
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- the *admissible sequence* of $A$: $c(A) = (c_1, c_2, \cdots, c_n)$, where $n = n(A)$ the number of simples, $c_i = l(P_i)$ and $P_{i+1} = P(\text{rad}P_i)$. 
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c($A$) is normalized if $c_n = 1$ (line algebra), $c_1 = \cdots = c_n$ (self-injective algebra), or $c_1 \leq c_j$ and $c_n = c_1 + 1$ (the interesting case).
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- $c(A)$ is normalized if $c_n = 1$ (line algebra), $c_1 = \cdots = c_n$ (self-injective algebra), or $c_1 \leq c_j$ and $c_n = c_1 + 1$ (the interesting case).
- From now on, assume that $A$ is non-self-injective and $c(A)$ is normalized.
Retractions of Nakayama algebras

$S = S_n$ is localizable: $0 \to P_1 \to P_n \to S_n \to 0$
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L(A) is connected Nakayama with n(L(A)) = n − 1 and c(L(A)) = (c'_1, c'_2, · · · , c'_{n−1}), where c'_j = c_j − [c_j+j−1 \over n].
S = S_n is localizable: $0 \rightarrow P_1 \rightarrow P_n \rightarrow S_n \rightarrow 0$

L(A) is connected Nakayama with $n(L(A)) = n - 1$ and $c(L(A)) = (c'_1, c'_2, \cdots, c'_{n-1})$, where $c'_j = c_j - \left[\frac{c_j + j - 1}{n}\right]$.

Similar consideration in [Zacharia 1983/Burgess-Fuller-Voss-Zimmermann 1985/Nagase 2011]
Retractions of Nakayama algebras

- $S = S_n$ is localizable: $0 \to P_1 \to P_n \to S_n \to 0$
- $L(A)$ is connected Nakayama with $n(L(A)) = n - 1$ and $c(L(A)) = (c'_1, c'_2, \cdots, c'_{n-1})$, where $c'_j = c_j - \left\lfloor \frac{c_j + j - 1}{n} \right\rfloor$.
- Similar consideration in [Zacharia 1983/Burgess-Fuller-Voss-Zimmermann 1985/Nagase 2011]
- set $r(A)$ to be the number of simples with infinite projective dimension
A retraction sequence
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**Theorem**

There is a sequence of homomorphisms between connected Nakayama algebras

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \rightarrow \cdots \xrightarrow{\eta_{r-1}} A_r$$

such that each $\eta_i$ is a left retraction, and $A_r$ is self-injective.

If $\text{gl.dim } A = \infty$, $r = r(A)$ and $A_r$ is unique up to isomorphism.
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**Corollary**

A triangle equivalence \( D_{\text{sg}}(A) \cong A_r\text{-mod} \), a truncated tube of rank \( n(A) - r(A) \).
Dichotomy for Nakayama algebras: $n(A) = 2$

for a non-self-injective algebra $A$, $c(A) = (c, c + 1)$
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**Proposition**

*We are in two cases.*

1. $c(A) = (2k, 2k + 1)$, $A$ is Gorenstein with self-injective dimension two;

2. $c(A) = (2k + 1, 2k + 2)$, $A$ is non-Gorenstein CM-free.
Trichotomy for Nakayama algebras: $n(A) = 3$

for a non-self-injective algebra $A$, $c(A) = (c, c + j, c + 1)$ for $j = -1, 0, 1$. 
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**Proposition**

1. **A Gorenstein:** \( c(A) = (2, 2, 3), (2, 4, 3), (3k, 3k, 3k + 1), (3k, 3k + 1, 3k + 1), (3k, 3k + 2, 3k + 1) \) or \( (3k + 1, 3k + 2, 3k + 2) \) for \( k \geq 1 \);

2. **A non-Gorenstein CM-free:** \( c(A) = (2, 3, 3), (3k + 1, 3k + 1, 3k + 2), (3k + 1, 3k + 3, 3k + 2), (3k + 2, 3k + 2, 3k + 3) \) or \( (3k + 2, 3k + 4, 3k + 3) \) for \( k \geq 1 \);

3. **A non-Gorenstein but not CM-free:** \( c(A) = (3k + 2, 3k + 3, 3k + 3) \) for \( k \geq 1 \).
Trichotomy for Nakayama algebras: $n(A) = 3$

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The proof uses some results in [Gustafson 1985].
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- in case (3), all indecomposable non-projective Gorenstein projective modules are $S_2^{[3m]}$, $1 \leq m \leq k$.
- The proof uses some results in [Gustafson 1985].
- more classification/ information on Gorenstein projective modules over Nakayama algebras is in [Ringel, 2012], using minimal projective modules and resolution quivers!
Thank You!

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