Highly irregular orbits for subshifts of finite type: large intersections and emergence

Yushi Nakano$^{1,*}$ and Agnieszka Zelerowicz$^2$

$^1$ Department of Mathematics, Tokai University, 4-1-1 Kitakaname, Hiraoka, Kanagawa, 259-1292, Japan
$^2$ Department of Mathematics, University of Maryland, College Park, MD 20742, United States of America

E-mail: yushi.nakano@tsc.u-tokai.ac.jp and azelerow@umd.edu

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Abstract
In their recent paper (2019 arXiv:1904.03424), the first author, Kiriki and Soma introduced a concept of pointwise emergence to measure the complexity of irregular orbits. They constructed a residual subset of the full shift with high pointwise emergence. In this paper we consider the set of points with high pointwise emergence for topologically mixing subshifts of finite type. We show that this set has full topological entropy, full Hausdorff dimension, and full topological pressure for any Hölder continuous potential. Furthermore, we show that this set belongs to a certain class of sets with large intersection property. This is a natural generalization of Färn and Persson (2011 Nonlinearity 24 1291–1309) to pointwise emergence and Carathéodory dimension.

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1. Introduction
Let $X$ be a compact metric space and $f : X \to X$ a continuous map. Let $\mathcal{P}(X)$ be the set of Borel probability measures on $X$ equipped with the weak topology. A point $x \in X$ is said to be irregular, if the time average along the forward orbit of $x$ does not exist, i.e. if the limit of empirical measures,

$$
\delta^n_x = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}, \quad n \geq 1,
$$

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does not exist in \( \mathcal{P}(X) \) (see [1, 31]). Such a point is also called historic [28, 29], non-typical [3] or divergent [13]. Although the set of irregular points (which we will call the irregular set and denote by \( I \)) is a \( \mu \)-zero measure set for any invariant measure \( \mu \) due to Birkhoff’s ergodic theorem, the set is known to be remarkably large for abundant dynamical systems. Pesin and Pitskel’ [25] obtained the first result for the largeness of the irregular set from thermodynamic viewpoint. In their paper, they showed that the irregular set for the full shift has full topological entropy and full Hausdorff dimension, that is,

\[
\text{h}_{\text{top}}(I) = \text{h}_{\text{top}}(X) \quad \text{and} \quad \text{dim}_H(I) = \text{dim}_H(X).
\]

Here \( h_{\text{top}}(Z) \) is the Pesin–Pitskel’ topological entropy for a (not necessarily compact) Borel set \( Z \) given in [25] (see subsection 4.1 for precise definition; this is a generalization of Bowen’s Hausdorff topological entropy of [10], and we refer to [19] for relation between entropies for a non-compact set). This thermodynamic largeness of irregular sets was extended to topologically mixing subshifts of finite type in [3] (together with the detailed study of the set of points at which Lyapunov exponent or local entropy fail to exist), to graph directed Markov systems in [17], to continuous maps with specification property in [13] (see also [31]), and to continuous maps with almost specification property in [32]. See also [1, 2, 4, 9, 11, 12, 20, 23, 28, 29, 33, 36] and references therein for the study of irregular sets from other viewpoints.

The irregular points considered in the above works were typically those, whose corresponding sequence of measures (1.1) oscillates between two (or finitely many) ergodic measures. Consequently, the space of accumulation points of (1.1) is finite dimensional. In this paper we consider irregular points with high complexity, that is points for which (1.1) oscillates between infinitely many ergodic measures. More precisely, given an infinite sequence of distinct ergodic measures, we consider points for which the set of accumulation points of (1.1) contains the infinite-dimensional simplex spanned by those measures. We obtain a lower bound on the dimension of the set of such points as the infimum over dimensions of measures in the sequence (see theorem 3.4).

Recently, Berger introduced a concept of metric emergence in [5] in order to ‘evaluate the complexity to approximate a system by statistics [5]’. Metric emergence quantifies such phenomena as the Newhouse phenomenon or KAM phenomenon. Even though the notion of emergence is relatively new, the study of metric emergence is already becoming a quite active research area [6–8, 14, 30]. Inspired by Berger’s work, the first author, Kiriki et al [22] introduced a concept of pointwise emergence to measure complexity of irregular orbits. The pointwise emergence \( \delta_x(\epsilon) \) at scale \( \epsilon > 0 \) at \( x \in X \) is defined by

\[
\delta_x(\epsilon) = \min \left\{ N \in \mathbb{N} \mid \text{there exists } \{\mu_j\}_{j=1}^N \subset \mathcal{P}(X) \text{ such that} \right. \\
\left. \times \limsup_{n \to \infty} \min_{1 \leq j \leq N} d \left( \delta^n_x, \mu_j \right) \leq \epsilon \right\}, \tag{1.2}
\]

where \( d \) is the first Wasserstein metric\(^3\) on \( \mathcal{P}(X) \) (see [34, 35] for its properties; we here merely recall that \( d \) is a metrization of the weak topology of \( \mathcal{P}(X) \)). According to [22, proposition 1.2],

\(^3\)For \( j = 1, 2 \), let \( p_j : X \times X \to X \) be the canonical projection to the \( j \)th coordinate, and \( (p_j)_* \pi \) the pushforward measure of a probability measure \( \pi \) on \( X \times X \) by \( p_j \). Let \( \Pi(\mu, \nu) \) be the set of probability measures \( \pi \) on \( X \times X \) such that \( (p_1)_* \pi = \mu \) and \( (p_2)_* \pi = \nu \). The first Wasserstein metric \( d \) is defined as \( d(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y) d\pi(x, y) \) for \( \mu, \nu \in \mathcal{P}(X) \).
$x$ is irregular if and only if
$$\lim_{\epsilon \to 0} \mathcal{E}_x(\epsilon) = \infty.$$ When $\min d \left( \delta^x, \mu_j \right)$ in (1.2) is replaced by $\int_X \min d \left( \delta^x, \mu_j \right) m(dx)$ with some $m \in \mathcal{P}(X)$, then the quantity given by (1.2) is called the metric emergence with respect to $m$. A fundamental relationship between metric and pointwise emergences is the following inequality:
$$\min_{x \in D} \mathcal{E}_x(\epsilon) \leq \mathcal{E}_m(m(D) \epsilon) \quad (1.3)$$ for every $\epsilon > 0$, Borel set $D$ and $m \in \mathcal{P}(X)$, see [22, proposition 1.4]. The pointwise emergence at $x \in X$ is called super-polynomial (or high) if
$$\limsup_{\epsilon \to 0} \frac{\log \mathcal{E}_x(\epsilon)}{-\log \epsilon} = \infty.$$ As is pointed out in [5], it is widely accepted among computer scientists that super-polynomial algorithms are impractical. From that perspective dynamical systems with high metric emergence are not feasible to be studied numerically. The set of points with high emergence can be considered as statistically very complex (see also [6, 8] for other motivations to study high emergence). Therefore, it is of great interest to investigate how large the set of points with high emergence is.

The main result of this paper is as follows.

**Theorem 1.1.** Let $X$ be a topologically mixing subshift of finite type of $\{1, 2, \ldots, \kappa\}^\mathbb{N}$ with $\kappa \geq 2^4$ Let $f : X \to X$ be the left-shift operation on $X$. Let $E$ be the set of points $x \in X$ satisfying
$$\lim_{\epsilon \to 0} \frac{\log \mathcal{E}_x(\epsilon)}{-\log \epsilon} = \infty.$$ Then,
$$h_{\text{top}}(E) = h_{\text{top}}(X) \quad \text{and} \quad \dim_H(E) = \dim_H(X).$$ In addition, for any Hölder continuous function $\varphi$, we have that $P_E(\varphi) = P_X(\varphi)$, That is, the set of points with high emergence carries full topological pressure.

Notice that $\dim_H(A) = h_{\text{top}}(A)/\beta$ for any $A \subset X$ in the setting of theorem 1.1, so $h_{\text{top}}(E) = h_{\text{top}}(X)$ is equivalent to $\dim_H(E) = \dim_H(X)$.

Our approach to proving theorem 1.1 is a generalization of ideas in [17]. In [17] the authors introduced classes of sets, $G^s$, $0 < s < \dim_H(X)$, such that: (a) every countable intersection of sets in a given class $G^s$ also belongs to $G^s$; (b) every set in $G^s$ has Hausdorff dimension at least $s$. This was later used to show that a certain subset of the set of irregular points has full Hausdorff dimension. We extend this result in two directions. We consider a general Carathéodory dimension structure (see section 2) introduced by Pesin in [24]. We then introduce classes of sets corresponding to this structure such that the analogue of (a) and (b) holds, under some conditions on the Carathéodory structure (see section 3). We then consider the set of points with high emergence and analyse when it belongs to such defined class of sets. As a result we obtain a more general version of theorem 1.1, which states that under certain conditions on the dimension structure.

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\footnote{We endow it with a standard metric $d(x, y) = \sum_{j=1}^{\infty} \frac{|x_j - y_j|}{\beta^j}$ for $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in X$ with some $\beta > 1$.}
Carathéodory structure, the set of points with high emergence has full Carathéodory dimension (see theorem 3.8).

**Remark 1.2.** It is possible to prove the first part of theorem 1.1 (the statement about the Hausdorff dimension) by modifying the construction given by Barreira–Schmeling in [3]. The advantage of our approach is that it gives a stronger result (see theorem 3.8). Moreover we develop tools that could potentially be used to study largeness of other sets in addition to the set of points with high emergence (see theorem 3.2).

**Remark 1.3.** It immediately follows from (the statement for the Hausdorff dimension in) theorem 1.1 that \( m_t^H(E) > 0 \) for any \( t < \dim_H(X) \), where \( E \) is the set of points with high pointwise emergence given in theorem 1.1 and \( m_t^H \) is the \( t \)-dimensional Hausdorff measure (see sections 2 and 4 for precise definition). By Frostman’s lemma, this implies that there is a Borel probability measure \( m \) such that the support of \( m \) is included in \( E \) (in particular, \( m(E) > 0 \)) and \( m(B(x, r)) \leq r^t \) for all \( x \in X \) and \( r > 0 \), where \( B(x, r) \) is the ball of radius \( r \) centred at \( x \). Therefore, by (1.3), we have

\[
\lim_{\epsilon \to 0} \frac{\log \mathcal{E}_m(\epsilon)}{-\log \epsilon} = \infty.
\]

We remark that Berger conjectured in [5] that a ‘typical’ diffeomorphism \( f \) on a compact manifold satisfies the above equation with the normalized Lebesgue measure \( \text{Leb} \) of the manifold in place of \( m \), i.e. \( \lim_{\epsilon \to 0} \log \mathcal{E}_m(\epsilon)/(-\log \epsilon) = \infty \).

**1.1. Structure of the paper**

In section 2 we recall Carathéodory dimension structure and introduce modifications to Carathéodory outer measures needed for our constructions. We also introduce conditions on the Carathéodory dimension structure needed for the results in this paper. In section 3 we introduce and study classes of sets with large intersection property. We state the general version of our main result (theorem 3.8), which is an immediate consequence of two results for large intersection classes (theorems 3.2 and 3.4). In section 4 we give examples of Carathéodory dimension structures satisfying the assumptions of theorem 3.8. As a corollary, we obtain theorem 1.1. In section 5, we give the proof of theorem 3.2. Section 6 is dedicated to the proof of theorem 3.4.

**2. Carathéodory dimension structure**

We recall the construction introduced in [24], called the Carathéodory dimension structure.

**2.1. Carathéodory dimension of sets and measures**

Let \( X \) be a set and \( \mathcal{F} \) a collection of subsets of \( X \) which we call *admissible* sets. Assume that there exist two set functions \( \eta, \psi : \mathcal{F} \to [0, \infty) \) satisfying the following conditions:

(A1) \( \emptyset \in \mathcal{F} \); \( \eta(\emptyset) = \psi(\emptyset) = 0 \) and \( \eta(U), \psi(U) > 0 \) for any \( U \in \mathcal{F}, U \neq \emptyset \);

(A2) For any \( \delta > 0 \) one can find \( \epsilon > 0 \) such that \( \eta(U) \leq \delta \) for any \( U \in \mathcal{F} \) with \( \psi(U) \leq \epsilon \);

(A3) There exists a sequence of positive numbers \( \epsilon_n \to 0 \) such that for any \( n \in \mathbb{N} \) one can find a finite subcollection \( \mathcal{G} \subset \mathcal{F} \) covering \( X \) such that \( \psi(U) = \epsilon_n \) for any \( U \in \mathcal{G} \).

Let \( \xi : \mathcal{F} \to [0, \infty) \) be a set function. We say that the collection of subsets \( \mathcal{F} \) and the functions \( \xi, \eta, \psi \), satisfying conditions (A1), (A2) and (A3) introduce a Carathéodory dimension structure or C-structure \( \tau \) on \( X \) and we write \( \tau = (\mathcal{F}, \xi, \eta, \psi) \).
For any subcollection $G \subset F$ denote by $\psi(G) := \sup \{\psi(U) | U \in G \}$. Given a set $Z \subset X$ and numbers $t \in \mathbb{R}$ and $\varepsilon > 0$, define

$$M^t_c(Z, \varepsilon) := \inf_{G, \sigma(G) \leq \varepsilon} \left\{ \sum_{U \in G} \xi(U) \eta(U)^t \right\},$$

where the infimum is taken over all finite or countable subcollections $G \subset F$ covering $Z$. Set

$$m^t_c(Z) := \lim_{\varepsilon \to 0} M^t_c(Z, \varepsilon).$$

If $m^t_c(\emptyset) = 0$, then the set function $m^t_c(\cdot)$ becomes an outer measure on $X$, which induces a measure on the $\sigma$-algebra of measurable sets. We call this measure the $t$-Carathéodory outer measure. In general, this measure may not be $\sigma$-finite or it may be a zero measure. The following is shown in [24].

**Proposition 2.1.** For any set $Z \subset X$ there exists a critical value $t_c \in \mathbb{R}$ such that $m^t_c(Z) = \infty$ for $t < t_c$ and $m^t_c(Z) = 0$ for $t > t_c$ (while $m^t_c(Z)$ may be 0, $\infty$, or a finite positive number).

We call $\dim_cZ = t_c$ the Carathéodory dimension of the set $Z$. If $X$ is a measurable space with a measure $\mu$, then the quantity

$$\dim_c \mu = \inf \{\dim_cZ : \mu(Z) = 1\} = \lim_{\delta \to 0} \inf \{\dim_cZ : \mu(Z) > 1 - \delta\}$$

is called the Carathéodory dimension of $\mu$.

2.2. Modification of Carathéodory outer measures

For the rest of the paper we restrict our attention to $C$-structures $\tau = (F, \xi, \eta, \psi)$ on the shift space $X$. Recall that $X \subset \{1, \ldots, \kappa\}^\mathbb{N}$ is a subshift of finite type, meaning that there is a matrix $M = (M_{i,j})_{1 \leq i, j \leq \kappa}$ such that each entry of $M$ is 0 or 1 and that $X$ consists of admissible words $x = (x_1x_2 \ldots) \in \{1, \ldots, \kappa\}^\mathbb{N}$ with respect to $M$. Here, $x = (x_1 \ldots x_n) \in \{1, \ldots, \kappa\}^n$, $n \in \mathbb{N} \cup \{\infty\}$, is called admissible if $M_{i,j,j+1} = 1$ for all $1 \leq j < n$. We denote the length $n$ of the word $x$ by $|x|$. Furthermore, for a given admissible word $u = (u_1 \ldots u_n)$, we define the cylinder $C(u)$ by $C(u) = \{x \in X | [x]_n = u\}$, where $[x]_n$ is the truncation $[x]_n = (x_1x_2 \ldots x_n)$ of $x = (x_1x_2 \ldots)$. Let the collection $F$ of admissible sets be the collection of cylinders,

$$F := \{\emptyset\} \cup \{C(u) | u \text{ is an admissible word}\}.$$

For a cylinder $C \in F$ and $t \in \mathbb{R}$ denote

$$q(C, t) := \xi(C) \eta(C)^t,$$

and denote by $l(C)$ the smallest number $n$ such that $C = C(u)$ with some admissible word $u$ of length $n$ (also called the length of $C$).

For our purpose we suggest a modification of the Carathéodory outer measure similarly as it was done in [17] for Hausdorff outer measure. The advantage of the new outer measure is that it is always finite.

Assume that a Carathéodory dimension structure $\tau = (F, \xi, \eta, \psi)$ on $X$ is given. For any set $Z \subset X$ and a number $t \in \mathbb{R}$ define

$$M^t_c(Z) := \inf_{G} \left\{ \sum_{U \in G} \xi(U) \eta(U)^t \right\},$$
where the infimum is taken over all finite or countable sub-collections \( G \subset F \) covering \( Z \).

In section 6 we will consider yet another collection of outer measures, \( N_{c,m}(t) \), depending on the parameter \( m \in \mathbb{N} \), defined identically as \( M_{c}(t) \), but with the family of admissible sets restricted to

\[
\mathcal{F}_{m} := \{ \emptyset \} \cup \{ C(u) \mid u \text{ is an admissible word,} \}
\]

\[
|u| = k \cdot m \text{ for some } k \in \mathbb{N} \subset \mathcal{F}.
\]

Remark 2.2. The measures \( M_{c}^{t}, N_{c,m}^{t} \), and \( m_{c}^{t} \) are particular examples of measures considered in [27]. Following the terminology introduced in [27], the set function \( q(., t) \) satisfies the definition of the pre-measure [27, definition 5], the measures \( M_{c}^{t} \) and \( N_{c,m}^{t} \), are obtained from this pre-measure by method I [27, theorem 4], while \( m_{c}^{t} \) is obtained from \( q(., t) \) by method II [27, theorem 15].

The main properties of \( M_{c}^{t} \) and \( N_{c,m}^{t} \) are summarized below.

**Theorem 2.3.** The set functions \( M_{c}^{t} \) and \( N_{c,m}^{t} \) satisfy the following properties:

(a) For all \( t \in \mathbb{R} \):

1. If \( M_{c}(0) = N_{c,m}(0) = 0 \), then \( M_{c}^{t} \) and \( N_{c,m}^{t} \) define outer measures;
2. \( M_{c}^{t}(Z) \leq m_{c}^{t}(Z) \) and \( M_{c}^{t}(Z) \leq N_{c,m}^{t}(Z) \) for every set \( Z \subset X \);

(b) For \( t > \dim_{c} Z \):

3. \( M_{c}^{t}(Z) = m_{c}^{t}(Z) = 0 \);

(c) For \( t < \dim_{c} Z \):

4. \( 0 < M_{c}^{t}(Z) \leq N_{c,m}^{t}(Z) < \infty \);

(d) For \( t = \dim_{c} Z \):

5. If \( 0 < m_{c}^{t}(Z) \leq \infty \), then \( 0 < M_{c}^{t}(Z) \leq N_{c,m}^{t}(Z) < \infty \);
6. If \( m_{c}^{t}(Z) = 0 \), then \( M_{c}^{t}(Z) = 0 \).

Remark 2.4. Observe that in the case (b3) and (d6) one may have \( N_{c,m}^{t}(Z) > 0 \).

**Proof.** Statement (a1) follows from theorem 4 in [27]. Statement (a2) follows directly from the definition of \( M_{c}^{t} \) and \( N_{c,m}^{t} \). To show statement (b3) it is enough to observe that by proposition 2.1, \( m_{c}^{t}(Z) = 0 \) for all \( t > \dim_{c} Z \). Then by statement (a2) of the theorem, \( 0 \leq M_{c}^{t}(Z) \leq m_{c}^{t}(Z) = 0 \).

To prove statement (c4) first observe that the outer measures \( M_{c}^{t} \) and \( N_{c,m}^{t} \) are always finite. That is because the set function \( q(., t) \) is finite and for every set \( Z \subset X \) one can find a finite cover \( \{ C_{i} \}_{i=1}^{K} \) by cylinders of length \( m \). Then we have

\[
M_{c}^{t}(Z) \leq N_{c,m}^{t}(Z) \leq \sum_{i=1}^{K} q(C_{i}, t) < \infty.
\]

On the other hand, \( M_{c}^{t}(Z) = 0 \) implies \( m_{c}^{t}(Z) = 0 \). To see this, observe that there are finitely many cylinders of a given length \( s \). Since \( 0 < q(C, t) < \infty \) for any cylinder \( C \), there are positive real numbers \( \{ \gamma_{s} \}_{s \geq 1} \) such that \( q(C, t) \geq \gamma_{s} > 0 \) for any cylinder of length \( s \). Assume that
\[ M_c'(Z) = 0. \]
For any large \( L \in \mathbb{N} \) choose \( \epsilon > 0 \) such that \( \epsilon < \min\{ \frac{1}{L}, \gamma_1, \ldots, \gamma_L \} \). There exists a cover \( \{ C_i \}_{i \geq 1} \) of \( Z \) by cylinders such that
\[
\sum_i q(C_i, t) < \epsilon.
\]

By the choice of \( \epsilon \), we must have that \( l(C_i) > L \) for each \( i \geq 1 \). Letting \( L \to \infty \) we obtain that \( m_c'(Z) = 0 \).

By proposition 2.1, \( m_c'(Z) = \infty \) for all \( t < \dim_c Z \). By (the contraposition of) the above argument, this implies that \( M_c'(Z) > 0 \). By statement (a2), this completes the proof of (c4).

The proof of (d5) is identical to the proof of (c4). Statement (d6) is a direct consequence of (a2). □

For simplicity of notation, if there is no confusion, we will simply write \( \mathcal{M}, m', M' \) and \( N'_m \).

Our arguments require the \( C \)-structure to satisfy the following additional conditions. We remark that all the conditions stated below are naturally satisfied by \( C \)-structures corresponding to topological entropy, Hausdorff dimension, and topological pressure of Hölder continuous potentials, which we prove in section 4.

(C1) There exists a uniform constant \( Q_1 > 0 \) such that for every \( t < \dim_c(X) \) and for every cylinder \( C \) there is a cylinder \( \tilde{C}' \) such that \( C \subseteq \tilde{C}' \) and one has:
\[
Q_1 q(C', t) \leq M'(C) \leq \min\{ q(C, t), q(\tilde{C}', t) \};
\]

(C2) For every \( t < \dim_c(X) \) there exists \( m = m(t) \in \mathbb{N} \) such that for every cylinder \( C \) whose length is a multiple of \( m \) there is a cylinder \( \tilde{C}'_m \) whose length is a multiple of \( m \) such that \( C \subseteq \tilde{C}'_m \), and one has:
\[
N'_m(C) = q(\tilde{C}'_m, t);
\]

(C3) There exists a uniform constant \( Q_3 > 1 \) such that for any two words \( u, v \), such that \( uv \) is an admissible word, and any \( t \in \mathbb{R} \) one has
\[
Q_3^{-1} q(C(u), t) q(C(v), t) \leq q(C(uv), t) \leq Q_3 q(C(u), t) q(C(v), t);
\]

(C4) For any two cylinders such that \( A \subseteq B \) one has \( \eta(A) \leq \eta(B) \).

3. Large intersection classes

As previously mentioned, theorem 1.1 follows from a stronger result that the set \( E \) in theorem 1.1 belongs to a certain class of sets, which are in some sense large.

We consider the following classes of sets, which are defined by generalizing classes introduced by Färm and Persson in [17] as modifications of Falconer’s intersection classes from [16].

**Definition 3.1.** Let \( \mathcal{G}'(X), t < \dim_c(X) \) be the class of \( G'_t \)-sets \( F \subset X \) such that
\[
M'(F \cap C) = M'(C)
\]
holds for all cylinders \( C \).
Our main results on these classes are the following theorems. The first theorem is a generalization of [17, theorem 1].

**Theorem 3.2.** Assume that the Carathéodory structure satisfies conditions (C1) and (C4). Then the classes \( G^t(X) \) are closed under countable intersections and the Carathéodory dimension of any set in one of these classes is at least \( t \).

As an important application of large intersection property, Färm and Persson calculated the Hausdorff dimension of the intersection of irregular sets over countably many different dynamical systems, see [18, proposition 1].

We prove theorem 3.2 in section 5. In the lemma below we observe that the claim about the Carathéodory dimension of sets in \( G^t(X) \) is a natural consequence of definition 3.1.

**Lemma 3.3.** If \( F \in G^t(X) \), then \( \dim_C(F) \geq t \).

**Proof.** Since \( t < \dim_C(X) \), by statement (c) in theorem 2.3, \( M^t(\mathcal{A}) > 0 \). Then there is a cylinder \( C \) such that \( M^t(C) > 0 \). By definition 3.1, \( M^t(F) > 0 \). By statement (b) in theorem 2.3, \( \dim_C(F) \geq t \). \( \square \)

Let \( A_x \) be the set of accumulation points of \( \{\delta^n_x\}_{n\geq 1} \). For a sequence \( \mathcal{J} = \{\mu(\ell)\}_{\ell \in \mathbb{N}} \) of probability measures on \( X \), we denote by \( \Delta(\mathcal{J}) \) the set of finite convex combinations of measures in \( \mathcal{J} \). Namely,

\[
\Delta(\mathcal{J}) = \bigcup_{L \geq 1} \Delta_L(\mathcal{J}), \quad \Delta_L(\mathcal{J}) = \{ \mu(\ell) \in A_L \},
\]

where \( A_L = \{ (t_0, t_1, \ldots, t_L) \in [0, 1]^{L+1} | \sum_{\ell=0}^L t_\ell = 1 \} \) and \( \mu(\ell) = \sum_{\ell=0}^L t_\ell \mu(\ell) \) for \( \ell = (t_0, t_1, \ldots, t_L) \in A_L \). We define the saturated set \( E(\mathcal{J}) \) of \( \mathcal{J} \) by

\[
E(\mathcal{J}) = \{ x \in X | \Delta(\mathcal{J}) \subset A_x \}
\]

(cf [15, 26]). For a probability measure \( \mu \) on \( X \), define the generic set \( G(\mu) \) by

\[
G(\mu) = \{ x \in X | \lim_{n \to \infty} \delta^n_x = \mu \}.
\]

The following theorem is a generalization of [17, theorem 2] and is the central result of this paper.

**Theorem 3.4.** Assume that the Carathéodory structure satisfies conditions (C1)–(C4). The following statements hold for any sequence \( \mathcal{J} = \{\mu(\ell)\}_{\ell \geq 0} \) of ergodic invariant probability measures:

(a) \( E(\mathcal{J}) \in G^t \) for all \( t < \inf \{ \dim_C(\mu(\ell)) | \ell \geq 0 \} \);

(b) \( \dim_C(E(\mathcal{J})) \geq \inf \{ \dim_C(\mu(\ell)) | \ell \geq 0 \} \).

The proof of theorem 3.4 occupies all of section 6.

**Remark 3.5.** It is worth pointing out that theorem 3.2 holds for any shift space (not necessarily a subshift of finite type) and any Carathéodory structure satisfying conditions (C1) and (C4). In the proof of theorem 3.4 (lemmas 6.3, 6.5 and 6.6) we use the fact that the underlying shift space is a subshift of finite type. By slightly modifying the arguments one should be able to extend this result to any shift with specification.

**Remark 3.6.** We remark that (b) of theorem 3.4 gives another proof of (a special version of) theorem 1.2 of Chen–Zhou [15] for the topological pressure of saturated sets, although their
method is different from ours. In particular, it is unclear whether the lower bound of (b) for the Hausdorff dimension and the large intersection property in (a) follow from their approach.

To conclude theorem 1.1, we also need the following proposition, which we prove on the next page.

**Proposition 3.7.** Let \( J = \{ \mu^{(l)} \}_{l \geq 0} \) be a linearly independent sequence of invariant probability measures on \( X \). Then, \( x \in E(J) \) implies that
\[
\lim_{\epsilon \to 0} \frac{\log \mathcal{E}_x(\epsilon)}{\log \epsilon} = \infty.
\]

The next theorem summarizes results in this section and gives the main result in this paper.

**Theorem 3.8.** Let \( X \) be a topologically mixing subshift of finite type of \( \{ 1, 2, \ldots, \kappa \}^\mathbb{N} \) with \( \kappa \geq 2 \). Let \( f : X \to X \) be the left-shift operation on \( X \). Let \( E \) be the set of points \( x \in X \) satisfying
\[
\lim_{\epsilon \to 0} \frac{\log \mathcal{E}_x(\epsilon)}{\log \epsilon} = \infty.
\]

Assume that the Carathéodory structure satisfies conditions (C1)–(C4) and the following condition:

(C5) For any \( t < \dim_c(X) \), there is a linearly independent sequence of invariant probability measures \( \{ \mu^{(l)} \}_{l \geq 0} \) on \( X \) such that \( \dim_c(\mu^{(l)}) > t \) for all \( l \geq 0 \).

Then, for any \( t < \dim_c(X) \), there is an \( E_t \subset E \) such that \( E_t \in G_t \). In particular,
\[
\dim_c(E) = \dim_c(X).
\]

**Proof.** Theorem 3.8 immediately follows from theorems 3.2, 3.4 and proposition 3.7.

**Proof of theorem 1.1.** We will see in section 4 that \( C \)-structures corresponding topological entropy, Hausdorff dimension, and topological pressure satisfy Conditions (C1)–(C4). Hence, theorem 1.1 immediately follows from theorem 3.8, if we prove (C5). Let \( \psi(x) : X \to \mathbb{R} \) be any Hölder continuous function not cohomologous to a constant. Observe that then for any two distinct values \( t_1, t_2 \), the corresponding potentials \( \varphi + t_1 \psi, \varphi + t_2 \psi \) are not cohomologous and by proposition 20.3.10 in [21] have distinct unique ergodic equilibrium measures \( \mu_{t_1} \) and \( \mu_{t_2} \) respectively. In addition, by [24, theorem 11.6], we have for any \( t \in \mathbb{R} \)
\[
|\dim_c(X) - \dim_c(\mu_t)| = |P(\varphi) - P_{\mu_t}(\varphi)| = |P(\varphi) - \int \varphi \, d\mu_t - h_{\mu_t}(f)|
\]
\[
\leq |P(\varphi) - \int \varphi + t \psi \, d\mu_t - h_{\mu_t}(f)| + |t||\psi|| = |P(\varphi - t\psi)| + |t||\psi|| \leq 2|t||\psi||,
\]
where in the last inequality we used continuity of the pressure [24, theorem 11.4] and \( \| \cdot \| \) denotes the supremum norm in the space of continuous functions.

**Proof of proposition 3.7.** For a subset \( A \) of \( \mathcal{P}(X) \), let \( N(\epsilon, A) \) denote the \( \epsilon \)-covering number of \( A \) with respect to the first Wasserstein metric \( d \). Then, it is straightforward to see that \( \mathcal{E}_x(\epsilon) = N(\epsilon, A_x) \), and recall that \( \Delta(J) \subset A_x \) for each \( x \in E(J) \). We will show that
\[
\lim_{\epsilon \to 0} \frac{\log N(\epsilon, \Delta(J))}{\log \epsilon} = \infty, \quad \text{(3.1)}
\]
which implies the conclusion by the above observations.

Let \( L \geq 1 \) be an integer. Since \( \mu^{(0)}, \ldots, \mu^{(L)} \) are linearly independent, \( \Delta_J(L) \) is an \( L \)-dimensional simplex. Therefore, it is easy to see that its box-counting dimension

\[
\lim_{\epsilon \to 0} \frac{\log N(\epsilon, \Delta_J(L))}{-\log \epsilon}
\]

is well-defined and equal to \( L \). It follows from this and the observation \( \Delta_J(L) \subset \Delta_J \) that

\[
\lim \inf_{\epsilon \to 0} \frac{\log N(\epsilon, \Delta_J)}{-\log \epsilon} \geq L.
\]

Since \( L \) is arbitrary, this implies (3.1) and completes the proof. \( \square \)

4. Applications

4.1. C-structure corresponding to topological pressure

In this section we fix a Hölder continuous function \( \varphi : X \to \mathbb{R} \) and consider the following C-structure \( \tau = (\mathcal{F}, \xi, \eta, \psi) \) on \( X \).

Given a cylinder \( C \), we define

\[
\xi(C) := \exp \left( S_{h(C)} \varphi(C) \right) := \exp \left( \sup_{x \in C} \sum_{k=0}^{h(C)-1} \varphi(f^k(x)) \right),
\]

\[
\eta(C) := e^{-h(C)}, \quad \psi(C) := \frac{1}{l(C)},
\]

and also set \( \eta(\emptyset) = \psi(\emptyset) = \xi(\emptyset) = 0 \). It is easy to see that the collection of subsets \( \mathcal{F} \) and the functions \( \xi, \eta, \psi \) satisfy conditions (A1), (A2) and (A3), and hence, introduce the Carathéodory dimension structure on \( X \).

The corresponding outer measures are given by

\[
\mathcal{M}_C^\prime(Z, 1/n) := \inf \left\{ \sum_i e^{S_{h(C_i)} \varphi(C_i) - h(C_i)l(C_i)} | l(C_i) \geq n, Z \subset \bigcup_i C_i \right\},
\]

\[
m_C^\prime(Z) := \lim_{n \to \infty} \inf \left\{ \sum_i e^{S_{h(C_i)} \varphi(C_i) - h(C_i)l(C_i)} | l(C_i) \geq n, Z \subset \bigcup_i C_i \right\}, \text{ and}
\]

\[
M_C^\prime(Z) := \inf \left\{ \sum_i e^{S_{h(C_i)} \varphi(C_i) - h(C_i)l(C_i)} | Z \subset \bigcup_i C_i \right\}.
\]

The corresponding Carathéodory dimension of a set \( Z \) is exactly the topological pressure \( P(\varphi, Z) \) on \( Z \) as defined by Pesin–Pitskel’ in [25]. We will denote \( P = P(\varphi) := P(\varphi, X) = \dim_c(X) \). We also observe that topological entropy corresponds to \( P(\varphi) \) for \( \varphi = 0 \).

By the lemma below, all the results in this paper apply to this structure.

**Lemma 4.1.** The above Carathéodory structure satisfies conditions (C1)–(C4).
Remark 4.2. In the proof of lemma 4.1 (more precisely in verifying conditions (C1), (C2), and (C3)) we used both the fact that X is a subshift of finite type and that \( \varphi \) is Hölder continuous. Our arguments can also be extended to any shift with specification and a function \( \varphi \) satisfying Bowen property (see [21, definition 20.2.5]). Deducing theorem 1.1 in this case would also require verifying condition (C5) in theorem 3.8.

Proof. First observe that \( M'(C) \leq q(B, t) \) for any cylinders \( C \) and \( B \) with \( C \subseteq B \). This follows from the definition of \( M' \). We now show that \( M'(C) \geq Qq(C', t) \) for some cylinder \( C' \) containing \( C \) with some positive number \( Q \) being independent of \( C \) and \( t \). Let \( \{ C_i \} \) be a cover of \( C \) by cylinders of length greater than \( l(C) \). By Hölder continuity of \( \varphi \), there exists \( Q_0 > 0 \) such that

\[
\sum_i e^{s(C_i) \varphi(C_i)} \geq Q_0 e^{s(C) \varphi(C)} \sum_i e^{s(C_{i-1} - l(C) \varphi(C_i))}.
\]

If \( C = C(u) \) for some admissible word \( u = u_1 \ldots u_t \) and \( M_{l_n} = 1 \) for some \( n \in \{ 1, \ldots, \kappa \} \), then \( f^{\kappa(C)}(C) \) covers \( C(a) \). In particular, the sets \( f^{\kappa(C)}(C) \) form a cover of \( C(a) \). Define \( M_{\min} := \min_{n \in \{ 1, \ldots, \kappa \}} M'(C(a)) > 0 \). We have that

\[
\sum_i q(C_i, t) \geq Q_0 e^{s(C) \varphi(C)} e^{-l(C)T} \sum_i q(f^{\kappa(C)}(C_i), t) \geq Q_0 q(C, t) M_{\min}.
\]

This proves (C1).

For (C2) let \( \{ C_i \} \) be a cover of a cylinder \( C \) such that \( l(C_i) \geq l(C) + m \) for some large \( m \), to be determined later. Let \( \{ A_j \} \) be a cover of \( C \) by cylinders of length \( l(C) + m \). Using similar reasoning as above, we have

\[
\sum_i q(C_i, t) \geq Q_0^2 e^{s(C) \varphi(C)} e^{-l(C)T} \sum_j e^{s(C_{i-1} \varphi(C_{i-1}))} e^{-l(C)T} \\
\times \sum_{C_i \subseteq A_j} e^{s(C_{i-1} - l(C) \varphi(C_i))} e^{-l(C) - l(C)T - mT}
\]

\[
\geq Q_0^2 q(C, t) M_{\min} \sum_j e^{s(C_{i-1} \varphi(C_{i-1}))} e^{-mT}.
\]

As a consequence of uniform counting estimates for the partition sums (see for example [21, proposition 20.3.2]), there is \( Q_1 > 0 \) such that

\[
\sum_j e^{s(C_{i-1} \varphi(C_{i-1}))} \geq Q_1 e^{nP}.
\]

Together we obtain,

\[
\sum_i q(C_i, t) \geq Q_0^2 q(C, t) M_{\min} Q_1 e^{m(P - t)}.
\]

Choosing \( m \) large enough so that \( Q_0^2 M'(X) Q_1 e^{m(P - t)} > 1 \) completes the proof of (C2).

Condition (C3) is a direct consequence of Hölder continuity of \( \varphi \), and condition (C4) follows directly from the definition of \( \eta \). □
4.2. C-structure corresponding to Hausdorff dimension

Results in this paper can be used in particular to study Hausdorff dimension of irregular sets. For this it is enough to set $\xi(C) = 1$ for every cylinder $C$ and define $\eta(C)$ as the diameter of $C$ (and $\psi(C) = 1/\ell(C)$ as in the previous example). It is automatic that such defined C-structure satisfies conditions (C1)–(C4).

5. Proof of theorem 3.2

**Lemma 5.1.** If $0 < c \leq 1$ and $F$ is a set such that
\[ M'(F \cap C) \geq c M'(C) \]
holds for all cylinders $C$, then
\[ M'(F \cap U) \geq c M'(U) \]
holds for open sets $U$.

**Proof.** The proof is identical to the proof of lemma 1 in [17] if we replace $M'_\infty$ with $M'$ and $d(\cdot, \cdot)$ with $q(C, t)$. We provide the argument for clarity.

Let $U \subset X$ be open. Then we can write $U$ as a countable union $U = \bigcup_i C_i$ of pairwise disjoint cylinders. Let $\{D_j\} \subset F$ be a cover of $F \cap U$. We can assume that this cover is disjoint.

Given $C_i$, if there are $D_j \subset C_i$, we may write
\[ \sum_{D_j \subset C_i} q(D_j, t) \geq M'(F \cap C_i) \geq c M'(C_i). \]

Here we used the fact that two cylinders are either disjoint or one of them is contained in the other. Hence if $D_j \subset C_i$ for some $C_i$, then all such sets $D_j$ form a cover of $C_i \cap F$.

We can construct a disjoint cover $\{\tilde{C}_k\}$ of $U$ by replacing each collection $C_i$ contained in some $D_j$ with the cylinder $D_j$. We then obtain
\[ \sum_j q(D_j, t) \geq c \sum_k M'(\tilde{C}_k) \geq c M'(U). \]

Taking the infimum over all covers $\{D_j\}$ finishes the proof. \[\square\]

**Lemma 5.2.** If $c > 0$ and $F$ is a set such that
\[ M^0(F \cap C) \geq c M^0(C) \]
holds for some $t_0 < \dim_q(X)$ and all cylinders $C$, then
\[ M'(F \cap C) \geq M'(C) \]
holds for all cylinders $C$, and $t \leq t_0$.

**Proof.** The proof that $M^0(F \cap C) \geq M^0(C)$ is identical to the proof of lemma 2 in [17] if we replace $M'_\infty$ with $M'$ and $d(\cdot, \cdot)$ with $q(C, t)$. It uses condition (C1). Hence this part of the lemma holds for every Carathéodory structure satisfying (C1). Here we provide the argument for clarity.
Let \( \{C_i\} \) be a collection of cylinders covering \( F \cap C \). We may assume that the cylinders are pairwise disjoint. Since \( M'(F \cap C) \) is finite, we may assume that \( \sum_i q(C_i, t_0) \) is finite. Therefore, for every \( \epsilon > 0 \), there exists \( m_0 \) such that
\[
\sum_{\{i:|C_i| \geq m_0\}} q(C_i, t_0) < \epsilon. \tag{5.1}
\]

Build a cover \( \{D_j\} \) of \( C \) by cylinders of length \( l(D_j) \leq m_0 \) in the following way. Either \( D_j = C_i \) for some \( C_i \) with \( l(C_i) < m_0 \) or \( C_i \cap F \subset D_j \cap F \) where \( l(C_i) \geq m_0 \) and \( D_j \cap F \subset \bigcup_{\{i:|C_i| \geq F \cap D_j \cap F\}} C_i \cap F \). Observe that by the assumption of the lemma for every cylinder \( D \subset C \) one has that \( F \cap D \neq \emptyset \) so such cover exists. In addition, in the latter case we can estimate
\[
\sum_{\{i:|C_i| \geq m_0\}} q(C_i, t_0) \geq M^0(F \cap D_j) \geq cM^0(D_j) \geq cQ_1 q(D_j^0, t_0)
\]
by (C1). It follows that
\[
\sum_{\{i:|C_i| < m_0\}} q(C_i, t_0) + c^{-1} Q_1^{-1} \sum_{\{i:|C_i| \geq m_0\}} q(C_i, t_0) \geq \sum_{j} q(D_j^0, t_0).
\]

Using this and (5.1) we obtain
\[
\sum_i q(C_i, t_0) = \sum_{\{i:|C_i| < m_0\}} q(C_i, t_0) + c^{-1} Q_1^{-1} \sum_{\{i:|C_i| \geq m_0\}} q(C_i, t_0)
+ (1 - c^{-1} Q_1^{-1}) \sum_{\{i:|C_i| \geq m_0\}} q(C_i, t_0) \geq M^0(C) + (1 - c^{-1} Q_1^{-1}) \epsilon.
\]

Letting \( \epsilon \to 0 \) and taking infimum over all covers \( \{C_i\} \) we conclude
\[M^0(F \cap C) \geq M^0(C).\]

We now turn to the case \( t < t_0 \). Let \( \{C_i\} \) be a collection of cylinders covering \( F \cap C \). We may assume that each \( C_i \) is contained in \( C \) so that \( \eta(C_i) \leq \eta(C) \) by condition (C4). We then have
\[
\sum_i q(C_i, t) = \sum_i \xi(C_i) \eta(C_i)^t = \sum_i \xi(C_i) \eta(C_i)^{t-t_0}
\geq \eta(C)^{t-t_0} \sum_i q(C_i, t_0) \geq \eta(C)^{t-t_0} M^0(C \cap F) \geq \eta(C)^{t-t_0} M^0(C)
\geq Q_1 \eta(C)^{t-t_0} q(\mathcal{C}^0, t_0)
\]
by (C1). Using (C4), we continue
\[
\geq Q_1 \eta(\mathcal{C}^0)^{t-t_0} \xi(\mathcal{C}^0) \eta(\mathcal{C}^0)^{t_0} = Q_1 q(\mathcal{C}^0, t) \geq Q_1 M'(C),
\]
where the last inequality follows from the definition of \( M' \) after observing that \( \mathcal{C}^0 \) covers \( C \).

Taking infimum over all covers \( \{C_i\} \) we conclude
\[M'(F \cap C) \geq Q_1 M'(C)\]
and by the first part of the lemma, \( M'(F \cap C) \geq M'(C) \). \( \square \)
The proof of theorem 3.2 is now a consequence of lemma 3.3 and the following lemma.

**Lemma 5.3.** If $F_i \in \mathcal{G}(X)$ for all $i \in \mathbb{N}$, then

$$M'(\bigcap_i F_i \cap U) = M'(U)$$

for all open sets $U$, and $\bigcap_i F_i \in \mathcal{G}(X)$.

**Proof.** The proof is very similar to the proof of proposition 2 in [17], which uses the increasing sets lemma for the outer measures [27, theorem 52].

Let first $\{F_i\}_{i=1}^{\infty}$ be a countable collection of open sets, with the property that $M'(F_i \cap U) \geq M'(U)$ holds for any open set $U$. Fix an open set $U \subset X$, $\epsilon > 0$, and a sequence $\{\epsilon_k\}_{k=0}^{\infty}$ of positive real numbers such that $\sum_{k=0}^{\infty} \epsilon_k < \epsilon$.

The main idea is to approximate $\bigcap_i F_i \cap U$ by a countable intersection of compact nested sets $\bigcap_{k=1}^{\infty} D_k$ such that $D_k \subset \bigcap_{i=1}^{k} F_i \cap U$ for $k \geq 1$.

We define the collection $\{D_i\}_{i=0}^{\infty}$ inductively in the following way. Let $D_0$ be an open subset of $U$ with the property that $D_0 \subset U$ and that

$$M'(D_0) > M'(U) - \epsilon_0.$$  

The fact that such a set exists follows from the increasing sets lemma [27, theorem 52]. To see that this result applies to our setting it is enough to observe that the set function $q(\cdot, t)$ satisfies the definition of the pre-measure [27, definition 5] and that $M'(\cdot)$ is constructed from this pre-measure by method I defined in [27, theorem 4].

Having defined $D_i$ for $i = 0, \ldots, k - 1$, we choose $D_k$ to be an open subset of $F_k \cap D_{k-1}$ (note that $F_k \cap D_{k-1} \subset \bigcap_{i=1}^{k} F_i \cap U$ is an open set) with the property that $D_k \subset F_k \cap D_{k-1}$ and such that

$$M'(D_k) > M'(F_k \cap D_{k-1}) - \epsilon_k.$$  

Observe that since each $D_k$ is an open set, we have that

$$M'(D_k) > M'(F_k \cap D_{k-1}) - \epsilon_k \geq M'(D_k) - \epsilon_k$$

$$> M'(F_{k-1} \cap D_{k-2}) - \epsilon_{k-1} - \epsilon_k > \cdots > M'(U) - \epsilon.$$  

In addition, $\bigcap_{i=0}^{\infty} D_i \subset \bigcap_{i=1}^{\infty} F_i \cap U$. Let $\{C_l\}$ be a cover of $\bigcap_{i=0}^{\infty} D_i$ by cylinders. Since $\{D_k\}$ is a nested sequence of compact sets and $\bigcup_l C_l$ is open, there is $m \in \mathbb{N}$ such that $D_m \subset \bigcup_l C_l$. Therefore

$$\sum_l q(C_l, t) \geq M'(D_m) \geq M'(U) - \epsilon.$$  

Letting $\epsilon \rightarrow 0$ we obtain

$$M'\left(\bigcap_{i=1}^{\infty} F_i \cap U\right) \geq M'(U).$$
Therefore we have shown that any countable intersection of open sets in $G(t) \subset G'(t)$. The proof is finished by observing that any countable intersection of $G_j$ sets can be expressed as a countable intersection of open sets.

\section{Proof of theorem 3.4}

In this section we fix an invariant ergodic probability measure $\mu$, $t < \dim_r (G(\mu))$, and $m(t)$, where $m(t)$ is defined in condition (C2). We then consider the corresponding outer measure $N'_m$, which is defined in subsection 2.2.

We say that a word $x$ is a subword of $y$, and write $x \preceq y$, if $C(y) \subset C(x)$. In addition, we say that $x$ is a proper subword of $y$, and write $x < y$, if the inclusion is strict. Furthermore, we denote the concatenation of words $x$ and $y$ by $xy$.

For $\mu \in \mathcal{P}(X)$, $n \in \mathbb{N}$ and $\epsilon > 0$, we define $E(\mu, n, \epsilon)$ by

\[ E(\mu, n, \epsilon) = \{ x \in X | \delta^n x \in B_{\epsilon}(\mu) \} , \]

where $B_{\epsilon}(\mu)$ is the ball of radius $\epsilon$ and with centre $\mu$.

\subsection{Preliminary lemmas}

We start this subsection with a simple but crucial observation. Namely, we note that there are finitely many words of length not exceeding $m$. Because of that there exists a constant $\alpha = \alpha(m, t) > 0$ such that $\alpha < q(C(u), t) < \alpha^{-1}$ for any word $u$ with $|u| \leq m$.

As a consequence of (6.1) and condition (C3) we obtain the following.

\begin{lemma}
\[ \text{The outer measures } M' and N'_m are equivalent. } \]
\end{lemma}

\begin{proof}
Clearly $M' \leq N'_m$. On the other hand, for any collection $\{ x' \}$ of words of lengths $l_i = k_i m + n_i$, with $k_i \in \mathbb{N}$ and $0 \leq n_i \leq m$, consider the corresponding collection $\{ x' \}$ of words of lengths $k_i m$ obtained by removing the last $n_i$ letters from the word $x'$. Then

\[ \sum_i q(C(x'), t) \leq Q \sum_i \frac{q(C(x'), t)}{q(C(x_{k_i + 1} \cdots x_{k_i + n_i}), t)} \leq \alpha^{-1} Q \sum_i q(C(x'), t). \]

In addition, $C(x') \subset C(x')$. Consequently, $N'_m \leq \alpha^{-1} Q; M'$.
\end{proof}

The next two lemmas correspond to lemmas 4 and 6 of [17] and will be used as inductive steps in the following subsection.

\begin{lemma}
Assume there are numbers $c, \epsilon > 0$ and a word $y$ of length $m$ satisfying

\[ N'_m (C(y) \cap E(\mu, N, \epsilon)) < c q(C(y), t) \]

for some $N > 4m/\epsilon$. Then for every word $z$ satisfying $|z| = km \leq cN/4$, and such that $zy$ is an admissible word, one has

\[ N'_m (C(zy) \cap E(\mu, N + |z|, \epsilon/2)) < c Q^2 q(C(zy), t). \]

\end{lemma}
Before we state the next lemma, we recall an important property of subshifts of finite type.

There exists \( \tau \in \mathbb{N} \) such that for any two admissible words \( u \) and \( v \) (of any length) there exists a word \( \omega \) of length \( \tau m \) such that the word \( uwv \) is admissible. For the rest of this paper we denote by \( \tau \) the positive integer with this property.

**Lemma 6.3.** Assume there are numbers \( c, \epsilon > 0 \) and a word \( y \) of length \( m \) satisfying
\[
N'_m \left( C(y) \cap E(\mu, N, \epsilon) \right) > c q(C(y), t),
\]
for some \( N > 2m/\epsilon \). Then for every word \( z \) satisfying \( |z| = km, (k + \tau)m \leq \epsilon N/2 \), one has
\[
N'_m \left( C(z) \cap E(\mu, N + \tau m + |z|, 2\epsilon) \right) > Q_3^{-(\tau + 1)} \alpha^{\tau + 1} c N'_m (C(z)),
\]
where \( \alpha \) is as in (6.1).

In order to show lemmas 6.2 and 6.3, we will use the following.

**Lemma 6.4.** If \( z \) is a word of length \( n_1 \) and \( x \in X \) satisfies \( zx \in E(\mu, n + n_1, \epsilon) \), then
\[
x \in E \left( \mu, n, \epsilon + \frac{2n_1}{n} \right).
\]
Conversely, if \( z \) is a word of length \( n_1 \), \( x \in E(\mu, n, \epsilon) \), and \( zx \) is an admissible word, then
\[
 zx \in E \left( \mu, n + n_1, \epsilon + \frac{2n_1}{n} \right).
\]

**Proof.** For each \( \varphi \in \text{Lip}^1(\mathbb{R}, [-1, 1]) \),
\[
\left| \int_X \varphi \, d\delta^n_x - \int_X \varphi \, d\delta^{n+n_1}_{zx} \right| \\
\leq \left| \left( \frac{1}{n} - \frac{1}{n + n_1} \right) \sum_{j=0}^{n+n_1-1} \varphi \circ f^j(zx) \right| + \left| \sum_{j=0}^{n-1} \varphi \circ f^j(zx) \right| \\
\leq \frac{2n_1}{n}.
\]
So, we have \( d(\delta^n_x, \delta^{n+n_1}_{zx}) \leq 2n_1/n \) by the Kantorovich–Rubinstein dual representation of the first Wasserstein metric (cf [22, (2.1)]), which immediately implies the conclusion. \( \square \)

**Proof of lemma 6.2.** By assumption, there are words \( (x_i)_{i=1}^I \) of lengths being multiples of \( m \) and such that
\[
C(y) \cap E(\mu, N, \epsilon) \subset \bigcup_{i=1}^I C(x_i) \tag{6.2}
\]
and
\[
\sum_{i=1}^I q(C(x_i), t) < cq(C(y), t).
\]
From (6.2) and lemma 6.4, it follows that
\[
C(zy) \cap E(\mu, N + |z|, \epsilon - 2|z|/N) \subset \bigcup_{i=1}^I C(zy).
\]
On the other hand, by condition (C3), we have
\[ q(C(zx^i), t) \leq Q_3 q(C(z), t) q(C(x^i), t) \leq Q_3 t \frac{q(C(z), t)}{q(C(y), t)} q(C(x^i), t). \]
Observing that the length of each word \( zx^i \) is a multiple of \( m \), we can conclude
\[ N_m^u \left( C(zy) \cap E(\mu, n + |z|, \epsilon - 2|z|/N) \right) \leq \sum_{i=1}^{l} q(C(zx^i), t) = Q_3 t \frac{q(C(z), t)}{q(C(y), t)} \sum_{i=1}^{l} q(C(x^i), t) < Q_3 t c q(C(z)). \]
Recall that by the assumption, \( |z| \leq \epsilon N/4 \), so that \( \epsilon - 2|z|/N \geq \epsilon/2 \). The conclusion follows. \( \square \)

**Proof of lemma 6.3.** Let \( u \) be a word of length \( \tau m \) such that the word \( zuy \) is admissible. We consider all possible covers of the set \( C(zuy) \cap E(\mu, N + \tau m + |z|, 2\epsilon) \) by cylinders of lengths of multiples of \( m \). There are three possibilities:

(a) \( C(zuy) \cap E(\mu, N + \tau m + |z|, 2\epsilon) \subseteq C(w) \), where:
   (1) \( w \preceq z \);
   (2) \( z < w \preceq zu \), that is, \( w = z\bar{u} \) with \( \bar{u} \preceq u \);

(b) There is a collection \( (\alpha_i)_{i=1}^{l} \) of words of lengths being multiples of \( m \) and such that \( C(zuy) \cap E(\mu, N + \tau m + |z|, 2\epsilon) \subseteq \bigcup_{i=1}^{l} C(\bar{w} x^i) \).

In case (a1) we necessarily have that \( C(z) \subseteq C(w) \). Consequently,
\[ q(C(w), t) \geq N_m^u(C(z)). \tag{6.3} \]

We now turn to case (a2). Dividing \( \bar{u} \) into subwords of lengths \( m \) and applying (6.1) to each segment, by condition (C3), we have that
\[ q(C(\bar{w}), t) \geq Q_3^{\tau - 1} \alpha^\tau. \]

Consequently, by condition (C3),
\[ q(C(w), t) = q(C(\bar{w}), t) \geq Q_3^{\tau - 1} q(C(z), t) q(C(\bar{w}), t) \geq Q_3^{\tau - 1} \alpha^\tau N_m^u(C(z)). \tag{6.4} \]

We now consider case (b). Let \( (\alpha_i)_{i=1}^{l} \) be a collection of words of lengths being multiples of \( m \) and such that
\[ C(zuy) \cap E(\mu, N + \tau m + |z|, 2\epsilon) \subseteq \bigcup_{i=1}^{l} C(\bar{w} x^i). \]

Observe that
\[ C(y) \cap E(\mu, N, \epsilon) \subseteq \bigcup_{i=1}^{l} C(y x^i). \]
By condition (C3), it follows that,
\[
\sum_q q(C(uyx^t), t) \geq Q^{-2}_3 q(C(z), t) q(C(u), t) \sum_q q(C(y^t), t)
\]
\[
= Q^{-2}_3 q(C(z), t) q(C(u), t) q(C(y), t) \sum_i q(C(y^t), t) q(C(y), t).
\]

We now estimate some of the terms on the line above. Dividing \( u \) into subwords of lengths \( m \) and applying (6.1) to each segment, by condition (C3), we have that \( q(C(u), t) \geq Q^{-1}_3 (r-1) \alpha^r \). In addition, \( q(C(y), t) > \alpha \) by (6.1).

Since \( \{ C(y^t) \}_{t=1}^t \) covers \( C(y) \cap E(\mu, N, \epsilon) \), we also have that
\[
\sum_q q(C(y^t), t) q(C(y), t) \geq \frac{N^{-1}_m(C(y) \cap E(\mu, N, \epsilon))}{q(C(y), t)} > c \] by the assumption of the lemma.

We conclude that
\[
\sum_q q(C(uyx^t), t) \geq Q^{-1}_3 (r-1) \alpha^r + 1 c q(C(z), t) \geq Q^{-1}_3 (r-1) \alpha^r + 1 c N^{-1}_m(C(z)). \quad (6.5)
\]

Inequalities (6.3), (6.4), and (6.5) together give that
\[
N^{-1}_m(C(z) \cap E(\mu, N + \alpha m + |z|, 2\epsilon)) \geq N^{-1}_m(C(uy) \cap E(\mu, N + \alpha m + |z|, 2\epsilon))
\]
\[
\geq Q^{-1}_3 (r-1) \alpha^r + 1 c N^{-1}_m(C(z)),
\]
which completes the proof.

\[\square\]

### 6.2. Key lemma

The following is the most important lemma in the proof of theorem 3.4.

**Lemma 6.5.** For each \( t < \dim_c(G(\mu)) \) and \( m \geq m(t) \), there is a constant \( \tilde{c} > 0 \) such that for any word \( z \) and any \( \epsilon > 0 \) one has
\[
\lim_{n \to \infty} N^{-1}_m(C(z) \cap E(\mu, n, \epsilon)) \geq \tilde{c} N^{-1}_m(C(z)). \quad (6.6)
\]

In addition, \( \tilde{c} \geq 2^{-1} \frac{Q^{-1}_3 (r-1)}{4} \).

**Proof.** Fix \( \epsilon > 0 \). We first show that (6.6) holds for words whose lengths are multiples of \( m \). Arguing by contradiction, we assume that there is a word \( z \) of length \( |z| = km \), for some \( k \in \mathbb{N} \) and a sequence \( \{a_n\} \subset \mathbb{N} \) increasing to infinity such that
\[
N^{-1}_m(C(z) \cap E(\mu, a_n + |z| + \tau m, 4\epsilon)) < \frac{\alpha^{r+1} Q_3 (r+3)}{4} N^{-1}_m(C(z))
\]
for all \( n \in \mathbb{N} \), where \( \alpha \) is the constant given in lemma 6.3. Applying lemma 6.3 with \( c = \frac{1}{4 Q_3} \), we obtain that for each word \( y \) of length \( m \),
\[
N^{-1}_m(C(y) \cap E(\mu, a_n, 2\epsilon)) \leq \frac{1}{4 Q_3} q(C(y), t) < \frac{1}{2 Q_3} q(C(y), t)
\]
for all \( n \in \mathbb{N} \), where \( \alpha \) is the constant given in lemma 6.3. Applying lemma 6.3 with \( c = \frac{1}{4 Q_3} \), we obtain that for each word \( y \) of length \( m \),
\[
N^{-1}_m(C(y) \cap E(\mu, a_n, 2\epsilon)) \leq \frac{1}{4 Q_3} q(C(y), t) < \frac{1}{2 Q_3} q(C(y), t)
\]
for all $n \in \mathbb{N}$ such that $a_n > 2(|z| + \tau m)/\epsilon$. By this estimate and lemma 6.2, for any word $w$ of length $|w| = jm$ with $j \in \mathbb{N}$ we have

$$N'_n(C(w) \cap E(\mu, a_n + |w| - m, \epsilon)) \leq \frac{1}{2} q(C(w), t)$$

(6.7)

for all $n \in \mathbb{N}$ such that $a_n \geq 4|w|/\epsilon$.

Fix a large number $N \in \mathbb{N}$. We first apply (6.7) to $w = z$. Choose $n_0 \in \mathbb{N}$ such that $a_{n_0} > \max\{N, 4|z|/\epsilon\}$ and denote $b_0 := a_{n_0} + |z| - m$. One can find a finite cover $(C_i)_{i=1}^I$ of $C(z) \cap E(\mu, b_0, \epsilon)$ by cylinders of lengths $l(C_i) = l_i m$ such that

$$\sum_{i=1}^{I} q(C_i, t) \leq \frac{2}{3} q(C(z), t).$$

For each $1 \leq i \leq I$, we again apply (6.7) to $C_i = \tilde{C}_i$. Choose $n_i \in \mathbb{N}$ such that $a_{n_i} > \max\{N, 4l_i m/\epsilon\}$ and denote $b_i := a_{n_i} + (l_i - 1)m$. There is a finite cover $(C_{i,j})_{j=1}^{J_i}$ of $C_i \cap E(\mu, b_i, \epsilon)$ by cylinders of lengths $l(C_{i,j}) = l_{i,j} m$ such that

$$\sum_{j=1}^{J_i} q(C_{i,j}, t) \leq \frac{2}{3} q(C_i, t).$$

Together we obtain that $\bigcup_{j=1}^{J_i} \bigcup_{j=1}^{J_i} C_{i,j}$ is a cover of $C(z) \cap \bigcap_{i=0}^{I} E(\mu, b_i, \epsilon)$ and

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} q(C_{i,j}, t) \leq \frac{2}{3} \sum_{i=1}^{I} q(C_i, t) \leq \left( \frac{2}{3} \right)^2 q(C(z), t).$$

Repeating this argument, we obtain for each $L \in \mathbb{N}$ a sequence $\{\tilde{b}_j\}_{j=1}^{B(L)}$ of numbers $\tilde{b}_j \geq N$ and a cover $\{\tilde{C}_j\}_{j=1}^{M(L)}$ of $C(z) \cap \bigcap_{i=0}^{B(L)} E(\mu, \tilde{b}_i, \epsilon)$ by cylinders of lengths being multiples of $m$ such that

$$\sum_{j=1}^{M(L)} q(\tilde{C}_j, t) \leq \left( \frac{2}{3} \right)^L q(C(z), t).$$

In addition, for each $L \in \mathbb{N}$ we have that

$$\bigcap_{n \geq N} E(\mu, n, \epsilon) \subset \bigcap_{i=0}^{B(L)} E(\mu, \tilde{b}_i, \epsilon).$$

Therefore, $N'_n(C(z) \cap \bigcap_{n \geq N} E(\mu, n, \epsilon)) = 0$.

Since $N$ was arbitrary, by observing that

$$G(\mu) = \bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} E(\mu, n, \epsilon),$$

(6.8)

we conclude that $N'_n(C(z) \cap G(\mu)) = 0$. We claim that $N'_n(G(\mu)) = 0$. Indeed, since $N'_n(C(z) \cap G(\mu)) = 0$, then for every small $\tilde{\epsilon} > 0$ there exists a cover $\{C(\omega_j')\}$ of $C(z) \cap G(\mu)$ with

$$\sum_{j} q(\omega_j', t) < \tilde{\epsilon}.$$
By the proof of theorem 2.3, there is \( L = L(\epsilon) > 0 \) such that each word \( \omega_j^\tau \) has length at least \( L \). In particular, choosing \( \epsilon > 0 \) small enough (thus ensuring that \( L \) is large enough), we can ensure that each word \( \omega_j^\tau \) is of the form \( zu_j \) where \( |u_j| = \tau m \).

Let \( x = (x_1, x_2, \ldots) \) be a point in \( G(\mu) \). One can find a word \( u \) of length \( \tau m \) such that \( x' = (zu_1, x_2, \ldots) \in X \). In fact, \( x' \in C(z) \cap G(\mu) \). Consequently, \( G(\mu) \subset f^\tau = f^{\tau m}(C(z) \cap G(\mu)) \).

In addition, taking the collection of cylinders \( \{D_j\} = \{C(zu_j)\} \) covering \( C(z) \cap G(\mu) \), the corresponding collection \( \{D_j\} = \{C(v_j)\} \) covers \( G(\mu) \). By condition (C3),

\[
q(D_j, t) = q(C(v_j), t) \leq Q_3 \frac{q(C(zu_j), t)}{q(C(zu), t)} \leq \frac{Q_3^{r+k}}{\alpha^{r+1}k} q(C(zu_j), t) < \frac{Q_3^{r+k}}{\alpha^{r+1}k}.
\]

Letting \( \epsilon \to 0 \), we obtain that \( N^\mu_m(G(\mu)) = 0 \). By theorem 2.3, we must have that \( \dim_c(G(\mu)) \leq t \). This contradicts with the assumption of the lemma. We therefore proved that for any word whose length is a multiple of \( m \) the inequality (6.6) holds with

\[
\tilde{c}_0 := \frac{Q_3^{r+1}}{4}.
\]

Let now \( x \) be a word of length \( |x| = km + l \) where \( k \in \mathbb{N} \) and \( 0 < l < m \). Let \( x, \bar{x} \) be words of lengths \( km \) and \( (k + 1)m \) respectively, such that \( x \prec x \prec \bar{x} \).

We first observe that, for any \( n \in \mathbb{N} \), \( C(x) \cap E(\mu, n, \epsilon) \subset C(\bar{x}) \cap E(\mu, n, \epsilon) \), so that

\[
\liminf_{n \to \infty} N^\mu_m(C(x) \cap E(\mu, n, \epsilon)) \geq \liminf_{n \to \infty} N^\mu_m(C(\bar{x}) \cap E(\mu, n, \epsilon))
\]

\[
\geq \tilde{c}_0 N^\mu_m(C(\bar{x})) = \tilde{c}_0 q(C(\bar{x}), t),
\]

where \( x' \preceq \bar{x} \) and \( |x'| = k'm \) for some \( k' \leq k + 1 \). If \( x' = \bar{x} \), then by (C3) and (6.1),

\[
q(C(x'), t) \geq Q_3^{-1} q(C(x), t) q(C(\bar{x}^{km+1} \ldots \bar{x}^{(k+1)m})),
\]

\[
\geq Q_3^{-1} \alpha q(C(x), t) \geq Q_3^{-1} \alpha N^\mu_m(C(x)).
\]

If now \( x' \preceq \bar{x} \), then we must have that \( C(x) \subset C(x') \). Consequently,

\[
q(C(x), t) \geq N^\mu_m(C(x)).
\]

We conclude that

\[
\liminf_{n \to \infty} N^\mu_m(C(x) \cap E(\mu, n, \epsilon)) \geq \tilde{c}_0 Q_3^{-1} \alpha N^\mu_m(C(x))
\]

\[
= \alpha r+2 \frac{Q_3^{r+1}}{4} N^\mu_m(C(x)),
\]

which completes the proof.

\[\square\]

6.3. Proof of theorem 3.4

Let \( \ell < \inf \{ \dim_c(\mu^{(\ell)}) | \ell \geq 0 \} \). Observe that for \( \ell \geq 0 \), \( \dim_c(G(\mu^{(\ell)})) \geq \dim_c(\mu^{(\ell)}) \) by Birkhoff’s ergodic theorem. Therefore, lemma 6.5 holds for \( \ell \) and each \( \mu^{(\ell)} \). Fix \( m \geq m(\ell) \).
Recall that \([x]_n\) is the truncation \([x]_n = (x_1, x_2, ..., x_n)\) of \(x = (x_1, x_2, ...)\) and 
\[d_\beta(x, y) = \sum_{j=1}^{\infty} |x_j - y_j|^{\beta/j} \] 
for \(x = (x_1, x_2, ...), \ y = (y_1, y_2, ...) \in X\). Note that if \(x \in X, \ n \in \mathbb{N}, y \in C([x]_n + Tm)\) and \(\varphi \in \text{Lip}_1(X, [-1, 1])\), then 
\[
\left| \int_X \varphi \, d\delta^y_n - \int_X \varphi \, d\delta^y_n \right| \leq \frac{1}{n} \sum_{j=0}^{n-1} d_\beta(f^j(x), f^j(y)) \leq \frac{1}{n(\beta - 1)} \sum_{j=0}^{n-1} \beta^{-(n + Tm - j)}.
\]

Let \(T\) be a positive integer such that \(1/(n(\beta - 1))\sum_{j=0}^{n-1} \beta^{-(n + Tm - j)} < \epsilon/4\) for any \(n \in \mathbb{N}\). By the Kantorovich–Rubinstein dual representation of the first Wasserstein metric, we have 
\[d(\delta^y_n, \delta^y_n) \leq \epsilon/4\] 
for any \(x \in X, n \in \mathbb{N}\) and \(y \in C([x]_n + Tm)\).

Recall that \(\tau\) denotes the specification constant, that is, for any two admissible words \(u\) and \(v\) (of any length) there exists a word \(w\) of length \(\tau m\) such that the word \(uwv\) is admissible. Finally, we denote \(T = (T_1 + \tau)m\).

**Lemma 6.6.** For any \(k \in \mathbb{N}, \ \{\ell_j\}_{j=1}^k \subset \mathbb{N}, \lambda_1, \lambda_2, ..., \lambda_k \in (0, 1]\) with \(\sum_{j=1}^k \lambda_j = 1, \ \epsilon > 0\) and \(m_0 \in \mathbb{N}\), we have that 
\[
\bigcup_{n \geq m_0} E \left( \sum_{j=1}^k \lambda_j \mu^{(\ell_j)}, n, \epsilon \right) \subset \mathcal{G}(X).
\]

**Proof.** Fix \(k \in \mathbb{N}, \ \{\ell_j\}_{j=1}^k \subset \mathbb{N}, \lambda_1, \lambda_2, ..., \lambda_k \in (0, 1]\) such that \(\sum_{j=1}^k \lambda_j = 1, \ \epsilon > 0\) and \(m_0 \in \mathbb{N}\). Fix also a cylinder \(C(z) \subset X\). It follows from lemma 6.5 that we can find a positive integer \(\bar{n} \geq m_0\) such that:

- \(|z| < \min\{\lambda, \bar{n}\};\)
- \(Tk/\bar{n} < \epsilon/4;\)
- \(N_m^j(C(z) \cap E(\mu^{(\ell_j)}, n, \epsilon/4)) \geq c_1 N_m^j(C(z))\) for each \(j = 1, ..., k\) and \(n \geq \lambda_j \bar{n}\) with some \(c_1 > 0\).

Denote 
\[
n_j = \lfloor \lambda_j \bar{n} \rfloor, \ \ N_j = Tj + \sum_{i=1}^{j} n_i \quad \text{for} \ j = 1, ..., k - 1, \ \text{and} \ N = N_{k-1} + n_k \quad (6.9)
\]

with \(N_0 = 0\). We consider the sets 
\[
E_0 := C(z) \cap E(\sum_{j=1}^k \lambda_j \mu^{(\ell_j)}, N, \epsilon), \text{ and} 
\]
\[
E_0 := C(z) \cap \left( \bigcap_{j=1}^k f^{-N_{j-1}} \left( E(\mu^{(\ell_j)}, n_j, \epsilon/2) \right) \right).
\]
Observe that for each $x \in X$,

$$\delta^N_x = \frac{\hat{N}}{N} \sum_{j=1}^k \frac{n_j}{\hat{N}} \delta_{\hat{N}^{\ell-1} x}^j,$$

and thus

$$d \left( \delta^N_x, \sum_{j=1}^k \lambda_j \mu(x) \right) \leq 1 - \frac{\hat{N}}{N} + \sum_{j=1}^N \left| \lambda_j - \frac{n_j}{\hat{N}} \right| + \sum_{j=1}^k \lambda_j d \left( \delta_{\hat{N}^{\ell-1} x}^j, \mu^{(x)} \right).$$

This implies that $E_0 \subset E$.

By the choice of $T_1$, for $j = 1, \ldots, k$ we have that, if $y^j \in E(\mu^{(x)}, n_j, \epsilon/4)$, then $C(y^j) \subset E(\mu^{(x)}, n_j, \epsilon/2)$. In addition, if $y^1, \ldots, y^k \in X$ are such that

$$y^j \in C(z) \cap E(\mu^{(x)}, n_1, \epsilon/4) \quad \text{and} \quad y^j \in E(\mu^{(x)}, n_j, \epsilon/4) \quad \text{for} \quad j = 2, \ldots, k,$$

then there are words $u^1, \ldots, u^{k-1}$ such that the length of each $u^j$ is $\tau m$ and that the word

$$w := [y^1]_{n_1 + T_1 m} [y^2]_{n_2 + T_1 m} u^2 \ldots [y^{k-1}]_{n_{k-1} + T_1 m} u^{k-1} [y^k]_{n_k + T_1 m}$$

is admissible. Denote by $W$ the collection of all words $\omega$ obtained this way. We then have that $C(w) \subset E_0$ for every $\omega \in W$.

From this and the definition of $N^2_m$, it follows that there is a positive number $c_2$ such that

$$N^2_m(E) \geq c_2 \left( \sum_{\omega \in W} C(\omega) \right) \geq c_2 \left( \sum_{\omega \in W} C(z) \cap E(\mu^{(x)}, n_1, \epsilon/4) \right) \geq c_2 \left( \sum_{\omega \in W} N^2_m(\mu^{(x)}, n_1, \epsilon/4) \right) \geq c_2 \left( \sum_{\omega \in W} N^2_m(\mu^{(x)}, n_2, \epsilon/4) \right) \ldots \geq c_2 N^2_m(\mu^{(x)}, \epsilon/4).$$

Since $N \geq m_0$, this completes the proof. \qed

Let $\{\nu^{(x)}\}_{x \in \Delta_0}$ be a countable dense subset of $\Delta(\mathcal{F})$, then it is straightforward to see that $E(\mathcal{F}) = \bigcap_{x \in \Delta(\mathcal{F})} E(\nu^{(x)})$. For each $\ell \in \mathbb{N}$, one can find $k \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in [0, 1]$ such that

$$\sum_{j=1}^k \lambda_j = 1 \quad \text{and} \quad \nu^{(x)} = \sum_{j=1}^k \lambda_j \mu^{(x)}.$$

It follows from lemma 6.6 that

$$E \left( \sum_{j=1}^k \lambda_j \mu^{(x)} \right) = \bigcap_{m_1 \in \mathbb{N}} \bigcap_{m_0 \in \mathbb{N}} E \left( \sum_{j=1}^k \lambda_j \mu^{(x)}, n, m_1^{-1} \right) \in G^i(X).$$

Therefore, theorem 3.4 immediately follows from theorem 3.2.

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