Extending Friedmann equations using fractional derivatives: the case of an expanding Universe without dark components.

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We present a toy model for extending Friedmann equations of relativistic cosmology using fractional derivatives. We do this by interchanging the integer derivatives by unknown fractional derivative orders on a few well known results of cosmology at the present epoch with the intention to explain the current observed acceleration of the Universe. In other words, we apply the Last Step Modification of fractional calculus to construct some useful fractional equations of cosmology. The statistical calibration of the unknown fractional derivative order and the fractional cosmographical parameters through SNIa data shows that this simple construction can explain the current accelerated expansion of the Universe without the use of any dark matter and/or dark energy components.

Keywords: Cosmology; SNIa; fractional calculus.

I. INTRODUCTION

The Hilbert-Einstein field equations, a success of general relativity works tremendously well at mass to length scales similar to the ones of the solar system [see e.g. 1], where the gravitational field is weak \(2 \leq 14\). At very large mass to length ratios, where the gravitational field is very strong, the detection of gravitational waves through the interaction of compact objects such as black holes and neutron stars has shown a remarkable good agreement with the predictions of general relativity.

When the Newtonian acceleration of a test particle reaches values \(a_0 \lesssim 10^{-10} \text{m/s}^2\), or equivalently when the mass to length ratios are much smaller than those of the solar system \([15]\), the Hilbert-Einstein field equations cannot fit an enormous amount of astrophysical and cosmological data unless: (a) extra dark matter and/or dark energy components are added or (b) the response of the curvature given by the matter and energy requires extensions or modifications. In other words, the Hilbert-Einstein field equations do not work at those acceleration scales unless (a) or (b) are adopted. In what follows, we will only deal with case (b) for the accelerated expansion of the Universe at the present epoch.

If the gravitational field equations are to be extended, the first intuitive attempt consists on assuming a general \(f(R)\) function of Ricci’s scalar \(R\) as the Lagrangian in the gravitational action. Despite some \(f(R)\) theories have interesting results \([10, 22]\), there is currently not a full \(f(R)\) Lagrangian which solves all the shortcomings between observations and the gravitational theory. Through the years, more general actions have been proposed, using for example functions of many scalars built with the Riemann tensor and even the ones in which couplings between Ricci’s curvature scalar \(R\) and the matter Lagrangian \(L_{\text{matter}}\), or the trace \(T\) of the energy momentum tensor \([24, 28]\) are adopted. These proposals are still being developed and investigated in full, and although interesting in principle, they have a small general inconvenient: the motion of free particles is not necessarily geodesic and as such, a fifth force appears naturally.

In Barrientos and Mendoza [30], Bernal et al. [31], Mendoza et al. [32], Barrientos and Mendoza [33], Barrientos, Bernal, and Mendoza [34], the authors proposed general gravitational actions with curvature-matter couplings in order to obtain a feasible explanation for the MOdified Newtonian Dinamics (MONDian) behavior of gravitational phenomena. The general conclusions reached by the works of Bernal et al. [31], Mendoza et al. [32], Barrientos, Bernal, and Mendoza [34] is that it is possible to recover the MONDian expression for the acceleration in the regime \(a < a_0\) for a pure metric \(f(R)\) theory provided a non-local action construction. Non-locality in these attempts is introduced as an extra scalar field with dimensions of mass that can be conveniently thought of as the causally-connected mass to each point in space-time.

The idea of a non-local gravity and its implications at solar system and cosmological scales have been recently revisited by the works of Mashhoon [35, 36], Chicone and Mashhoon [37, 38, 39], Blome et al. [40]. These theories claim that locality, which allows to treat an accelerated observer as an inertial one at each instant on his proper world-line in special relativity, has its limitations. Since the Einstein Equivalence Principle relates an observer in a local gravitational field with another accelerated one with no gravitation, those limitations are thus extended to gravitational theory. A general characteristic of non-locality is that fields are no longer given by its instant (or local) value but has a contribution attached to the history of the observer (in general terms the Lagrangian

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density is not localised and as such is not defined as a simple function of the space-time coordinates). Recent investigations have shown that these kind of proposals can simulate dark matter behaviour [41–43]. For Tully-Fisher scalings, MOND introduces a gravitational potential for a point mass source proportional to $\ln(r)$, which flattens rotation curves. This is also included in these non-local proposals, but they cannot fully explain MOND’s proportionality force to $\sqrt{G}$, where $G$ is the Newtonian constant of gravitation. For the non-local constructions, the relation between the acceleration and the gravitational constant turns out to be linear.

In the present article, we discuss another possible non-local approach in which fractional calculus is used. Fractional calculus, although still a curiosity for many, has proved to describe appropriately non-local space-time effects in a wide range of applications [44–51]. In general terms, fractional calculus extends the order of differentiation and integration from the natural numbers to the real ones (in fact to the complex numbers, but we are only going to work with real values in this work). In order to simplify the understanding of fractional calculus to the non-expert, we have included in the Appendix a simple introduction to fractional calculus.

Introducing of fractional derivatives in gravity is not a trivial task since there are several proposals in order to perform the required generalisations. From a pure mathematical point of view, fractional derivatives must induce a somehow fractional geometry in such a way that all the geometric entities involved in general relativity e.g. the connection, the covariant derivative, the Riemann tensor and the metric should be defined in terms of a fractional derivative order. Such an approach is known as the First Step Modification (FSM). A more practical and simple way consist in modifying the Hilbert-Einstein field equations for a given geometry, replacing the covariant derivative order by its analogous fractional derivative without going deeper in how such equation can be obtained. This last approach is usually called a Last Step Modification (LSM). In recent years the applicability of both approaches has been studied in gravity at a classical and and cosmological level [52–54]. An intermediate approach is to formulate a variational principle for a fractional order action. This latter approach is of particular interest for the scientific community, not exclusively of the gravitational area, since a fractional variational theory is general enough for applications in different scientific fields [55–60].

The introduction of fractional derivatives (with unknown derivative orders, which are to be calibrated by observations) in the Friedmann equations means that either the gravitational Lagrangian contains fractional derivatives, or the order of variation has a non-integer value (or both). In general terms, if fractional derivatives are to be allowed in physical equations, then the order of differentiation should also be a parameter in the action. In other words, the principle of least action for field equations should somehow contain fractional derivatives. If this is not the case, another possibility is that the variation of the action includes a fractional operator. In a more complicated scenario, both the variation and the Lagrangian could contain fractional derivatives. By itself, this constitutes a very profound subject outside the scope of this work. In this article we force the introduction of fractional derivatives into the standard Friedmann equation of cosmology and see whether we can fit the order of derivatives using SNIa observations. The goal of this article is to find out whether it would be possible to explain the accelerated expansion of the universe without introducing any dark matter and/or dark energy components into the cosmological density budget in the Friedmann equation for a dust flat universe as observed today.

The article is organised as follows. In Section II we describe a few key results of standard cosmology and cosmographical parameters. In Section III we extend the Friedmann equations and other relations -including cosmographical parameters to their fractional derivative counterparts. Since the derivative order is a free parameter of the proposal, using these fractional extensions we calibrate all free parameters using SNIa observations for the accelerated expansion of the Universe in Section IV. In Section V we present our statistical results and finally in Section VI we state final remarks of the model developed in this article.

II. STANDARD COSMOLOGY

In this section we briefly summarise a few standard concepts of relativistic cosmology that will be extended to their fractional derivatives counterparts later on. Many of the results presented in this section can be found elsewhere [see e.g. 61–64 and references therein].

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric describes an isotropic and homogeneous Universe, and is given by:

$$ds^2 = c^2 dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

where $k$ is the curvature and $a(t)$ is the cosmological scale factor, a function of the cosmic time $t$. The velocity of light is represented by $c$ and $\theta$ and $\phi$ represent the polar and azimuthal angles respectively. Substitution of the FLRW metric into the field equations of general relativity with a cosmological constant $\Lambda$ yields two independent expressions for the temporal 00 and radial 11 components respectively:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G \rho + \Lambda c^2}{3},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho - \Lambda c^2}{3}. $$
for a dust, i.e. pressure-less Universe. In the previous equations \( \rho \) represents the matter density. Using the standard definition for the Hubble parameter:

\[
H(t) := \frac{\dot{a}}{a},
\]

(4)
equation (2) turns into:

\[
H^2 = \frac{8\pi G \rho + \Lambda c^2}{3} - \frac{k c^2}{a^2}.
\]

(5)
The right hand of this equation contains all the information for a late-time Universe’s constituents: curvature \( k \), matter density \( \rho \) and cosmological constant \( \Lambda \). The previous equation means that:

\[
1 = \Omega_M + \Omega_\Lambda + \Omega_k.
\]

(6)

where:

\[
\Omega_M := \frac{8\pi G \rho}{3H^2}, \quad \Omega_\Lambda := \frac{c^2 \Lambda}{3H^2}, \quad \Omega_k := -\frac{k c^2}{H^2 a^2},
\]

(7)
represent the matter, dark energy and curvature density parameters respectively. Equation (6) is a convenient normalisation for all the energy constituents of the universe, so that their sum equals one.

We now define a few cosmographic parameters. To begin with, the deceleration parameter:

\[
q := -\frac{1}{H^2 a} \frac{\ddot{a}}{a}.
\]

(8)
can be rewritten as:

\[
q = \frac{\Omega_M}{2} - \Omega_\Lambda.
\]

(9)
by means of equation (3).

Derivating with respect to cosmic time \( t \) the second Friedmann equation (3) yields:

\[
\frac{\dddot{a}}{a} - 2 \frac{\ddot{a}}{a^2} - \frac{\dot{a}^2}{a^2} = -\frac{4\pi G \dot{\rho}}{3}.
\]

(10)
We now introduce another cosmographic parameter, namely the jerk:

\[
j := \frac{1}{H^3 a} \frac{\dddot{a}}{a}.
\]

(11)
In order to express the jerk in terms of the density parameters, we use the fact that the covariant divergence of the energy-momentum tensor vanishes, i.e. \( \nabla_\mu T^{\mu\nu} = 0, \)

For a matter dominated dust Universe, it follows that this relation yields: \( \dot{\rho} \propto a^{-3} \) and so, matter density is:

\[
\dot{\rho} = -3H \rho.
\]

(12)
Substitution of this last relation into equation (10) yields:

\[
j = \frac{3}{2} \Omega_M - q,
\]

(13)
which can be expressed in terms of energy densities only using relation (3):

\[
j = \Omega_M + \Omega_\Lambda
\]

(14)
An additional derivative with respect to cosmic time in equation (10) gives the following expression:

\[
\frac{\dddot{a}}{a} - 2 \frac{\ddot{a}}{a^2} + 2 \frac{\dot{a}^2}{a^2} - 4 \frac{\ddot{a}}{a^2} = -\frac{4\pi G \ddot{\rho}}{3}.
\]

(15)
The snap cosmographic parameter \( s \) is defined as:

\[
s := \frac{1}{H^3 a} \frac{\dddot{a}}{a}.
\]

(16)
So, using equation (15), and equation (12), together with \( \dot{H} = -H^2 (1 + q) \), it can be written as:

\[
s = -\frac{3}{2} \Omega_M (4 + q) + 2j + 2q + q^2,
\]

(17)
which in terms of the density parameters is:

\[
s = -3\Omega_M - \frac{1}{2} \Omega_M^2 + \frac{1}{2} \Omega_M \Omega_\Lambda + \Omega_\Lambda^2.
\]

(18)
Equations (9), (14) and (18) can be expressed in terms of the curvature density parameter \( \Omega_k \) instead of the dark energy density parameter \( \Omega_\Lambda \) using eq. (6). Since the current observations from Planck [66] strongly suggest that we are living in a flat universe \( k = 0 \), we will work with this value in what follows. Therefore, equation (9) simplifies to: \( 1 = \Omega_M + \Omega_\Lambda \) and so, the value for the jerk parameter in general relativity has a constant unitary value, i.e. \( j = 1 \).

III. FRACTIONAL FRIEDMANN EQUATION

In what follows, we explore the cosmological consequences of changing the integer time derivatives in Friedmann equations (2) and (3) for fractional derivatives. The idea is to calibrate the unknown derivative order
with SN1a supernova observations. Following this path, we write down the fractional Friedmann equations as:

\[
\left(\frac{D^\gamma a}{D\gamma t}\right)^2 = \kappa a^2 \left(\frac{8\pi G \rho + \Lambda c^2}{3}\right), \tag{19}
\]

\[
\frac{D^\gamma}{D\gamma t} \left(\frac{D^\gamma a}{D\gamma t}\right) = \kappa a \left(\frac{4\pi G \rho - \Lambda c^2}{3}\right), \tag{20}
\]

for a flat Universe. The constant $\kappa$, with dimensions of $t^{2(1-\gamma)}$ has been introduced into the fractional Friedmann equations in order to have dimensional coherence. The left-hand side of equation (20) is written as such since in general terms $D^\gamma D^\gamma \neq D^{2\gamma}$.

Since our target is to work with a Friedmann fractional model with no dark matter, we introduce Milgrom’s acceleration constant $a_0$ as a fundamental physical quantity for the description of gravitational phenomena at cosmological scales. With this and since the velocity of light $c$ and Newton’s gravitational constant $G$ are also fundamental, using Buckingham-Π theorem for the dimensional analysis \[67\], it follows that \[1\]:

\[
\kappa = \left(\frac{a_0}{c}\right)^{2(\gamma-1)}. \tag{21}
\]

Under the idea of fractional orders on the derivative, we can adapt the cosmographical parameters in a natural way as follows. To begin with, we define the fractional Hubble parameter as:

\[
H^\gamma := \frac{1}{a} \frac{D^\gamma a}{D\gamma t}, \tag{22}
\]

which means that equation (13) turns into:

\[
H^{2\gamma} = \left(\frac{a_0}{c}\right)^{2(\gamma-1)} \left(\frac{8\pi G \rho + \Lambda c^2}{3}\right). \tag{23}
\]

In order to express the previous equation in terms of the density parameter, we follow the procedures of Section \[13\] and so, dividing this last equation by $H^2$ it follows that:

\[
H = \frac{a_0}{c} (\Omega_M + \Omega_\Lambda)^{1/2(\gamma-1)}. \tag{24}
\]

In the previous equation, the definitions of the density parameters given in \[7\] were used. Nonetheless it is possible to define suitable new adequate density parameters in order to obtain an equation similar to (6), but this is not needed for the purpose of the present article.

By defining a fractional deceleration parameter as:

\[
q^\gamma := - \frac{1}{a H^2 \gamma} \frac{D^\gamma}{D\gamma t} \left(\frac{D^\gamma a}{D\gamma t}\right), \tag{25}
\]

the second fractional Friedmann equation (20) can be written as:

\[
-q^\gamma H^{2\gamma} = \kappa \left(-\frac{4\pi G \rho - \Lambda c^2}{3}\right), \tag{26}
\]

which after dividing by $H^2$ yields:

\[
-q^\gamma H^{2(\gamma-1)} = \kappa \left(\Omega_\Lambda - \frac{1}{2} \Omega_M\right), \tag{27}
\]

and so, using equation (24) we find:

\[
q = \left[\frac{\Omega_M - \Omega_\Lambda}{\Omega_M + \Omega_\Lambda}\right]^{1/\gamma}. \tag{28}
\]

In order to find expressions for the fractional jerk and snap cosmographic parameters as functions of the density parameters only, the natural way would be to follow an analogous procedure as the one described in Section \[11\]. This procedure will involve the cumbersome application of two (one for the jerk and another for the snap) consecutive fractional cosmic time derivatives in equation (20). In order to avoid that, we apply a Last Step Modification procedure in equations (10) and (15) with correct definitions for the fractional jerk and snap parameters. For simplicity and coherence, we will continue to use in what follows a Last Step Modification and so, the fractional equivalent of relation (10) is given by:

\[
\frac{1}{a} \frac{D^\gamma}{D\gamma t} \left(\frac{D^\gamma a}{D\gamma t}\right) - \frac{1}{a^2} \frac{D^\gamma}{D\gamma t} \left(\frac{D^\gamma a}{D\gamma t}\right) \frac{D^\gamma a}{D\gamma t} = -\kappa \frac{4\pi G \rho - \Lambda c^2}{3}. \tag{29}
\]

We define the fractional jerk parameter as:

\[
j^\gamma := \frac{1}{a H^3 \gamma} \frac{D^\gamma}{D\gamma t} \left(\frac{D^\gamma a}{D\gamma t}\right), \tag{30}
\]

and substitute this together with the definitions of the fractional deceleration (25) and Hubble parameter (22) into eq. (29) to obtain:

\footnote{Strictly speaking relation (21) is a proportionality relation and not an equality. However, since the target of this article is to see the feasibility of having fractional Friedmann equations in a real cosmological scenario, the constant of proportionality – which is of order one– that should appear in equation (21) can be absorbed in the definition of the cosmological density parameters.}
\[ \frac{D^\gamma \rho}{D\tau^\gamma} = -3\rho H^\gamma \]  

(32)

With all these assumptions, eq. \(31\) takes the following form:

\[ H^{2\gamma}(j^\gamma + q^\gamma) = \kappa(4\pi G \rho), \]

or:

\[ H^{2(\gamma-1)}(j^\gamma + q^\gamma) = \frac{3}{2} \Omega_M. \]

(34)

in terms of the matter density parameter. Direct substitution of equations (21), (24) and (28) yields:

\[ j = \left[ \Omega_M + \Omega_\Lambda \over \Omega_M + \Omega_\Lambda \right]^{1/\gamma} = 1. \]

(35)

Note that this is an expected result since we are using a Last Step Modification approach for the fractional cosmographical parameters.

By performing an analogous procedure as the one to obtain the fractional jerk parameter \([60]\), it follows that:

\[ s = \left[ -3\Omega_M - \frac{1}{3} \Omega_M^2 + \frac{1}{6} \Omega_M \Omega_\Lambda + \Omega_\Lambda^2 \over \Omega_M + \Omega_\Lambda \right]^{1/\gamma}. \]

(36)

IV. SNIA FITS

The accelerated expansion of the Universe was first inferred through cosmological observations of SNIa as standard candles at all redshifts. Perlmutter et al. \([68]\) used 42 high-redshift supernovae to construct an apparent magnitude-redshift diagram in order to obtain the best values for the matter density parameter \(\Omega_M\) with the introduction of a dark energy density parameter \(\Omega_\Lambda\) component.

The model independent relation between the luminosity distance \(d_L(z)\) as a function of redshift \(z\) and the apparent magnitude \(\mu\) is given by \([24, 64]\):

\[ \mu(z) = 5 \log \left( {H_0 d_L(z) \over c} \right) - 5 \log h(z) + 42.3856, \]

(37)

where \(H_0\) is the Hubble constant evaluated at the present epoch \(t_0\) and \(h := H_0/100\) km/s is the normalised Hubble constant, with a luminosity distance for a flat universe given by \([69]\):

\[ d_L(z) = \frac{c}{H_0} \left[ z + \frac{1}{2} (1 - q_0) z^2 - \frac{1}{6} (1 - q_0 - 3q_0^2 + j_0) z^3 + \frac{1}{24} (2 - 2q_0 - 15q_0^2 + 15q_0^3 + 5j_0 + 10q_0j_0 + s_0) z^4 + \right]. \]

where \(q_0\), \(j_0\) and \(s_0\) are the cosmographic parameters evaluated at the present epoch. Substitution of equations (9), (14) and (18) into the previous relation yields an expression for the distance modulus \(\mu\) as a function of \(z\), \(\Omega_M\) and \(\Omega_\Lambda\). Since the values for the distance modulus and redshift are given by observations, an statistical fit will return the best estimated values for both density parameters.

In order to see whether the fractional Friedmann model presented above can account for SNIa observations, instead of using equations (9), (14) and (18) for the cosmographic deacceleration, jerk and snap parameters, we use their fractional extension given in (28) \([55]\) and (30) \([36]\) when studying equation (38). This generalisation requires the Hubble parameter \(H_0\) to be given by equation (24). In order to explore whether a dark energy component would be required, we allowed fixed small values of \(\Omega_\Lambda \sim 0.01\). The SNIa distance modulus-redshift data was taken from the Supernova Cosmology Project (SCP) Union 2.1 \([70]\).

In order to perform the statistical fits, we used the free software \textit{gnuplot} (\texttt{www.gnuplot.info}) to obtain the best values for the free parameters of our model. The calibration can be directly performed since the constants \(c\), \(a_0\) and \(j_0\) are known and so the corresponding equations for \(H_0\), \(q_0\) and \(s_0\) become functions of the density parameters and the fractional order. We used the fit function command in \textit{gnuplot} for the calibration of the free parameters. This command uses non-linear and linear least squares methods and is able to fit the required function through the empirical data and provides the correlation matrix between the parameters, the number of iterations employed for the converged fit, the final sum of the squares of residuals (SSR), the best fit value for the parameters and its asymptotic standard error, the \(p\)-value for the \(\chi\)-square distribution and the root mean squares of residuals.

V. RESULTS

A. Free order fractional derivative.

As already mentioned, in all our fits the dark energy density parameter is fixed to a small arbitrarily value. This conclusion was obtained from an exhaustive exploration of the parameter space by \textit{gnuplot}. It turns out
TABLE I. Left: Best fit results for the fractional derivative model with two free parameters: the fractional derivative order $\gamma$ and the matter density parameter $\Omega_M$, both presented with their corresponding errors. The right panel shows the correlation matrix for the best fit values reported. The SSR for this model is: 892.82.

![Graph](image1)

**FIG. 1.** Apparent magnitude $\mu$ vs. redshift $z$ Hubble diagram from the Union 2.1 SNe Ia data (dots with their corresponding error bars) and the best fit from our model. The solid line represents the distance modulus $\mu(z)$ from the best fit to the data of the model.

That helping the program’s routine with a fixed value of $\Omega_\Lambda \sim 0.01$ a good convergence is obtained. As such, we decided to fix $\Omega_\Lambda = 0.01$ in all of the results presented in this article. This is a convenient result since it means that $\Omega_\Lambda \ll 1$ and for any practical purpose we can think that there is no need to introduce any dark energy component into the modelling. After a long search for initial values, we found that by providing starting given values to *gnuplot* around $\Omega_M = 9.5$ and $\gamma = 2.5$, the fit routine converged well, providing the results of Table I. Although the error in $\Omega_M$ is $\sim 60\%$, the mean plotted best distance modulus-redshift function in Figure 1 seems to be in quite a good agreement with the observations.

The envelope curves that represent the statistical errors above and below the solid mean curve in Figure 1 have not been drawn since they turn out to be quite wide due to the large error in $\Omega_M$.

**B. Fixed fractional derivative order.**

From the results obtained in the previous subsection, we see that the value $\gamma = 1.5$ is within an admissible range given by the best statistical fit. This value is of special interest for us since it has been recently shown that a fractional Poisson-like equation with this order of derivative yields the correct MONDian acceleration [71]. So, the next step on our modelling is to fix the fractional order of derivative to $3/2$. After a wide exploration of initial parameters, the best *gnuplot* convergence occurs by using $\Omega_M = 0.3$ as initial data. The best statistical fit for the matter density parameter is then:

$$\Omega_M = 9.1342 \pm 0.0020. \quad (38)$$

The SSR for this model is: 896.651. In Figure 2, the graph of $\mu$ vs. $z$ for the observational data and the function with the best fit values is shown. The envelope curve of uncertainties above and below the mean adjusted one are plotted in the figure, but they are so close to the mean solid line that are impossible to visualise.

**VI. FINAL REMARKS**

In this article we have generalised the standard Friedmann equations of cosmology using fractional derivatives by performing a Last Step Modification approach. The obtained fractional Friedmann equations and the formulae of the corresponding fractional cosmographic parameters in terms of the cosmological density parameters were used to explain the current accelerated expansion of a dust and zero curvature Universe, without the introduction of any dark matter and/or dark energy components. The statistical fitting of the free parameters in the model shows an excellent agreement with a fractional derivative order of $3/2$, which has recently been shown to be the required fractional order to explain MOdified Newtonian
Dynamics (MOND) phenomenology \[7\]. This result is in excellent agreements with the recent work by Barrientos, Bernal, and Mendoza \[54\] since the Universe at the present epoch is in the deep MOND regime\[4\].

The toy model presented in this article about the current expansion of the Universe using fractional derivatives represents a first exploration onto whether there could be any interesting aspects of gravitational theory to be more deeply investigated. We intend to go beyond this and construct a formalism in terms of fractional derivatives for geometrical objects in general relativity and in formulating a fractional calculus of variations with applications to gravitational theory in future works.

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\[2\] Milgrom noticed a coincidental relation between the acceleration \(a_0\) and \(H_0\); \(a_0 \approx cH_0\). Since the Hubble radius \(r_H = c/H_0\) and the Hubble mass is \(M_H = c^2/H_0^2\) it then follows that the Newtonian gravitational acceleration experienced by a test galaxy at a distance \(r_G\) of a point mass source \(M_H\) is \(\approx GM_H/r_G^2 \approx a_0\) [see e.g. \[7\] \[7\]]. In other words, the Newtonian gravitational acceleration at the present epoch in the Universe is approximately MONDian. As such, one would expect that if no dark matter components are introduced into the cosmological energy budget, then a MONDian description of the Universe at the present epoch is required.
interactions or memory effects are present. Before providing the general definition of fractional derivative, we give a few examples motivating this notion. Take for instance

\[ f(x) = x^k. \]

Computing the first derivative we obtain

\[ f'(x) = \frac{df}{dx} = kx^{k-1}. \]

The successive repetition of this procedure \( n \) times yields the \( n \)-th derivative:

\[ d^n f \frac{dx^n}{dx} = \frac{k!}{(k-n)!}x^{k-n}. \]

The above expression makes perfect sense for a real number \( k \), if we can define its factorial in a meaningful way. The \( \Gamma \) function provides such a generalization, and the above expression can be written as

\[ d^n f \frac{dx^n}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} x^{k-\alpha}, \]

where we have replaced \( n \) by \( \alpha \) to emphasize the fact that now the order of differentiation can be taken to be any real number.

A similar reasoning can be applied to other basic examples such as the exponential or trigonometric functions. In such a way, it is possible to define the fractional derivative or integral for functions expressed in their Taylor or Fourier series expansions. However, a more general and useful definition is based on the Riemann-Liouville representation formula of a function. More precisely, the integral operator of order \( \alpha \) is defined as

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) \frac{d\tau}{(t-\tau)^{1-\alpha}}. \]

Again, this definition can be justified by applying the standard integral operator \( m \) times and integrating by parts and then making sense of the definition for a real order of integration. Less intuitive, but more appropriate in formulating initial value problems than other definitions of fractional derivatives is the Caputo proposal given by

\[ D^\alpha f(t) = I^{m-\alpha}D^m f = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau) \, d\tau, \]

for \( m - 1 < \alpha < m, \ m \in \mathbb{N} \) and where \( f^{(m)} \) denotes the standard derivative of \( f \). For a systematic presentation of fractional calculus the reader is referred to \([14]\).

Appendix A: Fractional calculus: a simple introduction.

The question of whether it is possible to take half the derivative of a function goes back to a letter written by Leibniz to L'Hôpital in 1665. Since then the subject of fractional calculus, as it is now known, has attracted the interest of mathematicians like Abel, Liouville and Riemann, who developed the basis of the theory. In recent years, fractional derivatives have proved to be useful in many applications ranging from anomalous diffusion, finance to modeling of viscoelastic materials. As expected, when considering the order of differentiation to be an integer, the fractional derivative coincides with the standard definition. However, for non-integer values it has an intrinsic non-local character, which explains its applicability in complex phenomena in which long range