Abstract. Starting from Beris-Edwards system for the liquid crystal, we present a rigorous derivation of Ericksen-Leslie system with general Ericksen stress and Leslie stress by using the Hilbert expansion method.

1. Introduction

Liquid crystals are a state of matter that have properties between those of a conventional liquid and those of a solid crystal. One of the most common liquid crystal phases is the nematic. The nematic liquid crystals are composed of rod-like molecules with the long axes of neighboring molecules approximately aligned to one another. There are three different kinds of theories to model the nematic liquid crystals: Doi-Onsager theory, Landau-de Gennes theory and Ericksen-Leslie theory. The first is the molecular kinetic theory, and the later two are the continuum theory. In the spirit of Hilbert sixth problem, it is very important to explore the relationship between these theories.

Ball-Majumdar \cite{1} define a Landau-de Gennes type energy functional in terms of the mean-field Maier-Saupe energy. Majumdar-Zarnescu \cite{14} consider the Oseen-Frank limit of the static Q-tensor model. Their results show that the predictions of the Oseen-Frank theory and the Landau-De Gennes theory agree away from the singularities of the limiting Oseen-Frank global minimizer.

In \cite{11, 5}, Kuzzu-Doi and E-Zhang formally derive the Ericksen-Leslie equation from the Doi-Onsager equations by taking small Deborah number limit. In our recent work \cite{21}, we justify their formal derivation before the first singularity time of the Ericksen-Leslie system. In \cite{9, 23}, a systematical approach was proposed to derive the continuum theory from the molecular kinetic theory in static and dynamic case.

The goal of this work is to present a rigorous derivation from Landau-de Gennes theory to Ericksen-Leslie theory. Let us first give a brief introduction to two theories \cite{3, 4}.

1.1. Landau-de Gennes theory. In this theory, the state of the nematic liquid crystals is described by the macroscopic Q-tensor order parameter, which is a symmetric, traceless $3 \times 3$ matrix. Physically, it can be interpreted as the second-order moment of the orientational distribution function $f$, that is,

\[ Q = \int_{S^2} (m m - \frac{1}{3} I) f dm. \]

When $Q = 0$, the nematic liquid crystal is said to be isotropic. When $Q$ has two equal non-zero eigenvalues, it is said to be uniaxial and $Q$ can be written as

\[ Q = s (n n - \frac{1}{3} I), \quad n \in S^2. \]

When $Q$ has three distinct eigenvalues, it is said to be biaxial and $Q$ can be written as

\[ Q = s (n n - \frac{1}{3} I) + \lambda (n' n' - \frac{1}{3} I), \quad n, n' \in S^2, \quad n \cdot n' = 0. \]
The general Landau-de Gennes energy functional takes the form
\[
F(Q, \nabla Q) = \int_{\mathbb{R}^3} \left\{ \frac{-a}{2} \text{Tr}Q^2 - \frac{b}{3} \text{Tr}Q^3 + \frac{c}{4} \text{Tr}Q^4 \right\} \text{dx} + \frac{1}{2} \left( L_1 |\nabla Q|^2 + L_2 Q_{ij,k}Q_{ik,k} + L_3 Q_{ij,k}Q_{ik,j} + L_4 Q_{ij}Q_{kl,j}Q_{kl,j} \right) \text{dx}. \tag{1.1}
\]

Here \(a, b, c\) are material-dependent and temperature-dependent nonnegative constants and \(L_i (i = 1, 2, 3, 4)\) are material-dependent elastic constants. We refer to [3] [15] for more details.

There are several dynamic Q-tensor models to describe the flow of the nematic liquid crystal, which are either derived from the molecular kinetic theory for the rigid rods by various closure approximations such as [7, 8, 23], or directly derived by variational method such as Beris-Edwards model [2] and Qian-Sheng’s model [19]. In this work, we will use Beris-Edwards model, which takes the form

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \nabla \cdot (\sigma^s + \sigma^a + \sigma^d), \tag{1.2}
\]
\[
\nabla \cdot v = 0, \tag{1.3}
\]
\[
\frac{\partial Q}{\partial t} + v \cdot \nabla Q + Q \cdot \Omega - \Omega \cdot Q = \frac{1}{\Gamma} H + S_Q(D). \tag{1.4}
\]

Here \(v\) is the velocity of the fluid, \(p\) is the pressure, \(\Gamma\) is a collective rotational diffusion constant, \(D = \frac{1}{2}(\nabla v + (\nabla v)^T), \Omega = \frac{1}{2}(\nabla v - (\nabla v)^T)\); \(\sigma^s, \sigma^a\) and \(\sigma^d\) are symmetric viscous stress, anti-symmetric viscous stress and distortion stress respectively defined by

\[
\sigma^s = \eta D - S_Q(H), \quad \sigma^a = Q \cdot H - H \cdot Q, \quad \sigma^d_{ij} = -\frac{\partial F}{\partial Q_{kl,j}} Q_{kl,i},
\]

where \(\eta > 0\) is the viscous coefficient, \(H\) is the molecular field given by

\[
H(Q) = \frac{\delta F}{\delta Q} = -\frac{\partial F_b}{\partial Q} + \partial_i \left( \frac{\partial F_e}{\partial Q_{i,j}} \right),
\]

and \(S_Q(M)\) is defined by

\[
S_Q(M) = \xi \left( M \cdot (Q + \frac{1}{3} I) \right) \left( (Q + \frac{1}{3} I) \cdot M - 2(Q + \frac{1}{3} I)Q : M \right)
\]

for symmetric and traceless matrix \(M\), where \(\xi\) is a constant depending on the molecular details of a given liquid crystal.

We refer to [16] [17] for the well-posedness results of the Q-tensor model.

1.2. **Ericksen-Leslie theory.** The hydrodynamic theory of liquid crystals was established by Ericksen and Leslie in the 1960’s [6] [12]. In this theory, the configuration of the liquid crystals is described by a director field \(n \in S^2\). The general Ericksen-Leslie system takes the form

\[
v_t + v \cdot \nabla v = -\nabla p + \nabla \cdot \sigma, \tag{1.5}
\]
\[
\nabla \cdot v = 0, \tag{1.6}
\]
\[
n \times (h - \gamma_1 N - \gamma_2 D \cdot n) = 0. \tag{1.7}
\]

Here the stress \(\sigma\) is modeled by the phenomenological constitutive relation

\[
\sigma = \sigma^L + \sigma^E,
\]

where \(\sigma^L\) is the viscous (Leslie) stress

\[
\sigma^L = \alpha_1 (nn : D)nn + \alpha_2 nN + \alpha_3 Nn + \alpha_4 D + \alpha_5 nn \cdot D + \alpha_6 D \cdot nn \tag{1.8}
\]
with
\[ N = \mathbf{n}_t + \mathbf{v} \cdot \nabla \mathbf{n} - \Omega \cdot \mathbf{n}. \]
The six constants \( \alpha_1, \cdots, \alpha_6 \) are called the Leslie coefficients. While, \( \sigma^E \) is the elastic (Ericksen) stress
\[
\sigma^E_{ij} = -\frac{\partial E_F}{\partial n_{k,j}} n_{k,i},
\]
where \( E_F = E_F(\mathbf{n}, \nabla \mathbf{n}) \) is the Oseen-Frank energy with the form
\[
E_F = \frac{k_1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{k_2}{2} (\mathbf{n} \cdot (\nabla \times \mathbf{n}))^2 + \frac{k_3}{2} |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \frac{k_2 + k_4}{2} (\text{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2) .
\]
Here \( k_1, k_2, k_3, k_4 \) are the elastic constant. The molecular field \( \mathbf{h} \) is given by
\[
\mathbf{h} = -\frac{\delta E_F}{\delta \nabla \mathbf{n}} = \nabla \cdot \frac{\partial E_F}{\partial (\nabla \mathbf{n})} - \frac{\partial E_F}{\partial \nabla \mathbf{n}}.
\]
Finally, the Leslie coefficients and \( \gamma_1, \gamma_2 \) satisfy the following relations
\[
\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \\
\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5,
\]
where (1.11) is called Parodi’s relation derived from the Onsager reciprocal relation \[18]. These two relations will ensure that the system (1.5)–(1.7) has a basic energy law:
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} \frac{1}{2} |\mathbf{v}|^2 \mathbf{n} + E_F \right) = \int_{\mathbb{R}^3} \left( (\alpha_1 + \frac{\gamma_2^2}{\gamma_1}) (\mathbf{D} : \mathbf{n})^2 + \alpha_4 |\mathbf{D}|^2 \\
+ (\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1}) |\mathbf{D} \cdot \mathbf{n}|^2 + \frac{1}{\gamma_1} |\mathbf{n} \times \mathbf{h}|^2 \right) \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v} \mathbf{n} + E_F \right) = \int_{\mathbb{R}^3} \left( (\alpha_1 + \frac{\gamma_2^2}{\gamma_1}) (\mathbf{D} : \mathbf{n})^2 + \alpha_4 |\mathbf{D}|^2 \\
+ (\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1}) |\mathbf{D} \cdot \mathbf{n}|^2 + \frac{1}{\gamma_1} |\mathbf{n} \times \mathbf{h}|^2 \right) \mathbf{D} \cdot \mathbf{n} \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v} \mathbf{n} + E_F \right).
\]

We refer to \[13, 22\] for the well-posedness results of the Ericksen-Leslie system. In \[22\], we proved the well-posedness of the system under a natural physical condition on the Leslie coefficients, and in \[10, 20\], the authors proved the global existence of weak solution in 2-D case.

1.3. Main result: from Beris-Edwards system to Ericksen-Leslie system. Since the elastic constants \( L_i (i = 1, 2, 3, 4) \) are typically very small compared with \( a, b, c \), we introduce a small parameter \( \varepsilon \) and consider the following Landau-de Gennes energy functional
\[
\mathcal{F}_\varepsilon(Q, \nabla Q) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \left\{ -\frac{a}{2} \text{Tr} Q^2 + \frac{b}{3} \text{Tr} Q^3 + \frac{c}{4} \text{Tr} Q^4 \right\} \mathcal{F}_\varepsilon(Q) + \int_{\mathbb{R}^3} \left\{ L_1 |\nabla Q|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ij} Q_{kl,i} Q_{kl,j} \right\} \mathcal{F}_{\text{c}(Q)} \right). \]

In the case when \( L_4 \neq 0 \), the term \( Q_{ij} Q_{kl,i} Q_{kl,j} \) may cause the energy to be not bounded from below \[1\]. Therefore, we only consider the case \( L_4 = 0 \). Furthermore, we assume
\[
L_1 > 0, \quad L_1 + L_2 + L_3 > 0.
\]
which will ensure that the elastic energy is strictly positive (see Lemma \[2.2\]).

We introduce two operators
\[
\mathcal{J}(Q) \overset{\text{def}}{=} \frac{\delta F_b(Q)}{\delta Q} = -a Q - b |Q|^2 + c Q |Q|^3 + \frac{1}{3} b |Q|^2 I, \\
(\mathcal{L}(Q))_{kl} \overset{\text{def}}{=} -\partial_i \left( \frac{\partial F_c}{\partial Q_{kl,i}} \right) = -(L_1 \Delta Q_{kl} + \frac{1}{2} (L_2 + L_3) (Q_{km,ml} + Q_{lm,km} - \frac{2}{3} \delta_{kl} Q_{ij,ij})),
\]
and define the tensor $\sigma^d(Q, \tilde{Q})$ as

$$\\sigma^d_{ji}(Q, \tilde{Q}) \overset{\text{def}}{=} - \frac{\partial F^e_2}{\partial Q_{kl,j}} \tilde{Q}_{kl,i} = -(L_1Q_{kl,j}\tilde{Q}_{kl,i} + L_2Q_{km,m}\tilde{Q}_{kj,i} + L_3Q_{kij}\tilde{Q}_{kl,i}).$$

So, the molecular field and distortion stress can be written as

$$H_\varepsilon(Q) = -\frac{1}{\varepsilon} J(Q) - L(Q), \quad \sigma^d = \sigma^d(Q, Q).$$

We study the Beris-Edwards system with a small parameter $\varepsilon$:

$$\frac{\partial v^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon = -\nabla p^\varepsilon + \nabla \cdot (\sigma^\varepsilon_\varepsilon + \sigma^a_\varepsilon + \sigma^d_\varepsilon),$$

$$\nabla \cdot \mathbf{v}^\varepsilon = 0,$$

$$\frac{\partial Q^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon \cdot \nabla Q^\varepsilon + Q^\varepsilon \cdot \mathbf{Q}^\varepsilon - \mathbf{Q}^\varepsilon \cdot \mathbf{Q}^\varepsilon = \frac{1}{\varepsilon} \mathbf{H}_\varepsilon + S Q^\varepsilon (D^\varepsilon),$$

where $D^\varepsilon = \frac{1}{2}(\nabla v^\varepsilon + (\nabla v^\varepsilon)^T)$, $\mathbf{Q}^\varepsilon = \frac{1}{2}(\nabla v^\varepsilon - (\nabla v^\varepsilon)^T)$, and

$$\sigma^\varepsilon_\varepsilon = \eta D^\varepsilon - S Q^\varepsilon (H_\varepsilon), \quad \sigma^a_\varepsilon = Q^\varepsilon \cdot H_\varepsilon - H_\varepsilon \cdot Q^\varepsilon, \quad \sigma^d_\varepsilon = \sigma^d(Q^\varepsilon, Q^\varepsilon).$$

Our main result is stated as follows.

**Theorem 1.1.** Let $(n(t, x), \mathbf{v}(t, x))$ be a solution of the Ericksen-Leslie system (1.5)–(1.7) on $[0, T]$ with the coefficients $k_i(i = 1, 2, 3, 4)$ and $\alpha_i(i = 1, \cdots, 6)$ given by (3.15)–(3.16), which satisfies

$$v \in C([0, T]; H^k), \quad \nabla n \in C([0, T]; H^k) \quad \text{for} \quad k \geq 20.$$

Let $Q_0(t, x) = \delta(t) \mathbf{n}(t, x) \mathbf{n}(t, x) - I$ with $s = \frac{b + \sqrt{b^2 + 4ac}}{2c}$, and the functions $(Q_1, Q_2, Q_3, v_1, v_2)$ are determined by Proposition 3.8. Assume that the initial data $(Q_0^\varepsilon, v_0^\varepsilon)$ takes the form

$$Q_0^\varepsilon(x) = Q_0(0, x) + \varepsilon Q_1(0, x) + \varepsilon^2 Q_2(0, x) + \varepsilon^3 Q_3(0, x) + \varepsilon^3 Q_{0R}(x),$$

$$v_0^\varepsilon(x) = v_0(0, x) + \varepsilon v_1(0, x) + \varepsilon^2 v_2(0, x) + \varepsilon^3 v_{0R}(x),$$

where $(Q_{0R}, v_{0R})$ satisfies

$$\|v_{0R}\|_{H^2} + \|Q_{0R}\|_{H^3} + \varepsilon^{-1} \|\mathcal{P}^{\text{out}}(Q_{0R})\|_{L^2} \leq E_0.$$

Then there exists $\varepsilon_0 > 0$ and $E_1 > 0$ such that for all $\varepsilon < \varepsilon_0$, the system (1.16)–(1.18) has a unique solution $(Q^\varepsilon(t, x), v^\varepsilon(t, x))$ on $[0, T]$ which has the expansion

$$Q^\varepsilon(t, x) = Q_0(t, x) + \varepsilon Q_1(t, x) + \varepsilon^2 Q_2(t, x) + \varepsilon^3 Q_3(t, x) + \varepsilon^3 Q_{R}(t, x),$$

$$v^\varepsilon(t, x) = v_0(t, x) + \varepsilon v_1(t, x) + \varepsilon^2 v_2(t, x) + \varepsilon^3 v_{R}(t, x),$$

where $(Q_{R}, v_{R})$ satisfies

$$\mathcal{E}(Q_{R}^\varepsilon(t), v_{R}^\varepsilon(t)) \leq E_1.$$
Remark 1.1. It is known [22] that the energy (1.13) is dissipated or equivalently
\[
\beta_1(nn : D)^2 + \beta_2 D : D + \beta_3 |D \cdot n|^2 > 0
\]  
for any non-zero symmetric traceless matrix $D$ and unit vector $n$, if and only if
\[
\beta_2 > 0, \quad 2\beta_2 + \beta_3 > 0, \quad \frac{3}{2}\beta_2 + \beta_3 + \beta_1 > 0,
\]
where
\[
\beta_1 = \alpha_1 + \frac{\gamma_2^2}{\gamma_1}, \quad \beta_2 = \alpha_4, \quad \beta_3 = \alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1}.
\]
In [22], we proved the well-posedness of the system (1.3)–(1.7) under the condition (1.21) and in the case when $k_1 = k_2 = k_3$ and $k_4 = 0$. Wang-Wang [20] generalize our result to the case with general Oseen-Frank energy under the condition
\[
\min(k_1, k_2, k_3) > 0.
\]
By Remark 3.2, the energy for the Ericksen-Leslie system derived from the Beris-Edwards system is dissipated, and by (3.15) and (1.15), $\min(k_1, k_2, k_3) > 0$. Thus, it is well-posed.

Remark 1.2. The same result should be true for Qian-Sheng’s model in [19].

Let us conclude this section by presenting a sketch of the proof.

The first step is to make a formal expansion for the solution $(v^\varepsilon, Q^\varepsilon)$:
\[
Q^\varepsilon(t, x) = Q_0(t, x) + \varepsilon Q_1(t, x) + \varepsilon^2 Q_2(t, x) + \varepsilon^3 Q_3(t, x) + \varepsilon^3 Q_R(t, x),
\]
\[
v^\varepsilon(t, x) = v_0(t, x) + \varepsilon v_1(t, x) + \varepsilon^2 v_2(t, x) + \varepsilon^3 v_R(t, x).
\]
We find that $\mathcal{J}(Q_0) = 0$, and Proposition 2.1 ensures
\[
Q_0 = s(nn - \frac{1}{3}I),
\]
for some $n \in S^2$ and $s = \frac{b + \sqrt{b^2 + 24ac}}{4c}$. By studying the kernel of the linearized operator $\mathcal{H}_n$, it can be proved that $(v_0, n)$ is a solution of the Ericksen-Leslie system. The existence of $(Q_i, v_i)$ for $i \neq 0$ is also nontrivial, since they satisfy a system with the complicated dissipation relation.

The most difficult step is to show that the remainder $(v_R, Q_R)$ is uniformly bounded in $\varepsilon$, which satisfies (dropping good error terms)
\[
\frac{\partial v_R}{\partial t} = -\nabla p_R + \eta \Delta v_R + \nabla \cdot \left( \frac{1}{\varepsilon} S_{Q_0}(H_R) + \frac{1}{\varepsilon} Q_0 \cdot H_R + \frac{1}{\varepsilon} H_R \cdot Q_0 \right),
\]
\[
\frac{\partial Q_R}{\partial t} = -\frac{1}{\varepsilon} \mathcal{H}_n(Q_R) + S_{Q_0} D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R.
\]
This is a system with the singular terms of order $\frac{1}{\varepsilon}$. To deal with them, we introduce a key energy functional $\mathcal{E}$ defined by (1.19). Then we prove that $\mathcal{E}$ is uniformly bounded by the energy method, where main difficulty is to control the terms like
\[
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q_R, Q_R \rangle.
\]
A rough estimate gives
\[
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q_R, Q_R \rangle \leq C \varepsilon^{-1} \| Q_R \|_{L^2}^2 \leq C \varepsilon^{-1} \mathcal{E},
\]
which is obviously unacceptable. Surprisingly, it can be proved that for any $\delta > 0$
\[
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q_R, Q_R \rangle \leq C \delta \mathcal{E} + \delta \delta,
\]
where $\mathcal{F}$ is the dissipation part in the energy estimates. The proof relies on the fact that the linearized operator $\mathcal{H}_n$ is an 1-1 map outside its kernel, and its inverse $\mathcal{H}_n^{-1}$ can be explicitly given (see Proposition 2.3).

**Notations.** For any two vectors $m = (m_1, m_2, m_3), n = (n_1, n_2, n_3) \in \mathbb{R}^3$, we denote the tensor product by $m \otimes n = [m_i n_j]_{1 \leq i, j \leq 3}$. In the sequel, we use $mn$ to denote $m \otimes n$ for simplicity when no ambiguity is possible. $A \cdot B$ denotes the usual matrix/vector-matrix/vector product. $A : B$ denotes $\text{Tr}(AB) = A_{ij} B_{ji}$. The divergence of a tensor is defined by $\nabla \cdot \sigma = \partial_j \sigma_{ij}$. We also use $f_{ij}$ to denote $\partial_t f$ for simplicity.

2. Critical Points and the Linearized Operator

2.1. Critical Points of $F_b(\sigma)$. We say that a matrix $Q_0$ is a critical point of $F_b(\sigma)$ if $\mathcal{J}(Q_0) = 0$. We have the following characterization for critical points (see also [1] and references therein).

**Proposition 2.1.** $\mathcal{J}(Q) = 0$ if and only if

$$Q = s(nn - \frac{1}{3} I),$$

for some $n \in S^2$ and $s = 0$ or is a solution of $2cs^2 - bs - 3a = 0$, that is,

$$s_{1,2} = \frac{b \pm \sqrt{b^2 + 24ac}}{4c}.$$

**Proof.** Since $Q$ is symmetric and traceless, we may write

$$Q = \lambda_1 n_1 \otimes n_1 + \lambda_2 n_2 \otimes n_2 + \lambda_3 n_3 \otimes n_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and $n_1, n_2, n_3$ are the corresponding eigenvectors satisfying $n_i \cdot n_j = \delta_{ij}$. A direct computation gives

$$\mathcal{J}(Q) = \frac{b}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) I + \sum_{i=1}^{3} \left( - a \lambda_i - b \lambda_i^2 + c (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \lambda_i \right) n_i \otimes n_i.$$

So, $\mathcal{J}(Q) = 0$ if and only if $\mathcal{J}(Q) \cdot n_i = 0$ for $i = 1, 2, 3$, which is equivalent to

$$\rho_i \triangleq \frac{b}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - a \lambda_i - b \lambda_i^2 + c (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \lambda_i = 0 \quad \text{for } i = 1, 2, 3.$$

If $\lambda_i$ are all equal, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$, hence $Q = 0$. If not, we may assume $\lambda_1 \neq \lambda_2$ without loss of generality. Due to $\lambda_1 + \lambda_2 + \lambda_3 = 0$, we get

$$\rho_1 = \frac{b}{3} \left( \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2 \right) - a \lambda_1 - b \lambda_1^2 + c \left( \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2 \right) \lambda_1 = 0,$$

$$\rho_2 = \frac{b}{3} \left( \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2 \right) - a \lambda_2 - b \lambda_2^2 + c \left( \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2 \right) \lambda_2 = 0.$$

Let $r = \lambda_1 + \lambda_2, R = \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2$. From the fact $\rho_1 - \rho_2 = 0$, we infer

$$cR = a + br,$$

and from $\rho_1 + \rho_2 = 0$, we infer

$$\frac{2}{3} bR - ar - b(R - r^2) + cRr = 0,$$

which imply $\frac{b}{3} R = 2br^2$ or $b(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2) = 0$. Without loss of generality, we assume $\lambda_1 = -2\lambda_2$, then $\lambda_3 = \lambda_2$. Then using the identity $n_1 \otimes n_1 + n_2 \otimes n_2 + n_3 \otimes n_3 = I$, we get

$$Q = s(nn - \frac{1}{3} I),$$

where $s = -3\lambda_2$. We know from [2.1] that $s$ satisfies $2cs^2 - bs - 3a = 0$. $\square$
2.2. The linearized operator of $\mathcal{J}$. Given a critical point $Q_0 = s(nn - \frac{1}{3}I)$, the linearized operator $\mathcal{H}_{Q_0}$ of $\mathcal{J}$ around $Q_0$ is given by

$$\mathcal{H}_{Q_0}(Q) = aQ - b(Q_0 \cdot Q + Q \cdot Q_0 - \frac{2}{3}(Q_0 : Q)I) + c(|Q_0|^2Q + 2(Q_0 : Q)Q_0).$$

Putting $Q_0 = s(nn - \frac{1}{3}I)$ into the above formula and using the equation $2cs^2 - bs - 3a = 0$, we find that

$$\mathcal{H}_{Q_0}(Q) = bs(Q - (nn \cdot Q + Q \cdot nn) + \frac{2}{3}(Q : nn)I) + 2cs^2(Q : nn)(nn - \frac{1}{3}I). \quad (2.2)$$

In the sequel, we denote $\mathcal{H}_{Q_0}$ by $\mathcal{H}_{s,n}$ for the simplicity.

We denote by $\mathcal{Q}$ the Hilbert space of symmetric traceless matrix with the following inner product:

$$\langle Q^1, Q^2 \rangle \defeq \text{Tr}(Q^1Q^2),$$

which is a five-dimensional space. For a given $n \in S^2$, we define a two-dimensional space $\mathcal{Q}^{in}_n$ as

$$\mathcal{Q}^{in}_n \defeq \{ n \otimes n^\perp + n^\perp \otimes n \in \mathcal{Q} : n^\perp \in \mathcal{V}_n \}, \quad (2.3)$$

where $\mathcal{V}_n \defeq \{ n^\perp \in \mathbb{R}^3 : n^\perp \cdot n = 0 \}$. Let $\mathcal{Q}^{out}_n$ be the orthogonal complement of $\mathcal{Q}^{in}_n$ in $\mathcal{Q}$. We denote by $\mathcal{P}^{in}$ the projection operator from $\mathcal{Q}$ to $\mathcal{Q}^{in}_n$ and by $\mathcal{P}^{out}$ the projection operator from $\mathcal{Q}$ to $\mathcal{Q}^{out}_n$. Note that

$$|Q - (nn^\perp + n^\perp n)|^2 = |n^\perp - (I - nn)Q \cdot n|^2 + |Q|^2 - 2|Q \cdot n|^2 + 2(Q : nn),$$

which means that the left hand side attains minimum when $n^\perp = (I - nn)Q \cdot n$. Hence,

$$\mathcal{P}^{in}(Q) = n[(I - nn) \cdot Q \cdot n] + [(I - nn) \cdot Q \cdot n]n = (nn \cdot Q + Q \cdot nn) - 2(Q : nn)nn. \quad (2.4)$$

Moreover,

$$|\mathcal{P}^{in}(Q)|^2 = 2|Q \cdot n|^2 - 2(Q : nn)^2. \quad (2.5)$$

**Proposition 2.2.** Let $s = \frac{b + \sqrt{b^2 + 24ac}}{4c}$. Then for any $n \in S^2$, it holds that

- $\mathcal{H}_{s,n} : \mathcal{Q} \rightarrow \mathcal{Q}^{out}_n$, hence $\mathcal{H}_{s,n}\mathcal{Q}^{in}_n = 0$;
- There exists $c_0 = c_0(a,b,c) > 0$ such that for any $Q \in \mathcal{Q}^{out}_n$,

$$\mathcal{H}_{s,n}Q : Q \geq c_0|Q|^2.$$

**Proof.** It is easy to see that

$$A \defeq Q - (nn \cdot Q + Q \cdot nn) + \frac{2}{3}(Q : nn)I, \quad B \defeq (Q : nn)(nn - \frac{1}{3}I) \in \mathcal{Q}^{out}_n. \quad (2.6)$$

This gives the first point.

Take $c_0 = \min\{bs, 2cs^2 - bs\} > 0$. Then for any $Q \in \mathcal{Q}$, we have

$$\mathcal{H}_{s,n}(Q) : Q = bs(|Q|^2 - 2|Q \cdot n|^2 + (Q : nn)^2) + (2cs^2 - bs)(Q : nn)^2$$

$$\geq c_0(|Q|^2 - 2|Q \cdot n|^2 + 2(Q : nn)^2).$$

We infer from [25] that for $Q \in \mathcal{Q}^{out}_n$, we have $2|Q \cdot n|^2 - 2(Q : nn)^2 = 0$, hence,

$$\mathcal{H}_{s,n}(Q) : Q \geq c_0|Q|^2.$$

This proves the second point. □
Proposition 2.3. Let \( s = \frac{b + \sqrt{b^2 + 24ac}}{4c} \). Then \( \mathcal{H}_{s,n} \) is an 1-1 map on \( \mathbb{Q}^\text{out}_{n} \) and its inverse \( \mathcal{H}_{s,n}^{-1} \) is given by

\[
\mathcal{H}_{s,n}^{-1}(Q) = \frac{1}{bs} \left( Q - (nn \cdot Q + Q \cdot nn) + \frac{2}{3} (Q : nn)I \right) + \frac{4b + 2cs}{bs(4cs - b)} (Q : nn)(nn - \frac{1}{3}I).
\]

Proof. A direct computation gives

\[
\mathcal{H}_{s,n}(Q) : nn = -\frac{1}{3}(bs - 4cs^2)(Q : nn).
\]

From this fact, we deduce

\[
\mathcal{H}_{s,n}^{-1} \mathcal{H}_{s,n}(Q) = \frac{1}{bs} \left( \mathcal{H}_{s,n}Q - (nn \cdot \mathcal{H}_{s,n}Q + \mathcal{H}_{s,n}Q \cdot nn) + \frac{2}{3}(\mathcal{H}_{s,n}Q : nn)I \right)
\]

\[
+ \frac{4b + 2cs}{bs(4cs - b)} (\mathcal{H}_{s,n}Q : nn)(nn - \frac{1}{3}I)
\]

\[
= \left( Q - (nn \cdot Q + Q \cdot nn) + \frac{2}{3} (Q : nn)I \right) + \frac{2cs^2}{bs} (Q : nn)(nn - \frac{1}{3}I)
\]

\[
+ \frac{2}{bs} \left( \frac{1}{3} bs nn(Q : nn) - \frac{4}{3} cs^2(Q : nn)nn \right) - \frac{2}{9} \frac{1}{bs} (bs - 4cs^2)(Q : nn)I
\]

\[
- \frac{1}{3} \frac{4b + 2cs}{bs(4cs - b)} (bs - 4cs^2)(Q : nn)(nn - \frac{1}{3}I)
\]

\[
= \left( Q - (nn \cdot Q + Q \cdot nn) + 2(Q : nn)nn \right)
\]

\[
+ \left( - \frac{2cs^2}{3bs} - \frac{4}{3} - \frac{1}{3} \frac{4b + 2cs}{bs(4cs - b)} (bs - 4cs^2) \right) (Q : nn)(nn - \frac{1}{3}I)
\]

\[
= Q - (nn \cdot Q + Q \cdot nn) + 2(Q : nn)nn.
\]

Then it follows from (2.4) that for \( Q \in \mathbb{Q}^\text{out}_{n} \),

\[
\mathcal{H}_{s,n}^{-1} \mathcal{H}_{s,n}(Q) = Q.
\]

That means that \( \mathcal{H}_{s,n} \) is an 1-1 map on \( \mathbb{Q}^\text{out}_{n} \). \( \square \)

Remark 2.1. The construction of \( \mathcal{H}_{s,n}^{-1} \) is motivated by the fact (2.6). So, we hope that \( \mathcal{H}_{s,n}^{-1} \) has the form

\[
\mathcal{H}_{s,n}^{-1}(Q) = \alpha A + \beta B.
\]

Fortunately, we can choose suitable \( \alpha, \beta \) such that

\[
\mathcal{H}_{s,n}^{-1} \mathcal{H}_{s,n}(Q) = Q - (nn \cdot Q + Q \cdot nn) + 2(Q : nn)nn.
\]

Lemma 2.2. Under the assumption (1.15), there exists a positive constant \( L_0 \) depending only on \( L_1, L_2, L_3 \) such that

\[
\int_{\mathbb{R}^3} \mathcal{L}(Q) : Q dx \geq L_0 \| \nabla Q \|^2_{L2}.
\]
Proof. Let $Q_i = (Q_{i1}, Q_{i2}, Q_{i3})$. By integration by parts, we get

$$\int_{\mathbb{R}^3} \mathcal{L}(Q) : Q \, dx = \int_{\mathbb{R}^3} L_1 |\nabla Q|^2 + L_2 Q_{ki,i} Q_{kj,j} + L_3 Q_{ki,j} Q_{kj,i} \, dx$$

$$= \sum_{i=1}^3 \int_{\mathbb{R}^3} L_1 |\nabla Q_i|^2 + (L_2 + L_3) |\nabla \cdot Q_i|^2 \, dx$$

$$= \sum_{i=1}^3 \int_{\mathbb{R}^3} L_1 |\nabla \times Q_i|^2 + (L_1 + L_2 + L_3) |\nabla \cdot Q_i|^2 \, dx$$

$$\geq \min(L_1, L_1 + L_2 + L_3) \int_{\mathbb{R}^3} |\nabla Q|^2 \, dx.$$ 

This gives the lemma by taking $L_0 = \min(L_1, L_1 + L_2 + L_3)$. \hfill $\square$

Lemma 2.3. If $Q_i \in \mathcal{Q}_n^\text{in}(i = 1, 2, 3)$, then it holds that

$$\text{Tr}(Q_1 \cdot Q_2 \cdot Q_3) = 0.$$ 

Especially, if $Q_1, Q_2 \in \mathcal{Q}_n^\text{in}$, then $Q_1 \cdot Q_2 \in \mathcal{Q}_n^\text{out}$.

Proof. Assume that $Q_i = n_i n_i + n_i n$, where $n_i \cdot n = 0(i = 1, 2, 3)$. Then we have

$$\text{Tr}(Q_1 \cdot Q_2 \cdot Q_3) = \text{Tr} \left( (n_1 n_2 + n n(n_1 \cdot n_2)) \cdot (n n_3 + n_3 n) \right)$$

$$= (n_2 \cdot n_3)\text{Tr}(n_1 n) + (n_1 \cdot n_2)\text{Tr}(n n_3) = 0.$$

The second statement is obvious. \hfill $\square$

3. Hilbert Expansion

3.1. Hilbert expansion. Let $(v^\varepsilon, Q^\varepsilon)$ be the solution of (1.16)–(1.18). We perform the following so-called Hilbert expansion

$$v^\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3,$$  \hspace{1cm} (3.1)

$$Q^\varepsilon = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^3 Q_4,$$  \hspace{1cm} (3.2)

where $Q_i \in \mathcal{Q}(i = 0, 1, 2, 3)$ will be determined in what follows.

Let us first make some preliminaries. For $Q_i \in \mathbb{M}^{3 \times 3}(i = 1, 2, 3)$, we denote

$$B(Q_1, Q_2) = Q_1 \cdot Q_2 + Q_2^T \cdot Q_1^T - \frac{2}{3} I(\text{tr}(Q_1 : Q_2)),$$

$$C(Q_1, Q_2, Q_3) = Q_1(\text{tr}(Q_2 : Q_3) + Q_2(\text{tr}(Q_1 : Q_3) + Q_3(\text{tr}(Q_1 : Q_2)).$$

It is easy to see that

Lemma 3.1. For any $Q, \tilde{Q} \in \mathcal{Q}$, it holds that

$$\mathcal{J}(Q) = aQ - \frac{b}{2} B(Q, Q) + \frac{c}{3} C(Q, Q, Q),$$

$$\mathcal{H}_Q(\bar{Q}) = a\bar{Q} - bB(Q, \bar{Q}) + cC(Q, Q, \bar{Q}).$$
It follows from Lemma 3.1 that

\[ J(Q^\varepsilon) = J(Q_0) + \varepsilon \mathcal{H}_{Q_0}(Q_1) + a(\varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^3 Q_R) \]

\[ - \frac{b}{2} \sum_{m=0}^{6} \varepsilon^m \sum_{i+j=m, 1 \leq i, j \leq 3} B(Q_i, Q_j) \]

\[ - b \varepsilon^3 B(Q_0) + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3, Q_R) - \frac{b}{2} \varepsilon^6 B(Q_R, Q_R) \]

\[ + \frac{c}{3} \sum_{m=0}^{9} \varepsilon^m \sum_{i+j+k=m, \text{ at least two of } i, j, k \text{ are not zero}} C(Q_i, Q_j, Q_k) \]

\[ + \frac{c}{2} \varepsilon^3 \sum_{i,j=0}^{3} \varepsilon^{i+j} C(Q_R, Q_i, Q_j) \]

\[ + c \varepsilon^6 C(Q_R, Q_R, Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3) \]

\[ + c \varepsilon^9 C(Q_R, Q_R, Q_R). \]

Let \( Q^\varepsilon = Q_1 + \varepsilon Q_2 + \varepsilon^2 Q_3 \). We introduce the notations:

\[ B_1 = -\frac{b}{2} B(Q_1, Q_1) + c C(Q_0, Q_1, Q_1), \]

\[ B_2 = -b B(Q_1, Q_2) + 2c C(Q_0, Q_1, Q_2), \]

\[ B^\varepsilon = -\frac{b}{2} \sum_{i+j \geq 4, 1 \leq i, j \leq 3} \varepsilon^{i+j-4} B(Q_i, Q_j) \]

\[ + \frac{c}{3} \sum_{i+j+k \geq 4, \text{ at least two of } i, j, k \text{ are not zero}} \varepsilon^{i+j+k-4} C(Q_i, Q_j, Q_k). \]

Then we obtain the following expansion of \( J(Q^\varepsilon) \) in \( \varepsilon \):

\[ J(Q^\varepsilon) = J(Q_0) + \varepsilon \mathcal{H}_{Q_0}(Q_1) + \varepsilon^2 (\mathcal{H}_{Q_0}(Q_2) + B_1) + \varepsilon^3 (\mathcal{H}_{Q_0}(Q_3) + B_2) \]

\[ + \varepsilon^4 \mathcal{H}_{Q_0}(Q_R) + \varepsilon^4 J_R, \]

where

\[ J_R^\varepsilon = B^\varepsilon - b B(\bar{Q}^\varepsilon, Q_R) + c C(Q_R, \bar{Q}, Q_0) + \frac{c}{2} \varepsilon C(Q_R, \bar{Q}, \bar{Q}) \]

\[ - \frac{b}{2} \varepsilon^2 B(Q_R, Q_R) + \varepsilon \varepsilon^2 C(Q_R, Q_R, Q_0 + \varepsilon \bar{Q}^\varepsilon) + \varepsilon \varepsilon^5 C(Q_R, Q_R, Q_R). \]

We denote

\[ H_0 = \mathcal{H}_{Q_0}(Q_1) + \mathcal{L}(Q_0), \]

\[ H_1 = \mathcal{L}(Q_1) + \mathcal{H}_{Q_0}(Q_2) + B_1, \]

\[ H_2 = \mathcal{L}(Q_2) + \mathcal{H}_{Q_0}(Q_3) + B_2, \]

\[ D_i = \frac{1}{2}((\nabla v_i)^T + \nabla v_i), \quad \Omega_i = \frac{1}{2}(\nabla v_i - (\nabla v_i)^T). \]

Plugging the expansions (3.1)–(3.2) and (3.3) into (1.16)–(1.18), we conclude that

- The order \( O(\varepsilon^{-1}) \) system

\[ J(Q_0) = 0. \]
• The order $O(1)$ system

\[
\begin{align*}
\frac{\partial v_0}{\partial t} + v_0 \cdot \nabla v_0 &= -\nabla p_0 + \nabla \cdot (\eta D_0 + S_{Q_0}(H_0) - Q_0 \cdot H_0 + H_0 \cdot Q_0 + \sigma^d(Q_0, Q_0)), \\
\nabla \cdot v_0 &= 0,
\end{align*}
\] (3.5)

\[
\begin{align*}
\frac{\partial Q_0}{\partial t} + v_0 \cdot \nabla Q_0 + Q_0 \cdot \Omega_0 - \Omega_0 \cdot Q_0 &= -\frac{1}{\Gamma}(H_{Q_0}(Q_1) + L(Q_0)) + S_{Q_0}(D_0),
\end{align*}
\] (3.6)

• The order $O(\varepsilon)$ system

\[
\begin{align*}
\frac{\partial v_1}{\partial t} + v_0 \cdot \nabla v_1 &= -v_1 \cdot \nabla v_0 - \nabla p_1 + \nabla \cdot \left(\eta D_1 + S_{Q_0}(H_1) + \xi(B(Q_1, H_0) - 2Q_1(H_0 : Q_0) - 2Q_0(H_0 : Q_1)) - Q_1 \cdot H_0
\end{align*}
\] (3.7)

\[
\begin{align*}
\nabla \cdot v_1 &= 0,
\end{align*}
\] (3.8)

\[
\begin{align*}
\frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 &= -\frac{1}{\Gamma}(L(Q_1) + H_{Q_0}(Q_2) + B_1) + S_{Q_0}D_1 + \xi(B(D_0, Q_1)
\end{align*}
\] (3.9)

\[
\begin{align*}
- Q_1(Q_0 : D_0) - 2Q_0(Q_1 : D_0)) + \Omega_1 \cdot Q_0 + \Omega_0 \cdot Q_1
\end{align*}
\]

• The order $O(\varepsilon^2)$ system

\[
\begin{align*}
\frac{\partial v_2}{\partial t} + v_0 \cdot \nabla v_2 &= -v_2 \cdot \nabla v_0 - v_1 \cdot \nabla v_1 - \nabla p_2 + \nabla \cdot \left(\eta D_2 + S_{Q_0}(H_2) + \xi(B(Q_1, H_1)
\end{align*}
\] (3.10)

\[
\begin{align*}
+ B(Q_2, H_0) - 2Q_1(H_1 : Q_0) - 2Q_2(H_1 : Q_0) - 2Q_0(H_1 : Q_1)
\end{align*}
\]

\[
\begin{align*}
- 2Q_1(H_0 : Q_1)) - Q_0 \cdot H_2 + H_2 \cdot Q_0 - Q_1 \cdot H_1 - Q_2 \cdot H_0
\end{align*}
\] (3.11)

\[
\begin{align*}
+ H_1 \cdot Q_1 + H_0 \cdot Q_2 + \sigma^d(Q_2, Q_0) + \sigma^d(Q_1, Q_1) + \sigma^d(Q_0, Q_2))
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot v_2 &= 0,
\end{align*}
\] (3.12)

\[
\begin{align*}
\frac{\partial Q_2}{\partial t} + v_0 \cdot \nabla Q_2 &= -\frac{1}{\Gamma}(L(Q_2) + H_{Q_0}(Q_3) + B_2) + S_{Q_0}(D_2) + \xi(B(D_0, Q_2) + B(D_1, Q_1)
\end{align*}
\] (3.13)

\[
\begin{align*}
- 2Q_2(Q_0 : D_0) - 2Q_1(Q_1 : D_0 + D_1 : Q_0) - 2Q_0(Q_2 : D_0 + Q_1 \cdot D_1))
\end{align*}
\]

\[
\begin{align*}
+ \Omega_2 \cdot Q_0 + \Omega_0 \cdot Q_2 + \Omega_1 \cdot Q_1 - Q_0 \cdot \Omega_2 - Q_2 \cdot \Omega_0 - Q_1 \cdot \Omega_1
\end{align*}
\]

\[
\begin{align*}
- v_2 \cdot \nabla Q_0 - v_1 \cdot \nabla Q_1.
\end{align*}
\] (3.14)

3.2. Derivation of the Ericksen-Leslie system. Thanks to $J(Q_0) = 0$ and Proposition 2.1, $Q_0(t, x)$ takes the form

\[
Q_0(t, x) = s \left(n(t, x)n(t, x) - \frac{1}{3}I\right),
\] (3.15)

for some $n(t, x) \in S^2$ and we take $s = \frac{b + \sqrt{b^2 + 2ac}}{4c}$. For the sake of simplicity, we denote $H_{Q_0}$ by $H_n$ in the sequel. We will prove

**Proposition 3.1.** If $(v_0, Q_0)$ is a smooth solution of the system (3.5)–(3.7), then $(n, v_0)$ is necessary a solution of the Ericksen-Leslie system (1.3)–(1.7) with the elastic constants given by

\[
k_1 = k_3 = (2L_1 + L_2 + L_3)s^2, \quad k_2 = 2L_1s^2, \quad k_4 = L_3s^2,
\] (3.15)
and the Leslie coefficients given by
\[
\begin{align*}
\gamma_1 &= 2\Gamma s^2, \quad \gamma_2 = -\frac{2\Gamma_1 s(s + 2)}{3}, \\
\alpha_1 &= -\frac{2\Gamma_1 s^2(3 - 2s)(1 + 2s)}{3}, \quad \alpha_2 = \Gamma s^2 - \frac{\Gamma_1 s(2 + s)}{3}, \quad \alpha_3 = -\Gamma s^2 - \frac{\Gamma_1 s(2 + s)}{3}, \\
\alpha_4 &= \eta + \frac{4\Gamma_1 s^2}{9}, \quad \alpha_5 = \frac{\Gamma_1 s^2}{3} - \frac{\Gamma_1 s}{3}, \quad \alpha_6 = \frac{\Gamma_1 s^2}{3} + \Gamma s^2.
\end{align*}
\] (3.16)

Remark 3.2. The constants \(k_i\) can also be obtained by computing \(F_e(Q_0)\). Furthermore, it is easy to find that the Leslie coefficients satisfy the Parodi’s relation (1.11) and (1.12). On the other hand, it can be verified that \(\alpha_4 > 0\) and
\[
2\alpha_4 + \alpha_5 + \alpha_6 - \frac{\gamma_1^2}{\gamma_1} = 2\eta + \frac{2\Gamma_1 s^2}{9} \left((1 - s)^2 + 3s(4 - s) - (s + 2)^2\right) = 2\eta > 0,
\]
\[
\frac{3}{2}\alpha_4 + \alpha_5 + \alpha_6 + \alpha_1 = \frac{3}{2}\eta - \frac{2\Gamma_1 s^2}{3} \left((1 - s)^2 + s(4 - s) - s^2(3 - 2s)(1 + 2s)\right) = \frac{3}{2}\eta + \frac{2\Gamma_1 s^2}{3} (1 - s)^2 (1 + 2s)^2 > 0,
\]
which will ensure that the energy of the Ericksen-Leslie system is dissipated. It should be noticed that here the dissipation coefficient \(\alpha_5 + \alpha_6 - \frac{\gamma_1^2}{\gamma_1}\) in (1.13) is strictly negative when \(s \neq 1\).

Proposition 3.1 will follow from the following two lemmas.

Lemma 3.2. Let \(N = \frac{\partial n}{\partial t} + v_0 \cdot \nabla n - \Omega_0 \cdot n\). Then \(n\) satisfies
\[
n \times \left(h - \gamma_1 N - \gamma_2 D_0 \cdot n\right) = 0,
\]
where \(h = -\frac{\delta E(n, \nabla n)}{\delta n}\) and \(E(n, \nabla n)\) is the Oseen-Frank energy with the coefficients given by (3.15).

Proof. Since \(H_n(Q_1) \in Q_n^{\text{out}}\) by Proposition 2.2, it follows from (3.7) that
\[
\left(\frac{\partial Q_0}{\partial t} + v_0 \cdot \nabla Q_0 + Q_0 \cdot \Omega_0 - \Omega_0 \cdot Q_0 + \frac{1}{\Gamma} \mathcal{L}(Q_0) - S_{Q_0}(D_0)\right) : (nn^\perp + n^\perp n) = 0.
\]
Using (3.14), we get by some tedious computations that
\[
\frac{\partial Q_0}{\partial t} : (nn^\perp + n^\perp n) = s(n \frac{\partial n}{\partial t} + \frac{\partial n}{\partial t} n) : (nn^\perp + n^\perp n) = 2s \frac{\partial n}{\partial t} \cdot n^\perp,
\]
\[
v_0 \cdot \nabla Q_0 : (nn^\perp + n^\perp n) = s(n(v_0 \cdot \nabla n) + (v_0 \cdot \nabla n)n) : (nn^\perp + n^\perp n) = 2s(v_0 \cdot \nabla n) \cdot n^\perp,
\]
\[
(Q_0 \cdot \Omega_0 - \Omega_0 \cdot Q_0) : (nn^\perp + n^\perp n) = s(nn \cdot \Omega_0 - \Omega_0 \cdot nn) : (nn^\perp + n^\perp n) = -2s(\Omega_0 \cdot n) \cdot n^\perp.
\]
Using (3.14) again, we rewrite \(S_{Q_0}(D_0)\) as
\[
S_{Q_0}(D_0) = \xi \left(D_0 \cdot (snn + \frac{1 - s}{3} I) + (snn + \frac{1 - s}{3} I) \cdot D_0 - 2s(snn + \frac{1 - s}{3} I)(nn : D_0)\right)
\]
\[
= \xi \left(s(nD_0 \cdot n + D_0 \cdot nn) + \frac{2(1 - s)}{3} D_0 - 2s(snn + \frac{1 - s}{3} I)(nn : D_0)\right),
\]
from which, it follows that
\[
S_{Q_0}(D_0) : (nn^\perp + n^\perp n) = \xi \left(2s(D_0 \cdot n) \cdot n^\perp + \frac{4(1 - s)}{3} (D_0 \cdot n) \cdot n^\perp\right)
\]
\[
= \frac{2\xi(2 + s)}{3} (D_0 \cdot n) \cdot n^\perp.
\]
Moreover, we have
\[
-\mathcal{L}(Q_0) : (n n^\perp + n^\perp n) = 2\left( L_1 s \Delta(n_k n_l) + \frac{1}{2}(L_2 + L_3)\left[ s(n_k n_m)_{,ml} + s(n_m n_l)_{,mk} - \frac{2}{3} \delta_{kl} s(n_j n_j)_{,ij}\right] n_k n^\perp_l \right)
\]
\[
= 2s\left( L_1(\Delta n_k n_l + 2n_k n_l,i + n_k \Delta n_l) + \frac{1}{2}(L_2 + L_3)\left[ n_{k,m} n_{m,l} + n_{k,m} n_{l,m} + n_{l,m} n_{m,k} + n_{l,k} n_{m,m} \right] \right) n_k n^\perp_l
\]
\[
= 2s\left( L_1 \Delta n_l + \frac{1}{2}(L_2 + L_3)\left[ n_{k,m} n_{k,n} + n_{m,m} + n_{l,m} n_{k,n} + n_{i,m} n_{k,n} + n_{i,k} n_{l,m,m} \right] \right) n^\perp_l
\]
\[
= 2s\left( L_1 \Delta n_l + \frac{1}{2}(L_2 + L_3)\left[ - n_{k,m} n_{k,l} + n_{m,m} + \partial_m (n_k n_l, m) \right] \right) n^\perp_l.
\]

On the other hand, we have
\[
(h)_i = \left( -\frac{\delta}{\delta n_i} E(n, \nabla n) \right)_i = k_1 \partial_i(\partial_j n_j) + k_2(\Delta n_i - \partial_i(\partial_j n_j) + \partial_i n_k n_l \partial_l n_k - \partial_k(n_k n_l \partial_l n_i))
\]
\[
+ k_3(\partial_i n_k n_l \partial_l n_k + \partial_k(n_k n_l \partial_l n_i))
\]
\[
= k_2 \Delta n_i + (k_1 - k_2) \partial_i(\partial_j n_j) + (k_3 - k_2)(\partial_i n_k n_l \partial_l n_k + \partial_k(n_k n_l \partial_l n_i))
\]
\[
= s^2\left( 2L_1 \Delta n_i + (L_2 + L_3)[ - n_{k,m} n_{k,i} + n_{m,m} + \partial_m (n_k n_l, m) \right].
\]

This means that
\[
-\mathcal{L}(Q_0) : (n n^\perp + n^\perp n) = \frac{1}{s} h \cdot n^\perp.
\]

Summing up the above identities, we conclude that
\[
n^\perp \cdot \left( 2s^2 N - \frac{1}{s} h - \frac{2\xi s(2 + s)}{3} D_0 \cdot n \right) = 0.
\]

The lemma follows by the definition of \(\gamma_1\) and \(\gamma_2\).

Lemma 3.3. It holds that
\[
\sigma^L = \eta D_0 + S_{Q_0}(H_0) - Q_0 \cdot H_0 + H_0 \cdot Q_0,
\]
\[
\sigma^E = \sigma^d(Q_0, Q_0),
\]
where the coefficients of \(\sigma^L\) and \(\sigma^E\) are given by (3.13) and (3.10).

Proof. The key point is to calculate
\[
H_0 = \mathcal{H}_n(Q_1) + \mathcal{L}(Q_0).
\]

By (3.14) and the definition of \(Q_0^{in}\), it is easy to see that
\[
\frac{\partial Q_0}{\partial t} + v_0 \cdot \nabla Q_0 + Q_0 \cdot \Omega_0 - \Omega_0 \cdot Q_0 = s(n N + N n) \in Q_0^{in}.
\]

Then by (3.7), we get
\[
\mathcal{H}_n(Q_1) = P^0\left( -\mathcal{L}(Q_0) + \Gamma S_{Q_0}(D_0) \right).
\]

We can see from the proof of Lemma 3.2 that
\[
S_{Q_0}(D_0) \cdot n = \frac{2\xi(2 + s)}{3}(D_0 \cdot n), \quad h = -2s\mathcal{L}(Q_0) \cdot n,
\]
from which and (2.14), we infer that
\[
\mathcal{H}_n(Q_1) = \mathcal{P}_{\text{out}}(- \mathcal{L}(Q_0) + \Gamma S_{Q_0}(D_0))
\]
\[
= -\mathcal{L}(Q_0) - \frac{1}{s} \ln - \frac{1}{s} n \cdot h + \frac{2}{s} (h \cdot n) n n + \Gamma \left( S_{Q_0}(D_0) - \frac{\xi(2 + s)}{3} (n D_0 \cdot n + D_0 \cdot n) + \frac{2 \xi(2 + s)}{3} n n (D_0 : n n) \right),
\]
which along with Lemma 3.2 gives
\[
\mathcal{H}_n(Q_1) + \mathcal{L}(Q_0) = -\Gamma s (N n + n N) + \Gamma S_{Q_0}(D_0).
\]
Using (3.14), we rewrite \( S_{Q_0}(\cdot) \) as
\[
S_{Q_0}(M) = \xi \left( (s n n + \frac{1 - s}{3} I) \cdot M + M \cdot (s n n + \frac{1 - s}{3} I) - 2 (s n n + \frac{1 - s}{3} I) s (M : n n - \frac{1}{3} \text{Tr} M) \right).
\]
Then we obtain
\[
S_{Q_0}(N n + n N) = \xi \left( (s n n + \frac{1 - s}{3} I) \cdot (N n + n N) + (N n + n N) \cdot (s n n + \frac{1 - s}{3} I) \right)
\]
\[
= \xi s (N n + n N) + \frac{2(1 - s)}{3} \xi (N n + n N) = \frac{2 + s}{3} \xi (n N + N n),
\]
\[
S_{Q_0}(D_0) = \xi \left( s (n D_0 \cdot n + D_0 \cdot n n) + \frac{2(1 - s)}{3} D_0 - 2 s (s n n + \frac{1 - s}{3} I) (n n : D_0) \right).
\]
Then it can be deduced that
\[
(Q_0 + \frac{1}{3} I) : S_{Q_0}(D_0) = \xi \left( s n D_0 \cdot n + D_0 \cdot n n + \frac{2(1 - s)}{3} D_0 - 2 s (s n n + \frac{1 - s}{3} I) (n n : D_0) \right)
\]
\[
= \xi \left( n D_0 \cdot n + \frac{s(1 - s)}{3} D_0 \cdot n n + \frac{2(1 - s)^2}{9} D_0 - \frac{s^2(1 + 2s)}{3} n n (n n : D_0) - \frac{2s(1 - s)^2}{9} I (n n : D_0) \right),
\]
and
\[
(Q_0 + \frac{1}{3} I) : S(D_0, Q_0) = \xi \left( \frac{s(4 - s)}{3} - \frac{s^2(1 + 2s)}{3} - \frac{2s(1 - s)^2}{3} \right) (n n : D_0)
\]
\[
= \xi \frac{2s(1 - s)(1 + 2s)}{3} (n n : D_0).
\]
Hence, we get
\[
S_{Q_0}(S_{Q_0}(D_0)) = \xi^2 \left( \frac{s(4 - s)}{3} (n D_0 \cdot n + D_0 \cdot n n) + \frac{4(1 - s)^2}{9} D_0 - \frac{2s^2(1 + 2s)}{3} n n (n n : D_0)
\]
\[
- \frac{4s(1 - s)^2}{9} I (n n : D_0) - \frac{2s(1 - s)(1 + 2s)}{3} (n n : D_0) (s n n + \frac{1 - s}{3} I) \right)
\]
\[
= \xi^2 \left( \frac{s(4 - s)}{3} (n D_0 \cdot n + D_0 \cdot n n) + \frac{4(1 - s)^2}{9} D_0
\]
\[
- \frac{2s^2(3 - 2s)(1 + 2s)}{3} n n (n n : D_0) - \frac{8s(1 + s)^2(1 - s)^2}{9} I (n n : D_0) \right).
This gives by (3.17) that
\[
S_{Q_0}(H_0) = -\frac{\Gamma \xi s(s+2)}{3} (\nabla \cdot nN + nN) + \Gamma \xi^2 \left( \frac{s(s+2)}{3} (\nabla_0 \cdot n + D_0 \cdot n) + \frac{4(1-s)^2}{9} D_0 \right)
\]
\[
- \frac{2s^2(2-s)(1+2s)}{3} nN (\nabla (nD_0 - nD_0 \cdot n) - \frac{8s(s+2)(1-s)^2}{9} I(nN : D_0)).
\]
On the other hand, we have
\[
H_0 \cdot Q_0 - Q_0 \cdot H_0 = s\Gamma \left( -s(nN + nN) + S_{Q_0}(D_0) \right) \cdot (nN - \frac{1}{3} I)
\]
\[
- s\Gamma (nN - \frac{1}{3} I) \left( -s(nN + nN) + S_{Q_0}(D_0) \right)
\]
\[
= \Gamma s^2 (nN - nN) - \frac{1}{3} \xi s(s+2) (nD_0 \cdot n - D_0 \cdot n).
\]
Thus, we conclude that
\[
\eta D_0 + S_{Q_0}(H_0) - Q_0 \cdot H_0 + H_0 \cdot Q_0
\]
\[
= \alpha_1 (nN : D_0) nN + \alpha_2 nN + \alpha_3 nN + \alpha_4 D_0 + \alpha_5 nN \cdot D_0 + \alpha_6 D_0 \cdot nN + \text{pressure terms},
\]
with \(\alpha_i\) given by (3.16).

For the distortion stress, we have
\[
\sigma^d_{ij}(Q_0, Q_0) = -(L_1 Q_{0k,l,j} Q_{0k,l,i} + L_2 Q_{0k,l,l} Q_{0k,j,i} + L_3 Q_{0k,i,i} Q_{0k,j,i})
\]
\[
= -(L_1 s^2 (n_k n_i)_j (n_k n_i)_i + L_2 s^2 (n_k n_i)_j (n_k n_i)_i + L_3 s^2 (n_k n_i)_j (n_k n_i)_i)
\]
\[
= -(2L_1 s^2 n_k n_i n_j + L_2 s^2 (n_k n_i n_j + n_i n_j n_i) + L_3 s^2 (n_k n_i n_j + n_j n_i n_j)).
\]
Using the following facts
\[
(\nabla \cdot n)^2 = (\partial_i n_i)^2, \quad (n \cdot (\nabla \times n))^2 = \partial_i n_i \partial_i n_j - \partial_i n_j \partial_i n_j - n_i n_k \partial_i n_j \partial_k n_j,
\]
\[
|n \times (\nabla \times n)|^2 = n_i n_k \partial_i n_j \partial_k n_j, \quad \text{tr}(\nabla n)^2 - (\nabla \cdot n)^2 = \partial_i n_i \partial_j n_i - (\partial_i n_i)^2,
\]
we infer that
\[
\frac{\partial E(n, \nabla n)}{\partial n_{k,i}} = k_1 \delta_{k,j} \partial_i n_l + k_2 (\partial_j n_k - \delta_k n_j - n_i n_j \partial_i n_k) + k_3 n_i \partial_j n_k + (k_2 + k_4) (\partial_k n_j - \delta_k \partial_j n_i),
\]
hence,
\[
\frac{\partial E(n, \nabla n)}{\partial n_{k,j}} = k_1 \delta_{k,i} \partial_l n_i + k_2 \partial_i n_k (\partial_j n_k - \delta_k n_j - n_i n_j \partial_i n_k) + k_3 n_i \partial_j n_k \partial_l n_i
\]
\[
+ (k_2 + k_4) (\partial_i n_k \partial_j n_i - \partial_i n_l \partial_j n_i))
\]
\[
= 2L_1 s^2 n_k n_j + L_2 s^2 (n_k n_i n_j + n_i n_j n_i) + L_3 s^2 (n_k n_i n_j + n_j n_i n_i)
\]
\[
= -\sigma^d_{ij}(Q_0, Q_0),
\]
which means that \(\sigma^d(Q_0, Q_0)\) is the same as the Ericksen stress \(\sigma^E\).

3.3. Existence of Hilbert expansion. Let \((v_0, n)\) be a solution of (1.5)–(1.7) on \([0, T]\) and satisfy
\[
v_0 \in C([0, T]; H^k), \quad \nabla n \in C([0, T]; H^k) \quad \text{for} \quad k \geq 20.
\]
(3.18)
Hence, \(Q_0 \in C([0, T]; H^{k+1})\) by (3.14).

We write \(Q_1 = \mathcal{H}_n^{-1} ( -\mathcal{L}(Q_0) - \Gamma S(nN + nN) + \Gamma S_{Q_0}(D_0) ) \in C([0, T]; H^{k-1})\).
(3.19)
Next we solve \((v_1, Q_1^\top)\). Let us first derive the equations of \((v_1, Q_1^\top)\). We denote by \(L(\cdot)\) the linear function with the coefficients belonging to \(C([0, T]; H^{k - 1})\), and by \(R \in C([0, T]; H^{k - 3})\) some function depending only on \(n, v_0, Q_1^\top\). Set

\[
\mathcal{B}_1(Q, \tilde{Q}) = -b\left(Q \cdot \tilde{Q} - \frac{1}{3}(Q : \tilde{Q})I\right) + c(2(Q : Q_0)\tilde{Q} + (Q : \tilde{Q})Q_0).
\]

Then \(\mathcal{B}_1\) can be written as

\[
\mathcal{B}_1 = \mathcal{B}_1(Q_1, Q_1) = \mathcal{B}_1(Q^\top_1, Q^\top_1) + \mathcal{B}_1(Q^\top_1, Q^\top_1) + \mathcal{B}_1(Q^\top_1, Q^\top_1) + \mathcal{B}_1(Q^\top_1, Q^\top_1)
\]

\[
= \mathcal{B}_1(Q^\top_1, Q^\top_1) + L(Q^\top_1, v_1).
\]

It is easy to show that

\[
\mathcal{B}_1(Q^\top_1, Q^\top_1) \in Q_n^\text{out}.
\] (3.20)

**Lemma 3.4.** It holds that

\[
P^\text{out} \left( \frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 \right) = L(Q^\top_1) + R,
\]

\[
P^\text{in} \left( \frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 \right) = \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 + L(Q^\top_1) + R.
\]

**Proof.** Assume that \(Q^\top_1 = nn^\perp + n^\perp n\) with \(n^\perp \cdot n = 0\). Then we have

\[
\frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 = n\dot{m}^\perp + \dot{n}n^\perp + \dot{n}n + n^\perp \dot{n},
\]

where \(\dot{m} = \partial_t m + v_0 \cdot \nabla m\). Using the facts that

\[
n^\perp \cdot n = \dot{n} \cdot n = 0, \quad \dot{n}^\perp \cdot n + n^\perp \cdot \dot{n} = (\partial_t + v_0 \cdot \nabla)(n^\perp \cdot n) = 0,
\]

we get

\[
(I - nn) \cdot \left( \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 \right) \cdot n = \dot{n}^\perp + (n^\perp \cdot \dot{n})n.
\]

Then we infer from (2.4) that

\[
P^\text{in} \left( \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 \right) = n(\dot{n}^\perp + (n^\perp \cdot \dot{n})n) + (\dot{n}^\perp + (n^\perp \cdot \dot{n})n)n
\]

\[
= n\dot{n}^\perp + \dot{n} + L(Q^\top_1),
\]

from which, it follows that

\[
P^\text{out} \left( \frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 \right) = P^\text{out} \left( \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 \right) + R
\]

\[
= L(Q^\top_1) + R.
\]

Hence, we have

\[
P^\text{in} \left( \frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 \right) = \frac{\partial Q_1}{\partial t} + v_0 \cdot \nabla Q_1 - P^\text{out} \left( \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 \right)
\]

\[
= \frac{\partial Q^\top_1}{\partial t} + v_0 \cdot \nabla Q^\top_1 + L(Q^\top_1) + R.
\]

The proof is finished. \(\square\)
We denote
\[ A_1 = \mathcal{P}^{\text{in}}(\mathcal{L}(Q_1^T)), \quad A_2 = \mathcal{P}^{\text{out}}(\mathcal{L}(Q_1^T)), \]
\[ C_1 = \mathcal{P}^{\text{in}}(S_{Q_0}D_1 + \Omega_1 \cdot Q_0 - Q_0 \cdot \Omega_1), \quad C_2 = \mathcal{P}^{\text{out}}(S_{Q_0}D_1 + \Omega_1 \cdot Q_0 - Q_0 \cdot \Omega_1). \]
Since \( \mathcal{L}(Q_1) = \mathcal{L}(Q_1^T) + R \) and \( \mathcal{H}_{Q_0}(Q_2) \in \mathcal{Q}^\text{out} \), we take \( \mathcal{P}^{\text{out}} \) on both sides of (3.10) and use Lemma 3.4 and (3.20) to get
\[ \frac{\partial Q_1^T}{\partial t} + v_0 \cdot \nabla Q_1^T = -\frac{1}{\Gamma}A_1 + C_1 + L(Q_1^T, v_1) + R, \quad (3.21) \]
and take \( \mathcal{P}^{\text{out}} \) on both sides of (3.10) to get
\[ -\frac{1}{\Gamma}(A_2 + \mathcal{H}_{Q_0}(Q_2) + \mathcal{B}_1(Q_1^T, Q_1^T)) + C_2 + L(v_1, Q_1^T) + R = 0. \]
This also implies
\[ H_1 = \mathcal{L}(Q_1) + \mathcal{H}_{Q_0}(Q_2) + \mathcal{B}_1 = A_1 + \Gamma C_2 + L(v_1, Q_1^T) + R. \quad (3.22) \]
Plugging (3.22) into (3.21), we derive the equations of \((v_1, Q_1^T)\):
\[ \frac{\partial v_1}{\partial t} + v_0 \cdot \nabla v_1 = -\nabla p_1 + \nabla \cdot \left( \eta D_1 + S_{Q_0} (A_1 + \Gamma C_2) - Q_0 \cdot (A_1 + \Gamma C_2) \right) \]
\[ + (A_1 + \Gamma C_2) \cdot Q_0 + \sigma^d(Q_1, Q_0) + \sigma^d(Q_0, Q_1) + L(v_1, Q_1^T) + R, \quad (3.23) \]
\[ \nabla \cdot v_1 = 0, \]
\[ \frac{\partial Q_1^T}{\partial t} + v_0 \cdot \nabla Q_1^T = -\frac{1}{\Gamma}A_1 + C_1 + L(v_1, Q_1^T) + R. \quad (3.24) \]

The system (3.23)–(3.24) is just a linear system. To see its solvability, we present a priori estimate for the following energy
\[ E_k \overset{\text{def}}{=} \sum_{|\ell|=0}^{k-4} \left\langle \partial^\ell v_1, \partial^\ell v_1 \right\rangle + \left\langle \partial^\ell Q_1^T, \mathcal{L}(\partial^\ell Q_1^T) \right\rangle + \left\langle Q_1^T, Q_1^T \right\rangle. \]

We will show that
\[ \frac{d}{dt} E_k \leq C \left( E_k + \| R(t) \|_{H^{k-3}} \right), \]
which will ensure that the system (3.23)–(3.24) has a unique solution \((v_1, Q_1^T)\) on \([0, T]\) satisfying
\[ v_1 \in C([0, T]; H^{k-4}), \quad Q_1^T \in C([0, T]; H^{k-3}). \quad (3.25) \]
In what follows, we give an estimate for the term of \( \ell = 0 \) in \( E_k \), the proof for general case is similar. We set
\[ E_1 = \langle v_1, v_1 \rangle + \langle Q_1^T, \mathcal{L}(Q_1^T) \rangle + \langle Q_1^T, Q_1^T \rangle. \]
First of all, we get by (3.24) and Lemma 2.2 that for any \( \delta > 0 \),
\[ \frac{1}{2} \frac{d}{dt} \langle Q_1^T, Q_1^T \rangle = \left\langle -\frac{1}{\Gamma} \mathcal{L}(Q_1^T) + S_{Q_0}D_1 + \Omega_1 \cdot Q_0 - Q_0 \cdot \Omega_1, Q_1^T \right\rangle + \left\langle L(v_1, Q_1^T) + R, Q_1^T \right\rangle \]
\[ \leq \delta \| \nabla v_1 \|^2_{L^2} + C_\delta \| Q_1^T \|^2_{H^1} + \| R \|^2_{L^2}. \]
Using (3.23) again, we get 
\[
\frac{1}{2} \frac{d}{dt} \left( \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \langle \mathbf{Q}_1^T, \mathcal{L}(\mathbf{Q}_1^T) \rangle \right) = \langle \partial_t \mathbf{v}_1, \mathbf{v}_1 \rangle + \langle \partial_t \mathbf{Q}_1^T, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \eta \langle \mathbf{D}_1, \mathbf{D}_1 \rangle - \langle \mathbf{S}_{\mathbf{Q}_0}(\mathbf{A}_1 + \mathbf{C}_2) - \mathbf{Q}_0 \cdot (\mathbf{A}_1 + \mathbf{C}_2) + (\mathbf{A}_1 + \mathbf{C}_2) \cdot \mathbf{Q}_0, \nabla \mathbf{v}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
+ \sigma^d(\mathbf{Q}_1, \mathbf{Q}_0) + \sigma^d(\mathbf{Q}_0, \mathbf{Q}_1) + \mathbf{L}(\mathbf{v}_1, \mathbf{Q}_1^T) + R, \nabla \mathbf{v}_1 \rangle - \langle \mathbf{v}_0 \cdot \nabla \mathbf{Q}_1^T, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
- \frac{1}{\Gamma} \langle \mathcal{P}^{\text{in}}(\mathcal{L}(\mathbf{Q}_1^T)), \mathcal{L}(\mathbf{Q}_1^T) \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle + \langle \mathbf{L}(\mathbf{v}_1, \mathbf{Q}_1^T) + R, \mathcal{L}(\mathbf{Q}_1^T) \rangle. 
\]

The key point is that we find the following dissipation 
\[
- \langle \mathbf{S}_{\mathbf{Q}_0}(\mathbf{A}_1 + \mathbf{C}_2) - \mathbf{Q}_0 \cdot (\mathbf{A}_1 + \mathbf{C}_2) + (\mathbf{A}_1 + \mathbf{C}_2) \cdot \mathbf{Q}_0, \nabla \mathbf{v}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{S}_{\mathbf{Q}_0}(\mathbf{A}_1 + \mathbf{C}_2), \mathbf{D}_1 \rangle + \langle \mathbf{Q}_0 \cdot (\mathbf{A}_1 + \mathbf{C}_2) - (\mathbf{A}_1 + \mathbf{C}_2) \cdot \mathbf{Q}_0, \mathbf{D}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 \rangle + \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{Q}_0 \cdot \mathbf{Q}_1 - \mathbf{Q}_1 \cdot \mathbf{Q}_0 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 \rangle + \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{Q}_0 - \mathbf{Q}_0 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 \rangle + \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{Q}_0 - \mathbf{Q}_0 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \langle \mathbf{A}_1 + \mathbf{C}_2, \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
+ \langle \mathbf{P}^{\text{in}}(\mathcal{L}(\mathbf{Q}_1^T)), \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
- \Gamma \langle \mathbf{P}^{\text{out}}(\mathbf{Q}_0 \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1), \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
= - \Gamma \langle \mathbf{P}^{\text{out}}(\mathbf{Q}_0 \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1), \mathbf{S}_{\mathbf{Q}_0} \mathbf{D}_1 + \mathbf{Q}_1 - \mathbf{Q}_1 \rangle + \langle \mathbf{C}_1, \mathcal{L}(\mathbf{Q}_1^T) \rangle \leq 0. 
\]

For the other terms, we have 
\[
- \langle \sigma^d(\mathbf{Q}_1, \mathbf{Q}_0) + \sigma^d(\mathbf{Q}_0, \mathbf{Q}_1) + \mathbf{L}(\mathbf{v}_1, \mathbf{Q}_1^T) + \mathbf{R}, \nabla \mathbf{v}_1 \rangle + \langle \mathbf{L}(\mathbf{v}_1, \mathbf{Q}_1^T) + \mathbf{R}, \mathcal{L}(\mathbf{Q}_1^T) \rangle 
\]
\[
\leq \delta \| \nabla \mathbf{v}_1 \|^2_2 + C_\delta (\| \mathbf{v}_1 \|^2_2 + \| \mathbf{Q}_1^T \|_{H^1}^2 + \| \mathbf{R} \|_{H^1}), 
\]
and for any \( \mathbf{Q} \), 
\[
- \langle \mathbf{v}_0 \cdot \nabla \mathbf{Q}, \mathcal{L}(\mathbf{Q}) \rangle = \int_{\mathbb{R}^3} v_{ij} Q_{kl,j} \left( L_1 \Delta Q_{kl} + \frac{1}{2} (L_2 + L_3) (Q_{km,m} + Q_{lm,m} - \frac{2}{3} \delta_{kl} Q_{ij,ij}) \right) dx 
\]
\[
= \int_{\mathbb{R}^3} \left( - L_1 v_{ij} Q_{kl,m} + \frac{1}{2} (L_2 + L_3) (v_{ij} Q_{kl,m} + v_{ij} Q_{kl,kj}) \right) dx 
\]
\[
= \int_{\mathbb{R}^3} \left( - L_1 v_{ij} Q_{kl,m} + \frac{1}{2} (L_2 + L_3) (v_{ij} Q_{kl,m} + v_{ij} Q_{kl,kj}) \right) dx 
\]
\[
\leq C \| \mathbf{Q} \|_{H^1}^2. 
\]

Thus, we get 
\[
- \langle \mathbf{v}_0 \cdot \nabla \mathbf{Q}_1^T, \mathcal{L}(\mathbf{Q}_1^T) \rangle \leq C \| \mathbf{Q}_1^T \|_{H^1}^2. 
\]

Summing up, we obtain 
\[
\frac{d}{dt} E_1 \leq C (E_1 + \| \mathbf{R} \|_{H^1}). 
\]

This completes the proof of existence of \((\mathbf{v}_1, \mathbf{Q}_1)\).

Again, we write \( \mathbf{Q}_2 = \mathbf{Q}_2^+ + \mathbf{Q}_2^- \) with \( \mathbf{Q}_2^+ \in \mathcal{Q}_n^\text{in} \) and \( \mathbf{Q}_2^- \in \mathcal{Q}_n^\text{out} \). By (3.22), we can determine \( \mathbf{Q}_2^+ \) by 
\[
\mathbf{Q}_2^+ = \mathcal{H}^{-1}_n \left( - \mathcal{L}(\mathbf{Q}_1) - \mathbf{B}_1 + \mathbf{A}_1 + \mathbf{C}_2 + \mathbf{L}(\mathbf{v}_1, \mathbf{Q}_1^T) + \mathbf{R} \right) \in C([0,T]; H^{k-5}). 
\]

(3.26)
Then \((v_2, Q^T_2, Q_3)\) can be solved in a similar way as \((v_1, Q^T_1)\). We left it to the interested readers. Summing up, we prove

**Proposition 3.5.** Let \((v_0, n)\) be a solution of \((1.6)-(1.7)\) on \([0, T]\) and satisfy
\[
v_0 \in C([0, T]; H^k), \quad \nabla n \in C([0, T]; H^k) \quad \text{for} \quad k \geq 20.
\]
There exists the solution \((v_i, Q_i)\) \((i = 0, 1, 2)\) and \(Q_3 \in \mathbb{Q}^\text{out} \) of the system \((3.8)-(3.13)\) satisfying
\[
v_i \in C([0, T]; H^{k-4i}), \quad Q_i \in C([0, T]; H^{k+1-i})(i = 0, 1, 2), \quad Q_3 \in C([0, T]; H^{k-9}).
\]

4. **Uniform estimates for the remainder**

4.1. **The system for the remainder.** In this subsection, we derive the equations for the remainder \((v_\varepsilon^R, Q_\varepsilon^R)\) in the Hilbert expansion \((3.1)-(3.2)\). In what follows, we omit the superscript \(\varepsilon\) of \((v_\varepsilon^R, Q_\varepsilon^R)\).

By \((3.3)\) and the definitions of \(H_i\) \((i=0,1,2)\), the molecular field \(H(Q^\varepsilon)\) can be expanded into
\[
H(Q^\varepsilon) = \frac{1}{\varepsilon} \mathcal{J}(Q^\varepsilon) + \mathcal{L}(Q^\varepsilon) = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3 + \varepsilon^4 H_4 + \varepsilon^5 J_R,
\]
where \(H_R = H^\varepsilon_0(Q^R) \triangleq H_0(Q^R) + \varepsilon L(Q^R)\). We denote
\[
\tilde{v} = v_0 + \varepsilon v_1 + \varepsilon^2 v_2, \quad \tilde{D} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \quad \tilde{\Omega} = \Omega_0 + \varepsilon \Omega_1 + \varepsilon^2 \Omega_2.
\]

Thanks to \((3.4)-(3.13)\) and \((1.16)-(1.18)\), we obtain
\[
\begin{aligned}
\frac{\partial R}{\partial t} &= -\tilde{v} \cdot \nabla R - \nabla P_R + \eta \Delta R - \nabla \cdot \left(\frac{1}{\varepsilon} S_{Q_0}(H_R) - \frac{1}{\varepsilon} Q_0 \cdot H_R + \frac{1}{\varepsilon} H_R \cdot Q_0\right) + \nabla \cdot G_R + G'_R, \\
\nabla \cdot v_R &= 0, \\
\frac{\partial Q_0}{\partial t} &= -\frac{1}{\Gamma} H^\varepsilon_0(Q^R) + S_{Q_0}D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R + F_R.
\end{aligned}
\]
Let us give the precise formulation of \(F_R, G_R, G'_R\). The term \(G'_R\) takes the from
\[
G'_R = -v_1 \cdot \nabla v_2 - v_2 \cdot \nabla v_1 - \varepsilon v_1 \cdot \nabla v_2 - v_2 \cdot \nabla v_2 - v_R \cdot \nabla \tilde{v} - \varepsilon^3 v_R \cdot \nabla v_R.
\]
The term \(F_R\) consists of five parts
\[
F_R = F_1 + F_2 + F_3 + F_4 + F_5,
\]
where \(F_1\) is independent of \((v_R, Q_R)\):
\[
F_1 = -\frac{1}{\Gamma} \bar{B}^\varepsilon + \sum_{i+j \geq 3} \varepsilon^{i+j-3} \left(\xi B(D_i, Q_j) + \Omega_i \cdot Q_j - Q_j \cdot \Omega_i\right)
\]
\[
- 2\xi \sum_{i+j+k \geq 3} \varepsilon^{i+j+k-3} Q_i(D_j : Q_k) - v_0 \cdot \nabla Q_3 - v_1 \cdot \nabla (Q_2 + \varepsilon Q_3) - v_2 \cdot \nabla \tilde{Q} - \frac{\partial Q_3}{\partial t},
\]
and \(F_2, F_3\) linearly depend on \((v_R, Q_R)\):
\[
F_2 = \xi \left(B(\tilde{D}, Q_R) - 2Q_R \sum_{i=0}^{2} \sum_{j=0}^{3} \varepsilon^{i+j} D_i : Q_j - 2 \sum_{i=0}^{2} \sum_{j=0}^{3} \varepsilon^{i+j} Q_j(Q_R : D_i)\right) + \tilde{\Omega} \cdot Q_R
\]
\[
- Q_R \cdot \tilde{\Omega} - \frac{1}{\Gamma} \left(-bB(Q^\varepsilon, Q_R) + cC(Q_R, Q_R, Q_0) + \frac{c}{2} \varepsilon \C(Q_R, Q^\varepsilon, Q^\varepsilon)\right) - \tilde{v} \cdot \nabla Q_R,
\]
\[
F_3 = -v_R \cdot \nabla (Q_0 + \varepsilon \tilde{Q}) - \varepsilon \tilde{Q}^\varepsilon \cdot \Omega_R + \varepsilon \Omega_R \cdot \tilde{Q}^\varepsilon
\]
\[
+ \xi \left((\varepsilon \tilde{Q}^\varepsilon \cdot D_R + \varepsilon D_R \cdot \tilde{Q}^\varepsilon - \frac{2}{3} \xi \tilde{Q}^\varepsilon : D_R + \sum_{i+j \geq 1} \varepsilon^{i+j} Q_i(D_R : Q_j)\right),
\]
and \( F_4, F_5 \) nonlinearly depend on \((v_R, Q_R)\):

\[
F_4 = \varepsilon^3 \left( -v_R \cdot \nabla Q_R - Q_R \cdot \Omega_R + \Omega_R \cdot Q_R + \xi \left[ D_R \cdot Q_R + Q_R \cdot D_R - \frac{2}{3} I \right] (Q_R : D_R) \right.
- 2(Q_0 + \varepsilon \overline{Q}^e) (Q_R : D_R) - 2Q_R ((Q_0 + \varepsilon \overline{Q}^e) : D_R) - 2\varepsilon^3 Q_R (Q_R : D_R) \left. \right) ,
\]

\[
F_5 = - \frac{1}{\Gamma} \left( -b^e \mathcal{B}(Q_R, Q_R) + c\varepsilon^2 \mathcal{C}(Q_R, Q_R, Q_0 + \varepsilon \overline{Q}^e) + c\varepsilon^5 \mathcal{C}(Q_R, Q_R, Q_R) \right)
- \varepsilon\varepsilon^3 Q_R (Q_R : \tilde{D}^e).
\]

Similarly, \( G_R \) can be written as

\[
G_R = G_1 + G_2 + G_3 + G_4,
\]

where \( G_1 \) is given by

\[
G_1 = \xi \sum_{i+j \geq 3} \varepsilon^{i+j-3} \mathcal{B}(Q_i, H_j) - 2\xi \sum_{i+j \geq k \geq 3} \varepsilon^{i+j+k-3} Q_i (H_j : Q_k)
+ \sum_{i+j \geq 3} \varepsilon^{i+j-3} (Q_i \cdot H_j - H_j \cdot Q_i + \sigma^d(Q_i, Q_j)),
\]

and \( G_2, G_3 \) are given by

\[
G_2 = \xi \mathcal{B}(\overline{Q}^e, H_R) - 2\xi \sum_{i+j \geq 1} \varepsilon^{i+j-1} Q_i (H_R : Q_j) + \overline{Q}^e \cdot H_R - H_R \cdot \overline{Q}^e
+ \xi \sum_{i=0}^2 \varepsilon^i \mathcal{B}(Q_R, H_i) - 2\xi \sum_{i+j \geq 1} \varepsilon^{i+j} \left[ Q_i (H_j : Q_R) + Q_R (H_j : Q_i) \right]
+ \sum_{j=0}^2 \varepsilon^j \left[ Q_R \cdot H_j - H_j \cdot Q_R \right] + \sigma^d(Q_0 + \varepsilon \overline{Q}^e, Q_R) + \sigma^d(Q_R, Q_0 + \varepsilon \overline{Q}^e),
\]

\[
G_3 = \xi \mathcal{B}(Q_0 + \varepsilon \overline{Q}^e, J_R^e) - 2\xi \sum_{i,j=0}^3 \varepsilon^{i+j} Q_i (J_R^e : Q_j) + (Q_0 + \varepsilon \overline{Q}^e) \cdot J_R^e - J_R^e \cdot (Q_0 + \varepsilon \overline{Q}^e),
\]

and \( G_4 \) is given by

\[
G_4 = -2\varepsilon^2 (Q_0 + \varepsilon \overline{Q}^e + \frac{1}{3} I) (Q_R : (H_R + \varepsilon J_R^e)) - 2\varepsilon^2 Q_R ((Q_0 + \varepsilon \overline{Q}^e) : (H_R + \varepsilon J_R^e))
- 2\varepsilon^3 Q_R (Q_R : (H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^2 H_R + \varepsilon^3 J_R^e))
+ \varepsilon^3 (Q_R \cdot J_R^e - J_R^e \cdot Q_R) + \varepsilon^3 \sigma^d(Q_R, Q_R).
\]

4.2. A key lemma. For \( Q_1, Q_2 \in L^2(\mathbb{R}^3)^{3 \times 3} \), we define the inner product

\[
\langle Q_1, Q_2 \rangle \overset{\text{def}}{=} \int_{\mathbb{R}^3} Q_1(x) : Q_2(x) dx.
\]

The following lemma plays an important role in the energy estimates.
Lemma 4.1. For any $\delta > 0$, there exists a constant $C = C(\delta, \|\nabla_t x n\|_{L^\infty}, \|\nabla n\|_{L^\infty})$ such that for any $Q \in Q$, it holds that

$$
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q, Q \rangle \leq \delta \left( \frac{1}{\varepsilon} \mathcal{H}_n(Q) + \mathcal{L}(Q), \frac{1}{\varepsilon} \mathcal{H}_n(Q) + \mathcal{L}(Q) \right) + C_\delta \left( \frac{1}{\varepsilon} \mathcal{H}_n(Q) + \mathcal{L}(Q), Q \right) + \langle Q, Q \rangle,
$$

where $\mathcal{H}_n(Q) = \|Q\|_{L^4}$.

Proof. Let $Q = Q^\perp + Q^\top$, where $Q^\perp \in Q^\text{out}_n$ and $Q^\top \in Q^\text{in}_n$. Thus, we have

$$
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q, Q \rangle = \frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q^\top, Q^\top \rangle + 2 \varepsilon \langle \partial_t (nn) \cdot Q^\top, Q^\perp \rangle + \frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q^\perp, Q^\perp \rangle.
$$

Thanks to the fact that $\partial_t (nn) = nn + n, n \in Q^\text{in}_n$ and Lemma 2.2, the first term on the right hand side vanishes. By Proposition 2.2, the third term is bounded by

$$
\|n_t\|_{L^\infty} \leq C \|n_t\|_{L^\infty} \leq C \|H_n(Q), Q\|.
$$

For the second term, we infer from Proposition 2.3 that

$$
\frac{1}{\varepsilon} \langle \partial_t (nn) \cdot Q^\perp, Q^\perp \rangle = \left( \mathcal{H}_n^{-1}(\partial_t (nn) \cdot Q^\top), \frac{1}{\varepsilon} \mathcal{H}_n Q \right) = \left( \mathcal{H}_n^{-1}(\partial_t (nn) \cdot Q^\top), \frac{1}{\varepsilon} \mathcal{H}_n Q + \mathcal{L}(Q) \right) - \left( \mathcal{H}_n^{-1}(\partial_t (nn) \cdot Q^\top), \mathcal{L}(Q) \right) = C \|n_t\|_{L^\infty} \|Q^\top\|_{L^2} \|\mathcal{H}_n Q + \mathcal{L}(Q)\|_{L^2} + C_2 \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2,
$$

where $C_2$ depends on $\|\nabla_t x n\|_{L^\infty}$ and $\|\nabla n\|_{L^\infty}$. This gives the first inequality by Lemma 2.2.

Similarly, we have

$$
\frac{1}{\varepsilon} \langle Q : \partial_t (nn), Q : nn \rangle = \frac{1}{\varepsilon} \langle Q^\top : \partial_t (nn), Q^\top : nn \rangle + \frac{1}{\varepsilon} \langle Q^\perp : \partial_t (nn), Q^\perp : nn \rangle.
$$

Then the second inequality follows in the same way. \qed

4.3. Uniform energy estimates. Throughout this subsection, we assume that $v_i \in C([0, T]; H^{k-4i})$ for $i = 0, 1, 2$ and $Q_i \in C([0, T]; H^{k+1-4i})$ for $i = 0, 1, 2, 3$. We denote by $C$ a constant depending on $\sum_{i=0}^{2} \sup_{t \in [0, T]} \|v_i(t)\|_{H^{k-4i}}$ and $\sum_{i=0}^{3} \sup_{t \in [0, T]} \|Q_i(t)\|_{H^{k+1-4i}}$, and independent of $\varepsilon$.

We introduce the following energy functional

$$
\mathcal{E}(t) \stackrel{\text{def}}{=} \frac{1}{2} \int \left( |v_R|^2 + \frac{1}{\varepsilon} H_n(Q_R) : Q_R + |Q_R|^2 \right) + \varepsilon^2 \left( |\nabla v_R|^2 + \frac{1}{\varepsilon} H_n(\nabla Q_R) : \nabla Q_R \right) + \varepsilon^4 \left( |\Delta v_R|^2 + \frac{1}{\varepsilon} H_n(\Delta Q_R) : \Delta Q_R \right) dx,
$$

$$
\mathcal{F}(t) \stackrel{\text{def}}{=} \int \left( |\eta | \nabla v_R|^2 + \frac{1}{\varepsilon} H_n(\nabla Q_R) : H_n(Q_R) \right) + \varepsilon^2 \left( |\eta \Delta v_R|^2 + \frac{1}{\varepsilon} H_n(\Delta Q_R) : H_n(\nabla Q_R) \right) + \varepsilon^4 \left( |\eta \Delta v_R|^2 + \frac{1}{\varepsilon} H_n(\Delta Q_R) : H_n(\Delta Q_R) \right) dx.
$$

The uniform energy estimate is stated as follows.
Proposition 4.2. Let \((v_R, Q_R)\) be a smooth solution of the system (4.1)–(4.3) on \([0, T]\). Then for any \(t \in [0, T]\), it holds that

\[
\frac{d}{dt} E(t) + \mathcal{F}(t) \leq C \left( 1 + \mathcal{E} + \varepsilon^2 \mathcal{E} + \varepsilon^{14} \mathcal{E}^5 \right) + C \left( \varepsilon + \varepsilon^2 \mathcal{E}^\frac{1}{2} + \varepsilon^4 \mathcal{E} \right) \mathcal{F}.
\]

To prove the proposition, we need the following lemmas.

Lemma 4.3. It holds that

\[
\|Q_R\|_{H^1} + \|(\varepsilon^2 V^2 Q_R, \varepsilon^2 \nabla^2 Q_R)\|_{L^2} + \|(v_R, \varepsilon \nabla v_R, \varepsilon^2 \nabla^2 v_R)\|_{L^2} \leq C \mathcal{E}(t)^\frac{1}{4},
\]

\[
\left\| (\varepsilon^{-1} \mathcal{H}_n^\varepsilon(Q_R), \nabla \mathcal{H}_n^\varepsilon(Q_R), \varepsilon \Delta \mathcal{H}_n^\varepsilon(Q_R)) \right\|_{L^2} + \left\| (\nabla v_R, \varepsilon \nabla^2 v_R, \varepsilon^2 \nabla^3 v_R) \right\|_{L^2} \leq C \left( \mathcal{F}(t) + \mathcal{E}(t) \right)^{\frac{1}{2}}.
\]

Proof. The first inequality follows from Lemma 2.2. By the commutator estimate

\[
\|[\nabla, \mathcal{H}_n^\varepsilon(Q)]\|_{L^2} \leq C\|Q_R\|_{L^2}, \quad \|[\Delta, \mathcal{H}_n^\varepsilon(Q)]\|_{L^2} \leq C\|Q_R\|_{H^1},
\]

we have

\[
\|\nabla \mathcal{H}_n^\varepsilon(Q)\|_{L^2} \leq \|\mathcal{H}_n^\varepsilon(\nabla Q)\|_{L^2} + C\|Q_R\|_{L^2},
\]

\[
\|\varepsilon \Delta \mathcal{H}_n^\varepsilon(Q)\|_{L^2} \leq \|\varepsilon \mathcal{H}_n^\varepsilon(\Delta Q)\|_{L^2} + C\varepsilon\|Q_R\|_{H^1}.
\]

This gives the second inequality.

The following inequality will be useful for the estimates of \((F_R, G_R)\):

\[
\|fg\|_{H^k} \leq C\|f\|_{H^2}\|g\|_{H^k} \quad \text{for} \quad k = 0, 1, 2.
\]

Lemma 4.4. It holds that

\[
\|(F_R, \varepsilon \nabla F_R, \varepsilon^2 \Delta F_R)\|_{L^2} \leq C \left( 1 + \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E} + \varepsilon^2 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} + \varepsilon^3 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} + \varepsilon^4 \mathcal{E} \right).
\]

Proof. By Lemma 4.3, it is easy to see that

\[
\|(F_1, \varepsilon \nabla F_1, \varepsilon^2 \Delta F_1)\|_{L^2} \leq C,
\]

\[
\|(F_2, \varepsilon \nabla F_2, \varepsilon^2 \Delta F_2)\|_{L^2} \leq C\mathcal{E}^\frac{1}{2},
\]

\[
\|(F_3, \varepsilon \nabla F_3, \varepsilon^2 \Delta F_3)\|_{L^2} \leq C \left( \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E}^\frac{1}{2} \right),
\]

and by (4.8), we get

\[
\|(F_4, \varepsilon \nabla F_4, \varepsilon^2 \Delta F_4)\|_{L^2} \leq C \varepsilon \left( \mathcal{E} + \varepsilon \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} + \varepsilon^2 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} + \varepsilon^3 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} \right),
\]

\[
\|(F_5, \varepsilon \nabla F_5, \varepsilon^2 \Delta F_5)\|_{L^2} \leq C \varepsilon \left( \mathcal{E} + \varepsilon^2 \mathcal{E}^\frac{1}{2} \right).
\]

The lemma follows.

Lemma 4.5. It holds that

\[
\|(G_R, \varepsilon \nabla G_R, \varepsilon^2 \Delta G_R)\|_{L^2} \leq C \left( 1 + \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E} + \varepsilon^7 \mathcal{E}^\frac{1}{2} + \varepsilon^7 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} + \varepsilon^4 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} \right),
\]

\[
\|(G'_R, \varepsilon \nabla G'_R, \varepsilon^2 \Delta G'_R)\|_{L^2} \leq C \left( 1 + \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E}^\frac{1}{2} + \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} \right).
\]

Proof. The second inequality follows easily from (4.8) and Lemma 4.3. Obviously,

\[
\|(G_1, \varepsilon \nabla G_1, \varepsilon^2 \Delta G_1)\|_{L^2} \leq C.
\]

And by (4.8) and Lemma 4.3, we have

\[
\|(J^\varepsilon_{R, t}, \varepsilon \nabla J^\varepsilon_{R, t}, \varepsilon^2 \Delta J^\varepsilon_{R, t})\|_{L^2} \leq C \left( 1 + \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E} + \varepsilon^3 \mathcal{E}^\frac{1}{2} \mathcal{E}^\frac{1}{2} \right),
\]

which along with Lemma 4.3 gives

\[
\|(G_2, \varepsilon \nabla G_2, \varepsilon^2 \Delta G_2)\|_{L^2} \leq C \left( \varepsilon \mathcal{E}^\frac{1}{2} + \varepsilon^7 \mathcal{E}^\frac{1}{2} \right),
\]

\[
\|(G_3, \varepsilon \nabla G_3, \varepsilon^2 \Delta G_3)\|_{L^2} \leq C \left( 1 + \mathcal{E}^\frac{1}{2} + \varepsilon \mathcal{E} + \varepsilon^3 \mathcal{E}^\frac{1}{2} \right).\]
By (4.8), we get

$$\|G_4\|_{H^k} \leq C \varepsilon^2 \|\varepsilon Q_R\|_{H^2} (\|\varepsilon^{-1} \mathbf{H}_R\|_{H^k} + \|\mathcal{J}_R\|_{H^k} + \|Q_R\|_{H^k}) + C \varepsilon \|\varepsilon^2 \nabla Q_R\|_{H^2} \|\nabla Q_R\|_{H^k},$$

which implies that

$$\|(G_4, \varepsilon \nabla G_4, \varepsilon^2 \Delta G_4)\|_{L^2} \leq C (\varepsilon^2 + \varepsilon^3 \varepsilon \varepsilon^2 + \varepsilon^5 \varepsilon^2 + \varepsilon \varepsilon^4 \varepsilon^2 + \varepsilon^4 \varepsilon^2 \varepsilon^2).$$

Summing up, we conclude the first inequality.

Now we are in position to prove Proposition 4.2. The proof is split into four steps.

**Step 1. \(L^2\) estimate**

By (4.1)–(4.3) and Lemma 4.3, we get

$$\left\langle \frac{\partial Q_R}{\partial t}, Q_R \right\rangle + \frac{1}{\varepsilon} \left\langle \mathcal{H}_n(Q_R), Q_R \right\rangle = \left\langle S_{Q_0} D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R + F_R, Q_R \right\rangle$$

$$\leq C \|Q_R\|_{L^2} (\|\nabla v_R\|_{L^2} + \|F_R\|_{L^2})$$

$$\leq C (\varepsilon^2 + \|F_R\|_{L^2}),$$

and

$$\left\langle \frac{\partial v_R}{\partial t}, v_R \right\rangle + \left\langle \frac{\partial Q_R}{\partial t}, \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \right\rangle$$

$$= -\eta \langle \nabla v_R, \nabla v_R \rangle - \left\langle \frac{1}{\varepsilon} S_{Q_0} \mathcal{H}_n(Q_R), -\frac{1}{\varepsilon} Q_0 \cdot \mathcal{H}_n(Q_R) + \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \cdot Q_0 + G_R, \nabla v_R \right\rangle$$

$$+ \langle G'_R, v_R \rangle + \left\langle - \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) + S_{Q_0} D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R + F_R, \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \right\rangle$$

$$= -\eta \langle \nabla v_R, \nabla v_R \rangle - \frac{1}{\varepsilon} \langle \mathcal{H}_n(Q_R), \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \rangle - \langle G_R, \nabla v_R \rangle + \langle G'_R, v_R \rangle + \langle F_R, \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \rangle.$$

Here we used the following important cancelation relation

$$-\left\langle \frac{1}{\varepsilon} S_{Q_0} \mathcal{H}_n(Q_R), -\frac{1}{\varepsilon} Q_0 \cdot \mathcal{H}_n(Q_R) + \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \cdot Q_0, \nabla v_R \right\rangle$$

$$+ \left\langle S_{Q_0} D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R + F_R, \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \right\rangle = 0. \quad (4.10)$$

Thus, we obtain

$$\left\langle \frac{\partial v_R}{\partial t}, v_R \right\rangle + \left\langle \frac{\partial Q_R}{\partial t}, \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \right\rangle + \eta \langle \nabla v_R, \nabla v_R \rangle + \frac{1}{\varepsilon} \left\langle \mathcal{H}_n(Q_R), \frac{1}{\varepsilon} \mathcal{H}_n(Q_R) \right\rangle$$

$$\leq C (\|G'_R\|_{L^2} \varepsilon^2 + (\|G_R\|_{L^2} + \|F_R\|_{L^2}) \varepsilon^2). \quad (4.11)$$

**Step 2. \(H^1\) estimate**

Using (4.1)–(4.3) again, we get

$$\varepsilon^2 \left\langle \frac{\partial}{\partial t}, \v_R, \partial_i v_R \right\rangle + \varepsilon \left\langle \frac{\partial}{\partial t}, \partial_i Q_R, \mathcal{H}_n(\partial_i Q_R) \right\rangle + \varepsilon^2 \eta \langle \nabla \partial_i v_R, \nabla \partial_i v_R \rangle$$

$$= -\left\langle \partial_i [S_{Q_0} (\mathcal{H}_n(Q_R)) - Q_0 \cdot \mathcal{H}_n(Q_R) + \mathcal{H}_n(Q_R) \cdot Q_0 + \varepsilon G_R], \varepsilon \nabla \partial_i v_R \right\rangle - \varepsilon^2 \left\langle \partial_i \mathbf{v}_R, \nabla v_R, \partial_i v_R \right\rangle$$

$$+ \varepsilon \left\langle \partial_i G'_R, \varepsilon \partial_i v_R \right\rangle + \varepsilon \left\langle \partial_i \left[ \frac{1}{\varepsilon} \mathbf{H}_R + S_{Q_0} D_R + \Omega_R \cdot Q_0 - Q_0 \cdot \Omega_R + F_R \right], \mathcal{H}_n(\partial_i Q_R) \right\rangle.$$
The terms on the right hand sides are estimated as follows

\[ \langle \partial_t [S_{Q_0}(H_n^\varepsilon(Q_R)) - Q_0 \cdot H_n^\varepsilon(Q_R) + \partial_t H_n^\varepsilon(Q_R) \cdot Q_0, \varepsilon \nabla \partial_t \nu R] \rangle \leq \langle S_{Q_0}(\partial_t H_n^\varepsilon(Q_R)) - Q_0 \cdot \partial_t H_n^\varepsilon(Q_R) + \partial_t H_n^\varepsilon(Q_R) \cdot Q_0, \varepsilon \nabla \partial_t \nu R \rangle + C \|H_n^\varepsilon(Q_R)\|_{L^2} \|\varepsilon \Delta \nu R\|_{L^2}, \]

\[ \varepsilon \left( -\frac{1}{T}\partial_t H_n^\varepsilon(Q_R), H_n^\varepsilon(\partial_t Q_R) \right) \leq \frac{1}{T} \|H_n^\varepsilon(\partial_t Q_R)\|_{L^2}^2 + C \|Q_R\|_{L^2} \|H_n^\varepsilon(\partial_t Q_R)\|_{L^2}, \]

\[ \varepsilon \langle \partial_t [S_{Q_0}(D_R + \Omega_R - Q_0 - \Omega_R), H_n^\varepsilon(\partial_t Q_R) \rangle \leq \varepsilon \|\varepsilon \nabla \nu R\|_{L^2} \|H_n^\varepsilon(\partial_t Q_R)\|_{L^2}. \]

Thus by (4.10) and Lemma 4.3, we get

\[ \varepsilon^2 \langle \partial_t G_R, \nabla \partial_t \nu R \rangle \leq C \|\varepsilon \partial_t G_R\|_{L^2} \|\varepsilon \nabla \partial_t \nu R\|_{L^2}, \]

\[ \varepsilon^2 \langle \partial_t G'_R, \partial_t \nu R \rangle \leq C \|\varepsilon \partial_t G'_R\|_{L^2} \|\varepsilon \partial_t \nu R\|_{L^2}, \]

\[ \varepsilon \langle \partial_t F_R, H_n^\varepsilon(\partial_t Q_R) \rangle \leq C \|\varepsilon \partial_t F_R\|_{L^2} \|H_n^\varepsilon(\partial_t Q_R)\|_{L^2}. \]

**Step 3.** $H^2$ estimate

Since the proof is very similar to Step 2, we omit the details. We have

\[ \varepsilon^4 \left( \frac{\partial}{\partial t} \Delta \nu R, \Delta \nu R \right) + \varepsilon^3 \left( \frac{\partial}{\partial t} \Delta Q_R, H_n^\varepsilon(\Delta Q_R) \right) \leq -\varepsilon^4 \eta \langle \nabla \partial_t \nu R, \nabla \partial_t \nu R \rangle - \frac{\varepsilon^2}{T} \langle H_n^\varepsilon(\Delta Q_R), H_n^\varepsilon(\Delta Q_R) \rangle + C(\epsilon + \epsilon^2 \tilde{\delta}^2 + \epsilon \delta) + C \|\varepsilon^2 \Delta G'_R\|_{L^2} \|\epsilon^\frac{1}{2} \| + C \|\varepsilon^2 \Delta F_R\|_{L^2} \|\epsilon^\frac{1}{2} \|. \]

**Step 4.** The completion of energy estimate

Due to $Q_R : I = \text{Tr} Q_R = 0$, we have

\[ \frac{1}{\varepsilon} \frac{d}{dt} \langle Q_R, H_n^\varepsilon(Q_R) \rangle = \frac{2}{\varepsilon} \langle \partial_t Q_R, H_n^\varepsilon(Q_R) \rangle + \frac{1}{\varepsilon} \langle Q_R, bs(\partial_t (nn)) \cdot Q_R + Q_R \cdot \partial_t (nn) \rangle \]

\[ = -2cs^2 \left[ Q_R : \partial_t (nn) \right] (nn) - 2cs^2 (Q_R : nn) \partial_t (nn) \]

\[ = \frac{2}{\varepsilon} \left( \frac{\partial}{\partial t} Q_R, H_n^\varepsilon(Q_R) \right) + \frac{2}{\varepsilon} \langle Q_R, bs \partial_t (nn) \cdot Q_R - 2cs^2 Q_R : \partial_t (nn) (nn) \rangle. \]

We infer from Lemma 4.1 that

\[ \frac{2}{\varepsilon} \langle Q_R, bs \partial_t (nn) \cdot Q_R - 2cs^2 Q_R : \partial_t (nn) (nn) \rangle \leq \delta \|\frac{1}{\varepsilon} H_n^\varepsilon(Q_R)\|_{L^2}^2 + C \left( \frac{1}{\varepsilon} \langle H_n^\varepsilon(Q_R, Q_R) \rangle + \|Q_R\|_{L^2}^2 \right). \]
Therefore, we have
\[
\frac{1}{2\varepsilon^2} \frac{d}{dt} \langle Q_R, \mathcal{H}_n^\varepsilon(Q_R) \rangle \leq \frac{1}{\varepsilon} \langle \frac{\partial}{\partial t} Q_R, \mathcal{H}_n^\varepsilon(Q_R) \rangle + \delta \tilde{F} + C \mathcal{E}.
\]
Similarly, we can obtain
\[
\frac{\varepsilon}{2} \frac{d}{dt} \langle \partial_i Q_R, \mathcal{H}_n^\varepsilon(\partial_i Q_R) \rangle \leq \varepsilon \langle \frac{\partial}{\partial t} \partial_i Q_R, \mathcal{H}_n^\varepsilon(\partial_i Q_R) \rangle + \delta \tilde{F} + C \mathcal{E},
\]
\[
\frac{\varepsilon^3}{2} \frac{d}{dt} \langle \Delta Q_R, \mathcal{H}_n^\varepsilon(\Delta Q_R) \rangle \leq \varepsilon^3 \langle \frac{\partial}{\partial t} \Delta Q_R, \mathcal{H}_n^\varepsilon(\Delta Q_R) \rangle + \delta \tilde{F} + C \mathcal{E}.
\]
Summing up (4.9) and (4.11)–(4.13), we infer from Lemma 4.4 and Lemma 4.5 that
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}(t) + \tilde{F}(t) \leq C(1 + \mathcal{E} + \varepsilon^2 \mathcal{E} + \varepsilon^{14} \mathcal{E}^5) + (\delta + C \varepsilon + C \varepsilon^2 \mathcal{E}^2 + C \varepsilon^4 \mathcal{E}) \tilde{F}.
\]
Then the proposition follows by taking \( \delta \) small.

5. Proof of Theorem 1.1

Given the initial data \( (v_0, Q_0) \in H^2 \times H^3 \), it can be showed by the energy method [17] that there exists \( T_\varepsilon > 0 \) and a unique solution \( (v^\varepsilon, Q^\varepsilon) \) of the system (1.16)–(1.18) such that
\[
v^\varepsilon \in C([0, T_\varepsilon]; H^2) \cap L^2(0, T_\varepsilon; H^3), \quad Q^\varepsilon \in C([0, T_\varepsilon]; H^3) \cap L^2(0, T_\varepsilon; H^4).
\]
Thanks to \( H(Q), S_Q(D) \in Q \) for \( Q \in Q \), we have \( Q^\varepsilon \in Q \). Moreover, by Proposition 3.5, the solution has the expansion
\[
v^\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v^\varepsilon_R,
\]
\[
Q^\varepsilon = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^3 Q^\varepsilon_R.
\]
For the remainder \( (v^\varepsilon_R, Q^\varepsilon_R) \), we infer from Proposition 4.2 that
\[
\frac{d}{dt} \mathcal{E}(t) + \tilde{F}(t) \leq C(1 + \mathcal{E} + \varepsilon^2 \mathcal{E} + \varepsilon^{14} \mathcal{E}^5) + (\delta + C \varepsilon + C \varepsilon^2 \mathcal{E}^2 + C \varepsilon^4 \mathcal{E}) \tilde{F},
\]
for any \( t \in [0, T_\varepsilon] \). Thanks to the assumptions of Theorem 1.1, we know that \( \mathcal{E}(0) \leq CE_0 \). Thus, there exists \( \varepsilon_0, E_1 > 0 \) depending on \( T, v, n, E_0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( t \in [0, \min(T, T_\varepsilon)] \),
\[
\mathcal{E}(t) + \int_0^t \tilde{F}(s) ds \leq E_1.
\]
This in turn implies that \( T_\varepsilon \geq T \) by a continuous argument. Then Theorem 1.1 follows.

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**Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China**

*E-mail address*: wangw07@pku.edu.cn

**School of Mathematical Sciences and LMAM, Peking University, Beijing 100871, China**

*E-mail address*: pzwang@pku.edu.cn

**School of Mathematical Sciences and LMAM, Peking University, Beijing 100871, China**

*E-mail address*: zfzhang@math.pku.edu.cn