On global classical solutions to 1D compressible Navier-Stokes equations with density-dependent viscosity and vacuum

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1 | INTRODUCTION AND MAIN RESULTS

We consider the one-dimensional compressible Navier-Stokes equations which read as follows:

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x + [P(\rho)]_x - [\mu(\rho)u_x]_x &= \rho f.
\end{aligned}
\] (1)

Here, \( t \geq 0 \) is time, \( x \in \Omega = (0, 1) \) is the spatial coordinate, \( \rho \) and \( u \) represent respectively the fluid density and velocity. The pressure \( P \) is given by

\[ P(\rho) = A\rho^\gamma \quad (A > 0, \ \gamma > 1). \] (2)

Without loss of generality, we set \( A = 1 \). The viscosity \( \mu(\rho) \) satisfies

\[ 0 < \tilde{\mu} \leq \mu(\rho), \quad \forall \ \rho \geq 0, \] (3)

where \( \tilde{\mu} \) is a given positive constant. The function \( f = f(x) \) is the external force. We look for solutions \((\rho(x, t), u(x, t))\) to the initial-boundary-value problem with the boundary conditions:

\[ u(0, t) = u(1, t) = 0, \quad t \geq 0, \] (4)
and the initial conditions:

\[(\rho, \rho u)(x, 0) = (\rho_0, \rho_0 u_0)(x).\]  (5)

There is a huge literature on the studies of the global existence and large time behavior of solutions to the compressible Navier-Stokes equations. For the initial density away from vacuum, there are many results concerning the global existence and large-time dynamics of solutions to the one-dimensional (1D) problem, see previous studies and the references therein. When the vacuum is allowed initially, as emphasized in many papers related to compressible fluid dynamics, the viscosity depends on the temperature for the non-isentropic case and thus on the density for the isentropic case. For \(\mu(\rho) = \rho^\theta\), under different restrictions on the index \(\theta\) and the regularities of initial data, the global existence of solutions to 1D compressible Navier-Stokes equations is investigated in other works and the references therein. When \(\mu(\rho)\) admits a positive constant lower bound, the global well-posedness of solutions without initial vacuum to 1D problem is discussed extensively (see the literature and the references therein). Recently, in addition to (3), under some additional stringent condition on \(\mu \in C^2[0, \infty)\) as follows:

\[\mu(\rho) \leq C(1 + P(\rho)), \quad \forall \rho \geq 0,\]  (6)

Ding-Wen-Zhu proves the global existence of classical large solutions to problem (1) to (2) with initial vacuum. However, since the upper bound of density obtained in Ding et al depends crucially on time, nothing is known concerning the large-time behavior of solutions. For the initial density away from vacuum, Stras%kraba-Zlotnik proves the large time behavior of solutions \((\rho, u)\) to Equation (1). More precisely, they show that \((\rho, u)\) tend to the stationary case \((\rho_s, 0)\) as time tends to infinity, where the stationary density \(\rho_s\) is a solution to the following stationary problem:

\[
\begin{align*}
\{ & [P(\rho_s)]_x = \rho_s f, & x & \in (0, 1), \\
& \int_0^1 \rho_s dx = \int_0^1 \rho_0 dx.
\end{align*}
\]  (7)

In this paper, for general density-dependent viscosity (3) without the requirement on upper bound of \(P(\rho)\), we will study the global existence and large-time behavior of strong and classical solutions to problems (1) to (5) where the initial vacuum is admitted.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For \(1 \leq r \leq \infty, k \geq 1,\)

\[L^r = L^r(0, 1), \quad W^{k,r} = W^{k,r}(0, 1), \quad H^k = W^{k,2}(0, 1).\]

Moreover, without loss of generality, assume that the initial density \(\rho_0\) satisfies

\[\int_0^1 \rho_0 dx = 1.\]  (8)

Our first result concerns the global existence of strong solutions with large initial data.

**Theorem 1.** Suppose that \(f \in H^1, \mu \in C^1[0, \infty),\) and the initial data \((\rho_0, u_0)\) satisfy

\[0 \leq \rho_0 \in H^1, \quad u_0 \in H^1_0.\]  (9)

Then, there exists a unique strong solution \((\rho, u)\) to problems (1) to (5) satisfying for any \(0 < T < \infty,\)

\[
\begin{align*}
\rho & \in C\left(\{0, T]\}; H^1\right) \cap H^1(0, T; L^2), \\
u & \in L^\infty(0, T; H^1_0) \cap L^2(0, T; H^2), \\
t^{1/2}u & \in L^\infty(0, T; H^2), \quad t^{1/2}u_t \in L^2(0, T; H^1_0).\end{align*}
\]  (10)

Moreover, the density remains uniformly bounded for all time, that is,

\[\sup_{0 \leq t < \infty} \|\rho(\cdot, t)\|_{L^\infty} < \infty.\]  (11)
and the following large-time behavior holds:

\[
\lim_{t \to \infty} \|u(\cdot, t)\|_{H^s} = 0, \quad \forall \quad p \in [1, \infty). \tag{12}
\]

Then, we show that the strong solutions obtain in Theorem 1 are indeed classical ones, provided the initial data \((\rho_0, u_0)\) satisfy some additional conditions.

**Theorem 2.** In addition to (9), suppose that \(f \in H^1, \mu \in C^2[0, \infty), \) and the initial data \((\rho_0, u_0)\) satisfy

\[
\rho_0 \in H^2, \quad P(\rho_0) \in H^2, \quad u_0 \in H^2 \cap H^1_0, \tag{13}
\]

and the following compatibility condition:

\[
[\mu(\rho_0)u_{0x}]_x - [P(\rho_0)]_x = \rho_0^{1/2}g \tag{14}
\]

for some \(g \in L^2.\) Then, the strong solutions \((\rho, u)\) obtained by Theorem 1 become classical and satisfy for any \(0 < T < \infty,

\[
\begin{cases}
\rho, P(\rho) \in C([0, T]; H^2), \\
u \in C([0, T]; H^2) \cap L^2(0, T; H^3), \\
u_t \in L^2(0, T; H^1), \\
u_{tt} \in L^2(0, T; L^2), \\
\end{cases}
\tag{15}
\]

For the large-time behavior of the strong solutions, we first state some known results about the existence and uniqueness of positive solutions to the stationary problem (7), which have been discussed extensively under different conditions (see, for example, previous works\(^{6,7,28,29}\)).

**Lemma 1.** \(^{29}\) If \(f \in L^\infty\) satisfies

\[
\int_0^1 (1 - y^{-1})^{1/2} \left( \int_0^x f(y)dy - \min_{[0,1]} \int_0^x f(y)dy \right)^{1/2} dx < \int_0^1 \rho_0 dx, \tag{16}
\]

there exists a unique positive solution \(\rho_1\) to (7) which satisfies

\[
\rho_1 \in W^{1, \infty}, \quad 0 < K_1 \leq \rho_1 \leq K_2, \tag{17}
\]

where both \(K_1\) and \(K_2\) are positive constants depending on \(\|f\|_{L^\infty}.

Then, we have the following result concerning the large time asymptotic behavior of strong solutions.

**Theorem 3.** Under the same conditions as in Theorem 1, assuming that \(f\) satisfies (16), there are positive constants \(\alpha\) and \(C\) depending only on the initial data and \(\|f\|_{L^2}\) such that for any \(p \in [1, \infty)\) and any \(t > 1,

\[
\|\rho(\cdot, t) - \rho_1(\cdot)\|_{L^p} + \|u(\cdot, t)\|_{W^{1,p}} \leq Ce^{-\frac{\alpha}{t}}. \tag{18}
\]

Moreover, if there exists some point \(x_0 \in [0, 1]\) such that \(\rho_0(x_0) = 0,\) the spatial gradient of the density has to blow up as \(t \to \infty\) in the following sense,

\[
\lim_{t \to \infty} \|\rho_x(x, t)\|_{L^r} = \infty, \quad \forall \quad r \in (1, \infty). \tag{19}
\]

A few remarks are listed in order:

**Remark 1.** It should be noted here that the solutions \((\rho, u)\) obtained in Theorem 2 are actually classical. Indeed, by Sobolev embedding theorems, we have

\[
H^k(0, 1) \hookrightarrow C^{\frac{k-1}{2}}[0, 1], \quad \text{for} \quad k = 1, 2, 3,
\]
which together with (15) gives
\[
(\rho, \, P, \, u) \in C\left([0, \, T]; \, C^{1+\frac{3}{2}}[0, \, 1]\right), \quad \rho_t \in C\left([0, \, T]; \, C^{\frac{3}{2}}[0, \, 1]\right).
\] (20)

Furthermore, one can deduce from (15) that for any \(0 < \tau < T\),
\[
u \in L^\infty(\tau, \, T; \, H^1), \quad u_t \in L^\infty(\tau, \, T; \, H^1) \cap L^2(\tau, \, T; \, H^2), \quad u_{tt} \in L^2(\tau, \, T; \, H^1).
\]

This yields that for any \(0 < \alpha < 1/2\),
\[
u \in C\left([\tau, \, T]; \, C^{2,\alpha}[0, \, 1]\right), \quad u_t \in C\left([\tau, \, T]; \, C^\alpha[0, \, 1]\right)\]
(21)

Hence, it follows from (20) and (21) that \((\rho, \, u)\) are classical solutions to problems (1) to (5).

**Remark 2.** To obtain the global strong solutions in Theorem 1, we do not need the additional compatibility condition (14). Furthermore, in order to prove the global well-posedness of classical solutions in Theorem 2, we only require that the initial data satisfying the compatibility condition (14) for some \(g \in L^2\) which is in sharp contrast to Ding et al\(^9\) where \(g \in H^1\) is assumed. Therefore, our theorems weaken essentially those assumptions on the compatibility condition in Ding et al\(^9\).

**Remark 3.** The density-dependent viscosity considered here is much more general than those in Ding et al\(^9\). More precisely, the restrictions on \(\mu(\rho)\) in Ding et al\(^9\) are not only (3) but also an additional stringent condition (6), which plays a crucial role in the analysis\(^9\).

**Remark 4.** In Theorem 1, we obtain the time-independent upper bound of the density (11) and the large-time behavior of the velocity (12), which are in sharp contrast to Ding et al\(^9\) where the corresponding a priori estimates depend on time. Moreover, Theorem 1 also generalized the similar results in other studies\(^6,7\) without initial vacuum to the case that the initial density admits vacuum.

**Remark 5.** In contrast to the results in previous works\(^6,7\), where the spatial \(L^2\)-norm of \(\rho_x\) is proved to be bounded independently of time provided the density strictly away from vacuum, Theorem 3 shows that the spatial \(L^r\)-norm \((r > 1)\) of \(\rho_x\) will blow up as time goes to infinity when the initial density admits vacuum.

We now make some comments on the analysis of this paper. We begin with the local existence theorem (see Cho and Kim\(^8\) or Lemma 2) of classical solutions to problems (1) to (5) with the initial density strictly away from vacuum. Then, we prove that the local strong (classical) solution with vacuum also exists and thus extend the local existence time to be global. Hence, we need some global a priori estimates which do not depend on the lower bound of density. It turns out that the key issue is to derive both the time-independent lower order estimates and the time-dependent higher-order ones (see Section 2). It should be noted that the methods used in Ding-Wen-Zhu\(^9\) cannot be adapted here. Indeed, on the one hand, the analysis in Ding et al\(^9\) relies heavily on the special assumption on viscosity \(\mu(\rho)\) (see (6)), that is \(\mu(\rho)\) should be bounded by \(P(\rho)\) pointwisely. On the other hand, it seems difficult to study the large-time behavior of solutions since the a priori estimates obtained in Ding et al\(^9\) are all time-depending. To overcome these difficulties, motivated by Li-Xin\(^1, 3\),\(^1\)\(^4\),\(^1\)\(^4\),\(^1\)\(^6\) we succeed in obtaining the key uniform upper bound of the density by making full use of Zlotnik inequality (see Lemma 4), and bounding the \(L^2\)-norm of \(u_x\) according to the material derivative \(\dot{u}\) (see Lemma 5). The time-dependent higher-order estimates of \((\rho, \, u)\) are derived by some standard arguments and the time-weighted estimates due to Hoff\(^3\) (see also the literature\(^10,1\)\(^4,1\)\(^6\)). Next, with both the uniform upper bound of the density and the time-independent lower-order estimates at hand, we use the methods owing to Straškraba-Zlotnik\(^5,7\) and thus prove the following large-time behavior

\[
\lim_{t \to \infty} (\|u(\cdot, \, t)\|_{W^{1, \rho}} + \|\rho - \rho_x\|_{L^p}) = 0, \quad \forall \quad p \in [1, \, \infty).
\]

Finally, using a key testing function \(\rho_x^{-1} \int_0^x (\rho - \rho_x)dy\) motivated by Huang-Li-Xin\(^3\)\(^1\) and Li-Zhang-Zhao\(^1\)\(^6\), we derive the desired exponential decay rate with time (18) in Theorem 3 (see Section 4).
The rest of the paper is organized as follows: In Section 2, we will derive the necessary a priori estimates on smooth solutions. The main results, Theorems 1, 2, and 3, are proved in Sections 3 and 4, respectively.

# 2 A PRIORI ESTIMATES

In this section, we will establish some necessary a priori bounds for smooth solutions to problems (1) to (5) to extend the local classical solutions guaranteed by following Lemma 2, whose proof can be completed by similar arguments as in Cho and Kim.\(^8\)

**Lemma 2.** Assume that \( f \in H^2 \) and the initial data \((\rho_0, u_0)\) satisfy
\[
0 < \delta \leq \rho_0, \quad \rho_0 \in H^3, \quad P(\rho_0) \in H^3, \quad u_0 \in H^3 \cap H_0^1.
\]
Then, there exists a small time \( T_0 > 0 \) and a unique classical solution \((\rho, u)\) to problems (1) to (5) on \((0,1) \times (0,T_0)\) such that
\[
\begin{align*}
\rho, P(\rho) &\in C \left([0,T_0]; H^3 \right), \\
u &\in C \left([0,T_0]; H^3 \cap H_0^1 \right) \cap L^2(0,T_0; H^4), \\
u &\in C \left([0,T_0]; H_0^1 \right) \cap L^2(0,T_0; H^2).
\end{align*}
\]

## 2.1 A priori estimates (I): lower-order estimates

In this subsection, we will derive the time-independent lower order estimates of the solution \((\rho, u)\), use the convention that \( C \) denotes a generic positive constant depending on \( y, \mu, \|\rho_0\|_{H^3}, \|u_0\|_{H^3}, \) and \( \|f\|_{H^3} \) but independent of \( T \), and use \( C(a) \) to emphasize that \( C \) depends on \( a \).

First, we state the following Zlotnik inequality, whose proof can be found in Zlotnik.\(^28\)

**Lemma 3.** \(^28\)Let the function \( y \) satisfies
\[
y'(t) \leq g(y) + b'(t) \text{ on } [0,T], \quad y(0) = y^0,
\]
with \( g \in C(\mathbb{R}) \) and \( y, b \in W^{1,1}(0,T) \). If \( g(\infty) = -\infty \) and
\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
\]
for all \( 0 \leq t_1 < t_2 \leq T \) with some \( N_0 \geq 0 \) and \( N_1 \geq 0 \), then
\[
y(t) \leq \max \{ y^0, \tilde{\zeta} \} + N_0 < \infty \text{ on } [0,T],
\]
where \( \tilde{\zeta} \) is a constant such that
\[
g(\zeta) \leq -N_1 \text{ for } \zeta \geq \tilde{\zeta}.
\]

Then, using the Zlotnik inequality in Lemma 3, we will give the key time-independent upper bound for the density as follows.

**Lemma 4.** There is a positive constant \( C \) such that for any \((x,t) \in [0,1] \times [0,T]\),
\[
0 \leq \rho(x,t) \leq C. \quad (22)
\]

**Proof.** First, the standard energy estimate leads to
\[
sup_{0 \leq t \leq T} \int_0^1 \left( \frac{1}{2} \rho u^2 + \frac{1}{y-1} P \right) dx + \int_0^T \int_0^1 \mu(\rho) u_x^2 dx dt
\]
\[
\leq C + \sup_{0 \leq t \leq T} \int_0^1 \rho \int_0^x f dy dx
\]
\[
\leq C + \|f\|_{L^\infty} \sup_{0 \leq t \leq T} \int_0^1 \rho dx
\]
\[
\leq C. \quad (23)
\]
where in the last inequality, one has used the following fact:

\[ \int_0^1 \rho \, dx = \int_0^1 \rho \, dx = 1 \] (24)

going to (8) and (1)1.

Next, integrating (1)2 over \((0, x)\) gives

\[ -\mu(\rho)u_x = -P(\rho) + \int_0^x f \, dy - \frac{\partial}{\partial t} \int_0^x \rho u \, dy - \rho u^2 \]

\[ + (P(\rho) + \rho u^2 - \mu(\rho)u_x)(0, t), \] (25)

which in particular implies

\[ (P(\rho) + \rho u^2 - \mu(\rho)u_x)(0, t) = -\int_0^1 \mu(\rho)u_x \, dx + \int_0^1 P(\rho) \, dx - \int_0^x \int_0^1 f \, dy \, dx \]

\[ + \frac{\partial}{\partial t} \int_0^x \rho u \, dy + \int_0^1 \rho u^2 \, dx. \] (26)

Combining this with (25) shows that for \(D_t \triangleq \frac{\partial}{\partial t} + \mu \frac{d}{dx}\)

\[ -\mu(\rho)u_x + P(\rho) = \int_0^x \rho f \, dy - \int_0^1 \int_0^x \rho f \, dy \, dx + \int_0^1 \rho u^2 \, dx + \int_0^1 P(\rho) \, dx \]

\[ + D_t \left( \int_0^1 \rho \int_0^x \mu(s) s^{-2} ds \, dx + \int_0^x \int_0^1 \rho u \, dy \, dx - \int_0^x \rho u \, dy \right) \]

\[ \leq C + D_t (b_1(t) + b_2(t) + b_3(t)), \] (27)

where one has used (23) and the following fact

\[ -\int_0^1 \mu(\rho)u_x \, dx = D_t \int_0^1 \rho \int_0^x \mu(s) s^{-2} ds \, dx \]

due to (1)1.

Next, it follows from (1)1 that

\[ -\mu(\rho)u_x = D_t \int_0^x \mu(s) s^{-1} ds, \]

which together with (27) gives

\[ D_t \int_0^x \mu(s) s^{-1} ds \leq -P(\rho) + C + D_t (b_1(t) + b_2(t) + b_3(t)). \] (28)

Then, on the one hand, we have

\[ b_1(t) \leq 2^{-1} \int_1^{\max \{\sup \rho, 2\}} \mu(s) s^{-1} ds, \] (29)

where \(\Omega_T = [0, 1] \times [0, T]\). On the other hand, one deduces from (23) that

\[ |b_2(t)| + |b_3(t)| \leq C \int_0^1 \int_0^x \rho f \, dy \, dx + \int_0^1 \rho u^2 \, dx + \int_0^1 \rho \, dx \leq C. \] (30)

Finally, applying the Zlotnik inequality (Lemma 3 to 28–28), we get after using (29) and (30) that

\[ \int_0^1 \mu(s) s^{-1} ds \leq C + 2^{-1} \int_1^{\max \{\sup \rho, 2\}} \mu(s) s^{-1} ds, \]
which together with (3) implies (22) and completes the proof of Lemma 4.

Lemma 5. Let \((\rho, u)\) be smooth solutions of (1) to (5) on \([0, 1] \times [0, T]\), then

\[
\sup_{0 \leq t \leq T} \|u\|_{L^2}^2 + \int_0^T \|\rho^{1/2} \dot{u}\|_{L^2}^2 \, dt \leq C,
\]

where, and in what follows, we denote \(\dot{u} \triangleq u_t + u u_x\).

Proof. First, multiplying \((3)_2\) by \(\dot{u}\) and integrating the resulting equation by parts yield

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \mu(\rho) u_x^2 \, dx + \int_0^1 \rho \dot{u}^2 \, dx = \frac{d}{dt} \left( \int_0^1 P(\rho) u_x \, dx + \int_0^1 \rho f u \, dx \right)
- \frac{1}{2} \int_0^1 (\mu(\rho) - \mu'(\rho) \rho) u_x^2 \, dx + \gamma \int_0^1 P(\rho) u_x^2 \, dx - \int_0^1 \rho u^2 f_x \, dx
\leq \frac{d}{dt} \left( \int_0^1 P(\rho) u_x \, dx + \int_0^1 \rho f u \, dx \right) + C \|u_x\|_{L^2}^2 + C \|u\|_{L^\infty} \|f_x\|_{L^2}
\leq \frac{d}{dt} \left( \int_0^1 P(\rho) u_x \, dx + \int_0^1 \rho f u \, dx \right) + C \|u_x\|_{L^\infty} \|u_x\|_{L^2} + C \|u_x\|_{L^2}^2,
\]

where one has used (22) and the following direct calculations:

\[
\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2} \leq C \|u_x\|_{L^2}.
\]

Then, using \((1)_2\), \((3)_2\), \((23)_3\), and \((22)_2\), we have

\[
\|u_x\|_{L^\infty} \leq C \|\mu(\rho) u_x - P(\rho)\|_{L^\infty} + C \|P(\rho)\|_{L^\infty}
\leq C \|\mu(\rho) u_x - P(\rho)\|_{L^1} + C \|u\|_{L^1} \|u_x\|_{L^1} + C
\leq C \|\mu(\rho) u_x\|_{L^1} + C \|P(\rho)\|_{L^1} + C \|\rho u\|_{L^1} + C \|\rho f\|_{L^1} + C
\]

which together with (32) and Young’s inequality gives

\[
\frac{d}{dt} B(t) + \frac{1}{2} \int_0^1 \rho \dot{u}^2 \, dx \leq C \left( \int_0^1 \mu(\rho) u_x^2 \, dx \right)^2 + C \int_0^1 \mu(\rho) u_x^2 \, dx,
\]

where

\[
B(t) \triangleq \frac{1}{2} \int_0^1 \mu(\rho) u_x^2 \, dx - \int_0^1 P(\rho) u_x \, dx - \int_0^1 \rho f u \, dx
\]

satisfies

\[
\frac{1}{4} \left\| \mu(\rho) u_x \right\|_{L^2}^2 - C \leq B(t) \leq C \left\| \mu(\rho) u_x \right\|_{L^2}^2 + C
\]

owing to the following estimate:

\[
\int_0^1 P(\rho) u_x \, dx + \int_0^1 \rho f u \, dx \leq \frac{1}{4} \int_0^1 \mu(\rho) u_x^2 \, dx + C,
\]

due to (3), (23), and (22).

Hence, Gronwall’s inequality together with (35), (36), and (23) implies (31) and thus finishes the proof of Lemma 5.
Lemma 6. Let \((\rho, u)\) be smooth solutions of (1) to (5) on \((0, 1) \times [0, T]\), then

\[
\sup_{0 \leq t \leq T} \sigma \left( \|u\|_{W^{1,\infty}}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) + \int_0^T \sigma \|\dot{u}_x\|_{L^2}^2 dt \leq C,
\]

where, and in what follows, \(\sigma(t) \triangleq \min\{1, t\}\).

Proof. Operating \(\partial_t + (u \cdot \nabla)_x\) to (1)_2 yields that

\[
\rho \ddot{u}_t + \rho \dot{u} u_x - [\mu(\rho) \dot{u}_x]_x = -\gamma [P(\rho) \dot{u}]_x - [(\mu(\rho) + \mu'(\rho) \rho) \dot{u}_x^2]_x + \rho f_x,
\]

which multiplied by \(\dot{u}\) gives

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \rho |\dot{u}|^2 dx + \int_0^1 \mu(\rho) |\dot{u}_x|^2 dx = \gamma \int_0^1 P(\rho) \dot{u} u_x dx + \int_0^1 (\mu(\rho) + \mu'(\rho) \rho) \dot{u}_x^2 dx + \int_0^1 \rho f_x \dot{u} dx
\]

\[
\leq C \|u_x\|_{L^2} \|\dot{u}_x\|_{L^2} (1 + \|u_x\|_{L^\infty}) + C \|\dot{u}\|_{L^2} \|f_x\|_{L^2} \|\rho^{1/2} \dot{u}\|_{L^2}
\]

\[
\leq C \|u_x\|_{L^2} \|\dot{u}_x\|_{L^2} (1 + \|\rho^{1/2} \dot{u}\|_{L^2}) + C \|u_x\|_{L^2} \|\rho^{1/2} \dot{u}\|_{L^2}
\]

\[
\leq C \|u_x\|_{L^2} + C(\varepsilon) \|u_x\|_{L^2} + C \|\rho^{1/2} \dot{u}\|_{L^2},
\]

due to (34) and (31).

Then, integrating (39) multiplied by \(\sigma\) over \((0, T)\), we obtain after choosing \(\varepsilon\) suitably small and using (31) and (23) that

\[
\sup_{0 \leq t \leq T} \sigma \left( \int_0^1 \rho |\dot{u}|^2 dx + \int_0^T \sigma \int_0^1 \mu(\rho) |\dot{u}_x|^2 dx dt \right) \leq C,
\]

which together with (33), (34), and (31) leads to

\[
\sup_{0 \leq t \leq T} \sigma \|u_x\|_{L^\infty}^2 \leq C.
\]

Combining this with (40) gives (37). The proof of Lemma 6 is finished. \(\square\)

2.2 A priori estimates (II): higher-order estimates

In this subsection, we will prove the higher-order estimates of the smooth solutions \((\rho, u)\) to problems (1) to (5).

Lemma 7. For any given \(T > 0\), there exists a positive constant \(C\) depending on \(T, \gamma, \bar{\mu}, \|\rho_0\|_{H^s}, \|u_0\|_{H^s},\) and \(\|f\|_{H^s}\) such that

\[
\sup_{0 \leq t \leq T} \left( \|\rho_x\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 \right) \leq C,
\]

and

\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2} u\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|\rho u_{xx}\|_{L^2}^2 \right) + \int_0^T \left( \|u_{xx}\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 \right) dt \leq C.
\]

Proof. First, differentiating (1)_1 with respect to \(x\) gives

\[
\rho_{xx} + \rho_{xx} u + 2 \rho_x u_x + \rho u_{xx} = 0.
\]

Multiplying (43) by \(\rho_x\) and integrating the resulting equation by parts yield that

\[
\frac{d}{dt} \|\rho_x\|_{L^2} \leq C \|u_x\|_{L^\infty} \|\rho_x\|_{L^2} + C \|u_{xx}\|_{L^2},
\]

due to (22).
Next, it is easy to deduce from (1) that
\[ \mu(\rho)u_{xx} = \rho \dot{u} + P_x - \rho f - \mu'(\rho)\rho_x u_x, \] (45)
which together with (5), (22), (31), and (37) yields
\[ \|u_{xx}\| L^2 \leq C\|\rho^{1/2} \dot{u}\| L^2 + C\|\rho_x\| L^2 (\|u_x\| L^\infty + 1) + C \leq C\sigma^{-1/2} (1 + \|\rho_x\| L^2). \] (46)

Submitting (46) into (44) and using Gronwall's inequality, one gets
\[ \sup_{0 \leq t \leq T} \|\rho_x\| L^2 \leq C, \] (47)
which along with (1), and (31) leads to
\[ \|\rho_t\| L^2 \leq \|\mu\| L^\infty \|\rho_x\| L^2 + C\|u_x\| L^2 \leq C. \] (48)

The combination of (47) with (48) leads to (41).

Now, it follows from (22), (34), (31), (46), and (41) that
\[ \|\rho^{1/2} u_t\| L^2 \leq \|\rho^{1/2} \dot{u}\| L^2 + \|\rho^{1/2} \rho u\| L^2 \leq \|\rho^{1/2} \dot{u}\| L^2 + C, \]
\[ \|u_{xt}\| L^2 = \|(\dot{u} - \rho u_t)\| L^2 \leq \|\dot{u}\| L^2 + \|\rho u\| L^2 \leq \|\dot{u}\| L^2 + \|\rho u\| L^2 \]
\[ \leq \|\dot{u}_x\| L^2 + \|u_x\| L^\infty \|u_x\| L^2 \leq \|\dot{u}_x\| L^2 + \|\rho^{1/2} \dot{u}\| L^2 + C. \] (49)

Hence, (42) is a direct consequence of (49), (37), (46), and (31). We complete the proof of Lemma 7.

From now on, let \((\rho, u)\) be smooth solutions to problems (1) to (5) with the initial data satisfying the condition in Theorem 2 and \(\mu(\cdot) \in C^2[0, \infty)\). In the following, the general constant \(C\) may depend on \(T, \gamma, \bar{\mu}, \|\rho_0\| H^2, \|P(\rho_0)\| H^1, \|u_0\| H^2, \|f\| H^1, \) and \(\|g\| L^2\) with \(g\) given in (14).

**Lemma 8.** For any given \(T > 0\), there exists a positive constant \(C\) such that
\[ \sup_{0 \leq t \leq T} \left( \|u_{xx}\| L^2 + \|u\| W^{1,\infty} + \|\rho^{1/2} u_t\| L^2 \right) + \int_0^T \|u_{xx}\| L^2 dt \leq C. \] (50)

**Proof.** The compatibility condition (14) shows that
\[ \rho^{1/2} \dot{u}(x, t = 0) = g + \rho_0^{1/2} f \in L^2. \]

Integrating (39) over \((0, T)\), we obtain after using (31) and (23) that
\[ \sup_{0 \leq t \leq T} \int_0^1 \rho |\dot{u}|^2 dx + \int_0^T \int_0^1 \mu(\rho)|\dot{u}_x|^2 dx dt \leq C, \]
which together with (49), (33), (34), and (31) gives (50) and completes the proof of Lemma 8.

The following higher-order estimates of the solutions are used to guarantee the extension of local classical solutions to global, whose proof are similar to those in Ding et al., see also the literature considering the high-dimensional case. And we also sketch them here for completeness.

**Lemma 9.** For any given \(T > 0\), there exists a positive constant \(C\) such that
\[ \sup_{0 \leq t \leq T} \left( \|\rho xx\|^2 L^2 + \|P xx\|^2 L^2 + \|\rho xx\|^2 L^2 + \|P xx\|^2 L^2 \right) \]
\[ + \int_0^T \left( \|u_{xx}\|^2 L^2 + \|\rho_t\|^2 L^2 + \|P_t\|^2 L^2 \right) dt \leq C. \] (51)
Proof. Differentiating (43) with respect to $x$ gives

\[(\rho_{xx})_t + \rho_{xxx}u + 3\rho_{xx}u_x + 3\rho_xu_{xx} + \rho u_{xxx} = 0.\]  

(52)

Multiplying (52) by $\rho_{xx}$ and integrating the resulting equation by parts, it holds that

\[
\frac{1}{2} \frac{d}{dt} \|\rho_{xx}\|_{L^2}^2 = -\frac{5}{2} \int_0^1 \rho_{xx}^2 u_x dx - 3 \int_0^1 \rho_x \rho_{xx} u_{xx} dx - \int_0^1 \rho \rho_{xx} u_{xxx} dx
\]

\[
\leq C \|u_x\|_{L^2} \|\rho_{xx}\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2} \|\rho_{xx}\|_{L^2} + C \|u_{xxx}\|_{L^2} \|\rho_{xx}\|_{L^2}
\]

\[
\leq C \|\rho_{xx}\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C,
\]

(53)

where in the last inequality one has used (50), (41), and the following estimate:

\[
\|\rho_x\|_{L^\infty} \leq \|\rho_x\|_{L^2} + \|\rho_{xx}\|_{L^2} \leq C + C \|\rho_{xx}\|_{L^2}.
\]

(54)

Since $P(\rho)$ satisfies

\[P_t + P_xu + \gamma Pu_x = 0,
\]

(55)

following the same arguments as (53), one has

\[
\frac{1}{2} \frac{d}{dt} \|P_{xx}\|_{L^2}^2 \leq C \|P_{xx}\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C.
\]

(56)

Hence, the combination of (56) with (53) implies that

\[
\frac{1}{2} \frac{d}{dt} (\|\rho_{xx}\|_{L^2}^2 + \|P_{xx}\|_{L^2}^2) \leq C (\|\rho_{xx}\|_{L^2}^2 + \|P_{xx}\|_{L^2}^2) + C + C \|u_{xxx}\|_{L^2}^2.
\]

(57)

In order to estimate $\|u_{xxx}\|_{L^2}^2$, differentiating (45) with respect to $x$ gives

\[
\mu(\rho)u_{xxx} = \rho_xu_t + \rho_xu_{xt} + \rho u_{xx} + \rho uu_{xx} + Pu_{xx}
\]

\[
- \rho_xf - \mu'(\rho)\rho_xu_t - \mu'(\rho)\rho_xu_x - 2\mu'(\rho)\rho_xu_{xx}.
\]

(58)

Similar to (33), we also have

\[
\|u_t\|_{L^\infty} \leq C \|u_x\|_{L^2},
\]

(59)

which along with (58), (5), (22), (41), (54), and (50) implies that

\[
\|u_{xxx}\|_{L^2}^2 \leq C \|\rho_{xx}\|_{L^2}^2 \|u_t\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2}^2 \|u_x\|_{L^\infty} \|u_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^4
\]

\[
+ C \|u\|_{L^2}^2 \|u_{xx}\|_{L^2}^2 + C \|P_{xx}\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2}^2 \|f\|_{L^2}^2 + C \|f_x\|_{L^2}^2
\]

\[
+ C \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^2 \|\rho_{xx}\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2}^2
\]

\[
\leq C \left(1 + \|u_x\|_{L^2}^2 + \|\rho_{xx}\|_{L^2}^2 + \|P_{xx}\|_{L^2}^2\right).
\]

(60)

Submitting (60) into (57), one obtains after using Gronwall’s inequality and (50) that

\[
\sup_{0 \leq t \leq T} \left(\|\rho_{xx}\|_{L^2}^2 + \|P_{xx}\|_{L^2}^2\right) \leq C.
\]

(61)

This together with (43), (50), and (41) implies that

\[
\|\rho_{xx}\|_{L^2}^2 \leq \|\rho_{xx}\|_{L^2}^2 \|u_t\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^2 \leq C.
\]

(62)

Differentiating (1) with respect to $t$ gives

\[
\rho_t = -\rho_{xt}u - \rho Xu - \rho u_x - \rho u_{xt}
\]
which combined with (62), (41), (59), and (50) implies that

\[
\int_0^T \|\rho_t\|^2_{L^2} \, dt \leq \int_0^T \left( \|\rho_{xt}\|^2_{L^2} + \|\rho_{xt}\|^2_{L^\infty} + \|\rho_{xt}\|_{L^2} + C\|u_{xt}\|_{L^2} \right) \, dt \\
+ C \int_0^T \left( \|\rho_t\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2 \right) \, dt \\
\leq C \int_0^T (1 + \|u_{xt}\|_{L^2}^2) \, dt \\
\leq C. 
\]

(63)

Similarly, it holds

\[
\|P_{xt}\|^2_{L^2} + \int_0^T \|P_{tt}\|^2_{L^2} \, dt \leq C, 
\]

which together with (60), (50), (61), (62), and (63) implies (51). The proof of Lemma 9 is completed.

Lemma 10. For any given \( T > 0 \), there exists a positive constant \( C \) such that

\[
\sup_{0 \leq t \leq T} \left( t\|u_{xt}\|^2_{L^2} + t\|u_{xxx}\|^2_{L^2} \right) + \int_0^T \left( t\|\rho_{1/2} u_t\|^2_{L^2} + t\|u_{xxt}\|^2_{L^2} \right) \, dt \leq C. 
\]

(64)

Proof. Differentiating (1) with respect to \( t \) gives

\[
\rho_{tt} + \rho u_{xt} - \left[ \mu(\rho) u_x \right]_{xt} = -\rho_t (u_t + uu_x) - \rho u_{xt} - P_{xt} + \rho t f. 
\]

(65)

Multiplying (65) by \( u_t \) and integrating the resulting equation by parts lead to

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \mu u_t^2 dx + \|\rho^{1/2} u_t\|^2_{L^2} \\
= \frac{1}{2} \int_0^1 \mu u_t^2 dx - \int_0^1 \mu u_t u_{xt} dx + \int_0^1 \rho_t f u_t dx - \int_0^1 \rho_t u_{xt} dx - \int_0^1 P_{xt} u_t dx \\
\triangleq \sum_{i=1}^7 J_i. 
\]

(66)

The term \( J_i \) on the right hand of (66) can be estimated as follows. Using (41) and (51), it holds

\[
\|\rho_t\|_{L^\infty} \leq C\|\rho_t\|_{L^2} + C\|u_{xt}\|_{L^2} \leq C, 
\]

(67)

which implies that

\[
J_1 \leq C\|\mu_t\|_{L^\infty} \|u_{xt}\|_{L^2}^2 \leq C\|\rho_t\|_{L^\infty} \|u_{xt}\|_{L^2}^2 \leq C\|u_{xt}\|_{L^2}^2. 
\]

(68)

It deduces from (22), (41), and (67) that

\[
\|\mu_t\|_{L^2}^2 \leq \|\mu'(\rho)\rho_t^2\|_{L^2}^2 + \|\mu'(\rho)\rho_{xt}\|_{L^2}^2 \\
\leq C\|\rho_t\|_{L^\infty} \|\rho_t\|_{L^2}^2 + C\|\rho_{xt}\|_{L^2}^2 \\
\leq C + C\|\rho_t\|_{L^2}^2. 
\]
which along with (50) and (67) gives

\[
J_2 = -\frac{d}{dt} \int_0^1 \mu u u_{xl} dx + \int_0^1 \mu u u_{xx} dx + \int_0^1 \mu u_x^2 dx \\
\leq -\frac{d}{dt} \int_0^1 \mu u u_{xl} dx + C \|u_x\|_{L^\infty} \|u_{xl}\|_{L^2} + C \|\mu\|_{L^\infty} \|u_{xl}\|_{L^2}^2 \\
\leq -\frac{d}{dt} \int_0^1 \mu u u_{xl} dx + C \|u_{xl}\|_{L^2}^2 + C \|\rho u\|_{L^2}^2.
\]  

(69)

Next, it is easy to derive from (59) that

\[
J_3 = \frac{d}{dt} \int_0^1 \rho f u_t dx - \int_0^1 \rho u f u_t dx \\
\leq \frac{d}{dt} \int_0^1 \rho f u_t dx + C \|u_t\|_{L^\infty} \|\rho u\|_{L^2} \|f\|_{L^2} \\
\leq \frac{d}{dt} \int_0^1 \rho f u_t dx + C \|u_{xl}\|_{L^2}^2 + C \|\rho u\|_{L^2}^2.
\]  

(70)

For the term \(J_4\), we have

\[
J_4 = \int_0^1 (\rho u)_t u_{xl} dx = -\int_0^1 \rho u u_{xt} dx - \int_0^1 \rho u_t u_{xt} dx \\
= -\frac{d}{dt} \int_0^1 \rho u u_{xt} dx + \int_0^1 (\rho u)_t u_{xt} dx + \int_0^1 \rho u_t u_{xt} dx - \int_0^1 \rho u_t u_{xt} dx \\
= -\frac{d}{dt} \int_0^1 \rho u u_{xt} dx + J_{4,1} + J_{4,2} + J_{4,3}.
\]  

(71)

One deduces from (50), (41), and (59) that

\[
J_{4,1} = \int_0^1 \rho_t u u_{xt} dx + \int_0^1 \rho u_t u_{xt} dx \\
\leq C (\|u_t\|_{L^\infty} + \|u u_x\|_{L^\infty}) \|u_{xt}\|_{L^2} \|u_{xl}\|_{L^\infty} \|\rho_t\|_{L^2} + C \|\rho^{1/2} \dot{u}\|_{L^2} \|u_t\|_{L^\infty} \|u_{xl}\|_{L^2} \\
\leq C + C \|u_{xl}\|_{L^2}^2,
\]

\[
J_{4,2} = \int_0^1 \rho u u_{xt} u_{xl} dx + \int_0^1 \rho u u_x u_{xt} dx + \int_0^1 \rho u u_{xx} dx \\
\leq C \|\rho^{1/2} u\|_{L^\infty} \|u_{xt}\|_{L^2} \|u_{xl}\|_{L^2} \|\rho^{1/2} u_t\|_{L^2} \\
+ C \|\rho u\|_{L^\infty} \|u_t\|_{L^\infty} \|u_{xl}\|_{L^2} \|u_{xt}\|_{L^2} + C \|\rho u^2\|_{L^\infty} \|u_{xl}\|_{L^2} \\
\leq \epsilon \|\rho^{1/2} u_t\|_{L^2}^2 + C \|u_{xl}\|_{L^2}^2,
\]

\[
J_{4,3} \leq C \|\rho^{1/2} u\|_{L^\infty} \|u_{xl}\|_{L^2} \leq \epsilon \|\rho^{1/2} u_t\|_{L^2}^2 + C \|u_{xl}\|_{L^2}^2 + C.
\]

This combined with (71) yields

\[
J_4 \leq -\frac{d}{dt} \int_0^1 \rho u u_{xt} dx + \epsilon \|\rho^{1/2} u_t\|_{L^2}^2 + C \|u_{xl}\|_{L^2}^2 + C.
\]  

(72)

Moreover, by virtue of (50), it holds that

\[
J_5 + J_6 \leq C \|u_t\|_{L^\infty} \|\rho^{1/2} u_t\|_{L^2} \|\rho^{1/2} u\|_{L^2} + C \|\rho^{1/2} u\|_{L^\infty} \|\rho^{1/2} u u_t\|_{L^2} \|u_{xl}\|_{L^2} \\
\leq \epsilon \|\rho^{1/2} u_t\|_{L^2}^2 + C \|u_{xl}\|_{L^2}^2 + C.
\]  

(73)
Finally, we can estimate 

\[ J_7 = \int_0^1 \frac{d}{dt} \int_0^1 P_u u_{xx} dx - \int_0^1 P_{tt} u_{xx} dx \]

where

\[ \leq \frac{d}{dt} \int_0^1 P_u u_{xx} dx + C \| P_{tt} \|_{L^2}^2 + C \| u_{xx} \|_{L^2}^2. \]  

(74)

Submitting (68) to (74) into (66) and choosing \( \varepsilon \) suitably small, we have

\[ \frac{d}{dt} \Pi(t) + \| \rho^{1/2} u_{xx} \|_{L^2}^2 \leq C \| u_{xx} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 + C \| P_{tt} \|_{L^2}^2, \]  

(75)

where

\[ \Pi(t) \triangleq \int_0^1 \mu u_{xx}^2 dx + \int_0^1 \mu u_{x} u_{xx} dx + \int_0^1 \rho u^2 u_{xx} dx - \int_0^1 P_u u_{xx} dx - \int_0^1 \rho f u_{xx} dx \]

satisfies

\[ \frac{\mu}{2} \| u_{xx} \|_{L^2}^2 \leq \Pi(t) \leq C \| u_{xx} \|_{L^2}^2 + C \]

(76)

owing to the following estimates:

\[ \left| \int_0^1 \mu u_{xx} u_{xx} dx + \int_0^1 \rho u^2 u_{xx} dx - \int_0^1 P_u u_{xx} dx - \int_0^1 \rho f u_{xx} dx \right| \]

\[ \leq C \| \rho_{tt} \|_{L^2}^2 \| \rho_{tt} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| P_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| u_{tt} \|_{L^2}^2 \| \rho_{tt} \|_{L^2}^2 \| f \|_{L^2}^2 \]

\[ \leq \frac{\mu}{2} \| u_{xx} \|_{L^2}^2 + C, \]

where one has used (50), (41), and (59).

Hence, the Gronwall's inequality together with (75), (76), (50), and (51) gives

\[ \sup_{0 \leq t \leq T} t \| u_{xx} \|_{L^2}^2 + \int_0^T t \| \rho^{1/2} u_{tt} \|_{L^2}^2 dt \leq C, \]  

(77)

which along with (60) and (51) implies

\[ \sup_{0 \leq t \leq T} t \| u_{xx} \|_{L^2}^2 \leq C. \]

(78)

Next, it follows from (65) that

\[ \mu(\rho) u_{xx} = \rho_{tt} + \rho_{tt} u_{xx} + \rho u_{x} u_{xx} + P_{xx} - \rho f - \mu'((\rho)) \rho_{tt} u_{xx} - \mu'((\rho)) \rho_{tt} u_{xx} - \mu'((\rho)) \rho_{tt} u_{xx}. \]

Combining this with (59), (41), (50), (51), and (67) yields that

\[ \| u_{xx} \|_{L^2}^2 \leq C \| \rho^{1/2} u_{tt} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| u_{tt} \|_{L^2}^2 \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 \]

\[ + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| P_{xx} \|_{L^2}^2 + C \| f \|_{L^2}^2 \| \rho_{tt} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 \]

\[ + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 + C \| \rho_{tt} \|_{L^2}^2 \| u_{xx} \|_{L^2}^2 \]

\[ \leq C \| \rho^{1/2} u_{tt} \|_{L^2}^2 + C \| u_{xx} \|_{L^2}^2 + C, \]

which along with (77) implies that
\[
\int_0^T t\|u_{xtt}\|_{L^2}^2 dt \leq C. \tag{79}
\]

The combination of (77) to (78) with (79) yields (64) and completes the proof of Lemma 10. \qed

**Lemma 11.** For any given \( T > 0 \), there exists a positive constant \( C \) such that

\[
\sup_{0 \leq t \leq T} \left( t^2 \|\rho^{1/2} u_t\|_{L^2}^2 \right) + \int_0^T \left( t^2 \|u_t\|_{L^2}^2 + t^2 \|u_{xtt}\|_{L^2}^2 \right) dt \leq C. \tag{80}
\]

**Proof.** Differentiating (65) with respect to \( t \) gives

\[
\rho u_{tt} + \rho u_{xtt} - [\mu(\rho) u_{x}^2]_{x} = -2\rho u_t - 2\rho uu_{xt} - 2\rho u_{tx} + \rho \mu u_t + \rho \mu u_x - \rho u + u u_x - P_{xt} + \rho f. \tag{81}
\]

Multiplying (81) by \( u_t \) and integrating the resulting equations by parts yield that

\[
\frac{1}{2} \frac{d}{dt} \int \rho u_{tt}^2 dx + \int \mu(\rho) u_{xtt}^2 dx = -\int \rho u_t u_{tt} dx - 2 \int \rho u_{xt} u_{xtt} dx - 2 \int \rho u_{tt}^2 dx
\]

\[-2 \int \rho u u_{xt} u_{tt} dx - 2 \int \rho u_t u_{xt} u_{xtt} dx \]

\[+ \int \rho \mu(u_t + u_{x}) u_{tt} dx - \int \rho u_t u_{xx} u_{tt} dx - \int P_{xt} u_t dx\]

\[\triangleq \sum_{i=1}^{10} K_i.\]

Next, we will estimate the terms \( K_i (i = 1, \ldots, 10) \) on the right hand of (82) as follows. First, it follows from (67), (50), (51), and Young's inequality that

\[
K_1 + K_2 \leq C \|\mu_t\|_{L^2} \|u_{t}\|_{L^2} \|u_{xtt}\|_{L^2} + C \|\mu_t\|_{L^2} \|u_{xt}\|_{L^2} \|u_{xtt}\|_{L^2}
\]

\[\leq \epsilon \|u_{xtt}\|_{L^2}^2 + C \|\mu_t\|_{L^2}^2 + C \|u_{xtt}\|_{L^2}^2.\tag{83}
\]

Integrating by parts together with (1), (22), (50), and Young's inequality gives

\[
K_3 + K_{10} = 2 \int (\rho u_t) u_{tt}^2 dx - \int P_{xt} u_t dx
\]

\[= -4 \int \rho u_{xt} u_{xtt} dx + \int P_{xt} u_t dx\]

\[\leq C \|\rho^{1/2} u_t\|_{L^2} \|\rho^{1/2} u_{tt}\|_{L^2} \|u_{xtt}\|_{L^2} + C \|\mu_t\|_{L^2} \|u_{xtt}\|_{L^2}
\]

\[\leq \epsilon \|u_{xtt}\|_{L^2}^2 + C \|\rho^{1/2} u_t\|_{L^2}^2 + C \|\mu_t\|_{L^2}^2.\tag{84}
\]

Similar to (33), it holds that

\[
\|u_{tt}\|_{L^2} + \|u_{tt}\|_{L^\infty} \leq C \|u_{xtt}\|_{L^2},\tag{85}
\]

which along with (41), (50), and (59) implies that

\[
\sum_{i=1}^{7} K_i \leq C \|\rho_t\|_{L^2} \|u_t\|_{L^\infty} \|u_{xtt}\|_{L^2} \|u_{xtt}\|_{L^2} + C \|\mu_t\|_{L^2} \|u_t\|_{L^\infty} \|u_{xtt}\|_{L^2} \|u_{xtt}\|_{L^2}
\]

\[+ C \|\rho_t\|_{L^2} \|[u_{xt}\|_{L^\infty} \|u_{xtt}\|_{L^2} + \|u_{xt}\|_{L^2} + \|f\|_{L^2}\|u_t\|_{L^\infty}
\]

\[\leq C \|u_{xtt}\|_{L^2} \|u_{xtt}\|_{L^2} + C \|\rho_t\|_{L^2} (1 + \|u_{xtt}\|_{L^2}) \|u_{xtt}\|_{L^2}
\]

\[\leq \epsilon \|u_{xtt}\|_{L^2}^2 + C \|u_{xtt}\|_{L^2}^2 + C \|\rho_t\|_{L^2} (1 + \|u_{xtt}\|_{L^2}).\tag{86}
\]
Next, it deduces from (50) and (59) that
\[
K_\alpha + K_\beta \leq C \rho^{1/2} \|u_t\|_{L^2} \|\rho\|_{L^2} \|u_t\|_{L^2} \|u_{x\alpha}\|_{L^2} + C \|u_{x\alpha}\|_{L^2} \|\rho^{1/2} u_t\|_{L^2}^2.
\] (87)

Multiplying (82) by \(t^2\), we obtain after using (83) to (87) and choosing \(\epsilon\) suitably small that
\[
\frac{d}{dt} (t^2 \|\rho^{1/2} u_t\|_{L^2}^2) + t^2 \|u_{x\alpha}\|_{L^2}^2 \\
\leq C t^2 \|\rho^{1/2} u_t\|_{L^2}^2 + 2t \|\rho^{1/2} u_t\|_{L^2}^2 + Ct^2 \|u_{x\alpha}\|_{L^2}^2 + Ct^2 \|u_{x\alpha}\|_{L^2}^2 \\
+ Ct^2 |P_n|_{L^2}^2 + Ct^2 |\rho_n|_{L^2}^2 (1 + \|u_{x\alpha}\|_{L^2}^2) + Ct^2 \|u_{x\alpha}\|_{L^2}^2.
\] (88)

Since
\[
(\mu(\rho))_t = (\mu'(\rho)\rho)_t = \mu'(\rho)\rho_t + \mu'(\rho)\rho_{\alpha},
\]
which together with (22), (41), and (51) shows that
\[
\int_0^T \|\mu_t\|_{L^2}^2 dt \leq C \int_0^T \|\rho_t\|_{L^2}^2 \|\rho\|_{L^2}^2 dt + C \int_0^T \|\rho_n\|_{L^2}^2 dt \leq C(T).
\] (89)

Thus, Gronwall's inequality together with (88), (51), (64), and (89) implies that
\[
\sup_{0 \leq t \leq T} (t^2 \|\rho^{1/2} u_t\|_{L^2}^2) + \int_0^T t^2 \|u_{x\alpha}\|_{L^2}^2 dt \leq C(T).
\] (90)

This along with (85) gives (80) and finishes the proof of Lemma 11. \(\square\)

3 | PROOF OF THEOREMS 1.1 AND 1.2

Proof. With all the a priori estimates obtained in Section 2 at hand, we will divide the proof into three steps.

Step 1. We prove the local existence and uniqueness of the strong solutions when the initial density contains vacuum. That is, Theorem 1 holds for some \(T_0 > 0\).

Let \((\rho_0, u_0, f)\) be given as in Theorem 1, we construct
\[
\rho_0^\delta = \rho_0 + \delta, \quad u_0^\delta = u_0 * j_\delta, \quad f^\delta = f * j_\delta,
\] (91)

where \(j_\delta\) is the standard mollifying kernel of width \(\delta\) and \(0 \leq \rho_0^\delta \in C_0^\infty(0, 1)\) satisfies
\[
\rho_0^\delta \to \rho_0 \quad \text{in} \ H^1, \quad \text{as} \ \delta \to 0.
\] (92)

Thus, we have
\[
\rho_0^\delta \to \rho_0, \quad u_0^\delta \to u_0, \quad f^\delta \to f, \quad \text{in} \ H^1, \quad \text{as} \ \delta \to 0,
\] (93)

and
\[
\|\rho_0^\delta\|_{H^1} \leq C + C \|\rho_0\|_{H^1}, \quad \|u_0^\delta\|_{H^1} \leq C \|u_0\|_{H^1}, \quad \|f^\delta\|_{H^1} \leq C \|f\|_{H^1}.
\] (94)

By virtue of Lemma 2, the initial boundary problem (1) to (5) with the initial data \((\rho_0^\delta, u_0^\delta)\) has classical solutions \((\rho^\delta, u^\delta)\) on \((0, 1) \times [0, T_0]\). Furthermore, the estimates obtained in Lemmas 4 to 7 show that the solutions \((\rho^\delta, u^\delta)\) satisfy for any \(0 < T < +\infty\),
\[
\sup_{0 \leq t \leq T} \left( \left\| \left( \rho^\delta, \mu^\delta, P(\rho^\delta) \right) \right\|_{H^1} + \left\| \rho^\delta_t \right\|_{L^2} + \left\| \rho^\delta u^\delta_t \right\|_{L^2} + \left\| u^\delta_t \right\|_{H^1} \right) \\
+ \sqrt{t} \left\| \sqrt{\rho^\delta} u^\delta_t \right\|_{L^2} + \sqrt{t} \left\| \rho^\delta u^\delta_t \right\|_{L^2} \right) + \int_0^T \left( \left\| u^\delta_t \right\|_{H^1}^2 + t \left\| u^\delta_t \right\|_{L^2}^2 \right) dt \leq \tilde{C},
\]

where \( \tilde{C} \) is independent of \( \delta \). With the estimate (95) at hand, we find that the sequence \((\rho^\delta, u^\delta)\) converge, up to the extraction of subsequences, to some limit \((\rho, u)\) in the obvious weak sense. Then, letting \( \delta \to 0 \), we deduce from (95) that \((\rho, u)\) are strong solutions of problems (1) to (5) on \((0, 1) \times (0, T_0] \) satisfying

\[
\begin{cases}
\rho \in L^\infty(0, T_0; H^1), & \rho_t \in L^\infty(0, T_0; L^2), \\
u \in L^\infty(0, T_0; H^1) \cap L^2(0, T_0; H^2), & t^{1/2} u_t \in L^2(0, T_0; H^1), \\
t^{1/2} \rho^{1/2} u_t \in L^2(0, T_0; L^2). 
\end{cases} \tag{96}
\]

Then, the uniqueness of the strong solutions \((\rho, u)\) is guaranteed by the regularities (96). For the detailed proof, please see other works.\(^8,15\)

**Step 2.** We will extend the local existence time \( T_0 \) of strong solutions to be infinity and thus prove the global existence result.

Let \( T^* \) be the maximal time of existence for the strong solutions, then \( T^* \geq T_0 \). For any \( 0 < \tau < T \leq T^* \) with \( T \) finite, one deduces from

\[
u \in L^\infty(0, T; H^1_0) \cap L^2(0, T; H^2), \quad u_t \in L^2(\tau, T; H^1),
\]

that

\[
u \in C([\tau, T]; H^1_0). \tag{97}
\]

Furthermore, it follows from

\[
rho \in L^\infty(0, T; H^1), \quad \rho_t \in L^\infty(0, T; L^2)
\]

that

\[
rho \in C([0, T]; H^1). \tag{98}
\]

Defining

\[
(\rho^*, u^*) \equiv (\rho, u)(x, T^*) = \lim_{\tau \to T^*} (\rho, u)(x, \tau)
\]

we derive from (97) and (98) that \((\rho^*, u^*)\) satisfy the initial condition (9) at \( t = T^* \).

Hence, we take \((\rho^*, u^*)\) as the initial data at \( t = T^* \) and then use the local existence theory to extend the strong solutions beyond the maximum existence time \( T^* \). This contradicts the assumption on \( T^* \). We finally show that \( T^* \) could be infinity and prove the global existence of the strong solutions.

**Step 3.** It remains to prove (12). Direct calculations lead to

\[
\frac{d}{dt} \left\| u_x \right\|_{L^2}^2 = \left| 2 \int_0^1 u_x u_{xx} dx \right| \\
= \left| 2 \int_0^1 u_x (\ddot{u}_x - (u_{xx})_x) dx \right| \\
= \left| 2 \int_0^1 u_x \ddot{u}_x dx - \int_0^1 u_{xx}^2 dx \right| \\
\leq C \left\| \ddot{u}_x \right\|_{L^2}^2 + C (1 + \left\| u_x \right\|_{L^\infty}) \left\| u_x \right\|_{L^2}^2,
\]

which together with (23) and (37) yields
\[ \int_{1}^{\infty} \left( \|u_x\|_{L^2}^2 + \left| \frac{d}{dt} \|u_x\|_{L^2}^2 \right| \right) dt \leq C. \]

Thus, it holds
\[ \lim_{t \to \infty} \|u_x(\cdot, t)\|_{L^2}^2 = 0, \]
which combined with (37) gives
\[ \lim_{t \to \infty} \|u(\cdot, t)\|_{W^{1,p}} = 0, \quad \forall \ p \in [1, \infty). \tag{99} \]

The proof of Theorem 1 is finished. \[ \square \]

**Proof.** With the higher order estimates in Lemmas 8 to 11 at hand, the proof of Theorem 2 is similar to those of Theorem 1 and is omitted here for simplicity. \[ \square \]

### 4 | PROOF OF THEOREM 1.3

The proof of Theorem 3 is divided into two steps as follows.

**Step 1.** We will prove
\[ \lim_{t \to 0} \|\rho(\cdot, t) - \rho_s(\cdot)\|_{L^p} \to 0, \quad \forall \ p \in [1, \infty). \tag{100} \]

Considering the function
\[ P(t) = \int_{0}^{1} \left[ P(\rho) - \bar{P}(\rho) \right]^2 dx \]
with
\[ \bar{P}(\rho) = \bar{P}(\rho(\cdot, t)) = \int_{0}^{1} P(\rho) dx + \int_{0}^{\rho} \rho f dy - \int_{0}^{1} \int_{0}^{\rho} f dy dx, \]
we claim that
\[ \lim_{t \to \infty} \int_{t-1}^{t} \left( P(\tau) + \left| \frac{d}{d\tau} P(\tau) \right| \right) d\tau = 0. \tag{101} \]

With (101) at hand, one can derive the desired (100) with the same arguments as those in the literature.\(^5,7\) For reader’s convenience, we sketch them here for completeness. Indeed, for any \( t > 1 \) and \( s \in (t - 1, t) \), it holds
\[ P(t) \leq \int_{t-1}^{t} \left( P(s) + \left| \frac{d}{d\tau} P(s) \right| \right) d\tau, \]
which together with (101) implies that
\[ \lim_{t \to \infty} P(t) = \lim_{t \to \infty} \| P(\rho(\cdot, t)) - \bar{P}(\rho(\cdot, t)) \|_{L^2}^2 = 0. \tag{102} \]

Let \( t_n \to \infty \) be an arbitrary sequence, the uniform upper bound of the density (22) implies that there exist a function \( \tilde{\rho} \in L^\infty \) and a positive constant \( P_c \) such that for some subsequence \( \{ t'_n \} \subset \{ t_n \} \),
\[ \rho \left( \cdot, t'_n \right) \to \tilde{\rho}(\cdot) \text{ weakly * in } L^\infty, \quad \int_{0}^{1} P(\rho(\cdot, t'_n)) dx \to P_c. \tag{103} \]

Clearly, \( 0 \leq \tilde{\rho} \leq C \). The standard compactness argument together with (103) yields
\[
\begin{aligned}
P(\rho(\cdot,t_n')) \to P_s(\cdot) \equiv P_c + \int_0^1 \rho f \, dx - \int_0^1 \int_0^1 \rho f \, dy \, dx \quad \text{in } C([0,1]), \quad (104)
\end{aligned}
\]

which along with (102) leads to
\[
P(\rho(\cdot,t_n')) \to P_s(\cdot) \quad \text{in } L^2.
\]

Consequently, it holds that for some subsequence \(\{t_n''\} \subset \{t_n'\},\)
\[
P(\rho(x,t_n'')) \to P_s(x) \quad \text{ae in } (0,1).
\]

The continuity and monotonicity of \(P(\cdot)\) deduce that the inverse function of \(P,\) denoted by \(P^{-1},\) is continuous, and thus
\[
\rho(x,t_n'') \to \rho_s(x) \equiv P^{-1}(P_s(x)) \quad \text{ae in } (0,1),
\]

where \(\rho_s \in C([0,1])\) and \(\rho_s > 0.\) This together with Lebesgue dominated theorem and (22) implies that
\[
\rho(\cdot,t_n') \to \rho_s(\cdot) \quad \text{in } L^p, \quad \forall \ p \in [1,\infty), \quad (105)
\]

which along with (103) yields
\[
\rho_s = \bar{\rho}.
\]

According to the definition of \(P_s\) in (104), it holds
\[
P(\rho_s) = P_s = P_c + \int_0^1 \rho_s f \, dx - \int_0^1 \int_0^1 \rho_s f \, dy \, dx,
\]

which together with (105) and (24) yields that
\[
[P(\rho_s)]_x = \rho_s f, \quad \int_0^1 \rho_s \, dx = 1.
\]

Hence, we show that \(\rho_s\) is indeed the solution to the stationary problem (7) due to Lemma 1. And (100) is a direct consequence of (105).

Now, it remains to prove (101). Denoting
\[
\Psi = \int_0^1 (P(\rho) - P(\rho)) \, dx,
\]

which satisfies \(\Psi(0,t) = \Psi(1,t) = 0,\) using integration by parts and (12), we can rewrite \(P(t)\) as follows,
\[
\begin{aligned}
P(t) &= \int_0^1 (P(\rho) - \tilde{P}(\rho)) \, d\Psi = -\int_0^1 (P(\rho) - \tilde{P}(\rho)) x \, d\Psi \\
&= \int_0^1 (\rho u_t + (\rho u^2)_x - [\mu(\rho) u_x]_x) \, d\Psi \\
&= \frac{d}{dt} \int_0^1 \rho u \Psi \, dx - \int_0^1 \rho u \Psi_t \, dx - \int_0^1 (\rho u^2 - \mu(\rho) u_x) \, \Psi_x \, dx.
\end{aligned} \quad (106)
\]

It follows from (23) and (22) that
\[
|\Psi| + |\Psi_x| = \left| \int_0^1 \left( P(\rho) - \int_0^1 P(\rho) \, dx - \int_0^1 \int_0^1 \rho f \, dy \, dx + \int_0^1 \int_0^1 \rho f \, dy \, dx \right) \, dx \right| + \left| P(\rho) - \int_0^1 P(\rho) \, dx - \int_0^1 \rho f \, dx + \int_0^1 \int_0^1 \rho f \, dy \, dx \right| \leq C. \quad (107)
\]
and thus,

\[ \left| - \int_0^1 (\rho u^2 - \mu(\rho)u_x) \Psi_x \, dx \right| \leq C \| u_x \|_{L^2}^2 + C \| u_x \|_{L^2}. \]  

(108)

Since

\[ P_t + (Pu)_x + (\gamma - 1)Pu = 0, \]

one deduces from integration by parts and (1) that

\[
(P(\rho) - \bar{P}(\rho)) = -(Pu)_x - (\gamma - 1)Pu - \int_0^1 P_t \, dx - \int_0^x \rho \, d\gamma + \int_0^x \rho \, d\gamma
\]

\[
= -(Pu)_x - (\gamma - 1)Pu + (\gamma - 1) \int_0^x Pu \, dx + \int_0^x (\rho u)_y \, dy - \int_0^1 \int_0^x (\rho u)_y \, dy dx
\]

\[
= -(Pu)_x - (\gamma - 1)Pu + (\gamma - 1) \int_0^x Pu \, dx
\]

\[
+ \rho u - \int_0^x \rho u f \, dy - \int_0^1 \rho u f \, dx + \int_0^1 \int_0^x \rho u f \, dy dx.
\]

Combining this with (22) and (33) gives

\[
|\Psi| = \left| \int_0^x (P(\rho) - \bar{P}(\rho)) \, dx \right|
\]

\[
= \left| -\rho' u - (\gamma - 1) \int_0^x \rho u \, dx + (\gamma - 1) \int_0^1 \rho u \, dx \right|
\]

\[
= \left| \int_0^x \left( \rho u - \int_0^x \rho u f \, dy - \int_0^1 \rho u f \, dx + \int_0^1 \int_0^x \rho u f \, dy dx \right) \, dx \right|
\]

\[
\leq C (\bar{\rho}, \| f \|_{H^1}) \| u \|_{L^2} + C(\bar{\rho}) \| u \|_{L^2} + C(\bar{\rho}) \| u_x \|_{L^2}
\]

\[
\leq C \| u_x \|_{L^2}.
\]

One thus gets

\[
\int_0^1 \rho u \Psi_t \, dx \leq C \| u_x \|_{L^2} \| u \|_{L^2} \int_0^1 \rho \, dx \leq C \| u_x \|_{L^2}^2.
\]

(110)

Hence, on the one hand, it follows from (107), (106), (108), and (110) that

\[
\int_{t-1}^t \rho(\tau) \, d\tau = \int_0^1 \rho(\Psi \Psi)(t) - \int_0^1 \rho(\Psi \Psi)(t-1)
\]

\[
- \int_{t-1}^t \int_0^1 \rho(\Psi \Psi) \, dx \, d\tau - \int_{t-1}^t \int_0^1 (\rho \Psi^2 - \mu(\rho)u_x) \Psi_x \, d\tau
\]

\[
\leq C \| u \|_{L^2} \int_{t-1}^t \left( \| u_x \|_{L^2}^2 + \| u_x \|_{L^2} \right) \, d\tau
\]

\[
\leq C \| u_x \|_{L^2} + C \int_{t-1}^t \| u_x \|_{L^2}^2 \, d\tau + C \left( \int_{t-1}^t \| u_x \|_{L^2}^2 \, d\tau \right)^{1/2}.
\]

(111)

On the other hand, it deduces from (107) and (109) that
\[
\int_{t-1}^{t} \left| \frac{d}{dx} P(\rho) \right| dx = 2 \int_{t-1}^{t} \left| \frac{d}{dx} \Psi_{x}(P(\rho) - P(\rho)) \frac{d}{dx} \right| dx \leq C \int_{t-1}^{t} \|u_{x}\|_{L^{2}} \, dx \leq C \left( \int_{t-1}^{t} \|u_{x}\|_{L^{2}}^{1 \slash 2} \, dx \right)^{1 \slash 2}.
\]

(112)

Then, the desired (101) is deduced directly from (111), (112), (99), and (23).

**Step 2.** Now, we are in a position to prove (18). The method used here is motivated by Huang-Li-Xin\textsuperscript{31} and Li-Zhang-Zhao\textsuperscript{16}.

Thanks to (7), the momentum equation (1)\textsubscript{2} can be rewritten as

\[
\rho u_{t} + \rho uu_{x} + [P(\rho)]_{x} - \rho \rho_{s}^{-1} [P(\rho_{s})]_{x} = [\mu(\rho)u_{x}]_{x},
\]

which multiplied by \(u\), we obtain after using integration by parts and (1)\textsubscript{1} that

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{0}^{1} \rho u^{2} \, dx + \int_{0}^{1} G(\rho) \, dx \right) + \int_{0}^{1} \mu(\rho)u_{x}^{2} \, dx = 0.
\]

(113)

where

\[
G(\rho) \triangleq \int_{\rho_{s}}^{\rho} \int_{\rho_{s}}^{\rho} P'(\xi) \frac{d\xi}{\xi^{2}} \, d\rho = \frac{1}{\gamma - 1} (P(\rho) - P(\rho_{s}) - P'(\rho_{s})(\rho - \rho_{s})).
\]

(114)

Clearly, it follows from (22) and (17) that there are positive constants \(M_{1}\) and \(M_{2}\) depending only on \(\gamma, \bar{\rho}, K_{1}\), and \(K_{2}\) such that

\[
M_{1}(\rho - \rho_{s})^{2} \leq G(\rho) \leq M_{2}(\rho - \rho_{s})^{2}.
\]

(115)

Next, it follows from (1)\textsubscript{2} and (7) that

\[
-[P(\rho) - P(\rho_{s})]_{x} + \rho_{s}^{-1}(\rho - \rho_{s})P'(\rho_{s})(\rho_{s})_{x} = (\rho u)_{t} + (\rho u^{2})_{x} - [\mu(\rho)u_{x}]_{x}.
\]

(116)

Let \(\Phi(x, t) \triangleq \int_{0}^{x} (\rho - \rho_{s}) \, dy\), it holds that

\[
\Phi(0, t) = \Phi(1, t) = 0
\]

(117)

owing to (7) and (24). Multiplying (116) by \(\rho_{s}^{-1}\Phi(x, t)\) gives

\[
- \int_{0}^{1} \left[ \rho_{s}^{-1} (P(\rho) - P(\rho_{s})) \right]_{x} \Phi \, dx = \int_{0}^{1} \rho_{s}^{-2}(\rho_{s})_{x} \left[ P(\rho) - P(\rho_{s}) - P'(\rho_{s})(\rho - \rho_{s}) \right] \Phi \, dx
\]

\[
+ \int_{0}^{1} \rho_{s}^{-1}(\mu) \Phi \, dx + \int_{0}^{1} \rho_{s}^{-1}(\rho u_{x}) \Phi \, dx - \int_{0}^{1} \rho_{s}^{-1}[\mu(\rho)u_{x}]_{x} \Phi \, dx
\]

\[
= \sum_{i=1}^{4} J_{i}.
\]

First, integration by parts combined with (117) and (17) gives

\[
- \int_{0}^{1} \left[ \rho_{s}^{-1} (P(\rho) - P(\rho_{s})) \right]_{x} \Phi \, dx = \int_{0}^{1} \rho_{s}^{-1}P'(\rho)(\rho - \rho_{s})^{2} \, dx \geq C_{1}\|\rho - \rho_{s}\|_{L^{2}}^{2},
\]

(119)

where \(0 < \dot{\rho} \in (\min\{\rho, \rho_{s}\}, \max\{\rho, \rho_{s}\})\) and \(C_{1}\) is a positive constant depending on \(\gamma, \bar{\rho}, K_{1}\), and \(K_{2}\).

Next, the terms on the right hand of (118) can be estimated as follows. On the one hand, it follows from (100) that there is some \(T^{*} > 1\) such that for \(t > T^{*}\),
\[ \| \rho - \rho_s \|_{L^2} \leq \frac{C_1}{4}, \]

which along with (17) and (114) yields that

\[ J_1 \leq C \| \rho - \rho_s \|_{L^2} \| \rho - \rho_s \|_{L^2}^2 \leq \frac{C_1}{4} \| \rho - \rho_s \|_{L^2}^2. \]  

(120)

On the other hand, the integration by parts together with (17) and (22) implies that

\[
\dot{J}_2 = \frac{d}{dt} \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx - \int_0^1 \rho_s^{-1} \rho u \int_0^x \rho_i dy dx
\]

\[
= \frac{d}{dt} \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx + \int_0^1 \rho_s^{-1} \rho u^2 dx
\]

\[
\leq \frac{d}{dt} \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx + C \| u \|_{L^2}^2,
\]

(121)

and

\[
\dot{J}_3 + \dot{J}_4 = \int_0^1 \rho_s^{-2}(\rho_s) \rho u^2 \int_0^x (\rho - \rho_s) dy dx - \int_0^1 \rho_s^{-1} \rho u^2 (\rho - \rho_s) dx
\]

\[
+ \int_0^1 \rho_s^{-1} \mu(\rho) u_x (\rho - \rho_s) dx - \int_0^1 \rho_s^{-2}(\rho_s) \mu(\rho) u_x \int_0^x (\rho - \rho_s) dy dx
\]

\[
\leq C \| u \|_{L^2}^2 + C \| u \|_{L^2} \| \rho - \rho_s \|_{L^2}
\]

\[
\leq \frac{C_1}{4} \| \rho - \rho_s \|_{L^2}^2 + C \| u \|_{L^2}^2.
\]

(122)

Substituting (119) to (122) into (118) derives

\[
C_1 \| \rho - \rho_s \|_{L^2}^2 \leq 2 \frac{d}{dt} \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx + C \| u \|_{L^2}^2,
\]

(123)

Since

\[
2 \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx \leq C \| \sqrt{\rho u} \|_{L^2}^2 + C \| \rho - \rho_s \|_{L^2}^2.
\]

adding (123) multiplied by some suitably small \( \eta \) to (113) gives

\[
\frac{d}{dt} W(t) + \eta C_1 \| \rho - \rho_s \|_{L^2}^2 + \frac{1}{2} \int_0^1 \mu(\rho) u_x^2 dx \leq 0,
\]

(124)

where

\[
W(t) \triangleq \frac{1}{2} \int_0^1 \rho u^2 dx + \int_0^1 G(\rho) dx - 2\eta \int_0^1 \rho_s^{-1}(\rho u) \int_0^x (\rho - \rho_s) dy dx
\]

satisfies

\[
\frac{1}{4} \int_0^1 \rho u^2 dx + \frac{M_1}{2} \| \rho - \rho_s \|_{L^2}^2 \leq W(t) \leq C \| \sqrt{\rho u} \|_{L^2}^2 + C \| \rho - \rho_s \|_{L^2}^2.
\]

(125)

Furthermore, one has

\[
\int_0^1 \rho u^2 dx \leq C \| u \|_{L^2}^2 \leq \frac{C_1^{-1}}{2} \left\| \sqrt{\mu(\rho) u_x} \right\|_{L^2}^2.
\]

(126)
where $C_2 > 0$ is a positive constant depending on $\tilde{\mu}$.

The combination of (124) with (126) gives

$$
\frac{d}{dt} W(t) + a \| \rho - \rho_s \|^2_{L^2} + \alpha \int_0^1 \rho u^2 dx \leq 0, \tag{127}
$$

where $\alpha$ is a positive constant depending on $\eta$, $C_1$, and $C_2$. Hence, Gronwall's inequality combined with (125) and (127) shows

$$
\| \sqrt{\rho} u \|^2_{L^2} + \| \rho - \rho_s \|^2_{L^2} \leq Ce^{-\alpha t}, \quad \text{for} \quad t > T^*. \tag{128}
$$

In what follows, we will prove the exponential decay rate for the $L^2$-norm of $u_x$. First, multiplying (113) by $e^{\frac{\alpha}{2}t}$, we get after using (115) and (128) that

$$
\frac{d}{dt} \left( e^{\frac{\alpha}{2}t} \| \sqrt{\rho} u \|^2_{L^2} + 2e^{\frac{\alpha}{2}t} \int_0^1 G(\rho) dx \right) + e^{\frac{\alpha}{2}t} \| u_x \|^2_{L^2} \\
\leq Ce^{\frac{\alpha}{2}t} \| \sqrt{\rho} u \|^2_{L^2} + Ce^{\frac{\alpha}{2}t} \| \rho - \rho_s \|^2_{L^2} \\
\leq Ce^{-\frac{\alpha}{2}t}. \tag{129}
$$

Integrating (129) over $[T^*, t]$ gives for $t > T^*$,

$$
e^{\alpha t} \| \sqrt{\rho} u \|^2_{L^2} + e^{\alpha t} \| \rho - \rho_s \|^2_{L^2} + \int_{T^*}^t e^{\frac{\alpha}{2}t} \| u_x \|^2_{L^2} dt \leq C, \tag{130}
$$

due to (128).

Next, multiplying (116) by $\hat{u}$ and integrating the resulting equation by parts, it holds for $t > T^*$,

$$
\frac{d}{dt} \left( \| \sqrt{\mu(\rho) u_x} \|^2_{L^2} + \| \sqrt{\rho} u \|^2_{L^2} \right) \\
= -\frac{1}{2} \int_0^1 \left[ \mu(\rho) - \mu'(\rho) \rho \right] u_x^2 dx + \int_0^1 [P(\rho) - P(\rho_s)] u_x dx \\
+ \int_0^1 \rho_s^{-1}(\rho - \rho_s) P'(\rho_s) u_x dx \\
\leq C \| u_x \|^2_{L^2} + C \| \rho - \rho_s \|^2_{L^2} \| u_x \|^2_{L^2} \\
\leq C \| u_x \|^2_{L^2} + C \| \rho - \rho_s \|^2_{L^2} \| u_x \|^2_{L^2}, \tag{131}
$$

where in the last inequality one has used (37) and Poincaré inequality. Integrating (131) multiplied by $e^{\frac{\alpha}{2}t}$ over $[T^*, t]$, we thus obtain after using (130) and (37) that

$$
e^{\frac{\alpha}{2}t} \| u_x \|^2_{L^2} + \int_{T^*}^t e^{\frac{\alpha}{2}t} \| \sqrt{\rho} u \|^2_{L^2} dt \leq C, \quad \text{for} \quad t > T^*,
$$

which together with (130), (99), and (100) gives (18).

Finally, the proof of (19) is similar as that of Li and Xin\(^{13}\), theorem 1.2 (see also other studies\(^{10,31}\)). The proof of Theorem 3 is completed.

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REFERENCES

1. Beirão da Veiga H. Long time behavior for one-dimensional motion of a general barotropic viscous fluid. *Arch Ration Mech Anal.* 1989;108:141-160.
2. Kanel YI. On a model system of equations of one-dimensional gas motion. *Diff Eqs.* 1968;4:374-380.
3. Hoff D. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans Amer Math Soc.* 1987;303(1):169-181.
4. Serre D. Solutions faibles globales des quations de Navier-Stokes pour un fluide compressible. *C. R. Acad Sci Paris Sr. I Math.* 1986;303(13):639-642.
5. Serre D. On the one-dimensional equation of a viscous, compressible, heat-conducting fluid. *C. R. Acad Sci Paris Sr. I Math.* 1986;303(14):703-706.
6. Straškraba I, Zlotnik A. On a decay rate for 1D-viscous compressible barotropic fluid equations. *J Evol Eqs.* 2002;2:69-96.
7. Straškraba I, Zlotnik A. Global properties of solutions to 1D-viscous compressible barotropic fluid equations with density dependent viscosity. *Z Angew Math Phys.* 2003;54:593-607.
8. Cho Y, Kim H. On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. *Manuscript Math.* 2006;120:91-129.
9. Ding SJ, Wen HY, Zhu CJ. Global classical large solutions of 1D compressible Navier-Stokes equations with density-dependent viscosity and vacuum. *J Diff Eqs.* 2011;221:1696-1725.
10. Huang XD, Li J, Xin ZP. Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. *Comm Pure Appl Math.* 2012;65:549-585.
11. Li HL, Li J, Xin ZP. Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations. *Comm Math Phys.* 2008;281:401-444.
12. Li JK. Global well-posedness of non-heat conductive compressible Navier-Stokes equations in 1D. *arXiv*:1908.00514.
13. Li J, Xin ZP. Some uniform estimates and blowup behavior of global strong solution to the Stokes approximation equations for two-dimensional compressible flows. *J Diff Eqs.* 2006;221:275-308.
14. Li J, Xin ZP. Global well-posedness and large time asymptotic behavior of classical solution to the compressible Navier-Stokes equations with vacuum. *Ann PDE.* 2019;5(1). Art.7, 37pp.
15. Li J, Liang ZL. On classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum. *J Math Pures Appl.* 2014;102:640-671.
16. Li J, Zhang JW, Zhao JN. On the global motion of viscous compressible barotropic flows subject to large external potential forces and vacuum. *SIAM J Math Anal.* 2015;47(2):1121-1153.
17. Liu TP, Xin ZP, Yang T. Vacuum states of compressible flow. *Discrete Contin Dyn Syst.* 1998;4:1-32.
18. Ou YB, Zeng HH. Global strong solutions to the vacuum free boundary problem for compressible Navier-Stokes equations with degenerate viscosity and gravity force. *J Diff Eqs.* 2015;259:6803-6829.
19. Yang T, Yao ZA, Zhu CJ. Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. *Comm Partial Diff Eqs.* 2001;26:965-981.
20. Yang T, Zhu CJ. Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. *Comm Math Phys.* 2002;230:329-363.
21. Zeng HH. Global-in-time smoothness of solutions to the vacuum free boundary problem for compressible isentropic Navier-Stokes equations. *Nonlinearity.* 2015;28(2):331-345.
22. Fang DY, Zhang T. Compressible Navier-Stokes equations with vacuum state in the case of general pressure law. *Math Methods Appl Sci.* 2006;29:1081-1106.
23. Jiang S, Xin ZP, Zhang P. Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. *Methods Appl Anal.* 2005;12:239-251.
24. Okada M, Matusu-Necasová S, Makino T. Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity. *Ann Univ Ferrara Sez., VII (N.S.)* 2002;48:1-20.
25. Qin XL, Yao ZA, Zhao BJ. One dimensional compressible Navier-Stokes equations with density-dependent viscosity and free boundary. *Commun Pure Appl Anal.* 2008;7:373-381.
26. Amosov AA, Zlotnik AA. Global generalized solutions of the equations of the one-dimensional motion of a viscous heat-conducting gas. *Soviet Math Dokl.* 1989;38:1-5.
27. Kawohl B. Global existence of large solutions to initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas. *J Diff Eqs.* 1985;58:76-103.
28. Zlotnik AA. Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. *Diff Eqs.* 2000;36:701-716.
29. Zlotnik AA. Uniform estimates and stabilization of solutions to equations of one-dimensional motion of a multicomponent barotropic mixture. *Math Notes.* 1995;58:885-889.
30. Hoff D. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J Diff Eqs.* 1995;120(1):215-254.

31. Huang FM, Li J, Xin ZP. Convergence to equilibria and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows with large data. *J Math Pures Appl.* 2006;(9)8:471-491.

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