Reachability of turn sequences

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Abstract
A turn sequence of left and right turns is realized as a simple rectilinear chain of integral segments whose turns at its bends are the same as the turn sequence. The chain starts from the origin and ends at some point which we call a reachable point of the turn sequence. We investigate the combinatorial and geometric properties of the set of reachable points of a given turn sequence such as the shape, connectedness, and sufficient and necessary conditions on the reachability to the four signed axes. We also prove the upper and lower bounds on the maximum distance from the origin to the closest reachable point on signed axes for a turn sequence. The bounds are expressed in terms of the difference between the number of left and right turns in the sequence as well as, in certain cases, the length of the maximal monotone prefix or suffix of the turn sequence. The bounds are exactly matched or tight within additive constants for some signed axes.

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1 Introduction
We consider the problem of characterizing and approximating, for a given turn sequence, the set of points reachable by the sequence. A turn sequence consists of left turns $L$ and right turns $R$. For a turn sequence $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ where $\sigma_i \in \{L, R\}$, a rectilinear chain realizes the sequence $\sigma$ if the chain is simple (i.e., has no self-intersection), consists of integral segments, starts from the origin $o$, makes the sequence $\sigma$ of left turns ($L$) and right turns ($R$) in order, and ends at a point $p \in \mathbb{Z}^2$. We assume that the first segment of such chains is horizontal and heads to the east, i.e., the first turn occurs at $(t, 0)$ for some positive integer $t$. The endpoint $p$ is said to be reachable by $\sigma$. Let $A(\sigma)$ be the set of points in $\mathbb{Z}^2$ reachable by a turn sequence $\sigma$.

One may think of $A(\sigma)$ as those points reachable by a robot following the sequence $\sigma$ of turn commands when the distance the robot travels between turns is a positive but arbitrary integer. Where can the robot end up? How close the robots can reach a point from the origin by obeying the turn sequence?
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Related work

This type of question arose in the study of turtle graphics [14], in which a turtle obeys a sequence of commands such as “move forward 10 units” and “turn right 90 degrees”. The trace made by the turtle creates a chain or polygon on a display device. The metaphor helped children understand basic geometric shapes.

The problem of constructing chains or polygons with restricted angles shows up in curvature-constrained motion planning [1], angle-restricted tours [9, 10], and restricted orientation geometry [16]. Culberson and Rawlins [6] and Hartley [13] studied the problem of realizing a simple polygon by an angle sequence. An angle sequence is a sequence of the angles at vertices (say, in the counterclockwise direction) along the boundary of a polygon. They presented algorithms to construct the polygon from a given sequence of exterior angles whose sum is $2\pi$.

A rectilinear angle sequence has only two angles, $+90^\circ$ and $-90^\circ$, thus it is the same as the turn sequence considered in this paper. A turn sequence can be realized by a rectilinear simple polygon if and only if the number of left and right turns in the sequence differs by exactly four [6, 17]. Sack [17] represented a rectilinear polygon as a label-sequence of integer labels that are defined as the difference of the numbers of left and right turns at each edge from an arbitrary starting edge, and also presented a drawing algorithm that realizes a given label-sequence as a simple rectilinear polygon in linear time. Bae et al. [2] showed tight worst-case bounds on the minimum and maximum area of the rectilinear polygon that realizes a given turn sequence. Finding realizations that minimize the area or perimeter of the rectilinear polygon or the area of the bounding box of the polygon is NP-hard, however, the special case of monotone rectilinear polygons can be computed in polynomial time [8, 11].

A popular variant takes a sequence of angles defined by all vertices visible from each vertex as input [4, 5, 7]. The goal is to reconstruct a polygon from the information on angles and visibility. Another variant reconstructs a rectilinear polygon from a set of points, i.e., coordinates of the vertices, instead of angles, obtained by laser scanning devices [3].

For our problem, determining the reachable region from an angle (turn) sequence, there appears to be little that is known. Culberson and Rawlins [6] mention as future work “spline” problem: to draw a polygonal curve between two given points such that the turning angles of the curve form a given angle sequence. However, they do not suggest any approach to solving this problem.

Our contribution

We first show that the reachable set $A(\sigma)$ for a turn sequence $\sigma$ that contains a hook, i.e., two consecutive left turns or right turns, is a union of at most four halfplanes, not containing the origin, whose bounding lines are orthogonal to the four signed axes. If $\sigma$ has no hooks, it is called a staircase and has an easily computed reachable set. The proof is based on two crucial lemmas: the Stretching Lemma and the Axis Lemma in Section 2. The particular halfplanes are determined by the orientation of the hooks (Theorem 5 in Section 3). Using this characterization, we prove that both $A(\sigma)$ and its complement, i.e., the unreachable set $\mathbb{Z}^2 \setminus A(\sigma)$, are connected (Theorem 6 in Section 3). The boundary lines of the halfplanes forming $A(\sigma)$ for any turn sequence $\sigma$ with hooks are determined by the closest reachable points from the origin $o$ on the signed axes. Thus, it is important to know these points in order to calculate $A(\sigma)$ accurately. We give upper and lower bounds on the distance from $o$ to the closest points on each signed axis in Section 4. The upper bounds rely on an algorithm that provides an approximation to the reachable region (Theorem 9 in Section 4.2.1 and
Theorem 17 in Section 4.2.2). The lower bounds are derived from the rotation number \([12]\) of polygons, the sum of angle changes between two adjacent edges (Theorem 25 in Section 4.3 and Theorem 27 in Section 4.3.2). These bounds are expressed in terms of the difference between the number of left and right turns in \(\sigma\) and, for some signed axes and turn sequences with certain patterns, the length of the maximal prefix or suffix of the turn sequence that is monotone (i.e., does not contain a certain type of hook) or staircase (i.e., does not contain a hook). The bounds for some signed axis are exactly matched or tight to within an additive constant.

2 Cutting and Stretching

Let \(p\) be the endpoint of a chain \(C\) that realizes a turn sequence \(\sigma\). We can slightly modify \(C\) to show that other points are in the reachable region \(A(\sigma)\). Our mechanism for doing this relies on stretching (or lengthening) a subset of the segments in the chain \(C\) that are selected by a cut. A cut of a chain \(C\) is an \(x\)-monotone (or \(y\)-monotone) curve extending from \(-\infty\) to \(+\infty\) in the \(x\) (respectively, \(y\)) dimension that (1) separates \(o\) and \(p\) and (2) intersects only vertical (respectively, horizontal) segments of \(C\). We stretch \(C\) using the cut by lengthening every segment in \(C\) crossed by the cut by the same (positive) integral amount. As long as some segment in \(C\) crosses the cut, this creates a new chain \(C'\) starting at \(o\) that has the same turn sequence as \(C\), does not self-intersect (like \(C\)), and reaches points in \(A(\sigma)\) other than \(p\).

![Figure 1](image.png)

Figure 1 (a) Vertical cut (red) and horizontal cut (blue). (b) Stretching the chain by 2 using five horizontal segments intersected by a vertical cut. (c)-(d) Cuts for the reachable point \(p\) on y-axis and x-axis.

A vertical cut (the vertical line at \(x = 1/2\)) shown in Figure 1(a)-(b) intersects five horizontal segments of \(C\). Stretching \(C\) (lengthening the five horizontal segments) by \(d > 0\) units using this cut creates a chain \(C'\) with endpoint \(p' = p + (d, 0)\) that also belongs to \(A(\sigma)\). As a result, we observe that all the points on the horizontal ray \(p + (d, 0)\) with integer \(d > 0\) are reachable by \(\sigma\). Similarly, using the horizontal cut \(y = 1/2\) in Figure 1(a), all the points on the vertical ray \(p + (0, d)\) are also reachable.

We can generalize this stretching procedure to obtain the following lemma. We need some notation for it: For a non-zero value \(a\), \(\text{sgn}(a)\) represents its sign, \(+1\) or \(-1\). Let \(V^+(p)\) and \(H^+(p)\) be, respectively, the halfplanes of the vertical and horizontal lines through \(p\) that do not contain \(o\). Precisely, if \(p = (a, b)\) then \(V^+(p) := \{(a + \text{sgn}(a) \cdot i, j) \mid i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}\}\), and \(H^+(p) := \{(i, b + \text{sgn}(b) \cdot j) \mid i \in \mathbb{Z}, j \in \mathbb{Z}_{\geq 0}\}\). Let \(Q^+(p) := V^+(p) \cap H^+(p)\) be the quadrant at \(p\) that is diagonally opposite to the quadrant containing \(o\), see the shaded region in Figure 1(a).
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Lemma 1 (Stretching Lemma). For a turn sequence $\sigma$ with at least one turn,

1. if $p = (a, b) \in A(\sigma)$ for $a, b \neq 0$, then $Q^+(p) \subseteq A(\sigma)$,
2. if $p = (0, b) \in A(\sigma)$ for $b \neq 0$, then $H^+(p) \subseteq A(\sigma)$, and
3. if $p = (a, 0) \in A(\sigma)$ for $a \neq 0$, then $V^+(p) \subseteq A(\sigma)$.

Proof. Let $C$ be a chain with turn sequence $\sigma$ that reaches $p = (a, b)$. If $a \neq 0$ and $b \neq 0$, we can reach any point in $Q^+(p)$, i.e., $p + (\sigma(a) \cdot i, \sigma(b) \cdot j)$ for $i, j \in \mathbb{Z}_{\geq 0}$, by stretching $C$ by $i$ units using the vertical cut $x = \text{sgn}(a)/2$ and by $j$ using the horizontal cut $y = \text{sgn}(b)/2$. See Figure 1(a).

If $p$ is on the $y$-axis, i.e., $p = (0, b)$ for $b \neq 0$, as in Figure 1(c), we can reach any point in $H^+(p)$, i.e., $p + (i, \text{sgn}(b) \cdot j)$ for $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 0}$, by stretching $C$ by $j$ using the horizontal cut $y = \text{sgn}(b)/2$, and by $|i|$ using one of two $y$-monotone cuts $f(y)$ or $-f(y)$ depending on if the sign of $i$ is negative or positive. The cut $f(y)$ is $f(y) = 1/2$ except for the domain $y \in [-1/2, 1/2]$ where it is $f(y) = 2|y| - 1/2$. Since the origin is adjacent to a horizontal segment (to the east), $f(y)$ and $-f(y)$ intersect only horizontal segments and separate $p$ and $o$.

If $p$ is on the $x$-axis as in Figure 1(d), we can reach any point in $V^+(p)$, i.e., $p + (\sigma(a) \cdot i, j)$ for $i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$, by stretching $C$ by $i$ using the vertical cut $x = \text{sgn}(a)/2$, and by $|j|$ using one of two $x$-monotone cuts $g(x)$ or $-g(x)$ depending on if the sign of $j$ is negative or positive. Let $t$ be the $x$-coordinate of the first bend in $C$, which exists since $|\sigma| > 0$. The cut $g(x)$ is $g(x) = -1/2 - 2x$ for $x \in [-1/2, 0]$, $g(x) = -1/2$ for $x \in [0, t]$, $g(x) = -1/2 + 2(x - t)$ for $x \in [t, t + 1/2]$, and $g(x) = 1/2$ otherwise. Since the origin has no adjacent horizontal segment to its west and the point $(t, 0)$ has no adjacent horizontal segment to its east, both $g(x)$ and $-g(x)$ intersect only vertical segments and separate $p$ and $o$. This concludes the proof.

A staircase is a turn sequence of $n$ alternating left and right turns. One can easily observe using the Stretching Lemma (Lemma 1) that $A(\sigma)$ for a staircase $\sigma$ is identical to the quadrant $Q^+(p)$ where $p$ is the point reached by the realization of $\sigma$ with unit length segments. For a non-staircase sequence $\sigma$, we show that if $(a, b) \in A(\sigma)$ then $(a, 0) \in A(\sigma)$ or $(0, b) \in A(\sigma)$.

Lemma 2 (Axis Lemma). For any turn sequence $\sigma$, except a staircase, if $(a, b)$ is reachable by $\sigma$ then at least one of $(0, b)$ and $(a, 0)$ is reachable by $\sigma$.

Proof. If $a = 0$ or $b = 0$ there is nothing to show. Let $C$ be a chain with turn sequence $\sigma$ that reaches $(a, b)$ with non-zero $a$ and $b$. We show that at least one of $(0, b)$ and $(a, 0)$ is reachable by $\sigma$. We do this by constructing two cuts of the chain $C$ at least one of which will succeed in separating $(a, b)$ from the origin $o$ and will allow us to stretch $C$ to reach $(0, b)$ or $(a, 0)$.

Let $\epsilon$ be a small positive real number. We define the $\epsilon$-extended upper (resp., lower) side of a horizontal segment $(u, v)$ to be $(u + (-\epsilon, \epsilon), v + (\epsilon, \epsilon))$, i.e., the translation of the $(2\epsilon$-lengthened) segment up (resp., down) by $\epsilon$. Similarly, we define the $\epsilon$-extended right (resp., left) side of a vertical segment.

We define two cuts $f(y)$ and $g(x)$ for $C$. To simplify the description, we will assume that $a > 0$ and $b > 0$. The other cases are similar. See Figure 2, $g(x)$ is the $x$-monotone cut such that $g(x) = -\epsilon$ for $x \in (-\infty, 0]$, and $g(x)$ is never closer than $\epsilon$ (in $x$ or $y$-coordinate) to a horizontal segment of $C$. For $x > 0$, $g(x)$ is the maximum $y$-coordinate subject to these constraints. It follows that we can view $g(x)$ as a staircase curve whose finite length horizontal segments are subsegments of the (\epsilon-extended lower sides of) horizontal segments.
of $C$. Each such horizontal segment of $g(x)$ ends at $p_i + (\epsilon, -\epsilon)$ where $p_i$ is a bend point of $C$ at the right end of a horizontal segment.

Let $C'$ be the reflection of $C$ about the diagonal $x = y$, and $g'(x)$ be the $x$-monotone cut (defined above) of $C'$. $f(y)$ is the reflection of $g'(x)$ about the diagonal $x = y$. It follows that we can view $f(y)$ as a staircase curve whose finite length vertical segments are subsegments of the $(\epsilon$-extended left sides of) vertical segments of $C$. Each such vertical segment of $f(y)$ ends at $p_i + (-\epsilon, \epsilon)$ where $p_i$ is a bend point of $C$ at the upper end of a vertical segment. In fact, both $g(x)$ and $f(y)$ are staircase curves monotone to the $x$-axis and $y$-axis.

If $g(x)$ separates $o$ and $p$, i.e., $p$ is below $g(x)$, then we can stretch $C$ using the cut $g(x)$ so that $(a, 0)$ is reached since $o$ is above $g(x)$. If $f(y)$ separates $o$ and $p$, i.e., $p$ is above $f(y)$ then we can stretch $C$ using the cut $f(y)$ so that $(0, b)$ is reached. To prove the lemma, we must show that $p$ is below $g(x)$ or above $f(y)$. To do this we show the following claim:

**Claim 3.** Suppose that a chain $C$ realizing a non-staircase $\sigma$ reaches $p = (a, b)$ where $a$ and $b$ are positive integers. If $p$ is above $g(x)$ then $p$ is above $f(y)$.

**Proof.** For $i = 1, \ldots, n$, let $p_i = (x_i, y_i)$ be the point where $C$ makes the $i$th turn $\sigma_i$. Let $\sigma_j$ be the first turn where the staircase property is violated, so $\sigma_{j-1}\sigma_j = \text{LL}$ or $\sigma_{j-1}\sigma_j = \text{RR}$.

From $o$ to $p_j$, $f(y)$ and $g(x)$ follow opposite sides of every segment of $C$. Since $p = (a, b)$ is an integral point, if $a \leq x_j$ then $p$ above $g(x)$ implies $p$ above $f(y)$.

If $p_j$ is a left turn, $f(y)$ turns right at $p_j + (-\epsilon, \epsilon)$ while $g(x)$ continues upward. If $p_j$ is a right turn, $f(y)$ continues rightward while $g(x)$ turns left at $p_j + (\epsilon, -\epsilon)$. In either case, until $f(y)$ and $g(x)$ intersect again, if $p$ is above $g(x)$ then $p$ is above $f(y)$.

Suppose now that $f(y)$ and $g(x)$ intersect again. Since both are staircase curves, and $g(x)$ is above $f(y)$, this can occur only if a horizontal segment of $g(x)$ crosses a vertical segment of $f(y)$. Since both horizontal segments of $g(x)$ and vertical segments of $f(y)$ are subsegments of $(\epsilon$-extended sides of) segments of $C$, this intersection occurs within $\epsilon$ of a bend point $p_k$ which is the right endpoint of a horizontal segment of $C$ and the upper endpoint of a vertical segment of $C$. It follows that at $p_k + (\epsilon, -\epsilon)$, the curve $g(x)$ turns upward while at $p_k + (-\epsilon, \epsilon)$, the curve $f(y)$ turns rightward. Thus $g(x)$ and $f(y)$ intersect twice within $\epsilon$ of $p_k$ and $g(x)$ continues above $f(y)$. Since $p$ is an integral point and not a bend point of $C$, $p$ is not within $\epsilon$ of $p_k$ and the claim follows.

By this claim, if $p$ is above $g(x)$, then $p$ is above $f(y)$, so we can stretch $C$ using the cut $f(y)$.
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![Figure 3 Box algorithm: (a) \(\sigma_{i-1}\sigma_i = RL\). (b) \(\sigma_{i-1}\sigma_i = LR\). (c) \(\sigma_{i-1}\sigma_i = RR\). (d) \(\sigma_{i-1}\sigma_i = LL\).](image)

so that \((0, b)\) is reachable. Otherwise, since \(p\) is integral, \(p\) is below \(g(x)\), and we may stretch \(C\) using the cut \(g(x)\) so that \((a, 0)\) is reachable.

3 Reachable set \(A(\sigma)\)

We now show, using the two lemmas in the previous section, that the reachable set \(A(\sigma)\) is defined as the union of at most four halfplanes whose bounding lines are orthogonal to the four signed axes. For this, we define the Box algorithm and the TwoBox algorithm for realizing \(\sigma = \sigma_1 \cdots \sigma_n\) of length \(n\) as a simple rectilinear chain \(C\).

3.1 Box algorithm

The Box algorithm draws \(\sigma\) as a chain \(C\) in an incremental way, starting with a unit horizontal base segment \(C_0\) from \(o\) to \((1, 0)\) and constructing a chain \(C_i\) for the subsequence \(\sigma_1 \cdots \sigma_i\) from the chain \(C_{i-1}\) for \(\sigma_1 \cdots \sigma_{i-1}\) by adding a segment that realizes the \(i\)th turn \(\sigma_i\) for \(1 < i \leq n\). The chain \(C_i\) has \(i\) bend points, \(p_1, \ldots, p_i\); a starting point \(p_0 = o\), and an endpoint \(p(C_i) = p_{i+1}\). Each bend point \(p_j\) for \(1 \leq j \leq i\) corresponds to the turn \(\sigma_j\) on \(C_i\).

Each \(C_i\) satisfies three invariants: (I1) the smallest bounding box \(B_i\) of \(C_i\) has dimension \([\lceil \frac{i+1}{2} \rceil \times \lceil \frac{i+1}{2} \rceil]\) for odd \(i\) and \([\lceil \frac{i+1}{2} \rceil \times \lceil \frac{i+1}{2} \rceil]\) for even \(i\), (I2) the endpoint \(p(C_i) = p_{i+1}\) of \(C_i\) is at a corner of \(B_i\), and (I3) at least one side of the box that is incident to \(p_{i+1}\) is not occupied by any other segments of \(C_i\). Note that \(C_0\) fits in the (degenerated) box with width of one and height of zero, so the three invariants are clearly satisfied.

To get \(C_i\) from \(C_{i-1}\), we determine the position of \(p_{i+1}\), and connect it to \(p_i\), which is located at a corner of \(B_{i-1}\), with the segment \(e_i = p_ip_{i+1}\). See Figure 3. Without loss of generality, we assume that \(i\) is even, so \(e_i\) is horizontal because \(e_0\) is assumed to be horizontal. We further assume that \(p_i\) lies at one of the two corners on the upper side of the bounding box \(B_{i-1}\) of \(C_{i-1}\).

We have two cases: whether \(\sigma_i\) is different from \(\sigma_{i-1}\) or not. If they are different, then \(e_i\) can be a unit segment as in Figure 3(a)-(b). When they are same, as in Figure 3(c)-(d), we draw \(e_i\) as a horizontal segment whose length is the width of \(B_{i-1}\) plus one. The box \(B_i\) is one unit wider than \(B_{i-1}\). It is easy to check that \(C_{i}\) and \(B_i\) indeed satisfy the three invariants for both cases.

3.2 TwoBox algorithm

The TwoBox algorithm splits \(\sigma\) into two subsequences \(\sigma' = \sigma_1 \cdots \sigma_i\), and \(\sigma'' = \sigma_{i+1} \cdots \sigma_n\) for some \(1 \leq i < n\). We define \(\tilde{\sigma}''\) as the sequence of opposite turns of \(\sigma''\) in the reverse order, i.e., \(\tilde{\sigma}'' = \tilde{\sigma}_n \cdots \tilde{\sigma}_{i+1}\), where \(\tilde{\sigma}_j\) is the opposite turn from \(\sigma_j\).

The algorithm draws a chain \(C'\) for \(\sigma'\) and another chain \(C''\) for \(\tilde{\sigma}''\) by the Box algorithm. Their last segments should be connected to get the final chain \(C\) by connecting the two
endpoints \( p(C') \) and \( p(C'') \). See Figure 4. By the invariant (I2) of the Box algorithm, each endpoint lies at a corner of its box. We first place the boxes of \( C' \) and \( C'' \) so that their last segments are aligned along a common (horizontal or vertical) line. Such alignment can be achieved by rotating \( C'' \) (if necessary). Note that rotating a chain does not affect its turn sequence. After the alignment, we simply connect the two endpoints \( p(C') \) and \( p(C'') \) by a unit segment, then \( p_i \) of \( C' \) is finally connected with \( p_{i+1} \) of \( C'' \) as the \((i+1)\)th segment of \( C \). Note here that this connection is always possible due to the invariant (I2), and the two endpoints \( p(C') \) and \( p(C'') \) lie in the interior of the \((i+1)\)th segment of \( C \), thus both disappear.

A key property is that \( C' \) and \( C'' \) are separable either by a horizontal or vertical cut. The bounding box of the resulting chain \( C \) has dimensions each at most \( \lceil (n+1)/2 \rceil + 2 \) by the first invariant of the Box algorithm.

### 3.3 Hook patterns and axis reachability

By the Stretching Lemma (Lemma 1), if a turn sequence \( \sigma \) can reach a point \( p \) on an axis, then a halfplane \( H^+(p) \) or \( V^+(p) \) is also in \( A(\sigma) \). Thus, to figure out the shape of \( A(\sigma) \), it is important to know the closest reachable point from \( o \) on each signed axis. Of course, a turn sequence may not reach any point on some signed axis. We will show that axis reachability is determined by hook patterns in the sequence.

A hook pattern is either \( \text{LL} \) or \( \text{RR} \) in \( \sigma \). This is realized in a chain \( C \) by \( \sigma \) as three consecutive segments whose two bend points correspond to \( \text{LL} \) or \( \text{RR} \). We call the middle segment the hook segment in the chain \( C \). We use the term hook to indicate both a hook pattern and its associated segment in a chain.

We assign a direction to the segments of \( C \) along \( C \) from \( o \) to its endpoint of \( C \). According to its direction, we classify a hook into four different types, up, down, left, and right as shown in Figure 5. Up and down hooks are vertical segments, and left and right hooks are horizontal ones. Each hook has two subtypes according to its turns, \( \text{LL} \) or \( \text{RR} \). We illustrate all eight hook types in Figure 5.

The chain reverses its direction only at hooks. For instance, if \( C \) reaches a point \( p \) on the \(+y\)-axis, then it must have left the \(x\)-axis (at a point other than \( o \)) and headed to the \(+y\)-axis. This allows us to observe that a turn sequence without up hooks cannot reach any point on the \(+y\)-axis and indeed reveals the relation between the hook type and axis reachability. It is worth mentioning that the first hook in the chain is one of the four hooks, right-\( \text{RR} \), right-\( \text{LL} \), down-\( \text{RR} \), and up-\( \text{LL} \) because the first segment of any chain is the horizontal segment to the east. We also observe that the sequence of hooks in a turn sequence is restricted; for instance,
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**Figure 5** Four hook types: up, down, left, and right. Each type has an LL and RR subtype. A dashed arrow from a hook \( h \) to \( h' \) means that \( h' \) can be the next hook after \( h \) in the chain (they can be connected by a staircase of length \( \geq 0 \)). The four hooks marked * can be the first hook after \( o \).

the hook preceding a down-LL hook must be a left-LL hook or a down-RR hook. Figure [5] describes these constraints.

Using the relation on hook patterns with TwoBox algorithm, we can prove the necessary and sufficient conditions for the axis reachability.

**Lemma 4.** A turn sequence \( \sigma \) of \( n > 0 \) turns contains at least one up, down, right, or left hook pattern if and only if there is a reachable point on \(+y\)-axis, \(−y\)-axis, \(+x\)-axis, or \(−x\)-axis, respectively.

**Proof.** We first show that any turn sequence \( \sigma \) containing a down-RR hook can reach the \(−y\)-axis. We draw a chain \( C \) using the TwoBox algorithm by splitting \( \sigma \) into \( \sigma' = \sigma_1 \cdots \sigma_i \) and \( \sigma'' = \sigma_{i+1} \cdots \sigma_n \), where \( \sigma_i \sigma_{i+1} = \text{RR} \) is the first down-RR hook from \( o \) (see Figure 4(a)). Let \( p \) be the endpoint of the resulting chain. To get \( C' \), we translate \( C'' \) horizontally either by stretching the segment \( p_{i-1}p_i \) (if \( p \) is to the left of \( o \)) or the segment \( p_{i+1}p_{i+2} \) (if \( p \) is to the right of \( o \)) so that \( p \) has the same \( x \)-coordinate as \( o \). We then stretch the segment \( p(C')p(C'') \) until \( p \) is on the \(−y\)-axis. For the case that \( \sigma \) contains down-LL, we can apply a similar procedure to move \( p \) to the \(−y\)-axis. We thus conclude that if \( \sigma \) has a down hook, then it can reach a point on the \(−y\)-axis.

The same argument can be applied for the other hook types: If \( \sigma \) contains up, right, or left hooks, then there are reachable points on \(+y\)-axis, \(+x\)-axis, or \(−x\)-axis, respectively.

We now claim that the reverse is also true. In the following, we show that if there is a point on the \(−y\)-axis reachable by \( \sigma \), then \( \sigma \) must contain at least one down hook, either a down-LL hook or a down-RR hook. The other cases can be proved similarly.

For any chain \( C \) realizing \( \sigma \), we define a point \( q \) as the first intersection of \( C \) with the \(−y\)-axis. Let \( q' \) be the last intersection point of \( C \) with the \(+y\)-axis before \( q \). It is possible that \( q = p \) or \( q' = o \). Without loss of generality, we assume that the subchain \( C' \) of \( C \) from \( q' \) to \( q \) is to the right of the \(−y\)-axis. By the definition, \( C' \) cannot intersect with the \(y\)-axis except at \( q' \) and \( q \). Let \( e = ab \) be the rightmost vertical segment connecting two bends \( a \) and \( b \) of \( C' \), where \( a \) is below \( b \). Note that \( C' \) lies completely in the vertical slab between the \(y\)-axis and the line extending \( e \). We can easily see that \( e \) is a down hook from \( b \) to \( a \), i.e., \( C' \) is a subchain connecting \( q' \), \( b, a \), and \( q \) in this order. Otherwise, i.e., if it is an up hook from \( a \) to \( b \), then the subchain from \( b \) to \( q \) of \( C' \) must intersect the subchain from \( q' \) to \( a \) of \( C' \), which is a contradiction.

Let \((x^+_{\sigma}, 0)\) and \((x^-_{\sigma}, 0)\) be the closest reachable points by \( \sigma \) on \(+x\)-axis and \(−x\)-axis,
The signed coordinates of an axis has all four types of hooks, so such turn down hooks, respectively. The next obvious question is to find the four closest reachable points. This is a contradiction. The connectedness of the unreachable set of length zero) or a turn sequence with no hooks, $A(\sigma)$ is clearly connected. Suppose that $\sigma$ has hooks. By Theorem 5, $A(\sigma)$ is the union of the reachable set $A(\sigma)$ and the unreachable set $Z^2 \setminus A(\sigma)$ are both connected.

Proof. For an empty turn sequence (i.e., of length zero) or a turn sequence with no hooks, $A(\sigma)$ is clearly connected. Suppose that $\sigma$ has hooks. By Theorem 5, $A(\sigma)$ is the union of at most four halfplanes whose bounding lines are orthogonal to the signed axes. This implies that $A(\sigma)$ is connected except in the case that $A(\sigma)$ is the union of two parallel halfplanes. Suppose without loss of generality that their bounding lines are parallel to the $x$-axis. Then, by Theorem 5, $\sigma$ contains exactly two types of hooks: down and up. However, in order for the up and down hooks to be connected by the chain, $\sigma$ needs to have at least one left or right hook; see Figure 6 for an illustration of the ordering relation of the hooks in a chain. This is a contradiction. The connectedness of the unreachable set $Z^2 \setminus A(\sigma)$ immediately follows from the shape of $A(\sigma)$.

Remar 7. We observe that any turn sequence with five or more left turns than right ones (or with five or more right turns than left ones) has all four types of hooks, so such turn sequences can reach all four signed axes.

4 Closest reachable points on signed axes

The next obvious question is to find the four closest reachable points $x^+_{\sigma}, x^-_{\sigma}, y^+_{\sigma}$ and $y^-_{\sigma}$ for a given turn sequence $\sigma$ if they exist. We give upper and lower bounds on the distance from
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Let $o$ to the closest reachable points on the signed axes as a function of the number of left and right turns in $\sigma$, the size of the maximal monotone prefix or suffix of $\sigma$, and the maximal staircase prefix or suffix of $\sigma$.

To emphasize the number of the turns in $\sigma$, we use another notation $\sigma_{l,r}$ to denote a turn sequence with $l$ left turns and $r$ right turns, where $n = l + r$. We define the *excess number* $\delta$ of $\sigma_{l,r}$, denoted by $\delta(\sigma_{l,r})$, as the excess number of the left turns in the turn sequence, i.e., $\delta(\sigma_{l,r}) = l - r$. We define the *prefix excess number* $\delta_i$, as the excess number of the first $i$ turns in $\sigma_{l,r}$, i.e., $\delta_i = \delta(\sigma_1 \cdots \sigma_i)$, where $\delta_0 = 0$ and $\delta_{l+r} = l - r$. We call it *prefix number* in short.

We assign prefix numbers $\delta_0, \ldots, \delta_{l+r}$ to the segments of $C$ in the order from $o$ to $p$. Assume that a segment of $C$ is directed from $o$ to $p$. We call a segment of $C$ with prefix number $t$ a $\text{I}$-segment, and call a segment that is directed in the $z$-direction a $z$-segment for $z \in \{\pm x, \pm y\}$. We can easily observe that a $\text{I}$-segment is a $+x$-segment, $+y$-segment, $-x$-segment, or $-y$-segment if $t \equiv 0, 1, 2, 3$ (mod 4), respectively. For example, if $\sigma_{l,r} = \text{LLRR}$, we have $\delta_0 = 0, \delta_1 = 1, \delta_2 = 2, \delta_3 = 1, \delta_4 = 0$, and $\delta_5 = -1$. The fourth segment of $C$ has $\delta_3 = 1$, so it is a $\text{I}$-segment and also a $+y$-segment.

Let us define $x_{l,r}^i = \max_{\sigma_{l,r}} |x_{\sigma_{l,r}}^i|$ over all possible turn sequences $\sigma_{l,r}$ with $l$ left turns and $r$ right turns. Similarly, we define $x_{l,r}^-, y_{l,r}^+, \text{ and } y_{l,r}^-$ as the maximum distance from $o$ to the closest reachable points by any turn sequence $\sigma_{l,r}$ on the corresponding signed axes.

A subchain of a chain $C$ that realizes $\sigma_{l,r}$ is said to be $z$-monotone for $z \in \{\pm x, \pm y\}$ if it has no $-z$-segments. Let $m_z^o$ and $m_z^p$ denote the number of $z$-segments in the maximal $z$-monotone subchains respectively containing $o$ and $p$, where $z \in \{\pm x, \pm y\}$. Let $(z_1, z_2)$-staircase (or $(z_2, z_1)$-staircase) denote a staircase that is $z_1$-monotone and $z_2$-monotone, where $z_1, z_2 \in \{\pm x, \pm y\}$. Note that $z_1$- and $z_2$-directions are not the same, nor the opposite. For better understanding, we use the cardinal directions, NE-staircase for the north-east staircase, i.e., $(+y, +x)$-staircase, and NW-staircase, SE-staircase, and SW-staircase the $(+y, -x)$-staircase, $(-y, +x)$-staircase, and $(-y, -x)$-staircase, respectively. Let $z_1 m_{z_1}^o$ and $z_2 m_{z_2}^p$ denote the number of $z_2$-segments in the maximal $(z_1, z_2)$-staircases of $C$ containing $o$ and $p$, respectively.

Finally, we denote by $C[u,v]$ a subchain from $u$ to $v$ of $C$, where $u$ and $v$ are the points of $C$ and $u$ precedes $v$, that is, $u$ is closer to $o$ than $v$.

### 4.1 Upper and lower bounds on $y_{l,r}^-$ and $y_{l,r}^+$

We first explain the (upper and lower) bounds on $y_{l,r}^-$ and $y_{l,r}^+$ can be easily derived from the bounds on $x_{l,r}^-$ and $x_{l,r}^+$.

Let $C$ be a rectilinear chain that realizes a turn sequence $\sigma_{l,r} = \sigma_1 \cdots \sigma_n$ which reaches to a point $p = (0,-b)$ on the $-y$-axis for some $b > 0$. Define a new chain $C'$ by rotating $C$ by 90 degrees in counterclockwise direction. Turns are invariant to the rotation, thus the turn sequence of $C'$ remains unchanged. Then $p = (0,-b)$ on the $-y$-axis is mapped to a point $p' = (b,0)$ on the $+x$-axis, and the first (horizontal) segment of $C$ containing $(1,0)$ is mapped to a vertical segment containing $(0,1)$. We now augment $C'$ with a horizontal segment from $(-1,0)$ to $(0,0)$, which creates a new turn $L$ at $(0,0)$; this augmentation is always possible without causing self-intersections and changing the position of $p'$ by stretching along a vertical cut between $(-1,0)$ and $(0,0)$ if $(-1,0)$ is occupied by a part of $C'$. This resulting chain $C'$ realizes a new turn sequence $\sigma'_{l+1,r} = L \sigma_1 \sigma_2 \cdots \sigma_n$ such that it reaches to the point $p'' = (b+1,0)$ on the $+x$-axis when it starts from $(0,0)$. This shows that if $\sigma'_{l+1,r}$ reaches to a point $p'' = (a,0)$ on the $+x$-axis, then $\sigma_{l,r}$ reaches to a point $p = (0,-a+1)$ on the $-y$-axis. This implies that the bounds on the $-y$-axis are directly derived from the ones on
the $+x$-axis. Similarly, by rotating in clockwise direction, the bounds on the $+y$-axis are also derived from the ones on the $+x$-axis. Therefore, from now on, we consider the bounds only on the $-x$- and $+x$-axis.

### 4.2 Upper bounds on $x_{l,r}^+$ and $x_{l,r}^-$

We assume that $l - r \geq 0$.

The upper bounds on $x_{l,r}^-$ and $x_{l,r}^+$ can be obtained by giving an algorithm that draws a chain whose endpoint is on the target axis. We first introduce a simple algorithm, called LR-algorithm, which will be used as a subroutine in the drawing algorithms we propose. Similar algorithms were previously known for a given sequence of exterior angles, called the rl-algorithm [9], and for a given label-sequence [17].

**LR-algorithm**

The LR-algorithm runs in two phases. In the *reduction phase*, we find a pattern LR or RL in $\sigma_{1,r}$, and delete it, then we get a shorter sequence $\sigma_{l-1,r-1}$. We repeat this until there is no such pattern (see [18] for a similar procedure). Then the final sequence becomes $\sigma_{l-1,r-1}$ by the assumption that $l \geq r$. Note that $\sigma_{l-1,r-1} = L^{l-r}$ and is drawn as a spiral-like chain $C_{l-1,r-1}$ that wraps around $a$ in the counterclockwise direction. In the *reconstruction phase*, we draw the chain incrementally from $C_{l-1,r-1}$ for $\sigma_{l-1,r-1}$, by inserting back LR and RL patterns in the reverse order of the deletions in the reduction phase. We reconstruct the chain $C_{l-i+1,r-i+1}$ from the chain $C_{l-i-1,r-i-1}$ for $i = r, \ldots, 1$ by drawing the two additional segments (for inserting back an LR or RL) along a newly inserted empty row and column.

We now look closely at the chain $C_{l,r}$ drawn by the LR-algorithm when the excess number is small, i.e., $l - r = 0, 1, 2$. For $l - r = 0$, the final sequence in the reduction phase would be $\sigma_{0,0}$, thus the base chain is just a unit horizontal segment which connects two endpoints $a$ and $b$. Since rows and columns in the reconstruction phase are inserted between $a$ and $b$, those two endpoints still remain on the opposite sides of the bounding box until the end of the algorithm. A similar property holds for the other cases where $l - r = 1, 2$. The difference is that the sides where $a$ and $b$ lie are orthogonal for $l - r = 1$, and the same for $l - r = 2$.

From we can represent $C_{l,r}$ for $l - r = 0, 1, 2$ as a black box having one entry point $a$ and one exit point $b$ on its sides. These boxes will be used later as building blocks to draw chains with large excess numbers.

Let us consider $C_{l,r}$ in a different view when $l - r = 0$. If the last turn is $R$, then $C_{l,r}$ is a concatenation of $C_{l-1,r}$ and the last segment which are connected by $R$. Since $l - (r - 1) = 1$, we can draw the last segment (of $C_{l,r}$) along the top side (parallel to the first segment) of the bounding box of $C_{l-1,r}$. If the last turn is $L$, then we can draw the last segment, in a symmetric way, along the bottom side of the bounding box of $C_{l-1,r}$. We can draw any turn sequence with $l - r \leq 0$ in a similar way, so we can summarize these properties as follows.

**Lemma 8.** A turn sequence $\sigma_{l,r}$ with $l - r \in \{0, \pm 1, \pm 2\}$ has a realization by a chain $C_{l,r}$ with both endpoints on the sides of its bounding box. Moreover, if $l - r = 0$, then there is a realization in which the last segment is contained in one side of the bounding box.

**Notation and observation**

Consider a turn sequence $\sigma_{l,r}$. We define $i(w)$ and $j(w)$ as the indices of the last turn and the first turn whose excess number is exactly $w$, respectively. Then $0 \leq i(w), j(w) \leq l + r$ and $\delta_i(w) = \delta_j(w) = w$. We observe that for any $i(w) \leq l + r - 2$, if $\delta_i(w) < l - r$, then the
two consecutive turns $\sigma_{i(w)+1}$ and $\sigma_{i(w)+2}$ must be L, otherwise both must be R because otherwise, there would be $\delta_a = w$ for some $a > i(w)$, which is a contradiction that $i(w)$ is the index of the last segment with the excess number $w$. By the same reasoning, for any $j(w) \geq 2$, if $\delta_j(w) < 0$, the two previous turns $\sigma_{j(w)-1}$ and $\sigma_{j(w)-2}$ must be R, otherwise both must be L. We have the following observations.

\begin{itemize}
  \item \textbf{Observation 1.} For all $i(w) \leq l + r - 2$, if $\delta_i(w) < l - r$, then $\sigma_{i(w)+1}\sigma_{i(w)+2} = LL$. Otherwise, if $\delta_i(w) > l - r$, then $\sigma_{i(w)+1}\sigma_{i(w)+2} = RR$.

  \item \textbf{Observation 2.} For all $j(w) \geq 2$, if $\delta_j(w) < 0$, then $\sigma_{j(w)-2}\sigma_{j(w)-1} = RR$. Otherwise, if $\delta_j(w) > 0$, then $\sigma_{j(w)-2}\sigma_{j(w)-1} = LL$.\end{itemize}

4.2.1 Upper bounds on $x_{l,r}^-$

Let $C$ denote a chain that realizes a turn sequence $\sigma_{l,r}$. To reach the $-x$-axis, we know by Lemma 4 that $\sigma_{l,r}$ must have at least one left hook, i.e., there exists a $[3]$-segment or $[-3]$-segment in $C$. We will explain drawing algorithms that determine the endpoint $p$ on the $-x$-axis, which gives upper bounds on $x_{l,r}^-$, for the cases of $l - r = 0, 1, 2$, and $l - r \geq 3$.

\textbf{Case 1:} $l - r = 0$

If there exists a $[3]$-segment in $C$, then we can draw $C$ as shown in Figure 6(a). The black circle is $o$ and black square is $p$. The gray rectangles are bounding boxes enclosing subchains of $C$ which are drawn by LR-algorithm. The segments connecting the bounding boxes can be drawn without any self-intersection because entry and exit points of the boxes are on their sides by Lemma 8 and are properly positioned by stretching the boxes by horizontal or vertical cuts. Locating $[3]$-segment at $x = -1$ allows $p$ to be at $(-2,0)$, so we have that $x_{l,r}^- \leq 2$. If there exist a $[-3]$-segment in $C$, we can bound $x_{l,r}^- \leq 2$ similarly by reflecting $C$ with respect to the $x$-axis. Thus we have that $x_{l,r}^- \leq 2$.

\textbf{Case 2:} $l - r = 1$

We can draw $C$ in a similar way as Case 1.

\textbf{Case 2.1: There exist a $[-3]$-segment in $C$.} Draw $C$ as shown in Figure 6(b) so that the first $[-3]$-segment crosses the $x$-axis at $(-1,0)$. The part after this $[-3]$-segment can be drawing like Figure 6(b) so that $p$ is at $(-2,0)$. This is possible because the segment preceding the first $[-3]$-segment is a $[2]$-segment. We thus have that $x_{l,r}^- \leq 2$. 

\textbf{Case 2.2:} Draw $C$ as shown in Figure 6(c) so that $p$ is at $(-2,0)$, which is possible because the segment preceding the first $[-3]$-segment is a $[2]$-segment.
Case 2.2: There exist no \(-3\)-segment in \(C\) For this case, there must be a \(3\)-segment. Select the last \(3\)-segment of \(C\). Let \(u\) be its lower vertex. The drawing algorithm differs depending on whether \(C[u, p]\) contains a \(0\)-segment.

If \(C[u, p]\) contains no \(0\)-segments, then \(C[u, p]\) only contains \(1\)-segments and \(2\)-segments, which is indeed a NW-staircase, so the position of \(p\) is the width of the NW-staircase, i.e., the number of its horizontal segments, \(-x m_p^y\). See Figure (c).

Otherwise, if \(C[u, p]\) contains a \(0\)-segment, then select the last \(0\)-segment of the subchain. Its next two segments are \(1\)-segment and \(2\)-segment by Observation 1 which form a NW-staircase up to \(p\) as shown in Figure (d).

For both cases, the width of the maximal NW-staircase containing \(p\) affects the position of \(p\). Since all segments of the staircase can be drawn unit segments, we can locate \(p\) at \((-(-x m_p^y + 1), 0), \text{so } x_{l,r}^\pm \leq -x m_p^y + 1\).

Case 3: \(l - r = 2\)

We obtain a new chain \(C'\) by deleting the last segment from \(C\), then \(\delta(C') = 1\) if the last turn is \(L\) or \(3\) otherwise. After drawing \(C'\) by algorithms used for the cases \(l - r = 1\) or \(l - r = 3\) which will be explained below, adding a unit-length \(-x\)-segment at the end of \(C'\) gives a chain \(C\). Thus the upper bound for \(l - r = 2\) becomes the upper bounds for \(l - r = 1\) or \(l - r = 3\) plus one. We conclude that if \(\sigma_n = L\), then \(x_{l,r}^\pm \leq -x m_p^y + 2\), otherwise, if \(\sigma_n = R\), then \(x_{l,r}^\pm \leq 2\).

Case 4: \(l - r \geq 3\)

Consider the pairs \(\sigma_{i(w)+1}\sigma_{i(w)+2} = LL\) for \(w = 2, 6, 10, \ldots\). To get a chain \(C\), we split the turn sequence into subsequences at those pairs, draw them by the LR-algorithm, and merge them carefully.

At the first step, we take the subsequence \(\sigma_1 \cdots \sigma_{i(2)}\), and draw it as a chain \(C_1\) by the LR-algorithm. Since \(\delta_{i(2)} = 2\), \(C_1\) has entry and exit points on the left side of its bounding box \(B_1\) by Lemma 8. We can place \(B_1\) as in Figure (a) so that the entry point is on the \(+x\)-axis. We connect \(o\) with the entry point by a horizontal segment. Since \(\sigma_{i(2)+1}\sigma_{i(2)+2} = LL\) by Observation 1, we can draw the corresponding segments so that the vertical segment passes through \((-1,0)\) as in Figure (a). Note here that \(\delta_{i(2)+1} = 3\) and \(\delta_{i(2)+2} = 4\).

![Figure 7](image-url) The drawing steps for the turn sequence whose excess number is at least three.

The next steps are clear. We take the next subsequence \(\sigma_{i(2)+3} \cdots \sigma_{i(6)}\), draw its chain \(C_2\), place the bounding box \(B_2\) in the right of \(B_1\), and draw the next three segments corresponding...
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to $\sigma_{i(6)+1}\sigma_{i(6)+2}$, as in Figure 7(b), without any self-intersection. We repeat this until we place the bounding box $B_m$ of the chain $C_m$, where $m = \lfloor \frac{l-r+2}{4} \rfloor$. See Figure 7(c). Note that $m \geq 1$ from the assumption $l-r \geq 3$.

The last step is to place $B_{m+1}$ of the chain $C_{m+1}$ for the remaining subsequence $\sigma^* = \sigma_{i(4m-2)+1} \cdots \sigma_{l+r}$ and draw the last segments to reach the point $p = (-x,0)$ with $x = \lfloor \frac{l-r+2}{4} \rfloor$ for some positive integer $c$.

We need to handle this step in different ways according to the excess number $\delta(\sigma^*) \in \{0,1,2,3\}$. It is easy to place $B_{m+1}$ and draw the last segment when $\delta(\sigma^*) = 1$ or $\delta(\sigma^*) = 2$ (equivalently, $l-r \equiv 3$ or $0 \pmod{4}$); see Figure 8(a)-(b). The final chain reaches $p = (-m,0)$. When $\delta(\sigma^*) = 3$ (equivalently, $l-r \equiv 1 \pmod{4}$), we split it once again into two subsequences $\sigma'$ and $\sigma''$ with $\sigma^* = \sigma' \sigma''$ such that $\delta(\sigma') = 2$ and $\delta(\sigma'') = 1$. Draw $C'$ for $\sigma'$ and $C''$ for $\sigma''$ by the LR-algorithm, and place their boxes as in Figure 8(c) so that $p = (-m,0)$. For these three cases, we have that $x_{l,r} \leq m = \lfloor \frac{l-r+2}{4} \rfloor$.

![Figure 8](image.png)

**Figure 8** Handling the last subsequence $\sigma^*$. (a) When $\delta(\sigma^*) = 1$, the last segment enters $p = (-m,0)$ from the north. (b) When $\delta(\sigma^*) = 2$, the last segment enters $p = (-m,0)$ from the west along the negative x-axis. (c) When $\delta(\sigma^*) = 3$, $B_{m+1} = B' \cup B''$, so the last segment enters $p = (-m,0)$ from the south.

The last case that $\delta(\sigma^*) = 0$ (equivalently, $l-r \equiv 2 \pmod{4}$) should be handled more carefully. Let $\sigma'$ be the subsequence obtained by deleting the last turn $\sigma_n = \sigma_{l+r}$ from the original sequence $\sigma$. Suppose first that $\sigma_{l+r} = R$. Then $\delta(\sigma') = l-r+1$. We draw $\sigma'$ in the way explained above, then we can get a chain $C'$ as in Figure 8(a) because $m = \lfloor \frac{l-r+3}{4} \rfloor$ and the excess number of the $(m+1)$th subsequence is now one. The endpoint of $C'$ reaches $(-m,0)$.

To get $C'$, we simply extend $C'$ with a unit segment to the west from its endpoint, which makes a bend for $R$. Then $C'$ can reach $p = (-m-1,0)$, thus $x_{l,r}^- \leq \lfloor \frac{l-r+3}{4} \rfloor + 1 = \lfloor \frac{l-r+2}{4} \rfloor + 1$. Note that $l-r \equiv 2 \pmod{4}$. For the other case that the last turn is $L$, we can get $C'$ as in Figure 8(c) since $m = \lfloor \frac{l-r+1}{4} \rfloor$ and the excess number of the $(m+1)$th subsequence is three. We also add a unit segment from the endpoint to the west to reach $p = (-m-1,0)$, thus $x_{l,r}^- \leq \lfloor \frac{l-r+1}{4} \rfloor + 1 = \lfloor \frac{l-r+2}{4} \rfloor$.

**Theorem 9.** For any turn sequence $\sigma_{l,r}$ with $l-r \geq 0$, $x_{l,r}^-$ is bounded as follows:

| $l-r$       | $\leq 0$ | $= 1$ | $= 2$ | $\geq 3$ |
|------------|---------|------|------|--------|
| $x_{l,r}^-$ | 2       | $-x m_{p,y}^p + 1$ | $x m_{p,y}^p$ | $\lfloor \frac{l-r+2}{4} \rfloor + 1$ |

- If $\sigma_n = R$ and $l-r \equiv 2 \pmod{4}$
- Otherwise
4.2.2 Upper bounds on $x^+_{l,r}$

We draw a chain in a similar way as the algorithm we did for bounding $x^+_{l,r}$. Since the first segment of $C$ is a $+x$-segment, any $+x$-monotone prefix (possibly suffix) of the turn sequence, i.e., a turn sequence with no vertical hook, can prevent the chain from reaching a point close to $o$ on the $+x$-axis. This is the main difference with the case of $x^+_{l,r}$.

First, consider a special situation that the whole sequence $\sigma_{l,r}$ is $+x$-monotone, i.e., no vertical hook in $\sigma_{l,r}$. Note that this might happen only when $l-r=0$ or $l-r=1$. For this case, $m^o_{+x}=m^o_{-x}$. We can draw a chain with unit horizontal segments so that $p$ is at $(m^o_{+x},0)$. We thus have that $x^+_{l,r} \leq m^o_{+x}$.

Otherwise, i.e., $\sigma_{l,r}$ has at least one vertical hook, let $\sigma'=\sigma_1\cdots\sigma_g$ be the maximal $+x$-monotone prefix of $\sigma$ so that $\sigma_g\sigma_{g+1}$ is the first vertical hook. As shown in Figure 9, we have two cases: $\sigma_g\sigma_{g+1} = LL$ (up-LL hook) or $\sigma_g\sigma_{g+1} = RR$ (down-RR hook). We draw $C'$ for $\sigma'$ with unit segments, and $C''$ for the remaining subsequence $\sigma''$ by the winding scheme we used for bounding $x^+_{l,r}$. Note that $l-r-2 \leq \delta(\sigma'') \leq l-r+2$.

![Figure 9](image-url)

Figure 9 Reaching the $+x$-axis. Maximal $+x$-monotone prefixes followed by (a) LL and (b) RR.

For $\sigma_g\sigma_{g+1} = RR$, before applying the winding scheme, we draw the first part of $\sigma''$, that is, $\sigma_g\sigma_{g+2}\cdots\sigma_{l(2)}$ as a chain $C''$ using the LR algorithm, then connect it with $C'$ by RR, and extend its exit point to the bend corresponding to $\sigma_{l(2)+1}$. The remaining steps are the same as before.

The length of $\overline{op}$ is determined by the width of $C'$ plus the number of vertical segments in $C''$ intersected by the $+x$-axis; the latter could be increased by one if the last segment of $C$ is a $+x$-segment, i.e., $l-r \equiv 0 \pmod{4}$. The former is just the number of $+x$-segments of $C'$, denoted by $m^o_{+x}$, which is $\lceil |\sigma'|/2 \rceil$. The latter differs depending on the value of $l-r-2$. If $l-r-2 \geq 3$, then it is at most $\left\lfloor \frac{(l-r+2)}{4} \right\rfloor + 1 \leq \left\lfloor \frac{(l-r+2)}{4} \right\rfloor + 1 = \left\lceil \frac{l-r}{4} \right\rceil + 2$ by Theorem 9. Then we have that $x^+_{l,r} \leq m^o_{+x} + \left\lceil \frac{l-r}{4} \right\rceil + 2 \leq m^o_{+x} + \left\lceil \frac{l-r+2}{4} \right\rceil + 2$. We will explain the remaining case that $-2 \leq l-r-2 < 3$, i.e., $0 \leq l-r < 5$, later.

There is a way to improve this bound as follows. We simply switch the role of $o$ and $p$, and draw a rectilinear chain $C$ that realizes the turn sequence $\tilde{\sigma} = \tilde{\sigma}_1\cdots\tilde{\sigma}_{l}$ and starting from $p$, where $\tilde{\sigma}_i$ is the opposite turn from $\sigma_i$. Imagine that $p$ is the origin, then $o$ is now on the $-x$-axis. If the first segment from $p$ is not a $+x$-segment (equivalently, $l-r \not\equiv 2 \pmod{4}$), divide $\tilde{\sigma}$ into two subsequences $\tilde{\sigma}'$ and $\tilde{\sigma}''$, where $\tilde{\sigma}'$ is the maximal $-x$-monotone prefix of $\tilde{\sigma}$, and $\tilde{\sigma}''$ is the remaining subsequence. We draw $C'$ for $\tilde{\sigma}'$ with unit segments, and draw $C''$ for $\tilde{\sigma}''$, by the winding scheme we used for bounding $x^-_{l,r}$. Note that $l-r-3 \leq \delta(\tilde{\sigma}'') \leq l-r+3$.

The length of $\overline{op}$ is equal to the width of $C'$ plus the number of vertical segments in $C''$ intersected by the $-x$-axis plus one, because the last segment of $C$ is always a $-x$-segment.
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The former is the number of \(-x\)-segments of \(\bar{C}'\), which is the maximal \(-x\)-monotone subchain of \(\bar{C}\) containing \(p\). We can easily observe that it is equal to the number of \(+x\)-segments in the maximal \(+x\)-monotone subchain of \(C\) containing \(p\), denoted by \(m^p_{+x}\). The latter differs depending on the value of \(l - r - 3\). We will explain the case \(-3 \leq l - r - 3 < 3\), i.e., \(0 \leq l - r < 6\), later. If \(l - r - 3 \geq 3\), by Theorem 9 we can bound \(x^+_{l,r}\) as follows:

- If \(\sigma_1 = R\), i.e., \(\sigma_1 = L\), then
  \(x^+_{l,r} \leq m^p_{+x} + \frac{\delta(\sigma'') + 2}{4} \leq m^p_{+x} + \lfloor \frac{l - r + 2}{4} \rfloor + 1\).
- If \(\sigma_1 = L\), i.e., \(\sigma_1 = R\), then
  \(x^+_{l,r} \leq m^p_{+x} + \frac{\delta(\sigma'') + 2}{4} + 1 \leq m^p_{+x} + \lfloor \frac{l - r + 2}{4} \rfloor + 2\).

Otherwise, if the first segment from \(p\) is a \(+x\)-segment (equivalently, \(l - r \equiv 2 \mod 4\)), then we can apply the winding scheme used for bounding \(x^-_{l,r}\). By Theorem 9 we can bound \(x^+_{l,r}\) as follows:

- If \(l - r = 2\) and \(\sigma_1 = R\), i.e., \(\sigma_1 = L\), then
  \(x^+_{l,r} \leq m^o_{+y} + 2\).
- If \(l - r = 2\) and \(\sigma_1 = L\), i.e., \(\sigma_1 = R\), then
  \(x^+_{l,r} \leq 2\).
- If \(l - r \geq 6\) and \(\sigma_1 = R\), i.e., \(\sigma_1 = L\), then
  \(x^+_{l,r} \leq \lfloor \frac{l - r + 2}{4} \rfloor\).
- If \(l - r \geq 6\) and \(\sigma_1 = L\), i.e., \(\sigma_1 = R\), then
  \(x^+_{l,r} \leq \lfloor \frac{l - r + 2}{4} \rfloor + 1\).

We now have the following result.

\textbf{Lemma 10.} For any turn sequence \(\sigma_{l,r}\) with \(l - r = 2\) or \(l - r \geq 6\), \(x^+_{l,r}\) are bounded as follows:

| \(x^+_{l,r}\) | \(l - r = 2\) | \(l - r \geq 6\) and \(l - r = 2\) (mod 4) | \(l - r \geq 6\) and \(l - r \neq 2\) (mod 4) |
| --- | --- | --- | --- |
| \(x^+_{l,r} \leq \) | If \(\sigma_1 = L\) | \(x^+_{l,r} \leq m^o_{+y} + 2\) | \(\min\{m^o_{+y} + 2, m^p_{+x} + 1\} + \lfloor \frac{l - r + 2}{4} \rfloor\) |
| If \(\sigma_1 = R\) | 2 | \(\lfloor \frac{l - r + 2}{4} \rfloor + 1\) | \(\min\{m^o_{+y} + 2, m^p_{+x} + 2\} + \lfloor \frac{l - r + 2}{4} \rfloor\) |

We now have remaining cases \(l - r = 0, 1, 3, 4, 5\) which are not explained yet. We will explain each case in the order of \(l - r = 1, 0, 3, 5, 4\).

\textbf{Case 1:} \(l - r = 1\)

If \(\sigma\) is \(+x\)-monotone, i.e., no vertical hook, then it can reach \(p = (m^0_{+x}, 0)\) as mentioned earlier. We here suppose that \(\sigma\) has a vertical hook.

We divide \(\sigma\) into \(\sigma'\) and \(\sigma''\) as same as above; \(\sigma' = \sigma_1 \cdots \sigma_g\) is the maximal \(+x\)-monotone prefix of \(\sigma\) such that \(\sigma_g\sigma_{g+1}\) is the first vertical hook, and \(\sigma'' = \sigma_{g+1} \cdots \sigma_n\). Note that \(l - r - 2 \leq \delta(\sigma'') \leq l - r + 2\), i.e., \(-1 \leq \delta(\sigma'') \leq 3\). We now have two subcases according to the existence of the right hook in \(\sigma''\).

\textbf{Case 1.1:} \(\sigma''\) contains a right hook.

If \(\delta_{g+1} = -2\), i.e., \(\sigma_g\sigma_{g+1} = RR\), then \(\delta(\sigma'') = \delta_{i+r} - \delta_{g+1} = 3\). As in Figure 10(b), draw a chain \(C''\) below the \(-x\)-axis. The segment of \(C''\) is \(0\)-segment, so we can place \(p\) just right to \(C''\) on the \(-x\)-axis after a final left turn. This guarantees that \(x^+_{l,r} \leq m^o_{+x} + 1\). Otherwise, if \(\delta_{g+1} = 2\), i.e., \(\sigma_g\sigma_{g+1} = LR\), then \(\delta(\sigma'') = \delta_{i+r} - \delta_{g+1} = -1\). The method here is a symmetric and 180-degrees rotating version of the one to reach the \(-x\)-axis when \(l - r = 1\) in Figure 6(b)-(d). The difference is the existence of \(\bar{C}'\) segments. If \(\bar{C}'\) segment exists, then we draw four bounding boxes by \(\text{LR}\)-algorithm, place them as in Figure 10(a). This gives \(x^+_{l,r} \leq m^o_{+x} + 2\). For the other case that no \(\bar{C}'\) segment exists, the distance to \(p\) can be affected as well by the length of the
Figure 10 Case 1.1 that $\sigma''$ has a right hook and $\sigma_3\sigma_{g+1} = \text{LL}$: (a) $C$ has a $-1$-segment. (b) $C$ has no $-1$-segment.

staircase as in Figure 10(b). Since $-1$-segment exists in $\sigma''$, we draw the last $-1$-segment so that it passes the $+x$-axis in the right of $C''$. Two situations can occur depending on the existence of $2$-segments; if not exist, it directly reaches to $p$ via a NE-staircase (drawn as a black chain), otherwise it goes around then take a NE-staircase after the last $2$-segment (drawn as red dashed chain and box). For this case, the distance to $p$ is determined by the width of the maximal $+x$-monotone chain plus the width of the NE-staircase containing $p$. Thus, we have that $x^\tau_{l,r} \leq m^o_{+x} + m^p_{+y} + 1$.

Case 1.2: $\sigma''$ contains no right hook. In this case, we cannot apply the winding scheme used before. We first know that $\delta_{g+1} \neq -2$; otherwise $\delta(\sigma'') = \delta_{l+r} - \delta_{g+1} = 1 - (-2) = 3$, which means there exist a right hook in $\sigma''$, a contradiction. We now have that $\delta_{g+1} = 2$. Moreover, for all $g+1 < k < l+r$, $\delta_k \neq -1$ holds. Because if there exist such $k$ that $\delta_k = -1$, then $\delta_k - \delta_{g+1} = 3$, so a right hook exists. We conclude that after the turn $\sigma_{i(2)+1}$, there exist only $0$-segments and $1$-segments in $C$, and these segments form a NE-staircase. We can draw $C$ as shown in Figure 11(a), and we have $x^\tau_{l,r} \leq m^o_{+x} + m^p_{+y} + 1$. In particular, if $\sigma_1 = R$, then we have $x^\tau_{l,r} \leq m^o_{+x} + m^p_{+y} + 1$ because $m^o_{+y} = 0$.

Now, let us divide $\bar{\sigma}$ into $\bar{\sigma}' = \sigma_n \cdots \sigma_0$ and $\bar{\sigma}'' = \sigma_{h-1} \cdots \sigma_1$, which are the longest $+x$-monotone prefix of $\bar{\sigma}$, equivalently, the longest $+x$-monotone suffix of $\bar{\sigma}$, and the remaining subsequence, respectively. We have two subcases according to the existence of the left hook of $\sigma''$.

Case 1.3: $\sigma''$ contains a left hook Applying the same method we used in Case 1.1, we can have that if $\sigma_1 = L$, then $x^\tau_{l,r} \leq m^p_{+x} + m^o_{+y} + 2$, otherwise, if $\sigma_1 = R$, then $x^\tau_{l,r} \leq m^p_{+x} + m^o_{+y} + 2$.

Case 1.4: $\sigma''$ contains no left hook In this case, we need another drawing algorithm other than the winding scheme used for bounding $x^\tau_{l,r}$. If $\sigma_1 = L$, then we have $\delta_{h-2} = 2$ and $\delta_k \neq -1$ for all $1 < k < h - 2$. We know that before the turn $\sigma_{j(2)}$, there exist only $0$-segments and $1$-segments in $C$, and they form a NE-staircase. We can draw $C$ as shown in Figure 11(b), and we have $x^\tau_{l,r} \leq m^o_{+x} + m^p_{+y} + 1$.

Otherwise, if $\sigma_1 = R$, we have $\delta_{h-2} = -2$ and $\delta_k \neq 1$ for all $1 < k < h - 2$. We conclude that before the turn $\sigma_{j(2)}$, there exist only $0$-segments and $-1$-segments in $C$, and they
form a SE-staircase. We can draw $C$ as shown in Figure 11(c), and we get $x_{l,r}^+ \leq m_{-y}^0 + 1$.

We then have the following result for Case 1.

**Lemma 11.** For any $+x$-monotone sequence $\sigma_{l,r}$ with $l - r = 1$, $x_{l,r}^+ \leq m_{+x}^0$. For any non-$+x$-monotone sequence $\sigma_{l,r}$ with $l - r = 1$, if $\sigma_1 = L$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + m_{+x}^0 + 1, m_{+x}^0 + m_{+x}^0 + 2\},$$

if $\sigma_1 = R$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + m_{+y}^0 + 1, m_{+x}^0 + m_{+y}^0 + 2\}.$$

Consider a turn sequence $\sigma_{l,r}$ with $l - r = -1$ and a rectilinear chain $C$ that realizes $\sigma_{l,r}$ and whose endpoint $p$ is on the $+x$-axis. Let $C^*$ be the reflection of $C$ with respect to the $x$-axis with endpoint $p$. The turn sequence $\sigma^*$ of $C^*$ is obtained by reversing $\sigma$, so $\delta(\sigma^*) = 1$ and $p = p^*$, so the length of $\overline{op}$ is equal to the length of $\overline{op}^*$. Thus we have the following result.

**Lemma 12.** For any $+x$-monotone sequence $\sigma_{l,r}$ with $l - r = -1$, $x_{l,r}^+ \leq m_{+x}^0$. For any non-$+x$-monotone sequence $\sigma_{l,r}$ with $l - r = -1$, if $\sigma_1 = L$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + m_{+y}^0 + 1, m_{+x}^0 + m_{+y}^0 + 2\},$$

if $\sigma_1 = R$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + m_{+x}^0 + 1, m_{+x}^0 + m_{+x}^0 + 2\}.$$

Lemma 11 and Lemma 12 will be used for handling Case 2.

**Case 2:** $l - r = 0$

If we delete the last turn $\sigma_{l+r}$ from the original sequence $\sigma$, then its excess number becomes 1 or $-1$. Thus we can use Lemma 11 and Lemma 12 to bound $x_{l,r}^+$.

**Lemma 13.** For any $+x$-monotone sequence $\sigma_{l,r}$ with $l - r = 0$, $x_{l,r}^+ \leq m_{+x}^0$. For any non-$+x$-monotone sequence $\sigma_{l,r}$ with $l - r = 0$, if $\sigma_1 = L$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + \max\{m_{+y}^0, m_{+y}^0\}, m_{+x}^0 + m_{+y}^0\} + 2,$$

if $\sigma_1 = R$, then

$$x_{l,r}^+ \leq \min\{m_{+x}^0 + \max\{m_{+y}^0, m_{+y}^0\}, m_{+x}^0 + m_{+y}^0\} + 2.$$
Figure 12 Case 3 that \( l - r = 3 \): (a) Case 3.1, \( C \) has a \([-1]\)-segment. (b) Case 3.1, \( C \) has a \([5]\)-segment. (c) Case 3.2.

Case 3: \( l - r = 3 \)

We have two subcases depending on \( \sigma_1 \).

Case 3.1: \( \sigma_1 = L \). By Lemma 4, \( C \) contains a \([-1]\)-segment or a \([5]\)-segment to reach the \(+x\)-axis.

If \( C \) contains a \([-1]\)-segment, we can draw \( C \) as shown in Figure 12(a). If \( C \) has a \([2]\)-segment before the first \([-1]\)-segment, then it is drawn along the dashed red route, otherwise along the black one, which gives \( x_{i,r}^+ \leq +_x m_{+y}^a + 2 \).

If \( C \) contains a \([5]\)-segment, we can draw \( C \) as shown in Figure 12(b). If \( C \) has a \([2]\)-segment after the last \([5]\)-segment, then it is drawn along the dashed red route, otherwise along the black one. We have that \( x_{i,r}^+ \leq +_x m_{-y}^p + 2 \).

If \( C \) contains both \([-1]\)-segment and \([5]\)-segment, then we simply take the drawing that gives a smaller distance to \( p \), so we have that \( x_{i,r}^+ \leq \min\{+_x m_{+y}^a, +_x m_{-y}^p\} + 2 \).

Case 3.2: \( \sigma_1 = R \). We can draw \( C \) as shown in Figure 12(c). We have that \( x_{i,r}^+ \leq 2 \).

We then have the following result for Case 3.

Lemma 14. For any turn sequence \( \sigma_{i,r} \) with \( l - r = 3 \), \( x_{i,r}^+ \) is bounded as follows:

| \( x_{i,r}^+ \) | \(-1\)-segment | \([5]\)-segment | \(-1\) and \([5]\)-segments |
|---|---|---|---|
| exists? | \(+_x m_{+y}^a + 2\) | \(+_x m_{-y}^p + 2\) | \( \min\{+_x m_{+y}^a, +_x m_{-y}^p\} + 2 \) |
| \( \sigma_1 = L \) | \( +_x m_{+y}^a + 2 \) | \( +_x m_{-y}^p + 2 \) | \( \min\{+_x m_{+y}^a, +_x m_{-y}^p\} + 2 \) |
| \( \sigma_1 = R \) | \( 2 \) | | |

Case 4: \( l - r = 5 \)

We have two subcases depending on \( \sigma_1 \).

Case 4.1: \( \sigma_1 = L \) We can draw \( C \) as shown in Figure 13(a). Then we have that \( x_{i,r}^+ \leq 2 \).

Case 4.2: \( \sigma_1 = R \) We can draw \( C \) as shown in Figure 13(b)-(c). In Figure 13(b), if \( C \) has a \([-2]\)-segment before the first \([1]\)-segment, then it drawn along the dashed red route, otherwise along the black one. We have that \( x_{i,r}^+ \leq +_x m_{-y}^p + 2 \).
We now show a lower bound on the distance of the closest reachable point to the origin on the $x$-axis. Let $C$ be an arbitrary rectilinear chain that realizes a turn sequence $\sigma_{l,r}$ for $l - r \geq 0$ which is reachable to the $x$-axis, and $p$ be the endpoint of $C$ that is on the $x$-axis. First, we will bound the minimum number of vertical segments of $C$ that intersect $\overline{OP}$, which gives a lower bound on the distance. For this, we use the rotation number of a whisker-free polygon, which is introduced by Grünbaum [12]. A polygon is whisker-free if no two edges incident with a vertex of the polygon overlap in a segment of positive length.

We redefine the rotation number of a whisker-free polygon $P$ as follows.

**Definition 18.** Let $P$ be a whisker-free polygon that has $n$ vertices $v_1, v_2, \ldots, v_n = v_0$ and $n$ directed edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_n = e_0 = (v_n, v_1)$. Let $\alpha(v_i)$ denote the signed...
angle \( \angle \) between the direction vector of \( e_{i-1} \) and \( e_i \), and let \( d(v_i) = \alpha(v_i)/2\pi \) denote the deflection of \( v_i \) for \( i = 1, 2, \ldots, n \). The rotation number \( R(P) \) of \( P \) is the sum of deflections of vertices in \( P \), i.e., \( R(P) = \sum_{i=1}^{n} d(v_i) \).

Grünbaum \cite{12} also introduced the ordinary polygon; a polygon is ordinary if no three edges have a common point. It is clear that every ordinary polygon is whisker-free.

A rectilinear polygon \( P_C \) is defined by connecting \( o \) and \( p \) of \( C \), but it is not necessarily ordinary because some segments of \( C \) can be overlapped with \( \overline{op} \). If the first or last segment of \( C \) does, then define a subchain \( C' \) of \( C \) by deleting the overlapped segments. Note that \( P_{C'} \) obtained by connecting two endpoints of \( C' \) is now whisker-free, but not ordinary yet if there are segments of \( C' \) overlapped with the segment connecting the endpoints of \( C' \). We can make a new chain \( C'' \) from \( C' \) by translating the overlapped segments by small positive amount above or below the \( x \)-axis while keeping the continuity of \( C' \). Then \( P_{C''} \) obtained by connecting two endpoints of \( C'' \) becomes ordinary.

We will show that the minimum number of vertical segments of \( C \) intersecting \( \overline{op} \) is derived from the rotation number of the ordinary polygon made in this way. We now suppose that \( p \) is on the \( +x \)-axis.

### 4.3.1 The minimum number of vertical segments of \( C \) that cross \( \overline{op} \)

For any turn sequence \( \sigma_{l,r} \) with \( l - r \equiv 0 \pmod{4} \), we have \( \delta_{l+r-1} = l - r + 1 \equiv \pm 1 \pmod{4} \). The last segment of its chain \( C \) is a \( +x \)-segment coming to \( p \) from the west, that is, it is contained in the \( +x \)-axis. Deleting the last segment gives a shorter chain \( C^* \) whose \( l - r \equiv 1 \pmod{4} \) or \( l - r \equiv 3 \pmod{4} \). By bounding the length of the last segment of \( C \) into 1, we can make the lower bound on \( |\overline{op}| \) of \( C \) be bounded below by the lower bound on \( |\overline{op}^*| \) of \( C^* \) plus one, where \( p^* \) is the endpoint of \( C^* \). We here consider only the turn sequence \( \sigma_{l,r} \) with \( l - r \not\equiv 0 \pmod{4} \).

We have two cases \( \sigma_1 = \text{L} \) and \( \sigma_1 = \text{R} \).

#### Case 1: \( \sigma_1 = \text{L} \)

There are \( l - r \) more left turns than right turns in \( C \). Let \( k = \left\lfloor \frac{l - r}{4} \right\rfloor \geq 0 \). Then \( l - r \in \{4k + 1, 4k + 2, 4k + 3\} \). Let \( P \) be an ordinary polygon derived from \( C \) by connecting \( p \) of \( C \) and the second vertex \( q \) of \( C \) and removing the first segment \( \overline{pq} \) of \( C \). Let us count the number of left and right turns of \( P \). The first turn \( \text{L} \) at \( q \) disappears in \( P \). The last segment \( \overline{pq} \) of \( P \) creates two turns at \( p \) and \( q \); the turn \( \text{R} \) at \( q \) and the turn \( \text{L} \) or \( \text{R} \) at \( p \) depending on \( l - r \equiv 1 \pmod{4} \) or \( l - r \equiv 3 \pmod{4} \), respectively. Reflecting all these changes, we can conclude that \( P \) has exactly \( 4k \) more left turns than right ones. Since a left turn and a right turn contributes to a deflection by \( \frac{1}{4} \) and \( -\frac{1}{4} \) respectively, we have \( R(P) = 4k \times \frac{1}{4} = k = \left\lfloor \frac{l - r}{4} \right\rfloor \).

Whitney \cite{19} defined the rotation number of an oriented smooth closed curve, and Polyak \cite{15} rewrote it simply as the number of rotations made by the tangent vectors as traversing along the curve. Whitney \cite{19} also introduced a normal oriented smooth closed curve, and we redefine it as follows: An oriented smooth closed curve is normal if the curve has neither overlapping nor touching parts, and no three or more pieces of the curve intersect at a common point if it has self-intersections.

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1 An angle measured from \( e_{i-1} \) to \( e_i \) in counterclockwise direction is positive, for clockwise direction negative.
Let $S$ be an oriented smooth closed curve obtained by smoothing the vertices and their neighborhoods of $P$. Note that we can determine the smoothness so that no self-intersection is generated. Since $P$ and $S$ are topologically equivalent, we have $R(P) = R(S)$, where $R(S)$ denote the rotation number of $S$. Moreover $S$ is normal because $P$ is ordinary.

Let $I$ be the total number of self-intersections of $S$ (also of $P$). We show the relation between $R(S)$ and $I$ as follows.

Lemma 19. For a normal oriented smooth closed curve $S$, $I \geq ||R(S)|| - 1$.

Proof. A self-intersection of $S$ is the crossing between two different pieces of $S$. Let $c_1$ and $c_2$ be the two pieces of $S$ that cross each other such that $c_1$ precedes $c_2$. There are two types of self-intersections; it is the first type if $c_2$ crosses over $c_1$ from the left side to the right side of $c_1$, and otherwise the second type. Let $I^+$ and $I^-$ denote the number of the first and second type self-intersections, respectively. Then $I^+ + I^- = I$. Whitney [19] proved $R(S) = \mu + I^+ - I^-$, where $\mu$ is either $1$ or $-1$. By using this formula and the triangular inequality, we conclude that

$$I = I^+ + I^- = |I^+| + |I^-| \geq |I^+ - I^-| = |R(S) - \mu| \geq ||R(S)| - |\mu|| = ||R(S)| - 1||.$$

Because $R(P) = R(S)$ as we mentioned above, we have, by Lemma 19, that

$$I \geq |R(P) - 1| = \left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right|.$$

All self-intersections of $P$ are made on $\overline{pq}$, so at least $\left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right|$ vertical segments of $P$ cross $\overline{pq}$. As the second segment of $C$ which is vertical also intersects (in fact, touches) with $\overline{op}$, the number of vertical segments intersecting (or crossing) $\overline{op}$ is at least

$$\left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right| + 1.$$

Case 2: $\sigma_1 = R$

By the same argument as in Case 1, we have $R(P) = (4k + 4) \times \frac{1}{4} = k + 1 = \left\lfloor\frac{l - r}{4}\right\rfloor + 1$. By Lemma 19 at least $\left\lfloor\frac{l - r}{4}\right\rfloor + 1$ vertical segments of $C$ cross $\overline{pq}$, including the second segment of $C$.

We conclude this section with the following theorem.

Theorem 20. For any turn sequence $\sigma_{l,r}$ with $l - r \not\equiv 0 \pmod 4$ reachable to the $+x$-axis, any rectilinear chain $C$ that realizes $\sigma_{l,r}$ and reaches a point $p$ on the $+x$-axis intersects $\overline{op}$ in at least $\left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right| + 1$ vertical segments of $C$ if $\sigma_1 = L$; otherwise, if $\sigma_1 = R$, $\left\lfloor\frac{l - r}{4}\right\rfloor + 1$ vertical segments of $C$.

4.3.2 Lower bounds on $x_{l,r}^+$

By Theorem 20 there are at least $\left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right| + 1$ or $\left\lfloor\frac{l - r}{4}\right\rfloor + 1$ vertical segments lying between $o$ and $p$, which implies that the distance from $o$ to $p$ is at least $\left|\left\lfloor\frac{l - r}{4}\right\rfloor - 1\right| + 2$ or $\left\lfloor\frac{l - r}{4}\right\rfloor + 2$. Besides this term, we will show that for some turn sequence, as in the upper bound on $x_{l,r}^+$, the length of the maximal staircases from $o$ or to $p$ is also contributed to the lower bound on $x_{l,r}^+$. 
Figure 14 (a) New rectilinear polygon $P'$ bounded above by $\overline{st}$. (b) Vertical segments between $s$ and $t$ when $\ell$ passes $o_1$ and $p_f$.

Case 1: $l - r \equiv 1 \pmod{4}$

Subcase 1.1: $\sigma_1 = R$. It is obvious that $+_x m_{o_y} > 0$ and $+_x m_{p_y} > 0$. For the maximal SE-staircase $C_o$ that starts from $o$, let $o_f$ and $o_l$ denote the leftmost and rightmost vertical segments of $C$, respectively. Similarly, for the maximal NE-staircase $C_p$ that ends at $p$, let $p_f$ and $p_l$ denote the leftmost and rightmost vertical segments of $C$, respectively. See Figure 14(a).

We consider a horizontal line $\ell$ that intersects the interiors of two vertical segments of $C_o$ and $C_p$ each at points $s \in C_o$ and $t \in C_p$. Consider a rectilinear polygon $P'$ such that its boundary consists of the subchain $C[s, t]$ from $s$ to $t$ and a horizontal segment $\overline{st}$. The subchains $C[q, s]$ and $C[t, p]$ are staircases, and they have the same number of left and right turns. This means that $R(P') = R(P)$, where $P$ is a rectilinear polygon defined by connecting $pq$ as in Section 4.3.1. By Lemma 19, there are at least $R(P') - 1 = R(P) - 1 = \lfloor \frac{l - r}{4} \rfloor$ vertical segments that intersect $\overline{st}$, so the length of $\overline{st}$ is at least $\lfloor \frac{l - r}{4} \rfloor + 1$. The distance between $o$ and $p$ is expressed as the sum of four terms, as shown in Figure 14(b); (1) the length of $\overline{st}$, which is at least one, (2) the width $w_1$ of $C[q, s]$, (3) the length of $\overline{st}$, and (4) the width $w_2$ of $C[t, p]$. Their sum is at least $1 + w_1 + (\lfloor \frac{l - r}{4} \rfloor + 1) + w_2$.

The value of $w_1 + w_2$ gets bigger as $\ell$ moves lower because $w_1$ and $w_2$ are the width of the SE- and NE-staircases. Note here that $\overline{st}$ always intersects at least $\lfloor \frac{l - r}{4} \rfloor$ vertical segments regardless of the position of $\ell$. We also know that $\ell$ can intersect at least one of $o_f$ and $p_f$. If $\ell$ intersects both $o_1$ and $p_f$, then $w_1 + 1 = +_x m_{o_y}$ and $w_2 = +_x m_{p_y} - 1$, which is the maximum. If $\ell$ intersects only $o_1$, then $w_1 + w_2 + 1 \geq +_x m_{o_y}$. Otherwise, if $\ell$ intersects only $p_f$, then $w_1 + w_2 + 1 \geq +_x m_{p_y}$. Thus we have that $w_1 + w_2 + 1 \geq \min\{+_x m_{o_y}, +_x m_{p_y}\}$, so $w_1 + w_2 + 1 \geq \min\{+_x m_{o_y} + x m_{p_y}\} + \lfloor \frac{l - r}{4} \rfloor + 1 = \min\{+_x m_{o_y} + x m_{p_y}\} + \lfloor \frac{l - r + 2}{4} \rfloor + 1$ because $\lfloor \frac{l - r}{4} \rfloor = \lfloor \frac{l - r + 2}{4} \rfloor$ for $l - r \equiv 1 \pmod{4}$.

Subcase 1.2: $\sigma_1 = L$. Unlike Case 1.1, the maximal staircase $C_o$ (containing $o$) and the maximal staircase $C_p$ (containing $p$) are on the opposite sides of the $x$-axis, so there is no horizontal line $\ell$ that intersects $C_o$ and $C_p$ at the same time. This implies that the minimum number of vertical segments that cross $\overline{op}$, which is $\lfloor \frac{l - r}{4} \rfloor + 1 = \lfloor \frac{l - r + 2}{4} \rfloor + 1$, is only a term of the lower bound.
Reachability of turn sequences

For \( l - r \equiv 1 \pmod{4} \), we have the following bound.

Lemma 21. For any turn sequence \( \sigma_{l,r} \) with \( l - r \equiv 1 \pmod{4} \), if \( \sigma_1 = R \), then

\[
x^+_i \geq \min \{ +_x m^o_{-y}, +_x m^p_{+y} \} + \left\lceil \frac{l - r + 2}{4} \right\rceil + 1,
\]

if \( \sigma_1 = L \), then

\[
x^+_i \geq \left\lfloor \frac{l - r}{4} \right\rfloor - 1 + 2.
\]

Case 2: \( l - r \equiv 3 \pmod{4} \)

If \( \sigma_1 = L \), then the maximal staircases \( C_o \) and \( C_p \) are above the \( x \)-axis. Moreover, two staircases are monotone to the \(+x\)-axis. We can apply the same method used in Case 1 to bound \( x^+_i \). If \( \sigma_1 = R \), then two staircases are on the opposite sides of the \( x \)-axis, so only the minimum number of vertical segments that cross \( \sigma \) determines the lower bound.

Lemma 22. For any turn sequence \( \sigma_{l,r} \) with \( l - r \equiv 3 \pmod{4} \), if \( \sigma_1 = L \), then

\[
x^+_i \geq \min \{ +_x m^o_{-y}, +_x m^p_{+y} \} + \left\lceil \frac{l - r}{4} \right\rceil - 1 + 2 = \left\lceil \frac{l - r + 2}{4} \right\rceil + 1.
\]

if \( \sigma_1 = R \), then

\[
x^+_i \geq \left\lfloor \frac{l - r}{4} \right\rfloor + 2 = \left\lfloor \frac{l - r + 2}{4} \right\rfloor + 1.
\]

Case 3: \( l - r \equiv 2 \pmod{4} \)

For this case, the maximal staircase \( C_p \) is a SW-staircase or NW-staircase, i.e., goes to the west (to the \(-x\)-axis) while the maximal staircase \( C_o \) goes to the east. From this, we know that the lower bound is determined only by the minimum number of vertical segments that cross \( \sigma \). We can bound \( x^+_i \) as follows.

Lemma 23. For any turn sequence \( \sigma_{l,r} \) with \( l - r \equiv 2 \pmod{4} \), if \( \sigma_1 = L \), then

\[
x^+_i \geq \left\lfloor \frac{l - r}{4} \right\rfloor - 1 + 2 = \left\lfloor \frac{l - r + 2}{4} \right\rfloor - 2 + 2,
\]

if \( \sigma_1 = R \), then

\[
x^+_i \geq \left\lfloor \frac{l - r}{4} \right\rfloor + 2 = \left\lfloor \frac{l - r + 2}{4} \right\rfloor + 1.
\]

This lower bound is exactly matched with the upper bound in Lemma 10 for any \( l - r \equiv 2 \pmod{4} \) except when \( l - r = 2 \).

Case 4: \( l - r \equiv 0 \pmod{4} \)

The lower bound for this case can be easily derived from the one for the turn sequence of \( l - r \equiv 1 \pmod{4} \) or \( l - r \equiv 3 \pmod{4} \), obtained by deleting the last turn from the original sequence. Using Lemma 21 and Lemma 22 for the cases, we can bound \( x^+_i \) as follows.
Lemma 24. For any turn sequence $\sigma_{l,r}$ with $l - r \equiv 0 \pmod{4}$ and $l - r \geq 8$, if $\sigma_1 = L$, then
\[
x^+_l \geq \min \{+x_m^o + x_{m^p}, 2\} + \left\lfloor \frac{l - r + 2}{4} \right\rfloor,
\]
if $\sigma_1 = R$, then
\[
x^+_l \geq \left\lfloor \frac{l - r + 2}{4} \right\rfloor + 2.
\]
For any turn sequence $\sigma_{l,r}$ with $l - r = 0$, if $\sigma_1 = L$, then $x^+_l \geq \min \{+x_m^o + x_{m^p}, 2\} + 2$, otherwise, $x^+_l \geq \min \{+x_m^o + x_{m^p}, 2\} + 2$. For any turn sequence $\sigma_{l,r}$ with $l - r = 4$, $x^+_l \geq 3$.

We conclude this section with the following theorem.

Theorem 25. For any turn sequence $\sigma_{l,r}$ with $l - r \geq 0$, the lower bounds on $x^+_l$ can be summarized in Lemma 21, Lemma 22, Lemma 23, and Lemma 24 for $l - r \equiv 1, 2, 0 \pmod{4}$, respectively.

Remark 26. The lower bounds for $l - r = 5$ and for $l - r \equiv 2 \pmod{4}$ with $l - r \geq 6$ are exactly matched with their upper bounds.

4.3.3 Lower bounds on $x^-_{l,r}$

By the method we used in Section 4.3.1, we can also bound the minimum number of vertical segments of $C$ that cross $\overline{OP}$, where $p$ is on the $-x$-axis. The parameters that determine the bound are the last turn $\sigma_n$ (not the first turn $\sigma_1$) and the excess number $l - r$ only (not including the lengths of the maximal staircases). We here summarize the results without giving the detailed proofs.

Theorem 27. For any turn sequence $\sigma_{l,r}$ with $l - r \geq 0$, the lower bounds on $x^-_{l,r}$ are as follows:

| $l - r \equiv 0$ or $1 \pmod{4}$ | $l - r \equiv 2 \pmod{4}$ | $l - r \equiv 3 \pmod{4}$ |
|---|---|---|
| $\left\lfloor \frac{l - r + 2}{4} \right\rfloor - 1 \right\rfloor + 1$ | $\left\lfloor \frac{l - r + 2}{4} \right\rfloor - 2 \right\rfloor + 2$ if $\sigma_n = L$ | $\left\lfloor \frac{l - r + 2}{4} \right\rfloor + 1$ if $\sigma_n = R$ |

Remark 28. It is worthwhile to mention that unlike the bounds on the $+x$-axis, the lower bound on the $-x$-axis for any $l - r \geq 3$ is exactly matched with the upper bound. The bound is also tight for $l - r = 0$, but not for $l - r = 1, 2$.

5 Concluding remarks

In this paper, we characterize combinatorial and geometric properties on the reachable region by a turn sequence of left and right turns. For this, we first present the sufficient and necessary conditions on the reachability to the signed axes. We next obtain upper bounds on the maximum distance to the closest reachable point from the origin on the signed axes by describing drawing algorithms of the turn sequence, and prove the lower bounds by bounding the number of self-intersections of a (non-simple) rectilinear polygon induced by the turn sequence, which are almost tight within some additive constant for some signed axes. Interestingly, these bounds are expressed in terms of the difference of the number of left and right turns and the length of the maximal monotone prefix or suffix of the sequence.
We close this section with a list of open problems. First, the upper and lower bounds for some cases are not tight; for example, for the sequence with \( l - r = 2 \) and the last turn of \( L \), we have that \( 3 \leq x_{l,r} \leq -x_{l,r} + 2 \); the bounds are not tight within an additive constant. It remains open to narrow the gaps between the bounds or find the exact closest reachable point in polynomial time. Second, we can consider an interesting variant of characterizing the reachable region by a \textit{quad-turn sequence}, as a natural extension of the binary-turn sequence seen so far, which is a sequence consists of two different left turns \( L_1 \) and \( L_2 \), and two different right turns \( R_1 \) and \( R_2 \) as shown in Figure 15(a). For a quad-turn sequence \( \tau = \tau_1\tau_2\cdots\tau_n \) where \( \tau_i \in \{L_1, L_2, R_1, R_2\} \), a chain realizing this sequence is drawn in a triangular grid in Figure 15(a). This grid can be deformed to a right triangular grid like Figure 15(b). While the binary-turn sequence has the rotation number \( \lfloor \frac{l-r}{4} \rfloor \) as a term of the distance bound, the quad-turn sequence has a term of \( \lfloor \frac{l_1+2l_2-r_1-2r_2}{6} \rfloor \), where \( l_1, l_2, r_1, \) and \( r_2 \) denote the number of \( L_1, L_2, R_1, \) and \( R_2 \) in \( \tau \), respectively. We can also consider the \textit{hexagonal chain} drawing in the triangular grid for turn sequences that only contain \( L_1 \) and \( R_1 \) turns.

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