ASYMPTOTICS OF INTEGRALS OF SOME FUNCTIONS RELATED TO THE DEGENERATE THIRD PAINLEVÉ EQUATION

A. V. Kitaev* and A. Vartanian†

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It is shown how to calculate asymptotics of integrals over the positive semi-axis of two functions related to the degenerate third Painlevé equation \((dP_3)\). As an example, the corresponding results for the meromorphic solution of \(dP_3\) vanishing at the origin are presented. Bibliography: 9 titles.

Dedicated to Professor M. A. Semenov-Tian-Shansky on the occasion of his 70th anniversary

1. Introduction

The degenerate third Painlevé equation can be written in the following form (see [1,2]):

\[
\frac{d^2u}{d\tau^2} = \left(\frac{du}{d\tau}\right)^2 u - \frac{du}{d\tau} \tau + \frac{1}{\tau} (-8\epsilon u^2 + 2ab) + \frac{b^2}{u},
\]

(1)

where \(u = u(\tau)\), the primes denote differentiation with respect to \(\tau\), \(a \in \mathbb{C}\) and \(b > 0\) are parameters, and \(\epsilon = \pm1\). In most instances, the \(\tau\)-dependencies are suppressed; e.g., the notation \(u\) connotes \(u = u(\tau)\).

There is another form of Eq. (1), namely,

\[
b^2 \tau^2 \left( f'' - 2b^2 \right)^2 + (8f + i\epsilon b(2ai - 1))^2 \left( (f')^2 - 4b^2 f \right) = 0,
\]

(2)

where \(i^2 = -1\). Equation (2) coincides with the one given in [1] via the rescalings \(f \to i\epsilon b f/2\) and \(a \to a - i\). It occurs because of a slight difference in the definition of the function \(f\); more precisely, for any solution \(u\) of Eq. (1), define the functions (see [1, p. 1198])

\[
u_+ = \frac{i\epsilon b}{8u^2} \left( \tau(-u' - ib) - (2ai - 1)u \right)
\]

(3)

and

\[
u_- = \frac{i\epsilon b}{8u^2} \left( \tau(u' - ib) - (2ai + 1)u \right),
\]

which solve Eq. (1) for \(a = a_+ := a + i\) and \(a = a_- := a - i\), respectively. One proves that the function \(f = u_+ u\) solves Eq. (2), while the function \(i\epsilon b u_- / 2\) solves the equation for \(f\) presented on p. 1168 of [1]. Conversely, suppose that \(f\) is a solution of Eq. (2); then

\[
u = \frac{f'}{2i\epsilon} - \frac{\epsilon \tau (f'' - 2b^2)}{2(8f + i\epsilon b(2ai - 1))}
\]

(4)

solves Eq. (1), and \(u_+ = u - f'/(ib)\).

Due to the papers [3,4], there is another well-known class of equations quadratic with respect to the second derivative that are equivalent to the Painlevé equations, namely, the so-called \(\sigma\)-forms of the Painlevé equations, which are related to their Hamiltonian structures and the \(\tau\)-functions. In this paper, the \(\sigma\)-forms of Eq. (1) are not discussed.

*St.Petersburg Department of Steklov Institute of Mathematics, St.Petersburg, Russia, e-mail: kitaev@pdmi.ras.ru.
†College of Charleston, Department of Mathematics, Charleston, SC 29424, USA, e-mail: VartanianA@cofc.edu.
The equation that is equivalent to Eq. (1) is specified in the paper [5] (see p. 75 of [5]) as SD-III.b (5.66) under the conditions (5.68) and denoted by SD-III.A. One learns from this paper that Eq. (1) was first discovered by F. Bureau [6] via the direct Painlevé analysis; he also found a relation of this equation to Eq. (1), which is, without interpreting the functions \( u_\pm \) as Bäcklund transformations, equivalent to our formulas. The transformation (4) may, in fact, be new. It should be noted that the derivation of Eq. (2) in [1] is based on the Hamiltonian structure of Eq. (1) and, indirectly, its isomonodromy deformations.

The definition of the function \( f \) can be rewritten as

\[
 f = u_+ u = -\tau \frac{i e b}{8} \left( \frac{u'}{u} - \frac{1}{\tau} + i \left( \frac{2a}{\tau} + \frac{b}{u} \right) \right);
\]
equivalently,

\[
 f = -\frac{i e b}{8} \frac{d}{d\tau} \ln A(\tau), \quad A(\tau) := \frac{u}{\tau} e^{i\phi}, \quad \phi' := \frac{2a}{\tau} + \frac{b}{u}, \tag{5}
\]
where the functions \( A \) and \( \phi \) are introduced in Proposition 1.2 of [1] in connection with isomonodromy deformations. Integrating along a contour \( \mathcal{L}(\tau_0, \tau) \) connecting points \( \tau_0 \) to \( \tau \), one arrives, from the third equation in (5) and, after division by \( \tau \), the first equation in (5), at

\[
 \int_{\mathcal{L}(\tau_0, \tau)} \left( \frac{2a}{\tau} + \frac{b}{u(\tau)} \right) d\tau = \phi|_{\mathcal{L}(\tau_0, \tau)}, \quad \int_{\mathcal{L}(\tau_0, \tau)} \frac{f(\tau)}{\tau} d\tau = \frac{e b}{8} \left( \phi - i \ln \left( \frac{u}{\tau} \right) \right)|_{\mathcal{L}(\tau_0, \tau)}. \tag{6}
\]

The main goal of this paper is to explain how one can evaluate these integrals. Towards this end, one has to explain how to calculate the deviation of the functions \( \phi \) and \( u \) along \( \mathcal{L}(\tau_0, \tau) \).

In this paper, the aforementioned problem is considered asymptotically, that is, when the limits of integration belong to small neighborhoods of the singular points, 0 and \( \infty \), of Eq. (1). For this purpose, one requires asymptotics of the functions \( u \) and \( \phi \).

The asymptotics of the function \( u \) were studied in [1, 2]; the corresponding asymptotics for the function \( \phi \) can also be extracted from these papers. In order to do so, recall that in Proposition 1.2 of [1] there was one more function:

\[
 B(\tau) = -\frac{u}{\tau} e^{-i\phi};
\]
therefore, the function \( \phi \) can be presented as

\[
 \phi = -\frac{i}{2} \ln \left( -\frac{A}{B} \right) = -i \ln \left( \frac{\sqrt{-AB}}{B} \right). \tag{7}
\]

The final transformation of the above equation is necessary, because it is for the functions \( \sqrt{-AB} \) and \( B \) that asymptotic results are given in Proposition 4.3.1, Corollary 4.3.1, and Propositions 5.5 and 5.7 of [1]. It is important to note that in Appendix B of the subsequent paper [2], inconsistencies in the paper [1] were located and rectified. Furthermore, as explained in Sec. 7 of [7], due to the discrepancy in the definition of the canonical solutions and the corresponding linear ODE, one has to add to the asymptotics of the function \( \phi \), obtained with the help of the results in [1, 2], the term \( a \ln \tau \).

The integral analogous to the first one in (6), but for the second Painlevé equation, was calculated in [8]; in the latter case, however, an analog of Eq. (2) does not exist.
2. Meromorphic solution vanishing at the origin

In the previous section, to general scheme allowing one to calculate the integrals (6) was presented; however, for every particular solution and contour of integration, there are special questions that must be addressed. Here, one simple, yet interesting, example of such a calculation is considered. Note that in this section \( \epsilon = +1 \).

It is proved in [7] that for every \( a \in \mathbb{C} \setminus i\mathbb{Z} \) there exists a unique odd meromorphic solution of Eq. (1) such that \( u(0) = 0 \). The asymptotic calculation of the integrals for this solution is done by taking the simplest contour, \( \mathcal{L}(0, \tau) = [0, \tau], \tau \in \mathbb{R}_+, \tau \to +\infty \).

Consider the case \( a \in \mathbb{R} \setminus \{0\} \). For \( a = 0 \), a solution holomorphic in a neighborhood of \( \tau = 0 \) and vanishing at \( \tau = 0 \) does not exist. For \( a > 0 \), such a solution has an infinite number of poles on the real axis, which can be deduced from the results of [2]. Therefore, only the case \( a < 0 \) is considered below. Henceforth, by \( u(\tau) \) we mean only this special solution.

It is proved in [7] that \( u(\tau) \) has neither poles nor zeros on the real axis, except at the origin, where, by definition, \( u(0) = 0 \). It is straightforward to establish from Eq. (1) that \( u(\tau) \) is real for real \( \tau \), and \( u'(0) = -b/(2a) \). In this case, \( b > 0 \) and \( a < 0 \), so it is obvious that \( u(\tau) > 0 \) for \( \tau > 0 \) and \( u(\tau) < 0 \) for \( \tau < 0 \), since it is an odd function. Using the Taylor expansion for the function \( u(\tau) \) (see Eq. (23) of [7]), one finds that

\[
\lim_{\tau \to 0} \left( \frac{2a}{\tau} + \frac{b}{u} \right) = 0;
\]

therefore, the integral of the function \( 2a/\tau + b/u \) exists on the real segment \([0, \tau] \).

Since the function \( u \) is real, the functions \( u_\pm \) are complex conjugates, \( u_+ = \bar{u}_- \); moreover, Eq. (3) implies that \( u_+ \) does not have poles on the real axis. The function \( f(\tau)/\tau \) vanishes as \( \tau \to 0 \), since \( u(0) = u_+(0) = 0 \). Therefore, the integral of the function \( f(\tau)/\tau \) is properly defined on the real segment \([0, \tau] \).

Now, using Eq. (7) and Proposition 4.3.1 of [1] (with the corrections indicated above), one finds that

\[
\varphi(\tau) = \frac{3\pi}{2} + 3b^{1/3} \tau^{2/3} + 2a \ln(\tau^{2/3}) - \frac{\ln(2 + \sqrt{3})}{\pi} \ln \left( 1 - e^{2\pi a} \right) + 2a \ln 2 - a \ln(b^{1/3}) + \pi + i \ln \left( g_{11}^2 (1 - e^{2\pi a}) \right) + o(\tau^{-\delta}),
\]

where \( g_{11} \) is the monodromy parameter introduced in [1] (in this context, it might be viewed as the constant of integration), and \( \delta > 0 \). Equation (7) and Proposition (5.5) of [1] (with the additive correction term \( a \ln \tau \) give rise to the following result:

\[
\varphi(0) = \frac{3\pi}{2} - a \ln b + 2a \ln 2 + 2\text{Arg}(\Gamma(1 + ai)) + i \ln \left( g_{11}^2 (1 - e^{2\pi a}) \right),
\]

where \( \Gamma(\cdot) \) is the gamma function [9]. Subtracting Eq. (9) from Eq. (8), one arrives at

\[
\int_0^\tau \left( \frac{2a}{\tau} + \frac{b}{u(\tau)} \right) d\tau = 3b^{1/3} \tau^{2/3} + 2a \ln(b^{1/3} \tau^{2/3}) - \frac{\ln(2 + \sqrt{3})}{\pi} \ln \left( 1 - e^{2\pi a} \right) - \frac{\pi}{2} - 2\text{Arg}(\Gamma(1 + ai)) + o(\tau^{-\delta}).
\]

One recalls that the function \( \text{Arg}(\Gamma(1 + ai)) \) is defined as a continuous function of \( a \) such that \( \text{Arg}(\Gamma(1 + ai)) = \text{Arg}(\Gamma(1 + ai)) \) for \( a \in (-a_\pi, 0) \) where \( a_\pi = \pi/2 + 2.999389 \ldots \); when \( a \) decays from 0 to \(-a_\pi\), both arguments decay from 0 to \(-\pi\); for \( a = -a_\pi \), the function arg suffers a jump discontinuity of \( 2\pi \) (from \(-\pi\) to \(+\pi\)), and then continues to decay, while \( \text{Arg} \) continues to decay without this jump.
To calculate the second integral, one needs the following results:

\[
\lim_{\tau \to 0} \ln \left( \frac{u}{\tau} \right) = \ln b - \ln(-a) - \ln 2, \\
\ln \left( \frac{u}{\tau} \right)_{\tau \to +\infty} = -\frac{2}{3} \ln \tau + \frac{2}{3} \ln b - \ln 2 + O(\tau^{-1/3}).
\]

(11)

(12)

With the help of these estimates and Eq. (6), one deduces that

\[
\text{Re} \int_0^\tau \frac{f(\tau)}{\tau} d\tau = \frac{b}{S} \int_0^\tau \left( \frac{2a}{\tau} + \frac{b}{u(\tau)} \right) d\tau, \\
\text{Im} \int_0^\tau \frac{f(\tau)}{\tau} d\tau = \frac{b}{S} \left( \ln(b^{1/3} \tau^{2/3}) - \ln(-a) - O(\tau^{-1/3}) \right).
\]

(13)

(14)

In Eq. (14), a minus sign is indicated in the $O$-estimate in order to stress that it is exactly the same function as in Eq. (12). The results derived in [1] allow one to obtain the $O(\tau^{-1/3})$ terms in Eqs. (12) and (14) explicitly; in particular, let

\[
x = 3^{1/2} b^{1/3} \tau^{2/3} > 0 \quad \text{and} \quad q = \sqrt{\frac{-\ln(1 - e^{2\pi a})}{2\pi}} > 0;
\]

then

\[
O(\tau^{-1/3}) = -\frac{2q}{\sqrt{x}} \cos \left( 3x + q^2 \ln(3x) + \phi_0 + o(\tau^{-\delta}) \right),
\]

\[
\phi_0 = a \ln(2 + \sqrt{3}) + q^2 \ln(12) - \frac{\pi}{4} - \arg(\Gamma(iq^2)),
\]

(15)

where $\delta > 0$. This formula is an immediate consequence of the second equation in (6) and Eq. (179) of [7].

3. Numerical examples

In this section, several features of the results obtained in the previous section are illustrated.

![Fig. 1. Numerical plot of $u(\tau)$ for $a = -8$ and $b = 1/100$.](image)

It is known (see [1]) that $u(\tau)$ oscillates about the parabola in Fig. 1; however, this oscillatory structure is too fine to be observed for $a < -1$. 718
Fig. 2. Plot of $\int_0^\tau \left( \frac{2a}{\tilde{\tau}} + \frac{b}{u(\tilde{\tau})} \right) d\tilde{\tau}$ for $a = -8$ and $b = 1/100$. The upper line is the asymptotics, and the lower line is the numerical plot of the integral.

Fig. 3. Plot of $\text{Im} \int_0^\tau \frac{f(\tilde{\tau})}{\tilde{\tau}} d\tilde{\tau}$ for $a = -8$ and $b = 1/100$. The upper line is the numerical plot of the integral, and the lower line is the plot of its asymptotics (14).

Fig. 4. Numerical plot of $u(\tau)$ for $a = -1/8$ and $b = 1/100$. 
Fig. 5. Plot of $\int_0^{\tau} \left(\frac{2a}{\bar{\tau}} + b/u(\bar{\tau})\right) d\bar{\tau}$ for $a = -1/8$ and $b = 1/100$. The asymptotic and numerical values of the integral practically coincide for $\tau > 5$.

Fig. 6. Plot of $\text{Im} \int_0^{\tau} \left(\frac{f(\bar{\tau})}{\bar{\tau}}\right) d\bar{\tau}$ for $a = -1/8$ and $b = 1/100$. One sees oscillation of the numerical plot about the asymptotic line (14) without the correction term (15).

For the calculation of asymptotics via Eq. (10) in Fig. 2, it is important to note that $\text{Arg}(\Gamma(1 - 8i)) = \text{arg}(\Gamma(1 - 8i)) - 2\pi$.

According to Eq. (13), the plots in Figs. 2 and 5 for $\text{Re} \int_0^{\tau} \left(\frac{f(\bar{\tau})}{\bar{\tau}}\right) d\bar{\tau}$ coincide modulo the numeric factor $b/8$.

The correction term (15) is not observable in Fig. 3: this is the general situation for all values $a < -1$; it is obviously related to the analogous situation for $u(\tau)$.

For $-1 < a < 0$, oscillations of the solution (which are “hidden” for smaller values of $a$) are clearly seen.

Oscillations that are seen in Fig. 4 are not observable in Fig. 5 and, consequently, for $\text{Re} \int_0^{\tau} \left(\frac{f(\bar{\tau})}{\bar{\tau}}\right) d\bar{\tau}$.
Fig. 7. Plot of $\Im \int_0^\tau \frac{f(\tau)}{\tau} \, d\tau$ for $a = -1/8$ and $b = 1/100$. One notes that the numerical plot practically coincides with the asymptotic plot with the correction term (15).

Figures 6 and 7 illustrate, for $-1 < a < 0$, the importance of the correction term (15) for $\Im \int_0^\tau (f(\tau)/\tau) \, d\tau$.

Note that the value of the positive parameter $b$ is not important for observing the oscillations; for larger values of $b$, the oscillations become faster. The value $b = 1/100$ is chosen only for the purpose of obtaining clearer figures.

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