Integrability on generalized $q$-Toda equation and hierarchy

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In this paper, we construct a new integrable equation which is a generalization of $q$-Toda equation. Meanwhile its soliton solutions are constructed to show its integrable property. Further the Lax pairs of the generalized $q$-Toda equation and a whole integrable generalized $q$-Toda hierarchy are also constructed. To show the integrability, the Bi-Hamiltonian structure and tau symmetry of the generalized $q$-Toda hierarchy are given and this leads to the tau function.

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1. Introduction

The Toda lattice equation is a completely integrable system which has many important applications in mathematics and physics including the theory of Lie algebra representation, orthogonal polynomials and random matrix model [3, 19, 20, 23, 24]. Toda system has many kinds of reduction or extension, for example extended Toda hierarchy (ETH) [2], bigraded Toda hierarchy (BTH) [1]-[8] and so on. These generalized Toda hierarchies have important application in Gromov-Witten theory on $\mathbb{C}P^1$ and orbifold.

The $q$-calculus (also called quantum calculus) traces back to the early 20th century and attracted important works in the area of $q$-calculus [6, 7] and $q$-hypergeometric series. The $q$-deformation of classical nonlinear integrable system started in 1990’s by means of $q$-derivative $\partial_q$ instead of usual derivative with respect spatial variable in the classical system. Several $q$-deformed integrable systems have been presented, for example the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy is a subject of intensive study in the literatures [16]-[14]. The $q$-Toda equation was also studied in [17, 21] but not for a whole hierarchy. This paper will be devoted to the further studies on a generalized $q$-Toda equation (GQTE) and generalized $q$-Toda hierarchy (GQTH).

To show the complete integrability of nonlinear evolution, it is necessary to test whether the equation has Hirota bilinear equation, three-soliton solution, Lax pair, Bi-Hamiltonian structure

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and even tau symmetry. This paper will show the integrability on the Generalized $q$-Toda hierarchy from the above several directions.

2. $q$-difference operator and its generalization

As we all know, in common sense an integrable equation can always be rewritten in form of a Hirota bilinear equation using Hirota direct method. Therefore firstly we introduce some basic notation including Hirota derivatives as a preparation for introducing the Hirota bilinear equation of the generalized $q$-Toda equation.

Let $F$ be a space of differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$. The Hirota $D$-operator $D : F \times F \to F$ is defined as

\[ [D^m_x D^m_{x'}] \cdot g = \left[ (\partial_x \partial_{x'})^m (\partial_{x'} \partial_x)^m \right] f(x,t,...) g(x',t',...)|_{x=x'=t=...}. \tag{2.1} \]

Then one can find the following standard statement holds. Let $P(D)$ be an arbitrary polynomial in $D$ acting on two differentiable functions $f(x,t,...)$ and $g(x,t,...)$, then the following equations hold

\[ P(D) f \cdot g = P(-D) g \cdot f, \tag{2.2} \]

\[ P(D) f \cdot 1 = P(\partial) f, \quad P(D) 1 \cdot f = P(-\partial) f, \tag{2.3} \]

where $\partial$ is the usual differential operator with respect to spatial variable $x$. The virtue of exponential identity can appropriately be as following form in terms of the Hirota $D$-operator

\[ e^{\varepsilon D} f(x) g(x) = f(x+\varepsilon) g(x-\varepsilon). \tag{2.4} \]

If $\varepsilon$ is parameter and $f, g$ belong to continuously differentiable functions, like in [17], then define

\[ \sigma_\varepsilon(x) = e^{\varepsilon x \partial_x} x. \tag{2.5} \]

Then

\[ e^{\varepsilon x \partial_x} u(x) = u(e^{\varepsilon x \partial_x} u(x)) = u(\sigma_\varepsilon(x)), \quad \varepsilon > 0. \tag{2.6} \]

If $\sigma_\varepsilon(u(x)) = e^{\varepsilon x \partial_x} u(x) = u(x+\varepsilon)$, the system introduced later will lead to original Toda lattice. If $\sigma_\varepsilon(x) = e^{\varepsilon x \partial_x} x = e^{\varepsilon x} x$, it implies $e^{\varepsilon x \partial_x} u(x) = u(e^{\varepsilon x} x)$. Then the system will lead to $q$-Toda lattice in [17]. Considering that the vector field of the form $x(x)\partial_x = x^\theta x^\delta$ on $\mathbb{R}$, it will be the general generalized $q$-Toda lattice. In this paper, we only give the case $n = 2$, and we just name the leading system later the generalized $q$-Toda equation.

**Proposition 2.1.** The $q$-exponential identity acts on arbitrary continuous differentiable functions $f(x), g(x)$ as the rule

\[ e^{\varepsilon x^2 D_x} f(x) g(x) = \Lambda_\varepsilon f(x) \Lambda^{-1}_\varepsilon g(x), \quad x \in \mathbb{R} \tag{2.7} \]

where the forward and backward shift operators are separately represented by $\Lambda_\varepsilon$ and $\Lambda^{-1}_\varepsilon$, respective acting as

\[ \Lambda_\varepsilon f(x) = f\left(\frac{x}{1+\varepsilon}\right), \quad \Lambda^{-1}_\varepsilon g(x) = g\left(\frac{x}{1+\varepsilon}\right). \tag{2.8} \]
Proof. Making use of the change of variable $x^2D_x=-\frac{1}{x}$ is the idea to prove the identity, i.e.
\[
 e^{x^2D_x}f(x)g(x) = e^{x^2D_x}f(-\frac{1}{x})g(-\frac{1}{x}).
\] (2.9)

Using eq.(2.4) for the right hand side of eq.(2.9), we end up the proof with
\[
 e^{x^2D_x}f(x)g(x) = f(-\frac{1}{x+\varepsilon})g(-\frac{1}{x-\varepsilon}) = f(\frac{1}{x+\varepsilon})g(\frac{1}{x-\varepsilon}) = \Lambda_\varepsilon f(x)\Lambda_\varepsilon^{-1}g(x).
\]

To give the definition of the generalized $q$-Toda equation, we need the following central generalized difference operator.

Definition 2.1. The central $q$-difference operator $\Delta^2_x$ acts on an arbitrary function $f(x), x \in \mathbb{R}$, as
\[
 \Delta^2_x f(x) = f(\frac{x}{1-x\varepsilon}) + f(\frac{x}{1+x\varepsilon}) - 2f(x),
\] (2.10)

which is easily rewritten as $\Delta^2_x f(x) = (\Lambda_\varepsilon + \Lambda_\varepsilon^{-1} - 2)f(x)$.

In the next section, we will try to use the above defined generalize $q$-shift operator to define the generalized $q$-Toda equation.

3. The generalized $q$-Toda equation

The well-known Toda equation represents the motion of the one-dimensional particles by
\[
 \frac{d^2y_n}{dt^2} = e^{y_{n-1}-y_n} - e^{y_n-y_{n+1}}.
\] (3.1)

By introducing the force
\[
 U_n = e^{y_{n-1}-y_n} - 1,
\] (3.2)

the Toda equation eq.(3.1) turns out to be
\[
 \frac{d^2}{dt^2} \log(1+U_n) = U_{n+1} + U_{n-1} - 2U_n.
\] (3.3)

Similarly as Toda equation, we define the generalized $q$-Toda equation(GQTE) as follows
\[
 e^2\frac{d^2\phi(x)}{dt^2} = e^{\phi(\frac{x}{1+x\varepsilon})-\phi(x)} - e^{\phi(x)-\phi(\frac{x}{1-x\varepsilon})}.
\] (3.4)

By introducing the force
\[
 V = e^{\phi(\frac{x}{1+x\varepsilon})-\phi(x)} - 1,
\] (3.5)

then the GQTE becomes
\[
 e^2\frac{d^2}{dt^2} \log(1+V(x,t)) = \Delta^2_x V(x,t) = V(\frac{x}{1-x\varepsilon},t) + V(\frac{x}{1+x\varepsilon},t) - 2V(x,t).
\] (3.6)
It is necessary to introduce the dependent variable transformation as
\[ V(x,t) = \frac{d^2}{dt^2} \log f(x,t). \] (3.7)

Then the bilinear form for \( f(x,t) \) is evolved as
\[ V(x,t) = \frac{f_n f - f^2}{f^2} = \frac{f(\frac{x}{x-\varepsilon}, t) f(\frac{x}{x+\varepsilon}, t)}{f^2} - 1. \] (3.8)

Then the generalized \( q \)-Toda equation can be rewritten as a Hirota bilinear form in terms of Hirota D-operator as
\[ P(D) f(x,t) \cdot f(x,t) = \left[ D^2 - \left( e^{\varepsilon x D_x} + e^{-\varepsilon x D_x} \right) - 2 \right] f(x,t) \cdot f(x,t) = 0, \] (3.9)

by multiplying eq.(3.8) by \( 2f^2(x,t) \) and using the q-exponential identity eq.(2.7). Supposing function \( f \) has finite perturbation expansion around a formal perturbation parameter \( \varepsilon \) as
\[ f(x,t) = 1 + \varepsilon f^{(1)}(x,t) + \varepsilon^2 f^{(2)}(x,t) + \ldots \] (3.10)

Substituting eq.(3.10) into generalized \( q \)-Toda equation
\[ P(D) f(x,t) \cdot f(x,t) = 0, \] (3.11)

we have
\[ P(D) f(x,t) \cdot f(x,t) = P(D) [1 \cdot 1 + \varepsilon f^{(1)} \cdot f^{(1)} \cdot 1 + \varepsilon^2 (1 \cdot f^{(2)} + f^{(2)} \cdot 1 + f^{(1)} \cdot f^{(1)}) \]
\[ + \varepsilon^3 (1 \cdot f^{(3)} + f^{(3)} \cdot 1 + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)}) \]
\[ + \varepsilon^4 (1 \cdot f^{(4)} + f^{(4)} \cdot 1 + f^{(1)} \cdot f^{(3)} + f^{(3)} \cdot f^{(1)} + f^{(2)} \cdot f^{(2)}) + \ldots ] \] (3.12)

The coefficient of the first term \( \varepsilon^0 \) is trivial. For the coefficient of \( \varepsilon^1 \), we get
\[ P(D) 1 \cdot f^{(1)} + f^{(1)} \cdot 1 = 2P(\partial)f^{(1)} = 2[\partial^2 - \left( e^{\varepsilon x \partial_x} + e^{-\varepsilon x \partial_x} - 2 \right)] f^{(1)} = 0. \] (3.13)

Then the equation \( f^{(1)} \) has exponential type solution as
\[ f^{(1)}(x,t) = e^{\frac{-\alpha}{\beta} + \beta t + \eta}, \] (3.14)

where \( \alpha, \beta, \eta \) are arbitrary constants with the dispersion relation as
\[ \beta^2 = e^{\alpha \varepsilon} + e^{-\alpha \varepsilon} - 2. \] (3.15)

Comparing the coefficients of \( \varepsilon^2 \) in eq.(3.12) will yield
\[ P(D) 1 \cdot f^{(2)} + f^{(2)} \cdot 1 + f^{(1)} \cdot f^{(1)} = 2P(\partial)f^{(2)} + P(D)f^{(1)} \cdot f^{(1)} = 0, \] (3.16)

which implies exactly
\[ [D^2 - \left( e^{\varepsilon x D_x} + e^{-\varepsilon x D_x} \right) - 2] f^{(1)}(x,t) \cdot f^{(1)}(x,t) \]
\[ = -2[\partial^2 - \left( e^{\varepsilon x \partial_x} + e^{-\varepsilon x \partial_x} - 2 \right)] f^{(2)}(x,t). \] (3.17)

Since \( f^{(1)} \) given in eq.(3.14) satisfies the form of eq.(3.17) by considering eq.(3.15), it is logical to take all order terms as zero, i.e. \( f^{(j)} = 0, j \geq 2 \). Therefore without loss of generality, we let \( \varepsilon = 1 \).
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Then one-q-soliton is constructed by the virtue of eq.(3.14) and eq.(3.15) as

\[ V(x,t) = \beta^2 e^{-\frac{\alpha}{x} + \beta t + \eta} \left(1 + e^{-\frac{\alpha}{x} + \beta t + \eta}\right)^2. \]  

(3.18)

The solution of one-q-soliton \( V \) can be seen from Figure 1.

\[ \text{Fig. 1. One-q-soliton solution } V \text{ of generalized } q \text{-Toda equation with } \epsilon = 1.25, \alpha_1 = -5, \beta_1 = -1.1745. \]

We pick the starting solution of eq.(3.13) as the assumption of two-soliton solutions.

\[ f^{(1)} = 2 \cosh(-\frac{\alpha_1}{x} + \beta_1 t + \eta_1), \]  

(3.19)

where \( \alpha_i, \eta_i, i = 1, 2 \) are arbitrary constants with the related dispersion relation

\[ \beta_i^2 = e^{\alpha_i \epsilon} + e^{-\alpha_i \epsilon} - 2, i = 1, 2. \]  

(3.20)

Apparently the use of vector notation

\[ p_1 \pm p_2 = (\beta_1 \pm \beta_2, \alpha_1 \pm \alpha_2, \eta_1 \pm \eta_2), \]  

(3.21)

can lead to dispersion relation eq.(3.20) as \( P(p_i) = 0, i = 1, 2 \ldots \). Then we get

\[ -P(\partial) f^{(2)} = [(\beta_1 - \beta_2)^2 - (e^{(\alpha_1 - \alpha_2)\epsilon} + e^{(\alpha_2 - \alpha_1)\epsilon} - 2)]e^{-\frac{\alpha_1 + \alpha_2}{2} + (\beta_1 + \beta_2)\epsilon + \eta_1 + \eta_2}. \]  

(3.22)

Therefore, the form of \( f^{(2)} \) can be

\[ f^{(2)} = A(1,2)e^{-\frac{\alpha_1 + \alpha_2}{2} + (\beta_1 + \beta_2)\epsilon + \eta_1 + \eta_2}. \]  

(3.23)

Substituting such \( f^{(2)} \) into eq.(3.22) will help us determine the position of two-q-soliton as

\[ A(1,2) = \frac{P(p_1 - p_2)}{P(p_1 + p_2)}. \]  

(3.24)

Supposing \( f^{(3)} = 0 \), by the use of the dispersion relation eq.(3.20) the coefficient of \( \epsilon^3 \) vanishes trivially and so do the rest of \( \epsilon^j, j > 3 \). That means we have a good truncation up to \( \epsilon^3 \) which leads
to the two-q-soliton solution as
\[ f(x,t) = 1 + e^{-\alpha_1 x + \beta_1 t + \eta_1} + e^{-\alpha_2 x + \beta_2 t + \eta_2} + A(1,2)e^{-\frac{\alpha_1 + \alpha_2}{2} + (\beta_1 + \beta_2)t + \eta_1 + \eta_2}. \] (3.25)

Therefore, we illustrate the collision of two-q-solitons as Figure 2.

![Fig. 2. Two-q-soliton solution V of generalized q-Toda equation with \( \epsilon^0 = 1.25, \alpha_1 = -5, \alpha_2 = 6 \).](image)

To further derive three-soliton solution, we choose the starting solution of eq.(3.13) as the assumption as
\[ f^{(1)} = \sum_{i=1}^{3} e^{-\frac{\alpha_i}{2} x + \beta_i t + \eta_i}, \] (3.26)
where \( \alpha_i, \eta_i \) are arbitrary constants for \( i = 1, 2, 3 \). Similarly to the precious arguments, the coefficient of \( \epsilon^0 \) vanishes trivially. From the coefficient of \( \epsilon^1 \), we have the corresponding dispersion relation
\[ \beta_i^2 = e^{\alpha_i \epsilon} + e^{-\alpha_i \epsilon} - 2, i = 1, 2, 3. \] (3.27)

From the coefficient of \( \epsilon^2 \), we can obtain
\[-P(\partial) f^{(2)} = \sum_{i<j}^{(3)} [(\beta_i - \beta_j)^2 - (e^{(\alpha_i - \alpha_j) \epsilon} + e^{(\alpha_i - \alpha_j) \epsilon} - 2)] e^{-\frac{\alpha_i + \alpha_j}{2} + (\beta_i + \beta_j)t + \eta_i + \eta_j}. \] (3.28)

The equation eq.(3.28) implies the explicit form of \( f^{(2)} \)
\[ f^{(2)} = \sum_{i<j}^{(3)} A(i,j) e^{-\frac{\alpha_i + \alpha_j}{2} + (\beta_i + \beta_j)t + \eta_i + \eta_j}, \] (3.29)
with
\[ A(i,j) = \frac{P(p_i - p_j)}{P(p_i + p_j)} = \frac{(\beta_i - \beta_j)^2 - (e^{(\alpha_i - \alpha_j) \epsilon} + e^{(\alpha_i - \alpha_j) \epsilon} - 2)}{(\beta_i + \beta_j)^2 - (e^{(\alpha_i + \alpha_j) \epsilon} + e^{-(\alpha_i + \alpha_j) \epsilon} - 2)}. \] (3.30)
For the coefficient of \( \varepsilon^1 \), we have
\[
P(D) f^{(3)} + f^{(3)} \cdot f^{(1)} + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)} = 0.
\]
We can also rewrite them as
\[
-P(\partial) f^{(3)} = (A(1, 2) P(p_3 - p_1 - p_2) + A(1, 3) P(p_2 - p_1 - p_3) + A(2, 3) P(p_1 - p_2 - p_3)) \times \exp^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + (\beta_1 + \beta_2 + \beta_3) t + \eta_1 + \eta_2 + \eta_3}.
\]
Suppose that \( f^{(3)} \) is of the form
\[
f^{(3)} = A(1, 2, 3) \exp^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + (\beta_1 + \beta_2 + \beta_3) t + \eta_1 + \eta_2 + \eta_3},
\]
then one can find
\[
A(1, 2, 3) = -\frac{A(1, 2) P(p_3 - p_1 - p_2) + A(1, 3) P(p_2 - p_1 - p_3) + A(2, 3) P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}.
\]
Following the steps, one can find we can suppose the vanishing of \( f^{(4)} \) and it is a reasonable truncation to terms of \( \varepsilon^4 \), i.e. the form of the equation (3.12) becomes
\[
2P(D) f^{(1)} \cdot f^{(3)} + P(D) f^{(2)} \cdot f^{(2)} = 0,
\]
which means the following condition holds
\[
A(1, 2, 3) = A(1, 2) A(1, 3) A(2, 3).
\]
Then we can express the solution of three-q-soliton(see Figure 3) as
\[
f(x, t) = 1 + \sum_{i=1}^{3} \exp^{-\frac{\alpha_i}{2} + \beta_i t + \eta_i} + \sum_{i<j}^{3} A(i, j) \exp^{-\frac{\alpha_i + \alpha_j}{2} + (\beta_i + \beta_j) t + \eta_i + \eta_j} + A(1, 2) A(1, 3) A(2, 3) \exp^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + (\beta_1 + \beta_2 + \beta_3) t + \eta_1 + \eta_2 + \eta_3},
\]
The above three-soliton solutions show the great integrable possibility in a certain sense. To deeply prove the integrability, we will give the Lax pair of the generalized \( q \)-Toda equation and further generalize it to a whole integrable hierarchy in the next section.

4. The generalized \( q \)-Toda hierarchy

Now we will consider that the algebra of the shift operator \( \Lambda_x := e^{\epsilon L_x \frac{x}{x}} \). A Left multiplication by \( X \) is as \( X \Lambda_x \), \( (X \Lambda_x)(g)(x) := X(x) \circ g(\frac{x}{x}) \) with defining the product \( (X(x) \Lambda_x) \circ (Y(x) \Lambda_x) := X(x) Y(\frac{x}{x}) \Lambda_x \).

Now we introduce the following free operators \( W_0, \bar{W}_0 \)
\[
W_0 := \exp^{\sum_{j \neq \delta} A^j_\delta / \delta^j}, \tag{4.1}
\]
\[
\bar{W}_0 := \exp^{\sum_{j \neq \delta} A^\delta / \delta^j}, \tag{4.2}
\]
where \( t_j \in \mathbb{R} \) will play the role of continuous times.
We define the dressing operators $W, \bar{W}$ as follows
\[
W := S \circ W_0, \quad \bar{W} := \bar{S} \circ \bar{W}_0,
\]
where $S, \bar{S}$ have expansions as
\[
S = 1 + \omega_1(x) \Lambda_\varepsilon^{-1} + \omega_2(x) \Lambda_\varepsilon^{-2} + \cdots , \\
\bar{S} = \bar{\omega}_0(x) + \bar{\omega}_1(x) \Lambda_\varepsilon + \bar{\omega}_2(x) \Lambda_\varepsilon^2 + \cdots .
\]

The inverse operators $S^{-1}, \bar{S}^{-1}$ of operators $S, \bar{S}$ have expansions of the form
\[
S^{-1} = 1 + \omega'_1(x) \Lambda_\varepsilon^{-1} + \omega'_2(x) \Lambda_\varepsilon^{-2} + \cdots , \\
\bar{S}^{-1} = \bar{\omega}'_0(x) + \bar{\omega}'_1(x) \Lambda_\varepsilon + \bar{\omega}'_2(x) \Lambda_\varepsilon^2 + \cdots .
\]

The Lax operator $\mathcal{L}$ of the generalized $q$-deformed Toda hierarchy is defined by
\[
\mathcal{L} := W \circ \Lambda_\varepsilon \circ W^{-1} = \bar{W} \circ \Lambda_\varepsilon^{-1} \circ \bar{W}^{-1},
\]
and have the following expansions
\[
\mathcal{L} = \Lambda_\varepsilon + U(x) + V(x) \Lambda_\varepsilon^{-1}.
\]

In fact the Lax operators $\mathcal{L}$ are also be equivalently defined by
\[
\mathcal{L} := S \circ \Lambda_\varepsilon \circ S^{-1} = \bar{S} \circ \Lambda_\varepsilon^{-1} \circ \bar{S}^{-1}.
\]

### 4.1. Lax equations of the GQTH

In this section we will give the Lax equations of the GQTH. Let us firstly introduce some convenient notation such as the operators $B_j$ defined as $B_j := \frac{\partial^j}{\partial x^j}$. Now we give the definition of the generalized $q$-Toda hierarchy (GQTH).
**Definition 4.1.** The generalized $q$-Toda hierarchy is a hierarchy in which the dressing operators $S, \bar{S}$ satisfy following Sato equations

$$
\varepsilon \partial_t S = -(B_j)_- S, \quad \varepsilon \partial_t \bar{S} = (B_j)_+ \bar{S}.
$$

(4.9)

Then one can easily get the following proposition about $W, \bar{W}$.

**Proposition 4.1.** The dressing operators $W, \bar{W}$ are subject to following Sato equations

$$
\varepsilon \partial_t W = (B_j)_+ W, \quad \varepsilon \partial_t \bar{W} = -(B_j)_- \bar{W}.
$$

(4.10)

From the previous proposition one can derive the following Lax equations for the Lax operators.

**Proposition 4.2.** The Lax equations of the GQTH are as follows

$$
\varepsilon \partial_t \mathcal{L} = [(B_j)_+, \mathcal{L}].
$$

(4.11)

To see this kind of hierarchy more clearly, the generalized $q$-Toda equations as the $t_1$ flow equations will be given in the next subsection.

### 4.2. The generalized $q$-Toda equations

As a consequence Sato equations, after taking into account that $S$ and $\bar{S}$, the $t_1$ flow of $L$ in the form of $L = \Lambda_x + U + V \Lambda_x^{-1}$ is as

$$
\varepsilon \partial_t \mathcal{L} = [\Lambda_x + U, V \Lambda_x^{-1}],
$$

(4.12)

which lead to generalized $q$-Toda equation

$$
\varepsilon \partial_t U = V \left( \frac{x}{1 - \varepsilon x} \right) - V(x),
$$

(4.13)

$$
\varepsilon \partial_t V = U(x) V(x) - V(x) U \left( \frac{x}{1 + \varepsilon x} \right). \tag{4.14}
$$

From Sato equation we deduce the following set of nonlinear partial differential-difference equations

$$
\begin{align*}
\omega_1 (x) - \omega_1 \left( \frac{x}{1 + \varepsilon x} \right) &= \varepsilon \partial_t \Gamma (e^{\phi(x)}) e^{-\phi(x)}, \\
\varepsilon \partial_t \omega_1 (x) &= -e^{\phi(x)} e^{-\phi \left( \frac{x}{1-\varepsilon x} \right)}.
\end{align*}
$$

(4.15)

Observe that if we cross the two first equations, then we get the generalized $q$-Toda equation (3.5). To give a linear description of the GQTH, we introduce wave functions $\psi, \bar{\psi}$ defined by

$$
\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \bar{\chi},
$$

(4.16)

where

$$
\chi(z) := z^{-\frac{1}{\varepsilon}}, \quad \bar{\chi}(z) := z^{\frac{1}{\varepsilon}},
$$

(4.17)

and the “$\cdot$” means the action of an operator on a function. Note that $\Lambda_x \cdot \chi = z \chi$ and the following asymptotic expansions can be defined

$$
\begin{align*}
\psi &= (1 + \omega_1 (x) z^{-1} + \cdots) \psi_0 (z), \quad \psi_0 := z^{-\frac{1}{\varepsilon}} e^{\Sigma_{j=-1}^{0} \frac{j}{\varepsilon}} , \\
\bar{\psi} &= (\bar{\omega}_1 (x) + \bar{\omega}_1 (x) z + \cdots) \bar{\psi}_0 (z), \quad \bar{\psi}_0 := z^{\frac{1}{\varepsilon}} e^{-\Sigma_{j=0}^{0} \frac{j}{\varepsilon}}.
\end{align*}
$$

(4.18)
We can further get linear equations of the GQTH in the following proposition.

**Proposition 4.3.** The wave functions $\psi, \bar{\psi}$ are subject to following Sato equations

$$\mathcal{L} \cdot \psi = z \psi, \quad \mathcal{L} \cdot \bar{\psi} = z \bar{\psi},$$

(4.19)

$$\varepsilon \partial_j \psi = (B_j)_+ \cdot \psi, \quad \varepsilon \partial_j \bar{\psi} = -(B_j)_- \cdot \bar{\psi}.$$  

(4.20)

5. **Bi-Hamiltonian structure and tau symmetry**

To describe the integrability of the GQTH, we will construct the Bi-Hamiltonian structure and tau symmetry of the GQTH in this section. In this section, we will consider the GQTH on Lax operator

$$\mathcal{L} = \Lambda_\varepsilon + u + e^\varepsilon \Lambda_\varepsilon^{-1}.$$  

(5.1)

Then for $\bar{f} = \int f dx, \bar{g} = \int g dx$, we can define the Hamiltonian bracket as

$$\{ \bar{f}, \bar{g} \} = \int \sum_{\nu, w'} \frac{\delta f}{\delta w} \{ w, w' \} \frac{\delta g}{\delta w'} dx, \quad w, w' = u \text{ or } v.$$  

(5.2)

The bi-Hamiltonian structure for the GQTH can be given by the following two compatible Poisson brackets similar as [2, 8]

$$\{ \nu(x), \nu(y) \}_1 = \{ u(x), u(y) \}_1 = 0,$$

$$\{ u(x), \nu(y) \}_1 = \frac{1}{\varepsilon} \left[ e^{\varepsilon^2 \partial_y} - 1 \right] \delta(x - y),$$

(5.3)

$$\{ u(x), u(y) \}_2 = \frac{1}{\varepsilon} \left[ e^{\varepsilon^2 \partial_x} e^{\nu(x)} - e^{\nu(x)} e^{-\varepsilon^2 \partial_x} \right] \delta(x - y),$$

$$\{ u(x), \nu(y) \}_2 = \frac{1}{\varepsilon} u(x) \left[ e^{\varepsilon^2 \partial_y} - 1 \right] \delta(x - y),$$

$$\{ \nu(x), \nu(y) \}_2 = \frac{1}{\varepsilon} \left[ e^{\varepsilon^2 \partial_x} - e^{-\varepsilon^2 \partial_x} \right] \delta(x - y).$$

(5.4)

For any difference operator $A = \sum_k A_k \Lambda_\varepsilon^k$, define residue $\text{Res} A = A_0$. In the following theorem, we will prove the above Poisson structure can be as the the Bi-Hamiltonian structures of the GQTH.

**Theorem 5.1.** The flows of the GQTH are Hamiltonian systems of the form

$$\frac{\partial u}{\partial \lambda_j} = \{ u, H_j \}_1, \quad j \geq 0.$$  

(5.5)

They satisfy the following bi-Hamiltonian recursion relation

$$\{ \cdot, H_{n-1} \}_2 = n \{ \cdot, H_n \}_1.$$  

Here the Hamiltonians have the form

$$H_j = \int h_j(u, \nu; u_1, \nu_1; \ldots; \varepsilon) dx, \quad j \geq 1,$$  

(5.6)

with

$$h_j = \frac{1}{j!} \text{Res} \mathcal{L}^j.$$  

(5.7)
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**Proof.** The proof is similar as the proof in [2, 8]. Here we will prove that the flows \( \frac{\partial}{\partial t_n} \) are also Hamiltonian systems with respect to the first Poisson bracket.

Suppose
\[
B_n = \sum_k a_{nk} \Lambda_k^k,
\]
and from
\[
\frac{\partial L}{\partial t_n} = \{(B_n)_+, L\} = \{-(B_n)_-, L\},
\]
we can derive equation
\[
\varepsilon \frac{\partial u}{\partial t_n} = a_{n,1}(\frac{x}{1 - \varepsilon x}) - a_{n,1}(x),
\]
\[
\varepsilon \frac{\partial v}{\partial t_n} = a_{n,0}(\frac{x}{1 + \varepsilon x}) e^{\psi(x)} - a_{n,0}(x) e^{\psi(\frac{x}{1 + \varepsilon x})}.
\]

By the following calculation
\[
\frac{dh_n}{n!} = \frac{1}{n!} \text{Res} [L^n dL] = \frac{1}{n!} \text{Res} [L^n dL] = \text{Res} \left[ a_{n,0}(x) du + a_{n,1}(\frac{x}{1 + \varepsilon x}) e^{\psi(x)} dv \right],
\]
we can derive equation
\[
\delta H_n = a_{n,1}(x), \quad \delta H_n = a_{n,0}(\frac{x}{1 + \varepsilon x}) e^{\psi(x)}.
\]

This agree with Lax equation
\[
\frac{\partial u}{\partial t_n} = \{u, H_n\} = \frac{1}{\varepsilon} \left[ e^{\psi(x)} \frac{\partial}{\partial \psi} - 1 \right] \frac{\delta H_n}{\delta u} = \frac{1}{\varepsilon} \left( a_{n,1}(\frac{x}{1 - \varepsilon x}) - a_{n,1}(x) \right),
\]
\[
\frac{\partial v}{\partial t_n} = \{v, H_n\} = \frac{1}{\varepsilon} \left[ 1 - e^{\psi(x)} a_{n,0}(\frac{x}{1 + \varepsilon x}) e^{\psi(x)} - a_{n,0}(x) e^{\psi(\frac{x}{1 + \varepsilon x})} \right].
\]

From the above identities we see that the flows \( \frac{\partial}{\partial t_n} \) are Hamiltonian systems with the first Hamiltonian structure. The recursion relation follows from the following trivial identities
\[
\frac{n}{n!} L^n = L \frac{1}{(n-1)!} L^{n-1} = \frac{1}{(n-1)!} L^{n-1} L.
\]

Then we get,
\[
na_n(x) = a_{n-1,0}(\frac{x}{1 - \varepsilon x}) + ua_{n-1,1}(x) + e^{\psi(x)} a_{n-1,2}(\frac{x}{1 + \varepsilon x})
\]
\[
= a_{n-1,0}(x) + u(\frac{x}{1 - \varepsilon x}) a_{n-1,1}(x) + e^{\psi(\frac{x}{1 + \varepsilon x})} a_{n-1,2}(x).
\]

This further leads to
\[
\{u, H_{n-1}\} = \left\{ \Lambda_\varepsilon e^{\varepsilon(x)} - e^{\varepsilon(x)} \Lambda_\varepsilon^{-1} \right\} a_{n-1,0}(x) + u(x) [\Lambda_\varepsilon - 1] a_{n-1,1} \left( \frac{x}{1 + \varepsilon x} \right) e^{\varepsilon(x)}
\]

This is exactly the recursion relation on flows for \(u\). The similar recursion flow on \(v\) can be similarly derived. Theorem is proved till now.

Similarly as [2], the tau symmetry of the GQTH can be proved in the following theorem.

**Theorem 5.2.** The GQTH has the following tau-symmetry property:

\[
\frac{\partial h_m}{\partial t_n} = \frac{\partial h_n}{\partial t_m}, \quad m, n \geq 1.
\]  \hspace{1cm} (5.16)

**Proof.** Let us prove the theorem in a direct way

\[
\frac{\partial h_m}{\partial t_n} = \frac{1}{m! n!} \text{Res}[-(\mathcal{L}^m)_-, \mathcal{L}^m]
\]

\[
= \frac{1}{m! n!} \text{Res}[(\mathcal{L}^m)_+, (\mathcal{L}^n)_-]
\]

\[
= \frac{1}{m! n!} \text{Res}[(\mathcal{L}^m)_+, \mathcal{L}^m] = \frac{\partial h_n}{\partial t_m}.
\]  \hspace{1cm} (5.17)

Theorem is proved.

This property justifies the definition of the tau function for the GQTH as in the following proposition.

**Proposition 5.1.** The tau function of the GQTH can also be defined by the following expressions in terms of the densities of the Hamiltonians:

\[
h_n = \varepsilon (\Lambda_\varepsilon - 1) \frac{\partial \log \tau}{\partial t_n}, \quad n \geq 0.
\]  \hspace{1cm} (5.18)

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