The average search probabilities of discrete-time quantum walks

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Abstract
We study the average probability that a discrete-time quantum walk finds a marked vertex on a graph. We first show that, for a regular graph, the spectrum of the transition matrix is determined by the weighted adjacency matrix of an augmented graph. We then consider the average search probability on a distance regular graph, and find a formula in terms of the adjacency matrix of its vertex-deleted subgraph. In particular, for any family of
- Complete graphs, or
- Strongly regular graphs, or
- Distance regular graphs of a fixed parameter $d$, varying valency $k$ and varying size $n$, such that $k^{d-1}/n$ vanishes as $k$ increases,
the average search probability approaches $1/4$ as the valency goes to infinity. We also present a more relaxed criterion, in terms of the intersection array, for this limit to be approached by distance regular graphs.

Keywords Discrete-time quantum walks · Quantum search · Distance regular graphs

1 Introduction

Quantum walks can be turned into algorithms to search marked vertices in graphs. While this idea was formalized by Shenvi et al. [1], its first application dates back to 1996, when Grover [2] showed that finding a marked vertex in a looped $K_n$ takes $O(\sqrt{n})$ steps of a quantum walk. Since then, quantum walk search has been studied on various graphs, including hypercubes [1], Cartesian powers of cycles [3], strongly regular graphs [4], certain Johnson graphs [5–8], and more generally, regular locally arc-transitive graphs [9]. For most of these graphs, quantum walks arrive at the marked vertices faster than the classical random walks.
In this paper, we consider a related problem: given a graph and a marked vertex, what is the average probability, over any period of time, that a discrete-time quantum walk finds the marked vertex? This probability converges as the time period tends to infinity, and we will refer to its limit as the average search probability. Intuitively, the average search probability should depend on the parameters of the graph. However, as we will see later, for many parametric families of distance regular graphs with a fixed diameter, the average probability approaches $1/4$ as the valency increases, regardless of the defining parameters. This surprising phenomenon echoes with the fast search of quantum walks on highly regular graphs.

We will start with a general observation, on the spectrum of a quantum walk that incorporates an oracle (Theorem 3.2). We then compute the average search probability for a distance regular graph, which, by results from equitable partitions, is a function in the eigenprojections of the vertex-deleted subgraph (Theorem 6.2). Finally, we derive conditions for the average search probability to approach $1/4$ as the valency increases (Theorem 8.5), and show that this happens for any family of

- Complete graphs, or
- Strongly regular graphs, or
- Distance regular graphs of a fixed parameter $d$, varying valency $k$ and varying size $n$, such that $kd^{-1}/n$ vanishes as $k$ tends to infinity.

## 2 Quantum walks with oracles

Let $X$ be a $k$-regular graph on $n$ vertices, and let $a$ be a marked vertex of $X$. Our goal is to find $a$ using a discrete-time quantum walk on $X$.

Throughout, we will view each edge $\{u, v\}$ of $X$ as a pair of arcs $(u, v)$ and $(v, u)$. The states associated with $X$ are complex-valued functions on the arcs; hence, they form a vector space isomorphic to $\mathbb{C}^n \otimes \mathbb{C}^k$. Let $1_\ell$ be the all-ones vector in $\mathbb{C}^\ell$. The initial state of our quantum walks is the unit vector

$$x_0 = \frac{1}{\sqrt{nk}} 1_n \otimes 1_k.$$  

To find the marked vertex, we apply a unitary matrix $U$, called the transition matrix, iteratively to the initial state. More specifically, let $I_\ell$ and $J_\ell$ be the $\ell \times \ell$ identity matrix and $\ell \times \ell$ all-ones matrix, respectively, and let $E_{aa}$ be the square matrix with 1 in the $aa$-entry, and 0 elsewhere. Then, $U$ is the product of the following unitary operators on $\mathbb{C}^n \otimes \mathbb{C}^k$: the arc-reversal matrix $R$, which represents the permutation that swaps $(u, v)$ with $(v, u)$; the coin matrix

$$C = I_n \otimes \left(\frac{2}{k}J_k - I_k\right),$$

and the oracle

$$O_a = (I_n - 2E_{aa}) \otimes I_k.$$
For algorithmic meanings of these operators, see Shenvi et al. [1].

Now set

\[ U = RCO_a. \]

At time \( t \), our quantum walk will be in state \( U^t x_0 \) if it started with state \( x_0 \). For each arc \((u, v)\), let \( e_{(u, v)} \) be its characteristic vector in \( \mathbb{C}^{nk} \). As \( U^t x_0 \) is a unit vector, the entrywise product

\[ (U^t x_0) \circ (U^t x_0) \]

represents a probability distribution, and its \((a, v)\)-th entry,

\[ e_{(a, v)}^T \left( (U^t x_0) \circ (U^t x_0) \right), \]

gives the probability that the quantum walk lands on the particular arc \((a, v)\). We will call the following sum the search probability at time \( t \):

\[ \sum_{v \sim a} e_{(a, v)}^T \left( (U^t x_0) \circ (U^t x_0) \right). \]

An important question in quantum walk search is to determine, for certain family of graphs, the optimal time \( t \) at which the search probability is sufficiently large.

In this paper, we study a related concept called the average search probability. While

\[ (U^t x_0) \circ (U^t x_0) \]

does not converge, its time average converges, and the limit can be expressed using the spectral idempotents of \( U \). This was observed in Aharonov et al. [10], and we will state the result following the notation in Godsil and Zhan [11].

**Lemma 2.1** Let \( U \) be a unitary transition matrix and let \( x_0 \) be a unit vector. The time-averaged probability distribution

\[ \frac{1}{T} \sum_{t=0}^{T-1} (U^t x_0) \circ (U^t x_0) \]

converges as \( T \) goes to infinity. Moreover, if \( F_r \) is the orthogonal projection onto the \( r \)-th eigenspace of \( U \), then the limit is

\[ \sum_r (F_r x_0) \circ (F_r x_0). \]
For the rest of the paper, we will let $U$ and $x_0$ be

$$U = \text{RCO}_a, \quad x_0 = \frac{1}{\sqrt{n k}} \mathbf{1}_n \otimes \mathbf{1}_k,$$

and let the spectral decomposition of $U$ be

$$U = \sum_r e^{i \theta_r} F_r.$$

The average search probability of our quantum walk is

$$\sum_r \sum_{v \sim a} e^{T(a, v)} ((F_r x_0) \circ (F_r x_0)).$$

### 3 Spectral decomposition

In this section, we prove a spectral correspondence between $U$ and the weighted adjacency matrix of an augmented graph $\tilde{X}$ of $X$.

We first note that, upon reordering the rows and columns, $\text{CO}_a$ is a block-diagonal matrix, with the top left block indexed by the outgoing arcs of the marked vertex $a$:

$$\text{CO}_a = \begin{pmatrix}
I_k - \frac{2}{k} J_k \\
\frac{2}{k} J_k - I_k \\
\cdots \\
\frac{2}{k} J_k - I_k
\end{pmatrix}.$$

Since this is a reflection, we can write it as twice a projection minus the identity. To be more precise, let $K$ be the $k \times (k - 1)$ Vandermonde matrix:

$$K = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
e^{2 \pi i/k} & e^{4 \pi i/k} & e^{6 \pi i/k} & \cdots & e^{2(k-1)\pi i/k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e^{2(k-1)\pi i/k} & e^{4(k-1)\pi i/k} & e^{6(k-1)\pi i/k} & \cdots & e^{2(k-1)^2 \pi i/k}
\end{pmatrix}.$$  \hspace{1cm} (1)

Then, $KK^*/k$ is the projection onto the orthogonal complement of span$\{\mathbf{1}_k\}$. Thus, the first block of $\text{CO}_a$ can be rewritten as

$$I_k - \frac{2}{k} J_k = \frac{2}{k} KK^* - I_k.$$
Now let \( N \) be the block diagonal matrix

\[
N = \begin{pmatrix}
K & 1_k & & \\
& 1_k & & \\
& & \ddots & \\
& & & 1_k
\end{pmatrix}
\]

with exactly \( n \) blocks in the diagonal. It follows from our discussion that

\[
CO_a = \frac{2}{k} NN^* - I_{nk},
\]

that is, \( CO_a \) is a reflection about the column space of \( N \).

On the other hand, the arc-reversal matrix \( R \) is also a reflection. Thus, we may apply the following lemma to find the eigenspace, with

\[
P = \frac{1}{2}(R + I_{nk}), \quad L = \frac{1}{\sqrt{k}}N,
\]

to determine the spectrum of \( U \).

**Lemma 3.1** [12, Ch 2] Let \( P \) and \( Q \) be two projections, and write \( Q = LL^* \) for some matrix \( L \) with orthonormal columns. Let

\[
U = (2P - I)(2Q - I).
\]

Then, the eigenspaces of \( U \) are given as follows.

(i) The 1-eigenspace of \( U \) is the direct sum

\[
(\text{col}(P) \cap \text{col}(Q)) \oplus (\text{ker}(P) \cap \text{ker}(Q)).
\]

(ii) The \((-1)\)-eigenspace of \( U \) is the direct sum

\[
(\text{col}(P) \cap \text{ker}(Q)) \oplus (\text{ker}(P) \cap \text{col}(Q)).
\]

(iii) The remaining eigenspaces of \( U \) are completely determined by the eigenspaces of \( L^*(2P - I)L \). To be more specific, let \( \lambda \) be an eigenvalue of \( L^*(2P - I)L \) that lies strictly between \(-1\) and \(1\), and write \( \lambda = \cos(\theta) \) for some \( \theta \in \mathbb{R} \). The map

\[
z \mapsto ((\cos(\theta) + 1)I - (e^{i\theta} + 1)P)Lz
\]

is an isomorphism from the \( \lambda \)-eigenspace of \( L^*(2P - I)L \) to the \( e^{i\theta} \)-eigenspace of \( U \), and the map

\[
z \mapsto ((\cos(\theta) + 1)I - (e^{-i\theta} + 1)P)Lz
\]
is an isomorphism from the $\lambda$-eigenspace of $L^*(2P - I)L$ to the $e^{-i\theta}$-eigenspace of $U$.

The above lemma shows that the eigenspaces of $U$ are largely determined by those of $L^*(2P - I)L$. For our quantum walk,

$$P = \frac{1}{2}(R + I_{nk}), \quad L = \frac{1}{\sqrt{k}}N,$$

and so

$$L^*(2P - I_{nk})L = \frac{1}{k} \begin{pmatrix} 0 & K^* & 0 \\ K & A(X\setminus a) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the rows of $K$ are indexed by the neighbors of $a$. The right hand side, up to a scalar, is the Hermitian adjacency matrix of a weighted graph, obtained from $X$ by cloning the marked vertex $a$ and assigning appropriate $k$-th roots of unity to each arc.

We can say more about the eigenspaces of $U$.

**Theorem 3.2** Let $X$ be a $k$-regular graph on $n$ vertices. Let $a$ be the marked vertex, and $R$, $C$ and $O_a$ the corresponding arc-reversal matrix, coin matrix and oracle. Let $K$ and $N$ be defined as in (1) and (2). Then, the eigenspaces of the transition matrix, $U = RCO_a$, satisfy the following.

(i) There is a one-to-one correspondence between each conjugate pair of non-real eigenvalues $e^{\pm i\theta}$ of $U$, and each eigenvalue $\lambda$ of

$$\tilde{A} = \begin{pmatrix} 0 & K^* & 0 \\ K & A(X\setminus a) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3)$$

that lies strictly between $-k$ and $k$, and they are related by

$$\lambda = k \cos(\theta).$$

Moreover, if $F_{\theta}$ is the projection onto the $e^{i\theta}$-eigenspace of $U$, and $\tilde{E}_\lambda$ is the projection onto the $\lambda$-eigenspace of $\tilde{A}$, then

$$F_{\theta} = \frac{1}{2k \sin^2(\theta)}(N - e^{i\theta}RN)\tilde{E}_\lambda(N - e^{i\theta}RN)^*.$$

(ii) If $X$ is 2-connected, then the 1-eigenspace of $U$ is orthogonal to span$\{1_{nk}\}$.

**Proof** We first prove (i). The eigenvalue correspondence follows from Lemma 3.1. To compute $F_{\theta}$, it suffices to show that when

$$P = \frac{1}{2}(R + I_{nk}), \quad L = \frac{1}{\sqrt{k}}N,$$
the matrix

\[ ((\cos(\theta) + 1)I_{nk} - (e^{i\theta} + 1)P)L \]

is a scalar multiple of

\[ N - e^{i\theta}RN, \]

and that for any \( \lambda \)-eigenvector \( z \) of \( \tilde{A} \),

\[ |(N - e^{i\theta}RN)z|^2 = 2k \sin^2(\theta)|z|^2. \]

Indeed,

\[
\begin{align*}
((\cos(\theta) + 1)I_{nk} - (e^{i\theta} + 1)P)L &= \frac{1}{2\sqrt{k}} \left( (e^{-i\theta} + 1)I_{nk} - (e^{i\theta} + 1)R \right) N \\
&= \frac{e^{-i\theta} + 1}{2\sqrt{k}} (I_{nk} - e^{i\theta}R)N,
\end{align*}
\]

and

\[
\begin{align*}
|(N - e^{i\theta}RN)z|^2 &= z^*N^*(I_{nk} - e^{-i\theta}R)(I_{nk} - e^{i\theta}R)NZ \\
&= 2z^*(N^*N - \cos(\theta)N^*RN)z \\
&= 2z^*(kI_{n+k-2} - \cos(\theta)\tilde{A})z \\
&= 2(k - k \cos^2(\theta))|z|^2 \\
&= 2k \sin^2(\theta).
\end{align*}
\]

We now prove (ii). By Lemma 3.1, the 1-eigenspace of \( U \) is

\[ (\text{col}(R + I_{nk}) \cap \text{col}(NN^*)) \oplus (\ker(R + I_{nk}) \cap \ker(NN^*)). \]

We will show that the second space in the direct sum is orthogonal to \( \text{span}\{I_{nk}\} \), while the first space is trivial.

Since \( R \) reverses each arc, the column space of \( R + I_{nk} \) consists of vectors that are constant on each pair of opposite arcs \( (u, v) \) and \( (v, u) \); in particular, \( I_{nk} \) lies in this space. Thus, \( \ker(R + I_{nk}) \) is orthogonal to \( \text{span}\{I_{nk}\} \).

On the other hand, any vector \( x \) lying in the column space of \( NN^* \) sums to zero over the outgoing arcs of \( a \), and is constant on the outgoing arcs of any unmarked vertex; that is,

\[ \sum_{v \sim a} x_{(a,v)} = 0, \quad (4) \]

and for each \( u \neq a \),
\[ x_{(u,v)} = x_{(u,w)}, \quad \forall v \sim u, w \sim u. \] (5)

If, in addition, \( x \) lives in the column space of \( R + I_{nk} \), then for each arc \((u,v)\),

\[ x_{(u,v)} = x_{(v,u)}. \]

Now take any two neighbors \( b \) and \( c \) of \( a \). Since \( X \) is 2-connected, there is a cycle containing edges \{\( a, b \)\} and \{\( a, c \)\}, say

\[ a, b, v_1, v_2, \ldots, v_\ell, c, a. \]

Then,

\[ x_{(a,b)} = x_{(b,a)} = x_{(b,v_1)} = x_{(v_1,b)} = \cdots = x_{(c,a)} = x_{(a,c)}. \]

As our choice of \( b \) and \( c \) is arbitrary, this means that \( x \) is constant on all outgoing arcs of \( a \), which, together with (4) and (5), implies that \( x \) is the zero vector. \( \square \)

4 Equitable partitions

Given a graph \( X \), a partition \( \pi \) of its vertex set

\[ \pi = \{C_0, C_1, \ldots, C_d\} \]

is equitable if for any \( i \) and \( j \), there is a constant \( c_{ij} \) such that every vertex in \( C_i \) has \( c_{ij} \) neighbors in \( C_j \). Equivalently, if \( P \) denotes the characteristic matrix of \( \pi \), then \( \pi \) is equitable if and only if there is some matrix \( B \), called the quotient matrix, such that \( A(X)P = PB \).

The following are standard results on equitable partitions; for more background, see Godsil [13, Ch 5].

Lemma 4.1 [13, Ch 5] Let \( \pi \) be an equitable partition of \( X \). Let \( P \) be the characteristic matrix, and \( B \) the quotient matrix.

(i) If \( Bx = \lambda x \), then \( A(X)Px = \lambda Px \).

(ii) If \( A(X)y = \lambda y \), then \( y^T PB = \lambda y^T P \).

In particular, the eigenvectors of \( A(X) \) either sum to zero over each cell of \( \pi \), or are constant on each cell of \( \pi \).

Clearly, every equitable partition \( \pi \) for \( X \) gives rise to an equitable partition \( \pi \setminus \{C_i\} \) for the subgraph \( X \setminus C_i \), and the quotient matrix can be obtained from \( B \) by deleting the row and the column indexed by \( C_i \). This leads to the following observation.

Lemma 4.2 Let \( \pi \) be an equitable partition of \( X \), with \( C_i \) as one of its classes. For any integer \( m \geq 0 \), the vector \( A(X\setminus C_i)^m 1 \) is constant on the cells of \( \pi \setminus \{C_i\} \).
Proof Let $P$ be the characteristic matrix of $\pi$. Since $\pi$ is equitable, there is some matrix $B$ such that

$$A(X)P = PB.$$ 

Now let $Q$ be the matrix obtained from $P$ by deleting the column indexed by $C_i$, and $F$ the matrix obtained from $B$ by deleting the row and the column indexed by $C_i$. Then,

$$A(X \setminus C_i)Q = QF.$$ 

Thus for any positive integer $m$,

$$A(X \setminus C_i)^m Q = QF^m,$$

and so

$$A(X \setminus C_i)^m 1 = A(X \setminus C_i)^m Q 1 = QF^m 1,$$

which lies in the column space of $Q$. \hfill \Box

The next result is a consequence of Lemma 4.2.

Corollary 4.3 Let $X$ be a $k$-regular graph on $n$ vertices. Suppose $X$ has an equitable partition, where the singleton \{a\} and the neighborhood $N(a)$ are two of its classes. Let $K$ and $\tilde{A}$ be defined as in (1) and (3). For any integer $m \geq 0$,

$$\tilde{A}^m \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \begin{pmatrix} 0_{k-1} \\ A(X \setminus a)^m 1_{n-1} \end{pmatrix}.$$ 

Proof Clearly,

$$\tilde{A}^0 \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \begin{pmatrix} 0_{k-1} \\ A(X \setminus a)^0 1_{n-1} \end{pmatrix}.$$ 

Suppose for some integer $m \geq 1$,

$$\tilde{A}^{m-1} \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \begin{pmatrix} 0_{k-1} \\ A(X \setminus a)^{m-1} 1_{n-1} \end{pmatrix}.$$ 

Then,

$$\tilde{A}^m \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \tilde{A} \begin{pmatrix} 0_{k-1} \\ A(X \setminus a)^{m-1} 1_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & K^* & 0 \\ K & A(X \setminus a) & 0 \\ 0 & 0 & 0_{k-1} \end{pmatrix} \begin{pmatrix} 0_{k-1} \\ A(X \setminus a)^{m-1} 1_{n-1} \end{pmatrix}.$$
\[ (K^* 0) A(X\backslash a)^{m-1} 1_{n-1} \]

By Lemma 4.2, \( A(X\backslash a)^{m-1} 1_{n-1} \) is constant on \( N(a) \), and since \( K^* 1_k = 0 \), the top block in the last vector vanishes.

With \( \tilde{A} \) and \( A(X\backslash a) \) defined above, we now show a relation between their eigenspaces. To start, we cite a well-known result in linear algebra.

**Lemma 4.4** Let \( M \) be a normal matrix, with distinct eigenvalues \( \mu_r \) and eigenprojections \( E_r \). Let

\[ p_r(x) = \prod_{s \neq r} (x - \mu_s). \]

Then,

\[ p_r(M) = p_r(\mu_r) E_r. \]

In particular, \( E_r \) is a polynomial in \( M \).

An eigenvalue of a graph is called a main eigenvalue if its eigenspace is not orthogonal to \( \text{span}\{1\} \).

**Corollary 4.5** Let \( X \) be a \( k \)-regular graph on \( n \) vertices. Suppose \( X \) has an equitable partition, where the singleton \( \{a\} \) and the neighborhood \( N(a) \) are two of its classes. Let \( K \) and \( \tilde{A} \) be defined as in (1) and (3). Then, every main eigenvalue of \( X\backslash a \) is an eigenvalue of \( \tilde{A} \). Moreover, if \( \lambda \) is an eigenvalue of \( \tilde{A} \) with eigenprojection \( \tilde{E}_\lambda \), then

\[ \tilde{E}_\lambda \left( \begin{array}{c} 0_{k-1} \\ 1_{n-1} \end{array} \right) = \left( \begin{array}{c} 0_{k-1} \\ E_\lambda 1_{n-1} \end{array} \right) \]

if \( \lambda \) is also an eigenvalue of \( X\backslash a \) with eigenprojection \( E_\lambda \), and

\[ \tilde{E}_\lambda \left( \begin{array}{c} 0_{k-1} \\ 1_{n-1} \end{array} \right) = 0 \]

otherwise.

**Proof** Let \( \mu \) be a main eigenvalue of \( X\backslash a \) with eigenprojection \( E_\mu \). By Lemma 4.4,

\[ E_\mu = q(A(X\backslash a)) \]

for some polynomial \( q(x) \). It follows from Corollary 4.3 that

\[ q(\tilde{A}) \left( \begin{array}{c} 0_{k-1} \\ 1_{n-1} \end{array} \right) = \left( \begin{array}{c} 0_{k-1} \\ E_\mu 1_{n-1} \end{array} \right). \]
Now multiply both sides by $\tilde{A}$. We have

$$\tilde{A}q(\tilde{A}) \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \tilde{A} \begin{pmatrix} 0_{k-1} \\ E_\mu 1_{n-1} \end{pmatrix}.$$  

On the other hand,

$$\tilde{A}q(\tilde{A}) \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix} = \left( A(X\setminus a)q(A(X\setminus a))1_{n-1} \right) = \left( A(X\setminus a)E_\mu 1_{n-1} \right) = \mu \begin{pmatrix} 0_{k-1} \\ E_\mu 1_{n-1} \end{pmatrix}.$$

As $E_\mu 1_{n-1} \neq 0$,

$$\begin{pmatrix} 0_{k-1} \\ E_\mu 1_{n-1} \end{pmatrix}$$

is an eigenvector for $\tilde{A}$ with eigenvalue $\mu$. This proves the first statement.

To see the second statement, let $\lambda$ be an eigenvalue of $\tilde{A}$ with eigenprojection $\tilde{E}_\lambda$. Again, Lemma 4.4 tells us that $\tilde{E}_\lambda$ is a polynomial in $\tilde{A}$; moreover, this polynomial sends $\lambda$ to 1, and all other eigenvalues of $\tilde{A}$ to 0. Now, since each main eigenvalues of $X\setminus a$ is an eigenvalue of $\tilde{A}$, this polynomial also sends $A(X\setminus a)$ to its $\lambda$-eigenspace if $\lambda$ is indeed an eigenvalue of $A(X\setminus a)$, and to 0 otherwise.  

\section{Distance regular graphs}

One special type of equitable partitions arises in distance regular graphs. A graph is called distance regular if for any two vertices $u$ and $v$ at distance $m$, the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ is a constant which depends only on $i$, $j$ and $m$. As a result, for any vertex $a$, the distance partition relative to $a$ is equitable, and the quotient matrix is tridiagonal:

$$B = \begin{pmatrix}
a_0 & b_0 \\
c_1 & a_1 & b_1 \\
& \ddots & \ddots & \ddots \\
& & c_{d-1} & a_{d-1} & b_{d-1} \\
& & & c_d & a_d
\end{pmatrix}.$$  

(6)

Moreover, upon permuting the rows and columns, this quotient matrix does not depend on the choice of $a$. The list of parameters

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

is called the intersection array of the distance regular graph.
We cite some basic results on the intersection array. For references, see the book by Brouwer et al. [14, Sec 4.1], or the survey by van Dam et al. [15].

**Lemma 5.1** Let $X$ be a distance regular graph, with valency $k$ and intersection array \{ $b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d$ \}. The following holds.

(i) $a_0 = 0, b_0 = k$ and $c_1 = 1$.

(ii) For each $i$,

$$a_i = k - b_i - c_i,$$

with the convention that $c_0 = b_d = 0$.

(iii) For any vertex $a$, the number $k_i$ of vertices at distance $i$ from $a$ satisfies

$$b_i k_i = c_{i+1} k_{i+1}.$$

We now prove a lemma on a submatrix of the Laplacian matrix of a distance regular graph; this turns out useful when we study the average search probability in Sect. 6.

**Lemma 5.2** Let $X$ be a distance regular graph on $n$ vertices with valency $k$. Let $L(X)$ be the Laplacian matrix of $X$. For any vertex $a$, let $L(X) \setminus a$ denote the matrix obtained from $L(X)$ by removing the $a$-th row and the $a$-th column. Then, the entries in $(L(X) \setminus a)^{-1} 1_{n-1}$ increase in distance from $a$. Moreover, for any neighbor $v$ of $a$ in $X$,

$$e_v^T (L(X) \setminus a)^{-1} 1_{n-1} = \frac{n-1}{k}.$$

**Proof** It is a well-known result that $L(X) \setminus a$ is invertible. Let

$$y = (L(X) \setminus a)^{-1} 1_{n-1}.$$

As $X$ is distance regular, the distance partition relative to $a$ gives an equitable partition for $X \setminus a$, and so by Lemma 4.2 and the fact that $(L(X) \setminus a)^{-1}$ is a polynomial in $L(X) \setminus a$, the vector $y$ is constant on the cells of this partition. That is, there are integers $z_1, z_2, \ldots, z_d$ such that, for each vertex $u$ at distance $i$ from $a$ in $X$, the entry $y_u$ equals $z_i$.

Now we solve for $y$ in

$$(L(X) \setminus a)y = 1_{n-1}. \quad (7)$$

Pre-multiply both sides by $1_{n-1}^T$, and we get

$$\begin{pmatrix} 1_k & 0 \end{pmatrix}^T y = 1_{n-1}^T (L(X) \setminus a)y = 1_{n-1}^T 1_{n-1} = n - 1.$$
Therefore,
\[ z_1 = \frac{n - 1}{k}. \]

Moreover, expanding (7) gives
\[ k z_1 - a_1 z_1 - b_1 z_2 = 1, \]
and, assuming \( z_{d+1} = 0 \),
\[ k z_i - c_i z_{i-1} - a_i z_i - b_i z_{i+1} = 1, \quad i = 2, \cdots, d. \]

Since \( k = a_i + b_i + c_i \), this is equivalent to
\[ b_1 (z_2 - z_1) = z_1 - 1 \]
and
\[ b_i (z_{i+1} - z_i) + 1 = c_i (z_i - z_{i-1}), \quad i = 2, \cdots, d. \]

Let \( k_i \) denote the number of vertices at distance \( i \) from \( a \) in \( X \). Using Lemma 5.1 and induction, we see that
\[ z_{i+1} - z_i = \frac{k_{i+1} + \cdots + k_d}{k_i b_i} > 0. \]

Hence, \( y \) increases in distance from \( a \). \( \square \)

Finally, we cite a result on vertex connectivity, due to Brouwer and Koolen [16].

**Theorem 5.3** [16]. The vertex connectivity of a distance regular graph equals its valency.

### 6 Average search probabilities for distance regular graphs

We now consider the search problem on distance regular graphs. Let \( X \) be a distance regular graph on \( n \) vertices with valency \( k \geq 2 \). Let \( a \) be the marked vertex. Let \( U = \text{RCO}_a \) be as defined in Sect. 2, and \( F_\theta \) the projection onto its \( e^{i\theta} \)-eigenspace of \( U \). Recall that the average search probability is
\[ \frac{1}{nk} \sum_\theta \sum_{v \sim a} e^{T}_{(a,v)} \left( (F_\theta 1_{nk}) \circ (F_\theta 1_{nk}) \right). \]

We say an eigenvalue \( e^{i\theta} \) contributes to search if \( F_\theta 1_{nk} \neq 0 \).
Lemma 6.1 The eigenvalues of $U$ that contribute to search are $-1$ and $e^{\pm i\theta}$, where $	heta = \arccos(\lambda/k)$ for some eigenvalue $\lambda$ of $X \setminus a$. Moreover, if $E_\lambda$ is the projection onto the $\lambda$-eigenspace of $X \setminus a$, then for any neighbor $v$ of $a$ in $X$,

$$
e^{T}_{(a,v)} F_\theta 1_{nk} = \frac{1}{2 \sin^2(\theta)} (1 - e^{i\theta}) e_v^T E_\lambda 1_{n-1}.$$

Proof We apply Theorem 3.2. As $X$ is 2-connected, the eigenvalue 1 of $U$ does not contribute to search. Let $e^{i\theta}$ be a non-real eigenvalue of $U$. Let $\tilde{A}$ be defined as in (3). Then, $\lambda = k \cos(\theta)$ is an eigenvalue of $\tilde{A}$, and

$$F_{\theta} = \frac{1}{2k \sin^2(\theta)} (N - e^{i\theta} RN) \tilde{E}_\lambda (N - e^{i\theta} RN)^*,$$

where $\tilde{E}_\lambda$ is the projection onto the $\lambda$-eigenspace of $\tilde{A}$, and $N$ is defined as in (2). To see whether $e^{i\theta}$ contributes to search, we multiply both sides by $1_{nk}$. Since

$$N^* R 1_{nk} = N^* 1_{n+k-2} = k \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix},$$

we get

$$F_\theta 1_{nk} = \frac{1}{2 \sin^2(\theta)} (1 - e^{-i\theta}) (N - e^{i\theta} RN) \tilde{E}_\lambda \begin{pmatrix} 0_{k-1} \\ 1_{n-1} \end{pmatrix},$$

which, by Corollary 4.3, further reduces to

$$\frac{1}{2 \sin^2(\theta)} (1 - e^{-i\theta}) (N - e^{i\theta} RN) \begin{pmatrix} 0_{k-1} \\ E_\lambda 1_{n-1} \end{pmatrix},$$

where $E_\lambda$ is the projection to the $\lambda$-eigenspace of $X \setminus a$. Now, suppose $v$ is the $j$-th neighbor of $a$ in $X$. We have

$$e_j^T (N - e^{i\theta} RN) \begin{pmatrix} 0_{k-1} \\ E_\lambda 1_{n-1} \end{pmatrix} = (e_j^T (K 0) - e^{i\theta} e_v^T) \begin{pmatrix} 0_{k-1} \\ E_\lambda 1_{n-1} \end{pmatrix} = -e^{i\theta} e_v^T E_\lambda 1_{n-1},$$

from which the statement follows. \hfill \square

The above lemma allows us to express the average search probability completely in terms of the spectrum of $X \setminus a$.

Theorem 6.2 Let the spectral decomposition of $X \setminus a$ be

$$A(X \setminus a) = \sum \lambda E_\lambda.$$
Let \( v \) be any neighbor of \( a \) in \( X \). Then, the average search probability of the quantum walk on \( X \) is given by

\[
\frac{1}{n} \sum_{\lambda} \frac{k^3}{(k-\lambda)(k+\lambda)^2} (e_v^T E_\lambda 1_{n-1})^2 + \frac{1}{n} \left( 1 - \sum_{\lambda} \frac{k}{k+\lambda} e_v^T E_\lambda 1_{n-1} \right)^2.
\]

**Proof** The first term follows from Lemma 6.1 by taking the square of \( e_{(a,v)}^T F_{\theta} 1_{nk} \). For the second term, notice that

\[
F_\pi = I_{nk} - \sum_{\theta \neq \pi} F_{\theta},
\]

and so the \((-1)\)-eigenprojection of \( U \) is determined by the remaining eigenspaces of \( U \). Moreover, since the non-real eigenvalues of \( U \) come in conjugate pairs,

\[
F_{\theta} = \overline{F_{-\theta}}.
\]

This together with Lemma 6.1 yields the second term in the statement. \( \square \)

7 Tridiagonal matrices and orthogonal polynomials

In this section, we discuss the connection between eigenvectors of a generic tridiagonal matrix and eigenvectors of a graph whose quotient matrix coincides with this tridiagonal matrix. This will eventually provides us tools to study the limit of the average search probability on a family of distance regular graphs.

Let \( T \) be an \((m+1) \times (m+1)\) tridiagonal matrix,

\[
T = \begin{pmatrix}
\alpha_0 & \beta_0 & & \\
\gamma_1 & \alpha_1 & \beta_1 & \\
& \ddots & \ddots & \ddots \\
& & \gamma_{m-1} & \alpha_{m-1} & \beta_{m-1} \\
& & & \gamma_m & \alpha_m
\end{pmatrix},
\]

where the diagonal entries \( \alpha_i \) are nonnegative, and the off-diagonal entries \( \beta_i, \gamma_i \) are positive. Let \( T_i \) denote the leading \( i \times i \) principal matrix of \( T \). Then, \( T \) defines a sequence of monic polynomials,

\[
p_0(x) = 1 \\
p_1(x) = \det(x I - T_1) \\
p_2(x) = \det(x I - T_2) \\
\vdots \\
p_m(x) = \det(x I - T_m)
\]
\[ p_{m+1}(x) = \det(xI - T_{m+1}), \]
called the orthogonal polynomials associated with \( T \). Note that \( p_{m+1}(x) \) is the characteristic polynomial of \( T \).

We cite some standard results on orthogonal polynomials. The proofs can be found in Szego [17] or Chihara [18].

**Theorem 7.1** For each \( i \), the roots of \( p_i(x) \) are real and simple. Moreover, the roots of \( p_i(x) \) interlace those of \( p_{i+1}(x) \).

The orthogonal polynomials associated with \( T \) determine its eigenvectors.

**Theorem 7.2** Let \( \lambda \) be a root of \( p_m(x) \). Let \( z_\lambda \) be the vector defined by

\[
z_\lambda = \begin{pmatrix}
p_0(\lambda) \\
p_1(\lambda)/\beta_0 \\
\vdots \\
p_m(\lambda)/(\beta_0 \cdots \beta_{m-1})
\end{pmatrix},
\]

Then, \( z_\lambda \) is an eigenvector for \( T \) with eigenvalue \( \lambda \).

In Sect. 5, we saw that each distance regular graph has a quotient matrix of the form (6), and the entries are determined by the intersection array

\[ \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}. \]

In fact, we may consider the “reversed” tridiagonal matrix as well:

\[
S = \begin{pmatrix}
ad & cd \\
b_{d-1} & a_{d-1} & cd \\
& \ddots & \ddots & \ddots \\
& b_1 & a_1 & c_1 \\
& b_0 & a_0
\end{pmatrix},
\]

and, as we will see later, this form becomes handy when we relate the eigenvectors of a distance regular graph to that of its vertex-deleted subgraphs. The orthogonal polynomial associated with \( S \) is known as the dual orthogonal polynomials to those associated with (6), and has been studied in, for example, Vinet and Zhedanov [19].

From now on, assume \( X \) is a distance regular graph of diameter \( d \geq 2 \) on \( n \) vertices with valency \( k \). Fix a vertex \( a \), and let \( k_i \) be the number of vertices in \( X \) at distance \( i \) from \( a \). Let

\[
P = \begin{pmatrix}
\mathbf{1}_{k_d} \\
\mathbf{1}_{k_{d-1}} \\
\vdots \\
\mathbf{1}_{k_1} \\
\mathbf{1}_{k_0}
\end{pmatrix}.
\]
Then, $P$ is the characteristic matrix of the “reversed” distance partition, whose quotient matrix is defined in (9). As before, let $S_i$ denote the leading $i \times i$ principal matrix of $S$, and let $\{q_i(x)\}$ be the orthogonal polynomials associated with $S$, that is,

$$q_i(x) = \det(xI_i - S_i).$$

We also define a new sequence of polynomials by $u_0(x) = 1$ and

$$u_i(x) = \frac{q_i(x)}{c_d c_{d-1} \cdots c_{d-i+1}}, \quad i = 1, 2, \cdots d.$$

**Lemma 7.3** Let $\lambda$ be the largest eigenvalue of $X \setminus a$ and $E_\lambda$ the corresponding eigen-projection. For any neighbor $v$ of $a$ in $X$, we have

$$e_v^T E_\lambda 1_{n-1} \geq \frac{n-1}{n} (u_d-1(\lambda))^2 = \frac{n-1}{n} \left( \frac{q_d-1(\lambda)}{q_d-1(k)} \right)^2.$$

**Proof** The largest eigenvalue of a graph is a main eigenvalue, and so by Lemma 4.1, it must be an eigenvalue of its quotient matrix. Consider $X$ first. Since it is $k$-regular and $S$ is its quotient matrix, the vector

$$z_k = \begin{pmatrix} u_0(k) \\ u_1(k) \\ \vdots \\ u_{d-1}(k) \\ u_d(k) \end{pmatrix}$$

coincides with the all-ones vector, and $Pz_k$ is an eigenvector for $X$ with eigenvalue $k$. We have

$$n = |Pz_k|^2 = z_k^T P^T P z_k.$$

On the other hand, the submatrix $S_d$ is the quotient matrix of $X \setminus a$. Thus,

$$q_d(\lambda) = 0,$$

and

$$y_\lambda = \begin{pmatrix} u_0(\lambda) \\ u_1(\lambda) \\ \vdots \\ u_{d-1}(\lambda) \end{pmatrix}$$

is an eigenvector for $S_d$ with eigenvalue $\lambda$. Let $Q$ be the matrix obtained from $P$ by deleting the last column. Then, $Qy_\lambda$ is an eigenvector for $X \setminus a$ with eigenvalue $\lambda$. 

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Moreover, by interlacing, for each $i$ we have

$$u_i(k) \geq u_i(\lambda) > 0,$$

and so

$$|Qy\lambda|^2 = y\lambda^T Q^T Qy\lambda = y\lambda^T Q^T Qy\lambda + u_d(\lambda) \leq z_k^T P^T Pz_k = n.$$

Therefore,

$$e^T_v E\lambda_1 n-1 = \frac{1}{|Qy\lambda|^2} (e_{d-1}, Qy\lambda, 1_{n-1}, Qy\lambda) \geq \frac{1}{n} u_{d-1}(\lambda) (k_d u_0(\lambda) + k_{d-1} u_1(\lambda) + \cdots + k_1 u_{d-1}(\lambda)).$$

Now, since $y\lambda$ is an eigenvector for $S_d$, we have a recurrence relation:

$$b_{d-i} u_{i-1}(\lambda) + a_{d-i} u_i(\lambda) + c_{d-i} u_{i+1}(\lambda) = \lambda u_i(\lambda),$$

from which and $\lambda < k$ we get

$$b_{d-i} (u_{i-1}(\lambda) - u_i(\lambda)) < c_{d-i} (u_i(\lambda) - u_{i+1}(\lambda))$$

for $i \geq 1$, and

$$u_1(\lambda) < u_0(\lambda).$$

Thus, $u_i(\lambda) \geq u_{d-1}(\lambda)$ for all $i \leq d - 1$, and so

$$e^T_v E\lambda_1 n-1 \geq \frac{1}{n} (u_{d-1}(\lambda))^2 (k_d + \cdots + k_1) = \frac{n-1}{n} (u_{d-1}(\lambda))^2.$$

Finally, observe that $u_{d-1}(k) = 1$. \qed

Now we give a lower bound on the largest eigenvalue $\lambda$ of $X \setminus a$. A nice result of Eldridge et al. [20], derived from Courant-Fischer-Weyl min-max principle, turns out to be useful.

**Theorem 7.4** [20] Let $A$ and $B$ be two $n \times n$ real symmetric matrix. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $A$, with corresponding eigenvectors $x_1, x_2, \ldots, x_n$. Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ be the eigenvalues of $A + B$. Given $j \in \{1, 2, \cdots, n\}$, suppose

$$h \geq |\langle x, Bx \rangle|$$
for all unit vectors $x$ in $\text{span}\{x_1, x_2, \ldots, x_j\}$. Then for $i = 1, 2, \ldots, j$,

$$\lambda_i \geq \mu_j - h.$$ 

Applying this to the symmetrized quotient matrix

$$\hat{S} = \begin{pmatrix}
\frac{a_d}{\sqrt{b_d-1}c_d} & \frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{a_d}{\sqrt{b_d-1}c_d} & \frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{a_d}{\sqrt{b_d-1}c_d} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\sqrt{b_1}c_2}{a_1} & \frac{\sqrt{b_1}c_2}{a_1} & \frac{\sqrt{b_1}c_2}{a_1} & \cdots & \frac{a_1}{\sqrt{b_0}c_1} \\
\frac{\sqrt{b_0}c_1}{a_0} & \frac{\sqrt{b_0}c_1}{a_0} & \frac{\sqrt{b_0}c_1}{a_0} & \cdots & \frac{a_1}{\sqrt{b_0}c_1}
\end{pmatrix},$$

we find a lower bound for the largest eigenvalue of $X \setminus a$.

**Lemma 7.5** Let $\lambda$ be the largest eigenvalue of $X \setminus a$. Then,

$$\lambda \geq k - \frac{2k}{n}.$$ 

**Proof** It is not hard to see that $\hat{S}$ is similar to $S$, and

$$AP(P^TP)^{-1/2} = P(P^TP)^{-1/2}\hat{S}.$$ 

Therefore, $k$ is the largest eigenvalue of $\hat{S}$, with unit eigenvector

$$x = \frac{1}{\sqrt{n}}\begin{pmatrix} \sqrt{k} \\ \sqrt{k} \\ \vdots \\ \sqrt{k} \\ 1 \end{pmatrix}.$$ 

We consider the symmetrized quotient matrix of $X \setminus a$, which is the leading $d \times d$ principal submatrix of $\hat{S}$. Write

$$\begin{pmatrix}
\frac{a_d}{\sqrt{b_d-1}c_d} & \frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{a_d}{\sqrt{b_d-1}c_d} & \frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\frac{\sqrt{b_d-2}c_{d-1}}{a_d-2} & \frac{\sqrt{b_d-1}c_d}{a_d-1} & \frac{a_d}{\sqrt{b_d-1}c_d} & \cdots & \frac{\sqrt{b_d-j}c_{j-1}}{a_d-j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\sqrt{b_1}c_2}{a_1} & \frac{\sqrt{b_1}c_2}{a_1} & \frac{\sqrt{b_1}c_2}{a_1} & \cdots & \frac{a_1}{\sqrt{b_0}c_1} \\
\frac{\sqrt{b_0}c_1}{a_0} & \frac{\sqrt{b_0}c_1}{a_0} & \frac{\sqrt{b_0}c_1}{a_0} & \cdots & \frac{a_1}{\sqrt{b_0}c_1}
\end{pmatrix} = \hat{S} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \sqrt{k} \\ 0 & 0 & \sqrt{k} & 0 \end{pmatrix},$$

Clearly, $\lambda$ is the largest eigenvalue of the left hand side. Since

$$\left(\sqrt{k} \ 1\right) \begin{pmatrix} 0 & \sqrt{k} \\ \sqrt{k} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{k} \\ 1 \end{pmatrix} = 2k,$$
by Theorem 7.4, $\lambda \geq k - 2k/n$.

## 8 The limit of the average search probability

We would like to know how the average search probability, on a family of distance regular graphs, behaves as the valency grows. As before, let $X$ be a distance regular graph on $n$ vertices, with valency $k \geq 2$ and a fixed diameter $d$. Let $a$ be the marked vertex, and $v$ any neighbor of $a$. Let the spectral decomposition of $X \backslash a$ be

$$A(X \backslash a) = \sum_{\lambda} \lambda E_{\lambda}.$$  

Theorem 6.2 says that the average search probability on $X$ splits into two terms:

$$s_1 = \frac{1}{n} \sum_{\lambda} \frac{k^3}{(k - \lambda)(k + \lambda)^2} (e_v^T E_{\lambda} \mathbf{1}_{n-1})^2$$

$$s_2 = \frac{1}{n} \left( 1 - \sum_{\lambda} \frac{k}{k + \lambda} e_v^T E_{\lambda} \mathbf{1}_{n-1} \right)^2$$

We show that $s_2$ vanishes as the graph gets larger.

**Lemma 8.1** As $n$ goes to infinity, $s_2$ approaches 0.

**Proof** The eigenvalues $\lambda$ of $X \backslash a$ interlace those of $X$. In particular, $-k < \lambda < k$. Hence,

$$\frac{k}{k + \lambda} > \frac{1}{2},$$

and so

$$e_v^T \left( \sum_{\lambda} \frac{k}{k + \lambda} E_{\lambda} \right) \mathbf{1}_{n-1} > \frac{1}{2} e_v^T \left( \sum_{\lambda} E_{\lambda} \right) \mathbf{1}_{n-1} = \frac{1}{2}.$$  

The result now follows from the squeeze theorem.

Our next observation gives a lower bound for $s_1$.

**Lemma 8.2** We have

$$s_1 > \frac{1}{4} \frac{n - 1}{n} \sum_{\lambda} (E_{\lambda} J_{n-1} E_{\lambda})_{vv}.$$
**Proof** We can rewrite $s_1$ as

$$s_1 = \frac{1}{n} e_v^T \left( \sum_{\lambda} \frac{k^3}{(k-\lambda)(k+\lambda)^2} E_{\lambda} J_{n-1} E_{\lambda} \right) e_v,$$

which, by interlacing, is greater than

$$\frac{1}{4n} e_v^T \left( \sum_{\lambda} \frac{k}{k-\lambda} E_{\lambda} J_{n-1} E_{\lambda} \right) e_v.$$

Since

$$L(X) \setminus a = kI - A(X \setminus a) = \sum_{\lambda} (k-\lambda) E_{\lambda},$$

we have

$$\sum_{\lambda} \frac{k}{k-\lambda} E_{\lambda} J_{n-1} E_{\lambda} = k \left( \sum_{\lambda} \frac{1}{k-\lambda} E_{\lambda} \right) \left( \sum_{\lambda} E_{\lambda} J_{n-1} E_{\lambda} \right)$$

$$= k (L(X) \setminus a)^{-1} \left( \sum_{\lambda} E_{\lambda} J_{n-1} E_{\lambda} \right)$$

$$= k \sum_{\lambda} E_{\lambda} (L(X) \setminus a)^{-1} J_{n-1} E_{\lambda}.$$

The inequality now follows from Lemma 5.2. \qed

For readers who are familiar with continuous-time quantum walks, we notice that

$$\frac{1}{n-1} \sum_{\lambda} E_{\lambda} J_{n-1} E_{\lambda}$$

is the average state of the continuous-time quantum walk on $X \setminus a$, with $1_{n-1}/\sqrt{n-1}$ as the initial state. This reveals an interesting connection between the discrete-time quantum walk on a graph and the continuous-time quantum walk on its vertex-deleted subgraph. More discussion on average states can be found in Coutinho et al. [21].

We now find the limit for the search probabilities on complete graphs.

**Theorem 8.3** The average search probability on $K_n$ approaches $1/4$ as $n$ goes to infinity.

**Proof** The vertex deleted subgraph $K_{n-1}$ has two spectral idempotents:

$$E_{n-2} = \frac{1}{n-1} J_{n-1}, \quad E_{-1} = I_{n-1} - \frac{1}{n-1} J_{n-1}.$$
As they are orthogonal to each other, we have

$$\sum_{\lambda} E_{\lambda} J_{n-1} E_{\lambda} = E_{n-2} J_{n-1} E_{n-2} = J_{n-1}.$$ 

Hence, the average probability converges to 1/4. \qed

Any family of strongly regular graphs, that is, distance regular graphs of diameter two, also enjoy this probability.

**Theorem 8.4** The average search probability on a strongly regular approach 1/4 as the valency goes to infinity.

**Proof** Let $X$ be a strongly regular graph with intersection array $\{k, b_1; 1, c_2\}$. Clearly, $b_1 = k - a_1 - 1$, and $a_2 = k - c_2$. The main eigenvalues of $X \setminus a$ are eigenvalues of

$$\hat{S} = \begin{pmatrix} a_1 & \sqrt{(k-a_1-1)c_2} \\ \sqrt{(k-a_1-1)c_2} & k - c_2 \end{pmatrix}.$$ 

Solving $\hat{S}z = \lambda z$ yields

$$\lambda = \frac{1}{2} (k + a_1 - c_2) \pm \frac{1}{2} \sqrt{(k - a_1 + c_2)^2 - 4c_1}$$

and

$$z = \frac{1}{\sqrt{(k-a_1-1)c_2 + (\lambda - a_1)^2}} \begin{pmatrix} \sqrt{(k-a_1-1)c_2} \\ \lambda - a_1 \end{pmatrix}.$$ 

Thus, a unit eigenvector for $X \setminus a$ with eigenvalue $\lambda$ is

$$\frac{1}{\sqrt{(k-a_1-1)c_2 + (\lambda - a_1)^2}} \begin{pmatrix} (\sqrt{(k-a_1-1)c_2/\sqrt{k}})1_k \\ ((\lambda - a_1)/\sqrt{n-k-1})1_{n-k-1} \end{pmatrix}.$$ 

Let $E_{\lambda}$ be the $\lambda$-eigenprojection for $X \setminus a$ and $v$ is any neighbor of $a$ in $X$. Using the relation

$$(k - a_1 - 1)k = (n - k - 1)c_2,$$

we see that

$$e^T_v E_{\lambda} 1_{n-1} = \frac{(k - a_1 - 1)(\lambda - a_1 + c_2)}{(k-a_1-1)c_2 + (\lambda - a_1)^2}.$$ 

Thus, the average search probability is at least a quarter of

$$\sum_{\lambda} (E_{\lambda} J_{n-1} E_{\lambda})_{vv} = \frac{(k - a_1 + c_2)^2 - 4c_2 - 2(k - a_1) + 2}{(k - a_1 + c_2)^2 - 4c_2}.$$ 

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Dividing the right hand side by its numerator shows that it approaches 1 as \( k \) goes to infinity.

For distance regular graphs with larger diameters, we need the results from Sect. 7 to bound the limit.

**Theorem 8.5** Let \( d \geq 3 \) be a fixed positive integer. Let \( X \) be a distance regular graph of diameter \( d \). Assume the number of vertices \( n \), the valency \( k \), and the intersection numbers \( a_i, b_i, c_i \) are all functions in a parameter \( \tau \), and \( k(\tau) \) increases in \( \tau \). Further assume

\[
\lim_{\tau \to \infty} \frac{k(\tau)^{d-1}}{c_2(\tau) \cdots c_d(\tau)n(\tau)} = 0.
\]

Let \( a \) be any vertex of \( X \). As the valency goes to infinity, the average search probability on \( X \) approaches \( 1/4 \).

**Proof** Let the quotient matrix \( S \) of \( X \) be as defined in (9). Let \( \{q_i(x)\} \) be the associated sequence of orthogonal polynomials. Let \( n \) be the number of vertices in \( X \), and \( v \) any neighbor of \( a \). Let \( \lambda \) be the largest eigenvalue of \( X \setminus a \), with eigenprojection \( E_\lambda \).

Lemma 7.3 tells us that

\[
e_v^T E_\lambda 1_{n-1} \geq \frac{n-1}{n} \left( \frac{q_{d-1}(\lambda)}{q_{d-1}(k)} \right)^2.
\]

Since \( a_i, b_i, c_i \leq k \), by an inductive argument, the characteristic polynomial of any \( \ell \times \ell \) principal submatrix of \( S \),

\[
x^\ell + \alpha_1 x^{\ell-1} + \alpha_2 x^{\ell-2} + \cdots + \alpha_{\ell},
\]

satisfies \( \alpha_i = O(k^i) \). Thus

\[
q_{d-1} \left( k - \frac{2k}{n} \right) = q_{d-1}(k) + q'_{d-1}(k) \left( -\frac{2k}{n} \right) + q''_{d-1}(k) \left( -\frac{2k}{n} \right)^2 + \cdots = q_{d-1}(k) + \frac{O(k^{d-1})}{n} + \frac{O(k^{d-1})}{n^2} + \cdots + \frac{O(k^{d-1})}{n^{d-1}}.
\]

The result now follows from \( q_{d-1}(k) = c_2 c_3 \cdots c_d \).

A special case, with stronger assumptions, is given below.

**Theorem 8.6** Let \( d \geq 3 \) be a fixed positive integer. Let \( X \) be a distance regular graph of diameter \( d \). Assume the number of vertices \( n \), the valency \( k \), and the intersection numbers \( a_i, b_i, c_i \) are all functions in a parameter \( \tau \), and \( k(\tau) \) increases in \( \tau \). Further assume

\[
\lim_{\tau \to \infty} \frac{k(\tau)^{d-1}}{n(\tau)} = 0.
\]
Let a be any vertex of $X$. As the valency goes to infinity, the average search probability on $X$ approaches $1/4$.

We note that this criterion is met by many common families of distance regular graphs, including the Hamming graphs $H(d, τ)$, the Johnson graphs $J(τ, d)$, the Grassmann graphs $J_q(τ, D)$ with a fixed $q$, and the dual polar graphs with a fixed $e$.

9 Future work

The average probability we studied lies in the following vector,

$$\sum_r (F_r x_0) \circ (F_r x_0),$$

which, as we have seen, is the limit of

$$\frac{1}{T} \sum_{t=0}^{T-1} (U^t x_0) \circ (U^t x_0)$$

as $T$ goes to infinity. For graphs with high average search probabilities, it will be helpful to determine the mixing time $M_ε$, that is, the smallest $K$ such that for all $T > K$,

$$\left| \frac{1}{T} \sum_{t=0}^{T-1} (U^t x_0) \circ (U^t x_0) - \sum_r (F_r x_0) \circ (F_r x_0) \right| \leq ε.$$

This could potentially indicate that quantum search is fast on some graphs.

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Data availability The author can confirm that all relevant data are included in the article.

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