Dynamic model of real investment with lags

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Abstract. The dynamic optimal control problem of corporation's real investment in case of a limited budget is analyzed. Initial optimization problem is formulated as a system of linear integral equations of Volterra type. The offered model takes into account the accumulated effect of real investment and previous revenue on sales. It is supposed that the previous revenue indirectly affects the current one and reflects the influence of other factors. The accumulated impact of investment is distributed over a fix time interval. The total profit maximization problem under restrictions is stated. The existence theorem for the solutions of the total profit maximization problem is proved. The model is tested with the real data. In this case the Pontryagin's maximum principle and numerical methods of solving the integral and differential equations are used. The results of experiment are in accordance with the understanding of optimal investment strategy. Considering it the authors state that the offered model makes a solution which is possible to increase the total profit over the planning period.

1. Introduction
Generally, in order to analyze large economic facility work and assess its gross output production function is used [1]. It is expressed quantitatively and represents the technological relation between physical inputs and outputs of the goods.

Until recently, capital and labor were the most commonly used production factors. However today, economy is undergoing fundamental changes as a result of the rapid development of information technology. Enterprises are forced to survive in a modern economy where the global market is characterized by competition, diversity of products and services and short cycle life of the product [2].

Note that profit maximization is the main purpose of any business. As a rule, this is achieved through the cost savings. The introduction of advanced technology and automated systems in technological processes helps to significantly reduce costs of labor. Thus, the rule of labor as a production factor is substantially diminishing. And using this factor for modeling of production process is becoming irrelevant.

With regard to the capital it is necessary to note that researchers have only information on the balance of funds, which are often poorly reflecting the real state of economy. According to the neoclassical rationality principle, the effective productive assets have to be considered in the production functions representing the mathematical models of large production facilities [3]. Difficulties in assessing of the impact of effective productive assets on the output force the researchers [4] to replace the capital by the term "investment", which is more flexible, available to analysis and determined by current economic environment.

There are a number of models which describe production process by using investment as a production factor. In the Solow-Swan model investment is assumed to have the immediate impact on output.
However the process of installation, commissioning or modernization usually takes a long time. The revenue from sales of products, which are produced by using a new or modernized machine, has a "lagging" nature. In the work [5] the Solow-Swan model is considered taking into account distributed time lags. The type of the function which reflects the nature of distribution makes it possible to solve optimization problem by using simple analytical methods. However, the real economical conditions are more complicated. In order to improve the accuracy of assessment of production process it is required to use new approaches to solving optimization problems.

Investing activities of any business is the necessary financial costs that are aimed to develop production, improve quality of goods, extend a list of offered products and, as a result, increase sales and make a profit.

All large enterprises have different type of investment. But only real investment refers to the acquisition of property, plant, equipment and intangible assets. It is not as liquid as financial investment; "paper" assets can be diversified or sold in a relatively short period of time. Besides, new equipment or buildings require a large investment, so it is important to use well considered approaches to effective distribution real investment over the planning period.

Obviously, enterprises attempt to balance the output and consumer demand. This suggests that in the absence of information on production factors the demand factors can be used.

In addition, there are many internal and external factors that influence on sales, e.g. wage found, pricing, general market trend etc. It is impossible to estimate the impact of all of them. But assuming that non-investment factors are not changing quickly enough, we can use a previous revenue function for expression of cumulative non-real-investment effect.

Accordingly, the current revenue is changing under the impact of the previous revenue and the accumulated real investment effect.

An investment budget usually is limited to some amount of money. Thus, the problem of the best distribution of investments over the planning period can be considered as a dynamic optimal control problem. In this case, the total profit function can be chosen as an objective function which needs to be maximized.

The dynamic optimization models were considered by authors in the works [6–8].

In the papers [9, 10] continuous distributed lags were taken into account by the authors. The accumulated advertising and previous-sales effects were considered. Real investment can also be considered in frame of tasks with integral equations of Volterra type. Definitely, real investments influence on company’s production. Previous production can reflect aggregate non-real-investment effect on current output as well as accumulated previous-sales effect [9, 10]. However, there is no reason to take into account the accumulated effect because company production does not depend on consumer’s impression and we should use only point effect of previous output. It creates the new class of tasks with mixed impact, where both accumulated and point effects are considered.

In the practical application the revenue function is supposed to be linear with respect to the previous revenue and the accumulated effect of real investment. The practical application includes the algorithm for solving the problem of the best distribution of investments by using Pontryagin’s maximum principle [11] and numerical methods for solving integral and differential equations.

2. Problem statement
Let \( y(t) \) is the revenue function at the moment \( t \), \( I(t) \) is the investment function, \( v(t) \) is the function of investment effect which is accumulated on the fix interval \([\tau_1;\tau_2]\). \( G(\tau) \) describes the nature of this effect, \( \tau \) is the time lag, \( w(t) \) reflects the combined impact of non-real-investment factors:

\[
v(t) = \int_{\tau_1}^{\tau_2} G(\tau)I(t-\tau)\,d\tau; \tag{1}
\]

\[
w(t) = g(y(t-1)); \tag{2}
\]

\[
y(t) = f(v(t), w(t)). \tag{3}
\]
There are some assumptions about the functions $g(y), G(\tau)$ and $f(v,w)$:

1. The function $f(v,w)$ increases with respect to $v$.

2. Until the certain moment $\tau^*$ investment influence on revenue increases. After that, it begins to decrease until it disappears. Let the possibility of negative investment effect is excluded, i.e. real investment doesn't reduce production. Thus, the function $G(\tau)$ is non-negative and it has unique maximum $\tau = \tau^*$ which is the moment of the highest investment effect. If this function is differentiable, then assumption is equivalent to the following conditions:

$$G(\tau) \geq 0, \forall \tau \in [0;+\infty); \lim_{\tau \to +\infty} G(\tau) = 0; G'(\tau) \geq 0, \tau \in [0;\tau^*); G'(\tau) \leq 0, \tau \in [\tau^*;+\infty).$$

3. The function $g(y)$ is continuous for $y \geq 0$.

The financial result of the company for the planning period $[0;T]$ is the total profit or loss:

$$\Pi(T) = \int_{0}^{T} \pi(y(t), I(t)) dt = \int_{0}^{T} (f(v(t),w(t)) - I(t) - c(y(t),t)) dt.$$

Here $\pi(y(t), I(t))$ - profit or loss at the point $t$, $c(y(t),t) = c_1(y(t),t) + c_2$, $c$ - total costs, $c_1$ - variable costs excepting real investment, $c_2$ - fixed costs. Let other variable costs are in direct ratio to the volume of output with the rate $\mu$:

$$\Pi(T) = \int_{0}^{T} ((1-\mu) f(v(t),w(t)) - I(t) - c_2) dt. \quad (4)$$

Investment budget is limited in the following way:

$$0 \leq I(t) \leq b, \quad t \in [0;T]. \quad (5)$$

The optimization problem is formulated: to maximize the functional (4) under the restrictions (1), (2), (3), (5).

3. **Formulation of the problem as a system of Volterra integral equations**

Let $\bar{I}(t)$ and $\bar{y}(t)$ are continuous real-investment and revenue functions that are determined for $t < 0$, then $y(0) = \bar{y}(0)$. Let $G(\tau)$ is continuous for $\tau \geq 0$.

Denote by $\overline{G}(\tau), \varphi_1(t), \varphi_2(t)$ the following piecewise continuous functions:

$$\overline{G}(\tau) = \begin{cases} g(\bar{y}(t)), 0 \leq t < 1, \\
0, \quad \text{else},
\end{cases} \quad \varphi_1(t) = \begin{cases} \int_{t}^{\tau_1} G(t-s) \bar{I}(s) ds, 0 \leq t \leq \tau_1, \\
0, \quad t \geq \tau_2.
\end{cases}$$

Let $\delta(\cdot)$ - Dirac delta function, introduce the function $\overline{\delta}(\cdot; t) = \begin{cases} \delta(t), 0 \leq t < 1, \\
0, \quad t \geq 1.
\end{cases}$

Therefore, $v(t), w(t), \Pi(t)$ can be represented as Volterra type equations:

$$v(t) = \varphi_1(t) + \int_{0}^{t} \overline{G}(t-s) I(s) ds, \quad (6)$$

$$w(t) = \varphi_2(t) + \int_{0}^{t} \overline{\delta}(s; t) g(f(v(s),w(s))) ds, \quad (7)$$
\[ \Pi(t) = \int_0^t ((1 - \mu)f(v(s),w(s)) - I(s) - c_2) \, ds. \] (8)

And the maximization of \( \Pi(T) \) under conditions (5), (6), (7), (8) is equivalent to the maximization of (4) under conditions (5), (3), (1), (2).

4. Existence of the solution
Consider the question of the solution existence of total profit maximization under (5), (6), (7), (8).

A finite value of the accumulated profit \( \Pi(T) \) is a functional of control \( I(\cdot) \). Introduce \( J(I(\cdot)) = \Pi(T) \). Suppose that investment function \( I(t) \) is piecewise-continuous on the right in \([0; T]\).

Formulate the theorem of the solution existence of the optimization dynamic problem.

**Theorem 1.** Assume that the \( f(v,w) \) is continuous with respect to \( v, w \) and doesn't decrease monotonously with respect to \( v \), \( g(y) \) is continuous with respect to \( y \). Then there are two alternatives:

1. There exists \( \{I^*(t), 0 \leq t \leq T\} \) (5), a solution to the equations (6), (7), (8) \( \{v^*(t), w^*(t), \Pi^*(t), 0 \leq t \leq T\} \) \( J(I^*(\cdot)) \geq J(I(\cdot)) \) for any \( I(\cdot) : (5) \).

2. There exists a sequence of control functions \( \{I^n(t), 0 \leq t \leq T\} \) (5) and a value \( \bar{J} : J(I^n(\cdot)) \rightarrow \bar{J} \), \( n \rightarrow \infty \), \( J(I(\cdot)) \leq \bar{J} \) for any \( I(\cdot) : (5) \).

**Proof.** If \( G(t) \) is continuous, then \( v(t) \) is continuous in \([0; T]\). Taking into account (5), (6) we can state existence of the value \( b_1 > 0 \):

\[ 0 \leq \varphi_1(t) + \int_0^t G(t-s)I(s) \, ds \leq \max_{0 \leq t \leq T} \varphi_1(t) + \int_0^T G(t-s)b \, ds \leq b_1. \]

For all \( I(t) : (5) \) the function of accumulated investment effect satisfies the condition \( 0 \leq v(t) \leq b_1 \).

Consider the expression (2). For \( 0 \leq t \leq 1 \) \( w(t) = g(\tilde{y}(t-1)) \) is continuous in accordance with property of \( \tilde{y}(t) \). For \( 1 \leq t \leq 2 \) \( w(t) = g(f(v(t-1),w(t-1))) \) is continuous in accordance with continuity of \( v(t-1), w(t-1) \) with respect to \( t \in [1;2] \). Analogously, it is continuous on the next interval \([2;3]\) and etc.

Thus, for the investment strategy (5) the previous sale influence \( w(t) \) is continuous over the closed interval \([0; T]\) and consequently bounded by \( K : K = \max_{0 \leq t \leq T} w(t) \).

It is possible to demonstrate that the total profit is limited:

\[ J(I(\cdot)) = \int_0^T (1 - \mu)f(v(s),w(s)) - I(s) - c_2 \, ds \leq \int_0^T f(v(s),w(s)) \, ds \leq T \max_{(v,w) \in D} f(v,w), \]

where \( D = \{(v,w) : 0 \leq v \leq b_1, 0 \leq w \leq K\} \).

So, the range of the functional \( J(I(\cdot)) \) of the optimal control problem (5), (6), (7), (8) is limited. Denote this range as \( L \).

Suppose that \( \bar{J} = \sup L \). Evidently, \( \bar{J} \) exists and it is limited. If \( \bar{J} \in L \), then the first alternative is realized else the second alternative is realized [8]. □

**Remark.** If the second alternative of the Theorem 1 is realized, then there exists an approximated solution of the optimal control problem (5), (6), (7), (8) so that for any \( \varepsilon > 0 \) there exists such a control function \( I_\varepsilon(\cdot) : (5) \) and appropriate solutions of (6), (7), (8) that \( \bar{J} - J(I_\varepsilon(\cdot)) < \varepsilon \).

5. Maximum principle for optimization problem
Apply the maximum principle for the optimization problem. Introduce the vector \( F(s,t,h_1,h_2,h_3,I) : \)

\[
F(s,t,h_1,h_2,h_3,I) = \begin{pmatrix}
\bar{G}(t - s)I \\
\bar{\delta}(s; t) f(\varphi_1(s) + h_1, \varphi_2(s) + h_2) \\
(1 - \mu) f(\varphi_1(s) + h_1, \varphi_2(s) + h_2) - I - c_2
\end{pmatrix}.
\]

Denote the vector \( h(t) = (h_1(t), h_2(t), h_3(t))' \) : \( h(t) = \int_0^T F(s,t,h_1(s),h_2(s),h_3(s),I(s))ds \).

**Theorem 2.** Let \( \{ I'(s), \psi'(s), \psi''(s), \Pi'(s) \} \) is an optimal process of the problem of the maximization of \( \Pi(T) \) under conditions (5), (6), (7), (8), then there exist functions \( H_i(s), \psi = (\psi_1, \psi_2, \psi_3) \):

\[
H_i(s) = \psi_i(s) \bar{G}(0) - \psi_3(s) + \int_s^T \frac{\partial G(t - s)}{\partial t} \psi_i(t) dt,
\]

\[
\begin{cases}
\frac{d\psi_1(s)}{ds} = -\frac{\partial f(\varphi_1(s) + h_1, \varphi_2(s) + h_2)}{\partial h_1}(\psi_1(s) + \frac{d\psi_2(s + 1)}{ds}), \\
\frac{d\psi_2(s)}{ds} = -\frac{\partial f(\varphi_1(s) + h_1, \varphi_2(s) + h_2)}{\partial h_2}(\psi_2(s) + \frac{d\psi_3(s + 1)}{ds}), \\
\frac{d\psi_3(s)}{ds} = 0, \quad \psi_1(T) = \psi_2(T) = 0, \quad \psi_3(T) = 1,
\end{cases}
\]

that the optimal distribution of real investment has the following form:

\[
I'(s) = \begin{cases}
b, & H_1(s) > 0, \\
0, & H_1(s) < 0, \\
\bar{I}, & H_1(s) = 0, \ 0 \leq \bar{I} \leq b.
\end{cases}
\]

**Proof.** Introduce the modified Hamilton-Pontryagin function [11]:

\[
H(s,h,I,\psi) = (\psi(s), F(s,s,h,I)) + \int_s^T (\psi(t), F_i(t,s,h,I)) dt,
\]

and Cauchy problems for the adjoint variables \( \psi \):

\[
\frac{d\psi_i(s)}{ds} = -\frac{\partial H(s,h,I,\psi)}{\partial h_i}, \quad i = 1, 2, 3, \psi_1(T) = \psi_2(T) = 0, \quad \psi_3(T) = 1.
\]

Taking into account the property of Dirac delta function and \( F(s,t,h_1,h_2,h_3,I) \) we can get:

\[
H(s,h,I,\psi) = \psi_1(s) \bar{G}(0) I + \psi_3(s)(1 - \mu) f(\varphi_1(s) + h_1, \varphi_2(s) + h_2) - I - c_2
\]

\[
+ \int_s^T \psi_1(t) \frac{\partial G(t - s)}{\partial t} I dt + f(\varphi_1(s) + h_1, \varphi_2(s) + h_2) \frac{d\psi_3(s + 1)}{ds}.
\]

The Hamiltonian function is linear with respect to \( I \), it means that maximum can be obtained if only \( I \) takes the extreme values \( b \) or \( 0 \). The partial derivative of the Hamiltonian function with respect to \( I \) is determined:

\[
\frac{\partial H(s,h,I,\psi)}{\partial I} = \psi_1(s) \bar{G}(0) - \psi_3(s) + \int_s^T \psi_1(t) \frac{\partial G(t - s)}{\partial t} dt.
\]

The expression \( \frac{\partial H(s,h,I,\psi)}{\partial I} = H_1(s) \) and it defines the structure of optimal control (11).
Generally, the boundary problem with respect to $h(s), \psi(s)$ cannot be solved by analytical methods, so we need to apply numerical schemes.

6. Practical application

In order to test the dynamic model of real investment, the corporation semi-annual data set of revenue and real investment (in million Russian rubles) from consolidated financial statements (January 2007 - June 2017) is analyzed.

The Public Join Stock Company United Aircraft Corporation (PJSC UAC) was established to a Russian President’s Decree dated February the 20th, 2006 (its former name, before 2015, was OJSC UAC). The priority activity areas are development, production, sales, operation support, warranty and servicing, modernization, repair, and disposal of civil and military aircraft [12].

Correlation analysis is conducted to observe the relationship between investment and sales and identify the time lags. As a result, the upper and lower boundaries of the integral (1) are found in accordance with the highest values of the correlation coefficients: $\tau_1 = 2$, $\tau_2 = 5$.

Also regression analysis is used to understand the nature of influence of either physical investment (property, plant and equipment) or intangible assets on sales. It is possible to note that distribution of influence of intangible assets is fully in line with the properties of $G(\tau)$. As physical investment does not have a strong impact, intangible assets determine the pattern of real investment. It is in accordance with a modern economy where informational factors are becoming increasingly important.

Consider the case when the function $f(v, w)$ is linear with respect to $v$ and $w$, $g(y)$ is in direct ratio to $y$. That is, $y(t)$ can be written:

$$y(t) = \varphi_0(t) + \int_0^s \tilde{G}(t-s)I(s)d\tau + \varphi_1(s) + \int_0^s \tilde{a}(s; t)y(s)d\tau.$$

Based on the properties of the function $G(\tau)$, it is possible to define it:

$$G(\tau) = \exp(a_1\tau^3 + a_2\tau + a_3).$$

**Experimental Results.** Let the planning period $T = 3$ years. The limited number of investment budget $b$ is equal to 25902,5 million Russian rubles (maximal real investment in according to the biannual data). The parameter estimates are obtained by the method of least squares: $a_1 = -0.68009$, $a_2 = 6.4509$, $a_3 = -12.78169$, $\alpha = -0.56369$.

Runge-Kutta second order method was used for calculating the integrals (6), (8). The solutions are found for the different integration steps $h$. In each case the solutions have the same structure:

$$I(t) = \begin{cases} h, & 0 \leq t \leq t', \\ 0, & t' \leq t \leq T, \end{cases}$$

where $t'$ is a switch point of control. The table 1 demonstrates the obtained values of the total profit and the switch points for the different integration steps.

| $h$  | $\Pi(T)$, Russian rubles | $t^*$, years |
|------|--------------------------|-------------|
| 0.01000 | $1.26372 \times 10^6$ | 1.37000 |
| 0.00500 | $1.26240 \times 10^6$ | 1.37100 |
| 0.00250 | $1.26175 \times 10^6$ | 1.37130 |
| 0.01250 | $1.26142 \times 10^6$ | 1.37125 |

Note that the total profit over the past 3 years in according to the data set is 989012,5 million Russian rubles.
As it is shown in the table 1, the maximum of the total profit can be achieved if the corporation has the highest possible investments at the beginning of the planning period, which is a logical strategy. At first, it acquires property, plant, equipment and intangible assets. Products are sold after a while and previous investments continue to bring high profit.

7. Conclusion
A new dynamic model is offered in this paper. It is aimed to solve the problem of optimal distribution of enterprises real investment over the planning period. The influence of investment and non-investment factors on the revenue function takes place. In this case both constant and continuously distributed lags are taken into account.

The theorem of the solution existence is proved. The experiment by using the real data set is conducted. Based on the numerical scheme, the solution which satisfies necessary conditions of Pontryagin’s maximum principle are found. It is in accordance with the understanding of optimal investment strategy. Considering the results, it is possible to state that the offered model makes it possible to increase the total profit over the planning period. The model may be interesting for enterprises and integrated into decision support systems.

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