TOPOLOGY CHANGE IN CLASSICAL AND QUANTUM GRAVITY

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ABSTRACT

In these two lectures I describe the difficulties one encounters when trying to construct a framework in which to describe topology change in classical general relativity where one sticks to the assumption of an everywhere non-singular Lorentzian metric and how these difficulties can be circumvented in the Euclidean approach to quantum gravity.

Introduction

An important question both in classical and in quantum gravity theory is
whether the topology of space can change. In other words one may ask whether it is possible for the 4-dimensional spacetime manifold $M$ not to have the product topology of the real line times some spatial 3-manifold but rather something more complicated. Intuitively we are interested in processes variously described as:

- “trouser leg cosmology”
- “The birth of the universe from nothing”
- “The birth of twin universes”
- “The creation of a wormhole in an $S^3$ universe”.

Of course ideas this sort, involving as they do a definite three or four dimensional geometry can make sense at best classically or at the semi-classical level - no matter what the correct underlying theory of quantum gravity turns out to be. Thus by quantum gravity I shall mean semi-classical quantum gravity and more specifically I shall be describing an instanton approach within the framework of Euclidean Quantum Gravity.

Before turning in detail to the problem at hand it is worth pausing to remind ourselves of the range of validity of such semi-classical considerations. We know that some sort of new physics must set in at a scale which it is convenient to characterize by a temperature. Let us agree to call this temperature the Hagedorn temperature $T_{\text{Hagedorn}}$. In string theory for example this is given approximately by $(\alpha')^{-\frac{3}{2}}$ when $\alpha'$ is the string tension.

Now “conventional” Einstein quantum gravity effects of the sort that have been much discussed at this meeting using rough speaking semi-classical ideas and expansions around classical solutions of possibly modified Einstein equations set in at roughly the Planck temperature $T_{\text{Planck}} = G^{-\frac{1}{2}}$ where $G$ is Newton’s constant. In order to have a clear cut separation between these effects and those due to new physics - for example “stringy” effects - we need that $T_{\text{Planck}}/T_{\text{Hagedorn}}$ should be small, certainly less than unity and hopefully less than or of the order of $10^{-2}$, on the other hand, $T_{\text{Planck}}/T_{\text{Hagedorn}}$ is greater or of the order of unity little of what I have to say in this talk, or indeed little of much of the work in quantum gravity described at this meeting, makes much physical sense.

In string theory as it is at present the relation between Newton’s constant $G$ and the string tension $\alpha'$ is rather uncertain. Optimistic attempts at phenomenological models might work if $T_{\text{Planck}}/T_{\text{Hagedorn}}$ is roughly of order 10, but this is hardly
conclusive. The realistic position is that we simply do not know $T_{Hagedorn}/T_{Planck}$ - it might be greater than unity, it might be less than unity. It is worth remembering that the name $T_{Hagedorn}$ stems from a time when string theory was believed to be a theory of Hadron physics and its value compared with $T_{Planck}$ was thus tiny. The main point to be borne in mind is that all that I have to say here may or may not turn out to be physically relevant when, or if, we finally discover what the new physics really is.

With this caveat in mind here is the plan of what I want to cover:

1. **Topology Change in Classical Relativity.** By this I mean via a spacetime which carries an every-where non-singular Lorentz metric. This is an old topic. The basic conclusion is that this raises difficulties on purely kinematic grounds because of the necessity of non-time-orientability or of closed timelike curves. I shall also describe some new selection rules which Stephen Hawking and I have recently discovered which show that in some cases even if one is prepared to contemplate topology change via Lorentzian metrics with closed timelike curves a potentially even more disastrous problem arises, the impossibility of introducing 2-component spinors on the manifold $M$.

2. **Topology Change in Euclidean Quantum Gravity** By contrast with an approach based on every-where non-singular Lorentzian metrics in which one encounters great difficulties even at the purely kinematic level in describing topology change there are no such difficulties in the Euclidean Path Integral Approach. Not only is topology change kinematically allowed but there are actually classical complex paths in the path integral which mediate it. These classical paths correspond to what Hartle and I have called Real Tunneling Geometries. I will describe this attempt to formalize the idea of a complex classical path in the framework of the path integral approach to Quantum Gravity give a number of examples representing the production of pairs of black holes by strong external electromagnetic or cosmological fields. These classical solutions can serve as the basis for an Instanton calculation of the amplitude. Quantum corrections can be treated in a variety of ways but here I shall sketch an approach to the quantization of fluctuations about real tunneling geometries based on

Reflection Positivity which makes contact with the ideas of Euclidean Quantum Field theory.
1. Topology Change in Classical Gravity

By a classical topology change between to 3-manifolds $\Sigma_{\text{initial}}$ and $\Sigma_{\text{final}}$ we mean that there exists a compact connected 4-dimensional spacetime $M$ whose boundary $\partial M = \Sigma_{\text{initial}} \cup \Sigma_{\text{final}} = \Sigma$ with an everywhere non-singular Lorentz metric with respect to which the 3-manifolds are spacelike. Such a spacetime is called a Lorentz-cobordism between the initial and final 3-manifolds $\Sigma_{\text{initial}}$ and $\Sigma_{\text{final}}$ respectively. Note that neither the initial nor the final 3-manifold need be connected. In fact it is convenient to consider the spacetime as having just one spacelike boundary $\Sigma$ which is the union of all its components. We remark that the Lorentzian spacetime $M$ need not be, and in general will not be, geodesically complete. By Rohlin’s theorem there are many compact 4-manifolds $M$ whose boundary $\partial M = \Sigma$, where $\Sigma$ is a closed but not necessarily connected oriented 3-manifold. Topologically therefore topology change is always possible. However to have a spacetime we must endow $M$ with a Lorentzian metric.

We can always give $M$ a Riemannian metric $g^{R\alpha\beta}$ with signature $++\cdots\cdots$. By contrast however to give $M$ a Lorentzian metric $g^{L\alpha\beta}$ with signature $+++\cdots\cdots$ requires that the Euler characteristic $\chi(M)$ vanishes. As an aside let me remark, particularly in the context of studies of signature change, that the remaining possibility - signature $++\cdots\cdots$ is worthy of study and constitutes a sort of “Last Frontier” as far as 4-dimensional geometry is concerned. The analogous condition turns out to be that the Euler characteristic $\chi$ should be even and equal to the Hirzebruch signature $\tau$ modulo 4.

To return to the Lorentzian case. We can diagonalize $g^{L\alpha\beta}$ with respect to some arbitrary purely axillary Riemannian metric $g^{R\alpha\beta}$ to obtain a line field $\pm V^{\alpha}$ where $V^{\alpha}$ is the eigenvector with negative eigenvalue. Conversely given a line field $\pm V^{\alpha}$ which we may normalize with respect to $g^{R\alpha\beta}$ we may construct a Lorentzian metric in terms of

$$V_{\alpha} = g^{R\alpha\beta} V^{\beta}$$

by:

$$g^{L\alpha\beta} = g^{R\alpha\beta} - 2 V_{\alpha} V_{\beta}.$$ (2)

Time-orientability of the Lorentzian metric is equivalent to being able to remove the $\pm 1$ ambiguity remaining in the definition of $V^{\alpha}$ and obtaining a vector field $\mathbf{V}$ on $M$. To say that the boundary $\partial M = \Sigma$ is spacelike with respect to $g^{L\alpha\beta}$ amounts
to saying that the line field is transverse to $\Sigma$. A theorem of Hopf then implies that the necessary and sufficient condition for the existence of the line field is the vanishing of the Euler characteristic $\chi(M)$. According to the results of Misner, Rhinehart, Geroch, Yodzis, Sorkin etc this can always be achieved in 4-dimensions (but not in 2-dimensions) by taking connected sums of the original manifold $M$, which may not have vanishing Euler characteristic, with suitable closed 4-manifolds. The connected sum $M_1 \# M_2$ of two manifolds $M_1$ and $M_2$ is obtained by removing a 4-ball from both and and gluing together by identify the two $S^3$ boundary components so created. Under connected sum we have:
\[ \chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2 \] (3)

Thus for example:
- $M \# S^2 \times S^2$ has $\chi$ increased by 2,
- $M \# S^1 \times S^3$ has $\chi$ decreased by 2,
- $M \# \mathbb{CP}^2$ has $\chi$ increased by 1,
- $M \# \mathbb{RP}^4$ has $\chi$ decreased by 1.

We see that if $\chi$ is even we can use $S^2 \times S^2$ or $S^1 \times S^3$ to reduce $\chi$ to zero. These manifolds are familiar: $S^2 \times S^2$ corresponds to a (euclidean) black hole topology and $S^1 \times S^3$ to a (euclidean) wormhole. On the other hand if $\chi$ is odd we may have to use less benign manifolds like $\mathbb{CP}^2$ or $\mathbb{RP}^4$ to reduce $\chi$ to zero. The former has no spin structure and the latter is not orientable. We shall see that this can give rise to problems.

As an example consider the case of $\Sigma = S^1 \times S^2$ the 3-dimensional wormhole beloved of Wheeler. Since
\[ \Sigma = S^1 \times S^2 = \partial(S^1 \times B^3) \] (4)
where $S^1$ has angular coordinate $\psi$ such that $0 \leq \psi \leq 2\pi$ and the 3-ball $B^3$ has polar coordinates $(t, \theta, \phi)$ with $0 \leq t \leq 1, 0 < \theta \leq \pi, 0 < \phi \leq 2\pi$. We can give $S^1 \times B^3$ the flat Riemannian metric
\[ dS^2_R = d\psi^2 + dt^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2) \] (5)
and suitable vector field:
\[ \mathbf{V} = b(t) \frac{\partial}{\partial \psi} + a(t) \frac{\partial}{\partial t} \] (6)
where \(a^2 + b^2 = 1\) and \(a\) passes smoothly and monotonically from 0 to 1 as \(t\) runs from 1 to 1, with vanishing derivative at \(t = 0\). The resulting Lorentzian metric:

\[
ds_L^2 = d\psi^2 + dt^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2) - 2(bd\psi + a \, dt)^2
\]

is non-singular but in general geodesically incomplete. If \(t_c\) is such that \(a(t_c) = \frac{1}{\sqrt{2}}\) then \(t = t_c\) is a Cauchy horizon and for \(0 < t < t_c\) the spacetime contains closed timelike curves. Nevertheless we can claim from this example that the creation of a worm hole from nothing via a time orientable spacetime is certainly an allowed process at the purely kinematic level.

I did not specify earlier which components of \(\partial M\) lay in the past and which in the future, if \(M\) is time orientable. This is because of my next example which shows how to turn round the direction of time. Let \(M = \Sigma \times I\) where the 3-manifold \(\Sigma\) has metric \(g_{ij}(x^k), \, k = 1, 2, 3\) and the interval \(I\) has coordinate \(t, -1 \leq t \leq +1\). Choose as Riemannian metric on \(M\) the product metric

\[
ds_R^2 = g_{ij}(x)dx^idx^j + dt^2
\]

and if \(U(x^i)\) is a unit vector field on \(\Sigma\) (which always exists in 3-dimensions), choose for the vector field \(V\):

\[
V = b(t)U + a(t)\frac{\partial}{\partial t}
\]

where \(a(t)\) is the same function as used earlier but extended to \(-1 \leq t \leq 0\) by the requirement that \(a(t) = -a(-t)\), i.e. it be an odd function of \(t\). Thus \(V\) is outgoing on both components of the boundary of \(\partial M = (\Sigma \times 1) \cup (\Sigma \times -1)\). Again \(M\) has Cauchy horizons and closed timelike curves, and is incomplete in general. Nevertheless it shows that we need not specify the direction of time on any component of the boundary of a Lorentz cobordism because by attaching a suitable copy of this product manifold to that component we can change the future to the past or vice-versa.

We can also claim that pure kinematically the birth of twin universes from nothing is possible in the Lorentzian picture.

The traditional problem with these Lorentz-cobordisms, which will in general be geodesically incomplete, is encapsulated in the well known theorem of Geroch [1] to the effect that either
(1) there is no global time orientation or
(2) they contain closed timelike curves.

We have seen examples of (2), to see an example of (1) consider \( \Sigma = S^3 \) with \( M = \mathbb{RP}^4 - \{ pt \} \sim S^3 \times \mathbb{R} \). The metric could be of time-symmetric \( F - L - R - W \) form:

\[
ds_L^2 = -dt^2 + R^2(t)(d\psi^2 + \sin^2 \psi(d\theta^2 + \sin^2 \theta \phi^2))
\]

with \((t, \psi, \theta, \phi)\) identified with \((-t, \pi - \psi, \pi - \theta, \phi + \pi)\) and where \(R(t) = R(-t)\). If we restrict \(t\) to \((-1 \leq t \leq +1)\) we have just one boundary by virtue of the identification. Since the identification involution, call it \(J\), reverses the sense of time the quotient is not time-orientable. However the quotient does not contain any closed timelike curves since the 2-fold covering manifold does not contain any. Note that by a closed timelike curve we mean a closed curve whose tangent vector always points in the same half of the light cone. Thus if the worldline suffers a jump in velocity at some point the two tangent vectors at that point must lie in the same half of the light cone at that point. In fact it can happen in our example that two points identified under the involution \(J\) are timelike separated on the 2-fold covering manifold. However because of the reversal of time orientation these give a closed curve which sets of from some point into the future and returns from the future, i.e. in the same half of the light-cone. If \(R(t) = (3/\Lambda)^{1/2} \cosh((\Lambda/3)^{1/2}t)\) we obtain deSitter Spacetime and we are talking about the so-called “elliptic interpretation”. The involution \(J\) is then just the antipodal map. In De-Sitter spacetime a point and its antipode are never timelike separated, so we don’t have pathological closed curves of the sort described above in this case. Thus we can claim that the elliptic interpretation corresponds to the birth from nothing of a single \(S^3\) universe.

The lack of time-orientability leads to difficulties with introducing spinors on the Lorentzian spacetime and also with quantizing fields on the spacetime. One way of expressing this latter difficulty is to say that one is lead to consider real quantum mechanics. More geometrically the time-reversing involution \(J\) induces an anti-symplectic involution on the phase space of classical fields, i.e. on the space of Cauchy data. Thus attempts to quantize using conventional ideas are stymied. These difficulties are detailed in [2].

The question then arises can one find a time-orientable Lorentz-cobordism for a single \(S^3\) universe. This is possible, by removing a point from \(\mathbb{CP}^2 \# S^1 \times S^3\) for
example. However this cobordism is not a spin manifold. In fact this is a general problem: \textit{No Lorentz-cobordism for }$S^3\text{ can admit an }SL(2, \mathbb{C})\text{ spinor structure.}$

The \textbf{proof} is simple. If $M$ admits an $SL(2, \mathbb{C})$ spinor structure it must also admit a Spin (4) structure - i.e. be a spin manifold in the conventional sense, with vanishing second Stiefel-Whitney class $w_2 \in H^2(M; \mathbb{Z})$. We can fill in the boundary $\partial M = S^3$ with a 4-ball $B^4$ to get a closed 4-manifold $\tilde{M}$. The spin(4) structure extends to $\tilde{M}$, which moreover has Euler characteristic $\chi(\tilde{M}) = 1$. We have thus constructed a closed spin 4-manifold with odd Euler characteristic. However:

\textbf{Lemma:} Every closed spin 4-manifold has even Euler characteristic.

Thus we obtain a contradiction. To prove the lemma note that:

$$\chi(\tilde{M}) = 2 - 2b_1 + b_2^+ + b_2^-$$

where $b_1$ is the first Betti number of $\tilde{M}$ and $b_2^+$, $b_2^-$ the dimension of the spaces of harmonic 2 forms which are self-dual, respectively antiself-dual. Using results of Hirzebruch, Atiyah and Singer one has that the index of the Dirac operator $D$ on $\tilde{M}$ with respect to some (and hence every) Riemannian metric on $\tilde{M}$ is given by

$$\text{index } D = (b_2^+ - b_2^-) / 8$$

On a closed 4-manifold index $(D)$ is even and thus from $(2.11)$ we see that $\chi$ must be even.

Extending the argument in an obvious way we obtain a new \textit{selection rule for }$S^3\text{ universes.}$

The number of $S^3$ universes in any Lorentz cobordism admitting an $SL(2, \mathbb{C})$ spinor structure is conserved modulo 2. Another way to express this is to say that there is a $\mathbb{Z}_2$ \textbf{topological invariant} $u(\Sigma)$ for closed orientable 3-manifolds such that

$$u(\Sigma) = 0$$

if $\Sigma$ admits a spin-Lorentz-cobordism and

$$u(\Sigma) = 1$$

otherwise.

In fact one may prove that $u(\Sigma)$ behaves exactly as one expects of a conserved quantum number under disjoint union

$$u(\Sigma_1 \cup \Sigma_2) = u(\Sigma_1) + u(\Sigma_2) \mod 2$$
while under connected sum it satisfies

\[ u(\Sigma_1 \# \Sigma_2) = u(\Sigma_1) + u(\Sigma_2) + 1 \mod 2 \quad (16) \]

From the discussion above \( u(S^3) = 1 \) and our previous example for \( S^1 \times S^3 \) (which admits spin for both choices of spin structure on the boundary) shows that \( u(S^1 \times S^2) = 0 \). This shows that \textit{single wormholes cannot be created in the laboratory - they must be created in pairs}. In other words if we start with no wormholes, i.e. \( \Sigma_{\text{initial}} = S^3 \) and end with the connected sum of \( k \ S^1 \times S^2 \)'s, \( \Sigma_{\text{final}} = \#_k S^1 \times S^2 \), then since using these rules:

\[ u(\Sigma_{\text{initial}} \cup \Sigma_{\text{final}}) = 1 + (k - 1) \mod 2 \]

we must have \( k = 0 \mod 2 \) if our “laboratory” is to allow \( SL(2, \mathbb{C}) \) spinors.

One may express \( u(\Sigma) \) in terms of the \( \mathbb{Z}_2 \) homology groups of the boundary. It turns out to be the mod 2 Kervaire semi-characteristic, i.e.

\[ u(\Sigma) = \dim_{\mathbb{Z}_2}(H_0(\Sigma; \mathbb{Z}_2) \oplus H_1(\Sigma; \mathbb{Z}_2)) \mod 2 \]

where \( H_i(\Sigma; \mathbb{Z}_2) \) is the i’th cohomology group of \( \Sigma \) with \( \mathbb{Z}_2 \) coefficients. Thus for \( \mathbb{R}P^3 \cong SO(3) \) we have \( u(\mathbb{R}P^3) = 0 \) and this selection rule does not prevent such universes being born from nothing. More details will be given in a forthcoming paper with Stephen Hawking.

Before concluding this section I think it might be worth mentioning some results of Thurston on foliations which may be applied to the present case. Firstly the vanishing of the Euler characteristic of a closed manifold is the necessary and sufficient condition for the existence of a smooth foliation by co-dimension one leaves. If the manifold \( M \) has a boundary, which is more direct interest to us then Thurston proves that it admits a smooth co-dimension one foliation tangent to the boundary \( \partial M \) if and only if it admits a line field transverse to the boundary and either

1) \( M = \Sigma \times I \) or \( M = \Sigma \times_{\mathbb{Z}_2} I \) where \( \mathbb{Z}_2 \) acts as the involution \( J \) in our previous examples

or

2) For each boundary component \( \partial M_i \), \( H^i(\partial M_i; \mathbb{R}) \) is non-trivial.

Condition 2) may be paraphrased as saying that that there must be a wormhole of some sort on each boundary component. Thus, according to Thurston’s theorem the
cobordism for $S^1 \times S^2$ used above admits such a foliation. Superficially one might suppose that one could use these foliations to construct a Hamiltonian description of topology change in the Lorentzian regime but it must be borne in mind that the leaves need not all be diffeomorphic, and even if they are there will not exist a global time function. Nevertheless it would be interesting to investigate such foliations in more detail.

2. Topology Change in Quantum Gravity

The results described above for spin-Lorentz-cobordisms should be contrasted with the by now familiar picture of the creation of the universe from nothing obtained using ideas from quantum tunneling. Hartle and I [3] have formulated a general framework into which almost all known examples seem to fit, with the exception that our compactness assumption may need to be relaxed. The idea is to seek solutions of the relevant classical field equations (i.e. saddle points in the functional integral) associated with a (closed) 3-manifold $\Sigma$ with 3-metric $h_{ij}$ and

1) a (compact) Riemannian 4-manifold $M_R$ with boundary $\partial M_R = \Sigma$.

2) a Lorentzian 4-manifold $M_L$ for which $\Sigma$ is a (partial) Cauchy surface such that

3) $\Sigma$ is totally geodesic with respect to both $M_R$ and $M_L$, that is its second fundamental form $K_{ij} = 0$.

The motivation for this idea comes from the instanton or ”bounce” description of

1) False Vacuum Decay

or

2) Pair-creation by strong electric fields (Schwinger Process)

In both cases the stationary point of the classical Euclidean action has a symmetry under the reversal of imaginary time. The behaviour after tunneling is described by a solution of the classical equations in real time whose initial values coincide with the values of the solution in imaginary time at imaginary time zero. By joining the imaginary time trajectory for negative values of imaginary time to the real time trajectory for positive values of real time one obtains a complex classical path whose action will be complex, the imaginary part coming from the imaginary time portion, the real part coming from the real time portion. There is
no contribution from the mid-point at time zero because the velocities there vanish.

In General Relativity initial values correspond to giving the 3-metric on a spacelike surface \( \Sigma \) and its second fundamental form \( K_{ij} \) (i.e. its time derivative). If the data admit a moment of time symmetry at \( \Sigma \) the second fundamental form must vanish. In this case the boundary data may serve either as Cauchy data for the hyperbolic Lorentzian Einstein equations or as Dirichlet data for the elliptic Riemannian Einstein equations.

The basic gravitational example is for \( \Sigma = S^3 \). \( M_R \) is half of \( S^4 \) ie.

\[
d s^2_R = d\tau^2 + \left( \frac{3}{\Lambda} \right) \cos^2(\tau \sqrt{\frac{\Lambda}{3}}) \, d\Omega^2_3 \tag{17}
\]

with \(-\frac{\pi}{2} \leq \tau \sqrt{\frac{\Lambda}{3}} \leq 0\), and \( M_L \) is half of deSitter spacetime, i.e.

\[
d s^2_L = -dt^2 + \left( \frac{3}{\Lambda} \right) \cosh^2(t \sqrt{\frac{\Lambda}{3}}) \, d\Omega^2_3 \tag{18}
\]

with \( t \geq 0 \). Note that while the 4-metric is degenerate on the signature changing surface \( \Sigma \), given in our example by \( \tau = 0 = t \), considered as a curve of 3-metrics the vanishing of the second fundamental form on \( \Sigma \) means that there are no contributions to the variation of the action functional at \( \Sigma \) because the curve suffers no jump in slope. Put another way there is no distributional contribution to the Einstein tensor at \( \Sigma \). Thus real tunneling geometries may be considered as true critical points of the classical action functional. The compactness of \( M_R \) is convenient, for example when studying the No Boundary proposal of Hartle and Hawking [4] but not, I feel, an essential part of the idea. There are interesting examples involving black holes for which \( \Sigma \) is not closed and \( M_R \) is not compact as we shall see. As an aside: let me remark that as models of signature change rather than of tunneling one might also consider replacing \( M_R \) by a manifold with an ultra-hyperbolic metric of signature \(+ + --\), in which case \( \Sigma \) would have a timelike metric. There is, I believe, an argument for calling metrics with as many positive directions as negative directions “Kleinian”. Much of the present theory goes through if \( R \) is replaced by \( K \) under the understanding that \( K \) stands for Kleinian in that sense. Because Kleinian 4-metrics admit the idea of self-duality and because they arise in some (rather exotic) string theory this “last frontier” seems ripe for colonization.
Some pioneering attempts will be contained in [5]. Let me return however to the case of Riemannian tunneling.

Associated with the Riemannian manifold \( M_R \) and the Lorentzian manifold \( M_L \) are their doubles \( 2M_R \) and \( 2M_L \) respectively which are obtained by joining two copies of \( M_R \) on \( M_L \) respectively across their common boundary \( \Sigma \). Thus \( 2M_L \) is a spacetime admitting a “moment of time symmetry” while \( 2M_R \) is a compact Riemannian manifold admitting a isometric involution \( \theta \) (or reflection map) which interchanges the 2 halves and fixes \( \Sigma \), i.e.

\[
2M_R = M_R^+ \cup M_R^- \tag{18}
\]

with

\[
\theta M_R^\pm = M_R^\mp, \tag{19}
\]

and thus

\[
\theta \Sigma = \Sigma \tag{20}
\]

\[
\theta^2 = id. \tag{21}
\]

In fact \( 2M_R \) and \( 2M_L \) may be regarded as two real slices of the complexified manifold \( M_c \) of complex dimension 4 which carries a symmetric complex covariant tensor field of type \((2, 0)\) which restricts on \( 2M_R \) and \( 2M_L \) to the real metrics \( g^R_{\alpha\beta} \) and \( g^L_{\alpha\beta} \). The real slices \( 2M_R \) and \( 2M_L \) are stabilized by two anti-holomorphic involutions acting on \( M_c \), \( J_R \) and \( J_L \) respectively. Thus

\[
J_R(2M_R) = 2M_R \tag{22}
\]

\[
J_L(2M_L) = 2M_L \tag{23}
\]

Restricted to \( M_R \) \( J_L \) coincides with our previous reflection map \( \Sigma \) and \( J_R \), restricted to \( M_L \) corresponds to time-reversal.

As an example one may consider de Sitter spacetime as a complex quadric in \( \mathbb{C}^5 \). This is described in the article with Hartle. Here I will give a different example: the Schwarzschild solution. This may be complexified as an algebraic variety in \( \mathbb{C}^7 \) with complex coordinates \( Z^\alpha, \alpha = 1, \ldots 7 \) [6]. In terms of local Schwarzschild coordinates (which cover only a portion of the variety)

\[
Z^1 = r \sin \theta \cos \phi \tag{24}
\]
The algebraic variety is given by the 3-equations

\[
(Z^6)^2 - (Z^7)^2 + \frac{4}{3}(Z^5)^2 = 16M^2
\]  
(31)

\[
((Z^1)^2 + (Z^2)^2 + (Z^3)^2)(Z^5)^4 = 576M^6
\]  
(32)

\[
\sqrt{3}Z^4Z^5 + (Z^5)^2 = 24M^2
\]  
(31)

The Lorentzian section $2M_L$ is stabilized by

\[
J_L : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7)
\]  
(32)

and the Riemannian section $2M_L$ by

\[
J_R : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, -Z^7)
\]  
(33)

These intersect on the familiar 2-sheeted “Einstein Rosen bridge” $\Sigma$ with topology $S^2 \times \mathbb{R}$ given by $Z^7 = 0$. In terms of the real time coordinate $t$, $\Sigma$ is given by $t = 0$. In terms of the imaginary time coordinate $\tau = it$, which is periodic with period $8\pi M$, $\Sigma$ is given by $\tau = 0$ and $\tau = 4\pi M$.

In a tunneling context the Schwarzschild solution has been applied to the instability of hot flat space [7]. Of course neither $M_R$ nor $\Sigma$ is compact in the present case but physically this is not unreasonable since in practice one would be considering only a large but finite volume of spacetime and intending to compute a tunneling probability per unit time per unit volume, so at some stage in the calculation one
would have to take a suitable ratio to get a finite answer but this is a technicality which I won’t dwell on here.

Having isolated the essential features of the geometries of interest for tunneling one can proceed to investigate their properties in a systematic way. Some results in this direction are given in the paper with Hartle. For the moment I will restrict attention to just two results. One is what we called the “Unique Conception Theorem”. Suppose $M_R$ is compact and the Ricci tensor of the Riemannian metric $g_{\alpha\beta}^R$, is, considered as a quadratic form on tangent vectors, non-negative. Then the boundary $\Sigma$ must be connected. That is one cannot find classical solutions representing the birth from nothing of more than one universe if the positivity restriction on the Ricci curvature holds. It should be pointed out that there are perfectly reasonable Lagrangians for which the Ricci tensor does not satisfy these restrictions. In fact the mathematical fact behind the theorem is the same fact that allows one to prove that there are no 4-dimensional Riemannian solutions representing “wormholes” subject to an appropriate restriction on the Ricci curvature [8]. As is well known by considering axions etc such wormhole solutions can be found and similarly one could find solutions which nucleate the birth of more than one universe by considering axion fields.

The second result is on the topology of connected components of $\Sigma$. Again under suitable Positivity assumptions, the most important being a positive rather than negative cosmological constant, it follows that the Ricci scalar of $\Sigma$ must be positive. From this one may, following the work of Schoen, Yau and others deduce restrictions on the topology of $\Sigma$. The details are given in the paper with Hartle. Much of the motivation of that paper derived from quantum cosmology. However the formalism also covers the case of more localized topological fluctuations, as well as the ideas behind false vacuum decay. In the next section I shall describe some more examples relating topology change to black hole theory. Before doing so I want to mention a new result which is not contained in the paper with Hartle concerning the topology of homogeneous real tunneling metrics, i.e. those for which the double $2M_R$ admits the transitive isometric action of some Lie group.

It seems plausible that the solutions of any set of field actions with least action are homogeneous. By Bishop’s theorem [9] this is certainly true for Einstein metrics with $\Lambda > 0$. According to the classification by Ishihara [10] of 4-dimensional homogeneous Riemannian manifolds the universal covering space $2\tilde{M}_R$ of $2M_R$ must be
homeomorphic to one of \( \mathbb{R}^4, S^4, \mathbb{CP}^2, S^2 \times S^2, \mathbb{R} \times S^3 \) and \( S^2 \times \mathbb{R}^2 \). In fact the existence of the reflection map \( \theta \) rules out \( \mathbb{CP}^2 \) (since it admits no orientation reversing diffeomorphism). Thus, quite independently of any field equations, the topological possibilities for homogeneous real tunneling geometries are rather limited.

As I mentioned above, for the Einstein equations with \( \Lambda > 0 \), the lowest action solution is \( S^4 \) with its standard homogeneous metric.

I shall now give three striking examples of real tunneling geometries which support the idea that topology change, involving black holes, does occur at the semi-classical level. The three examples are derived from:

1. The Nariai - \( S^2 \times S^2 \), instanton
2. The Mellor-Moss instanton
3. The Melvin-Ernst instanton.

The word “instanton” is synonymous with “complete non-singular Riemannian solutions of the classical field equations”, and refers in the present case to the double \( 2M_R \).

1. **The Nariai Instanton** is just the standard Einstein metric in \( S^2 \times S^2 \) with its product metric. The reflection map \( \theta \) is just reflection in a meridian of the first \( S^2 \) factor. Thus \( \Sigma \equiv S^1 \times S^2 \), the \( S^1 \) factor being made up of a pair of meridians. The Nariai metric may thus be thought of as nucleating the birth of an \( S^1 \times S^2 \) universe. In view, however, of the horizons present in the Lorentzian section, i.e. 2-dimensional De Sitter spacetime \( \times S^2 \) I would also like to view it as the creation of a pair of black holes from a background cosmological field. Something like this interpretation has been given already by Perry and Perry and Ginsparg [11]. The interpretation gains support from comparison with the Mellor-Moss case to be described later. The basic idea is that a positive cosmological term causes pairs of particles to separate because of the mutual repulsion they experience. This repulsion becomes larger the further the particles are apart. Thus one expects a positive cosmological term to give rise to a pair creation and of course for conventional point particles this effect has been well understood for some time. Given that understanding it is reasonable to extend the idea to black holes. Now the Schwarzschild-De Sitter spacetime can be interpreted as containing two black holes in a background De Sitter universe. To do so one must identify points in the Penrose diagram so that each surface of constant time has one minimal 2-sphere and one maximal 2-sphere. The cosmological horizons intersect on the maximal 2-sphere.
and the black hole horizons on the minimal 2-sphere. The resulting spacetime has two static regions, each bounded by a cosmological horizon and a black hole horizon. One may regard the Nariai metric as a limiting case of the Schwarzschild-De Sitter spacetime as $9M^2\Lambda \geq 0$. The black hole horizons are as large as they can be and the cosmological horizons as small as they can be. In this limit the metric becomes a product metric and the two types of horizon becomes equivalent.

(2) **The Mellor-Moss Instanton** This is a particular member of the Reissner-Nordstrom de Sitter family of solutions of the Einstein-Maxwell equations with positive cosmological constant in which the mass parameter $M$ and the charge parameter $g$ (assumed purely magnetic for simplicity) are related by [12]

$$M = \frac{g}{\sqrt{4\pi G}}$$

(electromagnetic units are “rationalized”). The surface gravities of the cosmological and black hole horizons coincide in this limit and become:

$$\kappa = \sqrt{\frac{\Lambda}{3}} (1 - 4M \sqrt{\frac{\Lambda}{3}})^{\frac{1}{2}}$$

Thus the temperature of the two horizons is not zero, as it would be if $\Lambda = 0$, but it is less than the temperature of De Sitter spacetime. The Riemannian metric has an imaginary time coordinate $\tau = it$ which is periodic with period $\frac{2\pi}{\kappa}$ and thus $2M_R$ is topologically $S^2 \times S^2$ but with a warped product metric. The hypersurface $\Sigma$ has topology $S^1 \times S^2$ as before and corresponds to the meridians $\tau = 0$ and $\tau = \pi/\kappa$. On analytic continuation to $M_L$ these two meridians are associated with two static regions each bounded by a black hole horizon and a cosmological horizon. A common feature of the Mellor-Moss, the $S^2 \times S^2$ and the Schwarzschild cases is the necessity to consider $\Sigma$ as containing two “meridians”, $\tau = 0$ and $\tau = \pi/\kappa$.

The difference in action between the Mellor-Moss instanton and that of De Sitter spacetime (for the doubles $2M_R$ in both cases) is

$$\frac{M}{T_{DeSitter}}$$

where $T_{DeSitter} = \frac{1}{2\pi} \sqrt{\frac{4}{3}}$. Thus the probability of tunneling is proportional to:

$$\text{Probability} \propto \exp^{-\frac{M}{T_{DeSitter}}}$$
This seems an eminently reasonable answer. Note that the probability is proportional to \( \exp(-\text{Action}(2M_R)) \) because it is the (modulus)\(^2\) of the amplitude which is proportional to \( \exp(-\text{Action}(M_R)) \).

The special properties of extreme Reissner-Nordstrom black holes are widely recognized and have been much studied. In many ways they behave like solitons in more familiar flat space field theories. It seems very natural that they should be created in pairs by strong external fields, including cosmological fields. Heuristic phase space arguments indicate that the most likely process is that which entails the creation of particles with the least energy consistent with the conservation of any relevant charges. In the present case the relevant charge is electromagnetic. It is known that the extreme holes satisfy a Bogomolnyi bound \([13]\) and thus a condition like (4.1) is no surprise, though, strictly speaking, it does not imply that the created black holes have zero temperature. My next example adds further weight to this interpretation.

3. **The Melvin-Ernst Instanton** This is a solution of the Einstein-Maxwell equations representing a pair of charged black holes in a background Melvin type electromagnetic field. For convenience, as with the Mellor-Moss case we shall restrict attention to the purely magnetic case. The electrically charged case will also go through with minor modifications. In the absence of the background Melvin type field the solution is a special case of the charged \(C\)-metrics discussed many years ago by Kinnersley and Walker \([14]\). It is known that these metrics are singular - they contain “nodal” singularities. This is to be expected. If they did not one would be able to construct real tunneling geometries and hence probability amplitudes for the decay of flat Minkowski spacetime to a pair of black holes. This contradicts all our intuition, largely based on the Positive Mass theorem concerning the stability of Minkowski spacetime (a proof of the positive mass theorem in the presence of charged black holes may be found in \([13]\)). Examination of the \(C\)-metrics does indeed reveal that they have vanishing ADM mass. The \(C\)-metrics contain both a black hole event horizon and an acceleration or Rindler horizon. In general their surface gravities and hence their temperatures differ. Thus even if they were free of nodal singularities analytic continuation to give a non-singular Riemannian metric which is periodic in imaginary time with a single period is problematic.
In more detail the charged C-metric has the form:

\[ ds^2 = \frac{1}{A^2(x+y)^2} \left[ \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\alpha^2 - F(y)dt^2 \right] \] (38)

where

\[ G(x) = -F(-x) \] (39)

\[ = 1 - x^2 - 2GMAx^3 - G(g^2/4\pi)A^2x^4 \] (40)

\[ = -G(g^2/4\pi)A^2(x-x_1)(x-x_2)(x-x_3)(x-x_4) \] (41)

I have labelled the 4 real roots of \( G(x) \), \( x_1 \), \( x_2 \), \( x_3 \), \( x_4 \) in ascending magnitude (\( x_1 \), \( x_2 \) and \( x_3 \) are negative and \( x_4 \) is positive).

The range of the “radial” variable \( y \) is

\[ -x_3 \leq y \leq -x_2 \] (42)

with \( y = |x_3| \) being an acceleration horizon and \( y = |x_2| \) a black hole horizon. The range of the “angular” variable \( x \) is \( x_3 \leq \alpha \leq x_4 \). The 2-surfaces \( x = x_3 \) and \( x = x_4 \) are axes of symmetry for the angular Killing vector \( \frac{\partial}{\partial \alpha} \).

In order to understand what the coordinates used it is helpful to consider the case when the the mass parameter \( m \) vanishes. Then the metric is flat and one may transform to flat inertial coordinates using the formulae:

\[ X^1 \pm iX^2 = \frac{(1-x^2)^{\frac{1}{2}}}{A(x+y)} \exp(\pm i\alpha) \] (43)

\[ X^3 \pm X^0 = \frac{(y^2-1)^{\frac{1}{2}}}{A(x+y)} \exp(\pm t) \] (44)

Evidently the coordinate singularity at \( x = \pm 1 \) is a rotation axis while the coordinate singularity at \( y = \pm 1 \) corresponds to a pair of intersecting null hyperplanes forming the past and future event horizons for a family of uniformly accelerating worldlines. The points for which \( x + y = 0 \) correspond to infinity. A similar interpretation may be given in the case that \( M \neq 0 \) but there is in addition a Black Hole horizon. A detailed description was given by Kinnersley and Walker [14].
If $0 \leq \alpha \leq \Delta \alpha$ there will be angular deficits:

$$\frac{\delta_4}{2\pi} = \frac{\Delta \alpha - \Delta \alpha_4}{\Delta \alpha_4} ; \quad \frac{\delta_3}{2\pi} = \frac{\Delta \alpha - \Delta \alpha_3}{\Delta \alpha_3}$$

(45)

where

$$\Delta \alpha_4 = \frac{4\pi}{|G'(x_4)|}; \quad \Delta \alpha_3 = \frac{4\pi}{|G'(x_3)|}$$

(46)

Since (unless $MA = 0$) $\Delta \alpha_4 \neq \Delta \alpha_3$ it is not possible to eliminate both of these by choosing $\Delta \alpha$. One can eliminate $\delta_3$ in which case the black hole is pulled along by a string, or $\delta_4$ in which case it is pushed along by a rod.

However it is a striking fact [15,16,17] that if condition (4.1) holds then the two surface gravities become equal. This is not in itself a sufficient condition to provide a regular instanton because the problem of the nodal singularity remains. However some years ago Ernst [18] showed, using exact solution generating techniques, how this nodal singularity could be eliminated by appending a suitable electromagnetic field whose value is determined physically by the

condition that the acceleration induced by the electromagnetic field equals the force needed to acceleration a massive black hole. If no black hole is present the relevant solution is “Melvin’s magnetic universe”. If a black hole is present the solution is asymptotic to Melvin’s solution, and the strength of the applied field is unconstrained. If an accelerating charged black hole is present the strength of the appended field is determined in terms of the mass, charge and acceleration parameters of the solution.

The Melvin solution represents an infinitely long straight self-gravitating Faraday flux tube in equilibrium, the gravitational attraction being in equipoise with the transverse magnetic pressure. The metric is:

$$ds^2 = (1 + \pi GB^2 \rho^2)^2(-dt^2 + dz^2 + d\rho^2) + \rho^2 d\phi^2(1 + \pi GB^2 \rho^2)^{-2}$$

(47)

The magnetic field is given by:

$$F = \frac{B \rho d\rho \wedge d\phi}{(1 + \pi GB^2 \rho^2)^2}$$

(48)

The Melvin solution possesses a degree of uniqueness. For example Hiscock has shown the
**Theorem:** The only axisymmetric, static solution of the Einstein-Maxwell field equations without an horizon which is asymptotically Melvin is in fact the Melvin Solution.

In fact Hiscock also allows for a neutral or electrically charged black hole as well.

In fact one can show that the only translationally invariant static solution of the Einstein-Maxwell field equations without horizon which is asymptotically Melvin is in fact the Melvin solution.

**Proof:** assume the metric is static and has reflection invariance with respect to the $z-$direction. These two assumptions may easily be justified. The metric takes the form

$$ds^2 = -V^2 dt^2 + Y^2 dz^2 + g_{AB} dx^A dx^B$$

with $A = 1, 2$. The field equations are:

$$\nabla^A (VY \nabla_A \ln(V/Y)) = VY 8\pi G \left( T_{\hat{z}\hat{z}} + T_{\hat{0}\hat{0}} \right)$$

$$\nabla^A (VY \nabla_A \ln(VY)) = VY 8\pi GT_A^A$$

$$V^{-1} \nabla_A \nabla_B V + Y^{-1} \nabla_A \nabla_B Y = Kg_{AB} - 8\pi (T_{AB} - \frac{1}{2} g_{AB} (T_A^A + T_{\hat{z}\hat{z}} + T_{\hat{0}\hat{0}}))$$

where $K$ is the Gauss curvature of the 2-metric $g_{AB}$. The electromagnetic field is assumed to be of the form:

$$F = \frac{1}{2} F_{AB} dx^A dx^B.$$ 

It follows that $T_{\hat{0}\hat{0}} + T_{\hat{z}\hat{z}} = 0$ and hence:

$$\nabla^A (VY \nabla_A (V/Y)) = 0.$$  

Now $V/Y$ tends to one at infinity (asymptotic boost invariance) and so we may invoke the Maximum Principle to show that $V = Y$ everywhere. Thus the metric must be boost invariant. It now follows that

$$\nabla_A \nabla_B V = fg_{AB}$$

for some scalar $f$. Thus

$$K^A = \epsilon^{AB} \nabla_B V$$
is a Killing vector field of the 2-metric $g_{AB}$ and since $K^A \partial_A V = 0$ it is also a Killing vector field of the entire 4-metric. It is not difficult to see that this Killing vector field corresponds to rotational symmetry of the solution.

Having established the credentials of the Melvin solution as uniquely suitable model of a static magnetic field in general relativity we turn to looking for instanton solutions representing the creation of a black hole monopole anti-monopole pair. If there were no external magnetic field the obvious candidate instantons would be the magnetically charged C-metric for which

$$G(x) = 1 - x^2 - 2GMAx^3 - G(g^2/4\pi)A^2x^4.$$  

However this has nodal singularities. In fact since the metric is boost invariant it has zero ADM mass and thus it cannot be regular by the positive mass theorem generalised to include apparent horizons. However it was pointed out by Ernst [18] that the nodal singularity may be eliminated if one appends a suitable magnetic field. The resulting metric is of the same form as (38) but the first three terms are multiplied by and the last term divided by the factor:

$$(1 + GBgx/2)^4.$$  

If $M = 0 = g = A$ we get the Melvin solution but the limit must be taken carefully. The nodal singularity may be eliminated if $B$ is chosen so that

$$G'(x_3)/(1 + GBgx_3/2)^4 + G'(x_4)/(1 + Ggx_4/2)^4 = 0.$$  

This equation may be regarded as an equation for $B$ the magnetic field necessary to provide the force to accelerate the magnetically charged black hole. It is difficult to find an explicit solution in terms of $g$, $m$ and $A$ except when $GMA$ is small in which case one finds the physically sensible result:

$$gB \approx MA.$$  

In order to obtain an instanton which is regular on the Riemannian section obtained by allowing the time coordinate $t$ to be pure imaginary it is necessary that the $\tau = it$ is periodic with period given by the surface gravity. This leads to the condition that

$$G'(x_2) + G'(x_3) = 0.$$
It appears that the only way to satisfy this condition is to set:

\[ m = \frac{|g|}{\sqrt{4\pi G}} \]

Note that this equation implies that the horizons have a non-vanishing common surface gravity and hence temperature as in the Mellor-Moss case. It is not difficult to see that the topology of the Riemann section is \( S^2 \times S^2 \) with a point (corresponding to \( x + y = 0 \)) removed. In fact topologically one can obtain this manifold from \( \mathbb{R}^4 \), which is the topology of the Melvin solution, by surgery along an \( S^1 \). That is by cutting out a neighbourhood of a circle which has topology \( D^3 \times S^1 \) with boundary \( S^2 \times S^1 \) and replacing by \( S^2 \times D^2 \) which has the same boundary. This surgery is also what is needed to convert \( \mathbb{R}^3 \times S^1 \) to \( \mathbb{R}^2 \times S^2 \) i.e. to convert a manifold with the topology of "Hot Flat Space" to that with the topology of the Riemannian section of the Schwarzschild solution. This apparent connection between surgery along links and virtual black holes is an intriguing one and deserves to be investigated in more detail. The existence of the Melvi-Ernst instanton would seem to be rather important. It seems to imply for example that it would be inconsistent not to consider the effects of black hole monopoles since given strong enough magnetic fields they will be spontaneously created. Once they are created they should evolve by thermal evaporation to the extreme zero temperature soliton state. Another reason why I believe that this process is so important is that it seems to show that while one may have one’s doubts about the effects of wormholes because of the absence of suitable solutions of the classical equations of motion with positive definite signature, the solutions described here do indicate that some sort of topological fluctuations in the structure of spacetime must be taken into account in a satisfactory theory of gravity coupled to Maxwell or Yang-Mills theory.

I will now sketch how real tunneling geometries, which effectively exhaust the class of metrics which allow a Wick rotation, are especially well adapted to implementing the idea of Reflection Positivity used in flat space Euclidean Quantum Field theory. These ideas are not new - they resemble some ideas of Uhlmann [19] and I reviewed them briefly in my talk at the Jena GRG conference [20]. however since that time virtually nothing has been done (except [21]) on this. The time now seems ripe for developing them and I understand from Bernard Kay that he and Bob Wald also have some ideas in this direction.

The main point made by Uhlmann is that the geometric data needed for reflection positivity is a Riemannian manifold together with an isometric involution
\(\theta\) having exactly the properties that I listed earlier, i.e. such that the equations (18)-(21) hold. For Euclidean, i.e. flatspace, Quantum Field theory of course the manifold is 4-dimensional Euclidean space and \(\theta\) is reflection in a hyperplane of constant imaginary time. Given

this data one may construct, without even passing to the associated Lorentzian spacetime, the Hilbert Space of Quantum Mechanics in purely Riemannian geometric language. The Riemannian manifold \(2M_R\) may admit other isometries in addition to \(\theta\). If so the construction automatically builds in a degree of equivariance with respect to those isometries. The "standard" case is when \(2M_R = S^4\) with its round metric. The isometry group is \(O(5)\) and the map \(\theta\) commutes with an \(O(4)\) subgroup which stabilizes \(\Sigma\) as a set. As mentioned above, if we assume that the isometry group acts transitively on \(2M_R\) the possibilities are quite limited by Ishihara’s results.

I shall confine attention to the case of a free massive scalar field with mass \(m > 0\). We first construct the one particle Hilbert Space \(\mathcal{H}_1\). The whole Hilbert space \(\mathcal{H}\) is built up by taking the direct sum of the symmetric powers of \(\mathcal{H}_1\) under the tensor product. The \(k\)'th symmetric power is the \(k\)-particle Hilbert Space. Thus:

\[
\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_1 \otimes \mathcal{H}_1 \oplus \ldots
\]

I shall not dwell on function-analytic details so I will not specify very precisely the function spaces and their completions. What I am interested in are the basic geometric and physical ideas behind the construction. One begins by identifying \(\mathcal{H}_1\) as a vector space with \(L^2(M^+_R)\), i.e. with complex valued functions with support solely in \(M^+_R\). One may think of \(\mathcal{H}_1\) as being made up of "positive frequency functions". Recall that in flat Minkowski spacetime positive frequency functions may be characterized

as being holomorphic in the lower half complex t-plane. We have the obvious orthogonal direct sum:

\[
L^2(2M_R) = L^2(M^+_R) \oplus L^2(M^-_R)
\]

where the \(L^2\) norm is with respect to the Riemannian volume element \(\sqrt{g}d^4x\).

The involution \(\theta\) acts on functions by pullback, i.e. if \(f^+(x) \in L^2(M^+_R)\) then \(\theta^* f^+(x) = f^+(\theta^{-1}x) = f^+(\theta x) \in L^2(M^-_R)\). Note that \(\theta^*\) is a selfadjoint operator on
$\mathcal{L}^2(2M_R)$ which commutes with complex conjugation. For notational convenience I will drop the $*$ on $\theta^*$ from now on. Some other useful notation is to define the projections $\Pi_{\pm}$ onto $\mathcal{L}^2(M_R^\pm)$ and the projections $P_{\pm} = \frac{1}{2}(1 \pm \theta)$ onto even and odd functions with respect to $\theta$. Thus $\Pi_{\pm}P_{\pm}$ projects onto functions on $M_R^\pm$ satisfying Dirichlet boundary conditions on $\Sigma$, that is to say they vanish on $\Sigma$, while $\Pi_{\pm}P_+$ projects onto functions on $M_R^\pm$ satisfying Neumann boundary conditions on $\Sigma$. Although $\mathcal{L}^2(M_R^\pm)$ comes equipped with its defining Hilbert metric this does not give the correct norm for the one-particle Hilbert Space $\mathcal{H}_1$. To construct this, which we write as $\|f^+(x)\|^2$, we need to introduce an appropriate Green’s function or two-point function, $G(x,y)$, on $2M_R \times 2M_R$. For a free scalar field with mass $m$ we take the inverse of the Klein-Gordon operator $-\nabla^2_{g_R} + m^2$ which is a positive self-adjoint on $\mathcal{L}^2(2M_R)$, where $\nabla^2_{g_R}$ is the Laplacian with respect to the Riemannian metric $g_R$ and has a unique inverse $G = (-\nabla^2_{g_R} + m^2)^{-1}$. Clearly $G$ commutes with $\theta$. Two other Greens functions are of interest. They are defined on $\mathcal{L}^2(M_R^\pm)$ and satisfy Dirichlet, $G_D$, or Neumann, $G_N$, boundary conditions. Thus:

$$ G_D = (1 - \theta)G, $$

i.e.

$$ G_D = G(x,y) - G(x,\theta y), $$

and

$$ G_N = (1 + \theta)G $$

$$ = G(x,y) + G(x,\theta y). $$

We are now in a position to define $\|f^+(x)\|^2$ as

$$ \|f^+(x)\|^2 = \int_{2M_R \times 2M_R} \overline{f^+(\theta x)}G(x,y)f^+(y) $$

$$ = \int_{M_R^- \times M_R^-} \overline{\theta f^+(x)}G(x,y)f^+(y) $$

$$ + \int_{M_R^-} \overline{f^-(x)}\phi^+(x), $$

where, $f^- = \theta f^+ \in \mathcal{L}^2(M_R^\pm)$ and $\phi^+ \in \mathcal{L}^2(2M_R)$ is the potential due to the source $f^+ \in \mathcal{L}^2(M_R^\pm)$. 
To justify the notation $\|f^+(x)\|^2$ we must at least establish that the right hand side of (5.6) is indeed positive. If we had chosen an arbitrary two point function $G(x, y)$, even if it were pointwise positive such as the Gaussian function in Euclidean 4-space, this would not have been true so there is something non-trivial to be shown. The result depends on some special properties of the Klein Gordon operator. There are at least two ways to proceed. One is to follow de Angelis et al. [21] and show by means of Green’s identity and some manipulations that:

$$\int_{M^+_R \times M^+_R} \overline{f-} \phi^+ = 2 \int_{M^+_R} |\nabla \phi^+|^2 + m^2 |\phi^+|^2. \quad (56)$$

Another way, following Glimm and Jaffe [22] and used by Uhlmann [19] is to make use of the Dirichlet Principle. One re-writes (5.6) using (5.3) as

$$\|f^+(x)\|^2 = \int_{M^+_R} \overline{f^+(x)}(G(x, y) - G_D(x, y))f^+(y). \quad (57)$$

We may interpret $\|f^+(x)\|^2$ in terms of a simple 4-dimensional electrostatic model as the mutual potential energy of a charge distribution located entirely in $M^+_R$ and given by $f^+(x) \in \mathcal{L}^2(M^+_R)$ with an image charge distribution obtained by reflecting $f^+(x)$ in the ”conducting” hypersurface $\Sigma$ and taking the complex conjugate. According to (5.10) this is the difference between two terms of the form:

$$\int_{M^+_R} \overline{\phi f^+(x)} \quad (58)$$

where $\phi$ satisfies

$$(-\nabla^2_{gr} + m^2)\phi = f^+(x). \quad (59)$$

The first term in (57) corresponds to demanding as a boundary condition for (59) that the extension of $\phi$ to $2M_R$ is in $\mathcal{L}^2(2M_R)$ while the second term corresponds to imposing the Dirichlet boundary condition on $\phi$. Dirichlet’s Principle states that among all solutions $\phi$ of (59), that satisfying Dirichlet conditions has the least value for the integral (58). To prove this fact we follow Glimm and Jaffe and compare the positive and commuting operators $G^{-1}$ and $G_D^{-1}$. One may regard $G_D^{-1}$ as the restriction of $G_{-1}$ to functions in $\mathcal{L}^2(M^+_R)$ which in addition satisfy the condition that they vanish on $\Sigma$. It follows that as operators on $\mathcal{L}^2(M^+_R)$

$$G_D \leq G$$
and hence that \(\|f^+(x)\|^2\) is indeed positive. This second procedure is rather less direct than the one given previously. It does however have the advantage that it generalizes to other situations. It is can be used to obtain a proof in the case of flat Euclidean 4-space that the generator of imaginary time translations is a positive operator which serves as the physical Hamiltonian.

In the present case the Riemannian manifold \(2M_R\) cannot be expected to admit a translation Killing vector but it may well have some continous isometries belonging to some group \(K_R = \text{Isom}_0(2M_R, g_R)\) the identity component of the isometry group. The group \(K\) will act on \(L^2(2M_R)\) by pullback and via this the analytic continuation \(K_L\) should act on the physical Hilbert space \(\mathcal{H}\). In particular the physical vacuum should be invariant under \(K_L\). A rather general discussion of this topic has been given by [23] in the case that \(2M_R\) is a symmetric space. This would include the most important case which is \(S^4\) or DeSitter spacetime. There seems little doubt that the resulting vacuum state is the well known DeSitter invariant one, although this has not, to my knowledge, ever been checked in full detail.

The origin of DeSitter invariance described above is very similar to that given by D’Eath and Halliwell [24] in the context of Hawking and Hartle’s ”No boundary Proposal”. In the more general context of tunneling transitions it suggests a a natural candidate for the created quantum state after tunneling.

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