SAGITTA, LENSES, AND MAXIMAL VOLUME

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ABSTRACT. For a Riemannian manifold $M$ with a lower sectional curvature bound, we consider a metric invariant of $M$ that extends the definition of the radius of $M$. We obtain a uniform upper volume bound for the class of manifolds with a lower sectional curvature bound and an upper bound on this invariant. We show any $M$ in this class that has volume sufficiently close to this upper bound is diffeomorphic to the standard sphere $S^n$ or a standard lens space $S^n/\mathbb{Z}_m$ where $m \in \{2, 3, \ldots\}$ is no larger than an a priori constant.

The radius of a metric space $X$ is the number

$$\text{rad } X := \inf_{p \in X} \sup_{q \in X} \text{dist}(p, q)$$

and can be thought of as the radius of the smallest metric disk that covers $X$. Let $k \in \mathbb{R}$ and $S^n_k$ denote the simply connected space form of constant sectional curvature $k$. If $M^n$ is a Riemannian manifold with sectional curvature bounded below by $k$, it follows from usual volume comparison that

$$\text{vol } M \leq \text{vol } D^n_k(\text{rad } M)$$

where $D^n_k(r) \subset S^n_k$ denotes the disk of radius $r$. This gives a uniform upper volume bound for the class $\mathcal{M}^n_{k, r}$ of all Riemannian $n$-manifolds $M$ with $\text{sec } M \geq k$ and $\text{rad } M \leq r$.

In [2], Grove and Petersen showed if $r \leq \frac{1}{2} \text{diam } S^n_k$, then any $M \in \mathcal{M}^n_{k, r}$ with volume sufficiently close to $D^n_k(r)$, must topologically be a sphere or real projective space.

In this work, we define a parameterized geometric invariant, which we call the $r$-sagitta of $M$. This number, which we denote by $\text{sag } r M$, extends the definition of the radius of $M$ and allows us to refine the class $\mathcal{M}^n_{k, r}$ to a subclass with a uniform upper volume bound that improves the one considered in [2]. The definition of this invariant is technical, but is motivated from the following concept in the euclidean plane. Recall if $C$ is the circular arc of radius $r$ in the plane, the sagitta of $C$ is defined to be the distance between the center of $C$ and the center of the base of $C$.

Let $L^n_k(h, r)$ be the lens-like region formed by the intersection of two disks of radius $r$ in $S^n_k$, i.e.,

$$L^n_k(h, r) := D^n_k(\tilde{a}, r) \cap D^n_k(\tilde{b}, r) \subset S^n_k$$

where $|\tilde{a}\tilde{b}| = 2(r - h)$. Our main result is

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Theorem 1. Let $n \geq 2$, $k \in \mathbb{R}$, and $h, r \in (0, \frac{1}{2} \text{diam } S^n_k]$ be any real numbers with $h \leq r$. If $M$ is a compact, Riemannian $n$-manifold satisfying $k \leq \sec M$, and

\begin{equation}
\tag{*}
sag_r M \leq h \leq \text{rad } M \leq r
\end{equation}

then,

1. The volume of $M$ satisfies

$$\text{vol } M \leq \text{vol } L^n_k(h, r).$$

2. There is an $\varepsilon(n, k, h, r) > 0$, and an integer $c(n, k, h, r) \geq 2$ so that if

$$\text{vol } M > \text{vol } L^n_k(h, r) - \varepsilon,$$

then $M$ is diffeomorphic to either

(a) $S^n$,

(b) $\mathbb{R}P^n$ if $n$ is even, or

(c) A Lens space $S^n/\mathbb{Z}_m$ where $2 \leq m \leq c$ if $n$ is odd.

3. Each manifold $N$ in the conclusion of Part (2) admits a smooth Riemannian metric that satisfies $\sec N \geq k$ and (*), and has volume arbitrarily close to $\text{vol } L^n_k(h, r)$. However, there are no Riemannian manifolds $N$ that satisfy the equality $\text{vol } N = \text{vol } L^n_k(h, r)$ unless $k > 0$, $r = \frac{1}{2} \text{diam } S^n_k$, and $h$ divides $\text{diam } S^n_k$.

The number $c(n, k, h, r)$ above can be computed as the smallest integer larger than

$$\lim_{\rho \to 0} \frac{\text{vol } B(\hat{q}, \rho)}{\text{vol } (B(\hat{q}, \rho) \cap L^n_k(h, r))}$$

where $\hat{q}$ is any point in (the singular part of) $\partial L^n_k(h, r)$ (if $h < r$).

Intuitively, this is the largest number of $L^n_k(h, r)$s that can be joined together along their common $(n - 1)$-dimensional faces without violating volume constraints imposed by $S^n_k$.

The key ingredient in the definition of $\text{sag } M$, which we now describe, is based on the following geometric feature of $S^n_k$.

Let $m_k : \mathbb{R} \to \mathbb{R}$ denote the distance modifying function in $S^n_k$, i.e., the solution to $y'' + ky = 1$, with $y(0) = y'(0) = 0$. (see Equation (2) in Section 1 for more details). We show in Section 1

Proposition 1. Let $r \in (0, \text{diam } S^n_k]$ be a real number. Let $D^n_k(r) \subset S^n_k$ be a disk of radius $r$ with metric $\| \cdot \|$ induced from the metric on $S^n_k$. Let $\hat{p} \in D^n_k(r)$ and $\hat{q} \in \partial D^n_k(r)$ be points on a geodesic through the center of $D^n_k(r)$. Then, for any $\hat{x} \in D^n_k(r) \setminus \{\hat{q}\},$

\begin{equation}
\frac{m_k(\|\hat{p}\hat{x}\|) - m_k(\|\hat{p}\hat{q}\|)}{m_k(\|\hat{q}\hat{q}\|)} \leq \frac{m'_k(r - \|\hat{p}\hat{q}\|)}{m'_k(r)}
\end{equation}

and equality occurs if and only if $\hat{x} \in \partial D^n_k(r) \setminus \{\hat{q}\}$.

Remark 1. If $| \cdot |$ denotes the standard metric on $S^n_k$, then $\| \cdot \|$ coincides with $| \cdot |$ unless $k > 0$ and $r > \frac{1}{2} \text{diam } S^n_k$.

For any $k \in \mathbb{R}$ and $h \in (0, \text{diam } S^n_k)$, the function $r \mapsto m'_k(r - h)/m'_k(r)$ is strictly increasing on $(0, \text{diam } S^n_k)$. In particular, for any number $\lambda$ less than

$$\Lambda(k, h) := \lim_{r \to \text{diam } S^n_k} \frac{m'_k(r - h)}{m'_k(r)} = \begin{cases} 
\infty & \text{if } k > 0 \\
\frac{1}{e} \sqrt{|k|h} & \text{if } k \leq 0
\end{cases}$$
the equation
\[ \frac{m_k'(r - h)}{m_k'(r)} = \lambda \]
can be solved uniquely for \( r \). Because the right hand side of Equation (1) is independent of \( \tilde{x} \), we can make

**Definition 1.** If \( k \in \mathbb{R} \) is fixed and \( X \) is an \( n \)-dimensional Alexandrov space with \( \text{curv} \ X \geq k \), for \( p, q \in X \), we’ll say the \( k \)-eccentricity at \( p \) relative to \( q \) is given by
\[
\lambda_p(q) := \sup_{x \in M \setminus \{q\}} \frac{m_k(\text{dist}(p,x)) - m_k(\text{dist}(p,q))}{m_k(\text{dist}(q,x))}.
\]
If \( \text{dist}(p,q) \in (0, \text{diam} S^n_k) \) and \( \lambda_p(q) < \Lambda(k, \text{dist}(p,q)) \), we denote by \( \text{rad}_p(q) \) the unique solution \( r \) to the equation
\[
\frac{m_k'(r - \text{dist}(p,q))}{m_k'(r)} = \lambda_p(q),
\]
otherwise we set \( \text{rad}_p(q) = \infty \).

By critical point of a distance function we mean in the sense of [3], i.e., \( q \) is critical for the distance from \( p \) if the set of unit directions from \( q \) to \( p \) is a \( \pi/2 \)-net in the space of all directions at \( q \). Our first application of Proposition 1 is to give the following characterization of critical points.

**Proposition 2.** Let \( M \) be a Riemannian manifold with \( \text{sec} M \geq k \). If \( p, q \in M \), then \( \lambda_p(q) < \infty \) if and only if \( q \) is critical for the distance from \( p \).

It should be pointed out that when \( k > 0 \), for any \( h \), \( \Lambda(k, h) = \infty \). So, in positive curvature whenever \( \lambda_p(q) < \infty \), the number \( \text{rad}_p(q) \) is finite. Therefore, Proposition 2 gives a characteristic of critical points in positive curvature that isn’t necessarily present in an arbitrary lower curvature bound.

For any \( r \geq 0 \) and \( p \in M \), consider the closed subset
\[
A_{r,h}(p) := \{ q \in M \mid \text{rad}_p(q) \leq r \text{ and } \text{dist}(p,q) \leq h \}.
\]
In Section 1 we show the set of points at maximal distance from a point \( p \) that realizes the radius of \( M \) coincides with \( A_{r,h}(p) \) when \( r = h = \text{rad} M \). Moreover, we establish

**Proposition 3.** If \( p \) realizes the radius of \( M \), then the distance from \( A_{\text{rad} M, \text{rad} M}(p) \) is critical at \( p \).

Therefore, if \( \text{rad} M \leq r \), the set
\[
\{ p \in M \mid \text{the distance from the set } A_{r,h}(p) \text{ is critical at } p \ \text{for some } h \}
\]
is nonempty. This allows for

**Definition 2.** Suppose \( M \) is compact and satisfies \( \text{rad} M \leq r \). We let
\[
\text{sag}_r M := \min \{ h \mid p \text{ is critical for } \text{dist}(A_{r,h}(p), \cdot) \text{ for some } p \in M \}
\]
and call it the \( r \)-sagitta of \( M \).
Remark 2. If \( M \) contains a pair of points \( p, q \), with \( \text{rad}_p(q) < \infty \) and, in addition, \( p \) is critical to \( q \), then \( \text{sag}_{\text{rad}_p(q)} M \leq \text{dist}(p, q) \). In fact, because of its simplicity it’s tempting to take the infimum of distances between all points \( p, q \in M \) with \( \text{rad}_p(q) < \infty \) and \( p \) critical to \( q \) as our desired definition. However, if \( p, q \in M \) satisfies \( \text{dist}(p, q) = \text{rad}(p) = \text{rad}(M) \), though \( q \) must be critical to \( p \), \( p \) need not be critical to \( q \). Therefore, we prefer Definition 2 as it allows us to generalize the class of manifolds studied in [2].

Remark 3. For \( p \in M \), the star convex region of \( T_p M \) that contains the origin and is bounded by the tangent cut locus of \( p \) is particularly useful from a comparison point of view. When \( \text{rad}(M) \leq h \), there is a point \( p \in M \) where this region is contained in a disk of radius \( r \) centered at the origin of \( T_p M \). The point of Definition \( 1 \) is critical to \( p \), as we show in Proposition \( 2 \) that manifolds that satisfy (\( \ast \)) must have a point \( p \) where this region is contained in the intersection of a collection of disks of radius \( r \) in a configuration corresponding to the set of directions to \( A_r, h(p) \). This refines the case \( \text{rad}(M) \leq r \) as the distance from the origin of \( T_p M \) to the boundary of any disk in this collection is now \( h \leq r \).

We now give examples of Alexandrov spaces that satisfy the hypotheses of Theorem 1 but have the additional property that equality in Part 1 of Theorem 1 is achieved (see Section 3 for more details).

First assume \( n \geq 2, h, r \in (0, \frac{1}{2} \text{diam} S^n_k] \) with \( h \leq r \) and \( L_k^n(h, r) := D_k^n(\tilde{a}_1, r) \cap D_k^n(\tilde{a}_2, r) \) where \( 2(r-h) = |\tilde{a}_1\tilde{a}_2| \) (see Figure 1). If \( h < r \), we let \( H_0 \) be the totally geodesic hyperplane given by

\[
H_0 := \{ \tilde{u} \mid |\tilde{u}\tilde{a}_1| = |\tilde{u}\tilde{a}_2| \} \subset S^n_k,
\]

and if \( h = r \), we take \( H_0 \) to be any hyperplane through \( \tilde{a}_1 = \tilde{a}_2 \). Then we let \( S_k \) be the \((n-2)\)-sphere given by

\[
S_k := H_0 \cap \partial L_k^n(h, r)
\]

equipped with the constant curvature metric induced from \( L_k^n(h, r) \).

If \( h < r \), we set \( P \subset S^n_k \) to be any hyperplane which contains the geodesic through \( \tilde{a}_1 \) and \( \tilde{a}_2 \), otherwise if \( h = r \), we can take \( P = H_0 \).

Let

\[
R_{H_0} : S^n_k \to S^n_k
\]
\[
R_P : S^n_k \to S^n_k
\]

be reflections over the hyperplanes \( H_0 \) and \( P \), respectively.

Example 1. Take any \( m \in \{1, 2, \ldots, c(n, k, h, r)\} \) and assume \( n \) is odd if \( m > 2 \). Let

\[
\{\phi_m\}_{i \in I_m} \subset O(n-1)
\]

be all isometries of \( S^{n-2} \) of order \( m \) that, if \( m > 1 \), generate a cyclic group \( \mathbb{Z}_m \) that acts freely on \( S^{n-2} \). Since \( \mathbb{Z}_2 \) is the only group which acts freely on even dimensional spheres, \( m = 2 \) if \( n \) is even.

Any \( \phi_m \) in this collection, viewed as an isometry of \( S_k \), extends to an isometry of \( L_k^n(h, r) \). Therefore we set

\[
C_m(h, r, \phi_m) := L_k^n(h, r)/(\tilde{u} \sim R_{H_0} \circ \phi_m(\tilde{u})) \text{ with } \tilde{u} \in \partial L_k^n(h, r)
\]

and note \( C_m(h, r, \phi_m) \) is an Alexandrov space with curvature bounded below by \( k \), \( r \)-sagitta equal to \( h \), and volume equal to \( L_k^n(h, r) \).
Remark 4. If $m > 1$, each $C^n_k(h, r, \phi_m)$ has the topology of a usual Lens space $S^n/\mathbb{Z}_m$, i.e., a quotient by a finite sub-action of the Hopf action (see Section 3). If $m = 1$, the degenerate Lens space $C^n_k(h, r, \text{id})$ is topologically a sphere. We also note that $C^n_k(r, r, \text{id})$ is the “Curvature $k$ Purse” denoted $P^n_k$, and $C^n_k(r, r, -\text{id}) \cong \mathbb{R}P^n$ is the “Curvature $k$ Crosscap” denoted $C^n_k$ that were constructed in [2].

Example 2. Define

$$P^n_k(h, r) := L^n_k(h, r)/(\tilde{u} \sim R_P(\tilde{u}))$$

and note $P^n_k(h, r) \cong S^n$ is an Alexandrov space with curvature bounded below by $k$, $r$-sagitta equal to $h$, and volume equal to $L^n_k(h, r)$.

In Section 3 we establish the above propositions. In Section 2 we establish volume bound in Part 1 of Theorem 1. In Section 3 we prove the topological version of Part 2 by showing

**Theorem 2.** Let $n \geq 2$, $k \in \mathbb{R}$, and $h, r \in (0, \frac{1}{2} \text{diam } S^n_k]$ with $h \leq r$ be real numbers. If $\{M_i\}$ is a sequence of Riemannian $n$-manifolds with $\text{sec } M_i \geq k, \text{sag } M_i \leq h \leq \text{rad } M_i \leq r$ and $\text{vol } M_i \to \text{vol } L^n_k(h, r)$, then a subsequence of $\{M_i\}$ must converge to either $P^n_k(h, r)$ or $C^n_k(h, r, \phi_m)$ for some $\phi_m \in \{\phi_m\}_{m \in I_m}$.

In Section 4 we establish Part 3 of Theorem 1. The proof of the diffeomorphism conclusion of Part 2 of Theorem 1 is possible by exploiting the geometry of these limit spaces along the same lines that were achieved in [7]. However, we find that it is more convenient to defer to the following

**Theorem 3.** Let $X$ be an $n$-dimensional Alexandrov space with lower curvature bound $k$ and upper diameter bound $D$. Let $\{N_i\}_{i \in I}$ be a collection of isometrically embedded $(n - 2)$-dimensional Riemannian manifolds with smooth boundary such that if $N_i \cap N_j \neq \emptyset$, then $N_i \cap N_j = \partial N_i = \partial N_j$. Let $\{M_\alpha\}$ be a sequence of $n$-dimensional Riemannian manifolds with $\text{sec } M_\alpha \geq k$ and $\text{diam } M_\alpha \leq D$ that converge to $X$. If the space of directions of every point of $X \setminus (\cup_{i \in I} N_i)$ is isometric to $S_1^{n-1}$, then for all but finitely many $\alpha$ and $\beta$, $M_\alpha$ is diffeomorphic to $M_\beta$.

The proof of this result is established in [8] which is forthcoming.

To see that this theorem applies we now describe the singular structure of the limit spaces. Let $\pi_{m_i} : L^n_k(h, r) \to C^n_k(h, r, \phi_{m_i})$ and $p : L^n_k(h, r) \to P^n_k(h, r)$ be the projection maps.

**Figure 1.** Hyperplanes $H_0$ and $P$. 

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### Figure 1

Hyperplanes $H_0$ and $P$. 

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### Remark 4

If $m > 1$, each $C^n_k(h, r, \phi_m)$ has the topology of a usual Lens space $S^n/\mathbb{Z}_m$, i.e., a quotient by a finite sub-action of the Hopf action (see Section 3). If $m = 1$, the degenerate Lens space $C^n_k(h, r, \text{id})$ is topologically a sphere. We also note that $C^n_k(r, r, \text{id})$ is the “Curvature $k$ Purse” denoted $P^n_k$, and $C^n_k(r, r, -\text{id}) \cong \mathbb{R}P^n$ is the “Curvature $k$ Crosscap” denoted $C^n_k$ that were constructed in [2].

### Example 2

Define

$$P^n_k(h, r) := L^n_k(h, r)/(\tilde{u} \sim R_P(\tilde{u}))$$

and note $P^n_k(h, r) \cong S^n$ is an Alexandrov space with curvature bounded below by $k$, $r$-sagitta equal to $h$, and volume equal to $L^n_k(h, r)$.

In Section 3 we establish the above propositions. In Section 2 we establish volume bound in Part 1 of Theorem 1. In Section 3 we prove the topological version of Part 2 by showing

### Theorem 2

Let $n \geq 2$, $k \in \mathbb{R}$, and $h, r \in (0, \frac{1}{2} \text{diam } S^n_k]$ with $h \leq r$ be real numbers. If $\{M_i\}$ is a sequence of Riemannian $n$-manifolds with $\text{sec } M_i \geq k, \text{sag } M_i \leq h \leq \text{rad } M_i \leq r$ and $\text{vol } M_i \to \text{vol } L^n_k(h, r)$, then a subsequence of $\{M_i\}$ must converge to either $P^n_k(h, r)$ or $C^n_k(h, r, \phi_m)$ for some $\phi_m \in \{\phi_m\}_{m \in I_m}$.

In Section 4 we establish Part 3 of Theorem 1.

The proof of the diffeomorphism conclusion of Part 2 of Theorem 1 is possible by exploiting the geometry of these limit spaces along the same lines that were achieved in [7]. However, we find that it is more convenient to defer to the following

### Theorem 3

Let $X$ be an $n$-dimensional Alexandrov space with lower curvature bound $k$ and upper diameter bound $D$. Let $\{N_i\}_{i \in I}$ be a collection of isometrically embedded $(n - 2)$-dimensional Riemannian manifolds with smooth boundary such that if $N_i \cap N_j \neq \emptyset$, then $N_i \cap N_j = \partial N_i = \partial N_j$. Let $\{M_\alpha\}$ be a sequence of $n$-dimensional Riemannian manifolds with $\text{sec } M_\alpha \geq k$ and $\text{diam } M_\alpha \leq D$ that converge to $X$. If the space of directions of every point of $X \setminus (\cup_{i \in I} N_i)$ is isometric to $S_1^{n-1}$, then for all but finitely many $\alpha$ and $\beta$, $M_\alpha$ is diffeomorphic to $M_\beta$.

The proof of this result is established in [8] which is forthcoming.

To see that this theorem applies we now describe the singular structure of the limit spaces. Let $\pi_{m_i} : L^n_k(h, r) \to C^n_k(h, r, \phi_{m_i})$ and $p : L^n_k(h, r) \to P^n_k(h, r)$ be the projection maps.
When $m = 1$ and $\phi_m = \text{id}$, $\pi_1(S_0) \subset C_k^n(h, r, \text{id})$ is an isometric embedding of a round $(n - 2)$-sphere and every point in $C_k^n(h, r, \text{id}) \setminus \pi_1(S_0)$ has a euclidean space of directions.

For $m > 1$, $\pi_m(S_0) \subset C_k^n(h, r, m)$ is the quotient of $S_0$ by the free and orthogonal action of $\langle \phi_m \rangle = \mathbb{Z}_m$, thus is an isometric embedding of a constant curvature Lens space $S^{n-2}/\mathbb{Z}_m$. Again, every point of $C_k^n(h, r, \text{id}) \setminus \pi_m(S_0)$ has a euclidean space of directions.

Let $P \subset S_k^n$ be the hyperplane such that $P_k^n(h, r) = L_k^n(h, r)/((\ddot{u} \sim R_P(\ddot{u}))$, where $\ddot{u} \in \partial L_k^n(h, r)$. As $P$ contains a vector orthogonal to $H_0$,

$$p(S_0) = S_0/(\ddot{u} \sim R_P(\ddot{u})) \subset P_k^n(h, r)$$

is an isometrically embedded $(n - 2)$-disk of constant curvature. In addition, the $(n - 2)$-sphere $S_P := P \cap \partial L_k^n(h, r)$ decomposes into two $(n - 2)$-disks

$$D_i^P := P \cap \partial L_k^n(h, r) \cap \partial D_k^n(\ddot{a}_i, r), \quad i = 1, 2,$$

each with constant curvature induced from the metric on $L_k^n(h, r)$. Because $p|_{S_P} = \text{id}$, for $i = 1, 2$, we have $p(D_i^P) \subset P_k^n(h, r)$ is an isometric embedding. Moreover,

$$S^{n-3} \cong \partial p(S_0) = \partial p(D_i^P) = \partial p(D_i^P) \subset P_k^n(h, r).$$

Here, every point in $P_k^n(h, r) \setminus (p(S_0) \cup p(D_1^P) \cup p(D_2^P))$ has a euclidean space of directions.

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1. **The Segment Domain and Eccentricity**

In what follows we let $M$ denote a compact Riemannian $n$-manifold with $\text{sec} \geq k$. At any point $p \in M$ we can use a radial conformal change to equip $T_pM$ with a metric of constant curvature $k$. This allows us to identify $T_pM$ with $S_k^n$, which we will now do throughout.

Following [2], for $p \in M$, we’ll call the star convex region of $S_k^n = T_pM$ that is bounded by the tangent cut locus of $p$ and contains the origin the **segment domain** at $p$. We denote this set as $\text{seg}(p)$ and note

$$\text{seg}(p) = \{ v \in T_pM \mid \exp_p(tv) : [0, 1] \to M \text{ is a segment} \}.$$ 

By Toponogov’s Theorem, the surjective map $\exp_p : \text{seg}(p) \to M$ is $1$-Lipschitz and a diffeomorphism on the interior of $\text{seg}(p)$.

Given a geodesic $\tilde{\gamma} : \mathbb{R} \to S_k^n$ and a point $\ddot{a} \in S_k^n$, intuitively we’re interested in the behavior of the function $t \to |\ddot{a}\tilde{\gamma}(t)|$. However, the modified distance function $t \to m_k(|\ddot{a}\tilde{\gamma}(t)|)$ has a much nicer Hessian and thus allows for a clearer picture of
the geometry in $S^p_k$. Again, $m_k : \mathbb{R} \to \mathbb{R}$ is the solution to $y'' + k y = 1$, with initial conditions $y(0) = y'(0) = 0$. Explicitly, we can write

$$m_k(t) = \frac{t^2}{2} + \sum_{n=2}^{\infty} \frac{(-k)^{n-1}}{(2n)!} \frac{t^{2n}}{2^n} = \begin{cases} \frac{1}{k}(1 - \cos(\sqrt{k}t)) & \text{if } k > 0 \\ \frac{C}{2} & \text{if } k = 0 \\ \frac{1}{k}(\cosh(\sqrt{k}|t|) - 1) & \text{if } k < 0 \end{cases}.$$  

Let $r \in [0, \diam S^p_k]$ and let $\| \cdot \|$ denote the induced distance on $D^p_k(r)$ from $\| \cdot \|$ on $S^p_k$. Given three points $\tilde{a}, \tilde{x}, \tilde{y} \in D^p_k(r)$, the geometry of $D^p_k(r)$ is described by the equation

$$m_k(c) = m_k(a) + m_k(b) - k m_k(a) m_k(b) - m_k'(a) m_k'(b) \cos \alpha$$

where $c = \| \tilde{x} \tilde{y} \|, a = \| \tilde{a} \tilde{x} \|, b = \| \tilde{a} \tilde{y} \|$, and $\alpha = \angle \tilde{x} \tilde{a} \tilde{y}$. This is just the Law of Cosines in $S^p_k$ written in terms of $m_k$.

By applying (3) to any three points on a single geodesic in $S^p_k$, because $m_k$ is even and $m_k'$ is odd, for any $a, b \in \mathbb{R}$ we have the usual sum formulas

$$m_k(a + b) = m_k(a) + m_k(b) - k m_k(a) m_k(b) - m_k'(a) m_k'(b).$$

If $c = 0$, (3) also gives the Pythagorean Identities

$$m_k'(t)^2 = m_k(t) + m_k(t) - k m_k(t) m_k(t) = m_k(t)(1 + m_k''(t)).$$

**Proposition 4.** Let $\tilde{p}, \tilde{q}, \tilde{x} (\tilde{q} \neq \tilde{x})$ be the vertices of a triangle in disk $D^p_k(r)$ and suppose $\tilde{\beta}$ is the angle at $\tilde{q}$. Then

$$m_k(\| \tilde{p} \tilde{x} \|) - m_k(\| \tilde{p} \tilde{q} \|) \quad m_k(\| \tilde{x} \tilde{q} \|)$$

Proof. As $m_k'' + k m_k = 1$, by (3)

$$m_k(\| \tilde{p} \tilde{x} \|) = m_k(\| \tilde{p} \tilde{q} \|) + m_k(\| \tilde{x} \tilde{q} \|) - k m_k(\| \tilde{p} \tilde{x} \|) m_k(\| \tilde{p} \tilde{q} \|) - m_k'(h) m_k'(\| \tilde{q} \tilde{x} \|) \cos \tilde{\beta}$$

$$= m_k(\| \tilde{p} \tilde{q} \|) + m_k(\| \tilde{x} \tilde{q} \|) m_k''(\| \tilde{q} \tilde{x} \|) - m_k'(\| \tilde{p} \tilde{q} \|) m_k'(\| \tilde{x} \tilde{q} \|) \cos \tilde{\beta},$$

from which the result follows. \qed

We drop the “$k$” from $m_k$, and establish the

**Proof of Proposition 1.** Let $\tilde{\gamma} : \mathbb{R} \to S^p_k$ be a geodesic and $r \in (0, \diam S^p_k)$ a real number. Suppose $\tilde{a}, \tilde{p}, \tilde{q}$ are points on the image of $\tilde{\gamma}$ so that $\tilde{p} \in D^p_k(\tilde{a}, r)$ and $\tilde{q} \in \partial D^p_k(\tilde{a}, r)$. Set $h := \| \tilde{p} \tilde{q} \|$.

From (3),

$$m(\| \tilde{p} \tilde{q} \|) = m(h) = m(r - h) + m(r) - k m(r - h)m(r) - m'(r - h)m'(r).$$

Now take any $\tilde{x} \in \partial D^p_k(\tilde{a}, r) \setminus \{ \tilde{q} \}$ and let $\alpha := \angle \tilde{p} \tilde{a} \tilde{x}$. Then $\alpha = \angle \tilde{p} \tilde{a} \tilde{x}$ if $r - h > 0$ and $\alpha = \pi - \angle \tilde{p} \tilde{a} \tilde{x}$ if $r - h < 0$. Using the parity of $m, m'$, and cos, if necessary, (3) says

$$m(\| \tilde{p} \tilde{x} \|) = m(\| \tilde{a} \tilde{p} \|) + m(r) - k m(\| \tilde{a} \tilde{p} \|) m(r) - m'(\| \tilde{a} \tilde{p} \|) m'(r) \cos \alpha$$

$$= m(r - h) + m(r) - k m(r - h)m(r) - m'(r - h)m'(r) \cos \alpha.$$
It follows that,
\[ \frac{m(\|\hat{p}\|) - m(\|\hat{q}\|)}{m(\|\hat{q}\|)} = \frac{m'(r - \|\hat{q}\|)}{m'(r)}. \]

Now take any \( \hat{x} \in D^u_k(\hat{a}, r) \setminus \{\hat{q}\} \) and let \( \hat{\sigma} : [0, T] \to D^u_k(\hat{a}, r) \) be a maximal segment in \( D^u_k(\hat{a}, r) \) with \( \hat{\sigma}(0) = \hat{q} \) and \( \hat{\sigma}(t) = \hat{x} \) for some \( t \in (0, T] \). Let \( \hat{\beta} \) be the angle between \( \hat{\sigma} \) and \( \hat{\gamma} \) at \( \hat{q} \). By Proposition 4

(7) \[ \frac{m(\|\hat{p}\|) - m(h)}{m(\|\hat{q}\|)} = \frac{m''(h) - m'(h)\frac{m'(\|\hat{q}\|)}{m(\|\hat{q}\|)}\cos \hat{\beta}}{m'(r)} \]

Note that \( \hat{\beta} \leq \pi/2 \), otherwise, by first variation, there are values of \( \hat{\sigma} \) whose distance from \( \hat{a} \) is larger than \( \|\hat{a}\hat{q}\| = r \). On \([0, \text{diam} S^u_k]\) both \( m \) and \( m' \) are nonnegative, and since \( m \) is nondecreasing, \( m'/m \) is nonincreasing. Therefore, (7) is maximal if and only if \( \hat{x} \) is a value of \( \hat{\sigma} \) at maximal distance from \( \hat{q} \). The point at maximal distance from \( \hat{q} \) along \( \hat{\sigma} \) is in \( \partial D^u_k(\hat{a}, r) \). This shows

\[ \frac{m(\|\hat{p}\|) - m(\|\hat{q}\|)}{m(\|\hat{q}\|)} \leq \frac{m'(r - \|\hat{q}\|)}{m'(r)} \]

for all \( \hat{x} \in D^u_k(\hat{a}, r) \setminus \{\hat{q}\} \) and that equality occurs if and only if \( \hat{x} \in \partial D^u_k(\hat{a}, r) \setminus \{\hat{q}\} \).

Recall that if \( A \subset M \) and \( f := \text{dist}(A, \cdot) : M \to \mathbb{R} \) is the distance from \( A \), then \( f \) is directionally differentiable at any point \( q \not\in A \). Moreover, if \( U_q M \) is the unit tangent sphere at \( q \in M \), then for any \( v \in U_q M \), first variation says

\[ D_v(f) = -\cos \alpha_{\text{min}} \]

where \( \alpha_{\text{min}} \) is the distance in \( U_q M \) between \( v \) and the set \( \hat{h}_q^A \) of directions tangent to segments from \( q \) to \( A \). If it happens that \( \alpha_{\text{min}} \leq \pi/2 \), i.e., \( \hat{h}_q^A \) forms a \( \pi/2 \)-net in \( U_q M \), then \( q \) is called a critical point for the distance from \( A \).

**Proof of Proposition 2**. Assume \( p \neq q \). Take any segment \( \gamma_q : [0, T] \to M \) with \( \gamma_q(0) = q \). For any \( t \in (0, T) \), by the mean value theorem, for some \( d^* \in [\text{dist}(p, q), \text{dist}(p, \gamma_q(t))] \) and \( t^* \in (0, t) \) we have

\[ \frac{m(\text{dist}(p, \gamma_q(t))) - m(\text{dist}(p, q))}{m(\text{dist}(q, \gamma_q(t)))} = \frac{m'(d^*)}{m'(t^*)} \cdot \frac{\text{dist}(p, \gamma_q(t)) - \text{dist}(p, q)}{t^*}. \]

Using this and the first variation formula, if \( \lambda_p(q) < \infty \), then

\[ -\cos \alpha_{\text{min}} = \lim_{t \to 0^+} \frac{\text{dist}(p, \gamma_q(t)) - \text{dist}(p, q)}{t} \]

\[ \leq \lim_{t \to 0^+} \frac{m'(t^*)}{m'(d^*)} \lambda_p(q) \]

\[ = \frac{m'(0)}{m'(\text{dist}(p, q))} \lambda_p(q) \]

\[ = 0. \]

So \( q \) is a critical point for the distance from \( p \).

Conversely, suppose \( q \) is critical to \( p \). Take any \( x \in M \) different from \( q \) and any segment \( \sigma_{qx} \) from \( q \) to \( x \). It follows there is a segment \( \gamma_{qp} \) from \( q \) to \( p \) so that

\[ \beta := \angle(\gamma_{qp}(0), \sigma'_{qx}(0)) \leq \pi/2. \]
now let $\check{\gamma}_\check{a}p$ and $\check{\sigma}_\check{a}\check{x}$ be segments in $S^n_k$ where $|\check{q}p| = \text{dist}(p, q)$, $|\check{q}\check{x}| = \text{dist}(q, x)$, and $\check{\beta} := \angle(\check{\gamma}_\check{a}p(0), \check{\sigma}_\check{a}\check{x}(0)) = \beta \leq \pi/2$.

by the hinge version of Toponogov’s Theorem,

$$\text{dist}(p, x) \leq |\check{p}\check{x}|.$$ 

from Proposition IV with $\|\cdot\|$ replaced by $|\cdot|$, it follows that

$$\frac{m(\text{dist}(p, x)) - m(\text{dist}(p, q))}{m(\text{dist}(q, x))} \leq \frac{m(|\check{p}|) - m(|\check{q}p|)}{m(|\check{q}\check{x}|)} = \frac{m'(\check{p}) - m'(\check{q}p)}{m'(|\check{q}\check{x}|)} \cos \check{\beta} \leq m'(\check{q}p).$$

so $\lambda_p(q) < \infty$. \hfill \square

 recall that we use $\nabla^q_p$ to denote the set of unit tangent vectors in $T_pM$ which are tangent to segments from $p$ to $q$. the next Lemma is for notation.

**Lemma 1.** Suppose $\text{rad}_p(q) < \infty$, and let $\check{p}$ denote the origin of $\text{seg}(p)$. for any $v \in \nabla^q_p$, and $\check{q}v \in \text{seg}(p)$ such that $\exp_p(\check{q}v) = q$, there is a point $\check{a}_v \in S^n_k$ and a disk $D^*_k(\check{a}_v, \text{rad}_p(q)) \subset S^n_k$

centered at $\check{a}_v$, with radius $\text{rad}_p(q)$ and a geodesic $\check{\gamma}_v : \mathbb{R} \to S^n_k$ such that

1. $\check{\gamma}_v(0) = \check{a}_v$,
2. $\check{\gamma}(\text{rad}_p(q)) = \check{q}$, and
3. $\check{\gamma}_v(\text{rad}_p(q) - \text{dist}(p, q)) = \check{p}.

**Proof.** Take $v \in \nabla^q_p$ and let $r = \text{rad}_p(q)$. let $\check{\gamma}_v : \mathbb{R} \to S^n_k$ be a geodesic defined by $\check{\gamma}_v(t) = \exp_p((t - (r - \text{dist}(p, q)))v).

Take $D^*_k(\check{a}_v, \text{rad}_p(q)) \subset S^n_k$ to be the disk centered at $\check{a}_v := \gamma_v(0)$.

**Lemma 2.** if $\text{dist}(p, q) < \text{rad} M$ and $\text{rad}_p(q) < \infty$, then

$$\text{seg}(p) \subset \bigcap_{v \in \nabla^q_p} D^*_k(\check{a}_v, \text{rad}_p(q)).$$

**Proof.** let $\check{p}$ be the origin of $\text{seg}(p)$ and take any direction $v \in \nabla^q_p$. let $\check{a}_v, \check{q}v$, and $D^*_k(\check{a}_v, \text{rad}_p(q))$ be as in Lemma IV.

To show that $\text{seg}(p) \subset D^*_k(\check{a}_v, \text{rad}_p(q))$, by Proposition IV it will suffice to show for any $\check{x} \in \text{seg}(p),

$$\frac{m(|\check{p}x|) - m(|\check{p}\check{q}v|)}{m(|\check{q}\check{x}|)} \leq \frac{m'(\text{rad}_p(q) - \text{dist}(p, q))}{m'(|\check{q}\check{x}|)} = \lambda_p(q).$$

because $\text{dist}(p, q) < \text{rad} M$, it follows that

$$\lambda_p(q) = \sup_{x \in \text{M} \setminus \{q\}} \frac{\text{dist}(p, x)}{\text{dist}(q, x)} > 0.$$ 

in particular, (8) is satisfied if $|\check{p}x| \leq |\check{p}\check{q}v|$

Therefore, we can assume that $|\check{p}x| > |\check{p}\check{q}v|$. let $x \in M$ be such that $\exp_p(\check{x}) = x$.

by the hinge version of Toponogov’s Theorem,

$$\text{dist}(q, x) \leq |\check{q}x|.$$
Therefore,
\[
0 < \frac{m(|\tilde{p}\tilde{q}|) - m(|\tilde{p}\tilde{q}_c|)}{m(|\tilde{q}\tilde{x}|)} \\
\leq \frac{m(\text{dist}(p,x)) - m(\text{dist}(p,q))}{m(\text{dist}(q,x))} \\
\leq \lambda_p(q)
\]
as desired.

The next proposition explains the choice of parameters \(h\) and \(r\) and hypothesis \((*)\) in Theorem 1.

**Proposition 5.** Suppose that \(M\) is compact. If \(\text{dist}(p,q) < \text{rad} M\) and \(\text{rad}_p(q) < \infty\), then

1. \(\text{rad} M \leq \frac{1}{2} \text{diam} S^n_k\), and
2. \(\text{rad}_p(q) \in [\text{rad} M, \frac{1}{2} \text{diam} S^n_k]\).

If \(p\) realizes the radius of \(M\) and \(\text{dist}(p,q) = \text{rad} M\), then \(\text{rad}_p(q) \leq \text{rad} M\).

**Proof.** First we prove the upper bound of Part (2). For \(S \triangle\) in \(\tilde{\Delta}\) and Proposition 4,

\[
\text{dist}(p,x) = \text{dist}(\tilde{p}\tilde{x}) \leq \frac{1}{2} \text{diam} S^n_k \text{ for } \Delta
\]

Therefore, there must be a point \(\tilde{p}\) such that
\[
m''(\text{dist}(p,q)) < \frac{m(\text{dist}(p,x)) - m(\text{dist}(p,q))}{m(\text{dist}(q,x))}
\]

occurs in the \(k > 0\) case and is \(r = \frac{1}{2} \text{diam} S^n_k\).

Assume \(\text{rad}_p(q) > \frac{1}{2} \text{diam} S^n_k\). As \(r \mapsto m'(r - \text{dist}(p,q))/m'(r)\) is a strictly increasing function on \((0, \text{diam} S^n_k)\), it follows that \(m''(\text{dist}(p,q)) < \lambda_p(q)\).

Therefore, there must be a point \(x \in M \setminus \{q\}\) such that
\[
m''(\text{dist}(p,q)) < \frac{m(\text{dist}(p,x)) - m(\text{dist}(p,q))}{m(\text{dist}(q,x))} = m''(\text{dist}(p,q)) - m'(\text{dist}(p,q))\frac{m'(\text{dist}(q,x))}{m(\text{dist}(q,x))}\cos \tilde{\beta}_z,
\]

where the last equality comes from the fact that distances between vertices of \(\Delta\) are the same as those in \(\Delta\) and Proposition 4.

This says that \(\tilde{\beta}_z > \pi/2\). If \(\text{rad}_p(q) < \infty\), \(\lambda_p(q) < \infty\), by Proposition 2 we know \(\beta_x \leq \pi/2\) so \(\beta_x < \tilde{\beta}_z\). However, the triangle version of Toponogov’s Theorem says \(\beta_z \leq \beta_x\). So, \(\text{rad}_p(q) \leq \frac{1}{2} \text{diam} S^n_k\).

For the lower bound of Part (2), if \(\text{dist}(p,q) < \text{rad} M\), then Lemma 2 says, in particular, \(\text{seg} (p) \subset D^\text{\tilde{r}}_k(\text{rad}_p(q))\), so \(\text{rad} M \leq \text{rad}_p(q)\). This also gives Part (1).

For the last statement, if \(p\) realizes the radius of \(M\) and \(\text{dist}(p,q) = \text{rad} M\), then for all \(x \in M\), \(\text{dist}(p,x) \leq \text{dist}(p,q)\). By the monotonicity of \(r \mapsto m'(r - \text{dist}(p,q))/m'(r)\), this says that \(\lambda_p(q) \leq 0\) and so \(\text{rad}_p(q) \leq \text{dist}(p,q) = \text{rad} M\) completing the proof.

For a point \(p \in M\), let \(A(p)\) be the set of points at maximal distance from \(p\). If \(p\) realizes the radius of \(M\), it follows from Proposition 5 that \(A(p) = A_{\text{rad} M, \text{rad} M}(p)\).
Proof of Proposition 3. We show $p$ is a critical point for the distance from $A(p)$.

For any $x \in M$, let $\langle \cdot, \cdot \rangle$ denote the induced metric on $U_x M$. Set $R = \text{rad } M$ and suppose for a contradiction that there is a vector $g \in U_p M$ so that

$$\langle (g, \hat{A}(p)) \rangle > \pi/2.$$ 

As $\hat{A}_p(p) \subset U_p M$ is compact, there is a $\theta_0 > 0$ so that

$$\langle (g, \hat{A}(p)) \rangle \geq \pi/2 + \theta_0.$$ 

By definition of $A(p)$ and continuity of $\exp_p$, there is an $r_0 > 0$ so that any normal geodesic $\sigma_p$ emanating from $p$ that satisfies

$$\langle (g, \hat{\sigma}_p(0)) \rangle \leq \pi/2 + \theta_0/2$$

cannot be distance minimizing on $[0, R - r_0]$.

Let $\tilde{p} \in S_k^p$ and let $\tilde{\gamma}_\tilde{p}$ be a geodesic satisfying $\tilde{\gamma}_\tilde{p}(0) = \tilde{p}$. By first variation there is a $t_0 > 0$ so that if $\tilde{x} \in S_k^p$ makes an angle less than $\pi/2 - \theta_0/2$ with $\tilde{\gamma}_\tilde{p}$ at $\tilde{p}$, then

$$|\tilde{\gamma}_\tilde{p}(t)\tilde{x}| < |\tilde{p}\tilde{x}|$$

for all $t \in (0, t_0)$.

Now let $\tilde{\gamma}_p : \mathbb{R} \to M$ be a geodesic that satisfies $\gamma_p(0) = p$ and $\dot{\gamma}_p(0) = -g$. For any point $x \in M$ for which there is a segment $\gamma_{px}$ from $p$ to $x$ so that $\langle (-g, \dot{\gamma}_{px}(0)) \rangle < \pi/2 - \theta_0/2$, by the hinge version of Toponogov’s Theorem

$$\text{dist}(\gamma_p(t), x) < \text{dist}(p, x) \leq R$$

for all $t \in (0, t_0)$.

On the other hand we know any point $x \in M$ for which there is a segment $\gamma_{px}$ from $p$ to $x$ with $\langle (-g, \dot{\gamma}_{px}(0)) \rangle \geq \pi/2 - \theta_0/2$, satisfies

$$\text{dist}(p, x) < R - r_0.$$ 

By the triangle inequality, for any point $x \in M$ the point $\gamma_p(c)$ where $c \in (0, \min\{t_0, r_0\})$ satisfies $\text{dist}(\gamma_p(c), x) < R$ and so $M$ must have radius less than $R$. \qed

2. Volume Bound

Next we aim to prove the volume inequality in Part 1 of Theorem 1.

Given a point $\tilde{p} \in S_k^\tilde{p}$, let $S(\tilde{p}, c) := \{\tilde{q} \in S_k^\tilde{p} \mid ||\tilde{p}\tilde{q}|| = c\}$ denote the metric sphere of radius $c$ at $\tilde{p}$. If $\tilde{Q} \subset S(\tilde{p}, c)$ forms a $\pi/2$-net in $S(\tilde{p}, c)$, then for any $r > 0$,

$$\text{vol} \left( \bigcap_{\tilde{q} \in \tilde{Q}} D(\tilde{q}, r) \right) \leq \text{vol} (D(\tilde{a}_1, r) \cap D(\tilde{a}_2, r))$$

where $\tilde{a}_1, \tilde{a}_2 \in S(\tilde{p}, c)$ and $\angle \tilde{a}_1 \tilde{p} \tilde{a}_2 = \pi$. Equality in (9) occurs only when $\tilde{Q} = \{\tilde{q}_1, \tilde{q}_2\}$ with $\angle \tilde{q}_1 \tilde{p} \tilde{q}_2 = \pi$. This follows from Inequality (1.4) in [1], and is the complimentary version of the Inequality of the same name in [2].

Proof of Part 1 of Theorem 1. Take $h, r \in (0, \frac{1}{2}\text{diam } S_k^p]$ and assume that $\text{sag}_r M \leq h \leq \text{rad } M \leq r$. 

Take a point \( p \in M \) such that the distance from \( A_{r,h}(p) \) is critical at \( p \). Let \( \tilde{p} \) be the origin of \( \text{seg}(p) \). For each \( q \in A_{r,h}(p) \) and \( v_q \in \mathbb{R}^q \), let \( \tilde{\gamma}_{v_q} \) and \( \tilde{a}_{v_q} \) be as in Lemma [1]. By assumption, \( \mathbb{H}_p^{A_{r,h}(p)} \subset U_p M \) forms a \( \pi/2 \)-net and by Lemma [2]

\[
(10) \quad \text{seg}(p) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)).
\]

In addition, for each \( q \in A_{r,h}(p) \),
\[
\text{rad}_p(q) \leq r, \quad \text{dist}(p, q) \leq h,
\]

and for every \( v_q \in \mathbb{R}_p^q \),
\[
\tilde{a}_{v_q} = \tilde{\gamma}_{v_q}(0), \quad \tilde{p} = \tilde{\gamma}_{v_q}(\text{rad}_p(q) - \text{dist}(p, q)).
\]

For each \( q \in A_{r,h}(p) \) and \( v_q \in \mathbb{R}_p^q \), set
\[
\tilde{b}_{v_q} := \tilde{\gamma}_{v_q}(-(r - \text{rad}_p(q))),
\]
and
\[
\tilde{c}_{v_q} := \tilde{\gamma}_{v_q}((h - \text{dist}(p, q)) - (r - \text{rad}_p(q))).
\]

Then \( |\tilde{b}_{v_q} \tilde{a}_{v_q}| = r - \text{rad}_p(q) \), and by the Triangle Inequality,

\[
(11) \quad D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)) \subset D_k^n(\tilde{b}_{v_q}, r).
\]

Also, \( |\tilde{b}_{v_q} \tilde{p}| = r - \text{dist}(p, q) \) so,

\[
(12) \quad |\tilde{c}_{v_q} \tilde{p}| = r - h \leq |\tilde{b}_{v_q} \tilde{p}|.
\]

Note that

\[
(13) \quad \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{b}_{v_q}, r) \subset D(\tilde{p}, r).
\]

If not, then there is a point \( \tilde{x} \in \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{b}_{v_q}, r) \), with \( |\tilde{p} \tilde{x}| > r \). If \( \tilde{\sigma} \) is then a segment from \( \tilde{p} \) to \( \tilde{x} \), it would follow that the angle at \( \tilde{p} \) between \( \tilde{\sigma} \) and the segment from \( \tilde{p} \) to any \( \tilde{b}_{v_q} \) must be strictly less than \( \pi/2 \). This implies that \( \tilde{\sigma}'(0) \) is a non-zero gradient direction at \( \tilde{p} \) for the distance from \( A_{r,h}(p) \), contradicting the assumption.

It now follows from (10), (11), (12), (13), and the Triangle Inequality that

\[
(14) \quad \text{seg}(p) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{b}_{v_q}, r) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{c}_{v_q}, r).
\]
From Inequality (9) and Equation (12), we have for any \( \tilde{a}_1, \tilde{a}_2 \in S^n_k \) with \( |\tilde{a}_1 \tilde{a}_2| = 2(r - h) \),

\[
\text{vol } M \leq \text{vol (seg (p))} \\
\leq \text{vol } \left( \bigcap_{q \in \mathcal{A}_{r,h}(p)} D^n_k(\tilde{c}_q, r) \right) \\
\leq \text{vol } (D^n_k(\tilde{a}_1, r) \cap D^n_k(\tilde{a}_2, r)) \\
= \text{vol } L^n_k(h, r)
\]
as desired. \( \square \)

For future reference, we extract from Equations (12) and (14) in the proof above the following

**Proposition 6.** If \( h, r \in (0, \frac{1}{2}\text{diam } S^n_k) \) and \( \text{sag}_r M \leq h \leq \text{rad } M \leq r \), then there is a point \( p \in M \) such that if \( \tilde{p} \) is the origin of \( \text{seg } (p) \), there is a set \( \{ \tilde{c}_q \}_{q \in \mathcal{A}_{r,h}(p)} \subset S^n_k \) so that

\[
\text{seg } (p) \subset \bigcap_{q \in \mathcal{A}_{r,h}(p)} D^n_k(\tilde{a}_q, \text{rad }_p(q))
\]

Moreover, if \( h < r \), then \( \{ \tilde{c}_q \}_{q \in \mathcal{A}_{r,h}(p)} \) is a \( \pi/2 \)-net in the metric sphere \( S(\tilde{p}, r - h) \).

3. **Convergence and Topological Identification**

The goal of this section is to prove the topological version of Part 2 of Theorem 1 by proving Theorem 2. Most of the ideas in this section are taken directly from the analogous section in [2].

We fix \( n \geq 2 \) and real numbers \( k \in \mathbb{R}, h, r \in (0, \frac{1}{2}\text{diam } S^n_k) \), with \( h \leq r \). Fix a sequence \( \{ M_i \}_{i=1}^{\infty} \) of compact, Riemannian \( n \)-manifolds satisfying \( k \leq \text{sec } M_i \),

\[
\text{sag}_r M_i \leq h \leq \text{rad } M_i \leq r
\]
and

\[
\text{vol } M_i \to \text{vol } L^n_k(h, r).
\]

In each \( M_i \), take a point \( p_i \in M_i \) for which \( \mathcal{A}_{r,h}(p_i) \) is nonempty and critical at \( p_i \). By Gromov’s Compactness Theorem, \( M_i \to X \) where \( X \) is an \( n \)-dimensional Alexandrov space with curvature bounded below by \( k \). Let \( p := \lim p_i \). It follows that a subsequence of the sequence of domains \( \{ \text{seg } (p_i) \} \) converges to a compact subset \( \text{seg } (p) \subset S^n_k \), and a subsequence of the sequence of maps \( \{ \exp_{p_i} : \text{seg } (p_i) \to M_i \} \) converges to a surjective, 1-Lipschitz map \( \exp : \text{seg } (\tilde{p}) \to X \), (see [1] or [2] for details). The set \( \text{seg } (p) \subset S^n_k \) is star convex at a point \( \tilde{p} \in \text{seg } (p) \) and the map \( \exp \) maps segments emanating from \( \tilde{p} \) in \( \text{seg } (p) \) to segments emanating from \( p \) in \( X \). Conversely, any segment in \( X \) emanating from \( p \) is in the image under \( \exp \) of a segment in \( \text{seg } (p) \) that emanates from \( \tilde{p} \).

**Proposition 7.** \( \text{seg } (p) = L^n_k(h, r) \)
Proof. Set $A_{r,h}(p) := \lim_i A_{r,h}(p_i)$. From the definition of eccentricity, it follows that for each $q_i \in A_{r,h}(p_i)$ converging to $q \in A_{r,h}(p)$, $rad_{p_i}(q_i) \to rad_p(q)$. Therefore, because every $q_i \in A_{r,h}(p_i)$ satisfies $rad_{p_i}(q_i) \leq r$ and $dist(p_i, q_i) \leq h$, the same is true for every $q \in A_{r,h}(p)$. In addition, because for every $i$ the distance from the set $A_{r,h}(p_i)$ is critical at $p_i$, it follows that the distance from $A_{r,h}(p)$ is critical at $p$, i.e., $\hat{p}^\ast_{A_{r,h}(p)} \in \Sigma_m$ is a $\pi/2$-net. Therefore, since for every $i$, $\lim_{A_{r,h}(p_i)} \subset p_i$, it follows that
\[
\text{seg}(p_i) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)),
\]
it follows that
\[
\text{seg}(p) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)).
\]
By Proposition 6, we can select points $\{\tilde{c}_{v_q}\} \subset S(p, r - h)$ indexed over $v_q \in \bigcap_{A_{r,h}(p)}$ such that
\[
\bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{a}_{v_q}, \text{rad}_p(q)) \subset \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{c}_{v_q}, r).
\]
Because $\text{seg}(p_i) \to \text{seg}(p)$, it follows that,
\[
\text{vol } L_k^n(h, r) = \lim_i \text{vol } M_i
\leq \lim_i \text{vol } (\text{seg}(p_i))
\leq \text{vol } (\text{seg}(p))
\leq \left(\text{vol } \bigcap_{q \in A_{r,h}(p)} D_k^n(\tilde{c}_{v_q}, r)\right)
\leq \text{vol } L_k^n(h, r).
\]
From the equality statement in Inequality (\ref{inequality}), we must have $\{\tilde{c}_{v_q}\} = \{\tilde{c}_{v_{q_1}, \tilde{c}_{v_{q_2}}}\}$ where $|\tilde{c}_{v_{q_1}, \tilde{c}_{v_{q_2}}}| = 2(r - h)$. Therefore,
\[
\text{seg}(p) \subset D_k^n(\tilde{c}_{v_{q_1}}, r) \cap D_k^n(\tilde{c}_{v_{q_2}}, r) = L^n_k(h, r).
\]
From this and because $\text{vol } (\text{seg}(p)) = \text{vol } L^n_k(h, r)$, it follows that $\text{seg}(p) = L^n_k(h, r)$.

To continue, we’ll first fix notation for certain related geometric attributes of $L^n_k(h, r)$. Let $\tilde{a}_1, \tilde{a}_2, \tilde{q}_1, \tilde{q}_2 \in S^n_k$ such that $|\tilde{a}_1 \tilde{a}_2| = 2(r - h)$, $|\tilde{p}_\tilde{q}_1| = h$, and $|\tilde{a}_1 \tilde{q}_1| = r$. We assume that
\[
\text{seg}(p) = L^n_k(h, r) := D^n_k(\tilde{a}_1, r) \cap D^n_k(\tilde{a}_2, r).
\]
Denote the interior of $L^n_k(h, r)$ by $\hat{L}_k^n(h, r)$.

For $i = 1, 2$, let
\[
D_i^{n-1} := \partial D^n_k(\tilde{a}_i, r) \cap \partial L^n_k(h, r)
\]
and note $D_i^{n-1}$ is a disk in the metric sphere $\partial D^n_k(\tilde{a}_i, R)$ centered at $\tilde{q}_i$. Moreover, these disks have equal radii and $\partial L^n_k(h, R) = D_1^{n-1} \cup D_2^{n-1}$ (See Figure 3).

For distinct $i, j \in \{1, 2\}$, let
\[
B_i^{n-1} = \hat{D}_i^{n-1} := \{\tilde{x} \in S^n_k \mid |\tilde{x} \tilde{a}_i| < R \text{ and } |\tilde{x} \tilde{a}_j| = R\}
\]
be the interior of $D_i^{n-1}$.

\*\*\*\*

\small{\text{\textsuperscript{1}}\Sigma_m \text{ denotes the space of directions at } p \text{ which, with the angle metric, is an Alexandrov space with curvature bounded below by } 1.}
Let \( \tilde{s} : \mathbb{R} \to S^n_k \) be the geodesic through \( \tilde{a}_1 \) and \( \tilde{a}_2 \) such that \( \tilde{s}(-h) = \tilde{q}_2, \tilde{s}(0) = \tilde{p}, \) and \( \tilde{s}(h) = \tilde{q}_1. \) For each \( t \in [-h, h], \) let \( H_t \) be the totally geodesic hyperplane in \( S^n_k \) through \( \tilde{s}(t) \) and orthogonal to \( \tilde{s}'(t) \) and let \( S_t := H_t \cap \partial L^n_k(h, r) \) be the \((n-2)\)-dimensional metric sphere in \( H_t. \) Note that \( S_0 = \partial D^{n-1}_1 = \partial D^{n-1}_2 = D^{n-1}_1 \cap D^{n-1}_2. \)

We recall a crucial observation made in [2]. Let \( M \) be a compact, Riemannian \( n \)-manifold with \( \text{sec} M \geq k \in \mathbb{R}. \) Let \( p \in M \) be a point that realizes the radius of \( M \) and let \( Q \subset M \) and \( r : Q \to \mathbb{R}^+ \) a function. If \( \hat{p} \) is the origin of \( \text{seg} (p), \) and \( \hat{Q} := \exp^{-1}_p(Q) \subset S^n_k, \) the so called “Swiss Cheese” volume comparison given in [2] says,

\[
\text{vol} \left( M - \bigcup_{q \in Q} B(q, r(p)) \right) \leq \text{vol} \left( D^n_k(\hat{p}, \text{rad} M) - \bigcup_{\hat{q} \in \hat{Q}} B(\hat{q}, r \circ \exp_p(\hat{q})) \right).
\]

Now let \( \tilde{p}_i \) be the origin of \( \text{seg}(p_i). \) By Proposition 6 for each \( i, \)

\[
\text{seg}(p_i) \subset I(\tilde{p}_i, r) := \bigcap_{q \in A_{r, s}(p_i)} D^n_k(\tilde{c}_{s q_i}, r),
\]

and a straightforward modification of the above shows

\[
(15) \quad \text{vol} \left( M_i - \bigcup_{q \in Q} B(q, r(p)) \right) \leq \text{vol} \left( I(\tilde{p}_i, r) - \bigcup_{\tilde{q} \in \hat{Q}} B(\tilde{q}, r \circ \exp_p(\tilde{q})) \right).
\]

Up to needing Equation (15), the proofs of Parts (1) - (3) of the next Lemma are identical to the proofs of the analogous parts in Lemma 2.5 in [2]. Up to a modification of details, the proof of Part (4) is also the same. We give the proof of Part (4) and remark that it holds the main ingredient for the proof of all parts (cf. [4]).

**Lemma 3.** The map \( \exp_p : L^n_k(h, r) \to X \) satisfies

1. \( \exp|_{L^n_k(h, r)} \) is injective,
Proof of Part 4. Let $\tilde{q} \in S_0$ and $\rho > 0$. Let $B(\tilde{q}, \rho)$ be a metric ball in $S^n_k$ centered at $\tilde{q}$ of radius $\rho$. By Bishop-Gromov, the function
\[
\rho \to \frac{\text{vol} B(\tilde{q}, \rho)}{\text{vol} (B(\tilde{q}, \rho) \cap L^n_k(h, r))}
\]
is nondecreasing. Because this function is bounded below by 1, let $c(n, k, h, r)$ be the smallest integer larger than
\[
\lim_{\rho \to 0} \frac{\text{vol} B(\tilde{q}, \rho)}{\text{vol} (B(\tilde{q}, \rho) \cap L^n_k(h, r))}.
\]
By symmetry, $c$ is independent of $\tilde{q}$.

Let $q \in X$. For a contradiction, suppose there are $c+1$ distinct points $\{\tilde{x}^k\}_{k=1}^{c+1} \subset \partial L^n_k(h, r)$ such that $\exp_p(\tilde{x}^k) = q$. Choose $\rho > 0$ small enough so that $\{B(\tilde{x}^k, \rho)\}_{k=1}^{c+1} \subset S^n_k$ is a disjoint collection. For each $i$, let $\{\tilde{x}^k_i\}_{k=1}^{c+1} \subset \text{seg}(p_i)$ be chosen so that $\lim_i \{\tilde{x}_i^k\} = \{\tilde{x}^k\}$ and $\lim_{i} \exp_{p_i}(\tilde{x}_i^k) = \exp_p(\tilde{x}^k) = q$.

Note that for each $i$ and $k \in \{1, \ldots, c+1\}$,
\[
\text{vol} (B(\tilde{x}^k, \rho) \cap L^n_k(h, r)) \leq \text{vol} (B(\tilde{x}_i^k, \rho) \cap I(\tilde{p}_i, r)).
\]
If $i$ is large enough so that $\{B(\tilde{x}_i^k, \rho)\}$ are disjoint, by Equation (15), the display above, and the definition of $c$,
\[
\text{vol} M_i - \text{vol} \left( \bigcup_{k=1}^{c+1} B(\exp_{p_i}(\tilde{x}_i^k), \rho) \right) & \leq \text{vol} I(\tilde{p}_i, r) - \sum_{k=1}^{c+1} \text{vol} (B(\tilde{x}_i^k, \rho) \cap I(\tilde{p}_i, r)) \\
& \leq \text{vol} I(\tilde{p}_i, r) - \sum_{k=1}^{c+1} \text{vol} (B(\tilde{x}^k, \rho) \cap L^n_k(h, r)) \\
& \leq \text{vol} I(\tilde{p}_i, r) - \frac{c+1}{c} \text{vol} (B(\tilde{q}, \rho)).
\]
Because $\bigcup_{k=1}^{c+1} B(\exp_{p_i}(\tilde{x}_i^k), \rho) \to B(q, \rho)$, and $\text{vol} I(\tilde{p}_i, r) \to \text{vol} L^n_k(h, r)$, the above contradicts the assumption $\text{vol} M_i \to \text{vol} L^n_k(h, r)$.

Let $R_{H_0} : S^n_k \to S^n_k$ be reflection over the hyperplane $H_0$. Note that $R_{H_0}(D_i^{n-1}) = D_j^{n-1}$ for $i \neq j \in \{1, 2\}$. Equip $\partial L^n_k(h, r)$ with the induced length metric from $S^n_k$. Note this metric restricted to either $D_1^{n-1}$ or $D_2^{n-1}$ is Riemannian of constant curvature and with these metrics $D_1^{n-1}$ and $D_2^{n-1}$ are isometric.

**Lemma 4.** There is a positive integer $m$ and an isometry $\phi : \partial L^n_k(h, r) \to \partial L^n_k(h, r)$ which fix $\tilde{q}_1, \tilde{q}_2$, leaves $S_0$ invariant and satisfies $\phi^m = \text{id}$. Moreover, $\exp_p : L^n_k(h, r) \to X$ induces an isometry between $X$ and either

1. $L^n_k(h, r)/(\tilde{u} \sim \phi(\tilde{u}))$ provided $\phi$ is an involution, or
2. $L^n_k(h, r)/(\tilde{u} \sim R_{H_0} \circ \phi(\tilde{u}))$. 

\[
(2) \ \exp_p \text{ preserves the length of paths,} \\
(3) \ \exp_p |_{B^n_\rho} \text{ is at most } 2 \text{ to } 1, \text{ and} \\
(4) \ \text{there is a positive integer } c(n, k, h, r) \text{ such that } \exp_p |_{S_0} \text{ is no more than } c \text{ to } 1.
\]
Proof. As in [2], by Part 3 of Lemma 3 we can define a map $f : B_n^{n-1} \cup B_0^{n-1} \to \partial L^k_n(h,r)$ by

$$f(\tilde{u}) = \begin{cases} \tilde{u} & \text{if } \exp_p^{-1}(\exp_p(\tilde{u})) = \{\tilde{u}\} \\ \tilde{v} & \text{if } \exp_p(\tilde{u}) = \exp_p(\tilde{v}), \tilde{u} \neq \tilde{v} \end{cases}.$$  

As noted in [2], the map $f$ is continuous as a point of discontinuity would produce a bifurcation of geodesics in $X$. By Part 2 of Lemma 3 $f$ is 1-Lipschitz, so it uniquely extends to a continuous map $f : \partial L^k_n(h,r) \to \partial L^k_n(h,r)$.

Assume first that $r - h > 0$. In this case, $\lambda_p(\tilde{q}_1) = \lambda_p(\tilde{q}_2) > 0$. In particular, if $\tilde{x} \in D_i^{n-1}$, by Proposition 1 $m(|\tilde{p}|) = \lambda_p(\tilde{q}_1)m(\tilde{x}_{\tilde{p}}) - m(|\tilde{p}|)$. Therefore, if $\tilde{x}, \tilde{y} \in \partial L^k_n(h,r)$, then $|\tilde{p}| = |\tilde{q}|$ if and only if for some $t \in [-h,h]$, either $\tilde{x}, \tilde{y} \in S_t$ or $\tilde{x} \in S_t$ and $\tilde{y} \in S_{-t}$. By Parts 1 and 2 of Lemma 3 it follows that for all $t \in [0,h]$, either $f(S_t) = S_t$ or $f(S_t) = S_{-t}$. By continuity, it follows that $f(D_i^{n-1}) = D_j^{n-1}$ for either $i = j \in \{1,2\}$ or $i \neq j \in \{1,2\}$.

In the case $f(D_i^{n-1}) = D_j^{n-1}$ for $i \in \{1,2\}$ set $\phi = f$. By Part 3 of Lemma 5 it follows that $\phi^n = id$, in particular, $\phi$ must be an isometry. By Part 1 of Lemma 5, it follows that $X$ is isometric to $L^k_n(h,r)/(\tilde{u} \sim \phi(\tilde{u}))$.

In the case $f(D_i^{n-1}) = D_j^{n-1}$ for $i \neq j \in \{1,2\}$, set $\phi = R_{H_0} \circ f$. For each $\tilde{u} \in S_0$, $\exp_p^{-1}(\exp_p(\tilde{u})) = \{\phi^k(\tilde{u}) | k \in \mathbb{N}\}$, so it follows from Part 4 of Lemma 3 that $\phi^m = id$ for some positive integer $m \leq c(n,k,h,r)!$, in particular, $\phi$ must be an isometry. In this case $X$ is isometric to $L^k_n(h,r)/(\tilde{u} \sim R_{H_0} \circ \phi(\tilde{u}))$.

When $r = h$ we have $L^k_n(h,r) = D^k_0(r)$. This case is handled by Lemma 2.6 in [2] where they show that the identification must occur via an isometric involution. Up to an isometry of $D^k_0(r)$, the conclusion is the same. □

Lemma 5. Let $c(n,k,h,r)$ be as in Lemma 3. If $h < r$, there is a totally geodesic hyperplane $P \subset S^1_n$ that contains the geodesic through $\tilde{a}_1$ and $\tilde{a}_2$, such that if $R_P : S^1_n \to S^1_n$ is reflection over $P$, then $X$ is isometric to either

- (A) $P^k_n(h,r) = L^k_n(h,r)/(\tilde{u} \sim R_P(\tilde{u}))$, where $\tilde{u} \in \partial L^k_n(h,r)$,
- (B) $C^k_n(h,r,\phi_m) = L^k_n(h,r)/(\tilde{u} \sim R_{H_0} \circ \phi_m(\tilde{u}))$, where $\tilde{u} \in \partial L^k_n(h,r)$, or
- (C) $C^k_n(h,\phi_m) = L^k_n(h,r)/(\tilde{u} \sim R_{H_0} \circ \phi_m(\tilde{u}))$, where $\tilde{u} \in \partial L^k_n(h,r)$, and

- (a) $\phi_m : \partial L^k_n(h,r) \to \partial L^k_n(h,r)$ is an isometry which leaves $S_0$ invariant,
- (b) $\phi_m$ has order $m \in \{2,\ldots,c\}$, and
- (c) the cyclic group $\mathbb{Z}_m = \langle \phi_m \rangle$ acts freely and orthogonally on $S_0$.

Proof. Given that $X$ is the Gromov-Hausdorff limit of a sequence of Riemannian manifolds with an upper diameter bound, lower curvature bound, and lower volume bound, by Perelman’s Stability Theorem, it follows that $X$ is a topological manifold (see [3], Lemma 3.2).

By Lemma 4 there are two possibilities for the isometry type of $X$.

Case 1: $X$ is isometric to $L^k_n(h,r)/(\tilde{u} \sim \phi(\tilde{u}))$ where $\phi : \partial L^k_n(h,r) \to \partial L^k_n(h,r)$ is an isometric involution that fixes $\tilde{q}_1$ and $\tilde{q}_2$.

Identify $L^k_n(h,r)$ with the unit disk $D^m \subset \mathbb{R}^m = \mathbb{R} \oplus J$ and the isometry $\phi$ with a linear involution $\phi_2 : \mathbb{R}^m \to \mathbb{R}^m$ such that $\phi_2(J) = J$.

As in Lemma 2.7 of [2], because $X$ is a manifold, there are only two possibilities: $\phi_2 = -id$ or $\phi_2 = R_J$ where $R_J : \mathbb{R}^m \to \mathbb{R}^m$ is reflection over $J$. This follows by observing that because $\phi_2$ is an isometric involution we can assume for some $j$, $\phi_2|_{\mathbb{R}^j \times \{0\}} = id$ and $\phi_2|_{\{0\} \times \mathbb{R}^{n-j}} = -id$. It follows $X$ must be homeomorphic to the
j-fold suspension $\Sigma^j \mathbb{R} P^{n-j}$ which has the homology of a manifold only when $j = 0$ or $n - 1$.

Because the isometry $\phi$ in Lemma 1 fixes $\tilde{q}_1$ and $\tilde{q}_2$, it follows that $\phi = R_P$ where $P$ is a hyperplane that, if $h < r$, contains the geodesic through $\tilde{a}_1$ and $\tilde{a}_2$. This gives Part (A).

Case 2: $X$ is isometric to $L^k_0(h, r)/\langle \hat{u} \sim R_{H_0} \circ \phi(\hat{u}) \rangle$ where $\phi : \partial L^k_0(h, r) \rightarrow \partial L^k_0(h, r)$ is an isometry that has finite order and fix $\hat{q}_1$ and $\hat{q}_2$.

Let $C$ be the unit circle in $\mathbb{R}^2 = \mathbb{C}$. For $j \in \{1, 2, \ldots, m\}$, let $s_j := \{e^{i\theta} \in C \mid 2\pi \frac{j-1}{m} \leq \theta \leq 2\pi \frac{j}{m}\}$. Let $\mathbb{R}^{n+1} = \mathbb{C} \oplus J$. Identify $S_0 \subset H_0$ with the unit $(n-2)$-sphere $S^{n-2}_j$ in $J$ and the isometry $\phi$ with an isometry $\phi_m : J \rightarrow J$ that satisfies $(\phi_m)^m = \text{id}$ for some positive integer $m$. For each $j \in \{1, 2, \ldots, m\}$, let

$$D_j^0 := s_j \ast S^{n-2}_j,$$

and

$$D_j^{n-1} := \{e^{i\pi \frac{j-1}{m}}\} \ast S^{n-2}_j \text{ for } l = 1, 2,$$

where $\ast$ denotes the join relation.

Identify $L^k_0(h, r)$ with $D^0_j$ and $D_j^{n-1} \subset \partial L^k_0(h, r)$ with $D_j^{n-1}$ for $l = 1, 2$. Note for each $j$, $\partial D_j^0 = D_{j,1}^{n-1} \cup D_{j,2}^{n-1}$ and that

$$\partial D_{j,1}^{n-1} = \partial D_{j,2}^{n-1} = D_{j,1}^{n-1} \cap D_{j,2}^{n-1} = S_j^{n-2}.$$

Now let $\varphi_m : \mathbb{C} \oplus J \rightarrow \mathbb{C} \oplus J$ be an isometry of $\mathbb{R}^{n+1}$ such that

1. $\varphi_m|_{\mathbb{C}} = e^{2\pi i (\hat{u})}$, and
2. $\varphi_m|_J = \phi_m$.

It follows from the definition of $\varphi_m$ and Part (2) of Lemma 1 that

1. $\varphi_m$ has order $m$,
2. the cyclic group $\mathbb{Z}_m = \langle \varphi_m \rangle$ acts orthogonally on $S^n = C \ast S^{n-2}_j$,
3. $X$ is homeomorphic to the quotient $S^n/\langle \varphi_m \rangle$.

Let $\pi : S^n \rightarrow S^n/\langle \varphi_m \rangle$ be the projection map. If, in addition, the action of $\langle \varphi_m \rangle$ on $S^n$ is free, by Part (4) of Lemma 1 we have $m \in \{2, \ldots, c\}$ and so Part (C) holds. In particular, $X$ is homeomorphic to a Lens space $S^n/\mathbb{Z}_m$.

Therefore, let $x \in S^n$, $H = \langle \varphi_m \rangle_x$ be the isotropy group at $x$, and assume that $H \neq \{\text{id}\}$. Let

$$\langle S^n \rangle^H = \{y \in S^n \mid h(y) = y \text{ for all } h \in H\}$$

be the fixed point set of $H$. Because the action is linear, $(S^n)^H = S^r$ for some $r \in \{0, 1, \ldots, n\}$. Take $T_x^r$ to be the unit tangent sphere in $T_x(S^n)^H$ and let $S^{n-r}_{x-r}$ be the unit normal sphere to $(S^n)^H$ at $x$. Then $H$ acts orthogonally on $S^{n-r}_{x-r}$ and the space of directions at $\pi(x)$ is given by

$$\Sigma_{\pi(x)} S^n/\langle \varphi_m \rangle = S_x^r \ast (S^n_{x-r}/H).$$

Because $S^n/\langle \varphi_m \rangle \cong X$ is a manifold, $S^{n-r}_{x-r}/H$ must be an $(n-r)$-sphere. As $H$ is cyclic, it follows that $S^{n-r}_{x-r}$ must be a circle. In particular $(S^n)^H = S^{n-1}$. Because $\varphi_m|_{\mathbb{C}} = e^{2\pi i (\hat{u})}$, it follows that $\varphi_m|_J = \phi_m = \text{id}$. This gives Part (B) completing the proof.

**Proof of Part 2 of Theorem 1 (topological version).** The identification spaces $P^n_k(h, r)$ and $C^n_k(h, r, \text{id})$, topologically, are spheres. The proof of Lemma 5 shows $C^n_k(h, r, \phi_m)$ is homeomorphic to a quotient of $S^n$ by a free and orthogonal action of $\mathbb{Z}_m$ where $m \leq c(n, h, r)$, thus is a Lens space. The proof now follows...
from Gromov’s Compactness Theorem, Perelman’s Stability Theorem, and Lemma

4. Smooth Perturbation of the Limits

To construct Riemannian metrics that satisfy the hypotheses of Theorem \[1\] we give smooth perturbations of \(C^n_k(h, r, \phi_m)\) and \(P^n_k(h, r)\), for any \(\phi_m \in \{\phi_m\}_{i \in I_m}\) as in Example \[1\].

**Proof of Part 3 of Theorem \[1\]** For perturbations of \(C^n_k(h, r, \phi_m)\) and \(P^n_k(h, r)\)

we follow \[2\]. For the hyperplane \(J = H_0\) or \(P\), define \(L^{n, +}_k(J)(r)\) to be one side of \(L^n_k(h, r)\) separated by \(J\). Isometrically embed \(L^{n, +}_k(J)(r)\) into a totally geodesic \(S^n_k \subset S^{n+1}_k\) and take boundaries of smooth, symmetric, convex neighborhoods of \(L^{n, +}_k(J)(r)\).

For perturbations of \(C^n_k(h, r, \phi_m)\), since \(m \leq c(n, k, h, r)\) we can find \(m\) points \(\tilde{p}_1, \ldots, \tilde{p}_m\) on circle \(C\) of diameter no larger than \(\diam S^n_k\), such that \(|\tilde{p}_i \tilde{p}_{i+1}| = 2h\) (indices mod \(m\)).

Let \(R \in (0, \frac{1}{2}\diam S^n_k]\) be the intrinsic radii of the boundary disks \(D^{n-1}_k(h, r)\). Let \(k_{S_0} > 0\) be the constant curvature value of the metric on \(S_0 = D^{n-1}_k \cap D^{n-1}_k\) induced from \(D^n_k(\hat{a}_1, r)\). For \(\varepsilon > 0\), by using a doubly warped product metric on \([0, R - \varepsilon] \times C \times S^{n-2}\) we can construct a smooth metric \(g\) on \(S^n = C \times S^{n-2}\) such that induced metric on \(S^{n-2}\) has constant curvature \(k_{S_0} + \tau(\varepsilon)\) where \(\tau(\varepsilon) \searrow 0\) as \(\varepsilon \to 0\).

Smoothly deform \(g\) so that outside of an \(\varepsilon\)-neighborhood of \(\{p_1, \ldots, p_m\} \times S^{n-2}\) the deformed metric has constant curvature \(k\). This gives a smooth metric on \(S^n = S^n_1 \times S^{n-2}\) so that if \(Z_m\) acts on the \(S^1\) factor by taking \(\tilde{p}_i\) to \(\tilde{p}_{i+1}\), and by \(\langle \phi_m \rangle\) on the \(S^{n-2}\) factor we obtain a smooth Riemannian metric \(g_0\) on a Lens space \(S^n/Z_m\). The fundamental domain of this action on \(S^n_1\) converges to \(L^n_k(h, r)\) as \(\varepsilon \to 0\), so it follows that \(\text{vol} S^n_1/Z_m \to \text{vol} L^n_k(h, r)\).

\[\square\]

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