The asymptotic symmetry of de Sitter spacetime

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Abstract

We show how to construct a set of Euclidean conformal correlation functions on the boundary of a de Sitter space from an interacting bulk quantum field theory with a certain asymptotic behaviour. We discuss the status of the boundary theory w.r.t. the reflection positivity and conclude that no obvious physical holographic interpretation is available.
1 Introduction

It has been recently proposed a duality between a quantum theory on de Sitter space and a euclidean theory on its boundary [1] which should encode the de Sitterian quantum gravity degrees of freedom [1, 2]. In this paper we show that one can associate with a general (scalar) de Sitter quantum field theory satisfying suitable condition a conformal euclidean field theory on the boundary, here identified with a copy of the cone asymptotic to the de Sitter manifold in the embedding spacetime. However, the field theory that one gets this way does not in general satisfy reflection positivity, which is required to admit a physical interpretation. Therefore, the proposed construction can have in general a technical interest but no obvious holographic interpretation seems to be available.

2 Notations and geometry

Let us consider the vector space \( \mathbb{R}^{d+1} \) equipped with the Lorentz scalar product:

\[ X \cdot X' = X^0 X'^0 - X^1 X'^1 - \cdots - X^d X'^d. \]  

(1)

The \( d \)-dimensional dS universe can then be identified with the quadric

\[ dS_d = \{ X \in \mathbb{R}^d, \ X^2 = -R^2 \}, \]  

(2)

where \( X^2 = X \cdot X \), endowed with the induced metric

\[ ds^2 = \left( dX^0^2 - dX^1^2 - \cdots - dX^d^2 \right)_{dS_d}. \]  

(3)

The future cone is defined in the real Minkowski space \( \mathbb{R}^{d+1} \) as the subset

\[ V_+ = -V_- = \{ X \in \mathbb{R}^{d+1} : X^0 > 0, \ X \cdot X > 0 \} \]

and the future light cone as \( C_+ = \partial V_+ = -C_- \). The future cone induces the (partial) causal order defined by \( \preceq \), i.e. \( X \preceq Y \) if and only if \( Y - X \in \overline{V}_+ \) as vectors in the ambient space. The future and past shadows of a given event \( X \) in \( dS_d \) are given by

\[ \Gamma^+(X) = \{ Y \in dS_d : Y \geq X \}, \ \Gamma^-(X) = \{ Y \in dS_d : Y \leq X \}. \]

If \( X^2 = -R^2 \) and \( \eta^2 = 0 \), then \( (X + \eta)^2 = -R^2 \) is equivalent to \( x \cdot \eta = 0 \), and remains true if \( \eta \) is replaced with \( t \eta \) for any real \( t \). Hence the boundary set

\[ \partial \Gamma(X) = \{ Y \in dS_d : (X - Y)^2 = 0 \} \]  

(4)

of \( \Gamma^+(X) \cup \Gamma^-(X) \) is a cone (“light-cone”) with apex \( X \), the union of all linear generators of \( dS_d \) containing the point \( X \). Two events \( X \) and \( Y \) of \( dS_d \) are in space-like separated if \( Y \not\in \Gamma^+(X) \cup \Gamma^-(X) \), i.e. if \( X \cdot Y > -R^2 \).

The symmetry group of the de Sitter space-time, is the connected Lorentz group of the ambient Minkowski space, i.e. \( L_+^d = SO_0(1, d) \) leaving invariant each of the sheets of the cone \( C = C_+ \cup C_- \). \( L_+^d \) acts transitively on \( dS_d \).

We will also consider the complexification of \( dS_d \):

\[ dS_{d}^{(c)} = \{ Z = X + iY \in \mathbb{C}^d, \ Z^2 = -R^2 \}. \]  

(5)

In other terms, \( Z = X + iY \) belongs to \( dS_{d}^{(c)} \) if and only if \( X^2 - Y^2 = R^2 \) and \( X \cdot Y = 0 \). The complex Lorentz group \( L_+^d(\mathbb{C}) \) acts transitively on \( dS_{d}^{(c)} \).
The familiar forward and backward tubes are defined in complex Minkowski space as \( T_{\pm} = \mathbb{R}^{d+1} \pm i V_+ \), and we denote their intersection with the de Sitter manifold as follows:

\[
T_+ = T_+ \cap dS_d^{(c)}, \quad T_- = T_- \cap dS_d^{(c)}.
\] (6)

Since \( T_+ \cup T_- \) contains the "Euclidean subspace" of the complex Minkowski spacetime \( \mathbb{C}^{d+1} \), that is \( \mathbb{C}^{d+1} = \{ Z = (iY^0, X^1, \ldots, X^d) : (Y^0, X^1, \ldots, X^d) \in \mathbb{R}^{d+1} \} \) the subset \( T_+ \cup T_- \) of \( dS_d^{(c)} \) contains the "Euclidean sphere" \( S_d = \{ Z = (iY^0, X^1, \ldots, X^d) : Y^0)^2 + X^1)^2 + \ldots + X^d)^2 = R^2} \).

The de Sitter manifold admits a global parametrization \( X = X[\tau, \omega] \) whose "constant time" sections \( S(\tau) \) are spheres:

\[
\begin{cases}
X^0 = \sinh \tau \\
X^i = \cosh \tau \cdot \omega^i, \quad i = 1, \ldots, d, \\
\omega^2 = \omega^1 + \ldots + \omega^d = 1
\end{cases}
\] (7)

This parametrization has the advantage to globally describe the real de Sitter manifold. Another useful parametrization is the "horocyclic parametrization" \( X = X(x, v) \), obtained by intersecting \( dS_d \) with the hyperplanes \( X^0 + X^d = e^v \):

\[
\begin{cases}
X^0 = \sinh v + \frac{1}{2} e^v x^2 \\
X^i = e^v x^i \\
X^d = \cosh v - \frac{1}{2} e^v x^2
\end{cases}
\] (8)

For real values of the parameters it only covers the part \( II \) of the dS manifold which belongs to the half-space \( X^0 + X^d > 0 \) of the ambient space. Each slice \( \Pi_v \) (or "horosphere") is a (flat) paraboloid. The scalar product (1) and the dS metric can then be rewritten as follows:

\[
X \cdot X' = -\cosh(v - v') + \frac{1}{2} e^{v+v'} (x - x')^2,
\]

\[
ds^2 = dv^2 - e^{2v} dz^2
\] (9) (10)

Eq. (10) implies that

\[
(X(v, x) - X(v, x'))^2 = -e^{2v}(x - x')^2
\] (11)

This in turn implies that any slice \( \Pi_v \) is conformal to a Euclidean plane.

The de Sitter manifold has a boundary at timelike infinity. This can be easily understood by using a Penrose diagram. Another visualization can be obtained by taking the large \( \tau \) asymptotics in eq. (7):

\[
\begin{cases}
X^0 \approx \pm e^{\tau} \\
X^i \approx e^{\tau} \omega^i, \quad i = 1, \ldots, d, \\
e^2 = \omega^1 + \cdots + \omega^d = 1
\end{cases}
\] (12)

It follows that the ambient space light-cone

\[
C = C_+ \cup C_- = C_{1,d} = \{ \eta = (\eta^0, \eta^1, \ldots, \eta^{d+1}) : \eta^0^2 - \eta^1^2 - \cdots - \eta^{d+1}^2 = 0 \}
\]
can also be looked at as the boundary at timelike infinity of the de Sitter manifold. We will use the notation \( C_{1,d} \) to distinguish the boundary where the asymptotic theory will live from the light-cone itself, which has rather the interpretation of momentum space \( \mathbb{R}^d \). The invariance group of the cone \( C_{1,d} \), which is also a copy of \( SO(1, d) \), will be interpreted as the euclidean conformal group \( \mathbb{R}^d \).

By adapting the covering parametrization (6) of \( dS_d \) to the case of its asymptotic cone \( C_{1,d} = \{ \eta = (\eta^0, \eta^1, \ldots, \eta^{d+1}) : \eta^0^2 - \eta^1^2 - \cdots - \eta^{d+1}^2 = 0 \} \), one readily obtains the following parametrization:

\[
\begin{cases}
\eta^0 = r \\
\eta^i = r \omega^i, \quad i = 1, \ldots, d
\end{cases}
\] (13)
with $\omega^1 + \ldots + \omega^d = 1$ and $r \geq 0$, or in brief: $\eta = \eta[r, \omega]$.

By taking the intersection of $C_{1,d}$ with the family of hyperplanes with equation $\eta^0 + \eta^d = e^v$, one obtains the analogue of the horocyclic parametrization (3), namely:

$$\begin{cases}
\eta^0 = \frac{1}{2} e^v (1 - x^2) \\
\eta^i = e^v x^i \\
\eta^d = \frac{1}{2} e^v (1 + x^2)
\end{cases} \quad i = 1, \ldots, d - 1
$$

(14)

which implies the following identity (similar to (3)) between quadratic forms:

$$(\eta - \eta')^2 = -e^{v+v'}(x - x')^2
$$

(15)

By taking Eqs. (13) into account, one then sees that these formulae correspond to the embedding of Euclidean space into the the cone $C_{1,d}$ namely one has (in view of the identification $\eta^0 + \eta^d = e^v = r(\omega^d + 1)$):

$$x^i = \frac{\eta^i}{\eta^0 + \eta^d} = \frac{\omega^i}{\omega^d + 1}.
$$

(16)

In section (4) we shall prove that boundary theories obtained as a certain limit from theories living in the bulk, are $SO_0(1,d)$ symmetric (i.e. have the euclidean conformal invariance). In this sense, boundary theories have the same symmetry of bulk de Sitter theories.

### 3 dS Quantum Field Theory

One of the possible formulations of the AdS/CFT correspondence states a duality between a perturbative (tree level) theory on AdS and a non perturbative CFT on the boundary [6]. Nothing similar can at the moment be said for the de Sitter case and therefore we must consider general (non-perturbative) de Sitter theories.

A general approach to those theories has been developed in recent years based on very general principles [4, 7] and we give here a very short account of what is needed for the present purpose. Various consequences of these general principles have been derived in [7] and most of the well-known properties of the Wightman distributions in the Minkowskian case [8], including the PCT symmetry, hold without change in the de Sitterian case under the assumptions specified below (see [7] for a detailed account).

Before entering in the discussion, two important remarks are however in order:

1. the physical interpretation of the axioms specified below is the thermal one. A geodesic
   observer will perceive the “vacuum” as populated by a thermal bath of particles, but it has to be stressed that we are talking here of an interacting quantum field theory;

2. the Reeh-Schlieder property holds. This property says that the application to the vacuum vector of the polynomial field algebra of any open set in the de Sitter manifold yields a dense set of the Hilbert space of the theory. This reduces to irrelevance all the argument based on the presence of observer’s horizons that are used to discard the regions of the de Sitter universe that are not accessible classically.

Let us consider therefore a general QFT on $dS_d$; for simplicity we limit the present discussion to one scalar field $\Phi(X)$. According to the general reconstruction procedure [8], a theory is completely determined by the set of all $n$-point vacuum expectation values (or “Wightman functions”) of the field $\Phi$, given as distributions on the corresponding product manifolds $dS_d^n$:

$$W_n(X_1, \ldots X_n)$$

(17)
An important class of fields, which can be explicitly constructed in a Fock space, is the class of "generalized free fields"; these fields are completely determined by their two-point function \( W_2(X_1, X_2) \). In particular, the Klein-Gordon fields are those for which \( W_2(X_1, X_2) \) satisfies the corresponding field equation w.r.t. both points. Of course there are in general infinitely many inequivalent solutions to this problem (encoded in the choice of \( W_2 \)) and one has to select the meaningful ones on the basis of some physical principle.

Let us denote \( \mathcal{D}_n \) the space of functions on \( dS^2_0 \) infinitely differentiable and with compact support. As in the Minkowskian case, the Borchers algebra \( \mathcal{B} \) is defined as the tensor algebra over \( \mathcal{D} = \mathcal{D}_1 \). Its elements are terminating sequences of test-functions \( f = (f_0, f_1(X_1), \ldots, f_n(X_1, \ldots, X_n), \ldots) \), where \( f_0 \in \mathbb{C} \) and \( f_n \in \mathcal{D}_n \) for all \( n \geq 1 \), the product and \( \star \) operations being given by

\[
(fg)_n = \sum_{p, q \in \mathbb{N} \atop p+q=n} f_p \otimes g_q, \quad (f^*)_n(X_1, \ldots, X_n) = f_n(X_n, \ldots, X_1).
\]

The action of the de Sitter group on \( \mathcal{B} \) is defined by \( f \mapsto f_{(\Lambda_r)} \), where

\[
f_{(\Lambda_r)} = (f_0, f_{(\Lambda_r)}(X_1), \ldots, f_n(\Lambda_r), \ldots), \quad f_n(\Lambda_r)(x_1, \ldots, x_n) = f_n(\Lambda_r^{-1}x_1, \ldots, \Lambda_r^{-1}x_n),
\]

\( \Lambda_r \) denoting any (real) de Sitter transformation.

A quantum field theory is specified by a continuous linear functional \( W \) on \( \mathcal{B} \), i.e. a sequence \( \{W_n \in \mathcal{D}_n\}_{n \in \mathbb{N}} \) where \( W_0 = 1 \) and the \( \{W_n\}_{n>0} \) are distributions (Wightman functions) required to possess the following properties:

1. (Covariance). Each \( W_n \) is de Sitter invariant, i.e.

\[
\langle W_n, f_{(\Lambda_r)} \rangle = \langle W_n, f_n \rangle
\]

for all de Sitter transformations \( \Lambda_r \). (This is equivalent to saying that the functional \( W \) itself is invariant, i.e. \( W(f) = W(f_{(\Lambda_r)}) \) for all \( \Lambda_r \)).

2. (Locality)

\[
W_n(X_1, \ldots, X_j, X_{j+1}, \ldots, X_n) = W_n(X_1, \ldots, X_{j+1}, X_j, \ldots, X_n)
\]

if \( (X_j - X_{j+1})^2 < 0 \).

3. (Positive Definiteness). For each \( f \in \mathcal{B} \), \( W(f^*f) \geq 0 \). Explicitly, given \( f_0 \in \mathbb{C}, f_1 \in \mathcal{D}_1, \ldots, f_k \in \mathcal{D}_k \), then

\[
\sum_{n,m=0}^k \langle W_n+m, f_n^* \otimes f_m \rangle \geq 0.
\]

The latter property should be possibly relaxed to treat de Sitter gauge QFT.

As in the Minkowskian case the GNS construction yields a Hilbert space \( \mathcal{H} \), a unitary representation \( \Lambda_r \mapsto U(\Lambda_r) \) of \( SO_0(1, d) \), a vacuum vector \( \Omega \in \mathcal{H} \) invariant under \( U \), and an operator valued distribution \( \phi \) such that

\[
W_n(X_1, \ldots, X_n) = (\Omega, \phi(X_1) \ldots \phi(X_n) \Omega).
\]

The GNS construction also provides the vector valued distributions \( \Phi_n^{(b)} \) such that

\[
\langle \Phi_n^{(b)}, f_n \rangle = \int f_n(X_1, \ldots, X_n) \phi(X_1) \ldots \phi(X_n) \Omega d\sigma(X_1) \ldots d\sigma(X_n)
\]

and a representation \( f \mapsto \Phi(f) \) (by unbounded operators) of \( \mathcal{B} \) of which the field \( \phi \) is a special case: \( \phi(f_1) = \int \phi(X) f_1(X)d\sigma(X) = \Phi((0, f_1, 0, \ldots)) \). For every open set \( \mathcal{O} \) of \( dS_d \)
the corresponding polynomial algebra $\mathcal{P}(\mathcal{O})$ of the field $\phi$ is then defined as the subalgebra of $\Phi(\mathfrak{F})$ whose elements $\Phi(f_0, f_1, \ldots, f_n, \ldots)$ are such that for all $n \geq 1$ $\text{supp} f_n(x_1, \ldots, x_n) \subset \mathcal{O}^n$. The set $\mathcal{D} = \mathcal{P}(dS_d)\Omega$ is a dense subset of $\mathcal{H}$ and one has (for all elements $\Phi(f), \Phi(g) \in \mathcal{P}(dS_d)$):

$$\mathcal{W}(f^* g) = (\Phi(f)\Omega, \Phi(g)\Omega).$$

(24)

The properties 1-3 are literally carried over from the Minkowskian case, but no literal or unique adaptation exists for the usual spectral property. In the $(d+1)$-dimensional Minkowskian case, the latter is equivalent to the following: for each $n \geq 2$, $\mathcal{W}_n$ is the boundary value in the sense of distributions of a function holomorphic in the tube

$$\mathcal{T}_n = \{Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^{n(d+1)} : \text{Im} (Z_{j+1} - Z_j) \in V_+, 1 \leq j \leq n-1\}.$$  

(25)

In the case of the de Sitter space $dS_d$ (embedded in $\mathbb{R}^{d+1}$), a natural substitute for this property is to assume that $\mathcal{W}_n$ is the boundary value in the sense of distributions of a function holomorphic in

$$\mathcal{T}_n = dS_d^{(c)n} \cap \mathcal{T}_n.$$  

(26)

It has been shown that $\mathcal{T}_n$ is a domain and a tuboid $\mathbb{B} \mathbb{B}$, namely a domain which is bordered by the reals in such a way that the notion of “distribution boundary value of a holomorphic function from this domain” remains meaningful. It is thus possible to impose:

4. (Weak spectral condition). For each $n > 1$, the distribution $\mathcal{W}_n$ is the boundary value of a function $W_n$ holomorphic in the subdomain $\mathcal{T}_n$ of $dS_d^{(c)n}$.

4. Correspondence with conformal field theories on $\mathcal{C}_{1,d}$: dimensional boundary conditions at infinity.

In order to obtain correlation functions on the boundary of dS spacetime we are led to postulate a certain type of behavior at infinity for the Wightman functions which we call “dimensional boundary conditions at infinity”.

By making use of the coordinates $\mathbb{B}$ we say that a QFT on $dS_d$ is of asymptotic (complex) dimension $\Delta$ if the following limits exist in some sense:

5. (Dimensional boundary conditions at infinity)

$$\lim_{\tau \to +\infty} |\sinh \tau^n|^\Delta \mathcal{W}_n(X_1[\tau, \omega_1], \ldots, X_n[\tau, \omega_n]) = \mathcal{W}_n^\Delta(\omega_1, \ldots, \omega_n).$$

(27)

In words: we take the restriction of the $n$-point function to the manifold $S(\tau)^n$ (i.e. set all the times $\tau_j = \tau$), multiply by the indicated factor and take the limit to infinity.

The first thing to be shown is that the above condition is meaningful, since it is not true in general that a distribution $\mathcal{W}_n(X_1, \ldots, X_n)$ can be restricted to a submanifold of $dS_d^n$.

Our spectral condition 4. implies that this can be done at equal times for noncoinciding points. The argument is based on the existence of an analytic continuation of the Wightman $n$-point functions to corresponding primitive domains of analyticity.

Indeed for each permutation $\pi$ of $\{1, \ldots, n\}$, the permuted Wightman distribution

$$\mathcal{W}_n^{(\pi)}(X_1, \ldots, X_n) = \mathcal{W}_n(X_{\pi(1)}, \ldots, X_{\pi(n)})$$

(28)

is the boundary value of a function $W_n^{(\pi)}(Z_1, \ldots, Z_n)$ holomorphic in the ”permuted tuboid”

$$\mathcal{T}_n^{\pi} = \{Z = (Z_1, \ldots, Z_n) ; Z_k = X_k + i Y_k \in dS_d^{(c)}, 1 \leq k \leq n; Y_{\pi(j+1)} - Y_{\pi(j)} \in V^+, 1 \leq j \leq n-1\}.$$  

(29)

6
If two permutations \(\pi\) and \(\sigma\) differ only by the exchange of the indices \(j\) and \(k\), then \(\mathcal{W}_\pi\) and \(\mathcal{W}_\sigma\) coincide in
\[
\mathbb{R}_{jk} = dS^\mathbb{R}_n \cap \mathbb{R}_{jk}, \quad \mathbb{R}_{jk} = \{X \in \mathbb{R}^{n(d+1)} : (X_j - X_k)^2 < 0\}.
\]

In particular all the permuted Wightman distributions coincide in the intersection \(\Omega_n\) of all the \(\mathbb{R}_{jk}\), and it follows that they all are boundary values of a common function \(\mathcal{W}_n(\pi Z_1, \ldots Z_n)\), holomorphic in a primitive analyticity domain \(\mathcal{D}_n\). \(\mathcal{W}_n\) is the common analytic continuation of all the holomorphic functions \(W_\pi^n\) and the domain \(\mathcal{D}_n\) is the union of all the permuted tuboids \(T_\pi^n\) and of the local tuboids associated (by the edge-of-the-wedge theorem) with finite intersections of the \(\mathbb{R}_{jk}\). It is self-evident that any \(n\)-tuple \(\{X_1[\tau,\omega_1], \ldots X_n[\tau,\omega_n]\}\) such that \(X_i \neq X_j\) for any \(i \neq j\) belongs to such primitive domain of analyticity and actually it belongs to \(\Omega_n\); therefore the previous restriction can be considered.

Let us now consider a general QFT on \(dS_d\) whose Wightman functions \(\mathcal{W}_n\) satisfy dS invariance together with the properties described in the previous section, with the possible exception of the positive-definiteness property 3. In view of the asymptotics we can construct the following set of \(n\)-point distributions \(\mathcal{E}_n(\eta_1, \ldots, \eta_n)\) on \(\mathcal{C}_{1,d}^+\):
\[
\mathcal{E}_n(\eta_1, \ldots, \eta_n) = (r_1 \cdots r_n)^{-\Delta} W^\infty_n(\omega_1, \ldots, \omega_n).
\]

We are now going to establish that the \(SO_0(1,d)\)-invariance of the de Sitter \(n\)-point functions, together with the asymptotic boundary condition imply the \(SO_0(1,d)\)-invariance of the correlation functions \(\mathcal{E}_n\) now interpreted as euclidean conformal transformations of \(\mathcal{C}_{1,d}\):
\[
\mathcal{E}_n(g\eta_1, \ldots, g\eta_n) = \mathcal{E}_n(\eta_1, \ldots, \eta_n)
\]
for all \(g\) in \(SO_0(1,d)\). A part of this invariance is trivial in view of the limiting procedure: it is the invariance under the spatial orthogonal group \(SO(d)\) leaving \(\eta^0\) unchanged.

In order to show that the invariance condition \(32\) holds for all \(g\) in \(G\), it remains to show that it holds for all one-parameter subgroups of pseudo-rotations in the \((0,i)\)-planes with \(i = 1, \ldots, d\). Closely following the steps indicated in \([5]\) let us consider the case with e.g. \(i = 1\) and associate with the corresponding subgroup \(G_{0,1}\) of pseudo-rotations the following parametrization \(X = X\{\rho, \psi, u\}\) (with \(u = (u^2, \ldots, u^d)\)) of \(dS_d\):
\[
\begin{cases}
X^0 &= \rho \cosh \psi & \rho > 0 \\
X^i &= \rho \sinh \psi & i = 1, \ldots, d \\
X^i &= \sqrt{\rho^2 - 1} u^i & i = 2, \ldots, d \\
& & u^2 + \cdots + u^d = 1
\end{cases}
\]

Correspondingly we have the following parametrization \(\eta = \eta\{\rho, \psi, u\}\) for the cone \(\mathcal{C}_{1,d}^+\):
\[
\begin{cases}
\eta^0 &= \sigma \cosh \psi & \sigma > 0 \\
\eta^1 &= \sigma \sinh \psi \\
\eta^i &= \sigma u^i & i = 2, \ldots, d \\
& & u^2 + \cdots + u^d = 1
\end{cases}
\]

For \(g \in G_{0,1}\), the invariance condition \(32\) to be proven can be written as follows (with the simplified notation \(\mathcal{E}_n(\eta_1, \ldots, \eta_n) = \mathcal{E}_n(\eta_j)\)):
\[
\mathcal{E}_n(\eta_j\{\sigma_j, \psi_j + a, w_j\}) = \mathcal{E}_n(\eta_j\{\sigma_j, \psi_j, w_j\})
\]
for all \(a\). Now in view of the definition \((11)\) of \(\mathcal{W}_n(\eta_j)\) and of the relations between the sets of parameters \((r, \tau, \omega)\) and \((\rho, \psi, u)\) obtained by identification of the expressions \((13)\) and
The invariance condition (33) to be proven is equivalent to the following condition for the asymptotic forms of the dS $n$–point functions $W_n^\infty$ (for all $a$):

$$\prod_{1 \leq k \leq n} (\sigma_k \cosh \psi_k)^{-\Delta} W_n^\infty \left( \left[ \frac{\tanh \psi_j, \frac{u_j}{\cosh \psi_j}}{\cosh \psi_j} \right] \right) =$$

$$\prod_{1 \leq k \leq n} (\sigma_k (\cosh(\psi_k + a)))^{-\Delta} W_n^\infty \left( \left[ \frac{\tanh(\psi_j + a), \frac{u_j}{\cosh(\psi_j + a)}}{\cosh(\psi_j + a)} \right] \right).$$

(36)

Comparing the parametrizations (3) and (33) of dS we obtain the following relations:

$$\sinh \tau = \rho \cosh \psi, \quad \omega^j = \frac{\rho \sinh \psi u^j}{\sqrt{1 + \rho^2 \cosh^2 \psi}}, \quad \omega^j = \frac{\rho \sinh \psi u^j}{\sqrt{1 + \rho^2 \cosh^2 \psi}}$$

(37)

This implies that it is equivalent to take the limits in Eq. (27) for $\rho_j$ (instead of $\tau$) tending to infinity and at fixed value of $\psi_j$ and $u_j$, after plugging the expressions (37) of $\tau = \tau_j$ and $\omega_j$ into both sides of Eq. (27):

$$\lim_{\tau \to \infty} \left( \rho_1 \cdots \rho_n \right)^\Delta W_n (X_j \{ \rho_j, \psi_j, u_j \}) =$$

$$\prod_{1 \leq k \leq n} (\cosh \psi_k)^{-\Delta} W_n^\infty \left( \left[ \frac{\tanh \psi_j, \frac{\rho \sinh \psi_j}{\sqrt{1 + \rho^2 \cosh^2 \psi_j}}, \frac{\rho \sinh \psi u^j}{\sqrt{1 + \rho^2 \cosh^2 \psi}}}{\sqrt{1 + \rho^2 \cosh^2 \psi}} \right] \right) = 0.$$ 

(38)

If we now also consider the vanishing limit of the same difference after the transformation $\psi_j \to \psi_j + a$ has been applied, and take into account the fact that, by assumption, the first term of this difference has remained unchanged, we obtain the following relation:

$$\lim_{\tau \to +\infty} \left| \Pi_{1 \leq k \leq n} (\cosh \psi_k)^{-\Delta} W_n^\infty \left( \left[ \frac{\tanh \psi_j, \frac{\rho \sinh \psi_j}{\sqrt{1 + \rho^2 \cosh^2 \psi_j}}, \frac{\rho \sinh \psi u^j}{\sqrt{1 + \rho^2 \cosh^2 \psi}}}{\sqrt{1 + \rho^2 \cosh^2 \psi}} \right] \right) \right| -$$

$$\Pi_{1 \leq k \leq n} \cosh(\psi_k + a)^{-\Delta} W_n^\infty \left( \left[ \frac{\tanh(\psi_j + a), \frac{\rho \sinh(\psi_j + a)}{\sqrt{1 + \rho^2 \cosh^2(\psi_j + a)}}, \frac{\rho \sinh(\psi + a) u^j}{\sqrt{1 + \rho^2 \cosh^2(\psi + a)}}}{\sqrt{1 + \rho^2 \cosh^2(\psi + a)}} \right] \right) = 0.$$ 

(39)

In the latter, the limit can be taken separately in each term and the resulting equality yields precisely the required covariance relation (36).

We stress again that, thanks to our general setting and in particular the spectral condition 4., all the functions involved are of class $C^\infty$ with respect to all the variables $(\rho_j, \psi_j, u_j)$ and all the limits are taken in the sense of regular functions.

It would not have been possible to take restriction for points in general position; in particular to send for instance one point to minus infinity and the remaining points to plus infinity is not allowed since after a certain time all the points will enter the future cone of the point moving to minus infinity and the restriction would become meaningless.

The procedure we have described (expressed by Eqs. (27) and (31)) displays a general correspondence

$$W_n (X_1, \ldots, X_n) \to E_n (\eta_1, \ldots, \eta_n).$$

(40)

The degree of homogeneity (dimension) $\Delta$ of $E(\eta_1, \ldots, \eta_n)$ is equal to the asymptotic dimension of the dS field $\Phi(X)$. The correspondence (40) can be completed by constructing $n$–point functions $E_n$ on the Euclidean space $E^{d-1}$, expressed in terms of the $E_n$ by the following formulae (3, 41):

$$E_n (x_1, \ldots, x_n) = \Pi_{1 \leq j \leq n} (\eta_j^0 + \eta_j^1)^\Delta E_n (\eta_1, \ldots, \eta_n).$$

(41)

In the latter, the Euclidean variables $x_j$ are expressed in terms of the cone variables $\eta_j$ as in Eq. (10).
5 Discussion

Could we say that there exists a CFT associated to the so-constructed euclidean conformal correlation functions? If the original dS theory does not satisfy the positive-definiteness property the best that one can do in general is to use the GNS construction to build a linear space endowed with a (non-positive or indefinite metric) inner product and an operator valued distribution $O(\eta)$ having such euclidean correlation functions.

But even if one has positive-definiteness in the dS theory, i.e. if the dS theory has a direct physical interpretation, the construction does not guarantee that also the limiting euclidean theory is positive definite and the GNS construction gives an Hilbert space, because coinciding points are not in general under control (this is the unitarity property mentioned in [1]).

However this still does not settle the question of the physical meaning of the asymptotic theory. Indeed it is well known that the only meaningful notion of positivity for euclidean theories to admit a direct physical interpretation is the so-called Osterwalder-Schrader positivity or reflection positivity (and not positive-definiteness); it is the only condition which allows the reconstruction of quantities at real time through an appropriate Wick-rotation from the Euclidean $n$-point functions.

Unfortunately there is no way to show that the asymptotic theory has such a property, and actually in general it will not$^1$, even if in some lucky case this property may still hold. This generic situation will be illustrated by the following free field examples [1, 2].

6 Klein-Gordon fields

Let us consider now the de Sitter Klein-Gordon equation

$$\Box \phi + m^2 \phi = 0,$$

(42)

where $\Box$ is the Laplace-Beltrami operator relative to the de Sitter metric and $m^2$ is a complex number. It is possible to solve in a coordinate-independent way [3] by using the previous embedding of the de Sitter hyperboloid in the Minkowski ambient space. First of all one introduces plane waves solving the KG equation; these waves are similar to the Minkowskian exponentials but with the important difference that they are singular on $(d-1)$-dimensional light-like submanifolds of $dS_d$. The physically relevant global waves can be defined as analytic functions for $z$ in the tubular domains $T_+$ or $T_-$ of $dS_d^{(d)}$; for $Z \in T_+$ or $Z \in T_-$ we define

$$\psi_{i\nu}(Z,\xi) = (Z : \xi)^{-\frac{d-1}{2} + i\nu},$$

(43)

where $\nu$ is a complex number and $\xi = (\xi^0, \ldots, \xi^d)$ belongs to $C_+$. The phase is chosen to be zero when the argument is real and positive. Physical values of the parameter $\nu$ are real (principal series of representations) or purely imaginary with $|\nu| \leq \frac{d-1}{2}$ (complementary series of representations), corresponding to a real and positive $m^2$:

$$m^2 = \left( \frac{d-1}{2} \right)^2 + \nu^2 > 0.$$  

(44)

but we will study the limit for generic complex $\nu$. The corresponding QFT is completely encoded in the two-point function $W_{i\nu}(X,X')$ which should be a distribution on $dS_d \times dS_d$

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$^1$This is in contrast with what would happen with Euclidean sections of a Minkowskian theory, if a similar construction to the present one was performed. In that case reflection positivity would hold. The reason for this difference resides in the lack of translation invariance of the curved case.
satisfying the conditions of locality de Sitter invariance; positive-definiteness will hold only for physical values of \(m^2\). \(W(X, X')\) should solve the KG w.r.t. both variables:

\[
(\Box_X + m^2) W(X, X') = 0, \quad (\Box_{X'} + m^2) W(X, X') = 0.
\] (45)

There are infinitely many inequivalent solutions to this problem, but there is one preferred theory (for each value of the mass \(m\)) which is usually referred to as the “Euclidean” or Bunch-Davies vacuum \([11, 12, 4]\); what is perhaps not so well known is that these fields can be directly constructed in a manifestly de Sitter invariant way \([3]\) by exploiting the previous dS plane waves. Indeed it is possible to give a spectral analysis of the two-point functions very similar to the Fourier analysis usually done in the flat Minkowski case. This is constructed as follows: for \(Z \in \mathcal{T}_-\), \(Z' \in \mathcal{T}_+\), the Wightman function can be represented as a superposition of plane waves in the complex domain \(\mathcal{T}_+ \times \mathcal{T}_-\) \([4]\):

\[
W_v^d(Z, Z') = c_{d, \nu} \int_{\gamma} \psi^{(d)}_{\nu}(Z, \xi) \psi^{(d)}_{-\nu}(Z', \xi) d\mu_\nu(\xi)
\] (46)

with

\[
c_{d, \nu} = \frac{1}{2(2\pi)^d} \Gamma \left(\frac{d - 1}{2} + i\nu\right) \Gamma \left(\frac{d - 1}{2} - i\nu\right) e^{-\pi\nu}.
\] (47)

The integration can be performed along any basis submanifold \(\gamma\) of the cone \(C_+\) (i.e. a submanifold intersecting almost all the generatrices of the cone) w.r.t. a corresponding measure \(d\mu_\nu\) induced by the invariant measure on the cone. For instance, one can integrate on the manifold \(\gamma_d = \gamma^+_d \cup \gamma^-_d = \{\xi \in C_+ : \xi^d = 1\} \cup \{\xi \in C_+ : \xi^d = -1\}\), which is a pair of hyperboloids; in this case the measure \(d\mu_\nu\) looks like the Lorentz invariant measure on the mass shell. For the spherical basis \(\gamma_0 = \{\xi \in C^+ : \xi^0 = 1\}\) the measure \(d\mu_\nu\) is exactly the rotation invariant measure (on the sphere).

The function \(W_v^d\) manifestly solves the (complex) de Sitter Klein-Gordon equation in both variables, and is analytic in the domain \(\mathcal{T}_+ \times \mathcal{T}_-\). It can be shown that it is actually a function of the de Sitter invariant variable \((Z - Z')^2 = -2 - 2Z \cdot Z'\). This property allows the explicit computation

\[
W_v^d(Z, Z') = \frac{1}{2(2\pi)^d} \Gamma \left(\frac{d - 1}{2} + i\nu\right) \Gamma \left(\frac{d - 1}{2} - i\nu\right) (e^{\frac{d-2}{2}} - 1)^{-\frac{d^2-2}{4d+4} P_{-\frac{d^2-2}{4d+4}}(Z \cdot Z')}.
\] (48)

where \(P_{-\frac{d^2-2}{4d+4}}(\zeta)\) is Legendre function of the first kind \([13]\). At vanishingly short distances the Wightman function has the local Hadamard universal behaviour:

\[
W_v^d(Z, Z') \simeq \frac{\Gamma(d^2-2)}{2(2\pi)^d} \left| -(Z - Z')^2 \right|^{-\frac{d^2-2}{4d+4}}.
\] (49)

By equation \((45)\) one sees that \(W_v^d(z, z')\) is maximally analytic, i.e. can be analytically continued in the “cut-domain” \(dS^c_d \times dS^c_d \setminus \{(z, z') \in dS^c_d \times dS^c_d : (Z - Z')^2 \geq 0\}\). Furthermore, \(W_v^d(Z, Z')\) satisfies in this cut-domain the complex covariance condition: \(W_v^d(gZ, gZ') = W_v^d(Z, Z')\) for all \(g\) in the complex de Sitter group \([2]\).

As a function of the parameter \(\nu\) the two-point function \(W_\nu\) is analytic and symmetric:

\[
W_{-\nu}^d = W_\nu^d.
\]

---

2These properties are not restricted to Klein-Gordon fields and are actually true for any two-point Wightman function \(W\) satisfying our spectral condition \([4]\).
6.1 Boundary theories from KG fields with a complex mass

For large values of the argument $W_\nu^d(\zeta)$ has the following asymptotics [13]:

$$W_\nu^d(\zeta) \sim \frac{2^{-i\nu}\Gamma\left(\frac{d-1}{2} + i\nu\right)}{2(2\pi)^{d/2}} \zeta^{-\frac{d-1}{2} - i\nu} \quad \text{for } \text{Im} \nu > 0$$

$$W_\nu^d(\zeta) \sim \frac{2^{i\nu}\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{d/2}} \zeta^{-\frac{d+1}{2} + i\nu} \quad \text{for } \text{Im} \nu < 0$$

In the relevant case when $\text{Im} \nu = 0$, that corresponds to physical KG fields of the principal series, the two terms are of the same order and both contribute.

When $\text{Im} \nu > 0$ (resp. $\text{Im} \nu < 0$) the two-point function and thereby all the $n-$point functions of the corresponding Klein-Gordon field satisfy the dimensional boundary conditions at infinity with dimension $\Delta = \frac{d-1}{2} + i\nu$ (resp. $\Delta = \frac{d-1}{2} - i\nu$). Indeed, for $\text{Im} \nu > 0$

$$W_\nu^\infty(\omega, \omega') = \lim_{\tau \to \infty} (\sinh^2 \tau)^{\frac{d-1}{2} + i\nu} W_\nu(Z[\tau, \omega], Z'[\tau, \omega']) = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma(-i\nu)}{2(2\pi)^{d/2}} 2^{-i\nu}(1 - \omega' \cdot \omega)^{-\nu - \frac{d-1}{2}}.$$  

The two-point function of the conformal field $O(n)$ on the cone corresponding to (52) is then constructed by following the prescription of Eq. (13), which yields

$$E_{\nu}(\eta, \eta') = (rr')^{-\frac{d-1}{2} - i\nu} W_\nu^\infty(\omega, \omega') = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma(-i\nu)}{4\pi^{\frac{d+1}{2}}} \left(- (\eta - \eta')^2\right)^{-\nu - \frac{d-1}{2}}$$

Correspondingly, we can deduce from (53) the expression of the two-point function of the associated euclidean two-point function on $\mathbb{E}^{d-1}$; by taking Eqs. (11) and (13) into account, we obtain:

$$E_{\nu}^{d-1}(x, x') = e^{(\nu + \frac{d}{2} - i\nu)} W_\nu(\eta(v, x), \eta'(v', x')) = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma(-i\nu)}{4\pi^{\frac{d+1}{2}}} \left[(x - x')^2\right]^{-\nu - \frac{d-1}{2}}$$

Similar results hold for for $\text{Im} \nu < 0$.

6.2 Physical case: the complementary series

In this case one has $\nu = i\lambda$ with $0 < \lambda < \frac{d-1}{2}$.

$$E_{\lambda}^{d-1}(x, x') = \frac{\Gamma\left(\frac{d-1}{2} - \lambda\right)\Gamma(\lambda)}{4\pi^{\frac{d+1}{2}}} \left[(x - x')^2\right]^{-\frac{d-1}{2}}$$

This two-point function does satisfy positive definiteness exactly when $\lambda$ satisfies the above condition and we can construct an Hilbert space out of it in the usual GNS way [1].

One also checks easily that OS positivity holds when $\lambda \leq 1$. It is interesting to note that the two bounds coincide in the three dimensional case where the boundary theory has the full infinite-dimensional conformal invariance. In the two-dimensional case theories having the OS positivity arise from non-unitary de Sitter theories. It might also be possible to get other boundary CFT’s violating the bound $\lambda \leq 1$. These theories would arise however as limit of exotic de Sitter theories, which do not satisfy de Sitter locality.
6.3 Physical case: the principal series

These are dS Klein-Gordon theories corresponding to real values of $\nu$ and are the theories which have a standard flat limit (while theories of the complementary series disappear in that limit). Unfortunately they do not satisfy our asymptotic dimension property. The best one can do is to give a small imaginary part to $\nu$ and then apply the previous construction. One sees that a theory of the complementary series can be associated this way to two boundary theories which however have complex dimensions and satisfy neither positive definiteness nor OS positivity.

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