Gauge Formulation for Two Potential Theory of Dyons

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Abstract

Dual electrodynamics and corresponding Maxwell’s equations (in the presence of monopole only) are revisited from the symmetry of duality and gauge invariance. Accordingly, the manifestly covariant, dual symmetric and gauge invariant two potential theory of generalized electromagnetic fields of dyons has been developed consistently from $U(1) \times U(1)$ gauge symmetry. Corresponding field equations and equation of motion are derived from Lagrangian formulation adopted for $U(1) \times U(1)$ gauge symmetry for the justification of two four potentials of dyons.

Key words- Dual electrodynamics, duality, gauge invariance, monopoles and dyons

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1 Introduction

The asymmetry between electricity and magnetism has become clear at the end of 19th century with the formulation of Maxwell’s equations. and physicists were fascinated about the idea of magnetic monopoles. Dirac [1] put forward the idea of magnetic monopoles to symmetrize Maxwell’s equations and showed that the quantum mechanics of an electrically charged particle of charge $e$ and a magnetically charged particle of charge $g$ is consistent only if $eg = 2\pi n$, $n$ being an integer. Schwinger-Zwanziger [2] generalized this condition to allow for the possibility of particles (dyons) that carry both electric and magnetic charge. A quantum mechanical theory can have two particles of electric and magnetic charges ($e_1, g_1$) and ($e_2, g_2$) only if $e_1g_2 - e_2g_1 = 2\pi n$. Fresh interests in this subject have been enhanced by ’t Hooft-Polyakov [3] with the idea that the classical solutions having the properties of magnetic monopoles may be found in Yang - Mills gauge theories. Julia and Zee [4] extended the ’t Hooft-Polyakov theory of monopoles and constructed the theory of non Abelian dyons. It is now widely recognized that the standard model, which combines the gauge theory of strong interactions with the model of electroweak interaction, is a gauge theory that contains monopole and dyon solutions. The quantum mechanical

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excitation of fundamental monopoles include dyons which are automatically arisen from the semi-classical quantization of global charge rotation degree of freedom of monopoles. In view of the explanation of CP-violation in terms of non-zero vacuum angle of world, the monopoles are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. Renewed interests in the subject of monopole has gathered enormous potential importance in connection current grand unified theories, supersymmetric gauge theories and super strings. But unfortunately the experimental searches for these elusive particles have proved fruitless as the monopoles are expected to be super heavy and their typically masses are about two orders of magnitude heavier than the super heavy bosons mediating proton decay. However, a group of physicists are now claiming that they have found indirect evidences for monopoles and now it is being speculated that magnetic monopoles may play an important role in condensed matter physics. In spite of the enormous potential importance of monopoles (dyons) and the fact that these particles have been extensively studied, there has been presented no reliable theory which is as conceptually transparent and predictably tact-able as the usual electrodynamics and the formalism necessary to describe them has been clumsy and not manifestly covariant. On the other hand, the concept of electromagnetic (EM) duality has been receiving much attention in gauge theories, field theories, supersymmetry and super strings. So, keeping in view the recent potential importance of monopoles (dyons) and the applications of electromagnetic duality, in this paper, we have made an attempt to revisit the analogous consistent formulation of dual electrodynamics subjected by the magnetic monopole only. Gauge formulation has been adopted accordingly to derive the dual Maxwell’s equation, equation of motion and Bianchi identity for dual electric charge (i.e. magnetic monopole) from the minimum action principle. Accordingly, we have discussed the dual symmetric and manifestly covariant formulation of generalized fields of dyons in order to obtain the generalized Dirac-Maxwell’s (GDM) field equations and Lorentz force equation of motion of dyons in terms of two four potentials. Two potential theory of magnetic monopoles (dyons) have been justified from $U(1)\times U(1)$ gauge symmetry. Consequently, the gauge symmetric and dual invariant manifestly covariant theory has been reformulated consistently from the $U(1)\times U(1)$ gauge symmetry. It has been emphasized that the two $U(1)$ gauge group acts in different manner whereas the first $U^{(e)}(1)$ acts on the Dirac spinors while the other group $U^{(m)}(1)$ acts on Dirac iso-spinors. We have also developed accordingly the consistent Lagrangian formulation for the justification of two gauge potentials of dyons.

2 Dual Electrodynamics

Duality invariance is an old idea introduced a century ago in classical electromagnetism for the following Maxwell’s equations in vacuum i.e.

\[
\nabla \cdot \vec{E} = 0 \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}
\]

where $\vec{E}$ and $\vec{B}$ are respectively the electric and magnetic field strengths and for brevity we use natural units $c = \hbar = 1$, space-time four-vector $\{x^\mu\} = (t, x, y, z), \{x_\mu = \eta_{\mu\nu}x^\nu\}$ and $\{\eta_{\mu\nu} = +1, -1, -1, -1 = \eta^{\mu\nu}\}$ throughout the text. Maxwell’s equations are invariant not only under Lorentz and conformal transformations but are also invariant under the following duality transformations,
\[
\begin{align*}
\vec{E} = \vec{E} \cos \vartheta + \vec{B} \sin \vartheta \\
\vec{B} = -\vec{E} \sin \vartheta + \vec{B} \cos \vartheta
\end{align*}
\] (2)

where \(\vec{E}\) and \(\vec{B}\) are respectively the electric and magnetic field strengths. For a particular value of \(\vartheta = \frac{\pi}{2}\), equations (2) reduces to

\[
\begin{align*}
\vec{E} \rightarrow \vec{B}; \quad \vec{B} \rightarrow -\vec{E}
\end{align*}
\] (3)

which can be written as

\[
\begin{pmatrix}
\vec{E} \\
\vec{B}
\end{pmatrix} \Rightarrow
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
\vec{B}
\end{pmatrix}.
\] (4)

Consequently, Maxwell’s equations may be solved by introducing the concept of vector potential in either two ways \[13\].

**Case-I**: The conventional choice is being used as

\[
\begin{align*}
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi \\
\vec{B} &= \vec{\nabla} \times \vec{A}
\end{align*}
\] (5)

where \(\{A^\mu\} = (\phi, \vec{A})\) is described as the four potential. So, the dual symmetric and Lorentz covariant Maxwell’s equations (1) are written in as

\[
\begin{align*}
\partial_\nu F^{\mu\nu} &= 0 \\
\partial_\nu \tilde{F}^{\mu\nu} &= 0
\end{align*}
\] (6)

where \(F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu = A^{\mu,\nu} - A^{\nu,\mu}\) is anti-symmetric electromagnetic field tensor, \(\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\omega} F_{\lambda\omega}\) \((\forall \mu, \nu, \eta, \lambda = 0, 1, 2, 3)\) is the dual of electromagnetic field tensor and \(\varepsilon^{\mu\nu\lambda\omega}\) is the four index Levi-Civita symbol. \(\varepsilon^{\mu\nu\lambda\omega} = +1\) \((\mu\nu\lambda\omega = 0123)\) for cyclic permutation; \(\varepsilon^{\mu\nu\lambda\omega} = -1\) for any two permutations and \(\varepsilon^{\mu\nu\lambda\omega} = 0\) if any two indices are equal. Using equation (5), we may obtain the electric and magnetic fields as the components of anti-symmetric electromagnetic field tensors \(F^{\mu\nu}\) and \(\tilde{F}^{\mu\nu}\) given by

\[
\begin{align*}
F^{0j} &= E_j; \quad F^{jk} = \varepsilon^{jkl} B_l \quad (\forall j, k, l = 1, 2, 3) \\
\tilde{F}^{0j} &= B_j; \quad \tilde{F}^{jk} = \varepsilon^{jkl} E_l \quad (\forall j, k, l = 1, 2, 3)
\end{align*}
\] (7)

where \(\varepsilon^{jkl}\) is three index Levi-Civita symbol and \(\varepsilon^{jkl} = +1\) for cyclic, \(\varepsilon^{jkl} = -1\) for anti-cyclic permutations and \(\varepsilon^{jkl} = 0\) for repeated indices. The duality symmetry is lost if electric charge and current source densities enter to the conventional inhomogeneous Maxwell’s equations given by
\[
\begin{align*}
\vec{\nabla} \cdot \vec{E} &= \rho \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \vec{j} + \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]  
(8)

where \(\rho\) and \(\vec{j}\) are described as charge and current source densities which are the components of electric four-current \(\{j^\mu\} = (\rho, \vec{j})\) source density. So, the covariant form of Maxwell’s equations (8) is described as

\[
\partial_\nu F^{\mu\nu} = j^\mu
\]

\[
\partial_\nu \tilde{F}^{\mu\nu} = 0.
\]

Here, we may see that the pair \((\vec{\nabla} \cdot \vec{B} = 0; \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})\) of Maxwell’s equations (8) is described by \(\partial_\nu F^{\mu\nu} = 0\) in equation (9). It has become kinematical while the dynamics is contained in another pair \((\vec{\nabla} \cdot \vec{E} = \rho; \vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t})\) of Maxwell’s equations (8) which described as \(\partial_\nu F^{\mu\nu} = j^\mu\) in equation (9) and also reduces to following wave equation in the presence of Lorentz gauge condition \(\partial_\mu A^\mu = 0\) i.e

\[
\Box A^\mu = j^\mu
\]

(10)

where \(\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\) is the D’ Alembertian operator. So, a particle of mass \(m\), electric charge \(e\) moving with a velocity \(\{u^\nu\}\) in an electromagnetic field is subjected by a Lorentz force given by

\[
\frac{m d^2 x_\mu}{d\tau^2} = \frac{dp_\mu}{d\tau} = f_\mu = e F^{\mu\nu} u_\nu
\]

(11)

where \(\{\ddot{x}_\mu\}\) is the four-acceleration, \(f_\mu\) is four force and \(p_\mu\) is four momentum of a particle. Equation (11) is reduced to

\[
\vec{f} = \frac{d \vec{p}}{dt} = m \frac{d^2 \vec{x}}{dt^2} = e \left[ \vec{E} + \vec{\omega} \times \vec{B} \right]
\]

(12)

where \(\vec{p}, \vec{f}, \vec{x}\) and \(\vec{\omega}\) are respectively the three vector forms of momentum, force, displacement and velocity of a particle. Here we may observe that the Lorentz force equation of motion (11-12) are also not invariant under duality transformations (3-4).

**Case-II**: On the other hand, let us introduce [13] the another alternative way instead of equation (5) to write

\[
\begin{align*}
\vec{\nabla} \cdot \vec{E} &= \rho \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \vec{\nabla} \varphi - \vec{\nabla} \times \vec{C}
\end{align*}
\]

(13)
where a new potential \( \{ C^\mu \} = (\varphi, \vec{C}) \) is introduced \[13, 14\] as the alternative to \( \{ A^\mu \} \). Thus, we see that source free (homogeneous) Maxwell’s equations are same as those of equations \( \{ 1 \} \) but the inhomogeneous Maxwell’s equations \( \{ 8 \} \) are changed to

\[
\nabla \cdot \vec{E} = 0 \\
\nabla \cdot \vec{B} = \rho \\
\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \\
\n\nabla \times \vec{E} = -\vec{J} - \frac{\partial \vec{B}}{\partial t} 
\]

subjected by the introduction of a new four current source density \( \{ k^\mu \} = (\rho, \vec{\kappa}) \). In equation \( \{ 14 \} \) we see that the pair \((\nabla \cdot \vec{E} = 0; \nabla \times \vec{B} = \partial \vec{E}/\partial t)\) becomes kinematical while the dynamics is contained in the second pair \((\nabla \cdot \vec{B} = \rho; \nabla \times \vec{E} = -\vec{J} - \partial \vec{B}/\partial t)\). Equation \( \{ 14 \} \) may also be written in following covariant forms

\[
\partial_\nu F^{\mu\nu} = 0 \\
\partial_\nu \tilde{F}^{\mu\nu} = k^\mu
\]

where \( F^{\mu\nu} = \partial^\nu C^\mu - \partial^\mu C^\nu; \tilde{F}^{\mu\nu} = F^{\nu\mu}; \{ k^\mu \} = (\rho, \vec{\kappa}) \) and \( \{ k^\mu \} = (\rho, -\vec{\kappa}) \). Equation \( \{ 14 \} \) may also be obtained on applying the transformations \( \{ 3 \} \) and \( \{ 4 \} \) to equation \( \{ 8 \} \) followed by following duality transformations for potential, current and antisymmetric electromagnetic field tensors as

\[
A^\mu \rightarrow C^\mu; C^\mu \rightarrow -A^\mu \iff \begin{pmatrix} A^\mu \\ C^\mu \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^\mu \\ C^\mu \end{pmatrix} \\
\]

\[
j^\mu \rightarrow k^\mu; k^\mu \rightarrow -j^\mu \iff \begin{pmatrix} j^\mu \\ k^\mu \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} j^\mu \\ k^\mu \end{pmatrix} \\
\]

\[
F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu} \iff \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} 
\]

As such, we may identify the potential \( \{ C^\mu \} = (\varphi, \vec{B}) \) as the dual of potential \( \{ A^\mu \} \) and the current \( \{ k^\mu \} = (\rho, \vec{\kappa}) \) as the dual of current \( \{ j^\mu \} \). Correspondingly, the differential equations \( \{ 13 \} \) are identified as the dual Maxwell’s equations. So, accordingly, we may develop the electrodynamics of a charged particle with the charge dual to the electric charge (i.e. magnetic monopole). Applying the the electromagnetic duality to the Maxwell’s equations, we may establish the connection between electric and magnetic charge (monopole) \[16, 17\], in the same manner as an electric charge \( e \) interacts with electric field and the dual charge (magnetic monopole) \( g \) interacts with magnetic field, as,

\[
e \rightarrow g: g \rightarrow -e \iff \begin{pmatrix} e \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} 
\]

where \( g \) is described as the dual electric charge (charge of magnetic monopole). Hence, we may recall the dual electrodynamics as the dynamics of pure magnetic monopole. Consequently, the corresponding
dynamical variables associated therein are described as the dynamical variables in the theory of magnetic monopole. So, we may write the new electromagnetic field tensor $F_{\mu\nu}$ in place of $\tilde{F}_{\mu\nu}$ as

$$\tilde{F}_{\mu\nu} \mapsto F_{\mu\nu} = \partial_\nu C_\mu - \partial_\mu C_\nu \quad (\mu, \nu = 1, 2, 3)$$

which reproduces the following definition of magneto-electric fields of monopole as

$$F_{0i} = B_i, \quad F_{ij} = -\varepsilon^{ijk} E_k.$$  

Hence the covariant form of Maxwell’s equations (12) for magnetic monopole may now be written as

$$F_{\mu\nu,\nu} = \partial_\nu F_{\mu\nu} = k_\mu,$$
$$\tilde{F}_{\mu\nu,\nu} = \partial_\nu \tilde{F}_{\mu\nu} = 0$$

(20)

where $\{k_\mu\} = \{q, -m\}$ is the four-current density due to the presence of the magnetic charge $g$. Accordingly, the wave equation (15) for pure monopole is described as

$$\Box C_\mu = k_\mu$$

(21)

in presence of Lorentz gauge condition $\partial_\mu C^\mu = 0$. Accordingly, we may develop the classical Lagrangian formulation in order to obtain the field equation (dual Maxwell’s equations) and equation of motion for the dynamics of a dual charge (magnetic monopole) interacting with electromagnetic field. So, the Lorentz force equation of motion for a dual charge (i.e magnetic monopole) may now be written from the duality equations (3) and (4) as

$$m \ddot{\vec{x}}_\mu = \frac{dp_\mu}{d\tau} = \frac{d}{d\tau} \left( \vec{p} + \vec{F}_{\mu\nu} u^\nu \right)$$

(22)

where $\vec{p} = m \vec{v} = m \vec{u}$ is the momentum, and $\vec{F}$ is a force acting on a particle of charge $g$, mass $m$ and moving with the velocity $\vec{v}$ in electromagnetic fields. Equation (20) can be generalized to write it in the following four vector formulation as

$$m \frac{d^2 \vec{x}_\mu}{d\tau^2} = \frac{dp_\mu}{d\tau} = f_\mu = m \ddot{\vec{x}}_\mu = gF_{\mu\nu} u^\nu$$

(23)

where $\{u_\nu\}$ is the four velocity, $\{p_\mu\}$ is four momentum, $f_\mu$ is four force and $\{\ddot{x}_\mu\}$ is the four-acceleration of a particle carrying the dual charge (namely magnetic monopole).

3 Gauge Symmetry and Dual Electrodynamic

Let us define a Dirac field $\psi$ with dual (magnetic) charge $g$ for which the free Dirac Lagrangian is
\[ \mathcal{L}_0 = \bar{\psi} (i \gamma^\mu \partial_\mu + m) \psi (i = \sqrt{-1}) \]  

(24)

where \( \gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( \gamma^j = \begin{bmatrix} 0 & \tau^j \\ -\tau^j & 0 \end{bmatrix} \) are 4 × 4 complex Dirac matrices with 0, 1 and \( \tau^j \) are respectively the 2 × 2 null, unit and Pauli Matrices \( \forall j = 1, 2, 3 \). Also the gamma matrices satisfy the property

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu\nu} \]  

(25)

and \( \bar{\psi} = \psi^\dagger \gamma^0 \) with \((\dagger)\) denotes the Hermitian conjugation. So, the Dirac Lagrangian \( \mathcal{L}_0 \) is clearly invariant under the global gauge transformation

\[
\psi \mapsto - \psi' \mapsto - \exp\{ig\alpha(x)\} \psi \\
\bar{\psi} \mapsto - \bar{\psi}' \mapsto - \bar{\psi} \exp\{-i\alpha(x)\}
\]  

(26)

where \( \alpha \) is independent of space-time. So, like electromagnetism, we elevate this symmetry to invariance under local gauge transformation

\[
\psi \mapsto \psi' \mapsto \exp\{ig\alpha(x)\} \psi \\
\bar{\psi} \mapsto \bar{\psi}' \mapsto \bar{\psi} \exp\{-i\alpha(x)\}
\]  

(27)

so that the Lagrangian \( \mathcal{L}_0 \) transforms as

\[
\bar{\psi} (i \gamma^\mu \partial_\mu + m) \psi \mapsto \bar{\psi} (i \gamma^\mu \partial_\mu - g \gamma^\mu \partial_\mu \alpha + m) \psi.
\]  

(28)

Since the extra term \( g \gamma^\mu \partial_\mu \alpha \) looks like a gauge transformation of potential, we may couple the gauge field \( C_\mu \) with \( \psi \) so that the Lagrangian has the local gauge symmetry. We, thus, write the Lagrangian \( \mathcal{L}_0 \) as

\[
\mathcal{L} = \bar{\psi} (i \gamma^\mu \nabla_\mu + m) \psi
\]  

(29)

where the covariant derivative in equation \( \mathcal{L} \) is given by

\[
\nabla_\mu = \partial_\mu - igC_\mu.
\]  

(30)

Hence the Lagrangian \( \mathcal{L}_0 \) is invariant under the combined gauge transformation

\[
\psi(x) \mapsto \psi'(x) \mapsto \exp\{ig\alpha(x)\} \psi(x) \\
\bar{\psi}(x) \mapsto \bar{\psi}'(x) \mapsto \bar{\psi}(x) \exp\{-i\alpha(x)\} \\
C_\mu \mapsto C'_\mu = C_\mu + \partial_\mu \alpha(x)
\]  

(31)
and the covariant derivative is transformed as

$$\nabla'_\mu \psi' \mapsto \exp\{ig\chi(x)\}(\nabla_\mu \psi).$$  \hspace{1cm} (32)

As such, the Lagrangian for total dual quantum electrodynamics is subjected by

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi - C_\mu k^\mu$$  \hspace{1cm} (33)

where

$$k^\mu = g\bar{\psi}(x)\gamma^\mu \psi(x).$$  \hspace{1cm} (34)

Accordingly, we get

$$[\nabla_\mu, \nabla_\nu] \psi(x) = -igF_{\mu\nu}\psi(x)$$  \hspace{1cm} (35)

and with the use of Jacobi identity

$$[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0$$  \hspace{1cm} (36)

we get the Bianchi identity

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0$$  \hspace{1cm} (37)

which is the kinematical statement equivalent to $\tilde{F}_{\mu\nu,\nu} = \partial_\nu \tilde{F}_{\mu\nu} = 0$ for dual electrodynamics analogous to the kinematical statement $\partial_\nu \tilde{F}_{\mu\nu} = 0$ for usual electrodynamics. As such dual symmetry of electrodynamics requires the existence of dual electric charge (i.e. magnetic monopole).

## 4 Dual Symmetric Covariant Formulation of Dyons

Magnetic monopole has been introduced by Dirac\cite{1} in a different manner in order to symmetrize Maxwell’s equations \cite{8} from duality principle as

$$\nabla \cdot \vec{E} = \rho$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{k};$$
$$\nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}.$$  \hspace{1cm} (38)

We refer these differential equations as the generalized Dirac-Maxwell’s (GDM) equations. Since Dirac theory contains the controversial string variables, so it has been generalized \cite{2,4,16,17} to the theory of the particles carrying simultaneously the electric and magnetic charges namely dyons. As such, GDM
equations (38) are described as the field equations of dyons. The GDM equations (38) may directly be obtained on combining the Maxwell’s equations (8) and their dual equations (14). So, the electric field \( \vec{E} \) and magnetic field \( \vec{B} \) in GDM equations (38) are respectively described as the generalized electromagnetic fields of dyons by combining equations (5) and (13) in terms of the components of two four vector potentials \( \{ A^\mu \} \) and \( \{ C^\mu \} \) [15, 16, 17]

\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi - \vec{\nabla} \times \vec{C}
\]
\[
\vec{B} = -\frac{\partial \vec{C}}{\partial t} - \vec{\nabla} \Phi + \vec{\nabla} \times \vec{A}
\]

(39)

Generalized Dirac Maxwell’s (GDM) equations (38) are invariant not only under Lorentz and conformal transformations but are also invariant under the following duality transformations between electric \( E \) and magnetic \( B \) quantities i.e.

\[
E \mapsto E \cos \vartheta + B \sin \vartheta
\]
\[
B \mapsto B \cos \vartheta - E \sin \vartheta
\]

(40)

where \( E \equiv (e, \vec{E}, \rho, \vec{j}, \phi, \vec{A}) \) and \( B \equiv (g, \vec{B}, \varrho, \vec{k}, \varphi, \vec{C}) \). For a particular value of \( \vartheta = \frac{\pi}{2} \), equations (40) reduces to

\[
E \mapsto -B \quad B \mapsto -E.
\]

(41)

The generalized anti-symmetric dual invariant electromagnetic field tensors for dyons are written as

\[
F^{\mu \nu} = \partial^\nu A^\mu - \partial^\mu A^\nu - \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} (\partial_\lambda C_\sigma - \partial_\sigma C_\lambda) = F^{\mu \nu} - \tilde{F}^{\mu \nu}
\]
\[
\tilde{F}^{\mu \nu} = \partial^\nu C^\mu - \partial^\mu C^\nu + \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda) = F^{\mu \nu} + \tilde{F}^{\mu \nu}.
\]

(42)

The generalized electromagnetic fields of dyons (39) are then be defined as the components of generalized field tensors (42) as

\[
F^{0j} = E^j; \quad \tilde{F}^{jk} = \varepsilon^{jkl} B_l \quad (\forall j, k, l = 1, 2, 3)
\]
\[
\tilde{F}^{0j} = B^j; \quad F^{jk} = \varepsilon^{jkl} E_l \quad (\forall j, k, l = 1, 2, 3).
\]

(43)

Hence, the covariant form of dual symmetric GDM equations (38) is described [15, 16, 17] as

\[
F_{\mu \nu, \nu} = \tilde{F}_{\mu \nu, \nu} = j^\mu
\]
\[
\tilde{F}_{\mu \nu, \nu} = \tilde{F}_{\mu \nu, \nu} = k^\mu.
\]

(44)

Therefore, we may write the Lagrangian which follows the minimum action principle for generalized electromagnetic fields of dyons as
This Lagrangian yields the GDM field equations \((44)\) and the Lorentz force equation of motion for dyons as

\[
\mathcal{L}_{GEM} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu + C_\mu k^\mu. \tag{45}
\]

On the other hand, the dual parts of field tensors giving rise \(\partial_\nu \tilde{F}^{\mu\nu} = 0\) and \(\partial_\nu \tilde{F}^{\mu\nu} = 0\) describe the Bianchi identities like equation \((37)\) for electric and magnetic charges. If we define the four currents in terms of charge and velocity as

\[
j^\nu = e u^\nu; \quad k^\nu = g u^\nu, \tag{47}
\]

the dual invariant Lorentz force expression \((46)\) for dyon reduces \([15, 16, 17]\) to

\[
\vec{\mathcal{J}} = m \frac{d^2 \vec{r}}{dt^2} = e (\vec{E} + \vec{u} \times \vec{B}) + g (\vec{B} - \vec{u} \times \vec{E}) \tag{48}
\]

where \(\vec{u}\) is the velocity of a particle.

### 5 \(U(1) \times U(1)\) Gauge Formulation of Dyons

Let us start with the Lagrangian density \((24)\) within the introduction of four spinor Dirac field \(\Psi\) for dyons instead of two component spinor \(\psi\) as

\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{49}
\]

where \(\Psi_1\) and \(\Psi_2\) are two component spinors. Here \(\Psi_1\) is identified as the Dirac spinor for a electric charge (like electron) while the other spinor \(\Psi_2\) has been identified as the Dirac iso-spinors acting on the magnetic monopole. Thus the \(\Psi\) may be visualized as the bi-spinor for dyons in terms of its electric and magnetic counterparts. Each spinor \(\Psi_1\) and \(\Psi_2\) satisfy the free particle Dirac equation

\[
\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu + m)\hat{1}\Psi = \begin{pmatrix} \bar{\Psi}_1 \\ \bar{\Psi}_2 \end{pmatrix} \begin{pmatrix} (i\gamma^\mu \partial_\mu + m) & 0 \\ 0 & (i\gamma^\mu \partial_\mu + m) \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sum_{a=1}^{a=2} \bar{\Psi}_a (i\gamma^\mu \partial_\mu + m) \Psi_a \tag{50}
\]

where \(\hat{1}\) is \(2 \times 2\) unit matrix. So, the Unitary transformations taking part for the invariance of free particle Dirac equation for bi-spinor \(\Psi\) are the global \(U = U^{(e)}(1) \times U^{(m)}(1)\) two component spinors \(\Psi_1\) and \(\Psi_2\). In this case \(\Psi_1\) acts on unitary gauge group \(U^{(e)}(1)\) whereas the iso-spinor \(\Psi_2\) acts on the other unitary
group $U^{(m)}(1)$ with the symbols $(e)$ and $(m)$ are used for the electric and magnetic charges. Thus equation (50) is invariant under global gauge transformation

$$U = U^{(e)} \times U^{(m)} = \exp(i\Lambda_j \tau^j_a)$$

where

$$\tau^j_a = \tau^j_a b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tau^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\tau^j_a \tau^k_a] = \epsilon^{ikj} \tau^l_b = 0$$

because we have $j, k, l = 1, 2$. Accordingly the spinor transforms as

$$\Psi_1 \mapsto -\Psi_1' \mapsto U^{(e)} \Psi_1 = \exp\{i\Lambda_1\} \Psi_1$$

$$\Psi_2 \mapsto -\Psi_2' \mapsto U^{(m)} \Psi_2 = \exp\{i\Lambda_2\} \Psi_2$$

$$\Psi_1 \mapsto -\Psi_1' \mapsto U^{(e)} \Psi_1^{-1} = \Psi_1 \exp\{i\Lambda_1\}$$

$$\Psi_2 \mapsto -\Psi_2' \mapsto U^{(m)} \Psi_2^{-1} = \Psi_2 \exp\{i\Lambda_2\}$$

$$\Psi \mapsto -\Psi' \mapsto U^{(e)} U^{(m)}^{-1} = \begin{pmatrix} \exp\{i\Lambda_1\} & 0 \\ 0 & \exp\{i\Lambda_2\} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

In equations (51) and (53) $\Lambda_j(\forall j = 1, 2)$ are independent of space and time for global gauge transformations. If we elevate this symmetry to invariance under local gauge transformations where $\Lambda_j \mapsto \Lambda_j(x) (\forall j = 1, 2)$ in equations (51) and (53), the Lagrangian (50) transforms as

$$\Psi(i\gamma^\mu \partial_\mu + m) \mapsto \Psi(i\gamma^\mu D_\mu + m)$$

where partial derivative $\partial_\mu$ has been replaced by the covariant derivative $D_\mu$ as

$$D_\mu = D^b_\mu = \partial_\mu \delta^b_a + \beta_{\mu j} \tau^j_a$$

and $\tau^j_a$ are given by equation (52) along with

$$\beta_{\mu j} \mapsto \beta_{\mu 1} = A_\mu$$

$$\beta_{\mu j} \mapsto \beta_{\mu 2} = C_\mu.$$
\[ \beta_{\mu 1} = A_\mu \mapsto A'_\mu \mapsto \left[ U^{(e)} \right] A_\mu \left[ U^{(e)} \right]^{-1} + \frac{1}{e} \left[ U^{(e)} \right] \partial_\mu \left[ U^{(e)} \right]^{-1} \]
\[ \beta_{\mu 2} = C_\mu \mapsto C'_\mu \mapsto \left[ U^{(m)} \right] C_\mu \left[ U^{(m)} \right]^{-1} + \frac{1}{g} \left[ U^{(m)} \right] \partial_\mu \left[ U^{(m)} \right]^{-1} \]  
(58)

where

\[
\begin{bmatrix}
    U^{(e)} \\
    U^{(m)}
\end{bmatrix} \implies \exp \{ i \Lambda_1(x) \} \\
\exp \{ i \Lambda_2(x) \}.
\]  
(59)

As such, we may write the \( D_\mu \Psi \) as

\[
D_\mu \Psi = \begin{bmatrix}
    \partial_\mu - ieA_\mu & 0 \\
    0 & \partial_\mu - igC_\mu
\end{bmatrix}
\begin{pmatrix}
    \Psi_1 \\
    \Psi_2
\end{pmatrix}
\]  
(60)

which transforms as

\[
D_\mu \Psi \mapsto D'_\mu \Psi' \mapsto \left( \begin{array}{ccc}
    & 0 & \\
    0 & \end{array} \right)
\begin{pmatrix}
    \exp \{ i \Lambda_1(x) \} & 0 \\
    0 & \exp \{ i \Lambda_2(x) \}
\end{pmatrix}
\begin{pmatrix}
    (\partial_\mu - ieA_\mu) \Psi_1 \\
    (\partial_\mu - igC_\mu) \Psi_2
\end{pmatrix}
= U(D_\mu \Psi).
\]  
(61)

Hence, we get

\[
[D_\mu, D_\nu] \Psi(x) = \begin{bmatrix}
    -ieF^\mu_\nu & 0 \\
    0 & -igF^\mu_\nu
\end{bmatrix}
\begin{pmatrix}
    \Psi_1 \\
    \Psi_2
\end{pmatrix}
\]  
(62)

and it leads to the Jacobi identity

\[
[D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\nu]] = 0
\]  
(63)

along with the Bianchi identities

\[
D_\mu F^\mu_\nu + D_\nu F^\nu_\mu + D_\lambda F^\mu_\nu = 0
\]
\[
D_\mu F^\mu_\nu + D_\nu F^\nu_\mu + D_\lambda F^\mu_\nu = 0
\]  
(64)

As such, the total Lagrangian for generalized fields of dyons is described as

\[
\mathcal{L} = -\frac{1}{4} F^\mu_\nu F_{\mu\nu} - \frac{1}{4} F^\mu_\nu F_{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi - A_\mu j^\mu - C_\mu k^\mu
\]  
(65)

where

\[
j^\mu = e\bar{\psi}_1 \gamma^\mu \psi_1
\]  
(66)
and

\[ k^\mu = g \Psi_2 \gamma^\mu \Psi_2 \]  

are the four currents associated respectively with electric and magnetic charges on dyons. These four-currents obtained from the Dirac spinor \( \Psi_1 \) and the Dirac iso-spinor \( \Psi_2 \) satisfy the following conserved relations

\[ \partial_\mu j^\mu = j^\mu, \mu = 0 \]  

and

\[ \partial_\mu k^\mu = k^\mu, \mu = 0. \]  

Like other electric and magnetic dynamical variables, one can introduce duality transformations between the electric and magnetic gauges corresponding to orthogonal transformations in group space i.e.

\[ \Lambda_1 \Rightarrow \Lambda_1 \cos \vartheta + \Lambda_2 \sin \vartheta \]
\[ \Lambda_2 \Rightarrow \Lambda_2 \cos \vartheta - \Lambda_1 \sin \vartheta \]  

and with the use of constancy condition [16]

\[ \frac{g}{e} = \frac{C_\mu}{A_\mu} = \frac{k_\mu}{j_\mu} = \frac{\Lambda_2}{\Lambda_1} = \frac{F_{\mu\nu}}{F_{\mu\nu}} = \frac{\tilde{F}_{\mu\nu}}{\tilde{F}_{\mu\nu}} = -\tan \theta = \text{Constant} \]  

we get

\[ [D_\mu, D_\nu] \Psi(x) = \begin{pmatrix} -ieF_{\mu\nu} & 0 \\ 0 & -i g\tilde{F}_{\mu\nu} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \]  

where \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) are the generalized dual invariant electromagnetic fields of dyons and satisfy independently the Bianchi identity (64). Hence, the Lagrangian density (65) may now be written as

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \bar{\psi} (i \gamma^\mu \partial_\mu + m) \psi - A_\mu j^\mu - \frac{1}{2} C_\mu k^\mu \]  

where \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) transforms as

\[ U [D_\mu, D_\nu] U^{-1} \Psi(x) \Rightarrow \begin{pmatrix} -ie [U^{(m)}] F_{\mu\nu} [U^{(c)}]^{-1} & 0 \\ 0 & -i g [U^{(m)}] \tilde{F}_{\mu\nu} [U^{(m)}]^{-1} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \]
\[ \Rightarrow F_{\mu\nu} \mapsto [U^{(c)}]^{-1} F_{\mu\nu} [U^{(c)}] \]
\[ \Rightarrow \tilde{F}_{\mu\nu} \mapsto [U^{(m)}]^{-1} \tilde{F}_{\mu\nu} [U^{(m)}]. \]
So, it is observed that the Lagrangian density reproduces the dual symmetric and Lorentz covariant generalized Dirac Maxwell’s (GDM) field equations and Lorentz force equation of motion for two potential theory of dyons. Thus, with the use of Jacobi identity, we get the Bianchi identity for generalized electromagnetic field tensors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ as

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0$$
$$D_\mu \tilde{F}_{\nu\lambda} + D_\nu \tilde{F}_{\lambda\mu} + D_\lambda \tilde{F}_{\mu\nu} = 0.$$ (75)

As such, the classical theory of dyons has been verified and the incorporation of two four potentials in generalized electromagnetic fields of dyons has been justified in the frame work of $U(1) \times U(1)$ gauge theory where the first unitary Abelian gauge group $U^{(e)}(1)$ acts on the Dirac spinors due to the presence of the electric charge while second unitary Abelian gauge group $U^{(m)}(1)$ acts on the Dirac iso-spinors due to the presence of the magnetic charge on dyons. The activation of gauge group $U^{(m)}(1)$ on Dirac iso-spinor is advantageous so that it may further be extended to enlarge the gauge group to describe the non-Abelian correspondence of monopoles (dyons) in current grand unified and supersymmetric gauge theories associated with dyons. Consequently, $U(1) \times U(1)$ gauge group may further be extended in order to describe the built in duality between the $SU(2) \times U(1)$ gauge theory of electro-weak interaction and $SU(2) \times U(1)$ gauge theory of gravity described earlier by Dehnen et al so that one can be able to understand better the current grand unified theories, supersymmetry, super-gravity and super-strings.

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