ON FINITE GROUPS WITH GIVEN $IC\Phi$-SUBGROUPS

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Abstract. A subgroup $H$ of a group $G$ is said to be an $IC\Phi$-subgroup of $G$ if $H \cap [H, G] \leq \Phi(H)$. We analyze the structure of a finite group $G$ under the assumption that some given subgroups of $G$ are $IC\Phi$-subgroups of $G$. A new characterization of finite abelian groups and some new criteria for $2$-nilpotence and nilpotence of finite groups will be obtained. Moreover, we will obtain two criteria for a finite group to lie in a given solvably saturated formation containing the class of finite supersolvable groups.

1. INTRODUCTION

All groups in this paper are implicitly assumed to be finite. Our notation and terminology are standard. The reader is referred to [6, 10, 14] for unfamiliar definitions on groups and to [7] for unfamiliar definitions on classes of groups.

Given a group $G$ and a subgroup $H$ of $G$, we say that $H$ is an $IC\Phi$-subgroup of $G$ provided that $H \cap [H, G] \leq \Phi(H)$. This concept was introduced by Gao and Li in [5] and further investigated by the author in [12]. The papers [5] and [12] contain results that constrain the structure of a group $G$ under the condition that some given subgroups of $G$ are $IC\Phi$-subgroups of $G$. The following theorem is the main result of [12].

Theorem 1.1. ([12, Theorem 1.3]) Let $p$ be a prime dividing the order of a group $G$, and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that there is a subgroup $D$ of $P$ with $1 < |D| \leq |P|$ such that any subgroup of $P$ with order $|D|$ is an $IC\Phi$-subgroup of $G$. If $|D| = 2$ and $|P| \geq 8$, assume moreover that any cyclic subgroup of $P$ with order $4$ is an $IC\Phi$-subgroup of $G$. Then $G$ is $p$-nilpotent.

The goal of the present paper is to further study how the structure of a group is influenced by its $IC\Phi$-subgroups. Our results show, together with [5] and [12], that one often gets very rich information about a group when some of its subgroups are assumed to be $IC\Phi$-subgroups.

Groups some of whose primary subgroups are $IC\Phi$-subgroups. Let $Q_8$ denote the quaternion group with order $8$. Recall that a group $G$ is said to be $Q_8$-free if no section of $G$ is isomorphic to $Q_8$. Our first main result is the following improvement of Theorem 1.1 and [5, Theorem 3.1].

Theorem 1.2. Let $G$ be a $Q_8$-free group such that any subgroup of $G$ with order $2$ is an $IC\Phi$-subgroup of $G$. Then $G$ is $2$-nilpotent.

The condition in Theorem 1.2 that $G$ is $Q_8$-free is really necessary. For example, $Z(SL_2(3))$ is the only subgroup of $SL_2(3)$ with order $2$, and $Z(SL_2(3))$ is an $IC\Phi$-subgroup of $SL_2(3)$. But $SL_2(3)$ is not $2$-nilpotent.

Our second main result is a generalization of the following result of Gao and Li.

Theorem 1.3. ([5, Theorem 3.5]) Let $G$ be a group, and let $E$ be a normal subgroup of $G$ such that $G/E$ is supersolvable. If every maximal subgroup of every Sylow subgroup of $E$ is an $IC\Phi$-subgroup of $G$, then $G$ is supersolvable.

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To state our generalization of Theorem 1.3, we recall some definitions. A class of groups \( \mathcal{F} \) is said to be a formation if \( \mathcal{F} \) is closed under taking homomorphic images and subdirect products. A formation \( \mathcal{F} \) is said to be saturated if whenever \( G \) is a group with \( G/\Phi(G) \in \mathcal{F} \), we have \( G \in \mathcal{F} \). We say that a formation \( \mathcal{F} \) is solvably saturated if whenever \( G \) is a group and \( N \) is a solvable normal subgroup of \( G \) with \( G/\Phi(N) \in \mathcal{F} \), we have \( G \in \mathcal{F} \). Note that every saturated formation is solvably saturated.

The class of all supersolvable groups is denoted by \( \mathcal{U} \). It is well-known that \( \mathcal{U} \) is a saturated and hence a solvably saturated formation.

With these definitions at hand, we can now state our second main result.

**Theorem 1.4.** Let \( \mathcal{F} \) be a solvably saturated formation containing \( \mathcal{U} \), let \( G \) be a group, and let \( E \) be a non-trivial normal subgroup of \( G \) such that \( G/E \in \mathcal{F} \). Let \( t := \pi(E) \), and let \( p_1 < \cdots < p_t \) be the distinct prime divisors of \( |E| \). For each \( 1 \leq i \leq t \), let \( P_i \) be a Sylow \( p_i \)-subgroup of \( E \). Suppose that, for each \( 1 \leq i \leq t \), either \( P_i \) is cyclic or there is a subgroup \( D_i \) of \( P_i \) with \( 1 < |D_i| \leq |P_i| \) such that any subgroup of \( P_i \) with order \( |D_i| \) is an \( ICF \)-subgroup of \( G \). If \( p_1 = 2 \), \( |D_1| = 2 \) and \( P_1 \) is not \( Q_8 \)-free, assume moreover that any cyclic subgroup of \( P_1 \) with order 4 is an \( ICF \)-subgroup of \( G \). Then \( G \in \mathcal{F} \).

Theorem 1.3 is covered by Theorem 1.4. Also, the proof of Theorem 1.4 given here is shorter than the proof of Theorem 1.3 given in [5].

Our third main result shows that Theorem 1.4 remains true when we replace the assumption that the subgroups \( P_1, \ldots, P_t \) are Sylow subgroups of \( E \) by the assumption that they are Sylow subgroups of the generalized Fitting subgroup \( F^*(E) \) of \( E \).

**Theorem 1.5.** Let \( \mathcal{F} \) be a solvably saturated formation containing \( \mathcal{U} \), let \( G \) be a group, and let \( E \) be a non-trivial normal subgroup of \( G \) such that \( G/E \in \mathcal{F} \). Let \( t := \pi(F^*(E)) \), and let \( p_1 < \cdots < p_t \) be the distinct prime divisors of \( |F^*(E)| \). For each \( 1 \leq i \leq t \), let \( P_i \) be a Sylow \( p_i \)-subgroup of \( F^*(E) \). Suppose that, for each \( 1 \leq i \leq t \), either \( P_i \) is cyclic or there is a subgroup \( D_i \) of \( P_i \) with \( 1 < |D_i| \leq |P_i| \) such that any subgroup of \( P_i \) with order \( |D_i| \) is an \( ICF \)-subgroup of \( G \). If \( p_1 = 2 \), \( |D_1| = 2 \) and \( P_1 \) is not \( Q_8 \)-free, assume moreover that any cyclic subgroup of \( P_1 \) with order 4 is an \( ICF \)-subgroup of \( G \). Then \( G \in \mathcal{F} \).

All the above theorems are concerned with groups \( G \) such that some primary subgroups of \( G \) are \( ICF \)-subgroups of \( G \). It is natural to ask what we can say about the structure of a group \( G \) when every primary subgroup of \( G \) is an \( ICF \)-subgroup of \( G \). Clearly, any abelian group has this property. Also, one can check that any subgroup of \( Q_8 \) is an \( ICF \)-subgroup of \( Q_8 \). Our fourth main result characterizes the abelian groups as the \( Q_8 \)-free groups all of whose primary subgroups are \( ICF \)-subgroups.

**Theorem 1.6.** Let \( G \) be a group. Then the following are equivalent:

1. \( G \) is abelian.
2. \( G \) is \( Q_8 \)-free, and any subgroup of \( G \) is an \( ICF \)-subgroup of \( G \).
3. \( G \) is \( Q_8 \)-free, and any primary subgroup of \( G \) is an \( ICF \)-subgroup of \( G \).

**Groups all of whose maximal, 2-maximal or 3-maximal subgroups are ICF-subgroups.**

Let \( n \) be a positive integer, and let \( G \) be a group. A subgroup \( H \) of \( G \) is said to be \( n \)-maximal in \( G \) if there is a chain of subgroups \( H = H_0 < H_1 < \cdots < H_n = G \), where \( H_i \) is maximal in \( H_{i+1} \) for all \( 0 \leq i \leq n - 1 \).

There are many results in finite group theory that describe the structure of a group \( G \) under the assumption that, for a given positive integer \( n \), all \( n \)-maximal subgroups of \( G \) satisfy a given property.

Perhaps the most well-known result of this kind is due to Wielandt, who proved that a group \( G \) is nilpotent if every maximal subgroup of \( G \) is normal in \( G \) (see [10, Kapitel III, Hauptsatz 2.3]).
Huppert proved that a group $G$ is supersolvable if every 2-maximal subgroup of $G$ is normal in $G$ (see [9 Satz 23]) or if $|G|$ is divisible by at least three primes and every 3-maximal subgroup of $G$ is normal in $G$ (see [9 Satz 24]). Janko proved that a solvable group $G$ is supersolvable if every 4-maximal subgroup of $G$ is normal in $G$ and $|G|$ is divisible by at least four primes (see [11 Theorem 3]). Huppert’s and Janko’s results were strengthened by Asaad [1].

Mann [15] proved a number of structural results about groups whose $n$-maximal subgroups, for some positive integer $n$, are subnormal. In the last decade, a number of results have been obtained on groups whose $n$-maximal subgroups, for some positive integer $n$, satisfy certain properties generalizing subnormality, see for example [13, 16, 17, 18] (some of these results only deal with the case $n = 2$).

Other recent results on $n$-maximal subgroups and their influence on the structure of groups were obtained for example in [3, 4, 19].

As a development of the research on $n$-maximal subgroups, we will prove the following three theorems.

**Theorem 1.7.** Let $G$ be a group such that any maximal subgroup of $G$ is an ICΦ-subgroup of $G$. Then $G$ is nilpotent.

**Theorem 1.8.** Let $G$ be a group. Suppose that $G$ has a non-trivial 2-maximal subgroup and that any 2-maximal subgroup of $G$ is an ICΦ-subgroup of $G$. Then $G$ is nilpotent.

**Theorem 1.9.** Let $G$ be a group. Suppose that $G$ has a non-trivial 3-maximal subgroup and that any 3-maximal subgroup of $G$ is an ICΦ-subgroup of $G$. Then either $G$ is nilpotent or $G ≅ SL_2(3)$.

2. Preliminaries

In this section, we collect some results needed for the proofs of our main results.

**Lemma 2.1.** ([5 Lemma 2.1]) Let $G$ be a group, $H$ be an ICΦ-subgroup of $G$, and $N$ be a normal subgroup of $G$. Then the following hold:

1. If $H \leq K \leq G$, then $H$ is an ICΦ-subgroup of $K$.
2. If $N \leq H$, then $H/N$ is an ICΦ-subgroup of $G/N$.
3. If $H$ is a $p$-group for some prime divisor $p$ of $|G|$ and $N$ is a $p'$-group, then $HN/N$ is an ICΦ-subgroup of $G/N$.

**Lemma 2.2.** Let $G$ be a group possessing a proper non-trivial ICΦ-subgroup $H$. Then $G$ is not simple.

*Proof.* Since $H$ is an ICΦ-subgroup of $G$, we have $H \cap [H, G] \leq \Phi(H)$. If $G = [H, G]$, then it follows that $H \leq \Phi(H)$, which is impossible. Therefore, $[H, G]$ is a proper subgroup of $G$. Also, $[H, G]$ is normal in $G$. If $[H, G] \neq 1$, it follows that $G$ is not simple. If $[H, G] = 1$, then $H \leq Z(G)$, and again it follows that $G$ is not simple. \qed

**Lemma 2.3.** Let $G$ be a group, and let $H$ be an ICΦ-subgroup of $G$. Suppose that $G' \leq H$. Then $G$ is nilpotent.

*Proof.* We have $[G', G] \leq H \cap [H, G] \leq \Phi(H)$. Applying [10 Kapitel III, Hilfssatz 3.3], we conclude that $[G', G] \leq \Phi(G)$. It follows that

$$[(G/\Phi(G))', G/\Phi(G)] = [G'/\Phi(G)/\Phi(G), G/\Phi(G)] = [G', G]/\Phi(G)/\Phi(G) = 1.$$  

So the lower central series of $G/\Phi(G)$ terminates at 1. Consequently, $G/\Phi(G)$ is nilpotent, and [10 Kapitel III, Satz 3.7] implies that $G$ is nilpotent. \qed

**Lemma 2.4.** ([7 Theorem 3.4.11], [10 Kapitel III, Satz 5.2]) Let $G$ be a minimal non-nilpotent group. Then:
Lemma 2.5. Let $G$ be a $Q_8$-free minimal non-2-nilpotent group. Then $G$ has an elementary abelian Sylow 2-subgroup.

Proof. By [10, Kapitel IV, Satz 5.4], $G$ is minimal non-nilpotent. Lemma 2.4 (1) implies that $G$ has a normal Sylow 2-subgroup $P$.

Assume that $P$ is not elementary abelian. Then $P$ is non-abelian by Lemma 2.4 (3). Since $G$ is $Q_8$-free, we have that $P$ is $Q_8$-free. By a result of Ward, namely by [2, Theorem 56.1], any non-abelian $Q_8$-free 2-group has a characteristic maximal subgroup. Since $P$ is normal in $G$, it follows that there is a maximal subgroup $P_1$ of $P$ which is normal in $G$. We have $\Phi(P) \leq P_1$, and $P/\Phi(P)$ is a chief factor of $G$ by Lemma 2.4 (2). It follows that $\Phi(P) = P_1$. This implies that $P$ is cyclic and hence abelian. On the other hand, we have observed above that $P$ is non-abelian. This is a contradiction.

So we have that $P$ is elementary abelian, and the lemma follows. $\square$

Lemma 2.6. ([6, Chapter 7, Theorem 4.5]) Let $p$ be a prime number, and let $G$ be a group. If $N_G(H)/C_G(H)$ is a $p$-group for any non-trivial $p$-subgroup $H$ of $G$, then $G$ is $p$-nilpotent.

Lemma 2.7. ([14, 7.2.2]) Let $G$ be a non-trivial group, and let $p$ be the smallest prime divisor of $|G|$. Suppose that the Sylow $p$-subgroups of $G$ are cyclic. Then $G$ is $p$-nilpotent.

Lemma 2.8. ([21, Appendix C, Theorem 6.3]) Let $p$ be a prime number, and let $P$ be a normal $p$-subgroup of a group $G$ such that $G/C_G(P)$ is a $p$-group. Then $P \leq Z_\infty(G)$.

To state the next two lemmas, we recall some definitions. Let $\mathfrak{F}$ be a formation, let $G$ be a group, and let $H/K$ be a chief factor of $G$. Then $H/K$ is said to be $\mathfrak{F}$-central if $H/K \times G/C_G(H/K) \in \mathfrak{F}$. Note that $H/K$ is $\mathfrak{U}$-central if and only if $H/K$ is cyclic. A normal subgroup $N$ of $G$ is said to be $\mathfrak{F}$-central if any chief factor of $G$ below $N$ is $\mathfrak{F}$-central. The product of all $\mathfrak{F}$-central normal subgroups of $G$ is denoted by $Z_\mathfrak{F}(G)$.

Lemma 2.9. ([8, Lemma 3.3]) Let $\mathfrak{F}$ be a solvably saturated formation containing $\mathfrak{U}$. Let $G$ be a group and $E$ be a normal subgroup of $G$ such that $G/E \in \mathfrak{F}$ and $E \leq Z_\mathfrak{U}(G)$. Then $G \in \mathfrak{F}$.

Lemma 2.10. ([20, Theorem B]) Let $\mathfrak{F}$ be a formation. Let $G$ be a group, and let $E$ be a normal subgroup of $G$ such that $F^*(E) \leq Z_\mathfrak{F}(G)$. Then $E \leq Z_\mathfrak{F}(G)$.

Lemma 2.11. ([14, 5.3.7]) Let $p$ be a prime number, and let $P$ be a $p$-group such that $P$ has a unique subgroup with order $p$. Then either $P$ is cyclic, or $p = 2$ and $P$ is a generalized quaternion group.

Lemma 2.12. ([11, Lemma 2.1]) Let $G$ be a group such that the trivial subgroup 1 is a 2-maximal subgroup of $G$ and such that $G$ has no non-trivial 2-maximal subgroups. Then $|G| = pq$, where $p$ and $q$ are prime numbers (not necessarily distinct).

Lemma 2.13. ([11, Lemma 2.3]) Let $G$ be a group such that the trivial subgroup 1 is a 3-maximal subgroup of $G$ and such that $G$ has no non-trivial 3-maximal subgroups. Then $|G| = pqr$, where $p$, $q$, and $r$ are prime numbers (not necessarily distinct).

Lemma 2.14. ([6, Chapter 6, Theorem 1.5]) Let $G$ be a solvable group, and let $M$ be a maximal subgroup of $G$. Then $M$ has prime power index in $G$.

Lemma 2.15. ([14, 5.3.3]) $\text{Aut}(Q_8)$ is isomorphic to the symmetric group $S_4$.

Lemma 2.16. ([13, 8.6.10]) Let $G$ be a 2-closed group with order 24 such that the Sylow 2-subgroup of $G$ is isomorphic to $Q_8$. Then either $G \cong Q_8 \times C_3$ or $G \cong SL_2(3)$. 
3. Proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that the theorem is false, and let $G$ be a minimal counterexample.

Let $L$ be a proper subgroup of $G$. Then $L$ is $Q_8$-free since $G$ is $Q_8$-free. Also, by hypothesis, any subgroup of $L$ with order 2 is an $IC\Phi$-subgroup of $G$. Lemma 2.1 (1) implies that any subgroup of $L$ with order 2 is an $IC\Phi$-subgroup of $L$. Consequently, $L$ satisfies the hypotheses of the theorem, and so $L$ is 2-nilpotent by the minimality of $G$. It follows that $G$ is a minimal non-2-nilpotent group.

Let $P \in \text{Syl}_2(G)$. Then $P \neq 1$. By hypothesis, any subgroup of $P$ with order 2 is an $IC\Phi$-subgroup of $G$. By Lemma 2.5, $P$ is elementary abelian. In particular, $P$ has no cyclic subgroup with order 4. Applying Theorem 1.1 or [5, Theorem 3.1], we conclude that $G$ is 2-nilpotent. This contradiction completes the proof. □

4. Proofs of Theorems 1.4 and 1.5

Lemma 4.1. Let $p$ be a prime number, let $G$ be a group, and let $P$ be a non-trivial normal $p$-subgroup of $G$. Suppose that there is a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that any subgroup of $P$ with order $|D|$ is an $IC\Phi$-subgroup of $G$. If $p = 2$, $|D| = 2$ and $P$ is not $Q_8$-free, assume moreover that any cyclic subgroup of $P$ with order 4 is an $IC\Phi$-subgroup of $G$. Then $P \leq Z_\infty(G)$.

Proof. Let $q$ be a prime divisor of $|G|$ with $q \neq p$, and let $Q$ be a Sylow $q$-subgroup of $G$. Set $H := PQ$. We have $H \leq G$ since $P$ is normal in $G$. Note that $P$ is a Sylow $p$-subgroup of $H$. By Lemma 2.7 (1), any subgroup of $P$ with order $|D|$ is an $IC\Phi$-subgroup of $H$. Also, if $p = 2$, $|D| = 2$ and $P$ is not $Q_8$-free, we have that any cyclic subgroup of $P$ with order 4 is an $IC\Phi$-subgroup of $H$. Theorems 1.1 and 1.2 imply that $H$ is $p$-nilpotent. This implies that $H = P \times Q$, and so we have $Q \leq C_G(P)$. Since $q$ was arbitrarily chosen, it follows that $O^p(G) \leq C_G(P)$. Hence, $G/C_G(P)$ is a $p$-group. Lemma 2.8 implies that $P \leq Z_\infty(G)$. □

Proof of Theorem 1.4. Suppose that the theorem is false, and let $(G, E)$ be a counterexample such that $|G| + |E|$ is minimal.

Set $p := p_1$. We claim that $E$ is $p$-nilpotent. This follows from Lemma 2.7 when $P_1$ is cyclic. Assume now that $P_1$ is not cyclic. Then $P_1$ has a subgroup $D_1$ with $1 < |D_1| \leq |P_1|$ such that any subgroup of $P_1$ with order $|D_1|$ is an $IC\Phi$-subgroup of $G$. Also, if $p = 2$, $|D_1| = 2$ and $P_1$ is not $Q_8$-free, then any cyclic subgroup of $P_1$ with order 4 is an $IC\Phi$-subgroup of $G$. Applying Lemma 2.1 (1), Theorem 1.1 and Theorem 1.2, we conclude that $E$ is $p$-nilpotent, as claimed.

Assume that $O^{p'}(E) \neq 1$. From Lemma 2.1 (3), we see that $(G/O^{p'}(E), E/O^{p'}(E))$ satisfies the hypotheses of the theorem. So we have $G/O^{p'}(E) \in \mathfrak{F}$ by the minimality of $(G, E)$. Therefore, $(G, O^{p'}(E))$ also satisfies the hypotheses of the theorem. The minimality of $(G, E)$ implies that $G \in \mathfrak{F}$. This is a contradiction, and so we have $O^{p'}(E) = 1$.

We show now that $E \leq Z_\infty(G)$. Since $E$ is $p$-nilpotent and since $O^{p'}(E) = 1$, we have that $E = P_1$. If $P_1$ is cyclic, then it follows that $E = P_1 \leq Z_\infty(G)$. If $P_1$ is not cyclic, then the hypotheses of the theorem and Lemma 4.1 imply that $E = P_1 \leq Z_\infty(G)$ and thus $E \leq Z_\infty(G)$.

Now Lemma 2.9 implies that $G \in \mathfrak{F}$. This contradiction completes the proof. □

Proof of Theorem 1.5. Arguing as at the beginning of the proof of Theorem 1.4, we see that $F^*(E)$ is $p_1$-nilpotent. With $N_1 := O_{(p_1)^*}(F^*(E))$, we thus have $F^*(E)/N_1 \cong P_1$.

Assume that $t > 1$. Then $P_2 \in \text{Syl}_{p_2}(N_1)$, and again we can argue as at the beginning of the proof of Theorem 1.4 to see that $N_1$ is $p_2$-nilpotent. With $N_2 := O_{(p_2)^*}(N_1)$, we thus have $N_1/N_2 \cong P_2$.

Repeating this argumentation, we see that $G$ has a Sylow tower of supersolvable type, i.e. there is a chain $F^*(E) = N_0 > N_1 > \cdots > N_t = 1$ of normal subgroups of $F^*(E)$ such that $N_{i-1}/N_i \cong P_i$ for all $1 \leq i \leq t$. It follows that $F^*(E)$ is solvable.
As is well-known, $F^*(E)$ is generated by $F(E)$ together with the components of $E$. Every component of $E$ is a non-solvable subgroup of $E$. Since $F^*(E)$ is solvable, it follows that $E$ does not possess any components. Consequently, we have $F^*(E) = F(E).

Let $1 \leq i \leq t$. Since $F(E)$ is nilpotent and $P_i \in \text{Syl}_{p_i}(F(E))$, we have that $P_i$ is characteristic in $F(E)$. As $F(E) \leq G$, it follows that $P_i \leq G$. If $P_i$ is cyclic, then $P_i \leq Z_{\text{nil}}(G)$. If $P_i$ is not cyclic, then the hypotheses of the theorem and Lemma 4.1 imply that $P_i \leq Z_{\text{nil}}(G)$ and hence $P_i \leq Z_{\text{nil}}(G)$. Since $i$ was arbitrarily chosen, it follows that $F(E) \leq Z_{\text{nil}}(G).

Applying Lemmas 2.10 and 2.9 we conclude that $G \in \mathfrak{G}$. □

**Proof of Theorem 1.6**

**Lemma 4.2.** Let $p$ be a prime number, and let $P$ be a $p$-group such that any subgroup of $P$ is an $IC\Phi$-subgroup of $P$. If $p = 2$, assume moreover that $P$ is $Q_8$-free. Then $P$ is abelian.

**Proof.** Suppose that the lemma is false, and let $P$ be a minimal counterexample. We will derive a contradiction in three steps.

(1) $P'$ is minimal normal in $P$, and there are no minimal normal subgroups of $P$ other than $P'$.

Clearly $P \neq 1$, and so $P$ has a minimal normal subgroup, say $N$. We show that $N = P'$.

Let $N \leq H \leq P$. By hypothesis, $H$ is an $IC\Phi$-subgroup of $P$. Lemma 2.1 (2) shows that $H/N$ is an $IC\Phi$-subgroup of $P/N$. Since $H$ was arbitrarily chosen, it follows that any subgroup of $P/N$ is an $IC\Phi$-subgroup of $P/N$. Also, if $p = 2$, then $P/N$ is $Q_8$-free since $P$ is $Q_8$-free. Therefore, $P/N$ satisfies the hypotheses of the lemma, and so $P/N$ is abelian by the minimality of $P$.

It follows that $P' \leq N$. Noticing that $P' \neq 1$ since $P$ is not abelian, we conclude that $N = P'$, as required.

(2) We have $|P'| = p$, and there is no subgroup of $P$ with order $p$ other than $P'$.

We have $|P'| = p$ since $P'$ is minimal normal in $P$. Assume that there is a subgroup $Q$ of $P$ with $|Q| = p$ and $Q \neq P'$. Set $H := P'Q \leq P$. Note that $\Phi(H) = 1$.

By (1), $Q$ is not normal in $P$. So we have $Q \not\leq Z(P)$ and hence $H \not\leq Z(P)$. Thus $[H, P] \neq 1$. Clearly $[H, P] \leq P$ and $[H, P] \leq P'$. So we have $[H, P] = P'$ by (1).

By hypothesis, $H$ is an $IC\Phi$-subgroup of $P$. It follows that $P' = H \cap P' = H \cap [H, P] \leq \Phi(H) = 1$. This is a contradiction, and so there is no subgroup of $P$ with order $p$ other than $P'$.

(3) The final contradiction.

By (2), $P$ has precisely one subgroup with order $p$. Moreover, $P$ cannot be a generalized quaternion group since $P$ is $Q_8$-free by hypothesis. Lemma 2.11 implies that $P$ is cyclic and hence abelian. This final contradiction completes the proof. □

**Proof of Theorem 1.6** (1) ⇒ (2): Suppose that $G$ is abelian. Then any section of $G$ is abelian. In particular, $G$ is $Q_8$-free. Also, if $H$ is a subgroup of $G$, then $H \cap [H, G] = H \cap 1 = 1 \leq \Phi(H)$, so that $H$ is an $IC\Phi$-subgroup of $G$. Thus (2) holds.

(2) ⇒ (3): Clear.

(3) ⇒ (1): Suppose that $G$ is $Q_8$-free and that any primary subgroup of $G$ is an $IC\Phi$-subgroup of $G$. If $G = 1$, then there is nothing to show. Thus we assume that $G \neq 1$. Set $t := |\pi(G)|$, and let $p_1, \ldots, p_t$ be the distinct prime divisors of $|G|$. For each $1 \leq i \leq t$, let $P_i$ be a Sylow $p_i$-subgroup of $G$.

Let $1 \leq i \leq t$. Since any primary subgroup of $G$ is an $IC\Phi$-subgroup of $G$, we have that $P_i$ is an $IC\Phi$-subgroup of $G$. Theorem 1.1 implies that $G$ is $p_i$-nilpotent. Since $i$ was arbitrarily chosen, we have that $G$ is $p$-nilpotent for any prime divisor $p$ of $|G|$. Consequently, $G$ is nilpotent, and so we have $G = P_1 \times \cdots \times P_t$.

Let $1 \leq i \leq t$. Then any subgroup of $P_i$ is an $IC\Phi$-subgroup of $G$. Lemma 2.1 (1) implies that any subgroup of $P_i$ is an $IC\Phi$-subgroup of $P_i$. Also, $P_i$ is $Q_8$-free since $G$ is $Q_8$-free. Lemma 4.2 implies that $P_i$ is abelian.
Consequently, $G$ is a direct product of abelian groups, and so $G$ is abelian as well. \qed

**Proofs of Theorems 1.7, 1.8 and 1.9**

**Proof of Theorem 1.7** Suppose that the theorem is false, and let $G$ be a minimal counterexample.

Clearly, $G$ has a non-trivial maximal subgroup $M$. By hypothesis, $M$ is an $IC\Phi$-subgroup of $G$. Lemma 2.2 implies that $G$ is not simple.

Let $N$ be a proper non-trivial normal subgroup of $G$, and let $N \leq M \leq G$ such that $M/N$ is a maximal subgroup of $G/N$. Then $M$ is a maximal subgroup of $G$. So $M$ is an $IC\Phi$-subgroup of $G$. Lemma 2.1(2) implies that $M/N$ is an $IC\Phi$-subgroup of $G/N$. Since $M$ was arbitrarily chosen, it follows that any maximal subgroup of $G/N$ is an $IC\Phi$-subgroup of $G/N$. The minimality of $G$ implies that $G/N$ is nilpotent.

It follows that $(G/N)^{'} = G'/N/N$ is a proper subgroup of $G/N$. This implies that $G'$ is a proper subgroup of $G$. So there is a maximal subgroup $M$ of $G$ with $G' \leq M$. By hypothesis, $M$ is an $IC\Phi$-subgroup of $G$. Lemma 2.3 implies that $G$ is nilpotent. This is a contradiction to the choice of $G$, and so the proof is complete. \qed

**Proof of Theorem 1.8** Suppose that the theorem is false, and let $G$ be a (not necessarily minimal) counterexample.

Let $M$ be a maximal subgroup of $G$. By hypothesis, every maximal subgroup of $M$ is an $IC\Phi$-subgroup of $G$. Lemma 2.1(1) implies that any maximal subgroup of $M$ is an $IC\Phi$-subgroup of $M$. So $M$ is nilpotent by Theorem 1.7. Since $M$ was arbitrarily chosen, it follows that any maximal subgroup of $G$ is nilpotent. Consequently, $G$ is minimal non-nilpotent.

By Lemma 2.1(1), we have $|G| = p^aq^b$ with distinct prime numbers $p$, $q$ and positive integers $a$, $b$, where $G$ has a normal Sylow $p$-subgroup $P$ and cyclic Sylow $q$-subgroups.

We claim that $b = 1$. Assume, for sake of contradiction, that $b \geq 2$. Clearly, $G/P$ is cyclic with order $q^b$. Let $P \leq H < G$ such that $H/P$ is the unique 2-maximal subgroup of $G/P$. Then $H$ is a 2-maximal subgroup of $G$. By hypothesis, $H$ is an $IC\Phi$-subgroup of $G$, and we have $G' \leq P \leq H$. Lemma 2.3 implies that $G$ is nilpotent. This contradiction shows that $b = 1$, as claimed.

It follows that any maximal subgroup of $P$ is a 2-maximal subgroup of $G$. Consequently, any maximal subgroup of $P$ is an $IC\Phi$-subgroup of $G$. Also, we have $|P| > p$, since otherwise $G$ would not possess a non-trivial 2-maximal subgroup. Applying Theorem 1.1, we conclude that $G$ is $p$-nilpotent. Therefore, $G$ has a normal Sylow $q$-subgroup. Consequently, any Sylow subgroup of $G$ is normal in $G$. It follows that $G$ is nilpotent. This contradiction completes the proof. \qed

We need the following lemma to prove Theorem 1.9.

**Lemma 4.3.** Let $G$ be a group such that any 3-maximal subgroup of $G$ is an $IC\Phi$-subgroup of $G$. Then $G$ is solvable.

**Proof.** Suppose that the lemma is false, and let $G$ be a minimal counterexample. We will derive a contradiction in several steps.

1. $G$ has a non-trivial 2-maximal subgroup.

   Clearly, $G$ has a non-trivial maximal subgroup $M$. Any maximal subgroup of $M$ is a 2-maximal subgroup of $G$. Consequently, the set of 2-maximal subgroups of $G$ is not empty. If $1$ is the only 2-maximal subgroup of $G$, then $G$ is solvable as a consequence of Lemma 2.12 a contradiction. So $G$ has a non-trivial 2-maximal subgroup.

2. $G$ has a non-trivial 3-maximal subgroup.

   By (1), $G$ has a non-trivial 2-maximal subgroup, say $H$. Any maximal subgroup of $H$ is a 3-maximal subgroup of $G$. Consequently, the set of 3-maximal subgroups of $G$ is not empty. If $1$ is the only 3-maximal subgroup of $G$, then $G$ is solvable as a consequence of Lemma 2.13 a contradiction. So $G$ has a non-trivial 3-maximal subgroup.
(3) $G$ is not simple.

By (2), $G$ has a non-trivial 3-maximal subgroup. By hypothesis, any 3-maximal subgroup of $G$ is an ICΦ-subgroup of $G$. Consequently, $G$ has a proper non-trivial ICΦ-subgroup. Lemma 2.12 implies that $G$ is not simple.

(4) Any proper subgroup of $G$ is solvable.

Let $M$ be a maximal subgroup of $G$. It suffices to show that $M$ is solvable. If $M$ has no non-trivial maximal subgroup, then $M$ has prime order, and so $M$ is solvable.

Suppose now that $M$ has a non-trivial maximal subgroup. Then the set of 2-maximal subgroups of $M$ is not empty. If 1 is the only 2-maximal subgroup of $M$, then $M$ is solvable as a consequence of Lemma 2.12.

By hypothesis, any 2-maximal subgroup of $M$ is an ICΦ-subgroup of $G$. Lemma 2.12(1) implies that any 2-maximal subgroup of $M$ is an ICΦ-subgroup of $M$. So, if $M$ has a non-trivial 2-maximal subgroup, then Theorem 1.8 implies that $M$ is nilpotent and hence solvable.

(5) The final contradiction.

By (3), $G$ has a proper non-trivial normal subgroup, say $N$. Let $N \leq H \leq G$ such that $H/N$ is a 3-maximal subgroup of $G/N$. Then $H$ is a 3-maximal subgroup of $G$. So, by hypothesis, $H$ is an ICΦ-subgroup of $G$. Lemma 2.12(2) implies that $H/N$ is an ICΦ-subgroup of $G/N$. Since $H$ was arbitrarily chosen, it follows that any 3-maximal subgroup of $G/N$ is an ICΦ-subgroup of $G/N$. The minimality of $G$ implies that $G/N$ is solvable. Also, $N$ is solvable by (4). It follows that $G$ is solvable. This final contradiction completes the proof.

Proof of Theorem 1.9. Let $G$ be a non-nilpotent group such that $G$ has a non-trivial 3-maximal subgroup and such that any 3-maximal subgroup of $G$ is an ICΦ-subgroup of $G$. Our task is to show that $G$ is isomorphic to $SL_2(3)$. We accomplish the proof step by step.

(1) If $M$ is a non-nilpotent maximal subgroup of $G$, then there exist distinct prime numbers $p$ and $q$ such that $|M| = pq$ and such that $|G:M|$ is a power of $p$.

Let $M$ be a non-nilpotent maximal subgroup of $G$. Then $M$ has a non-trivial maximal subgroup, and so the set of 2-maximal subgroups of $M$ is not empty. By hypothesis, any 2-maximal subgroup of $M$ is an ICΦ-subgroup of $G$. Lemma 2.12(1) implies that any 2-maximal subgroup of $M$ is an ICΦ-subgroup of $M$. Since $M$ is not nilpotent, it follows from Theorem 1.8 that the trivial subgroup 1 is the only 2-maximal subgroup of $M$. Applying Lemma 2.12 we conclude that $|M| = pq$, where $p$ and $q$ are prime numbers. We have $p \neq q$ since $M$ is not nilpotent.

By Lemma 4.3, $G$ is solvable. So, by Lemma 2.11, the index $|G:M|$ is a power of a prime number $r$. Assume that $r \notin \{p, q\}$. As a solvable group, $G$ has a Sylow system. Hence, there exist $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$ and $R \in \text{Syl}_r(G)$ such that $P$, $Q$ and $R$ are pairwise permutable. Since $|P| = p$ and $|Q| = q$, we have that $R$ is maximal in $RQ$ and that $RQ$ is maximal in $G$. Consequently, any maximal subgroup of $R$ is a 3-maximal subgroup of $G$. Therefore, any maximal subgroup of $R$ is an ICΦ-subgroup of $G$. We have $|R| > r$, since otherwise $G$ would not possess a non-trivial 3-maximal subgroup. Applying Theorem 1.1, we conclude that $G$ is $r$-nilpotent. This implies that $M = O_r(G) \leq G$. Since $|G:M| = |R| > r$, it follows that $M$ cannot be maximal in $G$. This contradiction shows that $r \in \{p, q\}$, and without loss of generality, we may assume that $r = p$.

(2) $G$ is minimal non-nilpotent.

Assume that $G$ is not minimal non-nilpotent. Then $G$ has a non-nilpotent maximal subgroup $M$. By (1), there exist distinct prime numbers $p$ and $q$ such that $|M| = pq$ and such that $|G:M|$ is a power of $p$.

Let $P$ be a Sylow $p$-subgroup of $G$, and let $Q$ be a Sylow $q$-subgroup of $G$. Then $|G:P| = q$, and so we have $|P| \geq p^3$, since otherwise $G$ would not possess a non-trivial 3-maximal subgroup.
Assume that \( Q \trianglelefteq G \). Then \( M/Q \) is a minimal subgroup of \( G/Q \), and we have \( |G/Q| = |P| \geq p^3 \).

This is a contradiction to the maximality of \( M \) in \( G \). Consequently, \( Q \) is not normal in \( G \).

Any 2-maximal subgroup of \( P \) is a 3-maximal subgroup of \( G \). So, by hypothesis, any 2-maximal subgroup of \( P \) is an \( IC\Phi \)-subgroup of \( G \). If \( p \) is odd or if \( p = 2 \) and \( P \not\cong Q_8 \), then Theorems 1.1 and 1.2 imply that \( G \) is \( p \)-nilpotent, which is impossible since \( Q \) is not normal in \( G \). So we have \( p = 2 \) and \( P \cong Q_8 \).

Now let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) is solvable, \( N \) is a primary subgroup of \( G \). Also \( N \not\trianglelefteq Q \) since \( |Q| = q \) and \( Q \) is not normal in \( G \). So we have \( N \leq P \).

Since \( P \cong Q_8 \), we have that \( Z(P) \) is the only subgroup of \( P \) with order 2. Hence \( Z(P) \leq N \), and

the minimal normality of \( N \) in \( G \) implies that \( N = Z(P) \). It follows that \( N \) is the only subgroup of \( G \) with order 2. Consequently, \( N \in \text{Syl}_2(M) \), which easily implies that \( M \) is nilpotent. This is a contradiction, and so (2) holds.

(3) \( G \) has a normal Sylow 2-subgroup \( P \). We have \( P \cong Q_8 \) and \( |G : P| = 3 \).

In order to prove this, we use similar arguments as in the proof of Theorem 1.1.

By (2), \( G \) is minimal non-nilpotent. So, by Lemma 2.4 (1), we have \( |G| = p^e q^f \) with distinct prime numbers \( p \), \( q \) and positive integers \( a \), \( b \), where \( G \) has a normal Sylow \( p \)-subgroup \( P \) and cyclic Sylow \( q \)-subgroups.

Assume that \( b \geq 3 \). Clearly, \( G/P \) is cyclic with order \( q^b \). Let \( P \leq H < G \) such that \( H/P \) is the unique 3-maximal subgroup of \( G/P \). Then \( H \) is a 3-maximal subgroup of \( G \). By hypothesis, \( H \) is an \( IC\Phi \)-subgroup of \( G \), and we have \( G' \leq P \leq H \). Lemma 2.3 implies that \( G \) is nilpotent. This contradiction shows that \( b \leq 2 \).

Assume that \( b = 2 \). Then any maximal subgroup of \( P \) is 3-maximal in \( G \), and so we have that any maximal subgroup of \( P \) is an \( IC\Phi \)-subgroup of \( G \). We have \( |P| > p \), since otherwise \( G \) would not possess a non-trivial 3-maximal subgroup. Theorem 1.1 implies that \( G \) is \( p \)-nilpotent. But \( G \) is also \( q \)-nilpotent, and so \( G \) is nilpotent. This contradiction shows that \( b = 1 \).

It follows that any 2-maximal subgroup of \( P \) is 3-maximal in \( G \). Therefore, any 2-maximal subgroup of \( P \) is an \( IC\Phi \)-subgroup of \( G \). We have \( |P| \geq p^3 \), since otherwise \( G \) would not possess a non-trivial 3-maximal subgroup. If \( p \) is odd or if \( p = 2 \) and \( P \not\cong Q_8 \), then Theorems 1.1 and 1.2 imply that \( G \) is \( p \)-nilpotent, which leads to a contradiction as above. So we have \( p = 2 \) and \( P \cong Q_8 \).

If \( U \) is a non-trivial proper subgroup of \( P \), then \( U \) is a cyclic 2-group, and this implies that \( N_G(U)/C_G(U) \) is a 2-group. Since \( G \) is not 2-nilpotent, Lemma 2.6 implies that \( G/C_G(P) \) is not a 2-group. Since \( G/C_G(P) \) is isomorphic to a subgroup of \( \text{Aut}(P) \), and since \( \text{Aut}(P) \) has order 24 by Lemma 2.15, we conclude that \( |G/C_G(P)| \) is divisible by 3. Therefore, \( |G| \) is divisible by 3. Since \( P \) has prime index in \( G \), it follows that \( |G : P| = 3 \).

(4) Conclusion.

Applying Lemma 2.16, we deduce from (3) that \( G \cong SL_2(3) \). So we have reached the desired conclusion.

\[ \square \]

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