Explicit computation of the Drinfeld associator in the case of the fundamental representation of $gl(N)$

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Received 17 March 2012, in final form 2 August 2012
Published 31 August 2012
Online at stacks.iop.org/JPhysA/45/385204

Abstract

We solve the regularized Knizhnik–Zamolodchikov equation and find an explicit expression for the Drinfeld associator. We restrict ourselves to the case of the fundamental representation of $gl(N)$. Several tests of the results are presented. It can be explicitly seen that components of this solution for the associator coincide with certain components of the WZW conformal block for primary fields. We introduce the symmetrized version of the Drinfeld associator by dropping the odd terms. The symmetrized associator gives the same knot invariants, but has a simpler structure and is fully characterized by one symmetric function which we call the Drinfeld prepotential.

PACS numbers: 11.15.Yc, 02.10.Kn

1. Introduction

It is well known that the polynomial invariants of knots can be evaluated with the help of Chern–Simons theory [1]. We recall some essential facts and constructions.

The Chern–Simons action is given by $S_{CS} = \int tr_R (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$. Chern–Simons theory is a three-dimensional topological quantum field theory with the gauge group $G$; traces are taken in a fixed representation $R$ of this group. Throughout this paper we let $G = GL(N)$. One can consider the gauge invariant Wilson loop operators, which are defined as follows:

$$\langle W(K) \rangle = \frac{1}{Z} \int DA \exp \left( -\frac{i}{\hbar} S_{CS}(A) \right) W(C, A),$$

where

$$Z = \int DA \exp \left( -\frac{i}{\hbar} S_{CS}(A) \right), \quad W(C, A) = tr_R P \exp \oint_C A.$$
The vacuum expectation value (1) does not depend on the realization $C$ of the knot in $\mathbb{R}^3$, but only on the equivalence class of the knot $K$; therefore, $\langle W(K) \rangle$ defines a knot invariant. Then in order to evaluate integral (1) one fixes some particular gauge. There are two relevant gauges: the holomorphic gauge [2, 3] and the temporal gauge [4, 5].

If the temporal gauge is chosen, then integral (1) gives a polynomial invariant, namely it is proportional to the HOMFLY polynomial which specializes to the Jones polynomial if one fixes a dimension $N = 2$ gauge group. In the holomorphic gauge case, the vacuum expectation value (1) is equal to the celebrated Kontsevich integral. Due to the gauge invariance of Chern–Simons theory these answers should coincide, i.e. the Kontsevich integral computed for a particular knot (or link) is equal to the corresponding HOMFLY polynomial of this knot (or link). However, the Kontsevich integral is given by a complicated infinite power series ([6, 3]). It is impossible to sum this series in general and thus it is difficult to check this explicitly.

On the other hand, the Kontsevich integral for knots and links can be calculated as a trace (over a representation $R$ of the gauge group $G$) of a certain product of two basic operators: an $R$-matrix and a Drinfeld associator $\Phi$ [6, 7, 3]. The $R$-matrix has a very simple structure, while the associator looks almost as complicated as the Kontsevich integral itself. It is given by an infinite power series in two non-commuting variables $A$ and $B$ with multiple zeta values as coefficients. In order to compute the Kontsevich integral in this way, one has to sum the series for the associator. This paper addresses the problem of summing the associator series for the simplest case of the fundamental representation (see also [8, 9]). The key is the Knizhnik–Zamolodchikov (KZ) equation, which is an additional constraint satisfied by the correlation function of the conformal field theory associated with an affine Lie algebra at a fixed level (see [10]). The solutions to this equation are the correlation functions of primary fields (see [12]). However, in practice, they are rather difficult to find directly, except for the four-point functions, which are directly related to associators [6]. The only difference between associators and these four-point functions is that the associator is a solution of a regularized version of the equation. In addition, there are actually three slightly different versions of associators and their corresponding equations.

The problem of explicitly evaluating the Kontsevich integral may seem to be just a technical problem. However, we emphasize that the search for explicit expressions for the associator $\Phi$ is very important and is a conceptual problem. The point is that the associator plays a key role in many areas of mathematics and theoretical physics such as Hopf algebras, conformal field theory, quantum groups, braid theory and many others.

In this paper, we present an explicit expression for all forms of the associator in the case of the fundamental representation of $\mathfrak{gl}(N)$. The coefficients of one of these forms turn out to coincide with certain components of the WZW conformal block for primary fields. We apply the obtained results to a few simple knots to check that the expression for the Wilson loop in the two gauges mentioned above actually coincides.

The odd part of the associator is represented by the values of multiple zeta functions at some odd points, and therefore does not enter the expression for knot invariants. We show that dropping the odd terms leads to a symmetrized version of the Drinfeld associator whose structure is simpler: it is fully characterized by one function, which is even as a function of the Planck constant. We call it the Drinfeld prepotential.

Our paper is organized as follows. In section 2, we introduce the KZ equation and its regularized form, and then we list three different forms of this equation needed for the knot theory. In section 3, we provide explicit solutions to all forms of the equation from the previous section, for the case of the fundamental representation of $\mathfrak{gl}(N)$. In section 4, we apply the obtained results to several particular examples of knots and find that they lead to invariants.
coinciding with the HOMFLY polynomials. In the final section, we introduce the symmetrized version of the associator which does not contain odd zeta function values.

2. Regularized KZ equation for the associator

Associators appearing in combinatorial construction of Kontsevich integral coefficients are related to the so-called KZ equation. This is an equation coming from conformal field theory [12]. Consider the four-point function in WZW model:

\[ G(z) = \langle g(0) g(1) g(z) g(\infty) \rangle_{\text{WZW}}. \]

The KZ equation is the Ward identity for this four-point function:

\[ \frac{dG(z)}{dz} = \hbar \left( \frac{A}{z} + \frac{B}{1-z} \right) G(z), \tag{2} \]

where \( A \) and \( B \) have the form (we assume summation over repetitive indices)

\[ A = T^\alpha \otimes (T^\alpha)^* \otimes 1 \quad \text{and} \quad B = 1 \otimes T^\alpha \otimes (T^\alpha)^* \]

and \( G(z) \) is a rank-6 tensor of the same type as \( A \) and \( B \). Also note that everywhere where we write a product of two such rank-6 tensors, we assume the following:

\[ (AB)^{\alpha\beta\gamma}_{\delta\epsilon\chi} = A^{\alpha\beta\gamma}_{\mu\nu\rho} B^{\mu\nu\rho}_{\delta\epsilon\chi}. \tag{3} \]

The perturbative approach for solving equation (2) is discussed in the appendix. The solution in the first two orders of perturbative expansion is also given. The solutions of \( sl_N \) KZ equations at level zero are given in [14]. However, we are interested in the general situation and in the regularized KZ equation. Moreover, we need an explicit expression for the associator to apply the result to knot theory.

This solution \( G(z) \) diverges at points 0 and 1, so we make the following transformation to eliminate the divergences:

\[ G(z) = (1-z)^{-\hbar B} \Phi(z) z^{\hbar A}. \tag{4} \]

The regularized equation then has the form

\[ \frac{d\Phi}{dz} = \frac{\hbar}{z} (1-z)^{\hbar B} A (1-z)^{\hbar B} \Phi - \frac{\hbar}{z} \Phi A. \tag{5} \]

The solution of this equation taken at point \( z = 1 \) gives us the associator. In section 3, equation (5) is solved in the case of fundamental representation of \( gl(N) \).

2.1. Types of associators

Actually, to construct the coefficients of the Kontsevich integral combinatorially (see [6, 3]), one needs the \( R \)-matrix and three types of associators. These three types of associators correspond to three types of orientation of strands of a braid and lead to three slightly different forms of equation (5). They are listed in table 1.

3. Solutions

In this section, we solve the corresponding equation for each of three cases given in table 1. We restrict our considerations to the case of fundamental representation of the \( gl(N) \) algebra only.
where $4$ is a matrix with components equal to $0$ except the component $i \neq j, k \neq l$.

Let us consider type (a) from Table 1. In this case $A$ and $B$ take the following form:

$$A = T^a \otimes (T^a)^* \otimes 1, \quad B = 1 \otimes T^a \otimes (T^a)^*,$$

(6)

where, since we are considering the fundamental representation of $gl(N)$,

$$T^a = g_{ij}, \quad (T^a)^* = g_{ji},$$

(7)

where index $a$ corresponds to a pair of indices $i$ and $j$ each of which goes from $1$ to $N$, and $g_{ij}$ is a matrix with components equal to $0$ except the $(i, j)$th one, which is equal to $1$. Then we obtain the following relations for $A$ and $B$:

$$A^2 = B^2 = (AB)^3 = (BA)^3 = 1.$$

(8)

In the paper [8], it is proven that the associator $\Phi$ is given by the infinite series in $A$ and $B$ (47). However, there are relations (8) on $A$ and $B$; therefore, the associator can be written as a finite polynomial:

$$\Phi = \Phi_1 + \Phi_2 A + \Phi_3 B + \Phi_4 A B + \Phi_5 B A + \Phi_6 A B A,$$

(9)

where $\Phi_i$ are some coefficients and do not depend on $A$ and $B$. Now, let us substitute (9) into equation (5); then it turns into a system of differential equations, which can be represented in the matrix form as follows:

$$\frac{d\Phi}{dz} = h(z)\Phi,$$

(10)

where

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{pmatrix}.$$

(11)

Matrix $M$ has the form

$$M = \begin{pmatrix} 0 & s^2 & 0 & -cs & cs & -s^2 \\ s^2 & 0 & cs & -s^2 & 0 & -cs \\ 0 & -cs & 0 & c^2 & -c^2 & cs \\ cs & -s^2 & c^2 & 0 & -cs & -1 \\ -cs & 0 & -c^2 & cs & 0 & c^2 \\ -s^2 & cs & -cs & 1 & c^2 & 0 \end{pmatrix}.$$

(12)
where
\[ c := \cosh(h \log(1 - z)) = \frac{1}{2}((1 - z)^h + (1 - z)^{-h}), \]
\[ s := \sinh(h \log(1 - z)) = \frac{1}{2}((1 - z)^h - (1 - z)^{-h}). \]

In order to solve equation (10) one can change the basis in the following way:
\[ \tilde{\Phi} = S \Phi, \]
\[ S = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ -\sqrt{3} & -\sqrt{3} & \sqrt{3} & 0 & \sqrt{3} & 0 \\ -1 & -1 & 2 & -1 & 2 \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} & 0 & -\sqrt{3} & 0 \\ -1 & 1 & 1 & 2 & -1 & -2 \end{pmatrix}. \]

Then matrix \( M \) takes the form
\[ \tilde{M} = SMS^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}(1 - z)^{2h} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2}(1 - z)^{-2h} & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}(1 - z)^{-2h} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2}(1 - z)^{2h} & \frac{3}{2} \end{pmatrix}. \]

Hence, the matrix equation (10) turns into the following system of differential equations:
\[ \begin{align*}
& z \frac{d\tilde{\Phi}_3}{dz} = -\frac{1}{2}h\tilde{\Phi}_3 + \frac{\sqrt{3}}{2}h(1 - z)^{2h}\tilde{\Phi}_4, \\
& z \frac{d\tilde{\Phi}_4}{dz} = -\frac{3}{2}h\tilde{\Phi}_4 + \frac{\sqrt{3}}{2}h(1 - z)^{-2h}\tilde{\Phi}_5, \\
& z \frac{d\tilde{\Phi}_5}{dz} = \frac{1}{2}h\tilde{\Phi}_5 + \frac{\sqrt{3}}{2}h(1 - z)^{-2h}\tilde{\Phi}_6, \\
& z \frac{d\tilde{\Phi}_6}{dz} = \frac{3}{2}h\tilde{\Phi}_6 + \frac{\sqrt{3}}{2}h(1 - z)^{2h}\tilde{\Phi}_5.
\end{align*} \]

with the following initial conditions (\( \Phi(0) = 1 \)):
\[ \begin{align*}
& \tilde{\Phi}_3(0) = -\sqrt{3}, \\
& \tilde{\Phi}_4(0) = -1, \\
& \tilde{\Phi}_5(0) = \sqrt{3}, \\
& \tilde{\Phi}_6(0) = -1.
\end{align*} \]
Thus the system of equations ((18)–(20)) can be reduced to a system of hypergeometric equations:

\[
\begin{align*}
\ddot{\Phi}_3(0) &= \frac{3h^2}{1 + 2h}, \\
\ddot{\Phi}_4(0) &= \frac{\sqrt{3}h^2}{1 + 2h}, \\
\ddot{\Phi}_5(0) &= \frac{\sqrt{3}h^2}{1 + 2h}, \\
\ddot{\Phi}_6(0) &= -\frac{3h^2}{1 + 2h}.
\end{align*}
\]

(20)

Thus the system of equations ((18)–(20)) can be reduced to a system of hypergeometric equations:

\[
\begin{align*}
z(1 - z) \ddot{\Phi}_3 + (1 + 2h - z) \ddot{\Phi}_3 + h^2 \ddot{\Phi}_3 &= 0, \\
z(1 - z) \ddot{\Phi}_4 + (1 + 2h - (1 + 4h)z) \ddot{\Phi}_4 - 3h^2 \ddot{\Phi}_4 &= 0, \\
z(1 - z) \ddot{\Phi}_5 + (1 - 2h - z) \ddot{\Phi}_5 + h^2 \ddot{\Phi}_5 &= 0, \\
z(1 - z) \ddot{\Phi}_6 + (1 - 2h - (1 - 4h)z) \ddot{\Phi}_6 - 3h^2 \ddot{\Phi}_6 &= 0.
\end{align*}
\]

(21)

We obtain a system of differential equations of the second order. Solutions of this system considered with the initial conditions (19) and (20) are equivalent to the solutions of (18).

Solutions are given by the following hypergeometric functions:

\[
\begin{align*}
\ddot{\Phi}_1 &= 1, \\
\ddot{\Phi}_2 &= 0, \\
\ddot{\Phi}_3 &= -\sqrt{3} \, _2F_1 (h, -h, 1 + 2h; z), \\
\ddot{\Phi}_4 &= -2F_1 (h, 3h, 1 + 2h; z), \\
\ddot{\Phi}_5 &= \sqrt{3} \, _2F_1 (h, -h, 1 - 2h; z), \\
\ddot{\Phi}_6 &= -2F_1 (-h, -3h, 1 - 2h; z).
\end{align*}
\]

(22)

We have checked with the help of a computer algebra system that solutions (22) indeed obey the original equations (18) with the initial conditions (19).

Thus, we see that the coefficients of associator of this type are given by linear combinations of values of hypergeometric functions taken at the point \( z = 1 \), which can be rewritten in terms of the Gamma function and trigonometric functions. Note that the answer in this case does not depend on order \( N \) of the algebra \( gl(N) \):

\[
\begin{align*}
\ddot{\Phi}_1(1) &= 1, \\
\ddot{\Phi}_2(1) &= 0, \\
\ddot{\Phi}_3(1) &= -\sqrt{3} \, _2F_1 (h, -h, 1 + 2h; 1), \\
\ddot{\Phi}_4(1) &= -2F_1 (h, 3h, 1 + 2h; 1), \\
\ddot{\Phi}_5(1) &= \sqrt{3} \, _2F_1 (h, -h, 1 - 2h; 1), \\
\ddot{\Phi}_6(1) &= -2F_1 (-h, -3h, 1 - 2h; 1) \\
\dddot{\Phi}_1 &= -\frac{\sqrt{3}}{\Gamma(1 + 2h)\Gamma(1 + 3h)}, \\
\dddot{\Phi}_2 &= \frac{1}{\cos(\pi h)}, \\
\dddot{\Phi}_3 &= \frac{\Gamma(1 + 2h)^2}{\Gamma(1 + h)\Gamma(1 + 3h)}, \\
\dddot{\Phi}_4 &= \frac{\Gamma(1 - 2h)^2}{\Gamma(1 - h)\Gamma(1 - 3h)}, \\
\dddot{\Phi}_5 &= \frac{\Gamma(1 + 2h)^2}{\Gamma(1 + h)\Gamma(1 + 3h)}, \\
\dddot{\Phi}_6 &= \frac{1}{\cos(\pi h)}.
\end{align*}
\]

(23)

Thus, we obtained the explicit expressions for \( \ddot{\Phi} \). To obtain an expression for the associator one needs to take \( \ddot{\Phi} = S^{-1} \Phi \) and then use (9).
3.2. Type (b)

Now we briefly present results for type (b) from table 1. In this case, we have

\[ A = g_{ij} \otimes g_{ij} \otimes 1, \quad B = 1 \otimes g_{ij} \otimes g_{ij} \]  

(24)

Relations for \( A \) and \( B \) are the following ones:

\[
\begin{align*}
A^2 &= NA, \\
B^2 &= NB, \\
BAB &= B, \\
ABA &= A.
\end{align*}
\]

(25)

Then the associator takes the form

\[
\Phi = \Phi_1 + \Phi_2 A + \Phi_3 B + \Phi_4 AB + \Phi_5 BA.
\]

(26)

By analogy with the previous case we write

\[
\tilde{\Phi} = S \Phi,
\]

(27)

for

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & N & 0 & 1 & 0 \\
\frac{1}{N^2 - 1} & 0 & 0 & -1 & 0 \\
1 & 0 & N & 1 & 0 \\
\frac{1}{N^2} & \frac{1}{N} & \frac{1}{N} & 1 & 1 \\
\end{pmatrix}.
\]

(28)

Then we have

\[
\begin{align*}
\tilde{\Phi}_1 &= 1, \\
\tilde{\Phi}_2 &= zF_1(h, -h, 1 - Nh; z), \\
\tilde{\Phi}_3 &= \frac{1}{N^2 - 1} z F_1((N - 1)h, (N + 1)h, 1 + Nh; z), \\
\tilde{\Phi}_4 &= z F_1(h, -h, 1 + Nh; z), \\
\tilde{\Phi}_5 &= \frac{1}{N^2} z F_1(-(N - 1)h, -(N + 1)h, 1 - Nh; z)
\end{align*}
\]

\[
\begin{align*}
\Phi_1(1) &= 1, \\
\Phi_2(1) &= \frac{\Gamma(1 - Nh)^2}{\Gamma(1 - Nh - h) \Gamma(1 - Nh + h)}, \\
\Phi_3(1) &= \frac{\Gamma(1 + Nh)^2}{\Gamma(1 + Nh - h) \Gamma(1 + Nh + h)}, \\
\Phi_4(1) &= \frac{1}{N^2 - 1} \frac{\Gamma(1 + Nh)}{\sin(\pi Nh)}, \\
\Phi_5(1) &= \frac{N \sin(\pi Nh)}{N^2 \sin(\pi Nh)}.
\end{align*}
\]

(29)

Here we also see that the coefficients of associator of this type are given by linear combinations of values of hypergeometric functions taken at the point \( z = 1 \).

Note that the coefficients of the associator in this case coincide with certain components of the WZW conformal block for primary fields in the fundamental representation. Namely we have \( F_1^{(-)} = \tilde{\Phi}_2, \ F_1^{(+)} = \tilde{\Phi}_3 \) up to singular factors and substitution \( h = \frac{1}{2} \), where \( F_1^{(-)} \) and \( F_1^{(+)} \) are the components of the WZW conformal block for primary fields taken from section 15.3.2 of [12]. The other two coefficients \( \tilde{\Phi}_4 \) and \( \tilde{\Phi}_5 \) differ from the previous two just by substitution \( N \rightarrow -N \).

Again, to obtain the expression for the associator one needs to take \( \tilde{\Phi} = S^{-1} \tilde{\Phi} \) and then use (26).
3.3. Type (c)

Associators of type (c) allow us to calculate the simplest knots like in figure 1.

In this case, we have

\[ A = g_{ij} \otimes g_{ij} \otimes 1, \quad B = 1 \otimes g_{ij} \otimes g_{ij}. \]  

Relations for \( A \) and \( B \) are the following ones:

\[
\begin{align*}
A^2 &= 1, \\
B^2 &= NB, \\
BAB &= B.
\end{align*}
\]

Then the associator takes the form

\[ \Phi = \Phi_1 \Phi_2 A + \Phi_3 B + \Phi_4 AB + \Phi_5 BA + \Phi_6 ABA. \]

By analogy with the previous case we write

\[ \tilde{\Phi} = S \Phi, \]

for

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-N & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 1 & 1 \\
\frac{1}{2N} & \frac{1}{2N} & \frac{1}{2} & \frac{1}{2N} & \frac{1}{2} & \frac{1}{2N} \\
\frac{1}{2(N+1)} & \frac{1}{2(N+1)} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{N} & \frac{1}{N} & -1 & \frac{1}{N} & \frac{1}{N} \\
\frac{1}{N-1} & \frac{1}{N-1} & -1 & -1 & 0 & 1
\end{pmatrix}.
\]
Figure 2. Hump.

Then

\[
\begin{aligned}
\tilde{\Phi}_1 &= 1, \\
\tilde{\Phi}_2 &= -N, \\
\tilde{\Phi}_3 &= \frac{1}{2N} \left( F_1(h, -(N-1)h, 1 + 2h, z) \right) , \\
\tilde{\Phi}_4 &= \frac{1}{2(N+1)} \left( F_1(h, (N+1)h, 1 + 2h; z) \right), \\
\tilde{\Phi}_5 &= -\frac{1}{N} \left( F_1(-h, -(N+1)h, 1 - 2h; z) \right), \\
\tilde{\Phi}_6 &= \frac{1}{N-1} \left( F_1(-h, (N-1)h, 1 - 2h; z) \right) \\
\end{aligned}
\]

\[\Rightarrow \begin{aligned}
\tilde{\Phi}_1(1) &= 1, \\
\tilde{\Phi}_2(1) &= -N, \\
\tilde{\Phi}_3(1) &= \frac{1}{2N} \frac{4^h \Gamma(h + \frac{1}{2}) \Gamma(1 + Nh)}{\sqrt{\pi} \Gamma(1 + Nh + h)}, \\
\tilde{\Phi}_4(1) &= \frac{1}{2(N+1)} \frac{4^h \Gamma(h + \frac{1}{2}) \Gamma(1 - Nh)}{\sqrt{\pi} \Gamma(1 - Nh + h)}, \\
\tilde{\Phi}_5(1) &= -\frac{1}{N} \frac{4^{-h} \Gamma(-h + \frac{1}{2}) \Gamma(1 + Nh)}{\sqrt{\pi} \Gamma(1 + Nh - h)}, \\
\tilde{\Phi}_6(1) &= \frac{1}{N-1} \frac{4^{-h} \Gamma(-h + \frac{1}{2}) \Gamma(1 - Nh)}{\sqrt{\pi} \Gamma(1 - Nh - h)}. \\
\end{aligned} \tag{35}
\]

In this case as in the previous ones the coefficients of associator of this type are given by linear combinations of values of hypergeometric functions taken at the point \( z = 1 \).

4. Comparison to known formulas for knots

In this section, we apply the formulas for the associator obtained in section 3 to compute the Kontsevich integral for a few knots. We do this following [6, 15]. The computations are done in the case of the fundamental representation of \( gl(N) \). In this case, the Kontsevich integral is known to coincide with the HOMFLY polynomial for the given knot, and we use this fact to check the validity of our formulas for the associator. Namely we check that the Kontsevich integral computed with the help of these formulas indeed coincides with known expressions for the HOMFLY polynomial.

4.1. Hump

We start with the hump, which is a version of the unknot (for the definition of the hump see [6]). Recall that the associator is a tensor with six indices. Denote the associator of type (b) as \( \Phi_b \). The Kontsevich integral of the hump corresponds, by definition, to the contraction of the indices of \( \Phi_b \) corresponding to figure 2. We will refer to this contraction as the trace of the associator.

Writing this contraction in terms of the indices gives the following expression:

\[
KI_{\text{Hump}} = \text{tr}_{\text{Hump}} \Phi_b := \sum_{\alpha, \beta, \gamma} (\Phi_b)_{\gamma \gamma \alpha} \Phi_b^{\alpha \beta \beta}. \tag{36}
\]
Now let us expend the associator as in formula (26):

\[ \Phi_b = \Phi_1 + \Phi_2 A + \Phi_3 B + \Phi_4 A B + \Phi_5 B A. \]  

(37)

With the help of formula (24) one can compute traces (in the above-mentioned sense) of particular terms of (37) in the following way:

\[ \text{tr}_{\text{Hump}} A = \frac{1}{N} \text{tr}(\delta_{ij} g_{\mu}) = \frac{1}{N} \text{tr}(\delta_{ij} g_{\mu}) = N \]  

(38)

\[ \text{tr}_{\text{Hump}} A B = \frac{1}{N} \text{tr}(g_{ij} g_{mn} g_{\mu}) = 1 \]  

(39)

\[ \text{tr}_{\text{Hump}} B A = \frac{1}{N} \text{tr}(g_{ij} g_{mn} g_{\mu}) = N^2. \]  

(40)

Note that \( \text{tr}_{\text{Hump}} \) here stands for the above-mentioned trace of a rank-6 tensor, while \( \text{tr} \) stands for just the ordinary trace of a matrix. We also assume summation over repetitive indices.

Now substituting formulas (29), (28), (38)–(40) to (27) we obtain the following answer:

\[ K_{I_{\text{Hump}}} = 2 F_1 (-N + 1) \hbar, - (N + 1) \hbar, 1 - N \hbar; 1) = N \cdot \frac{e^{\pi i h} - e^{- \pi i h}}{e^{\pi i h} - e^{- \pi i h}} = \frac{N}{[N]}, \]  

(41)

where we introduce the following notation for a \( q \)-number:

\[ [N] = q^N - q^{-N} \quad q = e^{\pi i \hbar}. \]

This indeed coincides with the known result for the HOMFLY polynomial of the hump (which can be found e.g. in [15]).

4.2. Series \([2,2k + 1]\)

Consider the so-called torus knots of type \([2, 2k + 1]\), as shown in figure 1. Denote the associator of type (c) as \( \Phi_c \). With the help of this associator one can compute the Kontsevich integral for knots of this type [6, 15]:

\[ K_{I_{[2,2k+1]}} = \sum_{a, \beta, \gamma} (\Phi_c R^{2k+1} \Phi_c^{-1})^{\alpha \beta \gamma}, \]  

(42)

where \( R \) is in our notation equal to

\[ R = e^{\pi i h A}. \]  

(43)

Now expanding the exponential in series, substituting formula (32) for \( \Phi_c \) and taking into account relations (31), one can arrive at the formula

\[ K_{I_{[2,2k+1]}} = \kappa_1 + \kappa_2 A + \kappa_3 B + \kappa_4 A B + \kappa_5 B A + \kappa_6 A B A, \]  

(44)

where \( \kappa_1, \ldots, \kappa_6 \) are certain expressions in terms of the coefficients \( \Phi_1, \ldots, \Phi_6 \) from formula (32).

Then, after taking traces of the individual terms in a way analogous to formulas (38)–(40) and substituting \( \Phi_1, \ldots, \Phi_6 \) from formula (31), we obtain

\[ K_{I_{[2,2k+1]}} = N \frac{\cos(\pi h (2k + 1)) \sin(\pi h) \cos(\pi h N) + i \cdot \sin(\pi h (2k + 1)) \cos(\pi h) \sin(\pi h N)}{\cos(\pi h) \sin(\pi h N)}. \]  

(45)

Note that this answer is not normalized with respect to the unknot. Let us introduce \( q = e^{\pi i h} \) and normalize with respect to the unknot:

\[ K_{I_{[2,2k+1]}} / K_{I(\bigcirc)} = q^1 (q^{2k+2} - q^{-2N+2k} + q^{2N-2k} - q^{-2k-2}) / q^1 - 1. \]  

(46)
This expression is in perfect agreement with the known results. For example, it is known that in the Abelian case \( N = 1 \) the expectation value of the Wilson loop is given by \( q^{l(K)} \), where \( l(K) \) is the so-called linking number of the knot \( K \) [5]. The linking number can be computed as a sum of the orientations for the crossings of the knot, which takes values \( \pm 1 \). In our case—figure 1—the knot has \( 2k + 1 \) crossings with equal orientations, so the linking number is \( l = 2k + 1 \). It is an elementary exercise to check that for \( N = 1 \) our formula (46) indeed gives \( q^{2k+1} \). In the first nontrivial case \( N = 2 \), one can check that the result (46) for any choice of \( k \) is a Laurent polynomial in \( q \) which coincide precisely with the Jones polynomial for the torus knot of type \([2, 2k + 1]\). For a general \( N \), the answer (46) is a polynomial in \( A = q^{-N} \) and a Laurent polynomial in \( q \). This two-parametric polynomials again coincide with the known HOMFLY polynomials for these knots as computed for example in [16, 15]. Being a rather nontrivial computation, this provides a strong test of our formulas for the associator.

5. The Drinfeld prepotential

In the previous sections, we have computed the Drinfeld associator for the case of the fundamental representation explicitly and checked that it reproduces correct HOMFLY polynomials for the toric knots of type \([2, 2k + 1]\). In fact, as we explain below, the Drinfeld associator contains more information than is used for the knot invariants in the following sense: not all components of the Drinfeld associator contribute to the answer for the knot invariants.

Using the perturbation approach to the solution of the regularized KZ equation (as in the appendix), Le and Murakami found the expression for the associator as infinite sum in the non-commutative operators \( A \) and \( B \) (without any additional relations of the form (8), i.e. for all representations) with the coefficients in the form of the multiple zeta function values [8]:

\[
\Phi_3 = 1^\otimes 3 + \sum_{k=2}^\infty \left( \frac{\hbar}{2\pi i} \right)^k \sum_{m \geq 0} \sum_{p_0 \ldots p_m = 0} (-1)^m \tau(p_1, q_1, \ldots, p_m, q_m) \times \sum_{\substack{p_{m-1} \ldots p_n \geq 0 \geq 0 \geq 0 \ldots \geq 0 \geq 0 \geq 0}} (-1)^m \left( \prod_{i=1}^m \left( p_i \right) \left( q_i \right) \right) B[s_1 \ldots s_m] A[p_1 \ldots p_m] B[q_1 \ldots q_m],
\]

(47)

where \( p = (p_1, p_2, \ldots, p_m) \) is a vector with positive integer components. The length of the vector \( l(p) = m \) and \( |p| = \sum p_i \). For \( p \) and \( q \) with positive integers \( p > q \) means that \( p_i > q_i \) and \( p > 0 \) means that \( p_i > 0 \) for all \( i \). The coefficients \( \tau(p_1, q_1, \ldots, p_m, q_m) \) are expressed through the multiple zeta functions as follows:

\[
\tau(p_1, q_1, \ldots, p_m, q_m) = \zeta(1, 1, \ldots, 1, q_1 + 1, 1, \ldots, 1, q_2 + 1, \ldots, 1, q_n + 1)
\]

\[
\text{such that for example } \tau(1, 2) = \zeta(3) \text{ and } \tau(2, 1) = \zeta(1, 2).
\]

The zeta functions are defined as

\[
\zeta(m_1, m_2, \ldots, m_n) = \sum_{0 < k_1 < k_2 < \ldots < k_n} k_1^{-m_1} k_2^{-m_2} \ldots k_n^{-m_n}.
\]

(49)

The operators \( A \) and \( B \) depend on the gauge group and its representation:

\[
A = \Omega \otimes 1, \quad B = 1 \otimes \Omega, \quad \Omega = T_{1, \sigma}^\sigma \otimes T_{1, \sigma}^\sigma.
\]

(50)

Formula (47) gives exact answer for the Drinfeld associator in all orders of \( \hbar \); however, it is impractical for explicit computations of the knot invariants. For such computations, one needs to summate exactly series (47) in the presence of some relations among \( A \) and \( B \) given by a
representation of the gauge group. In the case considered here (fundamental representation of \( gl(N) \)), these relations are given by (8), (25) or (31) depending on the associator type. The main difficulty is that the values of the multiple zeta functions (49) are difficult to compute (see for example [17]) and there are no chances to summate series (47) exactly in all orders of \( h \) by ‘bare hands’ even in the presence of additional relations for \( A \) and \( B \).

Our approach implements the relations from the very beginning: using them, we reduce the KZ equation to the system of the hypergeometric equations. The solution of this system at point \( z = 1 \) gives the associator. In this way, we were able to summate the series (47) with nontrivial zeta functional coefficients explicitly, and the result has the form of simple gamma functional factors (23). Using a computer we checked explicitly that up to the order \( h^5 \) our result coincides precisely with the series coming from (47). This should hold for all higher orders as well, but we do not know how to show it explicitly due to the multiple zeta functions being difficult to work with, as mentioned above.

It is known that the knot invariants must be rational functions in \( q = \exp(i\pi h) \) for any choice of the gauge group and its representation. For example our answers for unknot (41) and torus knots of type \([2, 2k + 1]\) (45) are obviously of this type. Therefore, the \( h \)-series expansion of knot invariants has the form \( \sum_{k=0}^{\infty} a_k h^k \), where we denoted \( h = i\pi \hbar \), with the rational coefficients \( a_k \in \mathbb{Q} \).

The even values of the multiple zeta functions, i.e. \( \zeta(m_1, m_2, \ldots, m_n) \) for \( k = m_1 + m_2, \ldots + m_n \)-even number, are known to have the form \( a\pi^k \), where \( a \) is a rational number \( a \in \mathbb{Q} \). On the other hand, the odd zeta function values are always irrational (conjecturally are transcendental [17]). Moreover, the odd zeta values are algebraically independent over rational numbers, i.e. there is no polynomial relations among this values over \( \mathbb{Q} \) (the coefficients of polynomials are rational) [17].

It is clear from the above formula (47) that the odd (even) zeta values represent the odd (even) coefficients of the Drinfeld associator expanded in the Planck constant \( h \). Therefore, in order to make the coefficients of the \( h \)-series for invariants rational, the odd part of the associator must cancel in the answer. Indeed, the knot invariant is given by a trace (over the knot projection to the two-dimensional plane) of a product of associators and braiding matrices \( R \) as for example in (42). The braiding matrix is given by exponent (43), and its coefficients of \( h \)-expansion are rational for all representations. Therefore, the contribution of the odd part of associator to the knot invariant is given by polynomial combinations of odd zeta values with rational coefficients, and as mentioned above such a combination is rational only if all the coefficients of the polynomial are zero, i.e. the coefficients of odd zeta values cancel.

Calculations up to \( h^5 \) show the following: every time we are computing the knot invariant by taking the trace of a braiding, for example as in (42), the odd zeta values enter the answer with a group factor (i.e. trace words made from \( A \) and \( B \)) that are exactly zero due to the celebrated STU and IHX relations for chord diagrams [11]. Therefore, for knot theory, we can only use the even part of the associator. We found that dropping the odd terms in some clever way, the expression for associator possess valuable simplification. Indeed, consider the logarithm \( F \) of the associator

\[
\Phi = \exp(F(h)).
\]

Introduce then the symmetrized associator (i.e. we drop all odd terms)

\[
\Phi_s = \exp\left(\frac{1}{2}(F(h) + F(-h))\right).
\]

It should be clear from the discussion above that this ‘even’ associator leads to the same knot invariants, but its structure is simpler. For example, in the fundamental representation case, for the associator of type (a) specified by relations (8)

\[
A^2 = B^2 = (AB)^3 = (BA)^3 = 1,
\]
we obtain
\[ \Phi_s = \exp(\mathcal{F}_s(h^2)[A, B]) \]  
(54)

with
\[ \mathcal{F}_s(h^2) = \frac{i}{2\sqrt{N^2 - 1}} \ln \left( \frac{(i(q^N + 1)(1 - q)N - 2N\sqrt{(q^{N+1} - 1)(q^N - 1)})}{(1 + q)(1 - q^N)(\sqrt{N^2 - 1} + i)} \right), \quad q = e^{2\pi i \hbar}. \]  
(55)

Therefore, the symmetrized associator is characterized by one function \( \mathcal{F}_s \) that is even in \( \hbar \), and has rational coefficients of \( \hbar \)-series expansion. We call this function the Drinfeld prepotential due to the role that the logarithm of the four-point function plays in CFT, integrable systems and four-dimensional SUSY models [13].

It would be interesting to look for the differential equation for the symmetrized associator (52). As the form of the symmetrized associator appears to be simpler, it would simplify the calculation of knot invariants significantly. This problem, however, is beyond the scope of this paper and should be discussed elsewhere.

6. Conclusion

To conclude, we solved the regularized Knizhnik–Zamolodchikov equation and found the explicit expressions for associators in the case of the fundamental representation of \( \mathfrak{gl}(N) \). It turned out that the associator is a finite series in \( A \) and \( B \) with coefficients given by values of hypergeometric functions taken at point \( z = 1 \). Also, we presented some tests of our results.

One possible direction of further research is to consider higher representations. For that case, everything becomes more complicated due to more complicated relations for \( A \) and \( B \).

It would also be very interesting to understand the relation between conformal blocks and associator more deeply.

Acknowledgments

We are grateful to A Mironov and Ye Zenkevich for helpful remarks. We are especially grateful to A Morozov for stimulating discussions. Our work is partly supported by Ministry of Education and Science of the Russian Federation under contracts 14.740.11.0081 (P.D.B., A.Sl.) and 14.740.11.0347 (A.Sm.), by RFBR grants 10-02-00499 (P.D.B.) and 10-01-00536 (A.Sl., A.Sm.), by joint grants 12-02-92108-YaF (A.Sl.), 12-02-91000-ANF (A.Sl.), 11-01-92612-Royal Society(P.D.B., A.Sl., A.Sm.), by NWO grant 613.001.021 (P.D.B.), by RFBR-09-01-93106-NCN La (A.Sm.) and by Dynasty foundation.

Appendix. The solution of the KZ equation in the first two orders of perturbative expansion

In this appendix, we solve the KZ equation in the first two orders of perturbative expansion. For the KZ equation,
\[ \frac{dG(z)}{dz} = \hbar \left( \frac{A}{z} + \frac{B}{1-z} \right) G(z). \]

Let us try to find the solution as a series in \( \hbar \):
\[ G(z) = G_0(z) + \hbar G_1(z) + \hbar^2 G_2(z) + \cdots. \]
Equation (2) has a unique solution defined by the following boundary condition:

\[ G_0(0) = 1 = 1 \otimes 1 \otimes 1. \]

We are interested in the value \( G(1) \).

First order:

\[ \frac{dG_1(z)}{dz} = A \frac{1}{z} + B \frac{1}{1-z} \Rightarrow G_1(1) = A \int_0^1 \frac{dz}{z} + B \int_0^1 \frac{dz}{1-z}. \]

The integrals here are divergent, so we normalize them by

\[ \int_0^1 \to \int_0^{1-\epsilon}. \]

Then, we have

\[ G_1(z) = A \log \left( \frac{z}{\epsilon} \right) - B \log \left( \frac{1-z}{1-\epsilon} \right) \]

and, keeping only finite and singular terms at point \( z = 1 \),

\[ G_1(1) = -A \log(\epsilon) - B \log(\epsilon). \]

Second order:

\[ \frac{dG_2(z)}{dz} = \left( A \frac{1}{z} + B \frac{1}{1-z} \right) G_1(z) \]

\[ \quad = A^2 \frac{1}{z} \log z - AB \frac{1}{z} \log \left( \frac{1-z}{1-\epsilon} \right) + BA \frac{1}{1-z} \log \left( \frac{z}{\epsilon} \right) - B^2 \frac{1}{1-z} \log \left( \frac{1-z}{1-\epsilon} \right). \]  

(A.1)

Integrating over \( \epsilon, z \), we obtain

\[ G_2(z) = \frac{1}{2} A^2 \log^2 \frac{z}{\epsilon} - AB \left( \text{Li}_2(\epsilon) - \text{Li}_2(z) - \log(1-\epsilon) \log \frac{z}{\epsilon} \right) \]

\[ + BA \left( \text{Li}_2(\epsilon) - \text{Li}_2(z) - \log(1-z) \log \frac{z}{\epsilon} \right) + \frac{1}{2} B^2 \log^2 \frac{1-z}{1-\epsilon}. \]  

(A.2)

At point \( z = 1 - \epsilon \), for finite and singular terms we have

\[ G_2(1) = \frac{1}{4} A^2 \log^2 \epsilon + BA \log \epsilon \log \epsilon + \frac{1}{2} B^2 \log^2 \epsilon + [A, B] \text{Li}_2(1). \]  

(A.3)

Thus, up to the second order in \( \hbar \), \( G(1) \) has the following form (we drop the terms which tend to zero as \( \epsilon \to 0 \)):

\[ G(z) = (1 - \hbar B \log \epsilon + \frac{1}{2} \hbar^2 B^2 \log^2 \epsilon) \left( 1 + \hbar^2 [A, B] \text{Li}_2(1) \right) \left( 1 - \hbar A \log \epsilon + \frac{1}{2} \hbar^2 A^2 \log^2 \epsilon \right) + O(\hbar^3). \]  

(A.4)

References

[1] Witten E 1989 Quantum field theory and the Jones polynomial Commun. Math. Phys. 121 351
[2] Labastida J M F and Perez E 1998 Kontsevich integral for Vassiliev invariants from Chern–Simons perturbation theory in the light-cone gauge J. Math. Phys. 39 5183–98 (arXiv:hep-th/9710176)
[3] Dunin-Barkowski P, Sleptsov A and Smirnov A 2011 Kontsevich integral for knots and Vassiliev invariants arXiv:1112.5406
[4] Smirnov A 2009 Notes on Chern–Simons theory in the temporal gauge Proc. of Int. School of Subnuclar Phys. (Erice, Italy) pp 489–99 (arXiv:0910.5011)
[5] Morozov A and Smirnov A 2010 Chern–Simons theory in the temporal gauge and knot invariants through the universal quantum R-matrix Nucl. Phys. B 835 284–313
[6] Chmutov S and Duzhin S 2001 The Kontsevich integral Acta Appl. Math. 66 155–90
[7] Chmutov S, Duzhin S and Mostovoy J 2011 Introduction to Vassiliev Knot Invariants (Cambridge: Cambridge University Press) at press (arXiv:1103.5628)
[8] Le T T Q and Murakami J 1996 Kontsevich’s integral for the Kauffman polynomial Nagoya Math. J. 142 39–65
[9] Le T T Q and Murakami J 1995 Kontsevich’s integral for the HOMFLY polynomial and relations between values of multiple zeta functions Topol. Appl. 62 193–206
[10] Drinfeld V G 1991 On quasi-triangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ Leningr. Math. J. 2 829–60
[11] Bar-Natan D 1995 On the Vassiliev knot invariants Topology 34 423–72
[12] Di Francesco P, Mathieu P and Senechal D 1997 Conformal Field Theory (New York: Springer) p 890
[13] Morozov A and Shakirov S 2010 The matrix model version of AGT conjecture and CIV-DV prepotential J. High Energy Phys. JHEP08(2010)066 (arXiv:1004.2917)
[14] Nakayashiki A 1998 Integral and theta formulae for solutions of $sl_N$ Knizhnik–Zamolodchikov equation at level zero Publ. Res. Inst. Math. Sci. 34 439–86 (arXiv:math/9803086)
[15] Dunin-Barkowski P, Mironov A, Morozov A, Sleptsov A and Smirnov A 2011 Superpolynomials for toric knots from evolution induced by cut-and-join operators arXiv:1106.4305
[16] Lin X-S and Zheng H 2010 On the Hecke algebras and the colored HOMFLY polynomial Trans. Am. Math. Soc. 362 1–18 (arXiv:math/0601267)
[17] Zudilin V V 2003 Algebraic relations for multiple zeta values Usp. Mat. Nauk 58 332