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Anisotropic distribution functions for spherical galaxies

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Abstract A method is presented for finding anisotropic distribution functions for stellar systems with known, spherically symmetric, densities, which depends only on the two classical integrals of the energy and the magnitude of the angular momentum. It requires the density to be expressed as a sum of products of functions of the potential and of the radial coordinate. The solution corresponding to this type of density is in turn a sum of products of functions of the energy and of the magnitude of the angular momentum. The products of the density and its radial and transverse velocity dispersions can be also expressed as a sum of products of functions of the potential and of the radial coordinate. Several examples are given, including some of new anisotropic distribution functions. This device can be extended further to the related problem of finding two-integral distribution functions for axisymmetric galaxies.

Keywords celestial mechanics · stellar dynamics · galaxies

1 Introduction

There is a long history of studying the structure of galaxies by construction of self-consistent distribution functions for a stellar system with a known gravitational potential. The potential of the system determines the self-consistent mass density $\rho$ of the system via Poisson’s equation generated by the well-known Newtonian gravitational law, and also the structure of the stellar orbits according to Newton’s equations of motion. The system is constructed from the building blocks of the orbits that can lie within the potential. This is called the “from $\rho$ to $f$” approach for finding a self-consistent distribution function $f$ (Binney and Tremaine 1987, hereafter BT). The distribution function (hereafter DF) of the system represents how stars are distributed in the phase space of the system and the integration of the DF over velocity space yields the density. Therefore the problem of finding the DF is that of solving an integral equation; in the spherical case in particular this equation is of the first kind, in which the unknown DF $f$ only occurs inside the integral.

Some outstanding astronomers have contributed to this mathematical problem. Eddington (1916) showed that it can be solved for an isotropic DF that depends only on the energy in the spherical case, whose velocity dispersions in the radial and tangential directions must be equal, by first expressing the density as a function only of the potential, and then solving the Abel integral equation. It is also well known that the isotropic DF is unique for the potential of any spherical system.

Fricke’s (1952) expansion method can be obviously generalized to the spherical case (e.g., Camm 1952; Bouvier 1962, 1963; Ossipkov 1979a; Kent and Gunn 1982; Dejonghe 1986, 1987; Dejonghe and Merritt 1988;
Cuddeford 1991; Louis 1993), and similarly yields the result that DFs which are products of the two powers of the energy and the square of the angular momentum correspond to densities which are proportional to products of the potential and the spherical radial coordinate for spherical systems. Hence the DF for the system can be obtained by first expressing the density as a function of the potential and the spherical radial coordinate, and then expanding as a power series (Qian and Hunter 1995). Moreover, there may be an infinity of anisotropic DFs corresponding to any given mass density in spherical stellar systems (Dejonghe 1987).

A class of typical anisotropic DFs that depend on the energy and the magnitude of the angular momentum, whose velocity dispersions are anisotropic (but ellipsoidal), was independently found by Ossipkov (1979) and Merritt (1985), for a spherical density distribution. Such anisotropic DFs of Ossipkov-Merritt type are in fact analogues of Eddington’s formula, mentioned above. In a number of papers, different integral transformation techniques can be used to obtain the solution of the spherical (or axisymmetric) problem (e.g., Lynden-Bell 1962; Hunter 1975; Kalnajs 1976; Dejonghe 1986; Qian and Hunter 1995) but there is the same difficulty of requiring not only the validity of these transformations of the density but also the complex analyticity of a density-related integral kernel to complex arguments. The contour integral method of Hunter and Qian (1993) can be also used to find anisotropic DFs for spherical systems (Qian and Hunter 1995) but it is valid for densities that are analytic, and whose singularities satisfy some conditions.

It is worth mentioning that Kuzmin and Veltmann (1967a, 1973) and Veltmann (1961, 1965, 1979, 1981) developed some more general classes when DF is a product of an unknown function over any argument on a known function of another.

The fundamental integral equations of the problem are given in Sect. 2 and some new formulae for finding other anisotropic DFs for stellar systems with known spherically symmetric densities are presented in Sect. 3. These depend only on the two integrals and the magnitude of the angular momentum. This work constitutes the core of this paper. These formulae in fact come from an combination of the ideas of Eddington and Fricke (see above). Of course, they can be also regarded as simply an extension of Eddington’s formula. A type of anisotropic DF which is a sum of products of functions only of the energy and powers of the magnitude of the angular momentum is derived in Sect. 3.2 and another, which is a sum of products of functions only of a special variable and powers of the magnitude of the angular momentum, in Sect. 3.3. More general formulae are given in the last part of Sect. 3. Various formulae of the velocity dispersions for such models of these DFs are also shown in all the three parts of Sect. 3. Several examples are given in Sect. 4, including some of new anisotropic DFs. Different anisotropic DFs for the Plummer model are first given in Sect. 4.1. Then anisotropic DFs of the Hénon isochrone model appear in Sect. 4.2 and those of the γ-model are described in Sect. 4.3. Section 5 is a summary and conclusion.

2 The fundamental integral equations

For convenience of calculation, as in BT it is usual to introduce the relative potential \( \psi \) and the relative energy \( \varepsilon \) of a star in a stellar system, defined by \( \psi = -\Phi + \Phi_0 \) and \( \varepsilon = -E + \Phi_0 \), where \( \Phi \) and \( E \) are, respectively, the potential and the energy of a star, and \( \Phi_0 \) is a constant generally chosen to be such that there are only stars of \( \varepsilon > 0 \) in the system modelled by DFs. It is well known that the relative energy \( \varepsilon \) and the three components of the angular momentum vector \( \mathbf{L} \) are four isolating integrals for any orbit in a spherical potential. By the Jeans theorem, it follows that the DF of a steady-state spherical stellar system can be regarded as a non-negative function of these integrals, denoted by \( f = f(\varepsilon, \mathbf{L}) \). If the system is spherically symmetric in all its properties, \( f \) is independent of the direction of \( \mathbf{L} \) and depends only on its magnitude \( L \) (e.g., Shivshwarkar 1936; Ogorodnikov 1965), that is, \( f \) can be expressed as a non-negative function of the relative energy \( \varepsilon \) and the absolute value \( L \) of the angular momentum vector \( \mathbf{L} \), denoted by \( f = f(\varepsilon, L) \). If the stellar system itself provides the relative potential \( \psi = \psi(\mathbf{r}) \), the mass density \( \rho = \rho(\mathbf{r}) \) can be obtained from Poisson’s equation, and it is known that its distribution function \( f = f(\mathbf{r}, \mathbf{v}) \) satisfies

\[
-\nabla^2 \psi = 4\pi G \rho = 4\pi G \int f d^3v,
\]

where \( \mathbf{r} \) is a position vector, \( \mathbf{v} \) is a velocity vector, \( G \) is the gravitational constant. Let \( r \) be the modulus of the position vector \( \mathbf{r} \). For a spherically symmetric system, both the relative potential \( \psi = \psi(\mathbf{r}) \) and the mass density \( \rho = \rho(\mathbf{r}) \) can be, respectively, regarded as two functions of \( r \), that is, \( \psi = \psi(r) \) and \( \rho = \rho(r) \), and the distribution function \( f = f(\mathbf{r}, \mathbf{v}) \) can be expressed as a function of the relative energy \( \varepsilon \) and the momentum magnitude \( L \),...
thus equation (1) can be rewritten as
\[
-\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) = 4\pi G \rho = 4\pi G \int f(\varepsilon, L) \, d^3v.
\] (2)

Once \( f(\varepsilon, L) \) is known, \( \rho \) can be easily calculated by integration and \( \psi \) by solving Poisson’s equation for the spherical system. The inverse problem that is now investigated is how to derive the DF from the density \( \rho \) for any spherical system. Different classes of anisotropic DFs will be below shown, which are derived from spherical density profiles for galaxies, by combining some functions only of \( \varepsilon \) (or \( Q \equiv \varepsilon - L^2/(2r_a^2) \)) with some functions of the form \( L^{-2\beta_n} \) where \( r_a \) is a scaling radius, \( n \) is an integer greater than \(-2\) and \( \beta_n \) is a constant such that \( n\beta_n > -1 \).

3 Anisotropic DFs

In this section various formulae for anisotropic DFs are obtained from spherical density profiles of different forms and their radial and transverse velocity dispersions are expressed in a simple form. We will consider only infinite-size models, so \( \varepsilon = 0 \) will be an energy of escape from a system.

3.1 DFs of the form \( \sum_{n=0}^{m} L^{2n} h_n(\varepsilon) \)

The integral on the right side of (2) is first expressed in velocity space. Let \( \mathbf{v} = (v_r, v_\phi, v_\theta) \) be the velocity in the spherical coordinates \((r, \phi, \theta)\) of space and \( v_T = \sqrt{v_r^2 + v_\phi^2} \) be the transverse velocity. Then \( L = rv_T \) and, by \( \varepsilon = \psi - (v_r^2 + v_\phi^2)/2 \), (2) can be expressed as
\[
\rho(r) = 4\pi \int_0^\psi \left[ \int_0^{\sqrt{2\varepsilon}} f(\psi - \varepsilon, rv_T) rv_T \sqrt{2\varepsilon - r^2 v_T^2} \, dv_T \right] d\varepsilon
\] (3)

for the spherical density \( \rho = \rho(r) \) since the system has only stars with \( \varepsilon > 0 \), or say, \( f(\varepsilon, L) = 0 \) for \( \varepsilon \leq 0 \). This means that the mass density \( \rho(r) \) can be regarded as a function depending on the radial coordinate \( r \) and the relative potential \( \psi \). Let \( \rho(r) \) be below denoted by \( \rho(\psi, r) \). Assume that \( f(\varepsilon, L) = \sum_{n=0}^{m} L^{2n} h_n(\varepsilon) \) for \( \varepsilon > 0 \) and \( f(\varepsilon, L) = 0 \) for \( \varepsilon \leq 0 \), and that the density \( \rho(\psi, r) \) has the form of \( \rho(\psi, r) = \sum_{n=0}^{m} \tilde{\beta}_n(\psi) r^{2n} \) corresponding to the relative potential \( \psi \). Notice that such representation of the density is not unique. Similar models were discussed earlier by Bouvier (1962, 1963). By (3), it follows that
\[
\sum_{n=0}^{m} \tilde{\beta}_n(\psi) r^{2n} = \sum_{n=0}^{m} \frac{(2\pi)^{3/2} 2^n n! r_a^{2n}}{\Gamma(n+3/2)} \int_0^\psi h_n(\varepsilon)(\psi - \varepsilon)^{n+1/2} d\varepsilon.
\] (4)

Equation (4) holds if \( h_n(\varepsilon) \) \((n = 0, 1, 2, \ldots, m)\) satisfy the conditions
\[
\tilde{\beta}_n(\psi) = (2\pi)^{3/2} \frac{2^n n!}{\Gamma(n+3/2)} \int_0^\psi h_n(\varepsilon)(\psi - \varepsilon)^{n+1/2} d\varepsilon.
\] (5)

By taking the \((n+1)\)st derivative of both sides of equation (5) with respect to \( \psi \) and using Abel’s integral equation, it is found that equation (5) can be solved by
\[
h_n(\varepsilon) = \frac{1}{\sqrt{8\pi^2} 2^n n!} \left[ \int_0^\psi \frac{d^{n+2} \tilde{\beta}_n(\psi)}{d\psi^{n+2}} \frac{d\psi}{\sqrt{\psi - \varepsilon}} + \frac{1}{\sqrt{\varepsilon}} \left( \frac{d^{n+1} \tilde{\beta}_n(\psi)}{d\psi^{n+1}} \right)_{\psi=0} \right]
\] (6)
for \( n = 0, 1, 2, \ldots, m \). Hence it can be easily found from (6) that

\[
f(\varepsilon, L) = \frac{1}{\sqrt{8\pi^2}} \sum_{n=0}^{m} L^{2n} \frac{L^{2n} \hat{\rho}_n(\Psi)}{d^{n+2} \psi^{n+2}} \int_0^\Psi \frac{d^n \hat{\rho}_n(\Psi)}{\sqrt{\varepsilon - \overline{\psi}}} + \frac{1}{\sqrt{\varepsilon}} \left( \frac{d^{n+1} \hat{\rho}_n(\Psi)}{d\psi^{n+1}} \right)_{\Psi=0}
\]

(7)

for \( \varepsilon > 0 \), which is a class of anisotropic DFs generated from the spherical density of the form \( \rho(\psi, r) = \sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n} \). Furthermore, if \( (d^n \hat{\rho}_n(\psi)/d\psi^j)_{\Psi=0} = 0 \) for \( j = 0, 1, \ldots, m + 1 \), then, for \( \varepsilon > 0 \), (7) can be expressed as

\[
f(\varepsilon, L) = \frac{1}{\sqrt{8\pi^2}} \sum_{n=0}^{m} L^{2n} \frac{L^{2n} \hat{\rho}_n(\Psi)}{d^{n+2} \psi^{n+2}} \int_0^\Psi \hat{\rho}_n(\Psi) d\psi
\]

(8)

By (7), the velocity dispersions \( \sigma_1^2(\psi, r) \) and \( \sigma_0^2(\psi, r) \) can be also found to be of the following forms

\[
\sigma_1^2(\psi, r) = \frac{1}{\rho(\psi, r)} \int_0^\Psi \rho(\psi', r) d\psi'
\]

(9)

and

\[
\sigma_0^2(\psi, r) = \frac{1}{\rho(\psi, r)} \int_0^\Psi (\rho(\psi', r)) d\psi'
\]

(10)

for any DF derived from the spherical density of the form \( \rho(\psi, r) = \sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n} \). It can be also known that these dispersions (9) and (10) can be obtained directly according to the following velocity dispersion formulae (Dejonghe 1986, 1987)

\[
\sigma_1^2(\psi, r) = \frac{1}{\rho(\psi, r)} \int_0^\Psi \rho(\psi', r) d\psi'
\]

(11)

and

\[
\sigma_0^2(\psi, r) = \frac{1}{\rho(\psi, r)} \int_0^\Psi \frac{\partial [r^2 \rho(\psi', r)]}{\partial r^2} d\psi'.
\]

(12)

3.2 DFs of the form \( \sum_{n=0}^{m} L^{2n} g_n(Q) \)

A more general expression for the integral in the right side of (2) can also be derived. To do this, first put \( Q = \psi - (v_\psi^2 + v_\phi^2)/2 - \frac{r^2}{r_a^2} \), where \( v_\psi \) and \( v_\phi \) are the same as in Sect. 3.1 and \( r_a \) is a scaling radius. Obviously, \( Q = \varepsilon - L^2/(2r_a^2) \) and \( Q \rightarrow \varepsilon \) as \( r_a \rightarrow \infty \). Assume that the DF is of the form \( f = f(Q, L) \), and that the system has only stars with \( Q > 0 \), or equivalently, \( f = 0 \) for \( Q \leq 0 \). Then it is easy to see that

\[
\rho(r) = 4\pi \int_0^\Psi \left[ \int_0^{\sqrt{2Q/(1+r^2/r_a^2)}} \frac{f(\psi - Q, r, \sqrt{Q}) v_T}{\sqrt{2Q - (1 + r^2/r_a^2)v_T^2}} dv_T \right] dQ.
\]

(13)

This shows that the density \( \rho(r) \) can be also regarded as a function of the radial coordinate \( r \) and the relative potential \( \psi \). Suppose that \( f(Q, L) = \sum_{n=0}^{m} L^{2n} g_n(Q) \) for \( Q > 0 \), and that the density \( \rho(r) \) can be expressed as

\[
\rho(\psi, r) = \sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n}/(1 + r^2/r_a^2)^{n+1}.
\]

Then similarly by (13) it follows that

\[
\sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n}/(1 + r^2/r_a^2)^{n+1} = \sum_{n=0}^{m} \frac{(2\pi)^{3/2} 2^{2n} n! r^{2n}}{\Gamma(n+3/2)(1 + r^2/r_a^2)^{n+1}} \int_0^\Psi g_n(Q)(\psi - Q)^{n+1/2} dQ.
\]

(14)
Equation (14) holds if \( g_n(Q) \) \((n = 0, 1, 2, \ldots, m)\) satisfy

\[
\hat{\rho}_n(\psi) = \frac{(2\pi)^{3/2}2^n n!}{\Gamma(n+3/2)} \int_0^\psi g_n(Q)(\psi - Q)^{n+1/2} dQ.
\]  \( (15) \)

By taking the \((n + 1)\)st derivative of both sides of equation (15) with respect to \( \psi \), and using Abel’s integral equation, the solution of equation (15) is given by

\[
g_n(Q) = \frac{1}{\sqrt{8\pi^2 2^n n!}} \left[ \int_0^Q \frac{d^{n+2} \hat{\rho}_n(\psi)}{d\psi^{n+2}} \frac{d\psi}{\sqrt{Q - \psi}} + \frac{1}{\sqrt{Q}} \left( \frac{d^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}} \right)_{\psi=0} \right]
\]  \( (16) \)

for \( n = 0, 1, 2, \ldots, m \). Then it can be easily shown from (16) that

\[
f(Q, L) = \frac{1}{\sqrt{8\pi^2}} \sum_{n=0}^m L^{2n} \frac{d^{m+2} }{dQ^{m+2}} \int_0^Q \frac{dQ_n}{\sqrt{Q - \psi}} \hat{\rho}_n(\psi) d\psi
\]  \( (17) \)

for \( Q > 0 \), which is another class of anisotropic DFs generated from the spherical density of the form of \( \rho(\psi, r) = \sum_{n=0}^m \hat{\rho}_n(\psi) r^{2n}/(1 + r^2/r_a^2)^{n+1} \). Furthermore, if \( (d^j \hat{\rho}_n(\psi)/d\psi^j)_{\psi=0} = 0 \) for \( j = 0, 1, 2, \ldots, m + 1 \), then for \( Q > 0 \), (17) can be rewritten as

\[
f(Q, L) = \frac{1}{\sqrt{8\pi^2}} \sum_{n=0}^m \frac{L^{2n} d^{m+2} }{dQ^{m+2}} \int_0^Q \hat{\rho}_n(\psi) d\psi
\]  \( (18) \)

Of course, (17) and (18) coincide with (8) and (8), respectively. In other words, (7) and (8) are, respectively, limits of (17) and (18) when \( r_a \to \infty \).

Similar to those in Sect. 8.1, the velocity dispersions \( \sigma_1^2(\psi, r) \) and \( \sigma_2^2(\psi, r) \) can be also found to be of the following forms

\[
\sigma_1^2(\psi, r) = \frac{1}{\rho(\psi, r)} \sum_{n=0}^m \frac{r^{2n}}{(1 + r^2/r_a^2)^{n+1}} \int_0^\psi \hat{\rho}_n(\psi') d\psi'
\]  \( (19) \)

and

\[
\sigma_2^2(\psi, r) = \frac{1}{\rho(\psi, r)} \sum_{n=0}^m \frac{(n+1) r^{2n}}{(1 + r^2/r_a^2)^{n+2}} \int_0^\psi \hat{\rho}_n(\psi') d\psi'
\]  \( (20) \)

for any DF derived from the spherical density of the form \( \rho(\psi, r) = \sum_{n=0}^m \hat{\rho}_n(\psi) r^{2n}/(1 + r^2/r_a^2)^{n+1} \).

It is worth mentioning that if the sum contains only one term then such models have been studied by Cuddeford (1991). Therefore the above DFs can be indeed regarded as a simple generalization of the models given by Cuddeford. However, (17) and (18) are new formulae for our finding anisotropic DFs.

3.3 Miscellaneous DFs

One can also obtain more general formulae than (7) and (17). It can be shown that

\[
f(\varepsilon, L) = \sum_{n=0}^m B_n L^{2n} \left[ \int_0^\varepsilon \frac{d^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}} \frac{d\psi}{(\varepsilon - \psi)^{a_n}} + \frac{1}{\varepsilon} \left( \frac{d^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}} \right)_{\psi=0} \right]
\]  \( (21) \)

for \( \varepsilon > 0 \) is an anisotropic DF for the spherical density of the form \( \rho(\psi, r) = \sum_{n=0}^m \hat{\rho}_n(\psi) r^{2n} \), with \( n \beta_n > -1 \).

Here \( B_n = [(2\pi)^{3/2}2^n \Gamma(n \beta_n + 1) \Gamma(1 - a_n)^{-1}]^{-1} \), \( a_n = n \beta_n - a_n + 3/2 \) and \( a_n \) is a non-negative integer such that \( 0 \leq a_n < 1 \) for \( n = 0, 1, \ldots, m \).
Similarly, it can be readily shown that a further anisotropic DF is given by

\[
 f(Q, L) = \sum_{n=0}^{m} B_n L^{2n\beta_n} \left[ \frac{\int_{0}^{Q} d\psi^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}(Q-\psi)^{\alpha_n}} + \frac{1}{\varpi^{2}} \frac{d\hat{\rho}_n(\psi)}{d\psi^{n+1}} \right]_{\psi=0}
\]

(22)

for \( Q > 0 \), corresponding to the spherical density \( \rho(\psi, r) = \sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n\beta_n}/(1 + r^2/r_0^2)^{n\beta_n+1} \), where \( Q \) is given in Sect. 3.2 and \( B_n, \alpha_n, \alpha_n \) and \( \beta_n \) are the same as in (21).

Put \( \hat{Q} = \max(Q, 0) \). Then, it can be furthermore shown that the DFs of the form

\[
 f(\epsilon, Q, L) = \sum_{n=0}^{m} B_1n L^{2n\beta_n} \left[ \frac{\int_{0}^{\epsilon} d\psi^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}(\epsilon-\psi)^{\alpha_n}} + \frac{1}{\varpi^{2}} \frac{d\hat{\rho}_n(\psi)}{d\psi^{n+1}} \right]_{\psi=0}
\]

\[
 + \sum_{n=0}^{m} B_2n L^{2n\beta_n} \left[ \frac{\int_{0}^{\hat{Q}} d\psi^{n+1} \hat{\rho}_n(\psi)}{d\psi^{n+1}(\hat{Q}-\psi)^{\alpha_n}} + \frac{1}{\varpi^{2}} \frac{d\hat{\rho}_n(\psi)}{d\psi^{n+1}} \right]_{\psi=0}
\]

(23)

correspond to a spherically symmetric density of the form

\[
 \rho(\psi, r) = \sum_{n=0}^{m} \hat{\rho}_n(\psi) r^{2n\beta_n}/(1 + r^2/r_0^2)^{n\beta_n+1}
\]

(24)

with \( n\beta_n > -1 \), where \( B_n = [(2\pi)^{3/2} 2^n \beta_n \Gamma(n\beta_n + 1)\Gamma(1-\alpha_n)]^{-1} \), \( \alpha_n = n\beta_n - \alpha_n + 3/2 \) and \( \alpha_n \) is a non-negative integer such that \( 0 < \alpha_n < 1 \) for \( i = 1, 2 \) and \( n = 0, 1, \ldots, m \).

Finally, the velocity dispersions \( \sigma_z^2(\psi, r) \) and \( \sigma_t^2(\psi, r) \) can be also obtained as

\[
 \sigma_z^2(\psi, r) = \frac{1}{\rho(\psi, r)} \sum_{n=0}^{m} \left[ r^{2n\beta_n} \int_{0}^{\psi} \hat{\rho}_n(\psi') d\psi' + \frac{r^{2n\beta_n}}{(1 + r^2/r_0^2)^{n\beta_n+1}} \int_{0}^{\psi} \hat{\rho}_n(\psi') d\psi' \right]
\]

(25)

and

\[
 \sigma_t^2(\psi, r) = \frac{1}{\rho(\psi, r)} \sum_{n=0}^{m} \left[ (n\beta_n + 1) r^{2n\beta_n} \int_{0}^{\psi} \hat{\rho}_n(\psi') d\psi' + \frac{(n\beta_n + 1) r^{2n\beta_n}}{(1 + r^2/r_0^2)^{n\beta_n+1}} \int_{0}^{\psi} \hat{\rho}_n(\psi') d\psi' \right]
\]

(26)

for any DF derived from the spherical density given by (24).

4 Application to spherical densities

Some anisotropic DFs for spherically symmetric densities are given by using formulae obtained in Sect. 3. Different anisotropic Plummer models are given in Sect. 4.1, together with the ratios of their radial and transverse velocity dispersions. Then anisotropic Hénon isochrone models appear in Sect. 4.2 and anisotropic \( \gamma \)-models in Sect. 4.3. For convenience of calculation, dimensionless quantities are used for all the spherical models considered, and it is assumed in this section that \( Q = \epsilon - L^2/2 \) (or say, take \( r_a = 1 \) in the previous section).

4.1 Anisotropic Plummer models

The well-known Plummer model is given by the dimensionless potential-density pair:

\[
 \psi(r) = \frac{1}{\sqrt{1 + r^2}},
\]

(27)

\[
 \rho(r) = \frac{3/(4\pi)}{1 + r^2}^{5/2}.
\]

(28)
First, DFs of the form given in Sect. 3.1 are considered for the Plummer model. Eq. (28) can be expressed as

\[
\rho(\psi, \sigma, r) = \frac{1}{\pi^2 (1 + r^2)^{m+1}}
\]

where \(m\) is a positive integer. Then, by (8), the anisotropic Plummer model is given by

\[
f(\varepsilon, L) = \frac{3 \Gamma(2m+6+\varepsilon)}{2(2\pi)^{3/2}} \left[ \frac{\varepsilon}{(2m+9/2-n)(n-m)!} \right]^{1/2} \left( \frac{L^2}{2\varepsilon} \right)^n
\]

for \(\varepsilon > 0\). It can be easily proved that equation (29) is in agreement with that given by Dejonghe (1986). Figure 1 shows that the contours given by (29) resemble those of the even DFs for some axisymmetric models and that the larger the parameter \(m\), the more anisotropic the model. Furthermore, by (9) and (10), the velocity dispersion ratio \(\sigma^2_\psi(\psi, r)/\sigma^2_T(\psi, r)\) for the anisotropic model given by (29) can be obtained as follows:

\[
\sigma^2_\psi(\psi, r)/\sigma^2_T(\psi, r) = (1 + r^2)/(1 + (m+1)r^2).
\]

This is a very good analytical proof of the anisotropic property shown by Figure 1.

Fig. 1 The contours of the DFs given by (29) with \(m = 3, 5, 7\). Here and below, dashed curves are contours and the solid curve is the boundary of the physical domain. Successive contour levels differ by factors of 0.08.

Eq. (28) can also be rewritten as

\[
\rho(\psi, r) = \frac{1}{\pi} \left[ \frac{\epsilon}{\psi^2(1 + c^2 + \psi^2)} \right] \text{d}\psi
\]

for \(\epsilon > 0\). Eq. (30) can also be calculated analytically in terms of generalized hypergeometric functions. However, it is not generally easy to calculate numerically these hypergeometric functions. To evaluate the DFs, it is necessary to estimate the integrals in (30). Figure 2 shows the contours of the two different anisotropic DFs. Notice that \(c\) is an anisotropic parameter ranging from 0 to 1. The larger the parameter \(c\), the more anisotropic the model. When \(c = 0\), it degenerates to be an isotropic model; when \(c = 1\), it becomes the same anisotropic model as given by (29) with the parameter \(m = 1\). The velocity dispersion ratio \(\sigma^2_\psi(\psi, r)/\sigma^2_T(\psi, r)\) for the anisotropic model defined by (30) can be further given to be of the simple form \(\sigma^2_\psi(\psi, r)/\sigma^2_T(\psi, r) = (1 + c^2)/(1 + 2c^2)\).
Inserting (27) into (33) gives

\[
\frac{\sigma_r^2(r)}{\sigma_T^2(r)} = \frac{(1 + r^2)^2}{(1 - c)r^2}\frac{(1 - c)}{[1 + \ln \left( \frac{r^2}{1 + r^2} \right) ]}.
\]
By using the anisotropy parameter defined by Binney (1980), it can be also found from (34) with \( c = 1 \) that the galaxy is isotropic at the centre and becomes increasingly radially anisotropic with radius and that the velocity distribution is arbitrarily close to one made entirely of radial orbits at sufficiently large \( r \) (Cuddeford 1991).

### 4.2 Anisotropic Hénon isochrone models

Hénon’s (1959a, b) isochrone model has the dimensionless potential-density pair

\[
\psi(r) = \frac{1}{1 + \sqrt{1 + r^2}},
\]

\[
\rho(r) = \frac{1}{4\pi} \frac{3(1 + \sqrt{1 + r^2}) + 2r^2}{(1 + \sqrt{1 + r^2})^3(\sqrt{1 + r^2})^3}.
\]

(35)  
(36)

Some anisotropic DFs for such model were found by Kuzmin and Veltmann (1967b, 1973). Now let us show the other anisotropic DFs for the isochrone model. Eq. (36) can be expressed as

\[
\rho(\psi, r) = \frac{\psi^5(3 + 2\psi r^2)}{4\pi(1 - \psi)^3},
\]

(37)

thus, by (8), giving the anisotropic DF as

\[
f(\epsilon, L) = \frac{3}{2\pi^2} \left[ \int_0^\epsilon \frac{\psi^3(10 - 5\psi + \psi^2)d\psi}{(1 - \psi)^3\sqrt{\epsilon - \psi}} + L^2 \int_0^\epsilon \frac{\psi^3(20 - 15\psi + 6\psi^2 - \psi^3)d\psi}{(1 - \psi)^6\sqrt{\epsilon - \psi}} \right],
\]

(38)

which can be expressed in terms of elementary functions (See Appendix A). Figure 4(a) shows the contours of the DF given by (38).

---

**Fig. 4** The contours of the DFs for Hénon’s model. (a), (b) and (c) are for (38), (40) and (42), respectively. In (a), successive contour levels differ by factors of 0.2. In (b) and (c), successive contour levels differ by factors of 0.5.
4.3 Anisotropic elementary functions (See Appendix A). Figure 4(c) illustrates the contours of the DF given by (42).

It follows from (23) that the anisotropic DF can be obtained as

\[ f(\xi, Q, L) = \frac{3}{2^{7/2} \pi^3} \left[ \int_0^\xi \frac{\psi(3 - 3\psi + \psi^2) d\psi}{(1 - \psi^3)^2 \sqrt{\psi - \psi}} + \frac{L^2}{Q} \int_0^\xi \frac{\psi^3 (20 - 15 \psi + 6 \psi^2 - \psi^3) d\psi}{(1 - \psi)^{5/2} \sqrt{\psi - \psi}} \right], \tag{40} \]

where \( Q = \max(Q, 0) \). The integral in (40) can be calculated analytically in terms of elementary functions (See Appendix A). Figure 4(b) displays the contours of the DF given by (40).

The density (36) can be also expressed as

\[ \rho(\psi, r) = \frac{1}{4\pi} \left[ \frac{3\psi^3}{1 - \psi} + \frac{2\psi^6 r^2}{(1 - \psi^3)^2} \right], \tag{41} \]

and, corresponding to it, by (23), the anisotropic DF is

\[ f(\xi, Q, L) = \frac{3}{2^{7/2} \pi^3} \left[ 3 \int_0^\xi \frac{\psi^4 (5 - 4\psi + \psi^2) d\psi}{(1 - \psi^3)^2 \sqrt{\psi - \psi}} + \frac{L^2}{Q} \int_0^\xi \frac{\psi^3 (20 - 15 \psi + 6 \psi^2 - \psi^3) d\psi}{(1 - \psi)^{5/2} \sqrt{\psi - \psi}} \right], \tag{42} \]

where \( Q = \max(Q, 0) \). The two integrals at the right side of equation (42) can also be expressed in terms of elementary functions (See Appendix A). Figure 4(c) illustrates the contours of the DF given by (42).

4.3 Anisotropic \( \gamma \)-models

The dimensionless potential-density pair of the \( \gamma \)-model (Kuzmin, Veltmann, Tenjes 1986; Dehnen 1993; Saha 1993; Tremaine et al. 1994) is

\[ \psi(r) = \begin{cases} [1 - r^{\gamma - 1}/(r + 1)^{\gamma - 1}]/(2 - \gamma), & \gamma \neq 2 \\ \ln[(r + 1)/r], & \gamma = 2 \end{cases} \tag{43} \]

\[ \rho(r) = \frac{3 - \gamma}{4\pi} \frac{1}{r^{\gamma - 1}/(r + 1)^{\gamma - 1}}. \tag{44} \]

In the same way as for the Plummer model, by (8), it can be shown that the anisotropic DFs of the form of Sect.3.1 can be obtained if the density defined by (44) is expressed as

\[ \rho(\psi, r) = \frac{3 - \gamma}{4\pi} \frac{(1 - r)^{4 + 2m} (1 + r^2)^m}{y^7 \left(1 - y^2 + y^2 r^2\right)^m}, \tag{45} \]

where \( m \) is a positive integer and \( y \) is written as

\[ y = \begin{cases} [1 - (2 - \gamma) \psi]^{(1/2 - \gamma)}, & \gamma \neq 2 \\ e^{-\psi}, & \gamma = 2 \end{cases}. \tag{46} \]

In fact, it is known from (43) that \( y = r/(r + 1) \). These anisotropic DFs cannot be expressed in terms of elementary functions or generalized hypergeometric functions although they can be calculated numerically.

Another expression of the density for the \( \gamma \)-model is

\[ \rho(\psi, r) = \frac{3 - \gamma}{4\pi} \frac{(1 - y)^{4 + 2m} (1 + r^2)^{2m}}{y^7}, \tag{47} \]
Fig. 5 The contours of the anisotropic Hernquist DFs (49). Successive contour levels differ by factors of 0.1.

where $y$ is given in (46). It can be shown from (21) that the anisotropic DFs corresponding to (47) can be expressed generally in terms of elementary functions or generalized hypergeometric functions. In particular, when $\gamma = 1$, it is a model obtained by Kuzmin and Veltmann (1973) and Hernquist (1990) and its density can be expressed as

$$\rho(\psi, r) = \frac{1}{2\pi} \frac{\psi^{4+2m}}{1 - \psi} (1 + r)^{2m}. \quad (48)$$

For example, by use of (21), a DF of the mass density given by (48) with $m = 1$ is

$$f(\varepsilon, L) = \frac{1}{(2\pi^2)^{3/2}} \left[ \int_{\varepsilon}^{e} \frac{h(\psi)d\psi}{\sqrt{\varepsilon - \psi}} + 2\sqrt{2Lh(\varepsilon)} + \frac{3L^2}{2} \int_{\varepsilon}^{e} \frac{\psi^3(20 - 45\psi + 36\psi^2 + 10\psi^3)d\psi}{(1 - \psi)^4 \sqrt{\varepsilon - \psi}} \right] \quad (49)$$

for $\varepsilon > 0$, where $h(\psi) = \psi^4(15 - 24\psi + 10\psi^2)/(1 - \psi)^3$. The two integrals in eq. (49) can be expressed in terms of elementary functions (See (57) and (58) in Appendix A). The contours of the DF given by (49) are plotted in Figure 5.

By (18), the anisotropic DFs of the form in Sect. 3.2 can also be obtained in terms of elementary functions or generalized hypergeometric functions in the same way as the DFs of Ossipkov-Merritt type are given by Dehnen (1993). The detail derivation of these expressions is omitted here.

By the way, Baes and Dejonghe (2005) recently used the spherical $\gamma$-models to investigate the dynamical structure of isotropic spherical galaxies with a central black hole. Such work is still very significant.

5 Conclusions

Although real galaxies are hardly spherically symmetric, it is a very necessary and significant step to study the self-consistent anisotropic DFs for stellar systems with the known spherically symmetric density. This is not only because some important properties (e.g. surface density and cumulative mass) of the spherical model are similar to those of the models that can be generated from the spherically symmetric density by replacing the spherical radius $r$ by an axisymmetric or triaxial radius $m = \sqrt{x^2 + (y/q_2)^2 + (z/q_3)^2}$, but also because both spherical and non-spherical models have the same typical behaviours of the dynamical quantities in the limits of small and large radii. Such models are elliposoidal. Other classes of flattened models can be constructed by using the equipotential method (e.g., Kutusov and Ossipkov 1980; Jiang 2000).

Anisotropic DFs can be obtained for stellar systems with known spherically symmetric density as a sum of products of functions only of the potential and a special function (or power) only of the radial coordinate, i.e. these DFs are a sum of products of functions only of a special variable (or the energy) and a power only of the magnitude of the angular momentum. This comes from a combination of the ideas of Eddington and Fricke. It is an extension of Eddington’s classical solution for the isotropic DF of a known spherical density. Like his method, it requires that the density be expressed as a special function of the potential, and now also of the radial coordinate. Also, part of it is a sum of real integrals of functions only of the potential. Most of these integrals...
can be calculated analytically in terms of elementary functions or generalized hypergeometric functions (e.g. Kuzmin and Veltmann 1967b).

These formulae concerning anisotropic DFs are obtained by use of the Abel integral equation, and they are suitable for all such densities that allow the real integrals of the potential to be valid. The anisotropic DFs of Ossipkov-Merritt type are their examples in special cases. After the application to Plummer’s spheroidal model, it is found that the anisotropic DFs given by (29) are in fact the same as those given by Dejonghe (1986). Three different anisotropic DFs are given for Hénon’s (1959a, b) isochrone model and they can be expressed in terms of elementary functions. Many anisotropic DFs of $\gamma$-models, expressed in elementary functions or generalized hypergeometric functions, can also be obtained. One can further give formulae of the velocity dispersions for these anisotropic DFs.

These expressions can be extended further to axisymmetric systems. It is straightforward to find their analogues for axisymmetric systems and they can be used to obtain the even DF of Binney’s (BT) logarithmic potential although Evans (1993) derived it using Lynden-Bell’s (1962) method. For the well-known Lynden-Bell formulae for axisymmetric systems and they can be used to obtain the even DF of Binney’s (BT) logarithmic potential although Evans (1993) derived it using Lynden-Bell’s (1962) method. For the well-known Lynden-Bell (1962) model, these analogues degenerate into the method of Fricke (1952).

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Appendix A: Explicit expressions of integrals for the anisotropic DFs of both Hénon’s model and Hernquist’s model

Explicit expressions of integrals for the DFs of the Hénon model are given below. Using the identity

$$\frac{\psi^3(10 - 5\psi + \psi^2)}{(1 - \psi)^5} = \frac{6}{(1 - \psi)^3} \frac{15}{(1 - \psi)^4} + \frac{10}{(1 - \psi)^3} - 1,$$

the expression

$$\int_0^\epsilon \frac{\psi^3(10 - 5\psi + \psi^2)}{(1 - \psi)^3 \sqrt{\epsilon - \psi}} d\psi = \frac{(19 - 106\epsilon + 240\epsilon^2 - 48\epsilon^3)\sqrt{\epsilon}}{32(1 - \epsilon)^3}$$

$$+ \frac{15(3 - 12\epsilon + 16\epsilon^2)\arcsin(\sqrt{\epsilon})}{32(1 - \epsilon)^{5/2}} - 2\sqrt{\epsilon}$$

can be derived. Similarly, the identity

$$\frac{\psi^3(20 - 15\psi + 6\psi^2 - \psi^3)}{(1 - \psi)^6} = \frac{10}{(1 - \psi)^5} - \frac{24}{(1 - \psi)^3} + \frac{15}{(1 - \psi)^4} - 1,$$

gives

$$\int_0^\epsilon \frac{\psi^3(20 - 15\psi + 6\psi^2 - \psi^3)}{(1 - \psi)^6 \sqrt{\epsilon - \psi}} d\psi = \frac{(53 - 330\epsilon + 880\epsilon^2 - 352\epsilon^3 + 64\epsilon^4)\sqrt{\epsilon}}{64(1 - \epsilon)^3}$$

$$+ \frac{15(5 - 24\epsilon + 40\epsilon^2)\arcsin(\sqrt{\epsilon})}{64(1 - \epsilon)^{11/2}} - 2\sqrt{\epsilon}.$$

Note that $\frac{\psi^3(3 - 3\psi + \psi^2)}{(1 - \psi)^3} = \frac{1}{(1 - \psi)^3} - 1$. Then

$$\int_0^Q \frac{\psi(3 - 3\psi + \psi^2)}{(1 - \psi)^3 \sqrt{Q - \psi}} d\psi = \frac{(5 - 2Q)\sqrt{Q}}{4(1 - Q)^3} + \frac{3\arcsin(\sqrt{Q})}{4(1 - Q)^{5/2}} - 2\sqrt{Q}.$$

Using the identity

$$\frac{\psi^4(5 - 4\psi + \psi^2)}{(1 - \psi)^5} = \frac{2}{(1 - \psi)^3} - \frac{6}{(1 - \psi)^4} + \frac{5}{(1 - \psi)^3} - 2 + (1 - \psi),$$

$$\int_0^Q \frac{\psi^4(5 - 4\psi + \psi^2)}{(1 - \psi)^5 \sqrt{Q - \psi}} d\psi = \frac{(5 - 2Q)\sqrt{Q}}{4(1 - Q)^3} + \frac{3\arcsin(\sqrt{Q})}{4(1 - Q)^{5/2}} - 2\sqrt{Q}.$$
the result
\[
\int_0^\epsilon \psi'(5-4\psi+\psi^2)d\psi = \frac{(87-350\epsilon+464\epsilon^2-96\epsilon^3)}{96(1-\epsilon)^4}
\]
\[
+ \frac{5(7-24\epsilon+24\epsilon^2)}{32(1-\epsilon)^{9/2}} \arcsin(\sqrt{\epsilon}) - \frac{2\sqrt{\epsilon}(3+2\epsilon)}{3}
\]
(56)
is obtained.

Similarly, integrals for the anisotropic DFs of the Hernquist model can be also expressed as
\[
\int_0^\epsilon \psi'(15-24\psi+10\psi^2)d\psi = \frac{2}{35}\sqrt{\epsilon}(35+70\epsilon+112\epsilon^2+160\epsilon^3)
\]
\[
+ \frac{(5-2\epsilon)\sqrt{\epsilon}}{4(1-\epsilon)^2} + \frac{3\arcsin(\sqrt{\epsilon})}{4(1-\epsilon)^{5/2}}
\]
(57)
and
\[
\int_0^\epsilon \psi'(20-45\psi+36\psi^2-10\psi^3)d\psi = \frac{2}{3}\sqrt{\epsilon}(3+8\epsilon+116\epsilon^2)
\]
\[
+ \frac{(33-26\epsilon+8\epsilon^2)\sqrt{\epsilon}}{24(1-\epsilon)^3} + \frac{15\arcsin(\sqrt{\epsilon})}{24(1-\epsilon)^{7/2}}.
\]
(58)

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