AN EXPLICIT FORMULA FOR THE DETERMINANT OF THE ABELIAN INTEGRAL MATRIX

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Abstract. We consider a polynomial $h(x, y)$ in two complex variables of degree $n + 1 \geq 2$ with a generic higher homogeneous part. The rank of the first homology group of its nonsingular level curve $h(x, y) = t$ is $n^2$. To each 1-form in the variable plane and a generator of the homology group one associates the (Abelian) integral of the form along the generator. The Abelian integral is a multivalued function in $t$. For a fixed canonic tuple of $n^2$ monomial 1-forms we consider the multivalued $n^2 \times n^2$ matrix function in $t$ whose elements are the Abelian integrals of the forms along the generators. Its determinant does not depend on the choice of the generators in the homology group (up to change of sign, which corresponds to change of generator system that reverses orientation).

In 1999 Yu.S.Ilyashenko proved [1] that the determinant of the Abelian integral matrix is a polynomial in $t$ of degree $n^2$ whose zeroes are the critical values of $h$. We give an explicit formula for the determinant.

1. Introduction. Statement of result

The infinitesimal Hilbert 16-th problem is the following: what may be said about the number and the location of limit cycles of a two-dimensional polynomial vector field close to a Hamiltonian one?

Yu.S.Ilyashenko [1] have partially investigated the following question closely related to the infinitesimal Hilbert problem: to find an explicit form of the Picard-Fuchs equation for Abelian integrals.

In the present paper for a hamiltonian of degree $n + 1 \geq 2$ and $n^2$ canonic monomial 1-forms (2) we calculate explicitly the determinant of the corresponding Abelian integral matrix (3) (Theorem 1).

To state the main result, let us introduce some notations.

Let $(x, y)$ be coordinates in $\mathbb{C}^2$. Let $n \in \mathbb{N}$, $n \geq 1$, $h(x, y)$ be a polynomial of the degree $n + 1$,

$$H(x, y) = \sum_{s=0}^{n+1} h_s x^{n+1-s} y^s$$

be its homogeneous part of the degree $n + 1$. 

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Everywhere below we suppose that the homogeneous polynomial $H$ is *generic*: this means that it is not divisible by square of linear function, i.e., has $n + 1$ distinct zero lines. Let $a_1, \ldots, a_{n^2}$ be the critical values of the polynomial $h$, $t \in \mathbb{C} \setminus \{a_1, \ldots, a_{n^2}\}$. Then the first integer homology group of the nonsingular level curve $h(x, y) = t$ has dimension $n^2$.

Let $\{1, \ldots, n^2\} \rightarrow \{(l, m) \mid 0 \leq m, l \leq n - 1\}$ be the lexicographic numeration of the integer pairs $(l, m)$: $(0, 0), (0, 1), (0, 2)$... Define the following polynomials and differential 1-forms in $\mathbb{C}^2$:

\begin{equation}
(2) \quad e_j(x, y) = x^{l(j)} y^{m(j)}; \quad d(j) = \deg e_j = l(j) + m(j); \quad \omega_j = ye_j(x, y)dx.
\end{equation}

Let $t \neq a_i, \alpha = \{\alpha_r\}, r = 1, \ldots, n^2$, be a system of generators of the group $H_1(\{h(x, y) = t\}, \mathbb{Z})$. Define the Abelian integrals

\begin{equation}
(3) \quad I_{j, r}(t) = \int_{\alpha_r} \omega_j, \quad j, r = 1, \ldots, n^2.
\end{equation}

The system $\alpha$ is chosen to depend continuously on $t \neq a_i$; it is "multivalued": $\alpha$ transforms to another system of generators after going around a critical value. The new system is obtained from the initial one by unimodular linear transformation. The Abelian integrals (3) are multivalued holomorphic functions in $t$ with branching points $a_i$. They form an $n^2 \times n^2$ matrix with the indices $j, r$.

We calculate the determinant $\det(I_{j, r})(t)$.

**Remark 1.** The function $\det(I_{j, r})(t)$ is holomorphic in $t$ and *single-valued*. It does not depend (up to multiplication by -1) on the choice of the system $\alpha$ of generators: if two generator systems $\alpha$ and $\alpha'$ define the same orientation of the homology space, then the corresponding determinants are equal; otherwise they are opposite.

Earlier Yu.S.Ilyashenko proved [1] that the function $\det(I_{j, r})$ depends only on $H$:

\begin{equation}
(4) \quad \det(I_{j, r})(t) = C(H) \prod_{i=1}^{n^2} (t - a_i); \quad a_i \text{ are the critical values of } h,
\end{equation}

where $C(H) \neq 0$ (provided that $H$ is generic). Below we calculate the constant $C(H)$ from (4) as a function in $H$.

**Remark 2.** The nongeneric polynomials form an irreducible hypersurface in the space of all the homogeneous polynomials (1) (denote this hypersurface by $S$). For a generic $H$ the corresponding value $C(H)$ is well-defined up to sign. As it is shown below (Theorem 1), $C(H)$ is a double-valued function in $H$ with branching at $S$.

By $\Sigma(H)$ denote the discriminant of $H$, i.e., the polynomial in the coefficients of $H$ whose zero set is the hypersurface $S$ with the multiplicity 1.

**Remark 3.** Consider the decomposition

\begin{equation}
H(x, y) = h_0 \prod_{i=0}^{n}(x - b_iy)
\end{equation}

of $H$ into product of linear factors. The discriminant $\Sigma(H)$ is a degree $2n$ homogeneous polynomial in the coefficients (1) of $H$. It is equal to

\begin{equation}
(5) \quad \Sigma(H) = h_0^{2n} \prod_{0 \leq j < i \leq n} (b_i - b_j)^2.
\end{equation}
Let $k \in \mathbb{N}, k \leq n - 1$. Let us introduce the following four upper (lower) triangular $k \times k$-matrix linear functions in $H$:

$$A_{n,k} = \begin{pmatrix}
(n+1)h_0 & nh_1 & \ldots & (n-k+2)h_{k-1} \\
0 & (n+1)h_0 & \ldots & (n-k+3)h_{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & (n+1)h_0
\end{pmatrix},$$

$$B_{n,k} = \begin{pmatrix}
h_n & 0 & \ldots & 0 \\
2h_{n-1} & h_n & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
kh_{n-k+1} & (k-1)h_{n-k+2} & \ldots & h_n
\end{pmatrix},$$

$$C_{n,k} = \begin{pmatrix}
h_1 & 2h_2 & \ldots & kh_k \\
0 & h_1 & \ldots & (k-1)h_{k-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & h_1
\end{pmatrix},$$

$$D_{n,k} = \begin{pmatrix}
(n+1)h_{n+1} & 0 & \ldots & 0 \\
nh_n & (n+1)h_{n+1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(n-k+2)h_{n-k+2} & (n-k+3)h_{n-k+3} & \ldots & (n+1)h_{n+1}
\end{pmatrix}.$$  

Define the following $2k \times 2k$ matrix function:

$$E_{n,k} = \begin{pmatrix}
A_{n,k} & B_{n,k} \\
C_{n,k} & D_{n,k}
\end{pmatrix}.  \tag{6}$$

**Remark 4.** Let $n \in \mathbb{N}, n \geq 2, H, e_j, d(j), \omega_j$ be as at the beginning of the paper. Let $k \in \mathbb{N}, 1 \leq k \leq n - 1, E_{n,k}(H)$ be the matrix function from (6). Then $E_{n,k}(H)$ is degenerate, iff there exists a nonzero linear combination $\sum_{d(j)=n+k-1} e_j e_j$ that belongs to the gradient ideal of $H$ in the polynomial algebra (i.e., is equal to a linear combination of the partial derivatives of $H$ with the coefficients of homogeneous degree $k-1$ polynomials). Indeed, for any $l \leq k$ let us assign the monomial $x^{n+k-l}y^{l-1}$ ($x^{k-l}y^{n+l-1}$) to the $l$-th (respectively, $k+l$-th) column of $E_{n,k}$. Then by definition, for any $j \leq k$ the $j$-th ($k+j$-th) line of the matrix $E_{n,k}$ consists of the coefficients of the degree $n+k-1$ polynomial $x^{k-j}y^{j-1}\frac{\partial H}{\partial x}$ (respectively, $x^{k-j}y^{j-1}\frac{\partial H}{\partial y}$) at the monomials different from $e_i$. The last $2k$ polynomials are linearly independent and form a basis in the space of polynomials of the degree $n+k-1$ that belong to the gradient ideal of $H$. This follows from genericity of $H$. Therefore, $E_{n,k}(H)$ is nondegenerate, iff each linear combination of these $2k$ polynomials contains a monomial different from $e_i$ (or equivalently, no nontrivial linear combination of the monomials $e_i$ with $d(i) = n+k-1$ belongs to the gradient ideal of $H$).

**Theorem 1.** Let $h, H$ be as at the beginning of the paper, $I_{j,r}$ be the corresponding Abelian integrals (3), $C(H)$ be the corresponding function from (4), $E_{n,k}(H)$ be the matrix functions from (6). Then

$$C(H) = c_n(\Sigma(H))^{\frac{1}{2}} n^{-\frac{3}{2}} \prod_{k=1}^{n-1} \det E_{n,k}(H),  \tag{7}$$

$$c_n = (-1)^{\frac{n(3n-1)}{4}} \left( \frac{2\pi}{n} \right)^{\frac{n(n+1)}{2}} \frac{n^2+\frac{1}{4} n - \frac{1}{2} ((n+1)!)^n}{\prod_{m=1}^{n-1} (m + n + 1)!}.  \tag{8}$$

Theorem 1 is proved in Section 2.
Example 1. Let us check the statement of Theorem 1 in the simplest case, when \( n = 1, h(x, y) = H(x, y) = x^2 + y^2 \). Then \( C(H) = \pi \): by definition, the value \( C(H) \) is equal to the integral of the form \( ydx \) along the unit circle in the real \((x, y)\) plane; this integral is equal to the area of the unit disc. Now let us calculate \( C(H) \) by using (7) and (8). One has

\[
(9) \quad C(H) = c_1 (\Sigma(H))^{-\frac{n^2}{2}} \text{ by (7)},
\]

\[
\Sigma(H) = -4, \quad c_1 = 2\pi i \text{ by (5) and (8) respectively}.
\]

Substituting \((\Sigma(H))^{-\frac{n^2}{2}} = 2i\) and the value of \(c_1\) to (9) yields \( C(H) = \pi \), as above.

2. Calculation of \( C(H) \). Proof of Theorem 1

2.1. Scheme of the proof of Theorem 1.

Without loss of generality, everywhere below we suppose that \( h \equiv H \), so \( a_j = 0 \), \( C(H) = \det(I_{j,r})|_{t=1} \). Let us calculate the function \( C(H) \).

Firstly we show that \( C(H) \) has the type (7) with \( c_n \) independent on \( H \):

**Lemma 1.** For any \( n \in \mathbb{N}, n \geq 1 \), there exists a nonzero constant \( c_n \) satisfying (7) for any generic \( H \).

Lemma 1 is proved in Subsections 2.2-2.4.

Then in Subsection 2.5 we calculate the constant \( c_n \).

2.2. Homogeneity degrees and orders of zeroes. Proof of Lemma 1.

As it is shown below, Lemma 1 is implied by the two following statements.

**Proposition 1.** The function \( C(H) \) is homogeneous of degree \(-n^2\).

Proposition 1 is proved in Subsection 2.3.

**Lemma 2.** Let \( E_{n,k}(H) \) be the matrix functions from (6). The function \( C(H) \) is divisible by each polynomial \( \det E_{n,k} \). The ratio \( \frac{C(H)}{\prod_{k=1}^{n-1} \det E_{n,k}(H)} \) is a multi-valued holomorphic function with at most double branching at the hypersurface \( S = \{ \Sigma(H) = 0 \} \). No its branch vanishes outside \( S \).

Lemma 2 is proves in Subsection 2.4.

Let us prove the implication of Lemma 1 from Proposition 1 and Lemma 2. The function \( C(H) \) has at most polynomial growth, as \( H \) approaches \( S \) (by definition and a theorem of P.Deligne [2]). Recall that the surface \( S \) is irreducible. Therefore, the product of \( C(H) \) and appropriate power (that will be referred to, as \(-s\)) of the polynomial \( \Sigma(H) \) is a single-valued holomorphic function that does not vanish identically in \( S \) (by the singularity reducibility theorem). Hence, \( C(H) = c_n \prod_{k=1}^{n-1} \det E_{n,k}(H)(\Sigma(H))^s \) with \( c_n \) independent on \( H \). The degree of each \( \det E_{n,k} \) is equal to \( 2k \). This together with the previous statements on the homogeneity degrees of \( \Sigma(H) \) and \( C(H) \) (Remark 3 and Proposition 1) implies that \( s = \frac{1}{2} - n \). This proves Lemma 1 modulo Proposition 1 and Lemma 2.

2.3. Proof of Proposition 1. Let \( b \in \mathbb{C} \setminus 0 \). Let us compare \( C(H) \) and \( C(bH) \). By definition, for any \( t \in \mathbb{C} \) the value at \( t \) of the function \( \det(I_{j,r}) \) corresponding to a polynomial \( h = H \) is equal to the value at \( bt \) of that corresponding to \( bh \), i.e., \( \det(I_{j,r})(t) = C(H)t^{n^2} = C(bH)(bt)^{n^2} \). Therefore, \( C(bH) = b^{-n^2} C(H) \). This proves Proposition 1.
2.4. Proof of Lemma 2.

For the proof of Lemma 2 we consider \( \det(I_{j,r}) = (\det(I_{j,r}))_{t=1} \) as a function in variable polynomials \( e_j \) of degrees \( d(j) \geq n \): for each one of these \( j \) let us choose arbitrary homogeneous polynomials \( p_j(x,y) \) of degree \( d(j) \) and put

\[
\omega_j = p_j(x,y)y dx.
\]

(10)

For the other \( j \) define \( \omega_j \) to be the forms from (2). Let \( H \) be a generic homogeneous polynomial of degree \( n+1 \) (in the same sense, as at the beginning of the paper). Put \( p = (p_j| d(j) \geq n) \). Consider the value at \( t = 1 \) of the matrix function (3) corresponding to the new \( \omega_j \). We consider its determinant

\[
\det(I_{j,r})_{t=1} = C(H, p)
\]

(11)

as a function in the coefficients of the polynomials \( H \) and \( p_j \). By definition, \( C(H, (e_j| d(j) \geq n)) = C(H) \).

For the proof of Lemma 2 we construct an extension \( Pr(H, p) \) of the product

\[
\prod_{k=1}^{n-1} \det E_{n,k}(H)
\]

as a function in the coefficients of \( H \) and \( p_j \) with the following properties:

1) \( C(H, p) \) vanishes iff so does \( Pr(H, p) \);
2) for any fixed generic \( H \) the gradient of each one of the functions \( C(H, p) \), \( Pr(H, p) \) in the coefficients of all the polynomials \( p_j \) does not vanish in a Zariski open dense subset of the zero set \( Pr = 0 \).

This will imply that for any fixed generic \( H \) the ratio \( \frac{C}{Pr} \) is holomorphic in \( p \) and does not vanish (in particular, its value \( C(H) \) corresponding to \( p_j = e_j \) is finite and nonzero). This will prove Lemma 2.

In the proof of Lemma 2 we use the following

**Proposition 2.** The value \( C(H, p) \) of the function from (11) vanishes iff there exist a \( k, 1 \leq k \leq n-1 \), and a linear combination

\[
\sum_{d(j)=n+k-1} c_j \frac{\partial(y p_j)}{\partial y}
\]

(12)

with at least one \( c_j \neq 0 \)

that belongs to the gradient ideal of \( H \).

Proposition 2 was proved by Yu.S.Ilyashenko [1] in a particular case, and the proof remains valid in the general case.

**Remark 5.** For any \( n \geq 2, 1 \leq k \leq n-1 \), the number of the indices \( j \) with \( d(j) = n+k-1 \) is equal to \( n-k \). Indeed, it follows from (2) that the number of the monomials \( e_j \) of degree \( n+k-1 \) is equal to \( n-k \).

Let us construct the function \( Pr \). Firstly we define appropriate extension of each factor \( \det E_{n,k} \), \( 1 \leq k \leq n-1 \), that vanishes exactly iff a linear combination (12) belongs to the gradient ideal of \( H \). To do this, let us introduce the following

**Definition 1.** Let \( n \geq 1, H \) be a generic polynomial of degree \( n+1 \) (in the same sense, as at the beginning of the paper). Let \( 1 \leq k \leq n-1 \). Let \( q_j, j = 1, \ldots, n-k \), be \( n-k \) homogeneous polynomials of the degree \( n+k-1 \). For each \( j = 1, \ldots, n+k \) define the polynomial \( Y_j \) of the degree \( n+k-1 \) as follows: for \( j \leq n-k \) put \( Y_j = \partial (y q_j) \partial y \); for \( n-k+1 \leq j \leq n \) put \( Y_j = x^{n-j} y^{j-n+k-1} \frac{\partial H}{\partial x} \); for
n + 1 \leq j \leq n + k \) put \( Y_j = x^{k-j+n} y^{j-n-1} \frac{\partial H}{\partial y} \). Define \( A(k, H, q_1, \ldots, q_{n-k}) \) to be the \((n + k) \times (n + k)\)-matrix with columns numerated by the indices \( s, 1 \leq s \leq n + k \) (being assigned with the monomials \( x^{n+k-s} y^{s-1} \)) whose \( j \)-th line consists of the coefficients of the polynomial \( Y_j \).

**Remark 6.** The matrix function \( A \) from the previous Definition vanishes, iff a nonzero linear combination (12) belongs to the gradient ideal of \( H \).

Let \( 1 \leq k \leq n - 1 \), \( A \) be the matrix function from the previous Definition. The extension of the function \( \det E_{n,k} \) we are looking for is \( \det A(k, H, p_j | d(j) = n + k - 1) \). Its value at \( p_j = e_j \) is equal to \( \det E_{n,k}(H) \) up to multiplication by constant independent on \( H \) (see the next Remark).

The product

\[ \Pr(H, p) = \prod_{k=1}^{n-1} \det A(k, H, (p_j | d(j) = n + k - 1)) \]

is the function \( \Pr \) we are looking for: as it is shown below, it satisfies the statements 1) and 2) from the beginning of the Subsection.

**Remark 7.** Let \( e_j, 1 \leq j \leq n^2 \), be the monomials from (2), \( 1 \leq k \leq n - 1 \). The number of the indices \( j \) with \( d(j) = n + k - 1 \) is equal to \( n - k \): denote them \( j_1 < \cdots < j_{n-k} \). Let \( q_s = e_{j_s}, A = A(k, H, q_1, \ldots, q_{n-k}) \) be the corresponding matrix function from the previous Definition. Let \( E_{n,k} \) be the matrix function from (6). Then \( E_{n,k} \) coincides with the unique \( 2k \times 2k \)-minor of the matrix \( A \) such that the complementary minor has nonidentically-vanishing determinant, and the latter is equal to a nonzero constant that does not depend on \( H \). In particular, the value of the function \( \det A(k, H, q_1, \ldots, q_{n-k}) \) at \( q_s = e_{j_s} \) is equal to \( \det E_{n,k} \) up to multiplication by nonzero constant independent on \( H \).

Let \( C(H, p) \) be the function from (11). The value of the function \( \Pr \) at \( p_j = e_j \) is equal to \( \prod_{k=1}^{n-1} \det E_{n,k}(H) \) (up to multiplication by constant) by definition and the previous Remark. The function \( \Pr(H, p) \) vanishes iff so does \( C(H, p) \). Indeed, \( \Pr(H, p) = 0 \) iff there exists a \( k, 1 \leq k \leq n - 1 \), such that a nonzero linear combination (12) belongs to the gradient ideal of \( H \) (by definition and Remark 6). By Proposition 2, this is the case iff \( C(H, p) \) vanishes. This proves the statement 1) from the beginning of the Subsection. Now for the proof of Lemma 2 it suffices to prove the statement 2) from the same place.

Let us prove that the gradient of the function \( C \) does not vanish at a generic point of its zero set. A generic point \( (H, p = (p_j)_{d(j) \geq n}) \) of its zero set possesses the following property:

\[ (13) \quad H \text{ is generic; there exist a unique } k, 1 \leq k \leq n - 1, \text{ and a unique nontrivial linear combination (12) (up to multiplication by constant) such that the last combination belongs to the gradient ideal of } H. \]

The set of points satisfying (13) is Zariski open and dense in the zero set of \( C \). This follows from Proposition 2 and the statement that for each \( k = 1, \ldots, n - 1 \) the codimension of the gradient ideal of \( H \) in the space of homogeneous polynomials of the degree \( n+k-1 \) is equal to \( n-k \); then for a generic point of the set \( C(H, p) = 0 \) the corresponding system of \( n-k \) polynomials \( \frac{\partial(yp_j)}{\partial y} \) has rank \( n-k-1 \) modulo the gradient ideal, which means exactly that (13) holds. Indeed the intersection of the gradient ideal with the space of degree \( n+k-1 \) homogeneous polynomials
is equal to $2k$: by definition, it is generated by the $2k$ polynomials $x^i y^{k-1-i} \frac{\partial H}{\partial x}$, $x^i y^{k-1-i} \frac{\partial H}{\partial y}$, which are linearly independent (cf. Remark 4).

Let us fix a generic $H$ and show that grad $C$ (in the variables $p$) does not vanish at points satisfying (13). Fix a point satisfying (13). By definition, there is an index $l$ such that the corresponding coefficient $c_l$ from the linear combination (12) is nonzero. Let us fix such an $l$ and consider that $c_l = 1$ without loss of generality. Then the gradient of the function $C$ along the space of polynomials $p_l$ (with fixed $p_j$ corresponding to $j \neq l$) is nonzero. Indeed, let $q_l$ be a homogeneous polynomial of the degree $n+k-1$. The derivative of the function $C$ in $p_l$ in the direction $q_l$ is equal to its value $C(H, (p_j)_{d(j) \geq n, j \neq l}, q_l)$ at $H$ and the polynomials $p_j$ with $j \neq l$ and $q_l$. This value is nonzero for a generic $q_l$. This follows from Proposition 2 and the statement that for a generic $q_l$ there is no linear combination (12) (where $p_l$ is changed to $q_l$) that belongs to the gradient ideal of $H$. Indeed by definition, the codimension of the space of polynomials belonging to the gradient ideal in the space of homogeneous polynomials of the degree $n+k-1$ is equal to the number $n-k$ of the indices $j$ with $d(j) = n+k-1$. By (13) and the assumption that $c_l = 1$, no linear combination (12) with $c_l = 0$ belongs to the gradient ideal. The linear mapping $q \mapsto \frac{\partial (yq)}{\partial y}$ of the space of polynomials of a fixed positive degree is an isomorphism. The two last statements imply that for a generic $q_l$ the polynomials $\frac{\partial (yp_j)}{\partial y}$ with $j \neq l$ and $\frac{\partial (yq_l)}{\partial y}$ are linearly independent modulo the gradient ideal. This proves the statement 2) for the function $C$. The proof of the analogous statement on the function $P_r$ repeats the previous one. Lemma 2 is proved.

### 2.5. Calculation of $c_n$.

Everywhere below we suppose that $H(x, y) = x^{n+1} + y^{n+1}$. To find the constant $c_n$, we calculate the values $C(H)$, $\Sigma = \Sigma(H)$ and $\det E_{n,k}(H)$ at the above polynomial $H$. The constant $c_n$ is expressed via them by (7).

To sketch the calculations, let us introduce the following notations. Put

$$\varepsilon = e^{\frac{2\pi i}{n+1}}, \quad \sigma = \prod_{1 \leq l < k \leq n+1} (\varepsilon^k - \varepsilon^l)^2. \tag{14}$$

For any $j = 1, \ldots, n^2$ put

$$I_j = \int_0^1 x^{l(j)} (1-x^{n+1})^{\frac{m(j)+1}{n+1}} dx, \tag{15}$$

$$IP = \prod_{j=1}^{n^2} I_j. \tag{16}$$

In Subsection 2.5.1 we express $C(H)$ via $\sigma$ and $IP$: we show that

$$C(H) = \sigma^n IP. \tag{17}$$

In Subsection 2.5.2 we calculate $\sigma$ and $\Sigma$: we show that

$$\sigma = (-1)^{\frac{n(n+1)}{2}} (n+1)^{n+1}, \quad \Sigma = (-1)^n \sigma = (-1)^{\frac{n(n+1)}{2}} (n+1)^n. \tag{18}$$
In Subsection 2.5.3 we calculate $IP$: we show that

$$IP = \frac{(2\pi)^{n(n+1)/2} (n+1)^{-\frac{n^2+4n+3}{2}} ((n+1)!)^n}{\prod_{m=1}^{n-1} (m+n+1)!}.$$ \hfill (19)

It follows from definition that

$$\prod_{k=1}^{n-1} \det E_{n,k}(H) = (n+1)^{n(n-1)}; \hfill (20)$$

the matrices $E_{n,k}$ corresponding to the polynomial $H$ under consideration are diagonal with the diagonal elements equal to $n+1$; so, $\det E_{n,k} = (n+1)^{2k}$, which implies (20).

Now statement (8) of Theorem 1 follows from (17)-(20) and formula (7) proved before. This proves Theorem 1.

2.5.1. Calculation of $C(H)$. Proof of (17).

By definition, $C(H)$ is equal to the value of the determinant $\det(I_{j,r})$ at $t = 1$. Let us calculate the latter.

Let $F = \{H(x,y) = 1\}$. The fiber $F$ admits the action of the group $\mathbb{Z}_{n+1} \oplus \mathbb{Z}_{n+1} = \{(l,m)| l,m = 0,\ldots,n\}$ by multiplication by $\varepsilon^l$ and $\varepsilon^m$ of the coordinates $x$ and $y$ respectively.

We calculate the value $\det(I_{j,r})|_{t=1}$ for appropriate base $\alpha_1,\ldots,\alpha_{n^2}$ in $H_1(F,\mathbb{Z})$ (defined below) such that each $\alpha_j$ with $j > 1$ is obtained from $\alpha_1$ by the action of the element $(l(j),m(j)) \in \mathbb{Z}_{n+1} \oplus \mathbb{Z}_{n+1}$. (This basis is completely defined by choice of $\alpha_1$.)

To define $\alpha_1$, we consider the fiber $F$ as a covering over the $x$-axis having the branching points with the $x$-coordinates $\varepsilon^j$, $j = 0,\ldots,n$. It is the Riemann surface of the multivalued function $(1 - x^{n+1})^{1/(n+1)}$.

**Definition 2.** Let $(x,y)$ be coordinates in complex plane $\mathbb{C}^2$, $F = \{x^{n+1} + y^{n+1} = 1\}$. Consider the radial segments $[0,1]$ and $[0,\varepsilon]$ of the branching points 1 and $\varepsilon$ respectively in the $x$-axis; the former being oriented from 0 to 1, and the latter being oriented from $\varepsilon$ to 0. Their union is an oriented piecewise-linear curve (denote it by $\gamma$). Let $\gamma_0$ and $\gamma_1$ be its liftings to the covering $F$ such that $\gamma_0$ contains the point $(0,1)$ and $\gamma_1$ is obtained from $\gamma_0$ by multiplication of the coordinate $y$ by $\varepsilon$. The curves $\gamma_i$, $i = 0,1$, are oriented from their common origin $(\varepsilon,0)$ to their common end $(1,0)$. Define $\alpha_1 \in H_1(F,\mathbb{Z})$ to be the homology class represented by the union of the oriented curve $\gamma_0$ and the curve $\gamma_1$ taken with the inverse orientation.

**Proposition 3.** Let $F$, $\alpha_1$ be as in the previous Definition. Let $\alpha_j \in H_1(F,\mathbb{Z})$, $j = 2,\ldots,n^2$, be the homology classes obtained from $\alpha_1$ by the action of the element $(l(j),m(j)) \in \mathbb{Z}_{n+1} \oplus \mathbb{Z}_{n+1}$. The classes $\alpha_j$, $j = 1,\ldots,n^2$, generate the homology group.

**Proof.** Let $\Gamma = \cup_{j=0}^n [0,\varepsilon^j]$ be the union of the radial segments of the branching points of the fiber $F$ in the $x$-axis. Let $\widetilde{\Gamma} \subset F$ be the preimage of the set $\Gamma$ under the projection of $F$ to the $x$-axis. The set $\widetilde{\Gamma}$ is a deformation retract of the fiber $F$ by covering homotopy theorem (hence, the inclusion $\widetilde{\Gamma} \to F$ is a homotopy
equivalence). The group $H_1(\tilde{\Gamma}, \mathbb{Z})$ is generated by $\alpha_j$ by construction. Hence, this remains valid for the whole fiber $F$. This proves Proposition 3.

Let us calculate the value $\det(I_{j,r})|_{t=1}$ in the basis $\alpha_j$ from Proposition 3. To do this, we use the following

**Remark 8.** Let $\omega_j$ be the forms (2), $\alpha_j$ be as in Proposition 3, $I_{j,r}$ be the corresponding integrals from (3), $(l(j), m(j))$ be the lexicographic integer pair sequence from the beginning of the paper. For any $j, r = 1, \ldots, n^2$

$$I_{j,r} = \varepsilon^{l(r)(l(j)+1)+m(r)(m(j)+1)} I_{j,1}. \quad (21)$$

Formula (21) implies the following

**Corollary 1.** Let $\omega_j$ be the forms (2), $\alpha_j$ be as in Proposition 3, $(I_{j,r})$ be the corresponding matrix of the integrals from (3), $\det(I_{j,r})$ be its determinant. Put

$$I = \prod_{j=1}^{n^2} I_{j,1}. \quad (22)$$

Let $(l(j), m(j))$ be the lexicographic integer pair sequence from the beginning of the paper. Let $G = (g_{jr})$ be the $n^2 \times n^2$-matrix with the elements

$$g_{jr} = \varepsilon^{l(r)(l(j)+1)+m(r)(m(j)+1)}. \quad (23)$$

Then

$$\det(I_{j,r}) = I \det G. \quad (24)$$

Thus, to calculate $\det(I_{j,r})$, it suffices to calculate the expressions $I$ and $\det G$ from Corollary 1 (by (23)). They fill be calculated separately.

Firstly we calculate $\det G$. Below we show that

$$\det G = (n+1)^{-2n}\sigma^n. \quad (25)$$

Then we calculate $I$ (at the end of the Subsection). We show that

$$I = (n+1)^{2n}IP. \quad (26)$$

This will prove (17).

**Calculation of $\det G$. Proof of (24).** Define the following $n \times n$ matrix:

$$Q = (q_{jr}) = \begin{pmatrix}
1 & \varepsilon & \ldots & \varepsilon^{n-1} \\
1 & \varepsilon^2 & \ldots & \varepsilon^{2(n-1)} \\
& \ldots & \ldots & \ldots \\
1 & \varepsilon^n & \ldots & \varepsilon^{n(n-1)}
\end{pmatrix}. \quad (27)$$

To calculate $\det G$, we use the following periodicity property of the matrix $G$: its first $n$ columns are filled in by $n$ copies of the matrix $Q$, more precisely,

$$g_{j+s, r} = g_{jr} = q_{jr}. \quad (28)$$
For any $s = 1, \ldots, n$ by $Q_s$ denote the $ns \times ns$ matrix formed by the first $ns$ lines and columns of the matrix $G$ (thus, $Q_n = G$, $Q_1 = Q$). To find $\det G = \det Q_n$, we calculate the determinant $\det Q_s$ by induction in $s$.

We prove the following recurrent formula for $\det Q_s$:

\begin{equation}
\det Q_s = \left( \prod_{l=1}^{s-1} (\varepsilon^s - \varepsilon^l) \right)^n \det Q_{s-1}.
\end{equation}

**Proof of (27).** For the proof of (27), let us transform $Q_s$ by preserving its determinant as follows: for each $j \geq n+1$ we subtract the $(j-n)$-th line from the $j$-th one. This yields the new matrix (that will be denoted by $Q'_s$) whose first $n$ columns are formed by a single copy of the matrix $Q$ that takes the first $n$ lines and zeroes in the other places (by (26)). By definition,

\begin{equation}
(Q'_s)_{j \times r} = g_{j \times r} - g_{j-n \times r} \quad (\text{we put } g_{j \times r} = 0 \text{ for } j \leq 0).
\end{equation}

Let $Q''_s$ be the $n(s-1) \times n(s-1)$ matrix obtained from $Q'_s$ by throwing away its first $n$ columns and first $n$ lines. By definition,

\begin{equation}
\det Q_s = \det Q \det Q''_s.
\end{equation}

Now let us calculate $\det Q''_s$. By definition and (28),

\begin{equation}
(Q''_s)_{j \times r} = (Q'_s)_{j+n \times r+n} = g_{j+n \times r+n} - g_{j \times r+n}.
\end{equation}

By (22), for any $j, r = 1, \ldots, n^2 - n$

\[ g_{j+n \times r} = \varepsilon^{l(r)} g_{j \times r}; \quad g_{j \times r+n} = \varepsilon^{l(j)+1} g_{j \times r}. \]

This follows from (18) and the relations $l(j+n) = l(j)+1$, $m(j+n) = m(j)$ valid for all $j$ (the definition of the sequences $l(j)$ and $m(j)$). Hence, for any $j, r \leq n(s-1)$

\[ g_{j+n \times r+n} = \varepsilon^{l(r)+1}(l(j)+1) g_{j \times r}, \]

\[ (Q''_s)_{j \times r} = \varepsilon^{l(j)+1}(\varepsilon^{l(r)+1} - 1) g_{j \times r}. \]

In other terms, the matrix $Q''_s$ is obtained from the matrix $Q_{s-1}$ by multiplication of its $j$-th column by $\varepsilon^{l(j)+1}$ and $r$-th line by $\varepsilon^{l(r)+1} - 1$. Therefore,

\[
\det Q''_s = \left( \prod_{j=1}^{n(s-1)} \varepsilon^{l(j)+1} \right) \times \left( \prod_{r=1}^{n(s-1)} (\varepsilon^{l(r)+1} - 1) \right) \det Q_{s-1} = \left( \prod_{l=1}^{s-1} \varepsilon^l \right) \times \left( \prod_{k=1}^{s-1} (\varepsilon^k - 1) \right)^n \det Q_{s-1}.
\]

Putting together the product terms corresponding to $l$ and $k$ with $l + k = s$ in the previous formula and substituting the latter to (29) yields (27).

Let us calculate $\det G$. Put $\Delta = \sqrt{\sigma} = \prod_{1 \leq l < k \leq n+1} (\varepsilon^k - \varepsilon^l)$. By (27),

\[ \det G = (\det Q)^n \left( \prod_{1 \leq l < k \leq n} (\varepsilon^k - \varepsilon^l) \right)^n. \]
By van der Mond formula,

$$\text{(30)} \quad \det Q = \prod_{1 \leq l < k \leq n} (\varepsilon^k - \varepsilon^l); \quad \text{so,} \quad \det G = \left( \prod_{1 \leq l < k \leq n} (\varepsilon^k - \varepsilon^l) \right)^{2n}. $$

The product in (30) is equal to $(n + 1)^{-1} \Delta$ (this statement implies (24)). Indeed, by definition, this product is equal to

$$\Delta \left( \prod_{1 \leq l \leq n} (1 - \varepsilon^l) \right)^{-1}. $$

The previous statement on (30) follows from the formula

$$\text{(31)} \quad \prod_{1 \leq l \leq n} (1 - \varepsilon^l) = n + 1 :$$

by definition, its left-hand side is the value of the polynomial $\frac{x^{n+1} - 1}{x - 1} = \sum_{l=0}^{n} x^l$ at the point $x = 1$; hence it is equal to $n + 1$. Formula (24) is proved.

**Calculation of $I$. Proof of (25).** Let us express $I_{j,1}$ via the integral $I_j$ from (15). We show that

$$\text{(32)} \quad I_{j,1} = (1 - \varepsilon^{m(j)+1})(1 - \varepsilon^{l(j)+1})I_j. $$

This together with (31) will imply (25).

Let $\gamma_0, \gamma_1$ be the oriented curves from Definition 2. Then

$$\text{(33)} \quad I_{j,1} = \int_{\alpha_1} x^{l(j)} y^{m(j)+1} dx = \int_{\gamma_0} x^{l(j)} y^{m(j)+1} dx - \int_{\gamma_1} x^{l(j)} y^{m(j)+1} dx. $$

The second integral in the right-hand side of (33) is equal to the first one times $\varepsilon^{m(j)+1}$ (by definition). Analogously, the first integral in its turn is the integral along the segment $[0, 1]$ (which is equal to $I_j$) minus the one along the segment $[0, \varepsilon]$ oriented from 0 to $\varepsilon$. The integral along the last segment is equal to $\varepsilon^{l(j)+1}I_j$. This together with the two previous statements implies (32). Formula (25) is proved. The proof of formula (17) is completed.

**2.5.2. Calculation of $\sigma$ and $\Sigma$. Proof of (18).** Let us calculate $\sigma$. By definition,

$$\sigma = \left( \prod_{1 \leq l < k \leq n+1} (\varepsilon^k - \varepsilon^l) \right)^2 = (-1)^{\frac{n(n+1)}{2}} \prod_{1 \leq l < k \leq n+1} ((\varepsilon^k - \varepsilon^l)(\varepsilon^l - \varepsilon^k)) $$

$$= (-1)^{\frac{n(n+1)}{2}} \prod_{1 \leq k \leq n+1} \left( \prod_{1 \leq l \leq n+1; l \neq k} (\varepsilon^k - \varepsilon^l) \right) = (-1)^{\frac{n(n+1)}{2}} \prod_{1 \leq k \leq n+1} \left( \prod_{l=1}^{n} \varepsilon^k (1 - \varepsilon^l) \right).$$
Changing the second (inner) product in the right-hand side of the previous formula to \((n + 1)\varepsilon^n\) (by (31)) yields

\[
\sigma = (-1)^{n+1} \varepsilon^{\frac{n(n+1)}{2}} n^{n+1}.
\]

Substituting \(\varepsilon^{n+1} = -1\) to the right-hand side of the last formula, calculating the resulting power of -1 and changing it to its appropriate representative modulo 2 yields the first formula in (18).

Let us calculate \(\Sigma\). By (6),

\[
\Sigma = \left( \prod_{1 \leq l < k \leq n+1} \exp\left( -\frac{\pi i}{n+1} (\varepsilon^k - \varepsilon^l) \right) \right)^2 = \exp(n(n + 1)\frac{\pi i}{n+1})\sigma = (-1)^n \sigma.
\]

The second formula in (18) is proved.

2.5.3. Calculation of \(IP\). Proof of (19). To calculate \(IP = \prod_{j=1}^{n^2} I_j\), we firstly express it via appropriate values of \(B\)- and \(\Gamma\)- functions. Recall their definitions:

\[
B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,
\]

\[
\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx.
\]

The variable change \(u = x^{n+1}\) transforms integral (15) to

\[
\frac{1}{n+1} \int_0^1 u^{\frac{l(j)+1}{n+1}-1}(1-u)^{\frac{m(j)+1}{n+1}}du = \frac{1}{n+1} B\left(\frac{l(j)+1}{n+1}, \frac{m(j)+1}{n+1} + 1\right).
\]

Therefore,

\[
(34) \quad IP = (n + 1)^{-n^2} \prod_{0 \leq l, m \leq n-1} B\left(\frac{l+1}{n+1}, \frac{m+1}{l+1} + 1\right).
\]

To calculate the product in the right-hand side of (34), we use the following expression of \(B\)- function via \(\Gamma\)- function:

\[
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\]

Therefore, by (34),

\[
(35) \quad IP = (n + 1)^{-n^2} \frac{\left(\prod_{l=0}^{n-1} \Gamma\left(\frac{l+1}{n+1}\right)\right)^n \left(\prod_{l=0}^{n-1} \Gamma\left(\frac{l+1}{n+1} + 1\right)\right)^n}{\prod_{l,m=0}^{n-1} \Gamma\left(\frac{l+m+2}{n+1} + 1\right)}.
\]

To calculate the products in (35), we use the following identities for \(\Gamma\)- function [3]:

\[
\Gamma(n) = (n - 1)! \text{ for any } n \in \mathbb{N},
\]
\[(36)\quad \prod_{l=0}^{n} \Gamma\left(z + \frac{l}{n+1}\right) = (2\pi)^{\frac{\pi}{2}}(n+1)^{-\frac{1}{2}}((n+1)z)^{\frac{1}{2} -(n+1)^{2}} \Gamma((n+1)z) \quad \text{(Gauss-Legendre formula)}.
\]

One gets

\[(37)\quad \prod_{l=0}^{n-1} \Gamma\left(\frac{l+1}{n+1}\right) = (2\pi)^{\frac{\pi}{2}}(n+1)^{-\frac{1}{2}},
\]

\[(38)\quad \prod_{l=0}^{n-1} \Gamma\left(\frac{l+1}{n+1} + 1\right) = (2\pi)^{\frac{\pi}{2}}(n+1)^{-\frac{1}{2} -(n+2)}(n+1)!
\]

by applying (36) to \(z = \frac{1}{n+1}\) and \(z = \frac{n+2}{n+1}\) respectively and subsequent substitutions \(\Gamma(1) = 1, \Gamma(n+2) = (n+1)!\). Let us calculate the double product in (35). For any fixed \(m = 0, \ldots, n-1\)

\[
\prod_{l=0}^{n-1} \Gamma\left(\frac{l+m+2}{n+1} + 1\right) = (\Gamma\left(\frac{m+1}{n+1} + 1\right))^{-1}(2\pi)^{\frac{\pi}{2}}(n+1)^{-\frac{1}{2} -(m+n+2)}(m+n+1)!
\]

by (36) applied to \(z = \frac{m+1}{n+1} + 1\). Therefore,

\[
\prod_{l,m=0}^{n-1} \Gamma\left(\frac{l+m+2}{n+1} + 1\right) = \left(\prod_{m=0}^{n-1} \Gamma\left(\frac{m+1}{n+1} + 1\right)\right)^{-1}(2\pi)^{\frac{\pi}{2}}(n+1)^{-\frac{1}{2} -(m+n+2)} \prod_{m=0}^{n-1} (m+n+1)!
\]

Substituting formula (38) for the first product in the right-hand side of the last formula and summarizing the power of \(n+1\) yields

\[(39)\quad \prod_{l,m=0}^{n-1} \Gamma\left(\frac{l+m+2}{n+1} + 1\right) = (2\pi)^{\frac{n^2-n}{2}}(n+1)^{-\frac{3(n^2-1)}{2}} \prod_{m=1}^{n-1} (m+n+1)!.\]

Substituting (37)-(39) to (35) yields (19). The proof of Theorem 1 is completed.

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