Twisted Yangians and symmetric pairs

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ABSTRACT
We describe recent work on the twisted Yangians $Y(g, h)$ which arise as boundary remnants of Yangians $Y(g)$ in 1+1D integrable field theories, bringing out the special role played by the requirement that $(g, h)$ be a symmetric pair.

1. Introduction

In a series of recent papers we examined the boundary principal chiral model. The work began with the discovery that both the classically integrable boundary conditions and the vector-representation boundary $S$-matrices for the bulk $G$-model were classified and parametrized by the symmetric spaces $G/H$. These two points of view were linked by the discovery [2] that the boundary remnant of the bulk Yangian symmetry $Y(g)$ is the twisted Yangian $Y(g, h)$ [4, 5]. We went on to exploit our presentation of $Y(g, h)$ to give the spectral decompositions of a variety of reflection or ‘$K$’-matrices (the intertwiners of $Y(g, h)$-representations). In this talk we summarize how $Y(g)$ and $Y(g, h)$ fit together, emphasizing the special role of the requirement that $(g, h)$ be a symmetric pair. The focus is therefore on the algebra rather than the physics – an introduction to the latter may be found in [3].

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2In the PCM the symmetry is $Y_L(g) \times Y_R(g)$, but in this talk we shall just look at one copy of $Y(g)$. Our conclusions should apply to any integrable field theory with bulk $Y(g)$ symmetry.
We begin with a brief recapitulation of the structure of $Y(g)$, largely drawn from [5].

Suppose our 1+1D integrable quantum field theory to have a compact global symmetry group $G$, whose algebra $g$ is generated by conserved charges $Q^a_0$ with structure constants $f^a_{bc}$ and (trivial) coproduct $\Delta$,

$$(2.1) \quad [Q^a_0, Q^b_0] = if^a_{bc}Q^c_0 \quad \text{and} \quad \Delta(Q^a_0) = Q^a_0 \otimes 1 + 1 \otimes Q^a_0.$$  

The Yangian $Y(g)$ is the larger symmetry algebra generated by these and further non-local conserved charges $Q^a_1$, where

$$(2.2) \quad [Q^a_0, Q^b_1] = if^a_{bc}Q^c_1 \quad \text{and} \quad \Delta(Q^a_1) = Q^a_1 \otimes 1 + 1 \otimes Q^a_1 + \frac{1}{2}f^a_{bc}Q^b_0 \otimes Q^c_0.$$  

The requirement that $\Delta$ be a homomorphism fixes

$$(2.3) \quad f^{d[ab}[Q^c_1], Q^d_1] = \frac{i}{12}f^{api}f^{bqj}f^{crk}f^{ijk}Q^p_0Q^q_0Q^r_0,$$  

where $(\ )$ denotes symmetrization and $[\ ]$ anti-symmetrization on the enclosed indices, which have been raised and lowered freely with the invariant metric $\gamma$.

The Yangian is a deformation of the polynomial algebra $g[z]$; with $Q^a_1 = zQ^a_0$, the undeformed algebra would satisfy $(2.3)$ with the right-hand side zero – that is, $z^2$ times the Jacobi identity. In $Y(g)$, $(2.3)$ acts as a rigidity condition on the construction of higher $Q^a_n$ from the $Q^a_1$.

There is an (‘evaluation’) automorphism

$$(2.4) \quad L_\theta : \quad Q^a_0 \mapsto Q^a_0, \quad Q^a_1 \mapsto Q^a_1 + \theta \frac{c_A}{4i\pi}Q^a_0,$$  

where $c_A = C^g_2(g)$ is the value of the quadratic Casimir $C^g_2 \equiv \gamma_{ab}Q^a_0Q^b_0$ in the adjoint representation. (This normalization is chosen so that $\theta$ is the particle rapidity, as we shall see later.) Thus any representation $v$ of $Y(g)$ may carry a parameter $\theta$: the action of $Y(g)$ on $v^\theta$ is that of $L_\theta(Y(g))$ on $v^0$. The $i$th fundamental representation $v^\theta_i$ of $Y(g)$ is in general reducible as a $g$-representation, with one of its irreducible components (that with the greatest highest weight, where these are partially ordered using the simple roots) being the $i$th fundamental representation $V_i$ of $g$. In the simplest cases (which include all $i$ for $g = a_n$ and $c_n$), $v^\theta_i = V_i$ as a $g$-representation, and $Q^a_i = \theta \frac{c_A}{4i\pi}Q^a_0$ upon it.

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3For $g \neq sl(2)$. For the general condition see Drinfeld [6].
3. Twisted Yangians $Y(\mathfrak{g}, \mathfrak{h})$

A symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is a (here compact and simple) $\mathfrak{g}$ together with a (maximal) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ invariant under an involutive automorphism $\sigma$. We shall write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, so that $\mathfrak{h}$ and $\mathfrak{k}$ are the subspaces of $\mathfrak{g}$ with $\sigma$-eigenvalues $+1$ and $-1$ respectively, and

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}.$$  

We shall use $a, b, c, \ldots$ for general $\mathfrak{g}$-indices, $i, j, k, \ldots$ for $\mathfrak{h}$-indices and $p, q, r, \ldots$ for $\mathfrak{k}$-indices.

We define the twisted Yangian $Y(\mathfrak{g}, \mathfrak{h})$ to be the subalgebra of $Y(\mathfrak{g})$ generated by

$$Q^i_0$$

and

$$\tilde{Q}^p_1 \equiv Q^p_1 + \frac{1}{4}[C, Q^p_0],$$

where $C \equiv \gamma_{ij}Q^i_0Q^j_0$ is the quadratic Casimir operator of $\mathfrak{g}$, restricted to $\mathfrak{h}$.

$Y(\mathfrak{g}, \mathfrak{h})$ is also a deformation, this time of the subalgebra of (‘twisted’) polynomials in $\mathfrak{g}[z]$ invariant under the combined action of $\sigma$ and $z \mapsto -z$. Its defining feature is that $Y(\mathfrak{g}, \mathfrak{h})$ is a left co-ideal subalgebra\footnote{The analogous right co-ideal subalgebra may be obtained by reversing the sign of the second term in (3.2).} $\Delta(Y(\mathfrak{g}, \mathfrak{h})) \subset Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h})$. This is the crucial feature which allows the boundary states to form representations of $Y(\mathfrak{g}, \mathfrak{h})$ while the bulk states form representations of $Y(\mathfrak{g})$, and depends on the symmetric-pair property:

$$\Delta(\tilde{Q}^p_1) = \Delta\left(Q^p_1 + \frac{1}{4}[C, Q^p_0]\right)$$

$$= Q^p_1 \otimes 1 + 1 \otimes Q^p_1 + \frac{1}{4}[C, Q^p_0] \otimes 1 + 1 \otimes \frac{1}{4}[C, Q^p_0]$$

$$+ \frac{1}{2}f^p_{iq}Q^i_0 \otimes Q^q_0 + \frac{1}{2}f^p_{qi}Q^q_0 \otimes Q^i_0 + \frac{1}{2}[\gamma_{ij}Q^i_0 \otimes Q^j_0, Q^p_0 \otimes 1 + 1 \otimes Q^p_0]$$

$$= \tilde{Q}^p_1 \otimes 1 + 1 \otimes \tilde{Q}^p_1 + [\gamma_{ij}Q^i_0 \otimes Q^j_0, Q^p_0 \otimes 1]$$

$$= \tilde{Q}^p_1 \otimes 1 + 1 \otimes \tilde{Q}^p_1 + \frac{1}{2}[\Delta(C) - C \otimes 1 - 1 \otimes C, Q^p_0 \otimes 1].$$

holds essentially because for a symmetric pair the only non-zero structure constants are $f^{ijk}$ and $f^{ipq}$, and fails for a general subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Various particular cases of $(\mathfrak{g}, \mathfrak{h})$ have been studied before\footnote{The analogous right co-ideal subalgebra may be obtained by reversing the sign of the second term in (3.2).}: for example, section 3.5 of the first paper of\cite{4} describes $Y(gl(n), so(n))$. 

\cite{5}
4. Bulk and boundary scattering

In both the bulk and the boundary case the (twisted) Yangian symmetry allows one to determine the $S$-matrix up to an overall scalar factor. In the boundary case, this will lead us to another crucial implication of the symmetric-pair property.

In the bulk case, we first note that, in the scattering of two particle multiplets $u^\phi$ and $v^\theta$, the asymptotic state $u^\phi \otimes v^\theta$ is decomposable into a sum of $g$-representations on each of which the $S$-matrix acts as the identity (because of the $g$ symmetry). This state is in general irreducible as a $Y(g)$ representation, but it may become reducible at certain special values of $\phi - \theta$ (at which the $S$-matrix typically has a pole). In particular, the edge of the physical strip lies at $\phi - \theta = i\pi$. The crossing symmetry of the $S$-matrix requires that it project onto the scalar representation of $g$ at this value, while the $Q_a^1$ map the scalar into the adjoint representation for general $\phi - \theta$. The tensor product graph construction \cite{7,8} (which we shall not recap here; see \cite{5} for a brief introduction) makes it clear how the difference in the values of $C_g^2$ between these two representations fixes the pole value, and ensures through the factor $\frac{a}{4i\pi}$ in (2.4) that this pole is at $i\pi$.

The boundary $S$-matrix is determined as follows, in the simplest cases for which $v^\theta$ is an irreducible $g$-representation (hereafter ‘irrep’) $V$. Writing it (again up to an overall factor, when it is usually known as the ‘reflection’ or $K$-matrix)\footnote{In certain cases, in which non-self-conjugate $g$-representations branch to self-conjugate $h$-representations, $v^\theta$ may be conjugated by $K$ \cite{4}.} as $K_v(\theta) : v^\theta \rightarrow v^{-\theta}$ and intertwining the $Q_0^i$ (that is, from the physics point-of-view, requiring their conservation in boundary scattering processes) requires that

$$K_v(\theta)Q_0^i = Q_0^i K_v(\theta)$$

(in which by $Q_0^i$ we mean here its representation on $V$) and thus that $K_v(\theta)$ act trivially on $h$-irreducible components of $V$. So we have

$$K_v(\theta) = \sum_{W \subset V} \tau_W(\theta) P_W,$$

where the sum is over $h$-irreps $W$ into which $V$ branches, and $P_W$ is the projector onto $W$.

To deduce relations among the $\tau_W$ we intertwine the $\tilde{Q}_1^p$. Recall that, on a $g$-irreducible $v^\theta$, the action of $Q_1^p$ is given by $Q_1^p = \theta \frac{a}{4i\pi} Q_0^p$, so that

$$\langle W || K_v(\theta) \left( \theta \frac{a}{4i\pi} Q_0^p + \frac{1}{4} [C, Q_0^p] \right) || W' \rangle = \langle W || \left( -\theta \frac{a}{4i\pi} Q_0^p + \frac{1}{4} [C, Q_0^p] \right) K_v(\theta) || W' \rangle,$$
for $W, W' \subset V$. Thus when the reduced matrix element $\langle W||Q^\rho||W'\rangle \neq 0$ we have

\begin{equation}
\frac{\tau_{W'}(\theta)}{\tau_W(\theta)} = [\Delta], \quad \text{where} \quad [A] \equiv \frac{i \pi A}{c_A} + \frac{\theta}{c_A} - \frac{\theta}{c_A} \quad \text{and} \quad \Delta = C(W) - C(W').
\end{equation}

To find the $W, W'$ for which $\langle W||Q^\rho||W'\rangle \neq 0$ we recall that $k$ forms an irrep $K$ of $h$. A necessary condition for (4.1) to apply is then that $W \subset K \otimes W'$. Although not automatically sufficient, this is sufficiently constraining in simple cases to enable us to deduce $K_v(\theta)$ [2, 5].

We can describe $K_v(\theta)$ by using a graph, in which the nodes are the $h$-irreps $W$, linked by edges, directed from $W_i$ to $W_j$, and labelled by $\Delta_{ij}$, whenever $W_i \subset K \otimes W_j$. To calculate the labels, we first write $C = \sum_i c_i C_{h_i}^2$, where $h = \bigoplus h_i$, $h_i$ is a sum of simple factors $h_i$ (and $C_{h_i}^2$ is the quadratic Casimir of $h_i$). The point here is that $C$ was written in terms of generators of $g$: there will be non-trivial scaling factors $c_i$, which may be computed by taking the trace of the adjoint action of $C$ on $g$ (where we fix $\gamma$ to be the identity both on $g$ and on each $h_i$), yielding

$$c_i = \frac{c_A}{C_{h_i}^2(K) + C_{h_i}^2(K)}.$$

This has a highly non-trivial implication for the boundary $S$-matrix. The analogue of the crossing relation for bulk $S$-matrices is the ‘crossing-unitarity’ relation [9]. One requirement of this is that, at the edge $\theta = i\pi/2$ of the physical strip for the boundary $S$-matrix, $K$ project onto the scalar representation of $h$. In the graph described above, $\langle W||Q^\rho||1\rangle \neq 0$ only for $W = K$, and so we must have $C(K) = \frac{1}{2} c_A$, or

$$\frac{C(K)}{c_A} = \frac{1}{c_A} \sum_i c_i C_{h_i}^2(K) = \sum_i \left( \frac{C_{h_i}^2(K)}{C_{h_i}^2(K)} + \frac{\dim h_i}{\dim h_i} \right) = \frac{1}{2}.$$

That this holds, and does so only for symmetric pairs, is a result of [10] (also known as the ‘symmetric space theorem’ [11]) and we see once again the centrality of this property.
5. Concluding remarks

One reason why it is appropriate to emphasize the centrality of the symmetric-pair property is that in a closely-related and well-developed field, that of D-branes in group manifolds (a.k.a. the boundary WZW model), it appears not to be necessary; \( \sigma \) may be any automorphism \[12\]. (Whether the triality of \( d_4 \) enjoys some special status is unclear.)

A unifying principle for the exceptional algebras is the ‘magic square’, or equivalently the Cvitanovic-Deligne exceptional series \[13\], in terms of which both the boundary \( K \)-matrices \[5\] and bulk \( R \)-matrices \[14\] have a unified structure. Once again symmetric pairs are crucial in the construction \[15\].

As promised, we have focused here on algebraic structures rather than on physics, and so have not given examples of boundary \( S \)-matrices or the spectra of boundary bound states which may be deduced from them. Such calculations are generally very tough – they involve complex fusion/bootstrap calculations – and the results are typically more opaque than in bulk cases \[16\]. However, it is interesting to note that for the classical Grassmannians

\[
\begin{align*}
SU(N) & \rightarrow S(U(M) \times U(N-M))' \\
SO(N) & \rightarrow SO(M) \times SO(N-M)' \\
Sp(N) & \rightarrow Sp(M) \times Sp(N-M)
\end{align*}
\]

\( (N, M \text{ even}) \),

there seems to be a set of boundary states with masses

\[
m_a = m \sin \frac{a\pi}{h} \sin \frac{(p-a)\pi}{h},
\]

where \( m \) is a mass-scale and \( (p, h) = (M+1, N), (M, N-2), (M+2, N+2) \) for the three cases respectively (taking, without loss of generality, \( M \leq N/2 \)). In the \( N \rightarrow \infty \) limit, these are proportional to the values of the quadratic Casimir operator in the \( a \)th fundamental representations of (and so to the energy levels of the 0+1D principal chiral model defined on) \( SU(M), SO(M) \) or \( Sp(M) \) respectively.
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