DIMENSION OF THE TORELLI GROUP FOR $\text{Out}(F_n)$

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Abstract. Let $\mathcal{T}_n$ be the kernel of the natural map $\text{Out}(F_n) \to \text{GL}_n(\mathbb{Z})$. We use combinatorial Morse theory to prove that $\mathcal{T}_n$ has an Eilenberg–MacLane space which is $(2n - 4)$-dimensional and that $H_{2n-4}(\mathcal{T}_n, \mathbb{Z})$ is not finitely generated ($n \geq 3$). In particular, this recovers the result of Krstić–McCool that $\mathcal{T}_3$ is not finitely presented. We also give a new proof of the fact, due to Magnus, that $\mathcal{T}_n$ is finitely generated.

1. Introduction

There is a natural homomorphism from $\text{Out}(F_n)$, the group of outer automorphisms of the free group on $n$ generators, to $\text{GL}_n(\mathbb{Z})$, given by abelianizing the free group $F_n$. It is a theorem of Nielsen that this map is surjective [11]. We call its kernel the Torelli subgroup of $\text{Out}(F_n)$, and we denote it by $\mathcal{T}_n$:

$$1 \to \mathcal{T}_n \to \text{Out}(F_n) \to \text{GL}_n(\mathbb{Z}) \to 1$$

Main Theorem. For $n \geq 3$, we have:

1. $\mathcal{T}_n$ has a $(2n - 4)$-dimensional Eilenberg–MacLane space.
2. $H_{2n-4}(\mathcal{T}_n, \mathbb{Z})$ is infinitely generated.
3. $\mathcal{T}_n$ is finitely generated.

Part (3) of the main theorem is due to Magnus; we give our own proof in Section 5. We remark that $\mathcal{T}_1$ is obviously trivial and $\mathcal{T}_2$ is trivial by a classical result of Nielsen [11] (we give a new proof of the latter fact in Section 5).

The group $\mathcal{T}_n$, like any torsion free subgroup of $\text{Out}(F_n)$, acts freely on the spine for outer space (see Section 2), and therefore has an Eilenberg–MacLane space of dimension $2n - 3$, the dimension of this spine. Our theorem improves this upper bound on the dimension and shows that $2n - 4$ is sharp.

When $n = 3$, we obtain that $H_2(\mathcal{T}_3, \mathbb{Z})$ is not finitely generated, and this immediately implies the result of Krstić–McCool that $\mathcal{T}_3$ is not finitely presented [7].

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Historical background. The question of whether \( H_k(\mathcal{T}_n, \mathbb{Z}) \) is finitely generated, for various values of \( k \) and \( n \), is a long standing problem with few solutions. This question was explicitly asked by Vogtmann in her survey article [14]. We now give a brief history of related results, all of which are recovered by our main theorem.

Nielsen proved in 1924 that \( \mathcal{T}_3 \) is finitely generated [11]. Ten years later, Magnus proved that \( \mathcal{T}_n \) is finitely generated for every \( n \) [8].

Smillie–Vogtmann proved in 1987 that, if \( 2 < n < 100 \) or \( n > 2 \) is even, then \( H_\ast(\mathcal{T}_n, \mathbb{Z}) \) is not finitely generated [12] [13]. Their method is to consider the rational Euler characteristics of the groups in the short exact sequence defining \( \mathcal{T}_n \) (see [14]).

The Krстиć–McCool result that \( \mathcal{T}_3 \) is not finitely presented was proven in 1997, via completely algebraic methods [7]. It is a general fact that if the second homology of a group is not finitely generated, then the group is not finitely presented.

Large abelian subgroups. It follows from the second part of the main theorem that the first part is sharp; i.e., \( \mathcal{T}_n \) does not have an Eilenberg–MacLane space of dimension less than \( 2n - 4 \). A simpler proof that the cohomological dimension of \( \mathcal{T}_n \) is at least \( 2n - 4 \) is to simply exhibit an embedding of \( \mathbb{Z}^{2n-4} \) into \( \mathcal{T}_n \). There is a subgroup \( \mathbb{Z}^{2n-4} \cong G < \mathcal{T}_n \) consisting of elements with representative automorphisms given by:

\[
\begin{align*}
  x_1 &\mapsto x_1 \\
  x_2 &\mapsto x_2 \\
  x_3 &\mapsto [x_1, x_2]^{p_3} x_3 [x_1, x_2]^{q_3} \\
  &\quad \vdots \\
  x_n &\mapsto [x_1, x_2]^{p_n} x_n [x_1, x_2]^{q_n}
\end{align*}
\]

for varying \( p_i \) and \( q_i \) (the \( x_i \) are generators for \( F_n \)).

In Section 7, we prove that specific conjugates of \( G \) represent independent classes in \( H_{2n-4}(\mathcal{T}_n, \mathbb{Z}) \), thus proving the second part of the main theorem. These conjugates are exactly the generators of \( H_{2n-4}(\mathcal{M}_n, \mathbb{Z}) \), where \( \mathcal{M}_n \), called the “toy model”, is a particularly simple subcomplex of the Eilenberg–MacLane space \( \mathcal{Y}_n \) defined in Section 2. In Section 7, we prove that \( H_{2n-4}(\mathcal{M}_n, \mathbb{Z}) \) injects into \( H_{2n-4}(\mathcal{Y}_n, \mathbb{Z}) \). Since the homology of \( \mathcal{M}_n \) is not finitely generated in any dimension greater than 1, we are led to the following question:

**Question.** Does \( H_\ast(\mathcal{M}_n, \mathbb{Z}) \) inject into \( H_\ast(\mathcal{Y}_n, \mathbb{Z}) \)?

Mapping class groups. The term “Torelli group” comes from the theory of mapping class groups. Let \( \Sigma_g \) be a closed surface of genus \( g \geq 1 \). The mapping class group of \( \Sigma_g \), denoted \( \text{Mod}(\Sigma_g) \), is the group of isotopy classes of orientation preserving homeomorphisms of \( \Sigma_g \). The Torelli group, \( \mathcal{I}_g \), is the subgroup of \( \text{Mod}(\Sigma_g) \) acting trivially on the homology of \( \Sigma_g \). As \( \text{Mod}(\Sigma_g) \) acts on \( H_1(\Sigma_g, \mathbb{Z}) \) by symplectic automorphisms, \( \mathcal{I}_g \) is defined by:

\[
1 \to \mathcal{I}_g \to \text{Mod}(\Sigma_g) \to \text{Sp}_{2g}(\mathbb{Z}) \to 1
\]

It is a classical theorem of Dehn, Nielsen, and Baer that the natural map \( \text{Mod}(\Sigma_g) \to \text{Out}(\pi_1(\Sigma_g)) \) is an isomorphism. In this sense \( \mathcal{T}_n \) is the direct analog of \( \mathcal{I}_g \).
Our (lack of) knowledge of the finiteness properties of $\mathcal{I}_g$ mirrors that for $\mathcal{T}_n$. Using the fact that $\text{Mod}(\Sigma_1) \cong \text{SL}_2(\mathbb{Z}) = \text{Sp}_2(\mathbb{Z})$, it is obvious that $\mathcal{I}_2$ is trivial. Johnson showed in 1983 that $\mathcal{I}_{g}$ is finitely generated for $g \geq 3$ [6]. In 1986, McCullough–Miller showed that $\mathcal{I}_{2}$ is not finitely generated [9], and Mess improved on this in 1992 by showing that $\mathcal{I}_2$ is a free group of infinite rank [10]. At the same time, Mess further showed that $H_3(\mathcal{I}_3, \mathbb{Z})$ is not finitely generated. In Kirby’s problem list, Mess asked about finiteness properties in higher genus [1].

**Automorphisms vs. outer automorphisms.** Strictly speaking, Magnus and Krsti´c–McCool study the group $K_n$, by which we mean the kernel of $\text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z})$, where $\text{Aut}(F_n)$ is the automorphism group of the free group. By considering the short exact sequence

$$1 \to F_n \to K_n \to \mathcal{T}_n \to 1$$

we see that $K_n$ is finitely generated if and only if $\mathcal{T}_n$ is finitely generated. Moreover, it follows from our main theorem and the spectral sequence associated to this short exact sequence that $H_{2n-3}(K_n, \mathbb{Z})$ is not finitely generated and if $k$ is the smallest index so that $H_k(\mathcal{T}_n, \mathbb{Z})$ is not finitely generated, then $H_k(K_n, \mathbb{Z})$ is not finitely generated.

From a topological point of view, $\mathcal{T}_n$ is the more natural group to study.

In the literature, $\mathcal{T}_n$ is sometimes denoted by $IA_n$ for “identity on abelianization” (see, e.g. [14]). However, since Krsti´c–McCool use $IA_n$ to denote the kernel of $\text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z})$, we avoid this notation to eliminate the confusion. The notation $K_n$ comes from Magnus [8].

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### 2. An Eilenberg–MacLane space

In Section 2.1, we recall the definition of Culler–Vogtmann’s spine for Outer space. Then, in Section 2.2, we describe the quotient of this space by $\mathcal{T}_n$. This quotient is a $(2n-3)$-dimensional Eilenberg–MacLane space for $\mathcal{T}_n$.

A *rose* is a graph with one vertex. The *standard rose* in rank $n$, denoted $R_n$, is a particular rose which is fixed once and for all. We denote the standard generators of $F_n \cong \pi_1(R_n)$ by $x_1, \ldots, x_n$.

#### 2.1. Spine for Outer space.** Culler–Vogtmann introduced the *spine for Outer space*, which we denote by $X_n$, as a tool for studying $\text{Out}(F_n)$ [4]. This is a simplicial complex defined in terms of marked graphs.

A *marked graph* is a pair $(\Gamma, g)$, where $\Gamma$ is a finite metric graph (1-dimensional cell complex with a metric) with no separating edges and no vertices of valence less than 3 and $g : R_n \to \Gamma$ is a homotopy equivalence ($g$ is called the *marking*). We say that two marked graphs $(\Gamma, g)$ and $(\Gamma', g')$ are *equivalent* if $g' \circ g^{-1}$ is homotopic to an
isometry, where \( g^{-1} \) is any homotopy inverse of \( g \). We will denote the equivalence class \([\Gamma, g]\) by \((\Gamma, g)\).

The vertices of \( X_n \) are equivalence classes of marked graphs where all edges have length 1. A set of vertices 
\[
\{(\Gamma_1, g_1), \ldots, (\Gamma_k, g_k)\}
\]
is said to span a simplex if \( \Gamma_{i+1} \) is obtained from \( \Gamma_i \) by collapsing a forest in \( \Gamma_i \), and \( g_{i+1} \) is the marking obtained from \( g_i \) via this operation.

We can think of arbitrary points of \( X_n \) as marked metric graphs: for instance, as we move along an edge between two vertices in \( X_n \), the length of some edge in the corresponding graphs (more generally, the lengths of the edges in a forest) varies between 0 and 1.

There is a natural right action of \( \text{Out}(F_n) \) on \( X_n \). Namely, given \( \phi \in \text{Out}(F_n) \) and \((\Gamma, g) \in X_n\), the action is given by:
\[
(\Gamma, g) \cdot \phi = (\Gamma, g \circ \phi)
\]
(here we are using the fact that every element \( \phi \) of \( \text{Out}(F_n) \) can be realized by a homotopy equivalence \( R_n \to \Gamma \), also denoted \( \phi \), uniquely up to homotopy).

Culler–Vogtmann proved the following result [4]:

**Theorem 2.1.** For \( n \geq 2 \), the space \( X_n \) is contractible.

This theorem has the consequence that the virtual cohomological dimension of \( \text{Out}(F_n) \) is equal to \( 2n - 3 \), the dimension of \( X_n \).

The star of a rose in \( X_n \) is the union of the closed simplices containing the vertex corresponding to a rose. The key idea for Theorem 2.1 is to think of \( X_n \) as the union of stars of vertices corresponding to marked roses. We take an analogous approach in this paper.

2.2. **The quotient.** Baumslag–Taylor proved that \( \mathcal{T}_n \) is torsion free [2]. We also know that the action of \( \mathcal{T}_n \) on \( X_n \) is free: by the definition of the action, point stabilizers correspond to graph isometries, and isometries act nontrivially on homology. Finally, the action is simplicial, and so it follows that the quotient of \( X_n \) by \( \mathcal{T}_n \) is an Eilenberg–MacLane space for \( \mathcal{T}_n \):
\[
\mathcal{Y}_n = X_n / \mathcal{T}_n
\]

**Homology markings.** Since \( \text{Out}(F_n) \) identifies every pair of isometric graphs of \( X_n \), points of \( \mathcal{Y}_n \) can be thought of as equivalence classes of pairs \((\Gamma, g)\), where \( \Gamma \) is a metric graph (as before), and \( g \) is a homology marking; that is, \( g \) is an equivalence class of homotopy equivalences \( R_n \to \Gamma \), where two homotopy equivalences are equivalent if (up to isometries of \( \Gamma \)) they induce the same map \( H_1(R_n, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z}) \).

Via the marking \( g \), we can think of the (oriented) edges of \( \Gamma \) as elements of \( H^1(R_n) \cong \mathbb{Z}^n \) (if \( e \) is an edge and \( x \) is a simplicial 1-chain, then \( e(x) \) is the number of times \( e \) appears in \( x \)). As such, if we think of the generators \( x_1, \ldots, x_n \) of
\( \pi_1(R_n) \) as elements of \( H_1(R_n, \mathbb{Z}) \), then we can label each oriented edge \( a \) of \( \Gamma \) by the corresponding row vector:

\[
(a(g(x_1)), \ldots, a(g(x_n)))
\]

where \( a(g(x_i)) \) is the number of times \( g(x_i) \) runs over \( a \) homologically, with sign.

In this way, a point of \( Y_n \) is given by a labelled graph, and two such graphs represent the same point in \( Y_n \) if and only if there is a label preserving graph isomorphism between them (i.e. if the map induces the identity on cohomology). See Figure 1 for an example of a labelled graph. We remark that this example exhibits the fact that \( Y_n \) is not a simplicial complex—there are two edge collapses (and hence two edges in \( Y_n \)) taking this point to the rose with the identity marking.

![Figure 1. An example of a labelled graph.](image)

When convenient, we will confuse the points of \( Y_n \) with the corresponding marked graphs.

We will make use of the following generalities about marked graphs in \( Y_n \):

**Proposition 2.2.** Let \( (\Gamma, g) \) be a marked graph.

1. If an edge of \( \Gamma \) is collapsed, the labels of the remaining edges do not change.
2. Any two edges of \( \Gamma \) with the same label (up to sign) are parallel in the sense that the union of their interiors disconnects \( \Gamma \).
3. The sum of the labels of the (oriented) edges coming into a vertex of \( \Gamma \) is equal to the sum of the labels of the edges leaving the vertex.

We leave the proofs to the reader.

**Roses.** Let \( \Gamma \) be a rose with edges \( a_1, \ldots, a_n \), and let \( g : R_n \to \Gamma \) be a homology marking. Up to isometries of \( \Gamma \), the marking \( g \) gives an element of \( GL_n(\mathbb{Z}) \), called the marking matrix; the rows are exactly the labels of the edges.

Since all edges have length 1, the isometry group of \( \Gamma \) is generated by swapping edges and by reversing the orientations of edges; the former operation has the effect of switching rows of the matrix, and the latter corresponds to changing signs of rows. Thus, in this case, \( (\Gamma, g) \) gives rise to an element of \( W \setminus GL_n(\mathbb{Z}) \), where \( W = W_n \) is the signed permutation subgroup of \( GL_n(\mathbb{Z}) \), acting on the left. In fact, this gives a bijection between roses in \( Y_n \) and elements of \( W \setminus GL_n(\mathbb{Z}) \), as \( \text{Out}(F_n) \) acts transitively on the roses of \( X_n \).
The right action of $\text{Out}(F_n)$ on $X_n$ descends to a right action of $\text{GL}_n(\mathbb{Z})$ on $\mathcal{Y}_n$. In particular, the action on roses is given by the right action of $\text{GL}_n(\mathbb{Z})$ on $W \setminus \text{GL}_n(\mathbb{Z})$.

3. Stars of roses

As with $X_n$, we like to think of the quotient $\mathcal{Y}_n$ as the union of stars of roses. By definition, the star of a rose in $\mathcal{Y}_n$ is the image of the star of a rose in $X_n$. Thus, it consists of graphs which can be collapsed to a particular rose. We now discuss some of the basic properties of the star of a rose.

3.1. Labels in the star of a rose. We will need several observations about the behavior of labels in the star of a rose. The proofs of the various parts of the propositions are straightforward and are left to the reader. In each of the statements, let $\rho$ be a rose in $\mathcal{Y}_n$ represented by a marked graph $(\Gamma, g)$. Say that its edges $a_1, \ldots, a_n$ are labelled by $v_1, \ldots, v_n \in \mathbb{Z}^n$.

**Proposition 3.1.** If $(\Gamma', g')$ is a marked graph in $\text{St}(\rho)$, we have:

1. For each $i$, there is an edge of $\Gamma'$ labelled $\pm v_i$.
2. The edges not labelled $\pm v_i$ form a forest.
3. If each edge of $\Gamma'$ is labelled $\pm v_i$, then the union of edges labelled $\pm v_i$ (for any particular $i$) is a topological circle (see, e.g., Figure 1).
4. The label of any edge of $\Gamma'$ is of the form
   \[ \sum_{i=1}^{n} k_i v_i \]
   where $k_i \in \{-1, 0, 1\}$.

We have the following converse to the first two parts of the previous proposition:

**Proposition 3.2.** If $(\Gamma', g')$ is a marked graph which has, for each $i$, at least one edge of length 1 labelled $\pm v_i$, then $(\Gamma', g')$ is in $\text{St}(\rho)$.

We also have a criterion for when a marked graph is in the frontier of the star of a rose:

**Proposition 3.3.** A marked graph $(\Gamma', g')$ in $\text{St}(\rho)$ is in the frontier of $\text{St}(\rho)$ if and only if it has at least one edge of length 1 whose label is not $\pm v_i$ for any $i$. In this case, the given label is a label for some rose whose star contains $(\Gamma', g')$.

3.2. Ideal edges. Let $\rho = (\Gamma, g)$ be a rose whose edges $a_1, \ldots, a_n$ are labelled $v_1, \ldots, v_n$, as above. An ideal edge is any formal sum:

\[ \sum_{i=1}^{n} k_i a_i \]

where $k_i \in \{-1, 0, 1\}$, and at least two of the $k_i$ are nonzero. An ideal edge is a “direction” in $\text{St}(\rho)$ in the following sense: for any ideal edge, we can find a marked graph $(\Gamma', g')$ in the frontier of $\text{St}(\rho)$ where one of the edges of $(\Gamma', g')$ has the label
\[ \sum k_i v_i. \] If a marked graph in \( \text{St}(\rho) \) has an edge of length 1 with label \( \sum k_i v_i \), we say that the marked graph realizes the ideal edge \( \sum k_i a_i \).

**Lemma 3.4.** Given any ideal edge for a particular rose, there is a 1-edge blowup of \( \rho \) in the frontier of \( \text{St}(\rho) \) which realizes that ideal edge.

The lemma is proven by example. See Figure 2 for a picture of a 1-edge blowup realizing the ideal edge \( a_1 - a_3 + a_4 \) in rank 5 (apply Proposition 2.2(3)). Also, we see that there are many graphs satisfying the conclusion of the lemma—the other graphs are obtained by moving the loops labelled \( v_2 \) and \( v_5 \) arbitrarily around the graph.

![Figure 2. A 1-edge blowup realizing the ideal edge \( a_1 - a_3 + a_4 \).](image)

Our notion of an ideal edge is simply the homological version of the ideal edges of Culler–Vogtmann [4].

We say that an ideal edge \( \iota' \) is subordinate to the ideal edge \( \iota = \sum k_i a_i \) if \( \iota' \) is obtained by changing some of the \( k_i \) to zero. A 2-letter ideal edge is an ideal edge of the form \( k_i a_i + k_j a_j \). Two ideal edges are said to be opposite if one can be obtained from the other by changing the sign of exactly one coefficient. The following facts are used in Section 5:

**Lemma 3.5.** Let \( \rho = (\Gamma, g) \) be a rose whose edges \( a_1, \ldots, a_n \) are labelled \( v_1, \ldots, v_n \). Suppose that \( \iota \) and \( \iota' \) are ideal edges and that either:

1. \( \iota' \) is subordinate to \( \iota \), or
2. \( \iota \) and \( \iota' \) are 2-letter ideal edges which are not opposite.

In either case, there is a marked graph \( (\Gamma', g') \) in \( \text{St}(\rho) \) which simultaneously realizes \( \iota \) and \( \iota' \).

**Proof.** In each case, we can explicitly describe the desired graph. If \( \iota' \) is subordinate to the ideal edge \( \iota = \sum k_i a_i \), we start with a 1-edge blowup realizing \( \iota \) (Lemma 3.4), and then blow up another edge to separate the edges which appear in \( \iota' \) from those which do not. Figure 3 (left hand side) demonstrates this for \( \iota = v_1 + v_2 + v_3 + v_4 \) and \( \iota' = v_1 + v_2 \) in rank 4.

For the case of two 2-letter ideal edges which are not opposite, without loss of generality it suffices to demonstrate marked graphs which simultaneously realize
$v_1 + v_2$ with $-v_1 - v_2$, $v_2 + v_3$, or $v_3 + v_4$ (the arbitrary case is obtained by renaming/reorienting edges and by attaching extra 1-cells to any vertex). See Figure 3 (right hand side) for a demonstration. One can use Proposition 2.2(3) to verify the labels.

The reader may verify that opposite ideal edges are never simultaneously realized.

We remark that, in the framework established by Culler–Vogtmann, one can think of this lemma in terms of compatibility of partitions, in which case the proof is immediate; see [4].

3.3. Homotopy type. In the remainder of this section, we prove that the star of any rose retracts onto the subcomplex consisting of “cactus graphs”, and this subcomplex is homeomorphic to a union of $(n - 2)$-tori.

We define a rank $n$ cactus graph inductively as follows. A rank 1 cactus graph is a graph with 1 vertex and 1 edge (i.e. a circle with a distinguished point). In general, a rank $n$ cactus graph is obtained by gluing a rank 1 cactus graph to a rank $n - 1$ cactus graph along the vertex of the rank 1 cactus graph. The set of vertices of the new graph is the union of the sets of vertices of the original two graphs. We note that a rank $n$ cactus graph has exactly $n$ embedded circles, and every edge belongs to exactly one embedded circle (Figure 1 is an example).

Let $C(\rho)$ denote the space of cactus graphs in St($\rho$). Given any $\rho'$, there is a canonical homeomorphism $C(\rho) \rightarrow C(\rho')$, once we choose orderings of the edges of $\rho$ and $\rho'$. Thus, we can unambiguously use $C_n$ to denote the space of cactus graphs in the star of a rose in rank $n$. 

![Figure 3. Marked graphs simultaneously realizing subordinate ideal edges (left) and 2-letter ideal edges which are not opposite (right).](image_url)
In the remainder, assume that \( \rho = (\Gamma, g) \) is a rose in \( \mathcal{Y}_n \) with edges \( a_1, \ldots, a_n \) labelled \( v_1, \ldots, v_n \).

**Lemma 3.6.** \( \text{St}(\rho) \) strongly deformation retracts onto \( C(\rho) \).

**Proof.** For every marked graph in \( \text{St}(\rho) \), the set of edges whose label is not \( \pm v_i \) is a forest (Proposition 3.1(2)). We perform a strong deformation retraction of \( \text{St}(\rho) \) by shrinking the edges of each such forest in each marked graph in \( \text{St}(\rho) \).

Consider any marked graph \( (\Gamma', g') \) in the image of the retraction. By Propositions 3.1(1) and 3.1(3), there is a circle of edges labelled \( \pm v_i \) for each \( i \). We consider the “dual graph” obtained by assigning a vertex to each such circle (the *circle vertices*) and each intersection point (the *point vertices*) and we connect a point vertex to a circle vertex if the point is contained in the circle. It follows from Proposition 2.2(2) that this graph is a tree, and hence \( (\Gamma', g') \) is a cactus graph. \( \square \)

**Corollary 3.7.** For \( n \geq 2 \), the star of any rose \( \text{St}(\rho) \) in \( \mathcal{Y}_n \) is homotopy equivalent to a complex of dimension \( n - 2 \).

**Proof.** By the definition of cactus graphs, we can see that the dimension increases with slope 1 with respect to dimension, starting at \( n = 2 \). Since \( C_2 \) is a point, \( C_n \) is a complex of dimension \( n - 2 \). An application of Lemma 3.6 completes the proof. \( \square \)

We can filter \( C(\rho) \) by subsets according to the number of vertices in the cactus graphs:

\[ \{\rho\} = V_0 \subset V_1 \subset \cdots \subset V_{n-2} = C(\rho) \]

Each \( V_i \) consists of cactus graphs with \( i - 1 \) vertices.

Our goal now is to give a generating set for \( \pi_1(C(\rho)) \). Since \( V_2 \) is simple to understand, the following proposition will make it easy to do this.

**Proposition 3.8.** There is a cell structure on \( C(\rho) \) so that the \( i \)-skeleton is exactly \( V_i \).

**Proof.** We proceed inductively. The 0-skeleton is one point \( V_0 = \{\rho\} \).

Let \( i > 0 \). Any marked graph \( (\Gamma', g') \) in \( V_i - V_{i-1} \) lies in a unique \( i \)-cell \( C \). For each \( i \), let \( k_i \) be the number of edges of \( \Gamma' \) labelled \( \pm v_i \). If we reparameterize so that the sum of the lengths of the edges of \( \Gamma' \) labelled \( \pm v_i \) is 1, then we get a \( (k_i - 1) \)-simplex for each \( i \), and \( C \) is the product of these simplices.

The boundary of \( C \) is the set of points where some edge is assigned length 0. Clearly, \( \partial C \subset V_{i-1} \), and so the proposition follows. \( \square \)

For the remainder of this section, we use the cell structure given by Proposition 3.8, which is different from the cell structure inherited from \( \mathcal{Y}_n \).

Let \( V^1_i \) be the subset of \( V_i \) consisting of graphs with a vertex of valence 4 and a vertex of valence \( 2n - 2 \) (i.e. only a single loop is “travelling” around another loop).
We will see in Section 5 that the obvious generators for \( \pi_1(V_1^n) \) correspond to one of the two types of Magnus generators for \( \mathcal{T}_n \).

**Proposition 3.9.** The subcomplex \( V_1^1 \) contains a generating set for \( \pi_1(C(\rho)) \).

**Proof.** First, each of \( C_1 \) and \( C_2 \) is a single point. In rank 3, \( V_1^3 = V_1 \). Thus, in all of these cases, the proposition is vacuously true. For the remainder, assume \( n \geq 4 \).

As per Proposition 3.8, \( V_1 \) can be thought of as the 1-skeleton of the cell complex \( C(\rho) \). This subcomplex has 1 vertex (the rose \( \rho \)) and an edge for each combinatorial type of labelled graph with 2 vertices. We now need to show that any such standard loop \( \alpha \) in \( V_1 \) can be written in \( \pi_1(C(\rho)) \) as a product of loops in \( V_1^1 \). Our strategy is to show that \( V_2 \) is a union of 2-tori and that \( \pi_1(V_1^1) \) surjects onto \( \pi_1(V_2) \).

If \((\Gamma', g')\) is a point of \( V_2 - V_1 \), then there are two possibilities: the three vertices of \( \Gamma' \) either lie on the same circle or they do not; see Figure 4. If they do all lie on some “central circle”, then we obtain a 2-torus by fixing one vertex and letting the other two vertices “move around” the central circle (really we are changing lengths so as to give the appearance of this motion). In the other case, there are two central circles. By fixing the middle intersection point and letting the other two intersection points move around the respective circles, we again see a torus.

Consider a standard loop \( \alpha \) of \( V_1 \). At an interior point of \( \alpha \), there is a central circle with two vertices, and the two vertices have valence, say, \( p = p(\alpha) \) and \( q = q(\alpha) \). By definition of \( V_1 \), we have that \( p \) and \( q \) are even and at least 4; say \( p \leq q \). We thus have a filtration of \( V_1 \): \( \alpha \) is in \( V_1^k \) if \( (p - 2)/2 \leq k \). The number \( (p - 2)/2 \) is the number of loops glued to that vertex, other than the central circle.

Now, suppose that \( \alpha \) is a standard loop of \( V_1^k \) for some \( k \geq 2 \). At any interior point of \( \alpha \), we perform a blowup so that we end up with a graph in \( V_2 - V_1 \) of the first type (left side of Figure 4). Moreover, we choose the blowup so that (at least) one of the vertices has valence 4. The fundamental group of the corresponding torus is generated by a standard loop from \( V_1^{k-1} \) and a standard loop from \( V_1^1 \), and so \( \alpha \) can be written as a product of such loops. By induction, \( \alpha \) can be written as a product of loops from \( V_1^1 \). \qed

\[ \begin{array}{c}
\text{Figure 4. Two types of graphs in } V_2. \\
\end{array} \]
Remark. For completeness, we mention that the entire space $C_n$ can be thought of as a union of $(n - 2)$-tori, and the intersection between any two of these tori is a lower dimensional torus which is a product of diagonals of coordinate subtori. It is straightforward to prove this, given what we have already done. However, we will not need this fact.

4. Cohomological Dimension

We now give the argument for the first part of the main theorem, that $Y_n$ is homotopically $(2n - 4)$-dimensional. The basic strategy is to put an ordering on the stars of roses of $Y_n$ (we think of the ordering as a Morse function) and then to glue the stars of roses together in the prescribed order. This is in the same spirit as the proof of Culler–Vogtmann that $X_n$ is contractible.

4.1. Morse function. The ordering on roses will come from an ordering on matrices. We start with vectors. By the norm of an element $v = (a_1, \cdots, a_n)$ of $\mathbb{Z}^n$, we mean:

$$|v| = (|a_1|, \cdots, |a_n|) \in \mathbb{Z}_{+}^n$$

where the elements of $\mathbb{Z}_{+}^n$ are ordered lexicographically. Consider the matrix:

$$M = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

The norm of $M$ is:

$$|M| = (|v_n|, \ldots, |v_1|) \in (\mathbb{Z}_{+}^n)^n$$

where $(\mathbb{Z}_{+}^n)^n$ has the lexicographic ordering on the $n$ factors. We say that $M$ is a standard representative for an element of $W \setminus GL_n(\mathbb{Z})$ if $|v_n| < \cdots < |v_1|$ (i.e. if it is a representative with smallest norm). Note that two rows of a matrix in $GL_n(\mathbb{Z})$ cannot have the same norm, for otherwise these two rows would be equal after reducing modulo 2, and the resulting matrix would not be invertible.

We declare the norm of an element of $W \setminus GL_n(\mathbb{Z})$ to be the norm of a standard representative, and the norm of a rose in $Y_n$ to be the norm of the corresponding element of $W \setminus GL_n(\mathbb{Z})$.

In what follows, the following fact will be important:

**Lemma 4.1.** If the stars of two roses intersect, then the roses have different norms.

**Proof.** If $M$ and $M'$ are marking matrices for neighboring roses, then $M' = NM$, where each entry of $N$ is either $-1$, $0$, or $+1$ (apply Proposition 3.1(4)). Then, if $|M| = |M'|$, it follows that $N$ is the identity modulo 2, and so $N \in W$. \qed

The norm on roses turns the set of roses into a well-ordered set. We use this fact without mention in the transfinite induction arguments for Theorem 4.3 and Proposition 5.4.
4.2. The induction. We define an initial segment of \( Y_n \) to be a union of stars of a set of roses that is closed under taking smaller roses (i.e., a sublevel set of the “Morse function” given on stars of roses). Note that, in general, an initial segment consists of infinitely many roses. If we show that each initial segment is \((2n-4)\)-dimensional, it will follow by transfinite induction that \( Y_n \) has the same property.

To this end, we define the descending link of a rose in \( Y_n \) to be the intersection of its star with the union of all stars of roses of strictly smaller norm (by Lemma 4.1, we need not worry about roses of equal norm). The descending link of a rose \( \rho \), denoted \( \text{Lk}_< (\rho) \), is a subset of the frontier of its star. We will prove the following in Section 6:

**Proposition 4.2.** For \( n \geq 3 \), descending links are homotopically \((2n-5)\)-dimensional.

Given this, we can prove the first part of the main theorem:

**Theorem 4.3.** For \( n \geq 3 \), the complex \( Y_n \) is homotopy equivalent to a complex of dimension at most \( 2n - 4 \).

**Proof.** We proceed by transfinite induction on initial segments. The base step is Corollary 3.7.

Whenever we glue the star of a rose \( \text{St}(\rho) \) to an initial segment \( S \) in order to make a new initial segment \( \tilde{S} \), we can think of this as a diagram of spaces:

\[
S \leftarrow \text{Lk}_< (\rho) \rightarrow \text{St}(\rho)
\]

By the inductive hypothesis, \( S \) is homotopy equivalent to a \((2n-4)\)-dimensional space \( S' \). Denote by \( \text{St}(\rho)' \) the \((n-2)\)-complex homotopy equivalent to \( \text{St}(\rho) \) given by Proposition 3.7. By Proposition 4.2, the descending link \( \text{Lk}_< (\rho) \) is homotopy equivalent to a \((2n-5)\)-dimensional space \( \text{Lk}_< (\rho)' \). We choose maps \( \text{Lk}_< (\rho)' \rightarrow S' \) and \( \text{Lk}_< (\rho)' \rightarrow \text{St}(\rho)' \) so that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\text{St}(\rho)' & \rightarrow & \text{St}(\rho) \\
\uparrow & & \uparrow \\
\text{Lk}_< (\rho)' & \leftarrow & \text{Lk}_< (\rho) \\
\downarrow & & \downarrow \\
S' & \leftarrow & S
\end{array}
\]

It follows that the colimit of the diagram of spaces in the left column is homotopy equivalent to the colimit of the diagram of spaces in the right column (see e.g. [5, Proposition 4G.1]). The former, call it \( S'' \), is \((2n-4)\)-dimensional (consider the double mapping cylinder), and the latter is \( \tilde{S} \). By construction, the homotopy equivalence \( \tilde{S} \rightarrow \tilde{S} \) extends the homotopy equivalence \( S' \rightarrow S \).
By transfinite induction, we thus build a homotopy model $Z_n$ for $\mathcal{Y}_n$. By the inductive construction given above, $Z_n$ has a filtration by subcomplexes $\{Z^\alpha_n\}$, each equipped with a homotopy equivalence $h_\alpha : Z^\alpha_n \to \mathcal{Y}^\alpha_n$ for some some initial segment $\mathcal{Y}^\alpha_n$. What is more, the induced map $h : Z_n \to \mathcal{Y}_n$, when restricted to $Z^\alpha_n$, is precisely $h_\alpha$. It follows that $h$ is a homotopy equivalence (see, e.g., the discussion following [5, Proposition 4G.1]). Since $Z_n$ has dimension at most $2n - 4$ (by construction), we are done. □

Remark. If one wants to avoid transfinite induction, it is possible to alter the Morse function so that it is the same locally (i.e. Proposition 4.2 and its proof do not change) but the image of the Morse function is order isomorphic to the positive integers.

5. Finite Generation

In this section, we recall the definition of the Magnus generating set for $\mathcal{T}_n$, and explain how our point of view recovers the result that these elements do indeed generate $\mathcal{T}_n$ (Theorem 5.6 below).

Throughout the section (and the appendix), we denote an element $\phi$ of $\text{Out}(F_n)$ by:

$$[\Phi(x_1), \ldots, \Phi(x_n)]$$

where $x_1, \ldots, x_n$ are the generators of $F_n$, and $\Phi$ is a representative automorphism for $\phi$.

5.1. Magnus generators. Magnus proved that $\mathcal{T}_n$ is generated by:

$$K_{ik} = [x_1, \ldots, x_kx_ix_k^{-1}, \ldots, x_n]$$
$$K_{ikl} = [x_1, \ldots, x_lx_kx_l^{-1}, \ldots, x_n]$$

for distinct $i$, $k$, and $l$.

We can see the $K_{ik}$ as loops in the star of a rose in $\mathcal{Y}_n$. Consider the picture in Figure 1. As mentioned in Section 2.2, shrinking either of the parallel edges gives a path leading to the rose with the identity marking, and so this is a loop in the star of that rose in $\mathcal{Y}_n$. By considering what is happening on the level of homotopy (as opposed to homology), we see that this loop is exactly $K_{21}$ (see [4]). By attaching more loops at one of the vertices, and renaming the edges, we see that we can obtain any $K_{ik}$ in the star of the identity rose. In the stars of other roses, the analogously defined loops are conjugates of the $K_{ik}$. What is more, we have:

**Proposition 5.1.** The fundamental group of the star of the rose with the identity marking is generated by the $K_{ik}$.

The proposition follows immediately from the fact that the loops in the above discussion corresponding to the $K_{ik}$ are exactly the standard generators for $\pi_1(V_1)$ from Proposition 3.9.
5.2. **Proof of finite generation.** Our proof that the Magnus generators generate \( \pi_1(Y_n) \cong T_n \) rests on the following two topological facts about descending links which we prove in Section 6:

**Proposition 5.2.** Descending links are nonempty, except for that of the rose with the identity marking.

**Proposition 5.3.** Descending links are connected.

Combining Propositions 5.1, 5.2, and 5.3 with Van Kampen’s theorem and the transitivity of the action of \( \text{Out}(F_n) \) on stars of roses, we see that the fundamental group of any initial segment of \( Y_n \) is normally generated by the \( K_{ik} \). By transfinite induction, we have:

**Proposition 5.4.** \( T_n \) is normally generated by the \( K_{ik} \).

The group generated by the \( K_{ik} \) is not normal in \( \text{Out}(F_n) \), as any element of this subgroup is of the form:

\[
[g_1x_1g_1^{-1}, g_2x_2g_2^{-1}, \ldots, g_nx_ng_n^{-1}]
\]

Thus, to find a generating set for \( T_n \), we need to add more elements.

We have the following result of Magnus:

**Proposition 5.5.** For any \( n \), the group generated by

\[
\{K_{ik}, K_{ikt} : i \neq k < l \neq i\}
\]

is normal in \( \text{Out}(F_n) \).

It is now easy to prove the following, which is the third part of our main theorem:

**Theorem 5.6.** \( T_n \) is finitely generated. In particular, it is generated by \( \{K_{ik}, K_{ikt}\} \).

Proposition 5.5 is also one of the steps in Magnus’s proof that the \( K_{ik} \) and \( K_{ikt} \) generate \( K_n \) [8]. For completeness, we give Magnus’s proof of Proposition 5.5 in the appendix.

5.3. **Proof that \( T_2 \) is trivial.** Since there are two ways to blow up a rank 2 rose, it follows that the star of a rose in \( Y_2 \) is homeomorphic to an interval and that the frontier is homeomorphic to \( S^0 \). If we glue the stars of roses together inductively according to our Morse function as in Section 4, then at each stage we are gluing a contractible space (the star of the new rose) to a contractible space (the previous initial segment is contractible by induction) along a contractible space (Propositions 5.2 and 5.3 and the fact that the frontier is \( S^0 \)). It follows that each initial segment, and hence all of \( Y_2 \), is contractible; hence, \( T_2 = 1 \).

It is more illuminating to draw a diagram of \( X_2 = Y_2 \). It is a tree, with edges representing stars of roses. This tree is naturally dual to the classical Farey graph, with the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) corresponding to the unordered pair \( \left\{ \frac{b}{a}, \frac{d}{c} \right\} \). See Figure 5.
6. Descending links

Recall that the descending link \( \text{Lk}_\prec(\rho) \) of a rose \( \rho \) is the intersection of its star with the union of stars of roses of strictly smaller norm. The goal of this section is to prove Propositions 5.2, 5.3, and 4.2, that descending links are nonempty, connected, and homotopically \((2n - 5)\)-dimensional.

As in Section 3, let \( \rho \) be a rose represented by a marked graph \((\Gamma, g)\) whose edges \( a_i \) are labelled \( v_i \). We assume the \( a_i \) are ordered so that the marking matrix

\[
M = \begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix}
\]

is a standard representative.

6.1. Descending ideal edges. An ideal edge for \( \rho \) is called descending if any of the corresponding 1-edge blowups (Lemma 3.4) lies in \( \text{Lk}_\prec(\rho) \). Every edge of a marked graph in \( \text{St}(\rho) \) which is not labelled \( \pm v_i \) corresponds to some ideal edge; if the corresponding ideal edge is descending, we may say that the edge is descending.

We now give a criterion for checking whether or not a particular ideal edge is descending.

**Lemma 6.1.** Let \( \iota = a_1 + a_{i_1} + \cdots + a_{i_m} \) be an ideal edge. The following are equivalent:

1. \( \iota \) is descending
2. any of the corresponding 1-edge blowups lies in \( \text{Lk}_\prec(\rho) \)
3. all of the corresponding 1-edge blowups lie in \( \text{Lk}_\prec(\rho) \)
4. \( |v_1 + v_{i_1} + \cdots + v_{i_m}| < |v_1| \)

Similarly, \( \bar{\iota} = -a_1 + a_{i_1} + \cdots + a_{i_m} \) is descending if and only if \( |v_1 - (v_{i_1} + \cdots + v_{i_m})| < |v_1| \).
Proof. A 1-edge blowup which realizes the ideal edge \( \iota \) lies in \( m + 1 \) stars of roses (Proposition 3.2). Namely, for each of \( v_1, v_{i_1}, \ldots, v_{i_m} \), we get a new marking matrix by replacing that vector with:

\[
v_1 + v_{i_1} + \cdots + v_{i_m}
\]

and leaving all other row vectors the same. To see if \( \iota \) is descending, we look at the smallest of these matrices. We claim that the smallest is:

\[
N = \begin{pmatrix}
    v_1 + v_{i_1} + \cdots + v_{i_m} \\
v_2 \\
    \vdots \\
v_n
\end{pmatrix}
\]

Indeed, suppose we had replaced some other row vector, say \( v_i \), with \( v_1 + v_{i_1} + \cdots + v_{i_m} \), obtaining a matrix \( N' \). Now, forgetting the order of the rows, \( N \) and \( N' \) share \( n - 1 \) rows, and \( N \) has the row vector \( v_i \) whereas \( N' \) has the row vector \( v_1 \). By the assumption that \( M \) is a standard representative, we have \( |v_i| < |v_1| \). Now, if we put \( |N'| \) in standard form, it is easy to find a representative for the \( N \)-coset with smaller norm than the standard representative for \( N' \)—simply replace the row of \( N' \) consisting of \( v_1 \) with the vector \( v_i \). The norm of \( N \) is less than or equal to the norm of this representative, so the claim is proven.

Now both directions are easy: if \( |v_1 + v_{i_1} + \cdots + v_{i_m}| < |v_1| \) then \( |N| \) is obviously strictly less than \( |M| \) (the given representative has smaller norm) and so \( \iota \) is descending; conversely, if \( |v_1 + v_{i_1} + \cdots + v_{i_m}| \geq |v_1| \), then the given representative is in standard form and obviously has norm at least \( |M| \). (We remark that the last inequality must be strict by Lemma 4.1.)

The second statement follows by symmetry. \( \square \)

It is not hard to prove a stronger statement than the one given here. However, the relatively simple result given suffices for our purposes, and the generalities are notionally unpleasant.

Corollary 6.2. A marked graph in \( \text{St}(\rho) \) is in \( \text{Lk}_{\prec}(\rho) \) if and only if it realizes a descending ideal edge.

As a consequence of Lemma 6.1, we see that there exist pairs of marked graph which can never be simultaneously descending.

Lemma 6.3. If the ideal edge \( \iota = a_1 + a_{i_1} + \cdots + a_{i_m} \) is descending then \( \bar{\iota} = -a_1 + a_{i_1} + \cdots + a_{i_m} \) is not descending.

Generalizations of Lemma 6.1 lead to analogous generalizations of the current lemma.

Proof. To simplify notation, let \( w_0 = v_1 \), \( w_1 = v_{i_1} \), \( w_2 = v_{i_2} \), etc. We will denote particular entries in each of these row vectors by using double indices; i.e., \( w_{jk} \) is the \( k^{th} \) entry of the row vector \( w_j \).
Let \( k \) be the smallest number so that
\[
|w_{0k} + w_{1k} + w_{2k} + \cdots + w_{mk}| \neq |w_{0k}|
\]
Note that there is such a \( k \), for otherwise, the original matrix \( M \) would not be invertible (reduce modulo 2).

Applying Lemma 6.1, we see that \( \iota \) is descending if and only if
\[
|w_{0k} + w_{1k} + w_{2k} + \cdots + w_{mk}| < |w_{0k}|
\]
(we are using the minimality of \( k \)). It follows that \( w_{1k} + w_{2k} + \cdots + w_{mk} \neq 0 \) and that the sign of this sum differs from that of \( w_{0k} \). Thus, we have:
\[
|w_{0k} - (w_{1k} + w_{2k} + \cdots + w_{mk})| > |w_{0k}|
\]
and so \( \iota \) is not descending. By symmetry, we are done. \( \square \)

6.2. Proof of Propositions 5.2 and 5.3. As usual, let \( \rho \) be a rose represented by a marked graph \((\Gamma, g)\) with edges \( a_1, \ldots, a_n \) labelled by \( v_1, \ldots, v_n \), and assume that the edges are ordered so that the marking matrix
\[
M = \begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix}
\]
is a standard representative.

We first give the proof that descending links are nonempty:

Proof of Proposition 5.2. Let \( k \) be the first column of \( M \) which is not a coordinate vector (since \( M \) is a standard representative, it follows that the entries in the first \( k - 1 \) column vectors agree with the identity matrix up to sign). If we denote the \( j^{th} \) entry of \( v_i \) by \( v_{ij} \), then \( v_{kk} \) is nonzero. This follows from the fact that \( M \) is a standard representative and the fact that \( M \) is invertible.

Since the \( k^{th} \) column is not a coordinate vector (and since \( M \) is invertible), there is a \( j \), different from \( k \), so that \( v_{jk} \) is nonzero. If there is a \( j > k \) such that \( v_{jk} \neq 0 \), then, since \( M \) is a standard representative, \( |v_{jk}| \leq |v_{kk}| \), and \( a_k + \epsilon_j a_j \) is a descending ideal edge for some \( \epsilon_j = \pm 1 \). If \( v_{jk} = 0 \) for all \( j > k \), it follows that \( v_{kk} = \pm 1 \) (since \( M \) is invertible) and there is some \( j < k \) so that \( v_{jk} \neq 0 \). But then, again, \( a_j + \epsilon a_k \) is descending for some \( \epsilon = \pm 1 \).

Here is the proof that descending links are connected:

Proof of Proposition 5.3. We first claim that if \( \iota \) is any descending ideal edge, then there is a subordinate 2-letter ideal edge \( \iota' \) which is also descending; see Section 3.2 for definitions. It will then follow from Lemma 3.5 and Corollary 6.2 that there is a path in \( \text{Lk}_<(\rho) \) between the 1-edge blowup realizing \( \iota \) to the 1-edge blowup realizing \( \iota' \) (the graph simultaneously realizing \( \iota \) and \( \iota' \) is the midpoint of the path).

To prove the claim, we need some notation. First, recall the notations \( \rho, a_i, v_i, \) and \( M \) from above. Also, say (without loss of generality) that \( \iota = a_{i_1} + a_{i_2} + \cdots + a_{i_m} \),
and denote \( v_{ij} \) by \( w_{ij} \). Starting with the matrix with the \( w_{ij} \) as rows, we obtain a matrix \( M' \) by deleting all columns without a nonzero entry. The \( ij^{th} \) entry of \( M' \) is denoted \( w_{ij} \).

We proceed in two cases. If the first column of \( M' \) is not a coordinate vector, then at least two of the \( w_{i1} \) are nonzero, in particular, \( w_{11} \neq 0 \). Without loss of generality, say \( w_{11} > 0 \). Since \( \iota \) is descending, there must be a \( k \) so that \( w_{k1} < 0 \), and since \( M \) is a standard representative, we have \( |w_{k1}| \leq |w_{11}| \). It follows that \( a_{i1} + a_{ik} \) is descending, and this completes the proof of the first case.

If the first column of \( M' \) is a coordinate vector (i.e. \( w_{11} = \pm 1 \) and \( w_{k1} = 0 \) for \( k > 1 \)), then we look at the second column of \( M' \). Without loss of generality, assume \( w_{22} > 0 \). At this point there are three subcases. If \( w_{k2} = 0 \) for all \( k > 2 \), then \( a_{i1} + a_{i2} \) is descending, since \( \iota \) is descending. If there is a \( k > 2 \) so that \( w_{k2} < 0 \) then \( a_{i2} + a_{ik} \) is descending (since \( M \) is a standard representative). If \( w_{k2} \geq 0 \) for all \( k > 2 \) and \( w_{k2} \neq 0 \) for at least one \( k > 2 \), then, since \( \iota \) is descending, it follows that \( w_{12} < 0 \) and so \( a_{i1} + a_{ik} \) is descending for any \( k > 2 \) with \( w_{k} > 0 \).

We now claim that given any two descending 2-letter ideal edges, there is a path between the corresponding points in \( \text{Lk}<^1(\rho) \). This follows, as above, from Lemma 3.5 and Corollary 6.2, in addition to the fact that opposite 2-letter ideal edges cannot both be descending (Lemma 6.3). This completes the proof. □

6.3. Completely descending link. We now shift our attention to Proposition 4.2. Let \( \rho \) be a rose represented by a marked graph \( (\Gamma, g) \), and say that \( \Gamma \) has edges \( a_1, \ldots, a_n \) labelled by \( v_1, \ldots, v_n \).

The main argument for the proof (Section 6.4 below) is purely combinatorial, referring only to isomorphism types of labelled graphs. As things stand, however, we cannot describe \( \text{Lk}_<(\rho) \) in terms of combinatorial graphs without metrics. Indeed, given a marked graph in \( \text{Lk}_<(\rho) \), if we shrink the descending edges to have length less than 1 (while staying in the frontier by enlarging a nondescending edge), then the resulting marked graph is not in \( \text{Lk}_<(\rho) \) (Corollary 6.2).

To remedy this problem we perform a deformation retraction of \( \text{Lk}_<(\rho) \) onto the completely descending link, which we define to be the subset of \( \text{Lk}_<(\rho) \) consisting of marked graphs where each edge not labelled \( \pm v_i \) is descending. The deformation retraction is achieved by simply shrinking all edges which correspond to nondescending ideal edges. Recall that these edges form a forest (Proposition 3.1(2)), so there is no obstruction. We denote the completely descending link of \( \rho \) by \( \text{Lk}_{<}^{\rho} \).

**Lemma 6.4.** For any given rose \( \rho \), the completely descending link \( \text{Lk}_{<}^{\rho} \) is a strong deformation retract of the descending link \( \text{Lk}_<(\rho) \). In particular, the two are homotopy equivalent.

We see that \( \text{Lk}_{<}^{\rho} \) has the desired cell structure: a cell is given by a combinatorial type of labelled graph and the cells are parameterized by the lengths of the edges in the graph. To be more precise, let \( (\Gamma', g') \) be a marked graph in \( \text{Lk}_{<}^{\rho} \), and for each \( i \), let \( k_i \) be the number of edges of \( \Gamma' \) labelled \( \pm v_i \). For each \( i \) we thus get a
(\kappa_i - 1)-simplex by projecting
\{ (t_1, \ldots, t_{\kappa_i}) \in [0,1]^{\kappa_i} : t_j = 1 \text{ for some } j \}
to the simplex \( \Delta_i = \{ \sum t_i = 1 \} \). This projection is a homeomorphism. For each edge not labelled \( \pm v_i \), we allow its length to vary arbitrarily within \([0,1]\), as long as one such edge has length 1. If \( k_0 \) is the number of such edges, then, as above, we get a \((k_0 - 1)\)-simplex \( \Delta_0 \). Thus, the cell corresponding to \((\Gamma', g')\) has a cell structure given by the product:
\[ \Delta_0 \times \cdots \times \Delta_n \]
We now summarize some of the important features of this cell structure:

**Proposition 6.5.** Consider a cell \( C \) of \( \text{Lk}_{\lessdot}(\rho) \) as above.

1. Passing to faces of \( C \) corresponds to collapsing forests in \( \Gamma' \).
2. \( C \) is top-dimensional if and only if all vertices of \( \Gamma' \) have valence 3.
3. If \( \Gamma' \) has \( v \) vertices, then \( C \) has dimension \( v - 2 \).

6.4. **Proof of Proposition 4.2.** In this section we show that the completely descending link for any rose is homotopy equivalent to a complex of dimension \( 2n - 5 \) (Proposition 6.6). Since the completely descending link is a deformation retract of the descending link (Lemma 6.4), Proposition 4.2 follows as a corollary.

As usual, let \( \rho = (\Gamma, g) \) be a rose in \( \mathcal{Y}_n \), with edges \( a_1, \ldots, a_n \) labelled by \( v_1, \ldots, v_n \). If \((\Gamma', g')\) is any marked graph in \( \text{St}(\rho) \), we define the \( v_1 \)-loop as the image of \( a_i \) under a homotopy inverse of the collapsing map \( \Gamma' \to \Gamma \).

**Proposition 6.6.** Let \( n \geq 3 \). For any rose \( \rho \) in \( \mathcal{Y}_n \), there is a strong deformation retraction of \( \text{Lk}_{\lessdot}(\rho) \) onto a complex of dimension \( 2n - 5 \).

**Proof.** If any top-dimensional cell of \( \text{Lk}_{\lessdot}(\rho) \) has a free face in \( \text{Lk}_{\lessdot}(\rho) \), then there is a homotopy equivalence (deformation retraction) of \( \text{Lk}_{\lessdot}(\rho) \) which collapses away this cell. We perform this process inductively until we arrive at a subcomplex \( L \) where no top-dimensional cell has a free face.

We now suppose that \( L \) is \((2n - 4)\)-dimensional, i.e., it has at least one top-dimensional cell. Among these, choose a cell \( C \) where the total number of edges \( \ell \) of a \( v_1 \)-loop is minimal. Call the loop \( P \) and choose one of its edges labelled \( \pm v_1 \) and call it \( e \); see the leftmost diagram in Figure 6. Say that \( C \) is given by a marked graph \((\Gamma', g')\).

Firstly, note that \( \ell \) is not 1, since there are no graphs with separating edges in \( \mathcal{Y}_n \).

If we collapse any edge of \( P - e \) (middle of Figure 6), we move to a codimension 1 face of \( C \). There are two ways to move to a new top-dimensional cell, since there are two other blowups of the resulting valence 4 vertex. One way reduces the length of \( P \) (top right of Figure 6), so by the minimality assumption for \( C \), this is not a cell of \( L \). Since we are assuming \( C \) does not have any free faces, the other top-dimensional cell (bottom right of Figure 6), call it \( C' \), must be in \( L \). The marked graphs in \( C \) and \( C' \) have the same labels outside of \( P \); the difference is that the order of the edges leaving \( P \) has changed (Proposition 2.2(1) is applied twice).
Continuing in this way, we see that if we permute the edges leaving $P$ in any way, we arrive at cells which are necessarily part of $L$. In particular, the graph obtained by taking the edge which leaves $P$ at one endpoint of $e$ and moving it to the other endpoint of $e$ gives a descending cell $\bar{C}$.

We now argue that $C$ and $\bar{C}$ are opposite in the sense of Lemma 6.3. Consider either endpoint of $e$ in $\Gamma'$. This is a valence 3 vertex, as shown in Figure 7. By Proposition 2.2(3) and Proposition 3.1(4), the labels must be as in the left hand side of the figure. When we move the edge labelled $\sum k_i v_i$ to the other end of $e$ (as above), the labels must be as shown in the right hand side of the figure; the key point is that the labels and orientations do not change for $e$ and the edge being moved. It is then possible for us to determine the label for the third edge leaving the vertex where these edges meet. By Lemma 6.3 and Corollary 6.2, we have a contradiction. $\Box$

In this section we prove the second part of the main theorem, that $H_{2n-4}(\mathcal{T}_n, \mathbb{Z})$ is not finitely generated when $n \geq 3$. In order to do this, we define a subcomplex $\mathcal{M}_n$ of $\mathcal{Y}_n$, called the “toy model”, we find an explicit infinite basis for
7.1. **Description of the toy model.** Let \( \rho = (\Gamma, g) \) be the rose in \( \mathcal{Y}_n \) with the identity marking, let \( x_i \) denote the edges of the standard rose \( R_n \), and let \( a_i \) denote the corresponding edges of \( \Gamma \).

Consider the set of points \( \mathcal{M}_n \) to be the subset \( \mathcal{M}_n \) of \( \mathcal{Y}_n \) given by:

\[
\mathcal{M}_n = \bigcup_{p, q \in \mathbb{Z}} \mathcal{M}_n^0 \end{equation}
\]

\[
\begin{bmatrix}
1 & p_3 & p_4 & \cdots & p_n \\
0 & 1 & q_3 & q_4 & \cdots & q_n \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 1 
\end{bmatrix}
\]

**Another point of view.** We now give a different description of \( \mathcal{M}_n \), which will make it easier to find its homotopy type.

For each marked graph \( (\Gamma', g') \) of \( \mathcal{M}_n \), the union \( g'(x_1) \cup g'(x_2) \) is a rank 2 rose in \( \Gamma' \), and \( \Gamma' \) has \( n - 2 \) edges \( a_3, \ldots, a_n \) labelled \( v_3, \ldots, v_n \), where \( v_i \) is a coordinate vector with \(+1\) in the \( i^{th} \) spot. By considering the starting and ending points of \( a_3, \ldots, a_n \) as points in \( g'(x_1) \cup g'(x_2) \), a path in \( \mathcal{M}_n \) can be thought of as a path in the configuration space of \( n - 2 \) pairs of points in the universal abelian cover of \( g'(x_1) \cup g'(x_2) \), which is \( U = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \). To make this precise, for each metric graph \( (\Gamma', g') \) in \( \mathcal{M}_n \), we rescale the metric so that \( g'(x_1) \) and \( g'(x_2) \) both have length 1. After doing this, the endpoints of the \( a_i \) give a well-defined subset of the metric cover \( U \).

If, in the configuration space, we move the two points corresponding to the endpoints of some \( a_i \) by the same integral vector, then the corresponding point in \( \mathcal{M}_n \) does not change.

**Proposition 7.1.** The above construction defines a homeomorphism:

\[
(U^2)^{n-2}/(\mathbb{Z}^2)^{n-2} \rightarrow \mathcal{M}_n
\]

At this point, the proof is straightforward and is left to the reader.

A typical graph in \( \mathcal{M}_7 \) is shown in Figure 8. That graph is “maximally blown up” in the sense that it has the greatest number of valence 3 vertices possible in \( \mathcal{M}_7 \).

7.2. **Homotopy type of the toy model.** We start by focusing our attention on the rank 3 toy model \( \mathcal{M}_3 \). In general, we have \( \mathcal{M}_n \cong (\mathcal{M}_3)^{n-2} \), and so we will be able to deduce the finiteness properties of \( \mathcal{M}_n \) from those of \( \mathcal{M}_3 \).

Via Proposition 7.1, we can think of \( \mathcal{M}_3 \) as pairs of points in \( U \). However, it will simplify our analysis if we thicken \( U \) to a space \( V \), which we now define. First, for any integers \( p \) and \( q \), denote by \( D_{p,q} \) the open disk of radius \( r \) around \( (p, q) \in \mathbb{R}^2 \), for some fixed \( r \) close to zero. Then, define \( V = \mathbb{R}^2 - \cup D_{p,q} \).
The straight line retraction of $V$ onto $U$ gives a homotopy equivalence from $V^2/\mathbb{Z}^2$ to $U^2/\mathbb{Z}^2 \cong \mathcal{M}_3$. Thinking of $V^2/\mathbb{Z}^2$ as pairs of points in $V$, we immediately see the following features:

1. The diagonal of $V^2/\mathbb{Z}^2$ is a torus with one boundary component.
2. For each $(p, q)$, there is a 2-torus $Z_{p,q} = \partial D_{0,0} \times \partial D_{p,q}$.

We will now use Morse theory to argue that these features capture the homotopy type of $\mathcal{M}_3$. We consider the Morse function $d : V^2/\mathbb{Z}^2 \to \mathbb{R}$ which assigns to a point in $V^2/\mathbb{Z}^2$ the Euclidean distance between the pair of points in $V^2$.

![Figure 9. Critical points for the toy model in rank 3.](image)

We see that $d$ has the following features, depicted in Figure 9:

- The minset is the torus with one boundary component corresponding to the diagonal of $V^2/\mathbb{Z}^2$. 

![Figure 8. A maximally blown up graph in $\mathcal{M}_7$.](image)
There is a horizontal 1-cell of index 1 critical points corresponding to pairs of points lying diametrically opposite from each other on \( \partial D_{0,0} \).

For every \((p, q) \neq (0, 0)\), there is an index 2 critical point, corresponding to the two points of tangency of \( \partial D_{0,0} \) and \( \partial D_{p,q} \) with the unique circle tangent to both.

At all other points, there is a well-defined gradient flow, and so there are no other critical points.

We notice that each index two critical point is the maximum point of the corresponding \( Z_{p,q} \). Thus, at each critical point of index 2, a 2-cycle is added. One can also see that, at the horizontal 1-cell, there is another torus being added. Since there are no critical points of index greater than 2, these classes are nontrivial in \( H_2(M_3, \mathbb{Z}) \).

In higher rank, we define the torus \( Z_{p,q} \) to be the \((n - 2)\)-fold product \((\partial D_{0,0} \times \partial D_{p,q})^{n-2}\). As \( M_n \cong M_3^{n-2} \), we have:

**Proposition 7.2.** The \( Z_{p,q} \) freely generate \( H_{2n-4}(M_n, \mathbb{Z}) \).

To formalize the above argument, one can use Morse theory for manifolds with corners (see [3]). There is one technicality: the critical values of \( d \) are not isolated; this is easily overcome by replacing each \( D_{p,q} \) with an ellipse (or proving a more general Morse theory).

We remark that the image of \( \pi_1(Z_{p,q}) \cong \mathbb{Z}^{2n-4} \) in \( \pi_1(Y_n) \cong T_n \) is a conjugate of the subgroup \( G \) of \( T_n \) described in the introduction. To see this, one simply needs to understand the effect of blowups and blowdowns on the homotopy classes of marked graphs; see [4].

### 7.3. Independence in homology

We now set out to prove that the \( Z_{p,q} \) represent independent classes in \( Y_n \) (Theorem 7.7). In particular, this will prove the second part of the main theorem.

By understanding the homotopy equivalences

\[
(V^2)^{n-2}/(\mathbb{Z}^2)^{n-2} \to (U^2)^{n-2}/(\mathbb{Z}^2)^{n-2} \to M_n
\]

we can give a concrete description of the \( Z_{p,q} \) in terms of marked graphs. First of all, we have:

**Lemma 7.3.** Each \( Z_{p,q} \) is contained in the union of stars of roses with marking matrices of the form:

\[
\begin{bmatrix}
1 & 0 & p_3 & p_4 & \cdots & p_n \\
0 & 1 & q_3 & q_4 & \cdots & q_n \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

where each \( p_i \in [p - 1, p + 1] \) and \( q_i \in [q - 1, q + 1] \).
The main fact we will need about the stars of roses listed in Lemma 7.3 is that any
ideal edge of the form
\[ \pm v_1 \pm v_2 + \sum_{i \geq 3} k_i v_i \]
is ascending. We will also need the following observation about the \( Z_{p,q} \):

**Lemma 7.4.** For any \( Z_{p,q} \) and any rose \( \rho \), we can give \( Z_{p,q} \) the same cell structure
as \( \text{Lk}_{\leq}(\rho) \). In particular, if a marked graph in some \( Z_{p,q} \) has an edge of length
less than 1 corresponding to an ideal edge \( \iota \), then there is a point in that \( Z_{p,q} \) which
realizes \( \iota \).

For the remainder, fix a \( Z_{p,q} \), and let \( \rho \) be a rose of greatest norm which intersects
\( Z_{p,q} \). By Lemma 7.3, if \( p \) and \( q \) are both nonzero, then \( \rho \) is unique; if one of them
is zero, then there are \( 2^{n-2} \) choices for \( \rho \); and if \( p = q = 0 \), then there are \( 2^{2n-4} \)
choices. From Lemma 7.3, we deduce the following key fact:

**Lemma 7.5.** Any map from \( \{ Z_{p,q} \} \) to roses, sending \( Z_{p,q} \) to any rose \( \rho \) of maximal
norm with \( Z_{p,q} \cap \text{St}(\rho) \neq \emptyset \), is injective. In particular, given any finite subset of
\( \{ Z_{p,q} \} \), the rose of maximal norm intersecting this set has nonempty intersection
with exactly one torus in this set.

Let \( \text{Lk}_{\leq}(\rho) \) denote the descending link for \( \rho \) as defined in Section 4.2.

**Lemma 7.6.** The intersection \( Z_{p,q} \cap \text{Lk}_{\leq}(\rho) \) is homeomorphic to \( S^{2n-5} \).

**Proof.** We assume \( p, q > 0 \), with the other cases handled similarly.

Under the homotopy equivalence \( (V^2)^n-2/((\mathbb{Z}^2)^n-2) \to (U^2)^n-2/((\mathbb{Z}^2)^n-2) \), we can
identify \( Z_{p,q} \) with the configuration space of \( n-2 \) pairs of points in \( U \subset \mathbb{R}^2 \) where
the first point \( z_i \) in each pair lies on the coordinate square with vertices at \( (0,0) \)
and \( (1,1) \) and the second point \( z_i' \) in each pair lies on the square with vertices \( (p,q) \)
and \( (p+1,q+1) \).

The rose \( \rho \) is realized when each \( z_i \) is at \( (0,0) \) and each \( z_i' \) is at the point \( (p+1,q+1) \).
The points of \( Z_{p,q} \cap \text{Lk}_{\leq}(\rho) \) are exactly the set of points where each \( z_i \) is within a
distance of 1/2 from the origin, each \( z_i' \) is within 1/2 of \( (p+1,q+1) \), and at least
one \( z_i \) or \( z_i' \) has distance exactly 1/2.

In other words, each \( z_i \) and \( z_i' \) is allowed to move within a closed interval, and such
a configuration is in \( Z_{p,q} \cap \text{Lk}_{\leq}(\rho) \) if at least one of the points is on the boundary
of its interval. Thus, \( Z_{p,q} \cap \text{Lk}_{\leq}(\rho) \) is homeomorphic to \( \partial I^{2n-4} \cong S^{2n-5} \). \( \square \)

The following completes the proof of the main theorem:

**Theorem 7.7.** Let \( n \geq 3 \). The \( Z_{p,q} \) form an infinite set of independent classes
in \( H_{2n-4}(\mathcal{Y}_n,\mathbb{Z}) \cong H_{2n-4}(\mathcal{T}_n,\mathbb{Z}) \). In other words, \( H_{2n-4}(\mathcal{M}_n,\mathbb{Z}) \) injects into
\( H_{2n-4}(\mathcal{Y}_n,\mathbb{Z}) \).

**Proof.** Given a finite subset \( A \) of \( \{ Z_{p,q} \} \), let \( Z_{p,q} \) be an element which intersects a
rose \( \rho \) of highest norm (marking matrix as in Lemma 7.3, the \( i \)th row corresponds to
We know that there is a strong deformation retraction of $\text{Lk}_{<}(\rho)$ onto a complex of dimension $2n - 5$ (Lemma 6.4 plus Proposition 6.6). The goal is to show that the image of the sphere $Z_{p,q} \cap \text{Lk}_{<}(\rho)$ is embedded in this complex. Even better, we will show that the deformation retractions of Lemma 6.4 and Proposition 6.6 do not move the points of $Z_{p,q} \cap \text{Lk}_{<}(\rho)$.

To see why this proves this proposition, we consider the long exact sequence associated to the pair $(\text{St}(\rho), \text{Lk}_{<}(\rho))$:

$$
\cdots \to H_{2n-4}(\text{St}(\rho)) \to H_{2n-4}(\text{St}(\rho), \text{Lk}_{<}(\rho)) \to H_{2n-5}(\text{Lk}_{<}(\rho)) \to H_{2n-5}(\text{St}(\rho)) \to \cdots
$$

By excision, $Z_{p,q}$ corresponds to a class in $H_{2n-4}(\text{St}(\rho), \text{Lk}_{<}(\rho))$. The image in $H_{2n-5}(\text{Lk}_{<}(\rho))$ is the class $Z_{p,q} \cap \text{Lk}_{<}(\rho)$, which is nontrivial once we show $Z_{p,q} \cap \text{Lk}_{<}(\rho)$ is embedded in the $(2n - 5)$-dimensional deformation retract of $\text{Lk}_{<}(\rho)$ (Lemma 7.6). It follows that $Z_{p,q}$ is nontrivial in $H_{2n-4}(\text{St}(\rho), \text{Lk}_{<}(\rho))$ and hence, via excision, in $H_{2n-4}(\mathcal{S}', \mathcal{S})$ where $\mathcal{S}$ is the largest initial segment not containing $\rho$ and $\mathcal{S}' = \mathcal{S} \cup \text{St}(\rho)$. By Lemma 7.5, each element of $A$ other than $Z_{p,q}$ is trivial in $H_{2n-4}(\mathcal{S}', \mathcal{S})$, and so $Z_{p,q}$ is linearly independent from these, which is what we wanted to show.

Thus, we are reduced to showing that the two deformation retractions do not move the sphere $Z_{p,q} \cap \text{Lk}_{<}(\rho)$. We handle each in turn.

For the deformation retraction of $\text{Lk}_{<}(\rho)$ onto $\text{Lk}_{<}(\rho)$, we need to show that $Z_{p,q} \cap \text{Lk}_{<}(\rho)$ is already contained in $\text{Lk}_{<}(\rho)$. Suppose that there were a point of $Z_{p,q} \cap \text{Lk}_{<}(\rho)$ which were not contained in $\text{Lk}_{<}(\rho)$. By Lemma 7.4, there is a point which realizes an ascending ideal edge (Lemma 4.1), and this implies that $Z_{p,q}$ intersects some rose of higher norm, contradicting the choice of $\rho$.

We now focus on the deformation retraction of Proposition 6.6. A marked graph representing a maximal cell of $\text{Lk}_{<}(\rho)$ must have disjoint $v_1$- and $v_2$-loops. Indeed, there are no valence 4 vertices, so the overlap would have to contain an ascending edge by Proposition 2.2(3), the statement after Lemma 7.3, and Lemma 6.1. The codimension 1 cells which get collapsed are obtained from these maximal cells by collapsing an edge of the $v_1$-loop. Thus, the corresponding graphs still have disjoint $v_1$- and $v_2$-loops. On the other hand, in any graph of $Z_{p,q}$, the $v_1$-loop and the $v_2$-loop intersect in exactly 1 point. Thus, no points of $Z_{p,q}$ are moved during this retraction, so we are done. □

Appendix: Proof of Proposition 5.5

This appendix contains Magnus’s proof of Proposition 5.5. In this section, we freely use the notation of Section 5.

Let $K$ be the subgroup of $\text{Out}(F_n)$ generated by the $K_{ik}$ and $K_{ikl}$ for distinct $i$, $k$, and $l$. We now prove Proposition 5.5, that $K$ is normal in $\text{Out}(F_n)$. 

Proof of Proposition 5.5. We choose the following generating set for $\text{Out}(F_n)$:
\[
\begin{align*}
\delta_{12} &= [x_1x_2, x_2, \ldots, x_n] \\
\Omega_1 &= [x_1^{-1}, x_2, \ldots, x_n] \\
\Pi_{i-1} &= [x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_n]
\end{align*}
\]

It suffices to show that the conjugates of the $K_{ik}$ and $K_{ikl}$ by the chosen generators of $\text{Out}(F_n)$ (and their inverses) are elements of $K$.

We have the following simplifications:

1. Operations on disjoint sets of elements commute.
2. Since $\Omega_1$ and $\Pi_{i-1}$ have order 2, we don’t need to conjugate by their inverses.
3. We don’t need to conjugate by $\delta_{12}^{-1}$ since
   \[
   (\Pi_1\Omega_1\Pi_1)\delta_{12}(\Pi_1\Omega_1\Pi_1)^{-1} = \delta_{12}^{-1}
   \]
4. Since $K_{ikl} = K_{ikl}^{-1}$, we may assume $k < l$.
5. Any outer automorphism $\psi$ of the form
   \[
   [x_1, \ldots, g'x_ig'x_{i+1}\ldots, x_n]
   \]
   where $gg'$ is an element of the commutator subgroup of the subgroup $H$ of $F_n$ generated by $\{x_k : k \neq i\}$ is an element of $K$.

To see that $\psi \in K$, first note that, by postcomposing with a product of $K_{ikl}^{-1}$, we may assume that $g' = 1$. Now, we know that the commutator subgroup of $H$ is normally generated by the $[x_k, x_l]$, where $k$ and $l$ are both different from $i$. Therefore, it suffices to handle the case of
\[
g = h[x_k, x_l]h^{-1} = [hx_kh^{-1}, hx_lh^{-1}]
\]
where $h = x_{i_1} \cdots x_{i_p}$ is an arbitrary element of $H$. It is elementary to check that
\[
\psi = P^{-1}K_{ikl}P
\]
where
\[
P = \prod_{j \neq i} K_{ji_p} \cdots \prod_{j \neq i} K_{ji_2} \prod_{j \neq i} K_{ji_1}
\]

Given these simplifications, it is straightforward to check (case by case) that the conjugates by $\Pi_{i-1}$, $\Omega_1$, and $\delta_{12}$ of each $K_{ik}$ and $K_{ikl}$ are elements of $K$. There is one exception; we give Magnus’s computation for this difficult case here:
\[
\delta_{12}K_{211}\delta_{12}^{-1} = K_{12}K_{11}^{-1}K_{11}K_{211}K_{12l}K_{12l}^{-1}K_{2l}^{-1}
\]
\qed
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