PHASE RETRIEVAL FOR SPARSE SIGNALS

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ABSTRACT. The aim of this paper is to build up the theoretical framework for the recovery of sparse signals from the magnitude of the measurement. We first investigate the minimal number of measurements for the success of the recovery of sparse signals without the phase information. We completely settle the minimality question for the real case and give a lower bound for the complex case. We then study the recovery performance of the \(\ell_1\) minimization for the sparse phase retrieval problem. In particular, we present the null space property which, to our knowledge, is the first sufficient and necessary condition for the success of \(\ell_1\) minimization for \(k\)-sparse phase retrievable.

1. INTRODUCTION

The theory of compressive sensing has generated enormous interest in recent years. The goal of compressive sensing is to recover a sparse signal from its linear measurements, where the number of measurements is much smaller than the dimension of the signal, see e.g. [4–6,10]. The aim of this paper is to study the problem of compressive sensing without the phase information. In this problem the goal is to recover a sparse signal from the magnitude of its linear samples.

Recovering a signal from the magnitude of its linear samples, commonly known as phase retrieval or phaseless reconstruction, has gained considerable attention in recent years [1,2,7,8]. It has important application in X-ray imaging, crystallography, electron microscopy, coherence theory and other applications. In many applications the signals to be reconstructed are sparse. Thus it is natural to extend compressive sensing to the phase retrieval problem.

We first introduce the notation and briefly describe the mathematical background of the problem. Let \(\mathcal{F} = \{f_1, f_2, \ldots, f_m\}\) be a set of vectors in \(\mathbb{H}^d\) where \(\mathbb{H}\) is either \(\mathbb{R}\) or \(\mathbb{C}\). Assume that \(x \in \mathbb{H}^d\) such that \(b_j = |\langle x, f_j \rangle|\). The phase retrieval problem asks whether we can reconstruct \(x\) from \(\{b_j\}_{j=1}^m\). Obviously, if \(y = cx\) where \(|c| = 1\) then \(|\langle y, f_j \rangle| = |\langle x, f_j \rangle|\). Thus the best phase retrieval can do is to reconstruct \(x\) up to a unimodular constant.

Consider the equivalence relation \(\sim\) on \(\mathbb{H}: \mathbb{H}^d: x \sim y\) if and only if there is a constant \(c \in \mathbb{H}\) with \(|c| = 1\) such that \(x = cy\). Let \(\mathbb{H} / \sim\). We shall use \(\tilde{x}\) to denote the equivalent
class containing $x$. For a given $\mathcal{F}$ in $\mathbb{H}$ define the map $M_{\mathcal{F}}: \tilde{\mathbb{H}} \rightarrow \mathbb{R}_{+}^m$ by

\begin{equation}
M_{\mathcal{F}}(\tilde{x}) = [(\langle \tilde{x}, f_1 \rangle)^2, \ldots, (\langle \tilde{x}, f_m \rangle)^2]^\top.
\end{equation}

The phase retrieval problem asks whether a $\tilde{x} \in \tilde{\mathbb{H}}$ is uniquely determined by $M_{\mathcal{F}}(\tilde{x})$, i.e. $\tilde{x}$ is recoverable from $M_{\mathcal{F}}(\tilde{x})$. We say a set of vectors $\mathcal{F}$ has the phase retrieval property, or is phase retrievable, if $M_{\mathcal{F}}$ is injective on $\tilde{\mathbb{H}} = \mathbb{H}/\sim$.

It is known that in the real case $\mathbb{H} = \mathbb{R}$ the set $\mathcal{F}$ needs to have at least $m \geq 2d - 1$ vectors to have the phase retrieval property; furthermore a generic set of $m \geq 2d - 1$ elements in $\mathbb{R}$ will have the phase retrieval property, c.f. Balan, Casazza and Edidin [1]. In the complex case $\mathbb{H} = \mathbb{C}$ the same question remains open, and is perhaps the most prominent open problem in phase retrieval. It is known that $m \geq 4d - 2$ generic vectors in $\mathbb{C}^d$ has the phase retrieval property [1]. It is also shown that there exists a set $\mathcal{F}$ with $m = 4d - 4$ vectors having the phase retrieval property, c.f. Bodmann and Hammen [3]. The current conjecture is that phase retrieval property in $\mathbb{C}^d$ can only hold when $m \geq 4d - 4$, and furthermore any $m \geq 4d - 4$ generic vectors in $\mathbb{C}^d$ have the phase retrieval property.

The aforementioned results concern with the general phase retrieval problem in $\mathbb{H}^d$. In many applications, however, the signal $x$ is often sparse with $\|x\|_0 = k \ll d$.

We use the standard notation $\mathbb{H}_k^d$ to denote the subset of $\mathbb{H}^d$ whose elements $x$ have $\|x\|_0 \leq k$. Let $\tilde{\mathbb{H}}_k$ denote $\mathbb{H}_k^d/\sim$. A set $\mathcal{F}$ of vectors in $\mathbb{H}^d$ is said to have the $k$-sparse phase retrieval property, or is $k$-sparse phase retrievable, if any $\tilde{x} \in \tilde{\mathbb{H}}_k$ is uniquely determined by $M_{\mathcal{F}}(\tilde{x})$. In other words, the map $M_{\mathcal{F}}$ is injective on $\tilde{\mathbb{H}}_k$. One may naturally ask: how many vectors does $\mathcal{F}$ need to have so that $\mathcal{F}$ is $k$-sparse phase retrievable?

The best current results on the $k$-sparse phase retrieval property are proved by Li and Voroninski [14], which state that $k$-sparse phase retrieval property can be achieved by having $m \geq 4k$ and $m \geq 8k$ vectors for the real and complex case, respectively (see also [16]).

In Section 2, we prove sharper results for a set of vectors $\mathcal{F}$ to have the $k$-sparse phase retrieval property. In the real case $\mathbb{H} = \mathbb{R}$ we obtain a sharp result. We show that for any $k < d$ the set $\mathcal{F}$ must have at least $m \geq 2k$ elements to be $k$-sparse phase retrievable. Furthermore, any $m \geq 2k$ generic vectors will be $k$-sparse phase retrievable. In the complex case $\mathbb{H} = \mathbb{C}$ we proved that any $m \geq 4k - 2$ generic vectors have the $k$-sparse phase retrieval property. We conjecture that this bound is also sharp, namely for $k < d$ a set $\mathcal{F}$ in $\mathbb{C}^d$ needs at least $4k - 2$ vectors to have the $k$-sparse phase retrieval property.

A foundation of compressive sensing is built on the fact that the recovery of a sparse signal from a system of under-determined linear equations is equivalent to finding the extremal value of $\ell_1$ minimization under certain conditions. The $\ell_1$ minimization is extended to the phase retrieval in [15] and one also develops many algorithms to compute it (see [19, 20]). However, there have been few theoretical results on the recovery performance of $\ell_1$ minimization for sparse phase retrieval. If we take $k = d$, the null space property, which, to our knowledge, is the first sufficient and necessary condition for the success of $\ell_1$ minimization for $k$-sparse phase retrievable. If we take $k = d$, the null space property is reduced to a condition of the frame $\mathcal{F}$ under which $M_{\mathcal{F}}$ is injective on $\mathbb{C}^d/\sim$ and we present it in Section 4.
2. Minimal Sample Number for $k$-Sparse Phase Retrieval

In this section we study the problem of minimal number of samples (measurements) required for $k$-sparse phase retrieval. We shall introduce more notation here. Often it is convenient to identify a set of vectors $F = \{f_1, f_2, \ldots, f_m\}$ with the matrix $F = [f_1, f_2, \ldots, f_m]$ whose columns are the vectors $f_j$. When $F$ is a frame this is known as the frame matrix of $F$. We shall use the term frame matrix for $F$ regardless whether $F$ is a frame or not. Also for integers $n \leq m$ we use the notation $[n : m]$ to denote the set $\{n, n+1, \ldots, m\}$. For $x \in \mathbb{H}^d$, we set $|x| = [|x_1|, \ldots, |x_d|]$. Similar with before, we let

$$\mathbb{R}_k^d := \{x \in \mathbb{R}^d : \|x\|_0 \leq k\}.$$  

Our first theorem completely settles the minimality question for $k$-sparse phase retrieval in the real case $\mathbb{H} = \mathbb{R}$.

**Theorem 2.1.** Let $F = \{f_1, \ldots, f_m\}$ be a set of vectors in $\mathbb{R}^d$. Assume that $F$ is $k$-sparse phase retrievable on $\mathbb{R}^d$. Then $m \geq \min\{2k, 2d - 1\}$. Furthermore, a set $F$ of $m \geq \min\{2k, 2d - 1\}$ generically chosen vectors in $\mathbb{R}^d$ is $k$-sparse phase retrievable.

**Proof.** Note that the full sparsity case $k = d$ is already known: $m \geq 2d - 1$ vectors are needed for phase retrieval and a generic set of $F$ with $m \geq 2d - 1$ vectors will have the phase retrieval property. So we will focus only on $k < d$.

We first prove that $m \geq 2k$. Assume $F$ has $m < 2k$ elements. We prove $F$ does not have the $k$-sparse phase retrieval property by constructing $x, y \in \mathbb{R}_k^d$ with $|\langle x, f_j \rangle| = |\langle y, f_j \rangle|$ but $x \neq \pm y$.

We divide $F$ into two groups: $F_1 = \{f_j : j \in [1 : k]\}$ and $F_2 = \{f_j : j \in [k + 1 : m]\}$. Let the corresponding frame matrices be $F_1$ and $F_2$, respectively. Consider the subspace

$$W = \{[x_1, x_2, \ldots, x_{k+1}, 0, \ldots, 0]^\top \in \mathbb{R}^d : x_1, \ldots, x_{k+1} \in \mathbb{R}\}.$$  

For the first group $F_1$, there exists a $u \in W \setminus \{0\}$ such that $F_1^\top u = 0$, i.e. $\langle f_j, u \rangle = 0$ for all $1 \leq j \leq k$. This is because dim$(W) = k + 1$ and there are only $k$ equations. Note also that there are at most $k - 1$ vectors in the second group $F_2$ since $m - k < 2k - k = k$. Thus the solution space

$$\{v \in W : F_2^\top v = 0\}$$

has dimension at least 2. Hence, there exist linearly independent $\alpha, \beta \in W$ so that for all $t, s \in \mathbb{R}$

$$v = t\alpha + s\beta$$

satisfies

$$F_2^\top v = 0, \quad \text{i.e.} \quad \langle f_j, v \rangle = 0 \quad \text{for} \quad j \in [k + 1 : m].$$

Write $u = [u_1, u_2, \ldots, u_d]^\top$ (where $u_j = 0$ for $j > k + 1$). Since $\alpha$ and $\beta$ are linearly independent, we may without loss of generality assume $[\alpha_1, \alpha_2]^\top$ and $[\beta_1, \beta_2]^\top$ are linearly independent, where $\alpha = [\alpha_1, \ldots, \alpha_d]^\top$ and $\beta = [\beta_1, \ldots, \beta_d]^\top$. We first consider the case
where either $u_1 \neq 0$ or $u_2 \neq 0$. Then there exist $s_0, t_0 \in \mathbb{R}$ with $(s_0, t_0) \neq (0, 0)$ so that
\[
\begin{align*}
    u_1 &= t_0 \alpha_1 + s_0 \beta_1, \\
    -u_2 &= t_0 \alpha_2 + s_0 \beta_2.
\end{align*}
\]
Now set $\bar{v} = t_0 \alpha + s_0 \beta$ and
\[
x := u + \bar{v}, \quad y := u - \bar{v}.
\]
Clearly $x, y \in \mathbb{R}_d^k$ since $\text{supp}(x) \subseteq \{1, 3, \ldots, k+1\}$ and $\text{supp}(y) \subseteq \{2, 3, \ldots, k+1\}$. Moreover
\[
\langle f_j, x \rangle = \langle f_j, u \rangle + \langle f_j, \bar{v} \rangle = \begin{cases} 
    \langle f_j, u \rangle, & j \leq k \\
    \langle f_j, \bar{v} \rangle, & j > k,
\end{cases}
\]
and similarly
\[
\langle f_j, y \rangle = \langle f_j, u \rangle - \langle f_j, \bar{v} \rangle = \begin{cases} 
    \langle f_j, u \rangle, & j \leq k \\
    -\langle f_j, \bar{v} \rangle, & j > k.
\end{cases}
\]
It follows that $|\langle f_j, x \rangle| = |\langle f_j, y \rangle|$ for all $j$ but $x \neq \pm y$. Thus $F$ does not have the $k$-sparse phase retrieval property in $\mathbb{R}_d^k$.

We next prove that a set $F$ of $m \geq 2k$ generic vectors will have the $k$-sparse phase retrieval property. Let us first fix $I, J \subseteq [1 : N]$ with $\#I = \#J = k$. The goal is to prove that if $x, y \in \mathbb{R}_d^N$ with $\text{supp}(x) \subseteq I$ and $\text{supp}(y) \subseteq J$ satisfying
\[
(2.1) \quad |\langle f_j, x \rangle|^2 = |\langle f_j, y \rangle|^2, \quad j = 1, \ldots, m,
\]
then $x = \pm y$. Equation $(2.1)$ implies that for all $j$ we have
\[
(2.2) \quad \langle f_j, x - y \rangle \cdot \langle f_j, x + y \rangle = 0.
\]
Thus either $\langle f_j, x - y \rangle = 0$ or $\langle f_j, x + y \rangle = 0$. Without loss of generality, we assume that
\[
(2.3) \quad \begin{cases} 
    \langle f_j, x - y \rangle = 0, & j \in [1 : n] \\
    \langle f_j, x + y \rangle = 0, & j \in [n + 1 : m].
\end{cases}
\]
Set
\[
L := I \cap J \quad \text{and} \quad \ell := \#L.
\]
For convenience we write
\[
\begin{align*}
x &= u_x + v_x, \quad \text{supp}(u_x) \subseteq L, \text{ supp}(v_x) \subseteq I \setminus L, \\
y &= u_y + v_y, \quad \text{supp}(u_y) \subseteq L, \text{ supp}(v_y) \subseteq J \setminus L.
\end{align*}
\]
We abuse the notation a little by viewing $v_x \in \mathbb{R}^{k-\ell}$ since it is supported on $I \setminus L$ with $\#(I \setminus L) = k - \ell$. Similarly we view $v_y \in \mathbb{R}^{k-\ell}$ and $u_x, u_y \in \mathbb{R}^\ell$. Set

$$w_- := u_x - u_y, \quad w_+ := u_x + u_y,$$

and $z := \begin{bmatrix} v_x & v_y & w_- & w_+ \end{bmatrix} \in \mathbb{R}^{2k}$.

Using the notions above, we have

$$\langle f_j, x - y \rangle = \langle f_j, v_x \rangle - \langle f_j, v_y \rangle + \langle f_j, w_- \rangle,$$

$$\langle f_j, x + y \rangle = \langle f_j, v_x \rangle + \langle f_j, v_y \rangle + \langle f_j, w_+ \rangle.$$

Set $A := F^\top$ where $F$ is the frame matrix of $F$. Combining (2.3) and (2.4) now yields

$$(2.5) \quad \begin{bmatrix} A_{[1:n], I \setminus L} & -A_{[1:n], J \setminus L} & A_{[1:n], L} & 0 \\ A_{[n+1:m], I \setminus L} & A_{[n+1:m], J \setminus L} & 0 & A_{[n+1:m], L} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ w_- \\ w_+ \end{bmatrix} = 0,$$

where for any index sets $J_1, J_2$ we use the notation $A_{J_1, J_2}$ to denote the sub-matrix of $A$ with the rows indexed in $J_1$ and columns indexed in $J_2$. To show $x = \pm y$ we only need to show that the linear equations (2.5) force $v_x = 0, v_y = 0$ and either $w_- = 0$ or $w_+ = 0$.

We first consider the case $n \geq 2k - \ell$. In this case, we consider only the first set of equations (2.5)

$$(2.6) \quad \begin{bmatrix} A_{[1:n], I \setminus L} & -A_{[1:n], J \setminus L} & A_{[1:n], L} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ w_- \end{bmatrix} = 0.$$

Note that the matrix

$$\begin{bmatrix} A_{[1:n], I \setminus L} & -A_{[1:n], J \setminus L} & A_{[1:n], L} \end{bmatrix}$$

has dimensions $n \times (2k - \ell)$. The elements are generically chosen. Thus it has full rank $2k - \ell$. It follows that (2.6) has only trivial solution $v_x = 0, v_y = 0$ and $w_- = 0$. Hence $x = y$.

We next consider the case with $m - n \geq 2k - \ell$. Here we consider the second set of equations (2.5):

$$(2.7) \quad \begin{bmatrix} A_{[n+1:m], I \setminus L} & A_{[n+1:m], J \setminus L} & A_{[n+1:m], L} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ w_+ \end{bmatrix} = 0.$$

The same argument used for the case $n \geq 2k - \ell$ now applies to yield $v_x = 0, v_y = 0$ and $w_+ = 0$. Hence in this case $x = -y$.

We finally consider the case where $n < 2k - \ell$ and $m - n < 2k - \ell$. In this case we must have

$$2k - \ell > m - n \geq 2k - n,$$

and hence $n > \ell$. Similarly, we have $\ell < 2k - n \leq m - n$. We argue that the rank of the matrix in (2.5) is $2k$ when $F^\top$ is generic. Let $B$ denote the matrix in (2.5). If rank($B$) $< 2k$
then all $2k \times 2k$ sub-matrices of $B$ have determinant 0. Note that each determinant is either identically 0 or a nontrivial polynomial of the entries of $F$. Hence if there exists a single example of a matrix $B$ with $\text{rank}(B) = 2k$ then $\text{rank}(B) = 2k$ for a generic choice of $F$. We shall construct an example of such an $F$ with $\text{rank}(B) = 2k$. Set
\[
A_{[1:n],L} = \begin{bmatrix} I_\ell & 0 \\ 0 & I_\ell \end{bmatrix}, A_{[n+1:m],L} = \begin{bmatrix} I_\ell & 0 \\ 0 & I_\ell \end{bmatrix},
\]
\[
[A_{[1:n],I\setminus L}, -A_{[1:n],J\setminus L}] = \begin{bmatrix} 0 \\ H_1 \end{bmatrix}, [A_{[n+1:m],I\setminus L}, -A_{[n+1:m],J\setminus L}] = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},
\]
where $I_\ell$ denotes the $\ell \times \ell$ identity matrix. With this choice, for almost all $H_1 \in \mathbb{R}^{(n-\ell)\times (2k-2\ell)}$, $H_2 \in \mathbb{R}^{(m-n-\ell)\times (2k-2\ell)}$ we have $\text{rank}(B) = 2k$. The solution to (2.5) is thus trivial, namely $v_x = 0$, $v_y = 0$, $w_- = 0$ and $w_+ = 0$. Thus $x = y = 0$. The theorem is now proved.

We next consider the complex case. Similar to the real case we set
\[
\mathbb{C}^d_k := \{x \in \mathbb{C}^d : \|x\|_0 \leq k\}.
\]
Then we have

**Theorem 2.2.** A set $\mathcal{F}$ of $m \geq 4k - 2$ generically chosen vectors in $\mathbb{C}^d$ is $k$-sparse phase retrievable.

**Proof.** We shall identify $\mathcal{F}$ with $F$ where $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ is the corresponding frame matrix, $F = [f_{ij}]$. Following the technique in [1] we shall view $F$ as an element in $\mathbb{R}^{2md}$. The goal here is to show that the set of matrices $F$ that are not $k$-sparse phase retrievable has local real dimension strictly smaller than $2md$ provided $m \geq 4k - 2$.

For any subset of indices $I, J \subseteq [1 : m]$ with $|I| = |J| = k$ let $G_{I,J}$ denote the set of matrices in $\mathbb{C}^{d \times m}$ with the following property: There exist $x, y \in \mathbb{C}^d$ where $\text{supp}(x) \subset I$, $\text{supp}(y) \subset J$ and $x \neq cy$ with $|c| = 1$ such that $M_F(x) = M_F(y)$, i.e. $|\langle f_j, x \rangle| = |\langle f_j, y \rangle|$ for all $j$. Now if $M_F(x) = M_F(y)$, the for any $a, \omega \in \mathbb{C}$ with $|\omega| = 1$ we also have $M_F(\omega x) = M_F(a \omega y)$. Thus for any $F \in G_{I,J}$ we may find $x, y \in \mathbb{C}^d$ with $M_F(x) = M_F(y)$ such that

- $\text{supp}(x) \subset I$, $\text{supp}(y) \subset J$.
- The first nonzero entry of $x$ is 1.
- The first nonzero entry of $y$ is real and positive.

Let $X$ denote the subset of $\mathbb{C}^d$ consisting of elements $x \in \mathbb{C}^d$ whose first nonzero entry is 1. Let $Y$ denote the subset of $\mathbb{C}^d$ consisting of elements $y \in \mathbb{C}^d$ whose first nonzero entry, if it exists, is real and positive. Note that in essence $X$ can be viewed as the projective space $\mathbb{P}^{d-1} \setminus \{0\}$ and $Y$ can be viewed as the set $\mathbb{C}^d/\sim$. Let $\mathbb{C}^d_I$ denote the set of vectors $x \in \mathbb{C}^d$ such that $\text{supp}(x) \subseteq I$. Now consider the set of 3-tuples
\[
\mathcal{A}_{I,J} := \{(F, x, y)\}
\]
with the following properties:

- $x \in X \cap \mathbb{C}^d_I$ and $y \in Y \cap \mathbb{C}^d_J$.
\[ x \neq \omega y \text{ for any } \omega \in \mathbb{C} \text{ with } |\omega| = 1. \]

\[ \mathbf{M}_F(x) = \mathbf{M}_F(y). \]

Now the projection of \( A_{I,J} \) to the first component gives the full set \( G_{I,J} \). Each \((F, x, y) \in A_{I,J}\) gives rise to the constraints \(|\langle f_j, x \rangle| = |\langle f_j, y \rangle|\) for \( j \in [1 : m] \), which lead to the set of quadratic equations in \( \text{Re}(f_{ij}), \text{Im}(f_{ij}) \) (by viewing \( x, y \) as fixed)

\[
\sum_{k=1}^{N} f_{kj} x_k^2 = \sum_{k=1}^{N} f_{kj} y_k^2, \quad j = 1, \ldots, m.
\]

Note that all equations are independent and each is non-trivial because \( x \neq y \) in \( \mathbb{C}^d/\sim \). Thus for any fixed \( x, y \) the set of such \( A = [f_{ij}] \) satisfying \( (2.8) \) is a real algebraic variety of (real) codimension \( 2md - m \). Hence, \( A_{I,J} \) has local dimension everywhere at most

\[
2md - m + \dim_{\mathbb{R}}(X \cap \mathbb{C}^d_I) + \dim_{\mathbb{R}}(Y \cap \mathbb{C}^d_J)
\]

\[
= 2md - m + 2k - 2 + 2k - 1
\]

\[
= 2md - (m - 4k + 3).
\]

It follows from \( m \geq 4k - 2 \) that \( A_{I,J} \) has local (real) dimension at most \( 2md - 1 \). Now \( G_{I,J} \) is the projection of \( A_{I,J} \) onto the first component. Thus, \( G_{I,J} \) has dimension at most \( 2md - 1 \). In other words, a generic \( F \in \mathbb{C}^{d \times m} \) is not in \( G_{I,J} \).

Finally, the set of \( F \in \mathbb{C}^{d \times m} \) not having the \( k \)-sparse phase retrieval property for \( \mathbb{C}^d \) is the union of all \( G_{I,J} \) with \( \#I = \#J = k \). It is a finite union. The theorem is now proved.

**Remark.** Although the above theorem shows that in the complex case any \( m \geq 4k - 2 \) generically chosen vectors are \( k \)-sparse phase retrievable, it is unknown whether \( 4k - 2 \) is in fact the minimal number required. We conjecture that the minimal number of vectors needed for being \( k \)-sparse phase retrievable is indeed \( 4k - 2 \). Note that it is obvious that the conjecture holds for \( k = 1 \).

### 3. Null Space Property for Sparse Phase Retrieval

In this section, we investigate the performance of \( \ell_1 \) minimization for sparse phase retrieval with extending the null space property in compressed sensing to the phase retrieval setting. We first introduce the null space property in compressed sensing, and then extend it to the phase retrieval setting on \( \mathbb{R}_k^d \) and \( \mathbb{C}_k^d \), respectively.

**3.1. Null space property.** A key concept in compressive sensing is the so-called null space property of a matrix. For a given frame \( \mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathbb{H}_d^d \), we use \( F \) to denote the frame matrix. Let \( \mathcal{N}(F) \) denote the kernel of \( F^\top \), i.e.,

\[
\mathcal{N}(F) = \{ \eta : \langle f_j, \eta \rangle = 0, j = 1, \ldots, m \}.
\]

To state conveniently, when \( F = \emptyset \), we set \( \mathcal{N}(F) := \mathbb{H}_d^d \).
Proof. First we show (B) \( \| \eta_T \|_1 < \| \eta_{T^c} \|_1 \), where \( T^c \) is the complementary index set of \( T \) and \( \eta_T \) is the restriction of \( \eta \) to \( T \).

A fundamental result in compressed sensing is that a signal \( x \in \mathbb{H}^d_k \) can be recovered via the \( \ell_1 \)-norm minimization if and only if the sensing matrix \( A \) has the null space property of order \( k \). We state it as follows (see [9, 11–13, 17]):

**Theorem 3.1.** Let \( F \) be a set of vectors in \( \mathbb{H}^d_k \) and \( F \) be the associated frame matrix. Then \( F \) satisfies the null space property of order \( k \) if and only if it has

\[
\arg\min_{x \in \mathbb{H}^d} \{ \| x \|_1 : F^\top x = F^\top x_0 \} = x_0
\]

for every \( x_0 \in \mathbb{H}^d_k \).

3.2. The null space property for the real sparse phase retrieval. Our goal here is to extend Theorem 3.1 to the phase retrieval for the real signal. For a given frame \( \mathcal{F} = \{ f_1, \ldots, f_m \} \) and a subset \( S \) of \( [1 : m] \) we shall use \( \mathcal{F}_S \) to denote the set \( \mathcal{F}_S := \{ f_j : j \in S \} \). Similarly for the frame matrix we shall use \( \mathcal{F}_S \) to denote the corresponding frame matrix of \( \mathcal{F}_S \), i.e. the matrix whose columns are the vectors of \( \mathcal{F}_S \). We first consider the real case.

**Theorem 3.2.** Let \( \mathcal{F} = \{ f_1, f_2, \ldots, f_m \} \) be a set of vectors in \( \mathbb{R}^d \) and \( F \) be the associated frame matrix. The following properties are equivalent:

(A) For any \( x_0 \in \mathbb{R}^d_k \) we have

\[
\arg\min_{x \in \mathbb{R}^d} \{ \| x \|_1 : | F^\top x | = | F^\top x_0 | \} = \{ \pm x_0 \},
\]

where \( | F^\top x | = | \langle f_1, x \rangle, \ldots, \langle f_m, x \rangle \|_1 \).

(B) For every \( S \subseteq [1 : m] \) with \( \# S \leq k \), it holds

\[
\| u + v \|_1 < \| u \|_1
\]

for all nonzero \( u \in \mathcal{N}(F_S) \) and \( v \in \mathcal{N}(F_{S^c}) \) satisfying \( \| u + v \|_0 \leq k \).

**Proof.** First we show (B) \( \Rightarrow \) (A). Let \( b = [ b_1, \ldots, b_m ]^\top := | F^\top x_0 | \) where \( x_0 \in \mathbb{R}^d_k \). For a fixed \( \epsilon \in \{ 1, -1 \}^m \) set \( b_\epsilon := [ \epsilon_1 b_1, \ldots, \epsilon_m b_m ]^\top \). We now consider the following minimization problem:

\[
\min_{x} \| x \|_1 \quad \text{s.t.} \quad F^\top x = b_\epsilon.
\]

The solution to (3.2) is denoted as \( x_\epsilon \). We claim that for any \( \epsilon \in \{ 1, -1 \}^m \) we must have

\[
\| x_\epsilon \|_1 \geq \| x_0 \|_1
\]

if \( x_\epsilon \) exists (it may not exist), and the equality holds if and only if \( x_\epsilon = \pm x_0 \).

To prove the claim let \( \epsilon^* \in \{ 1, -1 \}^m \) such that \( b_{\epsilon^*} = F^\top x_0 \). Note that property (B) implies the classical null space property of order \( k \). To see this, for any nonzero \( \eta \in \mathcal{N}(F) \)
and $T \subseteq [1 : d]$ with $\#T \leq k$, set $u := \eta$ and $v := \eta_T - \eta_{T^c}$. Let $S = [1 : m]$. Then $u \in \mathcal{N}(F_S)$ and $v \in \mathcal{N}(F_{S^c})$. The hypothesis of (B) now implies

$$2\|\eta_T\|_1 = \|u + v\|_1 < \|u - v\|_1 = 2\|\eta_{T^c}\|_1.$$ 

Consequently we must have $x_{\epsilon^*} = x_0$ by Theorem 3.1. Now for any $\epsilon \in \{-1, 1\}^m \neq \pm \epsilon^*$, if $x_{\epsilon}$ doesn’t exist then we have nothing to prove. Assume it does exist. Set $S_\epsilon := \{ j : \epsilon_j = \epsilon_j^* \}$. Then

$$\langle f_j, x_\epsilon \rangle = \begin{cases} \langle f_j, x_0 \rangle & j \in S_\epsilon, \\ -\langle f_j, x_0 \rangle & j \in S_{\epsilon^c}. \end{cases}$$

Set $u := x_0 - x_\epsilon$ and $v := x_0 + x_\epsilon$. Clearly $u \in \mathcal{N}(F_{S_\epsilon})$ and $v \in \mathcal{N}(F_{S^c_{\epsilon^*}})$. Furthermore $u + v = 2x_0 \in \mathbb{R}^d_k$. By the hypothesis of (B) we must have

$$2\|x_0\|_1 = \|u + v\|_1 < \|u - v\|_1 = 2\|x_\epsilon\|_1.$$ 

This proves (A).

Next we prove (A) $\implies$ (B). Assume (B) is false, namely, there exist nonzero $u \in \mathcal{N}(F_S)$ and $v \in \mathcal{N}(F_{S^c})$ such that $\|u + v\|_1 \geq \|u - v\|_1$ and $u + v \in \mathbb{R}^d_k$. Now set

$$x_0 := u + v \in \mathbb{R}^d_k.$$ 

Clearly,

$$|\langle f_j, x_0 \rangle| = |\langle f_j, u + v \rangle| = |\langle f_j, u - v \rangle|, \quad j = 1, \ldots, m$$

since either $\langle f_j, u \rangle = 0$ or $\langle f_j, v \rangle = 0$. In other words, $|F^\top x_0| = |F^\top (u - v)|$. Note that $u - v \neq -x_0$, for otherwise we would have $u = 0$, a contradiction. It follows from the hypothesis of (A) that we must have

$$\|x_0\|_1 = \|u + v\|_1 < \|u - v\|_1.$$ 

This is a contradiction. \hfill \blacksquare

### 3.3. The null space property for the complex sparse phase retrieval

We now consider the complex case $\mathbb{H} = \mathbb{C}$. Throughout this subsection, we say that $S = \{ S_1, \ldots, S_p \}$ is a partition of $[1 : m]$ if

$$S_j \subseteq [1 : m], \quad \bigcup_{j=1}^p S_j = [1 : m] \quad \text{and} \quad S_j \cap S_{\ell} = \emptyset \quad \text{for all} \quad j \neq \ell.$$ 

To state conveniently, we set $\mathcal{S} := \{ c \in \mathbb{C} : |c| = 1 \}$ and

$$\mathcal{S}^m := \{ (c_1, \ldots, c_m) \in \mathbb{C}^m : |c_j| = 1, j \in [1 : m] \}.$$ 

Then we have:

**Theorem 3.3.** Let $\mathcal{F} = \{ f_1, f_2, \ldots, f_m \}$ be a set of vectors in $\mathbb{C}^d$ and $F$ be the associated frame matrix. The following properties are equivalent.
For any $x_0 \in \mathbb{C}_k^d$ we have
\begin{equation}
\arg\min_{\tilde{x} \in \mathbb{C}_k^d} \{\|x\|_1 : \|F^T \tilde{x}\| = \|F^T x_0\|\} = \tilde{x}_0,
\end{equation}
where $\|F^T \tilde{x}\| = \|(f_1, x), \ldots, (f_m, x)\|^T$ and $\tilde{x}_0$ denotes the equivalent class $\{cx_0 : c \in \mathbb{S}\}$ in $\mathbb{C}^d/\sim$ containing $x_0$.

Suppose that $S_1, \ldots, S_p$ is any partition of $[1 : m]$ and that $\eta_j \in \mathcal{N}(F_{S_j}) \setminus \{0\}$ satisfy
\begin{equation}
\frac{\eta_j - \eta_j'}{c_1 - c'} \in \mathbb{C}_k^d \setminus \{0\} \quad \text{for all} \quad \ell, j \in [2 : p],
\end{equation}
for some pairwise distinct $c_1, \ldots, c_p \in \mathbb{S}$. Then
\begin{equation*}
\|\eta_j - \eta_j'\|_1 < \|c_\ell \eta_j - c_j \eta_j'\|_1,
\end{equation*}
for all $j, \ell \in [1 : p]$ with $j \neq \ell$.

**Proof.** We first show (B) $\Rightarrow$ (A). Let $b = [b_1, \ldots, b_m]^T := |F^T x_0|$ where $x_0 \in \mathbb{C}_k^d$. For a fixed $\epsilon \in \mathbb{S}^m$ set $b_\epsilon := [\epsilon_1 b_1, \ldots, \epsilon_m b_m]^T$. We now consider the following minimization problem:
\begin{equation}
\min_{x} \|x\|_1 \quad \text{s.t.} \quad F^T x = b_\epsilon.
\end{equation}
The solution to (3.5) is denoted as $x_\epsilon$. We claim that for any $\epsilon \in \mathbb{S}^m$ we must have
\begin{equation*}
\|x_\epsilon\|_1 \geq \|x_0\|_1
\end{equation*}
if $x_\epsilon$ exists (it may not exist), and the equality holds if and only if $\tilde{x}_\epsilon = \tilde{x}_0$.

To prove the claim let $\epsilon^* \in \mathbb{S}^m$ such that $b_{\epsilon^*} = F^T x_0$. A similar argument as the proof of Theorem 3.2 shows that property (B) implies the classical null space property of order $k$. Consequently we must have $\tilde{x}_{\epsilon^*} = \tilde{x}_0$ by Theorem 3.1 Now we consider an arbitrary $\epsilon \in \mathbb{S}^m$. If $\tilde{\epsilon} = \tilde{\epsilon}^*$, then $\tilde{x}_{\epsilon^*} = \tilde{x}_0$. So, we only consider the case where $\tilde{\epsilon} \neq \tilde{\epsilon}^*$. If $x_\epsilon$ does not exist then we have nothing to prove. Assume it does exist. Set $c_j' := c_j/c_j^* \beta$ and $\eta_j' := c_j' x_\epsilon \beta - x_\epsilon$ for $1 \leq j \leq m$. We can use $c_j'$ to define an equivalence relation on $[1 : m]$, namely $j \sim \ell$ if $c_j' = c_\ell'$. This equivalence relation leads to a partition $\mathcal{S} = \{S_1, \ldots, S_p\}$ of $[1 : m]$. Now we set $c_j := c_j'$ where $\ell \in S_j$. Clearly all $c_j$, $1 \leq j \leq p$, are distinct and unimodular.

Now set $\eta_j := c_j x_\epsilon \beta - x_\epsilon$. Then we have
\begin{equation*}
\eta_j \in \mathcal{N}(F_{S_j}) \setminus \{0\}, \text{ for all } j \in [1 : p]
\end{equation*}
and
\begin{equation*}
\frac{\eta_j - \eta_j'}{c_1 - c'} \in \mathbb{C}_k^d, \quad \text{for all } j, \ell \in [2 : p].
\end{equation*}
By the hypothesis of (B) we must have
\begin{equation*}
|c_j - c'| \cdot \|x_0\|_1 = \|\eta_j - \eta_j'\|_1 < \|c_\ell \eta_j - c_j \eta_j'\|_1 = |c_j - c'| \cdot \|x_\epsilon\|_1,
\end{equation*}
which implies that
\begin{equation*}
\|x_0\|_1 < \|x_\epsilon\|_1.
\end{equation*}
This proves (A).
We next prove (A) ⇒ (B). Assume (B) is false, namely, there exist nonzero \( \eta_j \in \mathcal{N}(F_{S_j}) \), \( j \in [1 : p] \) satisfying (3.4) but
\[
\|\eta_{j_0} - \eta_{\ell_0}\|_1 \geq \|c_{j_0}\eta_{j_0} - c_{\ell_0}\eta_{\ell_0}\|_1
\]
for some distinct \( j_0, \ell_0 \in [1 : p] \). Note that (3.4) implies that
\[
(3.6) \quad \frac{\eta_j - \eta_0}{c_j - c_{\ell_0}} = \frac{\eta_m - \eta_n}{c_m - c_n} \in \mathbb{C}_d \setminus \{0\},
\]
for all \( j, \ell, m, n \in [1 : p] \) with \( j \neq \ell \) and \( m \neq n \). Without loss of generality, we assume that \( j_0 = 1, \ell_0 = 2 \), i.e.,
\[
(3.7) \quad \|\eta_1 - \eta_2\|_1 \geq \|c_2\eta_1 - c_1\eta_2\|_1.
\]
Set
\[
x_0 := \eta_1 - \eta_2,
\]
and (3.6) implies that \( x_0 \in \mathbb{C}_d \setminus \{0\} \). We claim that
\[
(3.8) \quad |\langle f_j, x_0 \rangle| = |\langle f_j, \eta_1 - \eta_2 \rangle| = |\langle f_j, c_2\eta_1 - c_1\eta_2 \rangle|, \quad \text{for all } j \in [1 : p].
\]
Note that \( x_0 \) is \( k \)-sparse. Combining (3.8), (3.7) and (3.3) now yields
\[
(c - c_2)x_0 = c\eta_1 - c\eta_2 = c_2\eta_1 - c_1\eta_2
\]
for some \( c \in \mathbb{S} \). Consequently we obtain
\[
(c - c_2)\eta_1 = (c - c_1)\eta_2,
\]
which implies that
\[
(3.9) \quad \eta_2 = \frac{c - c_2}{c - c_1}\eta_1.
\]
Here, note that \( c \notin \{c_1, c_2\} \), for otherwise we will have either \( \eta_1 = 0 \) or \( \eta_2 = 0 \). Combining (3.4) and (3.9) leads to
\begin{itemize}
  \item \( \eta_1 \) is \( k \)-sparse;
  \item for all \( j \in [2 : p] \), \( \eta_j \) and \( \eta_1 \) are linear dependent and hence \( \eta_1 \in \mathcal{N}(F_{S_j}) \).
\end{itemize}
And hence we have \( F^\top \eta_1 = 0 \). By the hypothesis of (A) and \( \eta_1 \in \mathbb{C}_d \) we have \( \eta_1 = 0 \). A contradiction.

We remain to prove (3.8). First, when \( j \in S_1 \cup S_2 \), (3.8) holds, since either \( \langle f_j, \eta_1 \rangle = 0 \) or \( \langle f_j, \eta_2 \rangle = 0 \). We consider the case where \( j \in S_3 \). Set \( y_0 := \frac{\eta_3}{c_1 - c_2} \). Then (3.6) implies that
\[
\frac{\eta_1 - \eta_3}{c_1 - c_3} = \frac{\eta_2 - \eta_3}{c_2 - c_3} = y_0
\]
and hence
\[
\eta_1 = (c_1 - c_3)y_0 + \eta_3,
\]
\[
\eta_2 = (c_2 - c_3)y_0 + \eta_3.
\]
Note that \( \langle f_j, \eta_3 \rangle = 0 \) with \( j \in S_3 \). Then
\[
|\langle f_j, c_2 \eta_1 - c_1 \eta_2 \rangle| = |\langle f_j, c_2 (c_1 - c_3) y_0 - c_1 (c_2 - c_3) y_0 \rangle| = |\langle f_j, c_3 (c_1 - c_2) y_0 \rangle| = |\langle f_j, \eta_1 - \eta_2 \rangle| = |\langle f_j, x_0 \rangle|.
\]

Using a similar argument, we easily prove the claim for \( j \in S_4, \ldots, S_p \).

**Remark.** Theorem 3.2 extends results for the null space property of order \( k \) in compressive sensing to phase retrieval. It will be very interesting for constructing matrix \( A \in \mathbb{R}^{m \times d} \) with \( m \approx k \log d \) satisfying (B) in Theorem 3.2.

### 4. Null space property for general phase retrieval

Theorem 3.2 and Theorem 3.3 present the null space property for the phase retrievable on \( \mathbb{R}^d \) and \( \mathbb{C}^d \), respectively. In phase retrieval, one is also interested in the condition under which \( F \) is phase retrievable on \( \mathbb{R}^d \) or \( \mathbb{C}^d \). For the real case, such a condition is presented in [1]:

**Theorem 4.1.** ([1]) Let \( F = \{f_1, f_2, \ldots, f_m\} \) be a set of vectors in \( \mathbb{R}^d \) and \( F \) be the associated frame matrix. The following properties are equivalent:

1. \( F \) is phase retrievable on \( \mathbb{R}^d \);
2. For every subset \( S \subseteq \{1, \ldots, m\} \), either \( \{f_j\}_{j \in S} \) spans \( \mathbb{R}^d \) or \( \{f_j\}_{j \in S^c} \) spans \( \mathbb{R}^d \).

We next consider the complex case. Motivated by Theorem 3.3, we can present the null space property under which \( F \) is phase retrievable on \( \mathbb{C}^d \). It can be considered as an extension of Theorem 4.1:

**Theorem 4.2.** Let \( F = \{f_1, f_2, \ldots, f_m\} \) be a set of vectors in \( \mathbb{C}^d \) and \( F \) be the associated frame matrix. The following properties are equivalent:

1. \( F \) is phase retrievable on \( \mathbb{C}^d \);
2. Suppose that \( S_1, \ldots, S_p \) is any partition of \([1 : m]\). There exists no \( \eta_j \in \mathcal{N}(F_{S_j}) \setminus \{0\}, j = 1, \ldots, p \), such that
   \[
   \frac{\eta_1 - \eta_\ell}{c_1 - c_\ell} = \frac{\eta_1 - \eta_j}{c_1 - c_j} \neq 0 \quad \text{for all } \ell, j \in [2 : p],
   \]
   for some pairwise distinct \( c_1, \ldots, c_p \in \mathbb{S} \).

**Proof.** We first prove (A) \( \Rightarrow \) (B). Assume (B) is false, namely, there exist nonzero \( \eta_j \in \mathcal{N}(F_{S_j}), j \in [1 : p] \), satisfying (4.10). Set
\[
x_0 := \eta_1 - \eta_2.
\]

Using a similar method as the proof of (3.8), we obtain that
\[
|\langle f_j, x_0 \rangle| = |\langle f_j, \eta_1 - \eta_2 \rangle| = |\langle f_j, c_2 \eta_1 - c_1 \eta_2 \rangle|, \text{ for all } j \in [1 : p].
\]

Then, according to (A) and the definition of phase retrievable, we have
\[
c x_0 = c \eta_1 - c \eta_2 = c_2 \eta_1 - c_1 \eta_2.
\]
for some unimodular constant $c \in \mathbb{S} \setminus \{c_1, c_2\}$, which implies that
\begin{equation}
\eta_2 = \frac{c - c_2}{c - c_1} \eta_1.
\end{equation}
Combining (4.10) and (4.11), we obtain that, for all $j \in [2 : p]$, $\eta_j$ and $\eta_1$ are linear dependent and hence $\eta_1 \in \mathcal{N}(F_{S_j})$. So, $F^\top \eta_1 = 0$. The (A) implies that $\eta_1 = 0$, a contradiction.

We next show (B) $\Rightarrow$ (A). Set $b = [b_1, \ldots, b_m]^\top := |F^\top x_0|$ where $x_0 \in \mathbb{C}^d \setminus \{0\}$. For a fixed $\epsilon \in \mathbb{S}^m$ set $b_\epsilon := [\epsilon_1 b_1, \ldots, \epsilon_m b_m]^\top$. We now consider the solution to
\begin{equation}
(4.12)
F^\top x = b_\epsilon.
\end{equation}
The solution to (4.12) is denoted as $x_\epsilon$. We claim that if $x_\epsilon$ exists then $\tilde{x}_\epsilon = \tilde{x}_0$, which implies (A). Recall that $\tilde{x}_0$ denotes the equivalent class $\{c x_0 : c \in \mathbb{S}\}$ in $\mathbb{C}^d/\sim$ containing $x_0$. To prove the claim let $\epsilon^* \in \mathbb{S}^m$ such that $b_{\epsilon^*} = F^\top x_0$. The (B) implies that the rank of $F$ is $d$. Consequently we must have $x_{\epsilon^*} = x_0$. Now we consider an arbitrary $\epsilon \in \mathbb{S}^m$. If $\tilde{\epsilon} = \epsilon^*$, then $\tilde{x}_\epsilon = \tilde{x}_0$. To this end, we only need prove that $x_\epsilon$ does not exist if $\tilde{\epsilon} \neq \epsilon^*$. Assume $x_\epsilon$ does exist. Set $c_j' := \epsilon_j/\epsilon^*_j$ and $\eta_j' := c_j' x_{\epsilon^*} - x_\epsilon$ for $1 \leq j \leq m$. We can use $c_j'$ to define an equivalence relation on $[1 : m]$, namely $j \sim \ell$ if $c_j' = c_\ell'$. This equivalence relation leads to a partition $S = \{S_1, \ldots, S_p\}$ of $[1 : m]$. Now we set $c_j := c_j'$ where $\ell \in S_j$. Clearly all $c_j$, $1 \leq j \leq p$, are distinct and unimodular. Now set $\eta_j := c_j x_{\epsilon^*} - x_\epsilon$. By definition for all $1 \leq j \leq p$ we have
\[\eta_j \in \mathcal{N}(F_{S_j}) \setminus \{0\}\]
and
\[\frac{\eta_j - \eta\epsilon}{c_1 - c_j} = \frac{\eta_j - \eta\epsilon}{c_1 - c_\ell} \neq 0 \quad \text{for all } j, \ell \in [2 : p],\]
which contradicts with (B). And hence $x_\epsilon$ does not exist if $\tilde{\epsilon} \neq \epsilon^*$. This proves (A).

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