Toppling mechanical characterization based on rheological test of the cantilever beam

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Abstract. This study aims to explain the time-sensitive deformation of rock mass toppling and analyze its mechanical characteristics. A mechanical model of the toppling rock beam is abstracted by performing a force analysis of the countertendency layered slope, and the bending rheological test of the cantilever beam is carried out. The flexural rheological model and constitutive equation of the rock cantilever are identified from the experimental results. The strain (ε = 0) at the ultimate depth of development of the toppling deformation and the strain acceleration (a ≥ a₂) during rock beam dumping bending are selected to obtain the development depth L_F and bending depth L_Z of the toppling deformation, and L_F ≥ L_Z, L_F > 0, L_Z > 0 is used to obtain the development conditions of the toppling deformation. The time-sensitive characteristics of toppling differ from those of other types of slope deformation. This study could serve as a reference for elucidating the mechanical mechanism and evaluating the stability of the toppling deformation.

1. Introduction

Toppling (Figure 1) is a mode of deformation and failure in layered rock mass, especially in anti-dip layered slopes. It is bent toward the free direction and gradually develops into the slope under the combined forces of in situ stress, gravity, and groundwater dynamic (hydrostatic) pressure. In the academia, the toppling pattern is classified into four modes: block toppling, compression toppling [1], shallow toppling [2], and deep toppling [3-4]. The block toppling and the compression toppling are often converted into concealed and less-threatening toppling dangerous rock masses, and the shallow toppling is manifested as a “rigid” rotation deformation controlled by the structural plane and with a small range of development. The deep toppling is mainly developed in soft rocks or in soft and hard rock slopes, and such a continuous and slow bending deformation with a large range of development may induce middle-deep instability and failure. Characterizing the mechanical properties of deep toppling is important to analyze the mechanical mechanism, identify the deformation factors, and establish the stability evaluation system suitable for the toppling deformation.
Early studies of the mechanical properties of the toppling deformation obtained qualitative results from toppling cases and actual investigations. With the development of relevant theories, scholars have gradually studied the mechanical properties of the toppling deformation from the perspective of theoretical derivation. For instance, Li Q considered the stratified rock of the west mountain regions of Hubei Province as an elastic plate and derived the criterion of bending instability and root cracking in middle-thin strata steeply brittle rocks; Xu Q et al. discussed the cusp catastrophe model of the bent fracture deformation of a countertendency layered slope; Chen H Q explored the criterion of toppling and breaking of a countertendency layered rock slope by using a beam-slab model; Zhang Y C established a toppling bending-rip model to reflect the occurrence of formation damage cantilever bending under its own weight, selected the arbitrary strata bar of the toppling slope, and calculated slope stability using mechanical means; basing from the limit equilibrium theory of blocks, Zheng Y performed the mechanical derivation of the toppling failure criterion under the earthquake condition of rocks with a large slenderness ratio; Liu H J abstracted the countertendency layered slope rock as a cantilever model, analyzed the bending fracture under the action of gravity stress and interlayer displacement resistance by mechanical means, and obtained the fracture depth. However, these studies are mostly based on the limit equilibrium theory or energy method and ignored the toppling deformation process. The deep toppling deformation usually occurs in soft rock or soft and hard rock slopes, and its time-sensitive characteristics and slow but long-lasting deformation distinguish it from other types of slope deformation. A mechanical characterization that ignores the aging of the toppling deformation is inadequate.

Huang R Q proposed that the essence of toppling deformation is the creep or rheological deformation of the strata in the long-term geological history and that the bending failure occurs with the extreme deformation of the rock beam. Then, Pang B used the Kelvin rheological model to introduce the time parameter into the mechanical calculation and took the strain of zero at the limit position of the toppling development as a criterion to obtain the development depth of the toppling deformation. However, calculation without any test is merely a tentative exploration that utilizes rheological theory to explain the time-sensitive characteristics of the toppling deformation. In view of the shortcomings above, this study performed a force analysis of the countertendency layered rock mass to conduct the rheological test by using the rock beam test model. The time-sensitive characteristics of the toppling deformation are explained by rheological theory, and then the mechanical properties of toppling are obtained. This study may serve as a reference for elucidating the mechanism and for evaluating the stability of the toppling deformation.

2. Cantilever rheological test

2.1. Modeling

Valley incision provides free conditions for strata deformation. The toppling deformation strata near the side of the free face are not restricted, and the side of the deep slope is impacted by the high confining pressure. Thus, the deformation and displacement space is small and fixed. In addition, the
toppling rock mass can be considered as a cantilever with a fixed end and a free end, and the countertoendency slope can be regarded as a superposition cantilever. On the basis of the slope generalization mechanical model (Figure 2) established by Liu H J [14], Pang B [16], and Wang J M [17], a stress analysis of the strata is carried out to establish a rheological test model that meets the stress boundary conditions of the strata (Figure 3).

The normal compressive stress and tangential shearing stress of the strata can be expressed as follows:

\[ \sigma_{ni} = \gamma h_i \left( \frac{1+n}{2} + \frac{1-n}{2} \cos 2\alpha \right) \]  

\[ \tau_i = \sigma_{ni} \tan \varphi + C \]

Therefore, the upper and lower boundaries of the strata subjected to the normal compressive stress difference are

\[ \sigma_{n1} - \sigma_{n2} = \frac{-m \cos (\alpha + \beta)}{\cos \beta} \gamma \]

From the above stress analysis: When the slope structure is determined, the normal compressive stress on the upper and lower interfaces is independent of the rock beam length and can be regarded as a uniform load. In addition, the upper and lower surfaces of the strata are subjected to the same tangential shearing stress in the opposite direction, and the effect is to reduce the toppling moment of the strata formation. The same effect can be achieved by appropriately reducing the uniform load in this test. In summary, this study regards the toppling rock mass as a cantilever with one end fixed and one end free. The mass is subjected to a uniform load to conduct a bending rheological test to analyze its mechanical properties.

Calculation of the rock beam bending creep is regarded as a plane strain problem by ignoring the influence of the rock beam width. Thus, the deformation process needs to meet the plane section assumption that the plane section of the beam is viewed as flat front and back bending deformation. On the basis of the conclusion of elasticity, the high-span ratio of the slender beam where the plane section assumption is met is \( H/L \leq 1/5 \). On the basis of the multi-point bending rheological test sample size (29.9 cm × 2.9 cm × 1.51 cm) proposed by Fan Q Z [18], the cantilever rheological test model size design is 30 cm × 3 cm × 0.4 cm, as shown in Figure 4. Rock samples are collected from the dam site of Yuqu River, Tibet. The lithology of the samples is slate with a saturated uniaxial compressive strength of 17–20 MPa and is classified as soft rock.
2.2. Load design

The cantilever rheological test is carried out by using weight loading and pasting the strain gauges on the root and middle of the rock beam to collect strain information. The ultimate load is identified by loading up to the rock beam fracture, and then the load size at all levels is determined. The results of the one trial loading test are listed in Table 1.

Table 1. Test result of loading to fracture at once.

| Sample number | Fracture load (g) | Maximum strain ($10^6$) |
|---------------|-------------------|-------------------------|
| 1<sup>st</sup> | 2500              | 438                     |
| 2<sup>nd</sup> | 3000              | 475                     |
| 3<sup>rd</sup> | 2500              | 449                     |
| Average       | 2667              | 454                     |

In consideration that the load formed by weight stacking is difficult to view as a uniform load, a 30 cm $\times$ 0.8 cm rock beam is used as the cushion block to transmit the uniform load in this test, and the cushion block weighs 175 g. The rheological properties of rock beams are studied in detail, and whether or not the deformation stages generate viscoelasticity and viscoplasticity deformation is determined. The rheological test is carried out by loading and unloading, and the loading scheme shown in Table 2.

Table 2. Loading plan of rheological test.

| Loading level | Loading weight (g) | Test time (h) |
|---------------|--------------------|---------------|
| 1             | 1175               | 21            |
| 2             | 1675               | 48            |
| 3             | 2175               | 43            |
| 4             | 2675               | 39            |
| 5             | 3175               | 40            |

The final scheme of the cantilever bending rheological test is shown in Figure 5.
2.3. Test phenomena and test data
After about 1200 h cantilever rheological tests on four rock beams, the rock beams showed visible cantilever bending deformation (Figure 6) and did not recover even after the revocation of the load.

Figure 6. Rock beam bending deformation in test.

The rheological curves of 2# and 3# beams are shown in Figures 7 and 8, respectively.

Figure 7. Rheological curve of 2# beam.
Figure 8. Rheological curve of 3# beam.

The specimen undergone transient bending deformation under the action of each load stage, and the deformation amount of each grade accounted for about 77% of the total strain of this grade. Then, the rock beam deformation ductility increased, and the average strain of the rock beam fracture eventually reached 5.12×10^{-4}, which is greater than that of the first load fracture test (4.54×10^{-4}). This result indicates that the rock can undergo a large deformation when the rock undergoes rheology to failure than the transient failure. The curve of the rock beam deformation rate obtained by the rheological curve is shown in Figures 9 to 11.

Figure 9. Rheological velocity curve of 2# beam.
Figure 10. Rheological velocity curve of 3# beam.
Figure 11. Strain rate under the fifth stage loading.

The above diagram combined with the rheological curve indicates that the cantilever bending creep deformation process of the rock beam can be divided into four stages:
(1) Transient deformation stage: At the moment of the load, the rock beam produces transient deformation, and its deformation rate occurs saltate from zero.
(2) Decay creep stage: After the transient deformation of the rock beam, some of the fissures above the neutral surface extend and open. However, the tensile stress does not reach the long-term tensile strength of the rock, and cracks will not develop and only increase with the degree of stretching to provide great resistance. At the same time, some of the micro-cracks below the neutral surface gradually compress and close, and the flexural rigidity increases. In these two factors, the beam rock deformation rate decreases, and the rheological curves show down concave.

(3) Steady creep stage: The main deformation of the rock beam in this stage is the elastic compression and tension of the particle spacing; in macroscopic view, the strain increases linearly and the strain acceleration fluctuates near zero (-1.02×10^{-4}/h-2<\alpha<1.45×10^{-4}/h-2) (Figure 12).

(4) Accelerating creep stage: The rock beam is curved obviously at this stage, and the cracks develop rapidly, leading to rock beam bending. The strain and strain rate increase rapidly, the rheological curve shows down concave, and the strain acceleration breaks the upper limit of the fluctuation at steady creep state.

![Figure 12. Strain acceleration curve.](image)

The rock beam cantilever rheological test revealed that when the rock beam is in steady creep stage, its strain acceleration $\alpha$ is expressed as fluctuating near zero ($a_1<\alpha<a_2$). When the rock beam enters the accelerating creep stage, deformation uncontrollably the rapid increase and eventually destroyed, strain acceleration quickly breaks the fluctuation threshold of the steady creep stage $a_2$. Combined with acceleration in the application of pre alarming just before landslide [19-20], this study uses strain acceleration $\alpha\geq a_2$ as the criterion of rock beam bending.

2.4. Model identification and curve fitting

Taking the 1# beam as an example, the rheological characteristics are analyzed in detail to determine its rheological model. As the rock beam is fractured at the end, the end strain is the main object in this study, considering the actual value of the study. The total creep strain of the rock beam is the result of the linear superposition of the creep produced by the load at all levels (Boltzmann’s superposition principle), the rheological curves under the load of 1# beam are decomposed as shown in Figure 13. The rock beam first generates the transient deformation after each load stage, the deformation increases slowly, and then the rheological curve shows a slow rise. After each unload stage, the rock beam deformation is partly restored instantaneously, and then the curve undergoes a nonlinear descent process and the final deformation is not recovered completely. The rock beam strain can be divided into transient strain and creep strain. At the same time, the deformation restored after unloading is elastic deformation, in which the instantaneous recovery part is transient elastic deformation, and the part after slow recovery is viscoelastic deformation.
Taking the first load of 1# beam as an example, at the moment of loading, the rock beam produces a transient strain value of $113 \times 10^{-6}$, and the transient recovered elastic strain at the time of unloading is $107 \times 10^{-6}$, indicating that at the moment of loading, the rock beam also produces an instant plastic strain of $6 \times 10^{-6}$. After unloading, the rock beam residual plastic strain value is still $33 \times 10^{-6}$. That is, the removal of the transient plastic strain, the rock beam in the subsequent rheological process, also produces a viscoplastic deformation value of $27 \times 10^{-6}$. Combined with the above analysis, the rheological deformation curve of the rock beam cantilever is generalized as shown in Figure 14.

where

\[ \text{strain} = \text{rheological strain} + \text{transient strain}, \]
\[ \text{Rheological strain} = \text{viscoelastic strain} + \text{viscoplastic strain}, \]
\[ \text{Transient strain} = \text{transient elastic strain (primary)} + \text{transient plastic strain (secondary)}. \]

On the basis of the above analysis, before the rock beam deformation enters the accelerated rheological stage, the rheological deformation of the cantilever beam is composed of four parts: transient elasticity, transient plasticity, viscoelasticity, and viscoplasticity. Meanwhile, in the rheological process, the transient plastic strain generated is minimal during loading because the load magnitude is less than the yield strength of the rock beam. Therefore, identification of the rheological model ignores transient plastic strain. Further, in addition to 1#, the 2–3# beams capture the rock beam to accelerate the rheological stage. Their rheological constitutive should also be included to reflect the accelerating deformation of the nonlinear rheological element.

In consideration that the basic rheological elements are linear elastic, the traditional rheological models obtained by their combination are also linear elastic, and the accelerated creep deformation of the rock cannot be identified. Therefore, basing from previous research results\cite{21-23}, the present study introduced a nonlinear viscoplastic body (Figure 15) to describe the rock accelerating creep stage.

where $\sigma_S$ is the long-term tensile strength of the rock \cite{23}.
Accordingly, the present study used Hooke body (H), Kelvin body (K), Bingham body (B), and the nonlinear viscoplastic body (N), which constitute a thin slab cantilever creep constitutive model, as shown in Figure 16.

**Figure 16.** Rheological model.

As shown in the figure above, the elements in each model are connected in series to meet

\[ \varepsilon = \varepsilon_H + \varepsilon_K + \varepsilon_B + \varepsilon_N \]  
(4)

\[ \sigma_H = \sigma_K = \sigma_B = \sigma_N \]  
(5)

The creep equation of the elastic element of the transient deformation (H) is

\[ \varepsilon_0 = \frac{\sigma_0}{E_0} \]  
(6)

The creep equation of the Kelvin model (K) is

\[ \varepsilon_K = \frac{\sigma_0 \left(1 - e^{-\frac{\sigma_1 t}{\eta_1}}\right)}{E_1} \]  
(7)

The creep equation of the Bingham model (B) is

\[ \varepsilon_B = \frac{(\sigma_0 - \sigma_1) t}{\eta_2} \]  
(8)

The creep equation of the nonlinear viscoplastic model (N) is

\[ \varepsilon_N = \frac{\sigma_0 - \sigma_2}{\eta_3} t^n \]  
(9)

Combined with (4)–(9), the conclusion is as follows:

When the specimen is subjected to low horizontal stress, the rock beam creep contains only three stages: transient strain, decay creep, and steady creep. When stress levels large enough to appear accelerated creep stage, and then destroyed. This is, the stress threshold \( \sigma_{s2} \) that triggers the plastic deformation should be smaller than the stress threshold \( \sigma_{s3} \) that triggers the accelerated creep, and the creep equation of the rheological model is as follows:

\[
\varepsilon = \begin{cases} 
\frac{\sigma_0}{E_1} \left(1 - e^{-\frac{\sigma_1 t}{\eta_1}}\right) & \sigma < \sigma_{s1} \\
\frac{\sigma_0}{E_1} \left(1 - e^{-\frac{\sigma_1 t}{\eta_1}}\right) + \frac{\sigma_0 - \sigma_{s2}}{\eta_2} t & \sigma_{s1} \leq \sigma < \sigma_{s2} \\
\frac{\sigma_0}{E_1} \left(1 - e^{-\frac{\sigma_1 t}{\eta_1}}\right) + \frac{\sigma_0 - \sigma_{s2}}{\eta_2} t + \frac{\sigma_0 - \sigma_{s3}}{\eta_3} & \sigma_{s2} \leq \sigma.
\end{cases}
\]  
(10)

The rheological properties that can be described by the above model are analyzed below:

(1) Loading moment, when \( t=0 \), rock beam generates transient strain. The value of the size can be expressed as

\[ \varepsilon_0 = \frac{\sigma_0}{E_0} \]  
(11)
(2) Then, the rock beam continues to bend and deform. However, when the load has not reached the acceleration rheological threshold $\sigma_{s3}$, the bending rheological model acceleration element is not activated. At this time, the rock beam creep equation is

$$\varepsilon = \frac{\sigma}{E_0} + \frac{\sigma}{E_1} \left(1 - e^{-\frac{E_1 t}{\eta_1}}\right) + \frac{\sigma - \sigma_{s2}}{\eta_2} t$$

(12)

The strain development rate is

$$\dot{\varepsilon} = \frac{E_1}{\eta_1} e^{-\frac{E_1 t}{\eta_1}} + \frac{\sigma - \sigma_{s2}}{\eta_2}$$

(13)

The strain $\varepsilon$ increases with time, and the part of the strain rate $\dot{\varepsilon}$ described by the Kelvin model decreases with time. When $t \to \infty$, $\dot{\varepsilon}_K \to 0$, the total strain rate of rock beam is constant. That is, when the accelerated rheological element is disabled, the deformation described by the rheological model experiences a process in which the strain rate decreases and then stabilizes, and the performance of the curve initially decreases and then increases, which is consistent with the test results.

(3) When the rock beam is subjected to a higher stress level, the rock beam eventually enters the accelerated rheological stage, and the $N$ body plays a role. The strain of the rock beam is

$$\varepsilon = \frac{\sigma}{E_0} + \frac{\sigma}{E_1} \left(1 - e^{-\frac{E_1 t}{\eta_1}}\right) + \frac{\sigma - \sigma_{s2}}{\eta_2} t + \frac{\sigma - \sigma_{s3}}{\eta_3} t^n$$

(14)

At this time, the strain rate of the rock beam is

$$\dot{\varepsilon} = \frac{E_1}{\eta_1} e^{-\frac{E_1 t}{\eta_1}} + \frac{\sigma - \sigma_{s2}}{\eta_2} + \frac{n(\sigma_0 - \sigma_{s3})}{\eta_3} t^{n-1}$$

(15)

(where $n$ is an indicator that reflects the speed of the rock accelerating rheology.)

At the higher stress level, the model $N$ body plays a role, the strain and strain rate increase exponentially, and the deformation performance shows an increase in strain rate. The strain–time curve shows an up concave, and the rock beam exhibits bending failure.

The above analysis indicates that the rheological characteristics described by the rheological model are in good agreement with the experimental phenomena, which proves that the model can better reflect the stress and strain state of the rock beam when the rock beam exhibits cantilever rheology. The curve fitting of the rheological model is performed by 1stOpt, a computational software platform (Figures 17–18).

Figure 17. Fitting curve of 2# beam. Figure 18. Fitting curve of 3# beam.

Fitting results in Table 3 show that most of the correlation coefficients are much larger than 0.98, which indicates that the model can describe the bending and rheological behavior of the strata.

| Beam number | Load level | $E_0$ (GPa) | $E_1$ (GPa) | $\eta_1$ (GPa·h) | $\eta_2$ (GPa·h) | $\eta_3$ (GPa·h) | $n$ | Correlation coefficient |
|-------------|------------|-------------|-------------|-----------------|-----------------|-----------------|-----|------------------------|
| 1#          | 1          | 79.646      | 0.365       | 252.933         | 659.252         | -               | -   | 0.98                   |
|             | 2          | 79.082      | 0.822       | 2258.69         | 95.644          | -               | -   | 0.99                   |
2.5. Rheological constitutive equation correction

Different from conventional compression or shear rheological tests, the specimen length affects the fixed end tensile stress of the rock beam, thereby affecting the rheological parameters of the cantilever beam. Meanwhile, rock beams of different lengths have different deformation transfer times. Therefore, the constitutive equation of the rheological model needs to be modified to describe the bending rheological behavior of rock beams of any length. The rheological behavior of rock beams with the same lithology and different length can be described by the rheological constitutive equation with the same form and different parameters. The strain generated by these rock beams at the same time would be a linear or nonlinear relationship with respect to the length $L$ of rock beams. Therefore, the strain obtained under this test condition is $\epsilon_0$. Considering the introduction of the correction coefficient $f(L)$ on the length of rock beams, the bending creep deformation of the rock beam with arbitrary length $L$ can be expressed by the following equation:

$$\epsilon_L = f(L) \epsilon_0$$

(16)

where $f(L_0) = 1$, $\epsilon_L = \epsilon_0$

As the length of the rock beam affects the end stress and deformation time, the strain of the $L$ length rock beam is expressed as

$$\epsilon_L = \frac{\sigma_L}{E_L} + \frac{\sigma_L - \sigma_L \gamma_n}{\eta_L} t_L + \frac{\sigma_L - \sigma_L \gamma_n}{\eta_L} t_L^n.$$

(17)

The stress analysis of single slabs in the counteract tendency slope by Liu H J [14], Pang B [16], and Wang J M [17] is directly quoted here. The tensile stress on the upper surface of the rock plate is

$$\sigma = K_1 L^2 + K_2 L,$$

(18)

where

$$K_1 = \frac{3\gamma \cos \alpha}{m} - \frac{3\gamma \cos (\alpha + \beta)}{m} \left(1 + n + \frac{1 - n}{2} \cos 2\alpha \right) - \frac{3\gamma (1 - n) \sin (2\alpha) \sin (\alpha + \beta)}{m \cos \beta},$$

(19)

$$K_2 = \frac{\gamma \cos (\alpha + \beta)}{m} \left(1 + n + \frac{1 - n}{2} \cos 2\alpha \right) \tan \varphi,$$

(20)

Combined with (16)–(20), the conclusion is

$$f(L) = \frac{K_1 L^2 + K_2 L}{E_L} + \frac{(K_1 L^2 + K_2 L)(1 - e^{-\frac{E_L L}{\eta_L + L^n}})}{E_L L^n}$$

(21)
The specific solution is as follows:

(1) Given that the toppling deformation time is extremely long, considering time $t \to \infty$, at this time by the rock beam length caused by the deformation time can be ignored;

(2) The rheological parameter $\eta_{L3}$ can be set as a function of length $L$; $\eta_{L3} = g(\eta_3, L)$, which can obtain large amounts of data by different confining pressures, in the curve fitting solution;

(3) When $t \to \infty$, the limit of $f(L)$ depends on the highest order of $t$. Since the parameter $n$ is used to describe the rocker to accelerate the rheological process, the rheological curve shows an up concave, i.e., $n > 1$;

(4) Xu W Y (2005) reported that the smaller the viscosity coefficient $\eta_3$, the more obvious the viscoplastic characteristics of the rock beam deformation, the faster the curve, the higher the rate of rock beam deformation, and $n$ with the confining pressure change research is still less. These results suggest that the rheological index $n$ is controlled by the lithologic conditions and the rock mass structure, and the effect caused by the length of the rock beam is reflected by $\eta_3$, which decreases with the increase in stress level, i.e., $\eta_L = \eta_0 = n$.

On the basis of the above analysis, $f(L)$ can be expressed as

$$ f(L) = \frac{(K_1L^2 + K_2L - \sigma_{L3})}{g(\eta_3, L, L)} \frac{\eta_{L3}}{\eta_3}. \tag{22} $$

In the above formula, the denominator is determined by this test and can be regarded as a constant. Thus, $A = \frac{\eta_1}{\sigma_0 - \sigma_3}$, and the correction coefficient $f(L)$ can be expressed as

$$ f(L) = A \frac{(K_1L^2 + K_2L - \sigma_{L3})}{g(\eta_3, L)}. \tag{23} $$

The cantilever bending rheology of any length $L$ in the cantilever rheological constitutive equation (Formula 23) can be expressed as

$$ \varepsilon = A \frac{(K_1L^2 + K_2L - \sigma_{s3})}{g(\eta_3, L)} \left( \frac{\sigma}{E_0} + \frac{\sigma - \sigma_{s2}}{E_1} - \frac{\sigma - \sigma_{s3}}{\eta_3} t^n \right). \tag{24} $$

3. Analysis on mechanical properties of the toppling deformation

3.1. Analysis on development depth of toppling deformation

The ultimate depth of the toppling bending deformation (i.e., no bending deformation) is the development depth of the toppling deformation, and then the rock surface tension strain should be zero. At the same time, the position is in the ultimate depth of the toppling deformation, the rock formation should be in the initial state of rheology, i.e., decaying the creep state, and the rock cantilever bending creep deformation can be described as

$$ \varepsilon = A \frac{(K_1L^2 + K_2L - \sigma_{s3})}{g(\eta_3, L)} \left[ \frac{1}{E_0} + \frac{1}{E_1} \right] \sigma. \tag{25} $$

Take the toppling deformation development time $t \to \infty$, and make $\varepsilon = 0$:

$$ \left[ A \frac{(K_1L^2 + K_2L - \sigma_{s3})}{g(\eta_3, L)} \left[ \frac{1}{E_0} + \frac{1}{E_1} \right] \sigma = 0 \right. \tag{26} $$
That is, $\sigma = K_1 L^2 + K_2 L = 0$

The development depth of the toppling deformation in a counterrdenity rock slope can be expressed as

$$L_F = -\frac{K_2}{K_1}$$  \hspace{1cm} (27)

3.2. Analysis on bending depth of the toppling deformation

Toppling deformation bending depth is defined as the depth of formation bending broken due to the toppling deformation. Results of the cantilever rheological test show that the strain acceleration $a$ increases rapidly when the rock beam is topped and bent and breaks the upper limit $a_2$ in the steady state creep stage. The rock beam bending criterion is

$$a = a_2$$  \hspace{1cm} (28)

Rock beam is about to bend into the accelerating rheological stage, i.e., the creep constitutive equation:

$$\varepsilon = \left[ A \left( \frac{K_1 L^2 + K_2 L}{g(\eta_3, L)} \right) - \sigma_{s3} \right] \left[ \frac{\sigma}{E_0} + \frac{\sigma}{E_1} + \frac{\eta_1}{E_1} \eta_2 t + \frac{\eta_3}{E_2} t^n \right].$$  \hspace{1cm} (29)

The strain acceleration is the second order of the strain versus time $t$:

$$a = \ddot{\varepsilon} = f(L) \left[ -\frac{E_1}{\eta_1^2} e^{-\frac{E_1}{\eta_1} t} \sigma_0 + \frac{n(n-1)(\sigma_0 - \sigma_{s2})}{\eta_3} t^{-2} \right]$$  \hspace{1cm} (20)

Similarly, take the time $t \to \infty$, the rock beam break criterion (Formula 28) into the above formula can obtain

$$f(L) = \lim_{t \to \infty} -\frac{E_1}{\eta_1^2} e^{-\frac{E_1}{\eta_1} t} \sigma_0 + \frac{n(n-1)(\sigma_0 - \sigma_{s2})}{\eta_3} t^{-\infty}. \hspace{1cm} (31)$$

The formula has two possible results:

1. When $1 < n < 2$, $n-2 < 0$, $\lim_{t \to \infty} -\frac{E_1}{\eta_1} e^{-\frac{E_1}{\eta_1} t} = \lim_{t \to \infty} \frac{n(n-1)(\sigma_0 - \sigma_{s2})}{\eta_3} t^{-2} = 0$, $f(L) = A \left( \frac{K_1 L^2 + K_2 L}{g(\eta_3, L)} \right) = \infty$, which is obviously not realistic.

2. When $n \geq 2$, $n-2 \geq 0$, $\lim_{t \to \infty} -\frac{E_1}{\eta_1} e^{-\frac{E_1}{\eta_1} t} = \lim_{t \to \infty} \frac{n(n-1)(\sigma_0 - \sigma_{s2})}{\eta_3} t^{-2} = \infty$, $f(L) = A \left( \frac{K_1 L^2 + K_2 L}{g(\eta_3, L)} \right) = 0$, so $(K_1 L^2 + K_2 L) - \sigma_{s3} = 0$. Thus, the counterrdenity rock slope toppling deformation bending depth can be expressed as

$$L_Z = -\frac{K_2 + (K_1^2 + 4K_1 \sigma_{s3})^{1/2}}{2K_1}.$$  \hspace{1cm} (32)

In contrast to the study by Liu H J (2016)[14], the calculation formula of rock beam bending depth was obtained using the static mechanical solution, and the depth of the bend belt obtained from the rheological theory is similar in form to static mechanics, but the bending criterion is made by stress equal to the tensile strength ($\sigma = \sigma_T$) to be equal to the acceleration creep trigger threshold, i.e., the rock long-term tensile strength ($\sigma = \sigma_{s3}$). From the perspective of the creep constitutive equation, the cantilever bending deformation experiences the transient deformation stage, the decay stage, the
steady creep stage, and the final failure of the accelerating creep stage. In addition, once the deformation enters the accelerating creep stage, the deformation would increase rapidly and generate bending failure quickly and uncontrollably. Once the tensile stress on the upper surface of the rock formation exceeds the long-term tensile strength $\sigma_{s3}$, the deformation of the formation quickly fails without control. Therefore, the rock bending criterion should be the tensile stress $\sigma$ equal to the long-term tensile strength $\sigma_{s3}$.

3.3. Discussion on development conditions of toppling deformation

The developmental conditions of toppling deformation in mathematical sense are discussed based on the formulas of development and bending depths:

$$y_1 = L_F = K_1 L^2 + K_2 L$$
$$y_2 = L_Z = (K_1 L^2 + K_2 L) - \sigma_{s3}$$

The function images are shown in Figures 19 and 20.

**Figure 19.** Function image of $y_1$ and $y_2$ ($K_1 > 0$).

**Figure 20.** Function image of $y_1$ and $y_2$ ($K_1 < 0$).

The toppling deformation part must be located in its development range; thus,

$$L_F \geq L_Z$$

From the function image, i.e., the intersection of $L_F$ and $x$ axis should be outside $L_Z$, and because $\sigma_{s3} > 0$, $K_1 < 0$.

In addition, the development and bending depths are more than zero, i.e., $L_F > 0, L_Z > 0$.

Based on the above analysis, the developed conditions of toppling deformation are

$$\cos \alpha \cos \beta < \cos(\alpha + \beta) \left(\frac{1+n}{2} + \frac{1-n}{2} \cos 2\alpha\right) + \frac{1}{2} (1-n) \sin(2\alpha) \sin(\alpha + \beta)$$

$$0 \leq \alpha + \beta \leq 90^\circ$$

From the above analysis, combined with geological understanding, the toppling deformation is controlled by the formation inclination angle $\alpha$, basal slope $\beta$, and Poisson’s ratio $\mu$ except for controlled free conditions. The slope lithology is not considered in the toppling deformation development conditions based on mathematical solution (such as severe, interlayer shear strength index, etc.). This scenario shows that the actual developmental conditions are the sub (intersection) set of the results of this study, and it does not mean that these factors do not affect the development process of the toppling deformation.

In addition, the developmental conditions of the toppling deformation are different from the actual ones (mainly in $0 \leq \alpha + \beta \leq 90^\circ$), but the calculation process and the solution ideas can provide ideas for follow-up studies.

4. Conclusion

Rheological theory is used to explain the time-sensitive characteristics of toppling deformation. Based on the force analysis of countercendency slope, the mechanical model of toppling deformation was
abstracted and the rheological test of cantilever was carried out to explore the mechanical properties of the toppling deformation. The main conclusions are as follows:

(1) The stress difference between the upper and lower surfaces of the toppling slope is independent of the working point’s position, and it can be abstracted as a cantilever with one fixed end and one free end, which undergoes bending creep deformation in the free direction under uniform load.

(2) A nonlinear viscoplastic element is introduced and connected in series with Hooke body (H), Kelvin body (K), and Bingham body (B) to describe the deformation characteristics of each stage of the rock beam. Then, the constitutive equation of cantilever bending creep deformation is derived. The correction coefficient \( f(L) \) of the rock beam length is introduced to obtain the constitutive equation of the cantilever bending creep deformation of any length rock beam.

(3) Taking the development time of the toppling deformation as infinite \((t \to \infty)\) and setting the strain as 0 \((\varepsilon = 0)\), we can obtain the calculation formula of the toppling development depth \(L_p\) of the counter tendency rock slope. Furthermore, setting the strain acceleration \(a\) as the upper limit value in the steady creep stage \(a_2\) \((a = a_2)\), we can obtain the calculation formula of the bending depth \(L_z\) of the toppling deformation.

(4) The development conditions of the toppling deformation are solved in a mathematical sense, and the relevant equations are obtained. The development of the toppling deformation is mainly affected by the inclination angle \(\alpha\), basal slope \(\beta\), and Poisson’s ratio \(\mu\). The impact of lithological conditions needs further study.

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