ON $K_2$ OF CERTAIN FAMILIES OF CURVES

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Abstract. We construct families of smooth, proper, algebraic curves in characteristic 0, of arbitrary genus $g$, together with $g$ elements in the kernel of the tame symbol. We show that those elements are in general independent by a limit calculation of the regulator. Working over a number field, we show that in some of those families the elements are integral. We determine when those curves are hyperelliptic, finding, in particular, that over any number field we have non-hyperelliptic curves of all composite genera $g$ with $g$ independent integral elements in the kernel of the tame symbol.

1. Introduction

Let $k$ be a number field, with ring of algebraic integers $\mathcal{O}_k$. The classical relation between the residue at $s = 1$ of the zeta-function $\zeta_k(s)$ and the regulator of $\mathcal{O}_k^* \simeq K_1(\mathcal{O}_k)$, was generalized by Borel to a relation between a regulator defined for $K_{2n-1}(k) = K_{2n-1}(\mathcal{O}_k)$ and $\zeta_k(n)$ for $n \geq 2$ [8]. Inspired by this, Bloch a few years later considered CM-elliptic curves $E$ over $\mathbb{Q}$, and proved a relation between a regulator associated to an element in $K_2(E)$ and the value of its $L$-function at 2. This was finally published in [6]. Beilinson in the meantime had made a very general conjecture about similar relations between regulators of certain $K$-groups of regular, projective varieties over number fields, and values of their $L$-functions at certain integers (see, e.g., [19]).

Below we shall briefly review Beilinson’s conjecture on $K_2$ of a curve over a number field. For more details we refer the reader to the first three sections of [12].

Let $F$ be a field. By a famous theorem of Matsumoto (see [17, Theorem 4.3.15]), the group $K_2(F)$ can be described explicitly as

$$F^* \otimes_\mathbb{Z} F^*/\langle a \otimes (1 - a), a \in F, a \neq 0, 1 \rangle,$$

where $\langle \cdots \rangle$ denotes the subgroup generated by the indicated elements. The class of $a \otimes b$ is denoted $\{a, b\}$, so that $K_2(F)$ is an Abelian group (written additively), with generators $\{a, b\}$ for $a$ and $b$ in $F^*$, and relations

$$\{a_1a_2, b\} = \{a_1, b\} + \{a_2, b\},$$
$$\{a, b_1b_2\} = \{a, b_1\} + \{a, b_2\},$$
$$\{a, 1 - a\} = 0 \quad \text{if } a \text{ is in } F, a \neq 0, 1.$$

These relations also imply $\{a, -a\} = 0$ and $\{a, b\} = -\{b, a\}$.

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Suppose that \( F \) is the function field of a regular, proper, irreducible curve \( C \) over a field \( k \). Then we let
\[
K_2^T(C) = \ker \left( K_2(F) \xrightarrow{T} \bigoplus_{x \in \text{Codim}^1(C)} k(x)^* \right),
\]
where \( C^{(1)} \) denotes the set of closed (codimension 1) points of \( C \), and the \( x \)-component of the map \( T \) is the \textit{tame symbol} at \( x \), defined on generators by
\[
T_x : \{a, b\} \mapsto (-1)\text{ord}_x(a)\text{ord}_x(b) \frac{a^{\text{ord}_x(b)}}{b^{\text{ord}_x(a)}}(x).
\]
For \( \alpha \) in \( K_2(F) \) we have the \textit{product formula}, \cite{Beilinson} Theorem 8.2
\[
\prod_{x \in \text{Codim}^1(C)} \text{Nm}_{k(x)/k}(T_x(\alpha)) = 1.
\]
Now assume that the base field \( k \) is a number field. Then Beilinson’s conjecture in its original statement (see \cite{Beilinson}) applied to the group \( K_2(C) \otimes \mathbb{Q} = K_2^T(C) \otimes \mathbb{Q} \), but in \cite{Beilinson} a (slightly stronger) formulation was given without tensoring with \( \mathbb{Q} \), and we shall use this approach here. Suppose that \( C \) has genus \( g \), and that \( k \) is algebraically closed in \( F \) (so that \( C \) is geometrically irreducible over \( k \)). Then Beilinson originally expected \( K_2^T(C) \otimes \mathbb{Q} \) to have dimension \( r = g - [k : \mathbb{Q}] \). However, computer calculations \cite{Weight} showed that this dimension could be larger and that an additional condition should be used, which led to a modification of the conjecture \cite{Beilinson}. For this, let us fix a regular, proper model \( C/O_k \) of \( C/k \), with \( O_k \) the ring of algebraic integers in \( k \). Then we define
\[
K_2^T(C) = \ker \left( K_2(F) \xrightarrow{T} \oplus_D \mathbb{F}(D)^* \right),
\]
where \( D \) runs through all irreducible curves on \( C \), and \( \mathbb{F}(D) \) is the residue field at \( D \). The component of \( T_C \) for \( D \) is given by the tame symbol corresponding to \( D \) similar to (1.1),
\[
\{a, b\} \mapsto (-1)^{v_D(a)v_D(b)} \frac{a^{v_D(b)}}{b^{v_D(a)}}(D),
\]
where \( v_D \) is the valuation on \( F \) corresponding to \( D \). Because the \( D \) that surject onto \( \text{Spec}(O_k) \) correspond exactly to the points \( x \) in \( C^{(1)} \) under localization, and the formula for the \( D \)-component of \( T_C \) localizes to the one in (1.1) for the corresponding point \( x \), it follows that \( K_2^T(C) \) is a subgroup of \( K_2^T(C) \). It was stated without proof on \cite{Beilinson} p. 344 that the image of \( K_2^T(C) \) inside \( K_2^T(F) \) modulo torsion is independent of the choice of \( C \), but in fact the subgroup \( K_2^T(C) \) of \( K_2^T(C) \) is independent of this choice (see Proposition \ref{prop:independence} below). We shall denote it by \( K_2^T(C)_{\text{int}} \), and call its elements \textit{integral}.

Let \( X \) be the complex manifold associated to \( C \otimes \mathbb{Q} \). It is a disjoint union of \([k : \mathbb{Q}]\) Riemann surfaces of genus \( g \), and complex conjugation acts on it through the action on \( C \) in \( C \otimes \mathbb{Q} \mathbb{C} \). We let \( H^1_{\text{DR}}(X, \mathbb{R})^- \) consist of those elements that are multiplied by \( -1 \) under the resulting action on \( X \). It is a real vector space of dimension \( r \). It can be paired with \( H_1(X, \mathbb{Z})^- \), the part of \( H_1(X, \mathbb{Z}) \) on which complex conjugation on \( X \) induces multiplication by \( -1 \), which is isomorphic with \( \mathbb{Z}^r \). Dividing Beilinson’s regulator map by \( i \) we obtain a map \( K_2^T(C)_{\text{int}}/\text{torsion} \to H^1_{\text{DR}}(X, \mathbb{R})^- \),
and combining this with the pairing we obtain the regulator pairing
\[
\langle \cdot, \cdot \rangle : H_1(X; \mathbb{Z})^- \times K^T_2(C)/\text{torsion} \rightarrow \mathbb{R}
\]
(1.3)
\[
(\gamma, \alpha) \mapsto \frac{1}{2\pi} \int_{\gamma} \eta(\alpha),
\]
with \(\eta(\alpha)\) obtained by writing \(\alpha\) as a sum of symbols \(\{a, b\}\), and mapping \(\{a, b\}\) to
(1.4)
\[
\eta(a, b) = \log |a| \text{d arg}(b) - \log |b| \text{d arg}(a),
\]
and \(\gamma\) is chosen such that \(\eta(\alpha)\) is defined. The pairing is well-defined [12, Section 3]. If \(\gamma_1, \ldots, \gamma_r\) form a basis of \(H_1(X; \mathbb{Z})^-\), and \(M_1, \ldots, M_r\) are in \(K^T_2(C)\) or \(K^T_2(C)/\text{torsion}\), we can define the regulator \(R(M_1, \ldots, M_r)\) by
(1.5)
\[
R = |\det(\langle \gamma_i, M_j \rangle)|.
\]
Beilinson expects \(K^T_2(C)_{\text{int}} \otimes \mathbb{Q}\) to have \(\mathbb{Q}\)-dimension \(r\), that \(R \neq 0\) if \(M_1, \ldots, M_r\) form a basis of it, and that \(R\) is related to the value of \(L(C, s)\) at \(s = 2\) (see [12, Conjecture 3.11]).

The proof of Proposition 3.1 shows that \(K^T_2(C)_{\text{int}}\) is a quotient of \(K_2(C)\), which is expected to be finitely generated by a conjecture of Bass. If that is the case, and \(M_1, \ldots, M_r\) form a \(\mathbb{Z}\)-basis of \(K^T_2(C)_{\text{int}}/\text{torsion}\), then \(R\) is also independent of the choice of this basis.

That \(K^T_2(C)_{\text{int}}\) is finitely generated is only known if \(g = 0\). Most of the work on the conjecture has been put into constructing \(r\) independent elements in \(K^T_2(C)_{\text{int}}/\text{torsion}\) and, if possible, relating the resulting regulator with the \(L\)-value either numerically or theoretically (see, e.g., [10, 11, 13, 12, 16]).

The goal of this paper is twofold. Firstly, we construct \(g\) elements in \(K^T_2(C)\) on certain families of curves of genus \(g\). The curves here are in general not hyperelliptic but include the curves in [12] as a special case. If the base field is a number field, we obtain families with \(g\) linearly independent elements in \(K^T_2(C)_{\text{int}}\). In particular, if the base field is \(\mathbb{Q}\) then we found as many linearly independent elements in \(K^T_2(C)_{\text{int}}\) as predicted by Beilinson’s conjecture.

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The structure of the paper is as follows. In Section 2 we construct elements of \(K^T_2\) on a family of curves, and prove some relations between these elements. In Section 3 we show the elements are in \(K^T_2(C)_{\text{int}}\) for some families, and find necessary and sufficient conditions for the curves in these families to be hyperelliptic. In Section 4 we prove a limit formula of the regulator and show the elements are linearly independent under some condition. We also give examples of elliptic curves over real quadratic fields with two independent integral elements.
2. Construction of elements in $K_2^T$ for certain curves

Let $C$ be a regular, proper, irreducible curve over a field, and let $F$ be its function field. The most obvious way to construct elements in $K_2^T(C)$ is to use two functions with only three zeroes and poles in total, as we now recall from [12, Construction 4.1].

Assume $P_1, P_2, P_3$ are distinct rational points of $C$ whose pairwise differences are torsion divisors. Thus, using indices modulo 3, there are rational functions $f_i$ with $\text{div}(f_i) = m_i(P_{i+1}) - m_i(P_{i-1})$, where $m_i$ is the order of $(P_{i+1}) - (P_{i-1})$ in the divisor group $\text{Pic}^0(C)$. We then define three elements of $K_2(F)$ by

$$\left\{ \frac{f_{i+1}}{f_{i+1}(P_{i+1})}, \frac{f_{i-1}}{f_{i-1}(P_{i-1})} \right\}.$$ 

Using the product formula (1.2), one sees that those elements are in $K_2^T(C)$.

Now let $L_1, \ldots, L_n$ be $n$ distinct non-parallel lines defined by $a_ix + b_iy + c_i$. In the following, we also denote the equation of the line by $L_i$ without any confusion. Consider the non-singular model $C$ of the projective closure of the affine curve given by $\prod_{i=1}^n L_i - 1$. It has $n$ distinct non-singular points corresponding to the points $P_i = [-b_i, a_i, 0]$ in $\mathbb{P}^2$, and $\text{div}(\frac{f_i}{L_i}) = n(P_i) - n(P_j)$. So, using these functions, we can construct elements in $K_2^T(C)$ as above.

The $P_i$ are hyperflexes of $C$ (in fact, this example was inspired by [14], where hyperflexes are used in order to construct elements in $K_2^T(C)$). However, in order to ensure the resulting elements in $K_2^T(C)$ are integral if our base field is a number field and all our coefficients are in the ring of integers, it is crucial that, modulo each prime ideal, all $P_i$ have different reductions. This bounds $N$, and hence the genus of $C$. For example, if we take $\mathbb{Q}$ as the base field, we would need that $N \leq 3$, hence $g \leq 1$. But it turns out that it is not crucial that the $P_i$ are hyperflexes, and that we can use several lines in $\mathbb{P}^2$ through each $P_i$. This gives the following construction.

**Construction 2.1.** Let $N \geq 2$, and for $i = 1, \ldots, N$, let $N_i \geq 1$. For $i = 1, \ldots, N$ and $j = 1, \ldots, N_i$, let $L_{i,j}$ be distinct lines given by equations $a_{i,j}x + b_{i,j}y + c_{i,j}$ that are not parallel for distinct $i$. Consider the affine curve defined by

$$f(x, y) = \lambda \prod_{i=1}^N \prod_{j=1}^{N_i} L_{i,j} - 1, \quad \lambda \neq 0. \tag{2.2}$$

Note that this curve is irreducible since $f$ is irreducible. Indeed, without loss of generality, suppose $L_{1,1} = x$ and $L_{2,1} = y$. If $f = (h_1 + 1)(h_2 - 1)$ with $h_1(0, 0) = h_2(0, 0) = 0$, it is easy to see that $x$ and $y$ both divide $h_1$ and $h_2$. Furthermore, one can show that all $L_{i,j}$ divide $h_1$ and $h_2$. Namely, we have $(h_1 + 1)(h_2 - 1) \equiv f \equiv -1$ modulo $L_{i,j}$. If $i = 1$, $h_1$ and $h_2$ are equivalent to functions of $y$ without constant term modulo $L_{i,j}$, which forces $h_1$ and $h_2$ to be $0$ modulo $L_{i,j}$. If $i \neq 1$, $h_1$ and $h_2$ are equivalent to functions of $x$ without constant term modulo $L_{i,j}$, which also forces $h_1$ and $h_2$ to be $0$ modulo $L_{i,j}$. Because the $L_{i,j}$ are pairwise coprime, $\prod_{i=1}^N \prod_{j=1}^{N_i} L_{i,j}$ divides $h_1$ and $h_2$, hence $h_1$ or $h_2$ equals $0$, showing that $f$ is irreducible.

Let $C$ be the non-singular curve obtained by taking the projective closure in $\mathbb{P}^2$ and blowing up the singularities, and let $F$ be its function field. Note that $C$ has points $\tilde{P}_{i,j}$ mapping to $P_i$, where $\tilde{P}_{i,j}$ corresponds to $L_{i,j}$. Although our construction of elements in $K_2^T(C)$ below will not be based on this, it turns out
that all \((\tilde{P}_{1,j}) - (\tilde{P}_{k,l})\) are torsion divisors. Namely, if \(d = \text{deg}(f) = \sum_{i=1}^{N} N_i\), then
\[
\text{div}\left(\frac{L_{1,1,1}}{T_{1,2,1}}\right) = \left[ d(\tilde{P}_{1,1,1}) + \sum_{i=1}^{N_1} ((\tilde{P}_{1,1,i}) - (\tilde{P}_{1,1,j})) \right] - \left[ d(\tilde{P}_{1,2,1}) + \sum_{i=1}^{N_2} ((\tilde{P}_{1,2,i}) - (\tilde{P}_{1,2,j})) \right].
\]
Taking \(i_1 = i_2 = i\) shows that \((\tilde{P}_{1,1,i}) - (\tilde{P}_{1,2,i})\) is a torsion divisor, and the result is clear.

**Remark 2.3.** If the base field has characteristic zero, then for fixed \(L_{i,j}\) the affine curve defined by (2.2) is non-singular except for finitely many values of \(\lambda\). In order to see this, it suffices to show that \(\partial_x f\) and \(\partial_y f\) are coprime, so that only finitely many points satisfy \(\partial_x f = \partial_y f = 0\). Because those points are independent of \(\lambda\), this excludes only finitely many values of \(\lambda\).

To prove this, suppose an irreducible \(h\) divides \(\partial_x f\) and \(\partial_y f\). The curve it defines can meet the \(L_{i,j}\) only at their points of intersection, since those are the singularities of the curve defined by \(\prod_{i=1}^{N} \prod_{j=1}^{N} L_{i,j}\).

Suppose \(h \neq 0\) at all of these points. Then \(h\) restricted to \(L_{i,j}\) has no zeroes, hence is a constant. The union of the \(L_{i,j}\) is connected as \(N \geq 2\), hence this is the same constant \(c\) for all lines. As the \(L_{i,j}\) are coprime, they all divide \(h - c\), which is impossible because of degrees.

If \(h = 0\) at one of these points, without loss of generality, suppose this is the case at the intersection of \(L_{1,1} = x\) and \(L_{2,1} = y\). Let \(u\) be the product of the lines that pass through the point \((0,0)\), and let \(\tilde{u}\) be the product of the other lines. Then \(h\) divides \(\tilde{u}\partial_x u + u\partial_x \tilde{u}\) as well as \(\tilde{u}\partial_y u + u\partial_y \tilde{u}\). By the definition of \(u\), it is a homogeneous polynomial, hence \(x\partial_x u + y\partial_y u = \text{deg}(u) u\). Therefore \(h\) divides \(u(\text{deg}(u) \tilde{u} + x\partial_x \tilde{u} + y\partial_y \tilde{u})\). Because \(\text{deg}(u) \tilde{u} + x\partial_x \tilde{u} + y\partial_y \tilde{u} \neq 0\) at \((0,0)\) by the definition of \(\tilde{u}\), we find \(h\) divides \(u\). But then \(h\) vanishes on one of the lines, which is impossible.

In order to simplify the notation, we denote \(a_i b_k - a_k b_i\) by \([i,k]\), so \([i,k] = -[k,i]\), and \([i,k] \neq 0\) if \(i \neq k\). We then define two types of elements in \(K_2(F)\) by
\[
R_{i,j,k,l,m,n} = \begin{cases} \frac{L_{i,j}}{L_{i,k}} & \text{if } i \neq l \\ \frac{L_{i,k}}{L_{i,l}} & \text{if } i = l \end{cases}, \quad T_{i,j,k,l,m,n} = \begin{cases} \frac{[i,m]}{[k,m]} & \text{if } i \neq k, l \neq m, n \neq i \\ [i,k] & [k,m] L_{i,j} & [m,k] L_{i,m} \end{cases}, \quad i, k, m \text{ distinct}.
\]

The \(R\)-element is constructed from two pairs of parallel lines (forming a parallelogram or rectangle), the \(T\)-element from three unparallel lines (forming a triangle).

Note that \(T_{i,j,k,l,m,n} = \frac{[i,m]L_{i,j}}{[k,m]L_{i,k}} \frac{[i,k]L_{m,n}}{[m,k]L_{i,m}} = -T_{k,i-j,m,n}\) is anti-symmetric for permuting the three pairs \((i,j), (k,l), (m,n)\). The \(R\)-element satisfies similar symmetries. Those will be stated in Lemma 2.7 below, together with some relations among the elements, but we first show all elements are in the kernel of the tame symbol.

**Lemma 2.6.** The \(R_{i,j,k,l,m,n}\) and \(T_{i,j,k,l,m,n}\) are in \(K_2^T(C)\).

**Proof.** First we compute the tame symbol of \(R_{i,j,k,l,m,n}\). Obviously, \(\frac{L_{i,j}}{L_{i,k}}\) and \(\frac{L_{i,k}}{L_{i,l}}\) only have zeroes and poles at \(\tilde{P}_{i,*}\) and \(\tilde{P}_{1,*}\), hence the tame symbol is trivial.
except at these points. Since \( \frac{L_{i,m}}{L_{i,n}}(P_t) = \frac{L_{k,m}}{L_{k,n}}(P_t) = 1 \), it is also trivial at these points.

Now consider \( T_{i,j,k,l;m,n} \). Denote \( \frac{[i,m]}{[k,m]} L_{i,j} \) and \( \frac{[i,k]}{[m,k]} L_{i,j} \) by \( h_1 \) and \( h_2 \) respectively. Then \( h_1 \) has zeroes and poles only at the \( P_{i,*} \) and \( P_{k,*} \), and \( h_2 \) has zeroes and poles only at the \( P_{i,*} \) and \( P_{m,*} \). The tame symbol is trivial at the \( P_{i,*} \) and \( P_{m,*} \) because \( h_1(P_m) = h_2(P_k) = 1 \). By anti-symmetry of the \( T \)-element, the same holds at the points \( P_{j,*} \).

We now give relations that will be used in Proposition 2.5 to reduce the number of generators for the subgroup \( V \) of \( K_3^E(\mathbb{C}) \) generated by the elements of types \( R \) and \( T \). Those are based on divisions and combinations of the polygons formed by lines. For example, if we consider two parallel lines \( L_{i,j} \), \( L_{i,k} \) and three parallel lines \( L_{i,m}, L_{i,n}, L_{j,p} \), we have three parallelograms such that one parallelogram is the combination of the other two. This gives the relation \( R_{i,j,k;l;m,n} = R_{i,j,k;l,m,n} + R_{i,j,k;l;m,n} \), which leads to relation (iii) in the next lemma. Similarly, if one has two horizontal lines, two vertical lines, and one diagonal line in general position, then one obtains a relation between four triangles and one rectangle, corresponding to relations (iv) and (v).

**Lemma 2.7.** We have the following relations among the \( R_{i,j,k;l;m,n} \) and \( T_{i,j,k;l;m,n} \).

(i) \( T_{i,j,k;l;m,n} = -T_{i,j,k;l;n,m} = -T_{i,j,m;n;k,l} \);
(ii) \( R_{i,j,k;l;m,n} = -R_{i,j,k;l;n,m} = -R_{i,m;n;i,j,k} \);
(iii) \( R_{i,j,k;l;m,n} = R_{i,j,k;l,q,m} - R_{i,j,k;l,q,n} - R_{i,j,k;l,q,n} + R_{i,j,k;l,q,m} ;
(iv) \( T_{i,j,k;l;m,n} = T_{i,j,k;l,m} - T_{i,j,k;l,m} - T_{i,j,k;l,m} + T_{i,j,k;l,m} ;
(v) \( T_{i,j,k;l;m,n} = T_{i,j,k;l,m} - T_{i,j,k;l,m} + T_{i,j,k;l,m} + T_{i,j,k;l,m} ;
(vi) \( T_{i,j,k;l;m,n} = T_{i,j,k;l,m} - T_{i,j,k;l,m} + T_{i,j,k;l,m} - T_{i,j,k;l,m} ;
(vii) \( T_{i,j,k;l;m,n} = T_{i,j,k;l,m} + T_{i,j,k;l,m} - T_{i,j,k;l,m} .

**Proof.** The first two parts are easy consequences of \( \{a^{-1}, b\} = \{a, b^{-1}\} = -\{a, b\} \) and \( \{a, b\} = -\{b, a\} \), and the third follows by working out \( \{a_1 a_2, b_1 b_2\} \) into the \( \{a, b\} \). Also, (v) is a consequence of (iv) and (i) by taking \( i, j, k, l, m, n, p, q \) in (iv) to be \( i, j, k, l, m, n, p, q \) in (iv) to be \( i, j, k, l, m, n \) respectively, and (vi) is a consequence of (vii) and (i). Thus it suffices to prove (iv) and (vii). Here we prove (vii), as (iv) is quite straightforward.

Note that \( T_{i,j;k;l,m,n} + T_{p,q;i,j;m,n} = T_{i,j;k,l,m,n} - T_{i,j;p,q;m,n} \) equals

\[
\begin{align*}
\frac{[i,m]}{[k,m]} L_{i,j} \times \frac{[i,k]}{[m,k]} L_{i,j} - \frac{[i,m]}{[p,m]} L_{i,j} \times \frac{[i,p]}{[m,p]} L_{i,j} = \\
\left\{ \frac{[i,m]}{[k,m]} L_{i,j} \times \frac{[i,k]}{[m,k]} L_{i,j} + \frac{[i,m]}{[p,m]} L_{i,j} \times \frac{[i,p]}{[m,p]} L_{i,j} \right\}.
\end{align*}
\]

Then \( T_{p,q;i,j;k,l;m,n} = T_{k,l;m,n;p,q} + T_{i,j;k,l;p,q} \) is equal to

\[
\left\{ \frac{[p,k]}{[k,i]} L_{p,q} \times \frac{[i,k]}{[m,k]} L_{i,j} + \frac{[m,k]}{[p,k]} L_{p,q} \times \frac{[i,k]}{[m,k]} L_{i,j} \right\}.
\]
Therefore $T_{i,j;k,l;m,n} + T_{p,q;i,j;m,n} - (T_{p,q;i,j;k,l} + T_{p,q;k,l;m,n})$ equals

\[
\left\{ \begin{array}{c}
[p, m][k, i] & [i, k] L_{m,n} \\
[k, m][p, i] & [m, k] L_{i,j}
\end{array} \right\} + \left\{ \begin{array}{c}
[i, m][p, k] L_{m,n} & [i, k][m, p] \\
[p, m][m, k] & [m, k][i, p]
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{c}
[p, m][k, i] & [i, k][p, m] \\
[k, m][p, i] & [i, m][p, k] \\
[p, m][k, i] & [p, m][k, i]
\end{array} \right\},
\]

which is trivial because $[k, m][p, i] - [i, m][p, k] = [p, m][k, i]$. This proves (vii). \(\Box\)

**Proposition 2.8.** Let $V$ be the subgroup of $K_2^T(C)$ generated by all the elements $R_{i,j;k,l;m,n}$ and $T_{i,j;k,l;m,n}$. Then $V$ is generated by the following elements:

- $R_{1,1;j,2,1;m}$, $1 < j \leq N_1, 1 < m \leq N_2$;
- $T_{1,1;k,l;m,n}$, $2 \leq k < m \leq N_1, 1 \leq l \leq N_k, 1 \leq n \leq N_m$;
- $T_{1,1;j,2,1;m,n}$, $2 < j \leq N_1, 3 \leq m \leq N_1, 1 \leq n \leq N_m$.

**Proof.** We use the identities in Lemma [22]. First we prove that any $R_{i,j;k,l;m,n}$ is a linear combination of these elements. We can suppose $i < l$ by (ii). If $i = 1$ and $l = 2$, it is a consequence of (iii) by letting $p = q = 1$. If $i = 1$ and $l \neq 2$, it is a consequence of (iv) by letting $p = 2$ and $q = 1$, and using (i). If $i \neq 1$, then we reduce to the case $i = 1$ by using (iv) with $p = q = 1$.

Now we consider $T_{i,j;k,l;m,n}$. By (i), we can suppose $i < k < m$. If $i = 1$ and $k = 2$, it is a consequence of (v) by letting $p = q = 1$. If $i = 1$ and $k \neq 2$, it is a consequence of (vi) by letting $p = r = 1$ and $q = 2$. If $i \neq 1$, it is a consequence of (vii) by letting $p = q = 1$. \(\Box\)

We now show that using all symbols in $K_2(F)$ with as entries functions with divisors supported in the $\bar{P}_{i,j}$ does not really lead to a larger subgroup of $K_2^T(C)$ than $V$.

**Proposition 2.9.** Let $A \subseteq K_2(F)$ be the subgroup generated by symbols $\{f_1, f_2\}$ where $|(f_i)| \subseteq \{\bar{P}_{i,j}\}$. Then there is a positive integer $a$ such that $a(A \cap K_2^T(C))$ is contained in the sum of $V$ and $K_2$ of the base field.

**Proof.** From our earlier discussion of the $\bar{P}_{i,j}$ it is clear that if $|(f_i)| \subseteq \{\bar{P}_{i,j}\}$, then there is a fixed positive integer such that $f_i$ raised to this power is a product of powers of the $L_{i,j}/L_{1,1}$ and a non-zero constant. Multiplying any element in $A$ by the square of this positive integer, and expanding, we see that the result can be expressed in terms of elements of type $T$, \((L_{i,j}/L_{1,1})^{k_1} \cdot (L_{k,l}/L_{1,1})^{k_2}\) with $k > 1$, \(c \cdot L_{m,n}/L_{i,j}\) where $c$ is a non-zero constant, and an element of $K_2$ of the base field.

Let us fix $j$. Then for $k_1, k_2 > 1$ we have

\[
\left\{ \begin{array}{c}
L_{i,j} \cdot L_{k_1, l_1} & L_{k_1, l_1} \\
L_{k_2, l_2} & L_{k_2, l_2}
\end{array} \right\} = \left\{ \begin{array}{c}
L_{i,j} \cdot L_{k_1, l_1} \\
L_{k_2, l_2}
\end{array} \right\} = \left\{ \begin{array}{c}
L_{i,j} \\
L_{k_2, l_2}
\end{array} \right\} \cdot \left\{ \begin{array}{c}
L_{k_1, l_1} \\
L_{k_2, l_2}
\end{array} \right\}.
\]

If $k_1 = k_2 = k$ this equals $R_{i,j,1;k,1;l,1}$, and, because it also equals $\left\{ \begin{array}{c}
L_{i,j} \\
L_{k_2, l_2}
\end{array} \right\} \cdot \left\{ \begin{array}{c}
L_{k_1, l_1} \\
L_{k_2, l_2}
\end{array} \right\}$, for $k_1 \neq k_2$ it is the sum of two $T$-elements, elements of the form \(c \cdot L_{m,n}/L_{i,j}\), and an element in $K_2$ of the base field.

From $\lambda \prod_{i=1}^{N} L_{i,j} = 1$ we obtain $\sum_{k=1}^{N} \sum_{l=1}^{N_k} \left\{ \begin{array}{c}
L_{i,j} \\
L_{k_1, l_1}
\end{array} \right\} = \left\{ \lambda, \frac{L_{i,j}}{L_{1,1}} \right\} + d \{L_{1,1}, -L_{1,1}\}$, where $d = \sum_{i=1}^{N} N_i$. Combining these two facts we see that
functions $L_m/L_{1,1}$ for $k > 1$ is the sum of $R$-elements, $T$-elements, elements of the form $\{c, \frac{L_{m,n}}{L_{i,j}}\}$, and an element in $K_2$ of the base field. Hence there is a fixed positive integer such that if we multiply an element in $A$ by this integer, then the result can be expressed in those four types of elements.

Now suppose this expression lies in $K^+_2(C)$. Collecting terms of the form $\{c, \frac{L_{m,n}}{L_{i,j}}\}$ for fixed $m$ and $n$, we may assume there is only such term $\{c_{m,n}, \frac{L_{m,n}}{L_{i,j}}\}$ for each pair $(m, n)$. It is trivial when $(m, n) = (1, 1)$. For the other pairs, the divisors of the functions $L_{m,n}/L_{1,1}$ are linearly independent as our earlier calculations show that there is only one relation among the divisors of the $L_{i,j}$, which must correspond to the identity $\lambda \prod_{i=1}^N L_{1,i} = 1$. Because elements of type $R$ and $T$ are in $K^+_2(C)$, it follows that each $c_{m,n}$ is a root of unity of order dividing some fixed positive integer. Multiplying the expression by this integer we obtain an element in the sum of $V$ and $K_2$ of the base field.

The completion in $\mathbb{P}^2$ of the affine curve defined by (2.2) has points $P_i$ at infinity, of multiplicity $N_i$ for $i = 1, \ldots, N$. If $N_i \geq 2$ then $P_i$ is a simple singular point. Suppose there are no other singular points, which in characteristic zero is in general the case by Remark 2.3. By the degree-genus formula, the genus $g$ of $C$ then equals

$$\sum_{i=1}^N N_i - 1 \left( \sum_{i=1}^N \frac{N_i}{2} \right) = \sum_{1 \leq i < j \leq N} N_i N_j - \sum_{1 \leq i \leq N} N_i + 1.$$  

By Proposition 2.8 the number of linearly independent elements we get is at most $(N_1 - 1)(N_2 - 1) + \sum_{2 \leq i < j \leq N} N_i N_j + (N_1 - 1) \sum_{3 \leq j \leq N} N_j = g$. So if we take $\mathbb{Q}$ as our base field, and we can show that these elements are integral by imposing some condition on the equation of the lines, and linearly independent, then we have as many elements as predicted by Beilinson’s conjecture. This is what we shall do in the next two sections, but in greater generality.

3. The case where $N = 2$ or 3

In this section, we work over an arbitrary base field of characteristic zero, and examine the special case where $N = 2$ or 3. We determine when the curve $C$ is (hyper)elliptic in those cases. If the base field is a number field, then we also show the integrality of the elements in (2.3) and (2.3) under certain conditions. We actually start with the latter as its proof is short.

Let $C$ be a regular, proper, geometrically irreducible curve over some number field, for which we defined $K^+_2(C)_{\text{int}}$ in Section 1. On p. 344, where the base field was $\mathbb{Q}$, it was stated without proof that $K^+_2(C)_{\text{int}}/ \text{torsion}$ (there denoted $K_2(C; \mathbb{Z})$) is independent of the choice of the regular, proper model $C$. In fact, we have the following result, which we include for the sake of completeness.

**Proposition 3.1.** The subgroup $K^+_2(C) \subseteq K^+_2(C)$ is independent of the choice of $C$.

**Proof.** One sees as on p. 13 that the image of $K_2(C)$ in $K_2(C)$ under localization is independent of $C$. Now consider the Gersten-Quillen spectral sequence $E_1^{p,q}(C) = \prod_{g \in \mathbb{C}^{(p)}} K_{p-q}(\mathbb{R}(g))$ which is compatible with the one for $C$ under localization. Then the image of $K_2(C)$ in the quotient $E^{0,-2}_\infty(C)$ of
proof. 
The elements $\alpha$, $\beta_j$, and $\gamma_k$ are algebraic integers, with $
abla_1 \geq \nabla_2 \geq \nabla_3$, and we take $\nabla_3 = 0$ if $\nabla = 2$. (Note that all $\alpha_i$ are distinct, all $\beta_j$ are distinct, and all $\gamma_k$ are distinct.) We again let $C$ denote the non-singular model of the closure in $\mathbb{P}^2$ of the affine curve defined by (2.2). In this case, we have $P_1 = [0,1,0]$, $P_2 = [1,0,0]$ and, if $\nabla = 3$, $P_3 = [1,1,0]$. If the affine curve is non-singular, then $C$ has genus $g = \nabla_1 \nabla_2 + \nabla_1 \nabla_3 + \nabla_2 \nabla_3 - \nabla_1 - \nabla_2 - \nabla_3 + 1$, also if $\nabla_3 = 0$. With our conventions $g \geq 1$ unless $\nabla_2 = 1$ and $\nabla_3 = 0$, and $g = 1$ occurs only for $\nabla_1 = \nabla_2 = 2$ and $\nabla_3 = 0$, or $\nabla_1 = \nabla_2 = \nabla_3 = 1$.

Theorem 3.3. Let all notation be as above, and assume $\lambda$, and all the $\alpha_i$, $\beta_j$, and $\gamma_k$ are algebraic integers, with $\lambda \neq 0$. Then the elements given by (2.4) and (2.5) are integral.

Proof. We first assume $\nabla_3 \geq 1$, so that by Lemma 2.4 it suffices to prove this for the elements $M = \{h_1, h_2\}$ with $h_1 = \frac{x + \alpha_i}{y - x + \gamma_k}$ and $h_2 = \frac{y + \beta_j}{y - x + \gamma_k}$, which is of $T$-type.

We can obtain a regular proper model $C$ of $C$ as follows. Let $O$ be the ring of algebraic integers in the base field. We start with the arithmetic surface $C'$ in $\mathbb{P}^2$ defined by the homogeneous polynomial $F$ of degree $\nabla_1 + \nabla_2 + \nabla_3$ associated to (3.2). We then blow up the closure of the singularities of the generic fibre of $C'$. The resulting surface has generic fibre $C$, and any singularities of the surface are contained in its fibres at prime ideals $\mathfrak{P}$ of $O$. Those are resolved through iterated blow-ups, resulting in our model $C$.

Now let $D$ be an irreducible component of some $C_{\mathfrak{P}}$. We have to show that $T_D(M) = 1$. The image of $D$ inside $C_{\mathfrak{P}}' \subset \mathbb{P}^2$ is either an irreducible component of $C_{\mathfrak{P}}'$, or a point of that curve. Note that if $\lambda$ is in $\mathfrak{P}$, then $C_{\mathfrak{P}}'$ is defined in $\mathbb{P}^2_{\mathfrak{P}/\mathfrak{P}}$ by the reduction of $F$ modulo $\mathfrak{P}$, which is still irreducible. If $\lambda$ is in $\mathfrak{P}$, then $C_{\mathfrak{P}}'$ only has the line at infinity as component. In either case the $h_1$ do not have a zero or a pole along the only irreducible component of $C_{\mathfrak{P}}'$. So if the image of $D$ in $C_{\mathfrak{P}}$ is not a point, then $v_D(h_1) = v_D(h_2) = 0$, and $T_D(M) = 1$. 


Now assume that $D$ maps to a point of $C'_m$. If it maps to an affine point of $C'_m$, then $\lambda$ is not in $\mathfrak{P}$, $h_1$ and $h_2$ are regular at that point and attain non-zero values. Therefore they are constant and non-zero on $D$, hence $v_D(h_1) = v_D(h_2) = 0$, and $T_D(M) = 1$. The same holds if $D$ maps to a point at infinity in $C'_m$ not equal to the reductions of $P_1$, $P_2$ or $P_3$. (Note this can only happen if $\lambda$ is in $\mathfrak{P}$.) The remaining case is when it maps to the reduction of one of the $P_j$. By Lemma 2.7, the $T$-elements are anti-symmetric for renumbering the $P_j$, so we may assume $D$ maps to $[0,1,0]$ in $C'_m$. But on $C'$ the function $h_2$ is regular and equal to 1 at this point, so $h_2$ is constant and equal to 1 along $D$. Therefore $T_D(M) = 1$ also in this case.

For $N_3 = 0$, all elements are of $R$-type, and the proof (using the same model $C$) is similar. $\square$

We now return to the case of an arbitrary ground field of characteristic zero. Since a lot of work has been done to find elements of $K_2$ of (hyper)elliptic curves \cite{13, 12}, we want to know if $C$ is (hyper)elliptic. If it is not, it means that we found a new class of curves that is geometrically more general.

Taking $N_3 = 1$, we obtain curves of genus $N_1N_2$. By the next proposition, for $N_2 = 1$ we obtain (hyper)elliptic curves of arbitrary positive genus, and for $N_1 \geq N_2 = 2$ we obtain non-hyperelliptic curves of arbitrary composite genus. In the non-hyperelliptic case not all primes occur as genus, but taking $N_2 = N_3 = 2$ already yields all primes congruent 1 modulo 6. Similarly, if $N_3 = 0$ and $N_2 \geq 3$, we find non-hyperelliptic curves of arbitrary composite genus $(N_1 - 1)(N_2 - 1)$.

**Proposition 3.4.** Suppose the affine curve defined by \eqref{eq:3.2} is non-singular and that $C$ has positive genus. Then $C$ is (hyper)elliptic if and only if either $N_2 = N_3 = 1$, or $N_2 = 2$ and $N_3 = 0$.

**Proof.** To prove this we shall use the following criterion of Max Noether (see \cite{15, 3} p.119). Suppose some $h_i(x,y)dx$ span the space of holomorphic differentials on $C$. Consider the quadratic combinations \{h_i(x,y)h_j(x,y)\}_{i,j}$ of functions on $C$. If $C$ has genus $g \geq 1$, then they generate a space of dimension at least $2g - 1$, and the curve is (hyper)elliptic if and only if this dimension equals $2g - 1$.

We first assume that $N_3 \geq 1$. We can change the coordinates such that all $\alpha_i, \beta_j, \gamma_k \neq 0$. We shall first show that the forms

$$\Omega_{i,j,k} = x^{i}y^{j}(x - y)^{k}dx \overline{\partial_{y}f(x,y)} \quad (0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1, 0 \leq k \leq N_3 - 1)$$

generate $H^0(C, \Omega_C)$.

Because $\frac{dx}{\partial_{x}f(x,y)} = -\frac{dy}{\partial_{x}f(x,y)}$ and $\partial_x f(x,y), \partial_y f(x,y)$ do not vanish at the same time by the non-singularity assumption, the $\Omega_{i,j,k}$ are regular except possibly at the points $P_{i,j}$ above $P_1, P_2$ and $P_3$. The order of $x, y$ and $x - y$ is 0, −1 and −1 at each of the points above $P_1$, −1, 0 and −1 at the points above $P_2$, and −1, −1 and 0 at the points above $P_3$ respectively. At the points above $P_1$, the order of $x + \alpha_i$ is $N_2 + N_3$ for exactly one of them and 0 for the rest. Hence the order of $\frac{dx}{\partial_{y}f(x,y)}$ is $N_2 + N_3 - 1$ at all points above $P_1$ and $-2$ at all points above $P_2$ or $P_3$. The order of $\frac{dy}{\partial_{y}f(x,y)}$ is $-1$, $N_1 + N_3$ and $N_1 + N_2$ at each of the points above $P_1, P_2$ and $P_3$ respectively, because $\partial_y f(x,y) = \sum_{j=1}^{N_2} \frac{1}{y + \beta_j} + \sum_{k=1}^{N_3} \frac{1}{y - x + \gamma_k}$.\[\square\]
the vector space generated by these forms. Consider the polynomials
\[ P_i := \sum_{j,k} \Omega_{i,j,k} U^j V^k \]
for 0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1, 0 \leq k \leq N_3 - 1 are in \( H^0(C, \Omega_C) \). Let \( V \) be the vector space generated by these forms. Consider the polynomials \( x^i y^j (y - x)^k \)
for 0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1, 0 \leq k \leq N_3 - 1. As all \( N_i \geq 1 \), there are \((N_1 - 1)(N_2 - 1)(N_3 - 1)\) linear relations between these polynomials. Since the degree of these polynomials is less than the degree of \( f(x,y) \), all the linear relations between the \( \Omega_{i,j,k} \) come from the linear relations between these polynomials. We conclude that the dimension of \( V \) is \( N_1 N_2 N_3 - (N_1 - 1)(N_2 - 1)(N_3 - 1) = g \), so \( V = H^0(C, \Omega_C) \).

In order to compute the dimension of the space spanned by the quadratic combinations, let us start with the space \( W \) of polynomials spanned by the
\[ x^i y^j (x - y)^k \quad (0 \leq i \leq 2N_1 - 2, 0 \leq j \leq 2N_2 - 2, 0 \leq k \leq 2N_3 - 2). \]
It has dimension \((2N_1 - 1)(2N_2 - 1)(2N_3 - 1) - (2N_1 - 2)(2N_2 - 2)(2N_3 - 2)\) as all \( N_i \geq 1 \). On \( W \) we have to impose the relation given by \( f \). If \( f \) divides some \( w \) in \( W \), so \( w = uf \), then \( H(w) = H(u) H(f) = x^{N_1} y^{N_2} (x - y)^{N_3} H(u) \), where \( H(\cdot) \) denotes the highest degree term of a polynomial. Since \( w \) is in \( W \), \( H(w) \) is also in \( W \), and we can deduce that \( H(u) \) is in the space \( U \) spanned by the
\[ x^i y^j (x - y)^k \quad (0 \leq i \leq N_1 - 2, 0 \leq j \leq N_2 - 2, 0 \leq k \leq N_3 - 2). \]
Similarly, the highest term of \( u - H(u) \) is in \( U \), so \( u \) is in \( U \). Hence the space of quadratic combinations of \( V \) has dimension equal to \( \dim(W) - \dim(U) \). Note that the dimension of \( U \) is \((N_1 - 1)(N_2 - 1)(N_3 - 1) - (N_1 - 2)(N_2 - 2)(N_3 - 2)\) if \( N_1, N_2, N_3 \geq 2 \) and 0 otherwise. In order to apply Noether’s criterion, we compute \( \dim(W) - \dim(U) - (2g - 1) \), and find
\[
\begin{cases}
N_1 N_2 + N_1 N_3 + N_2 N_3 - N_1 - N_2 - N_3 - 1, & \text{if } N_1, N_2, N_3 \geq 2, \\
2(N_1 - 1)(N_2 - 1), & \text{if } N_3 = 1.
\end{cases}
\]
Because \( N_1 \geq N_2 \geq N_3 \geq 1 \), this is 0 only if \( N_2 = N_3 = 1 \).

Similarly, if \( N_3 = 0 \) but \( N_2 \geq 2 \) (because we are assuming the genus is positive), the forms
\[ \Omega_{i,j} = \frac{x^i y^j dx}{\partial_y f(x,y)} \quad (0 \leq i \leq N_1 - 2, 0 \leq j \leq N_2 - 2) \]
generate \( H^0(C, \Omega_C) \). Then we can also consider the space of polynomials \( W \) and relations \( U \). It is easy to see that
\[
\dim(W) - \dim(U) - (2g - 1) = \begin{cases}
(N_1 - 1)(N_2 - 1) - 2, & \text{if } N_2 \geq 3, \\
0, & \text{if } N_2 = 2.
\end{cases}
\]
Because $N_1 \geq N_2$, this is 0 if and only if $N_2 = 2$.

In order to compare a curve $C$ satisfying the conditions in Proposition 3.4 with the (hyper)elliptic curves studied in [12], we describe it as a ramified cover of $\mathbb{P}^1$ of degree 2, starting from (3.2).

Firstly, if $N_2 = 2$, $N_3 = 0$, then we can assume that $\alpha_1 = \beta_1 = 0$ and $\beta_2 = 1$ by replacing $x$ with $x - \alpha_1$ and $y$ with $(\beta_2 - \beta_1)y - \beta_1$. Renumbering the non-zero $\alpha_i$ and scaling $\lambda$, we find $C$ is defined by

$$\lambda xy(y + 1) \prod_{i=1}^{g}(x + \alpha_i) - 1,$$

where $g = N_1 - 1$ is the genus of the curve, and all $\alpha_i$ are non-zero and distinct. Replacing $x$ with $\frac{1}{x}$ and $y$ with $\frac{y + x^{g+1}}{\lambda \prod_{i=1}^{g}(\alpha_i x + 1)}$, the equation of the curve becomes

$$y(y + 2x^{g+1} + \lambda \prod_{i=1}^{g}(\alpha_i x + 1)) + x^{2g+2}.$$

The element $\{\frac{y+1}{y}, \frac{x+\alpha_i}{x}\}$ of type (2.4) for $C$ corresponding to (3.5), is transformed into

$$M_i = \left\{\frac{y + x^{g+1} + \lambda \prod_{i=1}^{g}(\alpha_i x + 1)}{y + x^{g+1}}, \alpha_i x + 1\right\} = \left\{-\frac{x^{g+1}}{y}, \alpha_i x + 1\right\}$$

for the curve corresponding to (3.6). So $-2M_i$ is the element in [12, Construction 6.11] for the factor $\alpha_i x + 1$, which was considered in Example 10.8 of loc. cit. for $\lambda = \pm 1$.

Secondly, if $N_2 = N_3 = 1$, we can assume that $\beta_1 = \gamma_1 = 0$ by replacing $x$ with $x - \beta_1 + \gamma_1$ and $y$ with $y - \beta_1$. So $C$ is defined by

$$\lambda(y - x)y \prod_{i=1}^{g}(x + \alpha_i) - 1,$$

where $g = N_1$ and the $\alpha_i$ are distinct. Replacing $x$ with $\frac{1}{x}$ and $y$ with $\frac{y + x^{g+2}}{\lambda \prod_{i=1}^{g}(\alpha_i x + 1)} + \frac{1}{x}$, the equation of the curve becomes

$$y(y + 2x^{g+2} + \lambda \prod_{i=1}^{g}(\alpha_i x + 1)) + x^{2g+4}.$$

The element

$$\left\{\frac{y}{y - x} - \frac{x + \alpha_i}{y - x}\right\} = \left\{\frac{y}{y - x} - \frac{x + \alpha_i}{y - x}\right\} - \left\{\frac{y}{y - x} - \frac{x}{y - x}\right\} = \left\{\frac{y}{y - x} - \frac{x + \alpha_i}{x}\right\}$$

of type (2.5) for $C$ defined by (3.7), becomes $M_i = \{-\frac{x^{g+2}}{y}, \alpha_i x + 1\}$ for the curve defined by (3.8).

Since $C$ defined by (3.8) is studied in [12, Example 10.8], we concentrate on $C$ defined by (3.8). Write

$$A(x) = 2x^{g+2} + \lambda \prod_{i=1}^{g}(\alpha_i x + 1),$$
so that (3.8) becomes
\[
\left( y + \frac{A(x)}{2} \right)^2 = \frac{\lambda^2}{4} \prod_{i=1}^{g}(\alpha_i x + 1) \prod_{j=1}^{g+2}(\mu_j x + 1)
\]
where \(2g^2 + A(x) = \lambda \prod_{j=1}^{g+2}(\mu_j x + 1)\) for some \(\mu_j\), and we extend the base field to contain the \(\mu_j\) if necessary. With
\[
\tilde{M}_j = \left\{ -\frac{x^{g+2}}{y}, \mu_j x + 1 \right\},
\]
\[
\mathcal{M} = \{ -y, -x \},
\]
\[
\mathcal{N} = \left\{ -\frac{x^{g+2}}{y}, -\frac{x^{g+2}}{\lambda} \right\},
\]
we have the following proposition similar to Propositions 6.3 and 6.14 of [12].

**Proposition 3.9.** Let all notation be as above, with the \(\alpha_1, \ldots, \alpha_g\) distinct. Assume the resulting curve has genus \(g\) (or, equivalently, that \(\mu_1, \ldots, \mu_{g+2}\) are all distinct).

1. The \(\tilde{M}_j\) and \(\mathcal{N}\) are in \(K^T_2(C)\), and we have
\[
2 \sum_{i=1}^{g} M_i + 2 \sum_{j=1}^{g+2} \tilde{M}_i = 4\mathcal{N}
\]
\[
\sum_{i=1}^{g} M_i = \sum_{j=1}^{g+2} \tilde{M}_j.
\]
If \(\lambda^d = 1\) then \(d\mathcal{M}\) is in \(K^T_2(C)\), and \(2d\mathcal{N} = -2(g+2)d\mathcal{M}\).

2. If the base field is a number field, and \(\alpha_1, \ldots, \alpha_g\) are algebraic integers with \(\lambda\) a unit, then \(2\mathcal{N}\) and the \(2\tilde{M}_j\) are in \(K^T_2(C)_{\text{int}}\), and if \(\lambda^d = 1\) then \(d\mathcal{M}\) is in \(K^T_2(C)_{\text{int}}\).

**Remark 3.10.** By the earlier coordinate transformations and Lemma 2.6, the \(M_i\) are in \(K^T_2(C)\). If the base field is a number field and \(\alpha_1, \ldots, \alpha_g\) are algebraic integers with \(\lambda \neq 0\), then they are in \(K^T_2(C)_{\text{int}}\) by Theorem 4.8.

**Proof.** First we look at the points at infinity of the curve. For this, let \(x = 1/\tilde{x}\) and \(y = (\tilde{x}y - 1)/\tilde{x}^{g+2}\). Then equation (3.8) becomes
\[
(y^2 + \lambda(\tilde{x}\tilde{y} - 1) \prod_{i=1}^{g}(\tilde{x} + \alpha_i) = 0.
\]
If all \(\alpha_i \neq 0\) then there are two points at infinity, namely \((0, \pm \sqrt[\lambda]{\prod_{i=1}^{g} \alpha_i})\), which we denote by \(\infty\) and \(\infty'\). If some \(\alpha_i = 0\) then there is only one point at infinity, and we let \(\infty\) and \(\infty'\) both denote this point.

Let \(P_{\mu_j}\) be the point \((-\mu_j^{-1}, -\mu_j^{-1})^{g+2}\), and let \(O\) and \(O'\) be the points \((0, 0)\) and \((0, -\lambda)\) respectively. Then
\[
\text{div}(x) = (O) + (O') - (\infty) - (\infty'),
\]
\[
\text{div}(y) = (2g+4)(O) - (g+2)(\infty) - (g+2)(\infty'),
\]
\[
\text{div}(\mu_j x + 1) = 2(P_{\mu_j}) - (\infty) - (\infty'),
\]
\[
\text{div}\left(-\frac{x^{g+2}}{y}\right) = (g+2)(O') - (g+2)(O).
\]
Since \(-\frac{x^{g+2}}{y}\) and \(\mu_j x + 1\) only have zeros and poles at \(P_{\mu_j}, O, O', \infty\) and \(\infty'\), the tame symbol of \(\tilde{M}_j\) is trivial except at these points. We discuss the case when all \(\alpha_i \neq 0\), the other case being simpler. Then for these points we have

\[
TP_{\mu_j}(\tilde{M}_j) = (-1)^{0} \left( \frac{x^{g+2}}{y} \right)^{2} \bigr|_{P_{\mu_j}} = 1,
\]

\[
T_{O}(\tilde{M}_j) = (-1)^{0} \left( \frac{1}{(\mu_j x + 1)^{(g+2)}} \right) \bigr|_{O} = 1,
\]

\[
T_{O'}(\tilde{M}_j) = (-1)^{0} \left( \frac{1}{(\mu_j x + 1)^{(g+2)}} \right) \bigr|_{O'} = 1,
\]

\[
T_{\infty}(\tilde{M}_j) = (-1)^{0} \left( \frac{y}{x^{g+2}} \right) \bigr|_{\infty} = 1 - \frac{\tilde{x}\tilde{y}}{\infty} = 1,
\]

\[
T_{\infty'}(\tilde{M}_j) = (-1)^{0} \left( \frac{y}{x^{g+2}} \right) \bigr|_{\infty'} = 1 - \frac{\tilde{x}\tilde{y}}{\infty'} = 1.
\]

For \(\mathcal{M}\), we only need to calculate the tame symbol \(O, O', \infty\) and \(\infty'\), which gives

\[
T_{O}(\mathcal{M}) = (-1)^{2g+4} \left( \frac{-y}{(-x)^{2g+4}} \right) \bigr|_{O} = \lambda^{-1},
\]

\[
T_{O'}(\mathcal{M}) = (-1)^{0} \left( -y \right) \bigr|_{O'} = \lambda,
\]

\[
T_{\infty}(\mathcal{M}) = (-1)^{g+2} \left( \frac{-y}{(-x)^{(g+2)}} \right) \bigr|_{\infty} = \frac{1}{1 - \tilde{x}\tilde{y}} \bigr|_{\infty} = 1,
\]

\[
T_{\infty'}(\mathcal{M}) = (-1)^{g+2} \left( \frac{-y^{-1}}{(-x)^{(g+2)}} \right) \bigr|_{\infty'} = \frac{1}{1 - \tilde{x}\tilde{y}} \bigr|_{\infty'} = 1.
\]

Here the first calculation uses that \(-x^{2g+1}/y = y + 2x^{g+2} + \lambda \prod_{j=1}^{g} (\alpha_i x + 1)\) by (3.8). From this our claim for \(d\mathcal{M}\) is clear. One checks similarly that \(\mathcal{N}\) is in \(K_2^F(C)\). That \(2d\mathcal{N} = -2(g + 2)d\mathcal{M}\) is easily checked.

The remaining relations in (1) can be proved as in [12] Propositions 6.3 and 6.14. The statement about integrality can be proved as in [12] Theorem 8.3 except we use the model defined by (4.11) at the points at infinity. Note that the \(\mu_j\) are algebraic integers by our assumption on \(\lambda\) and the \(\alpha_i\). \(\square\)

4. LINEAR INDEPENDENCE OF THE ELEMENTS

In this section, we look at the curves \(C\) constructed in Section 2 as a family with one parameter \(t = 1/\lambda\). Taking our base field to be \(\mathbb{C}\), we give a description of part of a basis of \(H_{1}(C; \mathbb{Z})\) when \(t\) is close to zero. We compute the limit behaviour of the pairing (1.3) of those elements with the elements in Proposition 2.8 showing in particular that the latter are usually linearly independent. If we are working over \(\mathbb{R}\) then those elements form a basis of \(H_{1}(C; \mathbb{Z})\), and we obtain a non-vanishing result for the regulator. In particular, working over \(\mathbb{Q}\), we find examples of curves of genus \(g\) with \(g\) independent integral elements in \(K_2^F(C)\).

For fixed \(a_i, b_i\) and \(c_{ij}\), we view the desingularization of the projective closure of the curves in (2.2) as a family in the parameter \(t = 1/\lambda\). Note that the affine
part is now defined by
\[(4.1) \prod_{i=1}^{N} \prod_{j=1}^{N} L_{i,j} - t\]
for \(t \neq 0\), but we shall use the resulting curve for \(t = 0\) extensively. By Remark 2.8 if we take \(t\) in a small enough disc \(D\) around 0, then the resulting family \(X \to D\) has smooth fibres \(X_t = C_{1/t}\) when \(t \neq 0\), of genus \(g\) given by \(2.10\). The fibre \(X_0\) consists of \(\mathbb{P}^1\)s corresponding to the \(L_{i,j}\). Note that the points of intersection of those \(\mathbb{P}^1\)s correspond to the affine points of intersection of the \(L_{i,j}\).

For the remainder of this section we make the following assumption.

**Assumption 4.2.** No three of the \(L_{i,j}\) meet in an affine point.

We shall make certain elements in \(H_1(X_t; \mathbb{Z})\) with \(t \neq 0\) more explicit than in the general theory (see, e.g., [1, Exposé XV, Théorème 3.4]), which will enable us to carry out the necessary calculations in Lemma 4.5 below.

Changing coordinates, we may assume that for each value of \(x\) there are \(d\) or \(d-1\) points \((x,y)\) in \(X_0\), with the latter case occurring for the \(x\)-coordinate of the point of intersection of two lines. Let \(\pi\) be the projection from \(C \times D\) to \(\mathbb{A}^1 \times D\) obtained by mapping \((x,y,t)\) to \((x,t)\). Then for every affine intersection point \(s\) of two of the lines, we take a small loop in \(\mathbb{A}^1\) (interpreted as the complex plane) around the first coordinate of \(\pi(s)\). After taking the product with \(D\), and shrinking \(D\) if necessary to avoid ramification points, we can lift the result under \(\pi\) by the implicit function theorem, where we lift the loop for \(t = 0\) to one of the two lines passing through \(s\). Note that this results in a family of closed loops \(\gamma_s\) for \(t \in D\): the lift is unique, and for \(t = 0\) the loop is closed.

For the calculation of the limit behaviour of the pairing with the elements in Proposition 2.8 we construct \(g\) elements in \(H_1(X_t; \mathbb{Z})\) for \(t \neq 0\) in \(D\). In order to describe them, let \(P_{i,j;k,l}\) be the (affine) point of intersection of the lines \(L_{i,j}\) and \(L_{k,l}\) for \(i \neq k\), and consider the set
\[S = \{P_{i,j;k,l}| 1 < i < k \leq N, 1 < j \leq N, 1 < l \leq N, (i,j) \neq (1,1), (i,k,l) \neq (1,2,1)\} .\]

It consists of all points of intersection of the \(L_{i,j}\), except for the points of intersection of \(L_{1,1}\) with the other lines, as well as the points of intersection of \(L_{2,1}\) with the \(L_{1,j}\) with \(1 < j < N\). (For example, for \(N = 3\), if in Figure 4.1 the \(L_{1,j}\), \(L_{2,l}\), and \(L_{3,n}\) are the diagonal, horizontal, and vertical lines respectively, then the elements of \(S\) are the thick points.) There are \(\sum_{1 < i < k < N} N_i N_k \sum_{1 < k < N} N_k + 1 = g\) points in \(S\), and, by the discussion above, if \(D\) is small enough, then for every \(s\) in \(S\) we have a family of loops \(\gamma_s\) in the fibres of the \(X_t\) with \(t\) in \(D\).

**Lemma 4.3.** With notation and assumptions as above, for \(t \neq 0\) and \(|t|\) small enough:

1. \(\{\gamma_s\}_{s \in S}\) can be complemented to a basis of \(H_1(X_t; \mathbb{Z})\);
2. if all \(a_i, b_i, c_{i,j}\), and \(t\) are in \(\mathbb{R}\), then \(\{\gamma_s\}_{s \in S}\) is a basis of \(H_1(X_t; \mathbb{Z}^-)\).

**Proof.** If all \(a_i, b_i, c_{i,j}\) and \(t\) are real, then \(\gamma_s\) is anti-invariant under complex conjugation by uniqueness of the lift under \(\pi\). So the \(\gamma_s\) are in \(H_1(X_t; \mathbb{Z})^-\), and the second part of the lemma follows from the first as \(H_1(X_t; \mathbb{Z})^-\) has rank \(g\).

In order to prove the first part, we construct another family of loops \(\{\delta_s\}_{s \in S}\).

Note that, for \(s\) the affine point of intersection of two of the lines, \(l_1\) and \(l_2\), (4.1) is
of the form $l_1l_2h - t$ where $h(s) \neq 0$. One then sees easily that for $t \neq 0$ but small, the point $s$ splits into two ramification points, and we can parametrize one of the two ramification points for $t$ in a suitable circle sector of $D$. Using this, one sees that any loop $\delta$ in $X_0$ that is obtained by connecting distinct affine intersection points of lines using paths in the lines, can be extended to a continuous family of loops $\delta = \delta_t$ for $t$ in such a sector.

We apply this to loops on $X_0$ that we denote as

$(L_{1,1}, L_{k,l}, L_{m,n}), \quad 2 \leq k < m \leq N, 1 \leq l \leq N_k, 1 \leq n \leq N_m,$

$(L_{1,1}, L_{2,1}, L_{1,j}, L_{2,n}), \quad 1 < j \leq N_1, 1 < n \leq N_2,$

$(L_{1,1}, L_{2,1}, L_{1,j}, L_{m,n}), \quad 2 \leq j \leq N_1, 3 \leq m \leq N, 1 \leq n \leq N_m.$

Here we mean a loop obtained by starting with as vertices the intersection points of two consecutive lines in the tuple, as well as the intersection point of the first line with the last, and connecting two consecutive vertices by a path in the line containing both vertices.

It is easy to see that among those intersection points there is a unique $s$ in $S$, namely $P_{k,l;m,n}$ in the first case, $P_{1;j;2,n}$ in the second, and $P_{1;j;m,n}$ in the third. We can choose the connecting paths to avoid all other points in $S$, so that on $X_0$ we have a loop $\delta_s$ that contains $s$ but no other points in $S$, and on suitable sectors in $D$ all those $\delta_s$ can be deformed as above. Note that we defined exactly $g$ loops $\delta_s$.

For $s$ and $s'$ in $S$, and working in a suitable sector of $D$ all the time, $\gamma_s$ and $\delta_{s'}$ can only intersect for $t \neq 0$ if $s = s'$. Considering the different branches around $s$ in $X_0$, we see that $\gamma_s$ and $\delta_s$ meet exactly once in $X_t$ with $t \neq 0$. Changing the orientation of $\delta_s$ if necessary, we can assume that $\gamma_s \cap \delta_{s'}$ equals 1 if $s = s'$ and 0 otherwise. Since different $\gamma_s$ do not intersect, the intersection matrix of $\{\gamma_s\}_{s \in S} \cup \{\delta_s\}_{s \in S}$ is of the form

$$
\begin{pmatrix}
0 & I_g \\
-I_g & *
\end{pmatrix}.
$$

As this has determinant 1, the $\gamma_s$ and $\delta_s$ for $s$ in $S$ form a basis of $H_1(X_t; \mathbb{Z})$. \qed
We want to establish a limit formula for the integral over \( \gamma_t \) of the regulator 1-form \( \eta \) obtained from the elements constructed in Section 2. For this, consider an algebraic surface \( Y \) defined by \( xyh(x, y) - t \) around \((0, 0, 0)\), where \( h(0, 0) \neq 0 \). As above, one can construct a family of loops \( \gamma_t \) in \( Y_t \), where \( \gamma_0 \) is a clockwise loop around 0 in the \( x \)-axis in \( Y_0 \). Then we have the following simple fact.

**Fact 4.4.** Let \( U \subset \mathbb{C} \) be open and let \( \eta = \mu_1(x, t)dx + \mu_2(x, t)dt + \mu_3(x, t)d\overline{t} \) be a 1-form on \( \mathbb{R} \times U \), periodic in \( x \) with period \( p \). If \( d\eta = \nu_1(x, t)dx \land dt + \nu_2(x, t)dx \land d\overline{t} + \nu_3(x, t)dt \land d\overline{t} \), then for any \( c \in \mathbb{R} \) we have

\[
\frac{\partial}{\partial t} \int_{c}^{c+p} \eta = \frac{\partial}{\partial t} \int_{c}^{c+p} \mu_1(x, t)dx = \int_{c}^{c+p} \frac{\partial}{\partial t} \mu_1(x, t)dx = - \int_{c}^{c+p} \nu_1(x, t)dx.
\]

The first two identities in the fact are clear, and the last one holds because \( \nu_1(x, t) = -\frac{\partial}{\partial x} \mu_1(x, t) + \frac{\partial}{\partial x} \mu_2(x, t), \) and \( \int_{c}^{c+p} \frac{\partial}{\partial x} \mu_2(x, t)dx = \mu_2(c+p, t) - \mu_2(c, x) = 0 \).

**Lemma 4.5.** Let \( Y \) and \( \gamma_t \) be as above, and let \( u \) and \( v \) be holomorphic functions on \( h^2 \) that do not vanish at \((0, 0)\). With \( \eta \) as in 4.4, consider the function \( F(t) = \int_{\gamma_t} \eta((ux, vy^3)) \). If \( \gamma_t \) lifts a loop that is sufficiently small, then for \( t \in \mathbb{C} \) we have

\[
\lim_{|t| \to 0} \frac{F(t)}{\log|t|} = 2\pi ab.
\]

**Proof.** Write \( d\eta((ux, vy^3)) = \omega_1 \land \omega_2 \land d\overline{t} \) on a suitable smooth part of \( Y \setminus Y_0 \), with 1-forms \( \omega_1 \) and \( \omega_2 \). Because \( \gamma_t \) is lifted from a simple loop around 0 on the complex plane that can be parametrized by an interval in \( \mathbb{R} \), we can apply the fact above to deduce \( \frac{\partial F(t)}{\partial t} = -\int_{\gamma_t} \omega_1 \).

In order to calculate \( \omega_1 \), we notice that on \( Y \setminus Y_0 \) we have the identity

\[
\frac{h_x dx + h_y dy}{h} + \frac{dx}{x} + \frac{dy}{y} = \frac{dt}{t},
\]

hence

\[
\frac{h_x}{x} dx + \frac{h_y}{y} dy = \frac{dt}{t}
\]

for functions \( h_i(x, y) \) with \( h_i(0, 0) = 1 \), and \( \frac{dx}{x} \land \frac{dt}{t} = h_2 \frac{dx}{x} \land \frac{dy}{y} = \frac{h_2 dx}{x} \land \frac{dy}{y} \). Therefore

\[
d \log((ux)^a) \land d \log((vy)^b) = (d \log u + ad \log x) \land (d \log v + bd \log y)
\]

\[
= (ab + xh_3 + yh_4) \frac{dx}{x} \land \frac{dt}{t}
\]

with \( h_3(x, y) \) and \( h_4(x, y) \) holomorphic functions around \((0, 0)\).

Denote \( (ab + xh_3 + yh_4) \frac{dx}{x} \) by \( \omega \). Clearly,

\[
d\eta(f_1, f_2) = \text{Im}(d \log(f_1) \land d \log(f_2)) = \frac{1}{2i} \omega \land \frac{dt}{t} = \frac{1}{2i} \omega \land \frac{d\overline{t}}{\overline{t}},
\]

so that we can take \( \omega_1 = \frac{1}{2it} \omega \). Viewing \( y \) around the \( \gamma_t \) as a holomorphic function of \( x \) and \( t \), we have

\[
\frac{\partial F(t)}{\partial t} = -\int_{\gamma_t} \omega_1 = -\int_{\gamma_t} \frac{1}{2it} (ab + xh_3 + yh_4) \frac{dx}{x} = \frac{\pi ab}{t} + h_5(t)
\]

where \( h_5(t) \) is a holomorphic function around \( t = 0 \): \( \int_{\gamma_t} (xh_3 + yh_4) \frac{dx}{x} \) is holomorphic in \( t \) and vanishes for \( t = 0 \).

Since \( \frac{\partial \log|t|}{\partial t} = \frac{1}{t} \), we have \( \frac{\partial F(t) - 2\pi ab \log|t|}{\partial t} = h_5 \), which is bounded around \( t = 0 \). Both \( F(t) \) and \( \log|t| \) are real-valued, hence also \( \frac{\partial F(t) - 2\pi ab \log|t|}{\partial t} = \overline{h_5} \) is
bounded around $t = 0$. Therefore $F(t) - 2\pi ab \log|t|$ is bounded around 0, which proves the lemma.

We now return to the curves $X_t$ with $t \neq 0$ in $D$ and $D$ sufficiently small. Here we have classes $\alpha_1, \ldots, \alpha_g$ in $K_2^T(C)$/torsion from Proposition 2.8 and we pair those under the regulator pairing (1.3) with the $g$ loops $\gamma_s$ of Lemma 4.3. We then have the following limit result.

**Theorem 4.6.** Let $X_t$ be defined by (1.1), and assume no three $L_{i,j}$ meet at an affine point. If the $a_i$, $b_i$ and $c_{i,j}$ are fixed, then

$$
\lim_{t \to 0} \frac{\det (\langle \gamma_s, \alpha_j \rangle)}{\log^3|t|} = \pm 1.
$$

Before giving the proof of the theorem, we have the following immediate corollary, which is the main result of this section.

**Corollary 4.7.** Let $C$ be defined by (2.2) and assume no three of the $L_{i,j}$ meet in an affine point. If the $a_i$, $b_i$ and $c_{i,j}$ are fixed, and $|\lambda| \gg 0$, then the elements are independent in $K_2^T(C)$. In particular, for $C$ defined by (3.2) with $|\lambda| \gg 0$ and satisfying the condition of Theorem 3.3 we have $g$ independent elements in $K_2^T(C)_{\text{int}}$.

**Proof of Theorem 4.6.** To every $s' = P_{i,j;k,l}$ in $S$ with $i < k$, we associate an element $M_{s'}$ in $K_2^T(C)$, namely

$$
M_{s'} = \begin{cases}
R_{1,j,1;2,1} & \text{if } i = 1, k = 2, \\
T_{1,1;i,j;k,l} & \text{if } 2 \leq i < k \leq N, \\
T_{1,j,2;1;k,l} - T_{1,1,2;1;k,l} & \text{if } i = 1, k > 2.
\end{cases}
$$

Note that the elements above are the same as the elements in Proposition 2.8 in the first two cases, and in the last case it is an element minus an element of the second case. So the regulator of these elements is the same as the regulator of those in Proposition 2.8 and the theorem will be proved if we show that

$$
\lim_{t \to 0} \frac{\langle \gamma_s, M_{s'} \rangle}{\log|t|} = \begin{cases}
\pm 1, & \text{if } s = s', \\
0, & \text{otherwise}.
\end{cases}
$$

Recall that $\langle \gamma_s, M_{s'} \rangle = \frac{1}{2\pi} \int_{\gamma_s} \eta(M_{s'})$. By Lemma 4.3 for $s$ in $S$ we have

$$
\frac{1}{2\pi} \lim_{t \to 0} \frac{\int_{\gamma_s} \eta(L_{i,j}, L_{k,l})}{\log|t|} = \begin{cases}
\pm 1, & \text{if } L_{i,j} \text{ and } L_{k,l} \text{ intersect at } s, \\
0, & \text{otherwise}
\end{cases}
$$

Now take $s' = P_{i,j;k,l}$ in $S$ with $i < k$. Since $S$ does not contain the point in $L_{i,1} \cap L_{i,j}$ for $i > 1$, nor the point in $L_{2,1} \cap L_{1,j}$, we see by expanding $\eta(M_{s'})$ from (2.4) and (2.5) that for $i = 1, k = 2$,

$$
\frac{1}{2\pi} \lim_{t \to 0} \frac{\int_{\gamma_s} \eta(M_{s'})}{\log|t|} = \frac{1}{2\pi} \lim_{t \to 0} \frac{\int_{\gamma_s} \eta(L_{1,j}, L_{2,l})}{\log|t|} = \begin{cases}
\pm 1, & \text{if } s = s', \\
0, & \text{otherwise}
\end{cases}
$$

for $2 \leq i < k \leq N$,
and for $i = 1, k > 2$,
\[
\frac{1}{2\pi} \lim_{t \to 0} \frac{\int_{2 \gamma_j} \eta(M_{x'})}{\log |t|} = \frac{1}{2\pi} \lim_{t \to 0} \frac{\int_{2 \gamma_j} -\eta(L_{1,j}, L_{k,1})}{\log |t|} = \begin{cases} 
\pm 1, & \text{if } s = s', \\
0, & \text{otherwise}. 
\end{cases}
\]
So the theorem is proved. \qed

Remark 4.8. For $g \geq 1$, in Theorem 4.6 there are infinitely many different isomorphism classes of curves over the algebraic closure of the base field in the family.

Namely, for $g = 1$ the only two cases are $N = 2$ with both $N_i$ equal 2, and $N = 3$ and all $N_i$ equal 1. Using (3.6) and (3.8) it is easy to see that the $j$-invariant is a non-constant function of $t$ in either case.

For $g \geq 2$, note that, if a family of curves has a stable, singular fibre, and smooth general fibres of genus $g$, it cannot be isotrivial, since the image of the corresponding map to the moduli space passes through a general point and a point on the boundary and hence cannot be constant.

In our family the fibre $X_0$ is the union of lines meeting transversely. It is stable except if $N = 2$ and one of the $N_i$ equals 2, the other being larger than 2, or if $N = 3$ and precisely two of $N_1$, $N_2$ and $N_3$ equal 1. In either case we can contract the $(-2)$-curves in $X_0$. The resulting fibre now consists of two $\mathbb{P}^1$s meeting transversely at $g + 1$ points, which is stable. So if $g \geq 2$ the family is not isotrivial.

Remark 4.9. Using our techniques, we can also obtain two independent elements in $K_2^T(C)_{\text{int}}$ in certain families of elliptic curves over a given real quadratic field.

Namely, for an integer $a$ with $|a| > 5$, let $k$ be the real quadratic field $\mathbb{Q}(\sqrt{a^2 - 16})$. Note that every real quadratic field with discriminant $D$ occurs infinitely often, because $c^2 - Dd^2 = 1$ has infinitely many integer solutions.

Consider the elliptic curve over $k$ defined by $y^2 + (2x^2 + ax + 1)y + x^4$. Writing $4x^3 + ax + 1 = (\varepsilon_1 x + 1)(\varepsilon_2 x + 1)$ in $k[x]$, the elements
\[
\tilde{M}_i = 2 \left\{ \frac{y}{x^2}, \varepsilon_i x + 1 \right\} \quad (i = 1, 2)
\]
are in $K_2^T(C)_{\text{int}}$ by the proof of [12, Theorem 8.3]. Their regulator $R = R(a)$ satisfies
\[
\lim_{|a| \to \infty} \frac{R(a)}{\log^2 |a|} = 16.
\]
Namely, reading the transformation from (3.5) to (3.6) backwards, $C$ can be transformed into the curve defined by $x(x + a)g(y + 1) - 1$ and the elements become $2 \left\{ \frac{y}{x + 1}, \frac{x + \varepsilon_i}{x} \right\}$. Replacing $x$ with $ax$, and letting $t = \frac{x}{a}$, we obtain a curve defined by $x(x + 1)g(y + 1) - t$, with elements $2 \left\{ \frac{y}{x + 1}, \frac{x + \varepsilon_i}{x} \right\}$. Clearly, $\lim_{|a| \to \infty} \frac{\varepsilon_i}{a}$ equals 0 or 1, and the two different embeddings of $k$ into $\tilde{C}$ swap the two cases. Using a limit argument directly on $\int_{\gamma_i} \eta(\tilde{M}_i)$ it is easy to see that we may, in the limit of $R(a)$, we may replace $\varepsilon_i/a$ by its limit in $C$. The formula for $R(a)$ then follows from Lemmas 1.3 and 1.5.

As $C$ above is already defined over $\mathbb{Q}$, one could use the theory of quadratic twists and the modularity of $C$ in order to show the existence of two independent elements in $K_2^T(C)_{\text{int}}$ for $C/k$. That would be far less explicit, but it would also give the expected relation between the regulator and the $L$-function. We now give families for which this method cannot be used since the $j$-invariant is not rational.
Let $k$ be a real quadratic field with ring of algebraic integers $\mathcal{O}_k$. Fix $v \neq \pm 1$ in $\mathcal{O}_k$ as well as $p$ and $q$ in $\mathcal{O}_k$ with $pq = 4$, so that $(pv^n x + 1)(qv^{-n} x + 1) = 4x^2 + ax + 1$ with $a = pv^n + qv^{-n}$. Then for $pv^n \neq \pm 2$ we have an elliptic curve over $k$ defined by $y^2 + (2x^2 + ax + 1)y + x^4$, with elements $\widetilde{M}_1 = 2\left\{\frac{a}{2}, pv^n x + 1\right\}$ and $\widetilde{M}_2 = 2\left\{\frac{1}{2}, qv^{-n} x + 1\right\}$. Using this $a, \varepsilon_1 = pv^n$ and $\varepsilon_2 = qv^{-n}$ in the discussion above, one sees that the two elements are in $K_2^2(C)_{\text{int}}$ and that their regulator $R = R(n)$ satisfies
\[
\lim_{n \to \infty} \frac{R(n)}{n^2} = 16 \log^2|v|.
\]

Using explicit calculations one checks that the $j$-invariant is not a constant function of $n$, for $n \gg 0$ is not an algebraic integer, and, if the norm of $p$ in $\mathbb{Q}$ does not have absolute value $4$, for $n \gg 0$ is not rational. So for $n \gg 0$, the elliptic curve does not have complex multiplication, and if the norm of $p$ does not have absolute value $4$, the curve cannot be defined over $\mathbb{Q}$.

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