Gravitational radiation of accelerated sources

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Abstract

We investigate the gravitational radiation produced by a linearly accelerated source in general relativity. The investigation is performed by studying the vacuum C metric, which is interpreted as representing the exterior space-time of an uniformly accelerating spherically symmetric gravitational source, and is carried out in the context of the teleparallel equivalent of general relativity. For an observer sufficiently far from both the (modified) Schwarzschild and Rindler horizons, which is a realistic situation, we obtain a simple expression for the total emitted gravitational radiation. We also briefly discuss on the absolute or relative character of the accelerated motion.

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1 Introduction

In similarity to the expected radiation of a charged particle in classical electrodynamics, in general relativity an accelerated source is supposed to emit gravitational radiation. Although the magnitude of such radiation is expected to be very small, the understanding and description of this phenomenon is of great theoretical importance. The exterior space-time of an uniformly accelerated spherically symmetric gravitational source is described by the C metric, which has been the subject of renewed interest. The C metric is a vacuum solution of Einstein’s equation first obtained by Levi-Civita [1], and rediscovered subsequently by several authors, among whom Ehlers and Kundt [2], who established its present name. Its interpretation as an accelerated Schwarzschild black hole has been analyzed in Refs. [3, 4, 5]. By means of a maximal extension of coordinates [3, 5], the C metric may be taken to represent a pair of accelerated black holes. As a possible physical realization of a system that undergoes acceleration due to nongravitational forces, we may consider the motion of a star (the Sun, for instance) that results from anisotropic solar emission.

The relevant physical property of the C metric space-time may be analyzed in a straightforward way by considering the linearized form of the solution [6]. A particle in the linearized C metric space-time is subject to the gravitational attraction of the central source plus a uniform inertial force along the \( z \) direction, say. Therefore the inertial state of the physical sources does influence the gravitational field around it, a fact that is consistent with the principle of equivalence. According to the latter, gravitational and inertial forces are of the same nature. The constant, uniform gravitational field produced by the accelerated source gives rise (in the linearized form of the solution) to the gravitational Stark effect [6].

In this article we investigate the gravitational radiation emitted by an accelerated Schwarzschild black hole in the framework of the teleparallel equivalent of general relativity (TEGR). In our investigation we will adopt a Bondi type coordinate system that allows us to conclude that the full, nonlinearized form of the C metric may be understood as a nonlinear superposition of the Rindler and Schwarzschild space-times [6]. The C metric exhibits a timelike Killing vector \( \partial_t \), which indicates the static character of the metric. However, the metric is also radiative. As discussed in Ref. [3], this apparent paradox is dispelled by noting that the Killing vector above becomes spacelike beyond
the Rindler horizon. Therefore the metric is not globally static. Moreover, the Riemann tensor contains \( r^{-1} \) terms, which indicate the radiative character of the metric.

In the description of the C metric given by Ref. [6] the coordinate system is adapted to (centered on) the accelerated source. Therefore in our analysis we will perform a time dependent Lorentz boost in the opposite direction of the moving black hole that will determine a set of tetrad fields adapted to observers that are “approximately at rest” in space-time. In other words, with respect to the boosted set of tetrad fields the black hole does indeed display an accelerated motion, in an approximation to be explained. The energy and the gravitational radiation emitted by the black hole will be evaluated with respect to this set of observers. The surface integrals will be evaluated on a surface of integration that is simultaneously far from both (Schwarzschild and Rindler) modified horizons.

The article is organized as follows. In section II we present the C metric in Bondi type coordinates, and briefly discuss the relevant property of the linearized form of the metric. In section III we recall some relevant definitions that arise in the TEGR regarding the energy-momentum and the flux of energy-momentum of the gravitational field. In section IV we carry out, first, the construction of a set of (time independent) tetrad fields adapted to the accelerated black hole (i.e., for which the black hole is at rest). Then by means of a local Lorentz transformation we determine the tetrad fields which describe the accelerated motion of the source. After presenting the relevant calculations and approximations we arrive at a simple expression for the total gravitational radiation flux, which is the major result of the article. In section V we discuss on the relativity of acceleration. For this purpose, we examine whether we may attribute to the accelerated motion of the black hole in space-time a relative or absolute character.

Notation: space-time indices \( \mu, \nu, ... \) and SO(3,1) indices \( a, b, ... \) run from 0 to 3. Time and space indices are indicated according to \( \mu = 0, i, \, \, a = (0), (i) \).

The tetrad field \( e^a_\mu \) yields the definition of the torsion tensor: \( T^a_{\mu \nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu \). The flat, Minkowski space-time metric is fixed by \( \eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu \nu} = (- + + +) \).
2 The C metric

Except for a change in signature, we will adopt the notation of Ref. [6] in the description of the C metric. By means of a coordinate transformation the C metric in the form given by Ehlers and Kundt [2] may be written in Bondi type coordinates \((u, r, \theta, \phi)\) (which may be understood as an accelerated coordinate system) in terms of two functions, \(G(\theta)\) and \(H(r, \theta)\) [6]. We find it useful to define the function \(g(\theta)\) according to \(G(\theta) = g^2 \sin^2 \theta\). In terms of this function the C metric is written as

\[
\text{ds}^2 = -H \, du^2 - 2 \, du \, dr + 2Ar^2 \sin \theta \, dud\theta + \frac{r^2}{g^2} \, d\theta^2 + r^2 g^2 \, \sin^2 \theta \, d\phi^2 , \tag{1}
\]

where

\[
g^2 \sin^2 \theta = G(\theta) = \sin^2 \theta - 2mA \cos^3 \theta , \tag{2}
\]

\[
H(r, \theta) = 1 - \frac{2m}{r} - A^2 r^2 (\sin^2 \theta - 2mA \cos^3 \theta) \\
- 2Ar \cos \theta (1 + 3mA \cos \theta) + 6mA \cos \theta . \tag{3}
\]

The parameters \(m > 0\) and \(A > 0\) represent the mass and acceleration of the black hole, respectively. It is not difficult to show that \(G > 0\) provided the relation \(mA < 1/(3\sqrt{3})\) is satisfied. We will assume this relation to hold in the present analysis.

In the C metric space-time there are two horizons, the Schwarzshild and Rindler horizons located at \(r_S\) and \(r_R\), respectively. Let us define the acceleration length \(L_A = 1/(3\sqrt{3}A)\), and the functions

\[
U = -\frac{1}{\sqrt{3}} \cos \left( \frac{1}{3} \arccos \frac{m}{L_A} \right) , \]

\[
V = \frac{1}{\sqrt{3}} \sin \left( \frac{1}{3} \arccos \frac{m}{L_A} \right) .
\]

In terms of these quantities, \(r_S\) and \(r_R\) are given by [7]
\[ r_S = \frac{1}{A} \frac{\sqrt{3V - U}}{[1 + (\sqrt{3V - U}) \cos \theta]} \]

\[ r_R = \frac{1}{A} \frac{2U}{(1 + 2U \cos \theta)} \]

In the limit \( mA << 1 \) the quantities above reduce to [8]

\[ r_S \approx 2m (1 + 2Am \cos \theta) \]

\[ r_R \approx \frac{1}{A(1 - \cos \theta + Am \sin^2 \theta)} \]

The C metric may be interpreted as a nonlinear superposition of the Schwarzschild and Rindler space-times. This interpretation may be verified by investigating the limits of vanishing parameters. By making \( A = 0 \) the metric tensor above reduces to

\begin{equation}
    ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2 du dr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\end{equation}

This is precisely the Schwarzschild metric tensor written in terms of the retarded time \( u = t - r - 2m \ln(r/2m - 1) \).

On the other hand, by requiring \( m = 0 \) we obtain

\begin{equation}
    ds^2 = -(1 - 2Ar \cos \theta - A^2 r^2 \sin^2 \theta) du^2 - 2 du dr + 2Ar^2 \sin \theta dud\theta + r^2 d\theta^2 + r^2 \sin \theta^2 d\phi^2,
\end{equation}

This metric tensor can be transformed into Minkowski’s metric tensor by means of the coordinate transformation [3]

\begin{align*}
    \bar{t} &= (A^{-1} - r \cos \theta) \sinh Au + r \cosh Au, \\
    \bar{z} &= (A^{-1} - r \cos \theta) \cosh Au + r \sinh Au, \\
    \bar{x} &= r \sin \theta \cos \phi, \\
    \bar{y} &= r \sin \theta \sin \phi.
\end{align*}

The coordinate transformation above transforms Eq. (5) into
\[ ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2. \quad (7) \]

In the space-time represented by the coordinates \((\bar{t}, \bar{x}, \bar{y}, \bar{z})\) the locus \(r = 0\) determines a timelike curve given by

\[
\begin{align*}
\bar{t} &= A^{-1} \sinh Au, \\
\bar{z} &= A^{-1} \cosh Au, \\
\bar{x} &= \bar{y} = 0,
\end{align*}
\]

which represents one branch of the hyperbola \(\bar{z}^2 - \bar{t}^2 = 1/A^2\), along which takes place the motion with constant acceleration \(A\), parametrized in terms of \(u\). The Bondi type coordinate system \((u, r, \theta, \phi)\) covers only the half-space \(\bar{t} + \bar{z} > 0\). The maximal extension consists in filling the remaining half-space by replicating a time-reversed copy of the half-space \(\bar{t} + \bar{z} > 0\). In this case the hyperbola \(\bar{z}^2 - \bar{t}^2 = 1/A^2\) represents two uniformly accelerated particles, moving in opposite directions, and which are causally disconnected, because there is no overlapping of their fields in space-time. For further details, see Ref. [3].

The linearized form of the C metric is simply obtained by neglecting \(m^2, mA, A^2\) and higher-order terms [6]. By transforming \((u, r, \theta, \phi)\) to \((T, X, Y, Z)\) coordinates according to

\[
\begin{align*}
T &= u + r + 2m \ln \left( \frac{r}{2m} - 1 \right) \\
X &= r \sin \theta \cos \phi \\
Y &= r \sin \theta \sin \phi \\
Z &= r \cos \theta,
\end{align*}
\]

we find that in the linearized limit the \(g_{00}\) component of the metric tensor given by Eq. (1) reads

\[
\begin{align*}
-g_{00}(u, r, \theta, \phi) &= -g_{00}(T, X, Y, Z) \\
&\approx 1 - \frac{2m}{r} - 2Ar \cos \theta. \quad (10)
\end{align*}
\]
The $g_{00}$ component allows us to identify the Newtonian potential $\phi$ according to

$$-g_{00}(T, X, Y, Z) = 1 + 2\phi. \quad (11)$$

We obtain

$$\phi = -\frac{m}{r} - (r \cos \theta)A = -\frac{m}{r} - AZ. \quad (12)$$

The potential $\phi$ determines the Newtonian equation of motion for a point particle in this space-time,

$$\frac{d^2 X^i}{dT^2} = -\frac{\partial \phi}{\partial X^i}. \quad (13)$$

from what follows

$$\frac{d^2 r}{dT^2} = -\frac{m}{r^3} r + A\hat{z}. \quad (14)$$

Therefore in the linearized C metric space-time a point particle is subject to the usual central force plus an additional constant and uniform force that is due to the accelerated motion of the source, represented by $m$, along the $\theta = \pi$ direction.

In what follows we will need the inverse metric tensor associated to Eq. (1). It reads

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & H + A^2r^2g^2 \sin^2 \theta & Ag^2 \sin \theta & 0 \\ 0 & Ag^2 \sin \theta & \frac{g^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2g^2 \sin^2 \theta} \end{pmatrix}. \quad (15)$$

3 The TEGR and the gravitational energy-momentum

The TEGR is a reformulation of Einstein’s general relativity in terms of the tetrad field $e^a_\mu$ [9, 10, 11, 12, 13, 14, 15]. By considering the theory to be invariant under the global $SO(3,1)$ group of transformations of $e^a_\mu$, the
additional six degrees of freedom of the latter, with respect to the metric tensor $g_{\mu\nu}$, fix the reference frame adapted to particular observers.

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by

$$L(e_{a\mu}) = -k e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - L_m$$

$$\equiv -k e \Sigma^{abc} T_{abc} - L_m ,$$

(16)

where $k = 1/(16\pi G)$ and $e = \det(e^a_{\mu})$. The tensor $\Sigma^{abc}$ is defined by

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c) ,$$

(17)

and $T^a = T^b_{\ b} a^a$. The quadratic combination $\Sigma^{abc} T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence [13]. $L_m$ represents the Lagrangian density for matter fields. The field equations for the tetrad field read

$$e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e (\Sigma^{d\nu}_{\ a} T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}) = \frac{1}{4k} e T_{a\mu} ,$$

(18)

where $\delta L_m/\delta e^{a\mu} \equiv e T_{a\mu}$. It is possible to prove by explicit calculations that the left hand side of Eq. (18) is exactly given by $\frac{1}{2} e [R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e)]$.

The definition of the gravitational energy-momentum contained within an arbitrary volume $V$ of the three-dimensional spacelike hypersurface arises in the Hamiltonian formulation of the TEGR [16]. It reads

$$P^a = - \int_V d^3 x \partial_j \Pi^{aj} ,$$

(19)

where $\Pi^{aj} = -4k e \Sigma^{a0j}$ is the momentum canonically conjugated to $e_{aj}$. An essential feature of the gravitational energy-momentum $P^a = (E, P)$ is the covariance under global SO(3,1) transformations, in addition to the invariance under coordinate transformations of the three-dimensional spacelike hypersurface. Each configuration of tetrad fields establishes a reference frame adapted to an observer (see the discussion at the beginning of the following section).

Simple algebraic manipulations of Eq. (18) yield a continuity equation for the gravitational energy-momentum $P^a$ [17, 18, 19],
\[
\frac{dP^a}{dt} = -\Phi^a_g - \Phi^a_m, \tag{20}
\]
where
\[
\Phi^a_g = k \int_S dS_j [ee^a_{\mu}(4\Sigma^b c_j T_{bc\mu} - \delta^j_{\mu}\Sigma^{bcd} T_{bcd})], \tag{21}
\]
is a component of the gravitational energy-momentum flux, and
\[
\Phi^a_m = \int_S dS_j (ee^a_{\mu} T^{j\mu}), \tag{22}
\]
is the a component of the matter energy-momentum flux. S represents the spatial boundary of the volume V. Therefore in the vacuum space-time the gravitational energy flux \(\Phi_g^{(0)}\) is simply given by
\[
\Phi_g^{(0)} = -\frac{dE}{dt}, \tag{23}
\]
where \(E = P^{(0)}\). The analysis [17, 18] of some relevant and well known configurations of the gravitational field has supported the interpretation of \(\Phi_g^{(0)}\) as the gravitational energy flux.

4 Reference frames in the C metric space-time and the gravitational radiation

A given gravitational field configuration described by the metric tensor \(g_{\mu\nu}\) admits an infinity of tetrad fields, related to each other by means of a local SO(3,1) transformation. In order to examine how an observer is adapted to a particular set of tetrad fields, we consider its worldline in space-time. Let \(x^\mu(s)\) denote the worldline \(C\) of an observer, and \(u^\mu(s) = dx^\mu/ds\) its velocity along \(C\). We may identify the observer’s velocity with the \(a = 0\) component of \(e^\mu_a\), where \(e_a^\mu e^a_{\nu} = \delta^\mu_\nu\). Thus, \(u^\mu(s) = e_{(0)}^\mu\) along \(C\) [20]. The acceleration of the observer is given by
\[
a^\mu = \frac{Du^\mu}{ds} = \frac{De_{(0)}^\mu}{ds} = u^\alpha \nabla_\alpha e_{(0)}^\mu. \tag{24}
\]
The covariant derivative is constructed out of the Christoffel symbols. We see that \(e_a^\mu\) determines the velocity and acceleration along a worldline.
of an observer adapted to the frame. From this perspective we conclude that a given set of tetrad fields, for which \( e_{(0)}^\mu \) describes a congruence of timelike curves, is adapted to a particular class of observers, namely, to observers determined by the velocity field \( u^\mu = e_{(0)}^\mu \), endowed with acceleration \( a^\mu \).

In the case of asymptotically flat space-times, for instance, if \( e^{a}_{\mu} \rightarrow \delta_{\mu}^{a} \) in the asymptotic limit \( r \rightarrow \infty \), then we conclude that \( e^{a}_{\mu} \) is adapted to stationary observers at spacelike infinity.

Let us now consider the C metric. In order to determine \( e^{a}_{\mu} \) we will impose restrictions on \( e_{(0)}^\mu = g^{\mu\lambda} e_{(0)}^\lambda \). With the help of Eq. (15) we require

\[
e_{(0)}^1 = e_{(0)}^2 = e_{(0)}^3 = 0.
\]

(25)

The requirement of the equations above establishes a set of tetrad fields adapted to the moving black hole, i.e., in such reference frame the observer sees the black hole at rest.

It is not difficult to implement Eq. (25) in the construction of the tetrad field \( e^{a}_{\mu} \) that yields the C metric given by Eq. (1). After few algebraic manipulations we obtain

\[
e^{a}_{\mu}(u, r, \theta, \phi) =
\]

\[
= \begin{pmatrix}
B & C & -D & 0 \\
0 & C \sin \theta \cos \phi & \frac{1}{g} \cos \theta \cos \phi - D \sin \theta \cos \phi & -rg \sin \theta \sin \phi \\
0 & C \sin \theta \sin \phi & \frac{1}{g} \cos \theta \sin \phi - D \sin \theta \sin \phi & rg \sin \theta \cos \phi \\
0 & C \cos \theta & -\frac{1}{g} \sin \theta - D \cos \theta & 0
\end{pmatrix}
\]

(26)

where \( g \) is given by Eq. (2) and

\[
B = H^{1/2},
\]

\[
C = H^{-1/2},
\]

\[
D = H^{-1/2} Ar^2 \sin \theta
\]

(27)

The tetrad field above satisfies Eq. (25), and therefore it is adapted to the accelerated black hole. We further note that by requiring \( m = A = 0 \) the tetrad field above reduces to
\[ e^a_{\mu}(u, r, \theta, \phi) = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
0 & \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
0 & \cos \theta & -r \sin \theta & 0
\end{pmatrix}, \]

which is the flat space-time tetrad field in Bondi type coordinates. All components of the torsion tensor \( T^a_{\mu\nu} = \partial_{\mu} e^a_{\nu} - \partial_{\nu} e^a_{\mu} \) calculated out of the expression above vanish.

Given that Eq. (26) is time independent, we conclude that the resulting expression for \( P^{(0)} \) is likewise time independent. Therefore in view of Eq. (23) we have \( \Phi^{(0)} = 0 \). Thus from the standpoint of the set of observers for whom the black hole is at rest, as determined by Eq. (26), there is no gravitational radiation.

In the analysis of the linearized form of the C metric in section 2 we observed that the equation of motion given by Eq. (14), for a free point particle in the C metric space-time, is consistent with the picture of an accelerated black hole in the \(-z\) direction. Therefore we will consider a boost transformation \( \Lambda^a_b \) (a local Lorentz transformation) along the \(+z\) direction, and apply it on \( e^a_{\mu} \) given by Eq. (26). The boost is given by

\[ \Lambda^a_b = \begin{pmatrix}
\gamma & 0 & 0 & -\gamma v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\gamma v & 0 & 0 & \gamma
\end{pmatrix}, \quad (28) \]

where \( v = v(u) \) and \( \gamma = (1-v^2)^{-1/2} \). The resulting tetrad field \( \tilde{e}^a_{\mu} = \Lambda^a_b e^b_{\mu} \) reads (dropping the tilde)

\[ \tilde{e}^a_{\mu}(u, r, \theta, \phi) = \]

\[ \begin{pmatrix}
\gamma B & \gamma C - \gamma v C \cos \theta & -\gamma D - \gamma v (\frac{-z}{g} \sin \theta - D \cos \theta) & 0 \\
0 & C \sin \theta \cos \phi & \frac{z}{g} \cos \theta \cos \phi - D \sin \theta \cos \phi & -r g \sin \theta \sin \phi \\
0 & C \sin \theta \sin \phi & \frac{z}{g} \cos \theta \sin \phi - D \sin \theta \sin \phi & -r g \sin \theta \cos \phi \\
-\gamma v B & -\gamma v C + \gamma C \cos \theta & \gamma v D + \gamma (\frac{-z}{g} \sin \theta - D \cos \theta) & 0
\end{pmatrix} \quad (29) \]
We remark that the tetrad field above yields the complete C metric tensor, not its linearized form.

In the remaining of this section we will evaluate $T_{\alpha\mu\nu}$ and $\Sigma^{\lambda\mu\nu}$ out of Eq. (29). Let us note, however, that the tetrad field given by the equation above is adapted to the set of observers whose four-velocity field $e^{(0)}_{\mu} = g^{\mu\lambda}e_{(0)\lambda}$ in $(u, r, \theta, \phi)$ coordinates reads

$$e^{(0)}_{\mu} = \left(\gamma C(1 - v \cos \theta), \gamma v(B \cos \theta - Arg \sin \theta), -\gamma v\frac{g}{r} \sin \theta, 0\right). \tag{30}$$

In order to carry out the calculations we will establish two essential approximations. We will restrict considerations to the space-time region for which

$$\frac{m}{r} \ll 1 \quad \text{(31)}$$
$$Ar \ll 1 \quad \text{(32)}$$

Eqs. (31) and (32) ensure that an observer located at the radial position $r$ is far from the Schwarzschild and Rindler horizons, respectively, and may be taken as conditions for a weak field approximation. Let us consider a small parameter $\varepsilon$ and stipulate that

$$\frac{m}{r} \approx O(\varepsilon) \quad \text{(33)}$$
$$Ar \approx O(\varepsilon).$$

It follows that $(m/r)Ar = mA = O(\varepsilon^2)$. In this approximation we have $g = 1 + O(\varepsilon^2)$, $B = 1 + O(\varepsilon)$ and also $C = 1 + O(\varepsilon)$. It is easy to verify that in the approximation determined by Eqs. (31,32) the four-velocity given by Eq. (30) reduces to

$$e^{(0)}_{\mu} \approx (\gamma, 0, 0, \gamma v), \quad \text{(34)}$$

in $(T, X, Y, Z)$ coordinates, which we recognize as the usual four-velocity of observers boosted in the $+z$ direction in flat space-time.

Therefore in the approximation determined by Eqs. (31,32) the Lorentz boost given by Eq. (28) corresponds to an ordinary boost in flat space-time.
Since both \((u, r, \theta, \phi)\) and \((T, X, Y, Z)\) are accelerated coordinate systems, Eq. (34) establish the four-velocity of observers that are approximately at rest in space-time.

We return to Eq. (29) and proceed to evaluate the necessary quantities that lead to

\[
P^{(0)} = -\int_V d^3x \partial_j \Pi^{(0)j} = 4k \oint_S d\theta d\phi e \Sigma^{(0)01},
\]

according to Eq. (19). In the expression above \(S\) is a surface of constant radius \(r\) that satisfies Eqs. (31,32), \(e = \det(e^a_{\mu}) = r^2 \sin \theta\), and

\[
\Sigma^{(0)01} = e^{(0)}_0 \Sigma^{001} + e^{(0)}_1 \Sigma^{101} + e^{(0)}_2 \Sigma^{201}.
\]

Equation (35) determines the gravitational energy contained within a large surface of constant radius \(r\), which is also the radial position of an observer. It is not the total gravitational energy of the space-time, but since the surface \(S\) encloses the black hole, it allows the determination of the gravitational radiation, as we will see. In view of the fact that the radius \(r\) satisfies Eqs. (31,32), the surface of integration \(S\) is far from both the Schwarzschild and Rindler horizons.

We have the following nonvanishing, exact expressions for \(T_{\lambda\mu} = e^a_{\lambda} T_{a\mu}\):

\[
\begin{align*}
T_{012} &= B(\partial_i D + \partial_i C), \\
T_{102} &= C \partial_\theta B - \gamma^2 \dot{v} C \frac{r}{g} \sin \theta, \\
T_{201} &= -D \partial_\theta B + \gamma^2 \dot{v} \frac{r}{g} C \sin \theta, \\
T_{202} &= D \partial_\theta B, \\
T_{212} &= \frac{r}{g} \left( \frac{1}{g} - C \right), \\
T_{313} &= rg(g - C) \sin^2 \theta, \\
T_{323} &= -(1 - g^2)r^2 \sin \theta \cos \theta - rg(D - r \partial_\theta g) \sin^2 \theta.
\end{align*}
\]

where \(\dot{v} = dv/du\), and the functions \(B, C\) and \(D\) are defined by Eq. (27). Taking in account definition (17) we obtain
\[ \Sigma^{001} = -\frac{1}{2r} \left[ 2 - \left( g + \frac{1}{g} \right) C \right], \]  
\[ \Sigma^{101} = \frac{1}{2} \left[ \frac{g^2}{r^2} D \partial_\theta B + (1 - g^2) A \cos \theta, \right. \]  
\[ \left. + A \frac{q}{r} (D - r \partial_\theta g) \sin \theta \right], \]  
\[ \Sigma^{201} = \frac{g^2}{2r^2} B \partial_\theta C + \frac{g^2}{4r^2} \partial_r (BD) + \frac{g}{2r} A (g - C) \sin \theta. \]

The exact expression for the integrand in Eq. (35) is given by

\[ e \Sigma^{(0)01} = \gamma B \sin \theta \left\{ -\frac{r}{2} \left[ 2 - \left( g + \frac{1}{g} \right) C \right] \right\} \]  
\[ + \left( \gamma C - \gamma v \left[ C \cos \theta \right] \sin \theta \left[ \frac{1}{2} g^2 D \partial_\theta B \right. \right. \]  
\[ \left. + \frac{1}{2} (1 - g^2) A r^2 \cos \theta + \frac{1}{2} A r g (D - r \partial_\theta g) \sin \theta \right] \]  
\[ + \left( -\gamma D + \gamma v \frac{r}{g} \sin \theta + \gamma v D \cos \theta \right) \sin \theta \left[ \frac{1}{2} g^2 B \partial_\theta C \right. \]  
\[ \left. + \frac{1}{4} g^2 \partial_r (BD) + \frac{1}{2} A r g (g - C) \sin \theta \right]. \]

We proceed now to simplify the expression above by taking into account Eq. (33). Specifically, we wish to write it up to the first power in \( \varepsilon \). We note first that \( g = 1 + O(\varepsilon^2) \) and also \( g^{-1} = 1 + O(\varepsilon^2) \). Thus in the first order approximation we may take \( g \cong 1 \), and therefore \( e \Sigma^{(0)01} \) is simplified to

\[ e \Sigma^{(0)01} \cong -\gamma r (B - 1) \sin \theta \]  
\[ + \frac{1}{2} \gamma (C - v C \cos \theta) \sin \theta (D \partial_\theta B + A r D \sin \theta) \]  
\[ + \frac{1}{2} \gamma (-D + v r \sin \theta + v D \cos \theta) \sin \theta \left[ -C \partial_\theta B \right. \]  
\[ \left. + \frac{1}{2} \partial_r (BD) + A r (1 - C) \sin \theta \right]. \]

Next we expand the functions \( B \) and \( C \) up to \( O(\varepsilon^2) \). We find
\[ B \cong 1 - \frac{m}{r} - \left( Ar + \frac{1}{2} mA \right) \cos \theta - \frac{1}{2} \left( \frac{m}{r} \right)^2 \]
\[ \quad - \frac{1}{2} (Ar)^2 \sin^2 \theta - \frac{1}{8} (Ar)^2 \cos^2 \theta , \]
\[ C \cong 1 + \frac{m}{r} + \left( Ar + \frac{1}{2} mA \right) \cos \theta + \frac{1}{2} \left( \frac{m}{r} \right)^2 \]
\[ \quad + \frac{1}{2} (Ar)^2 \sin^2 \theta + \frac{1}{8} (Ar)^2 \cos^2 \theta . \]

We observe that

\[ B - 1 = \frac{m}{r} - Ar \cos \theta + O(\varepsilon^2) , \quad (41) \]
\[ \partial_\theta B = (Ar) \sin \theta + O(\varepsilon^2) . \]

Upon substitution of Eqs. (40,41) into Eq. (39) we obtain

\[ e \Sigma^{(0)01} = \gamma m \sin \theta + \gamma (Ar) \sin \theta \cos \theta + \gamma O(\varepsilon^2) . \quad (42) \]

Given that the integral of the second term in the expression above vanishes upon integration in \( \theta \) between 0 and \( \pi \), we finally integrate Eq. (35) and obtain

\[ P^{(0)} \cong \frac{1}{4\pi} \oint_S d\theta d\phi (\gamma m \sin \theta) = \gamma m . \quad (43) \]

The expression above corresponds to the the gravitational energy evaluated with respect to observers whose four-velocity is given by Eq. (30), which in first approximation reduces to Eq. (34). For such observers, the closed surface of integration \( S \) in Eq. (35) is sufficiently far from the black hole and from the Rindler horizon.

The total flux of gravitational radiation is determined by Eq. (23). We easily find

\[ \Phi_g^{(0)} = \dot{P}^{(0)} \cong m \dot{v} \gamma \dot{v} \gamma^3 . \quad (44) \]

For an uniformly accelerated motion we identify
\[ \dot{v} = \frac{1}{\gamma^3} A, \]

This identification is normally considered in the context of an uniformly accelerated particle in special relativity (see, for instance, Ref. [23]). Therefore we finally obtain for the total gravitational radiation flux

\[ \Phi_g^{(0)} \approx m v A. \quad (45) \]

Had we considered higher order terms in Eqs. (39-42), we would obtain additional terms of the type \( O(\varepsilon^3) \) in the expression above. This fact can be verified by noting that the time derivative of \( \gamma \) in the last term of Eq. (42), \( \gamma O(\varepsilon^2) \), yields a term of the type \( A O(\varepsilon^2) = (1/r)(Ar)O(\varepsilon^2) = (1/r)O(\varepsilon^3) \), which is negligible for large values of \( r \).

We remark that by making \( v \to -v \) we would be considering a totally different physical configuration, that has no particular relevance to our analysis. However, in view of Eq. (44), we note that \( \Phi_g^{(0)} \) would remain positive.

The gravitational radiation emitted by an uniformly accelerated source, as described by the C metric, has been previously investigated by Farhoosh and Zimmerman [4], and by Tomimatsu [21]. The idea of the approach by these authors is to bring the C metric into the Bondi-Metzner-Sachs form [22] and to take the time derivative of the mass aspect, that yields minus the square of news function. In this case the mass loss and the emitted radiation can be easily obtained. However the C metric does not display the asymptotic boundary conditions of the Bondi radiative solution, and therefore this scheme works only in a certain approximation (in Bondi’s space-time there is no Rindler horizon at large radial distances). Farhoosh and Zimmerman address the C metric gravitational radiation by constructing an inertial coordinate system out of the accelerated one. They conclude that to lowest order in acceleration, the mass loss is proportional to \((Am)^2\), in contrast to our result given by Eqs. (44,45), that is linear in \( Am \). We note, however, that we work with a unique coordinate system.

A possible small acceleration of the Sun has been considered recently as an explanation of the Pioneer anomaly [24, 25]. An anomalous acceleration on the motion of the Pioneer 10 and Pioneer 11 spacecrafts has been measured, which is directed towards the Sun. The suggested explanation [6] is based on a slight modification of Eq. (14) for the spacecrafts, possibly due to a
nongravitational force exerted on the Sun as a consequence of anisotropic solar emission. We note that the approximations given by Eqs. (31,32) are quite natural in this physical scenario. An observer on Earth located at the radial position $r$ from the center of the Sun is very distant from the Rindler horizon. The gravitational radiation emitted by the Sun and measured on Earth is evaluated through a solid angle of constant radius $r$.

5 The absolute character of acceleration

An intriguing issue in general relativity is the nature of accelerating or rotating motions. Let us consider two bodies in space-time, $A$ and $B$, that describe arbitrary motions. We assume, for simplicity, that these motions take place along a locally straight line. Let us consider the situation in which (i) $A$ is at rest, with $B$ moving relatively to $A$, and (ii) $B$ is at rest, with $A$ moving relatively to $B$. Are these two physical situations equivalent? If both situations are equivalent, we can assert that acceleration is relative. If not, we conclude that the acceleration of a physical system is absolute.

This issue has been addressed by Mashhoon [26], who investigated the relativity of rotation. He concluded that gravitomagnetic effects are an absolute measure of the rotation of a massive body, and therefore that rotation has an absolute character. Mashhoon considers a rotating astronomical body, the Earth for instance, and the heavens are described to a certain approximation by a spherical shell. It is observed that the two situations, the Earth considered at rest, at the center of the rotating heavens, and the Earth rotating at the center of the static heavens, produce physically inequivalent effects on test particles on the surface of the Earth.

We will conclude that acceleration is absolute by means of a totally different analysis. We consider the following two situations: (i) an observer linearly accelerated with respect to a static black hole, and (ii) a black hole linearly accelerated with respect to an observer at rest. In both situations we assume that the observer is somehow able to measure the gravitational radiation across an open spherical surface $S_r$, produced by the accelerated motion. The surface $S_r$ is established at a fixed distance $r$ from the center of the black hole, where the observer is located. The first situation described above has been investigated in Ref. [27].

Before we proceed, let us note that the concept of “static” black hole
makes sense provided we establish the usual asymptotic boundary conditions
\( g_{\mu\nu} \cong \eta_{\mu\nu} + h_{\mu\nu}(1/r) \) in the limit \( r \to \infty \). Moreover the absence of a Rindler horizon ensures the stationary character of the black hole. In Ref. [27] it was considered a moving observer in the presence of a static black hole. If the observer moves at constant proper velocity and is sufficiently far from the black hole, then the principle of relativity guarantees that the observer might be considered at rest in the presence of the moving black hole. We note that in this case the usual tetrad field boundary conditions for the black hole, \( e_{\alpha\mu} \cong \eta_{\alpha\mu} + (1/2)h_{\alpha\mu}(1/r) \), are not satisfied. The tetrad field for a Schwarzschild black hole in isotropic coordinates, that is boosted in the \( x \) direction, reads [27]

\[
e^a_{\mu}(t, x, y, z) = \begin{pmatrix}
\gamma M & -\gamma v N & 0 & 0 \\
-\gamma v M & \gamma N & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & N
\end{pmatrix},
\]

where

\[
M^2 = \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2},
\]

\[
N^2 = (1 + m/2\rho)^4,
\]

and \( x = \rho \sin \theta \cos \phi, \ y = \rho \sin \theta \sin \phi \) and \( z = \rho \cos \theta \). The variable \( \rho \) is related to the usual radial coordinate \( r \) by \( r = \rho(1 + m/2\rho)^2 \). In the asymptotic limit \( \rho \to \infty \) we have \( e_{(0)1} \to -v\gamma \) and \( e_{(1)0} \to -v\gamma \), which are not in agreement with the ordinary boundary conditions \( e_{\alpha\mu} \cong \eta_{\alpha\mu} + (1/2)h_{\alpha\mu}(1/r) \).

By considering an observer accelerated in the \( x \) direction, in the presence of a static Schwarzschild black hole, it has been shown that the gravitational radiation in the accelerated frame, across a surface \( S_r \), is given by [27]

\[
\Phi^{(0)}_g = \frac{\dot{P}^{(0)}}{4\pi} = \frac{mv\dot{v}\gamma^3}{4\pi} \sin \theta.
\]

The total gravitational radiation measured in the accelerated frame is

\[
\Phi^{(0)}_g = \dot{P}^{(0)} = mv\dot{v}\gamma^3.
\]

We note that the expression above is exact, in contrast to Eq. (44).
The situation in which the black hole is accelerated with respect to observers approximately at rest is the subject of section 4 above. In the context of the present analysis, the gravitational radiation across the same surface \( S_r \) is given by expression (46) plus correction terms. Taking into account Eq. (42), we see that up to first order in \( \varepsilon \) we have

\[
\Phi_g^{(0)} \approx \frac{1}{4\pi} \int_{S_r} d\theta d\phi \hat{v} \hat{v} \gamma^3 \sin \theta (m + Ar \cos \theta).
\]  

(48)

The expression above is expected to hold in the frame where the four-velocity of the observers is given by Eq. (34). However at the radial distance \( r \) such observers are assumed to be approximately at rest because Eq. (34) differs from Eq. (30) by \( O(\varepsilon) \) terms. Thus the \( O(\varepsilon) \) term in Eq. (48) might be questionable. However, as long as we depart from conditions (31,32), Eq. (48) will differ from Eq. (46) by several additional terms (according to Eq. (38)). Therefore we conclude that the detection of gravitational radiation in the two situations described above will be different. The measured values of the radiation can be used, in principle, to decide whether the black hole is static or accelerated.

6 Final Remarks

The analysis of the gravitational radiation emitted by accelerated particles in the framework of the C metric space-time has been previously addressed by, among others, Kinnersley and Walker [3], Farhoosh and Zimmerman [4], Bonnor [5] and Tomimatsu [21]. Since the C metric is a vacuum solution of Einstein’s field equations and describe the exterior space-time of an accelerated particle of mass \( m \), the radiation in question arises from a monopole term, very much like the similar phenomenon that takes place in electromagnetism. We recall that the analysis of the possible types of gravitational energy transfer have been investigated by Bondi [28], who concluded that in general relativity gravitational energy can be transferred by means of inductive or radiative transfer, in similarity to electromagnetism.

In this article we have likewise considered the C metric space-time and analyzed the issue of gravitational energy transfer by means of a radiative process. Since the C metric is a vacuum solution of Einstein’s field equations, the only available form of energy is gravitational energy, and thus we can only
have gravitational energy transfer.

We have constructed a set of tetrad fields that establish the four-velocity of observers that are approximately at rest in space-time, in the limiting case determined by conditions (31) and (32), which ensure that the observer is at great distance from the Schwarzschild and Rindler horizons. Under these conditions we evaluated the total gravitational radiation emitted by the accelerated black hole, which turns out to be the simple expression given by Eq. (44). Finally we concluded that the accelerated motion in space-time is not relative. In this respect we note that the deformation of the event horizon of a Schwarzschild black hole, as determined by $r_S$ presented in section 2, constitutes one further distinctive feature of the accelerated system. The physical properties of the horizon would, in principle, allow us to characterize a static or accelerated black hole.

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