Exceptional Seiberg-Witten Geometry with Massive Fundamental Matters

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Abstract

We propose Seiberg-Witten geometry for $N = 2$ gauge theory with gauge group $E_6$ with massive $N_f$ fundamental hypermultiplets. The relevant manifold is described as a fibration of the ALE space of $E_6$ type. It is observed that the fibering data over the base $\mathbb{CP}^1$ has an intricate dependence on hypermultiplet bare masses.
Recently several attempts have been made in extending the work of Seiberg and Witten on four-dimensional $N = 2$ supersymmetric gauge theory [1], [2] so as to include matter hypermultiplets in representations other than the fundamentals [3]-[6]. In our previous paper [3] we have refined the technique of $N = 1$ confining phase superpotentials toward the application to $N = 2$ or $N = 1$ gauge theories with A-D-E gauge groups with matter hypermultiplets in addition to an adjoint matter. The moduli space of $N = 1$ confining phase can also be studied in view of M-theory fivebrane, the result of which is in agreement with the field theory analysis [7], [8]. In [4] the fivebrane configurations are considered to obtain complex curves describing $N = 2$ SU($N_c$) gauge theory with matters in the symmetric or antisymmetric representation. In [5] the approach based on Type IIA string compactification on Calabi-Yau threefolds is used to find exact descriptions of $N = 2$ SO($N_c$) gauge theory with vectors and spinors.

In this paper we propose Seiberg-Witten geometry for $N = 2$ supersymmetric gauge theory with gauge group $E_6$ with massive $N_f$ fundamental hypermultiplets. To this aim we employ the technique of $N = 1$ confining phase superpotentials [3]-[14]. The ALE space description of Seiberg-Witten geometry for $N = 2$ SU($N_c$) and SO($2N_c$) QCD is recovered in this approach [3]. We will show in the following that an extension of [3] enables us to obtain exceptional Seiberg-Witten geometry with fundamental hypermultiplets. The resulting manifold takes the form of a fibration of the ALE space of type $E_6$.

Let us consider $N = 1$ $E_6$ gauge theory with $N_f$ fundamental matters $Q^i, \tilde{Q}_j$ ($1 \leq i, j \leq N_f$) and an adjoint matter $\Phi$. $Q^i, \tilde{Q}_j$ are in $27$ and $\overline{27}$, and $\Phi$ in $78$ of $E_6$. The coefficient of the one-loop beta function is given by $b = 24 - 6N_f$, and hence the theory is asymptotically free for $N_f = 0, 1, 2, 3$ and finite for $N_f = 4$. We take a tree-level superpotential

$$W = \sum_{k \in S} g_k s_k(\Phi) + \text{Tr}_{N_f} \gamma_0 \tilde{Q}Q + \text{Tr}_{N_f} \gamma_1 \tilde{Q}\Phi Q,$$

(1)

where $S = \{2, 5, 6, 8, 9, 12\}$ denotes the set of degrees of $E_6$ Casimirs $s_k(\Phi)$ and $g_k, (\gamma_0)^i_j (1 \leq i, j \leq N_f)$ are coupling constants. A basis for the $E_6$ Casimirs will be specified momentarily. When we put $(\gamma_0)^i_j = \sqrt{2}m^i_j$ with $[m, m^\dagger] = 0, (\gamma_1)^i_j = \sqrt{2}\delta^i_j$ and all $g_k = 0$, (1) is reduced to the superpotential in $N = 2$ supersymmetric Yang-Mills theory with massive $N_f$ hypermultiplets.
We now look at the Coulomb phase with $Q = \tilde{Q} = 0$. Since $\Phi$ is restricted to take the values in the Cartan subalgebra we express the classical value of $\Phi$ in terms of a vector

$$a = \sum_{i=1}^{6} a_i \alpha_i$$

with $\alpha_i$ being the simple roots of $E_6$. Then the classical vacuum is parametrized by

$$\Phi^{cl} = \text{diag} (a \cdot \lambda_1, a \cdot \lambda_2, \ldots, a \cdot \lambda_{27}),$$

where $\lambda_i$ are the weights for 27 of $E_6$. For the notation of roots and weights we follow [15]. We define a basis for the $E_6$ Casimirs $u_k(\Phi)$ by

$$u_2 = -\frac{1}{12} \chi_2, \quad u_5 = -\frac{1}{60} \chi_5, \quad u_6 = -\frac{1}{6} \chi_6 + \frac{1}{6 \cdot 12} \chi_2^2,$$

$$u_8 = -\frac{1}{40} \chi_8 + \frac{1}{180} \chi_2 \chi_6 - \frac{1}{2 \cdot 12} \chi_2^2, \quad u_9 = -\frac{1}{7 \cdot 6^2} \chi_9 + \frac{1}{20 \cdot 6^3} \chi_2^2 \chi_5,$$

$$u_{12} = -\frac{1}{60} \chi_{12} + \frac{1}{5 \cdot 6^3} \chi_6^2 + \frac{1}{5 \cdot 12^3} \chi_2 \chi_5^2$$
$$+ \frac{5}{2 \cdot 12^3} \chi_2^2 \chi_8 - \frac{1}{3 \cdot 6^4} \chi_2^3 \chi_6 + \frac{29}{10 \cdot 12^6} \chi_2^6,$$

where $\chi_n = \text{Tr} \Phi^n$. The standard basis $w_k(\Phi)$ are written in terms of $u_k$ as follows

$$w_2 = \frac{1}{2} u_2, \quad w_5 = -\frac{1}{4} u_5, \quad w_6 = \frac{1}{96} (u_6 - u_2^3),$$

$$w_8 = \frac{1}{96} \left( u_8 + \frac{1}{4} u_2 u_6 - \frac{1}{8} u_4^2 \right), \quad w_9 = -\frac{1}{48} \left( u_9 - \frac{1}{4} u_2^2 u_5 \right),$$

$$w_{12} = \frac{1}{3456} \left( u_{12} + \frac{3}{32} u_6^2 - \frac{3}{4} u_2^2 u_8 - \frac{3}{16} u_2^3 u_6 + \frac{1}{16} u_6^6 \right).$$

(5)

The basis $\{u_k\}$ and (4) were first introduced in [16]. In our superpotential (4) we then set

$$s_2 = w_2, \quad s_5 = w_5, \quad s_6 = w_6, \quad s_8 = w_8, \quad s_9 = w_9, \quad s_{12} = w_{12} - \frac{1}{4} w_6^2.$$

(6)

We will discuss later why this particular form is assumed.

The equations of motion are given by

$$\frac{\partial W(a)}{\partial a_i} = \sum_{k \in S} g_k \frac{\partial s_k(a)}{\partial a_i} = 0.$$  

(7)

*Our notation is slightly different from [3]. Here we use $a_i$ with lower index instead of $a_i$ in [3].
†The Casimirs $u_1, u_2, u_3, u_4, u_5, u_6$ in [16] are denoted here as $u_2, u_5, u_6, u_8, u_9, u_{12}$, respectively.
Thus, if we regard
where \( \mathcal{D} \) vacua only when the coupling constants are subject to the relation (10).

Let us focus on the classical vacua with an unbroken \( SU(2) \times U(1)^5 \) gauge symmetry. Fix the \( SU(2) \) direction by choosing the simple root \( \alpha_1 \), then we have the vacuum condition

\[
a \cdot \alpha_1 = 2a_1 - a_2 = 0.
\]  
(8)

It follows from (7), (8) that

\[
\frac{g_9}{g_{12}} = \frac{D_{1,9}}{D_{1,12}} = -\frac{1}{8}(2a_1a_5a_4 - a_4a_3^2 + a_3^2a_4 + a_2^2a_3 - a_3a_6^2 + a_3^2a_6 - 2a_1a_5^2 + 2a_1a_6^2 - 2a_1^2a_1 - a_5a_4^2 - 2a_1a_3a_6 + 2a_4a_3a_1),
\]

\[
\frac{g_8}{g_{12}} = \frac{D_{1,8}}{D_{1,12}} = -\frac{1}{48}(12a_1a_5^2a_4 - 6a_1^2a_5 - 6a_1a_5^2a_6 - 4a_1^3a_3 + 4a_1^2a_4 + 2a_1^3a_4 + 2a_1^3a_6 - a_1^4 - 3a_3a_6^2 - 3a_3a_4^2 - a_3^4 - 12a_1a_5a_4^2 - 2a_5a_4a_6^2 + 8a_1a_3a_6^2 + 3a_1^4 + 6a_1^2a_4a_3 - 8a_1a_3^2a_1 - 2a_5a_4a_3^2 - 2a_1^2a_3a_6 + 6a_1^2a_5a_4 - 2a_4a_3a_6 + 2a_5a_3^2a_3 - 2a_5a_3a_6 + 2a_4a_3^2a_6 - a_3^4 - 2a_5a_3a_4 - 2a_1a_3^2a_3 - 6a_1^2a_3^2 + 2a_5a_3^2a_6 + 2a_1a_3a_4 + 2a_1a_3^2a_6 - 8a_1a_3^2a_6 - 8a_4a_3a_1 - 4a_5a_1a_3 + 2a_5a_4a_3a_6 + 4a_5a_4a_1a_3 + 2a_3a_4^3 - 3a_3a_5^2 + 2a_4a_3^3 + 2a_4a_5 + 2a_6a_3),
\]

\[
\frac{g_6}{g_{12}} = \frac{D_{1,6}}{D_{1,12}} = \frac{1}{192}(4a_3^3a_1a_5^2 - 18a_3^2a_2a_5 + 13a_3^4a_1^2 - a_3^4a_5^2 - 7a_3^3a_5^3 + 9a_3^2a_6^2 + \cdots),
\]  
(9)

where \( D_{1,k} \) is the cofactor for a \((1,k)\) element of the \( 6 \times 6 \) matrix \([\partial s_i(a)/\partial a_j], i \in \mathcal{S}\) and \( j = 1, \ldots, 6 \) [3]. In (4) the explicit expression for \( g_6/g_{12} \) is too long to be presented here, and hence suppressed. Denoting \( y_1 = g_9/g_{12}, y_2 = g_8/g_{12}, y_3 = g_6/g_{12}, \) we find that the others are expressed in terms of \( y_1, y_2 \)

\[
\frac{g_2}{g_{12}} = \frac{D_{2,2}}{D_{2,12}} = y_1^2y_2, \quad \frac{g_5}{g_{12}} = \frac{D_{2,5}}{D_{2,12}} = y_1y_2.
\]

This means that our superpotential specified with Casimirs (3) realizes the \( SU(2) \times U(1)^5 \) vacua only when the coupling constants are subject to the relation (10).

Notice that reading off degrees of \( y_1, y_2, y_3 \) from (4) gives \([y_1] = 3, [y_2] = 4, [y_3] = 6\). Thus, if we regard \( y_1, y_2, y_3 \) as variables to describe the \( E_6 \) singularity, (3) and (10) may
be identified as relevant monomials in versal deformations of the $E_6$ singularity. In fact we now point out an intimate relationship between classical solutions corresponding to the symmetry breaking $E_6 \supset SU(2) \times U(1)^6$ and the $E_6$ singularity. For this we examine the superpotential (11) at classical solutions

$$W_{cl} = g_{12} \sum_{k \in S} \left( \frac{g_k}{g_{12}} \right) s_k^{cl}(a)$$

$$= g_{12} \left( s_2^{cl} y_1^2 y_2 + s_5^{cl} y_1 y_2 + s_6^{cl} y_3 + s_8^{cl} y_2 + s_9^{cl} y_1 + s_{12}^{cl} \right).$$

(11)

Evaluating the RHS with the use of (8)-(10) leads to

$$W_{cl} = -g_{12} \left( 2y_1^2 y_3 + y_2^2 - y_3^2 \right).$$

(12)

It is also checked explicitly that

$$-4y_1 y_3 = 2s_2^{cl} y_1 y_2 + s_5^{cl} y_2 + s_9^{cl},$$

$$-3y_2^2 = s_2^{cl} y_1^2 + s_5^{cl} y_1 + s_8^{cl},$$

$$-2y_1^2 + 2y_3 = s_6^{cl}.$$  

(13)

To illustrate the meaning of (11)-(13) let us recall the standard form of versal deformations of the $E_6$ singularity

$$W_{E_6}(x_1, x_2, x_3; w) = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12},$$

(14)

where the deformation parameters $w_k$ are related to the $E_6$ Casimirs via (5) [16]. Then what we have observed in (11)-(13) is that when we express $w_k$ in terms of $a_i$ as $w_k = w_k^{cl}(a)$ the equations

$$W_{E_6} = \frac{\partial W_{E_6}}{\partial x_1} = \frac{\partial W_{E_6}}{\partial x_2} = \frac{\partial W_{E_6}}{\partial x_3} = 0$$

(15)

can be solved by

$$x_1 = y_1(a), \quad x_2 = y_2(a), \quad x_3 = i \left( y_3(a) - y_1(a)^2 - s_6^{cl}(a) \right)$$

(16)

under the condition (8). This observation plays a crucial role in our analysis.

\footnote{We have observed a similar relation between the symmetry breaking solutions $SU(r + 1)$ (or $SO(2r)$) $\supset SU(2) \times U(1)^{r-1}$ and the $A_r$ (or $D_r$) singularity.}
When applying the technique of confining phase superpotentials we usually take all coupling constants \( g_k \) as independent moduli parameters. To deal with \( N = 1 \) \( E_6 \) theory with fundamental matters, however, we find it appropriate to proceed as follows. First of all, motivated by the above observations for classical solutions, we keep three coupling constants \( g'_6 = g_6/g_{12}, \ g'_8 = g_8/g_{12} \) and \( g'_9 = g_9/g_{12} \) adjustable while the rest is fixed as \( g'_2 = g_8g'_9, \ g'_5 = g'_6g'_9 \) with \( g'_k = g_k/g_{12} \). Taking this parametrization it is seen that the equations of motion are satisfied by virtue of (10) in the \( SU(2) \times U(1)^5 \) vacua (8). Note here that originally there exist six classical moduli \( a_i \) among which one is fixed by (8) and three are converted to \( g'_9 = y_1(a), \ g'_8 = y_2(a) \) and \( g'_6 = y_3(a) \), and hence we are left with two classical moduli which will be denoted as \( \xi_i \). Without loss of generality one may choose \( \xi_2 = s_2^d(a) \) and \( \xi_5 = s_5^d(a) \).

We now evaluate the low-energy effective superpotential in the \( SU(2) \times U(1)^5 \) vacua. \( U(1) \) photons decouple in the integrating-out process. The standard procedure yields the effective superpotential for low-energy \( SU(2) \) theory [9],[3]

\[
W_L = -g_{12} \left( 2y_1^2y_3 + y_2^3 - y_3^2 \right) \pm 2\Lambda_{YM}^3, \tag{17}
\]

where the second term takes account of \( SU(2) \) gaugino condensation with \( \Lambda_{YM} \) being the dynamical scale for low-energy \( SU(2) \) Yang-Mills theory. The low-energy scale \( \Lambda_{YM} \) is related to the high-energy scale \( \Lambda \) through the scale matching [3]

\[
\Lambda_{YM}^6 = g_{12}^2 A(a), \quad A(a) \equiv \Lambda^{24-6N_f} \prod_{s=1}^6 \det_{N_f} (\gamma_0 + \gamma_1 (a \cdot \lambda_s)), \tag{18}
\]

where \( \lambda_s \) are weights of 27 which branch to six \( SU(2) \) doublets respectively under \( E_6 \supset SU(2) \times U(1)^5 \). Explicitly they are given in the Dynkin basis as

\[
\begin{align*}
\lambda_1 &= (1, 0, 0, 0, 0, 0), & \lambda_2 &= (1, -1, 0, 0, 1, 0), \\
\lambda_3 &= (1, -1, 0, 1, -1, 0), & \lambda_4 &= (1, -1, 1, -1, 0, 0), \\
\lambda_5 &= (1, 0, -1, 0, 0, 1), & \lambda_6 &= (1, 0, 0, 0, 0, -1).
\end{align*}
\tag{19}
\]

Notice that \( \sum_{s=1}^6 \lambda_s = 3\alpha_1 \).
Let us first discuss the \( N_f = 0 \) case, i.e. \( E_6 \) pure Yang-Mills theory, for which \( A(a) \) in (18) simply equals \( \Lambda^{24} \). The vacuum expectation values are calculated from (17)

\[
\frac{\partial W_L}{\partial g_{12}} = \langle \tilde{W}(y_1, y_2, y_3; s) \rangle = -\left(2y_1^2y_3 + y_2^2 - y_3^2\right) \pm 2\Lambda^{12},
\]

\[
\frac{1}{g_{12}} \frac{\partial W_L}{\partial y_1} = \langle \frac{\partial \tilde{W}(y_1, y_2, y_3; s)}{\partial y_1} \rangle = -4y_1y_3,
\]

\[
\frac{1}{g_{12}} \frac{\partial W_L}{\partial y_2} = \langle \frac{\partial \tilde{W}(y_1, y_2, y_3; s)}{\partial y_2} \rangle = -3y_2^2,
\]

\[
\frac{1}{g_{12}} \frac{\partial W_L}{\partial y_3} = \langle \frac{\partial \tilde{W}(y_1, y_2, y_3; s)}{\partial y_3} \rangle = -2y_1^2 + 2y_3,
\]

where \( y_1, y_2, y_3 \) and \( g_{12} \) have been treated as independent parameters as discussed before and

\[
\tilde{W}(y_1, y_2, y_3; s) = s_2 y_1^2 y_2 + s_5 y_1 y_2 + s_6 y_3 + s_8 y_2 + s_9 y_1 + s_{12}.
\]

Define a manifold by \( W_0 = 0 \) with four coordinate variables \( z, y_1, y_2, y_3 \in \mathbb{C} \) and

\[
W_0 \equiv z + \frac{\Lambda^{24}}{z} - \left(2y_1^2y_3 + y_2^2 - y_3^2 + \tilde{W}(y_1, y_2, y_3; s)\right) = 0.
\]

It is easy to show that the expectation values (20) parametrize the singularities of the manifold where

\[
\frac{\partial W_0}{\partial z} = \frac{\partial W_0}{\partial y_1} = \frac{\partial W_0}{\partial y_2} = \frac{\partial W_0}{\partial y_3} = 0.
\]

Making a change of variables \( y_1 = x_1, y_2 = x_2, y_3 = -ix_3 + x_1^2 + s_6/2 \) in (22) we have

\[
z + \frac{\Lambda^{24}}{z} - \tilde{W}_E(x_1, x_2, x_3; w) = 0.
\]

Thus the ALE space description of \( N = 2 \) \( E_6 \) Yang-Mills theory \([17],[16]\) is obtained from the \( N = 1 \) confining phase superpotential.

We next turn to considering the fundamental matters. In the \( N = 2 \) limit we have

\[
A(a) = \Lambda^{24-6N_f} \cdot 8^{N_f} \prod_{i=1}^{N_f} f(a, m_i) \text{ with } f(a, m) = \prod_{s=1}^{6} (m + a \cdot \lambda_s).
\]

After some algebra we find

\[
f(a, m) = m^6 + 2\xi_2 m^4 - 8m^3 y_1 + \left(\xi_2^2 - 12y_2\right) m^2 + 4\xi_5 m - 4y_2\xi_2 - 8y_3.
\]
where we have used (3)-(9). Recall that, in viewing (17), we think of $(y_1, y_2, y_3, \xi_2, \xi_5, g_{12})$ as six independent parameters. Then the quantum expectation values are given by

$$\frac{1}{g_{12} \partial y_1} \langle \partial \tilde{W}(y_1, y_2, y_3; s) \rangle = -A(y_1, y_2, y_3; \xi, m),$$

$$\frac{1}{g_{12} \partial y_2} \langle \partial \tilde{W}(y_1, y_2, y_3; s) \rangle = -3y_2^2 + 2y_3 \pm 2\sqrt{A(y_1, y_2, y_3; \xi, m)},$$

$$\frac{1}{g_{12} \partial y_3} \langle \partial \tilde{W}(y_1, y_2, y_3; s) \rangle = -2y_1^2 + 2y_3 \pm 2\sqrt{A(y_1, y_2, y_3; \xi, m)}.$$  \hspace{0.5cm} (26)

Similarly to the $N_f = 0$ case one can check that these expectation values satisfy the singularity condition for a manifold defined by

$$z + \frac{1}{z} A(y_1, y_2, y_3; \xi, m) - \left(2y_1^2 y_3 + y_2^3 - y_3^2 + \tilde{W}(y_1, y_2, y_3; s)\right) = 0.$$  \hspace{0.5cm} (27)

Note that $s_k$ in $\tilde{W}$ are quantum moduli parameters. What about $\xi_2, \xi_5$ in the one-instanton factor $A$? Classically we have $\xi_i = s_i^{cl}$ as was seen before. The issue is thus whether the classical relations $\xi_i = s_i^{cl}$ receive any quantum corrections at the singularities.

If there appear no quantum corrections, $\xi_i$ in $A$ can be replaced by quantum moduli parameters $s_i$. Let us simply assume here that $\xi_i = s^{cl}_i = \langle s_i \rangle$ for $i = 2, 5$ in the $N = 1 SU(2) \times U(1)^5$ vacua. This assumption seems quite plausible as long as we have inspected possible forms of quantum corrections due to gaugino condensates.

Now we find that Seiberg-Witten geometry of $N = 2$ supersymmetric QCD with gauge group $E_6$ is described by

$$z + \frac{1}{z} A(x_1, x_2, x_3; w, m) - W_{E_6}(x_1, x_2, x_3; w) = 0,$$  \hspace{0.5cm} (28)

where a change of variables from $y_i$ to $x_i$ as in (24) has been made in (27) and

$$A(x_1, x_2, x_3; w, m) = \Lambda^{24 - 6N_f} \cdot 8^{N_f} \prod_{i=1}^{N_f} \left(m_i^6 + 2w_2m_i^4 - 8m_i^3x_1 + \left(w_2^2 - 12x_2\right) m_i^2 + 4w_5m_i - 4w_2y_2 - 8(x_1^2 - i x_3 + w_6/2)\right).$$  \hspace{0.5cm} (29)

The manifold takes the form of ALE space of type $E_6$ fibered over the base $\mathbb{CP}^1$. Note an intricate dependence of the fibering data over $\mathbb{CP}^1$ on the hypermultiplet masses. This is
in contrast with the ALE space description of $N = 2$ $SU(N_c)$ and $SO(2N_c)$ gauge theories with fundamental matters. In (28), letting $m_i \to \infty$ while keeping $\Lambda^{24-6N_f} \prod_{i=1}^{N_f} m_i^6 \equiv \Lambda_0^{24}$ finite we recover the pure Yang-Mills result (24).

As a non-trivial check of our proposal (28) let us examine the semi-classical singularities. In the semi-classical limit $\Lambda \to 0$ the discriminant $\Delta$ for (28) is expected to take the form $\Delta \propto \Delta_G \Delta_M$ where $\Delta_G$ is a piece arising from the classical singularities associated with the gauge symmetry enhancement and $\Delta_M$ represents the semi-classical singularities at which squarks become massless. When the $N_f$ matter hypermultiplets belong to the representation $\mathcal{R}$ of the gauge group $G$ we have

$$\Delta_M = \prod_{i=1}^{N_f} \det_{d \times d}(m_i \mathbb{1} - \Phi^d) = \prod_{i=1}^{N_f} P_R^G(m_i; u),$$

where $d = \dim \mathcal{R}$, $m_i$ are the masses, $\Phi^d$ denotes the classical Higgs expectation values and $P^R_G(x; u)$ is the characteristic polynomial for $\mathcal{R}$ with $u_i$ being Casimirs constructed from $\Phi^d$.

For simplicity, let us consider the case in which all the quarks have equal bare masses. Then we can change a variable $x_3$ to $\tilde{x}_3$ so that $A = A(\tilde{x}_3; w, m)$ is independent of $x_1$ and $x_2$. Eliminating $x_1$ and $x_2$ from (28) by the use of

$$\frac{\partial W_{E_6}}{\partial x_1} = \frac{\partial W_{E_6}}{\partial x_2} = 0,$$

we obtain a curve which is singular at the discriminant locus of (28). The curve is implicitly defined through

$$\mathbb{W}_{E_6} \left( \tilde{x}_3; w_i - \delta_{i,12} \left( z + \frac{A(\tilde{x}_3; w, m)}{z} \right) \right) = 0,$$

where $\mathbb{W}_{E_6}(\tilde{x}_3; w_i) = W_{E_6}(x_1(\tilde{x}_3, w_i), x_2(\tilde{x}_3, w_i), \tilde{x}_3; w_i)$ and $x_1(\tilde{x}_3, w_i), x_2(\tilde{x}_3, w_i)$ are solutions of (31). Now the values of $\tilde{x}_3$ and $z$ at singularities of this curve can be expanded in powers of $\Lambda^{24-6N_f}$. Then it is more or less clear that the classical singularities corresponding to massless gauge bosons are produced. Furthermore, if we denote as $R(\mathbb{W}, A)$ the resultant of $\mathbb{W}_{E_6}(\tilde{x}_3; w_i)$ and $A(\tilde{x}_3; w, m)$, then $R(\mathbb{W}, A) = 0$ yields another singularity condition of the curve in the limit $\Lambda \to 0$. We expect that $R(\mathbb{W}, A) = 0$ corresponds to the semi-classical massless squark singularities as is observed in the case
of $N = 2$ $SU(N_c)$ QCD [18],[14]. Indeed, we have checked this by explicitly computing $R(\overline{W}, A)$ at sufficiently many points in the moduli space. For instance, taking $w_2 = 2, w_5 = 5, w_6 = 7, w_8 = 9, w_9 = 11$ and $w_{12} = 13$ in the $N_f = 1$ case, we get

$$R(\overline{W}, A) = m^2 \left( 3 m^{10} + 12 m^8 + \cdots \right) \left( 26973 m^{27} + 258552 m^{25} + \cdots \right)^3 \left( m^{27} + 24 m^{25} + 240 m^{23} + 240 m^{22} + 2016 m^{21} + 3360 m^{20} + 16416 m^{19} + 34944 m^{18} + 88080 m^{17} + 216576 m^{16} + 448864 m^{15} + 607488 m^{14} + 2198272 m^{13} - 296000 m^{12} + 417792 m^{11} - 3407104 m^{10} + 7796224 m^9 + 10664448 m^8 - 31708160 m^7 + 41183232 m^6 - 21889792 m^5 + 15575040 m^4 - 17125120 m^3 - 38456320 m^2 - 3461120 m + 9798656 \right), \quad (33)$$

while the $E_6$ characteristic polynomial for $27$ is given by

$$P_{E_6}^{27}(x; u) = x^{27} + 12w_2x^{25} + 60w_2^2x^{23} + 48w_5x^{22} + \left( 96w_6 + 168w_2^3 \right) x^{21} + 336w_2w_5x^{20} + \left( 528w_2w_6 + 294w_4^2 + 480w_8 \right) x^{19} + \left( 1344w_9 + 1008w_2w_5 \right) x^{18} + \cdots. \quad (34)$$

We now find a remarkable result that the last factor of (33) precisely coincides with $P_{E_6}^{27}(m; u)$! Hence the manifold described by (28) correctly produces all the semi-classical singularities in the moduli space of $N = 2$ supersymmetric $E_6$ QCD.

If we choose another form of the superpotential (1), say, the superpotential with $s_i = w_i$ for $i \in S$ instead of (6) we are unable to obtain $\Delta_M$ in (30). As long as we have checked the choice made in (6) is judicious in order to pass the semi-classical test. At present, we have no definite recipe to fix the tree-level superpotential which produces the correct semi-classical singularities, though it is possible to proceed by trial and error. In fact we can find Seiberg-Witten geometry for $N = 2$ $SO(2N_c)$ gauge theory with spinor matters and $N = 2$ $SU(N_c)$ gauge theory with antisymmetric matters [19].

In our result (28) it may be worth mentioning that the gaussian variable $x_3$ of the $E_6$ singularity appears in the fibering term. An analogous structure is observed for $N = 2$ $SO(10)$ gauge theory with spinors and vectors in [5]. Their result reads

$$z + \frac{\Lambda^{2b}}{z} B(x_1, x_3; v) - W_{D_5}(x_1, x_2, x_3; v) = 0, \quad (35)$$
where \( b = 8 - (6 - n) - 2(4 - n) = 3n - 6 \) is the coefficient of the one-loop beta function and

\[
W_{D_6}(x_1, x_2, x_3; v) = x_1^4 + x_1 x_2^2 - x_3^2 + v_2 x_1^3 + v_4 x_1^2 + v_6 x_1 + v_8 + v_5 x_2,
\]

\[
B(x_1, x_3; v) = x_1^{6-n} \left( x_3 - \frac{1}{8} \left( 4v_4 - v_2^2 + 4v_2 x_1 + 8x_1^2 \right) \right)^{4-n}
\]

for \( SO(10) \) with massless \( (6 - n) \) fundamentals and \( (4 - n) \) spinors. Here \( v_i \) stand for the \( SO(10) \) Casimirs. It is tempting to suspect that the above \( E_6 \) and \( SO(10) \) results are related through the Higgs mechanism under the symmetry breaking \( E_6 \supset SO(10) \times U(1) \).

There is no difficulty in using our method to find Seiberg-Witten geometry in the form of ALE fibrations for \( N = 2 \) QCD with \( E_7 \) gauge group although more computer powers are obviously required. Finally, to compare the present results, it is desired to work out exceptional Seiberg-Witten geometry with fundamental matters in the framework of Calabi-Yau geometric engineering \([20]-[22]\).

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