UNIFORM DIOPHANTINE APPROXIMATION AND BEST APPROXIMATION POLYNOMIALS

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Abstract. Let $\zeta$ be a real transcendental number. We refine the upper bound for the classical exponent $\hat{w}_n(\zeta)$ concerning uniform polynomial approximation, for $n \geq 4$. For large $n$, our bound will be of order $2n - \frac{3}{2} + o(1)$, slightly improving the former best known bound of order $2n - \frac{5}{4} + o(1)$ due to Bugeaud and the author. We further establish a significantly stronger conditioned bound upon a certain presumably weak assumption on the structure of the best approximation polynomials. The proofs of the main results are based on the parametric geometry of numbers introduced by Schmidt and Summerer that stems from Minkowski’s Second Convex Body Theorem, and transference of the original approximation problem in dimension $n$ to suitable higher dimensions.

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1. Introduction

A classical problem in Diophantine approximation is to study the approximation to a real number $\zeta$ by algebraic real numbers $\alpha$ of bounded degree. For example, the famous Wirsing problem [31] asks if for any real number $\zeta$ and any $\epsilon > 0$ the estimate

$$|\alpha - \zeta| \leq H(\alpha)^{-n-1+\epsilon}$$

has infinitely many solutions in real algebraic numbers $\alpha$ of degree at most $n$. Here $H(\alpha) = H(P_\alpha)$ denotes the naive height of the minimal polynomial $P_\alpha$ of $\alpha$ over $\mathbb{Z}$ with coprime coefficients, that is the maximum modulus among the coefficients of $P_\alpha$. In case $n = 1$ this is an immediate consequence of Dirichlet’s Theorem. It is further known that the answer to Wirsing’s problem is positive for $n = 2$, see [32], as well as for any $n$ and Lebesgue almost all real numbers. The problem of making $|\zeta - \alpha|$ small naturally leads to the problem of making $|P(\zeta)|$ small among integer polynomials $P$, with the same degree and (naive) height restrictions as for $\alpha$. Indeed with the above notation it is straightforward to show that $|P_\alpha(\zeta)| \ll H(P_\alpha) \cdot |\alpha - \zeta|$, and conversely $|\alpha - \zeta| \ll H(P_\alpha)^n \cdot |P_\alpha(\zeta)|$ can be found in [31] Lemma A8]. This connection motivates to define the of the classical exponents of approximation $w_n(\zeta)$ and $\hat{w}_n(\zeta)$. They are given as the supremum of $w \in \mathbb{R}$ for which the system

$$H(P) \leq X, \quad 0 < |P(\zeta)| \leq X^{-w}. \quad (1)$$

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has a solution in an integer polynomial $P$ of degree at most $n$ for arbitrarily large $X$, and all large $X$, respectively. The exponents $w_n(\zeta)$ already date back to Mahler [13]. The most fundamental property of these exponents that stems from Dirichlet’s Theorem is

$$w_n(\zeta) \geq \hat{w}_n(\zeta) \geq n.$$  

Moreover we want to mention the obvious relations

$$w_1(\zeta) \leq w_2(\zeta) \leq \cdots, \quad \hat{w}_1(\zeta) \leq \hat{w}_2(\zeta) \leq \cdots.$$  

For $\zeta$ real algebraic of degree $d$, Schmidt’s Subspace Theorem yields

$$w_n(\zeta) = \hat{w}_n(\zeta) = \min\{d-1, n\}.$$  

In this paper we investigate the uniform exponents $\hat{w}_n(\zeta)$ for transcendental $\zeta$. Davenport and Schmidt [7, Theorem 2b], although not using our notation, established the upper bound $\hat{w}_n(\zeta) \leq 2n - 1$ for any transcendental real number $\zeta$. This might seem a little surprising in some sense, as for $\mathbb{Q}$-linearly independent vectors $\zeta$, the corresponding uniform exponent can attain any value in $[n, +\infty]$ when $n \geq 2$. Concrete examples of $\zeta \in \mathbb{R}^n$ with prescribed uniform exponent in the allowed interval can be readily derived from [21, Theorem 2.5], with a suitable choice of the occurring parameters $\eta$ and Mahler’s theorem on polar reciprocal bodies. The coordinates of $\zeta$ can be chosen Liouville numbers in Cantor’s middle third set. The deep theorem by Roy [20] yields another non-constructive proof. For $n = 2$, from [7, Theorem 1a] concerning the dual simultaneous approximation problem and Jarník’s identity [11] the estimate $\hat{w}_2(\zeta) \leq 2 \cdot 2 - 1 = 3$ can be improved to

$$\hat{w}_2(\zeta) \leq \frac{3 + \sqrt{5}}{2} = 2.6180\ldots.$$  

Surprisingly this was shown to be sharp by Roy [17]. Numbers with $\hat{w}_2(\zeta) > 2$ and the set of values taken by $\hat{w}_2(\zeta)$ when $\zeta$ runs through the transcendental real numbers, have been intensely investigated, see for example [3, 8, 9, 10, 18, 19]. Much less is known for $n \geq 3$, in fact it is still open if $\hat{w}_n(\zeta) > n$ can occur. However, the upper bound $2n - 1$ for general $n$ was recently improved to

$$\hat{w}_n(\zeta) \leq n - \frac{1}{2} + \sqrt{n^2 - 2n + \frac{5}{4}}, \quad n \geq 2,$$

in [5]. This is sharp again for $n = 2$, and of order $2n - \frac{3}{2} + \varepsilon_n$ with $\varepsilon_n > 0$ of order $O(n^{-1})$ for larger $n$. For $n = 3$, the stronger bound

$$\hat{w}_3(\zeta) \leq 3 + \sqrt{2} = 4.4142\ldots.$$  

was obtained in the same paper [5]. The purpose of the current paper is to refine (6). Our first central, and the only unconditioned, result is the following slight improvement.

**Theorem 1.1.** Let $n$ be a positive integer and $\zeta$ be a real number. Then

$$\hat{w}_n(\zeta) \leq \theta_n := \frac{3(n-1) + \sqrt{n^2 - 2n + 5}}{2}.$$  

For $n = 2$, the value $\theta_n$ again coincides with the sharp bound (5). For $n = 3$, it becomes the bound in (7). The latter is surprising, since the method in the proof of Theorem 1.1 differs significantly from the one in [5]. The proof in [5] partly relied on deep results of
Schmidt and Summerer [29, 30] concerning the minimum gap between the values \( w_n(\zeta) \) and \( \hat{w}_n(\zeta) \). Although our proof of Theorem 1.1 employs fundamental principles of the parametric geometry developed by Schmidt and Summerer in [28], our approach does not require the deeper results from [29, 30]. The two different proofs provide a cautious indication that (7) could be optimal (or at least close). For \( n \geq 4 \), we certainly do not expect \( \theta_n \) to be best possible, however it is a proper improvement of (6). We have \( \theta_n = 2n - 2 + \epsilon_n \) with \( \epsilon_n > 0 \) of order \( O(n^{-1}) \), thus the gain compared to (6) is roughly subtraction of \( \frac{1}{2} \) for \( n \) large. For small \( n \) we state the numerical values

\[ \theta_4 = 6.3028\ldots, \quad \theta_5 = 8.2361\ldots, \quad \theta_{10} = 18.1098\ldots. \]

2. Conditioned results

2.1. Known conditioned bounds. It is an obvious consequence from the proof of (6) in [5] that the bound can be readily improved as soon as the involved Schmidt-Summerer estimates from [29] concerning the quotient \( w_n(\zeta)/\hat{w}_n(\zeta) \) can be improved. The best possible estimates for this quotient one can expect were posed as a conjecture for general \( n \) and proved for \( n = 3 \) in [30], see also [16]. In fact this led to the stronger bound (7) compared to \( \hat{w}_3(\zeta) \leq 4.5615\ldots \) derived from (6). For \( n \geq 2 \), define \( w = w(n) \) as the unique solution of the polynomial identity

\[ \frac{(n-1)w}{w-n} - w + 1 = \left( \frac{n-1}{w-n} \right)^n \]

in the interval \([n, 2n-1)\). Then, upon the Schmidt-Summerer conjecture, the method in [5] yields

\[ \hat{w}_n(\zeta) \leq \max\{2n-2, w(n)\} \]

for any real \( \zeta \), and unless \( \zeta \) satisfies the relations \( w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta) \) in fact \( \hat{w}_n(\zeta) \leq w(n) \). For the related exponent \( \hat{w}_n^*(\zeta) \) concerning approximation by algebraic numbers of bounded degree, the according estimate \( \hat{w}_n^*(\zeta) \leq w(n) \) follows from the conjecture in [30] without further assumptions [25, Theorem 3.1]. For an exact definition of \( \hat{w}_n^*(\zeta) \) see for example [5]. It turns out that (10) is stronger than (8) as well for every \( n \geq 4 \). For \( n \in \{4, 5\} \), the conditional bounds are numerically given as

\[ \hat{w}_4(\zeta) \leq w(4) = 6.2874\ldots, \quad \hat{w}_5(\zeta) \leq w(5) = 8.2096\ldots. \]

Precisely for \( n \geq 10 \), we have \( w(n) < 2n - 2 \) and (10) becomes \( \hat{w}_n(\zeta) \leq 2n - 2 \). Using an equivalent different representation of \( w(n) \), the asymptotic formula

\[ w(n) = 2n - C + \epsilon_n, \quad C = 2.2564\ldots, \]

was carried out in [25, Theorem 3.1]. Here \( \epsilon_n > 0 \) decreases to 0 as \( n \to \infty \). Anyway, (11) is the best asymptotic upper bound one can hope to obtain for \( \hat{w}_n(\zeta) \) (or \( \hat{w}_n^*(\zeta) \)) solely with the method in [5].
2.2. New conditioned bounds. Our second main result of this paper Theorem 2.3 below yields an improvement on the upper bounds in (8) conditioned on some property of the structure of best approximation polynomials. We need some preparation before we can state it. First we recall best approximation polynomials.

**Definition 2.1.** For \( n \geq 1 \) and integer and \( \zeta \) a real number, an integer polynomial \( P \) of degree at most \( n \) will be called **best approximation polynomial associated to** \((n, \zeta)\) if it minimizes \( |P(\zeta)| \) among all non-identically zero integer polynomials of degree at most \( n \) and height at most \( H(P) \). Any pair \( n, \zeta \) thus gives rise to a uniquely determined sequence of best approximation polynomials we denote by \((P_k^{n, \zeta})_{k \geq 1}\).

The definition implies that
\[
H(P_1^{n, \zeta}) < H(P_2^{n, \zeta}) < \cdots, \quad |P_1^{n, \zeta}(\zeta)| > |P_2^{n, \zeta}(\zeta)| > \cdots,
\]
and there is no non-zero integer polynomial \( P \neq P_1^{n, \zeta} \) of degree at most \( n \) and height \( H(P) \leq H(P_1^{n, \zeta}) \) such that \( |P(\zeta)| < |P_1^{n, \zeta}(\zeta)| \). In Theorem 2.3 below we show that a putative large value of \( \hat{w}_n(\zeta) \) for given \( n \geq 1 \) and transcendental real \( \zeta \) implies a very biased structure of the best approximation polynomials \( P_k^{n, \zeta} \) associated to \((n, \zeta)\). Let us write
\[
P_k^{n, \zeta}(T) = h_{k,0} + h_{k,1}T + \cdots + h_{k,n}T^n, \quad k \geq 1, \quad h_{k,j} = h_{k,j}(n, \zeta) \in \mathbb{Z},
\]
and further let
\[
h_k^{n, \zeta} = (h_{k-1,0}, \ldots, h_{k-1,n}, h_{k,0}, \ldots, h_{k,n}, h_{k+1,0}, \ldots, h_{k+1,n}) \in \mathbb{Z}^{3n+3}, \quad k \geq 2,
\]
be the vector consisting of the coordinates of three consecutive best approximation polynomials glued together. Next for any even \( n \geq 2 \) we define a \((3\frac{n}{2} \times 3\frac{n}{2})\)-integer matrix \( \Lambda_n = \Lambda_n(\underline{x}) = \Lambda_n(x_1, \ldots, x_{3n+3}) \) with some circulant structure. For \( n = 2 \), when inserting \( \underline{x} = h_2^{2, \zeta} \) above, the columns of \( \Lambda_2(h_2^{2, \zeta}) \in \mathbb{Z}^{3 \times 3} \) consist precisely of the coordinate vectors of three consecutive best approximation polynomials \( P_{k-1}^{n, \zeta}(\zeta), P_k^{n, \zeta}(\zeta), P_{k+1}^{n, \zeta}(\zeta) \). For \( n = 4 \) it is given by
\[
\Lambda_4(h_4^{4, \zeta}) =  \\
\begin{pmatrix}
0 & h_{k,0} & 0 & h_{k+1,0} & 0 \\
0 & h_{k,1} & h_{k,0} & h_{k+1,1} & 0 \\
h_{k,2} & h_{k,1} & h_{k,0} & h_{k+1,2} & h_{k+1,1} \\
h_{k,3} & h_{k,2} & h_{k,1} & h_{k,0} & h_{k+1,3} \\
h_{k,4} & h_{k,3} & h_{k,2} & h_{k,1} & h_{k+1,4}
\end{pmatrix}, \quad k \geq 2.
\]
Similarly, for even \( n \geq 6 \) the matrix \( \Lambda_n(h_n^{n, \zeta}) \) arises by putting the vectors
\[
(h_{j,0}, h_{j,1}, \ldots, h_{j,n}, 0, \ldots, 0) \in \mathbb{Z}^{3n/2}, \quad j \in \{k-1, k, k+1\},
\]
corresponding to \( P_{k-1}^{n, \zeta}, P_k^{n, \zeta}, P_{k+1}^{n, \zeta} \) in the columns 1, \( \frac{n}{2} + 1 \) and \( n+1 \) respectively, and shifting each modulo \( 3n/2 \) from once up to \((\frac{n}{2} - 1)\)-times successively to obtain the remaining \( 3 \cdot (\frac{n}{2} - 1) \) columns. Define further
\[
\Phi_n(\underline{x}) = \det \Lambda_n(\underline{x}),
\]
which is a homogeneous polynomial in \( 3n + 3 \) variables of total degree \( 3n + 3 \).
Lemma 2.2. Let \( n \geq 2 \) be an even integer and \( \zeta \) be a transcendental real number. Then for any \( k \geq 2 \) the following claims are equivalent.

- We have
  \[
  \Phi_n\left( B_k^{n,\zeta} \right) \neq 0. 
  \]

- The identity
  \[
  A_k P_{k-1}^{n,\zeta} + B_k P_k^{n,\zeta} + C_k P_{k+1}^{n,\zeta} \equiv 0 
  \]
  in integer polynomials \( A_k, B_k, C_k \) each of degree at most \( \frac{n}{2} - 1 \) has only the trivial solution \( A_k \equiv B_k \equiv C_k \equiv 0 \).

- The space spanned by
  \[
  \bigcup_{0 \leq j \leq \frac{n}{2} - 1} \{ T_j P_{k-1}^{n,\zeta}, T_j P_k^{n,\zeta}, T_j P_{k+1}^{n,\zeta} \} 
  \]
  has full dimension \( 3 \frac{n}{2} \), i.e. the polynomials span the space of polynomials of degree at most \( \frac{3n}{2} - 1 \) isomorphic to \( \mathbb{R}^{3n/2} \) in a direct sum.

The proof relies on elementary linear algebra considerations and will be given in Section 4. If the equivalent conditions hold for some \( k \), it is not hard to conclude that the three polynomials \( P_{k-1}^{n,\zeta}, P_k^{n,\zeta}, P_{k+1}^{n,\zeta} \) have no common factor. Unfortunately, for any pair among them this is not clear. If that were true we could remove an inconvenient condition in Theorem 2.3 below, our main result of this section.

Theorem 2.3. Let \( n \) be an even integer and \( \zeta \) be a transcendental real number for which the equivalent conditions of Lemma 2.2 hold for infinitely many \( k \). For \( n \geq 4 \), we have

\[
\hat{w}_n(\zeta) \leq 2n - 2. 
\]

If \( n \geq 2 \) and additionally

\[
w_n(\zeta) > w_{n-1}(\zeta) 
\]

holds, then we have the significantly stronger estimate

\[
\hat{w}_n(\zeta) \leq \sigma_n := \frac{2n - 1 + \sqrt{2n^2 - 2n + 1}}{2}. 
\]

The value \( \sigma_n \) is just slightly smaller than \((1 + \frac{1}{\sqrt{2}}) \cdot n\), hence \( \text{(18)} \) is reasonably stronger than both the bound \( \text{(8)} \) of Theorem 1.1 and the conditioned bound in \( \text{(10)} \). In view of \( \text{(3)} \), we clearly obtain the same conditioned asymptotic estimate for odd \( n \) as well. For subtle technical reasons our proof of \( \text{(18)} \) requires the unpleasant condition \( \text{(17)} \), however we believe that it can be removed. Various alternative conditions instead of \( \text{(17)} \) can be stated that imply \( \text{(18)} \) as well, for example that \( P_k^{n,\zeta} \) and \( P_{k+1}^{n,\zeta} \) have no common factor for those indices \( k \) for which the conditions of Lemma 2.2 are satisfied. Observe that even \( \text{(16)} \) is stronger than the conditioned bounds \( \text{(10)} \) when \( n \in \{4,6,8\} \), in contrast to \( \text{(8)} \) which is weaker than \( \text{(10)} \) for any \( n \geq 4 \) as pointed out in Section 2.1. The first few values \( \sigma_n \) are numerically given as

\[
\sigma_2 = 2.6180\ldots, \quad \sigma_4 = 6, \quad \sigma_6 = 9.4051\ldots, \quad \sigma_8 = 12.8150\ldots.
\]
We discuss how strong the assumption of Theorem 2.3 is. If the condition fails, then \( \Phi_n \) has to vanish at the vectors \( \mathcal{L}_{i,m}^{n,\zeta} \) consisting of three consecutive best approximation polynomials, for every large \( k \geq k_0 \). Hence all these vectors \( \mathcal{L}_{i,m}^{n,\zeta} \), \( k \geq k_0 \), must belong to some fixed collection of lower dimension submanifolds of \( \mathbb{R}^{3n+3} \). It seems natural to expect this should not happen for any transcendental real \( \zeta \). Moreover, for \( n = 2 \) the condition can indeed be verified, and leads to another proof of [14]. For \( n = 2 \), we have \( 2n - 1 = 0 \), and thus [15] becomes that for any transcendental \( \zeta \) and infinitely many \( k \), three consecutive quadratic best approximation polynomials \( P_{k-1}^{2,\zeta}, P_k^{2,\zeta}, P_{k+1}^{2,\zeta} \) are linearly independent. Indeed, it has been known for a long time that this holds true more generally for any \( \zeta = (\zeta_1, \zeta_2) \) that is linearly independent with \( \{1\} \) over \( \mathbb{Q} \), and three successive best approximation linear forms \( L_{k-1}^{2,\zeta}, L_k^{2,\zeta}, L_{k+1}^{2,\zeta} \), defined similarly. More generally, for any \( \{\zeta_1, \ldots, \zeta_m\} \) with the according \( \mathbb{Q} \)-linear independence condition, there exist infinitely many \( k \) such that \( L_{k-1}^{m,\zeta}, L_k^{m,\zeta}, L_{k+1}^{m,\zeta} \) are linearly independent, as pointed out for example in the remark on page 77 in [28]. Equivalently, condition (14) of Theorem 2.3 cannot fail in the worst sense of constant polynomials \( A_k, B_k, C_k \) for all large \( k \). However, the analogous claim concerning four (or more) consecutive linear forms \( L_{k-1}^{m,\zeta}, L_k^{m,\zeta}, L_{k+1}^{m,\zeta}, L_{k+2}^{m,\zeta} \) is false for any \( m \geq 4 \), as shown by Moshchevitin [15]. However, it could potentially still be true when restricting to the Veronese curve \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \) we are concerned with.

2.3. Variations of Theorem 2.3. It is clear that the integer \( \frac{n}{2} - 1 \) within the conditions (14) and (15) is the smallest value for which they can possibly hold, hence (18) seems to be the end of the method. If we ask for restriction by some larger integer \( \frac{n}{2} - 1 < m < 2n - 1 \), we instead obtain some bound in between \( \theta_n \) and \( \sigma_n \) from (8) and (18) respectively, which increases as \( m \) grows.

Concretely, we will establish the generalization Theorem 2.8 of Theorem 2.3 below, based on generalizations of (15). We introduce some more notation.

**Definition 2.4.** Let \( n \geq 2 \) be an integer, \( \zeta \) be a transcendental real number and \( (P_i^{n,\zeta})_{i \geq 1} \) the sequence of best approximation polynomials associated to \( n, \zeta \). Say \( d_i \leq n \) denotes the degree of \( P_i^{n,\zeta} \). For any integer \( m \geq n \), let

\[
\mathcal{A}_{i,m}^{n,\zeta} := \bigcup_{0 \leq j \leq m-d_i} T^j \cdot P_i^{n,\zeta}(T), \quad i \geq 1,
\]

and for any \( k \geq 2 \), consider the unions of three consecutive polynomials

\[
\mathcal{B}_{k,m}^{n,\zeta} := \mathcal{A}_{k-1,m}^{n,\zeta} \cup \mathcal{A}_{k,m}^{n,\zeta} \cup \mathcal{A}_{k+1,m}^{n,\zeta}.
\]

We say the pair \( (n, \zeta) \) has the property \( \text{span}(m) \), if for infinitely many \( k \) the set \( \mathcal{B}_{k,m}^{n,\zeta} \) spans the \((m + 1)\)-dimensional space of polynomials of degree at most \( m \). Furthermore we say \( (n, \zeta) \) has the property \( \tilde{\text{span}}(m) \) if satisfies \( \text{span}(m) \) and additionally \( P_i^{n,\zeta} \) and \( P_k^{n,\zeta} \) within the definition can be chosen without common factor. Moreover, we denote by \( \Psi(n, \zeta) \) the smallest integer \( m \) with the property \( \text{span}(m) \), and \( \tilde{\Psi}(n, \zeta) \) the smallest integer \( m \) with the property \( \tilde{\text{span}}(m) \). Further we define \( \Psi(n) \) and \( \tilde{\Psi}(n, \zeta) \) to be the maximum of \( \Psi(n, \zeta) \) and \( \tilde{\Psi}(n, \zeta) \) respectively over transcendental \( \zeta \), that is

\[
\Psi(n) = \max_{\zeta \in \mathbb{R} \setminus \mathbb{Q}} \Psi(n, \zeta), \quad \tilde{\Psi}(n) = \max_{\zeta \in \mathbb{R} \setminus \mathbb{Q}} \tilde{\Psi}(n, \zeta).
\]
Remark 2.5. Clearly $\mathcal{B}^{n,\zeta}_{k,m}$ consists of polynomials of degree at most $m$, however it is a priori unclear if we can find sufficiently many linearly independent ones. The distinction between $\Psi$ and $\tilde{\Psi}$ is a subtle. Davenport and Schmidt showed that for any $n \geq 2$ and any transcendental real $\zeta$, two successive best approximation polynomials $P_{j-1,1}^n, P_j^n$ are coprime infinitely often [11 Theorem 2b], however given span$(m)$ we do not know whether this holds for those values $j = k$ with the property that $\mathcal{B}^{n,\zeta}_{k,m}$ spans the space of polynomials of degree at most $m$. Even for $n = 2$ this seems not obvious. The stronger property $\widehat{\text{span}}(m)$ guarantees that the space spanned by $\mathcal{A}^{n,\zeta}_{k-1,m} \cup \mathcal{A}^{n,\zeta}_{k,m} \setminus \mathcal{A}^{n,\zeta}_{k+1,m}$ has maximum possible dimension, so that only few multiples of $P_{k+1}^n$ need to be added to span the space of polynomials of degree $\leq m$.

Remark 2.6. Similar to Lemma 2.2 the condition $\Psi(n) = m_0 \geq \frac{3n}{2} - 1$ can be equivalently expressed by the vanishing of $2m_0 - 3n + 3 \geq 1$ sequences of $(m_0 + 1) \times (m_0 + 1)$-subdeterminants of $(m_0 + 1) \times 3(m_0 - n + 1)$-matrices whose entries are the coefficients of three successive best approximation vectors, arranged in certain circulant ways. Lemma 2.2 is recovered with $m_0 = \frac{3n}{2} - 1$ and a single sequence.

Note that if all $P_{k-1}^n, P_k^n, P_{k+1}^n$ have degree precisely $n$, then $\mathcal{B}^{n,\zeta}_{k,m}$ becomes

\begin{equation}
\mathcal{B}^{n,\zeta}_{k,m} = \bigcup_{0 \leq j \leq m-n} \left\{ T_j \cdot P_{k-1}^n(T), T_j \cdot P_k^n(T), T_j \cdot P_{k+1}^n(T) \right\}.
\end{equation}

This assumption holds in particular if $\zeta$ satisfies

\begin{equation}
\tilde{w}_n(\zeta) > w_{n-1}(\zeta),
\end{equation}

with irreducible polynomials $P_{k-1}^n, P_k^n, P_{k+1}^n$. For $n \geq 2$ and generic $\zeta$, we expect $\Psi(n, \zeta) = \tilde{\Psi}(n, \zeta) = \lceil \frac{3n}{2} - 1 \rceil$. The crucial question is by how much the quantities $\Psi(n, \zeta), \tilde{\Psi}(n, \zeta)$ can exceed this value for certain biased $\zeta$. Some rather easy estimates for $\Psi(n)$ and $\tilde{\Psi}(n)$ are summarized below.

Lemma 2.7. If $(n, \zeta)$ has the property span$(m_0)$ and $\widehat{\text{span}}(m_0)$ respectively for some $m_0$, then it has the respective property for all $m \geq m_0$. The quantities $\Psi(n, \zeta), \tilde{\Psi}(n, \zeta), \Psi(n), \tilde{\Psi}(n)$ are all well-defined and for any $n \geq 2$ we have

\begin{equation}
\Psi(n, \zeta) \leq \tilde{\Psi}(n, \zeta) \leq 2n - 1, \quad \frac{3n}{2} - 1 \leq \Psi(n) \leq \tilde{\Psi}(n) \leq 2n - 1.
\end{equation}

Moreover, we have $\Phi(2) = 2$. Furthermore, if (22) is satisfied, then $\Psi(n, \zeta) = \tilde{\Psi}(n, \zeta)$.

The proof of Lemma 2.7 will be carried out in Section 4. The problem to determine, or at least find good upper bounds, for $\Psi(n)$ and $\tilde{\Psi}(n)$, is at the very core of our strategy to find upper bounds for $\tilde{w}_n(\zeta)$. Our result reads as follows.

Theorem 2.8. Let $n \geq 2$ be an integer and $\zeta$ be a transcendental real number and assume $\tilde{\Psi}$ holds. Then for $\Psi = \tilde{\Psi}(n, \zeta)$ we have

\begin{equation}
\tilde{w}_n(\zeta) \leq D_n(\Psi) := \frac{2\tilde{\Psi} - n + 1 + \sqrt{4\tilde{\Psi}^2 + 17n^2 - 16\tilde{\Psi}n + 8\tilde{\Psi} - 18n + 5}}{2}.
\end{equation}
Similarly, for \( \Psi = \Psi(n, \zeta) \) we have
\[
\hat{w}_n(\zeta) \leq \mathcal{E}_n(\Psi) := \frac{\Psi + 1 - \sqrt{\Psi^2 - 4\Psi n + 8n^2 + 2\Psi - 12n + 5}}{2}.
\]

As for Theorem 2.3, there are again several alternative conditions to (17) with the same implications (21) and (25), and we believe none of them is in fact required. We point out that upon the condition (22) we have the stronger bound (24) anyway, in view of Lemma 2.7. Both \( D_n(t), \mathcal{E}_n(t) \) increase as functions of \( t \) and \( D_n(t) \leq \mathcal{E}_n(t) \) for \( t \geq 3\frac{n}{2} - 1 \). For \( n = 2 \) from (25) and \( \Phi(2) = 2 \) we get yet another proof of (5). The identities \( D_n(2n-1) = 2n-1 \) and \( \mathcal{E}_n(2n-1) = 2n-1 \) both confirm \( \hat{w}_n(\zeta) \leq 2n-1 \) from [7] again. When \( n \) is even and the generic case \( \tilde{\Psi} = \tilde{\Psi}(n, \zeta) = 3\frac{n}{2} - 1 \) occurs, we have \( D_n(\Psi) = \sigma_n \) by construction. For \( \tilde{\Psi}(n, \zeta) \sim \beta n \) with fixed \( \beta \geq 3/2 \), we obtain
\[
D_n(\tilde{\Psi}) = (\gamma + o(1))n, \quad \gamma = \beta - \frac{1}{2} + \frac{\sqrt{4\beta^2 - 16\beta + 17}}{2}.
\]
from (21), as \( n \to \infty \). Similarly, for \( \Psi(n, \zeta) \sim \beta n \) with fixed \( \beta \geq 3/2 \) we obtain
\[
\mathcal{E}_n(\Psi) = (\chi + o(1))n, \quad \chi = \frac{\beta + \sqrt{\beta^2 - 4\beta + 8}}{2}.
\]
The choice \( \beta = \frac{3}{2} \) yields \( \gamma = 1 + \frac{1}{\sqrt{2}} = 1.7071 \ldots \), the asymptotic order in Theorem 2.3, and \( \chi = 1.7808 \ldots \). Note \( \gamma < \chi < 2 \) as soon as \( \beta < 2 \). We also treat the case \( \tilde{\Psi}(n, \zeta) = 2n-d \) and \( \Psi(n, \zeta) = 2n-d \) with fixed \( d > 0 \), as \( n \to \infty \). For \( d = 2 \), that is \( \tilde{\Psi}(n, \zeta) = 2n-2 \), the estimate (24) naturally yields the bound \( \theta_n \) from (8). In general the bounds are of order
\[
D_n(2n-d) = 2n-d + o(1), \quad \mathcal{E}_n(2n-d) = 2n - \frac{d+1}{2} + o(1),
\]
as \( n \to \infty \), with positive error terms. Finally we compare Theorem 2.8 with the conditioned bounds (10) and (11) from Section 2.1 where it is useful to keep the notation \( \Psi(n, \zeta) = 2n-d \) and \( \Psi(n, \zeta) = 2n-d \) with fixed \( d > 0 \). It turns out that (24) is stronger as soon as \( d \geq 3 \), whereas (25) is stronger for \( d \geq 4 \), upon \( m = 2n-d \geq 3\frac{n}{2} - 1 \) or equivalently \( n \geq 2d-2 \).

3. Preliminaries

3.1. Successive minima and parametric geometry of numbers. We introduce the concept of parametric geometry of numbers following Schmidt and Summerer [28], where we slightly deviate in the notation and only treat the linear form case relevant to us. We prefer to state the results of this section for polynomials of degree at most \( m \) instead of \( n \) to avoid confusion later. Indeed in the proofs of Theorems 1.1 2.3 2.8, although their claims concern approximation in dimension \( n \), we will transition to an approximation problem in certain dimensions \( m > n \), corresponding to the present \( m \). For given \( m \geq 1 \), a real number \( \zeta \) and every \( 1 \leq j \leq m+1 \), define the functions \( \psi_{m,j}(Q) \) parametrized by \( Q \) as the maximum value \( \eta \) such that
\[
H(P) \leq Q^{\frac{j}{m+1}}, \quad |P(\zeta)| \leq Q^{-1 - \eta}
\]
has $j$ linearly independent solutions in integer polynomials $P$ of degree at most $m$. Define further
\[
\psi_{m,j}^* = \liminf_{Q \to \infty} \psi_{m,j}^*(Q), \quad \psi_{m,j} = \limsup_{Q \to \infty} \psi_{m,j}^*(Q).
\]
The system (27) can be equivalently formulated as a problem of determining the successive minima of a convex body (parametrized by $Q$) with respect to a fixed lattice, both in $\mathbb{R}^{m+1}$, see [28] for details. For convenience the derived functions
\[
L_{m,j}^*(q) := q\psi_{m,j}^*(q), \quad q = \log Q,
\]
were introduced in [28].

**Definition 3.1.** For given $m \geq 1$ and $\zeta$ a real number, we call the image of the functions
\[
(L_{m,1}^*(q), L_{m,2}^*(q), \ldots, L_{m,m+1}^*(q)),
\]
related to a polynomial $P \in \mathbb{Z}[T]$ of degree at most $m$. Hence the functions $L_{m,j}^*(q)$ are piecewise linear and have slope among $\{-1/m, 1\}$. Conversely, any best approximation polynomial $P_k$ associated to $m, \zeta$ as in Definition 2.1, induces the function $L_{m,1}^*(q)$ in some non-empty interval $I_k$, that is
\[
L_{P_k}^*(q) = \max \left\{ \log H(P) - \frac{q}{m}, \log |P(\zeta)| + q \right\} = L_{m,1}^*(q), \quad k \geq 1, \quad q \in I_k.
\]

Minkowski’s second Convex Body Theorem [14] implies the estimation
\[
\left| \sum_{j=1}^{m+1} L_{m,j}^*(q) \right| \leq C(m)
\]
for an absolute constant $C(m) > 0$, uniformly in the parameter $q$. As a consequence
\[
L_{m,m+1}^*(q) \geq \sum_{j=1}^{m} L_{m,j}^*(q) - C(k) \geq -mL_{m,m}^*(q) - C(m).
\]
More generally this argument yields
\[
L_{m,m+1}^*(q) \geq \left( -\frac{j}{m+1-j} \right) \cdot L_{m,j}^*(q) - O(1), \quad 1 \leq j \leq m,
\]
see also the introduction in [29]. It will be convenient to use the following parametric estimates connecting classical exponents with $L_{m,j}^*$.

**Lemma 3.2.** Let $m \geq 1$ be an integer, $\zeta$ a real number and $j \in \{1, 2, \ldots, m+1\}$. For linearly independent integer polynomials $P_1, \ldots, P_j$ of degree at most $m$, define $H, w$ by
\[
\max_{1 \leq i \leq j} H(P_i) = H, \quad \max_{1 \leq i \leq j} |P_i(\zeta)| = H^{-w}.
\]
Without loss of generality assume a labeling such that $|P_i(\zeta)| = H^{-w}$. Then for $q = \tilde{q} := \frac{m}{m+1}(\log H - w)$ the solution to $\log H - q/m = w + q$ we have

$$L_{m,j}(\tilde{q}) \leq L_{P_i}(\tilde{q}) = \tilde{q} \left( \frac{m+1}{m} \cdot \frac{1}{1+w} - \frac{1}{m} \right) = \tilde{q} \cdot \frac{m-w}{m(1+w)}.$$

**Proof.** Define the auxiliary function

$$\beta(q) := \max\{\log H - \frac{q}{m}, w + q\}, \quad q > 0.$$

We first claim that all graphs of the functions $L_{P_i}(q)$ lie below the graph of $\beta$, that is $\beta(q) \geq L_{P_i}(q)$ for $1 \leq i \leq j$ and $q > 0$. By $H(P_i) \leq H$ we have the inequality for $q = 0$. Then $\beta$ and all $L_{P_i}(q)$ decay with slope $-1/m$, so the claim is true for any $L_{P_i}$ and $q \geq q(i)$ where $q(i)$ is the point where $L_{P_i}$ starts to rise with slope 1, that is where equality of the expressions in (29) holds for $P = P_i$. By the property $|P_i(\zeta)| \leq H^{-w}$ we know that the rising phase $-\log |P_i(\zeta)| + q$ lies entirely below the one of $\beta$ which is $w + q$, with equality for $i = 1$ and $q$ large enough that $\beta$ and $L_{P_i}$ both rise. So the claim is true for $q \geq q_0 := \max q(i)$. If we had $\beta(q_1) < L_{P_i}(q_1)$ for some $i$ and some $q_1 > 0$, then by the above findings at $q_1$ the function $L_{P_i}$ must already rise with slope 1. However, since $\beta$ has slope at most 1, we would have the inequality $\beta(q) < L_{P_i}(q)$ for all $q \geq q_1$. This contradicts our second finding and proves the claim. It is easily verified that $\tilde{q}$ is the point where $\beta$ changes slope from $-1/m$ to 1, and that $\beta(\tilde{q})$ equals the right hand side in (33).

It is further readily verified that $L_{P_i}^*$ rises at $\tilde{q}$, and we infer the identities in (33) from the identity statement above. Finally, since the $P_i, 1 \leq i \leq j$, are linearly independent by assumption, we conclude $L_{m,j}(\tilde{q}) \leq \max_{1 \leq i \leq j} L_{P_i}(\tilde{q}) = L_{P_1}(\tilde{q}) = \beta(\tilde{q})$. The proof of (33) is thus complete. \hfill $\square$

We remark that Lemma 3.2 and its proof are closely connected to the identities

$$(w_m(\zeta) + 1) \left( \frac{1}{m} + \bar{\psi}_{m,1} \right) = (\tilde{w}_m(\zeta) + 1) \left( \frac{1}{m} + \bar{\psi}_{m,1} \right) = \frac{m+1}{m},$$

established in [28 Theorem 1.4]. A generalization for higher successive minima as in Lemma 3.2 was pointed out in [22].

3.2. Certain estimates for polynomials and Diophantine exponents. To shorten the proofs of the main results later, we provide some rather elementary facts on the relation between best approximation polynomials from Definition 2.1 and the exponents $w_n(\zeta), \tilde{w}_n(\zeta)$. We will be concerned with irreducibility of best approximation polynomials. First we recall that as pointed out in [31], for any given $n, \zeta$ and any $w < w_n(\zeta)$ there exist infinitely many irreducible integer polynomials of degree at most $n$ for which (11) holds. This follows from a general estimate on the height of products of polynomials, sometimes referred to as Gelfond’s Lemma. It states that for any positive integer $n$ there exists an absolute constant $K(n)$ such that

$$K(n)^{-1} H(P) H(Q) \leq H(PQ) \leq K(n) H(P) H(Q)$$

holds for all polynomials $P, Q \in \mathbb{Z}[T]$ of degree at most $n$. This estimate and the corollary above on irreducibility were also used by Wirsing in [31]. The same kind of argument,
already used in \cite{5}, yields that in case of \( w_n(\zeta) > w_{n-1}(\zeta) \), infinitely many among the best approximation polynomials associated to \((n, \zeta)\) are irreducible of degree precisely \( n \). Similarly, the stronger claim \( \hat{w}_n(\zeta) > w_{n-1}(\zeta) \) implies the same property for all best approximation polynomials of sufficiently large index/height, as already observed in the proof of \cite{23} Lemma 3.2]. For simplicity we put \( P_k = P_k^{n, \hat{w}_n} \) for the next result.

**Proposition 3.3.** Let \( n \geq 1 \) be an integer, \( \zeta \) be a transcendental real number and \( (P_k)_{k \geq 1} \) the sequence of best approximation polynomials associated to \((n, \zeta)\). For any \( \epsilon > 0 \) and all sufficiently large \( k \), upon denoting \( X_{k+1} = H(P_{k+1}) \) we have

\[
H(P_k) < H(P_{k+1}) = X_{k+1}, \quad |P_{k+1}(\zeta)| < |P_k(\zeta)| \leq X_{k+1}^{-\hat{w}_n(\zeta) + \epsilon}.
\]

**Proof.** We only have to show \( |P_k(\zeta)| \leq X_{k+1}^{-\hat{w}_n(\zeta) + \epsilon} \). Suppose otherwise \( |P_k(\zeta)| > X_{k+1}^{-\hat{w}_n(\zeta) + \epsilon} \). Since \( P_{k+1} \) is the best approximation polynomial succeeding \( P_k \), for any non-zero \( Q \) of degree at most \( n \) and height strictly less than \( X_{k+1} \) we have \( |Q(\zeta)| \geq |P_{k+1}(\zeta)| > X_{k+1}^{-\hat{w}_n(\zeta) + \epsilon} \). This obviously contradicts the definition of \( \hat{w}_n(\zeta) \) for large \( k \).

Proposition 3.3 tells us that if \( \hat{w}_n(\zeta) \) is large, infinitely often there are two linearly independent best approximation polynomials with small evaluation at \( \zeta \). The claim is closely related the more general to \cite{28} Theorem 1.1] and the derived inequalities \( \hat{\psi}_{n,j} \leq \hat{\psi}_{n,j+1} \) for \( 1 \leq j \leq n \) for \( \mathbb{Q} \)-linearly independent real vectors \((1, \zeta_1, \ldots, \zeta_n)\), taking \( j = 1 \).

**Lemma 3.4.** Let \( n \geq 1 \) be an integer, \( \zeta \) be a transcendental real number and \( (P_k)_{k \geq 1} \) be the sequence of best approximation polynomials associated to \((n, \zeta)\). Let \( \epsilon > 0 \). Then for all large \( k \geq k_0(\epsilon) \) we have

\[
H(P_{k+1}) \leq H(P_k)^{\frac{w_n(\zeta)}{\hat{w}_n(\zeta)} + \epsilon}.
\]

**Proof.** By definition of \( w_n(\zeta) \) for any \( \epsilon > 0 \) and large \( k \geq k_0(\epsilon) \) we have

\[
|P_k(\zeta)| \geq H(P_k)^{-w_n(\zeta) - \epsilon}.
\]

Let

\[
X_k = H(P_k)^{(w_n(\zeta) + \epsilon)/\hat{w}_n(\zeta) - \epsilon}.
\]

Then

\[
|P_k(\zeta)| \geq X_k^{-\hat{w}_n(\zeta) + \epsilon}.
\]

On the other hand, by definition of \( \hat{w}_n(\zeta) \) for large \( k \) there has to be polynomial \( Q_k \) of height less then \( X_k \) for which \( |Q_k(\zeta)| < X_k^{-\hat{w}_n(\zeta) + \epsilon} \leq |P_k(\zeta)| \). By definition of the best approximation polynomials we see that \( Q_k = P_{k+1} \) is a suitable choice. Hence

\[
H(P_{k+1}) \leq X_k = H(P_k)^{(w_n(\zeta) + \epsilon)/\hat{w}_n(\zeta) - \epsilon}.
\]

The claim follows since \( \epsilon \) can be chosen arbitrarily small, for \( \epsilon \) some modification of \( \epsilon \).

The next result will be applied to best approximation polynomials as well in order to simplify the proof of our main results, and establish the upper bounds in \cite{23} of Lemma 2.7.
Lemma 3.5. Let \( P(T), Q(T) \) be two integer polynomials of degrees \( a, b \) respectively, without common factor. Then the set of polynomials
\[
\Omega := \{ P, TP, \ldots, T^{b-1}P, Q, TQ, \ldots, T^{a-1}Q \},
\]
is linearly independent, and thus spans the space of polynomials of degree at most \( a+b-1 \) in a direct sum.

Proof. Assume the claim is false and \( \mathcal{P} \) is linearly dependent. Then by the structure of \( \mathcal{P} \), there exist polynomials \( A, B \) not both identically zero and of degree at most \( n-1 < n \) such that \( AP \equiv BQ \). However, since \( P \) and \( Q \) have no common factor and \( B \) has degree less than \( P \), we cannot have such an identity by the unique factorization in \( \mathbb{Z}[T] \).

Essentially the same claim was already implicitly used within the proof of [24, Theorem 2.1], where an equivalent proof using the resultant was given. For the immediate concern of Theorem 1.1 we state a direct corollary with \( a = b = n \).

Corollary 3.6. Let \( P, Q \) be two integer polynomials both of degree precisely \( n \) and without common factor. Then the set of polynomials
\[
\mathcal{P} := \bigcup_{0 \leq j \leq n-2} \{ T^jP, T^jQ \},
\]
is linearly independent and thus spans a \( 2n-2 \) dimensional hyperspace of the vector space of polynomials of degree at most \( 2n-2 \).

Finally we recall two facts from [5]. A special case of [5, Theorem 2.2] shows that the condition (17), that is \( w_n(\zeta) > w_{n-1}(\zeta) \), implies
\[
(37) \quad \hat{w}_n(\zeta) \leq n + (n-1) \frac{\hat{w}_{n-1}(\zeta)}{w_n(\zeta)}.
\]
It is hard to tell if (37) holds without the imposed condition. See [27] for estimates in the general case. Moreover we will need the estimate
\[
(38) \quad \min\{w_{n_1}(\zeta), \hat{w}_{n_2}(\zeta)\} \leq n_1 + n_2 - 1,
\]
valid for any positive integers \( n_1, n_2 \) and any transcendental real number \( \zeta \), as shown in [5, Theorem 2.3]. Notice the choice \( n_1 = n_2 = n \) recovers \( \hat{w}_n(\zeta) \leq 2n - 1 \) due to Davenport and Schmidt.

4. Proofs

We start with the proofs of the lemmata in Section 2.2.

Proof of Lemma 2.2. Write the polynomials \( A_k, B_k, C_k \) in coordinates as well, such that glued together they form vectors
\[
v_k = (a_k, 0, \ldots, a_{k,n/2-1}, b_k, 0, \ldots, b_{k,n/2-1}, c_k, 0, \ldots, c_{k,n/2-1}) \in \mathbb{Z}^{3n/2}.
\]
Multiplying out the product in (14) using (12) and assuming all coefficients to vanish yields a system of $3^k$ linear equations in $3^k$ variables. It can be checked that this system corresponds to $\Lambda_n(\hat{b}_k^\infty) \cdot \underline{v}^T = \underline{0}$, with the matrix $\Lambda_n(\hat{b}_k^\infty)$ from Section 2.2 and where $\underline{v}^T$ denotes the transpose of $\underline{v}$. From linear algebra this system has a non-trivial solution $\underline{v}$ if and only if the corresponding matrix $\Lambda_n(\hat{b}_k^\infty)$ is singular, or equivalently $\Phi_n(\hat{b}_k^\infty) = 0$. Hence the first two conditions are equivalent. Since any polynomial $P_{n+1} A_k, P_k^\infty C_k$ obviously lies in the span of the polynomials in (15), and conversely the latter form any possible linear combination as in (14), the last two assertions are equivalent as well.

\[ \Psi = \text{possible since the set of lemma. We next prove the lower bound in (23). Choose } \zeta \text{ polynomial has higher degree than any polynomial in the old span. We see that } \Omega \text{ is linearly independent by Lemma 3.5. Observe every element in } \Omega \text{ has degree at most } d_k + d_{k-1} - 1. \] 

Now when we add the polynomials in $\mathcal{Q}_k := \{ T^{d_k} P_{k-1}, \ldots, T^{m-1} P_{k-1} \}$ one by one to $\Omega_k$, the dimension of the span must increase in every step since the new polynomial has higher degree than any polynomial in the old span. We see that $\Omega_k \cup \mathcal{Q}_k$ consists of linearly independent polynomials of degree at most $m$ and has cardinality $d_k + d_{k-1} + m - (d_k + d_{k-1} - 1) = m + 1$. Hence indeed $\Psi(n) \leq 2n - 1$ and the quantities $\hat{\Psi}, \Psi$ are well-defined. A very similar inductive argument ensures the first claim of the lemma. We next prove the lower bound in (23). Choose $\zeta$ that satisfies (24). This is possible since the set of $\zeta$ with $w_{n-1}(\zeta) \geq n$ has Hausdorff dimension $n/(n+1) < 1$ by Baker and Schmidt [1] and Bernik [2], whereas $\hat{w}_n(\zeta) \geq n$, i.e. (2), holds for any transcendental real $\zeta$. The condition (22) ensures $d_{k-1} = d_k = d_{k+1} = n$ for $d_j$ the degree of $P_j^\infty$. The estimate $\Psi(n, \zeta) \geq 3 \frac{n}{2} - 1$ for such $\zeta$ follows as we have only $3(m-n+1) < m+1$ polynomials in the union in (21) if $m < 3 \frac{n}{2} - 1$. Thus in particular $\Psi(n) \geq 3 \frac{n}{2} - 1$. The last claim follows similarly from $d_{k-1} = d_k = d_{k+1} = n$ upon (22), and the linear independence of $\Omega_k$.

We remark that the quoted result [7, Theorem 2b] was a crucial observation to infer the bound $2n-1$ for $\hat{w}_n(\zeta)$ in that paper. Now we turn towards the proofs of the main results Theorems 1.1, 2.3, 2.8. The key idea of all proofs is to blow up the dimension of the problem from $n$ to some $m$ and observe that the assumption of large $\hat{w}_n(\zeta)$ conflicts with Minkowski’s Second Convex Body Theorem in this modified approximation problem, related to the combined Schmidt-Summerer graph in dimension $m$ from Definition 3.1. This contradiction will essentially yield the respective upper bounds. For Theorem 1.1 concretely we choose $m = 2n-2$. The method already requires some subtle application of
results from Section 3.2 to work properly. The technical problems increase as \( m < 2n - 2 \) decreases, thus leading only to conditioned bounds so far, see below.

**Proof of Theorem 1.1.** We proof the claim indirectly. Assume the claim of the theorem is false, that is there exists an integer \( n \) and transcendental real \( \zeta \) such that \( \hat{w}_n(\zeta) > \theta_n \). Since \( \theta_n > 2n - 2 \) we conclude \( \hat{w}_n(\zeta) > 2n - 2 \). Hence application of the relation (38) with \( n_1 = n, n_2 = n - 1 \) yields

\[
(40) \quad w_{n-1}(\zeta) \leq 2n - 2 < \theta_n < \hat{w}_n(\zeta).
\]

For simplicity write \( P_k = P_k^{n,\zeta} \) for \( (P_k^{n,\zeta})_{k \geq 1} \) the sequence of best approximation polynomials associated to \( (n, \zeta) \) as in Definition 2.1. As carried out in Section 3.2 Wirsing’s estimate (35) and (40) imply that for any large \( k \) the best approximation polynomial \( P_k \) is irreducible of degree precisely \( n \). Now consider the polynomial approximation problem for polynomials of degree at most \( 2n - 2 \) in the variable \( \zeta \), related to the combined Schmidt-Summerer graph associated to \( (2n - 2, \zeta) \). For any \( k \geq 1 \), let

\[
V_{k,j}(T) = T^j P_k(T), \quad 0 \leq j \leq n - 2,
\]

and further define the sets

\[
\mathcal{V}_k = \bigcup_{j=0}^{n-2} V_{k,j}, \quad \mathcal{P}_k = \mathcal{V}_{k-1} \cup \mathcal{V}_k.
\]

We see that \( \mathcal{P}_k \) consists of \( 2n - 2 \) polynomials with integer coefficients and degree at most \( 2n - 2 \). Since \( P_{k-1}, P_k \) are distinct and irreducible of degree precisely \( n \), we may apply Corollary 3.6 with \( P = P_{k-1}, Q = P_k \) and \( \mathcal{P} = \mathcal{P}_k \) and see that for any sufficiently large \( k \), the set \( \mathcal{P}_k \) is linearly independent and forms a basis of a \( 2n - 2 \) dimensional hyperspace of the space of polynomials of degree at most \( 2n - 2 \). Additionally consider the subsequent best approximation polynomial \( P_{k+1} \) and the set

\[
\mathcal{R}_k = \mathcal{P}_k \cup \mathcal{V}_{k+1} = \mathcal{V}_{k-1} \cup \mathcal{V}_k \cup \mathcal{V}_{k+1}.
\]

We identify a polynomial with its coefficient vector in \( \mathbb{Z}^{2n-1} \) in the sequel. We distinguish two cases.

Case 1: There exist arbitrarily large \( k \) such that \( \mathcal{R}_k \) spans the space of polynomials of degree at most \( 2n - 2 \). In other words for infinitely many \( k \) there exists some polynomial \( R_{k+1} \) in \( \mathcal{V}_{k+1} \) not included in the span of \( \mathcal{P}_k \). Let \( \epsilon \in (0, \theta_n - (2n - 2)) \) be arbitrary but fixed. Observe that

\[
(41) \quad H(P_{k-1}) < H(P_k) = H(V_{k,0}) = H(V_{k,1}) = \cdots = H(V_{k,n-2}), \quad k \geq 1.
\]

Moreover without loss of generality we may assume that \( \zeta \in (0, 1) \) and hence

\[
(42) \quad |P_k(\zeta)| = |V_{k,0}(\zeta)| = \max_{0 \leq j \leq n-2} |V_{k,j}(\zeta)|, \quad k \geq 1.
\]

For large \( k \), by (36) from Proposition 3.3 and (42), we have

\[
(43) \quad |P(\zeta)| \leq H(P_k)^{-\hat{w}_n(\zeta)} + \epsilon, \quad P \in \mathcal{P}_k.
\]
We consider the combined Schmidt-Summerer graph associated to \((m, \zeta)\) with \(m = 2n - 2\), as defined in Section 3.1. Keep in mind that (41) and (42) with (29) imply
\[
\max_{P \in \mathcal{P}_k} L^*_P(q) = L^*_{P_k}(q), \quad k \geq 1, \ q > 0.
\]

First look at the points \((q_k, L^*_{P_k}(q_k))\) where the graphs of \(L^*_{P_{k-1}}\) and \(L^*_{P_k}\) intersect, that is \(q_k\) is defined via
\[
L^*_{P_{k-1}}(q_k) = L^*_{P_k}(q_k).
\]

Since obviously at such points \(L^*_{P_{k-1}}\) rises and \(L^*_{P_k}\) decays, by (29) the value \(q_k\) is implicitly defined by
\[
\log H(P_k) - \frac{q_k}{2n - 2} = \log |P_{k-1}(\zeta)| + q_k,
\]
however, we will not need this directly. In view of (44) we know that
\[
L^*_P(q_k) \leq L^*_{P_{k-1}}(q_k) = L^*_{P_k}(q_k), \quad P \in \mathcal{P}_k,
\]
on the other hand the differences \(|L^*_P(q_k) - L^*_Q(q_k)|\) among \(P, Q \in \mathcal{P}_k\) is uniformly bounded for \(k \geq 1\).

From (45) and (46) and since the span of \(\mathcal{P}_k\) has full dimension \(2n - 2\), we infer \(L^*_{2n-2,2n-2}(q_k) \leq L^*_{P_k}(q_k)\). In fact, in view of (41) and (43), we may apply Lemma 3.2 to \(j = 2n - 2\), the polynomials in \(\mathcal{P}_k\) and a parameter \(w \geq \hat{w}_n(\zeta) - \epsilon\), with \(q = q_k\). From its claim (38) we obtain
\[
L^*_{2n-2,2n-2}(q_k) \leq L^*_{P_k}(q_k) \leq q_k \cdot \frac{2n - 2 - \hat{w}_n(\zeta)}{(2n - 2)(1 + \hat{w}_n(\zeta))} + \tilde{\epsilon}q_k
\]
where \(\tilde{\epsilon}\) is some variation of \(\epsilon\) which tends to 0 as \(\epsilon\) does. Observe the expression will be negative when \(\epsilon\) (or \(\tilde{\epsilon}\)) is small enough since \(\hat{w}_n(\zeta) > \theta_n > 2n - 2\) by assumption. Hence, with (31), for the last successive minimum function we conclude
\[
L^*_{2n-2,2n-1}(q_k) \geq (2 - 2n)q_k \cdot \frac{2n - 2 - \hat{w}_n(\zeta)}{(2n - 2)(1 + \hat{w}_n(\zeta))} - (2n - 2)\tilde{\epsilon}q_k + O(1).
\]

We now want to derive an upper bound for \(L^*_{2n-2,2n-1}(q_k)\) which is smaller under our assumption \(\hat{w}_n(\zeta) > \theta_n\), which will lead to the desired contradiction. By assumption of case 1 for infinitely many \(k\) there exists \(R_{k+1} \in \mathcal{V}_{k+1} = \mathcal{R}_k \setminus \mathcal{P}_k\) which does not lie in the span of \(\mathcal{P}_k\). Obviously \(H(R_{k+1}) = H(P_{k+1})\) by construction. To shorten the notation let \(H_l = H(P_l)\) for any integer \(l \geq 1\). Then Lemma 3.4 implies \(H_{k+1} \leq H_k^{w_n(\zeta)/\hat{w}_n(\zeta)+\epsilon}\). On the other hand, inequality (37), which can be applied since its condition \(w_n(\zeta) > w_{n-1}(\zeta)\) is satisfied by virtue of (40) and (2), can be reformulated to \(w_n(\zeta)/\hat{w}_n(\zeta) \leq (n-1)(\hat{w}_n(\zeta) - n)^{-1}\). Thus for large \(k\) we deduce
\[
H_{k+1} \leq H_k^{\nu+\epsilon}, \quad \nu = \frac{n - 1}{\hat{w}_n(\zeta) - n},
\]
Since \((q_k, L^*_{P_k}(q_k))\) lies in the graph of \(L^*_{P_k}\) which decays with slope \(-1/(2n - 2)\) in a neighborhood \(\hat{U}_k\) of \(q_k\) (since at \(q_k\) it meets \(L^*_{P_{k-1}}\) by (45) and \(H_k > H_{k-1}\)), we have
\[
\log H_k - \frac{q_k}{2n - 2} = L^*_{P_k}(q_k).
\]
Together with (47) we derive
\[
\log H_k \leq \frac{q_k}{2n - 2} + q_k \cdot \frac{2n - 2 - \tilde{w}_n(\zeta)}{(2n - 2)(1 + \tilde{w}_n(\zeta))} + \varepsilon q_k.
\]

It is not hard to see that the function \(L^*_{R_{k+1}}\) decays at \(q_k\) as well. We carry this out. We have \(L^*_{R_{k+1}}(q_k) > 0\) since otherwise we get a contradiction to (30) as all \(L^*_{2n-2,j}(q_k) < 0\) for \(1 \leq j \leq 2n - 1\) and the first \(2(m - n + 1)\) are negative by some fixed multiple of \(q_k\) (by assumption \(\{R_{k+1}\} \cup \mathcal{P}_k\) spans \(\mathbb{R}^{2n-1}\) and for \(P \in \mathcal{P}_k\) we have shown \(L^*_{P}(q_k) < -cq_k\) for some \(c > 0\)). On the other hand \(R_{k+1}\) induces an approximation of quality \(-\log |R_{k+1}(\zeta)| / \log H(R_k) > 2n - 2\), and by equating the two expressions in (29) in our present dimension \(m = 2n - 2\), this leads to \(L^*_{R_{k+1}}(r_{k+1}) < 0\) at the local minimum \(r_{k+1}\) of the function \(L^*_{R_{k+1}}\). Thus indeed we deduce that \(r_{k+1} > q_k\) and \(L^*_{R_{k+1}}\) decays at \(q_k\). Hence we also have
\[
\log H_{k+1} - \frac{q_k}{2n - 2} = L^*_{R_{k+1}}(q_k).
\]
Clearly \(L^*_{R_{k+1}}(q_k)\) is the maximum among \(L^*_{S}(q_k)\) for \(S \in \{R_{k+1}\} \cup \mathcal{P}_k\) since it is the only positive value. With (49) and since \(\{R_{k+1}, \mathcal{P}_k\}\) span \(\mathbb{R}^{2n-1}\) we infer
\[
L^*_{2n-2,2n-1}(q_k) \leq L^*_{R_{k+1}}(q_k) \leq (\nu + \varepsilon) \log H_k - \frac{q_k}{2n - 2},
\]
and further combination with (50) yields
\[
L^*_{2n-2,2n-1}(q_k) \leq (\nu + \varepsilon) \cdot (\tau_k + \varepsilon q_k) - \frac{q_k}{2n - 2},
\]
where
\[
\tau_k = \frac{q_k}{2n - 2} + q_k \cdot \frac{2n - 2 - \tilde{w}_n(\zeta)}{(2n - 2)(1 + \tilde{w}_n(\zeta))} = q_k \cdot \frac{2n - 1}{(2n - 2)(1 + \tilde{w}_n(\zeta))}.
\]
The bounds in the right hand sides of (48) and (51) depend on \(q_k\) and \(\tilde{w}_n(\zeta)\) only. Comparison of these two estimates and some rearrangements yield the estimate
\[
(2n - 1) \cdot (\tilde{w}_n(\zeta)^2 - 3(n - 1)\tilde{w}_n(\zeta) + 2n^2 - 4n + 1) + \varepsilon' \Phi(\tilde{w}_n(\zeta)) \leq 0,
\]
where \(\varepsilon'\) is again some variation of \(\varepsilon\) which tends to 0 as \(\varepsilon\) does and \(\Phi(\tilde{w}_n(\zeta))\) is some bounded expression when \(\tilde{w}_n(\zeta)\) is bounded. In the calculation we used that \(q_k\) cancels out since we may treat \(O(1)\) as \(o(q_k)\), and incorporated \(\tilde{w}_n(\zeta) - (2n - 2) < 0\) by assumption. Since we may take \(\varepsilon\) and thus \(\varepsilon'\) arbitrarily small, we see that the larger root of the quadratic polynomial, which is an upper bound for \(\tilde{w}_n(\zeta)\), will be arbitrarily close to \(\theta_n\). This contradicts our assumption \(\tilde{w}_n(\zeta) > \theta_n\). Thus the proof of case 1 is finished.

Case 2: For all large \(k\) the span of \(\mathcal{R}_k\) is not the entire space of polynomials of degree at most \(2n - 2\). Since \(\mathcal{P}_k\) is a hyperspace this means that the span of \(\mathcal{R}_k\) coincides with the span of \(\mathcal{R}_k\), in fact the span of all \(\mathcal{P}_k\) and \(\mathcal{R}_k\) (or their union) must be a constant hyperspace of \(\mathbb{R}^{2n-1}\) for all \(k \geq k_0\). Suppose this is the case. For \(k \geq 2\) and \(q_k\) defined as
in case 1, consider the intervals \( I_k := [q_k-1, q_k] \). Recall (40), for which we did not use any linear independence arguments of case 1. Moreover recall we showed in the proof of case 1 that our assumption \( \hat{w}_n(\zeta) > \theta_n \) implies
\[
\max_{P \in \mathcal{P}_k} L^*_P(q_k) = L^*_P(q_k) < L^*_{2n-2,2n-1}(q_k).
\]
On the other hand the function \( L^*_P \) decays in the interval \( I_k \) (in fact in \([0, q_k]\)) with slope \(-1/(2n - 2)\), and at \( q = q_k \) it meets the rising phase of \( L^*_P \). More generally the functions \( L^*_P \) for \( P \in \mathcal{V}_k \) decay in \([0, q_k]\) with slope \(-1/(2n - 2)\) and it obviously follows that
\[
\max_{P \in \mathcal{P}_k} L^*_P(q) = L^*_P(q), \quad q \in [0, q_k].
\]
In fact all values \( L^*_P(q) \) for \( P \in \mathcal{P}_k \) coincide in \([0, q_k]\) as their heights are equal. Hence, as the slope of \( L^*_{2n-2,2n-1} \) cannot be smaller than \(-1/(2n - 2)\), the estimate (52) implies that we have \( L^*_P(q) < L^*_{2n-2,2n-1}(q) \) for all \( P \in \mathcal{P}_k \) in the entire interval \( I_k \ni q \). On the other hand we have shown that the set \( \mathcal{P}_k = \mathcal{V}_{k-1} \cup \mathcal{V}_k \) spans a \((2n - 2)\)-dimensional hyperspace and thus
\[
L^*_{2n-2,2n-2}(q) \leq \max_{P \in \mathcal{P}_k} L^*_P(q) = L^*_P(q), \quad q \in I_k.
\]
Combining (52) and (53), we infer the strict inequality \( L^*_{2n-2,2n-2}(q) < L^*_{2n-2,2n-1}(q) \) for all \( q \in I_k \) when \( k \) is large. Since this argument holds for all large \( k \) and \( \bigcup_{j \geq k} I_j = [q_{k-1}, \infty] \), we derive
\[
L^*_{2n-2,2n-2}(q) < L^*_{2n-2,2n-1}(q), \quad q \geq \hat{q}.
\]
This contradicts [23] Theorem 1.1 which directly implies that \( L^*_{m,j}(q) = L^*_{m,j+1}(q) \) has arbitrarily large solutions \( q \) for any integer pair \((m, j)\) with \( m \geq 1 \) and \( 1 \leq j \leq m \), unless \( \zeta \) is algebraic of degree at most \( m \). Thus the assumption of case 2 cannot occur for transcendental \( \zeta \) when \( \hat{w}_n(\zeta) > \theta_n \), and this case is proved as well. For algebraic \( \zeta \) we know the better bounds from (4) anyway. \( \square \\

**Remark 4.1.** As quoted in the proof of Lemma 2.7 for any transcendental \( \zeta \) there exist infinitely many \( k \) such that two successive best approximation polynomials \( P^n_{k-1} \) and \( P^n_k \) have no common factor. However, for our method in case 2 to work, we had to guarantee this property for all large \( k \). Thus (40) was needed. On the other hand, the fact that the degrees of \( P^n_{k-1} \) and \( P^n_k \) are precisely \( n \) can be avoided by the argument used in the proof of \( \Psi(n) \leq 2n - 1 \) in Lemma 2.7

Some additional arguments may lead to a slight refinement of the bound in [8]. However, in dimension \( m = 2n - 2 \) as in the proof, we cannot expect something better than \( \hat{w}_n(\zeta) \leq 2n - 2 + o(1) \) with positive remainder term. If the essential arguments of the proof can be transferred to the situation of some dimension \( m < 2n - 2 \), with \( m \) not too small compared to \( n \), we would expect to obtain some better bound, as in Theorem 2.3 and Theorem 2.8. For example when \( m = 3 \frac{n}{2} - 1 \), the expected bounds turn out be just as in [15]. The difficulty when choosing \( m < 2n - 2 \) is to guarantee the linear independence of a sufficiently large subset of the polynomials defined similarly as \( \mathcal{P}_k \). This was an essential step to obtain reasonable bounds with Minkowski’s Second Convex Body Theorem (or parametric geometry of numbers). Indeed, if \( m < 2n - 2 \), the codimension
of the accordingly modified set $\mathcal{R}_k$ in the space of polynomials at most $m$ turns out to be $2n - 1 - m > 1$, and the argument of case 2 from the proof of Theorem 1.1 fails and yet we do not know how to modify it. In the proof of case 1 of Theorem 1.1 we showed that if $\Psi(n, \zeta) \leq 2n - 2$, then $\tilde{w}_n(\zeta)$ cannot exceed $\theta_n$. Some additional argument was needed in case 2 to derive the same estimate from $\Psi(n, \zeta) = 2n - 1$, provided $\zeta$ is transcendental. Concerning Theorem 2.3 its main assumption is essentially equivalent to $\Psi(n, \zeta) = 3\frac{n}{2} - 1$ and resolves the mentioned problems from case 2. The additional assumption (17) enters primarily to guarantee (37). If otherwise (17) fails, that is $w_n(\zeta) = w_{n-1}(\zeta)$, we derive (16) easily from (38) for $n \geq 4$, as already carried out in the deduction of [3, Theorem 2.1]. In the proof we will denote by $\epsilon_i$ certain small variations of $\epsilon$, and state beforehand that any of them tends to 0 as $\epsilon$ does.

**Proof of Theorem 2.3.** Suppose $\tilde{w}_n(\zeta) > \sigma_n$ holds for some transcendental real $\zeta$. Let $m = \frac{3n}{2} - 1$ and consider the combined Schmidt-Summerer graph associated to $(m, \zeta)$. For $(P_k)_{k \geq 1} = (P^n_k)_{k \geq 1}$, the sequence of best approximation polynomials associated to $(n, \zeta)$, we again define $(\mathcal{R}_k)_{k \geq 1}$ the sequence of points where $L^*_P$, that is $q_k$ is implicitly defined by

$$L^*_P(q_k) = \log H(P_k) - \frac{q_k}{m} = \log |P_{k-1}(\zeta)| + q_k.$$ 

Let further

$$V_{l,j}(T) = T^j P_l(T), \quad l \geq 1, \quad 0 \leq j \leq m - n,$$

and derive the sets $\mathcal{V}_l, \mathcal{P}_l$ and $\mathcal{R}_l$ very similarly as in the proof of Theorem 1.1 which are now of cardinality $m - n + 1, 2(m - n + 1)$ and $3(m - n + 1)$ respectively. For the same reason as in Theorem 1.1 we have (13). Again without loss of generality let $\zeta \in (0, 1)$, such that for any $l \geq 1$ the value $|P_l(\zeta)|$ maximizes $|P(\zeta)|$ among $P \in \mathcal{V}_l$. Now since (15) holds by assumption, and by the choice of $m$ we have $\mathcal{R}_k = 3(m - n + 1) = m + 1$, the set $\mathcal{R}_k$ is linearly independent and spans the space of polynomials of degree at most $m$. In particular $\mathcal{P}_k \subseteq \mathcal{R}_k$ is linearly independent and thus, very similar to (17), from (33) and (43) we obtain

$$L^*_{m,2(m-n+1)}(q_k) \leq L^*_{P_k}(q_k) \leq q_k \cdot \frac{m - \tilde{w}_n(\zeta)}{m(1 + \tilde{w}_n(\zeta))} + \epsilon_0 q_k.$$ 

With (32) applied for $j = 2(m - n + 1)$ we conclude

$$L^*_{m,m+1}(q_k) \geq \left( -\frac{2(m - n + 1)}{m + 1 - 2(m - n + 1)} \right) \cdot q_k \cdot \frac{m - \tilde{w}_n(\zeta)}{m(1 + \tilde{w}_n(\zeta))} + \epsilon_1 q_k + O(1).$$

Now again let $H_l = H(P_l)$ for any integer $l \geq 1$. We again infer

$$\log H_k - \frac{q_k}{m} = L^*_P(q_k),$$

and together with (54) we derive

$$\log H_k \leq \frac{q_k}{m} + q_k \cdot \frac{m - \tilde{w}_n(\zeta)}{m(1 + \tilde{w}_n(\zeta))} + \epsilon_2 q_k = q_k \left( \frac{m + 1}{m(1 + \tilde{w}_n(\zeta))} + \epsilon_2 \right).$$

Let $R \in \mathcal{V}_{k+1}$ arbitrary and put $r_{k+1}$ the local minimum of $L^*_R$. Now observe that $\sigma_n$ is larger than the dimension $m = \frac{3n}{2} - 1$. Hence the right hand side of (54) is negative, and
moreover in view of (33) at the local minimum $r_{k+1}$ of $L_R^*$ we also have $L_R^*(r_{k+1}) < 0$. On the other hand again $L_R^*(q_k) > 0$, as we carry out. Since by their definition for any $P, Q \in \mathcal{Y}_l$ we have $H(P) = H(Q)$ and the evaluations $P(\zeta)$ and $Q(\zeta)$ differ only by a multiplicative constant, it follows from (29) that $|L_P^*(q) - L_Q^*(q)| \leq 1$ uniformly on $q \in (0, \infty)$. Thus if we had $L_R^*(q_k) \leq 0$ then by the linear independence of $\mathcal{R}_k$ we have all $L_{m,j}^*(q_k) \leq 1$ for $1 \leq j \leq m+1$, and the first $2(m-n+1)$ values $1 \leq j \leq 2(m-n+1)$ are even bounded above by $(-c + o(1))q_k$ for some fixed $c > 0$ independent from $k$ in view of (54). Hence the sum of $L_{m,j}^*(q_k)$ over $1 \leq j \leq m$ is at most $(-c + o(1))q_k + O(1)$, contradicting (30) for large $k$. Hence the claim is shown. We conclude that $r_{k+1} > q_k$ and $L_R^*$ still decays at $q_k$, and hence

$$\log H_{k+1} - \frac{q_k}{m} = L_R^*(q_k), \quad R \in \mathcal{Y}_{k+1}.$$ 

For now assume $w_n(\zeta) > w_{n-1}(\zeta)$. Then we may apply (37), and as in the proof Theorem 1.1 with Lemma 3.4 we infer (49). Hence again since $\mathcal{R}_k$ spans the entire space of polynomials of degree at most $m$ from (50) we derive

$$(57) \quad L_{m,m+1}^*(q_k) \leq L_R^*(q_k) \leq (\nu + \epsilon) \cdot (\tau_k + \epsilon_2 q_k) - \frac{q_k}{m},$$

where

$$\tau_k = q_k \cdot \frac{m+1}{m(1 + \hat{w}_n(\zeta))}.$$ 

We combine (55) and (57), which leads after some computation to

$$(1 + m) \cdot (\hat{w}_n(\zeta)^2 + (n - 2m - 1)\hat{w}_n(\zeta) + (-4n^2 - 1 + 3mn - m + 4n)) + \epsilon_3 \Phi(\hat{w}_n(\zeta)) \leq 0,$$

where again $\Phi$ is bounded. We insert $m = \frac{3}{2}n - 1$ and obtain

$$(58) \quad \frac{3n}{4} \cdot (2\hat{w}_n(\zeta)^2 + (2 - 4n)\hat{w}_n(\zeta) + n^2 - n) + \epsilon_3 \Phi(\hat{w}_n(\zeta)) \leq 0.$$ 

For $\epsilon_3 = 0$, the quadratic inequality is satisfied precisely for $\hat{w}_n(\zeta) \leq \sigma_n$ with $\sigma_n$ in (18), such that our assumption of strict inequality $\hat{w}_n(\zeta) > \sigma_n$ yields a contradiction if we start with $\epsilon$ sufficiently small.

Finally, assume $w_n(\zeta) = w_{n-1}(\zeta)$. Then $\hat{w}_n(\zeta) \leq 2n - 2$ by (58) applied with $n_1 = n, n_2 = n - 1$, as in [5, Theorem 2.1]. Hence

$$\hat{w}_n(\zeta) \leq \max \left\{ 2n - 2, \frac{2n - 1 + \sqrt{2n^2 - 2n + 1}}{2} \right\}.$$ 

It can be readily checked that for $n = 4$ both bounds coincide and for $n \geq 6$ the bound $2n - 2$ is larger. \hfill \Box

The main ideas of the proof of Theorem 2.8 are again very similar. For convenience we state two easy propositions first. The first one, as well as its proof, is closely related to Lemma 2.7 and Lemma 3.5.

**Proposition 4.2.** Let $n \geq 2$ and integer and $\zeta$ a transcendental real number. Use the notation of Definition 2.4. For any integer $m \geq n$ define the set

$$\mathcal{A}_{k,m}^{n,\zeta} := \mathcal{A}_{k-1,m}^{n,\zeta} \cup \mathcal{A}_{k,m}^{n,\zeta} \subseteq \mathcal{P}_{k,m}^{n,\zeta}, \quad k \geq 2.$$
Then for all large \( k \) set \( \mathcal{C}^{n,\zeta}_{k,m} \) spans a vectorspace of dimension at least \( m - n + 2 \). If \( \tilde{\Psi}(n,\zeta) \leq m \leq 2m - 1 \), then for infinitely many \( k \) the set \( \mathcal{C}^{n,\zeta}_{k,m} \) spans a space of dimension at least \( 2(m - n + 1) \).

**Proof.** Without loss of generality assume \( d_{k-1} \leq d_k \) for \( d_{k-1} \) and \( d_k \) the degrees of \( P^{n,\zeta}_{k-1} \) and \( P^{n,\zeta}_k \) respectively, otherwise alter the set \( \Omega_k \) below accordingly. For the first claim, it suffices to consider the set

\[
\tilde{\Omega}_k := \left\{ P^{n,\zeta}_{k-1}, P^{n,\zeta}_k, TP^{n,\zeta}_k, \ldots, T^{m-d_k} P^{n,\zeta}_k \right\} \subseteq \mathcal{C}^{n,\zeta}_{k,m}
\]

Since \( P^{n,\zeta}_{k-1}, P^{n,\zeta}_k \) are linearly independent, it is easily seen that \( \tilde{\Omega}_k \) is as well, and has cardinality \( \tilde{\Omega}_k = m - d_k + 2 \geq m - n + 2 \). We need to show the second claim. By assumption \( m \geq \tilde{\Psi}(n,\zeta) \) we may assume that \( P^{n,\zeta}_{k-1} \) and \( P^{n,\zeta}_k \) have no common factor for certain arbitrarily large \( k \). For such \( k \), as pointed out in the proof of Lemma 2.7, the set \( \Omega_k \) in (59) of polynomials of degree at most \( d_{k-1} + d_k - 1 \) is linearly independent. We now distinguish two cases. Case 1: \( m \leq d_{k-1} + d_k - 1 \). Then

\[
\left\{ P^{n,\zeta}_{k-1}, TP^{n,\zeta}_{k-1}, \ldots, T^{m-d_k} P^{n,\zeta}_{k-1}, P^{n,\zeta}_k, TP^{n,\zeta}_k, \ldots, T^{m-d_k} P^{n,\zeta}_k \right\}
\]

is a subset of \( \Omega_k \), consisting of polynomials of degree at most \( m \). It is obviously linearly independent (as \( \Omega_k \) is) and has cardinality \( (m - d_k + 1) + (m - d_{k-1} + 1) \geq 2(m - n + 1) \) since \( d_i \leq n \). Case 2: \( m > d_{k-1} + d_k - 1 \). Then consider the set

\[
\Omega'_k := \left\{ P^{n,\zeta}_{k-1}, TP^{n,\zeta}_{k-1}, \ldots, T^{d_k-1} P^{n,\zeta}_{k-1}, P^{n,\zeta}_k, TP^{n,\zeta}_k, \ldots, T^{m-d_k} P^{n,\zeta}_k \right\}
\]

derived from \( \Omega_k \) by adding certain polynomials of higher degree. Similar to (59), it is easy to see that \( \Omega'_k \) is linearly independent again, as any polynomial in \( \Omega'_k \setminus \Omega_k \) has strictly larger degree than any polynomial in \( \Omega_k \) and the new degrees are all different. Moreover \( \Omega'_k \) has cardinality \( d_k + (m - d_k + 1) = m + 1 \geq 2(m - n + 1) \) since \( m \leq 2n - 1 \) by assumption. \( \square \)

**Remark 4.3.** The bound \( m - n + 2 \) of the first claim is sharp in case \( P^{n,\zeta}_{k-1}, P^{n,\zeta}_k \) are of degree \( n \) and have a common factor of maximum degree \( n - 1 \). Prescribing an upper bound \( d \) on the common factor results in lower dimension estimates in terms of \( d \) in the range between \( m - n + 2 \) and \( 2(m - n + 1) \).

Our second preparatory result is about extensions of linearly independent sets to bases, and almost a triviality.

**Proposition 4.4.** Let \( l \leq s \) be integers and \( v_1, \ldots, v_s \) be vectors that span a vectorspace \( \mathcal{S} \) of dimension \( t \leq s \). Assume \( v_1, \ldots, v_l \) are linearly independent. Then we can find \( t - l \) vectors \( w_1, \ldots, w_{t-l} \) among \( v_{l+1}, \ldots, v_s \) with the property that \( v_1, \ldots, v_l, w_1, \ldots, w_{t-l} \) span \( \mathcal{S} \).

**Proof.** Consider any maximum linear independent set containing \( v_1, \ldots, v_l \) by adding some remaining vectors \( v_j, l < j \leq k \). Clearly this set has cardinality at most \( t \), as it is linearly independent and its span is contained in \( \mathcal{S} \). On the other hand, it must have
Now we turn to the proof of Theorem 2.8.

**Proof of Theorem 2.8.** We start with (24). For simplicity let \( m = \tilde{\Psi}(n, \zeta) \). By definition of \( \tilde{\Psi}(n, \zeta) \), there exist arbitrarily large \( k \) for which two successive \( P_{n,\zeta}^{k-1}, P_{n,\zeta}^k \) are coprime and \( \mathcal{B}_{k,m}^{n,\zeta} \) defined via (19) and (20) span the space of polynomials of degree at most \( m \).

Fix such \( k \) large enough. By Proposition 4.2 the space spanned by \( \mathcal{C}_{n,\zeta}^{k,m} \subseteq \mathcal{B}_{k,m}^{n,\zeta} \) has dimension at least \( 2(m-n+1) \). Keep \( V_l, P_l, R_l \) for \( l \geq 1 \) from the proof of Theorem 2.3.

By assumption and Proposition 4.4 applied to \( s = 3(m-n+1), l = 2(m-n+1) \) and the vectors \( \{v_1, \ldots, v_s\} = \mathcal{R}_k \) and \( \{v_1, \ldots, v_l\} = \mathcal{R}_k \), there exist \( m+1-2(m-n+1) \) remaining polynomials among \( \mathcal{V}_{k+1} \) which together with \( \mathcal{P}_k \) spans the space of polynomials of degree at most \( m \). Moreover, (55) holds again by assumption, and we can again apply Lemma 3.3. Again combining these arguments yields (57). We then proceed as in the proof of Theorem 2.3 up to the point where we inserted \( m \) to obtain (58). We solve the general quadratic equation in terms of \( m, n \) and obtain the claimed bound (24).

Now we show (25). From the first claim of Proposition 4.2 the set \( \mathcal{C}_{n,\zeta}^{m,\zeta} \) spans a space of dimension at least \( m-n+2 \). We essentially proceed as in the proof of Theorem 2.3 again, but taking into account the lower cardinality in place of (54) and (55) we obtain

\[
L^*_{m,m-n+2}(q_k) \leq L^*_k(q_k) \leq q_k \cdot \frac{m - \hat{w}_n(\zeta)}{m(1 + \hat{w}_n(\zeta))} + \epsilon_0 q_k.
\]

and consequently

\[
L^*_{m,m+1}(q_k) \geq \left(1 - \frac{m-n+2}{m+1-(m-n+2)}\right) \cdot q_k \cdot \frac{m - \hat{w}_n(\zeta)}{m(1 + \hat{w}_n(\zeta))} + \epsilon_1 q_k + O(1).
\]

On the other hand we obtain (57) precisely as above. Again combination and some calculation yields the bound in (25). \( \square \)

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