DIVISION THEOREMS FOR SPACES OF SECTIONS OF EQUIVARIANT VECTOR BUNDLES

NIKOLAY KONOVALOV

Abstract. Let \( G \) be a semisimple complex Lie group and let \( X \) be a smooth complex projective \( G \)-variety. Let \( L \) be a \( G \)-equivariant line bundle over \( X \). Denote by \( \Gamma(X, L) \) the vector space of global holomorphic sections of the bundle \( L \). Let \( \text{Sing} \subset \Gamma(X, L) \) be the set of all sections whose zero locus is a singular subvariety of \( X \). We study the topology of the complement \( \Gamma_{\text{reg}}(X, L) := \Gamma(X, L) \setminus \text{Sing} \).

One can define the natural subring \( \text{Lk}(\Gamma_{\text{reg}}(X, L), \mathbb{Z}) \) only in terms of \( X \). The formula for the orbit map \( O^* \): \( \text{Lk}(X, L) \to H^*(G, \mathbb{Z}) \) is obtained. As a consequence we can generalize results of papers [10,11] on degeneration of the Leray spectral sequence in rational cohomology for the quotient map \( \Gamma_{\text{reg}}(X, L) \to \Gamma_{\text{reg}}(X, L)/G \).

Introduction

Let \( G \) be a semisimple complex Lie group and let \( X \) be a smooth complex projective \( G \)-variety. Let \( L \) be a \( G \)-equivariant line bundle over \( X \). Denote by \( \Gamma(X, L) \) the vector space of global holomorphic sections of the bundle \( L \). Let \( \text{Sing} \subset \Gamma(X, L) \) be the set of all sections such that its zero locus is a singular subvariety of \( X \). We study the topology of the complement \( \Gamma_{\text{reg}}(X, L) := \Gamma(X, L) \setminus \text{Sing} \). Suppose that the group \( G \) acts on \( \Gamma_{\text{reg}}(X, L) \) with finite stabilizers. One can ask the following question.

Question. Does the Leray-Serre spectral sequence for the quotient map \( \Gamma_{\text{reg}}(X, L) \to (\Gamma_{\text{reg}}(X, L))/G \):
\[
E_2^{p,q} = H^p(G, \mathbb{Q}) \otimes H^q(\Gamma_{\text{reg}}(X, L)/G, \mathbb{Q}) \Rightarrow H^{p+q}(\Gamma_{\text{reg}}(X, L), \mathbb{Q})
\]
degenerate at the second page?

This question was firstly posed by C. A. M. Peters and J. H. M. Steenbrink in paper [10] for \( X = \mathbb{P}^n \) and \( G = \text{SL}_{n+1} \). They obtained that the spectral sequence is degenerate at the second page for all \( G \)-equivariant line bundles \( L = O(d), d > 2 \). Due to [6], a positive answer to the question above for some \( G \)-equivariant line bundle \( L \) is called Division Theorem (or Phenomena) for the line bundle \( L \).

We follow the main idea of paper [10] and try to attack the problem by using the Leray-Hirsh theorem. Let \( Y \) be an irreducible subvariety of \( X \). Denote by \( \text{Sing}_Y \), the subset of \( \text{Sing} \) which consists of all sections with a singular zero point at \( Y \). We prove that the set \( \text{Sing}_Y \) is an irreducible subvariety of \( \text{Sing} \). By this reason, there exists the fundamental class \( [\text{Sing}_Y] \) in the Borel-Moore homology of \( \text{Sing}_Y \). By the Alexander duality, the homology class \( [\text{Sing}_Y] \) corresponds to the cohomology class \( \text{Lk}^{\text{Sing}}_Y \) in the cohomology ring of the topological space \( \Gamma_{\text{reg}}(X, L) \).

Let us fix some point \( s_0 \in \Gamma_{\text{reg}}(X, L) \). Denote by \( O: G \to \Gamma_{\text{reg}}(X, L) \) the orbit map for \( s_0 \), i.e. the map \( O \) acts as follows:
\[
O(g) = gs_0.
\]
Now the Leray-Hirsh theorem (or more exactly, Theorem 2 in [10]) guarantees the following statement. Suppose that the induced morphism \( O^*: H^*(\Gamma_{\text{reg}}(X, L), \mathbb{Q}) \to H^*(G, \mathbb{Q}) \) is surjective. Then we have a positive answer to the main problem.

Denote by \( \text{Lk}(X, L) \) the subring of the ring \( H^*(\Gamma_{\text{reg}}(X, L), \mathbb{Q}) \) generated by linking classes \( \text{Lk}^{\text{Sing}}_Y \) for all subvarieties \( Y \subset X \). Our main questions are following:

1. For what complex smooth projective \( G \)-variety \( X \) and for what \( G \)-equivariant line bundles \( L \) over \( X \), the ring \( \text{Lk}(X, L) \) maps onto the cohomological ring \( H^*(G, \mathbb{Q}) \) under the orbit map \( O^* \)?
2. Is it true that for general (in some sense) \( G \)-equivariant line bundles \( L \) over the variety \( X \) with regular \( G \)-action the ring \( \text{Lk}(X, L) \) maps onto the cohomological ring \( H^*(G, \mathbb{Q}) \) under the orbit map \( O^{**} \)?
There are several possible motivations for these questions. Although cohomology rings of Lie groups are well-known, a geometric description for primitive generators of the rational cohomology ring $H^*(G, \mathbb{Q})$ is known only for classical ones \cite{13}. In the case of positive answers, we obtain a geometric way to construct primitive generators in $H^*(G, \mathbb{Q})$ for any Lie group $G$.

The second motivation justifies the name for Division Theorems. Suppose that for a $G$-equivariant line bundle $L$ the ring $Lk(X, L)$ maps onto the ring $H^*(G, \mathbb{Q})$. Then one can easily derive the following algebra isomorphism:

$$H^*(\Gamma_{\text{reg}}(X, L), \mathbb{Q}) \cong H^*(G, \mathbb{Q}) \otimes H^*(\Gamma_{\text{reg}}(X, L)/G, \mathbb{Q}).$$

Therefore, one can say that the cohomology ring $H^*(\Gamma_{\text{reg}}(X, L), \mathbb{Q})$ can be “divided” by the cohomology ring $H^*(G, \mathbb{Q})$. One can also say something about the mixed Hodge structure on the cohomology groups of the space $\Gamma_{\text{reg}}(X, L)$. For a complex algebraic variety $V$, we denote by $P_{mHdg}(V)$ the mixed Hodge polynomial of $V$ (see \cite{6} for the definition). Then an easy corollary of the isomorphism above is the following equality:

$$P_{mHdg}(\Gamma_{\text{reg}}(X, L)) = P_{mHdg}(G) \cdot P_{mHdg}(\Gamma_{\text{reg}}(X, L)/G).$$

The third motivation concerns studying the action of the orbit map $O^*$ on integer cohomology. Since the cohomological classes $L^*_{\text{Sing}}(L)$ are naturally defined as integer cohomological classes, it makes sense to compute $O^*(L^*_{\text{Sing}}(L)) \in H^*(G, \mathbb{Z})$. A. Gorinov in paper \cite{6} successfully performed these computations in the case of $X = \mathbb{P}^n$ and $G = SL_{n+1}$. He also used it to obtain new bounds for orders of automorphism groups of smooth hypersurfaces in $\mathbb{P}^n$ of given degree; more precisely, he proved that the order of the automorphism group of any smooth hypersurface of given degree should divide certain explicit expression in the degree and dimension of the hypersurface. After, V. González-Aguilera and A. Liendo in paper \cite{5} showed that these bounds are sharp in “arithmetic” sense; any prime number in prime decomposition of the bound can be realized as the order of an automorphism of some smooth hypersurface of given degree. Therefore, we hope to apply orbit maps to finding new bounds of automorphism groups of more general algebraic varieties.

Our main result is Theorem \ref{2.2.7}, which describes the general formula for $O^*(L^*_{\text{Sing}}) \in H^*(G, \mathbb{Z})$.

**Theorem** (Theorem \ref{2.2.7}). Let $L$ be a 1-jet spanned $G$-equivariant line bundle over a smooth complex projective $G$-variety $X$ and let $Y$ be a closed irreducible subvariety of dimension $m$. Suppose that the Chern number $(c_m(J(L)), [Y])$ is nonzero. Then the following equality holds

$$O^*(L^*_{\text{Sing}}) = S(e_G(J(L))/[Y]).$$

Here $e_G(J(L)) \in H^*(X_{hG}, \mathbb{Z})$ is the equivariant Euler class (Definition \ref{1.4.5}) of the first jet bundle $J(L)$ for the line bundle $L$. The map $S : J^*(X, \mathbb{Q}) \to H^{*-1}(G \times X, \mathbb{Q})/M_{X}^{-1}$ (Definition \ref{1.4.3}) is between some natural ideal (Notation \ref{1.3.1}) of the ring $H^*(X_{hG}, \mathbb{Q})$ and some natural quotient (Definition \ref{1.1.7}). The main point is the map $S$ can be given in very explicit terms (Proposition \ref{1.4.10}). Therefore the classes $O^*(L^*_{\text{Sing}})$ are computable, at least theoretically.

By this theorem, we have an algorithm to answer the first question. For example, we can generalize the results of papers \cite{6,10} to $G = SO_{n+1}$ and $X$ is a quadric in $\mathbb{P}^n$. Computation for other varieties and groups are also feasible, but unfortunately, they require computer algebra systems.

As another consequence, we can answer the second question in a number of cases. For instance, we obtain the following statements.

**Theorem** (Theorem \ref{1.1.3}). Let $X = G/B$ be a complete flag variety for a simple complex Lie group $G$, the Cartan type of $G$ is not $D_{2n}$. Then for a general globally generated $G$-equivariant line bundle on the $G$-variety $X$ the ring $Lk(X, L)$ maps onto the cohomological ring $H^*(G, \mathbb{Q})$ under the orbit map $O^*$.

We can also deal with more exotic homogeneous varieties or exceptional Lie groups.

**Theorem** (Corollary \ref{1.1.8}). On the following $G$-varieties the ring $Lk(X, L)$ maps onto the cohomological ring $H^*(G, \mathbb{Q})$ under the orbit map $O^*$ for a general globally generated $G$-equivariant line bundle $L$.

1. $G = SO(2n+1)$, $X$ is the orthogonal Grassmannian $\text{OGr}(n+1, 2n+1)$.
2. $G = E_6$, $X$ is the Cayley plane $\text{OP}^2$.
3. $G = E_7$, $X$ is the Freudenthal variety.
The paper consists of four sections. In the first section, we develop the theory of secondary equivariant Euler classes. In the second one we apply our theory to the orbit map. As a result we are able to prove Theorem 1.1.4. The third section is devoted for examples. We apply Theorem 1.1.4 to the most simple cases: projective spaces and quadrics. In the last section we discuss division theorems for generic line bundles.

Acknowledgments. I am grateful to Alexey Gorinov for posing the problem and explaining many details of it. I would also like to thank Mikhail Finkelberg for useful conversations. The research was supported by the Simons Foundation and Independent University of Moscow.

1. Secondary characteristic classes

Let $G$ be a topological group and let $E$ be a $G$-equivariant vector bundle of real rank $r$ over a $G$-space $X$. Suppose that the bundle $E$ is orientable and its Euler class $e(E)$ is zero. In this case, we can define the characteristic class $sc_G(E)$, which lies in some natural quotient of the group $H^{r-1}(G \times X)$. In this section, we investigate different sides of this characteristic class. Our definitions and proofs are extremely elementary, but unfortunately, we could not find the obtained results in the literature.

1.1. Secondary Thom and Euler classes. Let $X$ be a topological space, and let $E$ be a vector bundle over $X$ of real rank $r$. We denote the complement of the zero section in the total space of $E$ by $\text{Tot}(E)$. Consider the projection map $p_0: \text{Tot}(E) \rightarrow X$. Let $\text{Tot}(E_x)$ be the fibre of the map $p_0$ over a point $x \in X$. Note that the fibre $\text{Tot}(E_x)$ is homotopy equivalent to the sphere $S^{r-1}$. Finally, let $R$ be a commutative ring with identity.

Definition 1.1.1. A cohomology class $a_E \in H^{r-1}(\text{Tot}(E), R)$ is called a secondary Thom class of $E$ if and only if its restriction to each fibre $\text{Tot}(E_x)$ is a generator of the $R$-module $H^{r-1}(\text{Tot}(E_x), R) \cong H^{r-1}(S^{r-1}, R)$.

Proposition 1.1.2. A secondary Thom class of a vector bundle $E$ exists if and only if $E$ is $R$-orientable and the Euler class $e(E)$ is zero.

Proof. Consider the exact sequence:

\begin{equation}
\xymatrix{ H^*(\text{Tot}(E), R) \ar[r]^{i} & H^*(\text{Tot}(E), R) \ar[r]^{\delta} & H^{*+1}(\text{Tot}(E), \text{Tot}(E), R) \ar[r]^{j} & H^{*+1}(\text{Tot}(E), R). }
\end{equation}

Here $i$ and $j$ denote the obvious maps $\text{Tot}(E) \rightarrow \text{Tot}(E)$ and $(\text{Tot}(E), \emptyset) \rightarrow (\text{Tot}(E), \text{Tot}(E))$ respectively. Exact sequence (1.1.3) is natural with respect to morphisms of vector bundles.

Assume that a secondary Thom class $a_E \in H^{r-1}(\text{Tot}(E), R)$ exists. Then $\delta(a_E)$ is a Thom class of the vector bundle $E$. Indeed, let us restrict exact sequence (1.1.3) to any point $x \in X$. By naturality we obtain the commutative diagram

\begin{equation}
\xymatrix{ H^{r-1}(\text{Tot}(E), R) \ar[r]^{\delta} \ar[d] & H^r(\text{Tot}(E), \text{Tot}(E), R) \ar[d] \ar[r]^{j} & H^{r+1}(\text{Tot}(E), R) \ar[d] \ar[r]^{j} & H^{r+1}(\text{Tot}(E), \text{Tot}(E), R) \ar[d] \ar[r]^{j} & H^{r+1}(\text{Tot}(E), \text{Tot}(E), R). }
\end{equation}

Since the map $\delta_x$ is an isomorphism, the restriction of $\delta_x(a_E)$ is a generator of $H^r(E_x, E_x \setminus \{0\}, R)$ for any point $x \in X$. So $E$ is $R$-orientable and the Euler class of $E$ is equal to $j^*(\delta(a_E)) = 0$.

Now assume that $E$ is $R$-orientable and the Euler class of $E$ is zero. Then there exists a Thom class $u_E \in H^r(\text{Tot}(E), \text{Tot}(E), R)$ and $j^*(u_E) = 0$. So there exists an element $a_E$ of the group $H^{r-1}(\text{Tot}(E), R)$ such that $\delta(a_E) = u_E$. A similar argument using diagram (1.1.4) shows that the element $a_E$ is a secondary Thom class.

Proposition 1.1.5. Let $f: Y \rightarrow X$ be a continuous map of topological spaces, and let $E'$ and $E$ be vector bundle of real rank $r$ over $Y$ and $X$ respectively. We suppose that $f$ is covered by a map $F: \text{Tot}(E') \rightarrow \text{Tot}(E)$ which is a linear isomorphism when restricted to each fibre. Suppose that $a_E \in H^{r-1}(\text{Tot}(E), R)$ is a secondary Thom class of the vector bundle $E$. Then $F^*(a_E)$ is a secondary Thom class of the vector bundle $E'$.
Proof. We need to prove that for every point \( y \in Y \) the restriction \( F^\ast(a_E)|_y \) is a generator of \( R \)-module \( H^{-1}(\text{Tot}_0(E)|_y, R) \). One can identify the fibre \( \text{Tot}_0(E)|_y \) with the fibre \( \text{Tot}_0(E)|_{f(y)} \). Under this identification the cohomology class \( F^\ast(a_E)|_y \) maps to the restriction \( (a_E)|_{f(y)} \). Since \( a_E \) is a secondary Thom class, the restriction \( (a_E)|_{f(y)} \) is a generator of the \( R \)-module \( H^{-1}(\text{Tot}_0(E)|_{f(y)}, R) \). So the same is true for the restriction \( F^\ast(a_E)|_y \).

\[ \square \]

**Definition 1.1.6.** Let \( E \) be an \( R \)-oriented vector bundle with the Thom class \( u_E \in H^\ast(\text{Tot}(E), \text{Tot}_0(E), R) \). A secondary Thom class \( a_E \) is called **R-oriented** if and only if \( \delta(a_E) = u_E \). Here \( \delta \) is the homomorphism from long exact sequence [1.1.3].

**Proposition 1.1.7.** An \( R \)-oriented secondary Thom class \( a_E \in H^{-1}(\text{Tot}_0(E), R) \) is uniquely defined up to the group \( \text{Im}(p_0^\ast) \). I.e., if \( a_E \) and \( a'_E \) are two \( R \)-oriented secondary Thom classes, then there exists a class \( b \in H^{-1}(X, R) \) such that \( a_E = a'_E + p_0^\ast(b) \).

Proof. Let \( u_E \) be the Thom class of the vector bundle \( E \). Since \( \delta(a_E) = u_E = \delta(a'_E) \), the difference \( a_E - a'_E \) belongs to the kernel of the homomorphism \( \delta \). By exact sequence [1.1.3] one can see that \( a_E - a'_E \in \text{Im}(i^\ast) \). The group \( \text{Im}(i^\ast) \) coincides with the image of the homomorphism \( p_0^\ast \).

\[ \square \]

**Proposition 1.1.8.** If the Euler class \( e(E) \) is zero, then the map \( p_0^\ast: H^\ast(X, R) \to H^\ast(\text{Tot}_0(E), R) \) is injective.

Proof. The Leray-Serre spectral sequence of the fibre bundle \( p_0: \text{Tot}_0 E \to X \) degenerates (i.e., all differentials are zero starting from \( d_2 \)) if and only if the Euler class \( e(E) \) is zero. Therefore, if the Euler class \( e(E) \) is zero, then the map \( p_0^\ast \) is injective.

Recall the definition of a linking class form [9]. Let \( M \) be a smooth oriented manifold without boundary of dimension \( N \), let \( X \subset M \) be a closed subset of \( M \). Then one can define Alexander duality isomorphism:

\[ (1.1.9) \quad D_{M, M \setminus X}: H^i(M \setminus X) \cong H_{N-i}^M(M \setminus X) \cong H_{n-i}(\hat{M}, \hat{X}). \]

Here \( \hat{M} \) and \( \hat{X} \) are one-point compactifications of spaces \( M \) and \( X \), respectively.

**Definition 1.1.10.** Let \( c \) be a homology class in \( H_{N-i}^M(X) \). A cohomology class \( \text{Lk}_{c, X, M} \in H^i(M \setminus X) \) is called a linking class w.r.t. \( c \) in \( M \) if and only if

\[ \delta(D_{M, M \setminus X}(\text{Lk}_{c, X, M})) = c. \]

Here \( \delta: H_{N-i}(\hat{M}, \hat{X}) \to H_{N-i-1}(\hat{X}) \) is the boundary homomorphism in the long exact sequence of pair \( (\hat{M}, \hat{X}) \).

Note that a linking class \( \text{Lk}_{c, X, M} \) is defined if and only if \( c \in \ker(H_{N-i}^M(X) \to H_{N-i-1}(M)) \) and unique up to the subgroup \( \text{Im}(\delta^i(M)) \subset H^i(M \setminus X) \).

**Proposition 1.1.11.** Let \( X \) be an oriented topological manifold, let \( E \) be an \( R \)-oriented vector bundle over \( X \) with \( e(E) = 0 \). Then \( \text{Tot}(E) \) is an oriented topological manifold and any \( R \)-oriented secondary Thom class \( a_E \) is a linking class \( \text{Lk}_{\ast(X), X, \text{Tot}(E)} \in H^i(\text{Tot}(E), R) \). Here we view \( X \) as the zero section inside \( \text{Tot}(E) \).

Proof. Straightforward.

Now let \( G \) be a topological group and let \( X \) be a topological space on which \( G \) acts continuously on the left. A **G-equivariant vector bundle on** \( X \) is a vector bundle \( E \) on \( X \) equipped with an action of \( G \) on \( \text{Tot}(E) \) which is compatible with the action of \( G \) on \( X \) and linear when restricted to each fibre. Let \( E \) be a \( G \)-equivariant \( R \)-oriented vector bundle over \( X \) of rank \( r \). Unless stated otherwise, we will assume that the \( G \)-action preserves the \( R \)-orientation of \( E \). Let \( \varphi: G \times X \to X \) be the action map, and let \( p_2: G \times X \to X \) be the projection map. Let \( \psi \) be an isomorphism of the vector bundles \( \varphi^\ast(E) \) and \( p_2^\ast(E) \).

**Remark 1.1.12.** Note that if \( E \) is a \( G \)-equivariant vector bundle on \( X \), then there is a preferred choice of \( \psi \): the total space of \( \varphi^\ast(E) \) (respectively, of \( p_2^\ast(E) \)) is the space of all triples \( \{(g, x, e) : g \in G, x \in X, e \in \text{Tot}(E)\} \) such that \( e \in E_gx \) (respectively, \( e \in E_2x \)). The isomorphism \( \psi: \varphi^\ast(E) \to p_2^\ast(E) \) is then given by \( (g, x, e) \mapsto (g, x, g^{-1}e) \).
We denote the action map $G \times \text{Tot}_0(E) \to \text{Tot}_0(E)$ and the projection $G \times \text{Tot}_0(E) \to \text{Tot}_0(E)$ as $\Phi$ and $P_2$ respectively. Note that there are homeomorphisms:

(1.1.13) \[
\psi_0: \text{Tot}_0(p_2^*E) \cong \text{Tot}_0(\varphi^*(E)),
\]

(1.1.14) \[
\text{Tot}_0(p_2^*E) \cong G \times \text{Tot}_0(E).
\]

**Lemma 1.1.15.** Assume the vector bundle $E$ is $R$-oriented and the action of $G$ on $E$ preserves the orientation. For any $R$-oriented secondary Thom class $a_E \in H^*(\text{Tot}_0(E), R)$ of a $G$-equivariant $R$-oriented vector bundle $E$ the difference $P'_2(a_E) - P_2^*(a_E) \in H^*(G \times \text{Tot}_0(E), R)$ belongs to the image of the homomorphism $(\text{id} \times p_0)^*$.

**Proof.** Using Proposition 1.1.7 we see that both the classes $\Phi^*(a_E)$ and $P'_2(a_E)$ are secondary Thom classes of the bundle $p_2^*(E)$. Let us check that they correspond to the same $R$-orientation, i.e., that they map to the same Thom class under the connecting homomorphism

$$H^*(G \times \text{Tot}_0(E), R) \cong H^*(\text{Tot}_0(p_2^*(E)), R) \to H^{*+1}(\text{Tot}(p_2^*(E)), \text{Tot}_0(p_2^*(E)), R).$$

Here we identify $\text{Tot}_0(p_2^*(E))$ with $G \times \text{Tot}_0(E)$ using (1.1.4). It suffices to check this over each point $(g, x)$ of $G \times X$. Due to our assumption that the action of $G$ is orientation-preserving, the maps $\Phi$ and $P_2$ induce the same orientation on the fibre of $p_2^*(E)$ over $(g, x)$. The lemma now follows from Proposition 1.1.7. \qed

**Notation 1.1.16.** Let $E$ be an $R$-oriented vector bundle of real rank $r$. Assume that the Euler class of $E$ is zero, and let $a_E$ be a secondary Thom class. Since the cohomology group $H^{r-1}(\text{Tot}_0(E), R)$ is isomorphic as an $R$-module to the direct sum $p_0^*(H^{r-1}, R) \oplus R[a_E]$, one can define a homomorphism of $R$-modules

$$\partial \Phi^*: H^{r-1}(\text{Tot}_0(E), R) \to H^{r-1}(G \times X, R)$$

using $(\text{id} \times p_0)^* \circ (\partial \Phi^*) = \Phi^* - P'_2$. The homomorphism $\partial \Phi^*$ is well-defined by Lemma 1.1.15 and by the fact that the homomorphism $(\text{id} \times p_0)^*$ is injective (see Proposition 1.1.8).

**Definition 1.1.17.** Let $X$ be a $G$-space. The graded $R$-module

$$M^*_X := \text{Im}((\varphi^* - p_2^*): H^*(X, R) \to H^*(G \times X, R))$$

will be called the indeterminacy submodule. We will omit the subscript if the space $X$ is clear from a setting.

**Corollary 1.1.18.** Let $a_E$ and $a'_E$ be $R$-oriented secondary Thom classes of the vector bundle $E$ of real rank $r$. Then the difference

$$\partial \Phi^*(a_E) - \partial \Phi^*(a'_E)$$

belongs to the indeterminacy submodule $M^{r-1}$.

**Proof.** Indeed, by Proposition 1.1.7 the difference $a_E - a'_E$ is equal to $p_0^*(b)$ for some $b \in H^{r-1}(X, R)$. Since the vector bundle $E$ is $G$-equivariant, we observe the following identity

$$(\Phi^* - P'_2)(a_E - a'_E) = (\Phi^* - P'_2)(p_0^*(b)) = (\text{id} \times p_0)^*((\varphi^* - p_2^*)(b)).$$

I.e. the difference $\partial \Phi^*(a_E - a'_E)$ is equal to $(\varphi^* - p_2^*)(b) \in M^{r-1}$. \qed

We now wish to describe the cohomology class $\partial \Phi^*(a_E) \in H^{r-1}(X, R)/M^{r-1}$ in more concrete terms.

**Definition 1.1.19.** Let $G$ be a topological group and let $X$ be a topological space with a continuous $G$-action map $\varphi: G \times X \to X$. Let us define the naive homotopy quotient $X \ast_G X$ of $X$ by the $G$-action as the homotopy coequalizer of morphisms $\varphi$ and $p_2$. Recall that the homotopy coequalizer is the homotopy colimit of the following diagram:

(1.1.20) \[
\begin{array}{ccc}
G \times X & \xrightarrow{\varphi} & X \\
\downarrow{p_2} & & \\
X
\end{array}
\]

Denote by $q_X$ (or just $q$, if the space $X$ is clear) the quotient map $X \to X \ast_G X$. 
Remark 1.1.21. Consider any homotopy coequiliser diagram

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} C.
\end{array}
\]

Then there exists the long exact sequence of cohomology groups:

\[
\ldots \to H^{s-1}(B, R) \xrightarrow{j^* - g^*} H^{s-1}(A, R) \xrightarrow{\delta} H^s(C, R) \xrightarrow{\varphi^*} \to H^s(A, R) \xrightarrow{j^* - g^*} H^s(B, R) \xrightarrow{\delta} H^{s+1}(C, R) \to \ldots
\]

Notice that the homomorphism \( \delta : H^{s-1}(A, R) \to H^s(C, R) \) in the long exact sequence above is induced by the natural map

\[(1.1.22) \quad \delta : C \to \Sigma(A_+).
\]

Moreover, the following sequence is a cofiber sequence

\[
B \xrightarrow{j} C \xrightarrow{\delta} \Sigma(A_+),
\]

the mapping cone of the map \( \delta \) is naturally homotopy equivalent to the space \( \Sigma(B_+) \), and the canonical map \( \Sigma(A_+) \to \text{Cone}(\delta) \) coincides with the following composition:

\[
\Sigma(A_+) \to \Sigma(A_+) \vee \Sigma(A_+) \xrightarrow{(\text{id}, - \text{id})} \Sigma(A_+) \vee \Sigma(A_+) \xrightarrow{(j, g)} \Sigma(B_+) \simeq \text{Cone}(\delta).
\]

Now we turn our attention to vector bundles over the naive homotopy quotient \( X \ast_G X \). Recall that the category of vector bundles over the space \( X \ast_G X \) is equivalent to the category of pairs \( (E, \psi) \), where \( E \) is a vector bundle over \( X \) and \( \psi : \varphi^* E \to p_2^* E \) is an isomorphism of vector bundles over \( G \times X \).

Notation 1.1.23. Suppose that \( (E, \psi) \) is a \( G \)-equivariant vector bundle over \( X \). Let \( E \ast_G X \) be the vector bundle over \( X \ast_G X \) corresponded to the pair \( (E, \psi) \).

As we have already mentioned in Remark 1.1.21, there exists the long exact sequence of cohomology groups associated to the coequiliser diagram \((1.1.20)\):

\[
(1.1.24) \quad \ldots \to H^{s-1}(X, R) \xrightarrow{\varphi^* - p_2^*} H^{s-1}(G \times X, R) \xrightarrow{\delta} H^s(X \ast_G X, R) \xrightarrow{\varphi^*} \to H^s(X, R) \xrightarrow{\varphi^* - p_2^*} H^s(G \times X, R) \xrightarrow{\delta} H^{s+1}(X \ast_G X, R) \to \ldots
\]

Definition 1.1.25. Suppose that \( E \) is a \( G \)-equivariant \( R \)-oriented vector bundle of real rank \( r \) over \( X \) and the Euler class of \( E \) is zero. A cohomology class \( se_G(E) \in H^{r-1}(G \times X, R) \) is called a secondary equivariant Euler class of the vector bundle \( E \) if and only if the following equality holds

\[
\delta(se_G(E)) = e(E \ast G E) \in H^r(X \ast_G X, R).
\]

Here \( E \ast_G E \) is the vector bundle defined in Notation 1.1.23 and \( \delta : H^{r-1}(G \times X, R) \xrightarrow{\delta} H^r(X \ast_G X, R) \) is the boundary homomorphism in long exact sequence \((1.1.21)\).

Remark 1.1.26. A secondary equivariant Euler class of the vector bundle \( E \) is unique up to the indeterminacy submodule \( M_{X}^{r-1} \). Indeed, by long exact sequence \((1.1.24)\) the kernel of the homomorphism \( \delta \) is precisely the submodule \( M_{X}^{r-1} \).

Proposition 1.1.27. Suppose that \( E \) is a \( G \)-equivariant \( R \)-oriented vector bundle of real rank \( r \) over \( X \) and \( e(E) = 0 \). Then for any \( R \)-oriented secondary Thom class \( a_E \in H^{r-1}(\text{Tot}_0(E), R) \) and for any secondary equivariant Euler class \( se_G(E) \in H^{r-1}(G \times X, R) \) the equality

\[
\partial \Phi^* (a_E) = se_G(E)
\]

holds up to the indeterminacy submodule \( M_{X}^{r-1} \).
Proof. It suffices to prove that $\delta(\partial \Phi^*(a_E)) = s e_G(E) = e(E \ast_G E)$. Let us choose a space model for the homotopy type $X \ast_G X$:

$$X \ast_G X \cong (G \times X \times I) / \{(g, x, 0) \sim gx, (g, x, 1) \sim x\}.$$  

Then the map $q_X : X \to X \ast_G X$ is an embedding, and it is equal to the composition of the canonical map $X \to G \times X \times I$ with the quotient map to $X \ast_G X$. We also choose similar space models for $G$-spaces $\text{Tot}(E)$ and $\text{Tot}_0(E)$.

Denote by $F_0$ the fiber of the projection map $p_0 : \text{Tot}_0(E \ast_G E) \to X \ast_G X$ over the point $x \in \text{Im}(q_X)$. Let $i_x : F_0 \to \text{Tot}_0(E)$ denote the embedding of the fiber. Since the vector bundle $E$ is $R$-oriented, the $R$-generator $a_F \in H^{r-1}(F_0, R)$ is already chosen. The Euler class $e(E \ast_G E)$ can be defined as $(p_0^\ast)^{-1}(\delta(a_F))$, where $\delta$ and $p_0^\ast$ are obvious homomorphisms in the diagram below

$$H^{r-1}(F_0, R) \xrightarrow{\delta} H^r(\text{Tot}_0(E \ast_G E), F_0, R) \xleftarrow{p_0^\ast} H^r(X \ast_G X, R).$$

Notice that the topological space $\text{Tot}_0(E \ast_G E)$ is equal to the space $\text{Tot}_0 E \ast_G \text{Tot}_0 E$. Consider the following diagram:

$$
\begin{array}{ccc}
H^{r-1}(F_0, R) & \xrightarrow{\delta} & H^r(\text{Tot}_0(E \ast_G E), F_0, R) \\
\downarrow{i_x} & & \uparrow{p_0^\ast} \\
H^{r-1}(\text{Tot}_0(E), R) & \xrightarrow{\delta} & H^r(\text{Tot}_0(E \ast_G E), \text{Tot}_0(E), R) \\
\downarrow{p_2^\ast - \Phi^*} & & \uparrow{\delta^*} \\
H^{r-1}(G \times \text{Tot}_0(E), R) & \xrightarrow{\sim} & H^r(\Sigma(G \times \text{Tot}_0(E)), R) \\
\downarrow{\delta^*} & & \uparrow{\delta^*} \\
H^r(\Sigma(G \times X), R) \\
\end{array}
$$

Here the homomorphism $\delta^*$ is induced by the morphism $\text{I.1.22}$. Notice that all squares in this diagram is immediately commutative except the left lower square. But the left lower square is commutative by the last statement of Remark $\text{I.1.24}$.

Now let us apply diagram $\text{I.1.28}$ to our situation. Since $e(E) = 0$, there exists the $R$-oriented secondary Euler class $a_E \in H^{r-1}(\text{Tot}_0(E), R)$ such that $i_x^\ast(a_E) = a_F$. Therefore,

$$e(E \ast_G E) = (p_0^\ast)^{-1}(\delta(a_F)) = j^* ((p_0^\ast)^{-1}(\delta(a_E))) = j^* \circ \delta^* ((p_0^\ast)^{-1}(P_2^\ast - \Phi^*(a_E))) = \delta(\partial \Phi^*(a_E)).$$

\square

1.2. Fibrewise symmetric joins. Let $f : X \to Y$ be a morphism of topological spaces.

Definition 1.2.1. The fibrewise symmetric join $X \ast_Y X$ of the space $X$ with itself over the space $Y$ with respect to the morphism $f$ is the homotopy colimit of the following diagram

$$X \times_Y X \xrightarrow{p_1, p_2} X.$$

Here the triple $(X \times_Y X, p_1, p_2)$ is defined as the homotopy pullback of the diagram below:

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{j} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

We denote the quotient map $X \to X \ast_Y X$ by $q_f$ (or just $q$ when it is clear which morphism $f$ is meant).

Example 1.2.3. If $Y$ is a one-point space, then the fibrewise symmetric join $X \ast_{pt} X$ is homotopy equivalent to $X \ast_{pt} X \cong S^2(X) := X \times X \times I / (x_0, y_0, 0) \sim (x_1, y, 0), (x, y_0, 1) \sim (x, y_1, 1), (x, y_0, 0) \sim (y_1, x, 1))$ for all $x_0, x_1, y_0, y_1, x, y \in X$.

Example 1.2.4. If $X$ is a one-point space and $Y$ is path connected, then the space $X \ast_Y X$ is homotopy equivalent to the space $S^1 \vee \Sigma Y \cong \Sigma(\Omega Y \vee)$.
By the definition of a homotopy coequaliser, for any topological space \( Z \) there is a natural bijection between the set of homotopy classes of maps \([X *_{Y} X, Z]\) and pairs \((g, H)\), where \( g \in [X, Z] \) and \( H \) is a homotopy between \( g \circ p_{1} \) and \( g \circ p_{2} \).

**Notation 1.2.5.** Let denote by \( f \ast f \in [X *_{Y} X, Y] \) the morphism which corresponds to the pair \((f, H)\), where \( H \) is the canonical homotopy given by diagram \[1.2.2\].

In particular, the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{q_{f}} & X *_{Y} X \\
\downarrow{f} & & \downarrow{f \ast f} \\
\ast & & Y
\end{array}
\]

(1.2.6)

**Example 1.2.7.** In the case of Example \[1.2.4\] the natural map \( f \ast f : S^{1} \vee \Sigma \Omega X \to X \) is homotopic to the composition of the pinch map \( S^{1} \vee \Sigma \Omega X \to \Sigma \Omega X \) and the usual counit map \( \Sigma \Omega X \to X \), \((t, \gamma) \mapsto \gamma(t)\), where \( t \in S^{1} \) and \( \gamma \in \Omega X \).

**Remark 1.2.8.** Suppose that the morphism \( f : X \to Y \) is a Serre fibration. Then the homotopy fibre of the map \( f : X *_{Y} X \to Y \) over a point \( y \in Y \) is homotopy equivalent to the symmetric join of the fibre of the map \( f \) over the point \( y \).

**Proposition 1.2.9.** Suppose that the following diagram is homotopy commutative

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{F} & & \downarrow{H} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

Then there exists a map \( F \ast G : X *_{Y} X \to X' *_{Y'} X' \) such that the diagram

\[
\begin{array}{ccc}
X *_{Y} X & \xrightarrow{f \ast g} & Y \\
\downarrow{F \ast G} & & \downarrow{H} \\
X' *_{Y'} X' & \xrightarrow{f' \ast g'} & Y'
\end{array}
\]

is homotopy commutative.

**Proof.** Straightforward \( \square \)

As usual we may associate the long exact sequence of cohomology group with coequaliser diagram \[1.2.1\]

\[
\ldots \to H^{*-1}(X, R) \xrightarrow{p_{2}^{*}-p_{1}^{*}} H^{*-1}(X \times_{Y} X, R) \xrightarrow{\delta} H^{*}(X *_{Y} X, R) \xrightarrow{q_{f}^{*}} \\
\to H^{*}(X, R) \xrightarrow{p_{2}^{*}-p_{1}^{*}} H^{*}(X \times_{Y} X, R) \xrightarrow{\delta} H^{*+1}(X *_{Y} X, R) \to \ldots
\]

(1.2.10)

**Proposition 1.2.11.** Let \( X \) and \( Y \) be connected topological spaces and let \( f : X \to Y \) be a morphism between them. Suppose that \( \ker(f^{*}) \subseteq I \subset H^{*}(X, R) \). Then the kernel of the map \((f \ast f)^{*} : H^{*}(X, R) \to H^{*}(Y *_{X} Z, R)\)

contains \( I^{2} \).

**Proof.** Let \( x, y \) be elements of the ideal \( I \). We need to show that \( x \sim y \in \ker(f \ast f)^{*} \). Since the morphism \((f \ast f)^{*} \) is a ring homomorphism it is enough to show that

\[
(f \ast f)^{*}(x) \sim (f \ast f)^{*}(y) = 0.
\]

Let us show that the element \((f \ast f)^{*}(x)\) belongs to the image of the boundary homomorphism \( \delta \) in exact sequence \[1.2.10\] By exactness, it suffices to show that \( q_{f}^{*}(f \ast g)^{*}(x) = 0 \). But

\[
q_{f}^{*}(f \ast g)^{*}(x) = f(x) = 0 \in H^{*}(X, R).
\]

So the element \((f \ast g)^{*}(x) \in \text{Im}(\delta)\). By the same argument \((f \ast g)^{*}(y) \in \text{Im}(\delta)\). Since the product of any two elements in the image of the homomorphism \( \delta \) is zero, the claim is proven. \( \square \)
Remark 1.2.12. The inclusion $I^2 \subseteq \ker(f * f)^*$ may be strict, \[9\].

Let $G$ be a topological group and $X$ be a topological $G$-space. Recall that $EG$ is a contractible topological space with a continuous free $G$-action.

Notation 1.2.13. Denote by $X_{hG}$ the homotopy quotient of $X$ by the group $G$. By the definition, the space $X_{hG}$ is the quotient $(X \times EG)/G$. Therefore, there exists the map $\alpha: X \to X_{hG}$ and the homotopy fibre of the map $\alpha$ is $G$.

Proposition 1.2.14. For any topological $G$-space $X$ there exists a natural homotopy equivalence

$$\kappa: X *_G X \xrightarrow{\sim} X *_{X_{hG}} X.$$

Here the fibrewise join is taken with respect to the map $\alpha: X \to X_{hG}$.

Proof. Since we are looking for a homotopy equivalence we can replace $X$ by $X \times EG$. So we can assume that the group $G$ acts freely on $X$. In this case the homotopy quotient $X_{hG}$ is homotopy equivalent to the classical group quotient $X/G$.

In order to construct the homotopy equivalence $\kappa$ it is enough to construct a homotopy equivalence $\mu: G \times X \to X \times_{X/G} X$ such that following diagrams are commutative

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\mu} & X \times_{X/G} X \\
| & | & | \\
\downarrow{p_2} & \xrightarrow{\text{id}} & \downarrow{p_2} \\
X & \xrightarrow{\text{id}} & X
\end{array}
\]

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\mu} & X \times_{X/G} X \\
| & | & | \\
\downarrow{\text{Act}} & \xrightarrow{\text{id}} & \downarrow{p_1} \\
X & \xrightarrow{\text{id}} & X
\end{array}
\]

Since the $G$-action is now assumed free, the quotient map $\alpha: X \to X/G$ is a Serre fibration. Therefore the homotopy pullback $X \times_{X/G} X$ is the usual pullback:

$$X \times_{X/G} X = \{(x, y) \in X \times X \mid \alpha(x) = \alpha(y)\}.$$ 

Define the map $\mu: G \times X \to X \times X$ by the rule $\mu(g, x) = (gx, x) \in X \times X$. Obviously, the image of the map $\mu$ is equal to $X \times_{X/G} X$ and diagrams (1.2.15) are commutative. So it is enough to prove that the map $\mu$ is a topological embedding.

\[\square\]

1.3. Leray-Serre spectral sequences. Let $p: E \to B$ be a Serre fibration, where $B$ is a simply connected topological space. Let $i: F \to E$ be an inclusion of the fibre of the morphism $p$. Suppose that the fibre $F$ is a connected topological space. Recall that in this case the cohomology group $H^q(E, R)$ has a descending Serre filtration $F^*H^q(E, R)$ for any $q$. Moreover, for any choice of a Serre filtration $F^qH^q(E, R) = \ker(i^*: H^q(E, R) \to H^q(F, R))$.

Notation 1.3.1. For convenient we will use the alternative notation for $F^1H^*(E, R)$. Let us denote by $I^*(p, R)$ the kernel of the natural map $i^*: H^*(E, R) \to H^*(F, R)$.

Notation 1.3.2. Denote by $I^1_1(p, R)$ the ideal of the ring $H^*(E, R)$ generated by the image of $p^*(\check{H}^*(B, R))$.

It is straightforward that the ideal $I_1^1(p, R)$ is contained in the ideal $I^1(p, R)$. But under mild assumptions on the spaces $F$ and $B$ we will see that the same kind of the opposite inclusion is also true.

Proposition 1.3.3. Suppose that the ring $R$ is a field. If the cohomology ring $H^*(B, R)$ is an integer domain, then the ideal $I^*(p, R)$ is contained in the ideal $I^1_1(p, R)$.

Proof. Let $x$ be an element of $I^q(p, R) = F^1H^q(E, R)$. We need to prove the claim that

$$x \in I^1_1(p, R).$$

Suppose that $x \in F^qH^q(E, R)$. Let us prove assertion (1.3.4) by descending induction.
If \( i = q \), then the group \( F^qH^q(E, R) \) coincides with the group \( \text{Im}(p^*: H^q(B, R) \to H^q(E, R)) \). So assertion [1.3.4] is true. Assume that assertion [1.3.4] is true for \( i = j + 1 \) and prove it for \( i = j \). It suffices to construct an element \( y \in I^q(p, R) \) such that \( x - y \in F^{j+1}H^q(E, R) \).

Let \( E^{q,*} \) be the cohomological Leray-Serre spectral sequence for the fibration \( p: E \to B \). Recall that the second page of the spectral sequence \( E^{2,1}_2 \) is equal to \( H^k(B, H^1(F, R)) \) and the infinite page \( E^{k,l}_\infty \) is equal to

\[
E^{k,l}_\infty = F^kH^{k+l}(E, R)/F^{k+1}H^{k+l}(E, R).
\]

Denote by \( \bar{x} \in F^jH^q(E, R)/F^{j+1}H^q(E, R) \) the image of the element \( x \in F^jH^q(E, R) \) in the quotient. Since the infinite page \( E^{q,*}_\infty \) is a subquotient of the second page \( E^{q,*}_2 \), there exists an element \( \bar{x} \in E^{j,q-j}_2 \) such that the image of the element \( \bar{x} \) in \( E^{q,*}_\infty \) is the element \( \bar{x} \).

Since \( R \) is a field, there exists the isomorphism \( H^k(B, H^1(F, R)) \cong H^k(B, R) \otimes H^1(F, R) \). Therefore the element \( \bar{x} \) can be decomposed in the following sum:

\[
\bar{x} = \sum_{i \in J} v_i \times u_i.
\]

Here \( J \) is a finite set, \( v_i \in E^{0,q-j}_2 = H^{q-j}(F, R), u_i \in E^{j,0}_2 = H^j(B, R) \), and \( \times : E^{2,1}_2 \otimes E^{k,l}_2 \to E^{k+l,1}_2 \) is the multiplication in the Serre spectral sequence.

We may assume that all summands in decomposition (1.3.5) persist to the infinite page. We claim that the elements \( v_i \) and \( u_i, i \in J \), also persist to \( E^{q,*}_\infty \). Let us prove by induction that the differentials \( d_s(v_i), i \in J \), are all zeros for any \( 2 \leq s \leq q \). Suppose that \( d_j(v_i) = 0 \) for \( j \leq s \leq q \), and there exists an element \( c \in E^{q,*}\infty \) such that

\[
d_{s+1}(v_i) = c,
\]

where \( c \in E^{q+1,j-q-j}_\infty \) and \( c \neq 0 \). We claim that in this case the element \( v_i \times u_i \) does not persist. Indeed, by the Leibniz rule

\[
d_{s+1}(v_i \times u_i) = \pm c \times u_i.
\]

Therefore,

\[
d_{s+1}(v_i \times u_i) = \sum c'_i \times c''_i \times u_i.
\]

Here \( c'_i \in E^{0,q-j-s}_2 = H^{q-j-s}(F, R), c''_i \in E^{s+1,0}_2 = H^{s+1}(B, R) \). The differential \( d_{s+1}(v_i \times u_i) \) is zero if and only if all products \( c'_i \times u_i \) are zeros. But we assumed that the cohomology ring \( H^*(B, R) \) is an integer domain.

So the element \( v_i \) is a permanent cycle. Let us prove that \( u_i, i \in J \), is also a permanent cycle. Assume the converse. Then there exists an integer \( s \) and \( c \in E^{q-j-s-1}_\infty \) such that \( d_s(c) = u_i \). But by the Leibniz rule \( d_s(v_i \times c) = v_i \times u_i \). Contradiction.

Now let us construct the element \( y \in I^q_1(p, R) \) such that \( x - y \in F^{j+1}H^q(E, R) \). Let \( [v_i] \) be some lifting of the element \( v_i \in E^{0,q-j}_2 \) to the group \( H^{q-j}(E, R), i \in J \). Similarly, let \( [u_i] \) be the image of the element \( u_i \in E^{j,0}_2 \) in the group \( H^j(E, R), i \in J \). Define:

\[
y = \sum_{i \in J} [u_i] \sim [v_i].
\]

By the construction \( y \in I^q_1(p, R), y \in F^{j}H^q(E, R) \) and the image of the element \( y \) in the quotient \( E^{q,j-q-j}_\infty \) is equal to the element \( \bar{x} \). \( \square \)

**Corollary 1.3.6.** Suppose that the ring \( R \) is a field. If the cohomology ring \( H^*(B, R) \) is an integer domain, then the ideals \( I^q_1(p, R) \) and \( I^q(p, R) \) coincide.

**Corollary 1.3.7.** Suppose that the ring \( R \) is a principal ideal domain and the topological space \( E \) is a finite type. If the cohomology ring \( H^*_B(B, R) \) is an integer domain, then for any integer \( q \) there exists an element \( r_q \in R \) such that

\[
r_qI^q(p, R) \subset I^q_1(p, R).
\]
Proposition 1.4.6. Let \( G \) be a connected topological group and let \( X \) be a connected topological \( G \)-space. Consider the fibre sequence

\[
X \xrightarrow{\alpha} X_{hG} \xrightarrow{\delta} BG.
\]

**Remark 1.3.8.** The elements \( r_q \in R \) can be expressed in terms of \( R \)-torsion parts of \( R \)-modules \( H^*(B, R) \) and \( H^*(F, R) \). Moreover, if the fibre \( F \) is a finite CW-complex, then there exists \( r \in R \) such that

\[
rI^*(p, R) \subset I^*_r(p, R).
\]

1.4. **Secondary equivariant Euler class II.** Let \( G \) be a connected topological group and let \( X \) be a connected topological \( G \)-space.\n
**Notation 1.4.2.** Denote by \( I^*(X, R) \) the ideal \( I^*(\beta, R) \). By Definition 1.3.4, it is the kernel of the natural map \( \alpha^* : H^*(X_{hG}, R) \to H^*(X, R) \).

Now we would like to define one of the main object of our study. So first, let us recall notation from previous sections:

\[
\bullet \quad M_X^* \subset H^*(G \times X, R) \text{ denotes the indeterminacy submodule (see Definition 1.2.17).}
\]

\[
\bullet \quad \delta : H^{*-1}(G \times X, R) \to H^*(X \ast_G X, R) \text{ is the map from exact sequence 1.2.10.}
\]

\[
\bullet \quad (\alpha \ast \alpha)^* : H^*(X_{hG}, R) \to H^*(X \ast_{hG} X, R) \text{ is the map defined in Notation 1.2.20.}
\]

\[
\bullet \quad \kappa \text{ is the homotopy equivalence from Proposition 1.2.14.}
\]

**Definition 1.4.3.** Define the homomorphism

\[
S : I^*(X, R) \to H^{*-1}(G \times X, R)/M^*_X
\]

by the following rule. Let \( x \) be an element of \( I^*(X, R) \subset H^*(X, R) \). Then \( S(x) \) is an element of the quotient \( H^{*-1}(G \times X, R)/M^*_X \) such that:

\[
\delta(S(x)) = \kappa^*((\alpha \ast \alpha)^*(x)).
\]

Let us fix the structure of a \( H^*(X_{hG}, R) \)-module on the graded abelian group \( H^*(G \times X, R) \) induced by the composition \( G \times X \xrightarrow{p} X \xrightarrow{\alpha} X_{hG} \). In addition, notice that the graded abelian group \( I^*(X, R) \) is an ideal of the ring \( H^*(X_{hG}, R) \).

**Lemma 1.4.4.** The homomorphism \( S \) is a well-defined homomorphism of \( H^*(X_{hG}, R) \)-modules. Moreover, it defines the natural transformation of functors from the category \( G \)-Sp of topological \( G \)-spaces with equivariant maps to the category of graded abelian groups \( \text{GrAb} \).

**Proof.** Show that the map \( S \) is well-defined. Since the map \( \delta \) induces an injective homomorphism between the groups \( H^{*-1}(G \times X, R)/M^*_X \) and \( H^*(X \ast_G X, R) \), it is enough to check that for any \( x \in I^*(X, R) \) the element \( \kappa^*(\alpha \ast \alpha)^*(x) \) belongs to the image of the map \( \delta \). By exact sequence 1.2.10 it enough to show that

\[
q^*_X(\kappa^*(\alpha \ast \alpha)^*(x)) = 0.
\]

Here \( q_X : X \to X \ast_G X \) is the map for Definition 1.1.19. Since \( x \in I^*(X, R) \) we obtain that

\[
q^*_X(\kappa^*(\alpha \ast \alpha)^*(x)) = \alpha^*(x) = 0.
\]

Since all ingredients in the definition of the map \( S \) are homomorphisms of \( H^*(X_{hG}, R) \)-modules and natural transformations of functors \( G \)-Sp \( \to \text{GrAb} \), the map \( S \) itself is also an \( H^*(X_{hG}, R) \)-homomorphism and a natural transformation.

**Definition 1.4.5.** Let \( E \) be a \( G \)-equivariant oriented vector bundle over a topological \( G \)-space \( X \) and suppose that \( G \)-action preserves the orientation. Then the homotopy quotient \( (\text{Tot}E)_{hG} \) naturally maps to the quotient \( X_{hG} \) and this map is a vector bundle. We define the **equivariant Euler class** \( e_G(E) \) of the vector bundle \( E \) by the following formula:

\[
e_G(E) = e(E_{hG}) \in H^*(X_{hG}, \mathbb{Z}).
\]

**Proposition 1.4.6.** Let \( E \) be a \( G \)-equivariant \( R \)-oriented vector bundle of real rank \( r \) over \( X \) such that \( e(E) = 0 \). Denote by \( e_G(E) \in H^*(X_{hG}, R) \) the equivariant Euler class of \( E \). Then \( e_G(E) \in \Gamma^*(X) \) and for any secondary equivariant Euler class \( se_G(E) \) the following equality holds up to the (big) indeterminacy submodule:

\[
se_G(E) = S(e_G(E)).
\]
Proof. We need to show that $\delta(S(e_G(E))) = e(E * G E)$, where $E * G E$ is the vector bundle defined in Notation 1.4.3. By the very definition of the map $S$ it is enough to prove that

$$\kappa^*(\alpha * \alpha)^*(e_G(E)) = e(E * G E).$$

Since the Euler class is functorial it is enough to construct the isomorphism of vector bundles over the space $X * G X$:

$$\kappa^* \circ (\alpha * \alpha)^*(E_{BG}) \cong E * G E.$$ 

But the existence of a such isomorphism is clear from constructions. \qed

Notation 1.4.7. Denote by $I_1^*(X, R)$ the ideal $I_1(\beta, R)$ of the ring $H^*(X_{BG}; R)$. By Definition 1.3.2 the ideal $I_1^*(X, R)$ is generated by the image of the abelian group $\tilde{H}^*(BG, R)$ under the homomorphism $\beta^*$.

Let us also define the following homomorphism

$$k: \tilde{H}^*(BG, R) \otimes H^*(X_{BG}, R) \to \tilde{H}^*(hG_{BG}, R),$$

$$x \otimes y \mapsto \beta^*(x) \bowtie y.$$  

Notice that the image of the map $k$ is precisely the ideal $I_1^*(X, R)$.

Proposition 1.4.8. Suppose that the coefficient ring $R$ is a principal ideal ring of characteristic zero and suppose that the topological group $G$ is homotopy equivalent to a finite CW-complex as well as the topological $G$-space $X$. Then there exists an element $r \in R$ such that

$$rI^*(X, R) \subset I_1^*(X, R).$$

Proof. Denote by $Q = \text{Quot}(R)$ the quotient field of the ring $R$. The field $Q$ is a field of characteristic zero. By Corollary 1.3.7 and Remark 1.3.8 it is sufficient to prove that the cohomology ring $H^*(BG, Q)$ is an integer domain. But this ring is the Koszul dual to the ring $H^*(G, Q)$. By the Milnor-Moore theorem the ring $H^*(G, Q)$ is isomorphic to an exterior algebra. So the cohomology ring $H^*(BG, Q)$ is isomorphic to a symmetric algebra. Hence it is an integer domain. \qed

Notation 1.4.9. Let $\gamma: \Sigma G \to BG$ be the adjoint morphism to the homotopy equivalence $G \xrightarrow{\sim} \Omega BG$. With an abuse of notation we denote the induced homomorphism

$$H^*(BG, R) \xrightarrow{\gamma^*} H^*(\Sigma G, R) \xrightarrow{\sim} H^{*, -1}(G, R)$$

by the same symbol $\gamma^*$. Then one can define the homomorphism

$$\tilde{S}: \tilde{H}^p(BG, R) \otimes H^q(X_{BG}, R) \to \tilde{H}^{p+q-1}(G \times X, R),$$

$$x \otimes y \mapsto \gamma^*(x) \boxtimes \alpha^*(y).$$

Proposition 1.4.10. The following diagram is commutative:

$$\begin{CD}
\tilde{H}^*(BG, R) \otimes H^*(X_{BG}, R) @>{\tilde{S}}>> H^{*-1}(G \times X, R) \\
@V{k}VV @VV{S}V \\
I^*(X, R) @>{S}>> H^{*-1}(G \times X, R)/M_X.
\end{CD}$$

Proof. Since all homomorphisms in the diagram are homomorphisms of $H^*(X_{BG}, R)$-modules, it is enough to check commutativity of the diagram for each generator of the module $\tilde{H}^*(BG, R) \otimes H^*(X_{BG}, R)$. In other words, it is enough to prove that

$$\tilde{S}(x \otimes 1) = S(k(x \otimes 1)) = S(\beta^*(x))$$

for any $x \in \tilde{H}^*(BG, R)$. By Definition 1.4.3 of the map $S$, it is enough to check that

$$(1.4.11) \quad \delta(\tilde{S}(x \otimes 1)) = \kappa^*(\alpha* \alpha)^*(\beta^*(x))$$

for any $x \in \tilde{H}^*(BG, R)$. 

Let us apply Proposition 1.2.9 to the following homotopy commutative diagram:

\[ X \xrightarrow{\alpha} pt \]
\[ \xrightarrow{\beta} X_{hG} \xrightarrow{\epsilon} BG. \]

As the result we obtain the homotopy commutative square:

\[ X \xrightarrow{\epsilon} pt \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG \]
\[ \xrightarrow{\beta} X_{hG} \xrightarrow{\epsilon} pt \]
\[ \xrightarrow{\beta * BG pt} \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG. \]

By Example 1.2.4 the space \( pt * BG pt \) is homotopy equivalent to the space \( S^1 \vee \Sigma \Omega BG \cong S^1 \vee \Sigma G \) and the composition \( \Sigma G \hookrightarrow pt * BG pt \to BG \) is the map \( \gamma \) defined in Notation 1.4.9. Denote by \( \tilde{\epsilon} : X *_{hG} X \to \Sigma G \) the composition of the map \( \epsilon \) and the pinch map \( S^1 \vee \Sigma G \to \Sigma G \). As a result we obtain the commutative diagram.

\[ X \xrightarrow{\epsilon} pt \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG \]
\[ \xrightarrow{\beta} X_{hG} \xrightarrow{\epsilon} pt \]
\[ \xrightarrow{\beta * BG pt} \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG. \]

Let \( \Delta : X *_{G} X \to \Sigma(G \times X) \) be the connecting morphism in the Puppe sequence for the coequaliser diagram 1.1.19 and let \( p_1 : G \times X \to G \) be the projection map. Observe that the induced homomorphism

\[ H^{* -1}(G \times X, R) \xrightarrow{\Delta^*} H^*(\Sigma(G \times X), R) \xrightarrow{\Delta^*} H^*(X *_{G} X, R) \]

coincides with the connecting homomorphism \( \delta \) in exact sequence 1.2.4. Furthermore, notice that the following diagram commutes

\[ X *_{G} X \xrightarrow{\epsilon \Delta} \Sigma G \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG \]
\[ \xrightarrow{\beta} X_{hG} \xrightarrow{\epsilon \Delta} \Sigma G \]
\[ \xrightarrow{\alpha * \alpha} X_{hG} \xrightarrow{\beta} BG. \]

The commutativity of the last diagram proves required equality 1.4.11. Indeed,

\[ \delta(\tilde{S}(x \otimes 1)) = \delta(\gamma^*(x) \otimes 1) = \Delta^*((\Sigma p_1)^* \circ \gamma^*(x)) = (\kappa^* \circ \tilde{\epsilon}^*)(\gamma^* x) = \kappa^* (\alpha * \alpha^*)(\beta^*(x)). \]

□

**Notation 1.4.12.** Let \( S_1 : I^*_1(X, R) \to H^{* -1}(G \times X, R)/M^{* -1}_X \) be the homomorphism induced by the homomorphism \( \tilde{S} \).

**Remark 1.4.13.** The homomorphism \( S_1 \) is very explicit. Indeed, if \( x \in I^*_1(X, R) \), then there exists the decomposition \( x = \sum a_i\beta^*(b_i) \), where \( a_i \in H^*(X_{hG}, R) \) and \( b_i \in H^*(BG, R) \). Then

\[ S_1(x) = \sum \gamma^*(b_i) \otimes \alpha^*(a_i). \]

By the only definition, the homomorphism \( S : I^*(X, R) \to H^{* -1}(G \times X, R)/M^{* -1} \) is quite mysterious for computations, but Proposition 1.4.10 says that its restriction to the ideal \( I^*_1(X, R) \) is equal to the homomorphism \( S_1 \). As a result, the homomorphism \( S|_{I^*_2} \) is computable. Moreover, by Proposition 1.4.8 the ideal \( I^*_1 \) is very close to the ideal \( I^* \). In the next section we will show that in some cases the homomorphism \( S \) itself is also computable.
Now we would like to discuss some functorial properties of the constructed homomorphism $S$. As we already see in Lemma 1.4.4, the homomorphism $S: I^*(X, R) \rightarrow H^{*-1}(G \times X, R)/M^*_{X^{-1}}$ is natural with respect to $G$-equivariant morphisms of topological spaces.

In the other words, for any $G$-equivariant morphism $f: X \rightarrow Y$ the following diagram commutes.

$$
\begin{array}{ccc}
I^*(Y, R) & \xrightarrow{S} & H^{*-1}(G \times Y, R)/M^*_{Y^{-1}} \\
\downarrow f^* & & \downarrow (\text{id} \times f)^* \\
I^*(X, R) & \xrightarrow{S} & H^{*-1}(G \times X, R)/M^*_{X^{-1}} 
\end{array}
$$

(1.4.14)

As a corollary of Proposition 1.4.10 we observe that the homomorphism $S_1$ is also natural, i.e. the following diagram is commutative.

$$
\begin{array}{ccc}
I^*_1(Y, R) & \xrightarrow{S_1} & H^{*-1}(G \times Y, R)/M^*_{Y^{-1}} \\
\downarrow f^* & & \downarrow (\text{id} \times f)^* \\
I^*_1(X, R) & \xrightarrow{S_1} & H^{*-1}(G \times X, R)/M^*_{X^{-1}} 
\end{array}
$$

(1.4.15)

Suppose that the group $G$ is a Lie group, the $G$-spaces $X$ and $Y$ are $R$-oriented manifolds and $G$-action preserves orientations. Then there exists the push-forward map:

$$f_1: H^*(X, R) \rightarrow H^*(Y, R).$$

Moreover, there exists the push-forward map in the equivariant cohomology theory:

$$f_{G,1}: H^*(BG, R) \rightarrow H^*(YG, R).$$

Since push-forward homomorphisms are homomorphisms of $H^*(BG, R)$-modules and compatible, $\alpha_Y \circ f_{G,1} = f_1 \circ \alpha_X$, we observe following inclusions:

$$f_{G,1}(I^*(X, R)) \subset I^*(Y, R),$$

$$f_{G,1}(I^*_1(X, R)) \subset I^*_1(Y, R).$$

**Proposition 1.4.16.** Let $G$ be a connected Lie group and let $f: X \rightarrow Y$ be a map of $G$-manifolds.

1. $f_1(M_X) \subset M_Y$
2. Following diagrams are commutative:

$$
\begin{array}{ccc}
I^*(X, R) & \xrightarrow{S} & H^{*-1}(G \times X, R)/M^*_{X^{-1}} \\
\downarrow f_{G,1} & & \downarrow (\text{id} \times f)_1 \\
I^*_1(X, R) & \xrightarrow{S_1} & H^{*-1}(G \times X, R)/M^*_{X^{-1}} 
\end{array}
$$

$$
\begin{array}{ccc}
I^*(Y, R) & \xrightarrow{S} & H^{*-1}(G \times Y, R)/M^*_{Y^{-1}} \\
\downarrow f_{G,1} & & \downarrow (\text{id} \times f)_1 \\
I^*_1(Y, R) & \xrightarrow{S_1} & H^{*-1}(G \times Y, R)/M^*_{Y^{-1}} 
\end{array}
$$

**Proof.** Straightforward from definitions. \qed

### 1.5. Smooth projective varieties

In the present section we consider more closely the case of smooth projective varieties. We will show that the homomorphism $S$ can be expressed in terms of the homomorphism $S_1$.

**Lemma 1.5.1.** Let $G$ be a complex linear algebraic group and let $X$ be a complex smooth projective $G$-variety. Denote by $\varphi: G \times X \rightarrow X$ the $G$-action morphism. Then homomorphisms $\varphi^*, p^*_{2}: H^*(X, \mathbb{Q}) \rightarrow H^*(G \times X, \mathbb{Q})$ coincide.

**Proof.** Recall that cohomology groups $H^i(X, \mathbb{Q})$ and $H^i(G \times X, \mathbb{Q})$ are mixed Hodge structures. Since $X$ is a smooth projective variety the Hodge structure on cohomology group $H^i(X, \mathbb{Q})$ is pure of weight $i$. If the group $G$ is non-trivial, then Hodge structure on $H^i(G \times X, \mathbb{Q})$ is impure of weights greater or equal $i$. Moreover, the $i$-th component of weight filtration $W_iH^i(G \times X, \mathbb{Q})$ is equal $p^*_2(H^i(X, \mathbb{Q}))$.

Since $\text{Act}: G \times X \rightarrow X$ is a morphism of complex (algebraic) varieties, the homomorphism $\text{Act}^*$ is strictly compatible with the weight filtration. This means that

$$\varphi^*(H^i(X, \mathbb{Q})) = \varphi^*(W_iH^i(X, \mathbb{Q})) \subset W_iH^i(G \times X, \mathbb{Q}) = p^*_2(H^i(X, \mathbb{Q})).$$


On the other hand, morphisms $\varphi$, $p_2: G \times X \to X$ have a common section $s: X \to G \times X$, $x \mapsto (e, x)$. Therefore if $\varphi^*(x) = p_2^*(y)$, then $x = y$. So the homomorphisms $\varphi^*$ and $p_2^*$ coincide. \hfill \Box

**Corollary 1.5.2.** Let $G$ be a complex linear algebraic group and let $X$ be a complex smooth projective $G$-variety. Then the indeterminacy submodule $M^*_X$ consists of torsion elements. In other words, the following inclusion holds

$$M^*_X \subset \text{Tors}(H^*(G \times X, \mathbb{Z})) \subset H^*(G \times X, \mathbb{Z}).$$

**Proof.** Recall that the indeterminacy submodule $M^* \otimes \mathbb{Q}$ is the image of the homomorphism $\varphi^* - p_2^*: H^*(X, \mathbb{Q}) \to H^*(G \times X, \mathbb{Q})$.

By Lemma 1.5.1 the homomorphism $\varphi^*$ is equal to $p_2^*$. Hence the group $M^* \otimes \mathbb{Q}$ is zero. \hfill \Box

**Notation 1.5.3.** Define two maps $S'$ and $S'_1$ as following compositions:

$$S': I^*(X, \mathbb{Z}) \xrightarrow{S} H^{*-1}(G \times X, \mathbb{Z})/M^{*-1} \to H^{*-1}_fr(G \times X),$$

$$S'_1: I^*_1(X, \mathbb{Z}) \xrightarrow{S'_1} H^{*-1}(G \times X, \mathbb{Z})/M^{*-1} \to H^{*-1}_fr(G \times X).$$

By Proposition 1.4.8 the quotient $I^*(X, \mathbb{Z})/I^*_1(X, \mathbb{Z})$ is finite, so let us denote by $N$ the order of the abelian group $I^*(X, \mathbb{Z})/I^*_1(X, \mathbb{Z})$.

**Corollary 1.5.4.** For any element $x \in I^*(X, \mathbb{Z})$ the following identity holds in $H^{*-1}_fr(G \times X)$

$$S'(x) = \frac{S'_1(Nx)}{N}.$$

**Corollary 1.5.5.** Let $G$ be a connected complex linear Lie group, let $X$ be a complex smooth projective $G$-variety and let $E$ be a $G$-equivariant $R$-oriented vector bundle over $X$ such that $e(E) = 0$. Then for any secondary equivariant Euler class $se_G(E)$ the following identity holds in $H^{*-1}_fr(G \times X)$

$$se_G(E) = S'(e_G(E)) = \frac{S'_1(Ne_G(E))}{N}.

**Proof.** Straightforward follows from Proposition 1.4.6 and Corollary 1.5.4. \hfill \Box

**Remark 1.5.6.** Recall that if $X$ is a one-point space and the group $G$ as above, then the homomorphism $S$ has a form:

$$S: \hat{H}^*(BG, \mathbb{Z}) \to H^{*-1}(G, \mathbb{Z}).$$

By Proposition 1.4.10 the homomorphism $S$ coincides with the homomorphism $\gamma^*$ (Notation 1.4.9). Well-known, the homomorphism $\gamma^*$ provides the isomorphism of graded vector spaces:

$$Q\hat{H}^*(BG, \mathbb{Q}) \to \text{Prim} H^*(G, \mathbb{Q})[1].$$

Here $QA$ is the module of indecomposable of a ring $A$ and $\text{Prim} H$ is the module of primitive elements of a Hopf algebra $H$.

We claim that if $G$ is a connected complex linear algebraic group and $X$ is a connected complex smooth projective $G$-variety, then the homomorphism

$$S: I^*(X, \mathbb{Z}) \to H^{*-1}(G \times X, \mathbb{Z})/M^{*-1}$$

induces the similar isomorphism. Let us briefly explain a such generalisation.

Indeed, by Proposition 1.2.11 ker$(S) \supset (I^*(X, \mathbb{Z}))^2$. By Corollary 1.5.2 the indeterminacy submodule $M^*_X \otimes \mathbb{Q}$ is zero. Moreover, by Proposition 1.4.10 we obtain the equality:

$$\text{Im}(S \otimes \mathbb{Q}) = \text{Im}(S_1 \otimes \mathbb{Q}) = (\text{Prim} H^*(G, \mathbb{Q})) \otimes H^*(X, \mathbb{Q}).$$

Here we use the smoothness and projectiveness assumptions. More precisely, we use that the restriction homomorphism $H^*(X_{hG}, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is surjective by P. Deligne [3].

Finally, the homomorphism $S$ induces the isomorphism of graded vector spaces:

$$I^*(X, \mathbb{Q})/(I^*(X, \mathbb{Q}))^2 \cong (\text{Prim} H^*(G, \mathbb{Q})) \otimes H^*(X, \mathbb{Q})[1].$$
2. Orbit map

This section is the central section of the paper. Given a complex projective $G$-variety $X$ with $G$-equivariant holomorphic vector bundle, here we define cohomology classes $\text{Lk}^{Y}_E \in H^*(\Gamma_{\text{reg}}(X, E), \mathbb{Z})$, connect them to the secondary Thom class of $E$ and finally describe how to compute $O^*(\text{Lk}^{Y}_E) \in H^*(G, \mathbb{Z})$. After, we apply this technique for computation $O^*(\text{Lk}^{Y}_{\text{Sing}}(L)) \in H^*(G, \mathbb{Z})$, where $L$ is a holomorphic $G$-equivariant vector bundle over $X$.

2.1. Linking classes and secondary Thom class.

Notation 2.1.1. Let $X$ be a complex projective variety and let $E$ be a holomorphic vector bundle over $X$. Consider the following subset of holomorphic global sections $\Gamma(X, E)$:

$$\Sigma(E) = \{ s \in \Gamma(X, E) \mid \exists x \in X, \text{ s.t } s(x) = 0 \}.$$

Let $\Gamma(X, E)_0$ be the complement of $\Sigma(E)$ in $\Gamma(X, E)$.

Let $Y$ be a subset of $X$. Then we can consider the following subset of $\Sigma(E)$:

$$\Sigma_Y(E) = \{ s \in \Gamma(X, E) \mid \exists x \in Y, \text{ s.t } s(x) = 0 \}.$$  

Lemma 2.1.2. Suppose that $E$ is a globally generated vector bundle and $Y$ is an irreducible closed subvariety of $X$. Then the set $\Sigma_Y(E)$ is also a closed irreducible subvariety of $\Gamma(X, E)$.

Proof. Consider the diagram

$$
\begin{array}{ccc}
X \times \Gamma(X, E) & \xrightarrow{Ev} & \text{Tot}(E) \\
\downarrow p_2 & & \\
\Gamma(X, E).
\end{array}
$$

(2.1.3)

Let $\mathcal{O}_Y \subset \text{Tot}(E)$ denote the variety $Y$ embedded in $\text{Tot}(E)$ as the zero section. Then

$$\Sigma_Y(E) = p_2(Ev^{-1}(\mathcal{O}_Y)).$$

Since the vector bundle $E$ is globally generated, the map $Ev: X \times \Gamma(X, E) \rightarrow \text{Tot}(E)$ is a fibre bundle. So the preimage $Ev^{-1}(\mathcal{O}_Y)$ is an irreducible variety. Now the claim follows from the fact that the image of an irreducible variety is irreducible. \hfill \square

Lemma 2.1.4. Let $Y$ be an irreducible closed subvariety of $X$ and let $E$ be a globally generated vector bundle.

1. $\text{codim}_{\Gamma(X, E)}(\Sigma_Y(E)) \geq \text{rk}(E) - \text{dim}(Y)$.
2. The equality in the estimate above takes place if and only if the Chern number $\langle c_{\text{dim}(Y)}(E), [Y] \rangle$ is non-zero.

Proof. First of all, since the vector bundle $E$ is globally generated, the restriction map $\Gamma(X, E) \rightarrow \Gamma(Y, E|_Y)$ is surjective. Hence

$$\text{codim}_{\Gamma(Y, E|_Y)}(\Sigma_Y(E|_Y)) = \text{codim}_{\Gamma(X, E)}(\Sigma_Y(E)).$$

So we may assume that $Y = X$.

1. Let us show the inequality $\text{codim}(\Sigma(E)) \geq \text{rk}(E) - \text{dim}(X)$. Consider diagram 2.1.3 Denote the preimage $ev^{-1}(\mathcal{O}_X)$ by $F \subset X \times \Gamma(X, E)$. The latter variety consists from pairs $(x, s) \in X \times \Gamma(X, E)$ such that $s(x) = 0$. Since the evaluation map is a fibre bundle, the variety $F$ is smooth of codimension $\text{rk}(E)$.

Set $D := \text{dim} \Gamma(X, E)$. Then the dimension of the variety $F$ is $D + \text{dim}(X) - \text{rk}(E)$. Since $\Sigma(E) = p_2(F)$, the dimension of the variety $\Sigma(E)$ is less or equal to $D + \text{dim}(X) - \text{rk}(E)$. In other words, the codimension is greater or equal to $\text{rk}(E) - \text{dim}(X)$.

2. Now suppose that $c_{\text{dim}(X)}(E) \neq 0$. Let us show the converse inequality $\text{codim}(\Sigma(E)) \leq \text{rk}(E) - \text{dim}(X)$. Since the Chern class $c_{\text{dim}(X)}(E)$ is non-zero, the vector bundle $E$ has exactly $\text{rk}(E) - \text{dim}(X)$ pointwise linearly independent global sections. In the other terms, there is an injective morphism $i: \mathcal{O}^{r-n} \rightarrow E$. Here, $r = \text{rk}(E)$ and $n = \text{dim}(X)$.
Denote the the cokernel of the morphism i by $\tilde{E}$. Consider the exact sequence:

$$0 \to \Gamma(X, \mathcal{O}^{-n}) \to \Gamma(X, E) \xrightarrow{i} \Gamma(X, \tilde{E}) \to 0.$$

Let $V$ be the image of the morphism $p$. Then the dimension of $V$ is equal to $D + n - r$. Let us show that $p(\Sigma(E)) = V$. It is enough to show that any section $s \in \Gamma(X, E)$ is a sum of a section $s' \in \Sigma(E)$ and a section $s'' \in \operatorname{Im}(i)$. Assume the converse. Then there exists a section $\tilde{s} \in \Gamma(X, E)$ such that $\tilde{s}$ and $\operatorname{Im}(i)$ are pointwise linearly independent. But it contradicts with $c_{\dim(X)}(E) \neq 0$. The proof of the opposite claim is very similar and we left it for a reader.

**Notation 2.1.5.** If $\Sigma_Y(E)$ is an irreducible closed subvariety of $\Sigma(E)$, then there exists the fundamental class of $\Sigma_Y(E) \in H^{BM}(\Sigma(E))$. By the Alexander duality it defines the cohomology class $Lk^\Sigma_Y(E) \in H^*(\Gamma(X, E); 0)$; see Definition [1.1.10].

**Proposition 2.1.6.** Let $E$ be a globally generated vector bundle over a projective variety $X$ and let $Y$ be an irreducible closed subvariety of $X$ such that $\langle c_{\dim(Y)}(E), [Y] \rangle \neq 0$. Then

1. There exists a constant $d_{Y,E} \in \mathbb{Z}$ such that for any secondary Thom class $a_E \in H^*(\operatorname{Tot}_0 E, \mathbb{Z})$ the following equality holds

$$d_{Y,E} Lk^\Sigma_Y = ev^*(a_E)/[Y].$$

Here $ev: \Gamma(X, E)_0 \times X \to \operatorname{Tot}_0(E)$ is the evaluation map $ev(s, x) = s(x)$.

2. The constant $d_{Y,E}$ is equal to the fraction:

$$\frac{\langle c_{\dim(Y)}(E), [Y] \rangle}{\deg(\Sigma_Y)}. $$

**Proof.** Notice that the following diagram commutes

$$\begin{array}{ccc}
\Gamma(Y, E|_Y)_0 \times Y & \xrightarrow{ev} & \operatorname{Tot}_0(E|_Y) \\
\downarrow & & \downarrow \\
\Gamma(X, E)_0 \times X & \xrightarrow{ev} & \operatorname{Tot}_0(E).
\end{array}$$

Therefore, we can assume that $Y = X$.

(1) Recall that the subvariety $F \subset \Gamma(X, E) \times X$ is the preimage of the zero section $\underline{0}_X \subset \operatorname{Tot}(E)$ under the evaluation map $Ev: \Gamma(X, E) \times X \to \operatorname{Tot}(E)$. Decompose the map $ev: \Gamma(X, E)_0 \times X \to \operatorname{Tot}_0(E)$ by the following way:

$$\Gamma(X, E)_0 \times X \xrightarrow{i} (\Gamma(X, E) \times X) \xrightarrow{j} F \xrightarrow{p} \operatorname{Tot}_0(E).$$

A secondary Thom class $a_E$ can be realised as a linking class of the homology class $[X]$ in the total space $\operatorname{Tot}(E)$ along the zero section (Proposition 1.1.11). Hence by Proposition 1 in [6]

$$ev^*(a_E) = f^*(g^*(Lk_{[X],\underline{0}_X})_i)) = f^*(Lk_{[F],\Gamma(X,E)\times X}) = Lk_{j_*[F],\Sigma(E)\times X,\Gamma(X,E)\times X}.$$

Here $j_*[F]$ is the image of the fundamental class $[F]$ under the embedding $j: F \to \Sigma(E, X)$. Now by Proposition 1 in [6]

$$ev^*(a_E)/[X] = (Lk_{j_*[F],\Sigma(E)\times X,\Gamma(X,E)\times X})/[X] = Lk_{D[X],j_*[F],\Sigma(E)\times X,\Gamma(X,E)} = Lk_{p_1, j_*[F], \Sigma(E)\times X, \Gamma(X, E)}.$$

Here $D[X]$ is the Poincaré dual class to the fundamental class $[X]$, which certainly equals to 1 in $H^0(X, \mathbb{Z})$, and $p_1: \Sigma(E) \times X \to \Sigma(E)$ is the projection map. The homology class $p_1 \cdot j_*[F]$ is equal to $d_{X,E}\Sigma(E)$, where $d_{X,E}$ is the degree of the map $p: F \to \Sigma(E)$.

(2) Let us compute the degree of the map $p: F \to \Sigma$. Recall that by the proof of Lemma 2.1.4, there exists a surjective map $p': \Sigma \to V$, $V$ is a vector space and $\dim(V) = \dim(\Sigma)$ (since $\langle c_{\dim(X)}(E), [X] \rangle \neq 0$). Moreover, the degree of the composition $p' \circ p: F \to V$ is equal to a number of point with multiplicities where general $\operatorname{rk}(E) - \dim(X)$ sections of the vector bundle $E$ are linear dependent. Since the vector bundle $E$ is globally generated, it means that $\deg(p' \circ p) = \langle c_{\dim(X)}(E), [X] \rangle$. Therefore, $\deg(p) = \frac{\langle c_{\dim(Y)}(E), [Y] \rangle}{\deg(\Sigma_Y)}$. □
Let \( G \) be a complex linear algebraic group and let \( X \) be a \( G \)-variety. Denote by \( \varphi: G \times X \to X \) the \( G \)-action. Suppose that \( E \) is a \( G \)-equivariant holomorphic vector bundle over \( X \). Denote by \( \Phi: G \times \text{Tot}(E) \to \text{Tot}(E) \) the \( G \)-action on total space. Then the following diagram is commutative.

\[
\begin{array}{ccc}
G \times \{s_0\} \times X & \xrightarrow{id \times x \times id} & G \times \Gamma(X, E)_0 \times X \\
& \downarrow A & \downarrow \Phi \\
\Gamma(X, E)_0 \times X & \xrightarrow{ev} & \text{Tot}_0(E).
\end{array}
\]

Here the map \( A: G \times \Gamma(X, E)_0 \times X \to G \times X \) given by the rule \((g, s, x) \mapsto (gs, gx)\) and the map \( \iota: \{s_0\} \to \Gamma(X, E)_0 \) is just the embedding of the fixed section.

**Definition 2.1.8.** Fix a section \( s_0 \in \Gamma(X, E)_0 \). The orbit map \( O: G \to \Gamma(X, E)_0 \) is defined by the rule \( O(g) = g(s_0) \).

**Corollary 2.1.9.** Suppose that \( E \) is a globally generated \( G \)-equivariant vector bundle over a projective \( G \)-variety \( X \), \( Y \) is an irreducible closed subvariety of \( X \), \( (\langle c_{\dim(Y)}(E), [Y] \rangle) \neq 0 \). Then for any secondary Thom class \( a_E \in H^*(\text{Tot}_0 E, \mathbb{Z}) \) the following equality holds up to a torsion

\[
d_{Y,E} O^*(\text{Lk}_Y^E) = (ev \circ A \circ \iota)^*(a_E)/[Y].
\]

**Proof.** Notice that the following diagram commutes

\[
\begin{array}{ccc}
G \times \{s_0\} \times X & \xrightarrow{id \times x \times id} & G \times \Gamma(X, E)_0 \times X \\
& \downarrow A & \downarrow \Phi \\
\Gamma(X, E)_0 \times X & \xrightarrow{ev} & \text{Tot}_0(E).
\end{array}
\]

Therefore for any cohomology class \( b \in H^*(\Gamma(X, E)_0 \times X, \mathbb{Z}) \)

\[
((A \circ \iota)^*b)/[Y] = (\varphi \circ \iota)^*b/[Y] = O^*(\varphi^*b/1 \boxplus [Y]) = O^*(b/\varphi_*(1 \boxplus [Y])) = O^*(b/[Y]).
\]

Let us substitute \( ev^*(a_E) \) for \( b \). By Proposition 2.1.4 \( ev^*(a_E)/[Y] = d_{Y,E} \text{Lk}_Y^E \). Consequently, we obtain the following equality

\[
(ev \circ A \circ \iota)^*(a_E)/[Y] = O^*(ev^*(a_E)/[Y]) = O^*(d_{Y,E} \text{Lk}_Y^E) = d_{Y,E} O^*(\text{Lk}_Y^E).
\]

**Definition 2.1.10.** Let \( G \) be a topological group and let \( X \) be a topological \( G \)-space. Then the **big indeterminacy submodule** \( bM^*_X \subset H^*(G \times X, \mathbb{Z}) \) is a sum of indeterminacy submodule \( M^*_X \) and \( p^*_2(H^*(X, \mathbb{Z})) \). As usual, we will omit the subscript, if the space \( X \) is clear.

**Lemma 2.1.11.** Suppose that \( X \) is a complex projective \( G \)-variety, \( E \) is a \( G \)-equivariant vector bundle over \( X \), \( e(E) = 0 \). Then for any secondary Thom class \( a_E \) the following equality holds up to the big indeterminacy submodule \( bM^*_X \subset H^*(G \times X, \mathbb{Z}) \):

\[
\partial \Phi^*(a_E) = (\Phi \circ (id \times ev) \circ \iota)^*(a_E) + bM^*_X.
\]

**Proof.** Denote by \( p_0: \text{Tot}_0(E) \to X \) the projection map and recall that the homomorphism

\[
(id \times p_0)^*: H^*(G \times X, \mathbb{Z}) \to H^*(G \times \text{Tot}_0(E), \mathbb{Z})
\]

is injective (Proposition 1.1.8). Then it is enough to check that

\[
(id \times p_0)^*(\partial \Phi^*(a_E)) = (id \times p_0)^*((\Phi \circ (id \times ev) \circ \iota)^*(a_E)) + (id \times p_0)^*bM^*.
\]

By Definition 1.1.10 the left hand side is equal to \( \Phi^*(a_E) - P^*_2(a_E) \). Set \( b := \Phi^*(a_E) - P^*_2(a_E) \). Then it is enough to prove that

\[
b = (id \times p_0)^*((id \times ev) \circ \iota)^*(b + P^*_2(a_E))) + (id \times p_0)^*M^*.
\]
Let \( q: G \times \Gamma(X, E)_0 \times X \to G \times X \) be the projection map. Notice that the following diagram commutes

\[
\begin{array}{ccc}
G \times \Gamma(X, E)_0 \times X & \xrightarrow{id \times xev} & G \times \operatorname{Tot}_0(E) \\
q \downarrow & & \downarrow \text{id} \times p_0 \\
G \times X & \xrightarrow{id} & G \times X.
\end{array}
\]

Moreover, the map \( \iota: G \times X \cong G \times \{ s_0 \} \times X \to G \times \Gamma(X, E)_0 \times X \) is a section of the projection map \( q \).

Since the class \( b \) has the form \((\text{id} \times p_0)^* (c)\), where \( c \in H^*(G \times X, \mathbb{Z}) \), we obtain that

\[
(id \times p_0)^* (((id \times xev) \circ \iota)^*(b)) = (id \times p_0)^* ((\iota^*(q^*(c)))) = (id \times p_0)^*(c) = b.
\]

It remains to show that \(((id \times xev) \circ \iota)^*(P^*_2(a_E)) \in (id \times p_0)^* M^*\). Indeed,

\[
((id \times xev) \circ \iota)^*(P^*_2(a_E)) = p^*_2 ((e ev \circ \iota)^*(a_E)) \in b M^*.
\]

So the lemma is proven. \(\Box\)

**Notation 2.1.12.** Let \( X \) be a smooth projective \( G \)-variety and let \( Y \) be an irreducible closed subvariety of \( X \). Denote by \( bM^T_{X,Y} \subset H^*(G \times X, \mathbb{Z}) \) the image of the big indeterminacy submodule \( bM^T_X \) under the slant product homomorphism \([Y]: H^*(G \times X, \mathbb{Z}) \to H^*(G, \mathbb{Z})\).

**Corollary 2.1.13.** Suppose that \( E \) is a globally generated \( G \)-equivariant vector bundle over smooth projective \( G \)-variety \( X \), \( e(E) = 0 \), \( Y \) is an irreducible closed subvariety of \( X \) and \( \langle \dim(Y)(E), [Y] \rangle \neq 0 \). Then the following equality holds:

\[
d_Y E \circ (L^T_{\Sigma Y}) = S(e_G(E))/[Y] + bM^T_{X,Y}^{-1}.
\]

Here \( \dim(E) \) is the complex rank of the vector bundle \( E \).

**Proof.** All equalities below hold up to a torsion subgroup.

\[
\begin{align*}
S(e_G(E))/[Y] &= se_G(E)/[Y] \quad \text{(Proposition 1.1.6)} \\
&= \partial \circ (\text{id} \times ev)(a_E)/[Y] \quad \text{(Proposition 1.1.27)} \\
&= (\text{Act} \circ (\text{id} \times ev) \circ \iota)^*(a_E)/[Y] + bM^T_{X,Y}^{\dim(E)} \quad \text{(Lemmas 2.1.11, 2.1.14)} \\
&= (ev \circ A \circ \iota)(a_E)/[Y] + bM^T_{X,Y}^{\dim(E)} \quad \text{(Commutative diagram 2.1.7)} \\
&= d_Y E \circ (L^T_{\Sigma Y}) + bM^T_{X,Y}^{-1} \quad \text{(Corollary 2.1.9)}
\end{align*}
\]

\(\Box\)

**Corollary 2.1.14.** Suppose that \( E \) is a globally generated \( G \)-equivariant vector bundle over smooth projective \( G \)-variety \( X \), \( e(E) = 0 \), \( Y \) is an irreducible closed subvariety of \( X \) and \( \langle \dim(Y)(E), [Y] \rangle \neq 0 \). Then the following equality holds up to a torsion:

\[
d_Y E \circ (L^T_{\Sigma Y}) = S(e_G(E))/[Y].
\]

**Proof.** By Corollary 2.1.13, it suffices to prove that submodule \( bM^T_{odd}X, Y \) is a torsion. By Corollary 1.5.2, it is enough to show that the image of submodule \( p^*_2(H^*(X, \mathbb{Z})) \) is zero. If \( x \in H^T_{odd}(X, \mathbb{Z}) \), then \( p^*_2(x)/[Y] = \langle x, [Y] \rangle = 0 \), since \([Y] \in H^T_{even}(X, \mathbb{Z})]\). \(\Box\)

**Corollary 2.1.15.** Suppose that \( E \) is a globally generated \( G \)-equivariant vector bundle over smooth projective \( G \)-variety \( X \), \( \dim(E) > \dim(X) \), \( Y \) is an irreducible closed subvariety of \( X \) and \( \langle \dim(Y)(E), [Y] \rangle \neq 0 \). Then the following equality holds in \( H^*(G, \mathbb{Z})\):

\[
d_Y E \circ (L^T_{\Sigma Y}) = S(e_G(E))/[Y].
\]

**Proof.** Since \( \dim(X) < \dim(E) \), the Euler class \( e(E) \) is zero. By Corollary 2.1.13, it suffices to prove that submodule \( bM^T_{X,Y}^{-1} \) is zero. But it is straightforward from definitions, since \( H^T_{odd}(X, \mathbb{Z}) = 0 \). \(\Box\)
2.2. Subset of singular sections.

**Notation 2.2.1.** Let \( X \) be a complex smooth projective variety and let \( L \) be a holomorphic line bundle over \( X \). Consider the following subset of holomorphic global sections \( \Gamma(X, L) \):

\[
\text{Sing}(L) = \{ s \in \Gamma(X, L) \mid \exists x \in X, \text{ s.t } x \text{ is a singular point of } Z(s) \}.
\]

Let \( \Gamma_{\text{reg}}(X, L) \) be the complement of \( \text{Sing}(L) \) in \( \Gamma(X, L) \).

Let \( Y \) be a subset of \( X \). Then we can consider the following subset of \( \text{Sing}(L) \):

\[
\text{Sing}_Y(L) = \{ s \in \Gamma(X, L) \mid \exists x \in Y, \text{ s.t } x \text{ is a singular point of } Z(s) \}.
\]

Let us denote by \( J(L) \) the first jet bundle of the line bundle \( L \) (we suppose that the definition is well-known, but one can find it in Chapter 16.7, [7]). Consider the natural injection \( j : \Gamma(X, L) \rightarrow \Gamma(X, J(L)) \).

By local arguments the preimage \( j^{-1}(\text{Sing}_Y(L)) \) is equal to the set \( \text{Sing}_Y(L) \). But this preimage may be very ill-behaved. In order to avoid any problems, we introduce the following definition.

**Definition 2.2.2.** A holomorphic line bundle \( L \) over the complex variety is called 1-jet spanned if and only if the following composition is surjective:

\[
T : \Gamma(X, L) \times X \xrightarrow{j \times \text{id}} \Gamma(X, J(L)) \times X \xrightarrow{\text{Ev}} \text{Tot}(J(L)).
\]

**Example 2.2.3.** It is not hard to see that line bundles \( O(d), d > 0 \) over the projective space \( \mathbb{P}^n \) are 1-jet spanned vector bundles.

**Proposition 2.2.4.** Suppose that \( L \) is a 1-jet spanned line bundle over \( X \) and \( Y \) is an irreducible closed subvariety of \( X \). Then the set \( \text{Sing}_Y(L) \) is also a closed irreducible subvariety of \( \Gamma(X, L) \). Moreover,

\[
\text{codim}_{\Gamma(X, E)}(\text{Sing}_Y(L)) \geq \text{codim}_X(Y) + 1.
\]

**Proof.** Consider the diagram

\[
\begin{align*}
X \times \Gamma(X, L) & \xrightarrow{T} \text{Tot}(J(L)) \\
\downarrow p_2 & \downarrow \\
\Gamma(X, L). & 
\end{align*}
\]

Let \( \mathcal{O}_Y \subset \text{Tot}(J(L)) \) be the variety \( Y \) embedded in \( \text{Tot}(J(L)) \) as the zero section. Then

\[
\text{Sing}_Y(L) = p_2(T^{-1}(\mathcal{O}_Y)).
\]

Since the line bundle \( L \) is 1-jet spanned, the map \( T : X \times \Gamma(X, L) \rightarrow \text{Tot}(J(L)) \) is surjective and so fibre bundle. Hence the preimage \( T^{-1}(\mathcal{O}_Y) \) is an irreducible variety. Now the claim follows from the fact that the image of an irreducible variety is irreducible.

Since the morphism \( T \) is a fibre bundle, we observe the equality:

\[
\text{codim}_{X \times \Gamma(X, L)}(T^{-1}(\mathcal{O}_Y)) = \text{codim}_{\text{Tot}(J(L))}(\mathcal{O}_Y) = \text{codim}_X(Y) + \text{dim}(X) + 1.
\]

Therefore,

\[
\text{codim}_{\Gamma(X, E)}(\text{Sing}_Y(L)) = \text{codim}_{\Gamma(X, E)}(p_2(T^{-1}(\mathcal{O}_Y))) \geq \\
\geq \text{codim}_{X \times \Gamma(X, L)}(T^{-1}(\mathcal{O}_Y)) - \text{dim}(X) = \\
= \text{codim}_X(Y) + 1.
\]

\[\square\]

**Proposition 2.2.5.** Let \( L \) be a 1-jet spanned line bundle over a complex variety \( X \) of dimension \( n \) and let \( Y \) be an irreducible closed subvariety of dimension \( m \). Suppose that the Chern number \( \langle c_m(J(L)), [Y] \rangle \) is non-zero. Then

1. \( \text{codim}_{\Gamma(X, L)}(\text{Sing}_Y(L)) = \text{codim}_X(Y) + 1; \)
2. \( \text{deg}(\text{Sing}_Y(J(L))) = \langle c_m(J(L)), [Y] \rangle; \)
3. The variety \( \Sigma_Y(J(L)) \) intersects the affine subspace \( j(\Gamma(X, L)) \subset \Gamma(X, J(L)) \) by the variety \( \text{Sing}_Y(L) \) with multiplicity one.
Moreover, the following diagram is commutative:
\[
\begin{array}{ccc}
O & \rightarrow & \Gamma_{\text{reg}}(X,L) \\
\downarrow & & \downarrow j \\
\Gamma(X,L)_0 & & \\
\end{array}
\]

Therefore, we observe that
\[
d\langle c_m(J(L)),[Y]\rangle = d \deg(\text{Sing}_Y(L)) = \deg(\Sigma_Y(L)) \leq \langle c_m(J(L)),[Y]\rangle.
\]

By the first part and by Proposition 2.2.6 there exists the following chain of inequalities:
\[
d\langle c_m(J(L)),[Y]\rangle = d \deg(\text{Sing}_Y(L)) = \deg(\Sigma_Y(L)) \leq \langle c_m(J(L)),[Y]\rangle.
\]

Therefore, \(d = 1\). \(\square\)

**Notation 2.2.6.** If \(\text{Sing}_Y(L)\) is an irreducible closed subvariety of \(\text{Sing}(L)\), then there exists the fundamental class of \([\text{Sing}_Y(L)] \in H^*_\text{BM}(\text{Sing}(L))\). By the Alexander duality it defines the cohomology linking class \(\text{Lk}^Y_{\text{sing}}(L) \in H^*(\Gamma_{\text{reg}}(X,L))\); see Definition 1.1.10 \(\text{Lk}^Y_{\text{sing}}(L) := \text{Lk}_{[\text{Sing}_Y(L)],\text{Sing}(L),\Gamma(L)}\). We will use the abbreviation \(\text{Lk}^Y_{\text{sing}}\) for \(\text{Lk}^Y_{\text{sing}}(L)\), if the line bundle \(L\) is clear.

**Theorem 2.2.7.** Let \(L\) be a 1-jet spanned \(G\)-equivariant line bundle over a smooth complex projective \(G\)-variety \(X\) and let \(Y\) be a closed irreducible subvariety of dimension \(m\). Suppose that the Chern number \(\langle c_m(J(L)),[Y]\rangle\) is nonzero. Then the following equality holds in \(H^*(G,\mathbb{Z})\):
\[
O^*(\text{Lk}^Y_{\text{sing}}) = S(e_G(J(L)))/[Y].
\]

**Proof.** Consider the mapping \(j: \Gamma_{\text{reg}}(X,L) \rightarrow \Gamma(X,J(L))_0\). By Proposition 2.2.5
\[
j^*(\text{Lk}^Y_{\text{sing}}(J(L))) = \text{Lk}^Y_{\text{sing}}(L).
\]

Moreover, the following diagram is commutative:
\[
\begin{array}{ccc}
G & \rightarrow & \Gamma_{\text{reg}}(X,L) \\
\downarrow & & \downarrow j \\
\Gamma(X,L)_0 & & \\
\end{array}
\]

Therefore, we observe that
\[
O^*(\text{Lk}^Y_{\text{sing}}(L)) = O^*(\text{Lk}^Y_{\text{sing}}(J(L))). \quad \text{By Corollary 2.2.8} \quad \text{the latter is equal to}
\]
\[
O^*(\text{Lk}^Y_{\text{sing}}(J(L))) = \frac{1}{d_{Y,E}} S(e_G(J(L)))/[Y].
\]

By Propositions 2.1.6 and 2.2.5 the constant \(d_{Y,E}\) is equal to 1. So finally,
\[
O^*(\text{Lk}^Y_{\text{sing}}(L)) = O^*(\text{Lk}^Y_{\text{sing}}(J(L))) = S(e_G(J(L)))/[Y].
\]

\(\square\)

**Corollary 2.2.8.** Let \(L\) and \(M\) be two 1-jet spanned \(G\)-equivariant line bundles over a smooth complex projective \(G\)-variety \(X\) and let \(Y\) be a closed irreducible subvariety of dimension \(m\). Suppose that

1. \(c_1(L) = c_1(M)\);
2. the Chern number \(\langle c_m(J(L)),[Y]\rangle\) is nonzero.

Then the following identity holds in \(H^*(G,\mathbb{Z})\):
\[
O^*(\text{Lk}^Y_{\text{sing}}(L)) = O^*(\text{Lk}^Y_{\text{sing}}(M)).
\]

**Corollary 2.2.9.** Let \(L\) be a 1-jet spanned line bundle over a smooth complex projective variety \(X\) of dimension \(n\) and let \(Z\) be a closed irreducible subvariety of dimension \(m\). Let \(i: Y \hookrightarrow X\) be a \(G\)-equivariant embedding of a smooth divisor on \(X\). Suppose the following:

1. \(\langle c_m(J(L)),[Z]\rangle \neq 0\), \(\langle c_{m-1}(J(L)),i^*[Z]\rangle \neq 0\);
(2) the cohomology class $c^G_1(L)$ is not a zero divisor in the ring $H^*(X_{hG}, \mathbb{Q})$;

(3) there exists constant $r \in \mathbb{Q}$, $r \neq 0$ such that $i_{\ast,G}1 = rc^G_1(L) \in H^*(X_{hG}, \mathbb{Q})$. Then the following identity holds in $H^*_R(G)$:

$$O^*(\text{Lk}^\text{Sing}_Z(L)) = \frac{1-r}{r}O^*(\text{Lk}^\text{Sing}_Z(i^\ast L)).$$

Proof. Let us compute the cohomology class $O^*(\text{Lk}^\text{Sing}_Z(i^\ast L))$. By Theorem 2.2.7

$$O^*(\text{Lk}^\text{Sing}_Z(i^\ast L)) = S(e_G(J(i^\ast L)))/i^\ast[Z] = (i_1S(e_G(J(i^\ast L)))/[Z].$$

By Proposition 1.4.16 one can see that $i_1S(e_G(J(i^\ast L))$ is equal to $S(i_G,e_G(J(i^\ast L)))$.

Denote by $C_{Y/X}$ the conormal bundle of $Y$ in $X$. Then there is the exact sequence of $G$-equivariant sheaves:

$$0 \rightarrow C_{Y/X} \otimes i^\ast L \rightarrow i^\ast J(L) \rightarrow J(i^\ast L) \rightarrow 0.$$

Therefore,

$$(2.2.10) \quad i^\ast e_G(J(L)) = e_G(J(i^\ast L)) \sim (c^G_1(C_{Y/X} \otimes i^\ast L)).$$

Since $Y$ is a divisor on $X$, the conormal bundle is equal to $i^\ast(\mathcal{O}(-Y))$. Moreover, $i_\ast,G1 = -c^G_1(\mathcal{O}(-Y))$ [Fulton].

So

$$(2.2.11) \quad c^G_1(C_{Y/X} \otimes i^\ast L) = i^\ast(e^G_1(L) - i_\ast,G1) = (1-r)i^\ast,c^G_1(L).$$

Substitute (2.2.11) for $c^G_1(C_{Y/X} \otimes i^\ast L)$ in (2.2.10)

$$i^\ast e_G(J(L)) = (1-r)e_G(J(i^\ast L)) \sim i^\ast,c^G_1(L).$$

Let us apply the homomorphism $i_{G,1}$ to the obtained identity. By the projection formula, one can observe:

$$e_G(J(L)) \sim i_{G,1}1 = (1-r)i_{G,1},e_G(J(i^\ast L)) \sim c^G_1(L).$$

By assumptions, $r,1 = rc^G_1(L)$ and $c^G_1(L)$ is not a zero divisor in the ring $H^*(X_{hG}, \mathbb{Q})$. Therefore, one can derive the following equality:

$$e_G(J(L)) = \frac{1-r}{r}i_{G,1},e_G(J(i^\ast L)).$$

The required identity follows from the identity above and Theorem 2.2.7. \qed

3. Examples

The following section is devoted for examples. Using Theorem 2.2.7 we can rediscover the result in Gotinov’s paper [6]. Moreover, we can also cope with the case of special orthogonal groups and quadrics. Through the section we will use the following notation.

Notation. Denote by $m(d,n,i)$ the following number:

$$m(d,n,i) = (d-1)^{n+1} + (-1)^{i+1}(d-1)^{n+1-i}.$$
Here $b_{n+1} = 0$. Denote by $a_i$ the element $b_i - b_{i-1} \in H^*(X_{hG}, \mathbb{Z})$. Then the ideal $I^*(\mathbb{P}(V), \mathbb{Z}) := \text{ker}(\alpha^*)$ is equal to the ideal $I^*_1(\mathbb{P}(V), \mathbb{Z}) := (a_2, a_3, \ldots, a_{n+1})$.

It is not hard to see that the equivariant Euler class $e_G(J(L)) = e_G(V^*(d - 1))$ is equal to the following sum:

$$e_G(V^*(d - 1)) = (d - 1)^{n+1} b_1^{n+1} + \sum_{i=2}^{n+1} (-1)^i (d - 1)^{n+1-i} a_i b_1^{n+1-i}.$$  \hfill (3.1.1)

Now it remains to compute the map $S: I^{2n+2}(X) \to H^{2n+1}(G \times X)/M^{2n+1}$. By Proposition 1.4.10 and Remark 1.4.13, it is enough to decompose the element $e_G(J(L)) \in I^*_1(\mathbb{P}(V), \mathbb{Z})$ by the generators $a_2, a_3, \ldots, a_{n+1}$. Notice that

$$b_1^{n+1} + \sum_{i=2}^{n+1} a_i b_1^{n+1-i} = b_1^{n+1} + \sum_{i=2}^{n+1} (b_i - b_{i-1}) b_1^{n+1-i} = 0.$$  \hfill (3.1.2)

Using (3.1.1) and (3.1.2) we observe that

$$e_G(V(d - 1)) = -\sum_{i=2}^{n+1} \left( (d - 1)^{n+1} + (-1)^i (d - 1)^{n+1-i} \right) a_i b_1^{n+1-i} = -\sum_{i=2}^{n+1} m(d, n, i) a_i b_1^{n+1-i}.$$  

Recall that the morphism $\gamma: \Sigma G \to BG$ is adjoint to the equivalence $G \simeq \Omega BG$. By Proposition 1.4.10 the following equality holds:

$$S(e_G(J(L))) = S(e_G(V^*(d - 1))) = \tilde{S} \left( -\sum_{i=2}^{n+1} m(d, n, i) c_i \otimes b_1^{n+1-i} \right)$$

$$= -\sum_{i=2}^{n+1} m(d, n, i) \gamma^*(c_i) \otimes e^{n+1-i} b_1$$

Finally, let us apply Theorem 2.2.7. For any $0 \leq k < n$:

$$O^*(\text{Lk}_{\mathbb{P}(V)}^{2n+3}) = S(e_G(J(L)))(\mathbb{P}^k) = -m(d, n, n - k + 1) \gamma^*(c_{n-k+1}) \in H^{2(n-k)+1}(G, \mathbb{Z}).$$  \hfill (3.1.3)

Moreover, $\gamma^*(c_{n-k+1})$ is unique up to a sign primitive generator of the ring $H^*(G, \mathbb{Z})$ in the degree $2(n-k)+1$. Compare with [Gorinov].

### 3.2. Odd-dimensional quadric

Let $V \cong \mathbb{C}^{2n+3}$ be a complex vector space and let $Q: V \to \mathbb{C}$ be a non-degenerate quadratic form on the space $V$. Denote by $C \subset V$ the affine quadric cone $C := \{ v \in V | Q(v) = 0 \}$. Let $G$ be the special orthogonal group $SO(V, Q) \cong SO(2n+3, \mathbb{C})$. Then the projective quadric $X = \mathbb{P}(C)$ is a smooth complex projective homogeneous $G$-variety of dimension $2n+1$.

The homology groups of $X$ is following:

$$H_i(\mathbb{P}(C), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 2k, 0 \leq k \leq 2n + 1; \\ 0, & i \text{ is odd} \end{cases}$$

The cohomology ring of $X$ is equal to

$$H^*(\mathbb{P}(C), \mathbb{Z}) = \mathbb{Z}[h, \Lambda]/(h^{2n+2}, h^n - 2\Lambda), \quad \text{deg}(h) = 2, \text{deg}(\Lambda) = 2n + 2.$$
and 1-jet spanned. Moreover, $L$ has a unique structure of a $G$-equivariant vector bundle. Let us compute the cohomology classes $O^*(L^\text{Sing}_Z) \in H^*(G, \mathbb{Z})$ for any $0 \leq k \leq 2n + 1$.

Let $F$ be the tautological line bundle over $X$, $F \cong \mathcal{O}_X(-1)$. Let $F^\perp$ be the tautological orthogonal vector bundle, $F^\perp \to V$, $F^\perp \cong (V/F)^*$. Moreover, there exists the monomorphism $\varphi: F \to F^\perp$ such that $\ker(Q_{F^\perp}) = \varphi(F)$.

The tangent bundle to the variety $X$ is $G$-isomorphic to the bundle $\text{Hom}(F, F^\perp/F)$. Therefore there are following $G$-equivariant isomorphisms:

$$J(L) \cong \text{Hom}(F^\perp, F) \otimes L \cong (F^\perp)^*(d - 1) \cong (V/F)(d - 1).$$

If $d > 1$, then for any $0 \leq k \leq 2n + 1$ the Chern number $\langle c_k(J(L)), [Z_k] \rangle$ is nonzero. So we can apply Theorem 2.2.7.

Let $P \subset G$ be the stabilizer of the point in $\mathbb{P}(C)$ by the $G$-action. Then $\mathbb{P}(C) \cong G/P$, where $P$ is a parabolic subgroup isomorphic to the group $SO(2n + 1) \times SO(2)$. Moreover, there exist following isomorphisms by [2]

$$H^n_*(BG, \mathbb{Z}) \cong \mathbb{Z}[p_1, p_2, \ldots, p_{n+1}], \quad \deg(p_i) = 4i,$$

$$H^n_*(X_{hG}, \mathbb{Z}) \cong H^n_*(BP, \mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, \ldots, q_n, c], \quad \deg(q_i) = 4i, \quad \deg(c) = 2,$$

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[h, \Lambda]/(h^{2n+2}, h^{n+1} - 2\Lambda).$$

The maps $\alpha^*: H^n_*(X_{hG}, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ and $\beta^*: H^n_*(BG, \mathbb{Z}) \to H^n_*(X_{hG}, \mathbb{Z})$ acts as

$$\alpha^*(q_i) = h^{2i}, \quad \text{for any } 1 \leq i \leq n,$$

$$\alpha^*(c) = h,$$

$$\beta^*(p_i) = q_i - c^2 q_{i-1}, \quad \text{for any } 1 \leq i \leq n + 1.$$

Here $q_{n+1} = 0$. Denote by $a_i$ the element $q_i - c^2 q_{i-1} \in H^n_*(X_{hG}, \mathbb{Z})$ for any $1 \leq i \leq n + 1$. Then the ideal $I^*(\mathbb{P}(V), \mathbb{Z}) := \ker(\alpha^*)$ is equal to the ideal $I^*_1(\mathbb{P}(C), \mathbb{Z}) := (a_1, a_2, \ldots, a_{n+1})$.

It is not hard to see that the equivariant Euler class $e_G(J(L)) = e_G((F^\perp)^*(d - 1)) = e_G(V(d - 1))/e_G(F(d - 1))$ is equal to the following sum:

$$e_G(J(L)) = \frac{1}{(d - 2)c} \left( (d - 1)^{2n+3} + \sum_{i=1}^{n+1} (d - 1)^{2n-2i+3} a_i c^{2n-2i+3} \right).$$

Now it remains to compute the map $S: I^{4n+4}(X) \to H^{4n+3}(G \times X)/M^{4n+3}$. By Proposition 1.4.10 and Remark 1.4.13, it is enough to decompose the element $e_G(J(L)) \in I^*_1(\mathbb{P}(C), \mathbb{Z})$ by the generators $a_1, a_2, \ldots, a_{n+1}$.

Notice that

$$c^{2n+3} + \sum_{i=1}^{n+1} a_i c^{2n-2i+3} = c^{2n+3} + \sum_{i=1}^{n+1} (q_i - c^2 q_{i-1}) c^{2n-2i+3} = 0.$$

Using (3.2.1) and (3.2.2) we observe that

$$e_G(J(L)) = -\frac{1}{d - 2} \sum_{i=1}^{n+1} \left( (d - 1)^{2n+3} - (d - 1)^{2n-2i+3} \right) a_i c^{2(n+1)-2i} =$$

$$= -\frac{1}{d - 2} \sum_{i=1}^{n+1} m(d, 2n + 2, 2i) a_i c^{2(n+1)-2i}.$$

Recall that the morphism $\gamma: \Sigma G \to BG$ is adjoint to the equivalence $G \simeq \Omega BG$. By Proposition 1.4.10 the following equality holds:

$$S(e_G(J(L))) = S\left( -\sum_{i=1}^{n+1} \frac{m(d, 2n + 2, 2i)}{d - 2} p_i \otimes c^{2(n+1)-2i} \right)$$
\[ = - \sum_{i=1}^{\frac{n+1}{2}} \frac{m(d, 2n + 2, 2i)}{d - 2} \gamma^*(p_i) \otimes \alpha\left(e^{2(n+1)-2i}\right) \]
\[ = - \sum_{i=1}^{\frac{n+1}{2}} \frac{2m(d, 2n + 2, 2i)}{d - 2} \gamma^*(p_i) \otimes \theta^{n-2i+1} \Lambda - \sum_{i=\lceil \frac{2n+1}{2} \rceil}^{n+1} \frac{m(d, 2n + 2, 2i)}{d - 2} \gamma^*(p_i) \otimes \theta^{2(n+1)-2i}. \]

Finally, let us apply Theorem 2.2.7

\[ O^*(\text{Lk}_{Z_k}^\text{Sing}) = \begin{cases} 
- \frac{m(d, 2n + 2, (n+1)-k)}{d - 2} \gamma^*(p_{(n+1)-k/2}), & k \text{ is even and } 0 \leq k \leq n \\
\frac{2m(d, 2n + 2, (n+1)-k)}{d - 2} \gamma^*(p_{(n+1)-k/2}), & k \text{ is even and } n + 1 \leq k \leq 2n \\
0, & k \text{ is odd} 
\end{cases} \]

Moreover, \( \gamma^*(p_{(n+1)-k/2}) \) is unique up to a sign a primitive generator of the ring \( H_{\ast}^n(SO(2n + 3), \mathbb{Z}) \) in the degree \( 4n - 2k + 3 \).

3.3. Even-dimensional quadric. Let \( V \cong \mathbb{C}^{2n+2} \) be a complex vector space and let \( Q : V \to \mathbb{C} \) be a non-degenerate quadratic form on the space \( V \). Denote by \( C \subset V \) the affine quadric cone \( C := \{ v \in V \mid Q(v) = 0 \} \). Let \( G \) be the special orthogonal group \( SO(V, Q) \cong SO(2n + 2, \mathbb{C}) \). Then the projective quadric \( X = \mathbb{P}(C) \) is a smooth complex projective homogeneous \( G \)-variety of dimension \( 2n \).

The homology groups of \( X \) are following:

\[ H_i(\mathbb{P}(C), \mathbb{Z}) = \begin{cases} 
\mathbb{Z}(Z_k), & i = 2k, \ 0 \leq k \leq 2n, k \neq n; \\
\mathbb{Z}(W_1) \oplus \mathbb{Z}(W_2), & i = 2n; \\
0, & i \text{ is odd} 
\end{cases} \]

The cohomology ring of \( X \) is equal to

\[ H^\ast(\mathbb{P}(C), \mathbb{Z}) = \mathbb{Z}[h, \Lambda_1, \Lambda_2]/(h^{2n+1}, h^n - (\Lambda_1 + \Lambda_2), h(\Lambda_1 - \Lambda_2)). \]

Here \( \deg(h) = 2 \), \( \deg(\Lambda_1) = \deg(\Lambda_2) = 2n \). Moreover,

\[ \langle h, W_j \rangle = \delta_{ij}. \]

Let \( i : \mathbb{P}(C) \hookrightarrow \mathbb{P}(V) \) be the induced embedding and let \( \mathcal{O}_X(1) \) be the very ample bundle \( i^*\mathcal{O}(1), h = c_1(\mathcal{O}_X(1)). \) Consider the line bundle \( L := \mathcal{O}_X(d) \) over the variety \( X \). If \( d > 0 \), then \( L \) is very ample and 1-jet spanned. Moreover, \( L \) has a unique structure of a \( G \)-equivariant vector bundle. Let us compute the cohomology classes \( O^*(\text{Lk}_{Z_k}^\text{Sing}) \in H^\ast(G, \mathbb{Z}) \) for any \( 0 \leq k \leq 2n \) and \( O^*(\text{Lk}_{W_i}^\text{Sing}) \in H^\ast(G, \mathbb{Z}), i = 1, 2 \).

Let \( F \) be the tautological line bundle over \( X, F \cong \mathcal{O}_X(-1) \). Let \( F^\perp \) be the tautological orthogonal vector bundle. The tangent bundle to the variety \( X \) is \( G \)-isomorphic to the bundle \( \text{Hom}(F, F^\perp / F) \). Therefore there are following \( G \)-equivariant isomorphisms:

\[ J(L) \cong \text{Hom}(F^\perp, F) \otimes L \cong (F^\perp)^* (d - 1) \cong (V / F)(d - 1). \]

If \( d > 1 \), then for any \( 0 \leq k \leq 2n \) the Chern number \( \langle c_k(J(L)), [Z_k] \rangle \) is nonzero as well as the Chern numbers \( \langle c_k(J(L)), [W_i] \rangle \) for any \( i = 1, 2 \). So we can apply Theorem 2.2.7

Let \( P \subset G \) be the stabilizer of the point in \( \mathbb{P}(C) \) by the \( G \)-action. Then \( \mathbb{P}(C) \cong G/P \), where \( P \) is a parabolic subgroup isomorphic to the group \( SO(2n) \times SO(2) \). Moreover, there exist following isomorphisms by [2]

\[ H_{\ast}^n(BG, \mathbb{Z}) \cong \mathbb{Z}[p_1, p_2, \ldots, p_n, \chi], \quad \deg(p_i) = 4i, \quad \deg(\chi) = 2n + 2, \]
\[ H_{\ast}^n(X_{hG}, \mathbb{Z}) \cong H_{\ast}^n(BP, \mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, \ldots, q_{n-1}, \chi_1, c], \quad \deg(q_i) = 4i, \quad \deg(\chi_1) = 2n, \quad \deg(c) = 2, \]
\[ H^\ast(X, \mathbb{Z}) = \mathbb{Z}[h, \Lambda_1, \Lambda_2]/(h^{2n+1}, h^n - (\Lambda_1 + \Lambda_2), h(\Lambda_1 - \Lambda_2)). \]

The maps \( \alpha^* : H_{\ast}^n(X_{hG}, \mathbb{Z}) \to H^\ast(X, \mathbb{Z}) \) and \( \beta^* : H_{\ast}^n(BG, \mathbb{Z}) \to H_{\ast}^n(X_{hG}, \mathbb{Z}) \) acts as

\[ \alpha^*(q_i) = h^{2i}, \quad \text{for any } 1 \leq i \leq n - 1, \]
\[ \alpha^*(c) = h, \]
\[ \alpha^*(\chi_1) = \Lambda_1 - \Lambda_2, \]
Here \( q_n = \chi_1^2 \). Denote by \( a_i \) the element \( q_i - c^2 q_{i-1} \in H_{2r}^*(X_{hG}, \mathbb{Z}) \) for any \( 1 \leq i \leq n \) and by \( b \) the element \( \chi_1 c \in H_{2r}^*(X_{hG}, \mathbb{Z}) \). Then the ideal \( I^*(\mathbb{P}(V), \mathbb{Z}) := \ker(\alpha^*) \) is equal to the ideal \( I_2^*(\mathbb{P}(C), \mathbb{Z}) := (a_1, a_2, \ldots, a_n, b) \).

It is not hard to see that the equivariant Euler class
\[
e_G(J(L)) = e_G((F^{-1})^*(d - 1)) = e_G(V(d - 1))/e_G(F(d - 1))
\]
is equal to the following sum:
\[
e_G(J(L)) = \frac{1}{(d - 2)c} \left( (d - 1)^{2n+2} c^{2n+2} - \chi_1^2 c^2 + \sum_{i=1}^{n} (d - 1)^{2n-2i+2} a_i c^{2n-2i+2} \right).
\]

Now it remains to compute the map \( S\): \( I^{2n+2}(X) \to H^{4n+1}(G \times X)/M^{4n+1} \). The indeterminacy submodule \( M^{4n+1} \) is equal to zero by degree reasons. By Proposition 1.4.19 and Remark 1.4.13, it is enough to decompose the element \( e_G(J(L)) \in I_2^*(\mathbb{P}(C), \mathbb{Z}) \) by the generators \( a_1, a_3, \ldots, a_n, b \). Notice that
\[
e^{2n+2} - b^2 + \sum_{i=1}^{n} a_i c^{2n-2i+2} = e^{2n+2} - \chi_1^2 c^2 + \sum_{i=1}^{n} (q_i - c^2 q_{i-1}) c^{2n-2i+2} = 0.
\]

Using 3.3.1 and 3.3.2 we observe that
\[
e_G(J(L)) = \frac{(d - 1)^{2n+2} - 1}{d - 2} b \chi_1 - \sum_{i=1}^{n} \frac{(d - 1)^{2n+2} - (d - 1)^{2n-2i+2}}{d - 2} a_i c^{2n-2i+1} =
\]
\[
= \frac{m(d, 2n + 1, 2n + 2)}{d - 2} b \chi_1 - \sum_{i=1}^{n} \frac{m(d, 2n + 1, 2i)}{d - 2} a_i c^{2n-2i+1}.
\]

Recall that the morphism \( \gamma\): \( \Sigma G \to BG \) is adjoint to the equivalence \( G \simeq \Omega BG \). By Proposition 1.4.10 the following equality holds:
\[
S(e_G(J(L))) = S \left( \frac{m(d, 2n + 1, 2n + 2)}{d - 2} \chi \otimes \chi - \sum_{i=1}^{n} \frac{m(d, 2n + 1, 2i)}{d - 2} p_i \otimes c^{2n-2i+1} \right)
\]
\[
= \frac{m(d, 2n + 1, 2n + 2)}{d - 2} \gamma(\chi) \boxtimes (\Lambda_1 - \Lambda_2) - \sum_{i=1}^{n} \frac{m(d, 2n + 1, 2i)}{d - 2} \gamma^*(p_i) \boxtimes \alpha(c^{2n-2i+1})
\]
\[
= \frac{m(d, 2n + 1, 2n + 2)}{d - 2} \gamma(\chi) \boxtimes (\Lambda_1 - \Lambda_2) - \sum_{i=1}^{n} \frac{m(d, 2n + 1, 2i)}{d - 2} \gamma^*(p_i) \boxtimes h^{2n-2i+1}.
\]

Finally, let us apply Theorem 2.2.7.
\[
O^*({\text{Lk}}_{\mathbb{Z}_k}^{\text{Sing}}) = \begin{cases} -\frac{m(d, 2n+1, 2n+1-k)}{d-2} \gamma^*(p_{n-(k-1)/2}), & k \text{ is odd and } 0 \leq k < n \\ -2 \frac{m(d, 2n+1, 2n+1-k)}{d-2} \gamma^*(p_{n-(k-1)/2}), & k \text{ is odd and } n + 1 \leq k \leq 2n \\ 0, & k \text{ is even} \end{cases}
\]

Furthermore, if \( n \) is even, then
\[
O^*({\text{Lk}}_{W_1}^{\text{Sing}}) = S(e_G(J(L)))/[W_1] = \begin{cases} \frac{m(d, 2n+1, 2n+2)}{d-2} \gamma^*(\chi), & i = 1; \\ -\frac{m(d, 2n+1, 2n+2)}{d-2} \gamma^*(\chi), & i = 2. \end{cases}
\]

If \( n \) is odd, then
\[
O^*({\text{Lk}}_{W_1}^{\text{Sing}}) = S(e_G(J(L)))/[W_1] = \begin{cases} \frac{m(d, 2n+1, 2n+2)}{d-2} \gamma^*(\chi) - \frac{m(d, 2n+1, n+1)}{d-2} \gamma^*(p_{n+1/2}), & i = 1; \\ \frac{m(d, 2n+1, 2n+2)}{d-2} \gamma^*(\chi) - \frac{m(d, 2n+1, n+1)}{d-2} \gamma^*(p_{n+1/2}), & i = 2. \end{cases}
\]

Moreover, the set of elements \( \{ \gamma^*(p_{n-(k-1)/2}) \mid k \text{ is odd, } 0 \leq k \leq 2n \} \cup \{ \gamma^*(\chi) \} \) is a set of primitive generators of the ring \( H_{2r}^*(SO(2n + 2), \mathbb{Z}) \).
3.4. Vector bundles of any rank. In previous sections we worked with discriminant complements for $G$-equivariant line bundles. But the same question can be asked for vector bundles of any rank. In the present remark we will explain how to deal with this case. First, let us recall the setting.

Notation 3.4.1. Let $X$ be a complex smooth projective variety and let $E$ be a holomorphic line bundle over $X$. Consider the following subset of holomorphic global sections $\Gamma(X, E)$:

$$\text{Sing}(E) = \{ s \in \Gamma(X, E) \mid \exists x \in X, \text{s.t. } x \text{ is a singular point of } Z(s) \}.$$  

Let $\Gamma_{\text{reg}}(X, E)$ be the complement of $\text{Sing}(E)$ in $\Gamma(X, E)$.

Let $Y$ be a subset of $X$. Then we can consider the following subset of $\text{Sing}(E)$:

$$\text{Sing}_Y(E) = \{ s \in \Gamma(X, E) \mid \exists x \in Y, \text{s.t. } x \text{ is a singular point of } Z(s) \}.$$  

Let $\mathbb{P}(E)$ be the projectivization of the vector bundle $E$. Let us denote by $O_{\mathbb{P}(E)}(1)$ the tautological bundle and denote by $\pi: \mathbb{P}(E) \to X$ the projection map. Recall that the direct image $\pi_* (O_{\mathbb{P}(E)}(1))$ is canonically isomorphic to the vector bundle $E$ itself. Hence, there exist the isomorphism

$$\Gamma \pi_* : \Gamma(\mathbb{P}(E), O_{\mathbb{P}(E)}(1)) \xrightarrow{\sim} \Gamma(X, E).$$  

The following lemma can be proved easily by local arguments. See Lemma 1.4 in [1].

Lemma 3.4.3. Under the isomorphism $\Gamma \pi_*$ the subset $\text{Sing}(O_{\mathbb{P}(E)}(1))$ maps to $\text{Sing}(E)$. Moreover, for any $Y \subset X$ the preimage of $\text{Sing}_Y(E)$ is equal to $\text{Sing}_{\pi Y}(O_{\mathbb{P}(E)}(1))$.

Definition 3.4.4. The vector bundle $E$ over $X$ is called 1-jet spanned if and only if the line bundle $O_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is 1-jet spanned.

Corollary 3.4.5. Suppose that $E$ is a 1-jet spanned vector bundle over $X$ and $Y$ is an irreducible closed subvariety of $X$. Then the set $\text{Sing}_Y(E)$ is also a closed irreducible subvariety of $\Gamma(X, E)$.

Proof. It follows directly from Lemma 3.4.3 and Proposition 2.2.4. \hfill \Box

Notation 3.4.6. If $\text{Sing}_Y(E)$ is an irreducible closed subvariety of $\text{Sing}(E)$, then there exists the fundamental class of $[\text{Sing}_Y(E)] \in H^*_{BM}(\text{Sing}(E))$. By the Alexander duality it defines the cohomology linking class $L_{Y}^{\text{Sing}}(E) \in H^*(\Gamma_{\text{reg}}(X, E))$; see Definition 1.1.10. $L_{Y}^{\text{Sing}}(E) := L_{[\text{Sing}_Y(E)], \text{Sing}(E), \Gamma(E)}$.

Let $G$ be a semisimple complex Lie group. Suppose that $X$ is a $G$-variety and $E$ is a $G$-equivariant vector bundle. Then $\mathbb{P}(E)$ has a natural structure of $G$-variety such that the morphism $\pi$ is $G$-equivariant.

The line bundle $O_{\mathbb{P}(E)}(1)$ also has a natural $G$-equivariant structure such that the canonical isomorphism $\pi_*(O_{\mathbb{P}(E)}(1)) \to E$ is a $G$-equivariant isomorphism. Therefore, the isomorphism $\Gamma \pi_*$ is $G$-equivariant as well. As a corollary we observe the following identity.

Lemma 3.4.7. Suppose that a $G$-equivariant vector bundle $E$ is 1-jet spanned and $Y$ is an irreducible subvariety of smooth projective variety $X$. Then the following identity holds:

$$O^*(L_{Y}^{\text{Sing}}(E)) = O^*(L_{[\pi Y]}(O_{\mathbb{P}(E)}(1))).$$

Corollary 3.4.8. Let $E$ be a 1-jet spanned $G$-equivariant vector bundle of rank $r$ over a smooth complex projective $G$-variety $X$ and let $Y$ be a closed irreducible subvariety of dimension $m$. Suppose that the Chern number $\langle c_{m+r-1}(J(O_{\mathbb{P}(E)}(1))), \pi^*[Y] \rangle$ is nonzero. Then the following equality holds up to a torsion:

$$O^*(L_{Y}^{\text{Sing}}(E)) = O^*(L_{[\pi Y]}^{\text{Sing}}(O_{\mathbb{P}(E)}(1))) = S(e_G(J(O_{\mathbb{P}(E)}(1))))/\pi^*[Y] = S(\pi_* e_G(J(O_{\mathbb{P}(E)}(1))))/\pi^*[Y].$$

Proof. It follows directly by Lemma 3.4.7 and Theorem 2.2.7. \hfill \Box

Remark 3.4.9. The condition $\langle c_{m+r-1}(J(O_{\mathbb{P}(E)}(1))), \pi^*[Y] \rangle \neq 0$ can be rewritten only in terms of Chern numbers of the bundle $E$ over the variety $X$.

Remark 3.4.10. The case of $X = \mathbb{P}^n$, $E$ is a direct sum of line bundles and $G = SL_{n+1}$ was considered by A. Gorinov in the paper [6].
4. Division theorems

In this section we discuss division theorems for spaces of sections of equivariant line bundles and show that division phenomena holds for a generic line bundle over particular \( G \)-varieties.

**Definition 4.0.1.** We say that a \( G \)-equivariant line bundle \( L \) over smooth projective \( G \)-variety \( X \) is satisfied the Division Phenomena for the line bundle \( L \) w.r.t. the group \( G \) iff the Leray-Serre spectral spectral for geometric \( G \)-quotient of \( \Gamma_{reg}(X, L) \):

\[
E^{p,q}_2 = H^p(G, Q) \otimes H^q(\Gamma_{reg}(X, L)/G, Q) \Rightarrow H^{p+q}(\Gamma_{reg}(X, L), Q)
\]

**Remark 4.0.3.** By the Leray-Hirsh theorem the Division Phenomena for a line bundle \( L \) w.r.t. the group \( G \) equivalent to the statement that restriction homomorphism \( O^*: H^*(\Gamma_{reg}(X, L), Q) \to H^*(G, Q) \) is surjective.

**Remark 4.0.4.** Notice that if the Division Phenomena holds for the line bundle \( L \) w.r.t. the group \( G \) then the then the group \( G \) acts on the discriminant complement \( \Gamma_{reg}(X, L) \) with finite stabilizers. Indeed, assume that there exists a point \( s_0 \in \Gamma_{reg}(X, L) \) with the stabilizer \( G_{s_0} \subset G \) such that \( \dim G_{s_0} > 0 \). Then the homomorphism \( O^* \) factors through the homomorphism

\[
q^*: H^*(G/G_{s_0}, Q) \to H^*(G, Q)
\]

induced by quotient map \( q: G \to G/G_{s_0} \). Since \( \dim(G/G_{s_0}) < \dim(G) \), the homomorphism \( q^* \) is not surjective. So we obtain the contradiction.

**Example 4.0.5.** The examples of the previous section show that if the variety \( X \) is a projective space of dimension \( n \) or a quadric of dimension \( n \), \( G = SL_{n+1} \) or \( G = SO_{n+2} \), respectively, and a line bundle \( L \) is \( \mathcal{O}_X(d) \), \( d > 2 \), then the Division Phenomena holds for \( L \) w.r.t. the group \( G \). Hence, the restriction homomorphism \( O^*: H^*(\Gamma_{reg}(X, L), Q) \to H^*(G, Q) \) is surjective.

**Example 4.0.6.** Let \( \rho: H \to G \) be a Lie group homomorphism. Let \( L \) be a \( G \)-equivariant line bundle over smooth projective \( G \)-variety \( X \). Suppose that \( L \) satisfies the Division Phenomena w.r.t the group \( G \). Then the line bundle \( L \) satisfies the Division Phenomena w.r.t. the group \( H \). Hence, the restriction homomorphism \( \rho^*: H^*(G, Q) \to H^*(H, Q) \) is surjective.

**Example 4.0.7.** Now we would like to present the counter-example for Division Phenomena. Let \( G \) be the projective space of dimension \( n \) or a quadric of dimension \( n \), \( G = SL_{n+1} \) or \( G = SO_{n+2} \), respectively, and a line bundle \( L \) is \( \mathcal{O}_X(d) \), \( d > 2 \), then the Division Phenomena holds for \( L \) w.r.t. the group \( G \). Hence, the restriction homomorphism \( O^*: H^*(\Gamma_{reg}(X, L), Q) \to H^*(G, Q) \) is surjective.

All in all, we would like to study conditions under that the restriction homomorphism

\[
O^*: H^*(\Gamma_{reg}(X, L), Q) \to H^*(G, Q)
\]

is surjective. Since the ring \( H^*(G, Q) \) is an exterior algebra, it suffices to show that the generators lies to the image of the homomorphism \( O^* \). But the cohomology groups \( H^*(\Gamma_{reg}(X, L), Q) \) and \( H^*(G, Q) \) equipped with a mixed Hodge structure and the homomorphism \( O^* \) is strictly compatible with it. Recall that a primitive generator of the ring \( H^*(G, Q) \) in degree \( k \) has weight \( k + 1 \). By the strict compatibility of the homomorphism \( O^* \), we are forced to consider more closely the groups \( W_{k+1}H^k(\Gamma_{reg}(X, L), Q) \). In all known examples, when the group \( W_{k+1}H^k(\Gamma_{reg}(X, L), Q) \) can be computed, one is generated by linking classes.
where all generators is primitive elements. Here 

$$G$$

ample Genericity theorems. 4.1.

Example 4.0.12. The metaconjecture above can be also checked by a computer algebra system in the

Example 4.0.13. Moreover, using computer algebra systems, is not hard to prove the metaconjecture in the following cases:

Example 4.0.11. As was already said, the metaconjecture is checked in the following cases:

- $$G = SL(V)$$ and $$X = \mathbb{P}(V)$$. Here $$V \cong \mathbb{C}^n$$. (See formula 3.1.3).
- $$G = SO(V, q)$$ and $$X$$ is a quadric in $$\mathbb{P}(V)$$. Here $$V \cong \mathbb{C}^n$$ and $$q$$ is a non-degenerate quadratic form on the vector space $$V$$. (See formula 3.2.3 and formulas 3.3.3, 3.3.4, and 3.3.5).

Moreover, using computer algebra systems, is not hard to prove the metaconjecture in the following cases:

- $$X = Gr(2, 5), G = SL_5$$.
- $$X = Gr(2, 6), G = SL_6$$.
- $$X = Gr(3, 6), G = SL_6$$.
- $$G$$ is a simple Lie group of rank 2, $$X$$ is the complete flag variety for the group $$G$$.

Example 4.0.12. The metaconjecture above can be also checked by a computer algebra system in the following few cases. The group $$G$$ is $$SL_n$$ and $$X = Gr(k, n)$$, where $$1 \leq n \leq 11$$.

Example 4.0.13. We use the word “metaconjecture”, because its premise is more than general. Let us list a few counter-examples:

- The group $$G$$ trivially acts on the variety $$X$$. Then for any line bundle $$L$$ over the variety $$X$$ the Division Phenomena does not hold. So the metaconjecture is not true in this case.
- Suppose the setting of Example 4.0.7: $$G = SL(V), X = \mathbb{P}(\Lambda^n V), V \cong \mathbb{C}^{2n}$$. As we already said there, the Division Phenomena does not hold for any line bundle over $$\mathbb{P}(\Lambda^n V)$$ w.r.t. the group $$SL(V)$$. So the metaconjecture is not true in this case.
- Let $$G = SL_2, X_1 = \mathbb{P}^1$$ equipped with the usual action, $$X_2 = \mathbb{P}^1$$ equipped with trivial action, $$X = X_1 \times X_2$$ with diagonal action, and let $$p_1, p_2$$ be the projection maps. Then the Division Phenomena does not hold for any line bundle of the form $$p_1^*(L)$$.

4.1. Genericity theorems. The metaconjecture above predicts that for all but a finite number enough ample $$G$$-equivariant line bundles the Division Phenomena holds. Statements of such kind is very strong. Nevertheless, we can easily prove in many cases that the Division Phenomena holds for generic line bundle.

Let $$G$$ be a simple complex Lie group, $$G$$ is not type $$D_{2n}$$. Consider the cohomology ring $$H^*(G, \mathbb{Q})$$ of the group $$G$$. By the Milnor-Moore theorem the ring $$H^*(G, \mathbb{Q})$$ is isomorphic to exterior algebra $$\Lambda^*_G[a_1, \ldots, a_r]$$, where all generators is primitive elements. Here $$r$$ is the rank of the group $$G$$ and the degree of the element $$a_i$$ is equal to $$2d_i + 1, 1 \leq i \leq r$$ and $$d_1 \leq d_2 \leq \ldots \leq d_r$$. Since the Cartan type of the group $$G$$ is not $$D_{2n}$$ all numbers $$d_i, 1 \leq i \leq r$$ are pairwise distinct.
Let $X$ be a smooth projective $G$-variety of dimension $d$. For the subvariety $Y \subset X$ of codimension $m$ denote by $P_Y \subset H^{2d+2}_G(X, \mathbb{Q})$ the kernel of the homomorphism

$$H^{2d+2}_G(X, \mathbb{Z}) \xrightarrow{\cdot \frac{[Y]}{[G]}} H^{2d+1}_G(G \times X) \xrightarrow{\cdot \mathbb{Q}} H^{2m+1}_G(G).$$

Define the linear subspace $P_i \subset H^{2d+2}_G(X, \mathbb{Q})$ by the rule

$$P_i = \bigcap_{\text{codim}(Y) = d_i} P_Y.$$ 

By Remark 1.5.6 the linear subspace $P_i$ is proper.

Denote by $c^G_1, c^G_2, \ldots, c^G_d$ the $G$-equivariant Chern classes of the cotangent bundle $\Omega_X$. Set $c^G_{d+1} = 0$.

Define the mapping $F_X : H^2_G(X, \mathbb{Q}) \to H^{2d+2}_G(X, \mathbb{Q})$ by the rule

$$F_X(x) = \sum_{i=0}^{d+1} x^{d+1-i} c^G_i.$$ 

Consider sets $H^2_G(X, \mathbb{Z})$ and $H^{2d+2}_G(X, \mathbb{Z})$ as affine spaces over $\mathbb{Z}$. Then the mapping $F_X$ is an algebraic morphism.

**Notation 4.1.1.** Denote by $N_J(X) \subset \text{Pic}^G(X)$ the monoid of 1-jet spanned line bundles. Denote by $\Delta(X) \subset N_J(X)$ the subset of 1-jet spanned line bundles such that the restriction homomorphism

$$O^* : \text{Lk}(X, L) \to H^*(G, \mathbb{Q})$$

is not surjective. Moreover, denote by $B(X) \subset N_J(X)$ the set of $G$-equivariant line bundles $L$ such that for all $Y \subset X$ the Chern number $\langle c(J(L)), [Y] \rangle$ is non-zero.

**Corollary 4.1.2.** The set $c^G_1(B(X))$ is Zariski open subset of the affine space $H^2_G(X, \mathbb{Z})$. The intersection $c^G_1(B(X) \cap \Delta(X)) \subset c^G_1(N_J(X)) \subset H^2_G(X, \mathbb{Z})$ is an intersection of Zariski closed subset in $H^2_G(X, \mathbb{Z})$ and $c^G_1(N_J(X))$.

**Proof.** By Theorem 2.2.7 the set $c^G_1(B(X) \cap \Delta(X))$ is equal to the intersection

$$c^G_1(B(X) \cap N_J(X)) \cap \left( \bigcup_{i=1}^r F_X^{-1}(P_i) \right).$$

Since the morphism $F$ is algebraic, the intersection $\left( \bigcup_{i=1}^r F^{-1}(P_i) \right)$ is Zariski closed. 

We say that the Division Phenomena holds for generic $G$-equivariant line bundle on the $G$-variety $X$ if and only if the set $c^G_1(B(X) \cap \Delta(X))$ is an intersection of $c^G_1(B(X) \cap N_J(X))$ and a proper Zariski closed subset of $H^2_G(X, \mathbb{Z})$.

**Theorem 4.1.4.** Let $X = G/B$ be a complete flag variety for the group $G$. Then the Division Phenomena holds for generic $G$-equivariant line bundle on the $G$-variety $X$.

First, we have to prove the following lemma.

**Lemma 4.1.5.** Let $V$ be a finite-dimensional vector space. Let $a_0 = 1$ and let $a_1, \ldots, a_d \in \text{Sym}^*(V)$ be elements of the symmetric algebra such that the map

$$F : V \to \text{Sym}^{d+1}(V),$$

$$v \mapsto \sum_{i=0}^{d+1} a_i v^{d+1-i}$$

is well-defined. Then the image of $F$ is not contained in any linear subspace of $\text{Sym}^{d+1}(V)$.
Proof. Denote by $Y$ the projective closure of the affine space $\text{Sym}^{d+1}(V)$, i.e. $Y$ is a projective space, there exists a open embedding $j: \text{Sym}^{d+1}(V) \to Y$ and the complement to $j$ is $\mathbb{P}(\text{Sym}^{d+1}(V))$.

Denote by $F_0(V)$ the following intersection:

$$F_0(V) := j^{-1}(\text{Im}(F)) \cap \mathbb{P}(\text{Sym}^{d+1}(V)).$$

Notice that if the subset $F(V)$ is contained in a linear subspace of $\text{Sym}^{d+1}(V)$, then the subset $F_0(V)$ is contained in a hyperplane in $\mathbb{P}(\text{Sym}^{d+1}(V))$. But the variety $F_0(V)$ is the image of the Veronese embedding of the projective space $\mathbb{P}(V)$ into the projective space $\mathbb{P}(\text{Sym}^{d+1}(V))$. Well-known, the Veronese variety does not lie in any hyperplane.

Proof of the theorem. Since $X = G/B$, $B$ is a Borel subgroup, the equivariant cohomology ring $H^*_G(X,\mathbb{Z})$ is the polynomial ring $H^*(BT,\mathbb{Z})$, where $T$ is a maximal torus of the group $G$. In particularly, $H^*_G(X,\mathbb{Z}) \cong \text{Sym}^*(H^2(BT,\mathbb{Z}))$. Hence, the algebraic map $F_X$ is satisfied the conditions of Lemma 4.1.5.

We have the equation 4.1.3:

$$B(X) \cap \Delta(X) = B(X) \cap N_f(X) \cap \left( \bigcup_{i=1}^r F_X^i(P_i) \right).$$

Since the map $F_X$ is algebraic the preimages $F_X^i(P_i)$ is Zariski closed. By Lemma 4.1.5 each $F^{-1}(P_i)$ is a proper subset.

Now we want to obtain a few more genericity theorems. For this we will use Theorem 2.2.7 and the results of the subsection 3.1.

Theorem 4.1.6. Let $V$ be a complex linear $G$-representation, $\rho: G \to SL(V)$ is the representation homomorphism. Let $i: X \hookrightarrow \mathbb{P}(V)$ be a regular closed $G$-equivariant embedding. Suppose the following:

1. The cohomology class $i_*G1$ is a non-zero divisor in the ring $H^*_G(\mathbb{P}(V), \mathbb{Q})$;
2. The restriction homomorphism $H^*(SL(V), \mathbb{Q}) \to H^*(G, \mathbb{Q})$ is surjective.

Then the set $c^G_i(\Delta(X))$ is proper.

Proof. Notice that for any $X$ the mapping $F_X$ goes zero to zero, $F_X(0) = 0$ and tangent planes to affine spaces $H^2_G(X,\mathbb{Z})$ and $H^{2d+2}_G(X,\mathbb{Z})$ in zero point are equal to affine spaces themselves, i.e. $T_0(H^2_G(X,\mathbb{Z})) = H^2_G(X,\mathbb{Z})$ and $T_0(H^{2d+2}_G(X,\mathbb{Z})) = H^{2d+2}_G(X,\mathbb{Z})$. So it suffices to prove that the image of $d_0F$ does not lie in the union $\bigcup_{i=1}^r P_i$. The mapping $d_0F$ acts by the rule:

$$d_0F(x) = xc^G_i.$$

Let $x = i^*(b_1)$, where $b_1 \in H^2_G(\mathbb{P}(V), \mathbb{Q})$ is a generator. It is enough to show that the set

$$Z = \{S_X(d_0F(x))/i^i(\mathbb{P}^k) | 0 \leq k \leq \dim(V) - 1 \} \subset H^*(G, \mathbb{Q})$$

is a generating set of the ring $H^*(G, \mathbb{Q})$.

Let us to compute the class $S_X(d_0F(x))/i^i(\mathbb{P}^k)$. By Proposition 4.1.16

$$S_X(d_0F(x))/i^i(\mathbb{P}^k) = S_{F(V)}(h \sim i_*(c^G_i))/[\mathbb{P}^k].$$

So we have to compute the class $i_*G(c^G_i)$. By the projection formula,

$$i_*G(c^G_i) \sim i_*G1 = i_*G(c^G_i(\Omega_X) \sim i_*i_*G1) = i_*G(c^G_i(\Omega_X) \sim c^G_m(N_{X/P(V)}) = \pm c^G_{\dim(V)-1}(T_{P(V)}) \sim i_*G1.$$

Here $m$ is the codimension of $X$ in $P(V)$, and $N_{X/P(V)}$ is the normal bundle of $X$ in $P(V)$. By assumptions, the cohomology class $i_*G1$ is not a zero divisor. So we can identify $i_*G(c^G_i) = \pm c^G_{\dim(V)-1}(T_{P(V)})$. Since the sign does not matter below, we presume $i_*G(c^G_i) = c^G_{\dim(V)-1}(T_{P(V)})$.

Let us to compute the cohomology class $S_X(d_0F(i^*b_1))/i^i(\mathbb{P}^k)$.

$$S_X(d_0F(i^*b_1))/i^i(\mathbb{P}^k) = S_{P(V)}(b_1 \sim i_*G(c^G_i))/[\mathbb{P}^k] = S_{P(V)}(b_1 \sim c^G_{\dim(V)-1}(T_{P(V)})/\mathbb{P}^k].$$
Let $H = SL(V)$. Assume the notations of subsection 3.1. By the Euler exact sequence,

$$c_{\text{dim}(V)-1}^H(T_P(V)) = c_{\text{dim}(V)-1}(V(1)) = \sum_{i=0}^{\text{dim}(V)-1} (\text{dim}(V) - i)b_1^{\text{dim}(V)-i-1}a_i.$$ 

By the latter equation and equation 3.1.2 we observe

$$b_1 \sim c_{\text{dim}(V)-1}^H(T_P(V))) = -\sum_{i=1}^{\text{dim}(V)-1} b_1^{\text{dim}(V)-i}a_i.$$ 

By Proposition 1.4.10 the set

$$Z_1 = \{S_{P(V)}^H(b_1 \sim c_{\text{dim}(V)-1}^H(T_P(V)))/[\mathbb{P}^k] \mid 0 \leq k \leq \text{dim}(V) - 1\} \subset H^*(G, \mathbb{Q})$$

is a generating set for the ring $H^*(H, \mathbb{Q})$.

Since $S_{P(V)}^G(b_1 \sim c_{\text{dim}(V)-1}^H(T_P(V)))/[\mathbb{P}^k] = \rho^*(S_{P(V)}^H(b_1 \sim c_{\text{dim}(V)-1}^H(T_P(V)))/[\mathbb{P}^k])$, the set $Z$ is equal to the image $\rho^*(Z_1)$. By assumptions, the homomorphism $\rho^*$ is surjective. Hence the set $Z$ is also the generating set for the ring $H^*(G, \mathbb{Q})$.

**Corollary 4.1.7.** Let $V$ be a complex linear minuscule $G$-representation, $\rho: G \to SL(V)$ is the representation homomorphism. Let $X$ be a closed $G$-orbit in $P(V)$. Suppose that the restriction homomorphism $H^*(SL(V), \mathbb{Q}) \to H^*(G, \mathbb{Q})$ is surjective. Then the set $c_A^G(\Delta(X))$ is proper.

**Corollary 4.1.8.** On the following $G$-varieties the Division Phenomena holds for generic $G$-equivariant line bundle.

1. $G = SO(2n+1)$, $X$ is the orthogonal grassmannian $OGr(n+1, 2n+1)$.
2. $G = E_6$, $X$ is the Cayley plane $\mathbb{OP}^2$.
3. $G = E_7$, $X$ is the Freudenthal variety.

**Remark 4.1.9.** Let $V$ be the complex vector space of dimension $n$. Then the vector space $\Lambda^k V$ is a minuscule $SL(V)$-representation for any $0 < k < n$. The closed orbit $X$ in the projectivization $P(V)$ is the Grassmann variety $Gr(k, V)$. In light of Corollary 4.1.7 one hope to prove the Division Theorem for generic line bundle over the Grassmann variety $Gr(k, V)$. Unfortunately, the homomorphism

$$\rho^*: H^*(SL(\Lambda^k V), \mathbb{Q}) \to H^*(SL(V), \mathbb{Q})$$

is not always surjective. Moreover, it is still the open question for what pairs $(k, n)$ the homomorphism $\rho^*$ is not surjective. Let us list few examples of such pairs: $(k, 2k), k \in \mathbb{N}$, $(2, 2^k), k \in \mathbb{N}$, $(3, 27), (3, 486), (4, 12)$. For more detailed discussions see 8.11.

**References**

[1] E. Ballico and J. A. Wisniewski, On Bánica sheaves and Fano manifolds, Compositio Mathematica 102 (1996), no. 3, 313-335.
[2] E. H. Brown, The Cohomology of $BSO_n$ and $BO_n$ with Integer Coefficients, Proceedings of the American Mathematical Society 85 (1982), no. 2, 283–288.
[3] P. Deligne, Théorie de Hodge: III, Publications Mathématiques de l’IHÉS 44 (1974), 5-77 (french).
[4] I. M. Gelfand, M. Kapranov, and A. Zelevinsky, Discriminants, resultants, and multidimensional determinants, 1st ed., Mathematics: theory & applications, Birkhäuser, 1994.
[5] V. González-Aguilera and A. Liendo, On the order of an automorphism of a smooth hypersurface, Isr. J. Math. 197 (2013), no. 1, 29–49.
[6] A. G. Gorinov, Division theorems for the rational cohomology of some discriminant complements (2005), preprint, available at arxiv:math.AT/0511593.
[7] A. Grothendieck, Éléments de géométrie algébrique: IV, Inst. Hautes Études Sci. Publ. Math. 20, 24, 28, 32 (1967), 5–361.
[8] B. Gordon, A. S. Fraenkel, and E. G. Straus, On the determination of sets by the sets of sums of a certain order, Pacific J. Math. 12 (1962), no. 1, 187–196.
[9] M. Farber and M. Grant, Robot motion planning, weights of cohomology classes, and cohomology operations, Proc. Amer. Math. Soc. 136 (2008), no. 9, 3339–3349.
[10] C. A. M. Peters and J. H. M. Steenbrink, Degeneration of the Leray spectral sequence for certain geometric quotients, Mosc. Math. J. 3 (2003), no. 3, 1085–1095.
[11] J. L. Selfridge and E. G. Straus, *On the determination of numbers by their sums of a fixed order*, Pacific J. Math. 8 (1958), no. 4, 847–856.

[12] E. A. Tevelev, *Projective Duality and Homogeneous Spaces*, 1st ed., Encyclopaedia of mathematical sciences, Invariant theory and algebraic transformation groups, vol. 133, Springer, 2005.

[13] V. A. Vassiliev, *Geometric realization of the homology of classical Lie groups, and complexes that are S-dual to flag manifolds*, Algebra i Analiz 3 (1991), no. 4, 113–120 (in Russian); English transl., St. Petersburg Math. J. 3 (1992), no. 4, 809–815.

Faculty of Mathematics,
National Research University Higher School of Economics,
6 Usacheva Str., Moscow, Russia, 119048
E-mail: nikolay.konovalov.p@gmail.com