Bubble tree compactification of moduli spaces of vector bundles on surfaces

Dimitri Markushevich\(^1\), Alexander S. Tikhomirov\(^2\), Günther Trautmann\(^3\)

1 Mathématiques - bât. M2, Université Lille 1, 59655 Villeneuve d’Ascq Cedex, France
2 Department of Mathematics, Yaroslavl State Pedagogical University, Respublikanskaya Str. 108, 150 000 Yaroslavl, Russia
3 Universität Kaiserslautern, Fachbereich Mathematik, Erwin-Schrödinger-Straße, 67663 Kaiserslautern, Germany

Received 9 November 2011; accepted 13 May 2012

Abstract: We announce some results on compactifying moduli spaces of rank 2 vector bundles on surfaces by spaces of vector bundles on trees of surfaces. This is thought as an algebraic counterpart of the so-called bubbling of vector bundles and connections in differential geometry. The new moduli spaces are algebraic spaces arising as quotients by group actions according to a result of Kollár. As an example, the compactification of the space of stable rank 2 vector bundles with Chern classes \( c_1 = 0, c_1 = 2 \) on the projective plane is studied in more detail. Proofs are only indicated and will appear in separate papers.

MSC: 14J60, 14D20, 14D21

Keywords: Instantons • Vector bundles • Coherent sheaves • Moduli spaces of sheaves • Donaldson–Uhlenbeck compactification • Monads • Serre construction • Fulton–MacPherson compactification

© Versita Sp. z o.o.

1. Introduction

In this article, we describe a conceptual scheme of a new construction of a compactification of moduli spaces of stable bundles on surfaces whose boundary consists of vector bundles on trees of surfaces, replacing the torsion free semistable sheaves appearing in the Gieseker–Maruyama compactification. In this description the long proofs of completeness,
separatedness, versality, properness are replaced by brief sketches, and the complete versions will appear in full detail in subsequent papers. In conclusion, we produce a concrete example of the compactification of the moduli space of stable bundles on the projective plane $\mathbb{P}_2$ with second Chern class $c_2 = 2$. In this example, we provide an alternative explicit construction of the same compactification and give a complete description of its boundary.

To some extent, the replacement of limit sheaves in a compactification by vector bundles on trees of bubbles is very natural. The bubbling phenomenon appeared in eighties and nineties in the description of degeneration processes in several conformally invariant problems of geometric analysis: minimal surfaces (Sacks–Uhlenbeck), harmonic maps (Parker), pseudoholomorphic curves in symplectic varieties (Parker–Wolfson, Rugang Ye), and Yang–Mills fields on 4-manifolds (Feehan [6], Taubes [23], Uhlenbeck [29]).

Donaldson–Uhlenbeck [5] constructed a (partial) compactification $\overline{YM}_{n}^{DU}$ of the moduli space $YM_n$ of instantons of charge $n$ on a 4-manifold $S$, where the instantons are defined as the ASD Yang–Mills connections on a vector bundle over a 4-manifold $S$ with Chern classes $c_1 = 0$, $c_2 = n$:

$$\overline{YM}_{n}^{DU} \subset YM_n \cup (YM_{n-1} \times X) \cup (YM_{n-2} \times S^2X) \cup \cdots \cup S^gX.$$ 

In [17] and [18] it was shown that in the case of $S = S^4$, $\overline{YM}_{n}^{DU}$ has a real semi-algebraic structure.

The boundary of $\overline{YM}_{n}^{DU}$ consists of “ideal instantons”, that is, singular connections whose curvature is a sum of a smooth part and of several delta-functions. A degenerating family of bundles and connections on them encloses more information than is kept by an “ideal instanton”. The bubble-tree compactification $\overline{YM}_{n}^{FTU}$ of Feehan–Taubes–Uhlenbeck (FTU) is a kind of blowup of the boundary of $\overline{YM}_{n}^{DU}$, encoding the way an ASD connection degenerates into an ideal instanton by a connection over a tree of surfaces, obtained from $S$ by successive gluings of spheres $S^4$ at a finite set of points.

When $S$ is a complex projective surface, Donaldson proved that the Kobayashi–Hitchin correspondence identifies $YM_n$ with the moduli space of $\mu$-stable vector bundles $M^\mu_n$ with $c_1 = 0$, $c_2 = n$ on $S$ [4]; see also [15] for further developments. Thus the natural question can be asked, whether the Uhlenbeck–Donaldson and the FTU compactifications also have an algebra–geometric interpretation. For the first one, the answer is known: Jun Li [14] endowed $\overline{YM}_{n}^{DU}$ with a structure of a quasi-projective scheme and defined a birational morphism $\overline{YM}_{n}^{DU} \to \overline{YM}_{n}^{DU}$, where the closure of $M^\mu_n$ is taken in the Gieseker–Maruyama moduli space of semistable sheaves on $S$. In [10, Section 8.2], the construction of the algebraic version of $\overline{YM}_{n}^{DU}$ is extended to the moduli spaces of bundles with arbitrary fixed Chern class $c_1$. This compactification is denoted as $M^\mu_{n,s}$ and is called the moduli space of $\mu$-semistable sheaves, though it does not corepresent any functor of families of $\mu$-semistable sheaves.

The main motivation of our work is to find an algebra–geometric analog of the FTU compactification, which would be a kind of blowup of $M^\mu_{n,s}$, the algebraic incarnation of $\overline{YM}_{n}^{DU}$. By analogy with the topological bubbles which are 4-spheres, we can introduce the notion of an algebraic bubble. Given a complex surface $S$ and its blowup $\tilde{S}$ at a point $p$ with the exceptional line $L$, an algebraic bubble is a complex projective plane $P = P_2(C)$, attached to $\tilde{S}$ along $L$. Algebraic bubble trees are obtained by iterating this construction. They appear as fibers of the semi-universal family over the compactified configuration spaces of Fulton–MacPherson [7].

In the realm of algebraic geometry, the degenerations are described in terms of flat families. It turns out that using flat deformations, one can replace singular sheaves $\mathcal{F}$ on $S$ by bundles on the trees of surfaces $S_T$, and that the thus obtained tree bundles together with bundles on the original surface $S$ fit into a separated algebraic space of finite type. The latter is a partial compactification of $M_n$, which we denote by $M^\mu_n$ and call the algebraic bubble tree compactification. The construction of $M^\mu_n$ is the main result of the paper, stated in Theorem 7.2. As in [10], we construct our compactification for bundles with arbitrary fixed $c_1$. In some particular cases, for example, if $S = \mathbb{P}_2$, we can assert that $M^\mu_n$ is a proper algebraic space (Theorem 7.3).

Our main result gives an answer to the question which served its motivation, though we still ignore whether $M^\mu_n$ possesses some natural properties that one might expect. First, there is a natural contraction morphism $\overline{YM}_{n}^{FTU} \to \overline{YM}_{n}^{DU}$, but our construction does not provide such a morphism between $M^\mu_n$ and $M^\mu_{n,s}$, though there is an obvious set-theoretic map contracting the bubbles. Next, any algebraic bubble tree defines, in an obvious way, a spherical bubble tree: just contract to points all the $\mathbb{P}_1$’s that are intersection lines of the bubbles. But it is not clear whether this can be extended to a map between $\overline{YM}_{n}^{FTU}$ and $M^\mu_{n,s}$ in either direction, that would set a correspondence between the bundles on algebraic
bubble trees and the ASD connections on the associated spherical bubble trees. Finally, we ignore the structure of the natural birational map between $\mathbb{M}_0^S$ and the Gieseker–Maruyama compactification.

The problem is that the stock of tree-like bundles which really appear in our compactification is not described in effective terms. We know a priori only that the limit bundles belong to some redundant stock, given by Definition 3.1, and the latter contains some bundles which are nontrivial on the intersections of bubbles. We do not know if they really occur as limits of stable bundles on $S$, but if they do, there is no way to push them forward along the topological contraction in order to get a bundle (and a connection) on a spherical bubble tree. This problem seems to be unavoidable in our approach, whose important ingredient is the Serre construction for flat families of rank 2 bundles parametrized by a curve. The usage of Serre's construction also explains, why our work is done only for rank 2 bundles: we do not have Serre's construction for higher rank. It would be interesting to find another approach which would only include boundary bundles, trivial on the intersections of components and which would work for any rank. M. Kontsevich informed us that he has worked out, jointly with N. Nekrasov, a construction of an algebro-geometric bubble–tree compactification, which is free from the above drawback and which, moreover, possesses a perfect tangent-obstruction theory (their manuscript is in preparation).

We will also briefly mention some other related work. N. Timofeeva produces two compactifications by vector bundles on possibly reducible surfaces. In [24], she uses the idea that the global sections of a rank 2 vector bundle $E$ over a surface $S$, upon a twist by $O_S(n)$ with $n > 0$, embed $S$ into a Grassmannian $G = \text{Gr}(N, 2)$, and a degeneration of $E$ to a non-locally-free sheaf can be replaced by a degeneration of the image of $S$ in $G$, providing a vector bundle on a possibly singular surface as a limit. In [25–27], the non-locally-free limits of vector bundles are replaced by vector bundles on blowups of Fitting ideals. Both approaches make it impossible to get a simple description of the modifications of surfaces occurring on the boundary, whilst our algebraic bubble trees have a very simple description. Tree-like bundles were also used with the opposite goal by D. Gieseker in [9] in order to construct bundles on $S$ by deforming bundles on trees to bundles on $S$. Over curves, Nagaraj and Seshadri [19, 20] considered bundles on a degenerating family of curves and compactified them by bundles on reducible curves, pasting in trees of rational components. Buchdahl [2, 3] studied, by differential geometric means, a compactification of degenerating bundles on $S$ by bundles on a blowup of points in $S$ (which is an irreducible surface, unlike our bubble trees).

In Section 2, starting from a smooth projective surface $S$, we define the tree-like surfaces $S_T$ with root $S$ and their associated graphs $T$, and fix the notation for the line bundles on the tree-like surfaces. In Section 3, we describe the stock of tree-like bundles, from which we will draw the limit bundles on the boundary of our compactification. We provide a formula for the Euler characteristic of such tree-like bundles and state the boundedness result for them. In Section 4, we define the families of tree-like bundles and several versions of their moduli functors. We also state the completeness and the separatedness for moduli functors and provide sketches of proofs. In Section 5, we exploit the fact that any family of tree-like surfaces is induced by the semi-universal family of tree-like surfaces over the Fulton–MacPherson compactified configuration space. We state the versality property for the maps to the Fulton–MacPherson spaces and use such maps to endow our moduli functor with a functorial ample line bundle in the sense of J. Kollár. In Section 6, we use the idea that a tree-like bundle $E_T$, after a sufficiently ample twist, defines an embedding of the base surface $S_T$ into a Grassmannian $\text{Gr}(N, 2)$ for some $N > 0$. We then define a locally closed subset $H$ in the Hilbert scheme of $S \times \text{Gr}(N, 2)$, whose GIT quotient by $\text{SL}(N)$ is expected to be the moduli space of tree-like bundles. In Section 7, we state the main results on the existence of moduli spaces. Section 8 contains a detailed description of the bubble tree compactification of $\mathbb{M}_{0,2}(2; 0, 2)$, the moduli space of stable rank 2 bundles on the projective plane with Chern classes $c_1 = 0, c_2 = 2$. We show that it is isomorphic to the blowup of the Veronese surface in $\mathbb{P}^3$. For this example, we apply a specific approach, different from the one used in the general case, but both approaches involve resolution of singularities of the Serre construction for rank 2 bundles in families, so the example illustrates some ingredients of the proof in the general case.

In this article, $S$ is always a smooth complex projective surface, endowed with a very ample polarization class $h$, and $\mathbb{M}_{0,2}(2; N, n)$ denotes the Gieseker–Maruyama moduli space of semistable torsion free sheaves $E$ of rank 2 on $S$ with fixed Chern classes $c_1(E) = N$ and $c_2(E) = n \geq N^2/4$, where "semistable" means "Gieseker semistable with respect to $h". The space we are compactifying is the open subspace $\mathbb{M}_{0,2}(2; N, n)$ of stable vector bundles, assumed nonempty. If not stated otherwise, all the schemes are locally of finite type over $\mathbb{C}$, and the base of any family we consider is always assumed to be a scheme.
2. Trees of surfaces

2.1. Trees.
A tree $T$ in this article is a finite graph, oriented by a partial order $<$ and satisfying the following conditions.

- There is a unique minimal vertex $a \in T$, the root of $T$.
- For any $a \in T$, $a \neq a$, there is a unique maximal vertex $b < a$, the predecessor of $a$, denoted by $a^{-}$.
- By $a^+ \overset{\text{def}}{=} \{ b \in T : b^- = a \}$ we denote the set of direct successors of $a \in T$. We let $T_{\text{top}}$ denote the vertices of $T$ without successor.

A weighted tree is a pair $(T, c)$ of a tree $T$ with a map $c$ which assigns to each vertex $a \in T$ an integer $n_a$, called the weight or charge of the vertex, subject to the conditions:

$$\begin{align*}
n_a &\geq 0 & \text{if} & & a \neq a, \\
\# a^+ &\geq 2 & \text{if} & & n_a = 0 \text{ and } a \neq a, \\
n_a &\geq C,
\end{align*}$$

where $C \leq 0$ is some constant depending on $S$, $h$ and $N$, as specified below in formula (3). The total weight or total charge of a weighted tree is the sum $\sum_{a \in T} n_a = n$ of all the weights. We denote by $T_n$ the set of all trees which admit a weighting of total charge $n$. It is obviously finite.

2.2. Trees of surfaces.
Let $S$ be a smooth complex projective surface with an ample invertible sheaf $O_S(h)$. Our trees of surfaces are reduced surfaces $S_T$, defined for any tree $T$ and whose components are indexed by the vertices of the tree $T$. They are constructed by the following data.

(i) For each vertex $a \notin T_{\text{top}} \cup \{a\}$, let $P_a$ be a copy of $\mathbb{P}_2(C)$ together with a line $l_a \subset P_a$ and a finite subset $Z_a = \{ x_a \in P_a \setminus l_a : b \in a^+ \}$, and let $S_a \xrightarrow{\phi_a} P_a$ be the blowup of $P_a$ along $Z_a$. We will denote the exceptional lines in $S_a$ by $l_b$, $b \in a^+$, and the inverse image of $l_b$ in $S_a$ by the same symbol $l_b$.

(ii) For $a = \alpha$, we set $S_{\alpha} \xrightarrow{\phi_{\alpha}} S$ to be the blowup of $S$ at a finite set $Z_\alpha = \{ x_\alpha \in S : a \in a^+ \}$.

(iii) For $a \in T_{\text{top}}$, $S_a$ is a copy of $\mathbb{P}_2(C)$.

(iv) For each $a > \alpha$, we fix some isomorphism $\tilde{l}_a \xrightarrow{\phi_a} l_a$.

A tree-like surface of type $T$ over $S$ or a $T$-surface is now defined as the result $S_T$ of gluing the above surfaces $S_a$ along the isomorphisms $\phi_a$ into a reduced connected normal crossing surface with components $S_a$. We write

$$S_T = \bigcup_{a \in T} S_a$$

and identify $\tilde{l}_a$ with $l_b$ for all $a \in T$. After this identification the lines $l_a$ are the intersections $l_a = S_a \cap S_{a^-}$.

By the construction of $S_T$, all or a part of its components can be contracted. In particular, there is the morphism

$$S_T \xrightarrow{\nu} S$$

(2)

which contracts all the components except $S_a$ to the points of the finite set $Z_a$. 
Note that:

- There are no intersections of the components other than the lines \( l_\alpha \).
- If \( a \in T_{\text{top}} \) then \( S_a \) is a plane \( \mathbb{P}_2(\mathbb{C}) \).
- If \( T = \{ a \} \) is trivial, then \( S_T = S \).
- After contracting the lines \( l_\alpha \) topologically (over \( \mathbb{C} \)), one obtains a tree of 4-sphere bubbles.

### 2.3. Line bundles on \( T \)-surfaces.

Let now \( L \) be a line bundle on a \( T \)-surface \( S_T \), and let the inverse image of the divisor class \( h \) on \( S_a \) also be denoted by \( h \). Then the restrictions of \( L \) to the components can be written as

\[
L|_{S_a} = \mathcal{O}_{S_a} \left( mh - \sum_{a \in \alpha^+} m_\alpha l_\alpha \right) \quad \text{and} \quad L|_{S_a} = \mathcal{O}_{S_a} \left( m_\alpha l_\alpha - \sum_{b \in \alpha^-} m_b l_b \right).
\]

\( L \) is called ample if each \( L|_{S_a} \) is ample for all \( a \in T \). This means that

\[
m, m_\alpha > 0, \quad m^2 > \sum_{a \in \alpha^+} m_\alpha^2 \quad \text{and} \quad m_\alpha^2 > \sum_{b \in \alpha^-} m_b^2
\]

for all \( a \in T \). Let \( m, r \) be positive integers. We will say that \( L \) is of type \((r, m, h)\) if its restriction to the root surface is given by \( L|_{S_a} = \mathcal{O}_{S_a} \left( rh - r \sum_{a \in \alpha^+} m_\alpha l_\alpha \right) \) for some \( m_\alpha, a \neq a \).

We define the multitype of a line bundle on a \( T \)-surface \( S_T \) to be the sequence

\[
m_T \overset{\text{def}}{=} \{ m_\alpha \}_{a \in T},
\]

where \( m_a = m \). The above inequalities imply

### 2.4. Lemma.

(i) For any \( m, r > 0 \) and any weighted tree \( (T, c), T \in T_n \), there are at most finitely many ample line bundles of type \((r, m, h)\).

(ii) Given \( n \), there is an integer \( m_0 \) such that for any \( m \geq m_0 \) and any \( (T, c), T \in T_n \), there is an ample line bundle on \( S_T \) of type \((1, m, h)\).

### 3. Tree-like bundles

#### 3.1. Definition.

Let \( S_T = \bigcup_{a \in T} S_a \) be a \( T \)-surface with tree \( T \in T_n \). A vector bundle \( E = E_T \) on \( S_T \), or the pair \((E_T, S_T)\), is called a tree of vector bundles or a \( T \)-bundle if the restrictions \( E_a = E|_{S_a}, a \in T \), satisfy the following conditions:

(i) \( E_a \) has rank 2, \( c_1(E_a) = 0 \) for \( a \neq a \), \( c_1(E_a) = a_\alpha^2 N \), and the second Chern classes \( c_2(E_a) = n_a, a \in T \), define a weighting of \( T \) in the sense of (1).

(ii) If \( T \neq \{ a \} \), then \( E_a \) is admissible as defined below for any \( a \in T \).

(iii) If \( T = \{ a \} \), then \( |E| \in M_{S,h}(2; N, n) \).
3.2. Definition.
Let $S_T = \bigcup_{a \in T} S_a$ be a $T$-surface with tree $T \subseteq T_n$. Assume that $T \neq \{\alpha\}$. Let $a \in T$, and let $E_a$ be a rank 2 vector bundle on $S_a$. We will say that $E_a$ is admissible if one of the following conditions is satisfied:

(i) In case $a \in T_{\text{top}}$, with $S_a$ a plane $P_a$, $E_a$ is an extension of type

$$0 \rightarrow \mathcal{O}_{P_a} \rightarrow E_a \rightarrow J_{x, P_a} \rightarrow 0, \quad x = \{\text{pt}\} \not\in l_a, \quad c_2(E_a) = 1,$$

or

(ii) in case $a \in T_{\text{top}}$, with $S_a$ a plane $P_a$, $E_a$ is an extension of type

$$0 \rightarrow \mathcal{O}_{P_a}(-1) \rightarrow E_a \rightarrow J_{Z, P_a}(1) \rightarrow 0,$$

where $\dim Z = 0$, $Z \cap l_a = \emptyset$, $c_2(E_a) = \text{length}(Z) - 1 \geq 2$.

(iii) In case $a \notin T_{\text{top}}$, $a \neq \alpha$, $|a^\ast| > 2$, $E_a$ is a non-split extension of type

$$0 \rightarrow \mathcal{O}_{S_a}\left(-l_a + \sum_{b \in a^\ast} l_b\right) \rightarrow E_a \rightarrow J_{Z, S_a}\left(l_a - \sum_{b \in a^\ast} l_b\right) \rightarrow 0,$$

where $\dim Z \leq 0$, $Z \subseteq S_a \setminus \left\{\left(\bigcup_{b \in a^\ast} l_b\right) \cup l_a\right\}$, $c_2(E_a) = \text{length } Z + |a^\ast| - 1 \geq 1$, or

(iv) $a \notin T_{\text{top}}$, $a \neq \alpha$, and $E_a$ is a non-split extension of the type

$$0 \rightarrow \mathcal{O}_{S_a}\left(-\sum_{b \in a^\ast} l_b\right) \rightarrow E_a \rightarrow \mathcal{O}_{S_a}\left(-\sum_{b \in a^\ast} l_b\right) \rightarrow 0, \quad c_2(E_a) = |a^\ast| \geq 1,$$

or

(v) $a \notin T_{\text{top}}$, $a \neq \alpha$, and $E_a = 2\mathcal{O}_{S_a}$.

(vi) In case $a = \alpha$, $E_a$ is a non-split extension of type

$$0 \rightarrow \mathcal{O}_{S_a}\left(-qh + \sum_{b \in a^\ast} l_b\right) \rightarrow E_a \rightarrow J_{Z, S_a}\left(qh - \sum_{b \in a^\ast} l_b + a^\ast N\right) \rightarrow 0,$$

$$c_2(E_a) \geq C \overset{\text{def}}{=} -q_0^2(h^2) - q_0[(h \cdot N)]. \quad (3)$$

for some subset $a^\ast \subseteq a^\ast$, where $0 \leq q \leq q_0$ for some integer $q_0$ depending on $S_a$, and where $\dim Z \leq 0$, $Z \subseteq S_a \setminus \left(\bigcup_{b \in a^\ast} l_b\right)$, length $Z \leq n + q_0^2(h^2) + q_0[(h \cdot N)]$.

Remark.
The above definitions single out a possibly redundant class of vector bundles including all the bundles which may occur in degenerations. The conditions (i)–(vi) guarantee boundedness of the family of tree-like bundles and replace the (semi)stability conditions, which are not obvious for tree-like bundles. Those $T$-bundles which indeed occur in degenerations will be called limit $T$-bundles, see Definition 4.5.

Notation.
Let $E = E_T$ be a $T$-bundle on $S_T$ for a tree $T \subseteq T_n$. We also write $E_T = \#_{a \in T} E_a$ and define its total second Chern class to be

$$c_2(E_T) \overset{\text{def}}{=} \sum_{a \in T} c_2(E_a) = \sum_{a \in T} n_a = n.$$

Two $T$-bundles $(E_T, S_T)$ and $(E'_T, S'_T)$ are called isomorphic if there exists an isomorphism $\phi: S_T \rightarrow S'_T$ over $S$ such that $E_T = \phi^* E'_T$.

In the following we need formulas for the Euler characteristics of line bundles and $T$-bundles. These follow by standard computations on the components of the trees.
3.3. Lemma.  
For any $T$-bundle $(E_T, S_T)$, $T \in T_n$, and any ample line bundle $L$ on $S_T$ of type $(1, m, h)$, the following Euler characteristics are independent of the tree and are given by the formulas

\[
\chi(E \otimes L) = m^2(h^2) + m(h \cdot (2N - K_S)) + \frac{1}{2} (N \cdot (N - K_S)) + 2\chi(O_S) - n = N_m, \tag{4}
\]

\[
\chi(L) = \frac{m^2}{2} (h^2) - \frac{m}{2} (h \cdot K_S) + \chi(O_S). \tag{5}
\]

Later we will consider the embeddings of $S_T$ into the Grassmannian $G = Gr(N_m, 2)$ of 2-dimensional quotient spaces of the space $\mathbb{C}^{N_m}$, defined by the global sections of an appropriate twist of tree-like bundles $E_T$ on $S_T$. Here $N_m$ denotes the integer given by (4). The universal rank 2 quotient bundle on $G$ will be denoted by $\Omega$. We have the following boundedness result.

3.4. Proposition (boundedness).  
For any $n$ there is an integer $m_0 > 0$ such that, for any $m \geq m_0$ and any $T \in T_n$, there is an ample line bundle $L$ of type $(1, m, h)$ on $S_T$ such that for any $T$-bundle $E_T$ on $S_T$,

(i) $h^i(E_T \otimes L) = 0$ for $i > 0$ and $h^0(E_T \otimes L) = \chi(E_T \otimes L) = N_m$;

(ii) the evaluation map $\mathbb{C}^{N_m} \otimes O_{S_T} \to E_T \otimes L$ is surjective;

(iii) the induced map $S_T \xrightarrow{i_C^*} G = Gr(\mathbb{C}^{N_m}, 2)$ is a closed embedding such that

\[
i_C^* \Omega \cong E_T \otimes L, \quad i_C^* O_C(1) \cong L^\otimes \otimes O_{S_T}(\sigma^* N);\]

(iv) for any $j, q > 0$, $h^j(i_C^* O_C(q)) = 0$, and the Hilbert polynomial $P_C(q) \overset{\text{def}}{=} \chi(i_C^* O_C(q))$ is given by the formula

\[
P_C(q) = 2q^2m^2(h^2) + 2q^2m(h \cdot N) - qm(h \cdot K_S) + \frac{1}{2} q^2(N^2) - \frac{1}{2} q(N \cdot K_S) + \chi(O_S(N)). \tag{6}
\]

Proof. This follows from Serre’s theorems A and B, the boundedness of the family of all admissible bundles of given type on each $S_T$, which follows easily from the definition, and from the boundedness of the family of $h$-semistable vector bundles [22] with $c_1 = N$ and $c_2 \leq n$ on $S$. (ii) follows from the above and [28, Lemma 5.13]. Formula (6) follows directly from Lemma 3.3.

4. Families of tree-like bundles

In this section we fix the definition of families of $T$-surfaces $S_T$ and $T$-bundles $(E_T, S_T)$ for trees $T \in T_n$ with fixed total charge $n$.

4.1. Definition ($T_n$-families of surfaces).

Let $X \xrightarrow{\pi} Y$ be a flat family of trees of surfaces over a scheme $Y$ locally of finite type, whose trees belong to $T_n$. Such a family is called a family of trees of surfaces of type $T_n$, or simply a $T_n$-family, if there exists a morphism $\sigma : X \to S \times Y$ such that the following holds:

(i) $\pi = pr_2 \circ \sigma$.

(ii) For each closed point $y \in Y$ the morphism $\sigma_y = \sigma | S_y : S_y \to S \times \{y\}$ is $\pi$-stable, where $S_y = \pi^{-1}(y)$, is the standard contraction [2].

(iii) There is a union of irreducible components $X^b$ of $X$ such that the restriction $\sigma | X^b$ is a birational morphism $X^b \to S \times Y$. 

4.2. Examples.

- The standard contraction $X = S_T \rightarrow S$ of a single $T$–surface for a tree $T \in T_n$ is a $T_n$–family over a point with $X^b = S_o, \sigma^b : S_o \rightarrow S$. This $T_n$–family is good only if $T = \{a\}$.

- Let $C$ be a smooth curve and let $X \rightarrow S \times C$ be the blowup of a point $(s,c)$. Then $X \rightarrow C$ is a good $T_1$–family.

- Let $X = S_T \times Y$ be the product of a $T$–surface with some scheme $Y$ locally of finite type, $T \in T_n, T \neq \{a\}$. Then $X \rightarrow Y$ is a $T_n$–family with $X^b = S_o \times Y$ which is not good.

For families over smooth curves we have the following

4.3. Theorem.

Let $X \rightarrow C$ be a good $T_n$–family of surfaces over a smooth curve $C$. Then $X$ has at most $A_k$–singularities, analytically locally trivial along the lines of intersection of components in the fibres of $X$.

The $A_k$–singularities in $T_n$–families over curves really appear in the construction of limit bundles in the proof of Completeness Theorem 4.9 as a result of certain contractions of tree-like surfaces.

4.4. Definition ($T_n$–families of bundles).

(a) For a fixed total second Chern class $n$, a $T_n$–family of tree-like bundles is given as a triple $(E/X/Y)$, where $X/Y$ denotes a $T_n$–family $X \rightarrow Y$ of surfaces and $E$ is a vector bundle on $X$, such that its restriction to all the components of all the fibers of $\pi$ over the closed points of $Y$ are admissible.

(b) Let now $X \rightarrow Y$ be a good $T_n$–family of surfaces. By Definition 4.1 (ii), there exists a maximal dense open subset $U$ of $Y$ with the property that the fibers $X_y = \pi^{-1}(y)$ are isomorphic to $S$ for all closed points $y \in U$. A triple $(E/X/Y)$ as in (a) will be called a good $T_n$–family of (tree-like) bundles if the restrictions of $E$ to the fibers $X_y$ over the closed points $y \in U$ are stable vector bundles from $M_{S,X}(2; X, n)$.

There is an obvious notion of isomorphism and equivalence for $T_n$–families. Two $T_n$–families of bundles $(E/X/Y)$ and $(E'/X'/Y)$ are called isomorphic, $$(E/X/Y) \simeq (E'/X'/Y),$$ if there is an isomorphism $X \xrightarrow{\phi} X'$ of $T_n$–families of surfaces such that $E \simeq \phi^*E'$. The families are called equivalent if there is an isomorphism $X \xrightarrow{\phi} X'$ and an invertible sheaf $L$ on $Y$ such that $(E \otimes \pi^*L/X/Y)$ and $(E'/X'/Y)$ are isomorphic.

4.5. Definition (limit bundles).

A $T$–bundle $(E_T, S_T)$ of total charge $n$ is called a limit $T$–bundle if there exists a (germ of a) smooth pointed curve $(C,0)$ and a good $T_n$–family of bundles $(E/X/C)$ as defined in Definition 4.4 (b), such that $X|_{C \setminus \{0\}} \xrightarrow{\phi} S \times (C \setminus \{0\})$ is an isomorphism, $E|_{(C \setminus \{0\})}$ is a family of stable vector bundles from $M_{S,X}(2; X, n)$ and $E_T \simeq E|_{X_0}$ on the fibre of $X$ over $0 \in C$. 

The morphism $\sigma : X \rightarrow S \times Y$ will also be called a standard contraction.

- A $T_n$–family of surfaces $X \xrightarrow{\sigma} Y$ is called a good $T_n$–family of surfaces if $\sigma$ is birational on the whole of $X$.

- A $T_n$–family of surfaces $X \xrightarrow{\sigma} Y$ is called trivial if $\sigma$ is an isomorphism.

- Two $T_n$–families $X \xrightarrow{\sigma} Y$ and $X' \xrightarrow{\sigma'} Y$ over the same base $Y$ are called isomorphic if there exists an isomorphism $X \xrightarrow{\phi} X'$ such that $\pi' \circ \phi = \pi$ and $\sigma' \circ \phi = \sigma$.

where $X \xrightarrow{\phi} S \times Y$ and $X' \xrightarrow{\phi'} S \times Y$ are the standard contractions.
A limit $T$-bundle is called sss-limit $T$-bundle if there exists a (germ of a) smooth pointed curve $(C, 0)$ and a $T_a$-family of bundles $(E/X/C)$ as defined in Definition 4.4 (a), such that $X|(C \setminus \{0\}) \xrightarrow{\sim} S \times (C \setminus \{0\})$ is an isomorphism, $E|(C \setminus \{0\})$ is a family of strictly semi-stable vector bundles from $M_{s,a}(2; X, n)$ and $E_T \simeq E|_0$ on the fibre of $X$ over $0 \in C$.

Let $M_a(pt)$ denote the set of all limit $T$-bundles with $T \in T_a$ and $M_a(pt)$ the set of their isomorphism classes. More generally, we give the following definition.

4.6. **Definition (moduli stack and moduli functor).**

For any $Y$ as above, we denote by $M_a(Y)$ the set of all $T_a$-families $(E/X/Y)$ of limit bundles. Given a morphism $f : Y' \to Y$, it is obvious how to define the pullback $f^*(E/X/Y)$ of a family $(E/X/Y)$ and it is easy to verify that this is again a $T_a$-family of limit bundles. Thus $M_a$ is a pseudofunctor $M_a : (\text{Sch}/\mathbb{C})^{\text{op}} \to (\text{Sets})$ in the language of stacks, where in our setting (Sch/C) denotes the category of complex schemes locally of finite type over $\mathbb{C}$. By definition, $M_a(pt)$ can be identified with $M_a(\text{Spec } \mathbb{C})$. To get a functor, we define $M_a(Y) = M_a(Y)/\sim$, where $\sim$ denotes equivalence.

4.7. **Open sub-pseudo-functors of $M_a$.**

Consider the open sub-pseudo-functors $M_a^0$, $M_a^1$, $M_a^0$ and $M_a^1$ of the pseudo-functor $M_a : (\text{Sch}/\mathbb{C})^{\text{op}} \to (\text{Sets})$, defined as follows:

- $M_a^0(Y) = \{ (E/X/Y) \in M_a(Y) : \text{for all } y \in Y, E_y \text{ is not an sss-limit bundle} \}$,
- $M_a^1(Y) = \{ (E/X/Y) \in M_a(Y) : X \to Y \text{ is a trivial } T_a\text{-family} \}$,
- $M_a^0(Y) = M_a^1(Y) \cup M_a^1(Y)$,
- $M_a^1(Y) = \{ (E/X/Y) \in M_a(Y) : X \to Y \text{ is a trivial } T_a\text{-family and for all } y \in Y, E_y \in M_{s,a}(2; X, n) \}$.

$Y \in \text{Sch}/\mathbb{C}$. Obviously, for any $Y \in \text{Sch}/\mathbb{C}$,

$$M_a^1(Y) = M_a^0(Y) \cap M_a^0(Y).$$

The pseudofunctors $M_a^0, M_a^1, M_a^0$ and $M_a^1$, and their associated functors $M_a^0, M_a^1, M_a^0$ and $M_a^1$ will be used in the construction of the moduli spaces $M_a^0, M_a^0$, see Section 7.

4.8. **Lift of families.**

In the proofs of the main results one mostly has to deal with good families of tree-like bundles over curves. We use the standard notation $(C, 0)$ for a curve $C$ with a marked point $0 \in C$ and denote by $C^*$ the punctured curve $C \setminus \{0\}$. By a finite covering of curves $\tau : (\tilde{C}, 0) \to (C, 0)$ we understand a finite morphism $\tau : \tilde{C} \to C$ such that $\tau(0) = 0$.

For a given curve $(C, 0)$ and a given family of tree-like bundles $F = (E/X/C) \in M_a(C)$ over $C$ we denote the restriction of $F$ onto $C^*$ by

$$F^* = (E^*/X^*/C^*) \in M_a(C^*),$$

where $X^* = C^* \times_C X$, $\pi^* = \pi|X^*$, $\sigma^* = \sigma|X^*$, $E^* = E|X^*$. Respectively, we denote the lift of $F = (E/X/C) \in M_a(C)$ to $\tilde{C}$ by

$$\tilde{F} = \tilde{C} \times C F = (\tilde{E}/\tilde{X}/\tilde{C}) \in M_a(\tilde{C}),$$

where $\tilde{X} = \tilde{C} \times C X$, and $X \xrightarrow{\tilde{\lambda}} \tilde{X} \xrightarrow{\pi} \tilde{C}$ denote the natural projections with $\tilde{E} = \tilde{\lambda}^* E$.

Moreover, the notation for the lift $\tilde{F} \xrightarrow{\phi} F'$ to $\tilde{C}$ of an isomorphism $F \xrightarrow{\phi} F'$ over $C$ should be self-explaining.

The proofs of properness and separatedness of the moduli space we are going to construct are reduced to the respective properties of the pseudofunctor $M_a$ over the smooth curves, basing upon the valuative criteria for properness and separatedness. Thus the following completeness and separatedness theorems for the pseudofunctor $M_a$ are key results for the existence of a proper moduli space of tree-like bundles. The proofs can be only roughly indicated in this note.
4.9. Completeness Theorem.

The pseudo-functor $\mathcal{M}_n$ is complete in the following sense. For any smooth pointed curve $(C, 0)$ and any family of tree-like bundles $(E/C \times S/C) \in \mathcal{M}_n(C)$, there is a finite covering

$$(\tilde{C}, 0) \xrightarrow{\sim} (C, 0)$$

and a family $(\tilde{E}/\tilde{X}/\tilde{C}) \in \mathcal{M}_n(\tilde{C})$ of tree-like bundles together with an isomorphism $\varphi$,

$$\begin{array}{c}
C \times S & \xrightarrow{\varphi} & \tilde{C}^* \times S \\
\downarrow & & \downarrow \\
\tilde{C} & \xrightarrow{\varphi^{-1}} & \tilde{C}^*
\end{array}$$

such that

$$\tilde{E} \mid \tilde{X}^* \cong \varphi^*(\tau \times \text{id}_S)^*E.$$

Sketch of the proof.

Step 1: Let $X = C \times S$. We use the description of reflexive rank 2 sheaves $F$ on $X$ via the Serre construction\(^1\) in the relative situation:

$$0 \to \mathcal{O}_X(-qh) \xrightarrow{\sim} F \to \mathcal{I}_Z(qh+N) \to 0,$$

where $Z$ is finite over $C$.

Consider the closure $\overline{M}_{s,b}^c(2; N, n)$ of the moduli space of stable vector bundles $\mathcal{M}_{s,b}^c(2; N, n)$ in the projective scheme of Gieseker–Maruyama of $\chi$-semistable sheaves. Using its construction and projectivity, we can replace $C$ by a finite covering and assume that there is a reflexive sheaf $E$ over the whole of $X$ such that:

- $E \mid C^* \times S$ is the given bundle;
- there is a number $q$ with an exact sequence

$$0 \to \mathcal{O}_X(-qh) \to E \to \mathcal{I}_Z(qh+N) \to 0;$$

- $Z$ is reduced, smooth over $C^*$, and $Z \to C$ is flat and finite.

Step 2: Smoothing $Z$ by blowups of points of $Z \cap \{(0) \times S\}$ in the 3-fold $X = C \times S$.

Step 3: Separating the components of $Z$ by lifting the families over finite base changes and by further blowing up, thus getting "bubbles" and separating the corresponding extension sequences of the Serre construction.

Step 4: Contracting superfluous bubbles in the 3-fold obtained in Step 3, thereby producing a 1-parameter family of tree-like surfaces whose total space is a three-dimensional variety having at worst curves of $A_k$-singularities.

Step 5: The case of a $T_0$-family $(E/X^*/C^*)$ such that all the fibres of $(X^*/C^*)$ are reducible. Since all the bundles $E_y$, $y \in C$, are limit bundles, after a possible shrinking of $C^*$, there exists a surface $Y_0$ containing $C^*$ and a $T_0$-bundle $(E_0/X_0/Y_0) \in \mathcal{M}_n(Y_0)$ such that, for $Y^* = Y_0 \setminus C^*$ one has $X_{0,Y_0} \times Y^* \simeq S \times Y^*$. Extend $Y_0$ to a surface $Y$ containing $C$ and, respectively, extend $(E_0 \mid S \times Y^*/S \times Y^*/Y^*)$ to a family of Gieseker semistable sheaves $(E_t/S \times Y/Y)$. As in Step 1, represent $E_t$ as a sheaf obtained by the Serre construction, applied to some subscheme $Z_t$ of $S \times Y$, finite over $Y$. Then take $Z = Z_t \times C$ and apply Steps 2–4 to the family $(E_t/S \times C/S \times C/C)$. As a result, possibly after a base change over $C$, we obtain a $T_0$-family $(\tilde{E}/\tilde{X}/\tilde{C})$ extending $(E/X^*/C^*)$.

\(^1\) For $S = \mathbb{P}_2$ one might use monads.
4.10. Separatedness Theorem.
The pseudofunctor $\mathcal{M}_0$ is separated in the following sense: Let $(C, 0)$ be a smooth pointed curve and let $F = (E/X/C), F' = (E'/X'/C) \in \mathcal{M}_0(C)$ be two families of tree-like bundles. Suppose that there is an isomorphism

$$X^* = C^* \times_X X \xrightarrow{\phi_0} C^* \times_X X' = X'$$

such that the restrictions $F^* = (E^*/X^*/C^*), F'^* = (E'^*/X'^*/C'^*) \in \mathcal{M}_0(C^*)$ of $F, F'$ over $C^*$ are isomorphic via $\phi_0$, i.e. there exists an isomorphism of vector bundles $\psi_0: E^* \xrightarrow{\sim} E'^*$. Then $\phi_0$ extends to an isomorphism $\phi: X \xrightarrow{\sim} X'$, and there exists an invertible sheaf $L$ on $C$ with an isomorphism $\rho: L \xrightarrow{\sim} \mathcal{O}_C$ such that the isomorphism $\psi_0 \oplus \pi'_0(\rho): E^* \oplus \pi'_0(L \mid C^*) \xrightarrow{\sim} \phi_0 E'^*$

extends to an isomorphism $E \oplus \pi' L \xrightarrow{\sim} \phi'^* E'$, which provides an isomorphism of the $T_n$-families of bundles $(E \oplus \pi' L/X/C) \simeq (E'/X'/C)$.

The proof is rather elaborate, and what follows gives a brief idea of it. The first step is blowing up $X, X'$ to obtain a model $\tilde{X}$ smooth over $C$ and dominating both $X, X'$. There is an isomorphism of the lifted vector bundles $\mu_0: \tilde{E}^* \xrightarrow{\sim} \tilde{E}'^*$ over $C^*$. We can twist $E$ by $\pi'(d)^k$ for some integer $k$, where $J$ is the ideal sheaf of $0 \in C$, so that $\mu_0$ extends to a sheaf morphism $\mu: \tilde{E} \xrightarrow{\sim} \tilde{E}'$. We assume that $k$ is chosen to be minimal with this property, so that the restriction of $\mu$ to the fiber over $0 \in C$ is a nonzero morphism of sheaves $\mu_0: \tilde{E}_0 \xrightarrow{\sim} \tilde{E}'_0$.

Our definitions imply that the determinants of $E, E'$ are lifts of line bundles from $C \times S$, and $\det \mu$ can be viewed as a section of $(\det \tilde{E})^{-1} \otimes \det \tilde{E}' \cong \pi' L$, where $L$ is a line bundle on $C$. This implies that the support of $\det \mu$ is a simple normal crossing surface, a union of components $S_0, \ldots, S_n$ of $\tilde{X}_0$. A combinatorial argument shows that $\det \mu$ cannot vanish only on a part of components, so if $\mu$ is not an isomorphism, then $\mu_0$ is degenerate on every component of $\tilde{X}_0$. It is quite obvious then that the image of $\mu_0$, restricted to $S_0$, is a rank 1 sheaf over every component $S_0$ of $\tilde{X}_0$.

The second step is the proof of a fact from commutative algebra, which, stated in geometric terms, reads as follows:

4.11. Lemma.
Let $\phi: E \xrightarrow{\sim} E'$ be an injective morphism of rank 2 locally free sheaves on a smooth irreducible threefold $X$, and assume that $D = \text{Supp coker } \phi$ is an effective divisor on $X$ having smooth irreducible components $D_i, i \in A$, such that $L_i = \text{coker } \phi \mid D_i$ is a rank 1 sheaf for each $i \in A$. Then $L_i$ is a line bundle on $D_i, i \in A$. In particular, for each $i \in A$ the bundle $E' \mid D_i$ has a quotient line bundle $L_i$.

As follows from Definition 3.2 (i)–(ii), there is always a component in $\tilde{X}_0$, namely, the inverse image of any top component of $X'$, on which the restriction of $\tilde{E}'$ has no invertible quotient. This proves that $\mu$ has to be an isomorphism.

The last step of the proof is an argument, showing that there is only one way to contract some of the components in $\tilde{X}$, on which $\tilde{E}$ is trivial, in order that the result of the contraction might satisfy the condition (1).

Remark.
The separatedness fails for the full functor $\mathcal{M}_0$ as there may be non-isomorphic $S$-equivalent vector bundles on $S$ which are limits of the same family of stable ones.

5. Fulton–MacPherson configuration spaces

The construction of the moduli space, up to technical details, follows the standard pattern. First we use Hilbert schemes and embeddings into Grassmannians for constructing a space $H$, which parametrizes all the objects we want to include in our moduli space, and then we quotient $H$ by a group action. First, to construct the parameter space $H$, we invoke the results of W. Fulton and R. MacPherson from their paper [7, Sections 1–3], where they introduce the configuration spaces $S[n]$ for any natural number $n$. 

1341
5.1. Notation.
Let \( n = \max \{ |T| - 1 : T \in T_n \} \), and let \( Y_{FM} = S[n] \) be the Fulton–MacPherson configuration space with the semi-universal family \( S[n]^+ \) over \( S[n] \) of the so-called "\( n \)-pointed stable configurations" over \( Y_{FM} \). Let

\[
\pi_{FM} : X_{FM} \overset{\text{def}}{=} S[n]^+ \overset{\sigma_{FM}}{\longrightarrow} S \times Y_{FM} \overset{pr_2}{\longrightarrow} Y_{FM}
\]

(7)
denote the standard morphisms, where \( \sigma_{FM} \) is a birational morphism which decomposes into a sequence of blowups with explicitly described smooth centers. The family (7) has a number of nice properties. In particular, both \( X_{FM} \) and \( Y_{FM} \) are smooth projective varieties and the following "versality" property holds:

5.2. Proposition (versality).
Let \( X \to S \times Y \to Y \) be a flat deformation of a standard contraction \( S_T \to S \) with base point \( y \in Y \). Then there is an open neighbourhood \( V(y) \subset Y \) of the point \( y \) and a morphism \( V(y) \overset{f}{\to} Y_{FM} \) such that \( V \to S \times V(y) \to V(y) \) is the pull back of \( X_{FM} \to S \times Y_{FM} \to Y_{FM} \) under \( f \).

In particular, for any \( T \in T_n \) and any tree of surfaces \( S_T \) there exists a point (not unique) \( y \in Y_{FM} \) such that

\[
S_y \overset{\text{def}}{=} \pi_{FM}^{-1}(y) \simeq S_T.
\]

Note that \( Y_{FM} \) represents a functor of configuration families, [7, Theorem 4], but there is no obvious way of deriving the above versality from that. A proof can be done by a detailed analysis of the Kodaira–Spencer map related to this deformation problem.

The birational morphism \( \sigma_{FM} : X_{FM} \to S \times Y_{FM} \) is by its construction decomposed into a sequence \( \sigma_i = \sigma_{1} \circ \cdots \circ \sigma_{R} \) of blowups with smooth centers, say, \( Z_i \subset (\sigma_{1} \circ \cdots \circ \sigma_{i-1})^{-1}(S_0 \times Y_{FM}) \), \( i = 1, \ldots, R \). By this construction, the divisors \( D_i = (\sigma_1 \circ \cdots \circ \sigma_R)^{-1}(Z_i) \) satisfy the property that, for any \( y \in Y_{FM} \), \( D_i \cap S_y \) is a subtree of the tree \( S_y = \pi_{FM}^{-1}(y) \). For a sequence of positive integers \( m_0, n_1, \ldots, n_R \) we define the invertible sheaves

\[
\mathcal{M}_0 = \mathcal{O}_S(m_0 h) \boxtimes \mathcal{O}_{Y_{FM}} \quad \text{on} \quad S \times Y_{FM} \quad \text{and}
\]

\[
\mathcal{M} = \sigma_{FM}^* \mathcal{M}_0 \otimes \mathcal{O}_{X_{FM}}(-\Sigma n_i D_i) \quad \text{on} \quad X_{FM}.
\]

Using the above property of the divisors and Serre’s theorems A and B, one can derive the following lemma.

5.3. Lemma.
For given \( n \), there is a sequence of positive integers \( m_0, n_1, \ldots, n_R \) such that

(i) \( \mathcal{M}_0 \) is \( pr_2 \)-very ample for any \( r \geq 1 \) and \( pr_2^* \mathcal{M}_0 \) is locally free,

(ii) \( \mathcal{M} \) is \( \pi_{FM} \)-very ample for any \( r \geq 1 \) and \( \pi_{FM}^* \mathcal{M} \) is locally free.

5.4. Remark.
(i) Note that, since \( Y_{FM} \) is a projective variety, then for each \( T \in T_n \) and each \( a \in T \) the number

\[
\epsilon(T, a) = \max \{ \epsilon \in \mathbb{N} : \text{there exists} \ y \in Y_{FM} \text{such that} \ S_y \text{ has} \ T \text{ as its graph and} \ m_a(\mathcal{M}^\epsilon|S_y) = \epsilon \}
\]

is clearly finite, where, as above, \( S_y = \pi_{FM}^{-1}(y) \) and we use the notation from 2.3.

(ii) From the definition of the line bundle \( \mathcal{M} \) it follows that, if \( y, y' \in Y_{FM} \) are two points such that the fibres as trees of surfaces \( S_y \) and \( S_{y'} \) have the same graph \( T \), then the line bundles \( \mathcal{M}^\epsilon|S_y \) and \( \mathcal{M}^\epsilon|S_{y'} \) have the same multitype: \( m_T(\mathcal{M}^\epsilon|S_y) = m_T(\mathcal{M}^\epsilon|S_{y'}) \). In particular,

\[
S_y \simeq S_{y'} \quad \text{implies} \quad \mathcal{M}^\epsilon|S_y \simeq \mathcal{M}^\epsilon|S_{y'}.
\]
We thus are led to the following notation for an arbitrary tree of surfaces $S_T$ with $T \in T_n$:  
\[ m_T(M) \overset{\text{def}}{=} m_T(M|S_y) \quad (8) \]
for any isomorphism $S_T \overset{\sim}{\rightarrow} S_y$, $y \in Y_{FM}$. This notation is coherent, for the right hand side of (8) does not depend on the choice of $y$.

Next, since the set of all pairs $(E_T, S_T) \in M_n(\mathbb{C})$ is bounded by Proposition 3.4, we may strengthen the result of Lemma 5.3 in the following way.

### 5.5. Proposition.

One can choose the numbers $m_0, n_1, \ldots, n_R$ in Lemma 5.3 in such a way that, for any $(E_T, S_T) \in M_n(\mathbb{C})$ and any isomorphism $\phi_T: S_T \overset{\sim}{\rightarrow} S_y$, $y \in Y_{FM}$, the following holds:

(i) Put $m^2 = r_0 m_0$. Then the line bundle $L = \phi_T^\ast(M^{n_0})$ on $S_T$ has type $(1, m, h)$, $h^1(E_T \otimes L) = 0$, $j > 0$, and $N_0 \overset{\text{def}}{=} h^0(E_T \otimes L) = \chi(E_T \otimes L)$ is given by (4).

(ii) For any isomorphism $\theta: \mathcal{C}^{N_0} \overset{\sim}{\rightarrow} H^0(E_T \otimes L)$, the induced map

\[ \theta(y): \mathcal{C}^{N_0} \otimes \mathcal{O}_{S_T} \xrightarrow{\theta(y) \otimes \text{id}} H^0(E_T \otimes L) \otimes \mathcal{O}_{S_T} \xrightarrow{\text{id} \otimes \text{ev}} E_T \otimes L \]

is surjective, and the induced morphism $i_{\theta(y)}: S_T \rightarrow G = \text{Gr}(N_0, 2)$ to the Grassmannian of 2-dimensional quotients of $\mathcal{C}^{N_0}$ is an embedding such that

\[ i^\alpha_{\theta(y)} \mathcal{O} = E_T \otimes L \quad \text{and} \quad i^\alpha_{\theta(y)} \mathcal{O}_C(1) \simeq L^2 \otimes \mathcal{O}_{S_T}(\sigma^2 \mathcal{N}), \quad (9) \]

where $\Omega$ is the universal rank 2 quotient sheaf on $G$.

(iii) $h^0(i^\alpha_{\theta(y)} \mathcal{O}_C(q)) = 0$ for all $j, q > 0$ and

\[ \chi(i^\alpha_{\theta(y)} \mathcal{O}_C(q)) = 2q^2 m^2(h^2) + 2q^2 m(h \cdot \mathcal{N}) - q m(h \cdot K_S) + \frac{1}{2} q (N^2) - \frac{1}{2} q (N \cdot K_S) + \chi(O_S(\mathcal{N})) \]

the same value as in (6).

**Proof.** (i) follows by a standard argument and Lemma 5.3, (ii) and (iv) follow directly from Lemma 5.3 and Riemann–Roch. (iii) follows from a lemma on embeddings into Grassmannians in [28, Lemma 5.13], and boundedness, Proposition 3.4.

Note that from (9) it follows immediately that

\[ m_T(i^\alpha_{\theta(y)} \mathcal{O}_C(1) \otimes \mathcal{O}_{S_T}(-\sigma^2 \mathcal{N})) = 2m_T(\phi_T^\ast(M^{n_0})) . \]

### 5.6. A functorial line bundle.

By the above the sheaf $\pi_M \mathcal{M}$ is locally free on $Y_{FM}$ of rank $r_0$. We consider the line bundle

\[ L_{FM} \overset{\text{def}}{=} M^{n_0} \otimes \pi_M^\ast(\det \pi_M \mathcal{M})^{-1} \]

on $X_{FM}$. Using this line bundle, one can construct for any $T_n$-family $(E/X/Y)$ a line bundle $L_Y$ on $X$ such that these line bundles are compatible with base change in the sense of Kollár [12, Definition 2.3], see Lemma 6.2 and Section 7. This defines a descent of the line bundles $L_Y$ to a line bundle $L_{FM}$ on the moduli space. One of the possible approaches to the proof of the projectivity of our moduli space would be to show that $L_{FM}$ is ample. To this end, one might verify the weak positivity property [30] for the bundles $L_Y$, but this seems to be difficult for the $T_n$-families $(E/X/Y)$ that are not good.
6. The Hilbert scheme construction

The parameter space for $T_n$-surfaces will be an open part of the Hilbert scheme $\text{Hilb}^{p_H}(S \times G)$ consisting of $T_n$-surfaces, where as above $G$ is the Grassmannian $\text{Gr}(N_n, 2)$, and $P_n$ will be determined below.

6.1. Definition.

Let $m = r_0 m_0$ be as in Proposition 5.5. An embedded $T_n$-surface is defined to be a closed embedding $S_T \hookrightarrow S \times G$ such that

(i) the composition $S_T \hookrightarrow S \times G$ is a closed embedding with $i_G^* \mathcal{O}_G(1) \otimes \mathcal{O}_{S_T}(-\sigma^* \mathcal{N})$ ample of type $(2, m, h)$,

(ii) the composition $S_T \to S$ is a standard contraction as defined in (2), which is an isomorphism if $T$ is a single vertex.

Let $\mathcal{O}_{S \times G}(1) = \mathcal{O}_S(mh - \mathcal{N}) \otimes \mathcal{O}_G(1)$ be the chosen very ample polarization of $S \times G$ and, for an embedded tree $S_T \hookrightarrow S \times G$, let $P_{i_H}(q) = \chi(i^* \mathcal{O}_{S \times G}(q))\text{ be the corresponding Hilbert polynomial.}$ From Definition 6.1 it follows immediately that $i^* \mathcal{O}_{S \times G}(1)$ is a very ample line bundle of type $(3, m, h)$ on $S_T$. Hence, $P_{i_H}(q)$ is given by the formula (5) with $q m$ substituted for $m$:

$$P_{i_H}(q) = \frac{9}{2} q^2 h^2 + \frac{3}{2} q m(h : K_S) + \chi(\mathcal{O}_S).$$

Consider now the Hilbert scheme $\text{Hilb}^{p_H}(S \times G)$ and let

$$H' \subset \text{Hilb}^{p_H}(S \times G)$$

be the open subscheme of all embedded $T_n$-surfaces in the sense of Definition 6.1, and let $H^i \subset H'$ be the open part of those $S_T \hookrightarrow S \times G$ for which $S_T \simeq S$ and $\{i^* \mathcal{O}_S(-mh) \otimes \mathcal{O}_G(1)|S_T\} \in M_n(\text{Spec } \mathbb{C})$.

Finally, let $\overline{H}$ be the closure of $H^i$ in $H'$ and define

$$H = \{ (S_T \hookrightarrow S \times G) \in \overline{H} : i^* \mathcal{O}_S(-\mathcal{N} \otimes \mathcal{O}_G(1)) = \phi^* \mathcal{M}_G \text{ for an isomorphism } \phi : S_T \to S_y \}
\text{ for some point } y \in Y_{FM}\text{ and } \{i^* \mathcal{O}_S(-mh) \otimes \mathcal{O}_G(1)|S_T\} \in M_n(\text{Spec } \mathbb{C}) \}.$$

Here the existence of such a point $y \in Y_{FM}$ follows from the versality of $Y_{FM}$ and the property that $i^* \mathcal{O}_S(-\mathcal{N} \otimes \mathcal{O}_G(1)) \simeq \phi^* \mathcal{M}_G$ does not depend on the choice of the point $y$ in view of Remark 5.4 (ii). Equivalently,

$$H = \{ (S_T \hookrightarrow S \times G) \in \overline{H} : m_T \mathcal{O}_S(-\mathcal{N} \otimes \mathcal{O}_G(1)|S_T) = 2m_T(\mathcal{M}_G) \text{ and } \{i^* \mathcal{O}_S(-mh) \otimes \mathcal{O}_G(1)|S_T\} \in M_n(\text{Spec } \mathbb{C}) \}.$$

Since $H$ is a locally closed subscheme of the Hilbert scheme, there is the semi-universal family $X_H$ of embedded $T_n$-surfaces with diagram

$$\begin{align*}
X_H & \xrightarrow{h} S \times G \times H \\
\xrightarrow{\phi} H.
\end{align*}$$

6.2. Lemma.

There is a line bundle $L_H$ on $X_H$ such that for any fibre $X_{H,z}$, $z = (S_T \hookrightarrow S \times G)$, and any isomorphism $\phi : S_T \to S_y$, $y \in Y_{FM}$,

$$L_H|X_{H,z} \simeq L_{\phi*G} \simeq \mathcal{M}_G|S_y$$

and that the Hilbert polynomials of the fibres $X_{H,z}$ are formed with respect to the line bundle $L_H \otimes i_G^* \mathcal{O}_S(-\mathcal{N} \otimes \mathcal{O}_G(1) \otimes \mathcal{O}_{H})$. 

The bundle $L_H$ plays also the role of a functorial polarization in the sense of Kollár. It can be obtained as follows. By the versality of $Y_{FM}$, there is an open covering $(H_i)$ of $H$ with morphisms $H_i \to Y_{FM}$ such that $X_H | H_i$ is the pullback of $X_{FM}$ under $H_i$. Then the pulled back bundles $L^\ast H_i$ can be glued to give the bundle $L_H$. Using the bundle $L_H$, we define the bundle

$$E_H \overset{\text{def}}{=} O_S \boxtimes Q \boxtimes O_H | (X_H \otimes (L_H)^{-1}).$$

Then $(E_H, X_H, H)$ is a $T_n$-family and belongs to $\mathcal{M}_n(H)$.

### 6.3. Remark (boundedness).

It follows from the results of the next section that any $T_n$-bundle occurs in the family $(E_H/X_H/H)$, proving that the functor $\mathcal{M}_n$ is bounded.

### 7. The coarse moduli space

Given an arbitrary family $(E/X/Y) \in \mathcal{M}_n(Y)$, one can construct a line bundle $L_Y$ on $X$ as in the case of the family over $H$ using the versality of $Y_{FM}$. This bundle is fibrewise isomorphic to $\mathcal{M}^n$ and has type $(1, m, h)$. If there is a morphism $\rho: Y' \to Y$, then the constructions of the line bundles $L_Y$ are compatible, so that $L_Y' \simeq \rho^\ast L_Y$. By Proposition 5.5, $\pi_* (E \otimes L_Y)$ is locally free of rank $N_a$, and we can consider the principal $G\!/\!/(N_a)$-bundle

$$\tilde{Y} \overset{\text{def}}{=} \text{Isom} \big( k^{N_a} \otimes O_Y, \pi_* (E \otimes L_Y) \big) \xrightarrow{\rho} Y$$

over $Y$ with the Cartesian diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\phi}} & \tilde{Y} \\
\downarrow \tilde{\rho} & & \downarrow \rho \\
X & \xrightarrow{\phi} & Y.
\end{array}$$

Denote the lifts of the bundles by $\tilde{E} = \tilde{\rho}^\ast E$, $L_{\tilde{Y}} = \tilde{\rho}^\ast L_Y$. In view of Proposition 5.5, we have on $\tilde{X}$ a universal epimorphism

$$\Theta: C^{N_a} \otimes O_{\tilde{X}} \xrightarrow{\cong} \tilde{\pi}^\ast \pi_* (\tilde{E} \otimes L_{\tilde{Y}}) \xrightarrow{\text{ev}} \tilde{E} \otimes L_{\tilde{Y}},$$

which induces an embedding $i_\Theta: \tilde{X} \hookrightarrow S \times G \times \tilde{Y}$ in the commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{i_\Theta} & S \times G \times \tilde{Y} \\
\downarrow \tilde{\rho_j} & & \downarrow \rho_j \\
Y & & \tilde{Y}
\end{array}$$

such that

$$\tilde{E} \otimes L_{\tilde{Y}} = i_\Theta^\ast (O_S \boxtimes Q \boxtimes O_{\tilde{Y}}).$$

Let now $L_Y \otimes i_\Theta^\ast (O_S \boxtimes O_{C(1)} \boxtimes O_Y)$ serve as the polarization for computing the Hilbert polynomial in $S \times G \times \tilde{Y}$. The restriction of $L_Y$ to each fibre of $\tilde{X}$ over a point $\tilde{y} \in \tilde{Y}$ is by its construction isomorphic to some $\mathcal{M}^n_a \otimes O_{S_Y} \otimes O_{C(1)}$ or to some $O_{S_Y} \otimes O_{C(1)}$. Hence the fibres of $\tilde{X}$ all have the Euler characteristic $P_{\tilde{Y}}(q)$, and the conditions of the definition of $H$ are satisfied for the fibers of $\tilde{X}$. By the universal property of the Hilbert scheme, there is a morphism $\phi$ in the following Cartesian diagram such that $\tilde{X}$ is the pull back of $X_H$,

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\phi}} & \tilde{Y} \\
\downarrow \phi & & \downarrow \phi \\
X_H & \xrightarrow{\pi_H} & H
\end{array}$$

(10)
and such that
\[ \tilde{E} \otimes L_{\overline{y}} \simeq \tau_0^*(\mathcal{O}_S \boxtimes Q \boxtimes O_{\overline{y}}) \simeq \tau_0^*(\text{id} \times \phi)^*(\mathcal{O}_S \boxtimes Q \boxtimes O_{\overline{y}}) \simeq \tilde{\phi}^* \tau_t^*(\mathcal{O}_S \boxtimes Q \boxtimes O_{\overline{y}}) \simeq \tilde{\phi}^*(E_t \otimes L_t). \]

By functoriality \( L_{\overline{y}} \simeq \tilde{\phi}^*L_H \), and hence \( \tilde{E} \simeq \tilde{\phi}^*E_H \).

The group \( SL(N_m) \) acts naturally on the Grassmannian \( G = \text{Gr}(N_m, 2) \) and induces an action on \( \text{Hilb}^h(S \times G) \), under which \( H \) is invariant. In order to find an algebraic structure on the quotient by the action of \( SL(N_m) \), we have to shrink \( H \).

Consider the open \( SL(N_m) \)-invariant subsets on the quotient by the action of \( SL(N_m) \), we have to shrink \( H \). For the proof we use the properness criterion via families over curves and Theorem 4.10 on separatedness.

**7.1. Proposition.**

The action \( SL(N_m) \times H^0 \to H^0 \) is proper and \( H^0 // SL(N_m) \) is isomorphic to \( M_{2,h}(2; N, n) \). From \[ \text{Propositions 3.1, 3.2}, \] we conclude that the points of \( H^0 // SL(N_m) \) represent exactly the \( S \)-equivalence classes of semistable vector bundles on \( S \). Since \( M_{2,h}(2; N, n) \) is projective, \( M^0_{2,h} \) is quasi-projective. In particular, \( M^0_{2,h} \) is separated.

**Theorem.**

Fix an excellent base scheme \( \Lambda \). Let \( G \) be an affine algebraic group scheme of finite type over \( \Lambda \) and \( X \) a separated algebraic space of finite type over \( \Lambda \). Let \( m: G \times X \to X \) be a proper \( G \)-action on \( X \). Assume that one of the following conditions is satisfied:

(i) \( G \) is a reductive group scheme over \( \Lambda \).

(ii) \( \Lambda \) is the spectrum of a field of positive characteristic.

Then a geometric quotient \( p_X: X \to X//G \) exists and \( X//G \) is a separated algebraic space of finite type over \( \Lambda \).

Applying now the case (i) of this theorem to our situation with \( \Lambda = \text{Spec} \mathbb{C}, G = SL(N_m), X = H^0 \), we obtain from Proposition 7.1 that \( M^0_{2,h} \equiv H^0 // SL(N_m) \) is a separated algebraic space of finite type over \( \mathbb{C} \) and \( p: H^0 \to H^0 // SL(N_m) = M^0_{2,h} \) is a geometric quotient.

Note that
\[ M^0_{2,h}(2; N, n) = M^0_{2,h} \cap M^0_{2,h} \]
is open in both \( M^n \), \( M'^n \), so we can glue them together along \( M^n \) into an algebraic space

\[
M^n = M^n \cup M'^n,
\]

which is of finite type, separated but not necessarily complete.

For an arbitrary \( T_k \)-family \( (E/X/Y) \) one has an \( SL(N_n) \)-equivariant diagram (10). Assume in addition that the family belongs to \( M^0(Y) \). Then there is the diagram

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\phi} & H^0 \\
\rho \downarrow & & \downarrow \rho_1 \\
Y & \xrightarrow{i} & M^0
\end{array}
\]

As \( \rho \) is a principal bundle map, it follows that there exists a morphism (of algebraic spaces) \( f: Y \rightarrow M^0 \) which extends (11) to a commutative diagram

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\phi} & H^0 \\
\rho \downarrow & & \downarrow \rho_1 \\
Y & \xrightarrow{f} & M^0
\end{array}
\]

The existence of this modular morphism \( f: Y \rightarrow M^0 \) and the existence of the semi-universal family \( (E^0/H^0) \) means that \( M^n \) corepresents the functor \( M^0 \), i.e. it is the wanted moduli space. The same argument applies to \( M'^n \) and \( M'^0 \).

Thus we have the following

7.2. Theorem.
There exists a separated algebraic space \( M^n \) (resp. \( M'^n \)) of finite type corepresenting the functor \( M^0 \) (resp. \( M'^0 \)).

There is a particular case in which we can say more. This is the case when \( M^n = M^0 \). This happens when \( M_{S,3}(2; N, n) \) contains no strictly semistable locally free sheaves, for example, when \( S = P^2 \). Then we can state:

7.3. Theorem.
Let \( S = P^2 \). Then \( M^n = M^0 \), and there exists a proper algebraic space \( M^0 \) corepresenting the functor \( M^0 \).

In the general case, we can only suggest a conjecture.

7.4. Conjecture.
Let \( S \) be any smooth projective surface. Then there exists a proper algebraic space \( M^0 \) corepresenting \( M^0 \).

It is not clear whether one should expect \( M^0 \) to be projective. However, in some examples one can present an explicit construction of \( M^0 \) as a projective variety. One such example is treated in the next section.

8. The bubble-tree compactification of \( M_{P^2}(2; 0, 2) \)

Let \( M(0, 2) = M_{P^2}(2; 0, 2) \) be the moduli space of semistable sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and rank 2. It is well known that \( M(0, 2) \) is isomorphic to the \( P^5 \) of conics in the dual plane, the isomorphism being given by \( [\mathcal{F}] \leftrightarrow C(\mathcal{F}) \), where \( C(\mathcal{F}) \) is the conic of jumping lines of \( [\mathcal{F}] \) in the dual plane, see \([1, 16, 21]\). For a more explicit description we use the following notation.

\[ i.e., \text{complete, separated and of finite type} \]
8.1. Beilinson resolutions.

In the sequel $S$ will be the projective plane: $S = P(V) = \mathbb{P}^2$, where $V$ is a fixed 3-dimensional vector space. We will write $m\mathcal{F}$ for $\mathbb{C}^m \otimes \mathcal{F}$, where $\mathcal{F}$ is a sheaf.

It is well known that any sheaf $\mathcal{F}$ from $M(0,2)$ has two Beilinson resolutions

$$0 \to 2\mathcal{O}_S(2) \xrightarrow{A} \mathcal{O}_S(1) \to \mathcal{F} \to 0, \quad 0 \to 2\mathcal{O}_S(-2) \xrightarrow{B} 4\mathcal{O}_S(-1) \to \mathcal{F} \to 0,$$

where the matrices $A$ (of vectors in $V$) and $B$ (of vectors in $V^*$) are related by the exact sequence

$$0 \to \mathbb{C}^2 \xrightarrow{A} \mathbb{C}^2 \otimes V \xrightarrow{B} \mathbb{C}^4 \to 0.$$

Recall that $\text{Hom}(\mathcal{O}_S(2), \mathcal{O}_S(1))$ is canonically isomorphic to $V$ with $v \in V$ acting by contraction. The matrix product $BA$ is zero, where the elements of the two matrices are multiplied by the rule that for $v \in V$ and $f \in V^*$, the product $fv$ is $f(v)$.

The matrices $A$ and $B$ are determined by $\mathcal{F}$ uniquely up to isomorphisms of the above resolutions. The first resolution implies that $\det A \in S^2 V$ is non-zero. It is the equation of the conic $C(\mathcal{F})$. The sheaf $\mathcal{F}$ is locally free if and only if $C(\mathcal{F})$ is smooth, or if and only if $\mathcal{F}$ is stable. If $C(\mathcal{F})$ decomposes into a pair of lines, then $A$ is equivalent to a matrix of the form

$$\begin{pmatrix} x & 0 \\ z & y \end{pmatrix},$$

and $\mathcal{F}$ is an extension

$$0 \to I_{(x)} \to \mathcal{F} \to I_{(y)} \to 0.$$

In this case $\mathcal{F}$ is $S$-equivalent to $I_{(x)} \oplus I_{(y)}$.

We will first present the $T_2$-bundles appearing in the compactification as limits of 1-parameter degenerations. We will start with the following explicit description of the blowup of $\mathbb{A}^1 \times S$ at a point.

8.2. A special blowup.

Let $(e_0, e_1, e_2)$ be a basis of $V$ and $(x_0, x_1, x_2)$ the corresponding homogeneous coordinates on $S$. The blowup

$$X \twoheadrightarrow \mathbb{A}^1 \times S$$

of the point $p = (0, (e_0))$ is the subvariety of $\mathbb{A}^1 \times S \times \mathbb{P}^2$ given by the equations

$$t x_0 u_1 - x_1 u_0 = 0, \quad t x_0 u_2 - x_2 u_0 = 0, \quad x_1 u_1 - x_2 u_0 = 0,$$

where the $u_v$ are the coordinates of the third factor $\mathbb{P}^2$. We consider the following divisors on $X$:

(i) $\tilde{S}$, the proper transform of $\{0\} \times S$, isomorphic to the blowup of $S$ at $p$;

(ii) $S_t$, the exceptional divisor of $\sigma$;

(iii) $H$, the lift of $\mathbb{A}^1 \times h$, where $h$ is a general line in $S$;

(iv) $L$, the divisor defined by $\mathcal{O}_X(L) = \text{pr}_2^* \mathcal{O}_{\mathbb{P}^2}(1)$.

The following divisors are linearly equivalent: $\tilde{S} \sim H - L$. We also let $x_v$, resp. $u_v$ denote the sections of $\mathcal{O}_X(H)$ resp. $\mathcal{O}_X(L)$ lifting the above coordinates. Using the equations of $X$, we see that the canonical section $s$ of $\mathcal{O}_X(\tilde{S})$ fits into the diagram

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{s} & \mathcal{O}_X(H - L) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(H) \biguplus \mathcal{O}_X(L) & \xrightarrow{(u_0, u_1, u_2)} & \mathcal{O}_X(H).
\end{array} \quad (12)$$
8.3. Example.

Using the notation from 8.2, we choose

\[ A = \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix}, \]

representing \( \mathcal{I}_0 = \mathcal{I}_0 \oplus \mathcal{I}_0 \), where \( \mathcal{I}_0 \) is the ideal sheaf of the point \( \langle e_0 \rangle \in P(V) \). Denote \( C = \Lambda^1(C) \) and let \( \mathcal{F} \) be the family of \( M(0,2) \)-sheaves over \( C \times S \) defined as the cokernel of the matrix

\[ A(t) = \begin{pmatrix} e_0 & -ta \\ -tb & e_0 \end{pmatrix} \]

with \( a = \alpha_1 e_1 + \alpha_2 e_2 \) and \( b = \beta_1 e_1 + \beta_2 e_2 \). Then \( \mathcal{F} \mid (\{t\} \times S) \) is locally free for \( t \neq 0 \). The second Beilinson resolution of \( \mathcal{F} \), see 8.1, is then

\[
0 \to 2\mathcal{O}_C \boxtimes \mathcal{O}_S(-2) \xrightarrow{\mathcal{B}(t)} 4\mathcal{O}_C \boxtimes \mathcal{O}_S(-1) \to \mathcal{F} \to 0
\]

with \( \mathcal{B}(t) \) given by

\[ \mathcal{B}(t) = \begin{pmatrix} x_1 & x_2 & \alpha_1 t x_0 & \alpha_2 t x_0 \\ \beta_1 t x_0 & \beta_2 t x_0 & x_1 & x_2 \end{pmatrix}. \]

Let now \( X \to C \times S \) be the blowup of \( C \times S \) at \( p = (0, \langle e_0 \rangle) \). We have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & \to & 2\mathcal{O}_S(-1) \\
0 & \to & 2\mathcal{O}_X(-2H) \xrightarrow{\sigma \mathcal{B}(t)} 4\mathcal{O}_X(-H) \xrightarrow{\sigma^* \mathcal{F}} 0 \\
0 & \to & 2\mathcal{O}_S(-H \to L) \xrightarrow{B_X} 4\mathcal{O}_X(-H) \xrightarrow{F} 0 \\
0 & \to & 2\mathcal{O}_S(-1) \xrightarrow{0} 0 \\
0 & \to & 0
\end{array}
\]

From the diagram (12) it follows that

\[ B_X = \begin{pmatrix} u_1 & u_2 & \alpha_1 u_0 & \alpha_2 u_0 \\ \beta_1 u_0 & \beta_2 u_0 & u_1 & u_2 \end{pmatrix}. \]

Here \( 2\mathcal{O}_S(-1) \) is the torsion of \( \sigma^* \mathcal{F} \), supported on the exceptional divisor, and \( F \) is its locally free quotient. Finally, let \( E \overset{\text{def}}{=} F(-\tilde{S}) \). One can verify that the restriction of \( E \) to \( \tilde{S} \) is trivial, \( E_{\tilde{S}} \simeq 2\mathcal{O}_{\tilde{S}} \), and that the restriction \( E_{S_1} \) belongs to \( M_{P_2}(2; 0, 2) \). Thus \( E \mid (\tilde{S} \cup S_1) \) is a \( T_2 \)-bundle with weighted tree of type (II), see (13), and is a flat degeneration of bundles in \( M(0, 2) \). Note that the vectors \( \alpha, b \) used in the construction of this degeneration correspond to two points on the double line \( C(\mathcal{F}_0) = \{ e_0^2 = 0 \} \) and thus determine a “complete conic” in classical terminology. Choosing different such pairs will lead to non-isomorphic \( T_2 \)-bundles.
8.4. Example.
Let now
\[ A = \begin{pmatrix} e_0 & 0 \\ -t^2 e_1 & e_2 \end{pmatrix} \]
and \( \mathcal{F}_0 = \mathcal{J}_0 \oplus \mathcal{J}_2 \), where \( \mathcal{J}_0 \), resp. \( \mathcal{J}_2 \), are the ideal sheaves of the points \( p_0 = (e_0) \), resp. \( p_2 = (e_2) \), in \( S = P(V) \). Here we consider the blowup \( X \to C \times S \) of \( C \times S \) at the two points \( p_0, p_2 \). Let \( \tilde{S} = S_0 \) again denote the proper transform of \( \{0\} \times S \), and let \( S_1 \) and \( S_2 \) denote the exceptional divisors of \( \sigma \). We embed \( X \) into \( C \times S \times \mathbb{P}_2 \times \mathbb{P}_2 \) with equations
\[ t_{x_0}u_1 - x_1u_0 = 0, \quad t_{x_0}u_2 - x_2u_0 = 0, \quad x_1u_2 - x_2u_1 = 0, \quad t_{x_1}v_0 - x_0v_2 = 0, \quad t_{x_1}v_1 - x_1v_2 = 0, \quad x_1v_0 - x_0v_1 = 0, \]
where \( u_v \) and \( v_v \) are the coordinates of the third and fourth factors \( \mathbb{P}_2 \) respectively. Define the sheaf \( \mathcal{F} \) over \( C \times S \) as the cokernel of the matrix
\[ A(t) = \begin{pmatrix} e_0 & -t^2 e_1 \\ -t^2 e_1 & e_2 \end{pmatrix}. \]
The corresponding matrix \( B(t) \) is then given by
\[ B(t) = \begin{pmatrix} x_1 & x_2 & 0 & t^2 x_0 \\ t^2 x_2 & 0 & x_0 & x_1 \end{pmatrix}. \]
In order to find the limit \( \mathbb{T}_2 \)-bundle on the \( \mathbb{T}_2 \)-surface \( \tilde{S} \cup S_1 \cup S_2 \), we proceed as in Example 8.3. Let \( x_v, u_v, v_v \) denote the sections of \( O_x(H), O_x(L_1), O_x(L_2) \) obtained by lifting the respective homogeneous coordinates, and let \( S_1, S_2, \ldots, S_{\ell} \in \Gamma \mathcal{O}_x(S_i) \) be the canonical sections with divisors \( S_1 \sim H - L_1 \) and \( S_2 \sim H - L_2 \). We obtain a torsion free sheaf \( F \) on \( X \) as the quotient in the exact triple
\[ 0 \to O_{S_1}(-1) \oplus O_{S_2}(-1) \to \sigma^* \mathcal{F} \to F \to 0 \]
with resolution
\[ 0 \to O_x(H - L_1) \oplus O_x(H - L_2) \xrightarrow{B_x} 4O_x(-H) \to F \to 0, \]
where
\[ B_x = \begin{pmatrix} u_1 & u_2 & 0 & tu_0 \\ tv_2 & 0 & v_0 & v_1 \end{pmatrix}. \]
The restrictions of \( F \) to \( \tilde{S}, S_1, S_2 \) are now
\[ F_{\tilde{S}} \simeq O_{\tilde{S}}(-\ell_1) \oplus O_{\tilde{S}}(-\ell_2), \]
where \( \ell_i = S_i \cap S, i = 1, 2 \), are the two exceptional curves on \( \tilde{S} \), and
\[ F_{S_1} \simeq O_{S_1} \oplus O_{S_1}(1), \quad F_{S_2} \simeq O_{S_2} \oplus O_{S_2}(1), \]
where \( q_1, \) resp. \( q_2 \), are the points \( \{u_1 = u_2 = 0\} \), resp. \( \{v_0 = v_1 = 0\} \).
Next, let \( E' \) be the elementary transform given by the exact sequence
\[ 0 \to E' \to F \to O_{S_1} \oplus O_{S_2} \to 0. \]
This sheaf turns out to be locally free on \( X \). Then the sheaf \( E \equiv E'(-\tilde{S}) \) has the restrictions
\[ E_{\tilde{S}} \simeq 2O_{\tilde{S}}, \quad E_{S_1}, \quad E_{S_2}, \]
where the Chern classes of \( E_{S_1}, E_{S_2} \) are \( c_1 = 0, c_2 = 1 \), and there are non-split extensions of the form
\[ 0 \to O_{S_1} \to E_{S_1} \to O_{\ell_1} \to 0, \quad 0 \to O_{S_2} \to E_{S_2} \to O_{\ell_2} \to 0. \]
Thus \( E \mid (\tilde{S} \cup S_1 \cup S_2) \) is a \( \mathbb{T}_2 \)-bundle with weighted tree of type (III), see (13). It is again a flat degeneration of vector bundles from \( M(0, 2) \).

The third example with weighted tree of type (IV) can be constructed by a similar but slightly more complicated procedure. The final result about the moduli space \( M_{\ell} \) is the following theorem.
8.5. **Theorem.**

Let $P(S^2V)$ denote the blowup of $P(S^2V)$ along the Veronese surface.

(i) The moduli space $M_2$, defined in Theorem 7.3 with $n = 2$, is isomorphic to $P(S^2V)$.

(ii) The isomorphism classes $[E_r, S_1], T \in T_2$, are in one-to-one correspondence with the points of $P(S^2V)$.

(iii) The weighted trees associated to the pairs $[E_r, S_1]$ that occur in $M_2$ are of one of the following four types:

$$\begin{align*}
\sigma : & & 1 & & 2 & & 0 & & 1 & & 1 & & 0 & & 0 \\
\text{(I)} & & \text{(II)} & & \text{(III)} & & \text{(IV)}
\end{align*}$$

The four types of weighted trees define a stratification of $P(S^2V)$ in locally closed subsets in the following way. Let $\Sigma_0$ be the exceptional divisor of $P(S^2V)$ and $\Sigma_1$ the proper transform of the cubic hypersurface of decomposable conics in $P(S^2V)$. Then

- the points of $P(S^2V) \setminus (\Sigma_0 \cup \Sigma_1)$ represent the bundles of type (I) on the original surface $S$;
- the bundles in $\Sigma_0 \setminus \Sigma_1$ are of type (II);
- the bundles in $\Sigma_1 \setminus \Sigma_0$ are of type (III);
- the bundles in $\Sigma_1 \cap \Sigma_0$ are of type (IV).

There is a construction of a complete family $(E/X/F)$ of $T_2$-bundles which contains all types of such bundles. This will be sketched next.

8.6. **Semi-universal family for $M(0, 2)$.**

From now on $H$ will denote the vector space $\mathbb{C}^2$ and $G$ will be the Grassmannian $Gr(H \otimes V, 2) = Gr(6, 2)$ of 2-dimensional subspaces of $H \otimes V$. Let $\mathfrak{U}$ denote the universal subbundle on $G$. For any subspace $C^2 \substack{\hookrightarrow \rightarrow \lambda^2 \otimes S^2 V \cong S^2 V}$ there is the determinant homomorphism $C^2 \cong \lambda^2 \otimes S^2 V \cong S^2 V$. We denote by $G^{\lambda^2}$ the open subset defined by $\lambda^2 y \neq 0$. By the description in 8.1, this open subset parametrizes all the sheaves in $M(0, 2)$, and there is a semi-universal family $\mathcal{F}$ on $G^{\lambda^2} \times S$ with resolution

$$0 \to \mathfrak{U} \otimes \Omega^2(2) \to H \otimes \mathcal{O}_G \otimes \Omega^1(1) \to \mathcal{F} \to 0,$$

where $\mathfrak{U}$ is the universal subbundle on $G$, restricted to $G^{\lambda^2}$. The map $y \mapsto \lambda^2 y$ defines a morphism $G^{\lambda^2} \to P(S^2V)$ which is the modular morphism of the family and at the same time is a good quotient by the natural action of $SL(H)$ on $G^{\lambda^2}$:

$$G^{\lambda^2}/\text{SL}(H) \simeq P(S^2V) \simeq M(0, 2).$$

Consider the subvarieties $\Delta_0, \Delta'_0, \Delta''_0, \Delta'''_0$ of $G^{\lambda^2}$ defined as follows. For each $y \in G^{\lambda^2}$, let $l_y \subset P(H \otimes V)$ be the corresponding line, and let $S$ be the image of the Segre embedding $P(H) \times P(V) \to P(H \otimes V)$. Then

$$\begin{align*}
\Delta_0 & \overset{\text{def}}{=} \{ y \in G^{\lambda^2} : l_y \subset S \}, \\
\Delta'_0 & \overset{\text{def}}{=} \{ y \in G^{\lambda^2} : l_y \cap S \text{ consists of two simple points} \}, \\
\Delta''_0 & \overset{\text{def}}{=} \{ y \in G^{\lambda^2} : l_y \cap S \text{ is a double point} \}, \\
\Delta'''_0 & \overset{\text{def}}{=} \{ y \in G^{\lambda^2} : l_y \cap S \text{ is a simple point} \}.
\end{align*}$$
One finds the following normal forms for the matrices $A$ defining the inclusions $\mathbb{C}^2 \rightarrow H \otimes V$ that represent the points $y \in G^{ss}$:

- $y \in \Delta_0$ if and only if $y$ is represented by a matrix $A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, 
- $y \in \Delta'_1$ if and only if $y$ is represented by a matrix $A = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}$ with independent vectors $x$ and $x'$, 
- $y \in \Delta''_1$ if and only if $y$ is represented by a matrix $A = \begin{pmatrix} x & 0 \\ z & x \end{pmatrix}$ with independent vectors $x$ and $z$, 
- $y \in \Delta'''_1$ if and only if $y$ is represented by a matrix $A = \begin{pmatrix} x & 0 \\ z & x' \end{pmatrix}$, where $(x, x', z)$ is a basis of $V$.

8.7. The Kirwan blowup.

Let $\tilde{G} \to G$ be the blowup of $G$ along $\Delta_0$ followed by the blowup along the proper transform $\tilde{\Delta}_1$ of $\Delta_1 \defeq \Delta'_1 \cup \Delta''_1$. Then $\tilde{G}$ is also acted on by $\text{SL}(H)$ and there is a suitable linearization of this action such that $\tilde{G}^s = G^{ss}$. The absence of strictly semistable points is a characteristic property of a Kirwan blowup [11], so that $\tilde{G}$ is a Kirwan blowup of $G$. The geometric quotient $\tilde{G}^s/\text{SL}(H)$ is isomorphic to $\hat{P}(S^2 V)$, and we have the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{G}^s & \xrightarrow{s} & \hat{P}(S^2 V) \\
\downarrow & & \downarrow \\
G^{ss} & \xrightarrow{s} & \hat{P}(S^2 V).
\end{array}
$$

We have the following exceptional divisors in $\tilde{G}^s$. Let $D_0 \defeq \tilde{s}^{-1}(\Delta_0)$, and let $D_1$ be the inverse image of $\tilde{\Delta}_1$ in $\tilde{G}^s$. Then

$$
D_i \defeq D_i \cap \tilde{G}^s = \tilde{s}^{-1}(\Sigma_i), \quad i = 1, 2,
$$

where $\Sigma_0$ and $\Sigma_1$ are the divisors in $\hat{P}(S^2 V)$ introduced in Theorem 8.5.

8.8. Universal family of Serre constructions.

We parametrized the first Beilinson resolutions

$$
0 \to 2\mathcal{O}_S(-1) \to 2\mathcal{O}^1_S(1) \to \mathcal{F} \to 0,
$$

of the sheaves from $M(0, 2)$ by the two-dimensional subspaces $U$ of $H \otimes V = H^0(S, H \otimes \mathcal{O}^1_S(2))$. Now we want to parametrize all the Serre constructions of these sheaves. A Serre construction for $\mathcal{F}$ is determined by a section $s$ of $\mathcal{F}(1)$, up to proportionality, and we are in one-to-one correspondence with three-dimensional subspaces $W$ of $H \otimes V$ such that $U \subset W$. Such a subspace provides an extension of the injection in the Beilinson resolution $2\mathcal{O}_S(-1) = U \otimes \mathcal{O}_S(-1) \hookrightarrow 2\mathcal{O}^1_S(1) = H \otimes \mathcal{O}^1_S(1)$ to an injection $W \otimes \mathcal{O}_S(-1) \hookrightarrow H \otimes \mathcal{O}_S(1)$, determined up to the action of $\text{GL}(W)$, and this extension defines two exact sequences

$$
0 \to W \otimes \mathcal{O}_S(-1) \to H \otimes \mathcal{O}^1_S(1) \to \mathcal{F} \to \mathcal{F}, \quad 0 \to (W/U) \otimes \mathcal{O}_S(-1) \to \mathcal{F} \to \mathcal{F}.
$$

Here $Z$ is a zero-dimensional subscheme of $S$ of length 3. The second sequence represents the Serre construction for $\mathcal{F}$. For a given $\mathcal{F}$, we can always find $W$ such that $Z$ is either the union of three distinct points, or an isotropic fat point $\text{Spec}: \mathcal{O}_S/m_x^2$, $x \in S$, of length 3. Let now relativize this construction over the whole of $G^s$. Let

$$
F_{2,3} \defeq \{(U, W) : U \subset W \subset H \otimes V\},
$$

where $U$ and $W$ are two-dimensional subspaces of $H \otimes V$. The set $F_{2,3}$ is then a fiber in the family $\mathcal{F}$, and it parametrizes all the Serre constructions for $\mathcal{F}$.
be the flag variety of 2- and 3-dimensional subspaces of $H \otimes V$, and

$$F = F_{2,3} \times \mathbb{G}^1.$$  

There are natural projections $\tilde{G} \xrightarrow{\gamma} F \xrightarrow{\rho} F_{2,3}$. We have the semi-universal family of Beilinson resolutions of the polystable sheaves from $M(0,2)$ over $\tilde{G}$. Lifting it to $F$, we obtain the exact sequence

$$0 \to U \boxtimes \Omega^2(2) \to H \otimes \mathcal{O}_F \boxtimes \Omega^1_s(1) \to \mathcal{F} \to 0.$$  

Shrinking $F$ to an appropriate open subset, mapped surjectively onto $\tilde{G}$, we obtain a semi-universal family of Serre constructions over $F$,

$$0 \to W/U \boxtimes \Omega^2(2) \to \mathcal{F} \to \mathcal{J}_{\mathcal{Z}, F \times S} \otimes \mathcal{O}_F \boxtimes \Omega^1_s(1) \to 0,$$

where $U$ and $W$ are the lifts of the tautological subbundles from $F_{2,3}$, and where $\mathcal{Z}$ is a flat family of zero-dimensional subschemes of $S$ of length 3 over $F$. We can shrink further $F$ in such a way that $\gamma$ remains surjective, so that the only singularities of $\mathcal{Z}$ are the quasi-transversal intersections of three smooth branches over the points of $D_0$.

**8.9. Semi-universal family for $M_2$.**

Let $D_0 = \gamma^{-1}(D_2^0)$ and $D_1 = \gamma^{-1}(D_1^0)$ be the lifted divisors in $F$, and let

$$B_0 \overset{\text{def}}{=} \mathcal{Z} \cap ((D_0 \times S))$$

be the codimension 3 intersection. This is the singular locus of $\mathcal{Z}$, where three branches intersect. Besides $B_0$, the singular locus $\operatorname{Sing} \mathcal{F}$ of $\mathcal{F}$ contains the points $(f, x_i) \in \mathcal{Z}$ such that $f \in D_1$, $x_i \in S$, $i = 1, 2$, $x_1 \neq x_2$ and $\mathcal{F}_{|_{f \times S}} \simeq \mathcal{J}_{x_1, S} \oplus \mathcal{J}_{x_2, S}$. At these points $\mathcal{Z}$ is smooth, but the local extension class of the Serre sequence degenerates. We are to resolve both types of singularities.

Let $X \xrightarrow{a_0} F \times S$ be the blowup of $B_0$. Let further $B_1$ be the closure of

$$a_0^{-1}(\operatorname{Sing} \mathcal{F} \cup ((F - D_0) \times S)).$$

Let $X \xrightarrow{a_1} X'$ be the blowup of $B_1$. Consider the composed morphism

$$\rho : X \xrightarrow{a_0 \times a_1} F \times S \xrightarrow{pr_2} F,$$

and let $E_0, E_1$ denote the exceptional divisors of the last two blowups. Then $X \xrightarrow{\rho} F$ is a family of $\mathcal{T}_2$-surfaces which includes the above examples. Moreover, the proper transform $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$ in $X$ is smooth. In order to replace it by a vector bundle, we consider the Serre construction on $X$ lifting (14). A computation shows that the local extension class of (14), when lifted to a local section of the invertible sheaf

$$e \in \mathcal{E} \mathcal{X}^1(\mathcal{J}_{\mathcal{Z}, X} \otimes \mathcal{O}_S^1(1), \rho^*(W/U \boxtimes \Omega^2_s(2))),$$

has a simple pole along $D_0 \cap \mathcal{Z}$ and is regular and nonvanishing everywhere else.

To transform the simple pole into a double one, we make the base change $\tilde{X} \to X$, which is a double covering branched at $D_0 + D_1$. It is defined locally over $X$. We add $e$ to mark the lifts to $\tilde{X}$ of all the objects defined on $X$. Then $\tilde{e}$ acquires a double pole along $\tilde{D}_0 \cap \tilde{\mathcal{Z}}$ and has no other singularities. Hence it defines a regular nowhere vanishing section of the invertible sheaf

$$\mathcal{E} \mathcal{X}^1(\mathcal{J}_{\mathcal{Z}, \tilde{X}} \otimes \mathcal{O}_S^1(1), \tilde{\rho}^*(W/U \boxtimes \Omega^2_s(2))(\tilde{D}_0)).$$

This implies that in the extension

$$0 \to \tilde{\rho}^*(W/U \boxtimes \Omega^2_s(2))(\tilde{D}_0) \to E \to \mathcal{J}_{\mathcal{Z}, \tilde{X}} \otimes \mathcal{O}_S^1(-\tilde{D}_0) \to 0$$

defined by $\tilde{e}$, the middle term $E$ is a vector bundle. This is the wanted family of $\mathcal{T}_2$-bundles. It is defined over $\tilde{X}$, which can be thought of as a DM stack with stabilizers of order $\leq 2$ whose associated coarse moduli space is $X$. The statements of Theorem 8.5 follow by considering the classifying map of our functor $M_2$ on this family towards the moduli space $M_2$. 


Acknowledgements

The first author acknowledges the support of the French Agence Nationale de Recherche VHSMOD-2009 Nr. ANR-09-BLAN-0104. Research of the second and third authors was supported by the DFG Schwerpunktprogramm 1094.

References

[1] Barth W., Moduli of vector bundles on the projective plane, Invent. Math., 1977, 42 63–91
[2] Buchdahl N.P., Sequences of stable bundles over compact complex surfaces, J. Geom. Anal., 1999, 9(3), 391–428
[3] Buchdahl N.P., Blowups and gauge fields, Pacific J. Math., 2000, 196(1), 69–111
[4] Donaldson S.K., Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc., 1985, 50(1), 1–26
[5] Donaldson S.K., Compactification and completion of Yang–Mills moduli spaces, In: Differential Geometry, Peñíscola, 1988, Lecture Notes in Math., 1410, Springer, Berlin, 1989, 145–160
[6] Feehan P.M.N., Geometry of the ends of the moduli space of anti-self-dual connections, J. Differential Geom., 1995, 42(3), 465–553
[7] Fulton W., MacPherson R., A compactification of configuration spaces, Ann. of Math., 1994, 139(1), 183–225
[8] Gieseker D., On the moduli of vector bundles on an algebraic surface, Ann. of Math., 1977, 106(1), 45–60
[9] Gieseker D., A construction of stable bundles on an algebraic surface, J. Differential Geom., 1988, 27(1), 137–154
[10] Huybrechts D., Lehn M., The Geometry of Moduli Spaces of Sheaves, 2nd ed., Cambridge Math. Lib., Cambridge University Press, Cambridge, 2010
[11] Kirwan F.C., Partial desingularisations of quotients of nonsingular varieties and their Betti numbers, Ann. of Math., 1985, 122(1), 41–85
[12] Kollár J., Projectivity of complete moduli, J. Differential Geom., 1990, 32(1), 235–268
[13] Kollár J., Quotient spaces modulo algebraic groups, Ann. of Math., 1997, 145(1), 33–79
[14] Li J., Algebraic geometric interpretation of Donaldson's polynomial invariants, J. Differential Geom., 1993, 37(2), 417–466
[15] Lübke M., Teleman A., The Kobayashi–Hitchin Correspondence, World Scientific, River Edge, 1995
[16] Maruyama M., Singularities of the curve of jumping lines of a vector bundle of rank 2 on $\mathbb{P}^2$, In: Algebraic Geometry, Tokyo, Kyoto, October 5–14, 1982, Lecture Notes in Math., 1016, Springer, Berlin–New York, 1983, 370–411
[17] Maruyama M., Trautmann G., On compactifications of the moduli space of instantons, Internat. J. Math., 1990, 1(4), 431–477
[18] Maruyama M., Trautmann G., Limits of instantons, Internat. J. Math., 1992, 3(2), 213–276
[19] Nagaraj D.S., Seshadri C.S., Degenerations of the moduli spaces of vector bundles on curves. I, Proc. Indian Acad. Sci. Math. Sci., 1997, 107(2), 101–137
[20] Nagaraj D.S., Seshadri C.S., Degenerations of the moduli spaces of vector bundles on curves. II, Proc. Indian Acad. Sci. Math. Sci., 1999, 109(2), 165–201
[21] Okonek C., Schneider M., Spindler H., Vector Bundles on Complex Projective Spaces, Progr. Math., 3, Birkhäuser, Boston, 1980
[22] Simpson C.T., Moduli of representations of the fundamental group of a smooth projective variety I, Inst. Hautes Études Sci. Publ. Math., 1994, 79, 47–129
[23] Taubes C.H., A framework for Morse theory for the Yang–Mills functional, Invent. Math., 1988, 94(2), 327–402
[24] Timofeeva N.V., Compactification of the moduli variety of stable 2-vector bundles on a surface in the Hilbert scheme, Math. Notes, 2007, 82(5–6), 667–690
[25] Timofeeva N.V., On a new compactification of the moduli of vector bundles on a surface, Sb. Math., 2008, 199(7–8), 1051–1070
[26] Timofeeva N.V., On a new compactification of the moduli of vector bundles on a surface. II, Sb. Math., 2009, 200(3–4), 405–427
[27] Timofeeva N.V., On a new compactification of the moduli of vector bundles on a surface. III: A functorial approach, Sb. Math., 2011, 202(3-4), 413–465

[28] Trautmann G., Moduli spaces in algebraic geometry (manuscript)

[29] Uhlenbeck K.K., Removable singularities in Yang–Mills fields, Comm. Math. Phys., 1982, 83(1), 11–29

[30] Viehweg E., Quasi-Projective Moduli for Polarized Manifolds, Ergeb. Math. Grenzgeb., 30, Springer, Berlin, 1995