Numerical approximation of the thermistor problem*

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ABSTRACT

We use a finite element approach based on Galerkin method to obtain approximate steady state solutions of the thermistor problem with temperature dependent electrical conductivity.

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1 Introduction

In this paper we develop a method to approximate steady-state solutions of the following one-dimensional thermistor problem:

\[
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( k(u) u_x \right) = \sigma(u) |\varphi_x|^2, \quad 0 < x < 1, \quad t > 0, \tag{1.1}
\]

subject to boundary and initial conditions,

\[
k(u) \frac{\partial u}{\partial x} = -\beta u \quad \text{on } \partial \Omega \times (0, T), \tag{1.2}
\]

\[
u(x, 0) = 0, \quad 0 \leq x \leq 1, \tag{1.3}
\]

and coupled with the electric potential equation:

\[
\left( \sigma(u) \varphi_x \right)_x = 0, \quad 0 < x < 1, \quad t > 0, \tag{1.4}
\]

\[
\frac{\partial \varphi}{\partial x} = \varphi(x, t) \quad \text{on } \partial \Omega, \tag{1.5}
\]

\[
\varphi(x, 0) = x, \quad 0 \leq x \leq 1. \tag{1.6}
\]

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The motivation for studying this kind of problem is that \((1.1)-(1.6)\) has important implications for a variety of technological processes. For example, it arises in the analytical study of phenomena associated with the occurrence of shear band in metal being deformed at high strain rates \([3]\); in the theory of gravitational equilibrium of polytropic stars \([9]\); in the investigation of the fully turbulent behavior of flows \([4]\); in modelling aggregation of cells via interaction with a chemical substance (chemotaxis) \([11]\); and specially in modelling electrical heating in a conductor \([12]\). In this case, \(u\) is the temperature of the conductor, \(\varphi\) is the electrical potential. Functions \(\sigma(u)\) and \(k(u)\) are, respectively, the electrical and thermal conductivities; \(\beta\) is the heat transfer coefficient. The condition \((1.3)\) is a condition of Robin-type. When \(\beta = 0\) it is called an adiabatic condition. Equation \((1.1)\) consists in the heat equation with Joule heating as a source; \((1.4)\) describes conservation of current in the conductor.

The thermistor problem has been extensively studied by several authors \([1, 5, 6, 7]\), where existence and uniqueness of solutions are given. Theoretical analysis, consisting in existence of solutions with the required regularity and which ensure error estimates of optimal order of convergence, are done in \([8]\). To construct a numerical approximation of the steady state solution we use a numerical method to approximate the solution of the parabolic problem. This approach has been used by \([2, 10]\) in the one-dimensional thermistor problem. Further, in these last works authors consider the thermal conductivity \(k\) equal to 1 and a particular electrical conductivity, then they obtain the exact solution \((\varphi(x, t) = x)\) of the conservation problem \((1.4)-(1.6)\) and so system \((1.1)-(1.6)\) of thermistor problem is reduced to the following single heat conduction problem:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial^2 x} = \sigma(u),
\]

subject to the boundary conditions \((1.2)-(1.3)\). In this paper, we propose to solve both equations \((1.1)\) and \((1.6)\) at the same time by using a finite element method and a fully Crank-Nicolson approach. The formulation of the finite element method is standard and is based on a variational formulation of the continuous problem. In Section 2 we give the variational formulation of problem \((1.1)-(1.6)\). An algorithm for solving the problem is then proposed in Section 3. In Section 4 numerical results are obtained for an appropriate test-problem.

2 Variational formulation of the problem

We divide the interval \(\Omega = [0, 1]\) into \(N\) equal finite elements \(0 = x_0 < x_1 < \ldots < x_N = 1\). Let \((x_j, x_{j+1})\) be a partition of \(\Omega\) and \(x_{j+1} - x_j = h = \frac{1}{N}\) the step length. By \(S\) we denote a basis of the usual pyramid functions:

\[
v_j = \begin{cases} 
\frac{1}{h}x + (1 - j) & \text{on } [x_{j-1}, x_j], \\
-\frac{1}{h}x + (1 + j) & \text{on } [x_j, x_{j+1}], \\
0 & \text{otherwise}.
\end{cases}
\]

As indicated above, it is convenient to proceed in two steps with the derivation and analysis of the approximate solution of \((1.1)-(1.6)\). First, we write the problem in weak or variational form. We multiply the parabolic equation by \(v_j\) (for \(j\) fixed), integrate over \((0, 1)\), and apply
Green’s formula on the left-hand side, to obtain
\[
\int_{\Omega} \frac{\partial u}{\partial t} v_j \, dx + \int_{\Omega} k(u) \nabla u \nabla v_j \, dx - \int_{\partial\Omega} k(u) \frac{\partial u}{\partial \nu} v_j \, ds = \int_{\Omega} \sigma(u) |\nabla \varphi|^2 v_j \, dx.
\]

Using the boundary condition we get
\[
\int_{\Omega} \frac{\partial u}{\partial t} v_j \, dx + \int_{\Omega} k(u) \nabla u \nabla v_j \, dx + \int_{\partial\Omega} \beta u v_j \, ds = \int_{\Omega} \sigma(u) |\nabla \varphi|^2 v_j \, dx. \tag{2.1}
\]

We also have
\[
\int_{\Omega} \sigma(u) \nabla \varphi \nabla v_j \, dx = \int_{\partial\Omega} \sigma(u) \frac{\partial \varphi}{\partial \nu} v_j \, ds. \tag{2.2}
\]

We now turn our attention to solve this system by discretization with respect to the time variable. We introduce a time step \(\tau\) and time levels \(t_n = n\tau\), where \(n\) is a nonnegative integer, and denote by \(u^n\) the approximation of \(u(t_n)\) to be determined. We use the backward Euler Galerkin method, which is defined by replacing the time derivative in (2.1) by a backward difference \(\frac{u^{n+1} - u^n}{\tau}\). So the approximations \(u^{n+1}\), \(\varphi^{n+1}\) admit unique representations
\[
u^{n+1} = \sum_{i=1}^N \alpha_i^{n+1} v_i, \quad \varphi^{n+1} = \sum_{i=1}^N \mu_i^{n+1} v_i,
\]
where \(\alpha_i^{n+1}\), \(\mu_i^{n+1}\) are unknown real coefficients to be determined. Then, after decoupling, we have that
\[
\int_{\Omega} \frac{u^{n+1} - u^n}{\tau} v_j \, dx + \int_{\Omega} k(u^n) \nabla u^{n+1} \nabla v_j \, dx + \int_{\partial\Omega} \beta u^{n+1} v_j \, ds = \int_{\Omega} \sigma(u^n) |\nabla \varphi^n|^2 v_j \, dx, \tag{2.3}
\]
and
\[
\int_{\Omega} \sigma(u^n) \nabla \varphi^{n+1} \nabla v_j \, dx = \int_{\partial\Omega} \sigma(u^n) \frac{\partial \varphi^{n+1}}{\partial \nu} v_j \, ds. \tag{2.4}
\]

3 Formulation of the numerical method

For scheme (2.4), we have
\[
\sum_{i=1}^N \int_{\Omega} \sigma(u^n) \frac{\partial v_i}{\partial x} \frac{\partial v_j}{\partial x} \, dx
\]

\[
= \mu_j^{n+1} \int_{x_{j-1}}^{x_j} \sigma(u^n) \frac{\partial v_{j-1}}{\partial x} \frac{\partial v_j}{\partial x} \, dx + \mu_j^{n+1} \int_{x_{j-1}}^{x_j} \sigma(u^n) \left( \frac{\partial v_j}{\partial x} \right)^2 \, dx + \mu_j^{n+1} \int_{x_{j-1}}^{x_j} \sigma(u^n) \frac{\partial v_j}{\partial x} \frac{\partial v_{j+1}}{\partial x} \, dx
\]

\[
= -\frac{\mu_j^{n+1}}{h^2} \int_{x_{j-1}}^{x_j} \sigma(u^n) \, dx + \frac{\mu_j^{n+1}}{h^2} \int_{x_{j-1}}^{x_j} \sigma(u^n) \, dx + \mu_j^{n+1} \int_{x_{j-1}}^{x_j} \sigma(u^n) \, dx
\]

\[
\approx -\frac{\mu_j^{n+1}}{2h} (\sigma(u^n(x_j)) + \sigma(u^n(x_{j-1}))) + \frac{\mu_j^{n+1}}{h} (\sigma(u^n(x_{j+1})) + \sigma(u^n(x_{j-1})))
\]

\[
- \frac{\mu_j^{n+1}}{2h} (\sigma(u^n(x_{j+1})) + \sigma(u^n(x_j)))
\]

\[
\approx -\frac{\mu_j^{n+1}}{2h} (\sigma(\alpha^n_j) + \sigma(\alpha^n_{j-1})) + \frac{\mu_j^{n+1}}{h} (\sigma(\alpha^n_{j+1}) + \sigma(\alpha^n_{j-1})) - \frac{\mu_j^{n+1}}{2h} (\sigma(\alpha^n_{j+1}) + \sigma(\alpha^n_j)).
\]
On the other hand, we have
\[
\int_{\partial \Omega} \sigma(u^n) \frac{\partial \varphi^{n+1}}{\partial \nu} v_j \, ds = \int_{\partial \Omega} \sigma(u^n) \varphi v_j \, ds
\]
\[
= \sigma(u^n(1)) \varphi(1)v_j(1) - \sigma(u^n(0)) \varphi(0)v_j(0)
\]
\[
= \left\{ \begin{array}{ll}
-\sigma(\alpha_0^n) \varphi(0) & \text{if } j = 0, \\
0 & \text{if } j = 1, \ldots N - 2, \\
0 & \text{if } j = N - 1.
\end{array} \right.
\]

Using boundary conditions (1.2) and initial condition (1.3), it follows that
\[
\mu_{j-1}^{n+1} = \mu_{j-1}^{n+1} - h\varphi(0),
\]
\[
\mu_N^{n+1} = h\varphi(1) + \mu_{N-1}^{n+1},
\]
\[
\alpha_{n-1}^{n+1} = \alpha_1^n + \left( \frac{h\beta}{k(\alpha_{n-1}^n)} - 1 \right) \alpha_0^n.
\]

Then, we have the resulting system of equations:
for \( j = 0 \),
\[
\left( \sigma(\alpha_0^n) + 3\sigma(\alpha_{n-1}^n) + 2\sigma(\alpha_1^n) \right) \mu_0^{n+1} - \left( \sigma(\alpha_{n-1}^n) + 2\sigma(\alpha_0^n) + \sigma(\alpha_1^n) \right) \mu_1^{n+1}
\]
\[
= -h\varphi(0)(3\sigma(\alpha_0^n) + \sigma(\alpha_{n-1}^n)); \quad (3.1)
\]
for \( j = 1, \ldots, N - 2 \),
\[
- \mu_{j-1}^{n+1} \left( \sigma(\alpha_j^n) + \sigma(\alpha_{j-1}^n) \right) + 2\mu_j^{n+1} \left( \sigma(\alpha_{j+1}^n) + \sigma(\alpha_j^n) \right) + \mu_{j+1}^{n+1} \left( \sigma(\alpha_{j+1}^n) + \sigma(\alpha_j^n) \right) = 0;
\]
\[
(3.2)
\]
for \( j = N - 1 \),
\[
- \left( \sigma(\alpha_{N-1}^n) + \sigma(\alpha_{N-2}^n) \right) \mu_{N-2}^{n+1} + \left( 2\sigma(\alpha_{N-2}^n) + \sigma(\alpha_{N-1}^n) - \sigma(\alpha_{N-1}^n) \right) \mu_{N-1}^{n+1}
\]
\[
= h\varphi(1) \left( \sigma(\alpha_N^n) + \sigma(\alpha_{N-1}^n) \right). \quad (3.3)
\]

Coming back to (2.3), the following may be stated in terms of the functions \((v_i)_i\): find the coefficients \(\alpha_i^{n+1}\) in \(u^{n+1} = \sum_{i=-1}^N \alpha_i^{n+1} v_i\) such that
\[
\sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} v_i v_j \, dx + \tau \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} k(u^n) \nabla v_i \nabla v_j \, dx + \tau \int_{\partial \Omega} \beta u^{n+1} v_j \, ds
\]
\[
= \sum_{i=-1}^N \alpha_i^n \int_{\Omega} v_i v_j \, dx + \tau \int_{\Omega} \sigma(u^n) \left| \nabla \varphi \right|^2 v_j \, dx. \quad (3.4)
\]

In matrix notation, this may be expressed as
\[
(A + \tau B) \alpha^{n+1} = f^n = f(n\tau),
\]
where
\[
A = (a_{ij}) \text{ with element } a_{ij} = \int_{\Omega} v_i v_j \, dx,
\]
\[
A = \begin{pmatrix}
\int_{\Omega} v_i v_j \, dx & \cdots & \int_{\Omega} v_i v_N \, dx \\
\int_{\Omega} v_0 v_j \, dx & \cdots & \int_{\Omega} v_0 v_N \, dx \\
\vdots & \ddots & \vdots \\
\int_{\Omega} v_{N-1} v_j \, dx & \cdots & \int_{\Omega} v_{N-1} v_N \, dx
\end{pmatrix}
\]
\[
B = \begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\]
\[ B = (b_{ij}) \text{ with } b_{ij} = \int_{\Omega} k(u^n) \nabla v_i \nabla v_j \, dx, \]

and

\[ \alpha^{n+1} \text{ is the vector of unknowns } (\alpha^{n+1}_i)_{i=-1}^N. \]

Since the matrix \( A \) and \( B \) are Gram matrices, in particular they are positive definite and invertible. Thus, the above system of ordinary differential equations has obviously a unique solution. We solve the system (3.1) for each time level. Estimating each term of (3.1) separately, we have:

\[
\sum_{i=-1}^{N} \alpha^{n+1}_i \int_{\Omega} v_i v_j \, dx = \sum_{i=-1}^{N} \alpha^{n+1}_i \int_{0}^{1} v_i v_j \, dx
\]

\[ = \alpha^{n+1}_{j-1} \int_{x_{j-1}}^{x_j} v_{j-1} v_j \, dx + \alpha^{n+1}_j \int_{x_{j-1}}^{x_{j+1}} v_j^2 \, dx + \alpha^{n+1}_{j+1} \int_{x_{j-1}}^{x_{j+1}} v_j v_{j+1} \, dx
\]

\[ + \alpha^{n+1}_{j-1} \int_{x_{j-1}}^{x_j} v_{j-1} v_j \, dx + \alpha^{n+1}_j \left( \int_{x_{j-1}}^{x_j} v_j^2 \, dx + \int_{x_{j-1}}^{x_{j+1}} v_j^2 \, dx \right) + \alpha^{n+1}_{j+1} \int_{x_{j}}^{x_{j+1}} v_j v_{j+1} \, dx.
\]

Using the expression of \( v_{j-1}, v_j \) and \( v_{j+1} \), we obtain

\[
\sum_{i=-1}^{N} \alpha^{n+1}_i \int_{\Omega} v_i v_j \, dx = \frac{h}{6} \alpha^{n+1}_{j-1} + \frac{2h}{3} \alpha^{n+1}_j + \frac{h}{6} \alpha^{n+1}_{j+1}.
\]

(3.5)

In the same way, we have

\[
\sum_{i=-1}^{N} \alpha^{n+1}_i \int_{\Omega} k(u^n) \nabla v_i \nabla v_j \, dx = \sum_{i=-1}^{N} \alpha^{n+1}_i \int_{\Omega} k(u^n) \frac{\partial v_i}{\partial x} \frac{\partial v_j}{\partial x} \, dx
\]

\[ = \alpha^{n+1}_{j-1} \int_{x_{j-1}}^{x_j} k(u^n) \frac{\partial v_{j-1}}{\partial x} \frac{\partial v_j}{\partial x} \, dx + \alpha^{n+1}_j \int_{x_{j-1}}^{x_{j+1}} k(u^n) \left( \frac{\partial v_j}{\partial x} \right)^2 \, dx
\]

\[ + \alpha^{n+1}_{j+1} \int_{x_{j}}^{x_{j+1}} k(u^n) \frac{\partial v_j}{\partial x} \frac{\partial v_{j+1}}{\partial x} \, dx,
\]

\[ = \frac{\alpha^{n+1}_{j-1}}{h^2} \int_{x_{j-1}}^{x_j} k(u^n) \, dx + \frac{\alpha^{n+1}_j}{h^2} \int_{x_{j-1}}^{x_{j+1}} k(u^n) \, dx - \frac{\alpha^{n+1}_{j+1}}{h^2} \int_{x_{j}}^{x_{j+1}} k(u^n) \, dx
\]

\[ \approx - \frac{\alpha^{n+1}_{j-1}}{2h} (k(u^n(x_{j-1})) + k(u^n(x_{j-1}))) + \frac{\alpha^{n+1}_j}{h} (k(u^n(x_{j+1})) + k(u^n(x_{j-1})))
\]

\[ - \frac{\alpha^{n+1}_{j+1}}{2h} (k(u^n(x_{j+1})) + k(u^n(x_{j}))
\]

\[ \approx - \frac{\alpha^{n+1}_{j-1}}{2h} k(\alpha^n_{j-1}) + \frac{\alpha^{n+1}_j}{h} (k(\alpha^n_{j+1}) + k(\alpha^n_{j-1})) - \frac{\alpha^{n+1}_{j+1}}{2h} (k(\alpha^n_{j+1}) + k(\alpha^n_{j})).
\]

On other hand, we similarly have

\[
\int_{\Omega} u^n v_j = \sum_{i=-1}^{N} \alpha^{n}_i \int_{\Omega} v_i v_j \, dx = \frac{h}{6} \alpha^{n}_{j-1} + \frac{2h}{3} \alpha^{n}_j + \frac{h}{6} \alpha^{n}_{j+1},
\]

\[
\int_{\Omega} \sigma(u^n) |\varphi^n_x|^2 v_j \, dx = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \sigma(u^n) |\varphi^n_x|^2 v_j(x) \, dx
\]

\[ \approx \frac{h}{2} \sum_{j=1}^{N-1} (\sigma(u^n(x_{j+1})) |\varphi^n_x(x_{j+1})|^2 v_j(x_{j+1}) + \sigma(u^n(x_j)) |\varphi^n_x(x_j)|^2 v_j(x_j))
\]

\[ \approx \frac{\sigma(\alpha^n_j)}{h} (-\mu^{n}_{j-1} + \mu^n_{j} + \mu^n_{j+1})^2.
\]
It also holds:

\[ \beta \int_{\partial \Omega = \{0,1\}} u^{n+1} v_j = \beta u^{n+1}(1)v_j(1) - \beta u^{n+1}(0)v_j(0) \]

\[ = \beta \alpha_N^{n+1} v_j(1) - \beta \alpha_0^{n+1} v_j(0) \]

\[ = \begin{cases} 
-\beta \alpha_0^{n+1} & \text{if } j = 0, \\
0 & \text{if } j = 1 \ldots N - 2, \\
0 & \text{if } j = N - 1.
\end{cases} \]

Using together (3.4) and (3.5), we get a system of \(N-1\) linear algebraic equations

\[
\begin{aligned}
\left( \frac{h}{6} - \frac{\tau}{2h} (k(\alpha_j^n) + k(\alpha_{j-1}^n)) \right) \alpha_j^{n+1} + \left( \frac{2}{3} h + \frac{\tau}{h} (k(\alpha_{j+1}^n) + k(\alpha_{j-1}^n)) \right) \alpha_j^{n+1} \\
+ \left( \frac{h}{6} - \frac{\tau}{2h} (k(\alpha_{j+1}^n) + k(\alpha_j^n)) \right) \alpha_{j+1}^{n+1} - \tau \beta \alpha_0^{n+1} v_j(0)
\end{aligned}
\]

(3.6)

Using the boundary conditions, we find

\[
\begin{aligned}
\alpha_{n-1}^{n+1} &= \alpha_0^{n+1} + \left( \frac{h \beta}{k(\alpha_0^n)} - 1 \right) \alpha_0^{n+1}, \\
\alpha_{n-1}^n &= \alpha_0^n + \left( \frac{h \beta}{k(\alpha_0^{n-1})} - 1 \right) \alpha_0^n, \\
\alpha_N^{n+1} &= \frac{k(\alpha_N^n)}{\beta h + k(\alpha_N^n)} \alpha_N^{n+1}, \\
\alpha_N^n &= \frac{k(\alpha_{n-1}^n)}{\beta h + k(\alpha_{n-1}^n)} \alpha_N^{n-1}.
\end{aligned}
\]

From the initial condition we get

\[ \alpha_0^0 = \alpha_N^0 = 0. \]

Let

\[
\begin{aligned}
a &= \left( \frac{h}{6} - \frac{\tau}{2h} (k(\alpha_0^n) + k(\alpha_{-1}^n)) \right), \\
b &= \left( \frac{2h}{3} + \frac{\tau}{h} (k(\alpha_1^n) + k(\alpha_{-1}^n)) \right),
\end{aligned}
\]

and

\[
c = \left( \frac{h}{6} - \frac{\tau}{2h} (k(\alpha_1^n) + k(\alpha_0^n)) \right).
\]

Substituting in (3.6), we obtain the following system of equations:

for \(j=0\),

\[
\left( a \left( \frac{\beta h}{k(\alpha_0^n)} - 1 \right) + b - \tau \beta \right) \alpha_0^{n+1} + (a + c) \alpha_1^{n+1}
\]

\[= \frac{h}{2} \left( 1 + \frac{h \beta}{3k(\alpha_{-1}^{n-1})} \right) \alpha_0^n + \frac{h}{3} \alpha_1^n + \frac{\tau}{h} \sigma(\alpha_0^n)(2\mu^n + h\varphi(0))^2; \quad (3.7)
\]

\[= (h - 2h \tau \beta (\alpha_0^n + \alpha_1^n - \tau \beta \alpha_0^n - h \sigma(\alpha_0^n)(2\mu^n + h\varphi(0))^2)).
\]
for \( j = 1, \ldots, N - 2 \),

\[
\begin{align*}
\frac{h}{6} - \frac{\tau}{2h}(k(\alpha_j^n) + k(\alpha_{j-1}^n)) \alpha_{j-1}^{n+1} + \left( \frac{2h}{3} + \frac{\tau}{h}(k(\alpha_{j+1}^n) + k(\alpha_j^n)) \right) \alpha_j^{n+1} \\
+ \left( \frac{h}{6} - \frac{\tau}{2h}(k(\alpha_{j+1}^n) + k(\alpha_j^n)) \right) \alpha_{j+1}^{n+1} \\
= \frac{h}{6} \alpha_j^n + \frac{2h}{3} \alpha_j^n + \frac{h}{6} \alpha_{j+1}^n + \frac{\tau}{h} \sigma(\alpha_j^n)(-\mu_{j-1}^n + \mu_j^n + \mu_{j+1}^n)^2;
\end{align*}
\]

for \( j = N - 1 \),

\[
\begin{align*}
d\alpha_{N-2}^{n+1} + \left( e + \frac{k(\alpha_{N-1}^n)}{\beta h + k(\alpha_N^n)} f \right) \alpha_{N-1}^{n+1} \\
= \frac{h}{6} \alpha_{N-2}^n + \frac{h}{6} \left( 4 + \frac{k(\alpha_{N-1}^n)}{\beta h + k(\alpha_N^n)} \right) \alpha_{N-1}^n + \frac{\tau}{h} \sigma(\alpha_{N-1}^n)(2\mu_{N-1}^n - \mu_{N-2}^n + h\sigma(1))^2,
\end{align*}
\]

where

\[
\begin{align*}
d &= \left( \frac{h}{6} - \frac{\tau}{2h}(k(\alpha_{N-1}^n) + k(\alpha_N^n)) \right), \\
e &= \left( \frac{2h}{3} + \frac{\tau}{h}(k(\alpha_N^n) + k(\alpha_{N-2}^n)) \right), \\
f &= \left( \frac{h}{6} - \frac{\tau}{2h}(k(\alpha_N^n) + k(\alpha_{N-1}^n)) \right).
\end{align*}
\]

4 An example

In this section we give an example of a model of the thermistor problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u_{xx} + \gamma |\varphi_x|^2 \\
\frac{\partial u}{\partial x} &= -\beta u \text{ on } \partial \Omega \\
u(x, 0) &= 0, \quad 0 < x < 1
\end{align*}
\]

(4.1)

\[
\begin{align*}
(\sigma(u)\varphi_x)_x &= 0 \\
\frac{\partial \varphi}{\partial x} &= 1 \text{ on } \partial \Omega \\
\varphi(x, 0) &= x, \quad 0 \leq x \leq 1.
\end{align*}
\]

(4.2)

The exact solution of the electrical potential problem (4.2) is \( \varphi(t, x) = x, \quad 0 \leq x \leq 1. \) Then, the diffusion equation (4.1) can be reduced to the form

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma.
\]

Using the proposed Galerkin finite element approach, we get the following system of algebraic equations:

for \( j = 0 \),

\[
(a_1(\beta h - 1) + b_1 - \tau \beta) \alpha_{0}^{n+1} + 2a_1 \alpha_{1}^{n+1} = \frac{h}{2}(1 + \beta h)\alpha_{0}^{n} + \frac{h}{3} \alpha_{1}^{n} + \gamma \tau h;
\]
for $j = 1, \ldots, N - 2$,

$$a_1 \alpha_{j-1}^{n+1} + b_1 \alpha_j^{n+1} + a_1 \alpha_{j+1}^{n+1} = \frac{h}{6} \alpha_{j-1}^n + \frac{2h}{3} \alpha_j^n + \frac{h}{6} \alpha_{j+1}^n + \gamma \tau h;$$

for $j = N - 1$,

$$a_1 \alpha_{N-2}^{n+1} + \left( b_1 + \frac{a_1}{\beta h + 1} \right) \alpha_{N-1}^{n+1} = \frac{h}{6} \alpha_{N-2}^n + \frac{h}{6} \left( 4 + \frac{1}{1 + \beta h} \right) \alpha_{N-1}^n + \gamma \tau h,$$

where

$$a_1 = \frac{h}{6} - \frac{\tau}{h}, \quad b_1 = \frac{2h}{3} + \frac{2\tau}{h}.$$

We now show some results from numerical experiments performed using our method and the computer algebra system Maple 10. According with physical situations, we choose values of $\beta$ and $\gamma$ verifying $\frac{1}{\beta} + \frac{1}{2} \leq \frac{1}{\tau}$. In particular, we fixed $\beta = 0.2$ and $\gamma = 0.1$. The calculation of the steady-state for the thermistor problem is an important issue regarding the applications of the model in the thermistor device. We obtained stable steady-state times for $\tau = 0.1, h = 0.01$ (see Fig. 1).

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References

[1] S. N. Antontsev and M. Chipot, The thermistor problem: existence, smoothness uniqueness, blowup, SIAM J. Math. Anal. 25 (1994), no. 4, 1128–1156.

[2] A. R. Bahadir, Steady-state solution of the PTC thermistor problem using a quadratic spline finite element method, Math. Probl. Eng. 8 (2002), no. 2, 101–109.

[3] J. W. Bebernes and A. A. Lacey, Global existence and finite-time blow-up for a class of nonlocal parabolic problems, Adv. Differential Equations 2 (1997), no. 6, 927–953.

[4] E. Caglioti, P.-L. Lions, C. Marchioro, M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Comm. Math. Phys. 143 (1992), no. 3, 501–525.

[5] G. Cimatti, The eddy current problem with temperature dependent permeability, Electron. J. Differential Equations 2003, No. 91, 5 pp. (electronic).

[6] A. El Hachimi and M. R. Sidi Ammi, Existence of weak solutions for the thermistor problem with degeneracy, in Proceedings of the 2002 Fez Conference on Partial Differential Equations, 127–137 (electronic), Electron. J. Differ. Equ. Conf., 9, Southwest Texas State Univ., San Marcos, TX, 2002.

[7] A. El Hachimi and M. R. Sidi Ammi, Semidiscretization for a nonlocal parabolic problem, Int. J. Math. Math. Sci. 2005, no. 10, 1655–1664.

[8] C. M. Elliott and S. Larsson, A finite element model for the time-dependent Joule heating problem, Math. Comp. 64 (1995), no. 212, 1433–1453.

[9] A. Krzywicki and T. Nadzieja, Some results concerning the Poisson-Boltzmann equation, Zastos. Mat. 21 (1991), no. 2, 265–272.

[10] S. Kutluay and A. Esen, Numerical solutions of the thermistor problem by spline finite elements, Appl. Math. Comput. 162 (2005), no. 1, 475–489.

[11] G. Wolansky, A critical parabolic estimate and application to nonlocal equations arising in chemotaxis, Appl. Anal. 66 (1997), no. 3-4, 291–321.

[12] X. Xu, On the existence of bounded temperature in the thermistor problem with degeneracy, Nonlinear Anal. 42 (2000), no. 2, Ser. A: Theory Methods, 199–213.