DETERMINATION OF INITIAL DATA FOR A REACTION-DIFFUSION SYSTEM WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we study a final value problem for a reaction-diffusion system with time and space dependent diffusion coefficients. In general, the inverse problem of identifying the initial data is not well-posed, and herein the Hadamard-instability occurs. Applying a new version of a modified quasi-reversibility method, we propose a stable approximate (regularized) problem. The existence, uniqueness and stability of the corresponding regularized problem are obtained. Furthermore, we also investigate the error estimate and show that the approximate solution converges to the exact solution in L^2 and H^1 norms. Our method can be applied to some concrete models that arise in biology, chemical engineering, etc.

1. Introduction. In this paper, we consider the question of finding the functions $u_k(x,t), (x,t) \in D_T$, satisfying the problem

$\begin{cases}
\partial_t u_1 - \nabla \cdot (\eta_1(x,t) \nabla u_1) = F_1(x,t; u_1; \ldots; u_m), & \text{in } D_T, \\
\vdots & \\
\partial_t u_m - \nabla \cdot (\eta_m(x,t) \nabla u_m) = F_m(x,t; u_1; \ldots; u_m), & \text{in } D_T, \\
u_k = 0, & \text{on } \Pi, \ k = 1, m,
\end{cases}

(1)$

with the final value condition

$u_k(x,T) = \varphi_k(x), \text{ in } \Omega, \ k = 1, m.

(2)$

where T is a positive number, $\Omega$ is an open, bounded and connected domain in $\mathbb{R}^n, n \geq 1$ with a smooth boundary $\Gamma$ and $D_T = \Omega \times ]0, T[, \ \Pi = \Gamma \times ]0, T[, \ m \geq 1$
is a natural integer, the functions \( \eta_k(x,t), \varphi_k(x) \), \( k = \frac{1}{m} \) are given and the source function \( F_k, k = \frac{1}{m} \) will be specified later. The problem (1) with an initial value condition has been studied so much and also has many applications in physical, chemical, biological and ecological sciences, see e.g. [10] and references therein. For example,

1. If \( m = 2 \), Problem (1) has application to chemical reactions [1, 8, 9]. With \( F_1(u_1; u_2) = f(u_1) - u_2, F_2(u_1; u_2) = a(u_1 - bu_2) \) for \( a > 0, b > 0, f(u) \) is a cubic function, Problem (1) is an extension of the FitzHugh-Nagumo model [10]. With \( F_1(u_1; u_2) = b_1(u_1 - u_2), F_1(u_1; u_2) = -b_1(u_1 - u_2) \) for positive constant \( b_1 \), Problem (1) has application in the binary mixture of rigid solids [7].

2. If \( m = 2 \), \( \eta_1(x,t) = d_1, \eta_2(x,t) = d_2 \) and

\[
\begin{align*}
F_1(x,t; u_1; u_2) &= -u_1 \left( m(x) - u_1 - cu_2 \right), \\
F_2(x,t; u_1; u_2) &= -u_2 \left( m(x) - bu_1 - u_2 \right),
\end{align*}
\]

system (1) is the two-species Lotka-Volterra competition-diffusion model [6, 11, 21] and references therein. This systems is a simple model of the population dynamics of species competing for some common resource. Here \( u_k(x,t), k = 1,2 \) represent the population densities of two competing species, \( d_1, d_2 \) are migration rates; the function \( m(x) \) represents their common intrinsic growth rate or carrying capacity, and \( b \) and \( c \) are interspecific competition coefficients.

3. If \( m = 3 \) and

\[
\begin{align*}
F_1(u_1, u_2, u_3) &= F_2(u_1, u_2, u_3) = -k^f u_1 u_2 + k^b u_3, \\
F_3(u_1, u_2, u_3) &= k^f u_1 u_2 - k^b u_3,
\end{align*}
\]

then system (1) represents the time-evolution of the concentrations \((u_1, u_2, u_3)\) of three chemical species. Here \( k^f, k^b > 0 \) are the rate constants for the forward and backward reaction. This system has many application in chemical reaction engineering (see more details in [2]).

4. If \( m = 4 \) and

\[
\begin{align*}
F_1(u_1, u_2, u_3, u_4) &= F_2(u_1, u_2, u_3, u_4) = -k_1 u_1 u_2 + k_2 u_3 u_4, \\
F_3(u_1, u_2, u_3, u_4) &= F_4(u_1, u_2, u_3, u_4) = k_1 u_1 u_2 - k_2 u_3 u_4,
\end{align*}
\]

for \( k_1, k_2 > 0 \) then system (3) describes quadratic chemical reactions [16].

Although the initial value problem for (1) is classical and has been so studied, however, the study of final value problem (called Initial inverse problem) for (1) is limited and open. The properties of the initial inverse problems are very different from the direct problem. For the direct problem, we often investigate the properties of the solution such as the existence, the blow up, the decay, etc. To our knowledge, the existence and uniqueness of solution of (1) is an open problem, and we do not investigate this problem here. Otherwise, our main purpose is described as follows. We assume that Problem (1)-(2) has a unique solution in a suitable space and we will investigate its approximation. In practice, we get the data \( \varphi_k, k = \frac{1}{m} \) by observed data. Hence, instead of \( \varphi_k, k = \frac{1}{m} \), we shall get an approximate data \( \varphi_k, k = \frac{1}{m} \) satisfying

\[
\| \varphi_k^1 - \varphi_1 \|_{L^2(\Omega)} + \| \varphi_k^2 - \varphi_2 \|_{L^2(\Omega)} + \cdots + \| \varphi_k^m - \varphi_m \|_{L^2(\Omega)} \leq \epsilon,
\]
where the constant $\epsilon > 0$ represents a bound on the measurement error. In fact, from a small perturbation of a physical measurement, the corresponding solution may have a large error. This makes the numerical computation delicate. Hence a regularization is needed. Now we dwell on backward problems:

- For $m = 1$, many papers are devoted to special cases of (1) restricted to $\eta = 1$ or $\eta = \eta(t)$, for example, [5, 13, 20] and references therein. As for the backward problem of a general parabolic equation in the case of $\eta = \eta(x, t)$, less study has been done.

- For $m = 2$, the first result on the backward problem for a system of parabolic equations is presented by P. Schaefer [19]. However, the regularization method with error estimate has not been studied in [19].

Noting that if the coefficient $\eta_k(x, t) = \eta_k(t)$ or $\eta_k(x, t) = \eta_k(x)$, $k = 1, m$ then system (1)-(2) is not difficult since it can be transformed to nonlinear integral equations. In this case, we follow spectral regularized method for solving it, such as nonlinear integral equation, truncation method, modified integral methods, quasi-boundary value method, etc. (See [13] and the references therein). However, with general form of $\eta(x, t)$, the system (1)-(2) cannot be transformed into a nonlinear integral equation as in the Fourier series. Hence, some classical methods and previous techniques are not applicable to approximate system (1). This case is more difficult for investigation and a new method is required. For the inverse problem, we assume that the solution of the system (1) exists. In this case its solution is not stable. In this paper, we propose a new modified quasi-reversibility method to regularize system (1) in case of global or local Lipschitz function $F$. The method of classical quasi-reversibility has now a quite long history since the pioneering book of Lattes and Lions [12]. The original idea of these authors was, starting from an ill-posed problem which satisfies the uniqueness property, to introduce a perturbation of such problem involving a small positive parameter $\epsilon$. In [12], the authors considered a linear homogeneous backward parabolic problem with space and time dependent coefficients in the following form:

$$\partial_t u + A(t)u = 0,$$

$$A(t) = -\nabla \cdot (\eta(x, t)\nabla \cdot )$$

and they suggested a regularized problem by perturbing directly $A(t)$ by $A(t) + \beta A^*(t)$ in order to obtain the following approximate problem:

$$\partial_t u^\beta + A(t)u^\beta + \beta A^*(t)u^\beta = 0,$$

and their results did not give a convergence rate. At first, we intend to apply the method of [12] for solving the system (1). We can show that the approximate problem is well-posed but we can not show that the convergence rates tend to zero. In this paper, we will not approximate directly the operator $\nabla \cdot (\eta(x, t)\nabla \cdot )$ as in [12]. We emphasize that our method is original and very different from the methods in [12]. The main idea of the paper is of approximating $R := R\Delta$, $R > 0$, by a bounded operator, in order to establish an approximation for the regularized problem. By using our new tool, we show that the regularized problem is well-posed; then we estimate the error between a solution of (1)-(2) and the approximate solution.

The paper is organized as follows. Section 2 introduces some preliminaries and notations and shows the ill-posedness of the system (1). In Section 3, a stability estimate is proved under an a priori condition on the exact solution and the globally Lipschitz source term. In Section 4, the analysis is extended to local Lipschitz source functions and perturbed time dependent coefficient. To show the well-posedness of
the regularized problem, we apply the Faedo–Galerkin and the compactness method. We propose a new idea in which the locally Lipschitz source functions $F_k, \ (k = \Gamma, \Omega)$ are approximated by a sequence $F_k(\ell)$ of globally Lipschitzian functions. Finally, in section 5, we introduce some specific systems which can be tackled by our method.

2. Ill-posedness of the Problem (1)-(2). In this section, we give an example which shows the instability of the solution of problem (1). Before doing so, we introduce some suitable Sobolev spaces, and fix some notation. Let us recall that the spectral problem

$$\begin{cases}
-\Delta \sigma_n(x) = \lambda_n \sigma_n(x), \quad \text{in } \Omega, \\
\sigma_n(x) = 0, \quad \text{in } \Gamma,
\end{cases}$$

(5)

admits a family of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$. See [4], p.335. We will use the Banach spaces $C([0,T];X), L_\infty(0,T;X)$ of real measurable functions $u : \bar{D}_T \rightarrow X$ ($X$ is also a Banach space), such that

$$\|u\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_X < \infty,$$

(6)

and

$$\|u\|_{L_p(0,T;X)} = \left( \int_0^T \|u(\cdot, t)\|_X^p \, dt \right)^{1/p} < \infty, \quad \forall t \in [0,T],$$

$$\|u\|_{L_\infty(0,T;X)} = \text{ess sup}_{t \in [0,T]} \|u(\cdot, t)\|_X < \infty.$$  

Recall that $H_1^0(\Omega)$ with norm and inner product

$$\|w\|_{H_1^0(\Omega)} = \|\nabla w\|_{L_2(\Omega)} = \left[ \int_\Omega |\nabla w(x)|^2 \, dx \right]^{1/2},$$

$$\langle w_1, w_2 \rangle_{H_1^0(\Omega)} = \int_\Omega \nabla w_1(x) \cdot \nabla w_2(x) \, dx.$$  

We introduce the abstract Gevrey class of functions of index $\delta > 0$, see, e.g. [3], defined by

$$V_\delta(\Omega) := \left\{ v \in L_2(\Omega) : \sum_{n=1}^\infty \exp (2\delta \lambda_n) \left| \langle v, \sigma_n(x) \rangle_{L_2(\Omega)} \right|^2 < \infty \right\},$$

which is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle_{V_\delta(\Omega)} := \left\{ \exp \left( \delta \sqrt{-\Delta} \right) u, \exp \left( \delta \sqrt{-\Delta} \right) v \right\}_{L_2(\Omega)}, \quad \text{for all } u, v \in V_\delta(\Omega);$$

its corresponding norm is

$$\|u\|_{V_\delta(\Omega)} = \left[ \sum_{n=1}^\infty \exp (2\delta \lambda_n) \langle u, \sigma_n(x) \rangle_{L_2(\Omega)}^2 \right]^{1/2},$$

Let $V(x) = (v_1(x), v_2(x), \ldots, v_m(x)) \in X^m := X \times X \times \ldots \times X$, we define $\| \cdot \|_{X^m}$ as follows

$$\|V\|_{X^m} = \left( \sum_{k=1}^m \|v_k\|_X^2 \right)^{1/2}.$$
Now, we return to the study of the ill-posedness of Problem (1). For sake of simplicity, in (1), we assume that \( \eta_k(x, t) = \eta_k(t) \geq r > 0 \), \( \forall t \in [0, T], k = 1, m \). For any \( p \in \mathbb{N}^* \), let the final data \( \varphi_k^{(p)} \), \( k = 1, m \) be as follows

\[
\varphi_1^{(p)}(x) = \ldots = \varphi_m^{(p)}(x) := \frac{\sigma_p(x)}{\lambda_p},
\]

and the source function

\[
F_k^{(p)}(x, t; u_1; \ldots; u_m) := \frac{\exp\left(-\lambda_p \int_0^T \eta_k(\mu) d\mu\right)}{2Tm} \sum_{i=1}^m \left( u_i \cdot \sigma_p(x) \right)_{L_2(\Omega)} \sigma_p(x), \quad k = 1, m.
\]

It is easy to see that the solution of problem (1) is then given by (for \( k = 1, m \))

\[
u_k^{(p)}(x, t) = \frac{\exp\left(\lambda_p \int_t^T \eta_k(\xi) d\xi\right)}{\lambda_p} \sigma_p(x) - \int_t^T \exp\left(\lambda_p \int_t^\mu \eta_k(\xi) d\xi\right) \left( F_k^{(p)}(x, \mu; u_1^{(p)}; \ldots; u_m^{(p)}), \sigma_p(x) \right)_{L_2(\Omega)} \sigma_p(x) d\mu.
\]

**Step 1.** The existence and uniqueness of the solution of the system (9).

At first, we show that (9) has a unique solution \( (u_1^{(p)}, u_2^{(p)}, \ldots, u_m^{(p)}) \) belong to \( [C([0, T]; L_2(\Omega))]^m \). For any \( (w_1, w_2, \ldots, w_m) \in [C([0, T]; L_2(\Omega))]^m \), we consider the following function

\[
I_k(w_1; w_2; \ldots; w_m)(t) = \mathbb{B}_1
\]

\[
- \int_t^T \exp\left(\lambda_p \int_t^\mu \eta_k(\xi) d\xi\right) \left( F_k^{(p)}(x, \mu; w_1; w_2; \ldots; w_m), \sigma_p(x) \right)_{L_2(\Omega)} \sigma_p(x) d\mu,
\]

for \( k = 1, m \) and

\[
J(w_1; w_2; \ldots; w_m)(t) = \left( I_1(w_1; w_2; \ldots; w_m)(t), I_2(w_1; w_2; \ldots; w_m)(t), \ldots, I_m(w_1; w_2; \ldots; w_m)(t) \right).
\]

Let any \( (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_m) \in [C([0, T]; L_2(\Omega))]^m \), using Hölder inequality, we have

\[
\left\| I_k(w_1; w_2; \ldots; w_m)(t) - I_k(\overline{w}_1; \overline{w}_2; \ldots; \overline{w}_m)(t) \right\|^2_{L_2(\Omega)} \leq \frac{1}{4Tm^2} \int_t^T \exp\left(2\lambda_p \int_t^\mu \rho_k(\xi) d\xi - 2\lambda_p \int_0^T \rho_k(\xi) d\xi\right) \times
\]

\[
\left( \sum_{k=1}^m \left( w_k(\mu) - \overline{w}_k(\mu), \sigma_p(x) \right)_{L_2(\Omega)} \right)^2 d\mu
\]
is estimated as follows

\[ \leq \frac{1}{4m} \sum_{k=1}^{m} \| w_k - \bar{w}_k \|_{C([0,T]; L_2(\Omega))}^2 \]

\[ = \frac{\| (w_1, w_2, ..., w_m) - (\bar{w}_1, \bar{w}_2, ..., \bar{w}_m) \|_{C([0,T]; L_2(\Omega))}^2}{4m}, \]

where we have used the inequality \((\sum_{k=1}^{m} \lambda_k^2)^{\frac{m}{2}} \leq m \sum_{k=1}^{m} \lambda_k^2\)

This implies that

\[
\left\| \left[ I_k(w_1; w_2; \ldots; w_m) - I_k(\bar{w}_1; \bar{w}_2; \ldots; \bar{w}_m) \right](t) \right\|_{L_2(\Omega)}^2
\leq \frac{4}{m} \left\| \left( w_1, w_2, \ldots, w_m \right) - \left( \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m \right) \right\|_{C([0,T]; L_2(\Omega))}^2.
\]

The latter inequality holds for all \(k = 1, m\). Thus, we obtain

\[
\left\| J(w_1; w_2; \ldots; w_m) - J(\bar{w}_1; \bar{w}_2; \ldots; \bar{w}_m) \right\|_{C([0,T]; L_2(\Omega))}^2
\leq \frac{4}{m} \left\| \left( w_1, w_2, \ldots, w_m \right) - \left( \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m \right) \right\|_{C([0,T]; L_2(\Omega))}^2.
\]

Hence \(J\) is a contraction. Using the Banach fixed-point theorem, we conclude that

\[ J(w_1; w_2; \ldots; w_m) = (w_1, w_2, \ldots, w_m), \]

has a unique solution

\[ \left( u_1^{(p)}, u_2^{(p)}, \ldots, u_m^{(p)} \right) \in \left[ C([0, T]; L_2(\Omega)) \right]^m. \]

**Step 2.** The instability of the solution of the system (1). The norm of \(B_1\) in \(L_2(\Omega)\) is estimated as follows

\[
\| B_1 \|^2_{L^2(\Omega)} = \frac{\exp \left( 2\lambda p \int_t^T \eta_k(\mu) d\mu \right)}{\lambda^2_p} \geq \frac{e^{2(T-t)\eta_p}}{\lambda^2_p}.
\]

The norm of \(B_2\) in \(L_2(\Omega)\) is bounded as follows

\[
\| B_2 \|^2_{L^2(\Omega)} \leq \frac{1}{4m^2T} \int_t^T \exp \left( 2\lambda p \int_t^\mu \eta_k(\xi) d\xi \right) \left\| \mathcal{F}_k^{(\mu)}(x, \mu; u_k^{(p)}; \ldots; u_m^{(p)}, \sigma_p(x)) \right\|_{L_2(\Omega)}^2 d\mu
\]

\[ \leq \frac{1}{4m^2T} \int_t^T m \sum_{k=1}^{m} \left\| u_k^{(p)} \right\|_{L_2(\Omega)}^2 d\mu \leq \frac{1}{4m} \left[ \sum_{k=1}^{m} \left\| u_k^{(p)} \right\|_{C([0,T]; L_2(\Omega))}^2 \right].
\]

(14)
Using the inequality \((c - d)^2 \geq \frac{c^2 - 2cd}{2}\) for any \(c, d \in \mathbb{R}\) and (9), (13), (14), we obtain for all \(k = 1, m\)

\[
\left\| u_k^{(p)}(\cdot, t) \right\|_{L^2(\Omega)}^2 \geq \frac{\| B_1 \|_{L^2(\Omega)}^2 - 2\| B_2 \|_{L^2(\Omega)}^2}{2} - \frac{e^{2(T-t)}r \lambda_p}{2 \lambda_p^2} - \frac{1}{4m} \sum_{k=1}^{m} \left\| u_k(p) \right\|_{C([0,T];L^2(\Omega))}^2.
\]

By computations analogous to the previous one for \(k = 1, m\) and summing up the obtained results, one has

\[
\sum_{k=1}^{m} \left\| u_k^{(p)}(\cdot, t) \right\|_{L^2(\Omega)}^2 \geq \frac{e^{2(T-t)}r \lambda_p}{2 \lambda_p^2} - \frac{1}{4} \sum_{k=1}^{m} \left\| u_k(p) \right\|_{C([0,T];L^2(\Omega))}^2.
\]

This implies that

\[
\sup_{0 \leq t \leq T} \sum_{k=1}^{m} \left\| u_k^{(p)}(\cdot, t) \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \sum_{k=1}^{m} \left\| u_k(p) \right\|_{C([0,T];L^2(\Omega))}^2 \geq \sup_{0 \leq t \leq T} \frac{e^{2T}r \lambda_p}{2 \lambda_p^2}.
\]

Therefore, we obtain

\[
\sum_{k=1}^{m} \left\| u_k^{(p)}(\cdot, t) \right\|_{C([0,T];L^2(\Omega))}^2 \geq \frac{2 me^{2T}r \lambda_p}{\lambda_p^2}.
\]

As \(p \to +\infty\), it follows from (8) and (15) that

\[
\sum_{k=1}^{m} \left\| u_k^{(p)}(\cdot, t) \right\|_{L^2(\Omega)}^2 = \frac{m}{\lambda_p^2} \to 0, \quad \sum_{k=1}^{m} \left\| u_k(p) \right\|_{C([0,T];L^2(\Omega))} \to \infty.
\]

Thus, the problem (1) is, in general, ill-posed in the Hadamard sense.

3. The inverse problem with global Lipschitz reaction terms. In this section, we present a regularized problem for approximating the system (1)-(2). We now make the following assumptions:

- **(A1)** Let \(r, R\) be positive constants. Suppose \(\eta_k : \mathcal{D}_T \to \mathbb{R}\), is a continuous function such that
  \[
  r < \eta_k(x, t) \leq R, \quad \forall (x, t) \in \mathcal{D}_T, \quad k = 1, m.
  \]

- **(A2)** Let \(F \in L_\infty(\mathcal{D}_T \times \mathbb{R}^m)\) satisfies the global Lipschitz condition
  \[
  |F(\cdot; u_1; \ldots; u_m) - F(\cdot; v_1; \ldots; v_m)| \leq K (|u_1 - v_1| + \ldots + |u_m - v_m|),
  \]
  for some constant \(K > 0\) independent of \(x, t, u_k, v_k, \quad k = 1, m\).

For the system (1)-(2), we choose the operator \(\mathcal{R} = R\Delta\), where \(R\) is defined in assumption (A1). Let \(\mathcal{B}_R^{e, \beta}\) be given by

\[
\mathcal{B}_R^{e, \beta} = \mathcal{R} + \mathcal{A}_R^{e, \beta},
\]

with

\[
\mathcal{A}_R^{e, \beta}(w) = \sum_{n=1}^{\infty} \log \left(1 + \beta e^{TR\lambda_n}\right)^{\frac{1}{2}} \left\langle w, \sigma_n(x) \right\rangle_{L^2(\Omega)} \sigma_n(x),
\]

for any function \(w \in L^2(\Omega)\). The function \(\beta := \beta(\epsilon)\) is such that

\[
\lim_{\epsilon \to 0^+} \beta = 0;
\]

it plays the role of a regularization parameter.
Using the modified quasi-reversibility method, we introduce the following regularized problem

\[
\begin{aligned}
\partial_t u_{1,\beta}^\epsilon - \nabla \cdot \bigl( \eta_1(x,t) \nabla u_{1,\beta}^\epsilon \bigr) - A_{1,\beta}^\epsilon (u_{1,\beta}^\epsilon) &= \mathcal{F}_1(x,t; u_{1,\beta}^\epsilon; \ldots; u_{m,\beta}^\epsilon), \quad \text{in } \mathcal{D}_T, \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\partial_t u_{m,\beta}^\epsilon - \nabla \cdot \bigl( \eta_m(x,t) \nabla u_{m,\beta}^\epsilon \bigr) - A_{m,\beta}^\epsilon (u_{m,\beta}^\epsilon) &= \mathcal{F}_m(x,t; u_{1,\beta}^\epsilon; \ldots; u_{m,\beta}^\epsilon), \quad \text{in } \mathcal{D}_T, \\
u_{k,\beta}^\epsilon &= 0, & & \text{on } \Pi, \\
u_{k,\beta}^\epsilon(x,T) &= \varphi_{k,\beta}(x), & & \text{in } \Omega,
\end{aligned}
\]  

where \( k = 1, \ldots, m \) and \( \varphi_{k,\beta} \in L_2(\Omega) \) satisfies (4).

The main tool for our proofs will be the following lemma.

**Lemma 3.1.** 1. For any \( w \in V_{TR}(\Omega) \), it holds

\[
\left\| A_{1,\beta}^\epsilon(w) \right\|_{L_2(\Omega)} \leq \frac{\beta}{T} \left\| w \right\|_{V_{TR}(\Omega)}. \tag{21}
\]

2. For any \( w \in L_2(\Omega) \), it holds

\[
\left\| B_{1,\beta}^\epsilon(w) \right\|_{L_2(\Omega)} \leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \left\| w \right\|_{L_2(\Omega)}. \tag{22}
\]

**Proof.** Using the inequality \( \log(1 + a) \leq a \), for any \( a > 0 \), we may estimate

\[
\left\| A_{1,\beta}^\epsilon(w) \right\|_{L_2(\Omega)}^2 = \frac{1}{T^2} \sum_{n=1}^\infty \log^2 \left( 1 + \beta e^{TR\lambda_n} \right) \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)}^2 \\
\leq \frac{\beta^2}{T^2} \sum_{n=1}^\infty e^{2TR\lambda_n} \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)}^2 \\
\leq \frac{\beta^2}{T^2} \left\| w \right\|_{V_{TR}(\Omega)}^2. \tag{23}
\]

From (18) and (19), it is easy to calculate that

\[
B_{1,\beta}^\epsilon(w) = \sum_{n=1}^\infty \left[ \log \left( 1 + \beta e^{TR\lambda_n} \right) - R\lambda_n \right] \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)} \sigma_n(x) \\
= \frac{1}{T} \sum_{n=1}^\infty \log \left( \beta + e^{-TR\lambda_n} \right) \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)} \sigma_n(x).
\]

Using Parseval’s equality, we deduce that

\[
\left\| B_{1,\beta}^\epsilon(w) \right\|_{L_2(\Omega)}^2 = \frac{1}{T^2} \sum_{n=1}^\infty \log^2 \left( \frac{1}{\beta + e^{-TR\lambda_n}} \right) \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)}^2 \\
\leq \frac{1}{T^2} \log^2 \left( \frac{1}{\beta} \right) \sum_{n=1}^\infty \left\langle w, \sigma_n(x) \right\rangle_{L_2(\Omega)}^2 \\
\leq \frac{1}{T^2} \log^2 \left( \frac{1}{\beta} \right) \left\| w \right\|_{L_2(\Omega)}^2. \tag{24}
\]

The proof is complete. \( \square \)

Before coming to the main theorem, let us set \( \mathcal{U} := (u_1, u_2, \ldots, u_m) \) the solution of system (1)-(2) and \( \mathcal{U}^{\epsilon,\beta} := (u_{1,\beta}^\epsilon, u_{2,\beta}^\epsilon, \ldots, u_{m,\beta}^\epsilon) \) the solution of system (20).
Theorem 3.2. Let \( \beta \in \left[0, 1 - e^{-T\lambda_1}\right] \) be such that
\[
\lim_{\epsilon \to 0^+} \beta = \lim_{\epsilon \to 0^+} \epsilon \beta^{-1} = 0. \tag{25}
\]
Then the system (20) has a unique solution \( U^{\epsilon, \beta} \in \mathbb{C}([0, T]; L_2(\Omega))^m \). Assume that the system (1)-(2) has a unique solution \( U \) satisfying
\[
U_k \in L_\infty(0, T; \mathcal{V}_{TR}(\Omega)); \quad \partial_t U_k \in L_\infty(0, T; L_2(\Omega)), \quad k = \overline{1, m},
\]
and
\[
\max \left\{ \| U_k \|_{L_\infty(0, T; \mathcal{V}_{TR}(\Omega))}, \| \partial_t U_k \|_{L_\infty(0, T; L_2(\Omega))} \right\} \leq E,
\]
for some known constant \( E > 0 \), then the following estimate holds true
\[
\| U^{\epsilon, \beta}(\cdot, t) - U(\cdot, t) \|_{L_2(\Omega)^m} \leq C(\epsilon, \beta, m, E, T) e^{(mK + 1/2)(T-t)} \beta^{+}, \quad \forall t \in [0, T]. \tag{27}
\]
Furthermore, there exists \( t_* \in [0, T] \) such that \( \lim_{\epsilon \to 0^+} t_* = 0 \) so that
\[
\| U^{\epsilon, \beta}(\cdot, t_*) - U(\cdot, 0) \|_{L_2(\Omega)^m} \leq C(\epsilon, \beta, m, E, T) e^{mKT + T/2} \sqrt{\frac{T}{\log \left( \frac{1}{\beta} \right)}}. \tag{28}
\]

Remark 3.1. The constant \( C(\epsilon, \beta, m, E, T) \) is well defined. From (83), we conclude that \( C(\epsilon, \beta, m, E, T) = \sqrt{\epsilon^2 \beta^{-2} + \frac{mE^2}{2}} \). Since \( \beta := \beta(\epsilon) \) satisfies (25) then its value has to be bounded.

Remark 3.2. If we choose \( \beta = \epsilon^\omega \) with \( \omega \in [0, 1] \) then the condition (25) is fulfilled as \( \epsilon \to 0^+ \) and in the estimates in (27) and (28) the order of estimates will be \( \epsilon^{\frac{m}{2}} \) and \( \log^{-1/2}(\epsilon^{-\omega}) \), respectively.

Proof. The proof of this theorem is divided into two steps.

Step 1. The existence and uniqueness of the solution to the regularized system (20).

Let \( g_k(x, t) = R - \eta_k(x, t), \ k = \overline{1, m} \). It is obvious that \( 0 < g_k(x, t) < R \). Then from (20), we obtain
\[
\partial_t u_k^{\epsilon, \beta} + \nabla \cdot \left( g_k(x, t) \nabla u_k^{\epsilon, \beta} \right) = F_k \left( x, t; u_1^{\epsilon, \beta}, \ldots, u_m^{\epsilon, \beta} \right) + B^R_k \left( u_k^{\epsilon, \beta} \right), \tag{29}
\]
for all \((x, t) \in D_T, \ k = \overline{1, m}\).

Let \( Z_k^{\epsilon, \beta}(x) = u_k^{\epsilon, \beta}(x, T - t) \). Then we have
\[
\partial_t Z_k^{\epsilon, \beta}(x, t) = -\partial_u Z_k^{\epsilon, \beta}(x, T - t),
\]
\[
\nabla \cdot \left( g_k(x, t) \nabla Z_k^{\epsilon, \beta} \right)(x, t) = \nabla \cdot \left( g_k(x, t) \nabla u_k^{\epsilon, \beta} \right)(x, T - t),
\]
and
\[
B^R_k(Z_k^{\epsilon, \beta})(x, t) = B^R_k(Z_k^{\epsilon, \beta})(x, T - t).
\]
This implies that $Z^{\epsilon, \beta}_k$ satisfies the following system

$$\begin{align*}
\partial_t Z^{\epsilon, \beta}_1 - \nabla \cdot \left( g_1(x, t) \nabla Z^{\epsilon, \beta}_1 \right) &= G_1(x, t; Z^{\epsilon, \beta}_1; \ldots; Z^{\epsilon, \beta}_m), \quad \text{in } \mathcal{D}_T, \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \\
\partial_t Z^{\epsilon, \beta}_m - \nabla \cdot \left( g_m(x, t) \nabla Z^{\epsilon, \beta}_m \right) &= G_m(x, t; Z^{\epsilon, \beta}_1; \ldots; Z^{\epsilon, \beta}_m), \quad \text{in } \mathcal{D}_T, \\
Z^{\epsilon, \beta}_k = 0, & \quad \text{on } \Pi, \ k = \overline{1, m}, \\
Z^{\epsilon, \beta}_k(x, 0) = \varphi^{\epsilon}_k(x), & \quad \text{in } \Omega, \ k = \overline{1, m},
\end{align*}$$

(30)

where the functions $G_k (k = \overline{1, m})$, are defined by

$$G_k(x, t; w_1; \ldots; w_m) = - F_k(x, t; w_1; \ldots; w_m) - B^{\epsilon, \beta}_R(w_k)(x, t),$$

(31)

for any $w_k \in C([0, T]; L^2(\Omega))$.

Put

$$U^{\epsilon, \beta} = (Z^{\epsilon, \beta}_1, \ldots, Z^{\epsilon, \beta}_m), \quad U^{\epsilon, \beta}_i = (\partial_t Z^{\epsilon, \beta}_1, \ldots, \partial_t Z^{\epsilon, \beta}_m).$$

In the space $H^1(\Omega)$, we take a basis $\{e_j\}_{j=1}^p$ and define the finite dimensional space

$$V_p = \text{span}\{e_1, e_2, \ldots, e_p\}.$$

Let $\varphi^{\epsilon}_{k,p}, \ k = \overline{1, m}$ be an element of $V_p$ such that

$$\varphi^{\epsilon}_{k,p} = \sum_{j=1}^p \delta_{pj} e_j \to \varphi^{\epsilon}_{k} \quad \text{strongly in } \ L^2(\Omega),$$

(32)

as $p \to +\infty$. We search the approximate solution of the problem (30) in the form

$$Z^{\epsilon, \beta}_{k,p}(t) = \sum_{j=1}^p c^{\epsilon, \beta}_{pj} (t) e_j, \ k = \overline{1, m},$$

(33)

where the coefficients $c^{\epsilon, \beta}_{pj} (t)$ satisfy the system of linear differential equations

$$\begin{align*}
\left. \frac{d}{dt} Z^{\epsilon, \beta}_{k,p}(t), e_j \right|_{L^2(\Omega)} + A_k \left( t; Z^{\epsilon, \beta}_{k,p}, e_j \right) \\
+ \left. \left( F_k(x, t; Z^{\epsilon, \beta}_1; \ldots; Z^{\epsilon, \beta}_m), e_j \right) \right|_{L^2(\Omega)} = \left. B^{\epsilon, \beta}_R(Z^{\epsilon, \beta}_{k,p})(t), e_j \right|_{L^2(\Omega)},
\end{align*}$$

(34)

for all $k = \overline{1, m}$. Here $A_k(t; u, v)$ is defined by

$$A_k(t; u, v) := \int_{\Omega} \rho_k(x, t) \nabla u \nabla v dx,$$

(35)

for any $u, v \in H^1(\Omega)$. The existence of a local solution of system (34) is guaranteed by Peano’s theorem [14].

Multiplying the $j$th equation of (34) by $c^{\epsilon, \beta}_{pj} (t)$ and summing up with respect to $j$, we obtain

$$\begin{align*}
\frac{d}{dt} \left\| Z^{\epsilon, \beta}_{k,p}(t) \right\|^2_{L^2(\Omega)} + A_k \left( t; Z^{\epsilon, \beta}_{k,p}, Z^{\epsilon, \beta}_{k,p} \right) \\
+ \left. \left( B^{\epsilon, \beta}_R(Z^{\epsilon, \beta}_{k,p})(t), Z^{\epsilon, \beta}_{k,p} \right) \right|_{L^2(\Omega)} \\
= \left. - F_k(x, t; Z^{\epsilon, \beta}_1; \ldots; Z^{\epsilon, \beta}_m), Z^{\epsilon, \beta}_{k,p} (t) \right|_{L^2(\Omega)}.
\end{align*}$$

(36)
By integrating (36) from $t$ to $T$, we obtain
\[
\left\| Z_{k,p}^{r,\beta}(t) \right\|_{L^2(\Omega)}^2 + 2 \int_0^t A_k \left( \mu; Z_{k,p}^{r,\beta}(\mu), Z_{k,p}^{r,\beta}(\mu) \right) \, d\mu \\
+ 2 \int_0^t \left\langle B^r_R (Z_{k,p}^{r,\beta})(\mu), Z_{k,p}^{r,\beta}(\mu) \right\rangle_{L^2(\Omega)} \, d\mu \\
= \left\| Z_{k,p}^{r,\beta}(0) \right\|_{L^2(\Omega)}^2 \\
+ 2 \int_0^t \left\langle - F_k(\mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu)), Z_{k,p}^{r,\beta}(\mu) \right\rangle_{L^2(\Omega)} \, d\mu. \tag{37}
\]

The assumption $\rho_k(\mathbf{x}, t) \geq (R - r)$ allows to write
\[
\int_0^t A_k \left( \mu; Z_{k,p}^{r,\beta}(\mu), Z_{k,p}^{r,\beta}(\mu) \right) \, d\mu = \int_0^t \int_{\Omega} \rho_k(\mathbf{x}, \mu) \left\| \nabla Z_{k,p}^{r,\beta}(\mu) \right\|^2 \, d\mathbf{x} \, d\mu \\
\geq (R - r) \int_0^t \left\| Z_{k,p}^{r,\beta}(\mu) \right\|^2_{H^1(\Omega)} \, d\mu. \tag{38}
\]

Using Hölder inequality and Lemma 3.1, we deduce that
\[
2 \int_0^t \left\langle B^r_R (Z_{k,p}^{r,\beta})(\mu), Z_{k,p}^{r,\beta}(\mu) \right\rangle_{L^2(\Omega)} \, d\mu \\
\leq 2 \int_0^t \left\| B^r_R (Z_{k,p}^{r,\beta})(\mu) \right\|_{L^2(\Omega)}^2 \, d\mu + \frac{1}{2} \int_0^t \left\| Z_{k,p}^{r,\beta}(\mu) \right\|_{L^2(\Omega)}^2 \, d\mu \\
\leq 2 \frac{1}{T} \log \left( \frac{1}{\beta} \right) \int_0^t \left\| Z_{k,p}^{r,\beta}(\mu) \right\|_{L^2(\Omega)}^2 \, d\mu + \frac{1}{2} \int_0^t \left\| Z_{k,p}^{r,\beta}(\mu) \right\|_{L^2(\Omega)}^2 \, d\mu, \tag{39}
\]
and
\[
2 \int_0^t \left\langle - F_k(\mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu)), Z_{k,p}^{r,\beta}(\mu) \right\rangle_{L^2(\Omega)} \, d\mu \\
\leq 2 \int_0^t \left\| F_k(\mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu)) \right\|_{L^2(\Omega)}^2 \, d\mu \\
+ \frac{1}{2} \int_0^t \left\| Z_{k,p}^{r,\beta}(\mu) \right\|_{L^2(\Omega)}^2 \, d\mu. \tag{40}
\]

Now we estimate the first term of (40). First, using (17), we get
\[
\left| F_k \left( \mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu) \right) \right| \\
\leq |F(\cdot, \mu; 0; \ldots; 0)| + K \left( \left| Z_{1,p}^{r,\beta}(\mu) \right| + \ldots + \left| Z_{m,p}^{r,\beta}(\mu) \right| \right). \tag{41}
\]

This implies that
\[
\left\| F_k(\mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu)) \right\|_{L^2(\Omega)}^2 = \int_\Omega \left| F_k(\mathbf{x}, \mu; Z_{1,p}^{r,\beta}(\mu), \ldots; Z_{m,p}^{r,\beta}(\mu)) \right|^2 \, d\mathbf{x} \\
\leq 2 \int_\Omega |F_k(\cdot, \mu; 0; \ldots; 0)|^2 \, d\mathbf{x} + 2Km \int_\Omega \left( \left| Z_{1,p}^{r,\beta}(\mu) \right|^2 + \ldots + \left| Z_{m,p}^{r,\beta}(\mu) \right|^2 \right) \, d\mathbf{x} \\
= 2 \left\| F_k(\cdot, \mu; 0; \ldots; 0) \right\|_{L^2(\Omega)}^2 + 2Km \sum_{k=1}^m \left\| Z_{k,p}^{r,\beta}(\mu) \right\|_{L^2(\Omega)}^2. \tag{42}
\]
This latter inequality together with (40) and (42) leads to
\[
2 \int_0^t \langle - \mathbf{F}_k(x, \mu; Z_{1,k,p}^\gamma(\mu); \ldots; Z_{m,k,p}^\gamma(\mu)), Z_{k,p}^\gamma(\mu) \rangle_{L_2(\Omega)} \, d\mu \\
\leq 4 \int_0^t \| \mathbf{F}(\cdot, \mu; 0; \ldots; 0) \|_{L_2(\Omega)}^2 \, d\mu + 4Km \int_0^t \sum_{k=1}^m \| Z_{k,p}^\gamma(\mu) \|_{L_2(\Omega)}^2 \, d\mu \\
+ \frac{1}{2} \int_0^t \| Z_{k,p}^\gamma(\mu) \|_{L_2(\Omega)}^2 \, d\mu.
\] (43)
Combining (36), (37), (38), (39), (43), we obtain
\[
\| Z_{k,p}^\gamma(t) \|_{L_2(\Omega)}^2 + (R - r) \int_0^t \| Z_{k,p}^\gamma(\mu) \|_{H_1(\Omega)}^2 \, d\mu \\
\leq \| \varphi_{k,p}^\gamma \|_{L_2(\Omega)} + \left(1 + \frac{1}{T} \log \left(\frac{2}{\beta}\right)\right) \int_0^t \| Z_{k,p}^\gamma(\mu) \|_{L_2(\Omega)}^2 \, d\mu \\
+ 4 \int_0^t \| \mathbf{F}_k(\cdot, \mu; 0; \ldots; 0) \|_{L_2(\Omega)}^2 \, d\mu + 4Km \int_0^t \sum_{k=1}^m \| Z_{k,p}^\gamma(\mu) \|_{L_2(\Omega)}^2 \, d\mu. \] (44)
Since (32), we deduce that
\[
\| \varphi_{k,p}^\gamma \|_{L_2(\Omega)} \leq C_k(\varepsilon), \forall p,
\] (45)
where \( C_k(\varepsilon) \) is a constant independent of \( p \). By summing up the obtained results (44), one has
\[
\sum_{k=1}^m \| Z_{k,p}^\gamma(t) \|_{L_2(\Omega)}^2 + (R - r) \int_0^t \sum_{k=1}^m \| Z_{k,p}^\gamma(\mu) \|_{H_1(\Omega)}^2 \, d\mu \\
\leq \sum_{k=1}^m C_k(\varepsilon) + 4 \int_0^T \sum_{k=1}^m \| \mathbf{F}_k(\cdot, \mu; 0; \ldots; 0) \|_{L_2(\Omega)}^2 \, d\mu \\
+ \left(4Km^2 + 1 + \frac{2}{T} \log \left(\frac{1}{\beta}\right)\right) \int_0^t \sum_{k=1}^m \| Z_{k,p}^\gamma(\mu) \|_{L_2(\Omega)}^2 \, d\mu. \] (46)
Let us set
\[
\mathbf{S}_p^\gamma(t) = \sum_{k=1}^m \| Z_{k,p}^\gamma(t) \|_{L_2(\Omega)}^2 + (R - r) \int_0^t \sum_{k=1}^m \| Z_{k,p}^\gamma(\mu) \|_{H_1(\Omega)}^2 \, d\mu,
\]
then we know from (46) that
\[
\mathbf{S}_p^\gamma(t) \leq \mathbb{D}(\varepsilon, T) + \left(4Km^2 + 1 + \frac{2}{T} \log \left(\frac{1}{\beta}\right)\right) \int_0^t \mathbf{S}_p^\gamma(\mu) \, d\mu. \] (47)
By using Gronwall’s inequality, we obtain that
\[
\mathbf{S}_p^\gamma(t) \leq \exp \left(4Km^2T + T + 2 \log \left(\frac{1}{\beta}\right)\right) \mathbb{D}(\varepsilon, T), \] (48)
for all \( p \in \mathbb{N} \). This implies that
\[
Z_{k,p}^\gamma \text{ is bounded in } L_\infty(0, T; L_2(\Omega)), \quad k = 1, m,
\] (49)
and
\[
Z_{k,p}^\gamma \text{ is bounded in } L_2(0, T; H_1(\Omega)), \quad k = 1, m,
\] (50)
Then we can extend the solution to the interval \([0, T]\). Now we need to pass to the limit when \(p \to +\infty\). Since \(\nabla \cdot \left( g_k(x, t) \nabla Z_{k,p}^{x,\beta} \right)\) defines an element of \(H_{-1}(\Omega)\), it is easy to see that

\[
\partial_t Z_{k,p}^{x,\beta} \text{ is bounded in } L_2(0, T; H_{-1}(\Omega)), \ k = 1, m, \tag{51}
\]

Due to the Banach–Alaoglu theorem, from (49), (50) and (51), we can extract subsequences which we denote by the same symbol \(Z_{k,p}^{x,\beta}\) such that

\[
\begin{align*}
Z_{k,p}^{x,\beta} &\to Z_k^{x,\beta} \quad \text{in } L_\infty(0, T; L_2(\Omega)) \text{ weak* convergence,} \\
Z_{k,p}^{x,\beta} &\to Z_k^{x,\beta} \quad \text{in } L_2(0, T; H_1(\Omega)) \text{ weak convergence,} \\
\partial_t Z_{k,p}^{x,\beta} &\to \partial_t Z_k^{x,\beta} \quad \text{in } L_2(0, T; H_{-1}(\Omega)) \text{ weak* convergence,}
\end{align*}
\tag{52}
\]

when \(p \to +\infty\) and \(k = 1, m\). Using Aubin–Lions compactness lemma, we obtain

\[
Z_{k,p}^{x,\beta} \to Z_k^{x,\beta} \quad \text{strongly in } L_\infty(0, T; L_2(\Omega)), \ k = 1, m, \tag{53}
\]

Hence, passing if necessary to a subsequence, still denoted by \(Z_{k,p}^{x,\beta}\), one has

\[
Z_{k,p}^{x,\beta} \to Z_k^{x,\beta} \quad \text{a.e in } D_T, \ k = 1, m, \tag{54}
\]

Since \(\mathcal{F}_k\) is globally Lipschitz, we also obtain

\[
\mathcal{F}_k(x, t; Z_{1,p}^{x,\beta}, \ldots; Z_{m,p}^{x,\beta}) \to \mathcal{F}_k(x, t; Z_1^{x,\beta}, \ldots; Z_m^{x,\beta}). \tag{55}
\]

Passing to the limit in (34) by (52), (53), (55), we have

\[
Z_k^{x,\beta} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_1(\Omega)),
\]

satisfying the system (30). Therefore, we can conclude that the system (20) has a unique solution \(U^{x,\beta}\) such that

\[
U^{x,\beta} \in \left[ L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_1(\Omega)) \right]^m.
\]

**Step 2. The uniqueness of solutions of (20).**

Let \(U^{x,\beta} = (u_1^{x,\beta}, \ldots, u_m^{x,\beta})\) and \(V^{x,\beta} = (v_1^{x,\beta}, \ldots, v_m^{x,\beta})\) be two solutions of System (20). Then we have

\[
\partial_t u_k^{x,\beta} + \nabla \cdot \left( g_k(x, t) \nabla u_k^{x,\beta} \right) = \mathcal{F}_k \left( x, t; u_1^{x,\beta}, \ldots, u_m^{x,\beta} \right) + \mathcal{B}_R^{\beta}(u_k^{x,\beta}), \quad \text{for } (x, t) \in D_T, \ k = 1, m, \tag{56}
\]

and

\[
\partial_t v_k^{x,\beta} + \nabla \cdot \left( g_k(x, t) \nabla v_k^{x,\beta} \right) = \mathcal{F}_k \left( x, t; v_1^{x,\beta}, \ldots, v_m^{x,\beta} \right) + \mathcal{B}_R^{\beta}(v_k^{x,\beta}), \quad \text{for } (x, t) \in D_T, \ k = 1, m. \tag{57}
\]

Put

\[
Y_k^{x,\beta}(x, t) = \exp (q_\beta(t - T)) \left[ u_k^{x,\beta}(x, t) - v_k^{x,\beta}(x, t) \right], \tag{58}
\]

where \(q_\beta > 0\) depends of \(\beta\) to be chosen later. It is easy to see that

\[
Y_k^{x,\beta}(x, T) = u_k^{x,\beta}(x, T) - v_k^{x,\beta}(x, T) = 0.
\]
A simple computation gives
\[
\partial_t Y_k^{\epsilon, \beta} + \nabla \cdot \left( \varrho_k(x,t) \nabla Y_k^{\epsilon, \beta} \right) - q_\beta Y_k^{\epsilon, \beta} = \mathcal{B}_R^{\epsilon, \beta}(Y_k^{\epsilon, \beta}) + \exp(q_\beta(t - T)) \left[ \mathcal{F}_k(x,t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) - \mathcal{F}_k(x,t; v_1^{\epsilon, \beta}; \ldots; v_m^{\epsilon, \beta}) \right].
\] (59)

Multiplying both sides of (59) by \( Y_k^{\epsilon, \beta} \), we get
\[
\frac{1}{2} \frac{d}{dt} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 = \left\langle \mathcal{B}_R^{\epsilon, \beta}(Y_k^{\epsilon, \beta}), Y_k^{\epsilon, \beta} \right\rangle_{L^2(\Omega)}
+ \left\langle \epsilon^{\beta(t-T)} \left[ \mathcal{F}_k(x,t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) - \mathcal{F}_k(x,t; v_1^{\epsilon, \beta}; \ldots; v_m^{\epsilon, \beta}) \right], Y_k^{\epsilon, \beta} \right\rangle_{L^2(\Omega)}. \]
Using the Hölder’s inequality and Lemma (3.1), we estimate \( E_1 \) as follows
\[
\left\| E_1 \right\| \leq \left\| -\mathcal{B}_R^{\epsilon, \beta}(Y_k^{\epsilon, \beta}) \right\|_{L^2(\Omega)} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)} \leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}. \] (61)

Using (17), the term \( E_2 \) is bounded by
\[
\left\| E_2 \right\| \leq \left\| \left[ \mathcal{F}_k(x,t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) - \mathcal{F}_k(x,t; v_1^{\epsilon, \beta}; \ldots; v_m^{\epsilon, \beta}) \right] \right\|_{L^2(\Omega)} \times
\leq \frac{1}{2} K^2 \left( \left\| Y_1^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)} + \ldots + \left\| Y_m^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)} \right)^2 + \frac{1}{2} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq \frac{K^2 m}{2} \sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq \frac{K^2 m}{2} \sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2, \]
where we have used the inequality \((\sum_{k=1}^{m} a_k)^2 \leq m \sum_{k=1}^{m} a_k^2\), for \( a_k \geq 0, \ k = 1, m \).

Combining (60), (61), (62), we obtain
\[
\frac{d}{dt} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \geq 2 \int_{\Omega} \varrho_k(x,t) \left\| \nabla Y_k^{\epsilon, \beta} \right\|^2 dx + 2q_\beta \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 - 2E_1 - 2E_2 \geq 2(R-r) \int_{\Omega} \left\| \nabla Y_k^{\epsilon, \beta} \right\|^2 dx + \left( 2q_\beta - \frac{2}{T} \log \left( \frac{1}{\beta} \right) - 1 \right) \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 - K^2 m \sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2. \] (63)
Choosing \( q_\beta = \frac{1}{T} \log \left( \frac{1}{\beta} \right) \) and integrating (63), we get
\[
\left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq \int_{t}^{T} \left\| Y_k^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 d\mu + K^2 m \int_{t}^{T} \sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 d\mu, \] (64)
where we have used \( Y_k^{\epsilon,\beta} (\cdot, T) = 0 \). The latter equality holds for all \( k = \frac{1}{m} \).

Therefore, summing up the obtained results, one has

\[
\sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq (1 + K^2 m^2) \int_t^T \sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 d\mu. \tag{65}
\]

Using Gronwall’s inequality, we see that

\[
\sum_{k=1}^{m} \left\| Y_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 = 0.
\]

This implies that \( v_k^{\epsilon, \beta} = 0 \), \( k = \frac{1}{m} \). So, we conclude that \( U^{\epsilon, \beta} = \mathcal{V}^{\epsilon, \beta} \).

**Step 3. Estimate of the error** \( \left\| U^{\epsilon, \beta} (\cdot, t) - U (\cdot, t) \right\|_{L^2(\Omega)} \).

Let us set

\[
w_n(t) = \langle w(x, t), \sigma_n (x) \rangle_{L^2(\Omega)}.
\]

For \( q_k (x, t) = R - \eta_k (x, t), \ k = \frac{1}{m} \), one has

\[
\partial_t u_k + \nabla \cdot \left( q_k (x, t) \nabla u_k \right) = \mathcal{F}_k (x, t; u_1; \ldots; u_m) + \Re (u_k), \quad (x, t) \in D_T, \ k = \frac{1}{m},
\]

\[
\partial_t u_k^{\epsilon, \beta} + \nabla \cdot \left( q_k (x, t) \nabla u_k^{\epsilon, \beta} \right) = \mathcal{F}_k \left( x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta} \right) + \mathcal{A}_R^{\beta} (u_k), \quad \epsilon, \beta \text{ to be chosen later.}
\]

Then, from the equalities \( (66) \) and \( (67) \), a simple computation gives

\[
\partial_t W_k^{\epsilon, \beta} + \nabla \cdot \left( q_k (x, t) \nabla W_k^{\epsilon, \beta} \right) = q_k W_k^{\epsilon, \beta} \mathcal{F}_k \left( x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta} \right) + \mathcal{A}_R^{\beta} (u_k),
\]

\[
\text{for all } (x, t) \in D_T \text{ and } W_k^{\epsilon, \beta} |_{\partial \Omega} = 0, \ W_k^{\epsilon, \beta} (x, T) = \varphi_k (x) - \varphi_k (x), \ k = \frac{1}{m}.
\]

We clearly have the equality

\[
\int_{\Omega} \nabla \cdot \left( q_k (x, t) \nabla W_k^{\epsilon, \beta} \right) W_k^{\epsilon, \beta} dx = - \int_{\Omega} q_k (x, t) \left\| \nabla W_k^{\epsilon, \beta} \right\|^2 dx,
\]

so we have \( (k = \frac{1}{m}, m) \)

\[
\frac{1}{2} \frac{d}{dt} \left\| W_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 = \int_{\Omega} q_k (x, t) \left\| \nabla W_k^{\epsilon, \beta} \right\|^2 dx - q_k \left\| W_k^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2
\]

\[
= \left\langle \mathcal{B}_R^{\beta} (W_k^{\epsilon, \beta}), W_k^{\epsilon, \beta} \right\rangle_{L^2(\Omega)} + \left\langle q_k (t-T) \mathcal{A}_R^{\beta} (u_k), W_k^{\epsilon, \beta} \right\rangle_{L^2(\Omega)} + \left\langle q_k (t-T) \mathcal{F}_k \left( x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta} \right), W_k^{\epsilon, \beta} \right\rangle_{L^2(\Omega)}. \tag{70}
\]

\[
\Rightarrow \lim_{m \to \infty} \left\| U^{\epsilon, \beta} (\cdot, t) - U (\cdot, t) \right\|_{L^2(\Omega)} = 0.
\]
Combining (70)-(72) and (74) yields
\[ 786 \]
\[ VO \text{ } VAN \text{ } AU, \text{ } MOKHTAR \text{ } KIRANE \text{ } AND \text{ } NGUYEN \text{ } HUY \text{ } TUAN \]
\[ q \]
where we have used the fact that \( \exp(2\beta) \leq 1 \). Second, using \( \log(1 + a) \leq a \), for any positive \( a \), the term \( \| \| \) can be estimated as follows
\[ \| \exp(q_\beta(t - T)) \|_L^2(\Omega) \leq \frac{1}{2} \left\| \exp(q_\beta(t - T)) \right\|_L^2(\Omega) + \frac{1}{2} \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \]
\[ \leq \frac{\beta^2}{2T^2} \left\| u_k(\cdot, t) \right\|_M^2(\Omega) + \frac{1}{2} \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega), \]  
(72)
where we have used the fact that \( \exp(2q_\beta(t - T)) \leq 1 \), \( \forall \in [0, T] \) and
\[ |\langle w_1, w_2 \rangle| \leq \frac{1}{2} \| w_1 \|_L^2(\Omega) + \frac{1}{2} \| w_2 \|_L^2(\Omega). \]  
(73)
Finally, using (17), we estimate
\[ |\| III \| | \]
\[ = |\exp(q_\beta(t - T)) \left[ \mathcal{F}_k(x, t; u_1^\epsilon, \beta, \ldots; u_m^\epsilon, \beta) - \mathcal{F}_k(x, t, u_1; \ldots; u_m) \right], W^\beta_\epsilon,\beta) \left| \right. \]
\[ \leq |\exp(q_\beta(t - T)) \left[ \mathcal{F}_k(\cdot, t; u_1^\epsilon, \beta, \ldots; u_m^\epsilon, \beta) - \mathcal{F}_k(\cdot, t; u_1; \ldots; u_m) \right] \left\| \right. \]
\[ \times \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \]
\[ \leq K \left( \left\| W^\epsilon,\beta \right\|_L^2(\Omega) + \ldots + \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \right) \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \]
\[ \leq K \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \sum_{k=1}^m \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega). \]  
(74)
Combining (70)-(72) and (74) yields
\[ \frac{1}{2} \frac{d}{dt} \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega)^2 - \int_\Omega q_k(x, t) \nabla W^\beta_\epsilon,\beta d^2 x - q_\beta \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega)^2 \]
\[ \geq - \frac{1}{T} \log \left( \frac{1}{\beta} \right) \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega)^2 - \left( \frac{\beta^2}{2T^2} \right) \left\| u_k(\cdot, t) \right\|_M^2(\Omega) \]
\[ - \frac{1}{2} \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega)^2 - K \left( \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega) \sum_{k=1}^m \left\| W^\beta_\epsilon,\beta \right\|_L^2(\Omega). \]  
(75)
Consequently,

\[
\begin{align*}
\left\| W_{k}^{e,\beta}(\cdot, T) \right\|^2_{L_2(\Omega)} - \left\| W_{k}^{e,\beta}(\cdot, t) \right\|^2_{L_2(\Omega)} + \int_{t}^{T} \frac{\beta^2}{T^2} \left\| u_k \right\|^2_{L_\infty(0,T;\mathcal{V}_{TR}(\Omega))} \ d\mu \\
\geq \int_{t}^{T} \left( 2q_{\beta} - \frac{2}{T} \log \left( \frac{1}{\beta} \right) \right) \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|^2_{L_2(\Omega)} \ d\mu \\
- \int_{t}^{T} 2K \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \sum_{m=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \ d\mu \\
+ 2 \int_{t}^{T} \int_{\Omega} \Theta_{k}(x, \mu) \left| \nabla W_{k}^{e,\beta}(x, \mu) \right|^2 \ d\mu \ d\mu \\
\geq \int_{t}^{T} \left( 2q_{\beta} - \frac{2}{T} \log \left( \frac{1}{\beta} \right) \right) \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|^2_{L_2(\Omega)} \ d\mu \\
- 2K \int_{t}^{T} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \sum_{m=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \ d\mu. 
\end{align*}
\]

This implies that

\[
\begin{align*}
\left\| W_{k}^{e,\beta}(\cdot, t) \right\|^2_{L_2(\Omega)} \\
\leq \left\| \varphi_{k} - \varphi_{k} \right\|^2_{L_2(\Omega)} + \frac{\beta^2}{T} \left\| u_k \right\|^2_{L_\infty(0,T;\mathcal{V}_{TR}(\Omega))} \\
+ \int_{t}^{T} \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_{\beta} \right) \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|^2_{L_2(\Omega)} \ d\mu \\
+ 2K \int_{t}^{T} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \sum_{m=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \ d\mu, \quad k = \overline{1,m}. 
\end{align*}
\]

By computations analogous to the previous one for \( k = \overline{1,m} \) and summing up the obtained results, one has

\[
\begin{align*}
\sum_{k=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, t) \right\|^2_{L_2(\Omega)} \\
\leq \sum_{k=1}^{m} \left\| \varphi_{k} - \varphi_{k} \right\|^2_{L_2(\Omega)} + \frac{\beta^2}{T} \sum_{k=1}^{m} \left\| u_k \right\|^2_{L_\infty(0,T;\mathcal{V}_{TR}(\Omega))} \\
+ \int_{t}^{T} \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_{\beta} \right) \sum_{k=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|^2_{L_2(\Omega)} \ d\mu \\
+ 2K \int_{t}^{T} \left( \sum_{k=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|_{L_2(\Omega)} \right)^2 \ d\mu \\
\leq \epsilon^2 + \frac{m\beta^2E^2}{T} \\
+ \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_{\beta} + 2mK \right) \int_{t}^{T} \sum_{k=1}^{m} \left\| W_{k}^{e,\beta}(\cdot, \mu) \right\|^2_{L_2(\Omega)} \ d\mu, \quad (78)
\end{align*}
\]

where we have used the inequality \((\sum_{k=1}^{m} a_k)^2 \leq m \sum_{k=1}^{m} a_k^2, \quad \text{ for } a_k \geq 0, \ k = \overline{1,m}.\)
Put \( \Psi^{\epsilon, \beta}(t) = \sum_{k=1}^{m} \left\| W^{\epsilon, \beta}_k (\cdot, t) \right\|_{L^2(\Omega)}^2 \); choosing \( q_\beta = \frac{1}{T} \log \left( \frac{1}{\beta} \right) > 0 \) and using (4) and (26), we observe that

\[
\Psi^{\epsilon, \beta}(t) \leq \epsilon^2 + \frac{m \beta^2 E^2}{T} + (2mK + 1) \int_t^T \Psi^{\epsilon, \beta}(\mu) d\mu. \tag{79}
\]

Gronwall’s inequality allows to obtain

\[
\Psi^{\epsilon, \beta}(t) \leq \left( \epsilon^2 + \frac{m \beta^2 E^2}{T} \right) \exp \left( (2mK + 1)(T - t) \right). \tag{80}
\]

This implies that

\[
\exp (2q_\beta (t - T)) \sum_{k=1}^{m} \left\| u^{\epsilon, \beta}_k (\cdot, t) - u_k (\cdot, t) \right\|_{L^2(\Omega)}^2 \\
\leq \left( \epsilon^2 + \frac{m \beta^2 E^2}{T} \right) \exp \left( (2mK + 1)(T - t) \right) \beta^2. \tag{81}
\]

Recalling that \( q_\beta = \frac{1}{T} \log \left( \frac{1}{\beta} \right) \), we obtain

\[
\sum_{k=1}^{m} \left\| u^{\epsilon, \beta}_k (\cdot, t) - u_k (\cdot, t) \right\|_{L^2(\Omega)}^2 \\
\leq \left( \epsilon^2 \beta^{-2} + \frac{m E^2}{T} \right) \exp \left( (2mK + 1)(T - t) \right) \beta^2. \tag{82}
\]

By putting

\[
C^2(\alpha, \beta, m, E, T) := \epsilon^2 \beta^{-2} + \frac{m E^2}{T}, \tag{83}
\]

we get

\[
\left\| U^{\epsilon, \beta} (\cdot, t) - U(\cdot, t) \right\|_{L^2(\Omega)^m} \\
\leq C(\alpha, \beta, m, E, T) \exp \left( (mK + 1/2)(T - t) \right) \beta^{2+}, \tag{84}
\]

which leads to (27).

To obtain the approximation of the solution \( U \) at \( t = 0 \), we choose a number \( t_\epsilon \in [0, T] \) such that \( \lim_{\epsilon \to 0} t_\epsilon = 0 \). Then we see that \( U^{\epsilon, \beta}(x, t_\epsilon) \) is an approximation of \( U(x, 0) \). Indeed, (82) implies that

\[
\left\| U^{\epsilon, \beta}(\cdot, t_\epsilon) - U(\cdot, 0) \right\|_{L^2(\Omega)^m} \\
\leq C(\alpha, \beta, m, E, T) \beta^{2+} e^{m KT/2 + t_\epsilon} \left\| U_\epsilon \right\|_{L^2(\Omega)^m},
\]

where we define \( U_\epsilon = (\partial_1 u_1, ..., \partial_1 u_m) \).

It is easy to show that for every \( \beta > 0 \), there exists a unique \( t_\epsilon \in [0, T] \) such that \( \lim_{\epsilon \to 0} t_\epsilon = 0 \) and \( t_\epsilon = \beta^{2+} \). This implies that \( \frac{\log t_\epsilon}{t_\epsilon} = \frac{1}{T} \log \beta \). Using the inequality that \( \log t > \frac{1}{T} \) for all \( t > 0 \), we obtain that \( t_\epsilon < \sqrt{T / \log (1/\beta)} \); it is straightforward
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to see then that (using (26))
\[
\|U^{\epsilon,\beta}(\cdot, t_\epsilon) - U(\cdot, 0)\|_{L^2(\Omega)} \leq \left( C(\alpha, \beta, m, E, T) e^{mKT + T/2} + mE \right) \sqrt{\frac{T}{\log \left( \frac{1}{\beta} \right)}}
\]
\[
\leq 2C(\alpha, \beta, m, E, T) e^{mKT + T/2} \sqrt{\frac{T}{\log \left( \frac{1}{\beta} \right)}}
\]
\[
\leq C(\alpha, \beta, m, E, T) e^{mKT + T/2} \sqrt{\frac{T}{\log \left( \frac{1}{\beta} \right)}},
\]
(85)

therein, without loss of generality, we may assume that \(mE \leq C(\alpha, \beta, m, E, T)\) and using the fact that \(e^{mKT + T/2} \geq 1\) then, the estimate (28) follows. This completes the proof of the theorem.

4. The inverse problem with locally Lipschitz reaction. Section 2 has addressed a problem in which \(F\) is a globally Lipschitz function, in this section we extend the analysis to a locally Lipschitz function \(F\). Results for the locally Lipschitz case are still very scarce. Hence, we have to find another regularization method to study the problem with the locally Lipschitz source which is similar to the latter source.

For emphasis and clarity, we outline our procedure. Keeping (A1), we replace the global Lipschitz assumption (A2) by the local Lipschitz (A4), specifically:

(A3) Suppose \(\eta^k : \mathcal{D}_T \to \mathbb{R}\), is a continuous function such that
\[
0 < r < \eta^k(x, t) \leq R, \quad \forall (x, t) \in \mathcal{D}_T, \quad k = 1, m,
\]
and
\[
\sum_{k=1}^m \|\eta^k - \eta_k\|_{C([0,T];L^2(\Omega))} \leq \epsilon.
\]
(86)

(A4) For each \(\ell > 0\), there exists \(K(\ell) > 0\) such that
\[
|F(x, t; u_1; \ldots; u_m) - F(x, t; v_1; \ldots; v_m)| \leq K(\ell) \left( |u_1 - v_1| + \ldots + |u_m - v_m| \right),
\]
if \(\max\{|u_1|, \ldots, |u_m|, |v_1|, \ldots, |v_m|\} \leq \ell\), (87)

where \((x, t) \in \mathcal{D}_T\) and
\[
K(\ell) := \sup \left\{ \left| \frac{F(x, t; u_1; \ldots; u_m) - F(x, t; v_1; \ldots; v_m)}{|u_1 - v_1| + \ldots + |u_m - v_m|} \right| : \max\{|u_1|, \ldots, |u_m|, |v_1|, \ldots, |v_m|\} \leq \ell, u_k \neq v_k, \quad k = 1, m, \quad (x, t) \in \mathcal{D}_T \right\} < \infty.
\]

We note that \(K(\ell)\) is increasing and \(\lim_{\ell \to +\infty} K(\ell) = +\infty\).

Now, we outline our ideas to construct a regularization. For all \(\ell > 0\), we approximate \(F\) by \(F(\ell)\) defined by
\[
F(\ell)(x, t; u_1; \ldots; u_m) := F(x, t; \tilde{u}_1; \ldots; \tilde{u}_m) \quad k = 1, m.
\]
(88)
where

\[ \tilde{u}_k := \begin{cases} -\ell, & \text{if } u_k \in ]-\infty, -\ell[, \\ u_k, & \text{if } u_k \in [-\ell, \ell[, \\ \ell, & \text{if } u_k \in ]\ell, +\infty[. \end{cases} \tag{89} \]

We still pay more attention to the system (20) from replacing \( \eta_k \) by \( \eta_k^\epsilon \) that satisfies (86). We are going to introduce the main idea to solve this problem with a special generalized case of source term defined by (88), we consider the problem:

\[
\begin{cases}
\partial_t u_1^{\epsilon, \beta} - \nabla \cdot (\eta_1^\epsilon(x, t) \nabla u_1^{\epsilon, \beta}) - A^{\epsilon, \beta}_R(u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) = F_1(\ell) (x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}), & \text{in } D_T, \\
\vdots & \\
\partial_t u_m^{\epsilon, \beta} - \nabla \cdot (\eta_m^\epsilon(x, t) \nabla u_m^{\epsilon, \beta}) - A^{\epsilon, \beta}_R(u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) = F_m(\ell) (x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}), & \text{in } D_T, \\
\end{cases}
\]

where \( k = \overline{1, m} \) and for each \( \epsilon > 0 \), we consider a parameter \( \ell^\epsilon \to +\infty \) as \( \epsilon \to 0^+ \).

Before stating the main theorem, we first consider the following lemma.

**Lemma 4.1.** Let \( F(\ell) \in L_{\infty}(D_T \times \mathbb{R}^m) \) given as (88). Then we have

\[
|F(\ell)(x; t; u_1; \ldots; u_m) - F(\ell)(x; t; v_1; \ldots; v_m)| \leq K(\ell) (|u_1 - v_1| + \ldots + |u_m - v_m|),
\]

\( \forall (x, t) \in D_T, \ u_k, v_k \in \mathbb{R}, \ k = \overline{1, m} \).

**Proof.** First, we show that for any \( u_k, v_k \in \mathbb{R} \) then

\[ |u_k - v_k| \leq |u_k - v_k|. \tag{91} \]

To prove (91), we divide all following cases.

**Case 1.** \( u_k < -\ell \)

- If \( v_k < -\ell \) then \( |\tilde{u}_k - \tilde{v}_k| = 0 \).
- If \( -\ell \leq v_k \leq \ell \) then
  \[ |\tilde{u}_k - \tilde{v}_k| = v_k + \ell < u_k - u_k = |u_k - v_k|. \]
- If \( v_k > \ell \) then
  \[ |\tilde{u}_k - \tilde{v}_k| = 2\ell \leq u_k - u_k = |u_k - v_k|. \]

**Case 2.** \( -\ell \leq u_k \leq \ell \)

- If \( v_k < -\ell \) then \( |\tilde{u}_k - \tilde{v}_k| = |u_k + \ell| = u_k + \ell < u_k - v_k = |u_k - v_k|. \)
- If \( -\ell \leq v_k \leq \ell \) then
  \[ |\tilde{u}_k - \tilde{v}_k| = |u_k - v_k|. \]
- If \( v_k > \ell \) then
  \[ |\tilde{u}_k - \tilde{v}_k| = \ell - u_k \leq v_k - u_k = |u_k - v_k|. \]

**Case 3.** \( u_k \geq \ell \)

- If \( v_k < -\ell \) then \( |\tilde{u}_k - \tilde{v}_k| = 2\ell < u_k - v_k = |u_k - v_k|. \)
- If \( -\ell \leq v_k \leq \ell \) then
  \[ |\tilde{u}_k - \tilde{v}_k| = \ell - v_k < u_k - v_k = |u_k - v_k|. \]
• If \( v_k > \ell \) then
\[
|\bar{u}_k - \bar{v}_k| = 0 \leq |u_k - v_k|.
\]
Now we return to the proof of Lemma (4.1). Indeed, since the fact that \( \bar{u}_k, \bar{v}_k \leq l \) and using (87), we have
\[
|\mathcal{F}(\ell)(x, t; u_1; \ldots; u_m) - \mathcal{F}(\ell)(x, t; v_1; \ldots; v_m)|
= |\mathcal{F}(x, t; \bar{u}_1; \ldots; \bar{u}_m) - \mathcal{F}(x, t; \bar{v}_1; \ldots; \bar{v}_m)|
\leq K(\ell) (|\bar{u}_1 - \bar{v}_1| + \ldots + |\bar{u}_m - \bar{v}_m|)
\leq K(\ell) (|u_1 - v_1| + \ldots + |u_m - v_m|),
\]
where we have used (91) for the last estimate. 

4.1. Error estimate in \( L_2 \)-norm. We then have the convergence result given by the following theorem.

**Theorem 4.2.** Let \( \beta := \beta(\epsilon) \) be as in (25), the system (90) has a unique solution \( \mathcal{U}^{\epsilon, \beta} \in [C([0, T]; L_2(\Omega))]^m \). Assuming that we can choose a sequence \( \ell^\epsilon > 0 \) such that \( \lim_{\epsilon \to 0^+} \ell^\epsilon = \infty \) and
\[
K(\ell^\epsilon) \leq \log \left( \log \left( \frac{1}{\beta} \right) \right)^{\frac{1}{r}}, \quad \forall \gamma > 0.
\]
Assume that the system (1)-(2) has a unique solution \( \mathcal{U} \) such that \( u_k \) satisfies (26) and \( u_k \in L_\infty(0, T; H_1(\Omega)), k = 1, m \) and also holds
\[
\max \left\{ E; \|u_k\|_{L_\infty(0, T; H_1(\Omega))} \right\} \leq \bar{E},
\]
for some known constant \( \bar{E} > 0 \), \( E \) is the constant defined in Theorem 3.2. Moreover, for \( 0 < t < T \), the error estimate over \( [L_2(\Omega)]^m \)-norm between the functional \( \mathcal{U}^{\epsilon, \beta} \) solutions to (90) and the sought solution \( \mathcal{U} \) to (1) is given by
\[
\|\mathcal{U}^{\epsilon, \beta}(\cdot, t) - \mathcal{U}(\cdot, t)\|_{[L_2(\Omega)]^m}
\leq C(\epsilon, \beta, m, r, \bar{E}, T) \beta^{\frac{r}{2}} \exp((T - t)/2) \log^{m\gamma} \left( \frac{1}{\beta} \right),
\]
where \( C(\epsilon, \beta, m, r, \bar{E}, T) := \sqrt{\epsilon^2 \beta^{-2} + \left( \frac{1}{r} + \frac{1}{r^2} \right)m \bar{E}^2} \), with the constant \( r \) is defined as in (97).

If \( \epsilon > 0 \) is small enough, let \( t_\epsilon \) be the (unique) solution of the equation \( t = \beta^{\frac{1}{r}} \) (there exists \( t_\epsilon \in (0, T) \) such that \( \lim_{\epsilon \to 0^+} t_\epsilon = 0 \)), we have
\[
\|\mathcal{U}^{\epsilon, \beta}(\cdot, t_\epsilon) - \mathcal{U}(\cdot, 0)\|_{[L_2(\Omega)]^m}
\leq C(\epsilon, \beta, m, r, \bar{E}, T) \exp(T/2) \log^{m\gamma} \left( \frac{1}{\beta} \right) \sqrt{T \log \left( \frac{1}{\beta} \right)}.\]

**Remark 4.1.**
1. If \( \beta := \beta(\epsilon) \in [0, 1 - e^{-T R \lambda_1}] \) satisfies (25) then since (95), we conclude that \( \|\mathcal{U}^{\epsilon, \beta}(\cdot, t) - \mathcal{U}(\cdot, t)\|_{[L_2(\Omega)]^m} \) tends to zero as \( \epsilon \to 0 \), for all \( t \in [0, T] \).
2. If we choose \( \gamma > 0 \) satisfying \( 0 < \gamma < \frac{m - 1}{2} \) then the right-hand side of (96) converges to zero, as \( \epsilon \to 0^+ \).
Proof. First, we note that the proof of the existence and uniqueness of the solutions to the problem (1) is completely the same as in Step 1 of Theorem 3.2. We pass to the error estimate between the regularized solution of system (90) and the exact solution of system (1).

For \((x, t) \in D_T\), we begin by establishing functions \(q_k(x, t), \rho_k(x, t)\) that satisfy
\[
 r \leq q_k(x, t), \rho_k(x, t) \leq R, \quad k = \frac{m}{\ell};
\]
and
\[
 \left(\begin{array}{c}
 q_k(x, t) \\
 \rho_k(x, t)
\end{array}\right) = \left(\begin{array}{c}
 R \\
 R
\end{array}\right) - \left(\begin{array}{c}
 \eta_k(x, t) \\
 \eta_k'(x, t)
\end{array}\right).
\]

By an argument analogous to that used for the proof of Theorem 3.2, we infer that
\[
 \left|X_1\right| = \frac{1}{2} \int_{\Omega} \left|W_{k, \beta}(x, t)\right|^2 - \int_{\Omega} \left|\nabla W_{k, \beta}(x, t)\right|^2 dx = \int_{\Omega} \left(\exp(q_{\beta}(t - T)) \mathcal{F}_k(\ell) \left(\mathcal{F}_k(x, t; u_1^{\epsilon, \beta}; \ldots; u_m^{\epsilon, \beta}) - \mathcal{F}_k(x, t; u_1; \ldots; u_m)\right), W_{k, \beta}\right)_{L_2(\Omega)}
\]
\[
 + \left(\exp(q_{\beta}(t - T)) \nabla \cdot \left((q_{\beta}(x, t) - q_{\beta}(x, t)) \nabla u_k\right), W_{k, \beta}\right)_{L_2(\Omega)}.
\]

First, estimating \(X_1\) and \(X_2\) is totally similar to (71) and (72) respectively.

Next, since \(\lim_{\epsilon \to 0+} \ell^\epsilon = +\infty\), for a sufficiently small \(\epsilon > 0\), there is an \(\ell^\epsilon > 0\) such that \(\ell^\epsilon \geq \|u_k\|_{L^\infty(0, T; L^2(\Omega))}, \quad k = \frac{m}{\ell}\). For this value of \(\ell^\epsilon\), we have \(\mathcal{F}_k(\ell^\epsilon)(x, t; u_1; \ldots; u_m) = \mathcal{F}_k(x, t; u_1; \ldots; u_m), \quad k = \frac{m}{\ell}\). Using the global Lipschitz property of \(\mathcal{F}(\ell)\) (see Lemma 4.1), one has
\[
 \left|X_3\right| \leq K(\ell^\epsilon) \left\|W_{k, \beta}(x, t)\right\|_{L_2(\Omega)} \sum_{k=1}^m \left\|W_{k, \beta}(x, t)\right\|_{L_2(\Omega)}.
\]

Finally, inequality (73) allows us to estimate (observe that \(\exp(q_{\beta}(t - T)) \leq 1\) for \(t \in [0, T], \quad k = \frac{m}{\ell}\))
\[
 \left|X_4\right| \\
 = \left\|u_k^2\right\|_{L^\infty(0, T; H^1(\Omega))} \int_{\Omega} (q_{\beta}(x, t) - q_{\beta}(x, t)) \left|\nabla u_k\right|^2 dx + \int_{\Omega} r \left|\nabla W_{k, \beta}\right|^2 dx \\
 \leq \frac{1}{4r} \left\|u_k^2\right\|_{L^\infty(0, T; H^1(\Omega))} \int_{\Omega} (q_{\beta}(x, t) - q_{\beta}(x, t))^2 dx + \int_{\Omega} r \left|\nabla W_{k, \beta}\right|^2 dx \\
 \leq \frac{1}{4} \left\|u_k^2\right\|_{L^\infty(0, T; H^1(\Omega))} \left\|q_{\beta} - q_{\beta}\right\|^2_{L_2(\Omega)} + r \int_{\Omega} \left|\nabla W_{k, \beta}\right|^2 dx.
\]
Combining (71), (72), (99), (100), (101) and direct computations, we obtain

\[ \left\| W_{k}^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \]
\[ \leq \left\| \varphi_k^\epsilon - \varphi_k \right\|_{L^2(\Omega)}^2 + \frac{\beta^2}{T} \left\| u_k \right\|_{L^\infty(0, T; V_{TR}(\Omega))}^2 \]
\[ + \frac{r^{-1} \left\| u_k \right\|_{L^\infty(0, T; H^1(\Omega))}}{2} \int_t^T \left\| \varphi_k^\epsilon - \varphi_k \right\|_{C([0, T]; L^2(\Omega))}^2 \, dt \]
\[ + \int_t^T \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_\beta \right) \left\| W_{k}^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 \, d\mu \]
\[ + 2K(\ell') \int_t^T \left( \sum_{k=1}^m \left\| W_{k}^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)} \right)^2 \, d\mu \]
\[ - 2 \int_t^T \int_\Omega \left( \varphi_k^\epsilon (x, \mu) - r \right) \left\| \nabla W_{k}^{\epsilon, \beta} \right\|_{\Omega}^2 \, dx \, d\mu, \quad k = 1, m. \quad (102) \]

Since \( \varphi_k^\epsilon (x, t) > r, \forall (x, t) \in D_T, k = 1, m \), it follows that
\[ \int_t^T \int_\Omega \left( \varphi_k^\epsilon (x, \mu) - r \right) \left\| \nabla W_{k}^{\epsilon, \beta} \right\|_{\Omega}^2 \, dx \, d\mu > 0. \]

Adding the results of (102) for \( k = 1, m \), and using (4), (86), (94) (noting that from (98), we infer

\[ \sum_{k=1}^m \left\| W_{k}^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}^2 \]
\[ \leq \sum_{k=1}^m \left\| \varphi_k^\epsilon - \varphi_k \right\|_{L^2(\Omega)}^2 + \frac{\beta^2}{T} \sum_{k=1}^m \left\| u_k \right\|_{L^\infty(0, T; V_{TR}(\Omega))}^2 \]
\[ + \sum_{k=1}^m \frac{r^{-1} \left\| u_k \right\|_{L^\infty(0, T; H^1(\Omega))}}{2} \int_t^T \left\| \varphi_k^\epsilon - \varphi_k \right\|_{C([0, T]; L^2(\Omega))}^2 \, d\mu \]
\[ + \int_t^T \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_\beta \right) \left\| W_{k}^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 \, d\mu \]
\[ + 2K(\ell') \int_t^T \left( \sum_{k=1}^m \left\| W_{k}^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)} \right)^2 \, d\mu \]
\[ \leq \epsilon^2 + \left( \frac{m\beta^2 \bar{E}^2}{T} + \frac{m\epsilon^2 T r^{-1} \bar{E}^2}{2} \right) \]
\[ + \left( \frac{2}{T} \log \left( \frac{1}{\beta} \right) + 1 - 2q_\beta + 2mK(\ell') \right) \int_t^T \sum_{k=1}^m \left\| W_{k}^{\epsilon, \beta} (\cdot, \mu) \right\|_{L^2(\Omega)}^2 \, d\mu. \quad (103) \]

Here we have used the fact that \( \sum_{k=1}^m \left\| \varphi_k^\epsilon - \varphi_k \right\|_{C([0, T]; L^2(\Omega))}^2 \leq \epsilon^2 \). By setting

\[ \varphi^{\epsilon, \beta}(t) = \sum_{k=1}^m \left\| W_{k}^{\epsilon, \beta} (\cdot, t) \right\|_{L^2(\Omega)}, \]
choosing $q_β = \frac{1}{T} \log \left( \frac{1}{β} \right) > 0$, and using Gronwall’s inequality we obtain

$$\sum_{k=1}^{m} \left\| u_k^\epsilon(\cdot, t) - u_k(\cdot, t) \right\|_{L^2(Ω)}^2 \leq \left[ e^{2β^2} + \left( \frac{1}{T} + \frac{r^{-1}e^2β^{-2}T}{2} \right) m\overline{E}^2 \right] \exp \left( (2mK(ℓ') + 1)(T - t) \right) \beta \frac{T}{2},$$

(104)

without loss of generality, from (25), we may assume that $C(ε, β, m, r, \overline{E}, T)$ is bounded. For all $γ > 0$, the function $f(β) = \sqrt{\frac{1}{T} \log \left( \log \left( \frac{1}{β} \right) \right)}$, satisfies $\lim_{ε \to 0^+} f(β) = +∞$ and $f(β) \leq \frac{1}{T} \log \left( \log \left( \frac{1}{β} \right) \right)$, where $β := β(ε)$ satisfies (25). Choose $Θ_β = \sup_{K - 1} \{ (-∞, f(β)) \}$ then $K(Θ_β) = f(β)$ and we obtain

$$\exp \left( K(Θ_β)T \right) \leq \log \left( \frac{1}{β} \right), \text{ for } γ > 0.$$

Therefore, $K^{-1} \{ -∞, f(β) \} \neq \emptyset$ and $Θ_β \in [0, +∞]$ is well defined. Moreover, $\lim_{ε \to 0^+} Θ_β = +∞$. In fact, if there exists a constant $C$ such that $Θ_β \leq C$ for $β$ in a neighbourhood of zero, by the fact that $K$ is non-decreasing, we have $K(C) \geq K(Θ_β) = f(β)$, this obviously contradicts with the fact that $\lim_{ε \to 0^+} f(β) = +∞$. Now, for $0 < ℓ' \leq Θ_β$ and $\lim_{ε \to 0^+} ℓ' = +∞$, we have

$$\exp \left( K(ℓ')T \right) \leq \log \left( \frac{1}{β} \right).$$

(105)

Introduce (105) into (104), we obtain (95). In order to obtain the approximation of the solution $U$ at $t = 0$, arguing as in Theorem 3.2 and using (105) to obtain (96).

4.2. Error estimate in $0_{H^1}$-norm. In this subsection, we give an error estimate in $0_{H^1}$-norm. The assumption $(A_4)$ is held fixed. Assume, furthermore, that

$$η_k(x, t) := η_k(t), \quad ∀(x, t) \in D_T,$$

and we make an alternative assumption

$(A_5)$ Suppose $η_κ^k : [0, T] → \mathbb{R}$ is a continuous function such that

$$0 < r < η_κ^k(t) \leq R, \quad ∀t \in [0, T], \quad k = \overline{1, m},$$

and

$$\sum_{k=1}^{m} \left\| η_κ^k - η_k \right\|_{C([0, T])} \leq ε.$$  

(106)
Let us assume that \( \varphi^c_k \in H^1(\Omega), k = 1, m \) and satisfies
\[
\|\varphi^c_1 - \varphi^c_\mathcal{O} \|_{H^1(\Omega)} + \|\varphi^c_2 - \varphi^c_2 \|_{H^1(\Omega)} + \ldots + \|\varphi^c_m - \varphi^c_m \|_{H^1(\Omega)} \leq \epsilon. \tag{107}
\]
The following theorem is a sharpening of the results in Theorem 4.2

**Theorem 4.3.** Assume the hypotheses in Theorem 4.2 hold. Then, the error estimate over \( \left[ 0, H^1(\Omega) \right]^m \)-norm between the functional \( U^{c,\beta} \) solutions to (90) and the sought solution \( U \) to (1) is given by
\[
\|U^{c,\beta} (\cdot, t) - U (\cdot, t)\|_{H^1(\Omega)}^m \leq \bar{C}(\epsilon, \beta, m, r, \bar{E}, T) \beta^{\frac{1}{2}} \left[ \log \left( \frac{1}{\beta} \right) \right]^\frac{m\gamma C_0}{\epsilon}, \tag{108}
\]
where \( \bar{C}(\epsilon, \beta, m, r, \bar{E}, T) = \sqrt{\epsilon^2 \beta^{-2} + \frac{6m}{\beta^2} \epsilon^2 \beta^{-2} E^2 + \frac{6m}{\beta^2} E^2} \).

If \( \epsilon > 0 \) is small enough, let \( t_\epsilon \) be the (unique) solution of the equation \( t = \beta^{\frac{1}{2}} \) (there exists \( t_\epsilon \in (0, T) \) such that \( \lim_{\epsilon \to 0^+} t_\epsilon = 0 \)), we have
\[
\|U^{c,\beta} (\cdot, t_\epsilon) - U (\cdot, 0)\|_{H^1(\Omega)}^m \leq C(\epsilon, \beta, m, r, \bar{E}, T) \exp(T/2) \left[ \log \left( \frac{1}{\beta} \right) \right]^\frac{m\gamma C_0}{\epsilon} \sqrt{\frac{T}{\log \left( \frac{1}{\beta} \right)}}. \tag{109}
\]

**Remark 4.2.** If we choose \( \gamma > 0 \) satisfying \( 0 < \gamma < \frac{\epsilon}{12mC_0} \), then the right-hand side of (109) converges to zero, as \( \epsilon \to 0^+ \).

**Proof.** Put
\[
\varrho_k(t) = R - \eta_k(t), \quad \varrho'_k(t) = R - \eta'_k(t), \quad t \in [0, T], k = 1, m.
\]
Let us set the following function
\[
W_k^{c,\beta}(x, t) = \exp(q_\beta(t - T)) \left[ u_k^{c,\beta}(x, t) - u_k(x, t) \right], \tag{110}
\]
where \( q_\beta > 0 \) depends of \( \beta \) to be chosen later. Then, from equalities (66) and (90), a simple computation gives
\[
\partial_t W_k^{c,\beta} + \partial_k(t) \Delta W_k^{c,\beta} - q_\beta W_k^{c,\beta} = \mathcal{B}^{c,\beta}(W_k^{c,\beta}) + \exp(q_\beta(t - T)) A_R^{c,\beta}(u_k)
\]
\[
+ \exp(q_\beta(t - T)) \left[ F_k(F(x, t; u_k^{c,\beta}; \ldots; u_m^{c,\beta})) - F_k(x, t; u_1; \ldots; u_m) \right]
\]
\[
+ [\varrho_k(t) - \varrho'_k(t)] \Delta u_k. \tag{111}
\]
By taking the inner product two sides of (111) with \( -\Delta W_k^{c,\beta} \), one deduces that
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla W_k^{c,\beta} (\cdot, t) \right\|^2_{L^2(\Omega)} - \int_{\Omega} \varrho_k(t) \left| \Delta W_k^{c,\beta} \right|^2 dx - q_\beta \left\| \nabla W_k^{c,\beta} (\cdot, t) \right\|^2_{L^2(\Omega)}
\]
\[
= \langle \mathcal{B}^{c,\beta}(W_k^{c,\beta}), -\Delta W_k^{c,\beta} \rangle_{L^2(\Omega)} + \langle \epsilon q_\beta(t - T) A_R^{c,\beta}(u_k), -\Delta W_k^{c,\beta} \rangle_{L^2(\Omega)}
\]
\[
= \varrho_1 + \varrho_2.
\]
\[
\begin{align*}
&\left( e^{q_3(t-T)} \left[ \mathcal{F}_k(\ell') \left( x, t; u_1^{e,\beta}, \ldots; u_m^{e,\beta} \right) - \mathcal{F}_k \left( x, t; u_1; \ldots; u_m \right) \right] - \Delta W_k^{e,\beta} \right)_{L^2(\Omega)} \\
&\quad + \left( -e^{q_3(t-T)} \left[ \varrho_k(t) - \varrho_k(t) \right] \Delta u_k - \Delta W_k^{e,\beta} \right)_{L^2(\Omega)}.
\end{align*}
\]

(112)

First, the term \( \Omega_1 \) is estimated as follows

\[
|\Omega_1| = \int_{\Omega} B_{R} \left( W_k^{e,\beta} \right) \Delta W_k^{e,\beta}(x, t) \, dx
\]

\[
= \frac{1}{T} \int_{\Omega} \sum_{n=1}^{\infty} \log \left( \frac{1}{\beta + \exp(-TR\lambda_n)} \right) \langle \nabla W_k^{e,\beta}(x, t), \sigma_n(x) \rangle_{L^2(\Omega)} \sigma_n(x) \, dx
\]

\[
\times \left[ \sum_{n=1}^{\infty} \langle \nabla W_k^{e,\beta}(x, t), \sigma_n(x) \rangle_{L^2(\Omega)} \sigma_n(x) \right] \, dx
\]

\[
= \frac{1}{T} \sum_{n=1}^{\infty} \log \left( \frac{1}{\beta + \exp(-TR\lambda_n)} \right) \int_{\Omega} \langle \nabla W_k^{e,\beta}(x, t), \sigma_n(x) \rangle_{L^2(\Omega)}^2 \, dx
\]

\[
\leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \left\| \nabla W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)} = \frac{1}{T} \log \left( \frac{1}{\beta} \right) \left\| W_k^{e,\beta}(\cdot, t) \right\|^2_{H^1_0(\Omega)}.
\] (113)

Second, using \( \log(1 + a) \leq a \), for any positive \( a \), the term \( \Omega_2 \) can be estimated as follows

\[
|\Omega_2| = \left| \left\langle \exp(q_3(t - T)) \mathcal{A}_{R}^{e,\beta}(u_k), -\Delta W_k^{e,\beta}(x, t) \right\rangle_{L^2(\Omega)} \right|
\]

\[
\leq \frac{3}{r} \left\| \exp(q_3(t - T)) \mathcal{A}_{R}^{e,\beta}(u_k) \right\|^2_{L^2(\Omega)} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{3}{r} \frac{\exp(2q_3(t - T))}{rT^2} \| u_k(\cdot, t) \|^2_{\mathcal{V}_{R\Omega}(\Omega)} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{3\beta^2}{rT^2} \| u_k \|^2_{L^2(0, T; \mathcal{V}_{R\Omega}(\Omega))} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}.
\] (114)

Using Hölder inequality, we have the following estimate

\[
|\Omega_3| \leq \frac{3}{r} \left\| \exp(q_3(t - T)) \left[ \mathcal{F}_k(\ell') \left( x, t; u_1^{e,\beta}, \ldots; u_m^{e,\beta} \right) - \mathcal{F}_k \left( x, t; u_1; \ldots; u_m \right) \right] \right\|^2_{L^2(\Omega)}
\]

\[
+ \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{3}{r} K^2(\ell') \left\| W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{3C_0}{r} K^2(\ell') \| W_k^{e,\beta}(\cdot, t) \|^2_{H^1_0(\Omega)} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}.
\] (115)

where we also employed Poincaré’s inequality \( \| \theta \|_{L^2(\Omega)} \leq C_0 \| \theta \|_{H^1_0(\Omega)} \) for \( C_0 > 0, \theta \in H^1_0(\Omega) \) and

\[
|\Omega_4| \leq \frac{3}{r} \left\| \exp(q_3(t - T)) \left[ \varrho_k(t) - \varrho_k(t) \right] \Delta u_k \right\|^2_{L^2(\Omega)} + \frac{r}{3} \left\| \Delta W_k^{e,\beta}(\cdot, t) \right\|^2_{L^2(\Omega)}
\]
Combining (113)–(116), we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \nabla W_{k}^{\epsilon, \beta} (\cdot, t) \|_{L^2(\Omega)}^2 &\geq \int_{\Omega} \left( \frac{\gamma(t)}{r} - \frac{3 C_0}{r} K^2(\epsilon) \right) \| W_{k}^{\epsilon, \beta} (\cdot, t) \|_{H^1(\Omega)}^2 \, d\mu \\
&\leq \| \nabla \varphi_k - \nabla \varphi_k \|_{L^2(\Omega)}^2 + \frac{6 C_0}{r} K^2(\epsilon) \int_t^T \| W_{k}^{\epsilon, \beta} (\cdot, \mu) \|_{H^1(\Omega)}^2 \, d\mu.
\end{align*}
\]
(117)

By choosing \( q_{\beta} = \frac{1}{T} \log \left( \frac{1}{\beta} \right) \) and integrating the latter estimate from \( t \) to \( T \), we obtain
\[
\| W_{k}^{\epsilon, \beta} (\cdot, t) \|_{H^1(\Omega)}^2 \leq \| \nabla \varphi_k - \nabla \varphi_k \|_{L^2(\Omega)}^2 + \frac{6 C_0}{r} K^2(\epsilon) \int_t^T \| W_{k}^{\epsilon, \beta} (\cdot, \mu) \|_{H^1(\Omega)}^2 \, d\mu,
\]
for all \( k = 1, m \). Therefore
\[
\begin{align*}
\sum_{k=1}^m \| W_{k}^{\epsilon, \beta} (\cdot, t) \|_{H^1(\Omega)}^2 &\leq \sum_{k=1}^m \| \varphi_k - \varphi_k \|_{H^1(\Omega)}^2 + \frac{6 C_0}{r} \sum_{k=1}^m \| u_k \|_{L^\infty(0,T;H^2(\Omega))}^2 \\
&\quad + \frac{6 \beta^2}{r T^2} \sum_{k=1}^m \| u_k \|_{L^\infty(0,T;W_T(\Omega))}^2 + \frac{6 C_0}{r} K^2(\epsilon) \int_t^T \| W_{k}^{\epsilon, \beta} (\cdot, \mu) \|_{H^1(\Omega)}^2 \, d\mu \\
&\quad \leq \epsilon^2 + \frac{6 m}{r} C_0^2 \frac{K^2(\epsilon)}{T} + \frac{6 \beta^2}{r} E^2 + \frac{6 m \beta^2}{r T^2} E^2 \\
&\quad + \frac{6 C_0}{r} K^2(\epsilon) \int_t^T \sum_{k=1}^m \| W_{k}^{\epsilon, \beta} (\cdot, \mu) \|_{H^1(\Omega)}^2 \, d\mu.
\end{align*}
\]
(119)

Applying Gronwall’s inequality to this yields
\[
\begin{align*}
\sum_{k=1}^m \| W_{k}^{\epsilon, \beta} (\cdot, t) \|_{H^1(\Omega)}^2 &\leq \left( \epsilon^2 + \frac{m \beta^2}{r} \frac{K^2}{T} + \frac{6 m \beta^2}{r T^2} E^2 \right) \exp \left( \frac{12 C_0}{r} K^2(\epsilon)(T-t) \right).
\end{align*}
\]
(120)
The same arguments as in Theorem 4.2, we will choose
theorems to some concrete models.

5. Applications to some specific systems. In this section, we shall apply our theorems to some concrete models.

- The Lotka-Volterra system. We consider the two-species system for Lotka-Volterra competition model (see [18])

\[
\begin{cases}
\frac{\partial u_1}{\partial t} - \text{div}(D_1(t, x)\nabla u_1) = r_1 u_1 - a_{11}(x)u_1^2 - a_{12}u_1u_2, \\
\frac{\partial u_2}{\partial t} - \text{div}(D_2(t, x)\nabla u_2) = r_2 u_2 - a_{21}(x)u_1u_2 - a_{22}u_2^2,
\end{cases}
\]

associated with the final conditions

\[
u_1(x, T) = \varphi_1(x), \quad u_2(x, T) = \varphi_2(x),
\]

and the boundary conditions

\[
u_1(0, t) = u_1(\pi, t) = 0, \quad u_2(0, t) = u_2(\pi, t) = 0.
\]

Here \(\Omega = [0, \pi]\), the diffusion coefficient \(D_1\) is a positive function, the intrinsic growth rate \(r_1 \in C([0, \pi])\) and the intraspecific competition coefficient \(a_{11}\) is positive and also belongs \(C([0, \pi])\). \(D_2 > 0\) is a positive function corresponding to the diffusion coefficient of the second species, \(r_2 \in C([0, \pi])\) is the second species intrinsic growth rate and \(a_{22} > 0\) corresponds to the second species intraspecific competition coefficient. We furthermore assume that \(a_{12}\) is constant and \(a_{21} \in C([0, \pi])\). Set

\[
F_1(u_1, u_2) = r_1 u_1 - a_{11}u_1^2 - a_{12}u_1u_2, \quad F_2(u_1, u_2) = r_2 u_2 - a_{21}u_1u_2 - a_{22}u_2^2.
\]

It is easy to check that \(F_1, F_2\) are locally Lipschitz functions which satisfy

\[
|F_i(u_1, u_2) - F_i(v_1, v_2)| \leq K_i(\ell) (|u_1 - v_1| + |u_2 - v_2|),
\]

if \(\max\{|u_1|, |u_2|, |v_1|, |v_2|\} \leq \ell, i = 1, 2,
\]

where

\[
K_i(\ell) = r_1 + 2\ell|a_{12}| + 2\ell \sup_{0 \leq x \leq \pi} |a_{11}(x)|,
\]

Introduce (122) into (121), we obtain (108). In order to obtain the approximation of the solution \(\bar{U}\) at \(t = 0\), arguing as in Theorem 4.2 and using (105) to obtain (109).

\[\square\]
and
\[ K_2(\ell) = r_1 + 2\ell |a_{22}| + 2\ell \sup_{0 \leq x \leq \pi} |a_{21}(x)|. \]

**The Brusselator.** Let us start with the Brusselator, appearing in the modeling of chemical morphogenetic processes (see [17])

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \text{div}(D_1(t, x) \nabla u_1) &= -u_1 u_2^2 + bu_2, \\
\frac{\partial u_2}{\partial t} - \text{div}(D_2(t, x) \nabla u_2) &= u_1 u_2^2 - (b + 1)u_2 + a,
\end{aligned}
\]

with the following conditions

\[
\begin{aligned}
u_1(x,t) = u_2(x,t) = 0, & \text{ on } \Pi, \\
u_1(x,T) = \varphi_1(x), u_2(x,T) = \varphi_2(x).
\end{aligned}
\]

Here \(u_1\) and \(u_2\) represent the concentrations of two chemical species, the activator and the inhibitor respectively; \(a\) and \(b\) are positive constants. This system exhibits rich dynamics, including oscillations, spatiotemporal chaos and turing instabilities. [15] The functions \(F_1(u_1, u_2) = -u_1 u_2^2 + bu_2\) and \(F_2(u_1, u_2) = u_1 u_2^2 - (b + 1)u_2 + a\) satisfy

\[
|F_i(u_1, u_2) - F_i(v_1, v_2)| \leq K_i(\ell) (|u_1 - v_1| + |u_2 - v_2|),
\]

if \(\max\{|u_1|, |u_2|, |v_1|, |v_2|\} \leq \ell, \ i = 1, 2,\)

where

\[ K_1(\ell) = 3\ell^2 + b, \]

and

\[ K_2(\ell) = 3\ell^2 + b + 1. \]

**The Rothe system.** We consider the reaction-diffusion system

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \text{div}(D_1(t, x) \nabla u_1) &= -k^f u_1 u_2 + k^b u_3, \\
\frac{\partial u_2}{\partial t} - \text{div}(D_2(t, x) \nabla u_2) &= -k^f u_1 u_2 + k^b u_3, \\
\frac{\partial u_3}{\partial t} - \text{div}(D_3(t, x) \nabla u_3) &= +k^f u_1 u_2 - k^b u_3,
\end{aligned}
\]

with the following conditions

\[
\begin{aligned}
u_1(x,t) = u_2(x,t) = u_3(x,t) = 0, & \text{ on } \Pi, \\
u_1(x,T) = \varphi_1(x), u_2(x,T) = \varphi_2(x), u_3(x,T) = \varphi_3(x).
\end{aligned}
\]

This system (129) is introduced in [2], it represents the time-evolution of the concentration \(u = (u_1, u_2, u_3)\) of three chemical species. Here, \(D_i, i = 1, 2, 3\) are diffusivities which depends on \(x\) and \(t\). The constants \(k^f, k^b \geq 0\) are the rate constants for the forward and backward reaction. Recall that \(k^f = 0\) or \(k^b = 0\) refers to the irreversible limit case which is included. The following functions

\[
F_1(u_1, u_2) = F_2(u_1, u_2) = -k^f u_1 u_2 + k^b u_3, \quad F_3(u_1, u_2) = k^f u_1 u_2 - k^b u_3,
\]

are locally Lipschitz functions since we know that

\[
|F_i(u_1, u_2) - F_i(v_1, v_2)| \leq K_i(\ell) (|u_1 - v_1| + |u_2 - v_2|),
\]

if \(\max\{|u_1|, |u_2|, |v_1|, |v_2|\} \leq \ell, \ i = 1, 2, 3,\)
where
\[ K_i(\ell) = 2k^f \ell + k^b \ell, \quad \text{for} \quad i = 1, 2, 3. \]

6. Conclusions. The system (1) is very important in applications related to reaction-diffusion problems. This study has achieved a major extension over the much more investigated with time and space dependent diffusion coefficients. The nonlinear inverse system has been solved by proposing a new version of a modified quasi-reversibility method. Convergence and stability estimates, as the noise level tends to zero, have been formulated and proved in $L_2$ and $H_1$ norms.

In order to increase the significance of the study, numerical results should be presented and discussed illustrating the theoretical findings in terms of accuracy and stability. However, in the case of system (1) with $F_k$, $k = 1, m$ are the nonlinear reactions take place, it goes beyond in this research due to the complications of such ill-posed problems, requiring huge efforts. Therefore, it should be considered carefully in a forthcoming work.

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