A Numerical Approach for the Design of RC Beams Subjected to Axial and Transverse Loads

Amal Wahbi 1,2, Duc Toan Pham 1, Ghazi Hassen 2, Denis Garnier 2, Patrick de Buhan 2

1 Centre Scientifique et Technique du Bâtiment (CSTB), 84 avenue Jean Jaurès, Champs-sur-Marne, 77447 Marne-la-Vallée Cedex 2, France
2 Laboratoire Navier, Ecole des Ponts ParisTech, Université Gustave Eiffel, CNRS, 6-8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France

amal.wahbi@cstb.fr

Abstract. The present contribution deals with a numerical approach for the design of RC beams subjected to axial and transverse loads. It is based on the finite-element implementation of the kinematic approach of the yield design (or limit analysis) theory combined with a “mixed modelling” where the concrete material is regarded as a classical two-dimensional continuum while the longitudinal reinforcements are modelled as one-dimensional elements working in tension-compression only. For the beams reinforced in shear, stirrups are incorporated in the analysis through a homogenization procedure. An optimization problem is formulated, then solved using conic quadratic optimization method. As a result, an upper bound estimate to the yield strength domain of RC beams may be drawn in the plane of axial and transverse loads. For illustrative purpose, calculations are conducted on typical RC beams with different longitudinal reinforcement degrees. Furthermore, it is shown that such numerical predictions prove to be in good agreement with the results derived from other numerical simulations of the same problem using a finite element-based limit analysis commercial software. In order to assess their practical validity, these predictions are also compared to some available experimental results published in the literature.

1. Introduction

In some practical cases such as columns of frames under high seismic action or continuous beams rigidly connected to columns (where axial loads may result from the restrain of thermal deformation of concrete members), reinforced concrete (RC) elements may be subjected to the combination of axial and transverse loads. Nowadays, the design of these elements in such conditions is still difficult since most of the actual design codes (see for example EN 1992-1-1 [1]; NF EN 1992-1-1/AN [2]; ACI 318-08 [3]) propose empirical or semi-empirical approaches which seem very conservative in the case of shear-tension loading, while they may be non-conservative in the case of shear-compression loading (Collins et al. [4], Wahbi et al. [5]). Besides, even though many experimental studies have been carried out over the last 60 years (see for example Bara [6]; Regan [7]; Jørgensen et al. [8]; Madsen et al. [9]; or recently Pham et al. [10, 11] and Wahbi et al. [5]), the apparent divergences between available results are still difficult to analyse.
Based on the finite-element implementation of the kinematic approach of the yield design (or limit analysis) theory (Salençon [12], de Buhan [13]) combined with a “mixed modelling” of reinforced concrete (see for example Averbuch and de Buhan [14]), this contribution presents a numerical approach for the design of RC beams subjected to axial and transverse loads. For illustrative purpose, calculations will be conducted on typical RC (reinforced concrete) beams with different longitudinal and transverse reinforcement degrees. Furthermore, it will be shown that such numerical predictions prove to be in good agreement with the results derived from other numerical simulations of the same problem using a finite element-based limit analysis commercial software. In order to assess their practical validity, these predictions are also compared to some available experimental results published in the literature.

2. Upper bound kinematic approach of yield design  
2.1. Statement of the yield design problem

The studied structure is a concrete beam of length $L$, depth $h$ and width $b$ as shown in figure 1. It is reinforced by longitudinal steel bars (of cross-sectional area $A_s$ and yield strength $f_y$) as well as by transverse reinforcements (stirrups of cross-sectional area $A_{sw}$ and yield strength $f_yw$) uniformly distributed along the beam axis, with a constant spacing $s$.

![Figure 1. RC beam subjected to transverse and axial loads](image)

The beam is simply supported on two simple supports (span $l$) allowing free horizontal displacements. It is subjected to a transverse vertical concentrated load $P$ applied at a distance $a \leq l/2$ (shear span) from the left support (see figure 1), in addition to axial loads $N$ applied at its both ends.

In what follows, an upper bound estimate to the resistance domain $(N, P)$ is derived from a finite element implementation of the kinematic approach of the yield design theory. A “mixed modelling” of the structure (see Averbuch and de Buhan [14] for more details) is adopted. The concrete material is regarded as a classical two-dimensional continuum while the longitudinal reinforcements are modelled as one-dimensional elements working in tension-compression only. For the beams reinforced in shear, stirrups are incorporated in the analysis through a homogenization procedure.

2.2. Material strength properties

The plain concrete material is assumed to be homogeneous and obey a modified Mohr-Coulomb failure condition with zero tension cut-off (Chen [15]; Nielsen and Hoang [16]) (the tensile strength of concrete is neglected due to the brittleness of concrete failure under tensile stresses). The corresponding strength criterion is given by the following expression:

$$f^c(\sigma) = \sup \left\{ \frac{1 + \sin \phi}{1 - \sin \phi} \sigma_m - \sigma_m - f_y; \sigma_m \right\} \leq 0 \tag{1}$$
where $\sigma_M$ and $\sigma_m$ are respectively the major and minor principal stresses while $\phi$ is the internal friction angle which may be taken as approximately equal to 37° for ordinary concrete (Nielsen and Hoang [16]). This criterion may be displayed in the Mohr-plane in the form of an intrinsic curve as shown in figure 2.

![Figure 2](image_url)

**Figure 2. Intrinsic curve of plain concrete with zero tension cut-off**

As regards the reinforcements, their strength properties are characterized by the following condition on the axial force (shear and bending resistance are neglected):

$$|T| \leq T_0 = A_s f_y$$

where $T$ is the axial force developed in the reinforcing bar while $T_0$ represents its tensile-compressive resistance, equal to the product of the steel uniaxial yield strength $f_y$ by its cross-sectional area $A_s$.

### 2.3. Principle of the kinematic approach of yield design and homogenization procedure

According to the principle of the kinematic approach of yield design, a necessary condition for the structure to be stable under the applied loads $(N, P)$ is that for every virtual kinematically admissible (K.A.) velocity field $U$ (i.e. satisfying the prescribed velocity boundary conditions), the virtual work of applied loads $P_{ext}$ remains smaller than the maximal resistance rate of work $P_{rm}$:

$$P_{ext} (N, P, U) \leq P_{rm} (U) \quad \forall U \text{ K.A.}$$ (3)

Assuming that $d$ is the strain-rate tensor associated with the velocity field $U$ at any point of the structure $S$ while $V$ is the jump in the velocity field that can occur across discontinuity lines $\Sigma$ of normal $n$, the maximum resisting work $P_{rm}$ in the latter equation (3) may be computed as:

$$P_{rm} (U) = \int_S \pi (d) \, dS + \int_\Sigma \pi (n, V) \, dl$$ (4)

where the support functions $\pi$ may be calculated as follows:

a) for the concrete and transverse reinforcements: it is assumed that the stirrups are sufficiently close (i.e. the spacing between the stirrups is sufficiently small when compared to the size of the reinforced area) so that an equivalent homogenous material can be adopted (figure 3). In such conditions, the corresponding support functions of the homogenized reinforced concrete take the contribution of both the concrete and the stirrups into account:

$$\pi^{\text{hom}} (d) = \pi^c (d) + \pi^{\text{st}} (d)$$

$$\pi^{\text{hom}} (n, V) = \pi^c (n, V) + \pi^{\text{st}} (n, V)$$ (5)
in which the support functions of the concrete write for a modified Mohr-Coulomb failure condition with zero tension cut-off under plane strain conditions:

\[
\pi^c \left( \underline{d} \right) = \begin{cases} 
\frac{f_c}{2} \left( \sum_{k=1}^{2} |d_k| - \text{tr} \underline{d} \right) & \text{if } \text{tr} \underline{d} \geq \left( \sum_{k=1}^{2} |d_k| \right) \sin \varphi \\
\frac{f_c}{2} \left( |V| - V_n \right) & \text{if } V_n \geq |V| \sin \varphi \\
\frac{f_c}{2} \left( |V| - V_n \right) & \text{if not}
\end{cases}
\]

(6)

\[
\pi^c \left( n, V \right) = \begin{cases} 
\frac{f_c}{2} \left( |V| - V_n \right) & \text{if } V_n \geq |V| \sin \varphi \\
\frac{f_c}{2} \left( |V| - V_n \right) & \text{if not}
\end{cases}
\]

and those of the transverse reinforcements:

\[
\pi^w \left( \underline{d} \right) = \sigma_0 |d_{\perp}| \\
\pi^w \left( n, V \right) = \sigma_0 |n \cdot V|
\]

(7)

with:
- \(d_k (k = 1, 2)\) are the principal strain rates in the plane of deformations \(Oxy\);
- \(\sigma_0 = (2A_{w_f} f_{yw}) / s\) is the resistance of the transverse reinforcement per unit of length;
- \(n_i\) is the projection of the normal to the surface of discontinuity onto the \(Oy\)-axis, which is the direction of the reinforcements;
- \(V_i\) is the projection of the discontinuity of the velocity field onto the \(Oy\)-axis.

**Figure 3.** Initial and equivalent homogenized problems

b) for the longitudinal reinforcements modelled as 1D elements:

\[
\pi^l \left( \underline{d} \right) = T_0 |d_{\perp}| \\
\pi^l \left( V \right) = T_0 |V|
\]

(8)
2.4. Numerical implementation

A triangular mesh of type T3 is used to discretize the equivalent homogenised medium, considering a linear velocity field with possible velocity jumps between every two adjacent elements. Using linear interpolation, the velocity field of an element \( j \) with nodes \( i = 1, 2, 3 \) (figure 4) may be written as:

\[
U_j(x, y) = \sum_{i=1}^{3} N_i(x, y) U_i^j
\]  

(9)

where

- \( U_j(x, y) \) is the velocity at a point \( (x, y) \) of element \( j \);
- \( U_i^j \) is the velocity of node \( i = 1, 2, 3 \) of element \( j \);
- \( N_i(x, y) (i = 1, 2, 3) \) are the linear interpolation functions [17].

**Figure 4.** Representation of a triangular element \( j \) of thomogenized medium (with nodes 1, 2 and 3)

In this case, the contribution of the homogenized material (concrete and stirrups) to the maximum resisting work is equal to the sum of the contributions associated to the strain rates inside the \( N \) triangular elements (of the homogenized material), and of the discontinuities of the velocity field at the \( N_z \) interfaces between each two adjacent elements:

\[
P_{RM}^{hom} = \sum_{j=1}^{N} \alpha_{hom}^{j} (d_j) S_j + \sum_{i=1}^{N_z} \alpha_{hom}^{n_i} (n_i, V_i) l_i
\]  

(10)

with the following notations:

- \( d_j (j = 1, \ldots, N) \) is the strain rate tensor of a triangular element \( j \) of the homogenized material (of surface \( S_j \)), computed from the velocities at its three nodes;
- \( V_i \) is the discontinuity of the velocity field across a discontinuity line \( i = 1, \ldots, N_z \) (of length \( l_i \)) between two adjacent elements, computed from the discontinuities of the velocity field at the nodes common to two adjacent elements (see figure 5).

**Figure 5.** Discontinuities of the velocity field at the nodes (1 and 2) common to two adjacent elements \( j_1 \) and \( j_2 \)
On the other hand, the longitudinal reinforcements are discretized into $N_r$ one dimensional elements with a linear velocity field and possible jumps of the velocity between the $N_{rd}$ nodes (that are common to two adjacent 1D elements of the reinforcement - figure 6):

$$P_{m} = \sum_{j=1}^{N_r} \pi' \left( d_j \right) l_j + \sum_{i=1}^{N_{rd}} \pi' \left( V_i \right)$$  \hspace{1cm} (11)

![Figure 6. Discontinuity of the velocity field at node $i$ between two adjacent 1D elements $j_1$ and $j_2$ of a longitudinal reinforcement bar](image)

It follows that the best upper bound to the resistance domain may be derived from inequality (3) by a minimization procedure using conic quadratic optimization methods [18], available in the optimization software Mosek [19].

3. Illustrative examples and comparison with other numerical simulations using a finite element-based limit analysis commercial software

For illustrative purpose, the above described calculation procedure is now implemented on three typical RC beams of total length $L = 6$ m, depth $h = 0.5$ m and width $b = 0.25$ m as follows:

- beam n°1: non-reinforced concrete beam (without longitudinal and transverse reinforcements);
- beam n°2: RC beam without shear reinforcement. It is reinforced by two symmetrical layers of two longitudinal bars of 32 mm diameter with a yield strength of $f_y = 700$ MPa and a concrete cover of 34 mm;
- beam n°3: RC equipped with shear reinforcements (vertical stirrups). It is reinforced by the same longitudinal reinforcements as beam n°2, in addition to 6 mm diameter stirrups that are regularly spaced by $s = 0.2$ m, with a yield strength of $f_{ys} = 665$ MPa.

The total span of the beams $l$ is equal to 5 m while the shear span $a$ is taken as three times the beam’s height ($a = 1.5$ m). Moreover, the constituent concrete has a compressive strength of $f_{c} = 30$ MPa.

The corresponding $(N, P)$ resistance domains obtained by the proposed calculation procedure are shown in figure 7, along with results from other numerical simulations using Optum G2 computer program [20], a finite element-based limit analysis commercial software. It can be seen that numerical simulations using Optum G2 provide results which are in good agreement with those predicted by the proposed method for all the three RC beams. It should be noted however that, while a homogenized criterion is considered in the proposed method, the transverse reinforcements should be individually modelled in Optum G2 using one dimensional plate elements. As a consequence, modelling and
discretizing every individual reinforcement of a beam containing a large number of stirrups (see figure 8 for example) may be very tedious and time consuming.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{Comparison between the resistance domains obtained by Optum G2 and the proposed method for beams n°1, 2 and 3}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure8.png}
\caption{Examples of a refined mesh of the beam (a) without and (b) with shear reinforcements}
\end{figure}

\section{Comparison with available experimental results}

In figures 9 and 10 below, the resistance domains predicted by the proposed method are compared to the results of two experimental campaigns performed by Jørgensen et al. [8] and Madsen et al. [9] on RC beams (without shear reinforcement) subjected to the same loading configuration as that described in this paper (see figure 1). The tests beams were subjected to a transverse load applied at the middle of the beam \((a = l/2)\), in addition to tensile loads (ranging from 0 to 50\% of the concrete compressive strength) in the tests of Jørgensen et al. [8], and compressive loads (up to 85\% of the concrete compressive strength) in the tests of Madsen et al. [9].

According to Nielsen and Hoang [16], a reduction factor \(\nu\) should be applied to the concrete compressive strength in order to take into account the reduction of the resistance due to cracking as follows:
where $f_c$ is expressed in MPa, $A_s$ is in $m^2$ and $b$ and $h$ are in m.

Indeed, such a reduction factor captures the effect of microcracking of the concrete material due to the shrinkage of the cement paste which is restrained by the unhydrated cement grains and the aggregate particles, as well as that of microcracking and macrocracking caused by the previously applied loadings (see Nielsen and Hoang [16] for more details).

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**Figure 9.** Comparison between the resistance domain obtained by the proposed method and the experimental results of Jørgensen et al. [8] with or without reducing the concrete compressive strength, for (a) $a = 0.45$ m and (b) $a = 0.55$ m.
Figure 10. Comparison between the resistance domain obtained by the proposed method and the experimental results of Madsen et al. [9] with or without reducing the concrete compressive strength.

It can be seen from the latter figures that the predicted numerical resistance domain is in good agreement with the experimental results of Madsen et al. [9] for both cases (with and without applying a reduction factor to the concrete compressive strength) whereas it greatly overestimates the shear capacity of the tests of Jørgensen et al. [8] when the reduction factor is not applied. It is due to the fact that the calculated reduction factor (according to the equation (12)) is still relatively high for the tests of Madsen et al. [9] ($\nu = 0.84$) while remaining much smaller for the tests of Jørgensen et al. [8] ($\nu = 0.40$ and 0.38 for the tests with $a = 0.45$ and 0.55 m respectively).

5. Conclusions

Relying on the finite-element implementation of the kinematic approach of the yield design (or limit analysis) theory combined with a “mixed modelling”, this paper presents a numerical approach for the design of RC beams subjected to axial and transverse loads. An upper bound to the resistance domain may be computed by a minimization procedure using conic quadratic optimization method. It has been shown that calculations conducted on typical RC beams with different longitudinal and transverse reinforcement degrees, provide results which are in good agreement with those derived from other numerical simulations of the same problem using a finite element-based limit analysis commercial software. Moreover, these predictions are also compared to some available experimental results published in the literature, showing that the use of the reduction factor of the concrete compressive strength due to cracking, may allow to give relatively accurate estimates of the beams shear capacity, with the same trend as that observed in the experimental results as regards the effect of the axial loads.

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