Integrable systems associated to open extensions of type A and D Dubrovin–Frobenius manifolds

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Abstract
We investigate the solutions to open WDVV equation, associated to type A and D Dubrovin–Frobenius manifolds. We show that these solutions satisfy some stabilization condition and associate to both of them the systems of commuting PDEs. In the type A we show that the system of PDEs constructed coincides with the dispersionless modified KP hierarchy written in the Fay form.

Keywords: Dubrovin–Frobenius manifolds, integrable hierarchies, open WDVV equation

1. Introduction

Given a Dubrovin–Frobenius manifold, there are several constructions of an integrable system associated to it (cf [DZ], [GM] and [B2]). In particular, it was proved that for type $A_N$ and $D_N$ Dubrovin–Frobenius manifolds all these constructions provide the corresponding Drinfeld–Sokolov hierarchies (cf [LRZ]). The $A_N$ Drinfeld–Sokolov hierarchy can be defined as the reductions of the KP and type $D_N$ Drinfeld–Sokolov hierarchy can be defined as a two-component reduction of the two-component BKP hierarchies.

In [BDbN] the authors proposed a new construction of a system of commuting PDEs associated to the family of $A$-type and $D$-type Dubrovin–Frobenius manifolds. One of the main properties of this construction was the following: consistency of the system of PDEs constructed was derived from WDVV equation on the potential of a Dubrovin–Frobenius manifold. Comparing to the previous approaches this construction associated the dispersionless KP hierarchy and the dispersionless one-component reduced two-component BKP hierarchy in $A$ and $D$ types respectively.

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1.1. Open WDVV equation

Motivated by studies of the open Gromov–Witten theories the new system of PDEs called open WDVV was introduced in [HS12]. Let \( F^c = F^c(t_1, \ldots, t_N) \) be a solution to WDVV equation with the metric \( \eta \). Namely for any fixed \( 1 \leq \alpha, \beta, \gamma, \delta \leq N \) holds

\[
\sum_{\gamma,\delta=1}^{N} \frac{\partial^3 F^c}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} \eta_{\mu\nu} \frac{\partial^3 F^c}{\partial t_{\mu} \partial t_{\nu} \partial t_{\delta}} = \sum_{\gamma,\delta=1}^{N} \frac{\partial^3 F^c}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} \eta_{\mu\nu} \frac{\partial^2 F^c}{\partial t_{\mu} \partial t_{\nu} \partial t_{\delta}}.
\]

For a fixed \( F^c \), the open WDVV equation on a function \( F^o = F^o(t_0, t_1, \ldots, t_N) \) depending on an additional variable \( t_0 \) is the 1-system of equations

\[
\sum_{\gamma,\delta=1}^{N} \frac{\partial^3 F^c}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} \eta_{\mu\nu} \frac{\partial^3 F^o}{\partial t_{\mu} \partial t_{\nu} \partial t_{\delta}} = \sum_{\gamma,\delta=1}^{N} \frac{\partial^3 F^c}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} \eta_{\mu\nu} \frac{\partial^2 F^o}{\partial t_{\mu} \partial t_{\nu} \partial t_{\delta}} + \frac{\partial^2 F^o}{\partial t_{\alpha} \partial t_{\gamma}} \frac{\partial^2 F^o}{\partial t_{\beta} \partial t_{\delta}}.
\] (1.1)

that should hold for any fixed \( 0 \leq \alpha, \beta, \gamma \leq N \).

Similarly to ‘classical’ WDVV equation, open WDVV equation is associativity equation of a product defined by

\[
\frac{\partial}{\partial t_{\alpha}} \circ \frac{\partial}{\partial t_{\beta}} = \sum_{\gamma,\delta=1}^{N} \frac{\partial^3 F^c}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} \eta_{\mu\nu} \frac{\partial^3 F^o}{\partial t_{\mu} \partial t_{\nu} \partial t_{\delta}} + \frac{\partial^2 F^o}{\partial t_{\alpha} \partial t_{\gamma}} \frac{\partial^2 F^o}{\partial t_{\beta} \partial t_{\delta}},
\]

\( 0 \leq \alpha, \beta \leq N \).

From this point of view the function \( F^o \), solving open WDVV equation, defines an open extension of a Dubrovin–Frobenius manifold given by \( F^c \).

For \( F^c \) being Dubrovin–Frobenius potential of type \( A_N \) or \( D_N \) the solutions to open WDVV equation were investigated in [BB2]. In particular, for the \( A_N \) and \( D_N \) Dubrovin–Frobenius potentials \( F^o_{A_N} \) and \( F^o_{D_N} \) the authors constructed the functions \( F^o_{A_N} \) and \( F^o_{D_N} \), being solutions to open WDVV equation. The function \( F^o_{A_N} \) appeared to be polynomial in \( t_0, t_1, \ldots, t_N \) and \( F^o_{D_N} \) polynomial in \( t_1, \ldots, t_N \), but Laurent polynomial in \( t_0 \).

The integrable hierarchies associated to the solutions of open WDVV equations were investigated in [BCT1, BCT2] and [A]. In all these references only \( A_N \) type was assumed.

1.2. System of commuting PDEs

Extending the approach of [BDbN] we associate to the families \( \{(F^c_{A_N}, F^o_{A_N})\}_{N \geq 1} \) and \( \{(F^c_{D_N}, F^o_{D_N})\}_{N \geq 4} \) the systems of commuting PDEs. Like in loc.cit. the first step on this way is the following stabilization result.

In the A case for any \( N < M \) and \( 1 \leq \alpha \leq N \) we have

\[
\frac{\partial F^o_{A_N}}{\partial t_0} \big|_{t_{N+1} = s_j} = \frac{\partial F^o_{A_M}}{\partial t_0} \big|_{t_{M+1} = s_j},
\]

assumed as polynomials in \( t_0, s_j, \ldots, s_{M+1} \). In the D case for any \( 4 \leq N < M \) and \( 1 \leq \alpha < N \) we have
Both systems of equations that we write express the higher derivatives \( \partial_{\alpha} \partial_{\beta} f \) of a solution \( f \) via \( \partial_{t_0} f, \partial_{t_1} f, \partial_{t_2} f, \ldots \) that may be viewed as ‘initial data’. It is expected that open and closed potentials of \( A_N \) and \( D_N \) respectively build up special solutions to these PDEs. We comment on this in proposition 3.2 and theorem 4.2.

For A type this is the system of PDEs on a function \( f = f(t_0, t_1, t_2, \ldots) \)

\[
\partial_{\alpha} \partial_{\beta} f = \frac{\partial^2 F_{\alpha}}{\partial t_0 \partial t_1} \bigg|_{t_0=\alpha=0, t_1=0}, \quad \kappa = \alpha + \beta + 1,
\]

\[
\partial_{\alpha} \partial_{\beta} f = \frac{\partial^2 F_{\alpha}}{\partial t_0 \partial t_1} \bigg|_{t_0=\alpha=0, t_1=0}.
\]

where the abbreviate \( \partial_r \equiv \partial/\partial t_r \). We show in section 3 that this system is well-defined and consistent. We also prove in theorem 3.5 that this system of equations coincides with the dispersionless modified KP hierarchy written in the Fay form. We also provide the construction extending new type A system to the full mKP hierarchy.

For D type we introduce the system of PDEs on \( f = f(t_0, t_1, t_2, \ldots) \)

\[
\partial_{\alpha} \partial_{\beta} f = \frac{\partial^2 F_{\alpha}}{\partial t_0 \partial t_1} \bigg|_{t_0=\alpha=0, t_1=0}, \quad \kappa = \alpha + \beta + 1,
\]

\[
\partial_{\alpha} \partial_{\beta} f = \frac{\partial^2 F_{\alpha}}{\partial t_0 \partial t_1} \bigg|_{t_0=\alpha=0, t_1=0}.
\]

where we abbreviate \( \partial_1 = \partial/\partial t_1 \). It is proved in theorem 4.2 that this system of equations is well-defined and consistent.

Like in [BDbN] we derive consistency of both A and D type PDEs from open WDVV equation above. Comparing to the previously cited references our work gives the first example of the system of PDEs constructed by the open WDVV solutions.

Following the relation of \( D_N \) type Drinfeld–Sokolov hierarchy and also D type hierarchy of [BDbN] to the two-component BKP hierarchy, one would expect that our D type system is connected to the modified BKP hierarchy. However the modified version of BKP hierarchy is not yet fully settled.

2. Open potentials of type A and D

The Dubrovin–Frobenius potentials \( F_{A_N} \) and \( F_{D_N} \) were first derived by Dubrovin via the geometry of the respective Coxeter groups (cf [D1]). Since then these potentials were constructed in the several different ways by the other authors (cf [DYZ, LYZ]). We neither give the construction of these Dubrovin–Frobenius manifolds nor provide the expansion of the respective potentials referencing the reader to [BDbN].
2.1. $A_N$ open potential

The genus zero $A_N$ open potential $F_{A_N}^o$ was first found in [BCT2]. It is a polynomial in $t_1, \ldots, t_N$ and $t_0$ defined by

$$\frac{\partial^{m+k} F_{A_N}^o}{\partial t_{\alpha_1} \cdots \partial t_{\alpha_m} (\partial t_0)^k} \bigg|_{t_0=0} = \begin{cases} (m+k-2)! & \text{if } \sum_{j=1}^m (N+2-\alpha_j) + k = N+2, \\ 0, & \text{otherwise.} \end{cases}$$

It is also connected to the $A_N$ unfolding coordinates by the following identity (see [B3])

$$\frac{\partial F_{A_N}^o}{\partial t_0} = \frac{t_0^{N+1}}{N+1} + \sum_{k=1}^N t_0^{k-1} v_k^A(t_1, \ldots, t_N), \quad (2.1)$$

where $v_k^A$ are defined by

$$v_b^A = \sum_{\sum_{k=1}^N (N+2-k)\alpha_k = N+2-b} \sum_{\alpha_1, \ldots, \alpha_N \geq 0} \frac{(\sum_{k=1}^N \alpha_k + b-2)!}{(b-1)!} \prod_{k=1}^N \frac{\bar{r}_k}{\alpha_k!} \prod_{k=1}^N \frac{\bar{r}_k}{b!} \quad (2.2)$$

for $|\alpha| = \sum_{k=1}^N \alpha_k$.

It follows immediately from the definition above that $F_{A_N}^o$ satisfies the following quasihomogeneity condition

$$\left( \sum_{k=1}^N (N+2-k) \frac{\partial}{\partial t_k} + t_0 \frac{\partial}{\partial t_0} \right) F_{A_N}^o = (N+2)F_{A_N}^o. \quad (2.3)$$

It was shown in [BB2] that $F_{A_N}^o$ is the only polynomial satisfying this quasihomogeneity condition, s.t. $(F_{A_N}^o, F_{A_N}^o)$ is a solution to open WDVV equation and $\partial_1 \partial_h F_{A_N}^o = 1, \partial_1 \partial_\alpha F_{A_N}^o = 0$ for $1 \leq \alpha \leq N$. We get

$$F_{A_N}^o = \frac{t_0^{N+2}}{(N+1)(N+2)} + \sum_{b=0}^N \sum_{\sum_{k=1}^N (N+2-k)\alpha_k = N+2-b} \frac{(\sum_{k=1}^N \alpha_k + b-2)!}{(b-1)!} \prod_{k=1}^N \frac{\bar{r}_k}{\alpha_k!} \prod_{k=1}^N \frac{\bar{r}_k}{b!} \quad (2.4)$$

2.2. $D_N$ open potential

The genus zero $D_N$ open potential $F_{D_N}^o$ is a polynomial in $t_1, \ldots, t_N$ and Laurent polynomial in $t_0$ defined by (cf [BB2, section 5.2])

$$F_{D_N}^o := \frac{t_0^{N-1}}{2^{N-2}(N-1)(2N-3)} + \frac{\bar{r}_0^N}{2t_0} + \sum_{k=1}^{N-1} \frac{v_k^D t_0^{2k-1}}{2^{N-1}(2k-1)}$$

with the functions $v_b^D$ given by (see [B3, corollary 4.6])

$$v_b^D = \sum_{\sum_{k=1}^N (N-k)\alpha_k = N-b} \frac{(\sum_{k=1}^N \alpha_k + 2b-3)!}{(2b-2)!} \prod_{k=1}^N \frac{\bar{r}_k}{\alpha_k!} \quad (2.4)$$
The Laurent polynomial $F^{o}_{DN}$ satisfies the following quasihomogeneity condition

$$\left(\sum_{k=1}^{N-1} 2(N-k)t_k \frac{\partial}{\partial t_k} + Nt_N \frac{\partial}{\partial t_N} + t_0 \frac{\partial}{\partial t_0}\right) F^{o}_{DN} = (2N-1)F^{o}_{DN}. \quad (2.5)$$

It is immediate to see that the variable $t_N$ plays a special role in $F^{o}_{DN}$. In particular, the functions $v^{D}_{b}$ do not depend on $t_N$ and the only non-polynomial summand of $F^{o}_{DN}$ is at the same time the only appearance of $t_N$ in the open potential. This variable is also special for $F^{c}_{DN}$. In particular, we have (see [BDbN, section 3.3])

$$F^{c}_{DN} = \frac{1}{2} t_1^2 t_{N-1} + \frac{1}{2} t_1 t^2_N + \frac{1}{2} t_1 \sum_{\alpha=2}^{N-1} t_\alpha t_{N-\alpha} + \phi(t_2, \ldots, t_{N-1}) + t_0^2 \frac{t^2_N}{2}, \quad (2.6)$$

for some polynomial $\phi$ that does not depend on $t_N$ again. This speciality of $t_N$ will result in the special form of the PDEs that we obtain in the D case.

3. Dispersionless open–closed system of type A

The following stabilization condition was proved in [BDbN, theorem 4.1]. For any $M > N \geq 1$ and $\alpha, \beta, \gamma, \kappa \geq 1$ such that $1 \leq \alpha, \beta \leq N, \alpha + \beta \leq N + 1$ and $1 \leq \kappa \leq N - 1$, we have

$$\frac{\partial^2 F^{o}_{\alpha N}}{\partial t_\alpha \partial t_\kappa} \bigg|_{t_N = \gamma, t_{N+1-\gamma} = x_\gamma} = \frac{\partial^2 F^{o}_{\alpha L}}{\partial t_\alpha \partial t_\beta} \bigg|_{t_N = \kappa, t_{M+1-\kappa} = x_\kappa},$$

understood as an equality of polynomials in $s_\gamma$. We show that it also holds for $F^{o}_{\alpha M}$.

**Proposition 3.1.** For any $\alpha \leq N < M$ and $1 \leq \alpha \leq N$ we have

$$\frac{\partial F^{o}_{\alpha M}}{\partial t_\alpha} \bigg|_{t_{N+1-\beta} = x_\beta} = \frac{\partial F^{o}_{\alpha L}}{\partial t_\alpha} \bigg|_{t_{M+1-\beta} = x_\beta},$$

assumed as polynomials in $t_0, s_1, \ldots, s_{M+1}$.

**Proof.** Note that the change of the variables above does not affect the variable $t_0$ and we can consider $F^{o}_{\alpha L}$ and $F^{o}_{\alpha M}$ as polynomials in $t_0$. Consider the free terms in $t_0$. We have for any $\kappa \geq 1$

$$\frac{\partial^{m+1} F^{o}_{\alpha L}}{\partial t_0 \partial t_{i_1+1-\beta_1} \cdots \partial t_{i_\kappa+1-\beta_\kappa}} \bigg|_{t_0 = 0} = \begin{cases} (m - 1)! & \text{if } m + \sum_{i=1}^{\kappa} \beta_i = \alpha, \\ 0 & \text{otherwise}. \end{cases}$$

Both the value and the condition above do not depend on $\kappa$ what proves the statement.

For any $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \mathbb{Z}_{\geq 0}$ take $N = \alpha + \beta + 1$ and denote $x := N + 1 - x$. Set.
Proof. Compatibility of the system written follows from open WDVV equation in the way it was proved in [BDbN, proposition 2.1]. The proof is parallel to the one given in the case \( \partial \alpha \partial \beta \) in the proof of theorem 4.2. This follows also from theorem 3.5.

Consider the system of PDEs.

\[
\partial_\alpha \partial_\beta f = \sum_{m \geq 1} \sum_{\tilde{\gamma}} R^{A}_{\alpha, \beta; \tilde{\gamma}} \prod_{k=1}^{m} \partial_\gamma \partial_\delta f, \quad (3.3)
\]

\[
\partial_\alpha f = \sum_{m \geq 1} \sum_{\tilde{\gamma}} R^{A, \text{ext}}_{\alpha; \tilde{\gamma}} \prod_{k=1}^{m} \partial_\gamma \partial_\delta f, \quad (3.4)
\]

One notes immediately that these PDEs coincide with the PDEs presented in introduction. In the next proposition let \( \int F_{\alpha}^{c} dt_0 \) stand for the polynomial \( P = P(t_0, t_1, \ldots, t_N) \), having zero constant term in \( t_0 \) and satisfying \( \partial_0 P = F_{\alpha}^{c} \).

**Proposition 3.2.** The system (3.3) and (3.4) is compatible. The function \( \tilde{f} = F_{\alpha}^{c} \) satisfies this system for \( \alpha + \beta \leq N + 1 \).

**Proof.** Compatibility of the system written follows from open WDVV equation in the same way it was proved in [BDbN, proposition 2.1]. The proof is parallel to the one given in the case in the proof of theorem 4.2. This follows also from theorem 3.5.

Because of the special dependance of \( F_{\alpha}^{c} \) and \( F_{\alpha}^{0} \) on the variable \( t_1 \) we have

\[
\partial_1 \partial_\gamma \tilde{f} = \begin{cases} 
\partial_1 \partial_\gamma F_{\alpha}^{c} = 0, & \text{if } \gamma > 0, \\
\partial_1 F_{\alpha}^{0} = t_0, & \text{if } \gamma = 0.
\end{cases}
\]

Let us show that applied to \( \tilde{f} \) assumed both (3.3) and (3.4) just provide the series expansions of \( \partial_\alpha \partial_\beta F_{\alpha}^{c} \) and \( \partial_\alpha F_{\alpha}^{0} \).

It follows from (2.3) that \( \partial_\delta \partial_\gamma F_{\alpha}^{c} \equiv 0 \) whenever \( \alpha + \beta \leq N + 1 \). The function \( F_{\alpha}^{c} \) does not depend on \( t_0 \) and therefore (3.3) holds for \( \tilde{f} \).

Similarly \( \partial_0 \partial_\gamma f = \partial_\gamma F_{\alpha}^{0} \) what proves (3.4) for \( \tilde{f} \). \( \square \)

### 3.1. The flows

For any \( f = f(t_0, t_1, t_2, \ldots) \) denote \( p_k := \partial_1 \partial_\gamma f \). The first flows of (3.4) read

\[
\partial_0 \partial_1 f = p_0,
\]

\[
\partial_0 \partial_2 f = \frac{p_1^2}{2} + p_1,
\]

\[
\partial_0 \partial_3 f = \frac{p_1^3}{3} + p_1 p_0 + p_2,
\]

\[
\partial_0 \partial_4 f = \frac{p_1^4}{4} + p_1 p_0^2 + p_2 p_0 + \frac{p_1^2}{2} + p_3.
\]
Denote by $P_f(g_1, \ldots, g_m)$ the number of all partitions $i_1, \ldots, i_m$ of $i$ and $j_1, \ldots, j_m$ of $j$, s.t. $i_k + j_k = \gamma_k + 1$ for all $k$. It was computed in [BDBN, corollary 5.1] that the flows of (3.3) read:

$$\partial_i \partial_j f = \sum_{m \geq 1} (-1)^{m-1} \sum_{\gamma_1 + \cdots + \gamma_m = i + j - m} P_f(\gamma_1, \ldots, \gamma_m) \prod_{k=1}^m \partial_1 \partial_{\gamma_k} f.$$  

(3.5)

**Proposition 3.3.** 
Equation (3.4) is equivalent to the following equality of the formal power series in $z$

$$\sum_{\alpha \geq 1} \partial_0 \partial_1 f \cdot z^\alpha = -\log \left[ 1 - \sum_{\alpha \geq 1} \partial_i \partial_0 f \cdot z^{\alpha+1} - z \cdot \partial_0 \partial_1 f \right].$$

**Proof.** Introduce the notation $x_\alpha := \partial_0 \partial_{\gamma_1-1} f$ and $y_\alpha := \partial_0 \partial_{\gamma_1} f$. For $m \geq 0$ set $\phi_{\alpha,m} := \sum x_{\gamma_1} \cdots x_{\gamma_m}$ where the summation is taken over all $\gamma_1 \geq 2$, s.t. $\gamma_1 + \cdots + \gamma_m = k$. Extend this definition to $m = 0$ by $\phi_{\alpha,0} := \delta_{\alpha,0}$. Then (3.4) can be written using (2.1) as

$$y_\alpha = \sum_{m=1}^\alpha \phi_{\alpha,m} + \sum_{m=0}^{\alpha-1} \sum_{k=m+1}^\alpha \frac{y_{\alpha-k}^{m+1-k}}{(\alpha+1-k)!} \frac{m!}{(m+1-k)!} \phi_{\gamma-k,1}. $$

For any polynomial $p = p(z)$ let $[z^\alpha]p$ stand for the coefficient of $z^\alpha$ in the polynomial assumed. For $\Phi := \sum_{k \geq 2} x_k z^k$ we have $\phi_{\alpha,m} = [z^\alpha][\Phi^m]$. Denote also $\Psi := \sum_{k \geq 1} x_k z^k / y_1^k$. Note that this power series is obtained from $\Phi$ by a formal rescaling of $z$ variable.

Equation above is equivalent to

$$y_\alpha = \frac{y_0^\alpha}{\alpha} = y_1^\alpha[z^\alpha] \left( \sum_{k=1}^{\alpha+1-k} \frac{(m + \alpha - k)!}{(\alpha + 1 - k)!} z^{\alpha+1-k} \Psi^m \right)$$

$$\Leftrightarrow y_\alpha = \frac{y_0^\alpha}{\alpha} = y_1^\alpha[z^\alpha] \left( \sum_{k=2}^{\alpha} \frac{z^{\alpha+1-k}}{(\alpha + 1 - k)} \frac{1}{(1 - \Psi)^{\alpha+1-k}} + \sum_{m \geq 1} \frac{\Psi^m}{m} \right)$$

$$\Leftrightarrow y_\alpha = y_1^\alpha[z^\alpha] \left( -\log \left( 1 - \frac{z}{1 - \Psi} \right) - \log(1 - \Psi) \right).$$

Rescaling formally $z$ on the both sides and collecting $y_\alpha$ into a power series the proposition follows. \(\square\)

The following proposition shows that the dependance of $\partial_i \partial_j f$ on $\partial_0 \partial_1 f$ is given via the Schur polynomials.

**Proposition 3.4.** Let $f$ satisfy (3.4) then we have

$$\partial_i \partial_j f = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m!} \sum_{\gamma_1 + \cdots + \gamma_m = i + j - m} \prod_{k=1}^m \partial_0 \partial_{\gamma_k} f.$$  

(3.6)

**Proof.** Assume the notation introduced in the proof of proposition 3.3. Exponentiating the equality of proposition 3.3 we get

$$\Phi(z) = 1 - \exp \left( -\sum_{\alpha \geq 1} y_\alpha z^\alpha \right) - y_1 z.$$
Expanding the exponent in a power series and combining the coefficients of $z^{\alpha}$ on the both sides we get exactly the desired equation. □

Our goal now is to relate the system (3.3) and (3.4) to dispersionless mKP hierarchy.

3.2. Dispersionless mKP hierarchy

Introduce the notation

$$D(z) := \sum_{k \geq 1} z^{-k} \frac{\partial}{\partial k}.$$ 

For a function $f = f(t_0, t_1, t_2, \ldots)$, the dispersionless limit of mKP hierarchy can be written in a Fay form as the following equality of the formal power series in $z^{-1}, w^{-1}$.

$$e^{D(z)D(w)} f = \frac{z \cdot e^{-D(z)} \partial_0 f - w \cdot e^{-D(w)} \partial_0 f}{z - w}, \quad (3.7)$$

The coefficient of $w^{-1}$ on the both sides gives the following equality

$$z - D(z) \partial_1 f = ze^{-D(z)} \partial_0 f. \quad (3.8)$$

Substituting it back in (3.7) gives

$$e^{D(z)D(w)} f = 1 - \frac{D(z) \partial_1 f - D(w) \partial_1 f}{z - w}. \quad (3.9)$$

This equation is exactly the Fay form of the dispersionless limit of KP hierarchy of the function $f$ assumed as a function of $t_1, t_2, \ldots$ with $t_0$ being fixed.

**Theorem 3.5.** The system of equations (3.3) and (3.4) coincides with the dispersionless mKP hierarchy after the change of the variables $t_k \to t_k/k, k \geq 1$.

**Proof.** After the change of the variables given, the system (3.3) coincides with the Fay form of dispersionless KP by (3.5) (see also [BDbN]). It follows immediately from proposition 3.4 that (3.8) holds for a solution to (3.4). Therefore substituting it in (3.3) one gets (3.4). □

3.3. $h$-deformation

Full mKP hierarchy can be obtained from its dispersionless limit by the following procedure extending the approach of [NZ].

Let $\tau_h(s, t) := \tau(h^{-1}s, h^{-1}t)$. Assume it to be expanded by $\log \tau_h = \sum_{k \geq 0} h^{-2k} F_k$. Denote $f := h^2 \log \tau_h$. Consider also the operators

$$\Delta(z) := h^{-1} \left( e^{hD(z)} - 1 \right), \quad \Delta^o := h^{-1} \left( e^{-h\partial_0} - 1 \right).$$

Note that $\Delta^o$ only involves the differentiations w.r.t. $t_0$ while $\Delta(z)$ only the differentiations w.r.t. $t_k, k \geq 1$.

The $h$-deformed version of (3.7) above is the following equation

$$e^{\Delta(z)\Delta(w)} f = \frac{z \cdot e^{-\Delta(z)\Delta^o f} - w \cdot e^{-\Delta(w)\Delta^o f}}{z - w}. \quad (3.10)$$

It coincides with the Fay form of mKP hierarchy (cf [T, section 3.2]).
Consider $\partial^h_b$ defined by

$$\partial^h_b := h^{-1}(e^{-h/b} - 1), \quad \sum_{k\geq 1} \frac{z^{-k}}{k} \partial^h_b = \Delta(z).$$

Comparing the coefficients of $z^{-a}w^{-b}$ on the both sides of (3.10) one gets exactly

$$\partial^h_b \partial f^h = \sum_{m\beta_1} R^h_{\alpha;\gamma_1,...,\gamma_n} \prod_{k=1}^m \partial \partial^h f,$$

$$\partial^h_b \partial f^h = \sum_{m\beta_1} R^h_{\alpha;\gamma_1,...,\gamma_n} \prod_{k=1}^m \partial \partial^h f.$$

This procedure recovers full mKP hierarchy from the dispersionless hierarchy that we’ve constructed from the family of pairs $\{F_{AN}, F_{AO}\}_{N \geq 1}$. This is an important open equation to establish the connection between full mKP hierarchy and all genera partition function of open intersection numbers.

### 4. Dispersionless open–closed system of type D

The following stabilization condition was proved in [BDbN].

For any $4 \leq N < M$ we have

$$\frac{\partial^2 F^c_{DN}(t)}{\partial \alpha \partial \beta} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0} = \frac{\partial^2 F^c_{DM}(t)}{\partial \alpha \partial \beta} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0}, \quad \forall \alpha + \beta < N,$$

$$\frac{\partial^2 F^c_{DN}(t)}{\partial \alpha \partial \beta} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0} = \frac{\partial^2 F^c_{DM}(t)}{\partial \alpha \partial \beta} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0}, \quad \forall \beta < N,$$

understood as an equality of polynomials in $s_*$. 

**Proposition 4.1.** For any $4 \leq N < M$ and $1 \leq \alpha < N$ we have

$$\frac{\partial F^c_{DN}}{\partial \alpha} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0} = \frac{\partial F^c_{DM}}{\partial \alpha} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0}, \quad \forall \alpha + \beta < N,$$

$$\frac{\partial F^c_{DN}}{\partial \alpha} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0} = \frac{\partial F^c_{DM}}{\partial \alpha} \bigg|_{\gamma_t = k_0, \; \gamma_0 = s_0}, \quad \forall \beta < N,$$

assumed as polynomials in $s_1, s_2, \ldots, s_M$ and Laurent polynomial in $t_0$.

**Proof.** Note that the change of the variables above does not affect the variable $t_0$ and we can consider $F^c_{DN}$ and $F^c_{DM}$ as Laurent polynomials in $t_0$.

The second equality is straightforward by the explicit form of $F^c_{DN}$. To show the first equality we only have to consider the summands of $F^c_{DN}$ and $F^c_{DM}$ involving the variable $t_0$. This follows from (2.4) and theorem 4.9 of [BDbN].

For any $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $\gamma_* \in \mathbb{Z}_{\geq 0}$ denote $N = \alpha + \beta - 1$ and $\hat{N} := -1, \; \hat{x} := N - x$ for $1 \leq x < N$. Set
It was proved in [BdBN] that \( R_{c}^{(1)} \) and \( R_{c}^{(2)} \) are well-defined. Note that despite \( F_{D_0}^{\alpha} \) being Laurent polynomial, according to the proposition above all coefficients \( R^{\alpha} \) are well-defined.

The data above does not collect all the information of \( F_{D_0}^{\alpha} \) that is stabilized as \( N \) grows. In particular we have \( \partial_{\gamma_0} \partial_{\gamma_1} F_{D_0}^{\alpha} = 1/t_0 \), for which \( t = 0 \) is not defined. However we still can add the corresponding flow (4.4) to the system of PDEs we build.

For a function \( f = f(t_0, t_1, t_2, \ldots) \) consider the system of PDEs.

\[
\begin{align*}
\partial_\alpha \partial_\beta f &= \sum_{m \geq 1} \sum_{\gamma_1 \cdots \gamma_m} R_{c}^{(1)}_{\alpha \beta \gamma_1 \cdots \gamma_m} \prod_{k=1}^{m} \partial_{\gamma_k} f, \quad (4.1) \\
\partial_{-1} \partial_\alpha f &= \partial_{-1} \partial_{\gamma_1} f \sum_{m \geq 1} \sum_{\gamma_1 \cdots \gamma_m} R_{c}^{(2)}_{\alpha \gamma_1 \cdots \gamma_m} \prod_{k=1}^{m} \partial_{\gamma_k} f, \quad (4.2) \\
\partial_\beta \partial_\alpha f &= \sum_{m \geq 1} \sum_{\gamma_1 \cdots \gamma_m} R_{c}^{(3)}_{\alpha \beta \gamma_1 \cdots \gamma_m} \prod_{k=1}^{m} \partial_{\gamma_k} f, \quad (4.3) \\
\partial_{\gamma_0} \partial_\alpha f &= \partial_{\gamma_0} \partial_{-1} f, \quad (4.4)
\end{align*}
\]

for all \( \alpha, \beta \geq 2 \) and \( \gamma_k \geq -1 \).

We show that this system of equations coincides with the system given in Introduction after setting \( t_{-1} = t_1 \).

In what follows let \( \int F_{D_0}^{\alpha} \, dt_0 \) stand for the Laurent polynomial \( p = p(t_0, t_1, \ldots, t_N) \), having zero constant term in \( t_0 \) and satisfying \( \partial_{\gamma_0} p = F_{D_0}^{\alpha} \).

**Theorem 4.2.**

(a) The numbers \( R_{c}^{(1)}_{\alpha \beta \gamma_1 \cdots \gamma_m} \) and \( R_{c}^{(2)}_{\alpha \gamma_1 \cdots \gamma_m} \) are the coefficients of the series expansions of \( \partial_\alpha \partial_\beta F_{D_0}^{\alpha} \) and \( \partial_\gamma \partial_\alpha F_{D_0}^{\alpha} \), respectively, written in the coordinates \( t_1, \ldots, t_N \).

(b) The system (4.1) and (4.2) and (4.3) and (4.4) is compatible.

(c) The function

\[
\tilde{f} = F_{D_0}^{\alpha}(t_1^{\alpha}, \ldots, t_N^{\alpha}) + \int F_{D_0}^{\alpha}(t_1^{\alpha}, \ldots, t_N^{\alpha}, t_0) \, dt_0
\]

satisfies this system for \( \alpha + \beta = N \).

**Proof.** Part (a) follows immediately from (2.6). After this in order to show (b) we may use the PDEs given in introduction. Compatibility of (4.1) and (4.2) was proved in [BdBN].

Let \( \kappa = \alpha + \beta - 1 \). In the following formulae we use \( \partial_\beta \partial_\gamma f = \partial_\gamma \partial_\beta f \). By the chain rule we have
we have
\[ \partial_b(\partial_{\alpha} \partial_{\beta} f) = \sum_{\delta=1}^{\kappa} \partial_\delta \partial_\delta f \cdot \frac{\partial^3 F_{\delta \alpha \beta}}{\partial t_{\delta} \partial t_{\alpha} \partial t_{\beta}}|_{t_{\delta} = \partial_\delta, \ t_{\alpha} = \partial_\alpha, \ t_{\beta} = \partial_\beta,} \]
\[ = \partial_1 \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} = \partial_1 \partial_1 \partial_1 f \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} \]
\[ + \sum_{\sigma=1}^{\kappa} \partial_\sigma \partial_\sigma f \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\sigma} \partial t_{\sigma}} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\sigma} \partial t_{\sigma}} \]

where we skip the variable substitution on the second line in order to simplify the formulae.

Similarly we have
\[ \partial_\gamma (\partial_{\alpha} \partial_{\beta} f) = \sum_{\delta=1}^{\kappa} \partial_\delta \partial_\delta f \cdot \frac{\partial^2 F_{\delta \alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\delta \alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}}|_{t_{\delta} = \partial_\delta, \ t_{\alpha} = \partial_\alpha, \ t_{\beta} = \partial_\beta,} \]
\[ = \partial_1 \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} = \partial_1 \partial_1 \partial_1 f \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} \]
\[ + \sum_{\sigma=1}^{\kappa} \partial_\sigma \partial_\sigma f \cdot \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\sigma} \partial t_{\sigma}} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\sigma} \partial t_{\sigma}} \]

The coefficient of \( \partial_1 \partial_1 \partial_1 f \) on both sides is the same if and only if
\[ \sum_{\delta=1}^{\kappa} \frac{\partial^2 F_{\delta \alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\delta \alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} = \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}}. \]

The metric \( \eta \) defined by \( F_{\delta \alpha \beta} \) has only the following non–zero entries \( \eta^{\alpha \kappa} = 1 \) and \( \eta^{\beta \kappa} = \delta^{\alpha + \beta \kappa} \). With our choice of \( \beta \) this is equivalent to \( \eta^{\alpha \beta} = \delta^{\alpha \beta} \).

By open WDVV equation the difference of lhs and rhs is equal to \( \frac{\partial^2 F_{\alpha \beta}}{\partial t_1 \partial t_1} \frac{\partial^2 F_{\alpha \beta}}{\partial t_{\alpha} \partial t_{\alpha}} \) that vanishes for \( \alpha + \beta \leq \kappa \) and \( \alpha, \beta \neq \kappa \) due to (2.5). For any \( \sigma \neq 0 \) the coefficients of \( \partial_1 \partial_1 \partial_1 f \) of the expressions above coincide by the same reasoning.

Consider now part (c). Because of the special dependance of \( F_{\alpha} \) and \( F_{\alpha} \) on the variable \( t_1 \) we have
\[ \partial_1 \partial_1 f = \begin{cases} \partial_1 \partial_1 F_{\alpha} = t_1, & \text{if } \gamma > 0, \\ \partial_1 \partial_1 F_{\alpha} = t_1, & \text{if } \gamma = -1, \\ \partial_1 F_{\alpha} = t_0, & \text{if } \gamma = 0. \end{cases} \]

To show (c) let us show that applied to \( \tilde{f} \) assumed both (3.3) and (3.4) just provide the series expansions of \( \partial_1 \partial_1 F_{\alpha} \) and \( \partial_1 F_{\alpha} \).

It follows from (2.3) that \( \partial_1 \partial_1 F_{\alpha} = 0 \) whenever \( \alpha + \beta \leq N + 1 \). The function \( F_{\alpha} \) does not depend on \( t_0 \) and therefore (3.3) holds for \( \tilde{f} \).

Similarly \( \partial_1 \partial_1 F_{\alpha} = \partial_1 F_{\alpha} \) what approves (3.4) for \( \tilde{f} \).

\[ \square \]
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Data availability statement

No new data were created or analysed in this study.

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References

[A] Alexandrov A 2015 Open intersection numbers, matrix models and MKP hierarchy J. High Energy Phys. JHEP03(2015)042

[BB1] Basalaev A and Buryak A 2019 Open WDVV equations and Virasoro constraints Arnold Math J. 5 145–86

[BB2] Basalaev A and Buryak A 2021 Open Saito theory for A and D singularities Int. Math. Res. Not. 2021 5460–91

[BdBN] Basalaev A, Dunin-Barkowski P and Natanzon S 2021 Integrable hierarchies associated to infinite families of Frobenius manifolds J. Phys. A: Math. Theor. 54 115201

[B1] Buryak A Y 2017 New approaches to integrable hierarchies of topological type Russ. Math. Surv. 72 841

[B2] Buryak A 2015 Double ramification cycles and integrable hierarchies Commun. Math. Phys. 336 1085–107

[B3] Buryak A 2020 Extended r-spin theory and the mirror symmetry for the $A_{r-1}$ singularity Moscow Math. J. 20 475–93

[BCT1] Buryak A, Clader E and Tessler R J 2018 Open r-spin theory: II. The analogue of Witten’s conjecture for r-spin disks (arXiv:1809.02536v4)

[BCT2] Buryak A, Clader E and Tessler R J 2019 Closed extended r-spin theory and the Gelfand–Dickey wave function J. Geom. Phys. 137 132–53

[D1] Dubrovin B 1996 Geometry of 2D topological field theories Lecture Notes in Math 160 (Springer) pp 120–348

[DZ] Dubrovin B and Zhang Y 2001 Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants (arXiv:math/0108160v1)

[DYZ] Dubrovin B, Liu S-Q and Zhang Y 2008 Frobenius manifolds and central invariants for the Drinfeld–Sokolov bihamiltonian structures Adv. Math. 219 780–837

[GM] Givental A and Milanov T 2005 Simple singularities and integrable hierarchies Breath of Symplectic and Poisson Geometry (Boston: Birkhauser) pp 173–201

[HS12] Horev A and Solomon J P 2012 The open Gromov–Witten–Welschinger theory of blowups of the projective plane (arXiv:1210.4034v1)

[LRZ] Liu S-Q, Ruan Y and Zhang Y 2015 BCFG Drinfeld–Sokolov hierarchies and FJRW-theory Invent. Math. 201 711–72

[LYZ] Liu S-Q, Yang D and Zhang Y 2013 Uniqueness theorem of W-constraints for simple singularities Lett. Math. Phys. 103 1329–45

[NZ] Natanzon S and Zabrodin A 2015 Symmetric solutions to dispersionless 2D Toda hierarchy, Hurwitz numbers and conformal dynamics Int. Math. Res. Not. 2082–110

[NZa] Natanzon S M and Zabrodin A V 2016 Formal solutions to the KP hierarchy J. Phys. A: Math. Theor. 49 145206

[T] Takasaki K 2011 Differential Fay identities and auxiliary linear problem of integrable hierarchies Adv. Stud. Pure Math. 61 387–441