Mathematical modeling of circular sandwich plate interaction with viscous liquid layer for predicting its hydroelastic response

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Abstract. The paper proposes a mathematical model for predicting the hydroelastic response of a mechanical system consisting of a circular sandwich plate forming the bottom wall of a narrow channel filled with a pulsating viscous liquid. The pressure pulsation in the liquid layer is caused by upper vibrating channel wall represented by a rigid disk. We needed to formulate and solve the hydroelasticity problem for developing the mathematical model. The hydroelasticity problem consisted of the Navier-Stokes equations, the continuity equation, the circular sandwich plate dynamic equations, and the corresponding boundary conditions. The viscous liquid layer movement was assumed as a creeping one. We studied the stationary axisymmetric hydroelastic oscillations problem under harmonic pulsation of liquid pressure, i.e. harmonic vibrations of the upper channel wall. Using the perturbation method, we obtained the system of two integro-differential equations for studying the radial and flexural vibrations of the sandwich plate. Hydroelastic oscillations of the main mode were considered as a special case. The program for calculating the hydroelastic response of circular sandwich plate in various frequency ranges was developed. As an example, we made calculations of the circular sandwich plate amplitude-frequency response. The developed model can be applied in monitoring and decision-making systems for complex mechanical objects.

1. Introduction

Constructions made of multilayered materials are widely used in engineering. The dynamic behavior models development of such structures under various loads and interactions with various environments is extremely important for the creation of automated monitoring and decision-making systems. The review of kinematic theories for studying the deformed state of multilayered structural elements was given in [1]. The statics’ and dynamics’ equations for three-layered beams and plates based on the theory of zigzag normal were presented in [2]. Hydroelastic vibrations problems of homogeneous plates are well represented in scientific literature. For example, hydroelastic vibrations of a circular plate were studied in [3] using the Rayleigh energy method. In references [4–6], the vibrations of the circular plate interacting with an ideal liquid were studied based on the solution of the hydroelasticity problem. Reference [7]...
was devoted the vibrations simulation of a rectangular plate immersed in an ideal incompressible liquid with a free surface. Natural vibrations of a plate floating on the free surface of an ideal incompressible heavy liquid of finite depth were studied in [8]. The plane problem for bending vibrations and stability of a plate that is part of an absolutely rigid wall of a channel transporting an ideal compressible fluid was carried out in [9].

The liquid viscosity influence on hydroelastic vibrations of the circular plate was taken into account in [10]. Reference [11] was devoted to the study of natural hydroelastic vibrations and stability of a rectangular plate, which is the wall of a channel filled with an ideal compressible liquid. In [12], free bending vibrations of cantilever plates partially immersed in an ideal incompressible liquid with a free surface were studied. The oscillations of the circular plate resting on an elastic foundation and interacting with a viscous liquid layer were studied in [13]. The dynamics and stability of the plate, which is part of the wall separating two viscous liquids, were considered in [14]. Nevertheless, there are much fewer studies of the multilayered plates interacting with liquid. We can point at the references [15–19], in which the interactions of composite plates and beams with liquid were studied. However, the above-mentioned papers did not study radial and bending vibrations of the circular three-layered plate taking into account an influence of normal and shear stresses of a viscous liquid. Therefore, the mathematical model of the circular sandwich plate hydroelastic response, taking into account the above aspects, is relevant for the development of monitoring and decision-making systems.

2. Statement of the Problem

Let us consider two coaxial paralleled disks forming a gap filled with a viscous fluid. The upper disk is a rigid one vibrating along the symmetry axis. The bottom disk is a three-layered structure composed of face sheets from steel and uncompressible lightweight core between them, i.e. a bottom disk is a circular sandwich plate. We introduce a cylindrical coordinate system with the center located in the core center. The geometric dimensions of the gap are shown in figure 1. Here \( R \) is a radius of upper and bottom disks, \( h_0 \) is a clearance between the gap walls in unperturbed state, \( z_m \) is the amplitude of the upper disk vibrations, \( 2c \) is a core thickness, \( h_1 \) is a top face sheet thickness, \( h_2 \) is a bottom face sheet thickness. Further, we assume that \( h_0 \ll R \) and \( z_m \ll h_0 \). The upper disk vibrates harmonically with frequency \( \omega \). The three-layered disk vibrates due to the pressure pulsation of liquid caused by its being squeezed by an upper rigid disk.

![Figure 1. The gap formed by two coaxial parallel disks: 1 – rigid disk, 2 – three-layered circular plate, 3 – viscous liquid.](image)

Taking into account inertia forces of the plate in the radial and normal directions, according to the reference [2], we obtained the dynamics’ equations for the circular sandwich plate in the
form:

\[
L_2 \left( a_1 u + a_2 \varphi - a_3 \frac{\partial w}{\partial r} \right) - M_0 \frac{\partial^2 u}{\partial t^2} = -q_{zr},
\]

\[
L_2 \left( a_2 u + a_4 \varphi - a_5 \frac{\partial w}{\partial r} \right) = 0,
\]

\[
L_3 \left( a_3 u + a_5 \varphi - a_6 \frac{\partial w}{\partial r} \right) - M_0 \frac{\partial^2 w}{\partial t^2} = -q_{zz},
\]

\[
L_2(g) = \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (rg) \right], \quad L_3(g) = \frac{1}{r} \frac{\partial}{\partial r} [rL_2(g)],
\]

\[
q_{zr} = \rho \nu \left( \frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z} \right) \quad \text{at } z = c + h_1,
\]

\[
q_{zz} = -p + 2 \rho \nu \frac{\partial V_z}{\partial z} \quad \text{at } z = c + h_1,
\]

\[
M_0 = \rho_1 h_1 + \rho_2 h_2 + \rho_3 2c,
\]

\[
a_1 = h_1 K_1^+ + h_2 K_2^+ + 2c K_3^+, \quad a_2 = c (h_1 K_1^+ - h_2 K_2^+),
\]

\[
a_3 = h_1 \left( c + \frac{1}{2} h_1 \right) K_1^+ - h_2 \left( c + \frac{1}{2} h_2 \right) K_2^+,
\]

\[
a_4 = c^2 \left( h_1 K_1^+ + h_2 K_2^+ + \frac{2}{3} c K_3^+ \right),
\]

\[
a_5 = c \left( h_1 \left( c + \frac{1}{2} h_1 \right) K_1^+ + h_2 \left( c + \frac{1}{2} h_2 \right) K_2^+ + \frac{2}{3} c^2 K_3^+ \right),
\]

\[
a_6 = h_1 \left( c^2 + c h_1 + \frac{1}{3} h_1^2 \right) K_1^+ + h_2 \left( c^2 + c h_2 + \frac{1}{3} h_2^2 \right) K_2^+ + \frac{2}{3} c^3 K_3^+,
\]

\[
K_k^+ = K_k + \frac{4}{3} G_k.
\]

Here \( u \) is a radial plate displacement; \( w \) is a plate deflection; \( \varphi \) is a rotation angle of deformed normal in the plate core; \( q_{zr} \) and \( q_{zz} \) are shear and normal liquid stresses acting on the top plate surface, respectively; \( V_r \) and \( V_z \) are liquid velocity projections on the coordinate system axes; \( G_k \) is a shear modulus of the \( k \)-th layer; \( K_k \) is a bulk modulus of the \( k \)-th layer; \( \rho_k \) is a density of the \( k \)-th layer material. The expressions for \( a_1, \ldots, a_6 \) were obtained in [2].

We assume that liquid movement between the gap walls is a creeping one [20]. Thus, dynamics’ equations for viscous liquid can be written as:

\[
1 \frac{\partial p}{\rho \partial r} = \nu \left( \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial z^2} - \frac{V_r}{r^2} \right),
\]

\[
1 \frac{\partial p}{\rho \partial z} = \nu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial z^2} \right),
\]

\[
\frac{\partial V_r}{\partial r} + \frac{1}{r} V_r + \frac{\partial V_z}{\partial z} = 0.
\]

Here \( p \) is a liquid pressure, \( \nu \) is a kinematic viscosity coefficient of the fluid, \( \rho \) is a fluid density.
The boundary conditions of Eqs. (1), (2) are:

\[ w = u = \varphi = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R, \]
\[ r \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = 0 \]
\[ V_r = 0, \quad V_z = \frac{dz^*}{dt} \quad \text{at} \quad z = h_0 + c + h_1, \]
\[ V_r = \frac{\partial u}{\partial t}, \quad V_z = \frac{\partial w}{\partial t} \quad \text{at} \quad z = c + h_1, \]
\[ p = p_0 \quad \text{at} \quad r = R, \]
\[ r \frac{\partial p}{\partial r} = 0 \quad \text{at} \quad r = 0, \]

where \( z^* = h_0 + z_m f(\omega t) \) is the rigid disk motion law, where \( \omega \) is the rigid disk vibration frequency.

3. Determining the Plate Response

Taking into account the gap narrowness, i.e. \( h_0/R \ll 1 \), we can assume that:

\[ V_z \ll V_r, \quad \frac{\partial^2 V_r}{\partial z^2} \gg \frac{\partial^2 V_z}{\partial r^2}, \quad \frac{\partial^2 V_k}{\partial z^2} \gg \frac{\partial^2 V_z}{\partial r^2}, \quad \frac{\partial^2 V_k}{\partial z^2} \gg \frac{\partial^2 V_z}{\partial z^2}, \]

as a result of Eqs. (2) are written as:

\[ \frac{1}{\nu \rho} \frac{\partial p}{\partial r} = \frac{\partial^2 V_r}{\partial z^2}, \]
\[ \frac{1}{\nu \rho} \frac{\partial p}{\partial z} = \frac{\partial^2 V_z}{\partial z^2}, \]
\[ \frac{\partial V_r}{\partial r} + \frac{1}{r} V_r + \frac{\partial V_z}{\partial z} = 0. \]

Considering the first and second equations of system (5) and taking into account that \( \frac{\partial^2 V_k}{\partial z^2} \gg \frac{\partial^2 V_k}{\partial z^2} \), we can write \( \frac{\partial p}{\partial z} \ll \frac{\partial p}{\partial r} \) and assume \( \frac{\partial p}{\partial z} = 0 \), therefore, Eqs. (5) take the form:

\[ \frac{1}{\nu \rho} \frac{\partial p}{\partial r} = \frac{\partial^2 V_r}{\partial z^2}, \]
\[ \frac{\partial p}{\partial z} = 0, \]
\[ \frac{\partial V_r}{\partial r} + \frac{1}{r} V_r + \frac{\partial V_z}{\partial z} = 0. \]

According to the above, solving Eqs. (6) with the corresponding boundary conditions (2) we obtained

\[ q_{zz} = -p_0 - \frac{\rho u R^2}{h_0^2} \left( 3(\xi^2 - 1)z_m \frac{df}{dt} + 12 \int_{\xi}^1 \int_{0}^{\xi} \frac{1}{\xi} \int_{\xi}^{1} \frac{\partial w}{\partial t} \, d\xi \right) \, d\xi, \]
\[ q_{zr} = \frac{\rho u R^2}{h_0^2} \left( 6 \int_{0}^{\xi} \frac{\xi \frac{\partial w}{\partial t} \, d\xi}{} - 3 \xi z_m \frac{df}{dt} \right), \]
\[ \xi = \frac{r}{R}. \]
Substituting Eqs. (7) in Eqs. (1), we obtained the system of integro-differential equations for studying radial and bending vibrations of the sandwich plate. Further, we study steady-state harmonic oscillations. Taking into account boundary conditions (3), elastic displacements $u$, and $w$, and rotation angle $\varphi$ can be represented as a series of eigenfunctions for the Sturm – Liouville problem:

$$w = w_m \sum_{k=1}^{\infty} R_k(t) \left[ \frac{J_0(\beta_k \xi)}{J_0(\beta_k)} - \frac{I_0(\beta_k \xi)}{I_0(\beta_k)} \right],$$

$$u = -u_m \sum_{k=1}^{\infty} \beta_k Q_k(t) \left[ \frac{J_1(\beta_k \xi)}{J_0(\beta_k)} + \frac{I_1(\beta_k \xi)}{I_0(\beta_k)} \right],$$

$$\varphi = -\varphi_m \sum_{k=1}^{\infty} \beta_k T_k(t) \left[ \frac{J_1(\beta_k \xi)}{J_0(\beta_k)} + \frac{I_1(\beta_k \xi)}{I_0(\beta_k)} \right],$$

where $J_0$ is a zero-order Bessel function, $J_1$ is a first-order Bessel function; $I_0$ is a modified zero-order Bessel function; $I_1$ is a modified first-order Bessel function; $\beta_k$ is a root of the transcendental equation $I_1(\beta_k)/I_0(\beta_k) = -J_1(\beta_k)/J_0(\beta_k)$ [2].

Finally, we obtained the following radial displacement $u$ and deflection $w$ of the three-layered circular plate for the main mode of oscillations, (i.e. for $k = 1$)

$$u = -\beta_1 \left[ \frac{J_1(\beta_1 \frac{R^3}{R})}{J_0(\beta_1)} + \frac{I_1(\beta_1 \frac{R^3}{R})}{I_0(\beta_1)} \right] \times \left( \frac{p_0 R^3}{b_{21}^4} \frac{2}{\beta_1^4} \frac{J_1(\beta_1)}{J_0(\beta_1)} \frac{b_{12}}{b_{11} b_{22}/b_{21}} - b_{12} \right)_{\omega=0} + z_m A_u(\omega) \sin(\omega t + \phi_u(\omega)),$$

$$w = -\left[ \frac{J_0(\beta_1 \frac{R^3}{R})}{J_0(\beta_1)} - \frac{I_0(\beta_1 \frac{R^3}{R})}{I_0(\beta_1)} \right] \times \left( \frac{p_0 R^3}{b_{21}^4} \frac{2}{\beta_1^4} \frac{J_1(\beta_1)}{J_0(\beta_1)} \frac{b_{11}}{b_{11} b_{22}/b_{21}} - b_{12} \right)_{\omega=0} - z_m A_w(\omega) \sin(\omega t + \phi_w(\omega)),$$

where the following notation is introduced

$$A_u(\omega) = \omega \sqrt{(K_{2z} K_{11} \omega - K_{1z} K_{21} \omega)^2 + (K_{1z} b_{22} - b_{12} K_{22})^2 \left( b_{11} b_{22} - b_{21} b_{12} \right)^2 + (b_{11} K_{21} \omega - b_{21} K_{11} \omega)^2},$$

$$A_w(\omega) = \omega \sqrt{(b_{11} K_{2z} - K_{1z} b_{21})^2 + (b_{11} K_{21} \omega - b_{21} K_{11} \omega)^2},$$

$$b_{11} = (a_1 - a_2^2/a_4), \quad b_{12} = (a_2 a_5/(a_4 R) - a_3/R),$$

$$b_{21} = (a_5 a_2/a_4 - a_3), \quad b_{22} = (a_6 R - a_2^2/(a_4 R) - M_0 \omega^2 R^3 \frac{d_{11}}{\beta_1^4}),$$

$$K_{11} = \frac{6 R^3 \rho \nu}{\beta_1^4} \frac{R^2}{R_0}, \quad K_{1z} = \frac{6 R^3 \rho \nu}{\beta_1^4} \frac{R^2}{R_0},$$

$$K_{2z} = \frac{12 \rho \nu R^5 d_{11}}{h_0 \beta_1^3}, \quad K_{21} = \frac{12 R^5 \rho \nu}{h_0 \beta_1^3},$$

$$d_{11}^1 = \frac{J_1^2(\beta_1)}{J_0^2(\beta_1)} - \frac{4 J_1(\beta_1)}{\beta_1 J_0(\beta_1)}, \quad d_{11}^3 = \frac{J_1^2(\beta_1)/J_0^2(\beta_1) - (4/\beta_1) J_1(\beta_1)}{\beta_1^2}.$$
4. Calculation results

The first terms in (9) determine the radial displacement and deflection of the sandwich plate due to static pressure, and the second terms are the radial displacement and deflection of the sandwich plate due to upper rigid disk vibrations. Therefore, expressions $A_u(\omega)$, $A_w(\omega)$ are dimensionless amplitude frequency responses of radial displacement and deflection of the sandwich plate. We developed the computer program for calculating sandwich plate hydroelastic response based on the mathematical model discussed above. We carried out simulation of $A_u(\omega)$, $A_w(\omega)$ for the following data: $R = 0.1$, $h_0/R = 0.08$, $h_1/R = 0.01$, $h_2/R = 0.015$, $2c/R = 0.02$, $\rho_0 = 10^3$ kg/m$^3$, $\rho_1 = \rho_2 = 2.7 \cdot 10^3$ kg/m$^3$, $\rho_3 = 2.15 \cdot 10^4$ kg/m$^3$, $K_1 = K_2 = 8 \cdot 10^3$ Pa, $K_3 = 4.7 \cdot 10^9$ Pa, $G_1 = G_2 = 2.67 \cdot 10^{10}$ Pa, $G_3 = 9 \cdot 10^7$ Pa, $\nu = 10^{-6}$ m$^2$/sec. The calculation results of $A_u(\omega)$, $A_w(\omega)$ are presented in figure 2. The calculation results showed the occurrence of two resonant frequencies with significant vibration amplitudes of the sandwich plate for both radial displacements and deflection.

5. Summary and Conclusion

We have formulated the mathematical model for studying the circular sandwich plate hydroelastic vibrations due to pressure pulsation in liquid caused by vibrations of the upper gap wall. Using this model, we developed a program for calculating and analyzing the sandwich plate amplitude-frequency response. Our mathematical modeling showed mutual influence of plate stiffness and inertia in radial and normal directions, as well as importance of taking into account shear stresses from the liquid layer. This outcome is based on the emergence of resonant frequencies determined by stiffness and inertia in radial and normal direction on both amplitude-frequency responses $A_u(\omega)$ and $A_w(\omega)$. Therefore, we came to the conclusion that studying hydroelastic oscillations of sandwich plates, in contrast to homogeneous plates [3-14], it is necessary to take into account both radial and normal inertia forces, as well as shear and normal viscous liquid layer stresses. Thus, the results obtained in this work can be used in the development of various mechanical engineering products and in the analysis of their work. For example, they can be used in monitoring and decision-making systems for complex mechanical
objects consisting of hydraulic systems, lubrication and cooling systems, as well as computer modeling in decision making on the acceptable frequency range of vibrations the above objects.

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