ON A MULTIPLIER OPERATOR INDUCED BY THE
SCHWARZIAN DERIVATIVE OF A UNIVALENT FUNCTION

JIANJUN JIN

Abstract. In this paper we study a multiplier operator which is induced by the Schwarzian derivative of a univalent function with a quasiconformal extension to the extended complex plane. We obtain a norm estimate for this operator. As an application, we can prove that the Brennan’s conjecture is true for a large class of quasidisks. We also establish a new characterization of the asymptotically conformal curves and the Weil-Petersson curves in terms of the multiplier operator.

1. Introductions

Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\hat{C} = \mathbb{C} \cup \{\infty\}$, $\Delta^* = \hat{C} \setminus \overline{\Delta}$ be the exterior of $\Delta$ and $S^1 = \partial \Delta = \partial \Delta^*$ be the unit circle. We use the notation $\Delta(z, r)$ to denote the disk centered at $z$ with radius $r$. We use $C_1(\cdot), C_2(\cdot), \cdots$ to denote some positive numbers which depend on the parameters in the bracket.

Let $A(\Delta)$ denote the class of all analytic functions in $\Delta$. We define $B(\Delta)$ as

$$B(\Delta) = \{h \in A(\Delta) : \|h\|_B := \sup_{z \in \Delta} |h(z)|(1 - |z|^2)^2 < \infty\}.$$ 

A closed subspace of $B(\Delta)$, denoted by $B_0(\Delta)$, is defined as

$$B_0(\Delta) = \{h \in B(\Delta) : \lim_{|z| \to 1^-} |h(z)|(1 - |z|^2)^2 = 0\}.$$ 

For $\alpha > 1$, we define the Hilbert space $\mathcal{H}_\alpha(\Delta)$ as

$$\mathcal{H}_\alpha(\Delta) = \{h \in A(\Delta) : \|h\|_{\alpha}^2 := (\alpha - 1) \int_{\Delta} |h(z)|^2(1 - |z|^2)^{\alpha - 2} dxdy < \infty\}.$$ 

Let $f$ be a univalent function in an open domain $\Omega$ of $\mathbb{C}$, i.e., $f$ is a one to one analytic function in $\Omega$. The Schwarzian derivative $S_f$ of $f$ (see [9, Chapter II]) is defined as

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)}\right)^2.$$ 

It is known that

$$|S_f(z)|(1 - |z|^2)^2 \leq 6, z \in \Delta,$$

for any univalent function $f$ in $\Delta$, hence $S_f \in B(\Delta)$.

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Let $f$ be a univalent function in $\Delta$. The \textit{multiplier operator} $M_f$, induced by the Schwarzian derivative of $f$, is defined as

$$M_f(h)(z) := S_f(z)h(z), \ h \in A(\Delta).$$

Let $t \in \mathbb{R}$. We define the \textit{integral means spectrum} $\beta_f(t)$ as the infimum of those numbers $\gamma > 0$ that there exists $C(f, \gamma) > 0$ such that

$$I_t(f', r) = \int_0^{2\pi} |f'(re^{i\theta})|^r|d\theta| \leq \frac{C(f, \gamma)}{(1 - r)^\gamma}, \ \text{for} \ r \in (0, 1).$$

The \textit{universal integral means spectrum} $B(t)$ is then the supremum of $\beta_f(t)$ taken over all univalent functions $f$ in $\Delta$.

The famous Brennan’s conjecture states that $B(-2) = 1$. It is known that the Brennan’s conjecture is true for some special types of domains, see [6, page 286]. In this note, we can show that the Brennan’s conjecture is true for a large class of quasidisks. This work is motivated by the paper [16]. Our arguments are very simple and clear.

We recall basic definitions and properties of the theory of quasiconformal mappings. We say a homeomorphism $f$ extends to a quasiconformal mapping in $\Omega$. If $g$ be a quasiconformal mapping from one open domain $\Omega$ to another, is a quasiconformal mapping if it has locally integral distributional derivatives and satisfies the Beltrami equation $\bar{\partial} f = \mu_f \partial f$ with $||\mu_f||_\infty = \sup_{z \in \Omega} |\mu_f(z)| < 1$. Here, the function $\mu_f$ is called the complex dilatation of $f$ and

$$\bar{\partial} f := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f, \ \partial f := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f.$$

Let $f$ be a quasiconformal mapping from one open domain $\Omega_1$ to another domain $\Omega_2$. If $g$ is another quasiconformal mapping from $\Omega_2$ to $\Omega_3$. Then the complex dilatations of $f$ and $g \circ f$ satisfy the following chain rule.

$$(1.1) \quad \mu_{g \circ f}(z) = \frac{\mu_f + (\mu_g \circ f(z))\kappa}{1 + \mu_f(\mu_g \circ f(z))\kappa}, \ \kappa = \frac{\bar{\partial} f}{\partial f}.$$ 

We say a Jordan curve $\Gamma$ in $\hat{\mathbb{C}}$ is a quasicleve if there is a quasiconformal mapping $f$ from $\hat{\mathbb{C}}$ to itself such that $f(S^1) = \Gamma$. The domain $f(\Delta)$ is called a quasidisk. For more detailed introduction for the theory of quasiconformal mappings, see [10] or [9].

The following result is the first result of our paper.

\textbf{Theorem 1.1.} For $\alpha > 1$. Let $f$ be a univalent function in $\Delta$, which can be extended to a quasiconformal mapping (still denoted by $f$) in $\mathbb{C}$. Let $||\mu_f||_\infty = k \in [0, 1)$. Then, for any $h \in \mathcal{H}_\alpha(\Delta)$, it holds that

$$||M_f(h)||_{\alpha+4}^2 = ||S_f(z)h(z)||_{\alpha+4}^2 \leq \frac{36(\alpha + 1)k^2}{(\alpha - 1)} ||h(z)||_{\alpha}^2.$$

\textbf{Remark 1.2.} We notice that, for a univalent function $f$ in $\Delta$, which can be extended to a quasiconformal mapping in $\hat{\mathbb{C}}$ with $||\mu_f||_\infty = k \in [0, 1)$, it holds that

$$\sum_{m=1}^{\infty} m \sum_{n=1}^{\infty} \gamma_{mn} \lambda_n^2 \leq k^2 \sum_{n=1}^{\infty} |\lambda_n|^2, \ \lambda_n \in \mathbb{C}.$$

Where $\gamma_{mn}$ are the Grunsky’s coefficients, see [11] Chapter 9. Then, by repeating the arguments of the proofs of Lemma 4 and Theorem 1 in [16] word by word, we can show that Theorem 1.1 is true.
We can use Theorem 1.1 to show that

**Theorem 1.3.** Let \( f \) be a univalent function in \( \Delta \), which can be extended to a quasiconformal mapping (still denoted by \( f \)) in \( \hat{\mathbb{C}} \). If \( \| \mu_f \|_\infty \leq \sqrt{\frac{5}{8}} \approx 0.79056 \), then the Brennan’s conjecture is true for the quasidisk \( f(\Delta) \).

**Remark 1.4.** Let \( t \in \mathbb{C} \). In [7], Hedenmalm proved that, for any univalent function \( f \), which can be extended to a quasiconformal mapping in \( \hat{\mathbb{C}} \) with \( \| \mu_f \|_\infty = k \in (0,1) \), it holds that

\[
\beta_f(t) \leq \frac{1}{4} k^2 |t|^2 (1 + 7k)^2, \quad \text{when } |t| \leq \frac{2}{k(1 + 7k)^2},
\]

and

\[
\beta_f(t) \leq k|t| - \frac{1}{(1 + 7k)^2}, \quad \text{when } |t| \geq \frac{2}{k(1 + 7k)^2}.
\]

We consider \( t = -2 \). (1) When \( k(1 + 7k)^2 \leq 1 \), i.e., \( k \in (0,k_0) \), here \( k_0 \approx 0.18726 \) is the real root of the equation \( k(1 + 7k)^2 = 1 \). Then we see that, in this case, \( \beta_f(-2) \leq 1 \) and the Brennan’s conjecture is true for the quasidisk \( f(\Delta) \).

(2) When \( k(1 + 7k)^2 \geq 1 \), i.e., \( k \in [k_0,1) \), we see that

\[
\beta_f(-2) \leq 2k - \frac{1}{(1 + 7k)^2} = \frac{2k(1 + 7k)^2 - 1}{(1 + 7k)^2}.
\]

Thus, if \( k_0 \leq k < 1 \) and \( \frac{2k(1 + 7k)^2 - 1}{(1 + 7k)^2} \leq 1 \), i.e., \( k_0 \leq k \leq k_1 \approx 0.52301 \), here \( k_1 \) is the real root of the equation \( \frac{2k(1 + 7k)^2 - 1}{(1 + 7k)^2} = 1 \), then the Brennan’s conjecture is true for the quasidisk \( f(\Delta) \).

Consequently, we see that, if \( 0 \leq k \leq k_1 \approx 0.52301 \), then the Brennan’s conjecture is true for the quasidisk \( f(\Delta) \). Our result is better than the one of Hedenmalm in [7].

The paper is organized as follows. We will present the proof of Theorem 1.3 in Section 2. In Section 3, we establish a new characterization of the asymptotically conformal curves and the Weil–Petersson curves in terms of the multiplier operator. In the final section, Section 4, we will add some remarks.

## 2. Proof of Theorem 1.3

We need the following result established by Shimorin in [16].

**Proposition 2.1.** If, for certain univalent function \( f \) in \( \Delta \), the inequality

\[
\| S_f(z)h(z) \|_{\alpha+4}^2 \leq \frac{36(\alpha + 1)(\alpha + 3)}{\alpha(\alpha + 2)} \| h(z) \|_{\alpha}^2
\]

holds for any \( \alpha > 2 \) and \( h \in H_\alpha(\Delta) \). Then \( \beta_f(-2) \leq 1 \).

**Remark 2.2.** Proposition 2.1 is Proposition 8 of [16] and it means that, if some univalent function \( f \) satisfies the conditions of Proposition 2.1, then \((f')^{-1}\) belongs to \( H_\alpha \) for any \( \alpha > 2 \).

The following lemma also will be used in the later.

**Lemma 2.3.** Let \( f \) be a univalent function in \( \Delta \), which can be extended to a quasiconformal mapping (still denoted by \( f \)) in \( \hat{\mathbb{C}} \). If \( j \) is another univalent function from \( \Delta \) to \( f(\Delta) \), then \( j \) also can be extended to a quasiconformal mapping (still denoted by \( j \)) in \( \hat{\mathbb{C}} \) with \( \| \mu_j \|_\infty = \| \mu_f \|_\infty \).


Proof. We first notice that $j^{-1} \circ f|_\Delta = \sigma|_\Delta$, here $\sigma$ is a Möbius transformation. Since $\sigma \circ f^{-1}$ is a quasiconformal mapping from $\hat{C}$ to itself, we see that $[\sigma \circ f^{-1}]^{-1}$ is a quasiconformal extension of $j$ to $\hat{C}$.

Noting that $j^{-1}|_{\hat{C}(\Delta)} = \sigma \circ f^{-1}|_{\hat{C}(\Delta)}$, we see that $\mu_{j^{-1}}(z) = \mu_{f^{-1}}(z)$, $z \in \mathbb{C} \setminus f(\Delta)$. On the other hand, we have

$$|\mu_f(z)| = |\mu_{f^{-1}}(f(z))| \quad \text{and} \quad |\mu_j(z)| = |\mu_{j^{-1}}(j(z))|, \quad z \in \mathbb{C} \setminus \Delta.$$ 

It follows that $\|\mu_j\|_\infty = \|\mu_f\|_\infty$. The lemma is proved.  

Now, we start to prove Theorem 1.3. By combining Theorem 1.1, Proposition 2.1 and Lemma 2.3, we see that, if $\mu_f$ is a quasiconformal extension of $f$ to $\hat{C}$.

We shall prove that $\|\mu_f\|_{\infty} \leq 36(\alpha+1)/(\alpha-1)$ for any $\alpha > 2$. This means that we have proved Theorem 1.3.

3. A NEW CHARACTERIZATION OF THE ASYMPTOTICALLY CONFORMAL CURVES AND THE WEIL–PETTERSON CURVES

Let $f$ be a univalent function from $\Delta$ to a bounded Jordan domain in $\mathbb{C}$. We say $f(S^1)$ is an asymptotically conformal curve, if $f$ can be extended to a quasiconformal mapping in $\hat{C}$ and whose complex dilatation $\mu_f$ satisfies that $\mu_f(z) \to 0$, $|z| \to 1^+$. We say $f(S^1)$ is a Weil-Petersson curve, if $f$ can be extended to a quasiconformal mapping in $\hat{C}$ and whose complex dilatation $\mu_f$ of $f$ satisfies that $\int_{\Delta} |\mu_f(z)|^2/(|z| - 1)^2 \, dxdy < \infty$. See [4], [12], [8], [13], [14].

It is known that $f(S^1)$ is an asymptotically conformal curve if and only if

$$S_f(z)(1 - |z|^2)^2 \to 0, \quad |z| \to 1^-.$$ 

Moreover, $f(S^1)$ is a Weil-Petersson curve if and only if

$$\int_{\Delta} |S_f(z)|^2(1 - |z|^2)^2 \, dxdy < \infty.$$ 

We shall prove that

Theorem 3.1. Let $\alpha > 1$. Let $f$ be a univalent function from $\Delta$ to a bounded Jordan domain in $\mathbb{C}$.

(I) $f(S^1)$ is an asymptotically conformal curve if and only if the multiplier operator $M_f$, acting from $H_\alpha(\Delta)$ to $H_{\alpha+4}(\Delta)$, is a compact. Moreover,

(II) $f(S^1)$ is a Weil-Petersson curve if and only if the multiplier operator $M_f$ belongs to the Hilbert-Schmidt class.
Proof of only if part of (I) of Theorem 3.7 Suppose that $f(S^1)$ is an asymptotically conformal curve. To show that $M_f$ is compact operator, it is sufficient to show that $M_f(\psi_n) \to 0$ for each sequence $(\psi_n)$ which converges to zero weakly. It is easy to check that $(\psi_n)$ converges to zero weakly if and only if $(\psi_n)$ is bounded and $(\psi_n)$ converges to zero locally.

On the other hand, we recall that $f(S^1)$ is an asymptotically conformal curve if and only if
\[ S_f(z)(1 - |z|^2)^2 \to 0, \ |z| \to 1^- . \]

Thus, for any $\varepsilon > 0$, there exists some $r \in (0, 1)$ such that $|S_f(z)(1 - |z|^2)^2| < \varepsilon$, when $|z| > r$. It follows that, for $\psi \in H_a(\Delta)$, it holds that
\[
\|M_f(\psi)\|_{\alpha+4}^2 = (\alpha + 3) \int_{\Delta} |S_f(z)\psi(z)|^2 (1 - |z|^2)^{\alpha+2} dxdy
\]
\[
= (\alpha + 3) \int_{\Delta} |S_f(z)|^2 (1 - |z|^2)^4 |\psi(z)|^2 (1 - |z|^2)^{\alpha-2} dxdy
\]
\[
\leq 36(\alpha + 3) \int_{|z|<r} |\psi(z)|^2 (1 - |z|^2)^{\alpha-2} dxdy + (\alpha + 3)\varepsilon \|\psi\|_2^2 .
\]

Consequently, we see that $M_f(\psi_n) \to 0$ for each sequence $(\psi_n)$ which converges to zero weakly. The only if part of (I) is proved. □

Proof of if part of (I) of Theorem 3.7 If $M_f$ is a compact operator, we consider the function
\[ \psi_a(z) = \frac{(1 - |a|^2)^{\frac{\alpha}{2}}}{(1 - az)^{\alpha}} . \]

It is easy to see that $\psi_a(z) \in H_a(\Delta)$ and $\psi_a(z)$ tends to zero locally uniformly in $\Delta$ when $|a| \to 1^-$. Then we conclude that $\psi_a$ converges to zero weakly. Thus, $M_f(\psi_a) \to 0$ as $|a| \to 1^-$. That is
\[
\lim_{|a| \to 1^-} \int_{|a| < 1} |S_f(z)|^2 \frac{(1 - |a|^2)^{\alpha}(1 - |z|^2)^{\alpha+2}}{|1 - az|^{2\alpha}} dxdy = 0 .
\]

For any $a \in \Delta$. Noting that $|S_f(z)|^2$ is a subharmonic function. Then there is a $r(a) \in (0, 1)$ such that the disk $\Delta(a, r(a)) := \{|z - a| \leq r(a)\}$ is contained in $\Delta$ and such that for every $r < r(a)$, we have
\[
|S_f(a)|^2 \leq \frac{1}{\pi r^2} \int_{|z-a|<r} |S_f(z)|^2 dxdy .
\]

Then, we can choose a (fixed) $l \in (0, 1)$ such that disk $\Delta(a, l(1 - |a|)) = \{|z - a| \leq l(1 - |a|)\}$ is contained in $\Delta(a, r(a))$. We see that, for $z \in \Delta(a, l(1 - |a|))$, it holds that
\[
(3.2) \ |1 - l(1 - |a|)| \leq 1 - |z| \leq (1 + l)(1 - |a|), \quad (1 - |a|) \leq |1 - az| \leq (2 + l)(1 - |a|) .
\]

It follows that
\[
(3.3) \ \frac{(1 - |a|^2)^{\alpha}(1 - |z|^2)^{\alpha}}{|1 - az|^{2\alpha}} \geq \frac{(1 - l)^{\alpha}}{(2 + l)^{2\alpha}}
\]
for $z \in \Delta(a, l(1 - |a|))$.

On the other hand, by \(3.2\) we have
\[
|S_f(a)|^2 (1 - |a|^2)^2 \leq \frac{4}{\pi l^2} \int_{|z-a|<l(1-|a|)} |S_f(z)|^2 dxdy .
\]
It follows from (3.3) that

\[(3.5) \quad |S_f(a)|^2(1 - |a|^2)^4 \leq \frac{(1 - l)^\alpha}{2^\alpha \pi l^2} \int_{|z-a|<l(1-|a|)} |S_f(z)|^2(1 - |z|^2)^2 \, dx \, dy.\]

Combining (3.4), (3.5), we get that there is a constant \(C(l, \alpha) > 0\) such that

\[(3.6) \quad |S_f(a)|^2(1 - |a|^2)^4 \leq C(l, \alpha) \int_{\Delta} \frac{(1 - |a|^2)^\alpha (1 - |z|^2)^{\alpha + 2}}{|1 - az|^{2\alpha}} \, dx \, dy.\]

Thus, from (3.1), we see that \(S_f(a)(1 - |a|^2)^2\to 0\), \(|a|\to 1^+\). This means \(f(S^1)\) is an asymptotically conformal curve. The if part of (I) is proved and we have shown that (I) is true. \(\square\)

Next, we will show (II) of Theorem 3.1 and finish the proof of this Proposition. For nonnegative number \(n\), let

\[e_n(z) = \sqrt{\frac{\Gamma(n + \alpha)}{n! \Gamma(\alpha)}} z^n, \quad z \in \Delta.\]

Here, \(\Gamma(s)\) stands for the usual Gamma function. It is easy to check that \(\{e_n\}\) is an orthonormal set in \(\mathcal{H}_\alpha(\Delta)\). We know that \(M_f\) belongs to the Hilbert-Schmidt class if and only if

\[
\sum_{n=0}^{\infty} \|M_f(e_n)\|_{\alpha+4}^2 < \infty.
\]

It follows from

\[
\frac{1}{(1 - |z|^2)^\tau} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha)}{n! \Gamma(\alpha)} |z|^{2n}, \quad \tau > 0, \quad z \in \Delta,
\]

that

\[
\sum_{n=0}^{\infty} \|M_f(e_n)\|_{\alpha+4}^2 = (\alpha + 3) \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha)}{n! \Gamma(\alpha)} \int_{\Delta} |S_f(z)|^2 \left(1 - |z|^2\right)^{\alpha + 2} \, dx \, dy
\]

\[
= (\alpha + 3) \int_{\Delta} |S_f(z)|^2 \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha)}{n! \Gamma(\alpha)} |z|^{2n} \left(1 - |z|^2\right)^{\alpha + 2} \, dx \, dy
\]

\[
= (\alpha + 3) \int_{\Delta} |S_f(z)|^2 (1 - |z|^2)^2 \, dx \, dy.
\]

On the other hand, we recall that \(f(S^1)\) is a Weil-Petersson curve if and only if

\[
\int_{\Delta} |S_f(z)|^2 (1 - |z|^2)^2 \, dx \, dy < \infty.
\]

Thus, we see that \(f(S^1)\) is a Weil-Petersson curve if and only if the multiplier operator \(M_f\) belongs to the Hilbert-Schmidt class. (II) is proved and the proof of Theorem 3.1 is complete.
4. Final Remarks

Remark 4.1. Let $f$ be a univalent function from $\Delta$ to a bounded Jordan domain in $\mathbb{C}$. We first remark that the Brennan's conjecture is also true for the quasidisk $f(\Delta)$ when $f(S^1)$ is an asymptotically conformal curve.

Assume that $f(S^1)$ is an asymptotically conformal curve and let $\alpha > 2$ be fixed. We will show that $(f')^{-1} \in H_{\alpha}(\Delta)$. The following lemma established in [10] will be used in our arguments.

Lemma 4.2. A function $h \in H_{\alpha}(\Delta)$ if and only if $h' \in H_{\alpha+2}(\Delta)$. Moreover, for any $\varepsilon$ such that $0 < \varepsilon < \alpha(\alpha + 1)$,

\begin{equation}
\|h\|^2_{\alpha+2} \leq \alpha(\alpha + 1 + \varepsilon)\|h\|^2_{\alpha} + C_1(h, \varepsilon);
\end{equation}

\begin{equation}
\|h\|^2_{\alpha} \leq \frac{1}{\alpha(\alpha + 1 - \varepsilon)}\|h'\|^2_{\alpha+2} + C_2(h, \varepsilon),
\end{equation}

where, the constants $C_1(h, \varepsilon)$ and $C_2(h, \varepsilon)$ depend only on finitely many first Taylor coefficients of the function $h$.

First, we have

\begin{equation}
- \frac{d^4}{dz^4} [(f')^{-1}] = \frac{d}{dz} [S_f(z)(f')^{-1}] + S_f(z) \frac{d}{dz} [(f')^{-1}].
\end{equation}

Since $f(S^1)$ is an asymptotically conformal curve, we have $S_f \in B_0(\Delta)$, i.e., $S_f(z)(1 - |z|^2)^2 \to 0$, $|z| \to 1^-$. Then we see that there is a constant $R > 0$, such that

\begin{equation}
\int \int_{A_R} |S_f(z)h(z)|^2(1 - |z|^2)^{\alpha+2} dxdy \leq \frac{1}{2(\alpha + 3)} \|h(z)\|^2_{\alpha}
\end{equation}

holds for any $\alpha > 2$ and $h \in H_{\alpha}(\Delta)$. Here $A_R := \Delta \setminus \Delta(0, R)$ is an annulus. The constant $\frac{1}{2(\alpha + 3)}$ in [10] is not essential, which is chosen just for the simplification of our arguments.

We consider $f_r(z) = f(rz)$, $r \in (0, 1)$. We see that $f'_r = rf'(rz)$ and $S_{f_r}(z) = r^2 S_f(rz)$. Then, for $r \in (0, 1)$, $h \in A_{\alpha}(\Delta)$, we get that

\begin{align*}
\int \int_{A_R} |S_{f_r}(z)h(rz)|^2(1 - |z|^2)^{\alpha+2} dxdy & \\
\leq & \int \int_{A_R} |S_f(rz)h(rz)|^2(1 - |z|^2)^{\alpha+2} dxdy \\
= & \int \int_{A_R} |S_f(rz) - S_f(z) + S_f(z)|^2 h(rz)^2(1 - |z|^2)^{\alpha+2} dxdy \\
\leq & \int \int_{A_R} 2[|S_f(rz)|^2 + |S_f(z)|^2] |h(rz)|^2(1 - |z|^2)^{\alpha+2} dxdy \\
\leq & \frac{1}{\alpha + 3} \|h(rz)\|^2 + \Pi(h, \alpha, r). \end{align*}

Here,

$$
\Pi(h, \alpha, r) = 2 \int \int_{A_R} |S_f(rz) - S_f(z)|^2 h(rz)^2(1 - |z|^2)^{\alpha+2} dxdy \\
\leq 2 \|S_f(rz) - S_f(z)\|^2 \|h(rz)\|^2_{\alpha}.
$$
Noting that $S_f \in B_0(\Delta)$, we get that from [18] that
\[
\|S_f(rz) - S_f(z)\|_{B} \to 0, \text{ as } r \to 1^{-}.
\]

Then, it is easy to see that
\[
\Pi(h, \alpha, r) \leq C_3(f, r)\|h(rz)\|_{\alpha}^2, \quad \text{here } C_3(f, r) \to 0, \text{ as } r \to 1^{-}.
\]

Here $C_3(f, r)$ can take $2\|S_f(rz) - S_f(z)\|_{B}$. Thus, we obtain that
\[
\|S_f(z)h(rz)\|_{\alpha + 4}^2 = (\alpha + 3) \int_{\Delta} |S_f(z)h(rz)|^2 (1 - |z|^2)^{\alpha + 2} dx dy \\
\leq \|h(rz)\|_{\alpha}^2 + (\alpha + 3)C_3(f, r)\|h(rz)\|_{\alpha}^2 + C_4(h, \alpha)
\]
(4.5)

Next, we suppose that $r > 1/2$. By (4.2) and (4.3), we have
\[
[a(a + 1)(a + 2)(a + 3)(a + 4)(a + 5) - \varepsilon]\|f'_r\|_{\alpha}^2
\]
\[
\leq \|d^3 [f'_r]^{-1}\|_{\alpha + 6}^2 + C_5(f, \varepsilon)
\]
(4.6)
\[
\|S_f(z)\|_{\alpha + 6}
\]
\[
\leq \sqrt{(a + 4)(a + 5) + \varepsilon}\|S_f(z)f'_r\|_{\alpha + 4} + C_6(f, \varepsilon)
\]
\[
\leq \sqrt{(a + 4)(a + 5) + \varepsilon \cdot \sqrt{1 + (a + 3)C_3(f, r)}\|f'_r\|_{\alpha}^2 + C_7(f, a) + C_6(f, \varepsilon)}
\]
\[
\leq \sqrt{(a + 4)(a + 5) + \varepsilon \cdot \sqrt{1 + (a + 3)C_3(f, r)}\|f'_r\|_{\alpha}^2 + C_8(f, a, \varepsilon)}.
\]

and
\[
\|S_f(z)\|_{\alpha + 6}
\]
\[
\leq \sqrt{1 + (a + 3)C_3(f, r)}\|d^2 [f'_r]^{-1}\|_{\alpha + 2}^2 + C_9(f, a)
\]
(4.8)
\[
\leq \sqrt{1 + (a + 3)C_3(f, r)}[a(a + 1) + \varepsilon]\|f'_r\|_{\alpha}^2 + C_{10}(f, a, \varepsilon).
\]

Combining (4.6), (4.7), (4.8), we obtain that
\[
[a(a + 1)(a + 2)(a + 3)(a + 4)(a + 5) - \varepsilon]\|f'_r\|_{\alpha}^2
\]
\[
\leq 2 \left\{ \sqrt{(a + 4)(a + 5) + \varepsilon \cdot \sqrt{1 + (a + 3)C_3(f, r)}} \right. \\
+ \sqrt{1 + (a + 3)C_3(f, r)}[a(a + 1) + \varepsilon] \right\} \|f'_r\|_{\alpha}^2 + C_{11}(f, a, \varepsilon).
\]

Let
\[
A(a, \varepsilon) = a(a + 1)(a + 2)(a + 3)(a + 4)(a + 5) - \varepsilon;
\]
\[
B(f, a, r, \varepsilon) = \sqrt{(a + 4)(a + 5) + \varepsilon \cdot \sqrt{1 + (a + 3)C_3(f, r)});
\]
\[
C(f, a, r, \varepsilon) = \sqrt[2]{1 + (a + 3)C_3(f, r)}[a(a + 1) + \varepsilon].
\]
When $\varepsilon$ small enough, and $r \to 1^-$ (so that $C_2(f, r) \to 0$), for fixed $\alpha > 2$, it is not hard to check that
\[
\mathbf{D}(f, \alpha, r, \varepsilon) = A(\alpha, \varepsilon) - 2B(f, \alpha, r, \varepsilon) + C(f, \alpha, r, \varepsilon))^2 > 0.
\]
It follows that
\[
\| (f')^{-1} \|_{\alpha}^2 \leq \frac{C_{11}(f, \alpha, \varepsilon)}{\mathbf{D}(f, \alpha, r, \varepsilon)}.
\]
On the other hand, by Fatou’s lemma, we have $\| (f')^{-1} \|_{\alpha}^2 \leq \lim_{r \to 1^-} \| (f_r')^{-1} \|_{\alpha}^2$. Consequently, we get that $(f')^{-1} \in \mathcal{H}_\alpha(\Delta)$ for any fixed $\alpha > 2$. Thus, we see that the Brennan’s conjecture is true for the quasidisk $f(\Delta)$ when $f(S^1)$ is an asymptotically conformal curve.

Remark 4.3. We finally consider the multiplier operator acting on the Hardy space $H^2(\Delta)$ in $\Delta$, i.e.,
\[
\mathbf{M}_f(h)(z) := S_f(z)h(z), \ h \in H^2(\Delta).
\]
The Hardy space $H^2(\Delta)$ (see [3]) is defined as
\[
H^2(\Delta) = \{ h \in A(\Delta) : \sup_{r \in (0,1)} \int_{0}^{2\pi} |h(re^{i\theta})|^2 d\theta < \infty \}.
\]
We define the operator $\mathbf{E}_f$ acting from $H^2(\Delta)$ to $L^2_\alpha(\Delta, (1 - |z|^2)^3)$ as
\[
\mathbf{E}_f(h)(z) = \iint_{\Delta} |\mathbf{M}_f(h)|^2 (1 - |z|^2)^3 \, dx \, dy = \iint_{\Delta} |S_f(z)h(z)|^2 (1 - |z|^2)^3 \, dx \, dy, \ h \in H^2(\Delta).
\]
Here, $L^2_\alpha(\Delta, (1 - |z|^2)^3)$ is defined as
\[
L^2_\alpha(\Delta, (1 - |z|^2)^3) = \{ h \in A(\Delta) : \iint_{\Delta} |h(z)|^2 (1 - |z|^2)^3 \, dx \, dy < \infty \}.
\]
Let $f$ be a univalent function from $\Delta$ to a Jordan domain in $\hat{C}$. We say $f(S^1)$ is a Bishop-Jones curve, if $f$ can be extended to a quasiconformal mapping in $\hat{C}$ and whose complex dilatation $\mu_f$ satisfies that $|\mu_f(z)|^2/(|z|^2 - 1) \, dx \, dy$ is a Carleson measure in $\Delta^*$. See [1], [15].

Let $f$ be a bounded univalent function from $\Delta$ to a bounded Jordan domain in $\hat{C}$. We say $f(S^1)$ is an asymptotically smooth curve, if $f$ can be extended to a quasiconformal mapping in $\hat{C}$ and whose complex dilatation $\mu_f$ satisfies that $|\mu_f(z)|^2/(|z|^2 - 1) \, dx \, dy$ is a compact Carleson measure in $\Delta^*$. See [12], [15].

Let $\Omega = \Delta$ or $\Delta^*$ and let $\nu$ be a positive measure on $\Omega$. Here, we say $\nu$ is a Carleson measure on $\Omega$ if
\[
\sup_{I \subseteq S^1} \frac{\nu(S_\Omega(I))}{|I|} < \infty.
\]
Moreover, we say a Carleson measure $\nu$ on $\Omega$ is compact if it satisfies that
\[
\lim_{|I| \to 0} \frac{\nu(S_\Omega(I))}{|I|} = 0,
\]
where $I$ is a subarc of $S^1$, and the Carleson box $S_\Delta(I)$ on $\Delta$ and $S_{\Delta^*}(I)$ on $\Delta^*$, based on $I$, are defined as
\[
S_\Delta(I) = \{ z = re^{i\theta} : 1 - |I| \leq r < 1, e^{i\theta} \in I \}
\]
and
\[
S_{\Delta^*}(I) = \{ z = re^{i\theta} : 1 < r \leq 1 + |I|, e^{i\theta} \in I \}.
\]
It is known that $f(S^1)$ is a Bishop-Jones curve if and only if $|S_f(z)|^2(1 - |z|^2)^3 dxdy$ is a Carleson measure in $\Delta$, and $f(S^1)$ is an asymptotically smooth curve if and only if $|S_f(z)|^2(1 - |z|^2)^3 dxdy$ is a compact Carleson measure in $\Delta$(here, $f$ is bounded). See [15].

Then we see from the well-known Carleson Embedding Theorem(see [3], [5], or [17]) that

**Proposition 4.4.** (1) Let $f$ be a univalent function from $\Delta$ to a Jordan domain in $\hat{\mathbb{C}}$, which can be extended to a quasiconformal mapping in $\hat{\mathbb{C}}$. Then, $f(S^1)$ is a Bishop-Jones curve if and only if $E_f$ is bounded from $H^2(\Delta)$ to $L^2(\Delta, (1 - |z|^2)^3)$.

(2) Let $f$ be a bounded univalent function from $\Delta$ to a bounded Jordan domain in $\mathbb{C}$. Then, $f(S^1)$ is an asymptotically smooth curve if and only if $E_f$ is compact from $H^2(\Delta)$ to $L^2(\Delta, (1 - |z|^2)^3)$.

We say a bounded Jordan curve $\Gamma$ in $\mathbb{C}$ is a Chord-arc curve if $\Gamma$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \Gamma$,

$$\frac{l(w_1 - w_2)}{|w_1 - w_2|} \leq C.$$ 

Where $l(w_1, w_2)$ is the length of the shorter arc of $\Gamma$ joining $w_1$ to $w_2$. It is known that Chord-arc curves are Bishop-Jones curves. The space of of Chord-arc curves have many applications in analysis, see [6], [2]. It is natural to study the following

**Problem 4.5.** Characterize $S_f(z)$ and $E_f$ when $f(S^1)$ is a Chord-arc curve.

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**School of Mathematics Sciences, Hefei University of Technology, Xuancheng Campus, Xuancheng 242000, P.R.China**

*Email address: jinjjhb@163.com, jin@hfut.edu.cn*