On uniqueness of semi-wavefronts
Diekmann-Kaper theory of a nonlinear convolution equation re-visited

Abstract Motivated by the uniqueness problem for monostable semi-wavefronts, we propose a revised version of the Diekmann and Kaper theory of a nonlinear convolution equation. Our version of the Diekmann-Kaper theory allows 1) to consider new types of models which include nonlocal KPP type equations (with either symmetric or anisotropic dispersal), nonlocal lattice equations and delayed reaction-diffusion equations; 2) to incorporate the critical case (which corresponds to the slowest wavefronts) into the consideration; 3) to weaken or to remove various restrictions on kernels and nonlinearities. The results are compared with those of Schumacher (J. Reine Angew. Math. 316: 54-70, 1980), Carr and Chmaj (Proc. Amer. Math. Soc. 132: 2433-2439, 2004), and other more recent studies.

Keywords Nonlinear convolution equation, nonlocal interaction, monostable nonlinearity, minimal wave, uniqueness, semi-wavefront

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1 Introduction

The main goal of this paper is to develop a version of the fundamental Diekmann and Kaper theory [10][11][12] (the DK theory for short) of a nonlinear convolution equation for the scalar integral equation

\[ \varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)g(\varphi(t-s),\tau)ds, \quad t \in \mathbb{R}, \]  

(1)
in the case of monostable nonlinearity $g$. Throughout the paper $(X,\mu)$ will denote a measure space with finite measure $\mu$, $K(s,\tau) \geq 0$ will be integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} K(s,\tau) ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, will be continuous in $\varphi$ for every fixed $\tau \in X$. When $X$ is just a single point (i.e. $\#X = 1$), equation (1) coincides with the nonlinear convolution equation from [12].

In a biological context, $\varphi$ is the size of an adult population, so we are interested in non-negative solutions of (1). Following the terminology of [22], we call a bounded continuous non-constant solution $\varphi : \mathbb{R} \to \mathbb{R}_+$ semi-wavefront if either $\varphi(-\infty) = 0$ or $\varphi(+\infty) = 0$. We will always assume $\varphi$ to satisfy $\varphi(-\infty) = 0$, since the other case can be easily transformed to this one via the change of variables $\zeta(t) = \varphi(-t)$, with equation (1) assuming the form

$$\zeta(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau) g(\zeta(t-s),\tau) ds, \quad K_1(s,\tau) := K(-s,\tau).$$

We would like to emphasize that the nonlinearity $g$ and semi-wavefronts are generally non-monotone [19] (nevertheless, typically semi-wavefronts are strictly increasing in some vicinity of $-\infty$ [1,13,37]). The non-monotonicity of waves complicates their analysis. For instance, the wave uniqueness is easier to establish within a subclass of monotone solutions [8,23,39].

Actually the ‘largely open uniqueness question’ [6] is central in our research where we follow the scheme elaborated in [12]. This means that after assuming the existence of a semi-wavefront to (1), we study its asymptotic behavior at infinity trying then to demonstrate the wave uniqueness (modulo translation). Similarly to other authors, we work mostly with the first positive eigenvalue $\lambda_l$ of the linearization of (1) at zero. As a consequence, our analysis excludes from the consideration so called “pushed” fronts [13,22,34] associated to the second positive eigenvalue $\lambda_r$. Analogously to [12], the existence of semi-wavefronts to (1) is not investigated here.

There are various motivations to study the above equation, mainly from the theory of traveling waves for nonlinear models (e.g. reaction-diffusion equations with delayed response [1,23,36,38,39], equations with non-local dispersal [2,4,7,8,28,33], lattice systems [6,16,26,30]). Only a few of these models take the simplest form with $\#X = 1$ of (1). Therefore our first goal is to show that the basic framework of [12] can be extended to include much broader class of convolution type equations than it was initially intended. Here is a simple step to create such a general direct extension of results in [12]. It would be interesting to consider further generalizations of (1) in order to include more applications (for example, equations with distributed delays considered in [16,17], see also [25,33,39]). However, we do not pursue this direction in our current work. After all, ours is not the first attempt to expand the DK theory. Schumacher has mentioned, while studying equation

$$c\varphi'(t) = g(\varphi, \mu_c \ast g(\varphi)),$$

the impossibility of transforming it into the form to which the DK theory could be applied [38, p.54]. Instead, Schumacher has developed an approach which is based on guidelines of the DK theory and, at the same time, which is
technically rather different from that in [12]. In particular, in order to extend
the DK uniqueness theorem, Schumacher has used a comparison method for
differential inequalities combined with Nagumo-point argument. In this
respect, his work [33] is very close to the recent contributions [6,7,8,26].

Similarly to [33], the present studies also follow the mainstream of the DK
ideology. In difference with [33] and trying to apply our results to delay-
ed equations (where in general the comparison argument does not work), we
preserve the original idea of the DK theory in the proof of uniqueness. Now,
from the technical point of view our approach to equation (1) differs from the
methods used by Diekmann and Kaper, Schumacher and Carr and Chmaj [4]
in many key points. Even though the logical sequence of results here basically
is the same as in [12], our proofs are essentially different. In particular, we
do not use the Titchmarsh theory of Fourier integrals [12,16] nor we use the
Ikehara Tauberian theorem [4,8,39] in order to obtain asymptotic expansions
of solutions (a necessary key component of each uniqueness proof). We have
found more convenient for our purpose the use of a suitable
\[L^2\]−variant of
the bootstrap argument (as it was suggested by Mallet-Paret in [31, p. 9-10]).

As a consequence of the DK strategy, we also present a non-existence
result and describe properties of the kernel \(K\) which is proved to satisfy
exponential convergence estimates (Mollison’s condition [5]). Here the fulfill-
ment of the Mollison’s condition means that the characteristic function
\[\chi(z) := 1 - \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds\]
is well defined for all \(z\) from some maximal non-degenerate interval (which
can be open, closed, half-closed, finite or infinite). One of the key results of
the theory says that, under rather mild assumptions on \(g,K\) the presence of a
semi-wavefront \(\varphi, \varphi(-\infty) = 0\), guarantees the existence of a minimal
positive zero \(\lambda_l\) to \(\chi(z)\). The spreading properties of some integro-
differential equations with ‘fat-tailed’ kernels were recently considered by Garnier [21].

Next, as it is known the DK and Schumacher uniqueness theorems do not
apply to the critical fronts (when \(\chi(\lambda_l) = \chi'(\lambda_l) = 0\). As an example, let us
consider the nonlocal KPP equation
\[u_t = J * u - u + g(u), \quad x \in \mathbb{R}, \quad g(0) = g(1) = 0, \quad f > 0 \text{ on } (0,1) \quad (2)\]
proposed in [28]. Here continuous birth function \(f\) is supposed to be dif-
ferentiable at 0, with \(g(s) = g'(0)s + O(s^{1+\alpha}), s \to 0^+\), for some \(\alpha > 0\),
and to satisfy the KPP condition [28] \(f'(s) \leq f'(0), s \in (0,1)\). Measurable
kernel \(J \geq 0, \int Jds = 1\) is allowed to be asymmetric and non compactly
supported. This agrees with the initial idea of Kolmogorov, Petrovsky and
Piskunov [28] who interpreted \(J(x)dx\) as the probability that an individual
passes a distance between \(x\) and \(x + dx\). It is easy to see that the DK theory
does not apply to (2). Under the above mentioned assumptions, Schumacher
[33, Example 2] has proved uniqueness of all non-critical wavefronts for (2).
Later on, Carr and Chmaj [4] achieved an important extension of the DK
theory for the special case of equation (2). By assuming several additional
conditions in [4] that \(J\) must be even, compactly supported and
\[|g(s) - g(t)| \leq g'(0)|t - s|, \quad s, t \geq 0, \quad (3)\]
they showed that the minimal wavefront \( \varphi(x + c_0 t) \) to (2) satisfying \( 0 \leq \varphi(s) \leq 1, s \in \mathbb{R}, \) is unique up to translation. Carr and Chmaj’s work has motivated the second goal of our research: to get an improvement of the DK theory that includes the critical semi-wavefronts. Theorem 3 below gives such an extension for general model (1). In the particular case of equation (2) our result (stated as Theorem 5) establishes the uniqueness of critical wavefronts under the same assumptions on \( J, f \) as in [33]. See Section 6.1 for more details, further discussion and references.

The necessity of the subtangential Lipschitz condition (3) [4,12,16,36] could be considered as a weak point of the DK uniqueness theorem, cf. [1,6,8,26,22,33]. For instance, as it was established recently by Coville, Dávila and Martínez [8], neither (3) nor \( g'(s) \leq g'(0), s \in (0, 1) \), is necessary to prove the uniqueness of non-stationary monotone traveling fronts to (2). Instead of that, it was supposed in [8] that generally asymmetric \( J \in C^1(\mathbb{R}) \) is compactly supported with \( J(a) > 0, J(b) > 0 \) for some \( a < 0 < b \), while \( g \in C^1(\mathbb{R}) \) has to satisfy \( g'(0)g'(1) < 0, g(s) \leq g'(0)s, s \geq 0, \) and \( g \in C^{1,\alpha} \) near 0. The proof in [8] follows ideas of [7] and is mainly based on the sliding methods proposed by Berestycki and Nirenberg [3] (see [7,8] for a comprehensive state-of-art overview about (2) and [5,30] for the further references).

The above discussion explains our third goal in this paper: to weaken various convergence and smoothness conditions of the DK theory, and especially condition (3). It is worthwhile to note that a similar task was also considered in [33]. The related improvements can be found in Theorems 3 and 4. In the latter theorem, we remove condition (3) by assuming a little more smoothness for \( g \) and exploiting the absence of zeros for \( \chi(z) \) in the vertical strip \( \lambda_l < \Re z < \lambda_r \) (see Lemma 2). Incidentally, Theorem 4 justifies the following principle for monostable equations: “fast positive semi-wavefronts are unique (modulo translation)”.

The main results of this paper are stated as Theorems 3, 4 below. We apply them to nonlocal integro-differential equations (Section 6.1), nonlocal lattice systems (Section 6.2), nonlocal (Section 6.3) and local (Section 6.4) reaction-diffusion equations with discrete delays. In Theorem 1 we give a short proof of the necessity of the Mollison’s condition for the existence of semi-wavefronts. Theorem 2 provides a non-existence result.

2 Mollison’s condition

In this section, we consider somewhat more general equation

\[
\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)g(\varphi(t-s), t-s, \tau)ds,
\]

where measurable \( g: \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R}^+ \) is continuous in the first two variables for every fixed \( \tau \in X \). We suppose additionally that, for some measurable \( p(\tau) \geq 0 \) and \( \delta > 0, \bar{s} \leq 0 \), it holds

\[
g(v, s, \tau) \geq p(\tau)v, \quad v \in (0, \delta), \quad s \leq \bar{s}, \quad \tau \in X.
\]
First, we present a simple proof of the necessity of the following Mollison’s condition (cf. [8]) for the existence of the semi-wavefronts:
\[ \int_{\mathbb{R}} \int_{X} K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds \text{ is finite for some } z \in \mathbb{R} \setminus \{0\}. \] (6)

**Theorem 1** Let continuous \( \varphi : \mathbb{R} \to [0, +\infty) \) satisfy (4) and suppose that \( \varphi(-\infty) = 0 \) and \( \varphi(t) \neq 0, t \leq t' \) for each fixed \( t' \). If (5) holds and
\[ \int_{X} \int_{\mathbb{R}} K(s,\tau)p(\tau)d\mu(\tau) \in (1, \infty), \] (7)
then \( \int_{-\infty}^{0} \varphi(s)e^{-sz}ds \) and \( \int_{\mathbb{R}} \int_{X} K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds \) are convergent for an appropriate \( \bar{x} > 0 \). Furthermore, \( \text{supp } K \cap (\mathbb{R}_+ \times X) \neq 0 \).

**Remark 1** Looking for heteroclinic solutions of the simple logistic equation \( x' = -\beta x + x(1 + \beta - x) \) with \( \beta > 0 \), we obtain an example of (1) where \( \text{supp } K \cap (\mathbb{R}_- \times X) = \emptyset \) under conditions of the above theorem.

**Proof** Since the support of \( K \) generally is unbounded, we will truncate \( K \) by choosing integer \( N \) such that
\[ \kappa := \int_{X} \int_{-N}^{N} K(s,\tau)p(\tau)d\mu(\tau) > 1, \text{ and } 0 \leq \varphi(t) < \delta, t < \bar{s} - N. \]

Integrating equation (4) between \( t' \) and \( t < \bar{s} - N \), we find that
\[ \int_{t'}^{t} \varphi(v)dv \geq \int_{X} \mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} g(\varphi(v-s),v-s,\tau)dvds \]
\[ \geq \int_{X} p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} \varphi(v-s)dvds \]
\[ = \int_{X} p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \left( \int_{t'}^{t} + \int_{t-s}^{t'} + \int_{t-s}^{t'} - s \right) \varphi(v)dvds, \]
from which
\[ \int_{t'}^{t} \varphi(v)dv \leq \frac{2\delta \int_{X} \int_{-N}^{N} |s|K(s,\tau)p(\tau)d\mu(\tau)}{\int_{X} \int_{-N}^{N} K(s,\tau)p(\tau)d\mu(\tau) - 1}, \quad t' < t < \bar{s} - N. \]

Hence, increasing function
\[ \psi(t) = \int_{-\infty}^{t} \varphi(s)ds \] (8)
is well defined for all \( t \in \mathbb{R} \) and
\[ \psi(t) \geq \int_{X} p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)\psi(t-s)ds \geq \kappa \psi(t-N), \quad t < \bar{s} - N. \]
Consider $h(t) = \psi(t)e^{-\gamma t}$ where $\kappa = e^{\gamma N}$, cf. [3]. For all $t < \bar{s} - N$ we have

$$h(t - N) = \psi(t - N)e^{-\gamma(t-N)} \leq \frac{1}{\kappa} \psi(t)e^{-\gamma t}e^{\gamma N} = h(t)$$

and $\gamma = N \ln \kappa > 0$. Hence $\sup_{t \leq 0} h(t) < \infty$ and $\psi(t) = O(e^{\gamma t})$, $t \to -\infty$. After taking $\bar{x} \in (0, \gamma)$ and integrating by parts, we obtain

$$\int_{-\infty}^{t} \varphi(s)e^{-2s}ds = \psi(t)e^{\bar{x}t} + \bar{x} \int_{-\infty}^{t} \psi(s)e^{-xs}ds$$

that proves the first statement of the theorem. Finally,

$$e^{-x_0} \psi(t) = \int_{X} d\mu(\tau) \int_{\mathcal{R}} e^{-x_0} K(s, \tau)e^{-z(t-s)} \psi_1(t - s, \tau)ds,$$

where $\psi_1(u, \tau) := \int_{u}^{\infty} g(\varphi(s), s, \tau)ds \geq p(\tau) \int_{-\infty}^{u} \varphi(s)ds$, $u \leq \bar{s} - N$. The latter yields

$$\int_{-\infty}^{s-N} e^{-x_0} \psi(v)dv = \int_{X} d\mu(\tau) \int_{\mathcal{R}} e^{-x_0} K(s, \tau) \int_{-\infty}^{s-N} e^{-z(v-s)} \psi_1(v - s, \tau)dvds \geq$$

$$\int_{X} p(\tau)d\mu(\tau) \int_{\mathcal{R}} e^{-x_0} K(s, \tau)ds \int_{-\infty}^{s-N} e^{-x_0} \psi(v)dv,$$

$$\mathcal{K}_{-}(\bar{x}) := \int_{X} p(\tau)d\mu(\tau) \int_{-\infty}^{0} e^{-x_0} K(s, \tau)ds \leq 1, \text{ (note that } \psi(s) > 0, s \in \mathcal{R}),$$

so that

$$\mathcal{K}_{-}(0) = \int_{X} p(\tau)d\mu(\tau) \int_{-\infty}^{0} K(s, \tau)ds \leq 1 < \int_{X} p(\tau)d\mu(\tau) \int_{\mathcal{R}} K(s, \tau)ds,$$

which completes the proof of the theorem.

Remark 2 Suppose that $|g(\varphi(s), s, \tau)| \leq C$ where $C$ does not depend on $s, \tau$. Then

$$|\varphi(t + h) - \varphi(t)| \leq C \int_{\mathcal{R}} |K_a(s + h) - K_a(s)|ds,$$

where $K_a(s) := \int_{X} K(s, \tau)d\mu(\tau) \in L_1(\mathcal{R})$. Since the translation is continuous in $L_1(\mathcal{R})$ [13] Example 5.4], we find that $\varphi(t)$ is uniformly continuous on $\mathcal{R}$. It is easy to see that the convergence of the integral $\int_{-\infty}^{0} \varphi(s)ds < \infty$ combined with the uniform continuity of $\varphi$ gives $\varphi(-\infty) = 0$. In this way, $\int_{-\infty}^{0} \varphi(s)ds < \infty$ implies that $\int_{-\infty}^{0} e^{-x\varphi(s)}ds < \infty$ for small positive $x$.

Remark 3 It is easy to see that the global non-negativity of $g$ is not necessary in the case of $K$ having bounded support (uniformly in $\tau \in X$).
Now, let $\varphi, K, g, \bar{x}$ be as in Theorem 1. Set

$$
\Phi(z) = \int_{\mathbb{R}} e^{-zs} \varphi(s) ds, \quad K(z) = \int_{\mathbb{R}} \int_X K(s, \tau) p(\tau) d\mu(\tau) e^{-sz} ds,
$$

and denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\varphi < \Re z < \gamma_\varphi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Evidently, $\sigma_\varphi, \sigma_K \leq 0$ and $\gamma_\varphi, \gamma_K \geq \bar{x} > 0$. Since $\varphi, K$ are both non-negative, by \[40, \text{Theorem 5b, p. 58}\], $\gamma_\varphi, \gamma_K, \sigma_\varphi, \sigma_K$ are singular points of $\Phi(z), K(z)$ (whenever they are finite). A simple inspection of the proof of Theorem 1 suggests the following

**Lemma 1** Assume $\varphi, g, K$ are as in Theorem 1. Then $\sigma_K \leq \sigma_\varphi < \gamma_\varphi \leq \gamma_K$. Furthermore, $K(\gamma_\varphi)$ is always a finite number.

**Proof** For all $z \in (0, \gamma_\varphi)$, $t \leq 0$, we have

$$
\psi(t) = \int_{-\infty}^t (\varphi(s) e^{-zs}) e^{zs} ds \leq e^{zt} \int_{-\infty}^0 \varphi(s) e^{-zs} ds,
$$

so that $\int_{-\infty}^0 \psi(s) e^{-zs} ds < \infty$ for each $z' \in (0, \gamma_\varphi)$ and, due to \[19\], we get

$$
K_{-}(z) := \int_X p(\tau) d\mu(\tau) \int_{-\infty}^0 e^{-zs} K(s, \tau) ds \leq 1
$$

for all $z \in (0, \gamma_\varphi)$. Hence, using the Beppo Levi monotone convergence theorem, we obtain that $K_{-}(\gamma_\varphi) \leq 1$. As a consequence, $K(\gamma_\varphi)$ is finite and $\gamma_K \geq \gamma_\varphi$.

**Corollary 1** Assume that

$$
\lim_{z \to \gamma_K} -\int_{\mathbb{R}} \int_X K(s, \tau) p(\tau) d\mu(\tau) e^{-sz} ds = +\infty.
$$

Then $\gamma_\varphi$ is a finite number and $\gamma_\varphi < \gamma_K$.

### 3 Abscissas of convergence

In this section, we investigate the abscissas of convergence for the bilateral Laplace transforms of $K$ and bounded non-negative $\varphi$ satisfying $\varphi(-\infty) = 0$, $\varphi(t) \not\equiv 0$, $t \leq t'$, for each fixed $t'$, and solving our main equation \[1\]. Now we are supposing that the continuous $g(\cdot, \tau) : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable at 0 with $g'(0+, \tau) > 0$ for each fixed $\tau$. Then the non-negative functions

$$
\lambda^+_\delta(\tau) := \sup_{u \in (0, \delta)} \frac{g(u, \tau)}{u}, \quad \lambda^-_\delta(\tau) := \inf_{u \in (0, \delta)} \frac{g(u, \tau)}{u}, \quad \delta > 0, \tau \in X,
$$

are well defined, measurable, monotone in $\delta$ and pointwise converging:

$$
\lim_{\delta \to 0^+} \lambda^+_\delta(\tau) = g'(0+, \tau).
$$
The characteristic function $\chi$ associated with the variational equation along the trivial steady state of (1) is defined by

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_X K(s, \tau) g'(0+, \tau) d\mu(\tau) e^{-sz} ds.$$ 

It is supposed to be negative at $t = 0$: $\chi(0) < 0$. Since condition (5) is obviously satisfied with $p(t) = \lambda_\delta(t)$ and

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}} \int_X K(s, \tau) \lambda_\delta^- (\tau) d\mu(\tau) ds = \int_{\mathbb{R}} \int_X K(s, \tau) g'(0+, \tau) d\mu(\tau) ds > 1$$

by the monotone convergence theorem, all results of Section 2 hold true for equation (1). Furthermore, we have the following

**Theorem 2** Assume $\chi(0) < 0$. Let $\varphi : \mathbb{R} \to [0, +\infty)$ be a semi-wavefront to equation (1). If $\varphi(-\infty) = 0$ and $\varphi(t) \neq 0$, $t \leq t'$ for each fixed $t'$, then $\chi(z)$ has a zero on $[0, \gamma_\varphi] \subset (0, \gamma_K] \subset \mathbb{R} \cup \{+\infty\}$.

**Remark 4** 1) If $\varphi(+\infty) = 0$ then a similar statement can be proved. Namely, in such a case $\chi(z)$ has a zero on $(0, \gamma_\varphi]$. 2) It should be noted that Theorem 2 also provides a non-existence result: if $\chi(x) < 0$ for all $x \in (0, \gamma_K]$ then equation (1) does not have any semi-wavefront vanishing at $-\infty$.

**Proof** For real positive $z \in (0, \gamma_\varphi)$ we consider the integrals

$$\Phi(z) = \int_{\mathbb{R}} e^{-sz} \varphi(s) ds, \quad G(z, \tau) := \int_{\mathbb{R}} e^{-sz} g(\varphi(s), \tau) ds, \quad K(z, \tau) := \int_{\mathbb{R}} e^{-sz} K(s, \tau) ds.$$ 

Since $\varphi$ is non-negative and bounded, and since $g'(0+, \tau) > 0$ exists, the convergence of $G(z, \tau)$ (for positive $z$) is equivalent to the convergence of $\Phi(z)$. Applying the bilateral Laplace transform to equation (1), we obtain that

$$\Phi(z) = \int_X K(z, \tau) G(z, \tau) d\mu(\tau).$$

(10)

Obviously, $K, G, \Phi$ are positive at each real point of the convergence.

Let us prove that $\chi(z)$ has a zero on $(0, \gamma_\varphi]$. First, we suppose that $\Phi(\gamma_\varphi) = \lim_{z \to \gamma_\varphi} \Phi(z) = \infty$. In such a case, we claim that

$$\lim_{z \to \gamma_\varphi} \frac{G(z, \tau)}{\Phi(z)} = g'(0, \tau).$$

Indeed, let $T_\delta$ be the rightmost non-positive number such that $\varphi(s) \leq \delta$ for $s \leq T_\delta$. Then

$$\lambda_\delta \int_{-\infty}^{T_\delta} e^{-sz} \varphi(s) ds \leq \int_{-\infty}^{T_\delta} e^{-sz} g(\varphi(s), \tau) ds \leq \lambda_\delta^+ \int_{-\infty}^{T_\delta} e^{-sz} \varphi(s) ds,$$

$$\int_{T_\delta}^{+\infty} e^{-sz} (g(\varphi(s), \tau) + \varphi(s)) ds \leq \sup_{z \in \mathbb{R}} (g(\varphi(s), \tau) + \varphi(s)) e^{-\gamma_\varphi T_\delta}.$$
As a consequence, for each positive $\delta > 0$,

$$\lambda_\delta^+ \leq \liminf_{z \to \gamma_\delta^-} \frac{G(z, \tau)}{\Phi(z)} \leq \limsup_{z \to \gamma_\delta^-} \frac{G(z, \tau)}{\Phi(z)} \leq \lambda_\delta^+,$$

that proves our claim.

Now, by using the Fatou lemma as $z \to \gamma_\delta$ in

$$\int_X \mathcal{K}(z, \tau) \frac{G(z, \tau)}{\Phi(z)} d\mu(\tau) \geq 1,$$

we obtain

$$1 - \chi(\gamma_\delta) = \int_X \mathcal{K}(\gamma_\delta, \tau) g'(0+, \tau) d\mu(\tau) \leq 1.$$

Therefore $\chi(\gamma_\delta) \geq 0$, and since $\chi(0) < 0$ we get the required assertion.

Hence, we may suppose that $\Phi(\gamma_\delta) = \lim_{z \to \gamma_\delta^-} \Phi(z) > 0$ is finite. Since $\varphi(t) \neq 0$, $t < t'$ for each fixed $t'$, in such a case $\gamma_\delta < \infty$. Due to Lemma 11 the value $\mathcal{K}(\gamma_\delta)$ is also finite. Set

$$\zeta(t) := \varphi(t) e^{-\gamma t}, \quad K_1(s, \tau) := e^{-\gamma s} K(s, \tau), \text{ where } \gamma := \gamma_\delta.$$

Then, for $t < T_\delta - N$, we have from (1) that

$$\int_{-\infty}^{t} \zeta(v) dv =$$

$$\int_{-\infty}^{t} \varphi(v) e^{-\gamma v} dv \geq \int_X d\mu(\tau) \int_{-N}^{N} K_1(s, \tau) \int_{-\infty}^{t} g(\varphi(v-s), \tau) e^{-\gamma (v-s)} dv ds \geq$$

$$\int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau) K_1(s, \tau) \int_{-\infty}^{t} \zeta(v-s) dv ds \geq$$

$$\int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau) K_1(s, \tau) ds \int_{-\infty}^{t-N} \zeta(v) dv.$$

Suppose now on the contrary that the characteristic equation

$$\chi(z) := 1 - \int_{R} \int_X K(s, \tau) g'(0+, \tau) d\mu(\tau) e^{-sz} ds = 0$$

has not real roots on $[0, \gamma_\delta]$. Then $\chi(0) < 0$ implies $\chi(\gamma) < 0$. As a consequence, in virtue of the monotone convergence theorem,

$$\lim_{\delta \to 0^+, N \to +\infty} \int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau) K_1(s, \tau) ds = 1 - \chi(\gamma) > 1.$$

Hence, for some appropriate $\delta, N > 0$, increasing function $\xi(t) = \int_{-\infty}^{t} \zeta(s) ds$ satisfies $\xi(t) \geq \kappa_\delta \xi(t-N), t < T_\delta - N$ with $\kappa_\delta > 1$. Arguing now as in the proof of Theorem 1 below (8) we conclude that the integral $\int_{-\infty}^{t} \zeta(s) e^{-zs}$ converges for all small positive $z$, contradicting to the definition of $\gamma_\delta$. 

On uniqueness of semi-wavefronts
Remark 5 It is clear that \( \chi(z) \) is concave on \((\sigma_K, \gamma_K)\), where \( \chi''(z) < 0 \). Since \( \chi(0) \) is negative, \( \chi \) can have at most two real zeros, and they must be of the same sign. We will denote them (if they exist) by \( \lambda_l \leq \lambda_r \). Under assumption of the existence of a semi-wavefront \( \varphi \) vanishing at \( -\infty \), \( \chi \) has at least one positive root \( \lambda_l \). Finally, it is clear that \( \chi \) is analytical in the vertical strip \( \Re z \in (0, \gamma_K) \).

Notation At this stage, it is convenient to introduce the following notation:

\[
\lambda_{rK} = \begin{cases} 
\lambda_r, & \text{if } \lambda_r \text{ exists} \\
\gamma_K, & \text{otherwise}
\end{cases}
\]

Lemma 2 Equation \( \chi(z) = 0 \) does not have roots in the open strip \( \Sigma := \Re z \in (\lambda_l, \lambda_{rK}) \). Furthermore, the only possible zeros on the boundary \( \Sigma \) are \( \lambda_l, \lambda_r \).

Proof Observe that if \( \chi(z_0) = 0 \) for some \( z_0 \in \Sigma \), then \( \chi(\Re z_0) > 0 \) since \( \chi \) is concave, \( \chi(\lambda_l) = 0 \) and \( \Re z_0 \in (\lambda_l, \min\{\lambda_r, \gamma_K\}) \). On the other hand, \( 1 = \left| \int_{\mathbb{R}} \int_X K(s, \tau)g'(0+, \tau)d\mu(\tau)e^{-sz_0}ds \right| \leq \int_{\mathbb{R}} \int_X K(s, \tau)g'(0+, \tau)d\mu(\tau)e^{-s\Re z_0}ds \)

and therefore \( \chi(\Re z_0) \leq 0 \), a contradiction. Now, if \( \chi(\lambda_l + i\omega) = 0 \) for some \( \omega \neq 0 \) then similarly

\[
1 = \chi(\lambda_l + i\omega) = |\chi(\lambda_l + i\omega)| \leq \chi(\lambda_l) = 1,
\]

so that

\[
\int_{\mathbb{R}} \int_X K(s, \tau)g'(0+, \tau)d\mu(\tau)e^{-s\lambda_l}(1 - \cos \omega s)ds = 0.
\]

Thus \( K(s, \tau)(1 - \cos \omega s) = 0 \) for almost all \( \tau \in X \), so that \( K(s, \tau) = 0 \) a.e. on \( X \times \mathbb{R} \), a contradiction.

4 A bootstrap argument

The main purpose of this section is to prove several auxiliary statements needed in the studies of the asymptotic behavior of solutions \( \varphi(t) \) at \( t = -\infty \). Usually proofs of the uniqueness are based on the derivation of appropriate asymptotic formulas with one or two leading terms (at \( t = -\infty \) as in [4, 12, 16, 39] or at \( t = +\infty \) as in [23]). Our approach is based on an asymptotic integration routine often used in the theory of functional differential equations, e.g. see [27, 31 Proposition 7.1] or [20]. Thus we use neither the Titchmarsh theory of Fourier integrals [35] nor the powerful Ikehara Tauberian theorem [4, 12]. First we will apply our methods to get an asymptotic formula for the integral \( \psi(t) := \int_{-\infty}^{t} \varphi(s)ds \). Since \( \psi \in C^1(\mathbb{R}) \) is strictly increasing and positive, this function is somewhat easier to treat than the solution \( \varphi(t) \).

Everywhere in the sequel, we assume all conditions of Section 3 on \( \varphi, K, g, \chi \). In particular, \( \chi(0) < 0 \). We also will use the following hypotheses (SB), (ECρ):
Take an arbitrary $\varepsilon > 0$ and let $N > 0$ be such that $$\int_{\{-N,N]\} |h(s)|e^{-sx}ds < 0.25\varepsilon, \quad x \in [a,b].$$ Since $e^{t}$ is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|x_1 - x_2| \leq \delta$, $s \in [-N,N]$ implies $|e^{-x_1s} - e^{-x_2s}| < 0.5\varepsilon/|h|_1$. But then $$|H(x_1, y) - H(x_2, y)| \leq 0.5\varepsilon + \int_{-N}^{N} |h(s)||e^{-x_1s} - e^{-x_2s}|ds < \varepsilon, \quad y \in \mathbb{R}.$$
Corollary 2  With $h$ as in Lemma [4], we have that $\lim_{y \to \infty} H(x,y) = 0$ uniformly on $x \in [a,b]$.

Proof Due to Lemma [4] for each $\varepsilon > 0$ there exists a finite sequence $a := x_0 < x_1 < x_2 < \ldots < x_m := b$ possessing the following property: for each $x$ there is $j$ such that $|H(x_j, y) - H(x,y)| < 0.5\varepsilon$ uniformly on $y$. Now, by the Riemann-Lebesgue lemma, $\lim_{y \to \infty} H(x_j, y) = 0$ for every $j$. Therefore, for all $j$ and some $M > 0$, we have that $|H(x_j, y)| < 0.5\varepsilon$ if $|y| \geq M$. This implies that $|H(x,y)| \leq |H(x_j, y) - H(x,y)| + |H(x_j, y)| < \varepsilon$, $|y| \geq M, x \in [a,b]$, and the corollary is proved.

As we know, the property $\varphi(-\infty) = 0$ implies the exponential decay $\psi(t) = O(e^{\alpha t})$ at $-\infty$ for each $z \in (0,\gamma_\phi)$. It is clear also that $\psi(t) = O(t)$ as $t \to +\infty$. Hence, for each fixed $z \in (0,\gamma_\phi)$, we can integrate equation (1) twice, to find that $\Psi(z) := \int_{\mathbb{R}} e^{-zv} \psi(v) dv$ satisfies

$$
\Psi(z) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-zs} \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v-s} g(\varphi(u), \tau) du dv ds =
\int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-zs} \int_{-\infty}^{\infty} g(\varphi(u), \tau) du dv ds =
\left( \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(0, \tau) e^{-zs} du ds \right) \int_{\mathbb{R}} e^{-zv} \psi(v) dv + \mathcal{R}(z),
$$

where

$$
\mathcal{R}(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-zs} ds \int_{-\infty}^{\infty} (g(\varphi(u), \tau) - g'(0, \tau) \varphi(u)) du dv.
$$

Therefore $\chi(z) \Psi(z) = \mathcal{R}(z)$. Set now

$$
G(z, \tau) := \int_{\mathbb{R}} e^{-zv} G(v, \tau) dv, \quad G(v, \tau) := \int_{-\infty}^{\infty} (g(\varphi(u), \tau) - g'(0, \tau) \varphi(u)) du.
$$

Lemma 5 Assume [13], (SB), (EC2$\varepsilon$) for some small $2\varepsilon \in (0,\gamma_K - \gamma_\phi)$. Then given $a, b \in (0,\gamma_\phi + \alpha\varepsilon)$ there exists $\rho > 0$ depending on $\varphi, a, b$ such that

$$
|G(z, \tau)| \leq \rho(\tau)/|z| := \rho(C(\tau) + d_2(\tau) + g'(0, \tau))/|z|, \quad \Re z \in [a,b] \subset (0,\gamma_\phi + \alpha\varepsilon).
$$

Proof For $x := \Re z \in (0,\gamma_\phi + \alpha\varepsilon), \ v \leq 0$, we have

$$
e^{-zv}|G(v, \tau)| \leq e^{-zv} C(\tau) \int_{-\infty}^{v} (\varphi(u))^{1+\alpha} du \leq e^{-zv} C^\alpha(\tau) \psi(v) e^{\alpha v},
$$

so that $e^{-z}|G(\cdot, \tau)| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. After integrating by parts, we obtain

$$
\int_{-N}^N e^{-zv} G(v, \tau) dv = \frac{G(-N, \tau)e^{-zN} - G(N, \tau)e^{-zN}}{z} +
$$

12 Maitere Aguerrea, Carlos Gomez and Sergei Trofimchuk
This yields

\[ \left| \int_{\mathbb{R}} e^{-z'u} G(u, \tau) du \right| = \frac{1}{|z|} \left| \int_{\mathbb{R}} e^{-z'u} (g(\varphi(u), \tau) - g'(0, \tau) \varphi(u)) du \right| \leq \frac{1}{|z|} \left( C e^{(Rz - \alpha)u} \varphi(u) du + |\varphi|_{\infty} (g(0, \tau) + d_2(\tau)) \int_{0}^{+\infty} e^{-Rz u} du \right). \]

**Corollary 3** In addition, assume that \( \int_{\mathbb{R} \times X} K(s, \tau) p(\tau) e^{-sx} d\mu ds \) converges for all \( x \in (0, \gamma_K) \). Then \( \gamma(\gamma_\phi) = 0 \) and, for appropriate \( \varepsilon_1 > 0, a, m \in \mathbb{R}, k \in \{0, 1\} \), and continuous \( r \in L^2(\mathbb{R}) \), it holds that

\[ \psi(t + m) = (a - t)^{k} e^{\gamma_\phi t} + e^{\gamma_\phi t} r(t), \quad t \in \mathbb{R}. \]

It should be noted that depending on the geometric properties of \( \gamma_\phi \), the value of \( \gamma_\phi \) can be minimal (the case of a pulled semi-wavefront \[13, 22, 34\]) or maximal (the case of a pushed semi-wavefront, ibid.) positive zero of \( \chi(z) \).

Observe that, due to the monotonicity of \( \psi \), we can also use here the Ikehara Tauberian theorem \[4\]. However it gives a slightly different result.

**Proof** Set \( z := x + iy \). For a fixed \( 0 < x < \gamma_\phi + \alpha \epsilon \) we have

\[ |\mathcal{R}(z)| = \left| \int_{X} G(z, \tau) \int_{\mathbb{R}} K(s, \tau) e^{-sz} d\mu ds \right| \leq \frac{1}{|z|} \int_{X} \rho(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-sz} d\mu ds, \]

so that \( \mathcal{R}(z) \) is regular in the strip \( 0 < \Re z < \gamma_\phi + \alpha \epsilon \). Thus we can deduce from \( \Psi(z) = \mathcal{R}(z)/\chi(z) \) that \( \gamma_\phi = \gamma_\phi \) (e.g. see \[12\] Lemma 4.4), the definition of \( \gamma_\phi \) is similar to that of \( \gamma_\phi \) must be a positive zero of \( \chi(z) \) and \( \Psi(\gamma_\phi) = \infty \).

It is clear that \( \mathcal{R}(x + i) \) is also bounded and square integrable on \( \mathbb{R} \) (for each fixed \( x \)). Take now \( \gamma', \gamma'' \) such that \( 0 < \gamma' < \gamma_\phi < \gamma'' < \gamma_\phi + \alpha \). Then we may shift the path of integration in the inversion formula for the Laplace transform (e.g. see \[31\] p. 10) to obtain

\[ \psi(t) = \frac{1}{2\pi i} \int_{\gamma' - i\infty}^{\gamma' + i\infty} e^{zt} \Psi(z) dz = -\text{Res}_{z=\gamma_\phi} \frac{e^{zt} \mathcal{R}(z)}{\chi(z)} + \frac{e^{\gamma_\phi t}}{2\pi i} \left\{ \int_{-\infty}^{+\infty} e^{ist} a_1(s) ds \right\}, \]

where the first term is different from 0 and \( a_1(s) = \mathcal{R}(\gamma'' + is)/\chi(\gamma'' + is) \) is square integrable on \( \mathbb{R} \). Here we recall that, by Corollary 3, \( \lim_{y \to \infty} \chi(x + iy) = 1 \) uniformly on \( x \in [\gamma', \gamma''] \). Since \( \chi''(x) > 0, x \in (0, \gamma_K) \), for some \( a, m \in \mathbb{R} \) we get \( \psi(t + m) = (a - t)^{k} e^{\gamma_\phi t} + e^{\gamma_\phi t} r(t) \).

**Lemma 6** Assume all conditions of Lemma 5 except \( \gamma_\phi < \gamma_K \). If

\[ 1 - \chi_1(x_0) := \int_{X} K(s, \tau) d_2(\tau) d\mu(\tau) e^{-sx_0} ds \leq 1, \]

for some \( x_0 \in (0, \gamma_K) \), then \( \gamma_\phi \) coincides with the minimal positive zero \( \lambda_1 \) of \( \chi(z) \).
Proof Since $d_2(\tau) \geq g'(0, \tau)$, we obtain that $x_0 \in [\lambda_t, \lambda_{rK}]$ and $\lambda_t < \gamma_K$.

Case I: $\gamma_\phi < \gamma_K$. Then, by Corollary 3, we have $\chi(\gamma_\phi) = 0$ so that $\gamma_\phi \in \{\lambda_t, \lambda_r\}$. Suppose that $\gamma_\phi > \lambda_t$, this implies $x_0 \leq \gamma_\phi = \lambda_r$. We have

$$\psi(z) = \left( \int_X d\mu(\tau) \int \mathcal{K}(s, \tau)d_2(z)e^{-zs}ds \int R e^{-zv}\psi(v)dv + \mathcal{R}_1(z) \right),$$

where

$$\mathcal{R}_1(z) := \int_X d\mu(\tau) \int \mathcal{K}(s, \tau)d_2(z)e^{-zs}ds \int R e^{-zv} \int_0^\gamma (g(u, \tau) - d_2(\tau)\varphi(u))dudv,$$

or, in a shorter form,

$$\chi_1(z) \Psi(z) = \mathcal{R}_1(z). \quad (14)$$

It is clear that $x_0 = \gamma_\phi = \lambda_r > \lambda_t$ implies immediately that $g'(0, \tau) = d_2(\tau)$ a.e. on $X$ and that $\chi_1(z) = \chi(z)$, $\mathcal{R}(z) = \mathcal{R}_1(z)$. As we have seen in the proof of Corollary 3, this guarantees that $\mathcal{R}_1(x_0)$ is a finite number. Of course, $\mathcal{R}_1(x_0)$ is also well defined if $x_0 < \gamma_\phi$. Now, it is clear that $\mathcal{R}_1(x_0) \leq 0$ because of $g(u, \tau) \leq d_2(\tau)u$, $u \geq 0$. We claim that, in fact, $\mathcal{R}_1(x_0) < 0$. Indeed, otherwise $g(u, \tau) = d_2(\tau)u$, $u \geq 0$, for almost all $\tau \in X$ that yields $d_2(\tau) = g'(0, \tau)$ and $\mathcal{R}_1(z) \equiv 0$ leading to a contradiction: $\psi(z) \equiv 0$ and $\psi(t) \equiv 0$.

Now, from $\mathcal{R}_1(x_0) < 0$, $\psi(x_0) > 0$, $\chi_1(x_0) \geq 0$, we deduce that $\psi$ must have a pole at $x_0 = \gamma_\phi < \gamma_K$. But then $\chi_1(\gamma_\phi) = \chi(\gamma_\phi)$ implies $\chi_1(z) \equiv \chi(z)$, $\mathcal{R}(z) = \mathcal{R}_1(z)$). Hence, $\lambda_t < \lambda_r = x_0 < \gamma_K$ and $\gamma_\phi = x_0$ is a simple pole of $\psi$. Therefore we can proceed as in the proof of Corollary 3 taking $0 < \gamma' < \gamma_\phi = \lambda_r < \gamma'' < \gamma_\phi + \alpha \epsilon$ to obtain

$$\psi(t) = \frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\psi(z)dz = - \text{Res}_{z=\lambda_r} \frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + e^{\gamma''t}r_1(t) =$$

$$= Ae^{\gamma't} + e^{\gamma''t}r_1(t), \quad \text{where } A := - \frac{\mathcal{R}(\lambda_r)}{\chi(\lambda_r)} < 0, \ r_1 \in L^2(\mathbb{R}),$$

contradicting to the positivity of $\psi$.

Case II: $\gamma_\phi = \gamma_K$. Since $x_0 < \gamma_K = \gamma_\phi$ and $\mathcal{R}_1(x_0) < 0$, we similarly deduce from (14) that $x_0$ is a singular point of $\psi(z)$, a contradiction.

5 The uniqueness theorems

To prove our uniqueness results we will need more strong property of $\varphi$ than the merely convergence of $\int_R e^{-zs}\varphi(s)ds$ for all $\Re z \in (0, \gamma_\phi$) (even combined, as in Section 4, with (EC) for some small $\epsilon > 0$). This property, assumed everywhere in the sequel, is (EC$_{\gamma_\phi}$). The nonlinearity $g$ is supposed to satisfy the hypothesis (SB).

The following assertion is crucial for extension of the Diekmann-Kaper theory on the critical case $\chi(\lambda_t) = \chi'(\lambda_r) = 0$. 
Lemma 7 Suppose that, for some $a, b > \delta > 0$, continuous $v : \mathbb{R} \to [0, 1]$ satisfies $v(t) = 1 + O(e^{at})$, $t \to -\infty$, $v(t) = O(e^{-bt})$, $t \to +\infty$, and

$$v(t) \leq \int_{\mathbb{R}} N(s)v(t-s)ds,$$

where measurable $N(s) \geq 0$, $s \in \mathbb{R}$, is such that

$$\int_{\mathbb{R}} N(s)ds = 1, \quad \int_{\mathbb{R}} sN(s)ds = 0, \quad \int_{\mathbb{R}} N(s)e^{xs}ds < \infty, \quad \text{for all } |x| \leq \delta.$$

Then $v(t) \equiv 0$.

Proof First we observe that, without restricting the generality, we may assume that $v \in C^2(\mathbb{R})$ with the finite norm $|v|_{C^2} := \sup_{s \in \mathbb{R}, j = 0, 1} |v^{(j)}(s)|$.

Indeed, if we set

$$w(t) := \int_{t}^{t+1} v(s)ds, \quad t \in \mathbb{R},$$

then $w \in C^1(\mathbb{R})$ has the same properties as $v$, $|w'(t)| < 1$, $t \in \mathbb{R}$, while $v(t) \equiv 0$ if and only if $w(t) \equiv 0$. For instance, if $v(t) \leq ce^{-bt}$ for $t \geq t_0$, $b > 0$, then $w(t) \leq ce^{-bt} \int_{t_0}^{t} e^{-bs}ds \leq ce^{-bt}$, $t \geq t_0$. Furthermore, $w'(t) = v(t+1) - v(t)$ behaves as $O(e^{at})$ at $-\infty$ and as $O(e^{-bt})$ at $+\infty$.

Applying the same procedure to $w$ once more, we obtain the desired smoothness property of $v$ with $v'(t), v''(t)$ satisfying

$$v'(t), v''(t) = O(e^{at}), \quad t \to -\infty, \quad v'(t), v''(t) = O(e^{-bt}), \quad t \to +\infty. \quad (15)$$

In any case, the bilateral Laplace transform $V(z)$ of $v(t)$ is well defined in the vertical strip $-b < \Re z < 0$.

Set now

$$f(t) := \int_{\mathbb{R}} N(s)v(t-s)ds - v(t) \geq 0.$$

It follows from this definition that $0 \leq f(t) \leq 1 - v(t)$ and therefore $f(t) = O(e^{at})$, $t \to -\infty$. Additionally, using (15), we obtain, for $j = 0, 1, 2$ and some positive $C, C' > 0$,

$$\int_{\mathbb{R}} N(s)|v^{(j)}(t-s)|ds \leq C \int_{\mathbb{R}} N(s)\epsilon^{\pm \delta(t-s)}ds = C\epsilon^{\pm \delta t} \int_{\mathbb{R}} N(s)e^{\mp \delta s}ds =: C'e^{\mp \delta t}.$$

Thus we conclude that the Laplace transform $F(z)$ of $C^2$-smooth function $f(t)$, $|f|_{C^2} < \infty$, is well defined in the strip $-\delta < \Re z < \delta$, where we have

$$|F(z)| \leq C_p \frac{p}{|z|^q}, \quad p \leq \Re z \leq q, \quad p, q \in (-\delta, \delta).$$

Hence, we can apply the Laplace transform to the equation

$$v(t) + f(t) = \int_{\mathbb{R}} N(s)v(t-s)ds,$$
to obtain that

\[ V(z) = \frac{F(z)}{N(z) - 1}, \quad -\delta < \Re z < 0, \]

where \( N(z) := \int_{\mathbb{R}} e^{-zs} N(s) ds \) of \( N \) is a regular function in the strip \( |\Re z| < \delta \).

Observe also that

\[ N(0) = 1, \quad N'(0) = 0, \quad N''(0) = \int_{\mathbb{R}} s^2 N(s) ds > 0. \]

Now, since \( V(z) \) is analytical in the strip \( \Pi := \{-\delta < \Re z < 0\} \), the function \( F(z)/(N(z) - 1) \) has the same property in \( \Pi \). On the other hand, for an appropriate \( \delta' \in (0, \delta) \) the quotient \( F(z)/(N(z) - 1) \) defines a meromorphic function in \( \Pi' := \{-\delta < \Re z < \delta'\} \), with a unique singularity (double pole) at \( z = 0 \). Note that Corollary 2 as well as the last argument in the proof of Lemma 2 are used at this stage. Since the Laplace transform \( V \) of \( v \in C^2(\mathbb{R}) \) is integrable along each vertical line inside of \( \Pi \), we may apply the inversion formula to get, for arbitrarily fixed \( c \in (-\delta, 0) \), \( r \in (0, \delta') \),

\[
v(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zt}F(z)\frac{dz}{N(z) - 1} = \text{Res}_{z=0} e^{zt}F(z)\frac{dz}{N(z) - 1}.
\]

Next, observe that if \( f(t) \equiv 0 \) then also \( F(z) \equiv 0 \) so that \( v(t) \equiv 0 \). Therefore the only case of the interest is when \( f(s') > 0 \) at some \( s' \in \mathbb{R} \) that implies \( F(0) > 0 \). Now, in such a case, we have that

\[
|\int_{r-i\infty}^{r+i\infty} e^{zt}F(z)\frac{dz}{N(z) - 1}| \leq c_0 e^{rt} \int_{\mathbb{R}} \frac{ds}{r^2 + s^2} \leq c_1 e^{rt}, \quad t \in \mathbb{R},
\]

while a direct calculation shows that

\[
\text{Res}_{z=0} e^{zt}F(z)\frac{dz}{N(z) - 1} = \frac{F(0)}{N''(0)} t + \frac{F'(0)}{N''(0)} - \frac{2F(0)N'''(0)}{3(N''(0))^2} =: At + B, \quad A > 0.
\]

In consequence, as \( t \to -\infty \),

\[
v(t) = At + B + O(e^{rt}), \quad \text{with } A, r > 0,
\]

which contradicts to the boundary condition \( v(-\infty) = 1 \).

Now we are ready to prove our first uniqueness result:

**Theorem 3** Assume (SB) except \( \gamma_\phi < \gamma_K \) as well as (EC\( \gamma_\phi \)) and suppose further that \( \chi(0) < 0 \), \( \chi(\gamma_K -) \neq 0 \),

\[
|g(u, \tau) - g(v, \tau)| \leq g'(0, \tau)|u - v|, \quad u, v \geq 0.
\]

Then equation (1) has at most one bounded positive solution \( \varphi \), \( \varphi(-\infty) = 0 \). Furthermore, \( \gamma_\phi \) coincides with the minimal positive zero \( \lambda_1 \) of \( \chi(z) \) and such a solution (if exists) has the following representation:

\[
\varphi(t + m) = (a - t)^k e^{\lambda t} + e^{(\lambda + \delta)t} r(t), \quad \text{with continuous } r \in L^2(\mathbb{R}),
\]

for some appropriate \( a, m \in \mathbb{R}, \delta > 0 \). Here \( k = 0 \) (respectively, \( k = 1 \) if \( \lambda_1 \) is a simple [respectively, double] root of \( \chi(z) = 0 \).
Remark 6 By Lemma 5, \( \chi(\gamma K-) \neq 0 \) yields \( \gamma \phi = \lambda t < \gamma K \). We assume this stronger assumption instead of \( \chi_0 < \gamma K \) since it is more easy to use. In the section of applications, the condition \( \chi(\gamma K-) \neq 0 \) is slightly modified in order to take into account the dependence of \( \chi, \gamma K \) on the wave velocity \( c \). Recall that we need \( \gamma \phi < \gamma K \) to apply the bootstrap argument.

Proof Step I: Asymptotic behavior at \(-\infty\). It is clear that equation (11) can be written as the linear inhomogeneous equation

\[
\varphi(t) = \int_X d\mu \int_R K(s, \tau)g(0, \tau)\varphi(t-s)ds + D(t), \quad t \in \mathbb{R},
\]

where all integrals are converging and

\[
D(t) := \int_X d\mu \int_R K(s, \tau)(g(\varphi(t-s), \tau) - g'(0, \tau)\varphi(t-s))ds \leq 0, \quad t \in \mathbb{R}.
\]

Take \( C(\tau), \sigma, \zeta(x) \) as in (SB). Observe that without restricting the generality, we can assume in (SB) that \( (1+\alpha)\gamma \phi < \gamma K \). Since equation (11) is translation invariant, we can suppose that \( \varphi(t) < \sigma \) for \( t < 0 \). Applying the bilateral Laplace transform to (17), we obtain that

\[
\chi(z)\Phi(z) = D(z).
\]

We claim that, due to conditions (SB) and (EC_\gamma), function \( D \) is regular in the strip \( \Pi_\alpha = \{ z : \Re z \in (0, (1+\alpha)\gamma \phi) \} \). Indeed, we have

\[
D(x + iy) = \int_R e^{-iyt}[e^{-xt}D(t)]dt.
\]

Given \( x := \Re z \in (0, (1+\alpha)\gamma \phi) \), we choose \( x' \) sufficiently close from the left to \( \gamma \phi \) to satisfy \(-x + (1+\alpha)x' > 0\). Then

\[
|e^{-xt}D(t)| \leq e^{-xt} \left[ \int_X C(\tau)d\mu \int_0^{+\infty} K(s, \tau)C_{-1+\alpha}^{1+\alpha}e^{(1+\alpha)ix'(t-s)}ds + \right.
\]

\[
+2|\varphi|\infty \int_X g'(0, \tau)d\mu \int_{-\infty}^{t} K(s, \tau)ds \right]
\]

\[
eq e^{-xt} \left[ e^{(1+\alpha)x't}C_{-1+\alpha}^{1+\alpha}\zeta((1+\alpha)x') + 2|\varphi|\infty \int_X g'(0, \tau)d\mu \int_{-\infty}^{t} K(s, \tau)ds \right] \quad=: \quad e^{(-x+(1+\alpha)x')t} A_1 + 2|\varphi|\infty \int_X g'(0, \tau)d\mu \int_{-\infty}^{t} K(s, \tau)e^{-(1+\alpha)x's}e^{(1+\alpha)ix's}ds \leq \quad e^{(-x+(1+\alpha)x')t} A_2 e^{(-x+(1+\alpha)x')t}, \quad t \in \mathbb{R}.
\]

Since clearly \( D(t) \) is bounded on \( \mathbb{R} \), the above calculation shows that \( e^{-xt}D(t) \) belongs to \( L^k(\mathbb{R}) \), for each \( k \in [1, \infty] \) once \( x \in (0, (1+\alpha)\gamma \phi) \). As a consequence, for each such \( x \) the function \( d_x(y) := D(x + i \cdot y) \) is bounded and square integrable on \( \mathbb{R} \).
By our assumptions, \(\chi(z)\) is also regular in the domain \(\Pi_\alpha\), while \(\Phi(z) = D(z)/\chi(z)\) is regular in \(\mathbb{R}z \in (0, \gamma_\phi)\) and meromorphic in \(\Pi_\alpha\). In virtue of Lemma 2, we can suppose that \(\Phi(z)\) has a unique singular point \(\gamma_\phi\) in \(\Pi_\alpha\) which is either simple or double pole.

Now, for some \(x'' \in (0, \gamma_\phi)\), using the inversion theorem for the Fourier transform, we obtain that for an appropriate sequence of integers \(N_j \to +\infty\)

\[
\varphi(t) = \frac{1}{2\pi i} \lim_{j \to +\infty} \int_{x''-iN_j}^{x''+iN_j} \frac{e^{zt}D(z)}{\chi(z)} \, dz
\]

almost everywhere on \(\mathbb{R}\), e.g. see [31, p. 9-10]. Next, if \(x \in (\gamma_\phi, (1 + \alpha \gamma_\phi))\)

\[
\int_{x''-iN_j}^{x''+iN_j} \frac{e^{zt}D(z)}{\chi(z)} \, dz = \left( \int_{x-1N}^{x+1N} + \int_{x''-iN}^{x''-iN} - \int_{x''+iN}^{x''} \right) \frac{e^{zt}D(z)}{\chi(z)} \, dz - 2\pi i \text{Res}_{z=\gamma_\phi} \frac{e^{zt}D(z)}{\chi(z)}.
\]

Since, by Corollary 2,

\[
\lim_{j \to +\infty} \max_{z \in [x'' \pm iN_j, x \pm iN_j]} |D(z)| + |1 - \chi(z)| = 0,
\]

we conclude that, for each fixed \(t \in \mathbb{R}\)

\[
\lim_{j \to +\infty} \int_{x'' \pm iN_j}^{x' \pm iN_j} \frac{e^{zt}D(z)}{\chi(z)} \, dz = 0.
\]

Therefore

\[
\varphi(t) = -\text{Res}_{z=\gamma_\phi} \frac{e^{zt}D(z)}{\chi(z)} + e^{zt} \int_{\mathbb{R}} \frac{e^{itx}d_x(y)}{\chi(x + iy)} \, dy.
\]

It should be noted here that \(D(\gamma_\phi) < 0\) since otherwise \(D(t) \equiv 0\) implying \(\chi(z)\Phi(z) = D(z) \equiv 0\) so that \(\Phi(z) \equiv 0\), a contradiction. Since

\[
\text{Res}_{z=\gamma_\phi} \frac{e^{zt}D(z)}{\chi(z)} = \frac{e^{zt}D(\gamma_\phi)}{\chi'(\gamma_\phi)}, \quad \text{if } \lambda_1 < \lambda_r,
\]

\[
\text{Res}_{z=\gamma_\phi} \frac{e^{zt}D(z)}{\chi(z)} = \frac{2e^{\gamma_\phi t}}{\chi''(\gamma_\phi)} \left( tD(\gamma_\phi) + D'(\gamma_\phi) - D(\gamma_\phi) \frac{\chi''(\gamma_\phi)}{3\chi'(\gamma_\phi)} \right), \quad \text{if } \lambda_1 = \lambda_r,
\]

we get the desired representation.

Step II: Uniqueness. By the contrary, suppose that \(\varphi_1\) and \(\varphi_2\) are different solutions of (1) in the sense that \(\varphi_1(t) \not\in \{\varphi_2(t + s), \ s \in \mathbb{R}\}\). Due to Step I we may suppose that \(\varphi_1, \varphi_2\) have the same main parts of their asymptotic representations:

\[
\varphi_j(t) = (a_j - t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi + b_j)t} r_j(t), \quad r_j \in L^2(\mathbb{R}).
\]

Therefore \(\omega(t) := \varphi_2(t) - \varphi_1(t) = e^{(\gamma_\phi + b_j)t} r_j(t), \ t \in \mathbb{R}, \ r \in L^2(\mathbb{R})\), in the case of \(\lambda_1 < \lambda_r\) and \(\omega(t) = (a_2 - a_1)e^{\gamma_\phi t} + e^{(\gamma_\phi + b_j)t} r_j(t), \ t \in \mathbb{R}, \ r \in L^2(\mathbb{R})\), in the case of \(\lambda_1 = \lambda_r\). Set

\[
w(t) := \int_{t-1}^t |\omega(s)| \, ds,
\]
it is clear that \( w \in C^1(\mathbb{R}) \) is bounded and has bounded derivative on \( \mathbb{R} \), in fact, 0 < \(|w'|\infty, |w|\infty \leq \max\{|\varphi_1|\infty, |\varphi_2|\infty \}. \) Furthermore, if \( \lambda_l < \lambda_r \) then

\[
w(t) = \int_{t-1}^{t} |e^{(\gamma_o+\delta)s}r(s)|\,ds \leq e^{(\gamma_o+\delta)t} \int_{t-1}^{t} |r(s)|\,ds \leq e^{(\gamma_o+\delta)t} \sqrt{\int_{t-1}^{t} r^2(s)\,ds},
\]

so that \( w(t) = e^{(\gamma_o+\delta)t}o(1) \) at \( t = -\infty \). Now, if \( \lambda_r = \lambda_r \), we know that

\[
\omega(t) = ae^{\gamma ot} + e^{(\gamma_o+\delta)t}r(t),
\]

where we can suppose that \( a \geq 0 \). Therefore

\[-ae^{\gamma ot}|r(t)| \leq |\omega(t)| - ae^{\gamma ot} \leq e^{(\gamma_o+\delta)t}|r(t)|,
\]

so that, in view of the above estimation of \( w(t) \), we get

\[
w(t) = \int_{t-1}^{t} |\omega(s)|\,ds = \frac{a(1-e^{-\gamma ot})}{\gamma_o}e^{\gamma ot} + e^{(\gamma_o+\delta)t}o(1), \ t \to -\infty.
\]

We have the following:

\[
\omega(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)(g(\varphi_2(t-s), \tau) - g(\varphi_1(t-s), \tau))\,ds,
\]

\[
|\omega(t)| \leq \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)|\omega(t-s)|\,ds,
\]

\[
\int_{t-1}^{t} |\omega(u)|\,du \leq \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) \int_{t-1}^{t} |\omega(u-s)|\,du\,ds,
\]

and, finally, \( w(t) \leq \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)w(t-s)\,ds \) \( \tag{18} \)

Case I (noncritical). If \( \chi'(\lambda) \neq 0 \), then \( \chi(\gamma') > 0 \) for some \( \gamma' \in (\gamma_o, \gamma_o + \delta) \).

After multiplying the both sides of 18 by \( e^{-\gamma ot} \) and setting \( v(t) := w(t)e^{-\gamma ot} \), we find that

\[
v(t) \leq \int_{\mathbb{R}} \left( \int_X g'(0, \tau) K(s, \tau)e^{-\gamma os} d\mu(\tau) \right) v(t-s)\,ds.
\]

Since \( v(t) \geq 0 \) and \( v(\pm \infty) = 0 \), there exists a finite \( t_m \) such that

\[
v(t_m) = |v|\infty = \max_{s \in \mathbb{R}} v(s).
\]

But then \( v(t_m) \leq \left( \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)e^{-\gamma os} ds \right) v(t_m) \), forcing \( 0 = v(t_m) \equiv v(t) \equiv w(t) \) in view of \( \chi(\gamma') > 0 \).
Case II (critical). Now, if \( \lambda_t = \lambda_r \), we set \( v(t) := u(t)e^{-\gamma \phi t} \), to conclude analogously that \( v(-\infty) = a(1 - e^{-\gamma \phi})/\gamma \), \( v(+\infty) = 0 \),

\[
v(t) \leq \int_{\mathbb{R}} \left( \int_{X} g'(0, \tau)K(s, \tau)e^{-\gamma \phi s}d\mu(\tau) \right) v(t-s)ds.
\]

After normalizing if necessary, we can assume that \( 0 \leq v(t) \leq 1 = \sup_{s \in \mathbb{R}} v(s) \) for all \( t \in \mathbb{R} \). If \( v(\hat{t}) = 1 \) for some finite rightmost \( \hat{t} \), then

\[
1 = v(\hat{t}) \leq \int_{\mathbb{R}} \left( \int_{X} g'(0, \tau)K(s, \tau)e^{-\gamma \phi s}d\mu(\tau) \right) v(\hat{t}-s)ds =: \int_{\mathbb{R}} N(s)v(\hat{t}-s)ds \leq \int_{\mathbb{R}} g'(0, \tau)d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)e^{-\gamma \phi s}ds = 1,
\]

which implies that \( N(s)v(\hat{t}-s) = N(s) \) a.e. and \( v(\hat{t}-s) = 1 \) for all \( s \) such that \( N(s) > 0 \). Now, since \( \int_{\mathbb{R}} N(s)ds = 1 \), \( \int_{\mathbb{R}} sN(s)ds = 0 \), there is a subset of \( \mathbb{R}^+ \) of positive measure where \( N(s) > 0 \). This means that \( \hat{t} \) does not possess the property to be the rightmost point where \( v(\hat{t}) = 1 \), a contradiction. Thus we have to analyze only the case when \( v(\hat{t}) = 1 \) for some finite rightmost \( \hat{t} \), then

\[
\int_{\mathbb{R}} N(s)e^{x\phi}ds = 1 - \chi(\gamma \phi - x) < \infty \quad \text{for all} \ |x| < \delta.
\]

Hence, \( v(t) \equiv 0 \), a contradiction.

Let us consider now the situation when the subtangential Lipschitz condition of Theorem 3 is not satisfied. In such a case, we prove the uniqueness under somewhat stronger hypotheses (SB*), (EC*):

**(SB*)** Either one of the following conditions holds

\[
|g(u, \tau) - g(v, \tau) - g'(0, \tau)(u - v)| \leq C(\tau)|u - v|^{1+\alpha}, \quad u, v \in (0, \sigma),
\]

\[
|g'(u, \tau) - g'(0, \tau)| \leq C(\tau)u^\alpha, \quad u \in (0, \sigma),
\]

for some \( \alpha, \sigma \in (0, 1] \) and measurable \( C(\tau) > 0 \) satisfying (\( \llcorner \)). Furthermore, there exist \( \hat{\sigma} \in (0, \gamma \phi) \) and measurable \( d_1(\tau) \) such that

\[
0 \leq K(s, \tau) \leq d_1(\tau)e^{\sigma s}, \quad s \in \mathbb{R}.
\]

**(EC*)** Either one of the following two assumptions is satisfied:

(i) Each solution of (\( \llcorner \)) is \( C^1 \)-smooth and if \( \varphi_1, \varphi_2 \in C^1(\mathbb{R}) \) satisfy (\( \llcorner \)) and the integral \( \int_{\mathbb{R}} e^{-\beta s}(\varphi_2(s) - \varphi_1(s))ds \) converges absolutely then the integral \( \int_{\mathbb{R}} e^{-\beta s}(\varphi_2'(s) - \varphi_1'(s))ds \) also converges absolutely.

(ii) There exists \( \delta_0 > 0 \) such that, for each \( x \in (\lambda_K - \delta_0, \lambda_K) \), it holds

\[
0 \leq K(s, \tau) \leq d_{2x}(\tau)e^{\sigma s}, \quad s \in \mathbb{R},
\]

for some \( \mu \)-measurable \( d_{2x}(\tau) \).
Theorem 4 Assume (SB*), (EC*) and suppose that
\[ |g(u,\tau) - g(v,\tau)| \leq \lambda(\tau)|u - v|, \quad u, v \geq 0, \tau \in X, \quad (19) \]
for some measurable \( \lambda(\tau) \) different from \( g'(0, \tau) \) and that the function
\[ \chi_1(z) = 1 - \int_X \int K(s, \tau, \lambda(\tau))d\mu(\tau)e^{-sz}ds \]
is well defined on \([0, \lambda_K)\). If, in addition, \( \lambda_d \in L^1(\mathbb{X}), j = 1, 2, \chi(0) < 0 \) and \( \chi_1(m) \geq 0 \) for some \( m \in (0, \lambda_K) \), then equation (11) has at most one bounded positive solution \( \varphi, \varphi(-\infty) = 0 \). Furthermore, \( \gamma_0 \) coincides with the minimal simple positive zero \( \lambda_0 \) of \( \chi(z) \) and, for appropriate \( m \in \mathbb{R}, \delta > 0, \varphi(t + m) = e^{\lambda t} + e^{(\lambda_0 + \delta)t}(t), \) with continuous \( r \in L^2(\mathbb{R}) \).

Proof Using Lemma 3 and the above conditions, we find that \( \lambda_1 = \gamma_0 < m < \lambda_K \leq \gamma_K \). Hence, due to Lemma 3 the assumptions of the theorem guarantee the fulfillment of the hypotheses (SB) and (EC*). Therefore all arguments of Step I in the proof of Theorem 4 can be repeated (with a unique change in the estimation of \( e^{-xt}D(t) \) where \( g'(0, \tau) \) is replaced with \( \lambda(\tau), \chi_1 \)). Thus each pair \( \varphi_1, \varphi_2 \) of solutions of (11) can be supposed to have the same main parts of their asymptotic representations: \( \varphi_j(t) = e^{\lambda_j t} + e^{(\lambda_0 + \delta_j) t}r_j(t), \) \( r_j \in L^2(\mathbb{R}) \). The further proof is divided in three steps.

Step I. Again, we consider bounded function \( \omega(t) := \varphi_2(t) - \varphi_1(t) = e^{(\lambda_0 + \delta_1) t}r(t), \) \( t \in \mathbb{R}, \) \( r \in L^2(\mathbb{R}) \). If \( \mathbb{R} \in (0, \lambda_1 + \delta) \), then \( \int_\mathbb{R} e^{-xs}\omega(s)ds \) converges absolutely and from condition (EC*)\( (i) \) we have
\[ |\omega(t)| = \left| \int_\mathbb{R} \omega'(s)ds \right| = \left| \int_{-\infty}^t e^{xs}\omega'(s)e^{-xs}ds \right| \leq \epsilon^t \int_\mathbb{R} e^{-xs}|\omega'(s)|ds =: C_xe^{\epsilon t}, \]
for all \( x \in (0, \lambda_1 + \delta) \) and \( t \in \mathbb{R} \). Similarly, we obtain from (SB*), (EC*)\( (ii) \) that
\[ |\omega(t)| = \left| \int_X d\mu \int_\mathbb{R} K(s, \tau) \left( g(\varphi_1(t-s), \tau) - g(\varphi_2(t-s), \tau) \right)ds \right| \leq \epsilon^t \int_X \lambda(\tau)d\mu \int_\mathbb{R} K(s, \tau)e^{-xs}e^{-(t-s)}|\omega(t-s)|ds \leq \epsilon^t \int_X \lambda(\tau)(d_1(\tau) + d_2, \lambda_1 + \delta)ds \int_\mathbb{R} e^{-xs}|\omega(s)|ds, \quad x \in (\epsilon, \lambda_1 + \delta), \quad t \in \mathbb{R}. \]

In each of these two cases, for every \( x \in (\epsilon, \lambda_1 + \delta) \) there exists an appropriate \( C_x > 0 \) such that \( |\omega(t)| \leq C_xe^{\epsilon t}, \quad t \in \mathbb{R} \). Set
\[ \Gamma = \sup\{x \geq \lambda_1 | \exists C_x : |\omega(t)| \leq C_xe^{\epsilon t}, \quad t \in \mathbb{R} \}, \]
we claim that \( \Gamma \geq \lambda_K \). Indeed, on the contrary, suppose that \( \Gamma < \lambda_K \) and let \( x_0 \in (\epsilon, \Gamma), \alpha > 0, \gamma_0 \in (\epsilon, \lambda_1) \) be such that \( \{x_0(1 + \alpha), x_0 + \alpha \gamma_0\} \subset (\Gamma, \lambda_K) \). Let \( x_0 \) be the minimal of these two numbers. We have that
\[ \omega(t) = \int_X d\mu \int_\mathbb{R} K(s, \tau)g'(0, \tau)\omega(t-s)ds + E(t), \quad t \in \mathbb{R}, \quad (20) \]
with bounded
\[ E(t) := \int_X d\mu \int_{\mathbb{R}} K(s, \tau) \left( g(\varphi_1(t-s), \tau) - g(\varphi_2(t-s), \tau) - g'(0, \tau) \omega(t-s) \right) ds. \]

Now, depending on assumptions chosen in (SB*), we have either
\[ |g(\varphi_1(s), \tau) - g(\varphi_2(s), \tau) - g'(0, \tau) \omega(s)| \leq C(\tau)|\omega(s)|^{1+\alpha} \leq C(\tau) \min \{ C_{x_0}^{1+\alpha} e^{x_0(1+\alpha)s}, (|\varphi_1|_\infty + |\varphi_2|_\infty)^{1+\alpha} \} \leq k_1 C(\tau)e^{x_\infty s}, \ s \in \mathbb{R}, \]
or
\[ |g(\varphi_1(s), \tau) - g(\varphi_2(s), \tau) - g'(0, \tau) \omega(s)| \leq C(\tau)|\omega(s)|(|\varphi_1(s)| + |\varphi_2(s)|)^\alpha \leq k_2 C(\tau) \min \{ C_{x_0} e^{(x_0+\alpha_\gamma)s}, (|\varphi_1|_\infty + |\varphi_2|_\infty)^{1+\alpha} \} \leq k_3 C(\tau)e^{x_\infty s}, \ s \in \mathbb{R}, \]
where \( k_i \) depend on \( x_0, \gamma_\ast \) and \( |\varphi_j|_\infty \) only. Hence,
\[ |E(t)| \leq 4e^{x_\infty} \max \{ |\varphi_1|_\infty, |\varphi_2|_\infty \} \int_X \lambda(\tau)d\mu \int_{-\infty}^t K(s, \tau)e^{-x_\infty s}ds \]
\[ + k_3 e^{x_\infty} \int_X C(\tau)d\mu \int_{-\infty}^{1+\infty} K(s, \tau)e^{-x_\infty s}ds \leq e^{x_\infty} \left( 4 \max \{ |\varphi_1|_\infty, |\varphi_2|_\infty \} (1 - \chi_1(x_*)) + k_3 \right) =: Ae^{x_\infty}, \ t \in \mathbb{R}. \]

Therefore \( e^{-x_\infty} E(t) \) belongs to \( L^k(\mathbb{R}) \), for each \( k \in [1, \infty) \) once \( x \in (0, x_*) \).

Using Lemma 2 we can repeat now the arguments of Step I of Theorem 3 (below the estimation of \( |e^{-x_\infty} D(t)| \)) to conclude that \( \omega(t) = e^{x_\infty r_x(t)} t \in \mathbb{R}, \ r_x \in L^2(\mathbb{R}) \), for each \( x \in (\lambda_1, x_*). \) This implies the absolute convergence of \( \int_{\mathbb{R}} e^{-x_\infty} \omega(s)ds \) for every \( x \in (\lambda_1, x_*). \) But as we have seen at the beginning of Step I, this yields \( |\omega(s)| \leq B_x e^{x_\infty}, \ s \in \mathbb{R}, \ x \in (\lambda_1, x_*) \) for appropriate \( B_x. \) Therefore \( \Gamma \geq x_* > \lambda_1, \) a contradiction. In this way, we have proved that
\[ |\omega(s)| \leq B_x e^{x_\infty}, \ s \in \mathbb{R}, \ x \in (\varepsilon, \min \{ \lambda_1, \gamma_\ast \}). \quad (21) \]

Step II. Suppose that \( \chi_1(m) > 0 \) for some \( m \in (0, \lambda_\gamma) \), it is clear that \( m > \lambda_1 \) and
\[ \kappa := \int_{\mathbb{R}} \int_X K(s, \tau)\lambda(\tau)d\mu(\tau)e^{-sm}ds < 1. \]

We now define \( \hat{\omega}(t) := |\omega(t)|e^{-mt} \geq 0, \ t \in \mathbb{R}. \) By (21), we obtain that \( \hat{\omega}(\pm \infty) = 0 \) and \( \hat{\omega}(t_m) = \max \{ \hat{\omega}(t), t \in \mathbb{R} \} \geq 0 \) for some \( t_m \in \mathbb{R}. \) Since
\[ \omega(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} \left( K(s, \tau)(g(\varphi_1(t-s), \tau) - g(\varphi_1(t-s), \tau))ds, \right) \]
we have
\[ \hat{\omega}(t_m) = |\omega(t_m)|e^{-mt_m} \leq \int_X \lambda(\tau)d\mu(\tau) \int_{\mathbb{R}} K(s, \tau)e^{-ms}\omega(t_m-s)|e^{-m(t_m-s)}ds \leq \]
\[
\bar{\omega}(t_m) \int_X \lambda(\tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-ms} ds = \bar{\omega}(t_m) \kappa.
\]

Hence, \( \bar{\omega}(\tau) = 0 \) and the uniqueness follows.

Step III. Suppose now that \( \chi_1(m) = \max_{s \in (0, \lambda_r)} \chi_1(s) = 0 \). Then additionally \( \chi'_1(m) = 0 \). Since \( \lambda(\tau) \) is different from \( g'(0, \tau) \), we have also that \( \lambda_t < m \). Furthermore, \( \bar{\omega}(t) := |\omega(t)|e^{-mt} \geq 0 \), \( t \in \mathbb{R} \) has the same properties as in Step II: \( \bar{\omega}(\pm \infty) = 0 \), \( \bar{\omega}(t_m) = \max_{s \in \mathbb{R}} \bar{\omega}(s) \geq 0 \) for some \( t_m \in \mathbb{R} \) and

\[
\bar{\omega}(t) \leq \int_{\mathbb{R}} \left( \int_X K(s, \tau) \lambda(\tau) e^{-ms} d\mu(\tau) \right) \bar{\omega}(t - s) ds.
\]

After normalizing, we may assume that \( 0 \leq \bar{\omega}(t) \leq 1 = \bar{\omega}(t_m) = 1 \), \( t \in \mathbb{R} \), for some finite rightmost \( t_m \). Then

\[
1 \leq \int_{\mathbb{R}} N_\lambda(s) \bar{\omega}(t_m - s) ds \leq \int_{\mathbb{R}} N_\lambda(s) ds = 1,
\]

where \( N_\lambda(s) := \int_X K(s, \tau) \lambda(\tau) e^{-ms} d\mu(\tau) \). This implies that \( N_\lambda(s) \bar{\omega}(t_m - s) = N_\lambda(s) \) a.e. and \( \bar{\omega}(t - s) = 1 \) for all \( s \) such that \( N_\lambda(s) > 0 \). Now, since \( \int_{\mathbb{R}} N_\lambda(s) ds = 1 \), \( \int_{\mathbb{R}} s N_\lambda(s) ds = 0 \), there is a subset of \( \mathbb{R}_+ \) of positive measure where \( N_\lambda(s) > 0 \). This means that \( t_m \) does not possess the property to be the rightmost point where \( \bar{\omega}(t_m) = 1 \), a contradiction. In consequence, \( \bar{\omega}(t) \equiv 0 \) that proves the uniqueness.

**Remark 7** It is enlightening to compare Theorem 4 and Theorem 2 in [33] where somewhat similar ideas were exploited. Indeed, from pure analytical estimations, without the use of asymptotic representations of solutions and without using the properties of \( \chi \) indicated in Lemma 2, Schumacher deduced that \( I' \geq \lambda_r \) (under assumptions made in [33]). In any case, monotonicity restrictions on the convolution term in [33] do not allow consider various interesting models (cf. Sections 6.3-6.4 below).

### 6 Applications

In this section, Theorems 3 and 4 are applied to several models which can be written as (11). This allows to improve or complement the uniqueness results in [1][4][5][12][16][36]. Everywhere in this section we assume that locally Lipschitzian \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), \( g(0) = 0 \), is differentiable at 0 with \( g'(0) > 0 \).

#### 6.1 A nonlocal integro-differential equation [1][8][9][20][28][33]

Consider the equation

\[
u_t = J * u - u + g(u), \tag{22}
\]

where \( J \geq 0 \), \( \int_{\mathbb{R}} J ds > 0 \). Let \( \gamma^# \) denote an extended positive real number such that \( \int_{\mathbb{R}} J(s) e^{-zs} ds \) is convergent for \( z \in [0, \gamma^#) \) and is divergent when \( z > \gamma^# \). As it can be easily deduced from Theorem 4, the existence of such \( \gamma^# \)
is automatically assured by the existence of positive semi-wavefronts \(u(t, x) = \phi(x + ct), \ \phi(-\infty) = 0\) to (22). Traveling wave profile \(\phi\) solves
\[
 c\phi' = J*\phi - \phi + g(\phi).
\]
In order to replace condition (3) with more weak
\[
g'(s) \leq g'(0) \text{ a.e. on } \mathbb{R}_+,
\]
we use the following trick. Set \(g\) by some positive \(M > 0\) compares two different solutions \(\phi\) with a constant \(c > 0\). Let us suppose that \(g(\phi)\) is a direct consequence of Theorem 3.

\[
 k \cdot (J*\phi)(t) + k \cdot g_\beta(\phi)(t) = (k \cdot J) * \phi(t) + k \cdot g_\beta(\phi)(t),
\]
where \(k(s) = c^{-1}e^{-s(1+\beta)/c}, s \geq 0\) and \(k = 0\) if \(s < 0\). Thus, equation (26) can be written as (4), with \(X = \{\tau_1, \tau_2\}\) and
\[
 K(s, \tau) = \begin{cases}
 k \cdot J(s), \ \tau = \tau_1, \\
 k(s), \ \tau = \tau_2, \\
 g(s, \tau) = \begin{cases}
 s, \ \tau = \tau_1, \\
 g_\beta(s), \ \tau = \tau_2.
\end{cases}
\end{cases}
\]

Finally, independently on the sign of \(c\), we find that
\[
 \chi(z, c) = 1 - \int_{\mathbb{R}} K(s, \tau_1)e^{-zs}ds - (g'(0) + \beta) \int_{\mathbb{R}} K(s, \tau_2)e^{-zs}ds = 1 - \frac{1}{1 + \beta + cz} \int_{\mathbb{R}} J(s)e^{-zs}ds - \frac{1 + g'(0) + \beta}{1 + \beta + cz} =: \tilde{\chi}(z, c).
\]

Let \(c_*\) be the minimal value of \(c\) for which
\[
 \tilde{\chi}(z, c) := 1 - g'(0) + cz - \int_{\mathbb{R}} J(s)e^{-zs}ds
\]
has at least one positive zero. It is easy to see that
\[
 c_* = \inf_{z > 0} \frac{1}{z} \left\{-1 + g'(0) + \int_{\mathbb{R}} J(s)e^{-zs}ds\right\}
\]
can be positive, negative (in these cases \(\inf\) can be replaced with \(\min\)) or zero. By Theorem 2, \(c \geq c_*\) for each admissible wave speed \(c\). The next result is a direct consequence of Theorem 3.
Theorem 5 Suppose \((24)\) together with \(1 = \int_R J(s)ds < g'(0)\) and
\[
|g(u) - g'(0)u| \leq Cu^{1+\alpha}, \quad u, v \in (0, \sigma) \quad \text{for some } \alpha, \sigma \in (0, 1],
\]
Then equation \((23)\) has at most one bounded positive solution \(\varphi, \varphi(-\infty) = 0, \) for each \(c \neq 0\) (if \(\check{x}(\gamma^# - c_\ast) \neq 0\)) or for each \(c \neq 0, c_\ast\) (if \(\check{x}(\gamma^# - c_\ast) = 0\)).

Proof Suppose that \(c > 0\) (the case \(c < 0\) is similar). We only have to check the assumptions (EC\(\gamma_0\)), (SB) except \(\gamma_0(c) < \gamma_K(c), \chi(0, c) < 0\) and \(\chi(\gamma_K - c) \neq 0\) of Theorem 3.

Step I. It is clear that \(g(\cdot, \tau)\) satisfies \((19)\), where \(g'(0, \tau_1) = 1, g'(0, \tau_2) = g'(0) + \beta\). Moreover, we have \(|g(u, \tau) - g'(0, \tau)u| \leq C(\tau)u^{1+\alpha}, \ u, v \in (0, \sigma)\), where \(C(\tau) = 0\) if \(\tau = \tau_1\) and \(C(\tau) = C\) if \(\tau = \tau_2\).

Step II. For each \(z > -\frac{c_0}{\beta} - \frac{1}{\beta c_0}\) we have \(\int_R k(s)e^{-zs}ds \geq \frac{1}{1 + \beta + cz} < +\infty\) so that \(\gamma_K(c) = \gamma^#\) because of \(\int_R k(s)e^{-zs}ds = \int_R J(s)e^{-zs}/(1 + \beta + cz)\). (Observe here that \(\gamma_K(c) = \min\{\gamma^#, -(1 + \beta)/c\}\) if \(c < 0\). However, if \(\gamma_K(c) = -(1 + \beta)/c\) then \(\chi(\gamma_K(c), c) = \infty\) so that \(\gamma_0(c) < \gamma_K(c)\) due to Corollary 1.)

Step III. If \(\varphi\) solves \((23)\), then \(\varphi \in C^1(R)\) and for each \(0 < z < \gamma_0\) we obtain
\[
c \int_R e^{-zs}|\varphi'(s)|ds \leq \int_R e^{-zs}J\varphi(s)ds + \int_R e^{-zs}\varphi(s)ds + \int_R e^{-zs}g(\varphi(s))ds \leq \int_R e^{-zs}J(s)ds + 1 + g'(0)\int_R e^{-zs}\varphi(s)ds < +\infty.
\]

Thus, by Lemma 3 condition (EC\(\gamma_0\)) is satisfied.

Step IV. We have \(\chi(0, c) = (1 - \int_R J(s)ds - g'(0))/(1 + \beta) < 0\). Now, if \(\gamma^# < +\infty\), then \(\check{x}(\gamma^# - c_\ast) \neq 0\) implies that \(\chi(\gamma^# - c_\ast) \neq 0\) and \(\gamma_0(c_\ast) = \lambda_0(c_\ast) < \gamma^#\). Since \(\chi(z, c)\) is strictly increasing in \(c\) for each fixed \(z > 0\), function \(\lambda_0(c)\) is strictly decreasing. Hence \(\gamma_0(c) = \lambda_0(c) < \gamma^#\) for each \(c \geq c_\ast\). Similar considerations show that \(\gamma_0(c) < \gamma^#\) for each \(c > c_\ast\) if \(\chi(\gamma^# - c_\ast) = 0\). Finally, in the case \(\gamma^# = +\infty\) we have that \(\chi(+\infty, c) \in (1, +\infty) \neq 0\), so that \(\chi(\gamma_K - c) \neq 0\) holds automatically.

Remark 8 Our approach allows to remove several restrictions on \(J\) and \(g\) assumed in the Carr and Chmaj uniqueness result [3, Theorem 2.1]. In the cited work \(g\) is supposed to satisfy [3] and \(J\) to be an even compactly supported function with \(\int_R Jds = 1\). These properties were essential in the proof of Theorem 2.1 in [3] even though [3] was not mentioned explicitly there. Similarly, conditions \(J \in C^1(R)\), \(J(a) > 0, J(b) > 0\) for some \(a < 0 < b\), and of \(J\) compactly supported were used by Coville et al. It was assumed in [3] that \(g'(0)g'(1) < 0\) together with \(g(u)/u \leq g'(0), \ u > 0\), instead of more restrictive \(g'(u) \leq g'(0), \ u > 0\). See also [3] for non-uniqueness of stationary traveling fronts (\(c = 0\)). Next, Schumacher [33], using a comparison method for differential inequalities combined with a Nagumo-point argument, established uniqueness of regular and non-critical semi-wavefronts to equation \((22)\) for general \(J\) and \(g\) satisfying \((24)\). The trick allowing to weaken the Lipschitz restriction [3] is due to Thieme and Zhao [36] (as far as we know). Usually it was applied under reversed inequality \(f'(s) \geq f'(0)\) to the second
(damping) term of equation, e.g. see also [17] and Section 6.3 for further generalizations. Here we show that this trick shows to be useful also in the case of birth functions. We would like to note that Theorem 5 remains true if we introduce a small delay \( h > 0 \) in the term \( g(\varphi(t-h)) \). Indeed, in such a case it suffices to replace \( k(s) \) with a positive fundamental solution \( v(s) \) of the scalar delayed equation \( cv'(s) = -v(s) - \beta v(s-h) \).

6.2 Nonlocal lattice equations \([6,16,26,30,41]\)

Now we consider semi-wavefronts \( w_j(t) = u(j+ct), u(-\infty) = 0 \), of the nonlocal lattice equation

\[
\frac{dw_j(t)}{dt} = D[w_{j+k}(t) - w_j(t)] - dw_j(t) + \sum_{k \in \mathbb{Z}} \beta(j-k)g(w_k(t-r)), \quad j \in \mathbb{Z},
\]

where \( \beta(k) \geq 0 \), \( \sum_{k \in \mathbb{Z}} \beta(k) = 1 \). Let \( \gamma^# \) be an extended positive real number such that \( \sum_{k \in \mathbb{Z}} \beta(k)e^{-zk} \) converges when \( z \in [0, \gamma^#) \) and is divergent when \( z > \gamma^# \). By Cauchy-Adamard formula, \( \gamma^# = -\lim \sup_{k \to +\infty} k^{-1} \ln \beta(-k) \), where by convention \( \ln(0) = -\infty \). The wave profile \( u \) satisfies

\[
cv'(x) = D[u(x+1)+u(x-1)-2u(x)]-du(x)+\sum_{k \in \mathbb{Z}} \beta(k)g(u(x-k-cr)). \quad (28)
\]

Again we take \( c > 0 \) for simplicity. Since \( u \) is bounded, we find that

\[
u(t) = \frac{1}{c} \int_{-\infty}^{t} e^{-\frac{\beta u(s)}{2}(t-s)} \left[ Du(s+1) + Du(s-1) + \sum_{k \in \mathbb{Z}} \beta(k)g(u(s-k-cr)) \right] ds
\]

\[
= D(H_{-1} + H_1) * u(t) + \sum_{k \in \mathbb{Z}} \beta(k)H_{k+cr} * g(u)(t), \quad (29)
\]

where

\[
H_\tau(t) = \begin{cases} e^{-\frac{\beta u(s)}{2}(t-\tau)}, & t \geq \tau, \\ 0, & t < \tau. \end{cases}
\]

Thus (29) can be written as (11), with \( X = \{ \tau_1, \tau_2 \} \) and

\[
K(s, \tau) = \begin{cases} D(H_{-1}(s) + H_1(s)), & \tau = \tau_1, \\ \sum_{k \in \mathbb{Z}} \beta(k)H_{k+cr}(s), & \tau = \tau_2. \end{cases}
\]

Next, \( \chi(z, c) = 1 - \int_{\mathbb{R}} K(s, \tau_1)e^{-zs}ds - g'(0) \int_{\mathbb{R}} K(s, \tau_2)e^{-zs}ds =
\]

\[
1 - \frac{2D \cosh(z)}{2D + d + cz} - \frac{g'(0)e^{-cz}}{2D + d + cz} \sum_{k \in \mathbb{Z}} \beta(k)e^{-kz} =: \tilde{\chi}(z, c).\]

Let \( c_* \) be the minimal value of \( c \) for which

\[
\chi(z, c) := d + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-cz}\sum_{k \in \mathbb{Z}} \beta(k)e^{-kz}
\]

has at least one positive zero. It is easily seen that \( c_* \) is well defined and is finite. By Theorem \( \ref{thm:pool} \), \( c \geq c_* \) for each admissible wave speed \( c \).

We are ready to apply our uniqueness results to (28).
Theorem 6  Suppose that $g$ satisfies \[(3), (27)\] and $g'(0) > d$. Then equation \[(25)\] has at most one bounded positive solution $u$, $u(-\infty) = 0$, for each $c \neq 0$ (if $\tilde{\chi}(\gamma^#, c_*) \neq 0$) or for each $c \neq 0, c_*$ (if $\tilde{\chi}(\gamma^#, c_*) = 0$).

Proof Step I. Obviously, $g(\cdot, \tau)$ verifies \[(3)\] with $g'(0, \tau_1) = 1$ and $g'(0, \tau_2) = g'(0)$. Moreover, we have $|g(u, \tau) - g'(0, \tau)u| \leq C(\tau)u^{1+\alpha}$, $u, v \in (0, \sigma)$, where $C(\tau_1) = 0$ and $C(\tau_2) = C$.

Step II. If $0 < z < \gamma#$, we get

$$
\int_{\mathbb{R} \times X} K(s, \tau) e^{-zs} ds d\mu = \int_{\mathbb{R}} D(H_{-1}(s) + H_1(s)) e^{-zs} ds + \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \beta(k) H_{k+c}(s) e^{-zs} ds = \frac{2D \cosh(z)}{2D + d + cz} + \frac{e^{-cz}}{2D + d + cz} \sum_{k \in \mathbb{Z}} \beta(k) e^{-kz}.
$$

Therefore $\gamma_K = \gamma#$ (if $c > 0$) and $\gamma_K = \min\{\gamma#, -(2D + d)/c\}$ (if $c < 0$).

Step III. If $u$ solves \[(25)\] with $c > 0$, then for each $0 < z < \gamma_\phi$ we obtain

$$
c \int_{\mathbb{R}} |u'(s)| e^{-zs} ds \leq D \int_{\mathbb{R}} (u(s + 1) + u(s - 1) + 2u(s)) e^{-zs} ds + d \int_{\mathbb{R}} u(s) e^{-zs} ds + g'(0) \int_{\mathbb{R}} u(s - k - cr) e^{-zs} ds =
$$

$$
\left(2D \cosh(z) + 1 + d + g'(0) e^{-cz} \sum_{k \in \mathbb{Z}} \beta(k) e^{-zk}\right) \int_{\mathbb{R}} u(s) e^{-zs} ds < +\infty.
$$

Thus, by Lemma 3 condition (EC$\gamma_\phi$) is satisfied.

Step IV. We have $\chi(0) = (d - g'(0))/(2D + d) < 0$. The proof of $\gamma_\phi(c) < \gamma#$ is the same as in Step IV of the previous section and is omitted.

Remark 9  Our approach allows to improve the uniqueness results of [16, Theorem 3.1], where additional conditions $\beta(k) = \beta(-k)$ and $\chi(\gamma K^-) = -\infty$ are assumed. Moreover, [16, Theorem 3.1] does not establish the uniqueness of the minimal wave. Similarly to Section 6.1, condition (3) in Theorem 6 can be replaced with more weak (24) if the nonlinear term is local and non-delayed. See [26], where a local and non-delayed variant of (28) was considered. Similarly to [7,8], and under the same conditions on $g$ as in [8], Guo and Wu prove their uniqueness result [26, Theorem 2] by means of the comparison argument. To establish the uniqueness in the degenerate case $(g'(0) - d)/(g'(1) - d) = 0$ (cf. Remark 3), about which is the main concern of [5], Chen et al. developed new interesting tools (magnification, compression, blow-up techniques, modified sliding method). Finally, we mention Ma and Zou uniqueness result from [30], where a local version of (28) is investigated. The Lipschitz condition (3) is not required in (30), it is supposed instead that $g'(s) \geq 0$, $g(s)/s \leq g'(0)$, $s > 0$. 

On uniqueness of semi-wavefronts 27
6.3 Nonlocal reaction-diffusion equation [17][24][32][36][38]

Here, we consider positive semi-wavefronts $u(t, x) = \phi(x + ct)$, $\phi(-\infty) = 0$, for non-local delayed reaction-diffusion equations

$$u_t(t, x) = u_{xx}(t, x) - f(u(t, x)) + \int_R k(w)g(u(t - h, x - w))dw, \quad h > 0,$$

where $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f(0) = 0$, is strictly increasing and $k \geq 0$. $\int_R kds = 1$, can be asymmetric (see [38] for further details concerning wave solutions in the presence of asymmetric non-local interaction). Let $\gamma^* > 0$ denote an extended positive real number such that $\int_R k(s)e^{-zs}ds$ converges when $z \in [0, \gamma^*)$ and diverges if $z > \gamma^*$. It is clear that profile $\phi$ must satisfy

$$y''(t) - cy'(t) - f(y(t)) + \int_R k(s)g(y(t - ch - s))ds = 0, \quad t \in \mathbb{R}.$$  \hspace{1cm} (31)

Equation (31) can be written as

$$y''(t) - cy'(t) - f(y(t)) + \int_R k_h(w)g(y(t - w))dw = 0, \quad t \in \mathbb{R},$$

where $k_h(w) = k(w - ch)$ and $f_h(s) = \beta s - f(s)$ for some $\beta > 0$.

Again, without restricting the generality, we may suppose that $f_\beta$ is a Lipschitzian function with $\text{Lip} f_\beta = \beta - \inf_{s \geq 0} f'(s)$. Indeed, our proof of uniqueness compares two solutions $\phi_1, \phi_2$. Since they are uniformly bounded by some positive $M > 0$, we can restrict our attention to a finite interval $[0, M]$. Let $\beta > f'(0) \geq 0$ be such that $f_\beta(s) = \beta s - f(s) \geq 0$ for all $s \in [0, M]$ and $\max_{s \in [0, M]} f'(s) \leq 2\beta - \inf_{s \geq 0} f'(s)$. But then

$$\left| \frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} \right| \leq \left( \beta - \inf_{s \geq 0} f'(s) \right), \quad s_1, s_2 \in [0, M].$$

Next, it is easy to see that the wave profile $\phi$ solves the equation

$$\phi(t) = \frac{1}{\sigma(c)} \left( \int_{-\infty}^t e^{\nu(t-s)}(\mathcal{G}\phi)(s)ds + \int_t^{+\infty} e^{\mu(t-s)}(\mathcal{G}\phi)(s)ds \right),$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}, \nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and $(\mathcal{G}\phi)(t) := \int_R k_h(s)g(\phi(t - s))ds + f_\beta(\phi(t))$. Equivalently,

$$\phi(t) = (\mathcal{K} * k_h) * g(\phi(t)) + \mathcal{K} * f_\beta(\phi(t)),$$

where

$$\mathcal{K}(s) = \sigma^{-1}(c) \begin{cases} e^{\nu s}, & s \geq 0, \\ e^{\mu s}, & s < 0. \end{cases}$$

We can invoke now Theorems[23][4] where $X = \{ \tau_1, \tau_2 \}$ and

$$K(s, \tau) = \begin{cases} (\mathcal{K} * k_h)(s), & \tau = \tau_1, \\ \mathcal{K}, & \tau = \tau_2, \end{cases} \quad g(s, \tau) = \begin{cases} g(s), & \tau = \tau_1, \\ f_\beta(s), & \tau = \tau_2. \end{cases}$$
Observe that $g(\cdot, \tau)$ meets $[19]$ with $\lambda(\tau_1) = g'(0)$, $\lambda(\tau_2) = 1 - \inf_{x \geq 0} f'(s)$. If $f'(0) \leq f'(v)$ for all $v \geq 0$, as in $[30]$, then $\beta - \inf_{x \geq 0} f'(s) = \beta - f'(0) = f'_\beta(0)$. We have also that

$$
\chi_1(z, c) = 1 - g'(0) \int_{\mathbb{R}} K(s, \tau_1) e^{-cz ds} - (\beta - \inf_{s \geq 0} f'(s)) \int_{\mathbb{R}} K(s, \tau_2) e^{-cz ds} = 1 - \frac{\beta - \inf_{x \geq 0} f'(s)}{\beta + cz - z^2} - \frac{g'(0)e^{-zch}}{\beta + cz - z^2} \int_{\mathbb{R}} k(s)e^{-cz ds} =: \tilde{\chi}_1(z).
$$

We see that $\gamma_K = \min\{\mu, \gamma^#\}$ so that $\gamma_\phi < \mu$. Let $c_\ast$ be the minimal value of $c$ for which

$$
\tilde{\chi}_1(z, c) := cz - z^2 + \inf_{s \geq 0} f'(s) - g'(0)e^{-zch} \int_{\mathbb{R}} k(s)e^{-cz ds}
$$

has at least one positive zero. This value is finite, well defined and does not depend on $\beta$. We will write $c_\ast$ instead of $c_\ast$ in the special case when $f'(0) \leq f'(v)$ for all $v \geq 0$. In such a case, we have $f'(0) = \inf_{x \geq 0} f'(s)$ and therefore $\chi_1 = \chi$. By Theorem $2$ $c \geq c_\ast$ for each admissible wave speed $c$.

**Theorem 7** Suppose $g$ satisfies $[5]$, $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, and $g, f \in C^{1, \alpha}$ in some neighborhood of $0$, and $g(0) = f(0) = 0$, $g'(0) < f'(0)$. Then equation $[30]$, has at most one positive semi-wavefront $u(t, x) = \phi(x + ct)$, $\phi(-\infty) = 0$, for each $c \geq c_\ast$ (if $\tilde{\chi}(\gamma^# - c_\ast) = 0$) or for each $c > c_\ast$ (if $\tilde{\chi}(\gamma^# - c_\ast) = 0$).

**Proof** Observe that $\beta \chi(0) = f'(0) - g'(0) < 0$, and $\chi_1(\gamma^# - c_\ast) \neq 0$ if $\tilde{\chi}_1(\gamma^# - c_\ast) \neq 0$. First let $c \geq c_\ast$, then $\chi_1(x, c) < \chi(x, c)$ so that $\chi_1(m, c) = 0$ for some $m \in (0, \lambda_{r_\beta})$. It is clear that $m = \lambda_{r_\beta}$ if and only if $m = \gamma^#$. Since $\chi_1(z, c)$ is strictly increasing in $c$ for each fixed positive $z$, this implies that $c = c_\ast$ and $\chi_1(\gamma^# - c_\ast) = 0$. Consequently, $m \in (0, \lambda_{r_\beta})$ for each $c \geq c_\ast$ (if $\tilde{\chi}(\gamma^# - c_\ast) \neq 0$) or for each $c > c_\ast$ (if $\tilde{\chi}(\gamma^# - c_\ast) = 0$).

Next, if $c = c_\ast$ then $\chi_1 = \chi$ and the inequality $\chi(\gamma^# - c_\ast) \neq 0$ guarantees that $\lambda_1(c_\ast) = \gamma_\phi(c_\ast) < \gamma^#$ for $c = c_\ast$. If $c > c_\ast$ then we have again $\lambda_1(c) = \gamma_\phi(c_\ast) < \gamma^#$ by monotone decreasing in $c$.

Step I. Since $|f'_\beta(0) u - f(x)| = |f'(0) u - f(u)|$, for an appropriate $C, \sigma$, it holds $|g(u, \tau) - g'(0, \tau) u| \leq C(\tau) u^{1 + \alpha}$, $u \in (0, \sigma)$.

Step II. We claim that for each $x \in (0, \gamma_K)$ and some $d_j(x)$ it holds

$$
0 \leq K(s, \tau_j) \leq d_j(x)e^{cz}, s \in \mathbb{R}.
$$

Indeed, if $j = 2$, we can even take $x = \mu, d_2 = 1/\sigma(c)$. Next, we have

$$
K(t, \tau_1) = \frac{1}{\sigma(c)} \left[ \int_{t - ch}^{t + \infty} e^{\mu(t - ch - v)} k(v) dv + \int_{-\infty}^{t - ch} e^{\mu(t - ch - v)} k(v) dv \right] \leq e^{-xh/\sigma(c)} \int_{\mathbb{R}} e^{-xv k(v) dv} e^{x t}.
$$

Since $\lambda_{r_\beta} \leq \gamma_K = \min\{\gamma^#, \mu\}$, the exponential estimations of $K$ in $(SB^\ast), (EC^\ast)(ii)$ are verified. This observation completes the proof of the theorem.
where $g$.

Here we study positive semi-wavefronts in (30) and its limit form (32) studied below.

6.4 Uniqueness of fast traveling fronts in delayed equations

Here we study positive semi-wavefronts $u(t,x) = \phi(x + ct), \phi(-\infty) = 0$, to

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \ x \in \mathbb{R},$$

(32)

where $g$ is a Lipschitzian function such that $|g'|_{L^{\infty}} > g'(0)$. Profile $\phi$ solves the delay differential equation

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t-hc)) = 0, \ t \in \mathbb{R}.$$  (33)

Similarly to Section 6.3 (where we take now $\beta = 0$), we find that $\phi$ satisfies

$$\phi(t) = \mathcal{K} * g(\phi)(t), \quad \mathcal{K}(s) = \frac{1}{\sigma(c)} \left\{ e^{\mu(s-ch)}, \ s \geq ch, \right.$$

$$ \left. e^{\mu(s-ch)}, \ s < ch, \right.$$  \hspace{1cm}

which is exactly the form considered in the DK theory (formally, we set $X = \{\tau\}, \mathcal{K}(s, \tau) = \mathcal{K}$ and $g(s, \tau) = g(s)$). Nevertheless, since $L > g'(0)$, the Diekmann-Kaper uniqueness theorem does not apply to (33).

In order to use Theorem 4 we first note that

$$\chi_1(z,c) = 1 - L \int_{\mathbb{R}} \mathcal{K}(s)e^{-sz}ds = 1 - \frac{L e^{-chc}}{1 + cz - z^2}.$$  

is well defined on $(\nu, \mu)$. Thus, $\gamma_K = \mu$ and since $\lim_{x \to \infty} \int_{\mathbb{R}} \mathcal{K}(s)e^{-sz}ds = +\infty$ we obtain that $\gamma_0 < \gamma_K$. The exponential estimations of $K$ in (SB*), (EC)**(ii) are also obviously verified.

Finally, let $c_\star$ be the minimal value of $c$ for which the equation $z^2 - cz - 1 + L e^{-chz} = 0$ has at least one positive root. This value is well defined and positive. It is easy to see that, for each $c > c_\star$ there exists $m > 0$ close to $\lambda_1$ from the right and such that $\chi_1(m) > 0$. Hence, we get the following

**Theorem 8** Suppose that $|g(s) - g(t)| \leq L|t-s|$, $s, t \geq 0$, and that $g \in C^{1,\alpha}$ in some neighborhood of 0 with $g'(0+) > 1$. Then, for every $c > c_\star$ equation (33) has at most one bounded positive solution $\phi$ vanishing at $-\infty$.

**Remark 11** Theorem 8 gives an alternative proof of the uniqueness result in [11, Theorem 1.1] where it was additionally assumed that $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and that $g'(0+) > 1$ in finite. Moreover, we give here a reasonably good lower bound $c_\star$ for the 'uniqueness' speeds. Observe that if $L = g'(0)$, then $c_\star$ coincides with the minimal speed of propagation $c_\star$.

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