SPECTRAL INVARIANCE FOR CERTAIN ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

ROBERT LAUTER, BERTRAND MONTHUBERT, AND VICTOR NISTOR

Abstract. We construct algebras of pseudodifferential operators on a continuous family groupoid \( G \) that are closed under holomorphic functional calculus, contain the algebra of all pseudodifferential operators of order 0 on \( G \) as a dense subalgebra, and reflect the smooth structure of the groupoid \( G \), when \( G \) is smooth. As an application, we get a better understanding on the structure of inverses of elliptic pseudodifferential operators on classes of non-compact manifolds. For the construction of these algebras closed under holomorphic functional calculus, we develop three methods: one using semi-ideals, one using commutators, and one based on Schwartz spaces on the groupoid.

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Introduction

Let \( M \) be a compact manifold and \( P \) an elliptic pseudodifferential operator of order \( m \geq 0 \) on \( M \). Assume that \( P \) is invertible as an unbounded operator on \( L^2(M) \) (the space of square integrable \( 1/2 \)-densities on \( M \)). A classical and very useful result states that then \( P^{-1} \) is also a pseudodifferential operator. For non-compact manifolds the situation is more complicated, essentially because we also want to control the behavior at infinity of the inverse.

For example, when \( M \) has cylindrical ends, a convenient class of pseudodifferential operators is that of \( b \)-pseudodifferential operators introduced by Melrose \([29, 30]\) (see also \([10]\)). Then it is known that the inverse \( P^{-1} \) of an elliptic \( b \)-pseudodifferential operator (defined in \( L^2 \)-sense) is not necessarily also a

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It is easily seen to be uniquely determined. If a also called an orthogonal generalized inverse or a Moore-Penrose inverse of \(a^*a\) of \(a^*a\), and the orthogonal projection \(p\) onto the kernel \(N(a^*a) = N(a)\) is given by the integral

\[
p := \frac{1}{2\pi i} \int_{\gamma} (z \text{id}_H - a^*a)^{-1}dz,
\]

where \(\gamma\) is a small circle around the origin in \(\mathbb{C}\) that does not intersect \(\sigma(a^*a)\). In that case, the operator

\[
\tilde{a} = (p + a^*a)^{-1}a^*
\]

is a relative inverse of \(a\). Because of \((\tilde{a}a)^* = \tilde{a}a\) and \((\tilde{a}a)^* = \tilde{a}a\) the operator \(\tilde{a}\) is also called an orthogonal generalized inverse or a Moore-Penrose inverse of \(a\) [37]; it is easily seen to be uniquely determined. If \(a : H \to H\) is additionally a Fredholm operator, then we have \(p = \text{id}_H - a\tilde{a}\), and \(q := \text{id}_H - a\tilde{a}\) is the orthogonal projection onto the orthogonal complement of \(R(a)\), i.e. \(\tilde{a}\) is a Fredholm inverse of \(a\). For more about generalized inverses we refer to [37].

In applications, we often know much more about the operator \(a\) than having closed range or being Fredholm, so it is natural to ask, which properties of \(a\) are inherited by the Moore-Penrose inverse \(\tilde{a}\). It is immediate from (1) and (2) that \(\tilde{a}\) belongs to the intersection of all subalgebras \(A \subseteq \mathcal{L}(H)\) that are symmetric with respect to the \(*\)-operation from \(\mathcal{L}(H)\) and closed under holomorphic functional calculus in \(\mathcal{L}(H)\); in particular, any property of \(a\) that can be covered by a symmetric subalgebra \(A\) of \(\mathcal{L}(H)\) that is closed under holomorphic functional calculus is true for the Moore-Penrose inverse as well. Thus, it is interesting to find algebras that are closed under holomorphic functional calculus. Without loss of generality, we can always assume that an algebra that is closed under holomorphic functional calculus is also symmetric.

Of particular importance for pseudodifferential and microlocal analysis are symmetric, continuously embedded Fréchet subalgebras of \(C^*\)-algebras that are closed under holomorphic functional calculus (\(\Psi^*\)-algebras [1]). Indeed, in contrast to the rather rigid \(C^*\)-topology, the Fréchet-topology allows a flexible treatment of \(C^\infty\)-phenomena within a functional analytic setting [13, 14]. On the other hand, stability under holomorphic functional calculus and symmetry still establishes a

\[1\] When \(n = 1\), the \(c_n\)-calculus is nothing but the \(b\)-calculus and, when \(n = 2\), it is usually called the “cusp-calculus,” see for instance [13] (which is based on earlier work of Melrose). Here, \(n\) should not be confused with the dimension of the manifold but determines the degree of degeneracy in direction to the boundary; more precisely, the \(c_n\)-calculus is modeled on the differential operators with degeneracies of the form \(x^n\partial_x\) at the boundary where \(x\) stands for the direction normal to the boundary.
strong relation between the structure of a $\Psi^*$-algebra and that of its $C^*$-closures leading to sometimes unexpected insights into the internal structure of a $\Psi^*$-algebra; for instance, the set of relatively invertible elements in $\Psi^*$-algebras has been shown to be a locally rational Fréchet manifold [11]. Starting from the seminal work [1] of Gramsch, the world of $\Psi^*$-algebras has been explored by many authors, and we refer the reader to [11, 12, 14, 15, 16, 19, 27] and the references given there for more details. Besides, the $K$-theory groups, $K_*(A)$, of a symmetric subalgebra of a $C^*$-algebra $B$ that is closed under holomorphic functional calculus coincide with those of its closure in $B$. Our basic example for an algebra that is closed under holomorphic functional calculus is the algebra of classical pseudodifferential operators of order 0 on a closed manifold.

In [24] the authors considered a pseudodifferential calculus on continuous family groupoids; this calculus generalizes the pseudodifferential calculus on $C^\infty,0$-foliations used by Connes to prove the index theorem for foliated spaces [7]. See [43, 44] for an introduction to the theory of pseudodifferential operators. In a slightly different context, Nistor, Weinstein, Xu [38] and Monthubert, Pierrot [36] have studied a pseudodifferential calculus on differentiable groupoids. As demonstrated by the examples in [35, 38] and the survey [23], the groupoid approach yields a pseudodifferential calculus for many interesting situations in analysis and geometry, especially on open manifolds and manifolds with singularities, in a unified way. Up to some support condition, this pseudodifferential calculus recovers the existing calculi, and in many cases goes beyond the results that are known in the literature. Let us only mention that the class of algebras that can be defined using groupoids include the ordinary pseudodifferential calculus, a $G$-equivariant pseudodifferential calculus on bundles of Lie groups, the $b$-calculus of Melrose and many of its cousins on manifolds with corners, the edge calculus on manifolds with fibered boundaries, the calculus of adiabatic pseudodifferential operators, and many others. On the other hand, we know for many of the different pseudodifferential calculi mentioned above that the algebra of operators of order 0 is not closed under holomorphic functional calculus. In fact, due to the support condition in the case of a general continuous family groupoid (which is a quite convenient condition that will insure that the composition is defined), the algebra of operators of order 0 is almost never closed under holomorphic functional calculus.

In the present paper, we develop a general strategy to embed the algebra $\Psi^{0,0}(\mathcal{G})$ of pseudodifferential operators of order 0 on a continuous family groupoid $\mathcal{G}$ into larger algebras $\mathcal{A}$ that are closed under holomorphic functional calculus and still share some of the interesting properties with the algebra $\Psi^{0,0}(\mathcal{G})$. (We shall denote by $\Psi^{m,0}(\mathcal{G})$ the space of order $m$ pseudodifferential operators on a continuous family groupoid $\mathcal{G}$.) In fact, it is one of the results of this paper that it usually suffices to embed the algebra $\Psi^{-\infty,0}(\mathcal{G})$ of operators of order $-\infty$ in an algebra $\mathcal{J}$ that is closed under holomorphic functional calculus. Up to some technical conditions, $\mathcal{A} := \Psi^{0,0}(\mathcal{G}) + \mathcal{J}$ is then an algebra that is closed under holomorphic functional calculus. For the construction of the algebra $\mathcal{J}$, we suggest three alternatives. The first one relates properties of an algebra to those of a two-sided ideal and its corresponding quotient, the second one is based on commutator methods from operator theory, whereas the third one, depends more on the geometry of the groupoid and requires the existence of a length function $\phi$ with polynomial growth on the groupoid. The role of $\phi$ is to define a Schwartz space $\mathcal{J}(\mathcal{G})$ on $\mathcal{G}$ that
replaces $C_c^\infty(G) = \Psi^{-\infty,0}(G)$. In this way, we control the behavior at infinity of the kernels of our pseudodifferential operators on a typical leaf $d^{-1}(x)$ of the groupoid.

The paper is organized as follows: In Section 1 we recall the notion of algebras closed under functional calculus, and consider and answer the question, whether an algebra is closed under functional calculus provided an ideal and the corresponding quotients are. In Section 2 we develop the operator theoretical methods (based on commutators) that are used in Section 3 to construct algebras $\mathcal{A}$ containing $\Psi^{0,0}(G)$ and closed under functional calculus. In Section 4 we introduce the $c_n$-calculi on manifolds with boundary and with corners. One can embed the $c_n$-calculi, $n \geq 2$, in $\Psi^*$-algebras that consist of smooth kernels, a result that is proved in Section 5. Section 6 is devoted to the study of the Schwartz space $\mathcal{S}(G)$ of a continuous family groupoid and the proof that it is closed under functional calculus. We also define length functions for the groupoids associated to various pseudodifferential calculi on manifolds with corners (the $b$-calculus, the "cusp"-calculus, or, more generally, the $c_n$-calculus).

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1. **Algebras closed under holomorphic functional calculus**

In this section we recall some basic facts about algebras that are closed under holomorphic functional calculus and describe a method to generate algebras closed under holomorphic functional calculus. This method is based on permanence properties of the closure under holomorphic functional calculus when passing to two-sided ideals, to quotients, or back from ideals and quotients to the algebra.

1.1. **Definitions.** We begin by recalling the definition of an algebra closed under holomorphic functional calculus.

**Definition 1.** Let $\mathcal{B}$ be a Banach algebra with unit $e$. A not necessarily unital subalgebra $A \subseteq \mathcal{B}$ is said to be closed under the holomorphic functional calculus in $\mathcal{B}$ provided for every $a = \lambda e + x \in \mathcal{C}e + A$ and all $f \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$, we have $f(a) \in \mathcal{C}e + A$.

Here $\mathcal{O}(\sigma_{\mathcal{B}}(a))$ stands for the algebra of germs of holomorphic functions on the spectrum $\sigma_{\mathcal{B}}(a)$ of $a$ with respect to the Banach algebra $\mathcal{B}$, and

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(z)(ze - a)^{-1}dz \in \mathcal{B}$$

(3)

is the operator given by the usual holomorphic functional calculus within the Banach algebra $\mathcal{B}$. For any algebra $A$ with unit, we shall denote by $A^{-1}$ the set of invertible elements of $A$.

**Remark 1.** The following observations are immediate.

(a) Let $\mathcal{B}$ be a unital $C^*$-algebra and $A \subseteq \mathcal{B}$ a symmetric subalgebra (i.e. closed under taking adjoints), then we have $\sigma_{\mathcal{B}}(a) = \sigma_{A_e}(a)$ where $A_e$ is the completion of $\mathcal{C}e + A$ with respect to the norm induced by $\mathcal{B}$, so Definition 1 recovers Definition 1, p. 285].
(b) An arbitrary intersection of algebras closed under holomorphic functional calculus in $B$ is again closed under holomorphic functional calculus in $B$.

(c) If $A \subseteq B$ is closed under the holomorphic functional calculus in $B$, then $A$ is spectrally invariant in $B$, i.e. for the groups of invertible elements we have

$$(Ce + A) \cap B^{-1} = (Ce + A)^{-1},$$

or, equivalently, $j^{-1}(B^{-1}) = (Ce + A)^{-1}$ if $j : Ce + A \rightarrow B$ denotes the natural inclusion.

It is often useful to consider algebras closed under holomorphic functional calculus that are in addition complete with respect to a finer topology. For example, the class of $\Psi^*_\ast$-algebras introduced by Gramsch [11] in connection with a perturbation theory for singular integral and pseudodifferential operators is, in fact, the appropriate setting to describe $C^\infty$-phenomena of microlocal analysis within a functional analytic framework [14].

**Definition 2.** Let $B$ be a unital $C^\ast$-algebra. A symmetric, spectrally invariant subalgebra $A \subseteq B, e \in A$, is called a $\Psi^*$-algebra in $B$ if, and only if, there exists a Fréchet topology $T_A$ on $A$ making the embedding $\iota : (A, T_A) \rightarrow (B, \|\cdot\|_B)$ continuous.

In case $e \notin A$, the algebra $A$ is said to be a non-unital $\Psi^*$-algebra provided $Ce \oplus A$ is a $\Psi^*$-algebra.

Following the usual convention, in the sequel a $\Psi^*$-algebra is always unital. A unital Fréchet algebra $A$ is said to be submultiplicative if the topology $T_A$ can be generated by a countable system $(q_j)_{j \in \mathbb{N}}$ of submultiplicative semi-norms, i.e. semi-norms satisfying $q_j(xy) \leq q_j(x)q_j(y)$ and $q_j(e) = 1$.

We recall a few basic facts about (non-unital) $\Psi^*$-algebra, most of them are obvious consequences of the definition.

**Proposition 1.** Let $B$ be a unital $C^\ast$-algebra and $A \subseteq B$ a subalgebra.

(a) If $A$ is a non-unital $\Psi^*$-algebra, then there is a Fréchet-topology $T_A$ on $A$ such that $(A, T_A) \rightarrow (B, \|\cdot\|_B)$ is continuous.

(b) If $A$ is a $\Psi^*$-algebra, then the group of invertible elements $A^{-1}$ is open and the inversion

$$(4) \quad A^{-1} \ni x \mapsto x^{-1} \in A$$

is continuous.

(c) If $A$ is a non-unital $\Psi^*$-algebra, then for any $a \in A$ there exists an analytic map $h : q_B(a) \rightarrow A$ where $q_B(a) \subseteq C$ is the resolvent set, such that

$$(\lambda e - a)^{-1} = \frac{1}{\lambda} e + h(\lambda).$$

(d) Any $\Psi^*$-algebra, unital or not, is closed under functional calculus. If $A$ is non-unital and $a = 1 + x \in Ce \oplus A$ is arbitrary, then $f(a) - f(\lambda)e \in A$ for all $f \in O(\sigma_B(a))$

**Proof.** For (b) it suffices to use an old result of Banach [3] which says that the inversion in a Fréchet algebra is continuous if and only if the group of invertible elements is a $G_\delta$-set.
Definition 3. Let $\mathcal{B}$ be a Banach algebra with unit $e$, and $\varphi : A \to \mathcal{B}$ be a morphism of algebras, which we assume to preserve the unit if $A$ has one. Then $A$ is called locally spectral invariant with respect to $\varphi$, if there exists $\varepsilon > 0$ such that we have

$$(e + \varphi(x))^{-1} \in \mathbb{C} + \varphi(A)$$

for all $x \in A$ with $\|\varphi(x)\|_{\mathcal{B}} < \varepsilon$. In that case, we say that $A$ has property $(P_A)$ in $\mathcal{B}$, and the morphism $\varphi$ is to be understood from the context. Moreover, $A$ is said to have property $(\tilde{P}_A)$ if $A$ is unital and $\varphi^{-1}(\mathcal{B}^{-1}) = A^{-1}$.

By [11, Lemma 5.3], properties $(P_A)$ and $(\tilde{P}_A)$ are closely related in many interesting cases.

Lemma 1. Let $\varphi : A \to \mathcal{B}$ be an injective morphism of unital algebras. Then

(a) If $\mathcal{B}$ is a unital Banach algebra and $\varphi(A)$ is dense in $\mathcal{B}$, then we have $(P_A) \iff (\tilde{P}_A)$.

(b) If $\mathcal{B}$ is a unital $C^*$-algebra and $\varphi(A)$ is symmetric in $\mathcal{B}$, then we have $(P_A) \iff (\tilde{P}_A)$.

In other words, $(\tilde{P}_A)$ is not stronger than the (apparently weaker) condition $P_A$, provided that either $A$ is dense in $\mathcal{B}$ or $A$ is symmetric.

We close this subsection with a lemma that helps to detect spectrally invariant subalgebras. It will be used in Section 6 to prove the spectral invariance of the Schwartz convolution algebra on a continuous family groupoid. We include here a proof that is closely related to the one of the previous Lemma, for the sake of completeness.

Lemma 2. Let $\mathcal{B}$ be a Banach algebra with unit $e$ and $\mathcal{A}$ a Banach algebra such that $e \in \mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A}$ is dense in $\mathcal{B}$, and $\varrho_{\mathcal{A}}(a) = \varrho_{\mathcal{B}}(a)$ holds for all $a \in \mathcal{A}$, where $\varrho_T(a)$ stands for the spectral radius of $a$ in a Banach algebra $T$.

Then we have

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1},$$

i.e. $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$ and $\mathcal{A}$ is closed under holomorphic functional calculus in $\mathcal{B}$.

Proof. Let $a \in \mathcal{A} \cap \mathcal{B}^{-1}$ be arbitrary. By the density of $\mathcal{A}$ in $\mathcal{B}$ there exists $y \in \mathcal{A}$ with $\|ay - e\|_{\mathcal{B}} \leq 1/2$, hence $\varrho_{\mathcal{A}}(ay - e) \leq 1/2$, and $ay$ is invertible in $\mathcal{A}$. The rest is clear.

The above statement generalizes right away to non-unital algebras.

1.2. Spectral invariance and ideals. We are now going to look more closely at the question of how spectral invariance of an algebra is related to that of its quotients, bearing in mind Proposition 1 (that a compatible Fréchet topology on a spectrally invariant subalgebra implies stability under holomorphic functional calculus). We start by describing the general setting.

From now on and throughout this section, $\mathcal{B}$ will be a Banach algebra with unit $e$, $\mathcal{A} \subseteq \mathcal{B}$ will be a subalgebra with $e \in \mathcal{A}$, $\mathcal{J} \subseteq \mathcal{B}$ will be a proper, closed two-sided ideal in $\mathcal{B}$, and $I \subseteq \mathcal{A}$ will be a two-sided ideal in $\mathcal{A}$ with $I \subseteq \mathcal{J}$. Then the map

$$(5) \quad \varphi : A/I \longrightarrow \mathcal{B}/\mathcal{J} : a + I \longmapsto a + \mathcal{J}$$
is a well-defined homomorphism of unital algebras. Also, note that \( \varphi \) is one-to-one if, and only if, \( \mathcal{J} \cap A = A \). Thus, we have a commutative diagram with exact rows

\[
0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0
\]

\[
0 \rightarrow \mathcal{J} \rightarrow \mathcal{B} \rightarrow \mathcal{B}/\mathcal{J} \rightarrow 0
\]

The next theorem relates the different properties \((P_1), (P_A), (P_{A/1})\), and \((\bar{P}_{A/1})\), to one another. A special case of part (a) can be found in \([12, \text{Section 5}]\), (b) is from \([12, \text{Appendix}]\).

**Theorem 1.** Let \( A, I, \mathcal{B}, \mathcal{J} \), and \( \varphi \) be as above (Equation \((5)\)). We write \((\bar{P})\) instead of \((\bar{P}_{A/1})\), for simplicity; the property \(P_1\) is considered with respect to the obvious morphism \( I \rightarrow \mathbb{C} \oplus \mathcal{J} \). Then we have

(a) If \( I \subseteq \mathcal{J} \) is dense, then \((P_1)\) together with \((\bar{P})\) imply \((P_A)\).

(b) If \( I \subseteq \mathcal{J} \) is dense, then \((P_A)\) implies \((P_{A/1})\).

(c) If \( I \subseteq \mathcal{J} \) and \( A \subseteq \mathcal{B} \) are dense, then \((P_A)\) implies \((\bar{P})\).

(d) \((P_A)\) implies \((P_1)\).

(e) \((\bar{P})\) implies \((P_{A/1})\); moreover, if \( \varphi : A/I \rightarrow \mathcal{B}/\mathcal{J} \) is one-to-one and \( \varphi(A/I) \) is dense in \( \mathcal{B}/\mathcal{J} \), then we have \((P_{A/1})\) if and only if we have \((\bar{P})\).

**Proof.** Let \( \varepsilon > 0 \) always be the constant from Definition 3.

(a) By the continuity of the inversion in the unital Banach algebra \( \mathcal{B}/\mathcal{J} \), we can find \( 0 < \delta < \varepsilon/2 < 1 \) such that \( \| (a+\mathcal{J})^{-1} - (e+\mathcal{J}) \|_{\mathcal{B}/\mathcal{J}} < \varepsilon/8 \), for all \( a \) in \( \mathcal{B} \) satisfying \( \| (b+\mathcal{J}) - (e+\mathcal{J}) \|_{\mathcal{B}/\mathcal{J}} < \delta \).

Consider now \( a \in A \) with \( \| a - e \|_{\mathcal{B}} < \delta \). From \( \| (a - e) + \mathcal{J} \|_{\mathcal{B}/\mathcal{J}} < \delta \) and \((\bar{P})\), we obtain that there exists \( a_1 \in A \) with \( a_1 - e =: x_1 \in I \) and \( \| a_1 - e + \mathcal{J} \|_{\mathcal{B}/\mathcal{J}} < \varepsilon/8 \).

Using the density of \( I \) in \( \mathcal{J} \), we find \( x_2 \in I \) with \( \| a_1 - e + x_2 \|_{\mathcal{B}} < \varepsilon/4 \), hence

\[
\| x_1 + ax_2 \|_{\mathcal{B}} \leq \| a \|_{\mathcal{B}} \| a_1 - e + x_2 \|_{\mathcal{B}} + \| a - e \|_{\mathcal{B}} < \varepsilon,
\]

and by \((P_1)\) we get \( z \in I \) with \( (e + x_1 + ax_2)^{-1} = e + z \), i.e.

\[
a(a_1 + x_2)(e + z) = e,
\]

which gives \( a \in A^{-1} \), and completes the proof of (a).

(b) Let \( 0 < \delta < \varepsilon/2 \), and \( a + I \in A/I \) be with \( \| \varphi(a + I) - (e + \mathcal{J}) \|_{\mathcal{B}/\mathcal{J}} < \delta \). By the density of \( I \) in \( \mathcal{J} \), there exists \( x \in I \) with \( \| a + x - e \|_{\mathcal{B}} < \varepsilon \), hence \((P_A)\) gives \( a_1 \in A \) with \( (a + x)a_1 = e = a_1(a + x) \). We thus obtain

\[
\varphi(a + I)^{-1} = \varphi(a_1 + I) \in \varphi(A/I),
\]

and hence (b) is proved.

(c) Let \( a + I \in A/I \) be with \( \varphi(a + I) \in (\mathcal{B}/\mathcal{J})^{-1} \). Thus, there exists \( b \in \mathcal{B} \) and \( y \in \mathcal{J} \) with \( ab - y - e = 0 \). By the density assumption we obtain \( x \in I \) and \( a_1 \in A \) with \( \| a a_1 - x - e \|_{\mathcal{B}} < \varepsilon \), hence \( a a_1 - x \in A^{-1} \) by \((P_A)\). Let \( a_2 \in A \) be such that \( e = (aa_1 - x)a_2 = aa_1a_2 - x a_2 \). Consequently, \( a_1a_2 + I \in A/I \) is a right-inverse of \( a + I \) in \( A/I \). Similarly, we obtain also a left-inverse of \( a + I \), which gives \( a + I \in (A/I)^{-1} \), and completes the proof of (c).

To prove (d), let \( \varepsilon > 0 \) and \( x \in I \) be with \( \| x \|_{\mathcal{B}} < \varepsilon \). Then \( e + x \) is invertible in \( A \) because of \((P_A)\). Then, \((P_1)\) is a consequence of \((P_A)\) and the identity

\[
(e + x)^{-1} - e = -x + (e + x)^{-1} x^2 \in I.
\]
Finally, (e) is a simple, straightforward computation using Lemma 1.

We are mostly interested in the question when \( A \) is spectrally invariant in \( B \). The following special case of Theorem 1 will be used in the sequel.

**Corollary 1.** Let \( B \) be a unital C*-algebra, and \( J, A, \) and \( I \) as above, but additionally symmetric with respect to the \(*\)-operation in \( B \). Assume that \( I \) is dense in \( J \). Then \( A \) is spectrally invariant in \( B \) provided

(a) \( \mathbb{C}e \oplus I \) is spectrally invariant in \( \mathbb{C}e \oplus J \), and

(b) \( \varphi^{-1}((B/J)^{-1}) = (A/I)^{-1} \).

2. Semi-ideals

We now turn to a method of constructing algebras closed under holomorphic functional calculus, or, more generally, \( \Psi^* \)-algebras. More precisely, in order to be able to deal with non-unital algebras we need to study composition with possibly unbounded operators and the semi-ideals generated by this process.

2.1. Definitions. We now introduce semi-ideals.

**Definition 4.** A subspace \( J \subseteq B \) of a unital algebra \( B \) is said to be a semi-ideal in \( B \) provided that we have \( xyb \in J \) for all \( x, y \in J \) and all \( b \in B \).

**Remark 2.**

(a) A left, right or two-sided ideal is obviously a semi-ideal.

(b) Let \( B \) be a unital algebra and \( J \subset B \) be a proper semi-ideal. Then we have \( e \notin J \) and \( J \cap B^{-1} = \emptyset \) because otherwise we would contradict the identities \( b = ebe \) and \( e = x(x^{-1})^2x \).

**Proposition 2.** Let \( A \subseteq B \) be a \( \Psi^* \)-algebra, and \( J \subset A \) be a proper semi-ideal.

(a) \( 0 \in \sigma_B(x) \) for all \( x \in J \).

(b) \( J \) is closed under the holomorphic functional calculus in \( B \).

**Proof.** The first statement is an immediate consequence of Remark 2(b) and the spectral invariance of \( A \) in \( B \). If \( a = \lambda e + x \in \mathbb{C}e \oplus J \) and \( f \in \mathcal{O}(\sigma_B(a)) \) are arbitrary, then we have \( \lambda \in \sigma_B(a) \) by (1), so for all \( \mu \notin \sigma_B(a) \) we have

\[
(\mu e - a)^{-1} = \frac{1}{\mu - \lambda}e + \frac{1}{(\mu - \lambda)^2}x + \frac{1}{(\mu - \lambda)^2}(\mu e - a)^{-1}x
\]

together with the definition (3) of \( f(a) \) lead to

\[
f(a) = f(\lambda)e + f'(\lambda)x + x \frac{1}{2\pi i} \int_{\gamma} \frac{f(\mu)}{(\mu - \lambda)^2} (\mu e - a)^{-1} d\mu \chi_e \in A
\]

which gives \( f(a) - f(\lambda)e \in J \), and completes the proof.

**Remark 3.** A careful inspection of the above proof shows that (a) holds for any spectrally invariant, unital subalgebra \( A \subseteq B \), and (b) remains true for any spectrally invariant, continuously embedded, sequentially complete, locally convex topological algebra \( A \) with continuous inversion and \( e \in A \).
If $B$ is in addition a $C^*$-algebra, we can assume without loss of generality that algebras closed under holomorphic functional calculus are symmetric. More precisely, we have the following result.

**Lemma 3.** If $A \subseteq B$ is closed under the holomorphic functional calculus in the $C^*$-algebra $B$, then $A_\ast := \{ a \in A : a^\ast \in A \}$ is symmetric and closed under the holomorphic functional calculus in $B$.

**Proof.** It is sufficient to note that we have $f(a)^\ast = f^\ast(a^\ast)$ if $f \in \mathcal{O}(\sigma_B(a))$ and $[f^\ast : z \mapsto f(\overline{z})] \in \mathcal{O}(\sigma_B(a^\ast))$. \hfill $\square$

### 2.2. A commutator method.

We shall use several procedures to construct subalgebras closed under holomorphic functional calculus. The first one leads to $\Psi^*$-algebras using commutator methods, whereas the second one produces a semi-ideal, hence also an algebra closed under functional calculus. We begin by recalling the construction of submultiplicative $\Psi^*$-algebras using commutators with closed, symmetric operators. These techniques were first used in [4] for the characterization of pseudodifferential operators on $\mathbb{R}^n$, and later on investigated systematically in [15], for instance. Here we follow the presentations in [15, 19]. Let us start with a description of the general setting.

Let $\mathcal{K}$ be a Hilbert space, $(\mathcal{A}, (\| \cdot \|_j)_{j \in \mathbb{N}})$ be a submultiplicative $\Psi^*$-algebra in $\mathcal{L}(\mathcal{K})$ with $\| \cdot \|_1 = \| \cdot \|_{\mathcal{L}(\mathcal{K})}$. Also, let $\mathfrak{T}$ be a finite set of densely defined, closed, symmetric operators $T : \mathcal{K} \supseteq \mathcal{D}(T) \longrightarrow \mathcal{K}$. By [19, Lemma 2.17] each $T \in \mathfrak{T}$ induces a closed $^*$-derivation

$$
\delta_T : \mathcal{A} \supseteq \mathcal{D}(\delta_T) \longrightarrow \mathcal{A}.
$$

Here, we have $a \in \mathcal{D}(\delta_T)$ if, and only if, $a(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$, there exists $\delta_T(a) \in \mathcal{A}$ with $\delta_T(a)\varphi = i(Ta\varphi - aT\varphi)$, for all $\varphi \in \mathcal{D}(T)$, and the same is also true for $a^\ast$.

Furthermore, as shown in [19, Section 2.2], the finite set $\{ \delta_T : T \in \mathfrak{T} \}$ leads to the following scale of symmetric subalgebras of $\mathcal{A}$

$$
\begin{align*}
\Psi^0(\mathfrak{T}) &:= \mathcal{A}, \\
\Psi^1(\mathfrak{T}) &:= \bigcap_{T \in \mathfrak{T}} \mathcal{D}(\delta_T), \\
\Psi^r(\mathfrak{T}) &:= \{ a \in \Psi^{r-1}(\mathfrak{T}) : \delta_T(a) \in \Psi^{r-1}(\mathfrak{T}) \text{ for all } T \in \mathfrak{T} \}, \quad r \geq 2.
\end{align*}
$$

Each of the algebras $\Psi^r(\mathfrak{T})$ is endowed with a system of semi-norms, namely,

$$
q_{0,j}(a) := \|a\|_j, \quad \text{for } a \in \Psi^0(\mathfrak{T}), j \in \mathbb{N}, \quad \text{and}
$$

$$
q_{r,j}(a) := q_{r-1,j}(a) + \sum_{T \in \mathfrak{T}} q_{r-1,j}(\delta_T(a)), \quad \text{for } a \in \Psi^r(\mathfrak{T}), r, j \in \mathbb{N}.
$$

Finally, we endow the algebra $\Psi^\infty(\mathfrak{T}) := \bigcap_{r=0}^\infty \Psi^r(\mathfrak{T})$ with the system of semi-norms $(q_{r,j})_{r,j \in \mathbb{N}}$.

Similarly, the set $\mathfrak{T}$ induces the scale of so-called $\mathfrak{T}$-Sobolev spaces by

$$
\begin{align*}
\mathcal{H}^0(\mathfrak{T}) &:= \mathcal{K}, \\
\mathcal{H}^1(\mathfrak{T}) &:= \bigcap_{T \in \mathfrak{T}} \mathcal{D}(T), \\
\mathcal{H}^r(\mathfrak{T}) &:= \{ x \in \mathcal{H}^{r-1}(\mathfrak{T}) \mathfrak{T}x \in \mathcal{H}^{r-1}(\mathfrak{T}) \text{ for all } T \in \mathfrak{T} \}, \quad r \geq 2.
\end{align*}
$$
As above, the spaces $\mathcal{H}^r(\mathfrak{T})$ are endowed with the iterated graph norms with respect to the system $\mathfrak{T}$, i.e. $p_0(x) := \|x\|_\infty$ and

$$p_r(x) := p_{r-1}(x) + \sum_{T \in \mathfrak{T}} p_{r-1}(Tx), \quad x \in \mathcal{H}^r(\mathfrak{T}), \quad r \geq 1.$$  

The intersection $\mathcal{H}^\infty(\mathfrak{T}) := \bigcap_{r=0}^\infty \mathcal{H}^r(\mathfrak{T})$ is endowed with the system of norms $(p_r)_{r \in \mathbb{N}}$. Moreover, let

$$\Psi^r(\mathfrak{T}) := \bigcap_{x \in J^0(\mathfrak{T})} \mathcal{H}^r(\mathfrak{T}),$$

Moreover, the projective limit $(\Psi^r(\mathfrak{T}))$ is endowed with the natural map

$$\Psi^r(\mathfrak{T}) \times \mathcal{H}^r(\mathfrak{T}) \to \mathcal{H}^r(\mathfrak{T}) : (a, \varphi) \mapsto a(\varphi)$$

is bilinear and continuous.

2.3. Commutators and semi-ideals. The second construction that we shall need associates to the algebra $A$ and the system $\mathfrak{T}$ a semi-ideal in $L(\mathcal{K})$.

Indeed, let $J_0(\mathfrak{T}) := A$, and denote by $J_1(\mathfrak{T})$ the space of all $x \in J_0(\mathfrak{T})$ such that, for all $T, T_1, T_2 \in \mathfrak{T}$, we have

(a) $x(\mathfrak{T}) \subseteq D(T)$ and $\omega^r_T(x) := Tx \in J_0(\mathfrak{T})$.  
(b) $x(\mathfrak{T}) \subseteq D(T_1)$, and there is $\omega^r_{T_1,T_2}(x) \in J_0(\mathfrak{T})$ such that $\omega^r_{T_1,T_2}(x)f = T_1xT_2f$ for all $f \in D(T_2)$.  

Moreover, let $J_{k+1}(\mathfrak{T})$ be the space of all $x \in J_k(\mathfrak{T})$ such that we have $\omega^r_T(x)$, $\omega^r_{T_1,T_2}(x) \in J_k(\mathfrak{T})$ for all $T, T_1, T_2 \in \mathfrak{T}$.

We endow the spaces $J_k(\mathfrak{T})$ with the following systems $(p_{j,k})_{j \in \mathbb{N}_0}$ of norms: let

$$p_{j,0}(x) := \|x\|_j, \quad x \in J_0(\mathfrak{T}),$$

and let

$$p_{j,k+1}(x) := p_{j,k}(x) + \sum_{T \in \mathfrak{T}} (p_{j,k}(\omega^r_T(x)) + p_{j,k}(\omega^r_{T_1,T_2}(x))) + \sum_{T_1, T_2 \in \mathfrak{T}} p_{j,k}(\omega^r_{T_1,T_2}(x)), \quad x \in J_{k+1}(\mathfrak{T}).$$

Moreover, the projective limit $J^\infty(\mathfrak{T}) := \bigcap_{k=0}^\infty J_k(\mathfrak{T})$ is endowed with the projective topology given by the system of norms $(p_{j,k})_{j,k \in \mathbb{N}_0}$.

Let us collect the main properties of this construction in the following theorem.

Theorem 3. Let $(\mathcal{A}, (\|\cdot\|_j)_{j \in \mathbb{N}})$ be a submultiplicative $\Psi^*\cdot$-algebra. The we have for $k \in \mathbb{N}_0 \cup \{\infty\}$:

(a) $(\mathcal{J}_k(\mathfrak{T}),(p_{j,k}))$ is a submultiplicative Fréchet algebra. The canonical embedding $\mathcal{J}_k(\mathfrak{T}) \hookrightarrow J_0(\mathfrak{T})$ is continuous.

(b) $\Psi^\infty(\mathfrak{T}) \mathcal{J}_k(\mathfrak{T}) \subseteq J_0(\mathfrak{T})$ and $J_k(\mathfrak{T}) \Psi^\infty(\mathfrak{T}) \subseteq J_k(\mathfrak{T})$; the two canonical bilinear maps,

$$\Psi^\infty(\mathfrak{T}) \times J_k(\mathfrak{T}) \to J_k(\mathfrak{T}) \quad \text{and} \quad J_k(\mathfrak{T}) \times \Psi^\infty(\mathfrak{T}) \to J_k(\mathfrak{T}),$$

are jointly continuous.

(c) $\mathcal{J}_k(\mathfrak{T})$ is a semi-ideal in the $\Psi^*$-algebra $\mathcal{A}$; in particular, $\mathcal{J}_k(\mathfrak{T})$ is closed under the holomorphic functional calculus in $L(H)$. Moreover, the canonical map

$$\mathcal{J}_k(\mathfrak{T}) \times \mathcal{A} \times \mathcal{J}_k(\mathfrak{T}) \to \mathcal{J}_k(\mathfrak{T})$$

is jointly continuous.
Proof. The proofs are by induction with respect to \( k \). Since the arguments for the steps from \( k \) to \( k + 1 \) are the same as for \( k = 1 \) the proofs of the steps are omitted.

For (a), let us first assume that we have \( x, y \in \mathcal{J}_1(\mathfrak{I}) \). Then we have \( xy \in \mathcal{J}_1(\mathfrak{I}) \) with

\[
\omega_T^r(xy) = \omega_T^r(x)y, \; \omega_T^l(xy) = x\omega_T^l(y), \; \text{and} \; \omega_{T_1,T_2}^{r,r}(xy) = \omega_{T_1}^r(x)\omega_{T_2}^r(y)
\]

for all \( T, T_1, T_2 \in \mathfrak{I} \); the submultiplicativity is now immediate whereas for the completeness of \( \mathcal{J}_1(\mathfrak{I}) \) we have to use the closedness of the operators \( T \in \mathfrak{I} \).

For (b), note that for \( a \in \Psi_1(\mathfrak{I}) \) and \( x \in \mathcal{J}_1(\mathfrak{I}) \) a straightforward computation gives \( ax \in \mathcal{J}_1(\mathfrak{I}) \) and \( xa \in \mathcal{J}_1(\mathfrak{I}) \) with

\[
\omega_T^r(ax) = a\omega_T^r(x) - i\delta_T(a)x \\
\omega_T^l(ax) = a\omega_T^l(x) \\
\omega_{T_1,T_2}^{r,r}(ax) = a\omega_{T_1,T_2}^{r,r}(x) - i\delta_{T_1}(a)\omega_{T_2}^r(x) \\
\omega_T^r(xa) = \omega_T^r(x)a \\
\omega_T^l(xa) = \omega_T^l(x)a - ix\delta_T(a) \\
\omega_{T_1,T_2}^{r,r}(xa) = \omega_{T_1,T_2}^{r,r}(x)a + i\omega_{T_1}^r(x)\delta_{T_2}(a)
\]

for all \( T, T_1, T_2 \in \mathfrak{I} \).

For the semi-ideal property (c) it suffices to note that for \( a \in \mathcal{A} \) and \( x, y \in \mathcal{J}_k(\mathfrak{I}) \) we have

\[
\omega_T^r(xay) = \omega_T^r(x)ay, \; \omega_T^l(xay) = x\omega_T^l(y), \; \text{and} \; \omega_{T_1,T_2}^{r,r}(xay) = \omega_{T_1}^r(x)\omega_{T_2}^r(xy)
\]

for all \( T, T_1, T_2 \in \mathfrak{I} \), which gives the joint continuity of (8) as well. \( \square \)

Remark 4. Note that it is not clear, and in general not true that the spaces \( \mathcal{J}_k(\mathfrak{I}) \) are symmetric subspaces of \( \mathcal{L}(\mathcal{K}) \). However, we easily obtain this property by considering the spaces

\[
\mathcal{J}_k(\mathfrak{I})_* := \{ x \in \mathcal{J}_k(\mathfrak{I}) : x^* \in \mathcal{J}_k(\mathfrak{I}) \}.
\]

It is straightforward to check that Theorem 3 remains true also for the smaller spaces \( \mathcal{J}_k(\mathfrak{I})_* \). By slight abuse of notation we will sometimes write \( \mathcal{J}_k(\mathfrak{I}) \) for the spaces \( \mathcal{J}_k(\mathfrak{I})_* \).

Corollary 2. We have that \( \mathcal{J}_m(\mathfrak{I}) \) is a non-unital \( \Psi^* \)-algebra, for any \( m \in \mathbb{N} \cup \{ \infty \} \).

3. An extended pseudodifferential calculus

Recall that the notion of a continuous family groupoid, defined in [39], generalizes that of \( \mathcal{C}^{\infty,0} \)-foliations as considered in [4]. More precisely, a continuous family groupoid is a locally compact topological groupoid such that \( \mathcal{G} \) is covered by some open subsets \( \Omega \) such that the following conditions are satisfied:

- each chart \( \Omega \) is homeomorphic to two open subsets of \( \mathbb{R}^k \times \mathcal{G}^{(0)}, T_d \times U_d \) and \( T_r \times U_r \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{R}^k \times \mathcal{G}^{(0)} & \xrightarrow{\pi} & \mathbb{R}^k \\
\downarrow & & \downarrow \\
T_d \times U_d & \xrightarrow{\pi} & T_r \times U_r
\end{array}
\]
The space measurable. Consequently, the set \( f, g \in C^\infty \) of \( \nu \) is of class \( C^\infty \), i.e., \( u \mapsto \phi(, u) \) is a continuous map from \( U_* \) to \( C^\infty(T, T'_x) \), \( x = d, r \).

In addition, one requires that the composition and the inversion be \( C^\infty \) morphisms. For the sake of simplicity, we will always assume that the space \( M := G^{(0)} \) of units is compact.

3.1. Groupoid algebras. To any continuous family groupoid \( G \), there is associated an algebra of pseudodifferential operators: namely, let \( \Psi^m,\omega(G) \) be the space of continuous, uniformly supported, invariant families of pseudodifferential operators \( (P_x)_{x \in M} \) on the fibers of the groupoid, see [2]. For later purposes, note that this implies in particular the following: Let \( \Omega \cong T \times U \) be a chart as in [3] and \( \varphi \in C^\infty(\Omega) \). Then there exists \( p \in C_c\left(U, S^m(T; \mathbb{R}^n)\right) \) such that for each \( x \in U \), \( \varphi P_x \varphi \) corresponds to the pseudodifferential operator \( p(x, y, D_y) \) on \( T \).

In this section, we are going to show how the methods of the previous section can be used to construct algebras closed under holomorphic functional calculus that contain the algebras \( \Psi^{-\infty,0}(G) \) and \( \Psi^0,0(G) \) of pseudodifferential operators on the groupoid and share some of their algebraic and analytic properties. As it might be expected, it is difficult to construct, in general, algebras closed under holomorphic functional calculus that retain all geometric properties of the given groupoid. Nevertheless, we do construct algebras closed under holomorphic functional calculus that retain at least some of the geometric properties of the groupoid.

First, let us fix some notations. Throughout this section, \( K_x : = L^2(G_x; r^*D_{\hat{x}}^+) \) stands for the completion of the space \( C^c(G_x; r^*D_{\hat{x}}^+) \) with respect to the sesquilinear pairing

\[
(f, g)_x := \int_{G_x} f(\gamma)\overline{g(\gamma)}.
\]

Moreover, the Hilbert spaces \( K_x \), \( x \in M \), can be glued together to a new, big Hilbert space \( \mathcal{H} \) containing all necessary information for us.

To be more precise, fix a positive density \( \nu \in C^\infty(M, \Omega) \) on \( M \), and denote the induced measure again by \( \nu \). Note that each \( f \in C^\infty(\Omega; r^*D_{\hat{x}}^+) \) induces a section

\[
\tilde{f} : M \rightarrow \prod_{x \in M} K_x : x \mapsto f|_{G_x} \in C^c(G_x; r^*D_{\hat{x}}^+) \subseteq K_x,
\]

where, as usual, “section” simply means \( \tilde{f}(x) \in K_x \) for all \( x \in M \). Now, for any \( f, g \in C^\infty(\Omega; r^*D_{\hat{x}}^+) \), the function \( M \ni x \mapsto (\tilde{f}(x), \tilde{g}(x))_x \) is continuous, hence \( \nu \)-measurable. Consequently, the set \( \mathcal{M} \) of all sections \( h : M \rightarrow \prod_{x \in M} K_x \), such that the map \( M \ni x \mapsto (\tilde{f}(x), h(x))_x \) is \( \nu \)-measurable for all \( f \in C^\infty(\Omega; r^*D_{\hat{x}}^+) \) induces the structure of a \( \nu \)-measurable field of Hilbert spaces on the family \( (K_x)_{x \in M} \) [4, Definition 2.1.3.1, Proposition 2.1.4.4]. The set \( \mathcal{H} \) of all \( h \in \mathcal{M} \) satisfying

\[
\int_M \| h(x) \|_{K_x}^2 \, d\nu(x) < \infty
\]

is in fact a Hilbert space and we write \( \mathcal{H} =: \int_M^\oplus K_x \, d\nu(x) \).

The space \( C^\infty(\Omega; r^*D_{\hat{x}}^+) \) is then a dense subspace of \( \mathcal{H} \).
Recall that an operator \( P \in \mathcal{L}(\mathcal{H}) \) is said to be \textit{decomposable} provided there exists a family \((\hat{P}(x))_{x \in M}\) of operators \( \hat{P}(x) \in \mathcal{L}(\mathcal{H}_x) \) such that, for any \( h \in \mathfrak{M} \)
\[
(Ph)(x) = \hat{P}(x)h(x),
\]
for all \( x \in M \), and \( [x \mapsto \|\hat{P}(x)\|_{\mathcal{L}(\mathcal{K}_x)}] \in L^\infty(M; \nu) \) \( \mathbb{U} \). Definition 2.2.3.2. As usual, we write in that case \( P = \int_M \hat{P}(x) d\nu(x) \). A straightforward computation gives \( \mathbb{U} \) Proposition 2.2.3.2
\[
\|P\|_{\mathcal{L}(\mathcal{H})} = \text{ess-sup}_x \|\hat{P}(x)\|_{\mathcal{L}(\mathcal{K}_x)}.
\]
The set of all decomposable operators is in fact a \( C^* \)-subalgebra, which we denote by \( \mathcal{L}_D(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H}) \).

Finally, for any \( \gamma \in \mathcal{G} \), the operators
\[
U_\gamma : C^\infty_c(\mathcal{G}_d(\gamma); \mathfrak{r}^*\mathfrak{D}^\frac{1}{2}) \to C^\infty_c(\mathcal{G}_{r(\gamma)}; \mathfrak{r}^*\mathfrak{D}^\frac{1}{2}), \quad U_\gamma(f)(\gamma') = f(\gamma'\gamma),
\]
extend by continuity to isometric isomorphisms \( U_\gamma : \mathcal{K}_d(\gamma) \to \mathcal{K}_{r(\gamma)} \), and hence induce an action of the groupoid \( \mathcal{G} \) on \( \mathcal{H} \). Decomposable operators \( P \in \mathcal{L}_D(\mathcal{H}) \) with \( \hat{P}(r(\gamma)) U_\gamma = U_\gamma \hat{P}(d(\gamma)) \) for all \( \gamma \in \mathcal{G} \) are called \textit{invariant} with respect to the action of the groupoid, and we denote the \( C^* \)-subalgebra of all invariant operators by \( \mathcal{L}_D^G(\mathcal{H}) \).

By the results of \( \mathbb{U} \) Section 3, the regular representations \( \pi_x, x \in M \), of the algebra \( \Psi^{0,0}(\mathcal{G}) \) fit together to a faithful \( * \)-representation
\[
\pi_r : \Psi^{0,0}(\mathcal{G}) \to \mathcal{L}_D^G(\mathcal{H}) : \pi_r(P) = \int_M \pi_x(P) d\nu(x)
\]
of unital algebras. In the sequel, we will identify \( \Psi^{0,0}(\mathcal{G}) \) with its image under \( \pi_r \) and construct subalgebras \( \mathcal{A} \subseteq \mathcal{L}(\mathcal{H}) \) closed under holomorphic functional calculus in \( \mathcal{L}(\mathcal{H}) \) and containing \( \Psi^{0,0}(\mathcal{G}) \) as a subalgebra. Because of Lemma 3 and the following Lemma, we can always assume that the algebras \( \mathcal{A} \) are symmetric subalgebras of \( \mathcal{L}_D^G(\mathcal{H}) \).

\textbf{Lemma 4.} Let \( \mathcal{A} \subseteq \mathcal{L}_D(\mathcal{H}) \) be closed under holomorphic functional calculus in \( \mathcal{L}(\mathcal{H}) \). Then \( \mathcal{A}^G := \mathcal{A} \cap \mathcal{L}_D^G(\mathcal{H}) \) is also \( \mathcal{G} \)-invariant and closed under holomorphic functional calculus in \( \mathcal{L}(\mathcal{H}) \).

\textbf{Proof.} This follows because \( \mathcal{L}_D^G(\mathcal{H}) \) is a \( C^* \)-subalgebra of \( \mathcal{L}(\mathcal{H}) \).

As a first step towards constructing algebras closed under holomorphic functional calculus, we are going to show that we can reduce the problem of finding such algebras \( \mathcal{A} \) essentially to the construction of algebras that contain \( \Psi^{-\infty,0}(\mathcal{G}) \) and are closed under the holomorphic functional calculus in \( \mathcal{L}(\mathcal{H}) \). Indeed, let
\[
\mathcal{J} := C^*_r(\mathcal{G}) = \overline{\Psi^{-\infty,0}(\mathcal{G})}_{\mathcal{L}(\mathcal{H})},
\]
\[
\mathcal{B} := \mathfrak{A}_r(\mathcal{G}) = \overline{\Psi^{0,0}(\mathcal{G})}_{\mathcal{L}(\mathcal{H})},
\]
and suppose that we have a subspace \( I = I^* \subseteq \mathcal{L}_D^G(\mathcal{H}) \) with the following properties
\[
\begin{align*}
(10) & \quad \Psi^{-\infty,0}(\mathcal{G}) \subseteq I \subseteq \mathcal{J}, \\
(11) & \quad I \text{ is a } \Psi^{0,0}(\mathcal{G})\text{-left and -right module, and} \\
(12) & \quad I \text{ has property } (P_I) \text{ in } \mathcal{B}.
\end{align*}
\]
Theorem 4. Let $I$ be as above and $A := \Psi^{0,0}(G) + I \subseteq \mathcal{L}_D^G(H)$. Then we have 
$$(\mathcal{C}id_H + A) \cap \mathcal{L}(H)^{-1} = (\mathcal{C}id_H + A)^{-1}.$$ 
In particular, if there exists a Fréchet topology on $I$ making the $\Psi^{0,0}(G)$-module action as well as the embedding $I \hookrightarrow \mathcal{L}(H)$ continuous, then $A$ is closed under the holomorphic functional calculus in $\mathcal{L}(H)$ and is a $\Psi^*$-algebra containing $\Psi^{0,0}(G)$.

Proof. Because of (11), the space $I \subseteq A$ is an ideal in the algebra $A$. By Lemma 3 and Theorem 1(a), it suffices to prove that $A/I$ has property $(\bar{P}_{A/I})$ in $B/J$. So, let $\bar{a} = a + I \in A/I$ be such that $\varphi(\bar{a}) \in (B/J)^{-1}$. Without loss of generality, we may assume $a \in \Psi^{0,0}(G)$. On the other hand, the homogeneous principal symbol map $\sigma_0$ induces an isomorphism $\bar{\sigma}_0 : B/J \rightarrow C(S^*(G))$, with $\sigma_0(a) \in C_c^{\infty,0}(S^*(G))^{-1}$. Using the exactness of the sequence

$$(13) \quad 0 \rightarrow \Psi^{-1,0}(G; E) \rightarrow \Psi^{m,0}(G; E) \xrightarrow{\sigma_m} C_c^{\infty,0}(S^*(G), \text{End}(E) \otimes P_m) \rightarrow 0$$

and the asymptotic completeness of $\Psi^{0,0}(G)$, we obtain $b \in \Psi^{0,0}(G)$ with 
$$id_H - ab \in \Psi^{-\infty,0}(G), \quad id_H - ba \in \Psi^{-\infty,0}(G).$$

Thus, $\bar{a}$ is invertible in $\Psi^{0,0}(G)/\Psi^{-\infty,0}(G)$, hence also in $\Psi^{0,0}(G)/(I \cap \Psi^{0,0}(G))$ because of $\Psi^{-\infty,0}(G) \subseteq I \cap \Psi^{0,0}(G)$. This gives the property $(\bar{P}_{A/I})$, and completes the proof. \hfill \square

The following Lemma enables us to apply the methods developed in the previous section to pseudodifferential operators on the groupoid $G$.

Lemma 5. Let $T \in \Psi^{m,0}(G; \mathcal{D}^\frac{1}{2})$ be arbitrary. Then the unbounded operator 
$$T : H \supseteq C_c^{\infty}(G; r^*\mathcal{D}^\frac{1}{2}) \rightarrow H$$

is closable. Moreover, if $T = T^* \in \Psi^{m,0}(G; \mathcal{D}^\frac{1}{2})$, then the closure of $T$ is symmetric.

For notational simplicity, we shall not distinguish between $T \in \Psi^{m,0}(G; \mathcal{D}^\frac{1}{2})$ and its minimal closed extension $\overline{T} : H \supseteq \mathcal{D}(\overline{T}) \rightarrow H$ in the sequel.

Proof. Let $(f_j)_{j \in \mathbb{N}}$ be a sequence in $C_c^{\infty}(G; r^*\mathcal{D}^\frac{1}{2})$ with $f_j \rightarrow 0$ in $H$ and $Tf_j \rightarrow f$ in $H$ for some $f \in H$. We have to show $f = 0$. Let $T^* \in \Psi^{m,0}(G; \mathcal{D}^\frac{1}{2})$ be the formal adjoint of $T$. Then we obtain, for all $h \in C_c^{\infty}(G, r^*\mathcal{D}^\frac{1}{2})$

$$< f, h >_H = \lim < T f_j, h >_H = \lim \int_M < T_x f_j | \varphi_x, h| \varphi_x >_{\mathcal{K}_x} d\nu(x) = \lim \int_M < f_j | \varphi_x, T_x^* h | \varphi_x >_{\mathcal{K}_x} d\nu(x) = \lim < f_j, T^* h >_H = 0.$$  

Since $C_c^{\infty}(G, r^*\mathcal{D}^\frac{1}{2})$ is dense in $H$, this gives $f = 0$. The rest is clear. \hfill \square

Let us briefly outline how the operator theoretic methods from the previous sections are used to embed the algebra $\Psi^{0,0}(G)$ of pseudodifferential operators of order 0 in an algebra that is closed under holomorphic functional calculus. 

We start with the construction of a submultiplicative $\Psi^*$-algebra $A_0 \subset \mathcal{L}(H)$ with $\Psi^{0,0}(G) \subseteq A_0$ by using a variant of the commutator methods described in Theorem 2. Without loss of generality, we can further assume that $A_0 \subseteq \mathcal{L}_D^G(H)$ and $\Psi^{0,0}(G)$
dense in $\mathcal{A}_0$ [19, Corollary 2.5]. Then, we take a finite set $\mathcal{T} \subseteq \Psi^{1,0}(G; D_{\mathbb{R}^n}^\perp)$ of formally self-adjoint first order pseudodifferential operators and construct the semi-ideal $\mathcal{J}_\infty(\mathcal{T})$ and the $\Psi^*$-algebra $\Psi^\infty(\mathcal{T})$ as described in Theorem 3. By Theorem 1 of [24] we have $\Psi^{-\infty,0}(G) \subseteq \mathcal{J}_\infty(\mathcal{T})$ and $\Psi^{0,0}(G) \subseteq \Psi^\infty(\mathcal{T})$, hence (11), (11), and (12) hold for the symmetrized semi-ideal $I := \mathcal{J}_\infty(\mathcal{T})$, thus $A_1 := \Psi^{0,0}(G) + \mathcal{J}_\infty(\mathcal{T}) +$ is a (submultiplicative) $\Psi^*$-algebra containing $\Psi^{0,0}(G)$ by Theorem 3. Again by [13, Corollary 2.5], we can even assume that $\Psi^{0,0}(G)$ is dense in $A_1$.

Since the arbitrary intersection of algebras closed under functional calculus in $\mathcal{L}(\mathcal{H})$ is still closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$, we can admit also countable subsets $\mathcal{T} \subseteq \Psi^{1,0}(G; D_{\mathbb{R}^n}^\perp)$, and then we obtain, by taking the intersection over all possible choices involved in the construction of the algebra $A_1$, an algebra $A_\infty$ that is independent of any choices, is closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$, and contains $\Psi^{0,0}(G)$. It remains to describe some properties of the elements of the algebra $A_\infty$.

3.2. Properties of the algebra $A_0$. Let $\chi : \Omega \to \mathbb{R} \times T \subseteq M \times \mathbb{R}^k$ be a chart of $G$ as in [14], and $\varphi_0, \psi_0 \in C^\infty,0(\Omega)$ be arbitrary. We define the local symbol of $a \in \mathcal{L}_D(\mathcal{H})$ on $\Omega$ with respect to $\varphi_0$ and $\psi_0$ by

$$\sigma_\Omega(a; \varphi_0, \psi_0)(x, y, \eta) := e^{-i\eta \cdot y_0}(x, y)\hat{\alpha}(x)[y \mapsto \varphi_0(x, y)e^{iy \cdot \eta}(y)$$

for $x \in U$, $y \in T$, and $\eta \in \mathbb{R}^k$. Thus, we get

$$\sup_{x, \eta} \int_T \left| \sigma_\Omega(a; \varphi_0, \psi_0)(x, y, \eta) \right|^2 dy \leq c(\varphi_0, \psi_0) \sup_x \left| \hat{\alpha}(x) \right|^2_{L^2(K_x)}$$

$$= c(\varphi_0, \psi_0) ||a||^2_{L^2(\mathcal{H})}.$$  

Since the operators $i\partial_{y_s}$, $iy_s\partial_{y_s}$, $r \neq s$, $iy_s\partial_{y_s} + \frac{1}{2}$, and $M_{y_r}$, after multiplication with some cut-off function can be realized as closed symmetric operators on $\mathcal{H}$, we can consider the corresponding submultiplicative $\Psi^*$-algebra $\tilde{A}_\Omega$ with respect to the $C^*$-algebra $\mathcal{L}_D(\mathcal{H})$ as in Theorem 3. A straightforward computation then yields $\Psi^{0,0}(G) \subseteq \tilde{A}_\Omega$. Observe that for $a \in \tilde{A}_\Omega$, the derivatives $\eta_r \partial_{\eta_r}, \sigma_\Omega(a; \varphi_0, \psi_0)$ and $\partial_{y_r} \sigma_\Omega(a; \varphi_0, \psi_0)$ of the local symbol of $a$ can be realized as the local symbol of the commutator of $a$ with one of the operators mentioned above, hence they satisfy an $L^2$ estimate similar to (14). As in [1] or [20], an application of Sobolev’s embedding theorem, leads to the following estimate.

**Lemma 6.** Let $\alpha, \beta \in \mathbb{N}^n$ be arbitrary. Then there exists a continuous semi-norm $q_{\alpha, \beta}$ on $\tilde{A}_\Omega$ such that

$$\sup_{x, y, \eta} \left| \eta > |\beta| \partial_{\eta_s}^\alpha \partial_{\eta_r}^\beta \sigma_\Omega(a; \varphi_0, \psi_0)(x, y, \eta) \right| \leq q_{\alpha, \beta}(a).$$

Let $A_\Omega$ be the closure of $\Psi^{0,0}(G)$ in the submultiplicative $\Psi^*$-algebra $\tilde{A}_\Omega \cap \mathcal{L}_D^0(\mathcal{H})$. By [13, Corollary 2.5], $A_\Omega$ is a $\Psi^*$-algebra in $\mathcal{L}_D^0(\mathcal{H})$, and $\Psi^{0,0}(G)$ is dense in $A_\Omega$. Note that the local symbols of elements in $A_\Omega$ are in the variable $x$ locally uniform limits of symbols in $\mathcal{C}(U, S^0(T; \mathbb{R}^k))$, hence they remain continuous in $x$ which gives the following proposition.

**Proposition 3.** Let $a \in A_\Omega$ be arbitrary and $\chi : \Omega \to \mathbb{R} \times T$ be as above. Then for all $\varphi, \psi \in C^\infty_c(\Omega)$ there exists a symbol $\sigma_\Omega(a; \varphi, \psi) \in \mathcal{C}(U, S^0(T; \mathbb{R}^k))$ such that
we have
\begin{equation}
\chi_*(\psi a \varphi) = \sigma_G(a; \varphi, \psi)(x, y, D_y).
\end{equation}

Choosing a (countable) cover \( G = \bigcup_{\Omega \in \mathcal{V}} \Omega \) of \( G \) by open charts \( \Omega \) as above, we can define \( \mathcal{A}_0 \) to be the closure of \( \Psi^{0,0}(\mathcal{G}) \) in the submultiplicative \( \Psi^* \)-algebra \( \bigcap_{\Omega \in \mathcal{V}} \mathcal{A}_\Omega \). Then \( \Psi^{0,0}(\mathcal{G}) \) is dense in \( \mathcal{A}_0 \), and each \( a \in \mathcal{A}_0 \) has a representation \([5]\) with respect to a symbol \( \sigma_G(a; \varphi, \psi) \in \mathcal{C}(U, S^0(T; \mathbb{R}^n)) \).

As explained above, in a next step we consider now the semi-ideal \( J_\infty(\mathfrak{T}) \).

### 3.3. Properties of the semi-ideal \( J_\infty(\mathfrak{T}) \)

The question which properties can be obtained by choosing the set \( \mathfrak{T} \) of closed, symmetric operators appropriately is more complicated because it includes in particular the analysis of pseudodifferential operators on non-compact manifolds, hence, we will be rather short at this point, and sketch only what is within reach.

Choose an at most countable set of sections \( S \in C^\infty(M, \mathcal{G}(\mathfrak{g})) \), \( S \in \mathfrak{S} \), such that for each \( x \in M \) the set \( \{ S(x) \in T_x \mathcal{G}_x : S \in \mathfrak{S}_x \} \) generates \( T_x \mathcal{G}_x \) as a real vector space for some finite subset \( \mathfrak{S}_x \subseteq \mathfrak{S} \), and let \( \mathfrak{T} \subseteq \mathfrak{S}^1(\mathcal{G}; \mathcal{D}^\sharp) \) be the corresponding set of right-invariant, \( d \)-vertical vector fields. After multiplying them with \( i \), the elements in \( \mathfrak{T} \) have symmetric, minimal closed extensions by Lemma \([6]\).

For simplicity, let us assume that \( \mathfrak{T} \) is finite; otherwise, we have to consider the projective limit of the corresponding semi-ideals with respect to an increasing sequence \( \mathfrak{T}_j \subseteq \mathfrak{T}_{j+1} \subseteq \mathfrak{T} \). Let \( \mathcal{H}^m(\mathfrak{T}) \), \( m \in \mathbb{N}_0 \) be the corresponding scale of \( \mathfrak{T} \)-Sobolev spaces. Note that the spaces \( \mathcal{H}^m(\mathfrak{T}) \) have a decomposition as a direct integral of the form
\[
\mathcal{H}^m(\mathfrak{T}) = \int_\mathcal{M} H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^* \mathcal{D}^\sharp) d\nu(x),
\]
where \( H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^* \mathcal{D}^\sharp) \) is the Sobolev spaces of order \( m \in \mathbb{N}_0 \) associated to the vector fields \( \mathfrak{T}_x := \mathfrak{T}_x|_{\mathcal{G}_x} \subseteq \mathfrak{S}^1(\mathcal{G}; \mathcal{D}^\sharp) \). We extend these scales of Sobolev spaces by duality to \( m \in \mathbb{Z} \), then the following result follows immediately from the definition of the ideal \( J_\infty(\mathfrak{T})_* \).

**Lemma 7.** Let \( a \in J_\infty(\mathfrak{T})_* \) be arbitrary. Then \( a \) induces for each \( m \in \mathbb{N} \) and each \( x \in M \) bounded operators \( a : \mathcal{H}^{-m}(\mathfrak{T}) \rightarrow \mathcal{H}^m(\mathfrak{T}) \) and
\[
\tilde{a}(x) : H^{-m}_{\mathfrak{T}_x}(\mathcal{G}_x, r^* \mathcal{D}^\sharp) \rightarrow H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^* \mathcal{D}^\sharp).
\]

Consequently, \( \tilde{a}(x) \) has a smooth kernel provided Sobolev’s embedding
\begin{equation}
H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^* \mathcal{D}^\sharp) \hookrightarrow C_b(\mathcal{G}_x, r^* \mathcal{D}^\sharp)
\end{equation}
holds for \( 2m > \dim \mathcal{G}_x \). Since everything takes place within the algebra \( \mathcal{A}_0 \), the smooth kernel depends in addition continuously on the parameter \( x \).

Note that \([10]\) holds for instance if the manifolds \( \mathcal{G}_x \) are of bounded geometry \([12]\) Appendix]. For the question, when the groupoid is of bounded geometry we refer to \([1]\). Let us denote by \( C_b^\infty(\mathcal{G}_x \times \mathcal{G}_x, \text{END}(\mathcal{D}^\sharp)) \) the space of all smooth sections \( \mathcal{G}_x \times \mathcal{G}_x \rightarrow \text{END}(\mathcal{D}^\sharp) \) that are uniformly bounded, as are all their covariant derivatives. A combination of Lemma \([7]\) and \([16]\) then yields a characterization of the Schwartz kernels of the operators \( \tilde{a}(x) \).
Proposition 4. Suppose that the manifolds $G_x$ are of bounded geometry for all $x \in M$, and let $a \in J_{\infty}(T^\ast)$ be arbitrary. Then $\hat{a}(x) \in L(K_x)$ is an operator with $C^\infty$-kernel $k_\hat{a}(x, \cdot, \cdot) \in C^\infty_b(G_x \times G_x, \text{END}(D^2))$.

We summarize the results of the above discussion in the following proposition.

Proposition 5. Suppose that for any $x \in M$, the fiber $G_x$ is a manifold of bounded geometry. Then there exists a $\Psi^\ast$-algebra $A_1$ containing $\Psi_{0,0}(G)$ as a dense subalgebra such that each $P \in A_1$ is given by a $G$-invariant family $(P_x)_{x \in M}$ of pseudodifferential operators $P_x$ on $G_x$.

Note that the definition of the $\Psi^\ast$-algebra $A_1$ depends on many choices.

Proof. This follows directly from the plan for the construction of $A_1$ after Lemma 5, Proposition 3 and Proposition 5.

The above results provide us with $\Psi^\ast$-algebras that are useful in practice, because they consist of pseudodifferential operators. These algebras will necessarily contain operators that are not properly supported (unless our manifold is compact without corners). Nevertheless, these algebras consist of bounded operators, so their Schwartz kernels must satisfy some decay conditions far from the diagonal. It is difficult in general to quantify these decay conditions. One possibility is to consider commutators with functions that approximate the distance function as in [17] or [42]. If the groupoid allows a length function with polynomial growth as in Section 4, we can improve this by introducing Schwartz spaces. Also, for certain explicitly given groupoids much more is possible; we have elaborated this in Section 5 for the case of generalized cusp-calculi on compact manifolds with corners.

4. Algebras on Manifolds with Corners

In this section we recall the constructions of various groupoids associated to manifolds with corners (see for instance [24, 35, 38]). We shall use these results to define length functions on some of these groupoids, which in turn is useful when defining Schwartz spaces associated to manifolds with corners, in Section 5. Note that we do not require that the manifolds used in this section have embedded hyperfaces. Also, we shall use these constructions in the particular case of manifolds with boundary in the next section in order to construct algebras with smooth kernels. The reader interested only in the next section, can skip this section at a first lecture, and only refer back to it when necessary.

Let $X$ be a manifold with corners, and $x$ a point of $X$; we denote by $F(x)$ the connected component of the set of points having the same codimension as $x$ which contains $x$, and by $N_x F(x) = T_x X / T_x F(x)$ the normal space to the boundary at $x$. One can define several groupoids associated to $X$, giving various pseudodifferential calculi, such as the $b$-calculus, the cusp-calculus, and its generalizations (the $c_\alpha$-calculus).

Let $G(X) = \{ (x, y, \alpha) \mid x, y \in X, \text{codim}(x) = \text{codim}(y), \alpha : N_y F(y) \xrightarrow{\sim} N_x F(x) \}$,

where $\alpha$ is given, through trivializations $N_y F(y) \simeq \mathbb{R}^k_+$ based on inward pointing normals and $N_x F(x) \simeq \mathbb{R}^k_+$, by a matrix which has one and only one non-zero element on each line and each column, and this element is positive. It is precisely
the product of a diagonal matrix with all terms strictly positive by a permutation matrix.

The groupoid structure of $\mathcal{G}(X)$ is given by $d, r : \mathcal{G}(X) \to X$ with $r(x, y, \alpha) = x$, and $d(x, y, \alpha) = y$; the composition law is induced by the composition of the isomorphisms, in the sense that $(x, y, \alpha)(y, z, \beta) = (x, z, \alpha\beta)$.

One can endow $\mathcal{G}(X)$ with several different differential structures such that the resulting groupoids are homeomorphic but not diffeomorphic. Let $\Omega$ and $\Omega'$ be two charts of $X$ of same codimension. Thus $\Omega \cong U \times \mathbb{R}^d$ and $\Omega' \cong U' \times \mathbb{R}^{d'}$, where $U$ and $U'$ are open subsets of some $\mathbb{R}^n$ and $A$ and $A'$ are the sets of local hyperfaces contained in $\Omega$ and $\Omega'$. The sets $A$ and $A'$ have the same cardinal, namely the codimension of $\Omega$. Fix a bijection $\sigma : A' \to A$.

If $t \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$, then let $B'_i = \{i \in A', t = 0\}$. The product of the matrix of $\sigma|_{B'_i}$ by the diagonal matrix consisting of the $\lambda_i$, for $i \in B'_i$, is denoted by $m_{\sigma, t, \lambda}$; it defines an isomorphism $\alpha_{\sigma, t, \lambda}$ through the trivializations induced by $\pi$ and $\pi'$:

$$\alpha_{\sigma, t, \lambda} : N\pi^{-1}(u', t)F(\pi'^{-1}(u', t)) \xrightarrow{\simeq} N\pi^{-1}(u, \sigma(\lambda t))F(\pi^{-1}(u, \sigma(\lambda t)))$$

$$\xrightarrow{\simeq} \mathbb{R}^{A'}_+ \xrightarrow{m_{\sigma, t, \lambda}} \mathbb{R}^{A'}_+$$

### 4.1. The $b$-calculus differential structure.

Using the notations above, consider the map

$$\psi_\sigma : U \times U' \times \mathbb{R}^{A'}_+ \times \mathbb{R}^{A'}_+ \to \mathcal{G}(X)$$

$$(u, u', t, \lambda) \mapsto (\pi^{-1}(u, \sigma(\lambda t)), \pi'^{-1}(u', t), \alpha_{\sigma, t, \lambda}).$$

This map is injective, and one thus obtains a $C^\infty$ structure on $\mathcal{G}(X)$, which turns it into a Lie groupoid, whose fibers are submanifolds without boundary; it is amenable.

**Definition 5.** The groupoid of the $b$-calculus, $\Gamma_1(X)$ is the union of the connected components containing the unit of each $d$-fiber of $\mathcal{G}(X)$.

**Remark 5.** It is instructive to consider the special case when $X$ is a manifold with connected boundary $\partial X$ and defining function $\rho$. Then one can prove that

$$\Gamma_1(X) \cong \{(x, y, \lambda) \in X \times X \times \mathbb{R}^+_+ \mid \rho(x) = \lambda \rho(y)\}.$$

The identification with the $b$-calculus is obtained by observing that $\partial M \times \partial M \times \mathbb{R}^+_+ \subset \Gamma_1(X)$ and that $\log \lambda = \log \rho(x) - \log \rho(y)$ if $(x, y, \lambda) \in \Gamma_1(X)$ and $x \notin \partial M$.

### 4.2. The cusp-calculus and $c_\alpha$-caluli differential structures.

Consider a continuous, strictly increasing map, smooth for $t \neq 0$, such that

$$\tau_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$$

$$t \mapsto \begin{cases} \frac{1}{\alpha}(-\log(t))^{-\frac{1}{\alpha}} & \text{if } t \in (0, 1/e) \\ 0 & \text{if } t = 0 \\ t & \text{if } t \geq 1 \end{cases}$$

If $U$ is an open subset of a Euclidean space, we will also denote by $\tau_\alpha : U \times \mathbb{R}^{A'_+} \to U \times \mathbb{R}^{A'_+}$ the map obtained by applying $\tau_\alpha$ to each coordinate of $\mathbb{R}^{A'_+}$. 
As above, one can define

$$\psi_{\sigma,n} : U \times U^\prime \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{G}(X)$$

$$(u,u^\prime,t,\lambda) \mapsto (\pi^{-1}(u,\sigma(\tau_n(\lambda t))), \pi^{-1}(u^\prime,\tau_n(t)), \alpha_{\sigma,t,\lambda})$$

which endows $\mathcal{G}(X)$ with a new differential structure.

**Definition 6.** Let $n \geq 2$. The groupoid of the $c_n$-calculus, $\Gamma_n(X)$, is the union of the connected components containing the unit of each $d$-fiber of the groupoid $\mathcal{G}(X)$ endowed with the structure defined by $\psi_{\sigma,n-1}$ (using $\tau_{n-1}$).

Recall that the $c_2$-calculus is also known as the **cusp-calculus** – see for instance [21, 28, 31, 33].

**Remark 6.** When $X$ is a manifold with connected boundary, endowed with a defining function of the boundary, $\rho$, then

$$\Gamma_{n+1}(X) \cong \{(u,v,\mu) \in X \times X \times \mathbb{R} | \mu \rho(u)^n \rho(v)^n = \rho(u)^n - \rho(v)^n\}$$

as smooth manifolds, which can be seen directly from the definition. Moreover, the structural morphisms of the groupoid $\Gamma_{n+1}$ become $d(u,v,\mu) = v$, $r(u,v,\mu) = u$, and $(u,v,\mu)(v,w,\lambda) = (u,w,\mu + \lambda)$. The Lie algebroid of this groupoid is seen to consist of the vector fields $X \in \Gamma(TM)$ such that $X(\rho(x)^{-n})$ is a smooth function on $M$.

### 4.3. Comparison of the $c_n$-pseudodifferential calculi

The groupoids defined above only differ by their differential structures; in fact, they are all homeomorphic. This is intuitively clear since $\tau_n$ is a homeomorphism (but it is NOT a diffeomorphism) which induces a homeomorphism of groupoids. An immediate application is the fact that the $b$-calculus and $c_n$-calculi have the same norm closure.

To keep notations simple, we will only consider here the case of a manifold with boundary. Then by considering a collar neighborhood of the boundary, one gets a partition

$$X = X_1 \cup X_2$$

with $\pi : X_1 \cong \partial X \times [0,1)$ and $X_2 = X \setminus \pi^{-1}(\partial X \times [0,1/e])$; the boundary defining function used here is

$$\rho(x) = \begin{cases} et & \text{if } x \in X \setminus X_2 \text{ with } \pi(x) = (u,t) \\ 1 & \text{if } x \in X_2 \end{cases}$$

($\rho$ is not a smooth function on $X$, but it is smooth on $\partial X$).

This allows us to define a homeomorphism

$$\Theta_{n+1} : \Gamma(X) \to \Gamma_{n+1}(X)$$

$$(x,y,\lambda) \mapsto (u,v,\mu)$$

with

$$u = \begin{cases} x & \text{if } x \in X_2 \\ \pi^{-1} \circ \tau_n \circ \pi(x) & \text{if } x \in X \setminus X_2 \end{cases}$$

$$v = \begin{cases} y & \text{if } y \in X_2 \\ \pi^{-1} \circ \tau_n \circ \pi(y) & \text{if } y \in X \setminus X_2 \end{cases}$$

$$\mu = \log(\lambda)$$
One can check then that \( \mu \rho(u)^n \rho(v)^n = \rho(u)^n - \rho(v)^n \) by considering separately the following cases:

(a) If \( x \in X \setminus X_2 \) and \( y \in X \setminus X_2 \), then \( u \in X \setminus X_2 \) and \( v \in X \setminus X_2 \), since \( \tau_n(t) \leq 1/e \) if \( t \leq 1/e \). Thus \( \rho(u) \neq 0 \), \( \rho(u)^{-n} = -\log(\lambda t) \) and \( \rho(v)^{-n} = -\log(t) \), which implies that \( \rho(v)^{-n} - \rho(u)^{-n} = \log(\lambda) \). If \( \rho(u) = 0 \), then \( \rho(v) = 0 \) and the equality is trivial.

(b) If \( x \in X \setminus X_2 \) and \( y \in X_2 \), then \( u \in X \setminus X_2 \) and \( v = y \in X_2 \). Thus \( \rho(u)^{-n} = -\log(\rho(x)/e) \), and \( \rho(v)^{-n} = 1 \). But \( \rho(x) = \lambda \rho(y) = \lambda \), so that \( \rho(v)^{-n} - \rho(u)^{-n} = \log(\lambda) \).

(c) If \( x \in X_2 \) and \( y \in X_2 \), then \( u = x \in X_2 \) and \( v = y \in X_2 \). Thus \( \rho(u) = \rho(v) = 1 \) and the equality is trivial.

5. Algebras of smooth kernels

In this section we shall study regularizing operators on certain differentiable groupoids on a manifold with boundary.

Recall that a differentiable groupoid is a continuous family groupoid \( \mathcal{G} \) such that the space of units \( M \), as well as the space of arrows \( \mathcal{G}^{(1)} = \mathcal{G} \), are differentiable manifolds (possibly with corners), all structural maps are differentiable, and the domain map \( d : \mathcal{G} \to M \) is a submersion of manifolds with corners. Note that the latter in particular implies that the fibers \( \mathcal{G}_x := d^{-1}(x) \) are smooth manifolds without corners that are in general non-compact. Pseudodifferential operators on differentiable groupoids have been considered in [23, 26], and [28] in more detail.

We shall use now the results of the previous sections to construct an algebra of regularizing operators that is closed under holomorphic functional calculus and whose kernels are smooth including on the boundary, for suitable \( \mathcal{G} \). This is non-trivial, in view of the results of [14, 19, 20], where it is proved that this is not possible for the \( b \)-calculus. We begin by formulating the problem more precisely.

Let \( \mathcal{G} \to M \) be a Hausdorff differentiable groupoid on a manifold with corners \( M \). We want to construct algebras \( A \) with the following properties:

(a) \( \Psi^{-\infty}(\mathcal{G}) = C^\infty_c(\mathcal{G}) \subset A \subset C^\infty(\mathcal{G}) \cap C^*(\mathcal{G}) \) and

(b) \( A \) is a (possibly non-unital) \( \Psi^* \)-algebra.

**Definition 7.** An algebra \( A \) satisfying properties (a) and (b) right above is called a \( \Psi^* \)-algebra of smooth kernels on \( \mathcal{G} \).

In [19] it is proved that there is no \( \Psi^* \)-algebra of smooth kernels on \( \mathcal{G} = \Gamma_1(M) \), where \( \Gamma_1(M) \) is the smooth groupoid associated to the \( b \)-calculus (see Section 3). However, we shall now show how to construct algebras of smooth kernels on \( \mathcal{G} \), if \( \mathcal{G} = \Gamma_n(M) \) are the groupoids defining the \( c_n \)-calculi on a manifold with boundary \( M \), provided that \( n \geq 2 \) (see Section 3 for the definition of \( \Gamma_n(M) \)). Some of us have learned that it is possible to construct algebras of smooth kernels on \( \Gamma_n(M) \) from Richard Melrose.

Let \( I := C^\infty(M \times M) \) be the space of smooth functions on \( M \times M \) that vanish to infinite order on the boundary (the boundary here is the union of hypersurfaces of \( M \times M \)). Then \( C^\infty(M \times M) \) is an algebra of smooth kernels, and hence an algebra of order \(-\infty\) pseudodifferential operators on \( M_0 := M \setminus \partial M \).

Moreover, these operators are bounded on \( L^2(M_0) \). We have to note here that the correct density on \( M_0 \) for the \( c_n \)-calculi is, on a tubular neighborhood of \( \partial M \),
of the form $hx^{-n}|dx||dy|$, where $x \in [0, \infty)$ is a boundary defining function on $M$ and $|dy|$ is a density on $\partial M$, and $h \in C^\infty(M)$.

**Lemma 8.** The space $I \subset C^*(\mathcal{G})$ is a non-unital $\Psi^*$-algebra.

**Proof.** Let $\Delta$ be the Laplace operator for some compatible metric (i.e. such that $x^n \partial x$ has length one). Take $\mathfrak{T} = \{\Delta, x^{-1}\}$, regarded as unbounded operators on $L^2(M)$, and apply the semi-ideal construction to $C^*(\Gamma_n(M))$ and $\mathfrak{T}$. This yields an algebra $J := \mathcal{J}_{-\infty}(\mathfrak{T})_*$ satisfying

$$J \subset \{ T \in C^*(\Gamma_n(M)) | x^{-i} \Delta^j T \Delta^k x^{-l} \text{ is bounded } \forall i, j, k, l \}.$$ 

Clearly, the opposite inclusion is also true by the definition of $\mathcal{J}_{-\infty}(\mathfrak{T})$.

Let $H^m(M)$ be the domain of $\Delta^{m/2}$, if $m \geq 0$, or $H^m(M)$ be the dual of $H^{-m}(M)$, if $m$ is negative. Also, let $H^{-\infty}(M) = \cup H^m(M)$ and $H^\infty(M) = \cap H^m(M)$.

Since every $T \in J$ maps $H^{-\infty}(M) \to H^\infty(M)$ and $J$ is symmetric, we obtain that $T$ is a smoothing operator. Thus, $T$ is an operator with integral kernel given by a smooth function $K(x, y)$. Since $x^{-i} K(x, y, x', y') x^{-j}$ also must define a bounded operator, we see that $K$ vanishes to infinite order at the boundary. Consequently, $J = I$, and Corollary 2 completes the proof.

**Remark 7.** In the proof above, we obtain the same conclusion by considering $\mathfrak{T} = \{x^{-1}, X_1, \ldots, X_m\}$, where $X_1, \ldots, X_m \in \Gamma(TM)$ is a system of generators for the vector fields corresponding to the $c_n$-calculus, that is, b-vector fields satisfying $X_j(x^{-n+1}) \in C^\infty(M)$.

We proceed now to describe the regularizing operators in the $c_n$-calculi on a manifold with boundary $M$ in a way that is most convenient for our purposes.

Let $(A, \| \|_{j \in \mathbb{N}})$ be a submultiplicative Fréchet algebra. Assume there is given an action $\alpha : \mathbb{R} \to \text{Aut}(A)$ of $\mathbb{R}$ by automorphisms on $A$ (so $\alpha_t \circ \alpha_s = \alpha_{t+s}$). If, for any $a \in A$, the map $\mathbb{R} \ni t \mapsto \alpha_t(a)$ is smooth and there exist polynomials $P_n$, independent of $n$, such that

$$\|\alpha_t(a)\|_n \leq P_n(|t|)\|a\|_n, \tag{18}$$

then we say that the action of $\mathbb{R}$ on $A$ is with polynomial growth. In that case, we can introduce on the Schwartz space $\mathcal{S}(\mathbb{R}, A)$ an algebra structure by

$$f \ast g(t) = \int_\mathbb{R} f(s)\alpha_s(g(t-s)) ds.$$ 

Moreover, $\mathcal{S}(\mathbb{R}, A)$ acquires a family of seminorms $\| \|_{n, i, j}$,

$$\|f\|_{n, i, j} = \int_{t \in \mathbb{R}} \|t^i \partial^j f(t)\|_n dt,$$

making it a Fréchet algebra, that is submultiplicative with respect to an equivalent system of seminorms that we now define.

First, we can assume that $P_n(t) = C_n(1 + t^{M_n})$, for some $C_n > 0$ and $M_n \in \mathbb{N}$. Then, a direct verification using the binomial expansion of $s^i = (s - t + t)^i$ and the
submultiplicativity of the seminorm $||| \cdot |||$ of $A$ shows that

$$
||f * g||_{n,i,j} = \int ||t^i \partial_t^j \int f(s) \alpha_s (g(t - s))ds||_n dt
\leq C_n \sum_{\beta + \gamma = i} C^\beta_n (||f||_{n,\beta,0} + ||f||_{n,\beta + M_n,0})||g||_{n,\gamma,j}.
$$

(Here $C_n^k = a!b!^{-1}(a-b)!^{-1}$ are the binomial coefficients.)

Let us adjoin a unit denoted $e$ to $\mathcal{S}(\mathbb{R}, A)$ and extend the norms $|| \cdot ||_{n,i,j}$ to $\mathcal{S}(\mathbb{R}, A)^+: = \mathcal{S}(\mathbb{R}, A) \oplus \mathbb{C} e$ by $||(f, \lambda e)||_{n,i,j} := ||f||_{n,i,j} + |\lambda|$. Equation (19) implies that left multiplication by $f$ on $\mathcal{S}(\mathbb{R}, A)^+$ is continuous in the norm $\sum_{\alpha \leq i} || \cdot ||_{n,\alpha,j}$. The corresponding operator norm, $|||f|||_{n,i,j}$ is then submultiplicative, by definition and satisfies

$$
|||f|||_{n,i,j} \leq ||f||_{n,i,j} + K_{n,i,j} \sum_{l \leq i + M_n} ||f||_{n,l,0}.
$$

(The first term appears due to the fact that we have adjoined a unit to $\mathcal{S}(\mathbb{R}, A)$. Also, $K_{n,i,j}$ is a constant that is independent of $f$.) On the other hand,

$$
||f||_{n,i,j} = ||f * e||_{n,i,j} \leq |||f|||_{n,i,j} ||e||_{n,i,j} = |||f|||_{n,i,j},
$$

which proves that the families of seminorms $|| | \cdot ||_{n,i,j}$ and $|| \cdot ||_{n,i,j}$ define the same topology on $\mathcal{S}(\mathbb{R}, A)$.

Let $M$ be a compact manifold with boundary $\partial M$. On $\partial M \times [0, \infty)$ we consider the vector field $X_n = (1 + x^n)^{-1} x^n \partial_x$, with $x \in [0, \infty)$. Our choice of this vector field is justified by the fact that

$$
X_n(x^{-n+1}) = (-n + 1)(1 + x^n)^{-1} \in C^\infty([0, \infty)),
$$

and hence it defines a vector field corresponding to the $c_n$-calculus, $n \geq 2$. Moreover, this vector field defines, by integration, an action of $\mathbb{R}$ on $\partial M \times [0, \infty)$, which is trivial on $\partial M$.

Let $A := \mathcal{S}(\partial M \times \partial M \times [0, \infty))$, with product

$$
(fg)(m_1, m_2, t) = \int_{\partial M} f(m_1, m, t)g(m, m_2, t) dm.
$$

Then $A$ is isomorphic, as an algebra, with the complete projective tensor product $\Psi^{-\infty}(\partial M) \otimes_\mathbb{Z} \mathcal{S}([0, \infty))$.

**Lemma 9.** Assume $n \geq 2$. Then the action of $\mathbb{R}$ on $A := \mathcal{S}(\partial M \times \partial M \times [0, \infty))$ integrating the action of the vector field $X_n = (1 + x^n)^{-1} x^n \partial_x$ is with polynomial growth.

**Proof.** Let $S^0(\mathbb{R})$ be the space of classical symbols of order zero on $\mathbb{R}$, with its natural Fréchet space structure. Consider the function

$$
f_n(x) = (-n + 1)^{-1} x^{-n+1} + x : (0, \infty) \to \mathbb{R}.
$$

Then $f_n$ is a bijection such that the induced map

$$
f_n^* : \mathcal{S}([0, \infty)) \to \sum_{k=0}^{n-2} S^{-k/(n-1)}(\mathbb{R})
$$

is an equivariant map, i.e. the action of $\mathbb{R}$ on $\mathcal{S}(\mathbb{R})$ being by translation. Moreover $f_n^*$ is an isomorphism onto its image, which consists of the linear combinations of
symbols that are of order \(-\infty\) on the positive semi-axis \([0, \infty) \subset \mathbb{R}\). In particular, the image of \(f_n^A\) is closed.

Since the action of \(\mathbb{R}\) by translation on \(S^k(\mathbb{R})\) is with polynomial growth, the given action of \(\mathbb{R}\) on \(\mathcal{S}(\mathbb{R})\) is also with polynomial growth.

Probably the above lemma is the only reason why we have to restrict to \(n \geq 2\) in order to construct algebras with smooth kernels on the \(c_n\)-calculus groupoid \(\Gamma_n(M)\).

Fix a smooth function \(\phi \in C^\infty([0, \infty))\), \(\phi(x) = 1\) if \(x\) is in a certain neighborhood of \(0\), \(\phi(x) = 0\) if \(x \geq 1\), and let

\[ A_1 = \phi \mathcal{S}(\mathbb{R}, \mathcal{A}) \phi := \phi \mathcal{S}(\mathbb{R}, \mathcal{S}(\partial M \times \partial M \times [0, \infty))) \phi, \]

where \(\mathcal{A} := \mathcal{S}(\partial M \times \partial M \times [0, \infty))\), as in the above lemma. The algebra \(A_1\) acts on \(L^2(\partial M \times [0, \infty))\).

Let \(\Gamma_n(M)\) be the groupoid defining the \(c_n\)-calculus. Then \(\Psi^{-\infty}(\Gamma_n(M))\) identifies with an algebra of bounded operators on \(L^2(M)\) (with the canonical induced measure).

We are ready to prove the following theorem.

**Theorem 5.** Let \(M\) be a compact, smooth manifold with connected boundary. Then \(\mathfrak{A} = A_1 + I \subset C^*(\Gamma_n(M))\) and is a non-unital \(\Psi^*\)-algebra with smooth kernels.

**Proof.** We shall use the results of the previous sections. First, however, we have to prove that \(\mathfrak{A} := A_1 + I\) is an algebra.

Indeed, by Theorem \([3]\) it is enough to check that \(A_1 \subset \Psi^\infty(\mathcal{I})\), where \(\mathcal{I} = \{x^{-1}, X_0, \ldots, X_m\}\) is chosen as in Remark \([3]\). We identify a tubular neighborhood of \(\partial M\) with a subset of \(\partial M \times [0, \infty)\). To prove this, we first notice that \(X_0, \ldots, X_m \in \Psi^1(\Gamma_n(M)), A_1 \subset \Psi^{-\infty}(\Gamma_n(M)),\)

and hence any product of the form

\[ X_{i_1} \ldots X_{i_r} a X_{i_{r+1}} \ldots X_{i_r}, \]

consists of bounded operators. Then, let us write by \(a_f\) the operator of convolution on \(\partial M \times [0, \infty)\) with the function \(f \in \mathcal{S}(\mathbb{R})\) (the action of \(\mathbb{R}\) is the one integrating the vector field \(X_m = (1 + x^n)^{-1} x^r \partial_x\)). Also, let us observe that \(\mathcal{A} = \mathcal{S}(\partial M \times \partial M \times [0, \infty))\) identifies with an algebra of operators on \(L^2(\partial M \times [0, \infty))\). Then the map

\[ \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{A} \ni f \otimes b \to a_f b \in \mathcal{S}(\mathbb{R}, \mathcal{A}) \]

is an isomorphism. We need to check that \(\delta, \delta(a) := [x^{-1}, a]\) maps \(\mathcal{S}(\mathbb{R}, \mathcal{A})\) to itself.

Clearly \(\delta(b) = 0\), for any \(b \in \mathcal{A}\). If \(f \in \mathcal{S}(\mathbb{R})\), then \(\delta(f) \in \mathcal{S}(\mathbb{R}, \mathcal{A})\).

This proves our claim that \(\mathcal{S}(\mathbb{R}, \mathcal{A})\) is stable with respect to \(\delta\). In conclusion, \(A_1 \subset \Psi^\infty(\mathcal{I})\) and hence \(A_1 I + IA_1 \subset I\) (see Theorem \([3]\)).

The algebra \(\mathfrak{A}\) has a Fréchet topology induced from the Frechet topologies of \(A_1\) and \(I\). To prove that it is a \(\Psi^*\)-algebra, we thus only need to prove that it is spectrally invariant. To this end, we shall use the results of Theorem \([1]\). Let \(\mathcal{B} = C^*(\Gamma_n(M))\), regarded as a subalgebra of the algebra of bounded operators on \(L^2(M)\). Also, let \(\mathcal{J}\) be the norm closure of \(x\mathcal{B}\). Let \(\mathcal{K}\) be the algebra of compact operators on \(L^2(\partial M)\). Then \(\mathcal{B}/\mathcal{J} \simeq C_0(\mathbb{R}, \mathcal{K})\) by standard results on groupoid \(C^*\)-algebras. (This statement can also be extracted from either of \([24\, 23]\), by using, for example the exact sequence associated to the invariant subset \(\partial M \times \{0\} \subset \partial M \times [0, \infty)\).) Let \(\varphi : \mathfrak{A}/I \to \mathcal{B}/\mathcal{J}\) be the induced morphism. Then the range of
this morphism is $\mathcal{S}(\mathbb{R}, C^\infty(\partial M \times \partial M)) \subset B/\mathcal{F} \simeq C_0(\mathbb{R}, K)$ and hence $\varphi$ is locally spectral invariant.

Theorem 3 or, more precisely Corollary 3, then shows that $\mathcal{A}$ is spectrally invariant. This completes the proof. 

We are planning to clarify the relation between our algebra $\mathcal{A}$ for the $\epsilon_2$-calculus and the cusp-calculus as defined by Melrose (an exposition can be found in [33]) in another paper.

6. The Schwartz space of a continuous family groupoid

In this section we define a notion of Schwartz space on a continuous family groupoid $G$, i.e. a space of rapidly decreasing functions as well as their derivatives. This was introduced in [34] in the case of differentiable groupoids. We prove, using also some methods introduced in [18], that this is a subalgebra of $C^*(G)$ stable under holomorphic functional calculus.

6.1. The Schwartz convolution algebra. Let $G$ be a Hausdorff, continuous family groupoid. Fix a 1-density on $A$. This will then give rise to a 1-density on each of the manifolds $G_x$, and hence to a smooth measure $\mu_x$ on each of $G_x$. Moreover, the measures $\mu_x$ are invariant with respect to right translations, and hence they form a Haar system.

**Definition 8.** Let $\mu$ be the Haar system on $G$ introduced above. A length function with polynomial growth on $G$ is a continuous function $\phi : G \to \mathbb{R}_+$ such that:

(a) $\phi(g_1g_2) \leq \phi(g_1) + \phi(g_2),$
(b) $\forall g \in G, \phi(g^{-1}) = \phi(g),$
(c) $\phi$ is proper,
(d) $\exists c, N, \forall x \in G^{(0)}, \forall r \in \mathbb{R}_+, \mu_x(\phi^{-1}([0, r]) \leq c(r^N + 1).$

The polynomial growth condition ensures that there exists $k_0 \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for any $k \geq k_0$, and for any $x \in G^{(0)},$

$$\int_{G_x} (1 + \phi(x))^k d\mu_x \leq C.$$ 

Let $v$ be a section of the Lie algebroid of $G$, $A(G)$; such a section defines a differential operator of order 1 on $G$. Thus if $v_1, \ldots, v_l$ are sections of $A(G)$, and if $f \in C_0(G, \Omega^\sharp)$, then $v_1 \ldots v_k \cdot f \cdot v_{k+1} \ldots v_l$ is a distribution on $G$. It belongs to $C_0(G, \Omega^\sharp)$ provided that there exists $g \in C_0(G, \Omega^\sharp)$ such that, for any $a \in C^\infty_0(G)$, $g \cdot a = (v_1 \ldots v_k \cdot f \cdot v_{k+1} \ldots v_l) \cdot a$.

**Definition 9.** Let $G$ be a continuous family groupoid and $\phi$ be a length function with polynomial growth on $G$. Define $\mathcal{S}^{k,d}(G, \phi) \equiv \{ f \in C_0(G, \Omega^\sharp), \forall v_1, \ldots, v_d \in C(A(G)), \forall i \leq d, v_1 \ldots v_i \cdot f \cdot v_{i+1} \ldots v_d \in C_0(G, \Omega^\sharp) \}$

and $\sup_{g \in G} |v_1 \ldots v_i \cdot f \cdot v_{i+1} \ldots v_d(g)| (1 + \phi(g))^k < \infty$. 

The Schwartz space of $G$ with respect to $\phi$ is $\mathcal{S}(G, \phi) = \bigcap_{k,d \in \mathbb{N}} \mathcal{S}^{k,d}(G, \phi).$
The space $\mathcal{S}^{k,d}(G, \phi)$ can be endowed with the norm

$$
\|f\|_{k,d} = \sup_{i \leq j \leq k} \sup_{v_i, v_j \in C(A(G))} \sup_{g \in V} \sup_{\|v_i\| \leq 1} \sup_{\|v_j\| \leq 1} |v_i \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l(g)(1 + \phi(g))^k.
$$

**Remark 8.**
- If $l \geq k$ and $f \in \mathcal{S}^{l,d}(G, \phi)$ then $f \in \mathcal{S}^{k,d}(G, \phi)$ and $\|f\|_{k,d} \leq \|f\|_{l,d}$.
- If $f \in \mathcal{S}^{k,d}(G, \phi)$ and $k \geq k_0$ then for any $x \in G^{(0)}$ and any $v_1, \ldots, v_l \in C(A(G))$ (with $l \leq d$), one has

$$
\int_{G_x} |v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l(g)|^2 \leq \|f\|_{k,d}^2 \int_{G_x} \frac{1}{(1 + \phi(g))^{2k}} d\mu_x
$$

so that $v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l \in L^2(G_x)$, and $\|v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l\|_{L^2(G_x)} \leq \sqrt{C} \|f\|_{k,d}$.

**Proposition 6.** For any $k, d \in \mathbb{N}$, with $k$ such that $\int_{G_x} \frac{1}{(1 + \phi(g))^{2k}} d\mu_x$ converges for any $x \in G^{(0)}$, $\mathcal{S}^{k,d}(G, \phi)$ is a dense subalgebra of $C^r_c(G)$, and there exists a constant $\lambda_k$ such that $\|\cdot\|_{C^r_c(G)} \leq \|\cdot\|_{k,d}$.

**Proof.** If $f_1, f_2 \in \mathcal{S}^{k,d}(G, \phi)$. Let $x = d(g)$, we then have

$$
|v_1 \cdots v_i \cdot f_1 \ast f_2 \cdot v_{i+1} \cdots v_l(g)| = \int_{G_x} |v_1 \cdots v_i \cdot f_1(gg^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g')| d\mu_x
$$

$$
\leq \int_{G_x} \sup_{\phi(g') \geq 2^{2k}} |v_1 \cdots v_i \cdot f_1(gg^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g')| d\mu_x
$$

$$
+ \int_{G_x} \sup_{\phi(gg'^{-1}) \geq 2^{2k}} |v_1 \cdots v_i \cdot f_1(gg'^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g')| d\mu_x
$$

$$
\leq \int_{G_x} \sup_{\phi(g') \geq 2^{2k}} \|f_1\|_{k,d} \frac{1}{(1 + \phi(g'))^{2k}} \|f_2\|_{k,d} \frac{1}{(1 + \phi(g'))^{2k}} d\mu_x
$$

$$
+ \int_{G_x} \sup_{\phi(gg'^{-1}) \geq 2^{2k}} \|f_1\|_{k,d} \frac{1}{(1 + \phi(gg'^{-1}))^{2k}} \|f_2\|_{k,d} \frac{1}{(1 + \phi(g'))^{2k}} d\mu_x
$$

$$
\leq \frac{2^k}{(1 + \phi(g))^{2k}} \|f_1\|_{k,d} \|f_2\|_{k,d} \left( \int_{G_x} \frac{1}{(1 + \phi(gg^{-1}))^{2k}} + \int_{G_x} \frac{1}{(1 + \phi(g'))^{2k}} \right) d\mu_x
$$

$$
\leq \frac{2^{k+1} C}{(1 + \phi(g))^{2k}} \|f_1\|_{k,d} \|f_2\|_{k,d}
$$

so that

$$
(21) \quad |v_1 \cdots v_i \cdot f_1 \ast f_2 \cdot v_{i+1} \cdots v_l(g)(1 + \phi(g))^k \leq 2^{k+1} C \|f_1\|_{k,d} \|f_2\|_{k,d}
$$

which implies that $f_1 \ast f_2 \in \mathcal{S}^{k,d}(G, \phi)$. Consequently, $\mathcal{S}^{k,d}(G, \phi)$ is an algebra.

To prove that $\mathcal{S}^{k,d}(G, \phi)$ is a subspace of $C^r_c(G)$, we need to show that if $f \in \mathcal{S}^{k,d}(G, \phi)$, and for any $x \in G^{(0)}$ and $\xi \in C_c(G_x)$, one has

$$
\|f \ast \xi\|_{L^2(G_x)} \leq \|f\|_{k,d} \|\xi\|_{L^2(G_x)}
$$
up to a constant.

Denote by \( \lambda_k \) the constant \( 2^{k+1}C \). Then the Cauchy-Schwarz inequality implies that
\[
\| f \xi \|_{L^2(G_x)}^2 \leq \| \xi \|_{L^2(G_x)} \| (f^* f) \xi \|_{L^2(G_x)}.
\]

By induction,
\[
\| f \xi \|_{L^2(G_x)}^{2^{n+1}} \leq \| \xi \|_{L^2(G_x)}^{2^{n+1}-1} \| (f^* f)^{2^n} \xi \|_{L^2(G_x)}.
\]

But if \( h \in \mathcal{S}^{k,d}(G, \phi) \),
\[
\| h \xi \|_{L^2(G_x)}^2 = \int_{G_x} (h(gg') \xi g')^2 d\mu_x \leq \| h \|_{k,d}^2 \int_{G_x} \left( \frac{\xi g'}{(1 + \phi(gg')^k)k} \right)^2 d\mu_x
\]

The inequality (21) also gives
\[
\| (f^* f)^{2^n} \|_{k,d} \leq \lambda_k^{2^{n+1}} \| f \|_{k,d}^{2^{n+1}},
\]
so that
\[
\| f \xi \|_{L^2(G_x)}^{2^{n+1}} \leq \| \xi \|_{L^2(G_x)}^{2^{n+1}-1} \lambda_k^{2^{n+1}} \| f \|_{k,d}^{2^{n+1}} \left( \int_{G_x} \frac{\xi g'}{(1 + \phi(gg')^k)k} \right)^2 d\mu_x.
\]

Taking \( n \to \infty \),
\[
\| f \xi \|_{L^2(G_x)} \leq \| \xi \|_{L^2(G_x)} \lambda_k \| f \|_{k,d},
\]

which implies that \( f \in C^*_r(G) \), and \( \| f \|_{C^*_r(G)} \leq \lambda_k \| f \|_{k,d} \).

6.2. The theorem. We are now ready to prove the main result of this section, the fact that the algebras \( \mathcal{S}(G, \pi) \) are closed under holomorphic functional calculus.

**Theorem 6.** The Schwartz space of \( G \) with respect to \( \phi \), \( \mathcal{S}(G, \phi) \), is closed under holomorphic functional calculus in \( C^*_r(G) \).

**Proof.** The methods of [13] extend without difficulty to our case.

As above, let \( k_0 \) be such that \( \int_{G_x} \frac{1}{(1 + \phi(g))^{k_0}} \) converges for any \( x \in G^{(0)} \).

**Lemma 10.** If \( l \geq k \geq k_0 \) then \( \mathcal{S}^{l,d}(G, \phi) \) is stable under holomorphic calculus in \( \mathcal{S}^{k,d}(G, \phi) \).

**Proof.** To prove this, by Lemma [13] we will show that
\[
\lim_{n \to \infty} \| f^n \|_{k,d}^{\frac{1}{k}} = \lim_{n \to \infty} \| f^n \|_{l,d}^{\frac{1}{l}}.
\]

Indeed, this implies that \( \mathcal{S}^{l,d}(G, \phi) \), which is a dense subalgebra of \( \mathcal{S}^{k,d}(G, \phi) \), is also full.
Now, as above, we have
\[ |v_1 \cdots v_i \cdot f^n \cdot v_{i+1} \cdots v_l(g)| \]
\[ = \left| \int_{g_1 \cdots g_n = g} (v_1 \cdots v_i \cdot f)(g_1)f(g_2) \cdots f(g_{n-1})(f \cdot v_{i+1} \cdots v_l)(g_n) \right| \]
\[ \leq \sum_{i=1}^{n} \int_{g_1 \cdots g_n = g} |v_1 \cdots v_i \cdot f(g_1)||f(g_2)||f(g_{n-1})||f \cdot v_{i+1} \cdots v_l(g_n)| \]
\[ \leq \sum_{i=1}^{n} \|f\|_{k,d}^{n-1} \cdot \|f\|_{l,d} \int_{g_1 \cdots g_n = g} \left( \frac{1}{1 + \phi(g_i)} \right)^{\frac{1}{n}} \prod_{j \neq i} \left( 1 + \phi(g_j) \right)^{\frac{1}{k}} \]
\[ \leq \|f\|_{k,d}^{n-1} \cdot \|f\|_{l,d} \left( \frac{1}{1 + \phi(g_i)} \right)^{\frac{1}{n}} \sum_{i=1}^{n} \int_{g_1 \cdots g_n = g} \left( \frac{1}{1 + \phi(g_i)} \right)^{\frac{1}{k}} \prod_{j \neq i} \left( 1 + \phi(g_j) \right)^{\frac{1}{k}} \]

As the latter integrals are lower than C,
\[ |f^n(g)|(1 + \phi(g))^l \leq \|f\|_{k,d}^{n-1} \cdot \|f\|_{l,d} \cdot nC \]

which gives
\[ \|f^n\|_{l,d}^\frac{1}{n} \leq n^{\frac{1}{n}} C \cdot \|f\|_{l,d} \]

thus
\[ \lim_{n \to \infty} \|f^n\|_{l,d}^\frac{1}{n} \leq \|f\|_{l,d} \]

Let's now apply this inequality to \( f^m \), we get
\[ \lim_{n \to \infty} \|f^{mn}\|_{l,d}^\frac{1}{n} \leq \|f^m\|_{l,d} \]

so that if \( m \to \infty \),
\[ \lim_{n \to \infty} \|f^m\|_{l,d} = \lim_{n, m \to \infty} \|f^{mn}\|_{l,d}^\frac{1}{n} \leq \lim_{m \to \infty} \|f^m\|_{l,d} \]

Because of \( \|f^m\|_{l,d} \leq \|f^m\|_{l,d} \) we get \( \lim_{n \to \infty} \|f^n\|_{l,d}^\frac{1}{n} = \lim_{n \to \infty} \|f^n\|_{l,d}^\frac{1}{n} \).

Lemma 11. If \( k \geq k_0 \), one has
\[ \mathcal{S}^{k,d}(\mathcal{G}, \phi) * C^*(\mathcal{G}) * \mathcal{S}^{k,d}(\mathcal{G}, \phi) \subset \mathcal{S}^{0,d}(\mathcal{G}, \phi) \]

Proof. If \( f_1, f_2 \in \mathcal{S}^{k,d}(\mathcal{G}, \phi) \) and \( f \in C^*(\mathcal{G}) \), then
\[ |v_1 \cdots v_i \cdot f_1 \ast f_2 \cdot v_{i+1} \cdots v_l(g)| = \left| \int_{\mathcal{G}} (v_1 \cdots v_i \cdot f_1)(g')(f_2 \cdot v_{i+1} \cdots v_l(g')) \right| \]

But if we denote by \( f_3 \) the function \( g' \mapsto (v_1 \cdots v_i \cdot f_1)(g') \), then the Cauchy-Schwarz inequality gives
\[ \|v_1 \cdots v_i \cdot f_1 \ast f_2 \cdot v_{i+1} \cdots v_l(g)\| \leq \|f_3\|_{L^2(\mathcal{G})} \|f_2 \cdot v_{i+1} \cdots v_l\|_{L^2(\mathcal{G})} \]
\[ \leq C \|f_3\|_{k,d} \|f_2\|_{C^0(\mathcal{G})} \|f_2\|_{L^2(\mathcal{G})} \]
\[ \leq C \|f_3\|_{k,d} \|f\|_{C^0(\mathcal{G})} C \|f_2\|_{k,d} \]

and hence \( f_1 \ast f_2 \in \mathcal{S}^{0,d}(\mathcal{G}, \phi) \).

Lemma 12. If \( k \geq k_0 \), then \( \mathcal{S}^{k,d}(\mathcal{G}, \phi) \) is stable under holomorphic functional calculus in \( C^*_r(\mathcal{G}) \).
Proof. If \( f \in \mathcal{S}^{k,k}(G, \phi) \), then the lemma \([1]\) implies that
\[
\|
f^n\|_{0,d} \leq C^2 \|f\|_{k,d} \|f^{n-2}\|_{C^*_{1}(G)} \|f\|_{k,d}
\]
thus
\[
\lim_{n \to \infty} \|f^n\|^{1/n}_{0,d} = \lim_{n \to \infty} \|f^n\|^{1/n}_{C^*_{1}(G)}
\]
(the inverse inequality is given by Proposition \([3]\)). To prove this lemma, it remains to show that
\[
\lim_{n \to \infty} \|f^n\|^{1/n}_{0,d} = \lim_{n \to \infty} \|f^n\|^{1/n}_{k,d}.
\]

If \( f_1, f_2 \in \mathcal{S}^{2k,d}(G, \phi) \), then
\[
|v_1 \ldots v_i \cdot f_1 * f_2 \cdot v_{i+1} \ldots v_l(g)| \leq \int_{\mathcal{G}_x} |v_1 \ldots v_i \cdot f_1(gg')| |f_2 \cdot v_{i+1} \ldots v_l(g')|
\]
\[
\leq \int_{\mathcal{G}_x} |f_1|^{2k,d} \|f_2\|_{0,d} \frac{(1 + \phi(gg'))^k}{(1 + \phi(g))^k}
\]
\[
+ \int_{\mathcal{G}_x} |f_1|^{0,d} |f_2|^{2k,d} \frac{(1 + \phi(g'))^k}{(1 + \phi(g))^k}
\]
\[
\leq C \frac{(1 + \phi(g))^k}{(1 + \phi(g'))^k} \|f_1\|_{0,d} \|f_2\|_{2k,d} + \|f_1\|_{2k,d} \|f_2\|_{0,d}
\]
hence \( \|f_1 f_2\|_{k,d} \leq C(\|f_1\|_{0,d} \|f_2\|_{2k,d} + \|f_1\|_{2k,d} \|f_2\|_{0,d}). \)

Applying this to \( f_1 = f_2 = f \), we get
\[
\|f^{2n}\|_{k,d} \leq 2C \|f\|_{0,d} \|f\|_{2k,d}.
\]

But by Lemma \([10]\),
\[
\lim_{n \to \infty} \|f^n\|^{1/n}_{2k,d} = \lim_{n \to \infty} \|f^n\|^{1/n}_{k,d},
\]
so that
\[
\lim_{n \to \infty} \|f^n\|^{1/n}_{k,d} \leq \lim_{n \to \infty} \|f^n\|^{1/n}_{0,d},
\]
which gives an equality as the opposite inequality comes from the Remark \([3]\). \( \square \)

We are ready now to complete the proof of Theorem \([1]\). The Schwartz space of \( G \) with respect to \( \phi \) is thus an intersection of subalgebras of \( C^*_{1}(G) \) which are stable under holomorphic functional calculus, hence \( \mathcal{S}(G, \phi) \) is stable under holomorphic functional calculus in \( C^*_{1}(G) \). \( \square \)

**Corollary 3.** Let \( G \) be a continuous family groupoid, and \( \phi \) be a length function with polynomial growth. Denote by \( \Psi^0_s(G) \) the sum of \( \Psi^0(G) \) and of the Schwartz space of \( G \) with respect to \( \phi \), \( \mathcal{S}(G, \phi) \). Then \( \Psi^0_s(G) \) is stable under holomorphic functional calculus.

Moreover, if \( P \in \Psi^0_s(G) \) is Fredholm, then it admits a parametrix \( Q \in \Psi^0_s(G) \).

**Proof.**

Let us first show that \( \Psi^0_s(G) \) is an algebra, which amounts to prove that if \( k_1, k_2 \in \mathcal{S}^{m,0}(G, \mathcal{G}^{(0)}) \) and \( f \in \mathcal{S}(G, \phi) \) then \( k_1 * f * k_2 \in \mathcal{S}(G, \phi) \).

The Lie algebroid of \( G \) being a \( C^\infty,0 \) fiber bundle, it is possible to choose a set \( v_1, \ldots, v_N \in C(A(G)) \) such that for any \( x \in G^{(0)} \), \( (v_1(x), \ldots, v_n(x)) \) generates
$T_2\mathcal{G}_2$. Now let $\Delta = \sum_{i=1}^{N} v_i^2$. This differential operator is of degree 2 and admits a parametrix.

If $k_1, k_2 \in I^m_{c,0}(\mathcal{G}, \mathcal{G}^{(0)})$ then for any $l$ there exist $k_1', k_2' \in I^{m-2l,0}_{c,0}(\mathcal{G}, \mathcal{G}^{(0)})$ and $r_1, r_2 \in C^2_c(\mathcal{G})$ such that $k_1 = k_1' * k_\Delta + r_1, k_2 = k_2' * k_\Delta + r_2$. But when $k \in I^{N-1,0}_{c,0}(\mathcal{G}, \mathcal{G}^{(0)})$, it is given by an absolutely convergent oscillatory integral, thus it belongs to $C^2_c(\mathcal{G})$. Hence if $l$ is large enough, $k_1'$ and $k_2'$ belong to $\mathcal{S}(\mathcal{G})$. But $k_1 * f * k_2 = k_1' * k_\Delta * f * k_\Delta * k_2' + k_1' * k_\Delta * f * r_2 + r_1 * f * k_\Delta * k_2' + r_1 * f * r_2$ so that $k_1 * f * k_2 \in \mathcal{S}(\mathcal{G})$ since $k_\Delta * f * k_\Delta \in \mathcal{S}(\mathcal{G})$ by definition and $\mathcal{S}(\mathcal{G})$ is an algebra.

The first part of this result is then a direct consequence of Theorem 4.

Consider the exact sequence

$$0 \to \mathcal{K} \to \mathfrak{A}(\mathcal{G}) \to \mathfrak{A}(\mathcal{G})/\mathcal{K} \to 0.$$ 

If $P \in \Psi^0_*(\mathcal{G})$ is Fredholm, then its image in $\mathfrak{A}(\mathcal{G})/\mathcal{K}$ is invertible, thus, Theorem 1 implies that its inverse lives in $\Psi^0_*(\mathcal{G})/(\Psi^0_*(\mathcal{G}) \cap \mathcal{K})$. This shows that there exists a parametrix in $\Psi^0_*(\mathcal{G})$.

As P. Piazza has kindly informed us, Schwartz-type spaces seem to be important also for the questions related to higher-signatures that were considered in [25, 26].

6.3. **Schwartz spaces associated to the $c_n$ groupoids.** The continuous family groupoids defined in Section 4 can be endowed with length functions in order to define, for each of them, a Schwartz space, thus to obtain an algebra of pseudodifferential operators closed under holomorphic functional calculus. To keep notations simple, we only consider here the case of a compact manifold with boundary. In the case of the $b$-calculus, we have

$$\Gamma(X) = \{(x, y, \lambda) \in X \times X \times \mathbb{R}_+^*, \rho(x) = \lambda \rho(y)\}.$$ 

Define $\phi(x, y, \lambda) = |\log(\lambda)|$. It was shown in [33] that this defines a length function.

Before considering the case of the $c_n$-calculus, we need an easy lemma:

**Lemma 13.** Let $\mathcal{G}$ and $\mathcal{G}'$ be two homeomorphic continuous family groupoids. Then any length function with polynomial growth on $\mathcal{G}$ induces such a function on $\mathcal{G}'$.

**Proof.** Denote by $f : \mathcal{G}' \to \mathcal{G}$ the homeomorphism, and assume that $\mathcal{G}$ has a length function with polynomial growth, $\phi$. Let $\phi' = \phi \circ f : \mathcal{G}' \to \mathbb{R}_+$. This function is clearly a length function with polynomial growth, since $f$ is a homeomorphism, and the Haar system on $\mathcal{G}'$ is induced by that on $\mathcal{G}$.

Recall from Section 4 that there is a homeomorphism $\Theta_n : \Gamma(X) \to \Gamma_n(X)$ where $\Gamma(X)$ is the groupoid of the $b$-calculus, and $\Gamma_n(X)$ is the groupoid of the $c_n$-calculus.

**Proposition 7.** Let $\phi_n = \phi \circ \Theta_n^{-1}$. Then $\phi_n$ is a length function. Under the identification

$$\Gamma_{n+1}(X) = \{(u, v, \mu) \in X \times X \times \mathbb{R}, \mu \rho(u)^n \rho(v)^n = \rho(u)^n - \rho(v)^n\},$$

one gets $\phi_n(u, v, \mu) = |\mu|$.

**Proof.** This is clear since $\Theta_n(x, y, \lambda) = (u, v, \log(\lambda))$. 

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Universität Mainz, Fachbereich 17-Mathematik, D-55099 Mainz, Germany
E-mail address: lauter@mathematik.uni-mainz.de

Laboratoire Emile Picard, Université Paul Sabatier (UFR MIG), 118 route de Narbonne, F-31062 Toulouse CEDEX 4
E-mail address: monthube@picard.ups-tlse.fr

Pennsylvania State University, Math. Dept., University Park, PA 16802
E-mail address: nistor@math.psu.edu